

Gauge theory on G_2 -manifolds

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Declaration of Originality

I hereby declare that material presented in this thesis is my own work except where indicated otherwise.

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Abstract

In their seminal paper [DT98] Donaldson–Thomas pointed out the possibility of an enumerative invariant for G_2 –manifolds obtained by counting certain connections, called G_2 –instantons. This putative invariant is sometimes referred to as the G_2 Casson invariant, since it should be formally similar to the Casson invariant for 3–manifolds. In this thesis I prove existence results for G_2 –instantons on G_2 –manifolds arising from Joyce’s generalised Kummer construction [Joy96b, Joy00] as well as the twisted connected sum construction [Kov03, CHNP12b]. These yield a number of concrete examples of G_2 –instantons and may, in the future, help to compute the G_2 Casson invariant. Moreover, I show how to construct families of G_2 –instantons that bubble along associative submanifolds. From this construction it follows that a naïve count of G_2 –instantons cannot yield a deformation invariant of G_2 –manifolds. Nevertheless, there can still be hope for a G_2 Casson invariant by counting G_2 –instantons as well as associative submanifolds (and objects in between) with carefully chosen weights. I present a promising proposal for the definition of these weights in the low energy $SU(2)$ –theory.

Introduction

A G_2 -manifold is a 7-manifold equipped with a torsion-free G_2 -structure or (essentially) equivalently a Riemannian metric with holonomy contained in G_2 . G_2 -manifolds appear as one of two exceptional cases in Berger's classification of holonomy groups of irreducible non-symmetric Riemannian manifolds [Ber55, Theorem 3]; they are Ricci flat and carry non-trivial parallel spinors which, so I am told, makes them relevant to high energy physics. G_2 -manifolds naturally carry two calibrations in the sense of Harvey–Lawson [HL82]: the associative and the coassociative calibrations; and they also admit interesting solutions to the Yang–Mills equations called G_2 -instantons.

Examples of G_2 -manifolds can be constructed from Riemannian manifolds with holonomy $SU(2)$ or $SU(3)$, which themselves can be constructed using Yau's solution of the Calabi conjecture [Yau77, Yau78]. The local existence of Riemannian metrics with holonomy equal to G_2 was established by Bryant [Bry87]. By fairly elementary means Bryant–Salamon [BS89] constructed the first examples of complete metrics with holonomy G_2 . Joyce [Joy96b] was the first to succeed in constructing compact G_2 -manifolds with holonomy equal to G_2 . He proceeds via a gluing construction, inspired by the folklore Kummer construction of hyperkähler metrics on $K3$, and obtains G_2 -manifolds which are in some sense close to flat G_2 -orbifolds of the form T^7/Γ with smooth singular sets of codimension 4 or 6. In his book [Joy00] he generalised this construction to accommodate more general and rather complicated singular sets. Based on an idea of Donaldson, Kovalev [Kov03] developed a further construction technique, which produces G_2 -manifolds with long necks, by gluing a pair of products of an asymptotically cylindrical Calabi–Yau 3-fold with S^1 in a non-trivial manner. This technique has been extended and generalised by Kovalev–Lee [KL11] and Corti–Haskins–Nordström–Pacini [CHNP12b] and can now be used to produce a vast number of examples of compact G_2 -manifolds.

It is an interesting question to ask which compact 7-manifolds do admit torsion-free G_2 -structures. While it is known that there are a number of topological obstructions, some of which I will discuss in Chapter 1, there are currently no useful general criteria that would guarantee the existence of a torsion-free G_2 -structure. Another interesting problem is to study the moduli space $\mathcal{M}$ of torsion-free G_2 -structures on a fixed compact 7-manifold Y . Joyce [Joy96b, Theorem C] proved that $\mathcal{M}$ is a smooth manifold of dimension $b^3(Y)$;

however, beyond that very little is known. Both of the above mentioned construction techniques produce G_2 -manifolds close to degenerate limits. In particular, in these cases $\mathcal{M}$ is always non-compact; although it is clear how to (at least partially) compactify it. There are many interesting and difficult questions to ask: What are the possible degenerations of a family of smooth G_2 -manifolds? How to detect these degenerations? Can we find meaningful compactifications $\bar{\mathcal{M}}$ of $\mathcal{M}$ in some generality? What can be said about the topology of $\mathcal{M}$ (or $\bar{\mathcal{M}}$)? Some aspects of these questions can be investigated using current technology; however, there are very few tools that make use of the special geometry of G_2 -manifolds.

In their hugely influential paper [DT98] Donaldson–Thomas noticed that it may be possible to carry over some of the gauge theoretic tools successfully employed in the study of 3-manifolds to the world of G_2 -manifolds. In particular, they point out the existence of a G_2 analogue of the Chern–Simons functional whose critical points are G_2 -instantons. Before being carried away by this observation into the land of fantasy one should ask some concrete questions: Are there non-trivial examples of G_2 -instantons? What do G_2 -instantons over generalised Kummer constructions or twisted connected sums look like? Can we find models for the moduli space of G_2 -instantons over generalised Kummer constructions or twisted connected sums? I will (partially) answer these questions in Chapters 2 and 3 using gluing techniques. Very optimistically, one could hope that the count of critical points of the G_2 Chern–Simons functional, i.e., the count of G_2 -instantons, would yield a deformation invariant for G_2 -manifolds. In Chapter 4, however, I will show that under certain circumstances G_2 -instantons can be spontaneously born out of associative submanifolds and thus a naïve count of G_2 -instantons will not yield an invariant which is invariant under deformations of the G_2 -structure; hence, the conjectural G_2 Casson invariant cannot be defined as a simple count of the number of G_2 -instantons. (The analogous problem in $\text{Spin}(7)$ -geometry is addressed in Chapter 5.) Nevertheless, there can still be hope that a G_2 Casson invariant can be constructed by counting G_2 -instantons as well as associative submanifolds (and objects in between) with carefully chosen weights. In Chapter 6 I will present a promising proposal for the definition of these weights in the low energy $\text{SU}(2)$ -theory.

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Chapter 1

Introduction to G_2 – and $\text{Spin}(7)$ –geometry

In this preparatory chapter I will review the background material on G_2 – and $\text{Spin}(7)$ –geometry necessary for understanding the main content of this thesis. A thorough study of G_2 – and $\text{Spin}(7)$ –geometry requires a good understanding of a number of underlying algebraic structures. Instead of confronting the reader with this material in bulk I have tried to keep it at a minimum and will introduce the necessary concepts and state the necessary result as needed. For detailed proofs I refer the reader to the notes Dietmar Salamon and I have written [SW10]. My discussion of $\text{Spin}(7)$ –geometry will be rather brief as it only plays a minor rôle in this thesis. The reader who wishes to learn more about this topic is advised to consult [HL82, Sal89, Har90, Joy00], which are also the main sources for the following exposition.

1.1 G_2 –manifolds

Definition 1.1. A 3–form ϕ on a 7–dimensional vector space V is called *non-degenerate* if for each $v \in V \setminus \{0\}$ the induced 2–form $i(v)\phi$ on $V/\langle v \rangle$ is symplectic. The space of non-degenerate 3–forms on V is denoted by $\mathcal{N}(V)$.

Example 1.2. The 3–form ϕ_0 on $\mathbf{R}^7$ defined by

$$(1.3) \quad \phi_0 := e^{123} - e^{145} - e^{167} - e^{246} + e^{257} - e^{347} - e^{356}$$

satisfies

$$(1.4) \quad i(u)\phi_0 \wedge i(v)\phi_0 \wedge \phi_0 = 6 g_0(u, v) \text{vol}_0$$

and, hence, is non-degenerate. Here $e^{i_1 \dots i_k}$ is a shorthand for $e^{i_1} \wedge \dots \wedge e^{i_k}$ and $(e^1, \dots, e^7)$ is the standard basis of $(\mathbf{R}^7)^*$, and g_0 and vol_0 denote the standard inner product and volume

form on $\mathbf{R}^7$, respectively.

Definition 1.5. A G_2 -structure on a 7-dimensional manifold Y is a non-degenerate 3-form $\phi \in \Gamma(\mathcal{N}(TY)) \subset \Omega^3(Y)$. A 7-manifold together with a G_2 -structure is called an *almost G_2 -manifold*.

Theorem 1.6 ([SW10, Theorem 3.2(iii)]). *Let ϕ be a non-degenerate 3-form on a 7-dimensional vector space V . Then there exists a basis of V in which ϕ takes the form ϕ_0 .*

Definition 1.7. G_2 is the subgroup of $\text{GL}(\mathbf{R}^7)$ preserving the 3-form ϕ_0 defined in (1.3).

Proposition 1.8 ([SW10, Theorem 8.1]). *G_2 is a simple, compact, connected and simply connected Lie group of dimension 14.*

By Theorem 1.6 the notion of G_2 -structure agrees with the notion of a G -structure for $G = G_2$, that is, a reduction of the structure group of the frame bundle of Y from $\text{GL}(7)$ to G_2 .

Using (1.4) it is quite easy to see that G_2 is a subgroup of $\text{SO}(7)$; hence, by Theorem 1.6, we can assign to every $\phi \in \mathcal{N}(V)$ an inner product g_ϕ and an orientation on V such that the analogue of (1.4) holds. In particular, every almost G_2 -manifold (Y, ϕ) is canonically equipped with a Riemannian metric g_ϕ , an orientation and thus also a Hodge- $*$ -operator $*_\phi$.

Definition 1.9. Let (Y, ϕ) be an almost G_2 -manifold. The *torsion* of the G_2 -structure ϕ is defined to be

$$\nabla_{g_\phi} \phi.$$

If $\nabla_{g_\phi} \phi = 0$, then ϕ is called *torsion-free* and (Y, ϕ) is called a G_2 -manifold.

This notion of torsion can be seen to agree with the notion of torsion of a G -structure for $G = G_2$. More importantly, however, it is clear from the definition that a G_2 -manifold (Y, ϕ) is essentially the same as a Riemannian manifold with holonomy $\text{Hol}(g_\phi) \subset G_2$.

Example 1.10. $(\mathbf{R}^7, \phi_0)$ is a G_2 -manifold.

Example 1.11. Let $(S, \omega_1, \omega_2, \omega_3)$ be a hyperkähler surface and denote by $(\delta^1, \delta^2, \delta^3)$ a parallel orthonormal frame on $\mathbf{R}^3$. Then

$$(1.12) \quad \phi := \delta^1 \wedge \delta^2 \wedge \delta^3 - \delta^1 \wedge \omega_1 - \delta^2 \wedge \omega_2 - \delta^3 \wedge \omega_3$$

defines a torsion-free G_2 -structure on $\mathbf{R}^3 \times S$. The metric and orientation induced by ϕ agree with the standard metric and orientation on $\mathbf{R}^3 \times S$.

Example 1.13. Let (Z, ω, Ω) be a Calabi-Yau 3-fold. Then

$$(1.14) \quad \phi := dt \wedge \omega + \text{Re } \Omega$$

defines a torsion-free G_2 -structure on $\mathbf{R} \times Z$. The metric and orientation induced by ϕ agree with the standard metric and orientation on $\mathbf{R} \times Z$.

The last example can also be seen as a consequence of the fact that the subgroup of G_2 stabilising a fixed non-trivial vector in $\mathbf{R}^7$ is isomorphic to $SU(3)$, which is also behind many other connections between the geometry of G_2 -manifolds and Calabi–Yau 3-folds.

All of the above examples have $\text{Hol}(g_\phi)$ strictly contained in G_2 .

Example 1.15. In [BS89] Bryant–Salamon found *explicit* torsion-free G_2 -structures on $\Lambda^+ S^4$, $\Lambda^+ \mathbf{P}^2$ and on $\mathbf{R}^4 \times S^3$, the total space of the spin bundle on S^3 , whose associated metrics are complete and have $\text{Hol}(g_\phi) = G_2$.

Examples of compact G_2 -manifolds with $\text{Hol}(g_\phi) = G_2$ are rather complicated to come by and there is to this day no good understanding of when one would expect a compact 7-manifold to admit a torsion-free G_2 -structure at all.

Proposition 1.16 ([SW10, Theorem 8.4]). *Let ϕ be a non-degenerate 3-form on a 7-dimensional vector space V and set $\psi := *_\phi \phi$. Then $\Lambda^* V^*$ splits into irreducible representations of G_2 as follows*

$$\begin{aligned}\Lambda^1 V^* &= \Lambda_7^1, \\ \Lambda^2 V^* &= \Lambda_7^2 \oplus \Lambda_{14}^2, \\ \Lambda^3 V^* &= \Lambda_1^3 \oplus \Lambda_7^3 \oplus \Lambda_{27}^3\end{aligned}$$

and correspondingly for $\Lambda^k V^* \cong \Lambda^{7-k} V^*$ with $k = 4, 5, 6$. Here $\dim \Lambda_d^k = d$ and

$$\begin{aligned}\Lambda_7^2 &:= \{\alpha : *(\alpha \wedge \phi) = 2\alpha\} = \{i(u)\phi : u \in V\} \cong \Lambda_7^1, \\ \Lambda_{14}^2 &:= \{\alpha : *(\alpha \wedge \phi) = -\alpha\} = \{\alpha : \alpha \wedge \psi = 0\} \cong \mathfrak{g}_2, \\ \Lambda_1^3 &:= \langle \phi \rangle, \\ \Lambda_7^3 &:= \{i(u)\psi : u \in V\} \cong \Lambda_7^1 \quad \text{and} \\ \Lambda_{27}^3 &:= \{\alpha : \alpha \wedge \phi = 0 \text{ and } \alpha \wedge \psi = 0\} \cong \text{Sym}_0(V) \cdot \phi.\end{aligned}$$

This induces an analogous splitting of $\Lambda^* T^* Y$ for every almost G_2 -manifold Y . By slight abuse of notation we will denote the corresponding summands by Λ_d^k as well. It is easy to see that the torsion of a G_2 -structure lies in $\Lambda^1 \otimes \Lambda_7^3 \cong \text{End}(TY) \cong \mathbf{R} \oplus \text{Sym}_0(TY) \oplus \Lambda_7^2 \oplus \Lambda_{14}^2$; hence, one can distinguish 16 classes of G_2 -structures with torsion. These classes were studied in detail by Fernández–Gray [FG82]. In particular, they proved the following.

Theorem 1.17 (Fernández–Gray [FG82, Theorem 5.2]). *A G_2 -structure ϕ on a 7-manifold Y is torsion-free if and only if*

$$(1.18) \quad d\phi = 0 \quad \text{and} \quad d\Theta(\phi) = 0.$$

Here $\Theta: \Gamma(\mathcal{N}(TY)) \rightarrow \Omega^4(Y)$ is defined by

$$\Theta(\phi) := *_\phi \phi.$$

Thus finding a torsion-free G_2 -structure on a 7-manifold Y is equivalent to solving the complicated *non-linear* p.d.e. (1.18). To check whether or not the resulting metric has holonomy G_2 one can use the following criterion.

Proposition 1.19 ([Joy00, Proposition 10.2.2]). *A compact G_2 -manifold (Y, ϕ) has holonomy $\text{Hol}(g_\phi) = G_2$ if and only if $\pi_1(Y)$ is finite.*

There are currently, broadly speaking, two methods for constructing G_2 -manifolds. Both of which rely on gluing techniques and produce G_2 -manifolds which are in some sense almost degenerate. The first is Joyce's generalised Kummer construction introduced in [Joy96b] and later extended in [Joy00] which produces G_2 -manifolds that are close to flat G_2 -orbifolds. The second is the twisted connected sum construction pioneered by Kovalev [Kov03] and extended and improved by Kovalev–Lee [KL11] and Corti–Haskins–Nordström–Pacini [CHNP12b]. The work of Corti–Haskins–Nordström–Pacini in particular provides us with an ample supply of examples of G_2 -manifolds. I will describe the geometry of these constructions in Sections 2.1 and 3.1.

Definition 1.20. Let Y be a compact 7-manifold. Denote by $\mathcal{X}$ the space of torsion-free G_2 -structures on Y and by $\text{Diff}_0(Y)$ the group of diffeomorphisms isotopic to the identity. Then the space $\mathcal{M} := \mathcal{X} / \text{Diff}_0(Y)$ is called the *moduli space of torsion-free G_2 -structures on Y* .

Theorem 1.21 (Joyce [Joy96b, Part I, Theorem C]). *$\mathcal{M}$ is a smooth manifold and the map $\pi: \mathcal{M} \rightarrow H^3(Y, \mathbf{R})$ defined by $\phi \cdot \text{Diff}_0(Y) \mapsto [\phi]$ is a local diffeomorphism.*

Theorem 1.22 (Joyce [Joy00, Proposition 10.4.5]). *Let Y be a compact 7-manifold. Then the map $\mathcal{M} \rightarrow H^3(Y, \mathbf{R}) \times H^4(Y, \mathbf{R}) \cong T^*H^3(Y, \mathbf{R})$ defined by*

$$\phi \cdot \text{Diff}_0(Y) \mapsto ([\phi], [*]\phi)$$

is a Lagrangian immersion.

I will now discuss some aspects of the topology of compact G_2 -manifolds. The first observation, which essentially goes back to Chern [Che57], is as follows.

Proposition 1.23. *If (Y, ϕ) is a G_2 -manifold, then the spaces of harmonic k -forms $\mathcal{H}^k$ split into summands $\mathcal{H}_d^k$ defined by*

$$(1.24) \quad \mathcal{H}_d^k := \{\alpha \in \Gamma(\Lambda_d^k) : \Delta\alpha = 0\}.$$

Moreover, $\mathcal{H}_1^k \cong \mathcal{H}^0$ and $\mathcal{H}_7^k \cong \mathcal{H}^1$.

If Y is compact, then this yields an analogous splitting of $H^k(Y, \mathbf{R})$ into summands $H_d^k(Y, \mathbf{R})$ with $H_1^k(Y, \mathbf{R}) \cong H^0(Y, \mathbf{R})$ and $H_7^k(Y, \mathbf{R}) \cong H^1(Y, \mathbf{R})$. In particular, a compact G_2 -manifold must have $b_3 > 0$. Therefore, while one can show that S^7 does admit a G_2 -structure, see Proposition 1.31, it cannot carry a torsion-free G_2 -structure.

An immediate consequence of Propositions 1.16 and 1.23 is the following.

Proposition 1.25. *If (Y, ϕ) is a compact G_2 -manifold, then $q_\phi: S^2 H^2(Y, \mathbf{R}) \rightarrow \mathbf{R}$ defined by*

$$(1.26) \quad q_\phi([\alpha], [\beta]) := \int_Y \alpha \wedge \beta \wedge \phi$$

is positive definite on $H_7^2(Y, \mathbf{R})$ and negative definite on $H_{14}^2(Y, \mathbf{R})$.

Proposition 1.27 (D.A. Salamon [Sal13, Remark 4.1(ii)]). *If M is a 3-manifold with first Betti number $b_1(M) \leq 2$ and S is a K3 surface, then $Y := M \times S$ does not admit a torsion-free G_2 -structure.*

Proof. By Künneth's formula $H^2(S, \mathbf{R})$ is a subspace of $H^2(Y, \mathbf{R})$ and the intersection form q on $H^2(S, \mathbf{R})$ agrees with q_ϕ up to a non-zero constant factor. Since $H_{14}^2(Y, \mathbf{R})$ has codimension $b_7^2 = b^1(Y) = b^1(M) \leq 2$ in $H^2(Y, \mathbf{R})$ and $2 \leq b^+(S) = 3 \leq b^-(S) = 19$ it must intersect $H^+(S, \mathbf{R})$ as well as $H^-(S, \mathbf{R})$ non-trivially. This contradicts Proposition 1.25. $\square$

Proposition 1.28. *If (Y, g) is a compact G_2 -manifold, then*

$$(1.29) \quad \langle p_1(Y) \cup [\phi], [Y] \rangle = -\frac{1}{8\pi^2} \int_Y |R(g_\phi)|^2.$$

In particular, $\langle p_1(M) \cup [\phi], [Y] \rangle < 0$ unless (Y, g) is flat.

Proof. By Chern–Weil theory $p_1(Y) = -\frac{1}{8\pi^2} [\text{tr}(R \wedge R)]$. Since g has holonomy contained in G_2 , $R \in S^2 \mathfrak{g}_2$ and thus, by Proposition 1.16, $*(R \wedge \phi) = -R$. This immediately implies (1.29). $\square$

Proposition 1.30. *Let Y be a compact 7-manifold with $b_1 = 0$. If ϕ is a torsion-free G_2 -structure, then $[\phi]$ lies in the cone*

$$C := \{\alpha \in H^3(Y, \mathbf{R}) : \alpha \wedge \beta \wedge \beta < 0 \text{ for all } \beta \in H^2(Y, \mathbf{R}) \setminus \{0\} \\ \text{and } \langle p_1(Y) \cup \alpha, [Y] \rangle < 0\}.$$

Proposition 1.31 (Gray [Gra69, Theorems 3.1 and 3.2], see also [LM89, p. 321]). *A 7-manifold admits a G_2 -structure if and only if it is spin. The spin bundle is isomorphic to $\Lambda_1^0 \oplus \Lambda_7^1$. In particular, if ϕ is torsion-free, then (Y, g) carries a non-trivial parallel spinor.*

Proposition 1.32. *If (M, g) is a spin manifold which possesses a non-trivial parallel spinor, then g is Ricci flat.*

This result is well-known, but may be folklore; I was unable to locate a reference. It can be proved by a direct computation using the formula for the curvature of the spin connection from [LM89, Theorem 4.15], the fact that this curvature acts trivially on the parallel spinor and some simple manipulations using the algebraic Bianchi identity.

Corollary 1.33. *G_2 -manifolds are Ricci flat.*

An application of the Cheeger–Gromoll splitting theorem now yields Proposition 1.19.

Theorem 1.34 (Baraglia [Bar09, Proposition 6.2.1]). *Let (Y, ϕ) be a compact G_2 -manifold with finite fundamental group. Then Y admits no locally trivial fibration $\pi: Y \rightarrow B$ onto a 3-dimensional base.*

This result was known to Bryant and Harvey in the case that Y is a product, but apparently has never been published. It is worthwhile to recall Baraglia’s proof.

Proof. By passing to universal covers and restricting to single connected components we may as well assume that Y , B and the fibre $F = \pi^{-1}(x)$, for some $x \in B$, are connected and simply-connected.

It follows that B has cohomology in degrees 0 and 3 while F has cohomology in degrees 0, 2 and 4. The second page of the Leray–Serre spectral sequence of π is given by $H^*(B) \otimes H^*(F)$ and the differential on page two is necessarily trivial. A moment’s thought shows that the differential on page three must be trivial as well, because otherwise $H^3(Y)$ would vanish. The spectral sequence collapses after page three and hence the maps $\pi^*: H^3(B) \rightarrow H^3(Y)$ and $i^*: H^2(Y) \rightarrow H^2(F)$ are isomorphisms. Here $i: F \rightarrow Y$ is the inclusion of the typical fibre.

It follows that $[\phi] = c \cdot \pi^*[\text{vol}_B]$ for some non-zero constant c . Since $\pi^*[\text{vol}_B]$ is Poincaré dual to the fibre F , the intersection form q on $H^2(F)$ is a constant multiple of q_ϕ , thus definite and, hence, by Donaldson’s theorem [Don83] isomorphic to $\pm \text{diag}(1, \dots, 1)$.

Since Y is spin and the normal bundle to F is trivial, $w_2(F) = i^*w_2(Y) = 0$ and thus F is spin as well. Wu’s formula

$$q(\alpha, \alpha) = q(\alpha, w_2(F)) = 0 \pmod{2}$$

and $q \cong \pm \text{diag}(1, \dots, 1)$ immediately imply that $H^2(F) = 0$. Now, $p_1(F) = i^*p_1(Y)$ and thus

$$\langle p_1(Y) \cup [\phi], [Y] \rangle = c \int_F p_1(F) = 3c\sigma(F) = 0$$

by Hirzebruch’s signature theorem. This contradicts Proposition 1.28. $\square$

The proof of Theorem 1.34 involves gauge theory (on 4–manifolds) via the use of Donaldson’s theorem. It would be interesting to try to find a way of proving Theorem 1.34 “directly” using gauge theory on G_2 –manifolds.

1.2 Spin(7)–manifolds

Definition 1.35. A 4–form Φ on an 8–dimensional vector space W is called *admissible* if there exists a basis of W in which it is identified with the 4–form Φ_0 on $\mathbf{R}^8$ defined by

$$(1.36) \quad \Phi_0 := e^0 \wedge \phi_0 + \psi_0$$

with ϕ_0 defined as in (1.3) and $\psi_0 := \Theta(\phi_0)$. Here we denote the standard basis of $(\mathbf{R}^8)^*$ by $(e^0, \dots, e^7)$. The space of admissible forms on W is denoted by $\mathcal{A}(W)$.

Remark 1.37. An intrinsic characterisation of admissible forms can be found in [SW10, Theorem 7.4 and Definition 7.5].

Definition 1.38. A Spin(7)–structure on an 8–dimensional manifold X is an admissible 4–form $\Phi \in \Gamma(\mathcal{A}(TX)) \subset \Omega^4(X)$. An 8–manifold together with a Spin(7)–structure is called an *almost Spin(7)–manifold*.

Definition 1.39. Spin(7) is the subgroup of $GL(\mathbf{R}^8)$ preserving the 4–form Φ_0 defined in (1.36).

Proposition 1.40 ([SW10, Theorem 9.1]). Spin(7) is a simple, compact, connected and simply connected Lie group of dimension 21.

Proposition 1.41 ([SW10, Theorem 7.4]). Spin(7) is a subgroup of $SO(8)$.

It follows that each almost Spin(7)–manifold is canonically equipped with a metric g_Φ and an orientation.

Definition 1.42. Let (X, Φ) be an almost Spin(7)–manifold. The *torsion* of the Spin(7)–structure Φ is defined to be

$$\nabla_{g_\Phi} \Phi.$$

If $\nabla_{g_\Phi} \Phi = 0$, then Φ is called *torsion-free* and (X, Φ) is called a Spin(7)–manifold.

Example 1.43. $(\mathbf{R}^8, \Phi_0)$ is a Spin(7)–manifold.

Example 1.44. If $(S, \omega_1, \omega_2, \omega_3)$ and (T, μ_1, μ_2, μ_3) are a pair of hyperkähler surfaces, then $(S \times T, \Phi)$ with

$$(1.45) \quad \Phi := \text{vol}_S - \sum_{i=1}^3 \omega_i \wedge \mu_i + \text{vol}_T$$

is a Spin(7)–manifold.

Example 1.46. If (Y, ϕ) is a G_2 -manifold, then $(\mathbf{R} \times Y, \Phi)$ with

$$(1.47) \quad \Phi := dt \wedge \phi + \psi$$

and $\psi := \Theta(\phi)$ is a $\text{Spin}(7)$ -manifold.

Underneath the last example lies the fact the subgroup of $\text{Spin}(7)$ stabilising a fixed non-trivial vector in $\mathbf{R}^8$ is isomorphic to G_2 . The inclusions $\text{SU}(3) \subset G_2 \subset \text{Spin}(7)$ underlie many relations between Calabi–Yau 3-folds, G_2 -manifolds and $\text{Spin}(7)$ -manifolds, which are analogous to those between Riemann surfaces, 3-manifolds and 4-manifolds.

Examples of compact $\text{Spin}(7)$ -manifolds with $\text{Hol}(g_\Phi) = \text{Spin}(7)$ have been constructed by Joyce, see [Joy96a, Joy99, Joy00].

Proposition 1.48 ([SW10, Theorem 9.5]). *Let Φ be an admissible 4-form on an 8-dimensional vector space W . Then $\Lambda^* W^*$ splits into irreducible representations of $\text{Spin}(7)$ as follows*

$$\begin{aligned} \Lambda^1 W^* &= \Lambda_8^1, \\ \Lambda^2 W^* &= \Lambda_7^2 \oplus \Lambda_{21}^2, \\ \Lambda^3 W^* &= \Lambda_8^3 \oplus \Lambda_{48}^3, \\ \Lambda^4 W^* &= \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4 \oplus \Lambda_{35}^4 \end{aligned}$$

and correspondingly for $\Lambda^k W^* \cong \Lambda^{8-k} W^*$ with $k = 5, 6, 7, 8$. Here $\dim \Lambda_d^k = d$ and

$$\begin{aligned} \Lambda_7^2 &:= \{\alpha : *(\alpha \wedge \Phi) = 3\alpha\} \cong \Lambda_7^4, \\ \Lambda_{21}^2 &:= \{\alpha : *(\alpha \wedge \Phi) = -\alpha\} \cong \mathfrak{spin}(7), \\ \Lambda_8^3 &:= \{i(u)\Phi : u \in W\}, \\ \Lambda_{48}^3 &:= \{\alpha : \alpha \wedge \Phi = 0\}, \\ \Lambda_1^4 &:= \langle \Phi \rangle, \\ \Lambda_7^4 &:= \mathfrak{so}(W) \cdot \Phi, \\ \Lambda_{27}^4 &:= \text{Sym}_0(W) \cdot \Phi \quad \text{and} \\ \Lambda_{35}^4 &:= \{\alpha : *\alpha = -\alpha\}. \end{aligned}$$

Remark 1.49. The action of $\text{Spin}(7)$ on Λ_7^2 gives rise to a double cover $\text{Spin}(7) \rightarrow \text{SO}(7)$; hence, the above definition of $\text{Spin}(7)$ agrees with the usual definition as the universal cover of $\text{SO}(7)$.

Proposition 1.48 induces an analogous splitting of $\Lambda^* T^* X$ for every almost $\text{Spin}(7)$ -manifold. By slight abuse of notation we will denote the corresponding summands by Λ_d^k as

well. We denote the projection onto Λ_d^k by

$$\pi_d: \Lambda^k T^* X \rightarrow \Lambda_d^k.$$

On a compact Spin(7)–manifold the splitting passes on to the harmonic forms and, hence, to cohomology. This gives rise to refined Betti numbers b_d^k .

The following two propositions are easy to check via straight-forward computation.

Proposition 1.50. *If Φ is the admissible 4–form on a product of two quaternionic lines S and T defined as in (1.45), then Λ_7^2 splits as*

$$\Lambda_7^2 = \Lambda_3^2 \oplus \Lambda_4^2,$$

where

$$\Lambda_3^2 = \bigoplus_{i=1}^3 \langle \omega_i - \mu_i \rangle \quad \text{and} \quad \Lambda_4^2 = \left\{ \langle L \cdot, \cdot \rangle \in S^* \otimes T^* : L \in \text{Hom}(S, T) \text{ satisfying } \sum_i J_i L I_i = -3L \right\}.$$

Here I_i and J_i denote the complex structures on S and T corresponding to ω_i and μ_i , respectively.

Proposition 1.51. *If Φ is the admissible 4–form on a product of $\mathbf{R}$ with a 7–dimensional vector space V equipped with a non-degenerate 3–form ϕ defined as in (1.47), then Λ_7^2 can be written as*

$$\Lambda_7^2 = \{ dt \wedge v^* + i(v)\phi : v \in V \}$$

and Λ_{21}^2 can be written as

$$\Lambda_{21}^2 = \{ dt \wedge *_V(\alpha \wedge \psi) - \alpha : \alpha \in \Lambda^2 V^* \}$$

where $\psi := \Theta(\phi)$.

Proposition 1.52 ([Joy00, Proposition 10.5.6]). *If Φ is a Spin(7)–structure on X , then X is spin and has a canonical spin structure with*

$$S^+ = \Lambda^0 \oplus \Lambda_7^2 \quad \text{and} \quad S^- = \Lambda_8^1.$$

Moreover, if Φ is torsion-free, then X admits a non-trivial parallel spinor.

Corollary 1.53. *Spin(7)–manifolds are Ricci flat.*

1.3 Calibrated submanifolds

An important concept in the theory of minimal submanifolds is that of calibrated geometries which was invented by Harvey and Lawson [HL82].

Definition 1.54. Let (M, g) be a Riemannian manifold. A k -form $\alpha \in \Omega^k(M)$ is called a *calibration of degree k* if it is closed and has $\text{comass}(\alpha) \leq 1$, i.e., $d\alpha = 0$ and for each $x \in M$ and each oriented k -plane E in $T_x M$ we have

$$\alpha|_E \leq \text{vol}_E.$$

If (M, g) is a Riemannian manifold with a calibration α of degree k , then a closed d -dimensional submanifold $N \subset M$ is called *calibrated with respect to α* if

$$\alpha|_{T_x N} = \text{vol}_{T_x N}$$

for each $x \in N$.

The importance of these definitions stems from the following simple, but incredibly useful observation.

Lemma 1.55 (Harvey–Lawson [HL82, Theorem 4.2]). *Let (M, g) be a Riemannian manifold with a calibration α and let N be an oriented submanifold of M calibrated with respect to α . Then N is volume minimising within its homology class.*

Proof. If N' is homologous to N , then

$$\text{vol}(N) = \int_N \text{vol}_N = \int_N \alpha = \int_{N'} \alpha \leq \int_{N'} \text{vol}_{N'} = \text{vol}(N'). \quad \square$$

Remark 1.56. The notion of being calibrated also makes sense for rectifiable currents and Lemma 1.55 applies mutatis mutandis.

Example 1.57. If (X, ω) is a Kähler manifold, then the $2k$ -forms $\frac{\omega^k}{k!}$ are calibrations and the corresponding calibrated submanifolds are the holomorphic submanifolds of X .

Example 1.58. If (X, ω, Ω) is a Calabi–Yau n -fold, then $\text{Re } \Omega$ is a calibration and the corresponding calibrated submanifolds are the special Lagrangian submanifolds of X (with phase zero).

1.3.1 Associative and coassociative submanifolds

The relevance of calibrated geometry for G_2 -geometry comes from the following result.

Theorem 1.59 (Harvey–Lawson [HL82, Chapter IV Theorems 1.4 and 1.16]). *If (Y, ϕ) is a G_2 -manifold, then ϕ and $\psi := \Theta(\phi)$ are calibrations.*

Moreover, $P \subset Y$ (resp. $Q \subset Y$) is calibrated by ϕ (resp. ψ) if and only if at each point $p \in P$ (resp. $p \in Q$) there exists a basis $(e_1, \dots, e_7)$ of $T_p Y$ with respect to which ϕ is given by (1.3) and (e_1, e_2, e_3) (resp. $(e_4, \dots, e_7)$) is a positive basis of $T_p P$ (resp. $T_p Q$).

Definition 1.60. Let (Y, ϕ) be a G_2 -manifold. Then ϕ is called the *associative calibration* and $\psi := \Theta(\phi)$ is called the *coassociative calibration*. An oriented 3-dimensional submanifold $P \subset Y$ that is calibrated by ϕ is called an *associative submanifold*. An oriented 4-dimensional submanifold $Q \subset Y$ that is calibrated by ϕ is called a *coassociative submanifold*.

Example 1.61. $\mathbf{R}^3 \times \{0\} \subset \mathbf{R}^3 \oplus \mathbf{R}^4 = \mathbf{R}^7$ is an associative submanifold and $\{0\} \times \mathbf{R}^4 \subset \mathbf{R}^3 \oplus \mathbf{R}^4 = \mathbf{R}^7$ is a coassociative submanifold of the G_2 -manifold $(\mathbf{R}^7, \phi_0)$ from Example 1.10.

Example 1.62. Examples of associative submanifolds arise as 3-dimensional fixed point sets of involutions of Y preserving ϕ and examples of coassociative submanifolds arise as 4-dimensional fixed point sets of involutions of Y mapping ϕ to $-\phi$. For concrete examples we refer the reader to Joyce [Joy96b, Part II, Section 4.2].

Example 1.63. The recent work of Corti–Haskins–Nordström–Pacini [CHNP12b] gives a large number of concrete examples of associative submanifolds in twisted connected sums.

Proposition 1.64. If P is an associative submanifold of a compact G_2 -manifold (Y, ϕ) , then

$$\text{vol}(P) = \int_P \phi = \langle \text{PD}[P] \cup [\phi] \rangle > 0.$$

Proposition 1.65. An orientable 3-dimensional submanifold P of a G_2 -manifold (Y, ϕ) can be oriented in such a way that it becomes associative if and only if at each point $p \in P$ and for each basis (e_1, e_2, e_3) of $T_p P$ the following holds: $\psi(e_1, e_2, e_3, \cdot) = 0$.

Using this fact, one can see that associative submanifolds are critical points of a functional, cf. [DT98, Section 8]: Fix an oriented submanifold P_0 of Y and consider the space of submanifolds $P \subset Y$ together with an equivalence class Q of 4-chains bounding $P - P_0$ with $[Q] \sim [Q']$ if $[Q - Q'] = 0 \in H_4(Y, \mathbf{Z})$. Then the critical points of the functional

$$(1.66) \quad (P, [Q]) \mapsto \int_Q \psi$$

are essentially the associative submanifolds of Y in the homology class of $[P_0]$.

In the following I will discuss some results due to McLean [McL98] concerning the deformation theory of associative and coassociative submanifolds. If P is an associative submanifold, then it follows from Theorem 1.59 that there is a natural identification

$$(1.67) \quad TP \cong \Lambda^+ N^* P$$

given by inserting tangent vectors to P into ϕ (and restricting to NP). Thinking of $\Lambda^+ N^*P$ as a sub-bundle of $\mathfrak{so}(NP)$ yields a Clifford multiplication $\gamma: TP \rightarrow \text{End}(NP)$. Denote by $\bar{\nabla}$ the connection on NP induced by the Levi-Civita connection on Y .

Definition 1.68. The *Fueter operator* $F_P = F_{P,\phi}: \Gamma(NP) \rightarrow \Gamma(NP)$ associated with P is defined by

$$F_{P,\phi}(n) := \sum_i \gamma(e_i) \bar{\nabla}_i n.$$

This is a self-adjoint elliptic operator of first order.

Remark 1.69. The Fueter operator F_P can be identified with a twisted Dirac operator as follows. Pick a spin structure $\mathfrak{s}$ on P . Because of the identification (1.67) there is a unique $\text{SU}(2)$ -bundle $\mathfrak{u}$ over P such that $\mathfrak{s} \times \mathfrak{u}$ is a spin structure on NP . The bundle $\mathfrak{u}$ also comes with a connection, such that the resulting connection on $\mathfrak{s} \times \mathfrak{u}$ is a spin connection. If S and U denote the quaternionic line bundles corresponding to $\mathfrak{s}$ and $\mathfrak{u}$, then $S \otimes_{\mathbb{C}} U$ has a natural real structure and its real part can be identified with NP . With respect to this identification F_P becomes the twisted Dirac operator $D: \Gamma(\text{Re}(S \otimes U)) \rightarrow \Gamma(\text{Re}(S \otimes U))$.

The importance of F_P is that it controls the infinitesimal deformation theory of the associative submanifold P .

Theorem 1.70 (McLean [McL98, Section 5]). *Let (Y, ϕ) be a compact G_2 -manifold and let $P \subset Y$ be an associative submanifold. Then there is an open subset $U \subset \ker F_P$ and a smooth map $\kappa: U \rightarrow \text{coker } F_P$ such that the moduli space of associative submanifolds near P is given by $\kappa^{-1}(0)$.*

Definition 1.71. An associative submanifold P is called *rigid/unobstructed* if F_P is injective/surjective.

Remark 1.72. Since F_P is self-adjoint and thus of index zero, the notions of rigidity and unobstructedness coincide.

If P is unobstructed, then the moduli space of associative submanifolds near P consists only of P ; in particular, it is smooth at P . The following is a simple consequence of the implicit function theorem and McLean's setup for the deformation theory of associative submanifolds in [McL98, Section 5].

Proposition 1.73. *Let Y be a compact 7-manifold and let $(\phi_t)_{t \in (-T, T)}$ be a family of torsion-free G_2 -structures on Y . Suppose that P is an unobstructed associative submanifold in (Y, ϕ_0) . Then there is a constant $T' \in (0, T]$ and a unique family of associative submanifolds $(P_t)_{t \in (-T', T')}$ in (Y, ϕ_t) with $P_0 = P$; moreover, each P_t is unobstructed.*

The deformation theory of coassociative submanifolds is in some sense tamer than that of associative submanifolds, because it is always unobstructed.

Theorem 1.74 (McLean [McL98, Section 4]). *Let (Y, ϕ) be a compact G_2 -manifold and let $Q \subset Y$ be a compact coassociative submanifold. Then the moduli space of coassociative submanifolds near Q is smooth and has dimension $b_2^+(Q)$.*

This theorem leads one to speculate about G_2 -manifolds which admit fibrations by possibly singular coassociative submanifolds and with typical fibre either $K3$ or T^4 . It is my understanding that one should expect that most of the G_2 -manifolds arising from the twisted connected sum construction (see Section 3.1) to have such a fibration. In this direction there is some as of yet unpublished work by Kovalev [Kov09]; however, to this day some key technical issues in his approach have not been resolved. The potential existence of such fibrations is also the basis of some even more speculative ideas about mirror symmetry for G_2 -manifolds along the lines of the SYZ approach to mirror symmetry for Calabi–Yau 3-folds, see, e.g., [Ach98, GYZ03].

1.3.2 Cayley submanifolds

Calibrated geometry also plays an important rôle in $\text{Spin}(7)$ -geometry.

Theorem 1.75 (Harvey–Lawson [HL82, Chapter IV Theorem 1.24]). *If (X, Φ) is a $\text{Spin}(7)$ -manifold, then Φ is a calibration. Moreover, $Q \subset X$ is calibrated by Φ if and only if at each point $p \in Q$ there exists a basis $(e_0, \dots, e_7)$ of $T_p X$ with respect to which Φ is given by (1.36) and $(e_0, \dots, e_3)$ is a positive basis of $T_p X$.*

Definition 1.76. Let (X, Φ) be a $\text{Spin}(7)$ -manifold. Then Φ is called the *Cayley calibration*. An oriented 4-dimensional submanifold $Q \subset X$ that is calibrated by Φ is called a *Cayley submanifold*.

If $Q \subset (X, \Phi)$ is a Cayley submanifold, then it follows from Theorem 1.75 that there is a natural identification

$$(1.77) \quad \Lambda^+ T^* Q \cong \Lambda^+ N^* Q.$$

We define a subbundle $\text{Hom}_\Phi(TQ, NQ) \subset \text{Hom}(TQ, NQ)$ by $L \in \text{Hom}_\Phi(TQ, NQ)$ if and only if

$$\sum_i I_i L I_i = -3L.$$

Here I_i runs through a local orthonormal basis of $\mathfrak{so}(TQ) \cong \mathfrak{so}(NQ)$. Up to multiplication by a normalising constant

$$\gamma L := L - \sum_i I_i L I_i$$

defines a projection of $\text{Hom}(TQ, NQ)$ onto $\text{Hom}_\Phi(TQ, NQ)$.

Definition 1.78. The *Fueter operator* $F_Q: \Gamma(Q, NQ) \rightarrow \Gamma(Q, \text{Hom}_\Phi(TQ, NQ))$ associated with Q is defined by

$$F_Q(n) := \gamma(\bar{\nabla}n).$$

Remark 1.79. If e_0 is a vector in TQ , then one can compose F_Q with evaluation on e_0 to obtain the operator

$$\text{ev}_{e_0} \circ F_Q(n) = \bar{\nabla}_{e_0}n - \sum_i I_i \bar{\nabla}_{e_i}n$$

where $e_i := I_i e_0$. It is therefore appropriate to think of F as a Dirac-type operator.

Remark 1.80. Suppose that Q is spin and $\mathfrak{s}$ is a spin structure on Q . Then the normal bundle NQ is also spin, since X is; moreover, there is a spin structure $\mathfrak{u}$ on NQ such that $S_Q^+ = S_{NQ}^+$ because of (1.77). If we set $U := S_{NQ}^-$, then it can be seen that $\text{Re}(S_Q^+ \otimes U) = NQ$, $\text{Re}(S_Q^- \otimes U) = \text{Hom}_\Phi(TQ, NQ)$ and that F_Q agrees with the twisted Dirac operator $D: \Gamma(\text{Re}(S_Q^+ \otimes U)) \rightarrow \Gamma(\text{Re}(S_Q^- \otimes U))$. For more details we refer the reader to [McL98, Section 6] and [Hay12, Section 3.2].

Theorem 1.81 (McLean [McL98, Section 6]). *Let (X, Φ) be a compact $\text{Spin}(7)$ -manifold and let $Q \subset X$ be a compact Cayley submanifold. Then there is an open subset $\mathcal{O} \subset \ker F_Q$ and a smooth map $\kappa: \mathcal{O} \rightarrow \text{coker } F_Q$ such that the moduli space of Cayley submanifolds near Q is given by $\kappa^{-1}(0)$. The index of F_Q is given by*

$$(1.82) \quad \text{index } F_Q = \frac{\sigma(Q) + \chi(Q)}{2} - [Q] \cdot [Q].$$

Here $\sigma(Q) := b^+(Q) - b^-(Q)$ denotes the signature of Q .

Remark 1.83. The index formula given by Joyce in [Joy00, Equation (10.32)] is incorrect and likely a misprint as it also contradicts his remarks at the bottom of p. 267.

Definition 1.84. A Cayley submanifold Q is called *unobstructed* if F_Q is surjective.

Proof of the index formula. We can assume that Q is spin. Then the index of F_Q agrees with the index of the twisted Dirac operator D_U . By the Atiyah–Singer index theorem

$$\text{index } D_U = \int_Q \hat{A}(Q) \text{ch}_2(U) = -\frac{1}{4} \sigma(Q) - \int_Q c_2(U).$$

This is the formula given by McLean. In order to obtain a more useful expression, we make use of the fact that if E and F are a pair of $\text{SU}(2)$ -bundles over a 4-manifold and $V = \text{Re}(E \otimes F)$, then

$$(1.85) \quad \begin{aligned} e(V) &= c_2(F) - c_2(E) \quad \text{and} \\ p_1(V) &= -2(c_2(E) + c_2(F)). \end{aligned}$$

To see this, note that there must be universal formulas of the form $e(V) = \alpha(c_2(E) - c_2(F))$ and $p_1(V) = \beta(c_2(E) + c_2(F))$, because $e(V)$ changes sign when E and F are interchanged since this changes the orientation on V , and $p_1(V)$ is independent of the order of E and F . The constants can be determined by a simple explicit computation for the spin bundles over $K3$. From these formulae it follows that

$$c_2(U) = -\frac{1}{4}(p_1(NQ) - 2e(NQ)).$$

To compute $p_1(NQ)$, we combine $S_Q^+ = S_{NQ}^+$ and (1.85) to obtain

$$p_1(NQ) + 2e(NQ) = -4c_2(S_{NQ}^+) = -4c_2(S_Q^+) = p_1(Q) + 2e(Q);$$

hence,

$$(1.86) \quad \int_Q p_1(NQ) = 3\sigma(Q) + 2\chi(Q) - 2[Q] \cdot [Q]$$

Therefore,

$$\int_Q c_2(U) = -\frac{3}{4}\sigma(Q) - \frac{1}{2}\chi(Q) + [Q] \cdot [Q],$$

which implies the claimed index formula. $\square$

Proposition 1.87. *Let X be a compact $\text{Spin}(7)$ -manifold. Suppose that Q is a compact Cayley submanifold in X which has self-intersection number zero, is diffeomorphic to a $K3$ surface whose induced metric is sufficiently close to a hyperkähler metric and suppose that the induced connection on NQ is almost flat. Then X is locally fibred by Cayley $K3$ surfaces near Q .*

Proof. Using the fact that Q and hence NQ is spin as well as (1.85) one can show that NQ is trivial. The Fueter operator F_Q thus agrees with the (untwisted) Dirac operator $D: \Gamma(S^+) \rightarrow \Gamma(S^-)$. On a hyperkähler $K3$ surface D is surjective and every non-zero element of $\ker D$ is nowhere vanishing; hence, the same is true on Q because its metric is sufficiently close to a hyperkähler metric. The existence of the local fibration now follows from (the proof of) Theorem 1.81. $\square$

1.4 Gauge theory

Let (Y, ϕ) be a G_2 -manifold, let $\psi := \Theta(\phi)$ and let E be a G -bundle over Y where G is a compact semi-simple Lie group. Denote by $\mathcal{A}(E)$ the space of connections on E and by $\mathfrak{g}_E := \text{Ad}(E)$ the adjoint bundle associated with E . We equip $\mathfrak{g}_E$ with the inner product arising from the negative of the Killing form on the Lie algebra $\mathfrak{g}$ associated with G .

Definition 1.88. A connection $A \in \mathcal{A}(E)$ on E is called a G_2 -instanton if it satisfies

$$(1.89) \quad *(F_A \wedge \phi) = -F_A.$$

Remark 1.90. If the Lie group G has a non-discrete centre $Z(G)$, it is sometimes useful to relax the above definition as follows. The adjoint bundle $\mathfrak{g}_E$ splits into a trivial bundle with fibre $\mathfrak{z}$, the centre of the Lie algebra, and its orthogonal complement $\mathfrak{g}_E^{(0)} = \text{Ad}(E \times_G G/Z(G))$ with respect to the Killing form. A connection $A \in \mathcal{A}(E)$ is then called a G_2 -instanton if the $\mathfrak{g}_E^{(0)}$ -component of F_A , instead of F_A , satisfies (1.89) and the $\mathfrak{z}$ -component of F_A is a $\mathfrak{z}$ -valued harmonic two-form. These G_2 -instantons are essentially equivalent to G_2 -instantons on $E \times_G G/Z(G)$.

The G_2 -instanton equations first appeared in the physics literature [CDFN83] and were later brought to wider attention by Donaldson–Thomas [DT98, Section 3]. Equation (1.89) can be thought of as a 7-dimensional version of the anti-self-duality condition familiar from dimension four. As we will discuss shortly, G_2 -instantons also have a striking similarity with flat connections over 3-manifolds.¹

Example 1.91. Flat connections are G_2 -instantons.

Example 1.92. Let X be a hyperkähler 4-manifold, let E be a G -bundle over X and let A be an *anti-self-dual (ASD) instanton* on E , that is, a connection on E whose curvature F_A is anti-self-dual. Then the pullback of A to the G_2 -manifold $\mathbf{R}^3 \times X$ from Example 1.11 is a G_2 -instanton:

$$*(F_A \wedge \phi) = *(F_A \wedge \delta^1 \wedge \delta^2 \wedge \delta^3) = *_X F_A = -F_A.$$

Here we used that $F_A \wedge \omega_i = 0$ and $*_X$ denotes the Hodge- $*$ -operator on X .

Example 1.93. Let Z be a Calabi–Yau 3-fold, let E be a G -bundle over X and let A be a *Hermitian–Yang–Mills (HYM) connection* on E , that is,

$$\Lambda F_A = 0 \quad \text{and} \quad F_A^{0,2} = 0$$

where Λ denotes the adjoint of $L := \omega \wedge \cdot$ or equivalently

$$F_A \wedge \omega \wedge \omega = 0 \quad \text{and} \quad F_A \wedge \text{Im } \Omega = 0.$$

Let $(Y := \mathbf{R} \times Z, \phi)$ be as in Example 1.13. Then $\psi := \Theta(\phi) = \frac{1}{2}\omega \wedge \omega - dt \wedge \text{Im } \Omega$ and using Proposition 1.16 it is not hard to see that the pullback of A to Y is a G_2 -instanton, cf. Proposition 1.98.

¹It was brought to my attention by Claude LeBrun at the BIRS workshop “Geometric Structures on Manifolds” that this can be seen as an instance of $7 = 3 + 4$.

Remark 1.94. If $\mathcal{E}$ is a holomorphic vector bundle, then a metric on $\mathcal{E}$ is called a *Hermitian–Yang–Mills (HYM) metric* if the associated Chern connection is HYM. If $\det \mathcal{E}$ is non-trivial, it is customary to work with a relaxed definition of HYM connection, cf. Remark 1.90, so that with a HYM metric one can associated a HYM connection (in the strict sense) on a $\mathbf{PU}(n)$ –bundle. By the work of Donaldson [Don85] and Uhlenbeck–Yau [UY86] a holomorphic vector bundle over a compact Kähler manifold admits a HYM metric if and only if it is polystable.

Example 1.95. The Levi–Civita connection on a G_2 –manifold (Y, ϕ) is a G_2 –instanton. To see that, observe that at each point we can think of the Riemannian curvature tensor R as an element of $S^2 \mathfrak{g}_2 \subset \Lambda^2 \otimes \mathfrak{gl}(7)$, since $\text{Hol}(g_\phi) \subset G_2$. It follows from Proposition 1.16 that $*(R \wedge \phi) = -R$.

Example 1.96. I will present construction techniques for G_2 –instantons on generalised Kummer constructions and twisted connected sums in Chapters 2 and 3, respectively. In particular, I will give concrete examples of G_2 –instantons on certain bundles with structure group $\text{SO}(3)$ on certain manifolds obtained from the generalised Kummer construction in Section 2.7.

Since ϕ is closed, it follows from the Bianchi identity that G_2 –instantons are Yang–Mills connections, that is, they satisfy $d_A^* F_A = 0$ and thus are critical points of the Yang–Mills functional $\text{YM}: \mathcal{A} \rightarrow \mathbf{R}$

$$\text{YM}(A) := \int_Y |F_A|^2 \, \text{dvol}.$$

The Yang–Mills functional is the natural energy functional on the space of connections. The following energy identity shows that G_2 –instantons are absolute minima of the Yang–Mills functional.

Proposition 1.97. *If $A \in \mathcal{A}(E)$, then*

$$\text{YM}(A) = \frac{1}{3} \int_Y |F_A + *(F_A \wedge \phi)|^2 \, \text{dvol} - 8\pi^2 \langle p_1(\mathfrak{g}_E) \cup [\phi], [Y] \rangle.$$

Proof. This is a simple consequence of Chern–Weil theory and Proposition 1.16, cf. the proof of (1.29). $\square$

It is sometimes convenient to rewrite (1.89) using the following result.

Proposition 1.98. *Let $A \in \mathcal{A}(E)$ be a connection on E . Then A is G_2 –instanton if and only if A satisfies*

$$(1.99) \quad F_A \wedge \psi = 0.$$

If Y is compact, then A is a G_2 -instanton if and only if there is a $\xi \in \Omega^0(Y, \mathfrak{g}_E)$ such that

$$(1.100) \quad *(F_A \wedge \psi) + d_A \xi = 0.$$

Proof. The first statement immediately follows from Proposition 1.16, and (1.99) implies (1.100) with $\xi = 0$. By the Bianchi identity and since $d\psi = 0$ it follows from (1.100) that $d_A^* d_A \xi = 0$. Hence, by integration by parts,

$$\int_Y |d_A \xi|^2 = \int_Y \langle d_A^* d_A \xi, \xi \rangle = 0.$$

Therefore, $d_A \xi = 0$ and (1.99) holds. $\square$

From Proposition 1.98 it becomes apparent that G_2 -instantons are rather similar to flat connections on 3-manifolds. In particular, if A_0 is a G_2 -instanton on E , then there is a G_2 Chern–Simons functional $CS^\psi : \mathcal{A}(E) \rightarrow \mathbf{R}$ defined by

$$(1.101) \quad CS^\psi(A_0 + a) := - \int_Y \left\langle a \wedge d_{A_0} a + \frac{1}{3} a \wedge [a \wedge a] \right\rangle \wedge \psi - \int_Y \langle a \wedge F_{A_0} \rangle \wedge \psi$$

whose critical points are precisely the G_2 -instantons on E .

Since equation (1.89) is invariant under the action of the group $\mathcal{G}$ of gauge transformations of E , we can consider the *moduli space* of G_2 -instantons on E over (Y, ϕ) :

$$\mathcal{M}(E, \phi) := \{A \in \mathcal{A}(E) : F_A \wedge \psi = 0\} / \mathcal{G}.$$

For the construction of local models of moduli spaces in gauge theory it is customary to work with local slices of the gauge group action. A particularly useful slicing condition is to require that $B \in \mathcal{A}(E)$ be in *Coulomb gauge* with respect to a fixed reference connection $A \in \mathcal{A}(E)$, that is, $d_A^*(B - A) = 0$. (The importance of the Coulomb gauge stems from the foundational work of Uhlenbeck [Uhl82a]. For a careful discussion of how the Coulomb gauge is used in the construction of moduli spaces we refer the reader to [DK90, Section 4.2]). For a fixed connection $A \in \mathcal{A}(E)$ we consider the system of equations

$$(1.102) \quad *(F_{A+a} \wedge \psi) + d_{A+a} \xi = 0 \quad \text{and} \quad d_A^* a = 0$$

for $\xi \in \Omega^0(Y, \mathfrak{g}_E)$ and $a \in \Omega^1(Y, \mathfrak{g}_E)$. This is simply (1.100) for $A+a$ instead of A together with the condition that $A+a$ be in Coulomb gauge with respect to A . The linearisation $L_A = L_{A,\phi} : \Omega^0(Y, \mathfrak{g}_E) \oplus \Omega^1(Y, \mathfrak{g}_E) \rightarrow \Omega^0(Y, \mathfrak{g}_E) \oplus \Omega^1(Y, \mathfrak{g}_E)$ of (1.102) is given by

$$(1.103) \quad L_{A,\phi} := \begin{pmatrix} 0 & d_A^* \\ d_A & *(\psi \wedge d_A) \end{pmatrix}$$

where $\psi := \Theta(\phi)$. This is a self-adjoint elliptic operator of first order. If $A \in \mathcal{A}(E)$ is a G_2 -instanton, then L_A controls the infinitesimal deformation theory of A as a G_2 -instanton.

Proposition 1.104. *Let (Y, ϕ) be a compact G_2 -manifold, let E be a G -bundle over Y and let $A \in \mathcal{A}(E)$ be a G_2 -instanton on E . Then there is an open subset $U \subset \ker L_A$ and a smooth map $\kappa: U \rightarrow \operatorname{coker} L_A$ such that the moduli space of G_2 -instantons near $[A]$ is given by $\kappa^{-1}(0)/\Gamma_A$. Here $\Gamma_A \subset \mathcal{G}(E)$ is the group of gauge transformations fixing A .*

Definition 1.105. If A is a G_2 -instanton, then we denote by

$$\mathcal{H}_A^0 := \ker L_A \cap \Omega^0(Y, \mathfrak{g}_E) \quad \text{and} \quad \mathcal{H}_A^1 := \ker L_A \cap \Omega^1(Y, \mathfrak{g}_E)$$

the space of infinitesimal automorphisms and the space of infinitesimal deformations/obstructions, respectively. A is called *irreducible* if $\mathcal{H}_A^0 = 0$ and *rigid/unobstructed* if $\mathcal{H}_A^1 = 0$. It is called *acyclic* if it is both irreducible and rigid/unobstructed.

Remark 1.106. The deformation complex of the G_2 -instanton equation (1.100) is given by

$$\Omega^0(Y, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(Y, \mathfrak{g}_E) \xrightarrow{\psi \wedge d_A} \Omega^6(Y, \mathfrak{g}_E) \xrightarrow{d_A} \Omega^7(Y, \mathfrak{g}_E).$$

Since this complex is self-adjoint, the notions of deformations and obstructions coincide.

If A is an acyclic G_2 -instanton on E , then the moduli space of G_2 -instantons near $[A]$ consists only of $[A]$; in particular, it is smooth at $[A]$. If A is only unobstructed, then the moduli space of G_2 -instantons near $[A]$ also only consists of $[A]$, but now $[A]$ should be viewed as a orbifold point.

Proposition 1.107. *Let Y be a compact 7-manifold and let $(\phi_t)_{t \in (-T, T)}$ be a family of torsion-free G_2 -structures on Y . Suppose that $A \in \mathcal{A}(E)$ is an unobstructed G_2 -instanton on a G -bundle E over (Y, ϕ_0) . Then there is a constant $T' \in (0, T]$ and a unique family of G_2 -instantons $(A_t)_{t \in (-T', T')}$ on E over (Y, ϕ_t) with $A_0 = A$; moreover, each A_t is unobstructed.*

Let us now move on and consider a $\operatorname{Spin}(7)$ -manifold (X, Φ) together with G -bundle E over X .

Definition 1.108. A connection $A \in \mathcal{A}(E)$ on E is called a $\operatorname{Spin}(7)$ -instanton if it satisfies

$$*(F_A \wedge \Phi) = -F_A$$

or equivalently

$$(1.109) \quad \pi_7(F_A) = 0.$$

This equation originated in the physics literature [CDFN83] and was introduced to a wider mathematical audience by Donaldson–Thomas [DT98, Section 3]. $\text{Spin}(7)$ -instantons were the topic of Lewis’ thesis [Lew98]; in particular, he constructed one non-trivial example on a $\text{SU}(2)$ -bundle over a $\text{Spin}(7)$ -manifold with full holonomy $\text{Spin}(7)$, cf. Section 5.6. Recently, a construction for $\text{Spin}(7)$ -instantons on $\text{Spin}(7)$ -manifolds arising from [Joy99] was given by Tanaka [Tan12].

In the situation of Example 1.46 where $X = \mathbf{R} \times Y$ and with E being the pullback of a bundle over Y one can see, using Proposition 1.51, that a connection $\mathbf{A} = A_t + \xi_t d_t$ on E over $\mathbf{R} \times Y$ satisfies (1.109) if and only if

$$\partial_t A_t - d_{A_t} \xi_t = *_Y(F_{A_t} \wedge \psi).$$

Hence, gauge equivalence classes of $\text{Spin}(7)$ -instantons on $\mathbf{R} \times Y$ are in one-to-one correspondence with negative gradient flow lines of the G_2 Chern–Simons functional CS^ψ on $\mathcal{A}(E)/\mathcal{G}(E)$ defined in (1.101). This is analogous to the situation in dimension three where negative gradient flow lines of the Chern–Simons functional over a 3-manifold M are in one-to-one correspondence with anti-self-dual instantons on $\mathbf{R} \times M$, see [Don02, Section 2.5.3]. It should be pointed out here that in a similar way one can view certain Cayley submanifolds in $\mathbf{R} \times Y$ as negative gradient flow lines of (1.66), see [SW10, Section 12.3].

If $A \in \mathcal{A}(E)$ is a connection on E , we define $L_A: \Omega^1(X, \mathfrak{g}_E) \rightarrow \Omega^0(X, \mathfrak{g}_E) \oplus \Omega_7^2(X, \mathfrak{g}_E)$ by

$$(1.110) \quad L_A(a) := (d_A^* a, \pi_7(d_A a)).$$

This is the linearisation of (1.109) supplemented with the Coulomb gauge condition and also agrees with the dual of the Dirac operator on X twisted by $\mathfrak{g}_E$.

Proposition 1.111. *If A is a $\text{Spin}(7)$ -instanton, then there is an open subset $U \subset \ker L_A$ and a smooth map $\kappa: U \rightarrow \text{coker } L_A$ such that the moduli space of $\text{Spin}(7)$ -instantons near A is given by $\kappa^{-1}(0)/\Gamma_A$. Here $\Gamma_A \subset \mathcal{G}(E)$ is the group of gauge transformations fixing A . The index of L_A is given by*

$$(1.112) \quad \begin{aligned} \text{index } L_A &= \dim \mathfrak{g} \cdot (b^1 - b^0 - b_7^2) \\ &+ \frac{1}{24} \int_X p_1(X) p_1(\mathfrak{g}_E) - \frac{1}{12} \int_X p_1(\mathfrak{g}_E)^2 - 2p_2(\mathfrak{g}_E). \end{aligned}$$

If E is a $\text{SU}(r)$ -bundle, then

$$(1.113) \quad \begin{aligned} \text{index } L_A &= (r^2 - 1)(b^1 - b^0 - b_7^2) \\ &- \frac{r}{12} \int_X c_2(E) p_1(X) - \int_X \left(1 + \frac{r}{6}\right) c_2(E)^2 - \frac{r}{3} c_4(E). \end{aligned}$$

Remark 1.114. The index formula given by Lewis [Lew98, Theorem 3.2] is incorrect. He mistakenly couples the Dirac operator to E instead of $\mathfrak{g}_E$.

Proof of the index formula. The existence of the Kuranishi map κ is standard; hence, we only prove the index formula. Using

$$\begin{aligned} \text{ch}_2(\mathfrak{g}_E \otimes \mathbf{C}) &= -c_2(\mathfrak{g}_E \otimes \mathbf{C}) \quad \text{and} \\ \text{ch}_4(\mathfrak{g}_E \otimes \mathbf{C}) &= \frac{1}{12}(c_2(\mathfrak{g}_E \otimes \mathbf{C})^2 - 2c_4(\mathfrak{g}_E \otimes \mathbf{C})) \end{aligned}$$

the index theorem yields

$$\begin{aligned} \text{index } L_A &= - \int_X \hat{A}(X) \text{ch}(\mathfrak{g}_E \otimes \mathbf{C}) \\ &= - \int_X \left(1 - p_1(X)/24 + \frac{1}{5670}(-4p_2(X) + 7p_1(X)^2) \right) \\ &\quad \cdot \left(\dim \mathfrak{g} + p_1(\mathfrak{g}_E) + \frac{1}{12}(p_1(\mathfrak{g}_E)^2 - 2p_2(\mathfrak{g}_E)) \right) \\ &= \dim \mathfrak{g} \cdot (b^1 - b^0 - b_7^2) \\ &\quad + \frac{1}{24} \int_X p_1(X) p_1(\mathfrak{g}_E) - \frac{1}{12} \int_X p_1(\mathfrak{g}_E)^2 - 2p_2(\mathfrak{g}_E). \end{aligned}$$

If E is a $\text{SU}(r)$ -bundle, then we can use

$$\text{ch}(\mathfrak{g}_E \otimes \mathbf{C}) = \text{ch}(E \otimes E^*) - 1 = r^2 - 1 - 2rc_2 + \frac{6+r}{6}c_2(E)^2 - \frac{r}{3}c_4(E). \quad \square$$

Definition 1.115. If A is a $\text{Spin}(7)$ -instanton, then we denote by

$$\begin{aligned} \mathcal{H}_A^0 &:= \ker L_A^* \cap \Omega^0(Y, \mathfrak{g}_E), \\ \mathcal{H}_A^1 &:= \ker L_A \cap \Omega^1(Y, \mathfrak{g}_E) \quad \text{and} \\ \mathcal{H}_{7,A}^2 &:= \ker L_A^* \cap \Omega_7^2(Y, \mathfrak{g}_E) \end{aligned}$$

the *space of infinitesimal automorphisms*, the *space of infinitesimal deformations* and the *space of infinitesimal obstructions*, respectively. A is called *irreducible* if $\mathcal{H}_A^0 = 0$ and *unobstructed* if $\mathcal{H}_{7,A}^2 = 0$.

If $\mathbf{A}(t, x) = \xi_t(x)dt + A_t(x)$ is a connection over $X = \mathbf{R} \times Y$ as above, then using the identification of $\Lambda_7^2 T^*X$ with T^*Y given by Proposition 1.51 the linearisation $L_{\mathbf{A}}$ can be written as

$$(1.116) \quad L_{\mathbf{A}} = \partial_t - L_{A_t, \xi_t}$$

where

$$L_{A,\xi} := \begin{pmatrix} 0 & d_A^* \\ d_A & *(\psi \wedge d_A) - [\xi, \cdot] \end{pmatrix}$$

which agrees with (1.103) if $\xi = 0$.

1.5 Enumerative invariants of G_2 -manifolds?

It was pointed out by Donaldson–Thomas in their influential paper [DT98] that counting associative submanifolds and/or G_2 -instantons could lead to interesting enumerative invariants for G_2 -manifolds analogous to the Casson invariant for 3-manifolds. Some formal credence is lent to this idea by the fact that both associative submanifolds and G_2 -instantons are critical points of a functional. Very roughly speaking one could try to define a number $n(\phi)$ by

$$(1.117) \quad n(\phi) := \# \mathcal{M}(\phi)$$

where $\mathcal{M}(\phi)$ is a moduli space of associative submanifolds in or G_2 -instantons over (Y, ϕ) and “ $\#$ ” is some operation that “counts” the moduli space (possibly with signs). There are many important technical issues with properly defining “ $\#$ ”. The central problem with trying to define an invariant via (1.117), however, has to do with the (potential) non-compactness of moduli spaces of associative submanifolds and G_2 -instantons. First of all, we need $\mathcal{M}(\phi)$ to be compact (and thus essentially finite) in order for $n(\phi)$ to be defineable and finite. Secondly, if we want $n(\phi)$ to be invariant under deformations of ϕ then we need the parameterised moduli spaces

$$\mathcal{M}(\{\phi_t : t \in [0, 1]\}) = \bigcup_{t \in [0, 1]} \mathcal{M}(\phi_t)$$

to be compact cobordisms from $\mathcal{M}(\phi_0)$ to $\mathcal{M}(\phi_1)$. There is reasonable evidence, that generically the compactness of $\mathcal{M}(\phi)$ will not be an issue. The parameterised moduli spaces, however, will be non-compact. For associative submanifolds a compactification can be constructed within the framework of geometric measure theory and for G_2 -instantons one can use the work of Uhlenbeck, Price, Nakajima and Tian, which I will describe at the beginning of Chapter 4. However, just having some compactification is far from sufficient. Instead one needs a rather detailed understanding of the degenerations that will occur generically. A typical degeneration of a family of associative submanifolds was described by Nordström in [Nor13] and I will recall some parts of his work in Section 6.3. In Chapter 4 I will describe a typical degeneration of a family of G_2 -instantons. As was pointed out by Joyce in [Joy12b], the degeneration behaviour of associative submanifolds described by Nordström together with the problem of finding consistent orientations for the moduli space of associative submanifolds essentially rules out the naïve way of constructing an

invariant as described above. Moreover, it follows from my work in Chapter 4, that just counting G_2 -instantons cannot yield an invariant in general. From the point of view of gauge theory it makes sense to try to construct a *counter term* to the naïve count of G_2 -instantons which takes into account contributions from associative submanifolds. For the low energy $SU(2)$ theory I will present a proposal for the construction of such a counter term in Chapter 6. Thus, while one cannot expect to obtain an invariant by counting only either associative submanifolds or G_2 -instantons, there may be some hope that one can construct a G_2 *Casson invariant* by considering contributions from *both* G_2 -instantons and associative submanifolds.

Chapter 2

G_2 –instantons on generalised Kummer constructions

In this chapter I will describe a method to construct G_2 –instantons on G_2 –manifolds arising from Joyce’s generalised Kummer construction [Joy96b]. To set up the framework for the construction, let me briefly review the geometry of Joyce’s construction: Equip T^7 with a flat G_2 –structure ϕ_0 and let Γ be a finite group of diffeomorphisms of T^7 preserving ϕ_0 . Then $Y_0 := T^7/\Gamma$ is a flat G_2 –orbifold. The singular set S of Y_0 can, in general, be quite complicated. In this chapter we restrict to *admissible* G_2 –orbifolds Y_0 . That is, we assume that each of the connected components S_j of S has a neighbourhood modelled on $(T^3 \times \mathbb{C}^2/G_j)/H_j$. Here G_j is a non-trivial finite subgroup of $SU(2)$ and H_j is a finite group acting by isometries on T^3 as well as on $\mathbb{C}^2/G_j$; moreover, there is a group extension $\tilde{H}_j$ of H_j by $\mathbb{Z}^3$ together with a free affine action of $\tilde{H}_j$ on $\mathbb{R}^3$ which is compatible with the action of $\mathbb{Z}^3$ on $\mathbb{R}^3$ and induces the action of H_j on T^3 . Suppose we are given *resolution data* $\mathbf{r} = \{(X_j, \rho_j)\}$ for Y_0 , that is, for each j , an ALE space X_j asymptotic to $\mathbb{C}^2/G_j$ together with an isometric action ρ_j of H_j on X_j which is asymptotic to the action of H_j on $\mathbb{C}^2/G_j$. Then using Joyce’s generalised Kummer construction [Joy96b] we can resolve the singularities in Y_0 and produce a compact 7–manifold Y together with a family of torsion-free G_2 –structures $(\phi_t)_{t \in (0, T)}$.

In order to construct G_2 –instantons over (Y, ϕ_t) we will require a set of *gluing data* $\mathbf{g}$ compatible with the resolution data $\mathbf{r}$ for Y_0 . The notion of gluing data will be defined carefully in Section 2.3. For now, it suffices to say that $\mathbf{g}$ consists of

- a G –bundle E_0 over Y_0 together with a flat connection θ and
- for each j , a G –bundle E_j over X_j together with a framed ASD instanton A_j

as well as various auxiliary data satisfying a number of compatibility conditions. Here we take G to be a compact connected semi-simple Lie group, e.g., $G = SO(3)$.

Theorem 2.1. *Let Y_0 be an admissible flat G_2 -orbifold, let $\mathbf{r}$ be resolution data for Y_0 and let $\mathbf{g}$ be compatible gluing data. Suppose that the flat connection θ is unobstructed (as a G_2 -instanton) and that the ASD instantons A_j are infinitesimally rigid, see Definition 2.50. Then there is a constant $T' \in (0, T]$ and a G -bundle E over Y as well as for each $t \in (0, T')$ a connection A_t on E that is an unobstructed G_2 -instanton over (Y, ϕ_t) . If θ is irreducible, then so are the connections A_t . The adjoint bundle $\mathfrak{g}_E$ associated with E satisfies*

$$(2.2) \quad p_1(\mathfrak{g}_E) = - \sum_j k_j \text{PD}[S_j] \quad \text{with} \quad k_j := \frac{1}{8\pi^2} \int_{X_j} |F_{A_j}|^2$$

and

$$(2.3) \quad \langle w_2(\mathfrak{g}_E), [\Sigma] \rangle = \langle w_2(\mathfrak{g}_{E_j}), [\Sigma] \rangle$$

for each $[\Sigma] \in H_2(X_j)^{H_j} \subset H_2(Y)$. Here $[S_j] \in H_3(Y, \mathbf{Q})$ is the rational homology class arising from S_j and $H_2(X_j)^{H_j}$ denotes the H_j -invariant part of $H_2(X_j)$, see Remark 2.15.

The proof of Theorem 2.1 is based on a gluing construction. The analysis involved is similar to work on $\text{Spin}(7)$ -instantons in Lewis' PhD thesis [Lew98], unpublished work of Brendle on the Yang–Mills equation in higher dimension [Bre03b] and Pacard–Ritoré's work on the Allen–Cahn equation [PR03]. From a geometric perspective my result can be viewed as a higher dimensional analogue of Kronheimer's work on ASD instantons on Kummer surfaces [Kro91].

It is not unreasonable to expect that under certain topological assumptions all G_2 -instantons on G_2 -manifolds arising from Joyce's generalised Kummer construction close to the degenerate limit come from a suitable generalisation of my construction. Optimistically, one could hope that this will some day make the (so far conjectural) G_2 Casson invariant accessible to computation. As a first step in this direction I will prove the following result.

Theorem 2.4. *Let Y_0 be an admissible G_2 -orbifold all of whose singularities S_j are modelled on $(T^3 \times \mathbf{C}^2/\mathbf{Z}_2)/H_j$ and let Y be the compact 7-manifold and let $(\phi_t)_{t \in (0, T)}$ be the family of torsion-free G_2 -structures obtained via Joyce's generalised Kummer construction from resolution data for Y_0 . Suppose that E is a $\text{SO}(3)$ -bundle over Y with the property that*

$$p_1(E) = - \sum_j \frac{\varepsilon_j}{2} \text{PD}[S_j]$$

where $\varepsilon_j = 1$ if $H_2(X_j)^{H_j} \subset H_2(Y)$ is non-trivial and $w_2(E)$ pairs non-trivially with $H_2(X_j)^{H_j}$ and $\varepsilon_j = 0$ otherwise. Suppose that $(A_t)_{t \in (0, T)}$ is a family of connections on E such that A_t is a G_2 -instanton over (Y, ϕ_t) . Then there exists a $\text{SO}(3)$ -bundle E_0 over Y_0 together with a flat connection θ , such that away from the singular set of Y_0 the family of connections (A_t) converges up to gauge transformations to θ in C_{loc}^∞ on $Y_0 \setminus S$ as t tends to zero.

In order to prove that Theorem 2.1 gives a complete description of the moduli space of G_2 -instantons under the hypothesis of Theorem 2.4 one additionally needs to control the behaviour of the family (A_t) on the resolution locus in Y_t and on the neck region between the resolution locus and the regular part of Y_0 . This appears to be technically very challenging. Moreover, one would need to either understand under which conditions the flat connection θ will be unobstructed or extend Theorem 2.1 to handle the case of obstructed θ as well. Instead of directly producing G_2 -instantons this latter generalisation of Theorem 2.1 would give a local description of the moduli space of G_2 -instantons in terms of a Kuranishi model.

In this chapter I have only considered the simplest case of Joyce's generalised Kummer construction. Joyce's construction in [Joy96b] can also handle G_2 -orbifolds with singular sets of codimension 6 which are resolved using ALE Calabi–Yau 3-folds. Using the work of Anda Degeratu and myself on rigid HYM connections over ALE Calabi–Yau 3-folds arising as moduli spaces of G -constellations on $\mathbb{C}^3$ [DW12] much of the work in this chapter can be extended to this situation. The latest version of Joyce's construction [Joy00] can handle very complicated singular sets which require QALE Calabi–Yau 3-folds in order to be resolved. Generalising the work in this chapter to this setting is a daunting task.

2.1 Joyce's generalised Kummer construction

I will now describe in more detail (a special case of) Joyce's construction: Equip $T^7 = \mathbb{R}^7/\mathbb{Z}^7$ with a flat G_2 -structure ϕ_0 , as in Example 1.10, and let Γ be a finite group of diffeomorphisms of T^7 preserving ϕ_0 . Then $Y_0 := T^7/\Gamma$ is a flat G_2 -orbifold. Denote by S the singular set of Y_0 and denote by $S_1, \dots, S_k$ its connected components.

Definition 2.5. Y_0 is called *admissible* if each S_j has a neighbourhood isometric to a neighbourhood of the singular set of $(T^3 \times \mathbb{C}^2/G_j)/H_j$. Here G_j is a non-trivial finite subgroup of $SU(2)$ and H_j is a finite group acting by isometries on T^3 as well as on $\mathbb{C}^2/G_j$; moreover, there is a group extension $\tilde{H}_j$ of H_j by $\mathbb{Z}^3$ together with a free affine action of $\tilde{H}_j$ on $\mathbb{R}^3$ which is compatible with the action of $\mathbb{Z}^3$ on $\mathbb{R}^3$ and induces the action of H_j on T^3 .

Let Y_0 be an admissible flat G_2 -orbifold. Then there is a constant $\zeta > 0$ such that if we denote by T the set of points at distance less than ζ to S , then T decomposes into connected components $T_1, \dots, T_k$ such that T_j contains S_j and is isometric to $(T^3 \times B_\zeta^4/G_j)/H_j$. On T_j we can write

$$\phi_0 = \delta^1 \wedge \delta^2 \wedge \delta^3 - \delta^1 \wedge \omega_1 - \delta^2 \wedge \omega_2 - \delta^3 \wedge \omega_3$$

where $(\delta^1, \delta^2, \delta^3)$ is an orthonormal triple of constant 1-forms on T^3 and $(\omega_1, \omega_2, \omega_3)$ is the triple of Kähler forms associated with the standard hyperkähler structure (g, I_1, I_2, I_3) on $\mathbb{C}^2 \cong \mathbb{H}$.

Definition 2.6. Let G be a finite subgroup of $SU(2)$. Then an *ALE space asymptotic to $\mathbb{C}^2/G$* is a hyperkähler 4-manifold $(X, \hat{g}, \hat{I}_1, \hat{I}_2, \hat{I}_3)$ together with a continuous map $\pi: X \rightarrow \mathbb{C}^2/G$ inducing a diffeomorphism from $X \setminus \pi^{-1}(0)$ to $(\mathbb{C}^2 \setminus \{0\})/G$ such that

$$(2.7) \quad \nabla^k(\pi_*\hat{g} - g) = O(r^{-4-k}) \quad \text{and} \quad \nabla^k(\pi_*\hat{I}_i - I_i) = O(r^{-4-k})$$

as $r \rightarrow \infty$ for $i = 1, 2, 3$ and $k \geq 0$. Here $r: \mathbb{C}^2/G_j \rightarrow [0, \infty)$ denotes the radius function.

We will remove the singularity in Y_0 along S_j by, roughly speaking, replacing each $\mathbb{C}^2/G_j$ with an ALE space asymptotic to $\mathbb{C}^2/G_j$. Due to work of Kronheimer [Kro89a, Kro89b] ALE spaces are very well understood.

Theorem 2.8 (Kronheimer [Kro89b, Theorems 1.1, 1.2 and 1.3]). *Let G be a non-trivial finite subgroup of $SU(2)$. Denote by X the real 4-manifold underlying the crepant resolution $\widetilde{\mathbb{C}^2/G}$. Then for each triple of cohomology classes $\alpha_1, \alpha_2, \alpha_3 \in H^2(X, \mathbb{R})$ satisfying*

$$(2.9) \quad (\alpha_1(\Sigma), \alpha_2(\Sigma), \alpha_3(\Sigma)) \neq 0 \in \mathbb{R}^3$$

for each $\Sigma \in H_2(X, \mathbb{Z})$ with $\Sigma \cdot \Sigma = -2$ there is a unique ALE hyperkähler structure on X for which the cohomology classes of the Kähler forms $[\omega_i]$ are given by α_i . Moreover, each ALE space asymptotic to $\mathbb{C}^2/G$ is diffeomorphic to $\widetilde{\mathbb{C}^2/G}$ and its associated triple of Kähler classes satisfies (2.9).

Remark 2.10. The crepant resolution $\widetilde{\mathbb{C}^2/G}$ can be obtained from $\mathbb{C}^2/G$ by a sequence of blow-ups. The exceptional divisor E of $X = \widetilde{\mathbb{C}^2/G}$ has irreducible components $\Sigma_1, \dots, \Sigma_k$. By the McKay correspondence [McK80], these components form a basis of $H_2(X, \mathbb{Z})$ and the matrix with coefficients $C_{ij} = -[\Sigma_i] \cdot [\Sigma_j]$ is the Cartan matrix associated with the Dynkin diagram corresponding to G in the ADE classification of finite subgroups of $SU(2)$.

Definition 2.11. A collection $\mathbf{r} = \{(X_j, \rho_j)\}$ consisting of, for each j , an ALE space X_j asymptotic to $\mathbb{C}^2/G_j$ together with an isometric action ρ_j of H_j on X_j which is asymptotic to the action of H_j on $\mathbb{C}^2/G_j$ is called *resolution data for Y_0* .

Suppose we are given resolution data $\mathbf{r} = \{(X_j, \rho_j)\}$. Denote by $\pi_j: X_j \rightarrow \mathbb{C}^2/G_j$ the resolution map for X_j . For $t > 0$ define

$$(2.12) \quad \pi_{j,t} := t\pi_j: X_j \rightarrow \mathbb{C}^2/G_j$$

and set

$$(2.13) \quad \tilde{T}_{j,t} := \left(T^3 \times \pi_{j,t}^{-1}(B_\zeta^4/G_j)\right)/H_j \quad \text{and} \quad \tilde{T}_t := \bigcup_j \tilde{T}_{j,t}.$$

Using $\pi_{j,t}$ we can replace each T_j in Y_0 by $\tilde{T}_{j,t}$ and thus obtain a compact 7-manifold Y_t .

Remark 2.14. The diffeomorphism type of Y_t is independent of $t > 0$. Hence, we will sometimes drop the label t and pretend to be working with a fixed 7-manifold Y . However, at various points it will be important to remember the precise way in which Y_t was constructed.

Remark 2.15. The rational (co)homology groups and the fundamental group of Y can relatively easily be computed from the above construction, the later being especially important in view of Proposition 1.19. In particular, it can be seen that every $[\Sigma] \in H_2(X_j, \mathbf{Z})$ invariant under the action of H_j yields a cohomology class $[\Sigma] \in H_2(Y, \mathbf{Z})$. Also each component S_j of the singular set gives rise to a *rational* homology class

$$(2.16) \quad [S_j] := \frac{1}{|H_j|} (\iota_{j,t})_* (T^3 \times \{x\}) \in H_3(Y, \mathbf{Q})$$

where $\iota_{j,t}: T^3 \times \pi_{j,t}^{-1}(B_\zeta^4/G_j) \rightarrow Y$ denotes the projection to $\tilde{T}_{j,t}$ followed by the inclusion into Y and x denotes a point in $\pi_{j,t}^{-1}(B_\zeta^4/G_j)$.

On $\tilde{T}_{j,t}$ there is a torsion-free G_2 -structure given by

$$\hat{\phi}_{j,t} := \delta^1 \wedge \delta^2 \wedge \delta^3 - t^2 \delta^1 \wedge \hat{\omega}_{j,1} - t^2 \delta^2 \wedge \hat{\omega}_{j,2} - t^2 \delta^3 \wedge \hat{\omega}_{j,3}.$$

Near the boundary of $\tilde{T}_{j,t}$ the 3-forms $\hat{\phi}_{j,t}$ and ϕ_0 are close to each other. In order to patch them together, note that there are 1-forms $\varrho_{j,t,i}$ on $(\mathbf{C}^2 \setminus \{0\})/G_j$ such that

$$t^2 (\pi_{j,t})_* \hat{\omega}_{j,i} = \omega_i + d\varrho_{j,t,i}$$

with $\nabla^k \varrho_{j,t,i} = t^4 O(r^{-3-k})$ for $k \geq 0$, see [Joy00, Theorem 8.2.3]. Now, fix a smooth non-decreasing function $\chi: [0, \zeta] \rightarrow [0, 1]$ such that $\chi(s) = 0$ for $s \leq \zeta/4$ and $\chi(s) = 1$ for $s \geq \zeta/2$ and set

$$\tilde{\omega}_{j,t,i} := t^2 \hat{\omega}_{j,i} - d(\chi(|\pi_{j,t}|) \cdot \pi_{j,t}^* \varrho_{j,t,i}).$$

Then $(\pi_{j,t})_* \tilde{\omega}_{j,t,i}$ and ω_i agree on $r^{-1}[\zeta/2, \infty)$ and we can define a 3-form $\tilde{\phi}_t \in \Omega^3(Y_t)$ by $\tilde{\phi}_t := \phi_0$ on $Y_0 \setminus T_t = Y_t \setminus \tilde{T}_t$ and by

$$\tilde{\phi}_t := \delta^1 \wedge \delta^2 \wedge \delta^3 - \delta^1 \wedge \tilde{\omega}_{j,t,1} - \delta^2 \wedge \tilde{\omega}_{j,t,2} - \delta^3 \wedge \tilde{\omega}_{j,t,3}$$

on $\tilde{T}_{j,t}$. Define the function $r_t: Y_t \rightarrow [0, \zeta]$ by

$$(2.17) \quad r_t(p) := \begin{cases} |\pi_{j,t}(y)| & \text{for } p = [(x, y)] \in \tilde{T}_{j,t} \\ \zeta & \text{for } p \in Y_t \setminus \tilde{T}_t \end{cases}$$

and set

$$(2.18) \quad R_{j,t} := \tilde{T}_{j,t} \cap r_t^{-1}[\zeta/4, \zeta/2] \quad \text{and} \quad R_t := \bigcup_j R_{j,t} = r_t^{-1}[\zeta/4, \zeta/2].$$

Outside R_t the 3-form $\tilde{\phi}_t$ defines a torsion-free G_2 -structure, while on $R_{j,t}$ it satisfies $\nabla^k(\tilde{\phi}_t - \hat{\phi}_{j,t}) = O(t^4)$ for $k \geq 0$ and similarly, for each fixed $\varepsilon > 0$, on $r_t^{-1}[\varepsilon, \zeta]$ we have $\nabla^k(\tilde{\phi}_t - \phi_0) = O(t^4)$ for $k \geq 0$. In particular, $\tilde{\phi}_t$ defines a G_2 -structure on Y_t provided $t > 0$ is sufficiently small.

We equip Y_t with the Riemannian metric $\tilde{g}_t := g_{\tilde{\phi}_t}$ associated with $\tilde{\phi}_t$.

Remark 2.19. Note that on the complement of $\tilde{T}_t$ the metric $\tilde{g}_t$ agrees with the flat metric g_0 on $(T^7/\Gamma) \setminus T$ and on $\tilde{T}_{j,t} \setminus R_{j,t}$ it agrees with the metric $g_{\hat{\phi}_{j,t}} = g_{\mathbf{R}^3} \oplus t^2 g_{X_j}$. Here $g_{\mathbf{R}^3}$ denotes the standard metric on $\mathbf{R}^3$ and g_{X_j} denotes the metric on X_j . Moreover, since the map $\phi \mapsto g_\phi$ is smooth, on $R_{j,t}$ we have $\nabla^k(\tilde{g}_t - g_{\mathbf{R}^3} \oplus t^2 g_{X_j}) = O(t^4)$ for $k \geq 0$ and, for each fixed $\varepsilon > 0$, on $r_t^{-1}[\varepsilon, \zeta]$ we have $\nabla^k(\tilde{g}_t - g_0) = O(t^4)$ for $k \geq 0$.

Theorem 2.20 (Joyce [Joy96b, Part I Theorem A and Theorem B; Part II Theorem 2.2.1]). *There are constants $T, c > 0$ and for each $t \in (0, T)$ a 2-form η_t on Y_t such that $\phi_t := \tilde{\phi}_t + d\eta_t$ defines a torsion-free G_2 -structure and*

$$(2.21) \quad \|d\eta_t\|_{L^\infty} \leq ct^{1/2}.$$

Remark 2.22. In view of Theorem 1.17 the above is tantamount to saying that one can solve the non-linear partial differential equation

$$(2.23) \quad d\Theta(\tilde{\phi}_t + d\eta_t) = 0$$

with estimates on $d\eta_t$. For small η_t , the dominant part of this equation is essentially the Laplacian on 2-forms. Now, as $t > 0$ decreases the size of $d\Theta(\tilde{\phi}_t)$ becomes smaller and smaller, but at the same time the mapping properties of the Laplacian degenerate. Solving (2.23) thus is a rather delicate balancing act.

For our application we need to slightly strengthen the estimate in Theorem 2.20. Let

$$w_t(x, y) := t + \min\{r_t(x), r_t(y)\}.$$

For $t \in (0, T)$ and a Hölder exponent $\alpha \in (0, 1)$ define

$$[f]_{C_{0,t}^{0,\alpha}(U)} := \sup_{d(x,y) \leq w_t(x,y)} w_t(x,y)^\alpha \frac{|f(x) - f(y)|}{d(x,y)^\alpha} \quad \text{and} \\ \|f\|_{C_{0,t}^{0,\alpha}(U)} := \|f\|_{L^\infty(U)} + [f]_{C_{0,t}^{0,\alpha}(U)}$$

for a tensor field f over $U \subset Y_t$. Here we use parallel transport to compare the values of f at various points of U . If U is unspecified, then we take $U = Y_t$. Sobolev norms with similar weights were previously used, e.g., in the work of Pacini [Pac12].

Proposition 2.24. *The constants $T, c > 0$ in Theorem 2.20 can be chosen such that for all*

$t \in (0, T)$ we have

$$\|d\eta_t\|_{C_{0,t}^{0,\alpha}} \leq ct^{1/2} \quad \text{and} \quad \|\Theta(\phi_t) - \Theta(\hat{\phi}_{j,t})\|_{C_{0,t}^{0,\alpha}(\tilde{T}_{j,t})} \leq ct^{1/2}.$$

For the proof of this result it will be helpful to note the following.

Proposition 2.25. *For each $\mu > 0$ and $K \in \mathbf{N}_0$ there exists a constant $\varepsilon > 0$ such that the following holds for all $t \in (0, T)$ and $p \in Y_t$: $R := \varepsilon(t + r_t(p))$ is less than the injectivity radius of $(Y_t, \tilde{g}_t)$ at p and if we identify $T_p Y$ isometrically with $\mathbf{R}^7$ and denote by $s_R: B_1 \rightarrow B_R(p)$ the map obtained by multiplication with R followed by the exponential map, then*

$$(2.26) \quad \left| \partial^k (R^{-2} s_R^* \tilde{g}_t - g_{\mathbf{R}^7}) \right| \leq \mu$$

for all $k \in \{0, \dots, K\}$. Here $g_{\mathbf{R}^7}$ denotes the standard metric on $\mathbf{R}^7$.

Proof. From Remark 2.19 it is clear that we can find $\varepsilon > 0$ such that the above statement holds for all $p \in r_t^{-1}[\zeta/8, \zeta]$. Moreover, for $p \in r_t^{-1}[0, \zeta/8]$ inequality (2.26) is equivalent to

$$\left| \partial^k \left(\tilde{R}^{-2} s_{\tilde{R}}^* (g_{\mathbf{R}^3} \oplus g_{X_j}) - g_{\mathbf{R}^7} \right) \right| \leq \mu$$

where $\tilde{R} := \varepsilon(1 + |\pi_j(y)|)$ and $p = [(x, y)]$. Because of (2.7) this holds for all $\varepsilon \leq \frac{1}{2}$ as long as $|\pi_j(y)|$ is sufficiently large, say, $|\pi_j(y)| > N$. For $|\pi_j(y)| \leq N$ it can be arranged to hold by choosing $\varepsilon > 0$ sufficiently small. $\square$

Proof of Proposition 2.24. Note that the second part follows from the first and the construction of $\tilde{\phi}_t$, because Θ is a smooth map. To obtain the estimate on $d\eta_t$ recall from Joyce's construction that η_t solves a non-linear partial differential equation that can be written schematically as

$$(2.27) \quad d^* d\eta_t + P(d\eta_t, \nabla d\eta_t) = G(d\eta_t, \dots) \quad \text{and} \quad d^* \eta_t = 0,$$

see [Joy96b, Part I Equation (33)]. The crucial points are that $P(x, y)$ is a smooth function which depends linearly on y and satisfies $P(0, y) = 0$ and that there is a constant $c > 0$ such that

$$(2.28) \quad \|G(d\eta_t, \dots)\|_{L^\infty} \leq ct^{1/2}.$$

Now, define

$$D_t \sigma := (d^* \sigma + P(d\eta_t, \nabla \sigma), d\sigma).$$

Since $d\eta_t$ is small provided $T > 0$ is small, this is a small perturbation of the operator $d^* \oplus d$. We extend D_t to an operator from $\Omega^*(Y_t)$ to itself by defining $D_t \sigma = (d^* \oplus d)\sigma$ for

$\sigma \in \Omega^k(Y_t)$ with $k \neq 3$, so that it becomes an elliptic operator. We will now prove that there are constants $c > 0$ and $\varepsilon \in (0, \frac{1}{2})$ such that for all $t \in (0, T)$ and each $p \in Y_t$ the following holds

$$(2.29) \quad R^\alpha[\sigma]_{C^{0,\alpha}(B_{R/2}(p))} \leq c(R\|D_t\sigma\|_{L^\infty(B_R(p))} + \|\sigma\|_{L^\infty(B_R(p))})$$

with $R := \varepsilon(t + r_t(p))$. From this the asserted bound on $[\mathrm{d}\eta_t]_{C^{0,\alpha}_{0,t}}$ follows at once using (2.21), (2.27) and (2.28), since on $B_{R/2}(p)$ we have $w_t \leq 2\varepsilon^{-1}R$.

For $\mu > 0$ choose $\varepsilon > 0$ according to Proposition 2.25 with $K = 1$. Let $s_R: B_1^7 \rightarrow B_R(p)$ be as in Proposition 2.25. We define a rescaled operator $\tilde{D}_{t,p}: \Omega^*(B_1) \rightarrow \Omega^*(B_1)$ by

$$\tilde{D}_{t,p}\sigma := (R^2 s_R^* \tau, s_R^* \theta)$$

for $\sigma \in \Omega^k(B_1)$ where $(\tau, \theta) := D_t(s_R^{-1})^* \sigma \in \Omega^{k-1}(B_1) \oplus \Omega^{k+1}(B_1)$. It follows from Theorem 2.20 and Proposition 2.25 that by choosing $T, \mu > 0$ sufficiently small, we can arrange that for all $t \in (0, T)$ and $p \in Y_t$ the rescaled operator $\tilde{D}_{t,p}$ is as close to $\mathrm{d} \oplus \mathrm{d}^*: \Omega^*(B_1) \rightarrow \Omega^*(B_1)$ as we wish. In particular, we can arrange that the family of operators $\tilde{D}_{t,p}$ is uniformly elliptic with coefficients uniformly bounded in C^1 . Hence, by standard elliptic theory, we can find a constant $c > 0$ independent of $t \in (0, T)$ and $p \in Y_t$ such that the following L^q estimate holds

$$\|\sigma\|_{W^{1,q}(B_{1/2})} \leq c \left(\|\tilde{D}_{t,p}\sigma\|_{L^q(B_1)} + \|\sigma\|_{L^q(B_1)} \right).$$

Combined with the Sobolev embedding $W^{1,q} \hookrightarrow C^{0,1-7/q}$ this yields

$$[\sigma]_{C^{0,\alpha}(B_{1/2})} \leq c \left(\|\tilde{D}_{t,p}\sigma\|_{L^\infty(B_1)} + \|\sigma\|_{L^\infty(B_1)} \right)$$

with $c > 0$ independent of $t \in (0, T)$ and $p \in Y_t$. This, however, is equivalent to the estimate (2.29) for the unscaled operator D_t . $\square$

Remark 2.30. Proposition 2.24 can be viewed as a quantification of Joyce's proof of the fact that η_t is smooth. In a similar fashion, one can also obtain estimates on higher Hölder norms of $\mathrm{d}\eta_t$.

Remark 2.31. The kind of argument used above goes back to work of Nirenberg–Walker [NW73, Theorem 3.1]. We will encounter this line of reasoning again in the proofs of Propositions 2.41 and 2.77 as well as in later chapters.

2.2 ASD instantons on ALE spaces

Let Γ be a finite subgroup of $\mathrm{SU}(2)$, let X be an ALE space asymptotic to $\mathbf{C}^2/\Gamma$ and let E be a G -bundle over X . We denote by $\mathcal{A}(E)$ the space of connections on E .

Definition 2.32. A *framing at infinity* of E is a bundle isomorphism $\Phi: E_\infty|_U \rightarrow \pi_* E|_U$ where E_∞ is a G -bundle over $(\mathbb{C}^2 \setminus \{0\})/\Gamma$ and U is the complement of a compact neighbourhood of the singular point in $\mathbb{C}^2/\Gamma$.

Let θ be a flat connection on a G -bundle E_∞ over $(\mathbb{C}^2 \setminus \{0\})/\Gamma$.

Definition 2.33. Let $\Phi: E_\infty|_U \rightarrow \pi_* E|_U$ be a framing at infinity of E . Then a connection $A \in \mathcal{A}(E)$ is called *asymptotic to θ at rate δ with respect to Φ* if

$$(2.34) \quad \nabla^k(\Phi^* A - \theta) = O(r^{\delta-k})$$

for all $k \geq 0$. Here ∇ is the covariant derivative associated with θ .

Definition 2.35. A *framed connection asymptotic to θ (at rate δ)* is a connection $A \in \mathcal{A}(E)$ on E together with a framing at infinity Φ of E such that A is asymptotic to θ at rate δ with respect to Φ . If no rate δ is specified, then we take $\delta = -3$.

Proposition 2.36. Let $A \in \mathcal{A}(E)$ be a Yang–Mills connection on E with finite energy, that is,

$$\int_X |F_A|^2 \, \text{dvol} < \infty,$$

then there is a G -bundle E_∞ over $(\mathbb{C}^2 \setminus \{0\})/\Gamma$ together with a flat connection θ and a framing $\Phi: E_\infty|_U \rightarrow \pi_* E|_U$ such that (2.34) holds with $\delta = -3$.

Proof. We extend the argument in [DK90, p. 96]. The topological space $\hat{X} := X \cup \{\infty\}$ can be given the structure of an orbifold whose atlas contains the charts of X as well as a uniformising chart at infinity $\varphi: B_\varepsilon/\Gamma \rightarrow \hat{X}$ which is constructed as follows. Fix an orientation reversing linear isometry σ of $\mathbb{R}^4$. Let Γ act on B_ε by $(g, x) \mapsto \sigma^{-1}(g \cdot \sigma(x))$ and define $\varphi(0) := \infty$ and $\varphi(x) = \pi^{-1}(\sigma(x)/|x|^2)$. If g denotes the metric on X , then the conformally equivalent metric $\hat{g} := (1 + |\pi|^2)^{-2}g$ extends to $\hat{X}$ as an orbifold metric. The metric is not necessarily smooth, but only $C^{3,\alpha}$; however, that does not cause any problems. One should think of $\hat{X}$ as a conformal compactification of X in the same way that S^4 is a conformal compactification of $\mathbb{R}^4$.

Since the Yang–Mills equation as well as the energy are conformally invariant, we can think of A as a finite energy Yang–Mills connection on $(\hat{X} \setminus \{\infty\}, \hat{g})$. By Uhlenbeck’s removable singularities theorem [Uhl82b, Theorem 4.1], the pullback of A to $B_\varepsilon \setminus \{0\}$ extends to a Γ -invariant Yang–Mills connection over all of B_ε . Hence, A extends to a Yang–Mills connection $\hat{A}$ on an orbifold G -bundle $\hat{E}$ over $\hat{X}$. Using radial parallel transport from ∞ we obtain a trivialisation of $\hat{E}$ over $\varphi(B_\varepsilon/\Gamma)$ in which the connection matrix representing $\hat{A}$ vanishes at $\infty = \varphi(0)$. Denote by $\rho: \Gamma \rightarrow G$ the monodromy representation associated with $\hat{E}|_\infty$. Associated with ρ there are a G -bundle E_∞ over $\varphi((B_\varepsilon \setminus \{0\})/\Gamma)$ and a flat connection θ on E_∞ . The above trivialisation of $\hat{E}$ over $\varphi(B_\varepsilon/\Gamma)$ amounts to a bundle

isomorphism $\Phi: E_\infty \rightarrow \hat{E}|_{\varphi(B_\varepsilon \setminus \{0\})/\Gamma}$ and the fact that the connection matrix representing $\hat{A}$ vanishes at $\infty = \varphi(0)$ implies that $\nabla^k(\varphi^*(\Phi^*(\hat{A}) - \theta)) = O(x^{1-k})$ for all $k \geq 0$. By considering the action of the inversion $x \mapsto \sigma(x)/|x|^2$ on k -fold derivatives of 1-forms one sees that $\nabla^k(\Phi^*A - \theta) = O(r^{-3-k})$. $\square$

Let us briefly discuss moduli spaces of framed ASD instantons on E asymptotic to θ . For a detailed discussion we refer the reader to Nakajima's beautiful article [Nak90]. Fix a framing at infinity Φ of E , a rate $\delta \in (-3, -1)$ and denote by $\mathcal{A}(E, \theta)$ the space of all connections asymptotic to θ at rate δ with respect to Φ . Similarly, define $\mathcal{G}(E)$ to be the group of gauge transformations asymptotic to a constant element of G at infinity at rate $\delta + 1$ with respect to Φ . Denote by $g_\infty: \mathcal{G}(E) \rightarrow G$ the homomorphism assigning to each gauge transformation its asymptotic value at infinity and let $\mathcal{G}_0(E) := \ker g_\infty \subset \mathcal{G}(E)$ be the based gauge group consisting of gauge transformations asymptotic to the identity. Then the space

$$M(E, \theta) := \{A \in \mathcal{A}(E, \theta) : F_A^+ = 0\} / \mathcal{G}_0(E)$$

is called the *moduli space of framed ASD instantons on E asymptotic to θ* .

Remark 2.37. The space $M(E, \theta)$ does not depend on the choice of $\delta \in (-3, -1)$. This is a consequence of Proposition 2.36.

Remark 2.38. If we denote by $\rho: \Gamma \rightarrow G$ the monodromy representation associated with θ and by $G_\rho := \{g \in G : g\rho g^{-1} = \rho\}$ the stabiliser of ρ , then $G_\rho \subset G \cong \mathcal{G}(E)/\mathcal{G}_0(E)$ acts on $M(E, \theta)$.

Theorem 2.39 (Nakajima [Nak90, Theorem 2.6 and Proposition 5.1]). *The moduli space $M(E, \theta)$ is a smooth hyperkähler manifold.*

Formally, this can be seen as an infinite-dimensional instance of a hyperkähler reduction [HKL87]. The space $\mathcal{A}(E, \theta)$ inherits a hyperkähler structure from X and the action of the based gauge group $\mathcal{G}_0$ has a hyperkähler moment map given by $\mu(A) = F_A^+$. To make this rigorous one needs to set up a suitable Kuranishi model for $M(E, \theta)$ along the lines of [DK90, Section 4.2.5]. This can be done using weighted Sobolev space completions of $\mathcal{A}(E, \theta)$ and $\mathcal{G}_0(E)$, see [Nak90, Section 2] for a detailed discussion. An important rôle is played by the operator $\delta_A: \Omega^1(X, \mathfrak{g}_E) \rightarrow \Omega^0(X, \mathfrak{g}_E) \oplus \Omega^+(X, \mathfrak{g}_E)$ defined by

$$(2.40) \quad \delta_A(a) := (d_A^*a, d_A^+a)$$

which governs the infinitesimal deformation theory of the ASD instanton A .

Proposition 2.41. *Let $A \in \mathcal{A}(E)$ be a finite energy ASD instanton on E . Then the following holds.*

1. *If $a \in \ker \delta_A$ decays to zero at infinity, i.e., $\lim_{r \rightarrow \infty} \sup_{\pi(x)=r} |a|(x) = 0$, then $\nabla_A^k a = O(|\pi|^{-3-k})$ for all $k \geq 0$.*

2. If $(\xi, \omega) \in \ker \delta_A^*$ decays to zero at infinity, then $(\xi, \omega) = 0$.

Remark 2.42. From the second part of this proposition one can deduce that the deformation theory of framed finite energy ASD instantons is always unobstructed; hence, $M(E, \theta)$ is a smooth manifold, see also [Nak90, Proposition 5.1]. By the first part the tangent space of $M(E, \theta)$ at $[A]$ agrees with the L^2 kernel of δ_A and thus the formal hyperkähler structure is indeed well-defined.

The proof of Proposition 2.41 rests on the following refined Kato inequality.

Proposition 2.43. *Let $A \in \mathcal{A}(E)$ be an ASD instanton on E . If $a \in \Omega^1(X, \mathfrak{g}_E)$ satisfies $\delta_A a = 0$, then*

$$(2.44) \quad |d|a|| \leq \sqrt{\frac{3}{4}} |\nabla_A a|$$

on the complement of the vanishing locus of a .

Proof. Recall that the Kato inequality follows from the Cauchy–Schwarz inequality

$$|\langle \nabla_A a, a \rangle| \leq |\nabla_A a| |\alpha|.$$

If $\delta_A a = 0$, then it is not hard to see that equality can only hold if $\nabla_A a = 0$. This shows that (2.44) holds with some constant $\varepsilon < 1$ instead of $\sqrt{3/4}$.

To see that one can take $\varepsilon = \sqrt{3/4}$ we follow an argument of Feehan [Fee01, Section 3]; however, also note that we could simply read off the value from the table given in [CGH00, Appendix]. We can write δ_A as a Dirac type operator

$$\delta_A a = \sum_i \gamma(e_i) \nabla_{e_i}^A a.$$

Here (e_i) is a local orthonormal frame and the Clifford multiplication γ is defined by $\gamma(v)a := (-i_v a, (v^* \wedge a)^+)$ where v^* denotes the dual of v with respect to the metric on X . For $x \in X$ with $a(x) \neq 0$ and $d|a|(x) \neq 0$ pick an orthonormal basis (e_i) of $T_x X$ with $e_1 := \nabla|a|/|\nabla|a||$. Since $\delta_A a = 0$ and $|\gamma(v)a| = |v||a|$, we have

$$|d|a||^2 = |\nabla_{e_1} a|^2 \leq |\nabla_{e_1}^A a|^2 = |\gamma(e_1) \nabla_{e_1}^A a|^2 = \left| \sum_{i \geq 2} \gamma(e_i) \nabla_{e_i}^A a \right|^2 \leq 3 \sum_{i \geq 2} |\nabla_{e_i}^A a|^2$$

and therefore

$$4|d|a||^2 = 4|\nabla_{e_1}^A a|^2 \leq 3 \sum_i |\nabla_{e_i}^A a|^2 = 3|\nabla_A a|^2.$$

This finishes the proof. $\square$

Proof of Proposition 2.41. First of all note that (1) implies (2), because if $\delta_A^*(\xi, \omega) = 0$, then $d_A^* d_A \xi = - * [F_A^+ \wedge \omega] = 0$ and $d_A^+ d_A \xi = [F_A^+, \xi] = 0$; therefore $d_A \xi = O(|\pi|^{-3})$.

Thus integration by parts yields $d_A \xi = 0$ and, hence, $\xi = 0$. Similarly, one shows that $\omega = 0$.

We will first explain why (1) for $k = 0$ implies the asserted estimates for $k > 0$ as well. The argument is similar to that in Proposition 2.24. For $x \in X$ set $R := \frac{1}{2}(1 + |\pi(x)|)$. We claim that there is a constant $c = c(k) > 0$ independent of $x \in X$ such that

$$(2.45) \quad R^k \|\nabla_A^k a\|_{L^\infty(B_{R/2}(x))} \leq c \|a\|_{L^\infty(B_R(x))}$$

for all $a \in \ker \delta_A$. This clearly implies (1) for $k > 0$ given the statement for $k = 0$. For $|\pi(x)|$ sufficiently large, say $|\pi(x)| > R_0$, the restriction of A to $B_R(x)$ is arbitrarily close to a flat connection by Proposition 2.36. We rescale to a ball of radius one and denote the rescaled connection by $\tilde{A}$ and the rescaling of δ_A by $\tilde{D}_x$. Then the family of operators $\tilde{D}_x$ is uniformly elliptic with coefficients uniformly bounded in C^1 . Therefore, there is a constant $c > 0$ independent of $x \in X$ such that the following Schauder estimates holds

$$\|\nabla_{\tilde{A}}^k a\|_{L^\infty(B_{1/2})} \leq c \left(\|\tilde{D}_x a\|_{C^{k,\alpha}(B_1)} + \|a\|_{L^\infty(B_1)} \right).$$

If a is in the kernel of $\tilde{D}_x$, the first term vanishes. Rescaling this inequality yields (2.45) for $a \in \ker \delta_A$ and $|\pi(x)| > R_0$. For $|\pi(x)| \leq R_0$, (2.45) follows from standard Schauder estimates.

Let us now prove (1) for $k = 0$. Recall, e.g., from [FU91, Equation (6.2.5)], that the operator $\tilde{\delta}_A: \Omega^1(X, \mathfrak{g}_E) \rightarrow \Omega^0(X, \mathfrak{g}_E) \oplus \Omega^+(X, \mathfrak{g}_E)$ defined by $\tilde{\delta}_A(a) := (d_A^* a, \sqrt{2} d_A^+ a)$ satisfies a Weitzenböck formula of the form

$$(2.46) \quad \tilde{\delta}_A^* \tilde{\delta}_A a = \nabla_A^* \nabla_A a + \{\text{Ric}, a\} + \{F_A^-, a\}.$$

Here $\{\cdot, \cdot\}$ denote certain universal bilinear forms, whose precise form, however, is not important for our purposes, and Ric denotes the Ricci tensor of X . In our situation, since X is hyperkähler and thus Ricci flat, this term vanishes. Now, suppose that $\delta_A a = 0$ and thus $\tilde{\delta}_A a = 0$. Then Proposition 2.43, the identity $\Delta|a|^2 + 2|\nabla_A a|^2 = 2\langle a, \nabla_A^* \nabla_A a \rangle$ and the Weitzenböck formula (2.46) yield the following estimate on the complement of the vanishing locus of a

$$\begin{aligned} 3\Delta|a|^{2/3} &\leq |a|^{-4/3} \left(\Delta|a|^2 + \frac{8}{3} |d|a|^2 \right) \\ &\leq |a|^{-4/3} (\Delta|a|^2 + 2|\nabla_A a|^2) \\ &= 2|a|^{-4/3} \langle a, \nabla_A^* \nabla_A a \rangle \\ &= 2|a|^{-4/3} \left(\langle \tilde{\delta}_A^* \tilde{\delta}_A a, a \rangle + \langle \{F_A^-, a\}, a \rangle \right) \\ &\leq O(|\pi|^{-4}) |a|^{2/3}. \end{aligned}$$

In the last step we used $\tilde{\delta}_A a = 0$ and $|F_A^-| = O(|\pi|^{-4})$, which is a consequence of Proposition 2.36.

Now, let $U := \{x \in X : a(x) \neq 0\}$ and set $f := |a|^{2/3}$. We will show that $f = O(|\pi|^{-2})$ which is equivalent to the desired decay estimate for a . It follows from the above that on U

$$\Delta f \leq \frac{cf}{1 + |\pi|^4}$$

for some constant $c > 0$. Since f is bounded, by [Joy00, Theorem 8.3.6(a)] there is a $g = O(|\pi|^{-1})$ such that

$$\Delta g = \begin{cases} (\Delta f)^+ & \text{on } U \\ 0 & \text{on } X \setminus U. \end{cases}$$

Here $(\cdot)^+$ denotes taking the positive part. Since g is superharmonic and decays to zero at infinity, the maximum principle implies that g is non-negative. The function $f - g$ is a subharmonic on U , decays to zero at infinity and is non-positive on the boundary of U ; hence, by the maximum principle $f \leq g = O(|\pi|^{-1})$. Now, $(\Delta f)^+ = O(|\pi|^{-5})$ on U and an application of [Joy00, Theorem 8.3.6(b)] shows that we could, in fact, have chosen g such that $g = O(|\pi|^{-2})$. It follows that $f = O(|\pi|^{-2})$ as desired. $\square$

The dimension of $M(E, \theta)$ can be computed using the following index formula.

Theorem 2.47 (Nakajima [Nak90, Theorem 2.7]). *Let A be a framed connection asymptotic to θ . Then the L^2 index of δ_A is given by*

$$(2.48) \quad \text{index } \delta_A = -2 \int_X p_1(\mathfrak{g}_E) + \frac{2}{|\Gamma|} \sum_{g \in \Gamma \setminus \{e\}} \frac{\chi_{\mathfrak{g}}(g) - \dim \mathfrak{g}}{2 - \text{tr } g}.$$

Here $p_1(\mathfrak{g}_E)$ is the Chern–Weil representative of the first Pontryagin class of E and $\chi_{\mathfrak{g}}$ is the character of Γ acting on $\mathfrak{g}$, the Lie algebra associated with G , via the monodromy representation $\rho: \Gamma \rightarrow G$ of θ . Moreover, if A is an ASD instanton, then $\text{index } \delta_A = \dim \ker \delta_A$.

Proof. Let us briefly explain how to derive (2.48) from Nakajima’s formula which can be written as

$$(2.49) \quad \begin{aligned} \text{index } \delta_A = & - \int_X (\dim \mathfrak{g} + p_1(\mathfrak{g}_E)) \text{ch}(S^+) \hat{A}(X) \\ & + \dim \mathfrak{g}^\Gamma + \frac{1}{|\Gamma|} \sum_{g \in \Gamma \setminus \{e\}} \chi_{\mathfrak{g}}(g) \frac{\text{tr } g}{2 - \text{tr } g}. \end{aligned}$$

Here $\mathfrak{g}^\Gamma$ denotes the Γ -invariant part of $\mathfrak{g}$, S^+ denotes the positive spin bundle on X , and $\text{ch}(S^+)$ and $\hat{A}(X)$ denote the Chern–Weil representatives of the Chern character of S^+ and the $\hat{A}$ -genus of X , respectively.

If A is the product connection on the trivial bundle rank 1 bundle and a lies in the L^2 kernel of δ_A , then it follows from the fact that X is Ricci flat and the Weitzenböck formula (2.46) that $\nabla^* \nabla a = 0$, and then by integration by parts, which is justified by Proposition 2.41, that $\nabla a = 0$. Since a lies in L^2 , it necessarily vanishes. Therefore $\dim \ker \delta_A = 0$ and (2.49) yields

$$\int_X \text{ch}(S^+) \hat{A}(X) = 1 + \frac{1}{|\Gamma|} \sum_{g \in \Gamma \setminus \{e\}} \frac{\text{tr } g}{2 - \text{tr } g}.$$

By plugging this back into (2.49) we obtain

$$\text{index } \delta_A = -2 \int_X p_1(\mathfrak{g}_E) + \dim \mathfrak{g}^\Gamma - \dim \mathfrak{g} + \frac{1}{|\Gamma|} \sum_{g \in \Gamma \setminus \{e\}} (\chi_{\mathfrak{g}}(g) - \dim \mathfrak{g}) \frac{\text{tr } g}{2 - \text{tr } g}.$$

Since

$$\frac{1}{|\Gamma|} \sum_{g \in \Gamma} (\chi_{\mathfrak{g}}(g) - \dim \mathfrak{g}) = \dim \mathfrak{g}^\Gamma - \dim \mathfrak{g},$$

this leads to the index formula (2.48) given above. $\square$

There is a rich existence theory for ASD instantons on ALE spaces. Gocho–Nakajima [GN92] observed that for each representation $\rho: \Gamma \rightarrow \text{U}(n)$ there is a bundle $\mathcal{R}_\rho$ over X together with an ASD instanton A_ρ asymptotic to the flat connection determined by ρ , and if σ is a further representation of Γ , then $A_{\rho \oplus \sigma} = A_\rho \oplus A_\sigma$. Kronheimer–Nakajima [KN90] took this as the starting point for an ADHM construction of ASD instantons on ALE spaces. One important consequence of their work is the following rigidity result.

Definition 2.50. An ASD instanton A is called *infinitesimally rigid* if the L^2 kernel of the linear operator δ_A is trivial.

Theorem 2.51 (Kronheimer–Nakajima [KN90, Lemma 7.1]). *For each $\rho: \Gamma \rightarrow \text{U}(n)$ the ASD instanton A_ρ is infinitesimally rigid.*

By combining this result applied to the regular representation with the index formula Kronheimer–Nakajima derive a geometric version of the McKay correspondence [KN90, Appendix A]. Let $\Delta(\Gamma)$ denote the Dynkin diagram associated to Γ in the ADE classification of the finite subgroups of $\text{SU}(2)$. Each vertex of $\Delta(\Gamma)$ corresponds to a non-trivial irreducible representation. We label these by $\rho_1, \dots, \rho_k$ and denote the associated bundles by $\mathcal{R}_j$ and the associated ASD instantons by A_j .

Theorem 2.52 (Kronheimer–Nakajima [KN90, Appendix A]). *The L^2 harmonic 2-forms $c_1(\mathcal{R}_j) = \frac{i}{2\pi} \text{tr } F_{A_j}$ form a basis of $L^2 \mathcal{H}^2(X) \cong H^2(X, \mathbf{R})$ and satisfy*

$$\int_X c_1(\mathcal{R}_i) \wedge c_1(\mathcal{R}_j) = -(C^{-1})_{ij}$$

where C is the Cartan matrix associated with $\Delta(\Gamma)$. Moreover, there is an isometry $\kappa \in \text{Aut}(H_2(X, \mathbf{Z}), \cdot)$ such that $\{c_1(\mathcal{R}_j)\}$ is dual to $\{\kappa[\Sigma_j]\}$, where Σ_j are the irreducible components of the exceptional divisor E of $\widehat{\mathbf{C}^2}/\Gamma$. If X is isomorphic to $\widehat{\mathbf{C}^2}/\Gamma$ as a complex manifold, then $\kappa = \text{id}$.

This result is very useful for computing the index of δ_A when A is constructed out of ASD instantons of the form A_ρ (by taking tensor products, direct sums, etc.).

Proposition 2.53. *Let X be an ALE space asymptotic to $\mathbf{C}^2/\mathbf{Z}_k$. Denote by $\rho_j: \mathbf{Z}_k \rightarrow U(1)$ the irreducible representation defined by $\rho_j(\ell) = \exp(\frac{2\pi i}{k} j \ell)$. For $n, m \in \mathbf{Z}_k$, let $E_{n,m}$ be the $\text{SO}(3)$ -bundle underlying $\mathbf{R} \oplus (\mathcal{R}_n^* \otimes \mathcal{R}_{n+m})$ and denote by $A_{n,m}$ the ASD instanton on $E_{n,m}$ induced by A_n and A_{n+m} . Then $A_{n,m}$ is infinitesimally rigid, asymptotic at infinity to the flat connection associated with ρ_m and*

$$\frac{1}{8\pi^2} \int_X |F_{A_{n,m}}|^2 = \frac{(k-m)m}{k}$$

as well as

$$w_2(\mathfrak{g}_{E_{n,m}}) = c_1(\mathcal{R}_{n+m}) - c_1(\mathcal{R}_n) \in H^2(X, \mathbf{Z}_2).$$

Proof. To see that $A_{n,m}$ is infinitesimally rigid apply Theorem 2.51 to $A_n \oplus A_{n+m}$ and observe that $\mathfrak{g}_{E_{n,m}} = \mathbf{R} \oplus (\mathcal{R}_n^* \otimes \mathcal{R}_{n+m})$ is a parallel subbundle of $\mathfrak{g}_{\mathcal{R}_n \oplus \mathcal{R}_{n+m}}$.

The energy of $A_{n,m}$ can be computed by noting that the first term in the index formula (2.48) is precisely twice the energy and the second term is given by $(-\frac{2}{k})$ -times

$$-\sum_{g \neq e} \frac{\chi_{\mathfrak{g}}(g) - \dim \mathfrak{g}}{2 - \text{tr } g} = \sum_{j=1}^{k-1} \frac{1 - \cos(2\pi m j/k)}{1 - \cos(2\pi j/k)} = (k-m)m.$$

The statement about the second Stiefel-Whitney class is clear. $\square$

2.3 Approximate G_2 -instantons

Throughout this section, let Y_0 be an admissible G_2 -orbifold, let $\mathbf{r} = \{(X_j, \rho_j)\}$ be resolution data for Y_0 and denote by $(Y_t, \phi_t)_{t \in (0, T)}$ the family of G_2 -manifolds obtained from $\mathbf{r}$ via Theorem 2.20. Denote by $\psi_t := \Theta(\phi_t)$ the coassociative calibration on Y_t . If θ is a flat connection on a G -bundle E_0 over Y_0 , then the monodromy of θ around S_j induces a representation $\mu_j: \pi_1(T_j, x_j) \cong G_j \rtimes \tilde{H}_j \rightarrow G$ of the orbifold fundamental group of T_j based at $x_j \in T_j \setminus S_j$.

Remark 2.54. For a general definition of orbifold fundamental group we refer the reader to [ALR07, Definition 1.50 and Section 2.2]. All orbifold fundamental groups $\pi_1(X)$ encountered in this chapter can be identified with the fundamental groups $\pi_1(X^{\text{reg}})$ of the regular part of the orbifold in question. This is a consequence of the next proposition which is a special case of [TT04, Lemma 5.2].

Proposition 2.55. *If B is an open ball in $\mathbf{R}^n$ and $S \subset B$ is a closed subset of Hausdorff codimension at least four, then $\pi_1(B \setminus S) = 1$.*

Proof. Note that for any given Lipschitz loop $\gamma : S^1 \rightarrow B \setminus S$, the set of $x \in B \setminus S$ such that the cone $C(\gamma, x) = \{\ell_{x, \gamma(s)}(t) : s, t \in [0, 1]\}$, where $\ell_{x, y}(t) = (1-t)x + ty$, is contained in $B \setminus S$ has Hausdorff codimension at least 2. Since any continuous loop is homotopic to a Lipschitz loop in $B \setminus S$ it follows that any continuous loop is homotopic to a constant one. $\square$

Definition 2.56. A collection $\mathbf{g} = ((E_0, \theta), \{(x_j, f_j)\}, \{(E_j, A_j, m_j)\})$ consisting of E_0 and θ as above as well as, for each j , the choice of

- a point $x_j \in T_j \setminus S_j$ together with a framing $f_j : (E_0)_{x_j} \rightarrow G$ of E_0 at x_j ,
- a G -bundle E_j over X_j together with a framed ASD instanton A_j asymptotic at infinity to the flat connection on the bundle $E_{\infty, j}$ over $(\mathbf{C}^2 \setminus \{0\})/G_j$ induced by the representation $\mu_j|_{G_j}$ and
- an action m_j of $\tilde{H}_j$ on E_j which lifts the action of $\tilde{H}_j$ induced by ρ_j

is called *gluing data compatible with $\mathbf{r} = \{(X_j, \rho_j)\}$* if the following compatibility conditions are satisfied:

- The action of $\tilde{H}_j$ on E_j preserves A_j .
- The action of $\tilde{H}_j$ on E_j is asymptotic at infinity, with respect to the framing associated with A_j , to the action of $\tilde{H}_j$ on $E_{\infty, j}$. Note that the lift of the action of $\tilde{H}_j$ on $E_{\infty, j}$ to the trivial bundle $G \times (\mathbf{R}^3 \times \mathbf{C}^2 \setminus \{0\})$ is given by $h \cdot (g, x) = (\mu_j(h) \cdot g, h \cdot x)$.

We should point out here that it is by far not always possible to extend a choice of (E_0, θ) and $\{(E_j, A_j)\}$ to compatible gluing data. This will become clear from the discussion in Section 2.7.

Before we proceed to construct approximate G_2 -instantons, we introduce weighted Hölder norms. It will become more transparent over the course of the next sections that these are well adapted to the problem at hand. We define weight functions by

$$w_t(x) := t + r_t(x) \quad \text{and} \quad w_t(x, y) := \min\{w_t(x), w_t(y)\}.$$

For $t \in (0, T)$, a Hölder exponent $\alpha \in (0, 1)$ and a weight parameter $\beta \in \mathbf{R}$ we define

$$\begin{aligned} [f]_{C_{\beta, t}^{0, \alpha}(U)} &:= \sup_{d(x, y) \leq w_t(x, y)} w_t(x, y)^{\alpha - \beta} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}, \\ \|f\|_{L_{\beta, t}^\infty(U)} &:= \|w_t^{-\beta} f\|_{L^\infty(U)} \quad \text{and} \\ \|f\|_{C_{\beta, t}^{k, \alpha}(U)} &:= \sum_{j=0}^k \|\nabla^j f\|_{L_{\beta-j, t}^\infty(U)} + [\nabla^j f]_{C_{\beta-j, t}^{0, \alpha}(U)}. \end{aligned}$$

Here f is a section of a vector bundle over $U \subset Y_t$ equipped with an inner product and a compatible connection. On tensor bundles associated with Y_t we use the metrics induced by $\tilde{g}_t$; however, in view of Proposition 2.24, we could equivalently use those induced by $\phi_t = \tilde{\phi}_t + d\eta_t$. We use parallel transport to compare the value of f at different points in Y . If U is not specified, then we take $U = Y_t$. We denote by $C_{\beta,t}^{k,\alpha}$ the Banach space $C^{k,\alpha}$ equipped with the norm $\|\cdot\|_{C_{\beta,t}^{k,\alpha}}$.

Remark 2.57. For fixed $t \in (0, T)$ and $\beta \in \mathbf{R}$, the norms $\|\cdot\|_{C_{\beta,t}^{k,\alpha}}$ and $\|\cdot\|_{C^{k,\alpha}}$ are equivalent, but not uniformly so as $t > 0$ tends to zero.

Note that, if $\beta = \beta_1 + \beta_2$, then

$$(2.58) \quad \|f \cdot g\|_{C_{\beta,t}^{k,\alpha}} \leq \|f\|_{C_{\beta_1,t}^{k,\alpha}} \cdot \|g\|_{C_{\beta_2,t}^{k,\alpha}}.$$

Also for $\beta > \gamma$ we have

$$(2.59) \quad \|f\|_{C_{\beta,t}^{k,\alpha}} \leq t^{\gamma-\beta} \|f\|_{C_{\gamma,t}^{k,\alpha}}.$$

Proposition 2.60. *Let $\mathbf{g}$ be gluing data compatible with $\mathbf{r}$. Then there is a constant $c > 0$ and for each $t \in (0, T)$ a G -bundle E_t over Y_t together with a connection $\tilde{A}_t$ satisfying*

$$(2.61) \quad \|F_{\tilde{A}_t} \wedge \psi_t\|_{C_{-2,t}^{0,\alpha}} \leq ct^{1/2}.$$

The adjoint bundle $\mathfrak{g}_{E_t}$ associated with E_t satisfies

$$(2.62) \quad p_1(\mathfrak{g}_{E_t}) = - \sum_j k_j \text{PD}[S_j] \quad \text{with} \quad k_j := \frac{1}{8\pi^2} \int_{X_j} |F_{A_j}|^2$$

and

$$(2.63) \quad \langle w_2(\mathfrak{g}_{E_t}), [\Sigma] \rangle = \langle w_2(\mathfrak{g}_{E_j}), [\Sigma] \rangle$$

for each $[\Sigma] \in H_2(X_j)^{H_j} \subset H_2(Y_t)$.

Proof. The choice of m_j defines a lift of the action of $\tilde{H}_j$ on $\mathbf{R}^3 \times X_j$ to the pullback of E_j to $\mathbf{R}^3 \times X_j$. Passing to the quotient yields a G -bundle over $(T^3 \times X_j)/H_j$ which we denote by E_j , by abuse of notation. It follows from the compatibility conditions that the pullback of A_j to $\mathbf{R}^3 \times X_j$ passes to the quotient and induces a connection on E_j which we denote by A_j , again by abuse of notation.

Fix $t \in (0, T)$. Let $R_{j,t} := \tilde{T}_{j,t} \cap r_t^{-1}[\zeta/4, \zeta/2]$ as in (2.18) with $\tilde{T}_{j,t}$ and r_t as defined in (2.13) and (2.17), respectively. By the compatibility conditions the monodromy of A_j along S_j on the fibre at infinity matches up with the monodromy of θ along $E_0|_{S_j}$. Thus, via parallel transport the framing of E_0 at x_j and the framing of E_j yield an identification of $E_0|_{R_{j,t}}$ with $E_j|_{R_{j,t}}$. Patching E_0 and the E_j via this identification yields the bundle E_t .

Under the identification of $E_0|_{R_{j,t}}$ with $E_j|_{R_{j,t}}$, we can write

$$(2.64) \quad A_j = \theta + a_j \quad \text{with} \quad \nabla^k a_j = t^{2+k} O(r_t^{-3-k}),$$

because of Remark 2.19 and Proposition 2.41. Fix a smooth non-increasing cut-off function $\chi: [0, \zeta] \rightarrow [0, 1]$ such that $\chi(s) = 1$ for $s \leq \zeta/4$ and $\chi(s) = 0$ for $s \geq \zeta/2$. Set $\chi_t := \chi \circ r_t$. After cutting off A_j to $\theta + \chi_t \cdot a_j$ it can be matched with θ and we obtain the connection $\tilde{A}_t$ on the bundle E_t .

To estimate $F_{\tilde{A}} \wedge \psi_t$ note that on $Y_t \setminus \tilde{T}_t$ the connection $\tilde{A}_t$ is flat. Thus we can focus our attention on $\tilde{T}_{j,t}$. By the definition of $\tilde{A}_t$ we have

$$F_{\tilde{A}_t} = \chi_t F_{A_j} + d\chi_t \wedge a_j + \frac{\chi_t^2 - \chi_t}{2} [a_j \wedge a_j].$$

The last two terms in this expression are supported in $R_{j,t}$ and of order t^2 in $C^{0,\alpha}$ by (2.64). By Example 1.92 and Proposition 2.24 we have

$$\|F_{A_j} \wedge \psi_t\|_{C_{-2,t}^{0,\alpha}(\tilde{T}_{j,t})} = \|F_{A_j} \wedge (\psi_t - \hat{\psi}_t)\|_{C_{-2,t}^{0,\alpha}(\tilde{T}_{j,t})} \leq ct^{1/2} \|F_{A_j}\|_{C_{-2,t}^{0,\alpha}}.$$

It follows from Proposition 2.36 and Remark 2.19 that

$$\nabla^k F_{A_j} = t^{2+k} O(r_t^{-4-k}).$$

This implies that $\|F_{A_j}\|_{C_{-4,t}^{0,\alpha}(\tilde{T}_{j,t})} \leq ct^2$ and, hence, $\|F_{A_j}\|_{C_{-2,t}^{0,\alpha}(\tilde{T}_{j,t})} \leq c$ by (2.59) with $c > 0$ independent of $t \in (0, T)$. Now, putting everything together yields (2.61).

Let $\iota_{j,t}: T^3 \times \pi_{j,t}^{-1}(B_\zeta^4/G_j) \rightarrow Y$ be as in Remark 2.15. Then $\iota_{j,t}^* \mathfrak{g}_{E_t}$ is isomorphic to the pullback of $\mathfrak{g}_{E_j}$ to $T^3 \times \pi_{j,t}^{-1}(B_\zeta^4/G_j)$. This implies (2.63) by naturality of Stiefel–Whitney classes. To compute $p_1(\mathfrak{g}_{E_t})$ we use Chern–Weil theory to represent it as $p_1(\mathfrak{g}_{E_t}) = -\frac{1}{8\pi^2} \text{tr}(F_{\tilde{A}_t} \wedge F_{\tilde{A}_t})$. We can write this as $p_1(\mathfrak{g}_{E_t}) = \sum_j p_j$ where p_j are compactly supported 4-forms on $\tilde{T}_{j,t}$. Recalling the definition of $[S_j]$ in (2.16) and considering the behaviour of Poincaré duality with respect to coverings we see that in order to prove (2.62) we have to show

$$\iota_{j,t}^* p_j = k_j \text{PD} [T^3 \times \{x\}] \in H_c^4(T^3 \times \pi_{j,t}^{-1}(B_\zeta^4/G_j), \mathbf{R}).$$

From our construction of $\tilde{A}_t$ it follows that the form $\iota_{j,t}^* p_j$ is the pullback of a compactly supported 4-form on X_j which we can write as $-\frac{1}{8\pi^2} \text{tr}(F_{\tilde{A}_j} \wedge F_{\tilde{A}_j})$ where $\tilde{A}_j = A_j + \alpha$ and, by slight abuse of notation, $\alpha = (1 - \chi_t)a_j$. Consequently, $\iota_{j,t}^* p_j$ is a multiple of $\text{PD} [T^3 \times \{x\}]$. To see that the multiplicity is precisely k_j we use the Chern–Simons

3-form, see [DK90, Equation (2.1.17)], to write

$$\text{tr}(F_{\tilde{A}_j} \wedge F_{\tilde{A}_j}) - \text{tr}(F_{A_j} \wedge F_{A_j}) = d \text{tr} \left(\alpha \wedge d_{A_j} \alpha + \frac{1}{3} \alpha \wedge [\alpha \wedge \alpha] \right).$$

By Proposition 2.41 the 1-form α decays sufficiently fast to conclude from Stokes' theorem that

$$-\frac{1}{8\pi^2} \int_{X_j} \text{tr}(F_{\tilde{A}_j} \wedge F_{\tilde{A}_j}) = -\frac{1}{8\pi^2} \int_{X_j} \text{tr}(F_{A_j} \wedge F_{A_j}) = \int_{X_j} \frac{1}{8\pi^2} |F_{A_j}|^2 = k_j.$$

This completes the proof. $\square$

Remark 2.65. If we identify all Y_t with one fixed Y , then the isomorphism type of the bundles E_t does not depend on $t \in (0, T)$. We can therefore think of them as one fixed G -bundle E over Y .

2.4 The model operator on $\mathbf{R}^3 \times \text{ALE}$

Because of Proposition 1.98, in order to prove Theorem 2.1 we need to find $\xi_t \in \Omega^0(Y_t, \mathfrak{g}_{E_t})$ and $a_t \in \Omega^1(Y_t, \mathfrak{g}_{E_t})$ such that

$$(2.66) \quad *_t \left(F_{\tilde{A}_t + a_t} \wedge \psi_t \right) + d_{\tilde{A}_t} \xi_t = 0$$

for $t \in (0, T')$ provided $T' \in (0, T]$ is sufficiently small. Here $*_t$ denotes the Hodge- $*$ -operator associated with ϕ_t . Equation (2.66) together with the Coulomb gauge condition $d_{\tilde{A}_t}^* a_t = 0$ can be written as

$$(2.67) \quad L_t \underline{a}_t + Q_t(\underline{a}_t) + *_t \left(F_{\tilde{A}_t} \wedge \psi_t \right) = 0.$$

Here we use the notation $\underline{a}_t := (\xi_t, a_t)$, the linear operator $L_t := L_{\tilde{A}_t}$ is defined as in (1.103) with $\phi = \phi_t$ and Q_t is defined by

$$(2.68) \quad Q_t(\underline{a}) := \frac{1}{2} *_t ([a \wedge a] \wedge \psi_t) + [a, \xi].$$

The key to solving (2.67) is a good understanding of the linearisation L_t . In this section, we study a model for L_t on $r_t^{-1}([0, \zeta))$.

Let X be an ALE space, let A be a G -bundle over X and let A be a finite energy ASD instanton on E . Fix an orthonormal triple $(\delta^1, \delta^2, \delta^3)$ of constant 1-forms on $\mathbf{R}^3$ and denote by $(\omega_1, \omega_2, \omega_3)$ the triple of Kähler forms associated with X . Consider $\mathbf{R}^3 \times X$ as a G_2 -manifold as in Example 1.11. Denote by $p_{\mathbf{R}^3}: \mathbf{R}^3 \times X \rightarrow \mathbf{R}^3$ and $p_X: \mathbf{R}^3 \times X \rightarrow X$ the projection onto the first and second factor, respectively. Slightly abusing notation, we denote the respective pullbacks of E and A to $\mathbf{R}^3 \times X$ via p_X by E and A as well. As in (1.103) we

define $L_A: \Omega^0(\mathbf{R}^3 \times X, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^3 \times X, \mathfrak{g}_E) \rightarrow \Omega^0(\mathbf{R}^3 \times X, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^3 \times X, \mathfrak{g}_E)$ by

$$L_A = \begin{pmatrix} 0 & d_A^* \\ d_A & *(\psi \wedge d_A) \end{pmatrix}$$

with ψ defined by

$$(2.69) \quad \psi := \frac{1}{2} \omega_1 \wedge \omega_1 - \delta^2 \wedge \delta^3 \wedge \omega_1 - \delta^3 \wedge \delta^1 \wedge \omega_2 - \delta^1 \wedge \delta^2 \wedge \omega_3.$$

Proposition 2.70. *If we identify $p_{\mathbf{R}^3}^* T^* \mathbf{R}^3$ with $p_X^* \Lambda^+ T^* X$ via $\delta^i \mapsto -\omega_i$ and accordingly*

$$\begin{aligned} \Omega^0(\mathbf{R}^3 \times X, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^3 \times X, \mathfrak{g}_E) \\ = \Omega^0(\mathbf{R}^3 \times X, p_X^* [(\mathbf{R} \oplus \Lambda^+ T^* X \oplus T^* X) \otimes \mathfrak{g}_E]), \end{aligned}$$

then the operator L_A can be written as $L_A = F + D_A$ where

$$F(\xi, \omega, a) = \sum_{i=1}^3 (-\langle \partial_i \omega, \omega_i \rangle, \partial_i \xi \cdot \omega_i, I_i \partial_i a) \quad \text{and} \quad D_A = \begin{pmatrix} 0 & \delta_A \\ \delta_A^* & 0 \end{pmatrix}.$$

Here $\delta_A: \Omega^1(X, \mathfrak{g}_E) \rightarrow \Omega^0(X, \mathfrak{g}_E) \oplus \Omega^+(X, \mathfrak{g}_E)$ denotes the linear operator defined in (2.40). Moreover,

$$(2.71) \quad L_A^* L_A = \Delta_{\mathbf{R}^3} + \begin{pmatrix} \delta_A \delta_A^* & \\ & \delta_A^* \delta_A \end{pmatrix}$$

where $\Delta_{\mathbf{R}^3} = -\sum_{i=1}^3 \partial_i^2$ and ∂_i denotes taking the derivative of a section of $p_X^* [(\mathbf{R} \oplus \Lambda^+ T^* X \oplus T^* X) \otimes \mathfrak{g}_E]$ in the direction of the i -th coordinate on $\mathbf{R}^3$.

Proof. It is a straight-forward computation to verify that $L_A = F + D_A$. It is also easy to see that $F^* F = \Delta_{\mathbf{R}^3}$ and that $F^* D_A + D_A^* F = 0$. This immediately implies (2.71). $\square$

To understand the properties of L_A we work with weighted Hölder norms. We define weight functions by

$$w(x) := 1 + |\pi \circ p_X(x)| \quad \text{and} \quad w(x, y) := \min\{w(x), w(y)\}.$$

Here $\pi: X \rightarrow \mathbf{C}^2/G$ denotes the resolution map associated with the ALE space X . For a

Hölder exponent $\alpha \in (0, 1)$ and a weight parameter $\beta \in \mathbf{R}$ we define

$$\begin{aligned} [f]_{C_\beta^{0,\alpha}(U)} &:= \sup_{d(x,y) \leq w(x,y)} w(x,y)^{\alpha-\beta} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}, \\ \|f\|_{L_\beta^\infty(U)} &:= \|w^{-\beta} f\|_{L^\infty(U)} \quad \text{and} \\ \|f\|_{C_\beta^{k,\alpha}(U)} &:= \sum_{j=0}^k \|\nabla^j f\|_{L_{\beta-j}^\infty(U)} + [\nabla^j f]_{C_{\beta-j}^{0,\alpha}(U)}. \end{aligned}$$

Here f is a section of a vector bundle over $U \subset \mathbf{R}^3 \times X$ equipped with an inner product and a compatible connection. We use parallel transport to compare the values of f at different points. If U is not specified, then we take $U = Y_t$. We denote by $C_\beta^{k,\alpha}$ the subspace of elements f of the Banach space $C_\beta^{k,\alpha}$ with $\|f\|_{C_\beta^{k,\alpha}} < \infty$ and equip it with the norm $\|\cdot\|_{C_\beta^{k,\alpha}}$.

Under the assumptions of Section 2.3 and with $\mathbf{g}$ denoting compatible gluing data suppose that $X = X_j$ and that $A = A_j$. Define $\tilde{l}_{j,t}: \mathbf{R}^3 \times \pi_{j,t}^{-1}(B_\zeta^4/G_j) \rightarrow \tilde{T}_{j,t}$ by

$$\tilde{l}_{j,t}(x, y) := [(tx, y)].$$

For a parameter $\beta \in \mathbf{R}$ and $\underline{a} = (\xi, a) \in \Omega^0(Y_t, \mathfrak{g}_{E_t}) \oplus \Omega^1(Y_t, \mathfrak{g}_{E_t})$ we define

$$(2.72) \quad s_{\beta,t}(\xi, a)(x, y) := t^{\beta-1}(t(\tilde{l}_{j,t})^*\xi, (\tilde{l}_{j,t})^*a).$$

Proposition 2.73. *There is a constant $c > 0$ such that for $t \in (0, T)$*

$$\begin{aligned} \|s_{\beta,t}\underline{a}\|_{C_\beta^{k,\alpha}(\mathbf{R}^3 \times \pi_{j,t}^{-1}(B_\zeta^4/G_j))} &\sim \|\underline{a}\|_{C_{\beta,t}^{k,\alpha}(\tilde{T}_{j,t})} \quad \text{and} \\ \|L_t \underline{a} - s_{\beta-1,t}^{-1} L_{A_j} s_{\beta,t} \underline{a}\|_{C_{\beta-1,t}^{0,\alpha}(\tilde{T}_{j,t})} &\leq ct^{1/2} \|\underline{a}\|_{C_{\beta,t}^{1,\alpha}(\tilde{T}_{j,t})} \end{aligned}$$

where $\sim$ means comparable uniformly in t .

Proof. The map $\tilde{l}_{j,t}$ pulls back the metric on $\tilde{T}_{j,t}$ associated with $\hat{\phi}_t$, that is $g_{\hat{\phi}_t} = g_{\mathbf{R}^3} \oplus t^2 g_{X_j}$, to $t^2(g_{\mathbf{R}^3} \oplus g_{X_j})$. This implies the first estimate in view of Remark 2.19. The second estimate is now immediate from the construction of $\tilde{A}_t$ and Proposition 2.24. $\square$

Proposition 2.74. *Let $\beta \in (-3, 0)$. Then $\underline{a} \in C_\beta^{1,\alpha}$ is in the kernel of $L_A: C_\beta^{1,\alpha} \rightarrow C_{\beta-1}^{0,\alpha}$ if and only if it is given by the pullback of an element of the L^2 kernel of δ_A to $\mathbf{R}^3 \times X$.*

The proof of Proposition 2.74 relies on the following lemma whose proof I will present in the next section.

Definition 2.75. A Riemannian manifold X is said to be of *bounded geometry* if it is complete, its Riemann curvature tensor is bounded from above and its injectivity radius is bounded from below. A vector bundle over X is said to be of *bounded geometry* if it has trivialisations over balls of a fixed radius such that the transition functions and all of their

derivatives are uniformly bounded. We say that a complete oriented Riemannian manifold X has *subexponential volume growth* if for each $x \in X$ the function $r \mapsto \text{vol}(B_r(x))$ grows subexponentially, i.e., $\text{vol}(B_r(x)) = o(\exp(cr))$ as $r \rightarrow \infty$ for every $c > 0$.

Lemma 2.76. *Let E be a vector bundle of bounded geometry over a Riemannian manifold X of bounded geometry and with subexponential volume growth, and suppose that $D: C^\infty(X, E) \rightarrow C^\infty(X, E)$ is a uniformly elliptic operator of second order whose coefficients and their first derivatives are uniformly bounded, that is non-negative, i.e., $\langle Da, a \rangle \geq 0$ for all $a \in W^{2,2}(X, E)$, and formally self-adjoint. If $a \in C^\infty(\mathbf{R}^n \times X, E)$ satisfies*

$$(\Delta_{\mathbf{R}^n} + D)a = 0$$

and $\|a\|_{L^\infty}$ is finite, then a is constant in the $\mathbf{R}^n$ -direction, that is $a(x, y) = a(y)$. Here, by slight abuse of notation, we denote the pullback of E to $\mathbf{R}^n \times X$ by E as well.

Proof of Proposition 2.74. Suppose $\underline{a} \in C_\beta^{1,\alpha}$ satisfies $L_A \underline{a} = 0$. Then $\underline{a}$ is smooth by elliptic regularity and satisfies $L_A^* L_A \underline{a} = 0$. By Definition 2.6 and by Proposition 2.36 both $\mathbf{R}^3 \times X$ and $\mathfrak{g}_E$ have bounded geometry. Moreover, by Proposition 2.70, $L_A^* L_A = \Delta_{\mathbf{R}^3} + D_A^* D_A$ and $D_A^* D_A$ is uniformly elliptic of second order and its coefficients and their first derivatives are uniformly bounded as can be seen from Proposition 2.36. Therefore, we can apply Lemma 2.76 to conclude that $\underline{a}$ is invariant under translations in the $\mathbf{R}^3$ -direction and, hence, by Propositions 2.41 and 2.70 must be the pullback of an element in the L^2 kernel of δ_A . $\square$

Proposition 2.77. *For $\beta \in \mathbf{R}$ there is a constant $c > 0$ such that*

$$\|\underline{a}\|_{C_\beta^{1,\alpha}} \leq c(\|L_A \underline{a}\|_{C_{\beta-1}^{0,\alpha}} + \|\underline{a}\|_{L_\beta^\infty}).$$

Proof. This is a standard result, see Remark 2.31. The desired estimate is local in the sense that is enough to prove estimates of the form

$$\|\underline{a}\|_{C_\beta^{1,\alpha}(U_i)} \leq c(\|L_A \underline{a}\|_{C_{\beta-1}^{0,\alpha}} + \|\underline{a}\|_{L_\beta^\infty})$$

with $c > 0$ independent of i , where $\{U_i\}$ is a suitable open cover of $\mathbf{R}^3 \times X$.

Fix $R > 0$ suitably large and set $U_0 := \{(x, y) \in \mathbf{R}^3 \times X : |\pi(x)| \leq R\}$. Then there clearly is a constant $c > 0$ such that the above estimate holds for $U_i = U_0$. Pick a sequence $(x_i, y_i) \in \mathbf{R}^3 \times X$ such that $r_i := |\pi(y_i)| \geq R$ and the balls $U_i := B_{r_i/8}(x_i, y_i)$ cover the complement of U_0 . On U_i , we have a Schauder estimate of the form

$$\begin{aligned} & \|\underline{a}\|_{L^\infty(U_i)} + r_i^\alpha [\underline{a}]_{C^{0,\alpha}(U_i)} + r_i \|\nabla_A \underline{a}\|_{L^\infty(U_i)} + r_i^{1+\alpha} [\nabla_A \underline{a}]_{C^{0,\alpha}(U_i)} \\ & \leq c(r_i \|L_A \underline{a}\|_{L^\infty(U_i)} + r_i^{1+\alpha} [L_A \underline{a}]_{C^{0,\alpha}(U_i)} + \|\underline{a}\|_{L^\infty(U_i)}) \end{aligned}$$

where $V_i = B_{r_i/4}(x_i, y_i)$ and $\underline{a} = (\xi, a)$. By arguing as in Propositions 2.24 and 2.41 one shows that the constant $c > 0$ can be chosen to work for all i simultaneously. Since on V_i we have $\frac{1}{2}r_i \leq w \leq 2r_i$, multiplying the above Schauder estimate by $r_i^{-\beta}$ yields the desired local estimate. $\square$

2.5 Proof of Lemma 2.76

Lemma 2.76, whose proof I will present in the current section, is an abstraction of various results that have appeared in the literature, e.g., in Pacard–Ritoré’s work on the Allen–Cahn equation [PR03, Corollary 7.5] and in Brendle’s unpublished work on the Yang–Mills equation in higher dimension [Bre03b, Proposition 3.3].

Let me first give a heuristic argument which is sometimes given as though it would constitute a proof. Denote by $\hat{a}$ the partial Fourier transform of a in the $\mathbf{R}^n$ –direction. Then $\hat{a}$ solves $(D + |k|^2)\hat{a} = 0$. But $D + |k|^2$ is invertible for $k \neq 0$. Thus $\hat{a}$ is supported on $\{0\} \times X$ and hence must be a linear combination of derivatives of various orders of $\Gamma(E)$ –valued δ –functions. Reversing the Fourier transform shows that a must be a polynomial in $\mathbf{R}^n$. But then it follows from the assumptions that a is constant in the $\mathbf{R}^n$ –direction. The actual proof will be slightly more pedestrian.

First we need to set-up some notation. We fix a point $p \in X$ and denote by $\rho: X \rightarrow [0, \infty)$ a smoothing of the distance from p , as in [Kor91, Proposition 4.1]. For $\delta \in \mathbf{R}$ we introduce a weight function $w_\delta := e^{-\delta\rho}$ and weighted Hilbert spaces $W_\delta^{s,2}(X, E)$ consisting of locally integrable sections f such that $w_\delta \cdot f$ lies in $W^{s,2}(X, E)$ with inner product defined by $\langle \cdot, \cdot \rangle_{W_\delta^{s,2}} := \langle w_\delta \cdot, w_\delta \cdot \rangle_{W^{s,2}}$. As usual we set $L_\delta^2(X, E) := W_\delta^{0,2}(X, E)$.

Proposition 2.78. *For each $k_0 > 0$ there is a constant $\varepsilon = \varepsilon(k_0) > 0$ such that for all $\delta \in (-\varepsilon, \varepsilon)$ and $k \in [k_0, \infty)$ the operator $D + k^2: W_\delta^{2,2}(X, E) \rightarrow L_\delta^2(X, E)$ is an isomorphism. Moreover, for $\ell \geq 0$ there is a constant $c_\ell = c_\ell(k_0) > 0$ such that*

$$(2.79) \quad \left\| \partial_k^\ell (D + k^2)^{-1} a \right\|_{W_\delta^{2,2}} \leq c_\ell (1 + k)^\ell \|a\|_{L_\delta^2}$$

for all $k \in [k_0, \infty)$ and $a \in L_\delta^2(X, E)$.

Proof. By standard elliptic theory we have

$$\|a\|_{W^{2,2}} \leq c(\|Da\|_{L^2} + \|a\|_{L^2}).$$

Since D is non-negative, we have

$$\|Da\|_{L^2} \leq \|(D + k^2)a\|_{L^2} \quad \text{and} \quad k^2\|a\|_{L^2} \leq \|(D + k^2)a\|_{L^2}.$$

Putting everything together yields

$$\|a\|_{W^{2,2}} \leq c(1 + 1/k_0^2) \|(D + k^2)a\|_{L^2}$$

for $k \in [k_0, \infty)$. This implies that $D + k^2: W^{2,2} \rightarrow L^2$ is an injective operator with closed range. It is also surjective, since its co-kernel can be identified with the L^2 kernel of $D + k^2$ which is trivial.

We now argue as in [Kor91, Proposition 4.4]. Via the Hilbert space isomorphism $W_\delta^{s,2} \cong W^{s,2}$ defined by multiplication with w_δ the operator $D + k^2: W_\delta^{2,2} \rightarrow L_\delta^2$ is equivalent to $D_\delta + k^2: W^{2,2} \rightarrow L^2$ where $D_\delta := w_\delta D w_\delta^{-1}$. We can write D_δ as

$$D_\delta = D + \delta P_\delta$$

with $P_\delta: W^{2,2} \rightarrow L^2$ bounded independent of δ . Therefore,

$$\|((D + k^2) - (D_\delta + k^2))(D + k^2)^{-1}a\|_{L^2} \leq |\delta|c(1 + 1/k_0^2)\|a\|_{L^2}.$$

If we choose $\varepsilon = \varepsilon(k_0) > 0$ sufficiently small, then for $\delta \in (-\varepsilon, \varepsilon)$ the factor on the right-hand side is less than $\frac{1}{2}$; thus, the series

$$(D_\delta + k^2)^{-1} := (D + k^2)^{-1} \sum_{i \geq 0} [((D + k^2) - (D_\delta + k^2))(D + k^2)^{-1}]^i$$

converges and the operator norm of $(D_\delta + k^2)^{-1}$ is bounded by $2c(1 + 1/k_0^2)$. This establishes (2.79) for $\ell = 0$. For $\ell > 0$, we have

$$\partial_k^\ell (D + k^2)^{-1} = \sum_{i=0}^{\ell} \sum_{j=2}^{\ell+1} c_{i,j,\ell} \cdot k^i [(D + k^2)^{-1}]^j$$

for universal constants $c_{i,j,\ell}$. Thus (2.79) for $\ell > 0$ can be reduced to the case $\ell = 0$. $\square$

Lemma 2.76 can now be proved using an argument similar to the one used by Brendle in [Bre03b, Proposition 3.3]. This is essentially the proof of the ingredients from classical distribution theory used in the heuristic proof adapted to our infinite dimensional setting.

Proof of Lemma 2.76. We proceed in 3 steps.

Step 1. Let $\chi \in \mathcal{S}(\mathbf{R}^n)$ be a fast decaying function whose Fourier transform $\hat{\chi}$ vanishes in $B_{k_0}(0)$ and let $b \in L_\delta^2(X, E)$ for some $\delta \in (-\varepsilon, \varepsilon)$ with $\varepsilon = \varepsilon(k_0)$. Then there exists $a \in \mathcal{S}(\mathbf{R}^n, W_\delta^{2,2}(X, E))$ such that $(\Delta_{\mathbf{R}^n} + D)a = \chi b$.

We construct $a \in \mathcal{S}(\mathbf{R}^n, W_\delta^{2,2}(X, E))$ using Fourier synthesis. By assumption $\hat{\chi}(k) =$

0 for $|k| \leq k_0$. For $|k| > k_0$ set

$$\hat{a}_k := (D + |k|^2)^{-1}b.$$

and define

$$a(x, y) := \int_{\mathbf{R}^n} e^{i\langle x, k \rangle} \hat{a}_k(y) \hat{\chi}(k) \, d\mathcal{L}^n(k).$$

Here $\mathcal{L}^n$ denotes the n -dimensional Lebesgue measure on $\mathbf{R}^n$. Then

$$(\Delta_{\mathbf{R}^n} + D)a(x, y) = b\chi.$$

Moreover, one can verify that $x \mapsto \|a(x, \cdot)\|_{W_{\delta}^{2,2}}$ is in $\mathcal{S}(\mathbf{R}^n)$ using a slight variation of the proof that the Fourier transform maps fast decaying functions to fast decaying functions and the estimate $\|\partial_k^\ell \hat{a}_k\|_{W_{\delta}^{2,2}} \leq c_\ell (1 + |k|)^\ell \|b\|_{L_{\delta}^2}$.

Step 2. Let $\chi \in \mathcal{S}(\mathbf{R}^n)$ with $\hat{\chi}(0) = 0$. Then there is a family $(\chi_\varepsilon)_{\varepsilon>0}$ of fast decaying functions such that $\hat{\chi}_\varepsilon$ vanishes on $B_\varepsilon(0)$ and $\lim_{\varepsilon \rightarrow 0} \|\chi_\varepsilon - \chi\|_{L^1} = 0$.

Pick a smooth function $\rho: \mathbf{R} \rightarrow [0, 1]$ such that $\rho(k) = 0$ for $|k| \leq 1$ and $\rho(k) = 1$ for $|k| \geq 2$. Set $\hat{\chi}_\varepsilon(k) := \rho(|k|/\varepsilon)\hat{\chi}(k)$ and denote its inverse Fourier transform by χ_ε . Then χ_ε clearly satisfies the first part of the conclusion. To see that the second part also holds, note that from $\hat{\chi}(0) = 0$ it follows that

$$\|\nabla^n(\hat{\chi}_\varepsilon - \hat{\chi})\|_{L^{2n/(2n-1)}} = O\left(\varepsilon^{\frac{1}{2}}\right)$$

and therefore

$$\begin{aligned} \|\chi_\varepsilon - \chi\|_{L^1} &\leq \|(1 + |x|)^{-n}\|_{L^{2n/(2n-1)}} \cdot \|(1 + |x|)^n(\chi_\varepsilon - \chi)\|_{L^{2n}} \\ &\leq c(\|\hat{\chi}_\varepsilon - \hat{\chi}\|_{L^{2n/(2n-1)}} + \|\nabla^n(\hat{\chi}_\varepsilon - \hat{\chi})\|_{L^{2n/(2n-1)}}) = O\left(\varepsilon^{\frac{1}{2}}\right) \end{aligned}$$

where $c > 0$ is a constant depending only on n . Here we used that the inverse Fourier transform is a bounded linear map from $L^{2n/(2n-1)}$ to L^{2n} and the Fourier transform's behaviour with respect to derivatives.

Step 3. Suppose that $(\Delta_{\mathbf{R}^n} + D)a = 0$. Then for $\sigma \in \mathcal{S}^n(\mathbf{R}^n)$, $\delta \in \mathbf{R}^n$ and $b \in C_c^\infty(X, E)$ we have

$$\int_{\mathbf{R}^n} \langle a(x, \cdot), b \rangle_{L^\infty, L^1} (\sigma(x + \delta) - \sigma(x)) \, d\mathcal{L}^n(x) = 0.$$

In particular, the conclusion of the lemma holds.

Set $\chi(x) := \sigma(x + \delta) - \sigma(x)$. Then $\hat{\chi}(0) = 0$. Let χ_ε be as in Step 2. According to Step 1, for each $\varepsilon > 0$ there is some small $\delta > 0$ and $c_\varepsilon \in \mathcal{S}(\mathbf{R}^n, W_{-\delta}^{2,2}(X, E))$ such that

$(\Delta_{\mathbf{R}^n} + D)c_\varepsilon = \chi_\varepsilon b$. By the assumptions on a and since X has subexponential volume growth we have

$$\begin{aligned} \int_{\mathbf{R}^n} \langle a(x, \cdot), b \rangle \chi(x) \, d\mathcal{L}^n(x) &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n} \langle a(x, \cdot), b \rangle \chi_\varepsilon(x) \, d\mathcal{L}^n(x) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n} \int_X \langle a(x, y), (\Delta_{\mathbf{R}^n} + D)c_\varepsilon \rangle \, d\mathcal{L}^n(x) \, d\text{vol}(y) \\ &= \lim_{\varepsilon \rightarrow 0} \int_{\mathbf{R}^n} \int_X \langle (\Delta_{\mathbf{R}^n} + D)a(x, y), c_\varepsilon \rangle \, d\mathcal{L}^n(x) \, d\text{vol}(y) \\ &= 0. \end{aligned}$$

Since σ , δ and b are arbitrary, it follows that a is invariant in the $\mathbf{R}^n$ -direction. This finishes the proof. $\square$

Remark 2.80. It is clear from the proof that in Lemma 2.76 one can replace the assumptions that X has subexponential volume growth and that $\|a\|_{L^\infty}$ is finite by the assumption that $\|a(x, \cdot)\|_{L^2_\delta}$ is bounded independent of $x \in \mathbf{R}^n$ for all $\delta > 0$.

2.6 Deforming to genuine G_2 -instantons

We continue with the assumptions of Section 2.3 and we suppose that the connection $\tilde{A}_t$ on G -bundle E_t over Y_t was constructed using Proposition 2.60 from a choice of compatible gluing data $\mathbf{g}$. In this section we will prove the following result which will complete the proof of Theorem 2.1 under the additional assumption that θ is irreducible. At the end of this section, I will explain how to remove this hypothesis.

Proposition 2.81. *Suppose that θ is acyclic and that each A_j is infinitesimally rigid. Then there are constants $T' \in (0, T]$ and $c > 0$ as well as, for each $t \in (0, T')$, $\underline{a}_t = (\xi_t, a_t) \in \Omega^0(Y_t, \mathfrak{g}_{E_t}) \oplus \Omega^1(Y_t, \mathfrak{g}_{E_t})$ such that*

$$(2.82) \quad *_t \left(F_{\tilde{A}_t + a_t} \wedge \psi_t \right) + d_{\tilde{A}_t} \xi_t = 0$$

and $\|\underline{a}_t\|_{C_{-1,t}^{1,\alpha}} \leq ct^{1/2}$. Moreover, the G_2 -instanton $A_t := \tilde{A}_t + a_t$ is acyclic.

As discussed in Section 2.4 it is crucial to understand the properties of the linear operator L_t . The key to proving Proposition 2.81 is the following result.

Proposition 2.83. *Given $\beta \in (-3, 0)$ there are constants $T' \in (0, T]$ and $c > 0$ such that for $t \in (0, T')$ we have*

$$\|\underline{a}\|_{C_{\beta,t}^{1,\alpha}} \leq c \|L_t \underline{a}\|_{C_{\beta-1,t}^{0,\alpha}}.$$

Before we move on to prove this, let us quickly see how this is used to establish Proposition 2.81. Recall the following elementary consequence of Banach's fixed point theorem.

Lemma 2.84 (Donaldson–Kronheimer [DK90, Lemma 7.2.23]). *Let X be a Banach space and let $T: X \rightarrow X$ be a smooth map with $T(0) = 0$. Suppose there is a constant $c > 0$ such that*

$$\|Tx - Ty\| \leq c(\|x\| + \|y\|)\|x - y\|.$$

Then if $y \in X$ satisfies $\|y\| \leq \frac{1}{10c}$, there exists a unique $x \in X$ with $\|x\| \leq \frac{1}{5c}$ solving

$$x + Tx = y.$$

Moreover, this $x \in X$ satisfies $\|x\| \leq 2\|y\|$.

Proof of Proposition 2.81 assuming Proposition 2.83. By Proposition 2.83 the linear operator $L_t: C_{-1,t}^{1,\alpha} \rightarrow C_{-2,t}^{0,\alpha}$ is injective and has closed range. Therefore its cokernel is isomorphic to the kernel of the dual operator L_t^* . By elliptic regularity any element in the kernel of L_t^* is smooth and thus, since L_t is formally self-adjoint, an element in the kernel of L_t , which is trivial. This shows that L_t is invertible. Denote its inverse by $R_t: C_{-2,t}^{0,\alpha} \rightarrow C_{-1,t}^{1,\alpha}$.

If we set $\underline{a}_t := R_t \underline{b}_t$, then (2.82) becomes

$$(2.85) \quad \underline{b}_t + Q_t(R_t \underline{b}_t) = - *_t (F_{\tilde{A}_t} \wedge \psi_t).$$

It follows from (2.58) and Proposition 2.83 that

$$\|Q_t(R_t \underline{b}_1) - Q_t(R_t \underline{b}_2)\|_{C_{-2,t}^{0,\alpha}} \leq c \left(\|\underline{b}_1\|_{C_{-2,t}^{0,\alpha}} + \|\underline{b}_2\|_{C_{-2,t}^{0,\alpha}} \right) \|\underline{b}_1 - \underline{b}_2\|_{C_{-2,t}^{0,\alpha}}$$

with a constant $c > 0$ independent of $t \in (0, T)$. Since by Proposition 2.60

$$\|F_{\tilde{A}_t} \wedge \psi_t\|_{C_{-2,t}^{0,\alpha}} \leq ct^{1/2},$$

Lemma 2.84 provides us, for each $t \in (0, T')$, with a solution $\underline{b}_t$ of (2.85) satisfying $\|\underline{b}_t\|_{C_{-2,t}^{0,\alpha}} \leq ct^{1/2}$ provided $T' \in (0, T]$ was chosen sufficiently small. Then $\underline{a}_t = (\xi_t, a_t) = R_t \underline{b}_t \in C_{-1,t}^{1,\alpha}$ is the desired solution of (2.82) and satisfies $\|\underline{a}_t\|_{C_{-1,t}^{1,\alpha}} \leq ct^{1/2}$.

It follows from elliptic regularity that a_t and thus $A_t := \tilde{A}_t + a_t$ is smooth. To see that A_t is acyclic, i.e., L_{A_t} is surjective, note that $\|R_t L_{A_t} - \text{id}\|_{C_{-1,t}^{1,\alpha}} \leq ct^{1/2}$ and thus L_{A_t} is invertible for $t \in (0, T')$ provided $T' \in (0, T]$ was chosen sufficiently small. $\square$

Before embarking on the proof of Proposition 2.83, it will be helpful to make a few observations. On $Y_t \setminus \tilde{T}_t$ the operators L_t and L_θ agree. For fixed $\varepsilon > 0$, the norms $\|\cdot\|_{C_{\beta,t}^{k,\alpha}(r_t^{-1}[\varepsilon, \infty))}$ are uniformly equivalent to the corresponding unweighted Hölder norms. Moreover, the restriction of L_t to $r_t^{-1}[\varepsilon, \infty)$ becomes arbitrarily close to L_θ restricted to $\{x \in Y_0 : d(x, S) > \varepsilon\}$ as t goes to zero. These observations and standard Schauder estimates combined with Propositions 2.73 and 2.77 yield the following Schauder estimate.

Proposition 2.86. *Given $\beta \in \mathbf{R}$ there is a constant $c > 0$ such that for all $t \in (0, T)$ we have*

$$\|\underline{a}\|_{C_{\beta,t}^{1,\alpha}} \leq c(\|L_t \underline{a}\|_{C_{\beta-1,t}^{0,\alpha}} + \|\underline{a}\|_{L_{\beta,t}^\infty}).$$

This reduces the proof of Proposition 2.83 to the following statement.

Proposition 2.87. *Given $\beta \in (-3, 0)$ there are constants $T' \in (0, T)$ and $c > 0$ such that for all $t \in (0, T')$ the following holds*

$$\|\underline{a}\|_{L_{\beta,t}^\infty} \leq c\|L_t \underline{a}\|_{C_{\beta-1,t}^{0,\alpha}}.$$

Proof. Suppose not. Then there exists a sequence $(\underline{a}_i)$ and a null-sequence (t_i) such that

$$\|\underline{a}_i\|_{L_{\beta,t_i}^\infty} = 1 \quad \text{and} \quad \|L_{t_i} \underline{a}_i\|_{C_{\beta-1,t_i}^{0,\alpha}} \leq \frac{1}{i}.$$

Hence, by Proposition 2.86, we have

$$(2.88) \quad \|\underline{a}_i\|_{C_{\beta,t_i}^{1,\alpha}} \leq 2c.$$

Pick $x_i \in Y_{t_i}$ such that

$$w_{t_i}(x_i)^{-\beta} |\underline{a}_i(x_i)| = 1.$$

After passing to a subsequence we can assume that one of the following three cases occurs. We will rule out all of them, thus proving the proposition.

Case 1. *The sequence (x_i) accumulates on the regular part of Y_0 : $\lim r_{t_i}(x_i) > 0$.*

Let K be a compact subset of $Y_0 \setminus S$. We can view K as a subset of Y_t . As t goes to zero, the metric on K induced from the metric on Y_t converges to the metric on Y_0 , similarly we can identify $E_0|_K$ with $E_t|_K$ and via this identification $\tilde{A}_t$ converges to θ on K . By (2.88) the sequence $(\underline{a}_i|_K)$ is uniformly bounded in $C^{1,\alpha}$. We can thus extract a convergent subsequence using Arzelà–Ascoli. Using a diagonal sequence argument over a sequence of compact sets (K_i) exhausting $Y_0 \setminus S$, we can pass to a further subsequence which converges in $C_{\text{loc}}^{1,\alpha/2}$ to a limit $\underline{a} \in \Omega^0(Y_0 \setminus S, \mathfrak{g}_{E_0}) \oplus \Omega^1(Y_0 \setminus S, \mathfrak{g}_{E_0})$. This limit satisfies

$$(2.89) \quad |\underline{a}| < c \cdot d(\cdot, S)^\beta$$

as well as

$$L_\theta \underline{a} = 0.$$

Since $\beta > -3$, it follows from (2.89) that $\underline{a}$ satisfies $L_\theta \underline{a} = 0$ in the sense of distributions on all of Y_0 and, therefore, is smooth by elliptic regularity. Because θ is assumed to be acyclic, $\underline{a}$ must be zero. However, by passing to a further subsequence we can arrange that

(x_i) converges to some point $x \in Y_0 \setminus S$. At this point we have $|\underline{a}|(x) = d(x, S)^\beta \neq 0$. This is a contradiction.

Case 2. *The sequence (x_i) accumulates on one of the ALE spaces: $\lim r_{t_i}(x_i)/t_i < \infty$.*

There is no loss in assuming that each x_i lies in $\tilde{T}_{j,t_i}$ for some fixed j . With s_{β,t_i} as in (2.72) we define $\tilde{\underline{a}}_i := s_{\beta,t_i} \underline{a}_i$ and denote by $\tilde{x}_i$ a lift of x_i to $\mathbf{R}^3 \times \pi_{j,t}^{-1}(B_\zeta^4/G_j)$. This rescaled sequence satisfies, in the notation of Section 2.4,

$$\|\tilde{\underline{a}}_i\|_{C_\beta^{1,\alpha}} \leq 4c \quad \text{and} \quad (1 + |\pi_j(\tilde{x}_i)|)^{-\beta} |\tilde{\underline{a}}(\tilde{x}_i)| \geq \frac{1}{2}$$

as well as

$$(2.90) \quad \|L_{A_j} \tilde{\underline{a}}_i\|_{C_{\beta-1}^{0,\alpha}} \leq 2/i.$$

Arguing as in the previous case, we can extract a subsequence of $(\tilde{\underline{a}}_i)$ which converges to a limit $\tilde{\underline{a}} \in C_\beta^{1,\alpha/2}$ in $C_{\text{loc}}^{1,\alpha}$ on $\mathbf{R}^3 \times X_j$. It follows from (2.90) that $\tilde{\underline{a}}$ satisfies

$$L_{A_j} \tilde{\underline{a}} = 0.$$

By Proposition 2.74, $\tilde{\underline{a}}$ must be zero since $\beta \in (-3, 0)$ and A_j is infinitesimally rigid. However, by translation we can arrange that the $\mathbf{R}^3$ -component of $\tilde{x}_i$ is zero and thus we can view $\tilde{x}_i$ as a point in X_j . Then the condition $\lim r_{t_i}(x_i)/t_i < \infty$ translates to $\lim |\pi_j(\tilde{x}_i)| < \infty$. Therefore, we can assume without loss of generality that $\tilde{x}_i$ converges to some point $\tilde{x} \in X_j$. But then $|\tilde{\underline{a}}(\tilde{x})| \geq \frac{1}{2}(1 + |\pi_j(\tilde{x})|)^\beta > 0$, which contradicts $\tilde{\underline{a}} = 0$.

Case 3. *The sequence (x_i) accumulates on one of the necks: $\lim r_{t_i}(x_i) = 0$ and at the same time $\lim r_{t_i}(x_i)/t_i = \infty$.*

As in the previous case, we rescale to obtain $(\tilde{\underline{a}}_i)$ and $(\tilde{x}_i)$, and we arrange it so that the $\mathbf{R}^3$ -component of $\tilde{x}_i$ is zero. Since $\lim r_{t_i}(x_i)/t_i = \infty$, we have $\lim |\pi_j(\tilde{x}_i)| = \infty$. Fix a sequence (R_i) tending to infinity such that $\varepsilon_i := R_i/|\pi_j(\tilde{x}_i)|$ goes to zero. Using $\pi_j: X \rightarrow \mathbf{C}^2/G$, we can think of the sets $\mathbf{R}^3 \times (\mathbf{C}^2 \setminus B_{R_i}^4)/G_j$ as subsets of $\mathbf{R}^3 \times X_j$. Restricting to these sets and rescaling everything by $1/|\pi_j(\tilde{x}_i)|$ we obtain, without changing notation, $\tilde{\underline{a}}_i \in \Omega^0(\mathbf{R}^3 \times (\mathbf{C}^2 \setminus B_{\varepsilon_i}^4)/G_j) \oplus \Omega^1(\mathbf{R}^3 \times (\mathbf{C}^2 \setminus B_{\varepsilon_i}^4)/G_j)$ and $\tilde{x}_i \in \mathbf{C}^2 \setminus B_{\varepsilon_i}^4$ satisfying

$$\|\tilde{\underline{a}}_i\|_{C_\beta^{1,\alpha}} \leq 8c \quad \text{and} \quad |\tilde{x}_i|^{-\beta} |\tilde{\underline{a}}_i(\tilde{x}_i)| \geq \frac{1}{4}$$

as well as

$$\|L \tilde{\underline{a}}_i\|_{C_{\beta-1}^{0,\alpha}} \leq 4/i.$$

Here the norms $\|\cdot\|_{C_\beta^{k,\alpha}}$ are defined like those in Section 2.4 except with the weight function now defined by $w(x, y) := |y|$ for $(x, y) \in \mathbf{R}^3 \times \mathbf{C}^2/G_j$. The operator L is defined by

$$L(\xi, a) := (d^*a, d\xi + *(\psi_0 \wedge da))$$

with $\psi_0 := \frac{1}{2}\omega_1 \wedge \omega_1 - \delta^2 \wedge \delta^3 \wedge \omega_1 - \delta^3 \wedge \delta^1 \wedge \omega_2 - \delta^1 \wedge \delta^2 \wedge \omega_3$ and $\omega_i \in \Omega^2(\mathbf{C}^2)$ as in Section 2.1.

As before, we can extract a subsequence converging in $C_{\text{loc}}^{1,\alpha/2}$ to a limit $\tilde{a}$ defined over $\mathbf{R}^3 \times (\mathbf{C}^2 \setminus \{0\})/G_j$ satisfying

$$(2.91) \quad |\tilde{a}| < cw^\beta$$

as well as

$$L\tilde{a} = 0.$$

Since $\beta > -3$, it follows from (2.91) that $\tilde{a}$ satisfies $L\tilde{a} = 0$ in the sense of distributions on all of $\mathbf{R}^3 \times \mathbf{C}^2/G_j$ and therefore $\tilde{a}$ is smooth by elliptic regularity. It also follows from (2.91) that $\tilde{a}$ is uniformly bounded: This is clear outside a tubular neighbourhood of $\mathbf{R}^3 \times \{0\}$. If B_1 is a ball of radius one centred at some point in $\mathbf{R}^3 \times \{0\}$, then (2.91) gives a uniform bound on $\|\tilde{a}\|_{L^p(B_1)}$, for some fixed $p \in (1, \infty)$. Using elliptic estimates this yields a uniform $W^{k,p}$ estimate on the ball of radius one-half; hence, using Sobolev embedding, uniform bounds on $\tilde{a}$. Because $L^*L = \Delta_{\mathbf{R}^3} + \Delta_{\mathbf{C}^2}$, it follows from Lemma 2.76 that $\tilde{a}$ is invariant under translations in the $\mathbf{R}^3$ -direction. Thus we can think of the components of $\tilde{a}$ as harmonic functions on $\mathbf{C}^2$. Since $\beta < 0$, they decay to zero at infinity and thus vanish identically. However, we know that $|\tilde{x}_i| = 1$ and thus a subsequence of $(\tilde{x}_i)$ converges to a point $\tilde{x} \in \mathbf{C}^2/G_j$ with $|\tilde{x}| = 1$ at which $|\tilde{a}|(\tilde{x}) \geq \frac{1}{4}$, contradicting $\tilde{a} = 0$. $\square$

Let me now explain how to remove the hypothesis that θ is irreducible. From the above proof we see that the main issue in that situation is that L_θ is not invertible. The operator $\tilde{L}_\theta: \Omega^0(Y_0, \mathfrak{g}_{E_0}) \oplus \Omega^1(Y_0, \mathfrak{g}_{E_0}) \oplus \mathcal{H}_\theta^0 \rightarrow \Omega^0(Y_0, \mathfrak{g}_{E_0}) \oplus \Omega^1(Y_0, \mathfrak{g}_{E_0}) \oplus \mathcal{H}_\theta^0$ defined by

$$(2.92) \quad \tilde{L}_\theta(a, \eta) := (L_\theta a + \eta, \pi a),$$

however, is invertible. Here π denotes the L^2 projection onto $\mathcal{H}_\theta^0$. Now, instead of (2.67) we solve

$$\tilde{L}_t(a_t, \eta_t) + Q_t(a_t) + *_t(F_{\tilde{A}_t} \wedge \psi_t) = 0$$

where

$$\tilde{L}_t(a, \eta) := (L_t a + \iota_t \eta, \pi_t a),$$

with $\iota_t: \mathcal{H}_\theta^0 \rightarrow \Omega^0(Y_t, \mathfrak{g}_{E_t})$ and $\pi_t: \Omega^0(Y_t, \mathfrak{g}_{E_t}) \rightarrow \mathcal{H}_\theta^0$ being appropriately cut-off versions of the inclusion of and L^2 projection to $\mathcal{H}_\theta^0$. Then $A_t := \tilde{A}_t + a_t$ still solves (2.82), i.e., is a

G_2 -instanton; however, it is not necessarily in Coulomb gauge with respect to $\tilde{A}_t$. Moreover, A_t is still unobstructed, but not necessarily irreducible.

2.7 Examples with $G = \mathrm{SO}(3)$

I will now explain how to use Theorem 2.1 to construct a few concrete examples of G_2 -instantons on the G_2 -manifolds from [Joy00, Section 12.3 and 12.4]. The flat G_2 -structure ϕ_0 on T^7 defined by

$$(2.93) \quad \phi_0 := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

is preserved by $\alpha, \beta, \gamma \in \mathrm{Diff}(T^7)$ defined by

$$\begin{aligned} \alpha(x_1, \dots, x_7) &:= (x_1, x_2, x_3, -x_4, -x_5, -x_6, -x_7), \\ \beta(x_1, \dots, x_7) &:= (x_1, -x_2, -x_3, x_4, x_5, \tfrac{1}{2} - x_6, -x_7) \quad \text{and} \\ \gamma(x_1, \dots, x_7) &:= (-x_1, x_2, -x_3, x_4, -x_5, x_6, \tfrac{1}{2} - x_7). \end{aligned}$$

It is easy to see that $\Gamma := \langle \alpha, \beta, \gamma \rangle \cong \mathbf{Z}_2^3$.

To understand the singular set S of T^7/Γ note that the only elements of Γ having fixed points are α, β and γ . The fixed point set of each of these elements consists of 16 copies of T^3 . The group $\langle \beta, \gamma \rangle$ acts freely on the set of T^3 fixed by α and $\langle \alpha, \gamma \rangle$ acts freely on the set of T^3 fixed by β , while $\alpha\beta \in \langle \alpha, \beta \rangle$ acts trivially on the set of T^3 fixed by γ . It follows that S consists of 8 copies of T^3 coming from the fixed points of α and β and 8 copies of $T^3/\mathbf{Z}_2$. Near the copies of T^3 the singular set is modelled on $T^3 \times \mathbf{C}^2/\mathbf{Z}_2$ while near the copies of $T^3/\mathbf{Z}_2$ it is modelled on $(T^3 \times \mathbf{C}^2/\mathbf{Z}_2)/\mathbf{Z}_2$ where the action of $\mathbf{Z}_2$ on $T^3 \times \mathbf{C}^2/\mathbf{Z}_2$ is given by

$$(x_1, x_2, x_3, \pm(z_1, z_2)) \mapsto (x_1, x_2, x_3 + \tfrac{1}{2}, \pm(z_1, -z_2)).$$

The 8 copies of T^3 can be desingularised by any choice of 8 ALE spaces asymptotic to $\mathbf{C}^2/\mathbf{Z}_2$. To desingularise the copies of $T^3/\mathbf{Z}_2$ we need to choose ALE spaces which admit an isometric action of $\mathbf{Z}_2$ asymptotic to the action of $\mathbf{Z}_2$ on $\mathbf{C}^2/\mathbf{Z}_2$ given by $\pm(z_1, z_2) \mapsto \pm(z_1, -z_2)$. Two possible choices are the resolution of $\mathbf{C}^2/\mathbf{Z}_2$ or a smoothing of $\mathbf{C}^2/\mathbf{Z}_2$. See [Joy00, pp. 313–314] for details.

We construct our examples on desingularisations of quotients of T^7/Γ . To this end we define $\sigma_1, \sigma_2, \sigma_3 \in \mathrm{Diff}(T^7)$ by

$$\begin{aligned} \sigma_1(x_1, \dots, x_7) &:= (x_1, x_2, \tfrac{1}{2} + x_3, \tfrac{1}{2} + x_4, \tfrac{1}{2} + x_5, x_6, x_7), \\ \sigma_2(x_1, \dots, x_7) &:= (x_1, \tfrac{1}{2} + x_2, x_3, \tfrac{1}{2} + x_4, x_5, x_6, x_7) \quad \text{and} \\ \sigma_3(x_1, \dots, x_7) &:= (\tfrac{1}{2} + x_1, x_2, x_3, x_4, \tfrac{1}{2} + x_5, \tfrac{1}{2} + x_6, x_7). \end{aligned}$$

The elements σ_j commute with all elements of Γ and thus act on T^7/Γ . Moreover, this action is free.

Example 2.94. Let $A := \langle \sigma_2, \sigma_3 \rangle$. By analysing how A acts on the singular set of T^7/Γ one can see that the singular set of $Y_0 := T^7/(\Gamma \times A)$ consists of one copy of T^3 , denoted by S_1 , and 6 copies of $T^3/\mathbf{Z}_2$, denoted by $S_2, \dots, S_7$. S_1 has a neighbourhood modelled on $T^3 \times \mathbf{C}^2/\mathbf{Z}_2$, while $S_2, \dots, S_6$ have neighbourhoods modelled on $(T^3 \times \mathbf{C}^2/\mathbf{Z}_2)/\mathbf{Z}_2$ where $\mathbf{Z}_2$ acts by $\pm(z_1, z_2) \mapsto \pm(z_1, -z_2)$ on $\mathbf{C}^2/\mathbf{Z}_2$. As before, S_1 can be desingularised by any choice of an ALE space asymptotic to $\mathbf{C}^2/\mathbf{Z}_2$. $S_2, \dots, S_6$ can be desingularised by the resolution of $\mathbf{C}^2/\mathbf{Z}_2$ or a smoothing of $\mathbf{C}^2/\mathbf{Z}_2$.

To compute the orbifold fundamental group $\pi_1(Y_0)$, note that it is isomorphic to the fundamental group $\pi_1(Y_0 \setminus S)$ of the regular part of Y_0 . Denote by $p: \mathbf{R}^7 \rightarrow Y_0$ the canonical projection. Then $p: p^{-1}(Y_0 \setminus S) \rightarrow Y_0 \setminus S$ is a universal cover. Up to conjugation we can therefore identify $\pi_1(Y_0)$ with the group of deck transformations

$$\pi_1(Y_0) = \langle \alpha, \beta, \gamma, \sigma_2, \sigma_3, \tau_1, \dots, \tau_7 \rangle \subset \text{Aff}(7) = \text{GL}(7) \ltimes \mathbf{R}^7.$$

Here we think of $\alpha, \beta, \gamma, \sigma_2, \sigma_3$ as elements of $\text{Aff}(7)$ defined by the formulae above and τ_i translates the i -th coordinate of $\mathbf{R}^7$ by one. The group $\pi_1(Y_0)$ is a non-split extension

$$0 \rightarrow \mathbf{Z}^7 \rightarrow \pi_1(Y_0) \rightarrow \Gamma \times A \rightarrow 0.$$

To work out the orbifold fundamental group $\pi_1(T_j)$ of T_j , again up to conjugation, one simply has to understand the subgroup of deck transformations preserving a fixed component of $p^{-1}(T_j) \subset p^{-1}(Y_0 \setminus S)$. In this way one can compute

$$\begin{aligned} \pi_1(T_1) &= \langle \alpha, \tau_1, \tau_2, \tau_3 \rangle, \\ \pi_1(T_2) &= \langle \beta, \sigma_3 \alpha, \tau_1, \tau_4, \tau_5 \rangle, & \pi_1(T_3) &= \langle \tau_3 \beta, \sigma_3 \alpha, \tau_1, \tau_4, \tau_5 \rangle, \\ \pi_1(T_4) &= \langle \gamma, \alpha \beta, \sigma_2, \tau_4, \tau_6 \rangle, & \pi_1(T_5) &= \langle \tau_3 \gamma, \tau_3 \alpha \beta, \sigma_2, \tau_4, \tau_6 \rangle, \\ \pi_1(T_6) &= \langle \tau_1 \tau_5 \gamma, \tau_5 \alpha \beta, \sigma_2, \tau_4, \tau_6 \rangle, & \pi_1(T_7) &= \langle \tau_1 \tau_3 \tau_5 \gamma, \tau_3 \tau_5 \alpha \beta, \sigma_2, \tau_4, \tau_6 \rangle. \end{aligned}$$

Here τ_2 does not appear explicitly in $\pi_1(T_j)$, for $j = 4, \dots, 7$, because $\sigma_2^2 = \tau_2 \tau_4$.

Denote by $V := \langle a, b, c \mid a^2 = b^2 = c^2 = 1, ab = c \rangle \cong \mathbf{Z}_2^2$ the Klein four-group. V can be thought of as a subgroup of $\text{SO}(3)$: $a = \text{diag}(1, -1, -1)$, $b = \text{diag}(-1, 1, -1)$ and $c = \text{diag}(-1, -1, 1)$. We define $\rho: \pi_1(Y_0) \rightarrow V \subset \text{SO}(3)$ by

$$\begin{aligned} \beta, \gamma, \tau_1, \dots, \tau_7 &\mapsto 1, \\ \alpha &\mapsto a, \quad \sigma_2 \mapsto a \quad \text{and} \quad \sigma_3 \mapsto b. \end{aligned}$$

To see that the flat connection θ induced by ρ is acyclic we use the following observation.

Proposition 2.95. *A flat connection θ on a G -bundle E_0 over a flat G_2 -orbifold Y_0 corresponding to a representation $\rho: \pi_1(Y_0) \rightarrow G$ is irreducible (resp. unobstructed) if and only if the induced representation of $\pi_1(Y_0)$ on $\mathfrak{g}$ (resp. $\mathbf{R}^7 \otimes \mathfrak{g}$) has no non-zero fixed vectors.*

Proof. Since Y_0 is flat as a Riemannian orbifold and θ is a flat connection

$$L_\theta^* L_\theta = \nabla_\theta^* \nabla_\theta.$$

Therefore, all elements in the kernel of L_θ are actually parallel sections of the bundle $\mathfrak{g}_{E_0} \oplus (T^*Y_0 \otimes \mathfrak{g}_{E_0})$ and these are in one-to-one correspondence with fixed vectors of the representation of $\pi_1(Y_0)$ on $\mathfrak{g} \oplus (\mathbf{R}^7 \otimes \mathfrak{g})$. $\square$

The elements σ_2 and σ_3 act trivially on $\mathbf{R}^7$ and their action on $\mathfrak{so}(3)$ has no common non-zero fixed vectors. Therefore the action of $\pi_1(Y_0)$ on $\mathfrak{g} \oplus (\mathbf{R}^7 \otimes \mathfrak{g})$ has no non-zero fixed vector and thus θ is acyclic.

The monodromy representation $\mu_j|_{G_j}: G_j = \mathbf{Z}_2 \rightarrow \mathrm{SO}(3)$ associated with the flat connection θ is non-trivial only for $j = 1$. Let $A_1 := A_{0,1}$ be the infinitesimally rigid ASD instanton on $E_1 := E_{0,1}$ given in Proposition 2.53. For $j = 2, \dots, 6$ we choose A_j to be the product connection on the trivial $\mathrm{SO}(3)$ -bundle E_j . We take m_1 to be trivial. For $j = 2, \dots, 6$ we can choose m_j accordingly to satisfy the compatibility conditions. Thus we obtain examples of G_2 -instantons on each of the desingularisations of Y_0 by appealing to Theorem 2.1.

Note that any choice of resolution data for $T^7/(\Gamma \times A)$ lifts to an A -invariant choice of resolution data for T^7/Γ . We can then carry out Joyce's generalised Kummer construction in a A -invariant way and lift up the G_2 -instanton constructed above. However, we could not have constructed this G_2 -instanton directly using Theorem 2.1, since the lift of θ to T^7/Γ is not acyclic.

Example 2.96. Here is a more complicated example. Let $Y_0 := T^7/(\Gamma \times A)$ be as before. Define $\rho: \pi_1(Y_0) \rightarrow V \subset \mathrm{SO}(3)$ by

$$\begin{aligned} \gamma, \tau_1, \dots, \tau_7 &\mapsto 1, \\ \alpha &\mapsto a, \quad \beta \mapsto b, \quad \sigma_2 \mapsto b \quad \text{and} \quad \sigma_3 \mapsto a. \end{aligned}$$

Again, the resulting flat connection θ is acyclic. For $j = 1, 2, 3$ let $A_j := A_{0,1}$ be the rigid ASD instanton on $E_j := E_{0,1}$. By adapting the framings of E_2 and E_3 , we can arrange that A_2 and A_3 are asymptotic at infinity to the flat connection with monodromy given by $b \in V$. For $j = 4, \dots, 7$ let A_j be the product connection on the trivial bundle E_j . To be able to extend this to compatible gluing data we need a lift of the action of $\mathbf{Z}_2$ on X_j to E_j preserving A_j and acting trivially on the framing at infinity for $j = 2, 3$. If X_j is a smoothing of $\mathbf{C}^2/\mathbf{Z}_2$, then the $\mathbf{Z}_2$ action on X_j does lift to E_j preserving A_j . However,

the action does not lift if X_j is the resolution of $\mathbf{C}^2/\mathbf{Z}_2$. The reason for this is that in the first case the action of $\mathbf{Z}_2$ on $H^2(X, \mathbf{R})$ is given by the identity, while in the second case it acts via multiplication by -1 , see [Joy00, pp. 313–314]. Thus we can only find compatible gluing data if we resolve both S_2 and S_3 using a smoothing of $\mathbf{C}^2/\mathbf{Z}_2$.

Here is a small modification of this example. Define $\rho: \pi_1(Y_0) \rightarrow V \subset \mathrm{SO}(3)$ by

$$\begin{aligned} \gamma, \tau_1, \dots, \tau_7 &\mapsto 1, \\ \alpha &\mapsto a, \quad \beta \mapsto b, \quad \sigma_2 \mapsto b \quad \text{and} \quad \sigma_3 \mapsto c. \end{aligned}$$

To find compatible gluing data, one simply has to compose the lift of the action of $\mathbf{Z}_2$ as above with multiplication by $b \in \mathcal{G}(E_j)$, for $j = 2, 3$.

Example 2.97. Let $B := \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ and $Y_0 := T^7/(\Gamma \times B)$. Then the singular set of Y_0 consists of 4 copies of $T^3/\mathbf{Z}_2$, denoted by $S_1, \dots, S_4$, each of which has a neighbourhood modelled on $(T^3 \times \mathbf{C}^2/\mathbf{Z}_2)/\mathbf{Z}_2$ where $\mathbf{Z}_2$ acts on $\mathbf{C}^2/\mathbf{Z}_2$ by $\pm(z_1, z_2) \mapsto \pm(z_1, -z_2)$. The orbifold fundamental group $\pi_1(Y_0)$ is given by

$$\pi_1(Y_0) = \langle \alpha, \beta, \gamma, \sigma_1, \sigma_2, \sigma_3, \tau_1, \dots, \tau_7 \rangle \subset \mathrm{Aff}(7).$$

Up to conjugation the fundamental groups of the neighbourhoods T_j of S_j are given by

$$\begin{aligned} \pi_1(T_1) &= \langle \alpha, \tau_4^{-1} \tau_5^{-1} \beta \sigma_1 \sigma_2 \sigma_3, \tau_1, \tau_2, \tau_3 \rangle, & \pi_1(T_2) &= \langle \beta, \sigma_3 \alpha, \tau_1, \tau_4, \tau_5 \rangle, \\ \pi_1(T_3) &= \langle \gamma, \alpha \beta, \sigma_2, \tau_4, \tau_6 \rangle, & \pi_1(T_4) &= \langle \tau_3 \gamma, \tau_3 \alpha \beta, \sigma_2, \tau_4, \tau_6 \rangle. \end{aligned}$$

Define $\rho: \pi_1(Y_0) \rightarrow V \subset \mathrm{SO}(3)$ by

$$\begin{aligned} \alpha, \beta, \sigma_3, \tau_1, \dots, \tau_7 &\mapsto 1, \\ \sigma_1 &\mapsto a, \quad \sigma_2 \mapsto b \quad \text{and} \quad \gamma \mapsto b. \end{aligned}$$

The induced flat connection θ is clearly acyclic. We choose A_1 and A_2 to be trivial and $A_3 = A_4 := A_{0,1}$, as in the previous example. As before, for $j = 3, 4$, we require S_j to be desingularised using a resolution of $\mathbf{C}^2/\mathbf{Z}_2$ in order to be able to find a lift of the $\mathbf{Z}_2$ action on X_j . Also note that, for $j = 3, 4$ we can choose m_j to satisfy the compatibility conditions, since $b \in V$ lies in $\mathcal{G}(E_j)$ and preserves A_j .

Again, the resulting G_2 -instanton can be lifted to appropriate σ_1 -invariant desingularisations of $T^7/(\Gamma \times A)$; however we could not have constructed the lifted G_2 -instanton directly, since the lift of θ to $T^7/(\Gamma \times A)$ it is not acyclic.

This list of examples is not exhaustive. The reader will have no difficulty to find more examples, e.g., by modifying the ones given above.

2.8 Proof of Theorem 2.4

The key to proving Theorem 2.4 is the following observation.

Proposition 2.98. *Let X denote an ALE space asymptotic to $\mathbf{C}^2/\mathbf{Z}_2$, let E be a $\mathrm{SO}(3)$ -bundle with non-trivial $w_2(E)$ and let $D \subset X$ be a discrete subset. Then every connection A on $E|_{X \setminus D}$ satisfies*

$$\mathrm{YM}(A) = \int_{X \setminus D} |F_A|^2 \geq 4\pi^2.$$

Equality is achieved if and only if A is gauge equivalent to the rigid ASD instanton $A_{0,1}$ from Proposition 2.53.

The proof requires some knowledge of Sedlacek's work on weak compactness in 4-dimensional gauge theory [Sed82]. For the reader's benefit I will briefly recall his main results.

Theorem 2.99 (Sedlacek [Sed82, Theorem 3.1]). *Let M be a Riemannian 4-manifold and let E be a G -bundle over M where G is a compact Lie group together with a fixed embedding $G \subset U(n) \subset \mathbf{C}^{n \times n}$. Suppose that (A_i) is a sequence of connections on E whose energy is uniformly bounded, i.e., $\mathrm{YM}(A_i) = \|F_{A_i}\|_{L^2} \leq c$ for some fixed finite constant c . Then, after passing to a subsequence, there exists a finite subset $D \subset M$ and a cover $\{U_\alpha\}$ of $M \setminus D$ consisting of small geodesic balls over which E is trivialised, such that on each U_α the sequence $(A_i|_{U_\alpha})$ converges up to gauge transformations to a connection $A_\alpha \in W^{1,2}(U_\alpha, T^*U_\alpha \otimes \mathfrak{g}_E)$ weakly in $W^{1,2}$ and on $U_{\alpha\beta} := U_\alpha \cap U_\beta$ the connections A_α and A_β are related by a transition function $g_{\alpha\beta} \in W^{2,2}(U_{\alpha\beta}, G)$.*

Sedlacek derived this result in a rather straight-forward manner from the work of Uhlenbeck [Uhl82a]. A proof of the same result using a different method can be found in Kessel's PhD thesis [Kes08, Chapter 3].

Definition 2.100. In the situation of Theorem 2.99 we say that the sequence (A_i) converges weakly to the $W^{1,2}$ connection $A = (A_\alpha)$ on the $W^{2,2}$ G -bundle $E = (g_{\alpha\beta})$.

In the case $G = \mathrm{SO}(3)$ the main result of [Sed82] can be stated as follows.

Theorem 2.101 (Sedlacek). *Let M be a Riemannian 4-manifold. Then for each $w \in H^2(M, \mathbf{Z}_2)$ the infimum*

$$m(w) := \inf\{\mathrm{YM}(A) : A \text{ is a connection on a } \mathrm{SO}(3)\text{-bundle } E \text{ with } w_2(E) = w\}$$

is attained.

Proof Sketch. Let (A_i) sequence of connections on $\mathrm{SO}(3)$ -bundles E_i with $w_2(E_i) = w$ and denote by $\tilde{A} = (\tilde{A}_\alpha)$ a limiting $W^{1,2}$ connection on a $W^{2,2}$ $\mathrm{SO}(3)$ -bundle $\tilde{E} = (\tilde{g}_{\alpha\beta})$.

Then each $\tilde{A}_\alpha$ is weakly Yang–Mills [Sed82, Theorems 4.1]; hence, each $\tilde{A}_\alpha$ and thus also each $\tilde{g}_{\alpha\beta}$ is smooth [Sed82, Proposition 4.2]. By Uhlenbeck’s removable singularities theorem [Uhl82b, Theorem 4.1] $\tilde{A}$ extends to a smooth connection A on a smooth bundle E over M . By lower semi-continuity of the Yang–Mills functional under weak convergence, $\text{YM}(A) \leq m(w)$. According to [Sed82, Theorem 5.5] the second Stiefel–Whitney class is preserved in this limit procedure, i.e., $w_2(E) = w$ and thus $\text{YM}(A) = m(w)$. $\square$

Proof of Proposition 2.98. We already know that the asserted minimum is achieved for $A = A_{0,1}$. Now, let A be any connection on a $\text{SO}(3)$ –bundle E with non-trivial $w_2(E)$. By Theorem 2.101 we may assume that the energy of A realises the infimum $m(w)$ where $w \in H^2(X, \mathbf{Z}_2)$ is the unique non-trivial element. By Proposition 2.36 it follows that in some framing A is asymptotic to a flat connection θ over $(\mathbf{C}^2 \setminus \{0\})/\Gamma$. By Theorem 2.47 the L^2 index of δ_A is given by

$$\text{index } \delta_A = -2 \int_X p_1(A) + \frac{\chi_{\mathfrak{so}(3)}(-1) - 3}{4}$$

Here $\chi_{\mathfrak{so}(3)}$ is the character of the representation of $\mathbf{Z}_2$ on $\mathfrak{so}(3)$ associated with θ . Depending on whether θ is trivial or not the second summand, which we denote by ε in the following, is either 0 or -1 . Since δ_A is quaternionic linear, the index of δ_A is divisible by four. We can write the energy of A as

$$\begin{aligned} \text{YM}(A) &= 2 \int_X |F_A^\pm|^2 \mp 8\pi^2 \int_X p_1(A) \\ &= 2 \int_X |F_A^\pm|^2 \pm 4\pi^2 (\text{index } \delta_A - \varepsilon). \end{aligned}$$

If $\text{index } \delta_A > 0$, then $\text{index } \delta_A \geq 4$ and thus $\text{YM}(A) \geq 16\pi^2$. If $\text{index } \delta_A < 0$, then $\text{index } \delta_A \leq -4$ and thus $\text{YM}(A) \geq 12\pi^2$. In both cases A cannot be a minimiser for the energy; therefore, $\text{index } \delta_A = 0$. If $\varepsilon = 0$, then $\int_X p_1(A) = 0$ and θ is trivial; however, this is impossible: The whole situation is conformally invariant and we may thus think of X as having a cylindrical end. Now, we can double X to obtain a compact manifold W and construct a $\text{SO}(3)$ –bundle F over this manifold which coincides with E over one half of W and is trivial over the second half of W . This bundle will have $p_1(F) = 0$ and $w_2(F)^2 = 2 \bmod 4$ which contradicts the identity $p_1(F) = w_2(F)^2 \bmod 4$, see [DK90, Equation (2.1.36)]. We therefore have $\varepsilon = -1$ which implies that $\text{YM}(A) \geq 4\pi^2$. Moreover, it follows that θ has precisely one non-trivial parallel section, A is anti-self-dual and, hence, $\dim \ker \delta_A = \text{index } \delta_A = 0$ and δ_A is invertible, say, from $W^{1,2}$ to L^2 . The fact that δ_A is invertible can be used to extend the non-trivial parallel section of θ to a non-trivial parallel section of E . Therefore E is of the form $\mathbf{R} \oplus \mathcal{L}$ and A comes from a $U(1)$ ASD instanton on $\mathcal{L}$; hence, it must be gauge equivalent as a $\text{SO}(3)$ ASD instanton to $A_{0,1}$. $\square$

The above combined with a contradiction argument based on Sedlacek's work yields the following.

Proposition 2.102. *In the situation of Proposition 2.98 suppose that (U_i) is an exhaustion of X by increasing compact sets. Then for each $\varepsilon > 0$, there is an $i_0 > 0$ such that for $i \geq i_0$ the following holds: If A is a smooth connection on $E|_{U_i}$, then*

$$\text{YM}(A) = \int_{U_i} |F_A|^2 \geq 4\pi^2 - \varepsilon.$$

Now, as in the hypothesis of Theorem 2.4, let Y_0 be an admissible G_2 -orbifold all of whose singularities S_j are modelled on $(T^3 \times \mathbb{C}^2/\mathbb{Z}_2)/H_j$ and let Y be compact 7-manifold and $(\phi_t)_{t \in (0,T)}$ be a family of torsion-free G_2 -structures on Y obtained via Joyce's generalised Kummer construction from resolution data for Y_0 . Suppose that E is a $\text{SO}(3)$ -bundle over Y with the property that

$$(2.103) \quad p_1(E) = - \sum_j \frac{\varepsilon_j}{2} \text{PD}[S_j]$$

where $\varepsilon_j = 1$ if $H_2(X_j)^{H_j} \subset H_2(Y)$ is non-trivial and $w_2(E)$ pairs non-trivially with $H_2(X_j)^{H_j}$ and $\varepsilon_j = 0$ otherwise. Suppose that $(A_t)_{t \in (0,T)}$ is a family of connections on E such that A_t is a G_2 -instanton over (Y, ϕ_t) .

By Proposition 1.97, (2.103) and Theorem 2.20 we have

$$\text{YM}(A_t) = 4\pi^2 \sum_j \varepsilon_j \text{vol}(S_j) + O(t^{1/2}).$$

Moreover, it follows from Proposition 2.102 that

$$\text{YM}\left(A_t|_{r_t^{-1}[0, \delta\zeta]}\right) \geq 4\pi^2 \sum_j \varepsilon_j \text{vol}(S_j) - o(1)$$

with r_t as in (2.17) and $\delta \in (0, 1/4]$ fixed, that is, the energy of A_t is mostly concentrated inside the resolution locus. Consequently, the energy outside the resolution locus goes to zero

$$(2.104) \quad \text{YM}\left(A_t|_{r_t^{-1}(\delta\zeta, \infty)}\right) = o(1).$$

Theorem 2.105 (Nakajima [Nak88, Lemma 3.1]). *Let M be a Riemannian n -manifold and let E be a G -bundle over M . Then there are constants $c, r_0, \varepsilon_0 > 0$, such that the following holds. Let A be a Yang–Mills connection on E . If $x \in M$ and $0 < r \leq r_0$ are such that*

$$(2.106) \quad \varepsilon := r^{4-n} \int_{B_r(x)} |F_A|^2 \, \text{dvol} \leq \varepsilon_0,$$

then

$$\sup_{y \in B_{\frac{r}{4}}(x)} |F_A|^2(y) \leq c\varepsilon r^{-4}.$$

Remark 2.107. This result is one of the cornerstones of compactness theory for the Yang–Mills equation in higher dimensions. Its proof heavily relies on Price’s monotonicity formula [Pri83] which says that the quantity $\varepsilon = \varepsilon(r)$ defined in (2.106) is (essentially) monotonically increasing in r .

It follows from (2.104) and Theorem 2.105 that on every fixed closed ball in $Y_0 \setminus S$ the L^∞ -norm of F_{A_t} will be arbitrarily small provided $t > 0$ is sufficiently small; hence, we can put A_t into Uhlenbeck gauge [Uhl82a, Theorem 1.3]. In this gauge the G_2 -instanton equation becomes elliptic and, using standard elliptic theory, one can show that on every closed ball in $Y_0 \setminus S$ there exist gauge transformations $(\tilde{g}_t)$ such that $(\tilde{g}_t^* A_t)$ converges to a flat connection on this ball. As in [DK90, Section 4.4.2] these local gauge transformations can be patched to yield global gauge transformations (g_t) such that $(g_t^* A_t)$ converges to a flat connection θ in $C_{\text{loc}}^\infty(Y_0 \setminus S)$.

From Proposition 2.55 it follows that θ extends to an orbifold flat connection on Y_0 . This completes the proof of Theorem 2.4. $\square$

Chapter 3

G_2 –instantons on twisted connected sums

In this chapter I will describe how to construct G_2 –instantons on G_2 –manifolds arising from Kovalev’s twisted connected sum construction [Kov03]. This project was begun by Sá Earp in his PhD thesis [SE09], see also [SE11a]. An extended version of this chapter will appear in a joint article by Sá Earp and myself [SEW13].

To begin with I will recall the idea behind Kovalev’s twisted sum construction using the terminology introduced by Corti–Haskins–Nordström–Pacini [CHNP12b]: A *building block* (Z, S) consists of a compact 3–fold Z and an anti-canonical $K3$ surface $S \subset Z$ with trivial normal bundle satisfying some additional properties specified in Definition 3.13. Given a hyperkähler structure $(\omega_I, \omega_J, \omega_K)$ on S such that $[\omega_I]$ is the restriction of a Kähler class on Z , one can give $V := Z \setminus S$ the structure of an asymptotically cylindrical (ACyl) Calabi–Yau 3–fold with asymptotic cross section $(S, \omega_I, \omega_J, \omega_K)$, see [HHN12]. Then $Y := S^1 \times V$ is an ACyl G_2 –manifold with asymptotic cross section $T^2 \times S$. Given a pair of building blocks $(Z_\pm, S_\pm)$ and hyperkähler structures $(\omega_{I,\pm}, \omega_{J,\pm}, \omega_{K,\pm})$ on $S_\pm$ as above, this yields a pair of ACyl G_2 –manifolds $Y_\pm$. Gluing Y_+ and Y_- in the naïve way can at most produce the product of a compact Calabi–Yau 3–fold with S^1 ; however, it was pointed out by Donaldson that if one interchanges the S^1 factors in $T^2 = S^1 \times S^1$, the resulting manifold can have finite fundamental group and thus carry a torsion-free G_2 –structure with full holonomy G_2 , see Proposition 1.19. In order to be able to match the G_2 –structures on $Y_\pm$ one requires a *hyperkähler rotation* $\tau: S_+ \rightarrow S_-$. The collection $\{(\omega_{I,\pm}, \omega_{J,\pm}, \omega_{K,\pm}), \tau: S_+ \rightarrow S_-\}$ is called *matching data* for the pair of building blocks $(Z_\pm, S_\pm)$. Given such data, it was shown by Kovalev in [Kov03] that one can construct a compact 7–manifold Y together with a family $(\phi_T)_{T \in (T_0, \infty)}$ of torsion-free G_2 –structures on Y .

By the work of Sá Earp [SE09, SE11a] holomorphic bundles $\mathcal{E}_\pm \rightarrow Z_\pm$ whose restriction to $S_\pm$ is stable yield HYM connections on $V_\pm$ and thus G_2 –instantons on $Y_\pm$. The following theorem gives sufficient conditions for being able to glue such a pair of G_2 –instantons to

obtain a G_2 -instanton over (Y, ϕ_T) .

Theorem 3.1. *Let $(Z_\pm, S_\pm)$ be a pair of building blocks for which there exists matching data $\{(\omega_{I,\pm}, \omega_{J,\pm}, \omega_{K,\pm}), \mathfrak{r}: S_+ \rightarrow S_-\}$. Denote by Y the compact 7-manifold and by $(\phi_T)_{T \in (T_0, \infty)}$ the family of torsion-free G_2 -structures obtained via the twisted connected sum construction from this pair of buildings blocks and matching data. Let $\mathcal{E}_\pm \rightarrow Z_\pm$ be holomorphic bundles such that $\mathcal{E}_\pm|_{S_\pm}$ is stable, $H^0(S_\pm, \text{End}_0(\mathcal{E}_\pm|_{S_\pm})) = 0$ and*

$$(3.2) \quad H^2(Z_\pm, \text{End}_0(\mathcal{E}_\pm)) = 0.$$

Denote by $A_{\infty, \pm}$ the unique ASD instanton on $\mathcal{E}_\pm|_{S_\pm}$ compatible with the holomorphic structure. Suppose that there exists an isomorphism of $U(n)$ -bundles $\mathfrak{r}: \mathcal{E}_+|_{S_+} \rightarrow \mathcal{E}_-|_{S_-}$ covering the hyperkähler rotation $\mathfrak{r}$ and denote by $\iota_\pm$ the map

$$H^1(Z_\pm, \text{End}_0(\mathcal{E}_\pm)) \rightarrow H^1(S_\pm, \text{End}_0(\mathcal{E}|_{S_\pm})) \cong H^1_{A_{\infty, \pm}} = \ker \delta_{A_{\infty, \pm}}$$

where $\delta_{A_{\infty, \pm}}$ is as in (2.40). If

$$(3.3) \quad \text{im}(\iota_+) \cap \text{im}(\mathfrak{r}^* \circ \iota_-) = \{0\},$$

then there exists a non-trivial $U(n)$ -bundle E over Y , a constant $T_1 \geq T_0$ and for each $T \geq T_1$ an acyclic G_2 -instanton A_T on E over (Y, ϕ_T) .

Remark 3.4. If $\dim H^1(S_\pm, \text{End}_0(\mathcal{E}|_{S_\pm})) = 0$, then (3.3) is trivially vacuous.

Remark 3.5. By Serre duality (3.2) is equivalent to $H^1(Z_\pm, \text{End}_0(\mathcal{E}_\pm)(-S_\pm)) = 0$ which means that there are no infinitesimal deformations of $\mathcal{E}_\pm$ that preserve the restriction to $S_\pm$.

Remark 3.6. A version of Theorem 3.1 without the hypothesis (3.2), but with the additional assumption $H^1(S_\pm, \text{End}_0(\mathcal{E}|_{S_\pm})) = 0$ was first claimed in [SE11b]; however, in this form the statement is likely not true and the proof given there has a mistake in [SE11b, Proposition 8].

It was noticed by Tyurin [Tyu08, p. 176 ff.] that if Z is a projective 3-fold, $\mathcal{E}$ is a holomorphic vector bundle over Z and S is an anti-canonical divisor, then the exact sequence

$$\begin{aligned} H^1(Z, \text{End}(\mathcal{E})(-S)) &\rightarrow H^1(Z, \text{End}(\mathcal{E})) \rightarrow H^1(S, \text{End}(\mathcal{E}|_S)) \\ &\rightarrow H^2(Z, \text{End}(\mathcal{E})(-S)) \rightarrow H^2(Z, \text{End}(\mathcal{E})) \end{aligned}$$

is self-dual with respect to Serre duality and, therefore, if S is a $K3$ surface and

$$H^1(Z, \text{End}(\mathcal{E})(-S)) = H^2(Z, \text{End}(\mathcal{E})) = 0,$$

then the image of $H^1(Z, \text{End}(\mathcal{E}))$ in $H^1(S, \text{End}(\mathcal{E}|_S))$ is a complex Lagrangian subspace with respect to Mukai's complex symplectic form. One can thus view (3.3) as a transverse

intersection condition for two subspaces which are Lagrangian with respect to *two different* symplectic structures.

In the analogy between G_2 -manifolds and 3-manifolds Calabi–Yau 3-folds take the place of Riemann surfaces. A G_2 -manifold Y obtained from gluing a pair of asymptotically cylindrical G_2 -manifolds $Y_{\pm}$ with asymptotic cross section a Calabi–Yau 3-fold Z can then be thought of as the analogue of a 3-manifold with a fixed Heegaard splitting. One can then ponder the idea of a G_2 *Atiyah–Floer conjecture*. I dare not make a precise formulation of what this conjecture should be; however, the contents of this chapter will shed some light on this conjectural conjecture.

3.1 The twisted connected sum construction

In this section I will review the twisted connected sum construction using the language introduced by Corti–Haskins–Nordström–Pacini [CHNP12b].

3.1.1 Gluing ACyl G_2 -manifolds

We begin with gluing matching pairs of ACyl G_2 -manifolds.

Definition 3.7. Let (Z, ω, Ω) be a compact Calabi–Yau 3-fold. A G_2 -manifold (Y, ϕ) is called *asymptotically cylindrical (ACyl) with asymptotic cross section* (Z, ω, Ω) if there exist a constant $\delta < 0$, a compact subset $K \subset Y$, a diffeomorphism $\pi: Y \setminus K \rightarrow \mathbf{R}_+ \times Z$ and a 2-form ρ on $\mathbf{R}_+ \times Z$ such that

$$\pi_*\phi = dt \wedge \omega + \operatorname{Re} \Omega + d\rho$$

and

$$\nabla^k \rho = O(e^{\delta t})$$

for all $k \in \mathbf{N}_0$. Here t denotes the coordinate on $\mathbf{R}_+$.

Definition 3.8. A pair of ACyl G_2 -manifolds $(Y_{\pm}, \phi_{\pm})$ with asymptotic cross sections $(Z_{\pm}, \omega_{\pm}, \Omega_{\pm})$ is said to *match* if there exists a diffeomorphism $f: Z_+ \rightarrow Z_-$ such that

$$f^*\omega_- = -\omega_+ \quad \text{and} \quad f^*\operatorname{Re} \Omega_- = \operatorname{Re} \Omega_+.$$

Let $(Y_{\pm}, \phi_{\pm})$ be a matching pair of ACyl G_2 -manifolds. Fix $T \geq 1$. Define $F: [T, T+1] \times Z_+ \rightarrow [T, T+1] \times Z_-$ by

$$F(t, z) := (2T + 1 - t, f(z)).$$

Denote by Y_T the compact 7-manifold obtained by gluing together

$$Y_{T,\pm} := K_{\pm} \cup \pi_{\pm}^{-1}((0, T+1] \times Z_{\pm})$$

via F . Fix a smooth non-increasing function $\chi: \mathbf{R} \rightarrow [0, 1]$ with $\chi(t) = 0$ for $t \leq 0$ and $\chi(t) = 1$ for $t \geq 1$. Define a 3-form $\tilde{\phi}_t$ on Y_T by

$$\tilde{\phi}_t = \phi_{\pm} - d[\pi_{\pm}^*(\chi(t - T + 1)\rho_{\pm})]$$

on $Y_{T,\pm}$. If $T \gg 1$, then $\tilde{\phi}_t$ defines a closed G_2 -structure on Y_T . Clearly, all the Y_T for different values of T are diffeomorphic; hence, we sometimes drop the T from the notation.

Theorem 3.9 (Kovalev [Kov03, Theorem 5.34]). *In the above situation there exists a constant $T_0 \geq 1$ and for each $T \in [T_0, \infty)$ there exists a 2-form η_T on Y_T such that $\phi_T := \tilde{\phi}_T + d\eta_T$ defines a torsion-free G_2 -structure and for some $\delta < 0$*

$$(3.10) \quad \|d\eta_T\|_{C^{0,\alpha}} = O(e^{\delta T}).$$

3.1.2 ACyl Calabi–Yau 3-folds from building blocks

The twisted connected sum construction is based on gluing ACyl G_2 -manifolds arising from ACyl Calabi–Yau 3-folds.

Definition 3.11. Let $(S, \omega_I, \omega_J, \omega_K)$ be a hyperkähler surface. We call a Calabi–Yau 3-fold (V, ω, Ω) *asymptotically cylindrical (ACyl) with asymptotic cross section $(S, \omega_I, \omega_J, \omega_K)$* if there exist a constant $\delta < 0$, a compact subset $K \subset V$, a diffeomorphism $\pi: V \setminus K \rightarrow \mathbf{R}_+ \times S^1 \times S$, a 1-form ρ and a 2-form σ on $\mathbf{R}_+ \times S^1 \times S$ such that

$$\begin{aligned} \pi_*\omega &= dt \wedge d\alpha + \omega_I + d\rho, \\ \pi_*\Omega &= (d\alpha - idt) \wedge (\omega_J + i\omega_K) + d\sigma \end{aligned}$$

and

$$\nabla^k \rho = O(e^{\delta t}) \quad \text{as well as} \quad \nabla^k \sigma = O(e^{\delta t})$$

for all $k \in \mathbf{N}_0$. Here t denotes the coordinate on $\mathbf{R}_+$ and α denotes the coordinate on S^1 .

If (V, ω, Ω) is an ACyl Calabi–Yau 3-fold with asymptotic cross section $(S, \omega_I, \omega_J, \omega_K)$, then $(S^1 \times V, d\beta \wedge \omega + \text{Re } \Omega)$ is an ACyl G_2 -manifold with asymptotic cross section $(T^2 \times S, d\alpha \wedge d\beta + \omega_K, (d\alpha - id\beta) \wedge (\omega_J + i\omega_I))$, see Example 1.13. Here β denotes the coordinate on S^1 .

Definition 3.12. Let $(S_{\pm}, \omega_{I,\pm}, \omega_{J,\pm}, \omega_{K,\pm})$ be a pair of hyperkähler $K3$ surfaces. A diffeo-

morphism $\mathfrak{r}: S^+ \rightarrow S^-$ is called a *hyperkähler rotation* if it satisfies

$$\mathfrak{r}^* \omega_{I,-} = \omega_{J,+}, \quad \mathfrak{r}^* \omega_{J,-} = \omega_{I,+} \quad \text{and} \quad \mathfrak{r}^* \omega_{K,-} = -\omega_{K,+}.$$

Let $V_{\pm}$ be a pair of asymptotically cylindrical Calabi–Yau 3-folds with asymptotic cross section $S_{\pm}$ and suppose that $\mathfrak{r}: S_+ \rightarrow S_-$ is a hyperkähler rotation. Then $Y_{\pm} := V_{\pm} \times S^1$ match via the diffeomorphism $f: T^2 \times S_+ \rightarrow T^2 \times S_-$ defined by

$$f(\alpha, \beta, x) := (\beta, \alpha, \mathfrak{r}(x)).$$

Note that it is this interchange of α and β that makes it possible for $\pi_1(Y_T)$ to be finite.

Definition 3.13 (Corti–Haskins–Nordström–Pacini [CHNP12a, Definition 5.1]). A *building block* is a smooth projective 3-fold Z together with a projective morphism $f: Z \rightarrow \mathbf{P}^1$ such that the following hold:

- The anticanonical class $-K_Z \in H^2(Z)$ is primitive.
- $S := f^{-1}(\infty)$ is a smooth $K3$ surface and $S \sim -K_Z$.
- If N denotes the image of $H^2(Z)$ in $H^2(S)$, then the embedding $N \hookrightarrow H^2(S)$ is primitive.
- $H^3(Z)$ is torsion-free.

Theorem 3.14 (Haskins–Hein–Nordström [HHN12, Theorem D]). *Let (Z, S) be a building block, let $(\omega_I, \omega_J, \omega_K)$ be a hyperkähler structure on S . If $[\omega_I] \in H^{1,1}(S)$ is the restriction of a Kähler class on Z , then there is an asymptotically cylindrical Calabi–Yau structure (ω, Ω) on $V := Z \setminus S$ with asymptotic cross section $(S, \omega_I, \omega_J, \omega_K)$.*

Remark 3.15. This result was first claimed by Kovalev in [Kov03, Theorem 2.4]; however, his proof is erroneous, see the discussion in [HHN12, Section 4.1].

Definition 3.16. *Matching data* for a pair of building blocks $(Z_{\pm}, S_{\pm})$ consists of hyperkähler structures $(\omega_{I,\pm}, \omega_{J,\pm}, \omega_{K,\pm})$ on $S_{\pm}$ such that $[\omega_{I,\pm}]$ is the restriction of a Kähler class on $Z_{\pm}$ together with a hyperkähler rotation $\mathfrak{r}: S_+ \rightarrow S_-$.

From the above it is clear that given a pair of building blocks and matching data one can construct a one-parameter family of G_2 -manifolds. This is called the *twisted connected sum construction*.

3.1.3 Constructing building blocks

Kovalev’s original construction [Kov03] was based on building blocks arising from Fano 3-folds. In their recent work [CHNP12b, CHNP12a] Corti–Haskins–Nordström–Pacini have been able to extend Kovalev’s construction to the much larger class of semi-Fano 3-folds.

Definition 3.17. A *weak Fano 3-fold* is a smooth projective 3-fold Y with big and nef anticanonical bundle $-K_Y$. A *semi-Fano 3-fold* Y is a weak Fano 3-fold whose anticanonical morphism is semi-small, i.e., it may contract divisors to curves, curves to points, but not divisors to points. Here the anticanonical morphism is the birational morphism corresponding to $-nK_Y$ for $n \gg 0$.

Proposition 3.18 (Corti–Haskins–Nordström–Pacini [CHNP12a, Propositions 4.24 and 5.7]). *Let Y be a semi-Fano 3-fold and let $|S_0, S_\infty| \subset |-K_Y|$ be a generic pencil with (smooth) base locus C and S_∞ generic. Denote by $Z := \text{Bl}_C Y$ the blow-up of Y along C . Then Z is a building block with $f: Z \rightarrow \mathbf{P}^1$ given by the strict transform of the pencil $|S_0, S_\infty|$.*

Definition 3.19. A building block obtained from a (semi-)Fano 3-fold by Proposition 3.18 is said to be of *(semi-)Fano type*.

In [KL11] Kovalev–Lee explained how to construct building blocks from $K3$ surfaces with non-symplectic involutions.

Definition 3.20. Let S be a $K3$ surface. A holomorphic involution $\tau: S \rightarrow S$ is called *non-symplectic* if τ^* acts as -1 on $H^{2,0}(S)$.

Proposition 3.21 (Kovalev–Lee [KL11, Section 4]). *Let S be a $K3$ surface with a non-symplectic involution τ . Then $Z := \text{Bl}_C((\mathbf{P}^1 \times S)/\langle \tau \rangle)$ is a building block. Here we extend τ to act on $\mathbf{P}^1$ by $[z : w] \rightarrow [-z : w]$ and C denotes the singular set of $(\mathbf{P}^1 \times S)/\langle \tau \rangle$.*

Remark 3.22. The fixed point set $\text{Fix}(\tau)$ of the action of $\mathbf{Z}_2$ on $\mathbf{P}^1 \times S$ consist of two copies of the fixed point set of the action of $\mathbf{Z}_2$ on S which is either empty or a disjoint union of smooth curves. It is often convenient to construct Z by first blowing-up $\mathbf{P}^1 \times S$ along $\text{Fix}(\tau)$ and then taking the quotient by the lifted action of $\mathbf{Z}_2$.

Definition 3.23. Building blocks obtained from $K3$ surfaces with non-symplectic involutions are said to be of *non-symplectic type*.

Let $L := -2E_8 \perp 3H$ denote the $K3$ lattice. Let S be a $K3$ surface with a non-symplectic involution τ . Fix a marking $h: L \rightarrow H^2(S, \mathbf{Z})$. Denote by L^τ the sublattice fixed by τ . For a non-symplectic building block N in Definition 3.13 is given by L^τ .

Remark 3.24. Attached to L^τ there are three invariants $r := \text{rk } L^\tau$, $a \in \mathbf{N}_0$ defined by $(L^\tau)^*/L^\tau = \mathbf{Z}_2^a$ and

$$\delta := \begin{cases} 0 & \text{if } x^2 \in \mathbf{Z} \text{ for each } x \in (L^\tau)^* \\ 1 & \text{otherwise.} \end{cases}$$

The invariants (r, a, δ) uniquely determine τ^* and L^τ up to the action of $O(L)$. The possible values (r, a, δ) which occur in the above setting have been classified by Nikulin. There are 75 possibilities. See [KL11, Appendix] for a summarising diagram.

For building blocks of non-symplectic type the matching problem has the following interesting interpretation, which I learned from Johannes Nordström. Suppose we have matching data for S_+ and S_- . Then via the hyperkähler rotation the non-symplectic involution τ_- on S_- corresponds to a real structure on S_+ . Thus $S := S_+$ carries a hyperkähler structure $(\omega_I, \omega_J, \omega_K)$ and *two* involutions $\tau_{\pm}$ which act on the hyperkähler structure by

$$\tau_{\pm}^* \omega_I = \pm \omega_I, \quad \tau_{\pm}^* \omega_J = \mp \omega_J \quad \text{and} \quad \tau_{\pm}^* \omega_K = -\omega_K.$$

Conversely, whenever we have this structure on a $K3$ surface we can construct two matching building blocks of non-symplectic type. In this case, the G_2 -manifold arising from the twisted connected sum construction can be seen as a resolution of $(T^3 \times S)/\langle \tilde{\tau}_+, \tilde{\tau}_- \rangle$ where $\tilde{\tau}_+(t_1, t_2, t_3, x) := (-t_1, t_2, \frac{1}{2} - t_3, \tau_+(x))$ and $\tilde{\tau}_-(t_1, t_2, t_3, x) := (\frac{1}{2} - t_1, t_2 - \frac{1}{2}, t_3, \tau_-(x))$. By taking S to be a Kummer surface with a real structure and a non-symplectic involution, one can obtain a deformation class of G_2 -manifolds that can be constructed using the twisted connected sum as well as the generalised Kummer construction, see also Kovalev–Nordström [KN10].

3.1.4 Matching building blocks

Using the Torelli theorem it was shown by Corti–Haskins–Nordström–Pacini in [CHNP12b, Proposition 6.2] that matching data in the sense of Definition 3.16 is equivalent to the following.

Definition 3.25. A set of *matching data* for a pair of building blocks $(Z_{\pm}, S_{\pm})$ consists of a triple (k_+, k_-, k_0) of classes in $L_{\mathbf{R}} := L \otimes \mathbf{R}$ for which there are markings $h_{\pm}: L \rightarrow H^2(S_{\pm}, \mathbf{Z})$ such that $\langle k_{\mp}, \pm k_0 \rangle$ is the period point of the marked $K3$ surface $(S_{\pm}, h_{\pm})$ and $h_{\pm}(k_{\pm})$ is the restriction of a Kähler class on $Z_{\pm}$.

Given matching data (k_+, k_-, k_0) for $(Z_{\pm}, S_{\pm})$ we can, in a unique way up to the action of $O(L)$, form the push-out diagram

$$(3.26) \quad \begin{array}{ccccc} & & N_+ & & \\ & \nearrow & & \searrow & \\ R & & & & W \longrightarrow L \\ & \searrow & & \nearrow & \\ & & N_- & & \end{array}$$

where $R = N_+ \cap N_-$, $W = N_+ + N_-$, all maps are primitive embeddings and W embeds into L . Since $N_{\pm} \subset \text{Pic}(S_{\pm})$, see [CHNP12b, Lemma 3.6], we have

- $k_{\pm} \in N_{\pm} \cap T_{\mp}$ with $T_{\pm} := N_{\pm}^{\perp}$,
- $k_0 \in T_+ \cap T_-$ and
- $\langle k_{\pm}, \pm k_0 \rangle$ lies in the Griffiths domain $D_{N_{\pm}}$.

Definition 3.27. Given a sublattice $N \subset L$, the *Griffiths domain* D_N of N is the set of oriented positive 2-planes $\Pi \subset N_{\mathbf{R}}^{\perp} \subset L_{\mathbf{R}}$.

Let me briefly recall Corti–Haskins–Nordström–Pacini’s *orthogonal gluing* method to find matching data.

Definition 3.28. The push-out (3.26) is called *orthogonal* if $N_{\pm}^{\perp} \subset N_{\mp}$.

Definition 3.29. Let $N \subset L$ be a primitive sublattice and let Amp be an open subcone of the positive cone in $N_{\mathbf{R}}$. Let $\mathcal{Z}$ be a deformation family of building blocks with $N(\mathcal{Z}) = N$. $\mathcal{Z}$ is called (N, Amp) -*generic* if there exists $U_{\mathcal{Z}} \subset D_N$

- which is the complement of a locally finite union of complex analytic submanifolds of positive codimension and
- for any $\Pi \in U_{\mathcal{Z}}$ and $k \in \text{Amp}$ there exists a building block $(Z, S) \in \mathcal{Z}$ and a marking $h: L \rightarrow H^2(S, \mathbf{Z})$ such that $h(\Pi) = H^{2,0}(S)$ and $h(k)$ is the restriction to S of a Kähler class on Z .

Corti–Haskins–Nordström–Pacini [CHNP12a, Proposition 6.9] proved that for a family of building blocks of semi-Fano type obtained from a full deformation type of semi-Fano 3-folds $\mathcal{Z}$ one can always find Amp such that $\mathcal{Z}$ is (N, Amp) -generic. If $\mathcal{Z}$ is a full deformation type of non-symplectic building blocks, then it is $(N = L^{\tau}, \text{Amp})$ -generic with Amp denoting the Kähler cone of a generic ample L^{τ} -polarised $K3$ surface, that is, all positive classes in $N_{\mathbf{R}}$ that are orthogonal to all (-2) -classes in N .

Proposition 3.30 (Corti–Haskins–Nordström–Pacini [CHNP12b, Proposition 6.18]). *Let $N_{\pm} \subset L$ be a pair of primitive sublattices of signature $(1, r_{\pm} - 1)$ and let $\mathcal{Z}_{\pm}$ be a pair of $(N_{\pm}, \text{Amp}_{\pm})$ -generic families of building blocks. Suppose we that $W := N_+ + N_-$ is an orthogonal push-out and suppose that $R := N_+ \cap N_-$ is negative-definite. Assume that*

$$(3.31) \quad \text{Amp}_{\pm} \cap W_{\pm} \neq \emptyset$$

where $W_{\pm} := N_{\pm} \cap T_{\mp}$. Then there exists $(Z_{\pm}, S_{\pm}) \in \mathcal{Z}_{\pm}$ and matching data for $Z_{\pm}$ compatible with the given embeddings $N_{\pm} \hookrightarrow L$.

Remark 3.32. If $\mathcal{Z}_{\pm}$ are of non-symplectic type, then (3.31) is equivalent to R not containing any (-2) -classes.

The orthogonal gluing now reduces finding a matching pair of building blocks, given a pair $\mathcal{Z}_\pm$ of $(N_\pm, \text{Amp}_\pm)$ -generic families of building blocks, to the following steps:

- Construct an orthogonal push-out $W = N_+ \perp_R N_-$ with R negative definite, e.g., $N_+ \perp N_-$, in which case $R = \{0\}$.
- Embed W primitively into L .
- Make sure that (3.31) holds.

For the second step the following result is very useful.

Theorem 3.33 (Nikulin [Nik79, Theorem 1.12.4 and Corollary 1.12.3]). *Let W be an even non-degenerate lattice of signature (t_+, t_-) and let L be an even unimodular lattice of signature (ℓ_+, ℓ_-) . Then there exists a primitive embedding $W \hookrightarrow L$ if $t_+ \leq \ell_+$, $t_- \leq \ell_-$ and*

- $2 \text{rk } W \leq \text{rk } L$ or
- $\text{rk } W + \ell(W) < \text{rk } L$.

Here $\ell(W)$ denotes the minimal number of generators of the discriminant group W^*/W .

3.2 Gluing G_2 -instantons over ACyl G_2 -manifolds

Definition 3.34. Let (Y, ϕ) be an ACyl G_2 -manifold with asymptotic cross section (Z, ω, Ω) . Let A_∞ be a HYM connection on a G -bundle E_∞ over Z . A G_2 -instanton A on a G -bundle E over Y is called *asymptotic to A_∞* if there exist a constant $\delta < 0$ and a bundle isomorphism $\bar{\pi}: E|_{Y \setminus K} \rightarrow E_\infty$ covering $\pi: Y \setminus K \rightarrow \mathbf{R}_+ \times Z$ such that

$$(3.35) \quad \nabla^k(\bar{\pi}_* A - A_\infty) = O(e^{\delta t})$$

for all $k \in \mathbf{N}_0$. Here by a slight abuse of notation we also denote by E_∞ and A_∞ their respective pullbacks to $\mathbf{R}_+ \times Z$.

Note, that we know from Example 1.93 that the pullback of a HYM connection from Z to $(\mathbf{R} \times Z, \phi)$ as in Example 1.13 is a G_2 -instanton.

Definition 3.36. Let (Y, ϕ) be an ACyl G_2 -manifold and let A be a G_2 -instanton on a G -bundle over (Y, ϕ) . For $\delta \in \mathbf{R}$ we define the *space of infinitesimal deformations of A of rate δ* by

$$\mathcal{T}_{A,\delta} := \left\{ \underline{a} \in \ker L_A : \nabla^k \bar{\pi}_* \underline{a} = O(e^{\delta t}) \text{ for all } k \in \mathbf{N}_0 \right\}$$

where L_A is as in (1.103) and $\underline{a} = (\xi, a)$. Set $\mathcal{T}_A := \mathcal{T}_{A,0}$.

The infinitesimal deformation spaces $\mathcal{T}_{A,\delta}$ associated with a G_2 -instanton A asymptotic to A_∞ are closely related to the infinitesimal automorphism and deformation spaces of the HYM connection A_∞ .

Definition 3.37. Let A be a HYM connection on a G -bundle E over a Kähler manifold Z . Then

$$\mathcal{H}_A^0 := \ker(\bar{\partial}_A: \Omega^0(Z, \mathfrak{g}_E) \rightarrow \Omega^{0,1}(Z, \mathfrak{g}_E))$$

is called the *space of infinitesimal automorphisms of A* and

$$\mathcal{H}_A^1 := \ker(\bar{\partial}_A \oplus \bar{\partial}_A^*: \Omega^{0,1}(Z, \mathfrak{g}_E) \rightarrow \Omega^{0,2}(Z, \mathfrak{g}_E) \oplus \Omega^0(Z, \mathfrak{g}_E))$$

is called the *space of infinitesimal deformations of A* .

Proposition 3.38. Let (Y, ϕ) be an asymptotically cylindrical G_2 -manifold and let A be a G_2 -instanton asymptotic to A_∞ . Then there are constants $\pm\delta_\pm > 0$ and a linear map $\iota: \mathcal{T}_{A,\delta_+} \rightarrow \mathcal{H}_{A_\infty}^0 \oplus \mathcal{H}_{A_\infty}^1$ such that

$$\nabla^k(\bar{\pi}_*\underline{a} - \iota(\underline{a})) = O(e^{\delta_-t})$$

for all $k \in \mathbb{N}_0$ and

$$\ker \iota = \mathcal{T}_{A,\delta_-}.$$

Moreover, $\mathcal{T}_{A,\delta} = \mathcal{T}_A$ for all $0 \leq \delta \leq \delta_+$.

The proof of Proposition 3.38 requires some preparation.

Proposition 3.39. If (Z, ω, Ω) is a compact Calabi–Yau 3-fold and A is a HYM connection on a G -bundle E over Z , then

$$\mathcal{H}_A^0 \oplus \mathcal{H}_A^1 \cong \ker D_A$$

where $D_A: \Omega^0(Z, \mathfrak{g}_E) \oplus \Omega^0(Z, \mathfrak{g}_E) \oplus \Omega^1(Z, \mathfrak{g}_E) \rightarrow \Omega^0(Z, \mathfrak{g}_E) \oplus \Omega^0(Z, \mathfrak{g}_E) \oplus \Omega^1(Z, \mathfrak{g}_E)$ is defined by

$$(3.40) \quad D_A := \begin{pmatrix} & & d_A^* \\ & & \Lambda d_A \\ d_A & -Id_A & -*(\text{Im } \Omega \wedge d_A) \end{pmatrix}.$$

Proof. If $s \in \mathcal{H}_A^0$ and $\alpha \in \mathcal{H}_A^1$, then $D_A(\text{Re } s, \text{Im } s, \alpha + \bar{\alpha}) = 0$. Conversely, if

$$d_A \xi - Id_A \eta - *(\text{Im } \Omega \wedge d_A a) = 0,$$

then applying d_A^* (resp. $d_A^* \circ I$), using that $I = \frac{1}{2} * (\omega \wedge \omega \wedge \cdot)$ on 1-forms, taking the L^2 inner product with ξ (resp. η) and integrating by parts yields $d_A \xi = 0$ (resp. $d_A \eta = 0$). Thus

$\xi + i\eta \in \mathcal{H}_A^0$ and

$$d_A^* a = 0 \quad \text{and} \quad \text{Im } \Omega \wedge d_A a = 0.$$

which implies $\alpha := a^{0,1} \in \mathcal{H}_A^1$. $\square$

It is a straight-forward computation to verify the following.

Proposition 3.41. *Let (Z, ω, Ω) be a compact Calabi–Yau 3-fold, let A be a HYM connection on a G -bundle E over Z , let $(Y := \mathbf{R} \times Z, \phi)$ be as in Example 1.13 and denote by $p: \mathbf{R} \times Z \rightarrow Z$ the canonical projection. Then the operator L_{p^*A} defined in (1.103) can be written as*

$$L_A = \tilde{I} \partial_t + D_A$$

where

$$\tilde{I} = \begin{pmatrix} & -1 \\ 1 & \\ & I \end{pmatrix}.$$

We can therefore write $L_A = \tilde{I}(\partial_t - \tilde{D}_A)$ where $\tilde{D}_A := \tilde{I}D_A$. It is also immediate that $\tilde{D}_A$ is still self-adjoint. Operators of this form are very well understood. In particular, we have the following results, see [Don02, Section 3.1] and [MP78].

Proposition 3.42. *Let X be a compact Riemannian manifold, let E be a vector bundle over X and let $D: C^\infty(X, E) \rightarrow C^\infty(X, E)$ be a self-adjoint linear elliptic differential operator of first order. Define the linear operator $L: C^\infty(\mathbf{R} \times X, E) \rightarrow C^\infty(\mathbf{R} \times X, E)$ by*

$$L := \partial_t - D.$$

Denote by λ_+ and λ_- the first positive and negative eigenvalue of D , respectively. If $a \in \ker L$ and

$$(3.43) \quad a = O(e^{\delta t}) \text{ as } t \rightarrow \infty$$

for some $\delta < \lambda_+$, then there exists $a_0 \in \ker D$ such that

$$a - a_0 = O(e^{\lambda_- t}) \text{ as } t \rightarrow \infty.$$

If $a \in L^\infty(\mathbf{R} \times X, E)$, then $a = a_0$.

Note that by standard elliptic theory (3.43) implies analogous decay estimates for $\nabla^k a$ for all $k \geq 0$.

Theorem 3.44 (Maz’ya–Plamenevskiĭ [MP78, Theorem 5.1]). *Assume the situation of Proposition 3.42. For $\delta \in \mathbf{R}$, $k \in \mathbf{Z}$ and $\alpha \in (0, 1)$ we define $\|\cdot\|_{C_\delta^{k,\alpha}} := \|e^{-\delta t} \cdot\|_{C^{k,\alpha}}$ where t denotes the coordinate function on $\mathbf{R}$ and denote by $C_\delta^{k,\alpha}(\mathbf{R} \times X, E)$ the closure*

of $C_0^\infty(\mathbf{R} \times X, E)$ with respect to this norm. Then L extends to a bounded linear operator $L: C_\delta^{k+1, \alpha} \rightarrow C_\delta^{k, \alpha}$ which is invertible if and only if $\delta \notin \text{spec}(D)$.

It is a consequence of Theorem 3.44 that $L: C_\delta^\infty \rightarrow C_\delta^\infty$ is invertible if $\delta \notin \text{spec}(D)$. Here $C_\delta^\infty := \bigcap_k C_\delta^{k, \alpha}$.

Proof of Proposition 3.38. Let $\lambda_\pm$ be the first positive/negative eigenvalue of $\tilde{D}_{A_\infty}$ and let $\delta < 0$ be as in Definition 3.34, but not too small. Pick $\delta_+ \in (0, \lambda_+)$ so that $\delta_- := \delta + \delta_+ \in (\lambda_-, 0)$ and . Given $a \in \mathcal{T}_{A, \delta_+}$, set $\tilde{a} := \chi(t)\bar{\pi}_*a_\pm$ with χ as in Section 3.1.1. Then $L_{A_\infty}\tilde{a} \in C_{\delta_-}^\infty(Z \times \mathbf{R}, \mathfrak{g}_{E_\infty})$. Thus by Theorem 3.44, there exists $b \in C_{\delta_-}^\infty$ such that $L_{A_\infty}(\tilde{a} - b) = 0$. By Proposition 3.42 $(\tilde{a} - b)_0 \in \ker D_{A_\infty}$ and $\tilde{a} - b - (\tilde{a} - b)_0 = O(e^{\lambda_- t})$ as t tends to infinity. This establishes the proposition with $\iota(a) := (\tilde{a} - b)_0$. $\square$

The main theorem of this section can now be stated as follows.

Theorem 3.45. *Let $(Y_\pm, \phi_\pm)$ be a pair of asymptotically cylindrical G_2 -manifolds that match via $f: Z_+ \rightarrow Z_-$ and denote by $(Y_T, \phi_T)_{T \in (T_0, \infty)}$ the resulting family of G_2 -manifolds arising from the twisted connected sum construction. Let $A_\pm$ be a pair of G_2 -instantons on $E_\pm$ over $(Y_\pm, \phi_\pm)$ asymptotic to $A_{\infty, \pm}$. Suppose that $\bar{f}: E_{\infty, +} \rightarrow E_{\infty, -}$ is a bundle isomorphism covering f such that $\bar{f}^*A_{\infty, -} = A_{\infty, +}$, the maps $\iota_\pm: \mathcal{T}_{A_\pm} \rightarrow \ker D_{A_{\infty, \pm}}$ constructed in Proposition 3.38 are injective and*

$$(3.46) \quad \text{im}(\iota_+) \cap \text{im}(\bar{f}^* \circ \iota_-) = \{0\} \subset \ker D_{A_{\infty, +}}.$$

Then there exists $T_1 \geq T_0$ and for each $T \geq T_1$ there exists an acyclic G_2 -instanton A_T on a G -bundle E_T over (Y_T, ϕ_T) .

Proof. The proof proceeds in three steps. We first produce an approximate G_2 -instanton $\tilde{A}_T$ by an explicit cut-and-paste procedure. This reduces the problem to solving the non-linear partial differential equation

$$(3.47) \quad *_T (F_{\tilde{A}_T + a} \wedge \psi_T) + d_{\tilde{A}_T + a} \xi = 0.$$

for $a \in \Omega^1(Y_T, \mathfrak{g}_{E_T}^{(0)})$ and $\xi \in \Omega^0(Y_T, \mathfrak{g}_{E_T}^{(0)})$ where $\psi_T := \Theta(\phi_T)$. Under the hypothesis of Theorem 3.45 we will show that we can solve the linearisation of (3.47) in a uniform fashion. The existence of a solution of (3.47) then follows from a simple application of Banach's fixed-point theorem.

Step 1. *There exists a $\delta < 0$ and for each $T \geq T_0$ there exists a connection $\tilde{A}_T$ on a G -bundle E_T over Y_T such that*

$$(3.48) \quad \|F_{\tilde{A}_T} \wedge \psi\|_{C^{0, \alpha}} = O(e^{\delta T}).$$

The bundle E_T is constructed by gluing $E_{\pm}|_{Y_{T,\pm}}$ via $\bar{f}$ and the connection $\tilde{A}_T$ is defined by

$$\tilde{A}_T := A_{\pm} - \bar{\pi}_{\pm}^*[\chi(t - T + 1)a_{\pm}]$$

over $Y_{T,\pm}$ where

$$a_{\pm} := \bar{\pi}_{\pm,*}A_{\pm} - A_{\infty,\pm}.$$

Then (3.48) is a straight-forward consequence of (3.10) and (3.35).

Step 2. Define a linear operator $L_T: C^{1,\alpha} \rightarrow C^{0,\alpha}$ by (1.103) with $A = \tilde{A}_T$ and $\phi = \phi_T$. Then there exist constants $\tilde{T}_1, c > 0$ such that for all $T \geq \tilde{T}_1$ the operator L_T is invertible and

$$(3.49) \quad \|L_T^{-1}\underline{a}\|_{C^{1,\alpha}} \leq ce^{\frac{|\delta|}{4}T}\|\underline{a}\|_{C^{0,\alpha}}$$

Step 2.1. There exists a constant $c > 0$ such that for all $T \geq T_0$

$$(3.50) \quad \|\underline{a}\|_{C^{1,\alpha}} \leq c(\|L_T\underline{a}\|_{C^{0,\alpha}} + \|\underline{a}\|_{L^\infty}).$$

This is an immediate consequence of standard interior Schauder estimates because of (3.10) and (3.35).

Step 2.2. There exist constants $\tilde{T}_1 \geq T_0$ and $c > 0$ such that for $T \in [\tilde{T}_1, \infty)$

$$(3.51) \quad \|\underline{a}\|_{L^\infty} \leq ce^{\frac{|\delta|}{4}T}\|L_T\underline{a}\|_{C^{0,\alpha}}.$$

Suppose not. Then there exist a sequence (T_i) tending to infinity and a sequence $(\underline{a}_i)$ such that

$$(3.52) \quad \|\underline{a}_i\|_{L^\infty} = 1 \quad \text{and} \quad \lim_{i \rightarrow \infty} e^{\frac{|\delta|}{4}T_i}\|L_{T_i}\underline{a}_i\|_{C^{0,\alpha}} = 0.$$

Then by (3.50)

$$(3.53) \quad \|\underline{a}_i\|_{C^{1,\alpha}} \leq 2c.$$

Hence, by Arzelà–Ascoli we can assume (by passing to a subsequence) that on compact subsets of $Y_{\pm}$ the sequence $\underline{a}_i|_{Y_{T_i,\pm}}$ converges in $C^{1,\alpha/2}$ to some $\underline{a}_{\infty,\pm}$ which is bounded and satisfies

$$L_{A_{\pm}}\underline{a}_{\infty,\pm} = 0$$

because of (3.10) and (3.35). Using standard elliptic estimates it follows that $\underline{a}_{\infty,\pm} \in \mathcal{T}_{A_{\pm}}$.

Proposition 3.54. In the above situation $\lim_{i \rightarrow \infty} \|\underline{a}_i|_{Y_{T_i,\pm}} - \underline{a}_{\infty,\pm}\|_{L^\infty(Y_{T_i,\pm})} = 0$.

The proof of this proposition will be given at the end of this section. Accepting it as a fact for now, it follows immediately that

$$\iota_+(\underline{a}_{\infty,+}) = \bar{f}^* \circ \iota_-(\underline{a}_{\infty,-}).$$

Now, by (3.46) it follows that $\iota_{\pm}(\underline{a}_{\infty,\pm}) = 0$; hence, $\underline{a}_{\infty,\pm} = 0$, since $\iota_{\pm}$ are injective.

However, by (3.53) there exist $x_i \in Y_{T_i}$ such that $|\underline{a}_{T_i}|(x_i) = 1$. By passing to a further subsequence and possibly changing the rôles of $+$ and $-$ we can assume that each $x_i \in Y_{T_i,+}$; hence, by Proposition 3.54, $\underline{a}_{\infty,+} \neq 0$ contradicting what was derived above.

Step 2.3. *We complete the proof of Step 2.*

Combining (3.50) and (3.51) yields

$$\|\underline{a}\|_{C^{1,\alpha}} \leq ce^{\frac{|\delta|}{4}T} \|L_T \underline{a}\|_{C^{0,\alpha}}.$$

Therefore, L_T is injective and hence also surjective, since L_T is formally self-adjoint.

Step 3. *There exists a constant $T_1 \geq T_0$ and for each $T \geq T_1$ a smooth solution $\underline{a} = \underline{a}_T$ of (3.47) such that $\lim_{T \rightarrow \infty} \|\underline{a}_T\|_{C^{1,\alpha}} = 0$.*

We can write (3.47) as

$$(3.55) \quad L_T \underline{a} + Q_T(\underline{a}) + \varepsilon_T = 0$$

where $Q_T(\underline{a}) := \frac{1}{2} *_{\tilde{A}_T} ([a \wedge a] \wedge \psi_T) + [a, \xi]$ and $\varepsilon_T := *_T(F_{\tilde{A}_T} \wedge \psi_T)$. We make the ansatz $\underline{a} = L_T^{-1} \underline{b}$. Then (3.55) becomes

$$(3.56) \quad \underline{b} + \tilde{Q}_T(\underline{b}) + \varepsilon_T = 0.$$

where $\tilde{Q}_T = Q_T \circ L_T^{-1}$. By (3.49)

$$\|\tilde{Q}_T(\underline{b}_1) - \tilde{Q}_T(\underline{b}_2)\|_{C^{0,\alpha}} \leq ce^{\frac{|\delta|}{2}T} (\|\underline{b}_1\|_{C^{0,\alpha}} + \|\underline{b}_2\|_{C^{0,\alpha}}) \|\underline{b}_1 - \underline{b}_2\|_{C^{0,\alpha}}$$

for some constant $c > 0$ independent of $T \geq \tilde{T}_1$. Now, Lemma 2.84 yields the desired solution of (3.56) and thus of (3.47) provided $T \geq T_1$ for a suitably large $T_1 \geq \tilde{T}_1$. By elliptic regularity $\underline{a}$ is smooth. $\square$

The proof of Proposition 3.54 requires a bit of preparation. Denote by $t_{\pm}: Y_{\pm} \rightarrow [1, \infty)$ a smooth function which agrees with the $\mathbf{R}$ -coordinate function on $\pi_{\pm}^{-1}([1, \infty) \times Z_{\pm}) \subset Y_{\pm}$. For $\gamma \in \mathbf{R}$, $k \in \mathbf{Z}$ and $\alpha \in (0, 1)$ we define $\|\cdot\|_{C_{\gamma}^{k,\alpha}} := \|e^{-\gamma t_{\pm}} \cdot\|_{C^{k,\alpha}}$ and denote by $C_{\gamma}^{k,\alpha}(Y_{\pm}, E)$ the closure of $C_0^{\infty}(Y_{\pm}, E)$ with respect to this norm.

Proposition 3.57. *There is a $\gamma_0 > 0$ such that for each $\gamma \in (0, \gamma_0)$ the linear operator $L_{A_\pm} : C_\gamma^{1,\alpha} \rightarrow C_\gamma^{0,\alpha}$ has a bounded right inverse.*

Proof. Using standard elliptic estimates and Theorem 3.44 one shows that $L_{A_\pm} : C_\gamma^{1,\alpha} \rightarrow C_\gamma^{0,\alpha}$ is Fredholm whenever $\gamma > 0$ is sufficiently small, cf. [HHN12, Proposition 2.4]. Also, the cokernel of $L_{A_\pm}$ can be identified to be $\mathcal{T}_{A_\pm, -\gamma}$ which is trivial by hypothesis. $\square$

Proof of Proposition 3.54. We restrict to the $+$ case; the $-$ case is identical. It follows from the construction of $\underline{a}_{\infty,+}$ that for each fixed compact subset $K \subset Y_{T_i,+}$

$$\lim_{i \rightarrow \infty} \|\underline{a}_i|_{Y_{T_i,+}} - \underline{a}_{\infty,+}\|_{L^\infty(K)} = 0.$$

To be able to strengthen this to a estimate on all of $Y_{T_i,+}$ the factor $e^{\frac{|\partial|}{4}T}$ in (3.52) is will be important, even though it is clearly not optimal.

With χ as in Section 3.1.1 define a cut-off function $\chi_T : Y_+ \rightarrow [0, 1]$ by $\chi_T(x) := 1 - \chi(t_+(x) - \frac{3}{2}T)$. For each sufficiently small $\gamma > 0$ we have

$$\|L_{A_+}(\chi_{T_i}\underline{a}_i)\|_{C_\gamma^{0,\alpha}(Y_+)} = O(e^{-\frac{3}{2}\gamma T_i})$$

because of (3.10), (3.35), (3.52) and (3.53). Using Proposition 3.57 we construct $\underline{b}_i \in C_\gamma^{1,\alpha}$ such that $\underline{a}_{\infty,+}^i := \chi_{T_i}\underline{a}_i + \underline{b}_i \in \mathcal{T}_{A_+,\gamma}$ and $\|\underline{b}_i\|_{C_{0,\gamma}^{1,\alpha}} = O(e^{-\frac{3}{2}\gamma T_i})$. Hence,

$$\|\underline{a}_i - \underline{a}_{\infty,+}^i\|_{L^\infty(Y_{T_i,+})} = O(e^{-\frac{1}{2}\gamma T_i}).$$

Moreover, $\lim_{i \rightarrow \infty} \|\underline{a}_{\infty,+}^i - \underline{a}_{\infty,+}\|_{L^\infty(K)} = 0$ and since both $\|\cdot\|_{L^\infty(K)}$ and $\|\cdot\|_{L^\infty(Y_+)}$ are norms on the finite dimensional vector space $\mathcal{T}_{A_+,\gamma} = \mathcal{T}_{A_+}$ it also follows that

$$\lim_{i \rightarrow \infty} \|\underline{a}_{\infty,+}^i - \underline{a}_{\infty,+}\|_{L^\infty(Y_+)} = 0.$$

Therefore,

$$\lim_{i \rightarrow \infty} \|\underline{a}_i - \underline{a}_{\infty,+}\|_{L^\infty(Y_{T_i,+})} = 0. \quad \square$$

3.3 Holomorphic bundles over building blocks and G_2 -instantons over ACyl G_2 -manifolds

Definition 3.58. Let (V, ω, Ω) be an ACyl Calabi–Yau 3-fold with asymptotic cross section $(S, \omega_I, \omega_J, \omega_K)$. Let A_∞ be an ASD instanton on a G -bundle E_∞ over S . A HYM connection A on a G -bundle E over V is called *asymptotic to A_∞* if there exist a constant $\delta < 0$ and a bundle isomorphism $\bar{\pi} : E|_{V \setminus K} \rightarrow E_\infty$ covering $\pi : V \setminus K \rightarrow \mathbf{R}_+ \times S^1 \times S$ such that

$$\nabla^k(\bar{\pi}_*A - A_\infty) = O(e^{\delta t})$$

for all $k \in \mathbb{N}_0$. Here by a slight abuse of notation we also denote by E_∞ and A_∞ their respective pullbacks to $\mathbf{R}_+ \times S^1 \times S$.

The following theorem of Sá Earp can be used to produce examples of HYM connections A asymptotic to ASD instantons A_∞ over ACyl Calabi–Yau 3-folds and, hence, by taking the product with S^1 examples of G_2 -instantons q^*A asymptotic to p^*A_∞ on the ACyl G_2 -manifold $S^1 \times V$. Here $p: T^2 \times S \rightarrow S$ and $q: S^1 \times V \rightarrow V$ denote the canonical projections.

Theorem 3.59 (Sá Earp [SE11a, Theorem 59]). *Let Z and S be as in Theorem 3.14 and let (V, ω, Ω) be the resulting asymptotically cylindrical Calabi–Yau 3-fold. Let $\mathcal{E}$ be a holomorphic vector bundle over Z and let h be a HYM metric on $\mathcal{E}|_S$. Then there exists a HYM metric H on $(\mathcal{E}|_{Z \setminus S}, \omega, \Omega)$ whose corresponding Chern connection is asymptotic to the Chern connection associated with h .*

Theorem 3.1 is now an immediate consequence of Theorems 3.45 and 3.59 and the following result.

Proposition 3.60. *In the situation of Theorem 3.59 suppose that $H^0(S, \text{End}_0(\mathcal{E}|_S)) = 0$ and denote by A and A_∞ the $\mathbf{PU}(n)$ -connections associated with H and h , respectively. Then*

$$(3.61) \quad \mathcal{H}_{p^*A_\infty}^1 = \mathcal{H}_{A_\infty}^1$$

and there exist injective linear maps $\kappa_-: \mathcal{T}_{q^*A, -\delta} \rightarrow H^1(Z, \text{End}_0(\mathcal{E}))$ for some small $\delta > 0$ and $\kappa: \mathcal{T}_{q^*A} \rightarrow H^1(Z, \text{End}_0(\mathcal{E}))$ such that the following diagram commutes:

$$(3.62) \quad \begin{array}{ccccc} \mathcal{T}_{q^*A, -\delta} & \xrightarrow{\quad} & \mathcal{T}_{q^*A} & \xrightarrow{\quad \iota \quad} & \mathcal{H}_{p^*A_\infty}^1 \\ \downarrow \kappa_- & & \downarrow \kappa & & \downarrow \cong \\ H^1(Z, \text{End}_0(\mathcal{E})(-S)) & \longrightarrow & H^1(Z, \text{End}_0(\mathcal{E})) & \longrightarrow & H^1(S, \text{End}_0(\mathcal{E}|_S)). \end{array}$$

Equation (3.61) is a direct consequence of $\mathcal{H}_{A_\infty}^0 = 0$ and the fact that all holomorphic functions on T^2 are constant. The proof of the remaining assertions requires some preparation.

Proposition 3.63. *If A is a HYM connection asymptotic to A_∞ over an ACyl Calabi–Yau 3-fold then there exists a $\delta_+ > 0$ such that for all $\delta < \delta_+$*

$$(3.64) \quad \mathcal{T}_{q^*A, \delta} = \left\{ \underline{a} \in \ker D_A : \nabla^k \bar{\pi}_* \underline{a} = O(e^{\delta t}) \text{ for all } k \in \mathbb{N}_0 \right\}$$

with D_A as in (3.40).

Proof. Note that we can write

$$(3.65) \quad L_A = \begin{pmatrix} & -\partial_\alpha & \\ \partial_\alpha & & \\ & & I\partial_\alpha \end{pmatrix} + D_A$$

where α denotes the coordinate on S^1 . For $\delta \leq 0$, (3.64) follows by an application of Lemma 2.76. By (the argument in the proof of) Proposition 3.38 both the left and the right hand side of (3.64) are independent of δ for $0 \leq \delta < \delta_+$ for some $\delta_+ > 0$. This completes the proof. $\square$

Proposition 3.66. *In the situation of Proposition 3.60 there exists a constant $\delta_+ > 0$ such that for all $\delta < \delta_+$*

$$\mathcal{T}_{q^*A,\delta} \cong \mathcal{H}_{A,\delta}^1 := \left\{ \alpha \in \mathcal{H}_A^1 : \nabla^k \bar{\pi}_* \alpha = O(e^{\delta t}) \text{ for all } k \in \mathbf{N}_0 \right\}.$$

Proof. If $\delta < 0$, then obviously

$$\mathcal{H}_{A,\delta}^0 := \left\{ \alpha \in \mathcal{H}_A^0 : \nabla^k \bar{\pi}_* \alpha = O(e^{\delta t}) \text{ for all } k \in \mathbf{N}_0 \right\} = 0$$

and it is clear that the proof of Proposition 3.39 goes through without modification since the necessary integration by parts is justified.

If $\delta < \delta_+$ with $\delta_+ > 0$ sufficiently small, cf. Proposition 3.38, and $(\xi, \eta, a) \in \mathcal{T}_{A,\delta}$, then $\iota(\xi, \eta, a) \in \{0\} \oplus \mathcal{H}_{A_\infty}^1$. Hence ξ and η decay exponentially and one can argue as in the proof of Proposition 3.39. Note also that again $\mathcal{H}_{A,\delta}^0 = 0$ $\square$

Proof of Proposition 3.60. In view of Proposition 3.66 we only need to establish (3.62) with $\mathcal{H}_{A,\delta}^1$ instead of $\mathcal{T}_{q^*A,\delta}$.

For $k \in \mathbf{N}_0$ and $\delta \in \mathbf{R}$ define a complex of sheaves $(\mathcal{A}_\delta^\bullet, \bar{\partial})$ on Z by

$$(3.67) \quad \mathcal{A}_\delta^i(U) = \left\{ \alpha \in \Omega^{0,i}(V \cap U, \mathcal{E}nd_0(\mathcal{E})) : \right. \\ \left. e^{-\delta t} \nabla^j \bar{\pi}_* \alpha \in L^\infty((0, \infty) \times Z, \mathfrak{g}_{E_\infty}) \text{ for all } k \in \mathbf{N}_0 \right\}.$$

There is an obvious map $\kappa_\delta: \mathcal{H}_{A,\delta}^1 \rightarrow H^1(\Gamma(\mathcal{A}_\delta^\bullet), \bar{\partial})$.

Proposition 3.68. *For every $\delta \in \mathbf{R} \setminus \mathbf{Z}$ the complex of sheaves $(\mathcal{A}_\delta^\bullet, \bar{\partial})$ is an acyclic resolution of $\mathcal{E}nd_0(\mathcal{E})(\lfloor \delta \rfloor S)$.*

Proof. The sheaves $\mathcal{A}_\delta^\bullet$ are acyclic because they are $C^\infty(Z)$ -modules and thus soft. Let $x \in Z$ and let $U \subset Z$ denote a small open neighbourhood of the point x . An element $s \in \ker(\bar{\partial}: \Gamma(U, \mathcal{A}_\delta^0) \rightarrow \Gamma(U, \mathcal{A}_\delta^1))$ corresponds to a holomorphic section of $\mathcal{E}nd_0(\mathcal{E})|_{U \cap (Z \setminus S)}$ such that $|z|^{-\delta} s$ stays bounded. Here z is a holomorphic function on U vanishing to first

order along $S \cap U$. Then $z^{-[\delta]}s$ is weakly holomorphic in U . By elliptic regularity $z^{-[\delta]}s$ extends across $U \cap S$ and thus s defines an element of $\Gamma(U, \mathcal{E}nd_0(\mathcal{E})([\delta]S))$. Conversely, it is clear that $\Gamma(U, \mathcal{E}nd_0(\mathcal{E})([\delta]S)) \subset \ker(\bar{\partial}: \Gamma(U, \mathcal{A}_\delta^0) \rightarrow \Gamma(U, \mathcal{A}_\delta^1))$. We are thus left with proving that $(\mathcal{A}_\delta^\bullet, \bar{\partial})$ is exact. It is easy to see that the usual argument for proving the $\bar{\partial}$ -Poincaré Lemma, see, e.g., [GH94, p. 25], goes through provided the linear operator

$$(3.69) \quad \bar{\partial}: C_\delta^\infty \Omega^0(S^1 \times \mathbf{R}) \rightarrow C_\delta^\infty \Omega^{0,1}(S^1 \times \mathbf{R})$$

is invertible. This, however, is a simple consequence of Theorem 3.44 since $\bar{\partial} = \partial_t + i\partial_\alpha$ and the spectrum of $i\partial_\alpha$ on $S^1 = \mathbf{R}/\mathbf{Z}$ is $\mathbf{Z}$. $\square$

By Proposition 3.68 we have linear maps $\kappa_\delta: \mathcal{H}_{A,\delta}^1 \rightarrow H^1(Z, \mathcal{E}nd_0(\mathcal{E})([\delta]S))$ for $\delta \in \mathbf{R} \setminus \mathbf{Z}$; hence, linear maps $\kappa_-: \mathcal{H}_{A,-\delta}^1 \rightarrow H^1(Z, \mathcal{E}nd_0(\mathcal{E})(-S))$ and $\kappa: \mathcal{H}_A^1 = \mathcal{H}_{A,\delta}^1 \rightarrow H^1(Z, \mathcal{E}nd_0(\mathcal{E}))$ for some small $\delta > 0$ making the following diagram commute:

$$\begin{array}{ccccc} \mathcal{H}_{A,-\delta}^1 & \xrightarrow{\quad} & \mathcal{H}_A^1 & \xrightarrow{\quad \iota \quad} & \mathcal{H}_{A_\infty}^1 \\ \downarrow \kappa_- & & \downarrow \kappa & & \downarrow \cong \\ H^1(Z, \mathcal{E}nd_0(\mathcal{E})(-S)) & \longrightarrow & H^1(Z, \mathcal{E}nd_0(\mathcal{E})) & \longrightarrow & H^1(S, \mathcal{E}nd_0(\mathcal{E}|_S)). \end{array}$$

The map κ_- is injective, because if $\kappa_- a = 0$, then $a = \bar{\partial}s$ for some $s \in \mathcal{A}_{-\delta}^0(Z)$ and thus

$$\int_{Z \setminus S} \|a\|^2 = \int_{Z \setminus S} \langle a, \bar{\partial}s \rangle = \int_{Z \setminus S} \langle \bar{\partial}^* a, s \rangle = 0.$$

Since $H^0(S, \mathcal{E}nd_0(\mathcal{E}|_S)) = 0$, the first map on the bottom is injective and a simple diagram chase proves that κ is injective. $\square$

Chapter 4

G_2 –instantons, associative submanifolds and Fueter sections

As I have already mentioned in Section 1.5 the main obstacle to defining a G_2 Casson invariant is that the moduli spaces of G_2 –instantons $\mathcal{M}(E, \{\phi_t\})$ need not be compact in general. There are a number of important results regarding the compactness problem in higher dimensional gauge theory: In [Uhl82a] Uhlenbeck showed the existence of good gauges in dimension n provided the $L^{n/2}$ –norm of the curvature is sufficiently small. Using Price’s monotonicity formula for the renormalised Yang–Mills energy [Pri83] Nakajima was able to show that a sequence of Yang–Mills connections over a compact Riemannian manifold with uniformly bounded Yang–Mills energy converges, after passing to a subsequence, outside a closed subset of Hausdorff codimension four [Nak88]. Tian analysed this subset in detail. He was able to show that it is rectifiable and that at each point of this subset the sequence of Yang–Mills connections loses energy via bubbling and/or develops a singularity [Tia00]. The question of which singularities are removable and which are not was addressed by Tao–Tian [TT04]. In the context of gauge theory on G_2 –manifolds the combined effort of these authors can be summarised as follows.

Theorem 4.1 (Uhlenbeck, Price, Nakajima, Tian, Tao–Tian). *Let Y be a compact 7–manifold, let (t_i) be a null–sequence, let (ϕ_{t_i}) be a sequence of torsion–free G_2 –structures on Y converging to a G_2 –structure ϕ_0 , let E be a G –bundle over Y where G is a compact Lie group and let (A_{t_i}) be a sequence of connections on E such that A_{t_i} is a G_2 –instanton over (Y, ϕ_{t_i}) . Then there exist subsets $\text{sing}(A_0) \subset S \subset Y$ and a connection A_0 on $E|_{Y \setminus S}$ such that after passing to a subsequence the following holds:*

- S is closed, countably $\mathcal{H}^3$ –rectifiable and satisfies $\mathcal{H}^3(S) < \infty$. Moreover, for $\mathcal{H}^3$ –a.e. $x \in S$, $\phi_0|_{T_x S}$ is a volume form on $T_x S$.

- A_0 is a G_2 -instanton over $(Y \setminus S, \phi_0)$ and the current $p_1(A_0)$ defined by

$$(4.2) \quad \langle p_1(A_0), \alpha \rangle := -\frac{1}{8\pi^2} \int_Y \text{tr}(F_{A_0} \wedge F_{A_0}) \wedge \alpha$$

is closed.

- The connection A_0 extends to a G_2 -instanton on a G -bundle $\tilde{E}$ over $(Y \setminus \text{sing}(A_0), \phi_0)$ which is isomorphic to E over $Y \setminus S$; moreover, $\mathcal{H}^3(\text{sing}(A_0)) = 0$.
- There exists a sequence of gauge transformations (g_{t_i}) of $E|_{Y \setminus S}$ such that $g_{t_i}^* A_{t_i}$ converges to A_0 in C_{loc}^∞ on $Y \setminus S$.
- For $\mathcal{H}^3$ -a.e. $x \in S$ there exists a non-trivial ASD instanton $\mathfrak{I}(x)$ on $T_x S^\perp$ whose pullback to $T_x Y$ is the limit of a blowing up of the sequence (A_{t_i}) around the point x .
- There is an upper semi-continuous function $\Theta: S \rightarrow [0, \infty)$ which satisfies

$$(4.3) \quad \Theta(x) \geq \text{YM}(\mathfrak{I}(x))$$

for $\mathcal{H}^3$ -a.e. $x \in S$ and is such that the rectifiable current $p_1(S, \Theta)$ defined by

$$\langle p_1(S, \Theta), \alpha \rangle := -\frac{1}{8\pi^2} \int_S \langle \phi_0|_{TS}, \alpha \rangle \Theta d\mathcal{H}^3$$

is closed. Moreover, the instanton charge density is conserved, i.e.,

$$(4.4) \quad p_1(\mathfrak{g}_E) = p_1(A_0) + p_1(S, \Theta).$$

- $S = \text{sing}(A_0) \cup \text{supp}(p_1(S, \Theta))$.

The function Θ measures the amount of energy density lost around a point $x \in S$. If at $x \in S$ there is not just one ASD instanton bubbling off in the direction transverse to S , but a whole bubbling tree of ASD instantons, then the inequality in (4.3) is necessarily strict. It is reasonable, to expect that if one replaced $\mathfrak{I}(x)$ by the entire tree of ASD instantons bubbling off, then (4.3) becomes an identity. It would then also follow that Θ takes values in $8\pi^2 \mathbf{Z}$ and thus that $p_1(S, \Theta)$ is an integral current. A special case of the analogous result for harmonic maps was proved by Lin–Rivière in [LR02].

Remark 4.5. In [Tia00, Chapter 5] Tian mentions some ideas for studying the singular set of A_0 based on a stratification according to the least number of directions a tangent instanton to A_0 at $x \in \text{sing}(A_0)$ splits off, cf. [Sim96, Chapter 3]. In particular, he conjectures that $\text{sing}(A_0)$ has Hausdorff dimension at most one. However, to this day there has been no significant progress in this direction as far as I know.

Remark 4.6. In certain very special situations one can rule out bubbling in Theorem 4.1 using an idea that goes back to Fintushel–Stern [FS84]. Let (Y, ϕ) be a compact 7–manifold and let E be a $\mathrm{SO}(3)$ –bundle over Y . If a sequence of G_2 –instantons on E bubbles, then we can write $p_1(\mathfrak{g}_E)$ as in (4.4). Since every $\mathrm{SO}(3)$ –bundle on S^4 lifts to a $\mathrm{SU}(2)$ –bundle and for a $\mathrm{SU}(2)$ –bundle E over a 4–manifold $p_1(\mathfrak{g}_E) = -4c_2(E)$, the energy of an $\mathrm{SO}(3)$ ASD instanton on S^4 is a multiple of $4 \cdot 8\pi^2$. Hence, assuming that $\mathfrak{I}(x)$ denotes the entire bubbling tree at $x \in S$ and equality holds in (4.3), it follows that $p_1(S, \Theta)$ is divisible by 4 in $H^4(Y, \mathbf{Z})$. Denote by $m(\phi)$ the infimal volume of an associative integral current in (Y, ϕ) . Because the first term on the right-hand side (4.4) pairs non-positively with $[\phi]$ it follows that $\langle p_1(\mathfrak{g}_E) \cup [\phi], [Y] \rangle \leq -4m(\phi)$. Conversely, if this inequality is violated, then there can be no bubbling. In general, $m(\phi)$ is hard to control precisely; however, we know that

$$m(\phi) \geq \min \left\{ \int_A \phi > 0 : A \in H_3(Y, \mathbf{Z}) \right\}.$$

In particular, if $[\phi]$ is (almost) integral, then $m(\phi) \geq 1(-\varepsilon)$.

Let us now assume that $\mathrm{sing}(A_0) = \emptyset$. One would expect that typically the subset $P := \mathrm{supp}(p_1(S, \Theta))$ is a smooth associative submanifold in (Y, ϕ_0) , the ASD instantons $\mathfrak{I}(x)$ can be arranged to form a smooth section $\mathfrak{I}$ of an *instanton moduli bundle* $\mathfrak{M}$ as described in Section 4.1 and equality holds in (4.3). It is important to understand which triples $(A_0, P, \mathfrak{I})$ arise as a limit of a sequence of G_2 –instantons. Donaldson–Segal [DS11] argued that $\mathfrak{I}$ should be a *Fueter section*. Here a section $\mathfrak{I}$ is called a Fueter section if it satisfies a certain non-linear Dirac equation associated to ϕ_0 and the restriction of A_0 to P called the *Fueter equation*, see Section 4.1. In the course of this chapter I will prove that this condition is indeed sufficient if A_0, P and $\mathfrak{I}$ satisfy certain “transversality conditions”.

Theorem 4.7. *Let Y be a compact 7–manifold equipped with a family of torsion-free G_2 –structures $(\phi_t)_{t \in (-T, T)}$. Suppose we are given*

- *an unobstructed G_2 –instanton A_0 on a G –bundle E_0 over (Y, ϕ_t) ,*
- *an unobstructed associative submanifold P in (Y, ϕ_0) and*
- *a Fueter section $\mathfrak{I}$ of an instanton moduli bundle $\mathfrak{M}$ associated with P and $E_0|_P$ which is unobstructed with respect to (ϕ_t) .*

Then there is a constant $\Lambda > 0$, a G –bundle E together with a family of connections $(\bar{A}_\lambda)_{\lambda \in (0, \Lambda]}$ and a continuous function $t: [0, \Lambda] \rightarrow (-T, T)$ such that $t(0) = 0$ and $\bar{A}_\lambda$ is a G_2 –instanton on E over $(Y, \phi_{t(\lambda)})$ for all $\lambda \in (0, \Lambda]$. Moreover, as λ tends to zero $\bar{A}_\lambda$ converges to A_0 on the complement of P and at each point $x \in P$ an ASD instanton in the equivalence class given by $\mathfrak{I}(x)$ bubbles off transversely.

Remark 4.8. The notion of unobstructedness for Fueter sections will be introduced in Definition 4.24.

Remark 4.9. In Theorem 4.7 we need to work with families of G_2 -structures, because the G_2 -instanton equation has index zero and, hence, one cannot expect to produce a family of G_2 -instantons depending on a gluing parameter $\lambda \in (0, \Lambda]$ over a *fixed* G_2 -manifold.

It is clear from Theorem 4.7 that the naïve definition of the conjectural G_2 Casson invariant as the signed count

$$n(E, \phi) := \# \mathcal{M}(E, \phi)$$

cannot be expected to be invariant under deformations of ϕ , in general. This was already anticipated by Donaldson–Segal [DS11]. They suggest that one should try to construct a counter term, say $m(E, \phi)$, as a *weighted* count of associative submanifolds and G_2 -instantons on bundles of “smaller” topological type than E , so that the sum $n(E, \phi) + m(E, \phi)$ is invariant under deformations. The crucial point here is to find out what these weights should be. I will discuss a promising candidate for the definition of these weights for the low energy $SU(2)$ -theory in Chapter 6.

One may be tempted to try to use Theorem 4.7 to produce bubbling families of G_2 -instantons. Unfortunately, I currently have no examples of the required input. There is, however, reason to hope that this situation will improve in the future with the help of recent work of Corti–Haskins–Nordström–Pacini [CHNP12b, Section 5] on associative submanifolds in G_2 -manifolds obtained by (a generalisation of) Kovalev’s twisted connected sum construction [Kov03] and possibly further progress in the work on G_2 -instantons over twisted connected sums [SE11a, SE11b] begun by Sá Earp, see also Chapter 3. It should be pointed out that using methods from this chapter we will construct concrete examples of families of $\text{Spin}(7)$ -instantons bubbling along Cayley submanifolds in Chapter 5, some of which were previously constructed in Lewis’ thesis [Lew98].

The proof of Theorem 4.7 is based on a gluing construction and the analysis involved is an extension of that in Chapter 2; as such it shares some similarities with Lewis’ construction of $\text{Spin}(7)$ -instantons [Lew98], unpublished work by Brendle on $\text{Spin}(7)$ -instantons [Bre03a], see also [Bre03b], and Pacard–Ritoré’s work on the Allen–Cahn equation [PR03].

4.1 Fueter sections of instanton moduli bundles over associative submanifolds

Let M be a fixed moduli space of framed ASD instantons on a G -bundle E over $\mathbf{R}^4$ as in Section 2.2. Clearly, $\Lambda^+ := \Lambda^+(\mathbf{R}^4)^* \cong \mathfrak{so}(4)$ acts $SO(4)$ -equivariantly on $\mathbf{R}^4$ and on $\mathbf{R} \oplus \Lambda^+$. It is a straight-forward computation to verify that the corresponding actions of Λ^+ on $\Omega^1(\mathbf{R}^4, \mathfrak{g}_E)$ and on $\Omega^0(\mathbf{R}^4, \mathfrak{g}_E) \oplus \Omega^+(\mathbf{R}^4, \mathfrak{g}_E)$ commute with the linear operator $\delta_I = (d_I^*, d_I^+): \Omega^1(\mathbf{R}^4, \mathfrak{g}_E) \rightarrow \Omega^0(\mathbf{R}^4, \mathfrak{g}_E) \oplus \Omega^+(\mathbf{R}^4, \mathfrak{g}_E)$ which controls the infinitesimal deformation theory of the ASD instanton I . This yields an $SO(4)$ -equivariant action of Λ^+ on TM .

Remark 4.10. If we fix an identification $\mathbf{R}^4 = \mathbf{H}$ and correspondingly $\Lambda^+ = \text{Im } \mathbf{H}$, then the above defines a hyperkähler structure on TM . However, for our purpose it is more natural not to fix such an identification.

M carries an action of $\mathbf{R}^4 \rtimes \mathbf{R}^+$ where $\mathbf{R}^4$ acts by translation and $\mathbf{R}^+$ acts by dilation, i.e., by pullback via s_λ where

$$s_\lambda(x) := \lambda x$$

for $\lambda \in \mathbf{R}^+$. Since the centre of mass of the measure $|F_I|^2 \text{dvol}$ is equivariant with respect to the $\mathbf{R}^4$ -action, we can write $M = \mathring{M} \times \mathbf{R}^4$ where $\mathring{M}$ is the space of instantons centred at zero. The action of Λ^+ preserves this product structure and Λ^+ acts on the factor $\mathbf{R}^4$ in the usual way.

Example 4.11. If E is the unique $\text{SU}(2)$ -bundle over S^4 with $c_2(E) = 1$, then up to scaling, translation and changing the framing at infinity there is a unique ASD instanton I on E , commonly called “the one-instanton”. Hence, $M = \mathring{M} \times \mathbf{R}^4 = \mathbf{R}^+ \times \text{SU}(2)/\mathbf{Z}_2 \times \mathbf{R}^4$. Moreover, we can write $\mathring{M} = (\text{Re}(S^+ \otimes \mathbf{C}^2) \setminus \{0\})/\mathbf{Z}_2$ where $\text{Spin}(4)$ acts in the natural way on S^+ , the positive spin representation associated with $\mathbf{R}^4$, and $\text{SU}(2)$ acts on $\mathbf{C}^2$.

Example 4.12. In general, if E is an $\text{SU}(n)$ -bundle over S^4 , then M can be understood rather explicitly in terms of the ADHM construction [DK90, Section 3.3].

Now, let (Y, ϕ) be a G_2 -manifold, let P be an associative submanifold in Y , let M be a fixed moduli space of framed ASD instantons over $\mathbf{R}^4$ and let E_∞ be a G -bundle over P together with a connection A_∞ . In the context of Theorem 4.7 we take $E_\infty := E_0|_P$ and $A_\infty := A_0|_P$.

Definition 4.13. The *instanton moduli bundle* $\mathfrak{M} \rightarrow P$ associated with P , E_∞ and M is defined by

$$\mathfrak{M} := (\text{Fr}(NP) \times E_\infty) \times_{\text{SO}(4) \times G} M.$$

Example 4.14. Let $M = (\text{Re}(S^+ \otimes \mathbf{C}^2) \setminus \{0\})/\mathbf{Z}_2 \times \mathbf{R}^4$ be the moduli space of framed ASD instantons from Example 4.11. If we pick $\mathfrak{s}$ and $\mathfrak{u}$ as in Remark 1.69, then

$$\mathfrak{M} = (\mathfrak{s} \times \mathfrak{u} \times E_\infty) \times_{\text{Spin}(4) \times \text{SU}(2)} M = (\text{Re}(S \otimes E_\infty) \setminus \{0\})/\mathbf{Z}_2 \times NP.$$

Here we used the fact that the $\text{SO}(4)$ action on M lifts to an action of $\text{Spin}(4)$.

Denote by $N_\infty P := \text{Fr}(NP) \times_{\text{SO}(4)} S^4$ the sphere-bundle obtained from NP by adjoining a section at infinity.

Theorem 4.15 (Donaldson–Segal [DS11] and Haydys [Hay12]). *To each section $\mathfrak{J} \in \Gamma(\mathfrak{M})$ we can assign a G -bundle $E = E(\mathfrak{J})$ over $N_\infty P$ together with a connection $I = I(\mathfrak{J})$ and a framing $\Phi: E|_\infty \rightarrow E_\infty$ such that:*

- For each $x \in P$ the restriction of I to $N_x P$ represents $\mathfrak{I}(x)$.
- The framing Φ identifies the restriction of I to the section at infinity with A_∞ .

The actions of $\mathbf{R}^+$ on $\mathbf{R}^4$ and M lift to fibre-wise actions on NP and $\mathfrak{M}$. The construction in Theorem 4.15 is equivariant with respect to these actions. In particular, $I(s_\lambda^* \mathfrak{I}) = s_\lambda^* I(\mathfrak{I})$. It will be convenient to use the shorthand notations

$$I_\lambda := I(s_{1/\lambda}^* \mathfrak{I}) \quad \text{and} \quad \mathfrak{I}_\lambda := s_{1/\lambda}^* \mathfrak{I}.$$

If a section $\mathfrak{I} \in \Gamma(\mathfrak{M})$ does arise from a sequence of G_2 -instantons bubbling along P , then it is reasonable to expect that in the limit as λ goes to zero the connection I_λ is “close to being a G_2 -instanton”. To make sense of that notion we define the 4-form ψ_0 on NP to be the zeroth order Taylor expansion of $\psi := \Theta(\phi) = *_\phi \phi$ off P . More explicitly, we can write ψ_0 as

$$(4.16) \quad \psi_0 := \text{vol}_{NP} - e^1 \wedge e^2 \wedge \omega_{e_3} - e^2 \wedge e^3 \wedge \omega_{e_1} - e^3 \wedge e^1 \wedge \omega_{e_2}.$$

Here (e_i) is a local positive orthonormal frame on P , (e^i) is its dual frame, vol_{NP} is the fibre-wise volume form on NP and $v \in TP \mapsto \omega_v \in \Lambda^+ N^* P$ is given by the identification (1.67). With this notation set up, the natural requirement is that

$$(4.17) \quad \lim_{\lambda \rightarrow 0} \lambda^{-2} s_\lambda^* (F_{I_\lambda} \wedge \psi_0) = F_I \wedge (\psi_0 - \text{vol}_{NP}) = 0.$$

If we introduce a bi-grading on k -forms on NP according to the splitting $TNP = \pi_1^* TP \oplus \pi_2^* NP$ corresponding to the connection on NP with $\pi_1: TP \rightarrow P$ and $\pi_2: NP \rightarrow P$ denoting the canonical projections, then it is easy to see that equation (4.17) splits into two parts. The first one is simply the condition that the anti-self-dual part of $F_I^{0,2}$ must vanish, while the second part is given by

$$F_I^{1,1} \wedge \psi_0 = 0.$$

This condition can be understood as a partial differential equation on $\mathfrak{I}$ as follows. Define the vertical tangent bundle $V\mathfrak{M}$ to $\mathfrak{M}$ by

$$V\mathfrak{M} := (\text{Fr}(NP) \times E_\infty) \times_{\text{SO}(4) \times G} TM.$$

If $\mathfrak{I}$ is a section of $\mathfrak{M}$, then the action of Λ^+ on M induces a Clifford multiplication $\gamma: TP \rightarrow \text{End}(\mathfrak{I}^* V\mathfrak{M})$ in view of the identification (1.67). Moreover, the connections on NP and E_∞ induce a connection on $\mathfrak{M}$ assigning to each section $\mathfrak{I}$ its covariant derivative $\nabla \mathfrak{I} \in \Omega^1(\mathfrak{I}^* V\mathfrak{M})$.

Definition 4.18. The *Fueter operator* $\mathfrak{F} = \mathfrak{F}_\phi$ associated with $\mathfrak{M}$ is defined by

$$\mathfrak{J} \in \Gamma(\mathfrak{M}) \mapsto \mathfrak{F}_\phi \mathfrak{J} := \sum_i \gamma(e_i) \nabla_i \mathfrak{J} \in \Gamma(\mathfrak{J}^* V \mathfrak{M})$$

where (e_i) is a local orthonormal frame on P . A section $\mathfrak{J} \in \Gamma(\mathfrak{M})$ is called a *Fueter section* if it satisfies

$$\mathfrak{F} \mathfrak{J} = 0.$$

Example 4.19. If M is as in Example 4.11, then the Fueter operator $\mathfrak{F}$ lifts to the twisted Dirac operator $D: \Gamma(\text{Re}(S \otimes (E_\infty \oplus U))) \rightarrow \Gamma(\text{Re}(S \otimes (E_\infty \oplus U)))$. Here U is as in Remark 1.69.

The Fueter operator $\mathfrak{F}$ is compatible with the product structure on $\mathfrak{M} = \mathring{\mathfrak{M}} \times NP$ corresponding to $M = \mathring{M} \times \mathbf{R}^4$. Its restriction to the second factor is given by the Fueter operator F_P associated with P .

Theorem 4.20 (Donaldson–Segal [DS11] and Haydys [Hay12]). *If $\mathfrak{J} \in \Gamma(\mathfrak{M})$, then we can identify $\Gamma(\mathfrak{J}^* V \mathfrak{M})$ with a subspace of $\Omega^1(NP, \mathfrak{g}_{E(\mathfrak{J})})$. With respect to this identification we have the identity*

$$\mathfrak{F} \mathfrak{J} = *_0 \left(F_{I(\mathfrak{J})}^{1,1} \wedge \psi_0 \right)$$

where $*_0$ is the Hodge–*–operator on NP . In particular, $I(\mathfrak{J})$ satisfies equation (4.17) if and only if $\mathfrak{J}$ is a Fueter section.

Definition 4.21. The *linearised Fueter operator* $F_{\mathfrak{J}} = F_{\mathfrak{J}, \phi}: \Gamma(\mathfrak{J}^* V \mathfrak{M}) \rightarrow \Gamma(\mathfrak{J}^* V \mathfrak{M})$ for $\mathfrak{J} \in \Gamma(\mathfrak{M})$ is defined by

$$F_{\mathfrak{J}, \phi}(\hat{\mathfrak{J}}) := \sum_i \gamma(e_i) \nabla_i \hat{\mathfrak{J}}$$

where (e_i) is a local orthonormal frame on P .

Example 4.22. If M is as in Example 4.11, then the linearised Fueter operator $F_{\mathfrak{J}}$ lifts to the twisted Dirac operator $D: \Gamma(\text{Re}(S \otimes_{\mathbf{C}} (E_\infty \oplus U))) \rightarrow \Gamma(\text{Re}(S \otimes_{\mathbf{C}} (E_\infty \oplus U)))$. In particular, it only depends on the spin structure $\mathfrak{s}$ and not on $\mathfrak{J}$.

The operator $F_{\mathfrak{J}}$ is self-adjoint and elliptic; however, it can never be invertible if $\mathfrak{J}$ is a Fueter section. This is because Fueter sections come in 1-parameter families $(\mathfrak{J}_\lambda)_{\lambda \in \mathbf{R}_+}$. In particular, taking the derivative at $\lambda = 1$ yields an element in the kernel of $F_{\mathfrak{J}}$. If $\hat{v} \in \Gamma(V \mathfrak{M})$ denotes the vector field generating the action of $\mathbf{R}^+$ on $\mathfrak{M}$, then we can succinctly write this element of the kernel as $\hat{v} \circ \mathfrak{J}$.

Let $(\phi_t)_{t \in (-T, T)}$ be a family of torsion-free G_2 -structures on Y , let A_0 be an unobstructed G_2 -instanton on a G -bundle E over (Y, ϕ_0) and let P_0 be an unobstructed associative submanifold in (Y, ϕ_0) . Then by Propositions 1.107 and 1.73 we obtain a family of G_2 -instantons $(A_t)_{t \in (-T', T')}$ over (Y, ϕ_t) and a family of associative submanifolds

$(P_t)_{t \in (-T', T')}$ in (Y, ϕ_t) for some $T' \in (0, T]$. Now, carry out the above construction with $P = P_t$, $E_\infty = E|_{P_t}$, $A_\infty = A_t|_{P_t}$ and a fixed moduli space M of framed finite energy ASD instantons on $\mathbf{R}^4$ to obtain a family of instanton moduli bundles $(\mathfrak{M}_t)_{t \in (-T', T')}$ along with a family of Fueter operators $(\mathfrak{F}_t)_{t \in (-T', T')}$. If $\mathfrak{I}_0$ is a Fueter section of $\mathfrak{M}_0$ with

$$\dim \ker F_{\mathfrak{I}_0} = 1,$$

then, using the implicit function theorem, we obtain a family $(\mathfrak{I}_t)_{t \in (-T', T')}$ of sections of $\mathfrak{M}_t$ satisfying

$$(4.23) \quad \mathfrak{F}_t \mathfrak{I}_t + \mu(t) \cdot \hat{v} \circ \mathfrak{I}_t = 0$$

where $\mu: (-T', T') \rightarrow \mathbf{R}$ is a smooth function vanishing at zero.

Definition 4.24. In the above situation we say that $\mathfrak{I}_0$ is *unobstructed* with respect to (ϕ_t) if

$$\left. \frac{\partial \mu}{\partial t} \right|_{t=0} \neq 0.$$

Example 4.25. If M is as in Example 4.11, then equation (4.23) can be viewed as the spectral flow of a family of twisted Dirac operators and $\mathfrak{I}_0$ is unobstructed if and only if this spectral flow has a regular crossing at $\mathfrak{I}_0$.

4.2 Overview of the proof of Theorem 4.7

Suppose that Y , $(\phi_t)_{t \in (-T, T)}$, A_0 , P and $\mathfrak{I} \in \Gamma(\mathfrak{M})$ are as in the hypothesis of Theorem 4.7. To simplify the exposition we work under the additional hypothesis that A_0 is irreducible, i.e., we assume that A_0 is acyclic. This assumption can be removed by arguing as before at the end of Section 2.6.

Convention 4.26. We fix constants $T' \in (0, T]$ and $\Lambda > 0$ such that all of the statements of the kind “if $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$, then ...” appearing in the following are valid. This is possible since there is only a finite number of these statements and each one of them is valid provided T' and Λ are sufficiently small. By $c > 0$ we will denote a generic constant whose value depends neither on $t \in (-T', T')$ nor on $\lambda \in (0, \Lambda]$ but may change from one occurrence to the next.

As discussed at the end of Section 4.1, A_0 and $P_0 := P$ give rise to

- a family $(A_t)_{t \in (-T', T')}$ of G_2 -instantons on E_0 over (Y, ϕ_t) ,
- a family of associative submanifolds $(P_t)_{t \in (-T', T')}$ in (Y, ϕ_t) and, hence,

- a family of instanton moduli bundles $(\mathfrak{M}_t)_{t \in (-T', T')}$ with $\mathfrak{M}_0 = \mathfrak{M}$ and Fueter operators $(\mathfrak{F}_t)_{t \in (-T', T')}$ together with sections $(\mathfrak{J}_t)_{t \in (-T', T')}$ satisfying $\mathfrak{J}_0 = \mathfrak{J}$ and

$$(4.27) \quad \mathfrak{F}_t \mathfrak{J}_t + \mu(t) \hat{v} \circ \mathfrak{J}_t = 0$$

where $\mu: (-T', T') \rightarrow \mathbf{R}$ is a smooth function vanishing at zero with

$$\left. \frac{\partial \mu}{\partial t} \right|_{t=0} \neq 0.$$

The proof of Theorem 4.7 proceeds via a gluing construction. As a first step we explain how to construct approximate solutions.

Proposition 4.28. *For each $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we can explicitly construct a G -bundle $E_{t,\lambda}$ together with a connection $A_{t,\lambda} = A_t \#_{\lambda} \mathfrak{J}_t$ from E_0 , $A_t \in \mathcal{A}(E_0)$ and $\mathfrak{J}_t$. The bundles $E_{t,\lambda}$ are pairwise isomorphic.*

Before we embark on the proof, let us set up some notation. Fix a constant $\sigma > 0$ such that for all $t \in (-T', T')$ the exponential map identifies a tubular neighbourhood of width 8σ of P_t in Y with a neighbourhood of the zero section in NP_t . For $I \subset \mathbf{R}$ we set

$$U_{I,t} := \{v \in NP_t : |v| \in I\} \quad \text{and} \quad V_{I,t} := \{x \in Y : r_t(x) \in I\}.$$

Here $r_t := d(\cdot, P_t): Y \rightarrow [0, \infty)$ denotes the distance from P_t . Fix a smooth-cut off function $\chi: [0, \infty) \rightarrow [0, 1]$ which vanishes on $[0, 1]$ and is equal to one on $[2, \infty)$. For $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we define $\chi_{t,\lambda}^-: Y \rightarrow [0, 1]$ and $\chi_t^+: Y \rightarrow [0, 1]$ by

$$\chi_{t,\lambda}^-(x) := \chi(r_t(x)/2\lambda) \quad \text{and} \quad \chi_t^+(x) := 1 - \chi(r_t(x)/2\sigma),$$

respectively.

Proof of Proposition 4.28. Via radial parallel transport we can identify $E(\mathfrak{J}_t)$ over $U_{(R,\infty),t}$ for some $R > 0$ with the pullback of $E(\mathfrak{J}_t)|_{\infty}$ to said region and similarly we can identify E_0 over $V_{[0,\sigma),t}$ with the pullback of $E_0|_{P_t}$. Hence, via the framing Φ we can identify $s_{1/\lambda}^* E(\mathfrak{J}_t)$ with E_0 on the overlap $V_{(\lambda,\sigma),t}$ for $\lambda \in (0, \Lambda]$. Patching both bundles via this identification yields $E_{t,\lambda}$.

To construct a connection on $E_{t,\lambda}$ note that on the overlap $I_{t,\lambda} := s_{1/\lambda}^* I(\mathfrak{J}_t)$ and A_t can be written as

$$I_{t,\lambda} = A_t|_{P_t} + i_{t,\lambda} \quad \text{and} \quad A_t = A_t|_{P_t} + a_t.$$

Here and in the following, by a slight abuse of notation, we denote by $A_t|_{P_t}$ the pullback of $A_t|_{P_t}$ to the overlap. We define $A_{t,\lambda}$ by interpolating between $I_{t,\lambda}$ and A_t on the overlap as

follows

$$(4.29) \quad A_{t,\lambda} := A_t|_{P_t} + \chi_{t,\lambda}^- a_t + \chi_t^+ i_{t,\lambda}. \quad \square$$

Remark 4.30. Note that in equation (4.29) we interpolate in such a way that there is a relatively large region in which $\chi_{t,\lambda}^-$ and χ_t^+ are equal to one simultaneously. This will turn out to be very useful later on, since it minimises the error introduced by derivatives of the cut-off functions, when measured at the appropriate scale, and the construction crucially hinges on this error to be rather small.

Now, in view of Proposition 1.98, the task at hand is to solve the equation

$$(4.31) \quad *_{\phi_t} (F_{A_{t,\lambda}+a} \wedge \psi_t) + d_{A_{t,\lambda}+a} \xi = 0$$

where $\psi_t := \Theta(\phi_t) = *_{\phi_t} \phi_t$, $t = t(\lambda)$, $a = a(\lambda)$ and $\xi = \xi(\lambda)$. If we could find an appropriate analytic setup in which $(a, \xi) = 0$ becomes closer and closer to being a solution of equation (4.31) while at the same time the linearisations $L_{t,\lambda} := L_{A_{t,\lambda}, \phi_t}$, as defined in (1.103), possess right inverses that can be controlled *uniformly* in t and λ , then it would not be too difficult to solve equation (4.31) for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$. Since the properties of $L_{t,\lambda}$ are closely linked among other things to those of $F_{\mathfrak{J}_t}$ and since $F_{\mathfrak{J}_0}$ has a one-dimensional cokernel, however, we will only be able to solve equation (4.31) “modulo the cokernel of $F_{\mathfrak{J}_0}$ ”. More precisely, when $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we will be able to solve the equation

$$(4.32) \quad L_{t,\lambda} \underline{a} + \eta \cdot \iota_{t,\lambda} \hat{v} \circ \mathfrak{J}_t + Q_{t,\lambda}(\underline{a}) + e_{t,\lambda} = 0$$

for $\underline{a} = (\xi, a) \in \Omega^0(Y, \mathfrak{g}_{E_{t,\lambda}}) \oplus \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}})$ and $\eta \in \mathbf{R}$ with $Q_{t,\lambda}$ and $e_{t,\lambda}$ defined by

$$(4.33) \quad Q_{t,\lambda}(\underline{a}) := \frac{1}{2} * ([a \wedge a] \wedge \psi_t) + [\xi, a].$$

and

$$e_{t,\lambda} := *(F_{A_{t,\lambda}} \wedge \psi_t) + \mu(t) \cdot \iota_{t,\lambda} \hat{v} \circ \mathfrak{J}_t,$$

respectively. Here the map $\iota_{t,\lambda}: \Gamma(\mathfrak{J}_t^* V \mathfrak{M}_t) \rightarrow \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}})$ is defined by

$$\iota_{t,\lambda} \hat{\mathfrak{J}} := \chi_t^+ s_{1/\lambda}^* \hat{\mathfrak{J}}$$

where we first identify $\hat{\mathfrak{J}} \in \Gamma(\mathfrak{J}_t^* V \mathfrak{M}_t)$ with an element of $\Omega^1(NP, E(\mathfrak{J}_t))$, then view the restriction of its pullback via s_λ^{-1} to $U_{[0,\sigma),t}$ as lying in $\Omega^1(V_{[0,\sigma),t}, \mathfrak{g}_{E_{t,\lambda}})$ and finally extended it to all of Y by multiplication with χ_t^+ . After solving (4.32) we are left with the residual scalar equation

$$\mu(t) + \eta(t, \lambda) = 0.$$

It will turn out that η and $\partial_t \eta$ go to zero as λ goes to zero. Since $\partial_t \mu(0) \neq 0$, finding $t = t(\lambda)$ such that equation (4.31) is satisfied is then a simple consequence of an implicit function theorem.

Let us now discuss some aspects of the analysis. First of all we will introduce appropriate weighted Hölder spaces in Section 4.3. One should think of these weighted spaces as a convenient framework to deal with different local scales simultaneously. In our case they are constructed to counteract the fact that the curvature of the connection $A_{t,\lambda}$ around P_t becomes larger and larger as λ goes to zero. We will see in Section 4.4 that the amount by which our approximate solutions $A_{t,\lambda}$ fail to be solutions of equation (4.31) “modulo the cokernel of $F_{\mathcal{J}_0}$ ” measured in our weighted Hölder norms goes to zero at a certain rate as λ goes to zero. The key difficulty then lies in analysing the linearisation $L_{t,\lambda}$. As is the case in most adiabatic limit constructions, the linearisation $L_{t,\lambda}$ is rather badly behaved on an infinite dimensional space: For every $\hat{\mathcal{J}} \in \Gamma(\mathcal{J}_t^* V \mathfrak{M}_t)$ the appropriate norm of $\iota_{t,\lambda} \hat{\mathcal{J}}$ is essentially independent of λ , while the appropriate norm of $L_{t,\lambda} \iota_{t,\lambda} \hat{\mathcal{J}}$ tends to zero as λ tends to zero. To overcome this issue it is convenient to split the problem at hand into a part coming from $\Gamma(\mathcal{J}_t^* V \mathfrak{M}_t)$ and the part orthogonal to it. We define $\pi_{t,\lambda} : \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}}) \rightarrow \Gamma(\mathcal{J}_t^* V \mathfrak{M}_t)$ by

$$(\pi_{t,\lambda} a)(x) := \sum_{\kappa} \int_{N_x P} \langle a, \iota_{t,\lambda} \kappa \rangle \kappa$$

for $x \in P_t$. Here κ runs through an orthonormal basis of $(V \mathfrak{M}_t)_{\mathcal{J}(x)}$ with respect to the inner product $\langle \iota_{t,\lambda} \cdot, \iota_{t,\lambda} \cdot \rangle$. Clearly $\pi_{t,\lambda} \iota_{t,\lambda} = \text{id}$ and hence $\bar{\pi}_{t,\lambda} := \iota_{t,\lambda} \pi_{t,\lambda}$ is a projection. We denote the complementary projection by $\rho_{t,\lambda} := \text{id} - \bar{\pi}_{t,\lambda}$. If we define

$$\mathfrak{A}_{t,\lambda} := \Omega^0(Y, \mathfrak{g}_{E_{t,\lambda}}) \oplus \ker \pi_{t,\lambda},$$

then we can write

$$\Omega^0(Y, \mathfrak{g}_{E_{t,\lambda}}) \oplus \Omega^1(Y, \mathfrak{g}_{E_{t,\lambda}}) = \mathfrak{A}_{t,\lambda} \oplus \Gamma(\mathcal{J}_t^* V \mathfrak{M}_t)$$

and decompose $L_{t,\lambda}$ accordingly into a 2-by-2 matrix of operators. We will see in Section 4.5 that the diagonal entries can be controlled in terms of certain models on $\mathbf{R}^7$, L_{A_0} and the linearised Fueter operator $F_{\mathcal{J}_0}$, while the off-diagonal terms are negligibly small. In Section 4.6 we discuss how to control the non-linearity $Q_{t,\lambda}$ in equation (4.32). The completion of the proof of Theorem 4.7 in Section 4.7 will then be rather straight-forward.

4.3 Weighted Hölder norms

For $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we define a family of weight functions $w_{\ell, \delta; t, \lambda}$ on Y depending on two additional parameters $\ell, \delta \in \mathbf{R}$ as follows

$$w_{\ell, \delta; t, \lambda}(x) := \begin{cases} \lambda^\delta (\lambda + r_t(x))^{-\ell - \delta} & \text{if } r_t(x) \leq \sqrt{\lambda} \\ r_t(x)^{-\ell + \delta} & \text{if } r_t(x) > \sqrt{\lambda} \end{cases}$$

and set $w_{\ell, \delta; t, \lambda}(x, y) := \min\{w_{\ell, \delta; t, \lambda}(x), w_{\ell, \delta; t, \lambda}(y)\}$. For a Hölder exponent $\alpha \in (0, 1)$ and $\ell, \delta \in \mathbf{R}$ we define (semi-)norms

$$\begin{aligned} \|f\|_{L_{\ell, \delta; t, \lambda}^\infty(U)} &:= \|w_{\ell, \delta; t, \lambda} f\|_{L^\infty(U)}, \\ [f]_{C_{\ell, \delta; t, \lambda}^{0, \alpha}(U)} &:= \sup_{\substack{x \neq y \in U: \\ d(x, y) \leq \lambda + \min\{r_t(x), r_t(y)\}}} w_{\ell - \alpha, \delta; t, \lambda}(x, y) \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \quad \text{and} \\ \|f\|_{C_{\ell, \delta; t, \lambda}^{k, \alpha}(U)} &:= \sum_{j=0}^k \|\nabla^j f\|_{L_{\ell - j, \delta; t, \lambda}^\infty(U)} + [\nabla^k f]_{C_{\ell - k, \delta; t, \lambda}^{0, \alpha}}. \end{aligned}$$

Here f is a section of a vector bundle over $U \subset Y$ equipped with an inner product and a compatible connection. We use parallel transport to compare the values of f at different points. If U is not specified, then we take $U = Y$. We will primarily use this norm for $\mathfrak{g}_{E_t, \lambda}$ -valued tensor fields.

Remark 4.34. The reader may find the following heuristic useful. Let f be a k -form on Y . Fix a small ball centred at a point $x \in P_t$, identify it with a small ball in $T_x Y = T_x P_t \oplus N_x P_t$ and rescale this ball by a factor $1/\lambda$. Upon pulling everything back to this rescaled ball the weight function $w_{-k, \delta; t, \lambda}$ becomes essentially $\lambda^k (1 + |y|)^{k - \delta}$, where y denotes the $N_x P_t$ -coordinate. Thus as λ goes to zero a uniform bound $\|f_\lambda\|_{L_{-k, \delta; t, \lambda}^\infty}$ on a family (f_λ) of k -forms ensures that the pullbacks of f_λ decay like $|y|^{-k + \delta}$ in the direction of $N_x P_t$. At the same time it forces f_λ not to blowup at a rate faster than $r_t^{-k - \delta}$ along P_t . The “discrepancy” in the exponents can be seen to be rather natural by considering the action of the inversion $y \mapsto \lambda y / |y|^2$.

Proposition 4.35. *If $(f, g) \mapsto f \cdot g$ is a bilinear form satisfying $|f \cdot g| \leq |f||g|$, then*

$$\|f \cdot g\|_{C_{\ell_1 + \ell_2, \delta_1 + \delta_2; t, \lambda}^{k, \alpha}} \leq \|f\|_{C_{\ell_1, \delta_1; t, \lambda}^{k, \alpha}} \|g\|_{C_{\ell_2, \delta_2; t, \lambda}^{k, \alpha}}.$$

Proof. This follows immediately from the above definition. $\square$

Corollary 4.36. *If $\delta < 0$, then there is a constant $c > 0$ which is independent of $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ such that*

$$\|f\|_{C_{\ell, \delta; t, \lambda}^{k, \alpha}} \leq c \lambda^{\delta/2} \|f\|_{C_{\ell, 0; t, \lambda}^{k, \alpha}} \quad \text{and} \quad \|f\|_{C_{\ell, 0; t, \lambda}^{k, \alpha}} \leq c \|f\|_{C_{\ell, \delta; t, \lambda}^{k, \alpha}}$$

Proof. Use $\|1\|_{C_{0,\delta;t,\lambda}^{k,\alpha}} \leq c\lambda^{\delta/2}$ and $\|1\|_{C_{0,-\delta;t,\lambda}^{k,\alpha}} \leq c$ for $\delta < 0$. $\square$

Proposition 4.37. *For $\ell \leq -1$ and $\delta \in \mathbf{R}$ such that $\ell + \delta \in (-3, -1)$ there is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\|\iota_{t,\lambda} \hat{\mathcal{J}}\|_{C_{\ell,\delta;t,\lambda}^{0,\alpha}} \leq c\lambda^{-1-\ell} \|\hat{\mathcal{J}}\|_{C^{0,\alpha}} \quad \text{and} \quad \|\pi_{t,\lambda} a\|_{C^{0,\alpha}} \leq c\lambda^{1+\ell-\alpha} \|a\|_{C_{\ell,\delta;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})}.$$

In particular, $\bar{\pi}_{t,\lambda} = \iota_{t,\lambda} \pi_{t,\lambda}$ and $\rho_{t,\lambda}$ are bounded by $c\lambda^{-\alpha}$ with respect to the $C_{\ell,\delta;t,\lambda}^{0,\alpha}$ -norms.

Proof. From Proposition 2.41 it follows at once that

$$\|s_{1/\lambda}^* \hat{\mathcal{J}}\|_{C_{-3,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} \leq c\lambda^2 \|\hat{\mathcal{J}}\|_{C^{0,\alpha}}.$$

The first inequality thus is a consequence of Proposition 4.35 since $\|\chi_t^+\|_{C_{3+\ell,\delta;t,\lambda}^{0,\alpha}} \leq c\lambda^{-3-\ell}$ for $\ell + \delta > -3$.

To prove the second inequality, note that by Proposition 2.41 for $\kappa \in (V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$ we have $|s_{1/\lambda}^* \kappa|(x) \leq c\lambda^2/(\lambda + |x|)^3 \|\kappa\|_{L^2}$ and thus

$$\begin{aligned} \int_{N_x P} \left\langle a, \chi_t^+ s_{1/\lambda}^* \kappa \right\rangle &\leq c \int_0^{\sqrt{\lambda}} \lambda^{2-\delta} (\lambda + r)^{\ell+\delta-3} r^3 dr \cdot \|a\|_{L_{\ell,\delta;t,\lambda}^\infty} \|\kappa\|_{L^2} \\ &\quad + c \int_{\sqrt{\lambda}}^\sigma \lambda^2 r^{\ell-\delta} (\lambda + r)^{-3} r^3 dr \cdot \|a\|_{L_{\ell,\delta;t,\lambda}^\infty} \|\kappa\|_{L^2} \\ &\leq c\lambda^{3+\ell} \|a\|_{L_{\ell,\delta;t,\lambda}^\infty} \|\kappa\|_{L^2} \end{aligned}$$

since $\ell \leq -1$ and $\ell + \delta < -1$. If κ is an element of an orthonormal basis of $(V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$ with respect to $\langle \iota_{t,\lambda} \cdot, \iota_{t,\lambda} \cdot \rangle$, then $\|\kappa\|_{L^2} \leq c/\lambda$ since for $\kappa_1, \kappa_2 \in (V\mathfrak{M}_t)_{\mathfrak{I}_t(x)}$

$$\lambda^2 \langle \kappa_1, \kappa_2 \rangle_{L^2} \sim \left\langle \chi_t^+ s_{1/\lambda}^* \kappa_1, \chi_t^+ s_{1/\lambda}^* \kappa_2 \right\rangle_{L^2}$$

where $\sim$ means comparable uniformly in t and λ . Therefore,

$$\|\pi_{t,\lambda} a\|_{L^\infty} \leq c\lambda^{1+\ell} \|a\|_{L_{\ell,\delta;t,\lambda}^\infty}.$$

The estimates on the Hölder norms follow by the same kind of argument. $\square$

4.4 Pregluing estimate

In the following we will need to differentiate various tensors over Y and NP_t depending on $t \in (-T, T)$. For tensors over Y we could simply differentiate using ∂_t ; however, ∂_t does not drag P_t along in a parallel fashion, which causes some additional error terms to show up, and also it is preferable to differentiate tensors over Y and NP_t in way that is consistent with

the identification $V_{I,t} = U_{I,t}$ for $I \subset [0, 2\sigma)$. Therefore we use a fixed set of connections constructed as follows: For each t we can write $P_t = \{\exp_p(v_t) : p \in P_0\}$ for some unique normal vector field $v_t \in \Gamma(P_0, NP_0)$; hence, the bundle $\coprod_{t \in (-T, T)} P_t \rightarrow (-T, T)$ comes with a canonical connection. Pick a connection on $\mathcal{N} := \coprod_{t \in (-T, T)} NP_t \rightarrow (-T, T)$ such that for each parallel path $t \mapsto p_t \in P_t$ its lift to the zero section $t \mapsto 0_{p_t} \in NP_t$ is also parallel. Moreover, we pick a connection on $Y \times (-T, T) \rightarrow (-T, T)$ which agrees with the connection on $\mathcal{N}$ on $\bigcup_{t \in (-T', T')} V_{[0, 2\sigma), t} = \bigcup_{t \in (-T', T')} U_{[0, 2\sigma), t}$ and with ∂_t on $Y \setminus \bigcup_{t \in (-T', T')} V_{[0, 4\sigma), t}$. These connections induce various connections on bundles of tensors over Y and NP_t ; we denote the associated covariant derivatives by ∇_t .

Proposition 4.38. *There is a constant $c > 0$ such that for $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\|e_{t,\lambda}\|_{C_{-2,0;t,\lambda}^{0,\alpha}} \leq c\lambda^2 \quad \text{and} \quad \|\nabla_t e_{t,\lambda}\|_{C_{-2,0;t,\lambda}^{0,\alpha}} \leq c\lambda^2.$$

The proof of this result requires some preparation.

Proposition 4.39. *In the tubular neighbourhood $V_{[0,\sigma);t}$ of P_t we can write $\psi_t := \Theta(\phi_t) = *_{\phi_t} \phi_t$ as*

$$\psi_t = \psi_{0;t} + \psi_{1;t} + \psi_{\geq 2;t}$$

where $\psi_{0;t}$ is defined as in equation (4.16), $\psi_{1;t}$ takes values in $\Lambda^2 T^* P_t \otimes \Lambda^+ N^* P_t$. Moreover, $\psi_{0;t}$, $\psi_{1;t}$ and $\psi_{\geq 2;t}$ depend continuously differentiably on t , and there is a constant $c > 0$ such that for all $t \in (-T', T')$ we have

$$\begin{aligned} & \|\psi_{0;t}\|_{C_{0,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma);t})} + \|\psi_{1;t}\|_{C_{1,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma);t})} + \|\psi_{\geq 2;t}\|_{C_{2,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma);t})} \leq c \quad \text{and} \\ & \|\nabla_t \psi_{0;t}\|_{C_{0,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma);t})} + \|\nabla_t \psi_{1;t}\|_{C_{1,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma);t})} + \|\nabla_t \psi_{\geq 2;t}\|_{C_{2,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma);t})} \leq c. \end{aligned}$$

Proof of Proposition 4.39. Let $P = P_t$ and $\psi = \psi_t$. If we pull the identity map of a tubular neighbourhood of P back to a tubular neighbourhood of the zero section of NP via the exponential map, then the Taylor expansion of its derivative around P can be expressed in the splitting $TNP = \pi_1^* TP \oplus \pi_2^* NP$ as

$$(x, y) \mapsto (x, y) + (\Pi_y(x), y) + O(|y|^2)$$

where Π is the second fundamental form of P in Y which we think of as a map from NP to $\text{End}(TP)$. This immediately yields the desired expansion of ψ near P , with ψ_1 taking values in $\Lambda^2 T^* P \otimes \Lambda^+ N^* P$, since we know that ψ is given by ψ_0 along P . Moreover, we have $\nabla^k \psi_1 = O(|y|^{1-k})$ and $\nabla^k \psi_{\geq 2} = O(|y|^{2-k})$ for $k = 0, 1$ which implies the first estimate and, since everything depends smoothly on t , also the second estimate. $\square$

The same reasoning also proves the following result.

Proposition 4.40. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\|*0 - *\|_{C_{1,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} + \|\nabla_t(*0 - *)\|_{C_{1,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} \leq c.$$

Proposition 4.41. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \|F_{I_{t,\lambda}}^{2,0} - F_{A_t|P_t}\|_{C_{-2,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} + \|\nabla_t(F_{I_{t,\lambda}}^{2,0} - F_{A_t|P_t})\|_{C_{-2,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} &\leq c\lambda^2, \\ \|F_{I_{t,\lambda}}^{1,1}\|_{C_{-3,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} + \|\nabla_t F_{I_{t,\lambda}}^{1,1}\|_{C_{-3,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} &\leq c\lambda^2 \quad \text{and} \\ \|F_{I_{t,\lambda}}^{0,2}\|_{C_{-4,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} + \|\nabla_t F_{I_{t,\lambda}}^{0,2}\|_{C_{-4,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} &\leq c\lambda^2. \end{aligned}$$

Proof. Theorem 4.15 asserts that the restriction of $I_t = I(\mathfrak{I}_t)$ to the section at infinity agrees with $A_t|_{P_t}$. Hence, $F_{I_t}^{2,0} - F_{A_t|P_t}$ vanishes to first order along the section at infinity which when viewed from the zero section in NP_t means that

$$|F_{I_t}^{2,0} - F_{A_t|P_t}| \leq \frac{c}{1 + |y|^2}.$$

The first estimate now follows from a simple scaling consideration and by realising that the above reasoning also applies to $\nabla_t(F_{I_t}^{2,0} - F_{A_t|P_t})$.

The last two estimates follow from Theorem 4.15, Proposition 2.41, the fact that the curvature of a finite energy ASD instanton decays at least like $|y|^{-4}$ and simple scaling considerations. $\square$

Proposition 4.42. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \|i_{t,\lambda}\|_{C_{-3,0;t,\lambda}^{0,\alpha}(V_{(\lambda,\sigma);t})} + \|d_{I_{t,\lambda}} i_{t,\lambda}\|_{C_{-4,0;t,\lambda}^{0,\alpha}(V_{(\lambda,\sigma);t})} &\leq c\lambda^2 \quad \text{and} \\ \|\nabla_t i_{t,\lambda}\|_{C_{-3,0;t,\lambda}^{0,\alpha}(V_{(\lambda,\sigma);t})} + \|\nabla_t(d_{I_{t,\lambda}} i_{t,\lambda})\|_{C_{-4,0;t,\lambda}^{0,\alpha}(V_{(\lambda,\sigma);t})} &\leq c\lambda^2 \end{aligned}$$

as well as

$$\begin{aligned} \|a_t\|_{C_{1,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} + \|d_{A_t|P_t} a_t\|_{C_{0,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} &\leq c \quad \text{and} \\ \|\nabla_t a_t\|_{C_{1,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} + \|\nabla_t(d_{A_t|P_t} a_t)\|_{C_{0,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} &\leq c. \end{aligned}$$

Proof. The first two estimates follow from the fact that we put I_t into radial gauge from the section at infinity in $N_\infty P_t$ and a simple scaling consideration, while the last two estimates follow from the fact that we put A_t into radial gauge from zero section in NP_t . $\square$

Proof of Proposition 4.38. We proceed in four steps. First we estimate an approximation

$\tilde{e}_{t,\lambda}$ of $e_{t,\lambda}$. Then we estimate the difference $e_{t,\lambda} - \tilde{e}_{t,\lambda}$ separately in the three subsets $V_{[0,\lambda);t}$, $V_{[\lambda,\sigma/2);t}$ and $V_{[\sigma/2,\sigma);t}$ constituting $V_{[0,\sigma);t}$ which contains the support of $e_{t,\lambda}$.

It will be convenient to use the following shorthand notation

$$\|f\|_{\ell,U} := \|f\|_{C_{\ell,0;t,\lambda}^{0,\alpha}(U)} + \|\nabla_t f\|_{C_{\ell,0;t,\lambda}^{0,\alpha}(U)}.$$

Note that if $(f, g) \mapsto f \cdot g$ is a bilinear map satisfying $|f \cdot g| \leq |f||g|$ and the Leibniz rule with respect to ∇_t , then it follows from Proposition 4.35 that $\|f \cdot g\|_{\ell_1+\ell_2,U} \leq \|f\|_{\ell_1,U} \cdot \|g\|_{\ell_2,U}$.

Step 1. *The term*

$$\tilde{e}_{t,\lambda} := * \left[\left(F_{I_{t,\lambda}} - F_{A_t|_{P_t}} \right) \wedge \psi_t \right] + \mu(t) \cdot \hat{v} \circ \mathfrak{I}_{t,\lambda}$$

satisfies $\|\tilde{e}_{t,\lambda}\|_{-2,V_{[0,\sigma);t}} \leq c\lambda^2$.

Because of Theorem 4.20, the fact that $F_{I_{t,\lambda}}^{0,2}$ is anti-self-dual and Proposition 4.39 we can write $\tilde{e}_{t,\lambda}$ on $V_{[0,\sigma);t}$ as

$$\begin{aligned} \tilde{e}_{t,\lambda} = * & \left[\left(F_{I_{t,\lambda}} - F_{A_t|_{P_t}} \right)^{2,0} \wedge \psi_t \right] + * \left[F_{I_{t,\lambda}}^{1,1} \wedge (\psi_{1;t} + \psi_{\geq 2;t}) \right] \\ & + * \left(F_{I_{t,\lambda}}^{0,2} \wedge \psi_{\geq 2;t} \right) + (* - *_0) \left(F_{I_{t,\lambda}}^{1,1} \wedge \psi_{0;t} \right). \end{aligned}$$

Using Propositions 4.39 and 4.41 as well as $\|1\|_{-1,V_{[0,\sigma);t}} \leq c$ we estimate $\|\tilde{e}_{t,\lambda}\|_{-2,V_{[0,\sigma);t}}$ by

$$\begin{aligned} & \left\| \left(F_{I_{t,\lambda}} - F_{A_t|_{P_t}} \right)^{2,0} \right\|_{-2,V_{[0,\sigma);t}} \cdot \|\psi_t\|_{0,V_{[0,\sigma);t}} \\ & + \left\| F_{I_{t,\lambda}}^{1,1} \right\|_{-3,V_{[0,\sigma);t}} \cdot \left(\|\psi_{1;t}\|_{1,V_{[0,\sigma);t}} + \|1\|_{-1,V_{[0,\sigma);t}} \cdot \|\psi_{\geq 2;t}\|_{2,V_{[0,\sigma);t}} \right) \\ & + \left\| F_{I_{t,\lambda}}^{0,2} \right\|_{-4,V_{[0,\sigma);t}} \cdot \|\psi_{\geq 2;t}\|_{2,V_{[0,\sigma);t}} \\ & + \|* - *_0\|_{1,V_{[0,\sigma);t}} \cdot \left\| F_{I_{t,\lambda}}^{1,1} \right\|_{-3,V_{[0,\sigma);t}} \cdot \|\psi_{0;t}\|_{0,V_{[0,\sigma);t}} \leq c\lambda^2. \end{aligned}$$

Step 2. *We prove that $\|e_{t,\lambda} - \tilde{e}_{t,\lambda}\|_{V_{[0,\lambda);t}} \leq c\lambda^2$.*

Since

$$\left\| F_{A_t|_{P_t}} \wedge \psi_t \right\|_{-2,V_{[0,\lambda);t}} \leq \|1\|_{-2,V_{[0,\lambda);t}} \cdot \left\| F_{A_t|_{P_t}} \wedge \psi_t \right\|_{0,V_{[0,\lambda);t}} \leq c\lambda^2,$$

it suffices to estimate $F_{A_{t,\lambda}} - F_{I_{t,\lambda}}$ in $V_{[0,\lambda);t}$. Now, in $V_{[0,\lambda);t}$ the curvature of $A_{t,\lambda}$ is given by

$$F_{A_{t,\lambda}} = F_{I_{t,\lambda}} + \chi_{t,\lambda}^- d_{I_{t,\lambda}} a_t + \frac{1}{2} (\chi_{t,\lambda}^-)^2 [a_t \wedge a_t] + d\chi_{t,\lambda}^- \wedge a_t.$$

Using Proposition 4.42 and the fact that the cut-off functions $\chi_{t,\lambda}^-$ were constructed so that

$\|\chi_{t,\lambda}^-\|_{0,V_{[0,\sigma]}} + \|\mathrm{d}\chi_{t,\lambda}^-\|_{-1,V_{[0,\sigma]}} \leq c$ we obtain

$$\begin{aligned} & \|F_{A_{t,\lambda}} - F_{I_{t,\lambda}}\|_{-2,V_{[0,\lambda]};t} \\ & \leq \|1\|_{-2,V_{[0,\lambda]};t} \cdot \|\chi_{t,\lambda}^-\|_{0,V_{[0,\lambda]};t} \cdot \|\mathrm{d}_{A_t}|_{P_t} a_t\|_{0,V_{[0,\lambda]};t} \\ & \quad + \|\chi_{t,\lambda}^-\|_{0,V_{[0,\lambda]};t} \cdot \|i_{t,\lambda}\|_{-3,V_{[\lambda/2,\sigma]};t} \cdot \|a_t\|_{1,V_{[0,\lambda]};t} \\ & \quad + \frac{1}{2} \|1\|_{-4,V_{[0,\lambda]};t} \cdot \|\chi_{t,\lambda}^-\|_{0,V_{[0,\lambda]};t}^2 \cdot \|a_t\|_{1,V_{[0,\lambda]};t}^2 \\ & \quad + \|1\|_{-2,V_{[0,\lambda]};t} \cdot \|\mathrm{d}\chi_{t,\lambda}^-\|_{-1,V_{[0,\lambda]};t} \cdot \|a_t\|_{1,V_{[0,\lambda]};t} \leq c\lambda^2. \end{aligned}$$

Step 3. We prove that $\|e_{t,\lambda} - \tilde{e}_{t,\lambda}\|_{V_{(\lambda,\sigma/2)};t} \leq c\lambda^2$.

This is an immediate consequence of $F_{A_t} \wedge \psi_t = 0$ and Proposition 4.42 since in $V_{[\lambda,\sigma/2];t}$ the curvature of $A_{t,\lambda}$ is given by $F_{A_{t,\lambda}} = F_{A_t} + [i_{t,\lambda} \wedge a_t] + F_{I_{t,\lambda}} - F_{A_t}|_{P_t}$.

Step 4. We prove that $\|e_{t,\lambda} - \tilde{e}_{t,\lambda}\|_{V_{[\sigma/2,\sigma]};t} \leq c\lambda^2$.

In $V_{[\sigma/2,\sigma];t}$ the curvature of $A_{t,\lambda}$ is given by

$$F_{A_{t,\lambda}} = F_{A_t} + \chi_t^+ \mathrm{d}_{A_t} i_{t,\lambda} + \frac{1}{2} (\chi_t^+)^2 [i_{t,\lambda} \wedge i_{t,\lambda}] + \mathrm{d}\chi_t^+ \wedge i_{t,\lambda}.$$

Since $\|\chi_t^+\|_{\ell,V_{[\sigma/2,\sigma]};t} + \|\mathrm{d}\chi_t^+\|_{\ell,V_{[\sigma/2,\sigma]};t} \leq c$, it follows that

$$\begin{aligned} & \|F_{A_{t,\lambda}} - F_{A_t}\|_{-2,V_{[\sigma/2,\sigma]};t} \\ & \leq \|\chi_t^+\|_{2,V_{[\sigma/2,\sigma]};t} \cdot \|\mathrm{d}_{I_{t,\lambda}} i_{t,\lambda}\|_{-4,V_{[\sigma/2,\sigma]};t} \\ & \quad + \|\chi_t^+\|_{0,V_{[\sigma/2,\sigma]};t} \cdot \|a_t\|_{1,V_{[\sigma/2,\sigma]};t} \cdot \|i_{t,\lambda}\|_{-3,V_{[\sigma/2,\sigma]};t} \\ & \quad + \frac{1}{2} \|\chi_t^+\|_{2,V_{[\sigma/2,\sigma]};t}^2 \cdot \|i_{t,\lambda}\|_{-3,V_{[\sigma/2,\sigma]};t}^2 \\ & \quad + \|\mathrm{d}\chi_t^+\|_{1,V_{[\sigma/2,\sigma]};t} \cdot \|i_{t,\lambda}\|_{-3,V_{[\sigma/2,\sigma]};t} \leq c\lambda^2. \end{aligned}$$

We are thus left with estimating

$$\|\iota_{t,\lambda} \hat{v} \circ \mathfrak{I}_t - \hat{v} \circ \mathfrak{I}_{t,\lambda}\|_{-2,V_{[\sigma/2,\sigma]};t} \leq c \|\chi_t^+ - 1\|_{1,V_{[\sigma/2,\sigma]};t} \cdot \|\hat{v} \circ \mathfrak{I}_{t,\lambda}\|_{-3,V_{[\sigma/2,\sigma]};t}.$$

To conclude the proof we observe that $\|\chi_t^+ - 1\|_{1,V_{[\sigma/2,\sigma]};t} \leq c$ and that

$$(4.43) \quad \|\hat{\mathcal{J}}_\lambda\|_{C_{-3,0;t,\lambda}^{k,\alpha}(V_{[0,\sigma]};t)} \leq c\lambda^2 \|\hat{\mathcal{J}}\|_{C^{k,\alpha}}$$

as a consequence of Proposition 2.41 and a simple scaling consideration. $\square$

4.5 Linear estimates

We denote by $\mathfrak{X}_{t,\lambda}$ and $\mathfrak{Y}_{t,\lambda}$ the Banach spaces $C^{1,\alpha} \oplus \mathbf{R}$ and $C^{0,\alpha} \oplus \mathbf{R}$ equipped with the norms

$$\begin{aligned} \|(\underline{a}, \eta)\|_{\mathfrak{X}_{t,\lambda}} &:= \lambda^{-\delta/2} \|\rho_{t,\lambda} \underline{a}\|_{C^{1,\alpha}_{-1,\delta;t,\lambda}} + \lambda \|\pi_{t,\lambda} \underline{a}\|_{C^{1,\alpha}} + \lambda |\eta| \quad \text{and} \\ \|(\underline{a}, \eta)\|_{\mathfrak{Y}_{t,\lambda}} &:= \lambda^{-\delta/2} \|\rho_{t,\lambda} \underline{a}\|_{C^{0,\alpha}_{-2,\delta;t,\lambda}} + \lambda \|\pi_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} + \lambda |\eta|, \end{aligned}$$

respectively. Here we fixed $\delta \in (-1, 0)$ and $0 < \alpha \ll |\delta|$. For concreteness one may take $\delta = -\frac{1}{2}$ and $\alpha = \frac{1}{256}$. It will become apparent in the course of this section that the choice of the relative weights between terms involving $\rho_{t,\lambda} \underline{a}$ and those involving $\pi_{t,\lambda} \underline{a}$ is quite natural. One consequence of the choice of the factor $\lambda^{-\delta/2}$ is that

$$(4.44) \quad \|e_{t,\lambda}\|_{\mathfrak{Y}_{t,\lambda}} \leq c \lambda^{2-\alpha},$$

because of Propositions 4.37 and 4.38 as well as Corollary 4.36. We consider the linear operator $\mathbf{L}_{t,\lambda}: \mathfrak{X}_{t,\lambda} \rightarrow \mathfrak{Y}_{t,\lambda}$ defined by

$$\mathbf{L}_{t,\lambda}(\underline{a}, \eta) := (L_{t,\lambda} + \eta \cdot \iota_{t,\lambda} \hat{v} \circ \mathfrak{I}_t, \langle \pi_{t,\lambda} \underline{a}, \hat{v} \circ \mathfrak{I}_t \rangle).$$

Proposition 4.45. *For all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ the linear operator $\mathbf{L}_{t,\lambda}$ is invertible, $\mathbf{L}_{t,\lambda}^{-1}$ depends continuously differentiably on t and continuously on λ and, moreover, there exists a constant $c > 0$, which is independent of $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$, such that*

$$(4.46) \quad \|\mathbf{L}_{t,\lambda}^{-1}(\underline{b}, \zeta)\|_{\mathfrak{X}_{t,\lambda}} \leq c \|(\underline{b}, \zeta)\|_{\mathfrak{Y}_{t,\lambda}} \quad \text{and}$$

$$(4.47) \quad \|\nabla_t \mathbf{L}_{t,\lambda}^{-1}(\underline{b}, \zeta)\|_{\mathfrak{X}_{t,\lambda}} \leq c \|(\underline{b}, \zeta)\|_{\mathfrak{Y}_{t,\lambda}}.$$

The key to this proposition is the following estimate which we will prove in the course of this section.

Proposition 4.48. *There exists a constant $c > 0$, which is independent of $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$, such that*

$$(4.49) \quad \|(\underline{a}, \eta)\|_{\mathfrak{X}_{t,\lambda}} \leq c \|\mathbf{L}_{t,\lambda}(\underline{a}, \eta)\|_{\mathfrak{Y}_{t,\lambda}}.$$

Proposition 4.50. *The family of operators $\mathbf{L}_{t,\lambda}: \mathfrak{X}_{t,\lambda} \rightarrow \mathfrak{Y}_{t,\lambda}$ depends continuously differentiably on t and continuously on λ and there exists a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \|\mathbf{L}_{t,\lambda}(\underline{a}, \eta)\|_{\mathfrak{Y}_{t,\lambda}} &\leq c \|(\underline{a}, \eta)\|_{\mathfrak{X}_{t,\lambda}} \quad \text{and} \\ \|\nabla_t \mathbf{L}_{t,\lambda}(\underline{a}, \eta)\|_{\mathfrak{Y}_{t,\lambda}} &\leq c \|(\underline{a}, \eta)\|_{\mathfrak{X}_{t,\lambda}}. \end{aligned}$$

Proof of Proposition 4.45. By Proposition 4.48 the operator $\mathbf{L}_{t,\lambda}$ is injective and has closed range. Hence, we can identify its cokernel with the kernel of $\mathbf{L}_{t,\lambda}^*$. Since $\mathbf{L}_{t,\lambda}$ is formally self-adjoint, it follows from elliptic regularity that the kernel of $\mathbf{L}_{t,\lambda}^*$ agrees with the kernel of $\mathbf{L}_{t,\lambda}^*$ and thus is trivial. Therefore, $\mathbf{L}_{t,\lambda}$ is invertible. Now, (4.46) follows at once from (4.49). Since $\mathbf{L}_{t,\lambda}$ depends continuously differentiably on t and continuously on λ , so does $\mathbf{L}_{t,\lambda}$. Since $\nabla_t \mathbf{L}_{t,\lambda}^{-1} = -\mathbf{L}_{t,\lambda}^{-1} \nabla_t \mathbf{L}_{t,\lambda} \mathbf{L}_{t,\lambda}^{-1}$, (4.47) follows from (4.46) and Proposition 4.50. $\square$

4.5.1 The model operator on $\mathbf{R}^7$

Let I be a finite energy ASD instanton on a G -bundle E over $\mathbf{R}^4$. By a slight abuse of notation we denote the pullbacks of I and E to $\mathbf{R}^7 = \mathbf{R}^3 \oplus \mathbf{R}^4$ by I and E as well. We define $L_I: \Omega^0(\mathbf{R}^7, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^7, \mathfrak{g}_E) \rightarrow \Omega^0(\mathbf{R}^7, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^7, \mathfrak{g}_E)$ by

$$\begin{pmatrix} 0 & d_I^* \\ d_I & *(\psi_0 \wedge d_I) \end{pmatrix}$$

where

$$\begin{aligned} \psi_0 &:= \frac{1}{2} \omega_1 \wedge \omega_1 - dx^{23} \wedge \omega_1 - dx^{31} \wedge \omega_2 - dx^{12} \wedge \omega_3 \quad \text{and} \\ \omega_1 &:= dx^{45} + dx^{67}, \quad \omega_2 := dx^{46} - dx^{47} \quad \text{and} \quad \omega_3 := dx^{47} + dx^{56}. \end{aligned}$$

We have already studied this operator in Section 2.4. I will briefly recall the main results here. Denote by $\pi_{\mathbf{R}^4}: \mathbf{R}^3 \oplus \mathbf{R}^4 \rightarrow \mathbf{R}^4$ the projection onto the second summand and define weight functions

$$(4.51) \quad w(x) := 1 + |\pi_{\mathbf{R}^4}(x)| \quad \text{and} \quad w(x, y) := \min\{w(x), w(y)\}.$$

For a Hölder exponent $\alpha \in (0, 1)$ and a weight parameter $\beta \in \mathbf{R}$ we define

$$\begin{aligned} [f]_{C_\beta^{0,\alpha}(U)} &:= \sup_{d(x,y) \leq w(x,y)} w(x, y)^{\alpha-\beta} \frac{|f(x) - f(y)|}{d(x, y)^\alpha}, \\ \|f\|_{L_\beta^\infty(U)} &:= \|w^{-\beta} f\|_{L^\infty(U)} \quad \text{and} \\ \|f\|_{C_\beta^{k,\alpha}(U)} &:= \sum_{j=0}^k \|\nabla^j f\|_{L_{\beta-j}^\infty(U)} + [\nabla^j f]_{C_{\beta-j}^{0,\alpha}(U)}. \end{aligned}$$

Here f is a section of a vector bundle over $U \subset \mathbf{R}^7$ equipped with an inner product and a compatible connection. We use parallel transport to compare the values of f at different points. If U is not specified, then we take $U = \mathbf{R}^7$. We denote by $C_\beta^{k,\alpha}$ the subspace of elements f of the Banach space $C^{k,\alpha}$ with $\|f\|_{C_\beta^{k,\alpha}} < \infty$ equipped with the norm $\|\cdot\|_{C_\beta^{k,\alpha}}$.

The linear operators L_I can serve as a model for $L_{t,\lambda}$ in the following sense: Fix $t \in (-T, T)$ and $x \in P_t$. Set $I := I(\mathfrak{J})|_{N_x P_t}$ and $E := E(\mathfrak{J}_t)|_{N_x P_t}$. Identify $T_x Y =$

$T_x P_t \oplus N_x P_t$ with $\mathbf{R}^7 = \mathbf{R}^3 \oplus \mathbf{R}^4$ in such a way that the summands are preserved and $\psi_t|_{T_x Y}$ is identified with ψ_0 . For $\varepsilon_1, \varepsilon_2 > 0$ we define

$$V_{\varepsilon_1, \varepsilon_2; t} := B_{\varepsilon_1}(x) \cap V_{[0, \varepsilon_2]; t}.$$

Using the exponential map we can identify $V_{\varepsilon_1, \varepsilon_2; t}$ with a small neighbourhood $\tilde{U}_{\varepsilon_1, \varepsilon_2}$ of the origin in $\mathbf{R}^7$. With respect to this identification a $\mathfrak{g}_{E_t, \lambda}$ -valued tensor field f on $V_{\varepsilon_1, \varepsilon_2; t}$ is identified with a $s_{1/\lambda}^* \mathfrak{g}_E$ -valued tensor field $\tilde{f}$ on $\tilde{U}_{\varepsilon_1, \varepsilon_2}$ and if $k \in \mathbf{N}$ is a scaling parameter, then with f we can associate a $\mathfrak{g}_E$ -valued tensor field $s_{k, \lambda} f$ on $U_{\varepsilon_1, \varepsilon_2; \lambda} := \lambda^{-1} \tilde{U}_{\varepsilon_1, \varepsilon_2}$ defined by

$$(s_{k, \lambda} f)(x, y) := \lambda^k \tilde{f}(\lambda x, \lambda y).$$

Proposition 4.52. *There are constants $c, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$, $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \|s_{k, \lambda} f\|_{L_{\ell+\delta}^\infty(U_{\varepsilon, \sqrt{\lambda}; \lambda_i})} &\sim \lambda^{k+\ell} \|f\|_{L_{\ell, \delta; t, \lambda}^\infty(V_{\varepsilon, \sqrt{\lambda}; t})}, \\ \|s_{k, \lambda} f\|_{C_{\ell+\delta}^{k, \alpha}(U_{\varepsilon, \sqrt{\lambda}; \lambda_i})} &\sim \lambda^{k+\ell} \|f\|_{C_{\ell, \delta; t, \lambda}^{k, \alpha}(V_{\varepsilon, \sqrt{\lambda}; t})} \quad \text{and} \\ \left\| L_{t, \lambda} \underline{a} - s_{2, \lambda}^{-1} L_I s_{1, \lambda} \underline{a} \right\|_{C_{-2, \delta; t, \lambda}^{0, \alpha}(V_{\varepsilon, \sqrt{\lambda}; t})} &\leq c(\varepsilon + \sqrt{\lambda}) \|\underline{a}\|_{C_{-1, \delta; t, \lambda}^{1, \alpha}(V_{\varepsilon, \sqrt{\lambda}; t})}. \end{aligned}$$

Here $\sim$ means comparable uniformly in $\varepsilon \in (0, \varepsilon_0]$, $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$.

We use Proposition 2.70 to write $L_I^* L_I$ as

$$(4.53) \quad L_I^* L_I = \Delta_{\mathbf{R}^3} + \begin{pmatrix} \delta_I \delta_I^* & \\ & \delta_I^* \delta_I \end{pmatrix}$$

and apply Lemma 2.76 to obtain the following result.

Proposition 4.54. *If $\underline{a} \in \Omega^0(\mathbf{R}^7, \mathfrak{g}_E) \oplus \Omega^1(\mathbf{R}^7, \mathfrak{g}_E)$ satisfies $L_I = 0$ and $\|\underline{a}\|_{L^\infty}$ is finite, then $\underline{a}$ is the pullback of an element in the kernel of δ_I .*

Furthermore, we have the following Schauder estimate from Proposition 2.77.

Proposition 4.55. *For $\beta \in \mathbf{R}$ there is a constant $c > 0$ such that*

$$\|\underline{a}\|_{C_\beta^{1, \alpha}} \leq c(\|L_I \underline{a}\|_{C_{\beta-1}^{0, \alpha}} + \|\underline{a}\|_{L_\beta^\infty}).$$

4.5.2 Schauder estimate

Proposition 4.56. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ the following estimate holds*

$$(4.57) \quad \|\underline{a}\|_{C_{-1, \delta; t, \lambda}^{1, \alpha}} \leq c \left(\|L_{t, \lambda} \underline{a}\|_{C_{-2, \delta; t, \lambda}^{0, \alpha}} + \|\underline{a}\|_{L_{-1, \delta; t, \lambda}^\infty} \right).$$

Proof. It suffices to show that there is a constant $c > 0$ such that for all $t \in (-T', T')$, $\lambda \in (0, \Lambda]$ and $x \in Y$ there exist open sets U and V such that

$$\|\underline{a}\|_{C_{-1,\delta;t,\lambda}^{1,\alpha}(U)} \leq c \left(\|L_{t,\lambda}\underline{a}\|_{C_{-2,\delta;t,\lambda}^{0,\alpha}(V)} + \|\underline{a}\|_{L_{-1,\delta;t,\lambda}^\infty(V)} \right).$$

For $x \in Y$ with $r_t(x) \leq \sqrt{\lambda}$ such an estimate follows from Propositions 4.52 and 4.55. For $x \in Y$ with $r_t(x) > \sqrt{\lambda}$ one can take $U = B_{r_t(x)/8}(x)$ and $V = B_{r_t(x)/4}(x)$ and argue as in the proof of Proposition 2.77. $\square$

4.5.3 Estimate of $\|\rho_{t,\lambda}\underline{a}\|_{L_{-1,\delta;t,\lambda}^\infty}$

Proposition 4.58. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ the following estimate holds*

$$(4.59) \quad \|\underline{a}\|_{L_{-1,\delta;t,\lambda}^\infty} \leq c \left(\|L_{t,\lambda}\underline{a}\|_{C_{-2,\delta;t,\lambda}^{0,\alpha}} + \|\bar{\pi}_{t,\lambda}\underline{a}\|_{L_{-1,\delta;t,\lambda}^\infty} \right).$$

Proof. If not, then there exist sequences (t_i) , (λ_i) and $(\underline{a}_i)$ such that $\lim_{i \rightarrow \infty} \lambda_i = 0$,

$$(4.60) \quad \begin{aligned} \|\underline{a}_i\|_{L_{-1,\delta;t_i,\lambda_i}^\infty} &= 1, \\ \lim_{i \rightarrow \infty} \|L_{t_i,\lambda_i}\underline{a}_i\|_{C_{-2,\delta;t_i,\lambda_i}^{0,\alpha}} &= 0 \quad \text{and} \\ \lim_{i \rightarrow \infty} \|\bar{\pi}_{t_i,\lambda_i}\underline{a}_i\|_{L_{-1,\delta;t_i,\lambda_i}^\infty} &= 0. \end{aligned}$$

After passing to a subsequence we can assume that (t_i) converges to a limit t . From Proposition 4.56 it follows that

$$(4.61) \quad \|\underline{a}_i\|_{C_{-1,\delta;t_i,\lambda_i}^{1,\alpha}} \leq c.$$

Pick a sequence (x_i) of points in Y such that

$$w_{-1,\delta;t_i,\lambda_i}(x_i)|\underline{a}_i|(x_i) = 1.$$

After passing to a subsequence we can assume that one of the following cases occurs. We will show that each of them leads to a contradiction, thus proving the proposition.

Case 1. *The sequence (x_i) accumulates away from P_t : $\lim_{i \rightarrow \infty} r_{t_i}(x_i) > 0$.*

By (4.61) the sequence $(\underline{a}_i)$ is uniformly bounded in $C^{1,\alpha}$ on each compact subset of $Y \setminus P_t$. Arzelà–Ascoli and a diagonal sequence argument thus yield a subsequence of $(\underline{a}_i)$ which converges to a limit $\underline{a}$ on $Y \setminus P_t$ in $C_{\text{loc}}^{1,\alpha/2}$. Since we can also arrange that the corresponding subsequence of (x_i) converges to a limit $x \in Y \setminus P_t$ for which $r_t(x)^{1+\delta}|\underline{a}|(x) = 1$, it follows

that $\underline{a}$ cannot vanish identically. However, $\underline{a}$ also satisfies

$$(4.62) \quad \|r_t^{1+\delta} \underline{a}\|_{L^\infty} \leq 1 \quad \text{and}$$

$$(4.63) \quad L_{A_t, \phi_t} \underline{a} = 0 \quad \text{on } Y \setminus P_t.$$

Since $\delta < 2$, it follows from (4.62) that $\underline{a}$ satisfies (4.63) on all of Y in the sense of distribution and hence is smooth by elliptic regularity. This contradicts the hypothesis that A_0 and hence A_t is acyclic, i.e., that L_{A_t, ϕ_t} has trivial kernel.

Case 2. *The sequence (x_i) quickly accumulates near P_t : $\lim_{i \rightarrow \infty} r_{t_i}(x_i)/\lambda_i < \infty$.*

After passing to a subsequence we can assume that (x_i) converges to a point $x \in P_t$. With the notation of the paragraph preceding Proposition 4.52, set

$$\underline{b}_i := s_{1, \lambda_i} \left(\underline{a}_i|_{V_{\sqrt{\lambda_i}, \sqrt{\lambda_i}; t}} \right).$$

This sequence satisfies

$$\|\underline{b}_i\|_{C_{-1+\delta}^{1, \alpha}(U_{\sqrt{\lambda_i}, \sqrt{\lambda_i}; \lambda_i})} \leq c \quad \text{and} \quad \lim_{i \rightarrow \infty} \|L_I \underline{b}_i\|_{C_{-2+\delta}^{0, \alpha}} = 0,$$

and if (y_i) denotes the sequence of points in $U_{\sqrt{\lambda_i}, \sqrt{\lambda_i}; \lambda_i}$ corresponding to the sequence (x_i) , then

$$w(y_i)^{1-\delta} |\underline{b}_i|(y_i) \geq \frac{1}{2}.$$

where $w = 1 + |\pi_{\mathbf{R}^4}|$ as in (4.51). Since the sequence of subsets $U_{\sqrt{\lambda_i}, \sqrt{\lambda_i}; \lambda_i} \subset \mathbf{R}^7$ is exhaustive, Arzelà–Ascoli and a diagonal sequence argument yield a subsequence of $(\underline{b}_i)$ which converges to a limit $\underline{b}$ on $\mathbf{R}^7$ in $C_{\text{loc}}^{1, \alpha/2}$. By translation we can arrange that the $\mathbf{R}^3$ -component of y_i is zero and thus $|y_i|$ is bounded. After passing to a further subsequence (y_i) converges to a limit $y \in \mathbf{R}^7$. At this point we must have $w(y)^{1-\delta} |\underline{b}|(y) \geq 1/2$ and thus $\underline{b}$ cannot vanish identically. It follows from Proposition 4.52 that $\underline{b}$ satisfies

$$\|w^{1-\delta} \underline{b}\|_{L^\infty} \leq 2 \quad \text{and} \quad L_I \underline{b} = 0.$$

Moreover, using (4.60) and arguing as in the proof of Proposition 4.37, making use of the hypothesis $\delta < 0$, one can show that each restriction of $\underline{b} = 0$ to a slice $\{x\} \times \mathbf{R}^4$ is L^2 -orthogonal to $\ker \delta_I$. This, however, contradicts Proposition 4.54.

Case 3. *The sequence (x_i) slowly accumulates near P_t : $\lim_{i \rightarrow \infty} r_{t_i}(x_i)/\lambda_i = \infty$.*

In a similar manner as in the previous case we set

$$\underline{b}_i := s_{1, \lambda_i} \left(\underline{a}_i|_{V_{\sqrt{\lambda_i}, \sigma; t}} \right)$$

and denote by (y_i) the sequence of points in $U_{\sqrt{\lambda_i}, \sigma; \lambda_i}$. Again, we can assume that the $\mathbf{R}^3$ -component of y_i is zero. After passing to a subsequence we can assume that one of the following two cases occurs.

Case 3.1. We have $|y_i| \leq 1/\sqrt{\lambda_i}$ for all $i \in \mathbf{N}$.

Set

$$\tilde{b}_i := |y_i|^{1-\delta} \underline{b}_i(|y_i| \cdot -) \quad \text{and} \quad \tilde{y}_i := y_i/|y_i|.$$

Again, Arzelà–Ascoli and a diagonal sequence argument yield a subsequence of $(\tilde{b}_i)$ converging to a limit $\tilde{b}$ on $\mathbf{R}^7 \setminus (\mathbf{R}^3 \times \{0\})$ which cannot vanish identically, since $|\tilde{y}|^{1-\delta} |\underline{a}|(\tilde{y}) \geq 1/4$ with $\tilde{y} := \lim_{t \rightarrow \infty} \tilde{y}_i$. However, $\tilde{b}$ also satisfies

$$(4.64) \quad \|\tilde{w}^{1-\delta} \tilde{b}\|_{L^\infty} \leq 4 \quad \text{and}$$

$$(4.65) \quad L\tilde{b} = 0 \quad \text{on } \mathbf{R}^7 \setminus (\mathbf{R}^3 \times \{0\}).$$

Here $\tilde{w} := |\pi_{\mathbf{R}^4}|$ and L is defined by

$$(4.66) \quad L\underline{a} := (d^*a, d\xi + *(\psi \wedge da)).$$

Since $\delta > -2$, it follows from (4.64) that $\tilde{b}$ solves (4.65) on all of $\mathbf{R}^7$ in the sense of distributions and hence is smooth by elliptic regularity. Moreover, using standard elliptic estimates one can show that $\tilde{b}$ is uniformly bounded near $\mathbf{R}^3$ and therefore by (4.64) on all of $\mathbf{R}^7$ since $\delta \leq 1$. Because $L^*L = \Delta_{\mathbf{R}^3} + \Delta_{\mathbf{R}^4}$, we can now apply Lemma 2.76 to conclude that $\tilde{b}$ is invariant under translations in the $\mathbf{R}^4$ -direction. We can thus think of $\tilde{b}$ as a vector of harmonic functions on $\mathbf{R}^4$. Since $\delta < 1$ it follows that the components of $\tilde{b}$ decay to zero at infinity and thus vanish by the maximum principle. This, however, contradicts the fact that $\tilde{b}$ cannot vanish identically.

Case 3.2. We have $|y_i| > 1/\sqrt{\lambda_i}$ for all $i \in \mathbf{N}$.

If we set

$$\tilde{b}_i := \lambda_i^\delta |y_i|^{1+\delta} \underline{b}_i(|y_i| \cdot -) \quad \text{and} \quad \tilde{y}_i := y_i/|y_i|,$$

then we obtain the desired contradiction by arguing, mutatis mutandis, as the previous case. The relevant constraint on δ is easily seen to be that $\delta \in (-1, 2)$. $\square$

4.5.4 Comparison with $F_{\mathfrak{I}_t}$

The connection on $\coprod_{t \in (-T, T)} NP_t$ induces connections on the bundles over $(-T, T)$ whose fibres are $C^{0, \alpha}(\mathfrak{I}_{t, \lambda}^* V\mathfrak{M}_t)$ and $C^{1, \alpha}(\mathfrak{I}_{t, \lambda}^* V\mathfrak{M}_t)$, respectively. We denote the corresponding covariant derivatives by ∇_t .

Proposition 4.67. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ the following estimate holds*

$$\begin{aligned} \|L_{t,\lambda}\iota_{t,\lambda}\hat{\mathcal{J}} - \iota_{t,\lambda}F_{\mathcal{J}_t}\hat{\mathcal{J}}\|_{C_{-2,0;t,\lambda}^{0,\alpha}} &\leq c\lambda^2\|\hat{\mathcal{J}}\|_{C^{1,\alpha}} \quad \text{and} \\ \|(\nabla_t L_{t,\lambda})\iota_{t,\lambda}\hat{\mathcal{J}} - \iota_{t,\lambda}(\nabla_t F_{\mathcal{J}_t})\hat{\mathcal{J}}\|_{C_{-2,0;t,\lambda}^{0,\alpha}} &\leq c\lambda^2\|\hat{\mathcal{J}}\|_{C^{1,\alpha}}. \end{aligned}$$

Proof. Consider the operator $\tilde{L}_t: \Omega^0(NP_t, \mathfrak{g}_{E_t}) \oplus \Omega^1(NP_t, \mathfrak{g}_{E_t}) \rightarrow \Omega^0(NP_t, \mathfrak{g}_{E_t}) \oplus \Omega^1(NP_t, \mathfrak{g}_{E_t})$ defined by

$$\tilde{L}_{t,\lambda}\underline{a} := (d_{I_{t,\lambda}}^* a, d_{I_{t,\lambda}}\xi + *_0(\psi_{0;t} \wedge d_{I_{t,\lambda}}a)).$$

If we identify $\hat{\mathcal{J}} \in \Gamma(\mathcal{J}_t^* V\mathfrak{M}_t)$ with an element of $\Omega^1(NP_t, \mathfrak{g}_{E_t})$, then since $\delta_{\mathcal{J}_t(x)}(\hat{\mathcal{J}}|_{NP_t}) = 0$ we have

$$\tilde{L}_{t,\lambda}\hat{\mathcal{J}} = \left(0, *_0 \left[\psi_{0;t} \wedge (d_{I_{t,\lambda}}\hat{\mathcal{J}})^{1,1} \right] \right)$$

which is the same as $s_{1/\lambda}^* \circ F_{\mathcal{J}_t} \circ s_\lambda^*(\hat{\mathcal{J}})$.

In order to prove the first estimate it thus suffices to control the following terms

$$\begin{aligned} L_{t,\lambda}\iota_{t,\lambda}\hat{\mathcal{J}} - \iota_{t,\lambda}F_{\mathcal{J}_t}\hat{\mathcal{J}} &= L_{t,\lambda}(\iota_{t,\lambda}\hat{\mathcal{J}} - \hat{\mathcal{J}}_\lambda) + (L_{t,\lambda} - \tilde{L}_{t,\lambda})\hat{\mathcal{J}}_\lambda + (s_{1/\lambda}^* \circ F_{\mathcal{J}_t}\hat{\mathcal{J}} - \iota_{t,\lambda}F_{\mathcal{J}_t}\hat{\mathcal{J}}) \\ &=: \text{I} + \text{II} + \text{III} \end{aligned}$$

on $V_{[0,\sigma];t}$. It is easy to see that

$$\|\text{I}\|_{C_{-2,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} + \|\text{III}\|_{C_{-2,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} \leq c\lambda^2\|\hat{\mathcal{J}}\|_{C^{1,\alpha}}$$

by using that fact that I and III are supported in $V_{[\sigma/2,\sigma];t}$ and the estimates

$$\|L_{t,\lambda}\underline{a}\|_{C_{-2,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} \leq c\|\underline{a}\|_{C_{-1,0;t,\lambda}^{1,\alpha}(V_{[0,\sigma];t})} \quad \text{and} \quad \|F_{\mathcal{J}_t}\hat{\mathcal{J}}\|_{C^{0,\alpha}} \leq c\|\hat{\mathcal{J}}\|_{C^{1,\alpha}}.$$

as well as

$$\begin{aligned} \|\iota_{t,\lambda}\hat{\mathcal{J}} - \hat{\mathcal{J}}_\lambda\|_{C_{-\ell,0;t,\lambda}^{k,\alpha}(V_{[\sigma/2,\sigma];t})} &\leq \|\chi_t^+ - 1\|_{C_{\ell+3,0;t,\lambda}^{k,\alpha}(V_{[\sigma/2,\sigma];t})} \cdot \|\hat{\mathcal{J}}_\lambda\|_{C_{-3,0;t,\lambda}^{k,\alpha}(V_{[\sigma/2,\sigma];t})} \\ &\leq c\lambda^2\|\hat{\mathcal{J}}\|_{C^{k,\alpha}}. \end{aligned}$$

To estimate II we expand it as

$$\begin{aligned} \text{II} &= * \left(\psi_t \wedge (A_{t,\lambda} - I_{t,\lambda}) \wedge \hat{\mathcal{J}}_\lambda \right) + * \left((\psi_{1;t} + \psi_{\geq 2;t}) \wedge d_{I_{t,\lambda}}\hat{\mathcal{J}}_\lambda \right) \\ &\quad + (* - *_0)(\psi_{0;t} \wedge d_{I_{t,\lambda}}\hat{\mathcal{J}}_\lambda) \\ &=: \text{II}_1 + \text{II}_2 + \text{II}_3. \end{aligned}$$

It follows from Proposition 4.42 that

$$(4.68) \quad \|A_{t,\lambda} - I_{t,\lambda}\|_{C_{1,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} = \|\chi_{t,\lambda}^- a_t + (\chi_t^+ - 1)i_{t,\lambda}\|_{C_{1,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} \leq c$$

which in conjunction with (4.43) yields

$$\|\Pi_1\|_{C_{-2,0;t,\lambda}^{0,\alpha}} \leq c\lambda^2 \|\hat{\mathcal{J}}\|_{C^{1,\alpha}}.$$

From Proposition 2.41 and simple scaling considerations it follows that

$$\|(d_{I_{t,\lambda}} \hat{\mathcal{J}}_\lambda)^{1,1}\|_{C_{-3,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} + \|(d_{I_{t,\lambda}} \hat{\mathcal{J}}_\lambda)^{0,2}\|_{C_{-4,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma];t})} \leq c\lambda^2 \|\hat{\mathcal{J}}\|_{C^{1,\alpha}}.$$

Since $\delta_{\mathfrak{I}_t(x)}(\hat{\mathcal{J}}|_{N_x P_t}) = 0$, we have

$$\psi_{0;t} \wedge (d_{I_{t,\lambda}} \hat{\mathcal{J}}_\lambda)^{0,2} = \psi_{1;t} \wedge (d_{I_{t,\lambda}} \hat{\mathcal{J}}_\lambda)^{0,2} = 0.$$

These facts together with Proposition 4.39 imply that

$$\|\Pi_2\|_{C_{-2,0;t,\lambda}^{0,\alpha}} + \|\Pi_3\|_{C_{-2,0;t,\lambda}^{0,\alpha}} \leq c\lambda^2 \|\hat{\mathcal{J}}\|_{C^{1,\alpha}}.$$

This finishes the proof of the first estimate. To prove the second estimate note that the individual terms of

$$\begin{aligned} (\nabla_t L_{t,\lambda} \iota_{t,\lambda}) \hat{\mathcal{J}} - \nabla_t F_{\mathfrak{I}_t} \hat{\mathcal{J}} &= (\nabla_t L_{t,\lambda}) (\iota_{t,\lambda} \hat{\mathcal{J}} - \hat{\mathcal{J}}_\lambda) + \nabla_t (L_{t,\lambda} - \tilde{L}_{t,\lambda}) \hat{\mathcal{J}}_\lambda \\ &\quad + (s_{1/\lambda}^* \circ \nabla_t F_{\mathfrak{I}_t} \hat{\mathcal{J}} - \iota_{t,\lambda} \nabla_t F_{\mathfrak{I}_t} \hat{\mathcal{J}}) \end{aligned}$$

can be estimated just as above. $\square$

Proposition 4.69. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\|\hat{\mathcal{J}}\|_{C^{1,\alpha}} \leq c \left(\|\pi_{t,\lambda} L_{t,\lambda} \iota_{t,\lambda} \hat{\mathcal{J}}\|_{C^{0,\alpha}} + \left| \langle \hat{\mathcal{J}}, \hat{v} \circ \mathfrak{I}_t \rangle \right| \right).$$

Proof. By hypothesis we have

$$\|\hat{\mathcal{J}}\|_{C^{1,\alpha}} \leq c \left(\|F_{\mathfrak{I}_t} \hat{\mathcal{J}}\|_{C^{0,\alpha}} + \left| \langle \hat{\mathcal{J}}, \hat{v} \circ \mathfrak{I}_t \rangle \right| \right)$$

for $t = 0$ and thus also for $t \in (-T', T')$. Together with

$$\|F_{\mathfrak{I}_t} \hat{\mathcal{J}}\|_{C^{0,\alpha}} \leq c \left(\|\pi_{t,\lambda} L_{t,\lambda} \iota_{t,\lambda} \hat{\mathcal{J}}\|_{C^{0,\alpha}} + \lambda^{1-\alpha} \|\hat{\mathcal{J}}\|_{C^{1,\alpha}} \right),$$

which is an immediate corollary of Propositions 4.37 and 4.67, this immediately implies the asserted estimate by rearranging. $\square$

4.5.5 Cross-term estimates

Proposition 4.70. *There is a constant $c > 0$ such that for all $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \|\rho_{t,\lambda} L_{t,\lambda} \iota_{t,\lambda} \hat{\mathcal{J}}\|_{C_{-2,0;t,\lambda}^{0,\alpha}} &\leq c\lambda^{2-\alpha} \|\hat{\mathcal{J}}\|_{C^{1,\alpha}} \quad \text{and} \\ \|\rho_{t,\lambda} (\nabla_t L_{t,\lambda}) \iota_{t,\lambda} \hat{\mathcal{J}}\|_{C_{-2,0;t,\lambda}^{0,\alpha}} &\leq c\lambda^{2-\alpha} \|\hat{\mathcal{J}}\|_{C^{1,\alpha}} \end{aligned}$$

as well as

$$\begin{aligned} \|\pi_{t,\lambda} L_{t,\lambda} \rho_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} &\leq c\lambda^{-\alpha} \|\rho_{t,\lambda} \underline{a}\|_{C_{-1,0;t,\lambda}^{1,\alpha}} \quad \text{and} \\ \|\pi_{t,\lambda} (\nabla_t L_{t,\lambda}) \rho_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} &\leq c\lambda^{-\alpha} \|\rho_{t,\lambda} \underline{a}\|_{C_{-1,0;t,\lambda}^{1,\alpha}}. \end{aligned}$$

Proof. The first two estimates are immediate consequences of Propositions 4.37 and 4.67 because

$$\rho_{t,\lambda} L_{t,\lambda} \iota_{t,\lambda} \hat{\mathcal{J}} = \rho_{t,\lambda} (L_{t,\lambda} \iota_{t,\lambda} \hat{\mathcal{J}} - \iota_{t,\lambda} F_{\mathcal{J}_t} \hat{\mathcal{J}})$$

and similarly for $\nabla_t L_{t,\lambda}$.

To prove the last two estimates first note that we can assume without loss of generality that $\underline{a}$ is supported in $V_{[0,\sigma]}$ and that $\underline{a} = \rho_{t,\lambda} \underline{a}$. Define $\tilde{\pi}_{t,\lambda} : \Omega^1(NP_t, \mathfrak{g}_{E_t}) \rightarrow \Gamma(\mathcal{I}_t^* V\mathfrak{M}_t)$ by

$$(\tilde{\pi}_{t,\lambda} a)(x) := \sum_{\kappa} \int_{N_x P_t} \langle a, s_{1/\lambda}^* \kappa \rangle s_{1/\lambda}^* \kappa$$

where, at each point $x \in P_t$, κ runs through an orthonormal basis of $(V\mathfrak{M}_t)_{\mathcal{I}_t(x)}$ with respect to $\langle s_{1/\lambda}^* \cdot, s_{1/\lambda}^* \cdot \rangle$ and set $\tilde{\rho}_{t,\lambda} := \text{id} - \tilde{\pi}_{t,\lambda}$. One can check $\tilde{\pi}_{t,\lambda} a = 0$ implies that $\tilde{\pi}_{t,\lambda} \tilde{L}_{t,\lambda} a = 0$ where $\tilde{L}_{t,\lambda}$ is as defined in the proof of Proposition 4.67. Therefore

$$\begin{aligned} \pi_{t,\lambda} L_{t,\lambda} \rho_{t,\lambda} \underline{a} &= \pi_{t,\lambda} (L_{t,\lambda} - \tilde{L}_{t,\lambda}) \rho_{t,\lambda} \underline{a} + (\pi_{t,\lambda} - \tilde{\pi}_{t,\lambda}) \tilde{L}_{t,\lambda} \rho_{t,\lambda} \underline{a} + \tilde{\pi}_{t,\lambda} \tilde{L}_{t,\lambda} (\rho_{t,\lambda} - \tilde{\rho}_{t,\lambda}) \underline{a} \\ &=: \pi_{t,\lambda} \text{I} + \text{II} + \text{III}. \end{aligned}$$

Define $\tilde{\tilde{\pi}}_{t,\lambda}$ like $\tilde{\pi}_{t,\lambda}$ but take the inner product with $\iota_{t,\lambda} \kappa$ instead of $s_{1/\lambda}^* \kappa$ and let κ run through an orthonormal basis with respect to $\langle \iota_{t,\lambda} \cdot, \iota_{t,\lambda} \cdot \rangle_{L^2}$. If $\tilde{\tilde{\text{II}}}$ and $\tilde{\tilde{\text{III}}}$ denote the same expressions as II and III but with $\tilde{\tilde{\pi}}_{t,\lambda}$ in place of $\tilde{\pi}_{t,\lambda}$ and $\text{id} - \tilde{\tilde{\pi}}_{t,\lambda}$ in place of $\tilde{\rho}_{t,\lambda}$, then $\tilde{\tilde{\text{II}}}$ and $\tilde{\tilde{\text{III}}}$ are supported in $V_{[\sigma/2,\sigma]}$ and one can argue as in the proof of Proposition 4.67 to show

$$\|\tilde{\tilde{\text{II}}}\|_{C^{0,\alpha}} + \|\tilde{\tilde{\text{III}}}\|_{C^{0,\alpha}} \leq c\lambda^{-\alpha} \|\underline{a}\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}}.$$

The eigenvalues of the quadratic form $\langle \iota_{t,\lambda} \cdot, \iota_{t,\lambda} \cdot \rangle$ with respect to $\langle s_{1/\lambda}^* \cdot, s_{1/\lambda}^* \cdot \rangle$ differ from one by $O(\lambda^4)$; hence, the differences between II and $\tilde{\tilde{\text{II}}}$ as well as between III and $\tilde{\tilde{\text{III}}}$ are negligibly small.

To estimate I we write it as

$$\begin{aligned} \mathbf{I} = & *(\psi_t \wedge (A_{t,\lambda} - I_{t,\lambda}) \wedge \rho_{t,\lambda} \underline{a}) + *(\psi_{1;t} + \psi_{\geq 2;t}) \wedge d_{I_{t,\lambda}} \rho_{t,\lambda} \underline{a} \\ & + (* - *_0)(\psi_0 \wedge d_{I_{t,\lambda}} \rho_{t,\lambda} \underline{a}). \end{aligned}$$

Using Propositions 4.37, 4.39 and 4.40 as well as and (4.68) it follows that

$$\|\pi_{t,\lambda} \mathbf{I}\|_{C^{0,\alpha}} \leq c\lambda^{-\alpha} \|\mathbf{I}\|_{C_{-1,0;t,\lambda}^{0,\alpha}(V_{[0,\sigma);t})} \leq c\lambda^{-\alpha} \|\underline{a}\|_{C_{-1,\delta;t,\lambda}^{1,\alpha}}.$$

This finishes the proof of the third estimate. The last estimate is proved along the same lines. $\square$

4.5.6 Proof of Proposition 4.48

Applying Propositions 4.56 and 4.58 to $\rho_{t,\lambda} \underline{a}$ and using Proposition 4.37 yields

$$\begin{aligned} \|\rho_{t,\lambda} \underline{a}\|_{C_{-1,\delta;t,\lambda}^{1,\alpha}} &\leq c \|L_{t,\lambda} \rho_{t,\lambda} \underline{a}\|_{C_{-2,\delta;t,\lambda}^{0,\alpha}} \\ &\leq c \left(\|\rho_{t,\lambda} L_{t,\lambda} \underline{a}\|_{C_{-2,\delta;t,\lambda}^{0,\alpha}} \right. \\ &\quad \left. + \|\rho_{t,\lambda} L_{t,\lambda} \bar{\pi}_{t,\lambda} \underline{a}\|_{C_{-2,\delta;t,\lambda}^{0,\alpha}} + \lambda^{1-\alpha} \|\pi_{t,\lambda} L_{t,\lambda} \rho_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} \right). \end{aligned}$$

By Proposition 4.69

$$\|\pi_{t,\lambda} \underline{a}\|_{C^{1,\alpha}} \leq c(\|\pi_{t,\lambda} L_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} + |\langle \pi_{t,\lambda} \underline{a}, \hat{v} \circ \mathfrak{I}_t \rangle| + \|\pi_{t,\lambda} L_{t,\lambda} \rho_{t,\lambda} \underline{a}\|_{C^{0,\alpha}}).$$

Recalling the definitions of $\|\cdot\|_{\mathfrak{X}_{t,\lambda}}$, $\|\cdot\|_{\mathfrak{Y}_{t,\lambda}}$ and $\mathbf{L}_{t,\lambda}$, and using Proposition 4.70 it follows that

$$\|(\underline{a}, \eta)\|_{\mathfrak{X}_{t,\lambda}} \leq c(\|\mathbf{L}_{t,\lambda}(\underline{a}, \eta)\|_{\mathfrak{Y}_{t,\lambda}} + \lambda^{1-\alpha} \|(\underline{a}, \eta)\|_{\mathfrak{X}_{t,\lambda}})$$

which yields (4.49) by rearranging. $\square$

4.5.7 Proof of Proposition 4.50

It is clear that $L_{t,\lambda}$ depends continuously differentiably on t and continuously on λ . By Propositions 4.67 and 4.70 as well as Corollary 4.36 and Proposition 4.37 we have

$$\begin{aligned} \|\rho_{t,\lambda} L_{t,\lambda} \underline{a}\|_{C_{-2,\delta;t,\lambda}^{0,\alpha}} &\leq c \left(\|L_{t,\lambda} \rho_{t,\lambda} \underline{a}\|_{C_{-2,\delta;t,\lambda}^{0,\alpha}} + \lambda^{1-\alpha} \|\pi_{t,\lambda} L_{t,\lambda} \rho_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} \right. \\ &\quad \left. + \|\rho_{t,\lambda} L_{t,\lambda} \bar{\pi}_{t,\lambda} \underline{a}\|_{C_{-2,\delta;t,\lambda}^{0,\alpha}} \right) \\ &\leq c \left(\|\rho_{t,\lambda} \underline{a}\|_{C_{-1,\delta;t,\lambda}^{1,\alpha}} + \lambda^{2+\delta/2-\alpha} \|\pi_{t,\lambda} \underline{a}\|_{C^{1,\alpha}} \right) \end{aligned}$$

and

$$\begin{aligned} \|\pi_{t,\lambda} L_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} &\leq \|F_{\mathfrak{J}_t} \pi_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} + \|\pi_{t,\lambda} L_{t,\lambda} \rho_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} + c\lambda^{1-\alpha} \|\pi_{t,\lambda} \underline{a}\|_{C^{0,\alpha}} \\ &\leq c \left(\|\pi_{t,\lambda} \underline{a}\|_{C^{1,\alpha}} + \lambda^{-\alpha} \|\rho_{t,\lambda} \underline{a}\|_{C_{-1,\delta;t,\lambda}^{1,\alpha}} \right). \end{aligned}$$

This yields $\|\mathbf{L}_{t,\lambda} \underline{a}\|_{\mathfrak{Y}_{t,\lambda}} \leq c \|\underline{a}\|_{\mathfrak{X}_{t,\lambda}}$. In a similar way one shows that $\|\nabla_t \mathbf{L}_{t,\lambda} \underline{a}\|_{\mathfrak{Y}_{t,\lambda}} \leq c \|\underline{a}\|_{\mathfrak{X}_{t,\lambda}}$. $\square$

4.6 Quadratic estimate

By a slight abuse of notation we denote by $Q_{t,\lambda}$ the quadratic form defined in (4.33) as well as the associated bilinear form.

Proposition 4.71. *The bilinear form $Q_{t,\lambda}$ depends continuously differentiably on t and continuously on λ and there exists a constant $c > 0$ such that for $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} &\|\rho_{t,\lambda} Q_{t,\lambda}(\underline{a}_1, \underline{a}_2)\|_{C_{-2,\delta;t,\lambda}^{0,\alpha}} \\ &\leq c\lambda^{-\alpha} \left(\|\rho_{t,\lambda} \underline{a}_1\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} \cdot \|\rho_{t,\lambda} \underline{a}_2\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} + \|\rho_{t,\lambda} \underline{a}_1\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} \cdot \|\pi_{t,\lambda} \underline{a}_2\|_{C^{0,\alpha}} \right. \\ &\quad \left. + \|\pi_{t,\lambda} \underline{a}_1\|_{C^{0,\alpha}} \cdot \|\rho_{t,\lambda} \underline{a}_2\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} + \|\pi_{t,\lambda} \underline{a}_1\|_{C^{0,\alpha}} \cdot \|\pi_{t,\lambda} \underline{a}_2\|_{C^{0,\alpha}} \right) \end{aligned}$$

and

$$\begin{aligned} &\|\rho_{t,\lambda} \nabla_t Q_{t,\lambda}(\underline{a}_1, \underline{a}_2)\|_{C_{-2,\delta;t,\lambda}^{0,\alpha}} \\ &\leq c\lambda^{-\alpha} \left(\|\rho_{t,\lambda} \underline{a}_1\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} \cdot \|\rho_{t,\lambda} \underline{a}_2\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} + \|\rho_{t,\lambda} \underline{a}_1\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} \cdot \|\pi_{t,\lambda} \underline{a}_2\|_{C^{0,\alpha}} \right. \\ &\quad \left. + \|\pi_{t,\lambda} \underline{a}_1\|_{C^{0,\alpha}} \cdot \|\rho_{t,\lambda} \underline{a}_2\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} + \|\pi_{t,\lambda} \underline{a}_1\|_{C^{0,\alpha}} \cdot \|\pi_{t,\lambda} \underline{a}_2\|_{C^{0,\alpha}} \right) \end{aligned}$$

as well as

$$\begin{aligned} &\lambda \|\pi_{t,\lambda} Q_{t,\lambda}(\underline{a}_1, \underline{a}_2)\|_{C^{0,\alpha}} \\ &\leq c\lambda^{-\alpha} \left(\|\rho_{t,\lambda} \underline{a}_1\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} \cdot \|\rho_{t,\lambda} \underline{a}_2\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} + \|\rho_{t,\lambda} \underline{a}_1\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} \cdot \|\pi_{t,\lambda} \underline{a}_2\|_{C^{0,\alpha}} \right. \\ &\quad \left. + \|\pi_{t,\lambda} \underline{a}_1\|_{C^{0,\alpha}} \cdot \|\rho_{t,\lambda} \underline{a}_2\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} + \lambda \|\pi_{t,\lambda} \underline{a}_1\|_{C^{0,\alpha}} \cdot \|\pi_{t,\lambda} \underline{a}_2\|_{C^{0,\alpha}} \right) \end{aligned}$$

and

$$\begin{aligned} & \lambda \|\pi_{t,\lambda} \nabla_t Q_{t,\lambda}(\underline{a}_1, \underline{a}_2)\|_{C^{0,\alpha}} \\ & \leq c\lambda^{-\alpha} \left(\|\rho_{t,\lambda} \underline{a}_1\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} \cdot \|\rho_{t,\lambda} \underline{a}_2\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} + \|\rho_{t,\lambda} \underline{a}_1\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} \cdot \|\pi_{t,\lambda} \underline{a}_2\|_{C^{0,\alpha}} \right. \\ & \quad \left. + \|\pi_{t,\lambda} \underline{a}_1\|_{C^{0,\alpha}} \cdot \|\rho_{t,\lambda} \underline{a}_2\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} + \lambda \|\pi_{t,\lambda} \underline{a}_1\|_{C^{0,\alpha}} \cdot \|\pi_{t,\lambda} \underline{a}_2\|_{C^{0,\alpha}} \right). \end{aligned}$$

Proof. The first two estimates are immediate consequences of Propositions 4.35 and 4.37. For the last two estimates we only have to explain why we get a factor λ (instead of λ^2) in front of $\|\pi_{t,\lambda} \underline{a}_1\|_{C^{0,\alpha}} \cdot \|\pi_{t,\lambda} \underline{a}_2\|_{C^{0,\alpha}}$. Note that

$$\left[*_0 \left(\iota_{t,\lambda} \hat{\mathcal{J}}_1 \wedge \iota_{t,\lambda} \hat{\mathcal{J}}_2 \wedge \psi_{0;t} \right) \right]^{0,1} = 0$$

on grounds of simple bi-degree considerations. Therefore, using Propositions 4.37, 4.39 and 4.40,

$$\begin{aligned} \|\pi_{t,\lambda} Q_{t,\lambda}(\iota_{t,\lambda} \hat{\mathcal{J}}_1, \iota_{t,\lambda} \hat{\mathcal{J}}_2)\|_{C^{0,\alpha}} & \leq c\lambda^{-\alpha} \left(\|(* - *_0)(\iota_{t,\lambda} \hat{\mathcal{J}}_1 \wedge \iota_{t,\lambda} \hat{\mathcal{J}}_2 \wedge \psi)\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} \right. \\ & \quad \left. + \|*_0 [\iota_{t,\lambda} \hat{\mathcal{J}}_1 \wedge \iota_{t,\lambda} \hat{\mathcal{J}}_2 \wedge (\psi_{1;t} + \psi_{\geq 2;t})]\|_{C_{-1,\delta;t,\lambda}^{0,\alpha}} \right) \\ & \leq c\lambda^{-\alpha} \|\hat{\mathcal{J}}_1\|_{C^{0,\alpha}} \cdot \|\hat{\mathcal{J}}_2\|_{C^{0,\alpha}}. \quad \square \end{aligned}$$

4.7 Conclusion of the proof of Theorem 4.7

Proposition 4.72. *There is a constant $c > 0$ and for $t \in (-T', T')$ and $\lambda \in (0, \Lambda]$ there are $\underline{a}(t, \lambda) \in C^{1,\alpha}(Y, (\Lambda^0 \oplus \Lambda^1) \otimes \mathfrak{g}_{E_{t,\lambda}})$ and $\eta(t, \lambda) \in \mathbf{R}$ depending continuously differentiable on t and continuously on λ such that the connection $\tilde{A}_{t,\lambda} := A_{t,\lambda} + a(t, \lambda)$ satisfies*

$$(4.73) \quad * \left(F_{\tilde{A}_{t,\lambda}} \wedge \psi_t \right) + d_{\tilde{A}_{t,\lambda}} \xi(t, \lambda) + (\mu(t) + \eta(t, \lambda)) \cdot \iota_{t,\lambda} \hat{v} \circ \mathfrak{I}_t = 0$$

and

$$\|\underline{a}(t, \lambda)\|_{\mathfrak{X}_{t,\lambda}} \leq c\lambda^{2-\alpha} \quad \text{and} \quad |\eta(t, \lambda)| + |\partial_t \eta(t, \lambda)| \leq c\lambda^{1-\alpha}.$$

The proof relies on the preceding analysis and the following simple consequence of Banach's fixed point theorem, cf. [DK90, Lemma 7.2.23].

Lemma 4.74. *Let X be a Banach space and let $T: X \rightarrow X$ be a smooth map with $T(0) = 0$. Suppose there is a constant $c > 0$ such that*

$$\|Tx - Ty\| \leq c(\|x\| + \|y\|)\|x - y\|.$$

Then if $y \in X$ satisfies $\|y\| \leq \frac{1}{10c}$, there exists a unique $x \in X$ with $\|x\| \leq \frac{1}{5c}$ solving

$$x + Tx = y.$$

The unique solution satisfies $\|x\| \leq 2\|y\|$. Moreover, if T and y depend continuously or continuously differentiablely on a parameter in an open subset of $\mathbf{R}^n$, then so does the solution x .

Proof of Proposition 4.72. We solve (4.73) with the additional constraints

$$d_{A_t, \lambda}^* a = 0 \quad \text{and} \quad \langle \pi_{t, \lambda} \underline{a}, \hat{v} \circ \mathfrak{I}_t \rangle = 0.$$

This can be written as

$$\mathbf{L}_{t, \lambda}(\underline{a}, \eta) + Q_{t, \lambda}(\underline{a}) + e_{t, \lambda} = 0.$$

With $(\underline{a}, \eta) = \mathbf{L}_{t, \lambda}^{-1}(\underline{b}, \zeta)$ this becomes

$$(4.75) \quad (\underline{b}, \zeta) + \tilde{Q}_{t, \lambda}(\underline{b}, \zeta) + e_{t, \lambda} = 0.$$

where $\tilde{Q}_{t, \lambda} := Q_{t, \lambda} \circ \mathbf{L}_{t, \lambda}^{-1}$. It follows from Propositions 4.45 and 4.71 that

$$\begin{aligned} & \|\tilde{Q}_{t, \lambda}(\underline{b}_1, \zeta_1) - \tilde{Q}_{t, \lambda}(\underline{b}_2, \zeta_2)\|_{\mathfrak{Y}_{t, \lambda}} \\ & \leq c\lambda^{-2-\delta/2-\alpha} (\|(\underline{b}_1, \zeta_1)\|_{\mathfrak{Y}_{t, \lambda}} + \|(\underline{b}_2, \zeta_2)\|_{\mathfrak{Y}_{t, \lambda}}) \|(\underline{b}_1, \zeta_1) - (\underline{b}_2, \zeta_2)\|_{\mathfrak{Y}_{t, \lambda}}. \end{aligned}$$

and we recall from (4.44) that

$$\|e_{t, \lambda}\|_{\mathfrak{Y}_{t, \lambda}} \leq c\lambda^{2-\alpha}.$$

Hence, we can solve (4.75) using Lemma 4.74 since $\delta \in (-1, 0)$ and $0 < \alpha \ll |\delta|$. The solution satisfies $\|(\underline{b}, \zeta)\|_{\mathfrak{Y}_{t, \lambda}} \leq c\lambda^{2-\alpha}$ and $(\nabla_t \underline{b}, \partial_t \zeta)$ solves the equation

$$(4.76) \quad (\nabla_t \underline{b}, \partial_t \zeta) + 2\tilde{Q}_{t, \lambda}((\underline{b}, \zeta), (\nabla_t \underline{b}, \partial_t \zeta)) + (\nabla_t \tilde{Q}_{t, \lambda})(\underline{b}, \zeta) + \nabla_t e_{t, \lambda} = 0.$$

Since $\|2\tilde{Q}_{t, \lambda}(\underline{b}, \cdot)\|_{\mathfrak{Y}_{t, \lambda}} \leq \frac{1}{2}\|\cdot\|_{\mathfrak{Y}_{t, \lambda}}$ and $\|(\nabla_t \tilde{Q}_{t, \lambda})(\underline{b}, \zeta) + \nabla_t e_{t, \lambda}\|_{\mathfrak{Y}_{t, \lambda}} \leq c\lambda^{2-\alpha}$, it follows that $\|(\nabla_t \underline{b}, \partial_t \zeta)\|_{\mathfrak{Y}_{t, \lambda}} \leq c\lambda^{2-\alpha}$. This implies the desired estimates on $(\underline{a}, \eta) = \mathbf{L}_{t, \lambda}^{-1}(\underline{b}, \zeta)$ and its derivative by Proposition 4.45. $\square$

The problem of finding $\bar{A}_\lambda$ is now reduced to constructing $t: [0, \Lambda] \rightarrow (-T', T')$ such that $t(0) = 0$ and

$$\mu(t(\lambda)) + \eta(t(\lambda), \lambda) = 0$$

for $t \in (0, \Lambda]$. But this is an immediate consequence of the following implicit function theorem, because $\mu(0) = 0$, $\partial_t \mu(0) \neq 0$ and $|\eta| + |\partial_t \eta| \leq c\lambda^{1-\alpha}$.

Proposition 4.77. *Let $\varepsilon, \delta > 0$ and suppose that $f: [-\varepsilon, \varepsilon] \times [0, \delta] \rightarrow \mathbf{R}$ is a continuous*

function such that for each $y \in [0, \delta]$ the function $f(\cdot, y)$ is strictly monotone and $f(0, 0) = 0$, then there is a $\delta' \in (0, \delta]$ and a continuous function $x: [0, \delta'] \rightarrow [-\varepsilon, \varepsilon]$ such that $x(0) = 0$ and $f(x(y), y) = 0$ for all $y \in [0, \delta']$.

The proof is elementary, but for lack of a good reference I give the details here.

Proof. Note that for each $y \in [0, \delta]$ there is at most one $x \in [-\varepsilon, \varepsilon]$ such that $f(x, y) = 0$. On the other hand there is such an x if $y \in \text{pr}_2(U^+) \cap \text{pr}_2(U^-)$ where $U^+ := f^{-1}[0, \infty) \neq \emptyset$, $U^- := f^{-1}(-\infty, 0] \neq \emptyset$ and $\text{pr}_2(U^+) \cap \text{pr}_2(U^-)$ denote the projection onto the second factor. Since $U^\pm$ are open sets and pr_2 is an open map the set $\text{pr}_2(U^+) \cap \text{pr}_2(U^-)$ is open as well and thus contains an interval of the form $[0, \delta']$ for some $\delta' \in (0, \delta]$. This defines the map $x: [0, \delta'] \rightarrow [-\varepsilon, \varepsilon]$.

If (y_i) is a sequence in $[0, \delta']$ converging to y , then $f(\lim x(y_i), y) = 0$ because f is continuous. However, there is precisely one $x \in [-\varepsilon, \varepsilon]$ such that $f(x, y) = 0$ and therefore $x(y) = \lim x(y_i)$. $\square$

The resulting connection $\bar{A}_\lambda := \tilde{A}_{t(\lambda), \lambda}$ will be smooth by elliptic regularity. That $\bar{A}_\lambda$ converges to A_0 on the complement of P_0 and that at each point $x \in P_0$ an ASD instanton in the equivalence class of $\mathfrak{I}(x)$ bubbles off transversely is clear, since we constructed $A_{t, \lambda}$ accordingly and $\bar{A}_\lambda$ is a small perturbation of $A_{t, \lambda}$. This concludes the proof of Theorem 4.7. $\square$

Chapter 5

Spin(7)–instantons, Cayley submanifolds and Fueter sections

One may be tempted to try to build an analogue of Donaldson theory for Spin(7)–manifolds. Again the main obstacle is that moduli spaces of Spin(7)–instantons need not be compact in general. The possible non-compactness phenomena are similar to those for G_2 –instantons described in Theorem 4.1. In particular, we expect to encounter bubbling along Cayley submanifolds. In this chapter I discuss the following Spin(7)–analogue of Theorem 4.7.

Theorem 5.1. *Let (X, Φ) be a compact Spin(7)–manifold. Suppose we are given*

- *an unobstructed Spin(7)–instanton A_0 on a G –bundle E_0 over X ,*
- *an unobstructed Cayley submanifold Q and*
- *an unobstructed Fueter section $\mathfrak{J}$ of an instanton moduli bundle $\mathfrak{M}$ associated with Q and $E_0|_Q$.*

Then there exists a constant $\Lambda > 0$ and a G –bundle E together with a family of unobstructed Spin(7)–instantons $(A_\lambda)_{\lambda \in (0, \Lambda]}$ on E . Moreover, as λ tends to zero A_λ converges to A_0 on the complement of Q and at each point $x \in Q$ an ASD instanton in the equivalence class given by $\mathfrak{J}(x)$ bubbles off transversely.

Remark 5.2. The notions of instanton moduli bundle and Fueter section, and what it means for a Fueter section to be unobstructed will be defined in Section 5.1.

The proof of Theorem 5.1 is very similar to the proof of Theorem 4.7 and we thus keep our discussion rather brief; After presenting the proof over the course of the next four sections, I will give concrete examples of situations in which Theorem 5.1 can be applied and mention a potential application to Spin(7)–manifolds with $K3$ Cayley fibrations (with singular fibres) in Section 5.6; in particular, I recover some results earlier obtained by Lewis [Lew98].

5.1 Fueter sections of instanton moduli bundles over Cayley submanifolds

Let Q be a Cayley submanifold in a $\text{Spin}(7)$ –manifold (X, Φ) . As in Section 4.1 we fix a moduli space M of framed ASD instantons on a G –bundle E over $\mathbf{R}^4$ and denote by E_∞ a G –bundle over Q together with a connection A_∞ .

Definition 5.3. The *instanton moduli bundle* $\mathfrak{M} \rightarrow Q$ associated with Q , E_∞ and M is defined by

$$\mathfrak{M} := (\text{Fr}(NQ) \times E_\infty) \times_{\text{SO}(4) \times G} M.$$

Example 5.4. If $M = (\text{Re}(S^+ \otimes \mathbf{C}^2) \setminus \{0\})/\mathbf{Z}_2 \times \mathbf{R}^4$, as in Example 4.11, and we pick spin structures as in Remark 1.80, then

$$\mathfrak{M} = (\mathfrak{s} \times \mathfrak{u} \times E_\infty) \times_{\text{Spin}(4) \times G} M = (\text{Re}(S^+ \otimes E_\infty) \setminus \{0\})/\mathbf{Z}_2 \oplus NQ$$

Denote by $N_\infty P := \text{Fr}(NQ) \times_{\text{SO}(4)} S^4$ the sphere-bundle obtained from NP by adjoining a section at infinity.

Theorem 5.5 (Donaldson–Segal [DS11] and Haydys [Hay12]). *To each section $\mathfrak{J} \in \Gamma(\mathfrak{M})$ we can assign a G –bundle $E = E(\mathfrak{J})$ over $N_\infty Q$ together with a connection $I = I(\mathfrak{J})$ and a framing $\Phi: E|_\infty \rightarrow E_\infty$ such that:*

- For each $x \in Q$ the restriction of I to $N_x Q$ represents $\mathfrak{J}(x)$.
- The framing Φ identifies the restriction of I to the section at infinity with A_∞ .

It is natural to impose the condition that

$$(5.6) \quad \lim_{\lambda \rightarrow 0} s_{1/\lambda}^* \pi_7^0(F_{I_\lambda}) = 0$$

where π_7^0 denotes the zeroth order Taylor expansion of π_7 off Q . It can be read off from Proposition 1.50 that equation (5.6) splits into two parts. The first part is just the condition that the restriction to the slices to be ASD, while the second part can be written as

$$(5.7) \quad \pi_7^0(F_I^{1,1}) = 0.$$

Equation (5.7) can be understood as a partial differential equation on $\mathfrak{J}$ as follows. Define the vertical tangent bundle $V\mathfrak{M}$ to $\mathfrak{M}$ by

$$V\mathfrak{M} := (\text{Fr}(NQ) \times E_\infty) \times_{\text{SO}(4) \times G} TM.$$

If $\mathfrak{J}$ is a section of $\mathfrak{M}$, then Φ selects a subbundle $\text{Hom}_\Phi(TQ, \mathfrak{J}^* V\mathfrak{M}) \subset \text{Hom}(TQ, \mathfrak{J}^* V\mathfrak{M})$ and there is a “Clifford multiplication” map $\gamma: \text{Hom}(TQ, V\mathfrak{M}) \rightarrow \text{Hom}_\Phi(TQ, V\mathfrak{M})$ as

discussed before in Section 1.3.2. Moreover, the connections on NP and E_∞ induce a connection on $\mathfrak{M}$ assigning to each section $\mathfrak{J}$ its covariant derivative $\nabla \mathfrak{J} \in \Omega^1(\mathfrak{J}^*V\mathfrak{M})$.

Definition 5.8. The *Fueter operator* $\mathfrak{F} = \mathfrak{F}_\Phi$ associated with $\mathfrak{M}$ is defined by

$$\mathfrak{J} \in \Gamma(\mathfrak{M}) \mapsto \mathfrak{F}_\Phi \mathfrak{J} := \gamma(\nabla \mathfrak{J}) \in \Gamma(\text{Hom}_\Phi(TQ, \mathfrak{J}^*V\mathfrak{M})).$$

A section $\mathfrak{J} \in \Gamma(\mathfrak{M})$ is called a *Fueter section* if it satisfies

$$\mathfrak{F} \mathfrak{J} = 0.$$

Example 5.9. If M is as in Example 5.4, then the Fueter operator $\mathfrak{F}$ lifts to the twisted Dirac operator $D: \Gamma(\text{Re}(S^+ \otimes (E_\infty \oplus U))) \rightarrow \Gamma(\text{Re}(S^- \otimes (E_\infty \oplus U)))$.

The Fueter operator $\mathfrak{F}$ is compatible with the product structure on $\mathfrak{M} = \mathring{\mathfrak{M}} \times NP$ corresponding to $M = \mathring{M} \times \mathbf{R}^4$. Its restriction to the second factor is given by the Fueter operator F_Q associated with Q .

Theorem 5.10 (Donaldson–Segal [DS11] and Haydys [Hay12]). *If $\mathfrak{J} \in \Gamma(\mathfrak{M})$, then we can identify $\Gamma(\text{Hom}_\Phi(TQ, \mathfrak{J}^*V\mathfrak{M}))$ with a subspace of $\Omega^2(NQ, \mathfrak{g}_{E(\mathfrak{J})})$. With respect to this identification we have the identity*

$$\mathfrak{F} \mathfrak{J} = \pi_7^0 \left(F_{I(\mathfrak{J})}^{1,1} \right)$$

where $*_0$ is the Hodge–*–operator on NP . In particular, $I(\mathfrak{J})$ satisfies equation (4.17) if and only if $\mathfrak{J}$ is a Fueter section.

Definition 5.11. The *linearised Fueter operator*

$$F_{\mathfrak{J}} = F_{\mathfrak{J}, \Phi}: \Gamma(\mathfrak{J}^*V\mathfrak{M}) \rightarrow \Gamma(\text{Hom}_\Phi(TQ, \mathfrak{J}^*V\mathfrak{M}))$$

for $\mathfrak{J} \in \Gamma(\mathfrak{M})$ is defined by

$$F_{\mathfrak{J}, \Phi}(\hat{\mathfrak{J}}) := \gamma(\nabla \hat{\mathfrak{J}}) \in \Gamma(\text{Hom}_\Phi(TQ, \mathfrak{J}^*V\mathfrak{M})).$$

Definition 5.12. A Fueter section $\mathfrak{J}$ is called *unobstructed* if the linearised Fueter operator $F_{\mathfrak{J}}$ is surjective.

Example 5.13. If M is as in Example 4.11, then the linearised Fueter operator $F_{\mathfrak{J}}$ lifts to the twisted Dirac operator $D: \Gamma(\text{Re}(S^+ \otimes (E_\infty \oplus U))) \rightarrow \Gamma(\text{Re}(S^- \otimes (E_\infty \oplus U)))$. In particular, it only depends on the spin structure $\mathfrak{s}$ and not on $\mathfrak{J}$. Using the Atiyah–Singer index theorem we can compute that in the current situation

$$(5.14) \quad \text{index } \mathring{F}_{\mathfrak{J}} = -\frac{1}{4}\sigma(Q) - \int_Q c_2(E_\infty)$$

where $\mathring{F}_{\mathfrak{J}}$ is the restriction of $F_{\mathfrak{J}}$ to $V\mathfrak{M}$.

5.2 Approximate $\text{Spin}(7)$ -instantons

Throughout the next three sections we assume the hypothesis of Theorem 5.1. For simplicity we will also assume that A_0 is irreducible. This assumption can be removed as in Section 2.6.

Proposition 5.15. *For each $\lambda \in (0, \Lambda]$ we can explicitly construct a G -bundle E_λ together with a connection $A_\lambda = A \#_\lambda \mathfrak{J}$ from E_0 , $A \in \mathcal{A}(E_0)$ and $\mathfrak{J}$. The bundles E_λ are pairwise isomorphic.*

Before we embark on the proof, let us set up some notation. Fix a constant $\zeta > 0$ such that the exponential map identifies a tubular neighbourhood of width ζ of Q in X with a neighbourhood of the zero section in NQ . For $I \subset \mathbf{R}$ we set

$$U_I := \{v \in NQ : |v| \in I\} \quad \text{and} \quad V_I := \{x \in X : r(x) \in I\}.$$

Here $r := d(\cdot, Q) : X \rightarrow [0, \infty)$ denotes the distance from Q . Fix a smooth-cut off function $\chi : [0, \infty) \rightarrow [0, 1]$ which vanishes on $[0, 1]$ and is equal to one on $[2, \infty)$. For $\lambda \in (0, \Lambda]$ we define $\chi_\lambda^- : X \rightarrow [0, 1]$ and $\chi_\lambda^+ : X \rightarrow [0, 1]$ by

$$\chi_\lambda^-(x) := \chi(r(x)/2\lambda) \quad \text{and} \quad \chi_\lambda^+(x) := 1 - \chi(r(x)/2\sigma),$$

respectively.

Proof of Proposition 5.15. Using radial parallel transport we can identify $E(\mathfrak{J})$ over $U_{(R, \infty)}$ for some $R > 0$ with the pullback of $E(\mathfrak{J})|_\infty$ to said region and similarly we can identify E_0 over $V_{[0, \sigma)}$ with the pullback of $E_0|_Q$. Hence, via the framing Φ we can identify $s_{1/\lambda}^* E(\mathfrak{J})$ with E_0 on the overlap $V_{(\lambda, \sigma)}$ for $\lambda \in (0, \Lambda]$. Patching both bundles via this identification yields E_λ .

To construct a connection on E_λ note that on the overlap $I_\lambda := s_{1/\lambda}^* I(\mathfrak{J})$ and A can be written as

$$I_\lambda = A|_Q + i_\lambda \quad \text{and} \quad A = A|_Q + a.$$

Here and in the following, by a slight abuse of notation, we denote by $A|_Q$ the pullback of $A|_Q$ to the overlap. We define A_λ by interpolating between I_λ and A on the overlap as follows

$$(5.16) \quad A_\lambda := A|_Q + \chi_\lambda^- a + \chi_\lambda^+ i_\lambda. \quad \square$$

In order to prove Theorem 5.1 we will solve the p.d.e.

$$(5.17) \quad (d_{A_\lambda}^* a, \pi_7(F_{A_\lambda + a})) = L_\lambda a + Q(a) + \pi_7(F_{A_\lambda}) = 0$$

with $L_\lambda := L_{A_\lambda}$, see (1.110), and $Q(a) := \frac{1}{2}\pi_7([a \wedge a])$ for $a \in \Omega^1(X, \mathfrak{g}_{E_\lambda})$. Recall, that we made the additional assumption that A_0 is irreducible. In case A_0 is reducible, we would only solve $d_{A_\lambda}^* a = 0$ “modulo $H_{A_0}^0$ ”.

Convention 5.18. We fix a constant $\Lambda > 0$ such that all of the statements of the kind “if $\lambda \in (0, \Lambda]$, then ...” appearing in the following are valid. This is possible since there are only a finite number of these statements and each one of them is valid provided Λ is sufficiently small. By $c > 0$ we will denote a generic constant whose value does not depend on $\lambda \in (0, \Lambda]$ but may change from one occurrence to the next.

For $\lambda \in (0, \Lambda]$ we define a family of weight functions $w_{\ell, \delta; \lambda}$ on X depending on two additional parameters $\ell \in \mathbf{R}$ and $\delta \in \mathbf{R}$ as follows

$$w_{\ell, \delta; \lambda}(x) := \begin{cases} \lambda^\delta (\lambda + r(x))^{-\ell - \delta} & \text{if } r(x) \leq \sqrt{\lambda} \\ r(x)^{-\ell + \delta} & \text{if } r(x) > \sqrt{\lambda} \end{cases}$$

and set $w_{\ell, \delta; \lambda}(x, y) := \min\{w_{\ell, \delta; \lambda}(x), w_{\ell, \delta; \lambda}(y)\}$. For a Hölder exponent $\alpha \in (0, 1)$ and $\ell, \delta \in \mathbf{R}$ we define (semi-)norms

$$\begin{aligned} \|f\|_{L_{\ell, \delta; \lambda}^\infty(U)} &:= \|w_{\ell, \delta; \lambda} f\|_{L^\infty(U)}, \\ [f]_{C_{\ell, \delta; \lambda}^{0, \alpha}(U)} &:= \sup_{\substack{x \neq y \in U: \\ d(x, y) \leq \lambda + \min\{r(x), r(y)\}}} w_{\ell - \alpha, \delta; \lambda}(x, y) \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \quad \text{and} \\ \|f\|_{C_{\ell, \delta; \lambda}^{k, \alpha}(U)} &:= \sum_{j=0}^k \|\nabla^j f\|_{L_{\ell - j, \delta; \lambda}^\infty(U)} + [\nabla^k f]_{C_{\ell - k, \delta; \lambda}^{0, \alpha}(U)}. \end{aligned}$$

Here f is a section of a vector bundle over $U \subset X$ equipped with an inner product and a compatible connection. We use parallel transport to compare the values of f at different points. If U is not specified, then we take $U = X$. We will primarily use this norm for $\mathfrak{g}_{E_\lambda}$ –valued tensor fields.

The norms $\|\cdot\|_{C_{\ell, \delta; \lambda}^{k, \alpha}}$ have the same general properties as the norms $\|\cdot\|_{C_{\ell, \delta; t, \lambda}^{k, \alpha}}$ introduced in Chapter 4; in particular, Proposition 4.35 and Corollary 4.36 hold mutatis mutandis. We introduce maps $\mu_\lambda: \Gamma(\mathcal{I}^* V \mathfrak{M}) \rightarrow \Omega^1(X, \mathfrak{g}_{E_\lambda})$ and $\nu_\lambda: \Gamma(\text{Hom}_\Phi(TQ, \mathcal{I}^* V \mathfrak{M})) \rightarrow \Omega_T^2(X, \mathfrak{g}_{E_\lambda})$ defined by

$$\mu_\lambda \hat{\mathcal{J}} := \chi^+ s_{1/\lambda}^* \hat{\mathcal{J}}$$

and

$$\nu_\lambda \hat{\mathcal{T}} := \pi_7(\chi^+ s_{1/\lambda}^* \hat{\mathcal{T}}),$$

respectively. Here we first identify $\hat{\mathcal{J}} \in \Gamma(\mathcal{I}^* V \mathfrak{M})$ with an element of $\Omega^1(NQ, E(\mathcal{I}))$, then view the restriction of its pullback via $s_{1/\lambda}$ to $U_{[0, \sigma]}$ as lying in $\Omega^1(V_{[0, \sigma]}, \mathfrak{g}_{E_\lambda})$ and finally extended it to all of X by multiplication with χ^+ ; similarly we proceed with $\hat{\mathcal{T}}$. Moreover,

we define $\pi_\lambda: \Omega^1(X, \mathfrak{g}_{E_\lambda}) \rightarrow \Gamma(\mathfrak{I}^* V \mathfrak{M})$ and $\sigma_\lambda: \Omega^2_7(X, \mathfrak{g}_{E_\lambda}) \rightarrow \Gamma(\text{Hom}_\Phi(TQ, \mathfrak{I}^* V \mathfrak{M}))$ by

$$(\pi_\lambda a)(x) := \sum_{\kappa} \int_{N_x P} \langle a, \mu_\lambda \kappa \rangle \kappa$$

and

$$(\sigma_\lambda \alpha)(x) := \sum_{\beta} \int_{N_x P} \langle \alpha, \nu_\lambda \beta \rangle \beta,$$

respectively. Here κ runs through an orthonormal basis of $V \mathfrak{M}_{\mathfrak{I}(x)}$ with respect to the inner product $\langle \mu_\lambda \cdot, \mu_\lambda \cdot \rangle$ and β runs through an orthonormal basis of $\text{Hom}_\Phi(T_x Q, V \mathfrak{M}_{\mathfrak{I}(x)})$ with respect to the inner product $\langle \nu_\lambda \cdot, \nu_\lambda \cdot \rangle$. Clearly, $\pi_\lambda \mu_\lambda = \text{id}$ and $\sigma_\lambda \nu_\lambda = \text{id}$; hence, $\bar{\pi}_\lambda := \mu_\lambda \pi_\lambda$ and $\bar{\sigma}_\lambda := \nu_\lambda \sigma_\lambda$ are projections. We denote the complementary projections by $\rho_\lambda := \text{id} - \bar{\pi}_\lambda$ and $\tau_\lambda := \text{id} - \bar{\sigma}_\lambda$. By arguing as in the proof of Proposition 4.37 one can show the following result.

Proposition 5.19. *For $\ell \leq -1$ and $\delta \in \mathbf{R}$ such that $\ell + \delta \in (-3, -1)$ there is a constant $c > 0$ such that for all and $\lambda \in (0, \Lambda]$ we have*

$$\|\mu_\lambda \hat{\mathfrak{I}}\|_{C^{0,\alpha}_{\ell,\delta;\lambda}} \leq c \lambda^{-1-\ell} \|\hat{\mathfrak{I}}\|_{C^{0,\alpha}} \quad \text{and} \quad \|\pi_\lambda a\|_{C^{0,\alpha}} \leq c \lambda^{1+\ell-\alpha} \|a\|_{C^{0,\alpha}_{\ell,\delta;\lambda}(V_{[0,\sigma]})}$$

as well as

$$\|\nu_\lambda \hat{\mathfrak{I}}\|_{C^{0,\alpha}_{\ell,\delta;\lambda}} \leq c \lambda^{-1-\ell} \|\hat{\mathfrak{I}}\|_{C^{k,\alpha}} \quad \text{and} \quad \|\sigma_\lambda \alpha\|_{C^{0,\alpha}} \leq c \lambda^{1+\ell-\alpha} \|\alpha\|_{C^{0,\alpha}_{\ell,\delta;\lambda}(V_{[0,\sigma]})}.$$

In particular, $\bar{\pi}_\lambda$, ρ_λ , $\bar{\sigma}_\lambda$ and τ_λ are bounded by $c \lambda^{-\alpha}$ with respect to the $C^{0,\alpha}_{\ell,\delta;\lambda}$ -norms.

We denote by $\mathfrak{X}_\lambda$ and $\mathfrak{Y}_\lambda$ the Banach spaces $C^{1,\alpha} \Omega^1(X, \mathfrak{g}_{E_\lambda})$ and $C^{0,\alpha} \Omega^0(X, \mathfrak{g}_{E_\lambda}) \oplus C^{0,\alpha} \Omega^2_7(X, \mathfrak{g}_{E_\lambda})$ equipped with the norms

$$\begin{aligned} \|a\|_{\mathfrak{X}_\lambda} &:= \lambda^{-\delta/2} \|\rho_\lambda a\|_{C^{1,\alpha}_{-1,\delta;\lambda}} + \lambda \|\pi_\lambda a\|_{C^{1,\alpha}} \quad \text{and} \\ \|(\xi, \alpha)\|_{\mathfrak{Y}_\lambda} &:= \lambda^{-\delta/2} \|\xi\|_{C^{0,\alpha}_{-2,\delta;\lambda}} + \lambda^{-\delta/2} \|\tau_\lambda \alpha\|_{C^{0,\alpha}_{-2,\delta;\lambda}} + \lambda \|\sigma_\lambda \alpha\|_{C^{0,\alpha}}, \end{aligned}$$

respectively. Here we fixed $\delta \in (-1, 0)$ and $0 < \alpha \ll |\delta|$. For concreteness one may take $\delta = -\frac{1}{2}$ and $\alpha = \frac{1}{256}$.

Proposition 5.20. *There exists a constant $c > 0$ such that for all $\lambda \in (0, \Lambda]$*

$$\|\pi_7(F_{A_\lambda})\|_{\mathfrak{Y}_\lambda} \leq c \lambda^{2-\alpha}.$$

We leave the details of the proof to the reader since it is completely analogous to that of Proposition 4.38 making use of the analogues of Propositions 4.40, 4.41, 4.42 and the following result whose proof proceeds in the same way as that of Proposition 4.39.

Proposition 5.21. *In the tubular neighbourhood $V_{[0,\zeta)}$ of Q we can write π_7 as*

$$\pi_7 = \pi_7^0 + \pi_7^1 + \pi_7^{\geq 2}.$$

where π_7^1 vanishes on $\Lambda^- N^* Q$ and there is a constant $c > 0$ which is independent of $\lambda \in (0, \Lambda]$ such that

$$\|\pi_7^0\|_{C_{0,0;\lambda}^{0,\alpha}(V_{[0,\zeta)})} + \|\pi_7^1\|_{C_{1,0;\lambda}^{0,\alpha}(V_{[0,\zeta)})} + \|\pi_7^{\geq 2}\|_{C_{2,0;\lambda}^{0,\alpha}(V_{[0,\zeta)})} \leq c.$$

5.3 Linear analysis

Proposition 5.22. *For $\lambda \in (0, \Lambda]$ the linear operator $L_\lambda: \mathfrak{X}_\lambda \rightarrow \mathfrak{Y}_\lambda$ has a right inverse $R_\lambda: \mathfrak{Y}_\lambda \rightarrow \mathfrak{X}_\lambda$ and there exists a constant $c > 0$ which is independent of $\lambda \in (0, \Lambda]$ such that*

$$\|R_\lambda(\xi, \alpha)\|_{\mathfrak{X}_\lambda} \leq c\|(\xi, \alpha)\|_{\mathfrak{Y}_\lambda}.$$

This is the key to proving Theorem 5.1. The proofs of Proposition 5.22 and Proposition 4.45 are similar in many details; however, in the current situation it is more convenient to produce R_λ by gluing various local right inverses “by hand”. We decompose L_λ as

$$L_\lambda = \begin{pmatrix} \mathfrak{R}_\lambda & \mathfrak{p}_\lambda \\ \mathfrak{q}_\lambda & \mathfrak{L}_\lambda \end{pmatrix}$$

where

$$\begin{aligned} \mathfrak{R}_\lambda &:= \bar{\sigma}_\lambda L_\lambda \bar{\pi}_\lambda, & \mathfrak{L}_\lambda &:= \tau_\lambda L_\lambda \rho_\lambda, \\ \mathfrak{p}_\lambda &:= \bar{\sigma}_\lambda L_\lambda \rho_\lambda, & \text{and } \mathfrak{q}_\lambda &:= \tau_\lambda L_\lambda \bar{\pi}_\lambda. \end{aligned}$$

In the course of this section we will show that $\mathfrak{R}_\lambda$ is essentially the linearised Fueter operator $F_{\mathcal{J}}$, which has a right inverse by assumption, and that local right inverses for $\mathfrak{L}_\lambda$ can be seen to exist by considerations of model operators on $\mathbf{R}^8$ and on the complement of Q , while $\mathfrak{p}_\lambda$ and $\mathfrak{q}_\lambda$ are negligibly small terms. An approximate right inverse $\tilde{R}_\lambda$ can then be constructed by carefully patching together the local right inverses. Finally, a simple deformation argument will yield R_λ .

5.3.1 The model operator on $\mathbf{R}^8$

Let I be a finite energy ASD instanton on a G -bundle E over $\mathbf{R}^4$. By a slight abuse of notation we denote the pullbacks of I and E to $\mathbf{R}^8 = \mathbf{R}^4 \oplus \mathbf{R}^4$ by I and E as well. We define $\mathbf{L}_I: \Omega^0(\mathbf{R}^8, \mathfrak{g}_E) \rightarrow \Omega^0(\mathbf{R}^8, \mathfrak{g}_E) \oplus \Omega_7^2(\mathbf{R}^8, \mathfrak{g}_E)$ by

$$\mathbf{L}_I(a) := (d_A^* a, \pi_7 d_A a).$$

Here π_7 is taken with respect to the standard Spin(7)–structure Φ_0 on $\mathbf{R}^8$, see (1.36). By (1.116) we can, with the appropriate identifications being made, write

$$\mathbf{L}_I = \partial_t - L_I$$

where we think of I as a G_2 –instanton on $\{0\} \oplus \mathbf{R}^3 \oplus \mathbf{R}^4$ and L_I is as in (1.103). In particular, using Proposition 2.70 we see that

$$(5.23) \quad \mathbf{L}_I \mathbf{L}_I^* = \mathbf{L}_I^* \mathbf{L}_I = \Delta_{\mathbf{R}^4} + \begin{pmatrix} \delta_I \delta_I^* & \\ & \delta_I^* \delta_I \end{pmatrix}$$

and, hence, we can argue using Lemma 2.76 as in Section 4.5.1.

For a Hölder exponent $\alpha \in (0, 1)$ and a weight parameter $\beta \in \mathbf{R}$ we define $[f]_{C_\beta^{0,\alpha}(U)}$, $\|f\|_{L_\beta^\infty(U)}$ and $\|f\|_{C_\beta^{k,\alpha}(U)}$ as in Section 4.5.1, but with weight functions defined by

$$(5.24) \quad w(x) := 1 + |\text{pr}_2(x)| \quad \text{and} \quad w(x, y) := \min\{w(x), w(y)\}$$

where $\text{pr}_2 : \mathbf{R}^4 \oplus \mathbf{R}^4 \rightarrow \mathbf{R}^4$ denotes the projection onto the second summand. If U is not specified, then we take $U = \mathbf{R}^8$. We denote by $C_\beta^{k,\alpha}$ the subspace of elements f of the Banach space $C^{k,\alpha}$ with $\|f\|_{C_\beta^{k,\alpha}} < \infty$ equipped with the norm $\|\cdot\|_{C_\beta^{k,\alpha}}$.

The linear operator $\mathbf{L}_I$ can serve as a model for L_λ in the following sense: Fix $x \in Q$. Set $I := I(\mathcal{J})|_{N_x Q}$ and $E := E(\mathcal{J})|_{N_x Q}$. Identify $T_x X = T_x Q \oplus N_x Q$ with $\mathbf{R}^8 = \mathbf{R}^4 \oplus \mathbf{R}^4$ in such a way that the summands are preserved and $\Phi|_{T_x X}$ is identified with Φ_0 . For $\varepsilon_1, \varepsilon_2 > 0$ we define

$$V_{\varepsilon_1, \varepsilon_2} := B_{\varepsilon_1}(x) \cap V_{[0, \varepsilon_2]}.$$

Using the exponential map we can identify $V_{\varepsilon_1, \varepsilon_2}$ with a small neighbourhood $\tilde{U}_{\varepsilon_1, \varepsilon_2}$ of the origin in $\mathbf{R}^8$. With respect to this identification a $\mathfrak{g}_{E_\lambda}$ –valued tensor field f on $V_{\varepsilon_1, \varepsilon_2}$ is identified with a $s_{1/\lambda}^* \mathfrak{g}_E$ –valued tensor field $\tilde{f}$ on $\tilde{U}_{\varepsilon_1, \varepsilon_2; \lambda}$, and if $k \in \mathbf{N}$ is a scaling parameter, then with f we can associate a $\mathfrak{g}_E$ –valued tensor field $s_{k, \lambda} f$ on $U_{\varepsilon_1, \varepsilon_2; \lambda} := \lambda^{-1} \tilde{U}_{\varepsilon_1, \varepsilon_2}$ defined by

$$(s_{k, \lambda} f)(x, y) := \lambda^k \tilde{f}(\lambda x, \lambda y).$$

Proposition 5.25. *There are constants $c, \varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]$ and $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \frac{1}{c} \lambda^{k+\ell} \|f\|_{L_{\ell, \delta; \lambda}^\infty(V_{\varepsilon, N\sqrt{\lambda}})} &\leq \|s_{k, \lambda} f\|_{L_{\ell+\delta}^\infty(U_{\varepsilon, N\sqrt{\lambda}; \lambda_i})} \leq c N^{-2\delta} \lambda^{k+\ell} \|f\|_{L_{\ell, \delta; \lambda}^\infty(V_{\varepsilon, N\sqrt{\lambda}})}, \\ \frac{1}{c} \lambda^{k+\ell} \|f\|_{C_{\ell, \delta; \lambda}^{k, \alpha}(V_{\varepsilon, N\sqrt{\lambda}})} &\leq \|s_{k, \lambda} f\|_{C_{\ell+\delta}^{k, \alpha}(U_{\varepsilon, N\sqrt{\lambda}; \lambda_i})} \leq c N^{-2\delta} \lambda^{k+\ell} \|f\|_{C_{\ell, \delta; \lambda}^{k, \alpha}(V_{\varepsilon, N\sqrt{\lambda}})} \quad \text{and} \\ \|L_\lambda a - s_{2, \lambda}^{-1} \mathbf{L}_I s_{1, \lambda} a\|_{C_{-2, \delta; \lambda}^{0, \alpha}(V_{\varepsilon, N\sqrt{\lambda}})} &\leq c(\varepsilon + \sqrt{\lambda}) \|a\|_{C_{-1, \delta; \lambda}^{1, \alpha}(V_{\varepsilon, N\sqrt{\lambda}})}. \end{aligned}$$

For $\beta < -1$ we define $\pi_I: C_\beta^{k,\alpha} \rightarrow C^{k,\alpha}(\mathbf{R}^4, \ker \delta_I)$ by

$$\pi_I(x) := \sum_{\kappa} \langle a(x, \cdot), \kappa \rangle_{L^2(\mathbf{R}^4)} \kappa$$

where κ runs through an L^2 orthonormal basis of $\ker \delta_I$ and set

$$\mathfrak{A}_\beta^{k,\alpha} := \ker \pi_I \cap C_\beta^{k,\alpha}.$$

The projection operators π_λ and σ_λ can be viewed as “global versions” of π_I . It follows from the discussion at the beginning of Section 4.1 that $\mathbf{L}_I$ defines a linear operator $\mathbf{L}_I: \mathfrak{A}_\beta^{1,\alpha} \rightarrow \mathfrak{A}_{\beta-1}^{0,\alpha}$.

The key result of this section is the following.

Proposition 5.26. *For $\beta \in (-2, -1)$ the linear operator $\mathbf{L}_I: \mathfrak{A}_\beta^{1,\alpha} \rightarrow \mathfrak{A}_{\beta-1}^{0,\alpha}$ is invertible.*

The proof rests on the following estimate.

Proposition 5.27. *For $\beta \in (-3, -1)$ there is a constant $c > 0$ such that for all $a \in \mathfrak{A}_\beta^{1,\alpha}$ the following holds*

$$\begin{aligned} \|a\|_{C_\beta^{1,\alpha}} &\leq c \|\mathbf{L}_I a\|_{C_{\beta-1}^{0,\alpha}} \quad \text{and} \\ \|a\|_{C_\beta^{1,\alpha}} &\leq c \|\mathbf{L}_I^* a\|_{C_{\beta-1}^{0,\alpha}}. \end{aligned}$$

Proof of Proposition 5.26 assuming Proposition 5.27. From Proposition 5.27 it follows that $\mathbf{L}_I: \mathfrak{A}_\beta^{1,\alpha} \rightarrow \mathfrak{A}_{\beta-1}^{0,\alpha}$ is injective and its image is closed. Thus we can identify its cokernel with the kernel of $\mathbf{L}_I^*: (\mathfrak{A}_{\beta-1}^{0,\alpha})^* \rightarrow (\mathfrak{A}_\beta^{1,\alpha})^*$. Since $\beta \in (-2, -1)$, the image of π_I is contained in $C_{\beta-1}^{0,\alpha}$ and thus $C_{\beta-1}^{0,\alpha} = \mathfrak{A}_{\beta-1}^{0,\alpha} \oplus \text{im } \pi_I$. Via this splitting we can extend any $b \in \ker \mathbf{L}_I^*$ to an element of $(C_{\beta-1}^{0,\alpha})^*$ which still satisfies $\mathbf{L}_I^* b = 0$. By elliptic regularity b is smooth and it follows from Lemma 2.76 that b is invariant under translations in the $\mathbf{R}^4$ -direction. Now b must be contained in $C_{-3-\beta}^{1,\alpha}$. Since $-3 - \beta \in (-3, -1)$, it follows that $b = 0$ by Proposition 5.27. Therefore $\mathbf{L}_I$ is also surjective; hence, invertible. $\square$

Proof of Proposition 5.27. By now we have seen the argument used to prove Proposition 5.27 a number of times, so we will go through the steps quickly and restrict to the case of $\mathbf{L}_I$ as the case $\mathbf{L}_I^*$ differs only by a slight change in notation. First it is easy to see that there are Schauder estimates

$$\|a\|_{C_\beta^{1,\alpha}} \leq c \left(\|\mathbf{L}_I a\|_{C_{\beta-1}^{0,\alpha}} + \|a\|_{L_\beta^\infty} \right)$$

with $c = c(\beta) > 0$. The crucial step is then to show that if $\beta \in (-3, -1)$ there is a constant $c > 0$ such that for all $a \in \mathfrak{A}_\beta^{1,\alpha}$ we have

$$\|a\|_{L_\beta^\infty} \leq c \|\mathbf{L}_I a\|_{C_{\beta-1}^{0,\alpha}}.$$

Again, this is proved by contradiction: Suppose the estimate does not hold. Then there exists a sequence $a_i \in \mathfrak{A}_\beta^{1,\alpha}$ such that

$$\|a_i\|_{L_\beta^\infty} = 1 \quad \|\mathbf{L}_I a_i\|_{C_{\beta-1}^{0,\alpha}} \leq \frac{1}{i}.$$

Hence, by the above Schauder estimate

$$\|a_i\|_{C_\beta^{1,\alpha}} \leq 2c.$$

Pick $(x_i, y_i) \in \mathbf{R}^4 \oplus \mathbf{R}^4$ such that

$$w(x_i, y_i)^{-\beta} |a_i(x_i, y_i)| = 1.$$

By translation we can assume that $x_i = 0$. Without loss of generality one of the following two cases must occur. We rule out both of them thus proving the estimate.

Case 1. *The sequence $|y_i|$ stays bounded.*

Let K be a compact subset of $\mathbf{R}^8$. When restricted to K , the elements a_i are uniformly bounded in $C^{1,\alpha}$. Thus, by Arzelà–Ascoli, we can assume (after passing to a subsequence) that a_i converges to a limit a in $C^{1,\alpha/2}$. Since K was arbitrary, this yields $a \in \Omega^1(\mathbf{R}^8, \mathfrak{g}_E)$ satisfying

$$|a|(x, y) < c(1 + |y|)^\beta$$

as well as

$$\mathbf{L}_I a = 0 \quad \text{and} \quad \pi_I a = 0.$$

It follows from Lemma 2.76 that $a = 0$. On the other hand we can assume that y_i converges to some point $y \in \mathbf{R}^4$ for which we would have $|a|(0, y) = w(0, y)^\beta \neq 0$. This is a contradiction.

Case 2. *The sequence $|y_i|$ goes to infinity.*

Define a rescaled sequence $\tilde{a}_i$ by

$$\tilde{a}_i(x, y) := |y_i|^{-\beta} (\xi_i, a_i)(|y_i|x, |y_i|y)$$

and set $\tilde{y}_i = y_i/|y_i|$. The rescaled sequence then satisfies

$$\|\tilde{a}_i\|_{C_\beta^{1,\alpha}} \leq 2c, \quad \|\mathbf{L}\tilde{a}_i\|_{C_{\beta-1}^{0,\alpha}} \leq 2/i \quad \text{and} \quad \tilde{w}(0, \tilde{y}_i)^{-\beta} |\tilde{a}_i(0, \tilde{y}_i)| \geq 1/2$$

where the norms $\|\cdot\|_{C_\beta^{k,\alpha}}$ are defined like those above, cf. Section 4.5.1, but with weight

function $w(x) = |\text{pr}_2(x)|$ instead of $w(x) = 1 + |\text{pr}_2(x)|$, and where L is defined by

$$\mathbf{L} := \partial_t - L$$

with L as in (4.66).

We can now pass to a limit using Arzelà–Ascoli as before to obtain $\tilde{a}$ defined over $\mathbf{R}^4 \times (\mathbf{R}^4 \setminus \{0\})$ satisfying

$$|\tilde{a}|(x, y) < c|y|^\beta \quad \text{and} \quad L\tilde{a} = 0.$$

Since $\beta > -3$, $L\tilde{a} = 0$ holds on all of $\mathbf{R}^8$ in the sense of distributions. Hence, by standard elliptic theory, $\tilde{a}$ extends to a bounded smooth solution of $L\tilde{a} = 0$ on $\mathbf{R}^8$. Since $L^*L = \Delta_{\mathbf{R}^4} + \Delta_{\mathbf{R}^4}$, it follows from Lemma 2.76 that $\tilde{a}$ is invariant in the $\mathbf{R}^4$ -direction. Therefore, we can think of the components of $\tilde{a}$ as harmonic functions on $\mathbf{R}^4$. Since they decay to zero at infinity as $\beta < 0$, they must vanish identically. On the other hand we know that $|\tilde{y}_i| = 1$ and thus without loss of generality $\tilde{y}_i$ converges to some point $\tilde{y}$ in the unit sphere for which $|\tilde{a}|(0, \tilde{y})| \geq \frac{1}{2}$, contradicting $\tilde{a} = 0$. $\square$

5.3.2 The model away from Q

Define weighted Hölder norms $\|\cdot\|_{C_\beta^{k,\alpha}}$ for tensor fields (with values in $\mathfrak{g}_{E_t}$) on $X \setminus Q$ as in Section 5.2 by

$$\begin{aligned} [f]_{C_\beta^{0,\alpha}} &:= \sup_{d(x,y) \leq w(x,y)} w(x,y)^{\alpha-\beta} \frac{|f(x) - f(y)|}{d(x,y)^\alpha}. \\ \|f\|_{L_\beta^\infty} &:= \|w^{-\beta} f\|_{L^\infty} \quad \text{and} \\ \|f\|_{C_\beta^{k,\alpha}} &:= \sum_{j=0}^k \|\nabla^j f\|_{L_{\beta-j}^\infty} + [\nabla^j f]_{C_{\beta-j}^{0,\alpha}}. \end{aligned}$$

with weight functions given by

$$w(x) := r(x) \quad \text{and} \quad w(x, y) := \min\{w(x), w(y)\}.$$

If we fix a constant $N > 0$, then over $V_{[\sqrt{\lambda}/N, \infty)}$ we can view a tensor field f with values in $\mathfrak{g}_{E_\lambda}$ as one which takes values in $\mathfrak{g}_E$ and vice versa.

Proposition 5.28. *There is a constant $c > 0$ such that for $\lambda \in (0, \Lambda]$ with respect to the above identification we have*

$$\frac{1}{c} \|a\|_{C_{-\ell+\delta}^{k,\alpha}(V_{[\sqrt{\lambda}/N, \infty)})} \leq \|a\|_{C_{\ell,\delta,\lambda}^{k,\alpha}(V_{[\sqrt{\lambda}/N, \infty)})} \leq cN^{-2\delta} \|a\|_{C_{-\ell+\delta}^{k,\alpha}(V_{[\sqrt{\lambda}/N, \infty)})}$$

and

$$\|L_\lambda a - L_A a\|_{C_{-2,\delta,\lambda}^{0,\alpha}(V_{[\sqrt{\lambda}/N,\infty)})} \leq c\sqrt{\lambda}/N \|a\|_{C_{-1,\delta,\lambda}^{1,\alpha}(V_{[\sqrt{\lambda}/N,\infty)})}.$$

Proposition 5.29. *For $\beta \in (-3, 0)$ the operator $L_A: C_\beta^{1,\alpha} \rightarrow C_{\beta-1}^{0,\alpha}$ has a right inverse R_A .*

Remark 5.30. If A was reducible, we would work with an extended operator $\tilde{L}_A$ analogous to $\tilde{L}_\theta$ in (2.92).

Proof of Proposition 5.29. Denote by $\pi: C_\beta^{1,\alpha} \rightarrow \ker L_A$ the L^2 –projection to the (smooth) kernel of L_A . This is well defined, because $\beta > -3$. We will shortly prove the estimates

$$\begin{aligned} \|a\|_{C_\beta^{1,\alpha}} &\leq c \left(\|L_A a\|_{C_{\beta-1}^{0,\alpha}} + \|\pi a\|_{L_\beta^\infty} \right) \quad \text{and} \\ \|a\|_{C_\beta^{1,\alpha}} &\leq c \|L_A^* a\|_{C_{\beta-1}^{0,\alpha}}. \end{aligned}$$

From the first estimate it follows immediately that the image of $L_A: C_\beta^{1,\alpha} \rightarrow C_{\beta-1}^{0,\alpha}$ is closed and its kernel is finite-dimensional (in fact, it can be seen to agree with the smooth kernel of L_A). To show that L_A has a right inverse it suffices to prove that $\text{coker } L_A = 0$. Let $b \in \ker L_A^* \cong \text{coker } L_A$. Then using elliptic regularity it can be seen that b represents an element in the kernel of $L_A^*: C_{-3-\beta}^{1,\alpha} \rightarrow C_{-4-\beta}^{0,\alpha}$. But then $b = 0$ by the second estimate.

Now we are left with proving the above estimates. We will only prove the first estimate, since the proof of the second estimate is similar, but slightly easier. First of all we have the following Schauder estimate

$$\|a\|_{C_{\beta,t}^{1,\alpha}} \leq c (\|L_A a\|_{C_{\beta-1,t}^{0,\alpha}} + \|a\|_{L_{\beta,t}^\infty}).$$

To prove that

$$\|a\|_{L_{\beta,t}^\infty} \leq c \left(\|L_t a\|_{C_{\beta-1,t}^{0,\alpha}} + \|\pi a\|_{L_{\beta,t}^\infty} \right)$$

one argues by contradiction. If a_i is a sequence of counterexamples as before, then we can assume that it either gives rise to a non-trivial element a in the kernel of $L_A: C_\beta^{1,\alpha} \rightarrow C_{\beta-1}^{0,\alpha}$ which also satisfies $\pi a = 0$ or localises in smaller and smaller neighbourhoods of Q . To see that the first case cannot occur observe that if $a \in C_\beta^{1,\alpha}$ solves $L_A a = 0$ on $X \setminus Q$, then it follows that $L_A a = 0$ on all of X in the sense of distributions and thus a extends smoothly to X , since $\beta > -3$. This contradicts $\pi a = 0$. Thus we must be in the second case. Rescaling a_i near Q as before yields a non-trivial harmonic function on $\mathbf{R}^4 \times \mathbf{R}^4 \setminus \{0\}$ which is bounded by a constant multiple of $|y|^\beta$. Since $\beta > -3$ the function extends to $\mathbf{R}^8$ and by Lemma 2.76 it is in variant in the $\mathbf{R}^4$ –direction. Hence, it corresponds to a decaying harmonic function on $\mathbf{R}^4$, since $\beta < 0$, and must vanish identically. So the second case does not occur as well, proving that the claimed estimate must hold. $\square$

5.3.3 Comparison of $\mathfrak{K}_\lambda$ with $F_{\mathfrak{J}}$

Proposition 5.31. *There is a constant $c > 0$ such that for all $\lambda \in (0, \Lambda]$ we have*

$$\|(L_\lambda \mu_\lambda - \nu_\lambda F_{\mathfrak{J}}) \hat{\mathfrak{J}}\|_{C_{-2,0;\lambda}^{0,\alpha}} \leq c \lambda^2 \|\hat{\mathfrak{J}}\|_{C^{1,\alpha}}.$$

Corollary 5.32. *There is a constant $c > 0$ such that for all $\lambda \in (0, \Lambda]$ we have*

$$\|(\sigma_\lambda L_\lambda \mu_\lambda - F_{\mathfrak{J}}) \hat{\mathfrak{J}}\|_{C^{0,\alpha}} \leq c \lambda^{1-\alpha} \|\hat{\mathfrak{J}}\|_{C^{1,\alpha}}.$$

Proof of Proposition 5.31. The argument is analogous to the one in the proof of Proposition 4.67; however, we now use the model operator $\tilde{L}_\lambda$ defined by

$$\tilde{L}_\lambda a := (d_{I_\lambda}^* a, \pi_7^0(d_{I_\lambda} a)).$$

If we view $\Gamma(\mathfrak{J}^* V \mathfrak{M})$ as a subspace of $\Omega^1(NQ, \mathfrak{g}_E)$, then on this subspace $\tilde{L}_\lambda$ agrees with the linearised Fueter operator $F_{\mathfrak{J}}$. We thus have to estimate the terms in the expression

$$\begin{aligned} L_\lambda \mu_\lambda \hat{\mathfrak{J}} - \nu_\lambda F_{\mathfrak{J}} \hat{\mathfrak{J}} &= L_\lambda (\mu_\lambda \hat{\mathfrak{J}} - \hat{\mathfrak{J}}_\lambda) + (L_\lambda - \tilde{L}_\lambda) \hat{\mathfrak{J}} + s_{1/\lambda}^* F_{\mathfrak{J}} \hat{\mathfrak{J}} - \nu_\lambda F_{\mathfrak{J}} \hat{\mathfrak{J}} \\ &=: \text{I} + \text{II} + \text{III} \end{aligned}$$

on $V_{[0,\zeta)}$. Again, it is clear that

$$\|\text{I}\|_{C_{-2,0;\lambda}^{0,\alpha}(V_{[0,\zeta)})} + \|\text{III}\|_{C_{-2,0;\lambda}^{0,\alpha}(V_{[0,\zeta)})} \leq c \lambda^2 \|\hat{\mathfrak{J}}\|_{C^{1,\alpha}}.$$

The key for the estimate of II is to notice that

$$\pi_7^0((d_{I_\lambda} \hat{\mathfrak{J}})^{0,2}) = \pi_7^1((d_{I_\lambda} \hat{\mathfrak{J}})^{0,2}) = 0,$$

because $\delta_{\mathfrak{J}(x)}(\hat{\mathfrak{J}}|_{N_x Q}) = 0$ and π_7^0 and π_7^1 vanish on $\Lambda^- NQ$. One can now prove that $\|\text{II}\|_{C_{-2,0;\lambda}^{0,\alpha}(V_{[0,\zeta)})} \leq c \lambda^2 \|\hat{\mathfrak{J}}\|_{C^{1,\alpha}}$ arguing as in the proof of Proposition 4.67. $\square$

5.3.4 Estimate of $\mathfrak{p}_\lambda$ and $\mathfrak{q}_\lambda$

Proposition 5.33. *For $\delta \in (-1, 0)$ there exists a constant $c > 0$ such that for all $\lambda \in (0, \Lambda]$ we have*

$$\begin{aligned} \|\sigma_\lambda \mathfrak{p}_\lambda a\|_{C^{0,\alpha}} &\leq c \lambda^{-\alpha} \|\rho_\lambda a\|_{C_{-1,\delta;\lambda}^{1,\alpha}} \quad \text{and} \\ \|\mathfrak{q}_\lambda a\|_{C_{-2,\delta;\lambda}^{0,\alpha}} &\leq c \lambda^{2+\delta/2-\alpha} \|\pi_\lambda a\|_{C^{1,\alpha}}. \end{aligned}$$

Proof. This proof proceeds along the lines of the proof of Proposition 4.70. First note that

the second estimate is an immediate consequence of Propositions 5.31 and 5.19, because

$$\mathfrak{q}_\lambda a = \tau_\lambda(L_\lambda \mu_\lambda - \nu_\lambda F_{\mathfrak{J}}) \mu_\lambda a,$$

since $\tau_\lambda \nu_\lambda = 0$. Now, to estimate $\mathfrak{p}_\lambda$ we define $\tilde{\pi}_\lambda : \Omega^2(NQ_{\mathfrak{g}_{E(\mathfrak{J}_\lambda)}}) \rightarrow \Gamma(\mathfrak{J}^* V \mathfrak{M}) \subset \Omega^1(NQ, \mathfrak{g}_{E(\mathfrak{J}_\lambda)})$ and $\tilde{\sigma}_\lambda : \Omega^2(NQ_{\mathfrak{g}_{E(\mathfrak{J}_\lambda)}}) \rightarrow \Gamma(\text{Hom}_\Phi(TQ, \mathfrak{J}^* V \mathfrak{M})) \subset \Omega^2(NQ, \mathfrak{g}_{E(\mathfrak{J}_\lambda)})$ by

$$\begin{aligned} (\tilde{\sigma}_\lambda a)(x) &:= \sum_{\kappa} \int_{N_x Q} \langle a, \kappa \rangle \kappa, \quad \text{and} \\ (\tilde{\sigma}_\lambda \alpha)(x) &:= \sum_{\beta} \int_{N_x Q} \langle \alpha, \beta \rangle \beta, \end{aligned}$$

respectively, where, at each point $x \in Q$, κ runs through an orthonormal basis of $V \mathfrak{M}_{\mathfrak{J}(x)}$ with respect to the inner product $\langle s_{1/\lambda}^*, s_{1/\lambda}^* \cdot \rangle$ and β runs through an orthonormal basis of $\text{Hom}_\Phi(T_x Q, V \mathfrak{M}_{\mathfrak{J}(x)})$ with respect to $\langle s_{1/\lambda}^*, s_{1/\lambda}^* \cdot \rangle$. We set $\tilde{\rho}_\lambda := \text{id} - \tilde{\pi}_\lambda$ and $\tilde{\tau}_\lambda := \text{id} - \tilde{\sigma}_\lambda$. One can check that $\tilde{\sigma}_\lambda \tilde{L}_\lambda \tilde{\rho}_\lambda = 0$. Therefore

$$\begin{aligned} \mathfrak{p}_\lambda a &= \bar{\sigma}_\lambda(L_\lambda - \tilde{L}_\lambda) \rho_\lambda a + (\bar{\sigma}_\lambda - \tilde{\sigma}_\lambda) \tilde{L}_\lambda \rho_\lambda a + \tilde{\sigma}_\lambda \tilde{L}_\lambda (\rho_\lambda - \tilde{\rho}_\lambda) a \\ &= \bar{\pi}_\lambda \text{I} + \text{II} + \tilde{\pi}_\lambda \text{III}. \end{aligned}$$

The terms I, II and III can now be handled as in the proof of Proposition 4.70. $\square$

5.3.5 Patching local inverses

Proof of Proposition 5.22. Fix $y \in \mathfrak{Y}_\lambda$ and set

$$u := \bar{\sigma}_\lambda y \quad \text{and} \quad v := \tau_\lambda y.$$

Step 1. *An approximate inverse for u .*

Denote by $G_{\mathfrak{J}}$ a fixed right inverse of $F_{\mathfrak{J}}$ and set

$$z := \mu_\lambda G_{\mathfrak{J}} \sigma_\lambda u.$$

Then by Proposition 5.19 we have

$$(5.34) \quad \|z\|_{\mathfrak{X}_\lambda} \leq c \|y\|_{\mathfrak{Y}_\lambda}$$

and by Corollary 5.32 and Proposition 5.33 we have

$$(5.35) \quad \|L_\lambda z - u\|_{\mathfrak{Y}_\lambda} \leq c \lambda^{1-\alpha} \|y\|_{\mathfrak{Y}_\lambda}.$$

Step 2. *Choice of cut-off functions.*

We construct an approximate inverse for v by finding local approximate inverses and then patching these together. This requires two kinds of cut-off functions. The first kind is constructed as follows: Let $\chi : [0, \infty) \rightarrow [0, 1]$ denote the smooth-cut off function chosen in Section 5.2 which vanishes on $[0, 1]$ and is equal to one on $[2, \infty)$. We define $\chi_\lambda : X \rightarrow [0, 1]$ by

$$\chi_\lambda(x) := \chi(r(x)/\sqrt{\lambda}).$$

Then

$$\|\chi_\lambda\|_{C_{0,0;\lambda}^{0,\alpha}} \leq c.$$

Fix a small constant $\varepsilon > 0$, a large constant $N \gg 1$, and note that in the following we can choose the constant $c > 0$ independent of ε and N . Throughout, we will make use of $\lambda \ll \varepsilon$ and $\lambda \ll 1/N$. We can pick a finite number of points $\{x_\gamma : \gamma \in \Gamma\} \subset Q$ such that the balls $B_\varepsilon(x_\gamma)$ cover all of Q and a partition of unity $1 = \sum_{\gamma \in \Gamma} \chi_\gamma$ subordinate to this cover such that

$$\|\chi_\gamma\|_{C_{0,0;\lambda}^{0,\alpha}(\text{supp}(1-\chi_\lambda))} \leq c\varepsilon^{-\alpha}.$$

We can now write

$$v = \sum_{\gamma \in \Gamma} v_\gamma + v_0$$

with

$$v_\gamma := (1 - \chi_\lambda)\chi_\gamma v \quad \text{and} \quad v_0 := \chi_\lambda v.$$

Although v_0 and the v_γ depend λ we choose not to make this dependence explicit in order not to clutter the notation any more. By construction we have

$$(5.36) \quad \sum_{\gamma} \|v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}} + \|v_0\|_{C_{-2,\delta;\lambda}^{0,\alpha}} \leq c\varepsilon^{-\alpha} \|v\|_{C_{-2,\delta;\lambda}^{0,\alpha}}.$$

The second kind of cut-off functions is constructed as follows: We choose $\beta_{\lambda,N}^\pm : X \rightarrow [0, 1]$ such that

$$\beta_{\lambda,N}^+(x) = \begin{cases} 1 & r(x) \leq 2\sqrt{\lambda} \\ 0 & r(x) \geq 2N\sqrt{\lambda} \end{cases}$$

and

$$\beta_{\lambda,N}^-(x) = \begin{cases} 0 & r(x) \leq \sqrt{\lambda}/N \\ 1 & r(x) \geq \sqrt{\lambda} \end{cases}$$

as well as

$$(5.37) \quad \|d\beta_{\lambda,N}^\pm\|_{C_{-1,0;\lambda}^{0,\alpha}} \leq c/\log(N) \quad \text{and} \quad \|\beta_{\lambda,N}^\pm\|_{C_{0,0;\lambda}^{0,\alpha}} \leq c.$$

This can be arranged by interpolating between 0 and 1 logarithmically, i.e., by defining $\beta_{\lambda,N}^+$ as an appropriate smoothing of $\log(2N\sqrt{\lambda}/r)/\log(N)$ in the intermediate region and similarly $\beta_{\lambda,N}^-$ as a smoothing of $\log(Nr/\sqrt{\lambda})/\log(N)$. Moreover, we choose $\tilde{\chi}_\gamma: Q \rightarrow [0, 1]$ such that $\tilde{\chi}_\gamma$ equals one on $B_\varepsilon(x_\gamma)$, $\tilde{\chi}_\gamma$ vanishes outside $B_{2\varepsilon}(x_\gamma)$ and satisfies

$$(5.38) \quad \|\tilde{\chi}_\gamma\|_{C_{-1,0;\lambda}^{1,\alpha}(\text{supp } \beta_{\lambda,N}^+)} \leq cN\sqrt{\lambda}/\varepsilon \quad \text{and} \quad \|\tilde{\chi}_\gamma\|_{C_{0,0;\lambda}^{0,\alpha}(\text{supp } \beta_{\lambda,N}^+)} \leq c.$$

Step 3. *Construction of local approximate inverses.*

Let I_γ be the ASD instanton obtained by restricting $I = I(\mathfrak{I})$ to $N_{x_\gamma}Q$. Using the identifications and the notation of Section 5.3.1 we define

$$\tilde{w}_\gamma := s_{1,\lambda}^{-1} \mathbf{L}_{I_\gamma}^{-1} \rho_{I_\gamma} s_{2,\lambda} v_\gamma \quad \text{and} \quad w_\gamma := \rho_\lambda \tilde{\chi}_\gamma \beta_{\lambda,N}^+ \tilde{w}_\gamma.$$

where $\rho_{I_\gamma} := \text{id} - \pi_{I_\gamma}$. Under the identifications employed in Section 5.3.1 the projections π_λ and σ_λ are identified. From $\sigma_\lambda v = 0$ one can deduce that

$$\|\pi_{I_\gamma} s_{2,\lambda} v_\gamma\|_{C_{-2-\delta}^{0,\alpha}} \leq c\varepsilon \|s_{2,\lambda} v_\gamma\|_{C_{-2-\delta}^{0,\alpha}} \leq c\varepsilon \|v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}}.$$

Using Proposition 5.25 we conclude that

$$(5.39) \quad \|\tilde{w}_\gamma\|_{C_{-1,\delta;\lambda}^{0,\alpha}(V_{2\varepsilon,\zeta})} \leq c \|s_{1,\lambda} \tilde{w}_\gamma\|_{C_{-1+\delta}^{1,\alpha}(U_{2\varepsilon,\infty;\lambda})} \leq c \|v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}}$$

and

$$(5.40) \quad \|L_\lambda \tilde{w}_\gamma - v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}(V_{2\varepsilon,\zeta})} \leq c\varepsilon \|v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}}$$

Since $\pi_{I_\gamma}(s_{1,\lambda} \tilde{w}_\gamma) = 0$, it follows that

$$\|\tilde{\pi}_{I_\gamma} s_{1,\lambda} \tilde{w}_\gamma\|_{C_{-1+\delta}^{1,\alpha}(U_{2\varepsilon,\infty;\lambda})} \leq c\varepsilon \|s_{1,\lambda} \tilde{w}_\gamma\|_{C_{-1+\delta}^{1,\alpha}(U_{2\varepsilon,\infty;\lambda})}$$

here $\tilde{\pi}_{I_\gamma}$ is defined like π_{I_γ} but with $\ker \delta_{I|_{N_{\exp x_\gamma}(\lambda \cdot -)Q}}^+$ instead of $\ker \delta_{I|_{N_{x_\gamma}Q}}^+$. Therefore,

$$(5.41) \quad \|\tilde{\pi}_\lambda w_\gamma\|_{C_{-1,\delta;\lambda}^{1,\alpha}} \leq c\varepsilon \|v_\gamma\|_{C_{-1,\delta;\lambda}^{1,\alpha}}$$

and it follows that

$$\sum_\gamma \|w_\gamma\|_{C_{-1,\delta;\lambda}^{1,\alpha}} \leq c \sum_\gamma \|v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}} \leq c\varepsilon^{-\alpha} \|v\|_{C_{-1,\delta;\lambda}^{1,\alpha}}.$$

By Proposition 5.28 $w_0 := \beta_{\lambda,N}^- R_A v_0$, with R_A as in Proposition 5.29, satisfies

$$(5.42) \quad \|w_0\|_{\mathfrak{X}_\lambda} \leq c \|v_0\|_{\mathfrak{Y}_\lambda}.$$

Combining all of the above we see that the $\tilde{R}_\lambda : \mathfrak{Y}_\lambda \rightarrow \mathfrak{X}_\lambda$ defined by

$$\tilde{R}_\lambda y := z + \sum_{\gamma} w_\gamma + w_0.$$

is bounded by $c\varepsilon^{-\alpha}$.

Step 4. $\tilde{R}_\lambda$ is an approximate right inverse to L_λ .

We need to estimate the three types of terms

$$\begin{aligned} \text{I} &:= \|L_\lambda z - u\|_{\mathfrak{Y}_\lambda}, \\ \text{II}_\gamma &:= \|L_\lambda w_\gamma - v_\gamma\|_{\mathfrak{Y}_\lambda} \quad \text{and} \\ \text{III} &:= \|L_\lambda w_0 - v_0\|_{\mathfrak{Y}_\lambda}. \end{aligned}$$

We have already treated I with (5.35). Now,

$$\begin{aligned} \text{II}_\gamma &= \|L_\lambda w_\gamma - v_\gamma\|_{\mathfrak{Y}_\lambda} \leq \lambda^{-\delta/2} \|L_\lambda \rho_\lambda \tilde{\chi}_\gamma \beta_{\lambda,N}^+ \tilde{w}_\gamma - v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}} \\ &\quad + \lambda^{-\delta/2} \|\bar{\sigma}_\lambda L_\lambda \rho_\lambda \tilde{\chi}_\gamma \beta_{\lambda,N}^+ \tilde{w}_\gamma - \bar{\sigma}_\lambda v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}} \\ &\quad + \lambda \|\sigma_\lambda L_\lambda \rho_\lambda \tilde{\chi}_\gamma \beta_{\lambda,N}^+ \tilde{w}_\gamma - \sigma_\lambda v_\gamma\|_{C^{0,\alpha}} \end{aligned}$$

Using (5.39), Proposition 5.33 and the fact that $\pi_\lambda v = 0$ the last two terms can be seen to be bounded by $c\lambda^{1-\alpha} \|v_\gamma\|_{\mathfrak{Y}_\lambda}$. To control the first term use the fact that on the support of v_γ we have $\tilde{\chi}_\gamma \beta_{\lambda,N}^+ = 1$, (5.37), (5.38), (5.39), (5.40) and (5.41) to derive

$$\begin{aligned} \|L_\lambda \rho_\lambda \tilde{\chi}_\gamma \beta_{\lambda,N}^+ \tilde{w}_\gamma - v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}} &\leq c \|L_\lambda \tilde{w}_\gamma - v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}(V_{2\varepsilon,\zeta})} \\ &\quad + c \|\text{d}(\tilde{\chi}_\lambda \beta_{\lambda,N}^+)\|_{C_{-1,0;\lambda}^{0,\alpha}(\text{supp } \beta_{\lambda,N}^+)} \|\tilde{w}_\gamma\|_{C_{-1,\delta;\lambda}^{0,\alpha}(V_{2\varepsilon,\zeta})} \\ &\quad + c \|\pi_\lambda \tilde{\chi}_\gamma \beta_{\lambda,N}^+ \tilde{w}_\gamma\|_{C_{-1,\delta;\lambda}^{1,\alpha}} \\ &\leq c(\varepsilon + 1/\log(N) + N\sqrt{\lambda}/\varepsilon) \|v_\gamma\|_{C_{-2,\delta;\lambda}^{0,\alpha}}. \end{aligned}$$

Similarly,

$$\text{III} \leq c(\sqrt{\lambda} + 1/\log(N)) \|y\|_{\mathfrak{Y}_\lambda}$$

Putting everything together we obtain

$$\|L_\lambda \tilde{R}_\lambda y - y\|_{\mathfrak{Y}_\lambda} \leq c\varepsilon^{-\alpha} (\varepsilon + 1/\log N + N\sqrt{\lambda}/\varepsilon) \|y\|_{\mathfrak{Y}_\lambda}.$$

By choosing ε small enough, N large enough and λ small enough we can make this factor in front of $\|y\|_{\mathfrak{Y}_\lambda}$ arbitrarily small.

Step 5. *Construction of R_λ .*

We can arrange that

$$\|L_\lambda \tilde{R}_\lambda y - y\|_{\mathfrak{Y}_\lambda} \leq \frac{1}{2} \|y\|_{\mathfrak{Y}_\lambda}.$$

for all $\lambda \in (0, \Lambda]$; hence, the series

$$R_\lambda := \tilde{R}_\lambda (L_\lambda \tilde{R}_\lambda)^{-1} = \tilde{R}_\lambda \sum_{k=0}^{\infty} (\text{id} - L_\lambda \tilde{R}_\lambda)^k$$

converges and constitutes a right inverse for L_λ . Clearly, R_λ is bounded uniformly with respect to $\lambda \in (0, \Lambda]$. $\square$

5.4 Conclusion of the proof of Theorem 5.1

In the model discussed in Section 5.3.1 if both $a, b \in \ker \delta_I$, then $\pi_I \pi_7([a \wedge b]) = 0$. Hence, one can argue as in Section 4.6 to show that

$$(5.43) \quad \|Q(a)\|_{\mathfrak{Y}_\lambda} \leq c\lambda^{-2-\delta/2} \|a\|_{\mathfrak{X}_\lambda}^2$$

If we set $\tilde{Q}_\lambda = Q \circ R_\lambda$, then (5.17) becomes

$$b + \tilde{Q}_\lambda(b) + \pi_7(F_{A_\lambda}) = 0.$$

In view of (5.43) and Propositions 5.20 and 5.22 this equation can be solved by appealing to Lemma 4.74. $\square$

5.5 Theorem 5.1 and the index formula

Proposition 5.44. *Let (X, Φ) be a compact $\text{Spin}(7)$ -manifold, let Q be a Cayley submanifold of X and let E_0 and E be $\text{SU}(2)$ -bundles over X which are related by*

$$c_2(E) = c_2(E_0) + \text{PD}[Q].$$

If A is a connection on E and A_0 is a connection on E_0 , then

$$(5.45) \quad \text{index } L_A = \text{index } L_{A_0} + \text{index } F_Q + \text{index } \mathring{F}_3 - \frac{5}{3} \int_Q e(\text{Re}(S_Q^+ \otimes E_0))$$

where L_A and L_{A_0} are as in (1.110) and $\mathring{F}_{\mathfrak{J}} = D: \Gamma(\text{Re}(S^+ \otimes E_0|_Q)) \rightarrow \Gamma(\text{Re}(S^- \otimes E_0|_Q))$ as in Example 5.13.

Proof. By Proposition 1.111 and (1.86) we have

$$\begin{aligned} \text{index } L_{A_\lambda} - \text{index } L_{A_0} &= -\frac{1}{6} \int_Q p_1(X) - \frac{4}{3} [Q] \cdot [Q] - \frac{8}{3} \int_Q c_2(E_0) \\ &= -\sigma(Q) - \frac{1}{3} \chi(Q) - [Q] \cdot [Q] - \frac{8}{3} \int_Q c_2(E_0). \end{aligned}$$

By (1.82) and (5.14) we have

$$\text{index } F_Q + \text{index } \mathring{F}_{\mathfrak{J}} = \frac{1}{4} \sigma(Q) + \frac{1}{2} \chi(Q) - [Q] \cdot [Q] - \int_Q c_2(E_0).$$

Using (1.85) and (1.86)

$$\int_Q e(\text{Re}(S_Q^+ \otimes E_0)) = \int_Q c_2(E_0) + \frac{3}{4} \sigma(Q) + \frac{1}{2} \chi(Q).$$

Verifying (5.45) is now straight-forward. $\square$

In the situation of Proposition 5.44 whenever Theorem 5.1 can be applied $e(\text{Re}(S_Q^+ \otimes E_0))$ vanishes and (5.45) can be taken as evidence that my construction gives a description of an open subset of the moduli space of $\text{Spin}(7)$ –instantons. (Note that the gluing parameter λ is already contained in $\text{index } \mathring{F}_{\mathfrak{J}}.$)

5.6 Local $K3$ Cayley fibrations and $\text{Spin}(7)$ –instantons

Proposition 5.46. *Let X be a compact $\text{Spin}(7)$ –manifold with $\text{Hol}(g_\Phi) = \text{Spin}(7)$. Suppose that Q is a Cayley submanifold in X which has self-intersection number zero, is diffeomorphic to a $K3$ surface whose induced metric is sufficiently close to a hyperkähler metric and suppose that the induced connection on NQ is almost flat. Then there exists a 5–dimensional family of $\text{Spin}(7)$ –instantons on a $\text{SU}(2)$ –bundle E over X .*

Moreover, if $Q_1, \dots, Q_k$ is a collection of k disjoint Cayley submanifolds as above, then there exists a $(8k - 3)$ –dimensional family of $\text{Spin}(7)$ –instantons over X .

Proof. Since $\text{Hol}(g_\Phi) = \text{Spin}(7)$, $b^1 = b_7^2 = 0$ [Joy00, Proposition 10.6.5] and thus the product connection θ on the trivial $\text{SU}(2)$ –bundle is unobstructed. We have $\text{index } L_\theta = -3$. It was shown in the proof of Proposition 1.87 that Q is unobstructed and locally deforms in a 4–dimensional family. If we choose M as in Example 5.4, then $\mathfrak{M} = S_Q^+$. If Q is equipped with a hyperkähler metric, the Dirac operator is surjective and has four-dimensional kernel; hence, the same is true provided the metric on Q is sufficiently close to a hyperkähler metric.

We can thus apply Theorem 5.1 and obtain a 5–dimensional family of $\text{Spin}(7)$ –instantons over X . A similar argument also proves the last assertion of the proposition. $\square$

In [Joy00, Example 14.3.3] Joyce gives an example of a $\text{Spin}(7)$ –manifold which contains two disjoint Cayley submanifolds Q_1 and Q_2 of the kind required by above. Applying Proposition 5.46 recovers the example of a $\text{Spin}(7)$ –instanton described in Lewis’ thesis [Lew98]. Since by Proposition 1.87 every Cayley submanifold as above gives rise to a local fibration of X by Cayley submanifold of this kind, we can use Proposition 5.46 to produce arbitrarily large families of $\text{Spin}(7)$ –instantons.

Suppose that X is a compact $\text{Spin}(7)$ –manifold together with a fibration $\pi: X \rightarrow B$ to a compact base whose generic fibre is a $K3$ Cayley submanifold. In view of the above one could hope (very optimistically) that one can show that the moduli space $\mathcal{M}$ of $\text{Spin}(7)$ –instantons on the $\text{SU}(2)$ –bundle E obtained by applying Proposition 5.46 to a generic fibre of π is smooth (or only mildly singular), 5–dimensional and can be compactified by adding B to the boundary. Then we can use $\mathcal{M}$ to construct a cobordism between B and the link of the singular set of $\mathcal{M}$ much as in the original proof of Donaldson’s theorem [Don83]. In particular, if $\mathcal{M} \cup B$ is smooth and compact, then B is null-cobordant and, hence, $\sigma(B) = 0$. There are of course a large number of problems with this idea beginning with the fact that there are currently no known examples of $\text{Spin}(7)$ –manifolds with $K3$ Cayley fibrations; however, it might serve as an indication what could be achieved using gauge theory on $\text{Spin}(7)$ –manifolds.

Chapter 6

Proposal for a counter term for the low energy $SU(2)$ -theory

Let Y be a compact 7-manifold and let E be an G -bundle over Y . Denote by $\mathcal{A}$ the space of connections on E and by $\mathcal{G}$ the group of gauge transformations of E . Given a torsion-free G_2 -structure on Y , we denote by $\mathcal{M}(E, \phi)$ the moduli space of G_2 -instantons on E over (Y, ϕ) . We fix a set Φ of torsion-free G_2 -structures on Y and make the following transversality assumption: For every $\phi \in \Phi$ and every $[A] \in \mathcal{M}(E, \phi)$, A is irreducible and unobstructed. It follows that all moduli spaces under consideration are smooth manifolds (or orbifolds). It may be possible to achieve transversality by perturbing ϕ and allowing ϕ to have some torsion using arguments similar to those used by Joyce for special Lagrangians with conical singularities [Joy04, Section 9]. In order to ensure that only “generic” non-compactness phenomena need to be considered it would be desirable to have a perturbation scheme that also provides some transversality “up to the boundary”. Frameworks for handling intricate transversality problems in a more direct fashion have been constructed or are currently under construction by Fukaya–Oh–Ohta–Ono [FOOO09a, FOOO09b], Hofer–Wysocki–Zehnder [HWZ07] and Joyce [Joy12a]. It is my understanding that in order to apply these machines one needs a detailed knowledge of *all* non-compactness phenomena that could possibly occur. While this maybe within reach for applications involving J -holomorphic curves, it currently appears to be in the far distance for higher dimensional gauge theories.

6.1 Orienting moduli spaces of G_2 -instantons

We make the following compactness assumption: For every $\phi \in \Phi$, $\mathcal{M}(E, \phi)$ is compact. Now, each $\mathcal{M}(E, \phi)$ is a finite set and we intend to define

$$(6.1) \quad n(E, \phi) := \sum_{[A] \in \mathcal{M}(E, \phi)} \varepsilon([A], \phi)$$

where $\varepsilon([A], \phi) = \pm 1$. To define ε , we follow the classical approach to construct coherent orientations on moduli spaces, see [Flo88]. Fix a reference connection $A_* \in \mathcal{A}(E)$ and a reference G_2 -structure ϕ_* (not necessarily torsion-free), such that L_{A_*, ϕ_*} is invertible. Given a connection $A \in \mathcal{A}(E)$ and a G_2 -structure ϕ we pick paths $(A_t)_{t \in [0,1]}$ and $(\phi_t)_{t \in [0,1]}$ from A_* to A and ϕ_* to ϕ , respectively. We can then consider the spectral flow $\text{SF}(L_{A_t, \phi_t})$ of the family of operators L_{A_t, ϕ_t} . If the parity of this spectral flow is independent of the choice of paths and, moreover, only depends on the gauge equivalence class $[A]$, then we can attach to $([A], \phi)$ the sign

$$(6.2) \quad \varepsilon([A], \phi) := (-1)^{\text{SF}(L_{A_t, \phi_t})}.$$

If we make a different choice of reference connection and reference G_2 -structure, then either all the $\varepsilon([A], \phi)$ stay the same or all of them change sign; that is to say, this approach produces coherent orientations on all the moduli spaces $\mathcal{M}(E, \phi)$.

Proposition 6.3. *The following assertions hold.*

1. *The spectral flow $\text{SF}(L_{A_t, \phi_t})$ is independent of the choice of paths $(A_t)_{t \in [0,1]}$ and $(\phi_t)_{t \in [0,1]}$.*
2. *If $G = \text{SU}(r)$, then $\text{SF}(L_{A_t, \phi_t}) \bmod 2$ only depends on the gauge equivalence class $[A]$ of A . In particular, $\varepsilon([A], \phi)$ is well-defined.*
3. *If $G = \text{SO}(3)$ and $p_1(E) = p_1(\mathfrak{g}_E) = \frac{1}{4}p_1(Y)$, then $\text{SF}(L_{A_t, \phi_t})$ depends only on the gauge equivalence class $[A]$ of A . In particular, $\varepsilon([A], \phi)$ is well-defined.*
4. *Suppose that ε is well-defined. If $f \in \text{Diff}_0(Y)$, then $\varepsilon([A], \phi) = \varepsilon([f^*A], f^*\phi)$.*

Proof. To prove (1), note that $\mathcal{A}(E)$ is contractible and, hence, $\text{SF}(L_{A_t, \phi_t})$ is clearly independent of the choice of $(A_t)_{t \in [0,1]}$. To see that it also does not depend on the choice of $(\phi_t)_{t \in [0,1]}$ it suffices to show that the spectral flow of L_{A, ϕ_t} along a loop of G_2 -structures $\{\phi_t : t \in S^1\}$ is even. It is well-known that this spectral flow coincides with the index of the operator $\mathbf{L} := \partial_t - L_{A, \phi_t} : \Omega^1(Y \times S^1, \mathfrak{g}_E) \rightarrow \Omega^1(Y \times S^1, \mathfrak{g}_E)$. This is simply the deformation operator associated with the $\text{Spin}(7)$ -instanton equation on $Y \times S^1$, see (1.116), and its index can be computed using the Atiyah–Singer index theorem

$$\text{index } \mathbf{L} = - \int_{Y \times S^1} \hat{A}(Y \times S^1) \text{ch}(\mathfrak{g}_E \otimes \mathbf{C}).$$

However, the integrand is the pullback of a class on Y and thus $\text{index } \mathbf{L} = 0$.

We prove (2). Let $A \in \mathcal{A}(E)$ be a connection and let $g \in \mathcal{G}$ be a gauge transformation of E . Fix a path $(A_t)_{t \in [0,1]}$ from A to g^*A . Denote by E_g the $\text{SU}(r)$ -bundle over $Y \times S^1$ obtained by identifying the pullback of E to $Y \times [0, 1]$ over the boundary components via g .

Then (A_t) defines a connection A_g on E_g and the spectral flow of $(L_{A_t, \phi})$ can be computed as the index of the operator $\mathbf{L}_{\mathfrak{g}_{E_g}} := \partial_t - L_{A_t, \phi} : \Omega^1(Y \times S^1, \mathfrak{g}_{E_g}) \rightarrow \Omega^1(Y \times S^1, \mathfrak{g}_{E_g})$:

$$\text{index } \mathbf{L}_{\mathfrak{g}_{E_g}} = \int_{Y \times S^1} -\frac{12+2r}{12} c_2(E_g)^2 + \frac{2r}{6} c_4(E_g) - \frac{2r}{24} c_2(E_g) p_1(Y),$$

see Proposition 1.111. The index of the analogue of $\mathbf{L}_{\mathfrak{g}_{E_g}}$ coupled to E_g instead of $\mathfrak{g}_{E_g}$ is

$$\text{index } \mathbf{L}_{E_g} = \int_{Y \times S^1} -\frac{1}{12} c_2(E_g)^2 + \frac{1}{6} c_4(E_g) - \frac{1}{24} c_2(E_g) p_1(Y),$$

and therefore

$$\text{index } \mathbf{L} = (12+2r) \text{index } \mathbf{L}_{E_g} - 2 \int_{Y \times S^1} c_4(E) + \frac{1}{2} \int_{Y \times S^1} c_2(E) p_1(Y).$$

By [CN13, Lemma 2.6], $p_1(Y)$ is divisible by four and hence $\text{index } \mathbf{L}_{\mathfrak{g}_{E_g}}$ is even.

We prove (3). If $G = \text{SO}(3)$, then $p_2(\mathfrak{g}_{E_g}) = 0$ and

$$p_1(\mathfrak{g}_{E_g}) = p_1(\mathfrak{g}_E) + [\text{dt}] \cup \gamma(g)$$

for some $\gamma(g) \in H^3(Y)$; hence

$$\text{index } \mathbf{L}_{\mathfrak{g}_{E_g}} = \frac{1}{24} \int_Y (p_1(Y) - 4p_1(\mathfrak{g}_E)) \gamma(g).$$

We prove (4). Denote by $T_f := Y \times [0, 1] / \sim$ with $(0, x) \sim (1, f(x))$ the mapping torus of f and denote the bundle over T_f induced by E with E_f . Then the relevant spectral flow is given by the index of $\mathbf{L}_{\mathfrak{g}_{E_f}}$. Since f is homotopic to the identity, there is a cobordism from T_f to $Y \times S^1$ together with a bundle which restricts to E_f respectively E on the ends; hence $\text{index } \mathbf{L}_{\mathfrak{g}_{E_f}} = \text{index } \mathbf{L}_{\mathfrak{g}_E} = 0$. $\square$

Suppose that $\varepsilon([A], \phi)$ is well-defined. If $f \in \text{Diff}(Y, \phi)$ preserves E , then f acts on $\mathcal{M}(E, \phi)$. If $\text{index } \mathbf{L}_{\mathfrak{g}_{E_f}}$ is odd, then for every $[A] \in \mathcal{M}(E, \phi)$, $[f^* A]$ has the opposite sign and, hence, the overall count $n(E, \phi)$ must be zero. Hence, the existence of such a diffeomorphism would obstruct the well-definedness of the conjectural G_2 Casson invariant.

6.2 Construction of the counter term

It is clear from the work presented in Chapter 4 that $n(E, \phi)$ cannot be invariant under deformations of ϕ , in general. The goal is now to produce a counter term $m(E, \phi)$ which precisely cancels the jumps in $n(E, \phi)$. (The trivial choice $m = -n$ does not lead to any interesting mathematics.) The jumps in $n(E, \phi)$ are caused by associative submanifolds P in Y and G_2 -instantons on G -bundles with “less” topology. We thus make the following

ansatz for the counter term $m(E, \phi)$:

$$m(E, \phi) := \sum w(P, [A], \phi)$$

where $w(P, [A])$ are weights and the sum is over a certain class of P and $[A]$, both of which are to be determined.

Remark 6.4. Instead of thinking of m as a counter term, one can also understand this procedure as extending the objects counted by $n(E, \phi)$ to a larger space which contains connections, submanifolds and mixtures of both (and possibly also some decorating data). In the world of differential geometry these objects are usually seen as different things; however, in algebraic geometry they all can be thought of as certain types of sheaves.

I will now explain an idea due to Donaldson [DS11, Section 6.2, first paragraph] for how to define w in the case of $G = \mathrm{SU}(2)$ and under the hypothesis that only charge one instantons bubble off. Let A_0 be a connection on a $\mathrm{SU}(2)$ -bundle E_0 , let P be an associative submanifold in (Y, ϕ) and let $\mathfrak{J} \in \Gamma(P, \mathfrak{M})$ be a section of an instanton bundle $\mathfrak{M}$ over P with M as in Example 4.14. For $\lambda \in (0, \Lambda]$, let $A_0 \#_\lambda \mathfrak{J}$ be as in Proposition 4.28. The counter-term $m(E, \phi)$ has to take into account those $(A_0, P, \mathfrak{J})$ such that $A_0 \#_\lambda \mathfrak{J}$ is a connection on the bundle $\mathrm{SU}(2)$ -bundle E . Since each element of $\Gamma(P, \mathfrak{M})$ lifts to a section of $\mathrm{Re}(S \otimes E_0 \setminus \{0\})$ for a unique spin structure $\mathfrak{s}$ on P and up to homotopy the sections of $\mathrm{Re}(S \otimes E_0 \setminus \{0\})$ are classified by the degree d , we have $\pi_0(\Gamma(P, \mathfrak{M})) \cong H_2(P, \mathbf{Z}_2) \times \mathbf{Z}$.

Proposition 6.5. *In the above situation assume that P connected and that F_P , $\mathring{F}_{\mathfrak{J}}$ and L_{A_0} all have trivial kernel, then*

$$\lim_{\lambda \downarrow 0} \varepsilon([A_0 \#_\lambda \mathfrak{J}], \phi)$$

depends only on $\mathfrak{s}$ and d . In particular, we can define

$$(6.6) \quad w(P, [A_0], \mathfrak{s}, d, \phi) := -\frac{1}{2} \lim_{\lambda \downarrow 0} \varepsilon([A_0 \#_\lambda \mathfrak{J}], \phi).$$

Remark 6.7. Note that $\mathring{F}_{\mathfrak{J}}$ depends only on $\mathfrak{s}$.

Remark 6.8. In general, the right-hand side of (6.6) gives the same value for $\mathfrak{J}_0$ and $\mathfrak{J}_1$ if they can be joined by a path $(\mathfrak{J}_t)$ in $\Gamma(P, \mathfrak{M})$ such that each $\mathring{F}_{\mathfrak{J}_t}$ is invertible.

Remark 6.9. If P is not connected, then the factor $-1/2$ should be adjusted to $(-1/2)^{\#\pi_0(P)}$.

Proof. We have to show that for $\lambda > 0$ sufficiently small the operator $L_{A_0 \#_\lambda \mathfrak{J}}$ has trivial kernel. This is straight-forward from the discussion in Section 4.5. One only has to realise that now, because $\mathring{F}_{\mathfrak{J}}$ is invertible, the second term in the estimate in Proposition 4.69 can be omitted. If $\mathfrak{J}_0$ and $\mathfrak{J}_1$ have the same associated spin structure $\mathfrak{s}$ and degree d they can be joined by a section $\mathfrak{J}_t$, and by compactness we can find $\lambda > 0$ such that $L_{A_0 \#_\lambda \mathfrak{J}_t}$ is invertible for all $t \in [0, 1]$. $\square$

I expect that in the situation of Proposition 6.5 $w(P, [A_0], \mathfrak{s}, d, \phi)$ is independent of d . Here is one possible approach to investigate this question: Let $\mathfrak{I}_0, \mathfrak{I}_1$ be any two Fueter sections as above with the same associated spin structure $\mathfrak{s}$. Let $(B_t)_{t \in [0,1]}$ be a path of connections from $A_0 \#_\lambda \mathfrak{I}_0$ to $A_0 \#_\lambda \mathfrak{I}_1$. We need to show that the spectral flow of L_{B_t} is even. (Note, we cannot join $\mathfrak{I}_0$ and $\mathfrak{I}_1$ by a path of sections of $\mathfrak{M}$ and, hence, the argument used above does not apply.) We promote (B_t) to a path of connections $(\tilde{B}_t)$ on the $SU(4)$ -bundle $E_0 \oplus \mathbf{C}^2$. It is not hard to see that the spectral flow of $L_{\tilde{B}_t}$ has the same parity as that of L_{B_t} . The moduli space of framed charge one $SU(4)$ ASD instantons can be seen to be $M_{SU(4)} = SU(4)/(SU(2) \times S^1)$; it is generated by the lift of “the one-instanton” whose stabiliser is $SU(2) \times S^1$. $M_{SU(4)}$ includes the moduli space of framed charge one $SU(2)$ ASD instantons $M_{SU(2)} = SU(2)/\mathbf{Z}_2$. The obstruction to joining $\mathfrak{I}_0$ to $\mathfrak{I}_1$ lies in $\pi_3(M_{SU(2)}) = \mathbf{Z}$. After lifting $\mathfrak{I}_0$ and $\mathfrak{I}_1$ to sections of $\mathfrak{M}_{SU(4)}$, the instanton bundle associated to $M_{SU(4)}$, this obstruction disappears because $\pi_3(M_{SU(4)}) = \pi_3((S^7 \times S^5)/S^1) = 0$. We can thus join the lifts of $\mathfrak{I}_0$ and $\mathfrak{I}_1$ by a path $\tilde{\mathfrak{I}}_t$ of section of $\mathfrak{M}_{SU(4)}$. By Proposition 6.3 we can assume that $\tilde{B}_t = \tilde{A}_0 \#_\lambda \tilde{\mathfrak{I}}_t$. The spectral flow of $L_{\tilde{B}_t}$ is given by the index of $\mathbf{L} = \partial_s - L_{\tilde{B}_s}$ over $Y \times \mathbf{R}$. (Here t and s are related by an appropriate reparameterisation of $(0, 1)$ as $\mathbf{R}$.) Using the arguments employed in Chapter 5 one can see that

$$\text{index } \mathbf{L} = \text{index } \hat{F}_{\tilde{\mathfrak{I}}}.$$

Here $\tilde{\mathfrak{I}}$ is the section of the instanton bundle $\mathfrak{M}$ over $P \times \mathbf{R}$ given by the path $\tilde{\mathfrak{I}}_s$. This reduces the question of how $w(P, [A_0], \mathfrak{s}, d, \phi)$ depends on d to analysing the index of $\hat{F}_{\tilde{\mathfrak{I}}}$.

In what follows I will assume that $w(P, [A_0], \mathfrak{s}, d, \phi)$ is independent of d ; if it does in fact depend on d , then further thought would be necessary. We define

$$w(P, [A_0], \phi) := \sum_s w(P, [A_0], \mathfrak{s}, \phi).$$

Proposition 6.10. *Let (ϕ_t) , A_0 , P and $\mathfrak{I}$ be as in the hypothesis of Theorem 4.7 and suppose that $\mathfrak{M}$ is as in Example 4.14. Let (A_t) , (P_t) and $(\mathfrak{I}_t)$ be as in Section 4.2. Denote by $\mathfrak{s}_t$ the spin structure on P_t associated with $\mathfrak{I}_t$. Then for $t > 0$ sufficiently small we have*

$$w(A_t, P_{-t}, \mathfrak{s}_{-t}, \phi_{-t}) = -w(A_t, P_t, \mathfrak{s}_t, \phi_t).$$

Proof sketch. Fix $\lambda > 0$ sufficiently small and set $L_{t,\lambda} := L_{A_t \#_\lambda \mathfrak{I}_t, \phi_t}$. Then a simplification of the argument in Proposition 4.38 shows that

$$\|L_{t,\lambda} \iota_{t,\lambda} \hat{v} \circ \mathfrak{I}_t + \mu(t) \iota_{t,\lambda} \hat{v} \circ \mathfrak{I}_t\|_{\mathfrak{H}_{t,\lambda}} \leq c\lambda^{2-\alpha}.$$

Hence, using Proposition 4.45 we can solve the equation

$$L_{t,\lambda}(\iota_{t,\lambda}\hat{v} \circ \mathfrak{I}_t + \underline{a}) + (\mu(t) + \eta_\lambda(t))(\iota_{t,\lambda}\hat{v} \circ \mathfrak{I}_t + \underline{a}) = 0.$$

for $\underline{a} = \underline{a}_\lambda(t)$ and $\eta = \eta_\lambda(t)$ with $\|\underline{a}\|_{\mathfrak{X}_{t,\lambda}} \leq c\lambda^{2-\alpha}$ and $|\eta_\lambda(t)| + |\partial_t \eta_\lambda(t)| \leq c\lambda^{1-\alpha}$. This gives one regular crossing solution to the spectral flow

$$L_{t,\lambda}\tilde{a}_\lambda(t) + \tilde{\mu}_\lambda(t)\underline{a}_\lambda(t) = 0$$

where $\tilde{a}_\lambda(t) := \iota_{t,\lambda}\hat{v} \circ \mathfrak{I}_t + \underline{a}_\lambda(t)$ and $\tilde{\mu}_\lambda(t) := \mu(t) + \eta_\lambda(t)$. To see that up to scaling this is the only solution to the spectral flow for $\lambda > 0$ crossing zero in $(-T', T')$, suppose that $(\underline{a}'_\lambda, \mu'_\lambda)$ is a further solution which satisfies $\|\underline{a}'_\lambda(t_0)\|_{\mathfrak{X}_{t_0,\lambda}} = 1$ and $\langle \underline{a}'_\lambda(t), \tilde{a}_\lambda(t) \rangle = 0$ for all t . Then for some $t = t_0$ we must have $L_{t_0,\lambda}\underline{a}'_\lambda(t_0) = 0$. However, then it follows from Proposition 4.48 that

$$1 = \|\underline{a}'_\lambda(t_0)\|_{\mathfrak{X}_{t_0,\lambda}} \leq c \left| \langle \pi_{t_0,\lambda}\underline{a}'_\lambda(t_0), \hat{v} \circ \mathfrak{I}_{t_0} \rangle \right|$$

which contradicts $\langle \underline{a}'_\lambda(t_0), \tilde{a}_\lambda(t_0) \rangle = 0$ if $\lambda > 0$ is sufficiently small. $\square$

It is reasonable to expect that generically the function $\lambda \mapsto t(\lambda)$ obtained from Theorem 4.7 takes on only positive (or negative) values for $\lambda > 0$ sufficiently small, that the spectral flow from $L_{\bar{A}_\lambda, \phi_{t(\lambda)}}$ to $L_{A_0 \#_\lambda \mathfrak{I}_{t(\lambda)}, \phi_{t(\lambda)}}$ is trivial and that P is connected. Under these hypotheses the jump in $n(E, \phi)$ at ϕ_0 is precisely compensated by the jump in $m(E, \phi)$ at ϕ_0 according to Proposition 6.10. This means that $m(E, \phi)$ with w as in (6.6) precisely compensates the jumps in $n(E, \phi)$ arising from low energy bubbling in the $SU(2)$ -theory.

6.3 Degenerations of associative submanifolds

While (under certain hypothesis) the definition of $m(E, \phi)$ counters the jumps in $n(E, \phi)$ it might quite well introduce new problems arising from the non-compactness of the moduli space of associative submanifolds. I will now explain why there can be hope that these phenomena do not create additional jumps in $m(E, \phi)$.

There are (at least) two model singularities of associative submanifolds which can occur in generic 1-parameter families of associative submanifolds, both of which have one singular point with tangent cone modelled on $S^2 \amalg S^2$ and T^2 , respectively. In the context of special Lagrangian 3-spheres these models have been studied by Joyce [Joy02].

Consider two families of associative submanifolds $(P_t^\pm)_{t \in \mathbf{R}}$ in a family $(Y, \phi_t)_{t \in \mathbf{R}}$ of G_2 -manifolds. By dimension considerations one can see that in this family one can expect to have a non-trivial intersection $P_t^+ \cap P_t^-$ only in isolated points and only for discrete values of t . Let us assume that this happens precisely at $t = 0$ at precisely one point. Nordström proved that out of this intersection a new one-parameter family $(P_\lambda^0)_{\lambda > 0}$ of associative submanifolds

in $(Y, \phi_{t(\lambda)})$, which is modelled on the connected sum $P_0^- \# P_0^+$, can spontaneously be created.

Theorem 6.11 (Nordström [Nor13]). *Let Y be a compact 7-manifold and let $(\phi_t)_{t \in (-\varepsilon, \varepsilon)}$ be a family of torsion-free G_2 -structures on Y . Suppose that $(P_t^\pm)$ is a pair of families of associative submanifolds in (Y, ϕ_t) such that $\mathbf{P}^\pm := \bigcup_{t \in (-\varepsilon, \varepsilon)} \{t\} \times P_t^\pm$ intersect transversely at precisely one point $(0, x) \in (-\varepsilon, \varepsilon) \times Y$. Then there exists $\Lambda > 0$, a continuous function $t: [0, \Lambda] \rightarrow (-\varepsilon, \varepsilon)$ with $t(0) = 0$ and for each $\lambda \in (0, \Lambda]$ an associative submanifold P_λ^0 in $(Y, \phi_{t(\lambda)})$ modelled on $P_0^+ \# P_0^-$. As λ tends to zero P_t^0 converges to $P_0^+ \cup P_0^-$ as integral currents.*

For simplicity let us assume that $t(\lambda) = \lambda$ or $t(\lambda) = -\lambda$. For each pair of spin structures $\mathfrak{s}_\pm$ on $P_0^\pm$ we can construct a spin structure $\mathfrak{s}_0 = \mathfrak{s}_+ \# \mathfrak{s}_-$ on $P_{\pm t}^0$ for $t > 0$. For $m(E, \phi_t)$ not to jump at $t = 0$ we need that for $t > 0$ sufficiently small

$$(6.12) \quad \begin{aligned} w(A_0, P_{-t}^+ \cup P_{-t}^-, \mathfrak{s}^+ \cup \mathfrak{s}^-, \phi_{-t}) \\ = w(A_0, P_t^+ \cup P_t^-, \mathfrak{s}^+ \cup \mathfrak{s}^-, \phi_t) \pm w(A_0, P_{\pm t}^0, \mathfrak{s}^0, \phi_{\pm t}) \end{aligned}$$

For simplicity let us assume that $P_t^\pm$ are connected. In this situation, I will outline a programme to show that (6.12) should read

$$\pm \frac{1}{4} = \mp \frac{1}{4} \pm \frac{1}{2}.$$

Consider the manifold $\mathbf{R} \times Y$ together with an almost $\text{Spin}(7)$ -structure Φ which is asymptotic to $dt \wedge \phi + \Theta(\phi)$ at infinity. Now, try to construct a Cayley submanifold Q in $\mathbf{R} \times Y$ which, for some small value of $t > 0$, is asymptotic to $P_{-t}^+ \cup P_{-t}^+$ at $-\infty$ and to $P_t^+ \cup P_t^+$ at $+\infty$ (or the other way around) by resolving the straight interpolation using a Lawlor neck. There is some promising work in progress in this direction by Jason Lotay. (In the case of special Lagrangians the analogous situation was studied by Joyce [Joy03, Section 9] and Lee [Lee04].) If this can in fact be done, we can construct an instanton moduli bundle $\mathfrak{M}$ as in Example 5.4 on Q and find a section $\mathfrak{J}$ asymptotic to $\mathfrak{J}_\pm$ at $\pm\infty$. Then, for sufficiently small $\lambda > 0$, the spectral flow from $L_{A_0 \#_\lambda \mathfrak{J}_\pm}$ is given by the L^2 index of the operator $\mathbf{L}_{A_0 \#_\lambda \mathfrak{J}}$. By a slight extension of the analysis in Chapter 5 one can show that

$$(6.13) \quad \text{index } \mathbf{L}_{A_0 \#_\lambda \mathfrak{J}} = \text{index } \mathbf{L}_{A_0} + \text{index } F_Q + \text{index } F_{\mathfrak{J}}.$$

Here the indices are taken to be L^2 indices. Now, the individual term on the right-hand side of (6.13) can be computed using the Atiyah–Patodi–Singer index theorem. It is not hard to see that the contributions from $\pm\infty$ cancel each other and thus we can use the analogues of (1.112), (1.82) and (5.14). A moment's thought shows that the only non-trivial contribution

arises from index F_Q and gives

$$\text{index } \mathbf{L}_{A_0 \# \lambda \mathfrak{J}} = \frac{1}{2} \chi(Q) - [Q] \cdot [Q] = 1 \pmod{2}$$

because $[Q] \cdot [Q]$ is even and $\chi(Q) = \pm 2$. This means that

$$w(A_0, P_{-t}^+ \cup P_{-t}^-, \mathfrak{s}^+ \cup \mathfrak{s}^-, \phi_{-t}) = -w(A_0, P_t^+ \cup P_t^-, \mathfrak{s}^+ \cup \mathfrak{s}^-, \phi_t).$$

Next, one has to show that if P_t^0 exists for positive/negative t then the spectral flow from $P_t^+ \cup P_t^-$ to P_t^0 is even/odd. If this can be done, then $m(E, \phi)$ will not jump at $t = 0$.

The second situation that is likely to arise in generic 1-parameter families of G_2 -structures is as follows: At $t = 0$ there is a associative submanifold $\hat{P}$ with a point singularity at $p \in \hat{P}$ which is modelled on a cone over T^2 :

$$L_0 = \{(0, z_1, z_2, z_3) \in \mathbf{R} \oplus \mathbf{C}^3 : \\ |z_1|^2 = |z_2|^2 = |z_3|^2, \text{Im}(z_1 z_2 z_3) = 0, \text{Re}(z_1 z_2 z_3) \geq 0\}.$$

This singularity model can be resolved in 3 ways:

$$\begin{aligned} L_t^1 &= \{(0, z_1, z_2, z_3) \in \mathbf{R} \oplus \mathbf{C}^3 : \\ &\quad |z_1|^2 - t = |z_2|^2 = |z_3|^2, \text{Im}(z_1 z_2 z_3) = 0, \text{Re}(z_1 z_2 z_3) \geq 0\}, \\ L_t^2 &= \{(0, z_1, z_2, z_3) \in \mathbf{R} \oplus \mathbf{C}^3 : \\ &\quad |z_1|^2 = |z_2|^2 - t = |z_3|^2, \text{Im}(z_1 z_2 z_3) = 0, \text{Re}(z_1 z_2 z_3) \geq 0\} \quad \text{and} \\ L_t^3 &= \{(0, z_1, z_2, z_3) \in \mathbf{R} \oplus \mathbf{C}^3 : \\ &\quad |z_1|^2 = |z_2|^2 = |z_3|^2 - t, \text{Im}(z_1 z_2 z_3) = 0, \text{Re}(z_1 z_2 z_3) \geq 0\}. \end{aligned}$$

Hence, it is reasonable to expect that out of $\hat{P}$ three families of associative submanifold $(P_\lambda^i)_{\lambda > 0}$ ($i = 1, 2, 3$) in $(Y, t_i(\lambda))$ are created. I expect that the analogue of Nordström's theorem applies in this situation.

Remark 6.14. In the special Lagrangian case studied by Joyce, the creation of certain of these families are obstructed by symplectic considerations; however, these don't apply in the G_2 -situation.

For simplicity we can assume that $t_i(\lambda) = \varepsilon_i \lambda$ where $\varepsilon_i = \pm 1$. In order for $m(E, \phi)$ not to jump at $t = 0$ we need that for $t > 0$ sufficiently small

$$(6.15) \quad \sum_i \varepsilon_i w(A_0, P_{\varepsilon_i t}^i, \phi_{\varepsilon_i t}) = \sum_i \sum_{\mathfrak{s}} \varepsilon_i w(A_0, P_{\varepsilon_i t}^i, \mathfrak{s}, \phi_{\varepsilon_i t}) = 0.$$

Each of the weights $\varepsilon_i w(A_0, P_{\varepsilon_i t}^i, \mathfrak{s}, \phi_{\varepsilon_i t})$ is $\pm \left(\frac{1}{2}\right)^{|\pi_0(\hat{P})|}$. A necessary condition for (6.15)

to hold is thus that

$$(6.16) \quad \sum_i \#H^1(P_\lambda^i, \mathbf{Z}_2) = 0 \bmod 2.$$

I will now show that this condition is always satisfied. An application of the Mayer–Vietoris theorem yields the exact sequence

$$0 \rightarrow H^1(P_\lambda^i, \mathbf{Z}_2) \rightarrow H^1(\hat{P} \setminus \{p\}, \mathbf{Z}_2) \oplus H^1(L_\lambda^i, \mathbf{Z}_2) \xrightarrow{f+g_i} H^1(T^2, \mathbf{Z}_2).$$

The image of f is one-dimensional; hence, $\text{im } f \cong \mathbf{Z}_2 \cdot k$ for some $k = (k_1, k_2) \neq (0, 0) \in \mathbf{Z}_2^2 \cong H^1(T^2, \mathbf{Z}_2)$. It can be checked that $g_i: \mathbf{Z}_2 \cong H^1(L_\lambda^i, \mathbf{Z}_2) \rightarrow \mathbf{Z}_2^2 \cong H^1(T^2, \mathbf{Z}_2)$ is given by

$$g_i(1) \mapsto \begin{cases} (1, 0) & \text{for } i = 1, \\ (0, 1) & \text{for } i = 2 \text{ and} \\ (1, 1) & \text{for } i = 3. \end{cases}$$

From the above exact sequence it now follows that $H^1(P_\lambda^i, \mathbf{Z}_2) \cong \ker f$ if $g_i(1) \neq k$ and $H^1(P_\lambda^i, \mathbf{Z}_2) \cong \ker f \oplus \mathbf{Z}_2$ if $g_i(1) = k$. Hence, (6.16) becomes

$$\# \ker f \cdot (1 + 1 + 2) = 0 \bmod 2$$

which clearly holds.

In this chapter we have seen that while $m(E, \phi)$ could fail to exhibit the desired jumping behaviour in numerous ways, it at least passes some of the most obvious sanity checks. Whether this is a good sign or simply the artefact of wishful thinking remains to be seen.

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