

Receptivity of a boundary-layer subject to vertical vibrations and the secondary instability of spanwise modulated shear flows

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Abstract

The viscous drag of a passenger aircraft is influenced significantly by the laminar-turbulent transition in the boundary-layer. Classically the transition process occurs when the turbulence develops as a result of the amplification of instability modes. In the flow past a unswept wing one such mode of instability is observed, Tollmien-Schlichting waves. We deal with the Tollmien-Schlichting waves or, more specifically, with the receptivity of the boundary layer with respect to them. If the surface becomes curved a different instability mode is created: Görtler vortices, which we will investigate as an inviscid secondary instability. Many disturbances can aid the generation of these instabilities but we concentrate on the vibrations of the wing caused by engine noise and the elasticity of the wing itself.

This thesis is separated into these two problems of boundary-layer flow over the vibrating wing surface. Firstly we focus on the generation of Tollmien-Schlichting waves due to wing surface vibrations and surface roughness. Piston theory is used to describe the response of the flow outside the boundary layer to wing surface vibrations. Then the perturbations in the Stokes' layer are analysed. The Stokes' layer itself cannot produce a Tollmien-Schlichting wave. Therefore, we will assume that there is a wall roughness. The analysis of the interaction of the Stokes' layer with a wall roughness can be analysed with the help of the Triple deck theory. This allows us to consider the downstream effects of our disturbances and under our flow regime, dependent on the size of wavelength of vibrations, we can Fourier transform and solve our problem for the disturbance pressure. Once we have inverted our solution back into real space with the use of Residue theory we are then able to calculate receptivity coefficients which can be compared to those of previous studies.

In the second problem we concentrate on the curved part of the wing where the flow is assumed to be slowly varying in the y direction. This generates Görtler vortices and we expect a sheared base flow with periodicity in the spanwise direction. Using a WKBJ approximation we can derive a multi-scale system of equations, which at leading order can be solved numerically to give eigenvalues and eigenfunctions representing pressure within the boundary-layer. We can only do so with the aid of a two-dimensional problem from which we fix an effective streamwise wavenumber, γ , and an effective maximum growth rate, $F(\gamma)$. We use this to create an initial value for our WKBJ eigenvalue, $\theta = \sqrt{\gamma^2 - \alpha^2}$. This restricts the values of the streamwise wavenumber, α , that we can calculate solutions for. This also means that when $\alpha \rightarrow \gamma$ we get a turning point and hence a breakdown of our WKBJ solutions. We derive and calculate solutions for $\alpha < \gamma$ and discuss the breakdown of these solutions.

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1. A.I. Ruban, T. Bernots and D. Pryce, Receptivity of the boundary layer to vibrations of the wing surface. *J. Fluid Mech*, 723:480-528, 2013.

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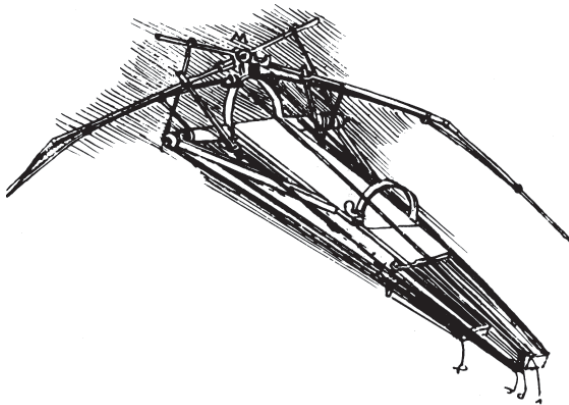
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Chapter 1

Introduction

1.1 Brief historical background: flight and theory

Throughout history man has shown a fascination with the phenomenon of flight. Early examples of this can be seen in the Greek mythology of Icarus and the sketches of ornithopters by Leonardo da Vinci (see figure 1.1). In the year 1783 man took this dream and made it a reality with the invention of hot air balloons by the Montgolfier brothers in Paris. The following centuries saw the rise of man's attempt to glide with the final aim to fly with the aid of an onboard motor. With this objective came many engineers, inventors but of course many casualties; notably Otto Lilienthal. This early German aviator was the first *airman*: someone who found the need to actually pilot the inventions himself to attain a "feel" for flight itself. Whilst experimenting a gust of wind caused his glider to stall and he died later of spinal injuries. Sad accidents like this happen across all of science, nevertheless many continued the work of Lilienthal; noting his successes as well as his fatal mistake. This ultimately leads us to the Wright brothers who are, of course, famous for the first successful powered flight in 1903. Orville and Wilbur spent many years before this experimenting on gliders, creating their own light weight motor, testing in their own wind tunnel and modifying their own fliers until they became the famous flying brothers we know of today.



(a) Da Vinci's sketches for an ornithopter.



(b) Ruben's portrayal of The Fall of Icarus.

Figure 1.1: Examples of the early intrigue into flight. Images courtesy of (a) Anderson (2005), (b) public domain.

However, it was Sir George Cayley who started this more scientific approach to flight. His work, which includes the design of the first aircraft wing in the late 19th century, earned him the title of the father of aeronautics (Anderson, 2005). He was the first to separate the notions of lift and propulsion on a flying machine, which is still seen in modern aircraft today. The list of Cayley's inventions is long, but a key notion he proposed was that of the four forces of aerodynamics; weight, lift, drag and thrust. His pioneering work was - for reasons unknown - mostly unseen by many of the engineers who followed him. If only they had looked into the easily accessible literature; powered and manned flight may have been realised many years sooner than it was. It would not be for 46 years after Cayley's death that the Wright brothers took to the skies. With this new ability to leave the earth it became more important to be able to measure and understand flow over a wings surface.

The earliest notable quantitative work into fluid flow is that of Sir Isaac Newton. His work recorded in the second book of *Principia Mathematica*, focussed mainly on the resistive forces acting on a body moving through a fluid. The mathematics contained in Newton's remarkable book encouraged applied mathematicians to describe fluid flow. In the works of Daniel Bernoulli, Jean d'Alembert and Leonhard Euler solutions to some simple fluid problems were given. Daniel Bernoulli is known for his self-named equation, which states that increasing the velocity of an inviscid fluid must simultaneously decrease the pressure or potential energy of the fluid, ensuring that the total energy per unit mass remains constant. Jean d'Alembert gives his name to a paradox relating to separation, which we will return to later. For now, we turn our attention to Leonhard Euler's work. The Swiss mathematician made many discoveries in so many areas of mathematics that it would take a whole thesis to describe them, but the one we are concerned with is the Euler equations. In 1757 Euler derived the equations for the conservation of mass and momentum in an inviscid fluid. With the addition of a conservation of energy equation - later derived by Laplace (1816) -

we have what has become known as the Euler equations:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \quad (1.1)$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p = 0, \quad (1.2)$$

$$\frac{\partial (\rho e)}{\partial t} + \nabla \cdot (\rho e \mathbf{u}) + p \nabla \cdot \mathbf{u} = 0, \quad (1.3)$$

where ρ is the fluid density, $\mathbf{u}$ is the vector of the fluid velocity, $\mathbf{0}$ is the zero vector, p is the pressure acting on the fluid and e is the internal energy of the fluid per unit mass. However these early beginnings in fluid dynamics were made under the assumption that the fluid is frictionless, which has little application to real fluid flow. The Euler equations, which only describe inviscid fluid flow, fail here.

It would be another hundred years before the Euler equations (1.1)-(1.3) were modified to include the effect of viscosity and so account for more realistic flows. This new system was derived independently by Claude-Louis Navier in 1822 and George Gabriel Stokes in 1845. The aptly named Navier-Stokes equations are still used as the basis for all studies of real flows with the inclusion of viscosity and body forces. Due to the nonlinearity of these partial differential equations, general smooth solutions in three dimensional have still not been proved to exist. However, under certain geometries, idealisations and flow conditions solutions can be found.

At this point we must mention the work of Osborne Reynolds - an english scientist who in 1883 investigated channel flow. He was the first to demonstrate the laminar-turbulent transition - where flow goes from having streamlines which are parallel to each other and any surface to one which is chaotic in nature - and to identify the critical value of a dimensionless parameter at which it occurs. This parameter became known as the Reynolds number and governs the viscous flows that the Navier-Stokes equations describe. It is defined as

$$Re = \frac{\rho_{\infty} V_{\infty} L}{\mu_{\infty}}, \quad (1.4)$$

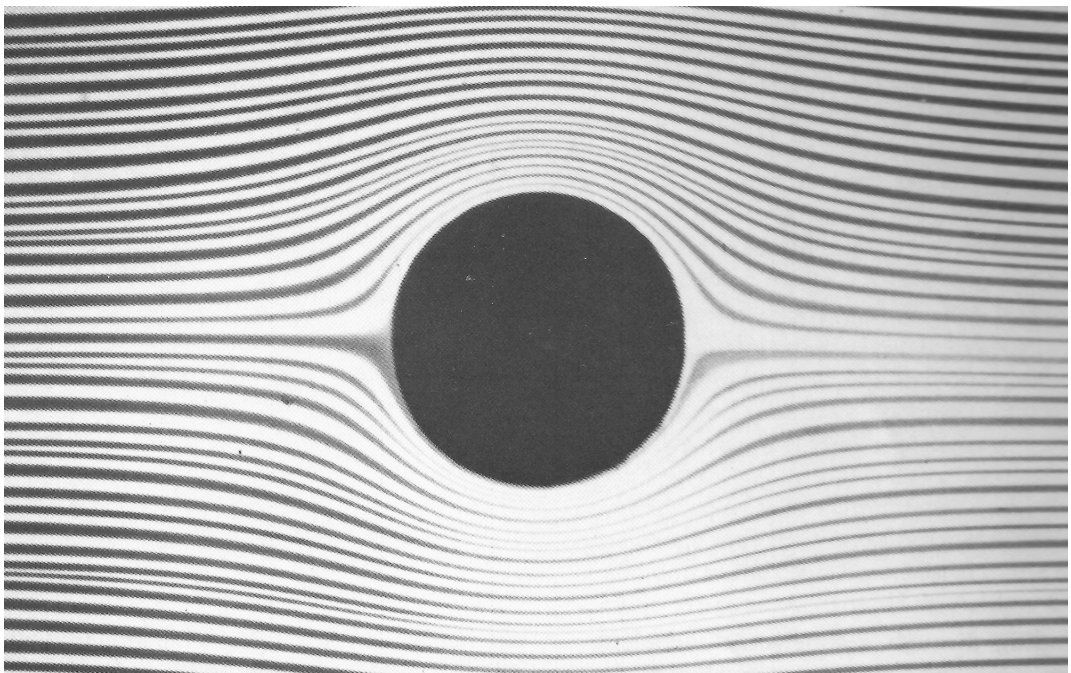
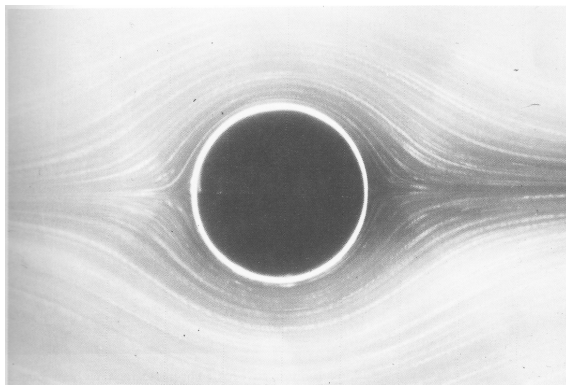
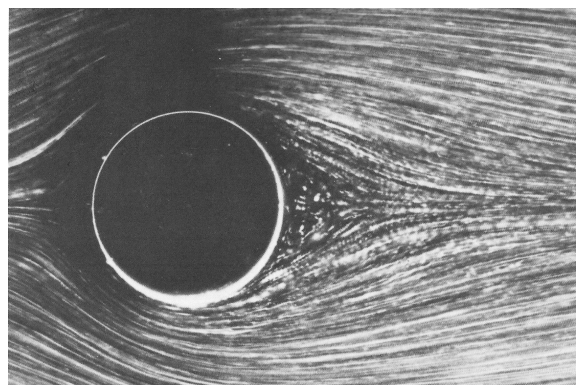


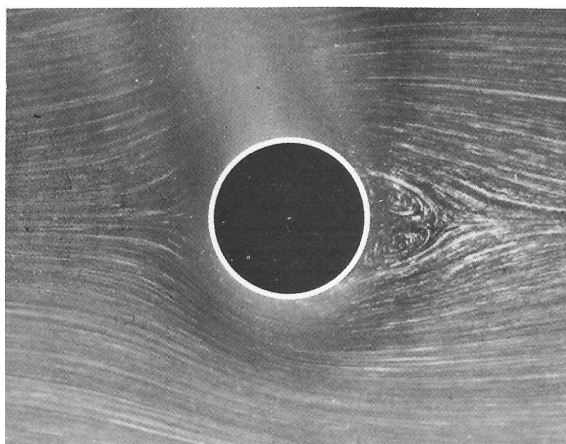
Figure 1.2: Hele-Shaw flow past a circle. Photograph by D.H. Peregrine (Dyke, 1982).



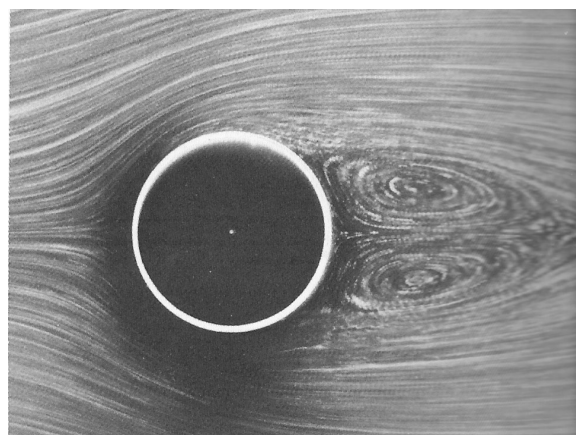
(a) Creeping flow: flow is symmetric in horizontal axis.



(b) Laminar flow: symmetry has been broken without separation occurring.



(c) Separation has begun behind cylinder.



(d) Pair of eddies can now be clearly seen downstream of the stagnation point.

Figure 1.3: Uniform flow past a circular cylinder for four different flow regimes. Photographs by Sadatoshi Taneda (Dyke, 1982).

where ρ_∞ is the fluid density, V_∞ is the fluid velocity, μ_∞ is the dynamic viscosity and L is the length scale which is problem dependent. The non-dimensional Navier-Stokes momentum equation can be written in the incompressible form ($\nabla \cdot \mathbf{u} = 0$) as

$$\frac{\partial \hat{\mathbf{u}}}{\partial \hat{t}} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} + \nabla \hat{p} = \frac{1}{Re} \nabla^2 \hat{\mathbf{u}} + \frac{\mathbf{f}L}{V_\infty}, \quad (1.5)$$

where $\mathbf{f}$ represents body forces. We may note that under the limit of Reynolds number tending to zero the inertial (right-hand side) terms dominate. Their balance against any change in pressure and body forces would give the aforementioned Stokes' equations for Stokes or creeping flows demonstrated in figure 1.2 and 1.3(a). The first photograph shows a circular plate held between two glass plates placed 1mm apart, the fluid is flowing from left to right at 1mm per second and so is an example of Hele-Shaw flow. Note the symmetry around the streamwise axis, which implies a lack of drag. Now whilst this is a thin film flow it does show the existence of flows that are symmetric around the streamwise axis. Flows of this form are called potential flows and are not realistic to air flow or three dimensional flows. Figures 1.3(a)-1.3(d) shows what we might intuitively expect: behind a moving body - here a circular cylinder - the symmetry is broken. For certain flow regimes separation may occur on the opposite side of the stagnation point eventually causing recirculation in the form of vortices, as seen most predominantly in figure 1.3(d). We would expect this to almost certainly apply a resistive force on the body.

As Reynolds number is increased the left-hand side terms in the Navier-Stokes equations must be taken into account. If we take Reynolds number to be very large then the time dependent, advective and pressure gradient terms dominate. This is the known inviscid limit and gives us the Euler equations once again. This relates back to the photographs of the circular cylinder and demonstrates exactly what the changes in the flow regime are in each image. Figure 1.3(a) has a small Reynolds number ($Re = 0.16$ in figure 1.3(a)) and so as idealised as it may be, it qualitatively verifies that the Navier-Stokes equations, under the required limit, can describe Stokes' flow. As for the more realistic flows seen in figures 1.3(b)-1.3(d) these have Reynolds numbers 1.54, 13.1 and 26 respectively. The point of separation occurs around $Re = 5$ and eventually the flow becomes unstable above

$Re = 40$. We have seen here that the Navier-Stokes equations describe both the inviscid and viscous limits. However, this theory is not without flaw. As observed by d'Alembert:

“an inviscid fluid offers no resistance to steady translational motion of a rigid body when the flow is everywhere irrotational”, which is paradoxical since *“rigid bodies do experience a resistance to motion through real fluid”* (Batchelor, 2000).

Fortunately, we have all the information used by Ludwig Prandtl to create boundary-layer theory and so provide a way to avoid d'Alembert's paradox in practice. In his paper Prandtl (1904), said to be one of the most important papers in fluid dynamics (Ackroyd et al., 2001), sets out the ideas of a boundary-layer. His new theory hypothesised that near the surface of a body a thin layer of viscous fluid obeyed the intuitive no-slip condition, i.e. that the fluid is not able to separate from the surface, and was described by a simplified version of the Navier-Stokes equations: the boundary-layer equations. Outside of this layer the flow can be described by the Euler equations and be thought of as inviscid. This then requires that a velocity at the outer edge of the boundary-layer must match the same velocity at the bottom of the inviscid region. Prandtl's theories began the final steps towards understanding how the drag we expect in real flows is created and how it relates to the surrounding flow, although his theories do not predict the separation of the flow to any degree of accuracy. Much of the subsequent work began to predict the separation behind bodies seen in experiments, which occurs where the pressure gradient is somehow increased to the point where drag created by pressure is much larger than that created by skin friction. Before this no-one understood the role of viscosity, the physical mechanism of separation or how to calculate skin friction drag; first explicitly calculated by one of Prandtl's students, Paul Richard Heinrich Blasius. Prandtl's timely work was the turning point for aerodynamics as well as fluid dynamics. We had finally understood that the need to calculate flow quantities and understand the flow of air over a wing was paramount.

Historically from this point on, designers of aircraft sought to build faster aeroplanes that could also reach greater altitudes. This could be said for most aircraft design of the 20th century until practicalities caught up with the industry and where flight had become accessible to many. Instead of faster and higher we became more concerned with safer, cleaner

for the environment, cheaper to build and run as well as improving reliability. With these constraints in mind we return to the idea alluded to by Reynolds in 1883, the transition from laminar to turbulent flow.

1.2 Laminar to Turbulent transition

From this point on we will consider flow over aerofoils or their simplification as flat plates. An aerofoil is the term that we will use to mean the shape of a wing looking from the end on, leading edge means the point at which the flow initially meets the aerofoil and the trailing edge means the last point downstream on the aerofoil. Laminar to turbulent transition prediction is necessary in order to understand the physical processes that may destabilise a fluid flow.

The drag of a passenger aircraft can be subdivided into three components; (i) the wave drag, which is due to the losses in the shock waves closing the supersonic regions that form on the upper and lower sides of the wing at transonic flight speed; (ii) the induced drag, which is due to the trailing vortices behind the wing; and (iii), the viscous drag. The latter is influenced significantly by the position of the laminar-turbulent transition in the boundary-layer.

We have already discussed how the cause of skin friction in the boundary-layer creates drag behind a body and this is equally applicable in aerofoils. The boundary-layer over an aerofoil starts off laminar near the leading edge but at some downstream position the boundary-layer starts to become unstable over a transitional region until it is completely dominated by unstable flow. This is called a turbulent boundary-layer. In practice air is high Reynolds number flow and in such circumstances turbulent flow is natural and only by removing all possible disturbances can the leading edge laminar boundary-layer be maintained.

Receptivity theory is a branch of fluid dynamics, the importance of which has been highlighted by various experimental observations. In particular, it appears that when the same

aerodynamic body was tested in different wind tunnels the laminar-turbulent transition would not take place at the same position on the wing surface. This apparent contradiction to intuition occurs despite the principal similarity of flow parameters, the Reynolds and Mach numbers, which were reproduced in the tunnels. It is now generally accepted that the difference in the position of the transition can be explained by seemingly less important factors as mentioned previously. The main conclusion is that the less turbulent the wind tunnel the longer the boundary-layer stays laminar. This phenomenon cannot be understood without taking into account the interaction of the boundary-layer with surrounding perturbation field. Analysis of possible forms of the interaction is the subject of the receptivity theory. Development of the theory is important in view of the fact that the level of the atmospheric turbulence in real flight conditions is lower than that in the wind tunnel tests, making the direct experimental simulation of the laminar-turbulent transition process a difficult proposition. The importance of the receptivity as an initial stage of the laminar-turbulent transition was first highlighted by Morkovin (1969).

Transition is said to take place at some critical downstream location then permitting the definition of a critical Reynolds number for which transition will occur. Even with years of study and a now large amount of literature on transition we have yet to find a mathematical model which can predict the critical Reynolds number on a flat plate (Saric et al., 2002). This is due to the various disturbances that can effect the transition of a boundary-layer: free-stream turbulence, pressure gradient, Reynolds number, Mach number, acoustics, surface roughness, surface temperature, curvature of streamlines and surface and any previous history of these in the flow. These varied disturbances establish different initial conditions which create transition and ultimately turbulence (as shown simply in figure 1.4). In this thesis we will be looking at route A on figure 1.4 and in particular focusing on a different, previously little studied disturbance; surface vibrations of the aerofoil coupled with its interaction with a surface roughness. The surface roughness and vibrations we assume already exist within the boundary-layer and so produce steady and unsteady perturbations to the assumed base flow. We will also discuss the Rayleigh wave second instability created by curvature of the wing and seen propagating within Görtler vortices.

We assume that Tollmien-Schlichting (T-S) waves (discussed in §1.3) are produced by our

disturbance and that they can then be described by the linearised Navier-Stokes equations. Following a possible neutral stability point a slowly varying flow over a curved surface can create a three dimensional secondary instability, which we study in the form of Görtler vortices (discussed in §1.4). From this point onwards the transition from laminar to turbulent flow will more than likely be over.

We have mentioned the two instabilities we expect to occur in our problem and we will now briefly concentrate on the conditions of their discoveries, occurrence within real flows and work surrounding them. First we briefly discuss T-S waves, which can arise over an aerofoil.

1.3 Tollmien-Schlichting wave instabilities and their analysis.

In the flow past a swept wing two modes of instability are observed: the cross-flow vortices and T-S waves. Cross-flow vortices dominate the transition process on an aerofoil with larger sweep angle, characteristic of long distance passenger carriers. T-S waves prevail in the case of smaller sweep angles, characteristic of regional aircraft.

In relatively smooth flows, such as a passenger aircraft flight, the initial amplitude of the T-S waves generated at the receptivity stage is insufficient to provoke immediate transition. The T-S waves must first amplify in the boundary-layer to trigger nonlinear effects, typical of the transition process. Consequently, a correct prediction of the location of the transition on the wing surface is impossible without accurate description of the receptivity process.

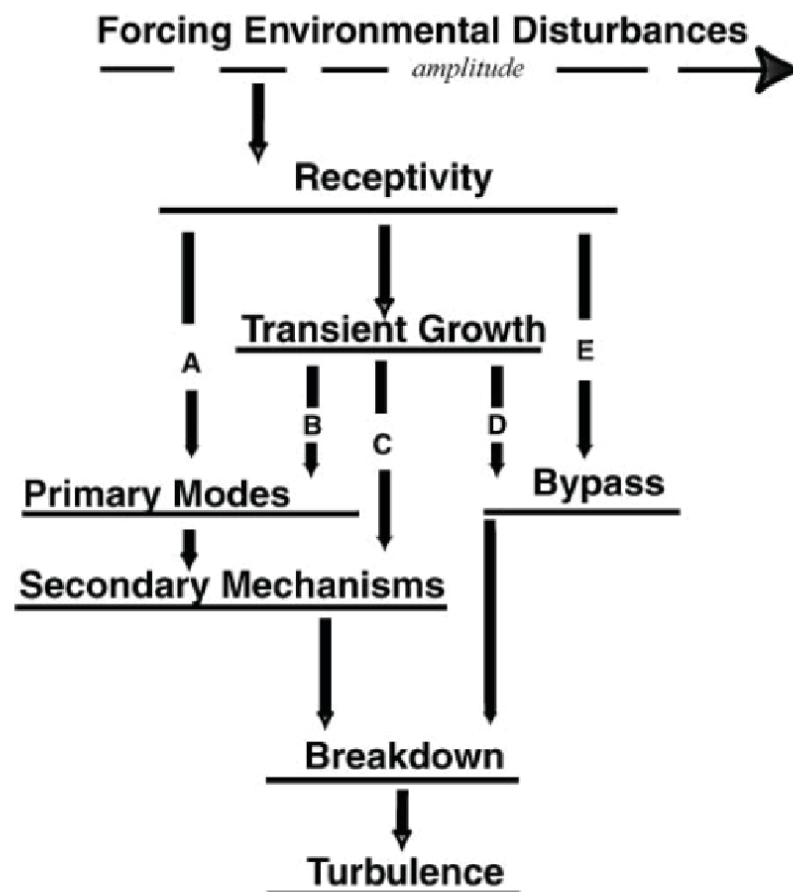


Figure 1.4: The paths from receptivity from various initial conditions to transition (Saric et al., 2002).

Numerous experimental studies starting with Schubauer and Skramsted (1948) have shown that some disturbances easily penetrate into the boundary-layer and turn into instability modes. This category includes, among others, acoustic waves, free-stream turbulence and local and distributed wall roughness. These perturbations have to satisfy restrictive resonance conditions which were first formulated by Kachanov et al. (1982). Unlike in a simple mechanical system, for example a pendulum, where the resonance is observed provided that the frequency of the external forcing is close to the natural frequency of the pendulum oscillations. In fluid flows, effective transformation of external disturbances into instability modes of the boundary-layer is only possible if the wavenumber and frequency of the external perturbations is in tune with the natural internal oscillations of the boundary-layer.

The theoretical study of the receptivity phenomenon has been developing in two main streams. The first of these relies on finite Reynolds number methods where the modified Orr-Sommerfeld equation is used with forcing terms representing the external perturbations. Papers by Zhigulev and Tumin (1987), Choudhari (1994), Ng and Crouch (1999) and Schrader et al. (2009) are examples of this. In the second stream are the studies that rely on the asymptotic analysis of the Navier-Stokes equations at large values of the Reynolds number. When dealing with the boundary-layer receptivity to T-S waves, this approach relies on the triple-deck theory.

1.3.1 Triple-deck theory

The triple-deck theory was first formulated for steady flows. It was derived simultaneously by Neiland (1969) and Stewartson and Williams (1969) in their studies of the self-induced separation of the boundary-layer in supersonic flow; and by Stewartson (1969) and Messiter (1970), who analysed the incompressible fluid flow near the trailing edge of a flat plate. In both flows the boundary-layer was shown to interact with the inviscid part of the flow. The region of the interaction extends along the body surface over the streamwise distance $\Delta x = \mathcal{O}(Re^{-3/8})$, and has a three-tiered structure shown in figure 1.5.

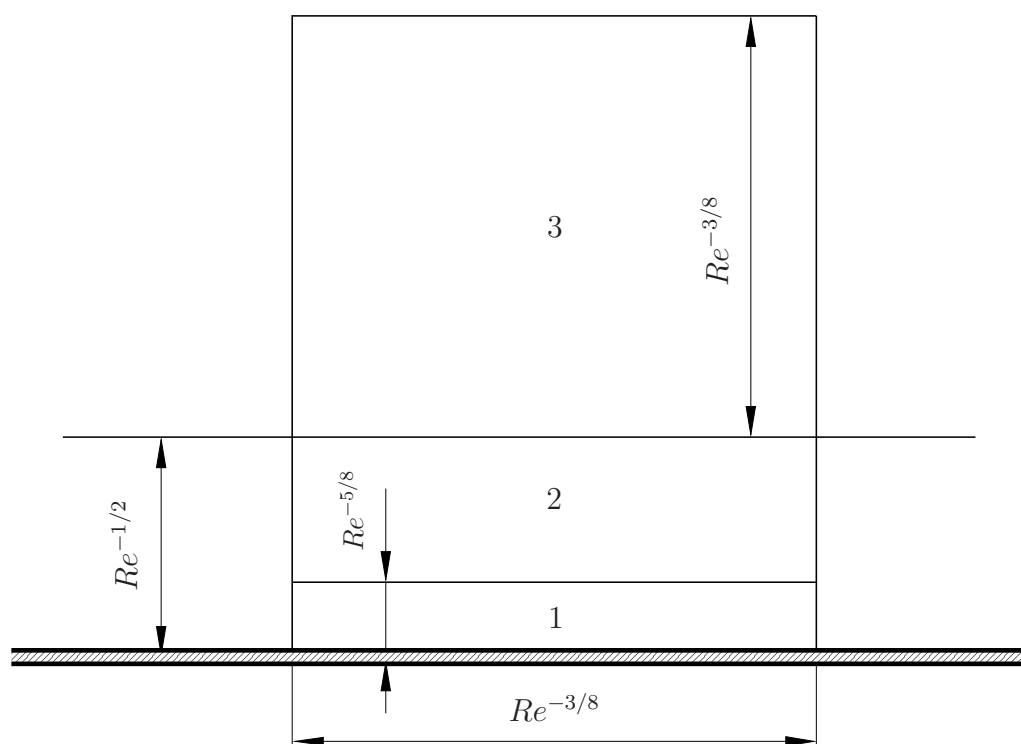


Figure 1.5: Three-tiered structure of the interaction region.

It is composed of the viscous sublayer (shown as region 1), the main part of the boundary-layer (region 2) and an inviscid potential flow (region 3) situated outside the boundary-layer. The characteristic thickness of the viscous sublayer is an $\mathcal{O}(LRe^{-5/8})$ quantity, so that it occupies an $\mathcal{O}(Re^{-1/8})$ portion of the boundary-layer and is comprised of the stream filaments that form the Stokes layer in the flow upstream of the roughness. The flow velocity in region 1 is $\mathcal{O}(Re^{-1/8})$ relative to the free-stream velocity, and due to the slow motion of gas here the flow exhibits high sensitivity to pressure variations. As a result the flow filaments change their thickness leading to a deformation of streamlines. This process is termed the displacement effect of the boundary-layer.

The main part of the boundary-layer, the middle tier of the triple-deck structure, is a continuation of the conventional boundary-layer developing along the aerofoil surface. Its thickness is estimated as $\mathcal{O}(LRe^{-1/2})$ and the velocity is an order V_∞ quantity. The flow in this tier is significantly less sensitive to the pressure variations. It does not produce any noticeable contribution to the displacement effect of the boundary-layer, which means that all the streamlines in the middle tier are parallel to each other and carry the deformation produced by the displacement effect of the viscous sublayer.

Finally, the upper tier is situated in the potential flow region outside the boundary-layer. It serves to convert the perturbations in the form of the streamlines into those of pressure. These are then transmitted through the main part of the boundary-layer back to the viscous sublayer. This process is self-sustained, and is called the viscous-inviscid interaction.

Following the pioneering works of Neiland (1969), Stewartson and Williams (1969) and Messiter (1970), many researchers were involved in the development of the theory. It became clear that the triple-deck model, which describes the interaction between the boundary-layer and inviscid flow outside the boundary-layer, plays a key role in a wide variety of fluid dynamic phenomena. An exposition of applications of the theory to different forms of the boundary-layer separation may be found in the monograph by Sychev, Ruban, Sychev & Korolev (Sychev et al., 1998).

Schneider (1974) was the first to extend the triple-deck theory to unsteady flows. He noticed that the most sensitive to unsteady perturbations is the flow in the viscous near-wall tier

(region 1) where the fluid velocity, u , is $Re^{1/8}$ times smaller than that at the outer edge of the boundary-layer. Taking this into account he deduced that the flow in region 1 becomes unsteady when the characteristic time is an order $Re^{-1/4}$ quantity; the flow in regions 2 and 3 still remains quasi-steady.

Later Smith (1979a,b) demonstrated that in subsonic flows the unsteady triple-deck theory describes the T-S waves at and near the lower branch of the neutral stability curve.¹ Smith's study was preceded by Lin's analysis of the asymptotic behaviour of the lower and upper branches of the neutral stability curve in incompressible boundary-layers (Lin, 1946). Lin used the Orr-Sommerfeld equation as a starting point in his work, aiming to solve this equation in the limit when the Reynolds number tended to infinity. When dealing with the lower branch he found that the solution develops a three-layered structure with the layers' thicknesses being the same as in figure 1.5. As a result of his analysis Lin also found the frequency of the oscillations and the wavenumber to be $\omega = \mathcal{O}(Re^{1/4})$ and $k = \mathcal{O}(Re^{3/8})$, respectively. The initial assumption that the flow perturbations could be described by the Orr-Sommerfeld equation precluded non-parallel and nonlinear effects to be a part of Lin's study. These restrictions were lifted when the full version of the triple-deck theory became available; see Smith (1979a,b).

The first paper where the triple-deck theory was used to study the receptivity of the boundary-layer was published by Terent'ev (1981). He considered an incompressible fluid flow past a flat plate with the basic steady flow given by the Blasius solution. He assumed that a short section of the plate surface performs periodic vibrations in the direction perpendicular to the wall. The frequency of the vibrations was chosen to be $\omega = \mathcal{O}(Re^{1/4})$ and the length of the vibrating section $\Delta x = \mathcal{O}(Re^{-3/8})$. This represents a simplified mathematical model of the classical experiments by Schubauer and Skramsted (1948) where the T-S waves were generated by a vibrating ribbon installed a small distance above the plate surface. Terent'ev was able to describe in detail the perturbation field around the vibrating part of the wall. In particular, he demonstrated that the T-S wave forms downstream of the oscillator. The

¹Neutral stability curves are the representation of wavenumber against Reynolds number which separates stable and unstable flows. See Lin (1946) p.285 for the general nature of a neutral stability curve for parallel flows.

amplitude of the wave can be found if the amplitude and the shape of the vibrating part of the wall are known.

Experimental observations further show that the boundary-layers are susceptible to acoustic noise. The asymptotic theory of the generation of T-S waves by acoustic noise was developed by Ruban (1984) and Goldstein (1985), independently. Their theory shows that if the acoustic field has a wide enough spectrum, then the receptivity process will extract from it a harmonic whose frequency is in tune with the frequency of the corresponding T-S wave. Of course, under this condition the wave length of the chosen acoustic wave appears to be much longer than the wave length of the T-S wave, which means that the second resonance condition - the tuning of the wavenumbers - is not satisfied. However, if the surface of the wing is not absolutely smooth - which is the case in all practical applications - then one also needs to look at the perturbations produced in the boundary-layer due to the wall roughnesses. These perturbations are steady and have a short length scale necessary for the resonance. Using the triple-deck model Ruban and Goldstein were able to find the amplitude of the T-S wave formed behind the roughness. It proved to be proportional to the amplitude of the acoustic wave and the Fourier transform of the roughness shape calculated for the value of the wavenumber. This coincides with the wavenumber of the T-S wave forming in the boundary-layer.

Along with acoustic waves, the free-stream turbulence is also known to have a significant influence on the laminar-turbulent transition in the boundary-layer. The asymptotic theory of the receptivity of the boundary-layer to the free-stream turbulence was given by Duck et al. (1996). They noticed that there is a significant difference in the way the boundary-layer interacts with acoustic waves and vorticity waves. The acoustic waves carry pressure perturbations that penetrate into the boundary-layer and lead to a formation of the Stokes layer near the body surface. The situation with the vorticity waves is different. They do not carry pressure perturbations and therefore are unable to penetrate into the boundary-layer. However, a wall roughness produces perturbations not only inside the boundary-layer but also in the upper tier of the triple-deck structure (region 3 in figure 1.5). Nonlinear interaction of the steady perturbations in region 3 with vorticity waves creates the forcing necessary for the T-S wave production.

For practical applications the case of distributed wall roughness is important. The theory of distributed receptivity was first presented by Wu (2001) for acoustic and vortical perturbations interacting with the boundary-layer, and then Kerimbekov and Ruban (2005) considered the T-S wave generation on an extended section of the body surface performing wave like vibrations with the characteristic wavenumber of order $\mathcal{O}(Re^{3/8})$. In both studies the flow analysis relied on multi-scale asymptotic approach suggested earlier by Ruban (1983) and Hall and Smith (1984) in their analysis of the effects of nonlinearity and non-parallelism on the growth of the T-S wave in the boundary-layer.

As already discussed these instabilities occur with wings of small sweep angle. Due to the scales and nature of the problem the interaction region covers a very small part of the aerofoil and we assume it to be like a flat plate when at rest. This is not a good assumption particularly further down the wing where curvature plays an important role. We now turn to instabilities which occur from the curvature of streamlines and specifically the curvature of the wing surface.

1.4 Instabilities due to curvature

The Taylor vortex instability is caused by curved surfaces and seen in flow between two rotating cylinders (Taylor, 1923). This has been thoroughly studied, both experimentally and theoretically. In this problem, when increasing the Taylor number sufficiently, a secondary instability that can occur is often referred to as a wavy vortex mode and causes the breakdown of Taylor vortices in the circumferential flow between cylinders of almost the same radius. Davey et al. (1968) describe this breakdown, which leads to the wavy vortex state and suggest it might well be of inviscid origin. Many experiments have also been performed that have lead to a deep understanding of the sequence of bifurcations to the Taylor problem. Note that in a Taylor vortex experiment the control parameter which governs the flow is constant in the steady regime. The mechanism we are interested in, however, is the Görtler vortex instability, which is known to be similar to the Taylor vortex instability. The Görtler mechanism is destroyed by an asymptotically small spanwise

mean flow, so it appears that the stationary vortex structures are associated with Rayleigh waves. It would be surprising if the wavy vortex instability in three dimensional Görtler vortices did not occur at $\mathcal{O}(1)$ vortex wavelengths. The main difference between T-S waves and Taylor vortices compared to Görtler vortices is the fact that $\mathcal{O}(1)$ -wavelength vortices evolve in a non-parallel manner and so a unique curve or growth rate is not justifiable (Hall, 1983 and 1988). In the Görtler problem, the downstream variable acts as the control parameter. This effectively means that we cannot restrict our attention to small increases in this parameter as we would for the Taylor problem, explaining why there are not as many experiments on Görtler vortices as there are on Taylor vortices.

Figure 1.6 shows a typical flow structure of Görtler vortices. The linear and nonlinear development of the Görtler and Taylor vortices are quite distinct. The Görtler vortex problem is one of spatial development of the boundary-layer, meaning that a self-consistent asymptotic description of the linear stages of the vortex development is not easy to obtain. Due to difficulties in performing Görtler experiments little progress has been made when attempting to comprehend the sequence of bifurcations compared to the Taylor vortex problem. Very little understanding of the growth of Görtler vortices can be inferred from the Taylor problem. In a three dimensional boundary-layer the wave disturbance - a stationary vortex instability - with an effective velocity profile having zero stress at the wall is possible.

We now consider the experimental investigations that followed from Görtler's original work (Görtler, 1940). In the mid 1940's Liepmann (1943, 1945) investigated the instability of boundary-layers on curved walls and came to the conclusion that transition to turbulence can be caused by Görtler vortices. The first to actually observe the vortices were Gregory and Walker (1956) who used the china-clay technique and induced the vortices using protrusions. Aihara (1962) demonstrated the existence of the vortices while Tani and Sakagami (1962) continued to attempt to observe the disturbances using

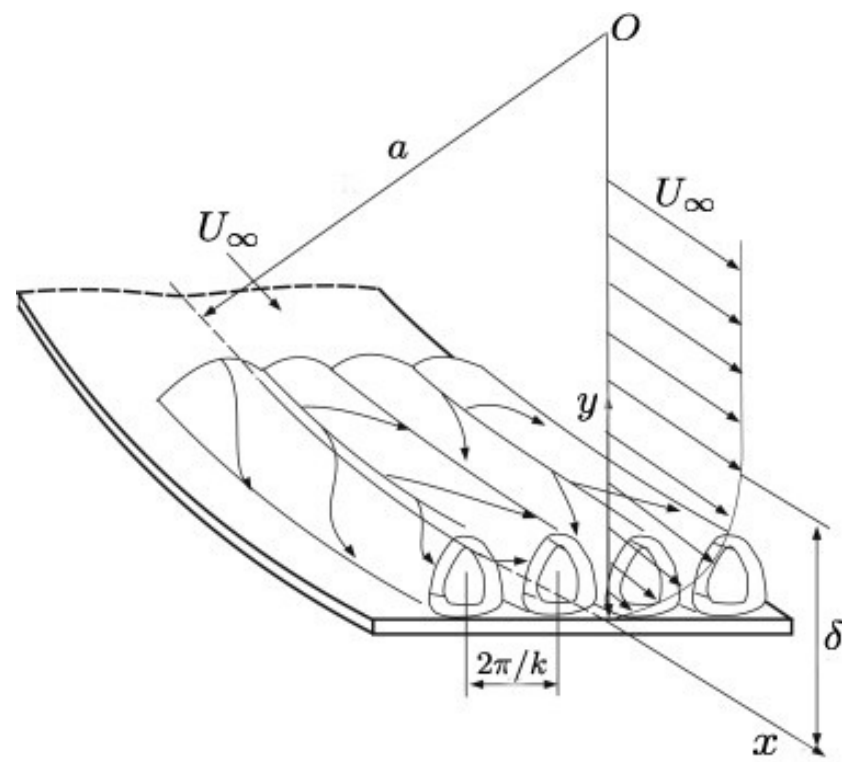


Figure 1.6: The flow pattern associated with Görtler vortices. Modified from figure 2 in Saric (1994)

smoke. The experiments of Tani (1962) and Tani and Sakagami (1962) concluded that the induced experimental vortex wavenumber is independent of the free-stream speed, the spanwise dimension and the streamwise location of the onset of vortex activity. The tellurium method² was used to carry out more detailed flow visualisations of Görtler vortices by Wortmann (1964, 1966) who was also able to show that leaning vortices could be induced in certain environments.

Of these early observations of Görtler vortices the most detailed came from Bippes (1972) and Bippes and Görtler (1972), who used the hydrogen bubble method and were able to measure the associated eigenfunctions. In the ensuing years several experiments were designed to see if the mechanism was significant on the curved part of modern laminar flow wings, including Pfenninger et al. (1980), Harvey and Pride (1982) and Allison and Dagenhart (1987). In particular, at NASA Langley experiments were carried out on a region of concave curvature on the underside of a wing beyond the leading edge by Mangalam et al. (1985, 1987).

In the 1972 experiment Bippes found that the wavenumber was dependent on the incoming disturbance. Swearingen and Blackwelder (1987) were able to answer the question of how the upstream conditions influence the induced vortices by performing low speed wind tunnel experiments to find the mechanism by initially fixing the vortex wavelength within their tunnel with tape. They also demonstrated that in the early stages of the vortex development the disturbance field is steady and takes the form of spanwise periodic counter rotation. These initial stages appear to be extremely sensitive to the upstream flow. This periodicity is often fixed by a certain type of wall forcing mechanism.

The experiments discussed above focus on showing that Görtler vortices can be created experimentally as predicted from the linear stability theory. They also confirmed that nonlinear effects will inhibit growth and lead to a finite amplitude equilibrium, since it was found that after the onset of instability the vortex disturbance increased slowly in the streamwise direction. Hence, given the absence of a threshold amplitude response

²A method for visualising the flow in which a cloud of tellurium is generated in a liquid through an irreversible electrolytic reaction, forming a tracer.

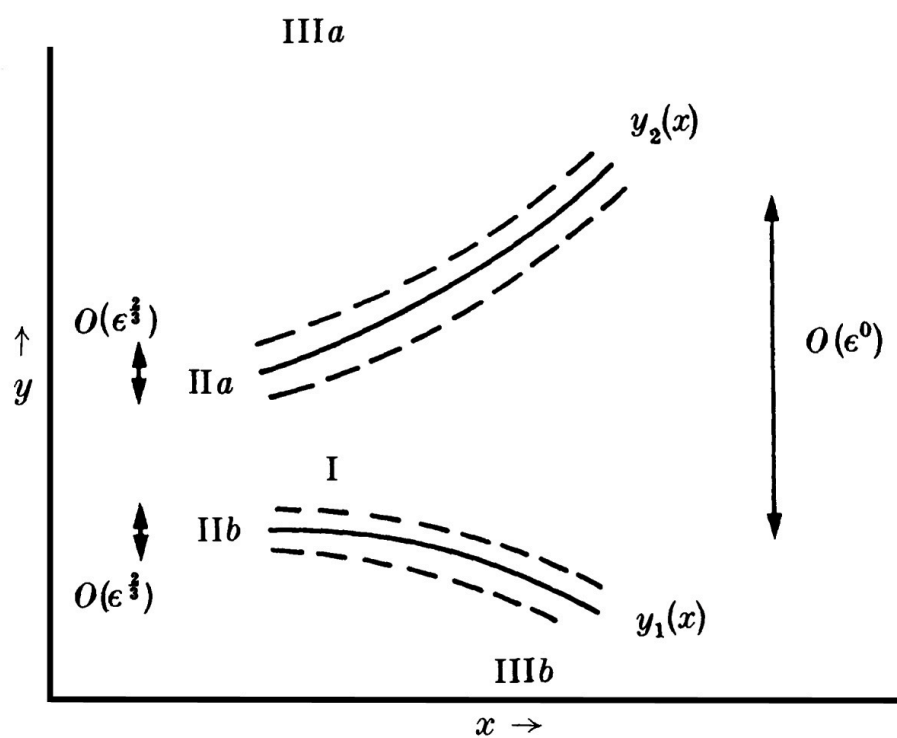


Figure 1.7: The different layers beyond the downstream position of neutral stability. Image scanned from Hall and Lakin (1988).

by the boundary-layer to longitudinal vortices, nonlinear effects are stabilising in the Görtler problem. Experiments have also shown that once the instability has settled, a finite-amplitude state is generated in the boundary-layer and evolves in the streamwise direction, then at a downstream location this steady state undergoes a secondary instability to a three dimensional unsteady disturbance. Another possibility is that the breakdown leaves the vortex boundaries flat, instead of wavy, but causes horseshoe vortices to be generated. T-S waves can also be involved in this breakdown process but only if the wall curvature is sufficiently small to delay the onset of Görtler vortices to a Reynolds number high enough that T-S waves are unstable. It appears that there are at least two separate modes of instability in longitudinal vortex structures.

For the Görtler problem, Hall and Seddougui (1989) theoretically described the onset of the wavy vortex structure, where non-parallel effects are not important on small-wavelength vortices, the results of which were consistent with the experiments of Peerhossaini and Wesfried (1988a,b). The theory showed that two wavy vortex modes exist in small-wavelength Görtler vortices and that they are localised in the normal direction between two thin shear layers above and below the region of vortex activity. The structure for this nonlinear small wavelength vortex system can be found in Hall and Lakin (1988) but a similar idea was put forward years previously by Malkus (1956). The postulation was that the mean part of a turbulent flow would organise itself so that any modes were marginally stable. In figure 1.7 vortex activity is concentrated in layer I and is bounded by transitional layers IIa and IIb of depth $\epsilon^{2/3}$, where ϵ refers to the non-dimensional wavelength of a vortex. The activity is reduced to zero outside of these layers by solutions inside IIa and IIb to the Airy's differential equation. The remainder of the boundary-layer flow is labelled as IIIa and IIIb, where the flow has no spanwise dependence and the velocity field satisfies the boundary-layer equations. These solutions must be matched to the IIa, b solutions.

In general, the activity within layer I will grow as the flow moves downstream to the point where it fills the boundary-layer only leaving a thin wall layer. The flow now has little relation to the flow that would exist without the generation of the vortices.

Bassom and Seddougui (1990) showed that some of the wavy modes in the Görtler problem

are inviscid, highlighting the similar result from the Taylor problem performed by Davey et al. (1968). Hall and Horseman (1991) looked into whether some of the experimentally observed breakdown routes of Görtler vortices owe their origin to an inviscid instability mechanism and found two modes of instability that correspond to those found in experiments.

1.5 Layout of thesis

Chapter 2 will address the receptivity of T-S waves due to vibration of the wings surface. We shall show that the boundary-layer displays high sensitivity to this form of excitation. We will also use piston theory to model the vibrations over a certain streamwise wavelength, and triple-deck theory to model the interaction of the wing vibrations with a roughness on the surface. By assuming the interaction remains within a thin Stokes layer near the surface and within the boundary-layer we then create a certain flow regime, with respect to the disturbance wavelength, which we focus our attention on.

Receptivity of this form was not highlighted by experimentalists because laminar-turbulent transition experiments are normally conducted on rigid surfaces. Meanwhile, a real aircraft wing has a flexible surface that is susceptible to elastic vibrations. Further, modern jet engine nacelles are known to exhibit oscillations.

We begin with the mathematical formulation of the problem given in §2.1, followed by the flow analysis outside the boundary layer. It is shown that unless the Mach number is very small the pressure perturbation field may be calculated using the piston theory (see, for example, Liepmann and Roshko (1967)). As the pressure perturbations penetrate into the boundary-layer, the Stokes layer forms near the body surface; this process is studied in §2.2. In order to simplify the flow analysis we assume that the amplitude of the wall vibrations is small. At this point we found that two different physical mechanisms effect the flow and depend on the size of the wavelength of wing vibrations, δ_x . If the wave length of the vibrations of the wing surface is an order $Re^{-1/8}$ quantity, then the oscillations of the streamwise velocity in the Stokes layer are caused by both the streamwise pressure

gradient and by the pressure oscillations in time. A feature of this mechanism is that the compressibility of the flow can be ignored and the pressure perturbations remain within the Stokes layer. When $\delta_x = \mathcal{O}(Re^{-1/8})$ or $\delta_x \gg Re^{-1/8}$ then the solution in the Stokes layer appears to be influenced significantly by the compressibility of the flow. The first mechanism linked to the streamwise pressure gradient can be seen in this thesis, while the second is included in Ruban et al. (2013). In §2.3 we analyse the flow in the triple-deck region which forms as a result of the interaction of the Stokes layer with a roughness on the wing surface. The solution of the triple-deck problem is presented in §2.4. We show that in the vicinity of the roughness the flow perturbation field is composed of discrete and continuous spectrums. We find that the continuous spectrum perturbations decay downstream of the roughness uniformly with respect to the frequency of oscillations, and so do all the perturbations of the discrete spectrum except the one which represents the T-S wave. The amplitude of this wave is found in an analytic form and the results are discussed in §2.5.

In chapter 3 we analyse the linear inviscid secondary instability of longitudinal spanwise modulated vortices, created due to surface curvature within the incompressible boundary-layer. An instability of this type has a wavelength that can be compared to the thickness of the boundary-layer. This means that a quasi-parallel approach to this problem can be used and from this a Rayleigh equation problem is obtained and solved. The work of Hall and Horseman (1991) provides the basis for this chapter; however, the analysis performed is just as relevant to a secondary instability of the vortex structures produced in the later stages of boundary-layer transition to turbulence (Hall and Horseman, 1991; Hall and Smith, 1991). The numerical and experimental works of (Hall and Sherwin, 2010; Swearingen and Blackwelder, 1987) are also used to compare to these already known results and our own.

We note that the nonlinear scheme of Hall (1988) can be used to determine the evolution of finite-amplitude $\mathcal{O}(1)$ vortices in a curved boundary-layer and solely within layer I, described above by Hall and Lakin (1988). We investigate the instability of the three dimensional state to an inviscid Rayleigh instability, which is a local stability problem. The formulation of the problem is given in §3.2. We then continue to separate the solution via an amplitude function. A numerical solution to the leading order differential equation is given in §3.5. We can then directly calculate our required eigenvalues, amplitude

function and finally full solutions to the three dimensional problem with the aid of a two dimensional problem for initial values. The solutions will be damped away from the centreline, as required for this problem, and we will discuss this. However, breakdown of the given solutions occurs with the coalescence of the effective two dimensional streamwise wavenumber, γ , and the three dimensional streamwise wave number, α , along the centreline $\bar{y}$ of the layer within Region 1 of depth $1/k$ as discussed within §3.6.

Finally, in chapter 4 we summarise the main results of the thesis and propose some extensions to the research herein.

Chapter 2

Receptivity of the boundary-layer due to vibrations of the wing surface

2.1 Formulation of the problem

A subsonic flow regime is considered with the Mach number outside the boundary-layer, M_∞ of $\mathcal{O}(1)$. The flow is investigated based on the asymptotic analysis of the Navier-Stokes equations at large values of the Reynolds number, defined by

$$Re = \frac{\rho_\infty V_\infty L}{\mu_\infty}, \quad (2.1)$$

using the same definitions for density, viscosity and fluid velocity as in (1.4). Let us consider an aerofoil placed in uniform subsonic two dimensional flow of a perfect gas. In what follows we shall use the body-fitted curvilinear orthogonal coordinate system $(\hat{x}, \hat{y})$ with $\hat{x}$ measured along the aerofoil surface from its leading edge, and $\hat{y}$ in the normal direction. The velocity components in these coordinates are denoted as $(\hat{u}, \hat{v})$. We further denote the gas density by $\hat{\rho}$, the pressure $\hat{p}$, the enthalpy $\hat{h}$ and the dynamic viscosity coefficient $\hat{\mu}$. The hat is used to signify that the corresponding variables are dimensional.

The flow analysis will be conducted using the Navier-Stokes equations written in the body-

fitted coordinates,

$$\hat{\rho} \left(\frac{\partial \hat{u}}{\partial \hat{t}} + \frac{\hat{u}}{1 + \kappa \hat{y}} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\kappa \hat{u} \hat{v}}{1 + \kappa \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\partial}{\partial \hat{y}} \left(\hat{\mu} \frac{\partial \hat{u}}{\partial \hat{y}} \right) + \dots, \quad (2.2a)$$

$$\hat{\rho} \left(\frac{\partial \hat{v}}{\partial \hat{t}} + \frac{\hat{u}}{1 + \kappa \hat{y}} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} - \frac{\kappa \hat{u}^2}{1 + \kappa \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{y}} + \frac{\partial}{\partial \hat{y}} \left(\hat{\mu} \frac{\partial \hat{v}}{\partial \hat{y}} \right) + \dots, \quad (2.2b)$$

$$\begin{aligned} \hat{\rho} \left(\frac{\partial \hat{h}}{\partial \hat{t}} + \frac{\hat{u}}{1 + \kappa \hat{y}} \frac{\partial \hat{h}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{h}}{\partial \hat{y}} \right) &= \frac{\partial \hat{p}}{\partial \hat{t}} + \frac{\hat{u}}{1 + \kappa \hat{y}} \frac{\partial \hat{p}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{p}}{\partial \hat{y}} \\ &+ \frac{1}{Pr} \frac{\partial}{\partial \hat{y}} \left(\hat{\mu} \frac{\partial \hat{h}}{\partial \hat{y}} \right) + \hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{y}} \right)^2 + \dots, \end{aligned} \quad (2.2c)$$

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} + \frac{1}{1 + \kappa \hat{y}} \frac{\partial (\hat{\rho} \hat{u})}{\partial \hat{x}} + \frac{1}{1 + \kappa \hat{y}} \frac{\partial}{\partial \hat{y}} \left[\hat{\rho} \hat{v} (1 + \kappa \hat{y}) \right] = 0, \quad (2.2d)$$

$$\hat{h} = \frac{\gamma}{\gamma - 1} \frac{\hat{p}}{\hat{\rho}}. \quad (2.2e)$$

Here $\kappa(\hat{x})$ denotes the aerofoil contour curvature, Pr is the Prandtl number, and γ is the specific heat ratio; for air $\gamma = 7/5$. For brevity we leave only the main viscous terms in the momentum and energy equations.

We shall assume that the surface of the aerofoil performs small amplitude vibrations the form of which will be now defined. An efficient generation of the T-S waves in the boundary-layer is expected to take place when a part of the spectrum of the wing vibrations comes in resonance with the T-S waves. For this to happen the period of the vibrations, T , is of the form

$$T \sim \frac{L}{V_\infty} Re^{-1/4}, \quad (2.3)$$

as deduced by Schneider (1974). Keeping this in mind we shall write the equation for the aerofoil contour as

$$\hat{y}_w = \varepsilon L f \left(\frac{\hat{t}}{\frac{L}{V_\infty} Re^{-1/4}}, \frac{\hat{x}}{L \delta_x} \right). \quad (2.4)$$

Here we use the coordinates, $(\hat{x}, \hat{y})$, fitted to the undisturbed aerofoil shape. Suitable restrictions on the amplitude of the aerofoil surface vibrations, ε , and on the characteristic wave length, δ_x , will be formulated in the course of the flow analysis.

2.1.1 Piston theory

As mentioned in "*Piston Theory - A New Aerodynamic Tool for the Aeroelastician*" (Ashley and Zaratarian, 1956) Piston theory is based on a physical model originally proposed by Hayes and Lighthill and in particular the application to aerofoils was developed by Landahl. The simplifications that the theory suggests means that closed-form solutions can be given for bending-torsion and flutter type problems for typical sections of aerofoils for a range of Reynolds numbers.

The term 'piston theory' refers to any method for calculating aerodynamic load on aircraft in which the local pressure generated by the body's motion is related to the local normal component of fluid velocity in the same that these quantities are related at the face of a piston moving in a one-dimensional channel (Ashley and Zaratarian, 1956). The situation in which you can apply this can be either for large flight Mach number or high reduced frequencies of unsteady motion and the latter is the one we focus on here.

We begin by analysing this situation we will derive assumptions on the flow quantities and we begin by looking into the pressure perturbations in the inviscid part of the flow. This will allow us to construct the asymptotic form of the solution therein. If the perturbations induced in the flow by the vibrating wing surface are small, then they will propagate with the speed of sound, a_∞ , and over the time (2.3) cover the distance

$$\hat{y} \sim a_\infty T \sim \frac{L}{M_\infty} Re^{-1/4}. \quad (2.5)$$

The gas velocity component normal to the aerofoil surface may be estimated by differentiating (2.4) with respect to time,

$$\hat{v} \sim \frac{\partial \hat{y}_w}{\partial \hat{t}} \sim V_\infty \varepsilon Re^{1/4}. \quad (2.6)$$

In order to evaluate the pressure perturbations produced by the vibrating wall we use the $\hat{y}$ -momentum equation (2.2b). We assume (subject to subsequent confirmation) that the local acceleration term, $\hat{\rho} \partial \hat{v} / \partial \hat{t}$, is dominant on the left hand side of this equation. Therefore it

must balance the pressure gradient on the right hand side of (2.2b),

$$\hat{\rho} \frac{\partial \hat{v}}{\partial \hat{t}} \sim \frac{\partial \hat{p}}{\partial \hat{y}}. \quad (2.7)$$

Assuming that the perturbations are small we can approximate the gas density as $\hat{\rho} \sim \rho_\infty$. The other quantities in the above equation can be estimated with the help of equations (2.3), (2.5) and (2.6), which leads to

$$\rho_\infty \frac{V_\infty \varepsilon Re^{1/4}}{\frac{L}{V_\infty} Re^{-1/4}} \sim \frac{\Delta \hat{p}}{\frac{L}{M_\infty} Re^{-1/4}}, \quad (2.8)$$

then solving for $\Delta \hat{p}$ gives

$$\Delta \hat{p} \sim \rho_\infty V_\infty^2 \frac{\varepsilon Re^{1/4}}{M_\infty}. \quad (2.9)$$

The principle balance in the energy equation (2.2c) can now be written as

$$\hat{\rho} \frac{\partial \hat{h}}{\partial \hat{t}} \sim \frac{\partial \hat{p}}{\partial \hat{t}}, \quad (2.10)$$

which we can rewrite using the perturbations of these quantities in the form

$$\Delta \hat{h} \sim \frac{\Delta \hat{p}}{\rho_\infty}, \quad (2.11)$$

hence using (2.9)

$$\Delta \hat{h} \sim V_\infty^2 \frac{\varepsilon Re^{1/4}}{M_\infty}. \quad (2.12)$$

We can estimate the density perturbations from the state equation (2.2e) to be

$$\hat{\rho} = \frac{\gamma}{\gamma - 1} \frac{\hat{p}}{\hat{h}}. \quad (2.13)$$

Introducing the perturbations of the thermodynamic function with respect to their values in

the free stream,

$$\rho_\infty + \Delta\hat{\rho} = \frac{\gamma}{\gamma - 1} \frac{p_\infty + \Delta\hat{p}}{h_\infty + \Delta\hat{h}} = \frac{\gamma}{\gamma - 1} \frac{p_\infty}{h_\infty} \frac{(1 + \Delta\hat{p}/p_\infty)}{(1 + \Delta\hat{h}/h_\infty)}. \quad (2.14)$$

Taking into account that

$$\rho_\infty = \frac{\gamma}{\gamma - 1} \frac{p_\infty}{h_\infty}, \quad (2.15)$$

and assuming $\Delta\hat{p}/p_\infty$ and $\Delta\hat{h}/h_\infty$ are small, we can conclude that

$$1 + \frac{\Delta\hat{\rho}}{\rho_\infty} = \left(1 + \frac{\Delta\hat{p}}{p_\infty}\right) \left(1 - \frac{\Delta\hat{h}}{h_\infty}\right) = 1 + \mathcal{O}(M_\infty \varepsilon Re^{1/4}). \quad (2.16)$$

Hence,

$$\Delta\hat{\rho} \sim \rho_\infty M_\infty \varepsilon Re^{1/4}. \quad (2.17)$$

Finally, it remains to find an estimate for the perturbations of the streamwise velocity component, $\hat{u}$. For this purpose we use the $\hat{x}$ -momentum equation (2.2a), which has a primary balance of

$$\hat{\rho} \frac{\partial \hat{u}}{\partial \hat{t}} \sim \frac{\partial \hat{p}}{\partial \hat{x}}. \quad (2.18)$$

Approximating the derivatives in this equation by finite differences gives

$$\rho_\infty \frac{\Delta \hat{u}}{T} \sim \frac{\Delta \hat{p}}{L \delta_x}, \quad (2.19)$$

and using (2.3) and (2.9) for the characteristic time and pressure perturbations, respectively, we find that

$$\Delta \hat{u} \sim V_\infty \frac{\varepsilon}{M_\infty \delta_x}. \quad (2.20)$$

In order for piston theory to be valid we require that relative variations of the thermodynamic quantities $\Delta\hat{p}/p_\infty$, $\Delta\hat{h}/h_\infty$ and $\Delta\hat{\rho}/\rho_\infty$, are small and that the distance perturbations travel during one period of oscillations, described by (2.5), is small compared to the characteristic wavelength of the aerofoil surface vibrations $L\delta_x$. These conditions impose restrictions on the flow parameters (the free-stream Mach number, M_∞ , the Reynolds num-

ber, Re , and the amplitude of the wall vibrations, ε). From the first, using (2.17) it follows that:

$$M_\infty \varepsilon Re^{1/4} \ll 1. \quad (2.21)$$

The second condition is satisfied if:

$$\delta_x \gg \frac{Re^{-1/4}}{M_\infty}. \quad (2.22)$$

If we further assume that the free-stream Mach number, M_∞ , is an order one quantity, conditions (2.21), (2.22) reduce to

$$\varepsilon Re^{1/4} \ll 1, \quad \delta_x \gg Re^{-1/4}. \quad (2.23)$$

Guided by (2.3), (2.5), (2.6)–(2.20) we represent the solution of the Navier-Stokes equations (2.2) in the region considered in the form

$$\left. \begin{aligned} \hat{u} &= V_\infty + V_\infty \frac{\varepsilon}{\delta_x} u'(t, \check{x}', y), & \hat{v} &= V_\infty \varepsilon Re^{1/4} v'(t, \check{x}', y), \\ \hat{\rho} &= \rho_\infty + \rho_\infty \varepsilon Re^{1/4} \rho'(t, \check{x}', y), & \hat{p} &= p_\infty + \rho_\infty V_\infty^2 \varepsilon Re^{1/4} p'(t, \check{x}', y), \\ \hat{h} &= h_\infty + V_\infty^2 \varepsilon Re^{1/4} h'(t, \check{x}', y), \end{aligned} \right\} \quad (2.24)$$

with the independent variables, t , $\check{x}'$ and y , being the non-dimensional time, streamwise and normal coordinates, respectively, introduced through the scalings

$$\hat{t} = \frac{L}{V_\infty} Re^{-1/4} t, \quad \hat{x} = L \delta_x \check{x}', \quad \hat{y} = L Re^{-1/4} y. \quad (2.25)$$

Substituting the expansions (2.24) and scalings (2.25) into the Navier-Stokes equations (2.2), and setting $Re \rightarrow \infty$ and $\varepsilon \rightarrow 0$ such that the conditions (2.23) are observed, we

arrive at the following set of equations

$$\frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial \tilde{x}'}, \quad (2.26a)$$

$$\frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial y}, \quad (2.26b)$$

$$\frac{\partial h'}{\partial t} = \frac{\partial p'}{\partial t}, \quad (2.26c)$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial v'}{\partial y} = 0, \quad (2.26d)$$

$$h' + \frac{\rho'}{(\gamma - 1)M_\infty^2} = \frac{\gamma}{\gamma - 1}p'. \quad (2.26e)$$

We can see that the $\hat{x}$ -momentum equation (2.26a) uncouples from the rest of the equations in (2.26). Hence it may be used to determine the streamwise velocity perturbation function, u' , once the pressure perturbation field, p' , is found. The equation for p'

$$M_\infty^2 \frac{\partial^2 p'}{\partial t^2} - \frac{\partial^2 p'}{\partial y^2} = 0, \quad (2.27)$$

is deduced by eliminating h' , ρ' and v' from the (2.26b)–(2.26e). To deduce a boundary condition for this equation we notice that the flow considered is inviscid, and therefore, has to satisfy the impermeability condition on the aerofoil surface

$$\hat{v} = \frac{\partial \hat{y}_w}{\partial \hat{t}} + \frac{\hat{u}}{1 + \kappa \hat{y}} \frac{\partial \hat{y}_w}{\partial \hat{x}} \quad \text{at} \quad \hat{y} = \hat{y}_w(\hat{t}, \hat{x}), \quad (2.28)$$

where κ is the curvature of the surface. Using (2.4) and (2.24) in (2.28) we have to leading order

$$v' \Big|_{y=0} = \frac{\partial f}{\partial t}. \quad (2.29)$$

Using the $\hat{y}$ -momentum equation (2.26b), the above can be converted into the following boundary condition for the normal pressure perturbation, p' , at the aerofoil surface.

$$\frac{\partial p'}{\partial y} \Big|_{y=0} = -\frac{\partial^2 f}{\partial t^2}. \quad (2.30)$$

The general solution of the equation (2.27) is

$$p' = \varphi(\xi; \check{x}') + \psi(\eta; \check{x}'), \quad (2.31)$$

where

$$\xi = t - M_\infty y, \quad \eta = t + M_\infty y, \quad (2.32)$$

and $\check{x}'$ plays the role of a parameter. Since we are interested in the perturbations emanating from the aerofoil surface we have to disregard the second term, $\psi(\eta; \check{x}')$, on the right hand side of (2.31). The function $\varphi(\xi; \check{x}')$ is then found from the boundary condition (2.30)

$$\varphi(\xi; \check{x}') = \frac{1}{M_\infty} \frac{\partial f}{\partial t}(\xi, \check{x}'). \quad (2.33)$$

Consequently, at any point in the flow field

$$p'(t, y; \check{x}') = \frac{1}{M_\infty} \frac{\partial f}{\partial t}(t - M_\infty y, \check{x}'). \quad (2.34)$$

Now, using (2.34) in (2.26) we find that

$$\left. \begin{aligned} u' &= -\frac{1}{M_\infty} \frac{\partial f}{\partial \check{x}'}(t - M_\infty y, \check{x}'), & v' &= \frac{\partial f}{\partial t}(t - M_\infty y, \check{x}'), \\ h' &= \frac{1}{M_\infty} \frac{\partial f}{\partial t}(t - M_\infty y, \check{x}'), & \rho' &= M_\infty \frac{\partial f}{\partial t}(t - M_\infty y, \check{x}'). \end{aligned} \right\} \quad (2.35)$$

The pressure on the aerofoil surface is found by setting $y = 0$ in (2.34). We have

$$p' \Big|_{y=0} = \frac{1}{M_\infty} \frac{\partial f}{\partial t}(t, \check{x}'). \quad (2.36)$$

We further assume that the vibrations of the aerofoil surface are time periodic, namely

$$f(t, \check{x}') = \exp(i(\check{\omega}t + \phi))\check{F}(\check{x}') + c.c., \quad (2.37)$$

where $\check{\omega}$ and ϕ are the real valued frequency and phase¹, $\check{F}(\check{x}')$ is a complex function of real

¹For ease and since it does not affect any derivatives the phase is dropped from any following exponentials

variable $\check{x}'$ and $c.c.$ denotes the complex conjugate of $\exp(i\check{\omega}t)\check{F}(\check{x}')$. Then, substituting (2.37) into (2.36) and then into the asymptotic expansion for $\hat{p}$ in (2.24) yields

$$\hat{p} = p_\infty + \rho_\infty V_\infty^2 \varepsilon Re^{1/4} \left\{ \frac{i\check{\omega}}{M_\infty} \check{F}(\check{x}') \exp(i\check{\omega}t) + c.c. \right\}. \quad (2.38)$$

2.2 Boundary-layer analysis

We now turn our attention to the boundary-layer on the aerofoil surface. If the flow was steady then the solution in the boundary-layer could be represented by the asymptotic expansions

$$\left. \begin{aligned} \hat{u} &= V_\infty U_0(x, Y) + \cdots, & \hat{v} &= V_\infty Re^{-1/2} V_0(x, Y) + \cdots, \\ \hat{p} &= p_\infty + \rho_\infty V_\infty^2 P_0(x, Y) + \cdots, & \hat{\rho} &= \rho_\infty \rho_0(x, Y) + \cdots, \\ \hat{h} &= V_\infty^2 h_0(x, Y) + \cdots, & \hat{\mu} &= \mu_\infty \mu_0(x, Y) + \cdots. \end{aligned} \right\} \quad (2.39)$$

Here the dimensionless coordinates, (x, Y) , are introduced by means of the transformations

$$\hat{x} = Lx, \quad \hat{y} = LRe^{-1/2}Y. \quad (2.40)$$

Substitution of (2.39) and (2.40) into the Navier-Stokes equations (2.2) leads to the classical

until required.

boundary-layer equations

$$\rho_0 \left(U_0 \frac{\partial U_0}{\partial x} + V_0 \frac{\partial U_0}{\partial Y} \right) = - \frac{\partial P_0}{\partial x} + \frac{\partial}{\partial Y} \left(\mu_0 \frac{\partial U_0}{\partial Y} \right), \quad (2.41a)$$

$$\frac{\partial P_0}{\partial Y} = 0, \quad (2.41b)$$

$$\rho_0 \left(U_0 \frac{\partial h_0}{\partial x} + V_0 \frac{\partial h_0}{\partial Y} \right) = U_0 \frac{\partial P_0}{\partial x} + \frac{1}{Pr} \frac{\partial}{\partial Y} \left(\mu_0 \frac{\partial h_0}{\partial Y} \right) + \mu_0 \left(\frac{\partial U_0}{\partial Y} \right)^2, \quad (2.41c)$$

$$\frac{\partial(\rho_0 U_0)}{\partial x} + \frac{\partial(\rho_0 V_0)}{\partial Y} = 0, \quad (2.41d)$$

$$h_0 = \frac{1}{(\gamma - 1)\rho_0} + \frac{\gamma}{\gamma - 1} \frac{P_0}{\rho_0}. \quad (2.41e)$$

When solving these equations one needs to pose two boundary conditions. Firstly, at the front stagnation point at the leading edge of the aerofoil

$$U_0 = 0, \quad h_0 = \frac{1}{2} + \frac{1}{(\gamma - 1)M_\infty^2} \quad \text{at} \quad x = 0, \quad (2.42)$$

with h_0 being the enthalpy. Secondly, the solution has to satisfy the no-slip condition on the aerofoil surface

$$U_0 = V_0 = 0 \quad \text{at} \quad Y = 0. \quad (2.43a)$$

We must also supplement these conditions with an appropriate thermal condition. If, for example, the temperature distribution on the aerofoil surface $\hat{T}_w(x)$, is known, then

$$h_0 = \frac{1}{(\gamma - 1)M_\infty^2} \frac{\hat{T}_w(x)}{T_\infty} \quad \text{at} \quad Y = 0. \quad (2.43b)$$

If, on the other hand, it is known that the aerofoil is thermally isolated, then (2.43b) is replaced by

$$\frac{\partial h_0}{\partial Y} = 0 \quad \text{at} \quad Y = 0. \quad (2.43c)$$

Finally, at the outer edge of the boundary-layer

$$U_0 = U_e(x), \quad h_0 = h_e(x) \quad \text{at} \quad Y = \infty, \quad (2.44)$$

where $U_e(x)$ and $h_e(x)$ are the tangential velocity and enthalpy on the aerofoil surface as given by the solution of the inviscid equations of motion.

Through these functions the solution to the boundary-value problem (2.41)–(2.44) depends on the aerofoil shape. In the general case a full solution would have to be constructed numerically. However, for the receptivity analysis we only need to know the behaviour of the fluid dynamic functions at the bottom of the boundary-layer near the aerofoil surface. It can be shown and that due to the flow being Blasius we have

$$\left. \begin{aligned} U_0(x, Y) &= \tau_w(x)Y + \mathcal{O}(Y^2), & h_0(x, Y) &= \rho_w(x) + \mathcal{O}(Y), \\ \rho_0(x, Y) &= \rho_w(x) + \mathcal{O}(Y), & \mu_0(x, Y) &= \mu_w(x) + \mathcal{O}(Y), \end{aligned} \right\} \quad \text{as} \quad Y \rightarrow 0, \quad (2.45)$$

where $\tau_w(x)$ denotes the skin friction, and $\rho_w(x)$, $\mu_w(x)$ are the gas density and dynamic viscosity coefficient on the aerofoil surface

We now study the perturbations produced in the boundary-layer due to the wall vibrations. When performing this task we continue to use the body fitted coordinates, which now move up and down with the aerofoil surface. As the coordinate system is no longer inertial, we must verify whether or not the inertial force changes the pressure distribution in the boundary-layer. The pressure at the outer edge of the boundary-layer is given by (2.38). The inertial force near the aerofoil surface is

$$\mathbf{F}_{\text{inertial}} = \hat{\rho} \frac{\partial^2 \hat{y}_w}{\partial \hat{t}^2} \mathbf{e}_y, \quad (2.46)$$

where $\mathbf{e}_y$ is a unit vector along the $\hat{y}$ -axis. Using the description of the aerofoil surface (2.4), in (2.46) we find

$$|\mathbf{F}_{\text{inertial}}| \sim \frac{\rho_\infty V_\infty^2}{L} \varepsilon Re^{1/2}. \quad (2.47)$$

We can also write an estimate the effect of the inertial force on the pressure variation across

the boundary layer as

$$\mathbf{F}_{\text{inertial}} \sim \frac{\partial \hat{p}}{\partial \hat{y}} \mathbf{e}_y. \quad (2.48)$$

By denoting $\Delta \hat{p}$ as the pressure variation due to the action of the inertial force, we can estimate the right hand side of (2.48) as

$$\frac{\partial \hat{p}}{\partial \hat{y}} \sim \frac{\Delta \hat{p}}{LRe^{-1/2}}. \quad (2.49)$$

Substituting (2.47) and (2.49) into (2.48) yields

$$\Delta \hat{p} \sim \rho_{\infty} V_{\infty}^2 \varepsilon, \quad (2.50)$$

which is $Re^{1/4}$ times smaller than the pressure (2.38) at the outer edge of the boundary-layer. Hence, in the flow considered, the inertial force can be disregarded.

2.2.1 Stokes layer

As the pressure perturbations (2.38) penetrate into the boundary-layer, they cause the Stokes layer to form near the aerofoil surface. The thickness of this layer is found by comparing the instantaneous acceleration term with the viscous forces in the longitude momentum equation (2.2a)

$$\hat{\rho} \frac{\partial \hat{u}}{\partial \hat{t}} \sim \frac{\partial}{\partial \hat{y}} \left(\hat{\mu} \frac{\partial \hat{u}}{\partial \hat{y}} \right). \quad (2.51)$$

Taking into account that $\hat{\rho} \sim \rho_{\infty}$, $\hat{\mu} \sim \mu_{\infty}$ and using equation (2.3) for the characteristic time, we find that

$$\hat{y} \sim LRe^{-5/8}. \quad (2.52)$$

Let us now estimate the perturbations, $\Delta \hat{u}$, of the streamwise velocity component. There are two physical mechanisms that lead to oscillations of the streamwise velocity in the Stokes layer. In the first one² the acceleration and deceleration of fluid particles are caused

²The calculations for this mechanism are the focus here, whilst the calculation for the second mecha-

by the pressure variations along the body surface. This process is expressed by the following balance in the $\hat{x}$ -momentum equation (2.2a)

$$\hat{\rho} \frac{\partial \hat{u}}{\partial \hat{t}} \sim \frac{\partial \hat{p}}{\partial \hat{x}}. \quad (2.53)$$

Using the results of the piston theory (2.38)

$$\Delta \hat{p} \sim \rho_\infty V_\infty^2 \varepsilon Re^{1/4}, \quad (2.54)$$

and the fact that the characteristic time and the length scales are

$$\hat{t} \sim \frac{L}{V_\infty} Re^{-1/4}, \quad \hat{x} \sim L \delta_x, \quad (2.55)$$

we find from (2.53) that

$$\Delta \hat{u} \sim V_\infty \frac{\varepsilon}{\delta_x}. \quad (2.56)$$

The second mechanism is attributed to the compressibility of the flow. When the pressure at the outer edge of the boundary-layer varies periodically in time, the gas in the Stokes layer undergoes periodic compression and expansion, which leads to an oscillatory motion of fluid particles in the direction normal to the wall. The characteristic velocity, $\hat{v}$, of this motion may be estimated using the following balance in the continuity equation (2.2d),

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} \sim \hat{\rho} \frac{\partial \hat{v}}{\partial \hat{y}}. \quad (2.57)$$

With the pressure perturbations given by (2.54), the density perturbations are estimated as

$$\Delta \hat{\rho} \sim \rho_\infty \varepsilon Re^{1/4}, \quad (2.58)$$

given that we will be taking the Mach number, M_∞ , to be order 1 from now on.

Using (2.58) and (2.3) the derivative $\partial \hat{\rho} / \partial \hat{t}$ on the left hand side of (2.57) may be

nism were performed by Tomas Bernots (Ruban et al., 2013). However, we still discuss both to obtain a distinguished flow regime.

estimated as

$$\frac{\partial \hat{\rho}}{\partial \hat{t}} \sim \frac{\rho_{\infty} V_{\infty}}{L} \varepsilon Re^{1/2}. \quad (2.59)$$

On the right hand side of (2.57) we substitute $\hat{\rho}$ with ρ_{∞} and evaluate the derivative $\partial \hat{v} / \partial \hat{y}$ keeping in mind that the thickness of the Stokes layer is given by (2.52). We have

$$\hat{\rho} \frac{\partial \hat{v}}{\partial \hat{y}} \sim \rho_{\infty} \frac{\hat{v}}{L Re^{-5/8}}. \quad (2.60)$$

It remains to substitute (2.59) and (2.60) into (2.57) and solve the resulting equation for $\hat{v}$. We find

$$\hat{v} \sim V_{\infty} \varepsilon Re^{-1/8}. \quad (2.61)$$

As the fluid particles migrate from a region with lower basic flow velocity $\hat{u}$ into a region with larger $\hat{u}$ and then back, an exchange between the instantaneous acceleration of the fluid particles and the convective acceleration takes place. This acceleration is represented by the following balance in the streamwise momentum equation (2.2a)

$$\frac{\partial \hat{u}}{\partial \hat{t}} \sim \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}}. \quad (2.62)$$

The derivative $\partial \hat{u} / \partial \hat{y}$ on the right hand side of (2.62) is approximated using the steady solution in the boundary-layer from (2.39) and (2.40) giving

$$\frac{\partial \hat{u}}{\partial \hat{y}} \sim \frac{V_{\infty}}{L Re^{-1/2}}, \quad (2.63)$$

which being substituted together with (2.61) into (2.62) yields

$$\frac{\Delta \hat{u}}{\frac{L}{V_{\infty}} Re^{-1/4}} \sim V_{\infty} \varepsilon Re^{-1/8} \frac{V_{\infty}}{L Re^{-1/2}}. \quad (2.64)$$

We conclude that

$$\Delta \hat{u} \sim V_{\infty} \varepsilon Re^{1/8}. \quad (2.65)$$

Comparing (2.56) with (2.65) one can see that the two physical mechanisms appear to be

equally important if

$$\delta_x = Re^{-1/8}. \quad (2.66)$$

One mechanism is defined by having a long wavelength, $\delta_x \gg Re^{-1/8}$, and here compressibility effects become of importance. The mechanism we will focus on here is one described with a shorter wavelength, $\delta_x \ll Re^{-1/8}$, where the disturbance is created by the interaction between the Stokes' layer and pressure perturbation only. Equation (2.66) defines a distinguished flow regime which we now study using regular asymptotic analysis.

Notice that the perturbations (2.56), (2.65) should be superimposed on the steady flow solution which, according to (2.39) and (2.45), is written in the Stokes layer as

$$\hat{u} = V_\infty Re^{-1/8} \tau_w(x) \check{Y}_* + \cdots, \quad \check{Y}_* = Re^{1/8} Y. \quad (2.67)$$

This suggests that in the Stokes layer the solution of the Navier-Stokes equations (2.2) has to be represented in the asymptotic form

$$\left. \begin{aligned} \hat{u} &= V_\infty \left\{ Re^{-1/8} \tau_w(x) \check{Y}_* + \varepsilon Re^{1/8} \check{u}(t, \check{x}', \check{Y}_*) + \cdots \right\}, \\ \hat{v} &= V_\infty \left\{ \varepsilon Re^{-1/8} \check{v}(t, \check{x}', \check{Y}_*) + \cdots \right\}, \\ \hat{p} &= p_\infty + \rho_\infty V_\infty^2 \left\{ P_0(x) + \varepsilon Re^{1/4} \check{p}(t, \check{x}') + \cdots \right\}, \\ \hat{h} &= V_\infty^2 \left\{ h_w(x) + \varepsilon Re^{1/4} \check{h}(t, \check{x}', \check{Y}_*) + \cdots \right\}, \\ \hat{\rho} &= \rho_\infty \left\{ \rho_w(x) + \varepsilon Re^{1/4} \check{\rho}(t, \check{x}', \check{Y}_*) + \cdots \right\}, \\ \hat{\mu} &= \mu_\infty \left\{ \mu_w(x) + \varepsilon Re^{1/4} \check{\mu}(t, \check{x}', \check{Y}_*) + \cdots \right\}, \end{aligned} \right\} \quad (2.68)$$

where higher order terms have been ignored and with the independent variables t , $\check{x}'$ and $\check{Y}_*$ introduced by

$$\hat{t} = \frac{L}{V_\infty} Re^{-1/4} t, \quad \hat{x} = L Re^{-1/8} \check{x}', \quad \hat{y} = L Re^{-5/8} \check{Y}_*. \quad (2.69)$$

Notice that the scale time t , and streamwise coordinate $\check{x}'$ are the same as in (2.25).

Substitution of (2.68) into (2.2) results in

$$\rho_w \frac{\partial \check{u}}{\partial t} + \rho_w \tau_w \check{v} = -\frac{\partial \check{p}}{\partial \check{x}'} + \mu_w \frac{\partial^2 \check{u}}{\partial \check{Y}_*^2}, \quad (2.70a)$$

$$\frac{\partial \check{p}}{\partial \check{Y}_*} = 0, \quad (2.70b)$$

$$\rho_w \frac{\partial \check{h}}{\partial t} = \frac{\partial \check{p}}{\partial t} + \mu_w \frac{\partial^2 \check{h}}{\partial \check{Y}_*^2}, \quad (2.70c)$$

$$\frac{\partial \check{\rho}}{\partial t} + \rho_w \frac{\partial \check{v}}{\partial \check{Y}_*} = 0, \quad (2.70d)$$

$$h_w(x) = \frac{1}{(\gamma - 1)M_\infty^2} \frac{1}{\rho_w(x)} \quad \text{and} \quad \check{h} = \frac{\gamma}{\gamma - 1} \frac{1}{\rho_w} \left(\check{p} - \frac{\check{\rho}}{\gamma M_\infty^2 \rho_w} \right). \quad (2.70e)$$

We start with the energy equation (2.70c). We seek the solution to this equation in the class of functions that are time periodic and do not show exponential growth as $\check{Y}_* \rightarrow \infty$. If the aerofoil surface is thermally isolated, i.e.

$$\left. \frac{\partial \check{h}}{\partial \check{Y}_*} \right|_{\check{Y}_*=0} = 0, \quad (2.71)$$

then the suitable solution is given by

$$\check{h} = \frac{\check{p}}{\rho_w}, \quad (2.72)$$

substituting this into (2.70e) gives

$$\check{\rho} = M_\infty^2 \rho_w \check{p}. \quad (2.73)$$

Then the continuity equation (2.70d) may be expressed in the form

$$\frac{\partial \check{v}}{\partial \check{Y}_*} = -M_\infty^2 \frac{\partial \check{p}}{\partial t}, \quad (2.74)$$

solving for the impermeability condition at the surface gives

$$\check{v} \Big|_{\check{Y}_*=0} = 0, \quad (2.75)$$

which we can write as

$$\check{v} = -M_\infty^2 \frac{\partial \check{p}}{\partial t} \check{Y}_*. \quad (2.76)$$

With (2.76), the $\hat{x}$ -momentum equation (2.70a) assumes the form

$$\rho_w \frac{\partial \check{u}}{\partial t} - \rho_w \tau_w M_\infty^2 \frac{\partial \check{p}}{\partial t} \check{Y}_* = -\frac{\partial \check{p}}{\partial \check{x}'} + \mu_w \frac{\partial^2 \check{u}}{\partial \check{Y}_*^2}. \quad (2.77)$$

According to (2.70b), the pressure does not change with $\check{Y}_*$ making the inviscid solution (2.38) applicable inside the Stokes layer. We can write (2.38) in the non-dimensional variables (2.68) as

$$\check{p} = \frac{i\check{\omega}}{M_\infty} \check{F}(\check{x}') \exp(i\check{\omega}t) + c.c.. \quad (2.78)$$

This suggests that the solution of the equation (2.77) should be sought in the form

$$\check{u}(t, \check{x}', \check{Y}_*) = \exp(i\check{\omega}t) U(\check{x}', \check{Y}_*) + c.c.. \quad (2.79)$$

Substitution of (2.78) and (2.79) into (2.77) results in the following equation for U

$$\mu_w \frac{\partial^2 U}{\partial \check{Y}_*^2} - i\check{\omega} \rho_w U = \frac{i\check{\omega}}{M_\infty} \frac{d\check{F}}{d\check{x}'} + \rho_w \tau_w M_\infty \check{\omega}^2 \check{F}(\check{x}') \check{Y}_*. \quad (2.80)$$

The solution of this equation which satisfies the no-slip condition on the aerofoil surface

$$U \Big|_{\check{Y}_*=0} = 0, \quad (2.81)$$

and does not grow exponentially at the outer edge of the Stokes layer has the form

$$U = \frac{d\check{F}/d\check{x}'}{M_\infty \rho_w} \left[\exp \left(-(1+i) \sqrt{\frac{\check{\omega} \rho_w}{2\mu_w}} \check{Y}_* \right) - 1 \right] + i\check{\omega} \tau_w M_\infty \check{F}(\check{x}') \check{Y}_*. \quad (2.82)$$

2.2.2 The main part of the boundary-layer

The form of the asymptotic expansions of the fluid dynamic function in the main part of the boundary-layer, where $\hat{y} \sim LRe^{-1/2}$, may be predicted with the help of the following procedure which is based on the principle of matching of asymptotic expansions. We will describe the procedure, using as an example the streamwise velocity component $\hat{u}$.

Substituting (2.82) into (2.79) and setting $\check{Y}_* \rightarrow \infty$, we find that at the outer edge of the Stokes layer the streamwise velocity can be expressed as

$$\check{u} = i\check{\omega}\tau_w(x)M_\infty\check{F}(\check{x}')\check{Y}_*\exp(i\check{\omega}t) + c.c.. \quad (2.83)$$

Comparing (2.83) with (2.78) one can see that

$$\check{u} = \tau_w(x)M_\infty^2\check{p}(t, \check{x}')\check{Y}_* + \dots \quad \text{as} \quad \check{Y}_* \rightarrow \infty. \quad (2.84)$$

We substitute (2.84) into the asymptotic expansion for $\hat{u}$ in (2.68) and find that the outer expansion of the inner solution has the form

$$\hat{u} = V_\infty \left\{ Re^{-1/8}\tau_w(x)\check{Y}_* + \varepsilon Re^{1/8}\tau_w(x)M_\infty^2\check{p}(t, \check{x}')\check{Y}_* + \dots \right\}. \quad (2.85)$$

If we now recast the above equation in terms of the outer variable, $Y = Re^{-1/8}\check{Y}_*$, then we find that the inner expansion of the outer solution has to have the asymptotic form

$$\hat{u} = V_\infty \left\{ \tau_w Y + \varepsilon Re^{1/4}\tau_w(x)M_\infty^2\check{p}(t, \check{x}')Y + \dots \right\}. \quad (2.86)$$

This suggests that the solution in the main part of the boundary-layer has represented by the asymptotic expansion

$$\hat{u} = V_\infty \left\{ U_0(x, Y) + \varepsilon Re^{1/4}U_1(t, \check{x}', Y) + \dots \right\}, \quad (2.87)$$

where function $U_0(x, Y)$ and $U_1(t, \check{x}', Y)$ are such that

$$\left. \begin{aligned} U_0 &= \tau_w(x)Y + \cdots, \\ U_1 &= \tau_w(x)M_\infty^2 \check{p}(t, \check{x}')Y + \cdots, \end{aligned} \right\} \quad \text{as } Y \rightarrow 0. \quad (2.88)$$

Repeating this procedure with the other fluid dynamic functions we find that in the main part of the boundary-layer the solution must have the asymptotic form

$$\left. \begin{aligned} \hat{u} &= V_\infty \left\{ U_0(x, Y) + \varepsilon Re^{1/4} U_1(t, \check{x}', Y) + \cdots \right\}, \\ \hat{v} &= V_\infty \left\{ \varepsilon V_1(t, \check{x}', Y) + \cdots \right\}, \\ \hat{p} &= p_\infty + \rho_\infty V_\infty^2 \left\{ P_0(x) + \varepsilon Re^{1/4} p_1(t, \check{x}', Y) + \cdots \right\}, \\ \hat{h} &= V_\infty^2 \left\{ h_0(x, Y) + \varepsilon Re^{1/4} h_1(t, \check{x}', Y) + \cdots \right\}, \\ \hat{\rho} &= \rho_\infty \left\{ \rho_0(x, Y) + \varepsilon Re^{1/4} \rho_1(t, \check{x}', Y) + \cdots \right\}, \end{aligned} \right\} \quad (2.89)$$

with

$$\hat{t} = \frac{L}{V_\infty} Re^{-1/4} t, \quad \hat{x} = L Re^{-1/8} \check{x}', \quad \hat{y} = L Re^{-1/2} Y. \quad (2.90)$$

The matching conditions with the solution in the Stokes layer are

$$\left. \begin{aligned} U_0 &= \tau_w(x)Y + \cdots, \\ U_1 &= \tau_w(x)M_\infty^2 \check{p}(t, \check{x}')Y + \cdots, \\ p_1 &= \check{p}(t, \check{x}') + \cdots, \\ V_1 &= -M_\infty^2 \frac{\partial \check{p}}{\partial t} Y + \cdots, \\ h_1 &= \frac{1}{\rho_w(x)} \check{p}(t, \check{x}') + \cdots, \\ \rho_1 &= M_\infty^2 \rho_w(x) \check{p}(t, \check{x}') + \cdots, \end{aligned} \right\} \quad \text{as } Y \rightarrow 0. \quad (2.91)$$

Substitution of (2.89) into the Navier-Stokes equations (2.2) results in

$$\frac{\partial U_1}{\partial t} + V_1 \frac{\partial U_0}{\partial Y} = 0, \quad (2.92a)$$

$$\frac{\partial p_1}{\partial Y} = 0, \quad (2.92b)$$

$$\frac{\partial h_1}{\partial t} + V_1 \frac{\partial h_0}{\partial Y} = \frac{1}{\rho_0} \frac{\partial p_1}{\partial t}, \quad (2.92c)$$

$$\frac{\partial \rho_1}{\partial t} + V_1 \frac{\partial \rho_0}{\partial Y} + \rho_0 \frac{\partial V_1}{\partial Y} = 0, \quad (2.92d)$$

$$h_0 = \frac{1}{(\gamma - 1)M_\infty^2 \rho_0}, \quad (2.92e)$$

$$h_1 = \frac{\gamma}{\gamma - 1} \frac{1}{\rho_0} \left(p_1 - \frac{1}{\gamma M_\infty^2 \rho_0} \right). \quad (2.92f)$$

Solving the set of equations (2.92) we find

$$\left. \begin{aligned} p_1 &= \frac{1}{M_\infty} \frac{\partial f}{\partial t}, & U_1 &= M_\infty Y \frac{\partial U_0}{\partial Y} \frac{\partial f}{\partial t}, & V_1 &= -M_\infty Y \frac{\partial^2 f}{\partial t^2}, \\ \rho_1 &= M_\infty \frac{\partial(\rho_0 Y)}{\partial Y} \frac{\partial f}{\partial t}, & h_1 &= \frac{\gamma}{\gamma - 1} \frac{1}{M_\infty \rho_0} \left[1 - \frac{1}{\gamma \rho_0} \frac{\partial(\rho_0 Y)}{\partial Y} \right] \frac{\partial f}{\partial t}. \end{aligned} \right\} \quad (2.93)$$

2.3 Formulation of the triple-deck problem

The Stokes layer on its own cannot produce the T-S wave because the characteristic length, $\delta_x \sim Re^{-1/8}$, of the flow oscillations in the Stokes layer is too large as compared to the wave length of the T-S wave. Hence, we shall assume that there is a roughness on the aerofoil surface centred at point $\hat{x} = \hat{x}_0$ whose shape is given by the equation,

$$\hat{y} = LRe^{-5/8} \delta \check{G} \left(\frac{\hat{x} - \hat{x}_0}{LRe^{-3/8}} \right). \quad (2.94)$$

Here we choose the streamwise extent of the roughness, $\hat{x} - \hat{x}_0$, to be an order $\mathcal{O}(LRe^{-3/8})$ quantity, the normal extent of the roughness is an $\mathcal{O}(LRe^{-5/8} \delta)$ quantity and the function $\check{G}$ represents the shape itself. The scalings are to make sure we satisfy the conditions of the resonance between the external perturbations and the T-S wave. We shall see that with

the roughness shape defined by (2.94) the flow is capable of extracting the mode with the wavenumber we require from the entire spectrum of steady perturbations.

It is well known that the flow past a roughness of this size is described by the triple-deck theory. According to this theory, when analysing the flow in the vicinity of the roughness one has to consider three regions: the viscous sublayer (region 1 in figure 2.1); the main part of the boundary-layer (region 2 in figure 2.1); and the outer deck (region 3 in figure 2.1), which lies in the inviscid potential flow outside the boundary-layer.

2.3.1 Interaction region solution

We can now use the form of the solution in the Stokes' layer (2.68) and its following solutions to derive the further terms we expect downstream, in the interaction region. Assuming that the amplitude of the pulsations in the Stokes layer, $\sigma = \varepsilon Re^{1/4}$, and the roughness height parameter, δ , are small, we can represent the solution in region 1 in the asymptotic form

$$\left. \begin{aligned} \hat{u} &= V_\infty Re^{-1/8} \left\{ \tau_w \check{Y}_* + \sigma \check{u}_S(t, \check{Y}_*) + \delta \check{U}_r(\check{x}_*, \check{Y}_*) + \sigma \delta \check{U}_*(t, \check{x}_*, \check{Y}_*) + \dots \right\}, \\ \hat{v} &= V_\infty Re^{-3/8} \left\{ \sigma \check{v}_S(t, \check{Y}_*) + \delta \check{V}_r(\check{x}_*, \check{Y}_*) + \sigma \delta \check{V}_*(t, \check{x}_*, \check{Y}_*) + \dots \right\}, \\ \hat{p} &= p_\infty + \rho_\infty V_\infty^2 \left\{ \sigma P_0(t) + Re^{-1/4} \sigma p_0(t) \check{x}_* + Re^{-1/4} \delta \check{P}_r(\check{x}_*) + \right. \\ &\quad \left. + Re^{-1/4} \sigma \delta \check{P}_*(t, \check{x}_*) + \dots \right\}, \\ \hat{h} &= V_\infty^2 \left\{ h_w + \sigma h_0(t) + \dots \right\}, \\ \hat{\rho} &= \rho_\infty \left\{ \rho_w + \sigma \rho_0(t) + \dots \right\}, \\ \hat{\mu} &= \mu_\infty \left\{ \mu_w + \sigma \mu_0(t) + \dots \right\}, \end{aligned} \right\} \quad (2.95)$$

with the independent variables, t , $\check{x}_*$ and $\check{Y}_*$, defined by

$$\hat{t} = \frac{L}{V_\infty} Re^{-1/4} t, \quad \hat{x} = \hat{x}_0 + L Re^{-3/8} \check{x}_*, \quad \hat{y} = L Re^{-5/8} \check{Y}_*. \quad (2.96)$$

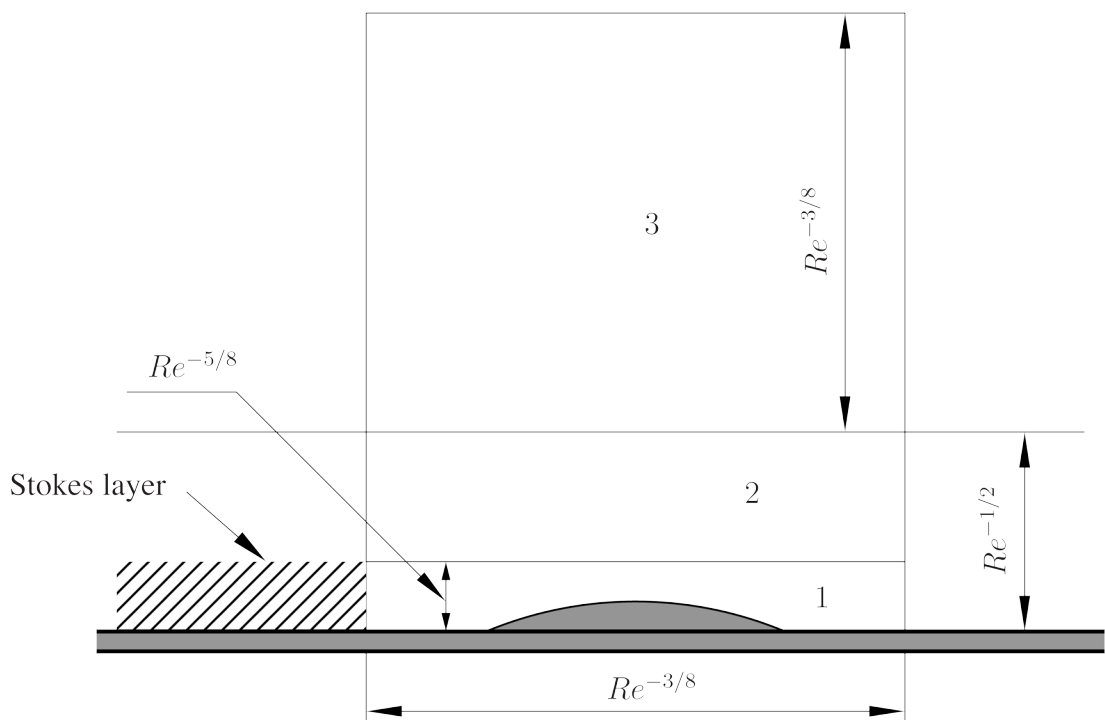


Figure 2.1: The flow in the vicinity of the wall roughness.

The leading order term, $\tau_w \check{Y}_*$, in the asymptotic expansion of the streamwise velocity component, $\hat{u}$, represents the unperturbed steady boundary-layer. If the Stokes layer and roughness were removed this would be the only remaining term. It is obtained by re-expanding the steady solution (2.39) and (2.45) in terms of the variables, $\check{x}_*$, $\check{Y}_*$, of region 1. In (2.95) the skin friction, τ_w , is a constant being equal to the value of the function $\tau_w(x)$ in (2.45) at the position of the roughness. The next two terms, $\sigma \check{u}_S(t, \check{Y}_*)$ and $\delta \check{U}_r(\check{x}_*, \check{Y}_*)$, represent the perturbations produced by the Stokes layer and the roughness, respectively. The function $\check{u}_S$ is obtained by setting $\check{x}'$ in (2.79) and (2.82) to its value $\check{x}'_0$ at the position of the roughness giving

$$\check{u}_S = \exp(i\check{\omega}t) \left\{ \frac{\frac{d\check{F}}{d\check{x}'}(\check{x}'_0)}{M_\infty \rho_w} \left[\exp \left(-(1+i) \sqrt{\frac{\check{\omega} \rho_w}{2\mu_w}} \check{Y}_* \right) - 1 \right] + i\check{\omega} \tau_w M_\infty \check{F}(\check{x}'_0) \check{Y}_* \right\} + c.c.. \quad (2.97)$$

Finally $\sigma \delta \check{U}_*(t, \check{x}_*, \check{Y}_*)$ represents the perturbations produced in the boundary-layer due to the interaction of the Stokes layer with the steady flow field around the roughness.

In the asymptotic expansion for $\hat{v}$ in (2.95) the $\sigma \check{v}_S$ term, represents the perturbations produced by the Stokes layer. It is obtained by substituting (2.78) into (2.76) and choosing $\check{x}' = \check{x}'_0$

$$\check{v}_S = M_\infty \check{\omega}^2 \check{F}(\check{x}'_0) \check{Y}_* \exp(i\check{\omega}t) + c.c.. \quad (2.98)$$

The next term, $\delta \check{V}_r$, describes the steady perturbations induced by the roughness. Lastly, the $\sigma \delta \check{V}_*$ term represents the perturbations produced in the boundary-layer due to the interaction of the Stokes layer with the steady flow field around the roughness.

The first two terms in the asymptotic expansion of the pressure $\hat{p}$ in (2.95) are obtained by setting $\check{x}' = \check{x}'_0 + Re^{-1/4} \check{x}_*$ in (2.38) and using the Taylor expansion of the function $\check{F}(\check{x}')$. This yields

$$P_0(t) = \frac{i\check{\omega}}{M_\infty} \check{F}(\check{x}'_0) \exp(i\check{\omega}t) + c.c., \quad (2.99)$$

$$p_0(t) = \frac{i\check{\omega}}{M_\infty} \frac{d\check{F}}{d\check{x}'}(\check{x}'_0) \exp(i\check{\omega}t) + c.c.. \quad (2.100)$$

The third pressure term, $Re^{-1/4}\delta\check{P}_r(\check{x}_*)$, describes the steady perturbations produced by the roughness, and the last written term, $Re^{-1/4}\sigma\delta\check{P}_*(t, \check{x}_*)$, is due to the interaction of the Stokes layer with the roughness.

For our purposes it is sufficient to keep only two terms in the asymptotic expansions (2.95) of the enthalpy, $\hat{h}$, density, $\hat{\rho}$, and viscosity coefficient, $\hat{\mu}$. In the leading order approximation these quantities are constant in region 1, with h_w , ρ_w and μ_w denoting the values of functions $h_w(x)$, $\rho_w(x)$ and $\mu_w(x)$ in the solution (2.45) of the boundary-layer equations (2.41) at the position, $x = x_0$, of the roughness. The next order terms, $\sigma h_0(t)$, $\sigma\rho_0(t)$, $\sigma\mu_0(t)$, are caused by the pressure pulsations in time. With $P_0(t)$ given by (2.99) we can find the functions $h_0(t)$ and $\rho_0(t)$ by substituting (2.95) into the energy equation (2.2c). This results in

$$\rho_w \frac{dh_0}{dt} = \frac{dP_0}{dt}. \quad (2.101)$$

It further follows from the state equation (2.2e) that

$$h_w = \frac{1}{(\gamma - 1)M_\infty^2} \frac{1}{\rho_w}, \quad \rho_w h_0 + h_w \rho_0 = \frac{\gamma}{\gamma - 1} P_0. \quad (2.102)$$

The time periodic solution of (2.101) is

$$h_0 = \frac{P_0}{\rho_w}, \quad (2.103)$$

which being substituted into (2.102) yields

$$\rho_0 = \rho_w M_\infty^2 P_0. \quad (2.104)$$

We continue with the substitution of (2.95) into the Navier-Stokes equations (2.2). This results in the following set of equations describing the steady flow past the roughness

$$\left. \begin{aligned} \rho_w \tau_w \check{Y}_* \frac{\partial \check{U}_r}{\partial \check{x}_*} + \rho_w \tau_w \check{V}_r &= -\frac{d\check{P}_r}{d\check{x}_*} + \mu_w \frac{\partial^2 \check{U}_r}{\partial \check{Y}_*^2}, \\ \frac{\partial \check{U}_r}{\partial \check{x}_*} + \frac{\partial \check{V}_r}{\partial \check{Y}_*} &= 0, \end{aligned} \right\} \quad (2.105a)$$

at order δ . These have to be solved subject to the boundary conditions

$$\left. \begin{aligned} \check{U}_r = \check{V}_r = 0 & \quad \text{at} \quad \check{Y}_* = 0, \\ \check{U}_r = \check{A}_r(\check{x}_*) & \quad \text{at} \quad \check{Y}_* = \infty, \\ \check{U}_r = 0 & \quad \text{at} \quad \check{x}_* = -\infty, \end{aligned} \right\} \quad (2.105b)$$

where $\check{A}_r(\check{x}_*)$ is the displacement function. Here we impose the no-slip condition on the surface of the roughness, the matching condition with the solution in region 2 (see figure 2.1), and the matching condition with the solution in the Stokes layer upstream of the roughness.

In the $\mathcal{O}(\sigma\delta)$ approximations the governing equations for the unsteady viscous sublayer are

$$\begin{aligned} \rho_w \frac{\partial \check{U}_*}{\partial t} + \rho_w \tau_w \check{Y}_* \frac{\partial \check{U}_*}{\partial \check{x}_*} + \rho_w \tau_w \check{V}_* = -\frac{\partial \check{P}_*}{\partial \check{x}_*} + \mu_w \frac{\partial^2 \check{U}_*}{\partial \check{Y}_*^2} \\ + \frac{1}{M_\infty} \frac{d\check{F}}{d\check{x}'}(\check{x}'_0) \check{Q}_1 \exp(i\check{\omega}t) + \mathcal{O}(M_\infty \check{F}) + c.c., \end{aligned} \quad (2.106a)$$

$$\frac{\partial \check{U}_*}{\partial \check{x}_*} + \frac{\partial \check{V}_*}{\partial \check{Y}_*} = 0, \quad (2.106b)$$

with

$$\check{Q}_1(\check{x}_*, \check{Y}_*) = (1+i) \sqrt{\frac{\check{\omega} \rho_w}{2\mu_w}} \check{V}_r e^{-(1+i)\sqrt{\frac{\check{\omega} \rho_w}{2\mu_w}} \check{Y}_*} - \left[e^{-(1+i)\sqrt{\frac{\check{\omega} \rho_w}{2\mu_w}} \check{Y}_*} - 1 \right] \frac{\partial \check{U}_r}{\partial \check{x}_*}, \quad (2.106c)$$

where $\check{Q}_1$ is connected to the first mechanism that is controlled by the pressure perturbations and any $\mathcal{O}(M_\infty)$ terms relate to the second mechanism where compressibility becomes an important factor as mentioned above. These equations have to be solved with the

boundary conditions

$$\left. \begin{aligned} \check{U}_* = \check{V}_* = 0 & \quad \text{at} \quad \check{Y}_* = 0, \\ \check{U}_* = \check{A}_*(t, \check{x}_*) & \quad \text{at} \quad \check{Y}_* = \infty, \\ \check{U}_* = 0 & \quad \text{at} \quad \check{x}_* = -\infty. \end{aligned} \right\} \quad (2.106d)$$

At both order δ and $\sigma\delta$ the approximations for pressure do not change across the boundary-layer

$$\frac{\partial \check{P}_r}{\partial \check{Y}_*} = \frac{\partial \check{P}_*}{\partial \check{Y}_*} = 0, \quad (2.107)$$

which may be confirmed by substituting (2.95) into the $\hat{y}$ -momentum equation (2.2b).

2.3.2 Middle deck solution

The solution in the middle deck (region 2) is sought in the asymptotic form

$$\left. \begin{aligned} \hat{u} &= V_\infty \left\{ U_{00}(Y) + \sigma U_1(t, Y) + Re^{-1/8} \sigma U_2(t, Y) + \right. \\ &\quad \left. + Re^{-1/8} \delta \tilde{U}_r(\check{x}_*, Y) + Re^{-1/8} \sigma \delta \tilde{U}_*(t, \check{x}_*, Y) + \dots \right\}, \\ \hat{v} &= V_\infty \left\{ Re^{-1/4} \sigma V_1(t, Y) + \right. \\ &\quad \left. + Re^{-1/4} \delta \tilde{V}_r(\check{x}_*, Y) + Re^{-1/4} \sigma \delta \tilde{V}_*(t, \check{x}_*, Y) + \dots \right\}, \\ \hat{p} &= p_\infty + \rho_\infty V_\infty^2 \left\{ \sigma P_0(t) + Re^{-1/4} \sigma p_0(t) \check{x}_* + \right. \\ &\quad \left. + Re^{-1/4} \delta \tilde{P}_r(\check{x}_*, Y) + Re^{-1/4} \sigma \delta \tilde{P}_*(t, \check{x}_*, Y) + \dots \right\}, \\ \hat{h} &= V_\infty^2 \left\{ H_{00}(Y) + \sigma H_1(t, Y) + \right. \\ &\quad \left. + Re^{-1/8} \delta \tilde{H}_r(\check{x}_*, Y) + Re^{-1/8} \sigma \delta \tilde{H}_*(t, \check{x}_*, Y) + \dots \right\}, \\ \hat{\rho} &= \rho_\infty \left\{ \rho_{00}(Y) + \sigma \rho_1(t, Y) + \right. \\ &\quad \left. + Re^{-1/8} \delta \tilde{\rho}_r(\check{x}_*, Y) + Re^{-1/8} \sigma \delta \tilde{\rho}_*(t, \check{x}_*, Y) + \dots \right\}. \end{aligned} \right\} \quad (2.108)$$

The terms in the first lines are obtained by re-expanding (2.89) in terms of the variables of region 2

$$\hat{t} = \frac{L}{V_\infty} Re^{-1/4} t, \quad \hat{x} = \hat{x}_0 + L Re^{-3/8} \check{x}_*, \quad \hat{y} = L Re^{-1/2} Y. \quad (2.109)$$

It follows from (2.93) that

$$\left. \begin{aligned} P_0 &= \frac{1}{M_\infty} \frac{\partial f}{\partial t}, & U_1 &= M_\infty Y \frac{dU_{00}}{dY} \frac{\partial f}{\partial t}, & V_1 &= -M_\infty Y \frac{\partial^2 f}{\partial t^2}, \\ \rho_1 &= M_\infty \frac{d(\rho_{00} Y)}{dY} \frac{\partial f}{\partial t}, & H_1 &= \frac{1}{M_\infty \rho_{00}} \left[1 - \frac{Y}{(\gamma - 1) \rho_{00}} \frac{d\rho_{00}}{dY} \right] \frac{\partial f}{\partial t}. \end{aligned} \right\} \quad (2.110)$$

The terms in the second lines in (2.108) represent the displacement effect of the viscous sublayer (region 1); see Sychev et al. (1998).

Substitution of (2.108) into the Navier-Stokes equations (2.2) shows that the steady perturbations produced by the wall roughness are described by

$$U_{00} \frac{\partial \tilde{U}_r}{\partial \check{x}_*} + \tilde{V}_r \frac{dU_{00}}{dY} = 0, \quad (2.111a)$$

$$\frac{\partial \tilde{P}_r}{\partial Y} = 0, \quad (2.111b)$$

$$U_{00} \frac{\partial \tilde{H}_r}{\partial \check{x}_*} + \tilde{V}_r \frac{dH_{00}}{dY} = 0, \quad (2.111c)$$

$$\rho_{00} \frac{\partial \tilde{U}_r}{\partial \check{x}_*} + U_{00} \frac{\partial \tilde{\rho}_r}{\partial \check{x}_*} + \rho_{00} \frac{\partial \tilde{V}_r}{\partial Y} + \tilde{V}_r \frac{d\rho_{00}}{dY} = 0, \quad (2.111d)$$

$$H_{00} = \frac{1}{(\gamma - 1) M_\infty^2 \rho_{00}}, \quad \text{and} \quad \tilde{H}_r = -\frac{H_{00}}{\rho_{00}} \tilde{\rho}_r. \quad (2.111e)$$

These equations are solved using the following elimination procedure. Substitution of (2.111e) into the energy equation (2.111c) results in

$$U_{00} \frac{\partial \tilde{\rho}_r}{\partial \check{x}_*} + \tilde{V}_r \frac{d\rho_{00}}{dY} = 0, \quad (2.112)$$

reducing the continuity equation to

$$\frac{\partial \tilde{U}_r}{\partial \check{x}_*} + \frac{\partial \tilde{V}_r}{\partial Y} = 0. \quad (2.113)$$

It remains to eliminate $\partial \tilde{U}_r / \partial \check{x}_*$ from (2.111a) and (2.113), and we conclude that

$$\frac{\partial}{\partial Y} \left(\frac{\tilde{V}_r}{U_{00}} \right) = 0. \quad (2.114)$$

The solution of this equation satisfying the matching condition with the solution in region 1

$$\left. \begin{aligned} \tilde{V}_r &= -\frac{d\check{A}_r}{d\check{x}_*}Y + \dots, \\ U_{00} &= \tau_w Y + \dots \end{aligned} \right\} \quad \text{as } Y \rightarrow 0 \quad (2.115)$$

is

$$\tilde{V}_r = -\frac{1}{\tau_w} \frac{d\check{A}_r}{d\check{x}_*} U_{00}. \quad (2.116)$$

From the order $\sigma\delta$ approximations we have to solve the following set of equations

$$U_{00} \frac{\partial \tilde{U}_*}{\partial \check{x}_*} + \tilde{V}_* \frac{dU_{00}}{dY} + U_1 \frac{\partial \tilde{U}_r}{\partial \check{x}_*} + \tilde{V}_r \frac{\partial U_1}{\partial Y} = 0, \quad (2.117a)$$

$$\frac{\partial \tilde{P}_*}{\partial Y} = 0, \quad (2.117b)$$

$$U_{00} \frac{\partial \tilde{H}_*}{\partial \check{x}_*} + \tilde{V}_* \frac{dH_{00}}{dY} + U_1 \frac{\partial \tilde{H}_r}{\partial \check{x}_*} + \tilde{V}_r \frac{\partial H_1}{\partial Y} = 0, \quad (2.117c)$$

$$\rho_{00} \frac{\partial \tilde{U}_*}{\partial \check{x}_*} + U_{00} \frac{\partial \tilde{\rho}_*}{\partial \check{x}_*} + U_1 \frac{\partial \tilde{\rho}_r}{\partial \check{x}_*} + \rho_{00} \frac{\partial \tilde{V}_*}{\partial Y} + \tilde{V}_* \frac{d\rho_{00}}{dY} + \tilde{V}_r \frac{\partial \rho_1}{\partial Y} = 0, \quad (2.117d)$$

$$H_{00} \tilde{\rho}_* + \rho_{00} \tilde{H}_* + H_1 \tilde{\rho}_r + \rho_1 \tilde{H}_r = 0. \quad (2.117e)$$

These may be manipulated in the same way as the equations (2.111). Alternatively one can notice that the flow in region 2 is inviscid, allowing us to use the entropy conservation law

$$\frac{\partial S}{\partial t} + \hat{u} \frac{\partial S}{\partial \hat{x}} + \hat{v} \frac{\partial S}{\partial \hat{y}} = 0. \quad (2.118)$$

For our purposes it is convenient to choose the entropy function, S , to be

$$S = \frac{\hat{p}/\rho_\infty}{(\hat{p}/p_\infty)^{1/\gamma}}. \quad (2.119)$$

Using the asymptotic expansions for $\hat{p}$ and $\hat{\rho}$ from (2.108) in (2.119) we find that the asymptotic expansion of the entropy function is written as

$$\begin{aligned} \hat{S} = & \rho_{00}(Y) + \sigma \left[\rho_1(t, Y) - M_\infty^2 \rho_{00}(Y) P_0(t) \right] + Re^{-1/8} \delta \tilde{\rho}_r(\check{x}_*, Y) + \\ & + Re^{-1/8} \sigma \delta \left[\tilde{\rho}_*(t, \check{x}_*, Y) - M_\infty^2 \tilde{\rho}_r(\check{x}_*, Y) P_0(t) \right] + \dots \end{aligned} \quad (2.120)$$

Substituting (2.120) into (2.118) and collecting $\mathcal{O}(Re^{1/4} \delta \sigma)$ terms, we arrive at the following equation

$$U_{00} \frac{\partial \tilde{\rho}_*}{\partial \check{x}_*} + \tilde{V}_* \frac{d\rho_{00}}{dY} + U_1 \frac{\partial \tilde{\rho}_r}{\partial \check{x}_*} + \tilde{V}_r \frac{\partial \rho_1}{\partial Y} = 0. \quad (2.121)$$

It can be used to eliminate $\partial \tilde{\rho}_* / \partial \check{x}_*$ from the continuity equation (2.117d), which results in

$$\frac{\partial \tilde{U}_*}{\partial \check{x}_*} + \frac{\partial \tilde{V}_*}{\partial Y} = 0. \quad (2.122)$$

We can now use (2.122) to eliminate $\partial \tilde{U}_* / \partial \check{x}_*$ from the streamwise momentum equation (2.117a). We find that

$$\frac{\partial}{\partial Y} \left(\frac{\tilde{V}_*}{U_{00}} \right) = \frac{U_1}{U_{00}^2} \frac{\partial \tilde{U}_r}{\partial \check{x}_*} + \frac{\tilde{V}_r}{U_{00}^2} \frac{\partial U_1}{\partial Y}. \quad (2.123)$$

In order to elucidate the physical meaning of the above equation we shall consider the slope of the streamlines in region 2. Using the asymptotic expansions for $\hat{u}$ and $\hat{v}$ in (2.108) it is deduced that

$$\frac{\hat{v}}{\hat{u}} = Re^{-1/4} \sigma \frac{V_1}{U_{00}} + Re^{-1/4} \delta \tilde{\vartheta}_r + Re^{-1/4} \sigma \delta \tilde{\vartheta}_* + \dots, \quad (2.124)$$

where

$$\tilde{\vartheta}_r = \frac{\tilde{V}_r}{U_{00}}, \quad \tilde{\vartheta}_* = \frac{\tilde{V}_*}{U_{00}} - \frac{U_1 \tilde{V}_r}{U_{00}^2}. \quad (2.125)$$

We know from (2.114) that $\tilde{\vartheta}_r$ does not change across region 2. Differentiating $\tilde{\vartheta}_*$ with

respect to Y and using equations (2.111a) and (2.111b) we find that

$$\frac{\partial \tilde{\vartheta}_*}{\partial Y} = \frac{\partial}{\partial Y} \left(\frac{\tilde{V}_*}{U_{00}} \right) - \frac{U_1}{U_{00}^2} \frac{\partial \tilde{U}_r}{\partial \check{x}_*} - \frac{\tilde{V}_r}{U_{00}^2} \frac{\partial U_1}{\partial Y}, \quad (2.126)$$

which, compared with (2.123), shows that $\tilde{\vartheta}_*$ also does not change across region 2. Therefore $\tilde{\vartheta}_*$ may be found through matching with the solution in region 1. It follows from the boundary condition for $\check{U}_*$ in (2.106d) and the continuity equation (2.106b) that, at the outer edge of the viscous sublayer, the normal velocity component exhibits the following behaviour

$$\check{V}_* = -\frac{\partial \check{A}_*}{\partial \check{x}_*} \check{Y}_* + \dots \quad \text{as} \quad \check{Y}_* \rightarrow \infty. \quad (2.127)$$

Consequently, at the bottom of region 2

$$\tilde{V}_* = -\frac{\partial \check{A}_*}{\partial \check{x}_*} Y + \dots \quad \text{as} \quad Y \rightarrow 0. \quad (2.128)$$

It further follows from (2.45), (2.110) and (2.116) that

$$\left. \begin{aligned} \tilde{V}_r &= -\frac{d\check{A}_r}{d\check{x}_*} Y + \dots, \\ U_{00} &= \tau_w(x) Y + \dots, \\ U_1 &= M_\infty \tau_w Y \frac{\partial f}{\partial t} + \dots, \end{aligned} \right\} \quad \text{as} \quad Y \rightarrow 0. \quad (2.129)$$

Using these on the right hand side of the equation for $\tilde{\vartheta}_*$ in (2.125) we conclude that

$$\tilde{\vartheta}_* = -\frac{1}{\tau_w} \frac{\partial \check{A}_*}{\partial \check{x}_*} + \frac{1}{\tau_w} \frac{d\check{A}_r}{d\check{x}_*} M_\infty \frac{\partial f}{\partial t}, \quad (2.130)$$

everywhere in region 2.

2.3.3 Outer deck solution

The perturbations in the outer deck (region 3 in figure 2.1) are produced by the vibrations of the wing surface and by the displacement effect of the boundary-layer. The former were studied in §2. When doing this we used stationary coordinates that did not move with the wing surface. We shall now write these as old coordinates, $(\hat{t}_{\text{old}}, \hat{x}_{\text{old}}, \hat{y}_{\text{old}}, \hat{u}_{\text{old}}, \hat{v}_{\text{old}})$. The triple-deck flow analysis is conducted in the coordinates which move with the wing surface. These will be written as the new coordinates, $(\hat{t}_{\text{new}}, \hat{x}_{\text{new}}, \hat{y}_{\text{new}}, \hat{u}_{\text{new}}, \hat{v}_{\text{new}})$. The transformation between the two coordinate systems is given by

$$\left. \begin{aligned} \hat{t}_{\text{old}} &= \hat{t}_{\text{new}}, & \hat{x}_{\text{old}} &= \hat{x}_{\text{new}}, & \hat{y}_{\text{old}} &= \hat{y}_w + \hat{y}_{\text{new}}, \\ \hat{u}_{\text{old}} &= \hat{u}_{\text{new}}, & \hat{v}_{\text{old}} &= \frac{\partial \hat{y}_w}{\partial \hat{t}} + \hat{v}_{\text{new}}, \end{aligned} \right\} \quad (2.131)$$

where the function $\hat{y}_w(\hat{t}, \hat{x})$ is given by (2.4).

In region 3 we introduce the dimensionless time and distance coordinates

$$\hat{t}_{\text{new}} = \frac{L}{V_\infty} Re^{-1/4} t, \quad \hat{x}_{\text{new}} = \hat{x}_0 + L Re^{-3/8} \check{x}_*, \quad \hat{y}_{\text{new}} = L Re^{-3/8} \check{y}_*. \quad (2.132)$$

Let us now re-expand the solution of the piston theory (§2) in terms of the variables (2.132) of region 3. We start with the vertical velocity component. According to (2.24) and (2.35),

$$\hat{v}_{\text{old}} = V_\infty \varepsilon Re^{1/4} \frac{\partial f}{\partial t}(t - M_\infty y, \check{x}'). \quad (2.133)$$

We now substitute (2.133) together with the wing surface equation (2.4) into the transformation of $\hat{v}$ in (2.131) and find

$$\hat{v}_{\text{new}} = V_\infty \varepsilon Re^{1/4} \left[\frac{\partial f}{\partial t}(t - M_\infty y, \check{x}') - \frac{\partial f}{\partial t}(t, \check{x}') \right]. \quad (2.134)$$

The coordinates (2.25) and (2.66) used in the piston theory are related to the coordinates

(2.132) of region 3 as

$$\begin{aligned} LRe^{-1/8}\check{x}' &= \hat{x}_0 + LRe^{-3/8}\check{x}_*, \\ LRe^{-1/4}y &= L\varepsilon f(t, \check{x}') + LRe^{-3/8}\check{y}_*. \end{aligned}$$

This gives

$$\check{x}' = \check{x}'_0 + Re^{-1/4}\check{x}_*, \quad y = Re^{-1/8}[\check{y}_* + \varepsilon Re^{3/8}f(t, \check{x}'_0)], \quad (2.135)$$

where $\check{x}'_0$ gives the position of the wall roughness in the piston theory variables.

It remains to substitute (2.135) into (2.134) and Taylor expand the resulting equation assuming that $\varepsilon Re^{3/8} \ll 1$. We find that the inner expansion of the outer solution is

$$\hat{v}_{new} = V_\infty \left\{ \varepsilon Re^{1/8} \left[-M_\infty \frac{\partial^2 f}{\partial t^2}(t, \check{x}'_0) \check{y}_* \right] + \dots \right\}. \quad (2.136a)$$

Similarly, it may be deduced that the rest of the fluid dynamics functions are expanded asymptotically as

$$\hat{u}_{new} = V_\infty \left\{ 1 + \varepsilon Re^{1/8} \left[-\frac{1}{M_\infty} \frac{\partial f}{\partial \check{x}'}(t, \check{x}'_0) \right] + \dots \right\}, \quad (2.136b)$$

$$\hat{p}_{new} = p_\infty + \rho_\infty V_\infty^2 \left\{ \varepsilon Re^{1/4} \left[\frac{1}{M_\infty} \frac{\partial f}{\partial t}(t, \check{x}'_0) \right] + \dots \right\}, \quad (2.136c)$$

$$\hat{h}_{new} = h_\infty + V_\infty^2 \left\{ \varepsilon Re^{1/4} \left[\frac{1}{M_\infty} \frac{\partial f}{\partial t}(t, \check{x}'_0) \right] + \dots \right\}, \quad (2.136d)$$

$$\hat{\rho}_{new} = \rho_\infty \left\{ 1 + \varepsilon Re^{1/4} \left[M_\infty \frac{\partial f}{\partial t}(t, \check{x}'_0) \right] + \dots \right\}. \quad (2.136e)$$

In order to evaluate the perturbations induced in region 3 by the displacement effect of the boundary-layer we need to calculate the streamline slope at the outer edge of region 2. Substituting (2.116) into the first of equations (2.125) we find

$$\tilde{\vartheta}_r = -\frac{1}{\tau_w} \frac{d\check{A}_r}{d\check{x}_*}. \quad (2.137)$$

If we now substitute (2.137) together with (2.130) into (2.124) and drop the old/new subscripts, then we find that in region 2

$$\begin{aligned} \frac{\hat{v}}{\hat{u}} = Re^{-1/4} \sigma \frac{V_1}{U_{00}} + Re^{-1/4} \delta \left\{ -\frac{1}{\tau_w} \frac{d\check{A}_r}{d\check{x}_*} \right\} + \\ + Re^{-1/4} \sigma \delta \left\{ -\frac{1}{\tau_w} \frac{\partial \check{A}_*}{\partial \check{x}_*} + \frac{1}{\tau_w} \frac{d\check{A}_r}{d\check{x}_*} M_\infty \frac{\partial f}{\partial t} \right\} + \dots \end{aligned} \quad (2.138)$$

We perform the flow analysis in the outer deck (region 3) in the coordinates with $\hat{x}$ -axis following the body contour without roughness. Consequently, when performing the matching of the streamline slope in regions 2 and 3 we need to add the slope of the body roughness to (2.138). Differentiating (2.94) we have

$$\frac{d\hat{y}}{d\hat{x}} = Re^{-1/4} \delta \frac{d\check{G}}{d\check{x}_*}. \quad (2.139)$$

We add this slope to (2.138) and use for V_1 an explicit expression; see (2.110). This results in

$$\begin{aligned} \frac{\hat{v}}{\hat{u}} = Re^{-1/4} \sigma \left\{ -M_\infty \frac{Y}{U_{00}(Y)} \frac{\partial^2 f}{\partial t^2} \right\} + Re^{-1/4} \delta \left\{ \frac{d\check{G}}{d\check{x}_*} - \frac{1}{\tau_w} \frac{d\check{A}_r}{d\check{x}_*} \right\} + \\ + Re^{-1/4} \sigma \delta \left\{ -\frac{1}{\tau_w} \frac{\partial \check{A}_*}{\partial \check{x}_*} + \frac{1}{\tau_w} \frac{d\check{A}_r}{d\check{x}_*} M_\infty \frac{\partial f}{\partial t} \right\} + \dots \end{aligned} \quad (2.140)$$

This confirms that the solution in the outer deck has the following asymptotic expansion

$$\left. \begin{aligned}
 \hat{u} &= V_\infty \left\{ 1 + \sigma Re^{-1/8} \left[-\frac{1}{M_\infty} \frac{\partial f}{\partial \check{x}'}(t, \check{x}_0') \right] + \dots \right. \\
 &\quad \left. + Re^{-1/4} \delta \tilde{u}_r(\check{x}_*, \tilde{y}_*) + Re^{-1/4} \delta \sigma \tilde{u}_*(t, \check{x}_*, \tilde{y}_*) + \dots \right\}, \\
 \hat{v} &= V_\infty \left\{ \sigma Re^{-1/8} \left[-M_\infty \frac{\partial^2 f}{\partial t^2}(t, \check{x}_0') \tilde{y}_* \right] + \dots \right. \\
 &\quad \left. + Re^{-1/4} \delta \tilde{v}_r(\check{x}_*, \tilde{y}_*) + Re^{-1/4} \delta \sigma \tilde{v}_*(t, \check{x}_*, \tilde{y}_*) + \dots \right\}, \\
 \hat{p} &= p_\infty + \rho_\infty V_\infty^2 \left\{ \sigma \left[\frac{1}{M_\infty} \frac{\partial f}{\partial t}(t, \check{x}_0') \right] + \dots \right. \\
 &\quad \left. + Re^{-1/4} \delta \tilde{p}_r(\check{x}_*, \tilde{y}_*) + Re^{-1/4} \delta \sigma \tilde{p}_*(t, \check{x}_*, \tilde{y}_*) + \dots \right\}, \\
 \hat{h} &= h_\infty + V_\infty^2 \left\{ \sigma \left[\frac{1}{M_\infty} \frac{\partial f}{\partial t}(t, \check{x}_0') \right] + \dots \right. \\
 &\quad \left. + Re^{-1/4} \delta \tilde{h}_r(\check{x}_*, \tilde{y}_*) + Re^{-1/4} \delta \sigma \tilde{h}_*(t, \check{x}_*, \tilde{y}_*) + \dots \right\}, \\
 \hat{\rho} &= \rho_\infty \left\{ 1 + \sigma \left[M_\infty \frac{\partial f}{\partial t}(t, \check{x}_0') \right] + \dots \right. \\
 &\quad \left. + Re^{-1/4} \delta \tilde{\rho}_r(\check{x}_*, \tilde{y}_*) + Re^{-1/4} \delta \sigma \tilde{\rho}_*(t, \check{x}_*, \tilde{y}_*) + \dots \right\},
 \end{aligned} \right\} \quad (2.141)$$

which can be checked with the use of a Taylor expansion and gathering similar terms to form (2.140). We have used the independent variables defined by

$$\hat{t} = \frac{L}{V_\infty} Re^{-1/4} t, \quad \hat{x} = \hat{x}_0 + L Re^{-3/8} \check{x}_*, \quad \hat{y} = L Re^{-3/8} \tilde{y}_*. \quad (2.142)$$

The $\mathcal{O}(Re^{-1/4}\delta)$ terms in (2.141) describe the steady flow perturbations produced by the wall roughness. Substitution of (2.141) into the Navier-Stokes equations results in the

following set of equations for the steady perturbations in region 3

$$\frac{\partial \tilde{u}_r}{\partial \tilde{x}_*} = -\frac{\partial \tilde{p}_r}{\partial \tilde{x}_*}, \quad (2.143a)$$

$$\frac{\partial \tilde{v}_r}{\partial \tilde{x}_*} = -\frac{\partial \tilde{p}_r}{\partial \tilde{y}_*}, \quad (2.143b)$$

$$\frac{\partial \tilde{h}_r}{\partial \tilde{x}_*} = \frac{\partial \tilde{p}_r}{\partial \tilde{x}_*}, \quad (2.143c)$$

$$\frac{\partial \tilde{\rho}_r}{\partial \tilde{x}_*} + \frac{\partial \tilde{u}_r}{\partial \tilde{x}_*} + \frac{\partial \tilde{v}_r}{\partial \tilde{y}_*} = 0, \quad (2.143d)$$

$$\tilde{h}_r + \frac{\tilde{\rho}_r}{(\gamma - 1)M_\infty^2} = \frac{\gamma}{\gamma - 1}\tilde{p}_r. \quad (2.143e)$$

In the next order approximation we have

$$\frac{\partial \tilde{u}_*}{\partial \tilde{x}_*} + M_\infty \frac{\partial f}{\partial t}(t, \tilde{x}'_0) \frac{\partial \tilde{u}_r}{\partial \tilde{x}_*} = -\frac{\partial \tilde{p}_*}{\partial \tilde{x}_*}, \quad (2.144a)$$

$$\frac{\partial \tilde{v}_*}{\partial \tilde{x}_*} + M_\infty \frac{\partial f}{\partial t}(t, \tilde{x}'_0) \frac{\partial \tilde{v}_r}{\partial \tilde{x}_*} = -\frac{\partial \tilde{p}_*}{\partial \tilde{y}_*}, \quad (2.144b)$$

$$\frac{\partial \tilde{h}_*}{\partial \tilde{x}_*} + M_\infty \frac{\partial f}{\partial t}(t, \tilde{x}'_0) \frac{\partial \tilde{h}_r}{\partial \tilde{x}_*} = \frac{\partial \tilde{p}_*}{\partial \tilde{x}_*}, \quad (2.144c)$$

$$\frac{\partial \tilde{\rho}_*}{\partial \tilde{x}_*} + \frac{\partial \tilde{u}_*}{\partial \tilde{x}_*} + \frac{\partial \tilde{v}_*}{\partial \tilde{y}_*} = -M_\infty \frac{\partial f}{\partial t}(t, \tilde{x}'_0) \left(\frac{\partial \tilde{u}_r}{\partial \tilde{x}_*} + \frac{\partial \tilde{v}_r}{\partial \tilde{y}_*} \right), \quad (2.144d)$$

$$\tilde{h}_* + \frac{\tilde{\rho}_*}{(\gamma - 1)M_\infty^2} + \frac{\partial f}{\partial t}(t, \tilde{x}'_0) \left(M_\infty \tilde{h}_r + \frac{1}{M_\infty} \tilde{\rho}_r \right) = \frac{\gamma}{\gamma - 1} \tilde{p}_*. \quad (2.144e)$$

The streamline slope in region 2 is given by the equation (2.140). In order to deduce the corresponding equation in region 3, we need to use the asymptotic expansions for $\hat{u}$ and $\hat{v}$ in (2.141). We find that

$$\frac{\hat{v}}{\hat{u}} = Re^{-1/8} \sigma \left\{ -M_\infty \frac{\partial^2 f}{\partial t^2} \tilde{y}_* \right\} + Re^{-1/4} \delta \tilde{v}_r + Re^{-1/4} \sigma \delta \tilde{v}_* + \dots \quad (2.145)$$

Setting $\tilde{y}_* \rightarrow 0$ in (2.145) and comparing the result with (2.140), we can conclude that

$$\tilde{v}_r \Big|_{\tilde{y}_*=0} = \frac{d\check{G}}{d\check{x}_*} - \frac{1}{\tau_w} \frac{d\check{A}_r}{d\check{x}_*}, \quad (2.146a)$$

$$\tilde{v}_* \Big|_{\tilde{y}_*=0} = -\frac{1}{\tau_w} \frac{\partial \check{A}_*}{\partial \check{x}_*} + \frac{1}{\tau_w} \frac{d\check{A}_r}{d\check{x}_*} M_\infty \frac{\partial f}{\partial t}. \quad (2.146b)$$

The set of equations (2.143) may be reduced to a single equation for the pressure, $\tilde{p}_r$. Indeed, solving the state equation (2.143e) for $\tilde{h}_r$ and substituting the result into the energy equation (2.143c) results in

$$\frac{\partial \tilde{p}_r}{\partial \tilde{x}_*} = M_\infty^2 \frac{\partial \tilde{p}_r}{\partial \tilde{x}_*}. \quad (2.147)$$

Now, we substitute (2.147) and the $\hat{x}$ -momentum equation (2.143a) into the continuity equation (2.143d). We find that

$$(M_\infty^2 - 1) \frac{\partial \tilde{p}_r}{\partial \tilde{x}_*} + \frac{\partial \tilde{v}_r}{\partial \tilde{y}_*} = 0. \quad (2.148)$$

It remains to differentiate (2.148) with respect to $\tilde{x}_*$ and, with the aid of $\hat{y}$ -momentum equation (2.143b), we find that the steady pressure, $\tilde{p}_r$, satisfies the equation

$$(1 - M_\infty^2) \frac{\partial^2 \tilde{p}_r}{\partial \tilde{x}_*^2} + \frac{\partial^2 \tilde{p}_r}{\partial \tilde{y}_*^2} = 0, \quad (2.149a)$$

subject to the boundary conditions

$$\frac{\partial \tilde{p}_r}{\partial \tilde{y}_*} \Big|_{\tilde{y}_*=0} = \frac{1}{\tau_w} \frac{d^2 \check{A}_r}{d\check{x}_*^2} - \frac{d^2 \check{G}}{d\check{x}_*^2}, \quad (2.149b)$$

$$\tilde{p}_r \rightarrow 0 \quad \text{as} \quad \check{x}_*^2 + \check{y}_*^2 \rightarrow \infty. \quad (2.149c)$$

The first of these is obtained from matching with the solution in region 2, and the second represents the condition of attenuation of the perturbations far from the roughness.

Similarly, the set of equations (2.144) may be reduced so that the unsteady perturbations in region 1 are represented by the $\mathcal{O}(Re^{-1/4}\delta\sigma)$ terms in (2.141). Our particular interest is

with the pressure, which satisfies the following equation

$$(1 - M_\infty^2) \frac{\partial^2 \tilde{p}_*}{\partial \check{x}_*^2} + \frac{\partial^2 \tilde{p}_*}{\partial \check{y}_*^2} = -(\gamma - 1) M_\infty^3 \frac{\partial f}{\partial t}(t, \check{x}_0) \frac{\partial^2 \tilde{p}_r}{\partial \check{x}_*^2}. \quad (2.150a)$$

We must solve the above with the boundary condition

$$\left. \frac{\partial \tilde{p}_*}{\partial \check{y}_*} \right|_{\check{y}_*=0} = \frac{1}{\tau_w} \frac{\partial^2 \tilde{A}_*}{\partial \check{x}_*^2} - \left\{ i\check{\omega} M_\infty \check{F}(\check{x}_0) \frac{d^2 \check{G}}{d\check{x}_*^2} \exp(i\check{\omega}t) + c.c. \right\}, \quad (2.150b)$$

and far field condition

$$\tilde{p}_* \rightarrow 0 \quad \text{as} \quad \check{x}_*^2 + \check{y}_*^2 \rightarrow \infty. \quad (2.150c)$$

2.3.4 Viscous-inviscid interaction problem

We have found that in order to describe the steady flow past a wall roughness one has to solve the equations (2.105) governing the flow in viscous sublayer (region 1) coupled with the equations (2.149) governing the flow in outer deck (region 3). We shall now write both sets of equations in canonical form. To simplify our systems of equations and remove values held constant within the flow i.e. ρ_w , μ_w , τ_w and M_∞ , we apply the following affine transformations to the variables in the viscous sublayer

$$\begin{aligned} \check{U}_r &= \frac{\mu_w^{1/4} \rho_w^{-1/2}}{\tau_w^{-1/4} \beta^{1/4}} U_r, & \check{V}_r &= \frac{\mu_w^{3/4} \rho_w^{-1/2}}{\tau_w^{-3/4} \beta^{-1/4}} V_r, & \check{P}_r &= \frac{\mu_w^{1/2}}{\tau_w^{-1/2} \beta^{1/2}} P_r, \\ \check{A}_r &= \frac{\mu_w^{1/4} \rho_w^{-1/2}}{\tau_w^{-1/4} \beta^{1/4}} A_r, & \check{x}_* &= \frac{\mu_w^{-1/4} \rho_w^{-1/2}}{\tau_w^{5/4} \beta^{3/4}} x_*, & \check{Y}_* &= \frac{\mu_w^{1/4} \rho_w^{-1/2}}{\tau_w^{3/4} \beta^{1/4}} Y_*, \end{aligned}$$

where $\beta = \sqrt{1 - M_\infty^2}$. We also transform the outer deck variables as

$$\tilde{p}_r = \frac{\mu_w^{1/2}}{\tau_w^{-1/2} \beta^{1/2}} p_r, \quad \tilde{y}_* = \frac{\mu_w^{-1/4} \rho_w^{-1/2}}{\tau_w^{5/4} \beta^{7/4}} y_*, \quad \check{G} = \frac{\mu_w^{1/4} \rho_w^{-1/2}}{\tau_w^{3/4} \beta^{1/4}} G. \quad (2.151)$$

As a result the viscous steady sublayer problem assumes the form

$$Y_* \frac{\partial U_r}{\partial x_*} + V_r = -\frac{dP_r}{dx_*} + \frac{\partial^2 U_r}{\partial Y_*^2}, \quad (2.152a)$$

$$\frac{\partial U_r}{\partial x_*} + \frac{\partial V_r}{\partial Y_*} = 0, \quad (2.152b)$$

with conditions

$$U_r = V_r = 0 \quad \text{at} \quad Y_* = 0, \quad (2.152c)$$

$$U_r = A_r(x_*) \quad \text{at} \quad Y_* = \infty, \quad (2.152d)$$

$$U_r = 0 \quad \text{at} \quad x_* = -\infty, \quad (2.152e)$$

and the outer deck steady problem turns into

$$\frac{\partial^2 p_r}{\partial x_*^2} + \frac{\partial^2 p_r}{\partial y_*^2} = 0, \quad (2.153a)$$

$$\left. \frac{\partial p_r}{\partial y_*} \right|_{y_*=0} = \frac{d^2 A_r}{dx_*^2} - \frac{d^2 G}{dx_*^2}, \quad (2.153b)$$

$$p_r \rightarrow 0 \quad \text{as} \quad x_*^2 + y_*^2 \rightarrow \infty. \quad (2.153c)$$

Similarly, to remove flow constants we apply the affine transformations

$$\begin{aligned} \check{U}_* &= \frac{\mu_w^{1/4} \rho_w^{-1/2}}{\tau_w^{-1/4} \beta^{1/4}} U_*, & \check{V}_* &= \frac{\mu_w^{3/4} \rho_w^{-1/2}}{\tau_w^{-3/4} \beta^{-1/4}} V_*, & \check{P}_* &= \frac{\mu_w^{1/2}}{\tau_w^{-1/2} \beta^{1/2}} P_*, \\ \check{A}_* &= \frac{\mu_w^{1/4} \rho_w^{-1/2}}{\tau_w^{-1/4} \beta^{1/4}} A_*, & t &= \frac{\mu_w^{-1/2}}{\tau_w^{3/2} \beta^{1/2}} t_*, & \check{\omega} &= \frac{\mu_w^{1/2}}{\tau_w^{-3/2} \beta^{-1/2}} \omega, \\ \check{F} &= \frac{\mu_w^{-1/2}}{\tau_w^{3/2} \beta^{1/2}} F, & \check{x}' &= \frac{\mu_w^{-3/4} \rho_w^{-1/2}}{\tau_w^{7/4} \beta^{1/4}} x', \end{aligned}$$

to the unsteady viscous sublayer problem (2.106), and

$$\tilde{p}_* = \frac{\mu_w^{1/2}}{\tau_w^{-1/2} \beta^{1/2}} p_*. \quad (2.154)$$

to the unsteady pressure perturbations in the outer deck, governed by (2.150).

This turns the viscous sublayer equations into

$$\begin{aligned} \frac{\partial U_*}{\partial t_*} + Y_* \frac{\partial U_*}{\partial x_*} + V_* &= -\frac{\partial P_*}{\partial x_*} + \frac{\partial^2 U_*}{\partial Y_*^2} \\ &+ \frac{1}{M_\infty} \frac{dF}{dx'}(x'_0) Q_1 \exp(i\omega t_*) + \mathcal{O}(M_\infty F) + c.c., \end{aligned} \quad (2.155a)$$

$$\frac{\partial U_*}{\partial x_*} + \frac{\partial V_*}{\partial Y_*} = 0, \quad (2.155b)$$

with

$$Q_1(x_*, Y_*) = (1+i) \sqrt{\frac{\omega}{2}} V_r e^{-(1+i)\sqrt{\frac{\omega}{2}} Y_*} - \left[e^{-(1+i)\sqrt{\frac{\omega}{2}} Y_*} - 1 \right] \frac{\partial U_r}{\partial x_*}, \quad (2.155c)$$

The boundary conditions for these equations assume the form

$$U_* = V_* = 0 \quad \text{at} \quad Y_* = 0, \quad (2.155d)$$

$$U_* = A_*(t_*, x_*) \quad \text{at} \quad Y_* = \infty, \quad (2.155e)$$

$$U_* = 0 \quad \text{at} \quad x_* = -\infty. \quad (2.155f)$$

The corresponding outer deck problem now has the form

$$\frac{\partial^2 p_*}{\partial x_*^2} + \frac{\partial^2 p_*}{\partial y_*^2} = -(\gamma - 1) \frac{M_\infty^3}{1 - M_\infty^2} \frac{\partial^2 p_r}{\partial x_*^2} \left\{ i\omega F(x'_0) \exp(i\omega t_*) + c.c. \right\}, \quad (2.156a)$$

$$\left. \frac{\partial p_*}{\partial y_*} \right|_{y_*=0} = \frac{\partial^2 A_*}{\partial x_*^2} - \left\{ i\omega M_\infty F(x'_0) \frac{d^2 G}{dx_*^2} \exp(i\omega t_*) + c.c. \right\}, \quad (2.156b)$$

$$p_* \rightarrow 0 \quad \text{as} \quad x_*^2 + y_*^2 \rightarrow \infty. \quad (2.156c)$$

2.4 Solution of the viscous-inviscid interaction problem

We use the Fourier transform technique for both the steady problem (2.152) (2.153) and the unsteady problem (2.155) and (2.156). As an example, we apply the Fourier trans-

form to the pressure, p_r :

$$\bar{p}_r(k, y_*) = \int_{-\infty}^{\infty} p_r(x_*, y_*) \exp(-ikx_*) dx_*, \quad (2.157)$$

where k is the streamwise wavenumber. In the subsequent analysis we do the same for the other flow quantities using analogous notation.

2.4.1 Steady problem

We start with the inviscid equation (2.153a) and apply the Fourier transform so that it reduces to the following ordinary differential equation

$$-k^2 \bar{p}_r + \frac{d^2 \bar{p}_r}{dy_*^2} = 0. \quad (2.158a)$$

The Fourier transforms of the boundary conditions (2.153b), (2.153c) are written as

$$\bar{p}_r = 0 \quad \text{at} \quad y_* = \infty, \quad (2.158b)$$

$$\frac{d\bar{p}_r}{dy_*} = k^2 [\bar{G}(k) - \bar{A}_r(k)] \quad \text{at} \quad y_* = 0, \quad (2.158c)$$

where $\bar{G}(k)$ and $\bar{A}_r(k)$ are the Fourier transforms of the roughness shape function, $G(x_*)$ and the displacement function $A_r(x_*)$, respectively.

The solution to (2.158) is written as

$$\bar{p}_r = |k| [\bar{A}_r(k) - \bar{G}(k)] \exp(-|k|y_*). \quad (2.159)$$

Setting $y_* = 0$ in the above equation gives the Fourier transform of the pressure in the viscous sublayer

$$\bar{P}_r = |k| [\bar{A}_r - \bar{G}(k)]. \quad (2.160)$$

Now we turn to the viscous problem (2.152). Applying the Fourier transform to (2.152a)

and (2.152b) we have

$$ikY_*\bar{U}_r + \bar{V}_r = -ik\bar{P}_r + \frac{d^2\bar{U}_r}{dY_*^2}, \quad (2.161a)$$

$$ik\bar{U}_r + \frac{d\bar{V}_r}{dY_*} = 0, \quad (2.161b)$$

while the boundary conditions (2.152c)–(2.152e) assume the form

$$\bar{U}_r = \bar{V}_r = 0 \quad \text{at} \quad Y_* = 0, \quad (2.161c)$$

$$\bar{U}_r = \bar{A}_r \quad \text{at} \quad Y_* = \infty. \quad (2.161d)$$

If we set $Y_* = 0$ in (2.161a) and use the solution for the pressure (2.160), then we find that

$$\left. \frac{d^2\bar{U}_r}{dY_*^2} \right|_{Y_*=0} = ik\bar{P}_r = ik|k|[\bar{A}_r - \bar{G}(k)]. \quad (2.162)$$

Now we differentiate (2.161a) with respect to Y_* and use the continuity equation (2.161b) to eliminate $\bar{V}_r$. This leads to the following equation for $\bar{U}_r$

$$ikY_* \frac{d\bar{U}_r}{dY_*} = \frac{d^3\bar{U}_r}{dY_*^3}. \quad (2.163a)$$

It should be solved with the boundary conditions

$$\bar{U}_r = 0 \quad \text{at} \quad Y_* = 0, \quad (2.163b)$$

$$\frac{d^2\bar{U}_r}{dY_*^2} = ik|k|[\bar{A}_r - \bar{G}(k)] \quad \text{at} \quad Y_* = 0, \quad (2.163c)$$

$$\bar{U}_r = \bar{A}_r \quad \text{at} \quad Y_* = \infty. \quad (2.163d)$$

The transformation of the independent variable $\zeta = \theta Y_*$ with $\theta = (ik)^{1/3}$ primarily to eliminate the ik multiplier in the (2.163c) but also turns (2.163a) into the Airy equation for the derivative $d\bar{U}_r/d\zeta$

$$\frac{d^3\bar{U}_r}{d\zeta^3} - \zeta \frac{d\bar{U}_r}{d\zeta} = 0. \quad (2.164a)$$

In the new variables the boundary conditions (2.163b)–(2.163d) assume the form

$$\bar{U}_r = 0 \quad \text{at } \zeta = 0, \quad (2.164b)$$

$$\frac{d^2 \bar{U}_r}{d\zeta^2} = (ik)^{1/3} |k| [\bar{A}_r - \bar{G}(k)] \quad \text{at } \zeta = 0, \quad (2.164c)$$

$$\bar{U}_r = \bar{A}_r \quad \text{at } \zeta = \infty. \quad (2.164d)$$

The general solution of the equation (2.164a) is written as

$$\frac{d\bar{U}_r}{d\zeta} = C_1 Ai(\zeta) + C_2 Bi(\zeta). \quad (2.165)$$

Here $Ai(\zeta)$ and $Bi(\zeta)$ are two complementary solutions of the Airy equation (see, for example Abramowitz and Stegun (1965)). Our task now is to find the constants C_1 and C_2 . We notice, first of all, that according to (2.164d) the function $\bar{U}_r$ should remain finite at large ζ , which is only possible if $d\bar{U}_r/d\zeta$ tends to zero as $\zeta \rightarrow \infty$. The behaviour of $Ai(\zeta)$ and $Bi(\zeta)$ at large ζ depends on the direction along which ζ tends to infinity in the complex ζ -plane. The latter in turn depends on a choice of the analytical branch of $\theta = (ik)^{1/3}$. For our purposes, it is convenient to make a branch cut in the complex k -plane along the positive imaginary semi-axis and define θ as

$$\theta = (ik)^{1/3} = \left(\exp(i\pi/2) |k| \exp(i\phi) \right)^{1/3} = |k|^{1/3} \exp(i(\pi/6 + \phi/3)), \quad (2.166)$$

with $\arg k \in (-\frac{3}{2}\pi, \frac{1}{2}\pi)$ and ϕ is the phase. Then for any real positive Y_* the argument of $\zeta = \theta Y_*$ will lie within the interval

$$\arg \zeta \in \left(\frac{1}{3}\pi, -\frac{1}{3}\pi \right),$$

such that the Airy function, $Ai(\zeta)$, appears to decay exponentially as $\zeta \rightarrow \infty$, while the $Bi(\zeta)$ function shows exponential growth for all k in the complex k -plane. Therefore, we

have to set $C_2 = 0$, which reduces (2.165) to

$$\frac{d\bar{U}_r}{d\zeta} = C_1 Ai(\zeta). \quad (2.167)$$

Substitution of (2.167) into (2.164c) yields

$$C_1 Ai'(0) = (ik)^{1/3} |k| [\bar{A}_r - \bar{G}(k)]. \quad (2.168)$$

In order to deduce the second equation relating the two unknown constants C_1 and $\bar{A}_r$ we integrate (2.167) with the initial condition (2.164b)

$$\bar{U}_r = C_1 \int_0^\zeta Ai(s) ds, \quad (2.169)$$

and after setting $\zeta = \infty$ in (2.169), use the boundary condition (2.164d). This gives

$$\bar{A}_r = \frac{1}{3} C_1. \quad (2.170)$$

It remains to solve the pair of equations (2.168) and (2.170) for C_1 . We find that

$$C_1 = \frac{3(ik)^{1/3} |k|}{(ik)^{1/3} |k| - 3Ai'(0)} \bar{G}(k). \quad (2.171)$$

With C_1 known the Fourier transform, $\bar{U}_r$, of the streamwise velocity can be calculated with the help of the equation (2.169). In order to find the Fourier transform of the lateral velocity, $\bar{V}_r$ one needs to integrate the equation (2.161b). Using the transformation $\zeta = (ik)^{1/3} Y_*$ this equation is written as

$$\frac{d\bar{V}_r}{d\zeta} = -(ik)^{2/3} \bar{U}_r. \quad (2.172)$$

Taking into account that $\bar{V}_r = 0$ at $\zeta = 0$, we have

$$\bar{V}_r = -(ik)^{2/3} \int_0^\zeta \bar{U}_r(\zeta') d\zeta'. \quad (2.173)$$

Summarising, it follows from (2.169), (2.171) and (2.173) that

$$\bar{U}_r = \bar{G}(k)\Phi(\zeta; k), \quad \bar{V}_r = \bar{G}(k)\Psi(\zeta; k), \quad (2.174)$$

where

$$\Phi(\zeta; k) = \frac{3(ik)^{1/3}|k|}{(ik)^{1/3}|k| - 3Ai'(0)} \int_0^\zeta Ai(s)ds \quad \text{and} \quad \Psi(\zeta; k) = -(ik)^{2/3} \int_0^\zeta \Phi(\zeta'; k)d\zeta'. \quad (2.175)$$

2.4.2 Unsteady problem

Now we turn our attention to solving the unsteady viscous-inviscid interaction problem (2.155) and (2.156). Being guided by the form of the forcing terms in (2.155a), (2.156a) and (2.156b) we represent the solution in the form

$$\left. \begin{aligned} U_* &= \left\{ \frac{1}{M_\infty} \frac{dF}{dx'}(x'_0)U_1(x_*, Y_*) + M_\infty F(x'_0)U_2(x_*, Y_*) \right\} \exp(i\omega t_*) + c.c., \\ V_* &= \left\{ \frac{1}{M_\infty} \frac{dF}{dx'}(x'_0)V_1(x_*, Y_*) + M_\infty F(x'_0)V_2(x_*, Y_*) \right\} \exp(i\omega t_*) + c.c., \\ P_* &= \left\{ \frac{1}{M_\infty} \frac{dF}{dx'}(x'_0)P_1(x_*) + M_\infty F(x'_0)P_2(x_*) \right\} \exp(i\omega t_*) + c.c., \\ A_* &= \left\{ \frac{1}{M_\infty} \frac{dF}{dx'}(x'_0)A_1(x_*) + M_\infty F(x'_0)A_2(x_*) \right\} \exp(i\omega t_*) + c.c., \\ p_* &= \left\{ \frac{1}{M_\infty} \frac{dF}{dx'}(x'_0)p_1(x_*, y_*) + M_\infty F(x'_0)p_2(x_*, y_*) \right\} \exp(i\omega t_*) + c.c.. \end{aligned} \right\} \quad (2.176)$$

This splits the flow perturbations into two parts. Both are linked to the pressure perturbations (2.38) produced by the wing surface vibrations. However, in the first part are the flow perturbations are due to the pressure along the boundary-layer, and the second are the flow perturbations are due to the pressure variations with time linked to the second mechanism not investigated here. The former is found by solving the following boundary-value

problem in region 1

$$i\omega U_1 + Y_* \frac{\partial U_1}{\partial x_*} + V_1 = -\frac{dP_1}{dx_*} + \frac{\partial^2 U_1}{\partial Y_*^2} + Q_1, \quad (2.177a)$$

$$\frac{\partial U_1}{\partial x_*} + \frac{\partial V_1}{\partial Y_*} = 0, \quad (2.177b)$$

with boundary and far field conditions

$$U_1 = V_1 = 0 \quad \text{at} \quad Y_* = 0, \quad (2.177c)$$

$$U_1 = A_1(x_*) \quad \text{at} \quad Y_* = \infty, \quad (2.177d)$$

$$U_1 = 0 \quad \text{at} \quad x_* = -\infty. \quad (2.177e)$$

The pressure P_1 in (2.177a) is not known in advance; it should be found by solving the boundary-value problem

$$\frac{\partial^2 P_1}{\partial x_*^2} + \frac{\partial^2 P_1}{\partial y_*^2} = 0, \quad (2.178a)$$

with the boundary and far field conditions

$$\frac{\partial P_1}{\partial y_*} = \frac{\partial^2 A_1}{\partial x_*^2} \quad \text{at} \quad y_* = 0, \quad (2.178b)$$

$$P_1 \rightarrow 0 \quad \text{as} \quad x_*^2 + y_*^2 \rightarrow \infty. \quad (2.178c)$$

for region 3.

The inviscid problem (2.178) is dealt with in the same way as the corresponding steady problem (2.153). It yields the following equation for the Fourier transform of the pressure in region 1

$$\bar{P}_1 = |k| \bar{A}_1. \quad (2.179)$$

The viscous problem (2.177) is written in terms of the Fourier transforms as

$$i(\omega + kY_*) \bar{U}_1 + \bar{V}_1 = -ik \bar{P}_1 + \frac{d^2 \bar{U}_1}{dY_*^2} + \bar{Q}_1, \quad (2.180a)$$

$$ik \bar{U}_1 + \frac{d\bar{V}_1}{dY_*} = 0, \quad (2.180b)$$

$$\bar{U}_1 = \bar{V}_1 = 0 \quad \text{at} \quad Y_* = 0, \quad (2.180c)$$

$$\bar{U}_1 = \bar{A}_1(k) \quad \text{at} \quad Y_* = \infty. \quad (2.180d)$$

Here

$$\bar{Q}_1 = (1+i)\sqrt{\frac{\omega}{2}} \exp\left(- (1+i)\sqrt{\frac{\omega}{2}}Y_*\right) \bar{V}_r - ik \left[\exp\left(- (1+i)\sqrt{\frac{\omega}{2}}Y_*\right) - 1 \right] \bar{U}_r. \quad (2.181)$$

The equations (2.180a) and (2.180b) may be manipulated in the same way as the corresponding steady flow equations (2.161a) and (2.161b). We start by eliminating $\bar{V}_1$, which is achieved through differentiation of (2.180a) with respect to Y_* . This leads to the equation

$$i(\omega + kY_*) \frac{d\bar{U}_1}{dY_*} = \frac{d^3\bar{U}_1}{dY_*^3} + \frac{d\bar{Q}_1}{dY_*}. \quad (2.182)$$

Since we are dealing now with the third order differential equation, an additional boundary condition for $\bar{U}_1$ is required. It is obtained by setting $Y_* = 0$ in (2.180a). Since $\bar{U}_1$, $\bar{V}_1$, $\bar{U}_r$ and $\bar{V}_r$ are all zero at $Y_* = 0$, we will have

$$\left. \frac{d^2\bar{U}_1}{dY_*^2} \right|_{Y_*=0} = ik\bar{P}_1 = ik|k|\bar{A}_1. \quad (2.183)$$

If we now introduce a new independent variable $z = z_0 + \theta Y_*$ where

$$z_0 = \frac{i\omega}{(ik)^{2/3}} \quad \text{and} \quad \theta = (ik)^{1/3}, \quad (2.184)$$

then the equation (2.182) and the boundary conditions (2.180c), (2.180d) and (2.183) assume the form

$$\frac{d^3\bar{U}_1}{dz^3} - z \frac{d\bar{U}_1}{dz} = \bar{G}(k)H_1(Y_*; k), \quad (2.185a)$$

where $\bar{G}(k)$ is the Fourier transform of the shape of the roughness. We have the conditions

$$\frac{d^2 \bar{U}_1}{dz^2} = (ik)^{1/3} |k| \bar{A}_1 \quad \text{at } z = z_0, \quad (2.185b)$$

$$\bar{U}_1 = 0 \quad \text{at } z = z_0, \quad (2.185c)$$

$$\bar{U}_1 = \bar{A}_1 \quad \text{at } z = \infty. \quad (2.185d)$$

Here

$$\begin{aligned} H_1 = \frac{\omega}{k} \exp \left(-(1+i) \sqrt{\frac{\omega}{2}} Y_* \right) \Psi(\zeta; k) \\ + (ik)^{1/3} \left[\exp \left(-(1+i) \sqrt{\frac{\omega}{2}} Y_* \right) - 1 \right] \frac{d\Phi}{d\zeta}, \end{aligned} \quad (2.186)$$

with functions $\Phi(\zeta; k)$ and $\Psi(\zeta; k)$ given by (2.175).

The general solution of the equation (2.185a) may be written as

$$\frac{d\bar{U}_1}{dz} = \bar{G}(k) \varphi_1(z) + C_1 Ai(z) + C_2 Bi(z), \quad (2.187)$$

where $\varphi_1(z)$ is the solution of the boundary-value problem

$$\varphi_1'' - z\varphi_1 = H_1 \quad (2.188)$$

which must be solved under the conditions

$$\varphi_1'(z_0) = 0, \quad \varphi_1(\infty) = 0. \quad (2.189)$$

We have chosen φ_1 to be zero at $z = \infty$ to ensure the boundary conditions are met in addition the boundary condition (2.185d) requires $d\bar{U}_1/dz$ to tend to zero as $z \rightarrow \infty$. This is only possible if $C_2 = 0$. Hence,

$$\frac{d\bar{U}_1}{dz} = \bar{G}(k) \varphi_1(z) + C_1 Ai(z). \quad (2.190)$$

The application of (2.185c) on (2.190) yields the first equation relating the two unknown constants, C_1 and $\bar{A}_1$

$$C_1 Ai'(z_0) = (ik)^{1/3} |k| \bar{A}_1. \quad (2.191)$$

In order to deduce $\bar{A}_1$ we need a second equation for C_1 , which we get by integrating (2.190) with the initial condition (2.185b):

$$\bar{U}_1 = \bar{G}(k) \int_{z_0}^z \varphi_1(z') dz' + C_1 \int_{z_0}^z Ai(z') dz'. \quad (2.192)$$

Then setting $z = \infty$ in (2.192), and use the boundary condition (2.185d) gives

$$\bar{A}_1 = \bar{G}(k) \int_{z_0}^{\infty} \varphi_1(z) dz + C_1 \int_{z_0}^{\infty} Ai(z) dz, \quad (2.193)$$

dropping the dashes due to integration being definite. Eliminating C_1 from (2.191) and (2.193), we find that

$$\bar{A}_1 = \frac{Ai'(z_0) \int_{z_0}^{\infty} \varphi_1(z) dz}{Ai'(z_0) - (ik)^{1/3} |k| \int_{z_0}^{\infty} Ai(z) dz} \bar{G}(k). \quad (2.194)$$

It remains to substitute the above equation into (2.179), which yields

$$\bar{P}_1(k) = \frac{Ai'(z_0) |k| \int_{z_0}^{\infty} \varphi_1(z) dz}{Ai'(z_0) - (ik)^{1/3} |k| \int_{z_0}^{\infty} Ai(z) dz} \bar{G}(k). \quad (2.195)$$

We transform the Fourier variables back to the physical ones by applying the inverse Fourier integral to (2.195). Following the same analysis for the $\mathcal{O}(M_\infty)$ terms in (2.176) will give the Fourier transformed pressure $\bar{P}_2$, details of this can be seen in Ruban et al. (2013).

2.4.3 Calculation of the receptivity coefficients

We shall now apply the inverse Fourier transformation to $\bar{P}_1(k)$, given by (2.195), which can be written as

$$P_1(x_*) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \frac{Ai'(z_0)|k| \int_{z_0}^{\infty} \varphi_1(z) dz}{Ai'(z_0) - (ik)^{1/3}|k| \int_{z_0}^{\infty} Ai(z) dz} \bar{G}(k) \exp(ikx_*) \right\} dk. \quad (2.196)$$

The integration in (2.196) is performed along the real axis. Our main interest is in the behaviour of the perturbations downstream of the roughness. For $x_* > 0$ it is convenient to consider an analytical extension of the integrand into the upper half plane and change the contour of integration, as explained later. When deforming the contour of integration we need to identify the poles of the integrand. These are given by the dispersion equation

$$Ai'(z_0) - (ik)^{1/3}|k| \int_{z_0}^{\infty} Ai(z) dz = 0. \quad (2.197)$$

This equation represents the large Reynolds number version of the Orr-Sommerfeld equation. It has been studied by various authors, and it is known that there are countably infinite number of roots of (2.197). The position of each root depends on the frequency ω . The trajectories of the first five roots, as ω changes from zero to infinity, are shown in figure 2.2. All the roots originate at $\omega = 0$ from the coordinate origin and all of them except the first one remain in the second quadrant for all $\omega \in (0, \infty]$, indicating that the corresponding perturbations in the boundary-layer decay with x . The behaviour of the first root is different. It stays in the second quadrant until the frequency reaches its critical value, $\omega_* = 2.29797$, and then it crosses the real axis at the point $k_* = 1.00049$ and remains in the third quadrant for all $\omega \in (\omega_*, \infty)$. This root represents the T-S wave; our task is to determine its amplitude.

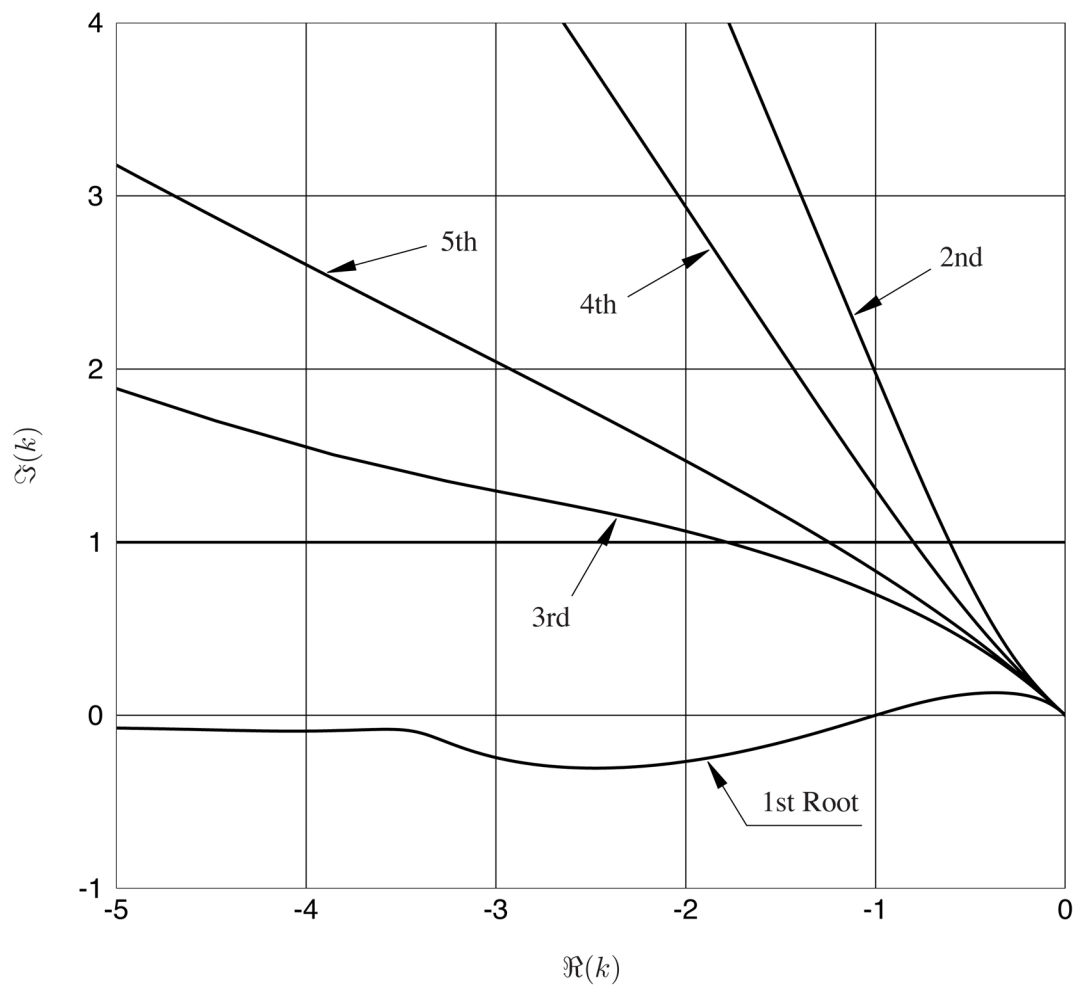


Figure 2.2: The first five roots of (2.197) in the complex k -plane.

Let us consider a value of the frequency, ω , which is smaller than the critical frequency, ω_* . For such ω all the roots of (2.197) are represented by points which lie in the second quadrant of the complex k -plane, as shown in figure 2.3. Remember that when introducing an analytical branch of the function $(ik)^{1/3}$ we had to make a branch cut along the positive imaginary axis in the k -plane. Also, the analytical extension of $|k|$ in the integrand in (2.196) requires the branch cut to be extended to the entire imaginary axis. Therefore, we shall split the integration interval into two parts, the negative real semi-axis and positive real semi-axis, shown in figure 2.3 as C_- and C_+ , respectively,

$$P_1(x_*) = \frac{1}{2\pi} \int_{-\infty}^0 \mathcal{M}(k; \omega, x_*) dk + \frac{1}{2\pi} \int_0^{\infty} \mathcal{N}(k; \omega, x_*) dk. \quad (2.198)$$

Here

$$\mathcal{M}(k; \omega, x_*) = - \frac{Ai'(z_0)k \int_{z_0}^{\infty} \varphi_1(z) dz}{Ai'(z_0) - i(ik)^{4/3} \int_{z_0}^{\infty} Ai(z) dz} \bar{G}(k) \exp(ikx_*), \quad (2.199)$$

and

$$\mathcal{N}(k; \omega, x_*) = \frac{Ai'(z_0)k \int_{z_0}^{\infty} \varphi_1(z) dz}{Ai'(z_0) + i(ik)^{4/3} \int_{z_0}^{\infty} Ai(z) dz} \bar{G}(k) \exp(ikx_*). \quad (2.200)$$

When calculating the first integral in (2.198) we shall close the contour of integration by adding to C_- a ray, C'_- , and a circular arc, C_R^- , of large radius R . We shall choose the angle between the ray C'_- and the negative real semi-axis such that only one root, k_1 , finds itself inside the combined contour. Using the Residue theorem and decomposition of the contour integration, we find

$$\int_{C_-} \mathcal{M} dk + \int_{C'_-} \mathcal{M} dk + \int_{C_R^-} \mathcal{M} dk = 2\pi i \text{Res}(\mathcal{M}, k_1). \quad (2.201)$$

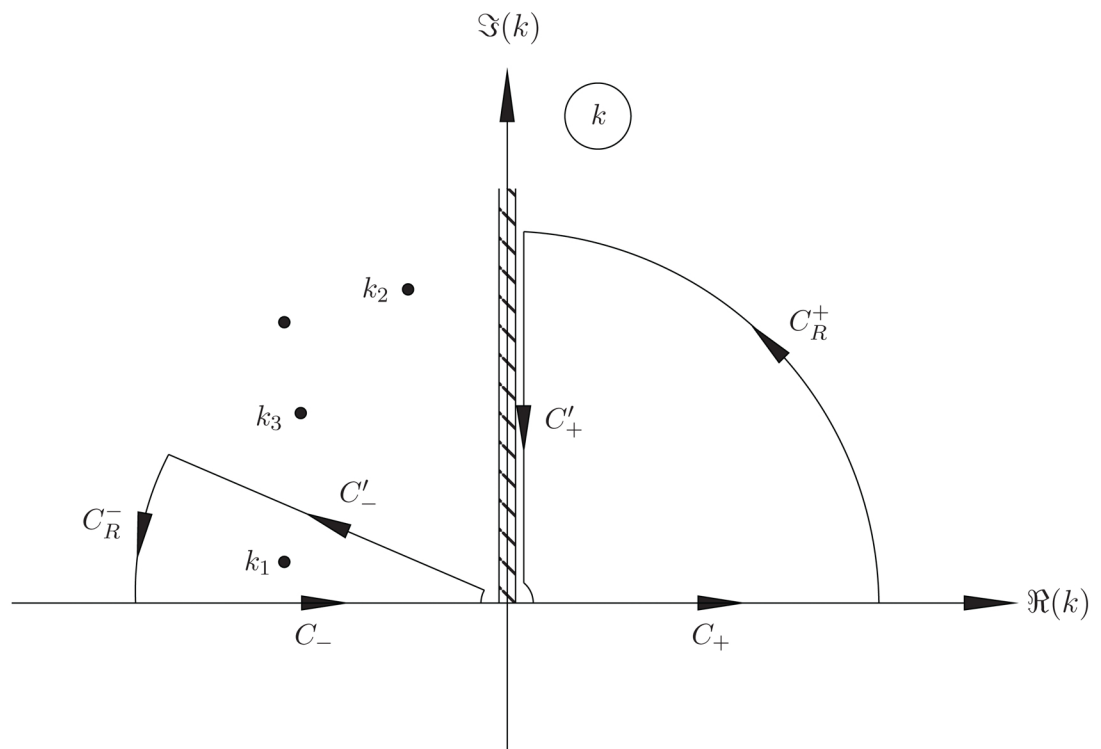


Figure 2.3: Deformation of the contour of integration in (2.198).

We note that in (2.199) we have the function

$$\mathcal{M} = \frac{N}{D}, \quad (2.202)$$

where

$$N(k) = -Ai'(z_0)k \int_{z_0}^{\infty} \varphi_1(z) dz \bar{G}(k) \exp(ikx_*), \quad (2.203)$$

and

$$D(k) = Ai'(z_0) - i(ik)^{4/3} \int_{z_0}^{\infty} Ai(z) dz, \quad (2.204)$$

which are both analytic in the neighbourhood of the simple pole at $k = k_1$. Hence we use an extension of the Residue theorem (see Ablowitz and Fokas (2003) p.210) that implies

$$\text{Res}(\mathcal{M}, k_1) = \frac{N(k_1)}{D'(k_1)}. \quad (2.205)$$

From (2.201) and (2.205) we now have

$$\begin{aligned} \int_{C_-} \mathcal{M} dk &= -2\pi i \frac{Ai'(z_0)k_1 \int_{z_0}^{\infty} \varphi_1(z) dz}{\frac{4}{3}(ik_1)^{1/3} \int_{z_0}^{\infty} Ai(z) dz - \frac{2}{3} \frac{z_0}{k_1} [z_0 + i(ik_1)^{4/3}] Ai(z_0)} \bar{G}(k_1) \exp(ik_1 x_*) - \\ &\quad - \int_{C'_-} \mathcal{M}(k; \omega, x_*) dk - \int_{C_R^-} \mathcal{M}(k; \omega, x_*) dk. \end{aligned} \quad (2.206)$$

On the right hand side of (2.206) the first term corresponds to the generated T-S wave, the second to the continuous spectrum of the perturbation field around the roughness and the last is an additional term. We now show that these latter two terms vanish in the boundary-layer behind the roughness, and only the T-S wave term remains. We shall show that the integral along the arc C_R^- (see figure 2.3) tends to zero as the arc radius, R , tends to infinity. In order to perform this task we need to return to the boundary-value problem (2.188) and (2.189) to study the asymptotic behaviour of its solution in the limit $k \rightarrow \infty$ and with

$\omega = \mathcal{O}(1)$. Setting $k \rightarrow \infty$ in (2.175) yields

$$\Phi = 3 \int_0^\zeta Ai(\zeta') d\zeta' \quad \text{and} \quad \Psi = -3(ik)^{2/3} \int_0^\zeta d\zeta' \int_0^{\zeta'} Ai(\zeta'') d\zeta''. \quad (2.207)$$

With ζ being an order one quantity, $Y_* = (ik)^{-1/3}\zeta$ is small, and therefore, the function (2.186) may be approximated as

$$H_1 = -3(1+i)\sqrt{\frac{\omega}{2}}\zeta Ai(\zeta) + \mathcal{O}[(ik)^{-1/3}]. \quad (2.208)$$

It remains to notice that $z_0 = i\omega/(ik)^{1/3}$ tend to zero as $k \rightarrow \infty$ with $\omega = \mathcal{O}(1)$, which reduces the boundary-value problem (2.188) and (2.189) to

$$\begin{aligned} \frac{d^2\varphi_1}{dz^2} - z\varphi_1 &= -3(1+i)\sqrt{\frac{\omega}{2}}\zeta Ai(\zeta), \\ \frac{d\varphi_1}{dz}(0) &= 0, \quad \varphi_1(\infty) = 0. \end{aligned}$$

Since ω , ζ and z are order one quantities the solution to this problem, φ_1 , is also order one and so is the integral

$$\int_0^\infty \varphi_1(z) dz. \quad (2.209)$$

By using the above and assuming that the function representing the wall roughness shape together with its modulus are integrable on the interval $x \in (-\infty, \infty)$, then we may state that there exists a positive constant B such that

$$|\bar{G}(k)| = \left| \int_{-\infty}^\infty G(x) \exp(-ikx) dx \right| \leq \int_{-\infty}^\infty |G(x)| dx < B, \quad (2.210)$$

we can then deduce from (2.199) that

$$\mathcal{M} = \mathcal{O}(k^{-1/3}) \quad \text{as} \quad k \rightarrow \infty. \quad (2.211)$$

From these results we have shown that the function $\mathcal{M}(k; \omega, x_*)$ satisfies the conditions of the Jordan's lemma (see Appendix A), which allows us to conclude that for all $x > 0$ the integral along C_R^- in (2.206) tends to zero as the arc radius $R \rightarrow \infty$.

Now we will show that the continuous spectrum integral

$$\int_{C_-'} \mathcal{M}(k; \omega, x_*) dk, \quad (2.212)$$

dies away further downstream. The integral (2.212) is a Laplace type integral. Its asymptotic behaviour may be determined with the help of the Watson lemma (see Appendix B). When employing the lemma one needs to know the behaviour of the integrand, $\mathcal{M}(k; \omega, x_*)$, at small values of k . Keeping this in mind we return to the boundary-value problem (2.188), and this time assume that we can take $k \rightarrow 0$ with $\omega = \mathcal{O}(1)$. It is convenient to recast the boundary-value problem (2.188) and (2.189) using $\zeta = z - z_0$ as the independent variable. We have

$$\frac{d^2 \varphi_1}{d\zeta^2} - (z_0 + \zeta) \varphi_1 = H_1(\zeta), \quad (2.213a)$$

$$\frac{d\varphi_1}{d\zeta}(0) = 0, \quad \varphi_1(\infty) = 0. \quad (2.213b)$$

With ζ being an order one quantity, $Y_* = \zeta/(ik)^{1/3}$ is large, and it follows from (2.186) and (2.175) that at leading order we have

$$H_1 = -\frac{(ik)^{2/3} k}{Ai'(0)} Ai(\zeta) + \dots \quad \text{as } k \rightarrow 0. \quad (2.214)$$

We further notice that under our new small k limit, $z_0 = i\omega/(ik)^{2/3}$ tends to infinity. Hence, the leading order solution of the equation (2.213a) is obtained by balancing the second term on the left hand side with the forcing term on the right hand side. We find that at leading order

$$\varphi_1 = -\frac{H_1}{z_0} = -\frac{(ik)^{7/3}}{\omega Ai'(0)} Ai(\zeta) + \dots \quad (2.215)$$

Consequently,

$$\int_{z_0}^{\infty} \varphi_1(z) dz = \int_0^{\infty} \varphi_1 d\zeta = -\frac{(ik)^{7/3}}{3\omega Ai'(0)} + \dots \quad (2.216)$$

We are ready now to analyse the equation (2.199) for the function $\mathcal{M}(k; \omega, x_*)$. As we have mentioned $z_0 \rightarrow \infty$ and so the known asymptotic formulae for the derivative and integral of the Airy function,

$$Ai'(z_0) = -\frac{z_0^{1/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}z_0^{3/2}\right) + \dots, \quad \int_{z_0}^{\infty} Ai(\zeta) d\zeta = \frac{z_0^{-3/4}}{2\sqrt{\pi}} \exp\left(-\frac{2}{3}z_0^{3/2}\right) + \dots, \quad (2.217)$$

may be used. It follows from (2.217) that the two terms in the denominator in (2.199) are related to one another as

$$\frac{Ai'(z_0)}{i(ik)^{4/3} \int_{z_0}^{\infty} Ai(z) dz} = -\frac{z_0}{i(ik)^{4/3}} = \frac{\omega}{k^2} \gg 1. \quad (2.218)$$

Keeping this in mind and using (2.216) in (2.199) we find that

$$\mathcal{M}(k; \omega, x_*) = \frac{(ik)^{10/3}}{3i\omega Ai'(0)} \bar{G}(0) \exp(ikx_*) + \dots \quad (2.219)$$

The Watson lemma converts (2.219) into the following asymptotic representation of the continuous spectrum integral (2.212)

$$\int_{C'_-} \mathcal{M}(k; \omega, x_*) dk = -\frac{(1-i\sqrt{3})}{6\omega Ai'(0)} \Gamma\left(\frac{13}{3}\right) \bar{G}(0) \frac{1}{x_*^{13/3}} + \dots \quad \text{as } x_* \rightarrow \infty. \quad (2.220)$$

The second integral in (2.198) may be treated in the similar way. We close the contour of integration as shown in figure 2.3. Since there are no roots of the dispersion equation in the first quadrant of the complex k -plane, the integral along the real positive semi-axis, C_+ may be substituted with the integral along imaginary semi-axis, C'_+ . The latter is evaluated with the help of Watson's lemma as was done with the integral (2.220) along C'_- . Using

Jordan's lemma on the integral along the arc C_R^+ , applying the same argument as with the integral along C_R^- , we find that as $x_* \rightarrow \infty$

$$\int_{C_+} \mathcal{N}(k; \omega, x_*) dk = - \int_{C_+'} \mathcal{N}(k; \omega, x_*) dk = - \frac{(1 + i\sqrt{3})}{6\omega Ai'(0)} \Gamma\left(\frac{13}{3}\right) \bar{G}(0) \frac{1}{x_*^{13/3}} + \dots \quad (2.221)$$

It remains to substitute (2.220) back into (2.206) and combine with (2.221) using the equation (2.198). This results in

$$P_1(x_*) = \frac{1}{2} \mathcal{K}_1(\omega) \bar{G}(k_1) \exp(ik_1 x_*) - \frac{i\Gamma(13/3)}{2\pi\sqrt{3}\omega Ai'(0)} \bar{G}(0) \frac{1}{x_*^{13/3}} + \dots \quad (2.222)$$

Here

$$\mathcal{K}_1(\omega) = -i \frac{2Ai'(z_0)k \int_{z_0}^{\infty} \varphi_1(z) dz}{\frac{4}{3}(ik)^{1/3} \int_{z_0}^{\infty} Ai(z) dz - \frac{2}{3} \frac{z_0}{k} [z_0 + i(ik)^{4/3}] Ai(z_0)} \Big|_{k=k_1}, \quad (2.223)$$

is a universal function of the frequency, ω . This function neither depends on the wing surface vibration function (2.4) nor does it depend on the wall roughness shape (2.94). We shall call $\mathcal{K}_1(\omega)$ the first receptivity coefficient. The results of a numerical calculation of $\mathcal{K}_1(\omega)$ are displayed in figure 2.4. We performed these in the following way. First, one needs to choose a real value of the frequency, ω . The corresponding values of k and z_0 are found from the solution of the dispersion equation (2.197); we are interested here in the first root. The function φ_1 was found by solving the boundary-value problem (2.188). We employed the Thomas algorithm (a tridiagonal matrix algorithm) for this purpose. Finally, the values of the Airy function, its derivative and the integral at the point z_0 in the complex z -plane were found by solving the initial-value problem for the Airy equation along the straight line connecting z_0 with the origin, $z = 0$, in the complex z -plane. We used the Runge-Kutta algorithm with the calculations starting from $z = 0$ where the values of the Airy function and its derivative are known (see, for example Abramowitz and Stegun (1965)). As any other complex quantity

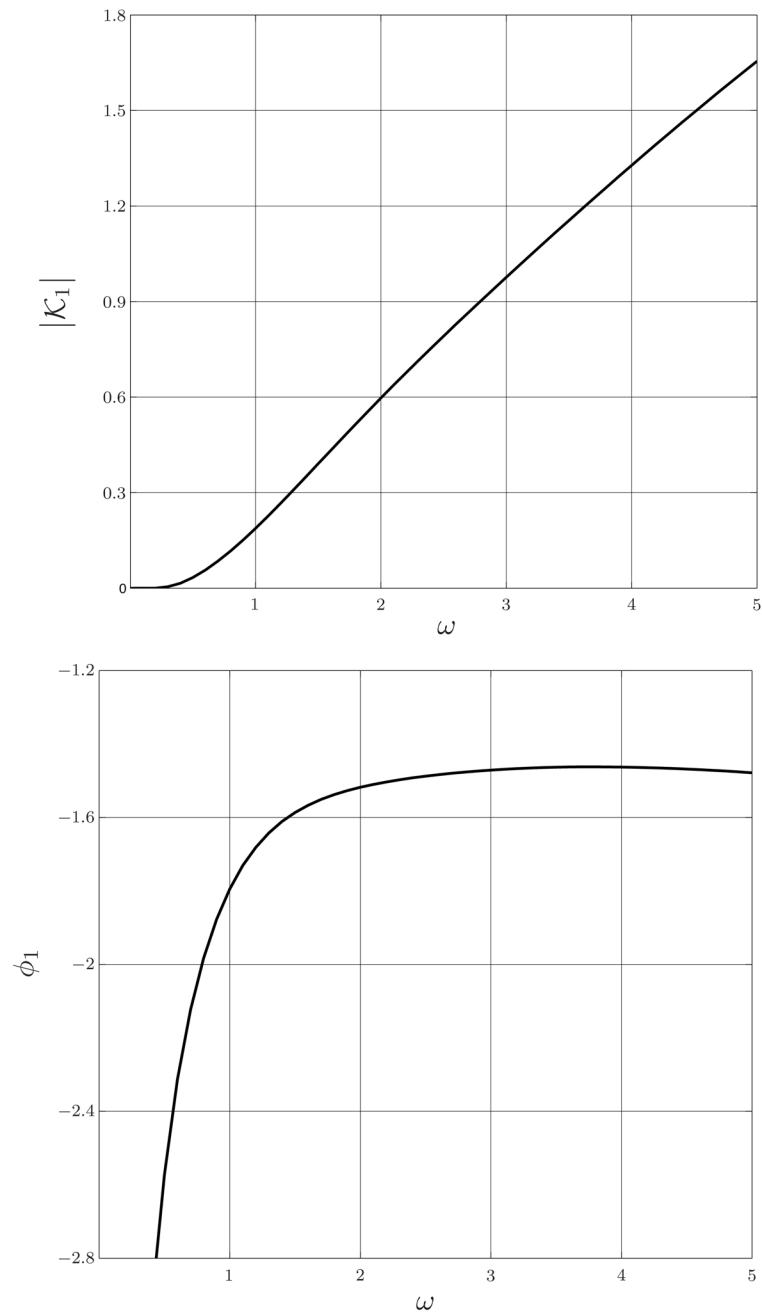


Figure 2.4: Dependence of the first receptivity coefficient, $\mathcal{K}_1$ on the frequency, ω . Top: Modulus of the first receptivity coefficient, $|\mathcal{K}_1|$. Bottom: Argument of first receptivity coefficient, ϕ_1 .

the receptivity coefficient, $\mathcal{K}_1(\omega)$ may be represented in the exponential form

$$\mathcal{K}_1 = |\mathcal{K}_1| \exp(i\phi_1). \quad (2.224)$$

Figure 2.4(a) shows the modulus, $|\mathcal{K}_1|$, of the receptivity coefficient, and its argument, ϕ_1 , is shown in figure 2.4(b). The analysis for the second part of the solution in (2.176) (order M_∞ terms) was performed in a similar manner to above, see Ruban et al. (2013) for details, to obtain the receptivity coefficient for the second mechanism, $\mathcal{K}_2(\omega)$.

Let us now substitute (2.223) into the equation for the pressure in (2.176). Disregarding the continuous spectrum contributions, we find that the pressure oscillations in the boundary-layer downstream of the roughness

$$\begin{aligned} P_*(t_*, x_*) = & \frac{\bar{G}(k_1)}{2M_\infty} \frac{dF}{dx'}(x'_0) |\mathcal{K}_1(\omega)| \exp(i(\omega t_* + k_1 x_* + \phi_1)) + c.c. + \\ & + \frac{\bar{G}(k_1) M_\infty F(x'_0)}{2} |\mathcal{K}_2(\omega)| \exp(i(\omega t_* + k_1 x_* + \phi_2)) + c.c.. \end{aligned}$$

This shows that the harmonic oscillations of the wing surface produce two T-S waves in the boundary-layer behind a wall roughness. Their amplitudes are

$$\left| \frac{\bar{G}(k_1)}{M_\infty} \frac{dF}{dx'}(x'_0) \mathcal{K}_1(\omega) \right| \quad \text{and} \quad \left| \bar{G}(k_1) M_\infty F(x'_0) \mathcal{K}_2(\omega) \right|, \quad (2.225)$$

respectively.

2.5 Results and discussion

In this chapter we present our analysis of the generation of T-S waves in the boundary-layer on the wing surface due to elastic vibrations of the wing. As the T-S waves are observed at relatively large values of the Reynolds number the asymptotic approach to the flow analysis is adopted. We assume that in the spectrum of the wing vibrations there are harmonics which come in resonance with the T-S wave on the lower branch of the stability curve; this happens when the frequency of the wing oscillations is an order $Re^{1/4}$ quantity.

We started our analysis with the flow outside the boundary-layer. We assumed that the free-stream Mach number is of an appropriate size and we found that since the frequency of the wing surface oscillations is high, the perturbations produced by wing surface vibrations can be described in the framework of piston theory.

As the pressure perturbations penetrate into the boundary-layer, a Stokes layer forms on the wing surface. With the frequency $\omega = \mathcal{O}(Re^{1/4})$ its thickness is estimated as an order $Re^{-5/8}$ quantity. In the flow considered, there are two physical mechanisms through which an oscillatory motion of the fluid in the Stokes layer is excited. The first one is the classical mechanism where the pressure gradient, being periodic a function of time, forces the fluid to oscillate in the direction along the wing surface. This process has been studied on numerous occasions before, including in receptivity theory (see, for example, Ruban (1984)). It can be observed in both compressible and incompressible flows. The second mechanism — that was not shown here in detail — can only be observed in compressible flows and was analysed in Ruban et al. (2013).

The two mechanisms prove to be equally important if the characteristic wave length of the elastic vibrations of the wing surface, δ_x , is an order $\mathcal{O}(LRe^{-1/8})$ quantity, with L being the wing cord. If $\delta_x \ll LRe^{-1/8}$, then the first mechanism defines the flow behaviour in the Stokes layer. If, on the other hand, $\delta_x \gg LRe^{-1/8}$, then the second mechanism is dominant. In the second case the velocity oscillations are not confined to the Stokes layer itself, but also permeate into the rest of the boundary-layer.

The Stokes layer on its own is incapable of producing the T-S waves since the resonance conditions require the external perturbations to have not only the same frequency as the T-S waves, but also the same wave length. Meanwhile, the characteristic wave length of the perturbation field in the Stokes layer is much longer than that of the T-S waves. However, the situation changes when the Stokes layer encounters a wall roughness, which exist in real aerodynamic flows. If the streamwise extent of the roughness is an order $Re^{-3/8}$ quantity, then efficient generation of the T-S waves becomes possible. We restricted our attention to the case when the Stokes layer interacts with an isolated roughness and the flow is described by the triple-deck theory. Solutions of the triple-deck problem were found in an analytic

form.

We focussed on the flow behaviour downstream of the roughness. We found that there are two T-S waves forming in the boundary layer behind the roughness. These are produced by the two physical mechanisms of excitation of the flow oscillations in the Stokes layer. Their amplitudes have been expressed in terms of the receptivity coefficients, which represent the efficiency of the T-S wave generation process. The first receptivity process is similar to the one studied earlier by Ruban (1984) and Goldstein (1985) where an acoustic wave impinges upon the boundary layer and interacts with a small wall roughness. In fact, if the acoustic receptivity coefficient is denoted as $\mathcal{K}_{\text{acoustic}}$, then

$$\mathcal{K}_1 = -2\omega\mathcal{K}_{\text{acoustic}}. \quad (2.226)$$

This is derived by applying similar analysis to equation number 4.8 in Ruban (1984) that we have here to (2.195) to obtain an equation for $\mathcal{K}_{\text{acoustic}}$ which can be compared to $\mathcal{K}$. Hence this first receptivity coefficient, $\mathcal{K}_1$ (see figure 2.4a), is larger than the acoustic receptivity coefficient, $\mathcal{K}_{\text{acoustic}}$ for all $\omega > \frac{1}{2}$. This suggests that the receptivity to the wing surface vibrations has to play a major role in the laminar-turbulent transition in the boundary-layer.

Chapter 3

Inviscid secondary instability of spanwise modulated shear flows.

3.1 Brief Introduction

In this chapter we will analyse the linear inviscid secondary instability of longitudinal spanwise modulated vortices, created due to surface curvature within an incompressible boundary-layer. This situation can be seen in many realistic airflows over wings e.g. towards the trailing edge of the wing. We note that the nonlinear scheme of Hall (1988) can be used to determine the evolution of finite-amplitude $\mathcal{O}(1)$ vortices in a curved boundary-layer and solely within the central layer denoted as region I in Hall and Lakin (1988). We investigate the instability of the three dimensional state to an inviscid Rayleigh instability, which is a local stability problem. An instability of this type has a wavelength that can be compared to the thickness of the boundary-layer. This means that a quasi-parallel approach to this problem can be used and from this a Rayleigh equation problem is obtained and solved.

The work of Hall and Horseman (1991) provides the basis for this chapter; however, the analysis performed is just as relevant to a secondary instability of the vortex structures produced in the latter stages of boundary-layer transition to turbulence (Hall and Horseman,

1991; Hall and Smith, 1991). The formulation of the problem is given in §3.2. We then continue to separate the solution via the superposition of an amplitude function and self-adjoint solution to pressure. A numerical solution to the leading order differential equation is given in §3.5. We can then directly calculate our required eigenvalues, amplitude function and finally full solutions to the three dimensional problem with the aid of a two dimensional problem for initial values. The solutions will be damped away from the centreline, as required for this problem, and we will discuss this. However, breakdown of the given solutions occurs with the coalescence of the effective two dimensional streamwise wavenumber, γ , and the three dimensional streamwise wave number, α , along the centreline $\bar{y}$ of the layer within region 1 of depth $1/k$ as discussed within §3.6. The numerical and experimental works of (Hall and Sherwin, 2010; Swearingen and Blackwelder, 1987) are also used to compare to these already known results and our own.

3.2 Formulation of the problem

We consider the flow of a viscous fluid, with viscosity ν , over a curved wall. The curvature is described along the wall by $\kappa(\hat{x})/a$, where a is a typical radius of curvature. We define the Reynolds number, Re , by

$$Re = \frac{U_\infty \ell}{\nu}, \quad (3.1)$$

where ℓ is a length scale in the streamwise direction, and U_∞ is the fluid speed far from the wall surface. We rescale with respect to the boundary-layer thickness and, for ease, use the notation $\delta = Re^{-1/2}$ to represent this. We will initially rescale the dimensional distances along and normal to the wall $(\tilde{x}, \tilde{y})$ by (ℓ, ℓ) . We also define the Görtler number G by

$$G = \frac{2\ell Re^{1/2}}{a} = \frac{2\ell}{a\delta}, \quad (3.2)$$

held fixed, since we know that instability occurs first for $G = \mathcal{O}(1)$ (Hall and Horseman, 1991).

To analyse the curvature of the wall at this downstream location we will use the unsteady non-dimensional body fitted Navier-Stokes equations [see (Rosenhead, 1963)]:

$$\begin{aligned} \frac{\partial \hat{V}_x}{\partial \hat{t}} + \frac{\hat{V}_x}{H_1} \frac{\partial \hat{V}_x}{\partial \hat{x}} + \hat{V}_y \frac{\partial \hat{V}_x}{\partial \hat{y}} + \hat{V}_z \frac{\partial \hat{V}_x}{\partial \hat{z}} + \frac{\kappa \hat{V}_x \hat{V}_y}{a H_1} = - \frac{1}{H_1} \frac{\partial \hat{p}}{\partial \hat{x}} \\ + \delta^2 \left[\frac{1}{H_1} \frac{\partial}{\partial \hat{x}} \left(\frac{1}{H_1} \frac{\partial \hat{V}_x}{\partial \hat{x}} \right) + \frac{\partial^2 \hat{V}_x}{\partial \hat{y}^2} + \frac{\partial^2 \hat{V}_x}{\partial \hat{z}^2} + \frac{\kappa}{a} \frac{\partial}{\partial \hat{y}} \left(\frac{\hat{V}_x}{H_1} \right) + \frac{\kappa}{a H_1^2} \frac{\partial \hat{V}_y}{\partial \hat{x}} + \frac{1}{H_1} \frac{\partial}{\partial \hat{x}} \left(\frac{\kappa \hat{V}_y}{a H_1} \right) \right], \end{aligned} \quad (3.3)$$

$$\begin{aligned} \frac{\partial \hat{V}_y}{\partial \hat{t}} + \frac{\hat{V}_x}{H_1} \frac{\partial \hat{V}_y}{\partial \hat{x}} + \hat{V}_y \frac{\partial \hat{V}_y}{\partial \hat{y}} + \hat{V}_z \frac{\partial \hat{V}_y}{\partial \hat{z}} - \frac{\kappa \hat{V}_x^2}{a H_1} = - \frac{\partial \hat{p}}{\partial \hat{y}} \\ + \delta^2 \left[\frac{1}{H_1} \frac{\partial}{\partial \hat{x}} \left(\frac{1}{H_1} \frac{\partial \hat{V}_y}{\partial \hat{x}} \right) + \frac{\partial^2 \hat{V}_y}{\partial \hat{y}^2} + \frac{\partial^2 \hat{V}_y}{\partial \hat{z}^2} + \frac{\kappa}{a} \frac{\partial}{\partial \hat{y}} \left(\frac{\hat{V}_y}{H_1} \right) - \frac{\kappa}{a H_1^2} \frac{\partial \hat{V}_x}{\partial \hat{x}} - \frac{1}{H_1} \frac{\partial}{\partial \hat{x}} \left(\frac{\kappa \hat{V}_x}{a H_1} \right) \right], \end{aligned} \quad (3.4)$$

$$\begin{aligned} \frac{\partial \hat{V}_z}{\partial \hat{t}} + \frac{\hat{V}_x}{H_1} \frac{\partial \hat{V}_z}{\partial \hat{x}} + \hat{V}_y \frac{\partial \hat{V}_z}{\partial \hat{y}} + \hat{V}_z \frac{\partial \hat{V}_z}{\partial \hat{z}} = - \frac{\partial \hat{p}}{\partial \hat{z}} \\ + \delta^2 \left[\frac{1}{H_1} \frac{\partial}{\partial \hat{x}} \left(\frac{1}{H_1} \frac{\partial \hat{V}_z}{\partial \hat{x}} \right) + \frac{\partial^2 \hat{V}_z}{\partial \hat{y}^2} + \frac{\partial^2 \hat{V}_z}{\partial \hat{z}^2} + \frac{\kappa}{a H_1} \frac{\partial \hat{V}_z}{\partial \hat{y}} \right], \end{aligned} \quad (3.5)$$

$$\frac{1}{H_1} \frac{\partial \hat{V}_x}{\partial \hat{x}} + \frac{\partial \hat{V}_y}{\partial \hat{y}} + \frac{\partial \hat{V}_z}{\partial \hat{z}} + \frac{\kappa \hat{V}_y}{a H_1} = 0, \quad (3.6)$$

which have been non-dimensionalised by

$$\tilde{t} = \frac{\ell}{U_\infty} \hat{t}, \quad \tilde{x} = \ell \hat{x}, \quad \tilde{y} = \ell \hat{y}, \quad \tilde{z} = \ell \hat{z}, \quad \text{and} \quad \tilde{\mathbf{V}} = U_\infty \hat{\mathbf{V}} \quad (3.7)$$

with the Lamé coefficient (Lamé, 1859) $H_1 = 1 + \frac{\kappa}{a} \hat{y}$.

Without longitudinal vortex structure in the flow we have a two dimensional steady boundary-layer flow $U_\infty(\bar{u}(X, Y), \delta\bar{v}(X, Y), 0)$ from

$$\frac{\partial \bar{u}}{\partial X} + \frac{\partial \bar{v}}{\partial Y} = 0, \quad (3.8)$$

$$\bar{u} \frac{\partial \bar{u}}{\partial X} + \bar{v} \frac{\partial \bar{u}}{\partial Y} = -\frac{d\bar{p}}{dX} + \frac{\partial^2 \bar{u}}{\partial Y^2}, \quad (3.9)$$

$$\bar{u} = \bar{v} = 0, \quad Y = 0, \quad (3.10)$$

$$\bar{u} \rightarrow 1, \quad Y \rightarrow \infty, \quad (3.11)$$

$$\bar{u} = 1, \quad \text{at } X = 0, \quad Y > 0, \quad (3.12)$$

where $(\hat{x}, \hat{y}) = (\ell X, \ell \delta Y)$ and $\bar{p}$ is the streamwise pressure gradient associated with this two dimensional boundary-layer flow that is undisturbed until a Görtler vortex develops at some downstream location. The solutions for this system are exactly those of the Blasius boundary-layer.

Suppose that the curvature of the wall induces a Görtler vortex velocity field and is defined by

$$\begin{aligned} \hat{\mathbf{V}}/U_\infty = \mathbf{u}_B(X, Y, Z) = & (\bar{u}, \delta\bar{v}, 0)(1 + \mathcal{O}(\delta)) \\ & + (U(X, Y, Z), \delta V(X, Y, Z), \delta W(X, Y, Z))(1 + \mathcal{O}(\delta)), \end{aligned} \quad (3.13)$$

where the spanwise variable is transformed by $\hat{z} = \delta \ell Z$ and we assume that the flow is periodic in this direction with wavelength $2\pi/k$. When we let the dimensionless pressure field associated with the velocities U, V, W be $P(X, Y, Z)$ then substituting all the above into the Navier-Stokes equations we obtain the non linear disturbance equations (Hall, 1982)

from which we can determine the vortex field

$$\frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 U}{\partial Z^2} - V \frac{\partial \bar{u}}{\partial Y} = \bar{u} \frac{\partial U}{\partial X} + U \frac{\partial \bar{u}}{\partial X} + \bar{v} \frac{\partial U}{\partial Y} + Q_1, \quad (3.14)$$

$$\frac{\partial^2 V}{\partial Y^2} + \frac{\partial^2 V}{\partial Z^2} - G\kappa \bar{u} U - \frac{\partial P}{\partial Y} = \bar{u} \frac{\partial V}{\partial X} + U \frac{\partial \bar{v}}{\partial X} + \bar{v} \frac{\partial V}{\partial Y} + V \frac{\partial \bar{v}}{\partial Y} + Q_2, \quad (3.15)$$

$$\frac{\partial^2 W}{\partial Y^2} + \frac{\partial^2 W}{\partial Z^2} - \frac{\partial P}{\partial Z} = \bar{u} \frac{\partial W}{\partial X} + \bar{v} \frac{\partial W}{\partial Y} + Q_3, \quad (3.16)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} + \frac{\partial W}{\partial Z} = 0, \quad (3.17)$$

$$U = V = W = 0, \quad Y = 0, \quad (3.18)$$

$$U \rightarrow 0, \quad V \rightarrow 0, \quad W \rightarrow 0, \quad Y \rightarrow \infty, \quad (3.19)$$

where

$$Q_1 = U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} + W \frac{\partial U}{\partial Z}, \quad (3.20)$$

$$Q_2 = U \frac{\partial V}{\partial X} + V \frac{\partial V}{\partial Y} + W \frac{\partial V}{\partial Z} + \frac{1}{2} G\kappa U^2, \quad (3.21)$$

$$Q_3 = U \frac{\partial W}{\partial X} + V \frac{\partial W}{\partial Y} + W \frac{\partial W}{\partial Z}. \quad (3.22)$$

This system does not include a streamwise pressure gradient and hence the vortex equations are parabolic in X . These non linear Görtler vortex equations were previously solved numerically by Hall (1988), the reader is referred to this paper for a discussion of the results. However these results can be expressed by writing the velocities as

$$U = U_0(x, y) + \sum_{n=1}^{\infty} U_n(x, y) \cos(nkz), \quad (3.23)$$

$$V = V_0(x, y) + \sum_{n=1}^{\infty} V_n(x, y) \cos(nkz), \quad (3.24)$$

$$W = \sum_{n=1}^{\infty} U_n(x, y) \sin(nkz). \quad (3.25)$$

Hall (1988) concluded that the non linear evolution of streamwise vortices produces inflectional profiles which will break down via a secondary Rayleigh instability. This has a very strong agreement with the experiments of Aihara and Kohama (1981), as mentioned in the

introduction, and the downstream velocity profiles of this form agree quantitatively with the experiments of Swearingen and Blackwelder (1987).

We continue this work by looking at the possibility of this Rayleigh instability of a non linear vortex flow being an inviscid travelling wave disturbance. It is known that for an inviscid disturbance the streamwise length scale in the Navier-Stokes equations is scaled by $X = \delta x$ and the disturbances vary over the non-dimensional and re-scaled time $\hat{t} = \frac{\ell \delta t}{U_\infty}$, keeping the other directional scalings ($Y = y$ and $Z = z$) we may perturb the basic state by

$$\hat{\mathbf{V}}/U_\infty = \mathbf{u}_B + \Delta(u(x, y, z), v(x, y, z), w(x, y, z)) \exp(i\hat{\alpha}x + \hat{\omega}t), \quad (3.26)$$

where Δ is sufficiently small for linearisation, $\hat{\alpha}$ is the streamwise wavenumber and $\hat{\omega}$ is the growth rate. We also take the pressure perturbation to be of the form

$$\Delta p(x, y, z) \rho U_\infty^2 \exp(i\hat{\alpha}x + \hat{\omega}t), \quad (3.27)$$

where ρ is the fluid density. We focus here on small surface curvature, so we can express curvature (and hence H_1) as

$$\frac{\kappa}{a} \ll 1 \Rightarrow H_1 \sim 1. \quad (3.28)$$

Taking the limit of $Re \rightarrow \infty$ the inviscid leading order disturbance equations at the local position x are

$$i\hat{\alpha} \left(\bar{U}_b + \frac{\hat{\omega}}{i\hat{\alpha}} \right) u + v \frac{\partial \bar{U}_b}{\partial y} + w \frac{\partial \bar{U}_b}{\partial z} = -i\hat{\alpha}p, \quad (3.29)$$

$$i\hat{\alpha} \left(\bar{U}_b + \frac{\hat{\omega}}{i\hat{\alpha}} \right) v = -\frac{\partial p}{\partial y}, \quad (3.30)$$

$$i\hat{\alpha} \left(\bar{U}_b + \frac{\hat{\omega}}{i\hat{\alpha}} \right) w = -\frac{\partial p}{\partial z}, \quad (3.31)$$

$$i\hat{\alpha}u + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (3.32)$$

where $\bar{U}_b = \bar{u} + U$, which can be thought of as the Blasius boundary-layer velocity plus the Taylor vortex mean flow and represents the total downstream velocity field. Since

viscous effects are negligible away from the wall (and any position where $\overline{U}_b = -\frac{\hat{\omega}}{i\hat{\alpha}}$), the boundary conditions for (3.29)-(3.32) are

$$v(0) = v(\infty) = 0. \quad (3.33)$$

To further analyse this system both analytically and numerically, equations (3.29)-(3.32) can be coupled by eliminating u , v and w ; acquiring the pressure equation identical to that of Hall and Horseman (1991)

$$\frac{\partial}{\partial y} \left(\frac{\frac{\partial p}{\partial y}}{(\overline{U}_b + \frac{\hat{\omega}}{i\hat{\alpha}})^2} \right) + \frac{\partial}{\partial z} \left(\frac{\frac{\partial p}{\partial z}}{(\overline{U}_b + \frac{\hat{\omega}}{i\hat{\alpha}})^2} \right) - \frac{\hat{\alpha}^2 p}{(\overline{U}_b + \frac{\hat{\omega}}{i\hat{\alpha}})^2} = 0. \quad (3.34)$$

This is the Rayleigh stability equation for pressure and can be solved as an eigenvalue problem $\hat{\alpha}$ for the eigenvalue $\hat{\omega} = \hat{\omega}(\hat{\alpha})$ with boundary conditions

$$p \rightarrow 0 \quad \text{as } |y| \rightarrow \infty, \quad (3.35)$$

and p must be periodic over $z = (-\pi, \pi)$. We may seek temporal modes ($\hat{\alpha}$ real, $\hat{\omega}$ complex) or spatial modes ($\hat{\alpha}$ complex, $\hat{\omega}$ purely imaginary). However, for neutral stability ($\hat{\alpha}$ real, $\hat{\omega}$ entirely imaginary) we can see that (3.34) has a singularity where $\overline{U} = -\frac{\hat{\omega}}{i\hat{\alpha}}$. We will only consider the non-critical case and focus on temporal modes hence $-\hat{\omega}_r$ and $-\hat{\omega}_i$ represent growth rate and frequency respectively. For a Görtler vortex flow the base flow $\overline{U}_b$, in the form discussed earlier from Hall (1988) in (3.23), is written as

$$\overline{U}_b = \overline{U}_0(x, y) + \sum_1^{\infty} \overline{U}_n(x, y) \cos(nkz), \quad (3.36)$$

so that equation (3.34) has solutions of the form

$$p = \sum_1^{\infty} p_n(x, y) \sin(nkz) \quad \text{or,} \\ p = \hat{p}_0(x, y) + \sum_1^{\infty} \hat{p}_n(x, y) \cos(nkz). \quad (3.37)$$

The sine solution leads to the wavy vortex boundaries observed experimentally (see (Swearingen and Blackwelder, 1987) for experiment and (Hall, 1988) for a short discussion of why the solutions are of this form), while the cosine solution leads to a time-dependent state in which the vortex boundaries remain flat. Experimentally, there is evidence of both these states but there exist many basic states that are of practical relevance, as mentioned in the introduction. We could progress the problem analytically by focusing on the case when the spanwise wavenumber k is small and this was studied by Hall and Horseman (1991) in detail. The conclusion was that, under this limit, we see the expected Rayleigh modes occur at the spanwise locations where the flow is most unstable and that the instability is associated with the inflexional nature of $\bar{U}$ in the y direction.

3.2.1 Spanwise wavenumber and slow normal variation

Unlike Hall and Horseman (1991), where the normal variation of the basic state is fixed, we will implement a slowly varying flow in the normal direction and large vortex wavenumber. We now look to understand more about the large spanwise k limit, whilst implementing a slow variation in the normal direction to the surface. Hall and Lakin (1988) provided an asymptotic description of the vortex driven mean state where the boundary-layer is split into three. Within the main part of the boundary-layer a finite amplitude vortex drives the mean flow and the base flow expands as

$$\bar{U}_b = \bar{U}_0(y) + k^{-1}\bar{U}_1(y) \cos(kz) + \dots \quad (3.38)$$

Note that $\bar{U}_0(y)$ is driven by the vortex and has no relationship with the state we would expect in the absence of a vortex. Hall and Lakin (1988) found that $\bar{U}_0$ is a monotonically increasing function of y in the region of vortex activity so a local analysis of this flow will not give meaningful results for instabilities obtained by wall curvature. A WKBJ (named after the theorists Wentzel (1926), Kramers (1926), Brillouin (1926) and Jeffreys (1925)) approximation of the solutions (see §3.3 for more detail) to the partial differential equation (3.34) can be used since we have a slowly spatially varying base flow. Hence we will use the WKBJ solutions for the normal part of our solution. Our given base flow (3.38) suggests

that the localised inviscid mode will be confined to a layer of depth $1/k$. More precisely we seek a trapped mode at the centre point $y = \bar{y}$. This is assumed to be the centreline of the layer where the vortex activity is concentrated, similar to the central layer defined in (Hall and Lakin, 1988). The trapped mode is kept within this layer of depth $1/k$ and so we write

$$\hat{Y} = k(y - \bar{y}). \quad (3.39)$$

We use the following expansions for the base flow and streamwise wavenumber

$$\bar{U}_b = \frac{1}{k} \bar{U}'_0(\bar{y}) \hat{Y} + \frac{1}{k} \cos(Z) \quad \text{and} \quad \hat{\alpha} = \alpha k + \dots, \quad (3.40)$$

where $Z = kz$.

Putting this altogether into (3.34) and dropping any extraneous hatted variables we attain

$$\frac{\partial}{\partial Z} \left(\frac{\frac{\partial p}{\partial Z}}{(\bar{U}_m + \frac{\omega}{i\alpha})^2} \right) + \frac{\partial}{\partial \hat{Y}} \left(\frac{\frac{\partial p}{\partial \hat{Y}}}{(\bar{U}_m + \frac{\omega}{i\alpha})^2} \right) - \frac{\alpha^2 p}{(\bar{U}_m + \frac{\omega}{i\alpha})^2} = 0, \quad (3.41)$$

with a modified base flow $\bar{U}_m = \bar{U}'_0(\bar{y}) \hat{Y} + \cos(Z)$. Since we are interested in a slowly varying flow in the normal direction around this centre point we take $\bar{U}'_0(\bar{y})$ to be small and we explicitly set

$$\bar{U}'_0(\bar{y}) = \epsilon \Rightarrow \bar{U}_m = \epsilon \hat{Y} + \cos(Z), \quad (3.42)$$

where ϵ is the small parameter. So we can rewrite (3.41) to get

$$\frac{\partial}{\partial Z} \left(\frac{\frac{\partial p}{\partial Z}}{(\bar{U} + \frac{\omega}{i\alpha})^2} \right) + \epsilon^2 \frac{\partial}{\partial Y} \left(\frac{\frac{\partial p}{\partial Y}}{(\bar{U} + \frac{\omega}{i\alpha})^2} \right) - \frac{\alpha^2 p}{(\bar{U} + \frac{\omega}{i\alpha})^2} = 0, \quad (3.43)$$

where $Y = \epsilon \hat{Y}$ and our final modified base flow is $\bar{U} = Y + \cos(Z)$. We introduce the phase integral seen below inside the exponential of our WKBJ approximations for the pressure in the normal direction. We then express our full solutions as

$$p(Y, Z) = [p_0(Y, Z) + \epsilon p_1(Y, Z) + \dots] \exp \left(-\frac{i}{\epsilon} \int_0^Y \theta(s) ds \right) + c.c., \quad (3.44)$$

where *c.c.* is the complex conjugate part of the solution. At leading order we obtain the differential equation

$$\frac{\partial}{\partial Z} \left(\frac{\frac{\partial p_0}{\partial Z}}{\left(\bar{U} + \frac{\omega}{i\alpha}\right)^2} \right) - \frac{(\alpha^2 + \theta^2)p_0}{\left(\bar{U} + \frac{\omega}{i\alpha}\right)^2} = 0, \quad (3.45)$$

with the condition

$$p_0 = 0 \quad \text{as} \quad |Y| \rightarrow \infty, \quad (3.46)$$

and periodicity in the spanwise direction. Equation (3.45) is the local Rayleigh problem to determine the eigenvalue $\theta = \theta(Y, \alpha, \omega)$. To analyse this further we consider the two dimensional ($Y = 0$, Z and α with fixed ω) problem and relate this to (3.45).

3.2.2 Two dimensional Rayleigh problem

In Green (1974) we can see solutions for high Reynolds number flows for the two dimensional Rayleigh stability equation of the form

$$\frac{d^2 f}{d\eta^2} - \alpha_*^2 f - \frac{\cos(\eta)f}{\left(\frac{\omega_*}{i\alpha_*} - \cos(\eta)\right)} = 0, \quad (3.47)$$

from which we can see some similarities to our equation with $Y = 0$ and $\theta = 0$. Green used a recurrence relation (similar to the one seen for solving Mathieu's equation) to obtain the growth rate seen in figure 3.1 by the highest thick line. By substituting $\theta = 0$ at $Y = 0$ into (3.45) we obtain

$$\frac{d}{dZ} \left(\frac{\frac{dp}{dZ}}{\left(\cos(Z) + \frac{\omega}{i\alpha}\right)^2} \right) - \frac{\alpha^2 p}{\left(\cos(Z) + \frac{\omega}{i\alpha}\right)^2} = 0, \quad (3.48)$$

this equation is not the same as that seen above however the solution to (3.48) will be seen to have similar results to those seen in figure 3.1. This means that the solutions of the 2D version of our equation have similar behaviours to the solutions for shear flows seen in (Green, 1974), as required.

Numerically solving (see §3.5) equation (3.48) for the growth rate $\omega(\alpha)$ we get the growth

curve shown in figure 3.3. These solutions agree very well with the accurate solutions given by Green (1974) as shown by the direct comparison in figure 3.2. Hence this two dimensional form of our problem exhibits the same properties as the flows focused on by Green and in particular describes a flow which satisfies the Rayleigh criterion for an inviscid instability. We are more concerned with the fully three dimensional problem but we can use this initial result to form the full solution.

We wish to be able to derive the three dimensional problem from the two dimensional problem. By selecting a value for the wavenumber (γ) in the two dimensional problem and setting the growth rate as the function $F(\gamma)$ the two dimensional equation (3.48) can now be written in the form

$$\frac{d}{dZ} \left(\frac{\frac{dp}{dZ}}{\left(\cos(Z) + \frac{F(\gamma)}{i\gamma} \right)^2} \right) - \frac{\gamma^2 p}{\left(\cos(Z) + \frac{F(\gamma)}{i\gamma} \right)^2} = 0. \quad (3.49)$$

We can re express the third dimension eigenvalue ($\theta(Y)$) in terms of both the 2D and 3D wavenumber around $Y = 0$ as

$$\theta(\alpha) = \sqrt{\gamma^2 - \alpha^2}, \quad (3.50)$$

where we have fixed γ the effective two dimensional streamwise wavenumber, from the two dimensional problem.

A quick substitution of (3.50) into (3.49) shows that we have almost connected the 2D and 3D problems at $Y = 0$. We already have $\gamma^2 = \alpha^2 + \theta^2$ in the second term but part of the denominator in the above is

$$\frac{F(\gamma)}{i(\alpha^2 + \theta^2)}, \quad (3.51)$$

and this needs to be adjusted to get the 3D equation (3.45) at $Y = 0$.

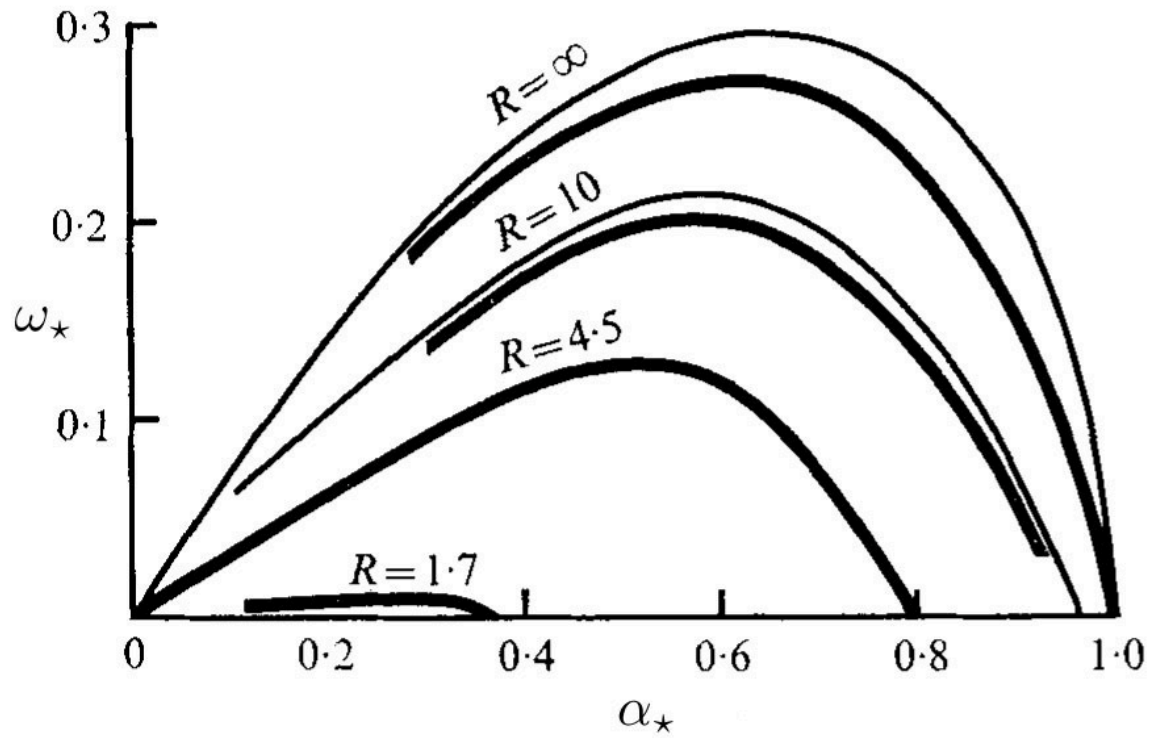


Figure 3.1: Growth rate ω_* of eddies in sheared flow against wavenumber for different Reynolds number. Thin lines are two terms from the Green's recurrence relation, thick lines are taking more terms in this relation. We are concerned with the highest line where $Re \rightarrow \infty$. Scanned and modified from Green (1974).

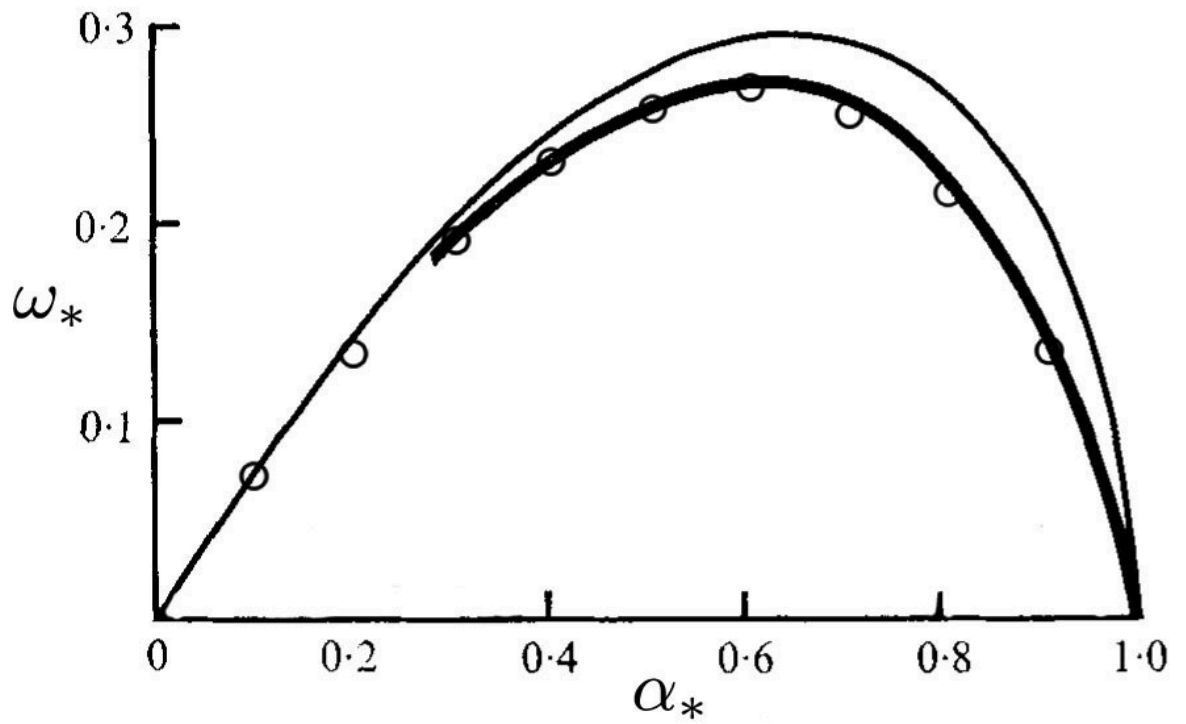


Figure 3.2: Comparison of two dimensional growth rate, ω_* , calculated in Green (1974) and numerically from the two dimensional form of our problem (3.48). Thick line represents the more accurate recurrence relation from Green and the circles are our calculations.

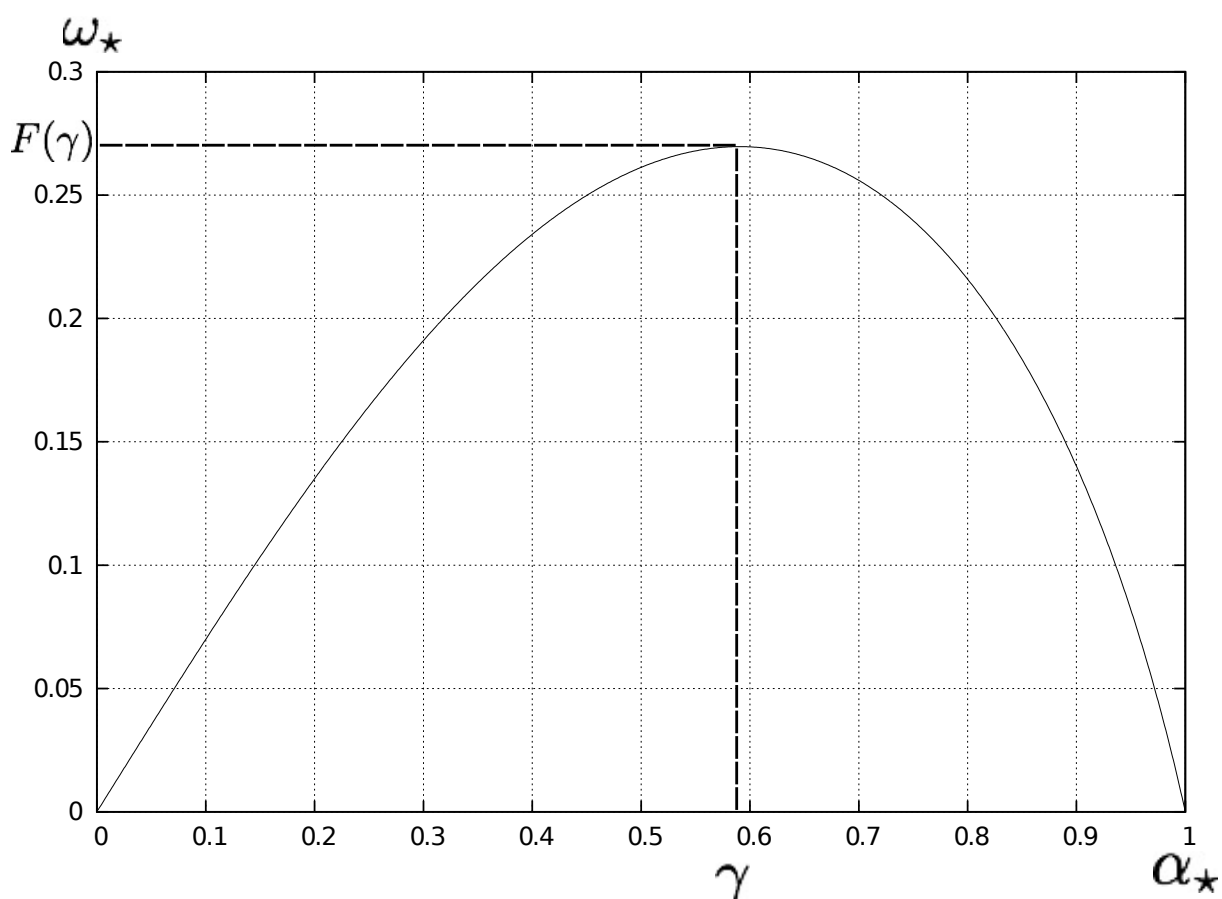


Figure 3.3: Growth curve from two dimensional instability and selection of $F(\gamma)$ for fixed $\gamma = 0.591$.

By writing

$$\omega_0 = \frac{i\alpha F(\gamma)}{i(\alpha^2 + \theta^2)} = \frac{\alpha F(\gamma)}{\gamma}, \quad (3.52)$$

we can substitute this into (3.49) with (3.50) to get have the correct form for our three dimensional problem at $Y = 0$:

$$\frac{d}{dZ} \left(\frac{\frac{dp}{dZ}}{\left(\cos(Z) + \frac{\omega_0}{i\alpha}\right)^2} \right) - \frac{(\alpha^2 + \theta^2)p}{\left(\cos(Z) + \frac{\omega_0}{i\alpha}\right)^2} = 0. \quad (3.53)$$

We have successfully formed the 3D problem from the 2D one. We have also acquired (3.50) and (3.52) for any streamwise wavenumber to obtain the correct initial values of ω_0 and θ for any streamwise wavenumber. This enables us to then extend the two dimensional solutions to three dimensional ones. The equation (3.52) also acts as a validation tool for any initial calculations made at the origin. This can be seen in the figures below which show the comparison of calculated values of ω_0 numerically and directly from (3.52) varying either γ (fig. 3.4) or α (fig. 3.5).

To solve the three dimensional problem we want to be able to obtain $\theta(\alpha, Y)$ for any choice of parameters. However, the WKBJ approximations will break down when $\theta = 0$, which occurs on $Y = 0$ and when $\alpha = \gamma$. We will focus on parameter values such that $\alpha > \gamma$ but we will first discuss this breakdown.

3.3 Breakdown of WKBJ approximation

Historically the WKBJ approximation was derived in quantum mechanics and used to solve the Schrödinger equation

$$\frac{d^2\psi}{dx^2} + \frac{\Theta^2}{\hbar^2}\psi(x) = 0, \quad (3.54)$$

where ψ is the wave function approximated by an oscillatory wave depending on a phase integral

$$\int_{x_1}^{x_2} \frac{\Theta}{\hbar} dx. \quad (3.55)$$

This approximation works well until you consider a harmonic oscillator. This particle is known to bounce back and forth (in a classical sense) between turning points where the points momentum acting on the particle is pushing it back from where it came. The analogy in wave mechanics is a region of negative energy on one side of the turning point and positive on the other. Within the negative (and forbidden) energy region the wave function decays exponentially and does not oscillate as it does in the positive region. A turning point, (x_j) , occurs in wave mechanics if

$$E_T - V(x_j) = \frac{\Theta^2(x_j)}{2m} = 0, \quad (3.56)$$

where E_T is the energy of the particle, $V(x)$ its potential energy and m its mass (Berry and Mount, 1972). So that the turning point, x_j , is where $\Theta^2 = 0$. When $\Theta^2 > 0$ the particle is in the positive energy region and where $\Theta^2 < 0$ it is in the forbidden negative region. Of course quantum mechanics predicts that exponential waves can exist in forbidden regions and the WKBJ solutions allow for that since Θ is imaginary there. Solutions can be found in both regions but around the turning point the WKBJ approximation fails due to Θ being zero and so we need to connect the two regions. There are many methods for solving this connection problem dependent on the type of turning point. One such method uses the technique of uniform approximation; mapping the solutions to a simpler equation (normally either Airy's equation or Weber differential equation) and another notable method is the complex method. This method traces a path in the complex x plane that is far enough from the turning points. Consequently, whilst it can connect the two regions it does not produce explicit wave functions near the turning point. This second method is also susceptible to the Stokes phenomenon which arises due to the phase integral and Θ becoming multivalued for complex x .

These approximations and consequential turning point problems can be used not only in quantum mechanics but also in fluid dynamical problems.

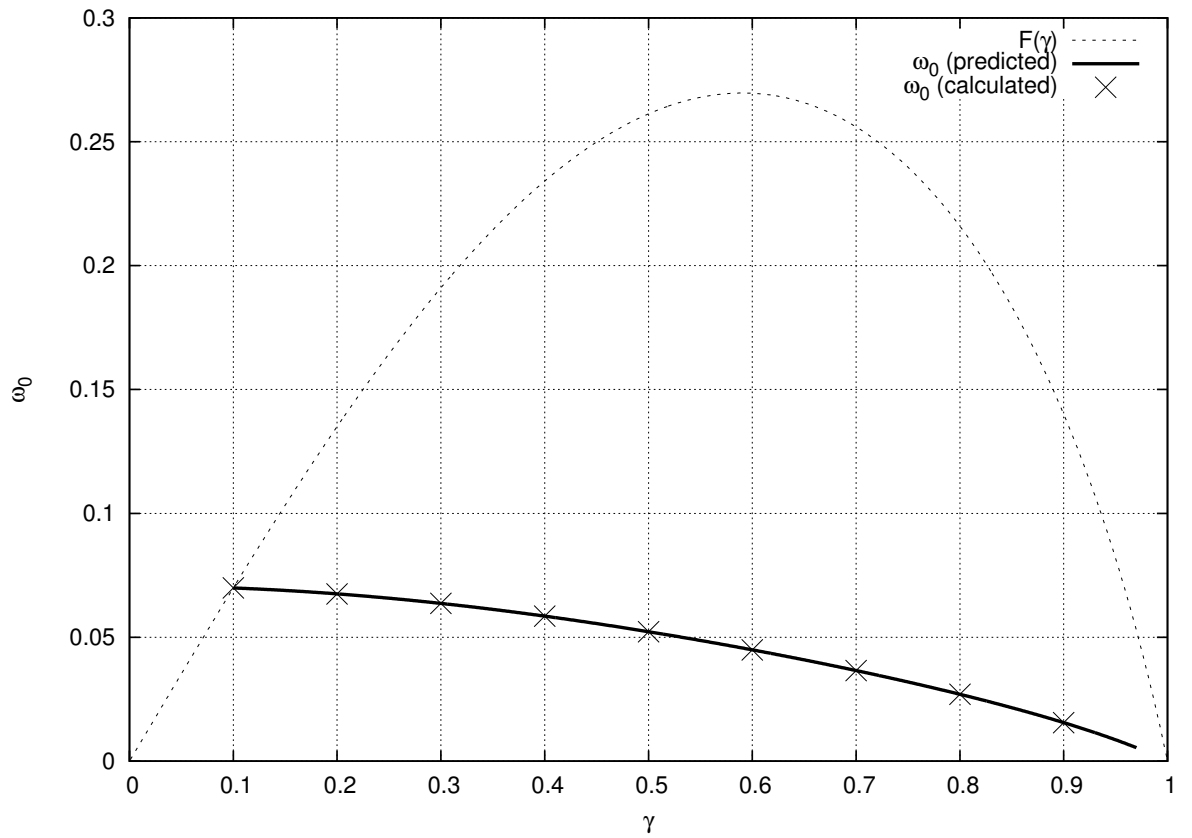


Figure 3.4: Comparison of growth rate, ω_0 , predicted in (3.52) against growth rate calculated when varying effective streamwise wavenumber, γ .

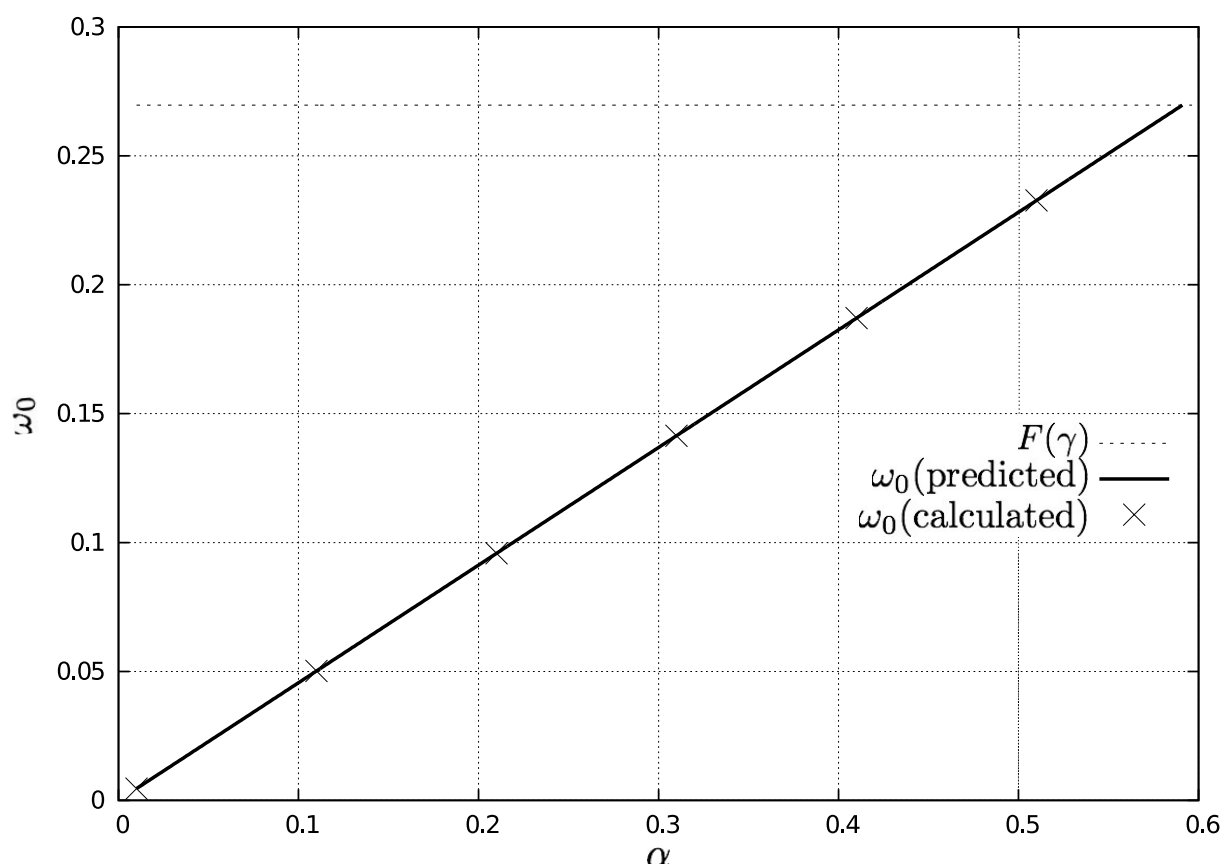


Figure 3.5: Comparison of growth rate, ω_0 , predicted in (3.52) against growth rate calculated when varying streamwise wavenumber, α .

Famously the paper by Soward and Jones (1982) looks at the flow between two concentric cylinders and locates the saddle point to find the global maximum Taylor number for given parameters, a similar analysis can be seen for the stability of a viscous, unsteady flow of a curved pipe in Papageorgiou (1987).

Our calculations are performed for $\alpha < \gamma$ and use the initial value for our eigenvalue, $\theta(0)$, at $\sqrt{\gamma^2 - \alpha^2}$ so that we can solve our partial differential equation for the given base flow. If we were to try to solve the same equation for $\alpha > \gamma$ we would have to adjust the initial value of the eigenvalue to $i\sqrt{\alpha^2 - \gamma^2}$, which would change the WKB approximation and result in a different form of solution but would still break down as $\alpha \rightarrow \gamma$. By construction, our turning point is located at

$$\theta(Y, \alpha) = 0 \quad \text{when} \quad Y = 0 \quad \text{and} \quad \alpha \rightarrow \gamma, \quad (3.57)$$

which implies that as we approach the effective spanwise wavenumber the three dimensional problem begins to collapse into the two dimensional one where the WKB solutions are no longer valid. We did attempt to find a way around the turning point by allowing Y to go complex using the complex method in the same way as Soward and Jones (1982), this however led to difficulties, which we will now discuss.

The leading order eigenvalue problem (3.45) has a solution provided that ω , Y , θ and α are related by some equation of the form

$$\mathcal{F}(\alpha, \theta, Y, \omega) = 0, \quad (3.58)$$

or a dispersion relation of the type

$$\omega = \omega_0(\theta, Y, \alpha) + \mathcal{O}(\epsilon) \quad (3.59)$$

for which we want to find a saddle point in the complex ω plane. We regard ω_0 as an analytic function of the two complex variables θ and Y and one real variable α , the increments of

ω_0 with respect to those variables may be derived from the differential

$$d\omega = \frac{\partial\omega_0}{\partial\theta}d\theta + \frac{\partial\omega_0}{\partial Y}dY + \frac{\partial\omega_0}{\partial\alpha}d\alpha. \quad (3.60)$$

At the saddle point given any arbitrary real values of dY , $d\theta$ there must exist some real value of $d\alpha$ for which $d\omega_0$ is zero. This result can only be achieved if

$$0 = \frac{\partial\omega_0}{\partial Y} \frac{dY}{d\alpha} + \frac{\partial\omega_0}{\partial\theta} \frac{d\theta}{d\alpha} + \frac{\partial\omega_0}{\partial\alpha} \quad (3.61)$$

$$\Rightarrow -1 = \frac{\frac{\partial\omega_0}{\partial Y}}{\frac{\partial\omega_0}{\partial\alpha}} \frac{dY}{d\alpha} + \frac{\frac{\partial\omega_0}{\partial\theta}}{\frac{\partial\omega_0}{\partial\alpha}} \frac{d\theta}{d\alpha}. \quad (3.62)$$

Hence,

$$\Im\left(\frac{\frac{\partial\omega_0}{\partial Y}}{\frac{\partial\omega_0}{\partial\alpha}}\right) = 0, \quad \Im\left(\frac{\frac{\partial\omega_0}{\partial\theta}}{\frac{\partial\omega_0}{\partial\alpha}}\right) = 0. \quad (3.63)$$

These conditions are met if

$$\Im\left(\frac{\partial\omega_0}{\partial Y}\right) = 0, \quad \Im\left(\frac{\partial\omega_0}{\partial\theta}\right) = 0. \quad (3.64)$$

However we will see that $\frac{\partial\omega_0}{\partial Y}$ is constant and $\frac{\partial\omega_0}{\partial\theta} \neq 0$ will occur and does not lead to acceptable neutrally stable solutions. A correct solution is found by isolating the value $\theta = \theta_0$ at which the conditions are met even if $\theta_0 \in \mathbb{C}$. The requirement that the corresponding values of ω_0 , α are real together with the real and imaginary parts of the conditions provides seven unknowns

$$\Re(\theta_0), \quad \Im(\theta_0), \quad \Re(Y_0), \quad \Im(Y_0), \quad \Re(\omega_0), \quad \Im(\omega_0), \quad \alpha.$$

Before we begin attempting to locate the turning point we can directly check the derivatives in the conditions above: $\frac{\partial\omega_0}{\partial Y}$ and $\frac{\partial\omega_0}{\partial\theta}$. We do this by expressing the leading order differential equation (3.45) in the form

$$\frac{\partial^2 p_0}{\partial Z^2} \left(Y + \cos(Z) + \frac{\omega_0}{i\alpha} \right) + 2 \sin(Z) \frac{\partial p_0}{\partial Z} - (\alpha^2 + \theta^2) p_0 \left(Y + \cos(Z) + \frac{\omega_0}{i\alpha} \right) = 0, \quad (3.65)$$

and differentiating through by Y and θ respectively. We get

$$\begin{aligned} \frac{\partial}{\partial Z} \left(\frac{\frac{\partial}{\partial Z} \left(\frac{\partial p_0}{\partial Y} \right)}{\left(\bar{U} + \frac{\omega_0}{i\alpha} \right)^2} \right) - \frac{(\alpha^2 + \theta^2) \left(\frac{\partial p_0}{\partial Y} \right)}{\left(\bar{U} + \frac{\omega_0}{i\alpha} \right)^2} = \\ \frac{1}{\left(\bar{U} + \frac{\omega_0}{i\alpha} \right)^3} \left(1 + \frac{1}{i\alpha} \frac{\partial \omega_0}{\partial Y} \right) \left((\alpha^2 + \theta^2) p_0 - \frac{\partial^2 p_0}{\partial Z^2} \right), \end{aligned} \quad (3.66)$$

$$\begin{aligned} \frac{\partial}{\partial Z} \left(\frac{\frac{\partial}{\partial Z} \left(\frac{\partial p_0}{\partial \theta} \right)}{\left(\bar{U} + \frac{\omega_0}{i\alpha} \right)^2} \right) - \frac{(\alpha^2 + \theta^2) \left(\frac{\partial p_0}{\partial \theta} \right)}{\left(\bar{U} + \frac{\omega_0}{i\alpha} \right)^2} = \\ \frac{2\theta p_0}{\left(\bar{U} + \frac{\omega_0}{i\alpha} \right)^2} + \frac{1}{i\alpha \left(\bar{U} - \frac{\omega_0}{i\alpha} \right)^3} \frac{\partial \omega_0}{\partial \theta} \left(\frac{\partial^2 p_0}{\partial Z^2} - p_0(\alpha^2 + \theta^2) \right). \end{aligned} \quad (3.67)$$

We notice that the left hand sides of both these are the linear operator $\mathcal{L}$ and so we can use the self-adjoint $\hat{p}_0$ from $p_0 = b_0(Y)\hat{p}_0(Y, \theta, Z)$ and collect b_0 and $\frac{db_0}{dY}$ terms in (3.66) to get two systems which can be separated as a complementary and particular solution

$$(1) : \quad \mathcal{L}(\hat{p}_0) = 0, \quad (3.68)$$

$$(2) : \quad \mathcal{L} \left(\frac{\partial \hat{p}_0}{\partial Y} \right) = \frac{1}{\left(\bar{U} + \frac{\omega_0}{i\alpha} \right)^3} \left(1 + \frac{1}{i\alpha} \frac{\partial \omega_0}{\partial Y} \right) \left((\alpha^2 + \theta^2) \hat{p}_0 - \frac{\partial^2 \hat{p}_0}{\partial Z^2} \right). \quad (3.69)$$

From the first equation we can simplify the second derivative and hence we have

$$\mathcal{L} \left(\frac{\partial \hat{p}_0}{\partial Y} \right) = \frac{2 \sin(Z)}{\left(\bar{U} + \frac{\omega_0}{i\alpha} \right)^4} \left(1 + \frac{1}{i\alpha} \frac{\partial \omega_0}{\partial Y} \right) \frac{\partial \hat{p}_0}{\partial Z}, \quad (3.70)$$

similarly for the θ -derivative from (3.67) we produce the equation

$$\mathcal{L} \left(\frac{\partial \hat{p}_0}{\partial \theta} \right) = \frac{2\theta \hat{p}_0}{\left(\bar{U} + \frac{\omega_0}{i\alpha} \right)^2} + \frac{2 \sin(Z)}{i\alpha \left(\bar{U} + \frac{\omega_0}{i\alpha} \right)^4} \left(\frac{\partial \omega_0}{\partial \theta} \right) \frac{\partial \hat{p}_0}{\partial Z}. \quad (3.71)$$

From these we can gain two solvability conditions by applying the self-adjoint $\hat{p}_0$ and

integrating over $Z \in [-\pi, \pi]$ namely:

$$\frac{\partial \omega_0}{\partial Y} = -i\alpha, \quad (3.72)$$

$$\frac{\partial \omega_0}{\partial \theta} = -\frac{\theta I_0}{J_0}, \quad (3.73)$$

where

$$I_0 = \int_{-\pi}^{\pi} \frac{\hat{p}_0^2}{(\bar{U} + \frac{\omega_0}{i\alpha})^2} dZ \quad \text{and} \quad J_0 = \int_{-\pi}^{\pi} \frac{\hat{p}_0 \sin(Z)}{(\bar{U} + \frac{\omega_0}{i\alpha})^4} \frac{\partial \hat{p}_0}{\partial Z} dZ. \quad (3.74)$$

The problem that we now face is that α is fixed throughout each calculation (as $\alpha = \gamma$ for the turning point) the Y derivative is fixed also and so constant. It follows that any attempt to minimise these quantities to achieve the conditions set out in (3.64) will not be possible. This is due to the determinant of the Jacobian matrix we would need to form to minimise the derivatives simultaneously in two coordinate directions will be zero always. This matrix contained the second order derivatives of ω_0 and hence the first order derivatives will be non-zero always. Consequently, we cannot change the value of Y to locate the saddle point as we wish and so cannot map a solution from the complex Y plane to the real one. If we scale around the point in the complex plane Y_0 where we expect to find a saddle point and consider the disc around this with radius $\epsilon^{1/2}$ and so write

$$\tilde{Y} = \frac{Y - Y_0}{\epsilon^{1/2}}, \quad (3.75)$$

we would eventually derive an amplitude function which has solutions that are parabolic cylinder functions and so exponential decays as $|Y| \rightarrow \infty$. However, the location of Y_0 that is needed in those solutions is not possible to obtain as outlined above.

We now continue our analysis by returning to Y real, and with the parameter space that we can use, $\alpha \neq \gamma$, namely $\alpha < \gamma$ here.

3.4 Amplitude function analysis

We expand the full growth rate as

$$\omega = \omega_0 + \epsilon\omega_1, \quad (3.76)$$

where ω_0 is the growth rate calculated from the two dimensional problem at the fixed effective streamwise wavenumber γ . Using (3.76) and (3.44) in (3.43) and collecting terms of $\mathcal{O}(1)$ and $\mathcal{O}(\epsilon)$ we now attain the partial differential equations

$$\mathcal{O}(1) : \quad \mathcal{L}(p_0) = 0, \quad (3.77)$$

$$\mathcal{O}(\epsilon) : \quad \mathcal{L}(p_1) = L(p_0), \quad (3.78)$$

where

$$\mathcal{L}(p_*) = \frac{\partial}{\partial Z} \left(\frac{\frac{\partial}{\partial Z} p_*}{(\bar{U} + \frac{\omega_0}{i\alpha})^2} \right) - \frac{(\alpha^2 + \theta^2)}{(\bar{U} + \frac{\omega_0}{i\alpha})^2} p_*, \quad (3.79)$$

and

$$L(p_0) = \frac{i}{(\bar{U} + \frac{\omega_0}{i\alpha})^2} \left[\frac{2\theta p_0}{(\bar{U} + \frac{\omega_0}{i\alpha})} \frac{\partial \bar{U}}{\partial Y} - p_0 \frac{d\theta}{dY} - 2\theta \frac{\partial p_0}{\partial Y} + \frac{2\omega_1}{\alpha (\bar{U} + \frac{\omega_0}{i\alpha})^2} \frac{\partial \bar{U}}{\partial Z} \frac{\partial p_0}{\partial Z} \right]. \quad (3.80)$$

We combine the above system with periodic Z conditions

$$p_0(Y, -\pi) = p_0(Y, \pi), \quad \left. \frac{\partial p_0}{\partial Z} \right|_{Z=-\pi} = \left. \frac{\partial p_0}{\partial Z} \right|_{Z=\pi}. \quad (3.81)$$

The equation (3.77) is the local Rayleigh problem to determine the eigenvalue $\theta = \theta(\alpha, Y)$ for the fixed growth rate ω_0 which can be calculated numerically or directly (with the use of (3.52)); see figures 3.6-3.8 in next section for calculated values of θ . Whilst (3.78) is an inhomogeneous version of (3.77) and therefore only has a solution if a solvability condition is satisfied. We write the solution to (3.77) in the form

$$p_0 = A(Y) \hat{p}_0(Y, Z), \quad (3.82)$$

where $A(Y)$ represents the amplitude function of the disturbance and $\hat{p}_0$ is the self-adjoint solution of the leading order problem¹. Multiplying by the adjoint and integrating over the Z -domain, or applying the solvability condition, on (3.78) we obtain

$$\frac{dA}{dY} + \left(\frac{1}{2\theta} \frac{d\theta}{dY} + \frac{1}{2I} \frac{dI}{dY} + \frac{\omega_1 J}{\alpha \theta I} \right) A = 0, \quad (3.83)$$

where

$$I(Y) = \int_{-\pi}^{\pi} \frac{\hat{p}_0^2}{\left(\bar{U} + \frac{\omega_0}{i\alpha}\right)^2} dZ \quad \text{and} \quad J(Y) = \int_{-\pi}^{\pi} \frac{\hat{p}_0 \sin(Z)}{\left(\bar{U} + \frac{\omega_0}{i\alpha}\right)^4} \frac{\partial \hat{p}_0}{\partial Z} dZ. \quad (3.84)$$

Writing

$$Q(Y) = \frac{1}{2\theta} \frac{d\theta}{dY} + \frac{1}{2I} \frac{dI}{dY} + \frac{\omega_1 J}{\alpha \theta I}, \quad (3.85)$$

and setting $A(0) = 1$ the solutions to the linear amplitude equation (3.83) are

$$A(Y) = \exp \left(- \int_0^Y Q(s) ds \right). \quad (3.86)$$

We now have a solution to the amplitude problem and can form full solutions for (3.77) numerically (see §3.5).

3.4.1 Eigenvalue solutions and decay

We choose $\gamma = 0.591$ ($F(\gamma) = 0.26966$) which corresponds to the maximum growth rate, as can be seen in figure 3.3. We numerically solve (3.77) and by varying Y we obtain the eigenvalues $\theta(\alpha, Y)$ for the problem from which the far field decay of the WKBJ approximation can be imposed. We see these solutions in both real and imaginary parts as well as in an Argand diagram in figures 3.6-3.8. We will discuss the behaviour of these solutions and how they solve the problem, in particular how they impose the decay that we require.

The θ solutions all have a fixed $\gamma = 0.591$ for values of $\alpha < \gamma$, figure 3.6 shows θ for

¹The leading order equation is written in a Sturm-Liouville form. This means we have $\mathcal{L}(p_0) = 0$, where $\mathcal{L}$ is a self adjoint operator since $\mathcal{L}(p_0) = \mathcal{L}(\hat{p}_0) = 0$

different values of α in an Argand diagram, demonstrating that as $|Y| \rightarrow \infty$ then $\theta \rightarrow \theta_i i$. We can validate this results by looking at the $|Y| \rightarrow \infty$ limit in (3.45). Under this limit we have

$$\frac{\partial^2 \hat{p}_0}{\partial Z^2} - (\alpha^2 + \theta^2) \hat{p}_0 = 0, \quad (3.87)$$

due to $1/\bar{U} \rightarrow 0$ as $Y \rightarrow \infty$. We expect solutions to be of the form

$$\hat{p}_0 \sim \sin(Z) \quad \text{or} \quad \cos(Z), \quad (3.88)$$

from the Y - damped sine and cosine solutions (3.37), solved with periodic conditions (3.81). This periodicity means that we can get a condition as $Y \rightarrow \infty$

$$\theta^2 \rightarrow -1 - \alpha^2, \quad (3.89)$$

or

$$\theta \rightarrow \pm i \sqrt{1 + \alpha^2}, \quad \text{as} \quad Y \rightarrow \mp \infty. \quad (3.90)$$

This result is confirmed in figure 3.6 where the $Y = 0$ part of the solution lies along the real axis and positive Y solutions are below and negative Y are above it. Hence the values for θ at infinity are those on the far left hand side. They agree with the above result for each α . Similarly, the results in figure 3.8 can be used to further verify this point.

At this stage we can also guarantee that the eigenvalue does impose decay, as expected, by numerically calculating the exponential WKBJ approximation, which is seen in figures 3.9 and 3.10. These two figures show that both the real and imaginary part of

$$E = \exp \left(-\frac{i}{\epsilon} \int_0^Y \theta(s) ds \right), \quad (3.91)$$

both decay to zero as $|Y| \rightarrow \infty$ as required. Hence both the eigenvalues and their respective eigenfunctions solve the problem for the parameters and for this base flow.

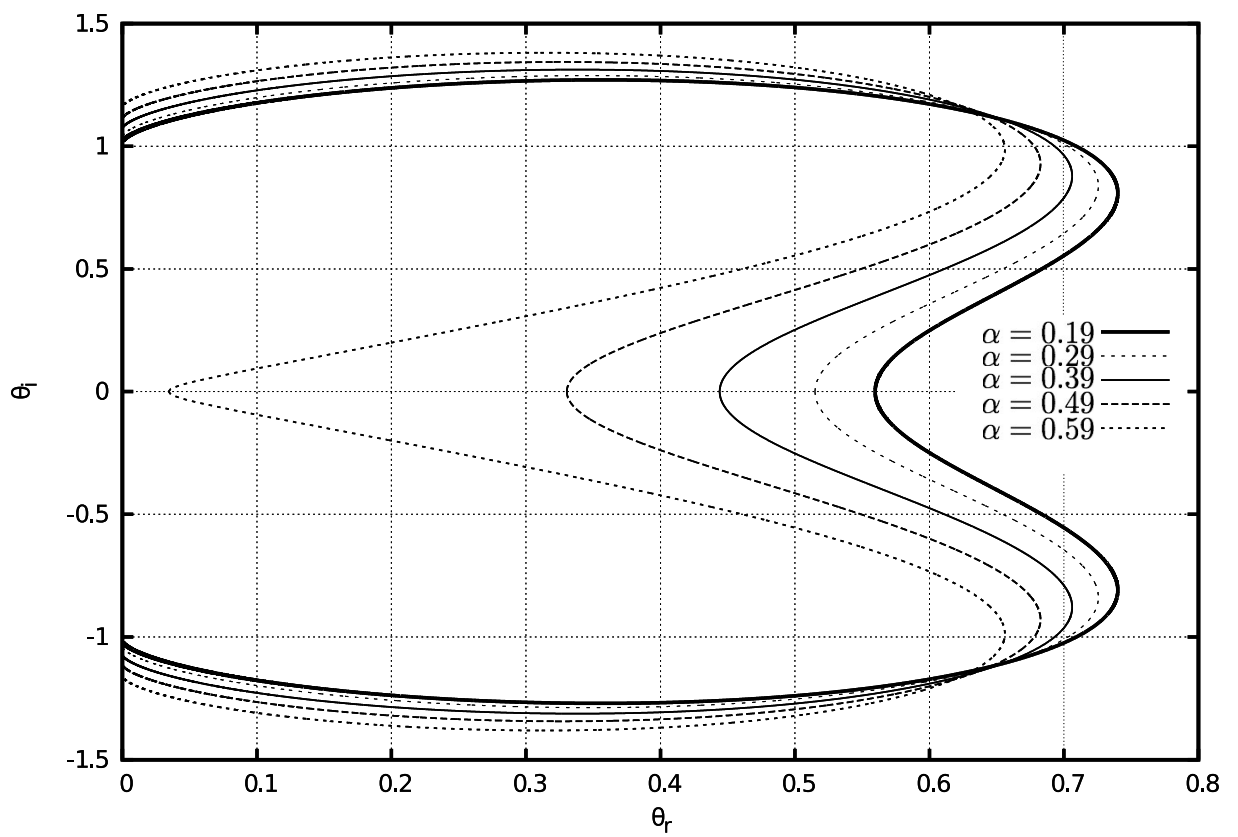


Figure 3.6: Argand diagram of eigenvalue for range of streamwise wavenumbers, α , and at a fixed effective streamwise wavenumber of $\gamma = 0.591$.

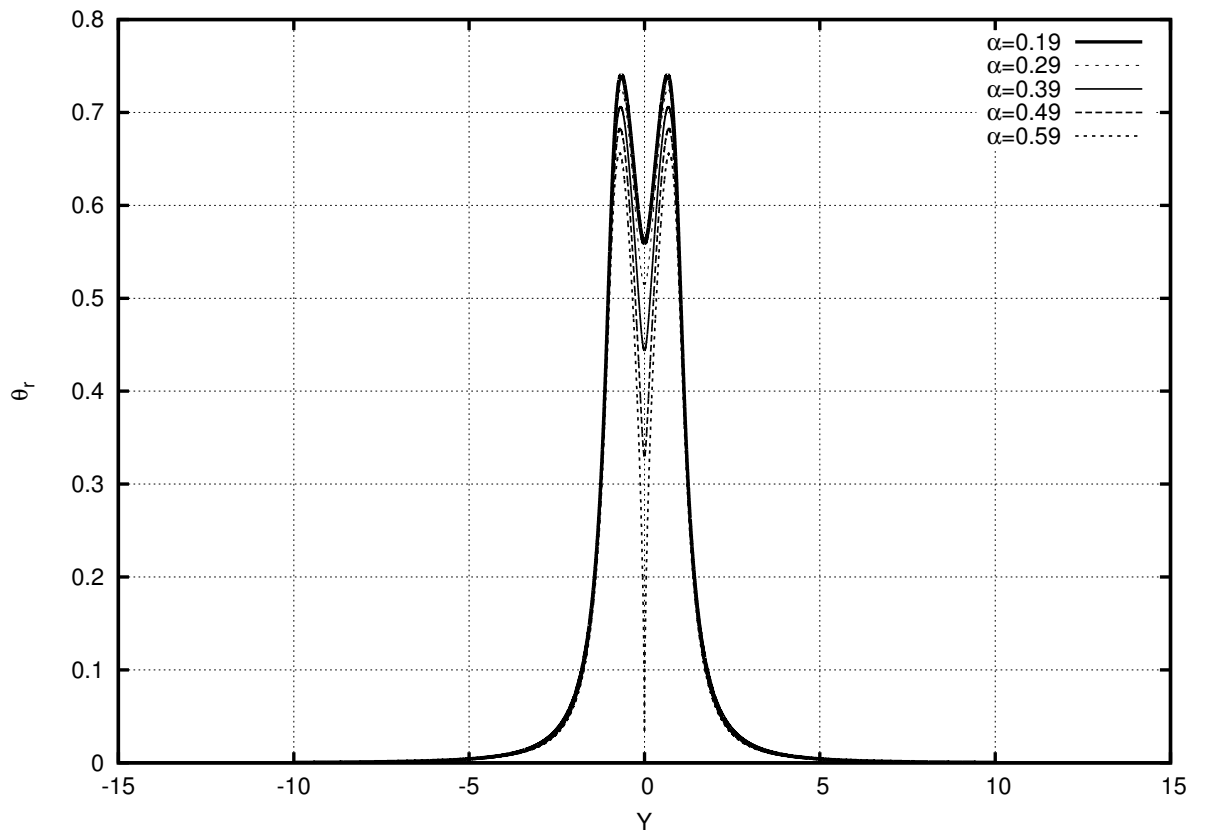


Figure 3.7: Real part of eigenvalue for range of streamwise wavenumbers, α , and at a fixed effective streamwise wavenumber, γ , of 0.591.

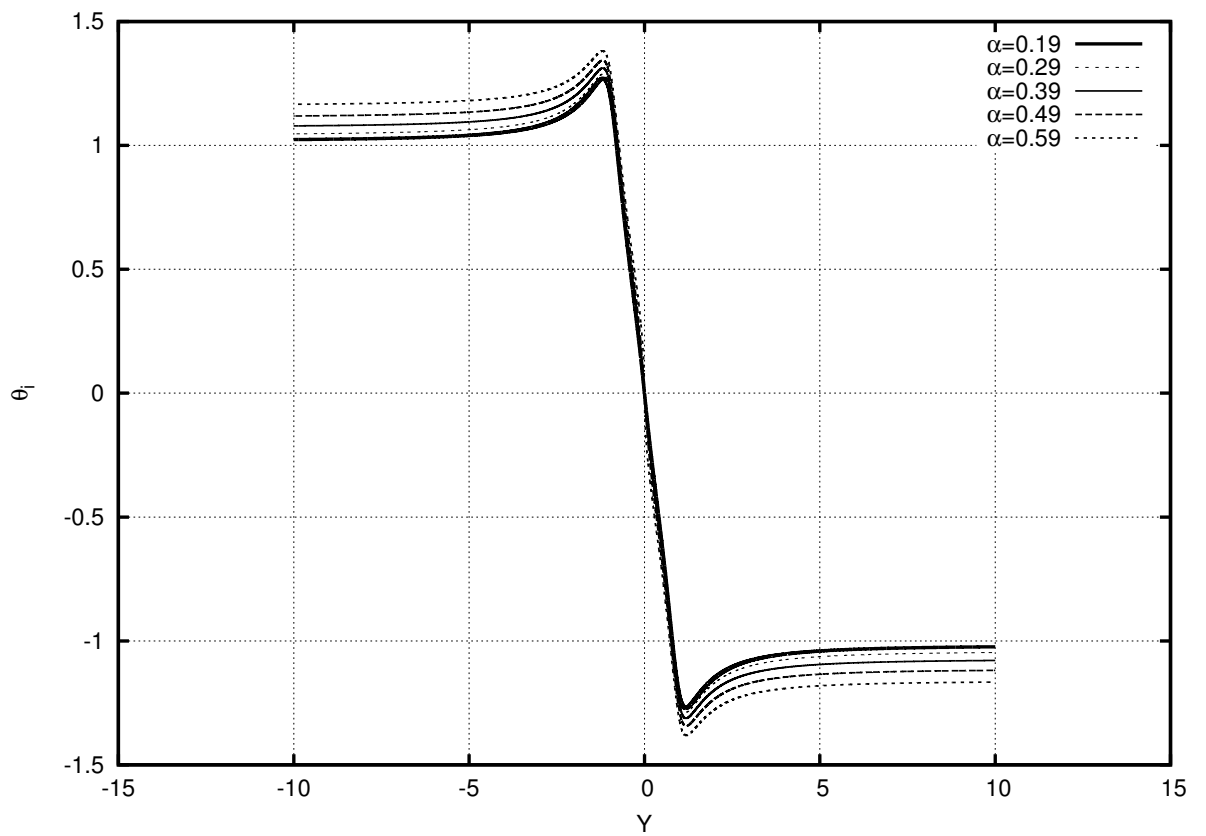


Figure 3.8: Imaginary part of eigenvalue for range of streamwise wavenumbers, α , and at a fixed effective streamwise wavenumber, γ , of 0.591.

Whilst calculating the θ values we obtain eigenfunctions of the form $\hat{p}_0(Y, Z)$. To use $\hat{p}_0$ in the solvability condition and to calculate $A(Y)$ we normalise the eigenfunctions by the constant

$$\|\hat{p}_0\| = \sqrt{\int_{-\pi}^{\pi} \hat{p}_0^2 dZ}. \quad (3.92)$$

before integrating:

$$I(Y) = \int_{-\pi}^{\pi} \frac{\hat{p}_0^2}{\left(\bar{U} + \frac{\omega_0}{i\alpha}\right)^2} dZ \quad \text{and} \quad J(Y) = \int_{-\pi}^{\pi} \frac{\hat{p}_0 \sin(Z)}{\left(\bar{U} + \frac{\omega_0}{i\alpha}\right)^4} \frac{\partial \hat{p}_0}{\partial Z} dZ. \quad (3.93)$$

to get $Q(Y)$ and consequently $A(Y)$ (see (3.86)). Finally, we can combine $A(Y)$, $\hat{p}_0(Y, Z)$ and the WKBJ exponential to form the expression

$$q(Y, Z) = \hat{p}_0 A(Y) \exp\left(-\frac{i}{\epsilon} \int_0^Y \theta(s) ds\right). \quad (3.94)$$

From which we construct the full solutions for a certain choice of predefined α and ω_1 to the pressure equation (3.77). For comparison, we normalise the solutions for different parameter values by $q_{max} = |\text{maximum}(q)|$, the maximum, absolute value of the eigenfunction. The resulting contour plots are shown in figures 3.11-3.17 discussed in §3.6.

It is increasingly difficult to obtain eigenvalues analytically and so we now turn to a numerical approach. These numerical solutions will also allow us to solve the amplitude equation (3.83) and then produce the full normalised eigenfunctions of the form (3.94).

3.5 Numerical scheme

For ease of notation and to demonstrate Runge-Kutta correctly we define the functions

$$F(Z) = p_0, \quad \text{and} \quad G(Z) = \frac{dp_0}{dZ}, \quad (3.95)$$

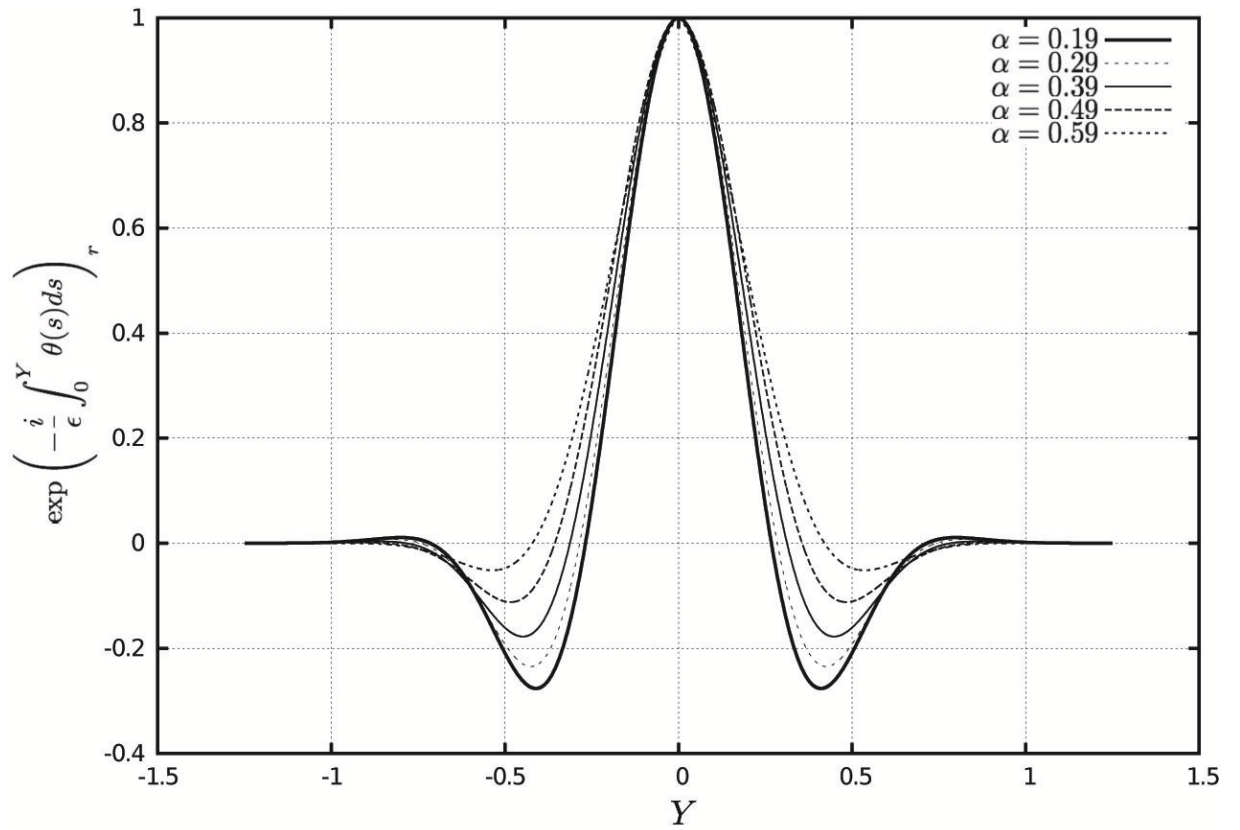


Figure 3.9: Real part of WKBJ exponential approximation for the range of streamwise wavenumber indicated, with the small parameter, ϵ , and effective streamwise wavenumber, γ , fixed at 0.1 and 0.591 respectively.

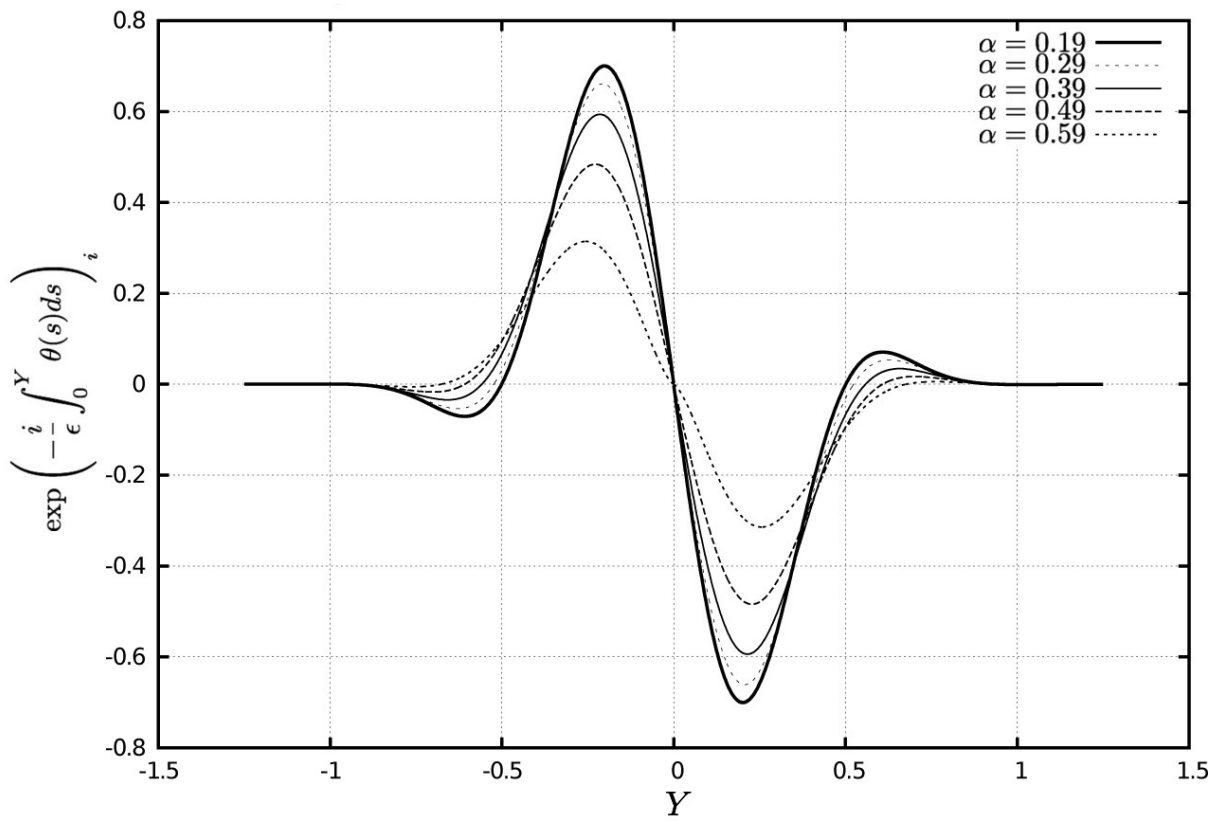


Figure 3.10: Imaginary part of WKBJ exponential approximation for the range of streamwise wavenumber indicated, with the small parameter, ϵ , and effective streamwise wavenumber, γ , fixed at 0.1 and 0.591 respectively.

hence

$$\begin{pmatrix} F' \\ G' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \alpha^2 + \theta^2 & -\frac{2 \sin(Z)}{Y + \cos(Z) + \frac{\omega_0}{i\alpha}} \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix}, \quad (3.96)$$

from a form of (3.77).

We define a grid in the Z and Y directions by

$$Z_j = \left(\frac{2j}{N} - 1 \right) \pi, \quad j = 0, 1, \dots, N, \quad (3.97)$$

$$Y_i = i\delta Y, \quad i = 0, 1, \dots, M. \quad (3.98)$$

To integrate the system (3.96) over the spanwise domain we used a Runge-Kutta fourth order integration scheme with $Y = Y_i$ (initially fixed at zero to make use of the two dimensional problem and then use a step-by-step algorithm to obtain the three dimensional solutions). Since we are implementing periodic conditions at $Z = \pm\pi$ we select an initial condition at $Z = -\pi$,

$$\begin{pmatrix} F_{00} \\ G_{00} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (3.99)$$

and apply the Runge-Kutta scheme to estimate a solution at $Z = \pi$:

$$\begin{pmatrix} F_{0N} \\ G_{0N} \end{pmatrix}. \quad (3.100)$$

We can then consider the difference between each solution at either end of the domain

$$\begin{pmatrix} F_{0N} - F_{00} \\ G_{0N} - G_{00} \end{pmatrix}, \quad (3.101)$$

for an initial θ or ω . We then use a second basis as our initial condition

$$\begin{pmatrix} F_{10} \\ G_{10} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad (3.102)$$

and repeat the above to form a similar vector to (3.101) and combining both basis' we have

the matrix

$$A_{RK4} \Big|_{\theta, \omega} = \begin{pmatrix} F_{0N} - F_{00} & F_{1N} - F_{10} \\ G_{0N} - G_{00} & G_{1N} - G_{10} \end{pmatrix}. \quad (3.103)$$

We now aim to drive $\det(A_{RK4})$ to zero, which will force the solution to be periodic as required. To do this we can, for example, perturb ω_0 by $\delta\omega_0$ and keep θ fixed to acquire a second matrix:

$$A_{RK4} \Big|_{\theta, \omega_0 + \delta\omega_0} = \begin{pmatrix} F_{1N} - F_{00} & F_{1N} - F_{10} \\ G_{0N} - G_{00} & G_{1N} - G_{10} \end{pmatrix}. \quad (3.104)$$

Using the same basis from above we can perform a Newton iteration on ω as follows

$$\omega_{NEW} = \omega_0 - \frac{\det(A_{RK4}(\theta, \omega_0))\delta\omega_0}{\det(A_{RK4}(\theta, \omega_0)) - \det(A_{RK4}(\theta, \omega_0 + \delta\omega_0))}. \quad (3.105)$$

This iteration is performed until

$$\left| \frac{\det(A_{RK4}(\theta, \omega_0))\delta\omega_0}{\det(A_{RK4}(\theta, \omega_0)) - \det(A_{RK4}(\theta, \omega_0 + \delta\omega_0))} \right| \rightarrow 0, \quad (3.106)$$

since these are complex valued determinants we use the absolute value. Alternatively, the same procedure can be performed for a perturbation in θ of $\delta\theta$ and ω_0 fixed.

We wish to solve this problem for $\theta(\alpha, Y)$ and so we begin by calculating our initial values of θ_0 and ω_0 seen in (3.52) and (3.50) at $Y = 0$ then we may begin a Newton iteration on θ whilst varying Y for fixed α . Using these Y calculations we can form the $Q(Y)$ (see (3.85)) for various values of ω_1 . Using finite differences for the derivative and trapezium rule for the integrands we then directly calculate the amplitude function using (3.86). Combining the amplitude functions with the eigenfunctions attained from the Runge-Kutta integration, Newton method and WKBJ approximations for the normal wave number we can obtain full solutions for our disturbance pressure. To calculate precise solutions such that $p \rightarrow 0$ as $Y \rightarrow \infty$, for $\epsilon = 0.1$, we found it was necessary to take $\delta Y = 5 \times 10^{-4}$ and $M = 4000$ with $N = 1000$.

3.6 Results and discussion

In this section we will outline and discuss the results that we have calculated. Initial calculations were performed to validate the code and problem description, for this purpose the two dimensional problem seen in Green (1974) was used. We first refer to figure 3.1 taken from the Green (1974) paper, which gives the growth rate from a similar ordinary differential equation for $Re \rightarrow \infty$. When calculating the growth rate for (3.49) using the numerical method outlined in the previous section we see that we have a qualitatively similar growth rate to Green. Using our maximum growth rate, $F(\gamma)$, taken from the two dimensional problem we can fix α or γ to directly calculate ω_0 from (3.52), we validate the numerical calculation by inputting $\theta = \sqrt{\gamma^2 - \alpha^2}$ on $Y = 0$ and fixing either γ or α then comparing the predicted ω_0 with the numerical one. This comparison is seen in figures 3.4 and 3.5 with error in the order of 10^{-7} . Hence we can correctly calculate (either numerically or from our derived expression) ω_0 using the value of θ at $Y = 0$, as long as $\alpha \neq \gamma$ to initiate the non-zero Y calculations. We fixed the value $\gamma = 0.591$ since it is the effective streamwise wavenumber for which we have the maximum growth rate and hence the most dangerous modes and solutions will be found in this vicinity. We chose the value $\alpha = 0.49$ for the streamwise wavenumber in the calculations that required it so that we avoided the turning point but still had solutions near this dangerous mode. Also we fixed our small parameter value $\epsilon = 0.1$ throughout all numerical calculations.

These solutions were created by solving the eigenvalue problem (3.77) to acquire θ and $\hat{p}_0$. This allowed us to calculate the amplitude function (3.86) and numerically calculate the WKBJ approximation (3.4.1). We can put all this together to form (3.94) and once normalised we calculate the absolute values and show these results in the contour plots seen in figure 3.11 onwards.

The main features that we can see in the results (figures 3.11-3.17) are that for our initial value ($\omega_1 = 0.0$) we have two columns of vortices that are contained within a vertical layer that is approximately of width 1. In the Y -direction the amplitude function along with the WKBJ approximation is forcing the damping of the pressure field far from the Y centreline

which refers to $Y = 0$, as required from the far field conditions. In the Z -direction we have the (absolute) sinusoidal and periodic solutions of the wave as predicted in (3.37), we also note the apparent symmetry in the $Z = 0$ line.

Figure 3.11 shows the pressure fields for ω_1 decreasing from 0 to -2.5 . Over this range of ω_1 we can see the pairs beginning to contract towards the centreline. As ω_1 continues decreasing to a large negative value, figure 3.12, the vortices collapse towards the centreline but do not disappear, so that by $\omega_1 = -20$ the layer of confinement has decreased by almost 80%.

Figures 3.13-3.16 show the pressure fields for ω_1 increasing firstly from 0 to 11.5. In figure 3.17 we compare the main states we notice over the total calculated range of positive growth rates (0 to 20). From $\omega_1 = 0$ to $\omega_1 = 2.0$ we notice that the vortices grow away from the centreline and especially at $\omega_1 = 2.5$ the vortex centres begin to be disturbed by the variation in Y . Increasing ω_1 further leads to this centre splitting in two (see figure 3.14 (b)) and this oscillates through the single row of vortices until approximately $\omega_1 = 11$, where we now have two rows and hence four vortices in a square arrangement. The lower pair are stronger in magnitude than the top pair and appear to remain that way. The wave that propagates in the Z direction can be seen by the offset centres contained within this period box, as expected from the inviscid wavy mode that leads to transition to turbulence. After splitting, the two pairs continue moving away from the centreline for larger ω_1 , each row is contained within two sub layers centred at $Y = \pm 0.5$ but both rows are still contained within $Y = \pm 0.85$. These two rows continue to concentrate around their new centres as ω_1 increases to larger values (figure 3.17).

Figures 3.11-3.17 all demonstrate how the amplitude function decays as $Y \rightarrow \pm\infty$ for all streamwise wavenumber but in figure 3.18 we see that as $\alpha \rightarrow \gamma$ the amplitude function blows up around the origin for $\alpha = 0.59$, implying a turning point. Figure 3.19 shows how the eigenfunction changes as

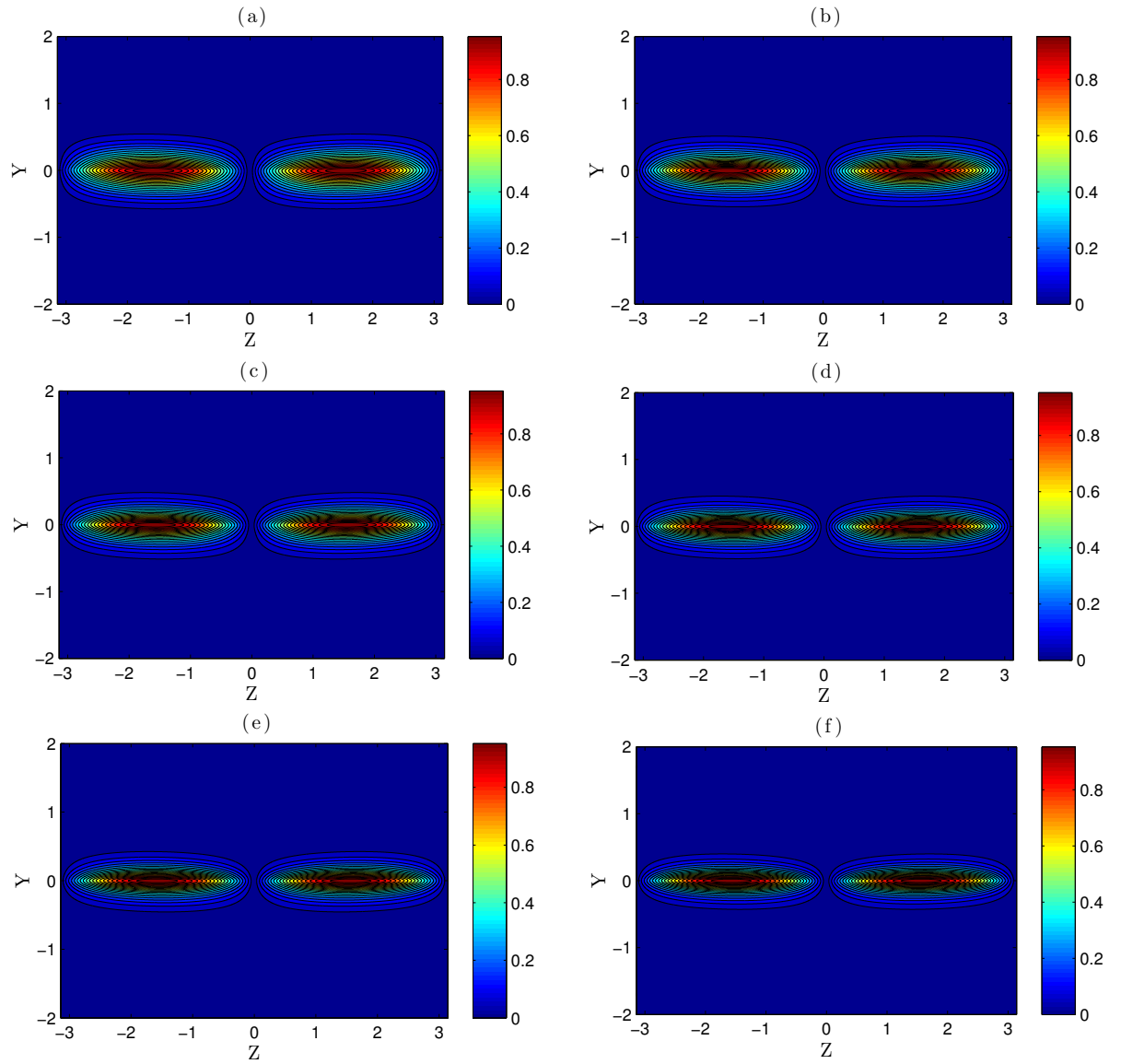


Figure 3.11: Absolute part of eigenfunctions for streamwise wavenumber of $\alpha = 0.49$ and growth rate correction values, ω_1 , of (a) 0.0, (b) -0.5 , (c) -1.0 , (d) -1.5 , (e) -2.0 and (f) -2.5 .

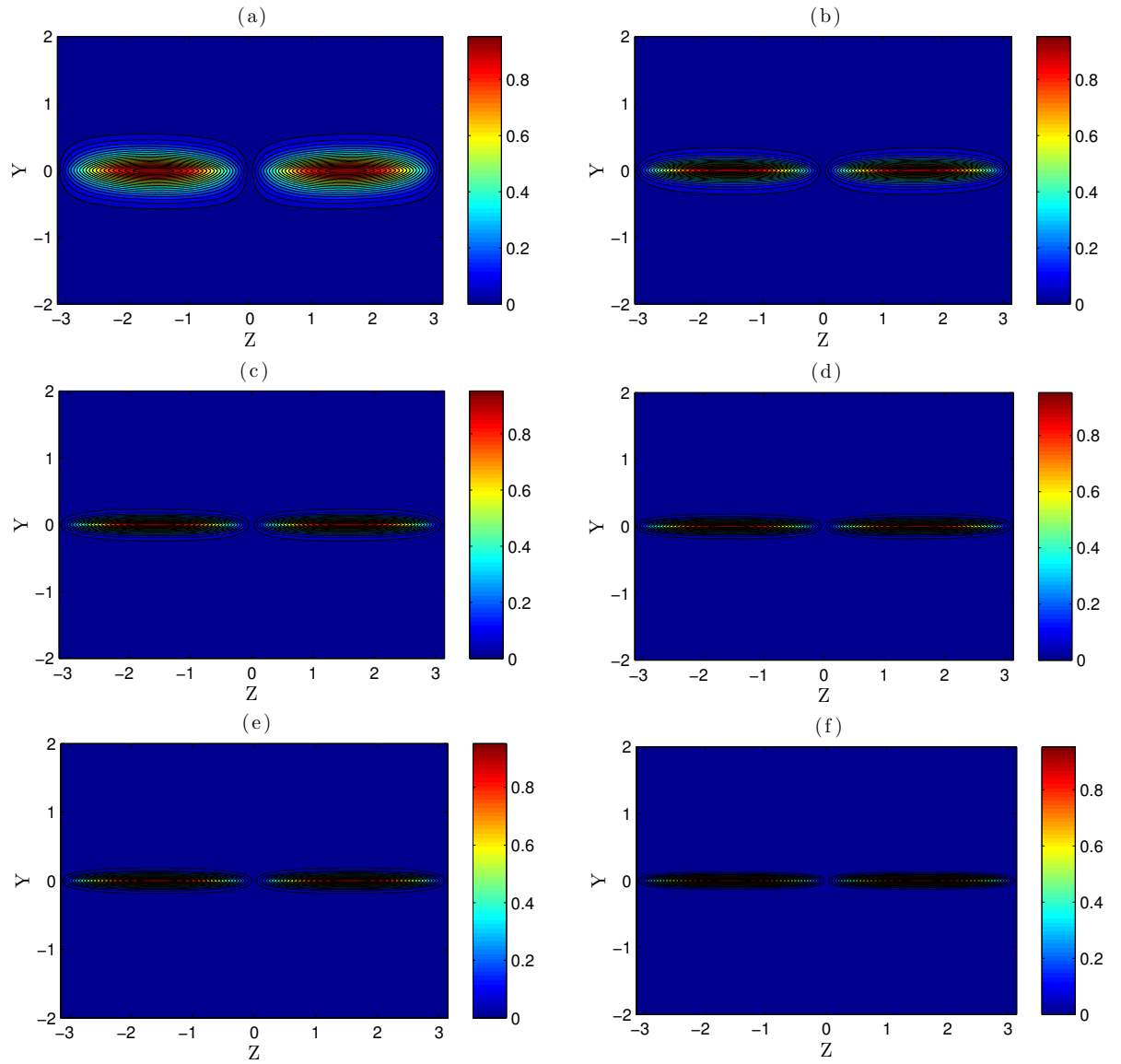


Figure 3.12: Absolute part of eigenfunctions for streamwise wavenumber of $\alpha = 0.49$ and growth rate correction values, ω_1 , of (a) 0.0, (b) -4.0, (c) -8.0, (d) -12.0, (e) -16.0 and (f) -20.0.

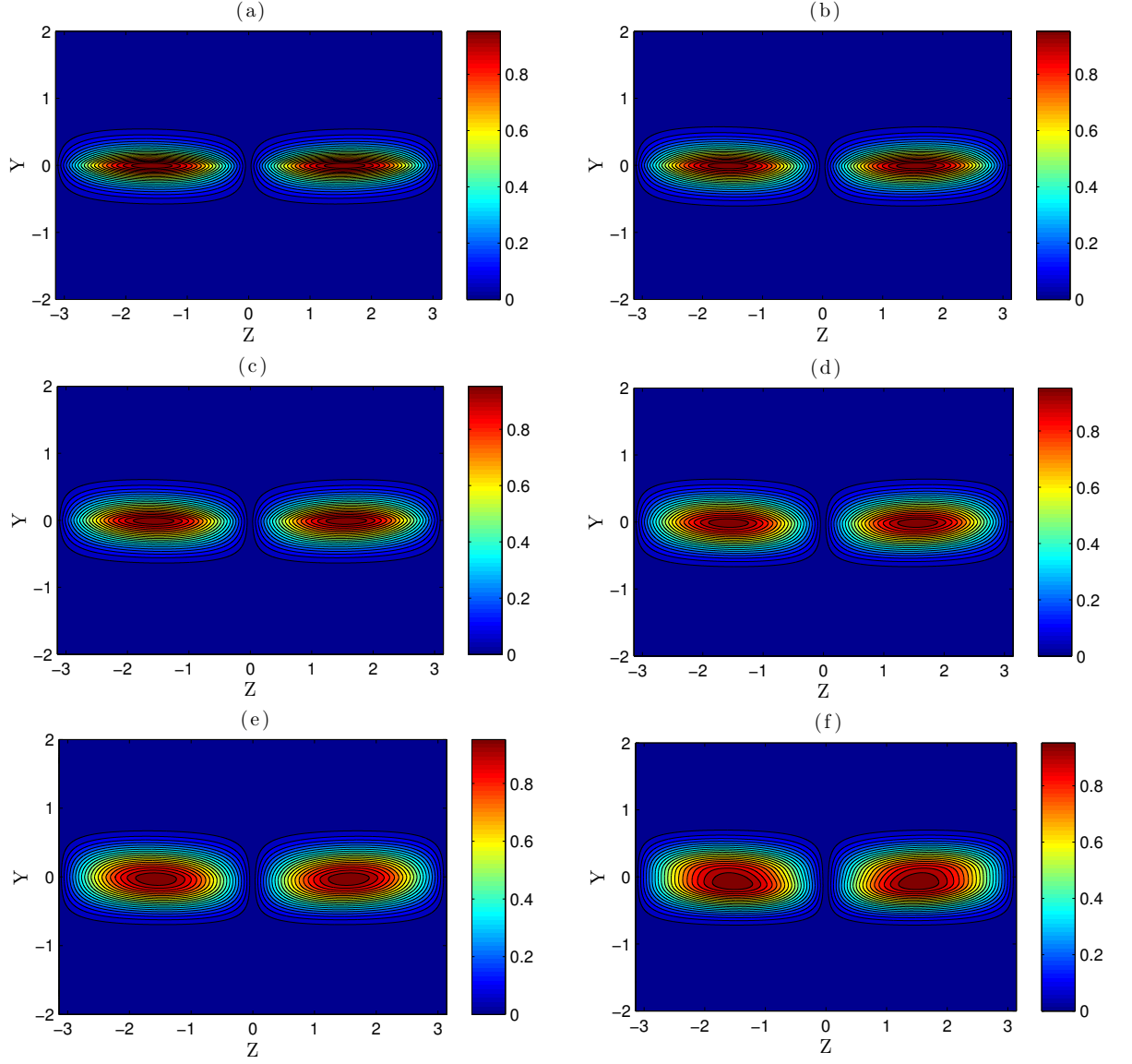


Figure 3.13: Absolute part of eigenfunctions for streamwise wavenumber of $\alpha = 0.49$ and growth rate correction values, ω_1 , of (a) 0.0, (b) 0.5, (c) 1.0, (d) 1.5, (e) 2.0 and (f) 2.5.

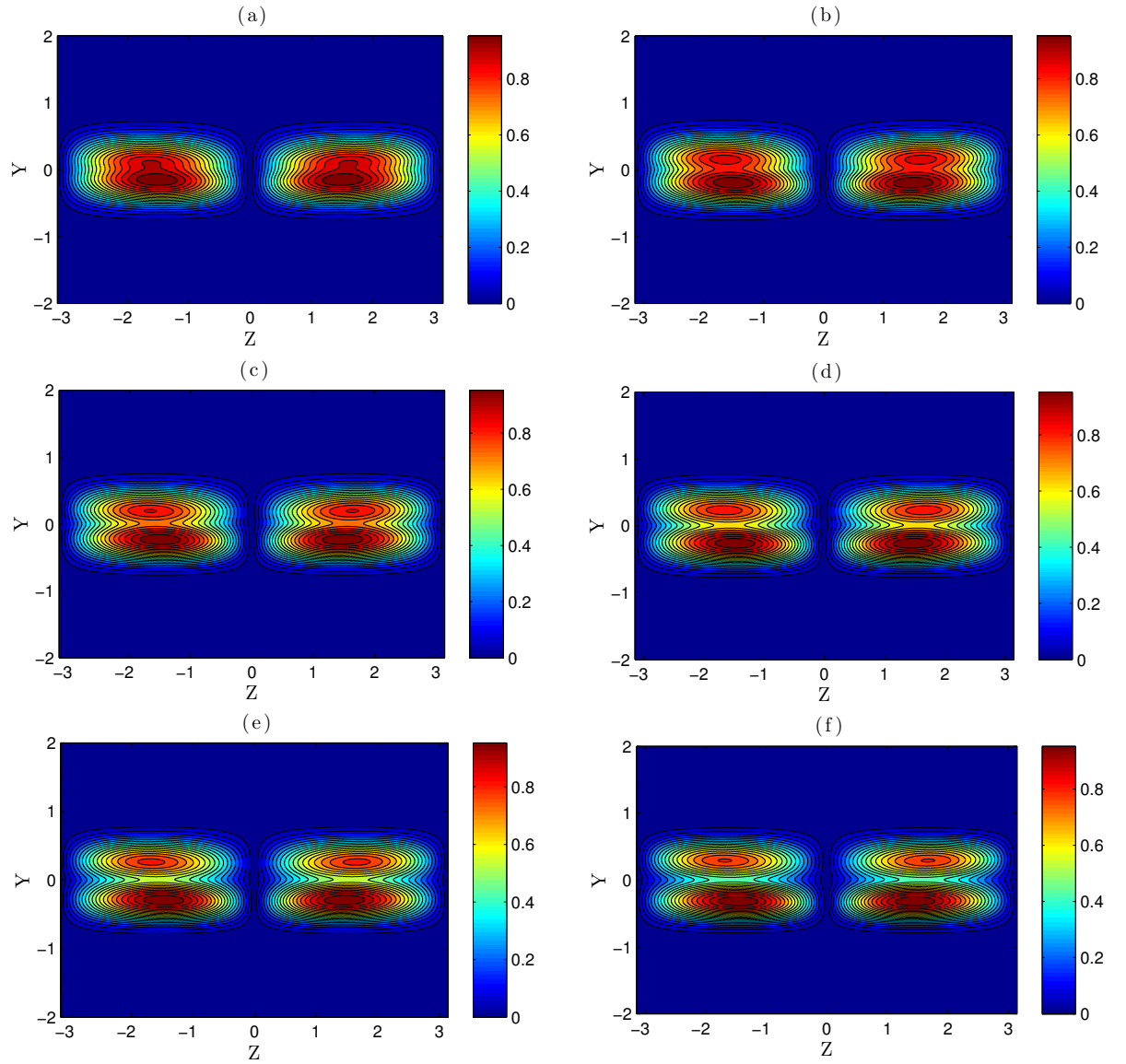


Figure 3.14: Absolute part of eigenfunctions for streamwise wavenumber of $\alpha = 0.49$ and growth rate correction values, ω_1 , of 3.0 (a), 4.0 (b), 4.5 (c), 4.5 (d), 5.0 (e) and 5.5.

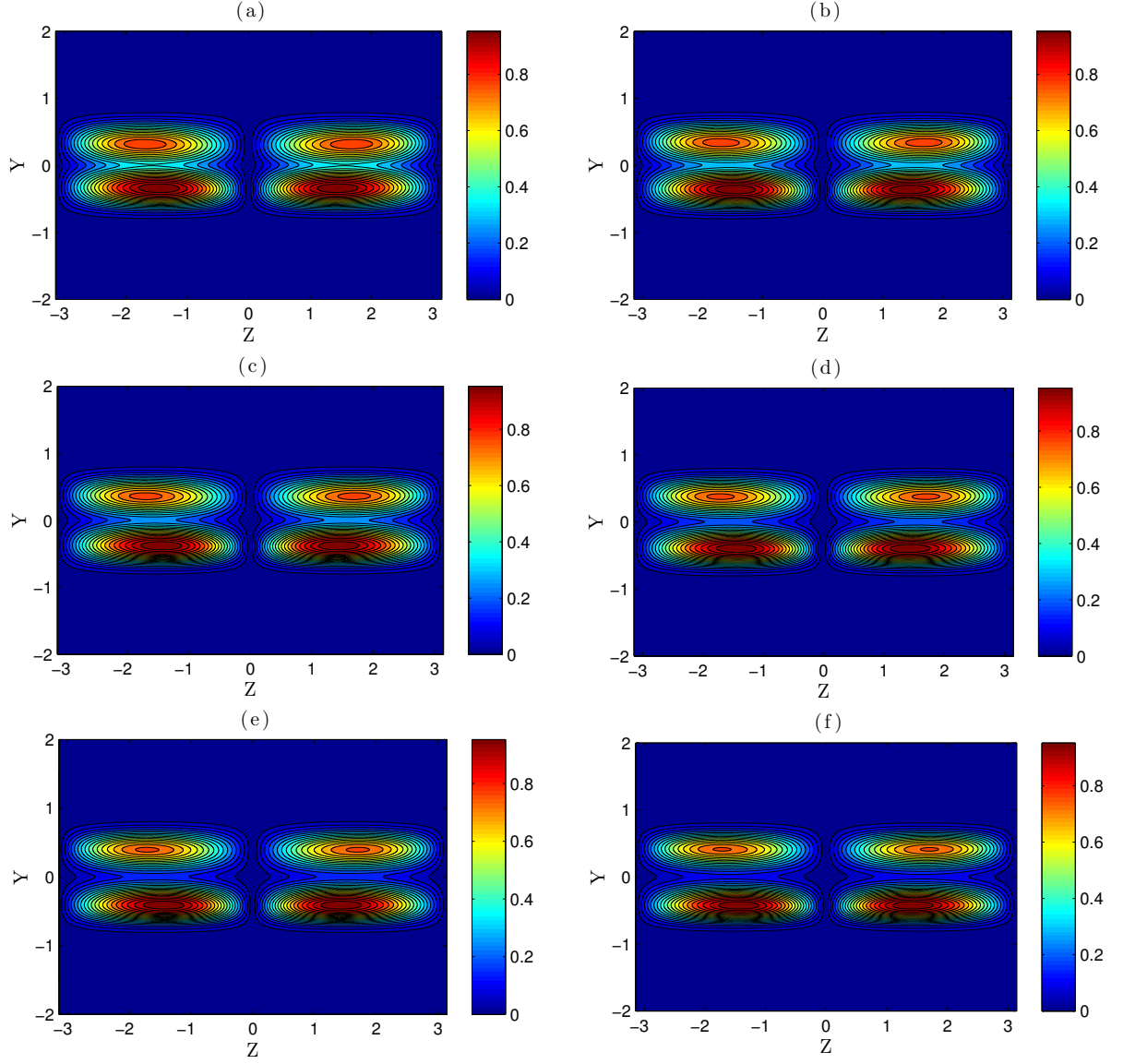


Figure 3.15: Absolute part of eigenfunctions for streamwise wavenumber of $\alpha = 0.49$ and growth rate correction values, ω_1 , of (a) 6.0, (b) 6.5, (c) 7.0, (d) 7.5, (e) 8.0 and (f) 8.5.

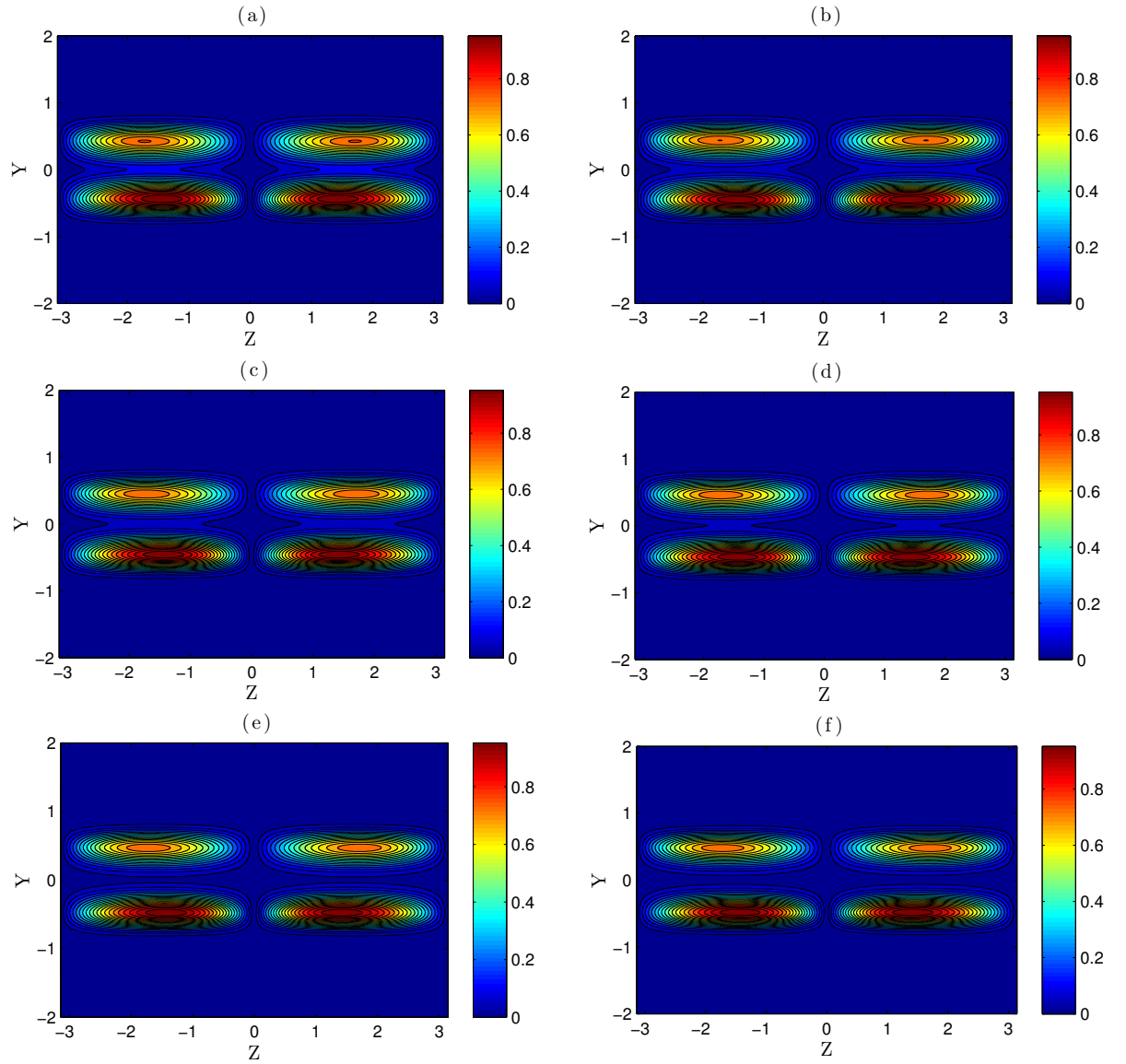


Figure 3.16: Absolute part of eigenfunctions for streamwise wavenumber of $\alpha = 0.49$ and growth rate correction values, ω_1 , of (a) 9.0, (b) 9.5, (c) 10.0, (d) 10.5, (e) 11.0 and (f) 11.5.

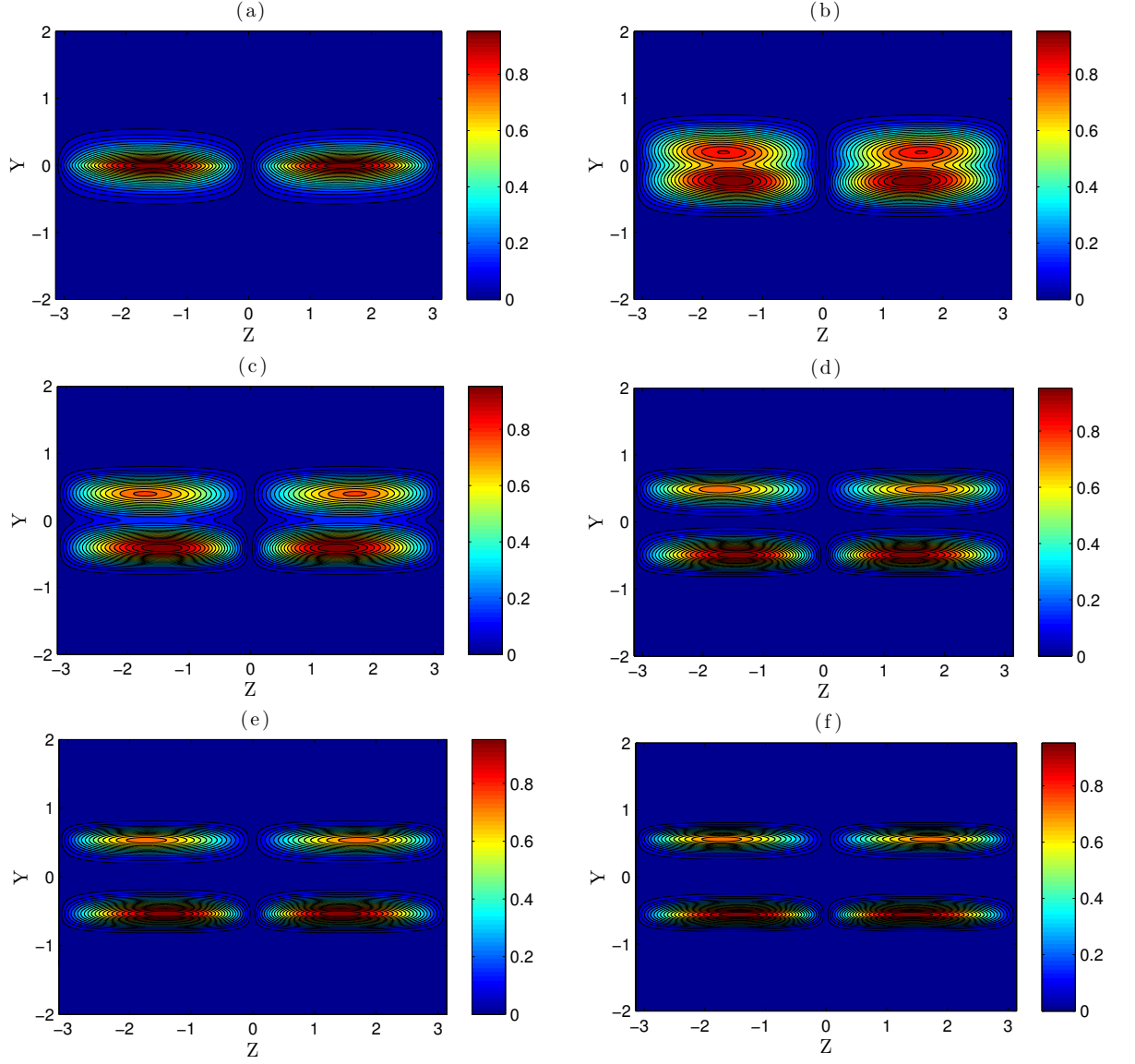


Figure 3.17: Absolute part of eigenfunctions for streamwise wavenumber of $\alpha = 0.49$ and growth rate correction values, ω_1 , of (a) 0.0, (b) 4.0, (c) 8.0, (d) 12.0, (e) 16.0 and (f) 20.0.

α is increased towards γ and demonstrates the expected variation from the flat mode to the wavy-mode. At first the solution starts to contract towards $Y = 0$ and then the amplitude function blows up (seen in figures 3.18 for $\alpha = 0.59$ and 3.19(e)) by which we mean that the solution is not valid and the WKBJ approximation falls down in this region. This could also be validated by looking at the un normalised functions and one would see that the maximum values for these functions would become unrealistic. When combining the amplitude function with the exponential one we obtain the full solution which will also break down as $\alpha \rightarrow \gamma$, seen explicitly in 3.19(e) for a fixed growth rate ($\omega_1 = 0$). Whilst several expansions have been investigated here as yet we have been unable to connect the $\alpha < \gamma$ and $\alpha > \gamma$ regions.

Some final validation of our results can be seen by comparison with the work of Hall and Sherwin (2010), which looked at the structures seen in streamwise vortices in shear flows. Figure 3.20 shows their results for two values of the streamwise wavenumber. We can see the real and imaginary parts of the pressure perturbations at the neutral stability point ($c = 0$ in Hall and Sherwin (2010) and $\omega = 0$ or $\omega_1 = -\omega_0/\epsilon$ in our problem). Specifically Hall and Sherwin (2010) completed the calculations for a Couette flow superimposed over a spanwise periodic base flow, not dissimilar to the flow used here. When comparing with similar calculations for the real and imaginary parts of q (figure 3.21) we must be careful due to the differing normalisations, streamwise wavenumber values and base flows. Noting immediately that we can only truly compare figure 3.20 (a) with figure 3.21 (c) due to their similar α value. Taking all this into account there are a lot of qualitative similarities. The real parts appears to keep the single vortex structure in general and the imaginary parts have two opposite strength vortices that are symmetric over a diagonal in both figures. Some notable differences come in the shape of the vortices and distortion for different α values that we see in 3.21. Nonetheless, an interesting and good comparison with the vortex-wave interaction approach seen in Hall and Sherwin (2010).

To conclude, we have been able to compute complex solutions to the Rayleigh stability equation for pressure within region 1 described by Hall and Lakin (1988) which has a sheared, spanwise periodic base flow in a slowly varying

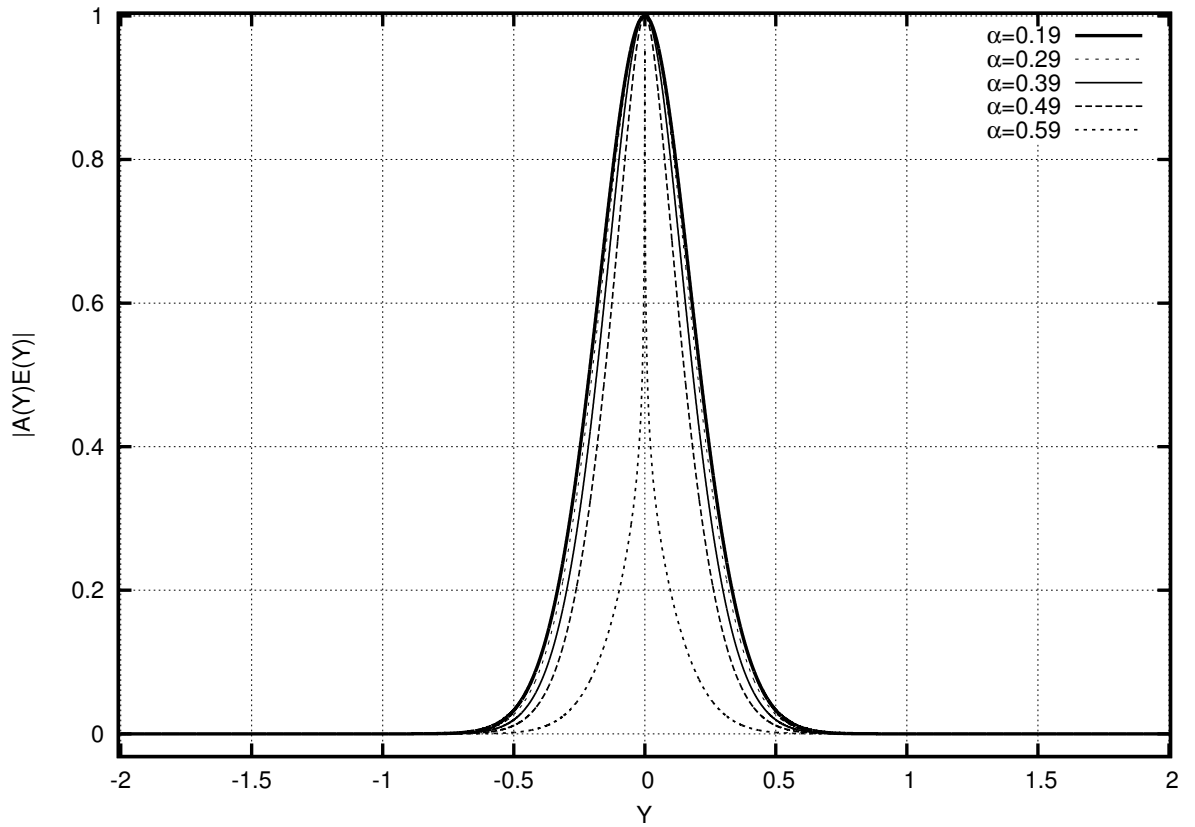


Figure 3.18: Absolute value of amplitude function multiplied with the WKB approximation for growth rate correction, ω_1 , fixed at zero with various streamwise wavenumber, α , approaching the fixed effective two dimensional streamwise wavenumber, $\gamma = 0.591$.

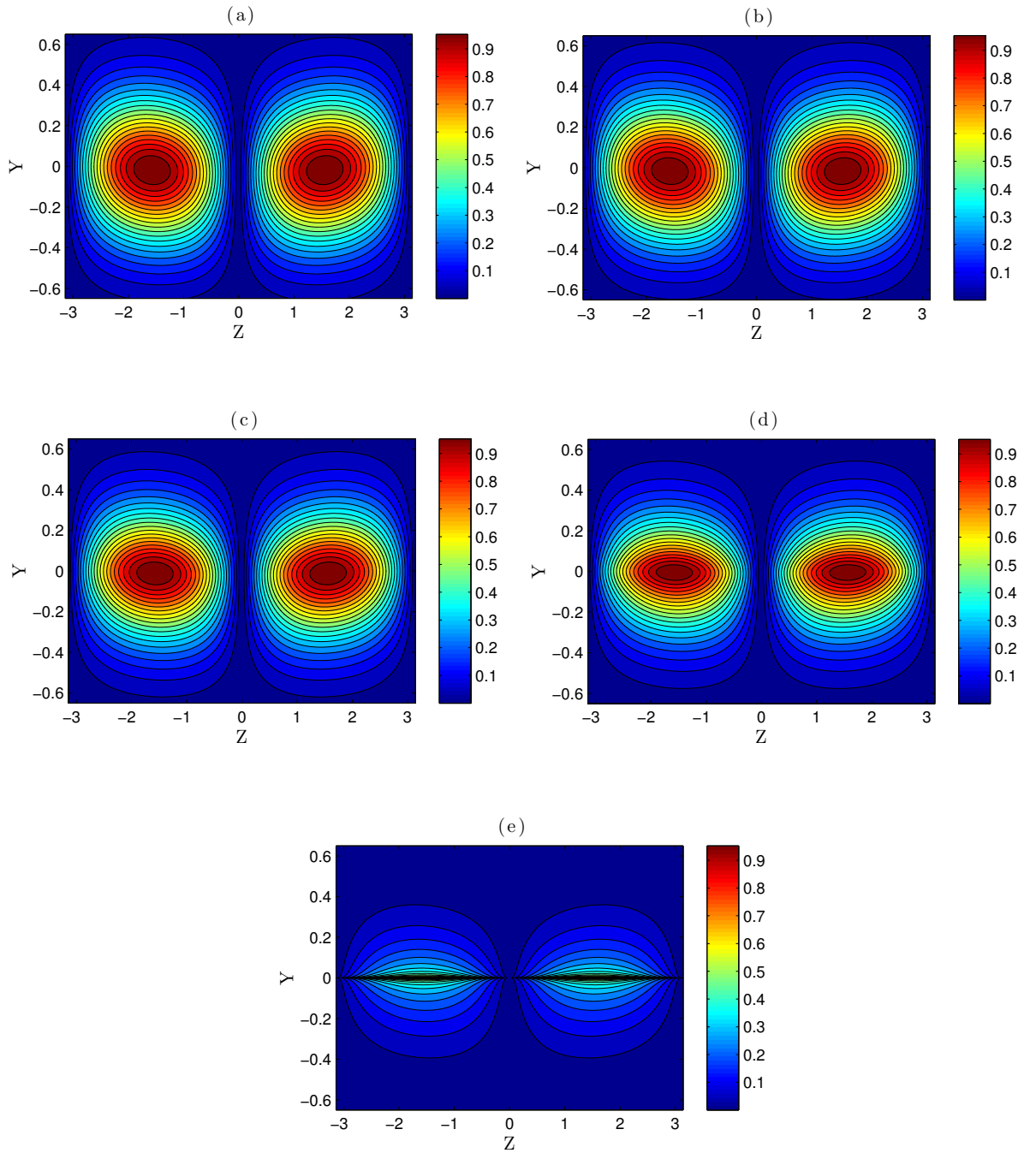


Figure 3.19: Absolute part of eigenfunctions for growth rate correction, ω_1 , fixed at zero with streamwise wavenumber, α , (a) 0.19, (b) 0.29, (c) 0.39, (d) 0.49 and (e) 0.59.

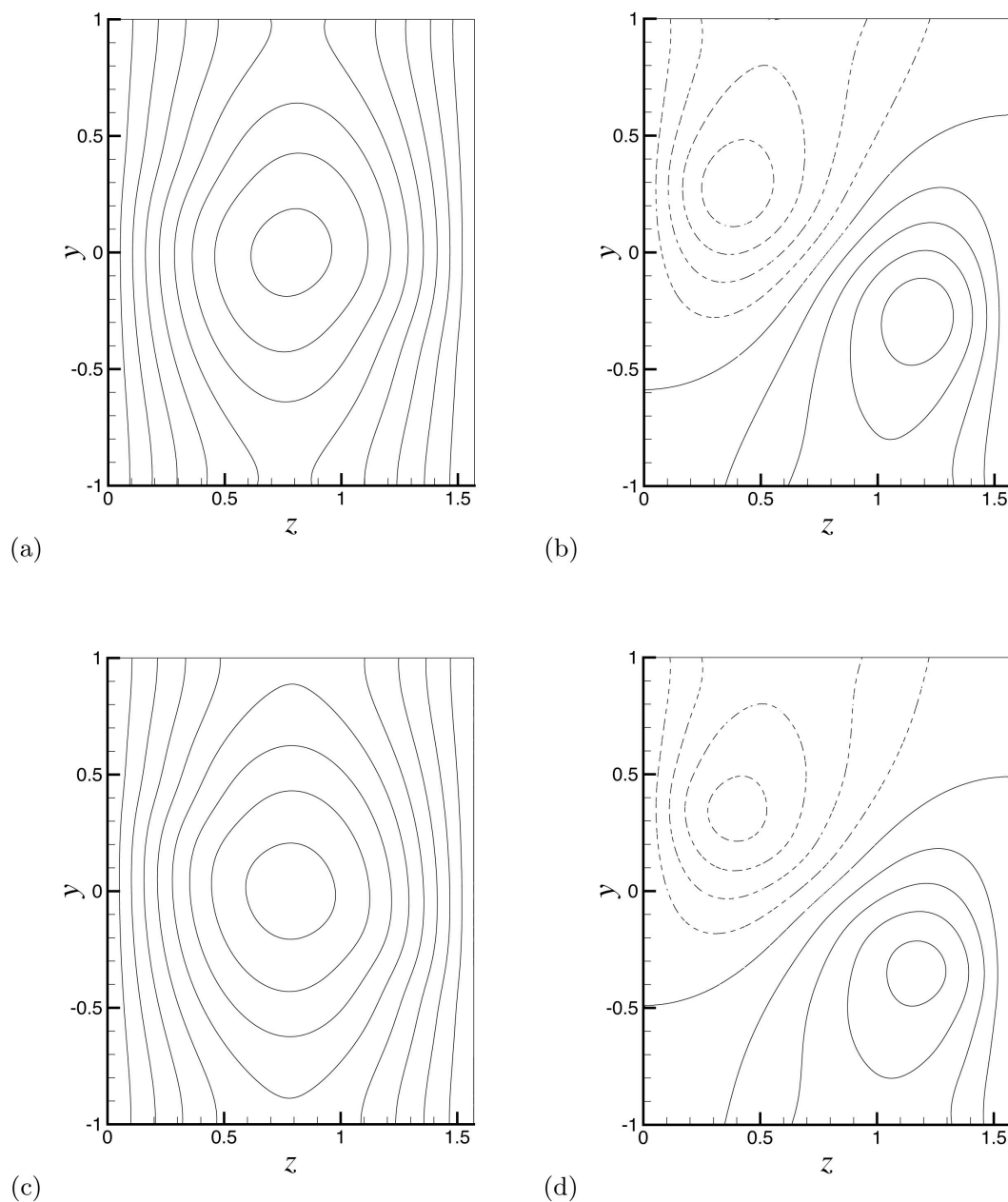


Figure 3.20: Isocontours for pressure perturbations at neutral point (wave speed $c = 0$) at $\alpha = 0.4$ ((a) real and (b) imaginary) , and for $\alpha = 1.0$ ((c) real and (d) imaginary). Scanned from Hall and Sherwin (2010)

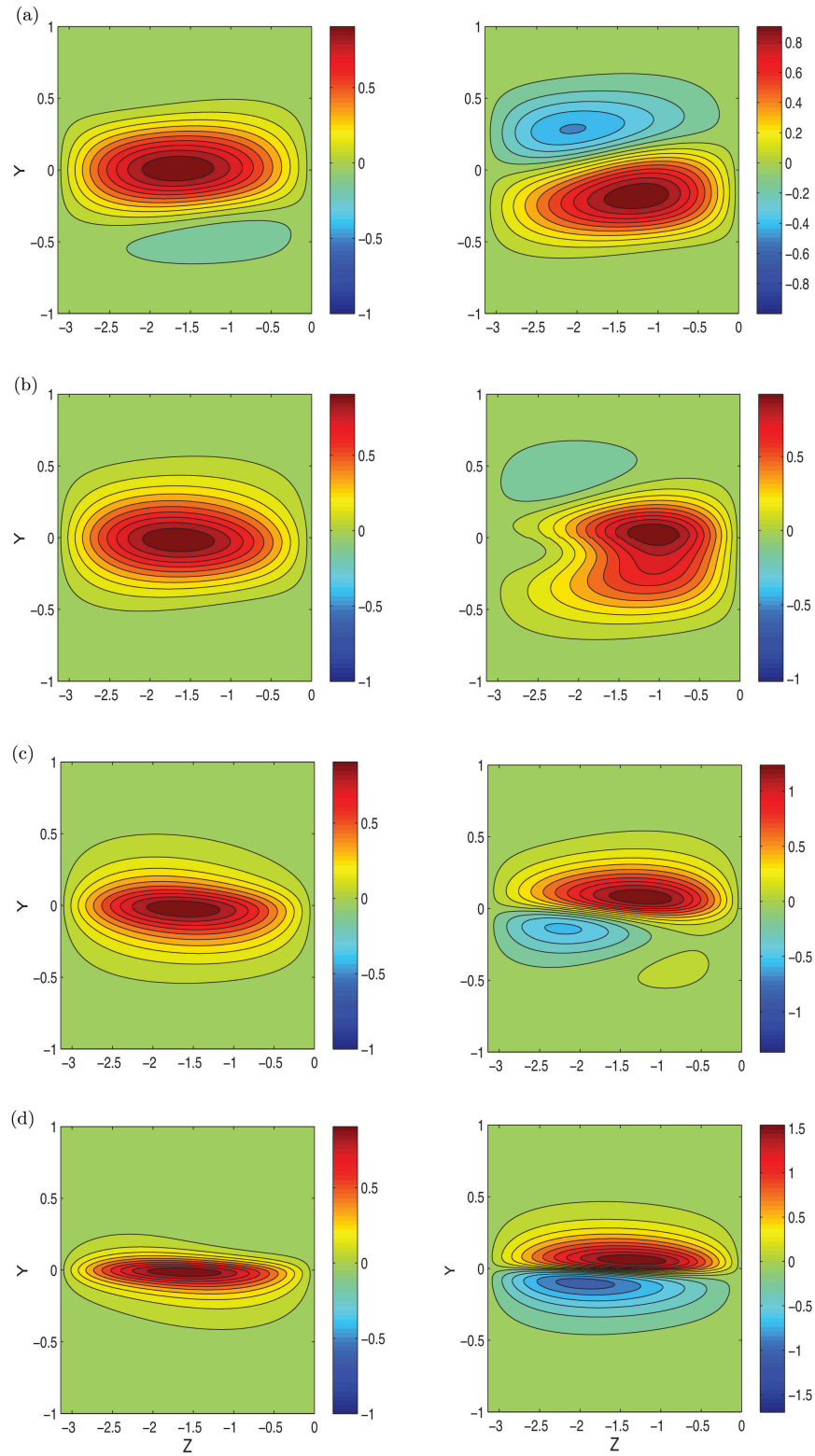


Figure 3.21: Real and imaginary contour plots of pressure perturbations at neutral point $\omega_1 = -\omega_0/\epsilon$ for streamwise wavenumber: (a) $\alpha = 0.19$, (b) $\alpha = 0.29$, (c) $\alpha = 0.39$ and (d) $\alpha = 0.49$. The left column is the real part and the right column the imaginary.

medium that meant we could take advantage of the WKBJ approximation in our solutions. We acquired solutions to the problem for a certain set of streamwise wavenumber (due to the breakdown of the WKBJ approximations near to the effective streamwise wavenumber) for this inviscid secondary instability, which are qualitatively similar to some of the solutions for odd modes given by Hall and Horseman (1991) and Hall and Sherwin (2010). We have solved our Rayleigh stability equation (3.45) by applying the WKBJ approximation to obtain results for a slowly varying medium for which we can say that these solutions have some relevance to a wide class of shear flows.

Chapter 4

Conclusions and future work

We have investigated both the receptivity and stability of the boundary-layer due to vibrations of the wing surface. Firstly we focused on the receptivity and generation of T-S waves in the boundary-layer over the wing in chapter 2. By assuming that the Reynolds number is large we studied the flow with the aid of asymptotic analysis of the Navier-Stokes equations. We also assumed that the wing vibrations were resonant with T-S waves frequency, $Re^{1/4}$. Our other assumptions were that the high frequency elastic vibrations have the wavelength of δ_x and that Mach number was larger than $Re^{-1/4}$. The latter condition meant we could use piston theory to model the wings vibrations this led us to discover two flow regimes that depend on the size of the disturbance wavelength, δ_x . As pressure perturbations permeate down to the surface of the wing they create a Stokes' layer and the two mechanisms are characterised by $\delta_x = \mathcal{O}(Re^{-1/8})$ or $\delta_x \gg Re^{-1/8}$ where compressibility effects then become important. Here we took $\delta_x \ll Re^{-1/8}$ so that the Stokes' layer contained all the elastic vibrations and any waves created by it. The vibrations within the Stokes' layer are not capable of producing T-S waves due to the perturbation field within this layer having a much larger wavelength than T-S waves. We then added a wall roughness to the Stokes' layer or streamwise length $Re^{-3/8}$, which means that T-S waves can now be produced. By using triple deck theory we then analysed the interaction of the flow around this roughness and were able to derive analytical forms of the solutions in that

region. We then focussed on the downstream behaviour, solving via a Fourier transform and consequent inversion of the flow and in particular the amplitude of our generated T-S waves. We were able to numerically calculate receptivity coefficients for this first mechanism, which is of a very similar form to that seen in the works of Ruban (1984) and Goldstein (1985).

We must note that the receptivity coefficients are not dependent on this shape or its Fourier transform. Indeed, we have not expressed the explicit shape of or the function that defines our roughness we are able to use any shape: a Gaussian bump, a step or a gap. As long as we could express it's Fourier transform we would be able to calculate the amplitudes of the disturbance downstream. This work set out in Ruban et al. (2013) is the first of its kind and due to the difficult nature of experiments in vibrating frames of reference there is no current experimental comparison that we can make. Whilst previous research has focussed on acoustic, vortical or external disturbances our work suggests that in fact wing vibrations may have a greater effect. Indeed the receptivity coefficients are larger than for example acoustic disturbances seen in Ruban (1984). We could consider adding an angle of attack to the wing. This, as mentioned in the introduction, would introduce cross-flow vortices and together with the vibrations would create a very interesting and yet complicated problem. In an experimental study on the excitation of cross-flow vortices by free stream vortices and surface roughness Borodulin et al. (2013) came to the conclusion that the receptivity of these cross-flow vortices is less than that of the corresponding acoustic-roughness mechanism studied previously. Whilst this result depends on the base flow and disturbance parameters it does suggest that even the addition of angle of attack to the wing may not produce as great an effect as we might wish. Nevertheless we believe that the wing vibrations and roughness element plays a greater role than previously thought.

In Chapter 3 we moved our focus to the boundary-layer further downstream where we might expect an inviscid secondary instability over a curved part of the wing. Hall and Lakin (1988) showed that the structure of a boundary layer over a curved surface after a neutral stability position took on a distinct formation (see figure 1.7). Our work concentrated on the generation of Görtler vortices in the central region of this formation. Our first assumption was that the disturbance would be damped to zero far away from the centre-

line. From the work of Hall and Horseman (1991) we were able to derive the Rayleigh stability equation for a sheared, spanwise periodic base flow as expected in the generation of Görtler vortices. Due to the nature of our problem we assumed that the flow was slowly varying in the y direction and this meant we could use WKBJ solutions to describe the flow in that direction and ultimately investigate this secondary instability. This allowed us to take advantage of the nature of the slowly varying flow and hence obtain at leading order a homogenous local Rayleigh problem for the eigenvalue θ and a non-homogenous version at the next order. Since we had a self-adjoint linear operator we were able to take advantage of this in applying a solvability condition on the order ϵ equation and were able to derive an analytic form of the amplitude equation.

Owing to the difficulty of the subsequent analysis, we proceeded by attempting to solve the leading order equation numerically to obtain the eigenvalue and amplitude functions for the problem. We did this by first using the two dimensional version of our problem and discovered similarities with the work of Green (1974) on eddies in shear flow. We were then able to use the effective two dimensional streamwise wavenumber, γ , to form an initial value of our eigenvalue $\theta(0) = \sqrt{\gamma^2 - \alpha^2}$ from which we could start the full three dimensional calculations. At this stage of the calculation we identified a turning point characterised by the parametric limit that for $\alpha \rightarrow \gamma$ and with $Y = 0$ then $\theta = 0$.

We attempted to address the breakdown of the WBKJ approximation near the turning point by applying the complex method seen in Soward and Jones (1982). However, owing to the structure of the base flow it is not possible to construct normal derivatives which would provide a framework for a global minimisation problem. Alternatively, rather than expanding in the y -direction one could consider a similar asymptotic expansion for the streamwise wavenumber $\alpha = \gamma + \epsilon^n \alpha_1$. With suitable scaling index, n , one could then derive the required parabolic cylinder functions for the eigenfunction around the turning point. This formalisation could might allow us to construct solutions for any streamwise wavenumber and was investigated but not reported here.

We were able to calculate solutions away from the turning point and concentrated on solutions for $\alpha < \gamma$ where we calculated θ and its eigenfunction p . These solutions resolved

the local Rayleigh problem for a certain shear flow, which can be applied to many realistic flows (Hall and Horseman, 1991). Furthermore they show that the wavy-type instability (that form part of the path from laminar to turbulence) can be created from this slowly varying flow over a curved surface with the help of the inviscid secondary Rayleigh type instability.

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Appendix A

Jordan's Lemma

We consider C_R the semi-arc of a closed contour with radius R . Jordan's lemma states that: if the only singularities of $F(z)$ are poles then

$$\lim_{R \rightarrow \infty} \int_{C_R} \exp(imz) F(z) dz = 0, \quad (\text{A.1})$$

provided that $m > 0$ and $|F(z)| \rightarrow 0$ as $R \rightarrow \infty$.

In our case we are interested in the behaviour of

$$\int_{C_R^-} \frac{Ai'(z_0) k \int_{z_0}^{\infty} \varphi_1(z) dz}{Ai'(z_0) - i(ik)^{4/3} \int_{z_0}^{\infty} Ai(z) dz} \bar{G}(k) \exp(ikx_*) dk \quad (\text{A.2})$$

where we have the closed contour C^- which has the semi-arc C_R^- as $k \rightarrow \infty$. Using (2.209) we can see that (A.2) is of the form

$$\int_{C_R^-} \mathcal{M} dk \sim k^{-1/3} \int_{C_R^-} \bar{G}(k) \exp(ikx_*) dk. \quad (\text{A.3})$$

This is where we use Jordan's Lemma, since $x_* > 0$ (considering downstream location)

and from (2.210) we know that $|\bar{G}(k)| \rightarrow 0$ as $R \rightarrow \infty$ hence we can write

$$\lim_{k \rightarrow \infty} \int_{C_R^-} \bar{G}(k) \exp(ikx_*) dk = 0, \quad (\text{A.4})$$

and so the integral along C_R^- tends to zero as the arc radius $R \rightarrow \infty$.

Appendix B

Watson's Lemma

Watson's lemma states that if we an integral of the form

$$F(s) = \int_0^\infty f(t) \exp(-st) dt \quad (\text{B.1})$$

the as $s \rightarrow \infty$ we can express this integral in the form

$$F(s) \sim \sum_{n=0}^N a_n \frac{\Gamma(\alpha + n + 1)}{s^{\alpha+n+1}} + \dots \quad (\text{B.2})$$

where

$$f(t) = t^\alpha \left[\sum_{n=0}^N a_n t^n + R_{n+1}(t) \right] \quad (\text{B.3})$$

around $t \sim 0$.

In our problem instead of t and s we have k and $-ix_*$ respectively. We have the integral

$$\int_{C'_-} \frac{(ik)^{10/3}}{3i\omega Ai'(0)} \bar{G}(0) \exp ikx_*. \quad (\text{B.4})$$

We can then write the leading order summation term of the function under the integral

as

$$a_0 = \frac{i^{7/3}}{3\omega Ai'(0)} \bar{G}(0) \quad (\text{B.5})$$

and set $\alpha = 10/3$, hence for small k we can use the asymptotic expansion

$$f(k) = k^{10/3} \left[\frac{i^{7/3}}{3\omega Ai'(0)} \bar{G}(0) + \dots \right]. \quad (\text{B.6})$$

By then applying (B.2) we have

$$\int_{C'_-} \mathcal{M} \sim -\frac{(1-i\sqrt{3})}{6\omega Ai'(0)} \Gamma\left(\frac{13}{3}\right) \bar{G}(0) \frac{1}{x_*^{13/3}} + \dots \quad (\text{B.7})$$

Appendix C

Zoomed in eigenfunctions from chapter 3

Here we have set out the eigenfunctions calculated in chapter 3 but zoomed into the vortices themselves. Giving more insight into the results and discussion laid out in the chapter.

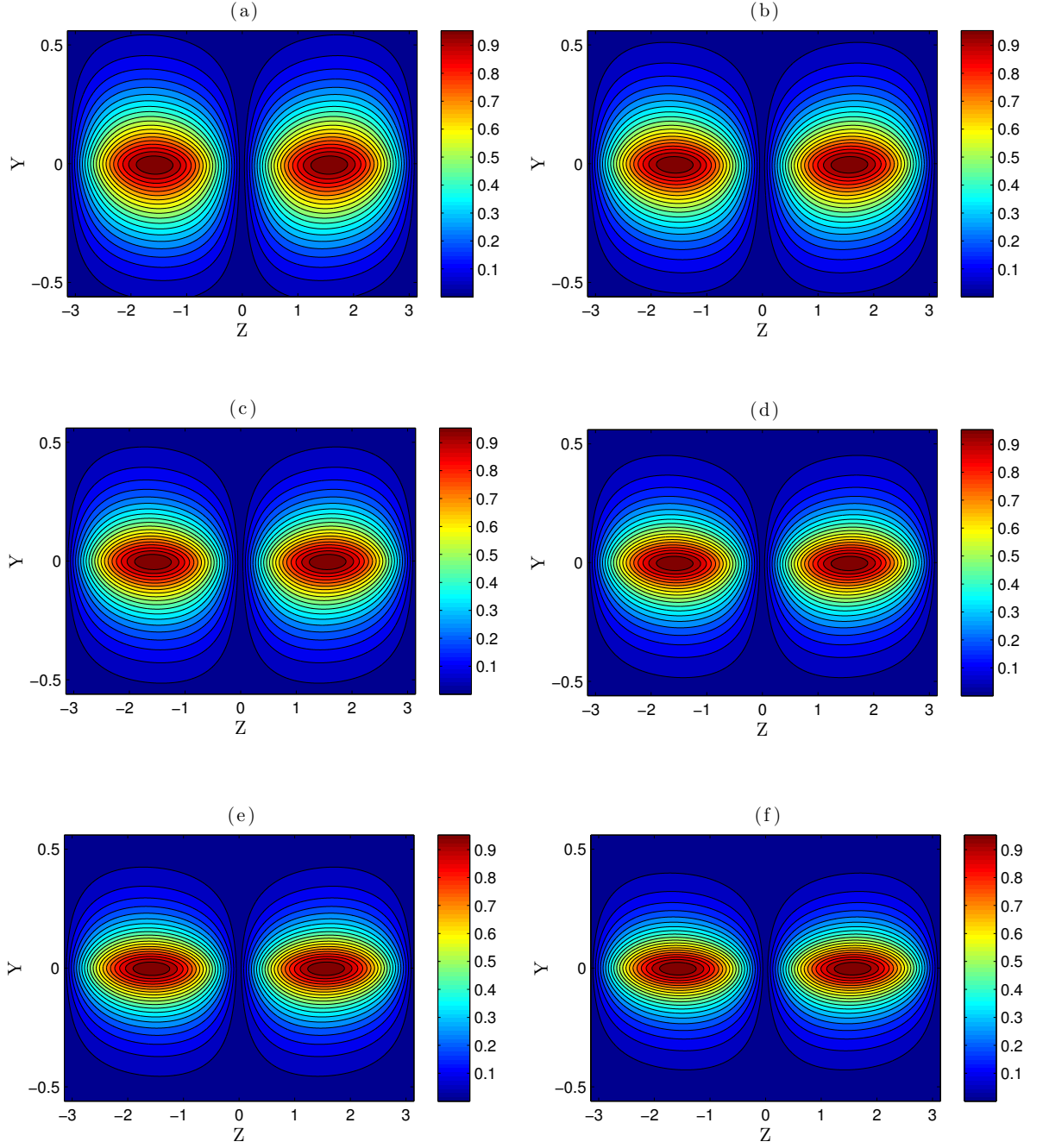


Figure C.1: Absolute part of eigenfunctions for streamwise wavenumber, α , of 0.49 and growth rate correction values, ω_1 , of (a) 0.0, (b) -0.5 , (c) -1.0 , (d) -1.5 , (e) -2.0 and (f) -2.5 .

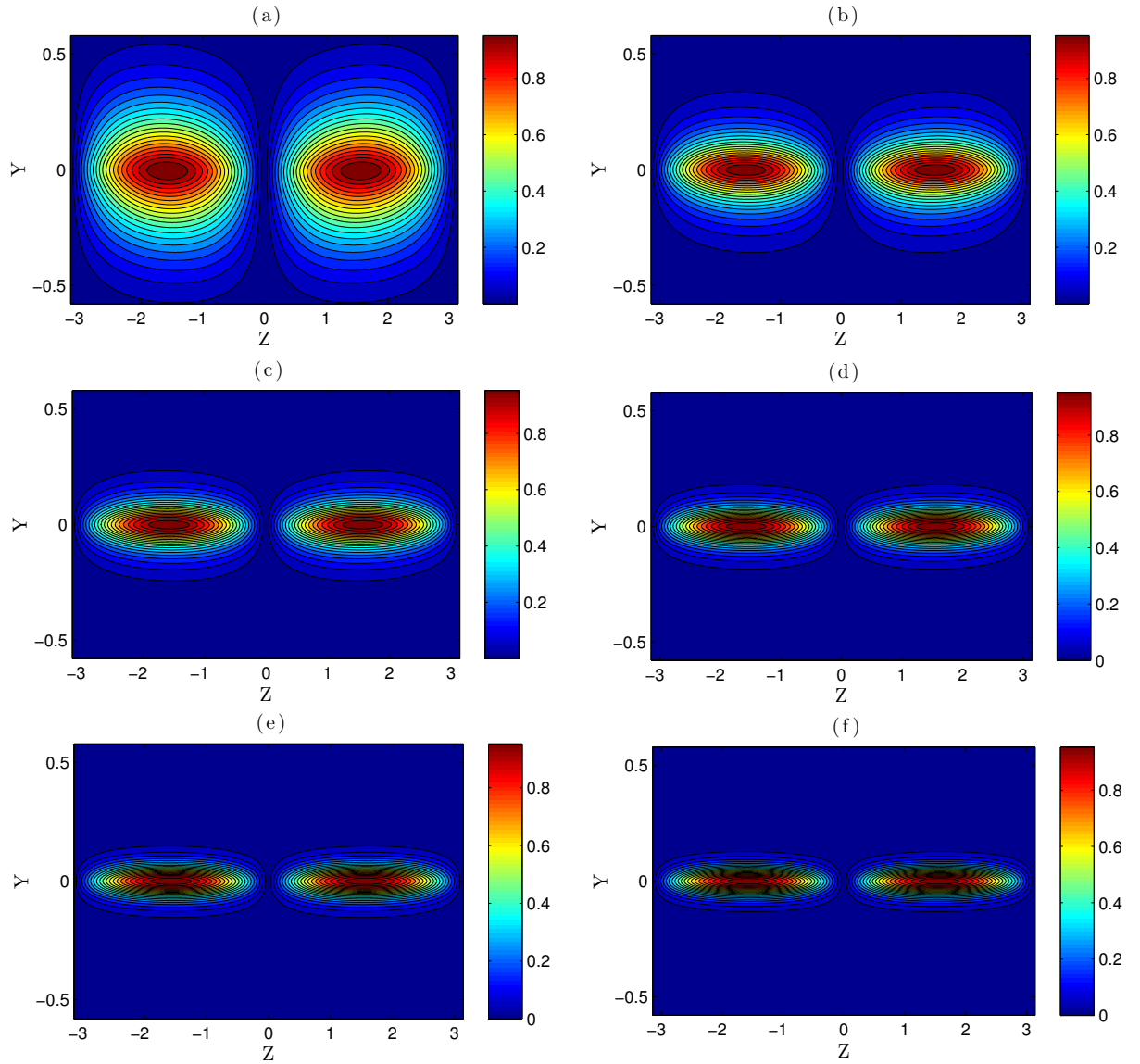


Figure C.2: Absolute part of eigenfunctions for streamwise wavenumber, α , of 0.49 and growth rate correction values, ω_1 , of (a) 0.0, (b) -4.0, (c) -8.0, (d) -12.0, (e) -16.0 and (f) -20.0.

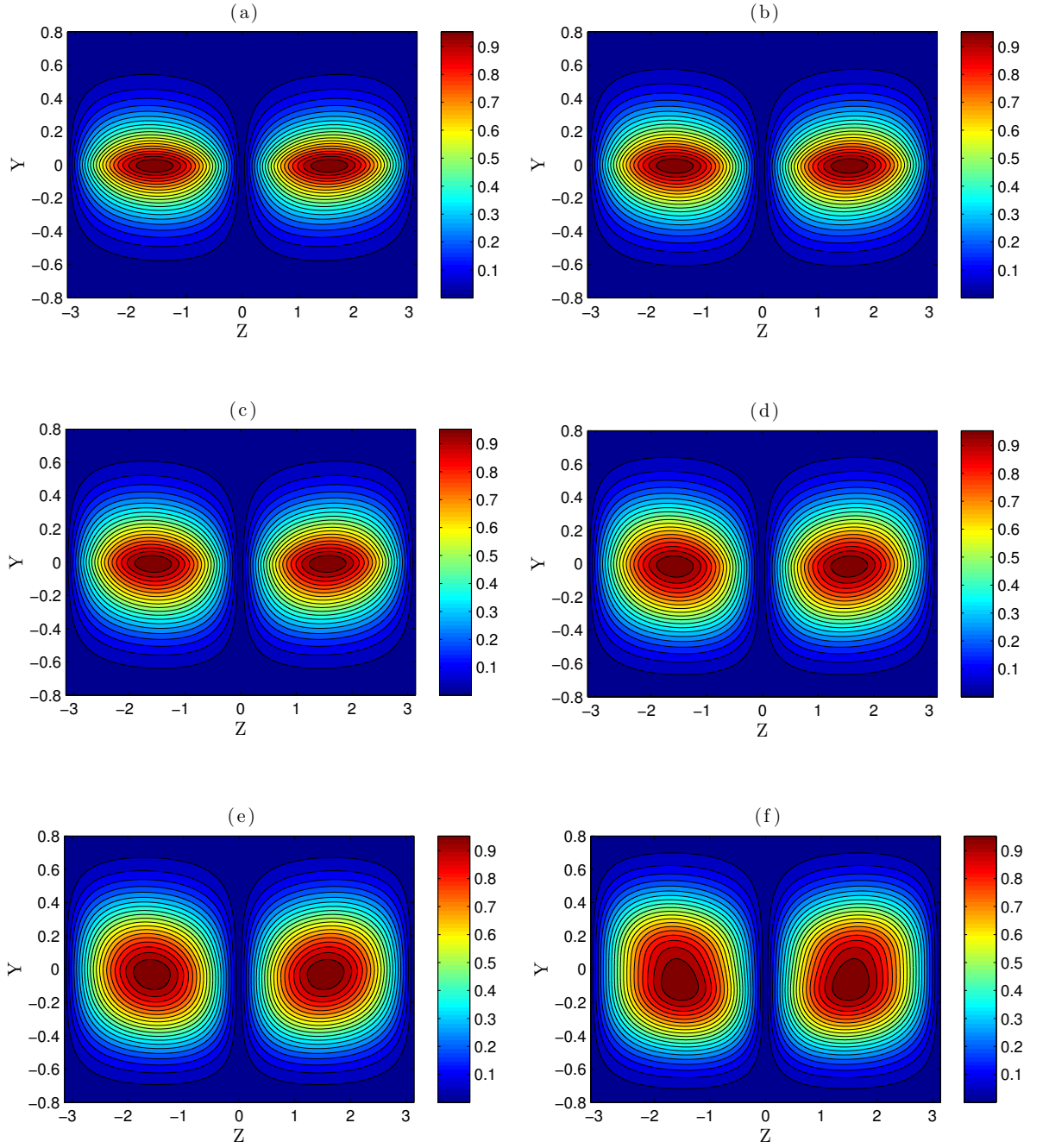


Figure C.3: Absolute part of eigenfunctions for streamwise wavenumber, α , of 0.49 and growth rate correction values, ω_1 , of (a) 0.0, (b) 0.5, (c) 1.0, (d) 1.5, (e) 2.0 and (f) 2.5.

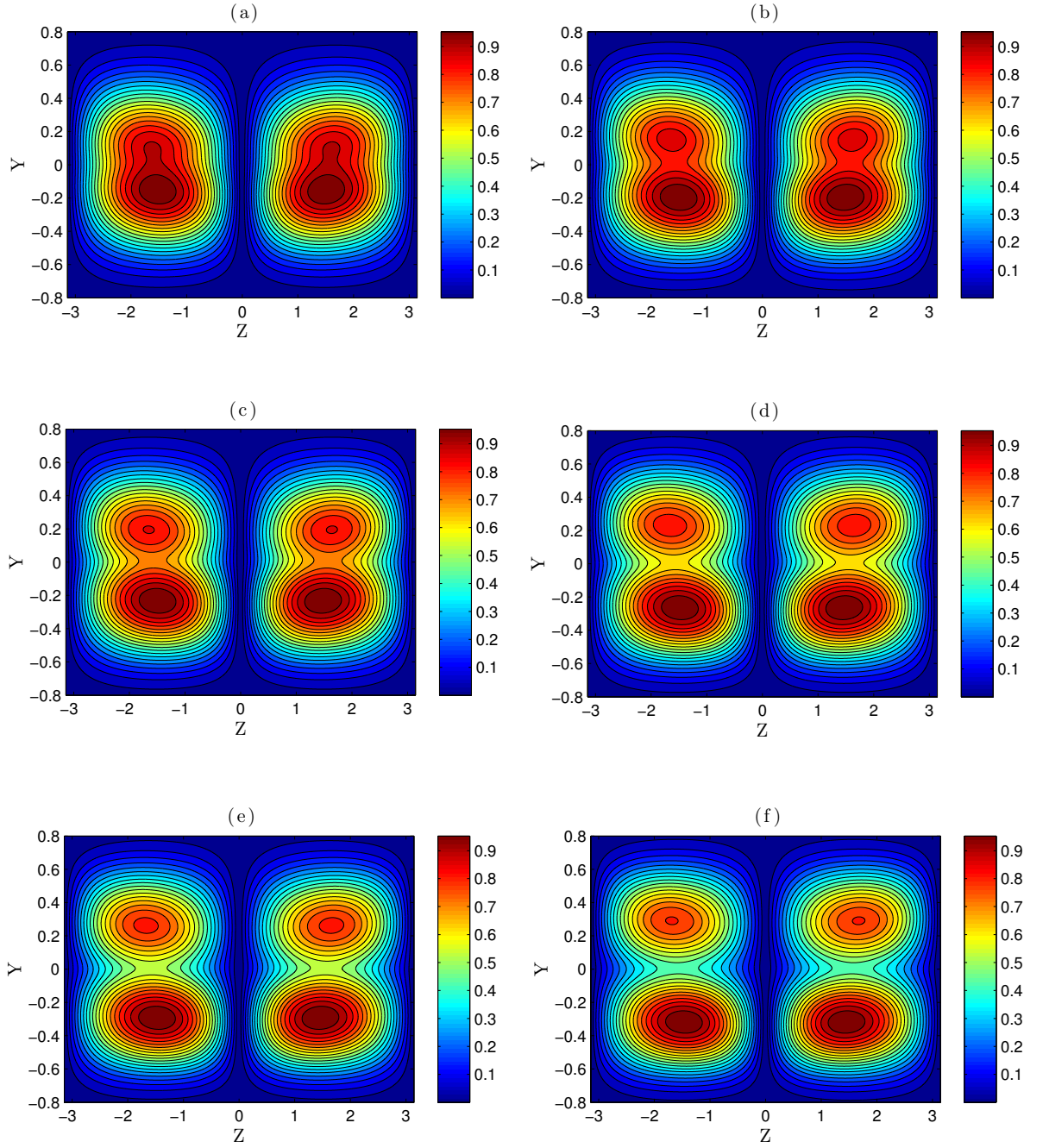


Figure C.4: Absolute part of eigenfunctions for streamwise wavenumber, α , of 0.49 and growth rate correction values, ω_1 , of (a) 3.0, (b) 4.0, (c) 4.5, (d) 4.5, (e) 5.0 and (f) 5.5.

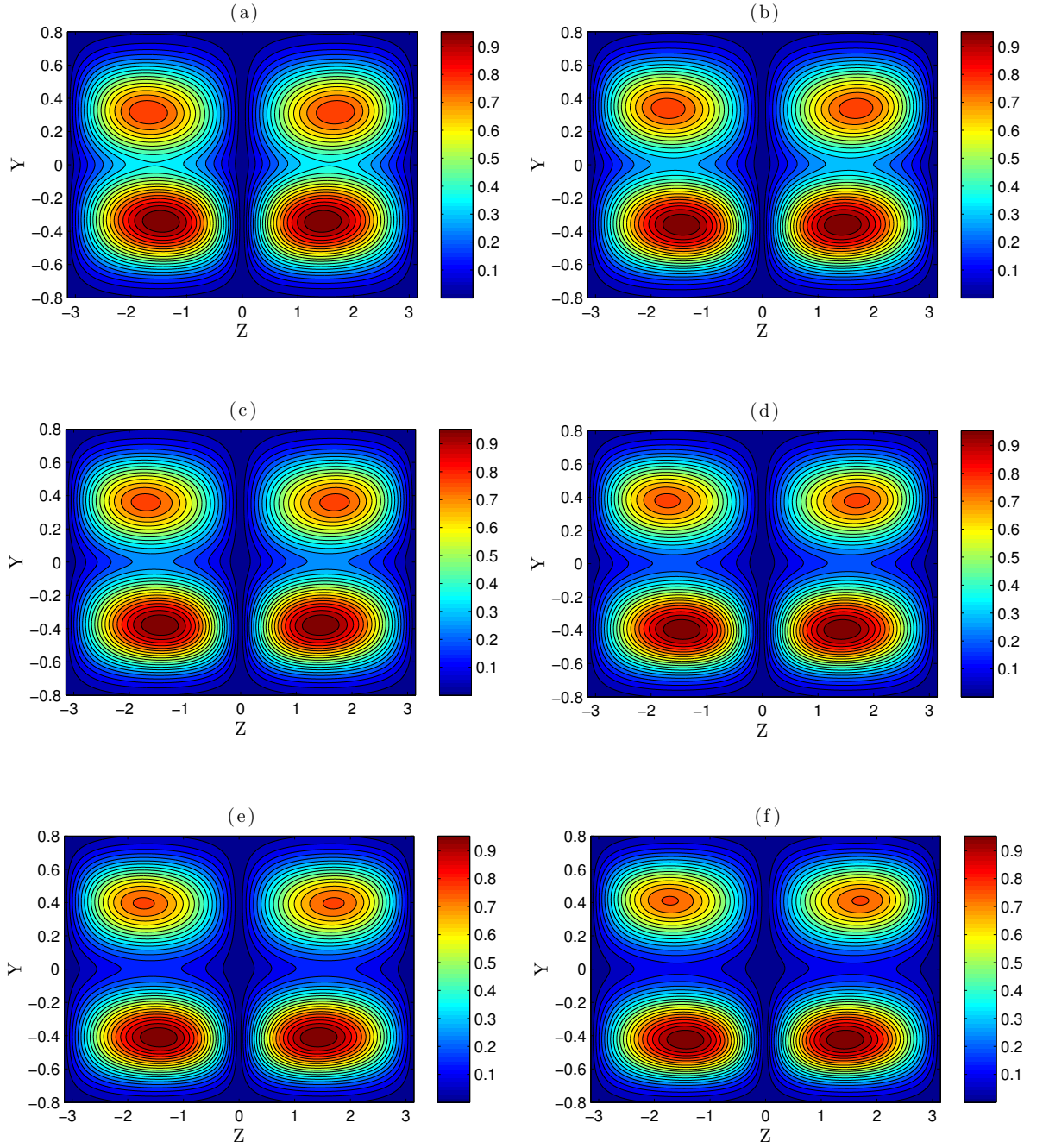


Figure C.5: Absolute part of eigenfunctions for streamwise wavenumber, α , of 0.49 and growth rate correction values, ω_1 , of (a) 6.0, (b) 6.5, (c) 7.0, (d) 7.5, (e) 8.0 and (f) 8.5.

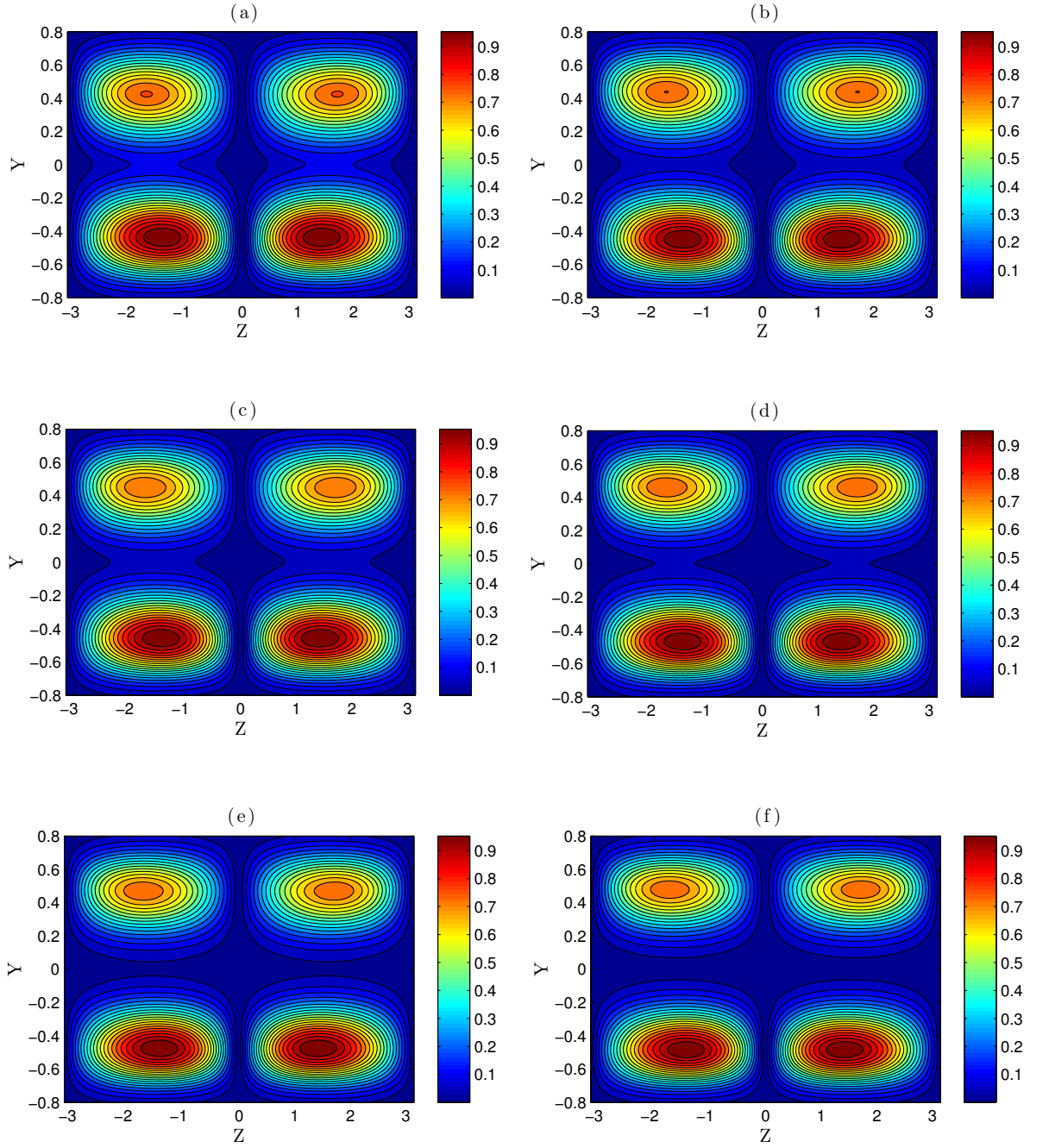


Figure C.6: Absolute part of eigenfunctions for streamwise wavenumber, α , of 0.49 and growth rate correction values, ω_1 , of (a) 9.0, (b) 9.5, (c) 10.0, (d) 10.5, (e) 11.0 and (f) 11.5.

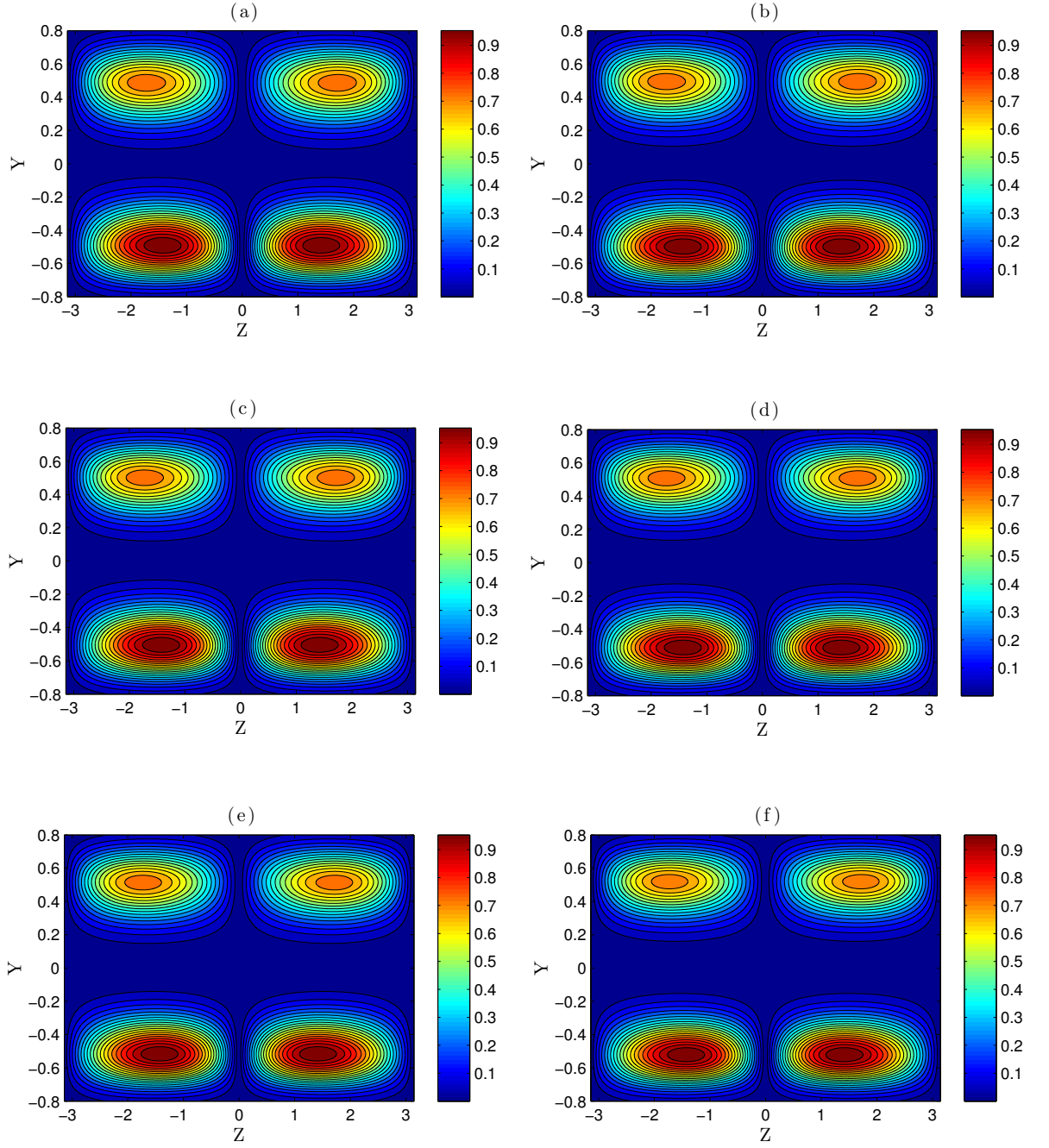


Figure C.7: Absolute part of eigenfunctions for streamwise wavenumber, α , of 0.49 and growth rate correction values, ω_1 , of (a) 10.0, (b) 11.5, (c) 12.0, (d) 12.5, (e) 13.0 and (f) 14.5.

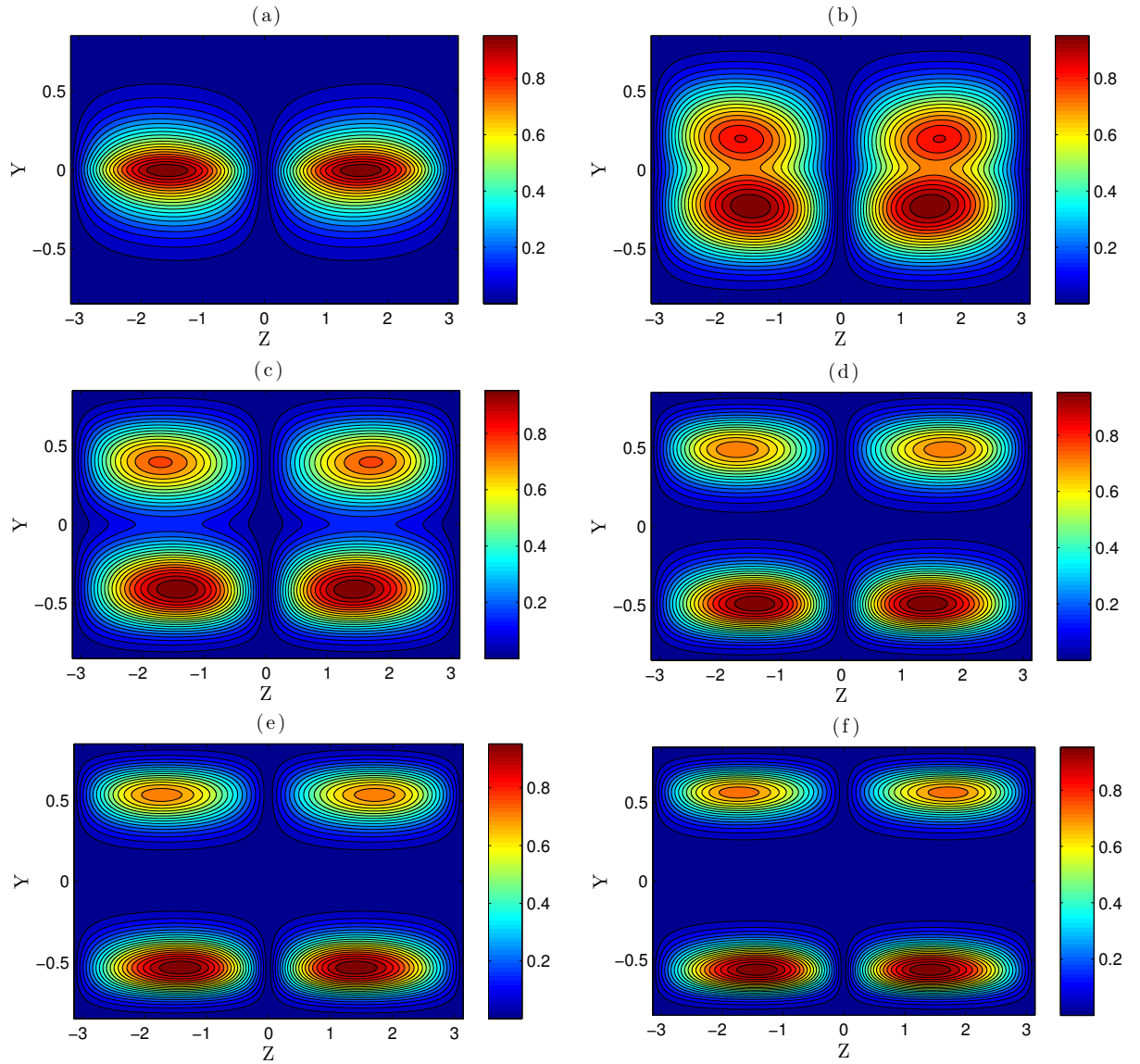


Figure C.8: Absolute part of eigenfunctions for streamwise wavenumber, α , of 0.49 and growth rate correction values, ω_1 , of (a) 0.0, (b) 4.0 , (c) 8.0 , (d) 12.0 , (e) 16.0 and (f) 20.0.