

Abstract of Thesis for Ph.D. degree in the Faculty of
Engineering.

submitted by E. H. Brown, Imperial College

"The Application of Photoelasticity to Stress Diffusion
Problems".

The thesis describes an analysis of the stresses in a plane sheet which stretches to infinity in all directions and is stiffened by a single semi-infinite stringer, i.e. one which stretches to infinity in one direction only. The stresses are induced by a point load applied in the line of the stringer at its end in the finite part of the plane; the reactions are at infinity.

The thesis is in two parts. The first gives an exact mathematical solution of the simplified problem in which the stringer is taken to be a "line-stiffener", i.e. to have zero width. The split plane is mapped onto the unit circle by a conformal transformation, and A.C. Stevenson's complex potentials appropriate to the problem are found as infinite series.

The second part is an account of photoelastic experiments on composite models made of C.R. 39 plastic and measuring 12" x 10". In addition to an explanation of the technique for making such models and a comparison of the experimental results with the theoretical ones, there is a description of the photoelastic bench which was designed and built as a part of this research.

THE APPLICATION OF PHOTOELASTICITY TO
STRESS-DIFFUSION PROBLEMS.

A Thesis

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The Application of Photoelasticity to Stress-Diffusion Problems.

The thesis discusses three problems of the stress-distribution in an infinite two-dimensional elastic sheet to which a semi-infinite elastic stiffener is continuously attached throughout its length (fig. 1). The first case has a load applied to the end of the stiffener, with its line of action along the stiffener and its reactions at infinity. In the other two cases the stiffener end is unloaded, but a uniform tension is applied to the sheet at infinity, in one case parallel to the stiffener, in the other perpendicular to it. Analytical and photoelastic techniques are presented, suitable to these and similar problems.

The problem of the diffusion of load into stiffened sheet has provoked attention from many elasticians (e.g. refs. 1, 2, 3, 4), and ingenious approximations have been suggested for solving practical problems, most of which have so far defied exact analysis. But the engineer - rarely a fanatical devotee of higher mathematics - is ever dissatisfied in using approximations made for mathematical expediency, however ingenious, until they are supported either by a solution which does not use them or by experiment. Or, preferably, both.

The present investigation is intended as a step towards a surer knowledge of practical structures, an exact solution with experimental confirmation of the most fundamental case of

diffusion: the stress-distribution near the end of a line stiffener, undisturbed by the effect of boundaries. The infinite structure considered, with its line stiffener, is still not an engineering proposition, but it yields to a solution free of approximations. The experimental investigation will suggest how nearly this solution approaches that of practical structures.

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PART I. ANALYSIS.

In this section Stevenson's theory of complex potentials (ref. 5) is used to find an exact solution for the stress-distribution when an infinite sheet is stiffened by a "line stiffener", a stiffener of finite area but having only a line of attachment to the sheet. Stevenson's basic theory is summarised in the Appendix.

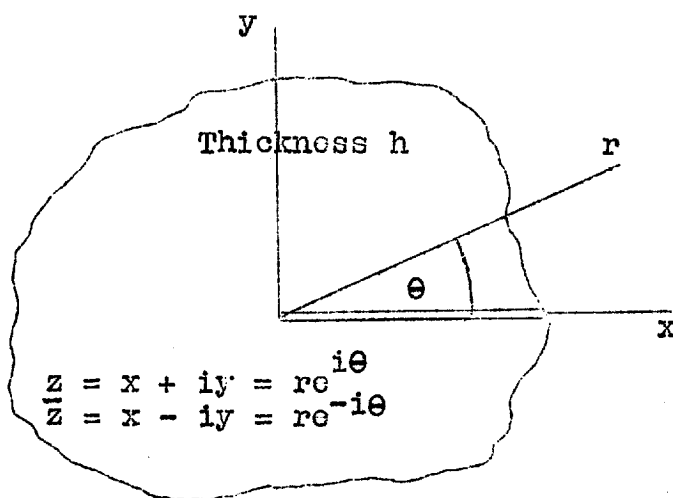


FIG. 1.

The coordinate system is shown in fig. 1. We use suffixes + and - to specify the upper and lower edges of the stiffener respectively. The plane $0 \leq \theta \leq 2\pi$ will be considered as one sheet of a Riemann surface, with a cut from the origin to infinity along the positive real axis, and $z = r_+$ and $z = \bar{r}_+$ will represent the same point $(r, 0)$, often to be written simply as $z = r$. But $z = r_-$ represents the point $(r, 2\pi)$, and

on the Riemann surface must be distinguished from $z = \bar{r}_-$, which is $(r, -2\pi)$. Other notation is as follows:-

h = thickness of sheet;

A = cross-sectional area of the stiffener, constant throughout its length;

ν = Poisson's ratio for the sheet;

G = modulus of rigidity for the sheet;

E = Young's modulus for the sheet;

E_s = Young's modulus for the stiffener;

$k = (3 - \nu) / (1 + \nu)$;

P = load in stiffener;

P_0 = externally applied load = load in stiffener at $x = 0$;

$\hat{x}\hat{x}$, $\hat{y}\hat{y}$,
 $\hat{x}\hat{y}$ = direct and shear stresses in the sheet;

$\hat{z}\hat{z} = \hat{x}\hat{x} + \hat{y}\hat{y}$;

$\hat{z}\hat{z} = \hat{x}\hat{x} - \hat{y}\hat{y} + 2i\hat{x}\hat{y}$;

σ = uniform tensile stress at infinity;

u, v = components of displacement;

$w = u + i v$;

$a(z), b(z)$ = complex potentials (see Appendix);

$\bar{P} = P_0 / h (1 + k)$;

$r_0 = \frac{2}{(1 + k)^2} \cdot \frac{AE_s}{Eh}$;

$\mu = AE_s / P_0 E$

Any complex quantity Φ may be expressed as $R\{\Phi\} + iI\{\Phi\}$, where $R\{\Phi\}$ and $I\{\Phi\}$ are both real quantities.

Other notation will be defined in the text as required.

1. Load Applied to the End of a Stiffener.

Edge conditions.

$$\text{On } \theta = 0 \quad v_+ = 0, \quad \dots\dots\dots (1)$$

$$2h \hat{xy}_+ = -dP/dr, \quad \dots\dots\dots (2)$$

$$P = AE_S du_+/dr. \quad \dots\dots\dots (3)$$

$$\text{On } \theta = 2\pi \quad v_- = 0, \quad \dots\dots\dots (4)$$

$$2h \hat{xy}_- = dP/dr, \quad \dots\dots\dots (5)$$

$$P = AE_S du_-/dr. \quad \dots\dots\dots (6)$$

The signs in eqs. (2) and (5) will be made clear by reference to fig. 2.

At infinity

$$\hat{xx} = \hat{xy} = \hat{yy} = 0, \text{ and } P = 0. \quad \dots\dots\dots (7)$$

$$\text{Finally} \quad P = P_0 \text{ when } r = 0. \quad \dots\dots\dots (8)$$

4.

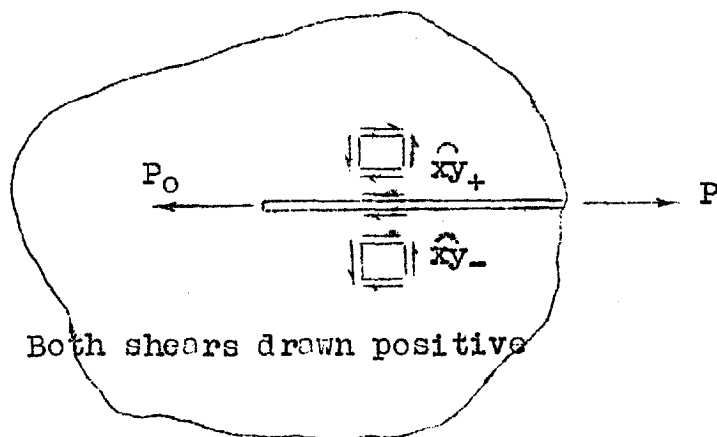


FIG. 2.

Eqs. (1) - (8) together with $u = 0$ at any arbitrarily chosen point - the origin, say, - completely specify the problem. The first step is to express them as restrictions on the complex potentials $a(z)$ and $b(z)$ (see Appendix).

Edge conditions in complex form.

Eqs. (1) and (4) hold for all r , and may be differentiated and combined with eqs. (3) and (6) to give

$$\frac{dw_+}{dr} = \frac{d\bar{w}_-}{dr} = \frac{P}{AE_S} \dots\dots\dots (9)$$

The choice of $\frac{d\bar{w}_-}{dr}$ rather than $\frac{dw_-}{dr}$ will seem less capricious later; the two are identical, since $dv_-/dr = 0$.

Using eq. (7) and the definition of $\bar{z}z$ we may integrate eq. (2) to find

$$P = h \int_r^\infty \mathcal{J} \{ \bar{z} z_+ \} dr,$$

and by eqs. (53) and (51) of the Appendix,

$$\begin{aligned} P &= h \int_r^\infty \mathcal{J} \{ \bar{b}'(r) - r \bar{a}''(r) \} dr, \\ &= h \left| \int \mathcal{J} \{ 4G w_+ + \bar{a}(r) - k a(r) \} \right|_r^\infty. \end{aligned}$$

Now $\oint \{w_+\} = v_+ = 0$, and $\oint \{\bar{a}(r)\} = - \oint \{a(r)\}$, so that

$$P = h(1+k) \oint \{a(r) - a(\infty)\} \text{ on } \theta = 0. \dots (10)$$

It may be seen from the expressions derived in the Appendix that the addition of a constant to the potentials $a(z)$ and $b(z)$ only affects the displacements, leaving the stresses unaltered. It is therefore legitimate to put $\oint \{a(0)\} = 0$, since the displacement at the origin may still be controlled by constants added to $b(z)$. Then if we write $\bar{P} = P_0/h(1+k)$, equation (8) becomes

$$\oint \{a(\infty)\} = -\bar{P} \text{ on } \theta = 0 \dots \dots \dots (11)$$

By the same argument as derived equation (10) from equation (2) we may convert equation (5) into

$$P = -h(1+k) \oint \{a(r_-) - a(\infty)\} \text{ on } \theta = 2\pi \dots \dots \dots (12)$$

Equations (7) become

$$\hat{z}\hat{z} = \hat{z}\hat{z} = 0, \quad P = 0 \text{ at infinity} \dots \dots \dots (13)$$

Equations (1) - (8) have now been replaced by equations (9) - (13), which in turn specify the problem completely.

For continuity at the origin $\oint \{a(0)\} = 0$ on $\theta = 2\pi$ as well as on $\theta = 0$, so that from equation (12) when $r_- = 0$, $\oint \{a(\infty)\} = +\bar{P}$ on $\theta = 2\pi$ (cf. equation (11)).

Using equations (10) and (11), (12) becomes

$$\oint \{a(r_-)\} = \bar{P} - P/h(1+k) = - \oint \{a(r)\},$$

or

$$\oint \{a(r_-)\} = \oint \{\bar{a}(r)\} \dots \dots \dots (14)$$

6.

For symmetry $\widehat{xx}_+ = \widehat{xx}_-$ and $\widehat{yy}_+ = \widehat{yy}_-$, so that from equation (52) of the Appendix

$$R\{a'(r_-)\} = R\{a'(\bar{r})\} = R\{\bar{a}'(r)\}.$$

Combining this with the result of differentiating equation (14) we have, since $dr_-/dr = 1$,

$$a'(r_-) = \bar{a}'(r) \quad \dots\dots\dots (15)$$

This conjugate relation must also hold between the second derivatives of $a(z)$, since $dr_-/dr = 1$. The first of equations (9) in terms of the complex potentials is

$$k a'(r) - \bar{a}'(r) - r\bar{a}''(r) + \bar{b}'(r) = k \bar{a}'(\bar{r}_-) - a'(r_-) - r a''(r_-) + b'(r_-)$$

giving a term-by-term correspondence and leaving

$$b'(r_-) = \bar{b}'(r) \quad \dots\dots\dots (16)$$

We may eliminate P and $a(\infty)$ between equations (10), (11), and the second of equations (9):

$$\bar{P} + \frac{1}{2}\{a'(r)\} = r_0 \{k a'(r) - \bar{a}'(r) - r\bar{a}''(r) + \bar{b}'(r)\} \quad \dots\dots\dots (17)$$

where

$$r_0 = \frac{AE_S}{4Gh(1+k)} = \frac{2}{(1+k)^2} \cdot \frac{AE_S}{Eh}$$

Equations (9) - (12) have now been reduced to equations (15) - (17), and we have to find two potential functions satisfying these three restrictions and giving zero stresses at infinity. Equations (15) and (16) show the nature of the discontinuity across

the stiffener: the imaginary part of both $a'(z)$ and $b'(z)$ changes sign when a closed circuit is made around the origin starting on $\theta = 0$ and finishing on $\theta = 2\pi$. This leads us to consider first a transformation of the type

$$z_1^2 = z,$$

which maps the plane $0 \leq \theta \leq 2\pi$ on the half-plane $0 \leq \theta_1 \leq \pi$. We then use a bilinear transformation to map this half-plane onto the unit circle.

Conformal transformations.

The first transformation is made non-dimensional by the form

$$r_0 z_1^2 = z. \quad \dots\dots\dots (18)$$

This maps the split z -plane onto the upper half of the z_1 -plane, and $\theta = 0 \rightarrow \theta_1 = 0$, $\theta = 2\pi \rightarrow \theta_1 = \pi$.

$$\begin{aligned} \text{If } a(z) &\rightarrow a_1(z_1), \\ \text{then } a'(z) &\rightarrow a_1'(z_1) / 2r_0 z_1, \\ \text{and } a''(z) &\rightarrow (1/2r_0 z_1)^2 \left\{ a_1''(z_1) - a_1'(z_1) / z_1 \right\}. \end{aligned}$$

The second transformation is

$$\zeta = \rho e^{i\psi} = \frac{1 + i z_1}{1 - i z_1}, \text{ or } z_1 = i. \frac{1 - \zeta}{1 + \zeta}. \quad \dots\dots\dots (19)$$

$$|\zeta|^2 = \zeta \bar{\zeta} = \frac{1 + r_1^2 - 2 r_1 \sin \theta_1}{1 + r_1^2 + 2 r_1 \sin \theta_1},$$

so that the region $0 < \theta_1 < \pi$ maps within the unit circle, and

$$\begin{aligned} \theta = 0 \rightarrow \theta_1 = 0 &\rightarrow \rho = 1, \quad 0 \leq \psi \leq \pi, \\ \theta = 2\pi \rightarrow \theta_1 = \pi &\rightarrow \rho = 1, \quad 0 \geq \psi \geq -\pi. \end{aligned}$$

argument is unchanged. This will be satisfied if we restrict $\alpha'(\xi)$ and $\beta'(\xi)$ to real functions : functions which take only real values, in the region considered, for real values of the argument.

It remains to transform equation (17), which becomes

$$\bar{P} + \mathcal{J}\{a_1(r_1)\} = (1/4r_1) \{ 2k a_1'(r_1) - \bar{a}_1'(r_1) - r_1 \bar{a}_1''(r_1) + 2\bar{b}_1'(r_1) \},$$

and after taking conjugates and some trouble,

$$\bar{P} + \mathcal{J}\{\alpha(e^{i\psi})\} = -i \cot \frac{\psi}{2} \cos^2 \frac{\psi}{2} \{ k e^{-i\psi} \alpha'(e^{-i\psi}) + \frac{e^{2i\psi}}{2} \alpha'(e^{i\psi}) - e^{i\psi} \beta'(e^{i\psi}) + \frac{i e^{2i\psi}}{2} \sin \psi \alpha''(e^{i\psi}) \} \dots (21)$$

The problem is now solved if we can find functions $\alpha(\xi)$ and $\beta(\xi)$ satisfying this functional differential equation, such that their derivatives are real functions and the stresses at infinity are zero.

The Solution.

First consider the behaviour of $a(z)$ when $r \rightarrow \infty$. It was shown earlier that $\mathcal{J}\{a(\infty)_{\theta=0}\} = -\bar{P} = \mathcal{J}\{\alpha(e^{i\pi})\}$, and that $\mathcal{J}\{a(\infty)_{\theta=2\pi}\} = +\bar{P} = \mathcal{J}\{\alpha(e^{-i\pi})\}$. In the ξ -plane therefore there is a discontinuity in $\alpha(\xi)$ on $\rho=1$ of $2i\bar{P}$ at $\psi = \pm\pi$. A suitable function would be $-2(\bar{P}/\pi) \log(1+\xi)$, which has an imaginary part $-2i\bar{P}\psi_1/\pi = -i\bar{P}\psi/\pi$ (see Fig. 4).

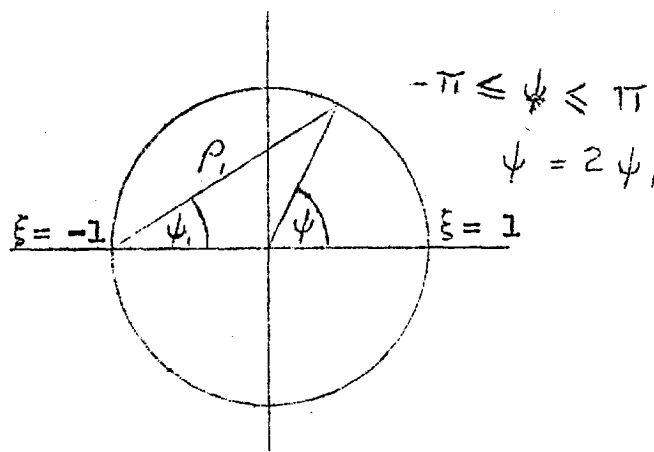


FIG. 4.

This discontinuity in $a(z)$ across the stiffener at infinity must not induce a similar discontinuity in w , and to balance it we must have in $\bar{b}(\bar{z})$ a term $-2k(\bar{P}/\pi) \log(1+\bar{\xi})$ (see Equation (51) of the Appendix). Then the combination $ka(z) + \bar{b}(\bar{z})$ has no discontinuity. We may now hope to express the remainder of the potentials as power series:

$$\left. \begin{aligned} \alpha(\xi) &= \sum_{n=0}^{\infty} A_n \xi^n - 2(\bar{P}/\pi) \log(1+\xi) \\ \beta(\xi) &= \sum_{n=0}^{\infty} B_n \xi^n - 2k(\bar{P}/\pi) \log(1+\xi) \end{aligned} \right\} \dots (22)$$

For the derivatives to be real functions it is only necessary that A_n and B_n be real for $n > 0$. Since $a(0) = \alpha(1)$ the earlier stipulation that $\mathcal{I}\{a(0)\} = 0$ requires that A_0 be real also. Thus neither A_0 nor B_0 enters equation (21), and their only function is to make $w = 0$ at $z = 0$. B_0 is sufficient for this, and we may immediately put $A_0 = 0$. Note that $\mathcal{I}\{\alpha(e^{i\pi})\}$ receives no

contribution from the series and retains the value $-\bar{P}$.

Substituting the expressions (22) in equation (21) and separating real and imaginary parts, we find

$$\sum_{n=1}^{\infty} \left[kn A_n + \frac{n}{4} \{ (n-1)A_{n-1} - (n+1)A_{n+1} \} - n B_n \right] \cos n\psi = (\bar{P}/2\pi) \cos \psi, \quad \dots\dots\dots (23)$$

and

$$\begin{aligned} \cot \frac{\psi}{2} \cos^2 \frac{\psi}{2} \sum_{n=1}^{\infty} \left[kn A_n - \frac{n}{4} \{ (n-1)A_{n-1} - (n+1)A_{n+1} \} + n B_n \right] \sin n\psi \\ + \sum_{n=1}^{\infty} A_n \sin n\psi \\ = (\bar{P}/\pi) \left\{ \psi - \pi + 2k \cos^2 \frac{\psi}{2} - \cos^4 \frac{\psi}{2} \right\}. \quad \dots\dots\dots (24) \end{aligned}$$

It is legitimate to differentiate infinite series term by term in this way provided the resulting series are uniformly convergent. This proviso will be met in the present instance if, as $n \rightarrow \infty$,

$$|A_n| < O(n^{-3}) \quad \text{and} \quad |B_n| < O(n^{-2}).$$

Multiply through equation (23) by $\cos m\psi$ ($m = 1, 2, 3, \dots$) and integrate from 0 to π , with the notation $\delta_{mn} = 1$ if $m = n$, $= 0$ if $m \neq n$:

$$m B_m = km A_m + \frac{m}{4} \{ (m-1)A_{m-1} - (m+1)A_{m+1} \} - (\bar{P}/2\pi) \delta_{m1} \quad \dots\dots\dots (25)$$

Equation (24) can now be considerably simplified by using this result to eliminate B_n :

$$\sum_{n=1}^{\infty} \left\{ 1 + 2kn \cot \frac{\psi}{2} \cos^2 \frac{\psi}{2} \right\} A_n \sin n\psi = (\bar{P}/\pi) \left\{ \psi - \pi + 2k \cos^2 \frac{\psi}{2} - \cos^4 \frac{\psi}{2} \right\}. \quad \dots\dots\dots (26)$$

If we put $k = 1$ this equation is identical with one derived by Buell (ref.6.) in finding an Airy stress function for a semi-infinite plate with a partially stiffened edge, and the same method may be used to evaluate the A'_n 's from it. Although elastic stresses are rarely susceptible to small changes in Poisson's ratio the temptation to put $k = 1$ and use Buell's arithmetical results is to be resisted: $k = 1$ implies a Poisson's ratio of 1 and a negative bulk modulus. Which is absurd.

Multiply equation (26) by $\sin \frac{\psi}{2} \sin m\psi$ (m integral), and integrate from 0 to π . Then

$$\sum_{n=1}^{\infty} g_{mn} A'_n = h_m, \quad \dots\dots\dots (27)$$

where
$$g_{mn} = \frac{3 kn (-1)^{m+n}}{\{4(m+n)^2 - 9\} \{4(m-n)^2 - 9\}} - \frac{2 + 3kn (-1)^{m+n}}{\{4(m+n)^2 - 1\} \{4(m-n)^2 - 1\}}$$

$$h_m = \frac{2k (-1)^m (4m^2 - 1) - 2(4m^2 - 9)}{(4m^2 - 1)^2 (4m^2 - 9)}$$

$$A'_n = n\pi A_n \sqrt{P}$$

When m takes the successive values 1, 2, 3, ... equation (27) gives an infinite system of linear simultaneous equations for the A'_n 's, and the coefficients of the system depend only on k , i.e., on Poisson's ratio. The numerical values were computed for $\nu = 0.3$, giving $k = 2.07692$, by putting $A'_{N+1} = A'_{N+2} = \dots\dots\dots = 0$ and discarding all equations for which $m > N$.

Table 1.

A_1'	A_2'	A_3'	A_4'	A_5'	A_6'	A_7'	A_8'	A_9'	A_{10}'	A_{11}'	A_{12}'
0.121607											
0.070247	0.120363										
0.044529	0.200976	-0.146497									
0.037900	0.227153	-0.209847	0.100284								
0.036483	0.234601	-0.232131	0.146656	-0.067071							
0.036346	0.236201	-0.238228	0.161962	-0.095852	0.039014						
0.036402	0.236449	-0.239809	0.166794	-0.106615	0.057824	-0.024272					
0.036452	0.236427	-0.240098	0.168036	-0.109880	0.064495	-0.035298	0.013684				
0.036480	0.236381	-0.240116	0.168340	-0.110912	0.066935	-0.039968	0.021069	-0.008878			
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0.036503	0.236327	-0.240047	0.168349	-0.111225	0.067987	-0.042438	0.025823	-0.016625	0.010273	-0.006229	0.002608

The solutions for the resulting finite systems for $N = 1, 2, 3, \dots$ are set out in Table 1. The systems up to and including $N = 9$ were solved by the Computing Section of the National Physical Laboratory, but the convergence is not very rapid, and it was necessary to proceed to $N = 12$. This final system was solved by the writer, using Relaxation and starting from the known solutions for $N = 9$.

Note that the convergence of the A_n 's is better than that of the A'_n 's, for $A_n = (\bar{P}/\pi) (A'_n/n)$. The convergence has not, of course, been formally demonstrated, but for large values of m $h_m = O(m^{-4})$, $g_{mm} = O(m^{-1})$, and g_{mn} decreases rapidly as we get away from the leading diagonal since $(m-n)^2$ grows, so that we may infer that, for large m , $A'_m = O(m^{-3})$, or $A_m = O(m^{-4})$. Then from equation (25) $B_m = O(m^{-4})$ also. In this case not only are the series convergent, but the differentiations leading to equations (23) and (24) are justified.

The values of B_m for $m > 0$ can now be found from equation (25), and B_0 from $w = 0$ at $z = 0$. Using equation (51) of the Appendix with $z = 0$ we find

$$4GW = \sum_{n=1}^{\infty} (kA_n + B_n) - 4k(\bar{P}/\pi) \log 2 + \bar{B}_0 = 0,$$

or

$$B_0 = 5.826 (\bar{P}/\pi).$$

For practical purposes the potentials are now established as

$$\begin{aligned} a(z) = \alpha(\xi) &= (\bar{P}/\pi) \left\{ \sum_{n=1}^{12} (A'_n/n) \xi^n - 2 \log(1 + \xi) \right\}, \\ b(z) = \beta(\xi) &= (\bar{P}/\pi) \left\{ \sum_{n=1}^{12} (B'_n/n) \xi^n - 2k \log(1 + \xi) + 5.826 \right\}, \end{aligned} \quad \dots\dots\dots (28)$$

where $B'_n = n\pi B_n/\bar{P}$.

Stresses.

We are interested in $\hat{x}\hat{x}$, $\hat{y}\hat{y}$ and $\hat{x}\hat{y}$ on $\theta = 0, 2\pi, \frac{\pi}{2}$ and π .

And P.

$\theta = 0$. The general method of finding the stresses is to use equations (28) with (52) and (53) of the Appendix to evaluate $\hat{z}\hat{z}$ and $\hat{z}\hat{z}$. However, more quickly convergent results may be obtained along the stiffener from the initial conditions. On $\theta = 0$ $\xi = e^{i\psi}$, where from equations (18) and (19) $\psi = 2 \tan^{-1} \sqrt{r/r_0}$ ($0 \leq \psi \leq \pi$), and $d\psi/dr = \sqrt{r/r_0}/r(1 + r/r_0)$. Then ψ is a function of the dimensionless parameter r/r_0 .

$a(r) = \alpha(e^{i\psi}) = (\bar{P}/\pi) \left\{ \sum_{n=1}^{12} (A'_n/n) e^{in\psi} - 2 \log(2 \cos \frac{\psi}{2}) - i\psi \right\}$,
and from equations (10) and (11)

$$P/P_0 = (1/\pi) \left\{ \sum_{n=1}^{12} (A'_n/n) \sin n\psi + \pi - \psi \right\}. \quad \dots\dots\dots (29)$$

To obtain dimensionless stress parameters we introduce

the factor $\mu = AE_s/P_0E = r_0(1+k)/2\bar{P}$. Then from equation (2)

$$\mu \hat{x}\hat{y}_+ = \frac{(1+k)^2}{4\pi} \cdot \frac{1 - \sum_{n=1}^{12} A'_n \cos n\psi}{(1 + r/r_0) \sqrt{r/r_0}} \quad \dots\dots\dots (30)$$

From the Appendix, Equation (52), $\hat{z}\hat{z} = 2R \{ a'(r) \} = 2R \left\{ \frac{d}{d\psi} \alpha(e^{i\psi}) \cdot \frac{d\psi}{dr} \right\}$,

so that

$$\mu (\hat{x}\hat{x}_+ + \hat{y}\hat{y}_+) = \frac{1+k}{\pi} \cdot \frac{\sqrt{r/r_0} - \sum_{n=1}^{12} A'_n \sin n\psi}{(1 + r/r_0) \sqrt{r/r_0}} \quad \dots\dots\dots (31)$$

From equation (3)

$$\mu (\hat{x}x_+ - \nu \hat{y}y_+) = \mu E \frac{du_+}{dr} = \mu PE/AE_S = P/P_0 \dots\dots\dots (32)$$

Equations (29), (31) and (32) combine to give $\mu \hat{x}x_+$ and $\mu \hat{y}y_+$

$\theta = 2\pi$. The stresses here are found from

$$\hat{x}x_- = \hat{x}x_+; \quad \hat{y}y_- = \hat{y}y_+; \quad \hat{x}y_- = -\hat{x}y_+.$$

For other values of θ we use the general expressions for $\hat{z}\bar{z}$ and $\hat{z}z$, transforming equations (52) and (53) to the most convenient form.

$$\mu \hat{z}\bar{z} = \frac{1+k}{\bar{P}} \cdot R\{\alpha'(\xi) \cdot r_0 d\xi/dz\},$$

$$\mu \hat{z}z = \frac{1+k}{2\bar{P}} \left[r_0 \frac{d\bar{\xi}}{dz} \left\{ \beta'(\bar{\xi}) - \frac{z}{r_0} \cdot r_0 \frac{d\bar{\xi}}{d\bar{z}} \cdot \alpha''(\bar{\xi}) \right\} - \frac{z}{r_0} \cdot r_0^2 \frac{d^2\bar{\xi}}{d\bar{z}^2} \cdot \alpha'(\bar{\xi}) \right],$$

$$\xi = (1 + i\sqrt{z/r_0}) / (1 - i\sqrt{z/r_0})$$

$$r_0 d\xi/dz = \frac{i}{(1 - i\sqrt{z/r_0})^2 \sqrt{z/r_0}}$$

$$r_0^2 d^2\xi/dz^2 = - \frac{3\sqrt{z/r_0} + i}{2(\sqrt{z/r_0} - iz/r_0)^3}$$

$$\left. \begin{aligned} (1/\bar{P}) \alpha'(\xi) &= (1/\pi) \left\{ \sum_{n=1}^{1/2} A'_n \xi^{n-1} - 2/(1+\xi) \right\}, \\ (1/\bar{P}) \alpha''(\xi) &= (1/\pi) \left\{ \sum_{n=1}^{1/2} (n-1) A'_n \xi^{n-2} + 2/(1+\xi)^2 \right\}, \\ (1/\bar{P}) \beta'(\xi) &= (1/\pi) \left\{ \sum_{n=1}^{1/2} B'_n \xi^{n-1} - 2k/(1+\xi) \right\}. \end{aligned} \right\} \dots\dots (33)$$

For a fixed value of θ these expressions are sufficient to

establish $\mu \hat{z}\hat{z}$ and $\mu \hat{z}\hat{z}$, and thus the complete stress system, as functions of the single parameter r/r_0 . The stresses on $\theta = \pi/2$ and $\theta = \pi$ were found by this means and are shown in Figs. 7 and 8. The load in the stiffener and the stresses on $\theta = 0$ are shown in Figs. 5 and 6.

Singularities.

It is seen from equations (28) that at $\zeta = -1$ (infinity in the z -plane) there is a logarithmic singularity in the potentials, giving a pole in the derivatives. Elsewhere the potentials and their derivatives w.r.t. ζ are regular. Further singularities arise from the transformation, and the combined effect on the stresses is shown below.

Stresses when $r/r_0 \rightarrow 0$.

$$\begin{aligned} \theta = 0 & \\ \mu \hat{xx}_+ &= \frac{1+k}{4} \left\{ 1 + \frac{3-k}{\pi} \left(1 - 2 \sum_1^{\infty} n A'_n \right) \right\} \mu \hat{xx}, \mu \hat{yy}, \mu \hat{xy} = 0 \quad (r/r_0)^{-\frac{1}{2}} \\ &= 0.937 \\ \mu \hat{yy}_+ &= \frac{1+k}{4} \left\{ -1 + \frac{1+k}{\pi} \left(1 - 2 \sum_1^{\infty} n A'_n \right) \right\} \mu \hat{xx}, \mu \hat{yy} = 0 \quad (r/r_0)^{-\frac{1}{2}} \\ &= -0.212 \\ \mu \hat{xy}_+ &= 0 \quad (r/r_0)^{-\frac{1}{2}} \end{aligned}$$

Stresses when $r/r_0 \rightarrow \infty$

$$\begin{aligned} \theta = 0 & \\ \mu \hat{xx}_+, \mu \hat{yy}_+ &= 0 \quad (r/r_0)^{-1} \\ \mu \hat{xy}_+ &= 0 \quad (r/r_0)^{-3/2} \end{aligned}$$

$$\begin{aligned} \theta = \pi/2 & \\ \mu \hat{xx}, \mu \hat{yy}, \mu \hat{xy} &= 0 \quad (r/r_0)^{-1} \\ \theta = \pi & \\ \mu \hat{xx}, \mu \hat{yy} &= 0 \quad (r/r_0)^{-1} \\ \mu \hat{xy} &\equiv 0 \end{aligned}$$

2. Uniform Tension at Infinity Parallel to Stiffener.

We first consider the case where such loads are applied as will produce a uniform tension $\hat{x}x = \sigma$ everywhere in the sheet.

$$\hat{x}x = \sigma, \quad \hat{x}y = \hat{y}y = 0, \quad \text{for all } z.$$

Then

$$\left. \begin{aligned} \hat{z}z &= 2R \{ a'(z) \} = \sigma \\ \hat{z}z &= \bar{b}'(\bar{z}) - z \bar{a}''(\bar{z}) = \sigma \end{aligned} \right\} \text{ for all } z \quad \begin{array}{l} \text{..... (34)} \\ \text{..... (35)} \end{array}$$

As z is isolated in equation (35) its coefficient must vanish:

$$\bar{a}''(\bar{z}) = 0.$$

The most general solution of (34) and (35) therefore is

$$a(z) = a_0 + \left(\frac{\sigma}{2} + i a_1\right) z, \quad (a_1 \text{ is a real constant})$$

$$b(z) = b_0 + \sigma z.$$

If we specify no displacement and no rotation at the origin, with $a(0) = 0$, we find

$$\left. \begin{aligned} a(z) &= \sigma z/2 \\ b(z) &= \sigma z \end{aligned} \right\} \text{..... (36)}$$

Now $\frac{P}{AE_S} = \frac{P_0}{AE_S} = \frac{du}{dr}$, a constant for all r . Using the potentials (36), together with the expressions for w from the Appendix, we find therefore that this state of stress is only possible if the uniform tension at infinity is combined with a tensile force at the end of the stiffener of

$$P_0 = \sigma AE_S/E \quad \text{..... (37)}$$

Then the required case of a uniform tension at infinity with an unloaded stiffener and is found by superimposing on the stress system (36) the system due to a load $P_0 = -\sigma AE_s/E$, whose reactions at infinity are infinitesimal and will not disturb the uniform state of tension there. This latter system of course is the one already solved, and combining (36) with (28) we find the appropriate potentials:

$$\left. \begin{aligned} a(z) &= \frac{\sigma}{2} \left[z - \frac{r_0(1+k)}{\pi} \left\{ \sum_{n=1}^{\infty} (A'_n/n) \xi^n - 2 \log(1+\xi) \right\} \right], \\ b(z) &= \sigma \left[z - \frac{r_0(1+k)}{2\pi} \left\{ \sum_{n=1}^{\infty} (B'_n/n) \xi^n - 2k \log(1+\xi) + 5.826 \right\} \right] \end{aligned} \right\} \dots\dots\dots (38)$$

The stresses, this time considered non-dimensionally as $\hat{z}\hat{z}/\sigma$, $\hat{z}\hat{z}/\sigma$, are also found by superposition. From (34) and (35) $\hat{z}\hat{z}/\sigma = \hat{z}\hat{z}/\sigma = 1$; for the other system $\hat{z}\hat{z}/\sigma = -\hat{z}\hat{z} AE_s/P_0 E = -\mu \hat{z}\hat{z}$, and $\hat{z}\hat{z}/\sigma = -\mu \hat{z}\hat{z}$, as plotted in Figs. 6 - 8. Thus $\hat{x}\hat{y}/\sigma$ and $\hat{y}\hat{y}/\sigma$ are given by figs. 6 - 8 and $PE/\sigma AE_s$ and $\hat{x}\hat{x}/\sigma$, which are 1 - the previous values, are plotted in Figs. 9 and 10.

3. Uniform Tension at Infinity Perpendicular to a Stiffener.

Proceeding exactly as in the last section we first find potentials to give $\hat{y}\hat{y} = \sigma$, $\hat{x}\hat{x} = \hat{x}\hat{y} = 0$, for all z . The result is

$$\left. \begin{aligned} a(z) &= \sigma z/2, \\ b(z) &= -\sigma z. \end{aligned} \right\} \dots\dots\dots (39)$$

Again $du_+/dr = P/AE_s$ is a constant for all r , and by expressing u_+ in terms of the potentials we find

$$P = P_0 = -\nu \sigma AE_s/E. \quad \dots\dots\dots (40)$$

Superimposing on the system (39) the system due to a load $P_0 = +\nu \sigma AE_s/E$, we find the potentials

$$\begin{aligned} a(z) &= \frac{\sigma}{2} \left[z + \frac{r_0(3-k)}{\pi} \left\{ \sum_{n=1}^{\infty} (A'_n/n) \xi^n - 2 \log(1+\xi) \right\} \right], \\ b(z) &= \sigma \left[-z + \frac{r_0(3-k)}{2\pi} \left\{ \sum_{n=1}^{\infty} (B'_n/n) \xi^n - 2k \log(1+\xi) + 5.826 \right\} \right], \end{aligned} \quad \dots\dots\dots (41)$$

This time $\hat{x}x/\nu\sigma$ and $\hat{x}y/\nu\sigma$ are given by Figs. 6 - 8 and $-PE/\nu\sigma AE_s$ is given by Fig 9. $\hat{y}y$ is not plotted; it can be obtained from the formula

$$\hat{y}y/\sigma = 1 + \nu \cdot \mu \hat{y}y_1 \quad \dots\dots\dots (42)$$

where $\hat{y}y_1$ is the stress found in Problem 1. On $\theta = 0$, $\hat{y}y/\sigma$ varies from 0.936 at $r/r_0 = 0$ to 1 at $r/r_0 = \infty$. and although for other values of θ $\hat{y}y/\sigma \rightarrow 0 (r/r_0)^{-\frac{1}{2}}$ when $r/r_0 \rightarrow 0$, nevertheless in the range $0.5 < r/r_0 < \infty$ it varies only between 1.06 and 1 for $\theta = \pi$ and between 0.98 and 1 for $\theta = \pi/2$. Thus except in the region very close to the end of the stiffener $\hat{y}y$ is practically constant.

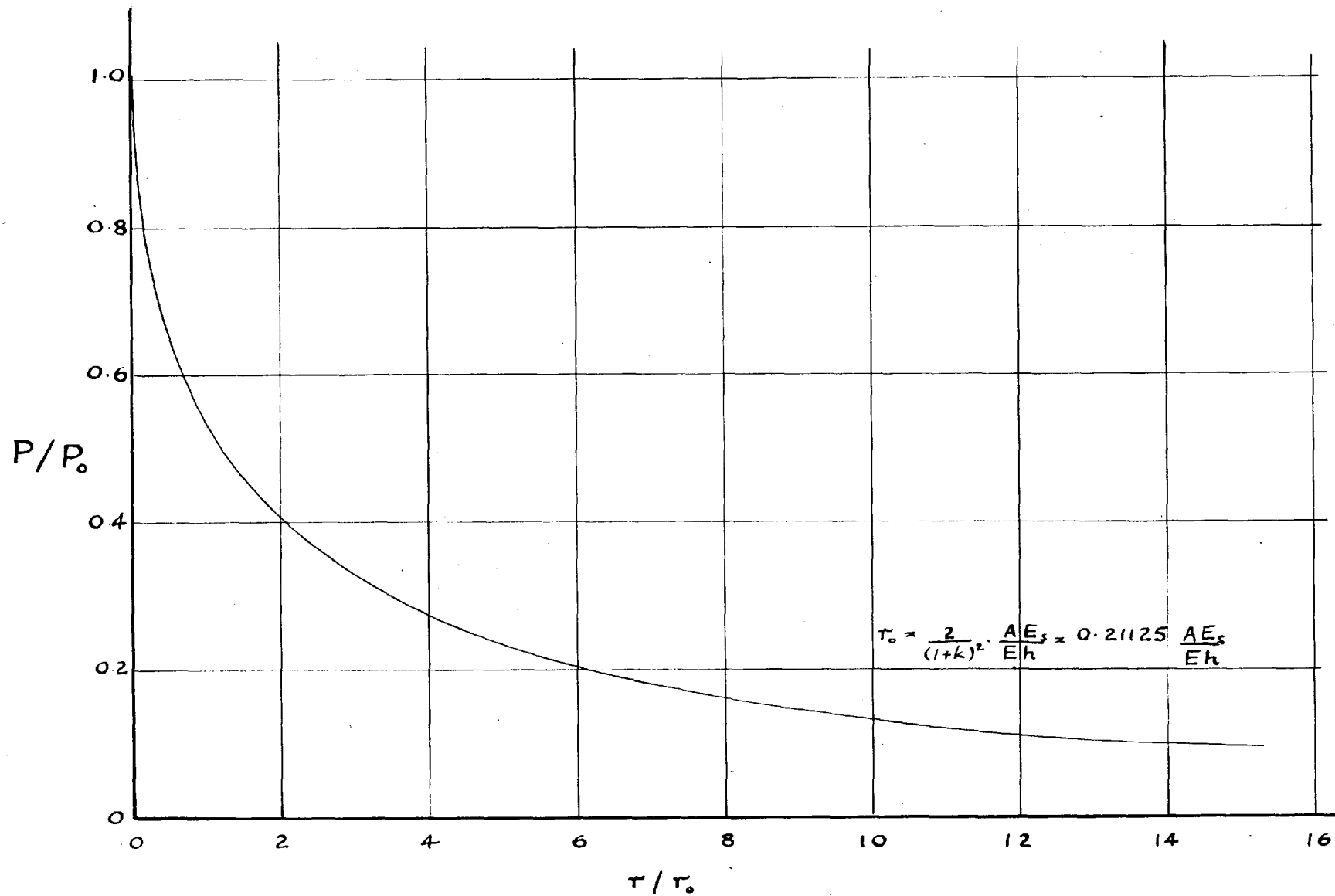


FIG. 5. Load in Stiffener for Problem 1.

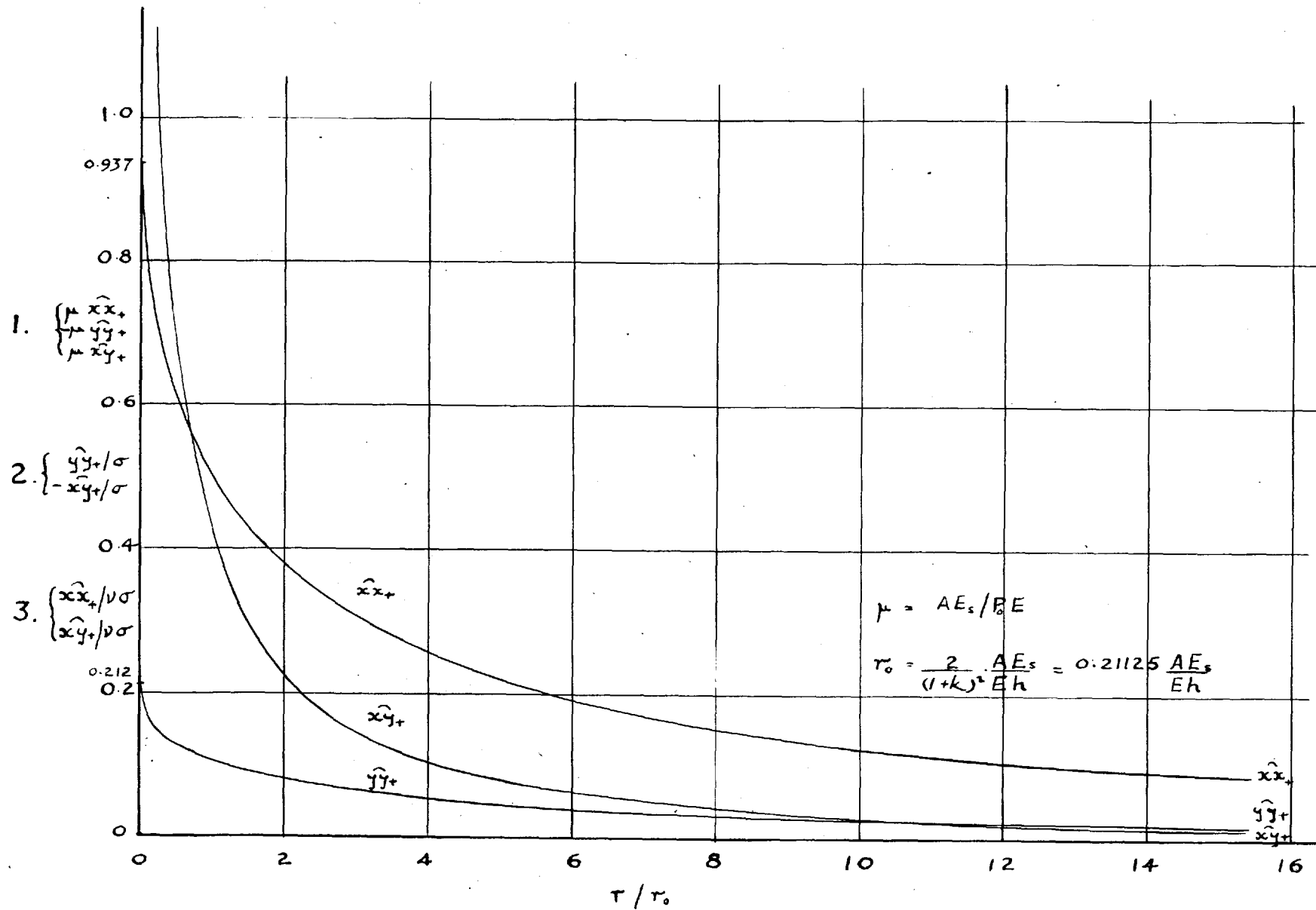


FIG. 6. Stresses on $\theta = 0$

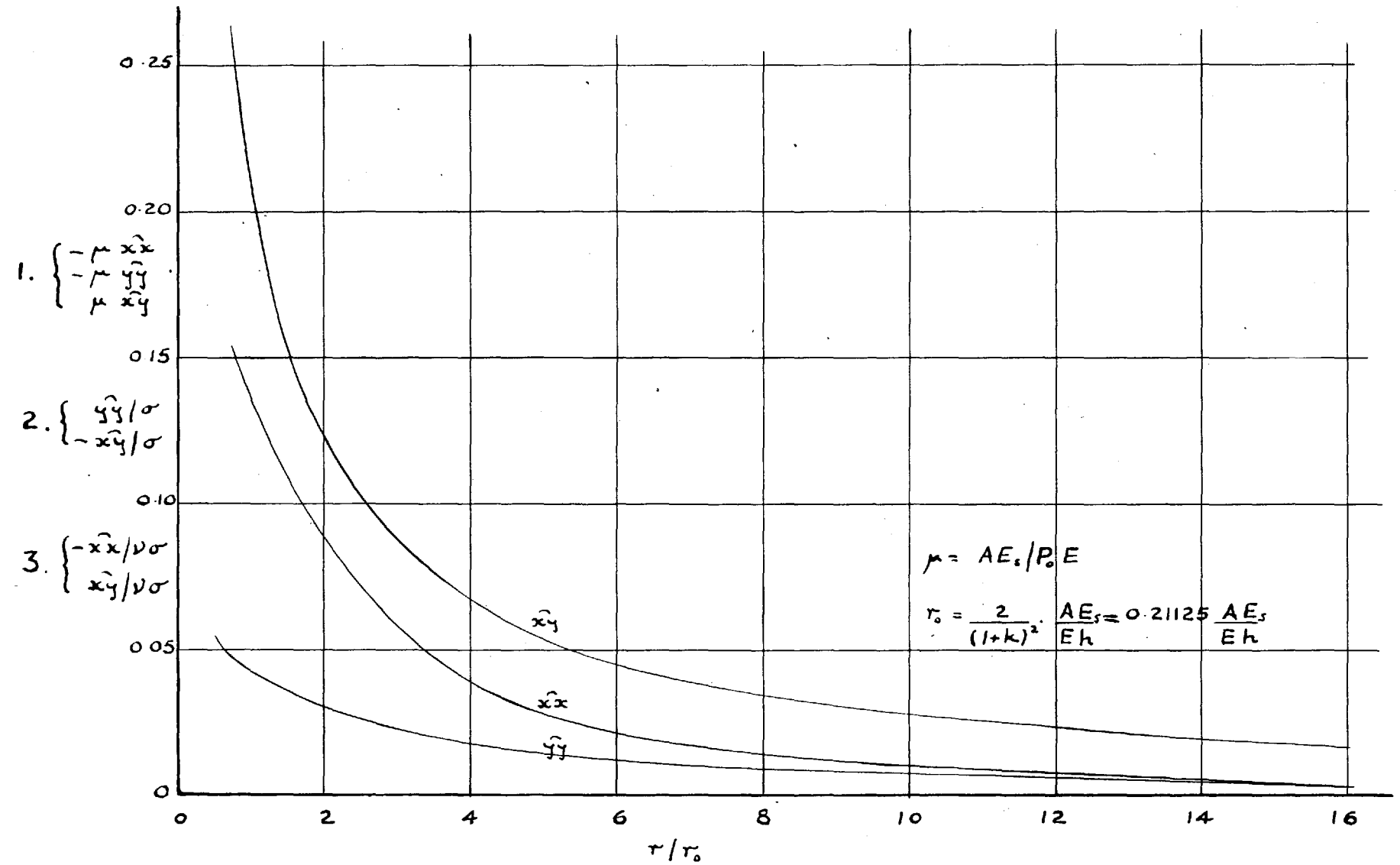


FIG. 7. Stresses on $\theta = \pi/2$

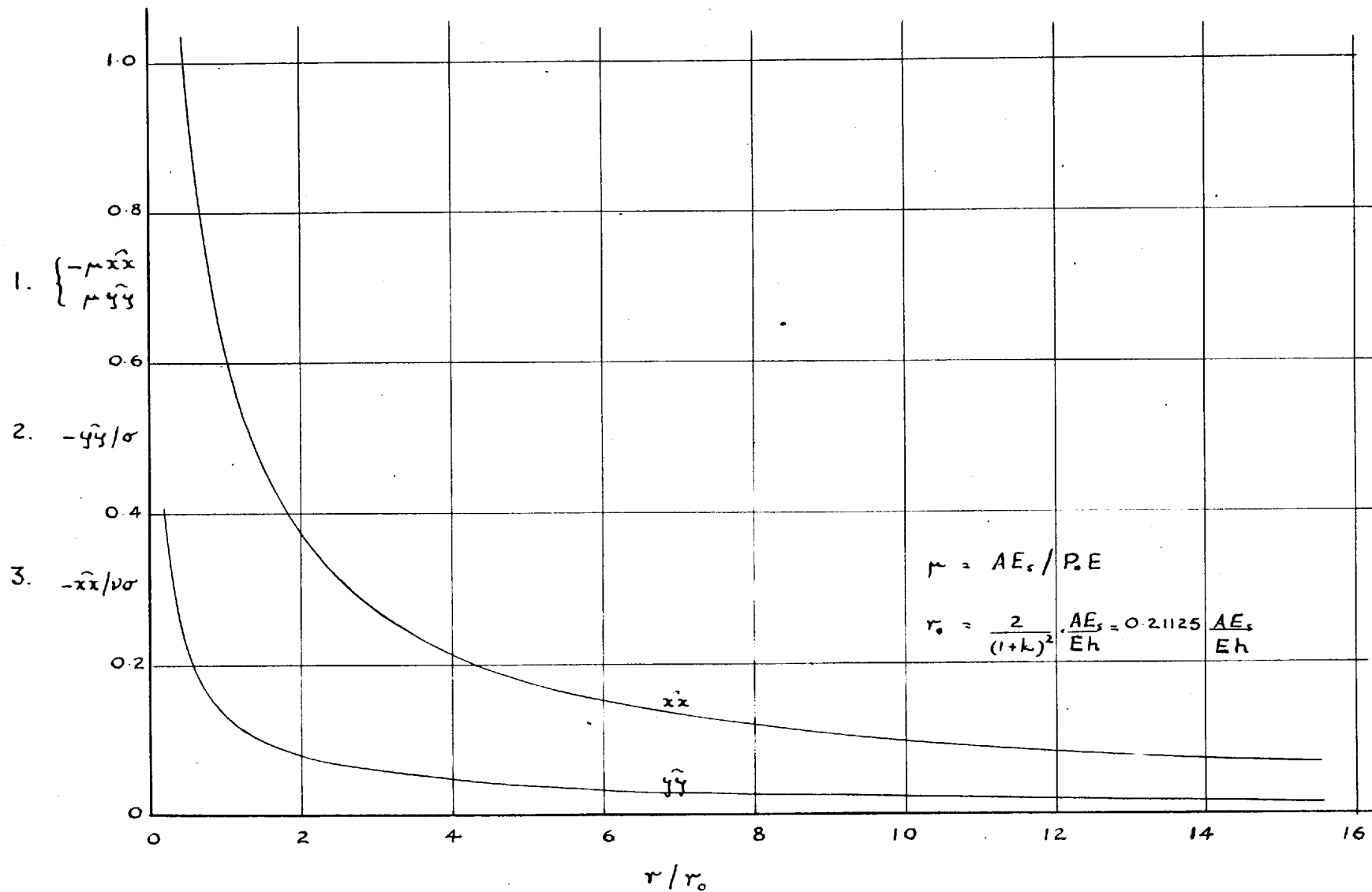


FIG. 8. Stresses on $\theta = \pi$

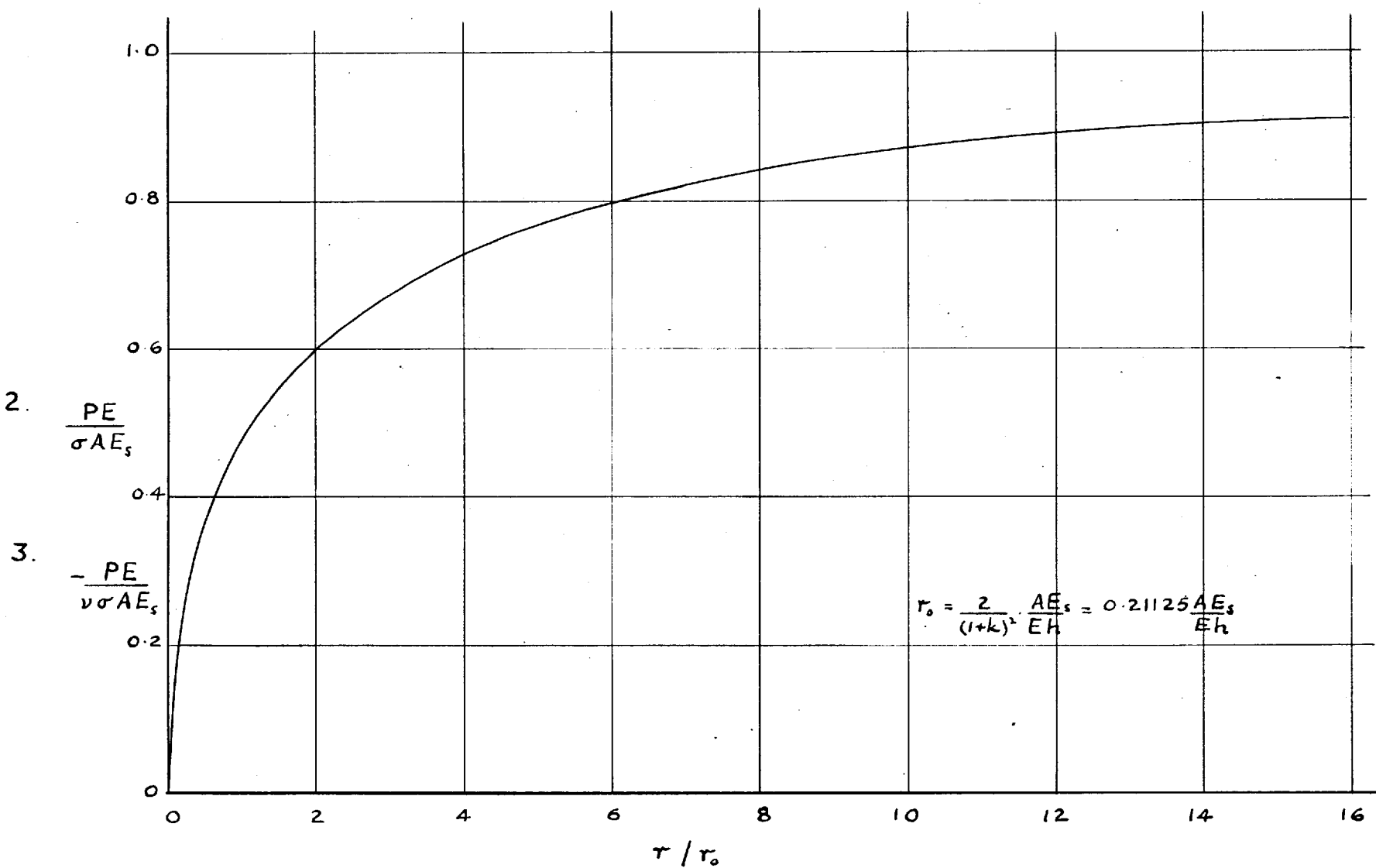


FIG. 9. Load in Stiffener for Problems 2 and 3

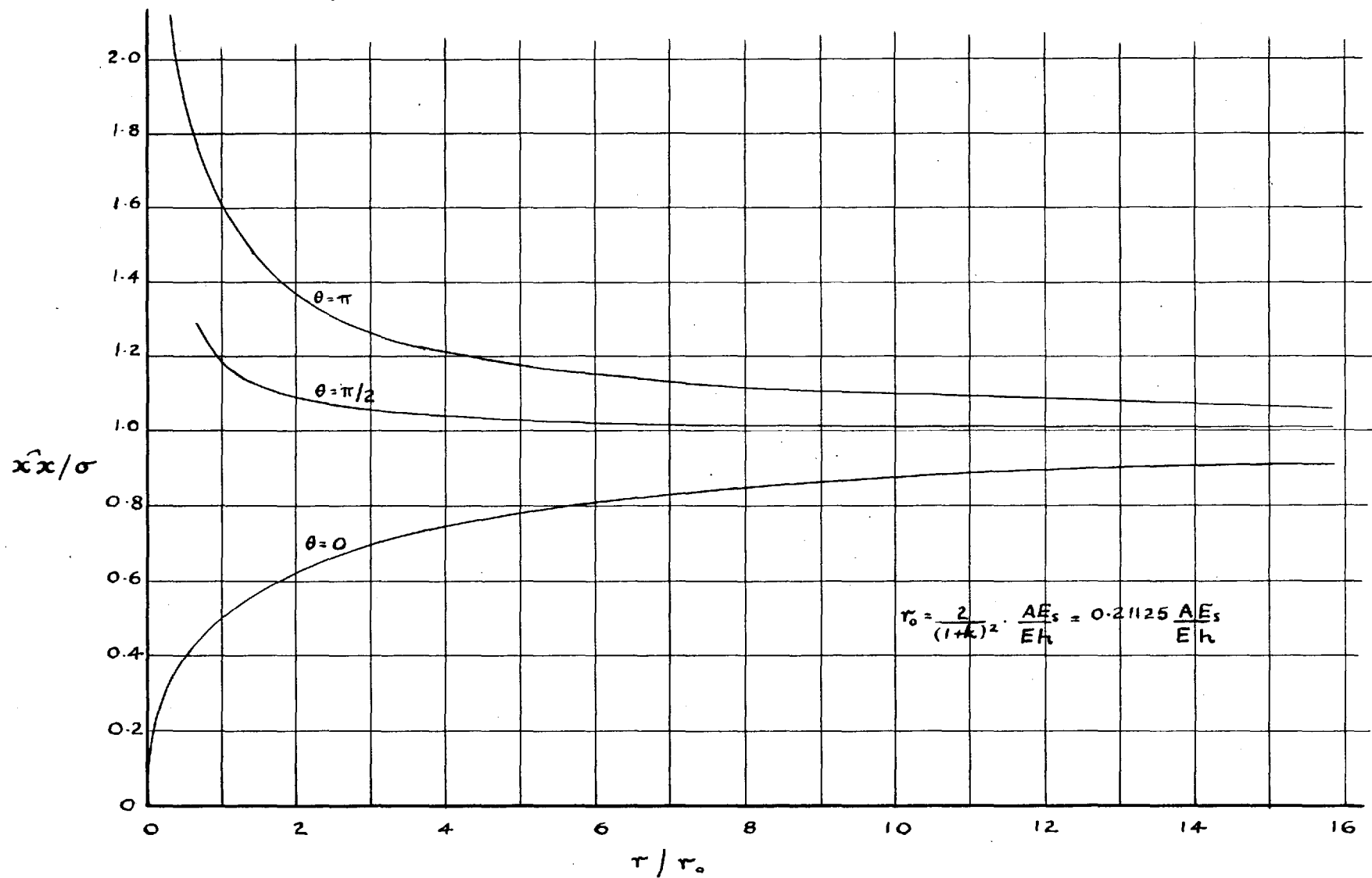


FIG. 10. $\hat{x}x/\sigma$ for Problem 2.

PART II. PRACTICAL WORK.

At the time this research was started no photoelasticity laboratory existed in Imperial College, and as there was no suitable polariscope on the English market it was resolved to purchase the necessary components and build the mounts in the College workshop. The first room available for the laboratory was rather small and narrow, and to some extent this influenced the design of the photoelastic bench, but it is hoped ultimately to modify the apparatus to take full advantage of the larger room in which it now stands.

Design of Apparatus.

A photograph of the photoelastic bench is given in Fig. 11. The polariscope was designed primarily for research, but the teaching of the College was also borne in mind and the light image can be ~~the~~ thrown onto a demonstration screen or a camera plate, as required.

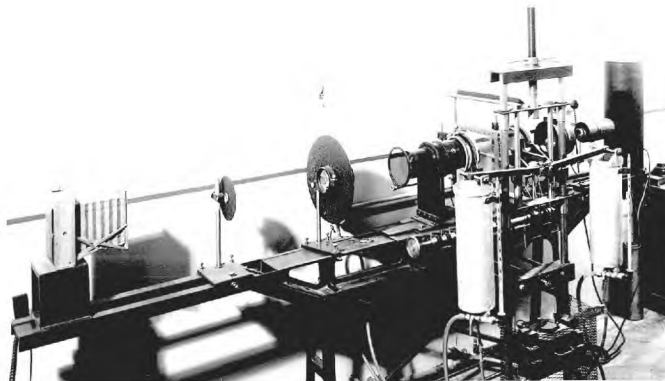


FIG. 11.

The Optical System.

Fig.12 shows the general layout diagrammatically. It is a transmission type system, where a parallel beam of light through the model is produced by a point source at the focus of a condenser lens. A 500 w. mercury arc lamp with an arc 6 mm. long, used as a point source, gives adequate results for most purposes, but a fringe pattern involving many lines in a small area may necessitate a better definition. At a high cost in intensity it can be contrived, by a pin-hole screen at the focus of the condenser with a second lens, to direct as much light from the arc as possible through the pin-hole. This produces a much better version of parallel light, and the apparatus has been built with a movable light source, so that an optional extra condenser lens and pin-hole screen can be inserted when required.

Note that no lenses come between the polaroid sheets, for stresses in the lenses would superimpose their effect upon the

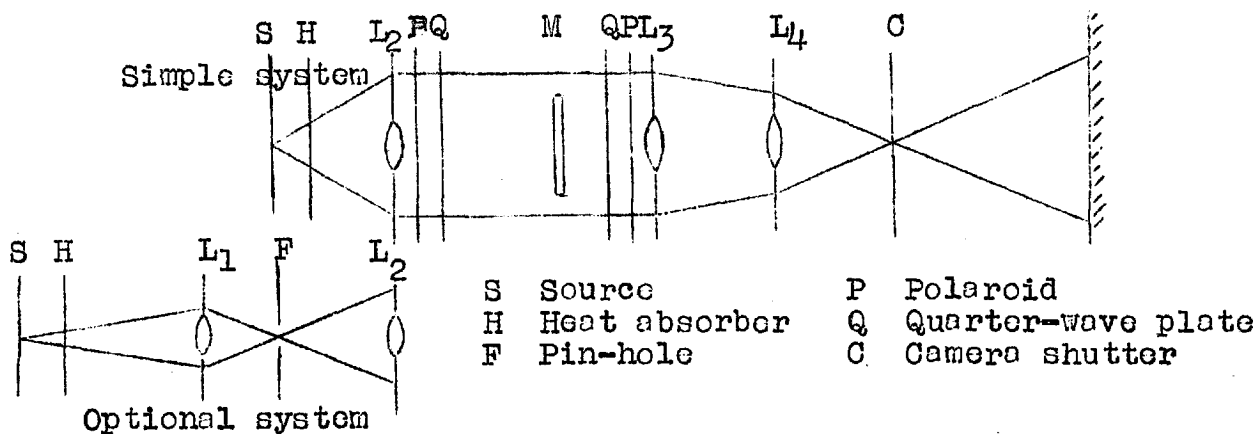


FIG. 12.

stress-pattern of the model.

The field which can be viewed at one instant is limited to the size of the analyser lens, which must be a compensated one of good optical quality as it transmits, and must not distort, the image of the model. Suitable lenses at a reasonable cost are very rare above 6" dia., and the present one has a diameter of about $5\frac{3}{4}$ ".

Choice of Lenses.

The analyser lens L_3 (Fig.12) is an f/6.3 telephoto lens with a 36" focal length; it was the largest compensated lens available.

The condenser lens L_2 need not be compensated, but the extreme edges of single glasses should not be used, for they give considerable aberration. The greatest intensity will be given, and the least space taken, by the lens with the shortest focal length. These requirements were met by two $7\frac{1}{8}$ " dia. plano-convex lenses, mounted with convex faces together and stopped down to the right diameter. The effective focal length is about 6". The combination makes up in aberration for what it gains in power, but in the original scheme (which had no optical pin-hole system) it was essential to save space wherever possible.

The camera lens L_4 is a compensated one, f/8, focal length 32", giving a diameter large enough to collect the entire beam whether the lens is placed to focus the image onto a camera plate

or onto the demonstration screen. It has a travel along the bench of 15".

The optional lens L_1 is required to collect as much light as possible and direct it through the pin-hole. Now the pin-hole must be at the focus of L_2 , and so must the light source when the pin-hole is not being used, so that S-F in Fig.12 represents the maximum travel of the light source. From practical considerations this is 15".

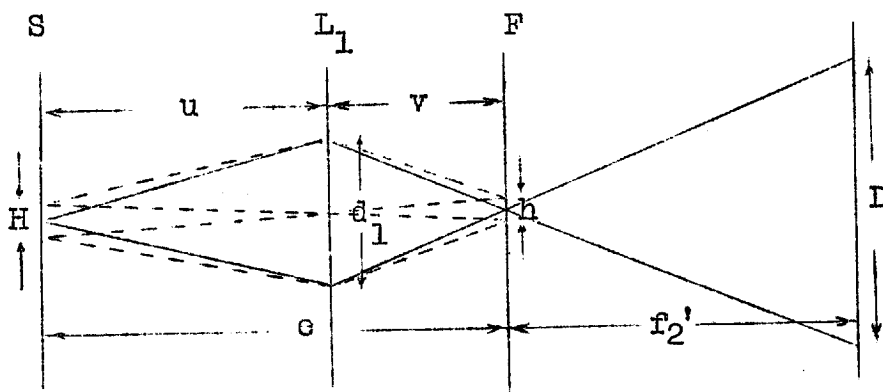


FIG. 13.

With the notation of Fig.13, $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$, and $u + v = e = 15"$.

Then

$$D/f_2' = d_1/v = (d_1/f_1) (1 - v/e)$$

In practice the ratio d/f for any lens is never much greater than 1, so that $D < f_2'$, where f_2' is the distance from the focus of the lens L_2 to its nearest face. $f_2' = 4\frac{1}{2}"$, so that the full

diameter of the condenser L_2 cannot be used in the high definition set-up. The factor $(1 - v/e)$ will be greatest with f_1 as small as possible; i.e., D will be greatest if L_1 is a small lens with the largest possible 'aperture' (d/f value).

The other consideration affecting the choice of L_1 is the magnification of the light source h/H (Fig.13), which must of course be less than 1 and as small as possible. Now $h/H = v/u = m$, say, and eliminating u and v by the formulae above we have

$$f_1 = me / (1 + m)^2$$

By differentiating twice we see that f_1 is a maximum when $m = 1$, and when $m < 1$ f_1 and m decrease together. Thus on this count also f_1 should be small.

The lens used has a diameter 1.8" and focal length 2.1".

Details of Polariscope Components.

The various components of the polariscope are supported on a 6 ft. long steel bench, which affords a 7" wide table with long edges slightly raised and machined to give two $\frac{3}{4}$ " wide runners. An existing part of the estate, and not designed to carry the present polariscope, the bench needed cantilever extensions at either end to give a final table 10' 6" long.

The lamp-house is a vertical 2 ft. length of 9" dia. asbestos-cement pipe, fitted with light-traps at either end to allow a through-draught. Its base-plate has a 15" travel

between guides clamped to the bench.

There are two lamps, mounted on the top plate of the light-trap with a shield between them, so that either can be brought into use without extinguishing the other by rotating a handle beneath the base-plate. The 500 w mercury arc lamp already referred to is mostly used with a filter for monochromatic work; the other is a 150 candlepower Pointolite, an arc lamp with tungsten electrodes giving a much fuller spectrum when polychromatic pictures are required. Although it has not the power of the other the Pointolite is particularly useful for rapid inspections, for it attains its full brilliance in a few seconds, whereas the mercury arc takes 5 minutes.

The window through which the light emerges is covered by a 3" square plate of heat-absorbing glass. It absorbs some of the heat.

The two $7\frac{1}{8}$ " dia. condenser lenses are mounted with convex faces adjacent in a short horizontal cylinder, with a stop to cut out light from the extreme edges of the lens, where the aberration is severe. There is also a light-shield, fitting close up to the lamp-house in its forward position; this is needed to prevent considerable reflection from the face of the lens onto the nearby wall.

The upper part of the central supporting pillar is threaded into the lower part, which rotates to raise the lenses

for vertical alignment. The lenses are prevented from turning by an inner connection to the base-plate.

This pillar also supports the optional pin-hole system, on a cantilever which may be swung out of the light beam when the system is not wanted. The lamp-house is then pulled forward and comes against a stop when the arc is at the focus of the main condenser lens. The change-over is simple and immediate.

The optional system consists of a 2" dia. lens, a $2\frac{1}{2}$ " dia. screen with a 1.5 mm. dia. central hole, and a tubular light-shield which fits close up to the lamp-house. The three components can independently be adjusted horizontally and vertically.

The polaroid sheets and quarter-wave plates are each sandwiched between two glass covering sheets in glycerine, which has almost the same refractive index as the glass and so reduces the effective number of reflecting surfaces. Between the two polaroids there are thus six sheets of glass, and the internal stresses in them originally resulted in an uneven initial field. A uniform field was only finally procured by using 1/16" glass sheets, annealed after cutting.

The mica quarter-wave plates are correct for mercury green light ($5461 \text{ \AA}.$); this wave-band of the mercury arc was chosen for its intensity, although the blue light would give a higher fringe order for a given model stress.

Each of these components is mounted in a steel ring, graduated in degrees and rotating in a support of three pulleys. The rings are fitted with a chain drive to gear-wheels on a shaft at bench-top level, and by a lever selector system any gear-wheel can be made fast either to the shaft or to an anchor-housing. Thus any or all of the polaroids and quarter-wave plates can be turned together by turning the shaft.

The analyser lens is the only component not adjustable for height, and is set with its axis 12" above the runners of the bench.

The Ilford mercury green chromatic filter, which is mounted on a cantilever from the analyser lens support, may be locked in position in the light beam or swung down out of the way.

The camera lens is fitted with a 15" dia. light-shield disc, and has its own base plate with a 15" travel along the bench.

The camera is the remains of a 13 x 18 cm. plate camera with the lens and frontpiece removed. It has a sliding mount on one of the bench extensions, and is quickly removable when it is desired to use the demonstration screen.

The camera shutter is a Compur one, with speeds ranging from 1/200 sec. to 1 sec., and is mounted separately, so that

for any camera position it can be placed where the beam diameter is least.

The Loading Frame.

The loading frame was designed to apply a load of up to 600 lb. in tension or compression to a specimen up to 12" square. The specimen, loaded, was to be capable of both horizontal and vertical movement, so that a 12" x 12" model could be brought into the field and examined piecemeal. Finally, the load was to be applied and removed smoothly and, if necessary, quickly.

This last requirement led to dead-weight loading by two 2-gallon cylinders of water, with a lever system to give a 15:1 mechanical advantage. The difficulty of the narrow original room was met by a double lever system, as shown in Fig. 14. The T-piece applies horizontal compression to the specimen from its upper leg or horizontal tension from the lower. The supports

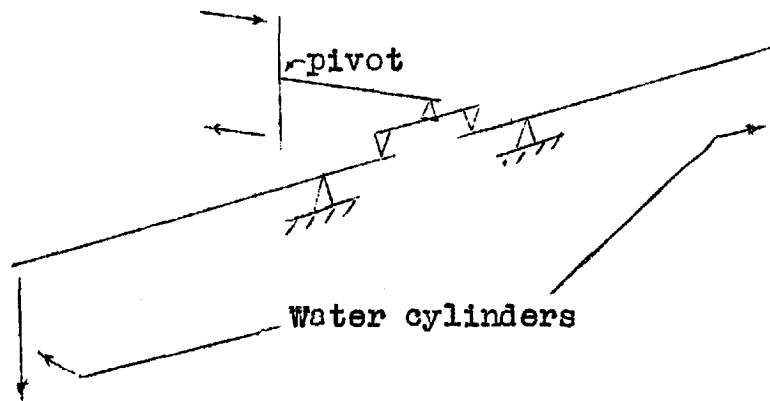


FIG. 14.

to the bridge-piece joining the two primary levers are small ball-races, which avoid friction when the levers separate slightly during rotation. All other pivots are hardened knife-edges.

Each water cylinder is fitted with a gauge glass, calibrated against a proving-ring in 20 lb. intervals of the load actually applied to the specimen. By interpolation it is easy to read the load correct to 2 lb. However, as the pivot and loading points of the primary levers are not colinear, and those of the T-piece do not enclose a right-angle, the inclination of the levers affects the mechanical advantage, and the gauge readings do not always represent the true load on the specimen. The levers rotate an equal distance on either side of their mean (at which the calibration is correct), so that the connection to the specimen can move $\pm 1/12$ " horizontally between stops and the cylinders can rise or fall $1\frac{1}{2}$ ". The extreme error is about $1\frac{1}{2}\%$, and the correction is made by multiplying the apparent load by $(1 + \Delta/15)$, where Δ is the deflection in inches of the inner end of the primary lever above the horizontal. $0 \leq |\Delta| \leq \frac{1}{4}$ approx.

The frame is not suitable for very light loads, for the gauge glass method of measurement is not sufficiently accurate.

The actual frame for mounting the model is 17" x 17", and is cantilevered over the bench. Together with the loading system it has a horizontal travel of 18" and a vertical one of

15", controlled by screw mechanisms connecting to the bench and to a crosshead in the main frame respectively. Thus not only can a loaded specimen be moved about for piecemeal viewing, but the entire frame can be drawn clear of the bench for mounting models.

The loading frame and all the mounts except the original bench were built in the College workshop. They are painted in black crystalline paint, crackle finished, with bright parts lacquered.

The Models.

Of the few good photoelastic materials available the plastic C.R.39 was chosen; it is the best material which can be cemented together to form composite specimens, and moreover it is supplied in glass-clear sheets and does not require polishing. Its principal defect is a considerable stress-optical creep, which for constant loading produces an increase of fringe order of about 10% in 15 minutes and 15% in half an hour (Ref.7.) The creep is closely proportional to stress, however, so that it simply makes the material fringe value (i.e., the photoelastic coefficient of the material) a function of the duration of loading. After 30 minutes the fringe pattern is reasonably settled.

The value of ν for C.R.39 remains unknown. Attempts were made to measure it but no technique was found to give a consistent answer.

Each model consisted of a 12" x 10" sheet, with a 6" long stiffener cemented on either face stretching from the middle of the sheet to one of the shorter edges. $\frac{1}{8}$ ", $\frac{3}{16}$ " and $\frac{1}{4}$ " thick sheets were used, with various stiffener cross-sections. The contact surfaces of the components were roughened with Durex Emery paper 240A and cleaned with acetone, and a thin layer of cement was applied to each surface; the components were carefully brought together, held under a pressure of about 3 lb./in.², baked for 8 hours in a temperature-controlled oven at 80° C, and allowed to cool slowly, usually overnight.

The cement is a mixture of the C.R.39 resin itself and a catalyst, supplied as a cement by the manufacturers of the plastic. It is unstable at room temperature, but may be satisfactorily kept for some weeks in a refrigerator below 32° F.

Design of Tests.

Photoelasticity provides two sets of contours on an image of a plastic model. One set, the isochromatic lines or fringes, is obtainable on its own, at one polariscope setting; it gives the values of the difference between the principal stresses. The other set, the isoclinic lines, is only obtainable superimposed on the fringe pattern; it gives the direction of the principal stresses, but one polariscope setting only gives a contour of points with one particular inclination of principal

axes, and for the contour appropriate to another inclination the polariscope must be reset.

Only problem 1 is suitable for photoelastic investigation. To give comparable fringe orders in the others $h\sigma$ would have to be comparable with hP_0/A , or the side length of the specimen with $5r_0$, since the load available is the same in each case. Such small models are impracticable because a boundary will occur at $r/r_0 = 2\frac{1}{2}$, which is an unacceptable substitute for infinity.

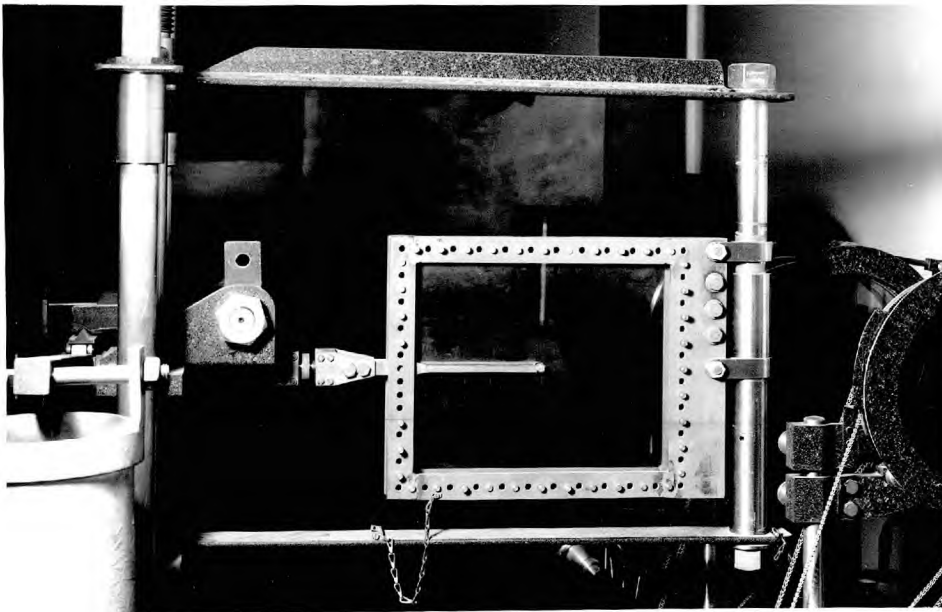


FIG. 15.

In problem 1 the reactions to the load were supplied by $\frac{1}{4}$ " dia. steel pins at 1" centres, which passed through the model on lines $\frac{1}{2}$ " inside each edge and supported it in the loading

Frame (Fig.15) The boundary is now at $r/r_0 = c. 50$. However, the problem stated earlier specified zero stresses at infinity and as the loading frame was made of $\frac{1}{8}$ " thick steel the present condition approximates to zero displacement on the lines of pins. To find the effect of these different conditions we first use the analysis to find u for the points $x = \pm 5\frac{1}{2}$ ", $y = 0$, and v for the points $x = 0$, $y = \pm 4\frac{1}{2}$ ". Take $r_0 = 0.1$ ", which is of the right order for all the specimens tested. Then from the analysis

$$u = 4.7 \bar{P}/\pi G \quad \text{and} \quad 4.4 \bar{P}/\pi G \quad \text{at } z = 5.5 \text{ and } -5.5 \text{ respectively}$$

$$v = -0.19 \bar{P}/\pi G \quad \text{and} \quad +0.19 \bar{P}/\pi G \quad \text{at } z = 4.5i \text{ and } -4.5i \text{ respectively}$$

If these displacements were constant along their respective boundaries the secondary stresses required to annul them in an unstiffened plate would be approximately

$$\mu \hat{xx} = -0.004, \quad \mu \hat{yy} = 0.005$$

If these stresses are now considered as applied at infinity the resulting stress-distributions can be found from Figs. 6 - 10 with $\sigma_x = -0.004/\mu$, $\sigma_y = 0.005/\mu$. The results are negligible compared with the primary stresses. On $\theta = 0$ the only appreciable secondary stress is $\hat{xx}_+$, which reaches 4% of the primary value at $r/r_0 = 10$, and increases in importance with increasing r/r_0 .

The effect of v -displacements on the lines $x = \pm 5\frac{1}{2}$ " and u -displacements on $y = \pm 4\frac{1}{2}$ " is even less serious, for the forces

required to annul them on each boundary individually form a statically equipollent system, and so by St. Venant's Principle the effect in the middle of the sheet will be very small.

The bogus reaction conditions have thus only a minor effect, and we may still hope for agreement between theory and experiment.

The first attempts to apply load to small pilot models were by steel L-shaped straining bars running along the stiffeners and gripping the end at the loading point as shown in Fig.16. This was unsuccessful. At quite low loads fan-shaped fractures spread down inside the sheet at about 20° to the face, and experiments with fillet radii on the end of the stiffener gave no better results. The stress-concentration in this region was clearly 3-dimensional, and to produce uniformity through the section the load had to be spread across the section at its point of application. There was thus no alternative to drilling a hole through the sheet at the end of the stiffener for a stiff loading beam in contact with the specimen throughout the stiffener-sheet-stiffener combination.

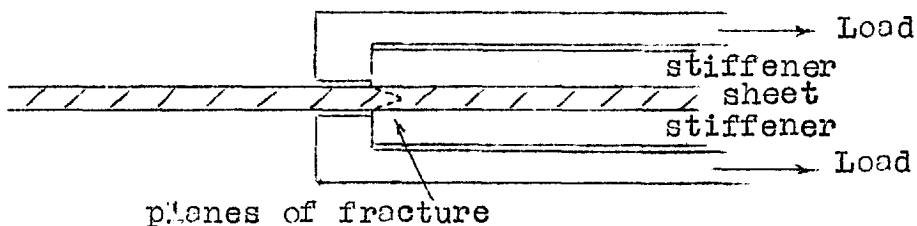


FIG. 16.

The arrangement now is shown in Fig. 17.

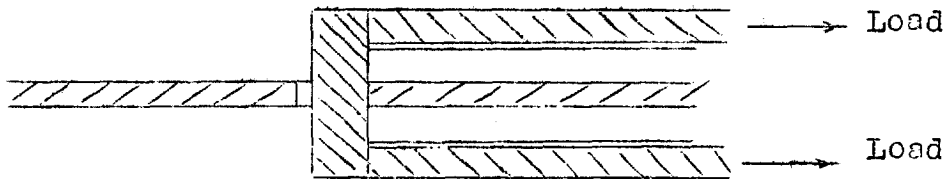


FIG. 17.

The pilot models however were still fickle in taking load. The failures were clean cracks through the sheet, starting between the stiffeners at the loading point and running out at about 60° to the direction of the applied load. This might happen with a maximum fringe value in the sheet anywhere between 4 and 12. (All the pilot models were $3/16$ " thick $A/h \doteq 0.5$). The difficulty was finally overcome by changing the loading beam from a circular rod to a shaped bar with a flatter contact surface, and filing the end of the stiffener to match. With this arrangement, after a little experimenting, a fringe order of 8 before fracture could be relied upon. This technique was now applied to the full size specimens.

The effect of the loading hole on the stress distribution can only be guessed at. For large values of r/r_0 it will be small, but in its own vicinity the hole may well be dominant.

The writer feels (but could not justify the feeling mathematically) that the stresses alongside the stiffener will be very little affected, and the experimental measurements have accordingly been limited to values on the upper and lower edges of the stiffener, which are fortunately the most interesting values. They will serve well as an experimental check on the analysis, and will show how far a theory based on a line stiffener in an infinite sheet can be applied to finite sheets with stiffeners of finite width.

The Tests.

The photographs of the fringe pattern gave, to an arbitrary enlargement, pictures of a $5\frac{3}{4}$ " dia. area of the middle of the specimen, and the scale of the pictures was fixed by two lines scratched on the sheets before test, 3" apart.

The test procedure was first to apply a 150 lb. or 200 lb. load, and hold it constant while the isoclinics were taken (isoclinics do not vary with load). The load was then increased by 50 lb. intervals to failure, with isochromatic photographs taken after each increment. Seven specimens were tested, and only two reached and held the machine's maximum load.

The details of the specimens are set out in Table II, P_0 being the maximum load at which an isochromatic photograph was taken. The values of A are calculated as (width of

stiffener) x (total thickness of stiffener + sheet + stiffener).

Specimen	h	Stiffener Width	Total Thickness	Failing Load	P_o
A	0.243 ins.	1/5 ins.	0.617 ins.	-	554 lb
B	0.185	3/10	0.308	308 lb.	300
C	0.263	1/4	0.497	450	400
D	0.239	1/3	0.358	-	550
E	0.185	1/6	0.552	420	400
F	0.244	1/3	0.606	590	590
G	0.131	1/3	0.618	500	500

Table II.

In the first five tests isoclinic photographs were taken for angles of 0, 10, 20, 30, 40, 45, 50, 60, 70 and 80° to the line of action of the load. The developed negatives showed that the isoclinics run into the stiffener almost tangentially, and they are impossible to read with any accuracy on the actual edge of it. On the remaining two specimens therefore a scale was scribed on the plastic against the stiffener, for isoclinic values to be found at particular points by looking at the model directly through the analyser lens. This is a much more accurate technique, but in the present instance it too failed, for the points

concerned were on the stiffener boundary, and it was impossible to say with confidence at what polariscope setting a line nearly parallel to the boundary just disappeared at such a point.

Consequently the only certain results for comparison with the theory are the isochromatic lines, giving the difference between the principal stresses, or the maximum shear stress.

The photographs were taken on Ilford Thin-film Half-tone Orthochromatic plates, which are slow and of very high contrast. They were developed in a hydroquinone contrast developer (formula Ilford I.D. - 13), and the results are firm black and white negatives with practically no greys.

The isochromatic photographs were taken using the mercury arc lamp and green filter and allowing an exposure of 2 secs. The isoclinics were photographed using the Pointolite lamp and no filter, with an exposure of 4 secs; this gives a degree of over-exposure at which the isochromatics develop out.

Results.

The difference between the principal stresses ($p - q$) along the edge of the stiffener for the seven specimens is shown in Figs. 18 - 20. The theoretical curve is also drawn in each figure for comparison; the values were found from the relation $p - q = \sqrt{2Z}$. The experimental curves were plotted from

$$\mu(p - q) = 2AE_s f n / P_0 E h,$$

where f is the material fringe value, taken as 42, and n is the fringe order.

The agreement between theory and experiment is in general good, especially for high values of r/r_0 , but specimens A and C (Fig. 18) show a considerable variation for $r/r_0 < 3$.

The most probable causes of the discrepancies are the finite width of the stiffener and the possibility of its initially plane cross-sections distorting under load. Both of these will have their greatest effect nearest to the load; the width of the stiffener $> r_0$ in every case, so that $\theta > 30^\circ$ for $r/r_0 < 1$, whereas the line-stiffener theory has $\theta = 0$ along the entire length of the stiffener.

Other possible sources of error are the loading hole, which again is most important for small r/r_0 , the effect of Poisson's ratio $\neq 0.3$ (unlikely to be serious), and the uncertain value of f . As has already been mentioned, f is inconstant. It falls about 15% in the first half-hour of loading. In each test the photograph from which the $(p - q)$ graph was drawn was taken last, within a few moments of failure. By this time isoclinics had been dealt with, plate-holders had been reloaded, and several other isochromatic photographs had been taken at increasing loads. It was never less than half an hour from the first application of load to the final photograph, so

that f , which for C.R.39 starts at 49 - 50, must have fallen to about 42 by the time failure occurred. In some experiments it is possible to find the instantaneous value of f , either by means of a calibrating strip of the same material in series with the load or by integrating numerically across some section to find the applied load. Neither of these methods is available in the present instance, so we must assume that $f = 42$ and admit to a possible error of 2 to 3%.

CONCLUSIONS.

The analysis is exact for an impossible, idealised structure, but it agrees well with experiments on Problem 1 with a variety of practicable specimens. When $r/r_0 > 4$ the agreement is very good on all models. Thus except for very low values of r/r_0 the solution, borne out by the experiments, will hold for practical stress-diffusion problems.

The solution predicts an infinite stress concentration at the end of a stiffener, whether a load is applied there or whether it is a free end in a tension field. Except insofar as the order of the singularity in this idealised problem may be a guide in other investigations, this result is of no interest. Practical structures, with three-dimensional stress-distributions and three-dimensional stiffeners, will be subject to some lesser concentration, as the load life of the brittle C.R.39 models showed. Nevertheless there will still be a stress concentration at an abrupt stiffener end, and the nearer a stiffener approaches a line contact with its sheet the nearer will be the stresses to those predicted here.

APPENDIX.Stevenson's Theory of Complex Potentials.

Stevenson showed (ref. 5) that problems of plane strain and generalised plane stress can be reduced to the choice of two analytic functions ('complex potentials') of a single complex variable, with no restrictions other than the externally applied conditions at the boundaries or elsewhere. It is not necessary, for example, to solve a biharmonic equation.

Stevenson's analysis is not couched in the terms and notation most familiar to engineers, so we derive his results afresh, using a different route but reaching expressions of only a slightly different form.

The equilibrium equations for generalised plane stress with no body forces are

$$\left. \begin{aligned} \frac{\partial \hat{x}x}{\partial x} + \frac{\partial \hat{x}y}{\partial y} &= 0, \\ \frac{\partial \hat{x}y}{\partial x} + \frac{\partial \hat{y}y}{\partial y} &= 0. \end{aligned} \right\} \dots\dots\dots (43)$$

Hooke's Law connects the stresses and strains:

$$\left. \begin{aligned} E\epsilon_x &= \hat{x}x - \nu \hat{y}y, \\ E\epsilon_y &= \hat{y}y - \nu \hat{x}x, \\ E\epsilon_{xy}/2(1+\nu) &= \hat{x}y. \end{aligned} \right\} \dots\dots\dots (44)$$

and the strains are expressed in terms of the displacements by

$$\left. \begin{aligned} e_x &= \partial u / \partial x, \\ e_y &= \partial v / \partial y, \\ e_{xy} &= \partial v / \partial x + \partial u / \partial y. \end{aligned} \right\} \dots\dots\dots (45)$$

Now writing $z \equiv x + iy$, $\bar{z} \equiv x - iy$, we have $\partial / \partial x \equiv \partial / \partial z + \partial / \partial \bar{z}$
 $\partial / \partial y \equiv i (\partial / \partial z - \partial / \partial \bar{z})$. We use two stress combinations to express the three stress components:

$$\left. \begin{aligned} \hat{z}\bar{z} &= \hat{x}x + \hat{y}y, \\ \hat{z}z &= \hat{x}x - \hat{y}y + 2i \hat{x}y. \end{aligned} \right\} \dots\dots\dots (46)$$

Clearly $\hat{z}\bar{z}$ and $\hat{z}z$ are sufficient to specify a state of stress completely.

The two equilibrium equations can now be expressed as the real and imaginary parts of a single complex equation:

$$\left(\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}} \right) (\hat{x}x + i \hat{x}y) + i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) (\hat{x}y + i \hat{y}y) = 0,$$

$$\text{or} \quad \frac{\partial \hat{z}\bar{z}}{\partial \bar{z}} + \frac{\partial \hat{z}z}{\partial z} = 0. \quad \dots\dots\dots (47)$$

We could at this point introduce a stress function $Z(z, \bar{z})$, which would solve (47) if $\partial Z / \partial z = \hat{z}\bar{z}$, $\partial Z / \partial \bar{z} = -\hat{z}z$. But functions of both z and $\bar{z}$ are not simple to manage.

From equations (44) and (46),

$$\left. \begin{aligned} \hat{z}\bar{z} &= \frac{E}{1-\nu} (e_x + e_y), \\ \hat{z}z &= \frac{E}{1+\nu} (e_x - e_y + i e_{xy}). \end{aligned} \right\} \dots\dots\dots (48)$$

We introduce the complex displacement $w \equiv u + iv$, $\bar{w} \equiv u - iv$, and find

$$e_x + e_y = \frac{\partial w}{\partial z} + \frac{\partial \bar{w}}{\partial \bar{z}} = 2R \left\{ \frac{\partial w}{\partial z} \right\}, \dots\dots\dots (49)$$

$$e_x - e_y + i e_{xy} = 2 \frac{\partial w}{\partial \bar{z}}. \dots\dots\dots (50)$$

Then, combining (48), (49) and (50), and substituting in (47), we have

$$\frac{\partial^2 \bar{w}}{\partial \bar{z}^2} + \frac{3-\nu}{1+\nu} \frac{\partial^2 w}{\partial z \partial \bar{z}} = 0.$$

Put $(3-\nu)/(1+\nu) = k$ and integrate w.r.t. $\bar{z}$:

$$k \frac{\partial w}{\partial \bar{z}} + \frac{\partial \bar{w}}{\partial \bar{z}} = A(z), \text{ an arbitrary function of } z.$$

The conjugate of this may be written

$$\frac{1}{k} \frac{\partial w}{\partial z} + \frac{\partial \bar{w}}{\partial \bar{z}} = \frac{1}{k} \bar{A}(\bar{z})$$

Subtract, and integrate w.r.t. z :

$$(k - 1/k)w = \int A(z) dz - \frac{z}{k} \bar{A}(\bar{z}) + \bar{B}(\bar{z}),$$

where $\bar{B}(\bar{z})$ is an arbitrary function of $\bar{z}$.

$$\text{Put } \frac{4G}{k^2 - 1} \int A(z) dz = a(z),$$

$$\text{and } \frac{4Gk}{k^2 - 1} \bar{B}(\bar{z}) = \bar{b}(\bar{z}).$$

Then

$$4G w = k a(z) - z \bar{a}'(\bar{z}) + \bar{b}(\bar{z}) \dots\dots\dots (51)$$

$$\begin{aligned}
\hat{\sigma}_{zz} &= \frac{E}{1-\nu} (e_x + e_y) = 2G \frac{(1+\nu)}{1-\nu} \left(\frac{\partial w}{\partial z} + \frac{\partial \bar{w}}{\partial \bar{z}} \right) \\
&= a' (z) + \bar{a}' (\bar{z}) \quad \dots\dots\dots (52)
\end{aligned}$$

$$\begin{aligned}
\hat{\sigma}_{zz} &= \frac{E}{1+\nu} (e_x - e_y + i e_{xy}) = 2G \frac{\partial w}{\partial \bar{z}} \\
&= \bar{b}' (\bar{z}) - z \bar{a}'' (\bar{z}), \quad \dots\dots\dots (53)
\end{aligned}$$

Equations (51), (52) and (53) give all the stresses and displacements in terms of two 'complex potentials', each a function of a single complex variable. The solution to any particular problem lies in choosing $a(z)$ and $b(z)$ to fulfil the external conditions.

It is also useful to express the rotation at any point in terms of the potentials. It is denoted by $\bar{\omega}$, where

$$\begin{aligned}
2\bar{\omega} &= \partial v / \partial x - \partial u / \partial y \\
&= -i (\partial w / \partial z - \partial \bar{w} / \partial \bar{z}) \\
&= 2 \Im \{ \partial w / \partial z \}.
\end{aligned}$$

$$\begin{aligned}
\therefore 4G\bar{\omega} &= \Im \{ k a' (z) - \bar{a}' (\bar{z}) \} \\
&= (1+k) \Im \{ a' (z) \}. \quad \dots\dots\dots (54)
\end{aligned}$$

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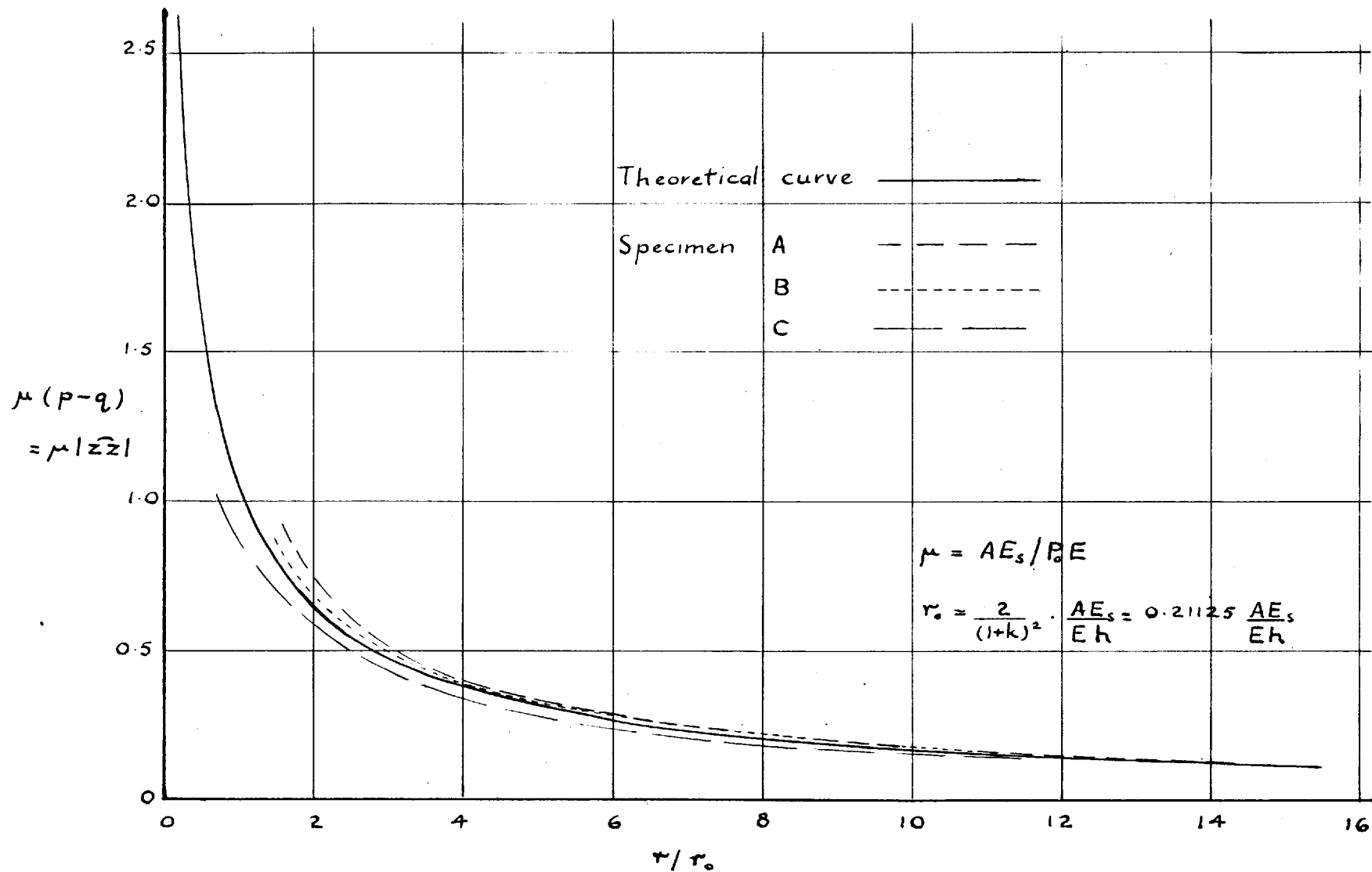


FIG. 18. Comparison with Theory - I.

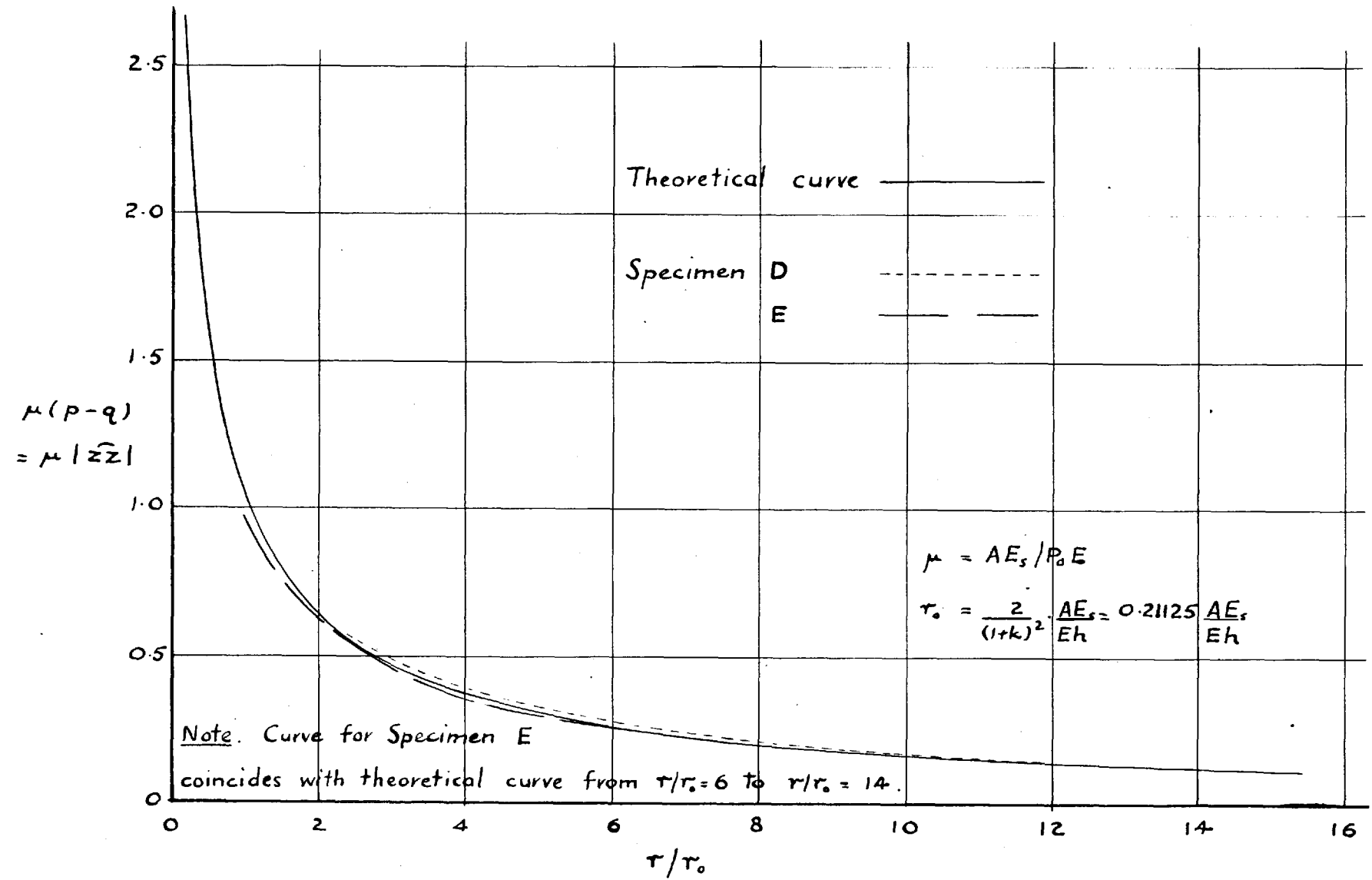


FIG. 19. Comparison with Theory - 2

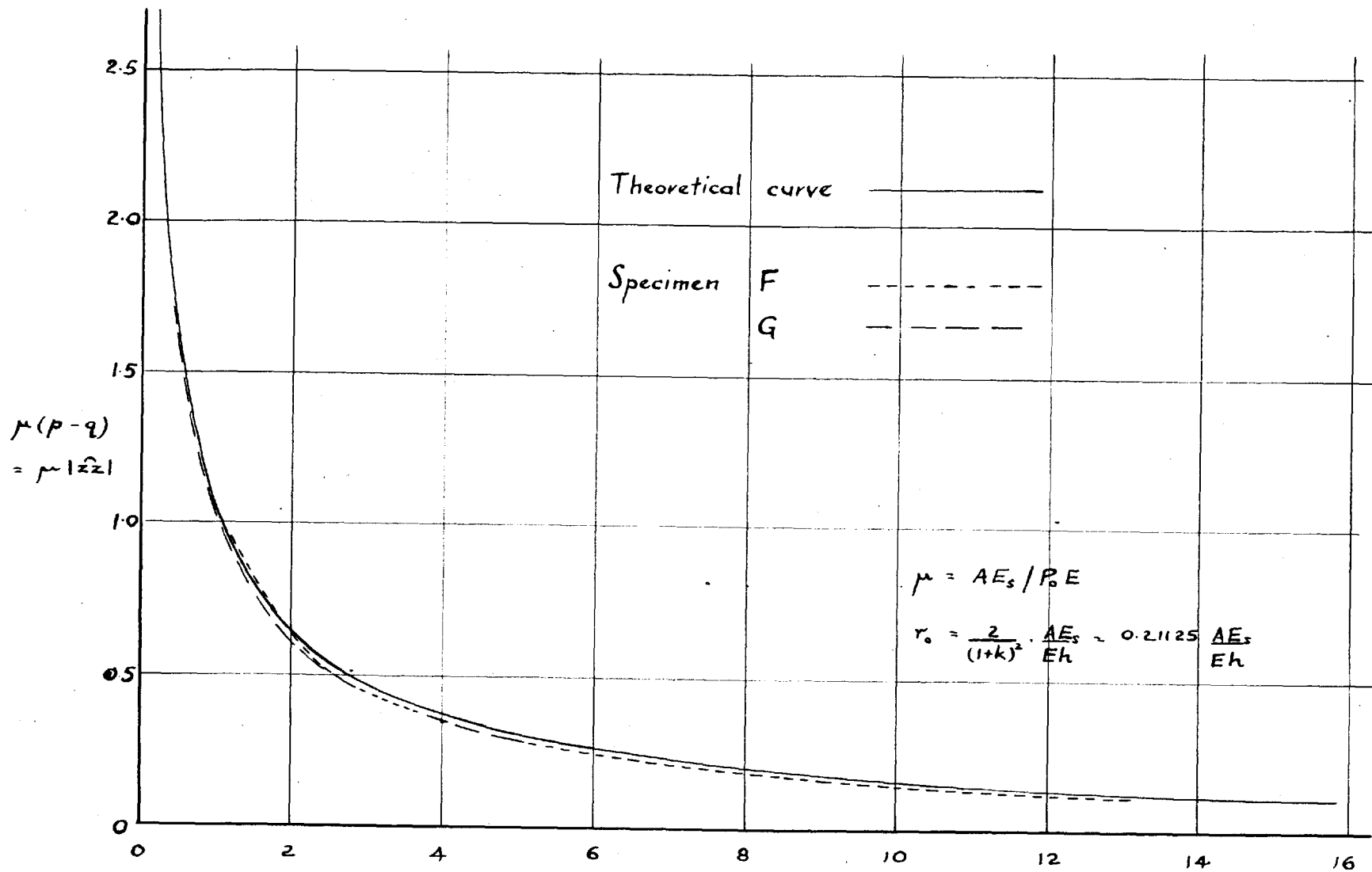


FIG. 20. Comparison with Theory - 3