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CALCULATION OF PROPAGATION CONSTANTS IN NON-UNIFORM WAVEGUIDES

by

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ABSTRACT

The first chapter analyses different theoretical and numerical methods used to determine the propagation characteristic of inhomogeneously filled waveguides. The following methods are discussed: field theory, network theory, variational and Rayleigh-Ritz procedures, reaction concept, finite-difference and finite-element methods, WKB approximation, point-matching method, subwaveguide method, transmission-line matrix and a method based on solution of equations of the form of Hill's equation.

In the second chapter, the method developed by the author is presented. Starting from Maxwell's equations, two coupled differential equations in H_z and E_z , are obtained for the rectangular waveguide case and for a dielectric constant which is a function of the space co-ordinate x . The axial field components are then expanded in terms of the waveguide zero-frequency modes. These modes are obtained as solutions of the previous differential equations as $\omega \rightarrow 0$. From this, using the orthogonal properties of the expanded modes, two matricial expressions are obtained.

In the third chapter it is shown that for the case of a rectangular waveguide filled with a dielectric slab placed at one side of the guide wall, the dominant mode, TE_{10} (or LSE_{10}), and higher order modes TE_{m0} are obtained from a relatively easy matricial equation. A computer program has been written, and the numerical results for propagation constants and cutoff frequencies are in good agreement with analytical results obtained by other authors.

In the last chapter, solutions for LSM_{mm} and LSE_{mm} modes are discussed and a computer program has been written for evaluating the propagation characteristics of these modes. The particular case of a

square waveguide half-filled with dielectric is again analysed. It is also shown how the method can be extended to the analysis of inhomogeneously loaded cylindrical waveguides and the cutoff frequency for the H_{01} mode for a particular configuration is evaluated using a computer program. Finally, suggestions for further applications of the method are presented.

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CHAPTER I

METHODS FOR CALCULATING PROPAGATION CHARACTERISTICS IN NON-UNIFORM WAVEGUIDES

1.1 Introduction

During the last thirty years many authors have been analyzing the problem of calculating the propagation characteristics of microwaves through waveguides containing dielectrics.

A survey of these methods and their most important characteristics is presented below. In some cases it has been considered worthwhile to present some significant mathematical steps leading to final results in order to provide a deeper understanding of the application of the method.

The distinction between analytical and approximate methods is sometimes blurred, but as a guide in the survey the criterion is used that an analytical method yields an expression of more or less closed form, whereas an approximate method leads to an algorithm.

1.2 Analytical Methods

1.2.1 Field Theory^{1,2}

The field components for any mode for the different regions in which the waveguide is subdivided, found from Maxwell's equations, involve constants which usually are evaluated in terms of the peak power flow through the guide and/or using boundary conditions.

Consider a waveguide filled with a medium of dielectric constant ϵ and permeability μ . Assume that the walls of the waveguide

are parallel to the z-axis.

It is possible to write the electric and magnetic field as follows :

$$\begin{aligned}\underline{E} &= (\underline{E}_t + \hat{k} E_z) \exp(j\omega t - \gamma z) \\ \underline{H} &= (\underline{H}_t + \hat{k} H_z) \exp(j\omega t - \gamma z)\end{aligned}\tag{1.1a}$$

where $\underline{E}_t$, $\underline{H}_t$ are the components of the field in the directions perpendicular to the longitudinal axis, $\hat{k}$ is the unit vector in the direction of z increasing. Note that both E_t and E_z (H_t and H_z) are functions of x and y.

For sinusoidal fields in a linear, isotropic and source free dielectric medium, Maxwell's equations can be written as

$$\begin{aligned}\text{curl } \underline{E} &= -j\omega\mu\underline{H} \\ \text{curl } \underline{H} &= j\omega\varepsilon\underline{E}\end{aligned}\tag{1.1b}$$

where $\partial/\partial t = j\omega$.

It is sometimes convenient, especially when dealing with boundary value problems, to write the vector differential operator ∇ as the sum of the two operators $\nabla = \nabla_t + \nabla_z$, where ∇_t is that portion of ∇ which represents differentiation with respect to the coordinates perpendicular to the axial direction, in this case x and y, and $\nabla_t \equiv -\gamma\hat{k}$. The curl equation for the electric field may then be expressed as

$$(\nabla_t - \gamma\hat{k}) \times (\underline{E}_t + \hat{k} E_z) = -j\omega\mu_0 (\underline{H}_t + \hat{k} H_z)\tag{1.1c}$$

where the exponential factors have been omitted.

Similarly, the curl equation for the magnetic field then becomes

$$(\nabla_t - \gamma\hat{k}) \times (\underline{H}_t + \hat{k} H_z) = j\omega\varepsilon (\underline{E}_t + \hat{k} E_z)\tag{1.1d}$$

The parameters ϵ and μ describe the nature of the space containing the two field vectors $\underline{E}$ and $\underline{H}$. It will be assumed that $\mu = \mu_0$ and that $\epsilon = \epsilon(x, y)$.

It is possible to write explicitly the relationships between the field components as follows:

$$\begin{aligned}\partial E_z / \partial y &= \gamma E_y - j\omega\mu H_x \\ \partial E_z / \partial x &= \gamma E_x + j\omega\mu H_y \\ j\omega\mu H_z &= \partial E_x / \partial y - \partial E_y / \partial x \\ \partial H_z / \partial y &= \gamma H_y + j\omega\epsilon E_x \\ \partial H_z / \partial x &= \gamma H_x - j\omega\epsilon E_y \\ j\omega\epsilon E_z &= \partial H_y / \partial x - \partial H_x / \partial y\end{aligned}$$

From the previous set of equations it is clear that if E_z and H_z are known, it is possible to deduce the four remaining components, E_x , E_y , H_x and H_y .

At a dielectric interface it is known that the axial component of the field, E_z and H_z , and the derivatives of the transverse components, E_t and H_t , with respect to a direction normal to the interface, must be continuous, even if ϵ and μ change discontinuously across the boundary.

Equations (1.1c) and (1.1d) may be split into two equations respectively;

$$\hat{k} \times \nabla_t E_z + \gamma \hat{k} \times E_t = j\omega\mu H_t \quad (1.2a)$$

$$\nabla_t \times E_t = -j\omega\mu \hat{k} H_z \quad (1.2b)$$

$$\hat{k} \times \nabla_t H_z + \gamma \hat{k} \times H_t = -j\omega\epsilon E_t \quad (1.2c)$$

$$\nabla_t \times H_t = j\omega\epsilon E_z \hat{k} \quad (1.2d)$$

Taking the vector product $\gamma \hat{k} \times$ eq. (1.2a)

$$\gamma \hat{k} \times \hat{k} \times \nabla_t E_z + \gamma^2 \hat{k} \times \hat{k} \times E_t = j\omega\mu \gamma \hat{k} \times H_t$$

$$\text{or} \quad -\gamma \nabla_t E_z - \gamma^2 E_t = j\omega\mu \gamma \hat{k} \times H_t$$

Using Eq.(1.2c), the following expression is obtained

$$(\gamma^2 + k_o^2 \epsilon_r) \underline{E}_t + \gamma \nabla_t \underline{E}_z = j\omega \mu \hat{k} \times \nabla_t H_z \quad (1.3a)$$

where $k_o^2 = \omega^2 \mu \epsilon_o = \omega^2 \mu_o \epsilon_o$ and $\epsilon_r = \epsilon / \epsilon_o$.

Taking the vector product of $\gamma \hat{k} \times$ Eq.(1.2c),

$$\gamma \hat{k} \times \hat{k} \times \nabla_t H_z + \gamma^2 \hat{k} \times \hat{k} \times H_t = -j\omega \epsilon \gamma \hat{k} \times \underline{E}_t$$

Using Eq. (1.2a), the following expression is obtained

$$(\gamma^2 + k_o^2 \epsilon_r) \underline{H}_t + \gamma \nabla_t H_z = -j\omega \epsilon \hat{k} \times \nabla_t H_z \quad (1.3b)$$

Both equations (1.3a) and (1.3b) can be written in a matricial form as

$$M(\hat{k} \times \nabla_t) \phi = \gamma \nabla_t A \phi + (\gamma^2 + k_o^2 \epsilon_r) A T \quad (1.3c)$$

where

$$M = \begin{bmatrix} \epsilon & 0 \\ 0 & \mu \end{bmatrix}$$

$$\phi = \begin{bmatrix} E_z \\ H_z \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} E_t \\ H_t \end{bmatrix}$$

For an homogenous region it is possible to obtain an expression, generally known as homogenous Helmholtz equation, that can be solved exactly for particular geometric configuration. But, in general, for inhomogeneous configurations, an analytical solution is not available. The propagation characteristics must then be evaluate from different expressions, such as a characteristic equation.

The characteristic equation is formed by applying proper boundary conditions at the interface between any two media

inside the guide. In general, this characteristic equation is transcendental and must be solved either graphically or using iterative methods. Its solution leads to the determination of the propagation constant.

This approach has been used by several authors^{3,4,5} to solve the problem for composite rectangular or cylindrical waveguides.

Some results so obtained will be discussed at a later stage when they will be compared with the results obtained by the method outlined in the next chapter, but it should be emphasised that it is not always possible to obtain a solution because of the complexity of the characteristic equation.

1.2.2 Network Theory

The composite structure is represented by a network comprising a variety of elementary lumped-constant circuits and transmission lines.

To be more specific, the composite structure is described in terms of a transmission region, characterized by a single dominant mode, and a discontinuity region, characterized by an infinite set of higher order rapidly evanescent modes, and the problem is then formulated in terms of conventional network concepts.

If the structure cross-section possesses an appropriate geometrical symmetry, the characteristics of the propagating modes may be determined by regarding one of the transverse directions perpendicular to the longitudinal axis as a transmission direction.

The transmission structure so defined is completely equivalent to the original guide: it is composed, in general of elementary discontinuities and waveguides and is terminated by the guide walls if any. The desired frequencies may be determined by simple network analysis of the transverse equivalent network representative of the desired mode in the above composite structure.

As an illustrative example, let us consider the structure depicted in Fig. 1.1. At the reference plane T the equivalent transverse network for the dominant mode in the guide is a junction

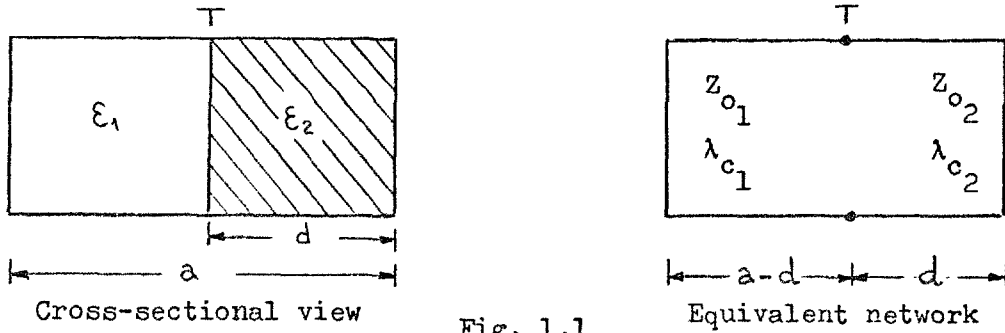


Fig. 1.1

of two short-circuited H-mode uniform transmission lines. The propagation wavelength λ_g of all TE_{on} modes is obtained^{6,7} in this case from the resonant condition

$$Z_{o1} \tan \frac{2\pi}{\lambda_{c1}} (a - d) = - Z_{o2} \tan \frac{2\pi}{\lambda_{c2}} d \quad (1.4a)$$

where

$$\frac{Z_{o1}}{Z_{o2}} = \frac{\lambda_{c1}}{\lambda_{c2}} = \sqrt{\frac{\epsilon_2 - (\lambda_o/\lambda_g)^2}{\epsilon_1 - (\lambda_o/\lambda_g)^2}}$$

and $\lambda_{c1} = \lambda_o / \sqrt{\epsilon_1 - (\lambda_o/\lambda_g)^2}$; $\lambda_{c2} = \lambda_o / \sqrt{\epsilon_2 - (\lambda_o/\lambda_g)^2}$

Equation (1.2) is exact provided the conductivity of the guide wall is infinite. The slab can be dissipative and characterized by a complex dielectric constant, and in this case λ_g is complex.

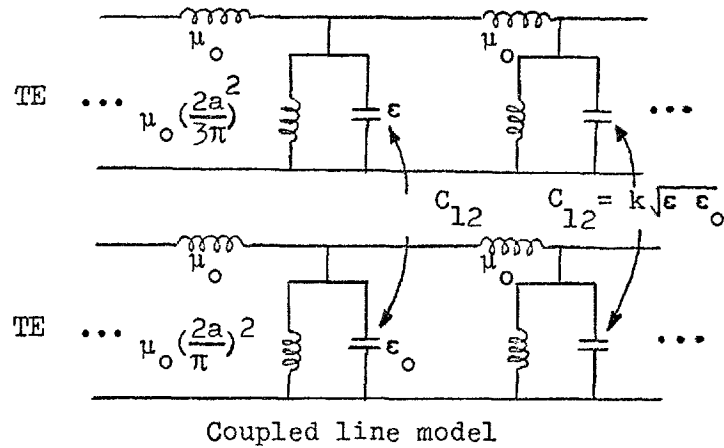
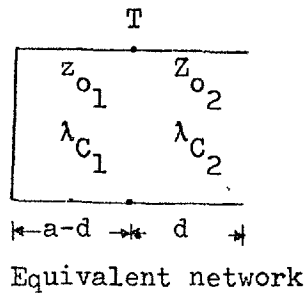
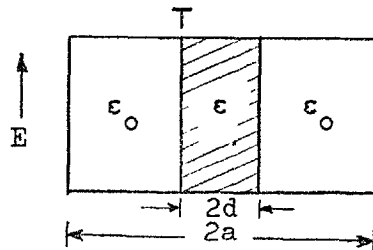
An alternative approach is to reformulate the problem in terms of a scattering matrix. The formulation of field problems either as network problem or as scattering problems provides a

fully equivalent and equally rigorous description of the far field in a microwave structure. It should, however, be noted that this method gives only the propagation constant. But a knowledge of the behaviour of fields requires that the solution of Maxwell's equations be subjected to the proper boundary conditions at the walls of the guide and at the interfaces separating the media.

Further extensions of the method above mentioned have been carried out. Carlin^{8,9} used coupled sets of transmission lines (not necessarily TEM) to obtain simple models of lossless waveguide structures, given that the physical structures are longitudinally uniform but transversely inhomogeneous. The uniform coupled line system is characterized by a series impedance coupling network per unit length whose impedance matrix per unit length is rational and Foster^{*} (lossless) together with a shunt admittance network per unit length whose admittance matrix per unit length is similarly rational and Foster. The problem of equivalent circuit representation for the prescribed waveguide geometry is to find the series and shunt network matrices per unit length, $Z(p)$ & $Y(p)$ respectively, with the complex frequency variable $p = \sigma + j\omega$, where ω is the angular frequency.

According to Carlin's work, it is possible to couple together simple TEM, TM and TE scalar lines, and these scalar lines can be used to construct models for lossless, longitudinally uniform guides. As an example, let us compare the equivalent networks and the characteristic equations of a waveguide filled with a dielectric slab using Carlin's method⁹ and Marcuvitz's method⁶, as shown in Fig. 1.2.

* (i.e. analytic in that $\text{Re } p = \text{Re}(\sigma + j\omega) > 0$, real where p is real, and paraskew hermitian, meaning that $-A^T(-j\omega) = A(j\omega)$)



$$Z_{o1} \tan \frac{2\pi}{\lambda_{C1}} (a - d) = Z_{o2} \cot \frac{2\pi}{\lambda_{C2}} d$$

$$\frac{Z_{o1}}{Z_{o2}} = \frac{\lambda_{C1}}{\lambda_{C2}} = \sqrt{\frac{\epsilon - (\lambda_o/\lambda_g)^2}{\epsilon_o - (\lambda_o/\lambda_g)^2}}$$

$$1 = \sqrt{\frac{\frac{\epsilon}{\epsilon_o} - x^2}{1 - x^2}} (\tan \pi (1 - \frac{d}{a}) \mu \sqrt{1 - x^2}) (\tan \pi \frac{d}{a} \mu \sqrt{\frac{\epsilon}{\epsilon_o} - x^2})$$

$$x = \lambda/\lambda_g, \quad \mu = 2a/\lambda$$

a) Marcuvitz's method

b) Carlin's method

Fig. 1.2 Comparison between Marcuvitz and Carlin's method.

Another very interesting method is due to Clarricoats and Oliver¹⁰. The transverse network representations for inhomogeneously filled waveguides supporting pure modes have been mentioned before, and they showed that a representation valid for a hybrid mode was possible, using a combination of a pure radial-line E mode and a pure radial-line H mode. They analyzed the circular waveguide containing two isotropic cylindrical regions (or two isotropic plasma regions) and the case for the unbounded rod with and without an axial inner conductor. The boundary conditions on the rod surface

impose a constraint on the total radial-line impedances and admittances. When the constraint is expressed mathematically, it leads to the characteristic equation for the propagation constant.

It is also shown that the β/ω dispersion curves reflect the constraint necessary to ensure that the transverse impedances or admittances are matched at any circumferential boundary.

Details of this paper will not be analyzed deeper because in the following chapters only rectangular waveguides will be considered; it has been considered worthy to mention it as an important reference for any work dealing with inhomogeneous filled cylindrical waveguides.

1.3 Approximate Methods

As was mentioned before, an exact solution of the equations encountered in problems on dielectric waveguides may be obtained for only a limited number of cases. For instance, for the scalar Helmholtz equation ($\nabla^2\phi + k^2\phi = 0$) it is possible to use, for some coordinate systems, the method of separation of variables. But if the surface on which the boundary conditions are to be satisfied is not one of these coordinate surfaces, or, if the boundary conditions are not the simple Dirichlet ($\phi = 0$) or Neumann ($\frac{\partial\phi}{\partial n} = 0$) types but of the mixed type $\frac{\partial\phi}{\partial n} + f(s)\phi = 0$, where generally f , a function of the surface coordinates, varies over the boundary, the method of separation of variables fails. This has led to theoretical techniques such as the integral equation method and other methods developed during the early 1940's. In the case of the

integral equation formulation, the transform technique may be used to obtain a solution only if the kernel of the integral equation is of a particular form. For example, the Fourier transform can be used if the kernel is a function of the difference of two variables and if the range of integration is from $-\infty$ to $+\infty$ or from 0 to $+\infty$. In such cases it is more convenient to use approximate methods for the solution.

Perturbation methods are useful whenever the problem under consideration closely resembles one which can be solved exactly⁴. This is the theory of the changes which occur in the eigenvalues and eigenfunctions in an eigenvalue problem due to small changes in the problem, such as surface or volume perturbations. When the deviations from the exactly soluble problem become large, the perturbation method is not useful any more. In this case it is more convenient to use the variational method. In contrast to the perturbational procedure, the variational procedure gives an approximation to the desired quantity itself, rather than to changes in the quantity. The variational procedure differs from other approximation methods in that the formula is stationary about the correct solution. This means that the formula is relatively insensitive to variations in an assumed field about the correct field. If the desired quantity is real, the variational formula may be an upper or lower bound to the desired quantity⁴. In order to determine the accuracy of a variational calculation, it is necessary to have a systematic procedure for improving on the original trial function. The method of inserting additional non-linear parameters will increase the accuracy, but it cannot be said that it will ultimately lead to the correct answer. One way to avoid this difficulty is to insert a linear combination of a

complete set of functions, the coefficients forming a set of linear variational parameters. For instance, if an assumed field is expressed as a series of functions with undetermined coefficients, then the coefficients can be adjusted by the Rayleigh-Ritz procedure. In fact, if a complete set of functions is used for the assumed field, it is sometimes possible to obtain an exact solution. The Rayleigh-Ritz procedure can be utilized to obtain approximations for the first N eigenvalues and eigenfunctions in an eigenvalue problem. Stationary formulas for electromagnetic problems can also be obtained by using the concept of reaction⁴.

When the boundary conditions are rather complicated, or in the case of coordinate systems where the analytical separation of variables technique fails, the approximate variational and perturbational technique are applied but in general information is only obtained about the eigenvalue, but none on the characteristics of the field itself. It has been shown¹¹ that finite-difference techniques in conjunction with matrix calculations on digital computers can be used more conveniently in such cases.

In the last decade, many different approaches have been used by several authors, using approximations based on more refined mathematical techniques or different basic concepts, such as the Wentzel, Kramers and Brillouin (WKB) approximation, the point-matching techniques, the moment method (unfortunately in the microwave literature the terms point-matching and moment method are used interchangeably contrary to what is done in other fields of science), and others that have been discussed in previous sections of this chapter.

1.3.1 The perturbation method

Many physical problems reduce to finding the eigenvalues λ_n and eigenfunctions $\phi_n = \phi_n(x)$ of an equation of the following form

$$L\phi_n + (\lambda_n - p)\phi_n = 0 \quad (1.5)$$

where L is a differential operator and $p = p(x)$, where x represents a set of coordinates. It is often necessary to find solutions under circumstances slightly different from conditions for which the solutions are known. In some problems, the boundary of the region is deformed slightly and the boundary conditions are assumed to be unaltered, whereas, in some cases, the boundary remains unchanged, but the boundary conditions for ϕ are slightly altered from the case when the solutions are known. The problem can be approached in several ways¹²;

(a) It can be solved by introducing additional terms to the differential operator and then employing the usual perturbation technique;

(b) it can also be approached by transforming the differential equation to an integral equation by using Green's function. The integral equation which includes the boundary conditions is then reduced to a standard form, the solution of which is expressed in a form suitable for obtaining its value to any approximation^{13,14}. This case has been extensively treated in Felsen and Marcuvitz's book, using the characteristic Green's function procedure. The method is formulated for the Sturm-Liouville differential operator describing propagation on a general non-uniform transmission line, even in presence of sources in the media. The singularities of the characteristic Green's function in the complex plane are explored, answering systematically the questions of mode completeness and

normalization for both discrete and continuous eigenfunctions. Using analytic continuation of spectral representations is possible to construct alternative field representations. Under general conditions, explicit construction of the field behaviour requires approximation procedures whose success relies on the ability to represent the solution to the given problem as a weak perturbation of a known solution to a related problem. If the unperturbed problem is described by a differential equation that exhibits in certain critical regions (near singularities, etc.) the same analytical behaviour as the desired problem, solution of the latter can be constructed by systematic techniques¹³. One aspect of this procedure involves reformulation of the differential equation problem as an integral equation whose kernel constitutes an unperturbed Green's function, and subsequent solution by iteration. A brief discussion of the method for a rectangular waveguide filled with a dielectric slab placed at one side of the guide (H modes in x) is given below.

The determination of the eigenfunctions ϕ_m and the eigenvalues λ_m in the domain $x_1 \leq x \leq x_2$ poses a problem of the Sturm-Liouville type (Appendix A):

$$\left[\frac{d}{dx} p(x) \frac{d}{dx} - q(x) + \lambda_m w(x) \right] \phi_m(x) = 0, \quad x_1 \leq x \leq x_2 \quad (1.6a)$$

subject to the homogeneous boundary conditions

$$p \frac{d\phi_m}{dx} + \alpha_{1,2} \phi_m = 0, \quad x = x_{1,2} \quad (1.6b)$$

where p, q and the weight function w are assumed to be piecewise continuous functions of x in $x_1 \leq x \leq x_2$. Multiplying (1.6a) by ϕ_m^* , where $*$ denotes the complex conjugate, and integrating between x_1 and x_2 , one finds

$$\lambda_m = \frac{\int_{x_1}^{x_2} dx \left(p \left| \frac{d\phi_m}{dx} \right|^2 + \int_{x_1}^{x_2} dx \, q \left| \phi_m \right|^2 - \alpha_1 \left| \phi_m(x_1) \right|^2 + \alpha_2 \left| \phi_m(x_2) \right|^2 \right)}{\int_{x_1}^{x_2} dx \, w \left| \phi_m \right|^2} \quad (1.7)$$

from which it follows that λ_m is real for real values of $p, q, w, \alpha_{1,2}$.

It is also possible to demonstrate that $\int_{x_1}^{x_2} w \phi_m \phi_n^* dx = 0, m \neq n$. (1.8)

For finite intervals and piecewise constant p, q and w , the mode spectrum associated with the general Sturm-Liouville eigenvalue problem of Eq. (1.6) and Eq. (1.7) is discrete. Hence the direct solution of these equations as well as the subsequent normalization of the eigenfunctions ϕ_m is straightforward. The present method exploits a close relation between eigenvalue solutions and the characteristic Green's function. In network terms, the eigenvalue problem defines the resonant voltage on a terminated non-uniform transmission line while the Green's function displays the similar resonant responses to excitation by a point voltage or current generator.

The characteristic Green's function $g(x, x'; \lambda)$ for the Sturm-Liouville problem of eq. (1.6) is defined by

$$\left[\frac{d}{dx} p(x) \frac{d}{dx} - q(x) + \lambda w(x) \right] g(x, x'; \lambda) = -\delta(x-x'),$$

$$x_1 < x' < x_2 \quad (1.9)$$

subject to the boundary conditions

$$\left(p \frac{d}{dx} + \alpha_{1,2} \right) g(x, x'; \lambda) = 0 \quad x = x_{1,2} \quad (1.10)$$

The parameter λ is arbitrary but so restricted as to assure a unique solution of Eq. (1.9).

A network representation of the characteristic Green's function problem is shown in Fig. 1.3:

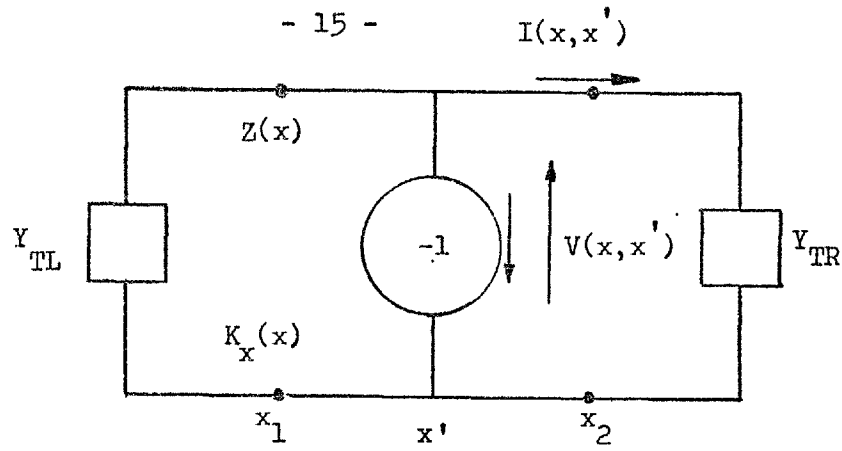


Fig. 1.3 Network representation of the characteristic Green's function.

where

$$-\frac{dV(x, x')}{dx} = j k_x(x) Z(x) I(x, x') \quad (1.11a)$$

$$-\frac{dI(x, x')}{dx} = j k_x(x) Y(x) V(x, x') - \delta(x - x') \quad (1.11b)$$

where $Z(x) = 1/Y(x)$ and $k_x(x)$ are the characteristic impedance and propagation constant respectively. For an H-mode transmission line $k_x Z = \omega \mu$, so that the corresponding second-order differential equation for $V(x, x')$ has the form

$$\left[\frac{d}{dx} \left(\frac{1}{\mu'(x)} \frac{d}{dx} \right) + k_o^2 \epsilon'(x) - \frac{K_T^2}{\mu'(x)} \right] V(x, x') = j\omega \mu_o \delta(x - x') \quad (1.12)$$

where $k_o^2 = \omega^2 \mu_o \epsilon_o$ and

$$\mu'(x) = \frac{\mu(x)}{\mu_o}, \quad \epsilon'(x) = \frac{\epsilon(x)}{\epsilon_o} \quad (1.12a)$$

By comparison,

$$p(x) = w(x) = \frac{1}{\mu'(x)}, \quad q(x) = -k_o^2 \epsilon'(x), \quad k_T^2 = -\lambda, \quad (1.13)$$

$$V(x, x') = j \omega \mu_o g(x, x'; \lambda)$$

The boundary conditions are rephrased by Eqs. (1.11a) and (1.13) in terms of

$$\frac{I}{V} = j \frac{p(dg/dx)}{\omega \mu_o g} \quad (1.14a)$$

and replaced by terminating admittances Y_{TR} and Y_{TL} and x_1 and x_2 :

$$Y_{TL} = - \frac{I(x_1, x')}{V(x_1, x')} = \frac{j\alpha_1}{\omega\mu_0}, \quad Y_{TR} = \frac{I(x_2, x')}{V(x_2, x')} = \frac{-j\alpha_2}{\omega\mu_0} \quad (1.14b)$$

$$\text{Also } g(x, x'; \lambda) \left| \begin{array}{l} x' + \Delta \\ x' - \Delta \end{array} \right. = 0, \quad p(x) \frac{d}{dx} g(x, x'; \lambda) \left| \begin{array}{l} x' + \Delta \\ x' - \Delta \end{array} \right. = -1 \quad (1.15)$$

Since the configuration in Fig. 1.4 can be viewed as a cavity (lossless if jY_{TR} , jY_{TL} , k_x^2 and Z^2 are real), it is evident that the voltage response g will be finite and well defined unless the parameter λ is chosen in such a way that a resonance can exist. For fixed values of $\alpha_{1,2}$ and $x_{1,2}$, resonances will exist for parameter values λ_m at which the corresponding voltage (or current) will be infinite. To assure a unique solution of the network problem it is necessary that $\lambda \neq \lambda_m$. For the lossless situation, denoted mathematically as the Hermitian case, where the resonant values λ_m are real, this restriction can be stated more weakly as $\text{Im } \lambda \neq 0$. If λ in Eq.(1.9) is now regarded as a general complex parameter, $g(x, x'; \lambda)$ is a regular function of λ in the complex λ plane except at points $\lambda = \lambda_m$, where it becomes infinite and possesses simple pole singularities. Since the resonant condition $\lambda = \lambda_m$ implies the persistence of a response even when the source is removed, the functional form of the resonant solution satisfies the homogeneous equation (1.5). Thus, information about the desired eigensolutions of Eq.(1.6) is contained in the singularities of the characteristic Green's function g , and the problem of determining all possible resonances (i.e. a complete set of eigenfunctions) is directly related to the complete investigation of the singularities of $g(x, x'; \lambda)$ in the complex λ plane. It has been assumed that the dimensions x_1 and x_2 are finite so that resonances, which occur for discrete values of λ_m , characterize simple pole singularities of g . If one of the dimensions

becomes infinite, or if p, q , or w possesses a singularity, the discrete resonances may coalesce into a continuous spectrum; in this instance, $g(x, x'; \lambda)$ possesses a branch-point singularity giving rise to the necessity of introducing a branch cut in the complex λ plane to ensure uniqueness of g .

To relate the complete eigenmode set ϕ_m in Eq.(1.6) to the characteristic Green's function in Eq.(1.9), it will be assumed that the mode set is known, so that

$$g(x, x'; \lambda) = \sum_m g_m(x'; \lambda) \phi_m(x) \quad , \quad x_1 < x < x_2 \quad (1.16)$$

$$\text{where} \quad g_m(x', \lambda) = \int_{x_1}^{x_2} w(\beta) \phi_m^*(\beta) g(\beta, x'; \lambda) d\beta \quad (1.17)$$

Utilizing the delta-function one obtains

$$g_m(x', \lambda) = - \frac{\phi_m^*(x')}{\lambda - \lambda_m} \quad (1.18)$$

so that

$$g(x, x'; \lambda) = - \sum_m \frac{\phi_m(x) \phi_m^*(x')}{\lambda - \lambda_m} \quad (1.19)$$

If Eq.(1.19) is integrated in the complex λ plane about a contour C enclosing all the singularities of g , one obtains

$$\oint_C g(x, x', \lambda) d\lambda = - 2\pi j \frac{\delta(x - x')}{w(x')} \quad (1.20)$$

and this solution can be extended by analytic continuation to apply as $\lambda \rightarrow \lambda_m$.

The characteristic H-mode Green's function can be constructed from the knowledge of the two solutions $V_L(x)$ and $V_R(x)$ of the homogeneous equation (1.9) satisfying the required boundary conditions at x_1 and x_2 , respectively

$$\left(\frac{d}{dx} p \frac{d}{dx} - q + \lambda w\right) V_L(x) = 0, \quad \left(p \frac{d}{dx} + \alpha_1\right) V_L = 0$$

(1.21a)

at $x = x_1$,

$$\left(\frac{d}{dx} p \frac{d}{dx} - q + \lambda w\right) V_R(x) = 0, \quad \left(p \frac{d}{dx} + \alpha_2\right) V_R = 0$$

(1.21b)

at $x = x_2$,

The solution for g satisfying Eq. (1.9) when $x \neq x'$ and the required continuity condition at $x = x'$ is thus

$$g(x, x'; \lambda) = A V_L(x <) V_R(x >) \quad (1.22a)$$

where $x <$ or $x >$ denote, respectively, the lesser and greater of the quantities x and x' . The constant A must be so determined as to satisfy the jump that occurs on $p \, dg/dx$ (i.e. the current) at $x = x'$:

$$A p(x') \left[V_L(x') \frac{d}{dx}, V_R(x') - V_R(x') \frac{d}{dx}, V_L(x') \right] = -1 \quad (1.22b)$$

Thus,

$$g(x, x', \lambda) = \frac{V_L(x <) V_R(x >)}{-p W(V_L, V_R)} \quad (1.23a)$$

Where W is the Wronskian determinant,

$$W(V_L, V_R) = \left(V_L \frac{dV_R}{dx} - V_R \frac{dV_L}{dx} \right) \quad (1.23b)$$

Let us consider specifically the composite cross section shown in Fig. 1.4

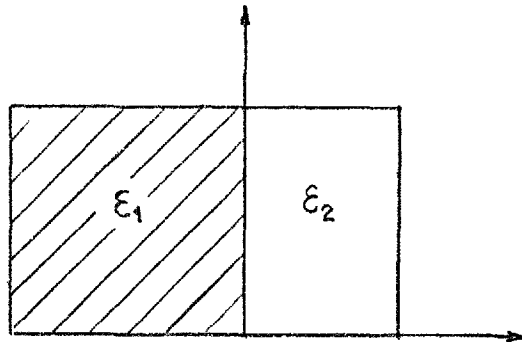


Fig. 1.4

The various media contain a piecewise constant lossless dielectric with

$$\epsilon(x) = \begin{cases} \epsilon_1, & -d < x < 0 \\ \epsilon_2, & 0 < x < a \end{cases}, \quad \epsilon_1 > \epsilon_2 \quad (1.24)$$

the discontinuous nature of which leads to a discontinuous representation of the eigenfunctions. A constant free-space permeability μ_0 is assumed, so $\mu'(x) = 1$, and the surfaces at $x = a, -d$ are assumed to be perfect conductors.

The network configuration of the H-mode characteristic Green's function problem is shown in Fig. 1.5. The relevant propagation constants and characteristic admittances are denoted, respectively, by k_{x1}, Y_1 and k_{x2}, Y_2 .

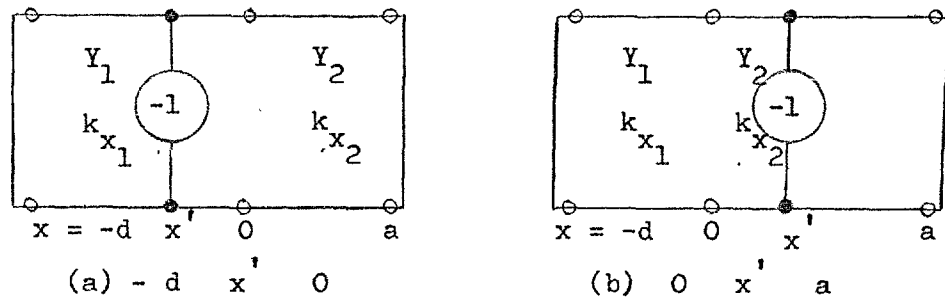


Fig. 1.5 Network representation for the H-mode characteristic Green's function:

The homogenous equation for the standing-wave functions C and S becomes

$$\begin{bmatrix} \frac{d^2}{dx^2} + k_x^2(x, \lambda) \end{bmatrix} \begin{matrix} C(x) \\ S(x) \end{matrix} = 0 \quad (1.25)$$

where

$$k_x^2(x, \lambda) = \begin{cases} k_{x1}^2(\lambda) = k_1^2 + \lambda, & -d < x < 0 \\ k_{x2}^2(\lambda) = k_2^2 + \lambda, & 0 < x < a \end{cases}, \quad k_{1,2}^2 = \omega^2 \mu_0 \epsilon_{1,2} > 0 \quad (1.25a)$$

If one chooses $x_0 = 0$, one gets from the boundary conditions

$$C(x) = \cos(k_{x_1} x), \quad S(x) = \frac{1}{k_{x_1}} \sin(k_{x_1} x) \quad -d < x < 0 \quad (1.26)$$

$$C(x) = \cos(k_{x_2} x), \quad S(x) = \frac{1}{k_{x_2}} \sin(k_{x_2} x) \quad 0 < x < a$$

Since $Y_{TL} = Y_{TR} = \infty$ for the perfectly conducting terminations at $x = -d, a$, it follows that

$$\omega \mu_0 Y_R(0) = -jk_{x_2} \cot(k_{x_2} a), \quad \omega \mu_0 Y_L(0) = -jk_{x_1} \cot(k_{x_1} d), \quad (1.27)$$

$k_x / \omega \mu_0$ being the H-mode characteristic admittance, and as

$$V_R(x, x_0) = C(x, x_0) - j \omega \mu_0 Y_R(x_0) S(x, x_0) \quad (1.28)$$

$$V_L(x, x_0) = C(x, x_0) + j \omega \mu_0 Y_L(x_0) S(x, x_0)$$

it follows that

$$V_R(x) = \begin{cases} V_{2R}(x) = \frac{\sin k_{x_2} (a - x)}{\sin k_{x_2} a} & 0 < x < a \\ V_{1R}(x) = \cos k_{x_1} x - \frac{k_{x_2}}{k_{x_1}} \cot k_{x_2} a \sin k_{x_1} x, & -d < x < 0 \end{cases} \quad (1.29a)$$

$$(1.29b)$$

$$V_L(x) = \begin{cases} V_{2L}(x) = \cos k_{x_2} x + \frac{k_{x_1}}{k_{x_2}} \cot k_{x_1} d \sin k_{x_2} x, & 0 < x < a \\ V_{1L}(x) = \frac{\sin k_{x_1} (x + d)}{\sin k_{x_1} d} & -d < x < 0 \end{cases} \quad (1.29c)$$

$$(1.29d)$$

Sometimes it is more convenient to use, instead of Eq. (1.29c),

$$V_{2L}(x) = \frac{1}{1 + \Gamma_{2L}(0)} \left[\exp(jk_{x_2} x) + \Gamma_{2L}(0) \exp(-jk_{x_2} x) \right], \quad 0 < x < a \quad (1.30)$$

where

$$\Gamma_{2L}(0) = \frac{Y_{02} - Y_L(0)}{Y_{02} + Y_L(0)} = \frac{k_{x_2} + jk_{x_1} \cot k_{x_1} d}{k_{x_2} - jk_{x_1} \cot k_{x_1} d}, \quad Y_{02} = \frac{k_{x_2}}{\omega \mu_0} \quad (1.30a)$$

The H-mode characteristic Green's function $g(x, x'; \lambda)$ can be written directly from (1.23a). Due to the discontinuous representation of $V(x)$ for $x > 0$ and $x < 0$, g is represented discontinuously about $x = 0$.

For the source location as in Fig. 1.5(a),

$$g(x, x'; \lambda) = \begin{cases} \frac{V_{1L}(x <) V_{1R}(x >)}{j \omega \mu_0 Y_S(0)} , & -d < x < 0, \quad -d < x' < 0 \quad (1.31a) \\ \frac{V_{1L}(x') V_{2R}(x)}{j \omega \mu_0 Y_S(0)} , & 0 < x < a, \quad -d < x' < 0; \quad (1.31b) \end{cases}$$

for the source location in Fig. 1.5(b),

$$g(x, x', \lambda) = \begin{cases} \frac{V_{1L}(x) V_{2R}(x')}{j \omega \mu_0 Y_S(0)} , & -d < x < 0, \quad 0 < x' < a \quad (1.31c) \\ \frac{V_{2L}(x <) V_{2R}(x >)}{j \omega \mu_0 Y_S(0)} , & 0 < x < a, \quad 0 < x' < a \quad (1.31d) \end{cases}$$

Eqs. (1.31) can be collected under the single formula

$$g(x, x'; \lambda) = \frac{V_{\beta L}(x <) V_{\beta R}(x >)}{j \omega \mu_0 Y_S(0)} , \quad Y_S(0) = Y_L(0) + Y_R(0) \quad (1.32)$$

where subscript β stands for 1 or 2 if the corresponding variable x or x' lies in the range $-d$ to 0 or 0 to a , respectively. To assure that the solution for g is unique, the restriction $\text{Im} \lambda \neq 0$ is implied.

The singularities of g in the complex λ -plane consist of real simple poles at the zeros of $Y_S(0)$. No branch point singularities exist at $\lambda = -k_{x_1}^2, x_2$, since $V_{\beta L, R}, Y_S(0)$ and therefore g are even functions of k_{x_1}, x_2 . From Eq. (1.27) the zeros λ_m of $Y_S(0, \lambda)$ are specified implicitly by the transcendental equation

$$k_{x_2} \cot k_{x_2} a = -k_{x_1} \cot k_{x_1} d \quad (1.33)$$

$$k_{x_2}^2 = \lambda + k_2^2 = \hat{\lambda}, \quad k_{x_1}^2 = \lambda + k_1^2 = \hat{\lambda} + h, \quad h = k_1^2 - k_2^2 > 0 \quad (1.33a)$$

For the real values of k_{x_1} and k_{x_2} (i.e. $\hat{\lambda} > 0$) Eq. (1.33) has an infinite number of solutions to be denoted by k_{1m}, k_{2m} (only positive roots k_{1m} and k_{2m} need be considered since negative values lead to the same λ_m). For imaginary values of k_{x_1} and k_{x_2} ($\hat{\lambda} < -h$), Eq. (10) becomes

$$\left| k_{x_2} \right| \coth(\left| k_{x_2} \right| a) = - \left| k_{x_1} \right| \coth(\left| k_{x_1} \right| d), \quad k_{x_1}, k_{x_2} \text{ imaginary.} \quad (1.34a)$$

Since the left-hand side of Eq.(1.34a) is positive while the right-hand side is negative, no solution exists. However, for real k_{x_1} and imaginary k_{x_2} ($-h < \hat{\lambda} < 0$), Eq. (10) can have roots $k_{1\alpha}, \left| k_{2\alpha} \right|$:

$$r_{\alpha} \cot r_{\alpha} = -t_{\alpha} \coth\left(\frac{a}{d} t_{\alpha}\right), \quad r_{\alpha}^2 + t_{\alpha}^2 = h d^2 = \left(1 - \frac{\epsilon_2}{\epsilon_1}\right) (k_{x_1} d)^2 \quad (1.34b)$$

where

$$k_{1\alpha} d \equiv r_{\alpha} > 0; \quad \left| k_{2\alpha} \right| d = t_{\alpha}, \quad k_{2\alpha} \text{ imaginary} \quad (1.34c)$$

Equation (1.34c) can be interpreted graphically as in Fig. 1.6.

It is noted that N roots exist for $(2N - 1)\pi/2 < \sqrt{h} d < (2N + 1)\pi/2$, with no solution possible when $\sqrt{h} d < \pi/2$.

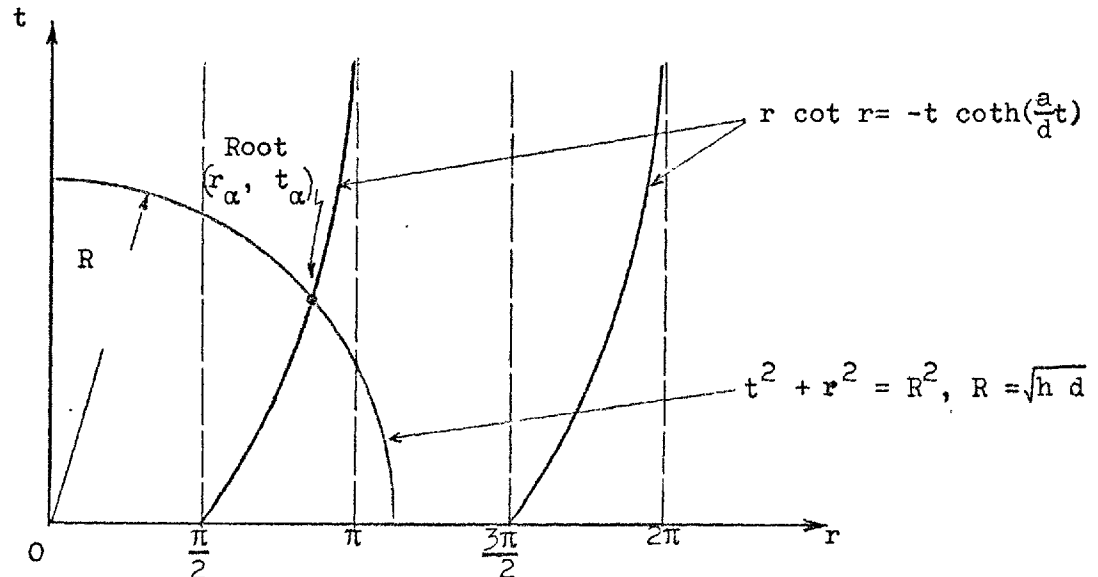


Fig. 1.6 Transcendental equation : graphical solution.

Although all the derivations of the above example imply a cumbersome work for such a relatively simple example, the same method can be applied to more complicated composite structures, such as semi-infinite and infinite regions filled with dielectric angular and radial transmission lines, continuous transitions, etc.

The eigenvalue problem can also be solved by introducing perturbation terms in ϕ and λ in Eq. (1.5), and then solving the differential equation. A brief discussion of this method when has discrete values and when the boundary is slightly deformed but the boundary conditions are assumed to be unaltered is given below.

In waveguides problems the differential operator is a Laplacian and the differential equation assumes the form

$$\nabla^2 \phi_n + (\lambda_n - p) \phi_n = 0 \quad (1.35)$$

which is assumed to be valid over a certain region S. Let the function $p(x)$ be modified to

$$P(x) = p(x) + \eta t(x)$$

due to slight deformation of the boundary, and as a consequence, change ϕ_n and λ_n respectively as follows:

$$\bar{\phi}_n(x) = \phi_n(x) + \eta \phi_n^{(1)}(x) + \eta^2 \phi_n^{(2)}(x) + \dots \quad (1.36)$$

$$T_n(x) = \lambda_n + \eta \lambda_n^{(1)} + \eta^2 \lambda_n^{(2)} + \dots$$

where the perturbed wave functions $\bar{\phi}_n$ and eigenvalues T_n are assumed to be analytic for all possible values of the parameter which takes only small values including zero. Equation (1.35) is then changed to:

$$\nabla^2 \bar{\phi}_n(x) + \{ T_n - p(x) - \eta t(x) \} \bar{\phi}_n(x) = 0 \quad (1.37)$$

Substituting equation (1.36) in the perturbed equation (1.37) and equating the coefficients of η , the following equations are obtained:

$$\nabla^2 \phi_n(x) + \{\lambda_n - p(x)\} \phi_n = 0 \quad (1.38a)$$

$$\nabla^2 \phi_n^{(1)}(x) + \{\lambda_n - p(x)\} \phi_n^{(1)}(x) + \{\lambda_n^{(1)} - t(x)\} \phi_n(x) = 0 \quad (1.38b)$$

$$\nabla^2 \phi_n^{(2)}(x) + \{\lambda_n - p(x)\} \phi_n^{(2)}(x) + \{\lambda_n^{(1)} - t(x)\} \phi_n^{(1)}(x) + \lambda_n^{(2)} \phi_n(x) = 0 \quad (1.38c)$$

Using Eqs. (1.38b) and (1.38c) and the expansions

$$\phi_n^{(1)}(x) = \sum_{q=0}^{\infty} \gamma_{n,q} \phi_q(x)$$

$$\phi_n^{(2)}(x) = \sum_{q=0}^{\infty} \delta_{n,q} \phi_q(x)$$

in terms of the unperturbed wave functions $\phi_q(x)$, the following equations are obtained:

$$\sum_{q=0}^{\infty} \gamma_{n,q} (\lambda_n - \lambda_q) \phi_q(x) + \{\lambda_n^{(1)} - t(x)\} \phi_n(x) = 0$$

$$\sum_{q=0}^{\infty} \delta_{n,q} (\lambda_n - \lambda_q) \phi_q(x) + \{\lambda_n^{(1)} - t(x)\} \sum_{q=0}^{\infty} \gamma_{n,q} \phi_q(x) + \lambda_n^{(2)} \phi_n(x) = 0 \quad (1.39)$$

which lead to the following relations:

$$\gamma_{n,n} = 0$$

$$\gamma_{n,m} = \frac{\alpha_{mn}}{\lambda_n - \lambda_m} \quad n \neq m$$

$$\delta_{n,n} = -\frac{1}{2} \sum_{q \neq n} \frac{\gamma_{n,q}^2}{(\lambda_n - \lambda_q)^2}$$

$$\delta_{n,m} = \frac{1}{\lambda_n - \lambda_m} \sum_{q \neq n} \frac{\alpha_{m,q} \alpha_{n,q}}{\lambda_n - \lambda_q} - \frac{\alpha_{n,n} \alpha_{m,n}}{(\lambda_n - \lambda_m)^2}$$

$$\lambda_n^{(1)} = \alpha_{n,n}$$

$$\lambda_n^{(2)} = \sum_{q \neq n} \frac{\alpha_{n,q}^2}{\lambda_n - \lambda_q}$$

where

$$\alpha_{n,m} = \int_S t(x) \phi_m(x) \phi_n(x) dx.$$

Since $\delta_{n,q} = \delta_{n,n} + \delta_{n,m}$ and $\gamma_{n,q} = \gamma_{n,n} + \gamma_{n,m} = \gamma_{n,m}$, $\phi_n^{(1)}$ and $\phi_n^{(2)}$ can be determined and hence $\bar{\Phi}_n(x)$ corresponding to T_n can be found. The method has been applied successfully for partially filled circular waveguides and for the case of material perturbations in waveguides at cutoff⁴, and also in dealing with the electromagnetic fields in anisotropic inhomogeneous media¹⁵.

1.3.2 Variational Methods

Many of the problems which arise in electromagnetic wave propagation may be formulated as problems in variational calculus^{17,18}. In differential calculus one is interested to find points at which functions of one or more variables possess maximum or minimum values, but in the calculus of variations the objective is to find the functional forms for which a given integral assumes maximum or minimum values. In other words, the calculus of variations deals with the problem of finding the paths of integration for which a given integral assumes maximum or minimum values. A variation indicated by the symbol δ signifies an infinitesimal change by analogy with the differentiation process, but this infinitesimal change is imposed externally on a set of variables and is not due to the actual change of an independent variable. Consider a function $y = f(x)$. Both δy and dy represent infinitesimal changes in y . But dy represents an infinitesimal change of the

function $f(x)$ caused by the infinitesimal change dx and δy refers to an infinitesimal change of y which produces a new function $y + \delta y$. The independent variable x does not take part in the process of variation. The problem of finding the position where a function has a relative maximum or minimum requires that the function must have stationary value at the point. If the rate of change of the function in every possible direction from that point vanishes, then the function is said to have a stationary value. The variational method is applicable to single and to multiple integrals, and to cases where the integrand is a function of one or more independent variables and their first or possibly higher order derivatives, and to cases where the limits of integration are variable.

If L is a function of the independent variables $x_i (i = 1, 2, 3, \dots, n)$ and the functional $u = u(x_i)$, $v = v(x_i)$ and their first derivatives

$$\dot{u}_i = \frac{\partial u_i}{\partial x_i}, \quad \dot{v}_i = \frac{\partial v_i}{\partial x_i},$$

and if it is assumed that L , u , v are continuous and $\dot{u}_i$, $\dot{v}_i$ are also continuous or piecewise continuous, then the essence of the variational process is to determine u and v , such that the definite integral in the n -dimensional space

$$I = \iiint \dots \int_N L(x_i, u, v, \dot{u}_i, \dot{v}_i) dx_i, \quad i = 1, 2, \dots, N \quad (1.40)$$

becomes stationary ($\delta I = 0$) when there are first-order variations δu , δv in u and v . The functions u and v satisfy the prescribed boundary conditions, so that the variations in u and v vanishes at the two extreme limits of integration. The necessary and sufficient condition that a function L of N variable shall be stationary at

a certain point is that the N partial derivatives of L with respect to all the N variables shall vanish at the point considered. In order that the integral I may be stationary the function L must satisfy independently the Euler-Lagrange partial differential equations as follows :

$$\begin{aligned} \frac{\partial L}{\partial u} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \frac{\partial L}{\partial \dot{u}_i} &= 0 \\ \frac{\partial L}{\partial v} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \frac{\partial L}{\partial \dot{v}_i} &= 0 \end{aligned} \quad (1.41)$$

The variational operator δ obeys the following laws:

$$\begin{aligned} a) \quad \delta(L_1, L_2) &= L_1 \delta L_2 + L_2 \delta L_1 \\ \delta\left(\frac{L_1}{L_2}\right) &= \frac{L_2 \delta L_1 - L_1 \delta L_2}{L_2^2}, \quad \text{etc.} \end{aligned}$$

which are analogous to the corresponding laws of differentiation.

b) The variational operator δ and the differential d/dx are commutative, i.e.

$$\frac{d}{dx} (\delta u) = \delta \left(\frac{du}{dx} \right)$$

c) The variation and the integration processes are commutative, i.e.

$$\delta I = \delta \int_{x_1}^{x_2} L dx = \int_{x_1}^{x_2} \delta L dx$$

i.e., the variation of a definite integral is equal to the definite integral of the variation.

In some physical problems which need the determination of several functions by a variational method, the variations cannot all be arbitrarily assigned but are restricted by some auxiliary conditions. If the stationary conditions are of the form

$$\delta \int_{x_1}^{x_2} L(x, u, v, \dot{u}, \dot{v}) dx = 0$$

where $u = u(x)$, $v = v(x)$. If u and v are not independent but are related by the restrictions

$$\phi(u, v) = 0$$

then in the variational integral the variation in u and v must be such that $\phi = 0$ always. The integral of ϕ between x_1 and x_2 also vanishes. Any multiple $\lambda\phi$ or ϕ may then be added to the function L under the integral sign without changing the value of the integral over the prescribed limits; λ is called the Lagrange's undetermined multiplier. This method reduces a variational problem with constraint to a variational problem without any constraint. In this method, the function L is modified by adding the left-hand side of the constraint equations after multiplying each by an undetermined multiplier λ . The modified problem is then treated as a variational problem without any auxiliary conditions. The resulting conditions, together with the prescribed constraints, determine the unknown and the undetermined multiplier λ .

The following equations:

$$\frac{d}{dx} \left(-\frac{L}{\dot{u}} \right) - \frac{\partial L}{\partial u} - \lambda \frac{\partial \phi}{\partial u} = 0$$

$$\frac{d}{dx} \left(-\frac{L}{\dot{v}} \right) - \frac{\partial L}{\partial v} - \lambda \frac{\partial \phi}{\partial v} = 0$$

with the auxiliary equation $\phi(u, v) = 0$, determine u, v and λ . For N -dimensional space, the conditional equations to find the functions

u, v and λ are

$$\frac{\partial}{\partial u} (L + \lambda \Phi) - \sum_{i=1}^N \frac{\partial}{\partial x_i} \frac{\partial}{\partial \dot{u}_i} (L + \lambda \Phi) = 0 \quad (1.42)$$

$$\frac{\partial}{\partial v} (L + \lambda \Phi) - \sum_{i=1}^N \frac{\partial}{\partial x_i} \frac{\partial}{\partial \dot{v}_i} (L + \lambda \Phi) = 0$$

with the auxiliary conditions

$$\Phi(x_i, u, v, \dot{u}_i, \dot{v}_i) = 0$$

In some cases the auxiliary condition may be of the form

$$\int_{x_1}^{x_2} G(x, u, \dot{u}) dx = k \text{ (a constant)}$$

If the variational integral is of the form

$$\delta I = \delta \int_{x_1}^{x_2} L(x, u, \dot{u}) dx$$

and is stationary ($\delta I = 0$), u and $u + \delta u$ satisfy the auxiliary condition and u satisfies the prescribed values at x_1 and x_2 , then δu is not arbitrary.

Suppose the relationship between arbitrary fixed functions $g(x)$, $f(x)$ is

$$\delta u = \eta f(x) + \alpha g(x)$$

both being continuously differentiable functions which vanish at the end points, η and α constants, then the necessary condition that $\delta I = 0$ is

$$\left\{ \frac{\partial L}{\partial u} - \frac{d}{dx} \left(\frac{\partial L}{\partial \dot{u}} \right) \right\} + \lambda \left\{ \frac{\partial G}{\partial u} - \frac{d}{dx} \left(\frac{\partial G}{\partial \dot{u}} \right) \right\} = 0 \quad (1.43)$$

The undetermined multiplier is given by the relation

$$\lambda = \frac{\int_{x_1}^{x_2} \left(\frac{\partial L}{\partial u} - \frac{d}{dx} \frac{\partial L}{\partial \dot{u}} \right) g dx}{\int_{x_1}^{x_2} \left(\frac{\partial G}{\partial u} - \frac{d}{dx} \frac{\partial G}{\partial \dot{u}} \right) g dx} \quad (1.44)$$

Before considering an example using this method, let us now briefly consider the advantages of a stationary formula over a nonstationary one. Figure 1.8⁴ shows the primary advantage. Given a resonant cavity formed by a perfect conductor enclosing a dielectric, and given a class of trial fields of the form

$$\underline{E}_{\text{trial}} = \underline{E} + \Delta \underline{E} = \underline{E} + p \underline{e}$$

where p is an arbitrary parameter, the parameter $\omega^2(p)$ may be determined from a stationary formula like⁴

$$\omega^2(p) = \frac{\iiint (\underline{E} + p \underline{e}) \cdot \nabla \times \mu^{-1} \nabla \times (\underline{E} + p \underline{e}) d\tau}{\iiint \epsilon (\underline{E} + p \underline{e}) \cdot (\underline{E} + p \underline{e}) d\tau}$$

and ω^2 will have a minimum or maximum at $p = 0$ (if it is complex it would have a saddle point at $p = 0$). This is shown in Fig. 1.8(a).

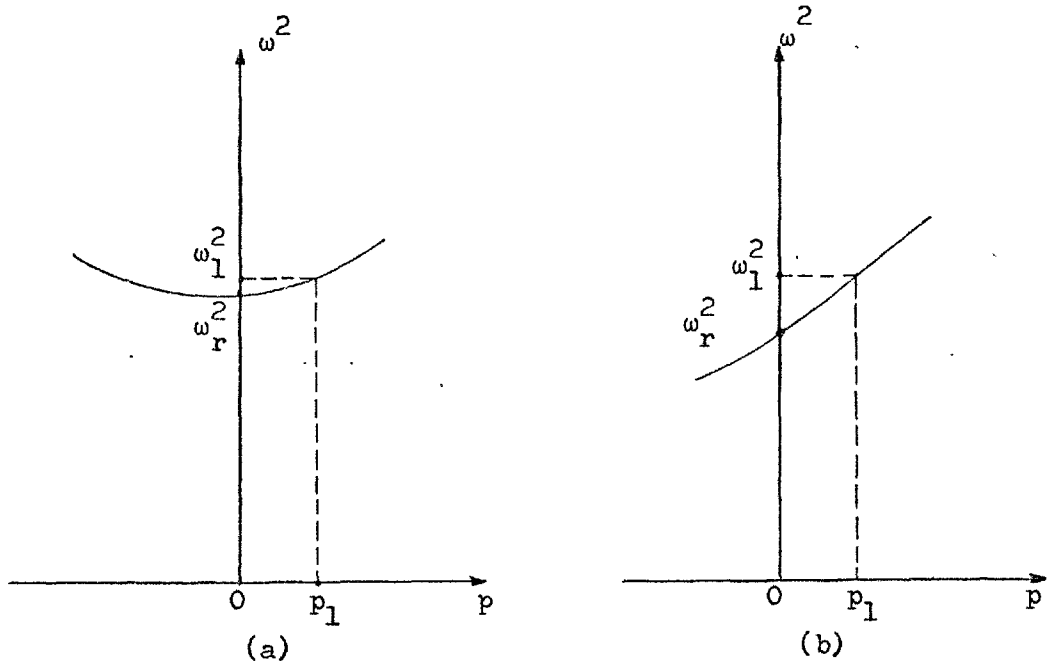


Fig. 1.7 ω^2 versus p for (a) a stationary formula and (b) a nonstationary formula.

The parameter ω^2 is determined from a nonstationary formula must have some definite slope at $p = 0$, as shown in Fig. 1.8(b). For a given error in the assumed field, say $\Delta \underline{E} = p_1 \underline{e}$, the corresponding error in the resonant frequency is determined from the figure as $\omega_1^2 = \omega_r^2$, where ω_r is the resonant frequency. It is evident from Fig. 1.7 that for small p_1 the stationary formula gives a smaller error in ω^2 than does the nonstationary formula. Thus, a parameter determined by a stationary formula is insensitive to small variations of the field about the true field.

As an illustrative example of the method discussed above, let us evaluate an expression of propagation constant in variational form^{18,19}.

The analysis will be confined to LSM waves. Figure 1.8 represents the inhomogeneously filled waveguide. A possible field is given by Equation (1.45)

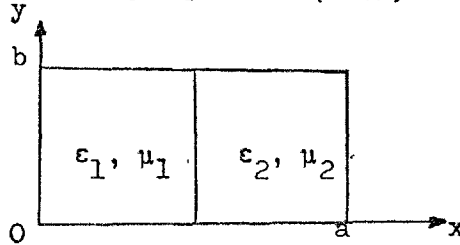


Fig. 1.8

$$\begin{aligned} E_x &= -\frac{\partial^2 \phi}{\partial z^2} - \frac{\partial^2 \phi}{\partial y^2}, \quad H_x = 0 \\ E_y &= -\frac{\partial^2 \phi}{\partial x \partial y}, \quad H_y = \epsilon \frac{\partial^2 \phi}{\partial t \partial x} \\ E_z &= \frac{\partial^2 \phi}{\partial x \partial z}, \quad H_z = \epsilon \frac{\partial^2 \phi}{\partial t \partial y} \end{aligned} \quad (1.45)$$

where ϕ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \mu \epsilon \frac{\partial^2 \phi}{\partial t^2} = 0$$

In order to satisfy the boundary conditions, ϕ must be of the form

$$\phi = \sum_{n=1}^{\infty} \phi_n \sin \frac{n\pi y}{b}$$

Thus we will consider the field generated by

$$\phi = f(x) \sin \frac{n\pi y}{b} \exp \{ j(\beta z - \omega t) \} \quad (1.46)$$

In order that ϕ should satisfy the wave equation, $f(x)$ must satisfy

$$\frac{d^2 f(x)}{dx^2} + (k^2 - \beta^2 - \frac{n^2 \pi^2}{b^2}) f(x) = 0 \quad (1.47)$$

and the field components are given by

$$E_x = (\beta^2 + \frac{n^2 \pi^2}{b^2}) f(x) \sin \frac{n \pi y}{b}, \quad H_x = 0,$$

$$E_y = \frac{n \pi}{b} \frac{df}{dx} \cos \frac{n \pi y}{b}, \quad H_y = j \omega \epsilon \beta f \sin \frac{n \pi y}{b},$$

$$E_z = j \beta \frac{df}{dx} \sin \frac{n \pi y}{b}, \quad H_z = j \omega \epsilon \frac{n \pi}{b} f \cos \frac{n \pi y}{b},$$

(having dropped the exponential factor). A solution of this type is valid in a region p provided k^2, μ, ϵ are given their appropriate values.

Multiplying (1.47) by $\epsilon_p f$ and integrating over $x_{p-1} < x < x_p$,

$$\int_{x_{p-1}}^{x_p} \epsilon_p \left\{ f \frac{d^2 f}{dx^2} + (k_p^2 - \beta^2 - \frac{n^2 \pi^2}{b^2}) f^2 \right\} dx = 0$$

Integrating the first term by parts

$$\int_{x_{p-1}}^{x_p} \epsilon_p \left\{ - \left(\frac{df}{dx} \right)^2 + (k_p^2 - \beta^2 - \frac{n^2 \pi^2}{b^2}) f^2 \right\} dx = - \left\{ \epsilon_p f \frac{df}{dx} \right\} \Big|_{x_{p-1}}^{x_p}$$

However $\epsilon f \frac{df}{dx}$ is of the form $A H_z E_y$, where A is a constant. Then,

$$\sum_{p=1}^m \int_{x_{p-1}}^{x_p} \left\{ - \left(\frac{df}{dx} \right)^2 + (k_p^2 - \beta^2 - \frac{n^2 \pi^2}{b^2}) f^2 \right\} dx = A \sum_{p=1}^m \left\{ H_z E_y \right\} \Big|_{x_{p-1}}^{x_p} \quad (1.48)$$

Since H_z, E_y are continuous at x_p , the product $H_z E_y$ is also continuous there; in addition $E_y = 0$ at $x = 0$ and $x = a$. The right-hand side of Eq. (1.48) then vanishes, and it is possible to write

$$\int_0^a \epsilon \left\{ (k^2 - \beta^2 - \frac{n^2 \pi^2}{b^2}) f^2 - \left(\frac{df}{dx} \right)^2 \right\} dx = 0 \quad (1.49)$$

It follows that

$$\beta^2 + \frac{n^2 \pi^2}{b^2} = \frac{\int_0^a \epsilon \{ k^2 f^2 - (df/dx)^2 \} dx}{\int_0^a \epsilon f^2 dx} \quad (1.50)$$

$$\text{Let } \eta^2 = k_{\max}^2 - k^2 \text{ and } \gamma^2 = k_{\max}^2 - \beta^2 - \frac{n^2 \pi^2}{b^2} \quad (1.51)$$

where $k_{\max}^2$ is the maximum value of k^2 .

It follows that

$$\gamma^2 = \frac{\int_0^a \epsilon \{ \eta^2 f^2 + (df/dx)^2 \} dx}{\int_0^a \epsilon f^2 dx} = \frac{Q(f)}{R(f)} \quad (1.52)$$

where $Q(f)$, $R(f)$ are positive definite, and $\gamma^2 > 0$.

Consider now the variation of the equation

$$\int_0^a \epsilon \{ (\gamma^2 - \eta^2) f^2 - \left(\frac{df}{dx} \right)^2 \} dx = 0.$$

The condition that γ^2 shall be insensitive to small variations

δf in f when $f + \delta f$ satisfies the boundary conditions, is given by

$$\int_0^a \epsilon (\gamma^2 - \eta^2) f \delta f dx - \int_0^a \epsilon \frac{df}{dx} \frac{d}{dx} (\delta f) dx = 0$$

Integrating by parts and using the fact that δf satisfies the boundary conditions gives

$$\int_0^a \left\{ \frac{d}{dx} \left(\epsilon \frac{df}{dx} \right) + \epsilon (\gamma^2 - \eta^2) f \right\} \delta f dx = 0$$

δf is, apart from the fact that it satisfies the boundary conditions, arbitrary. It follows that f is that solution of the equation

$$\frac{d}{dx} \left(\frac{df}{dx} \right) + \epsilon (\gamma^2 - \eta^2) f = 0 \quad (1.53)$$

which obeys the boundary conditions on the walls of the guide.

The problem has thus been reduced to the Sturm-Liouville eigenvalue problem (Appendix A).

Thus, if g is any function which is piecewise continuous in $0 < x < a$, it may be expanded in the form

$$g(x) = \sum_{n=1}^{\infty} a_n f_n(x)$$

where

$$a_n = \frac{1}{a} \int_0^a \epsilon g f_n dx$$

$$\text{Let } \gamma^2 = \frac{\int_0^a \epsilon \{ \eta^2 g^2 + (dg/dx)^2 \} dx}{\int_0^a \epsilon g^2 dx}$$

$$\text{and as } \int_0^a \epsilon f_n f_m dx = \delta_{mn}, \int_0^a \epsilon \left(\frac{df_n}{dx} \frac{df_m}{dx} + \eta^2 f_n f_m \right) dx = \gamma_m \gamma_n \delta_{mn},$$

it follows that

$$\gamma^2 = \frac{\sum_{n=1}^{\infty} a_n^2 \gamma_n^2}{\sum_{n=1}^{\infty} a_n^2} \quad (1.54)$$

If g is an approximation to f_m , γ^2 is an approximation to γ_n^2 and an error in the approximating function generates an error of the second order only in γ^2 , or in other words, an error of the order of 10 per cent in the assumed field gives an error of the order of only 1 per cent in γ^2 .

If one is interested only in the first or dominant mode (γ_1^2), and if g is an approximation to the first mode, γ^2 will be an approximation to γ_1^2 (in which the error is of the second order)

$$\text{and } \gamma^2 = \frac{\sum_{n=1}^{\infty} a_n^2 \gamma_1^2 + \sum_{n=2}^{\infty} a_n^2 (\gamma_n^2 - \gamma_1^2)}{\sum_{n=1}^{\infty} a_n^2} \geq \gamma_1^2$$

$$\text{whence } \beta_1^2 = k_{\max}^2 - \gamma_1^2 - \frac{n^2 \pi^2}{b^2} > k_{\max}^2 - \gamma^2 - \frac{n^2 \pi^2}{b^2}$$

Only when all the $a_n = 0$ for $n > 1$ will $\gamma^2 = \gamma_1^2$.

Thus, if g is an approximation to f , the value of β_1 is greater than the approximate value obtained and the difference is of the second order. If several approximate functions g are taken, the best approximation is that which gives the greatest approximate value to β_1^2 .

It is also relevant to mention the work by Thomas²⁰. He uses functional (as opposed to numerical) approximations in solving electromagnetic boundary value problems. Galerkin's method²³ is modified to simplify the choice of trial functions by permitting the use of trial functions which do not satisfy certain boundary conditions. A test problem is presented in this work²⁰ in which the dielectric loaded, rectangular waveguide is worked using both the modified and unmodified Galerkin's method with identical results. This method is then applied to the arbitrary waveguide. The cutoff frequencies and computer drawn contour plots are presented for circular, rectangular, triangular and star-shaped waveguides.

1.3.3 Rayleigh-Ritz method²¹

The method is a systematic procedure for determining approximately the eigenvalues and eigenfunctions to problems expressed in variational form. These are obtained by means of a variational integral, whose stationary values correspond to the true eigenvalues, when the true eigenfunctions are used in the integrand. In this method the problem of finding the maxima and minima of integrals is replaced by that of finding the maxima and minima of functions of several variables. This then reduces the problem to that of the calculus of functions.

Suppose one is required to determine the functional form of y so that the integral

$$I = \int_a^b F(x, y, \dot{y}, \ddot{y}, \dots) dx$$

becomes stationary. The procedure is to assume that it is possible the following expansion

$$y \simeq \sum_{i=1}^n C_i f_i(x) \quad \text{exists.}$$

The $f_i(x)$ are arbitrarily chosen, such that the expression satisfies the specific boundary condition for any choice of C_i , and the C_i are undetermined. Substituting for y and evaluating the integral, we obtain an expression in terms of C_i . Using the method of differential calculus, I is stationary if the C_i are such that $\partial I / \partial C_i = 0$. These n equations can then be solved to find the n parameters $C_1, C_2, \dots, C_n$, and hence y . In general an approximate result is obtained. The accuracy of the result depends on the choice of the functions $f_i(x)$. A closer approximation can be obtained by increasing the value of n . If a large number of $f_i(x)$ are used to obtain closer approximations, it is desirable to choose a sequence of functions which are complete. A function $f_i(x)$ is said to be complete if for any piecewise continuous function $F(x)$, a set of coefficients C_i can be chosen such that the following relation is satisfied.

$$\lim_{n \rightarrow \infty} \int_a^b \left\{ F(x) - \sum_{i=1}^n C_i f_i(x) \right\}^2 dx = 0$$

Frequently it is convenient to choose as the n th approximation a polynomial of degree i satisfying the prescribed boundary conditions. In certain cases, it is an advantage in numerical calculations to choose sine, cosine harmonics, Legendre polynomials and so on. This

This method avoids evaluation of many complicated expressions, especially for cases when a waveguide cross-section is divided into more than two regions by dielectric media of different dielectric constant.

Consider once more the problem analyzed in the previous section. In many practical cases the true eigenfunctions are very complex, and there is considerable advantage in working with a finite set of approximate eigenfunctions instead. In constructing this set of eigenfunctions, none of the true eigenfunctions is available to make the n th approximate eigenfunction also orthogonal. However, if the eigenfunctions given for the empty guide are used for the approximation, the approximate eigenfunctions are automatically orthogonal. Let us construct a set of N approximate eigenfunctions. It has been proved that

$$\gamma^2 = \frac{\int_0^a \epsilon \{ \eta^2 f^2 + (df/dx)^2 \} dx}{\int_0^a \epsilon f^2 dx} \quad (1.55)$$

where γ^2 is one of the eigenvalues of

$$\frac{d}{dx} \left(\epsilon \frac{df}{dx} \right) + \epsilon (\gamma^2 - \eta^2) f = 0 \quad (1.56)$$

when f is subjected to the appropriate boundary conditions.

Let $\phi_n(x)$ be some set of eigenfunctions which is complete and orthogonal over $0 < x < a$ and which satisfies the same boundary conditions as the f_n . Then f may be written in the form

$$f = \sum_{n=1}^{\infty} a_n \phi_n$$

as mentioned before a convenient set is that composed of the modes which exist in the empty guide. Consider the approximation to N terms of f ,

$$f(N) = \sum_{n=1}^N a_n \phi_n, \quad (1.57)$$

where the a_n are a set of coefficients to be determined, subject to the normalization condition

$$\sum_{n=1}^N \sum_{s=1}^N a_n a_s R_{ns} = 1$$

where $R_{sn} = R_{ns} = \int_0^a q \phi_n \phi_s dx$ (See Appendix A).

Substituting $f(N)$ in equation (1.52) it follows that an approximation to γ^2 is obtained of the form

$$\gamma^2 = \frac{\sum_{n=1}^N \sum_{s=1}^N a_n a_s Q_{ns}}{\sum_{n=1}^N \sum_{s=1}^N a_n a_s R_{ns}} \quad (1.58)$$

where the quadratic forms are clearly positive definite,

Thus, γ^2 is stationary for small variations, and if

$$F = \sum_{n=1}^N \sum_{s=1}^N (Q_{ns} - \gamma^2 R_{ns}) a_n a_s = 0 \quad (1.59)$$

$\partial F / \partial a_s = 0$ for all s . This implies

$$\sum_{n=1}^N Q_{ns} a_n - \gamma^2 \sum_{n=1}^N R_{ns} a_n - \frac{\partial \gamma^2}{\partial a_s} \sum_{n=1}^N \sum_{s=1}^N R_{ns} a_n a_s = 0$$

However, all the partial derivatives $\partial / \partial a_s$ for $n = 0, 1, \dots, N$ are equated to zero, and the following set of homogeneous equations is obtained

$$\sum_{n=1}^N a_n (Q_{ns} - \gamma^2 R_{ns}) = 0 \quad s = 0, 1, 2, \dots, N \quad (1.60)$$

This set of equations constitutes a matrix eigenvalue problem.

Because of the symmetry of the Q_{ns} and R_{ns} matrices, all the eigenvalues γ^2 are real, and the eigenvectors, whose components are the coefficients a_n , form an orthogonal set with respect to the weighting factors R_{ns} .

For (1.59) to hold, the determinant of the coefficients must vanish.

The vanishing of this determinant determines N roots for γ^2 , which are

the first N eigenvalues. For each root a set of coefficients (eigenvector) a_n may be determined. This set of coefficients is unique when subjected to the normalization condition. Since the totality of eigenvectors so determined forms an orthonormal set, it is clear that there is no need to impose any orthogonality restriction on the functions used. The orthogonality relations which hold are

$$\sum_{n=1}^N \sum_{s=1}^N a_n a_s R_{sn} = \delta_{ns}$$

Having found the coefficients a_n , the approximate eigenfunction is given by Eq. (1.57). If N was chosen as infinite, the eigenfunctions determined by the above method would be identical with the set of true eigenfunctions, since the set of functions ϕ_n is complete, and hence able to represent the set of $f(x)$. Choosing N as infinite would allow the possibility of finding the field distribution exactly. However, it is not possible, except for very few cases, to solve an infinite number of equations in an infinite number of unknowns. Thus it is necessary to use only a finite number N and the solution will be a compromise between small N , which involves comparatively simple numerical manipulations and comparatively low accuracy, and large N , which is more accurate but involves more troublesome calculations.

The following example¹⁹ shows the degree of approximation involved.

If a waveguide $0 < y < a$ is filled with dielectric (ϵ, μ) in $0 < y < t$ and dielectric (ϵ_0, μ) in $t < y < a$, the propagation constant may be calculated exactly in the form β/k_0 , where $k_0^2 = \omega^2 \epsilon_0 \mu$ and $k_0 = 2\pi/\lambda_0$, where λ_0 is the wavelength in the dielectric (ϵ_0, μ_0) . Now, E_x must vanish at $y = 0$, $y = a$, and so a suitable way of writing the field is

$$E_x = \sum_{n=1}^{\infty} A_n \sin(n\pi y/a)$$

and the appropriate partial approximations are

$$E_x(N) = \sum_{n=1}^N A_n \sin(n\pi y/a)$$

With the following values for the constants involved

$$\lambda_0 = a, \quad \epsilon/\epsilon_0 = 2.45, \quad t = a/2$$

the exact value is given by

$$\beta/k_0 = 1.3585$$

and the successive values for β/k_0 , for the various approximations, are

N	1	2	3
β/k_0	1.2145	1.3497	1.3546
% error	10.6	0.7	0.3

A method based on the Rayleigh-Ritz procedure, but which uses the Galerkin's method for solving the differential equations, has been presented by Baier²². Galerkin's method is a special version of the moment method²³. For the unknown fields trigonometrical series expansions with unknown coefficients are given, fulfilling automatically the boundary conditions. The series expansions are substituted in the differential equations of the problem and the integrations carried out by Galerkin's method, resulting in a matrix eigenvalue problem. The eigenvalues give the dispersion characteristics and the eigenvectors give the expansion coefficients of the fundamental mode and of higher order modes. The elements of the matrix are linear combinations of integrals of the type

$$\int_{x=0}^a \int_{y=0}^b (1/\epsilon_r) \cos(i\pi x/a) \cos(k\pi x/b) dx dy$$

with i and k integers. These integrals must be evaluated before the matrix can be constituted numerically. $1/\epsilon_r$ is assumed to be an even periodic function of x and y with the period lengths $2a$ and $2b$ respect-

ively. This assumption is allowed because $1/\epsilon$ can be freely disposed outside the range $0 \leq x \leq a$, $0 \leq y \leq b$. So, the integrals which constitute the matrix elements are, apart from constant factors, the Fourier coefficients of $1/\epsilon_r$. The matrix eigenvalue problem can be then formulated by first calculating the Fourier coefficients of $1/\epsilon_r$, substituting the series expansions for the fields and for $1/\epsilon_r$ in the differential equations, and finally making a comparison of coefficients.

The basic mathematical problem in terms of an eigenvalue problem, is described by the two equations that follow, where A_x and A_y are the components of the vector potential A along the x and y axis respectively.

$$\left\{ \frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \gamma^2 A_x \right\} / \epsilon_r + \frac{\partial(1/\epsilon_r)}{\partial x} \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right\} + \omega^2 \mu_0 \epsilon_0 A_x = 0$$

$$\left\{ \frac{\partial^2 A_y}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \gamma^2 A_y \right\} / \epsilon_r + \frac{\partial(1/\epsilon_r)}{\partial x} \left\{ \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \right\} + \omega^2 \mu_0 \epsilon_0 A_y = 0$$

Giving the γ^2 a certain value, prescribing the boundary conditions for A_x and A_y , the eigenvalues and eigenfunctions are then represented by ω^2 and A_x , A_y respectively. Every value ω^2 and the functions A_x and A_y belonging to it correspond to a different mode.

If ω^2 and A are evaluated, it is possible to obtain corresponding expressions for the component of the electric and magnetic field because of the relations

$$E = - \text{grad } \Phi - j\omega\mu_0 A$$

$$H = \text{curl } A$$

where $\Phi = j \text{ div } A / \omega\epsilon_0 \epsilon_r$.

In order to solve the eigenvalue problem, $1/\epsilon_r$, A_x and A_y are expressed in terms of convergent trigonometric series as follows

$$1/\epsilon_r = \sum_{i,k=-\infty}^{\infty} b_{ik} \exp \left\{ j\pi \left(\frac{ix}{a} + \frac{ky}{b} \right) \right\}$$

$$A_x = \sum_{m,n=-\infty}^{\infty} X_{mn} \exp \left\{ j \pi \left(\frac{mx}{a} + \frac{ny}{b} \right) \right\}$$

$$A_y = \sum_{m,n=-\infty}^{\infty} Y_{mn} \exp \left\{ j \pi \left(\frac{mx}{a} + \frac{ny}{b} \right) \right\}$$

$1/\epsilon_r$ is required to be an even real function of x and y , then

$$b_{ik} = b_{|i| |k|}$$

and these coefficients may be evaluated by Fourier analysis. A_x and A_y are calculated using boundary conditions, and it follows that A_x

has to be even with respect to x and odd with respect to y . Further A_y must be odd with respect to x and even with respect to y . These conditions are satisfied if

$$X_{mn} = \frac{n}{|n|} \cdot X_{|m| |n|} \quad (n \neq 0)$$

$$X_{m0} = 0$$

$$Y_{mn} = \frac{m}{|m|} Y_{|m| |n|} \quad (m \neq 0)$$

$$Y_{0n} = 0$$

By substitution of the convergent trigonometric series in the two basic equations, a system of linear equations is obtained. As the system is infinite, for numerical calculations only a finite number of modes can be considered.

In the paper under discussion²² the method is applied to a rectangular waveguide containing a longitudinal semicircular rod. The accuracy of the method has been checked by measurements and by the calculation of an example with an exactly known solution.

Another interesting work by Vorst et al²⁴ deals with a procedure to improve the use of the Rayleigh-Ritz method. A criterion is established, which is a measure of the cumulative improvement due to the addition of

more and more terms in the series expansion. Without calculating the exact roots of determinantal equations, the convergence is accelerated by skipping unnecessary intermediate steps. The computation time is drastically reduced because the final result is obtained after only a few (not more than 5 to 7) values of determinants of increasing order. The method is applied for evaluation of propagation constants in inhomogeneously loaded waveguides. Comparison with other approximate techniques suggest an advantage for this method.

The accuracy obtained by using the Rayleigh-Ritz method in the case of an E-plane dielectric slab, on the sidewall of a rectangular waveguide, for various dielectric constants and filling factors, has been investigated²⁵. It is shown that the error is much larger than the one obtained for a central loading, because of the coupling between even-and odd-order modes. The accuracy is a function of the compatibility between the field distribution for each mode taken in the expansion and the geometry of the loaded guide.

1.3.4 Reaction Concept

Stationary formulas can also be established by using the concept of reaction introduced by Rumsey²⁷. The use of this concept simplifies considerably the formulation of boundary value problems in electromagnetic theory⁴. If the volume distribution of electric $d\mathbf{J}^a$ and magnetic $d\mathbf{M}^a$ currents represents the source of a monochromatic electromagnetic field and the electric and magnetic fields generated by these source distributions are represented respectively by $\mathbf{E}^a$ and $\mathbf{H}^a$, and similarly $\mathbf{E}^b$ and $\mathbf{H}^b$ represent the electric and magnetic field due to another set of sources b of the same frequency and existing in the same linear medium, the reaction of field b on source a is defined as

$$\langle a, b \rangle = \int_V (\underline{E}^b \cdot d\underline{J}^a - \underline{H}^b d\underline{M}^a) \quad (1.61)$$

where the integration is performed in a volume containing the source.

If all the sources can be contained in a finite volume, the reciprocity theorem is

$$\langle a, b \rangle = \langle b, a \rangle \quad (1.62)$$

The linearity of the field equations is reflected in the identities

$$\langle a, b + c \rangle = \langle a, b \rangle + \langle a, c \rangle \quad (1.63a)$$

$$\langle Aa, b \rangle = A \langle a, b \rangle = \langle a, Ab \rangle \quad (1.63b)$$

where the notation Aa means the a field and source are multiplied by the number A .

Approximations to the desired reactions can be obtained by assuming trial fields (or sources) to approximate the true fields (or sources). It is then argued that the best approximation to a desired reaction is that obtained by equating reactions between trial fields to the corresponding reactions between trial and true fields. Suppose an approximation to the reaction $\langle C_a, C_b \rangle$ (where the symbol C stands for correct) is wanted. The approximation $\langle a, b \rangle$ is then best if it is subject to

$$\langle a, b \rangle = \langle C_a, b \rangle = \langle a, C_b \rangle \quad (1.64)$$

because all possible constraints have been imposed. In fact, Eq. (1.64) can imply that all trial sources look the same to themselves as to the correct sources do.

The reaction $\langle a, b \rangle$ obtained from (1.64) is also stationary for small variations of a and b about C_a and C_b . If

$$a = C_a + p_a e_a \quad b = C_b + p_b e_b$$

then $\langle a, b \rangle$ is stationary if

$$\left. \frac{\partial \langle a, b \rangle}{\partial p_a} \right|_{p_a=p_b=0} = \left. \frac{\partial \langle a, b \rangle}{\partial p_b} \right|_{p_a=p_b=0} = 0 \quad (1.65)$$

Substituting for a and b into Eqs. (1.64), it follows

$$\begin{aligned} \langle a, b \rangle &= \langle c_a, c_b \rangle + p_a \langle e_a, c_b \rangle + p_b \langle c_a, e_b \rangle + \\ &\quad p_a p_b \langle e_a, e_b \rangle \\ &= \langle c_a, c_b \rangle + p_b \langle c_a, e_b \rangle \\ &= \langle c_a, c_b \rangle + p_a \langle e_a, c_b \rangle \end{aligned}$$

Using the last two equations in the first equation,

$$\langle a, b \rangle = \langle c_a, c_b \rangle - p_a p_b \langle e_a, e_b \rangle$$

It is now evident that Eq. (1.65) is satisfied, proving the stationary character of $\langle a, b \rangle$.

For inhomogeneously loaded waveguides, it is possible to derive stationary formulas for propagation constants. These can be derived from the reaction concept but some modifications are necessary, because the sources are not of finite extent. The more direct, but less general, approach of constructing stationary formulas from the field equations⁴ will be taken.

Consider travelling-wave fields of the form

$$\begin{aligned} \underline{E} &= \underline{E}_t(x, y) \exp(-j\beta z) \\ \underline{H} &= \underline{H}_t(x, y) \exp(-j\beta z) \end{aligned}$$

The field equations become

$$\begin{aligned} \nabla \times \underline{E}_t + j\omega \mu \underline{H}_t &= j\beta \underline{\mu}_z \times \underline{E}_t \\ \nabla \times \underline{H}_t - j\omega \epsilon \underline{E}_t &= j\beta \underline{\mu}_z \times \underline{H}_t \end{aligned} \quad (1.66)$$

as can be verified by direct substitution. Scalar multiplication of the first of Eqs. (1.66) by $\underline{H}_t$, the second by $\underline{E}_t$, and taking the difference of the two resultant equations, yields

$$\nabla \cdot \underline{E}_t \times \underline{H}_t + j\omega (\mu H_t^2 + \epsilon E_t^2) = 2j\beta \underline{E}_t \times \underline{H}_t \cdot \underline{u}_z$$

This is now integrated over the cross section of the waveguide and rearranged to give

$$\beta = \frac{\iint (\omega \mu H_t^2 + \omega \epsilon E_t^2 - j \nabla \cdot \underline{E}_t \times \underline{H}_t) ds}{2 \iint \underline{E} \times \underline{H} \cdot \underline{u}_z ds}$$

Finally, the identity

$$\iint \nabla \cdot \underline{E}_t \times \underline{H}_t ds = \oint \underline{E} \times \underline{H} \cdot \underline{n} d\ell$$

will vanish if $\underline{n} \times \underline{E} = 0$ on C , hence

$$\beta = \omega \frac{\iint (\mu H_t^2 + \epsilon E_t^2) ds}{2 \iint \underline{E} \times \underline{H} \cdot \underline{u}_z ds} \quad (1.67)$$

This is stationary if the trial field satisfies $\underline{n} \times \underline{E}_t = 0$ on C .

For the $\underline{E}$ - field formulation, $\underline{H}_t$ is eliminated from Eqs. (1.66) and from a similar procedure,

$$\beta^2 = \frac{\iint \{ \mu^{-1} (\nabla \times \underline{E}_t)^2 - \omega^2 \epsilon E_t^2 \} ds}{\iint \mu^{-1} (\underline{u}_z \times \underline{E}_t)^2 ds} \quad (1.68)$$

which is stationary provided $\underline{n} \times \underline{E}_t = 0$ on C . Similarly, the $\underline{H}$ - field formula is

$$\beta^2 = \frac{\iint \{ \epsilon^{-1} (\nabla \times \underline{H}_t)^2 - \omega^2 \mu H_t^2 \} ds}{\iint \epsilon^{-1} (\underline{u}_z \times \underline{H}_t)^2 ds} \quad (1.69)$$

which is stationary with no boundary conditions required on $\underline{H}_t$.

Equations (1.67) and (1.68) can be extended to obviate the necessity of boundary conditions on $\underline{E}_t$ in the usual manner. Equations (1.67) to (1.69) remain stationary in the lossy case, for which $j\beta$ is replaced by $\gamma = \alpha + j\beta$.

For an example, consider the centered dielectric slab in a rectangular waveguide as shown in Fig. 1.9. As a trial field, take

$$\underline{E}_t = \underline{u}_y \sin(\pi x/a)$$

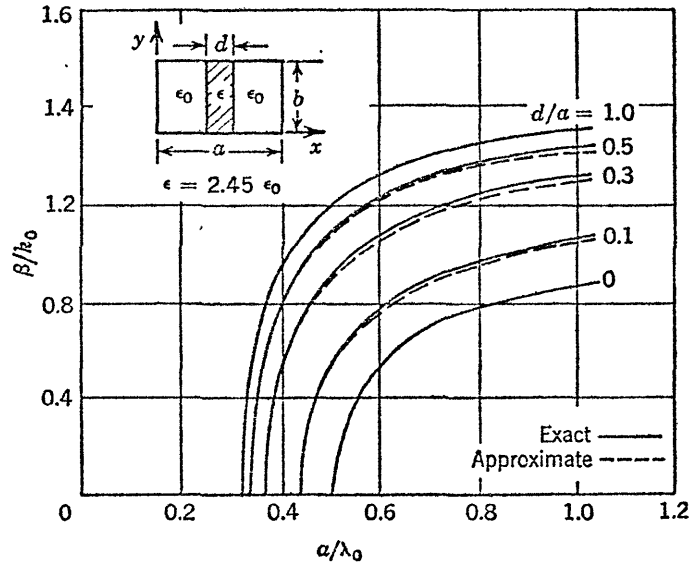


Fig.1.9. Comparison of approximate and exact propagation constants for the rectangular waveguide with centered dielectric slab, $\epsilon = 2.45 \epsilon_0$. (Berk²⁷).

Using Eq. (1.68), the result is²

$$\frac{\beta}{k_0} = \left\{ 1 + \frac{\epsilon - \epsilon_0}{\epsilon_0} \left[\frac{d}{a} + \frac{1}{\pi} \sin(\pi d/a) \right] - (\pi/k_0 a)^2 \right\}^{\frac{1}{2}} \quad (1.70)$$

The exact solution requires the solution of a transcendental equation. A comparison of values obtained from Eq. (1.70) with the exact value for β/k_0 is shown in Fig. 1.9 for the case $\epsilon = 2.45 \epsilon_0$.

1.3.5 Finite-difference method

The calculus of finite differences deals with the changes that occur in the values of a function $f(x)$ due to changes in the independent variable x . Finite differences form the basis of numerical differentiation, integration and solution of differential equations. In this method it is assumed that $f(x)$ has a specific numerical value $f(x_r)$ at each of a sequence of equally spaced values $x = x_r$. If the common interval between any two values $x_0, x_1, x_2 \dots$ is denoted by h , the

first difference $\Delta f(x)$ of the function $f(x)$ is defined as

$$\Delta f(x) = f(x + h) - f(x) \quad (1.71)$$

i.e. $\Delta f(x)$ gives the difference in values of the function for two neighbouring values of x , h units apart. The difference operator Δ can be regarded as acting upon $f(x)$ in the same way as the operator d/dx acts on $f(x)$ in the differentiation process. The operator Δ like the operator d/dx is linear with respect to scalar multiplication, i.e.

$$\Delta \{a f(x) + b g(x)\} = a \Delta f(x) + b \Delta g(x) \quad (1.72)$$

where a and b are constants. The Δ operation on the product of two functions $f(x)$ and $g(x)$ yields

$$\Delta \{f(x)g(x)\} = f(x + h)\Delta g(x) + g(x)\Delta f(x)$$

or,

$$\Delta \{f(x)g(x)\} = f(x)\Delta g(x) + g(x + h)\Delta f(x) \quad (1.73)$$

to the first order.

The Δ operation on the ratio of two functions gives

$$\Delta \frac{f(x)}{g(x)} = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)g(x+h)} \quad (1.74)$$

The second difference $\Delta^2 f(x)$ is defined as the difference of the first difference of $f(x)$ for two neighbouring values of x , h units apart, i.e.

$$\Delta^2 f(x) = \Delta \{ \Delta f(x) \} = \Delta f(x + h) - \Delta f(x) \quad (1.75)$$

Higher differences are defined similarly, and it is evident that any forward difference should be expressible in terms of the function values at various abscissae x_r . In general

$$\Delta^n f(x) = \Delta \{ \Delta^{n-1} f(x) \}, \quad n = 1, 2, \dots \quad (1.76)$$

Problems on inhomogeneous guides need the solution of partial differential equations. In the finite difference method, the partial differential equation is first converted to a difference equation which

is then solved to find the eigenvalues. A difference equation is one or more of the differences of the dependent variable. The difference quotient is defined as

$$\frac{f(x+h) - f(x)}{h}$$

which is in the limit, if $h \rightarrow 0$, the derivative of a simple function. So, if all the derivatives in a differential equation are replaced by the corresponding difference quotients, a difference equation is obtained. The numerical solution of difference equation is an extensive subject. Only partial difference quotients will be discussed. Let the x, y plane be divided into a network by two families of parallel lines, as shown in Figure 1.10

$$x = ah$$

$$a, b = 0, 1, 2, \dots$$

$$y = bh$$

The points of intersection of these lines are called lattice points or nodes. For each of the variables of a function $u(x, y)$, there is a forward and a backward difference quotient. Thus, with respect to x and y , the forward (u_x, u_y) and backward (u_x^-, u_y^-) quotients are respectively

$$u_x = \frac{u(x+h, y) - u(x, y)}{h}$$

$$u_y = \frac{u(x, y+h) - u(x, y)}{h}$$

$$u_x^- = \frac{u(x, y) - u(x-h, y)}{h}$$

$$u_y^- = \frac{u(x, y) - u(x, y-h)}{h}$$

The second difference quotients of $u(x, y)$ with respect to x and y can be defined as the difference quotient of the first difference quotients, i.e.

$$u_{x,x}^- = \frac{u_x - u_x^-}{h} = \frac{u(x+h, y) - 2u(x, y) + u(x-h, y)}{h^2} \quad (1.77)$$

$$u_{y,y}^- = \frac{u_y - u_y^-}{h} = \frac{u(x, y+h) - 2u(x, y) + u(x, y-h)}{h^2}$$

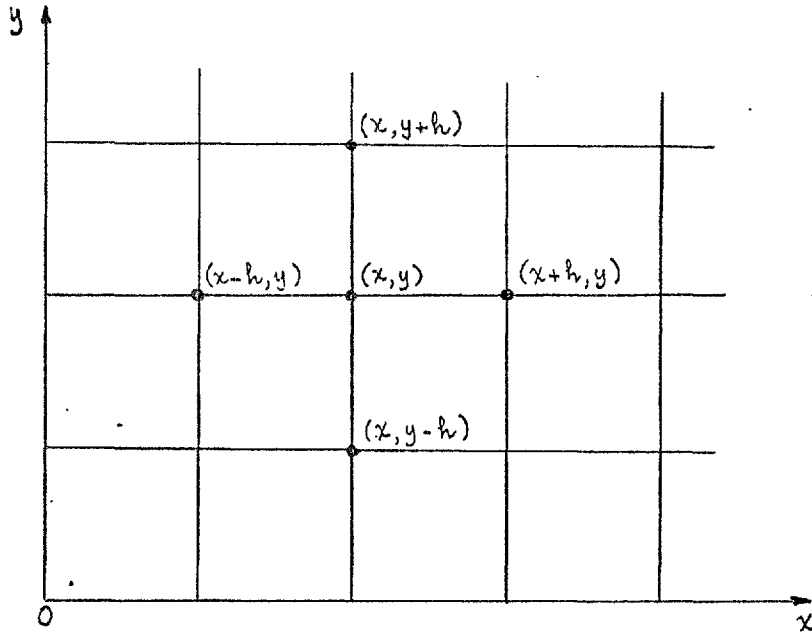


Fig. 1.10

These definitions replace the partial derivatives in a partial differential equation, resulting in a difference equation. The functions occurring in a difference equation are defined only at the nodes, so if a better description of the function is required, it is necessary to increase the number of nodal points. As an illustration, consider how a two dimensional Laplace's equation is transformed to an equivalent difference equation.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$u_{xx} \approx \frac{\partial^2 \phi}{\partial x^2}, \quad u_{yy} \approx \frac{\partial^2 \phi}{\partial y^2}$$

Using Eqs. (1.77), the following difference equation is obtained

$$\{u(x+h, y) - 2u(x, y) + u(x-h, y)\} + \{u(x, y+h) - 2u(x, y) + u(x, y-h)\} = 0$$

which yields

$$u(x, y) = \frac{1}{4} \{u(x+h, y) + u(x, y+h) + u(x-h, y) + u(x, y-h)\}$$

The value of $u(x, y)$ at any interior nodal point is the arithmetic mean of the values of $u(x, y)$ at the four lattice points surrounding the point in question.

However, the finite difference method is inherently inefficient and unwieldy when applied to dielectric loaded waveguides²⁸ because, in order to obtain sufficient accuracy, a prohibitively large matrix eigenvalue problem must be solved. On the other hand, the use of Rayleigh-Ritz procedure, although it has the advantage of only requiring the solution of a small matrix eigenvalue problem, has the disadvantage that each set of trial functions is limited to a particular geometry and requires the evaluation of lengthy analytic expressions.

A method of approximating a function space that does not suffer from these deficiencies is the finite-element method. In this method, the region of interest is divided into triangular elements and a polynomial approximation is made of the function in each triangle. In a limited sense, this has been attempted for dielectric loaded waveguides by Ahmed and Daly²⁹. They obtained a variational expression for the wavenumber k_o^2 , such as

$$J(\phi, \psi) = \sum_i \left\{ \tau_i \int_{S_i} |\nabla_t \phi_i|^2 dS_i + \bar{\beta}^2 \tau_i \epsilon_i \int_{S_i} |\nabla_t \psi_i|^2 dS_i + 2\tau_i \bar{\beta}^2 \int_{S_i} \underline{ez} (\nabla_t \psi_i \times \nabla_t \phi_i) dS - k_o^2 \left(\int_{S_i} |\phi_i|^2 dS_i + \bar{\beta}^2 \epsilon_i \int_{S_i} |\psi_i|^2 dS_i \right) \right\} \quad (1.78)$$

where $\phi_i = H_{zi}$, $\bar{\beta} \psi_i = (\epsilon_o/\mu_o)^{\frac{1}{2}} E_{zi}$, $\bar{\beta} = (\beta c/\omega)$, $\tau_i = (\bar{\beta}^2 - 1)/(\bar{\beta}^2 - \epsilon_i)$

They derived the finite-element equations for the general case of a point at the junction of four dielectric quadrants, and the equations for the special case of a rectangular waveguide with slab perpendicular to the electric field.

The configuration in question for the latter case is shown in Figure 1.11

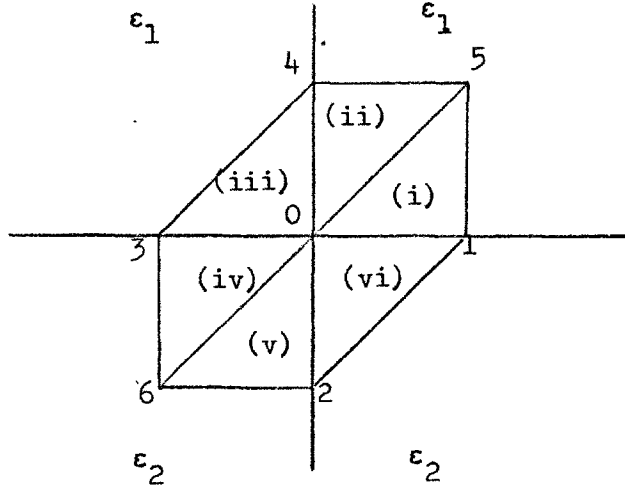


Fig. 1.11

The vertex 0 is surrounded by six right-triangular elements (i) - (vi), the points 1 - 6 forming the corners of a hexagon. As the permittivity is constant in each quadrant, there is no discontinuity in the derivatives of ϕ and ψ within each element. Thus ϕ and ψ can be uniquely specified in terms of their values at the vertices. Summing all contributions of the elements for the minimisation of the functional $J(\phi, \psi)$ at the point 0, the following equations are obtained

$$\frac{\partial J}{\partial \phi_0} = \sum_{p=1}^{vi} \left. \frac{\partial J}{\partial \phi_0} \right|_p = 0$$

$$\frac{\partial J}{\partial \psi_0} = \sum_{p=1}^{vi} \left. \frac{\partial J}{\partial \psi_0} \right|_p = 0$$

where the subscript p refers to a particular element.

In the first quadrant, with permittivity ϵ_1 , using a linear approximation to ϕ and ψ over each element, and following the standard procedure³⁰ for finite element discretisation, the partial contributions to $\partial J / \partial \phi_0$ by the elements (i) and (ii) are obtained as

$$\begin{aligned} \left. \frac{\partial J}{\partial \phi_0} \right|_{\epsilon_1} &= (\tau_{1/2})(-\phi_1 + 2\phi_0 - \phi_4) + (\tau_1 \bar{\epsilon}^2/2)(\psi_1 - \psi_4) \\ &\quad - (k_0^2 h^2/24)(\phi_1 + 4\phi_0 + \phi_4 + \phi_5) \end{aligned}$$

Similarly

$$\begin{aligned} \partial J / \partial \psi_0 \Big|_{\epsilon_1} &= (\bar{\beta}^2 \tau_1 \epsilon_1 / 2) (-\psi_1 + 2\psi_0 - \psi_4) + (\bar{\beta}^2 \tau_1 / 2) (\phi_4 - \phi_1) \\ &\quad - (k_0^2 h^2 / 24) \epsilon_1 \bar{\beta}^2 (\psi_1 + 4\psi_0 + \psi_4 + \psi_5) \end{aligned}$$

Repeating the procedure for the rest of the quadrants, and if $\epsilon_1 = 1$ and $\epsilon_2 = \epsilon$, then

$$\begin{aligned} \frac{1}{2}(1+\tau)(4\phi_0 - \phi_1 - \phi_3) - \phi_4 - \tau\phi_2 + \frac{1}{2}(1-\tau)\bar{\beta}^2(\psi_1 - \psi_3) &= \\ (k_0^2 h^2 / 12) (6\phi_0 + \sum_{i=1}^6 \phi_i) & \\ \frac{1}{2}(1+\tau\epsilon)(4\psi_0 - \psi_1 - \psi_3) - \psi_4 - \tau\epsilon\psi_2 + \frac{1}{2}(1-\tau)(\phi_3 - \phi_1) &= \\ (k_0^2 h^2 / 12) \{ 3(1+\epsilon)\psi_0 + (1+\epsilon)(\psi_1 + \psi_3)/2 + (\psi_4 + \psi_5) + \epsilon(\psi_2 + \psi_6) \} & \end{aligned}$$

The appropriate finite-element equations are written at each point of the cross-section. If these are n points, the resulting equations, in matrix form, are

$$A\theta = \lambda B\theta$$

where $\lambda = k_0^2 h^2 / 12$ and A and B are $2n \times 2n$ square-symmetric matrices. $2n$ equations arise because both ϕ and ψ must be used at each point. Coupling between ϕ and ψ occurs at points which border two or more dielectric regions. If $\bar{\beta} < 1$, matrices A and B are positive definite and diagonally dominant, whereas for $\bar{\beta} > 1$, this is not necessarily so. In the first case, successive over-relaxation methods lead to a converging solution for the lowest eigenvalue of the eigenvalue equation, but when $\bar{\beta} > 1$, the successive over-relaxation method fails, and other iterative methods become necessary. This condition sets severe limitations on the size of matrices which can be handled by the computer.

For the example analyzed, there is a good agreement between the results obtained with this method and the results derived by Marcuvitz.⁶

Owing to its variational nature, this method is inherently superior to the more conventional finite-difference method.

However, this work seems to be unnecessarily restricted to special geometries by imposing a regular mesh spacing and unnecessarily limited in computational efficiency by confining their polynomial approximation to first order. Two of the great advantages of the finite-element method are the freedom to fit any polygonal shape by choosing triangular-element shapes and sizes and the extremely accurate approximations provided by high order polynomials.

To avoid the restrictions mentioned previously, Csendes and Silvester³¹ have developed a general finite-element method.

It is a well known fact that the determination of the electromagnetic field reduces essentially to finding the two quantities E_z and H_z . If τ and z are orthogonal directions tangent to the interface, and n the normal to them, the interface conditions for the field are (see Eq. (1.3c) in section 1.2.1)

$$\frac{1}{k^2} M \frac{\partial \Psi}{\partial n} = \frac{\beta}{\omega k^2} J \frac{\partial \Psi}{\partial \tau} + \frac{1}{\omega} J \xi \quad (1.79)$$

where

$$\Psi = \begin{bmatrix} E_z \\ H_z \end{bmatrix}$$

$$\xi = \begin{bmatrix} E_\tau \\ H_\tau \end{bmatrix}$$

$$M = \begin{bmatrix} \epsilon & 0 \\ 0 & \mu \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$k^2 = \omega^2 \mu \epsilon - \beta^2$$

Note that ξ and $\partial \Psi / \partial n$ are continuous across a boundary even if ϵ and μ change discontinuously there. The axial field components satisfy the homogeneous Helmholtz equation which can be written in operator

form as

$$M \left(\frac{1}{k^2} \nabla^2 + 1 \right) \psi = 0$$

The functional

$$F(\phi) = -\frac{1}{k^2} \langle \phi | M \nabla^2 | \phi \rangle + \langle \phi | M | \phi \rangle \quad (1.80)$$

is stationary if and only if ϕ equals the true physical solution ψ .

Since the functional is stationary only at the true solution, at this stationary value Eq. (1.79) may be applied. By using a suitable integral definition of the scalar product, two appropriate vector identities³¹, and taking into account the boundary conditions, it is possible to obtain an expression for the functional that may be written as

$$F(\phi) = -\int_R \nabla \phi^T \frac{1}{k^2} M \cdot \nabla \phi \, dR + \int_R \phi^T M \phi \, dR + \frac{\beta}{\omega} \int_R (\nabla \phi^T \frac{1}{k^2} \mathbf{J} \times \nabla \phi) \hat{\alpha}_z \, dR \quad (1.81)$$

where the region of integration is over the whole waveguide cross section R , and the Eq. (1.81) is stationary at the true solution.

In order to solve Eq. (1.79) by determining the stationary condition of the functional, Eq. (1.81), the Rayleigh-Ritz method was applied. Seeking for solutions of the form

$$E_z = \sum_{i=1}^n e_i \alpha_i(x, y)$$

$$H_z = \sum_{i=1}^n h_i \alpha_i(x, y)$$

where the $\alpha_i(x, y)$ are a set of linearly independent functions. Substituting these values in (1.81), differentiating with respect to e_i and h_i , and setting the result equal to zero, $2n$ simultaneous equations are obtained

$$\sum_r \frac{1}{k_r^2} \sum_{i=1}^n e_r S_{k_i} e_i + \frac{\beta}{2\omega} U_{i_k} h_i = \sum_r e_r \sum_{i=1}^n T_{k_i} e_i$$

$$\sum_r \frac{1}{k_r^2} \sum_{i=1}^n \mu_r S_{k_i} h_i + \frac{\beta}{2\omega} \sum_{k_i} U_{k_i} e_i = \sum_r \mu_r \sum_{i=1}^n T_{k_i} h_i$$

where

$$S_{k_i} = \int_r \nabla \alpha_k \cdot \nabla \alpha_i d\tau$$

$$T_{k_i} = \int_r \alpha_k \alpha_i d\tau$$

$$U_{k_i} = \int_r \left(\alpha_k \frac{\partial \alpha_i}{\partial \tau} - \alpha_i \frac{\partial \alpha_k}{\partial \tau} \right) d\tau$$

These equations can be written as a matrix equation

$$V \Psi = \omega^2 T \Psi \quad (1.82)$$

where V and T are the partitioned matrices

$$V = \sum_r \frac{1}{\epsilon_r \mu_r - \delta^2} \left(\begin{array}{c|c} \epsilon_r S_{k_i} & \delta U_{k_i}/2 \\ \hline \delta U_{k_i}/2 & \mu_r S_{k_i} \end{array} \right) \quad (1.83)$$

$$T = \sum_r \left(\begin{array}{c|c} \epsilon_r T_{k_i} & 0 \\ \hline 0 & \mu_r T_{k_i} \end{array} \right); \quad \Psi = \begin{pmatrix} \epsilon_i \\ h_i \end{pmatrix}$$

where $\delta = \beta/\omega$. At any value of phase velocity, (1.82) may be solved for the frequency of propagation and field distribution in the waveguide.

Although the Rayleigh-Ritz expansion has been performed with an arbitrary set of trial functions, for each set of trial functions that is chosen the integrals in Eq. (1.79) must be evaluated and it is wise to choose trial functions that minimize such calculations. The basic idea of the finite-element method is to do this by splitting the region of integration into a number of simple elements. The integration over each element of some particular set of trial functions may then be reduced to the evaluation of a few parameters, and the calculation of the total integral may be performed by a simple combination of these

parameters. If triangular interpolation polynomials³² are used, the approximating functions are

$$\alpha_{ijk} = P_i(\nu_1) P_j(\nu_2) P_k(\nu_3)$$

where the ν_i are triangular coordinates and

$$P_m(z) = \prod_{i=1}^m \left\{ (N_z - i + 1)/i \right\}, \quad m \geq 1$$

$$P_0(z) = 1$$

The off-diagonal elements of V and T, S_{ik} and T_{ik} , have been used to solve the homogeneous Helmholtz equation and are therefore known and tabulated³⁴ for polynomial approximations up to the fourth order. To evaluate the remaining element U_{ik} , it is necessary to evaluate

$$\frac{\partial P_m(z)}{\partial z} = \sum_{i=1}^m \frac{P_m(z) N}{N - i - 1} \quad m \geq 1$$

$$\frac{\partial P_0(z)}{\partial z} = 0$$

and to use the relationship $\nu_p + \nu_q = 1$, valid on triangular-element edges.

After a little algebra, an expression for $\partial \alpha_{oik} / \partial \tau_i$ follows. The U_{ik} may then be evaluated to any order of polynomial approximation and for any triangular shape by substituting the expression for $\partial \alpha_{oik} / \partial \tau_1$ in the appropriate integral for U_{ik} and integrating around the perimeter of a triangle using triangular coordinates. The U matrix has been evaluated independently and concurrently by Daly³³.

The rest of the paper by Csendes and Silvester³¹ is dedicated to the applications, using a computer programme, to several cases of inhomogeneous waveguides and the results seem to indicate a high accuracy.

1.3.6 Modal Approximation Technique

Although the finite-element method explained in the previous section is easy to work with, it is not very efficient computationally as compared with Rayleigh-Ritz method using basis functions that closely resemble the expected solutions. Csendes and Silvester have developed a method³⁴ that expands the electromagnetic fields above cutoff in terms of the waveguide cutoff modes. The variational formulation of the problem is similar to that one in the previous section. The trial function are of the form:

$$E_z = \sum_{i=1}^m e_i \phi(i)$$

$$H_z = \sum_{k=1}^m h_k \phi(k)$$

where the $\{\phi(j)\}$ are a set of independent functions and the e_i and h_k are real numbers. The Rayleigh-Ritz equations are then obtained by equating the expressions $\partial F / \partial e_i$ and $\partial F / \partial h_k$ to zero.

Waveguide cutoff modes form an orthogonal set of functions that exist exactly over the waveguide cross section and satisfy all of the external boundary conditions. For rectangular homogeneous waveguides, they are products of sine and cosine functions; in circular waveguides the modes are composed of trigonometric and Bessel functions.

It is possible, therefore, to expand any function in a series of waveguide cutoff modes, provided that the function has the same region of definition and satisfies the same boundary conditions. In particular, it is possible to expand the electric and magnetic fields in a waveguide at any value of propagation constant in the cutoff modes. Because of the close physical relationship, the cutoff modes are near to the field functions, so that relatively few cutoff modes will provide a good approximation.

The waveguide cutoff modes may be represented numerically by using the Newton-Cotes interpolation polynomials³⁵. Consequently the waveguide cutoff modes may be written as

$$\phi^{(k)} = \sum_{t=1}^N \sum_{p=1}^N \phi_p^{(k)} \alpha_p(x,y)$$

where $\phi_p^{(k)}$ is the potential value of point p in triangle t and $\alpha_p(x,y)$ is the interpolation polynomial having a unit value at point p.

The cutoff modes of arbitrary dielectric loaded waveguides may be obtained by conventional finite-element analysis and if the triangular elements and polynomial order are chosen to be the same in mode expansion as in the finite element analysis, the finite-element solution will be exactly in the form of $\phi^{(k)}$.

Proceeding as in the previous section, a matrix eigenvalue equation is obtained in the form

$$\sum_{tpq} \frac{1}{\epsilon_t \mu_t - \delta^2} \begin{pmatrix} \epsilon_t \phi_p^{(i)} \phi_q^{(j)} S_{pq} & - \frac{\delta}{2} \phi_p^{(i)} \phi_q^{(k)} U_{pq} \\ - \frac{\delta}{2} \phi_p^{(j)} \phi_q^{(l)} U_{pq} & \mu_t \phi_p^{(k)} \phi_q^{(l)} S_{pq} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_i \\ h_k \end{pmatrix} = \omega^2 C \begin{pmatrix} \epsilon_i \\ h_k \end{pmatrix}$$

where

$$S_{pq} = \int \nabla \alpha_p \cdot \nabla \alpha_q \, d\mathbf{r}$$

$$U_{pq} = \int \left(\alpha_p \frac{\partial \alpha_q}{\partial \tau} - \alpha_q \frac{\partial \alpha_p}{\partial \tau} \right) d\mathbf{r}$$

and

$$\int_R \epsilon_t \phi^{(i)} \phi^{(i)} \, dR = 1$$

$$\int_R \mu_t \phi^{(k)} \phi^{(k)} \, dR = 1$$

Since the cutoff potential values $\phi^{(i)}$ and $\phi^{(k)}$ are known, this matrix equation can be easily assembled for any dielectric loaded waveguide and solved for any desired value of phase velocity. From these values, the dispersion characteristics of the waveguide can be obtained.

According to the authors, this method seems to be superior to the finite-element method analyzed in the previous section.

1.3.7 The Wentzel-Kramers-Brillouin (WKB) approximation

The WKB approximation for solving the Schroedinger equation has proved to be useful for solving electromagnetic wave equations, and Holmes³⁶ developed a method for analysis of wave propagation in dielectric filled rectangular wave-guides based on this approximation. TE_{mn} and TM_{mn} mode propagation in wholly filled waveguides are analyzed, as well the TE_{no} mode propagation in slab loaded guide.

Consider the wave equation

$$\frac{1}{f(x)} \frac{\partial}{\partial x} \left\{ f(x) \frac{\partial \phi(x)}{\partial x} \right\} + g^2(x) \phi(x) = 0$$

where $\phi(x)$ is the x variation of some field component and $f(x)$ and $g(x)$ are slowly varying functions of x. By substituting

$$\phi(x) = \phi_0 \exp \{ j \omega S(x) \}$$

in the wave equation, and choosing for $S(x)$ a series expansion as

$$S(x) = \sum_{n=1}^{\infty} \frac{S_n(x)}{\omega^n},$$

it is possible³⁷ to obtain an approximate solution for $\phi(x)$ as

$$\phi(x) = \left[\phi_0 \exp \left\{ \pm j \int_0^x g(x) dx \right\} / \sqrt{g(x)f(x)} \right]$$

If a dielectric slab occupies the region from x_1 to x_2 , in a rectangular waveguide, while the rest of the guide is empty, and if only the TE modes with no y variation will be considered, then the only field components are E_y , H_x and H_z . For this case, $\phi(x)$ is the x variation of E_y when $f(x) = 1$

$$\text{and} \quad g^2(x) = \gamma^2 + k_0^2 k_y(x)$$

Here γ is the propagation constant.

$$h(x, x_1) = \int_{x_1}^x g(x) dx$$

The appropriate solutions are

$$E_y = \begin{cases} E \sin (px) & 0 < x < x_1 \\ [B \sin \{ h(x, x_1) \} + C \sin \{ h(x_2, x_1) - h(x, x_1) \}] / \sqrt{g(x)} & x_1 < x < x_2 \\ D \sin \{ p(a-x) \} & x_2 < x < a \end{cases}$$

where $p^2 = \gamma^2 + k_0^2$, E , B , C , D are constants and h is defined as above.

By forcing E_y and $\partial E_y / \partial x$ to be continuous at the boundaries $x = x_1$

and $x = x_2$, a system of equations is obtained. Equating the determinant of the system of equations to zero, a transcendental equation is obtained,

$$g(x_1)g(x_2) \tan \{ h(x_2, x_1) \} \tan (px_1) \tan \{ p(a - x_2) \} - p^2 \tan \{ h(x_2, x_1) \} - pg(x_1) \tan (px_1) - pg(x_2) \tan \{ p(a - x_2) \} = 0$$

Another work based on the WKB approximation for TE mode propagation in the rectangular waveguide containing longitudinally inhomogeneous dielectric has been developed by Kao³⁸. Using the technique of separation of variables, it is shown that the general waveguide solution for the transverse variables (x, y) still holds true. An approximate method is used to solve the z -variable ordinary differential equation, by using an expansion in a Taylor Series. The relation of the Wronskian determinant of the solution and the accuracy of the solution is discussed: in fact, the deviation of the Wronskian determinant of the wave solutions at a certain point z from the initial value is indicative of the accuracy obtained. It also checks the power reflected and transmitted in the different regions.

1.3.8 Polarisation - Source Formulation

In this formulation, developed by Bates and Ng³⁹, all diffracting bodies are treated as sources of fields that travel undisturbed throughout space. The electric and magnetic field are represented by a column matrix $\underline{f}$,

$$\underline{f} = \begin{bmatrix} \underline{E} \\ \underline{H} \end{bmatrix}$$

According to the diffraction theory, the total field $\underline{f}$ can be split into an incident field $\underline{f}_o$ and a field $\underline{f}_p$, scattered or reradiated from inhomogeneities in the medium

$$\underline{f} = \underline{f}_o + \underline{f}_p$$

In this formulation all reflecting, refracting and diffracting bodies in a region are treated as perturbations of a uniform free space in which the speed c of electromagnetic waves is constant. Then $\underline{f}_o$ and $\underline{f}_p$ travel everywhere with the speed c , so they can be expressed in terms of the usual retarded integral over their sources. There is a source for $\underline{f}_p$ at each point in space where the medium is different from free space. Such sources are called polarisation sources because the amplitude of the source density at each point, apart from being a function of the difference in characteristics between the medium and the free space at that point, is also proportional to the total field there.

If discontinuous media are excluded, the vector wave equation for $\underline{E}$ and $\underline{H}$ can be written as

$$\nabla^2 \underline{f} - c^{-2} \ddot{\underline{f}} = -\underline{w}$$

where the vector column matrix $\underline{w}$, which depends linearly on $\underline{f}$, is called the polarisation-source density.

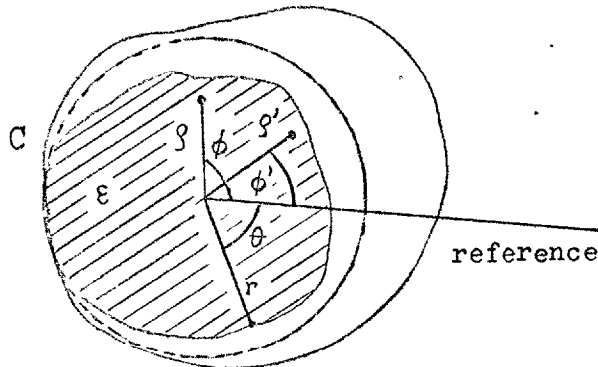
It is then possible to obtain a retarded integral solution of the wave equation. In Bates' paper, details of the formulation are given and they will not be reported here. In the final section, Bates and Ng find an expression for the determination of the cutoff characteristics of an arbitrary waveguide with circular dielectric tubes or rods, and give some numerical results.

Although this formulation does not seem to be of great help for calculating the propagation characteristics of inhomogeneously loaded waveguides, it has been pointed out to the author* that there

* Mr. H. Tosun, of Imperial College, in a private communication.

is the possibility of extending the method. The most important step will be outlined below and it is hoped that this can be applied sometimes in order to evaluate the practical difficulties.

Consider the waveguide of arbitrary cross section loaded with dielectric, as shown in Figure 1.12



Outside the waveguide, the incident field is zero³⁹, and this can be expressed in mathematical form as $E_z(\rho, \phi) = 0$, for all ϕ , when $\rho > r_{\max}$.

Fig. 1.12

The general expression is³⁹

$$E_z(\rho, \phi) = \int_S j\omega(\epsilon - \epsilon_0) E_z(\rho', \phi') H_0^{(2)}(k_0 R) dS + \frac{j}{4} \int_C (E_z \frac{\partial}{\partial n} H_0^{(2)}(k_0 R) - H_0^{(2)}(k_0 R) \frac{\partial E_z}{\partial n}) dC, \text{ where}$$

$\epsilon = \epsilon_0 \epsilon_r$, $H_0^{(2)}$ denotes the Hankel function of the second kind of zero order, and $\partial/\partial n$ is the partial derivative respect to the normal to C, and where

$$R^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos(\phi - \phi') \text{ for the first integral (over S)}$$

and

$$R^2 = \rho^2 + r^2 - 2\rho r \cos(\phi - \theta) \text{ for the second integral (on C).}$$

Applying the constraints, the following expression is obtained

$$0 = \iint_S j\omega(\epsilon - \epsilon_0) E_z(\rho', \phi') H_0^{(2)}(k_0 R) dS - \frac{j}{4} \int_C \frac{\partial E_z}{\partial n} H_0^{(2)}(k_0 R) dC \quad (1.81)$$

for $\rho > r_{\max}$.

It must be noted that starting from Eq. (1.81), it should be possible to take a different path and applying point-matching methods^{23,40}.

For $\rho > r$ it is possible to expand $H_0^{(2)}(k_0 R)$ in a series as follows

$$H_o^{(2)}(k_o R) = \sum_{m=-\infty}^{\infty} J_m(k_o \rho') H_m^{(2)}(k_o \rho) e^{jm(\phi - \phi')}$$

and substituting back in the integral equation (1.81),

$$\sum_{m=-\infty}^{\infty} j\omega H_m^{(2)}(k_o \rho) e^{jm\phi} \iint_S (\epsilon - \epsilon_o) E_z(\rho', \phi') J_m(k_o \rho) e^{-jm\phi'} dS' -$$

$$\frac{j}{4} \sum_{m=-\infty}^{\infty} H_m^{(2)}(k_o r) e^{jm\phi} \int_C \frac{\partial E_z}{\partial n} J_m(k_o r) e^{-jm\theta} dC = 0 \quad (1.82)$$

$$\text{Let } A_m = j\omega \iint_S (\epsilon - \epsilon_o) E_z(\rho', \phi') J_m(k_o \rho) e^{-jm\phi'} dS'$$

$$B_m = \frac{-j}{4} \int_C \frac{\partial E_z}{\partial n} J_m(k_o r) e^{-jm\theta} dC$$

then

$$\sum_{m=-\infty}^{\infty} (A_m + B_m) H_m^{(2)}(k_o \rho) e^{jm\phi} = 0$$

from which it follows that $A_m = -B_m$

Thus, Eq. (1.82) can be rewritten as

$$4\omega \iint_S (\epsilon - \epsilon_o) E_z(\rho', \phi') J_m(k_o \rho) e^{-jm\phi'} dS' = - \int_C \frac{\partial E_z}{\partial n} J_m(k_o r) e^{-jm\theta} dC \quad (1.83)$$

Let us suppose that it is possible to expand E_z in a series like

$$E_z(\rho', \phi') = \sum_{p=-\infty}^{\infty} A_p f_p(\rho') e^{jp\phi'}$$

where on C,

$$E_z \Big|_{\text{on } C} = \sum_{p=-\infty}^{\infty} A_p f_p(r) e^{jp\theta}$$

Thus,

$$\frac{\partial E_z}{\partial n} = \sum_{p=-\infty}^{\infty} A_p \frac{\partial}{\partial n} (f_p(r) e^{jp\theta})$$

and substituting in Eq. (1.83),

$$4\omega \sum_{p=-\infty}^{\infty} A_p \int_0^{2\pi} (\epsilon - \epsilon_0) e^{j(p-m)\phi'} \int_0^{r_{\max}} J_m(k_0 \rho') f_p(\rho') \rho' d\rho' =$$

$$- \sum_{p=-\infty}^{\infty} A_p \int_C \frac{\partial}{\partial n} (f_p(r) e^{jp\theta}) J_m(k_0 r) e^{-jm\theta} dC$$

Let

$$4\omega \iint_S (\epsilon - \epsilon_0) J_m(k_0 \rho') f_p(\rho') e^{j(p-m)\phi'} \rho' d\rho' d\phi' = C_{mp}$$

$$- \int_C \frac{\partial}{\partial n} (f_p(r) e^{jp\theta}) J_m(k_0 r) e^{-jm\theta} dC = D_{mp}$$

Then

$$\sum_{p=-\infty}^{\infty} A_p C_{mp} = \sum_{p=-\infty}^{\infty} A_p D_{mp}$$

and solutions are obtained from $\det (C_{mp} - D_{mp}) = 0$.

Obviously, the difficulties seem to be on the evaluation of C_{mp} rather than in D_{mp} , but it is probable that for standard configurations the evaluation can be straightforward.

1.3.9 Transmission-Line Matrix Method

Johns⁴¹ describes inhomogeneous waveguide structures using the transmission-line matrix method⁴². Propagation in a two-dimensional medium is represented by the voltages and currents on a Cartesian mesh of TEM transmission lines. At each node of the mesh, a submatrix of four numbers describes the magnitude of incident voltages along the four coordinate directions. The propagation of pulses is followed through the mesh and by observing the stream of pulses passing through some particular point, and taking the Fourier transform of the output

impulse function, it is possible to get the required information, in this case the phase constant β . In order to have an analog representation of the dielectric, on each node of the mesh there is an open-circuit stub of variable characteristic impedance. The stub length is $\Delta l/2$, half the distance between nodes. On each node, now, there are five pulses incident, four from the line and one from the stub. Pulses travel from one node to the next, and are transmitted and reflected. Each iteration in the computer represents a time interval of $\Delta l/C$. For each iteration and for each node, the new values of the five incident impulse amplitudes are calculated.

Although the method can only be used in cases where Maxwell's equations split into two independent sets of equations of three variables, Johns applied it successfully to several cases, like evaluation of the LSE mode in a rectangular cavity, cutoff of modes in waveguides with dielectric slab or ridge. The advantage in comparison with other methods it can be used on a small computer because of small storage requirements, and simplicity in the programme.

1.3.10 The Method of Sub-Waveguides

This method has been developed by Veszely⁴³ for analyzing inhomogeneous waveguides or waveguides of complicated cross-section. The procedure is to divide the cross-section of the waveguide into simpler parts: it is done by considering that in the "cut-lines" there are electric (sometimes magnetic) walls.

The starting point is writing as a functional, in terms of $\underline{E}_i$, $\underline{H}_i$, the electromagnetic fields in the i th subwaveguide. These fields can then be expanded in set of orthonormal

TE + TM Modes, in such a way that applying the Rayleigh-Ritz method, a system of equations is obtained.

From the dispersion characteristics of the subwaveguides it is possible to extract information about the dispersion characteristics of the original waveguide.

1.3.11 Method based on solutions to the Hill's equation⁴⁴

Casey⁴⁵ developed a method for the determination of electromagnetic fields in inhomogeneously filled rectangular waveguides.

If the material filling the rectangular waveguide is an inhomogeneous dielectric of permittivity $\epsilon(x)$, the LSE modes²¹ are obtained from

$$\nabla^2 \Phi + k^2(x) \Phi = 0$$

where $k^2(x) = \omega^2 \mu_0 \epsilon(x)$ and $\Phi(x, y, z) = f(x) \cos(n\pi y/b) \exp(-j\beta z)$

and $f(x)$ is a solution of

$$\frac{d^2 f}{dx^2} + \left\{ k^2(x) - \frac{n^2 \pi^2}{b^2} - \beta^2 \right\} f = 0 \quad (1.84)$$

subject to boundary conditions $f(a) = f(0) = 0$

The LSM modes are obtained from

$$\nabla^2 \Psi - \frac{1}{\epsilon} \frac{d\epsilon}{dx} \frac{\partial \Psi}{\partial x} + k^2(x) \Psi = 0$$

where $\Psi(x, y, z) = g(x) \sin(n\pi y/b) \exp(-j\beta z)$

and $g(x)$ satisfies

$$\frac{d^2 g}{dx^2} - \frac{1}{\epsilon} \frac{d\epsilon}{dx} \frac{dg}{dx} + \left\{ k^2(x) - \frac{n^2 \pi^2}{b^2} - \beta^2 \right\} g = 0 \quad (1.85)$$

subject to boundary conditions $g'(0) = g'(a) = 0$.

Eqs. (1.84) and (1.85) may be solved using

$$y = \pi x / 2a$$

$$u(y) = f(x)$$

$$v(y) = \epsilon^{-1/2}(x) g(x)$$

$$\lambda + 2 \sum_{n=1}^{\infty} g_n \cos 2n\gamma = \left(\frac{2a}{\pi} \right)^2 \left\{ k^2(x) - \left(\frac{n\pi}{b} \right)^2 - \beta^2 \right\}$$

$$\mu + 2 \sum_{n=1}^{\infty} h_n \cos 2n\gamma = \left(\frac{2a}{\pi} \right)^2 \varepsilon^{\frac{1}{2}}(x) \frac{d^2}{dx^2} \left\{ \varepsilon^{-\frac{1}{2}}(x) \right\}$$

yielding

$$\frac{d^2 u}{dy^2} + (\lambda + 2 \sum_{n=1}^{\infty} g_n \cos 2n\gamma) u = 0 \quad (1.86a)$$

$$\frac{d^2 v}{dy^2} + \left\{ \lambda - \mu + 2 \sum_{n=1}^{\infty} (g_n - h_n) \cos 2n\gamma \right\} v = 0 \quad (1.86b)$$

Eq. (1.86) is of the form of Hill's Equation. Solution may be obtained if $\sum_n g_n$ (or $\sum_n h_n$ for the LSM case) is absolutely convergent.

The boundary conditions are

$$u(0) = u(\pi/2) = 0 \quad (1.87a)$$

$$v'(0) = v'(\pi/2) = 0 \quad (1.87b)$$

when $\varepsilon'(0) = \varepsilon'(a) = 0$; if not Eq. (1.87) should be modified.

It has been proved⁴⁴ that Eq. (1.87a) is equivalent to

$$\det \left| \delta_{n,m} + \frac{g_{n-m} - g_{n+m}}{\lambda - 4n^2} \right|_{n,m=1,2,3,\dots} = 0$$

and this last equation constitutes the characteristic equation for the LSE mode. As it is expressed more or less directly in terms of the Fourier coefficients of the permittivity profile, the propagation characteristics can be evaluated numerically in a straight forward manner. A similar equation can be obtained for LSM modes. The last part of Casey's paper is devoted to the presentation of some examples.

1.3.12 Stochastic Methods

A technique based on the Monte Carlo method for solving TEM-mode transmission-line problems has been presented by Royer⁴⁶.

Although the method presented has been proved to be useful to evaluate the characteristic impedance of a guide with an inhomogeneous dielectric, when the dimensions of the guide are very small with respect to wavelength, it could be useful if used in conjunction with other methods where the knowledge of potentials on some region is necessary for further calculations.

Bevensee⁴⁷ uses a variation of the previous method, the so called number-diffusion process to obtain probabilistic solutions of the wave equation for a finite lossy transmission line sinusoidally excited.

These methods, although theoretically known for a long time, are still in a phase of practical development. Their great advantage is the use of few computational steps and low computer memory.

1.4 Conclusions

The numerical methods appearing in recent years on the analysis of inhomogeneously loaded waveguides, and which have been covered in previous sections, have in common the fact that each one is based on a different technique to reduce the problem to one or two dimensions.

Nothing more will be said about the analytical techniques since their applications are restricted to a few particular geometries.

In the author's opinion the numerical procedures that have the most general application are those classified as of Rayleigh-Ritz and finite-element type.

The direct Rayleigh-Ritz approximation method has been known for a long time but it was not applied as a standard procedure to wave-guide problems until relatively recently. For a large variety of waveguide shapes it is possible to construct the suitable approximating functions.

Approximating functions are typically chosen to be either polynomials in Cartesian coordinates or in polar coordinates²⁰: in the former case some difficulty may be encountered for some cross sections, while the second choice is best if the cross sections are star shaped with respect to a point.

For some boundary value problems an equivalent formulation in which a two-dimensional boundary value problem is replaced by an integral equation involving a contour integral has been given³⁹: in this case search techniques must be used for determining the eigenvalues. The drawback is that the matrix sizes are relatively small, and there is then the danger of missing some eigenvalue.

During the last few years, the finite-element method has gained considerable popularity in the solution of boundary value problems and at the present time is the method mostly used in the solution of waveguide problems. The method has the advantage that it combines the high accuracy and short computing time of the Rayleigh-Ritz method with the flexibility and simple usage of the finite-difference method. It is then expected that the use of this method will become increasingly important for solving waveguide problems.

The drawback of all methods discussed above is that the knowledge of the partially filled guide cutoff frequency is not sufficient to determine the propagation constant at other frequencies, which is necessary to solve the corresponding equations at each frequency.

APPENDIX A

Sturm - Liouville Problems

* A certain type of boundary value problem involves a differential equation of the form

$$\frac{d}{dx} \left(p \frac{dy}{dx} \right) + (\lambda w - q)y = 0 \quad (A - 1)$$

with some prescribed boundary conditions at the limits of the interval (a,b). In this equation p, w, q are continuous functions of x in the interval (a, b) and λ is a parameter independent of x. By a suitable choice of p, w, q, most of the important equations governing electromagnetic wave propagation can be obtained from Eq. (A - 1). For example, the equation reduces to a simple equation of periodic motion when $p = w = 1$, $q = 0$. But when $p = x$, $w = x$, $q = -n^2/xm$ Eq. (A - 1) leads to the Bessel equation, and when $p = 1 - x^2$, $w = 1$, $q = 0$, it reduces to the Legendre equation. Mathieu's equation, Gauss' hypergeometric equation, Chebyshev's equation, and Hermite and Laguerre polynomials are also special cases of the above equation.

In order to have a non-trivial solution of equation (A - 1) which will satisfy the boundary conditions at the end of the interval (a, b), λ must assume one of the eigenvalues λ_n . To each eigenvalue there is the corresponding eigenfunction ϕ_n . The solution may be expressed in terms of the eigenfunction as $y = c\phi_n(x)$, where c is an arbitrary constant. The boundary conditions of the Sturm-Liouville system include the following

- (i) At $x = a$ or $x = b$, either $y = 0$ or $\dot{y} = 0$ or a linear combination $\alpha y + \beta \dot{y} = 0$.
- (ii) If $p(x) = 0$ at $x = a$ or $x = b$, y and $\dot{y}$ must remain finite at that point and one imposes one of the conditions (i) at the other point.

- (iii) If $p(b) = p(a)$, then only $y(b) = y(a)$ and $\dot{y}(a) = \dot{y}(b)$ are required.

Any continuous function can be developed in terms of Sturm-Liouville functions which are the solutions of Eq. (A - 1).

The system possesses the following properties

- (i) There is an infinite set of eigenvalues λ_n and a corresponding number of eigenfunctions ϕ_n , such that $0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \dots$
- (ii) There exists a complete normalized set of eigenfunctions $\{\phi_n(x)\}$ in terms of which a well behaved function $f(x)$ can be a convergent series.
- (iii) The eigenfunctions possess orthogonality with respect to the weighting function q over the prescribed interval (a, b) , i.e.
- $$(\lambda_m - \lambda_n) \int_a^b q \phi_m \phi_n dx = 0 \quad \lambda_m \neq \lambda_n$$
- (iv) $\int_0^a q \phi_m \phi_n dx = \delta_{mn}$, where $\delta_{mn} = 1$ if $m = n$ and $\delta_{mn} = 0$ otherwise.
- (v) The solution to the problem may be formulated as the problem of finding the function which will make the variational integral stationary.

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CHAPTER 2

AN APPROXIMATE METHOD FOR THE CALCULATION OF PROPAGATION
CONSTANTS IN NON-UNIFORM WAVEGUIDES

2.1 Introduction^{1,3}

For sinusoidal fields in a linear, isotropic and source-free dielectric medium, Maxwell's equations can be written as

$$\nabla \times \underline{E} = - \mu \frac{\partial \underline{H}}{\partial t} \quad (2.1a)$$

$$\nabla \times \underline{H} = \epsilon \frac{\partial \underline{E}}{\partial t} \quad (2.1b)$$

where the two parameters μ and ϵ describe the nature of the space that contains the two field vectors $\underline{E}$ and $\underline{H}$.

In dealing with boundary value problems, it is sometimes convenient to express the vector differential operator ∇ (nabla) as

$$\nabla = \nabla_t + \nabla_n$$

where ∇_t is that component of ∇ which represents differentiation with respect to the coordinate of the plane of discontinuity and ∇_n is the component which differentiates with respect to the coordinates normal to this plane.

If the two field vectors $\underline{E}$ and $\underline{H}$ are also expressed in terms of their components, i.e.

$$\underline{E} = \underline{E}_t + \underline{E}_n$$

$$\underline{H} = \underline{H}_t + \underline{H}_n$$

and if it assumed that the time variation is given by $\exp(st)$, where $s = j\omega$, Eqs. (2.1) can be written as :-

$$\nabla_t \times (\underline{E}_t + \underline{E}_n) + \nabla_n \times (\underline{E}_t + \underline{E}_n) = - s\mu (\underline{H}_t + \underline{H}_n) \quad (2.2a)$$

$$\nabla_t \times (\underline{H}_t + \underline{H}_n) + \nabla_n \times (\underline{H}_t + \underline{H}_n) = s\epsilon (\underline{E}_t + \underline{E}_n) \quad (2.2b)$$

Consider now the structure depicted in Fig. 2.1, defined by a conducting surface C enclosing a region A, and where $\underline{n}$ is a unit vector normal, at each point, to C. It is assumed, without loss

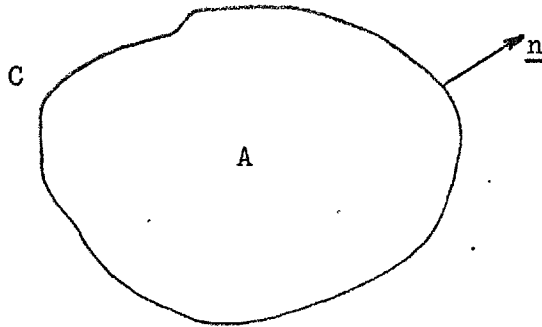


Fig. 2.1

of generality, that, within the guide, the relative permeability $\mu_r = 1$, and that the permittivity is a function of the coordinates in the region A but independent of the axial coordinate, perpen-

dicular to the section A. It is also assumed that the coordinates form a right-handed orthogonal system of coordinates.

An electromagnetic field, defined by the two field vectors $\underline{E}$ and $\underline{H}$, will satisfy Maxwell's equation and

$$\underline{n} \times \underline{E} = 0 \quad \text{on } C. \quad (2.3)$$

If the electric and magnetic field are expressed, respectively, as

$$\underline{E} = (\underline{E}_t + E_p \underline{a}_p) \exp(st - \gamma p) \quad (2.4a)$$

$$\underline{H} = (\underline{H}_t + H_p \underline{a}_p) \exp(st - \gamma p) \quad (2.4b)$$

where $\underline{E}_t$ ($\underline{H}_t$) is the component of the electric (magnetic) field in the region A. E_p (H_p) is the magnitude of the axial component and in general it will be a function of the coordinates on the section A. $\underline{a}_p$ is a unit vector that defines a direction perpendicular to the Section A in the direction of p increasing, $\exp(st)$ is the time variation and $\exp(-\gamma p)$ is the variation of the field along the axial direction defined by $\underline{a}_p$. γ is known as the propagation constant and in general it is defined as

$$\gamma = \alpha + j\beta \quad (2.5)$$

where α is the attenuation constant and β the phase constant. For a propagating mode γ must be imaginary, then $\gamma = j\beta$.

Substituting Eq. (2.4) in Maxwell's equations (2.1), and taking into account that $\nabla_n = \frac{a_p}{\partial p} = \nabla_p = -\gamma \frac{a_p}{\partial p}$

$$(\nabla_t - \gamma \frac{a_p}{\partial p}) \times (\underline{E}_t + \frac{a_p}{\partial p} \underline{E}_p) = - s_\mu (\underline{H}_t + \frac{a_p}{\partial p} \underline{H}_p) \quad (2.6a)$$

$$(\nabla_t - \gamma \frac{a_p}{\partial p}) \times (\underline{H}_t + \frac{a_p}{\partial p} \underline{H}_p) = s_\epsilon (\underline{E}_t + \frac{a_p}{\partial p} \underline{E}_p) \quad (2.6b)$$

where the exponential factors have been omitted. Solving

$$\nabla_t \times \underline{E}_t - \frac{a_p}{\partial p} \times \nabla_t \underline{E}_p - \gamma \frac{a_p}{\partial p} \times \underline{E}_t = - s_\mu (\underline{H}_t + \frac{a_p}{\partial p} \underline{H}_p) \quad (2.7a)$$

$$\nabla_t \times \underline{H}_t - \frac{a_p}{\partial p} \times \nabla_t \underline{H}_p - \gamma \frac{a_p}{\partial p} \times \underline{H}_t = s_\epsilon (\underline{E}_t + \frac{a_p}{\partial p} \underline{E}_p) \quad (2.7b)$$

where $\frac{a_p}{\partial p} \times \frac{a_p}{\partial p} = 0$ and $\nabla_t \times \frac{a_p}{\partial p} = - \frac{a_p}{\partial p} \times \nabla_t$.

Eqs. (2.7) can be rearranged as

$$\frac{a_p}{\partial p} \times \underline{E}_t + \frac{a_p}{\partial p} \times \nabla_t \underline{E}_p = s_\mu \underline{H}_t \quad (2.8a)$$

$$\nabla_t \times \underline{E}_t = - s_\mu \underline{H}_p \frac{a_p}{\partial p} \quad (2.8b)$$

$$\gamma \frac{a_p}{\partial p} \times \underline{H}_t + \frac{a_p}{\partial p} \times \nabla_t \underline{H}_p = - s_\epsilon \underline{E}_t \quad (2.8c)$$

$$\nabla_t \times \underline{H}_t = s_\epsilon \underline{E}_p \frac{a_p}{\partial p} \quad (2.8d)$$

Multiplying $\frac{a_p}{\partial p} \times$ Eq. (2.8a)

$$\frac{a_p}{\partial p} \times \frac{a_p}{\partial p} \times \underline{E}_t + \frac{a_p}{\partial p} \times \frac{a_p}{\partial p} \times \nabla_t \underline{E}_p = s_\mu (\frac{a_p}{\partial p} \times \underline{H}_t)$$

Using Eq. (2.8c)

$$\gamma^2 \frac{a_p}{\partial p} \times \frac{a_p}{\partial p} \times \underline{E}_t + \gamma \frac{a_p}{\partial p} \times \frac{a_p}{\partial p} \times \nabla_t \underline{E}_p = s_\mu (- s_\epsilon \underline{E}_t - \frac{a_p}{\partial p} \times \nabla_t \underline{H}_p)$$

$$\gamma^2 \frac{a_p}{\partial p} \times \frac{a_p}{\partial p} \times \underline{E}_t + s_\mu^2 \epsilon \underline{E}_t = - \gamma \frac{a_p}{\partial p} \times \frac{a_p}{\partial p} \times \nabla_t \underline{E}_p - s_\mu \frac{a_p}{\partial p} \times \nabla_t \underline{H}_p$$

Applying the vector identity $\underline{A} \times \underline{B} \times \underline{C} = (\underline{A} \cdot \underline{C})\underline{B} - (\underline{A} \cdot \underline{B})\underline{C}$,

$$\gamma^2 \{ (\frac{a_p}{\partial p} \cdot \underline{E}_t) \frac{a_p}{\partial p} - (\frac{a_p}{\partial p} \cdot \frac{a_p}{\partial p}) \underline{E}_t \} + s_\mu^2 \epsilon \underline{E}_t = - s_\mu \frac{a_p}{\partial p} \times \nabla_t \underline{H}_p - \gamma \{ (\frac{a_p}{\partial p} \cdot \nabla_t) \frac{a_p}{\partial p} - (\frac{a_p}{\partial p} \cdot \frac{a_p}{\partial p}) \nabla_t \} \underline{E}_p$$

where $\frac{a_p}{\partial p} \cdot \underline{E}_t = 0$, $\frac{a_p}{\partial p} \cdot \frac{a_p}{\partial p} = 1$, $\frac{a_p}{\partial p} \cdot \nabla_t = 0$

Then, a final expression is obtained as

$$\gamma^2 \underline{E}_t - s_\mu^2 \epsilon \underline{E}_t = s_\mu \frac{a_p}{\partial p} \times \nabla_t \underline{H}_p - \gamma \nabla_t \underline{E}_p \quad (2.9a)$$

Similarly, multiplying $\gamma \frac{a_p}{\partial p} \times$ Eq. (2.8c) and using Eq. (2.8a), an

equivalent expression relating $\underline{H}_t$, E_p and H_p as follows

$$\gamma^2 \underline{H}_t - s^2 \underline{\mu} \underline{H}_t = -s \underline{a}_p \times \nabla_t E_p - \gamma \nabla_t H_p \quad (2.9b)$$

Both equations (2.9) can be collected as a matrix equation as follows:

$$sM \underline{a}_p \times \nabla_t \phi = (s^2 \underline{\mu} \epsilon - \gamma^2) A \psi - \gamma A \nabla_t \phi \quad (2.10)$$

where

$$M = \begin{bmatrix} \epsilon & 0 \\ 0 & \mu \end{bmatrix}$$

$$\phi = \begin{bmatrix} E_p \\ H_p \end{bmatrix}$$

$$\psi = \begin{bmatrix} \underline{E}_t \\ \underline{H}_t \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

2.2 Expansion in rectangular coordinates

In rectangular coordinate, if the region A of Fig. 2.1 is defined as the x-y plane, and the axial direction perpendicular to it as the z-axis, then it follows that

$$\underline{E}_t = E_x \underline{i} + E_y \underline{j}$$

$$E_p = E_z$$

$$\underline{a}_p = \underline{k}$$

$$\nabla_t = \underline{i} \frac{\partial}{\partial x} + \underline{j} \frac{\partial}{\partial y}$$

where $\underline{i}$, $\underline{j}$ and $\underline{k}$ are, respectively, unit vectors in the directions of x, y, and z increasing. Replacing $s = j\omega$ and defining $k_0^2 = \omega^2 \underline{\mu}_0 \epsilon_0$, the matrix equation (2.10) can be rewritten as

$$j\omega M \frac{\partial \phi}{\partial y} = (\gamma^2 + k_o^2 \epsilon_r) A \psi_x + \gamma A \frac{\partial \phi}{\partial x} \quad (2.11a)$$

$$j\omega M \frac{\partial \phi}{\partial x} = -(\gamma^2 + k_o^2 \epsilon_r) A \psi_y - \gamma A \frac{\partial \phi}{\partial y} \quad (2.11b)$$

where M, A and ϕ are defined as before,

$$\psi_x = \begin{bmatrix} E_x \\ H_x \end{bmatrix} ; \quad \psi_y = \begin{bmatrix} E_y \\ H_y \end{bmatrix}$$

and $\epsilon_r = \epsilon/\epsilon_o$.

In a structure like the one depicted in Fig. 2.1, the electromagnetic fields are governed by the source-free Maxwell field equation that are, assuming again a sinusoidal variation in time and in the direction of propagation,

$$\frac{\partial E_z}{\partial x} = -\gamma E_x + j\omega\mu H_y$$

$$\frac{\partial E_z}{\partial y} = -\gamma E_y - j\omega\mu H_x$$

$$\frac{\partial E_x}{\partial y} - \frac{\partial E_y}{\partial x} = j\omega\mu H_z$$

$$\frac{\partial H_z}{\partial x} = -\gamma H_x - j\omega\epsilon E_y$$

$$\frac{\partial H_z}{\partial y} = -\gamma H_y + j\omega\epsilon E_x$$

$$\frac{\partial H_x}{\partial y} - \frac{\partial H_y}{\partial x} = -j\omega\epsilon E_z$$

From the previous set of equations it is evident that if it is possible to know E_z and H_z , all the other components can be determined. Hence, the next task will be the determination of an expression for E_z and H_z alone.

Consider the expressions (2.9) in rectangular coordinates,

$$(\gamma^2 + \omega^2 \mu \epsilon) \underline{E}_t = j\omega\mu \underline{k} \times \nabla_t H_z - \gamma \nabla_t E_z \quad (2.12a)$$

$$(\gamma^2 + \omega^2 \mu \epsilon) \underline{H}_t = -j\omega\epsilon \underline{k} \times \nabla_t E_z - \gamma \nabla_t H_z \quad (2.12b)$$

Now, let us multiply $\nabla_t \times$ Eq. (2.12a), and use the vector identity

$$\nabla \times (a \underline{B}) = \nabla a \times \underline{B} + a \nabla \times \underline{B}$$

(a is a scalar and $\underline{B}$ a vector), then the following expression is obtained

$$(\gamma^2 + \omega^2 \mu \epsilon) \nabla_t \times \underline{E}_t + \omega^2 \mu \epsilon_0 \nabla_t^{\epsilon_r} \times \underline{E}_t = j \omega \mu \nabla_t \times (\underline{k} \times \nabla_t H_z), \quad (2.13)$$

From Eq. (2.8b),

$$\nabla_t \times \underline{E}_t = -j \omega \mu H_z \underline{k}$$

and from Eq. (2.12a),

$$\underline{E}_t = (\gamma^2 + \omega^2 \mu \epsilon)^{-1} (j \omega \mu \underline{k} \times \nabla_t H_z - \gamma \nabla_t E_z)$$

Substituting in Eq. (2.13)

$$\begin{aligned} -j \omega \mu (\gamma^2 + \omega^2 \mu \epsilon) H_z \underline{k} + \omega^2 \mu \epsilon_0 \nabla_t^{\epsilon_r} \times (\gamma^2 + \omega^2 \mu \epsilon)^{-1} (j \omega \mu \underline{k} \times \nabla_t H_z - \gamma \nabla_t E_z) \\ = j \omega \mu \nabla_t \times (\underline{k} \times \nabla_t H_z) \end{aligned}$$

Multiplying by $\underline{k}$.

$$\begin{aligned} -j \omega \mu (\gamma^2 + \omega^2 \mu \epsilon)^2 H_z + \omega^2 \mu \epsilon_0 \underline{k} \cdot \nabla_t^{\epsilon_r} \times (j \omega \mu \underline{k} \times \nabla_t H_z - \gamma \nabla_t E_z) \\ = j \omega \mu (\gamma^2 + \omega^2 \mu \epsilon) \underline{k} \cdot \nabla_t \times (\underline{k} \times \nabla_t H_z) \end{aligned}$$

Using again the identity $\underline{A} \times \underline{B} \times \underline{C} = (\underline{A} \cdot \underline{C}) \underline{B} - (\underline{A} \cdot \underline{B}) \underline{C}$

$$\underline{k} \cdot \nabla_t^{\epsilon_r} \times \underline{k} \times \nabla_t H_z = \nabla_t^{\epsilon_r} \cdot \nabla_t H_z$$

$$\underline{k} \cdot \nabla_t \times \underline{k} \times \nabla_t H_z = \nabla_t^2 H_z$$

Then

$$\begin{aligned} -j \omega \mu \left\{ (\gamma^2 + \omega^2 \mu \epsilon)^2 H_z - \omega^2 \mu \epsilon_0 \nabla_t^{\epsilon_r} \cdot \nabla_t H_z + (\gamma^2 + \omega^2 \mu \epsilon) \nabla_t^2 H_z \right\} = \\ \omega^2 \mu \epsilon_0 \gamma \underline{k} \cdot (\nabla_t^{\epsilon_r} \times \nabla_t E_z) \\ (\gamma^2 + k_o^2 \epsilon_r) \nabla_t^2 H_z + (\gamma^2 + k_o^2 \epsilon_r) H_z - k_o^2 \nabla_t^{\epsilon_r} \cdot \nabla_t H_z = j \omega \epsilon_0 \gamma \underline{k} \cdot (\nabla_t^{\epsilon_r} \times \nabla_t E_z) \quad (2.14) \end{aligned}$$

From Eq. (2.12b), using the vector identities

$$\nabla \times (a \underline{B}) = \nabla a \times \underline{B} + a \nabla \times \underline{B}$$

$$\nabla \cdot (a \underline{B}) = \underline{B} \cdot \nabla a + a \nabla \cdot \underline{B}$$

the following expression is obtained

$$-(\gamma^2 + k_0^2 \epsilon_r) \nabla_t \times \underline{H}_t - k_0^2 \nabla_t \epsilon_r \times \underline{H}_t = j\omega \epsilon_0 \underline{k} \left\{ \nabla_t \cdot (\epsilon_r \nabla_t \underline{E}_z) \right\}$$

Using Eq. (2.8d)

$$\nabla_t \times \underline{H}_t = j\omega \epsilon_0 \epsilon_r \underline{E}_z \underline{k}$$

and Eq. (2.12b), we obtain

$$\begin{aligned} &-(\gamma^2 + k_0^2 \epsilon_r)^2 j\omega \epsilon_0 \epsilon_r \underline{E}_z \underline{k} - k_0^2 \nabla_t \epsilon_r \times (\gamma \nabla_t \underline{H}_z + j\omega \epsilon_0 \epsilon_r \underline{k} \times \nabla_t \underline{E}_z) = \\ &- j\omega \epsilon_0 (\gamma^2 + k_0^2 \epsilon_r) \underline{k} \left\{ \nabla_t \cdot (\epsilon_r \nabla_t \underline{E}_z) \right\} \end{aligned}$$

Multiplying by $\underline{k}$ and rearranging terms,

$$\begin{aligned} &(\gamma^2 + k_0^2 \epsilon_r) \left\{ \nabla_t \cdot (\epsilon_r \nabla_t \underline{E}_z) \right\} + (\gamma^2 + k_0^2 \epsilon_r)^2 \epsilon_r \underline{E}_z - k_0^2 \epsilon_r \nabla_t \epsilon_r \cdot \nabla_t \underline{E}_z \\ &- j\omega \epsilon_0 \gamma \underline{k} \cdot (\nabla_t \epsilon_r \times \nabla_t \underline{H}_z) \\ &(\gamma^2 + k_0^2 \epsilon_r) \epsilon_r \nabla_t^2 \underline{E}_z + (\gamma^2 + k_0^2 \epsilon_r)^2 \epsilon_r \underline{E}_z + \gamma^2 \nabla_t \epsilon_r \cdot \nabla_t \underline{E}_z = \\ &- j\omega \epsilon_0 \gamma \underline{k} \cdot (\nabla_t \epsilon_r \times \nabla_t \underline{H}_z) \end{aligned} \quad (2.15)$$

Equations (2.14) and (2.15) are two differential coupled equations

in $\underline{E}_z$ and $\underline{H}_z$.

To satisfy $\underline{n} \times \underline{E} = 0$ on C , it is required that

$$\underline{E}_z = 0 \text{ on } C \quad (2.16a)$$

$$\underline{n} \cdot \nabla_t \underline{H}_z = 0 \text{ on } C \quad (2.16b)$$

where $\underline{n}$ is a unit vector normal to C .

The problem is completely defined by Eqs. (2.14), (2.15) and (2.16).

2.3 Determination of expansion modes

It would appear that the solution of Eq. (2.14) and (2.15) is an impossible task. In order to evaluate the propagation constant, the two field components, $\underline{E}_z$ and $\underline{H}_z$, can be substituted by an appropriate set of trial functions. In the searching for trial functions that are the most naturally suited to the dielectric loaded waveguide problem, the zero-frequency modes appear to be a good solution. Waveguide zero-frequency modes form an orthogonal set of functions that exist exactly

over the waveguide cross-section and they satisfy all the boundary conditions. Their determination, for the most common geometrical structures, is relatively easy.

If zero order modes have been found, it is then possible to expand any function in a series of zero-frequency modes, provided that the function has the same region of definition and has to satisfy the same boundary conditions. In particular, it is possible to expand the axial components of the electric and magnetic field in a waveguide, at any value of propagation constant, in the above modes. Because of the close physical relationship between the trial field and the exact field, few terms of the expansion should provide a good approximation to our solution.

Consider then, in Eqs. (2.14) and (2.15), $\omega = 0$. Equation (2.15) will thus be

$$\nabla_t \cdot (\epsilon_r \nabla_t E_z) + \gamma^2 \epsilon_r E_z = 0 \quad (2.17a)$$

$$E_z = 0 \text{ on } C \quad (2.17b)$$

with a corresponding set of solutions

$$E_z = \phi_n, \quad \gamma = \gamma_{n,E}$$

such that ϕ_n satisfies Eq. (2.17),

$$\nabla_t \cdot (\epsilon_r \nabla_t \phi_n) + \gamma_{n,E}^2 \epsilon_r \phi_n = 0$$

$$\phi_n = 0 \text{ on } C.$$

Similarly, Eq. (2.14)

$$\nabla_t^2 H_z + \gamma^2 H_z = 0 \quad (2.18a)$$

$$\underline{n} \cdot \nabla_t H_z = 0 \text{ on } C \quad (2.18b)$$

with a corresponding set of solutions

$$H_z = \psi_n, \quad \gamma = \gamma_{n,H}$$

where again

$$\nabla_t^2 \psi_n + \gamma_{n,H}^2 \psi_n = 0$$

$$\underline{n} \cdot \nabla_t \psi = 0 \text{ on } C.$$

Note that, for $\omega = 0$, Eqs. (2.14) and (2.15) are uncoupled.

ϕ_n and ψ_n constitute a complete set of functions.

It is possible to express the axial components E_z and H_z as

$$E_z = \sum_{n=0}^{\infty} a_n \phi_n \quad (2.19a)$$

$$H_z = \sum_{n=0}^{\infty} b_n \psi_n \quad (2.19b)$$

Note that Eqs. (2.16) are satisfied.

The undetermined sets of coefficients a_n and b_n may be determined from Eqs. (2.14) and (2.15).

By substituting Eqs. (2.19) in Eqs. (2.14) and (2.15), and using

$$\nabla_t \cdot (\epsilon_r \nabla_t \phi_n) = -\gamma_{n,E}^2 \phi_n \epsilon_r$$

$$\nabla_t^2 \psi_n = -\gamma_{n,H}^2 \psi_n$$

the following expressions are obtained

$$\sum_n b_n \left\{ (\gamma^2 + k_o^2 \epsilon_r)(\gamma^2 + k_o^2 \epsilon_r - \gamma_{n,H}^2) \psi_n - k_o^2 \nabla_t^{\epsilon_r} \cdot \nabla_t \psi_n \right\} = j\omega \epsilon_o \gamma \sum_n a_n \underline{k} \cdot (\nabla_t^{\epsilon_r} \times \nabla_t \phi_n) \quad (2.20a)$$

$$\sum_n a_n \epsilon_r \left\{ (\gamma^2 + k_o^2 \epsilon_r)(\gamma^2 + k_o^2 \epsilon_r - \gamma_{n,E}^2) \phi_n - k_o^2 \nabla_t^{\epsilon_r} \cdot \nabla_t \phi_n \right\} = -j\omega \mu_o \gamma \sum_n b_n \underline{k} \cdot (\nabla_t^{\epsilon_r} \times \nabla_t \psi_n) \quad (2.20b)$$

In this section it is apparent that the modes corresponding to zero-frequency of a composite structure of regular shape are more easily determined than any other mode at any other value of frequency. This is because the electrical and magnetic fields in (2.14) and (2.16) are uncoupled at zero-frequency and, as a result, the expansion modes may be obtained from two separate Helmholtz equations with appropriate weighting factors in the dielectric regions.

This should result in a faster approximation to the true eigenvalues of the problem.

2.4 Modes in dielectric-slab-loaded rectangular guides

In Fig. 2.2 there are several illustrations of typical slab-loaded rectangular waveguides.

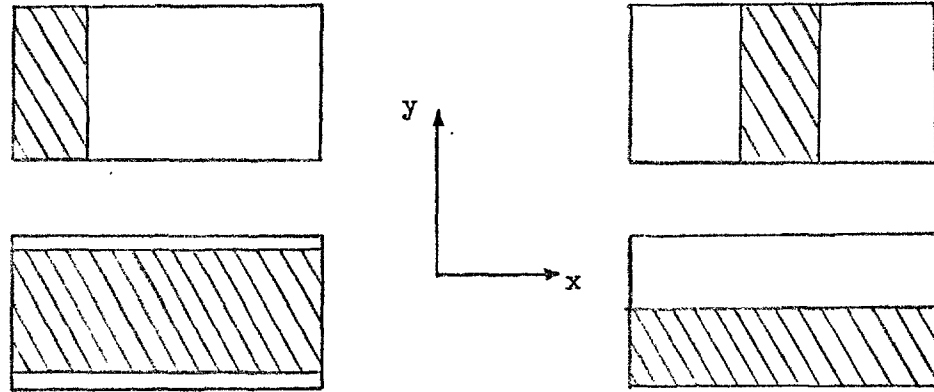


Fig. 2.2

For the cases illustrated it is relatively easy to find an expression for the propagation constants. When the configuration is different from those illustrated, it is difficult, in general, to find a handy solution and it must then be necessary to use an approximate method.

In general, in a dielectric loaded waveguide, like those in Fig. 2.2, the propagating modes are hybrids of E and H modes, except for the case when the electric field is parallel to the slab and there is no variation of the fields along the dielectric-air interface. In this particular case, the mode propagation is an H_{n0} mode.

The configurations of Fig. 2.2 are essentially the same: in some cases the air-dielectric interface lies in the yz plane, in other in the xz plane. Because of this, many authors used to derive the electromagnetic types from Hertzian potential functions. The resultant modes are usually classified as $E(TM_x)$ or $H(TE_x)$ modes with respect to the interface normal (i.e. x axis). If the mode has no component of electric field normal to the interface, then the electric field lies in the longitudinal interface plane, and the mode is commonly called a

longitudinal-section electric (LSE) mode. Instead, if there is no magnetic field component normal to the interface, the mode is called a longitudinal-section magnetic (LSM) mode.

It must be pointed out that, in the lossless case, for a guide filled with two dielectrics μ_1, ϵ_1 and μ_2, ϵ_2 , the cutoff frequencies ($\gamma = 0$) of the various modes always lie between those of the corresponding modes of a guide filled with a dielectric ϵ_1, μ_1 and those of a guide filled with a dielectric ϵ_2, μ_2 . The field patterns are similar to those of a hollow waveguide, except that the field tends to concentrate in the material of higher ϵ and μ .

In order to analyse the modes propagating in an inhomogeneously filled rectangular waveguide, consider the case where the parameters of the medium, μ and ϵ , are both functions of position. Starting from Maxwell's equations and following a procedure similar to that outlined in the previous sections, the following equation for the magnetic field is obtained

$$(\nabla^2 + k_0^2 \epsilon_r) \underline{H} = (\nabla \times \underline{H}) \times \frac{\nabla \epsilon}{\epsilon} - \nabla (\underline{H} \cdot \frac{\nabla \mu}{\mu}) \quad (2.21)$$

A similar equation can be obtained for the electric field just interchanging $\underline{H}$ with $\underline{E}$, ϵ with μ and μ with ϵ .

Consider now that μ and/or ϵ are only functions of y .

Thus,

$$\begin{aligned} \frac{\nabla \epsilon}{\epsilon} &= \nabla \ln \epsilon = \underline{j} \frac{\partial}{\partial y} (\ln \epsilon) \\ \frac{\nabla \mu}{\mu} &= \nabla \ln \mu = \underline{j} \frac{\partial}{\partial y} (\ln \mu) \end{aligned}$$

Expanding then Eq. (2.21) in rectangular coordinates, with x and y as transverse coordinates, and z as the direction of propagation, the following equations are obtained

$$(\nabla^2 + k_o^2 \epsilon_r) H_x = \frac{\partial}{\partial y} H_x \frac{\partial}{\partial y} \ln \epsilon - \frac{\partial}{\partial x} H_y \frac{\partial}{\partial y} \ln(\mu \epsilon) \quad (2.22a)$$

$$(\nabla^2 + k_o^2 \epsilon_r) H_y = - \frac{\partial}{\partial y} H_y \frac{\partial}{\partial y} \ln \mu - H_y \frac{\partial^2}{\partial y^2} \ln \mu \quad (2.22b)$$

$$(\nabla^2 + k_o^2 \epsilon_r) H_z = \frac{\partial}{\partial y} H_z \frac{\partial}{\partial y} \ln \epsilon - \frac{\partial}{\partial z} H_y \frac{\partial}{\partial y} \ln(\mu \epsilon) \quad (2.22c)$$

Looking for modes in which one of the components of magnetic field does not exist, there are three possibilities:

a) $H_z = 0$, then Eq.(2.22c) may be written

$$\frac{\partial}{\partial z} H_y \frac{\partial}{\partial y} \ln(\mu \epsilon) = 0$$

and this may occur when

1.- $\mu \epsilon = \text{constant}$. This is a trivial case.

2.- $H_y \neq f(z)$. It is impossible for a wave travelling in the z direction unless $H_y = H_z = 0$. This implies a TEM mode which cannot exist in a single connected region.

b) $H_x = 0$, and from Eq. (2.22a),

$$\frac{\partial}{\partial x} H_y \frac{\partial}{\partial y} \ln(\mu \epsilon) = 0$$

and this occurs for

1.- $\mu \epsilon = \text{constant}$. Trivial case as before.

2.- $H_y \neq f(x)$. This would correspond to solutions of the TE_{on} type with electric field along x . $H_y = 0$ is also a solution, but as H_z is missing, it is an impossible solution inside a waveguide.

For the electric field, an analogous analysis would lead to

$E_y \neq f(x)$ which implies $E_y = 0$ indicating that again a TE_{on} mode is acceptable.

c) $H_y = 0$. In this case, as Eq. (2.22b) is identically zero, the wave equations for the magnetic field are

$$(\nabla^2 + k_0^2 \epsilon_r) H_x = \frac{\partial}{\partial y} H_z \frac{\partial}{\partial y} \ln \epsilon$$

$$(\nabla^2 + k_0^2 \epsilon_r) H_z = \frac{\partial}{\partial y} H_x \frac{\partial}{\partial y} \ln \epsilon$$

If a solution for $\underline{H}$ is found, the electric field will be given by Maxwell's equation (2.1b)

It is then possible to conclude that longitudinal-section (LS) modes ($E_y = 0$ or $H_y = 0$) are the normal modes in the waveguide analysed, without any constraint.

By using a similar argument, it is possible to demonstrate that LS modes are not possible solutions when $\epsilon = f(z)$. In this case, TM and TE modes will be present; this implies that TE_{on} modes (which are both TE and LS types), are valid for waveguides in which the product is function of the direction of propagation and of one transverse coordinate.

A similar analysis for the electric field would lead to identical results if, in the preceding analysis, TM_{on} and TE_{on} are interchanged and the components of $\underline{H}$ are substituted for those of $\underline{E}$.

In the case of a slab-loaded waveguide, the dielectric constant changes abruptly but in a finite amount. It is just not possible⁶ to choose any mode type and try to match the transverse components at the boundary without the danger of inconsistencies in the derived eigenvalue equation. Matching only one component, electric or magnetic, across the interface is not a guarantee that the other transverse components will also be matched. This is because, apart from the simple dielectric and air region, there is a transition region: when this region is so small that it tends to vanish, the retardation across it is negligible and it is possible to equate the tangential fields through it.

2.5 Frequency dependence of the propagation constant

Carlin⁷ treated the problem of inhomogeneous waveguides as a zero-dimensional* problem, describing it with respect to specific types of

* The characterising functions are independent of spatial dimensions.

homogeneous ports.

Because of the restriction of homogeneity, Rhodes⁸ developed a more general theory, where physical constraints are applied to bounded guided-wave structures to formulate the basis of generalised uniform 1-dimensional network theory. Using the constraints of linearity and time invariance, and for a guiding structure bounded by a perfect magnetic or electric wall and a medium such that μ and ϵ are independent of the direction of the wave propagation z , it is possible to obtain a relationship between the propagation constant γ and the complex frequency p . This relationship is called the "boundary value equation". In general, it is a transcendental equation. As γ is a multivalued function, it must be represented on a Riemann surface which will consist of an infinite number of p plane sheets, each set representing a mode of propagation.

Some of these sheets will be connected by branch cuts. A mode set is defined as a collection of modes which are connected by branch points in $\text{Re}(p) > 0$.

By applying field linearity, reality and causality, it follows that in any finite inhomogeneous waveguide, a single mode of propagation cannot exist independently of other modes in the same mode set. In an infinite guide, the existence of a single mode is assured by mode orthogonality.

The constraints of reality and passivity show that the propagation constant γ belongs to a special class of functions termed "non zero real functions". For a dissipationless medium, γ becomes a "symmetrical non zero real function" with symmetry about the axis $p = j\omega$.

For dielectric-loaded rectangular waveguides, γ satisfies the following constraints :

(i) $\text{Re}(\gamma)$, $\text{Re}(\gamma/p) \neq 0$ for $\text{Re}(p) \neq 0$.

(ii) If $\gamma = \gamma_0$ is a solution for $p = p_0$,

$\gamma = \gamma_0^*$ is a solution for $p = p_0^*$, $\text{Re}(p) \neq 0$

γ exhibits reflection symmetry about cuts through both the real and imaginary axes of the Riemann surface, and all the infinities and zero exist along $\text{Re}(p) = 0$. γ may not be complex anywhere along $\text{Re}(p) = 0$, and it may be shown that

(i) γ must be real from $\omega = 0$ to $\omega = \pm \omega_c$.

(ii) γ possesses a branch point at $\omega = \pm \omega_c$ of the square-root variety at which $\gamma = 0$.

(iii) γ is imaginary for $|\omega| > |\omega_c|$ and tends monotonically to infinity as ω tends to infinity. At infinity, γ must possess a simple pole on every plane on the Riemann surface. The only zero is at a finite point along $p = j\omega$ which is a branch point of the square-root variety.

Along $p = j\omega$, corresponding to steady-state sinusoidal excitations, γ is purely real below the branch point (cutoff frequency) and purely imaginary above, as it is shown in Figure 2.3.

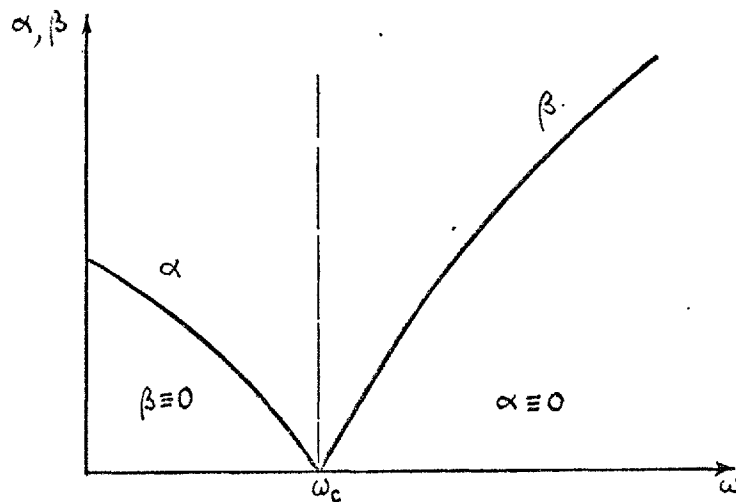


Fig. 2.3 Propagation characteristics for slab-loaded dielectric waveguides

If the macroscopic parameters are varied smoothly into a homogeneous state, the branch points that γ possesses on the axis, converge smoothly to infinity and are absorbed into the poles, such that the solutions for γ are of the form

$$\gamma = k(p^2 + \omega^2)^{\frac{1}{2}}$$

which are either TE or TM modes. Conversely, if the medium is varied from the homogeneous state, these branch points will reappear for complex values of p , but they will never touch the imaginary axis because of the constraint on the ratio γ/p .

It may then be shown that for LSE and LSM modes, a single-mode equivalent network representations are possible, with coupling existing only at the input and output ports.

The functional behaviour of the propagation constant has also been demonstrated in the frequency domain.

2.6 Determination of an expression for calculating propagation constants in dielectric-slab loaded rectangular waveguides

This section will be devoted to the analysis of a rectangular waveguide, the interior of which is filled with two different dielectrics, the distribution in all the cross-section being constant. One of the dielectrics will be air. In particular, the case of two dielectrics will be treated each half filling the guide longitudinally.

Consider then a rectangular waveguide, with infinitely conducting walls, with an axis parallel to the z -axis. The dimensions of the guide are a in the x -direction and b in the y direction. The whole length of the guide is filled, from $x = 0$ to $x = t$ with a non-absorbing dielectric of constant ϵ_0 (air), and from $x = t$ to $x = a$ with a non-absorbing dielectric of constant ϵ . The magnetic permeability of both media is taken equal to 1. It shall be taken that $\epsilon > \epsilon_0$. The described configuration is depicted in Fig. 2.4.

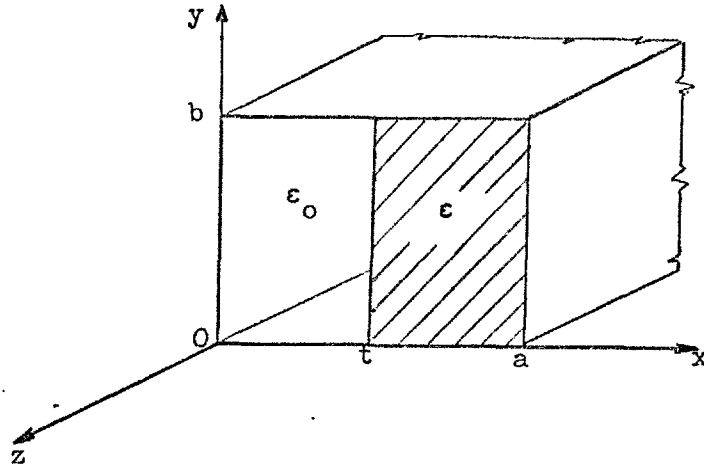


Fig. 2.4 System of coordinates and waveguide structure.

It was shown that the longitudinal components of the magnetic field and of the electric field satisfy respectively Eqs. (2.14) and (2.15), which are repeated below.

$$(\gamma^2 + k_0^2 \epsilon_r) \nabla_t^2 H_z + (\gamma^2 + k_0^2 \epsilon_r)^2 H_z - k_0^2 \nabla_t \epsilon_r \cdot \nabla_t H_z = j \omega \epsilon_0 \gamma \underline{k} \cdot (\nabla_t \epsilon_r \times \nabla_t E_z) \quad (2.23a)$$

$$(\gamma^2 + k_0^2 \epsilon_r) \epsilon_r \nabla_t^2 E_z + (\gamma^2 + k_0^2 \epsilon_r)^2 \epsilon_r E_z + \gamma^2 \nabla_t \epsilon_r \cdot \nabla_t E_z = - j \omega \mu_0 \gamma \underline{k} \cdot (\nabla_t \epsilon_r \times \nabla_t H_z) \quad (2.23b)$$

Here ϵ_r is a function of x only.

As before, supposing that the z and t - dependence of each field component is given by a factor $\exp(j\omega t - \gamma z)$, and assuming that the y - dependence is the same as for hollow waveguides³, it is possible to write two expressions, one for the magnetic field component and another for the electric field component, respectively,

$$H_z = f(x) \cos(l\pi y/b) \exp(j\omega t - \gamma z), \quad l = 0, 1, 2, \dots \quad (2.24a)$$

$$E_z = g(x) \sin(l\pi y/b) \exp(j\omega t - \gamma z), \quad l = 0, 1, 2, \dots \quad (2.24b)$$

With ϵ_r a function of x only, and substituting Eqs. (2.24) into Eqs. (2.23), the following expressions are obtained.

$$(\gamma^2 + k_o^2 \epsilon_r) \left\{ f''(x) + (\gamma^2 + k_o^2 \epsilon_r - p^2) f(x) \right\} - k_o^2 \epsilon_r' f'(x) = j\omega \epsilon_o \gamma p \epsilon_r' g(x) \quad (2.25a)$$

$$(\gamma^2 + k_o^2 \epsilon_r) \epsilon_r \left\{ g''(x) + (\gamma^2 + k_o^2 \epsilon_r - p^2) g(x) \right\} + \gamma^2 \epsilon_r' g'(x) = j\omega \mu_o \gamma p \epsilon_r' f(x) \quad (2.25b)$$

where ' and '' denote, respectively, the first and second order derivative with respect to x , p (not to be confused with the complex frequency of the previous section) represents $(l\pi/b)$, and $k_o^2 = \omega^2 \mu_o \epsilon_o$.

For zero-frequency ($k_o = 0$) Eqs. (2.25) are transformed to

$$f''(x) + (\gamma_H^2 - p^2) f(x) = 0 \quad (2.26a)$$

with boundary conditions given by

$$f'(0) = f'(a) = 0 \quad (2.26b)$$

$$\text{and } \epsilon_r \left\{ g''(x) + (\gamma_E^2 - p^2) g(x) \right\} + \epsilon_r' g'(x) = 0 \quad (2.26c)$$

with boundary conditions

$$g(0) = g(a) = 0 \quad (2.26d)$$

According to Section 2.3, it is possible to expand $f(x)$ and $g(x)$ in a series, as follows

$$f(x) = \sum_m b_m h_m(x) \quad (2.27a)$$

and

$$g(x) = \sum_n a_n e_n(x) \quad (2.27b)$$

where Eq. (2.27a) and Eq. (2.27b) will be solutions of Eq. (2.25a)

and Eq. (2.25b) respectively,

$$\sum_m b_m \left\{ (\gamma^2 + k_o^2 \epsilon_r) \left[h_m''(x) + (\gamma^2 + k_o^2 \epsilon_r - p^2) h_m(x) \right] - k_o^2 \epsilon_r' h_m'(x) \right\} = j\omega \epsilon_o \sum_n a_n \gamma p \epsilon_r' e_n(x) \quad (2.28a)$$

$$\sum_n a_n \left\{ (\gamma^2 + k_o^2 \epsilon_r) \left[e_n''(x) + (\gamma^2 + k_o^2 \epsilon_r - p^2) e_n(x) \right] + \gamma^2 \epsilon_r' e_n'(x) \right\} = j\omega \mu_o \sum_m b_m \gamma p \epsilon_r' h_m(x) \quad (2.28b)$$

Let us multiply Eqs. (2.28a) and (2.28b) by $h_s(x)$ and $e_s(x)$,

respectively, and integrate between $x = 0$ and $x = a$, in order to make use of the orthogonality properties of the modes in a waveguide.

A new pair of equations is so obtained

$$\sum_m \int_0^a b_m h_s(x) \left\{ (\gamma^2 + k_o^2 \epsilon_r) [h_m''(x) + (\gamma^2 + k_o^2 \epsilon_r - p^2) h_m(x)] - k_o^2 \epsilon_r h_m'(x) \right\} dx = j\omega \epsilon_o \sum_n \int_0^a a_n h_s(x) \gamma p \epsilon_r e_n(x) dx \quad (2.29a)$$

$$\sum_n \int_0^a a_n e_s(x) \left\{ (\gamma^2 + k_o^2 \epsilon_r) [e_n''(x) + (\gamma^2 + k_o^2 \epsilon_r - p^2) e_n(x)] + \gamma^2 \epsilon_r e_n'(x) \right\} dx = j\omega \mu_o \sum_m \int_0^a b_m e_s(x) \gamma p \epsilon_r h_m(x) dx \quad (2.29b)$$

Before analysing these expressions, it is necessary to find some suitable expressions for the field modes, namely h_m , e_n , h_s , e_s .

2.6.1 Magnetic field case

An appropriate solution for h_m seems to be a solution of the form

$$h_m(x) = A_m \cos(m\pi/a)x, \quad m = 1, 2, 3 \dots \quad (2.30)$$

where $\gamma_H^2 = (m\pi/a)^2 + p^2$. It is obvious, by direct substitution, that Eq. (2.30) satisfies Eq. (2.26a). It must also be noted that the boundary conditions are automatically fulfilled.

The constant A_m is evaluated using the orthogonal condition

$$\int_0^a h_m(x) h_n(x) dx = \delta_{mn}$$

where δ_{mn} is the Kronecker delta, and
$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Using these conditions, it is found that $A_m = (2/a)^{\frac{1}{2}}$.

Finally,

$$h_m(x) = (2/a)^{\frac{1}{2}} \cos (m\pi x/a) \quad (2.31)$$

2.6.2 Electric field case

For the electric field, an adequate solution will be

$$e_n(x) = \begin{cases} A_n \sin k_n x & 0 \leq x \leq t \\ B_n \sin k_n (a - x) & t \leq x \leq a \end{cases} \quad (2.32)$$

where Eqs. (2.26c) and (2.26d) are satisfied, with $k_n^2 = \gamma_E^2 - p^2$.

The boundary conditions at the interface ($x = t$) allow to write

$$A_n \sin k_n t = B_n \sin k_n (a - t)$$

$$A_n \cos k_n t = -\epsilon_r B_n \cos k_n (a - t)$$

where $\epsilon_r = \epsilon/\epsilon_0$ = relative dielectric constant of second medium.

Note that a transcendental equation can be obtained by dividing the previous expressions,

$$\epsilon \tan k_n t = \epsilon_0 \tan k_n (a - t)$$

Before evaluating the constants A_n and B_n , the particular case $t = a/2$ must be considered for which an inconsistency arises. For this case, the conditions at the interface can be written as

$$(A_n - B_n) \sin (k_n a/2) = 0 \quad (2.33a)$$

$$(A_n + \epsilon_r B_n) \cos (k_n a/2) = 0 \quad (2.33b)$$

Then, if

$$(i) \sin (k_n a/2) = 0, \text{ then } k_n a/2 = n\pi, \quad k_n = 2n\pi/a$$

$$\text{and } B_n = -(1/\epsilon_r) A_n$$

Thus, for this case

$$e_x(x) = \begin{cases} A_n \sin (2\pi n x/a) & 0 \leq x \leq t \\ -A_n (1/\epsilon_r) \sin \{ (2\pi n/a) (a - x) \} & t \leq x \leq a \end{cases} \quad (2.34)$$

It is known that the orthogonal condition for the electric field can be expressed as

$$\int_0^a \epsilon_r e_n(x) e_m(x) dx = \delta_{mn} \quad (2.35)$$

Thus, using Eq. (2.35), it is possible to obtain an expressions

for A_n ,

$$\begin{aligned} & \int_0^t A_n A_m \sin(2n\pi x/a) \sin(2m\pi x/a) dx + \int_t^a \epsilon_2 A_n A_m (1/\epsilon_2)^2 \\ & \cdot \sin \{ (2n\pi/a)(a-x) \} \sin \{ (2m\pi/a)(a-x) \} dx = \delta_{mn} \\ & A_n^2 \left\{ (t/2) + (a-t)/2\epsilon_2 \right\} = 1 \\ & A_n^2 \left\{ 2\epsilon_2 / [\epsilon_2 t + (a-t)] \right\}^{\frac{1}{2}} \end{aligned}$$

and, as $t = a/2$,

$$A_n = 2 \left\{ \epsilon_2 / (\epsilon_2 + 1)a \right\}^{\frac{1}{2}}$$

then

$$e_n(x) = \begin{cases} 2 \left\{ \epsilon_2 / (\epsilon_2 + 1)a \right\}^{\frac{1}{2}} \sin(2n\pi x/a) & 0 \leq x \leq a/2 \\ -2 \left\{ 1/\epsilon_2(\epsilon_2 + 1)a \right\}^{\frac{1}{2}} \sin \{ (2n\pi/a)(a-x) \} & a/2 \leq x \leq a \end{cases} \quad (2.36)$$

(ii) $\cos(k_n a/2) = 0$, then $k_n a/2 = (n - \frac{1}{2})\pi$, $k_n = (2n - 1)\pi/a$

and $A_n = B_n$.

Thus, for this case,

$$e_n(x) = \begin{cases} A_n \sin \{ (2n - 1)\pi x/a \} & 0 \leq x \leq t \\ A_n \sin \{ (2n - 1)\pi (a - x)/a \} & t \leq x \leq a \end{cases}$$

Again using the orthogonal relation Eq. (2.35), it is possible to evaluate A_n ; after some algebraic manipulations, it follows

$$A_n = 2 / [(1 + \epsilon_2)a]^{\frac{1}{2}}$$

such that

$$e_n(x) = \begin{cases} 2 \left\{ (1 + \epsilon_2)a \right\}^{-\frac{1}{2}} \sin \{ (2n - 1)\pi x/a \} & 0 \leq x \leq a/2 \\ 2 \left\{ (1 + \epsilon_2)a \right\}^{-\frac{1}{2}} \sin \{ (2n - 1)\pi(a - x)/a \} & a/2 \leq x \leq a \end{cases} \quad (2.37)$$

It is thus possible to assume that the expansion is a linear combination of cases (i) and (ii),

$$e(x) = \begin{cases} 2 \left[(\epsilon_2 + 1)a \right]^{-\frac{1}{2}} \left\{ \sum_{n \text{ odd}} \sin(n\pi x/a) + \sum_{n \text{ even}} (\epsilon_2)^{\frac{1}{2}} \sin(n\pi x/a) \right\} & 0 \leq x \leq a/2 \\ 2 \left[(\epsilon_2 + 1)a \right]^{-\frac{1}{2}} \left\{ \sum_{n \text{ odd}} \sin \{ n\pi(a-x)/a \} - \sum_{n \text{ even}} (1/\epsilon_2)^{\frac{1}{2}} \sin \{ n\pi(a-x)/a \} \right\} & a/2 \leq x \leq a \end{cases} \quad (2.38a)$$

If $t \neq a/2$

$$(A_n - B_n \cos k_n a) \sin k_n t + B_n \sin k_n a \cos k_n t = 0$$

$$(\epsilon_0 A_n + \epsilon B_n \cos k_n a) \cos k_n t + \epsilon B_n \sin k_n a \sin k_n t = 0$$

A similar analysis leads to

$$e(x) = \begin{cases} 2 / \{ \epsilon_2 t + (a - t) \}^{\frac{1}{2}} \left\{ \sum_{n \text{ odd}} \sin(n\pi x/a) + \sum_{n \text{ even}} (\epsilon_2)^{\frac{1}{2}} \sin(n\pi x/a) \right\} & 0 \leq x \leq t \\ 2 / \{ \epsilon_2 t + (a - t) \}^{\frac{1}{2}} \left\{ \sum_{n \text{ odd}} \sin \{ n\pi(a-x)/a \} + \sum_{n \text{ even}} (1/\epsilon_2)^{\frac{1}{2}} \sin n\pi(a-x)/a \right\} & t \leq x \leq a \end{cases} \quad (2.38b)$$

Before substituting these expansions, Eqs. (2.35) and (2.38a), into the integro-differential equations (2.29), let us analyse these. They can be written in a compact form as

$$\sum_m b_m G_{ms} = \sum_n a_n K_{ns} \quad (2.39a)$$

$$\sum_n a_n L_{ns} = \sum_m b_m M_{ms} \quad (2.39b)$$

where

$$G_{ms} = \int_0^a (\gamma^2 + k_0^2 \epsilon_r) \left\{ h_m''(x) + (\gamma^2 + k_0^2 \epsilon_r - p^2) h_m(x) \right\} h_s(x) dx -$$

$$\int_0^a k_0^2 \epsilon_r' h_m'(x) h_s(x) dx \quad (2.40a)$$

$$K_{ns} = i \omega \epsilon_0 \gamma^p \int_0^a \epsilon_r' e_n(x) h_s(x) dx \quad (2.40b)$$

$$L_{ns} = \int_0^a (\gamma^2 + k_0^2 \epsilon_r) \epsilon_r \left\{ e_n''(x) + (\gamma^2 + k_0^2 \epsilon_r - p^2) e_n(x) \right\} e_s(x) dx +$$

$$\gamma^2 \int_0^a \epsilon_r' e_n'(x) e_s(x) dx \quad (2.40c)$$

$$M_{ms} = j\omega \mu_0 \gamma p \int_0^a \epsilon_r' h_m(x) e_s(x) dx = (\mu_0/\epsilon_0) K_{sm} \quad (2.40d)$$

2.7 Evaluation of G_{ms}

In Eq. (2.39a), substitute h_m'' by $(p^2 - \gamma_{H,m}^2)h_m$ (Eq. (2.26a),

$$G_{ms} = \int_0^a (\gamma^2 + k_0^2 \epsilon_r) (\gamma^2 + k_0^2 \epsilon_r - \gamma_{H,m}^2) h_m(x) h_s(x) dx -$$

$$k_0^2 \int_0^a \epsilon_r' h_m'(x) h_s(x) dx \quad (2.41a)$$

where $\gamma_{H,m}^2 = (\pi/a)^2 + p^2$.

In Eq. (2.40a) there are integrals of the type

$$\int_0^a \epsilon_r^\alpha h_m(x) h_s(x) dx,$$

where α is an integer, positive value, such that $\alpha = 0, 1, 2, \dots$

The above-mentioned integral can be split into

$$1^\alpha \int_0^t h_m(x) h_s(x) dx + \epsilon_2^\alpha \int_t^a h_m(x) h_s(x) dx =$$

$$\epsilon_2^\alpha \int_{ms} + (1 - \epsilon_2^\alpha) \int_0^t h_m(x) h_s(x) dx$$

where $\epsilon_2 = \epsilon/\epsilon_0$ = relative dielectric constant of medium 2.

Using Eq. (2.26a), the following relationships can be established

$$(\gamma_{H,m}^2 - p^2) \int_0^t h_m(x) h_s(x) dx = - \int_0^t h_m''(x) h_s(x) dx =$$

$$= h_s(t) h_m'(t) + \int_0^t h_m'(x) h_s'(x) dx$$

and

$$(\gamma_{H,s}^2 - p^2) \int_0^t h_s(x) h_m(x) dx = - \int_0^t h_s''(x) h_m(x) dx =$$

$$- h_m(t) h_s'(t) + \int_0^t h_s'(x) h_m(x) dx$$

Finally, combining both relationships,

$$\int_0^t h_m(x) h_s(x) dx = \{ h_m(t) h_s'(t) - h_s(t) h_m'(t) \} / (\gamma_{H,m}^2 - \gamma_{H,s}^2)$$

The second term appearing in G_{ms}

$$\int_0^a \epsilon_r' h_m'(x) h_s(x) dx = (\epsilon_2 - 1) \int_0^a \delta(x-t) h_m'(x) h_s(x) dx =$$

$$(\epsilon_2 - 1) h_m'(t) h_s(t)$$

where $\epsilon_r = u(x) + (\epsilon_2 - 1) u(x-t) - \epsilon_2 u(x-a)$.

We must differentiate the two cases $m = s$ and $m \neq s$.

(i) $m = s$

$$G_{mm} = \int_0^a (\gamma^2 + k_o^2 \epsilon_r) (\gamma^2 + k_o^2 \epsilon_r - \gamma_{H,m}^2) h_m^2(x) dx -$$

$$k_o^2 \int_0^a \epsilon_r' h_m'(x) h_m(x) dx$$

where

$$h_m = (2/a)^{\frac{1}{2}} \cos(m\pi x/a)$$

There are integrals of the type

$$\int_0^t h_m^2(x) dx = (2/a) \int_0^t \cos^2(m\pi x/a) dx = \tau + \sin(2m\pi\tau)/2m\pi$$

where $\tau = t/a$.

There also integrals of the form

$$\int_0^a \epsilon_r^\alpha h_m^2(x) dx = I^\alpha \int_0^t h_m^2(x) dx + \epsilon_2^\alpha \int_0^a h_m^2(x) dx =$$

$$\epsilon_2^\alpha + (1 - \epsilon_2^\alpha) \int_0^t h_m^2 dx = \epsilon_2^\alpha + (1 - \epsilon_2^\alpha) \left\{ \tau + (\sin 2m\pi\tau)/2m\pi \right\} =$$

$$\tau + \epsilon_2^\alpha(1 - \tau) + (1 - \epsilon_2^\alpha) (\sin 2m\pi\tau)/2m\pi$$

And finally,

$$\int_0^a h_m^2(x) dx = 1$$

Thus,

$$\begin{aligned} G_{mm} &= \gamma^2 (\gamma^2 - \gamma_{H,m}^2) + k_o^2 (2\gamma^2 - \gamma_{H,m}^2) \{ \tau + \epsilon_2(1-\tau) + (1-\epsilon_2) \cdot \\ & (\sin 2m\pi\tau)/2m\pi \} + k_o^4 \{ \tau + \epsilon_2^2(1-\tau) + (1-\epsilon_2^2)(\sin 2m\pi\tau)/2m\pi \} + \\ & k_o^2 (\epsilon_2 - 1) (m\pi/a^2) \sin(2m\pi\tau) \\ G_{mm} &= \gamma^4 + \gamma^2 (2k_o^2 \{ \tau + \epsilon_2(1-\tau) + (1-\epsilon_2)(\sin 2m\pi\tau)/2m\pi \} - \gamma_{m,H}^2) + \\ & k_o^4 \{ \tau + \epsilon_2^2(1-\tau) + (1-\epsilon_2^2)(\sin 2m\pi\tau)/2m\pi \} + k_o^2 \{ (\epsilon_2-1)(m\pi/a^2) \sin(2m\pi\tau) - \\ & \gamma_{H,m}^2 [\tau + \epsilon_2(1-\tau) + (1-\epsilon_2)(\sin 2m\pi\tau)/2m\pi] \} \end{aligned} \quad (2.41b)$$

(ii) $m \neq s$

$$\begin{aligned} G_{ms} &= \int_0^a (\gamma^2 + k_o^2 \epsilon_r) (\gamma^2 + k_o^2 \epsilon_r - \gamma_{H,m}^2) h_m(x) h_s(x) dx - k_o^2 \int_0^a \epsilon_r' h_m'(x) h_s(x) dx = \\ & k_o^2 (2\gamma^2 - \gamma_{H,m}^2) \int_0^a \epsilon_r h_m(x) h_s(x) dx + k_o^4 \int_0^a \epsilon_r^2 h_m(x) h_s(x) dx - k_o^2 \int_0^a \epsilon_r' h_m'(x) h_s(x) dx = \\ & \frac{2}{\pi} (1-\epsilon_2) \left\{ \frac{m \cos(s\pi\tau) \sin(m\pi\tau) - s \cos(m\pi\tau) \sin(s\pi\tau)}{m^2 - s^2} \right\} k_o^2 \left\{ 2\gamma^2 - \gamma_{m,H}^2 + (1+\epsilon_2) k_o^2 \right\} \\ & - k_o^2 (1-\epsilon_2) \frac{2}{a} \frac{m\pi}{a} \sin(m\pi\tau) \cos(s\pi\tau) \\ G_{ms} &= \frac{2(1-\epsilon_2)}{\pi} k_o^2 \left\{ [2\gamma^2 - \gamma_{m,H}^2 + (1+\epsilon_2) k_o^2] \left[\frac{m \cos(s\pi\tau) \sin(m\pi\tau) - s \cos(m\pi\tau) \sin(s\pi\tau)}{m^2 - s^2} \right] \right. \\ & \left. - m(\pi/a)^2 \sin(m\pi\tau) \cos(s\pi\tau) \right\} \end{aligned} \quad (2.41c)$$

where again $\tau = t/a$.

2.8 Evaluation of L_{ns}

$$L_{ns} = \int_0^a \epsilon_r (\gamma^2 + k_o^2 \epsilon_r) (\gamma^2 + k_o^2 \epsilon_r - \gamma_{E,n}^2) e_n(x) e_s(x) dx - k_o^2 \int_0^a \epsilon_r \epsilon_r' e_n'(x) e_s(x) dx$$

(i) $n = s$

For this case,
$$\int_0^a \epsilon_r e_n^2(x) dx = 1$$

$$\begin{aligned} \int_0^a \epsilon_r^\alpha e_n^2(x) dx &= \int_0^t e_n^2(x) dx + \epsilon_2 \int_t^a e_n^2(x) dx = \int_0^t e_n^2(x) dx + \epsilon_2^{\alpha-1} (1 - \int_0^t e_n^2(x) dx) \\ &= \epsilon_2^{\alpha-1} + (1 - \epsilon_2^{\alpha-1}) \int_0^t e_n^2(x) dx = \epsilon_2^{\alpha-1} + (1 - \epsilon_2^{\alpha-1}) A_n^2 \int_0^t \sin^2(n\pi x/a) dx \end{aligned}$$

where

$$A_n = \begin{cases} 2 \left\{ (1 + \epsilon_2)a \right\}^{-\frac{1}{2}} & \text{for } n \text{ odd} \\ 2 \left\{ \epsilon_2 / (1 + \epsilon_2)a \right\}^{\frac{1}{2}} & \text{for } n \text{ even} \end{cases} \quad (2.42)$$

$$\begin{aligned} \int_0^a \epsilon_r^\alpha e_n^2(x) dx &= \epsilon_2^{\alpha-1} + (1 - \epsilon_2^{\alpha-1}) A_n^2 \left\{ (t/2) - \sin(2n\pi\tau)/(4n\pi/a) \right\} \\ &= \epsilon_2^{\alpha-1} + (1 - \epsilon_2^{\alpha-1}) A_n^2 (a/4n\pi) \left\{ 2n\pi\tau - \sin(2n\pi\tau) \right\} \\ &= \epsilon_2^{\alpha-1} + (1 - \epsilon_2^{\alpha-1}) \beta^2 (1 + \epsilon_2)^{-1} \left\{ 2n\pi\tau - \sin(2n\pi\tau) \right\} / n\pi \end{aligned}$$

$$\text{where } \beta = \begin{cases} 1 & \text{if } n \text{ odd} \\ (\epsilon_2)^{\frac{1}{2}} & \text{if } n \text{ even} \end{cases} \quad (2.43)$$

and $\alpha > 1$.

$$\begin{aligned} \int_0^a \epsilon_r'(\epsilon_r e_n'(x)) e_n(x) dx &= \int_0^a (\epsilon_2 - 1) \delta(x - t) (\epsilon_r e_n'(x)) e_n(x) dx \\ &= (\epsilon_2 - 1) (\epsilon_r e_n'(x)) \Big|_{x=t} \cdot e_n(x) \Big|_{x=t} = (n\pi/a) A_n^2 (\epsilon_2 - 1)^{\frac{1}{2}} \sin(2n\pi\tau) \end{aligned}$$

where A_n is still defined as in Eq. (2.42).

Finally,

$$\begin{aligned} L_{nn} &= \gamma^4 + k_0^2 (2\gamma^2 - \gamma_{n,E}^2) \left\{ \epsilon_2 + (1 - \epsilon_2)(1 + \epsilon_2)^{-1} \beta^2 (2n\pi\tau - \sin 2n\pi\tau) / n\pi \right\} + \\ & k_0^4 \left\{ \epsilon_2^2 + (1 - \epsilon_2^2)(1 + \epsilon_2)^{-1} \beta^2 (2n\pi\tau - \sin 2n\pi\tau) / n\pi \right\} + k_0^2 (n\pi/a) A_n^2 * \\ & (1 - \epsilon_2)^{\frac{1}{2}} \sin(2n\pi\tau) \end{aligned} \quad (2.44)$$

where β is defined in Eq. (2.43).

(ii) $n \neq s$

There are integrals of the type

$$\int_0^a \epsilon_r e_n(x) e_s(x) dx = (1 - \epsilon_2^{\alpha-1}) \int_0^t e_n(x) e_s(x) dx =$$

$$(1 - \epsilon_2^{\alpha-1}) A_n A_s \int_0^t \sin(n\pi x/a) \sin(s\pi x/a) dx =$$

$$(1 - \epsilon_2^{\alpha-1}) A_n A_s \frac{1}{2} \left\{ \frac{\sin(n-s)\pi\tau}{n-s} - \frac{\sin(n+s)\pi\tau}{n+s} \right\}$$

Thus,

$$L_{ns} = (1 - \epsilon_2^{\alpha-1}) A_n A_s \frac{1}{2} k_o^2 \left\{ (2\gamma^2 - \gamma_{n,E}^2) + (1 + \epsilon_2) k_o^2 \right\} \left\{ \frac{\sin(n-s)\pi\tau}{n-s} - \frac{\sin(n+s)\pi\tau}{n+s} \right\} +$$

$$k_o^2 A_n A_s (n\pi/2a) (1 - \epsilon_2) \sin(s\pi\tau) \cos(n\pi\tau) \quad (2.45)$$

2.9 Evaluation of M_{ms} and K_{ns}

$$M_{ms} = j\omega\mu_o \gamma p \int_0^a \epsilon_r h_m(x) e_s(x) dx = j\omega\mu_o \gamma p (\epsilon_2 - 1) h_m(t) e_s(t) =$$

$$M_{ms} = j\omega\mu_o \gamma p (\epsilon_2 - 1) (2/a)^{\frac{1}{2}} \cos(m\pi\tau) A_s \sin(s\pi\tau) \quad (2.46)$$

$$M_{mm} = j\omega\mu_o \gamma p (\epsilon_2 - 1) (2/a)^{\frac{1}{2}} A_m \frac{1}{2} \sin(2m\pi\tau) \quad (2.47)$$

$$K_{ms} = (\epsilon_o/\mu_o) M_{sm} = j\omega\epsilon_o \gamma p (\epsilon_2 - 1) (2/a)^{\frac{1}{2}} A_m \sin(m\pi\tau) \cos(s\pi\tau) \quad (2.48)$$

$$K_{mm} = j\omega\epsilon_o \gamma p (\epsilon_2 - 1) (2/a)^{\frac{1}{2}} A_m \frac{1}{2} \sin(2m\pi\tau) \quad (2.49)$$

2.10 Determination of γ .

Once the expressions for G_{ms} , L_{ns} , K_{ns} and M_{ms} are known, Eqs. (2.39) must be used in order to evaluate γ , for a given frequency (or a given value of k_o).

Thus

$$\sum_m b_m G_{ms} = \sum_n a_n K_{ns}$$

$$\sum_n a_n L_{ns} = \sum_m b_m M_{ms}$$

can be written, omitting suffixes, as

$$\begin{bmatrix} G & -K \\ M & -L \end{bmatrix} \begin{bmatrix} b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (2.50)$$

G is a square matrix, then

$$b = G^{-1} Ka$$

Hence

$$(MG^{-1}K-L)a = (0)$$

Thus

$$\det \begin{bmatrix} G & -K \\ M & -L \end{bmatrix} = 0 \quad \text{implies} \quad (2.51)$$

$$\det (MG^{-1}K-L) = 0 \quad (2.52)$$

Solutions for γ may be obtained either from Eq. (2.51) or Eq. (2.52).

The advantage of Eq. (2.52) is that this determinant is smaller than the determinant of Eq. (2.51), the disadvantage being on the cost of inverting G.

In the following Chapter some numerical examples will be shown and all the possible complications will be pointed out.

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NUMERICAL DETERMINATION OF PROPAGATION CONSTANTS IN A DIELECTRIC-
LOADED RECTANGULAR WAVEGUIDE FOR THE DOMINANT MODE

3.1 Structure configuration

The cross-section of the structure that will be analysed is shown in Fig. 3.1

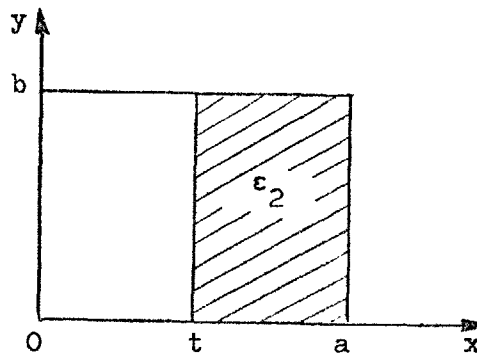


Fig. 3.1

where the thickness of the slab is $(a - t)$ and its relative dielectric constant is ϵ_2 .

It is known^{1,2} that for this configuration, the dominant mode is the H_{10} mode or longitudinal-section electric mode LSE_{10} , characterised by the electric field parallel to the slab and no variations of the fields along the dielectric-air interface.

In order to simplify the problem, the values of the dimensions and of the parameters will be as follows,

$$a = b = \pi$$

$$t = a/2 = \pi/2$$

$$\epsilon_2 = 2.45$$

Thus, the problem is really the determination of the propagation constants of a square waveguide filled with a dielectric slab, of relative dielectric constant equals to 2.45, and placed at one side of the waveguide wall.

3.2 Determination of an expression for evaluating χ .

Instead of using the general expressions derived in the previous chapter, it will be much easier, for this particular case, to derive a particular expression.

The starting point will be Eq. (2.25a), with $p = 0$, giving

$$(\gamma^2 + k_o^2 \epsilon_r) f''(x) + (\gamma^2 + k_o^2 \epsilon_r)^2 f(x) - \epsilon_r' k_o^2 f'(x) = 0 \quad (3.1)$$

where

$$H_z = f(x) \cos(py) \exp(j\omega t - \gamma z).$$

From Eq. (3.1), rearranging terms,

$$\frac{(\gamma^2 + k_o^2 \epsilon_r) f''(x) - \epsilon_r' k_o^2 f(x)}{(\gamma^2 + k_o^2 \epsilon_r)^2} = -f(x)$$

$$f(x) = - \frac{d}{dx} \left\{ \frac{f'(x)}{(\gamma^2 + k_o^2 \epsilon_r)} \right\} \quad (3.2)$$

Integrating both sides of Eq. (3.2) between 0 and x , and remembering that $f'(0) = f'(a) = 0$,

$$\int_0^x f(x) dx = - \frac{f'(x)}{\gamma^2 + k_o^2 \epsilon_r} \quad (3.3)$$

and

$$\int_0^a f(x) dx = 0 \quad (3.4)$$

Eq. (3.4) implies that if for $f(x)$ it is possible an expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n \cos \gamma_n x, \quad (3.5)$$

then $a_0 = 0$; thus

$$f(x) = \sum_{n=1}^{\infty} a_n \cos \gamma_n x$$

where $\gamma_n = n\pi/a = n$,

$$f(x) = \sum_{n=1}^{\infty} a_n \cos nx \quad (3.6)$$

and, for numerical evaluations, the $f(x)$ expansion is truncated such that

$$f(x) = \sum_{n=1}^N a_n \cos nx \quad (3.6b)$$

Rearranging Eq. (3.3),

$$f'(x) = -(\gamma^2 + k_o^2 \epsilon_r) \int_0^x f(x) dx$$

Using Eq. (3.6), and solving

$$\sum_{n=1}^{\infty} a_n \cdot n \sin nx = \sum_{n=1}^{\infty} (\gamma^2 + k_o^2 \epsilon_r) a_n \cdot (1/n) \sin nx \quad (3.7)$$

Now, multiply Eq. (3.7) by $\sin mx$ and integrate between 0 and $a = \pi$,

$$\sum_{n=1}^{\infty} a_n \cdot n \int_0^a \sin nx \sin mx dx = \sum_{n=1}^{\infty} (a_n/n) \int_0^a (\gamma^2 + k_o^2 \epsilon_r) \sin nx \sin mx \cdot dx \quad (3.8)$$

Due to the orthogonal condition

$$\int_0^a \sin nx \sin mx dx = \frac{a}{2} \delta_{mn}$$

it is possible to obtain the following relation

$$m a_m = (a_m/m) \gamma^2 + k_o^2 \sum_n (a_n/n) P_{mn}$$

where

$$P_{mn} = \frac{2}{a} \int_0^a \epsilon_r \sin nx \sin mx dx \quad (3.9)$$

Thus, it is possible to write a final expression, in matrix form, as follows

$$(\gamma^2 I - D^2 + k_o^2 D P D^{-1}) [a] = (0) \quad (3.10)$$

where:-

I is the identity matrix defined as that matrix with the main diagonal filled with 1's and the other elements equal to zero.

D is a diagonal matrix defined as $\text{diag} [1, 2, 3, \dots, n]$, although in general it could be defined as $\text{diag} [\gamma_{H,1}, \gamma_{H,2}, \dots, \gamma_{H,n}]$.

P is a general matrix with elements P_{mn} as defined by Eq. (3.9).

Eq. (3.10) is still susceptible to further simplification as follows

$$[\gamma^2 I - D^2 + k_o^2 D P D^{-1}] [a] = D [\gamma^2 I - D^2 + k_o^2 P] D^{-1} [a] = [0] \quad (3.11)$$

Solutions for this equation can be obtained from

$$\det(\gamma^2 I - D^2 + k_o^2 P) = 0 \quad (3.12)$$

Notice that Eq. (3.12) is of the form

$$\det(\lambda I - A) = 0$$

which is the standard equation for the solution of an eigenvalue problem, where the λ are the eigenvalues to be obtained. For our case, $\lambda = \gamma^2$ and $A = D^2 - k_o^2 P$. It is for this reason that Eq. (3.12) can be solved by computer using standard subroutines for eigenvalue problems. Before doing so, an expression for P_{mm} and P_{mn} will be obtained,

$$P_{mm} = (2/a) \int_0^a \epsilon_r \sin^2 mx \, dx = \tau + \epsilon_2(1 - \tau) + (\epsilon_r - 1)(\sin 2m\pi\tau)/(2m\pi)$$

$$\text{where } \tau = t/a = 1/2$$

$$P_{mn} = (2/a) \int_0^a \epsilon_r \sin mx \sin nx \, dx = (2/a)(1 - \epsilon_2) \int_0^t \sin mx \sin nx \, dx =$$

$$(1 - \epsilon_2/a) \left\{ \frac{\sin(m-n)t}{m-n} - \frac{\sin(m+n)t}{m+n} \right\}$$

and for our particular case

$$P_{mm} = 0.5 + 0.5 * 2.45 = 1.725 \quad (3.13)$$

$$P_{mn} = - (1.45/\pi) \left\{ \frac{\sin (m - n)\pi/2}{m - n} - \frac{\sin (m + n)\pi/2}{m + n} \right\} \quad (3.14)$$

The cutoff frequency can be evaluated by equating $\gamma = 0$ in Eq. (3.12), then k_c is obtained from

$$\begin{aligned} \det(k_c^2 P - D^2) &= 0 \\ \text{or} \quad \det(k_c^2 I - D^2 P^{-1}) &= 0 \end{aligned} \quad (3.15)$$

3.3 Properties of $(\gamma^2 I - D^2 + k_o^2 P)$

For the case we are considering, Eq. (3.12) satisfies the condition

$$M = (\gamma^2 I - D^2 + k_o^2 P) = M^T$$

where M^T is the transpose, if γ^2 is real. In fact it is so because M is a real and symmetric matrix. Also, because M is real and symmetric, it is also Hermitian^{3,4}.

It is well known that the eigenvalues of an Hermitian matrix are real, and eigenvectors belonging to different eigenvalues are orthogonal.

Obviously, as I and D^2 are always real and symmetrical, and since the eigenvalue problem is of the form

$$\det (\lambda I - A),$$

where $\lambda = \gamma^2$ and $A = D^2 - k_o^2 P$, the matrix A is determined. Since P is symmetric and real, A is a symmetric matrix, so that $a_{ij} = a_{ji}$ and $A = A^T$. If X_i and X_j are eigenvectors that belong to the eigenvalues λ_i and λ_j , where $\lambda_i \neq \lambda_j$, then X_i and X_j are orthogonal vectors, then their scalar product vanishes, or

$$X_i^T X_j = X_j^T X_i = 0 \quad (3.16)$$

By definition, X_i satisfies the eigenvector equation

$$AX_i = \lambda_i X_i \quad (3.17)$$

Taking the transpose of Eq. (3.17) and using the reversal law of transposed products, it follows

$$X_i^T A = \lambda_i X_i^T \quad (3.18)$$

since $A = A^T$. Postmultiplying Eq. (3.18) by X_j ,

$$X_i^T A X_j = \lambda_i X_i^T X_j \quad (3.19)$$

The vector X_j satisfies the eigenvector equation

$$A X_j = \lambda_j X_j \quad (3.20)$$

Subtracting Eq. (3.20) from Eq. (3.18)

$$0 = (\lambda_i - \lambda_j) X_i^T X_j \quad (3.21)$$

Since $\lambda_i \neq \lambda_j$, the following relation is obtained

$$X_i^T X_j = 0 \quad (3.22)$$

Then X_i and X_j are orthogonal.

As A is a real symmetric matrix, all of its eigenvalues are real.

If X is a typical eigenvector of A , then

$$A X = \lambda X \quad (3.23)$$

The complex conjugate of this equation is

$$A X^* = \lambda^* X^* \quad (3.24)$$

where the symbol $*$ denotes the complex conjugate. As A is real, is unchanged. The transpose of Eq. (3.24) is

$$X^{T*} A = \lambda^* X^{T*} \quad (3.25)$$

Postmultiplying Eq. (3.24) by X , the following relation is obtained

$$X^{T*} A X = \lambda^* X^{T*} X \quad (3.26)$$

Premultiplying Eq. (3.23) by X^{T*} ,

$$X^{T*} A X = \lambda X^{T*} X \quad (3.27)$$

Subtracting Eq. (3.27) from Eq. (3.26)

$$0 = (\lambda^* - \lambda) X^{T*} X \quad (3.28)$$

If X consists of the elements $x_1, x_2, x_3, \dots, x_n$, then $X^{T*}X$ is the sum of the square of the absolute values of the elements of X .

Hence, $X^{T*}X$ is a nonzero positive quantity and, therefore, satisfies Eq. (3.28),

$$\text{i.e. } \lambda^* = \lambda \quad (3.29)$$

Eq. (3.29) implies that λ is real. Since λ is a typical eigenvalue, all the eigenvalues of A are real numbers.

The eigenvalues of $C = B^{-1}AB$ are equal to the eigenvalues of A .

The characteristic polynomial of C is given by

$$(C - \lambda I) = (B^{-1}AB - \lambda I) = B^{-1}(A - \lambda I)B$$

Therefore the characteristic polynomial of C , $q(\lambda)$, is given by

$$q(\lambda) = \det(C - \lambda I) = (\det B^{-1}) \det (A - \lambda I) \det B$$

and since $\det B^{-1} = 1/\det(B)$,

$$q(\lambda) = \det(A - \lambda I) = p(\lambda)$$

where $p(\lambda)$ is the characteristic polynomial of A , then it is evident that the matrices A and C have the same characteristic equations

$$q(\lambda) = p(\lambda) = 0$$

Thus the eigenvalues of C and A are equal. The step from Eq. (3.10) to Eq. (3.11) is then justified, and all the eigenvalues obtained from Eq. (3.12) are real.

3.4 Program description

In this section information is given which should enable the reader to apply the programme included in the Appendix to specific configurations of interest. Examples of its use are given in Section 3.4. The programme is written in Fortran IV and has been applied with a CDC 6400 computer.

3.4.1 Data input

The values of the parameters are given as separate instructions. E_1 and E_2 are the relative dielectric constants in medium 1 (air) and 2 respectively. A is the width of the waveguide and B

is the height. The value of B is not necessary for the evaluation of the propagation constant of the dominant mode, but it will be necessary when, as shown in Section 3.4, the cutoff frequencies of higher order modes H_{no} are to be calculated. T is the width of guide filled with air.

The next data statements are:-

The values of the incremental step for k_0 , DELKO; the maximum number of terms used in the expansion, (i.e. the value of N in Eq. (3.66)) and indicated as KJ; finally, the number of points on the k_0 axis (or different frequencies) where the propagation constant should be evaluated.

3.4.2 Dimensions

The required dimensions of all the matrices and arrays are related to KJ, the maximum number of terms used in the expansion for the field. For the program annexed in the Appendix, it was supposed that the maximum value for KJ is 10 and the matrices are accordingly dimensioned.

Table 1 lists the matrices that must be dimensioned and indicates how the dimensions relate to KJ.

Table 1

a. Main Program

<u>Matrix</u>	<u>Dimension</u>
P, DI, Q, DD	KJ, KJ
DET, Z, D, R, PIPI, ER, EI, ITS	KJ x KJ
BETA	2KJ

b. Subroutine MINV

<u>Matrix</u>	
A, L, M	KJ x KJ

3.4.3 Description of the program

As was mentioned earlier, the program will be used to evaluate cutoff frequencies and propagation constants for different frequencies. As the value of the cutoff frequency is used for determining at which frequencies the propagation constant is to be evaluated, the first part of the programme will calculate the cutoff frequency of the dominant mode of the particular configuration, according to Eq. (3.15). The determination of the cutoff frequency, using different terms in the expression, or in other words, expanding each time the matrices D^2 and P^{-1} , is done inside the DO loop 50, where KON indicates the number of terms used in each expansion.

Note that in this DO loop, the matrices D^2 and P^{-1} in Eq. (3.15) correspond to DI and Q, respectively, in the loop. In DO 21, the elements of the matrices D^2 and P are obtained. In DO 22 the matrix P is stored columnwise in a matrix Z, and is then inverted using the subroutine MINV. This subroutine is a standard one in a subroutine Package⁵, and the reader is referred to the reference for more details. In DO 33 the inverted column matrix Z is transformed again to a square matrix and stored in Q.

DO 24 is used to multiply the two matrices DI and Q. Hence subroutine FO2AFF is used to evaluate the eigenvalues of the product matrix. The general declarative statement of this subroutine is FO2AFF (A, IA, N, ER, EI, INT, IFAIL) where

- A is a two dimensional array $N \times N$ that contains the matrix which eigenvalues are to be calculated. The original matrix is destroyed in the computation: in our case DD is equal to A.
- IA is an integer which states the first dimension of A as declared (in our case, $IA = 10$ because the dimension of DD is (10,10)).

- N is an integer, equal to the order of the matrix. It is a varying value given, in the programme, by KON.
- ER is an array of dimension N, in which the real part of the eigenvalues are stored.
- EI is an array equal to ER, in which the imaginary parts of the eigenvalues are stored.

INT(N) is an array used as working space.

If two simultaneous eigenvalues are found, the first will be indicated with a positive sign and the second with a negative one.

IFAIL (LZ in the programme) is an integer that can be either 0 or 1. On the entry stage, a 0 means a failure of entry; a 1, means error detected, but no message is written. On the exit, a 0 means success and a 1 means that no eigenvalue was found after thirty iterations.

In the following steps, the smallest eigenvalue is stored in PIPI(1), and this is the value of the cutoff frequency for the dominant mode in the waveguide analysed.

Different values of cutoff frequency, each corresponding to a given number of terms in the expansion, are printed.

The best approximation to the cutoff frequency is stored in VKIN. After this all the digits after the first significant digit are eliminated and this number is stored in VKO (which stands for value of k_0). For instance, if the cutoff frequency is 0.74596639, the number stored in VKO will be 0.7000. Adding to this number the value of the incremental step stored in DELKO, the first value of k_0 , at which the propagation constant is to be calculated, is obtained.

The propagation constants at different frequencies and using a varying number of terms in the expansion are evaluated in the DO loop 1.

The first value of k_0 is fixed and stored in VKO, and printed. Then, in DO 2, the elements of the matrices D^2 and P, that appear in Eq. (3.12), are evaluated. The resultant matrix corresponding to $(-D^2 + k_0^2 P)$ is stored in a matrix called P. This matrix is then stored columnwise into the array D, and its eigenvalues are then calculated by means of the subroutine EIGEN⁶. An improved version of the subroutine that appears in the reference has been used. The calculated eigenvalues are stored in the array BETA using DO 3, and the number of terms used in the expansion and the corresponding value of β^2 are printed. Likewise, the associated value of the eigenvector, stored in the array R, is printed.

The process is repeated, for the same value of k_0 , increasing each time the number of terms in the expansion. When the maximum number of terms is reached, a new value of k_0 is fixed and the process is repeated.

3.5 Examples

3.5.1 $a = \pi, \quad t = a/2, \quad \epsilon_2 = 2.45$

The output of the program in the Appendix corresponds to this case. The results are compared with analytical results obtained from solving the corresponding transcendental equation for this case and with the curves obtained from Harrington⁷. Table 2 shows the results for the cutoff frequencies and the differences between them and the analytical solution. In Table 3 are the results for the analytical value of β for different values of k_0 and the corresponding values of β calculated with the programme, using 1 to 5 terms in the expansions. Differences between the analytical value and the values using 5 terms in the expansion are also shown.

Table 2

Cutoff Frequencies

Analytical Solution	$k_c = 0.745925$	Differences
Number of terms in the expansion		
1	0.761387	0.015462
2	0.74650705	0.000582
3	0.74648319	0.0005582
4	0.74604502	0.0001202
5	0.74604417	0.00011917
6	0.74596639	0.0000414

From Table 2 and 3 it is evident that a good approximation is rapidly achieved with a few terms. It can also be seen that as the frequency increases more terms should be used in order to get an approximation as good as the one obtained at lower frequencies.

To give an idea of the computing time, on the CDC 6400 computer, 5 different values of cutoff frequency and the evaluation of the propagation constant at 10 different frequencies, and at each frequency evaluating 5 different values of the propagation constant (taking 1 to 5 modes), takes 3.4 seconds. If the programme is not listed and the propagation constants are evaluated at 12 different frequencies, the time is about 3.5 seconds.

Table 3
Propagation Constants

Value of k_0	β						Difference between columns 2 and 7
	Analytical	1 term	2 terms	3 terms	4 terms	5 terms	
0.8	0.39664	0.3224903	0.39350427	0.393649	0.3956722	0.3956774	0.0009626
0.9	0.693827	0.6302777	0.691308	0.691513	0.6933226	0.69333	0.000497
1.0	0.92296	0.8514693	0.9199623	0.9203058	0.922322	0.922335	0.000128
1.1	1.12695	1.042713	1.123346	1.1239171	1.1262505	1.126271	0.000679
1.2	1.31793	1.2181954	1.3138542	1.3147694	1.3174727	1.3175064	0.000694
1.3	1.5023	1.3839256	1.4968404	1.4982475	1.501347	1.5014	0.0009
1.4	1.68175	1.543049	1.6750539	1.6771284	1.680631	1.68071	0.00104
1.5	1.85817	1.6974245	1.850036	1.8529766	1.8568734	1.8569857	0.001184
1.6	2.03255	1.8482424	2.0226976	2.026718	2.030988	2.0311436	0.001406
1.7	2.20526	1.996309	2.1935933	2.1989173	2.2035287	2.2037385	0.0015215

3.5.2 Waveguide completely empty ($t = a$) and completely filled ($t = 0$)

Both cases can be solved exactly, so exact results can be used as a comparison. In fact, Eq. (3.12) for these cases, reduces the same exact equation and obviously the results obtained match perfectly with the analytic ones.

Table 4 and 5 show the results obtained.

Table 4
Cutoff Frequency and Propagation Constants
of the Empty Guide

k_o	β
1.00	0
1.10	0.4582576
1.20	0.6633249
1.30	0.8306624
1.40	0.9797959
1.50	1.1180339
1.60	1.2489996
1.70	1.3747727
1.80	1.496663
1.90	1.6155494
2.00	1.7320508
2.10	1.8466185
2.20	1.9595918

Computer time for evaluating Table 4 : 2.9 sec.

Table 5

Cutoff Frequency and Propagation Constants
of a Guide Filled with a Dielectric of Relative
Dielectric Constant = 2.45

k_o	β
0.63887656	0.
0.7	0.4477723
0.8	0.75365774
0.9	0.99221973
1.0	1.20415945
1.1	1.40160622
1.2	1.58996855
1.3	1.77214559
1.4	1.94987179
1.5	2.12426457
1.6	2.2960836
1.7	2.46586698
1.8	2.63400835

Computer time for evaluating Table 5 : 2.8 sec.

3.5.3 $t = a/4$ and $t = 3a/4$

In Tables 6 and 7 results are found for the case
when the slab occupies $\frac{3}{4}$ of the width of the guide and $\frac{1}{4}$, respectively.

Table 6

Cutoff Frequency and Propagation Constants
of a Guide Filled with a Slab of Relative
Dielectric Constant = 2.45 and Occupying
 $\frac{3}{4}$ of the width

k_0	β
0.6553772	0.
0.7	0.3760836
0.8	0.7020235
0.9	0.94449508
1.0	1.15733046
1.1	1.3546684
1.2	1.542547
1.3	1.72412516
1.4	1.90125892
1.5	2.07512597
1.6	2.24651478
1.7	2.4159749
1.8	2.58390168

Number of terms used in the expansion = 5 .

Computer time for evaluating results in Table 6 = 3.64 sec.

Table 7

Cutoff Frequencies and Propagation Constants
of a Guide Filled with a Slab of Relative
Constant = 2.45 and Occupying $\frac{1}{4}$ of the width

k_o	β
0.92989087	0.
1.0	0.4012898
1.1	0.6433766
1.2	0.83392935
1.3	1.00367868
1.4	1.1627602
1.5	1.3161177
1.6	1.466699
1.7	1.6164902
1.8	1.76690976
1.9	1.91896154
2.0	2.0732875
2.1	2.23019568

Number of terms used in the expansion = 5.

Computer time for evaluating results in Table 7 = 3.72 sec.

All the results of Sections 3.5.1 to 3.5.3 are plotted in Fig. 3.2.

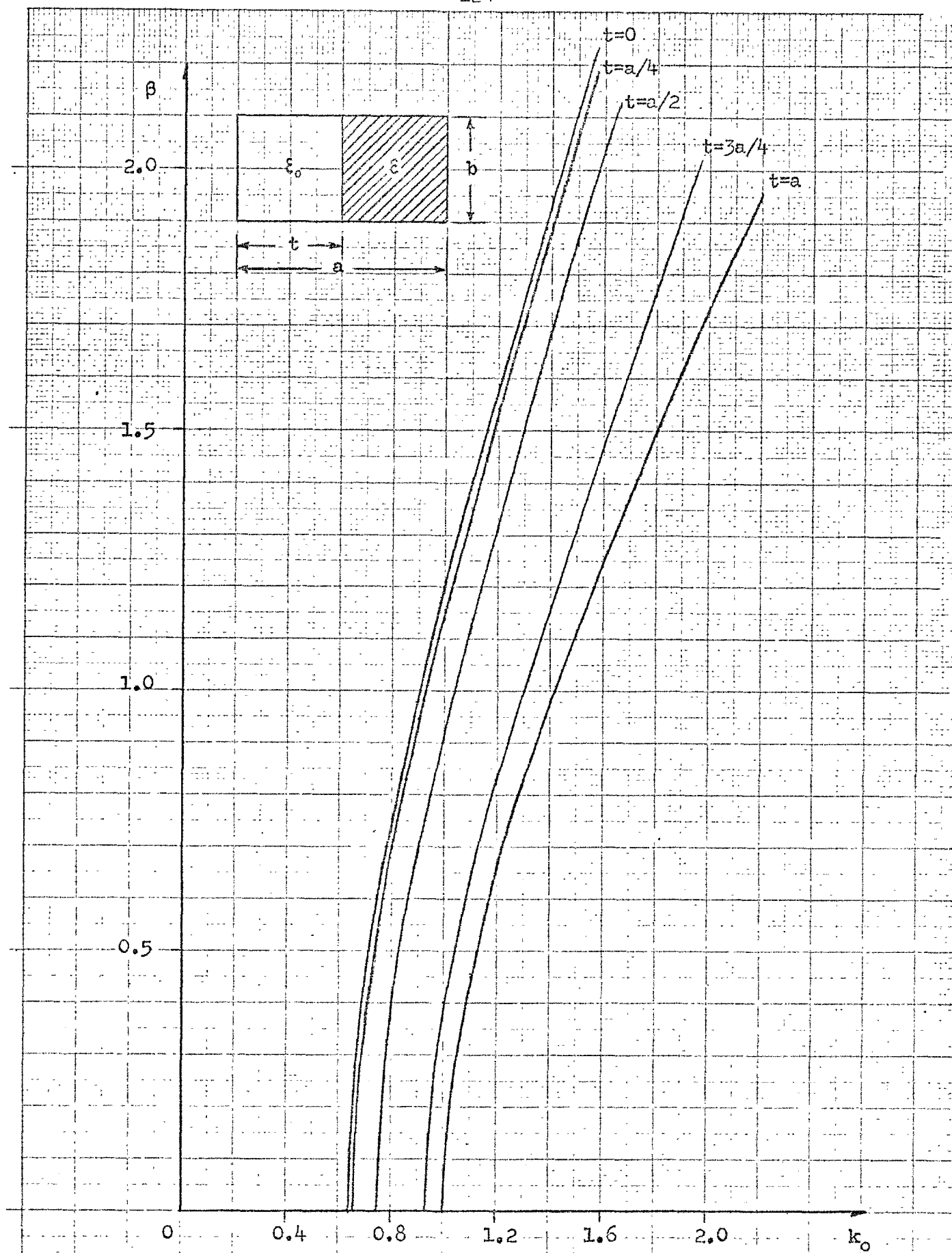


Fig.3.2. Propagation constant for a rectangular waveguide partially filled with dielectric, $\epsilon = 2.45 \epsilon_0$.

3.5.4 $\epsilon_2 = 3, t = a/2$

This case was analysed in order to compare the results obtained by this method with the results obtained by Casey⁸. The data read from Casey's graph are very approximate because of the size of it.

Table . 8

Casey's method		This method	
k_o	β	k_o	β
0.685	0	0.6881418	0
0.7	0.185	0.7	0.19625
0.8	0.62	0.8	0.61538
1.0	1.12	1.0	1.110073
1.2	1.53	1.2	1.527906
1.4	1.94	1.4	1.923186

3.5.5 Cutoff frequencies for higher order LSE modes .

For $\gamma = 0$ ($\beta = 0$), the expressions for K and M in Eq. (2.50) become zero, such that G and L are zero independently. This characteristic allows us to evaluate the cutoff frequencies for higher order modes LSE_{nl}.

Eq. (3.15) can still be used, but now the matrix D^2 will be affected by the quantity $p = l\pi/b$; so,

$$D^2 = \text{diag}(d_1^2, d_2^2, \dots, d_n^2)$$

where $d_n^2 = (n\pi/a)^2 + p^2$, where $n = 1, 2, \dots$

The case that will be analysed is that of a square waveguide ($a = b$) half-filled with a dielectric slab of relative dielectric constant $\epsilon_2 = 2.45$. For this case, if $a = \pi$ and we are interested in the

cutoff frequency of the mode LSE_{11} .

$$d_n^2 = n^2 + 1, \text{ such that } d_1^2 = 2, d_2^2 = 5, \text{ etc.}$$

The results obtained are in Table 9.

Table 9
Cutoff frequency for LSE_{11} mode

	k_0
Analytic solution	1.0369
1 term	1.07676
2 terms	1.03834
3 terms	1.0381366
4 terms	1.0371013
5 terms	1.0370939
6 terms	1.0369082

3.6 Amplitudes of the modes

For the case analysed in Section 3.5.1, when $t = a/2$, the eigenvalues corresponding to β^2 were evaluated and these values give information about the amplitude of the expansion modes, according to Eq. (3.11). In other words, it is possible to evaluate the a_n 's of the expression

$$f(x) = \sum_{n=1}^N a_n \cos nx.$$

All eigenvectors are automatically given by the computer programme if wanted. For a particular frequency ($k_0 = 1$) the following eigenvalues were obtained.

Table 10

Number of terms	1	2	3	4	5
Eigen-vectors	1.0	0.981113	0.9808833	0.9808384	0.9808326
		-0.1934354	-0.1943963	-0.1940336	-0.19405835
			0.0088376	0.0077985	0.00779793
				0.0156905	0.01571682
					-0.00097442
Amplitude of $f(0)$	1.0	0.7876947	0.79532463	0.81029384	0.80931457

It may be seen that the most important modes are the first two. In Fig. 3.3 the total amplitude of $f(x)$ is plotted versus the number of terms in the expansion. It should be noted how the total amplitude tends toward the value $(2/\pi)^{\frac{1}{2}} = 0.797885$. In the same figure the differences between the amplitude for each n and the value $(2/\pi)^{\frac{1}{2}}$ are tabulated.

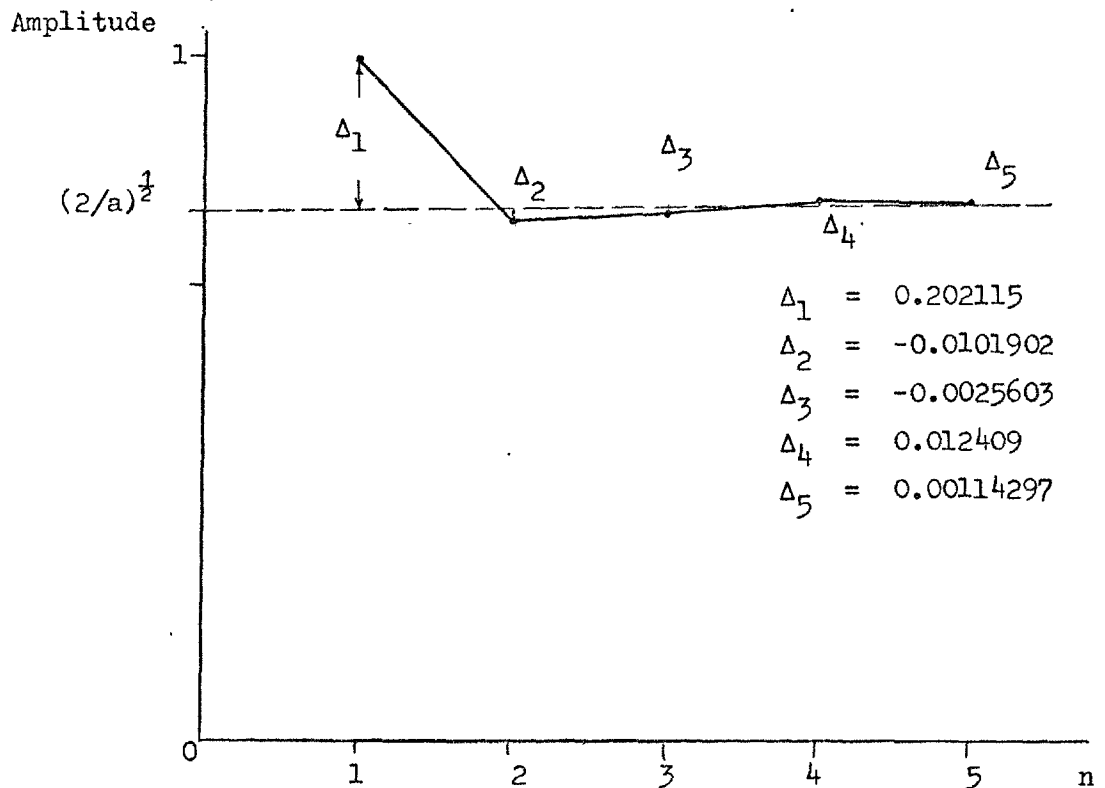


Fig. 3.3 Amplitude of $f(x)$ as function of the number of terms in the expansion

3.7 Other eigenvalues

In the solution of the eigenvalue problem, if N terms are used in the expansion, we will obtain N values for β^2 . For instance, for the case analysed in Section 3.5.1., with $N = 5$ and $k_0 = 1$, the 5 eigenvalues are

0.85070155
-2.3686235
-7.2630827
-14.299926
-23.294070

The reader will notice that as these values correspond to β^2 , only the first one will correspond to a propagation constant and the other four to an attenuation constant. There is then no possibility of confusion.

Unfortunately, this is not always the case. In fact, for some values of k_0 and N , it is possible to get more than a possible propagation constant ($\beta > 0$). Obviously, only one of these values will correspond to the propagation constant of the mode under analysis and the other values could correspond to higher order modes or to extraneous roots. For instance, for the same case treated in Section 3.5.1, with $N = 5$ and $k_0 = 1.7$, the 5 values for β^2 are,

4.8564633
0.40738049
-3.9498531
-11.213766
-20.173975

Although this kind of ambiguity will not arise in the output of the programme, where only the propagation constant is given, it

has been mentioned because the occurrence of multivalued propagation constants has been analysed by Rhodes⁹. He showed that for the same frequencies there can be different values of γ (real or imaginary).

All these facts show how much work must still be done on this subject and how many questions are still not answered. For instance, if from a given configuration a set of values for γ is obtained, is it possible, given a set of values for γ , at a fixed frequency, to obtain a correspondingly unique configuration? Although this analysis is out of the scope of this work, it was considered noteworthy to mention it.

3.8 Conclusions

A new method and a user-oriented computer programme have been presented and described for calculating propagation constants for the dominant mode (TE_{on}) in dielectric-slab loaded rectangular waveguides. Detailed instructions for using this programme were given and an example is included to illustrate its use. Data for other examples are also given, and all the results show a good agreement with the analytic results. The evaluation of the amplitudes of the expanded modes is also discussed, as well as the implications of obtaining different eigenvalues as solutions of the problem.

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APPENDIX

PROGRAM: PUT (INPUT, OUTPUT, TAPE1=INPUT, TAPE3=OUTPUT)
DETERMINATION OF PROPAGATION CONSTANTS IN RECTANGULAR WAVEGUIDES
FILLED WITH DIELECTRIC AS IN FIGURE

```

+ *****
| *          *          *          *          *          *
| *          *          *          *          *          *
| *          *          *          *          *          *
| *          *          *          *          *          *
| *          *          *          *          *          *
| *          *          *          *          *          *
| *          *          *          *          *          *
| *          *          *          *          *          *
| *          *          *          *          *          *
| *          *          *          *          *          *
| *          *          *          *          *          *
| *          *          *          *          *          *
+ *****
+ .....A.....+

```

WHERE E1 AND E2 ARE THE RELATIVE DIELECTRIC CONSTANTS, A IS THE HEIGHT
AND A THE WIDTH.

```

DIMENSION P(10,10), ET(100), EI(100), ITS(100)
DIMENSION P(10,10), ET(100), EI(100), ITS(100), D(100)
DIMENSION D(10,10), DT(10,10), Z(100)
DIMENSION DET(100)
DO 100 I=1,10

```

INPUT PARAMETERS

```

E1=1.0
E2=2.45
A=PT
T=0.5*A

```

```

WRITE(3,777)E1,E2,A,T
777 FORMAT(1H1,5X,*E1=*,F4.2,2X,*E2=*,F4.2,2X,*A=*,F10.8,2X,*T=*,F10.
1)

```

TAU=T/A
DETERMINATION OF CUTOFF FREQUENCY
INITIAL VALUE FOR KO

DO 21 KS TO EVALUATE $DET((KO**2))-DT**2*O**(-1)$, WHERE T IS THE
UNIT MATRIX, DI THE DIAGONAL MATRIX AND O IS DEFINED IN THE TEXT AS

```

DO 21 I=1,KON
DO 21 J=1,KON
Z=0
ZM=0

```

```

G1=7M*DT//
GM=7M*DT//A

```

IF (E1-E2) 41,41,40

41 O(I,J)=E1*TAU+E2*(1.0-TAU)+(E2-E1)*SIN(2.0*7M*PI*TAU)/(2.0*7M*PI)

O(I,I)=GM*GM

O(I,I)=0.0

40 O(I,J)=((E1-E2)/A)*(SIN((GM-GN)*T)/(GM-GN)-SIN((GM+GN)*T)/(GM+GN))

DT(I,J)=0.0

21 CONTINUE

KL=KON*KON

L=0

DO 22 I=1,KON

DO 22 J=1,KON

L=L+1

22 Z(L)=O(J,I)
CALL MINV(Z,KON,DET)

TJH=0

DO 33 I=1,KON

DO 33 J=1,KON

TJH=TJH+1

37 O(T,J)=Z(TJH)
DO LOOP 24 IS USED TO PERFORM THE MULTIPLICATION OF TWO MATRICES

DO 24 I=1,KON

DO 24 J=1,KON

SUM=0.0

DO 23 K=1,KON

23 SUM=SUM+O(I,K)*O(K,J)

24 O(I,J)=SUM

LZ=L

```

C SUBROUTINE F2AFF IS TO EVALUATE THE EIGENVALUES OF A REAL UNSYM-
C METRIC, IN THIS CASE OF SQUARE TIMES Q INVERSE
47. CALL F2AFF(10,11,KCN,E2,E1,108,L7)
48. IF(L2.NE.0)GO TO 672
49. DO 713 I=1,KCN
50. IF(E1(I).NE.0.)GO TO 713
51. - TS LOCATING POINT EQUALITY TO BE EXPECTED
52. PIP1(I)=0.1 (E(I))
53. IF(I.LE.1)GO TO 713
54. IF(PIF(I).LE.PIP1(I))GO TO 713
55. HOLD=PIF(I)
56. PIP1(I)=PIPT(I)
57. PIP1(I)=HOLD
58. 713 CONTINUE
59. WRITE(3,500)PIPT(1)
60. 500 FORMAT(1HJ,10X,*OUTOFF F= QUANTITY=*,F12.8)
61. CONTINUE
62. VKIN=PIPT(1)
63. IF(VKIN-1.0)60,61,61
64. 61 VKIN=VKIN*10.0
65. KK=VKIN
66. GO TO 62
67. 61 KK=VKIN*10.0
68. 62 VKO=FL012(H1)/10.0
69. INITIALIST P FOR KO
70. INITIALIST L STEP FOR KO
71. DELKO=0.1
72. C MAXIMUM NUMBER OF TERMS USED IN THE EXPANSTION
73. *****
60. C NUMBER OF TERMS USED IN FIRST EXPANSTION
70. C *****
71. C DIFF. POINT VALUES OF KO USED
72. TETKO=12
73. DO 1 TJ=1,TEIKO
74. KK=1
75. VKO=VKO+DELKO
76. VKOC=VKO*VKO
77. KK=1+(3,300)VKO
78. 300 FORM(1HJ,10X,*KO=*,F5.2)
79. 11 KK=KK+1
80. NPJ=KK+1
81. C EVALUATION OF MATRICES C SQUARE AND P
82. P(2)=K A, KK
83. P(2)=K A, KK
84. Z=0
85. ZH=0
86. G=7*PT//
87. 3 G=ZM*PI/A
88. IF(P-N)1,30,10
89. 1 G(1,N)=(E1-E2)/A*(SIN((GM-GN)*T)/(GM-GN)-SIN((GM+GN)*T)/(GM+GN))
90. 30 CONTINUE
91. L=0
92. DO 6 I=1,KK
93. DO 6 J=1,KK
94. L=L+1
95. 6 D(L)=P(J,I)
96. KKS=KK*KK
97. C EVALUATION OF THE EIGENVALUES
98. CALL EIGEN(0,7,KK,KKS,0)
99. JJ=1
100. DO 3 IT=1,KK
101. 3 ETA(II)=G(JJ)
102. JJ=JJ+1
103. WRITE(3,701)
104. 701 FORM(1HJ,20X,*NUMBER OF TERMS*,5X,* ETA SQUARE *)
105. 303 FORM(1HJ,10X,12,15X,14.8)
106. IF(KK.LT.1)GO TO 47
C. UTTION - S1 TEMENT IS GO TO 47
107. 47 WRITE(3,706)
108. 306 FORM(1HJ,20X,*EIGENVECTORS ASSOCIATED TO BETA SQUARE*)
109. WRITE(3,705)PI(1)
110. 705 FORM(1HJ,20X,F14.6)
111. IF(KK.EQ.KJ)GO TO 1
112. GO TO C. 11
113. 1 CONTINUE
114. 673 CONTINUE
COMMENT - CONTINUE STATEMENT IS NOT DO TERMINATOR
115. STOP
116. END
C A IS THE INPUT MATRIX,DESTROYED IN THE COMPUTATION AND REPLACED BY
C RESULT AT INVERSE.
C M2 IS THE DIAGONAL OF MATRIX A, D IS THE RESULTANT DETERMINANT,
C L AND M ARE WORKING VECTORS OF LENGTH N2

```

```

117.      SUBROUTINE MINV(A,N7,P)
118.      DIMENSION L(100),M(100)
119.      N=N7
120.      C=1.0
121.      NK=N
122.      DO 10 K=1,N
123.      NK=NK+1
124.      L(K)=K
125.      M(K)=K
126.      KK=NK+K
127.      TGA=A(KK)
128.      DO 20 J=K,N
129.      TZ=N*(J-1)
130.      DO 20 T=K,N
131.      JZY=TZ+T
132.      IF (A(KK)-A(S(A(JZY))) 15,20,20
133.      15 TGA=A(JZY)
134.      L(K)=J
135.      M(K)=J
136.      20 CONTINUE

```

```

137      C INTERCHANGE ROWS
138.      J=L(K)
139.      25 IF (J-K) 35,35,25
140.      KT=K-N
141.      DO 30 I=1,N
142.      KI=KT+I
143.      HOLD=A(KI)
144.      JJ=KT-K+J
145.      A(KI)=A(JI)
146.      A(JI)=HOLD
147.      C INTERCHANGE COLUMNS
148.      35 T=M(K)
149.      IF (T-K) 45,45,38
150.      30 JP=N*(T-1)
151.      DO 40 J=1,N
152.      JK=NK+J
153.      JT=JP+J
154.      HOLD=A(JK)
155.      A(JK)=A(JT)
156.      A(JT)=HOLD
157.      C DIVIDE COLUMN BY MINUS PIVOT VALUE OF PIVOT ELEMENT CONTAINED IN TGA
158.      45 IF (TGA) 46,46,48
159.      46 D=1.0
160.      RETURN
161.      48 DO 55 I=1,N
162.      IF (M-K) 57,55,57
163.      5 IK=NK+I
164.      A(IK)=A(IK)/(-TGA)
165.      55 CONTINUE
166.      C ROWS BY PIVOT
167.      DO 65 I=1,N
168.      JK=NK+I
169.      HOLD=A(IK)
170.      JZY=T-N
171.      DO 65 J=1,N
172.      JZY=JZY+M
173.      IF (T-K) 67,65,67
174.      67 IF (J-K) 67,65,62
175.      60 KJ=JZY-T+K
176.      A(JZY)=HOLD*A(KJ)+A(JZY)
177.      65 CONTINUE
178.      C ROWS BY PIVOT
179.      KJ=K-N
180.      DO 75 J=1,N
181.      KJ=KJ+N
182.      IF (J-K) 77,75,77
183.      7 A(KJ)=A(KJ)/TGA
184.      75 CONTINUE
185.      D=D*TGA
186.      C PLACE PIVOT BY GETTING L

```

```

181.      A7KK)=1./BTGA
182.      CONTINUE
183.      C      FINAL ROW AND COLUMN INTERCHANGE
184.      K=N
185.      150 K=(K-1)
186.      IF(K)150,150,195
187.      155 I=L(K)
188.      IF(I-K)120,120,190
189.      120 JQ=N*(K-1)
190.      JP=N*(I-1)
191.      DO 110 J=1,N
192.      JK=JQ+J
193.      HOLD=A(JK)
194.      JT=JP+J
195.      A(JK)=A(JT)
196.      11 A(JT)=HOLD
197.      120 J=J+1
198.      IF(J-K)190,190,125
199.      125 KT=K-N
200.      DO 130 T=1,N
201.      KI=KI+N
202.      HOLD=A(KT)
203.      JT=KI-K+J
204.      A(KT)=A(JT)
205.      17 A(JT)=HOLD
206.      GO TO 190
207.      155 RETURN
208.      END

```

C A-ORIGINAL MATRIX, REAL AND SYMMETRIC, DESTROYED IN COMPUTATION
 C RESULTANT EIGENVALUES DEVELOPED IN DIAGONAL OF MATRIX A IN
 C DESCENDING ORDER.
 C E-RESULTANT MATRIX OF EIGENVECTOR (COLUMNWISE, SAME SEQUENCE AS
 C EIGENVALUES).
 C N-ORDER OF MATRICES A AND R.
 C MV-INPUT CODE
 C 1-COMPUTE EIGENVALUES AND EIGENVECTOR, 1 ONLY EIGENVALUES
 C ALSO-WORKING SPACE DIMENSION EQUAL TO N SQUARE.

```

209.      SUBROUTINE EIGEN(A,R,NL,NCO,MV)
210.      DIMENSION A(NCO),R(NCO)
211.      N=NL
212.      C TRANSFER OF A TO CORRECT UPPER TRIANGULAR FORM.
213.      5 DO 5 T=1,N
214.      DO 5 J=1,I
215.      MT=MI+1
216.      MARK=(I-1)*N+J
217.      A(MT)=A(MARK)
218.      C GENERATE IDENTITY MATRIX
219.      IF(MV-1)10,25,10
220.      10 I=N
221.      DO 20 J=1,N
222.      IO=IO+1
223.      DO 20 I=1,N
224.      IU=IO+1
225.      A(IU)=1
226.      IF(I-J)20,15,20
227.      15 A(IJ)=1.0
228.      20 CONTINUE
229.      C COMPUTE INITIAL AND FINAL NORMS (ANORM AND ANORMX)
230.      25 ANORM=0.
231.      DO 35 T=1,N
232.      DO 35 J=1,N
233.      IF(T-J)30,35,70
234.      30 JA=I+(J-J-J)/2
235.      ANORM=ANORM+A(JA)*A(JA)
236.      35 CONTINUE
237.      IF(ANORM)165,165,40
238.      40 ANORM=1./414*SQRT(ANORM)
239.      ANORMX=ANORM*1.2E-6/ELCAT(N)
240.      C INITIALIZE INDICATORS AND COMPUTE THRESHOLD. THE

```

```

237.      INDEX
238.      THRESHOLD
239.      45 THRESH=THRESH/FLOAT(N)
240.      50 L=L+1
241.      55 M=L+1
242.      C 60 IF (M-N) 135,140,135
243.      60 KQ=(M*N-M)/2
244.      LQ=(L*L-L)/2
245.      62 IF (ABS(A(LM))-THRESH) 135,65,65
246.      65 INQ=1
247.      LL=L+LQ
248.      MM=M+MQ
249.      X=A(LM)*A(LL)-A(MM)
250.      63 Y=-A(LM)/SQRT(A(LM)*A(LM)+X*X)
251.      IF (X) 70,75,75
252.      70 Y=-Y
253.      75 SINX=Y/SQRT(2.0*(1.0+(SQRT(1.0-Y*Y))))
254.      SINX2=SINX*SINX
255.      78 COSX=SQRT(1.0-SINX2)
256.      COSX2=COSX*COSX
257.      SINCOS=SINX*COSX
258.      C 80 TEST FOR LAST COLUMN
259.      ILQ=A*(L-1)
260.      JQ=N*(M-1)
261.      125 I=1,N
262.      10=(I+J-1)/2
263.      IF (I-L) 80,115,80
264.      80 IF (J-M) 85,115,80
265.      85 I=I+MQ
266.      90 GO TO 95
267.      90 IM=J+MQ
268.      95 IF (T-L) 100,105,105
269.      100 IL=I+LQ
270.      105 IL=L+IQ
271.      110 X=A(IL)*COSX-A(IM)*SINX
272.      A(IM)=A(IL)*SINX+A(IM)*COSX
273.      A(IL)=X
274.      115 IF (M-N) 120,125,120
275.      120 ILP=IL+1
276.      IMP=IM+1
277.      X=R(IL)*COSX-R(IMP)*SINX
278.      R(IMP)=R(IL)*SINX+R(IMP)*COSX
279.      R(IL)=X
280.      125 CONTINUE
281.      X=2.0*A(LL)*SINCOS
282.      Y=A(LL)*COSX2+A(MM)*SINX2-X
283.      X=A(LL)*SINX2+A(MM)*COSX2+X
284.      A(LL)=(A(LL)-A(MM))*SINCOS+X*(COSX2-SINX2)
285.      A(LL)=Y
286.      A(MM)=X
287.      C 85 TEST FOR COMPLETION
288.      C 90 TEST FOR M= LAST COLUMN
289.      C 95 TEST FOR L= SECOND FROM LAST COLUMN
290.      140 IF (L-(N-1)) 145,150,145
291.      145 L=L+1
292.      150 GO TO 55
293.      152 IF (M-N) 160,155,160
294.      155 INQ=0
295.      160 GO TO 50
296.      C 165 COMPARISON THRESHOLD WITH FINAL NORM
297.      C 170 SORT EIGENVALUES AND EIGENVECTORS

```

```

207. 165 IO=-N
208. DO 185 T=1,N
209. IO=IO+N
210. LL=(I+J*J)/2
211. JO=N*(J-2)
212. DO 185 J=I,N
213. JO=JO+J
214. MM=(J+J*J)/2
215. IF (A(LL)-A(MM)) 177,185,185
216. 177 X=A(LL)
217. A(LL)=A(MM)
218. A(MM)=X
219. IF (MM-1) 175,185,175
220. 175 DO 185 K=1,N
221. ILR=IO+K
222. IJR=JO+K
223. X=A(ILR)
224. A(ILR)=A(IJR)
225. A(IJR)=X
226. 185 CONTINUE
227. C RETURN TO NORMAL STACKING OF MATRIX
228. DO 195 T=1,N
229. LL=N-I
230. MARK=LL*(N+1)+1
231. J=(LL+1)*(LL+2)/2
232. 195 A(MARK)=A(J)
233. RETURN
234. END

```

COMMENT - STATEMENT NUMBER 62 NEVER USED
 COMMENT - STATEMENT NUMBER 68 NEVER USED
 COMMENT - STATEMENT NUMBER 78 NEVER USED

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 CUTOFF FREQUENCY= .74614417
 CUTOFF FREQUENCY= .74596639

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	EIGENVECTORS ASSOCIATED TO BETA SQUARE
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	.99713762E+00
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KO= 1.00

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BETA SQUARE

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NUMBER OF TERMS

BETA SQUARE

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.99111301E+00

NUMBER OF TERMS

BETA SQUARE

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EIGENVECTORS ASSOCIATED TO BETA SQUARE

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NUMBER OF TERMS

BETA SQUARE

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EIGENVECTORS ASSOCIATED TO BETA SQUARE

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BETA SQUARE

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EIGENVECTORS ASSOCIATED TO BETA SQUARE

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NUMBER OF TERMS

BETA SQUARE

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EIGENVECTORS ASSOCIATED TO BETA SQUARE

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NUMBER OF TERMS

BETA SQUARE

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BETA SQUARE

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NUMBER OF TERMS BETA SQUARE
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EIGENVECTORS ASSOCIATED TO BETA SQUARE

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      EIGENVECTORS ASSOCIATED TO BETA  SQUARE
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      NUMBER OF TERMS      BETA  SQUARE
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      EIGENVECTORS ASSOCIATED TO BETA  SQUARE
      .02024241E+00
      NUMBER OF TERMS      BETA  SQUARE
      4      34473707E+01
      EIGENVECTORS ASSOCIATED TO BETA  SQUARE
      .02022697E+00
      NUMBER OF TERMS      BETA  SQUARE
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      EIGENVECTORS ASSOCIATED TO BETA  SQUARE

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NUMBER OF TERMS

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K0= 1.70

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NUMBER OF TERMS

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CHAPTER 4

Other Applications

4.1 Introduction

In this chapter other applications of the method described in Chapter 2 are discussed. Some numerical applications are analysed as well as the possibility of an extension of the method for dealing with different structures, in particular cylindrical waveguides and microstrips.

4.2 A numerical example : the LSM₁₁ mode

It is a well known fact¹ that apart from the H_{no} mode (LSE₁₀) analysed in the previous chapter, the modes of propagation for guides loaded with dielectric slabs are combinations of E and H mode. Our attention will be devoted to a half-filled dielectric loaded square waveguide as shown in Figure 4.1.

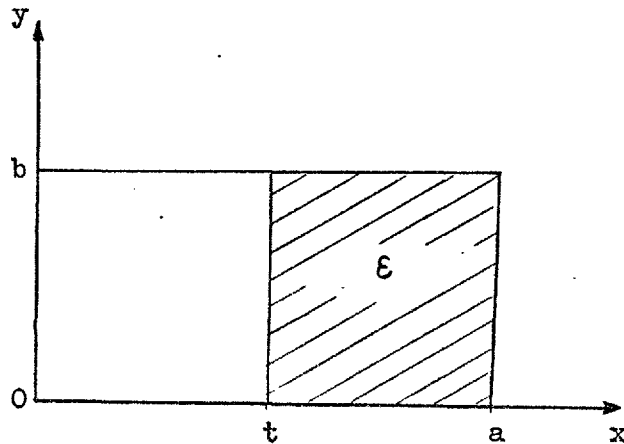


Fig.4.1.

As was mentioned at the end of Chapter 2, the values of the propagation constants can be obtained from

$$\det(GL - MK) = 0 \quad (4.1)$$

where G, L, M and K are defined by Eqs. (2.40).

Although Eq. (4.1) does not look very complicated, its numerical evaluation is rather difficult. In fact, we are dealing with matrices whose elements are polynomials in β^4 , β^2 and β^0 , where β is the unknown to be determined. Obviously when we are dealing with matrices of the order greater than 3, approximate methods must be applied.

The method we followed was first to evaluate the cutoff frequency for the mode under analysis. This will provide the starting point for the initial value for β . In fact, knowing the cutoff frequency, a slightly greater frequency is chosen and β is allowed to vary between 0 and another value that we choose as 0.5. For increasing values of β , Eq. (4.1) is evaluated, seeking for a change of sign of the determinant. If such a change occurs, the determinant must be zero for a value between the last two β estimates. This implies that there is a value of β that satisfies Eq. (4.1). It is then possible to subdivide the interval, seeking for a better approximation for β . Then a linear approximation can be taken as shown in Fig. 4.2.

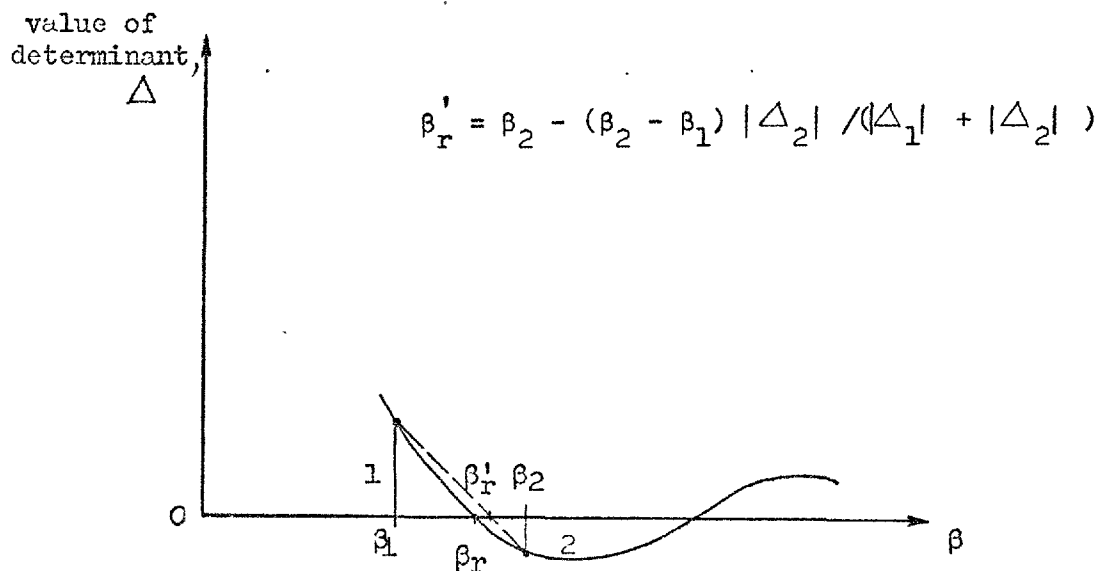


FIG.4.2. Evaluation of roots of Eq.(4.1).

Unfortunately, this method is not very accurate as was reported by several authors^{2,3}. There is also the danger of including in the solutions extraneous roots, without any physical meaning. It is then evident that a more accurate method for determining the roots of Eq. (4.1) should be found. Section 4.3.4 suggests alternative methods to tackle the problem.

As was mentioned before, the case to be analysed is that of the LSM₁₁ mode for the case of a square waveguide of width π , half-filled with a dielectric of relative dielectric constant equal to 2.45. The programme and the results are described in the next section.

4.3 Programme description

In the first instructions, the parameters of the configuration to be analysed are specified and some constants are evaluated. The arrays are properly dimensioned: for this programme the maximum order of the matrices is 10. The arrays AL, AG, AM, AK will be used to store, respectively, the elements of the matrices L, G, M and K. The functions of the other arrays will be specified in the description.

4.3.1 Cutoff frequency evaluation

The evaluation of the cutoff frequency of the mode is performed in the first part of the programme. From the theoretical analysis, it follows that when $\beta = 0$, the cutoff frequency may be evaluated from $\det(GL) = 0$ or from $\det(L) = 0$. In fact, for $\beta = 0$, the equations for the axial electric and magnetic field components are uncoupled. It is then possible to evaluate the cutoff frequency from $\det(L|_{\beta=0}) = 0$. By using DO 1, the elements of L for $\beta = 0$, are evaluated. As these elements are polynomials in k_o^2 and k_o^0 , what we have done is to evaluate the coefficients of k_o^2 and k_o^0 . The

order of the matrix L is given by NT.

Once all the coefficients of k_o^2 and k_o^0 are evaluated, the value of k_o that makes the $\det(L) = 0$ is determined by using DO 5. An initial value for k_o^2 is given and stored in VIN, where $VIN \leq (1/\epsilon_2)$. The values of the elements of $\det(L)$ for k_o^2 (VKOS in the programme) equal to VIN are evaluated and stored in an array D of dimensions (NT, NT) inside the DO 3. When this process has been completed, the determinant is evaluated using the subroutine F 03 AAF which will be described in Section 4.3.2. The value of the determinant is stored in an array DET(K), where K is an integer varying between 1 and a number NUF; $NUF \geq \{1 - (1/\epsilon_2)\} / VINT$, where VINT is the value of the increment for the next value of k_o^2 .

The value of k_o^2 is now incremented by a quantity VINT and a new value for the determinant evaluated. This value of the determinant is multiplied by the previous one and checked if the product is less than 1. If the product is less than 1, this implies a change of sign in the determinant. Then it is checked whether the absolute value of one of the two determinants is less than 10^{-5} . If so, the value for k_o^2 at cutoff will be the one used for evaluating that determinant. If not, the interval between the two last k_o^2 is subdivided in three parts and the process is repeated.

Finally, a value for $k_{o_cutoff}^2$ is evaluated using the expression

$$k_{o_cutoff}^2 = k_{o_2}^2 - (k_2^2 - k_{o_1}^2) |\Delta_2| / (|\Delta_1| + |\Delta_2|)$$

where Δ_1 and Δ_2 are the values of the determinant for $k_{o_1}^2$ and $k_{o_2}^2$ respectively.

Using a matrix of 3rd order, the result obtained for the cutoff frequency was

$$k_o = 0.76138699$$

compared with the analytical value of 0.761386. Obviously for this case the agreement is quite good.

Before analyzing the second part of the programme, used for evaluating the propagation constants, a description of the subroutine FO3AAF is presented.

4.3.2 Subroutine FO3AAF⁴

The general form of the subroutine is

FO3AAF(D, N, NT, DI, P, IFAIL).

It calculates the determinant of D using the Crout factorisation $D = LU$ where L is lower triangular and U is unit upper triangular. The determinant of D is the product of the diagonal elements of L. Double precision accumulation of inner products is used throughout.

Description of parameters.

- . D is a two dimensional real array which on entry contains the real matrix and on exit, unless an error has occurred, will contain the Crout factorisation $D = LU$.
- . N is the name of an integer containing the order of the matrix as dimensioned the initial time.
- . NT is the name of an integer containing the actual order of the matrix.
- . DI is the name of a real variable. On exit it will contain the value of the determinant of D unless an error has occurred.
- . P is a one-dimensional working array, of dimension $N \times N$.
- . IFAIL is the name of an integer variable. On entry the value of IFAIL determines the mode of failure in the routine. On exit, IFAIL indicates the successful use of the routine or acts as an error indicator.

If, on entry, IFAIL = 0 (hard failure), the programme will terminate with a failure message if any error is detected.

If, on entry, IFAIL = 1 (soft failure), control returns to the calling sequence within the programme if any error is detected by the routine.

No failure message will be printed.

On exit, IFAIL = 0 for a successful call of the routine.

IFAIL = 1, the matrix is singular, possibly due to rounding errors.

IFAIL = 2, the value of the determinant is outside the range of the 6400 element.

Auxiliary subroutines used: FO3AFF, FOLARF and POLAAF.

The computation of a matrix of order 10 took approximately 0.2 seconds.

4.3.3 Evaluation of propagation constants

The evaluation of the propagation constants, at each frequency of interest, is done in the DO loop 7. The number of different frequencies where β is evaluated, is determined by IFIKO. The value of the initial frequency is stored in VKO, and the increment in VKO is given by DELKO. The initial value for β is stored in BETIN. The number of terms used in the expansion is determined by KK and its maximum value is given by NTS.

In DO 9, the elements of the matrices G, L, M, K are evaluated for the corresponding β and k_0 , and stored in the arrays AG, AL, AM and AK respectively. As explained in Section 4.1, the value of β is changed on an iterative basis until a change in the sign of the determinant in Eq. (4.1) is found. In DO 28 the operation (GL - MK) is performed, and using the subroutine FO3AFF the $\det(\text{GL} - \text{MK})$ is evaluated and stored in the array DET of dimension K. If $K = 1$, the value of β (BETA) is incremented by an amount DELB and the process is repeated. The new determinant so obtained is multiplied by the previous determinant. If the product is greater than 0, which means that both determinants have the same sign, the

process is continued, evaluating a new determinant for a new value of β . If the product is not greater than 0, it implies that the two determinants have different signs and in this interval of β 's, there is a particular value of β that makes the determinant null. Then the interval is subdivided by 10 and the complete process is repeated until the absolute value of one of the determinants is less than 10^{-5} , or until the process has been performed 4 times. In the former case, the value of β corresponds to the same β that makes the determinant $\leq 10^{-5}$, and in the latter, the value of β is obtained using a linear approximation as shown in Fig. 4.2. The values of k_0 and β are then printed, the number of terms incremented, and the whole process repeated.

4.3.4 Results

For this case, and for the configuration shown in Fig. 4.1 with $\epsilon = 2.45 \epsilon_0$, the propagation constants for the LSM₁₁ mode have been evaluated using the programme in the Appendix. The numerical results are shown as output of the programme and in Fig. 4.3 they are compared with the analytical results.

It may be seen that the convergence is quite good for the cutoff frequency, but not as good for the propagation constant. Most of the problem appears to be due to the numerical evaluation of the roots of Eq. (4.1).

It was earlier (Chapter 2) suggested that a better approximation, but with an increase in the computer time, could possibly be achieved using the expression

$$\det(MG^{-1}K - L) = 0 \quad (4.2)$$

because this determinant is smaller than the one of Eq. (4.2).

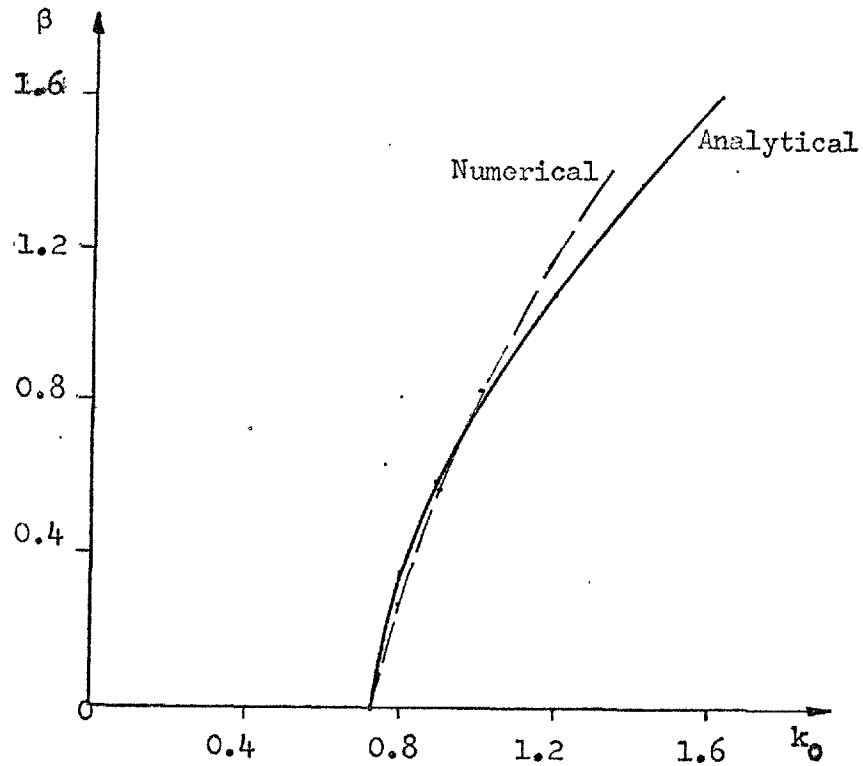


Fig.4.3. Propagation constants of LSM₁₁ mode in a square waveguide half filled with a slab of $\epsilon_r=2.45$.

Another interesting possibility would be the evaluation of the roots of Eq. (4.1) but this seems a cumbersome task unless some kind of simplification could be achieved.

4.4 H-modes in inhomogeneously loaded cylindrical waveguides

Consider the structure shown in Fig. 4.4 consisting of a perfect conducting cylinder of radius R and a concentric dielectric rod of radius r and dielectric constant ϵ .

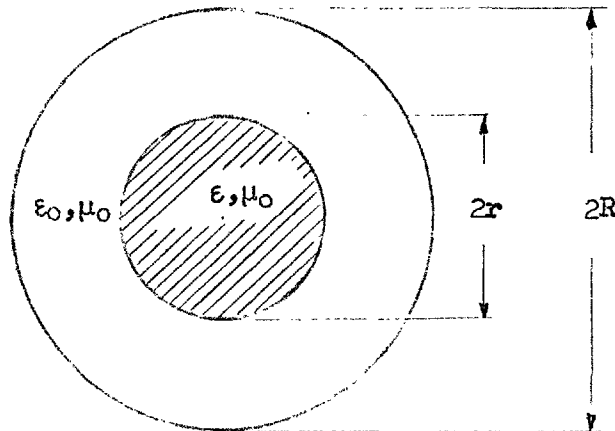


Fig.4.4. Dielectric rod in a cylindrical waveguide: cross-section.

Let us rewrite Eq. (2.14).

$$(\gamma^2 + k_o^2 \epsilon_r) \nabla_t^2 H_z + (\gamma^2 + k_o^2 \epsilon_r)^2 H_z - k_o^2 \nabla_t \epsilon_r \cdot \nabla_t H_z = j\omega \epsilon_o \gamma k_o (\nabla_t \epsilon_r \times \nabla_t E_z) \quad (4.3)$$

It is known that in cylindrical coordinates⁽⁵⁾

$$\nabla_t^2 \omega = \underline{u}_\rho \frac{\partial \omega}{\partial \rho} + \underline{u}_\phi \frac{1}{\rho} \frac{\partial \omega}{\partial \phi} \quad (4.4a)$$

$$\nabla_t^2 \omega = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \omega}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \omega}{\partial \phi^2} \quad (4.4b)$$

where $\underline{u}_\rho$ and $\underline{u}_\phi$ are unit vectors in the radial and azimuthal directions respectively.

Using the relations (4.4a) and (4.4b), the Eq. (4.3) is rewritten

as

$$(\gamma^2 + k_o^2 \epsilon_r) \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial H_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 H_z}{\partial \phi^2} \right\} + (\gamma^2 + k_o^2 \epsilon_r)^2 H_z - k_o^2 \nabla_t \epsilon_r \cdot \nabla_t H_z = j\omega \epsilon_o \gamma k_o (\nabla_t \epsilon_r \times \nabla_t E_z) \quad (4.5)$$

As ϵ is only a function of ρ , such that Eq. (4.5) is

$$(\gamma^2 + k_o^2 \epsilon_r) \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial H_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 H_z}{\partial \phi^2} \right\} + (\gamma^2 + k_o^2 \epsilon_r)^2 H_z - k_o^2 \frac{d\epsilon_r}{d\rho} \frac{\partial H_z}{\partial \rho} = j\omega \epsilon_o \gamma k_o \left(\frac{d\epsilon_r}{d\rho} \underline{u}_\rho \times \nabla_t E_z \right) \quad (4.6)$$

The solution for H_z for H-modes is of the form⁽⁶⁾

$$H_z = f(\rho) \cos m\phi \quad (4.7)$$

where the variation with z , given by $\exp(-\gamma z)$ is implicit.

Substituting in Eq. (4.6),

$$(\gamma^2 + k_o^2 \epsilon_r) \left\{ \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f(\rho)}{\partial \rho} \right) - \frac{m^2}{\rho^2} f(\rho) \right\} \cos m\phi + (\gamma^2 + k_o^2 \epsilon_r) f(\rho) \cos m\phi - k_o^2 \frac{d\epsilon_r}{d\rho} \frac{df(\rho)}{d\rho} \cos m\phi = j\omega \epsilon_o \gamma k_o \left(\frac{d\epsilon_r}{d\rho} \underline{u}_\rho \times \nabla_t E_z \right) \quad (4.8)$$

For $\omega = 0$ ($k_o = 0$), Eq. (4.8) is

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f(\rho)}{\partial \rho} \right) + \left(\gamma_H^2 - \frac{m^2}{\rho^2} \right) f(\rho) = 0 \quad (4.9)$$

If we are only interested in the H_{01} mode, $m = 0$ such that

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f(\rho)}{\partial \rho} \right) + \gamma_H^2 f(\rho) = 0 \quad (4.10)$$

or

$$f''(\rho) + \frac{1}{\rho} f'(\rho) + \gamma_H^2 f(\rho) = 0 \quad (4.11)$$

where $f(\rho) = A J_0(\gamma_{H,n} \rho)$, subject to the boundary conditions

$$f'(R) = A J_0'(\gamma_{H,n} R) = 0 \quad (4.12)$$

which implies $J_1(\gamma_{H,n} R) = 0$ and, omitting the subindex H

$$\gamma_n R = p_n, \text{ or}$$

$$\gamma_n = \frac{p_n}{R}, \quad n = 1, 2, 3 \dots \quad (4.13)$$

where the p_n 's are the zeros of the Bessel function of the first kind, or order 1.

Then $f(\rho)$ can be expanded in a series

$$f(\rho) = \sum_n a_n f_n(\rho) \quad (4.14)$$

The following relation holds

$$\iint_S f_n(\rho) f_m(\rho) \rho d\rho d\phi = \delta_{mn} \quad (4.15)$$

or

$$A_m A_n 2\pi \int_0^R \rho J_0(\gamma_n \rho) J_0(\gamma_m \rho) d\rho = \delta_{mn} \quad (4.16)$$

$$\text{As } \int_0^R \rho J_0^2(\gamma_n \rho) d\rho = \frac{R^2}{2} \{J_0'(\gamma_n R)\}^2 + \frac{R^2}{2} \{J_0(\gamma_n R)\}^2 \quad (4.17)$$

but $J_0'(\gamma_n R) = 0$, so

$$A_n^2 \frac{R^2}{2} \{J_0(\gamma_n R)\}^2 = 1$$

$$A_n = 1/[R J_0(\gamma_n R) (\pi)^{\frac{1}{2}}] \quad (4.18)$$

Then, Eq. (4.8) can be written as

$$\sum_n a_n \left\{ (\gamma^2 + k_o^2 \epsilon_r)^2 f_n(\rho) + (\gamma^2 + k_o^2 \epsilon_r) \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f_n(\rho)}{\partial \rho} \right) - k_o^2 \epsilon_r' f_n'(\rho) \right\} = 0 \quad (4.19)$$

Multiplying the previous equation by $f_m(\rho)$ and integrating over the cross-section,

$$\sum_n a_n \int_0^R \left\{ (\gamma^2 + k_o^2 \epsilon_r)^2 f_n(\rho) f_m(\rho) + (\gamma^2 + k_o^2 \epsilon_r) \frac{1}{\rho} f_m(\rho) \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f_n(\rho)}{\partial \rho} \right) - k_o^2 \epsilon_r' f_n' f_m(\rho) \rho d\rho \right\} = 0 \quad (4.20)$$

From Eq. (4.10),

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f_n(\rho)}{\partial \rho} \right) = -\gamma_n^2 f_n(\rho)$$

so Eq. (4.20) is transformed to

$$\begin{aligned} \sum_n a_n \int_0^R \left\{ [(\gamma^2 + k_o^2 \epsilon_r)^2 - (\gamma^2 + k_o^2 \epsilon_r) \gamma_n^2] f_n(\rho) f_m(\rho) - k_o^2 \epsilon_r' f_n'(\rho) f_m(\rho) \rho d\rho \right\} = 0 \\ \sum_n a_n \left\{ \gamma^2 (\gamma^2 - \gamma_n^2) \int_0^R \rho f_n(\rho) f_m(\rho) d\rho + k_o^4 \int_0^R \epsilon_r^2 \rho f_n(\rho) f_m(\rho) d\rho + \right. \\ \left. k_o^2 (2\gamma^2 - \gamma_n^2) \int_0^R \epsilon_r \rho f_n(\rho) f_m(\rho) d\rho - k_o^2 \int_0^R \epsilon_r' f_n'(\rho) f_m(\rho) \rho d\rho \right\} = 0 \end{aligned} \quad (4.21)$$

The integral

$$I = \int_0^R \epsilon_r' f_n'(\rho) f_m(\rho) \rho d\rho$$

can be integrated by parts to give

$$I = \epsilon_r \rho f_m(\rho) f_n'(\rho) \Big|_0^R - \int_0^R \epsilon_r \left\{ f_m(\rho) f_n''(\rho) + \rho f_m'(\rho) f_n'(\rho) + \rho f_m(\rho) f_n''(\rho) \right\} d\rho$$

and, as $f_n'(R) = 0$, and using Eq. (4.11),

$$\begin{aligned} I &= - \int_0^R \epsilon_r \left\{ \rho f_m(\rho) \left[-\gamma_n^2 f_n(\rho) - \frac{1}{\rho} f_n'(\rho) \right] + f_m(\rho) f_n'(\rho) + \right. \\ &\quad \left. \rho f_m'(\rho) f_n'(\rho) \right\} d\rho \\ &= \int_0^R \epsilon_r \rho (\gamma_n^2 f_m(\rho) f_n(\rho) - f_m'(\rho) f_n'(\rho)) d\rho \end{aligned} \quad (4.22)$$

Substituting Eq. (4.22) into Eq. (4.21),

$$\sum_n a_n \left\{ \gamma^2 (\gamma^2 - \gamma_n^2) \int_0^R \rho f_n(\rho) f_m(\rho) d\rho + k_o^4 \int_0^R \epsilon_r^2 \rho f_n(\rho) f_m(\rho) d\rho + \right. \\ \left. 2k_o^2 (\gamma^2 - \gamma_n^2) \int_0^R \epsilon_r \rho f_n(\rho) f_m(\rho) d\rho + k_o^2 \int_0^R \epsilon_r \rho f_m'(\rho) f_n'(\rho) d\rho \right\} = 0 \quad (4.23)$$

Let us evaluate the integrals that appear in Eq. (4.23).

$$\int_0^R \rho f_n(\rho) f_m(\rho) d\rho = \delta_{mn} / 2\pi \quad (4.24a)$$

$$\int_0^R \epsilon_r \rho f_n(\rho) f_m(\rho) d\rho = A_m A_n \int_0^R \epsilon_r \rho J_o(\gamma_n \rho) J_o(\gamma_m \rho) d\rho = \\ = A_m A_n \int_0^r \epsilon_r \rho J_o(\gamma_n \rho) J_o(\gamma_m \rho) d\rho + \int_r^R 1 \cdot \rho J_o(\gamma_n \rho) J_o(\gamma_m \rho) d\rho \quad (4.24b)$$

where

$$\epsilon_r = \begin{cases} \epsilon_r & 0 < \rho < r \\ 1 & r < \rho < R \end{cases}$$

Using the relation

$$\int x J_n(\alpha x) J_n(\beta x) dx = \frac{x \{ \alpha J_o(\beta x) J_o'(\alpha x) - \beta J_o(\alpha x) J_o'(\beta x) \}}{\beta^2 - \alpha^2}$$

Eq. (4.24b) is transformed to

$$A_m A_n \left\{ (\epsilon_r^\alpha - 1) \frac{r \left[\gamma_n J_o(\gamma_m r) J_o'(\gamma_n r) - \gamma_m J_o(\gamma_n r) J_o'(\gamma_m r) \right]}{\gamma_m^2 - \gamma_n^2} + \right. \\ \left. \frac{R \left[\gamma_n J_o(\gamma_m R) J_o'(\gamma_n R) - \gamma_m J_o(\gamma_n R) J_o'(\gamma_m R) \right]}{\gamma_m^2 - \gamma_n^2} \right\}$$

but the last term is zero because $J_o'(\gamma_n R) = J_o'(\gamma_m R) = 0$

So, if $m \neq n$

$$\int_0^R \epsilon_r^\alpha \rho f_n(\rho) f_m(\rho) d\rho = \frac{(\epsilon_r^\alpha - 1)r \left\{ \gamma_n J_0(\gamma_m r) J_0'(\gamma_n r) - \gamma_m J_0(\gamma_n r) J_0'(\gamma_m r) \right\}}{R^2 \pi J_0(\gamma_n R) J_0(\gamma_m R) (\gamma_m^2 - \gamma_n^2)} \quad (4.25)$$

If $m = n$

$$\begin{aligned} \int_0^R \epsilon_r^\alpha \rho f_n^2(\rho) d\rho &= A_n^2 \int_0^R \epsilon_r^\alpha \rho J_0^2(\gamma_n \rho) d\rho = \\ &= A_n^2 \left\{ \int_0^r \epsilon_r^\alpha \rho J_0^2(\gamma_n \rho) d\rho + \int_r^R \rho J_0^2(\gamma_n \rho) d\rho \right\} \\ &= A_n^2 \left\{ (\epsilon_r^\alpha - 1) \frac{r^2}{2} \left[(J_0'(\gamma_n r))^2 + J_0^2(\gamma_n r) \right] + \frac{R^2}{2} J_0^2(\gamma_n R) \right\} \end{aligned}$$

Then, for $m = n$

$$\int_0^R \epsilon_r^\alpha \rho f_n^2(\rho) d\rho = \frac{(\epsilon_r^\alpha - 1)r^2 \left\{ [\gamma_n J_1(\gamma_n r)]^2 + J_0^2(\gamma_n r) \right\} + \frac{1}{2\pi R^2} J_0^2(\gamma_n R)}{2\pi R^2 J_0^2(\gamma_n R)} + \frac{1}{2\pi} \quad (4.26)$$

The last integral is of the form

$$\int_0^R \epsilon_r \rho f_m'(\rho) f_n'(\rho) d\rho$$

If $m \neq n$

$$A_n A_m \int_0^R \epsilon_r \rho J_0^2(\gamma_m \rho) J_0'(\gamma_n \rho) d\rho = A_n A_m \gamma_m \gamma_n \int_0^R \epsilon_r \rho J_1(\gamma_m \rho) J_1(\gamma_n \rho) d\rho,$$

where $J_0'(\gamma_m \rho) = -\gamma_m J_1(\gamma_m \rho)$, where the ' indicates the first derivative with respect to ρ .

$$\begin{aligned} &= A_n A_m \gamma_m \gamma_n \left\{ \int_0^r \epsilon_r \rho J_1(\gamma_m \rho) J_1(\gamma_n \rho) d\rho + \int_r^R \rho J_1(\gamma_m \rho) J_1(\gamma_n \rho) d\rho \right\} \\ &= A_n A_m \gamma_m \gamma_n \left\{ (\epsilon_r - 1)r \frac{\gamma_n J_1(\gamma_m r) J_1'(\gamma_n r) - \gamma_m J_1(\gamma_n r) J_1'(\gamma_m r)}{\gamma_m^2 - \gamma_n^2} \right\} \\ &= \frac{\gamma_m \gamma_n (\epsilon_r - 1) r \left\{ \gamma_n J_1(\gamma_m r) J_1'(\gamma_n r) - \gamma_m J_1(\gamma_n r) J_1'(\gamma_m r) \right\}}{\pi R^2 J_0(\gamma_n R) J_0(\gamma_m R) (\gamma_m^2 - \gamma_n^2)} \quad (4.27) \end{aligned}$$

But $J_1'(\gamma_n r) = \frac{1}{2} \gamma_n \{ J_0(\gamma_n r) - J_2(\gamma_n r) \}$, then

$$= \gamma_m \gamma_n (\gamma_n^{-1} r) \frac{\left\{ \gamma_n^2 J_1(\gamma_m r) [J_0(\gamma_n r) - J_2(\gamma_n r)] - \gamma_m^2 J_1(\gamma_n r) [J_0(\gamma_m r) - J_2(\gamma_m r)] \right\}}{2 \pi R^2 J_0(\gamma_n R) J_0(\gamma_m R) (\gamma_m^2 - \gamma_n^2)} \quad (4.28)$$

If $m = n$

$$\begin{aligned} \int_0^R \epsilon_r \rho f_n'(\rho) \rho^2 d\rho &= A_n^2 \int_0^R \epsilon_r \rho J_0'(\gamma_n \rho) \rho^2 d\rho = A_n^2 \gamma_n^2 \int_0^R \epsilon_r \rho J_1^2(\gamma_n \rho) d\rho \\ &= A_n^2 \gamma_n^2 (\epsilon_r - 1) \frac{r^2}{2} \left\{ [J_1'(\gamma_n r)]^2 + \left(1 - \frac{1}{\gamma_n^2 r^2}\right) J_1^2(\gamma_n r) \right\} + \\ &A_n^2 \gamma_n^2 \frac{R^2}{2} \left\{ J_1'(\gamma_n R) \right\}^2 \\ &= A_n^2 \gamma_n^2 (\epsilon_r - 1) \frac{r^2}{2} \left\{ \frac{1}{4} \gamma_n^2 [J_0^2(\gamma_n r) - 2J_0(\gamma_n r) J_2(\gamma_n r) + J_2^2(\gamma_n r)] + \right. \\ &\left. \left(1 - \frac{1}{\gamma_n^2 r^2}\right) J_1^2(\gamma_n r) \right\} + A_n^2 \gamma_n^4 \frac{R^2}{8} \left\{ J_0^2(\gamma_n R) - 2J_0(\gamma_n R) J_2(\gamma_n R) + \right. \\ &\left. J_2^2(\gamma_n R) \right\} \\ &= \frac{1}{\pi 2 R^2 J_0^2(\gamma_n R)} \left\{ \gamma_n^4 (\epsilon_r - 1) \frac{r^2}{4} [J_0^2(\gamma_n r) - 2J_0(\gamma_n r) J_2(\gamma_n r) + J_2^2(\gamma_n r)] + \right. \\ &\left. (\gamma_n^2 r^2 - 1)(\epsilon_r - 1) J_1^2(\gamma_n r) + \gamma_n^4 \frac{R^2}{4} [J_0^2(\gamma_n R) - 2J_0(\gamma_n R) J_2(\gamma_n R) + \right. \\ &\left. J_2^2(\gamma_n R)] \right\} \quad (4.29) \end{aligned}$$

Finally, Eq. (4.23) can be written as a matrix equation

$$\left\{ \gamma^2 (\gamma^2 - \gamma_n^2) I + 2\pi k_0^4 Q + 4\pi k_0^2 (\gamma^2 - \gamma_n^2) + 2\pi k_0^2 R \right\} \{a\} = \{0\} \quad (4.30)$$

where I is the unit matrix, Q_{mn} is given by Eq. (4.25) with $\alpha = 2$,

Q_{mm} is given by Eq. (4.26) with $\alpha = 2$, P_{mn} and P_{mm} are given by

Eqs. (4.25) and (4.26) with $\alpha = 1$, respectively. R_{mn} and R_{mm} are

given by Eqs. (4.28) and (4.29) respectively.

Solution of Eq. (4.3) can be obtained from

$$\det \left\{ \gamma^2 (\gamma^2 - \gamma_n^2) I + 2\pi k_o^4 Q + 4\pi k_o^2 (\gamma^2 - \gamma_n^2) P + 2\pi k_o^2 R \right\} = 0 \quad (4.31)$$

4.4.1 Cutoff frequency

The cutoff frequency for the H_{01} mode can be evaluated from Eq. (4.31) with $\gamma = 0$, obtaining

$$\det (k_o^2 Q - 2 \gamma_n^2 P + R) = 0 \quad (4.32a)$$

or

$$\det \left\{ k_o^2 I - (2 \gamma_n^2 P - R) Q^{-1} \right\} = 0 \quad (4.32b)$$

where, in general, $\det(Q) \neq 0$.

A short programme has been written for evaluating the cutoff frequency of the H_{01} mode, and is shown in Appendix B. A description of the programme can be found in the next section.

For the particular case of a circular waveguide of radius 1 with a concentric circular rod of radius 0.2 and relative dielectric constant 20, the value read from curves given by Waldron⁽⁷⁾ was 2.51327, and the numerical result using the programme in Appendix B using four terms in the expansion, was 2.58558.

4.4.2 Description of the programme

The maximum number of terms to be used in the expansion is given by NS. The waveguide radius R is stored in RE and the dielectric rod radius r in RI. The values of the zeros of $J_0'(x)$ and stored in G(NS).

By using DO 1, the Bessel's functions $J_0(\gamma_n r)$, $J_0(\gamma_n R)$, $J_1(\gamma_n r)$, $J_2(\gamma_n r)$ and $J_2(\gamma_n R)$ are calculated using the subroutine

BESJ and stored in the arrays BJOI(NS), BJOE(NS), BJ1I(NS), BJ2I(NS) and BJ2E(NS), respectively.

In the DO loop 2, the elements of the matrices P, Q, and R of Eq. 4.3 b) are calculated. They are stored respectively in the arrays P(NS,NS), Q(NS,NS) and R(NS,NS).

Inside the same DO loop, the difference $2 \gamma_n^2 P - R$ is evaluated and stored in the array DIFF(NS,NS).

Using DO loops 5 and 9 and the subroutine MINV, Q^{-1} is evaluated and stored in Q(NS,NS).

The multiplication $(2 \gamma_n^2 P - R)Q^{-1}$ is performed using the DO loops 11 and 13, and the result stored in the array DD(NS,NS). As Eq. (4.3 b) has the same form of the standard eigenvalue equation, the values of k_o^2 are obtained by evaluating the eigenvalues of the array DD(NS,NS). This is done using the subroutine FO2AFF.

Finally the number of terms used in the expansion and the values of the eigenvalues are printed.

Subroutines MINV and FO2AFF have been discussed previously, so no further comment will be made on them.

Subroutine BESJ⁸ computes the J Bessel function for a given argument and integer order. It is called as BESJ(X,N,BJ,D,1ER), where:

- . X is the argument of the J Bessel function desired.
- . N is the order of the J Bessel function desired.
- . BJ is the resultant J Bessel function.
- . D is the required accuracy.
- . 1ER is an error code, such that

- i) $LER = 0$, no error .
- ii) $LER = 1$, N is negative .
- iii) $LER = 2$, X is negative or zero .
- iv) $LER = 3$, the range of N compared to X is not correct (see below).

N must be greater or equal to zero, but it must be less than

$$20 + 10 * X - X ** 2 / 3 \quad \text{for } X \leq 15$$

$$90 + X / 2 \quad \text{for } X > 15$$

For the particular case analysed, the computer time for running the programme was about 2.6 seconds.

4.5 Conclusions

In this chapter further applications of the method developed in Chapter 2 have been presented.

In the first part, the method was applied for evaluating the propagation characteristics of a dielectric loaded rectangular waveguide. In particular, the LSM_{11} mode was analysed. A computer programme was written, and the numerical results have shown a very good agreement with the analytical values for the cutoff frequency. Good agreement is observed in the region between the frequency of cutoff and the frequency for which $k_0 = 1.0$. For other values, the numerical values start diverging from the analytical solution. No attempt has been made to optimise the programme.

In the second part, the method is applied to dielectric loaded cylindrical waveguides, in particular for the H_{01} mode. A computer programme has been written for evaluating the cutoff frequency of the H_{01} mode for a nearly arbitrary variation in permittivity across the waveguide. The numerical results show a good agreement with results obtained either analytically or by other authors.

4.6 Future work

The first task should be the optimisation of the programme for determining any mode in a dielectric loaded rectangular waveguide. Related to this, there is also the problem of determining which eigenvalue corresponds to the mode under analysis, what is the meaning of the others, and what information, if any, can be extracted from this set of eigenvalues.

The second task would be to analyse the case of dielectric loaded cylindrical waveguides, obtaining a general expression both for H and E modes, and comparing the accuracy of the numerical results obtained with the analytical ones.

Another application could be the analysis of junctions of empty and dielectric loaded waveguides, a problem encountered in many practical situations.

Finally, the last objective would be trying to apply the basic ideas of this method to the analysis of microstrips. In fact, a shielded microstrip can be regarded as the problem of a rectangular dielectric-slab loaded waveguide, with a metallic strip placed upon the dielectric slab. Of equal interest is the problem of open microstrip.

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APPENDIX A

```

1.  PROGRAM P01A (INPUT, OUTPUT, TAP=1=INPUT, TAP=3=OUTPUT)
2.  DIMENSION AL(10,10), EL(10,10), AG(10,10), AH(10,10), AK(10,10)
3.  DIMENSION D(10,10), PROC(10,10), DET(100), PEPE(100), PAPA(100)
4.
5.  PI=ACOS(-1.)
6.  P=PI
7.  P=PI
8.  E1=1.4
9.  E2=2.45
10. T=A/2.
11. TAN=T/A
12. NT=3
13. CL=-(2.0*(E2-E1)*SQRT(E2))/((E1+E2)*PI)
14. GME=1.0
15. IND=2
16. C EVALUATION OF COEFFICIENTS OF K0
17. DO 1 I=1,NT
18. DO 1 J=1,NT
19. ZJ=J-1
20. ZI=I-1
21. GME=(Z1*PI/A)**2+(OFI*PI/3)**2
22. T=TA=TAU*180.
23. ET=1.
24. KAPPA=ZT
25. IF(((KAPPA/2)*2-KAPPA).EQ.0) ET=E2
26. IF(T.EQ.0) GO TO 2
27. S=SIGN((ZI-ZJ)*T*ET/A)/(ZI-ZJ)-SIGN((ZI+ZJ)*T*ET/A)/(ZI+ZJ)
28. T=SIGN(ZJ*PI*ET/A)*COS(ZI*PI*ET/A)
29. AL(I,J)=CL*(E1+E2)*S
30. EL(I,J)=CL*(T-S*GME)
31. GO TO 1
32.
33. 2 AL(I,T)=(E2**2-(E2**2-E1**2)*ET/(E1+E2))
34. EL(I,T)=(E2-E1)*ET/(E1+E2)-E2*GME
35. 1 CONTINUE
36. C EVALUATION OF THE CUTOFF FREQUENCIES
37. VJN=0.3
38. VINT=0.05
39. TAPUL=1.
40. NUF=10
41. 6 VKOS=VJN
42. DO 5 K=1,NUF
43. DO 3 I=1,NT
44. DO 3 J=1,NT
45. C(I,J)=L(I,J)*VKOS+BL(I,J)
46. L2=1
47. CALL F03AAF(D,10,NT,DI,PEPE,L2)
48. DET(K)=DT
49. IF(K.EQ.1) GO TO 5
50. IF((DET(K)*DET(K-1)).LE.0.) GO TO 4
51. 5 VKOS=VKOS+VINT
52. TAPUL=TAPUL+1.
53. NUF=30
54. VINT=VINT/3.
55. IF(TAPUL.LE.1.) GO TO 6
56. GO TO 999
57. 4 IF(ABS(DET(K)),LE.1E-5) IND=1
58. IF(ABS(DET(K-1)),LE.1E-5) IND=0
59. IF(IND.LE.2) GO TO 8
60. VIN=VKOS-VINT
61. VINI=(VKOS-VIN)/11.
62. GO TO 6
63. 8 IF(IND.EQ.1) VKO=SQRT(VKOS)
64. IF(IND.EQ.2) VKO=SQRT(VKOS-VINT)
65. ZULU=VINI*ABS(DET(K))/(ABS(DET(K))+ABS(DET(K-1)))
66. IF(IND.EQ.1) VKO=SQRT(VKOS-ZULU)
67. WRITE(3,301) NT, VKO
301 FORMAT(1H1, *NUMBER OF TERMS=*, I2, 5X, *CUTOFF FREQUENCY=*, F12.
68. IF(VKO-1.0) 10, 12, 12
69. 10 VKO=VKO*10.0

```

```

68. MM=VKO
69. GO TO 14
70. 12 MM=VKO*10
71. 14 VKO=FLOAT(MM)/10.
72. DELK=0.1
73. NIS=5
74. IFTKO=10
75. BETIN=0.1
76. DO 7 IJ=1, IFTKO
77. KK=3
78. VKO=VKO+DELK
79. VKOS=VKO*VKO
80. WRITE(3,302) VKO
81. 302 FORMAT(1H0,10X,*KO=*,F5.2)
82. 26 K=0
83. DELP=0.02
84. TAPUL=1.0
85. BETA=BETIN
86. KK=KK+1
87. 20 K=K+1
88. DO 9 I=1, KK
89. DO 9 J=1, KK
90. ZJ=J-1
91. ZI=I-1
92. ZN=J-1
93. ZK=I-1
94. GMS=(ZJ*PI/A)**2+(OR*PI/B)**2
95. GFS=(ZK*PI/A)**2+(OR*PI/B)**2
96. TETA=TAU*180.
97. LV=1.0
98. KAPPA=ZT
99. IF((KAPPA/2)**2-KAPPA).EQ.0)BT=E2
100. IF(T.EQ.J)GO TO 16
101. S=SIN((ZI-ZJ)*TETA)/(ZI-ZJ)-SIN((ZI+ZJ)*TETA)/(ZI+ZJ)
102. C=COS((ZI-ZJ)*TETA)/(ZI-ZJ)+COS((ZI+ZJ)*TETA)/(ZI+ZJ)
103. U=(ZK*COS(ZN*TETA)*SIN(ZK*TETA)-ZN*COS(ZK*TETA)*SIN(ZN*TETA))/ZK
104. 1 ZK-ZN*ZN
105. V=-ZK*SIN(ZK*TETA)*COS(ZN*TETA)
106. Z=VKOS*(OR*PI/E)*((2-E1)*SQRT(2.0*BT/(E2+E1))*COS(ZK*TETA)*SIN
107. 1*TETA)/PI
108. AG(I,J)=(2.0*(E1-E2)*VKOS/PI)*((-2.0*BETA*BETA-GMS+(E1+E2)*VKOS)*
109. 1V)
110. AL(I,J)=(L*VKOS*((-2.0*BETA*BETA-GMS+(E1+E2)*VKOS)*S+
111. 1T))
112. AK(J,J)=BETA*Z
113. AK(J,J)=BETA*Z/VKOS
114. GO TO 9
115. 16 AG(I,T)=(GFS/BETA)**2-BETA*LV*((E1+E2)*VKOS-GMS)+(E1*E1+2*E
116. 1*VKOS*VKOS*TAU-(E1+E2)*TAU*GFS*VKOS
117. 1AL(I,I)=BETA*BETA**2-BETA*BETA*VKOS*2.0*(E2-(E2-E1)*BT/(E1+E2))
118. 1KOS*GMS*(E2-(E2-E1)*BT/(E1+E2))+VKOS*VKOS*(E2*E2-(E2*E2-E1*E1)*
119. 2/(E1+E2))+2.0*TA*BETA*GMS
120. AG(I,I)=1.
121. AK(I,I)=0.
122. 9 CONTINUE
123. DO 28 M=1, KK
124. DO 28 N=1, KK
125. SUM=0.
126. RUM=0.
127. DO 30 MK=1, KK
128. SUM=SUM+AG(I,MK)*AL(MK,N)
129. 30 RUM=RUM+AK(MK,N)*AK(MK,N)
130. 28 PROD(M,N)=SUM-RUM
131. LZ=1
132. CALL F13AAF(FPROD,10, KK, 01, PEPE, LZ)
133. DET(K)=0
134. IF(K.EQ.1)GO TO 11
135. IF(DET(K)*DET(K-1).GT.0)GO TO 11
136. TAPUL=TAPUL+1.
137. IF(ABS(DET(K)).LE.1.0E-5)GO TO 40
138. IF(ABS(DET(K-1)).LE.1.0E-5)GO TO 41
139. IF(TAPUL.GT.3)GO TO 42
140. BETA=BETA-DELP
141. DELP=DELP/10.
142. K=0
143. GO TO 21
144. 11 BLTA=BETA+DELP
145. IF(K.GT.25)GO TO 1000
146. GO TO 20
147. 40 WRITE(3,510)KK,BETA
148. 50 FORMAT(1H0,*NUMBER OF TERMS=*,I2,* BETA=*,F14.6)
149. GO TO 43
150. 41 BETA=BETA-DELP

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144.      WRITE(3,500) KK,BETA
145.      GO TO 43
146.      42 ZULU=DELTA*ABS(DELTA(K))/(ABS(DELTA(K))+ABS(DELTA(K-1)))
147.      BETA=DELTA-ZULU
148.      WRITE(3,500) KK,BETA
149.      43 IF((KK+1).EQ.NTS)GO TO 7
150.      BETA=DELTA-0.04
151.      GO TO 26
152.      7 BETA=BETA+0.1
153.      GO TO 900
154.      900 WRITE(3,305)
155.      305 FORMAT(1H0,*NO CUTOFF FREQUENCY FOUND*)
156.      GO TO 900
157.      1300 WRITE(3,306)
158.      306 FORMAT(1H0,*NO VALUE FOR BETA FOUND*)
159.      900 CONTINUE
COMMENT - CONTINUE STATEMENT IS NOT DO TERMINATOR
160.      STOP
161.      END

```

NUMBER OF TERMS= 3	CUTOFF FREQUENCY=	.76138699
K0= .80		
NUMBER OF TERMS= 4 BETA=	27.983E+00	
K0= .90		
NUMBER OF TERMS= 4 BETA=	.577546E+00	
K0= 1.00		
NUMBER OF TERMS= 4 BETA=	.831343E+00	
K0= 1.10		
NUMBER OF TERMS= 4 BETA=	.119470E+01	
K0= 1.20		
NUMBER OF TERMS= 4 BETA=	1.76674E+01	
K0= 1.30		
NUMBER OF TERMS= 4 BETA=	.152261E+01	
K0= 1.40		
NUMBER OF TERMS= 4 BETA=	.174955E+01	

APPENDIX B

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1. PROGRAM DUA (INPUT, OUTPUT, TAPE1=INPUT, TAPE3=OUTPUT)
2. DIMENSION G(10), PJ1I(10), PJ2I(10), PJ1E(10), PJ2E(10)
3. DIMENSION ZP(10,1), ZP(10,10), W(10,10), V(10,10), GIF(10), QO(100)
4. DIMENSION P(10,10), O(10,10), F(10,10), DIFF(10,10), ITS(100), FR(100)
5. DIMENSION ET(100), OC(10,10), ORESO(10)
6. PI=ACOS(-1.0)
7. NS=4
8. G(1)=3.8317
9. C(2)=7.0156
10. G(3)=10.1735
11. G(4)=13.3237
12. E=20.
13. RE=1.
14. PI=2.
15. DO 75 MKL=1, NS
16. 35 EP(MKL)=0.0
17. DO 1 J=1, NS
18. X=G(I)*0.1
19. Y=G(I)*0.1
20. CALL RESU(X,0,PJ,1.0E-5,IER)
21. PJ1I(I)=PJ
22. CALL RESU(Y,0,PJ,1.0E-5,IFR)
23. PJ2I(I)=PJ
24. CALL RESU(X,1,PJ,1.0E-5,IER)
25. PJ1I(I)=PJ
26. CALL RESU(X,2,PJ,1.0E-5,IFR)
27. PJ2I(I)=PJ
28. CALL RESU(Y,2,PJ,1.0E-5,IER)
29. 1 PJ2I(I)=PJ
30. DO 2 J=1, NS
31. DO 2 J=1, NS
32. IF(I.EQ.J) GO TO 3
33. A=(E-1.0)*PI/(RE*RE*PI)
34. B=2*PJ1I(I)*PJ2I(J)*(G(J)*G(J)-G(I)*G(I))
35. DRESO(I)=-G(I)*PJ1I(I)
36. DRESO(J)=-G(J)*PJ1I(I)
37. F(I,J)=A*(G(I)*PJ1I(J)*DRESO(I)-G(J)*PJ1I(I)*DRESO(J))/3
38. Q(I,J)=(C+1.0)*P(I,J)
39. R(I,J)=(A*G(I)*G(J)/I)*(G(I)*G(J)*PJ1I(J)*(PJ1I(I)-PJ2I(I))-G(J)*
1(J)*PJ1I(I)*(PJ1I(J)-PJ2I(J)))/2.0
40. DIFF(I,J)=R(I,J)-2.0*G(I)*G(J)*P(I,J)
41. GO TO C
42. 7 ZP(I,J)=PI*PI*((G(I)*PJ1I(I))**2+PJ1I(I)**2)/(2.0*PI*RE*RE*PJ1E(I)
1*PJ2E(I))
43. P(I,I)=(E-1.0)*ZP(I,I)+(1.0/(2.0*PI))
44. Q(I,I)=(E*E-1.0)*ZP(I,I)+(1.0/(2.0*PI))
45. ZP(I,I)=1.0/(2.0*PI*RE*RE*PJ1E(I)*PJ2E(I))
46. W(I,I)=(PJ1I(I)-PJ2I(I))**2
47. V(I,I)=(PJ1E(I)-PJ2E(I))**2
48. GIF(I)=(G(I)*G(I))**2
49. R(I,I)=ZP(I,I)*(CTF(I)*(E-1.0)*(PI*PI/4.0)*W(I,I)+(G(I)*G(I)*PI*P
1-1.0)*(E-1.0)*PJ1I(I)*PJ1I(I)+GIF(I)*(RE*RE/4.0)*V(I,I))
50. DIFF(I,I)=P(I,I)-2.0*G(I)*G(I)*P(I,I)
51. 2 CONTINUE
52. K=4
53. 7 N=1
54. DO 5 L=1,K
55. DO 5 M=1,K
56. N=N+1
57. 5 QO(N)=Q(N,L)
58. KK=K-1
59. CALL MINV(QO, KK, 0)
60. K=1
61. DO 9 L=1,K
62. DO 9 M=1,K
63. N=N+1
64. 9 QO(N,L)=QO(N)
65. DO 11 L=1,K
66. DO 11 M=1,K
67. SUM=1.
68. DO 13 N=1,K
69. 13 SUM=SUM+DIFF(L,N)*Q(N,M)
70. 11 QO(L,M)=-SUM
71. L7=1
72. CALL F02AFF(CD,10, KK, ER, EI, ITS, LZ)
73. IF(L7.NE.0) GO TO 900
74. WRITE(3,300)K,L,R(1),FR(2),FR(3),FR(4)
75. 300 FORMAT(1H,*,NO. OF TAPES=*,I3,*,K0 SQUARE=*,4E16.0)
76. IF(K.EQ.(NS+1)) GO TO 900
77. K=K+1
78. GO TO 7
79. 900 CONTINUE
80. STOP
81. - CONTINUE STATEMENT IS NOT DO TERMINATOR

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82.
83. C A IS THE INPUT MATRIX, DESTROYED IN THE COMPUTATION AND REPLACED BY
C RESULTANT INVERSE.
C NZ IS THE ORDER OF MATRIX A, Q IS THE RESULTANT DETERMINANT,
C L AND M ARE WORKING VECTORS OF LENGTH NZ.
84. SUPERROUTINE MINV(A,NZ,Q)
85. DIMENSION A(100),L(100),M(100)
86. N=NZ
87. D=1.0
88. NK=-N
89. DO 20 K=1,N
90. NK=NK+N

91. L(K)=K
92. M(K)=K
93. KK=NK+K
94. BIGA=A(KK)
95. DO 20 J=K,N
96. IZ=M*(J-1)
97. DO 20 I=K,N
98. JZY=IZ+I
99. IF (ABS(BIGA)-ABS(A(JZY))) 15,20,20
100. 15 BIGA=A(JZY)
101. L(K)=I
102. M(K)=J
103. 20 CONTINUE
C INTERCHANGE ROWS
104. J=L(K)
105. IF (J-K) 35,35,25
106. 25 KI=K-N
107. DO 30 I=1,N
108. KI=KI+N
109. HOLD=-A(KI)
110. JI=KI-K+J
111. A(KI)=A(JI)
112. A(JI)=HOLD
C INTERCHANGE COLUMNS
113. 35 I=M(K)
114. IF (I-K) 45,45,38
115. 38 JP=N*(I-1)
116. DO 40 J=1,N
117. JK=NK+J
118. JI=JP+J
119. HOLD=-A(JK)
120. A(JK)=A(JI)
121. A(JI)=HOLD
C DIVIDE COLUMN BY MINUS PIVOT (VALUE OF PIVOT ELEMENT CONTAINED IN BIG
122. 45 IF (BIGA) 48,46,48
123. 46 D=1.0
124. RETURN
125. 48 DO 55 I=1,N
126. IF (I-K) 50,55,50
127. 50 IK=NK+I
128. A(IK)=A(IK)/(-BIGA)
129. 55 CONTINUE
C REDUCE MATRIX
130. DO 65 I=1,N
131. JK=NK+I
132. HOLD=A(IK)
133. JZY=I-N
134. DO 65 J=1,N
135. JZY=JZY+N
136. IF (I-K) 60,65,60
137. 60 IF (J-K) 62,65,62
138. 62 KJ=JZY-I+K
139. A(JZY)=HOLD*A(KJ)+A(JZY)
140. 65 CONTINUE
C DIVIDE ROW BY PIVOT
141. KJ=K-N
142. DO 75 J=1,N
143. KJ=KJ+N
144. IF (J-K) 70,75,70
145. 70 A(KJ)=A(KJ)/BIGA
146. 75 CONTINUE
C REPLACE PIVOT BY RECIPROCAL
147. A(KK)=1.0/BIGA
148.

```

```

C      FINAL ROW AND COLUMN INTERCHANGE
150.      K=N
151.      100 K=(K-1)
152.      IF (K) 150,150,105
153.      105 J=L(K)
154.      IF (I-K) 120,120,108
155.      108 J0=N*(K-1)
156.      JF=N*(I-1)
157.      DO 110 J=1,N
158.      JK=J0+J
159.      HOLD=A(JK)
160.      JT=JF+J
161.      A(JK)=-A(JT)
162.      110 A(JI)=HOLD
163.      120 J=N(K)
164.      IF (J-K) 100,100,125
165.      125 KI=K-N
166.      DO 130 I=1,N
167.      KI=KI+N
168.      HOLD=A(KI)

169.      JT=KI-K+J
170.      A(KI)=-A(JT)
171.      130 A(JI)=HOLD
172.      GO TO 100
173.      150 RETURN
173.      END

```

```

175
176.
177.      SUBROUTINE : PSJ(Z,N,PJ,D,IER)
178.      PJ=0.
179.      IF (N) 10,20,20
180.      10 IER=1
181.      RETURN
182.      20 IF (Z) 30,30,31
183.      30 IF 2=2
184.      K=1
185.      71 IF (Z-15.) 32,32,34
186.      32 NTEST=20.+15.*Z-Z**2/3.
187.      GO TO 36
188.      34 NTEST=9. +7/2.
189.      36 IF (N-NTEST) 40,38,38
190.      38 IER=4
191.      RETURN
192.      40 IER=0
193.      M1=N+1
194.      CORR=V=0.0
195.      IF (Z-5.) 50,60,60
196.      50 MA=7+6.
197.      GO TO 70
198.      60 MA=1.4*7+60./7
199.      70 M2=N+FIX(Z)/4+2
200.      M7=FC=MAX0(MA,M2)
201.      MMAX=N+1
202.      100 DO 100 M=ZZERO,MMAX,3
203.      FM1=1.0E-23
204.      FM=0.
205.      ALPHA=3.
206.      IF (M-(M/2)*2) 120,110,120
207.      110 JT=-1
208.      GO TO 130
209.      120 JT=1
210.      130 M2=M-2
211.      DO 160 K=1,M2
212.      MK=M-K
213.      BMK=2.*FLOAT(MK)*FM1/7-FM
214.      FM=FM1
215.      FM1=FMK
216.      IF (MK-N-1) 150,140,150
217.      140 PJ=BMK
218.      150 JT=-JT
219.      S=1+JT
220.      160 ALPHA=ALPHA+BMK*S
221.      BMK=2.*FM1/7-FM
222.      IF (N) 180,170,180
223.      170 PJ=FMK
224.      180 ALPHA=ALPHA+FMK
225.      CJ=CJ/ALPHA
226.      IF (ABS(CJ-CORR=V)-ABS(D*PJ)) 200,200,190
227.      190 CORR=V
228.      IER=3
229.      200 RETURN
230.      END

```