Interpolation and Model Reduction of Nonlinear Systems in the Loewner Framework

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A thesis presented to Imperial College London in fulfilment of the thesis requirement for the degree of Doctor of Philosophy in Electrical Engineering

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Abstract

This thesis studies the problem of interpolation and model order reduction for dynamical systems, with the primary objective being the development of an enhancement of the Loewner framework for general families of nonlinear differential-algebraic systems. First, an interconnection-based interpretation of the Loewner framework for linear time-invariant systems is developed. This interpretation does not rely on frequency domain notions, yielding a natural approach for enhancement of the Loewner framework to more complex systems possessing nonlinear dynamics. Next, the interconnection-based interpretation is used to develop the framework, first for systems of nonlinear ordinary differential equations, then for systems of nonlinear differential-algebraic equations, and interpolants are constructed using the so-called tangential data mappings and Loewner functions. Following this, parameterized families of systems interpolating the tangential data mappings are given. The problem of constructing interpolants from tangential data mappings and Loewner functions is considered in the most general scenario, and a dynamic extension approach to interpolant construction is developed. As a result,

all systems matching the tangential data mappings, and having dimension at least as large as that of the auxiliary interpolation systems, are parameterized under mild conditions. Hence, if an interpolant exists while possessing additional desired properties, then it is contained in the dynamically extended family of interpolants. Finally, the use of behaviourally equivalent representations of a system is investigated with the goal of selecting a representation having less stringent conditions guaranteeing the existence of solution to partial differential equations. This is accomplished for a class of semi-explicit nonlinear differential-algebraic systems by making use of the explicit algebraic constraints to simplify the model of the system.

Statement of Originality

I hereby declare that I am the sole author of this thesis and that this work is the result of my own endeavours. Any ideas or results from the works of other people have been appropriately referenced.

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> "You can trust us to stick to you through thick and thin – to the bitter end. And you can trust us to keep any secret of yours – closer than you keep it yourself. But you cannot trust us to let you face trouble alone, and go off without a word."
>
> *The Lord of the Rings* by J. R. R. Tolkien

For family and friends who have accompanied me along the way. The support and companionship are the most important things I've had on this journey.

Acknowledgements

Earning a doctoral degree is not an easy task but, as daunting as it may seem, the experience has been overwhelmingly positive. This is due, in no small part, to the incredible amount of support that I received along the way. In my personal experience, earning the degree has required not only a significant amount of time and effort, but also guidance from mentors and patience from loved ones.

I am deeply grateful for my supervisor, Alessandro Astolfi. His incredible knowledge and endless support are truly inspirational, and working with him has been a wonderful experience.

I am also very thankful for my former M.A.Sc. supervisors at the University of Waterloo, Daniel Miller and Christopher Nielsen, who I feel did an exceptional job preparing me for further work in research.

I also wish to express my thanks to my reviewers and examiners, Giordano Scarciotti and Nathan van de Wouw, for their time and effort reviewing and evaluating this work.

I've met many wonderful people since moving to the UK. This includes

former residents of EEE 1107, people in the KIOS room, and many other members of the Control and Power Group at Imperial College London. I am particularly thankful for Alberto, Angelo, Eugenie, and Natasha, who have supported me during tumultuous times.

I am thankful for my long-time friends in Canada. Colin and Mickey have always been there for me since we were kids. Richard, who encouraged me to think about interesting problems, was a particular inspiration in my decision to pursue the M.A.Sc. and Ph.D. degrees.

Finally, it is without question that none of what I have accomplished would be possible without the love and support of my family, and I am eternally grateful for the way they have influenced me to be the person I am.

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Chapter 1

Introduction

In this chapter the motivations and objectives of the thesis are stated, relevant literature is reviewed, and the thesis organization is described. The chapter is structured as follows. In Section 1.1 the motivations for model order reduction are discussed and the objectives of the thesis are stated. In Section 1.2 and Section 1.3 brief reviews of the literature are given for model order reduction and for the Loewner framework, respectively. In Section 1.4 the organization of the thesis is stated, and in Section 1.5 publications associated to the results of this thesis are discussed.

1.1 Motivations and Objectives

Mathematical models of high complexity are prevalent in engineering, mathematics, and the sciences. Often, systems of interest are those the behaviour

of which influences, and is influenced by, the external environment that they operate within. These mathematical models may be appropriately described by dynamical systems, *i.e.* systems having internal memory in the form of a state, for which the future evolution of the state depends on the input to the system, the initial conditions of the system, and the mathematical equations which describe the system's behaviour in time.

Such mathematical models describing dynamical systems often appear in the form of systems of differential-algebraic equations (DAEs). Systems of DAEs have been the subject of great interest for several decades, and are also referred to as implicit systems, singular systems, or descriptor systems. DAEs consist of both ordinary differential equations (ODEs) and algebraic constraints. Due to the inclusion of algebraic constraints, the analysis of DAEs is a complicated task that involves further issues that do not need to be considered when dealing with a system described only by ODEs. Despite this additional complexity, the study of DAEs is of great interest for a number of reasons.

An important motivation for the study of DAE models is that they often arise naturally when considering the interconnection of multiple systems, *e.g.* electro-mechanical systems, or when considering systems having physical constraints which may be described naturally with algebraic equations defining a restriction of the state, *e.g.* the dynamics of a pendulum may be described in terms of its angular position relative to an equilibrium via two first order differential equations, or, alternatively, the dynamics may be

described in translational coordinates via differential equations along with an algebraic equation describing the restriction of the end of the pendulum to the unit circle. Furthermore, it may take a significant amount of effort to manipulate a model described by DAEs into one described by ODEs, and, while doing so may result in a simpler model, if the algebraic constraints are completely disregarded this can result in a loss of information as the algebraic equations may describe physical structure or quantities of interest. For example, consider two systems described by ODEs. The interconnection of these systems can be described by the inclusion of an algebraic constraint which restricts the input of one system to be equal to the output of the other, and it may be that retaining knowledge of how the systems are interconnected is important.

When studying dynamical systems, models of high complexity arise naturally for a number of reasons, for example, due to the modelling of large-scale network systems (such as power distribution networks, see *e.g.* [140], [145], [69], [40]), due to the need for high precision models (such as when performing spatial discretization of partial differential equations describing fluid dynamics or flexible structures, see *e.g.* [146], [139]), or due to the use of automated modelling and analysis software (such as modelling methods used for physical verification of very large scale integration circuits, see *e.g.* [72], [141], [110]). Complexity of a model typically appears in the form of the model having high dimension, *i.e.* requiring many state variables and equations to model the behaviour of the system accurately, and in the form of

the model having nonlinear equations when a linearized description of the system does not adequately model the system's dynamic behaviour.

Analysis and simulation of high complexity mathematical models is often difficult and costly, typically requiring significant amounts of time and computational resources. It may be difficult to ascertain important properties of such systems. Furthermore, when applied to models of high complexity, some methods of control system design may result in a controller of proportionately high, or worse, complexity, which is difficult and costly to design and implement, *e.g.* state feedback and observer design which scale with dimension, model predictive control in which the future behaviour of the model must be calculated in real time, adaptive controllers which may need to estimate many system parameters while also constructing a control input.

Given a model of high complexity, the aim of *model reduction* is to make analysis, design, and simulation for the system feasible by constructing a simplified model which approximates the behaviour of the high complexity model. The dimension of a system yields a quantifiable measure of a system's complexity, hence often the considered approach is that of *model order reduction*, where a reduced order model, one of lower dimension, is constructed to approximate a model of higher dimension's behaviour.

A number of approaches to accomplish model order reduction of dynamical systems have been developed. Some approaches, such as *balanced truncation*, seek to construct an approximate model by disregarding states which have relatively little influence on the input/output behaviour of the higher-

dimensional model. This may require intimate knowledge of a system's internal model. Other approaches are based on the notion of interpolation, such as *moment matching*, in that the reduced order model faithfully reproduces the input/output behaviour of the higher-dimensional model subject to specific operating conditions and stability properties. Due to their ability to reproduce the system behaviour exactly under particular scenarios, interpolatory methods are also strong candidates for data-driven system identification tools, as it might be possible to collect the input/output behaviour to be reproduced from experiments.

The Loewner framework provides another toolset for the interpolation, and model order reduction, of LTI DAEs. This thesis is concerned with, in particular, the Loewner framework, and the development of an enhancement of the framework to general systems of nonlinear DAEs. The objectives of this thesis are therefore as follows.

- The development of an interpolation and model order reduction framework capable of treating general nonlinear systems of DAEs. This is accomplished by enhancing, for nonlinear systems, the objects in the Loewner framework for LTI systems.

- Given nonlinear enhancements of the tangential interpolation data and the Loewner objects, determine a nonlinear DAE model which interpolates these objects. This is accomplished by directly constructing a model using the given objects.

- The construction of parameterized families of systems which interpolate the tangential data and Loewner objects. This allows equiping an interpolant with additional desired properties, and also allows constructing interpolants in the absence of any strict conditions on the Loewner objects. This is accomplished, first, by inclusion of a feedforward term as a free design parameter in the aforementioned interpolant, and, then, by dynamically extending the aforementioned interpolant to parameterize all nonlinear systems which interpolate the objects under mild conditions.

While a data-driven approach has not yet been completed, the results of this thesis set the stage for a powerful data-driven interpolation and model order reduction framework that is capable of treating general systems of nonlinear DAEs by faithfully reproducing nonlinear system behaviours under specific operating conditions.

1.2 Model Reduction of Dynamical Systems

Given a high-dimensional mathematical model of a system, the primary goal of model order reduction is to determine a lower-dimensional model which approximates the higher-dimensional system. The system of lower-dimension, referred to as the reduced order model, typically approximates the original system by preserving some of the properties that the original system possesses, such as stability or the steady-state response to particular signals,

by neglecting parts of the system which are relatively uninfluenced by the input signal and have relatively little influence on the output signal, or by minimizing the approximation error (in some sense to be specified) of the reduced order model relative to the original system.

A variety of approaches to accomplish model order reduction have been developed. Broadly speaking, such methods are typically categorized as either being based upon the singular value decomposition (SVD), or being based upon the concept of Krylov projectors and moment matching [5].

SVD-based methods used to accomplish model order reduction typically yield error bounds and allow for the preservation of stability and structural properties. The approach of balanced truncation hinges upon transforming a system into a *balanced representation* in which two positive definite matrices of interest are made to be diagonal and equal, see *e.g.* [99], [97], [37], [53], [101], [90]. Classically, the controllabilty and observability Gramians have been used in the Lyapunov balancing approach [5], [67]. In the balanced representation it follows that states which take a large amount of input energy to control also contribute little energy to the output and vice versa, hence these states are appropriate choices to truncate when reducing the order of the system. Other positive definite matrices have also been considered for use in balanced truncation, see *e.g.* stochastic balancing [63], [62], [24], and others [50], [102], [148], [147], [32]. Balancing has been extended to nonlinear systems, see *e.g.* [122], [123], [61], [70], [71], [46], [26], to time-varying systems, see *e.g.* [142], [89], [113], [91], and to linear DAEs, see *e.g.*

[107], [138], [73], [41], [68], [25]. The approach of Hankel-norm approximation involves determining an approximant such that the associated error system is optimal in the Hankel-norm [1], [85], [52], [17], [18], [112]. The Hankel-norm approximation approach has been developed for nonlinear systems, see *e.g.* [42], [43], [44], [45], for time-varying systems, see *e.g.* [35], and for linear DAEs, see [30].

Model reduction by moment matching is an approach in which, for linear systems, a reduced order model matches the transfer matrix, and some of its derivatives, for a higher-dimension model at particular points in the complex plane, see *e.g* [8], [64], [51], [48], [47], [16], [49], [14], [76], [78], [119], [75], [114], [104], [103]. Given a state-space model for the higher-dimensional system, the problem of determining a moment matching model can be solved iteratively and in a numerically efficient manner [96], [5]. Following the development of a time-domain interpretation of moments in [13], the approach of moment matching has been enhanced for many classes of nonlinear systems, see *e.g.* [13], [14], [21], [120], [77], [116], [115], [105], [121]. The moment matching approach has also been developed for linear DAEs, see *e.g.* [65], [64], [117], [124].

Some model order reduction methods have been enhanced for special classes of nonlinear DAEs, see *e.g.* bilinear systems [9], [22], [3], [36], [57], [12], and quadratic bilinear systems [2], [23], [54], [74], and for models of circuits [137], [93]. Relatively few model order reduction methods have been developed for more general families of nonlinear DAEs. A method based on

balanced truncation has been developed for semi-explicit nonlinear DAEs in [136]. Model reduction by moment matching has been developed for nonlinear DAEs in [118]. A technique for model reduction of nonlinear DAEs has also been developed using a piecewise-linear approximation method, see *e.g.* [143], [144], [27].

1.3 The Loewner Framework

The objective of the rational interpolation problem for LTI systems is to construct an internal or external description of a system the transfer matrix of which is consistent with finite sets of tangential data in the complex plane, and the generalized realization problem for LTI systems entails the construction of a minimal state-space model of a system given the transfer matrix of the system [17], [5]. The generalized realization problem is closely related to the partial realization problem [81], [60], [29], [19]. The Loewner framework was developed as a toolset used for the solution of the rational interpolation problem and the generalized realization problem for LTI DAEs, see *e.g.* [4], [47], [94], [10], [83]. The framework provides another approach to accomplish model order reduction via the interpolation, by a lower-dimensional realization, of tangential data obtained from a higher-dimensional system.

The core object of the Loewner framework is the Loewner matrix, a divided-difference matrix constructed from two sets of points in the complex plane: the right tangential data and the left tangential data. The sets

of tangential data are obtained by sampling, along particular directions, a rational matrix function evaluated at particular points in the complex plane. The Loewner matrix is related to the Hankel matrix [39], [20], and has been used to solve rational interpolation problems, see *e.g.* [6], [7], [8]. The Loewner matrix has a fine structure that allows its factorization into the tangential generalized controllability and observability matrices. Once obtained, the tangential generalized controllability and observability matrices can be used to construct LTI descriptor system realizations interpolating the sets of tangential data [94], [10].

The Loewner framework has been enhanced, for LTI systems, to solve the generalized realization problem with an internal delay in [125]. A post-processing procedure to obtain stable approximate models is proposed in [56]. The Loewner framework has been used for model reduction of parameterized linear systems in [79]. In [106] a data-driven model reduction approach using the Loewner framework is given for linear systems, in which frequency-response data are inferred from trajectories of the input and output signals. An approach preserving the original DAE structure in the reduced order model is given in [59], [11]. As a result of the interpretation of moments in the time-domain [14], it has been shown in [124] that the Loewner framework for LTI systems can be viewed as a special case of a two-sided moment-matching procedure [75].

Classically, the Loewner framework for linear systems has been interpreted in the context of interpolating sets of data in the Laplace domain,

and the Loewner framework of [94] has been enhanced for some classes of nonlinear systems using a generalized transfer function (Laplace domain) approach, in which interpolation is accomplished using the kernels of a truncated Volterra series expansion, see *e.g.* results for bilinear systems [9], [82], quadratic-bilinear systems [54], linear switched systems [55], and bilinear time-delay systems [57]. The proposed proceedure has the benefit that it can be made data-driven by sampling the generalized transfer matrices, however the nonlinearities that the enhanced framework is applied to are limited, so when working with state-space models the representation may need to be significantly manipulated into an appropriate form using a procedure such as the Carleman linearization approach [111]. The approach has also been used to enhance the framework to linear parameter-varying systems in [58]. A similar approach has been pursued in [93].

1.4 Thesis Organization and Contributions

The organization and primary contributions of the thesis are as follows. In Chapter 2, notation and background material regarding systems of differential-algebraic equations, nonlinear systems, and the Loewner framework for interpolation and model order reduction, are reviewed. In Chapter 3, an interconnection-based interpretation of the Loewner framework for linear time-invariant differential-algebraic systems is developed. Classically, important objects in the Loewner framework have been interpreted in the Laplace

domain, however the interpretation developed herein does not require any frequency domain notions. In Chapter 4, the interconnection-based interpretation of the previous chapter is used to enhance the Loewner framework to nonlinear systems of ordinary differential equations. In Chapter 5, the interconnection-based interpretation is used to enhance the Loewner framework to a general family of nonlinear differential-algebraic systems, and a family of interpolants parameterized by a feedforward term is constructed. In Chapter 6, the construction of interpolants is considered for the scenario in which the presented family of interpolants from the previous chapter yields an underdetermined system. In Chapter 7, the construction of interpolants is considered in the most general scenario, a dynamic extension approach to interpolant construction is given, and it is shown that a resulting family of dynamically extended interpolants parameterizes all interpolants of sufficiently large dimension matching the tangential data mappings. In Chapter 8, the use of simplified behaviourally equivalent representations for obtaining the Loewner functions is considered for a class of semi-explicit nonlinear differential-algebraic systems with the goal of selecting a representation having less stringent conditions guaranteeing existence of solution to partial differential equations. In Chapter 9, a summary of the contributions in this thesis is stated and some important directions for further research are given. In Appendix A, it is shown that real-valued interpolants can be constructed in the Loewner framework as long as the auxiliary systems and the system generating the tangential data mappings are obtained from coor-

dinates transformations of real-valued systems. In Appendix B, it is shown that the existence of a unique solution in the class of analytic functions can be guaranteed for a class of partial differential-algebraic equations by using a generalized non-resonance condition.

1.5 Publications

The interconnection-based interpretation of the Loewner framework for linear time-invariant systems, given in Chapter 3, is based on the conference paper [126]. The enhancement of the Loewner framework for nonlinear systems of ordinary differential equations, given in Chapter 4, is based on the conference paper [15] and the journal publication [130]. The enhancement of the Loewner framework for nonlinear differential-algebraic systems and the family of interpolants parameterized by a feedforward mapping, given in Chapter 5, are based upon the conference paper [129]. The regularization approach for underdetermined interpolants, given in Chapter 6, is based upon the conference paper [133]. The dynamic extension approach for construction of interpolants and the parameterization of all interpolants of sufficiently high dimension, given in Chapter 7, is developed in the journal submission [132]. The investigation of behaviourally equivalent representations of systems in the Loewner framework, given in Chapter 8, is based upon the conference paper [131].

The enhancement of the Loewner framework for linear time-varying sys-

tems, developed in [127], is omitted from this thesis. Results pertaining to online estimation of the Loewner matrices in [126] and [128] are also omitted.

Further research based on the results of this thesis, including the parameterization of all nonlinear DAE moment matching interpolants, the construction of interpolants in the Loewner framework with second-order structure, and an enhancement of the Loewner framework to linear port-Hamiltonian systems, are developed in the conference submissions [135], [134], and [98].

Chapter 2

Preliminaries

In order to lay a foundation for results in the following chapters of the thesis, preliminary definitions and results regarding the analysis of differential-algebraic systems and the Loewner framework for interpolation and model reduction are reviewed in this chapter.

The structure of this chapter is as follows. In Section 2.1 frequently used notation is introduced. In Section 2.2 regular matrix pencils and generalized Sylvester equations are reviewed. In Section 2.3 systems of differential-algebraic equations are discussed, and basic system properties, such as regularity, controllability, and observability, are reviewed. In Section 2.4 the centre manifold theory and Hadamard's lemma are reviewed. Finally, in Section 2.5 existing results related to the Loewner framework are discussed.

2.1 Notation

Standard mathematical notation is used throughout the thesis. This is briefly reviewed in what follows for the reader's benefit.

The union and intersection of two sets S and U are denoted by $S \cup U$ and $S \cap U$, respectively. The empty set is denoted by $\emptyset$. The set of natural numbers is denoted by $\mathbb{N}$. The fields of real numbers and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The set of complex numbers having zero real part is denoted by $\mathbb{C}_0$, and the set of complex numbers having negative real part is denoted by $\mathbb{C}^-$. The set of vectors having n rows, with entries belonging to a field $\mathbb{F}$, is denoted by $\mathbb{F}^n$. The set of matrices having n rows and m columns, with entries belonging to a field $\mathbb{F}$, is denoted by $\mathbb{F}^{n\times m}$.

Consider a matrix $A \in \mathbb{F}^{n\times m}$. The transpose of A is denoted by $A^\top$, and the complex conjugate transpose of A is denoted by A^*. The column vector with the i-th entry equal to 1 and all other entries equal to zero is denoted by e_i. The entry in the i-th row and j-th column of the matrix A, *i.e.* $e_i^\top A e_j$, is denoted by $A_{i,j}$. For a square matrix $A \in \mathbb{F}^{n\times n}$, the determinant of A is denoted by $\det(A)$, and the spectrum, or the set of eigenvalues, of A is denoted by $\sigma(A)$. The matrix $B \in \mathbb{F}^{n\times n}$ such that $B_{i,i} = b_i \in \mathbb{F}$ and $B_{i,j} = 0$ if $i \neq j$ is denoted by $\mathrm{diag}[b_1,\ \ldots,\ b_n]$. The 2-norm of a vector, or the induced 2-norm of a matrix, A is denoted by $\|A\|_2$.

The set of functions $f(\cdot)$ for which the first k derivatives $f'(\cdot)$, $f''(\cdot)$, $\ldots$, $f^{(k)}(\cdot)$ exist and are continuous is denoted by C^k. For two functions $f(\cdot)$

and $g(\cdot)$ having compatible domain and range, respectively, the composition $f\big(g(\cdot)\big)$ is denoted by $f \circ g$. The Jacobian of a function $f(\cdot)$ is denoted by $\frac{\partial f}{\partial x}$, and the Jacobian of $f(\cdot)$ evaluated at $\bar{x}$ is denoted by $\big(\frac{\partial f}{\partial x} \circ \bar{x}\big)$.

2.2 Regular Matrix Pencils and Generalized Sylvester Equations

Consider, for a pair of square matrices (A, E), the generalized eigenvalue problem [33]

$$Av = \lambda Ev, \quad \lambda \in \mathbb{C}, \quad \|v\|_2 \neq 0, \tag{2.1}$$

where the spectrum of the matrix pencil $\lambda E - A$, the set containing all $\lambda \in \mathbb{C}$ such that $\det(\lambda E - A) = 0$, is denoted by $\sigma(A, E)$. The problem can also be stated in the form

$$\gamma Av = \alpha Ev, \quad \gamma, \alpha \in \mathbb{C}, \quad (\gamma \neq 0) \vee (\alpha \neq 0), \quad \|v\|_2 = 1, \tag{2.2}$$

with the advantage that the roles of A and E are the same in (2.2). This allows defining the notion of an eigenvalue at infinity, $\lambda = \infty$, for the eigenvalue problem (2.1). Particularly, defining the equivalence relation for quotients

$$(\alpha, \gamma) \equiv (\delta, \beta) \iff \alpha\beta - \gamma\delta = 0,$$

the spectrum of the pencil $\alpha E - \gamma A$ is given by the equivalence classes of pairs (α, γ) satisfying $\det(\alpha E - \gamma A) = 0$. When (α, γ), $\gamma \neq 0$, satisfies (2.2), it follows that $\lambda = \alpha/\gamma$ satisfies the generalized eigenvalue problem (2.1) and $\lambda = \alpha/\gamma \in \sigma(A, E)$. Otherwise, when $(\alpha, 0)$ satisfies (2.2), it is said that the matrix pencil (A, E) has an eigenvalue at $\lambda = \infty$.

The following definition of regularity for matrix pairs, given in [34], is required to guarantee the existence and uniqueness of solutions for generalized Sylvester equations. This is also important in the theory of linear differential-algebraic equations.

Definition 1. *Given two matrices $E \in \mathbb{C}^{n\times n}$ and $A \in \mathbb{C}^{n\times n}$, the pencil (A, E) is called* regular *if there exists a constant scalar $\lambda \in \mathbb{C}$ such that $det(\lambda E - A) \neq 0$.*

The generalized Sylvester equation, a matrix equation of the form

$$AXB - CXD = E, \tag{2.3}$$

with $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$, $D \in \mathbb{R}^{m\times m}$, $X \in \mathbb{R}^{n\times m}$, and $E \in \mathbb{R}^{n\times m}$, is encountered frequently in the study of linear descriptor systems. Recall that there exists a unique solution, $X \in \mathbb{R}^{n\times m}$, to the generalized Sylvester equation (2.3) if, and only if, the matrix pencil $A - \lambda C$ is regular, the matrix pencil $D - \lambda B$ is regular, and $\sigma(A, C) \cap \sigma(D, B) = \emptyset$ [33], [38].

2.3 Differential-Algebraic Systems of Equations

The most general form of systems of DAEs is given by

$$F(t, x, \dot{x}, u) = 0, \tag{2.4}$$

$$G(t, x, y, u) = 0, \tag{2.5}$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$, output $y(t) \in \mathbb{R}^p$, and continuous mappings $F : \mathcal{I} \times \mathcal{D}_x \times \mathcal{D}_{\dot{x}} \times \mathcal{D}_u \to \mathbb{R}^q$ and $G : \mathcal{I} \times \mathcal{D}_x \times \mathcal{D}_y \times \mathcal{D}_u \to \mathbb{R}^p$, where $\mathcal{D}_x \subseteq \mathbb{R}^n$, $\mathcal{D}_{\dot{x}} \subseteq \mathbb{R}^n$, $\mathcal{D}_y \subseteq \mathbb{R}^p$, and $\mathcal{D}_u \subseteq \mathbb{R}^m$ are open, and $\mathcal{I} = [t_0, t_f] \subseteq \mathbb{R}$. The state $x : \mathcal{I} \to \mathbb{R}^n$ is a continuously differentiable mapping, and $\dot{x}(\cdot)$ denotes the derivative of $x(\cdot)$ with respect to $t \in \mathcal{I}$. In the absence of additional assumptions or structure, the analysis of general systems taking the form (2.4)-(2.5) is a challenging problem involving further issues of well-posedness and consistency that systems of ODEs do not suffer from. The following definitions are taken from [95].

Definition 2 ([95])**.** *A function $x : \mathcal{I} \to \mathbb{R}^n$ is called a* solution *of the system* (2.4) *if $x \in C^1$ and x satisfies* (2.4) *pointwise for a given input function u. It is called a* solution of the initial value problem *consisting of* (2.4) *and*

$$x(t_0) = x_0, \tag{2.6}$$

if x is a solution of (2.4) *and satisfies* (2.6). *An initial condition* (2.6) *is called* consistent *if the corresponding initial value problem has at least one solution.*

Generalized notions of solution, including discontinuous and impulsive behaviour, are considered using the theory of distributions [34]. It is also important to consider if the system (2.4)-(2.5) is solvable for every input and initial condition consistent with the input, as a recurring theme in interpolation and model order reduction of dynamical systems is the approximation of a system output signal when particular input signals are considered.

Definition 3 ([95])**.** *A system of DAEs* (2.4) *is called* consistent *if there exists an input function u for which the system* (2.4) *has a solution x. It is called* regular *(locally with respect to a given pair $(\bar{x}, \bar{u})$ satisfying* (2.4)*) if it has a unique solution for every sufficiently smooth input function u in a neighbourhood of $\bar{u}$ and every initial value in a neighbourhood of $\bar{x}(t_0)$ that is consistent for the system with the input function u.*

Issues regarding solvability, consistency, and regularity for nonlinear systems of the form (2.4)-(2.5) are technically complex, see *e.g.* [86], [87], [88]. Linear time-invariant (LTI) systems of DAEs have been the subject of great interest in the last several decades. In the LTI setting, systems of DAEs are

described by equations of the form

$$E\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \tag{2.7}$$

$$y(t) = Cx(t) + Du(t), \tag{2.8}$$

where $x(t) \in \mathbb{R}^n$ is the state (or pseudostate), $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, and the initial condition x_0 is "admissible" for the given system and input signal, *i.e.* for a sufficiently smooth input $u(\cdot)$, the system (2.7)-(2.8) with initial condition x_0 has a differentiable solution $x(\cdot)$. Note that the matrix E is not full rank in general, hence the system (2.7)-(2.8) cannot be put into a standard state-space form, and has to be described by both differential equations and algebraic constraints.

For DAEs of the form (2.7)-(2.8), regularity is a global property and relies on the notion of regularity of the matrix pair (A, E). Indeed, regularity of the matrix pair (A, E) ensures the existence of a unique solution to the system (2.7)-(2.8) for any sufficiently differentiable input $u(\cdot)$ and admissible initial condition x_0. Hence, the system (2.7)-(2.8) is called regular if the associated matrix pair (A, E) is regular.

Using the Laplace transform, the system (2.7)-(2.8) can be interpreted in the Laplace domain as

$$(sE - A)X(s) = Ex_0 + BU(s),$$
$$Y(s) = CX(s) + DU(s),$$

and if the system is regular, then one can obtain the transfer function description

$$Y(s) = \big(C(sE - A)^{-1}B + D\big)\, U(s) + C(sE - A)^{-1}Ex_0. \tag{2.9}$$

The pencil $sE-A$ is called asymptotically stable if all of the finite eigenvalues in $\sigma(A, E)$ belong to $\mathbb{C}^-$, in which case, for $u(t) = 0$, the solutions of the system tend to zero.

2.3.1 Equivalence and Canonical Forms

Consider two differential-algebraic systems of the form (2.7)-(2.8), that is

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned}$$

and

$$\begin{aligned} \widetilde{E}\dot{\widetilde{x}}(t) &= \widetilde{A}\widetilde{x}(t) + \widetilde{B}\widetilde{u}(t), \\ \widetilde{y}(t) &= \widetilde{C}\widetilde{x}(t) + \widetilde{D}\widetilde{u}(t), \end{aligned}$$

denoted by Σ and $\widetilde{\Sigma}$, respectively. Then the systems Σ and $\widetilde{\Sigma}$ are called *restricted system equivalent* (RSE) if there exist nonsingular matrices $P, Q \in$

$\mathbb{C}^{n\times n}$ such that

$$\widetilde{E} = PEQ, \qquad \widetilde{A} = PAQ, \qquad \widetilde{C} = CQ, \qquad \widetilde{D} = D.$$

If Σ and $\widetilde{\Sigma}$ are RSE, then $x(t_0) = Q\widetilde{x}(t_0)$, and $u(t) = \widetilde{u}(t)$ for all $t \geq t_0$, imply that the states of Σ and $\widetilde{\Sigma}$ are related by $x(t) = Q\widetilde{x}(t)$ for all $t \geq t_0$. Note finally that if two systems are RSE then the systems have the same transfer function, although the converse does not necessarily hold.

Analytic solutions for the initial value problem associated to a DAE may be given in terms of the representation (2.7)-(2.8) using the Drazin inverse, see *e.g.* [88]. Alternatively, one can consider some commonly encountered canonical forms.

Every regular system of the form (2.7)-(2.8) is RSE to a system in the Weierstrass canonical form. That is, every regular system can be transformed into an equivalent representation given by the equations

$$\dot{x}_1(t) = A_1 x_1(t) + B_1 u(t), \qquad y_1(t) = C_1 x_1(t), \tag{2.10}$$

$$N\dot{x}_2(t) = x_2(t) + B_2 u(t), \qquad y_2(t) = C_2 x_2(t), \tag{2.11}$$

$$y(t) = y_1(t) + y_2(t) + Du(t), \qquad N^k = 0, \quad N^{k-1} \neq 0,$$

with states $x_1(t) \in \mathbb{R}^{n_1}$ and $x_2(t) \in \mathbb{R}^{n_2}$ such that $n_1 + n_2 = n$, matrices $A_1 \in \mathbb{R}^{n_1\times n_1}$, $N \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{n_1\times m}$, $B_2 \in \mathbb{R}^{n_2\times m}$, $C_1 \in \mathbb{R}^{p\times n_1}$, $C_2 \in \mathbb{R}^{p\times n_2}$, and with $k \in \mathbb{N}$ the index of nilpotency of the matrix N. The system

given by (2.10) is typically referred to as the *slow subsystem*, and the system given by (2.11) is typically referred to as the *fast subsystem*. The system in Weierstrass canonical form has the closed-form solution

$$x_1(t) = \mathrm{e}^{A_1(t-t_0)}x_1(t_0) + \int_{t_0}^{t} \mathrm{e}^{A_1(t-\tau)}B_1u(\tau)d\tau,$$
$$x_2(t) = -\sum_{i=0}^{k-1} N^i B_2 u^{(i)}(t),$$

and it can be seen that, because the first $k-1$ time derivatives of $u(\cdot)$ appear in $x_2(\cdot)$, the state associated to a system of differential-algebraic equations may be discontinuous, or impulsive, if the input is not sufficiently smooth. Let $q := \mathrm{rank}(E)$. Another canonical form is obtained by noting that there exist nonsingular matrices Q and P such that $QEP = \mathrm{diag}(I_q, 0)$, hence the system (2.7)-(2.8) is RSE to a representation, often referred to as *semi-explicit*, given by the equations

$$\dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t),$$
$$0 = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t),$$
$$y(t) = C_1x_1(t) + C_2x_2(t) + Du(t),$$

with states $x_1(t) \in \mathbb{R}^q$ and $x_2(t) \in \mathbb{R}^{n-q}$. If the matrix A_{22} is nonsingular then the system can be simplified to a set of ODEs having the usual closed-form solution.

2.3.2 Controllability, Observability, and Minimality

There are multiple notions of controllability for linear systems of DAEs, including C-controllability, R-controllability, and impulse controllability [34]. C-controllability corresponds to controllability of both the fast and slow subsystems (2.10) and (2.11), while R-controllability corresponds only to controllability of the slow subsystem (2.10). Impulse controllability characterizes the ability to generate impulses using admissible control signals, and is important for eliminating discontinuous behaviour that may be detrimental to a system. In what follows, the discussion of controllability is restricted to C-controllability, which is a necessary condition for the representation (2.7)-(2.8) to be minimal.

Definition 4. *The system* (2.7)-(2.8) *is called controllable (or C-controllable) if, for any* $t_1 > 0$, $x(0) \in \mathbb{R}^n$, *and* $w \in \mathbb{R}^n$, *there exists a control input* $u(t) \in C^{k-1}$ *such that* $x(t_1) = w$.

C-controllability of a DAE can be verified using a condition similar to that of the PBH test for controllability of ODEs.

Theorem 1. *The system* (2.7)-(2.8) *is controllable if, and only if,*

$$rank \begin{bmatrix} sE - A & B \end{bmatrix} = n, \ \forall s \in \mathbb{C},$$

and

$$rank \begin{bmatrix} E & B \end{bmatrix} = n.$$

Similarly, there are multiple notions of observability for differential-algebraic systems, including C-observability, R-observability, and impulse observability. C-observability corresponds to observability of both the fast and the slow subsystems (2.10) and (2.11), while R-observability corresponds only to observability of the slow subsystem (2.10). Impulse observability characterizes the ability to determine impulsive behaviour in the state using impulsive behaviour of the output signal and discontinuous behaviour in the input signal. The definition of C-observability is given as follows.

Definition 5. *The system* (2.7)-(2.8) *is observable (or C-observable) if the initial condition* $x(0)$ *may be uniquely determined by* $u(t)$, $y(t)$, $0 \leq t < \infty$.

C-observability of a DAE can be verified using a condition similar to that of the PBH test for observability of ODEs.

Theorem 2. *The system* (2.7)-(2.8) *is observable if, and only if,*

$$rank \begin{bmatrix} sE - A \\ C \end{bmatrix} = n, \ \forall s \in \mathbb{C},$$

and

$$rank \begin{bmatrix} E \\ C \end{bmatrix} = n.$$

The transfer function (2.9) of a DAE depends only on the C-controllable and C-observable parts of the system (2.7)-(2.8), so a system is minimal if, and only if, it is both C-controllable and C-observable [34].

2.4 Some Tools for the Analysis of Nonlinear Systems

To aid the analysis of nonlinear systems, some additional tools are required. In particular, some key results regarding the centre manifold theory and Hadamard's lemma are reviewed in this section.

2.4.1 The Centre Manifold Theory

One approach to simplify the analysis of a nonlinear system of ODEs is to consider the behaviour of the system when restricted to an invariant manifold. The centre manifold theory is often used for this purpose, provided the system satisfies some structural properties. The following definitions and theorems

are recalled from [31]. Consider the system of ODEs given by

$$\dot{x}(t) = N(x(t)), \tag{2.12}$$

with state $x(t) \in \mathbb{R}^n$. A set $S \subset \mathbb{R}^n$ is said to be a local invariant manifold for (2.12) if, for all $x_0 \in S$, the solution $x(t)$ of (2.12), with $x(0) = x_0$, is in S for all $|t| < T$, where $T > 0$. If $T = \infty$, then it is said that S is an invariant manifold.

In order to introduce the notion of centre manifold, consider the system given by the equations

$$\dot{x}(t) = Ax(t) + f\big(x(t), y(t)\big), \tag{2.13}$$

$$\dot{y}(t) = By(t) + g\big(x(t), y(t)\big), \tag{2.14}$$

with states $x(t) \in \mathbb{R}^n$ and $y(t) \in \mathbb{R}^m$, matrices A and B such that $\sigma(A) \subset \mathbb{C}_0$ and $\sigma(B) \subset \mathbb{C}^-$, and functions $f(\cdot)$ and $g(\cdot)$ belonging to C^2 and such that $f(0,0) = 0$, $f'(0,0) = 0$, $g(0,0) = 0$, and $g'(0,0) = 0$.

With some abuse of terminology, "the manifold $y = h(x)$" refers to the manifold given by the set $S = \{(x^\top, y^\top)^\top \in \mathbb{R}^{m+n} : y = h(x)\}$. In general, if $y = h(x)$ is a (local) invariant manifold for (2.13)-(2.14) and $h(\cdot)$ is smooth, then $y = h(x)$ is called a (local) centre manifold if $h(0) = 0$ and $h'(0) = 0$. The system of equations (2.13)-(2.14) restricted to the centre manifold describes, locally, the asymptotic behaviour of solutions of the unrestricted

system modulo the effects of exponentially decaying transient terms.

Theorem 3 ([31]). *There exists a centre manifold for* (2.13)-(2.14)*,* $y = h(x)$*,* $|x| < \delta$*, where* $h(\cdot)$ *is* C^2*.*

As a consequence of Theorem 3, it follows that under the same conditions ensuring existence of the centre manifold for the system (2.13)-(2.14), one can also ensure the existence of a mapping $h \in C^2$ satisfying, locally, the partial differential equation, with boundary condition, of the form

$$\frac{\partial h}{\partial x}\Big(Ax + f\big(x, h(x)\big)\Big) = Bh(x) + g\big(x, h(x)\big),$$
$$h(0) = 0, \quad h'(0) = 0. \tag{2.15}$$

The flow on the centre manifold is governed by the system

$$\dot{u}(t) = Au(t) + f\big(u(t), h(u(t))\big), \tag{2.16}$$

with state $u(t) \in \mathbb{R}^n$, and the asymptotic behaviour of "small" solutions of (2.13)-(2.14) are determined by the system (2.16), as shown in the following theorem.

Theorem 4 ([31]). *Suppose that the zero solution of* (2.16) *is stable (asymptotically stable, unstable). Then the zero solution of* (2.13)-(2.14) *is stable (asymptotically stable, unstable). Furthermore, suppose that the zero solution of* (2.16) *is stable. Let* $(x(t), y(t))$ *be a solution of* (2.13)-(2.14) *with* $(x(0), y(0))$ *sufficiently small. Then there exists a solution* $u(\cdot)$ *of* (2.16) *and*

constants $\gamma \in \mathbb{R}$, $c_x \in \mathbb{R}^n$, *and* $c_y \in \mathbb{R}^m$, *such that as* $t \to \infty$,

$$\|x(t) - u(t)\| \leq c_x e^{-\gamma t},$$
$$\big\|y(t) - h\big(u(t)\big)\big\| \leq c_y e^{-\gamma t},$$

where $\gamma > 0$.

If the zero solution of the system (2.16) is stable, then it follows, by Theorem 4, that the invariant manifold $y = h(x)$ is locally attractive.

Determining an explicit closed-form solution to the PDE (2.15) is equivalent to solving (2.13)-(2.14), which is impossible in general. Despite this, it is possible to approximate the centre manifold to any degree of accuracy.

Theorem 5 ([31])**.** *Let* $\varphi : \mathbb{R}^n \to \mathbb{R}^m$ *be a* C^1 *mapping in a neighbourhood of the origin with* $\varphi(0) = 0$ *and* $\varphi'(0) = 0$. *Define an operator*

$$(M\varphi)(x) = \frac{\partial \varphi}{\partial x}\Big(Ax + f\big(x, \varphi(x)\big)\Big) - B\varphi(x) - g\big(x, \varphi(x)\big),$$

and note that $(Mh)(x) = 0$. *Suppose that*[1] *as* $x \to 0$, $|(M\varphi)(x)| = O\left(|x|^q\right)$ *where* $q > 1$. *Then as* $x \to 0$, $|h(x) - \varphi(x)| = O\left(|x|^q\right)$.

Theorem 5 suggests that one can define a series of functions which converge to the formal solution of the PDE (2.15) in a neighbourhood of the origin.

[1]The statement $|f(x)| = O(\cdot)$ is the big O notation denoting the limiting behaviour of the function $f(\cdot)$.

2.4.2 Hadamard's Lemma

Hadamard's lemma can be used to provide convenient representations for smooth nonlinear mappings. The lemma is stated as follows [100].

Lemma 1 ([100]). *Let $f : U \to \mathbb{R}$ be a smooth function defined on a starlike neighbourhood U of a point z in an n dimensional Euclidean space. Then $f(\cdot)$ can be expressed, for all $x \in U$, in the form*

$$f(x) = f(z) + \sum_{i=1}^{n} (x_i - z_i) g_i(x),$$

where each $g_i(\cdot)$ is a smooth function on U, $z = (z_1, \ldots, z_n)$, and $x = (x_1, \ldots, x_n)$.

2.5 The Loewner Framework

In the paper [94] the authors have provided a toolset, known as the Loewner framework, for the construction of generalized state-space representations described by equations of the form[2]

$$E\dot{x}(t) = Ax(t) + Bu(t), \tag{2.17}$$

$$y(t) = Cx(t) + Du(t), \tag{2.18}$$

[2]Signals consisting, pointwise in time, of vectors having entries in the field of complex numbers, $\mathbb{C}$, are considered for ease of presentation. Matrices having complex entries are assumed to be obtained from a linear coordinates transformation of corresponding objects having entries belonging to the field of real numbers, $\mathbb{R}$. A discussion addressing why this assumption is not restrictive can be found in Appendix A.

having state $x(t) \in \mathbb{C}^n$, input $u(t) \in \mathbb{C}^m$, and output $y(t) \in \mathbb{C}^p$, and matrices $E \in \mathbb{C}^{n\times n}$, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{p\times n}$, and $D \in \mathbb{C}^{p\times m}$ that are consistent with sets of so-called tangential interpolation data, *i.e.* matrix data obtained by sampling the system's associated transfer function directionally. The situation in which the tangential interpolation data is generated by sampling an underlying system of the form (2.17)-(2.18) is also considered. The following assumptions ensure that the constructed interpolant, or the underlying system to be interpolated, is *well-posed*, by which it is meant that solutions exist and are unique for sufficiently smooth inputs.

Assumption 1 (Regularity)**.** *The system of equations* (2.17)-(2.18) *is a regular realization, i.e. the determinant of the matrix pencil* $sE - A$ *is not identically zero.*

In the Loewner framework for linear time-invariant systems the notion of tangential data for interpolation is based on sampling the transfer matrix of a system, *i.e.* tangential data are related to the input-output response of the system and does not rely on a particular system representation. Hence, for simplicity when considering generalized state-space models, the following assumption holds.

Assumption 2 (Minimality)**.** *The system of equations* (2.17)-(2.18) *is a minimal realization, i.e. the triple* (E, A, B) *is controllable, and the triple* (E, C, A) *is observable.*

2.5.1 The Tangential Interpolation Problem

The notion of tangential data is required in order to pose the interpolation problem within the Loewner framework. Tangential data consist of two disjoint sets of data: the right tangential interpolation data, and the left tangential interpolation data. The right tangential interpolation data are represented by the set

$$\{(\lambda_i, r_i, w_i) \mid \lambda_i \in \mathbb{C},\ r_i \in \mathbb{C}^m,\ w_i \in \mathbb{C}^p,\ i = 1, \ldots, \rho\}, \tag{2.19}$$

and the left tangential interpolation data are represented by the set

$$\left\{(\mu_j, \ell_j, v_j) \mid \mu_j \in \mathbb{C},\ \ell_j^* \in \mathbb{C}^p,\ v_j^* \in \mathbb{C}^m,\ j = 1, \ldots, v\right\}. \tag{2.20}$$

In what follows, the right tangential data are represented, equivalently, in matrix form as

$$\begin{aligned}
\Lambda &= \operatorname{diag}\Big[\lambda_1,\ \ldots,\ \lambda_\rho\Big] \in \mathbb{C}^{\rho\times\rho}, \\
R &= \begin{bmatrix} r_1 & \ldots & r_\rho \end{bmatrix} \in \mathbb{C}^{m\times\rho}, \\
W &= \begin{bmatrix} w_1 & \ldots & w_\rho \end{bmatrix} \in \mathbb{C}^{p\times\rho},
\end{aligned}$$

and the left tangential data are represented in matrix form as

$$M = \operatorname{diag}\begin{bmatrix}\mu_1, & \ldots, & \mu_v\end{bmatrix} \in \mathbb{C}^{v\times v},$$
$$L = \begin{bmatrix}\ell_1^* & \ldots & \ell_v^*\end{bmatrix}^* \in \mathbb{C}^{v\times p},$$
$$V = \begin{bmatrix}v_1^* & \ldots & v_v^*\end{bmatrix}^* \in \mathbb{C}^{v\times m}.$$

The following assumption ensures the existence and uniqueness of solution to a number of generalized Sylvester equations encountered in the thesis.

Assumption 3 (Disjoint Spectra)**.** *The matrix pencils $sE - A$, $sI - \Lambda$, and $sI - M$ have no common generalized eigenvalues, i.e.*

$$\sigma(\Lambda) \cap \sigma(A, E) = \emptyset, \quad \sigma(A, E) \cap \sigma(M) = \emptyset, \quad \sigma(\Lambda) \cap \sigma(M) = \emptyset.$$

When considering tangential data generated by an underlying system of the form (2.17)-(2.18), the following assumption ensures that the data is sampled from every input and output channel at least once.

Assumption 4. *The matrix R has full row rank and the matrix L has full column rank.*

As a result of Assumption 4, the cardinality of the right tangential interpolation data set (2.19) is greater than or equal to the number of inputs of the system (2.17)-(2.18), *i.e.* $\rho \geq m$, and the cardinality of the left tangential interpolation data set (2.20) is greater than or equal to the number of outputs

of the system (2.17)-(2.18), *i.e.* $v \geq p$.

Given the sets of tangential data (2.19)-(2.20) the goal of the generalized realization problem is to determine a differential-algebraic realization of the form (2.17)-(2.18) such that the corresponding rational transfer matrix

$$H(s) = C(sE - A)^{-1}B + D, \quad \det(sE - A) \neq 0 \ \forall \ s \in \sigma(\Lambda) \cap \sigma(M),$$

interpolates the right and left tangential data. That is, $H(s)$ satisfies the right tangential interpolation conditions

$$H(\lambda_i)r_i = w_i, \quad i = 1, ..., \rho,$$

and the left tangential interpolation conditions

$$\ell_j H(\mu_j) = v_j, \quad j = 1, ..., v.$$

2.5.2 Objects in the Loewner Framework

The primary tools used to accomplish the interpolation objective in the Loewner framework are the Loewner matrix, $\mathbb{L}$, and the shifted Loewner matrix, $\sigma\mathbb{L}$, which are defined in terms of the tangential data (2.19) and (2.20), entry-wise, as

$$\mathbb{L}_{j,i} = \frac{v_j r_i - \ell_j w_i}{\mu_j - \lambda_i}, \qquad \sigma\mathbb{L}_{j,i} = \frac{\mu_j v_j r_i - \lambda_i \ell_j w_i}{\mu_j - \lambda_i},$$

or

$$\mathbb{L} = \begin{bmatrix} \dfrac{v_1 r_1 - \ell_1 w_1}{\mu_1 - \lambda_1} & \cdots & \dfrac{v_1 r_\rho - \ell_1 w_\rho}{\mu_1 - \lambda_\rho} \\ \vdots & \ddots & \vdots \\ \dfrac{v_v r_1 - \ell_v w_1}{\mu_v - \lambda_1} & \cdots & \dfrac{v_v r_\rho - \ell_v w_\rho}{\mu_v - \lambda_\rho} \end{bmatrix},$$

and

$$\sigma\mathbb{L} = \begin{bmatrix} \dfrac{\mu_1 v_1 r_1 - \lambda_1 \ell_1 w_1}{\mu_1 - \lambda_1} & \cdots & \dfrac{\mu_1 v_1 r_\rho - \lambda_\rho \ell_1 w_\rho}{\mu_1 - \lambda_\rho} \\ \vdots & \ddots & \vdots \\ \dfrac{\mu_v v_v r_1 - \lambda_1 \ell_v w_1}{\mu_v - \lambda_1} & \cdots & \dfrac{\mu_v v_v r_\rho - \lambda_\rho \ell_v w_\rho}{\mu_v - \lambda_\rho} \end{bmatrix},$$

respectively. It is easy to see that if the rational transfer matrix $H(s)$ generates the data, then the shifted Loewner matrix is exactly the Loewner matrix associated to the transfer matrix $sH(s)$. In addition, the Loewner matrix is the unique solution, by Assumption 3, of the Sylvester equation

$$\mathbb{L}\Lambda - M\mathbb{L} = LW - VR, \tag{2.21}$$

and the shifted Loewner matrix is the unique solution, by Assumption 3, of the Sylvester equation

$$\sigma\mathbb{L}\Lambda - M\sigma\mathbb{L} = LW\Lambda - MVR. \tag{2.22}$$

Furthermore, these matrices satisfy the relationships[3]

$$\sigma\mathbb{L} - \mathbb{L}\Lambda = VR, \qquad \sigma\mathbb{L} - M\mathbb{L} = LW, \tag{2.23}$$

which are important relationships to preserve when enhancing the Loewner framework to more complicated classes of systems.

When considering the situation in which the tangential data is generated by an underlying system of the form (2.17)-(2.18), the Loewner and shifted Loewner matrices can be decomposed into matrices with a system theoretic interpretation. Consider the tangential generalized controllability matrix, $X \in \mathbb{C}^{n\times\rho}$, defined as

$$X := \begin{bmatrix} (\lambda_1 E - A)^{-1}Br_1 & \dots & (\lambda_\rho E - A)^{-1}Br_\rho \end{bmatrix},$$

and the tangential generalized observability matrix, $Y \in \mathbb{C}^{v\times n}$, defined as

$$Y := \begin{bmatrix} \ell_1 C(\mu_1 E - A)^{-1} \\ \vdots \\ \ell_v C(\mu_v E - A)^{-1} \end{bmatrix}.$$

[3]This can be seen by noting that $\sigma\mathbb{L} = \mathbb{L}\Lambda + VR = M\mathbb{L} + LW$ uniquely solves the Sylvester equation (2.22).

These matrices are the unique solutions, by Assumption 3, of the generalized Sylvester equations

$$EX\Lambda = AX + BR, \tag{2.24}$$

and

$$YA = MYE - LC, \tag{2.25}$$

respectively. The tangential data can thus be represented as

$$W = CX + DR, \qquad V = YB + LD, \tag{2.26}$$

and it follows, again by uniqueness of solution to the Sylvester equations (2.21) and (2.22), that

$$\mathbb{L} = -YEX, \qquad \sigma\mathbb{L} = -YAX + LDR. \tag{2.27}$$

In order to construct an interpolant of the data it is sufficient to know the matrices $\mathbb{L}$, $\sigma\mathbb{L}$, W, and V. However since by assumption, R has full row rank and L has full column rank, by (2.26) one needs only to be concerned with determining $\mathbb{L}$ and $\sigma\mathbb{L}$.

2.5.3 Construction of Interpolants

Square Loewner Matrices

Following [94], if the number of right tangential interpolation points is equal to the number of left tangential interpolation points, *i.e.* $\rho = v$, if the Loewner matrices, $\mathbb{L}$ and $\sigma\mathbb{L}$, and the tangential data matrices, W and V, are known, and if the matrix pencil $s\mathbb{L}-\sigma\mathbb{L}$ has full rank for all $s \in \sigma(\Lambda)\cup\sigma(M)$, then a system that interpolates the tangential data given by the sets (2.19) and (2.20) can be given by the system of differential-algebraic equations

$$\begin{aligned}\mathbb{L}\dot{\omega}(t) &= \sigma\mathbb{L}\omega(t) - Vu_r(t),\\ y_r(t) &= W\omega(t),\end{aligned}$$

with state $\omega(t) \in \mathbb{C}^\rho$, input $u_r(t) \in \mathbb{C}^m$, and output $y_r(t) \in \mathbb{C}^p$, and if, in addition, the Loewner matrix is nonsingular then the system is the unique strictly proper interpolant of degree ρ and the implicit system can be rearranged into the explicit system of ordinary differential equations

$$\dot{\omega}(t) = \mathbb{L}^{-1}\sigma\mathbb{L}\omega(t) - \mathbb{L}^{-1}Vu_r(t), \tag{2.28}$$

$$y_r(t) = W\omega(t). \tag{2.29}$$

Note now that a feedforward term can be leveraged to provide a family of differential-algebraic interpolants matching the tangential data. Consider the

system given by the equations

$$\mathbb{L}\dot{\omega}(t) = (\sigma\mathbb{L} - L\overline{D}R)\omega(t) - (V - L\overline{D})u_r(t), \tag{2.30}$$

$$y_r(t) = (W - \overline{D}R)\omega(t) + \overline{D}u_r(t), \tag{2.31}$$

with state $\omega(t) \in \mathbb{C}^{\rho}$, input $u_r(t) \in \mathbb{C}^m$, and output $y_r(t) \in \mathbb{C}^P$, and where the matrix $\overline{D} \in \mathbb{C}^{p\times m}$ is any matrix such that the matrix pencil $s\mathbb{L} - (\sigma\mathbb{L} - L\overline{D}R)$ is full rank for all $s \in \sigma(\Lambda) \cup \sigma(M)$. Then the system (2.30)-(2.31) interpolates the tangential data given by (Λ, R, W) and (M, L, V). To see that this is true, for any $\overline{D}$ satisfying the regularity condition consider the tangential generalized controllability and observability matrices, $\overline{X}$ and $\overline{Y}$, respectively, the Loewner and shifted Loewner matrices, $\overline{\mathbb{L}}$ and $\sigma\overline{\mathbb{L}}$, respectively, and the tangential data matrices, $\overline{W}$ and $\overline{V}$, generated when the transfer function associated to the system (2.30)-(2.31) is evaluated at the interpolation points given by Λ, R, and M, L. Then, because $\sigma(\sigma\mathbb{L} - L\overline{D}R, \mathbb{L}) \cap \sigma(\Lambda) = \emptyset$ and $\sigma(\sigma\mathbb{L} - L\overline{D}R, \mathbb{L}) \cap \sigma(M) = \emptyset$, it follows that $\overline{X}$ and $\overline{Y}$ are given as the unique solutions to the generalized Sylvester equations

$$\mathbb{L}\overline{X}\Lambda = (\sigma\mathbb{L} - L\overline{D}R)\overline{X} - (V - L\overline{D})R,$$

and

$$\overline{Y}(\sigma\mathbb{L} - L\overline{D}R) = M\overline{Y}\mathbb{L} - L(W - \overline{D}R),$$

respectively. Recalling (2.23), it follows that $\overline{X} = I$ and $\overline{Y} = -I$, hence

$$\begin{aligned}\overline{\mathbb{L}} &= -\overline{Y}\mathbb{L}\overline{X} = \mathbb{L},\\ \sigma\overline{\mathbb{L}} &= -\overline{Y}(\sigma\mathbb{L} - L\overline{D}R)\overline{X} + L\overline{D}R = \sigma\mathbb{L},\end{aligned}$$

and

$$\begin{aligned}\overline{W} &= (W - \overline{D}R)\overline{X} + \overline{D}R = W,\\ \overline{V} &= -\overline{Y}(V - L\overline{D}) + L\overline{D} = V.\end{aligned}$$

Thus, the system (2.30)-(2.31) interpolates the tangential data given by (Λ, R, W) and (M, L, V) for any $\overline{D}$ such that the matrix pencil $s\mathbb{L} - (\sigma\mathbb{L} - L\overline{D}R)$ is full rank for all $s \in \sigma(\Lambda) \cup \sigma(M)$.

Wide Loewner Matrix With Full Rank

The authors of [94] also present a family of interpolants for the scenario in which the Loewner matrix, $\mathbb{L} \in \mathbb{C}^{v\times\rho}$, is wide and has full row rank, *i.e.* $v \leq \rho$ and $\text{rank}(\mathbb{L}) = v$. Let $\mathbb{L}^{\#}$ be any right inverse of $\mathbb{L}$, that is $\mathbb{L}\mathbb{L}^{\#} = I_v$.

Then the system given by the equations

$$\dot{\omega}(t) = \big(\Lambda + \mathbb{L}^{\#}(V - L\overline{D})R\big)\omega(t) - \mathbb{L}^{\#}(V - L\overline{D})u_r(t), \tag{2.32}$$

$$y_r(t) = (W - \overline{D}R)\omega(t) + \overline{D}u_r(t), \tag{2.33}$$

parameterizes, in $\overline{D} \in \mathbb{C}^{p\times m}$, a family of proper interpolants matching the tangential data (Λ, R, W) and (M, L, V) for any matrix $\overline{D}$ such that the matrix $\Lambda + \mathbb{L}^{\#}(V - L\overline{D})R$ has no eigenvalues belonging to $\sigma(\Lambda) \cup \sigma(M)$. This can be seen by noting that, associated to the tangential interpolation points given by the matrices (Λ, R, M, L), the system (2.32)-(2.33) has tangential generalized controllability matrix $\overline{X} = I$ and tangential generalized observability matrix $\overline{Y} = -\mathbb{L}$, resulting in the tangential data matrices $\overline{W} = W$ and $\overline{V} = V$.

SVD Approach for More General Scenarios

The problem of finding an interpolant of dimension $k \leq \min\{v, \rho\}$ when $\text{rank}(\mathbb{L}) = n \leq k$ (*i.e.* when the Loewner matrix contains redundant data) is also considered in [94]. Suppose that for some $x \in \sigma(\Lambda) \cup \sigma(M)$ there exists

a matrix $\overline{D} \in \mathbb{C}^{p\times m}$ such that the condition

$$\begin{aligned}\operatorname{rank}\big(x\mathbb{L} - (\sigma\mathbb{L} - L\overline{D}R)\big) &= \operatorname{rank}\begin{bmatrix}\mathbb{L} & \sigma\mathbb{L} - L\overline{D}R\end{bmatrix}\\ &= \operatorname{rank}\begin{bmatrix}\mathbb{L}\\ \sigma\mathbb{L} - L\overline{D}R\end{bmatrix}\\ &= k,\end{aligned} \tag{2.34}$$

is satisfied, and the short singular value decomposition (SVD)

$$x\mathbb{L} - (\sigma\mathbb{L} - L\overline{D}R) = Y\Sigma X,$$

is performed to construct matrices $Y \in \mathbb{C}^{v\times k}$, $X \in \mathbb{C}^{k\times\rho}$, and $\Sigma \in \mathbb{C}^{k\times k}$ with Σ nonsingular. Then an interpolant of the tangential data is given by the equations

$$\begin{aligned}Y^*\mathbb{L}X^*\dot{\omega}(t) &= Y^*(\sigma\mathbb{L} - L\overline{D}R)X^*\omega(t) - Y^*(V - L\overline{D})u_r(t),\\ y_r(t) &= (W - \overline{D}R)X^*\omega(t) + \overline{D}u_r(t),\end{aligned}$$

with state $\omega(t) \in \mathbb{C}^k$, input $u_r(t) \in \mathbb{C}^m$, and output $y_r(t) \in \mathbb{C}^p$, which is a minimal realization of an interpolant. The rank condition (2.34) is generically satisfied as the amount of data increases, however there exist situations in which an interpolant exists but the short SVD procedure does not yield an interpolant. Furthermore, the authors of [94] have considered only the

construction of state-space realizations of interpolants having dimension $k \leq \min\{v, \rho\}$.

Building Real Loewner Matrices, Recursive Interpolation, and Post-Processing Methods

When the underlying system from which the tangential data matrices are collected is real-valued, then it is important to construct an interpolant which is also real-valued, in which case the matrices Λ, M, W, and V must contain complex conjugate data. An approach to construct real-valued Loewner objects when the interpolation frequencies are imaginary-valued and these matrices don't contain complex conjugate data is presented in [10], wherein the tangential data matrices are extended by adding the necessary complex conjugate interpolation points and performing a similarity transformation. In particular, suppose the sets of right and left tangential data (2.19)-(2.20) are given such that $\lambda_i = j\lambda_i^R$, $\lambda_i^R \in \mathbb{R}$, $r_i \in \mathbb{R}^m$, for $i = 1, \ldots, \rho$, and $\mu_j = j\mu_j^R$, $\mu_j^R \in \mathbb{R}$, $\ell_j^\top \in \mathbb{R}^p$, for $j = 1, \ldots, v$. Then one can build the real-valued block-diagonal matrices

$$\Lambda^R = \text{diag}\left(\begin{bmatrix} 0 & \lambda_1^R \\ -\lambda_1^R & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & \lambda_\rho^R \\ -\lambda_\rho^R & 0 \end{bmatrix} \right),$$

and

$$M^R = \operatorname{diag}\left(\begin{bmatrix} 0 & \mu_1^R \\ -\mu_1^R & 0 \end{bmatrix}, \cdots, \begin{bmatrix} 0 & \mu_v^R \\ -\mu_v^R & 0 \end{bmatrix}\right),$$

the real-valued matrices

$$V^R = \begin{bmatrix} \operatorname{Re}(v_1^\top) \\ -\operatorname{Im}(v_1^\top) \\ \vdots \\ \operatorname{Re}(v_v^\top) \\ -\operatorname{Im}(v_v^\top) \end{bmatrix}, \quad L^R = \begin{bmatrix} \ell_1^\top \\ 0 \\ \vdots \\ \ell_v^\top \\ 0 \end{bmatrix},$$

$$W^R = \begin{bmatrix} \operatorname{Re}(w_1) & \operatorname{Im}(w_1) & \dots & \operatorname{Re}(w_\rho) & \operatorname{Im}(w_\rho) \end{bmatrix},$$

$$R^R = \begin{bmatrix} r_1 & 0 & \dots & r_\rho & 0 \end{bmatrix},$$

and solve, for $\mathbb{L}^R$ and $\sigma\mathbb{L}^R$, the Sylvester equations

$$\mathbb{L}^R\Lambda^R - M^R\mathbb{L}^R = L^R W^R - V^R R^R,$$

and

$$\sigma\mathbb{L}^R\Lambda^R - M^R\sigma\mathbb{L}^R = L^RW^R\Lambda^R - M^RV^RR^R.$$

Then the real-valued objects R^R, W^R, L^R, V^R, $\mathbb{L}^R$, $\sigma\mathbb{L}^R$, can be used in place of the objects R, W, L, V, $\mathbb{L}$, $\sigma\mathbb{L}$, respectively, to build the real-valued interpolants given in the previous sections.

The authors of [92] present an approach to build an interpolant recursively by constructing an interpolant from multiple sets of right and left tangential data. In particular, consider the first set of right and left tangential data, given by (Λ_1, R_1, W_1) and (M_1, L_1, V_1), respectively, with associated Loewner matrix given by $\mathbb{L}_1$, and consider the second set of right and left tangential data, given by (Λ_2, R_2, W_2) and (M_2, L_2, V_2), respectively, with associated Loewner matrix given by $\mathbb{L}_2$. Then a rational transfer function which interpolates both sets of right and left tangential data is given by

$$H(s) = \begin{bmatrix} W_1 & W_2 \end{bmatrix} \begin{bmatrix} s\mathbb{L}_1 - \mathbb{L}_1\Lambda_1 - V_1R_1 & -L_1W_2 \\ -V_2R_1 & s\mathbb{L}_2 - \mathbb{L}_2\Lambda_2 - V_2R_2 \end{bmatrix}^{-1} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}.$$

Note that in the construction of this transfer function the Loewner matrices $\mathbb{L}_1$ and $\mathbb{L}_2$ must be square.

In [59] the authors present a post-processing approach to preserve the DAE structure in an interpolant in the Loewner framework. This is accomplished by removing from the tangential data the contribution by the coef-

ficients of the polynomial part of the underlying system's transfer function, constructing a modified interpolant associated to the new data, projecting the modified interpolant to a lower dimension, and then adding polynomial terms associated to the parts of the data that were initially removed. The approach ensures that the resulting model preserves the polynomial terms of the original transfer function while giving up exact matching of the tangential data. Similarly, at the cost of giving up exact matching of the tangential data, several post-processing methods to construct stable reduced order models in the Loewner framework are presented in [56].

2.5.4 Enhancements of the Loewner Framework

Some notable enhancements of the Loewner framework have been developed for more complicated families of systems than those given by the equations (2.17)-(2.18). These enhancements are briefly discussed in this section.

Linear Time Delay Systems

In [125] the authors consider the scenario in which the sets of tangential interpolation data (2.19)-(2.20), with $\rho = v$, are collected by sampling a linear time-delay system of the form

$$E\dot{x}(t) = A_1 x(t) + A_2 x(t-\tau) + Bu(t), \qquad y(t) = Cx(t), \qquad (2.35)$$

with state $x(t) \in \mathbb{C}^n$, input $u(t) \in \mathbb{C}^m$, output $y(t) \in \mathbb{C}^p$, time-delay $\tau \geq 0$, and transfer function

$$H(s) = C\left(sE - A_1 - e^{-\tau s}A_2\right)^{-1} B,$$

with $\det\left(sE_\rho - A_{1,\rho} - e^{-\tau s}A_{2,\rho}\right) \neq 0$ for all $s \in \sigma(\Lambda) \cup \sigma(M)$. Then the tangential data is such that $w_i = H(\lambda_i)r_i$, $v_i = \ell_i H(\mu_i)$, $i = 1, \ldots, \rho$. Given the underlying system (2.35), the tangential generalized controllability matrix is defined as the unique solution to the generalized Sylvester equation

$$EX\Lambda = A_1 X + A_2 X e^{-\tau\Lambda} + BR,$$

so that $W = CX$, and the tangential generalized observability matrix is defined as the unique solution to the generalized Sylvester equation

$$YA_1 + e^{-\tau M}YA_2 = MYE - LC,$$

so that $V = YB$. A family of systems which interpolate the tangential data is given by

$$E_\rho\dot{\omega}(t) = \left(E_\rho\Lambda - A_{2,\rho}e^{-\tau\Lambda} - VR\right)\omega(t) + A_{2,\rho}\omega(t-\tau) + Vu_r(t), \quad (2.36)$$

$$y_r(t) = W\omega(t), \quad (2.37)$$

where $E_\rho \in \mathbb{C}^{\rho\times\rho}$ and $A_{2,\rho} \in \mathbb{C}^{\rho\times\rho}$ are any two matrices satisfying

$$E_\rho\Lambda - ME_\rho + e^{-\tau M}A_{2,\rho} - A_{2,\rho}e^{-\tau\Lambda} = VR - LW, \tag{2.38}$$

such that $sE_\rho - \left(E_\rho\Lambda - A_{2,\rho}e^{-\tau\Lambda} - VR\right) - e^{-\tau s}A_{2,\rho}$ is nonsingular for all $s \in \sigma(\Lambda)\cup\sigma(M)$. The equation (2.38) is solved only in terms of the tangential data and the time-delay τ, hence the interpolant given by (2.36)-(2.37) can be constructed solely in terms of the tangential data and the time-delay.

Bilinear, Quadratic Bilinear, Linear Switched, and LPV Systems

The Loewner framework has been enhanced for the treatment of bilinear systems, [9], quadratic-bilinear systems [54], linear switched systems [55], bilinear time-delay systems [57], and LPV systems [58], using a generalized transfer function approach. Each enhancement utilizes the same underlying idea, so, for the sake of brevity, the general approach is described here.

First, for the treated class of nonlinear systems the authors consider the associated Volterra series, which is, in general, an infinite sum of multivariable convolution integrals of the form

$$y(t) = \sum_{i=1}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} h_i(\tau_1,\ldots,\tau_i)u(t-\tau_1)\ldots u(t-\tau_i)d\tau_1\ldots d\tau_i,$$

for single-input single-output (SISO) systems. Locally, the Volterra series describes the input-output behaviour of a system, and the series is interpreted as an enhancement, to nonlinear systems, of the convolution integral solution

for linear systems.

Generalized transfer functions, which are evaluated for multituple frequencies, are defined as the multivariable Laplace transforms of the Volterra series kernels, *i.e.* generalized transfer functions are defined having the form

$$\begin{aligned} H_1(s_1) &:= \mathcal{L}\{h_1(t_1)\}, \\ H_2(s_1, s_2) &:= \mathcal{L}\{h_2(t_1, t_2)\}, \\ &\vdots \\ H_n(s_1, \ldots, s_n) &:= \mathcal{L}\{h_n(t_1, \ldots, t_n)\}, \\ &\vdots \end{aligned}$$

which have closed-form solutions for the considered classes of systems.

Considering the first 2ρ generalized transfer functions, generalized Loewner matrices, $\mathbb{L}$ and $\sigma\mathbb{L}$, and tangential data matrices, W and V, are defined, entrywise, as

$$\begin{aligned} \mathbb{L}_{j,i} &:= \frac{H_{j+i-1}(\mu_1, \ldots, \mu_j, \lambda_{i-1}, \ldots, \lambda_1)}{\mu_j - \lambda_i} \\ &\quad - \frac{H_{j+i-1}(\mu_1, \ldots, \mu_{j-1}, \lambda_i, \ldots, \lambda_1)}{\mu_j - \lambda_i}, \\ \sigma\mathbb{L}_{j,i} &:= \frac{\mu_j H_{j+i-1}(\mu_1, \ldots, \mu_j, \lambda_{i-1}, \ldots, \lambda_1)}{\mu_j - \lambda_i} \\ &\quad - \frac{\lambda_i H_{j+i-1}(\mu_1, \ldots, \mu_{j-1}, \lambda_i, \ldots, \lambda_1)}{\mu_j - \lambda_i}, \end{aligned}$$

$$W_{1,i} := H_i(\lambda_i, \lambda_{i-1}, \ldots, \lambda_1), \qquad V_{j,1} := H_j(\mu_1, \ldots, \mu_{j-1}, \mu_j),$$

with $1 \leq i, j \leq \rho$, along with any additional required objects for the considered class of systems, for example, in [9] the additional matrix Ψ is defined entrywise as

$$\Psi_{j,i} := H_{j+i}(\mu_1, \ldots, \mu_{j-1}, \mu_j, \lambda_i, \lambda_{i-1}, \ldots, \lambda_1).$$

Tangential generalized controllability and observability matrices, $\mathcal{R}$ and $\mathcal{O}$, respectively, are defined such that matrix factorizations of the generalized Loewner matrices in terms of $\mathcal{R}$, $\mathcal{O}$, and the underlying system matrices, can be accomplished, and these matrices are shown to satisfy Sylvester-like matrix equations after arranging the interpolation points into a particular form, *i.e.* for the SISO bilinear system, considered in [9], of the form

$$E\dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t), \quad y(t) = Cx(t),$$

the matrices can be factored as

$$\mathbb{L} = -\mathcal{O}E\mathcal{R}, \quad \sigma\mathbb{L} = -\mathcal{O}A\mathcal{R}, \quad \Psi = \mathcal{O}N\mathcal{R}, \quad W = C\mathcal{R}, \quad V = \mathcal{O}B,$$

where $\mathcal{R}$ and $\mathcal{O}$ satisfy

$$A\mathcal{R} + N\mathcal{R}S_R + BR = E\mathcal{R}\Lambda, \qquad \mathcal{O}A + S_L\mathcal{O}N + LC = M\mathcal{O}E,$$

with Λ and M matrices defined with a special structure based on the se-

lected multituple interpolation frequencies, and R, L, S_R, and S_L matrices containing ones and zeros with a special structure.

Finally, the generalized Loewner matrices are used to construct a reduced order model in the considered class of systems which satifies particular interpolation conditions on the first 2ρ generalized transfer functions associated to the Volterra series kernels.

The approaches of [9], [54], [55], [57], and [58], have the benefit that data-driven methods to construct the generalized Loewner matrices are easily designed; experiments are simple to perform and the generalized Loewner matrices can be calculated from data entrywise. However, the considered classes of nonlinear systems are somewhat limited, objects with no association to objects in the linear setting (such as Ψ in [9]) must be defined in order to construct interpolants for particular types of nonlinearities considered, and, due to interpolating only some of the first 2ρ generalized transfer functions, the input-output response of the constructed interpolants does not exactly match the response of the original system.

Chapter 3

An Interconnection-Based Interpretation of the Loewner Framework for Linear Differential-Algebraic Systems

While frequency domain tools provide a simple and powerful approach when considering the analysis of linear systems and systems possessing "mild nonlinearities", a more widely applicable enhancement of the Loewner framework should avoid such notions. To this end, the purpose of this chapter is to present a novel interconnection-based interpretation of the Loewner matrices. This is based on a conceptual experimental setup consisting of the cascade interconnection of the underlying system (the system to be in-

terpolated) with two auxiliary systems encoding the sets of right and left interpolation points into their dynamics. New objects, the left and right Loewner matrices, are introduced, and the selection of a particular set of coordinates reveals that the left and right Loewner matrices are the input and output "gains" of a parallelized representation of the cascade interconnection. While not necessary for the construction of an interpolant, the left and right Loewner matrices greatly aid in understanding the interconnection-based interpretation of the Loewner framework, and these objects are also crucial for justifying the definitions of the Loewner objects and interpolation in the nonlinear setting.

The interconnection-based interpretation of the Loewner matrices developed in this chapter does not require the notion of frequency response, thus the interpretation will be leveraged in the following chapters to enhance the classical objects in the Loewner framework to general systems of nonlinear and/or time-varying differential-algebraic equations.

The results of this chapter are based on [126], in which strictly proper linear time-invariant systems of ordinary differential equations are considered. Herein, a more general formulation is considered: the systems are described by linear time-invariant differential-algebraic equations having a feedforward term, *i.e.* when the system (2.7)-(2.8) has $E \neq I$ and $D \neq 0$.

The structure of this chapter is as follows. In Section 3.1 new objects related to the Loewner and shifted Loewner matrices, the left and right Loewner matrices, are introduced. In Section 3.2 a conceptual experimental

setup yielding an interconnection-based interpretation of the Loewner and shifted Loewner matrices is presented, and a definition of interpolation forgoing frequency domain notions is provided. Finally, in Section 3.3 some concluding remarks are given.

3.1 The Left and Right Loewner Matrices

Some additional objects are defined to facilitate presenting an interconnection-based interpretation of the Loewner matrices. The existence and uniqueness of these matrices are guaranteed by Assumption 3. The left Loewner matrix, $\mathbb{L}^\ell \in \mathbb{C}^{v\times\rho}$, is defined as the unique solution, by Assumption 3, to the Sylvester equation

$$M\mathbb{L}^\ell - \mathbb{L}^\ell\Lambda = VR, \tag{3.1}$$

and the right Loewner matrix, $\mathbb{L}^r \in \mathbb{C}^{v\times\rho}$, is defined as the unique solution, by Assumption 3, to the Sylvester equation

$$\mathbb{L}^r\Lambda - M\mathbb{L}^r = LW. \tag{3.2}$$

Similarly, the left shifted Loewner matrix, $\sigma\mathbb{L}^\ell \in \mathbb{C}^{v\times\rho}$, is defined as the unique solution, by Assumption 3, to the Sylvester equation

$$M\sigma\mathbb{L}^\ell - \sigma\mathbb{L}^\ell\Lambda = MVR, \tag{3.3}$$

and the right shifted Loewner matrix, $\sigma\mathbb{L}^r \in \mathbb{C}^{v\times\rho}$, is defined as the unique solution, by Assumption 3, to the Sylvester equation

$$\sigma\mathbb{L}^r\Lambda - M\sigma\mathbb{L}^r = LW\Lambda. \tag{3.4}$$

By uniqueness of solution, again by Assumption 3, to each of the Sylvester equations (2.21) and (2.22) it readily follows that

$$\mathbb{L} = \mathbb{L}^\ell + \mathbb{L}^r, \qquad \sigma\mathbb{L} = \sigma\mathbb{L}^\ell + \sigma\mathbb{L}^r. \tag{3.5}$$

Furthermore, from (3.1) and (3.2) it can be noted that

$$M(M\mathbb{L}^\ell) - (M\mathbb{L}^\ell)\Lambda = MVR, \qquad (\mathbb{L}^r\Lambda)\Lambda - M(\mathbb{L}^r\Lambda) = LW\Lambda,$$

hence, by uniqueness of solutions to (3.3) and (3.4), it follows that

$$\sigma\mathbb{L}^\ell = M\mathbb{L}^\ell, \qquad \sigma\mathbb{L}^r = \mathbb{L}^r\Lambda.$$

Remark 1. The left and right Loewner matrices are not explicitly required when constructing an interpolant in the Loewner framework, but rather the existence of these objects enhances understanding on how an interpolant in the framework fulfills its purpose. The interpretation that is obtained via these objects does not require frequency domain notions and can be readily used to define interpolants for nonlinear systems.

3.2 A Conceptual Experimental Setup

The development of the interconnection-based interpretation of the Loewner matrices relies on a conceptual experimental setup. To this end, consider two auxiliary systems encoding the tangential interpolation points and given by the equations[1]

$$\dot{\zeta}_r(t) = \Lambda\zeta_r(t) + \Delta(t), \tag{3.6}$$

$$v(t) = R\zeta_r(t), \tag{3.7}$$

and

$$\dot{\zeta}_\ell(t) = M\zeta_\ell(t) + L\chi(t), \tag{3.8}$$

$$\eta(t) = \zeta_\ell(t), \tag{3.9}$$

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$. Consider now a cascade interconnection of the system (2.17)-(2.18) and the auxiliary systems (3.6)-(3.7) and (3.8)-(3.9) defined by the interconnection equations $v = u$ and $\chi = y$ shown in Figure 3.1. The interconnected system has the differential-algebraic state-

[1]Similarity transformations of the auxiliary systems (3.6)-(3.7) and (3.8)-(3.9) produce the exact same interconnected system, hence the auxiliary systems can be realized under modest assumptions. This is discussed in depth in Appendix A.

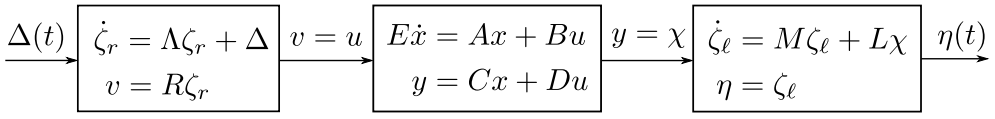

Figure 3.1: The cascade interconnected system (3.10)-(3.11).

space representation

$$\begin{bmatrix} \dot{\zeta}_r \\ E\dot{x} \\ \dot{\zeta}_\ell \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ BR & A & 0 \\ LDR & LC & M \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \Delta, \tag{3.10}$$

$$\eta = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix}, \tag{3.11}$$

with state $\begin{bmatrix} \zeta_r^* & x^* & \zeta_\ell^* \end{bmatrix}^*$, input Δ, and output η. The interconnected system given by (3.10)-(3.11) encodes all information on the Loewner matrices and tangential generalized controllability and observability matrices generated by the underlying system (2.17)-(2.18) at the tangential interpolation points given by the systems (3.6)-(3.7) and (3.8)-(3.9). However, it is not readily apparent in what way these matrices make an appearence in the dynamic behaviour of the interconnected system. In order to determine the relationship between the Loewner objects and the interconnected system one can select a new set of coordinates that makes explicit the encoding of $\mathbb{L}^\ell$, $\mathbb{L}^r$, X, and Y in the differential-algebraic state-space representation. In particular, the

new choice of coordinates allows reinterpreting the series interconnected system (3.10)-(3.11) as a parallel interconnection in which the diagonal blocks of the autonomous system are preserved, *i.e.* the diagonal blocks are given by Λ, (E, A), and M.

Theorem 6 (Loewner Coordinates)**.** *Consider the interconnected system* (3.10)-(3.11) *and the coordinates transformation*

$$\begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -X & I & 0 \\ \mathbb{L}^\ell & YE & I \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix}.$$

Then the system in the new coordinates has the differential-algebraic state-space representation

$$\begin{bmatrix} \dot{z}_r \\ E\dot{z}_c \\ \dot{z}_\ell \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} + \begin{bmatrix} I \\ -EX \\ \mathbb{L}^\ell \end{bmatrix} \Delta, \tag{3.12}$$

$$\eta = \begin{bmatrix} \mathbb{L}^r & -YE & I \end{bmatrix} \begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix}. \tag{3.13}$$

Proof. The transformed representation (3.12)-(3.13) for the interconnected system given by the equations (3.10)-(3.11) is calculated directly. Multiplying $z_c = x - X\zeta_r$ by E on the left side and taking the derivative with respect to

time yields

$$\begin{aligned} E\dot{z}_c &= E\dot{x} - EX\dot{\zeta}_r \\ &= Ax + BR\zeta_r - EX\Lambda\zeta_r - EX\Delta \\ &= Az_c - (EX\Lambda - AX - BR)\zeta_r - EX\Delta. \end{aligned}$$

The tangential generalized controllability matrix, X, is the unique solution, by Assumption 3, to the generalized Sylvester equation (2.24), hence

$$E\dot{z}_c = Az_c - EX\Delta.$$

Taking the derivative of $z_\ell = \zeta_\ell + YEx + \mathbb{L}^\ell\zeta_r$ with respect to time yields

$$\begin{aligned} \dot{z}_\ell &= \dot{\zeta}_\ell + YE\dot{x} + \mathbb{L}^\ell\dot{\zeta}_r \\ &= M\zeta_\ell + LCx + LDR\zeta_r + YAx + YBR\zeta_r + \mathbb{L}^\ell\Lambda\zeta_r + \mathbb{L}^\ell\Delta \\ &= Mz_\ell + (YA - MYE + LC)x + (\mathbb{L}^\ell\Lambda - M\mathbb{L}^\ell + (YB + LD)R)\zeta_r + \mathbb{L}^\ell\Delta. \end{aligned}$$

The tangential generalized observability matrix, Y, is the unique solution, by Assumption 3, to the generalized Sylvester equation (2.25) and the left Loewner matrix, $\mathbb{L}^\ell$, is the unique solution satisfying the Sylvester equation (3.1), hence

$$\dot{z}_\ell = Mz_\ell + \mathbb{L}^\ell\Delta.$$

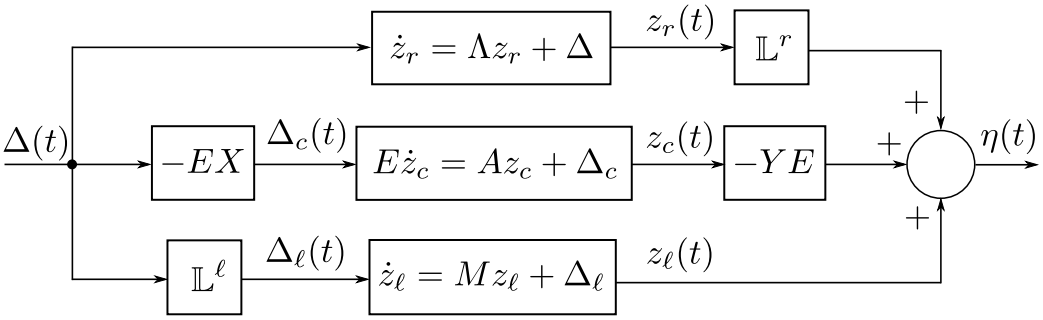

Figure 3.2: The "parallelized" interconnected system (3.12)-(3.13).

Finally, the output η in the new coordinates is

$$\begin{aligned}\eta = \zeta_\ell &= -\mathbb{L}^\ell \zeta_r - YEx + z_\ell \\ &= (-YEX - \mathbb{L}^\ell) z_r - YEz_c + z_\ell.\end{aligned}$$

Recalling (2.27) and (3.5), it follows that

$$\eta = \mathbb{L}^r z_r - YEz_c + z_\ell. \qquad \square$$

Theorem 6 yields the following interconnection-based interpretation of the Loewner matrices: the left Loewner matrix, the right Loewner matrix, the tangential generalized controllability matrix, and the tangential generalized observability matrix are the input and output "gains" of the interconnected system when observed in coordinates in which the output is produced by three systems running in parallel, as in Figure 3.2, in which the autonomous behaviour of each parallel subsystem in (3.12)-(3.13), associated to Λ, A,

and M, is the same as the autonomous behaviour of each system in the series interconnection (3.10)-(3.11). This interpretation, which is both simple and powerful, is the primary motivation underlying particular choices of definitions for Loewner objects when enhancing the framework to more complicated classes of systems in later chapters.

Recalling that the shifted Loewner matrices associated to the transfer function $H(s)$ are exactly the Loewner matrices associated to the transfer function $sH(s)$, a similar interpretation for the shifted Loewner matrices can be deduced. Consider now an interconnected system given by the interconnection equations $u = v$ and[2] $\chi = \dot{y}$. The resulting interconnected system has the differential-algebraic state-space representation

$$\begin{bmatrix} \dot{\zeta}_r \\ E\dot{x} \\ \dot{\zeta}_\ell - LC\dot{x} \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ BR & A & 0 \\ LDR\Lambda & 0 & M \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ LDR \end{bmatrix} \Delta, \tag{3.14}$$

$$\eta = \begin{bmatrix} 0 & 0 & I \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix}. \tag{3.15}$$

The following is a modification of Theorem 2 in[3] [126].

[2]There is no issue with taking the time derivative of y as long as the trajectory of ζ_r is smooth, the matrix pencil $sE - A$ is regular, and the initial condition $x(t_0)$ is consistent. By linearity the derivative could also be taken at the output v or at the output η.

[3]In [126] the dual interpretation for the shifted Loewner matrices was demonstrated by applying a time derivative to the output of each auxiliary system, (3.6)-(3.7) and (3.8)-(3.9), then constructing the corresponding interconnected system with (2.17)-(2.18). Thus, the approach in [126] uses two time derivatives. Here, only a single derivative is used so a

Theorem 7 (Shifted Loewner Coordinates)**.** *Consider the interconnected system* (3.14)-(3.15) *and the coordinates transformation*

$$\begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} = \begin{bmatrix} I & 0 & 0 \\ -X & I & 0 \\ \sigma\mathbb{L}^\ell - LDR & YA & I \end{bmatrix} \begin{bmatrix} \zeta_r \\ x \\ \zeta_\ell \end{bmatrix}.$$

Then the system in the new coordinates has the differential-algebraic state-space representation

$$\begin{bmatrix} \dot{z}_r \\ E\dot{z}_c \\ \dot{z}_\ell \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} + \begin{bmatrix} I \\ -EX \\ \sigma\mathbb{L}^\ell \end{bmatrix} \Delta,$$

$$\eta = \begin{bmatrix} \sigma\mathbb{L}^r & -YA & I \end{bmatrix} \begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix}.$$

Proof. The proof follows a similar procedure to that in the proof of Theorem 6, however the time derivative of $z_\ell = (\sigma\mathbb{L}^\ell - LDR)\zeta_r + YAx + \zeta_\ell$ is

result corresponding to the relationship between $(\mathbb{L}, H(s))$ and $(\sigma\mathbb{L}, sH(s))$ is obtained.

now

$$\begin{aligned}
\dot{z}_\ell &= (\sigma\mathbb{L}^\ell - LDR)\dot{\zeta}_r + YA\dot{x} + \dot{\zeta}_\ell \\
&= (\sigma\mathbb{L}^\ell - LDR)(\Lambda\zeta_r + \Delta) + MYE\dot{x} + (\dot{\zeta}_\ell - LC\dot{x}) \\
&= (\sigma\mathbb{L}^\ell - LDR)(\Lambda\zeta_r + \Delta) + MY(Ax + BR\zeta_r) \\
&\quad + (LDR\Lambda\zeta_r + M\zeta_\ell + LDR\Delta) \\
&= (\sigma\mathbb{L}^\ell\Lambda + MYBR)\zeta_r + \sigma\mathbb{L}^\ell\Delta + MYAx + M\zeta_\ell \\
&= M(\sigma\mathbb{L}^\ell - (V - YB)R)\zeta_r + MYAx + M\zeta_\ell + \sigma\mathbb{L}^\ell\Delta \\
&= M((\sigma\mathbb{L}^\ell - LDR)\zeta_r + YAx + \zeta_\ell) + \sigma\mathbb{L}^\ell\Delta \\
&= Mz_\ell + \sigma\mathbb{L}^\ell\Delta,
\end{aligned}$$

and the output η becomes

$$\begin{aligned}
\eta = \zeta_\ell &= -(\sigma\mathbb{L}^\ell - LDR)z_r - YA(z_c + Xz_r) + z_\ell \\
&= (-YAX + LDR - \sigma\mathbb{L}^\ell)z_r - YAz_c + z_\ell \\
&= \sigma\mathbb{L}^r z_r - YAz_c + z_\ell,
\end{aligned}$$

thus completing the proof. □

Using the interconnection-based interpretation of the Loewner objects obtained from the interconnection (3.10)-(3.11) along with its parallelized representation (3.12)-(3.13), a definition of interpolation can now be given. This definition avoids the use of frequency domain notions, and thus is read-

ily enhanced for the purpose of defining interpolation in more complicated settings.

Definition 6 (Loewner Equivalence)**.** *Let* Σ *and* $\overline{\Sigma}$ *be two systems with left and right Loewner matrices* $\mathbb{L}^\ell$, $\mathbb{L}^r$, *and* $\overline{\mathbb{L}}^\ell$, $\overline{\mathbb{L}}^r$, *respectively, associated to the generating matrices* Λ, R, M, *and* L. *Then,* Σ *and* $\overline{\Sigma}$ *are called Loewner equivalent at* (Λ, R, M, L) *if* $\mathbb{L}^\ell = \overline{\mathbb{L}}^\ell$ *and* $\mathbb{L}^r = \overline{\mathbb{L}}^r$.

By considering Theorem 6, the fact that two systems are Loewner equivalent at (Λ, R, M, L) has an obvious implication regarding the behaviour of their corresponding cascade interconnections with the auxiliary systems (3.6)-(3.7) and (3.8)-(3.9). If two systems, Σ and $\overline{\Sigma}$, are Loewner equivalent at (Λ, R, M, L), then in the Loewner coordinates represented in Figure 3.2 the responses of the top and bottom branches are exactly matched for corresponding initial conditions. Any difference in behaviour of the corresponding cascade interconnected system outputs, η and $\overline{\eta}$, respectively, is associated to the behaviour of the centre branches. Furthermore, if the systems Σ and $\overline{\Sigma}$ possess asymptotically stable equilibrium points at the origin and the input Δ converges to zero (such as when Δ sets an initial condition or represents a transient disturbance) then the respective output trajectories of the interconnected systems differ only by a transient and converge to the same trajectory that is characterized entirely by the Loewner matrices and the states of the auxiliary systems (3.6)-(3.7) and (3.8)-(3.9). Thus, under mild conditions it follows that Loewner equivalent systems induce similar output responses when interconnected with the same auxiliary systems. This is demonstrated

by considering the system (3.12)-(3.13) restricted to the invariant (when the matrix pencil[4] $sE - A$ is regular and $\Delta = 0$) manifold $z_c = 0$, or $x = X\zeta_r$, which has the simplified dynamics

$$\begin{aligned}\dot{z}_r &= \Lambda z_r + \Delta,\\ \dot{z}_\ell &= M z_\ell + \mathbb{L}^\ell \Delta,\\ \eta &= \mathbb{L}^r z_r + z_\ell.\end{aligned}$$

It should be noted that Loewner equivalence pertains to the behaviour of the interconnected system (3.10)-(3.11) and, for Loewner equivalent systems Σ and $\overline{\Sigma}$ with transfer functions $H(\cdot)$ and $\overline{H}(\cdot)$, respectively, right tangential data matrices W and $\overline{W}$, respectively, and left tangential data matrices V and $\overline{V}$, respectively, implies the matching conditions

$$LH(\lambda_i)r_i = LWe_i = L\overline{W}e_i = L\overline{H}(\lambda_i)r_i, \qquad i = 1, \ldots, \rho,$$

and

$$\ell_j H(\mu_j)R = e_j^\top VR = e_j^\top \overline{V}R = \ell_j \overline{H}(\mu_j)R, \qquad j = 1, \ldots, v,$$

[4]As discussed in Section 2.3 and [88], the condition that $sE - A$ is a regular matrix pencil is necessary and sufficient for the property that for every sufficiently smooth input, the differential-algebraic equation is solvable and the solution is unique for every consistent initial value. Thus, if $\Delta(t) = 0$ for $t \geq t_0$ and $z_c(t_0) = 0$, then $z_c(t) = 0$ and $\dot{z}_c(t) = 0$, for all $t \geq t_0$, is a unique solution.

because

$$LW = \mathbb{L}^r\Lambda - M\mathbb{L}^r = \overline{\mathbb{L}}^r\Lambda - M\overline{\mathbb{L}}^r = L\overline{W},$$

and

$$VR = M\mathbb{L}^\ell - \mathbb{L}^\ell\Lambda = M\overline{\mathbb{L}}^\ell - \overline{\mathbb{L}}^\ell\Lambda = \overline{V}R.$$

Thus, in its most general form, Loewner equivalence is a weaker condition than that of matching the same tangential data W and V. However, under mild conditions Loewner equivalence is necessary and sufficient for matching the tangential data.

Theorem 8 (Matching Conditions)**.** *Let Σ and $\overline{\Sigma}$ be two systems with Loewner matrices $\mathbb{L}^\ell$, $\mathbb{L}^r$, $\mathbb{L}$, $\sigma\mathbb{L}$, and $\overline{\mathbb{L}}^\ell$, $\overline{\mathbb{L}}^r$, $\overline{\mathbb{L}}$, $\sigma\overline{\mathbb{L}}$, and tangential data W, V, and $\overline{W}$, $\overline{V}$, respectively, associated to the generating matrices Λ, R, M, and L. If $\sigma(\Lambda) \cap \sigma(M) = \emptyset$, R has full row rank, and L has full column rank then the following statements are equivalent:*

i) $\mathbb{L}^\ell = \overline{\mathbb{L}}^\ell$ *and* $\mathbb{L}^r = \overline{\mathbb{L}}^r$,

ii) $\mathbb{L} = \overline{\mathbb{L}}$ *and* $\sigma\mathbb{L} = \sigma\overline{\mathbb{L}}$,

iii) $W = \overline{W}$ *and* $V = \overline{V}$.

Proof. Clearly, if $\mathbb{L}^\ell = \overline{\mathbb{L}}^\ell$ and $\mathbb{L}^r = \overline{\mathbb{L}}^r$ then

$$\overline{\mathbb{L}} = \overline{\mathbb{L}}^\ell + \overline{\mathbb{L}}^r = \mathbb{L}^\ell + \mathbb{L}^r = \mathbb{L},$$

and

$$\sigma\overline{\mathbb{L}} = M\overline{\mathbb{L}}^\ell + \overline{\mathbb{L}}^r\Lambda = M\mathbb{L}^\ell + \mathbb{L}^r\Lambda = \sigma\mathbb{L},$$

hence $i) \Rightarrow ii)$, trivially. If $\mathbb{L} = \overline{\mathbb{L}}$ and $\sigma\mathbb{L} = \sigma\overline{\mathbb{L}}$ then

$$L\overline{W} = \sigma\overline{\mathbb{L}} - M\overline{\mathbb{L}} = \sigma\mathbb{L} - M\mathbb{L} = LW,$$

and

$$\overline{V}R = \sigma\overline{\mathbb{L}} - \overline{\mathbb{L}}\Lambda = \sigma\mathbb{L} - \mathbb{L}\Lambda = VR,$$

and $ii) \Rightarrow iii)$, because R has full row rank and L has full column rank. If $W = \overline{W}$ and $V = \overline{V}$ then

$$M\overline{\mathbb{L}}^\ell - \overline{\mathbb{L}}^\ell\Lambda = \overline{V}R = VR = M\mathbb{L}^\ell - \mathbb{L}^\ell\Lambda,$$

and

$$\overline{\mathbb{L}}^r\Lambda - M\overline{\mathbb{L}}^r = L\overline{W} = LW = \mathbb{L}^r\Lambda - M\mathbb{L}^r,$$

hence

$$M(\overline{\mathbb{L}}^{\ell} - \mathbb{L}^{\ell}) - (\overline{\mathbb{L}}^{\ell} - \mathbb{L}^{\ell})\Lambda = 0, \quad (\overline{\mathbb{L}}^{r} - \mathbb{L}^{r})\Lambda - M(\overline{\mathbb{L}}^{r} - \mathbb{L}^{r}) = 0,$$

and $iii) \Rightarrow i)$, because $\sigma(\Lambda) \cap \sigma(M) = \emptyset$. □

Based on the definition of Loewner equivalence, a reduced order model in the Loewner sense is formally defined as follows.

Definition 7 (**Reduced Order Model**). *Let Σ and $\overline{\Sigma}$ be two systems of order n and v, respectively. $\overline{\Sigma}$ is called a reduced order model of Σ in the Loewner sense if Σ and $\overline{\Sigma}$ are Loewner equivalent at (Λ, R, M, L) and $v < n$.*

3.3 Conclusion

In this chapter an interconnection-based interpretation of the Loewner matrices has been presented. To this end, new objects, the left and right Loewner matrices, have been defined, the introduction of which is instrumental for revealing how the Loewner matrices are encoded into a conceptual experimental setup. From this interpretation a definition of interpolation which does not rely on frequency domain notions has been presented. Due to its system theoretic nature the interpretation can be leveraged to extend the definitions of Loewner "objects" to nonlinear and time-varying systems, and hence to extend previously studied tangential interpolation methods to more general classes of systems.

Chapter 4

Interpolation of Nonlinear Differential Systems

In this chapter the Loewner framework is extended to general nonlinear input-affine systems of ordinary differential equations using the state-space interpretation of the Loewner matrices developed in Chapter 3. This is accomplished by considering auxiliary systems encoding interpolation points and, along with the underlying system-to-be-interpolated, constructing a nonlinear enchancement of the conceptual experimental setup discussed in Chapter 3. Generalizations of the Loewner matrices, the Loewner functions, are introduced. These functions have definitions motivated by the construction of a "parallelizing" coordinates transformation for the cascade interconnection. The Loewner functions are then used to build models which produce the exact same Loewner functions corresponding to the same auxiliary

systems, thus achieving interpolation in the Loewner sense. Locally, under mild assumptions, the original model and the interpolant produce the same steady-state response when cascade interconnected with the same auxiliary systems, provided that the response exists. Similar to the linear setting, the Loewner framework for nonlinear systems resembles the two-sided moment matching procedure of [77].

This chapter is structured as follows. In Section 4.1 the class of systems that the Loewner framework is enhanced for in this chapter is introduced. In Section 4.2 the notion of Loewner matrices is generalized to that of *Loewner functions* for nonlinear systems interconnected with linear auxiliary systems; a special set of coordinates is introduced, the *Loewner coordinates*; and an interpolant is presented, achieving interpolation on the basis of the Loewner functions and matching *tangential data functions*. In Section 4.3 the results are further enhanced to allow for nonlinear auxiliary systems. In Section 4.4 a demonstrative example is provided wherein a reduced order model is constructed for the averaged model of a DC-to-DC Ćuk converter, and in Section 4.5 an example is provided where a system is constructed which interpolates nonlinear tangential data functions associated to the output response induced when a system's input is driven by a Van der Pol oscillator. Finally, in Section 4.6 some concluding remarks are drawn.

4.1 Problem Formulation

To take the first step towards enhancing the Loewner framework to more general classes of systems, this chapter focuses on the interpolation and model order reduction of nonlinear systems described by equations of the form

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t), \tag{4.1}$$

$$y(t) = h(x(t)), \tag{4.2}$$

with state $x(t) \in \mathbb{C}^n$, input $u(t) \in \mathbb{C}^m$, and output $y(t) \in \mathbb{C}^p$, and mappings $f : \mathbb{C}^n \to \mathbb{C}^n$, $g : \mathbb{C}^n \to \mathbb{C}^{n\times m}$, and $h : \mathbb{C}^n \to \mathbb{C}^p$ of appropriate dimensions, and such that $f(0) = 0$, $h(0) = 0$, and $f(\cdot)$ is differentiable. Let $A := \frac{\partial f}{\partial x}(0)$. For ease of presentation, as done in Chapter 3, complex valued mappings and signals are considered and are assumed to have been obtained from linear coordinates transformations of corresponding real valued mappings and signals. In addition, with some abuse of terminology, it is said, for example, that the zero equilibrium of $\dot{x} = f(x)$, with $x(t) \in \mathbb{C}^n$ and $f : \mathbb{C}^n \to \mathbb{C}^n$, is locally asymptotically stable if the zero equilibrium of the underlying "real" system is locally asymptotically stable.

The following assumption is made to relate the Loewner functions to the response of the system when interconnected with auxiliary systems. It is also used to prove the existence of solution for various PDEs arising in the Loewner framework via the centre manifold theory.[1]

[1]It will also be demonstrated that existence of the various objects in the Loewner

Assumption 5. *The unforced system $\dot{x} = f(x)$ is locally exponentially stable at the origin, i.e. all eigenvalues of A belong to $\mathbb{C}^-$.*

4.2 Linear Auxiliary Systems

To begin leveraging the state-space intepretation of the Loewner matrices given in Chapter 3, consider again constructing two linear auxiliary systems of the form

$$\dot{\zeta}_r(t) = \Lambda\zeta_r(t) + \Delta(t), \tag{4.3}$$

$$v(t) = R\zeta_r(t), \tag{4.4}$$

and

$$\dot{\zeta}_\ell(t) = M\zeta_\ell(t) + L\chi(t), \tag{4.5}$$

$$\eta(t) = \zeta_\ell(t), \tag{4.6}$$

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$, and with matrices $\Lambda \in \mathbb{C}^{\rho\times\rho}$, $R \in \mathbb{C}^{m\times\rho}$, $M \in \mathbb{C}^{v\times v}$, and $L \in \mathbb{C}^{v\times p}$. The matrices Λ, R, M, L, are taken to be the compact representations of the right and left tangential interpolation points (2.19) and (2.20) as in the previous chapter.

framework can be guaranteed using nonresonance conditions, hence the Loewner objects can still be obtained for unstable systems. However, in this case the connection with the steady-state response of the interconnected system is lost.

Bounded signals should be considered in order to relate the response of the system to the Loewner functions, hence the following assumption is made.

Assumption 6. *The matrices Λ and M have all eigenvalues on the imaginary axis, and these eigenvalues have geometric multiplicity one.*

Consider now the interconnection of the system (4.1)-(4.2) with the auxiliary systems (4.3)-(4.4) and (4.5)-(4.6), defined via the interconnection equations $u = v$ and $\chi = y$. The resulting interconnected system has state-space representation given by the equations

$$\begin{bmatrix} \dot{\zeta}_r \\ \dot{x} \\ \dot{\zeta}_\ell \end{bmatrix} = \begin{bmatrix} \Lambda\zeta_r \\ f(x) + g(x)R\zeta_r \\ M\zeta_\ell + Lh(x) \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \Delta, \tag{4.7}$$

$$\eta = \zeta_\ell, \tag{4.8}$$

with state $\begin{bmatrix} \zeta_r^* & x^* & \zeta_\ell^* \end{bmatrix}^*$, input Δ, and output η.

4.2.1 Loewner Functions

Nonlinear enhancements of the tangential generalized controllability and observability matrices, and nonlinear enhancements of the tangential data matrices and Loewner matrices, can now be defined. These enhancements, from now on collectively referred to as *Loewner functions*, or even more generally as *Loewner objects*, are defined in terms of the functions and matrices appearing in the interconnected system (4.7)-(4.8). It is important to note that

the following statements regarding the existence of Loewner functions, some of which are defined as the solutions to partial differential equations (PDEs), are all local.

The tangential generalized controllability function, $X : \mathbb{C}^{\rho} \to \mathbb{C}^{n}$, is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial X}{\partial \zeta_r}\Lambda\zeta_r = f(X(\zeta_r)) + g(X(\zeta_r))R\zeta_r, \qquad X(0) = 0. \tag{4.9}$$

The following proposition follows as a direct consequence of Assumptions 5 and 6 and of the centre manifold theory, see [31].

Proposition 1 (Existence of X)**.** *Consider the PDE* (4.9) *with the boundary condition* $X(0) = 0$*. Suppose Assumption 5 and Assumption 6 hold. Then there exists a function* $X : \mathbb{C}^{\rho} \to \mathbb{C}^{n}$ *satisfying the partial differential equation* (4.9) *with the given boundary condition.*

The tangential generalized observability function, $Y : \mathbb{C}^{n} \to \mathbb{C}^{v}$, is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial Y}{\partial x}f(x) = MY(x) - Lh(x), \qquad Y(0) = 0. \tag{4.10}$$

To prove existence of solution for the PDE with boundary condition (4.10) the following definition from [84] is leveraged.

Definition 8 ([84, Def. 2])**.** *Given an* $n \times n$ *matrix* F*, with spectrum* $\sigma(F) = \lambda = (\lambda_1, \ldots, \lambda_n)$*, and constants* $C > 0$ *and* $v > 0$*, it is said that*

a complex number μ is of type (C, v) with respect to $\sigma(F)$ if, for any vector $m = (m_1, m_2, \ldots, m_n)$ of nonnegative integers, it follows that

$$|\mu - (m \cdot \lambda)| \geq \frac{C}{|m|^v},$$

where $|m| = \sum m_i > 0$.

The purpose of introducing Definition 8 is to give a *nonresonance condition* which asserts the existence and uniqueness of an analytic solution to various PDEs possessing a "Sylvester-like" structure. The type (C, v) condition can be thought of as a nonlinear enhancement of the well-known disjoint spectrum condition, which asserts the existence of a unique solution, for Sylvester equations. The following proposition follows by a direct application of the main theorem of [84].

Proposition 2 (Existence of Y)**.** *Consider the PDE* (4.10) *with the boundary condition $Y(0) = 0$ and suppose that $f(\cdot)$ and $h(\cdot)$ are analytic. Suppose there exist constants $C > 0$ and $v > 0$ such that all eigenvalues of M are of type (C, v) with respect to $\sigma(A)$. Then there exists a function $Y : \mathbb{C}^n \to \mathbb{C}^v$ satisfying the partial differential equation* (4.10) *with the given boundary condition.*

Remark 2. An alternative proof of existence of solution to the PDE (4.10) utilizing the centre manifold theory and an auxiliary object can be found in [130], however this proof requires an invertibility condition for the auxiliary object.

Remark 3. A proof of existence of solution to the PDE with boundary condition (4.9) that does not require any stability assumptions can be given by asserting that the eigenvalues of A are of type (C, v) with respect to $\sigma(\Lambda)$. The proof, however, requires a slightly more general existence result than the enhancement of the main theorem of [84] given later in this chapter, and a more general result that can be found in Appendix B.

Having defined the tangential generalized controllability and observability functions, nonlinear enhancements of the right and left tangential data matrices, $W : \mathbb{C}^{\rho} \to \mathbb{C}^{p}$ and $V : \mathbb{C}^{\rho} \to \mathbb{C}^{v \times m}$, are given by

$$W(\zeta_r) := h\big(X(\zeta_r)\big), \qquad V(\zeta_r) := \left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) g\big(X(\zeta_r)\big), \quad (4.11)$$

respectively. The nonlinear Loewner function, $\mathbb{L} : \mathbb{C}^{\rho} \to \mathbb{C}^{v}$, is defined in terms of the tangential generalized controllability and observability functions as

$$\mathbb{L}(\zeta_r) := -Y\big(X(\zeta_r)\big),$$

and the nonlinear shifted Loewner function, $\sigma\mathbb{L} : \mathbb{C}^{\rho} \to \mathbb{C}^{v}$, is defined in terms of the tangential generalized controllability and observability functions as

$$\sigma\mathbb{L}(\zeta_r) := -\left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) f\big(X(\zeta_r)\big).$$

The left Loewner function, $\mathbb{L}^{\ell} : \mathbb{C}^{\rho} \to \mathbb{C}^{v}$, is defined as the solution, provided

it exists, to the PDE with boundary condition

$$\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Lambda\zeta_r - M\mathbb{L}^\ell(\zeta_r) = -V(\zeta_r)R\zeta_r, \qquad \mathbb{L}^\ell(0) = 0, \tag{4.12}$$

and the right Loewner function, $\mathbb{L}^r : \mathbb{C}^\rho \to \mathbb{C}^v$, is defined as

$$\mathbb{L}^r(\zeta_r) := \mathbb{L}(\zeta_r) - \mathbb{L}^\ell(\zeta_r).$$

The following proposition regarding the existence of the left Loewner function follows, again, by a direct application of the main theorem of [84].

Proposition 3 (Existence of $\mathbb{L}^\ell$)**.** *Consider the PDE* (4.12) *with the boundary condition* $\mathbb{L}^\ell(0) = 0$ *and suppose that* $V(\cdot)$ *is analytic. Suppose there exist constants* $C > 0$ *and* $v > 0$ *such that all eigenvalues of* M *are of type* (C, v) *with respect to* $\sigma(\Lambda)$*. Then there exists a function* $\mathbb{L}^\ell : \mathbb{C}^\rho \to \mathbb{C}^v$ *satisfying the partial differential equation* (4.12) *with the given boundary condition.*

The following proposition gives some useful relationships between the Loewner objects introduced thus far.

Proposition 4. *Given the functions* $X(\cdot)$*,* $Y(\cdot)$*,* $W(\cdot)$*,* $V(\cdot)$*,* $\mathbb{L}(\cdot)$*,* $\sigma\mathbb{L}(\cdot)$*,* $\mathbb{L}^\ell(\cdot)$*, and* $\mathbb{L}^r(\cdot)$*, and the matrices* Λ*,* R*,* M*, and* L*, the following hold.*

i) The Loewner function, $\mathbb{L}(\cdot)$*, satisfies the PDE with boundary condition*

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\Lambda\zeta_r - M\mathbb{L}(\zeta_r) = LW(\zeta_r) - V(\zeta_r)R\zeta_r, \qquad \mathbb{L}(0) = 0. \tag{4.13}$$

ii) The right Loewner function, $\mathbb{L}^r(\cdot)$, satisfies the PDE with boundary condition

$$\frac{\partial \mathbb{L}^r}{\partial \zeta_r}\Lambda\zeta_r - M\mathbb{L}^r(\zeta_r) = LW(\zeta_r), \qquad \mathbb{L}^r(0) = 0.$$

iii) The shifted Loewner function satisfies the equation

$$\sigma\mathbb{L}(\zeta_r) = M\mathbb{L}(\zeta_r) + LW(\zeta_r). \tag{4.14}$$

iv) The shifted Loewner function satisfies the equation

$$\sigma\mathbb{L}(\zeta_r) = \frac{\partial \mathbb{L}}{\partial \zeta_r}\Lambda\zeta_r + V(\zeta_r)R\zeta_r. \tag{4.15}$$

v) The shifted Loewner function satisfies the equation

$$\sigma\mathbb{L}(\zeta_r) = M\mathbb{L}^{\ell}(\zeta_r) + \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\Lambda\zeta_r.$$

Proof. Multiplying (4.9) by $-\left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right)$ on the left yields

$$\begin{aligned}
\frac{\partial \mathbb{L}}{\partial \zeta_r}\Lambda\zeta_r &= -\left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right)\frac{\partial X}{\partial \zeta_r}\Lambda\zeta_r \\
&= -\left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) f\big(X(\zeta_r)\big) - \left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) g\big(X(\zeta_r)\big)R\zeta_r \\
&= \sigma\mathbb{L}(\zeta_r) - V(\zeta_r)R\zeta_r,
\end{aligned}$$

which proves iv). Substituting $X(\zeta_r)$ into (4.10) yields

$$\begin{aligned}\sigma\mathbb{L}(\zeta_r) &= -\left(\frac{\partial Y}{\partial x}\circ X(\zeta_r)\right) f\big(X(\zeta_r)\big) \\ &= -MY\big(X(\zeta_r)\big) + Lh\big(X(\zeta_r)\big) \\ &= M\mathbb{L}(\zeta_r) + LW(\zeta_r),\end{aligned}$$

which proves iii). Equating iii) and iv) yields

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\Lambda\zeta_r + V(\zeta_r)R\zeta_r = M\mathbb{L}(\zeta_r) + LW(\zeta_r),$$

and noting that $X(0) = 0$, $Y(0) = 0$, and $W(0) = 0$ proves i). Subtracting (4.12) from (4.13) yields

$$\begin{aligned}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} - \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\right)\Lambda\zeta_r - M\big(\mathbb{L}(\zeta_r) - \mathbb{L}^\ell(\zeta_r)\big) &= \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\Lambda\zeta_r - M\mathbb{L}^r(\zeta_r) \\ &= LW(\zeta_r),\end{aligned}$$

and noting that $W(0) = 0$ proves ii). Finally, from iv) and (4.12) it follows that

$$\begin{aligned}\sigma\mathbb{L}(\zeta_r) &= \frac{\partial \mathbb{L}}{\partial \zeta_r}\Lambda\zeta_r + V(\zeta_r)R\zeta_r \\ &= \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\Lambda\zeta_r + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Lambda\zeta_r + V(\zeta_r)R\zeta_r\right) \\ &= \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\Lambda\zeta_r + M\mathbb{L}^\ell(\zeta_r),\end{aligned}$$

proving v). $\square$

Remark 4. If the system (4.1)-(4.2) is linear then the solution to the PDEs (4.9), (4.10), and (4.12) becomes $X(\zeta_r) = X\zeta_r$, $Y(x) = Yx$, and $\mathbb{L}^\ell(\zeta_r) = \mathbb{L}^\ell\zeta_r$, where X, Y, and $\mathbb{L}^\ell$ are the solutions to the Sylvester equations (2.24), (2.25), and (3.1). Thus the linear Loewner objects are recovered.

4.2.2 Loewner Coordinates

Similar to the linear setting considered in Chapter 3, the relationship between the Loewner functions and the interconnected system (4.7)-(4.8) is exposed by selecting an appropriate set of coordinates.

Theorem 9 (Loewner Coordinates)**.** *Consider the interconnected system* (4.7)-(4.8) *and the coordinates transformation*

$$\begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} = \begin{bmatrix} \zeta_r \\ x - X(\zeta_r) \\ \zeta_\ell + Y(x) + \mathbb{L}^\ell(\zeta_r) \end{bmatrix}.$$

Then the system in the new coordinates has the state-space realization

$$\begin{bmatrix} \dot{z}_r \\ \dot{z}_c \\ \dot{z}_\ell \end{bmatrix} = \begin{bmatrix} \Lambda & 0 & 0 \\ 0 & \widetilde{A}(z_c, z_r) & 0 \\ 0 & \widetilde{G}(z_c, z_r) & M \end{bmatrix} \begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} + \begin{bmatrix} I \\ -\left(\frac{\partial X}{\partial \zeta_r} \circ z_r\right) \\ \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r} \circ z_r\right) \end{bmatrix} \Delta,$$

$$\eta = \mathbb{L}^r(z_r) - \widetilde{Y}(z_c, z_r)\, z_c + z_\ell,$$

where $z_r(t) \in \mathbb{C}^\rho$, $z_c(t) \in \mathbb{C}^n$, $z_\ell(t) \in \mathbb{C}^v$, *and where* $\widetilde{A} : \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{n\times n}$, $\widetilde{G} : \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{v\times n}$, *and* $\widetilde{Y} : \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{v\times n}$ *are obtained via application of Hadamard's Lemma.*

Proof. The proof proceeds by direct differentiation. Taking the time derivative of z_c yields

$$\begin{aligned}
\dot{z}_c &= \dot{x} - \frac{\partial X}{\partial \zeta_r}\dot{\zeta}_r \\
&= \Big(f\big(z_c + X(\zeta_r)\big) - f\big(X(\zeta_r)\big)\Big) \\
&\quad + \Big(g\big(z_c + X(\zeta_r)\big) - g\big(X(\zeta_r)\big)\Big)R\zeta_r - \frac{\partial X}{\partial \zeta_r}\Delta \\
&= \widetilde{A}(z_c, z_r)z_c - \left(\frac{\partial X}{\partial \zeta_r} \circ z_r\right)\Delta,
\end{aligned}$$

where $\widetilde{A}(\cdot)$ is obtained via Hadamard's lemma and it is such that

$$\begin{aligned}
\widetilde{A}(z_c, z_r)z_c = &\Big(f\big(z_c + X(z_r)\big) - f\big(X(z_r)\big)\Big) \\
&+ \Big(g\big(z_c + X(z_r)\big) - g\big(X(z_r)\big)\Big)Rz_r.
\end{aligned}$$

For z_ℓ, taking the derivative yields

$$\begin{aligned}
\dot{z}_\ell &= \dot{\zeta}_\ell + \frac{\partial Y}{\partial x}\dot{x} + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\dot{\zeta}_r \\
&= Mz_\ell + \left(\frac{\partial Y}{\partial x}f(x) - MY(x) + Lh(x)\right) \\
&\quad + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Lambda\zeta_r - M\mathbb{L}^\ell(\zeta_r) + \frac{\partial Y}{\partial x}g(x)R\zeta_r\right) + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Delta.
\end{aligned}$$

By the PDEs defining $Y(\cdot)$ and $\mathbb{L}^{\ell}(\cdot)$, that is (4.10) and (4.12), this becomes

$$\begin{aligned}\dot{z}_{\ell} &= Mz_{\ell} + \left(\frac{\partial Y}{\partial x} \circ \big(z_c + X(\zeta_r)\big)\right) g\big(z_c + X(\zeta_r)\big) R\zeta_r \\ &\quad - \left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) g\big(X(\zeta_r)\big) R\zeta_r + \frac{\partial \mathbb{L}^{\ell}}{\partial \zeta_r}\Delta \\ &= Mz_{\ell} + \widetilde{G}(z_c, z_r)z_c + \left(\frac{\partial \mathbb{L}^{\ell}}{\partial \zeta_r} \circ z_r\right)\Delta,\end{aligned}$$

where $\widetilde{G}(\cdot)$ is obtained via Hadamard's lemma and it is such that

$$\begin{aligned}\widetilde{G}(z_c, z_r)z_c &= \left(\frac{\partial Y}{\partial x} \circ \big(z_c + X(z_r)\big)\right) g\big(z_c + X(z_r)\big) Rz_r \\ &\quad - \left(\frac{\partial Y}{\partial x} \circ X(z_r)\right) g\big(X(z_r)\big) Rz_r.\end{aligned}$$

Finally, it follows that

$$\begin{aligned}\eta &= z_{\ell} - Y(z_c + X(\zeta_r)) - \mathbb{L}^{\ell}(\zeta_r) \\ &= \big(\mathbb{L}(\zeta_r) - \mathbb{L}^{\ell}(\zeta_r)\big) - \Big(Y\big(z_c + X(\zeta_r)\big) - Y\big(X(\zeta_r)\big)\Big) + z_{\ell} \\ &= \mathbb{L}^{r}(z_r) - \widetilde{Y}(z_c, z_r)z_c + z_{\ell},\end{aligned}$$

where $\widetilde{Y}(\cdot)$ is obtained via Hadamard's lemma and it is such that

$$\widetilde{Y}(z_c, z_r)z_c = Y\big(z_c + X(z_r)\big) - Y\big(X(z_r)\big),$$

yielding the desired result. □

Note that, by Assumption 5, for any sufficiently small initial conditions $x(0)$ and $\zeta_r(0)$, the solution of the interconnected system approaches the centre manifold $x = X(\zeta_r)$ exponentially fast, hence z_c approaches zero provided Δ is sufficiently small and converges to zero. On the centre manifold, *i.e.* when $x = X(\zeta_r)$ or $z_c = 0$, the interconnected system has the simplified dynamics

$$\begin{aligned}\dot{z}_r &= \Lambda z_r + \Delta,\\ \dot{z}_\ell &= M z_\ell + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r} \circ z_r\right)\Delta,\end{aligned}$$

and

$$\eta = \mathbb{L}^r(z_r) + z_\ell,$$

hence the system restricted to the centre manifold contains only information on the Loewner functions. The simplified dynamics allow for an interpretation of the Loewner functions in a similar fashion to the interpretation of the Loewner matrices presented in Chapter 3.

4.2.3 Loewner Equivalent Model

Given the definitions of the Loewner functions, and given the simplified interconnected system dynamics when restricted to the manifold $z_c = 0$, the concepts of interpolation and of model order reduction in the Loewner sense for nonlinear systems can be developed. Furthermore, a nonlinear system

which interpolates the Loewner functions defined by the PDEs (4.9), (4.10), and (4.12) can be constructed. This nonlinear interpolant is reminiscent of the linear interpolants matching the tangential data in [94] and [127]. The frequency domain interpretations of the tangential data (2.19) and (2.20) hold little meaning in the nonlinear context, so, when referring to nonlinear systems, the definition of interpolation in the Loewner sense is given as an extension of the definition of Loewner equivalence for linear systems, *i.e.* as an enhancement of Definition 6 in Chapter 3.

Definition 9 (Loewner Equivalence)**.** *Let* Σ *and* $\overline{\Sigma}$ *be two systems described by equations of the form* (4.1)-(4.2) *admitting left and right Loewner functions* $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, *and* $\overline{\mathbb{L}}^{\ell}(\cdot)$, $\overline{\mathbb{L}}^{r}(\cdot)$, *respectively, associated to the matrices* Λ, R, M, *and* L. *Then* Σ *and* $\overline{\Sigma}$ *are called Loewner equivalent at* (Λ, R, M, L) *if* $\mathbb{L}^{\ell}(\zeta_r) = \overline{\mathbb{L}}^{\ell}(\zeta_r)$ *and* $\mathbb{L}^{r}(\zeta_r) = \overline{\mathbb{L}}^{r}(\zeta_r)$ *in a neighbourhood of the origin.*

Consistently, it is said that a nonlinear system interpolates another nonlinear system (in the Loewner sense) at (Λ, R, M, L) if the two systems are Loewner equivalent at (Λ, R, M, L). That is, for the same matrices Λ, R, M, L, the interpolating model possesses the exact same left and right Loewner functions as the system it interpolates.

Just as in the linear setting of Chapter 3, the property of Loewner equivalence at (Λ, R, M, L) has a strong implication on the steady-state behaviour of the system when interconnected with the auxiliary systems (4.3)-(4.4) and (4.5)-(4.6). By Theorem 9, and assuming the foregoing stability conditions hold, *i.e.* Assumptions 5 and 6, if Δ is sufficiently small, bounded, and

converges to zero, and if the plant state x has not left the region of attraction of the origin (*i.e.* x still approaches the centre manifold $x = X(\zeta_r)$), then it is easy to see that the steady-state response, provided it exists, of the system interconnected with the auxiliary systems is dependent entirely on the auxiliary system states and the left and right Loewner functions. Thus, if two locally exponentially stable systems are Loewner equivalent at (Λ, R, M, L) then there exist initial conditions[2] such that the two systems interconnected with the auxiliary system have the same steady-state behaviour, provided it exists.

Along with the notion of Loewner equivalence, which is useful for the purpose of interpreting the Loewner functions, the following definition of interpolation related to the tangential data mappings is also considered.

Definition 10 (Interpolant of the Tangential Data)**.** *Consider the set of right tangential data, given by* Λ*,* R*,* $W(\cdot)$*, and the set of left tangential data, given by* M*,* L*,* $V(\cdot)$*. Let* Σ *be a system described by equations of the form* (4.1)- (4.2) *and admitting right and left tangential data mappings* $\overline{W}(\cdot)$ *and* $\overline{V}(\cdot)$*, respectively, associated to the matrices* Λ*,* R*,* M*, and* L*. Then* Σ *is called an interpolant of the tangential data, or is said to match the tangential data, if* $W(\cdot) = \overline{W}(\cdot)$ *and* $V(\cdot) = \overline{V}(\cdot)$*.*

Remark 5. It is important to note that the property of Loewner equivalence at (Λ, R, M, L) is weaker than that of matching the tangential data functions $W(\cdot)$ and $V(\cdot)$ in the nonlinear multiple-input setting because different

[2]These initial conditions correspond to points on the manifold $x = X(\zeta_r)$.

left tangential data mappings can induce the same left Loewner function. However, in contrast with the two-sided moment matching problem of [77] where multiple interconnected systems are considered, the interpretation of the Loewner objects using the property of Loewner equivalence is simple as it requires only one interconnected system and it motivates defining the tangential data mapping $V(\cdot)$ as a function of ζ_r rather than x which is more conducive to a state-agnostic data-driven approach. Regardless, the Loewner equivalent interpolants presented in this work always match the tangential data functions $W(\cdot)$ and $V(\cdot)$.

A reduced order model in the Loewner sense can now be defined using the definition of Loewner equivalence.

Definition 11 (Reduced Order Model)**.** *Let Σ and $\overline{\Sigma}$ be two systems of order n and v, respectively. $\overline{\Sigma}$ is called a reduced order model of Σ in the Loewner sense if Σ and $\overline{\Sigma}$ are Loewner equivalent at (Λ, R, M, L) and $v < n$.*

Given that the Loewner functions resulting from the interconnected system (4.7)-(4.8) are known, a nonlinear system resembling (2.28)-(2.29) which is Loewner equivalent at (Λ, R, M, L) to (4.1)-(4.2) can be constructed.

Theorem 10. *Consider the interconnected system* (4.7)-(4.8) *with $\rho = v$. Let $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $V(\cdot)$, and $W(\cdot)$ be the associated Loewner functions. Assume that $\frac{\partial \mathbb{L}}{\partial \zeta_r}$ is nonsingular in a neighbourhood of the origin. Define the*

system

$$\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right) \dot{\omega} = \sigma \mathbb{L}(\omega) - V(\omega) u_r, \tag{4.16}$$

$$y_r = W(\omega), \tag{4.17}$$

with state $\omega(t) \in \mathbb{C}^{\rho}$*, input* $u_r(t) \in \mathbb{C}^m$*, and output* $y_r(t) \in \mathbb{C}^p$*. Then the system* (4.16)-(4.17) *is Loewner equivalent at* (Λ, R, M, L) *to the system* (4.1)-(4.2) *and matches the tangential data functions* $W(\cdot)$ *and* $V(\cdot)$*.*

Remark 6. While the left and right Loewner functions are not explicitly used in the construction of the presented interpolant, their existence provides straightforward justification of how the interpolant in the nonlinear setting works (namely, via the parallelized representation and the definition of Loewner equivalence). Thus, in the nonlinear setting one does not need to explicitly solve the PDE (4.12) to determine $\mathbb{L}^{\ell}(\cdot)$ and $\mathbb{L}^{r}(\cdot)$. That being said, for LTV plants the left and right Loewner functions are explicitly required within the formulation given in [127] when defining the Loewner equivalent interpolant.

Proof. Let $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{\mathbb{L}}^{\ell}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\mathbb{L}}^{r}(\cdot)$, $\overline{W}(\cdot)$, and $\overline{V}(\cdot)$ be the Loewner functions for the system (4.16)-(4.17) interconnected with the auxiliary systems (4.3)-(4.4) and (4.5)-(4.6). As the Jacobian of the Loewner function is nonsingular in a neighbourhood of the origin, begin by rearranging (4.16)

into the explicit state-space form

$$\dot{\omega} = \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{-1} \sigma \mathbb{L}(\omega) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{-1} V(\omega) u_r,$$

for which the Loewner functions have been defined. It follows that the functions $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, and $\overline{\mathbb{L}}^{\ell}(\cdot)$ are solutions to the PDEs, with boundary conditions,

$$\begin{aligned} \frac{\partial \overline{X}}{\partial \zeta_r} \Lambda \zeta_r = &\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right)^{-1} \sigma \mathbb{L}\big(\overline{X}(\zeta_r)\big) \\ &- \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right)^{-1} V\big(\overline{X}(\zeta_r)\big) R \zeta_r, \qquad \overline{X}(0) = 0, \end{aligned} \tag{4.18}$$

$$\frac{\partial \overline{Y}}{\partial \omega} \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{-1} \sigma \mathbb{L}(\omega) = M \overline{Y}(\omega) - L W(\omega), \qquad \overline{Y}(0) = 0, \tag{4.19}$$

and

$$\begin{aligned} \frac{\partial \overline{\mathbb{L}}^{\ell}}{\partial \zeta_r} \Lambda \zeta_r = &\left(\frac{\partial \overline{Y}}{\partial \omega} \circ \overline{X}(\zeta_r)\right) \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right)^{-1} V\big(\overline{X}(\zeta_r)\big) R \zeta_r \\ &+ M \overline{\mathbb{L}}^{\ell}(\zeta_r), \qquad \overline{\mathbb{L}}^{\ell}(0) = 0, \end{aligned} \tag{4.20}$$

while $\overline{\mathbb{L}}(\cdot)$ and $\overline{\mathbb{L}}^{r}(\cdot)$ are defined as

$$\overline{\mathbb{L}}(\zeta_r) := -\overline{Y}\big(\overline{X}(\zeta_r)\big),$$

and

$$\overline{\mathbb{L}}^{r}(\zeta_r) := \overline{\mathbb{L}}(\zeta_r) - \overline{\mathbb{L}}^{\ell}(\zeta_r),$$

and $\overline{W}(\cdot)$ and $\overline{V}(\cdot)$ are defined as

$$\overline{W}(\zeta_r) := W\big(\overline{X}(\zeta_r)\big), \tag{4.21}$$

and

$$\overline{V}(\zeta_r) := -\left(\frac{\partial \overline{Y}}{\partial \omega} \circ \overline{X}(\zeta_r)\right)\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right)^{-1} V\big(\overline{X}(\zeta_r)\big). \tag{4.22}$$

To prove that (4.16)-(4.17) is a Loewner equivalent model, it is shown that $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\mathbb{L}(\omega)$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ are solutions[3] to the set of PDEs (4.18), (4.19), (4.20). Rearranging (4.18) yields

$$\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right)\frac{\partial \overline{X}}{\partial \zeta_r}\Lambda\zeta_r = \sigma\mathbb{L}\big(\overline{X}(\zeta_r)\big) - V\big(\overline{X}(\zeta_r)\big)R\zeta_r,$$

while letting $\overline{X}(\zeta_r) = \zeta_r$ yields

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\Lambda\zeta_r = \sigma\mathbb{L}(\zeta_r) - V(\zeta_r)R\zeta_r,$$

which holds by (4.15). Thus $\overline{X}(\zeta_r) = \zeta_r$ satisfies (4.18). Letting $\overline{Y}(\omega) =$

[3]If type (C, v) conditions hold, then it follows that the solutions of the PDEs are the unique solutions in the space of analytic functions.

$-\mathbb{L}(\omega)$ in (4.19) yields

$$-\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{-1} \sigma \mathbb{L}(\omega) = -M\mathbb{L}(\omega) - LW(\omega),$$

or

$$\sigma \mathbb{L}(\omega) = M\mathbb{L}(\omega) + LW(\omega),$$

which holds by (4.14). Thus $\overline{Y}(\omega) = -\mathbb{L}(\omega)$ satisfies (4.19). Finally, letting $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\mathbb{L}(\omega)$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ in (4.20) yields

$$\begin{aligned}\frac{\partial \mathbb{L}^{\ell}}{\partial \zeta_r}\Lambda\zeta_r &= M\mathbb{L}^{\ell}(\zeta_r) - \frac{\partial \mathbb{L}}{\partial \zeta_r}\Big(\frac{\partial \mathbb{L}}{\partial \zeta_r}\Big)^{-1} V(\zeta_r)R\zeta_r \\ &= M\mathbb{L}^{\ell}(\zeta_r) - V(\zeta_r)R\zeta_r,\end{aligned}$$

which holds by the PDE (4.12) defining $\mathbb{L}^{\ell}(\cdot)$. Thus $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\mathbb{L}(\omega)$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ satisfy (4.20). Hence, $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\mathbb{L}(\omega)$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ are solutions to the set of PDEs (4.18), (4.19), (4.20), and it also follows that

$$\overline{\mathbb{L}}(\zeta_r) = -\overline{Y}\big(\overline{X}(\zeta_r)\big) = \mathbb{L}(\zeta_r),$$

and

$$\overline{\mathbb{L}}^{r}(\zeta_r) = \overline{\mathbb{L}}(\zeta_r) - \overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}(\zeta_r) - \mathbb{L}^{\ell}(\zeta_r) = \mathbb{L}^{r}(\zeta_r),$$

and thus the system (4.16)-(4.17) is Loewner equivalent at (Λ, R, M, L) to the system (4.1)-(4.2). Finally, substituting $\overline{X}(\zeta_r) = \zeta_r$ and $\overline{Y}(\omega) = -\mathbb{L}(\omega)$ into (4.21) and (4.22) yields

$$\overline{W}(\zeta_r) = W(\zeta_r),$$

and

$$\overline{V}(\zeta_r) = \frac{\partial \mathbb{L}}{\partial \zeta_r} \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \right)^{-1} V(\zeta_r) = V(\zeta_r),$$

respectively, hence the system (4.16)-(4.17) matches the tangential data functions $W(\cdot)$ and $V(\cdot)$. □

Remark 7. Reduced order models in the Loewner sense at (Λ, R, M, L) can be constructed for the system (4.1)-(4.2) using Theorem 10 by simply setting the dimensions of the auxiliary systems smaller than the dimension of the underlying system, *i.e.* setting $\rho = v < n$, and then determining the nonlinear Loewner functions associated to the system (4.1)-(4.2) interconnected with the systems (4.3)-(4.4) and (4.5)-(4.6).

4.3 Nonlinear Auxiliary Systems

Having enhanced the Loewner framework to be able to study nonlinear input-affine systems of ODEs in the previous section, in this section a more general scenario is considered in which the system (4.1)-(4.2) is interconnected with

two nonlinear auxiliary systems of the form

$$\dot{\zeta}_r(t) = \lambda(\zeta_r(t)) + \Delta(t), \tag{4.23}$$

$$v(t) = r(\zeta_r(t)), \tag{4.24}$$

and

$$\dot{\zeta}_\ell(t) = m(\zeta_\ell(t)) + \ell(\chi(t)), \tag{4.25}$$

$$\eta(t) = \zeta_\ell(t), \tag{4.26}$$

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$, and with functions $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, and $\ell(\cdot)$ of appropriate dimensions, and such that $\lambda(0) = 0$, $r(0) = 0$, $m(0) = 0$, $\ell(0) = 0$, and $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, and $\ell(\cdot)$ are differentiable. Let $\Lambda := \left(\frac{\partial \lambda}{\partial \zeta_r} \circ 0\right)$ and $M := \left(\frac{\partial m}{\partial \zeta_\ell} \circ 0\right)$, with Assumption 6 still holding.

Within the context of the interconnection-based interpretation of the Loewner framework given in Chapter 3, considering nonlinear auxiliary systems of the form (4.23)-(4.24) and (4.25)-(4.26) can be thought of as generalizing the notion of the tangential interpolation points in (2.19)-(2.20). To motivate the introduction of these systems consider the Van der Pol oscillator, see e.g. [66]. The limit cycle of the oscillator is stable, however its linearization at the origin is unstable. If one wanted to determine an interpolant for (4.1)-(4.2) when its input is excited by the output of a Van der Pol

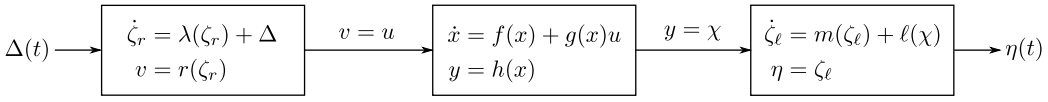

Figure 4.1: The interconnected system (4.27)-(4.28).

oscillator then the linearization at the origin would not be appropriate due to the instability of the approximation. Furthermore, choosing instead a linear auxiliary system with poles on the imaginary axis to more appropriately approximate the oscillator would amount to ignoring nonlinear behaviours that one might be interested in capturing.

Consider now the interconnection of the system (4.1)-(4.2) with the auxiliary systems (4.23)-(4.24) and (4.25)-(4.26) defined by the interconnection equations $u = v$ and $\chi = y$. The resulting interconnected system, depicted in Figure 4.1, has the state-space representation

$$\begin{bmatrix} \dot{\zeta}_r \\ \dot{x} \\ \dot{\zeta}_\ell \end{bmatrix} = \begin{bmatrix} \lambda(\zeta_r) \\ f(x) + g(x)r(\zeta_r) \\ m(\zeta_\ell) + \ell\big(h(x)\big) \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \Delta, \tag{4.27}$$

$$\eta = \zeta_\ell, \tag{4.28}$$

with state $\begin{bmatrix} \zeta_r^* & x^* & \zeta_\ell^* \end{bmatrix}^*$, input Δ, and output η.

4.3.1 Loewner Functions

The tangential generalized controllability and observability functions and the Loewner functions can now be further enhanced in order to accommodate the nonlinear auxiliary systems (4.23)-(4.24) and (4.25)-(4.26). The enhanced functions are, again, defined in terms of the mappings that constitute the interconnected system (4.27)-(4.28). The tangential generalized controllability function, $X : \mathbb{C}^{\rho} \to \mathbb{C}^{n}$, is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial X}{\partial \zeta_r}\lambda(\zeta_r) = f\big(X(\zeta_r)\big) + g\big(X(\zeta_r)\big)r(\zeta_r), \qquad X(0) = 0. \tag{4.29}$$

The following claim is a direct consequence of Assumptions 5 and 6 and of the centre manifold theory, see [31].

Proposition 5 (Existence of X)**.** *Consider the PDE* (4.29) *with the boundary condition* $X(0) = 0$*. Suppose Assumption 5 and Assumption 6 hold. Then there exists a function* $X : \mathbb{C}^{\rho} \to \mathbb{C}^{n}$ *satisfying the partial differential equation* (4.29) *with the given boundary condition.*

The tangential generalized observability function, $Y : \mathbb{C}^{n} \to \mathbb{C}^{v}$, is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial Y}{\partial x}f(x) = -m\big(-Y(x)\big) - \ell\big(h(x)\big), \qquad Y(0) = 0. \tag{4.30}$$

In order to prove the existence of a solution to the PDE (4.30), consider an

alternative proof requiring the construction of an auxiliary object as briefly mentioned in Section 4.2. Consider a system described by the equations

$$\dot{\zeta}_\ell = m(\zeta_\ell) + \ell(h(x)), \tag{4.31}$$

$$\dot{x} = f(x). \tag{4.32}$$

By the centre manifold theory and Assumptions 5 and 6 there exists a mapping $x = \overline{Y}(-\zeta_\ell)$ satisfying the PDE with boundary condition

$$-\left(\frac{\partial \overline{Y}}{\partial \zeta_\ell} \circ (-\zeta_\ell)\right)\Big(m(\zeta_\ell) + \ell\big(h(\overline{Y}(-\zeta_\ell))\big)\Big) = f\big(\overline{Y}(-\zeta_\ell)\big),$$

$$\overline{Y}(0) = 0. \tag{4.33}$$

Proposition 6 (Existence of Y)**.** *Consider the PDE* (4.30) *with the boundary condition* $Y(0) = 0$*. Suppose Assumptions 5 and 6 hold. Suppose that the map* $\overline{Y}(\cdot)$*, solving the PDE* (4.33)*, has a local differentiable left inverse around the origin. Then there exists a function* $Y : \mathbb{C}^n \to \mathbb{C}^v$ *satisfying the partial differential equation* (4.30) *with the given boundary condition.*

Proof. Recall that $\overline{Y}(\cdot)$ satisfies the PDE

$$f\big(\overline{Y}(-\zeta_\ell)\big) = -\left(\frac{\partial \overline{Y}}{\partial \zeta_\ell} \circ (-\zeta_\ell)\right)\Big(m(\zeta_\ell) + \ell\big(h(\overline{Y}(-\zeta_\ell))\big)\Big),$$

with boundary condition $\overline{Y}(0) = 0$. Let $Y(\cdot)$ be the local left inverse of $\overline{Y}(\cdot)$,

which exists by assumption, that is

$$Y\big(\overline{Y}(-\zeta_\ell)\big) = -\zeta_\ell,$$

in a neighbourhood of the origin. Taking the time derivative of $Y(x) = Y\big(\overline{Y}(-\zeta_\ell)\big) = -\zeta_\ell$ along the trajectories of the system (4.31)-(4.32) yields

$$\frac{\partial Y}{\partial x}\dot{x} = -\left(\frac{\partial Y}{\partial x} \circ \overline{Y}(-\zeta_\ell)\right)\left(\frac{\partial \overline{Y}}{\partial \zeta_\ell} \circ (-\zeta_\ell)\right)\dot{\zeta}_\ell = -\dot{\zeta}_\ell,$$

and substituting (4.31) and (4.32) into this equation yields

$$\frac{\partial Y}{\partial x} f(x) = -\Big(m(\zeta_\ell) + \ell\big(h(x)\big)\Big) = -m\big(-Y(x)\big) - \ell\big(h(x)\big).$$

Thus, noting that $Y(0) = 0$, the left inverse of $\overline{Y}(\cdot)$, *i.e.* $Y(\cdot)$, solves the PDE (4.30) in a neighbourhood of the origin with the given boundary condition. □

Remark 8. The proofs of existence of solutions for the PDEs in Proposition 5 and Proposition 6 rely on the centre manifold theory, and solutions to the involved PDEs can be iteratively approximated to any degree of accuracy, see [31].

Having further enhanced the tangential generalized controllability and observability functions, the right and left tangential data functions, $W : \mathbb{C}^\rho \to \mathbb{C}^p$ and $V : \mathbb{C}^\rho \to \mathbb{C}^{v\times m}$, respectively, remain defined as in Section 4.2,

that is

$$W(\zeta_r) := h\big(X(\zeta_r)\big), \qquad V(\zeta_r) := \left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) g\big(X(\zeta_r)\big).$$

Similarly, the nonlinear Loewner and shifted Loewner functions remain defined as in Section 4.2. That is, the Loewner function, $\mathbb{L} : \mathbb{C}^\rho \to \mathbb{C}^v$, is defined in terms of the tangential generalized controllability and observability functions as

$$\mathbb{L}(\zeta_r) := -Y\big(X(\zeta_r)\big),$$

and the shifted Loewner function, $\sigma\mathbb{L} : \mathbb{C}^\rho \to \mathbb{C}^v$, is defined as

$$\sigma\mathbb{L}(\zeta_r) := -\left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) f\big(X(\zeta_r)\big).$$

To accommodate the nonlinear auxiliary systems, the left Loewner function, $\mathbb{L}^\ell : \mathbb{C}^\rho \to \mathbb{C}^v$, is now defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\lambda(\zeta_r) = -m\big(-\mathbb{L}^\ell(\zeta_r)\big) - V(\zeta_r)r(\zeta_r), \qquad \mathbb{L}^\ell(0) = 0, \tag{4.34}$$

and the right Loewner function, $\mathbb{L}^r : \mathbb{C}^\rho \to \mathbb{C}^v$, remains defined as

$$\mathbb{L}^r(\zeta_r) := \mathbb{L}(\zeta_r) - \mathbb{L}^\ell(\zeta_r).$$

A proof that a solution for the PDE (4.34) exists can now be given by extending the proof of the main theorem in [84]. Note that this theorem requires analycity of $\lambda(\cdot)$, $m(\cdot)$, and $V(\cdot)r(\cdot)$ as the proof makes use of a Taylor series expansion.

Proposition 7 (Existence of $\mathbb{L}^\ell$)**.** *Consider the PDE* (4.34) *with the boundary condition* $\mathbb{L}^\ell(0) = 0$ *and suppose that* $\lambda(\cdot)$, $m(\cdot)$, $V(\cdot)$, *and* $r(\cdot)$ *are analytic. Suppose there exist constants* $C > 0$ *and* $v > 0$ *such that all eigenvalues of* M *are of type* (C, v) *with respect to* $\sigma(\Lambda)$*. Then there exists a function* $\mathbb{L}^\ell : \mathbb{C}^\rho \to \mathbb{C}^v$ *satisfying the partial differential equation* (4.34) *with the given boundary condition.*

To prove Proposition 7, the following preliminary result, which is an enhancement of the main theorem of [84], is required.

Lemma 2. *Assume that* $\kappa : \mathbb{C}^\rho \to \mathbb{C}^\rho$, $h : \mathbb{C}^\rho \to \mathbb{C}^p$, *and* $\beta : \mathbb{C}^p \to \mathbb{C}^v$ *are, locally, analytic vector fields such that* $\kappa(0) = 0$, $h(0) = 0$, *and* $\beta(0) = 0$, *and* $\epsilon : \mathbb{C}^v \to \mathbb{C}^v$ *is a globally analytic vector field such that* $\epsilon(0) = 0$*. Let* $K = \frac{\partial \kappa}{\partial x}(0)$ *and* $E = \frac{\partial \epsilon}{\partial z}(0)$*. Suppose that* K *and* E *are diagonalizable, and suppose there exist constants* $C > 0$ *and* $v > 0$ *such that all eigenvalues of* E *are of type* (C, v) *with respect to* $\sigma(K)$*. Then, locally around* $x = 0$*, there exists a unique analytic solution,* $\theta : \mathbb{C}^\rho \to \mathbb{C}^v$*, to the PDE with boundary condition*

$$\frac{\partial \theta}{\partial x}\kappa(x) = \epsilon\big(\theta(x)\big) - \beta\big(h(x)\big), \qquad \theta(0) = 0.$$

Proof. The proof of Lemma 2 is similar to the proof given in [84], but here the nonlinear term $\epsilon\big(\theta(\cdot)\big)$ is included. To begin, let $H = \frac{\partial h}{\partial x}(0)$, $B = \frac{\partial \beta}{\partial y}(0)$, $T = \frac{\partial \theta}{\partial x}(0)$, and, by analycity, expand the functions comprising the partial differential equation using the Taylor series as

$$\theta(x) = Tx + \sum_{i=2}^{\infty} \theta^{(i)}(x), \qquad \kappa(x) = Kx + \sum_{i=2}^{\infty} \kappa^{(i)}(x),$$
$$\beta\big(h(x)\big) = BHx + \sum_{i=2}^{\infty} \beta^{(i)}(x), \qquad \epsilon\big(\theta(x)\big) = E\theta(x) + \sum_{i=2}^{\infty} \epsilon^{(i)}\big(\theta(x)\big),$$

where $\theta^{(i)}(\cdot)$, $\kappa^{(i)}(\cdot)$, $\beta^{(i)}(\cdot)$, and $\epsilon^{(i)}(\cdot)$ denote the terms of order i in the Taylor series expansions of $\theta(\cdot)$, $\kappa(\cdot)$, $\beta\big(h(\cdot)\big)$, and $\epsilon(\cdot)$, respectively. Let $\sigma(K) = \{\lambda_1, \ldots, \lambda_n\}$ be the spectrum of K and let $\sigma(E) = \{\mu_1, \ldots, \mu_\rho\}$ be the spectrum of E. For simplicity, and similarly to [84], it is assumed that K and E are diagonal, however this is not necessary[4]. Substituting the series expansions of the functions into the PDE yields

$$\begin{aligned}
&\Big(T + \sum_{i=2}^{\infty} \frac{\partial \theta^{(i)}}{\partial x}\Big)\Big(Kx + \sum_{i=2}^{\infty} \kappa^{(i)}(x)\Big) \\
&\quad = \Big(E\theta(x) + \sum_{i=2}^{\infty} \epsilon^{(i)}\big(\theta(x)\big)\Big) - \Big(BHx + \sum_{i=2}^{\infty} \beta^{(i)}(x)\Big) \\
&\quad = \Big(ET - BH\Big)x + \sum_{i=2}^{\infty} \Big(E\theta^{(i)}(x) - \beta^{(i)}(x)\Big) \\
&\qquad + \sum_{i=2}^{\infty} \epsilon^{(i)}\big(\theta(x)\big). \qquad (4.35)
\end{aligned}$$

[4] A much more general result which includes the undiagonalizable scenario can be found in Appendix B.

Note that the series expansion of $\epsilon^{(m)}\big(\theta(\cdot)\big)$ contains terms of degree $d \geq m$. Let $\deg\big(\epsilon^{(m)}(\theta(\cdot)), p\big)$ denote the terms of degree p from the series expansion of $\epsilon^{(m)}\big(\theta(\cdot)\big)$. The terms of degree $d = 1$ from the analytically expanded PDE (4.35) are

$$TKx = \big(ET - BH\big)x.$$

With some abuse of the summation notation when $d = 2$ (the summation on the LHS is taken to be 0 in this case), the terms of degree $d \geq 2$ in (4.35) are

$$\begin{aligned} T\kappa^{(d)}(x) + \frac{\partial \theta^{(d)}}{\partial x} Kx + \sum_{k=2}^{d-1} \frac{\partial \theta^{(k)}}{\partial x} \kappa^{(d+1-k)}(x) \\ = E\theta^{(d)}(x) - \beta^{(d)}(x) + \sum_{k=2}^{d} \deg\big(\epsilon^{(k)}(\theta(x)), d\big). \end{aligned}$$

Rearranging, this becomes

$$\frac{\partial \theta^{(d)}}{\partial x} Kx = E\theta^{(d)}(x) - \overline{\beta}^{(d)}(x),$$

with

$$\begin{aligned} \overline{\beta}^{(d)}(x) := \beta^{(d)}(x) + T\kappa^{(d)}(x) + \sum_{k=2}^{d-1} \frac{\partial \theta^{(k)}}{\partial x} \kappa^{(d+1-k)}(x) \\ - \sum_{k=2}^{d} \deg\big(\epsilon^{(k)}(\theta(x)), d\big). \end{aligned}$$

It is important to note that $\overline{\beta}^{(d)}(\cdot)$ only contains coefficients from the series

expansion of $\theta(\cdot)$ associated to terms of degree less than d. Therefore, the functions $\overline{\beta}^d(\cdot)$ and $\theta^d(\cdot)$ are further expanded as

$$\overline{\beta}^{(d)}(x) = \sum_{k=1}^{n} \sum_{|m|=d} \overline{\beta}_{k,m} e_k x^m,$$

and

$$\theta^{(d)}(x) = \sum_{k=1}^{n} \sum_{|m|=d} \theta_{k,m} e_k x^m,$$

where $m = (m_1, \ldots, m_n)$, $x^m = x_1^{m_1} \ldots x_n^{m_n}$, $\overline{\beta}_{k,m}$ and $\theta_{k,m}$ denote the kth row of the coefficient corresponding to x^m in the series expansions, and e_k denotes a vector of all zeros except a one in the kth row. Since K and E are diagonal, $\sigma(K) = \{\lambda_1, \ldots, \lambda_n\}$, and $\sigma(E) = \{\mu_1, \ldots, \mu_n\}$, without loss of generality regarding the ordering of the eigenvalues it follows that (4.35) becomes

$$-\sum_{k=1}^{n} \sum_{|m|=d} \overline{\beta}_{k,m} e_k x^m = \sum_{k=1}^{n} \sum_{|m|=d} \theta_{k,m} e_k \frac{\partial(x^m)}{\partial x} Kx - \sum_{k=1}^{n} \sum_{|m|=d} \mu_k \theta_{k,m} e_k x^m.$$

Note also that

$$\frac{\partial(x^m)}{\partial x} Kx = (m_1\lambda_1 + \cdots + m_n\lambda_n) x^m = (m \cdot \lambda) x^m,$$

with $\lambda = (\lambda_1, \ldots, \lambda_n)^\top$, hence

$$-\sum_{k=1}^{n}\sum_{|m|=d}\overline{\beta}_{k,m}e_k x^m = \sum_{k=1}^{n}\sum_{|m|=d}\theta_{k,m}e_k(m\cdot\lambda)x^m - \sum_{k=1}^{n}\sum_{|m|=d}\mu_k\theta_{k,m}e_k x^m.$$

By matching coefficients for powers of x in the series expansion, the equation

$$\theta_{k,m}(m\cdot\lambda)x^m - \mu_k\theta_{k,m}x^m = -\overline{\beta}_{k,m}x^m,$$

is obtained for each power of x, hence the coefficient $\theta_{k,m}$ can be selected as

$$\theta_{k,m} = \big(\mu_k - (m\cdot\lambda)\big)^{-1}\overline{\beta}_{k,m}.$$

Because the eigenvalues of E are of type (C, v) with respect to $\sigma(K)$, the coefficient $\theta_{k,m}$ exists and is unique for all k and m. Solving for each k and for each x^m gives $\theta^{(d)}(\cdot)$, and then determining $\theta^{(d)}(\cdot)$ for $d = 2, 3, \ldots,$ yields a solution to the PDE, that is the function $\theta(\cdot)$. □

Remark 9. The proof of existence of a solution to the PDE in Lemma 2 is constructive, *i.e.* existence of solution is proven by actually building a particular solution. As such, one can obtain an explicit solution to the PDE by following the steps of the proof, calculating Taylor series expansions of mappings and constructing each $\bar{\beta}^{(d)}(\cdot)$ term, when the involved functions are analytic and the type (C, v) condition holds. Scalability for higher-dimensional systems is not straightforward, however this is not within the scope of this thesis.

Proof of Prop. 7. It is sufficient to substitute $\mathbb{L}^{\ell}(\cdot)$, $\lambda(\cdot)$, $m(\cdot)$, and $V(\cdot)r(\cdot)$ into Lemma 2 to complete the proof of Proposition 7. Note that the conditions in Lemma 2 are satisfied by the supposition that all eigenvalues of M are of type (C, v) with respect to $\sigma(\Lambda)$. □

Remark 10. A more general result than Lemma 2, meant for the setting of nonlinear differential-algebraic systems of equations, found in Appendix B, can be applied to prove existence of solutions for the PDEs (4.29) and (4.30) defining $X(\cdot)$ and $Y(\cdot)$, respectively, when appropriate type (C, v) conditions hold. It should be noted, however, that Lemma 2 can immediately be applied to prove existence of solution for the PDE (4.30), and to prove existence of solution for the PDE (4.29) when $g(\cdot)$ is constant (which is always locally achievable via a coordinates transformation, if the distribution spanning $g(\cdot)$ is involutive). Consequently, if all mappings are analytic and the type (C, v) conditions hold, then explicit solutions can be constructed for the considered PDEs via Lemma 2.

Remark 11. Existence conditions based on Lemma 2 have the advantage that the considered PDEs have solutions even for mappings corresponding to unstable systems. That is, unstable nonlinear systems can be analyzed in the nonlinear Loewner framework without stability assumptions by instead requiring appropriate type (C, v) conditions. However, when considering unstable systems the relation of the Loewner functions to the output response of the interconnected system (4.27)-(4.28) is lost.

Similar to Section 4.2, the following useful relationships hold for the Loewner objects.

Proposition 8. *Given the functions* $X(\cdot)$, $Y(\cdot)$, $W(\cdot)$, $V(\cdot)$, $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, *and* $\ell(\cdot)$, *the following hold.*

i) The Loewner function, $\mathbb{L}(\cdot)$, *satisfies the PDE with boundary condition*

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) - m\big(\mathbb{L}(\zeta_r)\big) = \ell\big(W(\zeta_r)\big) - V(\zeta_r)r(\zeta_r), \quad \mathbb{L}(0) = 0. \tag{4.36}$$

ii) The right Loewner function, $\mathbb{L}^{r}(\cdot)$, *satisfies the PDE with boundary condition*

$$\frac{\partial \mathbb{L}^{r}}{\partial \zeta_r}\lambda(\zeta_r) - m\big(\mathbb{L}(\zeta_r)\big) - m\big(-\mathbb{L}^{\ell}(\zeta_r)\big) = \ell\big(W(\zeta_r)\big), \qquad \mathbb{L}^{r}(0) = 0.$$

iii) The shifted Loewner function satisfies the equation

$$\sigma\mathbb{L}(\zeta_r) = m\big(\mathbb{L}(\zeta_r)\big) + \ell\big(W(\zeta_r)\big). \tag{4.37}$$

iv) The shifted Loewner function satisfies the equation

$$\sigma\mathbb{L}(\zeta_r) = \frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) + V(\zeta_r)r(\zeta_r). \tag{4.38}$$

v) The shifted Loewner function satisfies equation

$$\sigma\mathbb{L}(\zeta_r) = -m\big(-\mathbb{L}^{\ell}(\zeta_r)\big) + \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\lambda(\zeta_r).$$

Proof. The proof proceeds in the same fashion as that of Proposition 4. Multiplying (4.29) by $-\big(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\big)$ on the left yields

$$\begin{aligned}
\frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) &= -\left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right)\frac{\partial X}{\partial \zeta_r}\lambda(\zeta_r) \\
&= -\left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) f\big(X(\zeta_r)\big) - \left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) g\big(X(\zeta_r)\big) r(\zeta_r) \\
&= \sigma\mathbb{L}(\zeta_r) - V(\zeta_r) r(\zeta_r),
\end{aligned}$$

which proves *iv*). Substituting $X(\zeta_r)$ into (4.30) yields

$$\begin{aligned}
\sigma\mathbb{L}(\zeta_r) &= -\left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) f\big(X(\zeta_r)\big) \\
&= m\big(-Y(X(\zeta_r))\big) + \ell\big(h(X(\zeta_r))\big) \\
&= m\big(\mathbb{L}(\zeta_r)\big) + \ell\big(W(\zeta_r)\big),
\end{aligned}$$

which proves *iii*). Equating *iii*) and *iv*) yields

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) + V(\zeta_r) r(\zeta_r) = m\big(\mathbb{L}(\zeta_r)\big) + \ell\big(W(\zeta_r)\big),$$

and noting that $X(0) = 0$, $Y(0) = 0$, and $W(0) = 0$ proves *i*). Subtract-

ing (4.34) from (4.36) yields

$$\begin{aligned}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} - \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\right)\lambda(\zeta_r) - m\big(\mathbb{L}(\zeta_r)\big) - m\big(-\mathbb{L}^\ell(\zeta_r)\big) \\ = \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\lambda(\zeta_r) - m\big(\mathbb{L}(\zeta_r)\big) - m\big(-\mathbb{L}^\ell(\zeta_r)\big) \\ = \ell\big(W(\zeta_r)\big),\end{aligned}$$

and noting $W(0) = 0$ proves ii). Finally, from iv) and (4.34) it follows that

$$\begin{aligned}\sigma\mathbb{L}(\zeta_r) &= \frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) + V(\zeta_r)r(\zeta_r) \\ &= \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\lambda(\zeta_r) + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\lambda(\zeta_r) + V(\zeta_r)r(\zeta_r)\right) \\ &= \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\lambda(\zeta_r) - m\big(-\mathbb{L}^\ell(\zeta_r)\big),\end{aligned}$$

proving v). □

Remark 12. As noted in Remark 4, if the system (4.1)-(4.2) is linear, and the auxiliary systems (4.23)-(4.24) and (4.25)-(4.26) are linear, then the solution to the PDEs (4.29), (4.30), and (4.34) becomes $X(\zeta_r) = X\zeta_r$, $Y(x) = Yx$, and $\mathbb{L}^\ell(\zeta_r) = \mathbb{L}^\ell\zeta_r$, where X, Y, and $\mathbb{L}^\ell$ are the solutions to the Sylvester equations (2.24), (2.25), and (3.1). Thus the linear Loewner objects are recovered.

4.3.2 Loewner Coordinates

As in Theorem 9 and Theorem 6, the relationship between the Loewner functions and the behaviour of the interconnected system (4.27)-(4.28) can be exposed by selecting a particular set of coordinates.

Theorem 11. *Consider the system* (4.27)-(4.28)*. The coordinates transformation*

$$\begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} := \begin{bmatrix} \zeta_r \\ x - X(\zeta_r) \\ \zeta_\ell + Y(x) + \mathbb{L}^\ell(\zeta_r) \end{bmatrix},$$

is such that the system in the new coordinates is described by the equations

$$\begin{bmatrix} \dot{z}_r \\ \dot{z}_c \\ \dot{z}_\ell \end{bmatrix} = \begin{bmatrix} \lambda(z_r) \\ \widetilde{A}\left(z_c, z_r\right) z_c \\ \widetilde{M}\left(z_\ell, z_r\right) z_\ell + \widetilde{G}\left(z_\ell, z_c, z_r\right) z_c \end{bmatrix} + \begin{bmatrix} I \\ -\left(\dfrac{\partial X}{\partial \zeta_r} \circ z_r\right) \\ \left(\dfrac{\partial \mathbb{L}^\ell}{\partial \zeta_r} \circ z_r\right) \end{bmatrix} \Delta + \begin{bmatrix} 0 \\ 0 \\ m(\mathbb{L}^r(z_r)) - m(\mathbb{L}(z_r)) - m(-\mathbb{L}^\ell(z_r)) \end{bmatrix}, \tag{4.39}$$

$$\eta = \mathbb{L}^r(z_r) - \widetilde{Y}\left(z_c, z_r\right) z_c + z_\ell. \tag{4.40}$$

with $z_r(t) \in \mathbb{C}^\rho$, $z_c(t) \in \mathbb{C}^n$, $z_\ell(t) \in \mathbb{C}^v$, *and* $\widetilde{A} : \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{n \times n}$, $\widetilde{G} : \mathbb{C}^v \times \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{v \times n}$, $\widetilde{M} : \mathbb{C}^v \times \mathbb{C}^\rho \to \mathbb{C}^{v \times v}$, *and* $\widetilde{Y} : \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{v \times n}$.

Proof. The proof proceeds by direct differentiation. Taking the time derivative of z_c yields

$$\begin{aligned}\dot{z}_c &= \dot{x} - \frac{\partial X}{\partial \zeta_r}\dot{\zeta}_r \\ &= \Big(f\big(z_c + X(\zeta_r)\big) + g\big(z_c + X(\zeta_r)\big)r(\zeta_r)\Big) \\ &\quad - \Big(f\big(X(\zeta_r)\big) + g\big(X(\zeta_r)\big)r(\zeta_r)\Big) - \frac{\partial X}{\partial \zeta_r}\Delta \\ &= \widetilde{A}(z_c, z_r)z_c - \left(\frac{\partial X}{\partial \zeta_r} \circ z_r\right)\Delta,\end{aligned}$$

where $\widetilde{A}(\cdot)$ is obtained via Hadamard's lemma and it is such that

$$\begin{aligned}\widetilde{A}(z_c, z_r)z_c = &\Big(f\big(z_c + X(z_r)\big) + g\big(z_c + X(z_r)\big)r(z_r)\Big) \\ &- \Big(f\big(X(z_r)\big) + g\big(X(z_r)\big)r(z_r)\Big).\end{aligned}$$

For z_ℓ, taking the time derivative yields

$$\begin{aligned}\dot{z}_\ell &= \dot{\zeta}_\ell + \frac{\partial Y}{\partial x}\dot{x} + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\dot{\zeta}_r \\ &= \Big(m\big(z_\ell - Y(x) - \mathbb{L}^\ell(\zeta_r)\big) - m\big(-Y(x)\big) - m\big(-\mathbb{L}^\ell(\zeta_r)\big)\Big) \\ &\quad + \left(\frac{\partial Y}{\partial x}f(x) + \ell\big(h(x)\big) + m\big(-Y(x)\big)\right) \\ &\quad + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\lambda(\zeta_r) + \frac{\partial Y}{\partial x}g(x)r(\zeta_r) + m\big(-\mathbb{L}^\ell(\zeta_r)\big)\right) + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Delta,\end{aligned}$$

By the PDE-based definitions of $Y(\cdot)$ and $\mathbb{L}^\ell(\cdot)$, (4.30) and (4.34), this be-

comes

$$\begin{aligned}\dot{z}_\ell &= \Big(m(z_\ell - Y(x) - \mathbb{L}^\ell(\zeta_r)) - m(-Y(x)) - m(-\mathbb{L}^\ell(\zeta_r))\Big) \\ &\quad + \left(\frac{\partial Y}{\partial x} g(x) - \left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) g(X(\zeta_r))\right) r(\zeta_r) + \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\Delta \\ &= \widetilde{M}(z_\ell, z_r) z_\ell + \widetilde{G}(z_\ell, z_c, z_r) z_c + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r} \circ z_r\right)\Delta \\ &\quad + m\big(\mathbb{L}^r(z_r)\big) - m\big(\mathbb{L}(z_r)\big) - m\big(-\mathbb{L}^\ell(z_r)\big),\end{aligned}$$

where $\widetilde{M}(\cdot)$ and $\widetilde{G}(\cdot)$ are obtained via Hadamard's lemma and they are such that

$$\widetilde{M}(z_\ell, z_r) z_\ell = m\big(z_\ell + \mathbb{L}^r(z_r)\big) - m\big(\mathbb{L}^r(z_r)\big),$$

and

$$\begin{aligned}\widetilde{G}(z_\ell, z_c, z_r) z_c = &\Big(m\big(z_\ell - Y(z_c + X(z_r)) - \mathbb{L}^\ell(z_r)\big) - m\big(z_\ell + \mathbb{L}^r(z_r)\big)\Big) \\ &+ \Big(m\big(\mathbb{L}(z_r)\big) - m\big(-Y(z_c + X(z_r))\big)\Big) \\ &\left(\frac{\partial Y}{\partial x} \circ (z_c + X(z_r))\right) g\big(z_c + X(z_r)\big) r(z_r) \\ &- \left(\frac{\partial Y}{\partial x} \circ X(z_r)\right) g\big(X(z_r)\big) r(z_r).\end{aligned}$$

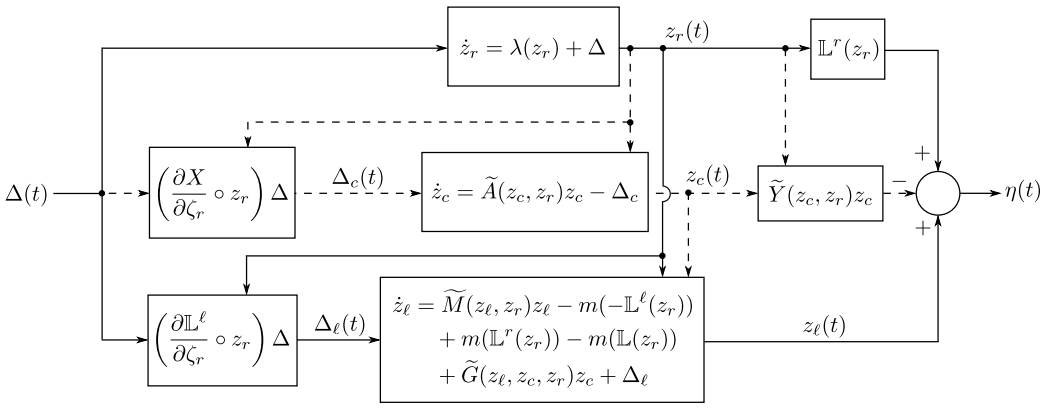

Figure 4.2: The transformed, parallel interconnected, system (4.39)-(4.40).

Finally, it follows that

$$\begin{aligned}\eta &= z_\ell - Y\big(z_c + X(\zeta_r)\big) - \mathbb{L}^\ell(\zeta_r) \\ &= \mathbb{L}^r(\zeta_r) - \Big(Y\big(z_c + X(\zeta_r)\big) - Y\big(X(\zeta_r)\big)\Big) + z_\ell \\ &= \mathbb{L}^r(z_r) - \widetilde{Y}(z_c, z_r)z_c + z_\ell,\end{aligned}$$

where $\widetilde{Y}(\cdot)$ is obtained via Hadamard's lemma and it is such that

$$\widetilde{Y}(z_c, z_r)z_c = Y\big(z_c + X(z_r)\big) - Y\big(X(z_r)\big),$$

yielding the desired result. □

By Assumption 5, for any sufficiently small initial conditions $x(0)$ and $\zeta_r(0)$, solutions of the interconnected system (4.27)-(4.28) approach the centre manifold $x = X(\zeta_r)$ exponentially fast, hence the state in the centre branch, z_c, approaches zero provided the input Δ converges to zero, and

the system has a converging input converging state property. On the centre manifold, that is for $x = X(\zeta_r)$, or $z_c = 0$, the interconnected system has the simplified dynamics

$$\begin{aligned}
\dot{z}_r &= \lambda(z_r) + \Delta, \\
\dot{z}_\ell &= \Big(m\big(z_\ell + \mathbb{L}^r(z_r)\big) - m\big(\mathbb{L}(z_r)\big) - m\big(-\mathbb{L}^\ell(z_r)\big)\Big) \\
&\quad + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r} \circ z_r\right) \Delta,
\end{aligned}$$

and

$$\eta = \mathbb{L}^r(z_r) + z_\ell.$$

As is the case with linear auxiliary systems in Section 4.2, the system restricted to the centre manifold contains only information on the Loewner functions and the nonlinear auxiliary systems. The transformed system is depicted in Figure 4.2.

4.3.3 Loewner Equivalent Model

Similar to Section 4.2, given the definitions of the Loewner functions, and given the simplified interconnected system dynamics when restricted to the manifold $z_c = 0$, the concepts of interpolation and of model order reduction in the Loewner sense for nonlinear systems can now be introduced. In addition, a nonlinear system, reminiscent of the linear systems in [94] and [127],

which interpolates the Loewner functions defined by the PDEs (4.29), (4.30), and (4.34), is constructed.

As in Definition 9, the concept of Loewner equivalence can be easily enhanced for the more general scenario in which the *objects* defining the auxiliary systems are mappings rather than matrices.

Definition 12 (Loewner Equivalence)**.** *Let* Σ *and* $\overline{\Sigma}$ *be two systems described by equations of the form* (4.1)-(4.2) *admitting left and right Loewner functions* $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, *and* $\overline{\mathbb{L}}^{\ell}(\cdot)$, $\overline{\mathbb{L}}^{r}(\cdot)$, *respectively, associated to the functions* $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, *and* $\ell(\cdot)$. *Then* Σ *and* $\overline{\Sigma}$ *are called Loewner equivalent at* (λ, r, m, ℓ) *if* $\mathbb{L}^{\ell}(\zeta_r) = \overline{\mathbb{L}}^{\ell}(\zeta_r)$ *and* $\mathbb{L}^{r}(\zeta_r) = \overline{\mathbb{L}}^{r}(\zeta_r)$ *in a neighbourhood of the origin.*

Again, it is said that a nonlinear system interpolates another nonlinear system (in the Loewner sense) at (λ, r, m, ℓ) if the two systems are Loewner equivalent at (λ, r, m, ℓ), *i.e.* for the same mappings $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, and $\ell(\cdot)$ the interpolating system possesses the exact same left and right Loewner functions in a neighbourhood of the origin.

As in Chapter 3 and Section 4.2, the property of Loewner equivalence, under mild conditions, has a strong implication on the steady-state behaviour of a system when interconnected with the auxiliary systems (4.23)-(4.24) and (4.25)-(4.26). Particularly, by Theorem 11, and given that the foregoing stability assumptions, that is Assumptions 5 and 6, hold, then if Δ is sufficiently small, bounded, and converges to zero, the steady-state response of the system interconnected with the auxiliary systems is dependent entirely on the auxiliary system states and the left and right Loewner functions. Thus, if

two locally exponentially stable systems are Loewner equivalent at (λ, r, m, ℓ) then there exist initial conditions such that the two systems interconnected with the auxiliary systems have the same steady-state behaviour, provided it exists.

The notion of an interpolant of the tangential data mappings can also be enhanced by generalizing Definition 10.

Definition 13 (Interpolant of the Tangential Data)**.** *Consider the set of right tangential data, given by $\lambda(\cdot)$, $r(\cdot)$, $W(\cdot)$, and the set of left tangential data, given by $m(\cdot)$, $\ell(\cdot)$, $V(\cdot)$. Let Σ be a system described by equations of the form* (4.1)- (4.2) *and admitting right and left tangential data mappings $\overline{W}(\cdot)$ and $\overline{V}(\cdot)$, respectively, associated to the mappings $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, and $\ell(\cdot)$. Then Σ is called an interpolant of the tangential data, or is said to match the tangential data, if $W(\cdot) = \overline{W}(\cdot)$ and $V(\cdot) = \overline{V}(\cdot)$.*

Model order reduction in the Loewner sense can now be defined. As with the enhancement of the notion of Loewner equivalence, this is a more general version of Definition 11 in which the *objects* defining the auxiliary systems are mappings rather than matrices.

Definition 14 (Reduced Order Model)**.** *Let Σ and $\overline{\Sigma}$ be two systems of order n and v, respectively. $\overline{\Sigma}$ is called a reduced order model of Σ in the Loewner sense if Σ and $\overline{\Sigma}$ are Loewner equivalent at (λ, r, m, ℓ) and $v < n$.*

A nonlinear system which is Loewner equivalent at (λ, r, m, ℓ) to (4.1)-(4.2) can now be constructed, given that the Loewner functions of (4.27)-

(4.28) are known. This is a more general version of Theorem 10.

Theorem 12. *Consider the interconnected system* (4.27)-(4.28) *with* $\rho = v$. *Let* $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $V(\cdot)$, *and* $W(\cdot)$ *be the associated Loewner functions. Assume that* $\frac{\partial \mathbb{L}}{\partial \zeta_r}$ *is nonsingular in a neighbourhood of the origin. Define the system*

$$\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right) \dot{\omega} = \sigma\mathbb{L}(\omega) - V(\omega)u_r, \tag{4.41}$$

$$y_r = W(\omega), \tag{4.42}$$

with state $\omega(t) \in \mathbb{C}^{\rho}$, *input* $u_r(t) \in \mathbb{C}^{m}$, *and output* $y_r(t) \in \mathbb{C}^{p}$. *Then the system* (4.41)-(4.42) *is Loewner equivalent at* (λ, r, m, ℓ) *to the system* (4.1)-(4.2) *and matches the tangential data functions* $W(\cdot)$ *and* $V(\cdot)$.

Proof. Let $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{\mathbb{L}}^{\ell}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\mathbb{L}}^{r}(\cdot)$, $\overline{W}(\cdot)$, and $\overline{V}(\cdot)$ be the Loewner functions for the system (4.41)-(4.42) interconnected with the auxiliary systems (4.23)-(4.24) and (4.25)-(4.26). By assumption, the Jacobian of the Loewner function is nonsingular in a neighbourhood of the origin, so begin by rearranging (4.41) into the explicit state-space form

$$\dot{\omega} = \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{-1} \sigma\mathbb{L}(\omega) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{-1} V(\omega)u_r,$$

for which the Loewner functions have been defined. The functions $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$,

and $\overline{\mathbb{L}}^{\ell}(\cdot)$ are solutions to the PDEs, with boundary conditions,

$$\frac{\partial \overline{X}}{\partial \zeta_r}\lambda(\zeta_r) = \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right)^{-1} \sigma \mathbb{L}\big(\overline{X}(\zeta_r)\big) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right)^{-1} V\big(\overline{X}(\zeta_r)\big) r(\zeta_r), \qquad \overline{X}(0) = 0, \tag{4.43}$$

and

$$\frac{\partial \overline{Y}}{\partial \omega}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{-1} \sigma \mathbb{L}(\omega) = -m\big(-\overline{Y}(\omega)\big) - \ell\big(W(\omega)\big), \qquad \overline{Y}(0) = 0, \tag{4.44}$$

and

$$\frac{\partial \overline{\mathbb{L}}^{\ell}}{\partial \zeta_r}\lambda(\zeta_r) = \left(\frac{\partial \overline{Y}}{\partial r} \circ \overline{X}(\zeta_r)\right)\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right)^{-1} V\big(\overline{X}(\zeta_r)\big) r(\zeta_r) - m\big(-\overline{\mathbb{L}}^{\ell}(\zeta_r)\big), \qquad \overline{\mathbb{L}}^{\ell}(0) = 0, \tag{4.45}$$

while $\overline{\mathbb{L}}(\cdot)$ and $\overline{\mathbb{L}}^{r}(\cdot)$ are defined as

$$\overline{\mathbb{L}}(\zeta_r) := -\overline{Y}\big(\overline{X}(\zeta_r)\big),$$

and

$$\overline{\mathbb{L}}^{r}(\zeta_r) := \overline{\mathbb{L}}(\zeta_r) - \overline{\mathbb{L}}^{\ell}(\zeta_r),$$

and $\overline{W}(\cdot)$ and $\overline{V}(\cdot)$ are defined as

$$\overline{W}(\zeta_r) := W\big(\overline{X}(\zeta_r)\big), \tag{4.46}$$

and

$$\overline{V}(\zeta_r) := -\left(\frac{\partial \overline{Y}}{\partial \omega} \circ \overline{X}(\zeta_r)\right)\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right)^{-1} V\big(\overline{X}(\zeta_r)\big). \tag{4.47}$$

To prove that (4.41)-(4.42) is a Loewner equivalent model, it is shown that $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\mathbb{L}(\omega)$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ is a solution to the PDEs (4.43), (4.44), (4.45). Rearranging (4.43) yields

$$\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right)\frac{\partial \overline{X}}{\partial \zeta_r}\lambda(\zeta_r) = \sigma \mathbb{L}\big(\overline{X}(\zeta_r)\big) - V\big(\overline{X}(\zeta_r)\big)r(\zeta_r),$$

while letting $\overline{X}(\zeta_r) = \zeta_r$ yields

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) = \sigma \mathbb{L}(\zeta_r) - V(\zeta_r)r(\zeta_r),$$

which holds by (4.38). Thus $\overline{X}(\zeta_r) = \zeta_r$ satisfies (4.43). Letting $\overline{Y}(\omega) = -\mathbb{L}(\omega)$ in (4.44) yields

$$-\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{-1} \sigma \mathbb{L}(\omega) = -m\big(\mathbb{L}(\omega)\big) - \ell\big(W(\omega)\big),$$

or

$$\sigma \mathbb{L}(\omega) = m\big(\mathbb{L}(\omega)\big) + \ell\big(W(\omega)\big),$$

which holds by (4.37). Thus $\overline{Y}(\omega) = -\mathbb{L}(\omega)$ satisfies (4.44). Finally, letting

$\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\mathbb{L}(\omega)$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ in (4.45) yields

$$\begin{aligned}\frac{\partial \mathbb{L}^{\ell}}{\partial \zeta_r}\lambda(\zeta_r) &= -m\big(-\mathbb{L}^{\ell}(\zeta_r)\big) - \frac{\partial \mathbb{L}}{\partial \zeta_r}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\right)^{-1} V(\zeta_r) r(\zeta_r) \\ &= -m\big(-\mathbb{L}^{\ell}(\zeta_r)\big) - \left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) g\big(X(\zeta_r)\big) r(\zeta_r),\end{aligned}$$

which holds by the PDE (4.34) defining $\mathbb{L}^{\ell}(\cdot)$. Thus $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\mathbb{L}(\omega)$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ satisfies (4.45). Hence, $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\mathbb{L}(\omega)$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ are a solution to the set of PDEs (4.43), (4.44), (4.45), and it also follows that

$$\overline{\mathbb{L}}(\zeta_r) = -\overline{Y}\big(\overline{X}(\zeta_r)\big) = \mathbb{L}(\zeta_r),$$

and

$$\overline{\mathbb{L}}^{r}(\zeta_r) = \overline{\mathbb{L}}(\zeta_r) - \overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}(\zeta_r) - \mathbb{L}^{\ell}(\zeta_r) = \mathbb{L}^{r}(\zeta_r),$$

and thus the system (4.41)-(4.42) is Loewner equivalent for (λ, r, m, ℓ) to the system (4.1)-(4.2). Finally, substituting $\overline{X}(\zeta_r) = \zeta_r$ and $\overline{Y}(\omega) = -\mathbb{L}(\omega)$ into (4.46) and (4.47) yields

$$\overline{W}(\zeta_r) = W(\zeta_r),$$

and

$$\overline{V}(\zeta_r) = \frac{\partial \mathbb{L}}{\partial \zeta_r}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\right)^{-1} V(\zeta_r) = V(\zeta_r),$$

respectively, hence the system (4.41)-(4.42) matches the tangential data functions $W(\cdot)$ and $V(\cdot)$. □

4.4 Example - A Reduced Order Model for the DC-to-DC Ćuk Converter

The averaged model of the DC-to-DC Ćuk converter is given by the system of equations (see *e.g.* [14], [109])

$$\begin{aligned}
L_1\frac{di_1}{dt} &= -(1-u)v_2 + \overline{E}, & C_2\frac{dv_2}{dt} &= (1-u)i_1 + i_3 u,\\
L_3\frac{di_3}{dt} &= -v_4 - v_2 u, & C_4\frac{dv_4}{dt} &= i_3 - Gv_4,\\
y_{FOM} &= v_4,
\end{aligned}$$

where $i_1(t) \in \mathbb{R}^+$, $v_2(t) \in \mathbb{R}^+$, $i_3(t) \in \mathbb{R}^-$, and $v_4(t) \in \mathbb{R}^-$ describe currents and voltages, and $L_1 \in \mathbb{R}$, $C_2 \in \mathbb{R}$, $L_3 \in \mathbb{R}$, $C_4 \in \mathbb{R}$, $\overline{E} \in \mathbb{R}$, and $G \in \mathbb{R}$ are positive parameters, and $u(t) \in (0,1)$ is a continuous control signal used to control the switch position in the converter. Defining states $x_1 := i_1$, $x_2 := v_2 - \overline{E}$, $x_3 := i_3$, and $x_4 := v_4$, one obtains the implicit state-space

model

$$E\dot{x} = Ax + (B_0 + B_1 x)\, u, \qquad y_{FOM} = Cx, \tag{4.48}$$

where

$$x := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad E := \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}, \qquad A := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -G \end{bmatrix},$$

$$B_0 := \overline{E} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \qquad B_1 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C := \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

The matrix E is nonsingular, thus this representation can be rearranged into the state-space form

$$\dot{x} = E^{-1}Ax + E^{-1}\,(B_0 + B_1 x)\, u, \qquad y_{FOM} = Cx.$$

Consider now the problem of determining a reduced order model for the DC-to-DC Ćuk converter associated to the auxiliary systems (4.3)-(4.4) and

(4.5)-(4.6) with matrices given by

$$\Lambda = 0, \qquad R = 1, \qquad M = m, \qquad L = 1,$$

for some $m \in \mathbb{R}$. Then the tangential generalized controllability function, $X(\cdot)$, defined as the solution to the PDE with boundary condition (4.9), is the solution to the equation

$$0 = E^{-1}AX(\zeta_r) + E^{-1}\big(B_0 + B_1X(\zeta_r)\big)\zeta_r, \qquad X(0) = 0.$$

Thus, the tangential generalized controllability function is

$$X(\zeta_r) = -\left(A + B_1\zeta_r\right)^{-1} B_0\zeta_r = \begin{bmatrix} G\zeta_r/(\zeta_r - 1) \\ -1 \\ G \\ 1 \end{bmatrix} \frac{\overline{E}\zeta_r}{\zeta_r - 1},$$

and the right tangential data mapping, $W(\cdot)$, defined via (4.11), is

$$W(\zeta_r) = CX(\zeta_r) = \frac{\overline{E}\zeta_r}{\zeta_r - 1}.$$

The tangential generalized observability function, $Y(\cdot)$, defined as the solution to the PDE with boundary condition (4.10), is the solution to the

equation

$$\frac{\partial Y}{\partial x}E^{-1}Ax = mY(x) - Cx, \qquad Y(0) = 0.$$

This PDE is solved by the linear mapping $Y(x) = Yx$, where Y is the unique solution, for $m \notin \sigma(E^{-1}A)$, of the Sylvester equation

$$YE^{-1}A = mY - C,$$

hence

$$Y = C(mE - A)^{-1}E.$$

That is,

$$Y(x) = \left(1 + mGL_3 + m^2L_3C_4\right)^{-1}L_3\begin{bmatrix}0 & 0 & 1 & mC_4\end{bmatrix}x,$$

and the left tangential data mapping, $V(\cdot)$, defined via (4.11), is

$$\begin{aligned} V(\zeta_r) &= \left(\frac{\partial Y}{\partial x} \circ X(\zeta_r)\right) E^{-1}\big(B_0 + B_1X(\zeta_r)\big) \\ &= YE^{-1}\big(B_0 + B_1X(\zeta_r)\big) \\ &= \frac{\overline{E}}{(1 + mGL_3 + m^2L_3C_4)(\zeta_r - 1)}. \end{aligned}$$

The Loewner function, $\mathbb{L}(\cdot)$, obtained as the solution to the PDE with boundary condition (4.13), and the shifted Loewner function, $\sigma\mathbb{L}(\cdot)$, obtained via (4.15), are constructed from the tangential data mappings as

$$\begin{aligned}\mathbb{L}(\zeta_r) &= m^{-1}\big(V(\zeta_r)\zeta_r - W(\zeta_r)\big)\\ &= -\frac{\overline{E}L_3(G+mC_4)\zeta_r}{(1+mGL_3+m^2L_3C_4)(\zeta_r-1)},\end{aligned}$$

and

$$\begin{aligned}\sigma\mathbb{L}(\zeta_r) &= V(\zeta_r)\zeta_r\\ &= \frac{\overline{E}\zeta_r}{(1+mGL_3+m^2L_3C_4)(\zeta_r-1)},\end{aligned}$$

respectively. Note that

$$\frac{\partial\mathbb{L}}{\partial\zeta_r} = \frac{\overline{E}L_3(G+mC_4)}{(1+mGL_3+m^2L_3C_4)(\zeta_r-1)^2},$$

and

$$\left(\frac{\partial\mathbb{L}}{\partial\zeta_r}\right)^{-1} = \frac{(1+mGL_3+m^2L_3C_4)(\zeta_r-1)^2}{\overline{E}L_3(G+mC_4)}.$$

It follows that an interpolant of the tangential data mappings and the Loewner functions generated by the DC-to-DC Ćuk converter (4.48) at (Λ, R, M, L),

obtained using Theorem 10, is given by

$$\dot{\omega} = \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{-1} \sigma \mathbb{L}(\omega) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{-1} V(\omega) u,$$

$$y_{ROM} = W(\omega), \qquad \omega \neq 1,$$

or

$$\begin{aligned} \dot{\omega} &= \frac{\omega(\omega-1)}{L_3(G+mC_4)} - \frac{\omega-1}{L_3(G+mC_4)} u \\ &= \frac{(\omega-1)}{L_3(G+mC_4)}(\omega-u), \end{aligned} \tag{4.49}$$

$$y_{ROM} = \frac{\overline{E}\omega}{\omega-1}, \qquad \omega \neq 1, \tag{4.50}$$

with state $\omega(t) \in \mathbb{R}$, input $u(t) \in \mathbb{R}$, and output $y_{ROM}(t) \in \mathbb{R}$. If $m \in \mathbb{R}$ is selected such that $-\frac{G}{C_4} < m < 0$ then both the interpolant (4.49)-(4.50) and the auxiliary system

$$\dot{\zeta}_\ell = m\zeta_\ell + \chi, \qquad \eta = \zeta_\ell,$$

have a locally exponentially stable equilibrium point at the origin.

If a system interpolating the tangential data mappings is asymptotically stable, and if $u = r(\zeta_r)$, then for sufficiently small initial conditions it follows that $\lim_{t\to\infty} \big(y - W(\zeta_r)\big) = 0$. It is harder to show the effect of matching $V(\cdot)$ in the output response of the system without invoking the mappings

$X(\cdot)$ and $Y(\cdot)$. However, note that for any system of the form (4.1)-(4.2)

$$\begin{aligned}\ell(y) = \ell\big(h(x)\big) &= -\frac{\partial Y}{\partial x} f(x) - m\big(-Y(x)\big) \\ &= -\frac{\partial Y}{\partial x}\dot{x} + \frac{\partial Y}{\partial x} g(x)u - m\big(-Y(x)\big) \\ &= -\dot{\overbrace{Y(x)}} - m\big(-Y(x)\big) + \frac{\partial Y}{\partial x} g(x)u,\end{aligned}$$

hence

$$\ell(y) + \dot{\overbrace{Y(x)}} + m\big(-Y(x)\big) = \frac{\partial Y}{\partial x} g(x)u.$$

Recalling that if $u = r(\zeta_r)$, and the system is locally asymptotically stable, then for sufficiently small initial conditions it follows that

$$\lim_{t\to\infty} \big(x - X(\zeta_r)\big) = 0,$$

therefore

$$\lim_{t\to\infty} \left(\ell(y) + \dot{\overbrace{Y(x)}} + m\big(-Y(x)\big) \right) = V(\zeta_r) r(\zeta_r),$$

and if the system is single-input and $r(\zeta_r) \neq 0$ then

$$\lim_{t\to\infty} \left(\frac{\ell(y) + \dot{\overbrace{Y(x)}} + m\big(-Y(x)\big)}{r(\zeta_r)} \right) = V(\zeta_r).$$

In order to demonstrate the effect of interpolating $V(\cdot)$, define the mappings

$$V_{t,FOM}(t) := \frac{\ell\big(y_{FOM}(t)\big) + \overbrace{Y\big(x(t)\big)}^{\cdot} + m\big(-Y(x(t))\big)}{r\big(\zeta_r(t)\big)}, \quad r\big(\zeta_r(t)\big) \neq 0,$$

and

$$V_{t,ROM}(t) := \frac{\ell\big(y_{ROM}(t)\big) - \overbrace{\mathbb{L}\big(\omega(t)\big)}^{\cdot} + m\big(\mathbb{L}(\omega(t))\big)}{r\big(\zeta_r(t)\big)}, \quad r\big(\zeta_r(t)\big) \neq 0,$$

so that, if the tangential data mappings are matched and the systems (4.48) and (4.49)-(4.50) are asymptotically stable with sufficiently small initial conditions, then

$$\lim_{t\to\infty} y_{FOM}(t) = \lim_{t\to\infty} y_{ROM}(t) = W\big(\zeta_r(t)\big),$$

$$\lim_{t\to\infty} V_{t,FOM}(t) = \lim_{t\to\infty} V_{t,ROM}(t) = V\big(\zeta_r(t)\big).$$

Consider now the full order model of the DC-to-DC Ćuk converter, given by (4.48), and the reduced order model, given by (4.49)-(4.50), with the parameters

$$L_1 = C_2 = L_3 = C_4 = \overline{E} = 1, \qquad G = 2, \qquad m = -1.5,$$

so (4.48) becomes

$$\dot{x} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} x + \left(\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x \right) u,$$

$$y_{FOM} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} x,$$

and (4.49)-(4.50) becomes

$$\dot{\omega} = 2(\omega - 1)(\omega - u),$$
$$y_{ROM} = \frac{\omega}{\omega - 1}.$$

Figures 4.3 and 4.4 show the responses of the full order model and the reduced order model to a piecewise constant input signal. It can be seen that the responses of the systems are consistent with the tangential data mappings.

4.5 Example - An Interpolant of Nonlinear Tangential Data Mappings

Consider constructing an interpolant for nonlinear tangential data mappings associated to auxiliary systems of the form (4.23)-(4.24) and (4.25)-(4.26). The right tangential data mappings, associated to the output response of an

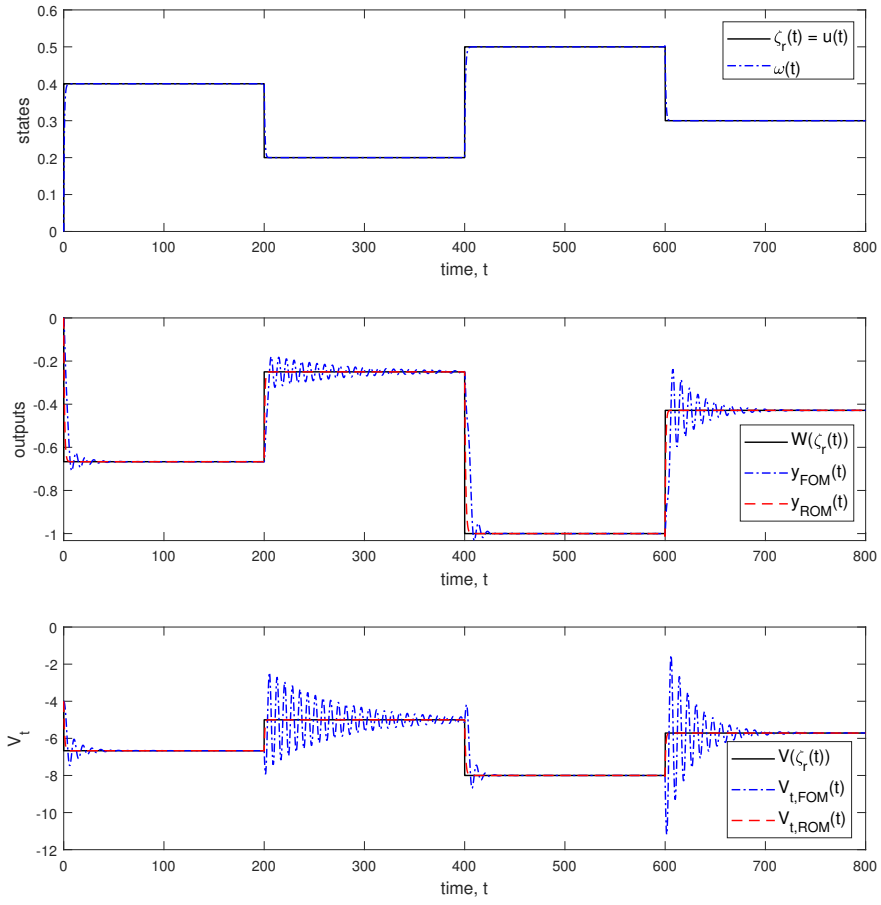

Figure 4.3: Responses of the full order model (4.48) and the reduced order model (4.49)-(4.50).

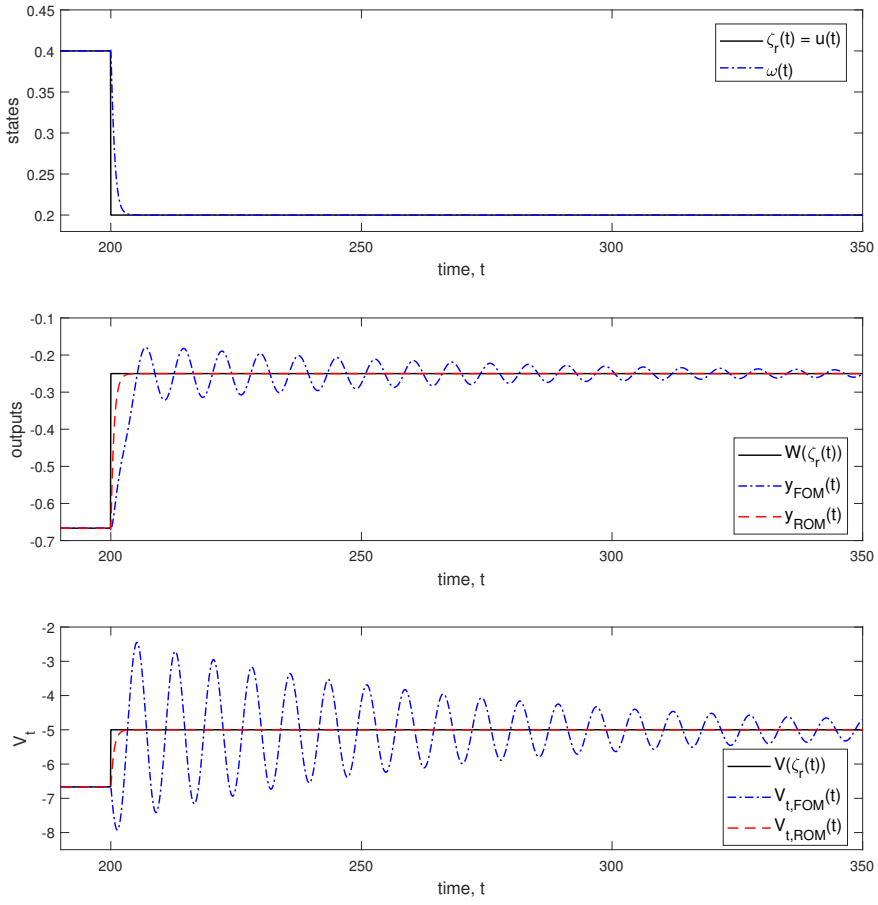

Figure 4.4: Responses of the full order model (4.48) and the reduced order model (4.49)-(4.50) for $t \in [190, 350]$.

underlying system when interconnected with a Van der Pol oscillator, are given by

$$\lambda(\zeta_r) = \begin{bmatrix} \zeta_{r,2} \\ \mu(1-\zeta_{r,1}^2)\zeta_{r,2} - \zeta_{r,1} \end{bmatrix}, \qquad r(\zeta_r) = \frac{1}{10}\zeta_{r,2}, \qquad \mu = 2,$$

and

$$W(\zeta_r) = -\zeta_{r,1} - \zeta_{r,1}^3 + 4\zeta_{r,2}.$$

The left tangential data mappings are given by

$$m(\zeta_\ell) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \zeta_\ell, \qquad \ell(\chi) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \chi,$$

and

$$V(\zeta_r) := \begin{bmatrix} 20 \\ -30 \end{bmatrix} + \begin{bmatrix} 0 \\ 10 \end{bmatrix} \zeta_{r,1}^2 + \begin{bmatrix} -20.03 \\ -0.04 \end{bmatrix} \zeta_{r,1}\zeta_{r,2} + \begin{bmatrix} 0 \\ -0.07 \end{bmatrix} \zeta_{r,2}^2 + \begin{bmatrix} 20 \\ 0.02 \end{bmatrix} \zeta_{r,1}^4 + \begin{bmatrix} 0.04 \\ 0.04 \end{bmatrix} \zeta_{r,1}^3\zeta_{r,2} + \begin{bmatrix} 0 \\ 0.06 \end{bmatrix} \zeta_{r,1}^2\zeta_{r,2}^2.$$

The Loewner function, $\mathbb{L}(\cdot)$, and the shifted Loewner function, $\sigma\mathbb{L}(\cdot)$, must be determined in order to construct a system which interpolates the tangential data mappings. The Loewner function, determined as the unique solution to

the PDE with boundary condition (4.36), is given by

$$\mathbb{L}(\zeta_r) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \zeta_{r,1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \zeta_{r,2} + \begin{bmatrix} 0.001 \\ 0 \end{bmatrix} \zeta_{r,1}^3 + \begin{bmatrix} 1 \\ 0.001 \end{bmatrix} \zeta_{r,1}^2 \zeta_{r,2} + \begin{bmatrix} 0.001 \\ 0.001 \end{bmatrix} \zeta_{r,1} \zeta_{r,2}^2 + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix} \zeta_{r,2}^3,$$

and the Jacobian of the Loewner function is

$$\frac{\partial \mathbb{L}}{\partial \zeta_r} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0.003 & 1 \\ 0 & 0.001 \end{bmatrix} \zeta_{r,1}^2 + \begin{bmatrix} 2 & 0.002 \\ 0.002 & 0.002 \end{bmatrix} \zeta_{r,1} \zeta_{r,2} + \begin{bmatrix} 0.001 & 0 \\ 0.001 & 0.003 \end{bmatrix} \zeta_{r,2}^2.$$

The shifted Loewner function, obtained via (4.37), is

$$\sigma\mathbb{L}(\zeta_r) = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \zeta_{r,1} + \begin{bmatrix} 5 \\ -1 \end{bmatrix} \zeta_{r,2} + \begin{bmatrix} -1 \\ -0.001 \end{bmatrix} \zeta_{r,1}^3 + \begin{bmatrix} 0.001 \\ -1 \end{bmatrix} \zeta_{r,1}^2 \zeta_{r,2} + \begin{bmatrix} 0.001 \\ -0.001 \end{bmatrix} \zeta_{r,1} \zeta_{r,2}^2 + \begin{bmatrix} 0.001 \\ 0 \end{bmatrix} \zeta_{r,2}^3.$$

An interpolant of the tangential data mappings and the Loewner functions at (λ, r, m, ℓ) is obtained using Theorem 12.

In order to show the effect of matching $V(\cdot)$, define the mapping

$$V_{t,r}(t) := \ell\big(y_r(t)\big) - \overbrace{\mathbb{L}\big(\omega(t)\big)}^{\cdot} + m\big(\mathbb{L}(\omega(t))\big),$$

so that if $\lim_{t\to\infty} \omega(t) = \zeta_r(t)$, then

$$\lim_{t\to\infty} V_{t,r}(t) = V\big(\zeta_r(t)\big) r\big(\zeta_r(t)\big).$$

Figure 4.5 shows the response of the interpolant when interconnected with the auxiliary system (4.23)-(4.24) with initial conditions $\zeta_r(0) = (-1, -0.05)^\top$ and $\omega(0) = (0, 0)^\top$. It can be seen that the response of the interpolant is consistent with the tangential data mappings.

4.6 Conclusion

In this chapter a new method for model order reduction for nonlinear systems has been presented. This method extends the state-space interpretation of the Loewner matrices, which are classically interpreted in the frequency domain, developed for linear systems in Chapter 3 and [126], to nonlinear input-affine systems of ordinary differential equations. New objects, the Loewner functions, which are solutions to partial differential equations and are generalizations of the Loewner matrices have been defined. In addition, given the Loewner functions for an underlying nonlinear system a particular reduced order model which interpolates the Loewner functions of the

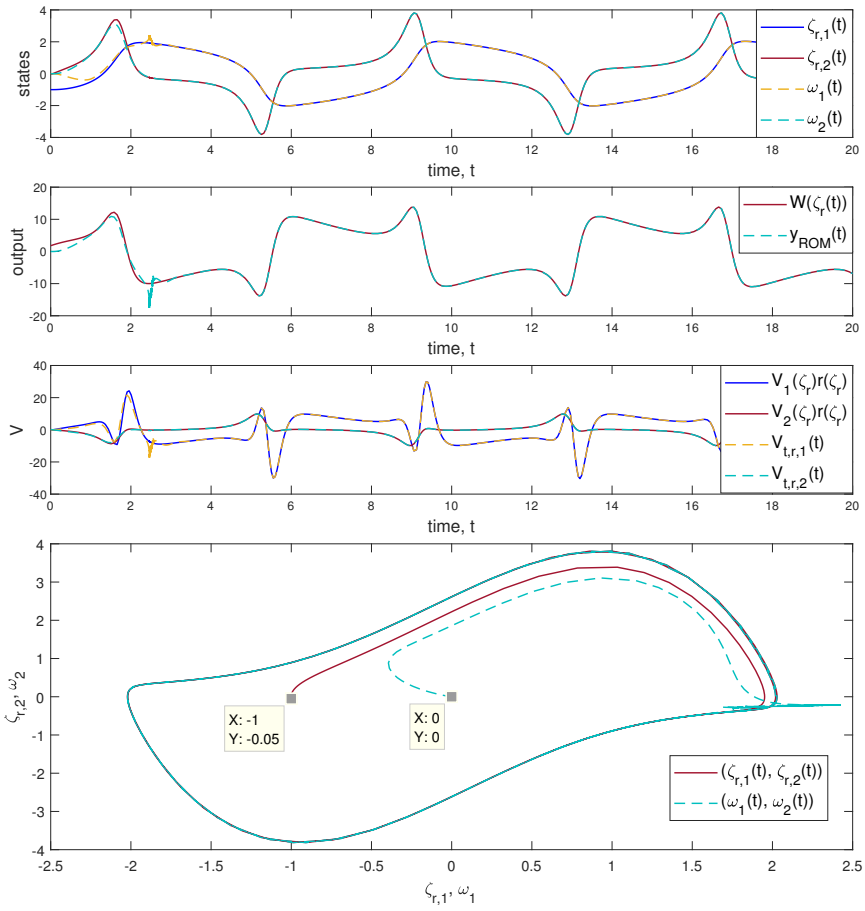

Figure 4.5: Response of the interpolant interconnected with the auxiliary system (4.23)-(4.24).

underlying system can be constructed. Locally, and under mild conditions, the two systems produce the same steady-state response, provided it exists, when interconnected with the same auxiliary systems corresponding to the Loewner functions. Two examples have been provided, with the first example demonstrating the construction of a reduced order model for a nonlinear underlying system, and the second example demonstrating the construction of an interpolant using tangential data mappings associated to nonlinear auxiliiary systems.

Chapter 5

Interpolation of Nonlinear Differential-Algebraic Systems

In this chapter the nonlinear Loewner framework of Chapter 4 is further generalized for the treatment of nonlinear input-affine differential-algebraic systems possessing a feedforward term. Additionally, a parameterized family of interpolants is provided by introducing a feedforward term. The enhancement presented in this chapter is accomplished by further generalizing the definitions of the Loewner functions in such a way that they can be used to construct a "parallelizing" coordinates transformation. The Loewner functions can then be used to construct an interpolant which yields the same Loewner functions and therefore achieves interpolation of the underlying system in the Loewner sense. Locally, the original model and the interpolant produce the same response when interconnected with the same auxiliary sys-

tems associated to the Loewner functions. The enhancement in this chapter relies on the interconnection-based interpretation of Chapter 3 and builds upon the results of Chapter 4.

This chapter is structured as follows. In Section 5.1 the class of nonlinear differential-algebraic systems that the Loewner framework is enhanced to treat in this chapter is presented. In Section 5.2 the definitions of the Loewner functions are enhanced to handle nonlinear descriptor systems possessing a feedforward term, and a set of coordinates, the *Loewner coordinates*, is introduced which reveals the relationship between the Loewner functions and the response of the system. In Section 5.3 a differential-algebraic system is provided which, given the Loewner functions of the underlying system, interpolates the underlying system in the Loewner sense. In Section 5.4 a family of nonlinear Loewner equivalent interpolants is parameterized via the introduction of a feedforward term. In Section 5.5 a demonstrative example is provided wherein the averaged model of a DC-to-DC Ćuk converter is treated in implicit form, and a feedforward mapping is utilized to enforce local exponential stability of the origin for the reduced order model. Finally, in Section 5.6 some concluding remarks are given.

5.1 Problem Formulation

The approach of Chapter 4 has a drawback in that, for Theorems 10 and 12, the Jacobian of the Loewner function must be nonsingular in a neighbour-

hood of the origin. Furthermore, as in [94] and Chapter 3, in the Loewner framework for linear systems one typically studies differential-algebraic systems, or descriptor systems, of the form

$$E\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t),$$

where E is generally a singular matrix, so the system contains algebraic constraints. Furthermore, (E, A) is a regular matrix pencil, so the system is *regular* or *solvable*.

In the following sections a more general class of nonlinear input-affine descriptor systems possessing a feedforward term is considered. This class of systems is described by the equations

$$\dot{\overbrace{E\big(x(t)\big)}} = \left(\frac{\partial E}{\partial x} \circ x(t)\right)\dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t), \quad x(0) = x_0, \tag{5.1}$$
$$y(t) = h\big(x(t)\big) + d\big(x(t)\big)u(t), \tag{5.2}$$

with state $x(t) \in \mathbb{C}^n$, input $u(t) \in \mathbb{C}^m$, output $y(t) \in \mathbb{C}^p$, and mappings $f : \mathbb{C}^n \to \mathbb{C}^n$, $g : \mathbb{C}^n \to \mathbb{C}^{n\times m}$, $h : \mathbb{C}^n \to \mathbb{C}^p$, $d : \mathbb{C}^n \to \mathbb{C}^{p\times m}$, and $E : \mathbb{C}^n \to \mathbb{C}^n$, such that $f(0) = 0$, $h(0) = 0$, and $E(0) = 0$, and an initial condition x_0 which is *consistent* with the input function $u(\cdot)$ in that, locally, there exists at least one continuously differentiable solution for (5.1)-(5.2). It is assumed that $f(\cdot)$ and $E(\cdot)$ are, locally, differentiable, and, in a neighbourhood of the

origin, the Jacobian of $E(\cdot)$ has constant rank but it is not invertible. Hence, systems possessing both differential and algebraic equations are considered and the system (5.1)-(5.2) cannot be manipulated into a standard state-space form.

It is assumed that any system of the form (5.1)-(5.2) is *regular* (with respect to the solution $x(t) = 0$ and $u(t) = 0$). In the nonlinear setting, when saying a system is regular it is meant that locally, in a neighbourhood $\mathbb{X} \times \mathbb{U} \subset \mathbb{C}^n \times \mathbb{C}^m$ of $x = 0$ and $u = 0$, continuously differentiable solutions of the system exist and are unique for every sufficiently smooth input $u(t) \in \mathbb{U}$ and every initial condition $x_0 \in \mathbb{X}$ such that the initial condition x_0 is consistent. Furthermore, it is assumed that the trajectory of the auxiliary system (4.23)-(4.24) is such that $r\big(\zeta_r(t)\big) \in \mathbb{U}$, so the interconnection of a regular system of the form (5.1)-(5.2) with the auxiliary system (4.23)-(4.24), via $u = v$, yields an interconnected system for which there exists locally unique solutions.

Remark 13. The existence and uniqueness theory for general nonlinear implicit systems is technically complex and beyond the scope of this chapter, see *e.g.* [108], [86], [87], or Hypothesis 4.2 and Theorem 4.11 in [88].

Remark 14. As with linear systems having algebraic constraints, the motion of (5.1)-(5.2) need not be continuous, and the states of the system may exhibit discontinuities and impulsive behaviour if the initial conditions are not consistent or the input signal is not sufficiently smooth, *e.g* [88]. This issue is important only when establishing trajectories based results.

The purpose of the following sections is to further enhance the Loewner framework for systems of the form (5.1)-(5.2). First, the Loewner functions of Chapter 4 are generalized to include the "descriptor function", $E(\cdot)$, and the feedforward term, $d(\cdot)$, and a Loewner equivalent model is presented forgoing the assumption that $\frac{\partial \mathbb{L}}{\partial \zeta_r}$ is nonsingular in a neighbourhood of the origin[1]. Then, a feedforward term is introduced to the interpolant, thus parameterizing a family of Loewner equivalent interpolants and giving some degrees of freedom in the selection of the reduced order model.

5.2 Interconnection With Auxiliary Systems

Consider now, as in Chapter 4, two nonlinear auxiliary systems of the form

$$\dot{\zeta}_r(t) = \lambda\big(\zeta_r(t)\big) + \Delta(t), \tag{5.3}$$

$$v(t) = r\big(\zeta_r(t)\big), \tag{5.4}$$

and

$$\dot{\zeta}_\ell(t) = m\big(\zeta_\ell(t)\big) + \ell\big(\chi(t)\big), \tag{5.5}$$

$$\eta(t) = \zeta_\ell(t), \tag{5.6}$$

[1]The Loewner functions are reinterpreted in the context of differential-algebraic systems, hence the proof that the interpolant is Loewner equivalent no longer requires inverting the Jacobian of the Loewner function, and the interpolant can contain algebraic constraints.

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$, and with functions $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, and $\ell(\cdot)$ of appropriate dimensions, and such that $\lambda(0) = 0$, $r(0) = 0$, $m(0) = 0$, $\ell(0) = 0$, and $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, and $\ell(\cdot)$ are differentiable. Let $\Lambda := \left(\frac{\partial \lambda}{\partial \zeta_r} \circ 0\right)$ and $M := \left(\frac{\partial m}{\partial \zeta_\ell} \circ 0\right)$, with Assumption 6 still holding.

An additional mapping, derived from $\ell(\cdot)$, is required due to the addition of the feedforward term in (5.2) and the non-affine choice of $\ell(\cdot)$ in (5.5). By Hadamard's lemma, there exists a function $\overline{\ell} : \mathbb{C}^p \times \mathbb{C}^p \to \mathbb{C}^{v \times p}$ such that

$$\overline{\ell}(b, a)a = \ell(b + a) - \ell(b), \tag{5.7}$$

where

$$\overline{\ell}(b, a) := \int_0^1 \left(\frac{\partial \ell}{\partial \chi} \circ (b + \tau a)\right) d\tau.$$

Furthermore, it follows that

$$\overline{\ell}\big(b, -a\big) = \overline{\ell}\big(b - a, a\big), \tag{5.8}$$

because, with $v := 1 - \tau$,

$$\begin{aligned}\overline{\ell}\big(b - a, a\big) &= \int_0^1 \left(\frac{\partial \ell}{\partial \chi} \circ \big(b - a(1-\tau)\big)\right) d\tau \\ &= \int_0^1 \left(\frac{\partial \ell}{\partial \chi} \circ (b - va)\right) dv \\ &= \overline{\ell}\big(b, -a\big).\end{aligned}$$

Consider the scenario in which the system (5.1)-(5.2) is interconnected with the auxiliary systems (5.3)-(5.4) and (5.5)-(5.6) via the interconnection equations $u = v$ and $\chi = y$. This interconnection, depicted in Figure 5.1, has the differential-algebraic representation

$$\begin{bmatrix} \dot{\zeta}_r \\ \overbrace{E(x)}^{\cdot} \\ \dot{\zeta}_\ell \end{bmatrix} = \begin{bmatrix} \lambda(\zeta_r) \\ f(x) + g(x) r(\zeta_r) \\ m(\zeta_\ell) + \ell\big(h(x) + d(x) r(\zeta_r)\big) \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \Delta, \tag{5.9}$$

$$\eta = \zeta_\ell, \tag{5.10}$$

with state $\begin{bmatrix} \zeta_r^* & x^* & \zeta_\ell^* \end{bmatrix}^*$, input Δ, and output η. The Loewner functions are further enhanced in association with the interconnected system (5.9)-(5.10).

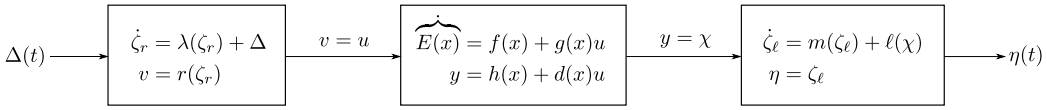

Figure 5.1: The interconnected system (5.9)-(5.10).

5.2.1 Loewner Functions

One can now define further generalizations for the objects arising in the nonlinear Loewner framework in order to accomodate differential-algebraic systems of the form (5.1)-(5.2). These new definitions, which now take into account the additional mappings $E(\cdot)$ and $d(\cdot)$, are defined in terms of the functions comprising the interconnected system (5.9)-(5.10). First, the tangential generalized controllability function, $X : \mathbb{C}^\rho \to \mathbb{C}^n$, is defined as the solution, provided it exists, to the PDE with boundary condition

$$\begin{aligned} \frac{\partial\Big(E\big(X(\zeta_r)\big)\Big)}{\partial\zeta_r}\lambda(\zeta_r) &= \left(\frac{\partial E}{\partial x} \circ X(\zeta_r)\right)\frac{\partial X}{\partial\zeta_r}\lambda(\zeta_r) \\ &= f\big(X(\zeta_r)\big) + g\big(X(\zeta_r)\big)r(\zeta_r), \\ X(0) &= 0, \end{aligned} \tag{5.11}$$

and the tangential generalized observability function, $Y : \mathbb{C}^n \to \mathbb{C}^v$, is defined as the solution, provided it exists, to the PDE with boundary condition

$$\left(\frac{\partial Y}{\partial x} \circ E(x)\right) f(x) = -m\big(-Y(E(x))\big) - \ell\big(h(x)\big),$$
$$Y(0) = 0. \tag{5.12}$$

Enhancements of the right and left tangential data functions, $W(\cdot)$ and $V(\cdot)$, respectively, are defined as

$$W(\zeta_r) := h\big(X(\zeta_r)\big) + d\big(X(\zeta_r)\big) r(\zeta_r), \tag{5.13}$$

and

$$V(\zeta_r) := \left(\frac{\partial Y}{\partial x} \circ E\big(X(\zeta_r)\big)\right) g\big(X(\zeta_r)\big)$$
$$+ \overline{\ell}\Big(h\big(X(\zeta_r)\big),\ d\big(X(\zeta_r)\big) r(\zeta_r)\Big) d\big(X(\zeta_r)\big), \tag{5.14}$$

and enhancements of the Loewner function, $\mathbb{L}(\cdot)$, and the shifted Loewner function, $\sigma\mathbb{L}(\cdot)$, are defined as

$$\mathbb{L}(\zeta_r) := -Y\big(E(X(\zeta_r))\big),$$

and

$$\sigma\mathbb{L}(\zeta_r) := -\left(\frac{\partial Y}{\partial x} \circ E\big(X(\zeta_r)\big)\right) f\big(X(\zeta_r)\big) + \bar{\ell}\Big(h\big(X(\zeta_r)\big),\ d\big(X(\zeta_r)\big)r(\zeta_r)\Big)d\big(X(\zeta_r)\big)r(\zeta_r),$$

respectively. The left Loewner function, $\mathbb{L}^{\ell}(\cdot)$, and the right Loewner function, $\mathbb{L}^{r}(\cdot)$ are unchanged from Chapter 4. That is, the left Loewner function is defined as the solution, provided it exists, to the PDE with boundary condition

$$\frac{\partial \mathbb{L}^{\ell}}{\partial \zeta_r}\lambda(\zeta_r) = -m\big(-\mathbb{L}^{\ell}(\zeta_r)\big) - V(\zeta_r)r(\zeta_r), \qquad \mathbb{L}^{\ell}(0) = 0, \tag{5.15}$$

and the right Loewner function is defined as

$$\mathbb{L}^{r}(\zeta_r) := \mathbb{L}(\zeta_r) - \mathbb{L}^{\ell}(\zeta_r).$$

As in Section 4.3, the following useful relationships hold for the Loewner functions.

Proposition 9. *Given the functions $X(\cdot)$, $Y(\cdot)$, $W(\cdot)$, $V(\cdot)$, $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, and $\ell(\cdot)$, the following hold.*

i) The Loewner function, $\mathbb{L}(\cdot)$, satisfies the PDE with boundary condition

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) - m\big(\mathbb{L}(\zeta_r)\big) = \ell\big(W(\zeta_r)\big) - V(\zeta_r)r(\zeta_r), \quad \mathbb{L}(0) = 0. \tag{5.16}$$

ii) The right Loewner function, $\mathbb{L}^r(\cdot)$, satisfies the PDE with boundary condition

$$\frac{\partial \mathbb{L}^r}{\partial \zeta_r}\lambda(\zeta_r) - m\big(\mathbb{L}(\zeta_r)\big) - m\big(-\mathbb{L}^\ell(\zeta_r)\big) = \ell\big(W(\zeta_r)\big), \quad \mathbb{L}^r(0) = 0.$$

iii) The shifted Loewner function satisfies the equation

$$\sigma\mathbb{L}(\zeta_r) = m\big(\mathbb{L}(\zeta_r)\big) + \ell\big(W(\zeta_r)\big). \tag{5.17}$$

iv) The shifted Loewner function satisfies the equation

$$\sigma\mathbb{L}(\zeta_r) = \frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) + V(\zeta_r)r(\zeta_r). \tag{5.18}$$

v) The shifted Loewner function satisfies the equation

$$\sigma\mathbb{L}(\zeta_r) = -m\big(-\mathbb{L}^\ell(\zeta_r)\big) + \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\lambda(\zeta_r).$$

Proof. The proof proceeds in the same fashion as that of Proposition 8, however properties of the function $\overline{\ell}(\cdot)$ stated earlier are now required. Mul-

tiplying (5.11) by $-\big(\frac{\partial Y}{\partial x} \circ E\big(X(\zeta_r)\big)\big)$ on the left yields

$$\begin{aligned}
\frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) &= -\left(\frac{\partial Y}{\partial x} \circ E\big(X(\zeta_r)\big)\right)\left(\frac{\partial E}{\partial x} \circ X(\zeta_r)\right)\frac{\partial X}{\partial \zeta_r}\lambda(\zeta_r) \\
&= -\left(\frac{\partial Y}{\partial x} \circ E\big(X(\zeta_r)\big)\right)\Big(f\big(X(\zeta_r)\big) + g\big(X(\zeta_r)\big)r(\zeta_r)\Big) \\
&= -\left(\frac{\partial Y}{\partial x} \circ E\big(X(\zeta_r)\big)\right) f\big(X(\zeta_r)\big) \\
&\qquad + \overline{\ell}\Big(h\big(X(\zeta_r)\big),\ d\big(X(\zeta_r)\big)r(\zeta_r)\Big)d\big(X(\zeta_r)\big)r(\zeta_r) \\
&\quad - \left(\frac{\partial Y}{\partial x} \circ E\big(X(\zeta_r)\big)\right) g\big(X(\zeta_r)\big)r(\zeta_r) \\
&\qquad - \overline{\ell}\Big(h\big(X(\zeta_r)\big),\ d\big(X(\zeta_r)\big)r(\zeta_r)\Big)d\big(X(\zeta_r)\big)r(\zeta_r) \\
&= \sigma\mathbb{L}(\zeta_r) - V(\zeta_r)r(\zeta_r),
\end{aligned}$$

which proves *iv*). Substituting $X(\zeta_r)$ into (5.12) yields

$$\begin{aligned}
\sigma\mathbb{L}(\zeta_r) &= -\left(\frac{\partial Y}{\partial x} \circ E\big(X(\zeta_r)\big)\right) f\big(X(\zeta_r)\big) \\
&\qquad + \overline{\ell}\Big(h\big(X(\zeta_r)\big),\ d\big(X(\zeta_r)\big)r(\zeta_r)\Big)d\big(X(\zeta_r)\big)r(\zeta_r) \\
&= m\big(-Y(E(X(\zeta_r)))\big) + \ell\big(h(X(\zeta_r))\big) \\
&\qquad + \overline{\ell}\Big(h\big(X(\zeta_r)\big),\ d\big(X(\zeta_r)\big)r(\zeta_r)\Big)d\big(X(\zeta_r)\big)r(\zeta_r),
\end{aligned}$$

and by recalling (5.7) it follows that

$$\begin{aligned}\sigma\mathbb{L}(\zeta_r) &= m\big(\mathbb{L}(\zeta_r)\big) + \ell\big(h(X(\zeta_r))\big) \\ &\qquad + \ell\big(h(X(\zeta_r)) + d(X(\zeta_r))r(\zeta_r)\big) - \ell\big(h(X(\zeta_r))\big) \\ &= m\big(\mathbb{L}(\zeta_r)\big) + \ell\big(W(\zeta_r)\big),\end{aligned}$$

which proves *iii*). Equating *iii*) and *iv*) yields

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) + V(\zeta_r)r(\zeta_r) = m\big(\mathbb{L}(\zeta_r)\big) + \ell\big(W(\zeta_r)\big),$$

and noting that $X(0) = 0$, $Y(0) = 0$, and $W(0) = 0$ proves *i*). Subtracting (5.15) from (5.16) yields

$$\begin{aligned}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} - \frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\right)\lambda(\zeta_r) - m\big(\mathbb{L}(\zeta_r)\big) - m\big(-\mathbb{L}^\ell(\zeta_r)\big) \\ = \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\lambda(\zeta_r) - m\big(\mathbb{L}(\zeta_r)\big) - m\big(-\mathbb{L}^\ell(\zeta_r)\big) \\ = \ell\big(W(\zeta_r)\big),\end{aligned}$$

and noting that $W(0) = 0$ proves *ii*). Finally, from *iv*) and (5.15) it follows

that

$$\begin{aligned}\sigma\mathbb{L}(\zeta_r) &= \frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) + V(\zeta_r)r(\zeta_r) \\ &= \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\lambda(\zeta_r) + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r}\lambda(\zeta_r) + V(\zeta_r)r(\zeta_r)\right) \\ &= \frac{\partial \mathbb{L}^r}{\partial \zeta_r}\lambda(\zeta_r) - m\big(-\mathbb{L}^\ell(\zeta_r)\big),\end{aligned}$$

proving v). □

Remark 15. The additional mappings $E(\cdot)$ and $d(\cdot)$ do not directly appear in the relationships for $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $\mathbb{L}^\ell(\cdot)$, $\mathbb{L}^r(\cdot)$ in Proposition 9. That is, the Loewner functions $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $\mathbb{L}^\ell(\cdot)$, and $\mathbb{L}^r(\cdot)$ can still be constructed using only the tangential data functions, $W(\cdot)$ and $V(\cdot)$, and the auxiliary system mappings, $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, and $\ell(\cdot)$.

5.2.2 Loewner Coordinates

Using the Loewner functions associated to the interconnected system (5.9)-(5.10), a specific set of coordinates can be chosen which reveals the relationship between the Loewner functions and the response of the interconnected system. Consider the new coordinates

$$\begin{bmatrix} z_r \\ z_c \\ z_\ell \end{bmatrix} := \begin{bmatrix} \zeta_r \\ x - X(\zeta_r) \\ \zeta_\ell + Y\big(E(x)\big) + \mathbb{L}^\ell(\zeta_r) \end{bmatrix}. \tag{5.19}$$

The system in the new coordinates, depicted in Figure 5.2, is described by the equations

$$\dot{z}_r = \lambda(z_r) + \Delta, \tag{5.20}$$

and

$$\left(\frac{\partial E}{\partial x} \circ \big(z_c + X(z_r)\big)\right) \dot{z}_c = \widetilde{A}(z_c, z_r) z_c \\ - \left(\frac{\partial E}{\partial x} \circ \big(z_c + X(z_r)\big)\right) \left(\frac{\partial X}{\partial \zeta_r} \circ z_r\right) \Delta, \tag{5.21}$$

and

$$\begin{aligned} \dot{z}_\ell = {} & \widetilde{M}(z_\ell, z_r) z_\ell \\ & + \Big(m\big(\mathbb{L}^r(z_r)\big) - m\big(\mathbb{L}(z_r)\big) - m\big(-\mathbb{L}^\ell(z_r)\big)\Big) \\ & + \widetilde{G}(z_\ell, z_c, z_r) z_c + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r} \circ z_r\right) \Delta, \end{aligned} \tag{5.22}$$

and

$$\eta = \mathbb{L}^r(z_r) - \widetilde{Y}(z_c, z_r) z_c + z_\ell, \tag{5.23}$$

where $\widetilde{A}(\cdot)$, $\widetilde{M}(\cdot)$, $\widetilde{G}(\cdot)$, and $\widetilde{Y}(\cdot)$ are obtained via application of Hadamard's lemma. Particularly, the function $\widetilde{A} : \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{n \times n}$ is such that

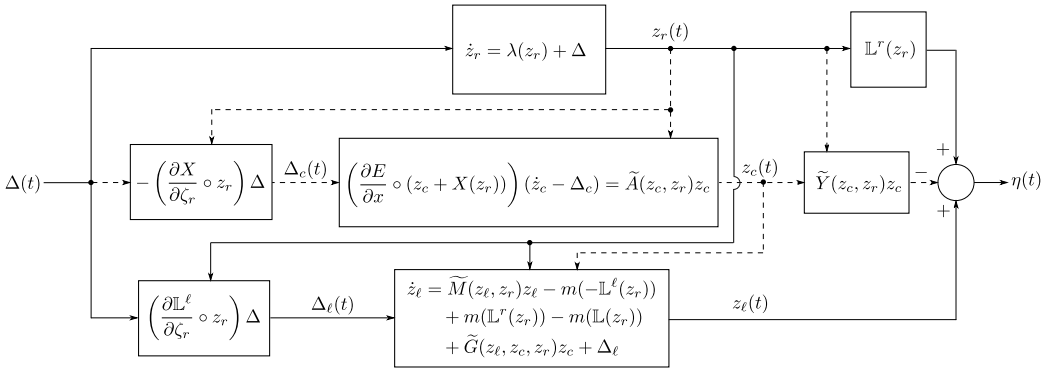

Figure 5.2: The transformed representation of the interconnected system (5.9)-(5.10) in the new coordinates (5.19), given by the equations (5.20), (5.21), (5.22), (5.23).

$$\widetilde{A}(z_c, z_r)z_c = \Big(f(x) - f\big(X(z_r)\big)\Big) + \Big(g(x) - g\big(X(z_r)\big)\Big)r(z_r) - \left(\frac{\partial E}{\partial x} - \left(\frac{\partial E}{\partial x} \circ X(z_r)\right)\right)\left(\frac{\partial X}{\partial \zeta_r} \circ z_r\right)\lambda(z_r),$$

the function $\widetilde{M} : \mathbb{C}^v \times \mathbb{C}^\rho \to \mathbb{C}^{v \times v}$ is such that

$$\widetilde{M}(z_\ell, z_r)z_\ell = m\big(z_\ell + \mathbb{L}^r(z_r)\big) - m\big(\mathbb{L}^r(z_r)\big),$$

the function $\widetilde{G} : \mathbb{C}^v \times \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{v\times n}$ is such that

$$
\begin{aligned}
&\widetilde{G}(z_\ell, z_c, z_r)z_c = \\
&\quad \Big(m\big(z_\ell - Y(E(x)) - \mathbb{L}^\ell(z_r)\big) - m\big(z_\ell + \mathbb{L}^r(z_r)\big)\Big) \\
&\quad + \Big(m\big(\mathbb{L}(z_r)\big) - m\big(-Y(E(x))\big)\Big) \\
&\quad + \left(\left(\frac{\partial Y}{\partial x} \circ E(x)\right) g\big(x\big) - \left(\frac{\partial Y}{\partial x} \circ E\big(X(z_r)\big)\right) g\big(X(z_r)\big)\right) r(z_r) \\
&\quad + \Big(\overline{\ell}\Big(h(x),\ d(x)r(z_r)\Big)d(x) \\
&\qquad - \overline{\ell}\Big(h\big(X(z_r)\big),\ d\big(X(z_r)\big)r(z_r)\Big)d\big(X(z_r)\big)\Big)r(z_r),
\end{aligned}
$$

and the function $\widetilde{Y} : \mathbb{C}^n \times \mathbb{C}^\rho \to \mathbb{C}^{v\times n}$ is such that

$$
\widetilde{Y}(z_c, z_r)z_c = Y\big(E(x)\big) - Y\big(E(X(z_r))\big).
$$

If the system in the new coordinates is, locally, regular with respect to the origin (which follows by the assumption that $X\big(\zeta_r(t)\big) \in \mathbb{X}$ and $r\big(\zeta_r(t)\big) \in \mathbb{U}$), then $z_c(0) = 0$ and $\Delta(t) = 0$ for $t \geq 0$ imply that the solution $z_c(t) = 0$ for $t \geq 0$ is unique. Hence, the manifold given by $z_c = 0$, or $x = X(\zeta_r)$, is invariant. It is now easy to see that the interconnected system restricted to the invariant manifold $x = X(z_r)$, or $z_c = 0$, has simplified dynamics given

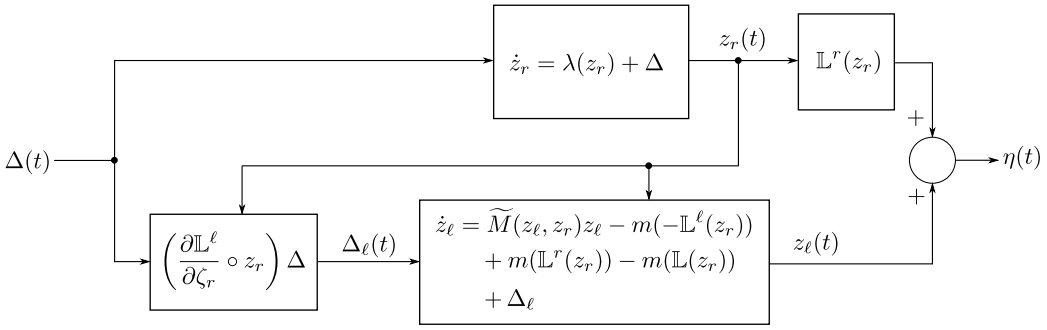

Figure 5.3: The simplified dynamics of the interconnected system (5.9)-(5.10) when restricted to the manifold $x = X(\zeta_r)$, given by the equations (5.24), (5.25), (5.26).

by the equations

$$\dot{z}_r = \lambda(z_r) + \Delta, \tag{5.24}$$

$$\dot{z}_\ell = m\big(z_\ell + \mathbb{L}^r(z_r)\big) - m\big(\mathbb{L}(z_r)\big) - m\big(-\mathbb{L}^\ell(z_r)\big) + \left(\frac{\partial \mathbb{L}^\ell}{\partial \zeta_r} \circ z_r\right) \Delta, \tag{5.25}$$

$$\eta = \mathbb{L}^r(z_\ell) + z_\ell, \tag{5.26}$$

thus the system restricted to the manifold $x = X(\zeta_r)$ only contains information given by the Loewner functions and the auxiliary systems. The simplified dynamics that result from restricting the interconnected system to the manifold $x = X(\zeta_r)$ are depicted in Figure 5.3.

Remark 16. The simplied dynamics of the interconnected system in the setting of differential-algebraic systems are identical to the simplified dynamics

for nonlinear systems of ordinary differential equations given in Chapter 4, thus the previous definitions of Loewner equivalence and interpolation of the tangential data given in Chapter 4 are still appropriate in the differential-algebraic systems context.

Remark 17. One explanation for the simplified dynamics being identical for DAEs and ODEs is that the Loewner framework is based solely on input-output behaviour (with respect to the interconnection with auxiliary systems); the Loewner functions can be constructed only from tangential data and thus do not depend on any particular representation of internal system dynamics. As long as the motion of the internal system is sufficiently smooth, when simply observing input-output trajectories of the system one cannot distinguish an internal DAE representation from an internal ODE representation because regular DAEs with admissible initial conditions can be reduced to lower order ODEs.

5.3 Loewner Equivalent Model

A nonlinear system can now be constructed which is Loewner equivalent at (λ, r, m, ℓ) to (5.1)-(5.2), and which interpolates the tangential data mappings, given that the Loewner functions for the interconnected system (5.9)-(5.10) are known. The following theorem is a more general version of Theorem 12.

Theorem 13. *Consider the interconnected system* (5.9)-(5.10) *with* $\rho = v$. *Let* $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $V(\cdot)$, *and* $W(\cdot)$ *be the associated Loewner functions, and suppose that the Jacobian of* $\mathbb{L}(\cdot)$ *has constant rank in a neighbourhood of the origin. Define the system*

$$\dot{\widetilde{\mathbb{L}(\omega)}} = \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right) \dot{\omega} = \sigma\mathbb{L}(\omega) - V(\omega)u_r, \tag{5.27}$$

$$y_r = W(\omega), \tag{5.28}$$

with state $\omega(t) \in \mathbb{C}^{\rho}$, *input* $u_r(t) \in \mathbb{C}^{m}$, *and output* $y_r(t) \in \mathbb{C}^{p}$. *Then the system* (5.27)-(5.28) *is Loewner equivalent at* (λ, r, m, ℓ) *to the system* (5.1)-(5.2) *and matches the tangential data functions* $W(\cdot)$ *and* $V(\cdot)$.

Remark 18. In Theorem 13 the presented Loewner equivalent model no longer requires that the Jacobian of the Loewner function be nonsingular in a neighbourhood of the origin. That is, rather than an implicitly defined set of ODEs, the system (5.27)-(5.28) is a more general descriptor system which may contain algebraic constraints and cannot necessarily be put into standard state-space form. This follows as a result of the further generalization of the tangential generalized controllability and observability functions and the Loewner functions in Section 5.2.

Proof. Let $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{V}(\cdot)$, $\overline{W}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\mathbb{L}}^{\ell}(\cdot)$, $\overline{\mathbb{L}}^{r}(\cdot)$, and $\sigma\overline{\mathbb{L}}(\cdot)$ be the set of Loewner functions for the system (5.27)-(5.28) interconnected with the auxiliary systems (5.3)-(5.4) and (5.5)-(5.6). The function $\overline{X}(\cdot)$ is the solution

to the PDE with boundary condition

$$\frac{\partial \mathbb{L}\big(\overline{X}(\zeta_r)\big)}{\partial \zeta_r}\lambda(\zeta_r) = \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right) \frac{\partial \overline{X}}{\partial \zeta_r}\lambda(\zeta_r)$$
$$= \sigma \mathbb{L}\big(\overline{X}(\zeta_r)\big) - V\big(\overline{X}(\zeta_r)\big) r(\zeta_r), \quad \overline{X}(0) = 0, \quad (5.29)$$

and letting $\overline{X}(\zeta_r) = \zeta_r$ yields

$$\frac{\partial \mathbb{L}}{\partial \zeta_r}\lambda(\zeta_r) = \sigma \mathbb{L}(\zeta_r) - V(\zeta_r) r(\zeta_r),$$

which holds by (5.18). The function $\overline{Y}(\cdot)$ is the solution to the PDE with boundary condition

$$\left(\frac{\partial \overline{Y}}{\partial \omega} \circ \mathbb{L}(\omega)\right) \sigma \mathbb{L}(\omega) = -m\big(-\overline{Y}(\mathbb{L}(\omega))\big) - \ell\big(W(\omega)\big),$$
$$\overline{Y}(0) = 0, \quad (5.30)$$

and letting $\overline{Y}(\omega) = -\omega$ yields

$$-\sigma \mathbb{L}(\omega) = -m\big(\mathbb{L}(\omega)\big) - \ell\big(W(\omega)\big),$$

which holds by (5.17). The function $\overline{\mathbb{L}}^{\ell}(\cdot)$ is the solution to the PDE with

boundary condition

$$\frac{\partial \overline{\mathbb{L}}^{\ell}}{\partial \zeta_r}\lambda(\zeta_r) = -m\big(-\overline{\mathbb{L}}^{\ell}(\zeta_r)\big) + \left(\frac{\partial \overline{Y}}{\partial \omega} \circ \mathbb{L}\big(\overline{X}(\zeta_r)\big)\right) V\big(\overline{X}(\zeta_r)\big) r(\zeta_r),$$
$$\overline{\mathbb{L}}^{\ell}(0) = 0, \tag{5.31}$$

and letting $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\omega$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ obtains

$$\frac{\partial \mathbb{L}^{\ell}}{\partial \zeta_r}\lambda(\zeta_r) = -m\big(-\mathbb{L}^{\ell}(\zeta_r)\big) - V(\zeta_r) r(\zeta_r),$$

which holds by (5.15). Thus $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\omega$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ is a solution to the system of PDEs (5.29), (5.30), and (5.31). It follows that the Loewner function and the right Loewner function are

$$\overline{\mathbb{L}}(\zeta_r) = -\overline{Y}\big(\mathbb{L}(\overline{X}(\zeta_r))\big) = \mathbb{L}(\zeta_r),$$
$$\overline{\mathbb{L}}^{r}(\zeta_r) = \mathbb{L}^{r}(\zeta_r),$$

respectively, hence $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ and $\overline{\mathbb{L}}^{r}(\zeta_r) = \mathbb{L}^{r}(\zeta_r)$, and (5.27)-(5.28) is Loewner equivalent to (5.1)-(5.2) at (λ, r, m, ℓ). Finally, note that

$$\sigma\overline{\mathbb{L}}(\zeta_r) = -\left(\frac{\partial \overline{Y}}{\partial \omega} \circ \mathbb{L}\big(\overline{X}(\zeta_r)\big)\right) \sigma\mathbb{L}\big(\overline{X}(\zeta_r)\big) = \sigma\mathbb{L}(\zeta_r),$$

and

$$\overline{W}(\zeta_r) = W\big(\overline{X}(\zeta_r)\big) = W(\zeta_r),$$

and

$$\overline{V}(\zeta_r) = -\left(\frac{\partial \overline{Y}}{\partial \omega} \circ \mathbb{L}\big(\overline{X}(\zeta_r)\big)\right) V\big(\overline{X}(\zeta_r)\big) = V(\zeta_r),$$

hence the system (5.27)-(5.28) also matches the tangential data functions $W(\cdot)$ and $V(\cdot)$. □

5.4 Parameterization of Interpolants via a Feedforward Term

The authors of [94] utilize a feedforward term to provide the parameterized family of systems, (2.30)-(2.31), each element of which interpolates the linear descriptor system (2.17)-(2.18). Such a parameterization provides extra degrees of freedom when the goal is to select an interpolant with some desired property, *e.g.* stability or passivity. Having enhanced the Loewner framework to treat nonlinear differential-algebraic systems possessing a feedforward term, in this section a similar parameterization is achieved for the nonlinear interpolant (5.27)-(5.28) using an input-affine feedforward term.

Theorem 14. *Consider the interconnected system* (5.9)-(5.10) *with* $\rho = v$.

Let $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $V(\cdot)$, *and* $W(\cdot)$ *be the associated Loewner functions, and suppose the Jacobian of* $\mathbb{L}(\cdot)$ *has constant rank in a neighbourhood of the origin. Define the system*

$$\begin{aligned}\dot{\widetilde{\mathbb{L}(\omega)}} &= \Big[\sigma\mathbb{L}(\omega) - \overline{\ell}\Big(W(\omega),\ -\overline{d}(\omega)r(\omega)\Big)\overline{d}(\omega)r(\omega)\Big] \\ &\qquad - \Big[V(\omega) - \overline{\ell}\Big(W(\omega),\ -\overline{d}(\omega)r(\omega)\Big)\overline{d}(\omega)\Big]u_r, \qquad (5.32)\\ y_r &= \Big[W(\omega) - \overline{d}(\omega)r(\omega)\Big] + \overline{d}(\omega)u_r, \qquad (5.33)\end{aligned}$$

with state $\omega(t) \in \mathbb{C}^{\rho}$, *input* $u_r(t) \in \mathbb{C}^{m}$, *and output* $y_r(t) \in \mathbb{C}^{p}$, *and with* $\overline{d} : \mathbb{C}^{\rho} \to \mathbb{C}^{p\times m}$. *Then the system* (5.32)-(5.33) *is Loewner equivalent at* (λ, r, m, ℓ) *to the system* (5.1)-(5.2) *and matches the tangential data functions* $W(\cdot)$ *and* $V(\cdot)$.

Proof. Let $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{V}(\cdot)$, $\overline{W}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\mathbb{L}}^{\ell}(\cdot)$, $\overline{\mathbb{L}}^{r}(\cdot)$, and $\sigma\overline{\mathbb{L}}(\cdot)$ be the set of Loewner functions for the system (5.32)-(5.33) interconnected with the auxiliary systems (5.3)-(5.4) and (5.5)-(5.6). To prove that (5.32)-(5.33) is Loewner equivalent at (λ, r, m, ℓ) to (5.1)-(5.2) it is shown that $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\omega) = -\omega$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ is a solution to the interpolant's system of PDEs corresponding to (5.11), (5.12), and (5.15). The function $\overline{X}(\cdot)$ is the

solution to the PDE with boundary condition

$$\begin{aligned}
&\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \overline{X}(\zeta_r)\right) \frac{\partial \overline{X}}{\partial \zeta_r} \lambda(\zeta_r) \\
&\quad = \sigma \mathbb{L}\big(\overline{X}(\zeta_r)\big) - V\big(\overline{X}(\zeta_r)\big) r(\zeta_r) \\
&\qquad - \overline{\ell}\Big(W\big(\overline{X}(\zeta_r)\big),\ -\overline{d}\big(\overline{X}(\zeta_r)\big) r\big(\overline{X}(\zeta_r)\big)\Big) \overline{d}\big(\overline{X}(\zeta_r)\big) r\big(\overline{X}(\zeta_r)\big) \\
&\qquad + \overline{\ell}\Big(W\big(\overline{X}(\zeta_r)\big),\ -\overline{d}\big(\overline{X}(\zeta_r)\big) r\big(\overline{X}(\zeta_r)\big)\Big) \overline{d}\big(\overline{X}(\zeta_r)\big) r(\zeta_r), \\
&\overline{X}(0) = 0,
\end{aligned}$$

and letting $\overline{X}(\zeta_r) = \zeta_r$ yields

$$\frac{\partial \mathbb{L}}{\partial \zeta_r} \lambda(\zeta_r) = \sigma \mathbb{L}(\zeta_r) - V(\zeta_r) r(\zeta_r),$$

which holds by (5.18). The function $\overline{Y}(\cdot)$ is the solution to the PDE with boundary condition

$$\begin{aligned}
&\left(\frac{\partial \overline{Y}}{\partial \omega} \circ \mathbb{L}(\omega)\right) \left(\sigma \mathbb{L}(\omega) - \overline{\ell}\Big(W(\omega),\ -\overline{d}(\omega) r(\omega)\Big) \overline{d}(\omega) r(\omega)\right) \\
&\quad = -m\big(-\overline{Y}(\mathbb{L}(\omega))\big) - \ell\big(W(\omega) - \overline{d}(\omega) r(\omega)\big), \\
&\overline{Y}(0) = 0,
\end{aligned}$$

and letting $\overline{Y}(\omega) = -\omega$ yields

$$\begin{aligned}
&\sigma\mathbb{L}(\omega) - m\big(\mathbb{L}(\omega)\big) \\
&\qquad = \overline{\ell}\Big(W(\omega),\ -\overline{d}(\omega)r(\omega)\Big)\overline{d}(\omega)r(\omega) + \ell\big(W(\omega) - \overline{d}(\omega)r(\omega)\big) \\
&\qquad = \ell\big(W(\omega)\big),
\end{aligned}$$

which holds by (5.7) and (5.17). It follows that

$$\overline{\mathbb{L}}(\zeta_r) = -\overline{Y}\big(\mathbb{L}(\overline{X}(\zeta_r))\big) = \mathbb{L}(\zeta_r),$$

and

$$\begin{aligned}
\overline{W}(\zeta_r) &= \Big(W\big(\overline{X}(\zeta_r)\big) - \overline{d}\big(\overline{X}(\zeta_r)\big)r\big(\overline{X}(\zeta_r)\big)\Big) + \overline{d}\big(\overline{X}(\zeta_r)\big)r(\zeta_r) \\
&= W(\zeta_r).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\sigma\overline{\mathbb{L}}(\zeta_r) &= m\big(\overline{\mathbb{L}}(\zeta_r)\big) + \ell\big(\overline{W}(\zeta_r)\big) \\
&= m\big(\mathbb{L}(\zeta_r)\big) + \ell\big(W(\zeta_r)\big) \\
&= \sigma\mathbb{L}(\zeta_r),
\end{aligned}$$

and $\overline{V}(\cdot)$ is given by

$$\begin{aligned}\overline{V}(\zeta_r) = &-\left(\frac{\partial \overline{Y}}{\partial \omega} \circ \mathbb{L}\big(\overline{X}(\zeta_r)\big)\right) V\big(\overline{X}(\zeta_r)\big) \\ &+ \left(\frac{\partial \overline{Y}}{\partial \omega} \circ \mathbb{L}\big(\overline{X}(\zeta_r)\big)\right) \overline{\ell}\Big(W\big(\overline{X}(\zeta_r)\big),\ -\overline{d}\big(\overline{X}(\zeta_r)\big) r\big(\overline{X}(\zeta_r)\big)\Big) \overline{d}\big(\overline{X}(\zeta_r)\big) \\ &+ \overline{\ell}\Big(W\big(\overline{X}(\zeta_r)\big) - \overline{d}\big(\overline{X}(\zeta_r)\big) r\big(\overline{X}(\zeta_r)\big),\ \overline{d}\big(\overline{X}(\zeta_r)\big) r(\zeta_r)\Big) \overline{d}\big(\overline{X}(\zeta_r)\big).\end{aligned}$$

Then, by (5.8),

$$\begin{aligned}\overline{V}(\zeta_r) &= V(\zeta_r) - \overline{\ell}\Big(W(\zeta_r),\ -\overline{d}(\zeta_r) r(\zeta_r)\Big) \overline{d}(\zeta_r) \\ &\qquad + \overline{\ell}\Big(W(\zeta_r) - \overline{d}(\zeta_r) r(\zeta_r),\ \overline{d}(\zeta_r) r(\zeta_r)\Big) \overline{d}(\zeta_r) \\ &= V(\zeta_r).\end{aligned}$$

The function $\overline{\mathbb{L}}^{\ell}(\cdot)$ is the solution to the PDE with boundary condition

$$\begin{aligned}\frac{\partial \overline{\mathbb{L}}^{\ell}}{\partial \zeta_r} \lambda(\zeta_r) &= -m\big(-\overline{\mathbb{L}}^{\ell}(\zeta_r)\big) - \overline{V}(\zeta_r) r(\zeta_r), \\ \overline{\mathbb{L}}^{\ell}(0) &= 0,\end{aligned}$$

and letting $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ yields

$$\frac{\partial \mathbb{L}^{\ell}}{\partial \zeta_r} \lambda(\zeta_r) = -m\big(-\mathbb{L}^{\ell}(\zeta_r)\big) - V(\zeta_r) r(\zeta_r),$$

which holds by (5.15), hence it follows that

$$\begin{aligned}\overline{\mathbb{L}}^{r}(\zeta_r) &= \overline{\mathbb{L}}(\zeta_r) - \overline{\mathbb{L}}^{\ell}(\zeta_r) \\ &= \mathbb{L}(\zeta_r) - \mathbb{L}^{\ell}(\zeta_r) \\ &= \mathbb{L}^{r}(\zeta_r).\end{aligned}$$

Thus, $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ and $\overline{\mathbb{L}}^{r}(\zeta_r) = \mathbb{L}^{r}(\zeta_r)$, and (5.32)-(5.33) is Loewner equivalent to (5.1)-(5.2) at (λ, r, m, ℓ). Furthermore, the system (5.32)-(5.33) matches the tangential data functions $W(\cdot)$ and $V(\cdot)$. □

Note that the function $\overline{d} : \mathbb{C}^{\rho} \to \mathbb{C}^{p\times m}$ is unconstrained (as long as the system (5.32)-(5.33) is regular) so the interpolant (5.32)-(5.33) now has $p \times m$ free parameters.

5.5 Example - Treating the DC-to-DC Ćuk Converter in Implicit Form, and Enforcing LES Using a Feedforward Term

Recall, from Section 4.4, the averaged model of the DC-to-DC Ćuk converter in the implicit form

$$E\dot{x} = Ax + (B_0 + B_1 x)\, u, \qquad y_{FOM} = Cx, \tag{5.34}$$

where

$$x := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad E := \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}, \qquad A := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -G \end{bmatrix},$$

and

$$B_0 := \overline{E} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \qquad B_1 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C := \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

This implicit model can now be treated directly without rearranging the model into an explicit state-space form. Consider again constructing an interpolant associated to the auxiliary systems (5.3)-(5.4) and (5.5)-(5.6) with mappings given by

$$\lambda(\zeta_r) = 0, \qquad r(\zeta_r) = \zeta_r, \qquad m(\zeta_\ell) = m\zeta_\ell, \qquad \ell(\chi) = \chi, \qquad m \in \mathbb{R}.$$

Then the tangential generalized controllability function, $X(\cdot)$, defined as the solution to the PDE with boundary condition (5.11), is the solution to the

equation

$$0 = AX(\zeta_r) + \big(B_0 + B_1 X(\zeta_r)\big)\zeta_r, \qquad X(0) = 0,$$

so it follows that

$$X(\zeta_r) = -(A + B_1\zeta_r)^{-1} B_0 \zeta_r = \begin{bmatrix} G\zeta_r/(\zeta_r - 1) \\ -1 \\ G \\ 1 \end{bmatrix} \frac{\overline{E}\zeta_r}{\zeta_r - 1},$$

and the right tangential data mapping, $W(\cdot)$, defined via (5.13), is

$$W(\zeta_r) = CX(\zeta_r) = \frac{\overline{E}\zeta_r}{\zeta_r - 1}.$$

The tangential generalized observability function, $Y(\cdot)$, defined as the solution to the PDE with boundary condition (5.12), is the solution to

$$\left(\frac{\partial Y}{\partial x} \circ Ex\right) Ax = mY(Ex) - Cx, \qquad Y(0) = 0.$$

This PDE is solved by the linear mapping $Y(x) = Yx$, where Y is the unique solution, by assumption, to the generalized Sylvester equation

$$YA = mYE - C,$$

hence

$$Y = C(mE - A)^{-1}.$$

That is,

$$Y(x) = (1 + mGL_3 + m^2L_3C_4)^{-1} \begin{bmatrix} 0 & 0 & 1 & mL_3 \end{bmatrix} x,$$

and the left tangential data mapping, $V(\cdot)$, defined via (5.14), is

$$\begin{aligned} V(\zeta_r) &= \left(\frac{\partial Y}{\partial x} \circ EX(\zeta_r) \right) \big(B_0 + B_1 X(\zeta_r) \big) \\ &= YB_0 + YB_1X(\zeta_r) \\ &= \frac{\overline{E}}{(1 + mGL_3 + m^2L_3C_4)(\zeta_r - 1)}. \end{aligned}$$

Note that while the tangential generalized observability function is different than in Section 4.4, the tangential generalized controllability function, and the right and left tangential data mappings, are identical to what they were before. It follows that the Loewner function, $\mathbb{L}(\cdot)$, and the shifted Loewner function, $\sigma\mathbb{L}(\cdot)$, are unchanged. That is

$$\mathbb{L}(\zeta_r) = -\frac{\overline{E}L_3(G + mC_4)\zeta_r}{(1 + mGL_3 + m^2L_3C_4)(\zeta_r - 1)},$$

and

$$\sigma\mathbb{L}(\zeta_r) = \frac{\overline{E}\zeta_r}{(1 + mGL_3 + m^2L_3C_4)(\zeta_r - 1)}.$$

An interpolant of the tangential data mappings and the Loewner functions generated by the DC-to-DC Ćuk converter (5.34) at (λ, r, m, ℓ), obtained using Theorem 13 and given in implicit form, is

$$\dot{\widetilde{\mathbb{L}(\omega)}} = \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)\dot{\omega} = \sigma\mathbb{L}(\omega) - V(\omega)u, \qquad y_r = W(\omega),$$

or

$$\begin{aligned}\frac{\overline{E}L_3(G + mC_4)}{(1 + mGL_3 + m^2L_3C_4)}\frac{\dot{\omega}}{(\omega - 1)^2} & \\ = \frac{\overline{E}}{(1 + mGL_3 + m^2L_3C_4)}\frac{\omega}{(\omega - 1)} & \\ - \frac{\overline{E}}{(1 + mGL_3 + m^2L_3C_4)}\frac{1}{(\omega - 1)}u, & \quad (5.35)\end{aligned}$$

$$y_r = \frac{\overline{E}\omega}{\omega - 1}, \qquad \omega \neq 1, \quad (5.36)$$

with state $\omega(t) \in \mathbb{R}$, input $u(t) \in \mathbb{R}$, and output $y_r(t) \in \mathbb{R}$. Note that, by arranging the model into an explicit state-space form, the interpolant (5.35)-(5.36) is equivalent to the interpolant (4.49)-(4.50) of Section 4.4.

If $m < -\frac{G}{C_4}$ then the reduced order model (5.35)-(5.36) does not have a locally exponentially stable equilibrium point at the origin. This issue can

now be resolved by making use of the free parameters given by constructing an interpolant with a feedforward mapping. Another interpolant of the tangential data mappings and the Loewner functions, obtained using Theorem 14, is given by

$$\begin{aligned}
\overset{\cdot}{\mathbb{L}(\omega)} &= \sigma\mathbb{L}(\omega) - \overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)r(\omega) \\
&\qquad - \Big(V(\omega) - \overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)\Big)u \\
&= \sigma\mathbb{L}(\omega) - \overline{d}(\omega)\omega - \big(V(\omega) - \overline{d}(\omega)\big)\,u, \\
y_{ROM} &= W(\omega) - \overline{d}(\omega)r(\omega) + \overline{d}(\omega)u \\
&= W(\omega) - \overline{d}(\omega)\omega + \overline{d}(\omega)u,
\end{aligned}$$

with state $\omega(t) \in \mathbb{R}$, input $u(t) \in \mathbb{R}$, and output $y_{ROM}(t) \in \mathbb{R}$, where the free mapping, $\overline{d}(\cdot)$, is selected such that

$$\overline{d}(\omega) := \frac{c\overline{E}L_3(G + mC_4) + \overline{E}(\omega - 1)}{(1 + mGL_3 + m^2L_3C_4)(\omega - 1)^2}, \qquad c \in \mathbb{R}, \qquad \omega \neq 1.$$

Then the reduced order model is given by

$$\begin{aligned}
&\frac{\overline{E}L_3(G+mC_4)}{(1+mGL_3+m^2L_3C_4)}\frac{\dot{\omega}}{(\omega-1)^2} \\
&\qquad = \frac{\overline{E}}{(1+mGL_3+m^2L_3C_4)}\frac{\omega}{(\omega-1)} \\
&\qquad\quad - \frac{(c\overline{E}L_3(G+mC_4)+\overline{E}(\omega-1))}{(1+mGL_3+m^2L_3C_4)}\frac{\omega}{(\omega-1)^2} \\
&\qquad\quad - \left(\frac{\overline{E}}{(1+mGL_3+m^2L_3C_4)}\frac{1}{(\omega-1)}\right. \\
&\qquad\qquad \left. - \frac{(c\overline{E}L_3(G+mC_4)+\overline{E}(\omega-1))}{(1+mGL_3+m^2L_3C_4)}\frac{1}{(\omega-1)^2}\right)u,
\end{aligned}$$

$$y_{ROM} = \frac{\overline{E}}{(\omega-1)}\left(\omega - \frac{(cL_3(G+mC_4)+\omega-1)}{(1+mGL_3+m^2L_3C_4)}\frac{(\omega-u)}{(\omega-1)}\right).$$

Note that, by rearranging, this model is equivalent to

$$\frac{\dot{\omega}}{(\omega-1)^2} = -\frac{c\omega}{(\omega-1)^2} + \frac{c}{(\omega-1)^2}u, \tag{5.37}$$

$$y_{ROM} = \frac{\overline{E}}{(\omega-1)}\left(\omega - \frac{(cL_3(G+mC_4)+\omega-1)}{(1+mGL_3+m^2L_3C_4)}\frac{(\omega-u)}{(\omega-1)}\right), \tag{5.38}$$

which is an interpolant of the tangential data mappings and the Loewner functions at (λ, r, m, ℓ) that has a locally exponentially stable equilibrium point at the origin for $c > 0$.

The effect of matching the tangential data mappings $W(\cdot)$ and $V(\cdot)$ at

(λ, r, m, ℓ) can be demonstrated by noting, as a generalization of the discussion in Section 4.4, that

$$\begin{aligned}
\ell\big(h(x)\big) &= -\left(\frac{\partial Y}{\partial x} \circ E(x)\right) f(x) - m\big(-Y(E(x))\big) \\
&= -\left(\frac{\partial Y}{\partial x} \circ E(x)\right) \dot{\overbrace{E(x)}} + \left(\frac{\partial Y}{\partial x} \circ E(x)\right) g(x)u - m\big(-Y(E(x))\big) \\
&= -\dot{\overbrace{Y(E(x))}} - m\big(-Y(E(x))\big) + \left(\frac{\partial Y}{\partial x} \circ E(x)\right) g(x)u,
\end{aligned}$$

for any system of the form (5.1)-(5.2), hence

$$\begin{aligned}
&\ell(y) + \dot{\overbrace{Y\big(E(x)\big)}} + m\big(-Y(E(x))\big) \\
&\qquad = \ell\big(h(x) + d(x)u\big) + \dot{\overbrace{Y\big(E(x)\big)}} + m\big(-Y(E(x))\big) \\
&\qquad = \overline{\ell}\big(h(x),\ d(x)u\big)d(x)u + \ell\big(h(x)\big) + \dot{\overbrace{Y\big(E(x)\big)}} + m\big(-Y(E(x))\big) \\
&\qquad = \left[\left(\frac{\partial Y}{\partial x} \circ E(x)\right) g(x) + \overline{\ell}\big(h(x),\ d(x)u\big)d(x)\right] u. \qquad (5.39)
\end{aligned}$$

When $u = r(\zeta_r)$, and when the systems (5.34) and (5.37)-(5.38) are locally asymptotically stable, then for sufficiently small initial conditions it follows that $\lim_{t\to\infty}\big(x(t) - X(\zeta_r(t))\big) = 0$ and $\lim_{t\to\infty}\big(\omega(t) - \zeta_r(t)\big) = 0$. If, in

addition, the systems are single-input, define the mappings

$$V_{t,FOM}(t) := \frac{\ell\big(y_{FOM}(t)\big) + \overbrace{Y\big(E(x(t))\big)}^{\cdot} + m\big(-Y(E(x(t)))\big)}{r\big(\zeta_r(t)\big)},\ r\big(\zeta_r(t)\big) \neq 0,$$

and

$$V_{t,ROM}(t) := \frac{\ell\big(y_{ROM}(t)\big) - \overbrace{\mathbb{L}(\omega(t))}^{\cdot} + m\big(\mathbb{L}(\omega(t))\big)}{r\big(\zeta_r(t)\big)},\ r\big(\zeta_r(t)\big) \neq 0.$$

Hence, if the tangential data mappings are matched, and if the systems are locally asymptotically stable, then for sufficiently small initial conditions it follows that

$$\lim_{t\to\infty} y_{FOM}(t) = \lim_{t\to\infty} y_{ROM}(t) = W\big(\zeta_r(t)\big),$$

and, by (5.39),

$$\lim_{t\to\infty} V_{t,FOM}(t) = \lim_{t\to\infty} V_{t,ROM}(t) = V\big(\zeta_r(t)\big).$$

Consider the full order model of the DC-to-DC Ćuk converter (5.34), and the reduced order model (5.37)-(5.38), with the parameters

$$L_1 = C_2 = L_3 = C_4 = \overline{E} = 1, \qquad G = 2.$$

If $m < -\frac{G}{C_4} = -2$ then the interpolant (5.35)-(5.36) (with no feedforward mapping) is not locally exponentially stable at the origin. The reduced order model (5.37)-(5.38) (with a feedforward mapping) is

$$\frac{\dot{\omega}}{(\omega-1)^2} = \frac{-c\omega}{(\omega-1)^2} + \frac{cu}{(\omega-1)^2}$$

$$y_{ROM} = \frac{1}{(\omega-1)}\left(\omega - \frac{\big((2+m)c+\omega-1\big)(\omega-u)}{(1+2m+m^2)(\omega-1)}\right).$$

Consider $m = -3$, so that the interpolant (5.37)-(5.38) is

$$\frac{\dot{\omega}}{(\omega-1)^2} = \frac{-c\omega}{(\omega-1)^2} + \frac{cu}{(\omega-1)^2},$$

$$y_{ROM} = \frac{1}{(\omega-1)}\left(\omega - \frac{(\omega-c-1)(\omega-u)}{4(\omega-1)}\right).$$

Finally, selecting $c = \frac{1}{5} > 0$ yields

$$\frac{\dot{\omega}}{(\omega-1)^2} = \frac{-\omega+u}{5(\omega-1)^2},$$

$$y_{ROM} = \frac{1}{(\omega-1)}\left(\omega - \frac{(\omega-\frac{6}{5})(\omega-u)}{4(\omega-1)}\right),$$

which is an interpolant of the tangential data mappings and the Loewner functions at (λ, r, m, ℓ) with a locally exponentially stable equilibrium point at the origin.

Figures 5.4 and 5.5 show the responses of the full order model and the

reduced order model to a piecewise constant input signal. The responses of the systems are consistent with the tangential data mappings.

5.6 Conclusion

This chapter has presented an extension of the Loewner framework for nonlinear input-affine systems of ordinary differential equations, developed in Chapter 4, to nonlinear differential-algebraic systems possessing a feedforward term. The Loewner objects are generalized, once again, to allow for the construction of a "parallelizing" coordinates transformation when the differential-algebraic system is interconnected with two auxiliary systems. Using the Loewner functions, an interpolant which yields the same Loewner functions when interconnected with the same two auxiliary systems is constructed. Locally, and under mild conditions, the response of the plant interconnected with the auxiliary systems and the response of the interpolant interconnected with the auxiliary systems are the same, provided the responses exist. A feedforward term is added to the interpolant in order to parameterize a family of interpolants, providing extra degrees of freedom which can be used when the goal is to construct an interpolant equiped with particular properties. Finally, to demonstrate the results of the chapter, the averaged model of a DC-to-DC Ćuk converter is considered again; the model is treated in implicit form, and the feedforward term is leveraged to enforce a local exponential stability property for the reduced order model.

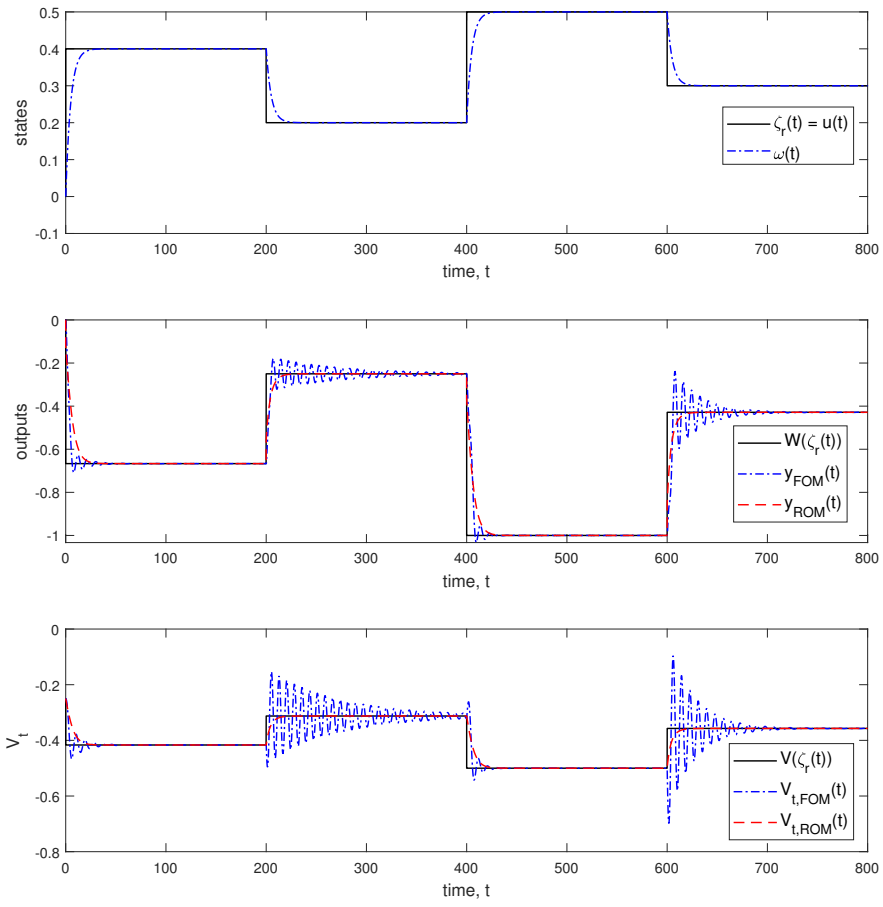

Figure 5.4: Response of the full order model (5.34) and of the reduced order model (5.37)-(5.38).

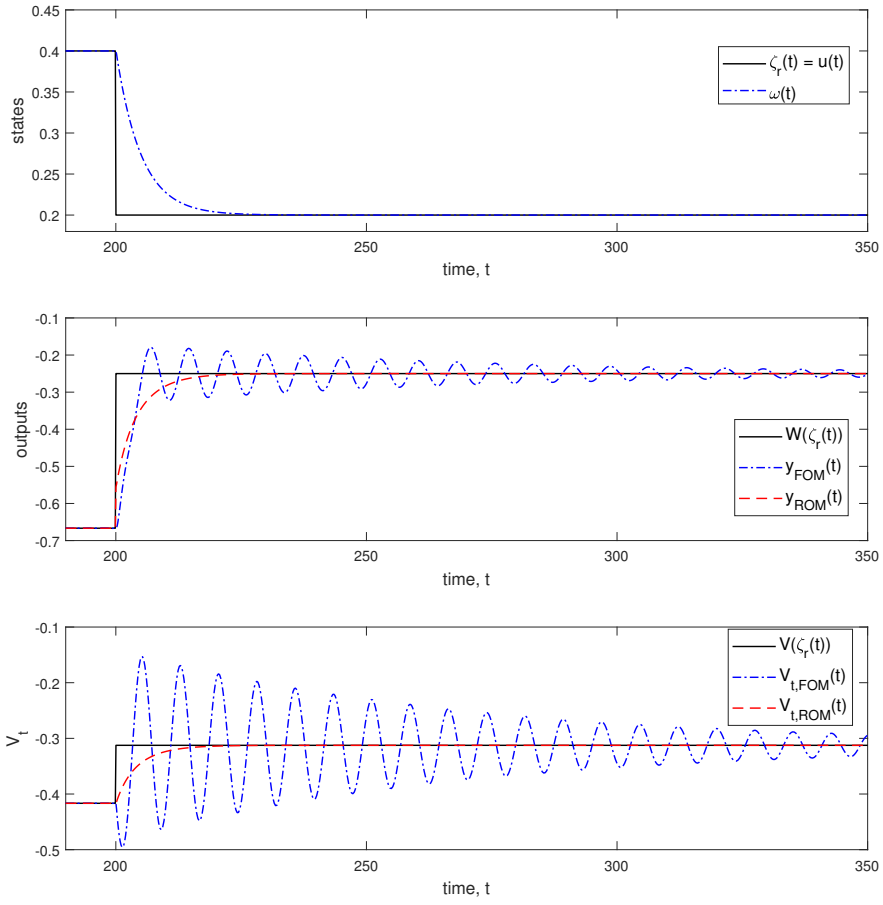

Figure 5.5: Response of the full order model (5.34) and of the reduced order model (5.37)-(5.38) for $t \in [190, 350]$.

Chapter 6

Regularization of Underdetermined Nonlinear Interpolants

In prior chapters the construction of nonlinear interpolants has been considered only for scenarios in which the two auxiliary systems encoding the interpolation points have the same dimension. If this is not the case then the Loewner functions have domain and codomain with different dimensions, and the families of interpolants presented thus far contain systems having either more variables than equations, representing *underdetermined* systems, or more equations than variables, representing *overdetermined* systems.

The assumption that the two auxiliary systems have the same dimension is restrictive in practice. For instance, it may be desirable to select the

auxiliary systems such that the interconnected systems of Chapters 4 and 5 correspond to a real operating environment, *i.e.* the auxiliary systems may be selected as actual signal generators and filters that a system of interest is interacting with, and these choices may have differing dimension. Furthermore, for the purposes of analysis, design, and simulation, it is important to construct an interpolant for which, at least locally, solutions exist and are unique, and an interpolant that has this property is referred to as *well-posed*. Solutions for underdetermined systems are generally not unique, and solutions for overdetermined systems generally do not exist. Hence, it is of great practical importance to address the problem of constructing well-posed interpolants in the Loewner framework when the families of interpolants presented thus far are not well-posed.

In this chapter the first steps are taken toward the construction of interpolants possessing, in general, the same number of equations and variables, and the framework of Chapter 5 is further enhanced by weakening the assumption that the auxiliary systems have the same dimension. In particular, the scenario in which the families of interpolants given in Theorems 13 and 14 in Chapter 5 are underdetermined is addressed. The family of interpolants is enhanced in a natural way via dynamic extension, in which new equations are added to the systems in such a way that the properties required to be an interpolant in the Loewner framework are preserved. The procedure results in a new family of interpolants that contain the same number of equations as variables and may be locally well-posed. To guarantee the existence of a

locally well-posed system contained in this family, it is sufficient (but not necessary) to assert that the Loewner function be surjective with full row rank Jacobian in a neighbourhood of the origin. Under this condition, a particular choice of system in the family of interpolants is a nonlinear enhancement of the interpolant provided in [94] when the Loewner matrix has full row rank.

This chapter is structured as follows. In Section 6.1 a brief overview of the results and theorems in the Loewner framework of Chapter 5 is given for the scenario in which the considered systems are allowed to be underdetermined or overdetermined, and the possibility of constructing underdetermined or overdetermined interpolants is discussed. In Section 6.2 underdetermined families of Loewner equivalent systems are considered and, given the underdetermined interpolant, a family of augmented systems which maintain the property of Loewner equivalence is constructed; a particular well-posed interpolant is presented for the scenario in which the Loewner function is surjective with full row rank Jacobian in a neighbourhood of the origin. Finally, in Section 6.3 some concluding remarks are given.

6.1 Problem Formulation

In Chapter 5 the Loewner framework has been enhanced for the treatment of nonlinear input-affine differential-algebraic systems of the form[1]

$$\dot{\overbrace{E\big(x(t)\big)}} = \left(\frac{\partial E}{\partial x} \circ x(t)\right) \dot{x}(t) = f\big(x(t)\big) + g\big(x(t)\big)u(t), \tag{6.1}$$

$$y(t) = h\big(x(t)\big) + d\big(x(t)\big)u(t), \tag{6.2}$$

with state $x(t) \in \mathbb{C}^n$, input $u(t) \in \mathbb{C}^m$, output $y(t) \in \mathbb{C}^p$, and functions $f : \mathbb{C}^n \to \mathbb{C}^k$, $g : \mathbb{C}^n \to \mathbb{C}^{k\times m}$, $h : \mathbb{C}^n \to \mathbb{C}^p$, $d : \mathbb{C}^n \to \mathbb{C}^{p\times m}$, and $E : \mathbb{C}^n \to \mathbb{C}^k$, such that $f(0) = 0$, $h(0) = 0$, and $E(0) = 0$, and an initial condition x_0 which is consistent with the input function $u(\cdot)$ in that, locally, there exists at least one continuously differentiable solution for (6.1)-(6.2). Furthermore, it is assumed that $f(\cdot)$, $g(\cdot)$, $h(\cdot)$, $d(\cdot)$, and $E(\cdot)$ are, locally, differentiable, and, in a neighbourhood of the origin, the Jacobian of $E(\cdot)$ has constant rank but it is not invertible, so the system (6.1)-(6.2) cannot be put into a standard state-space form. Chapter 5 considered the scenerio wherein the system possesses the same number of equations as variables, *i.e.* $k = n$. In this chapter, in order to consider the treatment of interpolants when the auxiliary systems do not have the same dimension, the definitions of the Loewner objects must be considered for the more general situation

[1]It is possible to produce underdetermined or overdetermined interpolants in the Loewner framework, hence when defining objects the possibility that $k \neq n$ is taken into consideration here. The main result of this chapter is the regularization of an interpolant when it has dimensions such that $k < n$.

in which $k \neq n$. Hence, when considering a system of the form (6.1)-(6.2) generating the tangential data functions it is assumed that the system is regular (with respect to the solution $x(t) = 0$ and $u(t) = 0$), however when constructing an interpolant of the form (6.1)-(6.2) with $k \neq n$ the system cannot be regular in general, as either no solution for the system exists or any particular solution for the system is not unique.

As in Chapter 5, objects in the Loewner framework for systems of the form (6.1)-(6.2) are defined in terms of the interconnection with two additional systems of the form

$$\dot{\zeta}_r(t) = \lambda\big(\zeta_r(t)\big) + \Delta(t), \tag{6.3}$$

$$v(t) = r\big(\zeta_r(t)\big), \tag{6.4}$$

and

$$\dot{\zeta}_\ell(t) = m\big(\zeta_\ell(t)\big) + \ell\big(\chi(t)\big), \tag{6.5}$$

$$\eta(t) = \zeta_\ell(t), \tag{6.6}$$

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$, and with functions $\lambda : \mathbb{C}^\rho \to \mathbb{C}^\rho$, $r : \mathbb{C}^\rho \to \mathbb{C}^m$, $m : \mathbb{C}^v \to \mathbb{C}^v$, and $\ell : \mathbb{C}^p \to \mathbb{C}^v$ such that $\lambda(0) = 0$, $r(0) = 0$, $m(0) = 0$, $\ell(0) = 0$, and $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, and $\ell(\cdot)$ differentiable.

The interconnection of the system (6.1)-(6.2) with the systems (6.3)-(6.4)

and (6.5)-(6.6), defined via the interconnection equations $u = v$ and $\chi = y$, has the representation

$$\begin{bmatrix} \dot{\zeta}_r \\ \dot{E}(x) \\ \dot{\zeta}_\ell \end{bmatrix} = \begin{bmatrix} \lambda(\zeta_r) \\ f(x) + g(x)r(\zeta_r) \\ m(\zeta_\ell) + \ell\big(h(x) + d(x)r(\zeta_r)\big) \end{bmatrix} + \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \Delta, \tag{6.7}$$

$$\eta = \zeta_\ell, \tag{6.8}$$

with state $\begin{bmatrix} \zeta_r^\top & x^\top & \zeta_\ell^\top \end{bmatrix}^\top$, input Δ, and output η. With respect to the Loewner objects in Chapter 5, some of the definitions for the Loewner objects associated to (6.7)-(6.8) change, superficially, for the scenario in which $k \neq n$. In particular, the tangential generalized observability function, $Y : \mathbb{C}^k \to \mathbb{C}^v$, is defined as the solution to the PDE with boundary condition

$$\left(\frac{\partial Y}{\partial \kappa} \circ E(x)\right) f(x) = -m\big(-Y(E(x))\big) - \ell\big(h(x)\big),$$
$$Y(0) = 0, \tag{6.9}$$

where $Y(\cdot)$ is now a function of a variable $\kappa \in \mathbb{C}^k$, rather than $x \in \mathbb{C}^n$, in order to accommodate the differing number of equations and variables in (6.1)-(6.2), *i.e.* the domain of $Y(\cdot)$ must be consistent with the range of $E(\cdot)$ in order to treat underdetermined and overdetermined systems, and $\dim(E(x)) \neq \dim(x)$ for any $x \in \mathbb{C}^n$, hence the new input variable κ is introduced. The left tangential data function, $V : \mathbb{C}^\rho \to \mathbb{C}^{v \times m}$, is modified

accordingly as

$$V(\zeta_r) := \Big(\frac{\partial Y}{\partial \kappa} \circ E\big(X(\zeta_r)\big)\Big)\, g\big(X(\zeta_r)\big) + \overline{\ell}\Big(h\big(X(\zeta_r)\big),\ d\big(X(\zeta_r)\big)r(\zeta_r)\Big) d\big(X(\zeta_r)\big),$$

and the shifted Loewner function, $\sigma\mathbb{L} : \mathbb{C}^{\rho} \to \mathbb{C}^{v}$, is modified accordingly as

$$\sigma\mathbb{L}(\zeta_r) := -\Big(\frac{\partial Y}{\partial \kappa} \circ E\big(X(\zeta_r)\big)\Big)\, f\big(X(\zeta_r)\big) + \overline{\ell}\Big(h\big(X(\zeta_r)\big),\ d\big(X(\zeta_r)\big)r(\zeta_r)\Big) d\big(X(\zeta_r)\big)r(\zeta_r).$$

Considering the underlying system (6.1)-(6.2) with $k \neq n$, all other objects of Chapter 5 are unchanged, and the results of Chapter 5 hold with the exact same proofs. Thus, given the functions $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $V(\cdot)$, and $W(\cdot)$ associated to the system (6.1)-(6.2) interconnected with the systems (6.3)-(6.4) and (6.5)-(6.6), when $\rho \neq v$ the system of equations

$$\begin{aligned} \dot{\widetilde{\mathbb{L}(\omega)}} &= \Big(\sigma\mathbb{L}(\omega) - \overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)r(\omega)\Big) \\ &\quad - \Big(V(\omega) - \overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)\Big)u_r, \qquad (6.10) \\ y_r &= \Big(W(\omega) - \overline{d}(\omega)r(\omega)\Big) + \overline{d}(\omega)u_r, \qquad (6.11) \end{aligned}$$

with state $\omega(t) \in \mathbb{C}^{\rho}$, input $u_r(t) \in \mathbb{C}^{m}$, and output $y_r(t) \in \mathbb{C}^{p}$, with $\overline{d} : \mathbb{C}^{\rho} \to \mathbb{C}^{p\times m}$ a free parameter, still matches the tangential data functions and it is still Loewner equivalent at (λ, r, m, ℓ) to the system (6.1)-(6.2) in the sense

that the Loewner functions are matched, and the system still matches the tangential data functions $W(\cdot)$ and $V(\cdot)$. However, the system (6.10)-(6.11) is not well-posed because the number of equations differs from the number of variables, *i.e.* $\rho \neq \nu$.

The condition in Theorem 14 of Chapter 5 that the dimensions of the systems (6.3)-(6.4) and (6.5)-(6.6) be the same, *i.e.* $\rho = v$, is restrictive. When $\rho > v$ the proposed interpolant (6.10)-(6.11) is *underdetermined*, in the sense that it possesses more variables than constraints, and when $\rho < v$ the proposed interpolant is *overdetermined*, in the sense that it possesses more constraints than variables.

The purpose of the remainder of this chapter is to take the first step towards weakening the assumption that $\rho = v$ by constructing interpolating systems in the particular scenario in which $\rho > v$. The construction begins by considering the underdetermined system of equations (6.10)-(6.11) as a basis to which new dynamics, $\rho - v$ new equations, are added in such a way that the "dynamically extended interpolant" still possesses the property of Loewner equivalence at (λ, r, m, ℓ) to the system (6.1)-(6.2), and still matches the tangential data functions $W(\cdot)$ and $V(\cdot)$. The scenario in which $\rho < \nu$ is treated in the following chapter via an approach inspired by the results in the remainder of this chapter.

6.2 Extension of the Underdetermined Interpolant

Consider now the scenario in which $\rho > v$, so the system of equations (6.10)-(6.11) is underdetermined, and the Loewner function, $\mathbb{L} : \mathbb{C}^{\rho} \to \mathbb{C}^{v}$, has constant rank in a neighbourhood of the origin. Using the Loewner functions associated to the interconnected system (6.7)-(6.8) another family of nonlinear systems can be constructed with the property that each system in the family is Loewner equivalent at (λ, r, m, ℓ) to (6.1)-(6.2) and matches the tangential data functions $W(\cdot)$ and $V(\cdot)$. The following theorem considers the underdetermined system of equations (6.10)-(6.11) as a foundation, however new dynamics are added in such a way that the properties of Loewner equivalence at (λ, r, m, ℓ) to (6.1)-(6.2) and matching of tangential data functions are preserved.

Theorem 15. *Consider the interconnected system* (6.7)-(6.8) *with* $\rho > v$. *Let* $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $V(\cdot)$, *and* $W(\cdot)$ *be the associated Loewner functions. Furthermore, consider functions* $P : \mathbb{C}^{\rho} \to \mathbb{C}^{\rho-v}$, $N : \mathbb{C}^{\rho} \to \mathbb{C}^{\rho-v}$, *and* $T : \mathbb{C}^{\rho} \to \mathbb{C}^{(\rho-v)\times m}$, *where* $P(\cdot)$ *is a differentiable mapping such that the Jacobian of* $(\mathbb{L}(\cdot)^{\top}, P(\cdot)^{\top})^{\top}$ *has constant rank in a neighbourhood of the origin, and*

$$N(\omega) = \frac{\partial P}{\partial \omega}\lambda(\omega) + T(\omega)r(\omega). \tag{6.12}$$

Define the system

$$\begin{bmatrix} \dot{\overbrace{\mathbb{L}(\omega)}} \\ \dot{\overbrace{P(\omega)}} \end{bmatrix} = \begin{bmatrix} \sigma\mathbb{L}(\omega) - \overline{\ell}\Big(W(\omega),\ -\overline{d}(\omega)r(\omega)\Big)\overline{d}(\omega)r(\omega) \\ N(\omega) \end{bmatrix} - \begin{bmatrix} V(\omega) - \overline{\ell}\Big(W(\omega),\ -\overline{d}(\omega)r(\omega)\Big)\overline{d}(\omega) \\ T(\omega) \end{bmatrix} u_r, \tag{6.13}$$

$$y_r = \Big(W(\omega) - \overline{d}(\omega)r(\omega)\Big) + \overline{d}(\omega)u_r, \tag{6.14}$$

with state $\omega(t) \in \mathbb{C}^\rho$*, input* $u_r(t) \in \mathbb{C}^m$*, and output* $y_r(t) \in \mathbb{C}^p$*, and* $\overline{d} : \mathbb{C}^\rho \to \mathbb{C}^{p\times m}$*. Then the system* (6.13)-(6.14) *is Loewner equivalent at* (λ, r, m, ℓ) *to the system* (6.1)-(6.2) *and matches the tangential data functions* $W(\cdot)$ *and* $V(\cdot)$*.*

Remark 19. The dynamically extended system (6.13)-(6.14) has the same number of equations as variables, ρ. Furthermore, the functions $\overline{d}(\cdot)$, $P(\cdot)$, and $T(\cdot)$ are unconstrained, with the exception of $P(\cdot)$ being differentiable and the Jacobian of $(\mathbb{L}(\cdot)^\top, P(\cdot)^\top)^\top$ having constant rank in a neighbourhood of the origin, giving the possibility of constructing a regular, or well-posed, interpolant.

Proof. Let $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{V}(\cdot)$, $\overline{W}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\mathbb{L}}^\ell(\cdot)$, $\overline{\mathbb{L}}^r(\cdot)$, and $\sigma\overline{\mathbb{L}}(\cdot)$ be the set of Loewner functions for the system (6.13)-(6.14) interconnected with the systems (6.3)-(6.4) and (6.5)-(6.6). To prove that the extended interpolant (6.13)-(6.14) is Loewner equivalent at (λ, r, m, ℓ) to the system (6.1)-(6.2) it is

shown that $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\kappa_1, \kappa_2) = -\kappa_1$ where $\kappa_1 \in \mathbb{C}^v$ and $\kappa_2 \in \mathbb{C}^{\rho - v}$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ is a solution to the interpolant's system of PDEs corresponding to (5.11), (6.9), and (5.15). The function $\overline{X}(\cdot)$ is the solution to the PDE with boundary condition

$$\begin{aligned}
&\frac{\partial \begin{bmatrix} \mathbb{L}\big(\overline{X}(\zeta_r)\big) \\ P\big(\overline{X}(\zeta_r)\big) \end{bmatrix}}{\partial \zeta_r} \lambda(\zeta_r) \\
&\qquad = \left(\begin{bmatrix} \sigma \mathbb{L}(\zeta) - \overline{\ell}\Big(W(\zeta),\ -\overline{d}(\zeta) r(\zeta)\Big) \overline{d}(\zeta) r(\zeta) \\ N(\zeta) \end{bmatrix} \circ \overline{X}(\zeta_r) \right) \\
&\qquad\qquad - \left(\begin{bmatrix} V(\zeta) - \overline{\ell}\Big(W(\zeta),\ -\overline{d}(\zeta) r(\zeta)\Big) \overline{d}(\zeta) \\ T(\zeta) \end{bmatrix} \circ \overline{X}(\zeta_r) \right) r(\zeta_r), \\
&\overline{X}(0) = 0,
\end{aligned}$$

and letting $\overline{X}(\zeta_r) = \zeta_r$ yields

$$\begin{bmatrix} \frac{\partial \mathbb{L}}{\partial \zeta_r} \\ \left(\frac{\partial P}{\partial \omega} \circ \zeta_r \right) \end{bmatrix} \lambda(\zeta_r) = \begin{bmatrix} \sigma \mathbb{L}(\zeta_r) - V(\zeta_r) r(\zeta_r) \\ N(\zeta_r) - T(\zeta_r) r(\zeta_r) \end{bmatrix},$$

which holds by (5.18) and (6.12). The function $\overline{Y}(\cdot)$ is the solution to the

PDE with boundary condition

$$\Big(\Big[\tfrac{\partial\overline{Y}}{\partial\kappa_1} \quad \tfrac{\partial\overline{Y}}{\partial\kappa_2}\Big] \circ \big(\mathbb{L}(\omega), P(\omega)\big)\Big) \begin{bmatrix} \sigma\mathbb{L}(\omega) - \overline{\ell}\Big(W(\omega),\ -\overline{d}(\omega)r(\omega)\Big)\overline{d}(\omega)r(\omega) \\ N(\omega) \end{bmatrix}$$
$$= -m\Big(-\overline{Y}\big(\mathbb{L}(\omega), P(\omega)\big)\Big) - \ell\big(W(\omega) - \overline{d}(\omega)r(\omega)\big),$$
$$\overline{Y}(0,0) = 0.$$

Setting $\overline{Y}(\kappa_1, \kappa_2) = -\kappa_1$ yields

$$\Big[-I \quad 0\Big] \begin{bmatrix} \sigma\mathbb{L}(\omega) - \overline{\ell}\Big(W(\omega),\ -\overline{d}(\omega)r(\omega)\Big)\overline{d}(\omega)r(\omega) \\ N(\omega) \end{bmatrix}$$
$$= -m\big(\mathbb{L}(\omega)\big) - \ell\big(W(\omega) - \overline{d}(\omega)r(\omega)\big),$$

and utilizing (5.7) one obtains

$$\begin{aligned} \sigma\mathbb{L}(\omega) &= m\big(\mathbb{L}(\omega)\big) + \ell\big(W(\omega) - \overline{d}(\omega)r(\omega)\big) \\ &\quad + \overline{\ell}\Big(W(\omega),\ -\overline{d}(\omega)r(\omega)\Big)\overline{d}(\omega)r(\omega) \\ &= m\big(\mathbb{L}(\omega)\big) + \ell\big(W(\omega)\big), \end{aligned}$$

which holds by (5.17). It now follows readily that

$$\overline{\mathbb{L}}(\zeta_r) = -\overline{Y}\Big(\mathbb{L}\big(\overline{X}(\zeta_r)\big), P\big(\overline{X}(\zeta_r)\big)\Big) = \mathbb{L}(\zeta_r),$$

and

$$\begin{aligned}\overline{W}(\zeta_r) &= W\big(\overline{X}(\zeta_r)\big) - \overline{d}\big(\overline{X}(\zeta_r)\big) r\big(\overline{X}(\zeta_r)\big) + \overline{d}\big(\overline{X}(\zeta_r)\big) r(\zeta_r) \\ &= W(\zeta_r).\end{aligned}$$

Additionally, it follows by (5.17) that

$$\begin{aligned}\sigma\overline{\mathbb{L}}(\zeta_r) &= m\big(\overline{\mathbb{L}}(\zeta_r)\big) + \ell\big(\overline{W}(\zeta_r)\big) \\ &= m\big(\mathbb{L}(\zeta_r)\big) + \ell\big(W(\zeta_r)\big) \\ &= \sigma\mathbb{L}(\zeta_r).\end{aligned}$$

The function $\overline{V}(\cdot)$ is given by

$$\begin{aligned}\overline{V}(\zeta_r) = &\left(\begin{bmatrix} \frac{\partial \overline{Y}}{\partial \kappa_1} & \frac{\partial \overline{Y}}{\partial \kappa_2} \end{bmatrix} \circ \big(\mathbb{L}(\overline{X}(\zeta_r)), P(\overline{X}(\zeta_r))\big)\right) \\ &\times \left(\begin{bmatrix} \overline{\ell}\Big(W(\zeta),\ -\overline{d}(\zeta) r(\zeta)\Big)\overline{d}(\zeta) - V(\zeta) \\ -T(\zeta) \end{bmatrix} \circ \overline{X}(\zeta_r)\right) \\ &+ \overline{\ell}\Big(W\big(\overline{X}(\zeta_r)\big) - \overline{d}\big(\overline{X}(\zeta_r)\big) r\big(\overline{X}(\zeta_r)\big),\ \overline{d}\big(\overline{X}(\zeta_r)\big) r(\zeta_r)\Big)\overline{d}\big(\overline{X}(\zeta_r)\big).\end{aligned}$$

This simplifies to

$$\begin{aligned}\overline{V}(\zeta_r) = V(\zeta_r) &- \overline{\ell}\Big(W(\zeta_r),\ -\overline{d}(\zeta_r) r(\zeta_r)\Big)\overline{d}(\zeta_r) \\ &+ \overline{\ell}\Big(W(\zeta_r) - \overline{d}(\zeta_r) r(\zeta_r),\ \overline{d}(\zeta_r) r(\zeta_r)\Big)\overline{d}(\zeta_r),\end{aligned}$$

and by (5.8) it follows that

$$\overline{V}(\zeta_r) = V(\zeta_r).$$

The function $\overline{\mathbb{L}}^{\ell}(\cdot)$ is the solution to the PDE with boundary condition

$$\begin{aligned}\frac{\partial \overline{\mathbb{L}}^{\ell}}{\partial \zeta_r}\lambda(\zeta_r) &= -m\big(-\overline{\mathbb{L}}^{\ell}(\zeta_r)\big) - \overline{V}(\zeta_r)r(\zeta_r),\\ \overline{\mathbb{L}}^{\ell}(0) &= 0.\end{aligned}$$

Letting $\overline{X}(\zeta_r) = \zeta_r$, $\overline{Y}(\kappa_1, \kappa_2) = -\kappa_1$, and $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ yields

$$\frac{\partial \mathbb{L}^{\ell}}{\partial \zeta_r}\lambda(\zeta_r) = -m\big(-\mathbb{L}^{\ell}(\zeta_r)\big) - V(\zeta_r)r(\zeta_r),$$

which holds by (5.15). It follows that

$$\begin{aligned}\overline{\mathbb{L}}^{r}(\zeta_r) &= \overline{\mathbb{L}}(\zeta_r) - \overline{\mathbb{L}}^{\ell}(\zeta_r)\\ &= \mathbb{L}(\zeta_r) - \mathbb{L}^{\ell}(\zeta_r)\\ &= \mathbb{L}^{r}(\zeta_r).\end{aligned}$$

Therefore, $\overline{\mathbb{L}}^{\ell}(\zeta_r) = \mathbb{L}^{\ell}(\zeta_r)$ and $\overline{\mathbb{L}}^{r}(\zeta_r) = \mathbb{L}^{r}(\zeta_r)$, and (6.13)-(6.14) is Loewner equivalent at (λ, r, m, ℓ) to (6.1)-(6.2) and matches the tangential data functions $W(\cdot)$ and $V(\cdot)$. □

The authors of [94] provide the linear interpolant (2.32)-(2.33) for the sce-

nario in which the Loewner matrix has full row rank, *i.e.* when the Loewner matrix has a right inverse. One should expect this interpolant to belong to the family of interpolants given in Theorem 15. Indeed, a particular choice of system in Theorem 15 yields a nonlinear enhancement of the interpolant (2.32)-(2.33).

Corollary 16. *Consider the interconnected system* (6.7)-(6.8)*. Let* $\mathbb{L}^{\ell}(\cdot)$*,* $\mathbb{L}^{r}(\cdot)$*,* $\mathbb{L}(\cdot)$*,* $\sigma\mathbb{L}(\cdot)$*,* $V(\cdot)$*, and* $W(\cdot)$ *be the associated Loewner functions. Suppose that the Loewner function,* $\mathbb{L}(\cdot)$*, is surjective with full row rank Jacobian in a neighbourhood of the origin, and thus has a right inverse,* $\mathbb{L}^{\#}(\cdot)$*, such that* $\mathbb{L}\big(\mathbb{L}^{\#}(\kappa)\big) = \kappa$*. Then the system*

$$\dot{\omega} = \left(\lambda(\omega) + \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{\#} \Big(V(\omega) - \overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)\Big)r(\omega)\right) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ \omega\right)^{\#} \Big(V(\omega) - \overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)\Big)u_r, \tag{6.15}$$

$$y_r = \big(W(\omega) - \overline{d}(\omega)r(\omega)\big) + \overline{d}(\omega)u_r, \tag{6.16}$$

with state $\omega(t) \in \mathbb{C}^{\rho}$*, input* $u_r(t) \in \mathbb{C}^{m}$*, and output* $y_r(t) \in \mathbb{C}^{p}$*, where* $\overline{d} : \mathbb{C}^{\rho} \to \mathbb{C}^{p\times m}$*, is Loewner equivalent at* (λ, r, m, ℓ) *to the system* (6.1)-(6.2) *and matches the tangential data functions* $W(\cdot)$ *and* $V(\cdot)$*.*

Proof. Consider the interpolant (6.13)-(6.14) in Theorem 15, and let $P(\cdot)$ be any differentiable function such that $(\mathbb{L}(\omega)^{\top}, P(\omega)^{\top})^{\top}$ is bijective with

nonsingular Jacobian in a neighbourhood of the origin. Set

$$T(\omega) := \frac{\partial P}{\partial \omega}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right)^{\#}\Big(V(\omega) - \overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)\Big),$$

so that

$$\begin{aligned} N(\omega) = {} & \frac{\partial P}{\partial \omega}\lambda(\omega) \\ & + \frac{\partial P}{\partial \omega}\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right)^{\#}\Big(V(\omega) - \overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)\Big)r(\omega). \end{aligned}$$

Then, by using (5.18) and rearranging, the interpolant (6.13)-(6.14) becomes

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = {} & \begin{bmatrix} \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right) \\ \frac{\partial P}{\partial \omega} \end{bmatrix} \\ & \times \Bigg(\dot{\omega} - \lambda(\omega) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right)^{\#} V(\omega)\big(r(\omega) - u_r\big) \\ & \qquad + \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right)^{\#} \overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)r(\omega) \\ & \qquad - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right)^{\#} \overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)u_r\Bigg), \qquad (6.17) \\ y_r = {} & \big(W(\omega) - \overline{d}(\omega)r(\omega)\big) + \overline{d}(\omega)u_r, \qquad (6.18) \end{aligned}$$

and because $(\mathbb{L}(\omega)^\top, P(\omega)^\top)^\top$ has a full rank Jacobian in a neighbourhood of the origin, (6.17)-(6.18) is equivalent to (6.15)-(6.16). □

Remark 20. As a consequence of Corollary 16, to guarantee existence of a

locally well-posed interpolant in the family of systems given by Theorem 15 it is sufficient (but not necessary) to assert that the Loewner function be surjective with a full row rank Jacobian in a neighbourhood of the origin.

Remark 21. A dynamically extended family of interpolants in the linear setting follows immediately from the nonlinear family given in Theorem 15. When the systems (6.1)-(6.2), (6.3)-(6.4), and (6.5)-(6.6) are linear time-invariant, *i.e.* $\lambda(\zeta_r) = \Lambda\zeta_r$, $r(\zeta_r) = R\zeta_r$, $m(\zeta_\ell) = M\zeta_\ell$, $\ell(\chi) = L\chi$, $E(x) = Ex$, $f(x) = Ax$, $g(x) = B$, $h(x) = Cx$, $d(x) = D$, the Loewner functions become $X(\zeta_r) = X\zeta_r$, $Y(\kappa) = Y\kappa$, $\mathbb{L}^\ell(\zeta_r) = \mathbb{L}^\ell\zeta_r$, $\mathbb{L}^r(\zeta_r) = \mathbb{L}^r\zeta_r$, $\mathbb{L}(\zeta_r) = \mathbb{L}\zeta_r$, $\sigma\mathbb{L}(\zeta_r) = \sigma\mathbb{L}\zeta_r$, $V(\zeta_r) = V$, $W(\zeta_r) = W\zeta_r$. Setting $P(\omega) = P\omega$, $N(\omega) = N\omega$, and $T(\omega) = T$, the dynamically extended family of interpolants given by (6.13)-(6.14) of Theorem 15 becomes

$$\begin{bmatrix} \mathbb{L} \\ P \end{bmatrix} \dot{\omega} = \begin{bmatrix} \sigma\mathbb{L} - L\overline{D}R \\ N \end{bmatrix} \omega - \begin{bmatrix} V - L\overline{D} \\ T \end{bmatrix} u_r,$$
$$y_r = (W - \overline{D}R)\omega + \overline{D}u_r,$$
$$N = P\Lambda + TR,$$

and the interpolant (6.15)-(6.16) becomes the interpolant (2.32)-(2.33) given in [94] provided the Loewner matrix has full row rank.

In the linear setting, if one selects a square matrix F with $\sigma(F) \cap \sigma(\Lambda) = \emptyset$

and solves the Sylvester equation

$$P\Lambda + TR = FP,$$

then the added dynamics become

$$\begin{aligned} P\dot{\omega} &= N\omega - Tu_r, \\ &= FP\omega - Tu_r. \end{aligned}$$

The following lemma makes clear that, under mild conditions, the added dynamics can always be chosen such that the manifold $P\omega = 0$ is invariant and attractive (when $u_r = 0$), with P full row rank, while preserving the Loewner equivalence and matching the tangential data functions; $\rho - v$ poles of the interpolant can be assigned freely.

Lemma 3. *Suppose (R, Λ) is observable and consider scalars $\mu_i \in \mathbb{C} \setminus \sigma(\Lambda)$, $i \in \{1, 2, \ldots, \rho - v\}$. Then there exist matrices $P \in \mathbb{C}^{(\rho-v)\times\rho}$, $T \in \mathbb{C}^{(\rho-v)\times m}$, and $F \in \mathbb{C}^{(\rho-v)\times(\rho-v)}$ such that*

$$P\Lambda + TR = FP, \tag{6.19}$$

where $\sigma(F) = \{\mu_1, \ldots, \mu_{\rho-v}\}$ and P has full row rank.

Proof. Suppose (R, Λ) is observable. Then there exists a nonsingular matrix Q, and matrices $\overline{\Lambda}_1 \in \mathbb{C}^{(\rho-v)\times(\rho-v)}$, $\overline{\Lambda}_2 \in \mathbb{C}^{(\rho-v)\times v}$, $\overline{\Lambda}_3 \in \mathbb{C}^{v\times v}$, $\overline{R}_1 \in \mathbb{C}^{m\times(\rho-v)}$,

$\overline{R}_2 \in \mathbb{C}^{m\times v}$, such that

$$\Lambda = Q \begin{bmatrix} \overline{\Lambda}_1 & \overline{\Lambda}_2 \\ 0 & \overline{\Lambda}_3 \end{bmatrix} Q^{-1}, \quad R = \begin{bmatrix} \overline{R}_1 & \overline{R}_2 \end{bmatrix} Q^{-1},$$

where $(\overline{R}_1, \overline{\Lambda}_1)$ is observable. The Sylvester equation (6.19) becomes

$$PQ \begin{bmatrix} \overline{\Lambda}_1 & \overline{\Lambda}_2 \\ 0 & \overline{\Lambda}_3 \end{bmatrix} Q^{-1} + T \begin{bmatrix} \overline{R}_1 & \overline{R}_2 \end{bmatrix} Q^{-1} = FP,$$

or

$$PQ \begin{bmatrix} \overline{\Lambda}_1 & \overline{\Lambda}_2 \\ 0 & \overline{\Lambda}_3 \end{bmatrix} + T \begin{bmatrix} \overline{R}_1 & \overline{R}_2 \end{bmatrix} = FPQ.$$

Define $\begin{bmatrix} \overline{P}_1 & \overline{P}_2 \end{bmatrix} := PQ$, $\overline{P}_1 \in \mathbb{C}^{(\rho-v)\times(\rho-v)}$, $\overline{P}_2 \in \mathbb{C}^{(\rho-v)\times v}$, so that

$$\begin{bmatrix} \overline{P}_1 & \overline{P}_2 \end{bmatrix} \begin{bmatrix} \overline{\Lambda}_1 & \overline{\Lambda}_2 \\ 0 & \overline{\Lambda}_3 \end{bmatrix} + T \begin{bmatrix} \overline{R}_1 & \overline{R}_2 \end{bmatrix} = F \begin{bmatrix} \overline{P}_1 & \overline{P}_2 \end{bmatrix},$$

or

$$\overline{P}_1\overline{\Lambda}_1 + T\overline{R}_1 = F\overline{P}_1, \tag{6.20}$$

$$\overline{P}_1\overline{\Lambda}_2 + \overline{P}_2\overline{\Lambda}_3 + T\overline{R}_2 = F\overline{P}_2. \tag{6.21}$$

By observability of $(\overline{R}_1, \overline{\Lambda}_1)$, choose T such that $\sigma(\overline{\Lambda}_1 + T\overline{R}_1) = \{\mu_1, \ldots, \mu_{\rho-v}\}$, and select $F := \overline{\Lambda}_1 + T\overline{R}_1$. It follows that (6.20) becomes

$$(\overline{P}_1 - I)\overline{\Lambda}_1 - F(\overline{P}_1 - I) = 0,$$

which is uniquely solved by $\overline{P}_1 = I$ because $\sigma(\Lambda) \cap \sigma(F) = \emptyset$. Finally, (6.21) becomes

$$\overline{\Lambda}_2 + \overline{P}_2\overline{\Lambda}_3 + T\overline{R}_2 = F\overline{P}_2,$$

which has a unique solution $\overline{P}_2$ because $\sigma(\Lambda) \cap \sigma(F) = \emptyset$. It follows that the unique solution of (6.19), $P = \begin{bmatrix} I & \overline{P}_2 \end{bmatrix} Q^{-1}$, has full row rank. Hence, by construction, there exists a matrix T, a full row rank matrix P, and a matrix F with $\sigma(F) = \{\mu_1, \ldots, \mu_{\rho-v}\}$ satisfying the Sylvester equation (6.19). $\square$

Remark 22. A similar result is obtained, locally, in the nonlinear setting by solving the PDE

$$\frac{\partial P}{\partial \omega}\lambda(\omega) + T(\omega)r(\omega) = F\big(P(\omega)\big), \qquad F(0) = 0,$$

and employing the centre manifold theory or type (C, v) nonresonance conditions, with $F(\cdot)$ chosen such that the system

$$\dot{\overbrace{P(\omega)}} = F\big(P(\omega)\big) - T(\omega)u_r,$$

has desired stability properties.

6.3 Conclusion

This chapter has presented an approach for regularization of underdetermined interpolants in the Loewner framework for nonlinear DAEs, building on the results in Chapter 5. The result relies on the construction of a dynamically extended family of interpolants preserving the properties of Loewner equivalence and matching of tangential data functions. It is sufficient (but not necessary) that the Loewner function be surjective with a full row rank Jacobian in a neighbourhood of the origin to guarantee the existence of a locally well-posed interpolant. Finally, it has been shown that the $\rho - v$ poles introduced in the extended interpolant can be assigned freely if an observability condition holds for one of the auxiliary systems.

Chapter 7

Construction and Parameterization of Nonlinear Interpolants in the Loewner Framework

In Chapters 4 and 5 the construction of nonlinear interpolants in the Loewner framework has been considered. The approach, however, has a limitation in that the two auxiliary systems used for interpolation are required to have the same dimensions.

For the linear setting, [94] has provided two methods to construct interpolants for the scenarios in which the Loewner matrix is not square, *i.e.* for the scenarios in which the auxiliary systems have different dimensions. The

first method, reviewed in Chapter 2, requires that the Loewner matrix be full row rank. The second method, also reviewed in Chapter 2, requires that a more general rank condition hold, and the construction procedure relies on the use of the singular value decomposition yielding an approach that may not be the most natural or convenient for enhancement to nonlinear systems. Furthermore, the approach of [94] does not consider building interpolants of larger dimension which would introduce extra degrees of freedom when the construction of an interpolant with additional properties, such as stability or regularity, is desired.

An approach to nonlinear interpolant construction in the case in which the Loewner functions produce an underdetermined interpolant is provided in Chapter 6. This approach uses an underdetermined interpolant and adds constraints in a natural way such that the properties of Loewner equivalence and matching of tangential data functions are preserved. The approach, however, only considers adding new equations, rather than adding both new equations and new states, and the existence of a well-posed system belonging to the constructed family is difficult to guarantee in general, with the Jacobian of the Loewner function having full row rank in a neighbourhood of the origin being a sufficient condition.

In this chapter the applicability of the nonlinear Loewner framework given in Chapters 4 and 5 is greatly enhanced by further extending, to the most general case, the approach for well-posed interpolant construction of Chapter 6. Particularly, the situation wherein an ill-posed family of interpolants

of the form given in Chapter 5 is known is considered and the family of interpolants is augmented in a natural way by adding an arbitrary number of new equations and variables such that the properties of Loewner equivalence and matching of tangential data functions are preserved. The approach results in a new family of interpolants, and existence of locally well-posed interpolants can be guaranteed for sufficiently high dimension which provides an approach to construct interpolants for both the underdetermined and the overdetermined scenarios within the same formalism. The results are specialized to the linear setting and it is proven that under mild conditions the family of dynamically extended systems contains, or parameterizes, all interpolants of the associated dimension. Taken with the complementary results of [94], the parameterization provides a more broad theory for the construction of linear interpolants. Inspired by the linear parameterization, an enhanced parameterization is given for the nonlinear setting.

This chapter is structured as follows. In Section 7.1 the problem of constructing well-posed Loewner equivalent nonlinear interpolants is formally stated. In Section 7.2 the problem is considered in the most general setting in which the sizes of the auxiliary systems are arbitrary, and a family of augmented systems which maintain the properties of Loewner equivalence and matching of tangential data functions based on an ill-posed interpolant is presented. Two particular well-posed interpolants are presented for scenarios in which the Loewner function is surjective, or injective, with full rank Jacobian, respectively. In Section 7.3 the linearized forms of the results in the

preceeding section are presented, and it is proven that under mild conditions the dynamically extended family of systems contains, or parameterizes, all interpolants of the associated dimension. In Section 7.4 an enhanced parameterization is presented for the nonlinear setting. In Section 7.5 an example is considered for the linear setting wherein it is not possible to construct a stable interpolant of minimal order; a system of higher dimension must be considered, and a general approach to choosing such an interpolant is demonstrated by "embedding" an observer and estimated state-feedback controller using free parameters. In Section 7.6 an example is considered wherein a reduced order model is constructed for the averaged model of a DC-to-DC Ćuk converter when the auxiliary systems do not have the same dimension. Finally, in Section 7.7 some concluding remarks are given.

7.1 Problem Formulation

In Chapters 5 and 6 the Loewner framework has been developed for nonlinear input-affine differential-algebraic systems of the form

$$\widetilde{\dot{E}(x)} = \frac{\partial E}{\partial x}\dot{x} = f(x) + g(x)u, \tag{7.1}$$

$$y = h(x) + d(x)u, \tag{7.2}$$

with state $x(t) \in \mathbb{C}^n$, input $u(t) \in \mathbb{C}^m$, and output $y(t) \in \mathbb{C}^p$, and mappings $f : \mathbb{C}^n \to \mathbb{C}^k$, $g : \mathbb{C}^n \to \mathbb{C}^{k\times m}$, $h : \mathbb{C}^n \to \mathbb{C}^p$, $d : \mathbb{C}^n \to \mathbb{C}^{p\times m}$, and

$E : \mathbb{C}^n \to \mathbb{C}^k$, such that $f(0) = 0$, $h(0) = 0$, and $E(0) = 0$, and initial condition x_0 which is consistent with the input function $u(\cdot)$ in that, locally, there exists at least one continuously differentiable solution for (7.1)-(7.2). It is assumed that $f(\cdot)$, $g(\cdot)$, $h(\cdot)$, $d(\cdot)$, and $E(\cdot)$ are, locally, differentiable and, in a neighbourhood of the origin, the Jacobian of $E(\cdot)$ has constant rank but is not invertible, hence systems consisting of both differential and algebraic equations are considered and the system (7.1)-(7.2) cannot be put in general into a standard state-space form. As in Chapter 6, when constructing an interpolant in the Loewner framework it is possible to produce an underdetermined or overdetermined system, hence when defining objects the possibility that $k \neq n$ is taken into consideration in order to generalize the notion of Loewner objects for such systems. The consideration of this situation is of great importance in practice and it is the primary concern of Chapter 6 and this chapter. The original system to be interpolated, from which the tangential data mappings have first been collected, is considered to be regular (with respect to the solution $x(t) = 0$ and $u(t) = 0$) and is not underdetermined or overdetermined so that the obtained set of tangential data objects is unique, *i.e.* the system generating the tangential data functions is well-posed, and $k = n$.

As in Chapters 5 and 6, to discuss the Loewner objects for differential-algebraic systems of the form (7.1)-(7.2) consider two auxiliary systems de-

fined by the equations

$$\dot{\zeta}_r = \lambda(\zeta_r) + \Delta, \tag{7.3}$$

$$v = r(\zeta_r), \tag{7.4}$$

and

$$\dot{\zeta}_\ell = m(\zeta_\ell) + \ell(\chi), \tag{7.5}$$

$$\eta = \zeta_\ell, \tag{7.6}$$

with states $\zeta_r(t) \in \mathbb{C}^\rho$ and $\zeta_\ell(t) \in \mathbb{C}^v$, inputs $\Delta(t) \in \mathbb{C}^\rho$ and $\chi(t) \in \mathbb{C}^p$, and outputs $v(t) \in \mathbb{C}^m$ and $\eta(t) \in \mathbb{C}^v$, and with functions $\lambda : \mathbb{C}^\rho \to \mathbb{C}^\rho$, $r : \mathbb{C}^\rho \to \mathbb{C}^m$, $m : \mathbb{C}^v \to \mathbb{C}^v$, and $\ell : \mathbb{C}^p \to \mathbb{C}^v$ such that $\lambda(0) = 0$, $r(0) = 0$, $m(0) = 0$, $\ell(0) = 0$, and $\lambda(\cdot)$, $r(\cdot)$, $m(\cdot)$, and $\ell(\cdot)$ differentiable.

In Chapter 6 the problem of constructing well-posed nonlinear interpolants when $\rho \geq v$ is considered. The proposed approach provides only a sufficient condition for the existence of a well-posed interpolant and, if the condition does not hold, then existence of a well-posed interpolant in the constructed family of systems is not guaranteed. The remainder of this chapter considers the general case of regularizing a nonlinear interpolant, or constructing a well-posed interpolant, when $\rho \neq v$ by considering the addition of new equations and/or variables to the interpolant of Theorem 14. This "dynamic extension" is perfomed while adhering to conditions which

preserve the properties of Loewner equivalence at (λ, r, m, ℓ) and matching of the tangential data functions. This yields a parameterization of a family of interpolants with dimension $k \geq \max\{\rho, v\}$ such that both the scenarios of regularizing an underdetermined interpolant and an overdetermined interpolant are dealt with simultaneously, *i.e.* the construction of well-posed interpolants is performed for $\rho = v$, $\rho < v$, and $\rho > v$ within the same formalism. The dimension of the constructed model can be increased arbitrarily and, as a result, an upper bound on the minimum number of states k required to guarantee the existence of a well-posed interpolant is given by $k \leq \rho + v$. Furthermore, the addition of new equations and/or variables may allow for the selection of particular properties of an interpolant, *e.g.* stability and regularity. The proposed approach begins, as in Chapter 6, by constructing the interpolant (6.10)-(6.11). Both new equations and new variables are then added in such a way that the properties of Loewner equivalence at (λ, r, m, ℓ) to (7.1)-(7.2) and matching of tangential data functions are preserved.

7.2 Regularization of Nonlinear Interpolants

Consider the dimensions of the auxiliary systems, ρ and v, arbitrary. Given the Loewner objects associated to the interconnected system (6.7)-(6.8) one can "augment" the interpolant (6.10)-(6.11) in Theorem 14 to produce more families of interpolants by adding new constraints and/or variables in such a way that the property of Loewner equivalence at (λ, r, m, ℓ) to the sys-

tem (7.1)-(7.2) is preserved. Consider now the augmented system described by the equations (7.8), (7.9), (7.10), (7.11), (7.12), (7.13), and (7.14) at the top of the next page, with state $\omega(t) \in \mathbb{C}^{\rho}$ and $\gamma(t) \in \mathbb{C}^{k-\rho}$, input $u_r(t) \in \mathbb{C}^m$, and output $y_r(t) \in \mathbb{C}^p$, where $Q : \mathbb{C}^{\rho} \times \mathbb{C}^{k-\rho} \to \mathbb{C}^v$, $P : \mathbb{C}^{\rho} \times \mathbb{C}^{k-\rho} \to \mathbb{C}^{k-v}$, $Z : \mathbb{C}^{\rho} \times \mathbb{C}^{k-\rho} \to \mathbb{C}^v$, $N : \mathbb{C}^{\rho} \times \mathbb{C}^{k-\rho} \to \mathbb{C}^{k-v}$, $U : \mathbb{C}^{\rho} \times \mathbb{C}^{k-\rho} \to \mathbb{C}^{v \times m}$, $T : \mathbb{C}^{\rho} \times \mathbb{C}^{k-\rho} \to \mathbb{C}^{(k-v) \times m}$, $H : \mathbb{C}^{\rho} \times \mathbb{C}^{k-\rho} \to \mathbb{C}^p$, and $K : \mathbb{C}^{\rho} \times \mathbb{C}^{k-\rho} \to \mathbb{C}^{p \times m}$, and $k \geq \max\{\rho, v\}$, where $Q(\cdot)$ and $P(\cdot)$ are differentiable and the Jacobian of

$$\begin{bmatrix} \mathbb{L}(\omega) + Q(\omega, \gamma) \\ P(\omega, \gamma) \end{bmatrix}, \tag{7.7}$$

has constant rank in a neighbourhood of the origin. The following theorem shows that the augmented system is still an interpolant in the Loewner sense.

Theorem 17. *Consider the interconnected system* (6.7)-(6.8). *Let* $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $V(\cdot)$, *and* $W(\cdot)$ *be the associated Loewner objects. Then the system* (7.8)-(7.14), *with* $P(\cdot)$ *and* $Q(\cdot)$ *chosen such that the Jacobian of* (7.7) *has constant rank in a neighbourhood of the origin, is Loewner equivalent at* (λ, r, m, ℓ) *to the system* (7.1)-(7.2) *and matches the tangential data functions* $W(\cdot)$ *and* $V(\cdot)$.

Proof. Let $X_e(\cdot)$, $Y_e(\cdot)$, $V_e(\cdot)$, $W_e(\cdot)$, $\mathbb{L}_e(\cdot)$, $\mathbb{L}_e^{\ell}(\cdot)$, $\mathbb{L}_e^{r}(\cdot)$, and $\sigma\mathbb{L}_e(\cdot)$ be the Loewner functions for the system (7.8)-(7.14) interconnected with the auxiliary systems (7.3)-(7.4) and (7.5)-(7.6). To prove that the extended inter-

$$\underbrace{\begin{bmatrix}\dot{\mathbb{L}}(\omega) + \dot{Q}(\omega,\gamma)\\ \dot{P}(\omega,\gamma)\end{bmatrix}}_{\dot{E}_e(\omega,\gamma)\,:=} = \underbrace{\begin{bmatrix}\sigma\mathbb{L}(\omega) - \overline{\ell}\Big(W(\omega)+H(\omega,\gamma), -K(\omega,\gamma)r(\omega)\Big)K(\omega,\gamma)r(\omega) + Z(\omega,\gamma)\\ N(\omega,\gamma)\end{bmatrix}}_{f_e(\omega,\gamma):=}$$

$$- \underbrace{\begin{bmatrix}V(\omega) - \overline{\ell}\Big(W(\omega)+H(\omega,\gamma), -K(\omega,\gamma)r(\omega)\Big)K(\omega,\gamma) + U(\omega,\gamma)\\ T(\omega,\gamma)\end{bmatrix}}_{-g_e(\omega,\gamma):=} u_r, \tag{7.8}$$

$$y_r = \underbrace{\Big(W(\omega)+H(\omega,\gamma)-K(\omega,\gamma)r(\omega)\Big)}_{h_e(\omega,\gamma):=} + \underbrace{K(\omega,\gamma)}_{k_e(\omega,\gamma):=} u_r, \tag{7.9}$$

$$Q(\omega,0) = 0, \tag{7.10}$$

$$U(\omega,0) = 0, \tag{7.11}$$

$$H(\omega,0) = 0, \tag{7.12}$$

$$\begin{aligned} Z(\omega,\gamma) = {} & \Big(m\big(\mathbb{L}(\omega)+Q(\omega,\gamma)\big) - m(\mathbb{L}(\omega))\Big) \\ & + \Big(\ell\big(W(\omega)+H(\omega,\gamma)\big) - \ell\big(W(\omega)\big)\Big), \end{aligned} \tag{7.13}$$

$$N(\omega,0) = \left(\frac{\partial P}{\partial \omega}\circ(\omega,0)\right)\lambda(\omega) + T(\omega,0)r(\omega), \tag{7.14}$$

---

polant (7.8)-(7.14) is Loewner equivalent at (λ, r, m, ℓ) to the system (7.1)-(7.2) it is shown that $X_e(\zeta_r) = (\zeta_r^*, 0^*_{(k-\rho)\times\rho})^*$, $Y_e(\kappa_1, \kappa_2) = -\kappa_1$ where $\kappa_1 \in \mathbb{C}^v$, $\kappa_2 \in \mathbb{C}^{k-v}$, and $\mathbb{L}_e^\ell(\zeta_r) = \mathbb{L}^\ell(\zeta_r)$ is a solution to the interpolant's system of PDEs corresponding to (5.11), (6.9), and (5.15). The tangential generalized controllability function, $X_e(\cdot)$, is the solution to the PDE with boundary condition

$$\frac{\partial\left(E_e(\omega,\gamma)\circ X_e(\zeta_r)\right)}{\partial\zeta_r}\lambda(\zeta_r) = \left(f_e(\omega,\gamma)\circ X_e(\zeta_r)\right) + \left(g_e(\omega,\gamma)\circ X_e(\zeta_r)\right)r(\zeta_r),$$
$$X_e(0) = 0,$$

where $E_e(\cdot)$, $f_e(\cdot)$, and $g_e(\cdot)$ are defined as in (7.8), and letting $X_e(\zeta_r) = (\zeta_r^*,\ 0^*_{(k-\rho)\times\rho})^*$ yields

$$\frac{\partial E_e(\zeta_r,0)}{\partial\zeta_r}\lambda(\zeta_r) = f_e(\zeta_r,0) + g_e(\zeta_r,0)r(\zeta_r),$$

or

$$\begin{bmatrix} \frac{\partial\mathbb{L}}{\partial\zeta_r}\lambda(\zeta_r) + \left(\frac{\partial Q}{\partial\omega}\circ(\zeta_r,0)\right)\lambda(\zeta_r) \\ \left(\frac{\partial P}{\partial\omega}\circ(\zeta_r,0)\right)\lambda(\zeta_r) \end{bmatrix}$$
$$= \begin{bmatrix} \sigma\mathbb{L}(\zeta_r) + Z(\zeta_r,0) \\ N(\zeta_r,0) \end{bmatrix} - \begin{bmatrix} V(\zeta_r) + U(\zeta_r,0) \\ T(\zeta_r,0) \end{bmatrix} r(\zeta_r),$$

which holds by (5.18) and (7.10)-(7.14). The tangential generalized observ-

ability function, $Y_e(\cdot)$, is the solution to the PDE with boundary condition

$$\left(\begin{bmatrix} \frac{\partial Y_e}{\partial \kappa_1} & \frac{\partial Y_e}{\partial \kappa_2} \end{bmatrix} \circ E_e(\omega,\gamma)\right) f_e(\omega,\gamma) = -m\left(-Y_e \circ E_e(\omega,\gamma)\right) - \ell\left(h_e(\omega,\gamma)\right),$$

$$Y_e(0) = 0,$$

where $h_e(\cdot)$ is defined as in (7.9). Setting

$$Y_e(\kappa_1,\kappa_2) = -\kappa_1 = -\begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \\ \kappa_2 \end{bmatrix},$$

yields

$$\begin{bmatrix} I & 0 \end{bmatrix} f_e(\omega,\gamma) = m\left(\begin{bmatrix} I & 0 \end{bmatrix} E_e(\omega,\gamma)\right) + \ell\left(h_e(\omega,\gamma)\right).$$

Using (5.7) this simplifies to

$$\begin{aligned} \sigma\mathbb{L}(\omega) + Z(\omega,\gamma) = m\big(\mathbb{L}(\omega) + Q(\omega,\gamma)\big) \\ + \ell\big(W(\omega) + H(\omega,\gamma)\big), \end{aligned}$$

which holds by (5.17) and (7.13). The equation for $W_e(\zeta_r)$ becomes then

$$\begin{aligned} W_e(\zeta_r) &= h_e(\zeta_r,0) + k_e(\zeta_r,0)r(\zeta_r) \\ &= W(\zeta_r) + H(\zeta_r,0), \end{aligned}$$

where $k_e(\cdot)$ is defined as in (7.9), and, by (7.12),

$$W_e(\zeta_r) = W(\zeta_r).$$

The equation for $V_e(\zeta_r)$ is

$$V_e(\zeta_r) = -\begin{bmatrix} I & 0 \end{bmatrix} g_e(\zeta_r, 0) + \overline{\ell}\Big(h_e(\zeta_r, 0),\ k_e(\zeta_r, 0) r(\zeta_r)\Big) k_e(\zeta_r, 0),$$

and using (5.8) and (7.11) this simplifies to

$$V_e(\zeta_r) = V(\zeta_r) + U(\zeta_r, 0) = V(\zeta_r).$$

If follows that the Loewner function is

$$\begin{aligned} \mathbb{L}_e(\zeta_r) &= -Y_e \circ E_e \circ X_e(\zeta_r) \\ &= \begin{bmatrix} I & 0 \end{bmatrix} E_e(\zeta_r, 0) \\ &= \mathbb{L}(\zeta_r) + Q(\zeta_r, 0) = \mathbb{L}(\zeta_r), \end{aligned}$$

and the shifted Loewner function is

$$\begin{aligned}\sigma\mathbb{L}_e(\zeta_r) &= m\left(\mathbb{L}_e(\zeta_r)\right) + \ell\left(W_e(\zeta_r)\right)\\ &= m\left(\mathbb{L}(\zeta_r)\right) + \ell\left(W(\zeta_r)\right)\\ &= \sigma\mathbb{L}(\zeta_r).\end{aligned}$$

The left Loewner function, $\mathbb{L}_e^\ell(\cdot)$, is the solution to the PDE with boundary condition

$$\frac{\partial\mathbb{L}_e^\ell}{\partial\zeta_r}\lambda(\zeta_r) = -m\left(-\mathbb{L}_e^\ell(\zeta_r)\right) - V_e(\zeta_r)r(\zeta_r), \qquad \mathbb{L}_e^\ell(0) = 0.$$

Letting $X_e(\zeta_r) = (\zeta_r^*,\ 0^*)^*$, $Y_e(\kappa_1,\kappa_2) = -\kappa_1$, and $\mathbb{L}_e^\ell(\zeta_r) = \mathbb{L}^\ell(\zeta_r)$ yields

$$\begin{aligned}\frac{\partial\mathbb{L}^\ell}{\partial\zeta_r}\lambda(\zeta_r) &= -m\left(-\mathbb{L}^\ell(\zeta_r)\right) - V_e(\zeta_r)r(\zeta_r)\\ &= -m\left(-\mathbb{L}^\ell(\zeta_r)\right) - V(\zeta_r)r(\zeta_r),\end{aligned}$$

which holds by (5.15). It follows that

$$\begin{aligned}\mathbb{L}_e^r(\zeta_r) &= \mathbb{L}_e(\zeta_r) - \mathbb{L}_e^\ell(\zeta_r)\\ &= \mathbb{L}(\zeta_r) - \mathbb{L}^\ell(\zeta_r) = \mathbb{L}^r(\zeta_r).\end{aligned}$$

Thus, the system (7.8)-(7.14) is Loewner equivalent at (λ, r, m, ℓ) to the system (7.1)-(7.2) and matches the tangential data functions $W(\cdot)$ and $V(\cdot)$.

□

Remark 23. The functions $K(\cdot)$, $P(\cdot)$, $Q(\cdot)$, $U(\cdot)$, $T(\cdot)$, and $H(\cdot)$ in Theorem 17 are "free", with $P(\cdot)$ and $Q(\cdot)$ having only to satisfy some mild differentiability conditions, and the requirements that $Q(\cdot, 0) = 0$, $U(\cdot, 0) = 0$, and $H(\cdot, 0) = 0$.

Remark 24. The conditions (7.13) and (7.14) on $Z(\cdot)$ and $N(\cdot)$, respectively, closely resemble the relations (5.17) and (5.18). This is not surprising as the mappings $Z(\cdot)$ and $N(\cdot)$ play a similar role to $\sigma\mathbb{L}(\cdot)$ in satisfying the PDEs in the proof of Theorem 17.

Theorem 17 is of great practical importance when constructing interpolants in the Loewner framework for nonlinear systems as it removes the requirement $\rho = v$. Also of significance is that if the Loewner objects are already known, then generating interpolants of higher complexity does not require solving any further PDEs as long as (7.10)-(7.14) hold. Theorem 17 allows for an approach to the treatment of both underdetermined and overdetermined interpolants, and the ability to construct interpolants of an arbitrarily given dimension $k \geq \max\{\rho, v\}$ further allows for an upper bound on the minimum number of states required to guarantee that a well-posed interpolant exists, namely $\rho + v$. To see this, consider the scenario in which $\mathbb{L}(\omega) = 0$. It may be that a locally well-posed differential-algebraic interpolant of dimension $k < \rho + v$ exists by considering a regularity condition and retaining algebraic constraints, however one could instead augment the

system in such a way that the equations are purely differential by choosing $Q(\omega, \gamma) = \gamma$ and $P(\omega, \gamma) = \omega$. Such a choice makes

$$\begin{bmatrix} \mathbb{L}(\omega) + Q(\omega, \gamma) \\ P(\omega, \gamma) \end{bmatrix} = \begin{bmatrix} \gamma \\ \omega \end{bmatrix}$$

a bijective map on $\mathbb{C}^{\rho+v}$ with nonsingular Jacobian, and thus a locally well-posed interpolant of dimension $k \leq \rho + v$ always exists. The upper bound on the minimum number of states required to guarantee the existence of a locally well-posed interpolant can be enhanced by considering the situation in which $\operatorname{rank} \frac{\partial \mathbb{L}}{\partial \zeta_r} = r$ in a neighbourhood of the origin. By the same argument, there always exists a locally well-posed interpolant of dimension $k \leq \rho + v - r$. Suppose, for example, that

$$\mathbb{L}(\omega) = \begin{bmatrix} I_r & 0_{r\times(\rho-r)} \\ 0_{(v-r)\times r} & 0_{(v-r)\times(\rho-r)} \end{bmatrix} \omega.$$

Then choosing $\gamma = (\gamma_1, \ldots, \gamma_{v-r})^\top$,

$$Q(\omega, \gamma) = \begin{bmatrix} 0_{r\times(v-r)} \\ I_{(v-r)} \end{bmatrix} \gamma,$$

and

$$P(\omega, \gamma) = \begin{bmatrix} 0_{(\rho-r)\times r} & I_{(\rho-r)} \end{bmatrix} \omega,$$

yields

$$\begin{bmatrix} \mathbb{L}(\omega) + Q(\omega, \gamma) \\ P(\omega, \gamma) \end{bmatrix} = \left[\begin{array}{cc|c} I_r & 0 & 0 \\ 0 & 0 & I_{(v-r)} \\ \hline 0 & I_{(\rho-r)} & 0 \end{array}\right] \begin{bmatrix} \omega \\ \hline \gamma \end{bmatrix},$$

which is a bijective map on $\mathbb{C}^{\rho+v-r}$ with nonsingular Jacobian. From this discussion the following corollary, the proof of which relies on a simple application of the constant rank theorem [80, pp. 472] on the Loewner function, holds.

Corollary 18. *Consider the interconnected system* (6.7)-(6.8)*. Let* $\mathbb{L}^{\ell}(\cdot)$*,* $\mathbb{L}^{r}(\cdot)$*,* $\mathbb{L}(\cdot)$*,* $\sigma\mathbb{L}(\cdot)$*,* $V(\cdot)$*, and* $W(\cdot)$ *be the associated Loewner objects. Suppose rank* $\frac{\partial \mathbb{L}}{\partial \zeta_r} = r$ *in a neightbourhood of the origin. Then there exists an integer* $k \in \{\max\{\rho, v\}, \rho + v - r\}$ *such that a locally well-posed interpolant of dimension* k *of the form* (7.8)-(7.14) *exists.*

7.2.1 Right Invertible (Surjective) Loewner Function

As discussed in Chapter 6, when the Loewner function, $\mathbb{L}(\zeta_r)$, is surjective with full row rank Jacobian in a neighbourhood of the origin then there exists a right inverse, $\mathbb{L}^{\#}(\psi)$, such that $\mathbb{L}\big(\mathbb{L}^{\#}(\psi)\big) = \psi$, and for the purposes of regularizing the system one can simply set[1] $\gamma = 0$ and $K(\omega, \gamma) = \overline{d}(\omega)$, yielding $P(\omega, \gamma) = P(\omega, 0)$, and $Q(\cdot)$, $Z(\cdot)$, $U(\cdot)$, and $H(\cdot)$ identically equal

[1]Setting $\gamma = 0$ in the system (7.8)-(7.14) corresponds to the scenario in which only new equations, and no new variables, are added.

to zero. Setting

$$T(\omega,0) := \left(\frac{\partial P}{\partial \omega}\circ(\omega,0)\right)\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right)^{\#} \times \Big(V(\omega) - \overline{\ell}\big(W(\omega), -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)\Big),$$

yields

$$N(\omega,0) = \left(\frac{\partial P}{\partial \omega}\circ(\omega,0)\right)\lambda(\omega) + \left(\frac{\partial P}{\partial \omega}\circ(\omega,0)\right)\left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right)^{\#} \times \Big(V(\omega) - \overline{\ell}\big(W(\omega), -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)\Big)r(\omega),$$

and the interpolant of Theorem 17 becomes (7.15)-(7.16).

$$\begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix}\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\\ \frac{\partial P}{\partial \omega}\circ(\omega,0)\end{bmatrix} \times \Bigg(\dot{\omega} - \lambda(\omega) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right)^{\#} V(\omega)\big(r(\omega)-u_r\big) + \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right)^{\#}\overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)r(\omega) - \left(\frac{\partial \mathbb{L}}{\partial \zeta_r}\circ\omega\right)^{\#}\overline{\ell}\big(W(\omega),\ -\overline{d}(\omega)r(\omega)\big)\overline{d}(\omega)u_r\Bigg), \quad (7.15)$$

$$y_r = \Big(W(\omega) - \overline{d}(\omega)r(\omega)\Big) + \overline{d}(\omega)u_r. \quad (7.16)$$

If $P(\omega, 0)$ is chosen such that $(\mathbb{L}(\omega)^*,\ P(\omega,0)^*)^*$ is bijective with nonsingular Jacobian in a neighbourhood of the origin, then this is equivalent to the system (6.15)-(6.16). The system (6.15)-(6.16) is in a standard state-space form and makes explicit the encoding of the auxiliary system (7.3)-(7.4) and the tangential data function $W(\cdot)$ into the interpolant, *i.e.* setting $u_r = r(\omega)$ yields

$$\dot{\omega} = \lambda(\omega), \quad y_r = W(\omega).$$

7.2.2 Left Invertible (Injective) Loewner Function

A similar result can be obtained for the situation in which the Loewner function is injective with full column rank Jacobian in a neighbourhood of the origin. Particularly, when the Loewner function, $\mathbb{L}(\zeta_r)$, is injective with full column rank Jacobian then there exists a left inverse, $\mathbb{L}^{\#}(\psi)$, such that $\mathbb{L}^{\#}\big(\mathbb{L}(\zeta_r)\big) = \zeta_r$, and for the purposes of regularizing the system one can set the functions[2] $P(\cdot)$, $N(\cdot)$, and $T(\cdot)$ to zero. Choosing $Q(\cdot)$ such that

$$\psi := \mathbb{L}(\omega) + Q(\omega, \gamma),$$

[2]Setting $P(\omega,\gamma) = 0$, $N(\omega,\gamma) = 0$, and $T(\omega,\gamma) = 0$ in the system (7.8)-(7.14) makes the newly added $k - v$ equations $0 = 0$. This, effectively, corresponds to the scenario in which only new variables, and no new equations, are added.

is bijective with nonsingular Jacobian in a neighbourhood of the origin, and setting

$$H(\omega,\gamma) = W\big(\mathbb{L}^{\#}(\psi)\big) - W(\omega) + \overline{d}\big(\mathbb{L}^{\#}(\psi)\big)\big(r(\omega) - r(\mathbb{L}^{\#}(\psi))\big),$$

$$\begin{aligned} U(\omega,\gamma) = {} & V\big(\mathbb{L}^{\#}(\psi)\big) - V(\omega) \\ & + \overline{\ell}\Big(W\big(\mathbb{L}^{\#}(\psi)\big) + \overline{d}\big(\mathbb{L}^{\#}(\psi)\big)\big(r(\omega) - r(\mathbb{L}^{\#}(\psi))\big), \\ & \qquad - \overline{d}\big(\mathbb{L}^{\#}(\psi)\big) r(\omega)\Big)\overline{d}\big(\mathbb{L}^{\#}(\psi)\big) \\ & - \overline{\ell}\Big(W\big(\mathbb{L}^{\#}(\psi)\big),\ -\overline{d}\big(\mathbb{L}^{\#}(\psi)\big) r\big(\mathbb{L}^{\#}(\psi)\big)\Big)\overline{d}\big(\mathbb{L}^{\#}(\psi)\big), \end{aligned}$$

and

$$K(\omega,\gamma) = \overline{d}\big(\mathbb{L}^{\#}(\psi)\big),$$

yields the interpolant

$$\begin{aligned} \dot{\psi} = {} & \Big(m(\psi) + \ell\big(W(\mathbb{L}^{\#}(\psi)) - \overline{d}(\mathbb{L}^{\#}(\psi)) r(\mathbb{L}^{\#}(\psi))\big)\Big) \\ & - \Big(V\big(\mathbb{L}^{\#}(\psi)\big) - \overline{\ell}\big(W(\mathbb{L}^{\#}(\psi)),\ -\overline{d}(\mathbb{L}^{\#}(\psi)) r(\mathbb{L}^{\#}(\psi))\big)\overline{d}\big(\mathbb{L}^{\#}(\psi)\big)\Big) u_r, \end{aligned} \tag{7.17}$$

$$y_r = \Big(W\big(\mathbb{L}^{\#}(\psi)\big) - \overline{d}\big(\mathbb{L}^{\#}(\psi)\big) r\big(\mathbb{L}^{\#}(\psi)\big)\Big) + \overline{d}\big(\mathbb{L}^{\#}(\psi)\big) u_r. \tag{7.18}$$

Similar to the case in which the Loewner function is surjective with full row rank Jacobian, this system is in a standard state-space form and makes explicit the encoding of the auxiliary system (7.5)-(7.6) into the interpolant, *i.e.* setting $u_r = 0$ yields

$$\dot{\psi} = m(\psi) + \ell(y_r).$$

7.3 Regularization of Linear Interpolants

Specializing the results of Section 7.2 to the linear differential-algebraic setting yields an alternative approach to that given in [94] when considering the construction of well-posed interpolants. Consider again the linear auxiliary systems

$$\dot{\zeta}_r = \Lambda\zeta_r + \Delta, \tag{7.19}$$

$$v = R\zeta_r, \tag{7.20}$$

and

$$\dot{\zeta}_\ell = M\zeta_\ell + L\chi, \tag{7.21}$$

$$\eta = \zeta_\ell, \tag{7.22}$$

and the linear system

$$E\dot{x} = Ax + Bu, \tag{7.23}$$

$$y = Cx + Du, \tag{7.24}$$

generating the tangential data.

Considering Theorem 17 and noting that, by the conditions (7.10)-(7.14), for linear systems $Q(\omega,\gamma) = Q\gamma$, $P(\omega,\gamma) = P\omega + G\gamma$, $N(\omega,\gamma) = N\omega + F\gamma$, $U(\omega,\gamma) = 0$, $T(\omega,\gamma) = T$, $H(\omega,\gamma) = H\gamma$, and $K(\omega,\gamma) = \overline{D}$, it follows that the interpolant (7.8)-(7.14) becomes

$$\begin{bmatrix} \mathbb{L} & Q \\ P & G \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \sigma\mathbb{L} - L\overline{D}R & Z \\ N & F \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} - \begin{bmatrix} V - L\overline{D} \\ T \end{bmatrix} u_r, \tag{7.25}$$

$$y_r = \begin{bmatrix} W - \overline{D}R & H \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} + \overline{D}u_r, \tag{7.26}$$

with

$$Z = MQ + LH, \tag{7.27}$$

$$N = P\Lambda + TR, \tag{7.28}$$

where the matrices P, Q, G, $\overline{D}$, F, T, and H are free parameters. Hence, the following result is obtained directly from Theorem 17.

Corollary 19. *Consider the interconnected system resulting from the sys-*

tem (7.23)-(7.24) *and the auxiliary systems* (7.19)-(7.20) *and* (7.21)-(7.22) *with the interconnection equations* $u = v$ *and* $\chi = y$. *Let* $\mathbb{L}^\ell$, $\mathbb{L}^r$, $\mathbb{L}$, $\sigma\mathbb{L}$, V, *and* W *be the associated Loewner objects. Then the system* (7.25), (7.26), (7.27), *and* (7.28) *is Loewner equivalent at* (Λ, R, M, L) *to the system* (7.23)-(7.24) *and matches the tangential data matrices* W *and* V.

Similarly, the following result is obtained directly from Corollary 18.

Corollary 20. *Consider the interconnected system resulting from the system* (7.23)-(7.24) *and the auxiliary systems* (7.19)-(7.20) *and* (7.21)-(7.22) *with the interconnection equations* $u = v$ *and* $\chi = y$. *Let* $\mathbb{L}^\ell$, $\mathbb{L}^r$, $\mathbb{L}$, $\sigma\mathbb{L}$, V, *and* W *be the associated Loewner objects. Suppose rank* $\mathbb{L} = r$. *Then there exists an integer* $k \in \{\max\{\rho, v\}, \rho + v - r\}$ *such that a well-posed interpolant of dimension* k *of the form* (7.25)-(7.28) *exists.*

7.3.1 Right Invertible Loewner Matrix

Similar to Section 7.2.1, consider the scenario in which the Loewner matrix, $\mathbb{L}$, has full row rank, *i.e.* $\mathbb{L} \in \mathbb{C}^{v \times \rho}$, $\rho \geq v$, rank $\mathbb{L} = v$. Let $\mathbb{L}^\#$ be any right inverse of $\mathbb{L}$. Setting Q, G, Z, F, and H equal to zero, letting P be any matrix such that $[\mathbb{L}^* \ P^*]^*$ is nonsingular, and setting

$$T := P\mathbb{L}^\# \left(V - L\overline{D}\right),$$

yields

$$N = P\big(\Lambda + \mathbb{L}^{\#}\left(V - L\overline{D}\right)R\big).$$

Thus the interpolant of Corollary 19 becomes

$$\begin{bmatrix}\mathbb{L}\\ P\end{bmatrix}\dot{\omega} = \begin{bmatrix}\mathbb{L}\\ P\end{bmatrix}\left(\Lambda + \mathbb{L}^{\#}\left(V - L\overline{D}\right)R\right)\omega - \begin{bmatrix}\mathbb{L}\\ P\end{bmatrix}\mathbb{L}^{\#}\left(V - L\overline{D}\right)u_r,$$

$$y_r = \left(W - \overline{D}R\right)\omega + \overline{D}u_r,$$

which is equivalent to

$$\dot{\omega} = \left(\Lambda + \mathbb{L}^{\#}\left(V - L\overline{D}\right)R\right)\omega - \mathbb{L}^{\#}\left(V - L\overline{D}\right)u_r,$$

$$y_r = \left(W - \overline{D}R\right)\omega + \overline{D}u_r,$$

that is the interpolant (2.32)-(2.33) given in [94] for the scenario in which the Loewner matrix has full row rank.

7.3.2 Left Invertible Loewner Matrix

In [94] a similar result regarding construction of an interpolant for the scenario in which the Loewner matrix has full column rank has not been explicitly presented. Such a result is easily obtained from Corollary 19. Let $\mathbb{L}^{\#}$ be any left inverse of $\mathbb{L}$. Setting P, G, N, F, and T equal to zero, letting Q be

any matrix such that $[\mathbb{L} \;\; Q]$ is nonsingular, and setting

$$H := \left(W - \overline{D}R\right) \mathbb{L}^{\#} Q,$$

yields

$$Z = \left(M + L\left(W - \overline{D}R\right) \mathbb{L}^{\#}\right) Q.$$

The interpolant of Corollary 19 thus becomes

$$\begin{bmatrix} \mathbb{L} & Q \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \dot{\gamma} \end{bmatrix} = \left(M + L\left(W - \overline{D}R\right) \mathbb{L}^{\#}\right) \begin{bmatrix} \mathbb{L} & Q \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} - \left(V - L\overline{D}\right) u_r,$$

$$y_r = \left(W - \overline{D}R\right) \mathbb{L}^{\#} \begin{bmatrix} \mathbb{L} & Q \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} + \overline{D} u_r,$$

and defining the new set of coordinates $\psi := \mathbb{L}\omega + Q\gamma$ yields

$$\begin{aligned} \dot{\psi} &= \left(M + L\left(W - \overline{D}R\right) \mathbb{L}^{\#}\right) \psi - \left(V - L\overline{D}\right) u_r, \\ &= M\psi + L y_r - V u_r, \\ y_r &= \left(W - \overline{D}R\right) \mathbb{L}^{\#} \psi + \overline{D} u_r. \end{aligned}$$

7.3.3 Parameterization of All Linear Interpolants

Suppose that R has full row rank and L has full column rank. Then the family of systems given by (7.25)-(7.28) allows for the parameterization of all Loewner equivalent interpolants having an arbitrarily given dimension $k \geq \max\{\rho, v\}$ and possessing full column rank tangential generalized controllability matrix and full row rank tangential generalized observability matrix, as is shown in the following theorem.

Theorem 21. *Consider the interconnected system resulting from the system* (7.23)-(7.24) *and the auxiliary systems* (7.19)-(7.20) *and* (7.21)-(7.22) *with the interconnection equations* $u = v$ *and* $\chi = y$. *Let* $\mathbb{L}^{\ell}$, $\mathbb{L}^{r}$, $\mathbb{L}$, $\sigma\mathbb{L}$, V, *and* W *be the associated Loewner objects. Let* k *be any integer such that* $k \geq \max\{\rho, v\}$. *Then the system* (7.25)-(7.28) *parameterizes all systems of order* k *that are Loewner equivalent at* (Λ, R, M, L) *to the system* (7.23)-(7.24) *and possess full rank tangential generalized controllability and observability matrices.*

Proof. Let $k \geq \max\{\rho, v\}$, and let

$$J\dot{\eta} = K\eta + Su, \qquad y = U\eta + \Delta u, \tag{7.29}$$

with state $\eta(t) \in \mathbb{C}^k$, input $u(t) \in \mathbb{C}^m$, and output $y(t) \in \mathbb{C}^p$, be any system of dimension k having full column rank tangential generalized controllability matrix, $\widehat{X}$, and full row rank tangential generalized observability matrix, $\widehat{Y}$,

which are solutions to the generalized Sylvester equations

$$J\widehat{X}\Lambda = K\widehat{X} + SR, \quad \widehat{Y}K = M\widehat{Y}J - LU,$$

such that

$$\mathbb{L} = -\widehat{Y}J\widehat{X}, \quad \sigma\mathbb{L} = -\widehat{Y}K\widehat{X} + L\Delta R.$$

It then follows that

$$\widehat{V}R = \sigma\mathbb{L} - \mathbb{L}\Lambda = VR, \quad L\widehat{W} = \sigma\mathbb{L} - M\mathbb{L} = LW,$$

so $\widehat{V} = V$ and $\widehat{W} = W$ because R has full row rank and L has full column rank. Furthermore, $\widehat{V} = \widehat{Y}S + L\Delta$ and $\widehat{W} = U\widehat{X} + \Delta R$. Consider now the system (7.25)-(7.28) and select

$$P = \beta J\widehat{X}, \quad T = -\beta S, \quad Q = -\widehat{Y}J\theta,$$
$$H = U\theta, \quad G = \beta J\theta, \quad F = \beta K\theta, \quad \overline{D} = \Delta,$$

where β and θ are any matrices such that $[-\widehat{Y}^* \ \beta^*]^*$ and $[\widehat{X} \ \theta]$ are nonsingular.

Then

$$\begin{bmatrix} \mathbb{L} & Q \\ P & G \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \sigma\mathbb{L} - L\overline{D}R & Z \\ N & F \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} - \begin{bmatrix} V - L\overline{D} \\ T \end{bmatrix} u_r,$$

$$y_r = \begin{bmatrix} W - \overline{D}R & H \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} + \overline{D}u_r,$$

can be rewritten as

$$\begin{bmatrix} -\widehat{Y} \\ \beta \end{bmatrix} J \begin{bmatrix} \widehat{X} & \theta \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} -\widehat{Y} \\ \beta \end{bmatrix} \left(K \begin{bmatrix} \widehat{X} & \theta \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} + Su_r \right),$$

$$y_r = U \begin{bmatrix} \widehat{X} & \theta \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} + \Delta u_r.$$

Since the matrix $[-\widehat{Y}^* \ \beta^*]^*$ is nonsingular this is equivalent to

$$J \begin{bmatrix} \widehat{X} & \theta \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \dot{\gamma} \end{bmatrix} = K \begin{bmatrix} \widehat{X} & \theta \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} + Su_r,$$

$$y_r = U \begin{bmatrix} \widehat{X} & \theta \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} + \Delta u_r,$$

which, defining the new coordinates $\overline{\eta} := \widehat{X}\omega + \theta\gamma$, yields

$$J\dot{\overline{\eta}} = K\overline{\eta} + Su_r, \qquad y_r = U\overline{\eta} + \Delta u_r.$$

Thus, the family of systems given by (7.25)-(7.28) contains the system (7.29).

□

Remark 25. As a consequence of Theorem 21, to build any interpolant of an arbitrarily given order $k \geq \max\{\rho, v\}$ it suffices to solve the generalized Sylvester equations (or build the Loewner objects) just once. Every other interpolant of the same order having full rank tangential generalized controllability and observability matrices may then be constructed in a computationally cheap algebraic fashion using the equations (7.25)-(7.28).

Remark 26. Consider the problem of constructing an interpolant possessing some desired properties. If an interpolant of order $k \geq \max\{\rho, v\}$ having full rank tangential generalized controllability and observability matrices and possessing the desired properties exists, then it is realizable in the form (7.25)-(7.28).

Remark 27. The approach provided by [94, Theorem 5.2] considers the construction of interpolants having dimension $k \leq \min\{\rho, v\}$, whereas the results of this section provide a method for the construction of interpolants having dimension $k \geq \max\{\rho, v\}$. Hence, the results are complementary, and taken together they provide a more broad framework for the construction of interpolants in the linear setting.

7.4 Parameterization of All Nonlinear Interpolants

A slightly modified generalization of the parameterization result in Theorem 21 can be given. While the interpolant (7.8)-(7.14) matches the tangential data functions $W(\cdot)$ and $V(\cdot)$, the property of Loewner equivalence alone is weaker in some situations than matching the tangential data[3], hence the following result considers parameterization of systems matching the functions $W(\cdot)$ and $V(\cdot)$.

Theorem 22. *Consider the interconnected system* (6.7)-(6.8). *Let* $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $V(\cdot)$, *and* $W(\cdot)$ *be the associated Loewner objects, and suppose* $\mathbb{L}(\cdot)$ *is the unique analytic solution of the PDE* (5.16). *Let* k *be any integer such that* $k \geq \max\{\rho, v\}$. *Then the system* (7.8)-(7.14) *parameterizes all systems of order* k *that match the tangential data* $V(\cdot)$ *and* $W(\cdot)$ *at* (λ, r, m, ℓ), *while possessing tangential generalized controllability and observability functions with full row and column rank Jacobians in a neighbourhood of the origin, respectively.*

[3]Loewner equivalence is necessary and sufficient for matching the tangential data matrices in the linear setting when R and L have full row and column rank, respectively, and it is equivalent to matching the tangential data functions for single-input nonlinear systems when $r(\cdot)$ and $\ell(\cdot)$ are surjective and injective, respectively. In the nonlinear multiple-input setting this may not be the case since $r(\cdot)$ and $V(\cdot)$ may be such that $V(\zeta_r)r(\zeta_r) = \widehat{V}(\zeta_r)r(\zeta_r) \not\Rightarrow V(\zeta_r) = \widehat{V}(\zeta_r)$.

Proof. Let $k \geq \max\{\rho, v\}$, and let

$$\dot{\widehat{J(\eta)}} = \Gamma(\eta) + S(\eta)u, \qquad y = G(\eta) + \Delta(\eta)u, \tag{7.30}$$

with state $\eta(t) \in \mathbb{C}^k$, input $u(t) \in \mathbb{C}^m$, and output $y(t) \in \mathbb{C}^p$, be any well-posed[4] system of the form (7.1)-(7.2) of dimension k, and having tangential generalized controllability function, $\widehat{X}(\cdot)$, with full column rank Jacobian and tangential generalized observability function, $\widehat{Y}(\cdot)$, with full row rank Jacobian, which are solutions to the PDEs

$$\frac{\partial\Big(J\big(\widehat{X}(\zeta_r)\big)\Big)}{\partial\zeta_r}\lambda(\zeta_r) = \Gamma\big(\widehat{X}(\zeta_r)\big) + S\big(\widehat{X}(\zeta_r)\big)r(\zeta_r), \qquad \widehat{X}(0) = 0,$$

$$\left(\frac{\partial\widehat{Y}}{\partial\eta} \circ J(\eta)\right)\Gamma(\eta) = -m\left(-\widehat{Y}\big(J(\eta)\big)\right) - \ell\big(G(\eta)\big), \qquad \widehat{Y}(0) = 0,$$

such that

$$\begin{aligned}\widehat{V}(\zeta_r) &= \left(\frac{\partial\widehat{Y}}{\partial\eta} \circ J\big(\widehat{X}(\zeta_r)\big)\right) S\big(\widehat{X}(\zeta_r)\big) \\ &\quad + \overline{\ell}\left(G\big(\widehat{X}(\zeta_r)\big),\ \Delta\big(\widehat{X}(\zeta_r)\big)r(\zeta_r)\right)\Delta\big(\widehat{X}(\zeta_r)\big) \\ &= V(\zeta_r),\end{aligned}$$

[4]The system (7.1)-(7.2) is well-posed if $k = n$ and solutions (locally) exist and are unique.

and

$$\widehat{W}(\zeta_r) = G\big(\widehat{X}(\zeta_r)\big) + \Delta\big(\widehat{X}(\zeta_r)\big)r(\zeta_r) = W(\zeta_r).$$

Then the corresponding Loewner function, $\widehat{\mathbb{L}}(\cdot)$, is $\widehat{\mathbb{L}}(\zeta_r) = -\widehat{Y}\big(J(\widehat{X}(\zeta_r))\big)$ and satisfies the PDE

$$\begin{aligned}\frac{\partial\widehat{\mathbb{L}}}{\partial\zeta_r}\lambda(\zeta_r) - m\big(\widehat{\mathbb{L}}(\zeta_r)\big) &= \ell\big(\widehat{W}(\zeta_r)\big) - \widehat{V}(\zeta_r)r(\zeta_r)\\ &= \ell\big(W(\zeta_r)\big) - V(\zeta_r)r(\zeta_r).\end{aligned}$$

Because $\mathbb{L}(\cdot)$ is the unique solution of the PDE (5.16), it follows that

$$\widehat{\mathbb{L}}(\zeta_r) = -\widehat{Y}\big(J(\widehat{X}(\zeta_r))\big) = \mathbb{L}(\zeta_r),$$

and

$$\begin{aligned}\widehat{\sigma\mathbb{L}}(\zeta_r) = &-\left(\frac{\partial\widehat{Y}}{\partial\eta}\circ J\big(\widehat{X}(\zeta_r)\big)\right)\Gamma\big(\widehat{X}(\zeta_r)\big)\\ &+\bar{\ell}\left(G\big(\widehat{X}(\zeta_r)\big),\ \Delta\big(\widehat{X}(\zeta_r)\big)r(\zeta_r)\right)\Delta\big(\widehat{X}(\zeta_r)\big)r(\zeta_r)\\ &= \sigma\mathbb{L}(\zeta_r).\end{aligned}$$

Let $\theta(\cdot)$ be any function such that $\widehat{X}(\omega) + \theta(\omega,\gamma)$ has a nonsingular Jacobian at the origin and $\theta(\omega,0) = 0$, and let $\beta(\cdot)$ be any function such that $[-\widehat{Y}(\eta)^*\ \beta(\eta)^*]^*$ has a nonsingular Jacobian in a neighbourhood of the

origin. Consider now the system (7.8)-(7.14), choose a set of coordinates $\overline{\eta} := \widehat{X}(\omega) + \theta(\omega, \gamma)$, and select

$$P(\omega, \gamma) := \beta\big(J(\overline{\eta})\big),$$

$$T(\omega, \gamma) := -\left(\frac{\partial \beta}{\partial \eta} \circ J(\overline{\eta})\right) S(\overline{\eta}),$$

$$\begin{aligned} Q(\omega, \gamma) &:= -\widehat{Y}\big(J(\overline{\eta})\big) + \widehat{Y}\big(J(\widehat{X}(\omega))\big) \\ &= -\widehat{Y}\big(J(\overline{\eta})\big) - \mathbb{L}(\omega), \end{aligned}$$

$$N(\omega, \gamma) := \left(\frac{\partial \beta}{\partial \eta} \circ J(\overline{\eta})\right) \Gamma(\overline{\eta}),$$

$$\begin{aligned} H(\omega, \gamma) &:= \Big(G(\overline{\eta}) - G\big(\widehat{X}(\omega)\big)\Big) + \Big(\Delta(\overline{\eta}) - \Delta\big(\widehat{X}(\omega)\big)\Big)\, r(\omega) \\ &= G(\overline{\eta}) + \Delta(\overline{\eta}) r(\omega) - W(\omega), \end{aligned}$$

$$
\begin{aligned}
U(\omega,\gamma) := &\left(\frac{\partial \widehat{Y}}{\partial \eta} \circ J(\overline{\eta})\right) S(\overline{\eta}) - \left(\frac{\partial \widehat{Y}}{\partial \eta} \circ J\big(\widehat{X}(\omega)\big)\right) S\big(\widehat{X}(\omega)\big) \\
&+ \overline{\ell}\Big(G(\overline{\eta}),\ \Delta(\overline{\eta}) r(\omega)\Big)\Delta(\overline{\eta}) \\
&- \overline{\ell}\Big(G\big(\widehat{X}(\omega)\big),\ \Delta\big(\widehat{X}(\omega)\big) r(\omega)\Big)\Delta\big(\widehat{X}(\omega)\big) \\
= &\left(\frac{\partial \widehat{Y}}{\partial \eta} \circ J(\overline{\eta})\right) S(\overline{\eta}) - V(\omega) + \overline{\ell}\Big(G(\overline{\eta}),\ \Delta(\overline{\eta}) r(\omega)\Big)\Delta(\overline{\eta}),
\end{aligned}
$$

and

$$
K(\omega,\gamma) := \Delta(\overline{\eta}).
$$

It follows that $Q(\omega,0) = 0$, $U(\omega,0) = 0$, $H(\omega,0) = 0$,

$$
\begin{aligned}
N(\omega,0) &= \left(\frac{\partial \beta}{\partial \eta} \circ J\big(\widehat{X}(\omega)\big)\right) \Gamma\big(\widehat{X}(\omega)\big) \\
&= \left(\frac{\partial \beta}{\partial \eta} \circ J\big(\widehat{X}(\omega)\big)\right) \left(\frac{\partial J}{\partial \eta} \circ \widehat{X}(\omega)\right) \frac{\partial \widehat{X}}{\partial \omega} \lambda(\omega) \\
&\quad - \left(\frac{\partial \beta}{\partial \eta} \circ J\big(\widehat{X}(\omega)\big)\right) S\big(\widehat{X}(\omega)\big) r(\omega) \\
&= \left(\frac{\partial P}{\partial \omega} \circ (\omega,0)\right) \lambda(\omega) + T(\omega,0) r(\omega),
\end{aligned}
$$

and

$$
\begin{aligned}
Z(\omega,\gamma) &= \Big(m\big(\mathbb{L}(\omega)+Q(\omega,\gamma)\big)-m\big(\mathbb{L}(\omega)\big)\Big) \\
&\qquad + \Big(\ell\big(W(\omega)+H(\omega,\gamma)\big)-\ell\big(W(\omega)\big)\Big) \\
&= m\left(-\widehat{Y}\big(J(\overline{\eta})\big)\right)-\sigma\mathbb{L}(\omega)+\ell\Big(G(\overline{\eta})+\Delta(\overline{\eta})r(\omega)\Big) \\
&= -\left(\frac{\partial\widehat{Y}}{\partial\eta}\circ J(\overline{\eta})\right)\Gamma(\overline{\eta})-\sigma\mathbb{L}(\omega) \\
&\qquad + \overline{\ell}\Big(G(\overline{\eta}),\ \Delta(\overline{\eta})r(\omega)\Big)\Delta(\overline{\eta})r(\omega).
\end{aligned}
$$

Then the system (7.8)-(7.14) becomes

$$
\begin{bmatrix} \dot{\overbrace{-\widehat{Y}\big(J(\overline{\eta})\big)}} \\ \dot{\overbrace{\beta\big(J(\overline{\eta})\big)}} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial\widehat{Y}}{\partial\eta}\circ J(\overline{\eta}) \\ \dfrac{\partial\beta}{\partial\eta}\circ J(\overline{\eta}) \end{bmatrix}\Big(\Gamma(\overline{\eta})+S(\overline{\eta})u_r\Big),
$$

$$
y_r = G(\overline{\eta})+\Delta(\overline{\eta})u_r,
$$

which is equivalent to

$$
\dot{\overbrace{J(\overline{\eta})}} = \Gamma(\overline{\eta})+S(\overline{\eta})u_r, \qquad y_r = G(\overline{\eta})+\Delta(\overline{\eta})u_r,
$$

because the Jacobian of $[-\widehat{Y}(\eta)^*\ \ \beta(\eta)^*]^*$ is nonsingular. Thus, the system (7.8)-(7.14) parameterizes the system (7.30). □

7.5 Example - When A Stable Interpolant Can Only Be Constructed by Dynamic Extension

Since, for any $\overline{D} \in \mathbb{C}^{p\times m}$, the poles of the interpolant (2.30)-(2.31), *i.e.* the poles of the system

$$\begin{aligned}\mathbb{L}\dot{\omega} &= (\sigma\mathbb{L} - L\overline{D}R)\omega - (V - L\overline{D})u,\\ y &= (W - \overline{D}R)\omega + \overline{D}u,\end{aligned}$$

are given by the generalized eigenvalues of $(\mathbb{L}, \sigma\mathbb{L} - L\overline{D}R)$, then selecting a stable interpolant using the free parameter $\overline{D}$ is only possible if one can solve a static output feedback problem for the system of equations

$$\mathbb{L}\,\dot{\overline{\omega}} = \sigma\mathbb{L}\,\overline{\omega} + L\,\overline{u}, \qquad \overline{y} = -R\,\overline{\omega}.$$

It is well-known that there exist controllable and observable systems which are not static output feedback stabilizable, hence there exist sets of tangential interpolation data for which a stable interpolant of minimal dimension cannot be constructed. Consider now the set of right tangential interpolation data

given in the form of the real-valued matrices

$$\Lambda = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad W = \begin{bmatrix} 3 & -1 \end{bmatrix},$$

and the set of left tangential interpolation data given in the form of the real-valued matrices

$$M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad V = \begin{bmatrix} 3 \\ -1 \end{bmatrix},$$

respectively. The Loewner matrix and the shifted Loewner matrix resulting from the tangential data, determined as the unique solutions to the Sylvester equations (2.21) and (2.22), respectively, are

$$\mathbb{L} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma\mathbb{L} = \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix}.$$

By Theorem 21, all interpolants of dimension $k = 2$, parameterized by the feedforward term $\overline{D}$, are given by

$$\mathbb{L}\dot{\omega} = (\sigma\mathbb{L} - L\overline{D}R)\omega - (V - L\overline{D})u,$$

$$y_{ROM1} = (W - \overline{D}R)\omega + \overline{D}u,$$

or

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & -(1+\overline{D}) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1+\overline{D} \end{bmatrix} u, \tag{7.31}$$

$$y_{ROM1} = \begin{bmatrix} 3 & -(1+\overline{D}) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} + \overline{D}u. \tag{7.32}$$

Unfortunately, it is easy to see that for any $\overline{D} \in \mathbb{R}$ the resulting interpolant is unstable because

$$\det(s\mathbb{L} - \sigma\mathbb{L} + L\overline{D}R) = s^2 + (1+\overline{D})s - 2.$$

Hence, it is only possible to obtain a stable interpolant if it is of dimension larger than 2.

It turns out that an observer and state-feedback controller can be "embedded" into an interpolant of higher dimension. Consider the parameterization of interpolants in Theorem 21, with dimension $k = 4$, given by

$$\begin{bmatrix} \mathbb{L} & Q \\ P & G \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \sigma\mathbb{L} - L\overline{D}R & MQ + LH \\ P\Lambda + TR & F \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} - \begin{bmatrix} V - L\overline{D} \\ T \end{bmatrix} u,$$

$$y = \begin{bmatrix} W - \overline{D}R & H \end{bmatrix} \begin{bmatrix} \omega \\ \gamma \end{bmatrix} + \overline{D}u,$$

and select

$$P = 0_{2\times 2}, \quad Q = 0_{2\times 2}, \quad G = \mathbb{L}, \quad F = \sigma\mathbb{L} - L\overline{D}R - TR + LH,$$

where

$$T = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} \in \mathbb{R}^{2\times 1},$$

and

$$H = \begin{bmatrix} H_1 & H_2 \end{bmatrix} \in \mathbb{R}^{1\times 2},$$

are the observer and state-feedback gains, respectively. This yields

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & -(1+\overline{D}) & H_1 & H_2 \\ 0 & T_1 & 0 & 1-T_1 \\ 0 & T_2 & 2+H_1 & H_2 - 1 - \overline{D} - T_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1+\overline{D} \\ -T_1 \\ -T_2 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 3 & -(1+\overline{D}) & H_1 & H_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} + \overline{D}u.$$

Defining a new set of coordinates corresponding to the state estimation errors,

$e := \gamma - \omega$, yields the representation

$$\begin{bmatrix} \mathbb{L} & 0 \\ 0 & \mathbb{L} \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \sigma\mathbb{L} - L\overline{D}R + LH & LH \\ 0 & \sigma\mathbb{L} - L\overline{D}R - TR \end{bmatrix} \begin{bmatrix} \omega \\ e \end{bmatrix} - \begin{bmatrix} V - L\overline{D} \\ T - V + L\overline{D} \end{bmatrix} u,$$

$$y = \begin{bmatrix} W - \overline{D}R + H & H \end{bmatrix} \begin{bmatrix} \omega \\ e \end{bmatrix} + \overline{D}u,$$

or

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2+H_1 & H_2-(1+\overline{D}) & H_1 & H_2 \\ 0 & 0 & 0 & 1-T_1 \\ 0 & 0 & 2 & -(T_2+1+\overline{D}) \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1+\overline{D} \\ 3-T_1 \\ -(T_2+1+\overline{D}) \end{bmatrix} u, \tag{7.33}$$

$$y = \begin{bmatrix} H_1+3 & H_2-(1+\overline{D}) & H_1 & H_2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ e_1 \\ e_2 \end{bmatrix} + \overline{D}u. \tag{7.34}$$

If the free parameters are selected such that

$$H_1 < -2, \qquad H_2 < 1+\overline{D}, \qquad T_1 > 1, \qquad T_2 > -(1+\overline{D}),$$

then the interpolant of the tangential data and Loewner matrices given by (7.33)-(7.34) is exponentially stable.

For example, consider setting

$$\overline{D} = 0, \qquad H_1 = -10, \qquad H_2 = -1, \qquad T_1 = 2, \qquad T_2 = 4,$$

in (7.33)-(7.34) to obtain

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & -1 & -10 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 4 & -8 & -6 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ -2 \\ -4 \end{bmatrix} u, \tag{7.35}$$

$$y_{ROM2} = \begin{bmatrix} 3 & -1 & -10 & -1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}, \tag{7.36}$$

or

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -8 & -2 & -10 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -5 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 1 \\ -5 \end{bmatrix} u,$$

$$y_{ROM2} = \begin{bmatrix} -7 & -2 & -10 & -1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ e_1 \\ e_2 \end{bmatrix},$$

which is a controllable and observable system with poles at -0.4384, $-1 \pm j2.6458$, and -4.5616, and which matches the tangential data by construction.

Alternatively, consider setting

$$\overline{D} = 0, \qquad H_1 = -3, \qquad H_2 = -1, \qquad T_1 = 2, \qquad T_2 = 4,$$

in (7.33)-(7.34) to obtain

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\gamma}_1 \\ \dot{\gamma}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & -1 & -3 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 4 & -1 & -6 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma_1 \\ \gamma_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ -2 \\ -4 \end{bmatrix} u, \tag{7.37}$$

$$y_{ROM3} = \begin{bmatrix} 3 & -1 & -3 & -1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma_1 \\ \gamma_2 \end{bmatrix}, \tag{7.38}$$

or

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{e}_1 \\ \dot{e}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & -2 & -3 & -1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & -5 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ e_1 \\ e_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 1 \\ -5 \end{bmatrix} u,$$

$$y_{ROM3} = \begin{bmatrix} 0 & -2 & -3 & -1 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ e_1 \\ e_2 \end{bmatrix},$$

which is controllable and observable, has poles at -0.4384, -1, -1, and -4.5616, and which matches the tangential data by construction.

It turns out that an exponentially stable interpolant of dimension $k = 3$ can also be constructed. Consider, for $k = 3$ in Theorem 21, the parameterized family of interpolants given by

$$\begin{bmatrix} 1 & 0 & Q_1 \\ 0 & 1 & Q_2 \\ P_1 & P_2 & G \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 0 & 1 & Q_2 \\ 2 & -(1+\overline{D}) & H - Q_1 \\ 2P_2 & T - 2P_1 & F \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma \end{bmatrix} + \begin{bmatrix} -3 \\ 1+\overline{D} \\ -T \end{bmatrix} u,$$

$$y = \begin{bmatrix} 3 & -(1+\overline{D}) & H \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma \end{bmatrix} + \overline{D}u,$$

where P_1, P_2, Q_1, Q_2, $\overline{D}$, T, H, F, $G \in \mathbb{R}$. Selecting the parameters

$$P_1 = P_2 = Q_1 = Q_2 = \overline{D} = 0, \qquad G = 1,$$

yields

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & H \\ 0 & T & F \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ -T \end{bmatrix} u, \qquad y = \begin{bmatrix} 3 & -1 & H \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma \end{bmatrix},$$

and by setting

$$F = 0.75, \qquad T = -10, \qquad H = 2,$$

one obtains

$$\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 2 & -1 & 2 \\ 0 & -10 & 0.75 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \\ 10 \end{bmatrix} u, \tag{7.39}$$

$$y_{ROM4} = \begin{bmatrix} 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \gamma \end{bmatrix}, \tag{7.40}$$

which is controllable and observable, has poles at $-0.0815 \pm j4.1508$ and -0.087, and which matches the tangential data by construction.

The individual tangential interpolation data points, obtained by considering the auxiliary systems in their diagonalized representations (see Appendix A), are given by

$$\begin{aligned} (\lambda_1, r_1, w_1) &= (2j,\ -0.7071,\ 0.7071 - j2.1213), \\ (\lambda_2, r_2, w_2) &= (-2j,\ -0.7071,\ 0.7071 + j2.1213), \\ (\mu_1, \ell_1, v_1) &= (j,\ -j0.7071,\ 2.1213 + j0.7071), \\ (\mu_2, \ell_2, v_2) &= (-j,\ j0.7071,\ 2.1213 - j0.7071), \end{aligned}$$

meaning that the transfer function associated to each interpolant should

satisfy

$$\begin{aligned}
-0.7071H(2j) = 2.2361e^{-j1.249} \quad &\Rightarrow \quad H(2j) = 3.1624e^{j1.8926}, \\
-0.7071H(-2j) = 2.2361e^{j1.249} \quad &\Rightarrow \quad H(-2j) = 3.1624e^{-j1.8926}, \\
-j0.7071H(j) = 2.2361e^{j0.3218} \quad &\Rightarrow \quad H(j) = 3.1624e^{j1.8926}, \\
j0.7071H(-j) = 2.2361e^{-j0.3218} \quad &\Rightarrow \quad H(-j) = 3.1624e^{-j1.8926}.
\end{aligned}$$

Note that

$$20 \log_{10}(3.1624) \approx 10, \qquad 1.8926 \text{ rad} \approx 108.4 \text{ deg}.$$

Bode plots of ROM1, given by the equations (7.31)-(7.32) with $\overline{D} = 0$, ROM2, given by the equations (7.35)-(7.36), ROM3, given by the equations (7.37)-(7.38), and ROM4, given by the equations (7.39)-(7.40), are shown in Figures 7.1 and 7.2. The Bode plots indicate that all four constructed interpolants match the tangential interpolation data at the frequencies $s = j$ and $s = 2j$, where ROM1 is an unstable system of dimension 2, ROM2 and ROM3 are exponentially stable systems of dimension 4, and ROM4 is an exponentially stable system of dimension 3.

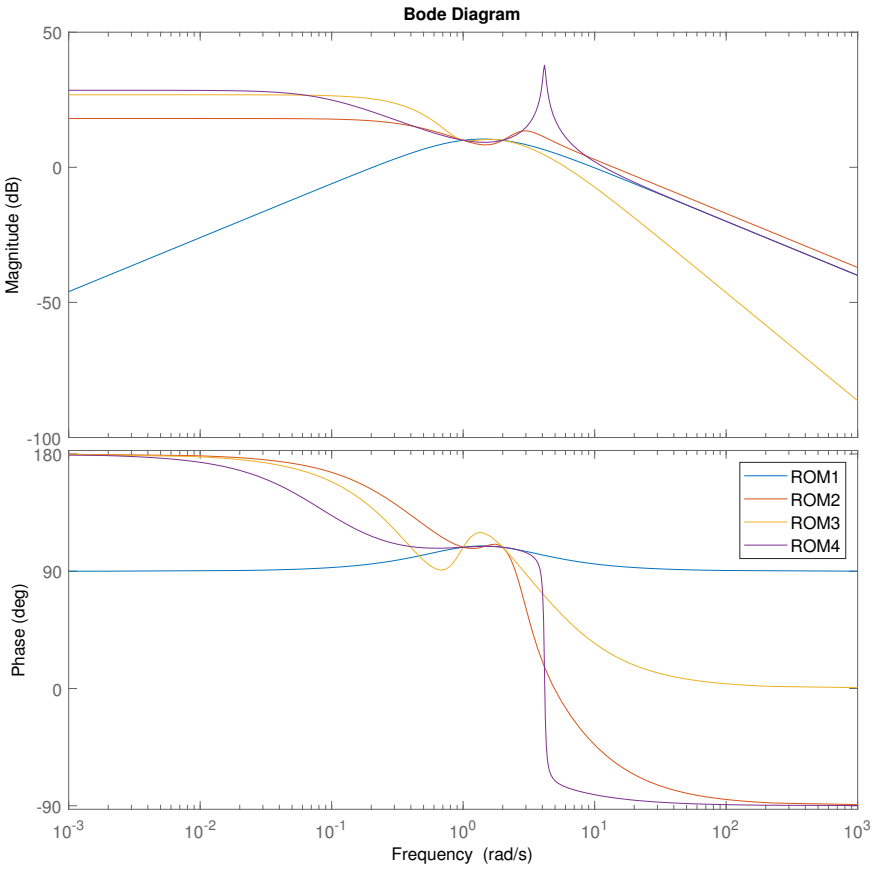

Figure 7.1: Bode plots of ROM1, ROM2, ROM3, and ROM4.

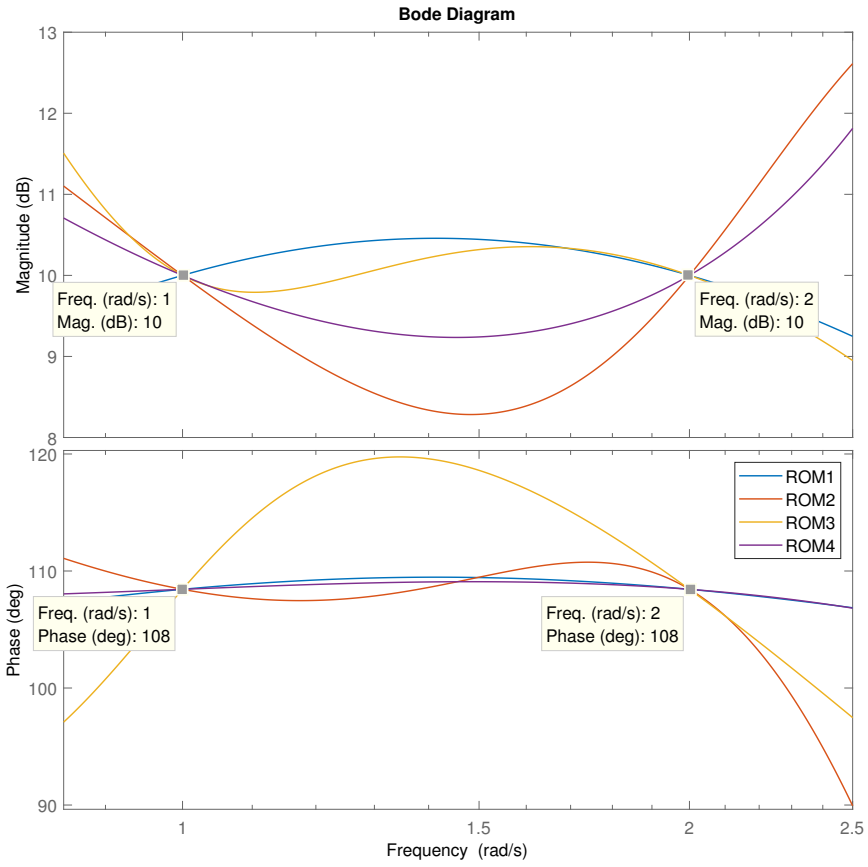

Figure 7.2: Bode plots of ROM1, ROM2, ROM3, and ROM4.

7.6 Example - An Interpolant for the DC-to-DC Ćuk Converter When $\rho \neq \nu$

Recall, from Sections 4.4 and 5.5, the averaged model of the DC-to-DC Ćuk converter in the implicit form

$$E\dot{x} = Ax + (B_0 + B_1 x)\, u, \qquad y_{FOM} = Cx, \tag{7.41}$$

where

$$x := \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \qquad E := \begin{bmatrix} L_1 & 0 & 0 & 0 \\ 0 & C_2 & 0 & 0 \\ 0 & 0 & L_3 & 0 \\ 0 & 0 & 0 & C_4 \end{bmatrix}, \qquad A := \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -G \end{bmatrix},$$

and

$$B_0 := \overline{E} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \qquad B_1 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad C := \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}.$$

It is now possible to construct an interpolant for this system at (λ, r, m, ℓ) when $\rho \neq \nu$. Consider constructing a reduced order model associated to the

auxiliary systems (7.3)-(7.4) and (7.5)-(7.6) with mappings given by

$$\lambda(\zeta_r) = 0, \qquad r(\zeta_r) = \zeta_r, \qquad m(\zeta_\ell) = \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix} \zeta_\ell, \qquad \ell(\chi) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \chi.$$

Then the tangential generalized controllability function, $X(\cdot)$, defined as the solution to the PDE with boundary condition (5.11), and the right tangential data mapping, $W(\cdot)$, defined via (5.13), are unchanged from Section 5.5. That is

$$X(\zeta_r) = \begin{bmatrix} G\zeta_r/(\zeta_r - 1) \\ -1 \\ G \\ 1 \end{bmatrix} \frac{\overline{E}\zeta_r}{\zeta_r - 1}, \qquad W(\zeta_r) = \frac{\overline{E}\zeta_r}{\zeta_r - 1}.$$

The tangential generalized observability function, $Y(\cdot)$, defined as the solution to the PDE with boundary condition (5.12), is the solution to

$$\left(\frac{\partial Y}{\partial x} \circ Ex\right) Ax = MY(Ex) - LC, \qquad Y(0) = 0.$$

This PDE is solved by the linear mapping $Y(x) = Yx$, where Y is the unique solution to the generalized Sylvester equation

$$YA = MYE - LC.$$

That is, with $k := L_3^2 C_4^2 - 8L_3C_4 + 4G^2L_3^2 + 16$,

$$Y(x) = \frac{2}{k}\begin{bmatrix} 0 & 0 & 2(4 - L_3C_4) & 2GL_3^2 \\ 0 & 0 & 4GL_3 & L_3(L_3C_4 - 4) \end{bmatrix} x,$$

and the left tangential data mapping, $V(\cdot)$, defined via (5.14), is

$$\begin{aligned} V(\zeta_r) &= \left(\frac{\partial Y}{\partial x} \circ EX(\zeta_r)\right)\big(B_0 + B_1X(\zeta_r)\big) \\ &= \begin{bmatrix} 4 - L_3C_4 \\ 2GL_3 \end{bmatrix} \frac{4\overline{E}}{k(\zeta_r - 1)}. \end{aligned}$$

The Loewner function, $\mathbb{L}(\cdot)$, obtained as the solution to the PDE with boundary condition (5.16), and the shifted Loewner function, $\sigma\mathbb{L}(\cdot)$, obtained via (5.18), are constructed from the tangential data mappings as

$$\begin{aligned} \mathbb{L}(\zeta_r) &= M^{-1}\big(V(\zeta_r)\zeta_r - LW(\zeta_r)\big) \\ &= -\begin{bmatrix} 8G \\ L_3C_4^2 - 4C_4 + 4L_3G^2 \end{bmatrix} \frac{2\overline{E}L_3\zeta_r}{k(\zeta_r - 1)}, \end{aligned}$$

and

$$\begin{aligned}\sigma\mathbb{L}(\zeta_r) &= V(\zeta_r)\zeta_r \\ &= \begin{bmatrix} 4 - L_3C_4 \\ 2GL_3 \end{bmatrix} \frac{4\overline{E}\zeta_r}{k(\zeta_r - 1)},\end{aligned}$$

respectively. Note that the Jacobian of the Loewner function is

$$\frac{\partial \mathbb{L}}{\partial \zeta_r} = \begin{bmatrix} 8G \\ L_3C_4^2 - 4C_4 + 4L_3G^2 \end{bmatrix} \frac{2\overline{E}L_3}{k(\zeta_r - 1)^2}.$$

The Jacobian of the Loewner function is not a square matrix function, so the interpolant in Theorem 13 of Chapter 5 can not be used as it would, in general, be overdetermined. The Jacobian of the Loewner function has full column rank in a neighbourhood of the origin, hence the interpolant (7.17)-(7.18) can be considered. A local left inverse for the Loewner function, $\mathbb{L}^{\#}(\cdot)$ such that $\mathbb{L}^{\#}(\mathbb{L}(\zeta_r)) = \zeta_r$, is required in order to construct the interpolant (7.17)-(7.18). A particular left inverse is given by

$$\mathbb{L}^{\#}(\zeta_\ell) := \frac{k\overline{\mathbb{L}}^{\#}\zeta_\ell}{2\overline{E}L_3 + k\overline{\mathbb{L}}^{\#}\zeta_\ell},$$

where

$$\overline{\mathbb{L}}^{\#} := \left(64G^2 + (L_3C_4^2 - 4C_4 + 4L_3G^2)^2\right)^{-1} \begin{bmatrix} 8G & L_3C_4^2 - 4C_4 + 4L_3G^2 \end{bmatrix}.$$

Consider the parameters

$$L_1 = C_2 = L_3 = C_4 = \overline{E} = 1, \qquad G = 2,$$

so that

$$X(\zeta_r) = \begin{bmatrix} 2\zeta_r/(\zeta_r - 1) \\ -1 \\ 2 \\ 1 \end{bmatrix} \frac{\zeta_r}{(\zeta_r - 1)}, \qquad W(\zeta_r) = \frac{\zeta_r}{\zeta_r - 1},$$

$$Y(x) = \frac{2}{25} \begin{bmatrix} 0 & 0 & 6 & 4 \\ 0 & 0 & 8 & -3 \end{bmatrix} x, \qquad V(\zeta_r) = \frac{4}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \frac{1}{(\zeta_r - 1)},$$

$$\mathbb{L}(\zeta_r) = -\frac{2}{25} \begin{bmatrix} 16 \\ 13 \end{bmatrix} \frac{\zeta_r}{(\zeta_r - 1)}, \qquad \sigma\mathbb{L}(\zeta_r) = \frac{4}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \frac{\zeta_r}{(\zeta_r - 1)},$$

$$\frac{\partial \mathbb{L}}{\partial \zeta_r} = \frac{2}{25} \begin{bmatrix} 16 \\ 13 \end{bmatrix} \frac{1}{(\zeta_r - 1)^2}, \qquad \mathbb{L}^{\#}(\zeta_\ell) = \frac{\begin{bmatrix} 16 & 13 \end{bmatrix} \zeta_\ell}{34 + \begin{bmatrix} 16 & 13 \end{bmatrix} \zeta_\ell}.$$

The system of equations given in Theorem 13 of Chapter 5 is

$$\frac{2}{25} \begin{bmatrix} 16 \\ 13 \end{bmatrix} \frac{\dot{\omega}}{(\omega - 1)^2} = \frac{4}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix} \frac{(\omega - u)}{(\omega - 1)}, \qquad y_r = \frac{\omega}{\omega - 1},$$

which is *consistent* because the solution $(\dot{\omega}(t), \omega(t), u(t)) = (0, \omega(t_0), \omega(t_0))$

always exists, however it is not *regular* due to the system of equations being overdetermined: the input $u(\cdot)$ must be constant, so there does not exist a unique solution for every sufficiently smooth input function in a neighbourhood of $u(t) = \omega(t_0)$ and every initial value in a neighbourhood of $\omega(t_0)$ that is consistent for the input function $u(\cdot)$.

The system of equations (7.17)-(7.18) has the form

$$\begin{aligned}\dot{\psi} &= M\psi + LW\big(\mathbb{L}^{\#}(\psi)\big) - L\overline{d}\big(\mathbb{L}^{\#}(\psi)\big)R\mathbb{L}^{\#}(\psi) \\ &\quad - \Big(V\big(\mathbb{L}^{\#}(\psi)\big) - L\overline{d}\big(\mathbb{L}^{\#}(\psi)\big)\Big)u, \\ y_{ROM} &= W\big(\mathbb{L}^{\#}(\psi)\big) - \overline{d}\big(\mathbb{L}^{\#}(\psi)\big)R\mathbb{L}^{\#}(\psi) + \overline{d}\big(\mathbb{L}^{\#}(\psi)\big)u,\end{aligned}$$

hence

$$\begin{aligned}\begin{bmatrix}\dot{\psi}_1 \\ \dot{\psi}_2\end{bmatrix} &= \begin{bmatrix}-8/17 & 2/17 \\ -1/2 & 0\end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} - \begin{bmatrix}1 \\ 0\end{bmatrix}\overline{d}\big(\mathbb{L}^{\#}(\psi)\big)\mathbb{L}^{\#}(\psi) \\ &\quad + \left(\frac{4}{25}\begin{bmatrix}3 \\ 4\end{bmatrix} + \frac{2}{425}\begin{bmatrix}48 & 39 \\ 64 & 52\end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} + \begin{bmatrix}1 \\ 0\end{bmatrix}\overline{d}\big(\mathbb{L}^{\#}(\psi)\big)\right)u, \\ y_{ROM} &= -\frac{1}{34}\begin{bmatrix}16 & 13\end{bmatrix}\begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix} - \overline{d}\big(\mathbb{L}^{\#}(\psi)\big)\mathbb{L}^{\#}(\psi) + \overline{d}\big(\mathbb{L}^{\#}(\psi)\big)u,\end{aligned}$$

is an interpolant of the tangential data mappings and the Loewner functions

at (λ, r, m, ℓ) for any mapping $\overline{d}(\cdot)$. Setting the free parameter $\overline{d}(\cdot) = 0$ yields

$$\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \end{bmatrix} = \begin{bmatrix} -8/17 & 2/17 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \left(\frac{4}{25} \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \frac{2}{425} \begin{bmatrix} 48 & 39 \\ 64 & 52 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} \right) u, \tag{7.42}$$

$$y_{ROM} = -\frac{1}{34} \begin{bmatrix} 16 & 13 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}, \tag{7.43}$$

with state $\psi(t) \in \mathbb{R}^2$, input $u(t) \in \mathbb{R}$, and output $y_{ROM}(t) \in \mathbb{R}$, which is a *regular* interpolant of the tangential data mappings and the Loewner functions at (λ, r, m, ℓ) with a locally exponentially stable equilibrium point at the origin.

With mappings $V_{t,FOM}(\cdot)$ and $V_{t,ROM}(\cdot)$ defined in the same way as Section 5.5, Figures 7.3 and 7.4 show the responses of the full order model and the reduced order model to a piecewise constant input signal. The responses of the systems are consistent with the tangential data mappings.

7.7 Conclusion

In this chapter an approach for the regularization of ill-posed nonlinear interpolants in the Loewner framework has been developed. This method eliminates the restrictive requirements for interpolant construction given in Chapters 4 and 5, namely that $\rho = v$ and that rank $\frac{\partial \mathbb{L}}{\partial \zeta_r}(0) = \rho$. The ap-

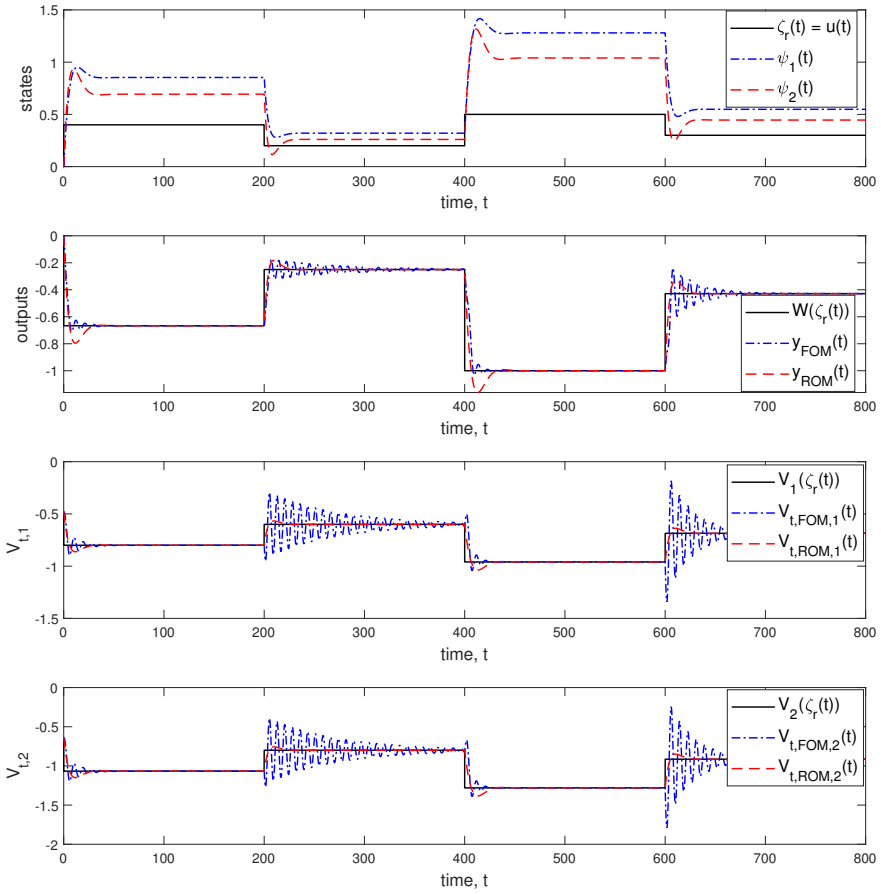

Figure 7.3: Response of the full order model (7.41) and of the reduced order model (7.42)-(7.43).

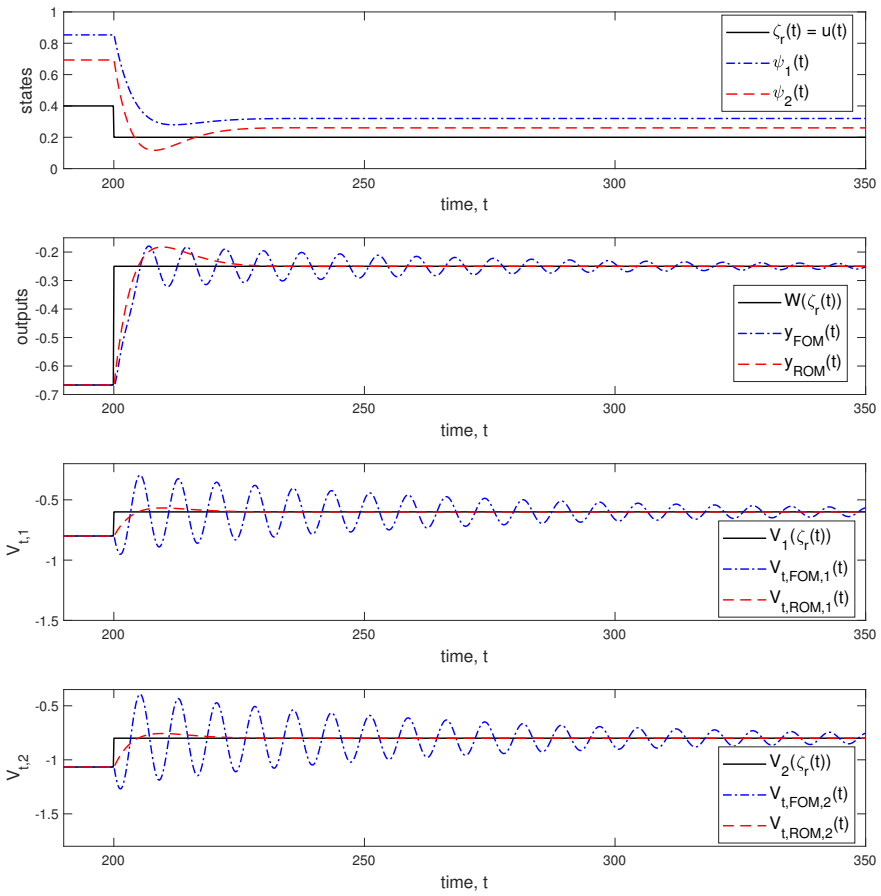

Figure 7.4: Response of the full order model (7.41) and of the reduced order model (7.42)-(7.43) for $t \in [190, 350]$.

proach is based on taking the ill-posed interpolant and augmenting it with additional constraints and/or variables in such a way that the properties of Loewner equivalence and matching of tangential data are preserved. Under mild conditions this family parameterizes all interpolating systems of dimension larger than or equal to the dimension of the larger auxiliary system, *i.e.* of dimension larger than or equal to $\max\{\rho, v\}$. Thus, the family of systems allows one to build any other larger interpolant in a simple algebraic fashion without solving any additional partial differential equation. The linear results, taken along with the results given in [94] regarding construction of interpolants with dimension less than or equal to $\max\{\rho, v\}$, provide a complete method for generating interpolants of any dimension for which an interpolant exists. Two demonstrative examples are given, with the first example demonstrating how the parameterized family of interpolants of higher order can be used to construct stable interpolants when a stable interpolant of minimal order doesn't exist, and with the second example demonstrating the construction of a reduced order model for a nonlinear underlying system when the auxiliary systems do not have the same dimension.

Chapter 8

On Equivalence of the Loewner Functions for Behaviourally Equivalent DAEs

In the nonlinear settings of Chapters 4 and 5 the tangential data functions and the Loewner functions are defined in terms of the tangential generalized controllability and observability functions. The tangential generalized controllability and observability functions are defined as the solutions to a set of PDEs, and in the setting of nonlinear systems of ODEs, developed in Chapter 4, simple conditions ensuring the existence of unique analytic solutions for the PDEs have been given in terms of type (C, v) nonresonance conditions. However, in the setting of nonlinear systems of DAEs developed in Chapter 5, conditions regarding the existence of solutions for the PDEs

have not been discussed.

In the nonlinear DAE setting, conditions ensuring the existence of a solution for the PDE defining the tangential generalized controllability function can be stated, simply, in terms of an enhanced notion of type (C, v) nonresonance condition for matrix pencils, see Appendix B. In stark contrast, conditions guaranteeing the existence of a solution for the PDE defining the tangential generalized observability function are much more involved. This difficulty arises when not all variables are "locally accessible" to the solution of the PDE, and the consistency of the PDE must be considered. As a result, simple existence conditions cannot be given in general. That is, the PDE defining the tangential generalized observability function may be too difficult, or impossible, to solve, and it may be necessary to leverage the algebraic constraints of the system to simplify the PDE by eliminating some variables. Indeed, in the differential-algebraic setting a system's algebraic constraints can be used to obtain "behaviourally equivalent" representations of the system[1], and some representations may be more difficult to work with than others.

This chapter considers the PDEs defining the tangential generalized controllability and observability functions for a simplified class of nonlinear differential-algebraic systems for which the algebraic constraints are explicit and are independent of the input. Three additional systems which are be-

[1]When referring to "behaviourally equivalent" representations of a system, it is meant that for consistent initial conditions the representations yield the same input-output response despite the fact that they are not related to each other by a diffeomorphism.

haviourally equivalent to the original system are considered with the goal of simplifying the construction of the Loewner functions. These four systems possess different tangential generalized observability functions, some of which have more strict existence conditions than others. Despite this, the Loewner functions are defined based on the input-output behaviour of the system interconnected with auxiliary systems and should not depend on the system having a particular internal representation. Indeed, it is shown that even though each system possesses a different tangential generalized observability function, each representation still results in the same tangential data functions, the same Loewner functions, and the same families of interpolants, provided that the functions exist. As a result, if the PDE defining the tangential generalized observability function associated to the original system is too difficult, or impossible, to solve, then, without restricting the resulting family of interpolants, one can manipulate the algebraic constraints of the system to produce a behaviourally equivalent representation with associated PDEs having less stringent existence conditions.

This chapter is structured as follows. In Section 8.1 conditions ensuring the existence of solution for a family of PDEs are stated, the class of systems considered for the remainder of the chapter is introduced, and difficulties regarding existence of solutions to the tangential generalized observability PDE are discussed. In Section 8.2, three differential-algebraic systems which are behaviourally equivalent to the class of systems being considered are introduced. It is shown that all four systems possess the exact same tangential

data functions and Loewner functions, and therefore the exact same family of Loewner equivalent interpolants. Finally, in Section 8.3 some concluding remarks are given.

8.1 Preliminaries

The following theorem, which is a special case of the existence of solutions result in Appendix B, is used when discussing the existence of unique solutions in the space of analytic functions for various PDEs.

Theorem 23. *Assume that* $\kappa : \mathbb{C}^\rho \to \mathbb{C}^\rho$, $\alpha : \mathbb{C}^v \to \mathbb{C}^{v\times m}$, $\beta : \mathbb{C}^\rho \to \mathbb{C}^m$, *and* $\epsilon : \mathbb{C}^v \to \mathbb{C}^v$ *are, locally, analytic vector fields such that* $\kappa(0) = 0$, $\beta(0) = 0$, $\epsilon(0) = 0$. *Let* $K = \frac{\partial \kappa}{\partial x}(0)$ *and* $E = \frac{\partial \epsilon}{\partial z}(0)$. *Suppose there exist constants* $C > 0$ *and* $v > 0$ *such that all eigenvalues of* E *are of type* (C, v) *with respect to* $\sigma(K)$. *Then, locally around* $x = 0$, *there exists a unique analytic solution,* $\theta(\cdot)$, *to the PDE*

$$\frac{\partial \theta}{\partial x}\kappa(x) = \epsilon\big(\theta(x)\big) + \alpha\big(\theta(x)\big)\beta(x), \qquad \theta(0) = 0.$$

Theorem 23 is an enhancement of Lemma 2 in Chapter 4 as it allows $\alpha(\cdot)$ to be a function of $\theta(\cdot)$ and does not require K to be diagonalizable. The former enhancement is dealt with, as in Chapter 4, by managing some extra terms in the series expansion, while the latter requires the selection of a non-trivial ordering in which to solve coefficients in the series expansion of $\theta(\cdot)$.

8.1.1 Problem Formulation

The remainder of this chapter considers a simplified class of systems of the form (5.1)-(5.2) in which the algebraic constraints are represented explicitly and are independent of the input u. Particularly, consider DAEs described by equations of the form

$$\Sigma : \begin{cases} \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1(x_1, x_2) \\ x_2 - \varphi(x_1) \end{bmatrix} + \begin{bmatrix} g_1(x_1, x_2) \\ 0 \end{bmatrix} u, \\ \qquad y = h(x_1, x_2) + d(x_1, x_2)u, \\ x_2(t_0) = \varphi\big(x_1(t_0)\big), \end{cases}$$

with states $x_1(t) \in \mathbb{C}^{n_1}$ and $x_2(t) \in \mathbb{C}^{n-n_1}$, input $u(t) \in \mathbb{C}^m$, output $y(t) \in \mathbb{C}^p$, and functions $f_1 : \mathbb{C}^{n_1} \times \mathbb{C}^{n-n_1} \to \mathbb{C}^{n_1}$, $g_1 : \mathbb{C}^{n_1} \times \mathbb{C}^{n-n_1} \to \mathbb{C}^{n_1 \times m}$, $h : \mathbb{C}^{n_1} \times \mathbb{C}^{n-n_1} \to \mathbb{C}^p$, $d : \mathbb{C}^{n_1} \times \mathbb{C}^{n-n_1} \to \mathbb{C}^{p \times m}$, and $\varphi : \mathbb{C}^{n_1} \to \mathbb{C}^{n-n_1}$, such that $f_1(0,0) = 0$, $h(0,0) = 0$, $\varphi(0) = 0$. Furthermore, it is assumed that $f_1(\cdot)$, $g_1(\cdot)$, $h(\cdot)$, $d(\cdot)$, and $\varphi(\cdot)$ are differentiable. If the Jacobian of $E(\cdot)$ in (5.1)-(5.2) has constant rank n_1 in a neighbourhood of the origin, and if the algebraic constraints can be obtained without taking derivatives and are independent of the input, then the representation Σ may be obtained, locally, from (5.1)-(5.2) by performing a coordinates transformation obtained via application of the constant rank theorem and the implicit function theorem [28], [80].

Let F, Λ, and M be the Jacobians of $f_1\big(x_1, \varphi(x_1)\big)$, $\lambda(\zeta_r)$, and $m(\zeta_\ell)$ eval-

uated at the origin, respectively. The following assumption, which enables the use of Theorem 23, is used throughout this chapter.

Assumption 7. *The eigenvalues of F are of type (C, v) with respect to $\sigma(\Lambda)$, and the eigenvalues of M are of type (C, v) with respect to $\sigma(F)$.*

Consider now the problem of determining the right and left tangential data functions, $W(\cdot)$ and $V(\cdot)$, and the Loewner and shifted Loewner functions, $\mathbb{L}(\cdot)$ and $\sigma\mathbb{L}(\cdot)$, for the system Σ interconnected with the auxiliary systems (5.3)-(5.4) and (5.5)-(5.6). In order to construct these functions the tangential generalized controllability function, $X(\cdot)$, and the tangential generalized observability function, $Y(\cdot)$, must be determined as solutions to the PDEs (5.11) and (5.12), respectively. For the function $X(\cdot)$ it is generally easy to ascertain the existence of a unique analytic solution by leveraging a type (C, v) non-resonance condition[2], similar to that of Assumption 7, and applying Theorem 23, as in Chapter 4. In stark contrast, the function $Y(\cdot)$ is generally difficult to determine without simplifying the problem. The difficulties arise due to the fact that if $E(\cdot)$ is not bijective then $Y(\cdot)$ "does not have access" to all independent variables in the PDE (5.12); the PDE (5.12) may be too general and one may need to leverage the algebraic constraints of the DAE in order to simplify the problem.

Thus, the goal of this chapter is to show that it is sufficient to solve a simplified set of PDEs by presenting relationships between the Loewner

[2]For the tangential generalized controllability PDE (5.11) a simple existence condition may be stated without the selection of a particular set of coordinates, *i.e.* stated directly in terms of $E(\cdot)$, $f(\cdot)$, and $\lambda(\cdot)$, see Appendix B.

functions associated to Σ and the Loewner functions associated to three simplified systems that are behaviourally equivalent to Σ.

8.2 Equivalent Representations

Towards the goal of simplifying the PDE (5.12) consider, in addition to Σ, three additional DAE systems. The system $\overline{\Sigma}$ is defined as

$$\overline{\Sigma}: \begin{cases} \begin{bmatrix} \dot{x}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_1\big(x_1, \varphi(x_1)\big) \\ x_2 - \varphi(x_1) \end{bmatrix} + \begin{bmatrix} g_1\big(x_1, \varphi(x_1)\big) \\ 0 \end{bmatrix} u, \\ \qquad y = h\big(x_1, \varphi(x_1)\big) + d\big(x_1, \varphi(x_1)\big)u, \\ x_2(t_0) = \varphi\big(x_1(t_0)\big), \end{cases}$$

which is derived from Σ by substituting $x_2 = \varphi(x_1)$ into $f_1(\cdot)$, $g_1(\cdot)$, $h(\cdot)$, and $d(\cdot)$. The system $\widehat{\Sigma}$ is defined as

$$\widehat{\Sigma}: \begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} I \\ \frac{\partial \varphi}{\partial x_1} \end{bmatrix} \Big(f_1\big(x_1, \varphi(x_1)\big) + g_1\big(x_1, \varphi(x_1)\big)u \Big), \\ \qquad y = h(x_1, x_2) + d(x_1, x_2)u, \\ x_2(t_0) = \varphi\big(x_1(t_0)\big), \end{cases}$$

which is derived from Σ by differentiating the algebraic constraint and rewriting the resulting equations into a state-space form. Finally, the system $\widetilde{\Sigma}$ is

defined as

$$\widetilde{\Sigma}: \begin{cases} \overbrace{\begin{bmatrix} x_1 \\ x_2 - \varphi(x_1) \end{bmatrix}}^{\dot{}} = \begin{bmatrix} f_1(x_1, x_2) \\ 0 \end{bmatrix} + \begin{bmatrix} g_1(x_1, x_2) \\ 0 \end{bmatrix} u, \\ y = h(x_1, x_2) + d(x_1, x_2)u, \\ x_2(t_0) = \varphi\big(x_1(t_0)\big), \end{cases}$$

which is obtained from Σ by differentiating the algebraic state without writing the resulting equations into a state-space form.

Given that the systems have consistent initial conditions, it is clear that Σ, $\overline{\Sigma}$, $\widehat{\Sigma}$, and $\widetilde{\Sigma}$ exhibit equivalent input-output behaviour. The tangential data functions and the Loewner functions encode information on the response of a system to interconnection with auxiliary systems so Σ, $\overline{\Sigma}$, $\widehat{\Sigma}$, and $\widetilde{\Sigma}$ will produce the same functions $W(\cdot)$, $V(\cdot)$, $\mathbb{L}(\cdot)$, and $\sigma\mathbb{L}(\cdot)$, and thus the same interpolant as in Theorem 14. Despite this, it is easy to see that there is no diffeomorphism relating all four models, as such a mapping would need to change the rank, when evaluated at the origin, of the Jacobian of $E(\cdot)$ in (5.1)-(5.2). Given that the solution, if it exists, to the tangential generalized observability PDE (5.12) is the most difficult function to obtain, and given that each representation of the system yields a different tangential generalized observability PDE, the goal of this section is to show that it is sufficient to solve the set of PDEs given by any of the representations Σ, $\overline{\Sigma}$, $\widehat{\Sigma}$, and $\widetilde{\Sigma}$.

The following theorem makes clear the relationship between the tangential generalized controllability functions and the right tangential data functions associated to each system.

Theorem 24. *Consider each of the systems* Σ, $\overline{\Sigma}$, $\widehat{\Sigma}$, *and* $\widetilde{\Sigma}$ *interconnected with the auxiliary system* (5.3)-(5.4)*, and suppose that Assumption 7 holds. Let* $X(\cdot)$, $\overline{X}(\cdot)$, $\widehat{X}(\cdot)$, *and* $\widetilde{X}(\cdot)$ *be the resulting tangential generalized controllability functions, respectively, provided they exist locally in a neighbourhood of the origin, and let* $W(\cdot)$, $\overline{W}(\cdot)$, $\widehat{W}(\cdot)$, *and* $\widetilde{W}(\cdot)$ *be the resulting right tangential data functions, respectively. Then*

$$X(\zeta_r) = \overline{X}(\zeta_r) = \widehat{X}(\zeta_r) = \widetilde{X}(\zeta_r),$$

and

$$W(\zeta_r) = \overline{W}(\zeta_r) = \widehat{W}(\zeta_r) = \widetilde{W}(\zeta_r),$$

in a neighbourhood of the origin.

Proof. The function $X(\zeta_r) = (X_1(\zeta_r)^\top,\ X_2(\zeta_r)^\top)^\top$ is defined as the solution,

provided it exists, to the PDE with boundary condition

$$\begin{bmatrix} \dfrac{\partial X_1}{\partial \zeta_r} \\ 0 \end{bmatrix} \lambda(\zeta_r) = \begin{bmatrix} f_1\big(X_1(\zeta_r), X_2(\zeta_r)\big) \\ X_2(\zeta_r) - \varphi\big(X_1(\zeta_r)\big) \end{bmatrix} + \begin{bmatrix} g_1\big(X_1(\zeta_r), X_2(\zeta_r)\big) \\ 0 \end{bmatrix} r(\zeta_r),$$

$$X(0) = 0. \tag{8.1}$$

By Assumption 7 the eigenvalues of F are of type (C, v) with respect to $\sigma(\Lambda)$, so by Theorem 23 there exists a solution to the PDE with boundary condition

$$\frac{\partial X_1}{\partial \zeta_r} \lambda(\zeta_r) = f_1\big(X_1(\zeta_r), \varphi(X_1(\zeta_r))\big) + g_1\big(X_1(\zeta_r), \varphi(X_1(\zeta_r))\big) r(\zeta_r),$$

$$X_1(0) = 0,$$

hence a solution $X(\cdot)$ exists and has the form

$$X(\zeta_r) = \begin{bmatrix} X_1(\zeta_r) \\ X_2(\zeta_r) \end{bmatrix} = \begin{bmatrix} X_1(\zeta_r) \\ \varphi\big(X_1(\zeta_r)\big) \end{bmatrix}. \tag{8.2}$$

The functions $\overline{X}(\cdot)$, $\widehat{X}(\cdot)$, and $\widetilde{X}(\cdot)$ are the solutions to the PDEs, with

boundary conditions,

$$\begin{bmatrix} \dfrac{\partial \overline{X}_1}{\partial \zeta_r} \\ 0 \end{bmatrix} \lambda(\zeta_r) = \begin{bmatrix} f_1\big(\overline{X}_1(\zeta_r), \varphi(\overline{X}_1(\zeta_r))\big) \\ \overline{X}_2(\zeta_r) - \varphi\big(\overline{X}_1(\zeta_r)\big) \end{bmatrix} + \begin{bmatrix} g_1\big(\overline{X}_1(\zeta_r), \varphi(\overline{X}_1(\zeta_r))\big) \\ 0 \end{bmatrix} r(\zeta_r),$$

$$\overline{X}(0) = 0, \tag{8.3}$$

$$\begin{bmatrix} \dfrac{\partial \widehat{X}_1}{\partial \zeta_r} \\ \dfrac{\partial \widehat{X}_2}{\partial \zeta_r} \end{bmatrix} \lambda(\zeta_r) = \begin{bmatrix} I \\ \dfrac{\partial \varphi}{\partial x_1} \circ \widehat{X}_1(\zeta_1) \end{bmatrix} f_1\big(\widehat{X}_1(\zeta_r), \widehat{X}_2(\zeta_r)\big) + \begin{bmatrix} I \\ \dfrac{\partial \varphi}{\partial x_1} \circ \widehat{X}_1(\zeta_1) \end{bmatrix} g_1\big(\widehat{X}_1(\zeta_r), \widehat{X}_2(\zeta_r)\big) r(\zeta_r),$$

$$\widehat{X}(0) = 0, \tag{8.4}$$

and

$$\begin{bmatrix} \dfrac{\partial \widetilde{X}_1}{\partial \zeta_r} \\ \dfrac{\partial \widetilde{X}_2}{\partial \zeta_r} - \dfrac{\partial \varphi\big(\widetilde{X}_1(\zeta_r)\big)}{\partial \zeta_r} \end{bmatrix} \lambda(\zeta_r) = \begin{bmatrix} f_1\big(\widetilde{X}_1(\zeta_r), \widetilde{X}_2(\zeta_r)\big) \\ 0 \end{bmatrix} + \begin{bmatrix} g_1\big(\widetilde{X}_1(\zeta_r), \widetilde{X}_2(\zeta_r)\big) \\ 0 \end{bmatrix} r(\zeta_r),$$

$$\widetilde{X}(0) = 0, \tag{8.5}$$

respectively. With $X(\cdot)$ of the form (8.2), it is easy to see that setting

$$X(\zeta_r) = \overline{X}(\zeta_r) = \widehat{X}(\zeta_r) = \widetilde{X}(\zeta_r),$$

in (8.3), (8.4), and (8.5) solves each PDE. Finally, it is easy to see that

$$\begin{aligned} W(\zeta_r) &= h\big(X_1(\zeta_r), X_2(\zeta_r)\big) + d\big(X_1(\zeta_r), X_2(\zeta_r)\big) r(\zeta_r) \\ &= h\big(\overline{X}_1(\zeta_r), \varphi(\overline{X}_1(\zeta_r))\big) + d\big(\overline{X}_1(\zeta_r), \varphi(\overline{X}_1(\zeta_r))\big) r(\zeta_r) && = \overline{W}(\zeta_r) \\ &= h\big(\widehat{X}_1(\zeta_r), \widehat{X}_2(\zeta_r)\big) + d\big(\widehat{X}_1(\zeta_r), \widehat{X}_2(\zeta_r)\big) r(\zeta_r) && = \widehat{W}(\zeta_r) \\ &= h\big(\widetilde{X}_1(\zeta_r), \widetilde{X}_2(\zeta_r)\big) + d\big(\widetilde{X}_1(\zeta_r), \widetilde{X}_2(\zeta_r)\big) r(\zeta_r) && = \widetilde{W}(\zeta_r), \end{aligned}$$

thus proving the result. □

In the proof of Theorem 24 the conditions for existence of a solution

to each of the tangential generalized controllability PDEs (8.1), (8.3), (8.4), and (8.5) are the same. This is not the case for the tangential generalized observability PDEs, for which conditions ensuring existence of a solution for one representation are easier to satisfy than conditions ensuring existence of solution for others. For example, existence of the tangential generalized observability function can be guaranteed for the system $\overline{\Sigma}$ with Assumption 7, allowing one to construct a solution by following the procedure in the proof of Lemma 2 in Chapter 4, while existence conditions pertaining to the system Σ are more strict. The following theorem exposes the relationship between the tangential generalized observability functions for each system interconnected with the auxiliary system (5.5)-(5.6).

Theorem 25. *Consider each of the systems Σ, $\overline{\Sigma}$, $\widehat{\Sigma}$, and $\widetilde{\Sigma}$ interconnected with the auxiliary system* (5.5)-(5.6)*, and suppose that Assumption 7 holds. Let $Y(\cdot)$, $\overline{Y}(\cdot)$, $\widehat{Y}(\cdot)$, and $\widetilde{Y}(\cdot)$ be the resulting tangential generalized observability functions, respectively, provided they exist locally in a neighbourhood of the origin, and let $V(\cdot)$, $\overline{V}(\cdot)$, $\widehat{V}(\cdot)$, and $\widetilde{V}(\cdot)$ be the resulting left tangential data functions, respectively. Then*

$$Y(x_1, 0) = \overline{Y}(x_1, 0) = \widehat{Y}(x_1, \varphi(x_1)) = \widetilde{Y}(x_1, 0),$$

and

$$V(\zeta_r) = \overline{V}(\zeta_r) = \widehat{V}(\zeta_r) = \widetilde{V}(\zeta_r),$$

in a neighbourhood of the origin.

Proof. The function $\overline{Y}(\kappa_1, \kappa_2)$ is defined as the solution, provided it exists, to the PDE with boundary condition

$$\left(\frac{\partial \overline{Y}}{\partial \kappa} \circ (x_1, 0)\right) \begin{bmatrix} f_1\big(x_1, \varphi(x_1)\big) \\ x_2 - \varphi(x_1) \end{bmatrix} = -m\left(-\overline{Y}(x_1, 0)\right) - \ell\big(h(x_1, \varphi(x_1))\big),$$

$$\overline{Y}(0, 0) = 0,$$

and setting $\overline{Y}(\kappa_1, \kappa_2) = \overline{Y}(\kappa_1, 0)$ yields

$$\frac{\partial \overline{Y}(x_1, 0)}{\partial x_1} f_1\big(x_1, \varphi(x_1)\big) = -m\left(-\overline{Y}(x_1, 0)\right) - \ell\big(h(x_1, \varphi(x_1))\big), \quad (8.6)$$

for which existence of a solution can be guaranteed by Theorem 23 since, by Assumption 7, the eigenvalues of M are of type (C, v) with respect to $\sigma(F)$. The functions $Y(\kappa_1, \kappa_2)$, $\widehat{Y}(\kappa_1, \kappa_2)$, and $\widetilde{Y}(\kappa_1, \kappa_2)$ are defined as the solutions, provided they exist, to the PDEs, with boundary conditions,

$$\left(\frac{\partial Y}{\partial \kappa} \circ (x_1, 0)\right) \begin{bmatrix} f_1(x_1, x_2) \\ x_2 - \varphi(x_1) \end{bmatrix} = -m\big(-Y(x_1, 0)\big) - \ell\big(h(x_1, x_2)\big),$$

$$Y(0, 0) = 0, \quad (8.7)$$

$$\left(\frac{\partial \widehat{Y}}{\partial \kappa} \circ (x_1, x_2)\right) \begin{bmatrix} I \\ \dfrac{\partial \varphi}{\partial x_1} \end{bmatrix} f_1(x_1, x_2) = -m\big(-\widehat{Y}(x_1, x_2)\big) - \ell\big(h(x_1, x_2)\big),$$

$$\widehat{Y}(0,0) = 0, \tag{8.8}$$

and

$$\left(\frac{\partial \widetilde{Y}}{\partial \kappa} \circ \big(x_1, x_2 - \varphi(x_1)\big)\right) \begin{bmatrix} f_1(x_1, x_2) \\ 0 \end{bmatrix} = -m\big(-\widetilde{Y}(x_1, x_2 - \varphi(x_1))\big) - \ell\big(h(x_1, x_2)\big),$$

$$\widetilde{Y}(0,0) = 0. \tag{8.9}$$

Existence of a solution to the PDE (8.7) cannot be guaranteed by Theorem 23 because $Y(\cdot)$ does not have access to all variables in the equation. The PDEs (8.8) and (8.9) can be put into the form required by Theorem 23, however the non-resonance condition needed to guarantee existence of solution is stronger than that of Assumption 7. Note that for each system one has $x_2(t) = \varphi\big(x_1(t)\big)$ for all t because $x_2(t_0) = \varphi\big(x_1(t_0)\big)$. Considering (8.6) and restricting each of the PDEs (8.7), (8.8), and (8.9) along this trajectory

yields

$$\begin{aligned}
\ell\big(h(x_1,\varphi(x_1))\big) &= -\frac{\partial Y(x_1,0)}{\partial x_1} f_1\big(x_1,\varphi(x_1)\big) - m\big(-Y(x_1,0)\big) \\
&= -\frac{\partial \overline{Y}(x_1,0)}{\partial x_1} f_1\big(x_1,\varphi(x_1)\big) - m\big(-\overline{Y}(x_1,0)\big) \\
&= -\frac{\partial \widehat{Y}\big(x_1,\varphi(x_1)\big)}{\partial x_1} f_1\big(x_1,\varphi(x_1)\big) - m\big(-\widehat{Y}(x_1,\varphi(x_1))\big) \\
&= -\frac{\partial \widetilde{Y}(x_1,0)}{\partial x_1} f_1\big(x_1,\varphi(x_1)\big) - m\big(-\widetilde{Y}(x_1,0)\big),
\end{aligned}$$

and, because the eigenvalues of M are of type (C,v) with respect to $\sigma(F)$, it follows that

$$Y(x_1,0) = \overline{Y}(x_1,0) = \widehat{Y}\big(x_1,\varphi(x_1)\big) = \widetilde{Y}(x_1,0),$$

provided that the functions exist. Finally, recalling from Theorem 24 that each system induces the same tangential generalized controllability function, $X(\cdot)$, when interconnected with the auxiliary system (5.3)-(5.4), it follows that

$$\begin{aligned}
V(\zeta_r) = &\left(\frac{\partial Y(x_1,0)}{\partial x_1} \circ \big(X_1(\zeta_r),0\big)\right) g_1\big(X_1(\zeta_r),X_2(\zeta_r)\big) \\
&+ \overline{\ell}\Big(h\big(X_1(\zeta_r),X_2(\zeta_r)\big),\ d\big(X_1(\zeta_r),X_2(\zeta_r)\big)r(\zeta_r)\Big) d\big(X_1(\zeta_r),X_2(\zeta_r)\big),
\end{aligned}$$

$$\overline{V}(\zeta_r) = \left(\frac{\partial \overline{Y}(x_1, 0)}{\partial x_1} \circ \big(X_1(\zeta_r), 0\big) \right) g_1\big(X_1(\zeta_r), X_2(\zeta_r)\big)$$
$$+ \overline{\ell}\Big(h\big(X_1(\zeta_r), X_2(\zeta_r)\big),\ d\big(X_1(\zeta_r), X_2(\zeta_r)\big) r(\zeta_r)\Big) d\big(X_1(\zeta_r), X_2(\zeta_r)\big),$$

$$\widehat{V}(\zeta_r) = \left(\frac{\partial \widehat{Y}\big(x_1, \varphi(x_1)\big)}{\partial x_1} \circ X_1(\zeta_r) \right) g_1\big(X_1(\zeta_r), X_2(\zeta_r)\big)$$
$$+ \overline{\ell}\Big(h\big(X_1(\zeta_r), X_2(\zeta_r)\big),\ d\big(X_1(\zeta_r), X_2(\zeta_r)\big) r(\zeta_r)\Big) d\big(X_1(\zeta_r), X_2(\zeta_r)\big),$$

and

$$\widetilde{V}(\zeta_r) = \left(\frac{\partial \widetilde{Y}(x_1, 0)}{\partial x_1} \circ \big(X_1(\zeta_r), 0\big) \right) g_1\big(X_1(\zeta_r), X_2(\zeta_r)\big)$$
$$+ \overline{\ell}\Big(h\big(X_1(\zeta_r), X_2(\zeta_r)\big),\ d\big(X_1(\zeta_r), X_2(\zeta_r)\big) r(\zeta_r)\Big) d\big(X_1(\zeta_r), X_2(\zeta_r)\big),$$

hence

$$V(\zeta_r) = \overline{V}(\zeta_r) = \widehat{V}(\zeta_r) = \widetilde{V}(\zeta_r),$$

proving the result. □

When constructing the functions $W(\cdot)$, $V(\cdot)$, $\mathbb{L}(\cdot)$, and $\sigma\mathbb{L}(\cdot)$ the input of the tangential generalized observability function is always restricted as

in Theorem 25. This leads to the following result implying that, despite possibly having different tangential generalized observability functions, each representation yields the same family of interpolants given in Theorem 14.

Theorem 26. *Consider each of the systems* Σ, $\overline{\Sigma}$, $\widehat{\Sigma}$, *and* $\widetilde{\Sigma}$ *interconnected with the auxiliary systems* (5.3)-(5.4) *and* (5.5)-(5.6), *and suppose Assumption 7 holds. Let* $X(\cdot)$, $Y(\cdot)$, $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\widehat{X}(\cdot)$, $\widehat{Y}(\cdot)$, $\widetilde{X}(\cdot)$, *and* $\widetilde{Y}(\cdot)$ *be the resulting tangential generalized controllability and observability functions, respectively, provided they exist locally in a neighbourhood of the origin, and let* $\mathbb{L}(\cdot)$, $\sigma\mathbb{L}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\sigma\mathbb{L}}(\cdot)$, $\widehat{\mathbb{L}}(\cdot)$, $\widehat{\sigma\mathbb{L}}(\cdot)$, $\widetilde{\mathbb{L}}(\cdot)$, *and* $\widetilde{\sigma\mathbb{L}}(\cdot)$ *be the Loewner and shifted Loewner functions, respectively. Then*

$$\mathbb{L}(\zeta_r) = \overline{\mathbb{L}}(\zeta_r) = \widehat{\mathbb{L}}(\zeta_r) = \widetilde{\mathbb{L}}(\zeta_r),$$

and

$$\sigma\mathbb{L}(\zeta_r) = \overline{\sigma\mathbb{L}}(\zeta_r) = \widehat{\sigma\mathbb{L}}(\zeta_r) = \widetilde{\sigma\mathbb{L}}(\zeta_r),$$

in a neighbourhood of the origin.

Proof. The result is obtained by constructing each of the Loewner functions and applying Theorems 24 and 25. By Theorem 24 it follows that

$$X(\zeta_r) = \begin{bmatrix} X_1(\zeta_r) \\ \varphi(X_1(\zeta_r)) \end{bmatrix} = \overline{X}(\zeta_r) = \widehat{X}(\zeta_r) = \widetilde{X}(\zeta_r),$$

and by Theorem 25 it follows that

$$Y(x_1, 0) = \overline{Y}(x_1, 0) = \widehat{Y}\big(x_1, \varphi(x_1)\big) = \widetilde{Y}(x_1, 0).$$

Hence,

$$\begin{aligned}
\mathbb{L}(\zeta_r) = -Y\big(X_1(\zeta_r), 0\big) = -\overline{Y}\big(\overline{X}_1(\zeta_r), 0\big) \qquad &= \overline{\mathbb{L}}(\zeta_r) \\
= -\widehat{Y}\big(\widehat{X}_1(\zeta_r), \varphi(\widehat{X}_1(\zeta_r))\big) \qquad &= \widehat{\mathbb{L}}(\zeta_r) \\
= -\widetilde{Y}\big(\widetilde{X}_1(\zeta_r), \widetilde{X}_2(\zeta_r) - \varphi(\widetilde{X}_1(\zeta_r))\big) \qquad &= \widetilde{\mathbb{L}}(\zeta_r).
\end{aligned}$$

By Theorem 24, $W(\zeta_r) = \overline{W}(\zeta_r) = \widehat{W}(\zeta_r) = \widetilde{W}(\zeta_r)$, hence

$$\begin{aligned}
\sigma\mathbb{L}(\zeta_r) &= m\big(\mathbb{L}(\zeta_r)\big) + \ell\big(W(\zeta_r)\big) \\
&= m\big(\overline{\mathbb{L}}(\zeta_r)\big) + \ell\big(\overline{W}(\zeta_r)\big) = \overline{\sigma\mathbb{L}}(\zeta_r) \\
&= m\big(\widehat{\mathbb{L}}(\zeta_r)\big) + \ell\big(\widehat{W}(\zeta_r)\big) = \widehat{\sigma\mathbb{L}}(\zeta_r) \\
&= m\big(\widetilde{\mathbb{L}}(\zeta_r)\big) + \ell\big(\widetilde{W}(\zeta_r)\big) = \widetilde{\sigma\mathbb{L}}(\zeta_r),
\end{aligned}$$

proving the result. □

Remark 28. As a result of Theorems 24, 25, and 26, the systems Σ, $\overline{\Sigma}$, $\widehat{\Sigma}$, and $\widetilde{\Sigma}$ produce the exact same family of Loewner equivalent interpolants, provided that the solutions to the PDEs exist, despite each system yielding a different tangential generalized observability PDE, and therefore a different

mapping solving the PDE, of the form (5.12).

Remark 29. Conditions for existence and uniqueness of analytic solution to the PDEs (8.6), (8.8), and (8.9) defining the tangential generalized observability functions $\overline{Y}(\cdot)$, $\widehat{Y}(\cdot)$, and $\widetilde{Y}(\cdot)$ can be easily obtained via type (C, v) non-resonance conditions. Hence, if the PDE (8.7) defining $Y(\cdot)$ associated to the original representation Σ is difficult, or impossible, to solve then one can instead work with the simplified systems $\overline{\Sigma}$, $\widehat{\Sigma}$, or $\widetilde{\Sigma}$ without any change to the obtained family of interpolants.

8.3 Conclusion

In this chapter the Loewner functions associated to four behaviourally equivalent DAE systems have been considered with the goal of simplifying the PDE defining the tangential generalized observability function. Particularly, it has been shown that, provided solutions to the PDEs exist, all four systems yield the exact same family of Loewner equivalent interpolants given by Theorem 14 despite possibly having different tangential generalized observability functions. Thus, if the observability function associated with one representation is difficult or impossible to obtain, it is shown that one may instead solve a simplified set of PDEs without changing the resulting family of interpolants.

Chapter 9

Conclusion

9.1 Thesis Contributions

The Loewner framework has been enhanced towards the goal of developing a data-driven interpolation and model order reduction framework for general nonlinear systems. This enhancement has been accomplished in multiple steps. First, an interconnection-based interpretation of the Loewner framework has been developed for linear time-invariant differential-algebraic systems. The interconnection-based interpretation has then been leveraged to enhance the framework, first for nonlinear input-affine systems of ordinary differential equations, then for general systems of nonlinear differential-algebraic equations. Following this, families of systems interpolating the tangential data mappings have been parameterized, and the construction of well-posed interpolants via dynamic extension has been developed. As a re-

sult of the dynamic extension approach to interpolant construction, all interpolants of dimension greater than or equal to the dimensions of the auxiliary interpolation systems are parameterized. Hence, if an interpolant with some additional desired properties exists, then it is contained in the parameterized family. Finally, the use of simplified representations when determining the Loewner functions for a class of nonlinear semi-explicit differential-algebraic systems has been discussed with the goal of selecting a behaviourally equivalent representation having less stringent conditions guaranteeing existence of solution to partial differential equations.

9.2 Future Work

Exploiting the results of this thesis, there is a plethora of further research directions to be investigated. Of great importance is the completion of a data-driven approach, including an experimental procedure for obtaining the tangential data mappings so that interpolants can be constructed without having to solve the partial differential equations defining the tangential generalized controllability and observability mappings. The left tangential data mapping, $V(\cdot)$, is challenging to obtain, and the development of an experimental procedure to accomplish this would be a significant result.

Following the methodology of Chapters 4 and 5, the Loewner framework may be enhanced for more general families of DAEs, including nonlinear time-varying systems and nonlinear discrete-time systems. The enhancement for

nonlinear discrete-time systems would be particularly interesting when taken with the development of the data-driven approach where data is most likely to be sampled periodically. An enhancement of the framework for systems with multiple time-delays could further lead to an enhancement of the framework for infinite-dimensional systems, however initial investigations have shown that the approach of Chapters 4 and 5 is not sufficient on its own, and further steps need to be taken. Another research direction is the enhancement of the framework for more general implicit systems, where the "descriptor function" $E(\cdot)$ is allowed to be a more general mapping of the system's state, the state's time-derivative, and the system's input.

Another direction to be investigated is using the extra degrees of freedom in the parameterized families of interpolants introduced in Chapters 6 and 7 to assign additional desirable properties to the constructed interpolants. Such properties could include the asymptotic stability of an equilibrium point, the assignment of desirable transient behaviour to input signals, or structural properties relating the interpolant to some family of physical systems, *e.g.* interpolants possessing the structural properties of a mechanical system or an electrical system.

Finally, it would be interesting to further generalize the behaviourally equivalent representation results of Chapter 8. For example, by taking the parameterization results of Chapter 7 it may be possible to assign a convenient parameterized internal representation for experimentally collecting tangential data in a data-driven approach.

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Appendix A

On the Use of Complex Valued Signals

The restriction of Λ and M to diagonal matrices seems, at first, prohibitive. When discussing the interconnection of physical systems, or the time-domain response of the cascade interconnections in an experimental setup, many important interpolation points, such as those on the imaginary axis of the complex plane, would not be achievable under such an assumption. Furthermore, it is desirable to construct interpolants using only real-valued objects. In this appendix it is shown that diagonality of the auxiliary system matrices is not actually required, and that such interpolation points can, in fact, be achieved using real-valued systems. As a result of what follows, the interpolants presented in Chapters 2, 4, and 5, have equivalent real-valued realizations as long as the auxiliary systems have associated real-valued realizations.

A.1 Linear DAEs (Chapter 2 and Chapter 3)

Let $P \in \mathbb{C}^{\rho\times\rho}$ and $Q \in \mathbb{C}^{v\times v}$ be nonsingular matrices. Consider auxiliary systems of the form (3.6)-(3.7) and (3.8)-(3.9) defined by the equations

$$\dot{\zeta}_r = \underbrace{P\Lambda P^{-1}}_{=:\overline{\Lambda}}\zeta_r + \Delta, \tag{A.1}$$

$$v = \underbrace{RP^{-1}}_{=:\overline{R}}\zeta_r, \tag{A.2}$$

and

$$\dot{\zeta}_\ell = \underbrace{QMQ^{-1}}_{=:\overline{M}}\zeta_\ell + \underbrace{QL}_{=:\overline{L}}\, y, \tag{A.3}$$

$$\eta = \zeta_\ell, \tag{A.4}$$

where $\Lambda \in \mathbb{C}^{\rho\times\rho}$, $R \in \mathbb{C}^{m\times\rho}$, $M \in \mathbb{C}^{v\times v}$, and $L \in \mathbb{C}^{v\times p}$ are complex-valued matrices, and $\overline{\Lambda}$, $\overline{R}$, $\overline{M}$, and $\overline{L}$ are real-valued matrices, and therefore *implementable*. Let X, Y, W, V, $\mathbb{L}$, and $\sigma\mathbb{L}$ denote the Loewner objects associated with the auxiliary systems (3.6)-(3.7) and (3.8)-(3.9) interconnected with a real-valued system of the form (2.17)-(2.18). The real-valued matrices in (A.1)-(A.2) and (A.3)-(A.4) will now be used to construct a new set of real-valued Loewner objects, which are represented in terms of the former complex-valued Loewner objects. Letting $\overline{X}$, $\overline{Y}$, $\overline{W}$, $\overline{V}$, $\overline{\mathbb{L}}$, and $\overline{\sigma\mathbb{L}}$ denote the new set of Loewner matrices associated with the auxiliary systems (A.1)-(A.2) and (A.3)-(A.4) interconnected with the system (2.17)-(2.18), solving

the associated generalized Sylvester equations yields

$$\overline{X} = XP^{-1}, \quad \overline{W} = WP^{-1}, \quad \overline{Y} = QY, \quad \overline{V} = QV,$$

from which the real-valued Loewner matrices

$$\overline{\mathbb{L}} = Q\mathbb{L}P^{-1}, \quad \overline{\sigma\mathbb{L}} = Q\sigma\mathbb{L}P^{-1},$$

are obtained. Constructing a model of the form (2.30)-(2.31) using the real-valued objects yields the system

$$\begin{aligned} \overline{\mathbb{L}}\dot{\omega} &= (\overline{\sigma\mathbb{L}} - \overline{L}\,\overline{D}\,\overline{R})\omega - (\overline{V} - \overline{L}\,\overline{D})u_r \qquad \text{(A.5)} \\ &= Q(\sigma\mathbb{L} - L\overline{D}R)P^{-1}\omega - Q(V - L\overline{D})u_r = Q\mathbb{L}P^{-1}\dot{\omega}, \\ y_r &= (\overline{W} - \overline{D}\,\overline{R})\omega + \overline{D}u_r \qquad \text{(A.6)} \\ &= (W - \overline{D}R)P^{-1}\omega + \overline{D}u_r, \end{aligned}$$

where the feedforward matrix $\overline{D}$ is a free parameter. This system is simplified to

$$\begin{aligned} \mathbb{L}P^{-1}\dot{\omega} &= (\sigma\mathbb{L} - L\overline{D}R)P^{-1}\omega - (V - L\overline{D})u_r, \\ y_r &= (W - \overline{D}R)P^{-1}\omega + \overline{D}u_r. \end{aligned}$$

The new interpolant (A.5)-(A.6) is obtained by a coordinates transformation of the complex-valued interpolant (2.30)-(2.31), hence the auxiliary systems (3.6)-(3.7) and (3.8)-(3.9), and the auxiliary systems (A.1)-(A.2) and (A.3)-(A.4), interconnected with the system (2.17)-(2.18) produce the same interpolant, albeit in different coordinates, with the system (A.5)-(A.6) being real-valued if $\overline{D}$ is a real-valued matrix.

A.2 Nonlinear ODEs with Linear Auxiliary Systems (Chapter 4)

In a similar fashion, consider the set of Loewner functions associated with the auxiliary systems (A.1)-(A.2) and (A.3)-(A.4) interconnected with a real-valued system of the form (4.1)-(4.2), denoted by $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{W}(\cdot)$, $\overline{V}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\mathbb{L}}^{\ell}(\cdot)$, $\overline{\mathbb{L}}^{r}(\cdot)$, and $\overline{\sigma\mathbb{L}}(\cdot)$. Let $X(\cdot)$, $Y(\cdot)$, $W(\cdot)$, $V(\cdot)$, $\mathbb{L}(\cdot)$, $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, and $\sigma\mathbb{L}(\cdot)$ denote the set of Loewner functions associated with the interconnected system (3.6)-(3.7), (3.8)-(3.9), and (4.1)-(4.2). Then, by solving the associated PDEs, it follows that

$$\overline{X}(\zeta_r) = X(P^{-1}\zeta_r), \quad \overline{W}(\zeta_r) = W(P^{-1}\zeta_r),$$

$$\overline{Y}(x) = QY(x), \quad \overline{V}(\zeta_r) = QV(P^{-1}\zeta_r),$$

from which the real-valued Loewner functions

$$\overline{\mathbb{L}}(\zeta_r) = Q\mathbb{L}(P^{-1}\zeta_r), \quad \overline{\mathbb{L}}^{\ell}(\zeta_r) = Q\mathbb{L}^{\ell}(P^{-1}\zeta_r),$$

$$\overline{\mathbb{L}}^{r}(\zeta_r) = Q\mathbb{L}^{r}(P^{-1}\zeta_r), \quad \overline{\sigma\mathbb{L}}(\zeta_r) = Q\sigma\mathbb{L}(P^{-1}\zeta_r),$$

are obtained. Note also that

$$\frac{\partial\overline{\mathbb{L}}}{\partial\zeta_r} = Q\left(\frac{\partial\mathbb{L}}{\partial\zeta_r}\circ P^{-1}\zeta_r\right)P^{-1}.$$

Constructing a model of the form (4.16)-(4.17) using the real-valued objects yields the system

$$\left(\frac{\partial\overline{\mathbb{L}}}{\partial\zeta_r}\circ\omega\right)\dot{\omega} = \overline{\sigma\mathbb{L}}(\omega) - \overline{V}(\omega)u_r,$$
$$y_r = \overline{W}(\omega).$$

This is simplified to

$$\left(\frac{\partial\mathbb{L}}{\partial\zeta_r}\circ P^{-1}\omega\right)P^{-1}\dot{\omega} = \sigma\mathbb{L}(P^{-1}\omega) - V(P^{-1}\omega)u_r,$$
$$y_r = W(P^{-1}\omega),$$

which is obtained from the complex-valued interpolant (4.16)-(4.17) via a coordinates transformation.

A.3 Nonlinear ODEs with Nonlinear Auxiliary Systems (Chapter 4)

Consider now auxiliary systems of the form (4.23)-(4.24) and (4.25)-(4.26) defined by the equations

$$\dot{\zeta}_r = \underbrace{P\lambda(P^{-1}\zeta_r)}_{=:\overline{\lambda}(\zeta_r)} + \Delta, \tag{A.7}$$

$$v = \underbrace{r(P^{-1}\zeta_r)}_{=:\overline{r}(\zeta_r)}, \tag{A.8}$$

and

$$\dot{\zeta}_\ell = \underbrace{Qm(Q^{-1}\zeta_\ell)}_{=:\overline{m}(\zeta_\ell)} + \underbrace{Q\ell(y)}_{=:\widetilde{\ell}(y)}, \tag{A.9}$$

$$\eta = \zeta_\ell, \tag{A.10}$$

where $\lambda : \mathbb{C}^\rho \to \mathbb{C}^\rho$ and $m : \mathbb{C}^v \to \mathbb{C}^v$ are such that $\frac{\partial \lambda}{\partial \zeta_r}(0) = \Lambda$ and $\frac{\partial m}{\partial \zeta_\ell}(0) = M$ are complex-valued, and $\overline{\lambda}(\cdot)$, $\overline{r}(\cdot)$, $\overline{m}(\cdot)$, and $\widetilde{\ell}(\cdot)$ are real-valued maps. The set of Loewner functions associated with the auxiliary systems (A.7)-(A.8) and (A.9)-(A.10) interconnected with a real-valued system of the form (4.1)-(4.2) is denoted by $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{W}(\cdot)$, $\overline{V}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\mathbb{L}}^\ell(\cdot)$, $\overline{\mathbb{L}}^r(\cdot)$, and $\overline{\sigma\mathbb{L}}(\cdot)$. Let $X(\cdot)$, $Y(\cdot)$, $W(\cdot)$, $V(\cdot)$, $\mathbb{L}(\cdot)$, $\mathbb{L}^\ell(\cdot)$, $\mathbb{L}^r(\cdot)$, and $\sigma\mathbb{L}(\cdot)$ denote the set of Loewner functions associated with the interconnected system (4.23)-(4.24), (4.25)-(4.26), and (4.1)-(4.2). Then, by solving the asso-

ciated PDEs, it follows that

$$\overline{X}(\zeta_r) = X(P^{-1}\zeta_r), \quad \overline{W}(\zeta_r) = W(P^{-1}\zeta_r),$$

$$\overline{Y}(x) = QY(x), \quad \overline{V}(\zeta_r) = QV(P^{-1}\zeta_r),$$

from which the real-valued Loewner functions

$$\overline{\mathbb{L}}(\zeta_r) = Q\mathbb{L}(P^{-1}\zeta_r), \quad \overline{\mathbb{L}}^{\ell}(\zeta_r) = Q\mathbb{L}^{\ell}(P^{-1}\zeta_r),$$

$$\overline{\mathbb{L}}^{r}(\zeta_r) = Q\mathbb{L}^{r}(P^{-1}\zeta_r), \quad \overline{\sigma\mathbb{L}}(\zeta_r) = Q\sigma\mathbb{L}(P^{-1}\zeta_r),$$

are obtained. Note also that

$$\frac{\partial\overline{\mathbb{L}}}{\partial\zeta_r} = Q\left(\frac{\partial\mathbb{L}}{\partial\zeta_r}\circ P^{-1}\zeta_r\right)P^{-1}.$$

Constructing a model of the form (4.41)-(4.42) using the real-valued objects yields the system

$$\left(\frac{\partial\overline{\mathbb{L}}}{\partial\zeta_r}\circ\omega\right)\dot{\omega} = \overline{\sigma\mathbb{L}}(\omega) - \overline{V}(\omega)u_r,$$
$$y_r = \overline{W}(\omega).$$

This is simplified to

$$\left(\frac{\partial \mathbb{L}}{\partial \zeta_r} \circ P^{-1}\omega\right) P^{-1}\dot{\omega} = \sigma\mathbb{L}(P^{-1}\omega) - V(P^{-1}\omega)u_r,$$

$$y_r = W(P^{-1}\omega),$$

which is obtained from the complex-valued interpolant (4.41)-(4.42) via a coordinates transformation.

A.4 Nonlinear DAEs (Chapter 5)

Consider the set of Loewner functions associated with the auxiliary systems (A.7)-(A.8) and (A.9)-(A.10) interconnected with a real-valued system of the form (5.1)-(5.2), denoted by $\overline{X}(\cdot)$, $\overline{Y}(\cdot)$, $\overline{W}(\cdot)$, $\overline{V}(\cdot)$, $\overline{\mathbb{L}}(\cdot)$, $\overline{\mathbb{L}}^{\ell}(\cdot)$, $\overline{\mathbb{L}}^{r}(\cdot)$, and $\overline{\sigma\mathbb{L}}(\cdot)$. Let $X(\cdot)$, $Y(\cdot)$, $W(\cdot)$, $V(\cdot)$, $\mathbb{L}(\cdot)$, $\mathbb{L}^{\ell}(\cdot)$, $\mathbb{L}^{r}(\cdot)$, and $\sigma\mathbb{L}(\cdot)$ denote the set of Loewner functions associated with the interconnected system (4.23)-(4.24), (4.25)-(4.26), and (5.1)-(5.2). Then, by solving the associated set of PDEs, it follows that

$$\overline{X}(\zeta_r) = X(P^{-1}\zeta_r), \quad \overline{W}(\zeta_r) = W(P^{-1}\zeta_r),$$

$$\overline{Y}(x) = QY(x), \quad \overline{V}(\zeta_r) = QV(P^{-1}\zeta_r),$$

from which the real-valued Loewner functions

$$\overline{\mathbb{L}}(\zeta_r) = Q\mathbb{L}(P^{-1}\zeta_r), \quad \overline{\mathbb{L}}^{\ell}(\zeta_r) = Q\mathbb{L}^{\ell}(P^{-1}\zeta_r),$$

$$\overline{\mathbb{L}}^{r}(\zeta_r) = Q\mathbb{L}^{r}(P^{-1}\zeta_r), \quad \overline{\sigma\mathbb{L}}(\zeta_r) = Q\sigma\mathbb{L}(P^{-1}\zeta_r),$$

are obtained. Note also that

$$\frac{\partial\overline{\mathbb{L}}}{\partial\zeta_r} = Q\left(\frac{\partial\mathbb{L}}{\partial\zeta_r}\circ P^{-1}\zeta_r\right)P^{-1}.$$

Constructing a model of the form (5.32)-(5.33) using the real-valued objects yields the system

$$\begin{aligned}
\dot{\overline{\mathbb{L}}(\omega)} &= \left[\overline{\sigma\mathbb{L}}(\omega) - Q\overline{\ell}\Big(\overline{W}(\omega),\ -\widetilde{d}(\omega)\overline{r}(\omega)\Big)\widetilde{d}(\omega)\overline{r}(\omega)\right] \\
&\qquad - \left[\overline{V}(\omega) - Q\overline{\ell}\Big(\overline{W}(\omega),\ -\widetilde{d}(\omega)\overline{r}(\omega)\Big)\widetilde{d}(\omega)\right]u_r \qquad \text{(A.11)} \\
&= Q\left[\sigma\mathbb{L}(P^{-1}\omega) - \overline{\ell}\Big(W(P^{-1}\omega),\ -\widetilde{d}(\omega)r(P^{-1}\omega)\Big)\widetilde{d}(\omega)r(P^{-1}\omega)\right] \\
&\qquad - Q\left[V(P^{-1}\omega) - \overline{\ell}\Big(W(P^{-1}\omega),\ -\widetilde{d}(\omega)r(P^{-1}\omega)\Big)\widetilde{d}(\omega)\right]u_r \\
&= Q\dot{\overline{\mathbb{L}(P^{-1}\omega)}}, \\
y_r &= \left[\overline{W}(\omega) - \widetilde{d}(\omega)\overline{r}(\omega)\right] + \widetilde{d}(\omega)u_r \qquad \text{(A.12)} \\
&= \left[W(P^{-1}\omega) - \widetilde{d}(\omega)r(P^{-1}\omega)\right] + \widetilde{d}(\omega)u_r,
\end{aligned}$$

where the feedforward mapping $\widetilde{d}(\cdot)$ is a free parameter. Setting the mapping $\widetilde{d}(\omega) := \overline{d}(P^{-1}\omega)$, this is simplified to

$$\begin{aligned}\dot{\overbrace{\mathbb{L}(P^{-1}\omega)}} &= \Big[\sigma\mathbb{L}(P^{-1}\omega) \\ &\qquad - \overline{\ell}\Big(W(P^{-1}\omega),\ -\overline{d}(P^{-1}\omega)r(P^{-1}\omega)\Big)\overline{d}(P^{-1}\omega)r(P^{-1}\omega)\Big] \\ &\quad - \Big[V(P^{-1}\omega) - \overline{\ell}\Big(W(P^{-1}\omega),\ -\overline{d}(P^{-1}\omega)r(P^{-1}\omega)\Big)\overline{d}(P^{-1}\omega)\Big]u_r,\end{aligned}$$

$$y_r = \Big[W(P^{-1}\omega) - \overline{d}(P^{-1}\omega)r(P^{-1}\omega)\Big] + \overline{d}(P^{-1}\omega)u_r,$$

which is obtained from the complex-valued interpolant (5.32)-(5.33) via a coordinates transformation. For any mapping $\widetilde{d}(\cdot)$ there is an associated choice of mapping $\overline{d}(\cdot)$, and if the mapping $\overline{d}(\cdot)$ is selected such that $\widetilde{d}(\cdot)$ is a real-valued mapping, then the interpolant (A.11)-(A.12) is real-valued.

Appendix B

On the Solutions to Some Partial Differential-Algebraic Equations

Consider the problem of determining a solution, $X : \mathbb{C}^\rho \to \mathbb{C}^n$, to the partial differential-algebraic equation with boundary condition

$$\frac{\partial E\big(X(\zeta)\big)}{\partial \zeta}\lambda(\zeta) = f\big(X(\zeta), r(\zeta)\big), \qquad X(0) = 0, \tag{B.1}$$

with functions $E : \mathbb{C}^n \to \mathbb{C}^n$, $\lambda : \mathbb{C}^\rho \to \mathbb{C}^\rho$, $f : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n$, and $r : \mathbb{C}^\rho \to \mathbb{C}^m$ obtained via linear coordinates transformation of real-analytic functions, and such that $E(0) = 0$, $\lambda(0) = 0$, $f(0,0) = 0$, and $r(0) = 0$. It is assumed that, in a neighbourhood of the origin, the Jacobian of $E(\cdot)$ has

constant rank but is not invertible.

In order to state conditions ensuring the existence of a solution for the PDE given by (B.1), consider the following definition introducing the notion of type (C, v) conditions for matrix pencils, which can be seen as a generalization of Definition 2 in [84].

Definition 15. *Given a $\rho \times \rho$ matrix Λ, with spectrum $\sigma(\Lambda) = \lambda = \{\lambda_1, \ldots, \lambda_\rho\}$, a pair of $n \times n$ matrices E and F, and constants $C > 0$ and $v > 0$, it is said that the matrix pair (E, F) is of type (C, v) with respect to $\sigma(\Lambda)$ if for any vector $m = (m_1, m_2, \ldots, m_\rho)$ of nonnegative integers it follows that the matrix $((m \cdot \lambda)E - F)$ is invertible and*

$$\left\| \Big((m \cdot \lambda)E - F \Big)^{-1} \right\| \leq \frac{|m|^v}{C},$$

where $|m| = \sum m_i > 0$.

Conditions ensuring the existence of a solution to the PDE (B.1) can now be given in terms of Definition 15 and the mappings $E(\cdot)$, $\lambda(\cdot)$, $f(\cdot)$, and $r(\cdot)$.

Theorem 27. *Consider the PDE* (B.1) *with the boundary condition $X(0) = 0$ and suppose that $\lambda(\cdot)$ and $r(\cdot)$ are obtained via linear coordinates transformation of, locally, real-analytic functions, and $E(\cdot)$, $f(\cdot)$, and $g(\cdot)$ are obtained via linear coordinates transformation of, globally, real-analytic functions. Let $E := \frac{\partial E}{\partial x}(0)$, $F := \frac{\partial f}{\partial x}(0, 0)$, $G := \frac{\partial f}{\partial u}(0, 0)$, and $\Lambda := \frac{\partial \lambda}{\partial \zeta}(0)$, and suppose that there exist constants $C > 0$ and $v > 0$ such that the matrix pair*

(E, F) is of type (C, v) with respect to $\sigma(\Lambda)$. Then, locally around $\zeta = 0$, there exists a unique analytic solution, $X : \mathbb{C}^\rho \to \mathbb{C}^n$, satisfying the partial differential equation (B.1) *with the given boundary condition.*

Proof. By analycity expand the functions comprising the PDE (B.1), including the solution $X(\cdot)$, using the Taylor series as

$$E(x) = Ex + \sum_{i=2}^{\infty} E^{(i)}(x), \qquad X(\zeta) = X\zeta + \sum_{i=2}^{\infty} X^{(i)}(\zeta),$$
$$\lambda(\zeta) = \Lambda\zeta + \sum_{i=2}^{\infty} \lambda^{(i)}(\zeta), \qquad f(x,u) = Fx + Gu + \sum_{i=2}^{\infty} f^{(i)}(x,u),$$
$$r(\zeta) = R\zeta + \sum_{i=2}^{\infty} r^{(i)}(\zeta),$$

where $E^{(i)}(x)$, $X^{(i)}(\zeta)$, $\lambda^{(i)}(\zeta)$, $f^{(i)}(x,u)$, and $r^{(i)}(\zeta)$ denote the terms of order i in the Taylor series expansions of $E(x)$, $X(\zeta)$, $\lambda(\zeta)$, $f(x,u)$, and $r(\zeta)$, respectively. Substituting the series expansions of the functions in the PDE yields

$$\left(E + \sum_{i=2}^{\infty} \left(\frac{\partial E^{(i)}}{\partial x} \circ X(\zeta)\right)\right) \left(X + \sum_{i=2}^{\infty} \frac{\partial X^{(i)}}{\partial \zeta}\right) \left(\Lambda\zeta + \sum_{i=2}^{\infty} \lambda^{(i)}(\zeta)\right)$$
$$= FX\zeta + F\sum_{i=2}^{\infty} X^{(i)}(\zeta) + \sum_{i=2}^{\infty} f^{(i)}\big(X(\zeta), r(\zeta)\big) + G\left(R\zeta + \sum_{i=2}^{\infty} r^{(i)}(\zeta)\right),$$

and rearranging yields

$$
\begin{aligned}
0 = &\Big(EX\Lambda - FX - GR\Big)\zeta \\
&+ \sum_{i=2}^{\infty}\left(E\frac{\partial X^{(i)}}{\partial \zeta}\Lambda\zeta - FX^{(i)}(\zeta) + EX\lambda^{(i)}(\zeta) + \left(\frac{\partial E^{(i)}}{\partial x}\circ X(\zeta)\right)X\Lambda\zeta \right. \\
&\qquad\qquad \left. - f^{(i)}\big(X(\zeta), r(\zeta)\big) - Gr^{(i)}(\zeta)\right) \\
&+ \sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\left(E\frac{\partial X^{(i)}}{\partial \zeta}\lambda^{(j)}(\zeta) + \left(\frac{\partial E^{(j)}}{\partial x}\circ X(\zeta)\right)\left(X\lambda^{(i)}(\zeta) + \frac{\partial X^{(i)}}{\partial \zeta}\Lambda\zeta\right)\right) \\
&+ \sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty}\left(\frac{\partial E^{(i)}}{\partial x}\circ X(\zeta)\right)\frac{\partial X^{(j)}}{\partial \zeta}\lambda^{(k)}(\zeta). \qquad \text{(B.2)}
\end{aligned}
$$

A solution to the PDE is constructed explicitly by matching coefficients for powers of ζ in (B.2). The terms of degree $d = 1$ from the analytically expanded PDE (B.2) are

$$0 = (EX\Lambda - FX - GR)\zeta,$$

which is a generalized Sylvester equation having a unique solution, X, because the matrix pair (E, F) is of type (C, v) with respect to $\sigma(\Lambda)$. For an analytic function $y(\zeta)$, let $\deg\big(y(\zeta), p\big)$ denote the degree p terms of ζ in $y(\zeta)$.

For degree $d > 1$ the terms of degree d are

$$E\frac{\partial X^{(d)}}{\partial \zeta}\Lambda\zeta - FX^{(d)}(\zeta) = \overline{\beta}^{(d)}(\zeta),$$

where

$$\begin{aligned}
\overline{\beta}^{(d)}(\zeta) := {} & Gr^{(d)}(\zeta) - EX\lambda^{(d)}(\zeta) \\
& + \sum_{i=2}^{\infty} \Bigg(\deg\left(f^{(i)}(X(\zeta), r(\zeta)), d\right) \\
& \qquad - \deg\left(\left(\frac{\partial E^{(i)}}{\partial x} \circ X(\zeta)\right) X\Lambda\zeta, d\right)\Bigg) \\
& - \sum_{i=2}^{\infty}\sum_{j=2}^{\infty} \Bigg(\deg\left(E\frac{\partial X^{(i)}}{\partial \zeta}\lambda^{(j)}(\zeta), d\right) \\
& \qquad + \deg\left(\left(\frac{\partial E^{(j)}}{\partial x} \circ X(\zeta)\right)\left(X\lambda^{(i)}(\zeta) + \frac{\partial X^{(i)}}{\partial \zeta}\Lambda\zeta\right), d\right)\Bigg) \\
& - \sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\sum_{k=2}^{\infty} \deg\left(\left(\frac{\partial E^{(i)}}{\partial x} \circ X(\zeta)\right)\frac{\partial X^{(j)}}{\partial \zeta}\lambda^{(k)}(\zeta), d\right),
\end{aligned}$$

which, with some abuse of the summation notation when $d = 2$ (in which case the double and triple summations are taken to be zero) and $d = 3$ (in

which case the triple summation is taken to be zero), simplifies to

$$\begin{aligned}
\overline{\beta}^{(d)}(\zeta) = {} & Gr^{(d)}(\zeta) - EX\lambda^{(d)}(\zeta) \\
& + \sum_{i=2}^{d} \Bigg(\deg\left(f^{(i)}(X(\zeta), r(\zeta)), d\right) \\
& \qquad - \deg\left(\left(\frac{\partial E^{(i)}}{\partial x} \circ X(\zeta)\right) X\Lambda\zeta, d\right) \Bigg) \\
& - \sum_{i=2}^{d-1} \sum_{j=2}^{d-1} \Bigg(\deg\left(E\frac{\partial X^{(i)}}{\partial \zeta}\lambda^{(j)}(\zeta), d\right) \\
& \qquad + \deg\left(\left(\frac{\partial E^{(j)}}{\partial x} \circ X(\zeta)\right)\left(X\lambda^{(i)}(\zeta) + \frac{\partial X^{(i)}}{\partial \zeta}\Lambda\zeta\right), d\right) \Bigg) \\
& - \sum_{i=2}^{d-2} \sum_{j=2}^{d-2} \sum_{k=2}^{d-2} \deg\left(\left(\frac{\partial E^{(i)}}{\partial x} \circ X(\zeta)\right) \frac{\partial X^{(j)}}{\partial \zeta}\lambda^{(k)}(\zeta), d\right),
\end{aligned}$$

where the summations only contain a finite number of terms after eliminating terms of $\deg(\cdot, d)$ that are equal to zero. It is important to note that $\overline{\beta}^{(d)}(\cdot)$ contains only coefficients in the series expansion of $X(\cdot)$ associated to terms of degree less than d, thus $\overline{\beta}^{(d)}(\cdot)$ is not a function of $X^{(d)}(\cdot)$ and $\overline{\beta}^{(d)}(\cdot)$ is completely determined given that the functions $X^{(i)}(\cdot)$, $i = 1, \ldots, d-1$, are known. Hence, the remainder of the proof is concerned with solving the simplified PDE

$$E\frac{\partial X^{(d)}}{\partial \zeta}\Lambda\zeta - FX^{(d)}(\zeta) = \overline{\beta}^{(d)}(\zeta), \tag{B.3}$$

for each $d > 1$. Suppose now that the coefficients in the series expansion of $X(\cdot)$ up to degree $d-1$ are known, and let

$$X^{(d)}(\zeta) = \sum_{\|m\|_1 = d} X_m \zeta^m, \qquad X_m \in \mathbb{C}^n,$$
$$\overline{\beta}^{(d)}(\zeta) = \sum_{\|m\|_1 = d} \overline{\beta}_m \zeta^m, \qquad \overline{\beta}_m \in \mathbb{C}^n,$$

where $m = (m_1, \ldots, m_\rho)^\top$, $m_i \in \mathbb{N} \cup \{0\}$ $\forall$ $i \in \{1, \ldots, \rho\}$, and $\zeta^m = \zeta_1^{m_1} \ldots \zeta_\rho^{m_\rho}$, so the PDE (B.3) becomes

$$\sum_{\|m\|_1 = d} E X_m \frac{\partial \zeta^m}{\partial \zeta} \Lambda \zeta - \sum_{\|m\|_1 = d} F X_m \zeta^m = \sum_{\|m\|_1 = d} \overline{\beta}_m \zeta^m.$$

Suppose, without loss of generality, that Λ has an arbitrary Jordan block structure, so

$$\Lambda = \begin{bmatrix} \lambda_1 & n_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & n_2 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_{\rho-1} & n_{\rho-1} \\ 0 & 0 & 0 & \cdots & 0 & \lambda_\rho \end{bmatrix}, \qquad n_i \in \{0, 1\} \ \forall \ i = 1, \ldots, \rho - 1.$$

Then it follows that

$$\begin{aligned}\frac{\partial \zeta^m}{\partial \zeta}\Lambda\zeta &= m_1\lambda_1\zeta^m + \cdots + m_{\rho-1}\lambda_{\rho-1}\zeta^m + m_\rho\lambda_\rho\zeta^m \\ &\quad + m_1 n_1 \zeta_1^{m_1-1}\zeta_2^{m_2+1}\ldots\zeta_{\rho-1}^{m_{\rho-1}}\zeta_\rho^{m_\rho} + \ldots \\ &\quad + m_{\rho-1}n_{\rho-1}\zeta_1^{m_1}\zeta_2^{m_2}\ldots\zeta_{\rho-1}^{m_{\rho-1}-1}\zeta_\rho^{m_\rho+1} \\ &= (m\cdot\lambda)\zeta^m + m\cdot\begin{bmatrix} n_1\zeta_1^{m_1-1}\zeta_2^{m_2+1}\ldots\zeta_{\rho-1}^{m_{\rho-1}}\zeta_\rho^{m_\rho} \\ \vdots \\ n_{\rho-1}\zeta_1^{m_1}\zeta_2^{m_2}\ldots\zeta_{\rho-1}^{m_{\rho-1}-1}\zeta_\rho^{m_\rho+1} \\ 0 \end{bmatrix},\end{aligned}$$

where $\lambda = (\lambda_1, \ldots, \lambda_\rho)^\top$. The PDE (B.3) becomes

$$\begin{aligned}\sum_{\|m\|_1=d}\Big(\big((m\cdot\lambda)E - F\big)X_m - \overline{\beta}_m\Big)\zeta^m = \\ -\sum_{\|m\|_1=d} m\cdot\begin{bmatrix} n_1\zeta_1^{m_1-1}\zeta_2^{m_2+1}\ldots\zeta_{\rho-1}^{m_{\rho-1}}\zeta_\rho^{m_\rho} \\ \vdots \\ n_{\rho-1}\zeta_1^{m_1}\zeta_2^{m_2}\ldots\zeta_{\rho-1}^{m_{\rho-1}-1}\zeta_\rho^{m_\rho+1} \\ 0 \end{bmatrix} EX_m.\end{aligned}$$

As a result of this, the coefficient X_m is a function of any $X_{\overline{m}}$ such that, for $i \in \{1, \ldots, \rho-1\}$,

$$\overline{m}_i = m_i + 1, \quad \overline{m}_{i+1} = m_{i+1} - 1, \quad \overline{m}_j = m_j \;\forall\; j \in \{1,\ldots,\rho\} \setminus \{i, i+1\}.$$

A potential issue with such dependencies arises if a circular dependency exists, as such a dependency would change the nonresonance condition required to guarantee existence of a solution. To prove that no such dependency exists, consider now constructing an ordered set in which to solve the coefficients X_m. Define the set

$$M = \Big\{ m : 0 \leq m_i \leq d,\ m_i \in \mathbb{N} \cup \{0\}\ \forall\ i \in \{1, \ldots, \rho\},\ m_1 + \cdots + m_\rho = d \Big\},$$

and define an order on the set M such that for any elements $m, \overline{m} \in M$

(i) $m = \overline{m}$ if $m_i = \overline{m}_i\ \forall\ i \in \{1, \ldots, \rho\}$,

(ii) $m > \overline{m}$ if

$$\begin{aligned} &(m_1 > \overline{m}_1) \\ &\vee\ (m_1 = \overline{m}_1, m_2 > \overline{m}_2) \\ &\vee \ldots \\ &\vee\ (m_1 = \overline{m}_1, \ldots, m_{\rho-1} = \overline{m}_{\rho-1}, m_\rho > \overline{m}_\rho), \end{aligned}$$

(iii) $m < \overline{m}$ otherwise.

This order is well-defined as exactly one of $(i), (ii), (iii)$ holds for any $m, \overline{m} \in M$, and $m > \overline{m}$, $\overline{m} > \overline{\overline{m}}$ implies that $m > \overline{\overline{m}}$.

Equiped with the ordered set M, existence of a solution to the PDE can be proven by explicitly determining the coefficients of $X^{(d)}(\cdot)$, *i.e.* X_m for all

$\|m\|_1 = d$, in the correct order. At any step in the procedure, let S be the set of $\overline{m} \in M$ such that the coefficient $X_{\overline{m}}$ is known, and let U be the set of $m \in M$ such that the coefficient X_m is unknown. It follows that

$$M = S \cup U, \quad S \cap U = \emptyset,$$

and $|M|$ is finite and not empty for any particular $d = \|m\|_1$. Furthermore, S and U inherit the well-defined ordered set property of M. To proceed, partition the procedure for solving the coefficients into three cases: the base case where $S = \emptyset$ and $U = M$, the final (trivial) step where $S = M$ and $U = \emptyset$, and intermediate steps $S \neq \emptyset$ and $U \neq \emptyset$.

First, suppose that $S = \emptyset$ and $U = M$. Consider the element $m = (d, 0, \ldots, 0) \in U$, and note that $m = \sup(U)$. It can be seen that X_m depends on no $X_{\overline{m}}$ such that $\overline{m} \in U$, $\overline{m} \neq m$, because no $\overline{m}$ can be chosen such that, for any $i \in \{1, \ldots, \rho - 1\}$,

$$\overline{m}_i = m_i + 1, \ \overline{m}_{i+1} = m_{i+1} - 1, \ \overline{m}_j = m_j \ \forall \ j \in \{1, \ldots, \rho\} \setminus \{i, i+1\},$$

without setting one of $\overline{m}_i < 0$. Collecting the coefficients of $\zeta^m = \zeta_1^d$ in the PDE (B.3) yields

$$\Big((m \cdot \lambda)E - F\Big)X_m - \overline{\beta}_m = 0,$$

so the coefficient $X_{(d,0,\ldots,0)}$ is uniquely determined to be

$$X_{(d,0,\ldots,0)} = \Big(d\lambda_1 E - F\Big)^{-1}\overline{\beta}_{(d,0,\ldots,0)}.$$

The coefficient $X_{(d,0,\ldots,0)}$ is now known, so in the next step one can consider

$$S_{\text{new}} = \{(d,0,\ldots,0)\}, \qquad U_{\text{new}} = M \setminus \{(d,0,\ldots,0)\}.$$

Suppose now that $S \neq \emptyset$. If $U = \emptyset$ then $S = M$ and $X^{(d)}(\cdot)$ is uniquely determined, thus only $U \neq \emptyset$ needs to be considered. Let $m \in U$ be the largest element of U, so $m \geq \widetilde{m}$ for all $\widetilde{m} \in U$. The order on M is well-defined and $|U|$ is finite, so $m = \sup(U) \in U$ exists and is the unique upper bound of U. The coefficient X_m is a function only of $X_{\overline{m}}$ such that, for some $i \in \{1,\ldots,\rho-1\}$,

$$\overline{m}_i = m_i + 1,\ \overline{m}_{i+1} = m_{i+1} - 1,\ \overline{m}_j = m_j\ \forall\ j \in \{1,\ldots,\rho\} \setminus \{i, i+1\},$$

where $m_i, \overline{m}_i \geq 0$ and $\overline{m} \in M$. It follows that determining X_m depends only on having determined $X_{\overline{m}}$ such that $\overline{m} > m$. At least one such $\overline{m}$ exists for any $m \neq (d,0,\ldots,0)$, but because m is an upper bound of U and $\overline{m} > m$ it must be that $\overline{m} \in M \setminus U = S$, so $X_{\overline{m}}$ is known and collecting the coefficients

of ζ^m yields

$$\Big((m\cdot\lambda)E-F\Big)X_m=\overline{\beta}_m-\sum_{\overline{m}\in D(m)}\overline{m}_i n_i E X_{\overline{m}}.$$

where

$$D(m):=\Big\{\overline{m}\in S:(\exists i\in\{1,\dots,\rho-1\})\ \overline{m}_i=m_i+1,\ \overline{m}_{i+1}=m_{i+1}-1\Big\}.$$

Because the matrix pair (E,F) is of type (C,v) with respect to $\sigma(\Lambda)$ the coefficient X_m is uniquely determined to be

$$X_m=\Big((m\cdot\lambda)E-F\Big)^{-1}\left(\overline{\beta}_m-\sum_{\overline{m}\in D(m)}\overline{m}_i n_i E X_{\overline{m}}\right).$$

The coefficient X_m of ζ^m in $X^{(d)}(\cdot)$ is now determined, so in the next step one can take

$$S_{\text{new}}=S\cup\{m\},\quad U_{\text{new}}=U\setminus\{m\}.$$

Repeating this procedure until $U=\emptyset$ yields the solution $X^{(d)}(\cdot)$ to the PDE (B.3). Applying the procedure inductively for $X^{(i)}(\cdot)$, $i>d$, yields the formal series solution to the PDE (B.1). Local analycity of the solution follows by an argument similar to that of [84]. □