

Frequency-Domain Bounds for the Multiconductor Telegrapher's Equation

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Abstract—We establish mathematical bounds on the chain, ABCD and immittance matrices of a multiconductor transmission line, based on the Telegrapher's equation. Closed-form expressions for those matrices are also presented. Existing results that hold on the imaginary axis are extended to the complex plane, without reliance on a diagonalizability assumption that is prevalent in the literature. Therefore, the results remain valid under arbitrary arrangements of the conducting wires, which include electrical faults. The analysis ultimately reveals important system-theoretic implications of the Telegrapher's equation, which are of general relevance to control, power systems, and signal processing involving multiconductor transmission lines.

Index Terms—Circuit theory, control theory, distributed parameter systems, infinite-dimensional systems, linear systems, network theory, transmission lines.

I. INTRODUCTION

A. Problem Statement and Motivation

THE multiconductor Telegrapher's equation serves as a foundational tool for modelling the behaviour of electrical signals in n -phase transmission lines. Consider such a uniform line of length $d > 0$, as illustrated in Figure 1. Let $v(t, \zeta) \in \mathbb{R}^n$ denote the n -phase line voltages, relative to ground, at time $t \geq 0$ and position $\zeta \in (0, d)$ along the line. Let $i(t, \zeta) \in \mathbb{R}^n$ denote the corresponding line currents, where current in the direction of increasing ζ is defined as positive.

The relationship between currents and voltages along the interior of the transmission line is modelled by the

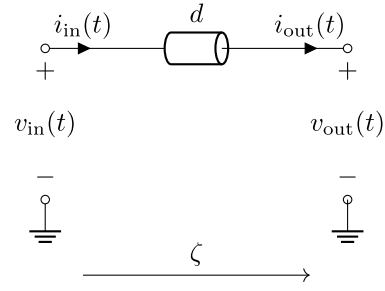


Fig. 1. Transmission line element of length $d > 0$.

Telegrapher's distributed parameter model [1, (3.27)]: for all $t > 0$, $\zeta \in (0, d)$,

$$\frac{\partial x}{\partial \zeta}(t, \zeta) = - \begin{bmatrix} 0 & \mathbf{L} \\ \mathbf{C} & 0 \end{bmatrix} \frac{\partial x}{\partial t}(t, \zeta) - \begin{bmatrix} 0 & \mathbf{R} \\ \mathbf{G} & 0 \end{bmatrix} x(t, \zeta), \quad (1)$$

with state $x(t, \zeta) := \begin{bmatrix} v(t, \zeta) \\ i(t, \zeta) \end{bmatrix}$ and boundary conditions

$$\lim_{\zeta \rightarrow 0^+} x(t, \zeta) = \begin{bmatrix} v_{\text{in}}(t) \\ i_{\text{in}}(t) \end{bmatrix}, \quad \lim_{\zeta \rightarrow d^-} x(t, \zeta) = \begin{bmatrix} v_{\text{out}}(t) \\ i_{\text{out}}(t) \end{bmatrix}, \quad (2)$$

that prescribe the voltages $v_{\text{in}}(t), v_{\text{out}}(t) \in \mathbb{R}^n$ and currents $i_{\text{in}}(t), i_{\text{out}}(t) \in \mathbb{R}^n$ at the two ends of the line. The diagonal elements of $\mathbf{L}, \mathbf{C}, \mathbf{R}$ and $\mathbf{G}$ correspond, respectively, to the inductance, capacitance, line-resistance and conductance-to-ground of each phase, per unit length. The off-diagonal terms result from electromagnetic interactions between different phases, as well as the external environment.

Understanding the frequency-domain bounds associated with the multiconductor Telegrapher's equation is essential for evaluating the structural properties and physical realizability of the model. From the Telegrapher's equation, various two-port network parameter matrices [2, Chapter 2.4] can be derived, each of which serves as a transfer matrix between different tuples of port voltages and currents. These transfer matrices are notoriously non-rational, because they describe infinite-dimensional line dynamics. In the multiconductor case, their derivation is further complicated by the fact that matrix multiplication is generally not commutative.

Existing expressions for the transfer matrices in the literatures [1], [3], [4], and [5] restrict their domains of study to the imaginary axis, and assume diagonalizability of the matrices $(\mathbf{L}\hat{j}\omega + \mathbf{R})(\mathbf{C}\hat{j}\omega + \mathbf{G})$ or $(\mathbf{C}\hat{j}\omega + \mathbf{G})(\mathbf{L}\hat{j}\omega + \mathbf{R})$ at all frequencies $\omega \in \mathbb{R}$. This assumption is widely invoked [6], [7], [8], [9], as it can be guaranteed under specific geometric

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arrangements of the conducting wires [1, Chapter 7.2.2]. Yet, it can break down in several scenarios, including electrical faults and various other situations, as documented in the literature [10]. Moreover, restricting the domain of the network parameter matrices to the imaginary axis is problematic, since their effective use in control or signal processing requires *a priori* knowledge of their behavior over at least the right half of the complex plane.

Treatment of multiconductor transmission lines in the literature, without assuming diagonalizability, is scant. In [11], the authors derived voltage and current solutions to the multiconductor Telegrapher's equation, and also the *surge admittance* of the line, but they did not derive or analyze the chain, ABCD and immittance matrices. To the best of our knowledge, the only other work that treats multiconductor network parameter matrices with such generality is [12]. However, [12] does not provide closed-form expressions; instead, it establishes the existence and analyticity of the scattering and immittance matrices in the open right half-plane. It also shows that the scattering matrix is bounded-real, while the immittance matrices are positive-real. Since positive-realness does not imply boundedness, the results in [12] cannot be directly applied to derive growth bounds. Our earlier work on fault localisation [13] only concerned a two-conductor (single phase) transmission line, and just made empirical observations without establishing any mathematical guarantees.

Growth bounds characterize the behaviour of the entries of the transmission matrices over the complex plane. Among the numerous properties ensured by such bounds are those of causality and input-output stability. In physical domains, systems comply with the principle of causality, i.e., the system state is a function of past and present inputs, but not future ones. In signal processing domains, however, acausality can arise depending on the assignments of inputs and outputs, as illustrated in this paper. To satisfy causality between the given inputs and outputs, the magnitude of a system's transfer function must be bounded and analytic in the right half-plane of the complex frequency variable. Such bounded analyticity ensures that the corresponding time-domain impulse response is causal, meaning that no output can occur before its associated input. In addition, to guarantee stable and bounded input-output behavior, the magnitude of the transfer function must also be bounded along the imaginary axis. Such boundedness in the frequency domain prevents unbounded signal amplification in the time domain, ensuring the system remains physically realizable and stable.

B. Contributions

The preceding discussion highlights the necessity for a rigorous analysis of the multiconductor case, which forms the primary objective of this work. Specifically, we propose *closed-form expressions* for the chain, ABCD, impedance and admittance matrices of a multiconductor line over a complex right-half plane, *without reliance on diagonalizability*. Growth bounds in analytical form over the right-half plane are also established, and their key implications for network stability, causality, and transmission delays are discussed in detail. Our results are valid for any number of conductors, and place no restrictions on the line constants beyond their fundamental physical properties.

C. Organization

After establishing in Section II the fundamental notations and definitions, and deriving, from the multiconductor Telegrapher's equation, the chain and ABCD matrices of the transmission line, Section III presents the principal contributions of the paper. In particular, a blockwise decomposition of the ABCD matrix is introduced, enabling the explicit determination of the A, B, C and D parameters, together with the derivation of growth bounds for both the chain and ABCD matrices. In addition, the admittance matrix is expressed as a function of the A, B, C and D parameters, and its corresponding growth bound is established. Section III further derives the impedance matrix by considering a dual transmission line and provides the associated growth bounds. The proofs of the results presented in Section III are then reported in Sections IV, V, and VI. Finally, Section VII concludes the paper by summarizing the main findings and outlining their prospective application to fault localization.

II. BASIC NOTIONS AND NOTATIONS

1) *Sets and Numbers*: The subset relation is denoted by $\subseteq$, and its strict version by $\subset$. Let $\mathbb{C}$ denote the complex numbers, $j \in \mathbb{C}$ the imaginary unit, $\mathbb{R}$ the reals, $j\mathbb{R}$ the imaginary axis, $\mathbb{Z}$ the integers, and $\mathbb{N} = \{0, 1, \dots\}$ the naturals. Given $\alpha \in \mathbb{R}$, define $\mathbb{C}_\alpha^+ := \{s \in \mathbb{C} \mid \text{Re}(s) > \alpha\}$, with $\mathbb{C}^+ := \mathbb{C}_0^+$. Let $\bar{S}$ denote the closure of $S \subseteq \mathbb{C}$ relative to $\mathbb{C}$, and define $S^+ := S \cap \mathbb{C}^+$. The complex conjugate of $s \in \mathbb{C}$ is $\bar{s}$.

2) *Linear Algebra*: Denote the $n \times n$ identity matrix by $\mathbf{I}_n$. The k th singular value of $A \in \mathbb{C}^{n \times n}$ is $\sigma_k(A)$, its spectrum $\text{spec}(A) := \{\lambda \in \mathbb{C} \mid \det(A - \lambda \mathbf{I}_n) = 0\}$, its conjugate transpose A^* , and its Hermitian part $\mathcal{H}(A) := \frac{A + A^*}{2}$. If $\text{spec}(A) \subset \mathbb{R}$, then $\lambda_{\min}(A) := \min \text{spec}(A)$ and $\lambda_{\max}(A) := \max \text{spec}(A)$. If A is Hermitian, $A > 0$ asserts its positive definiteness, and $A \geq 0$ positive semi-definiteness. The Frobenius matrix norm is denoted by $\|\cdot\|_F$. Let $\|\cdot\|_p$ return the p -norm of a vector in $\mathbb{C}^n$, and for a matrix, the corresponding (p, p) -induced norm. To streamline notation, $\|\cdot\| := \|\cdot\|_2$. The numerical range of $A \in \mathbb{C}^{n \times n}$ is given by

$$W(A) := \{x^* A x \mid x \in \mathbb{S}^n\}, \quad (3)$$

where the Euclidean unit sphere is

$$\mathbb{S}^n := \{x \in \mathbb{C}^n \mid \|x\| = 1\}. \quad (4)$$

Definition 1: [Principal square root] A *principal square root* of $A \in \mathbb{C}^{n \times n}$ is a matrix $X \in \mathbb{C}^{n \times n}$ such that both $\text{spec}(X) \subset \mathbb{C}^+$ and $X^2 = A$. When such an X is unique, denote it by $\sqrt{A}$.

Note that a principal square root is always invertible, because its spectrum cannot contain 0.

3) *Functions*: Given a function $f : [0, \infty) \rightarrow \mathbb{R}^n$ denoted by a lower-case letter, the corresponding upper-case letter is reserved for its Laplace transform

$$F(s) := \mathcal{L}[f](s) := \int_0^\infty f(t)e^{-st} dt.$$

Conversely, $f(t) = \mathcal{L}^{-1}[F](t)$.

Definition 2: [H-infinity] Let $\mathcal{H}_\infty^{m \times n}$ be the space of functions H such that:

- 1) $H : S \rightarrow \mathbb{C}^{m \times n}$ for some $S \subseteq \mathbb{C}$;
- 2) H is analytic on $\mathbb{C}^+ \subseteq S$;
- 3) $\sup\{\|H(s)\| \mid s \in \mathbb{C}^+\} < \infty$.

The space $\mathcal{H}_\infty := \bigcup_{m=1}^\infty \bigcup_{n=1}^\infty \mathcal{H}_\infty^{m \times n}$ is the set of transfer functions of causal $\mathcal{L}_2$ -stable linear time-invariant (LTI) systems [14, Chapter 3.4.3]. Continuity allows the open right-half plane bound in Clause 3 to be extended to the imaginary axis, as per the next result.

Proposition 1: If $H \in \mathcal{H}_\infty^{m \times n}$ is continuous over $\overline{\mathbb{C}^+} \subseteq \text{dom } H$, then $\sup\{\|H(s)\| \mid s \in \overline{\mathbb{C}^+}\} < \infty$.

A. Chain Matrix Derivation

To derive the transfer function for the transmission line in Figure 1, associated with the distributed parameter model (1) with boundary conditions (2), from $\begin{bmatrix} v_{\text{in}} \\ i_{\text{in}} \end{bmatrix}$ to $\begin{bmatrix} v_{\text{out}} \\ i_{\text{out}} \end{bmatrix}$, we consider the following solution space.

Definition 3 (Solution space): Let $\mathcal{X}$ be the set of continuously differentiable functions $x : \mathbb{R}^+ \times (0, d) \rightarrow \mathbb{R}^{2n}$ that satisfy all the following:

- 1) initial rest: $\forall \zeta \in (0, d)$, $\lim_{t \rightarrow 0^+} x(t, \zeta) = 0$;
- 2) exponential order in time: there exists $\alpha \in \mathbb{R}$ such that

$$\forall \zeta \in (0, d), \exists M, T \geq 0, \\ \forall t \geq T, \|x(t, \zeta)\|, \left\| \frac{\partial x}{\partial t}(t, \zeta) \right\| \leq M e^{\alpha t};$$

- 3) interchange of temporal Laplace transformation and spatial differentiation: for all $s \in \mathbb{C}_\alpha^+$ and $\zeta \in (0, d)$,

$$\int_0^\infty \frac{\partial x}{\partial \zeta}(t, \zeta) e^{-st} dt = \frac{\partial}{\partial \zeta} \int_0^\infty x(t, \zeta) e^{-st} dt;$$

- 4) interchange of Laplace transformation and limits: for every $s \in \mathbb{C}_\alpha^+$ and $p \in \{0^+, d^-\}$,

$$\lim_{\zeta \rightarrow p} \int_0^\infty x(t, \zeta) e^{-st} dt = \int_0^\infty \lim_{\zeta \rightarrow p} x(t, \zeta) e^{-st} dt.$$

These restrictions on the solution space permit derivation of the transfer function according to [15, Section 39]. Clauses 2–4 of Definition 3, in particular, formalise assumptions [15, W_1 – W_3 , Section 39.1].

Suppose there exists a solution $x \in \mathcal{X}$ to (1) that satisfies the boundary conditions (2). At each $\zeta \in (0, d)$, the Laplace transform

$$X(s, \zeta) := \int_0^\infty x(t, \zeta) e^{-st} dt$$

converges on $\mathbb{C}_\alpha^+$, by Clause 2. The same holds for the Laplace transform of $\frac{\partial x}{\partial t}$. Take Laplace transforms of both sides of (1) to obtain

$$\frac{\partial X}{\partial \zeta}(s, \zeta) = - \begin{bmatrix} 0 & \mathbf{L}s + \mathbf{R} \\ \mathbf{C}s + \mathbf{G} & 0 \end{bmatrix} X(s, \zeta), \quad (5)$$

under Clauses 1–3 of Definition 3. Since (5) is now a linear ordinary differential equation in ζ , its solution satisfies

$$\forall \zeta, \zeta_0 \in (0, d) \quad \forall s \in \mathbb{C}_\alpha^+, \quad X(s, \zeta) = \Xi(s, \zeta_0 - \zeta) X(s, \zeta_0),$$

where $\Xi : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C}^{2n \times 2n}$ is the matrix exponential

$$\Xi(s, \zeta) := \exp \left(\zeta \begin{bmatrix} 0 & \mathbf{L}s + \mathbf{R} \\ \mathbf{C}s + \mathbf{G} & 0 \end{bmatrix} \right). \quad (6)$$

Taking $\zeta_0 \rightarrow 0$ and $\zeta \rightarrow d$, Clause 4 gives

$$\begin{bmatrix} V_{\text{out}}(s) \\ I_{\text{out}}(s) \end{bmatrix} = \Xi(s, -d) \begin{bmatrix} V_{\text{in}}(s) \\ I_{\text{in}}(s) \end{bmatrix}. \quad (7)$$

Thus, the mapping $s \mapsto \Xi(s, -d)$ is the desired transfer function, with $\Xi(s, -d)$ being the *chain matrix* [1, Section 6.5.1] of a transmission line of length d .

Given that the matrix exponential defined in (6) satisfies $\Xi(s, d) = \Xi(s, -d)^{-1}$, which is the inverse of the chain matrix, it follows from (7) that the matrix in (6) satisfies

$$\begin{bmatrix} V_{\text{in}}(s) \\ I_{\text{in}}(s) \end{bmatrix} = \Xi(s, d) \begin{bmatrix} V_{\text{out}}(s) \\ I_{\text{out}}(s) \end{bmatrix}, \quad (8)$$

making $\Xi(s, d)$ the *ABCD matrix* [16] of the line.

Remark 1: [Port conventions] Given (1) and the direction of the ζ -coordinate, the port labels in Figure 1 are chosen such that the transfer function from input port to output port is the chain matrix. Likewise, every other network parameter matrix corresponds to a different designation of inputs and outputs.

III. MAIN RESULTS

This section presents the principal results of the paper, comprising formal statements that are subsequently followed by in-depth discussions. To streamline the presentation, the detailed proofs for each statement have been derived in Sections IV, V, and VI.

Before presenting the main results, we introduce an instrumental assumption regarding the structural properties of $\mathbf{L}, \mathbf{C}, \mathbf{R}$ and $\mathbf{G}$ defining the Telegrapher's equation (1).

Assumption 1: [Constants] The matrices $\mathbf{L}, \mathbf{C}, \mathbf{R}, \mathbf{G} \in \mathbb{R}^{n \times n}$ are symmetric, with $\mathbf{L}, \mathbf{C} > 0$.

The symmetry of $\mathbf{L}, \mathbf{C}, \mathbf{R}$ and $\mathbf{G}$ is established in [1, Chapter 3] from first principles, whereas the positive definiteness of $\mathbf{L}$ and $\mathbf{C}$ is established in [1, Section 3.5].

With this in mind, and recalling the notion of principal square root of a matrix (see Definition 1), we can present the first result, which concerns the closed-form expression of the ABCD matrix of the Telegrapher's equation.

Theorem 1: [ABCD matrix] Under Assumption 1, for any $s \in \mathbb{C}_\alpha^+$ and $d \in \mathbb{R}$,

$$\Xi(s, d) = \begin{bmatrix} A_d(s) & B_d(s) \\ C_d(s) & D_d(s) \end{bmatrix},$$

where

$$\alpha := -\min \left\{ \frac{\lambda_{\min}(\mathbf{G})}{\lambda_{\max}(\mathbf{C})}, \frac{\lambda_{\min}(\mathbf{G})}{\lambda_{\min}(\mathbf{C})}, \frac{\lambda_{\min}(\mathbf{R})}{\lambda_{\max}(\mathbf{L})}, \frac{\lambda_{\min}(\mathbf{R})}{\lambda_{\min}(\mathbf{L})} \right\}, \quad (9a)$$

$$A_d(s) := \cosh \left(d \sqrt{(\mathbf{L}s + \mathbf{R})(\mathbf{C}s + \mathbf{G})} \right), \quad (9b)$$

$$B_d(s) := \mathbf{Z}(s) \sinh \left(d \sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})} \right), \quad (9c)$$

$$C_d(s) := \mathbf{Y}(s) \sinh \left(d \sqrt{(\mathbf{L}s + \mathbf{R})(\mathbf{C}s + \mathbf{G})} \right), \quad (9d)$$

$$D_d(s) := \cosh \left(d \sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})} \right). \quad (9e)$$

and

$$\mathbf{Z}(s) := (\mathbf{L}s + \mathbf{R}) \sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})}^{-1}, \quad (10a)$$

$$\mathbf{Y}(s) := (\mathbf{C}s + \mathbf{G}) \sqrt{(\mathbf{L}s + \mathbf{R})(\mathbf{C}s + \mathbf{G})}^{-1}. \quad (10b)$$

The above blockwise decomposition of the closed-form expression for the ABCD matrix is valid on the set $\mathbb{C}_\alpha^+$, which contains the imaginary axis when $\mathbf{G}, \mathbf{R} > 0$ (i.e., the conductor system dissipates energy). In this case, it is hence

evident that Theorem 1 serves as a direct generalisation of the various results in [1, Section 7.2], which evaluate $\Xi(s, -d)$ along the imaginary axis under assumptions that render $(\mathbf{L}\hat{j}\omega + \mathbf{R})(\mathbf{C}\hat{j}\omega + \mathbf{G})$ and $(\mathbf{C}\hat{j}\omega + \mathbf{G})(\mathbf{L}\hat{j}\omega + \mathbf{R})$ diagonalizable. Indeed, given that both factors are assumed symmetric, if one of these products is diagonalizable, it follows that the other must also be diagonalizable.

The closed-form representation of the ABCD matrix in (6) is essential for establishing the corresponding growth bound, as presented in the following Theorem.

Theorem 2: [Growth of ABCD matrix] Under Assumption 1,

$$\|\Xi(s, d)\| \leq \kappa e^{\frac{|d|}{\nu}(\operatorname{Re}(s) + \theta)} \quad (11)$$

for all $s \in \mathbb{C}$ and $d \in \mathbb{R}$, where

$$\theta := \frac{\max\{\|\mathbf{R}\|, \|\mathbf{G}\|\}}{\max\{\|\mathbf{L}\|, \|\mathbf{C}\|\}} \geq 0, \quad (12a)$$

$$\nu := \frac{1}{\kappa \max\{\|\mathbf{L}\|, \|\mathbf{C}\|\}} > 0 \quad (12b)$$

$$\kappa := \sup \left\{ \left\| \exp \left(\begin{bmatrix} 0 & \hat{j}\omega \mathbf{L} \\ \hat{j}\omega \mathbf{C} & 0 \end{bmatrix} \right) \right\| \mid \omega \in \mathbb{R} \right\} \in [1, \infty). \quad (12c)$$

The bound on the growth of the ABCD matrix presented in Theorem 2 yields several physical implications.

1) *Stability:* The bound (11) is independent of the imaginary part of s . Thus, the chain and ABCD matrices are bounded along every vertical line in the complex plane, including the imaginary axis.

Corollary 1: [$\mathcal{L}_2$ -Stability] For any $d > 0$,

$$\sup\{\|\Xi(\hat{j}\omega, d)\| \mid \omega \in \mathbb{R}\} < \infty.$$

In other words, both chain and ABCD matrices are elements of $\mathcal{L}_\infty(\mathbb{R})$, which establishes $\mathcal{L}_2$ -stability of their corresponding time-domain operators [14, Theorem 3.25]. For the chain matrix, this guarantees the energy at the output port in Figure 1 is bounded linearly by the energy at the input port. A linear bound in the opposite direction follows from the $\mathcal{L}_2$ -stability of the ABCD matrix.

2) *Causality:* The bound (11) grows exponentially with the real part of s . Whilst it is only an upper bound, consideration of the two-conductor ($n = 1$) case reveals that the chain and ABCD matrices do indeed grow exponentially as $\operatorname{Re}(s) \rightarrow \infty$. Hence, these transfer functions are not in $\mathcal{H}_\infty$, and are therefore not causal [14, Chapter 3.4.3].

To understand the implications of this, consider two identical transmission lines. If the input signals to each are equal up to some time t , network acausality means that the outputs may differ even before time t . Differences in the input after time t can thus “travel back in time” to affect the output before time t . Of course, transmission lines do not really permit electrons to travel back in time (to the best of the authors’ knowledge). The mathematical fact that these transfer matrices are acausal indicates that the signals labelled “inputs” should not be viewed as “causing” the signals labelled “outputs”. Rather, both inputs and outputs satisfy a joint constraint in their product signal space [17], and information flows both ways. If the propagation of information were instantaneous, then causality would still be preserved. However, since the speed of propagation is finite, disturbances at an “output” terminal take

time to reach an “input” terminal, so that the output is seen to change before the input. Thus, the acausality here reflects the physical propagation delays in the transmission line. In fact, we now show that the rate of exponential growth in (11) yields a lower bound on the propagation speed.

Remark 2: [Lead-time] For any fixed $d \in \mathbb{R}$, an ABCD matrix can be written as the product

$$\Xi(s, d) = H_d(s) e^{\frac{|d|}{\nu} s}$$

of a pure time advance with the causal transfer function

$$H_d(s) := e^{-\frac{|d|}{\nu} s} \Xi(s, d),$$

where $H_d \in \mathcal{H}_\infty$ by Theorem 2. The exponent $\frac{|d|}{\nu}$ in (11) thus bounds the time by which the output can lead the input, thereby quantifying the degree of acausality in the system. This, in turn, makes ν a lower bound on the speed of information propagation. The same relationships clearly hold for the chain parameter matrix, by Proposition 2.

We now analyze the admittance matrix associated with the Telegrapher’s equation (1). We preliminary note that, in contrast to (7) and (8), the admittance matrix $Y(s, d)$ must satisfy

$$\begin{bmatrix} I_{\text{in}}(s) \\ -I_{\text{out}}(s) \end{bmatrix} = Y(s, d) \begin{bmatrix} V_{\text{in}}(s) \\ V_{\text{out}}(s) \end{bmatrix} \quad (13)$$

by definition, with $-i_{\text{out}}$ directed into the line as required.

With this in mind, we can present the following theorem concerning the properties of the admittance matrix.

Theorem 3: [Telegrapher’s admittance matrix] Under the hypotheses of Theorem 1, the admittance matrix $Y : \mathbb{C}_\alpha^+ \times \mathbb{R}^+ \rightarrow \mathbb{C}^{2n \times 2n}$ given by

$$Y(s, d) = \begin{bmatrix} D_d(s)B_d(s)^{-1} & -B_d(s)^{-1} \\ -B_d(s)^{-1} & D_d(s)B_d(s)^{-1} \end{bmatrix} \quad (14)$$

has the following properties:

- 1) Y is continuous.
- 2) For every $d \in \mathbb{R}^+$, the map $s \mapsto Y(s, d)$ is analytic.
- 3) For any $\beta \in [0, \infty) \cap (\alpha, \infty)$ and $\delta > 0$, there exists $M > 0$ such that

$$\forall s \in \overline{\mathbb{C}_\beta^+} \quad \forall d \geq \delta, \quad \|Y(s, d)\| \leq M e^{\frac{\delta d}{\nu} \operatorname{Re}(s + \theta)}. \quad (15)$$

Besides providing the closed-form of the admittance matrix, Theorem 3 also gives the growth bound (15). Indeed, Theorem 3 suffices to establish $\mathcal{L}_2$ -stability of the admittance matrix.

Corollary 2: If $\mathbf{G}, \mathbf{R} > 0$, then for any $d > 0$,

$$\sup\{\|Y(\hat{j}\omega, d)\| \mid \omega \in \mathbb{R}\} < \infty.$$

Recalling Remark 2, the lower bound on propagation speed $\frac{\nu}{n}$ implied by Theorem 3 is more conservative than the lower bound ν implied by Theorem 2.

Following the same line of reasoning, impedance matrices share all the properties established in Theorem 3 for admittance matrices. This is stated formally by the following two results, analogous to Theorem 3 and Corollary 2.

Theorem 4: [Telegrapher’s impedance matrix] Under the hypotheses of Theorem 1, the impedance matrix $Z : \mathbb{C}_\alpha^+ \times \mathbb{R}^+ \rightarrow \mathbb{C}^{2n \times 2n}$ given by $Z(s, d) := Y(s, d)^{-1}$, has the following properties:

- 1) Z is continuous.
- 2) For every $d \in \mathbb{R}^+$, the map $s \mapsto Z(s, d)$ is analytic.
- 3) For any $\beta \in [0, \infty) \cap (\alpha, \infty)$ and $\delta > 0$, there exists $M > 0$ such that

$$\forall s \in \overline{\mathbb{C}_\beta^+} \quad \forall d \geq \delta, \quad \|Z(s, d)\| \leq M e^{\frac{\alpha d}{\beta} \operatorname{Re}(s+\theta)}.$$

Corollary 3: If $\mathbf{G}, \mathbf{R} > 0$, then for any $d > 0$,

$$\sup\{\|Z(\hat{\omega}, d)\| \mid \omega \in \mathbb{R}\} < \infty.$$

We have presented bounds on all relevant transfer functions of a possibly non-balanced transmission line. These bounds allow us to characterize how both magnitudes and propagation delays vary along the line. Beyond their practical utility, the results are of theoretical interest in circuit theory, as they establish fundamental properties, and the existence of solutions to the Telegrapher's equation, even under atypical conditions.

The results are also of practical significance. For example, fault localization is a natural domain of application. The transfer-function expressions derived here enable the construction of models for transmission networks under weak assumptions on line characteristics, including those that hold under faulty conditions. In such regimes, even defining the propagation delay of transient faults is non-trivial. The bounds we provide on propagation delays and signal amplification can serve as key inputs to a fault-localization procedure, as reported in [18].

In the sections that follow, we present rigorous proofs establishing the claims made in this section.

IV. PROOF OF THEOREM 1 AND THEOREM 2

To begin with, the following result demonstrates that the chain and ABCD matrices are unitarily similar.

Proposition 2: For any $s \in \mathbb{C}$ and $d \in \mathbb{R}$,

$$\Xi(s, d) = \begin{bmatrix} -\mathbf{I}_n & 0 \\ 0 & \mathbf{I}_n \end{bmatrix}^* \Xi(s, -d) \begin{bmatrix} -\mathbf{I}_n & 0 \\ 0 & \mathbf{I}_n \end{bmatrix},$$

and thus $\|\Xi(s, d)\| = \|\Xi(s, -d)\|$.

Proof: Let $M := \begin{bmatrix} 0 & \mathbf{L}s + \mathbf{R} \\ \mathbf{C}s + \mathbf{G} & 0 \end{bmatrix}$ and $U := \begin{bmatrix} -\mathbf{I}_n & 0 \\ 0 & \mathbf{I}_n \end{bmatrix}$. Then, $-M = U M U^*$. Thus, $\Xi(s, -d) = \exp(-dM) = \exp(U d M U^*) = U \exp(dM) U^* = U \Xi(s, d) U^*$. ■

In light of this, properties derived for the ABCD matrix can be applied directly to the chain matrix, and *vice versa*.

A. Blockwise Decomposition of ABCD Matrix (Theorem 1)

In order to derive the other network parameter matrices, the ABCD matrix must be partitioned into four square blocks, which are themselves the A, B, C and D network parameters.

Proposition 3: Let $A, B \in \mathbb{C}^{n \times n}$. If there exist invertible $P, Q \in \mathbb{C}^{n \times n}$ such that $P^2 = AB$ and $Q^2 = BA$, then

$$\exp \left(\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \right) = \begin{bmatrix} \cosh P & A Q^{-1} \sinh Q \\ B P^{-1} \sinh P & \cosh Q \end{bmatrix}.$$

Proof: Let $M := \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}$. The matrix exponential, $\exp(M) = \sum_{k=0}^{\infty} \frac{M^k}{k!}$, converges for all square matrices. Observe that

$$M^{2k} = \begin{bmatrix} (AB)^k & 0 \\ 0 & (BA)^k \end{bmatrix}, \quad (16)$$

$$M^{2k+1} = \begin{bmatrix} 0 & A(BA)^k \\ B(AB)^k & 0 \end{bmatrix}, \quad (17)$$

for any $k \in \mathbb{N}$. Hence, splitting the summation into even and odd powers gives

$$\begin{aligned} \exp(M) &= \sum_{k=0}^{\infty} \left(\frac{M^{2k}}{(2k)!} + \frac{M^{2k+1}}{(2k+1)!} \right) \\ &= \sum_{k=0}^{\infty} \left(\frac{\begin{bmatrix} (AB)^k & 0 \\ 0 & (BA)^k \end{bmatrix}}{(2k)!} + \frac{\begin{bmatrix} 0 & A(BA)^k \\ B(AB)^k & 0 \end{bmatrix}}{(2k+1)!} \right) \\ &= \sum_{k=0}^{\infty} \begin{bmatrix} \frac{(AB)^k}{(2k)!} & \frac{A(BA)^k}{(2k+1)!} \\ \frac{B(AB)^k}{(2k+1)!} & \frac{(BA)^k}{(2k)!} \end{bmatrix}. \end{aligned}$$

Suppose $P, Q \in \mathbb{C}^{n \times n}$ are such that $P^2 = AB$ and $Q^2 = BA$. The matrix hyperbolic functions are given by the globally convergent [19, Section 4.3] power series

$$\begin{aligned} \cosh P &= \sum_{k=0}^{\infty} \frac{P^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(AB)^k}{(2k)!}, \\ \cosh Q &= \sum_{k=0}^{\infty} \frac{Q^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(BA)^k}{(2k)!}, \\ \sinh P &= \sum_{k=0}^{\infty} \frac{P^{2k+1}}{(2k+1)!} = P \sum_{k=0}^{\infty} \frac{(AB)^k}{(2k+1)!}, \\ \sinh Q &= \sum_{k=0}^{\infty} \frac{Q^{2k+1}}{(2k+1)!} = Q \sum_{k=0}^{\infty} \frac{(BA)^k}{(2k+1)!}. \end{aligned}$$

If P, Q are also invertible, then the result follows. ■

Clearly, before Proposition 3 can be applied to $\Xi(s, d)$, invertible matrix square roots of $(\mathbf{L}s + \mathbf{R})(\mathbf{C}s + \mathbf{G})$ and $(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})$ must be found.

Now, a square root of a Hermitian matrix is easily obtainable via diagonalization, but even though $\mathbf{L}, \mathbf{C}, \mathbf{R}, \mathbf{G}$ are symmetric, $\mathbf{L}s + \mathbf{R}$ and $\mathbf{C}s + \mathbf{G}$ are not Hermitian when $s \notin \mathbb{R}$. They do, however, fall within another convenient class of matrices.

Definition 4: [Accretive matrix] A square complex matrix M is *accretive* if its Hermitian part $\mathcal{H}(M) > 0$.

Proposition 4: If $A, B \in \mathbb{C}^{n \times n}$ are accretive, then the unique principal square root $\sqrt{AB}$ exist.

Proof: Any real eigenvalues of a product of accretive matrices are strictly positive [20, Theorem 1]. Thus, $\operatorname{spec}(AB) \cap (-\infty, 0] = \emptyset$. Since AB has no eigenvalues in $(-\infty, 0]$, it has a unique principal square root [19, Theorem 1.29]. ■

To see that $\mathbf{L}s + \mathbf{R}$ and $\mathbf{C}s + \mathbf{G}$ are both accretive over a right half-plane, first consider the linear combination of two Hermitian matrices.

Lemma 1: If $A, B \in \mathbb{C}^{n \times n}$ are Hermitian and $t \in \mathbb{R}$, then

$$\lambda_{\min}(At) + \lambda_{\min}(B) \leq \lambda_{\min}(At + B).$$

Proof: Recalling that $\mathbb{S}^n$ is the unit sphere by (4), we have

$$\begin{aligned} \lambda_{\min}(At + B) &= \min\{x^*(At + B)x \mid x \in \mathbb{S}^n\} \\ &\geq \min\{x^*Atx \mid x \in \mathbb{S}^n\} + \min\{x^*Bx \mid x \in \mathbb{S}^n\} \\ &= \lambda_{\min}(At) + \lambda_{\min}(B). \end{aligned}$$

■

This lower bound is now used to show that the sum of a positive definite matrix, with an arbitrary Hermitian matrix, remains positive definite if the former is sufficiently large.

Lemma 2: Let $A, B \in \mathbb{C}^{n \times n}$ be Hermitian. If $A > 0$, then $At + B > 0$ for all $t > -\min \left\{ \frac{\lambda_{\min}(B)}{\lambda_{\max}(A)}, \frac{\lambda_{\min}(B)}{\lambda_{\min}(A)} \right\}$.

Proof: Let $t > \max \left\{ -\frac{\lambda_{\min}(B)}{\lambda_{\max}(A)}, -\frac{\lambda_{\min}(B)}{\lambda_{\min}(A)} \right\}$. Lemma 1 implies

$$\lambda_{\min}(At + B) \geq \lambda_{\min}(At) + \lambda_{\min}(B).$$

Suppose $A > 0$, by which $\lambda_{\max}(A) \geq \lambda_{\min}(A) > 0$. If $t \geq 0$, then $\lambda_{\min}(At) = t\lambda_{\min}(A) > -\lambda_{\min}(B)$, and it follows that $\lambda_{\min}(At + B) > 0$. Otherwise, $\lambda_{\min}(At) = t\lambda_{\max}(A) > -\lambda_{\min}(B)$, which yields the same result. ■

These results for Hermitian matrices can now be applied to $\mathbf{L}s + \mathbf{R}$ and $\mathbf{C}s + \mathbf{G}$ by taking their Hermitian parts.

Corollary 4: Under Assumption 1, $\mathbf{L}s + \mathbf{R}$ is accretive for all $s \in \mathbb{C}_\rho^+$, and $\mathbf{C}s + \mathbf{G}$ is accretive for all $s \in \mathbb{C}_\gamma^+$, where

$$\gamma := -\min \left\{ \frac{\lambda_{\min}(\mathbf{G})}{\lambda_{\max}(\mathbf{C})}, \frac{\lambda_{\min}(\mathbf{G})}{\lambda_{\min}(\mathbf{C})} \right\}, \quad (18a)$$

$$\rho := -\min \left\{ \frac{\lambda_{\min}(\mathbf{R})}{\lambda_{\max}(\mathbf{L})}, \frac{\lambda_{\min}(\mathbf{R})}{\lambda_{\min}(\mathbf{L})} \right\}. \quad (18b)$$

Proof: For any $s \in \mathbb{C}$, $\mathcal{H}(\mathbf{L}s + \mathbf{R}) = \mathbf{L}\text{Re}(s) + \mathbf{R}$ is symmetric, and if $\text{Re}(s) > \rho$, is also positive definite by Lemma 2. Similarly, if $\text{Re}(s) > \gamma$, then $\mathcal{H}(\mathbf{C}s + \mathbf{G}) = \mathbf{C}\text{Re}(s) + \mathbf{G} > 0$. ■

In contrast to [1, Chapter 7], which relies on diagonalizability to construct the square roots, their existence can now simply be guaranteed by changing the domain of evaluation from the imaginary axis to a right half-plane.

Corollary 5: Under Assumption 1, for any $s \in \mathbb{C}_\rho^+ \cap \mathbb{C}_\gamma^+$, the unique principal square roots $\sqrt{(\mathbf{L}s + \mathbf{R})(\mathbf{C}s + \mathbf{G})}$ and $\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})}$ both exist.

Finally, piecing together the preceding results, we obtain the desired blockwise decomposition of the ABCD matrix of Theorem 1 as proved below.

Proof (Proof Theorem 1): Observe that $\mathbb{C}_\rho^+ \cap \mathbb{C}_\gamma^+ = \mathbb{C}_\alpha^+$, since $\alpha = \max\{\rho, \gamma\}$. The rest is a direct consequence of Proposition 3 and Corollary 5. ■

B. Growth Bounds (Theorem 2)

Bounding the chain and ABCD matrices over the whole complex plane first involves bounding the exponent $\left[\begin{smallmatrix} 0 & \mathbf{L}s + \mathbf{R} \\ \mathbf{C}s + \mathbf{G} & 0 \end{smallmatrix} \right]$ along the real axis.

Lemma 3: Under Assumption 1,

$$\left\| \begin{bmatrix} 0 & \mathbf{L}\varsigma + \mathbf{R} \\ \mathbf{C}\varsigma + \mathbf{G} & 0 \end{bmatrix} \right\| \leq c_1|\varsigma| + c_0 \quad (19)$$

for all $\varsigma \in \mathbb{R}$, where

$$c_0 := \max\{\|\mathbf{R}\|, \|\mathbf{G}\|\} \geq 0, \quad (20a)$$

$$c_1 := \max\{\|\mathbf{L}\|, \|\mathbf{C}\|\} > 0. \quad (20b)$$

Proof: Applying [21, Lemma 2.10],

$$\begin{aligned} \left\| \begin{bmatrix} 0 & \mathbf{L}\varsigma + \mathbf{R} \\ \mathbf{C}\varsigma + \mathbf{G} & 0 \end{bmatrix} \right\| &\leq \left\| \begin{bmatrix} 0 & \|\mathbf{L}\varsigma + \mathbf{R}\| \\ \|\mathbf{C}\varsigma + \mathbf{G}\| & 0 \end{bmatrix} \right\| \\ &\leq \left\| \begin{bmatrix} 0 & \|\mathbf{L}\|\varsigma + \|\mathbf{R}\| \\ \|\mathbf{C}\|\varsigma + \|\mathbf{G}\| & 0 \end{bmatrix} \right\|. \end{aligned}$$

Since $\left\| \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \right\| \leq \max\{a, b\}$ for $a, b \geq 0$, the result follows. ■

To bound the magnitude of the ABCD matrix along the imaginary axis, temporarily consider the lossless case ($\mathbf{R} = \mathbf{G} = 0$).

Lemma 4: Under Assumption 1,

$$\kappa := \sup \left\{ \left\| \exp \left(\begin{bmatrix} 0 & j\omega \mathbf{L} \\ j\omega \mathbf{C} & 0 \end{bmatrix} \right) \right\| \mid \omega \in \mathbb{R} \right\} \in [1, \infty). \quad (21)$$

If, moreover, $\mathbf{CL}$ is normal, then

$$\kappa \leq 1 + \frac{\max\{\|\mathbf{L}\|, \|\mathbf{C}\|\}}{\sqrt{\lambda_{\min}(\mathbf{CL})}}. \quad (22)$$

Proof: To see that $\kappa \geq 1$, consider $\omega = 0$.

Its finiteness is established next. The products $\mathbf{CL}$ and $\mathbf{LC}$ are both diagonalizable, and have strictly positive eigenvalues by [22, Corollary 7.6.2]. Therefore, they have unique principal square roots by [19, Theorem 1.29]. In particular, there exists an invertible $Q \in \mathbb{C}^{n \times n}$ such that

$$\Lambda := Q^{-1}\mathbf{LC}Q = Q^*\mathbf{CL}Q^{-*} > 0 \quad (23)$$

is real and diagonal, by which

$$\sqrt{\mathbf{LC}} = Q\sqrt{\Lambda}Q^{-1}, \quad \sqrt{\mathbf{CL}} = Q^{-*}\sqrt{\Lambda}Q^*. \quad (24)$$

Applying Proposition 3, $\exp \left(\begin{bmatrix} 0 & j\omega \mathbf{L} \\ j\omega \mathbf{C} & 0 \end{bmatrix} \right) =$

$$\begin{aligned} &\begin{bmatrix} \cosh(j\omega \sqrt{\mathbf{LC}}) & \mathbf{L}\sqrt{\mathbf{CL}}^{-1} \sinh(j\omega \sqrt{\mathbf{CL}}) \\ \mathbf{C}\sqrt{\mathbf{LC}}^{-1} \sinh(j\omega \sqrt{\mathbf{LC}}) & \cosh(j\omega \sqrt{\mathbf{CL}}) \end{bmatrix} = \\ &\begin{bmatrix} \cos(\omega \sqrt{\mathbf{LC}}) & j\mathbf{L}\sqrt{\mathbf{CL}}^{-1} \sin(\omega \sqrt{\mathbf{CL}}) \\ j\mathbf{C}\sqrt{\mathbf{LC}}^{-1} \sin(\omega \sqrt{\mathbf{LC}}) & \cos(\omega \sqrt{\mathbf{CL}}) \end{bmatrix} = \\ &\begin{bmatrix} Q \cos(\omega \sqrt{\Lambda}) Q^{-1} & j\mathbf{L}\sqrt{\mathbf{CL}}^{-1} Q^{-*} \sin(\omega \sqrt{\Lambda}) Q^* \\ j\mathbf{C}\sqrt{\mathbf{LC}}^{-1} Q \sin(\omega \sqrt{\Lambda}) Q^{-1} & Q^{-*} \cos(\omega \sqrt{\Lambda}) Q^* \end{bmatrix}. \end{aligned}$$

Since every component of the above matrix is bounded, $\kappa < \infty$.

If $\mathbf{CL}$ is normal, then a unitary Q can be chosen in (23), in which case $Q^* = Q^{-1}$ and $Q^{-*} = Q$. Now $\|\sin(\omega \sqrt{\Lambda})\|, \|\cos(\omega \sqrt{\Lambda})\| \leq 1$ for all ω . Since the spectral norm is unitarily invariant, [21, Lemma 2.10] yields

$$\left\| \exp \left(\begin{bmatrix} 0 & j\omega \mathbf{L} \\ j\omega \mathbf{C} & 0 \end{bmatrix} \right) \right\| \leq \left\| \frac{1}{\|\mathbf{C}\sqrt{\mathbf{LC}}^{-1}\|} \frac{\|\mathbf{L}\sqrt{\mathbf{CL}}^{-1}\|}{1} \right\|.$$

It is also clear from (24) that

$$\|\sqrt{\mathbf{LC}}^{-1}\| = \|\sqrt{\mathbf{CL}}^{-1}\| = \lambda_{\min}(\mathbf{CL})^{-\frac{1}{2}}.$$

Recalling $\left\| \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \right\| \leq \max\{a, b\}$ for $a, b \geq 0$, (22) follows. ■

Combining the preceding bounds, we obtain an exponential bound on $\Xi(s, d)$ over the whole complex plane, as presented in Theorem 2 and proved below.

Proof (Proof Theorem 2): For any $d, \varsigma, \omega \in \mathbb{R}$,

$$\Xi(\varsigma + j\omega, d) = \exp((A + \Delta)|d|),$$

where $A := j\omega \text{sgn}(d) \begin{bmatrix} 0 & \mathbf{L} \\ \mathbf{C} & 0 \end{bmatrix}$ and $\Delta := \text{sgn}(d) \begin{bmatrix} 0 & \mathbf{L}\varsigma + \mathbf{R} \\ \mathbf{C}\varsigma + \mathbf{G} & 0 \end{bmatrix}$. By Lemma 4, $\|e^{A|d|}\| \leq \kappa < \infty$ for all $d \in \mathbb{R}$. It follows from [23, Proposition 4.2.18] that $\|\exp((A + \Delta)|d|)\| \leq \kappa e^{\kappa\|\Delta\||d|}$, where $\|\Delta\| \leq c_1|\varsigma| + c_0$ by Lemma 3. The exponent is thus $\kappa c_1 \left(|\varsigma| + \frac{c_0}{c_1} \right) |d|$, and the result follows from (20). ■

V. PROOF OF THEOREM 3

A. Derivation

In contrast to (7) and (8), the admittance matrix $Y(s, d)$ must satisfy

$$\begin{bmatrix} I_{\text{in}}(s) \\ -I_{\text{out}}(s) \end{bmatrix} = Y(s, d) \begin{bmatrix} V_{\text{in}}(s) \\ V_{\text{out}}(s) \end{bmatrix}$$

by definition, with $-i_{\text{out}}$ directed into the line as required. Applying the two-port conversion formula [24, (13)],

$$Y(s, d) = \begin{bmatrix} D_d(s)B_d(s)^{-1} & C_d(s) - D_d(s)B_d(s)^{-1}A_d(s) \\ -B_d(s)^{-1} & B_d(s)^{-1}A_d(s) \end{bmatrix}. \quad (25)$$

To simplify this, observe

$$\begin{aligned} \Xi(s, -d)\Xi(s, d) &= \begin{bmatrix} A_d(s) & -B_d(s) \\ -C_d(s) & D_d(s) \end{bmatrix} \begin{bmatrix} A_d(s) & B_d(s) \\ C_d(s) & D_d(s) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I}_n & 0 \\ 0 & \mathbf{I}_n \end{bmatrix}, \end{aligned}$$

by which $0 = A_d(s)B_d(s) - B_d(s)D_d(s)$, so $B_d(s)^{-1}A_d(s) = D_d(s)B_d(s)^{-1}$. Additionally, $\mathbf{I}_n = -C_d(s)B_d(s) + D_d(s)^2$, so $B_d(s)^{-1} = -C_d(s) + D_d(s)^2B_d(s)^{-1} = -C_d(s) + D_d(s)B_d(s)^{-1}A_d(s)$. Thus, the line admittance matrix is given by (14), i.e.,

$$Y(s, d) = \begin{bmatrix} D_d(s)B_d(s)^{-1} & -B_d(s)^{-1} \\ -B_d(s)^{-1} & D_d(s)B_d(s)^{-1} \end{bmatrix}.$$

The invertibility of $B_d(s)$ is verified in the next two results

Lemma 5: For any $X \in \mathbb{C}^{n \times n}$, $\sinh(X)$ is singular if and only if $\text{spec}(X) \cap \{k\pi j \mid k \in \mathbb{Z}\} \neq \emptyset$.

Proof: Suppose $\sinh(X)$ is singular. Then there exists a nonzero $z \in \mathbb{C}^n$ such that $\sinh(X)z = \left(\frac{\exp(X) - \exp(-X)}{2}\right)z = 0$. It follows that $\exp(X)z = \exp(-X)z$, by which $\exp(2X)z = z$. Thus, 1 is an eigenvalue of $\exp(2X)$. Since $\text{spec}(\exp(2X)) = \{e^\lambda \mid \lambda \in \text{spec}(2X)\}$ by [25, Proposition 11.2.3], there exists $\lambda \in \text{spec}(2X)$ such that $e^\lambda = 1$. It follows that $\lambda \in \{2k\pi j \mid k \in \mathbb{Z}\}$. Thus, $\frac{\lambda}{2} \in \text{spec}(X) \cap \{k\pi j \mid k \in \mathbb{Z}\}$. To establish the converse, retrace the same argument in the opposite direction. ■

Corollary 6: Under the hypotheses of Theorem 1, the matrix $B_d(s)$ is invertible for all $s \in \mathbb{C}_\alpha^+$ and $d \in \mathbb{R} \setminus \{0\}$.

Proof: Let $s \in \mathbb{C}_\alpha^+$. Then the principal square root $\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})}$ exists by Corollary 5. Since its spectrum is contained in $\mathbb{C}^+$, $\text{spec}(d\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})}) \cap j\mathbb{R} = \emptyset$ for any $d \in \mathbb{R} \setminus \{0\}$, and so $\sinh(d\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})})$ is invertible. Moreover, $\text{spec}(\mathbf{L}s + \mathbf{R}) \subset \mathbb{C}^+$ by Corollary 4, so $\mathbf{L}s + \mathbf{R}$ is invertible, and thus $\mathbf{Z}(s)$ is invertible. ■

Remark 3: [Short-circuit] The admittance matrix $Y(s, d)$ cannot be defined at $d = 0$, because this constitutes a short-circuit.

B. Growth Bounds

To bound the admittance matrix in (14), observe that $D_d(s)$ is already bounded by (11), as a submatrix of $\Xi(s, d)$. The challenge is then to find an upper bound on $B_d(s)^{-1}$, which via the next result, translates to a lower bound on the determinant of $B_d(s)$.

Proposition 5: For any invertible $A \in \mathbb{C}^{n \times n}$,

$$\|A^{-1}\| \leq \frac{\|A\|_F^{n-1}}{|\det(A)|}.$$

Proof: The bound

$$\sigma_n(A) \geq \left(\frac{n-1}{\|A\|_F^2}\right)^{\frac{n-1}{2}} |\det A|$$

is proved in [26]. Recalling $0^0 = 1$, $k_n := (n-1)^{\frac{n-1}{2}}$ is non-decreasing over $n \geq 1$, so $k_n \geq k_1 = 1$ for all $n \geq 1$. Thus,

$$\sigma_n(A) \geq \frac{|\det A|}{\|A\|_F^{n-1}},$$

and the result follows because $\|A^{-1}\| = \sigma_n(A)^{-1}$. ■

Recall, now, the expression

$$B_d(s) = \mathbf{Z}(s) \sinh\left(d\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})}\right) \quad (9c)$$

in Theorem 1. We first focus on lower bounding the determinant of $\sinh(d\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})})$, which can be achieved by bounding the spectrum of $\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})}$ away from the imaginary axis. This in turn demands bounds on the spectrum of the matrix product $(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})$, for which there are no straightforward results (unless mutual diagonalisability of the two factors is assumed). The spectra of products of general linear operators is treated in [27], which in Theorem 1, gives

$$\sigma(A^{-1}B) \subseteq W(B)/W(A) \quad (26)$$

$$:= \left\{ \frac{z_2}{z_1} \mid z_2 \in W(B), z_1 \in W(A) \right\},$$

for complex square matrices A and B , provided $0 \notin W(A)$. We therefore turn to bounds on the numerical ranges $W((\mathbf{C}s + \mathbf{G})^{-1})$ and $W(\mathbf{L}s + \mathbf{R})$.

Proposition 6: For any $A \in \mathbb{C}^{n \times n}$,

$$\begin{aligned} \lambda_{\min} \mathcal{H}(A) &= \min \text{Re}(W(A)) \\ &= \min\{\text{Re}(x^*Ax) \mid x \in \mathbb{C}^n, \|x\| \geq 1\}. \end{aligned}$$

Proof: For any $x \in \mathbb{C}^n$,

$$\begin{aligned} \text{Re}(x^*Ax) &= \frac{x^*Ax + \overline{x^*Ax}}{2} \\ &= \frac{x^*Ax + x^*A^*x}{2} = x^* \mathcal{H}(A)x. \end{aligned}$$

Since $\mathcal{H}(A)$ is Hermitian,

$$\begin{aligned} \min \text{Re}(W(A)) &= \min\{\text{Re}(x^*Ax) \mid x \in \mathbb{S}^n\} \\ &= \min\{x^* \mathcal{H}(A)x \mid x \in \mathbb{S}^n\} = \lambda_{\min} \mathcal{H}(A). \end{aligned}$$

Similarly,

$$\begin{aligned} &\min\{\text{Re}(x^*Ax) \mid x \in \mathbb{C}^n, \|x\| \geq 1\} \\ &= \min\{x^* \mathcal{H}(A)x \mid x \in \mathbb{C}^n, \|x\| \geq 1\} \\ &= \min\{x^* \mathcal{H}(A)x \mid x \in \mathbb{S}^n\} = \lambda_{\min} \mathcal{H}(A). \end{aligned}$$

Proposition 6 encloses the numerical range within a right-half plane: $W(A) \subset \overline{\mathbb{C}_{\lambda_{\min} \mathcal{H}(A)}^+}$. To deal with $W((\mathbf{C}s + \mathbf{G})^{-1})$, the same must now be done for an inverse matrix.

Lemma 6: For any invertible $A \in \mathbb{C}^{n \times n}$,

$$\{\text{Re}(z^{-1}) \mid z \in W(A^{-1}) \setminus \{0\}\} \subset \overline{\mathbb{C}_{\lambda_{\min} \mathcal{H}(A)}^+}.$$

Proof: Suppose $z \in W(A^{-1}) \setminus \{0\}$. Then there exists $y \in \mathbb{S}^n$ such that $z = y^* A^{-1} y$. Setting $x := A^{-1} y$,

$$x^* A^* x = y^* A^{-*} A^* A^{-1} y = y^* A^{-1} y = z,$$

so

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} = \frac{\overline{x^* A^* x}}{|z|^2} = \frac{x^* A x}{|z|^2} = w^* A w,$$

where

$$w := \frac{x}{|z|} = \frac{A^{-1} y}{y^* A^{-1} y}.$$

Since $\|y\| = 1$, the Cauchy-Schwarz implies $\|w\| \geq 1$. Thus, $\operatorname{Re}(z^{-1}) = \operatorname{Re}(w^* A w) \geq \lambda_{\min} \mathcal{H}(A)$, by Proposition 6. ■

An additional result is now required to exclude the origin from the numerical range of the inverse.

Lemma 7: For any invertible $A \in \mathbb{C}^{n \times n}$,

$$0 \in W(A) \iff 0 \in W(A^{-1}).$$

Proof: Suppose $w \in W(A)$. Then $w = x^* A x$ for some $x \in \mathbb{S}^n$. Since A is invertible, $Ax \neq 0$, so set $y = \frac{Ax}{\|Ax\|} \in \mathbb{S}^n$. Then

$$y^* A^{-1} y = \frac{x^* A^* A^{-1} A x}{\|Ax\|^2} = \frac{x^* A^* x}{\|Ax\|^2} = \frac{\overline{x^* A x}}{\|Ax\|^2} = 0,$$

by which $0 \in W(A^{-1})$. ■

Before appeal to (26), a right half-plane bound on the square root of a scalar product is required to deal with the spectrum of $\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})}$.

Lemma 8: If $w^2 = z_1 z_2$, where $z_1, z_2, w \in \overline{\mathbb{C}^+}$, then

$$\operatorname{Re}(w) \geq \min\{\operatorname{Re}(z_1), \operatorname{Re}(z_2)\}.$$

Proof: Since $z_1, z_2 \in \overline{\mathbb{C}^+}$, there exist $r_1, r_2 \geq 0$ and $\theta_1, \theta_2 \in [\frac{-\pi}{2}, \frac{\pi}{2}]$ such that $z_1 = r_1 e^{j\theta_1}$ and $z_2 = r_2 e^{j\theta_2}$, so $w = \sqrt{r_1 r_2} e^{j(\frac{\theta_1 + \theta_2}{2})}$. Clearly $\frac{\theta_1 + \theta_2}{2} \in [\frac{-\pi}{2}, \frac{\pi}{2}]$, by which $\cos(\frac{\theta_1 + \theta_2}{2}) \geq 0$, and therefore

$$\operatorname{Re}(w) = \sqrt{r_1 r_2} \cos(\frac{\theta_1 + \theta_2}{2}) \geq \min\{r_1, r_2\} \cos(\frac{\theta_1 + \theta_2}{2}).$$

Since the cosine function is concave over $[\frac{-\pi}{2}, \frac{\pi}{2}]$,

$$\begin{aligned} \operatorname{Re}(w) &\geq \min\{r_1, r_2\} \frac{\cos \theta_1 + \cos \theta_2}{2} \\ &\geq \min\{r_1, r_2\} \cdot \min\{\cos \theta_1, \cos \theta_2\}, \end{aligned}$$

from which the result follows. ■

Williams' result (26) now enables the generalisation of Lemma 8 to matrix products.

Lemma 9: If $A, B \in \mathbb{C}^{n \times n}$ are accretive,

$$\operatorname{spec} \sqrt{AB} \subseteq \overline{\mathbb{C}_b^+},$$

where $b = \min\{\lambda_{\min} \mathcal{H}(A), \lambda_{\min} \mathcal{H}(B)\}$.

Proof: First recall, the existence of $\sqrt{AB}$ is guaranteed by Proposition 4. By hypothesis, $\mathcal{H}(A) > 0$, so $\lambda_{\min} \mathcal{H}(A) > 0$. Proposition 6 then implies $0 \notin W(A)$, and Lemma 7 in turn implies $0 \notin W(A^{-1})$. Suppose $\lambda \in \operatorname{spec} \sqrt{AB}$. Then $\lambda^2 \in \operatorname{spec} AB$, and it follows from [27, Theorem 1] that

$$\exists z_1 \in W(A^{-1}) \exists z_2 \in W(B), \lambda^2 = \frac{z_2}{z_1}.$$

Observe that $\operatorname{Re}(z_2) \geq \lambda_{\min} \mathcal{H}(B)$ by Proposition 6, and $\operatorname{Re}(z_1^{-1}) \geq \lambda_{\min} \mathcal{H}(A)$ by Lemma 6, so the result then follows from Lemma 8. ■

This can finally be applied to $\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})}$. Below, its spectrum is bounded away from the imaginary axis when s is also sufficiently far to the right.

Lemma 10: Under the hypotheses of Theorem 1, for any $\epsilon \in \mathbb{R}^+ \cap [-\alpha, \infty)$,

$$\bigcup_{s \in \overline{\mathbb{C}_{\alpha+\epsilon}^+}} \operatorname{spec} \sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})} \subseteq \overline{\mathbb{C}_{b\epsilon}^+},$$

where $b = \min\{\lambda_{\min}(\mathbf{L}), \lambda_{\min}(\mathbf{C})\} > 0$.

Proof: Let $s \in \overline{\mathbb{C}_{\alpha+\epsilon}^+}$, where $\epsilon > 0$ is such that $\alpha + \epsilon \geq 0$. Recall from (18) and (9a) that

$$\rho, \gamma \leq \alpha < \alpha + \epsilon \leq \operatorname{Re}(s).$$

Thus, $\mathbf{L}s + \mathbf{R}$ and $\mathbf{C}s + \mathbf{G}$ are accretive by Corollary 4. Applying Lemma 1,

$$\begin{aligned} \lambda_{\min} \mathcal{H}(\mathbf{L}s + \mathbf{R}) &= \lambda_{\min}(\mathbf{L} \operatorname{Re}(s) + \mathbf{R}) \\ &\geq (\alpha + \epsilon) \lambda_{\min}(\mathbf{L}) + \lambda_{\min}(\mathbf{R}) \\ &\geq \epsilon \lambda_{\min}(\mathbf{L}) > 0, \end{aligned} \quad (27)$$

because $\alpha \geq -\frac{\lambda_{\min}(\mathbf{R})}{\lambda_{\min}(\mathbf{L})}$ by (9a). Similarly,

$$\begin{aligned} \lambda_{\min} \mathcal{H}(\mathbf{C}s + \mathbf{G}) &= \lambda_{\min}(\mathbf{C} \operatorname{Re}(s) + \mathbf{G}) \\ &\geq (\alpha + \epsilon) \lambda_{\min}(\mathbf{C}) + \lambda_{\min}(\mathbf{G}) \\ &\geq \epsilon \lambda_{\min}(\mathbf{C}) > 0, \end{aligned} \quad (28)$$

because $\alpha \geq -\frac{\lambda_{\min}(\mathbf{G})}{\lambda_{\min}(\mathbf{C})}$ by (9a). The result then follows by Lemma 9. ■

As previously mentioned, the preceding right half-plane bound on the spectrum of $\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})}$ turns into a lower bound on the determinant of $\sinh(d \sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})})$.

Lemma 11: Under the hypotheses of Theorem 1, let $\delta > 0$ and $\epsilon \in \mathbb{R}^+ \cap [-\alpha, \infty)$. Then for all $s \in \overline{\mathbb{C}_{\alpha+\epsilon}^+}$ and $d \geq \delta$,

$$\left| \det \sinh \left(d \sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})} \right) \right| \geq (\delta b \epsilon)^n,$$

where $b = \min\{\lambda_{\min}(\mathbf{L}), \lambda_{\min}(\mathbf{C})\} > 0$.

Proof: Let $s \in \overline{\mathbb{C}_{\alpha+\epsilon}^+}$, $d \geq \delta$, and let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of $\sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})}$. Then by [19, Theorem 1.13], $\sinh(d\lambda_k)$ are the eigenvalues of $\sinh(d \sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})})$, so

$$\left| \det \sinh \left(d \sqrt{(\mathbf{C}s + \mathbf{G})(\mathbf{L}s + \mathbf{R})} \right) \right| = \prod_{k=1}^n |\sinh(d\lambda_k)|.$$

Now $\sinh$ is strictly increasing, and that $\sinh t \geq t$ for all $t \geq 0$, which is evident from its Taylor series. Applying the identity $|\sinh z|^2 = \sinh^2(\operatorname{Re} z) + \sin^2(\operatorname{Im} z)$ in [28, (4.5.54)],

$$|\sinh(d\lambda_k)| \geq \sinh(d|\operatorname{Re} \lambda_k|) \geq \sinh(\delta \epsilon_b) \geq \delta b \epsilon,$$

by Lemma 10. ■

We finally have the desired lower bound on the determinant of the right-hand factor of $B_d(s)$ in (9c). A lower bound on the determinant of the left-hand factor $\mathbf{Z}(s)$ is required next. This can be obtained by considering a generic first-order transfer matrix.

Lemma 12: Let $A, B, P, Q \in \mathbb{R}^{n \times n}$, and define the transfer function $H : S \rightarrow \mathbb{C}^{n \times n}$ as

$$H(s) := (As + B)(Ps + Q)^{-1},$$

where $S := \{s \in \mathbb{C} \mid \det(Ps + Q) \neq 0\}$. If $P, Q > 0$ then

$$\sup\{|\det H(s)| \mid s \in \overline{\mathbb{C}^+}\} < \infty.$$

Proof: Since $P > 0$, $Ps + Q = P(s + P^{-1}Q)$, where $P^{-1} > 0$. Noting that $\text{spec}(P^{-1}Q) \subset \mathbb{R}^+$ by [22, Corollary 7.6.2(a)], the set of poles of H is then

$$\{s \in \mathbb{C} \mid \det(sI + P^{-1}Q) = 0\} = \text{spec}(-P^{-1}Q) \subset \mathbb{R}^-.$$

Thus, H is a proper rational transfer matrix with poles in the open left-half plane, which implies $H \in \mathcal{H}_\infty^{n \times n}$ [21, Section 4.3]. There exists $K > 0$ such that $\|H(s)\| \leq K$ for all $s \in \overline{\mathbb{C}^+}$ by Proposition 1. It follows that $|\det H(s)| \leq K^n$, because the determinant is the product of the eigenvalues, and the spectral norm bounds the spectral radius. ■

The previous result is an upper bound when the denominator matrices P, Q are positive definite. If the numerator matrices are as well, then we get a lower bound.

Corollary 7: If $A, B, P, Q > 0$, then

$$\inf\{|\det H(s)| \mid s \in \overline{\mathbb{C}^+}\} > 0.$$

Proof: Apply Lemma 12 to $H(s)^{-1}$. ■

This result can now be applied to $\mathbf{Z}(s)$, recalling its definition in (10a). Observe, below, that $\beta = \alpha + \epsilon \geq 0$ for some $\epsilon > 0$.

Lemma 13: Under the hypotheses of Theorem 1, for any $\beta \in [0, \infty) \cap (\alpha, \infty)$,

$$\inf\{|\det \mathbf{Z}(s)| \mid s \in \overline{\mathbb{C}_\beta^+}\} > 0.$$

Proof: For any $s \in \mathbb{C}_\beta^+$, both $\mathbf{L}s + \mathbf{R}$ and $\mathbf{C}s + \mathbf{G}$ are accretive, and therefore invertible. Thus,

$$\begin{aligned} |\det \mathbf{Z}(s)| &= \frac{|\det(\mathbf{L}s + \mathbf{R})|}{\sqrt{|\det(\mathbf{L}s + \mathbf{R}) \det(\mathbf{C}s + \mathbf{G})|}} \\ &= \sqrt{\frac{|\det(\mathbf{L}s + \mathbf{R})|}{|\det(\mathbf{C}s + \mathbf{G})|}} \\ &= \sqrt{|\det((\mathbf{L}s + \mathbf{R})(\mathbf{C}s + \mathbf{G})^{-1})|} \\ &= \sqrt{|\det H(s)|}, \end{aligned}$$

where $H(s) := (\mathbf{L}(s + \beta) + \mathbf{R})(\mathbf{C}(s + \beta) + \mathbf{G})^{-1}$. Now $\mathbf{L}, \mathbf{C} > 0$ by Assumption 1, and $\mathbf{L}\beta + \mathbf{R} > 0$ and $\mathbf{C}\beta + \mathbf{G} > 0$ by Lemma 2. The result then follows from Corollary 7. ■

Having lower bounded the left-hand factor determinant, combining this with Lemma 11 yields the desired lower bound on $\det B_d(s)$.

Corollary 8: Under the hypotheses of Theorem 1, for any $\delta > 0$ and $\beta \in [0, \infty) \cap (\alpha, \infty)$,

$$\inf\{|\det B_d(s)| \mid s \in \overline{\mathbb{C}_\beta^+}, d \geq \delta\} > 0.$$

The preceding analysis leads to the result presented in Theorem 3, which provides a growth bound on the admittance matrix. The formal result is proved below.

Proof: [Proof (Theorem 3)] Recall the parameter definitions in (21) and (12). Analyticity of both B_d and D_d , as well as continuity with respect to d , follow from Theorem 1 because they are properties of the matrix exponential. Corollary 6 implies $B_d(s)$ is invertible on the domain of Y , and matrix inversion preserves both properties, so they hold for Y too.

For the third property, observe the admittance matrix factors as

$$Y(s, d) = \begin{bmatrix} D_d(s) & -\mathbf{I}_n \\ -\mathbf{I}_n & D_d(s) \end{bmatrix} \begin{bmatrix} B_d(s)^{-1} & 0 \\ 0 & B_d(s)^{-1} \end{bmatrix},$$

thereby

$$\|Y(s, d)\| \leq (\|D_d(s)\| + 1)\|B_d(s)^{-1}\|.$$

Proposition 5 and Corollary 8 imply the existence of $K > 0$ such that, for all $s \in \overline{\mathbb{C}_\beta^+}$ and $d \geq \delta$,

$$\|B_d(s)^{-1}\| \leq K\|B_d(s)\|^{n-1},$$

recalling that $\|B_d(s)\|_F \leq \sqrt{n}\|B_d(s)\|$ by [25, Fact 9.8.12(ix)]. Thus,

$$\|Y(s, d)\| \leq K(\|D_d(s)\| + 1)\|B_d(s)\|^{n-1},$$

and since $\|B_d(s)\|, \|D_d(s)\| \leq \|\Xi(s, d)\|$,

$$\|Y(s, d)\| \leq K(\|\Xi(s, d)\|^n + \|\Xi(s, d)\|^{n-1}).$$

Theorem 2 implies

$$\forall s \in \overline{\mathbb{C}_\beta^+} \quad \forall d \geq \delta, \quad \|\Xi(s, d)\| \leq \kappa e^{\frac{d}{\nu}(\text{Re}(s) + \theta)}.$$

Since $\kappa \geq 1$, and $d, \nu > 0$,

$$\kappa^{n-1} e^{(n-1)\frac{d}{\nu}(\text{Re}(s) + \theta)} \leq \kappa^n e^{n\frac{d}{\nu}(\text{Re}(s) + \theta)},$$

and therefore $\|Y(s, d)\| \leq 2K\kappa^n e^{n\frac{d}{\nu}(\text{Re}(s) + \theta)}$. ■

VI. PROOF OF THEOREM 4

Given the transmission line model, consider a *dual line* with inductance $\mathbf{L}' := \mathbf{C}$, capacitance $\mathbf{C}' := \mathbf{L}$, resistance $\mathbf{R}' := \mathbf{G}$ and conductance $\mathbf{G}' := \mathbf{R}$. From Theorem 1, it is clear that the ABCD matrices of the dual are related to those of the primal via

$$\begin{aligned} A'_d(s) &= D_d(s), & B'_d(s) &= C_d(s), \\ C'_d(s) &= B_d(s), & D'_d(s) &= A_d(s). \end{aligned}$$

The impedance matrix of the primal is given by

$$\mathbf{Z}(s, d) = \begin{bmatrix} A_d(s)C_d(s)^{-1} & A_d(s)C_d(s)^{-1}D_d(s) - B_d(s) \\ C_d(s)^{-1} & C_d(s)^{-1}D_d(s) \end{bmatrix},$$

in terms of its ABCD parameters [24, (9)]. Comparing this with (25) reveals that

$$\mathbf{Z}(s, d) = \begin{bmatrix} 0 & -\mathbf{I}_n \\ \mathbf{I}_n & 0 \end{bmatrix} Y'(s, d) \begin{bmatrix} 0 & -\mathbf{I}_n \\ \mathbf{I}_n & 0 \end{bmatrix},$$

by which $\|\mathbf{Z}(s, d)\| = \|Y'(s, d)\|$ for all s and d .

Remark 4: [Duality] The impedance matrix of a transmission line is unitarily similar to the admittance matrix of its dual.

Impedance matrices thus share all the properties established in Theorem 3 for admittance matrices. This is stated formally in Theorem 4 and proved below.

Proof (Proof Theorem 4): In view of Remark 4, it only remains to show that the dual κ', θ', ν' are identical in value to the primal κ, θ, ν . Since

$$\begin{bmatrix} 0 & \mathbf{L} \\ \mathbf{C} & 0 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{I}_n \\ \mathbf{I}_n & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{C} \\ \mathbf{L} & 0 \end{bmatrix} \begin{bmatrix} 0 & \mathbf{I}_n \\ \mathbf{I}_n & 0 \end{bmatrix},$$

it follows that $\exp(j\omega \begin{bmatrix} 0 & \mathbf{L} \\ \mathbf{C} & 0 \end{bmatrix})$ and $\exp(j\omega \begin{bmatrix} 0 & \mathbf{L}' \\ \mathbf{C}' & 0 \end{bmatrix})$ are unitarily similar, and so share the same spectral norm. Equation (21) then implies $\kappa' = \kappa$, by which $(\theta', \nu') = (\theta, \nu)$ from (12). ■

VII. CONCLUDING REMARKS

The chain, ABCD, admittance and impedance matrices of a multiconductor transmission line have been derived, as transfer functions parametrised by the length of the line. This relies on the linearity and time-invariance of the underlying dynamics [14, Chapter 3.4.1]. Mathematically, each transfer function corresponds to a different linear operator, with different pairs of voltages and currents designated as inputs and outputs. None of the transfer functions are rational, implying they all describe infinite-dimensional dynamics. This is consistent with their derivation from the Telegrapher's equation, a linear time-invariant PDE. As a result, this line model exhibits a finite speed of information propagation, a feature that cannot be reproduced by any finite-dimensional model.

Frequency-domain bounds for each of the transfer matrices have also been derived. The bounds have implications for network stability, causality, and transmission delays. The chain and ABCD matrices are bounded over the imaginary axis, which establishes their $\mathcal{L}_2$ -stability. This means the energy of the output signals is bounded linearly by the energy of the input signals. The impedance and admittance matrices are, likewise, shown to be $\mathcal{L}_2$ -stable when the line conductance and resistance matrices are positive-definite.

An $\mathcal{L}_2$ -stable LTI system is causal if and only if its transfer function lies in $\mathcal{H}_\infty$ [14, Chapter 3.4.3], which requires boundedness over the complex right half-plane. However, the bounds derived here permit all four transfer matrices to grow exponentially as $\text{Re}(s) \rightarrow \infty$. The chain, ABCD, admittance, and impedance matrices of a distributed parameter transmission line are, in fact, not causal. This is due to the bidirectional flow of information between inputs and outputs. The finite speed of information propagation is lower bounded by the rate of exponential growth in the frequency domain.

The physical properties mentioned above could be expected of a transmission line on purely physical grounds. Our contribution is to demonstrate that they are system-theoretic consequences of using the Telegrapher's equation to model the line. Applications of our results in the domain of fault localization are reported in [18].

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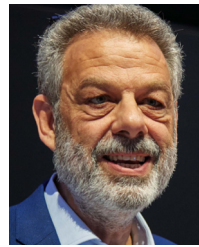
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