SOME ASPECTS OF THE THEORY OF DISORDERED SOLIDS

by

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ABSTRACT

We present here some contributions to the theory of disordered solids. More precisely, we investigate the effects of disorder on the normal modes of certain model Hamiltonians. We ask (and partially answer) such questions as whether the normal modes are localised within a finite region of the solid or not, and whether the damping of the modes caused by the disorder can be found.

A review of concepts and models in current use in the theory of disordered solids is presented, and a new model for a spin glass is introduced. Then work on the calculation of the dispersion relation of the Mattis-Heisenberg model in the R.P.A. for various strengths of the disorder is presented. The damping of the long-wavelength modes is found for low concentrations of antiferromagnetic impurities and for the case when there is no overall magnetisation. This is followed by a contribution to the precise characterisation of localised states and localisation edges for a large class of self-adjoint operators. We discuss the possibility of the existence of a singular continuous part of the spectrum of the operator and the physical significance of this for stochastic Hamiltonians.

The self-consistent theory of localisation of Abou-Chacra et al is extended to investigate the localisation of magnons. The success of the adaptation is partial, but a demonstration that the magnons of the Mattis-Heisenberg model in one dimension are all localised for arbitrarily small disorder is found.

Finally, results of computer calculations of the localisation properties of the magnon modes in disordered magnets are presented. This is followed by a conclusion which draws together some of the themes developed in the previous chapters.

à Marie-Ermeline

PREFACE

The work presented here was performed between October 1977 and May 1980 while the author was a Ph.D. student in the solid state theory group at Imperial College of Science and Technology. At the same time the author was employed by the Institut Laue-Langevin in Grenoble in the Theoretical Physics group.

The work presented in this thesis is that of the author except where specific references are made to the work of others.

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CHAPTER 1

INTRODUCTION

In this thesis we present an account of contributions made to the theory of disordered solids. The physics of disordered solids has been of wide interest, and much progress in our understanding of effects and mechanisms peculiar to disorder has been made recently. (For reviews see Thouless [1], Weaire and Kramer [2] and K. Binder [3] and [4]).

The motivation for much of the theoretical work has come from the insight of P.W. Anderson, who introduced the concept of disorder causing the localisation of excitations [5] and who, with S.F. Edwards, introduced a model of a spin-glass [6] which has become the accepted model in this discipline. The concept of localisation is a central concern of this thesis : in Chapter 5 we advance a characterisation of localised and extended states, in Chapter 6 we investigate the use of the Bethe lattice method introduced by Abou-Chacra, Anderson and Thouless [7] for the problem of magnon localisation in disordered magnets, and in Chapter 7 results of computer calculations on disordered magnets are presented. The work on spin glasses has led to interest in various random Heisenberg models. In particular the Mattis-Heisenberg model (Sherrington [8], [9], [10] and Chapter 2 § 4) is studied closely in this thesis.

The question of central interest in the spin glass problem is : what is the nature of the spin glass phase transition? This is a lively but still controversial field with many differing speculations put forward supported by a greater or lesser amount of analysis. As a contribution to the understanding of the Edwards-Anderson model for a spin-glass [6], Sherrington and Kirkpatrick proposed a soluble model of a spin glass which is an infinite range Edwards-Anderson-Ising model [11] and [12].

The Sherrington Kirkpatrick model has so far defied all attempts at a solution except at high temperatures where Thouless, Anderson and Palmer [13] showed that the Sherrington Kirkpatrick analysis is correct. There have been many attempts to either solve or explain certain properties of this model, Sommers [14], Bray and Moore [15] and Parisi [16] among others. The phase transition controversy will not, however, be contributed to in this thesis, we shall be interested exclusively in the properties of the magnons in the non-interacting approximation, that is, elementary excitations from the ground state.

As we mentioned above, properties of the Mattis-Heisenberg model will be of central concern here. As this model has its thermodynamic properties given by those of a Heisenberg ferromagnet (Chapter 2, §4), we must go to the theory of the magnons to find a problem of interest because of the disorder. Chapters 3 and 4 are exclusively concerned with the Mattis-Heisenberg model. In these chapters the theory of the long-wavelength dynamic response of the Model is developed. This subject has been a major concern of Sherrington [8], [9] and [10], who has developed most of the theory of the excitations in this model. The reason for studying the Mattis-Heisenberg model, rather than the Edwards-Anderson model, is that it has a well defined classical ground state, in contrast with the Edwards-Anderson model, about which the elementary excitations can be treated in the harmonic approximation. Chapters 6 and 7 have sections concerned with the localisation of Mattis magnons. In Chapter 6 we use the Bethe lattice approximation as developed by Abou-Chacra et al [7], [17] and [18]. In Chapter 7 results of computer calculations are reported. Although the Mattis-Heisenberg model is much simpler to treat than the Edwards-Anderson model it does furnish us with an example of a disordered Heisenberg magnet whose excitations can be successfully analysed. It is hoped that it may

give some inspiration when dealing with more difficult models.

As was mentioned above, Chapter 5 is concerned with the characterisation of localised and extended states as they appear in the physics of disordered solids. The treatment is very general, as we often make statements which are true for a wide class of self-adjoint operators. However, a demonstration is given (as a proof of relevance) that the Anderson model for the localisation problem [5] is, in particular, an operator for which the R.A.G. theorem holds (Chapter 5, § 4, the R.A.G. theorem). This leads to many results for the rigorous theory of localisation. A distinct difference in style is noticeable between Chapter 5 and the rest of this thesis. This is due to the use of Functional Analysis. This may be unfortunate for the unity of presentation, but the insight into the physics of localisation achieved by this style of presentation is hoped to compensate for this.

A final word about the overall attitude that is adopted in this thesis. It could be summed up by the phrase "model physics", that is, we are concerned with the properties of certain model Hamiltonians (see Chapter 2) with which we isolate a property and look at its effect. Usually this property is disorder, and we are looking at its effect on the spectrum of the magnons of the Heisenberg model. Direct applications to experimental measurements and materials are not discussed.

CHAPTER 2

CONCEPTS AND MODELS USED IN THE THEORY OF DISORDERED SOLIDS

§ 1. Introduction

In this chapter we shall introduce the various concepts used throughout this thesis. These are concepts which have become central to the understanding of disordered solids. The concepts can be illustrated by simple models, the physics of which is thought to be dominated by one of the concepts. For example, the study of the Edwards-Anderson model of a spin-glass gave birth to the concept of frustration [19] . This frustration is a consequence of competing interactions which cannot all be satisfied. This seems to be what destroys the singlet ground state of the ordered Heisenberg magnet and creates a large number of low energy long lived metastable states [20] .

Although frustration was developed from the study of the Edwards-Anderson model it is now often considered as a condition for a disordered magnet to be a "good" model for a spin-glass. Thus, we have the situation where the Mattis model [21] and the Mattis-Heisenberg model (Sherrington [8] and § 6) are not considered to be good models for spin glasses because there is no frustration. On the other hand, we have models which have too much frustration to be considered as realistic models of a spin-glass. An example of this is the Villain model [22] which has every plaquette frustrated (see § 3).

Another concept which shall be of greater importance than frustration in this thesis is localisation. This concept is due to Anderson [5] and work on the subject of the localisation of electron states in a disordered solid (similarly for magnons and phonons, but to a lesser degree) has occupied a considerable number of people since 1958.

The concept of localisation is simple to imagine: a particle moving in a random potential may become trapped within a finite region of space by the barriers caused by the disorder. Localisation will be discussed in § 2.

§§ 4, 5, 6, 7 and 8 will each be concerned with the definition of a model Hamiltonian. A review of the work done previously on these models will not be given. Properties of the models which are of relevance for our purposes will be discussed, however.

§ 2. Localisation and Band Broadening

Localisation theory is an extremely active branch of the theory of disordered solids. The concept was introduced into solid state theory by Anderson [5] . For a review of progress made since that time see Weaire and Kramer [2] or, for a more mathematical approach, see Kunz and Souillard [23] .

The notion is a simple one. Firstly, consider a particle moving in a periodic potential. We know from Bloch's theorem that the particle states are running waves; we say that such states are extended. Now, consider turning on the disorder (for example, introduce impurities which break the crystalline symmetry). It is the contention of Anderson that once the disorder becomes sufficiently strong then all the states of the model are localised.

Given a selfadjoint operator with extended states it is known from a deep theorem of Weyl and von Neumann [24] p. 523 that one can always find a small perturbation which makes all the states localised (this is further discussed in Chapter 5).

So, we consider a particle which moves in a potential V. The formal Hamiltonian is written as

$$H = -\Delta + V(W) \qquad (1)$$

where W is a measure of the randomness. W is, usually, the width of the distribution density of the associated stochastic process. We make the choice that for W = 0, V (W = 0) is a periodic potential. We ask, what happens as we increase W from zero?

If we consider only one band of the spectrum of H, we shall be able to discuss the effects of localisation and band broadening with reference to figure 1.

The number of exact results are small in localisation theory and restricted to one dimension where it is known that all the states are localised no matter how small the disorder is, with the exception of zero disorder. This result was put forward as a conjecture by Mott and Twose [25] but was proved only recently by Goldsheid, Molchanov and Pastur [26], 1977. There are more results available on the spectrum; Kunz and Souillard [23] have found the spectrum of the finite difference operator, which is an approximation to (1), in arbitrary positive integer dimension.

An effect that is common to many stochastic one particle Hamiltonians is band broadening. It is known that the band broadens with increasing W. So if we compare fig. 1(b) with fig. 1(a) we see that the band broadens by an amount proportional to W (see Chapter 5, § 5 for a deeper discussion), also we notice that the states associated with the band edges become localised. In fig. 1(b) we see two localisation edges. The localisation edge is the boundary between those energies for which the states are localised and those energies for which the states are extended.

Fig. 1(c) shows the band when the disorder is strong enough to localise all states. The minimum value of W for which all states are localised is called the critical value of W for the Anderson transition.

We must stress that we have taken the simplest example for the illustration of the effects in this section, more complicated behaviour in the band can be imagined and is known to exist for certain binary alloys, see Abou-Chacra and Thouless [17].

§ 3. Frustration

The concept of frustration has been developed chiefly by G. Toulouse, [27] and [28]. This concept has become central to the qualitative and quantitative

understanding of spin glass behaviour. Frustration is most easily discussed with reference to an Ising model defined on a simple square lattice, but the only requirement for the frustration effect is competing exchange interactions. Consider the Ising model illustrated in fig. 2(a). We write the Hamiltonian as

$$H = -J_{12} S_1 S_2 - J_{23} S_2 S_3 - J_{34} S_3 S_4 + J_{41} S_4 S_1 \qquad (2)$$

We can see from equation (2) and fig. 2(a) that for any arrangement of the spins there will always be one bond unsatisfied.

Illustrated in figure 2(b) and (c) we have two triangles which are frustrated. Now, we can obtain lattices from the plaquettes shown in figure 2 and we can easily write down an Ising Hamiltonian which has a certain density of frustrated plaquettes. The usefulness of the concept of a density of frustrated plaquettes was demonstrated by Toulouse and Vannimenus [29] who performed numerical calculations to find some ground state properties of the two dimensional Edwards-Anderson-Ising model.

The effects of frustration without disorder were investigated long before the work of Toulouse. If fig. 2(b) is taken as an elementary plaquette for the two dimensional anti-ferromagnetic Ising model defined on a triangular lattice, then the work of Newell and Montroll [30] shows that there is no finite temperature phase transition for this model. Fig. 2(c), when generalised to a lattice, gives the Union Jack lattice considered by Vaks, Larken and Ovchinnikov [31] . Villain [22] introduced a model of a spin glass in which the disorder could be transformed away (as far as the thermodynamics is concerned), leaving an Ising model defined on a simple square lattice with every plaquette frustrated. This model was shown, by Villain, to be equivalent to the Dimer problem [32] which is known to have no phase transition in two dimensions [33] .

From the cases of the Villain model and the antiferromagnet on a triangular

lattice we can see that frustration has the effect of suppressing the phase transition. It must be remembered that these models have every plaquette frustrated; such total frustration cannot be obtained for an Edwards-Anderson model with the disorder governed by the distribution density given by equation (11).

So, in two dimensions, we have two soluble spin-glasses. For both of them the disorder can be transformed away; the Mattis model is not frustrated but it has a phase transition in two dimensions given by the phase transition of the Ising model [34] and the Villain model, which has no transition and is equivalent to the Dimer problem.

A large body of opinion and physical insight holds that we want models for spin glasses which have frustration given by the disorder, and the disorder can not be transformed away in a trivial manner. The Edwards-Anderson model (see section 5) is such a model, and so is the binary alloy model discussed in § 6.

The concept of frustration will be used in § 6 of this chapter and in Chapter 6 where it is shown that frustration is eliminated when the Edwards-Anderson model is defined on a Bethe lattice.

§ 4. The Mattis-Heisenberg model

The Mattis-Heisenberg model is an extension of the Mattis model introduced in [21]. The Mattis model is an Ising model with separable random exchange which can take values of $\pm$ of some exchange strength J. The Mattis-Heisenberg model is a model with three dimensional vector spins. Both models have thermodynamic properties given trivially by that of the pure system by a change of spin variables which transforms the models to and from the pure systems. We write the Hamiltonian as

$$H = -\frac{1}{2} \sum_{ij} J_{ij} \underline{S}_i \underline{S}_j \epsilon_i \epsilon_j \qquad (3)$$

where the sums are over all sites in the lattice (simple cubic, simple square, chain and

Bethe lattice are the choices of lattice which we shall be using). The non-random part of the exchange is

$$\overline{J_{ij}} = J > 0 \qquad \text{if i, j are nearest neighbours} \qquad (4.a)$$

$$\overline{J_{ij}} = 0 \qquad \text{otherwise} \qquad (4.b)$$

The random part of the exchange is determined by the probability distribution density for the set $\{\epsilon_i\}_i$ of statistically independent, identically distributed random variables. The choice of distribution density made for the Mattis-Heisenberg model is

$$P(\epsilon) = (1-c)\delta(\epsilon-1) + c\,\delta(\epsilon+1) \qquad (5)$$

This form of random exchange is said to be separable. That is, that each site has a statistically independent random variable associated with it. This means that we can make a transformation $\underline{\underline{T}}_i = \epsilon_i \underline{\underline{S}}_i$ which eliminates the disorder. This contrasts with the Edwards-Anderson model (see § 5) which has non-separable disorder. It is an important consequence of the choice of disorder for the Mattis-Heisenberg model that there is a well defined classical ground state $(\langle S_i^z \rangle_{c.g.s.} = S\epsilon_i)$.

As we can see from the above discussion, the equilibrium statistical mechanics of the Mattis-Heisenberg model is equivalent to the study of the equilibrium statistical mechanics of the pure (not disordered) Heisenberg ferromagnet. However, in this thesis we shall be considering the spin waves (or magnons) in the random phase approximation (R.P.A.) and the theory of Mattis-magnons [35] is not equivalent to the study of the ferromagnetic magnons.

As we are interested in studying the magnons of the Mattis-Heisenberg model in the non-interacting approximation, we shall proceed to construct an effective Hamiltonian for the magnons. This will be accomplished by performing an expansion of the spin vectors,

about their classical ground state direction, as boson creation and annihilation operators. The derivation of this Hamiltonian is shown in Appendix A, and the derivation of Sherrington [8] is followed. The Holstein-Primakoff similarity transformations are used.

$$S_i^z = \epsilon_i (S - a_i^\dagger a_i) \tag{6.a}$$

$$S_i^+ \doteq (2S)^{1/2}\{(1+\epsilon_i)a_i/2 + (1-\epsilon_i)a_i^\dagger/2\} \tag{6.b}$$

$$S_i^- \doteq (2S)^{1/2}\{(1+\epsilon_i)a_i^\dagger/2 + (1-\epsilon_i)a_i/2\} \tag{6.c}$$

All of the symbols have their usual meanings. The Hamiltonian is

$$\mathcal{H}^{eff} = -\tfrac{1}{2}S^2\sum_{ij} J_{ij} + S\sum_{ij} a_i^\dagger a_i J_{ij}$$

$$-\frac{S}{2}\sum_{ij} J_{ij}\left[(1+\epsilon_i\epsilon_j)a_i^\dagger a_j - \tfrac{1}{2}(1-\epsilon_i\epsilon_j)(a_i^\dagger a_j^\dagger + a_i a_j)\right] \tag{7}$$

This is the Hamiltonian given by Sherrington [8]. We see that if we always have $\epsilon_i\epsilon_j = 1$, then we have the R.P.A. Hamiltonian for a Heisenberg ferromagnet and if $\epsilon_i\epsilon_j = -1$ always then we would have a Heisenberg antiferromagnet. For c = 0 in equation (5) we obtain the case of $\epsilon_i\epsilon_j = 1$ always, but the probability of having a pure antiferromagnet is zero in the thermodynamic limit. We can think of the Hamiltonian as a disordered X-Y model; this was pointed out by Ching Leung and Huber [36]. If we have $c = \frac{1}{2}$ in equation (5) and replace the random elements of (7) by their average values (i.e. $\overline{\epsilon_i} = 0$) we obtain

$$\mathcal{H}^{eff}_{X-Y} = -\tfrac{1}{2}S\sum_{ij} J_{ij} + S\sum_{ij} a_i^\dagger a_i J_{ij}$$

$$-\frac{S}{2}\sum_{ij}\overline{J_{ij}}\left[a_i^+ a_j - \tfrac{1}{2}\left(a_i^+ a_j^+ + a_i a_j\right)\right]$$

which is the R.P.A. effective Hamiltonian for the X-Y model. As this model gives a long wavelength dispersion going as q (linear dispersion) we might expect to obtain this for the disordered model (7). Indeed, this is what we find for $c = \frac{1}{2}$, equation (4.32).

The Hamiltonian (7) is written in a form symmetric in the ferromagnetic and antiferromagnetic contributions. While this is ideal for the case of $c = \frac{1}{2}$, it is not the best form for $c \ll 1$ or $c \sim 1$ which are the limits of considering a ferromagnetic host to isolated antiferromagnetically coupled impurities. The low concentration problem is studied in Chapter 2. We obtain the required Hamiltonian by the following transformation

$$\epsilon_i \epsilon_j \longrightarrow (\epsilon_i \epsilon_j - 1) + 1 \qquad (8)$$

This transformation is trivial, so we get by a rearrangement of the terms in the Hamiltonian

$$\mathcal{H}^{eff} = -\tfrac{1}{2}S^2\sum_{ij}\overline{J_{ij}} + S\sum_{ij}\overline{J_{ij}}\, a_i^+ a_i - S\sum_{ij}\overline{J_{ij}}\, a_i^+ a_j$$

$$-\frac{S}{2}\sum_{ij}\overline{J_{ij}}\,(\epsilon_i\epsilon_j - 1)\left[a_i^+ a_j + \tfrac{1}{2}\left(a_i^+ a_j^+ + a_i a_j\right)\right] \qquad (9)$$

The first line is the R.P.A. form of the Heisenberg ferromagnet Hamiltonian. The term on the second line removes a ferromagnetically coupled spin and replaces it with an antiferromagnetically coupled spin.

Now we have obtained the R.P.A. forms of the Mattis-Heisenberg Hamiltonian which will be used in Chapters 3, 4 and 7. The Mattis-Heisenberg model will be

studied in Chapters 6 and 7 by other approximations which will be explained in the appropriate place.

§ 5. The Edwards-Anderson Model

In the paper by Edwards and Anderson [6] a model Hamiltonian was proposed, the properties of which, it is hoped, account for the main features of certain experiments done on spin-glasses [37]. This paper started a spin-glass industry to which many theorists have made contributions [38]. There has been an enormous amount of work done on the statistical mechanics of the model, and attempts to construct and understand a mean field theory for the model, such as the Sherrington Kirkpatrick model [11], [12]. Reviews of this work on equilibrium properties can be found in Young [39] and Binder [3], [4].

The model is a nearest-neighbour Heisenberg model with random exchange.

$$H = -\frac{1}{2} \sum_{ij} J_{ij} \underline{S}_i \cdot \underline{S}_j \qquad (10)$$

where we choose three dimensional spin vectors, the indices run over all sites of the lattice. J_{ij} is zero except for when i and j are nearest neighbours, in which case we have a set of statistically independent, identically distributed random variables $\{J_{ij}\}_{ij}$, with probability distribution density

$$P(J) = (1-c)\,\delta(K-J) + c\,\delta(K+J) \qquad (11)$$

where c varies between zero and one, and K is greater than zero. Edwards and Anderson [6] chose a Gaussian distribution for the J's in the original paper on this subject, but the distribution density (11) is often chosen because it has the essential feature of producing competing interactions. This model will be used in Chapter 6 with the distribution density given by (11).

The model defined by (10) and (11) is (unless c = 1 or 0) frustrated. Now, this statement must be qualified because there are choices of lattice for which the Edwards-Anderson is not frustrated. These are lattices without closed, non-intersecting lattice paths such as the Bethe lattice which is studied in Chapter 6. The Bethe lattice has a special simple case when the connectivity is one; this gives the one-dimensional chain.

§ 6. Binary alloy spin glasses

The term binary alloy spin glass is used here for a disordered Heisenberg model which has exchange disorder due to two species of spin associated randomly with the sites of the lattice.

If we call one species A and the other B and call $J_{AB} = J_{BA}$, the exchange between two unlike spins which are nearest neighbours, J_{AA} is the exchange between two nearest neighbour A spins and J_{BB} is the exchange between two nearest neighbour B spins. There are no other possibilities. Also, we would like to make our choice so that we obtain frustrated plaquettes.

The choices of exchange will be the simplest which would give us our spin glass. $J_{ij} = \pm J$ where J is a constant greater than zero, and i and j can be either A or B. A model we have met before is given by

$$J_{AB} = -J \; ; \quad J_{BB} = J \; ; \quad J_{AA} = J. \tag{12}$$

Inspection shows that this is the Mattis (or Mattis-Heisenberg) exchange. As was shown in § 4, this model has no frustrated plaquettes, and this is true for both close packed and loosely packed lattices. Now, if we reverse the signs in (12) we get

$$J_{AB} = J \; ; \; J_{BB} = J_{AA} = -J \tag{13}$$

and we do obtain, for this model, frustrated plaquettes if we define our Hamiltonian on a closely packed lattice. It is, however, an over-frustrated model (see § 3) as every elementary plaquette is frustrated.

Now what we are looking for is a model which has frustration due to the same causes as that of the Edwards Anderson model, that is, the choice of disorder. Such a model is one with exchange constants

$$J_{AB} = -J \; ; \; J_{BB} = -J \; ; \; J_{AA} = +J. \tag{14}$$

The Heisenberg Hamiltonian corresponding to this choice of exchange is

$$H = -\frac{1}{2} \sum_{ij} J_{ij} (2\epsilon_i \epsilon_j - 1) \underline{S}_i \cdot \underline{S}_j \tag{15}$$

where the set $\{\epsilon_i\}_i$ of identically distributed statistically independent random variables are given by the probability distribution density

$$P(\epsilon) = c\,\delta(\epsilon) + (1-c)\,\delta(\epsilon - 1) \tag{16}$$

It can be seen that this Hamiltonian has frustrated plaquettes due to the disorder. It helps in our understanding of this Hamiltonian if we notice it can be written as the sum of two Heisenberg magnets.

$$H = H_{DF} + H_{AF} \tag{17}$$

H_{DF} is the dilute ferromagnet (§ 7) with exchange strength 2J and H_{AF} is the pure Heisenberg antiferromagnet with exchange -J. This lets us see some of the geometry of the disorder. H_{DF} gives us the percolation problem for the existence of

the ferromagnetic ground state, but in this case the dilute ferromagnet is coupled to an antiferromagnetic host to which it is coupled ferromagnetically. This means that the two components of the alloy are separated by a boundary of frustrated plaquettes. Now we have an exact result for the critical percolation concentration for the existence of a ferromagnetic ground state in the dilute ferromagnet, and this gives an upper bound for the concentration c for which there is a bulk magnetisation for the spin glass model (15). We have from [40] and [41] that the percolation concentration for the two dimension simple square lattice site percolation problem is 0.59. This gives an upper bound on c_c, the concentration for which we have a ferromagnetic ground state in two dimensions

$$c_c \leqslant 0.41 \tag{18}$$

This can be seen to be an upper bound if we consider the picture of the infinite cluster near percolation proposed by De Gennes [42]. This picture considers bulk clusters connected by chains of sites. In model (15) these chains will be bordered by frustrated plaquettes. These plaquettes will have the effect of stopping the direction of the magnetisation in each of the clusters from being the same, and thus destroying the magnetisation.

Thus, we have in model (15) an interesting model of a spin glass which could be of greater relevance to certain physical situations than the Edwards-Anderson model. The model could, however, be studied by exactly the same methods as have been used previously to study the Edwards-Anderson model. For example, a study of the ground state properties using a method like that developed by Grinstein et al [43] would provide a useful and illuminating comparison with the Edwards-Anderson model.

§ 7. The Dilute Ferromagnet

The dilute Heisenberg ferromagnet is one of the easier disordered magnets to think about. It has no frustration; it does not have any antiferromagnetic bonds. It is not a model which will be of central importance to this thesis, we shall simply define it and refer the reader to Chapter 7 for our report of the computer calculations of its properties.

The Hamiltonian is

$$H = -\frac{1}{2} \sum_{ij} J_{ij}\, \epsilon_i \epsilon_j\, \underline{S}_i \cdot \underline{S}_j \qquad (19)$$

where the set $\{\epsilon_i\}_i$ are statistically independent, identically distributed random variables with a distribution density,

$$P(\epsilon) = c\,\delta(\epsilon - 1) + (1-c)\delta(\epsilon) \qquad (20)$$

It is possible to use standard Holstein-Primakoff methods to obtain an R.P.A. boson approximation to (19).

$$H = S \sum_{ij} J_{ij} \epsilon_i \epsilon_j (a_i^\dagger a_i + a_i^\dagger a_j) \qquad (21)$$

This is the approximation which is usually used when studying the localisation of non-interacting magnons [44], [45]. It is the finding of the above references that localisation is weak except near the percolation concentration.

§ 8. Finite Difference Operators

Finite difference operators are the class of operators which are most often studied in the physics of random solids. The earliest studies were on a harmonic random chain (see Dyson [46] and Schmidt [47]), and this continued to be of interest in the

seventies (Casher and Lebowitz [48], and Ishii [49]).

An important interpretation of random finite difference operators is the tight binding approximation for the electron moving in a random potential which is the Anderson model ([5] and Chapter 5, § 5). Also, the equations of motion for magnons from Heisenberg models in the R.P.A. approximation take the form of a finite difference operator.

We shall consider the finite difference operator to be an approximation to a particle moving in a potential V for which the Schrodinger equation is

$$i\frac{d}{dt}\phi(x) = -\Delta\phi(x) + V(x)\phi(x) \tag{22}$$

where x is a position vector in $\mathbb{R}^d$ the d dimensional real Euclidean space, $\phi(x) \in L^2(\mathbb{R}^d)$. The finite difference approximation to (22) is

$$i\frac{d}{dt}\phi(x) = V(x)\phi(x) - \sum_{y:|x-y|=1}\phi(y) + z\phi(x) \tag{23}$$

for $\phi \in l^2(Z^d)$ where Z^d is the d dimension hypercubic lattice of lattice spacing 1 and the coordination number z = 2d. $V(x)$ is given by the distribution of the stochastic process which defines the Hamiltonian. We shall recast equation (23) in more familiar notation in Chapter 5, § 5, where it shall be discussed as the Anderson model for localisation.

If we want to consider the finite difference operators that spin wave theory gives rise to, we must allow for a random hopping term in the equation of motion.

$$i\frac{d}{dt}\phi(x) = (V(x) + z)\phi(x) - \sum_{y:|x-y|=1} T(|x-y|)\phi(y) \tag{24}$$

where all the terms are the same except that we have introduced a random variable T, which is defined by a stochastic process, which is usually (but not always : see equation (21) and think of the finite difference operator which can be constructed from it) independent of the process which defines V.

It can easily be seen that if we define our model in a finite box in Z^d we have the model used for most numerical work on the normal modes of disordered solids [50] .

§ 9. Figure Captions for Chapter 2

Figure 1 (a) Shown is a band of the Hamiltonian for a particle moving in a periodic potential. All states associated with energies within the band are extended.

(b) This picture shows a band for an electron moving in a periodic potential which is perturbed by a stochastic potential. The effects of the stochastic perturbation are band broadening and the appearance of localised states, associated with energies near the band edges. E_L^1 and E_L^2 denote the localisation edges.

(c) Here we show a picture of the band for the same model but with stochastic perturbation sufficiently strong to cause the localisation of all states in the band.

Figure 2 (a) This picture shows a frustrated elementary plaquette as is found for a Villain model or an Edwards-Anderson model defined on a simple square lattice.

(b) Shown here we have a triangular frustrated plaquette which is found in an antiferromagnet defined on a triangular lattice.

(c) Shown is the elementary plaquette of the Union Jack lattice, as studied by Vaks et al [31] .

Figure 1

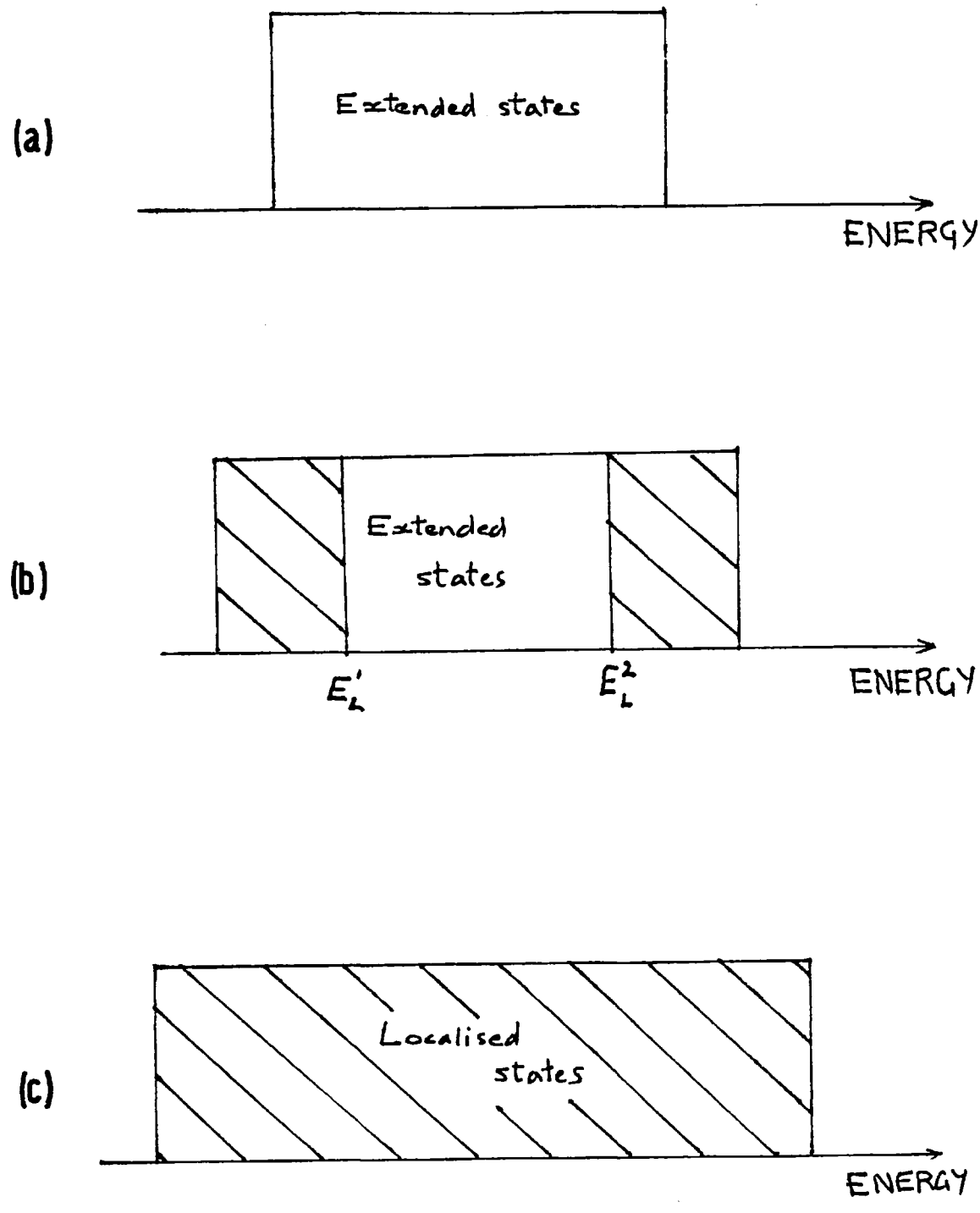
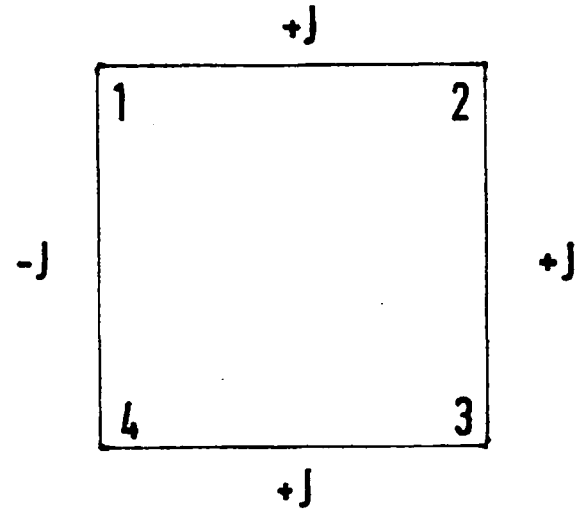
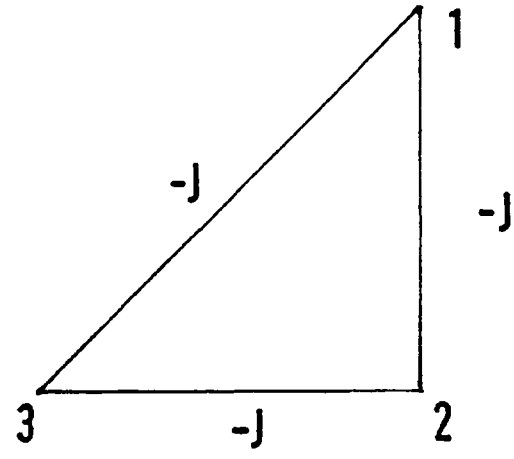

(a)
Extended states
ENERGY
(b)
Extended
states
E_L'
E_L^2
ENERGY
(c)
Localised
states
ENERGY

Figure 2

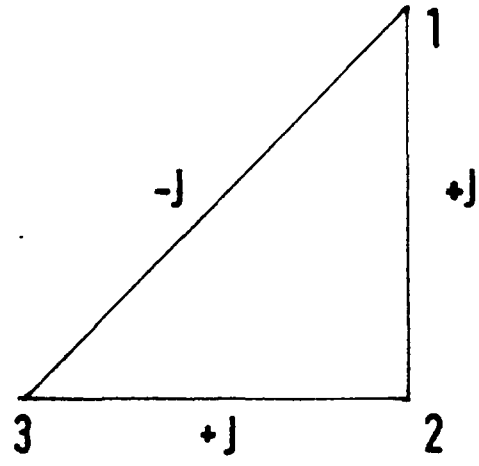
(a)



(b)



(c)



CHAPTER 3

LONG-WAVELENGTH DYNAMIC RESPONSE OF THE MATTIS-HEISENBERG MODEL : WEAK DISORDER

§ 1. Introduction

The Mattis-Heisenberg model given by equation (2.3) is here considered in the limit where disorder is small. That is, c in equation (2.5) is small, or equivalently, (1 - c) is small. With this restriction on the disorder the problem of studying the dynamics reduces to an isolated impurity problem. And so, because we are considering an isolated antiferromagnetic impurity in a ferromagnetic host, our methods are in direct line of descent from the works by Wolfram and Callaway, 1963 [51], Izyumov, 1966 [52], and Wang and Callen [53].

We shall use what is called by Wang and Callen the spin wave approximation; they attacked the use of this approximation by Ishii et al [54], but if we consider our approximation to be good for large spins and sufficiently low temperatures, we can see that use in [54] of the approximation to study spin one half impurities is well outside the region of validity.

Edwards and Jones [55] developed a diagrammatic perturbation theory for the impurity problem and applied it to the diluted Heisenberg ferromagnet.

The quantities which we shall calculate are the dominant long-wavelength, small energy and small c dispersion relation with the dominant imaginary correction which gives a measure of the damping of the spin-wave modes due to the disorder, the spectral density function can be calculated easily once the dispersion relation is known.

§ 2. The calculation of the self-energy

We are considering the Mattis-Heisenberg model in the limit where it is a

ferromagnet with isolated anti-ferromagnetic impurities. We study this model by an equation of motion method developed by Zubarev [56]. The effective Hamiltonian appropriate for our problem is given by equation (2.9). The Zubarev Green's function which we shall study is the retarded G.F. with $E = \omega + i\delta$ ($\delta > 0$).

$$\Delta(q,q';E) = \langle\langle a_q^+ + a_q \; ; \; a_{q'}^+ + a_{q'} \rangle\rangle_r(E) \quad (1)$$

where the a's are boson operators and the q's are reciprocal space vectors. This Green's function is proportional to $\langle\langle S_q^x \; ; \; S_{q'}^x \rangle\rangle_r$ where S^x is the x spin component. The equation of motion technique gives (see App. B. equation (12))

$$\Delta(q,q';E) = \Delta^\circ(E;q)\delta(q-q')$$
$$-\frac{S}{2}\Delta^\circ(E;q)\int_{B.Z.}\frac{dq''}{(2\pi)^d} J(q,q'')\Delta(q'',q';E) \quad (2)$$

where the free Green's function is

$$\Delta^\circ(E;q) = \frac{2S(J(0)-J(q))}{E^2 - S^2(J(0)-J(q))^2} \quad (3)$$

The potential J (q, q") is defined in terms of the parameters of the Mattis-Heisenberg model as

$$J(q,q'') = \sum_{ij} J_{ij}(\epsilon_i\epsilon_j - 1)e^{i(q\cdot R_i - q''\cdot R_j)} \quad (4)$$

where the R's are lattice vectors.

As we are working on an isolated impurity problem the simplification of the potential is in real space. So we Fourier transform equation (2)

$$\Delta_{ij} = \Delta^\circ_{ij} + \frac{S}{2}\sum_{kk'}\Delta^\circ_{ik}J_{kk'}(1-\epsilon_k\epsilon_{k'})\Delta_{k'j} \quad (5)$$

where the indices take values on the lattice and we do not write explicitly the E dependence.

We can now define the T-matrix and the self-energy respectively by

$$\Delta_{ij} = \Delta^{\circ}_{ij} + \sum_{kk'} \Delta^{\circ}_{ik} T_{kk'} \Delta^{\circ}_{k'j} \qquad (6)$$

and

$$\overline{\Delta}_{ij} = \Delta^{\circ}_{ij} + \sum_{kk'} \Delta^{\circ}_{ik} \Sigma_{kk'} \overline{\Delta}_{k'j} \qquad (7)$$

where bar denotes the operation of taking an average over the disorder (be careful not to confuse the sigma for the summation with the sigma for the self-energy).

It can be shown that for very small c the self energy is given by the concentration of antiferromagnetic impurities multiplied by the single impurity T-matrix. This result was shown previously by Izyumov [52], but see Elliott et al [87] for a more general discussion.

So let us call the site with the impurity o. We define

V_{ij} = - SJ/2 when i or j (but not both) are site o

V_{ij} = 0 otherwise

That is, $V_{ij} = J_{ij}(1 - \epsilon_i \epsilon_j) S/2$ but ϵ_i is negative only if i = 0 and likewise for ϵ_j.

From equations (5) and (6)

$$T_{ij} = V_{ij} + \sum_{kk'} V_{ik} \Delta^{\circ}_{kk'} T_{k'j} \qquad (8)$$

In the following sections we shall use the notation that nearest neighbour sites to the site o are denoted by δ, δ' and δ''. Thus we can write all the elements of the T-matrix by the following three equations :

$$T_{oo} = \sum_{\delta\delta'} V_{o\delta} \Delta^{\circ}_{\delta\delta'} T_{\delta' o} + \sum_{\delta} V_{o\delta} \Delta^{\circ}_{\delta o} T_{oo} \qquad (9.a)$$

$$T_{o\delta} = V_{o\delta} + \sum_{\delta'} V_{o\delta'} \Delta^{o}_{\delta' o} T_{o\delta} + \sum_{\delta'\delta''} V_{o\delta'} \Delta^{o}_{\delta'\delta''} T_{\delta''\delta} \qquad (9.b)$$

$$T_{\delta\delta'} = V_{\delta o} \Delta^{o}_{oo} T_{o\delta'} + \sum_{\delta''} V_{\delta o} \Delta^{o}_{o\delta''} T_{\delta''\delta'} \; . \qquad (9.c)$$

We consider a lattice where every lattice site has 2d (where d is the dimension of the lattice) nearest neighbours, and we generalise cubic symmetry to arbitrary dimension. This allows us to introduce the notation

$$T_o = T_{oo} \; ; \; T_1 = T_{o\delta} = T_{\delta o} \; ; \; T_2 = T_{\delta\delta'} \qquad (10.a)$$

$$\Delta_o = \Delta^{o}_{oo} \; ; \; \Delta_1 = \Delta^{o}_{o\delta} = \Delta^{o}_{\delta o} \; ; \; \Delta_2 = z^{-1} \sum_{\delta'} \Delta^{o}_{\delta\delta'} \qquad (10.b)$$

where z = 2d; the coordination number.

The elements of the T-matrix can now be written in terms of the elements of the free Green's functions.

$$T_o = z^2 J^2 S^2 \Delta_2 D^{-1} \qquad (11.a)$$

$$T_1 = SJ(1 - zJS\Delta_1) D^{-1} \qquad (11.b)$$

$$T_2 = J^2 S^2 \Delta_o D^{-1} \qquad (11.c)$$

where

$$D = (1 - zJS\Delta_1)^2 - z^2 J^2 S^2 \Delta_2 \Delta_o \; . \qquad (11.d)$$

So if the elements of the free Green's function can be evaluated, then we have an expression for the T-matrix. $\Delta^{o}(q, E)$ can be expressed as (from equation (3))

$$\Delta^{o}(q) = (E - S(J(o) - J(q)))^{-1} + (E - S(J(o) - J(q)))^{-1} \; . \qquad (12)$$

Define

$$\Delta_0 = G_0(E) + G_0(-E) \tag{13.a}$$

where

$$G_0(E) = \int_{B.Z.} (E - S(J(0) - J(q)))^{-1} \frac{dq^d}{(2\pi)^d} \tag{13.b}$$

and B.Z. denotes the Brillouin zone.

This procedure can be generalised to give Δ_1 and Δ_2

$$\Delta_1 = G_1(E) + G_1(-E) \tag{14.a}$$

where

$$G_1(E) = \int_{B.Z.} \frac{dq^d}{(2\pi)^d} \frac{\cos q.\underline{\delta}}{E - S(J(0) - J(q))} \tag{14.b}$$

and

$$\Delta_2 = G_2(E) + G_2(-E) \tag{15.a}$$

where

$$G_2(E) = \int_{B.Z.} \frac{dq^d}{(2\pi)^d} \frac{\cos q.(\underline{\delta} - \underline{\delta}')}{E - S(J(0) - J(q))} \tag{15.b}$$

$G_2(E)$ and $G_1(E)$ can both be expressed in terms of $G_0(E)$, E and constants. From equations (13), (14) and (15)

$$G_1(E) = (SJz)^{-1} + \left(1 - \frac{E}{SJz}\right) G_0(E) \tag{16.a}$$

$$G_2(E) = \left(1 - \frac{E}{SJz}\right) G_1 \ . \qquad (16.b)$$

These expressions give

$$\Delta_1 = \Delta_0 - \frac{E}{SJz}\left(G_0(E) - G_0(-E)\right) + 2(JSz)^{-1} \qquad (17.a)$$

$$\Delta_2 = \frac{-2E}{JSz}\left(G_0(E) - G_0(-E)\right) + 2(JSz)^{-1} + \left(1 + \frac{E^2}{S^2J^2z^2}\right)\Delta_0 \ . \qquad (17.b)$$

Equation (11.d) takes a simple form when equations (13.a), (17.a) and (17.b) are substituted into the expression for D.

$$D = \left(1 - 2E\,G_0(E)\right)\left(1 + 2E\,G_0(-E)\right) \ . \qquad (18)$$

If we Fourier transform the T-matrix to reciprocal space variables

$$T(q) = T_0 + 2\sum_{\underline{\delta}} T_1 \cos q\cdot\underline{\delta} + \sum_{\underline{\delta},\underline{\delta}'} T_2 \cos\left(q\cdot(\underline{\delta} - \underline{\delta}')\right) \ . \qquad (19)$$

Now we can look at the dominant long-wavelength contribution to $T(\underline{q})$

$$T(q) = T_0 + 2zT_1 + z^2T_2 - 2T_1(qa)^2 - 2zT_2(qa)^2 \qquad (20)$$

where a is the lattice spacing (a length). Substitute equations (11) into equation (20)

$$\begin{aligned} DT(q) = {} & z^2J^2S^2\Delta_2 + 2zJS\left(1 - zJS\,\Delta_1\right) \\ & + z^2J^2S^2\Delta_0 - 2(qa)^2\left(SJ\left(1 - 2JS\Delta_1\right)\right. \\ & \left. + zJ^2S^2\Delta_0\right) \ . \end{aligned} \qquad (21)$$

Substitute equations (13), (14) and (15) into equation (21)

$$DT(q)= z^2J^2S^2(\Delta_o-\frac{2E}{JSz}(G_o(E)-G_o(-E))+\frac{2}{JSz}+\frac{E^2\Delta_o}{S^2J^2z^2})$$
$$+2zJS(1-zJS(\Delta_o-\frac{E}{SJz}(G_o(E)-G_o(-E))+2/SJz))$$
$$+z^2J^2S^2\Delta_o-2(qa)^2(SJ(1-zJS(\Delta_o-\frac{E}{SJz}(G_o(E)$$
$$-G_o(-E)+\frac{2}{SJz}))+zJ^2S^2\Delta_o). \qquad (22)$$

Equation (22) simplifies to give to the dominant order in q and E

$$T(q)=2JS(qa)^2+E^2\Delta_o-2JSE(qa)^2(G_o(E)-G_o(-E)). \qquad (23)$$

We use the relationship between the isolated single impurity T-matrix and the self-energy which is valid for the low concentration problem

$$\Sigma(q) = cT(q) + O(c^2) . \qquad (24)$$

Thus we obtain the self-energy to dominant order in c. To obtain the self-energy to dominant order in c, E and q, we substitute equation (23) into equation (24)

$$\Sigma(q) = c\{2JS(qa)^2+E^2\Delta_o$$
$$-2SJE(qa)^2(G_o(E)-G_o(-E))\} . \qquad (25)$$

The Green's functions $G_o(E)$ and $G_o(-E)$ are evaluated in appendices D and E. Appendix D deals with the case when d is an odd integer and appendix E the case when d is real, greater than zero but not an odd integer.

So in the following sections we shall consider separately the cases when d is allowed to take odd integer values and when it is not. But firstly we consider the dominant real contribution to the dispersion relation and this is independent of the dimension.

§3. The dominant real contribution to the dispersion relation

We consider in this section the dispersion relation obtained by setting to zero the denominator of the expression for the full Green's function averaged over the disorder. From equation (7) we obtain by the Fourier transform

$$\overline{\Delta(q;E)} = \Delta^{\circ}(q;E) + \Delta^{\circ}(q;E)\Sigma(q)\overline{\Delta(q;E)} \qquad (26)$$

where all terms take their long-wavelength form. We make a trivial rearrangement

$$\overline{\Delta(q;E)} = \frac{\Delta^{\circ}(q;E)}{1-\Delta^{\circ}(q;E)\Sigma(q)} \; . \qquad (27)$$

Equation (3) is used (with the approximation $(J(o) - J(q)) \longrightarrow J(qa)^2$) to give

$$\overline{\Delta(q;E)} = \frac{2SJ(qa)^2}{E^2 - S^2J^2(qa)^4 - 2SJ(qa)^2\Sigma(q)} \; . \qquad (28)$$

Thus, the dispersion relation is given by

$$E = SJq^2a^2\left(1 + 2\Sigma(q)/SJq^2a^2\right)^{1/2} . \qquad (29)$$

We only consider terms up to linear order in c, so equation (29) can be approximated by

$$E = SJq^2a^2 + \Sigma(q) + O(c^2) \; . \qquad (30)$$

Now because $\Sigma(q)$ is a complex number for any non-zero q we must treat E as a complex number, let us write

$$E(q) = \omega(q) + i\epsilon(q) \; . \qquad (31)$$

The dominant real part is evaluated from equations (30) and (25)

$$\omega(q) = SJq^2a^2\,(1+2c)\ . \tag{32}$$

We shall postpone the discussion of this result until we have obtained the full body of our results; we shall postpone all discussion of results until the final section of this chapter.

§ 4. The dominant imaginary contribution to the self energy

The dominant imaginary contribution to the self energy gives by equations (30) and (31) the imaginary contribution to the dispersion relation.

We obtain from equations (D.(3)) and (25)

$$\mathrm{Im}\{\Sigma(q)\} = -ic\hat{C}_d\left\{\frac{\pi\omega^{\frac{d}{2}+1}}{2(JS)^{d/2}} - \frac{2SJ\ q^2a^2\pi\omega^{d/2-1}}{2(JS)^{d/2}}\right\}. \tag{33}$$

The energy in equation (33) is replaced by its dominant real value $\omega(q)$ (see equation (32)). Thus, we have our dominant imaginary contribution

$$\epsilon(q) = \tfrac{1}{2}\,c\,\hat{C}_d\,\pi SJ(qa)^{d+2}. \tag{34}$$

§ 5. The spectral density function $S(q,\omega)$

The spectral density function $S(q,\omega)$ is obtained here for the Mattis-Heisenberg model. It is this quantity which is measured in neutron scattering experiments. We refer the interested reader to Marshall and Lovesey [57] for details of the theory of neutron scattering.

We shall evaluate $S(q,\omega)$ in the long-wavelength limit. The spectral density function is obtained from the imaginary part of the full Green's function

$$S(q,\omega) = \lim_{\delta \to 0^+} \left[-\frac{1}{2\pi} \mathrm{Im} \left\{ \overline{\Delta(q;\omega+i\delta)} \right\} \right]. \qquad (35)$$

The self-energy will be written as

$$\Sigma(q) = 2cSJ q^2 a^2 + ic\pi \chi(d) SJ q^{d+2} a^{d+2} \qquad (36)$$

where χ (d) is determined by equation (33) .

We use equation (38) to obtain the dominant long-wavelength small energy low

concentration spectral density function

$$S(q,\omega) = \frac{2cS^2J^3\chi(d)(qa)^{d+6}}{(\omega^2 - S^2J^2(qa)^4(1+4c))^2 + (2\pi S^2J^2c\chi(d)(qa)^{d+4})^2} \quad (37)$$

Now in the final section we shall discuss the results obtained in this chapter.

§ 6. Discussion

The results of this chapter are summarised by the expression for the spectral density function equation (37), this is also the directly measurable quantity. We see that the function has a maximum at energies given by equation (32) and peaks have widths which disappear as q^{d+2} as q tends to zero. This means that the peaks become sharper as q tends to zero.

It is of interest to compare this result to that obtained by Edwards and Jones [55] who obtain the same q dependence as us, but they perform their calculation for the weakly diluted Heisenberg ferromagnet. The results differ in the coefficients obtained for the terms in q. This is not surprising because we expect antiferromagnetic impurities to increase the spin wave stiffness, as they do (see equation (32)), while dilution decreases the spin wave stiffness, as is shown in [55] .

It should be pointed out that this calculation for the Mattis-Heisenberg model is valid for the binary alloy introduced in Chapter 2 § 6. When c is small in equation (2.16) we have a model of a ferromagnetic host with isolated antiferromagnetic impurities which is identical to the Mattis-Heisenberg model in the small c limit which we have used here. Notice that if we were to take into account terms of order c^2 or higher, then the models would no longer be equivalent because the model given by (2.15) will have frustrated plaquettes whereas the Mattis-Heisenberg model is never frustrated.

CHAPTER 4

THE LONG-WAVELENGTH DYNAMIC RESPONSE OF THE MATTIS-HEISENBERG MODEL FOR STRONG DISORDER

§ 1. Introduction

Once again we treat the Mattis-Heisenberg model as introduced in Chapter 2, § 4. The first five sections reproduce the results of Sherrington [9] and [10]. The approach is somewhat different because we start with the quadratic approximation to the Hamiltonian (2.3) and (A.5), which has been studied before by Sherrington [8], who introduced the Hamiltonian.

The results of [10] are rederived here because they are essential to the extension given in this chapter to arbitrary disorder. Also, there was a question (Nozières : remark in a seminar) about the effective Hamiltonian for the analysis in [10] once we get into the details of the calculation. So the analysis of an effective Hamiltonian is presented here.

The results are presented for three dimensions only. This is sufficient to illustrate the method and because the Mattis-Heisenberg model has no phase transition in one and two dimensions, [8], we want our results to be valid at low but not necessarily zero temperatures. We obtain the effective Hamiltonian (App. A) by linearising the spin operators about the classical ground state to obtain the effective Hamiltonian as a quadratic form in Bose operators. Because the model (2.3) has antiferromagnetic exchange terms present we know that the classical ground state would be a bad approximation for the ground state of the Quantum Mattis-Heisenberg model except when the spin length is large. This can be thought of as a semi-classical approximation. It has been pointed out by Ching et al [36] that in its bilinear form the Mattis-Heisenberg model resembles a random X-Y model. The X-Y model has q like dispersion

at long-wavelengths and this q dependence of the dispersion is found in section 3, equation (32). However, in no limit do we have a pure X-Y model, so there is an imaginary correction to the dispersion relation which goes as q^2 for $c = \frac{1}{2}$. This is due to the disorder and causes damping of the spin-waves.

§2. The derivation of the Dyson equation: Green's function and scattering

The Green's function which we shall study is the $\langle\langle S_i^x ; S_j^x \rangle\rangle_r$ Zuburev Green's function [56] in the R.P.A. approximation. The transformation defined by equations (2.6) can be used to give

$$S_i^x = \tfrac{1}{2}(2S)^{1/2}(a_i^+ + a_i) . \qquad (1)$$

As we shall be studying the Hamiltonian (App. A.(5)) we consider the Green's function which is proportional to $\langle\langle S_i^x ; S_j^x \rangle\rangle_r$

$$\Delta_{ij}(E) = \langle\langle (a_i^+ + a_i) ; (a_j^+ + a_j) \rangle\rangle_r (E) . \qquad (2)$$

The spectral density function $S(q, E)$ is defined by (Marshall and Lovesey [57] ; and (3.38))

$$S(q,E) = -\frac{1}{2\pi S} \mathrm{Im} \{ \overline{\Delta(q ; E)} \} \qquad (3)$$

where the bar denotes the average over the disorder. The Fourier transform is performed after the average in real space, which restores translational invariance.

The Dyson equation is derived in App. B and is given by equation (App. B.8)

$$\Delta(E ; q, q') = \Delta^0(E ; q)\, \delta(q - q') \qquad (4)$$
$$- \frac{\Delta^0(E ; q)}{(2\pi)^d} \frac{S}{2} \int_{B.Z.} dq_2\, J(q, q_2) \Delta(q_2, q')$$

where d is the dimension of the lattice, B.Z. is the Brillouin Zone, $\Delta(E;\underline{q},\underline{q})$ is the Fourier transform of equation (2) and $J(\underline{q}, \underline{q}_2)$ is given by

$$J(q,q_2) = \sum_{ij} J_{ij}\,\epsilon_i\epsilon_j e^{i(q\cdot R_i - q_2\cdot R_j)} \quad . \tag{5}$$

Equation (4) can be Fourier transformed to the real space representation,

$$\Delta_{ij}(E) = \Delta^0_{ij}(E) - \frac{S}{2}\sum_{kl} \Delta^0_{ik}(E) J_{kl}\epsilon_k\epsilon_l\,\Delta_{lj}(E) \tag{6}$$

where

$$\Delta^0_{ij} = \frac{1}{(2\pi)^d}\int dq\,\Delta^0(E;q)\,e^{-iq\cdot(\underline{R}_i-\underline{R}_j)} \tag{7}$$

where $\Delta^0(E\,;\,\underline{q})$ in equation (4) is given by (App. B.10) :

$$\Delta^0(E;q) = \frac{2S(J(0)-J(q))}{E^2 - S^2J(0)(J(0)-J(q))} \quad . \tag{8}$$

The free propagator (8) has some low energy properties which we shall exploit in subsequent sections of this chapter. Consider an energy so small that the following is true :

$$E^2 \ll S^2J(0)(J(0)-J(q)) \quad . \tag{9}$$

This means that a geometric expansion of $\Delta^0(E\,;\,\underline{q})$ is allowed.

$$\Delta^0(E;q) = -\frac{2}{SJ(0)}\left(1 + \frac{E^2}{S^2J(0)(J(0)-J(q))} + O(E^4)\right). \tag{10}$$

Now we shall Fourier transform (10) to real space and consider the local free propagator

$$\Delta^0_{ii}(E) = -\frac{2}{SJ(0)}\left[1 + \left(\frac{E}{S}\right)^2 J(0)^{-2} W + O(E^4)\right] \tag{11}$$

where W is Watson's integral (Mattis [59] p.146) ;

$$W = (2\pi)^{-3} \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \frac{dq_x dq_y dq_z}{1 - \frac{1}{3}(\cos q_x + \cos q_y + \cos q_z)}$$

$$= 1.516386 \qquad (12)$$

where we use this value throughout because we are only interested in three dimensions. The method is not limited to three dimensions because of this (see Sherrington [10] equation (3.11)).

The important point to notice is that when E is so small that terms of order E^2 are negligible, we have a simplification of $\Delta^0_{ij}(E) \longrightarrow \Delta^0_{ij}(0)$

$$\Delta^0_{ij}(0) = \frac{-2}{zJ(0)}\delta_{ij} \; . \qquad (13)$$

That is in the limit of E tends to zero, the free propagator is diagonal.

We now go on to define the self-energy and the T-matrix. From equation (4) we define

$$\Delta(E;q,q') = \Delta^0(E;q)\delta(q-q') + \frac{z}{2}\Delta^0(E;q)T(q,q')\Delta^0(E;q') \qquad (14)$$

where T(q, q') is the T-matrix in the reciprocal space representation. The averaged Green's function is written as,

$$\overline{\Delta(q,q')} = \Delta^0(q)\delta(q-q') - \frac{z}{2}\Delta^0(q)\int \frac{dq_2}{(2\pi)^3}\overline{J(q,q_2)\Delta(q_2,q')} \; . \qquad (15)$$

The operation of averaging over the disorder restores the translational invariance, so we have from equation (14)

$$\overline{\Delta(E;q)} = \Delta^{\circ}(q) + \tfrac{S}{2}\Delta^{\circ}(q)\,\overline{T(E;q)}\,\Delta^{\circ}(E;q)\,. \qquad (16)$$

The self-energy is defined to be the quantity $\Sigma(E;\underline{q})$ which satisfies the relationship

$$\overline{\Delta(E;q)} = \Delta^{\circ}(E;q) + \Delta^{\circ}(E;q)\,\Sigma(E;q)\,\overline{\Delta(E;q)}\,. \qquad (17)$$

This equation is rearranged to give

$$\overline{\Delta(E;q)} = \frac{2S(J(0)-J(q))}{E^2 - S(J(0)-J(q))(SJ(0)+2\Sigma(E;q))}\,. \qquad (18)$$

The self-energy can in turn be expressed as a function of the averaged T-matrix

$$\Sigma(E;q) = \frac{\overline{T(E;q)}}{1 + \overline{T(q;E)}\,\Delta^{\circ}(E;q)}\,. \qquad (19)$$

The T-matrix shall be studied in real space, so, by the Fourier transform,

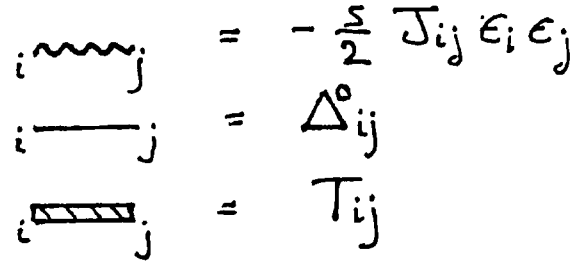
$$\Delta_{ij}(E) = \Delta^{\circ}_{ij}(E) + \sum_{k,l}\Delta^{\circ}_{ik}(E)T_{kl}(E)\Delta^{\circ}_{lj}(E) \qquad (20)$$

we shall obtain an expansion for T_{kl} in the low energy limit which can be averaged.

§ 3. The dominant real part of the dispersion relation : c = ½

We are going to evaluate the dominant real part of the self-energy when c (equation (2.5)) is equal to half. This is the only concentration for which the overall magnetisation disappears for the Mattis-Heisenberg model in its classical ground state.

The T-matrix T_{ij} will be expanded in a series which we shall represent graphically as

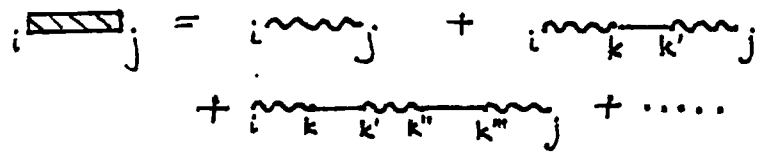


$$= -\tfrac{S}{2}J_{ij}\epsilon_i\epsilon_j \qquad (21.a)$$

$$= \Delta^{\circ}_{ij} \qquad (21.b)$$

$$= T_{ij} \qquad (21.c)$$

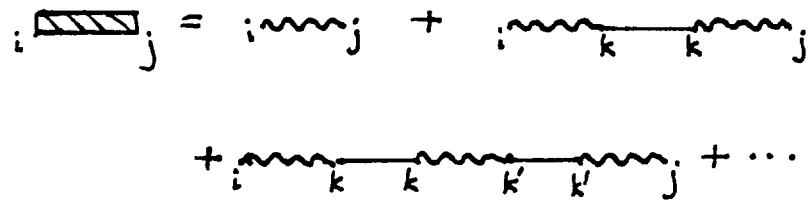
So, by using equations (6) and (20) we obtain the infinite series



(22)

where summation over internal indices is assumed.

$\overline{T_{ij}(E)}$ must be evaluated in the limit of $E \to 0$. We use the fact from (13) that $\Delta^{o}_{ij}(0)$ is diagonal to evaluate $\overline{T_{ij}(0)}$. The series (22) becomes



(23)

The set $\{\epsilon_i\}$ of statistically independent identically distributed random variables has some useful properties when $c = \frac{1}{2}$ in equation (2.5). We have from (2.5)

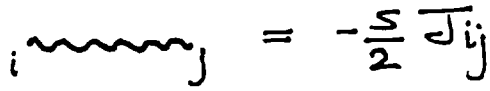
$$\overline{\epsilon^{2n}} = 1 \tag{24.a}$$

$$\overline{\epsilon^{2n+1}} = 1 - 2c = \delta \tag{24.b}$$

for all $n = 0, 1, 2, \ldots$; where the bar denotes the average. We can now perform the average. The averaging process introduces two new lines in our diagrammatic representation of the series.

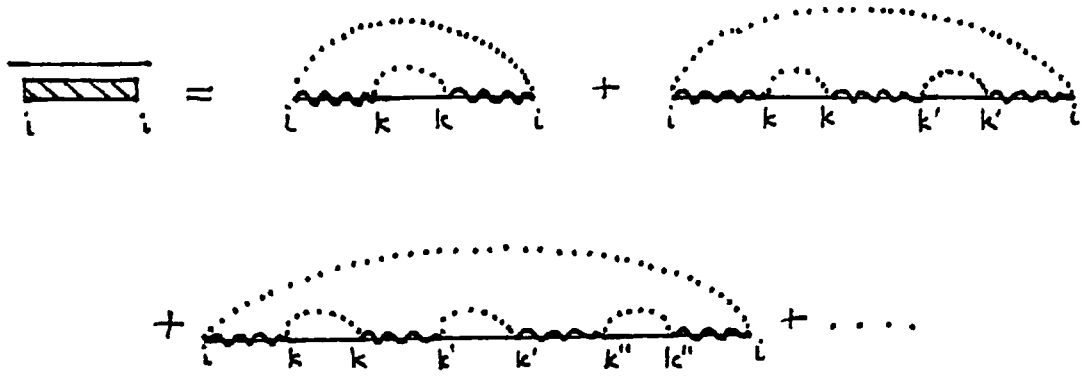


(25.a)



(25.b)

The averaged T-matrix is given by (for $E = 0$),



(26)

This series is summable and we have from equation (13)

$$\overline{T(0)} = (2\pi)^{-3} \int_{B.Z.} dq \sum_{n=1}^{\infty} \left(\Delta^{0}_{ii}(0)\right)^{n} \left(-\frac{S}{2} J(q)\right)^{n+1} . \qquad (27)$$

We can use the properties of the geometric series to sum inside the integral of (27)

$$\overline{T(0)} = (2\pi)^{-3} \int_{B.Z.} dq \frac{S^2}{4} \frac{\Delta^{0}_{ii}(0) J(q)^2}{1 + \frac{S}{2} \Delta^{0}_{ii}(0) J(q)} . \qquad (28)$$

This expression can be rearranged using the fact that $J_{ii} = 0$

$$\overline{T(0)} = \left\{\Delta^{0}_{ii}(0)\right\}^{-1} \left[(2\pi)^{-3} \int_{B.Z.} dq \left(1 + \frac{S}{2} \Delta^{0}_{ii}(0) J(q)\right)^{-1} - 1\right] . \qquad (29)$$

For the evaluation of this expression see Appendix F. We have equation (F.2)

$$\overline{T(0)} = \frac{-SJ(0)}{2} (W - 1) . \qquad (30)$$

where W = 1.516386 ([59] p. 146).

Substitute equation (30) into equation (19) for the self-energy to obtain,

$$\Sigma(0) = -SJ(0)(W-1)/2W . \qquad (31)$$

The above equation is used in the expression for the Green's function (18) to give the dispersion relation by equating the denominator to zero.

$$E^2 = S^2 J(0)^2 q^2 (SJ(0) - SJ(0)(1 - W^{-1}))$$

giving

$$E = SJ(0)\, q\, W^{-1/2}. \qquad (32)$$

Equation (32) gives the dominant part of the dispersion relation for $c = \frac{1}{2}$ in the small energy long wavelength limit. If we take the μ of Sherrington ([10] equation (3.16)) to be J(0) as it is for our choice of nearest neighbour exchange, then our equation (32) is the same as his equation (3.19). (We chose a = 1 from the beginning).

Notice that E is a linear function of q. This is a direct consequence of there being no overall magnetisation. This shall be seen in § 6 where the dominant part of the dispersion relation will be seen to go as q^2 for q sufficiently small.

§ 4. The dominant imaginary part of the dispersion relation for $c = \frac{1}{2}$

To evaluate the imaginary part of the self-energy in the low energy small wavelength limit we must keep a non-zero energy dependence in the free propagator. This would mean that we should lose the simplicity of the diagrams of (26) and would be forced to consider all the diagrams for $\Delta^0_{ij}(E)$ not diagonal and isolate the dominant energy part. This would be a very much more difficult problem. We can, however, keep the series of diagrams (26) and it can be argued ([10] § 3 and App.2) that these give the dominant contribution. We must consider $\Delta^0_{ii}(E)$ to the order E^2. This is given by equation (11)

$$\Delta^0_{ii}(E) = \frac{-2}{SJ(0)}\left[1 + \left(\frac{E}{S}\right)^2 J(0)^{-2} W\right] + O(E^4)$$

So we obtain from equation (29)

$$\overline{T(E)} = \{\Delta^{\circ}_{ii}(E)\}^{-1}\left[(2\pi)^{-3}\int_{B.Z.} dq\,(1+\tfrac{s}{2}\Delta^{\circ}_{ii}(E)J(q))^{-1} - 1\right] \quad . \quad (33)$$

This expression is evaluated in Appendix F equation (F.6).

$$\overline{T(E)} = \frac{-SJ(0)}{2}(W-1) - \frac{i\lambda(E)W^{1/2}E}{8\pi} \quad . \quad (34)$$

The equation above is substituted into equation (19) to give to dominant order in energy the self-energy

$$\Sigma(E) = \frac{-SJ(0)}{2W}\left(W-1+\frac{i\lambda(E)W^{1/2}E}{4\pi J(0)S}\right) \quad . \quad (35)$$

Equation (35) gives, when substituted into equation (18) and the denominator is put equal to zero, the dispersion relation ;

$$E = SJ(0)qW^{-1/2}\left(1-\frac{i\lambda(E)EW^{1/2}}{J(0)4\pi S}\right)^{1/2} \quad (36)$$

let

$$E(q) = \omega(q) + i\,\epsilon(q)$$

then we obtain an expression for $\epsilon(q)$

$$\epsilon(q) = \frac{-SJ(0)\lambda(E)\,q^2}{8\pi W^{1/2}} \quad . \quad (37)$$

This is the dominant imaginary part. We notice that it goes as q^2 and so the damping on the modes becomes less as $q \longrightarrow 0$, and so the very long wavelength modes have long life times.

We can see that $\omega(q)$ as a function of q is given by equation (32)

$$\omega(q) = S\mathcal{J}(0)\, q\, W^{-1/2} \tag{38}$$

The equation (32) was evaluated by assuming that the imaginary part of $E(q)$ was small enough to be neglected.

§5. The spectral density function for long-wavelengths and $c = \frac{1}{2}$

The spectral density function $S(q, E)$ which is evaluated at small q and small E is derived from the results of sections 3 and 4. The equation (3) gives $S(q, E)$ as being proportional to the imaginary part of $\overline{\Delta}(q, E)$. So, we have

$$S(q,E) = \frac{2\mathcal{J}(0)\, q^2 [\epsilon(q) E]/2\pi}{[E^2 - \omega(q)]^2 + [2\epsilon(q)E]^2} \tag{39}$$

which has a peak structure which sharpens as q is reduced to zero.

$$\lim_{q \to 0} \frac{\epsilon(q)}{\omega(q)} = 0 \tag{40}$$

It must be remembered that this is a zero temperature calculation. The damping which we have found is a result of the disorder only. This should be compared with the results of Chapter 3 that the damping goes as q^5 in three dimensions. As the damping due to finite temperature effects and spin wave interaction goes as q^4, we see that in this case for $c = \frac{1}{2}$ we have a dominant order damping which would be of greater effect (for sufficiently low temperatures and sufficiently long wavelengths).

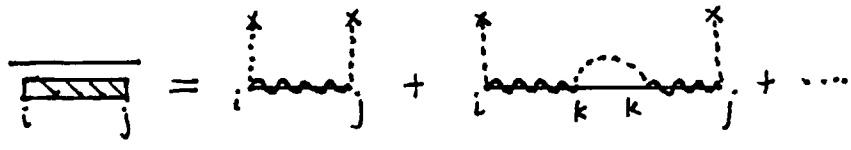
§6. The dominant real part of the dispersion relation for arbitrary c except c = $\frac{1}{2}$

The results of Sherrington [9] and [10] will be extended in this section to general c. That is, we shall calculate the dominant long-wavelength and small energy real contribution to the dispersion relation for all possible strengths of the disorder.

We shall use the diagrammatic method developed in §3. We shall also use the same approximation as before by taking $\Delta_{ii}(0)$ for all the free propagators. A new symbol must be introduced in the diagrammatic notation; this is because for $c \neq \frac{1}{2}$ the average $\overline{\epsilon} \neq 0$. So, we introduce

$$\overline{\epsilon} = \delta \equiv \quad i \quad (41)$$

where $\delta = 1 - 2c$ from equation (24.b). So, in the E = 0 approximation we have for $\overline{T_{ij}(0)}$ the series of diagrams



(42)

as well as diagrams (26); where i = j. This gives for the full E = 0 T-matrix

$$\overline{T(0;q)} = -\frac{S}{2}\delta^2 J(q) \sum_{n=0}^{\infty} \left(-\Delta^0_{ii}(0)\frac{S}{2}J(q)\right)^n$$

$$+ \frac{(1-\delta^2)}{(2\pi)^3} \int_{B.Z.} dq_1 \sum_{n=1}^{\infty} \left(\Delta^0_{ii}(0)\right)^n \left(-\frac{S}{2}J(q_1)\right)^{n+1}.$$

This series is summed to give

$$\overline{T(0;q)} = \frac{-\delta^2 \frac{S}{2} J(q)}{1+\frac{S}{2}\Delta^0_{ii} J(q)} + \frac{(1-\delta^2)}{(2\pi)^3}\frac{S^2}{4}\int_{B.Z} \frac{dq_1 \Delta^0_{ii}(0) J(q_1)^2}{1+\frac{S}{2}\Delta^0_{ii} J(q_1)} \quad (43)$$

The second term in equation (43) (R.H.S.) is given by equation (28) and so from equation (30) we have

$$\overline{T(0;q)} = \frac{-\delta^2 \frac{5}{2} J(q)}{1+\frac{5}{2}\Delta^0_{ii}(0)J(q)} - \tfrac{1}{2}(1-\delta^2)5J(0)(W-1). \quad (44)$$

We now use this equation for the T-matrix in equation (19) to obtain an expression for the self-energy.

$$\Sigma(0;q) = \frac{-\frac{5J(0)}{2}\left((W-1)+\delta^2\left(\frac{J(q)}{J(0)-J(q)} - (W-1)\right)\right)}{W+\delta^2\left(\frac{J(q)}{J(0)-J(q)} - W+1\right)} \cdot (45)$$

Equation (45) is substituted into equation (18); the denominator of which is equated to zero to give

$$E^2 = S^2 J(0)\left(J(0)-J(q)\right)\Bigg(1 - \left[\frac{(W-1)+\delta^2\left(\frac{J(q)}{J(0)-J(q)} - (W-1)\right)}{W+\delta^2\left(J(q)/(J(0)-J(q)) - (W-1)\right)}\right]\Bigg). \quad (46)$$

Now this expression can be evaluated in various limits. Consider $\delta^2 = 1$; this should be a pure ferromagnet and we look for the well-known dispersion relation as a check on the method. Equation (46) with $\delta^2 = 1$ becomes

$$E = S\left(J(0) - J(q)\right) \quad (47)$$

and this is the standard results for a Heisenberg ferromagnet in R.P.A.

We now go on to consider some more interesting consequences of equation (46). We shall consider the long-wavelength approximation so that $(J(0) - J(q)) = J(0)q^2$

and consider the regime $q^2/\delta^2 << 1$. This is a forbidden regime only for $\delta^2 = 0$, that is when $c = \frac{1}{2}$.

$$E^2 = \frac{s^2 J(0)^2 q^4}{\delta^2}((1 + Wq^2) + O(q^2/\delta^2)) \qquad (48)$$

This equation is interesting because if we consider $\delta^2 \sim 1$ we are in the low concentration limit considered in Chapter 3. In this limit we have

$$E^2 = s^2 J(0)^2 q^4/(1-2c)^2$$

to dominant order. Now consider the dominant order in c, which is taken to be very much smaller than 1,

$$E = sJ(0)q^2(1+2c) . \qquad (49)$$

This is the same as equation (3.32) (we have chosen a = 1 in this chapter). This result shows that the method can reproduce the results of an independent method for low concentrations. It should be remembered that the method shown in this chapter was originally formulated for $c = \frac{1}{2}$ only. Equation (48) tells us that the very long wavelength q dispersion is true for $c = \frac{1}{2}$ only. For any other value of c we can find a regime (at long enough wavelength) where we have q^2 dispersion.

§7. The dominant imaginary correction to the dispersion relation for general c

We should like to be able to find the dominant imaginary correction to the dispersion relation given by equation (46). This is not, however, an easy extension of the process shown in §4. If we go ahead and use the approximation $\Delta_{ij,ij}(E) = \Delta_{ii,ii}(E)\,\delta_{ij,ij}$ to obtain $\epsilon(q)$ we find a dispersion relation which goes as q^6 at long wavelengths with the exception of the concentration $c = \frac{1}{2}$. This cannot be correct

because we know from equation (3.34) that $\epsilon(q)$ goes as q^5 in the limit when c is very small. If the q^6 result is correct it would mean that for a concentration of impurities which is not small the damping would be weaker, at sufficiently long wavelengths, than the damping obtained for very small c. This result goes against all physical intuition, and so we must reject the correction which goes as q^6 because it does not give a reasonable result.

The answer to our problem must be that by using the diagonal approximation to the free propagator we do not obtain the relevant set of diagrams to sum. The solution is to consider diagrams with off-diagonal free propagators; $\Delta^{\circ}_{ij}(E)$ $(i \neq j)$. This allows more complicated diagrams to contribute to the series for the T-matrix. This complication means that we can no longer find a simple summation of the series. So the dominant imaginary correction has only been found in the cases of $c = \frac{1}{2}$ and c very small.

§ 8. The spectral density function S (q, E) for $c \neq \frac{1}{2}$

We do not have much to say about this function except that it shall have a peaked structure and the peaks are at energies given by equation (48) for $c \neq \frac{1}{2}$. We can say nothing about the widths of the peaks except when $c \ll 1$ or $c \sim 1$ or $c = \frac{1}{2}$ (look at Chapter 3 and §§ 4 and 5 of this chapter for the discussion of the widths which have been found).

§ 9. Discussion

The results that we have reached in this chapter have been obtained before by Sherrington [9] and [10] except for those reported in § 6 which are reported for the first time here.

It is now possible to give a review of the results obtained for the long-wavelength

dynamic response of the Mattis-Heisenberg model. We notice that for $c \sim 1$, $c \ll 1$ and $c = \frac{1}{2}$ we can find arbitrarily long-lifed modes by going to sufficiently long wavelengths. The damping for intermediate c is not known, but it would be physically reasonable to imagine that it should be stronger for $c \sim 1$ or $c \ll 1$ but weaker than that found for $c = \frac{1}{2}$.

We should like to compare our results with those obtained by Ching et al [36] and [58]. They find for $c = \frac{1}{2}$ the long-wavelength peaks in $S(\underline{q}, E)$ at energies given by our formula (32) and once we take account of their computational procedures there is agreement with our results for the width given by equation (37). Those comparisons were made by Sherrington [10]. It is unfortunate that no data is available for concentrations other than $c = \frac{1}{2}$ because it is to this regime that our new results are relevant. The completion of this work depends on whether it is possible to find the dominant imaginary contribution to the dispersion relations for all concentrations. This is something which can only be decided in the future.

CHAPTER 5

THE CHARACTERISATION OF LOCALISED AND EXTENDED STATES

§1. Introduction

In this chapter we shall not be concerned with calculations of quantities, but with the precise definition of the quantities we want to calculate. It has been a problem with localisation theory since Anderson's work [5] that localisation of the wavefunction of a particle moving in an infinite disordered potential is intuitively grasped with ease, but giving mathematical precision to this concept has been much more difficult. The step from the qualitative concept of localisation to the mathematical criterion used in evaluating localisation properties of the wavefunction, at best, carry heuristic conviction, usually. The criterion introduced by Anderson [5] is based on the convergence or not of the renormalised perturbation series for the local self-energy and this is used also by Economou and Cohen [60] who argue its relation to spectral properties of the Hamiltonian in the thermodynamic limit.

The spectral characterisation of Economou and Cohen will be of concern to us here because it states a widely held view among localisation theorists. They consider the Anderson model and they argue that if the support of the density of states (i.e. the spectrum) is a continuous interval on the real line, then the states associated with it are extended, and if the spectral points are a dense but denumerable subset of the interval, then the associated states are localised. We have an indication that this view neglects some of the more sophisticated possibilities that spectral theory offers us from the work of Casher and Lebowitz [48], who show for the disordered mass harmonic chain that, although there are no intervals of the real line for which each point is an element of the spectrum, the spectrum is not a denumerable set of points, either. The fault with [60] is that a complete spectral theory was not developed.

Ishii [49] introduced some concepts and theorems from spectral analysis of linear operators into the analysis of the Anderson localisation problem. He proved for the Anderson model, defined on the semi-infinite chain, the absence of fast diffusion with one probability, but he could not prove the absence of slow diffusion and thus that all states are localised with probability one.

Work by Pastur [61], Goldsheid and Molcanov [62], [63] lead to rigorous results for random Jacobi matrices which include as a special case the one dimensional Anderson model. This includes the first proof that all states are localised in one dimension, [62] and [63]. The interpretation of their results seems to need the fact that their operator obeys the R.A.G. theorem; see § 4.

Another approach to the localisation problem is to study the D.C. conductivity. This method was applied to electrons moving in a one-dimensional random potential by Halperin [64], who presented an argument to show the D.C. conductivity is zero for arbitrarily weak disorder. Recently a rigorous argument was given by Kunz and Souillard [23] who proved that the D.C. conductivity is zero when all the states are exponentially localised.

In this chapter we shall consider an approach to the problem which is based on functional analysis, but we shall derive consequences, from the analysis, for two well known localisation criteria; the disappearance of the imaginary part of the local Green's function for almost every energy (see § 6) and the inverse participation ratio which was introduced by Bell and Dean [65].

These criteria compare with each other in an interesting way. The first was shown not to be a localisation criterion (see Ishii [49], Kimball [66] and § 6), whereas the second is shown here in § 7 and § 8 to be a good localisation criterion for the case of the operator defined on an infinite lattice. The proof of the results

depends on a theorem which characterises localised and extended states by the spectral properties of the operator. We shall call this theorem the R.A.G. theorem, which may be thought of as the earlier contributions to the RAGE theorem (see [67] Vol. III, page 341). The contributors to the RAG theorem are Ruelle [68], Amrein and Georgescu [69] and a statement, with discussion, of their theorem will be presented in § 4.

The localisation edge is made into a precise concept by the use of some simple properties of spectral projection operators (see § 2 for a definition). These properties enable us to define an operator which is restricted to a certain subset of the spectrum of the full operator (see § 8).

The little Functional Analysis which is necessary for the derivations in this chapter is presented in the next section (§ 2). This can only be a bare outline, so we supply detailed references to texts and proofs.

§2. Spectral Analysis

We shall consider the spectral analysis of a bounded self-adjoint operator H which acts on a separable Hilbert space $\mathcal{H}$. There is a spectral representation of such operators which is a generalisation of the well known matrix algebra result that a Hermitian matrix can be diagonalised and that its eigenvalues are real. First, we must define projection valued measures (see [67] Volume 1, page 234).

Definition 1. Let H be a bounded self-adjoint operator and Λ a Borel set of $\mathbb{R}$. $P_\Lambda \equiv \chi_\Lambda(H)$ is called a spectral projection of H where $\mathbb{R}$ is the real line and χ_Λ is the characteristic function of the set Λ

Definition 2. A family of spectral projections $\{P_\Lambda\}$ of a bounded self-adjoint operator H obeying (a) - (c) is called a bounded projection-valued measure (p.v.m.).

(a) Each P_Λ is an orthogonal projection.

(b) $P_\emptyset = 0$; $P_{(-a,\, a)} = 1$ for some a.

(c) If $\Lambda = \bigcup_{n=1}^{\infty} \Lambda_n$ with $\Lambda_n \cap \Lambda_m = \emptyset$ for all $n \neq m$, then

$$P_\Lambda = \underset{N \to \infty}{s-\lim} \left(\sum_{n=1}^{N} P_{\Lambda_n} \right)$$

where $\emptyset$ denotes the empty set, and I the identity operator.

Notice that if $\{P_\Lambda\}$ is a p.v.m. then we have that for $\phi \in \mathcal{H}$, $(\phi, P_\Lambda \phi)$ is an ordinary measure.

Thus, we have a form of the spectral measure theorem ([67], Volume 1, page 235).

Theorem I (spectral theorem - p.v.m. form). There is a one to one correspondence between (bounded) self-adjoint operators H and (bounded) projection valued measures $\{P_\Lambda\}$ given by

$$H \mapsto \{P_\Lambda\} = \{\chi_\Lambda(H)\}$$

$$\{P_\Lambda\} \mapsto H = \int_{\mathbb{R}} \lambda \, dP_\lambda$$

where Λ is a Borel subset of $\mathbb{R}$ and $\chi_\Lambda(H)$ is the characteristic function of H on Λ. From the definition of P_λ, where λ is a point in $\mathbb{R}$ we have a spectral measure $u(\lambda)$ given by $\left(P_\lambda = P_{(-\infty,\lambda)} = \chi_{(-\infty,\lambda)}(H)\right)$.

$$u(\lambda) = (\phi, P_\lambda \phi) \quad , \quad \phi \in \mathcal{H} \, . \tag{1}$$

The Lebesgue decomposition theorem ([67], Volume 1, page 22) tells us that $u(\lambda)$ is uniquely decomposable as :-

$$u(\lambda) = u(\lambda)_{AC} + u(\lambda)_S \tag{2.a}$$

$$u(\lambda) = u(\lambda)_P + u(\lambda)_C \tag{2.b}$$

$$u(\lambda) = u(\lambda)_{SC} + u(\lambda)_{AC} + u(\lambda)_P \tag{2.c}$$

where the subscripts will now be explained. $u(\lambda)_p$ is pure point relative to Lebesgue measure, that is

$$u(X)_p = \sum_{x \in X} u(\{x\})$$

for any Borel set X. $u(\lambda)_s$ is singular relative to Lebesgue measure;

$$\frac{du(\lambda)}{d\lambda} = 0 \qquad \text{a.e.}$$

where a.e. means almost everywhere, that is, with a possible exception of a set of measure zero. $u(\lambda)_{sc}$ is both singular and continuous relative to Lebesgue measure. $u(\lambda)_c$ is continuous relative to Lebesgue measure. $u(\lambda)_{A.C.}$ is absolutely continuous with respect to Lebesgue measure.

If $u(\lambda) = u(\lambda)_p$ we say that $\phi \in \mathcal{H}_p$; if $u(\lambda) = u(\lambda)_{sc}$; $\phi \in \mathcal{H}_{sc}$ etc.. That is, the Hilbert space on which the operator H is defined is completely reduced by the action of H into three mutually orthogonal subspaces.

$$\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_{AC} = \mathcal{H}_p \oplus \mathcal{H}_c = \mathcal{H}_s \oplus \mathcal{H}_{AC} \qquad (3)$$

where, if P_p is the spectral projection which projects onto the complete set of eigenfunctions of H,

$$\mathcal{H}_p = P_p \mathcal{H}$$

and similarly we have P_c, $P_{s.c.}$ etc.

The resolvent of the operator H will be denoted $\mathcal{R}(z)$ where $z \in \mathbb{C}$ (the set of complex numbers). The spectral representation theorem can be used to write the resolvent as

$$\mathcal{R}(z) = (z - H)^{-1} = \int_{\mathbb{R}} \frac{dP_\lambda}{(z - \lambda)} \quad . \qquad (4)$$

For values of z for which $R(z)$ is not defined, we say that z is in the pure point spectrum of H $\sigma_p(H)$; when $R(z)$ is defined but unbounded, then z is in the continuous spectrum of H , $\sigma_c(H)$.

Another theorem of interest comes from perturbation theory for linear operators, as it is found in the monograph by Kato [24] . The theorem is due to original work by Weyl and was later generalised by Von Neumann ([24] page 523).

Theorem II (Weyl - Von Neumann). Let H be a self-adjoint operator in a separable Hilbert space $\mathcal{H}$. For any $\varepsilon > 0$, there exists a self-adjoint operator $A \in \mathcal{B}_2(\mathcal{H})$ with $\|A\|_2 < \varepsilon$ such that H + A has a pure point spectrum.

where $\mathcal{B}_2(\mathcal{H})$ is the Schmidt class of operators ([24] pages 262, 519) and $\|A\|_2$ is the Schmidt norm ([24], page 262).

This theorem was generalised by Kuroda, who realised that the presence of the Schmidt norm condition in Th. II is not essential. We have ([24] , page 525) the following theorem.

Theorem III. In Theorem II, A can be chosen in such a way that $|||A||| < \varepsilon$. Here $|||\ |||$ is any (unitarily invariant) cross norm, with the single exception of the trace norm or its equivalent.

This finishes our development of the Functional Analysis central to this chapter. It is not, however, self contained, and the interested reader is referred to [24] and [67] for background and proofs.

It can easily be seen that the expectation value of the resolvent (equation (4)) gives the one particle Green's function when H is interpreted as the Hamiltonian operator and $z = E + i0^+$ is interpreted as the energy. We shall begin to study our operators from this point of view now.

§3. The theorem of Wiener

The central purpose of this section is the statement of a theorem by Wiener on the L^2-mean value of the Fourier transform of a measure. However, to underline the applications of this theorem, we outline the interpretation which we give to the measure and its Fourier transform. Consider a bounded self-adjoint operator H (the Hamiltonian); its time evolution operator is given by exp (- itH) where $t \in \mathbb{R}$ is the time variable. We have from Theorem I

$$-i\,(\phi, \exp(-itH)\,\phi) = -i\int_{\mathbb{R}} e^{-it\lambda}\, du(\lambda) = G(t) \qquad (5)$$

where $u(\lambda) = (\phi, P_\lambda \phi)$. So we can consider the function G(t) given by equation (5) as being the time dependent single particle Green's function.

So, we state Wiener's theorem (for a proof we refer to [67] volume III, page 340).

Theorem IV (Wiener's theorem). Let u be a finite Baire measure on $\mathbb{R}$ and let

$$G(t) = \int_{\mathbb{R}} e^{-i\lambda t}\, du(\lambda)$$

be its Fourier transform (up to a factor of $(2\pi)^{-\frac{1}{2}}$). Then

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G(t)|^2 dt = \sum_{x\in\mathbb{R}} |u(\{x\})|^2 \qquad (6)$$

We see that if H has no pure point spectrum, then the limit is zero. So the L^2-mean value of G(t) provides important information on the spectrum of H, and as a consequence (see § 4) provides information on the localisation of the states of H.

§4. The RAG theorem

The RAG theorem is the earlier part of the RAGE theorem named after Ruelle, Amrein, Georgescu and Enss ([67], Volume III, p.340). The theorem was developed in the context of non-relativistic quantum scattering theory to characterise bound states and scattering states which are given a geometric definition by equation (7), by the Hilbert subspace ($\mathcal{H}_{sub}$) with which the states are associated, that is the continuous or the pure point Hilbert subspaces.

The statement of the theorem shall follow that of Amrein and Georgescu [69].

Two subspaces of $\mathcal{H}$ are defined; M_o is the sub-space of bound states,

$$M_o = \{f \in \mathcal{H} \mid \lim_{r\to\infty} \sup_{t\in\mathbb{R}} \|(I-F_r)V_t f\|^2 = 0\} \qquad (7.a)$$

and M_{co} is the subspace of scattering states.

$$M_{co} = \{f \in \mathcal{H} \mid \lim_{s\to 0^+} s\int_0^\infty e^{-st}\, dt\, \|F_r V_t f\|^2 = 0, \forall r = 1,2,\ldots.\}. \qquad (7.b)$$

The set $\{F_r\}$ are the orthogonal projections acting on the separable Hilbert space $\mathcal{H}$. (In subsequent applications of the theorem F_r is the projection on to the subspace of states spatially localised in a given spherical region, $|x| < r$. V_t denotes the unitary evolution operator exp (- iHt). $\{H\}'$ (in i) of theorem V) denotes the commutant of H, that is to say that S_n commutes with H and belongs to the set of all linear operators on $\mathcal{H}$ that are bounded and defined everywhere.

We can now state the RAG theorem.

Theorem V (Ruelle, Amrein and Georgescu). Let H be a self-adjoint operator and $\{F_r\}$, r = 1,2, ... , be a family of orthogonal projections such that $\text{s-}\lim_{r\to\infty} F_r = I$ (the identity operator). Suppose there exists a family $\{S_n\}$, n = 1, 2, ..., of linear operators acting in $\mathcal{H}$, a separable Hilbert space, such that

i) $S_n \in \{H\}'$ for all n

ii) The sequence $\{S_n\}$ converges strongly to an operator S, such that

$$\underset{n\to\infty}{s\text{-}\lim} S_n = S$$

iii) The range of S is dense in $\mathcal{H}$

iv) $F_r S_n P_c$ is compact for all r, $n < \infty$

(P_c is the spectral projection onto $\mathcal{H}_c$)

Then $\mathcal{H}_p = M_o$ and $\mathcal{H}_c = M_\infty$

where $\mathcal{H}_p$ and $\mathcal{H}_c$ are as defined in (3).

We can see by examination of equations (7 a, b) that M_o is the subspace of functions localised in a region of space of diameter r, where r is taken to infinity at the end. So all localised states of H are in M_o which is identified with $\mathcal{H}_p$ by the theorem, and M_∞ is the subspace of extended states. The operator H must, however, have the properties demanded by the theorem. We show this in §5 for the Anderson model.

§5. The Anderson Model

The finite difference operator was defined in chapter 2, § 8. We shall now rewrite this operator in the form familiar in localisation theory, that is, as the Anderson model [5].

$$(Ha)(x) = \epsilon(x)a(x) - \sum_y t(|x-y|)a(y) \qquad (8)$$

for $a \in \ell^2(Z^d)$ where Z^d is the d dimensional generalisation of the simple cubic lattice, filling all space and with unit lattice spacing. The hopping term can be taken to be a random variable, but we shall consider the example where it is not taken to be random;

$$t(|x-y|) = 1 \text{ if } |x-y| = 1$$

$$t(|x-y|) = 0 \text{ otherwise}$$

x, y are points in Z^d. $\epsilon(x)$ is the random potential defined by the probability distribution density $\rho(\epsilon(x))$ for each $x \in Z^d$. The distribution density which we choose as an example is the rectangular distribution

$$\rho(\epsilon) = \chi_{(-\frac{W}{2}, \frac{W}{2})}(\epsilon) W^{-1}$$

which was used by Anderson [5]. The Schrodinger equation which we obtain from (8) can be seen to be bounded. The bounds are given by

$$\inf(\sigma(H)) = -2d + \inf\{\text{Support}(\rho)\} \tag{9}$$

$$\sup(\sigma(H)) = 2d + \sup\{\text{Support}(\rho)\} \tag{10}$$

So we have that (8) is a bounded self-adjoint operator acting on a separable Hilbert space, and from Th. I that (8) has a spectral representation of p.v.m. form. If we use the family of spectral projections to define a S_n (see Th. V) as

$$S_n = P_n - P_{-n} \tag{11}$$

we see that this S_n has all the properties of S_n in Th. V i), ii) and iii). If we now consider part iv), $F_r S_n P_c = F_r P_c S_n$ using (11) and the commutation property of the projection operators. The operator defined by (8) is compact when restricted to a finite lattice, therefore $F_r P_c S_n$ is compact. Thus, by Th. V we have :-

The random, finite difference operator (the Anderson model) (8) has all its localised states in $\mathcal{H}_p$ and all its extended states in $\mathcal{H}_c$ which are mutually orthogonal subspaces of $\mathcal{H}$, the cartesian sum of which gives $\mathcal{H}$.

We have as the result of this section that the Anderson model for localisation has its localised and extended states characterised by the RAG theorem as belonging to $\mathcal{H}_p$ and $\mathcal{H}_c$ respectively.

It is an interesting aspect of the random operator (8) that it is the sum of two operators :-

$$(H_1 a)(x) = \epsilon(x) a(x)$$

has a purely pure point spectrum, which is dense in the support of the probability distribution density supp $\{\rho(\epsilon)\}$, and

$$(H_2 a)(x) = -\sum_y t(|x-y|)\, a(y)$$

which has purely absolutely continuous spectrum ([67], Volume IV, page 309). Now we know from the theorem of Weyl and Von Neumann (Theorems II and III) that we can find an operator which perturbs H_2 and has arbitrarily small norm, which gives a pure point spectrum for the sum of the two operators. It has been proved [63] that H_1 satisfies these conditions for H_2 in one dimension. The norm of H_1 is proportional to the size of the interval which is the support of $\rho(\epsilon)$; $\text{supp}\{\rho(\epsilon)\} = \left[\frac{W}{2}, \frac{W}{2}\right]$. $d|\text{supp}\,\rho(\epsilon)| = W$.

The result is : for all $W > 0$ the one dimensional Anderson model has purely pure point spectrum, that is all states are localised. So, we have shown that the result of [63] provides an example of an operator which obeys the theorem of Weyl and Von Neumann.

§ 6. The imaginary part of the one particle local Green's function

Let us denote by δ_x the element of $\ell^2(Z^d)$ such that $\delta_x(y) = \delta_{xy}$ then we have if P_λ is the spectral measure of H defined by equation (8),

$$d(\delta_x, P_\lambda \delta_x) = d\mu(\lambda) \tag{12}$$

This allows us to write the one particle local Green's function as

$$G_{xx}(E+is) = \int_{\mathbb{R}} \frac{d\mu(\lambda)}{E+is-\lambda} \tag{12}$$

which comes directly from the expectation value of $\mathcal{R}(E+is)$ in δ_x (equation (4)). The imaginary part of $G_{xx}(E+is)$ can be written as

$$\mathrm{Im}\{G_{xx}(E+is)\} = -s\int_{\mathbb{R}} \frac{du(\lambda)}{(E-\lambda)^2+s^2} \qquad (13)$$

Now it has been proved for a large number of choices of potential for model (8), by Ishii [49] (d=1)

$$\lim_{s\to 0^+} \mathrm{Im}\{G_{xx}(E+is)\} = 0 \quad \text{a.e.} \qquad (14)$$

This is equivalent to saying that $\mathcal{H}_{AC}$ is empty. It does not mean that $\mathcal{H}_c$ is empty because (14) holds when $\mathcal{H}_{s.c.} \neq \emptyset$ - see Ishii [49] and Kimball [66]. Equation (14) is a localisation criterion only for an operator with no singular continuous spectrum. That the existence of singular continuous spectrum is possible for some models was suggested by Casher and Lebowitz (who studied the disordered mass harmonic chain) and examples of models with singular continuous spectrum have been constructed by Pearson [70] (see the example in § 9).

We see from this section that we need to define a good localisation criterion, which works even in the presence of the singular continuous spectrum. This will be the purpose of the next two sections (§7 and § 8).

§7. The inverse participation ratio

The inverse participation ratio was introduced by Bell and Dean [65] and has become a criterion for localisation, which is popular for its ability to measure directly the localisation properties we are interested in. It was introduced for an operator defined on a finite lattice which consequently is compact, and so any function belonging to the Hilbert space on which H acts can be written as a linear combination of eigenvectors. Let us call the d-dimensional finite lattice Ω. We

have by analogy with § 6

$$\delta_x^{\Omega} = \sum_{\alpha=1}^{N} A_\alpha \phi_{x\alpha} \tag{15}$$

where N is the rank of the operator H, and $\{\phi\}_{\alpha=1}^{N}$ is its set of eigenvectors.

The inverse participation ratio is defined to be

$$P_x^{\Omega} = \sum_{\alpha=1}^{N} |A_\alpha|^4 \tag{16}$$

and the local Green's function is

$$G_{xx}^{\Omega}(E+is) = \sum_{\alpha=1}^{N} \frac{|A_\alpha|^2}{(E+is-\epsilon_\alpha)} \tag{17}$$

where $\{\epsilon_\alpha\}_{\alpha=1}^{N}$ is the set of eigenvalues of H. Thus, we can use (17) to write P_x^{Ω} in terms of the local Green's function

$$P_x^{\Omega} = \lim_{s\to 0^+} s\int_{\mathbb{R}} dE\, |G_{xx}^{\Omega}(E+is)|^2 \tag{18}$$

and by Fourier transforming $G_{xx}^{\Omega}(E + is)$ we obtain

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G_{xx}^{\Omega}(t)|^2 dt = P_x^{\Omega}. \tag{19}$$

We want to consider the infinite lattice Z^d. We can write the Hamiltonian defined in the box Ω as $H^{\Omega} = F_{\Omega} H F_{\Omega}$ where F_{Ω} is a projection operator which acts on $\mathcal{H}$ and projects out the functions localised in the box Ω. If we take what we shall call the strong thermodynamic limit, [71] $\overset{s\text{-}\lim}{\Omega\to Z^d} H^{\Omega} = H$ we obtain the infinite volume inverse participation ratio

$$P_x = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |G_{xx}(t)|^2 dt \tag{20}$$

where

$$\rho_{xx}(t) = -i\,(\delta_x, e^{-iHt}\delta_x) = -i\int_{\mathbb{R}} e^{-i\lambda t}\,d\mu(\lambda) \qquad (21)$$

so we obtain from Wiener's theorem (Th. IV)

$$P_x = \sum_{y\in\mathbb{R}} |\mu(\{y\})|^2 \,. \qquad (22)$$

Thus, the inverse participation ratio is only zero if there is no pure point spectrum. That is

$$P_x = 0 \iff \delta_x \in \mathcal{H}_c\,, \qquad (23.a)$$

$$\delta_x \in \mathcal{H}_p \Rightarrow P_x > 0. \qquad (23.b)$$

We can see that as it stands the inverse participation ratio is a condition for the existence of the pure point Hilbert subspace $\mathcal{H}_p$. We want to find a condition for the disappearance of the continuous Hilbert subspace $\mathcal{H}_c$. This will be done by a modification of the inverse participation ratio which we shall perform in §8.

§ 8. The localisation edge and the Anderson transition

The concepts that we discuss in this section were developed with the theory of localisation from [5] to [1]. We shall discuss the theory of localisation edges and the Anderson transition with reference to the Anderson model (equation (8)). We shall write our Hamiltonian as H(W) where W has the same meaning as in the definition of $\rho(\epsilon)$ in § 5.

We know that, when W = 0, H(0) has purely absolutely continuous spectrum ([67] Vol. IV, p. 309) and that all states are extended; by the R.A.G. theorem, if W is very much greater than one, we expect all of the states to be localised because

the diagonal term of the Anderson model would dominate the hopping term, and the diagonal term has pure point spectrum.

We denote a Borel subset of $\mathbb{R}$ by Λ when it is a subset of $\sigma(H)$, which is the spectrum of H. Now we choose Λ so that if $\phi \in \mathcal{H}$ is associated with energies in Λ only, then it is an extended state, whereas if $\phi \in \mathcal{H}$ is associated with energies in $\sigma(H(W)) \setminus \Lambda$ only, then ϕ is a localised state. So the boundary values of Λ are the localisation edges. $\Lambda \equiv \sigma_c(H(W))$ by the R.A.G. theorem. The usual picture of two localisation edges is shown in Chapter 2, figure 1(b), but our definition is more general and takes account of the possible existence of more than two localisation edges.

The Anderson transition is defined to be at the smallest value of W for which there are no extended states. To be able to find the Anderson transition or the localisation edges we need to find a good localisation criterion. As we saw in §7, the inverse participation ratio is a criterion for the existence of the pure point spectrum. This criterion can be refined to look at only a subset of the spectrum. To do this we use a theorem (Roman [72] p.644).

Theorem VI. If H is a self-adjoint operator, then its spectral family P_λ completely reduces H.

This means that if M_λ is a subspace of $\mathcal{H}$ onto which P_λ projects, then M_λ and $M_\lambda^\perp$ are both invariant under H. The restriction of H to M_λ is given by

$$H_\lambda \equiv HP_\lambda \quad , \quad \lambda \in \mathbb{R}. \tag{24}$$

If Λ is a Borel subset of $\mathbb{R}$, then an operator can be defined such that P_Λ projects onto M_Λ, which is invariant under H.

$$H_\Lambda \equiv HP_\Lambda \quad , \quad \Lambda \subset \mathbb{R} \; . \tag{25}$$

H_Λ is self-adjoint and bounded, $\sigma(H_\Lambda) \subset \Lambda$; by the properties of orthogonal spectral projections.

The spectral representation theorem (p.v.m. form) can be used to give

$$H_\Lambda \equiv \left(\int_{\mathbb{R}} \lambda \, d(P_\lambda)\right) P_\Lambda = \int_{\mathbb{R}} \lambda \, d(P_\lambda P_\Lambda)$$

$$\Rightarrow H_\Lambda = \int_\Lambda \lambda \, dP_\lambda \,. \tag{26}$$

Thus, H_Λ is related to the spectral representation of H by a simple restriction of the range of the integral to the set Λ. H_Λ has all the properties which are necessary for the results of § 7 to remain valid for it.

We can divide $\mathbb{R}$ into two non-intersecting sets, Λ and $\mathbb{R}\setminus\Lambda$. Thus, by equation (26) the operator H can be reduced by this partition of the real line.

$$H = H_\Lambda + H_{\mathbb{R}\setminus\Lambda} \,. \tag{27}$$

Now it is possible to define the localisation edges. Let Λ be a Borel subset of $\sigma(H(W))$, so that $\sigma(H_\Lambda(W)) = \sigma_c(H_\Lambda(W))$ and $\sigma(H_{\mathbb{R}\setminus\Lambda}(W)) = \sigma_p(H_{\mathbb{R}\setminus\Lambda}(W))$. We have $\sigma(H_\Lambda) = \sigma_c(H)$ and $\sigma(H_{\mathbb{R}\setminus\Lambda}) = \sigma_p(H)$. We define the localisation edges as the set of points on the real line

$$\{E_\alpha\} = \overline{\Lambda} \setminus \Lambda^\circ \tag{28}$$

where E_α is one of the edges, the bar denotes the closure of the set, and $^\circ$ denotes the interior of the set.

The Anderson transition is given when Λ shrinks to a set of measure zero (i.e. no open interval can be found which is a subset of $\sigma_c(H(W))$).

We can now define a modified inverse participation ratio. We use H_Λ as

defined by equation (26),

$$P_x^{\Lambda} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |(\delta_x, e^{-iH_\Lambda t}\delta_x)|^2 dt \qquad (29)$$

So we can say that the operator H has purely pure point spectrum when no open interval of $\sigma(H)$ can be found such that $P_x^{\Lambda} = 0$. The Anderson transition takes place at the smallest value of W for which $P_x^{\Lambda} \neq 0$ for all open subintervals Λ of $\sigma(H)$.

So we have obtained a precise characterisation of localised and extended states, and we have shown that the inverse participation ratio, when it is weighted to look at intervals of the spectrum, is a good localisation criterion which we compare with (14) which is not a good localisation criterion.

§9. Consequences for the theory of localisation

The results of this chapter have put the theory of localisation in solid state physics onto a firm rigorous base. There have been other results in this field of research mainly in one dimension, where the absence of $\mathcal{H}_c$ has been proved for the Anderson model for a large choice of probability densities [62] and [63]. This means that the heuristic arguments of Mott and Twose [25] have been justified because we know from the results of this chapter that if $\mathcal{H}_c$ is empty, then all states are localised.

It would be of interest to know whether the singular continuous Hilbert subspace plays any part in localisation theory or whether it is a technical difficulty, as it is when discussed in quantum scattering theory ([67] Vols. III and IV). We should like to know how a particle in a singular continuous state behaves, and this subject has been investigated by Pearson [70] . It is known in quantum scattering theory that a particle moving in a potential which satisfies the no bump condition (Lavine

[73]) has no singular continuous spectrum. So Pearson considered a potential with a series of bumps, and found under some conditions on the separation of the bumps the Hamiltonian of the system has singular continuous spectrum. The bump condition which we have described gives us some hope that disordered solids may provide systems in which we can observe singular continuous states. The behaviour of a particle in a singular continuous state is that it is not localised in a given region of space, but it takes an infinite amount of time to escape from a region. This can be illustrated by saying that if $\phi \in \mathcal{H}_{sc}$ then

$$\int_0^\infty |(\phi, e^{-iHt}\phi)|^2 dt = \infty \qquad (30)$$

but

$$\lim_{s \to 0^+} \int_0^\infty e^{-st} |(\phi, e^{-iHt}\phi)|^2 dt = 0 \qquad (30.b)$$

Pearson provides an example which may be of interest to solid state theorists and we shall reproduce it here :

Example : If $V(r)$ is any locally L' periodic potential, plus a sequence of bumps, that is a periodic lattice with impurities, then $\frac{-d^2}{dr^2} + V$ has a singular continuous spectrum (for energies in the absolutely continuous spectrum of the unperturbed Hamiltonian without bumps).

It must be remembered that there is a special condition on the spacing of these bumps (see [70] Proposition 3, p.35) which means that their density must decrease sufficiently rapidly as one moves along the real line (or chain).

Now we shall advance a conjecture. The problem we want to consider is an electron moving in a two-dimensional tight-binding random potential (the Anderson model defined on Z^2). There have been many recent claims about localisation in two

dimensions; Abrahams et al [74], Opperman and Wegner [75] and Wegner [76] have put forward various arguments which lead to the conclusion that all states are localised for arbitrarily weak disorder. However, numerical calculations appear to show that there is a transition when all extended states vanish, and that it is necessary to have a sufficiently strong disorder to localise all of the states. For example, the calculation by Weaire and Srivastiva [77] of the weighted inverse participation ratio and the real-space renormalisation group calculation of Lee [78] give the Anderson transition at the same critical value of the disorder in two dimensions.

Now, none of the above results are rigorous, and the computer calculations are performed on finite sized samples. However, an interesting possibility suggests itself; if, for arbitrarily weak disorder, the Hilbert subspace $\mathcal{H}_{AC}$ disappears, but $\mathcal{H}_{S.C.}$ is not empty, then at some sufficiently large disorder $\mathcal{H}_{S.C.}$ disappears too, and we have the localisation transition.

The above picture suggests an interesting analogue between the localisation problem and that of phase transitions in two dimensional spin systems. The disorder parameter plays the same role as the temperature ($W \longleftrightarrow T$); all states being localised correspond to the paramagnetic phase; the existence of extended states corresponds to an ordered phase; the $\mathcal{H}_{S.C.}$ states correspond to a topologically ordered phase (such as the Kosterlitz-Thouless phase [79]) and the $\mathcal{H}_{A.C.}$ states correspond to a phase with long range order (such as a ferromagnetic phase).

It must be stressed that the last paragraph is a speculative picture of localisation which draws upon the much better understood theory of phase transitions in lattice gases and spin systems. If we could make the above analogue precise, then we could use the wealth of knowledge developed in rigorous statistical mechanics [80] to clarify the problems in localisation theory which seem, however, to be much more difficult.

CHAPTER 6

THE LOCALISATION OF MAGNONS DEFINED ON A BETHE LATTICE

§ 1. Introduction

The localisation problem for an electron moving in a random potential which is defined to be on a Bethe lattice was first formulated by Abou-Chacra, Anderson and Thouless [7]. Their purpose was to construct a self-consistent theory of localisation as the Bethe-Peierls approximation is for the Ising model [81]. The method was developed in a series of papers (see Abou-Chacra et al [7], [17] and [18]) and we shall follow their method closely as we adapt it for the magnon problem.

The Bethe lattice is used because a great simplification in Feenberg perturbation theory [82] is obtained (see the expansion for the local self energy given by equation (4) and the surrounding discussion). The Bethe lattice has yet another advantage for the Edwards-Anderson model because when this model is defined on a Bethe lattice it has no frustration (for a discussion of frustration, see Chapter 2, § 3).

We shall complete our introduction with a brief survey of the method developed by Abou-Chacra et al. They study the Anderson model (see [5] and Chapter 5 § 5), which can be written as a time independent Schrodinger equation

$$E a_i = \epsilon_i a_i - \sum_j V_{ij} a_j \qquad (1)$$

where i and j take values on some infinite lattice Ω, the variables ϵ and V are chosen by some stochastic processes, in general, and the sum is over the nearest neighbour sites of i. The one particle Green's function is defined by the equation

$$(E - \epsilon_i) G_{ik}(E) + \sum_j V_{ij} G_{jk}(E) = \delta_{ik} \qquad (2)$$

from which a local self-energy can be defined,

$$S_i(E) = G_i^{\circ}(E)^{-1} - \{G_{ii}(E)\}^{-1} \qquad (3)$$

where $G_i^{\circ}(E) = (E - \epsilon_i)^{-1}$. The local self-energy can be expressed as a sum of terms each of which corresponds to a self avoiding closed path (this is the Feenberg expansion [82] for $S_i(E)$).

$$S_i(E) = \sum_{j \neq i} V_{ij} \{ G_j^{0-1} - S_j^{(i)}(E) \}^{-1} V_{ji}$$
$$- \sum_{j \neq i} \sum_{k \neq j,i} V_{ik} \{ G_k^{0-1} - S_k^{(ij)} \}^{-1} V_{kj}$$
$$\{ G_j^{0-1} - S_j^{(i)} \}^{-1} V_{ji} + \ldots . \qquad (4)$$

where $S_i^{(i)}$ is the self-energy for a Hamiltonian given by equation (1) but with ϵ_i equal to infinity and for $S_k^{(ij)}$ both ϵ_i and ϵ_j are equal to infinity. It is obvious that if the potential is infinite at a given site, then that site is forbidden to the particle.

The series in equation (4) is simplified by defining our model on a Bethe lattice. As we can think of the first term in the series as being given by fig. (1.a), the second term by fig. (1.b), and the third term by fig. (1.c), we can see that for a Bethe lattice only the first term is available, and so the full series is

$$S_i(E) = \sum_{j \neq i} V_{ij} \{ G_j^{0-1} - S_j^{(i)}(E) \}^{-1} V_{ji} \qquad (5)$$

Notice that this is not in fact a self-consistent equation because $S_i(E)$ is not the same as $S_i^{(i)}(E)$. The equation for $S_i^{(i)}$ is

$$S_j^{(i)} = \sum_{k \neq i,j} V_{jk} \{ G_k^{0-1} - S_k^{(ij)}(E) \}^{-1} V_{kj} \qquad (6)$$

Both equation (5) and equation (6) have the same structure. The argument [7] is that if we calculate the probability distribution density of the self energy, then we can replace $S_i^{(i)}$ by S_i in equation (5); so we obtain the self-consistent equation

$$S_i(E) = \sum_{j \neq i} V_{ij} \{ G_j^0(E)^{-1} - S_j(E) \}^{-1} V_{ji} \qquad (7)$$

Abou-Chacra et al [7] use this equation to obtain information on the distribution density of the real and imaginary parts of $S_i(E)$. The reason for choosing this quantity to study is because of the localisation criterion chosen, which is that if the imaginary part of the local self-energy is zero on the real energy axis with probability one, then all states are localised with probability one. We have not been able to give a rigorous analysis of this localisation criterion in Chapter 5, but we must be careful to consider the possibility that it is only a condition for the absence of the absolutely continuous spectrum [83].

The method of Abou-Chacra et al is followed closely in the next two sections. So, to avoid repetition, we shall develop the method for the magnon problems of §§ 2 and 3.

§2. The Edwards-Anderson model

In this section we consider the localisation of Edwards-Anderson magnons defined on a Bethe lattice using the spin-wave approximation. This method can be used for the Edwards-Anderson model when it is defined to be on a Bethe lattice because the classical ground state, about which we linearise the Green's function equations of motion, is easily found. To find the linearised equations of motion, we need to know the ground state expectation value of the z-component of the spin at each site $\langle S_o^z \rangle$.

Let us define

$$\eta_{ij} = J_{ij} / |J_{ij}| \tag{8}$$

where J_{ij} is the random exchange interaction of the Edwards-Anderson model (see Chapter 2 § 5). Choose without any loss of generality the site o to have

$$\langle S_o^z \rangle = S. \tag{9}$$

We shall show that because the Bethe lattice is a tree the classical ground state direction for a spin at site i, n lattice spacings away from o , is obtained by multiplying all of the linking η's together. That is define

$$\langle S_i^z \rangle = S \lambda_i \quad . \tag{10}$$

Then the λ's are a set of random variables which are defined as :-

$$\lambda_i = \prod_{o}^{i-1} \eta_{n\, n+1} \quad . \tag{11}$$

We can further illustrate this by figure (2) which shows the path from site o to site i.

We can now construct the linearised equations of motion for the Zubarev Green's functions [56], $\langle\langle S_i^+ ; S_j^- \rangle\rangle$ and $\langle\langle S_i^- ; S_j^+ \rangle\rangle$ (which we shall call G_{ij}^+ and G_{ij}^- respectively).

The equations of motion are derived in Appendix G, where they are obtained in the time independent form (equations (G.4) and (G.5)).

$$E G_{ij}^+ = 2\lambda_i S \delta_{ij} + S J \sum_{\ell} \{ \lambda_i G_{\ell j}^+ - \lambda_\ell G_{\ell j}^+ \} \tag{12}$$

$$-E G_{ij}^- = 2\lambda_i S \delta_{ij} + S J \sum_{\ell} \{ \lambda_i G_{\ell j}^- - \lambda_\ell G_{\ell j}^- \} \quad . \tag{13}$$

It is not the λ's which are the statistically independent random variables, but the η's.

We obtain a closed equation in terms of the η's by deriving a more complicated expression which combines equations (12) and (13). An equation is constructed for the two by two matrices constructed from G^{+}_{ij} and G^{-}_{ij} in the following manner;

$$\underline{\underline{G}}_{ij} = \begin{bmatrix} \ll S_i^{+};\, S_j^{-} \gg & 0 \\ 0 & \ll S_i^{-};\, S_j^{+} \gg \end{bmatrix} \tag{14}$$

and

$$\underline{\underline{\Lambda}}_i = \frac{1}{2} \begin{bmatrix} (1+\lambda_i) & (1-\lambda_i) \\ (1-\lambda_i) & (1+\lambda_i) \end{bmatrix} . \tag{15}$$

The Green's function matrix with which we shall work is given by

$$\underline{\underline{\tilde{G}}}_{ij} = \underline{\underline{\Lambda}}_i \, \underline{\underline{G}}_{ij} \, \underline{\underline{\Lambda}}_j . \tag{16}$$

Thus we obtain from equations (12) and (13)

$$\tfrac{1}{2}\Big[\Big\{\frac{E}{S}\,\underline{\underline{\sigma}}^{z} - J(0)\,\underline{\underline{1}}\Big\}\underline{\underline{\tilde{G}}}_{ij} + \frac{J}{2}\sum_{l} \underset{\sim}{Q}_{il}\,\underline{\underline{\tilde{G}}}_{lj}\Big] = \delta_{ij}\,\underline{\underline{1}} \tag{17}$$

where $\underline{\underline{\sigma}}^{z}$ is the z component of the set of Pauli matrices, the $\underline{\underline{1}}$ is the two by two unit matrix, $J(0) = z\,|\,J\,|$ and $\underset{\sim}{Q}_{il}$ is given by

$$\underset{\sim}{Q}_{il} = \begin{bmatrix} (\eta_{il}+1) & (\eta_{il}-1) \\ (\eta_{il}-1) & (\eta_{il}+1) \end{bmatrix} . \tag{18}$$

From equation (17) we want to obtain an equation for $\tilde{G}_{ii}(E)$ (the local Green's function matrix). The free Green's function is defined by

$$\underline{\underline{G}}^{0}(E) = 2\Big\{\frac{E}{S}\,\underline{\underline{\sigma}}^{z} - J(0)\,\underline{\underline{1}}\Big\}^{-1} \tag{19}$$

and the local self-energy $\underline{\underline{S}}_i(E)$ is defined by

$$\underline{\underline{G}}^{\circ}(E)^{-1} - \underline{\underline{S}}_i(E) = \{ \underline{\underline{\tilde{G}}}_{ii}(E) \}^{-1} \quad . \tag{20}$$

$\underline{\underline{G}}^{\circ}(E)$ and $\underline{\underline{\tilde{G}}}_{ii}(E)$ are diagonal matrices, therefore $\underline{\underline{S}}_i(E)$ is also diagonal, so we can obtain, by analogue with equation (7), the local self-energy by a self-consistent equation

$$\underline{\underline{S}}_i(E) = \sum_{j \neq i} \underline{\underline{V}}_{ij} \{ \underline{\underline{G}}^{\circ -1} - \underline{\underline{S}}_j(E) \}^{-1} \underline{\underline{V}}_{ij} \tag{21}$$

where $\underline{\underline{V}}_{ij} = \frac{J}{4} \underline{\underline{Q}}_{ij}$

We use the above self-consistency equation to generate the probability distribution for the real and imaginary parts of the self-energy matrix, and we follow Abou-Chacra et al in taking as our localisation criterion the condition that the imaginary part of the local self-energy is zero with probability one for localised states.

We shall now introduce a notational change. Consider the matrix $\underline{\underline{V}}$ in equation (21). The off-diagonal elements are always equal; call them V_{OD} and the diagonal elements are always equal; call them V_D. We shall from now on put

$$V_D^2 = V \quad \text{and} \quad V_{OD}^2 = (-V + \frac{J^2}{4}) \tag{22}$$

where the V is given by the probability distribution density,

$$P(V) = c\,\delta(V - \frac{J^2}{4}) + (1-c)\,\delta(V). \tag{23}$$

The energy E and the local self-energy are split into their real and imaginary parts

$$\underline{\underline{S}}_i(E) = \underline{\underline{H}}_i(E) - i\,\underline{\underline{\Delta}}_i(E) \; ; \; \frac{E}{2S} = R + i\eta \quad . \tag{24}$$

We can easily obtain equations for $\underline{\underline{H}}_i(E)$ and $\underline{\underline{\Delta}}_i(E)$ from equation (21),

$$\underline{\underline{H}}_i = \sum_{j \neq i} \underline{\underline{V}}_{ij} \left[\left(R\underline{\underline{\sigma}}^{-2} - \frac{J(0)}{2}\underline{\underline{1}} - \underline{\underline{H}}_j \right)^2 + \left(\eta \underline{\underline{\sigma}}^{-2} + \underline{\underline{\Delta}}_j \right)^2 \right]^{-1} \left(R\underline{\underline{\sigma}}^{-2} - \frac{J(0)}{2}\underline{\underline{1}} - \underline{\underline{H}}_j \right) \underline{\underline{V}}_{ji} \tag{25}$$

$$\underline{\underline{\Delta}}_i = \sum_{j \neq i} \underline{\underline{V}}_{ij} \left(\underline{\underline{\sigma}}^{-2}\eta + \underline{\underline{\Delta}}_j \right) \left[\left(R\underline{\underline{\sigma}}^{-2} - \frac{J(0)}{2}\underline{\underline{1}} - \underline{\underline{H}}_j \right)^2 + \left(\eta \underline{\underline{\sigma}}^{-2} + \underline{\underline{\Delta}}_j \right)^2 \right]^{-1} \underline{\underline{V}}_{ji} \tag{26}$$

Now define,

$$\underline{\underline{X}} = \underline{\underline{V}}_j \left(R\underline{\underline{\sigma}}^{-2} - \frac{J(0)}{2}\underline{\underline{1}} - \underline{\underline{H}}_j \right) \left[\left(R\underline{\underline{\sigma}}^{-2} - \frac{J(0)}{2}\underline{\underline{1}} - \underline{\underline{H}}_j \right)^2 + \left(\eta \underline{\underline{\sigma}}^{-2} + \underline{\underline{\Delta}}_j \right)^2 \right]^{-1} \underline{\underline{V}}_j \tag{27}$$

$$\underline{\underline{Y}} = \underline{\underline{V}}_j \left(\underline{\underline{\sigma}}^{-2}\eta + \underline{\underline{\Delta}}_j \right) \left[\left(R\underline{\underline{\sigma}}^{-2} - \frac{J(0)}{2}\underline{\underline{1}} - \underline{\underline{H}}_j \right)^2 + \left(\eta \underline{\underline{\sigma}}^{-2} + \underline{\underline{\Delta}}_j \right)^2 \right]^{-1} \underline{\underline{V}}_j \, . \tag{28}$$

The diagonal matrices are to have their elements denoted by

$$\underline{\underline{Z}} = \begin{bmatrix} Z^1 & 0 \\ 0 & Z^2 \end{bmatrix} \, . \tag{29}$$

The non-diagonal matrix $\underline{\underline{V}}$ is to be notated as we explained above

$$\underline{\underline{V}} = \begin{bmatrix} V_D & V_{OD} \\ V_{OD} & V_D \end{bmatrix} \, . \tag{30}$$

We use the above notation to write the following relationships among the matrix elements, from equations (27) and (28) :-

$$X^1 = \frac{\left(R - \frac{J(0)}{2} - H_j^1 \right) V_D^2}{\left(R - \frac{J(0)}{2} - H_j^1 \right)^2 + \left(\eta + \Delta_j^1 \right)^2} + \frac{\left(-R - \frac{J(0)}{2} - H_j^2 \right) V_{OD}^2}{\left(-R - \frac{J(0)}{2} - H_j^2 \right)^2 + \left(-\eta + \Delta_j^2 \right)^2} \tag{31}$$

$$X^2 = \frac{(R-\frac{J(0)}{2}H_j')V_{0D}^2}{(R-\frac{J(0)}{2}H_j')^2+(\eta+\Delta_j')^2} + \frac{(-R-\frac{J(0)}{2}H_j^2)V_D^2}{(-R-\frac{J(0)}{2}H_j^2)^2+(-\eta+\Delta_j^2)^2} \quad (32)$$

$$Y' = \frac{(\eta+\Delta_j')V_D^2}{(R-\frac{J(0)}{2}H_j')^2+(\eta+\Delta_j')^2} + \frac{(-\eta+\Delta_j^2)V_{0D}^2}{(-R-\frac{J(0)}{2}H_j^2)^2+(-\eta+\Delta_j^2)^2} \quad (33)$$

$$Y^2 = \frac{(\eta+\Delta_j')V_{0D}^2}{(R-\frac{J(0)}{2}H_j')^2+(\eta+\Delta_j')^2} + \frac{(-\eta+\Delta_j^2)V_D^2}{(-R-\frac{J(0)}{2}H_j^2)^2+(-\eta+\Delta_j^2)^2}. \quad (34)$$

We can now construct an integral equation for the distribution of (X^1, X^2, Y^1, Y^2); we obtain from equations (31), (32), (33) and (34)

$$g(X^1,X^2,Y^1,Y^2) = \int dV P(V)\, dH'dH^2d\Delta'd\Delta^2 f(H',H^2,\Delta',\Delta^2)$$
$$\delta(X^1-(31))\,\delta(X^2-(32))\,\delta(Y-(33))\,\delta(Y^2-(34)) \quad (35)$$

where $P(V)$ is that distribution density which is given by equation (23), $f(H^1,H^2,\Delta',\Delta^2)$ is the distribution density we want to obtain, and the numbers within the delta functions are the right hand sides of the so-numbered equations.

We shall call F the Fourier transform of f and G that of g. We shall define the Fourier transform by

$$G(k_1,\tilde{k}_1,k_2,\tilde{k}_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\alpha\, d\tilde{\alpha} \int_0^{\infty}\int_0^{\infty} dy\, d\tilde{y}\, g(x,\tilde{x},y,\tilde{y})$$
$$\exp[-i(k_1x+\tilde{k}_1\tilde{x}+k_2y+\tilde{k}_2\tilde{y})] \quad (36)$$

$$F(q_1,\tilde{q}_1,q_2,\tilde{q}_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, d\tilde{\alpha} \int_0^{\infty}\int_0^{\infty} dy\, d\tilde{y}\, f(x,\tilde{x},y,\tilde{y})$$
$$\exp[-i(q_1x+\tilde{q}_1\tilde{x}+q_2y+\tilde{q}_2\tilde{y})]. \quad (37)$$

If we were considering a one dimensional system, we should have the relationship

$$F(k_1,\tilde{k}_1,k_2,\tilde{k}_2) = G(k_1,\tilde{k}_1,k_2,\tilde{k}_2). \quad (38)$$

On a crystalline lattice the relationship would be very complicated, but on a Bethe

lattice the connectivity K (the coordination number minus one) enters to give a simple relationship

$$F(k_1, \tilde{k}_1, k_2, \tilde{k}_2) = \{G(k_1, \tilde{k}_1, k_2, \tilde{k}_2)\}^K \tag{39}$$

Equations (35), (36), (37) and (39) give

$$F(k_1, \tilde{k}_1, k_2, \tilde{k}_2) = \{(2\pi)^{-4} \int dV\, P(V)\, d\mathrm{H}\, d\tilde{\mathrm{H}}\, d\Delta\, d\tilde{\Delta}$$

$$dq_1\, d\tilde{q}_1\, dq_2\, d\tilde{q}_2\, F(q_1, \tilde{q}_1, q_2, \tilde{q}_2) \exp(i[q_1 \mathrm{H} + \tilde{q}_1 \tilde{\mathrm{H}} +$$

$$q_2 \Delta + \tilde{q}_2 \tilde{\Delta} - k_1(31) - \tilde{k}_1(32) - k_2(33) - \tilde{k}_2(34)])\}^K \tag{40}$$

So, we have derived the full integral equation for the distribution density F, it is, however, intractable in its full form given by equation (40).

We now try to simplify equation (40) by using our localisation condition to look at the stability of the localised states. If we assume we are in a region of our parameter space where all states are localised, then Δ and $\tilde{\Delta}$ will disappear with probability one as η tends to zero. So we shall assume that η, Δ and $\tilde{\Delta}$ are all very small, this simplifies equations (31), (32), (33) and (34) in the following way.

$$x = (R - \tfrac{J(0)}{2} - \mathrm{H})^{-1} V + (-R - \tfrac{J(0)}{2} - \tilde{\mathrm{H}})^{-1} (\tfrac{J^2}{4} - V) \tag{41}$$

$$\tilde{x} = (R - \tfrac{J(0)}{2} - \mathrm{H})^{-1} (\tfrac{J^2}{4} - V) + (-R - \tfrac{J(0)}{2} - \tilde{\mathrm{H}})^{-1} V \tag{42}$$

$$y = (\eta + \Delta) V (R - \tfrac{J(0)}{2} - \mathrm{H})^{-2} + (-\eta + \tilde{\Delta}) (\tfrac{J^2}{4} - V) (-R - \tfrac{J(0)}{2} - \tilde{\mathrm{H}})^{-2} \tag{43}$$

$$\tilde{y} = (\eta + \Delta)(\tfrac{J^2}{4} - V) (R - \tfrac{J(0)}{2} - \mathrm{H})^{-2} + (-\eta + \tilde{\Delta}) V (-R - \tfrac{J(0)}{2} - \tilde{\mathrm{H}})^{-2} \tag{44}$$

We substitute the right hand sides of equations (41), (42), (43) and (44) for the right hand sides of equation (32), (33), (34) and (35) in equation (40), which then simplifies

to

$$\{F(k_1,\tilde{k}_1,k_2,\tilde{k}_2)\}^{1/K} = (2\pi)^{-2}\int dV P(V)\, dH\, d\tilde{H}\, dq_1$$

$$d\tilde{q}_1 F\left(q_1,\tilde{q}_1, -\frac{(k_2 V+\tilde{k}_2(\frac{J^2}{4}-V))}{(R-\frac{J(0)}{2}-H)^2} - \frac{(k_2(\frac{J^2}{4}-V)+\tilde{k}_2 V)}{(-R-\frac{J(0)}{2}-\tilde{H})^2}\right)$$

$$\exp\left(i\left[q_1 H+\tilde{q}_1\tilde{H}-k_1\left[(R-\tfrac{J(0)}{2}-H)^{-1}V+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}(\tfrac{J^2}{4}\right.\right.\right.$$

$$\left.\left.\left.-V)\right]-k_1\left[(R-\tfrac{J(0)}{2}-H)^{-1}(\tfrac{J^2}{4}-V)+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}V\right]-k_2\left[\frac{\eta V}{(R-\frac{J(0)}{2}-H)^2}\right.\right.\right.$$

$$\left.\left.\left.+\frac{\eta(V-J^2/4)}{(-R-\frac{J(0)}{2}-\tilde{H})^2}\right]-\tilde{k}_2\left[\frac{\eta(\frac{J^2}{4}-V)}{(R-\frac{J(0)}{2}-H)^2}-\frac{\eta V}{(-R-\frac{J(0)}{2}-\tilde{H})^2}\right]\right]. \qquad (45)$$

We introduce two small variables, $-is$ and $-i\tilde{s}$ which replace k_2 and $\tilde{k}_2$ and we expand to the first order in s and $\tilde{s}$,

$$F(k_1,\tilde{k}_1,-is,-i\tilde{s}) = F_0(k_1,\tilde{k}_1) + s^{\beta} B(k_1,\tilde{k}_1)$$

$$+ \tilde{s}^{\tilde{\beta}}\ \tilde{B}(k_1,\tilde{k}_1)+\ldots. \qquad (46)$$

where $F_0(k_1,\tilde{k}_1)$ is obtained self-consistently by

$$F_0(k_1,\tilde{k}_1) = \{(2\pi)^{-2}\int dV P(V)\, dH\, d\tilde{H}\, dq_1\, d\tilde{q}_1 F_0(q_1,\tilde{q}_1)$$

$$\exp i\left(q_1 H+\tilde{q}_1\tilde{H}-k_1\left[(R-\tfrac{J(0)}{2}-H)^{-1}V+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}(\tfrac{J^2}{4}-V)\right]\right.$$

$$\left.-\tilde{k}_1\left[(R-\tfrac{J(0)}{2}-H)^{-1}(\tfrac{J^2}{4}-V)+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}V\right]\right)\}^K. \qquad (47)$$

From equations (45), (46) and (47) we construct a self-consistent equation for F

$$F(k_1,\tilde{k}_1,-is,-i\tilde{s}) = \{(2\pi)^{-2}\int dV P(V)\, dH\, d\tilde{H}\, dq_1\, d\tilde{q}_1$$

$$\left[F_0(q_1,\tilde{q}_1)+\left(\frac{sV+\tilde{s}(\frac{J^2}{4}-V)}{(R-\frac{J(0)}{2}-H)^2}\right)^{\beta}B(q_1,\tilde{q}_1)+\left(\frac{s(\frac{J^2}{4}-V)+\tilde{s}V}{(-R-\frac{J(0)}{2}-\tilde{H})^2}\right)^{\tilde{\beta}}\right.$$

$$\left.\tilde{B}(q_1,\tilde{q}_1)\right]\exp i\left(q_1H+\tilde{q}_1\tilde{H}-k_1\left[(R-\tfrac{J(0)}{2}-H)^{-1}V+(-R-J(0)/2\right.\right.$$

$$\left.-\tilde{H})^{-1}(\tfrac{J^2}{4}-V)\right]-\tilde{k}_1\left[(R-\tfrac{J(0)}{2}-H)^{-1}(\tfrac{J^2}{4}-V)\right.$$

$$\left.\left.+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}V\right]\right)\Big\}^{K}. \qquad (48)$$

We can further simplify equation (48) to dominant order in s and $\tilde{s}$,

$$F(k_1,\tilde{k}_1,-is,-i\tilde{s})=F_0(k_1,\tilde{k}_1)-K\left\{F_0(k_1,\tilde{k}_1)\right\}^{(K-1)/K}$$

$$(2\pi)^{-2}\int dVP(V)\,dH\,d\tilde{H}\,dq_1\,d\tilde{q}_1\left\{\left[\frac{sV+\tilde{s}(\frac{J^2}{4}-V)}{(R-\frac{J(0)}{2}-H)^2}\right]^{\beta}B(q_1,\tilde{q}_1)\right.$$

$$\left.+\left[\frac{(s(\frac{J^2}{4}-V)+\tilde{s}V)}{(-R-\frac{J(0)}{2}-\tilde{H})^2}\right]^{\tilde{\beta}}\tilde{B}(q_1,\tilde{q}_1)\right\}\exp i\left[q_1H+\tilde{q}_1\tilde{H}\right.$$

$$-k_1\left[(R-\tfrac{J(0)}{2}-H)^{-1}V+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}(\tfrac{J^2}{4}-V)\right]-\tilde{k}\left[(R-J(0)/2\right.$$

$$\left.\left.-H)^{-1}(\tfrac{J^2}{4}-V)+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}V\right]\right]. \qquad (49)$$

Introduce

$$A(y,\tilde{y})=(2\pi)^{-2}\int B(k_1,\tilde{k}_1)\exp\left[-ik_1y+i\tilde{k}_1\tilde{y}\right]dk_1\,d\tilde{k}_1 \qquad (50)$$

$$\tilde{A}(y,\tilde{y})=(2\pi)^{-2}\int\tilde{B}(k_1,\tilde{k}_1)\exp\left[-ik_1y+i\tilde{k}_1\tilde{y}\right]dk_1\,d\tilde{k}_1 \qquad (51)$$

We can obtain self-consistent expressions for the A's from equations (46) and

and (49).

$$s^{\beta}A(y,\tilde{y})+\tilde{s}^{\tilde{\beta}}\tilde{A}(y,\tilde{y}) = -(2\pi)^{-2}\int dk_1 d\tilde{k}_1 K\{F_0(k_1,\tilde{k}_1)\}^{(K-1)/K}$$

$$\int dV\,P(V)\,dH\,d\tilde{H}\Big\{\Big[\frac{(sV+\tilde{s}(\frac{J^2}{4}-V))}{(R-\frac{J(0)}{2}-H)^2}\Big]^{\beta}A(H,\tilde{H})+$$

$$\Big[\frac{(s(\frac{J^2}{4}-V)+\tilde{s}V)}{(-R-\frac{J(0)}{2}-\tilde{H})^2}\Big]^{\tilde{\beta}}\tilde{A}(H,\tilde{H})\Big\}\exp i\Big[-k_1\big((R-\tfrac{J(0)}{2}-H)^{-1}V$$

$$+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}(\tfrac{J^2}{4}-V)\big)+k_1 y+\tilde{k}_1\tilde{y}-\tilde{k}_1\big((R-\tfrac{J(0)}{2}-H)^{-1}(\tfrac{J^2}{4}-V)$$

$$+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}V\big)\Big]. \qquad (52)$$

This is not a simple generalisation of the equation obtained by Abou-Chacra et al [7]. For the case of one dimension, that is K = 1, equation (52) simplifies to give

$$s^{\beta}A(y,\tilde{y})+\tilde{s}^{\tilde{\beta}}\tilde{A}(y,\tilde{y}) = -\int dV P(V)\,dH\,d\tilde{H}$$

$$\Big\{\Big[\frac{(sV+\tilde{s}(\frac{J^2}{4}-V))}{(R-\frac{J(0)}{2}-H)^2}\Big]^{\beta}A(H,\tilde{H})+\Big[\frac{(s(\frac{J^2}{4}-V)+\tilde{s}V)}{(-R-\frac{J(0)}{2}-\tilde{H})^2}\Big]^{\tilde{\beta}}$$

$$\tilde{A}(H,\tilde{H})\Big\}\delta\big(y-\big[(R-\tfrac{J(0)}{2}-H)^{-1}V$$

$$+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}(\tfrac{J^2}{4}-V)\big]\big)\,\delta\big(\tilde{y}-\big[(R-J(0)/2$$

$$-H)^{-1}(\tfrac{J^2}{4}-V)+(-R-\tfrac{J(0)}{2}-\tilde{H})^{-1}V\big]\big). \qquad (53)$$

Unfortunately, this equation too has defied further analysis. The source of the difficulty is that we have to deal with two by two matrices, and this is directly responsible for the complexity of equations (52) and (53). So, we have derived the stability equations for the localisation edge for the magnons of the Edwards-Anderson model, but the complexity of the equations is such that no further progress has been made. So the question of the localisation properties of the Edwards-Anderson model is left unanswered by this analysis. We shall see, however, in the next section that the situation with the Mattis-Heisenberg model is different, and that we obtain a theory which is of the same order of difficulty as the random binary alloy potential for the Anderson model as it was obtained by Abou-Chacra and Thouless [17].

§3. The Mattis-Heisenberg model

The methods of Abou-Chacra et al are applied in this section to the spin-wave equations of motion for the Mattis-Heisenberg model. This model can be treated by these methods because it reduces to a problem of the same order of difficulty as the binary alloy potential for the one electron problem ([17] and [18]).

The equation of motion for the one magnon Zubarev's Green's function $\ll S_i^+ ; S_j^- \gg$ is derived in Appendix H using the technique of Zubarev [56]. Thus, we obtain the equation (from equation (H.7))

$$\sum_l \left\{ \frac{\epsilon_i E}{2S} - \tfrac{1}{2} J(0) \right\} \delta_{il} + \tfrac{1}{2} \epsilon_i \epsilon_l J_{il} \Big\} \ll S_l^+ ; S_j^- \gg = \delta_{ij} \qquad (54)$$

This equation has the peculiar property that it is like an equation for a random harmonic oscillator problem but with the possibility of having a negative mass. The method of Abou-Chacra et al can be applied exactly as it is developed in [7] and [17]. It may be thought that the random hopping term could cause some

difficulties, but we show (equation (57)) that it cancels out of the calculation.

From equation (54) we can define a local self-energy $S_i(E)$ by

$$\frac{E\epsilon_i}{2s} - \frac{Jz}{2} - S_i(E) = \{G_{ii}(E)\}^{-1} \qquad (55)$$

This equation should be compared with equation (3) and equation (2.3) of [7] and the close similarity should be noticed.

Feenberg perturbation theory [82] gives by analogue with equation (7)

$$S_i(E) = \frac{J^2}{4}\sum_j \epsilon_i\epsilon_j \left\{\frac{\epsilon_j E}{2s} - \frac{Jz}{2} - S_j(E)\right\}^{-1}\epsilon_j\epsilon_i \qquad (56)$$

Now, because for the Mattis-Heisenberg model the ϵ's take the values ± 1 randomly (see chapter 2, § 4) $\epsilon_i^2 = 1$ which gives from (56)

$$S_i(E) = \frac{J^2}{4}\sum_j \left(\frac{E\epsilon_j}{2s} - \frac{Jz}{2} - S_j(E)\right)^{-1} \qquad (57)$$

We now define $E = R + i\eta$ and $S_i(R + i\eta) = E_i - i\Delta_i$. We start to construct the distribution density for E and Δ and our localisation condition is that the imaginary part of the self energy is zero with probability one for energies R in the region of the spectrum for which the associated states are localised. The equations for E_i and Δ_i can be found from equation (57)

$$E_i = \frac{J^2}{4}\sum_j \frac{(R\epsilon_j/2s - Jz/2 - E_j)}{(R\epsilon_j/2s - Jz/2 - E_j)^2 + (\frac{\eta\epsilon_j}{2s} + \Delta_j)^2} \qquad (58)$$

$$\Delta_i = \frac{J^2}{4}\sum_j \frac{(\eta\epsilon_j/2s + \Delta_j)}{(\frac{R\epsilon_j}{2s} - \frac{Jz}{2} - E_j)^2 + (\frac{\eta\epsilon_j}{2s} + \Delta_j)^2} \qquad (59)$$

and we further define

$$X = \frac{J^2/4\,\left(\frac{R\epsilon}{2S} - \frac{Jz}{2} - E\right)}{\left(\frac{R\epsilon}{2S} - \frac{Jz}{2} - E\right)^2 + \left(\frac{\eta\epsilon}{2S} + \Delta\right)^2} \tag{60}$$

$$Y = \frac{J^2/4\,\left(\frac{\eta\epsilon}{2S} + \Delta\right)}{\left(\frac{R\epsilon}{2S} - \frac{Jz}{2} - E\right)^2 + \left(\frac{\eta}{2S} + \Delta\right)^2} \tag{61}$$

We know from [7] that if we try to construct the full probability distribution density for E and Δ using equation (58), (59), (60) and (61), we shall obtain an equation which cannot be solved (see equation (4.6) of [7]). So, we assume that we are in the regime of localised states and test for the stability of the assumption. If we have localised states, then Δ tends to zero as η tends to zero, so we consider both η and Δ to be small, and this simplifies both equations (60) and (61)

$$X = \frac{J^2}{4}\left(\frac{R\epsilon}{2S} - \frac{Jz}{2} - E\right)^{-1} \tag{62}$$

$$Y = \frac{J^2}{4}\,\frac{(\eta\epsilon/2S + \Delta)}{\left(\frac{R\epsilon}{2S} - \frac{Jz}{2} - E\right)^2} \tag{63}$$

From the unknown distribution density $f(E, \Delta)$ we write

$$g(\alpha, y) = \int d\epsilon P(\epsilon)\, dE\, d\Delta f(E,\Delta)\delta\left(\alpha - \frac{J^2}{4}\left(\frac{R\epsilon}{2S} - \frac{Jz}{2} - E\right)^{-1}\right)\delta\left(y - \frac{J^2}{4}\left(\frac{\eta}{2S} + \Delta\right)\left(\frac{R\epsilon}{2S} - \frac{Jz}{2} - E\right)^{-2}\right) \tag{64}$$

Define the Fourier transform of $g(x, y)$ by

$$G(k_1, k_2) = \int_{-\infty}^{\infty}\int_{0}^{\infty} dx\, dy \exp(-ik_1 x - ik_2 y)\, g(\alpha, y) \tag{65}$$

and call F the Fourier transform of f which is defined as in equation (65).

We obtain from equations (64) and (65)

$$G(k_1,k_2) = (2\pi)^{-1} \int d\epsilon\, P(\epsilon)\, dx\, dk_1'\, F(k_1', J^2/4x^2) \exp\left(ik_1'\left(\frac{R\epsilon}{2S} - \frac{Jz}{2} - x\right) - \frac{ik_1 J^2}{4x} - \frac{ik_2 J^2 \eta \epsilon}{8Sx^2}\right) \quad (66)$$

where $x = \frac{R\epsilon}{2S} - \frac{Jz}{2} - E$; it should not be confused with the x in equation (65).

$F(k_1, k_2)$ is related to $G(k_1, k_2)$ by the connectivity K of the lattice.

$$F(k_1,k_2) = \{G(k_1,k_2)\}^K. \quad (67)$$

So, by taking equations (66) and (67) together we have a self consistent equation for F, but to obtain an equation which can be solved we must make some further simplifications.

We put $\eta = 0$, and we assume that we are at an energy in the spectrum where the associated states are localised. Thus, we obtain a self consistent equation for F_o defined by

$$F_o(k_1) = \left\{\frac{1}{2\pi}\int d\epsilon P(\epsilon) dx\, dk_1' F_o(k_1') \exp\left(ik_1'\left(\frac{R\epsilon}{2S} - \frac{Jz}{2} - x\right) - ik_1 J^2/4x\right)\right\}^K. \quad (68)$$

We shall now take the Fourier transform of the distribution density $P(\epsilon)$

$$P(\epsilon) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} dk \exp(ik\epsilon)\, p(k). \quad (69)$$

This equation is substituted into equation (68), to give

$$F_o(k_1) = \left\{(2\pi)^{-1} \int dx\, dk_1' \, p\left(-\frac{ik_1' R}{2S}\right) F_o(k_1') \exp\left(ik_1'\left(\frac{-Jz}{2} - x\right) - ik_1 J^2/4x\right)\right\}^K. \quad (70)$$

Now we want to find the limit of the stability of our assumption that the states are localised. This is done by replacing k_2 by $-is$ in $F(k_1, k_2)$ where s is very small

and expand $F(k_1, -is)$

$$F(k_1,-is) = F_0(k_1) - s^\beta B(k_1) + \dots . \tag{71}$$

We use equations (66), (67) and (71) to obtain

$$F(k_1,-is) = F_0(k_1) - (2\pi)^{-1} K s^\beta \left(\frac{J}{2}\right)^{2\beta} \{F_0(k_1)\}^{1-1/K}$$

$$\int dx\, dk_1' \, p\left(-\frac{k_1' R}{2s}\right) B(k_1') |x|^{-2\beta} \exp\left(ik_1'\left(-\frac{Jz}{2} - x\right) - ik_1 J^2/4x\right) \tag{72}$$

We equate terms of order s^β to obtain a self-consistent equation for B

$$B(k_1) = (2\pi)^{-1} K \left(\frac{J}{2}\right)^{2\beta} \{F_0(k_1)\}^{1-1/K} \int dx\, dk_1' \, p\left(-\frac{k_1' R}{2s}\right)$$

$$B(k_1') |x|^{-2\beta} \exp\left(ik_1'\left(-\frac{Jz}{2} - x\right) - \frac{ik_1 J^2}{4x}\right) . \tag{73}$$

We shall define two distributions A and Q

$$A(x) = (2\pi)^{-1} \int p\left(\frac{-k_1 R}{2s}\right) B(k_1) \exp\left(-ik_1 x - \frac{ik_1 Jz}{2}\right) dk_1 \tag{74}$$

$$Q(x) = (2\pi)^{-1} \int p\left(\frac{-k_1 R}{2s}\right) \{F_0(k_1)\}^{1-1/K} \exp(-ik_1 x - ik_1 Jz/2)\, dk_1 . \tag{75}$$

These expressions give when they are substituted into equation (73)

$$A(y) = K \left(\frac{J}{2}\right)^{2\beta} \int A(x) Q\left(y + \frac{J^2}{4x}\right) |x|^{-2\beta} dx \tag{76}$$

where Q(x) is the probability distribution density of

$$x = \frac{R\epsilon_i}{2s} - \frac{Jz}{2} - \sum_j{}' \frac{J^2/4}{\frac{R\epsilon_j}{2s} - \frac{Jz}{2} - E_j} \tag{77}$$

and $\sum_j'$ denotes the sum over K-1 neighbours of i. These expressions should be compared with equations (6.5) and (6.6) of [7]. The similarity is not surprising because we have followed their method very closely.

The value of β is taken to be $\frac{1}{2}$ because of this formal similarity, and we obtain the integral equation

$$A(y) = \frac{K\mathcal{J}}{2} \int A(x) Q(y + \mathcal{J}^2/4x) |x|^{-1} dx . \qquad (78)$$

The condition for finding the localisation edge is that a solution to this equation must exist. That is, this equation (78) has a solution at the limit of stability of localised states.

We can now obtain the result that all states are localised in one dimension, that is, when K = 1. The demonstration of this result will follow that given by Abou-Chacra and Thouless [17]. We square both sides of equation (78) and integrate with respect to y

$$\int A(y)^2 dy = \frac{K^2\mathcal{J}^2}{4} \iiint Q(u)\, A\left(\frac{\mathcal{J}^2}{4(u-y)}\right) Q(v)\, A\left(\frac{\mathcal{J}^2}{4(v-y)}\right) \frac{du\, dv}{|u-y|\,|v-y|}\, dy . \qquad (79)$$

We use the Schwartz inequality to give

$$\int A(y)^2 dy \leqslant \frac{K^2\mathcal{J}^2}{4} \iint Q(u)Q(v) \left\{ \int A\left(\frac{\mathcal{J}^2}{4(u-y)}\right) \frac{dy}{(u-y)^2} \int A\left(\frac{\mathcal{J}^2}{4(v-z)}\right) \frac{dz}{(v-z)^2} \right\}^{1/2} du\, dv . \qquad (80)$$

Q(x) is a normalised distribution density, so we have

$$\int A(y)^2 dy \leqslant K^2 \int A(y)^2 dy . \qquad (81)$$

In one dimension $K = 1$, so we must have equality, but it is a property of the Schwartz inequality that equality holds only when $Q(x)$ is a delta function. $Q(x)$ is a delta function only in the limit of zero disorder; so we have that for arbitrarily small disorder in one dimension, the Mattis-Heisenberg has all R.P.A. magnon states localised with probability one.

The next result we want to obtain is the breakdown of the localisation assumption in the limit when there is no disorder. The calculation is performed in Appendix I, and the results are that the localisation edges are at

$$E = 0 \text{ and } E = 2z\,JS \tag{82}$$

as is given by equation (1.8). This is a good result for a lattice such as the simple cubic because these energies are the band edges for the magnons of a Heisenberg ferromagnet on a simple cubic lattice, for example, when $z = 6$. A Bethe lattice gives a different result for the pure ferromagnet's band width which is narrower than that for a standard lattice, particularly the band edge does not reach down to zero energy, which is a great worry when we remember that Goldstone's theorem tells us that $E = 0$ must be a generalised eigenvalue of the Heisenberg system. However, our calculation on the Bethe lattice for the localisation properties obtains $E = 0$ as the zero disorder localisation edge which would normally be the pure band edge.

There are great difficulties involved in trying to find out whether a solution to equation (78) exists for $K > 1$ and $0 < c < 1$ in the distribution density. The Mattis-Heisenberg model has a two delta function distribution function, as has the binary alloy problem. So, the Fredholm method is not applicable to equation (78) because we have a $Q(x)$ which is a pathalogically fluctuating function of x. The other method would be a small concentration expansion in the antiferromagnetic impurity concentration, but we would not expect this to give much information for

the Mattis-Heisenberg model because we do not expect to have strong localisation effects for small c in other than one dimension (see Chapter 7 for a further discussion of the extent of the localisation in this model). The reason why Abou-Chacra et al [18] were able to show strong localisation for their low concentration expansion is that they were able to make the impurity energy arbitrarily large, whereas the corresponding quantity in the Mattis-Heisenberg model is fixed (that is $\epsilon = \pm 1$).

So we did not give sufficiently high priority to the low concentration expansion approach to carry it out. Therefore, the main result of this section is the demonstration that all states are localised in one dimension with probability one. The self-consistent condition for the localisation edge has been derived for all K, and could be solved numerically if enough computer time is spent on it.

§ 4. Figure Captions for Chapter 6 and Figures

Figure 1 (a) We show here the only type of self avoiding path on a Bethe lattice.

(b) This shows a closed three step self avoiding path.

(c) This shows a closed four step self avoiding path.

Figure 2 We show here the only self-avoiding path between the site o and the site i.

Figure 1

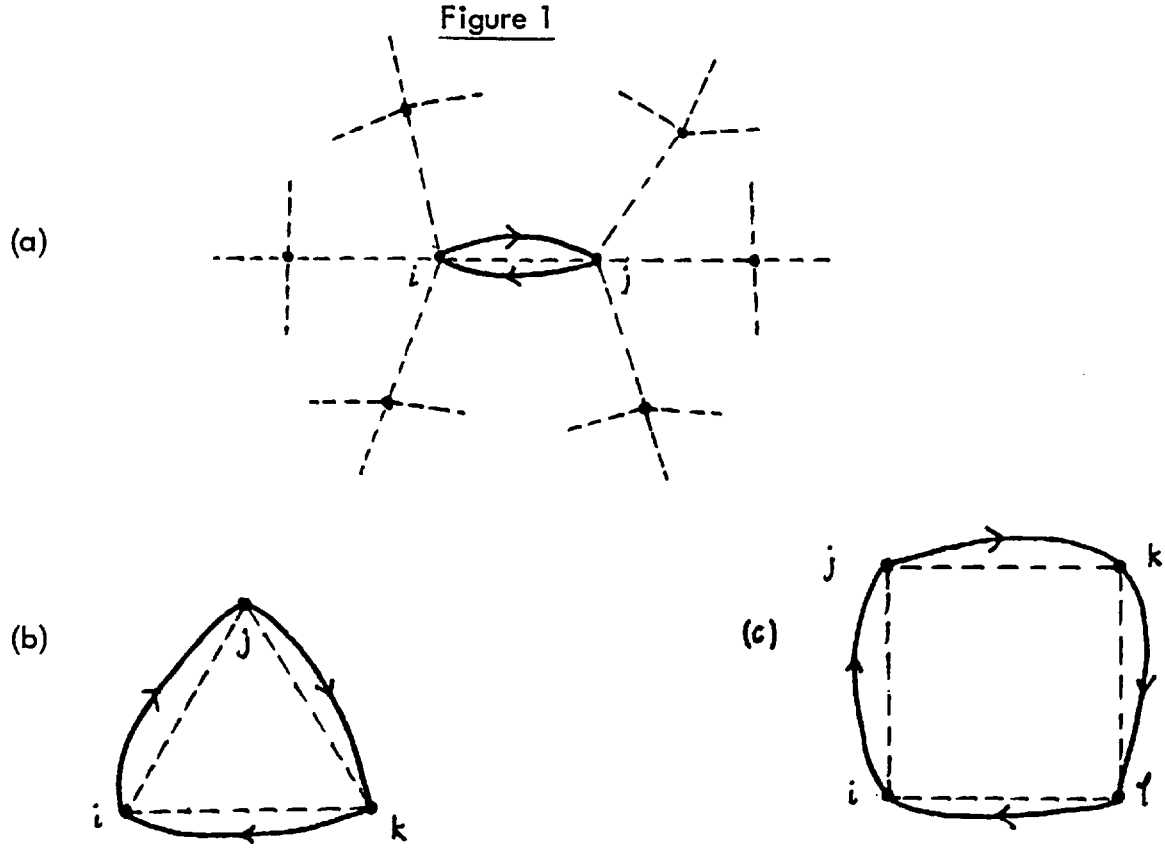
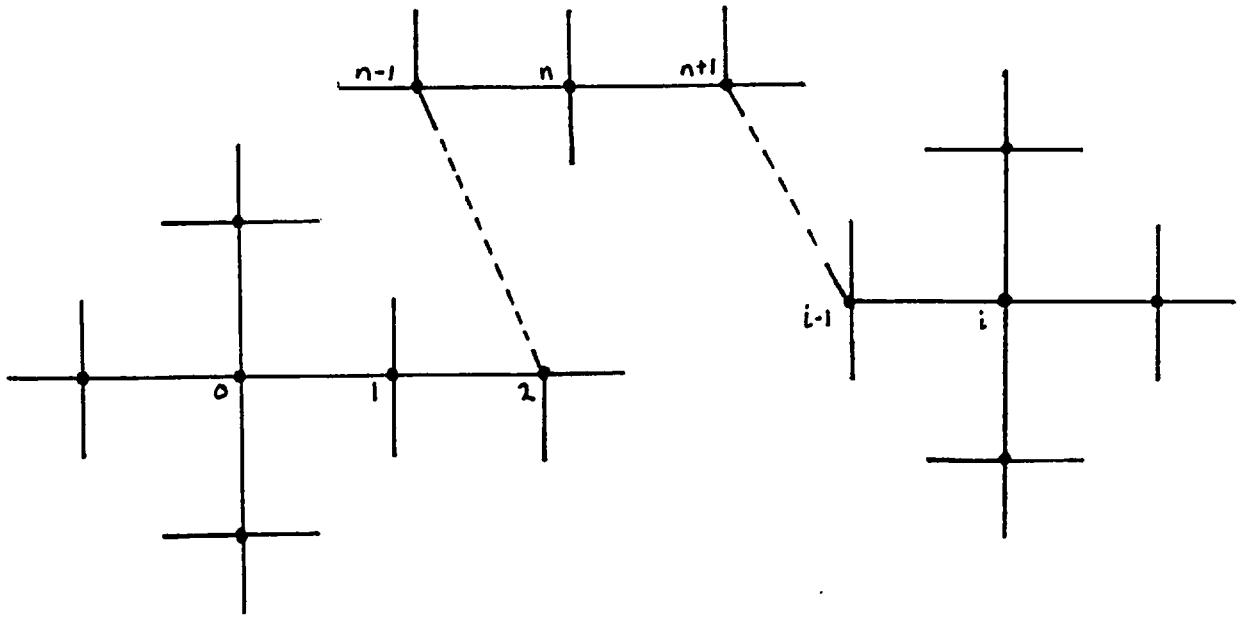


Figure 2



CHAPTER 7

THE LOCALISATION OF MAGNONS - NUMERICAL CALCULATIONS OF THE INVERSE PARTICIPATION RATIO

§1. Introduction

Most of the numerical work on the localisation problem has been concerned with electron localisation, usually in the context of the Anderson model (see Chapter 5 §5 for a discussion of this model). This is because it is widely believed that the Anderson model is the simplest non-trivial model to demonstrate the full range of localisation effects. This seems to be true but as we shall see, in the magnon problem we study finite difference operators, so there is a formal similarity between the magnon and the electron problems. We also find some physically interesting characteristics which arise more naturally in the magnon problem. These characteristics are correlated disorder (see equations (10), (14) and (16)), and next nearest neighbour hopping (equation (10)). One of the effects of the correlation we should expect is to leave the zero energy mode extended, because we know from Goldstone's theorem that the zero energy mode of any Heisenberg model must be extended.

There has been some previous work on the localisation of magnons. Walker and Walstedt [84] studied an R.K.K.Y. interaction spin glass and found a fairly strong localisation effect. There is, however, an interval of the spectrum centred on the zero energy mode for which the associated eigenstates are extended. We would like to see whether the localisation effects on non-frustrated models have the same degree of localisation or not.

We shall be calculating the inverse participation ratio by two independent methods : the method of Weaire and Williams [85] and a method using the N.A.G.-FO2AGF library routine, which FINDS ALL THE EIGENVALUES AND EIGEN-

VECTORS OF A REAL UNSYMMETRIC MATRIX.

We shall outline, for the original electron problem, the method as it was developed by Weaire and Srivastiva [86] and [77]. The adaptation to the magnon problem will follow in § 2 and § 3.

The Anderson model (Chapter 5, § 5) can be formulated as a time dependent Schrodinger equation

$$i \frac{d}{dt} a_i = \epsilon_i a_i - \sum_{<j>} a_j \qquad (1)$$

where $\{\epsilon_i\}_{i=1}^{N}$ is a set of N statistically independent random variables and $\{a_i\}_{i=1}^{N}$ is a set of site amplitudes. If we adopt as our basis set that given in Chapter 5 § 6 we can write a_i as

$$a_i \delta_i = \sum_{\alpha=1}^{N} A_\alpha^i \phi_\alpha \qquad (2)$$

where $\{\phi_\alpha\}_{\alpha=1}^{N}$ is a complete set of eigenvectors for equation (1). The inverse participation ratio to the site i of all the eigenvectors is given by

$$P_i = \sum_{\alpha=1}^{N} |A_\alpha^i|^4 \qquad (3)$$

The site averaged inverse participation ratio is given by

$$\langle P \rangle = \frac{1}{N} \sum_i \sum_\alpha |A_\alpha^i|^4 \qquad (4)$$

When all the states are extended each $|A_\alpha^i|^2 \sim 1/N$, so $\langle P \rangle \sim 1/N$, but when some states are localised, then $\langle P \rangle \gtrsim 1/N$ and tends to a non zero limit as N tends to infinity (there are possible exceptions to this - see the discussion in Chapter 5). To obtain a criterion for localisation edges we must construct an inverse participation ratio which is weighted to be sensitive to states associated with only a sub-interval of

the spectrum. The philosophy is the same as that adopted in Chapter 5 § 8; a weighting function is introduced, equation (5), which acts on our randomly chosen initial state $\Psi^0(0)$.

$$\Psi^0(0) = \sum_{i=1}^{N} a_i(0)\,\delta_i$$

where the $a_i(0)$'s are chosen by an exponential probability distribution. Let $\Psi^m(0)$ denote the weighted initial state which is obtained from $\Psi^0(0)$ by

$$\Psi^m(0) = \left[1 - E_{max}^{-2}\,(H-E_0)^2\right]^m \Psi^0(0) \qquad (5)$$

where E_{max} is the largest eigenvalue of the operator H and H denotes the operator defined by the right hand side of equation (1). The function $\Psi^m(0)$ is used to start the Runge-Kutta integration of the state and so we obtain the time dependence of the site amplitudes. We write the time dependence explicitly as

$$a_i(t) = \sum_{\alpha=1}^{N} A_\alpha^i \exp(-iE_\alpha t) \qquad (6)$$

and use equation (6) to obtain an equation for the inverse participation ratio (under the assumption, known to be true, for a wide class of distributions for the disorder, that there are no degenerate states with probability one)

$$\sum_{\alpha=1}^{N} |A_\alpha^i|^4 = 2\left(\overline{|a_i|^2}\right)^2 - \overline{|a_i|^4} \qquad (7)$$

where the bar denotes the time average. The inverse participation ratio of a normalised eigenvector which is labelled by α, which has squared modulus P_α^i on the i^{th} site, is defined as $\sum_{i=1}^{N} (P_\alpha^i)^2$. There is no correlation between the inverse participation ratio of an eigenvector and its weight in our randomly constructed wavevector, so it follows that, in the thermodynamic limit

$$\frac{N}{2}\sum_{i=1}^{N}\sum_{\alpha=1}^{N}|A_\alpha^i|^4 = N^{-1}\sum_{\alpha=1}^{N}\sum_{i=1}^{N}(P_\alpha^i)^2 \qquad (8)$$

We have that the weighted inverse participation is calculated by the formula

$$P(E_0) = \frac{N}{2}\sum_{i=1}^{N}\left(2\left(\overline{|a_i|^2}\right)^2 - \overline{|a_i|^4}\right) \qquad (9)$$

It can be seen in [77] that this method gives extremely good results for the Anderson transition in the electron problem, and this is the reason for its choice as a method for the investigation of the magnon problem.

§ 2. The application of the method of Weaire and Williams to the Mattis-Heisenberg model

We want to use a method which needs an equation of motion. The equation of motion for the site amplitudes is derived in Appendix J and is given by equation (J.3)

$$-\frac{d^2}{dt^2}a_i = z^2 a_i - \sum_j \left\{ z\left(1+\epsilon_i\epsilon_j\right) - \sum_k \epsilon_k\epsilon_j \right\} a_j \qquad (10)$$

where the sum on j is over the nearest neighbours of i, the sum on k is over the nearest neighbours of each j, and ϵ takes the value +1 with probability c and the value of -1 with probability of (1 - c). Notice that we have a next nearest neighbour term and that the differential is of second order in equation (10). We should prefer to have a first order equation with only diagonal and nearest neighbour terms, but if we take the root of equation (10) and write it in a form which is equivalent to equation (1), we obtain an unsymmetric equation which is unstable when integrated by the Runge-Kutta method. Equation (10) is manifestly symmetric and is stable when integrated by the modified

Runge-Kutta method for second order differential equations. The other modification which is necessary is to the weighting operator, which becomes

$$\Psi^m(0) = \left[1 - E_{max}^{-4}\,(H - E_o^2)^2\right]^m \Psi^o(0) \qquad (11)$$

where Ψ^o and Ψ^m have the same meanings as in the previous section. It is easily seen that $E_{max} = 2z$ (z is the coordination number of the lattice and 2z is the maximum generalised eigenvalue in the thermodynamic limit!). H is obtained from the right hand side of equation (10), and E_o is the central energy of the weighting. The averaged inverse participation ratio is found by using equation (9). We must know the half width of the weighting operator if we are going to interpret the data we obtain. Let us call the half-width at the half height Δ and we derive from equation (11)

$$\Delta = 2z\left(1 - \left(\tfrac{1}{2}\right)^{1/m}\right)^{1/4} \qquad (12)$$

The problem with this function is that Δ decreases very slowly as we increase m; for m = 11 $\Delta \sim 0.5(2z)$, that is, all energies are given strong weighting, so m = 11 is the smallest m which can be used (m = 4 was used in [77]). We could not afford the time to use more than m = 20, so we had $\Delta \sim 0.3\,(2z)$. With such a width we are sampling at least 0.6 of the band with a strong weighting.

We should expect with such a crude weighting to see only strong localisation effects, such as a transition in which all states become localised. We studied a system of 31 x 32 sites on a square lattice with periodic boundary conditions. E_o was varied across the band, and the disorder parameter c was varied between zero and one. We found no localisation effects,* but because of the crudeness of the weighting operator this is only evidence for the absence of a localisation transition. That is, three factors conspire to make the localisation effects unobservable : i) the size of Δ, ii) the weakness of the localisation, and iii) the finite size of the system.

* see the comment on this section on page 100

It is difficult, from this negative result, to obtain estimates of what proportion of the spectrum has localised eigenstates associated with it, but we expect that a region around the zero energy has extended states for arbitrarily strong disorder.

We need an independent method of investigating the localisation properties of this model, and this is provided by the work reported in § 4.

§ 3. The application of the method of Weaire and Williams to the dilute Heisenberg ferromagnet

The dilute Heisenberg ferromagnet is discussed briefly in Chapter 2, § 7, where the R.P.A. Hamiltonian is given by

$$H = -S \sum_{\langle ij \rangle} \overline{J_{ij}}\, \epsilon_i \epsilon_j \left(a_i^+ a_i + a_i^+ a_j \right) \qquad (13)$$

The construction of the equation of motion for this model from (13) is trivial

$$i \frac{d}{dt} a_i = \sum_{\langle j \rangle} \epsilon_i \epsilon_j \left(a_i - a_j \right) \qquad (14)$$

where J and S are put equal to one, and ϵ takes the values 1 or 0 randomly. Equation (14) is a first order equation like equation (1), and so no modification to the method developed in [77] and outlined in § 1 is necessary. We should note that the diagonal disorder and the hopping disorder are strongly correlated. It is thought that the major effect of such correlations is to decrease the tendency of the states to localise, [45] , and the result we obtain does not contradict this.

We looked at a 31 x 32 simple square lattice with periodic boundary conditions. The number of sites occupied was varied through the percolation concentration (see [41]) and we studied all regions of the band. No evidence of localisation was found, not even as we pass through the critical percolation concentration. Now, we know that the magnons

must be localised when we are below the percolation concentration, so what is the answer to this paradox? The answer lies in our method of calculating the averaged inverse participation ratio. Consider the case when we are well above the percolation point. The inverse participation must go as N^{-1} as N tends to infinity.

Now, we shall consider the case when there are an infinite number (of the order of N) of finite clusters. There are less than a fraction p_c of sites occupied, and they form X clusters of size n_i where $\sum_{i=1}^{X} n_i \leq N$. A cluster with n_i sites has n_i normal-modes and so the inverse participation ratio of normal-mode ϕ_α will be of order n_i^{-1} if it is localised in the i^{th} cluster. But consider the averaged inverse participation ratio $\langle P \rangle$. We have that

$$\langle P \rangle \sim N^{-1} \sum_{\alpha} 1/n^{\alpha}$$

where n^{α} is the number of sites in the cluster in which the α^{th} normal-mode is localised. The number of normal-modes belonging to a cluster of n^{α} sites is n^{α}, so $\sum_{\alpha} 1/n^{\alpha} = n^{\alpha}(1/n^{\alpha}) = X$. Therefore $\langle P \rangle \sim XN^{-1}$, and so if we are to know whether $\langle P \rangle$ is sensitive to the percolation transition or not, we need to know how X varies with c and N.

If we had used a fine enough weighting function we should have been able to find out whether there are any localised modes in the infinite cluster as the critical point is approached from above. We used the weighting operator shown in equation (5) but with the H obtained from equation (14). The maximum energy in the band is $E_{max} = 2z$, and we used m = 20.

As we said, this method is not ideally suited to an investigation of the localisation of the modes in this model. This is because the main cause of localisation is the percolation transition. However, the weakness of the localisation of the modes

above the critical concentration (that is when there is an infinite cluster) is in agreement with the results of the work by Fujiiwara [44] and Ziman and Elliott [45].

§ 4. The application of the N.A.G. library F02 AGF routine to the Mattis-Heisenberg model

The purpose of the work presented here is to check and clarify the results reported in § 2. We want to obtain an averaged inverse participation ratio which has an arbitrarily fine width to the weighting given to the different eigenmodes. The straightforward way of doing this is to find all the eigenmodes of the equation of motion, and this is what we do. The most convenient equation to use is the time independent normal mode equation which is derived in appendix J, equation (J.4)

$$\epsilon_i E\, a_i = z a_i - \sum_{<j>} \epsilon_i \epsilon_j a_j \qquad (15)$$

(If we obtain a time independent equation from equation (10) and then take the square root of both sides, in an operator sense, we obtain equation (15)).

We write a program to obtain the inverse participation ratio of each eigenmode of equation (15) defined on a one-dimensional or simple square lattice with periodic boundary conditions. When we construct the matrix from equation (15), we find it more convenient to move all of the disorder over to the right hand side,

$$E\, a_i = z\, \epsilon_i a_i - \sum_{<j>} \epsilon_j a_j \qquad (16)$$

where we use the fact that ϵ is ± 1 randomly. The matrix obtained from the right hand side of equation (16) is unsymmetric; therefore, the appropriate sub-routine for finding the complete sets of the eigenvalues and the eigenmodes is the N.A.G.

library F02 AGF routine. We can write each eigenmode as

$$\phi_\alpha = \sum_{i=1}^{N} A_\alpha^i \delta_i \qquad (17)$$

where $\{\delta_i\}_{i=1}^{N}$ is the complete set of basis vectors for the site representation, $\delta_i(j) = \delta_{ij}$. The inverse participation ratio of an eigenmode measures the order of the number of sites to which the eigenmode contributes (or in which it participates) and we write it in the notation of equation (17)

$$P_\alpha = \sum_i |A_\alpha^i|^4 \qquad (18)$$

The N.A.G. subroutine gives the eigenmodes in the representation shown in equation (17), so we use equation (18) in our program to evaluate the set $\{P_\alpha\}_{\alpha=1}^{N}$

Although we are interested in checking and clarifying the results of §2, the results for the one-dimensional lattice are also of interest, and we shall discuss these results first. We ran the program for c = 0.1 and c = 0.5 for 25 and 50 spins in the chain. The reason for running the program for two sizes of system for each value of c was to see how each P_α changed as the size of the system increased. If P_α decreased at least as quickly as N^{-1} (where N is the number of spins) as N increases, then the state is said to be extended. From our examination of the results we find that one state is unambiguously extended, and that is the zero energy mode. We notice that although all the states seem to be localised (or at least our evidence points in this direction), with the exception of the zero energy mode in one dimension the P_α's for c = 0.5 are on the whole much larger than those for c = 0.1. That is, as the disorder becomes stronger, the eigenmodes are more strongly localised.

In the two dimensional case, we ran the program for c = 0.5 for 7 x 7 and 10 x 10 spins (see the results in figure 1), and for c = 0.75 for 7 x 7 and 10 x 10 sites.

The reason for using the two sizes of sample is that the change in the inverse participation ratios as we increase N (the number of spins) gives us information on whether an eigenmode is localised or extended. The histograms shown in figures 1 and 2 are constructed by taking the average of the inverse participation ratio over the eigenvalues in each interval of one tenth of the band width. We notice, in figure 1, the change in behaviour after the third interval as we go from 49 to 100 sites. The averaged inverse participation ratio in the first three tenths of the band decrease substantially as N is increased (an indication that the states are predominantly extended), whereas in the following fourth interval there is a slight increase in the averaged inverse participation ratio. So, we can say that, for $c = 0.5$, the best candidate which we have for the localisation edge is 2.4 (on the scale shown in figure 1).

We ran the program only once for each c and N, so we do not have a set of systems with which we obtain an average over the disorder. The reason for not obtaining an average over the disorder in this manner is one of cost, and we thought it was better to obtain the largest possible system with the resources and method, rather than obtaining good statistics on systems which would be too small to show the localisation effects. So it seems that for $c = 0.5$ (the strongest disorder which is possible for the Mattis-Heisenberg model) we have a region from zero energy to about one quarter of the band width for which the associated states seem to be predominantly extended. This conclusion is not at variance with the results of § 2.

As we wanted to obtain some notion as to how the localisation properties change as we decrease the disorder, we performed the same calculation but with $c = 0.75$. If we look at figure 2, we see that this time, as we increase N, the first four tenths of the histogram have an averaged inverse participation ratio which decreases, whereas in the fifth interval it increases. This means that an energy about 3.2 is our best candidate

for a localisation edge. Notice also that we have a substantial decrease of the averaged inverse participation ratio for the eighth and ninth intervals, which suggests that the modes associated with these energies are predominantly extended. The data for the tenth interval is more ambiguous, so we cannot decide with the present data whether the modes are localised or not.

We should point out that within each interval of energies over which we average the P_ℓ 's the values of the $P_{\alpha\ell}$'s vary widely. There is evidence that some of the modes in regions of the spectrum which we have designated as predominantly localised may well be extended. The localised and extended modes may be so mixed that we can only speak of a localisation edge in the sense of dominant behaviour. So we have from figure 2 that the evidence is that for $c = 0.75$ less than half the modes are localised (compare this with three-quarters of the modes for $c = 0.5$).

In Chapter 6 §3 we mentioned that the Mattis-Heisenberg model is analagous (in R.P.A.) to the one-electron binary alloy problem on a formal level. We see in figure 2 evidence for a block of localised states in the band between energies 3.2 and 5.6, and similar behaviour was found in [18] for an electron moving in a binary-alloy-tight-binding potential. The electron calculation was on a Bethe lattice, but the comparison is nonetheless an interesting one, because it indicates that the discrete probability distribution density may be the necessary condition for such behaviour.

§ 5. A comparison of the results

The Mattis-Heisenberg model does have localised modes for the magnon equation of motion in two dimensions. The result of §2 is that the modes are too weakly localised for the effect to be observable by the method of Weaire and Williams, and so the results of §4 are needed to show that some of the modes are localised. The results of §4 also gave us more insight into the effects of the disorder

such as is found in equation (15), even though the systems which were studied were one tenth of the size of those studied in § 2.

The F02 AGF method (§ 4) showed that there is a substantial amount of localisation but no transition to all states being localised in two dimensions. In one dimension almost all states seem to be localised; the calculations have, however, shown clearly that the zero energy mode is extended ($P_o = 0.02$; $N = 50$, and $P_o = 0.04$ for $N = 25$).

In one dimension only the zero energy mode seems to be extended. In two dimensions, however, we have a large set of modes which appear to be extended. For $c = 0.5$ about one quarter of the modes are extended, and for $c = 0.75$ about one half of the modes seem to be extended. It should be noticed that for $c = 0.5$ the extended modes are associated with energies about the zero energy mode, whereas for $c = 0.75$ the extended modes are associated with a set of energies near the band edge too.

We have seen that the normal modes of a two dimensional random finite difference operator (as given by equation (15)) are not all localised for arbitrarily weak disorder. In fact, the evidence says that some states are never localised, even for $c = 0.5$ which is the strongest disorder which we can have for this model. This merely underlines the point that statements about all states being localised in two dimensions (see Chapter 5 § 9) are dependent on the model and the distribution of the stochastic process.

The normal mode equation of motion for the dilute Heisenberg ferromagnet (equation (14)) seems to have no strong localisation effect either. This model was studied only by the method of Weaire and Williams. The reason why it was not studied by the F02 AGF method is that obtaining good results for the Mattis-Heisenberg model was given priority. We simply note that the results of § 3 are consistent with previous results, [44] and [45] .

If we want to consider the results of this chapter as a contribution to a general theory of stochastic finite difference operators, we see that the processes considered here are discrete (i.e. the probability distribution densities are sums of two delta-functions) and the diagonal disorder is correlated to the off-diagonal disorder.

Now we see that the effect of such disorder is to decrease the tendency for the modes to localise. This tendency must be due, to a great extent, to the correlations, because if we consider a disordered Heisenberg model with a continuous distribution density, then we should still expect at least the zero energy mode to be extended in all dimensions because of the Goldstone theorem and the symmetry of any Heisenberg model. So, from the point of view of the study of the theory of stochastic operators we see that the theory of correlated disorder and discrete distribution densities are subjects worthy of systematic investigation, both numerically and analytically, because we see evidence for delocalisation mechanisms in these properties.

§ 6. Figure Captions for Chapter 7 and Figures

Figure 1 (a) This figure shows the inverse participation ratio averaged over each tenth of the bandwidth (P) against the energy for the Mattis-Heisenberg model on a 7 x 7 square lattice with periodic boundary conditions. The concentration is c = 0.5, which is the strongest disorder possible for this model.

(b) P is the inverse participation ratio which is averaged over each tenth of the bandwidth. The Mattis-Heisenberg model is defined on a 10 x 10 square lattice with periodic boundary conditions. The concentration c = 0.5.

Figure 2 (a) This figure shows the P which is constructed as above for the Mattis-Heisenberg model on a 7 x 7 square lattice with periodic boundary conditions. c = 0.75.

(b) $c = 0.75$. The lattice is a 10 x 10 square lattice with periodic boundary conditions. P is constructed as shown above for the Mattis-Heisenberg model.

Comments on § 2

It is perhaps useful to explain what is meant in § 2 by the statement "We found no localisation effects". We calculated the time averaged inverse participation ratio (over a certain number, T, of integration steps). We then printed the averaged value up to that time, then we continued the process in time for the same number T steps and again we printed the value obtained and so we continued for as long as possible. This allowed us to estimate, by extrapolating to infinite time, the time averaged inverse participation ratio. The extrapolated value should change as we move from a choice of parameters E_0 and c for which the states are extended to a choice for which the states are localised. We would expect the averaged inverse participation ratio to increase. We did not see this and this is what was meant by our statement above.

Figure 1

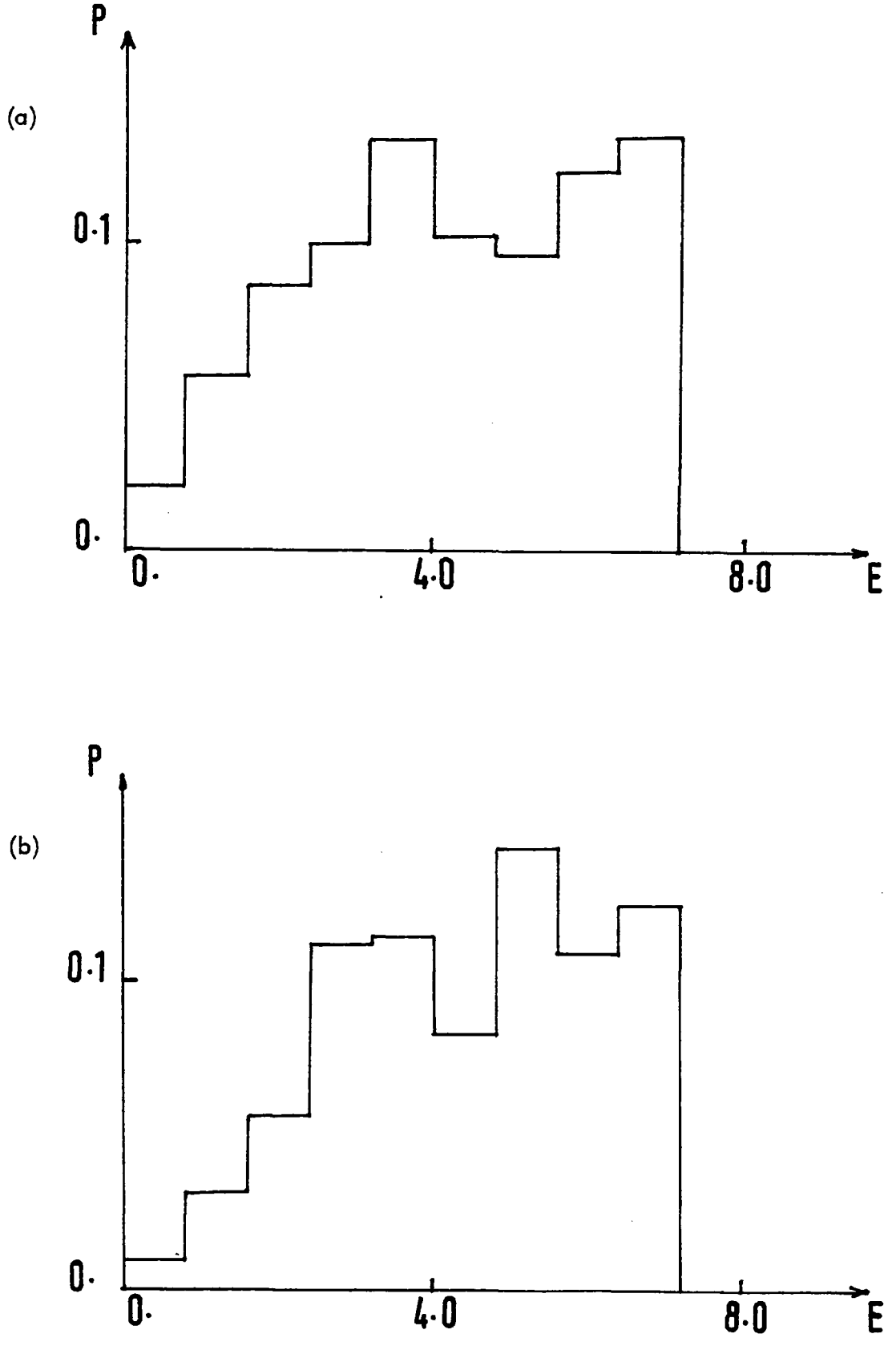
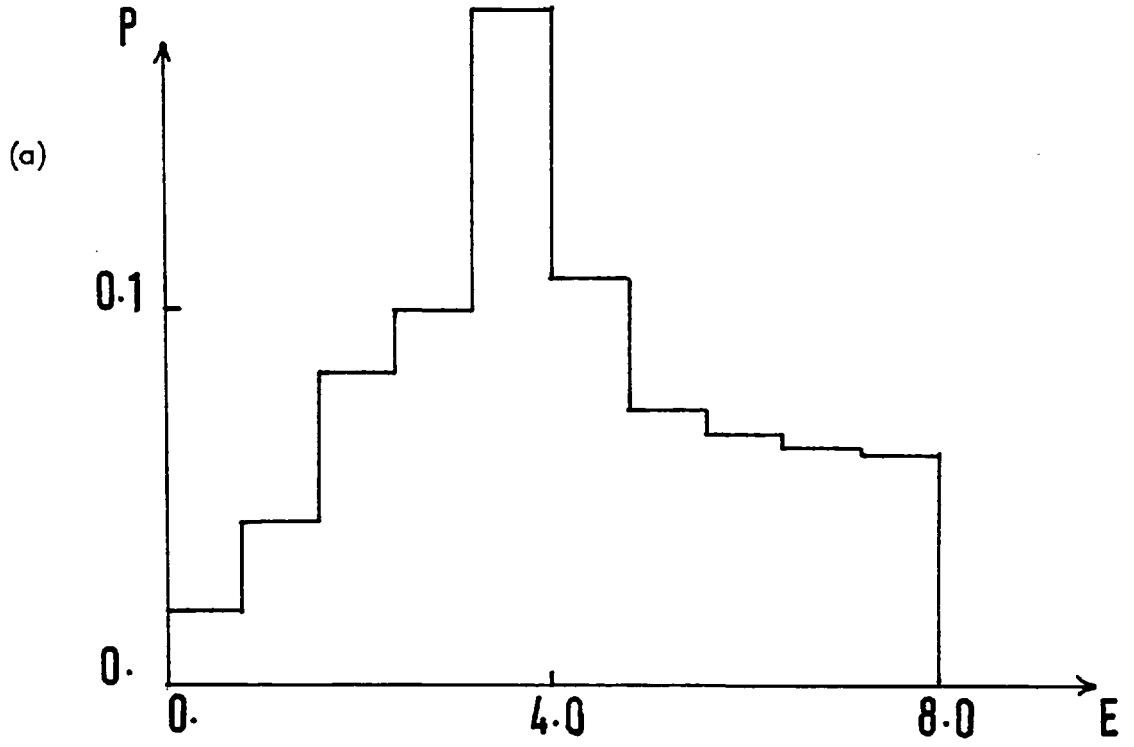
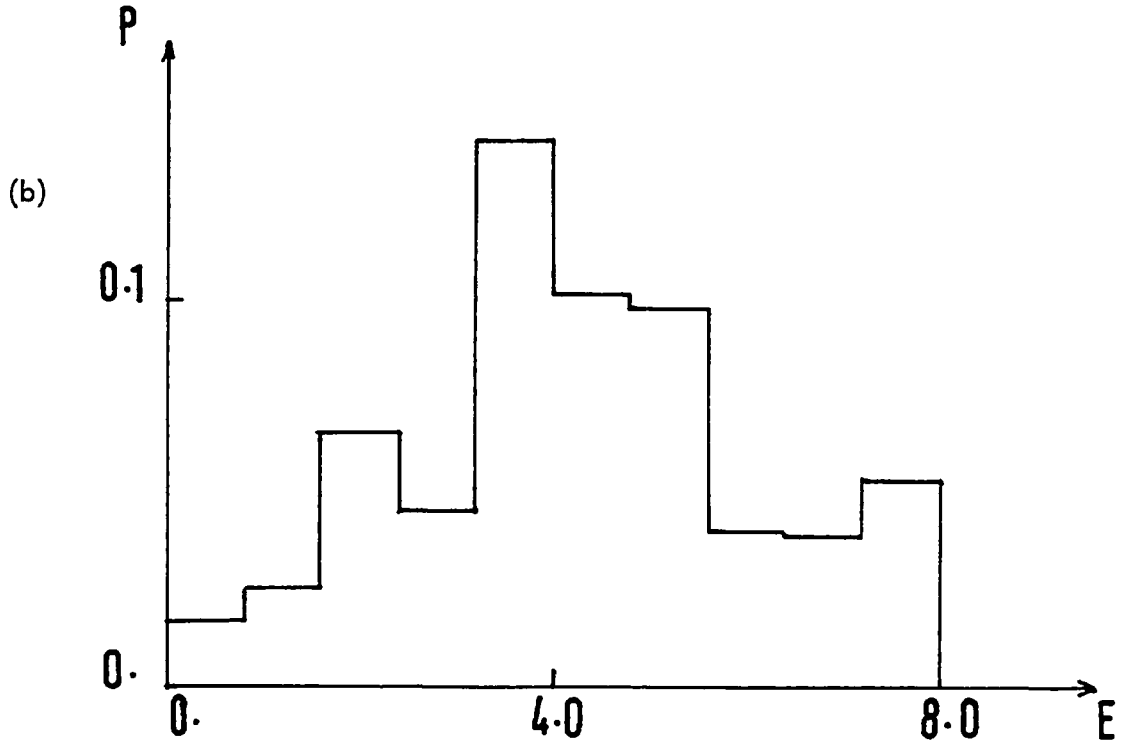


Figure 2





CHAPTER 8

THE CONCLUSION

The effects of disorder on the modes of one particle equations of motion have been investigated. Our main concern has been with the investigation of non-interacting magnons for disordered Heisenberg magnets, but we have also considered the characterisation of localised states for general stochastic operators, and, in particular, the Anderson model for an electron moving in a random potential.

The calculations in Chapters 3 and 4 of the long-wavelength dynamic response of the Mattis-Heisenberg model are important because we calculate the experimentally measurable quantities, that is the spectral density function. The results to be obtained for this domain (long-wavelength, small energy and low temperature) are almost complete in the work presented in Chapters 3 and 4. The only quantity left to be found in this regime is the width of the peaks of the spectral density function for concentrations c (see equation (2.5) for a definition of c) which are not very small ($c \ll 1$), nor almost one ($c \sim 1$), nor equal to one half ($c = 0.5$). An interesting bonus of the work in Chapter 3 is that the results are valid for the binary alloy Heisenberg spin glass discussed in Chapter 2 § 6, for c (c is defined by equation (2.16)) very small ($c \ll 1$). Thus, we have information on the spectral density function for a spin-glass which can have frustrated plaquettes, but we can only use the results of Chapter 3 in the limit where the frustrated plaquettes are not observable. So, the calculation originally for the Mattis-Heisenberg model has given us a small foothold in the theory of the long-wavelength dynamic response of a spin-glass.

The localisation of the non-interacting magnon modes is a less well studied problem than the localisation of an electron. This is because we obtain hopping disorder which is much more difficult to deal with than purely diagonal disorder, and also the

hopping disorder is usually correlated with the diagonal disorder. The reason why this is more difficult from the point of view of the analysis of the equations in one dimension is that if we use a transfer matrix technique to solve the problem, then the transfer matrices are no longer statistically independent random variables, but are correlated to their neighbours. This is a more complicated stochastic process for which no exact results are known at the moment.

There could be a simplification which would allow an analysis of the Mattis-Heisenberg model in one-dimension if we use equation (J.4) and develop a continued fraction method in which the hopping term enters squared, then we eliminate the hopping disorder because $(\epsilon_i \epsilon_j)^2 = 1$ always. This was, in fact, used in our analysis in Chapter 6.

We have obtained a non-rigorous but exact result (this description of their method was used by Abou-Chacra, Anderson and Thouless [7], but there are points of the analysis which need further justification) for the Mattis-Heisenberg model. This result says that all states are localised in one dimension with the possible exception of a set of states associated with a set of energies of measure zero. This result is supported by numerical calculations reported in Chapter 7 § 4 which show that all states seem to be localised with the exception of the zero energy mode. So all the evidence points to the conclusion that the normal modes associated with almost all energies are localised in one dimension for the Mattis-Heisenberg model.

The results in two dimensions are more complicated and we refer the reader to Chapter 7 for a detailed discussion. We notice that the combined effects of the correlated disorder and the discrete probability distribution density are to decrease the tendency for the modes to localise. This should be compared with the claims made recently (see Chapter 5 § 9) on localisation in two dimensions. We learn from this that such statements as "all modes are localised in two dimensions for arbitrarily

small disorder" must be placed in the context of a model or an ensemble of models.

Magnon equations such as equation (J.4) could be studied by analytical methods, and because the equation is similar to an equation for the modes of a disordered mass harmonic chain, the rigorous method for obtaining information on the spectral properties developed by Casher and Lebowitz [48] may be adapted to deal with the Mattis-Heisenberg problem. The fact that equation (J.4) has some masses negative, if it is interpreted as a harmonic oscillator chain problem, may lead to technical difficulties because Casher and Lebowitz seem to use the positivity of the mass in their method.

The results presented in Chapter 5 can be thought of as posing precisely the well known problems of localisation theory. This precision has more than one pay-off; our understanding of the phenomena of localisation has been improved, the recent work by mathematicians on stochastic processes can be used, formal ideas developed in quantum scattering theory can be used, developing localisation theory from a general point of view could lead to a contribution to the other trapping and confinement problems in theoretical physics, and we have a rigorous foundation on which we can develop our theory and calculate the properties of specific models.

The work on the magnon modes has provided evidence that correlations in the disorder, and discrete stochastic processes may provide important mechanisms for the delocalisation of modes. So this suggests that a systematic investigation of different stochastic processes and various correlations would be an interesting program for research in probability theory. There are, to my knowledge, few rigorous results in this field at present, and certainly there are no such results for correlated disorder and discrete probability distribution densities.

The work of Anderson in 1958 introduced the localisation problem into solid state theory. Since then much progress has been made in the understanding of the

phenomena. There rests, however, a great deal of work to do in this discipline. Some areas of future research are outlined above, but the logical next step, now that so much progress has been made in the one dimension electron problem, is to tackle the two dimensional problem and to try to obtain some rigorous results. A period of almost twenty years has passed since the heuristic arguments for localisation in one dimension were proposed by Mott and Twose. Only recently have rigorous results been obtained for this problem, proving that all states are localised. No complete proof has as yet been published. It is hoped that we do not have to wait a further twenty years for rigorous results for the two dimensional problem. Much of the work in the past twenty years has been towards developing a sufficiently deep understanding of the localisation problem to pose the questions precisely. Now that the precision is achieved, we can expect more rapid progress.

APPENDIX A

THE R.P.A. EFFECTIVE HAMILTONIAN FOR THE MATTIS-HEISENBERG MODEL

The Mattis-Heisenberg model given by equation (2.3) can be rewritten as

$$H = -\tfrac{1}{2} \sum_{ij} J_{ij}\, \epsilon_i \epsilon_j \left(\tfrac{1}{2} S_i^+ S_j^- + \tfrac{1}{2} S_i^- S_j^+ + S_i^z S_j^z \right) \qquad (1)$$

where all of the symbols have their standard meaning. We make a Holstein-Primakoff transformation to Boson creation and annihilation operators introduced by Sherrington [8]. For all i in the lattice we have

$$S_i^z = \epsilon_i \left(S - a_i^+ a_i \right) \qquad (2.a)$$

$$S_i^+ \doteq (2S)^{1/2} \left\{ (1+\epsilon_i)\, a_i/2 + (1-\epsilon_i)\, a_i^+/2 \right\} \qquad (2.b)$$

$$S_i^- \doteq (2S)^{1/2} \left\{ (1+\epsilon_i)\, a_i^+/2 + (1-\epsilon_i)\, a_i/2 \right\} \qquad (2.c)$$

where all spins $\{S_i\}_i$ have the same length S. The equations (2) are substituted into equation (1) to give an effective Hamiltonian for the magnons. We expect this transformation to be a good approximation for the non-interacting magnon dynamics at sufficiently low temperatures in three dimensions. We know that the Mattis-Heisenberg does not have a non-zero temperature phase transition in one and two dimensions [8]. So, for one and two dimensional lattices the approximation defined in equations (2) is only valid at zero temperature. The other assumption which we need for the approximation to be good is that the spin is large, that is, the spin vector is almost a classical spin vector. So, with these restrictions on the validity of our analysis we have the effective Hamiltonian from equation (1)

$$H \doteqdot -\tfrac{1}{2}\sum_{ij} J_{ij}\epsilon_i\epsilon_j \{\tfrac{1}{2}(2S[(1+\epsilon_i)a_i/2+(1-\epsilon_i)a_i^\dagger/2]$$
$$[(1+\epsilon_j)a_j^\dagger/2+(1-\epsilon_j)a_j/2]+2S[(1+\epsilon_i)a_i^\dagger/2$$
$$+(1-\epsilon_i)a_i/2][(1+\epsilon_j)a_j/2+(1-\epsilon_j)a_j^\dagger/2])$$
$$+\epsilon_i\epsilon_j(S-a_i^\dagger a_i)(S-a_j^\dagger a_j)\}. \qquad (3)$$

This expression simplifies to give

$$H \doteqdot -\tfrac{1}{2}S^2\sum_{ij}J_{ij}+S\sum_{ij}J_{ij}a_i^\dagger a_i$$
$$-\tfrac{1}{2}\sum_{ij}J_{ij}a_i^\dagger a_i a_j^\dagger a_j - \tfrac{S}{2}\sum_{ij}J_{ij}[(1+\epsilon_i\epsilon_j)a_i^\dagger a_j$$
$$-(1-\epsilon_i\epsilon_j)(a_i^\dagger a_j^\dagger + a_i a_j)/2]. \qquad (4)$$

As we stated above, we shall only be considering sufficiently low temperatures so that the magnons can be considered to be non-interacting; that is, we use an independent particle approximation. This means that we shall neglect the two particle interaction term $a_i^+ a_i a_j^+ a_j$. The final approximation gives us our effective Hamiltonian, which has been studied previously by Sherrington [8], [9] and [10]; Ching, Leung and Huber [36], and Ching and Huber [58].

$$H^{\text{eff}} = -\tfrac{1}{2}S^2\sum_{ij}J_{ij}+S\sum_{ij}J_{ij}a_i^\dagger a_i$$
$$-\tfrac{S}{2}\sum_{ij}J_{ij}[(1+\epsilon_i\epsilon_j)a_i^\dagger a_j-(1-\epsilon_i\epsilon_j)$$
$$(a_i^\dagger a_j^\dagger + a_i a_j)/2] \qquad (5)$$

This Hamiltonian is studied in Chapters 3, 4 and 7. It is further discussed in Chapter 2 §4, where it is given by equation (2.7).

APPENDIX B

THE GREEN's FUNCTION EQUATIONS FOR THE MATTIS-HEISENBERG MODEL

The equation for the Zubarev Green's function

$$\Delta_{nm}(E) = \ll (a_n^+ + a_n);\ (a_m^+ + a_m) \gg_r \qquad (1)$$

will be calculated by the equation of motion method developed in [56]. The equation is calculated using the effective Hamiltonian derived in Appendix A equation (A.5). Notice that we can see from the transformations used from spin operators to boson operators in Appendix A that

$$a_i^+ + a_i = S^{-1}(2S)^{1/2} S_i^x. \qquad (2)$$

This means that $\Delta_{nm}(E)$ is proportional to $\ll S_n^x ;\ S_m^x \gg_r$. So it is easy, from this relation, to see the physical relevance of $\Delta_{nm}(E)$ given by (1) and we do not have to do any decoupling approximation in our calculation if we want to calculate $\Delta_{nm}(E)$ using the Hamiltonian (A.5).

The derivation of the Dyson equation for the Green's function from the effective Hamiltonian H^{eff} is now presented in some detail.

$$E\Delta_{nm}(E) = \ll [(a_n^+ + a_n), H^{eff}];\ (a_m^+ + a_m) \gg_r. \qquad (3)$$

Evaluation of the commutator gives

$$E\Delta_{nm}(E) = J(0)S \ll (a_n - a_n^+);\ (a_m^+ + a_m) \gg_r$$

$$+ S \sum_k J_{nk} \ll (a_k^+ - a_k);\ (a_m^+ + a_m) \gg_r \qquad (4)$$

where $J(0) = \sum_j J_{ij}$. To obtain a closed form expression for Δ_{nm} we must go to second order in E.

$$E^2\Delta_{nm}(E) = -J(0)S(-2\delta_{nm} + S\sum_{j'} J_{nj'}\epsilon_n\epsilon_{j'}\Delta_{j'm} - J(0)S\Delta_{nm}) + S\sum_k J_{nk}(-2\delta_{km} + S\sum_j J_{kj}\epsilon_k\epsilon_j\Delta_{jm} - J(0)S\Delta_{km}). \quad (5)$$

We rearrange this equation as

$$\sum_j \{[(\frac{E}{S})^2 - J(0)^2]\delta_{nj} + J(0)J_{nj}\}\Delta_{jm} = \frac{2}{S}(J(0)\delta_{nm} - J_{nm}) + \sum_{jk}(J_{nj}J_{jk}\epsilon_j\epsilon_k - J_{nj}J_{nk}\epsilon_n\epsilon_k)\Delta_{km}. \quad (6)$$

We find that there is a great simplification when the Fourier transform is performed. The Fourier transform is defined by

$$\Delta(E;q,q') = \sum_{ij}\Delta_{ij}(E)e^{i(q\cdot R_i - q'\cdot R_j)} \quad (7)$$

Thus, we obtain from equation (6)

$$\Delta(E;q,q') = \Delta^0(E;q)\delta(q-q') - \frac{\Delta^0(E;q)}{(2\pi)^d}\frac{S}{2}\int_{B.Z.} dq_2 J'(q,q_2)\Delta(q_2,q') \quad (8)$$

where d is the dimension of the lattice, and B.Z. denotes Brillouin zone,

$$J'(q,q') = \sum_{ij} J_{ij}\epsilon_i\epsilon_j e^{i(q R_i - q' R_j)} \quad (9)$$

and

$$\Delta^{\circ}(q;E) = \frac{2S(J(0)-J(q))}{E^2 - S^2 J(0)(J(0)-J(q))} \tag{10}$$

We Fourier transform equation (8) back into real space

$$\Delta_{ij}(E) = \Delta^{\circ}_{ij}(E) - \frac{S}{2}\sum_{k\ell}\Delta^{\circ}_{ik} J_{k\ell}\epsilon_k\epsilon_\ell \Delta_{\ell j}(E) \tag{11}$$

Thus, we have obtained our equation for $\Delta_{ij}(E)$.

Another equation can be constructed for this Green's function. This will be in a form more convenient for the problem considered in Chapter 3. The method is similar to that used to derive equation (11) and so it shall not be reported here. The equation is

$$\Delta(E;q,q') = \Delta^{(0)}(q;E)\,\delta(q-q') - \frac{S}{2}\Delta^{(0)}(q;E)\int_{B.Z.}\frac{dq''}{(2\pi)^d} V(q,q'')\Delta(E;q'',q') \tag{12}$$

where

$$V(q,q') = \sum_{ij} J_{ij}\,(\epsilon_i\epsilon_j - 1)\,e^{i(q\cdot R_i - q'\cdot R_j)} \tag{13}$$

and

$$\Delta^{(0)}(q;E) = \frac{2S(J(0)-J(q))}{E^2 - S^2(J(0)-J(q))^2} \tag{14}$$

Notice that equation (14) is not the same as equation (10). We Fourier transform equation (12)

$$\Delta_{ij}(E) = \Delta^{(0)}_{ij}(E) - \frac{S}{2}\sum_{kk'}\Delta^{(0)}_{ik}(E)\,J_{kk'}(\epsilon_k\epsilon_{k'}-1)\Delta_{k'j}(E). \tag{15}$$

We shall show that equation (12) with $\Delta^{(0)}$ given by equation (14) is equivalent to equation (8) with Δ^{0} given by equation (10). Firstly notice that from equation (13) we have

$$V(q,q') = J'(q,q') - J(q)\delta(q-q')$$

substitute this expression into equation (12)

$$\Delta(E;q,q') = \Delta^{(0)}(q)\,\delta(q-q') + \frac{s}{2}\Delta^{(0)}(q)J(q)\Delta(E;q,q')$$

$$-\frac{s}{2}\Delta^{(0)}(q)\int_{B.Z.}\frac{dq''}{(2\pi)^d}J'(q,q'')\Delta(E;q'',q')$$

$$\Leftrightarrow \Delta(E;q,q') = \frac{\Delta^{(0)}(q)\delta(q-q')}{(1-\frac{s}{2}\Delta^{(0)}(q)J(q))}$$

$$-\frac{s}{2}\frac{\Delta^{(0)}(q)}{(1-\frac{s}{2}\Delta^{(0)}(q)J(q))}\int_{B.Z}\frac{dq''}{(2\pi)^d}J'(q,q'')\Delta(E;q'',q') \qquad (16)$$

where by equation (14)

$$\frac{\Delta^{(0)}(q)}{(1-\frac{s}{2}\Delta^{(0)}(q)J(q))} = \frac{2s(J(0)-J(q))}{E^2-s^2J(0)(J(0)-J(q))} \qquad (17)$$

equation (17) should be compared with equation (10). So we have shown that by a change of variables the equations (8) and (12) are equivalent.

APPENDIX D

THE EVALUATION OF THE DOMINANT IMAGINARY PART OF THE SELF-ENERGY IN CHAPTER THREE

The equation for the self-energy in the long-wavelength limit, equation (3.25), shows us that we can evaluate the imaginary part of the self-energy from the terms $cE^2\Delta_0(E)$ (where $\Delta_0(E) = G_0(E) + G_0(-E)$) and $c\{-2JE(qa)^2(G_0(E) - G_0(-E))\}$. We take $E = \omega + i\delta$ where both ω and δ are real and $\delta > 0$ (see page 22). We are interested in the complex poles of $\Delta(\omega+i\delta,q)$, defined by equation (3.28), in the limit as δ tends to zero; evaluated for small $|E|$ and q. The poles give the quasi-excitations and we find $\mathrm{Im}\,\xi \ll \mathrm{Re}\,E$ for small q.

$G_0(E)$ and $G_0(-E)$ are defined by equations (3.13a) and (3.13b); equation (3.13a) gives us

$$G_0(E) = \int_{B.Z.} \frac{dq^d/(2\pi)^d}{E - S(J(0) - J(q))} \qquad (1)$$

where B.Z. denotes the Brillouin zone. We make the long-wavelength approximation for equation (1) as we are only interested in the dominant long-wavelength behaviour.

$$G_0(\omega+i\delta) = \hat{C}_d \int_0^\infty \frac{q^{d-1}\,dq}{\omega + i\delta - SJ(qa)^2}$$

where $\hat{C}_d = \{(2\pi)^{d/2}\Gamma(d/2)\}^{-1} 2^{1-d/2}$

We now use the well known formula

$$\lim_{\delta \downarrow 0} \frac{1}{\omega - SJ(qa)^2 + i\delta} = \mathcal{P}\left(\frac{1}{\omega - SJ(qa)^2}\right) - i\pi\delta(\omega - SJ(qa)^2)$$

where $\mathcal{P}$ denotes the Cauchy principal value (see[67] Volume 1 page 137 for a discussion of this relation from the point of view of the theory of distributions).

Thus, we obtain for the imaginary part of $G_0(E)$

$$\lim_{\delta \downarrow 0} \mathrm{Im}\, G_0(\omega + i\delta) = \frac{-i\pi \hat{C}_d}{SJ} \int_0^\infty \delta(\hat{\omega} - (qa)^2)\, q^{d-1} dq$$

where $\hat{\omega} = \omega/SJ$. Therefore

$$\lim_{\delta \downarrow 0} \mathrm{Im}\, G_0(\omega + i\delta) = -\frac{i\pi \hat{C}_d}{2SJ} (\hat{\omega})^{d/2-1}. \qquad (2)$$

$G_0(-E)$ makes no contribution to the imaginary part of the self-energy. So we obtain the imaginery part of $\Sigma(q)$ from equations (2) and(3.25).

$$\mathrm{Im}\, \Sigma(q) = \frac{-ic\pi \hat{C}_d}{2(SJ)^{d/2}} \left(\omega^{\frac{d}{2}+1} - 2SJ(qa)^2 \omega^{d/2} \right). \qquad (3)$$

We replace ω in equation (3) by its dominant long-wavelength value which is given by equation (3.32); $\omega(q) = SJq^2a^2$. Thus we obtain for the dominant (long-wavelength, small concentration) imaginary part of the self-energy

$$\mathrm{Im}\, \Sigma(q) = ic\frac{\pi}{2} \hat{C}_d SJ (qa)^{d+2} \qquad (4)$$

APPENDIX F

THE EVALUATION OF EQUATION (4.29) AND (4.33) IN THE LONG WAVE-LENGTH LIMIT

The equation to be evaluated is that of the averaged T matrix of Chapter 4 § 3. This is given by equation (4.29)

$$\overline{T(0)} = \{\Delta^0_{ii}(0)\}^{-1}\left[(2\pi)^{-3}\int_{B.Z.} dq\,\left(1+\tfrac{S}{2}\Delta^0_{ii}(0)J(q)\right)^{-1} - 1\right].$$

The integral which must be evaluated is

$$(2\pi)^{-3}\int_{B.Z.} dq\,\left(1+\tfrac{S}{2}\Delta^0_{ii}(0)J(q)\right)^{-1} = W. \qquad (1)$$

This is, when the equation (4.13) is used, simply the Watson integral ([59] p. 146). So we have that

$$\overline{T(0)} = \frac{SJ(0)}{2}(W-1). \qquad (2)$$

Notice that in the evaluation of (2) no longwavelength assumption was used.

If we introduce the energy into equation (4.29) we obtain equation (4.33)

$$\overline{T(E)} = \{\Delta^0_{ii}(E)\}^{-1}\left[(2\pi)^{-3}\int_{B.Z.} dq\,\left(1+\tfrac{S}{2}\Delta^0_{ii}(E)J(q)\right)^{-1} - 1\right].$$

The integral to be evaluated is

$$I = (2\pi)^{-3}\int_{B.Z.} dq\,\left(1-\frac{J(q)}{J(0)}\left(1+\left(\frac{E}{S}\right)^2 J(0)^{-2}W\right)\right)^{-1}. \qquad (3)$$

The dominant small energy real part of this integrand is given by equation (1). So, we must evaluate the integral in the longwavelength limit to obtain the dominant

imaginary part of the integral, I becomes

$$I = \frac{(4\pi^2)^{-1}}{(\hat{E}^2+1)} \int_{-\pi}^{\pi} \frac{dq\, q^2}{q^2 - \hat{E}^2} \tag{4}$$

where $\hat{E}^2 = \left(\frac{E}{S}\right)^2 J(0)^{-2} W$ and we have used the fact that $E^2 << 1$.

$$Im\{I\} = \frac{i\lambda(E)\hat{E}}{4\pi(\hat{E}+1)} \tag{5}$$

where $\lambda(E) = \text{sign}\{Im\, E\}$.

Thus we have for the dominant parts of $\overline{T(E)}$, we write

$$\overline{T(E)} = -\frac{SJ(0)}{2}(W-1) - \frac{i}{8\pi}\lambda(E) W^{\frac{1}{2}} E. \tag{6}$$

This gives the evaluation of equation (4.33) in the limits of small E and small $|q|$.

APPENDIX G

THE LINEARISED GREEN'S FUNCTION EQUATIONS FOR THE EDWARDS-ANDERSON MODEL DEFINED ON A BETHE-LATTICE

The Edwards-Anderson model in a lattice with closed paths does not have an easily found classical ground state or long lifed metastable state about which we could define spin-wave oscillations. There is a classical ground state for the model defined on a Bethe lattice, as we have discussed in Chapter 6 § 2. We shall now derive the Green's function equations which are linearised about this ground state.

The Edwards-Anderson model (Chapter 2 § 5) can be reformulated as

$$H = -\frac{1}{2}\sum_{ij} J_{ij}\left\{\frac{1}{2}S_i^+ S_j^- + \frac{1}{2}S_i^- S_j^+ + S_i^z S_j^z\right\} . \qquad (1)$$

The method of derivation for the Green's function follows the equation of motion method developed by Zubarev [56]. We have for $\langle\langle S_i^+ ; S_j^- \rangle\rangle$

$$i\frac{d}{dt}\langle\langle S_i^+ ; S_j^- \rangle\rangle = 2\langle S_i^z\rangle \delta_{ij} + \sum_{\langle l\rangle} J_{il}\left\{\langle\langle S_l^z S_i^+ ; S_j^- \rangle\rangle - \langle\langle S_i^z S_l^+ ; S_j^- \rangle\rangle\right\} . \qquad (2)$$

A decoupling approximation is used to obtain an equation in closed form, $\langle\langle S_l^z S_i^+ ; S_j^- \rangle\rangle \rightarrow \langle S_l^z\rangle \langle\langle S_i^+ ; S_j^- \rangle\rangle$ where $\langle S_l^z\rangle$ takes its classical ground state value $S\lambda_l$ and λ_l is defined by equation (6.11). So we obtain the linearised equation

$$i\frac{d}{dt}\langle\langle S_i^+ ; S_j^- \rangle\rangle = 2\lambda_i S\delta_{ij} + SJ\sum_{\langle l\rangle}\left\{\lambda_l\langle\langle S_i^+ ; S_j^- \rangle\rangle - \lambda_l\langle\langle S_l^+ ; S_j^- \rangle\rangle\right\} . \qquad (3)$$

We Fourier transform to obtain

$$E\langle\langle S_i^+ ; S_j^- \rangle\rangle = 2\lambda_i S\delta_{ij} + SJ\sum_{\langle l \rangle}\{\lambda_i \langle\langle S_i^+ ; S_j^- \rangle\rangle - \lambda_l \langle\langle S_l^+ ; S_j^- \rangle\rangle\}. \quad (4)$$

To obtain the equation for $\langle\langle S_i^- ; S_j^+ \rangle\rangle$ we take the conjugate of equation (3) and then we Fourier transform again

$$-E\langle\langle S_i^- ; S_j^+ \rangle\rangle = 2\lambda_i S\delta_{ij} + SJ\sum_{\langle l \rangle}\{\lambda_i \langle\langle S_i^- ; S_j^+ \rangle\rangle - \lambda_e \langle\langle S_l^- ; S_j^+ \rangle\rangle\}. \quad (5)$$

APPENDIX H

THE LINEARISED GREEN'S FUNCTION EQUATIONS FOR THE MATTIS-HEISENBERG MODEL

The Mattis-Heisenberg model can be rewritten as

$$H = -\tfrac{1}{2}\sum_{ij} J_{ij}\left\{\tfrac{1}{2}S_i^+S_j^- + \tfrac{1}{2}S_i^-S_j^+ + S_i^zS_j^z\right\}\epsilon_i\epsilon_j . \qquad (1)$$

The equation of motion for the Green's function $\langle\langle S_i^+ ; S_j^- \rangle\rangle$ will now be derived by the method of Zubarev.

$$i\frac{d}{dt}\langle\langle S_i^+ ; S_j^- \rangle\rangle = 2\langle S_i^z\rangle\delta_{ij} + \sum_l J_{il}\epsilon_i\epsilon_l \left\{\langle\langle S_l^zS_i^+ ; S_j^- \rangle\rangle - \langle\langle S_i^zS_l^+ ; S_j^- \rangle\rangle\right\} . \qquad (2)$$

We close the above equation by making the following decoupling approximation

$$\langle\langle S_l^zS_i^+ ; S_j^- \rangle\rangle = \langle S_l^z\rangle\langle\langle S_i^+ ; S_j^- \rangle\rangle = \epsilon_l S\langle\langle S_i^+ ; S_j^- \rangle\rangle \qquad (3)$$

which gives, when substituted into equation (2)

$$i\frac{d}{dt}\langle\langle S_i^+ ; S_j^- \rangle\rangle = 2\epsilon_i S\delta_{ij} + S\sum_l J_{il}\left\{\epsilon_i\langle\langle S_i^+ ; S_j^- \rangle\rangle - \epsilon_l\langle\langle S_l^+ ; S_j^- \rangle\rangle\right\}. \qquad (4)$$

We perform a Fourier transform to obtain

$$E\langle\langle S_i^+ ; S_j^- \rangle\rangle = 2\epsilon_i S\delta_{ij} + S\sum_l J_{il}\left\{\epsilon_i\langle\langle S_i^+ ; S_j^- \rangle\rangle - \epsilon_l\langle\langle S_i^+ ; S_j^- \rangle\rangle\right\} . \qquad (5)$$

In Chapter 6 § 2 we shall use this equation in its local form, which we shall obtain from equation (5)

$$\frac{\epsilon_i E}{2S}\langle\langle S_i^+ ; S_j^- \rangle\rangle = \delta_{ij} + \frac{1}{2}\sum_l J_{il}\{\langle\langle S_i^+ ; S_j^- \rangle\rangle - \epsilon_i\epsilon_l\langle\langle S_l^+ ; S_j^- \rangle\rangle\} \qquad (6)$$

and we rearrange equation (6) to get

$$\sum_l \{(\frac{\epsilon_i E \delta_{il}}{2S} - \frac{1}{2}J(0))\delta_{il} + \frac{1}{2}\epsilon_i\epsilon_l J_{il}\}\langle\langle S_l^+ ; S_j^- \rangle\rangle = \delta_{ij} \qquad (7)$$

which is the Green's function equation in its local form.

APPENDIX I

THE LOCALISATION OF MATTIS-MAGNONS ON A BETHE LATTICE - THE LIMIT OF ZERO DISORDER

We want to show that the condition for the localisation edge in Chapter 6 § 3 gives the band edges in the limit of zero disorder. Consider equation (6.57)

$$S_i(E) = \frac{J^2}{4} \sum_j \left(\frac{E}{2S} \epsilon_j - \frac{Jz}{2} - S_j(E) \right)^{-1}$$

which becomes in the limit of zero disorder,

$$S = \frac{K J^2/4}{\frac{E}{2S} - \frac{Jz}{2} - S} \qquad (1)$$

where K is the connectivity of the Bethe lattice. We obtain for S, from equation (1)

$$S = \frac{1}{2}\left(\frac{E}{2S} - \frac{Jz}{2}\right) - \frac{1}{2}\left\{ \left(\frac{E}{2S} - \frac{Jz}{2}\right)^2 - KJ^2 \right\}^{1/2}. \qquad (2)$$

In the limit of zero disorder the distribution density Q(x) is a delta function,

$$Q(x) = \delta(x - a) \qquad (3)$$

where, from equation (6.77)

$$a = \frac{E}{2S} - \frac{Jz}{2} - \frac{(K-1)}{K} S \quad . \qquad (4)$$

We substitute equation (3) into equation (6.78) to obtain

$$A(y) = \frac{KJ}{2|a-y|} A\left(J^2/4(a-y)\right). \qquad (5)$$

The above equation has a delta-function solution

$$\delta(y-y_0) = \frac{kJ}{2|a-y|}\,\delta\left(\frac{J^2}{4(a-y)} - y_0\right)$$

$$= \frac{2K|a-y|}{J}\,\delta\left(y - a + \frac{J^2}{4y_0}\right) \qquad (6)$$

and this gives a solution

$$y_0 = a - J^2/4y_0 \;;\; |y_0| = kJ/2 \qquad (7)$$

from which we find

$$E = 0 \quad \text{and} \quad E = 2zJS \qquad (8)$$

where $z = K + 1$. The peculiar thing about the above result is that it does not give the band edges for the ferromagnetic spin waves on a Bethe lattice, but the result which would be expected for a lattice such as the simple cubic. That we obtain the result of equation (8) is fortunate because we should want, from Goldstone's theorem, that $E = 0$ is a generalised eigenvalue of any Heisenberg model.

APPENDIX J

THE SITE AMPLITUDE EQUATION OF MOTION FOR THE MATTIS-HEISENBERG MODEL

We construct our equation of motion from the one magnon Green's function equation shown in Appendix B equation (B.5). The method we use is shown in Ishii ([49] pages 118 and 119). We have our set of site amplitudes $\{ a_i \}_{i=1}^{N}$ and we define the Laplace transform

$$f_i(s) = \int_0^\infty dt\, e^{-st} a_i(t) \tag{1}$$

where

$$f_i(s) = i \Delta_{ij}(is) . \tag{2}$$

The Δ is the Green's function defined by equation (B.1) and the site j is an arbitrary member of the set of N sites. The equation for Δ (equation (B.5)) gives through the relationship (2) an equation for the f's and then through equation (1) we obtain, after some manipulation, an equation of motion for the site amplitudes,

$$-\frac{d^2}{dt^2} a_i = J(0)^2 S^2 a_i - S^2 \sum_j \{ J(0) J_{ij} (1 + \epsilon_i \epsilon_j) - \sum_k J_{ik} J_{kj} \epsilon_k \epsilon_j \} a_j .$$

As we shall be using this equation to perform a numerical integration we shall choose $J(0) = z$ and $J_{ij} = S = 1$.

$$-\frac{d^2}{dt^2} a_i = z^2 a_i - \sum_j \{ z(1 + \epsilon_i \epsilon_j) - \sum_k \epsilon_k \epsilon_j \} a_j . \tag{3}$$

If we Fourier transform equation (3) we obtain

$$E^2 a_i = z^2 a_i - \sum_j \{ z(1+\epsilon_i \epsilon_j) - \sum_k \epsilon_k \epsilon_j \} a_j .$$

This equation has a square root given by

$$\epsilon_i E a_i = z a_i - \sum_j \epsilon_i \epsilon_j a_j \qquad (4)$$

where we have used the fact that $\epsilon = \pm 1$. The other root is

$$-\epsilon_i E a_i = z a_i - \sum_j \epsilon_i \epsilon_j a_j \ . \qquad (5)$$

Equation (3) will be used in Chapter 7 § 2 and equation (4) will be used in Chapter 7 § 4. Equation (5) is solved when equation (4) is solved, so it is not necessary to solve both equations independently.

NOTES AND REFERENCES

1. THOULESS D.J., 1974, Physics Reports, 13, #3, 93
2. KRAMER B., WEAIRE D., 1980, Reprint, Topics in Applied Physics, Vol. 36, Amorphous Semiconductors, Ed. Brodsky M.H.
3. BINDER K., 1977, Festkörperprobleme, 17, 55
4. BINDER K., 1980, Invited paper at the E.P.S. Conference of Condensed Matter
5. ANDERSON P.W., 1958, Phys. Rev., 109, 1492
6. EDWARDS S.F., ANDERSON P.W., 1975, J. Phys. F., 5, 965
7. ABOU-CHACRA R., ANDERSON P.W., THOULESS D.J., 1973, J. Phys. C., 6, 1734
8. SHERRINGTON D., 1977, J. Phys. C, 10, L7
9. SHERRINGTON D., 1978, Phys. Rev. Lett., 41, 1321
10. SHERRINGTON D., 1979, J. Phys. C, 12, 5171
11. SHERRINGTON D., KIRKPATRICK S., 1975, Phys. Rev. Lett., 35, 1972
12. KIRKPATRICK S., SHERRINGTON D., 1978, Phys. Rev. B, 17, 4384
13. THOULESS D.J., ANDERSON P.W., PALMER R.G., 1977, Phil. Mag., 35, 593
14. SOMMERS H.J., 1978, Z. Physik B, 31, 301
15. BRAY A.J., MORE M.A., 1978, Phys. Rev. Lett., 41, 1068
16. PARISI G., 1979, Phys. Rev. Lett., 43, 1754
17. ABOU-CHACRA R., THOULESS D.J., 1974, J. Phys. C, 7, 65
18. ABOU-CHACRA R., CHAMOUN E., HOMAIDAN N., 1978, J. de Phys., 39, 624
19. See references [27], [28] and [29] .
20. SHERRINGTON D., 1978, Seminar at the I.L.L. Grenoble
21. MATTIS D.C., 1976, Phys. Lett., A 56, 421
22. VILLAIN J., 1977, J. Phys. C, 10, 1717

23. KUNZ H., SOUILLARD B., 1980, C.N.R.S. preprint. "Sur le spectre des operateurs aux differences finies aléatoires"

24. KATO T., 1966, Perturbation theory for linear operators, Springer-Verlag

25. MOTT N.F., TWOSE W.D., 1961, Adv. Phys., 10, 107

26. GOLDSHEID I.Ya., MOLCHANOV S.A., PASTUR L.A., 1977, Funkts. Anal. y Prilozhen, 11, 1

27. TOULOUSE G., 1977, Comm. on Phys., 2, 115

28. TOULOUSE G., 1979, in Non-perturbative aspects in quantum field theory, Phys. Rep., 49, 267

29. VANNIMENUS J., TOULOUSE G., 1977, J. Phys. C., 10, L537

30. NEWELL G.F., MONTROLL E.W., 1953, Rev. Mod. Phys., 25, 353

31. VAKS V.G., LARKIN A.I., OVCHINNIKOV Yu.N., 1966, Soviet Physics JETP, 22, # 4, 820

32. VILLAIN J., 1977, J. Phys. C., 10, 4793

33. LIEB E.H., in Statistical mechanics and quantum field theory, Les Houches 1970, Edited by C. De Witt, R. Stora; GORDON and BREACH

34. Ref. [33] has a section in which the statistical mechanics of the Ising model are discussed

35. Mattis-magnons means the magnons obtained from the R.P.A. to the Mattis-Heisenberg model. The term is due to D. Sherrington

36. CHING W.Y., LEUNG K.M., HUBER D.L., 1977, Phys. Rev. Lett., 39, 729

37. The experimental work in spin glasses is now vast, so we shall not give individual references, but we refer the interested reader to :- MYDOSH J.A., 1978, J. Mag . Magn. Mat., 7, 237

38. As examples of approaches to the problem of the phase transition of the Edwards-Anderson model we give the following references :-

LUBENSKY T.C., 1975, Phys. Rev. B, 11, #9, 3573, used the ϵ -expansion;

HERTZ J.A., 1978, Phys. Rev. B, 18, #9, 4875, used gauge theory methods;

SOUTHERN B.W., YOUNG A.P., 1977, J. Phys. C, 10, 2179 used a real space renormalization group method; and see references in [3], [4] and [39].

39. YOUNG A.P., 1979, J. Appl. Phys., 50, #3, 1691

40. STINCHCOMBE R.B., 1979, J. Phys. C., 12, 4533

41. SYKES M.F., GAUNT D.S., GLEN M., 1976, J. Phys. A, 9, 97

42. DE GENNES P.G., 1976, J. de Physique, 37, L1

43. GRINSTEIN G., JAYAPRAKASH C., WORTIS M., 1978, Phys. Rev. B, 19, # 1, 260

44. FUJIIWARA T., 1978, J. Phys. C, 11, 2293

45. ZIMAN T.A.L., ELLIOT R.J., 1978, J. Phys. C, 11, L847

46. DYSON F.J., 1953, Phys. Rev., 92, 1331

47. SCHMIDT H., 1957, Phys. Rev., 105, 425

48. CASHER A., LEBOWITZ J.L., 1971, J. Math. Phys., 12, 1707

49. ISHII K., 1973, Supp. Prog. Theor. Phys., 53, 77

50. For a further discussion on numerical work on disordered solids (investigation of normal modes) see Chapter 7 and references [36], [44], [58], [77], [78], [84], [85] and [86]

51. WOLFRAM T., CALLAWAY J., 1963, Phys. Rev., 30, 2207

52. IZYUMOV Y., 1966, Proc. Phys. Soc., 87, 505

53. WANG Y., CALLEN H., 1967, Phys. Rev., 160, 358

54. ISHII H., KANGMORI J., NAKAMURA T., 1965, Progr. Theoret. Phys., 33, 795

55. EDWARDS S.F., JONES R.C., 1971, J. Phys. C, 4, 2109

56. ZUBAREV D.N., 1960, Sov. Phys. Usp., 3, 320

57. MARSHALL W., LOVESEY S.W., 1971, Theory of Thermal Neutron Scattering (LONDON : Oxford University Press)

58. CHING W.Y., HUBER D.L., 1979, Preprint

59. MATTIS D.C., 1965, The Theory of Magnetism, Harper and Row, page 146

60. ECONOMOU E.N., COHEN M.H., 1972, Phys. Rev. B, 5, #8, 2931

61. PASTUR L.A., 1973, Russian Math. Surv., 28, 1

62. GOLDSHEID I.Ja., MOLCHANOV S.A., 1976, Soviet Math. Dokl., 17, 1369

63. GOLDSHEID I. Ja., MOLCHANOV S.A., PASTUR L.A., 1977, Funkts. Anal. y Prilozhen, 11, 1

64. HALPERIN, B.I., 1967, Adv. Chem. Phys., 13, 123

65. BELL R.J., DEAN P., 1970, Discuss. Faraday Soc., 50, 55

66. KIMBALL J.C., 1978, J. Phys. C., 11, 4347

67. REED M., SIMON B., 1972 - 1979. Methods of modern mathematical physics, Volumes I to IV, ACADEMIC PRESS

68. RUELLE D., 1969, Nuov. Cim., 61A, 655

69. AMREIN W.O., GEORGESCU V., 1973, Helv. Phys. Acta., 46, 635

70. PEARSON D.B., 1978, Commun. Math. Phys., 60, 13

71. Notice that the thermodynamic limit given by $\lim_{\Omega \to \mathbb{Z}^d} \rho_\alpha^\Omega \leq \rho_\alpha$, but for equation (5.20) we have taken a "strong" thermodynamic limit and so we only consider the infinite lattice.

72. ROMAN P., 1975, Some Modern Mathematics, (PERGAMON PRESS INC.)

73. LAVINE R., 1973, J. Funct. Anal., 12, 30

74. ABRAHAMS E., ANDERSON P.W., LICCIARDELLO D.C., RAMAKRISHNAN T., 1979, Phys. Rev. Lett., 42, 673

75. OPPERMANN R., WEGNER F., 1979, Z. Physik B, 34, 327

76. WEGNER F., 1979, Z. Physik B, 35, 207

77. WEAIRE D., SRIVASTIVA V., 1977, J. Phys. C, 10, 4309

78. LEE P., 1979, Phys. Rev. Lett., 42, 1492

79. KOSTERLITZ J.M., THOULESS D.J., 1973, J. Phys. C., 6, 1181

80. The subject of rigorous statistical mechanics has a very large number of publications, so we refer to :-

RUELLE D., 1969, Statistical Mechanics, W.A. BENJAMIN INC.

GRIFFITHS R.B., 1971, Phase transitions in Statistical Mechanics and Quantum Field Theory, Les Houches (1970), GORDON and BREACH

And for more recent references see RUELLE D., 1978, Thermodynamic Formalism, ADDISON-WESLEY

81. DOMB C., 1960, Adv. Phys., 9, 149

82. FEENBERG E., 1948, Phys. Rev., 74, 206

83. KUNZ H., private communication (verbal) 23.6.1980, who tells me that there is no singular continuous spectrum for the Anderson model defined on the Bethe lattice, so the absence of the absolutely continuous spectrum means that all states are localised (see the R.A.G. theorem in Chapter 5)

84. WALKER L.R., WALSTEDT R.E., 1977, Phys. Rev. Lett., 38, 514

85. WEAIRE D., WILLIAMS A.R., 1977, J. Phys. C, 10, 1239

86. WEAIRE D., SRIVASTIVA V., 1977, Comm. on Phys., 2, 73.

87. ELLIOTT R. J., KRUMHANSL J.A., LEATH P.L., 1974, Rev. Mod. Phys., 46, 465.