

Orbitally shaken shallow fluid layers. Part 2. An improved wall shear stress model

Paola Alpresa,¹ Spencer Sherwin,¹ Peter Weinberg,¹ and Maarten van Reeuwijk^{1, a)}

Imperial College London, SW7 2AZ London, UK

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A new model for the analytical prediction of wall shear stress distributions at the base of orbitally shaken shallow fluid layers is developed. This model is a generalisation of the classical extended Stokes solution and will be referred to as the PT-Stokes model. The model is validated using a large set of numerical simulations covering a wide range of flow regimes representative of those used in laboratory experiments. It is demonstrated that the model is in much better agreement with the simulation data than the classical Stokes solution, improving the prediction in 63% of the studied cases. The central assumption of the model — which is to link the wall shear stress with the surface velocity — is shown to hold remarkably well over all regimes covered. A MATLAB-implementation of the PT-Stokes model is available online at <http://dx.doi.org/10.5281/zenodo.1186255>.

^{a)}Electronic mail: m.vanreeuwijk@imperial.ac.uk

I. INTRODUCTION

Orbital shakers are often used to impose fluid dynamic shear stresses on mechanosensitive cells cultured on the base of a container filled with a nutrient-rich medium. For example, the shaking of the medium is used to reproduce the effect of shear stresses exerted by the blood on endothelial cells and hence to study the activation of atherosclerotic pathways. In Part 1 of this study¹, we showed that the flows produced by orbital shakers in shallow fluid layers in cylindrical containers are of notable complexity, in part due to the free surface which often has a discontinuity in the form of a breaking wave. This makes the prediction of the wall shear stress (WSS) a challenging matter. Only few numerical studies have been carried that calculate precisely the WSS vector field in orbital shakers because of the difficulties in modelling the flow²⁻⁴. Shear stresses have also been measured *in situ*, for example by using Doppler or particle image velocimetry^{3,5}; but these methods have limited accuracy and can be labour intensive when mapping complex vector fields.

Given the difficulty in determining wall shear stresses directly, many experimental studies⁵⁻¹⁰ have relied on estimating the WSS vector $\boldsymbol{\tau}_w$ magnitude using the extended solution of Stokes' second problem¹¹:

$$\tau_w^{St} = \rho a \omega^{3/2} \nu^{1/2} \quad (1)$$

where the superscript *St* denotes "Stokes", a and ω are the radius and speed of rotation, respectively, and ρ and ν are the fluid density and kinetic viscosity, respectively. Equation (1) originates from the classic Stokes second problem configuration which consists of a half-space $z > 0$ bounded by a plate at $z = 0$ that oscillates uniaxially in its plane as $\mathbf{v}_u = A \cos(\omega t) \mathbf{e}_x$, where A and $\mathbf{e}_x$ are the oscillation amplitude and direction, respectively, and t represents time. Since the governing equations are linear, the theory can be extended to a plate oscillating in two perpendicular directions out of phase by $\pi/2$ ¹¹: $\mathbf{v}_o = A [-\sin(\omega t) \mathbf{e}_x + \cos(\omega t) \mathbf{e}_y]$.

The main differences between Stokes' second problem and the orbital shaker configuration concern the boundary conditions: the cylindrical container placed on the orbital shaker has a finite liquid height and side walls whereas the Stokes problem is unbounded and assumes that the fluid is unperturbed in the far field. As a result, Stokes' solution is unable to capture the spatial dependence of $\boldsymbol{\tau}_w$ observed in culture containers on orbital shakers³ — fundamental information needed for the study of endothelial cells — and, thus, does not take into account the configuration parameters fluid height d and container radius R , which

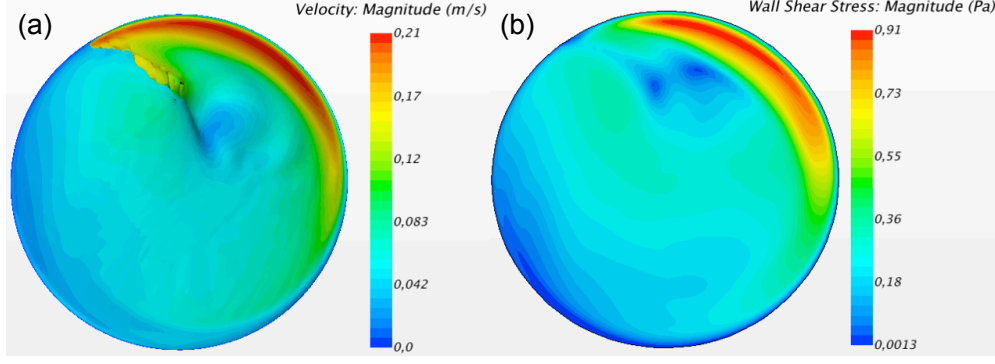


FIG. 1. (a) Surface velocity $\mathbf{v}_s$ magnitude map, and (b) WSS τ_w magnitude map at the bottom of the container for the reference case (see Part 1¹).

have been shown in Part 1¹ of this study to have a notable influence on the flow. Stokes' solution, consequently, can be expected to be valid only in the centre of the container, away from the side walls.

The validity of (1) has been investigated using numerical simulations. Thomas *et al.*¹² tested its validity at the centre of the container for six cases with increasing speed of rotation ω ; τ_w and τ_w^{St} magnitudes followed similar trends, increasing with ω , but Stokes' solution predicted notably higher values of WSS in all cases. Salek, Sattari, and Martinuzzi³ observed that, for low frequency shaking, the fluid height d had a significant influence on the mean τ_w (linear dependence), and that, for high frequency shaking, Stokes' solution correctly predicted the value at the centre, but there was high spatial variability from centre to the edge. In fact, Chakraborty *et al.*¹³ suggested that Stokes' solution is only suitable for a particular set of dimensionless numbers. In Thomas *et al.*¹⁴, the Stokes solution was used to validate the CFD model for six cases within the Stokes validity domain given in Chakraborty *et al.*¹³. Velocities averaged in an circular area of $0.2R$ radius had a mean discrepancy of 7.5%, and the vertical profiles of the horizontal velocity components at the centre of the container followed similar trends, with the maximum and minimum peaking around the same vertical distance. Furthermore, Chakraborty *et al.*¹⁵ reported agreement to within 1% in the centre of the container for a case with a viscosity an order of magnitude higher than the typical experimental value.

In this study, we use the set of 33 simulations introduced in Part 1 to examine the validity of the Stokes solution. For low-forcing cases, we will show that (1) severely overpredicts the

shear stresses observed in the numerical simulations by a factor of 10, demonstrating the limitations of the model and the need for improvement of the WSS model. We present an improved model of the Stokes solution that takes into account the boundaries of the orbital shaker problem. The crux of the model is to link $\boldsymbol{\tau}_w$ to the surface velocity $\mathbf{v}_s$. Fig. 1, which displays results from the reference case simulation introduced in Part 1¹, shows that the surface velocity and the wall shear stress are strongly correlated. The novel model, which we will refer to as the PT-Stokes WSS model, is a generalisation of the extended Stokes model, in which the surface velocity is predicted by potential theory (PT).

The paper is organised as follows. In Section II, the theory for the PT-Stokes WSS model is presented. Section III reviews the 33 simulations that will provide the wall shear stress data. Section IV explores the validity of the PT-Stokes WSS model for cases with low forcing (for which potential flow can be assumed to be valid), demonstrating that the boundary layer profile and $\boldsymbol{\tau}_w$ magnitude are predicted correctly with exception of the edge of the container, and that the $\boldsymbol{\tau}_w$ angle is adequately predicted near the centre. In Section V, the applicability of the PT-Stokes model to higher-forcing cases is explored, and a good correlation is found with the simulation data in the centre and intermediate regions except for resonant cases. A direct comparison of the the Stokes and PT-Stokes WSS models demonstrates that the new model performs significantly better than the classic WSS prediction. Section VI contains the main conclusions of the study.

II. THEORY

In this section we develop a novel WSS model that is a generalisation of the extended classic Stokes model, taking into account the container boundaries. This model will be referred to as the PT-Stokes WSS model. The flow will be shown to be organised in a relatively thin boundary layer which contains an Ekman spiral (this layer will also be referred to as the inner layer or near field) and an irrotational layer away from the bottom wall (the outer layer or far field), see Fig. 2. It will be shown that the surface velocity $\mathbf{v}_s^{\text{PT}}$ can be directly related to the wall shear stress $\boldsymbol{\tau}_w$ and direction by

$$\tau_w = \rho V_s^{\text{PT}} \sqrt{\nu \omega}, \quad (2)$$

$$\alpha_\tau = \alpha_v^{\text{PT}} + 45^\circ. \quad (3)$$

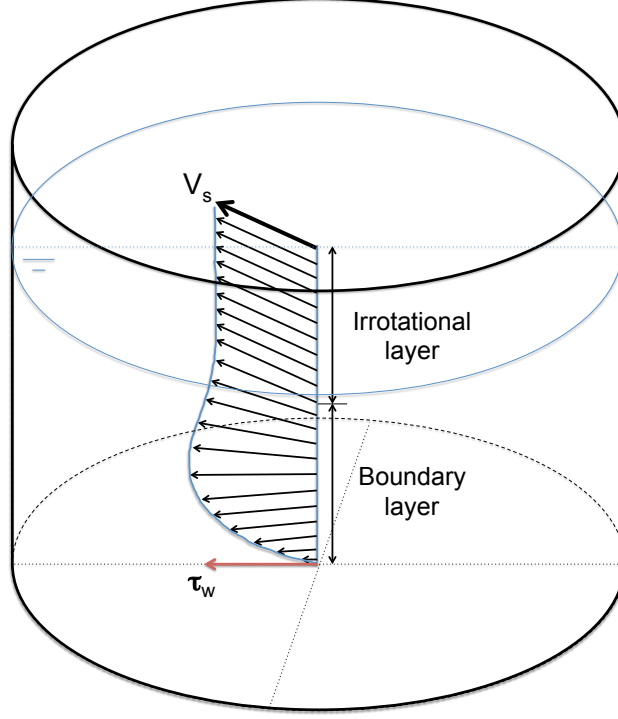


FIG. 2. Definition sketch showing the structure of the velocity profile and the angle difference between the surface velocity and wall shear stress.

Here, ρ and ν are the fluid density and kinetic viscosity, respectively, V_s is the surface velocity magnitude, and α_τ and α_v are the τ_w and $\mathbf{v}$ angles with respect to the x -axis at $t \bmod 2\pi/\omega = 0$.

A. Relationship of the wall shear stress to the surface velocity

In order to arrive at a local description of the flow structure, we will assume that the flow in the outer layer is homogeneous and unbounded. We assume that the fluid layer is shallow and thus $\Gamma = d/R \ll 1$, where d is the fluid layer thickness at rest and R is the radius of the orbital shaker. The boundary conditions at the container bottom and the free surface are given by, respectively,

$$\text{at } z = 0 : u = 0, v = 0, \quad (4)$$

$$\text{at } z = h : u = V_s \cos(\omega t + \alpha_v), v = V_s \sin(\omega t + \alpha_v). \quad (5)$$

Using the assumption $\Gamma \ll 1$, it follows that horizontal gradients will be much smaller than vertical gradients and that the vertical velocity is much smaller than the horizontal velocity in the far field. In the near field, viscous forces dominate. Under these assumptions, the Navier-Stokes equations simplify to

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} + a\omega^2 \cos(\omega t), \quad (6)$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2} + a\omega^2 \sin(\omega t), \quad (7)$$

where u and v are the velocity components in the x and y -directions, p is pressure. The trigonometric terms on the right-hand side of the equations are the inertial forcing terms; they originate from the acceleration caused by the circular translation of the platform on which the container stands: $u_o = -a\omega \sin(\omega t)$ and $v_o = a\omega \cos(\omega t)$ (see Part 1¹). The momentum equation in the vertical direction has been omitted: consistent with the assumption of shallowness it provides the hydrostatic pressure distribution in the z -direction. Here we note that the pressure gradient appears in the equations not because of a hydrostatic balance but due to the expected geostrophic balance in the far field due to the non-inertial forcing.

Denoting the boundary layer thickness by δ we expect that in the far field region ($z \gg \delta$), the velocity field $\mathbf{v} = \mathbf{v}_s$ is independent of z . Under these circumstances, (6) and (7) provide a closed form expression for the pressure gradients:

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} \Big|_{z \gg \delta} = -V_s \omega \sin(\omega t + \alpha_v) - a\omega^2 \cos(\omega t), \quad (8)$$

$$-\frac{1}{\rho} \frac{\partial p}{\partial y} \Big|_{z \gg \delta} = V_s \omega \cos(\omega t + \alpha_v) - a\omega^2 \sin(\omega t). \quad (9)$$

The standard boundary layer assumption that the pressure gradient does not vary across a boundary layer then immediately suggests that the boundary layer equations for this problem are given by

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial z^2} - V_s \omega \sin(\omega t + \alpha_v), \quad (10)$$

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial z^2} + V_s \omega \cos(\omega t + \alpha_v). \quad (11)$$

Solving the two independent partial differential equations above using the boundary condi-

tions (4)-(5) results in

$$u(z, t) = V_s \cos(\omega t + \alpha_v) - V_s \cos\left(\omega t + \alpha_v - \sqrt{\frac{\omega}{2\nu}} z\right) e^{-\sqrt{\frac{\omega}{2\nu}} z}, \quad (12)$$

$$v(z, t) = V_s \sin(\omega t + \alpha_v) - V_s \sin\left(\omega t + \alpha_v - \sqrt{\frac{\omega}{2\nu}} z\right) e^{-\sqrt{\frac{\omega}{2\nu}} z}. \quad (13)$$

Here, $\sqrt{\nu/\omega}$ is a diffusive length scale commonly referred to as the viscous penetration depth¹⁶, and the characteristic boundary layer thickness δ is expected to be of this order of magnitude.

The velocity magnitude V and the angle α with respect to the far field velocity $\mathbf{v}_s$ are given by

$$\frac{V(z)}{V_s} = \sqrt{1 + e^{-2\sqrt{\frac{\omega}{2\nu}} z} - 2 \cos\left(-\sqrt{\frac{\omega}{2\nu}} z\right) e^{-\sqrt{\frac{\omega}{2\nu}} z}}, \quad (14)$$

$$\alpha(z) = \arctan\left(\frac{v}{u}\right) - (\omega t + \alpha_v), \quad (15)$$

as depicted in Fig. 3a-b; both V and α are independent of t . The velocity profile has its maximum velocity $V_{max} = 1.07V_s$ located at $z_{max} = 3.2\sqrt{\nu/\omega}$. The velocity direction changes 45° across the boundary layer. This change in the velocity direction creates a spiral-like distribution of the velocity projection on the plane $\{x, y\}$ (Fig. 3c) which is known as the Ekman spiral in the atmospheric and ocean sciences¹⁷. The spiral originates from zero velocity at the bottom, and develops clockwise with z up to the far-field velocity $\mathbf{v}_s$. The profile is fully developed at around $z = 5.8\sqrt{\nu/\omega}$, where $V = 1.01V_s$ and $\alpha \sim 0$. The spiral shape is invariant in time and rotates with frequency $\omega/2\pi$.

The WSS vector $\boldsymbol{\tau}_w$ is obtained using (12) and (13) with result

$$\frac{\tau_x}{\rho} = \nu \frac{\partial u}{\partial z} \Big|_{z=0} = V_s \sqrt{\frac{\nu\omega}{2}} (\cos(\omega t + \alpha_v) - \sin(\omega t + \alpha_v)), \quad (16)$$

$$\frac{\tau_y}{\rho} = \nu \frac{\partial v}{\partial z} \Big|_{z=0} = V_s \sqrt{\frac{\nu\omega}{2}} (\cos(\omega t + \alpha_v) + \sin(\omega t + \alpha_v)). \quad (17)$$

These equations are most conveniently expressed in terms of the WSS magnitude τ_w and direction α_τ :

$$\tau_w = V_s \sqrt{\rho\mu\omega}, \quad (18)$$

$$\tan \alpha_\tau = \frac{\tau_y}{\tau_x} = \frac{(\cos \alpha_v + \sin \alpha_v)}{(\cos \alpha_v - \sin \alpha_v)} \Rightarrow \alpha_\tau = \alpha_v + 45^\circ. \quad (19)$$

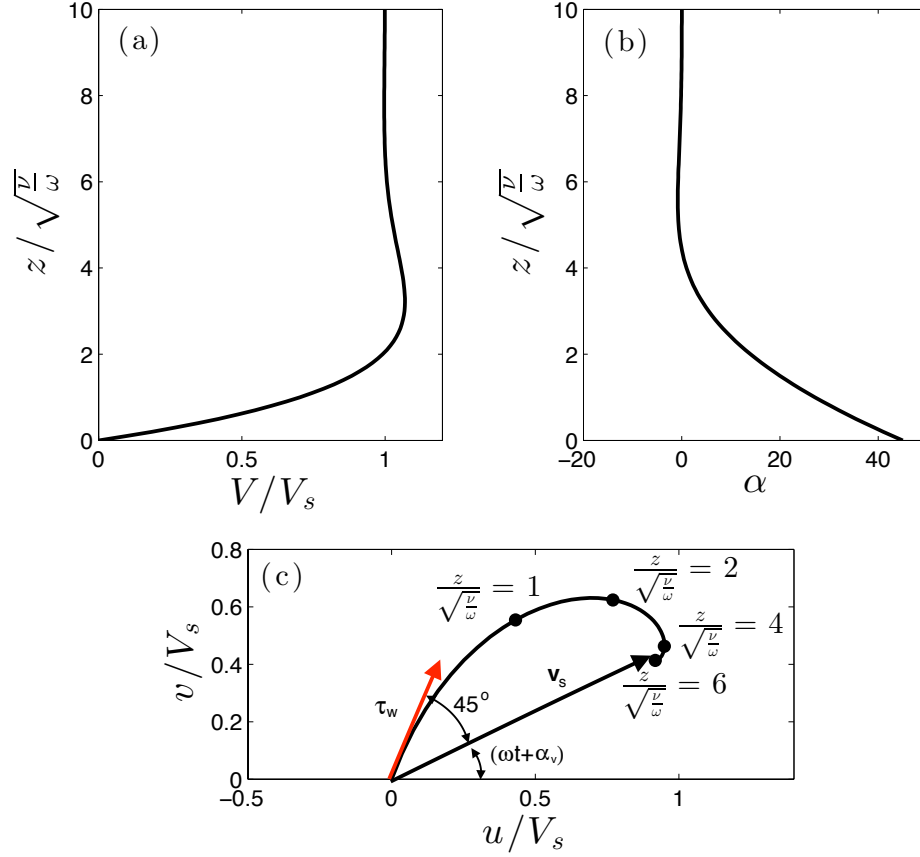


FIG. 3. Boundary layer profile. (a) Velocity magnitude V as a function of z . (b) Angle α of the velocity $\mathbf{v}$ with respect to $\mathbf{v}_s$ as a function of z . (c) Ekman spiral describe by the projection of the velocity vectors on the plane $\{x, y\}$.

where V_s and α_v are the surface velocity magnitude and angle, respectively. The boundary layer model predicts that the magnitude of τ_w is proportional to V_s , and that the angle between τ_w and $\mathbf{v}_s$ is 45° , with τ_w tangent to the spiral at its origin (see Fig. 3c).

The relationship between the characteristic length $\sqrt{\frac{\nu}{\omega}}$ and surface height h will determine the suitability of the model. If $h \ll \sqrt{\frac{\nu}{\omega}}$, viscous effects would extend up to the free surface, affecting the inviscid character of the flow outside the boundary layer and the PT-Stokes model cannot be used; if $h \gg \sqrt{\frac{\nu}{\omega}}$, the free surface velocities are undisturbed and the PT-Stokes model is expected to be a good approximation.

This derivation of the boundary layer velocity field is a generalisation of the extended Stokes solution¹¹. Indeed, the extended Stokes solution can be recovered for the particular case where the velocity in the far field is quiescent due to the unbounded problem configuration.

Thus, if regarded from a non-inertial frame of reference where the plate is static, the fluid in the far field is predicted to move homogeneously as $\mathbf{v}_s = -\mathbf{v}_o = A [\sin(\omega t)\mathbf{e}_x - \cos(\omega t)\mathbf{e}_y]$. In the present formulation, this is equivalent to $V_s = a\omega$ and $\alpha_v = -\pi/2$; and Stokes' solution can be obtained from (12) and (13), and the commonly used Stokes WSS prediction (1) from (18). It is for this reason that we will keep referring to the boundary layer as a Stokes boundary layer.

B. PT-Stokes wall shear stress model

The approximate model for the boundary layer structure of orbitally shaken shallow fluid layers derived in the previous section requires a closure in terms of the surface velocity before it can be used for predictive purposes. In Part 1¹, we showed that potential theory provides a reasonably good description of flows orbitally shaken in cylindrical containers, suggesting a surface velocity (in cylindrical coordinates)

$$u_{rs}^{\text{PT}}(r, \theta, t) = -2a\omega \sin(\omega t - \theta) \sum_{n=1}^{\infty} \left[\frac{\epsilon_{1n}}{(\epsilon_{1n}^2 - 1)} \frac{1}{\left(\frac{\omega_{1n}}{\omega}\right)^2 - 1} \frac{J_1'(\epsilon_{1n} \frac{r}{R})}{J_1(\epsilon_{1n})} \frac{\cosh(\epsilon_{1n} \frac{h}{R})}{\cosh(\epsilon_{1n} \frac{d}{R})} \right], \quad (20)$$

$$u_{\theta s}^{\text{PT}}(r, \theta, t) = 2a\omega \cos(\omega t - \theta) \sum_{n=1}^{\infty} \left[\frac{1}{\frac{r}{R} (\epsilon_{1n}^2 - 1)} \frac{1}{\left(\frac{\omega_{1n}}{\omega}\right)^2 - 1} \frac{J_1(\epsilon_{1n} \frac{r}{R})}{J_1(\epsilon_{1n})} \frac{\cosh(\epsilon_{1n} \frac{h}{R})}{\cosh(\epsilon_{1n} \frac{d}{R})} \right], \quad (21)$$

where h is the free surface elevation relative to the container bottom; $J_1(r)$ is a Bessel function of the first kind, of order 1; ϵ_{1n} is the n th root of $J_1'(s)$ (where the prime denotes differentiation with respect to the argument); $\lambda_{1n} = \epsilon_{1n}/R$; and ω_{1n} is the natural frequency of the n th mode given by the dispersion relationship $\omega_{1n}^2 = g\lambda_{1n} \tanh(\lambda_{1n}d)$. The value of ϵ_{1n} for the first five modes is [1.841, 5.331, 8.536, 11.706, 14.864]. In Cartesian form, (20) and (21) are expressed as

$$u_s^{\text{PT}} = u_{rs}^{\text{PT}} \cos \theta - u_{\theta s}^{\text{PT}} \sin \theta, \quad (22)$$

$$v_s^{\text{PT}} = u_{rs}^{\text{PT}} \sin \theta + u_{\theta s}^{\text{PT}} \cos \theta. \quad (23)$$

There are fundamental differences between the surface velocity predicted by the potential flow model and the Stokes solution. As discussed before, the Stokes solution assumes a quiescent far field, which in the non-inertial frame of reference is given by

$$u_s^{\text{St}}(t) = a\omega \sin(\omega t), \quad (24)$$

$$v_s^{\text{St}}(t) = -a\omega \cos(\omega t). \quad (25)$$

	V_s	$\omega t + \alpha_v$	
Stokes' solution (St)	$\sqrt{(u_s^{\text{St}})^2 + (v_s^{\text{St}})^2} = a\omega$	$\omega t - \pi/2$	(28)
Potential Theory (PT)	$\sqrt{(u_s^{\text{PT}})^2 + (v_s^{\text{PT}})^2}$	$\arctan\left(\frac{v_s^{\text{PT}}}{u_s^{\text{PT}}}\right)$	(29)

TABLE I. Surface velocity magnitude V_s and x-axis angle ($\omega t + \alpha_v$) predicted by the analytical models.

For instance, at the centre of the container at the given instant $t \bmod 2\pi/\omega = 0$ and assuming $h = d$:

$$\mathbf{v}_s^{\text{PT}}(r = 0, \theta = 0, t = 0) = a\omega \sum_{n=1}^{\infty} \left[\frac{\epsilon_{1n}}{(\epsilon_{1n}^2 - 1)} \frac{1}{(\frac{\omega_{1n}}{\omega})^2 - 1} \frac{1}{J_1(\epsilon_{1n})} \right] \mathbf{e}_y, \quad (26)$$

$$\mathbf{v}_s^{\text{St}}(t = 0) = -a\omega \mathbf{e}_y. \quad (27)$$

Typically, $\omega_{1n}/\omega > 1$ giving a positive summation factor; hence, the velocities predicted by the potential theory and the Stokes solution at the centre have an opposite direction. This discrepancy highlights the limitations imposed by the Stokes model in the far field velocity. The surface velocities predicted by each model are summarised in Table I in the PT-Stokes WSS model notation.

Equations (18) and (19) together with the surface velocity prediction of the potential theory (29) in Table I form the PT-Stokes WSS model for orbitally shaken shallow fluid layers that will be examined in this paper.

III. SIMULATIONS

This study uses the computational model and simulation case studies introduced in detail in Part 1¹. The cases cover a wide range of flow behaviours including wave breaking, multi-crest waves, and the bottom being uncovered. The simulation configurations are summarised in Table II using the dimensionless groups

$$F = \frac{a\omega^2}{g}, \quad E = \frac{a}{R}, \quad \Gamma = \frac{d}{R}, \quad Re = \frac{R^2\omega}{\nu}. \quad (30)$$

As explained in Part 1¹, the dimensionless forcing F represents the ratio between the inertial force per unit mass and the gravitational acceleration. The eccentricity E represents the

ratio between the orbital forcing radius a and the container radius R . The shallowness Γ represents the ratio between the horizontal and the vertical scales of the container. The Reynolds number represents the ratio between the local acceleration and the viscosity forces. A dimensionless height,

$$\tilde{d} = \frac{d}{\sqrt{\nu/\omega}}, \quad (31)$$

has been included in the table which represents the ratio between the fluid height at rest and the viscous penetration depth (Chakraborty *et al.*¹⁵ refer to it as the Stokes number). This dimensionless number characterises whether the bottom viscous effects reach the free surface (which occurs when $\tilde{d} < 1$) which would render the model unsuitable. This quantity can be rewritten in term of the dimensionless groups as $\tilde{d} = \Gamma Re^{1/2}$.

All the simulations are in the shallow water regime ($\Gamma \ll 1$). The dimensionless height $\tilde{d}$ is 7.9 in most cases, hence the model assumption ($d \gg \sqrt{\nu/\omega}$) is satisfied fairly well. For the Re cases, $\tilde{d}$ varies from 2.8 ($Re1$) to 11.2 ($Re5$); the $Re1$ case is the most extreme case, and the viscous effects are expected to reach the free surface.

IV. VALIDATION OF THE PT-STOKES WALL SHEAR STRESS MODEL

The aim of this section is to verify in detail the extent to which the PT-Stokes WSS model predicts $\boldsymbol{\tau}_w$ produced by orbitally shaken shallow fluid layers in cylindrical containers. This validation is carried out in two parts: first, the suitability of the boundary layer model is validated by observing in the simulations the velocity profile, the boundary layer characteristic thickness, and the relationship of $\boldsymbol{\tau}_w$ to $\mathbf{v}_s$; second, the analytical surface velocities and the $\boldsymbol{\tau}_w$ predicted by the PT-Stokes WSS model are compared to data from the simulations. The low-forcing cases (L cases), where $F \ll 1$ (see Table II), are used in order to limit the non-linear effects, which are not predictable by the linear theory, that especially affect the behaviour of the free surface. For each case, the domain has been sampled radially and azimuthally as shown in Fig. 4. The sample points have been divided into three regions: centre, $r < 0.2R$ (purple points); intermediate, $0.4R < r < 0.6R$ (orange points); and edge, $r = 0.96R$ (green points). As we described in Part 1¹, the flow forced response is periodic, thus, for a point at a given radial distance, sampling in the azimuthal direction is equivalent to sampling in time.

Case	E	F	Re	Γ	$\tilde{d}$
$Re1$	0.45	0.13	240	0.18	2.8
$Re2$	"	"	480	"	4.0
$Re3$	"	"	959	"	5.6
$Re4$	0.45	0.13	1918	0.18	7.9
$Re5$	"	"	3836	"	11.2
$EF1$	0.07	0.02	1981	0.18	7.9
$EF2$	0.15	0.04	"	"	"
$EF3$	0.29	0.08	"	"	"
$EF4$	0.58	0.16	"	"	"
$EF5$	1.16	0.32	"	"	"
$E1$	0.09	0.13	1981	0.18	7.9
$E2$	0.18	"	"	"	"
$E3$	0.37	"	"	"	"
$E4$	0.74	"	"	"	"
$E5$	1.48	"	"	"	"
$E6$	4.61	"	"	"	"
$E7$	9.22	"	"	"	"
$Fl1$	0.09	0.25	1981	0.18	7.9
$Fl2$	"	0.27	"	"	"
$Fl3$	"	0.30	"	"	"
$Fl4$	"	0.36	"	"	"
$Fm1$	0.45	0.03	1981	0.18	7.9
$Fm2$	"	0.06	"	"	"
$Fm3$	"	0.08	"	"	"
$Fm4$	"	0.24	"	"	"
$Fh1$	1.48	0.10	1981	0.18	7.9
$Fh2$	"	0.11	"	"	"
$Fh3$	"	0.15	"	"	"
$L1$	0.05	0.0009	511	0.18	4.1
$L2$	0.09	0.0018	"	"	"
$L3$	0.27	0.0030	383	"	3.5
$L4$	0.14	0.0060	767	"	6.3
$L5$	0.27	0.0120	"	"	"

TABLE II. Summary of all simulation cases analysed in this study. The configuration is given in the dimensionless groups defined in (30). In addition, the dimensionless height $\tilde{d}$ is given, which provides an indicator of the validity of the model.

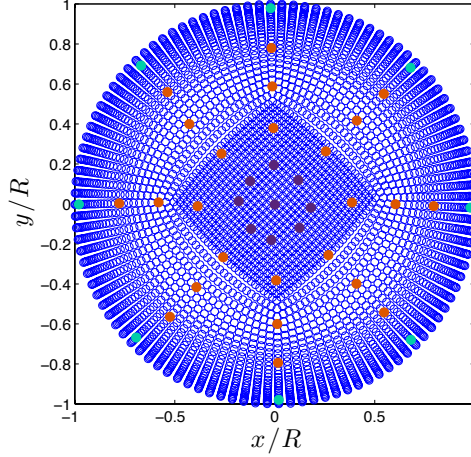


FIG. 4. Mesh points at a z -constant plane. The sample points are divided into three regions: centre (purple points); intermediate (orange points); and edge (green points).

A. Relationship of the wall shear stress to the surface velocity

In the boundary layer model, the thickness of the boundary layer is characterised by $\sqrt{\frac{\nu}{\omega}}$. At this z level, the theory (14) predicts that $V(z = \sqrt{\frac{\nu}{\omega}}) = 0.7V_s = 0.66V_{max}$. By means of the inverse operation, we define the boundary layer thickness δ in the simulations as the z level where the simulated velocity is 0.66 of the simulated maximum velocity, i.e. $\delta \leftarrow V(z = \delta) = 0.66V_{max}$. Fig. 5 shows a comparison of δ obtained from the simulations to the theoretical value $\sqrt{\nu/\omega}$ at each sample point. According to expectation we find that $\delta \sim \sqrt{\nu/\omega}$ in the centre region. As we move to the edge, the scatter increases, denoting a discrepancy in the shape of velocity profile. Interestingly, δ has a lower bound of $0.6\sqrt{\nu/\omega}$.

The velocity profile obtained in the simulations has been normalised by the characteristic thickness δ and plotted in the vertical direction up to the free surface for each point in the different regions of the container (Fig. 6); this allows for comparison to the theoretical profile shape (14) which is indicated by a discontinuous line. In the centre region, the theoretical profile correctly describes the velocity shape except for some points where the maximum peaks at the free surface instead of $z_{max} = 3.2\delta$, as predicted by the theory (§ II A). As we move to the edge, the scatter increases. In the intermediate region, the theoretical profile correctly describes the velocity slope at the bottom with very few exceptions; however for $z > \delta$, the profiles show multiple behaviours. In the edge region, most profiles have a shape

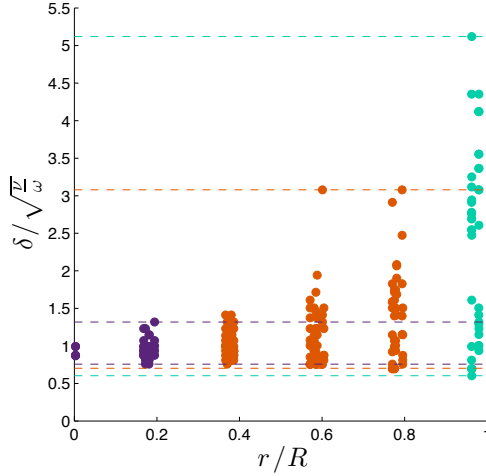


FIG. 5. Boundary layer thickness δ normalised by $\sqrt{\nu/\omega}$ for each sample point for the L cases. Colour-coding depends on container region as shown in Fig. 4.

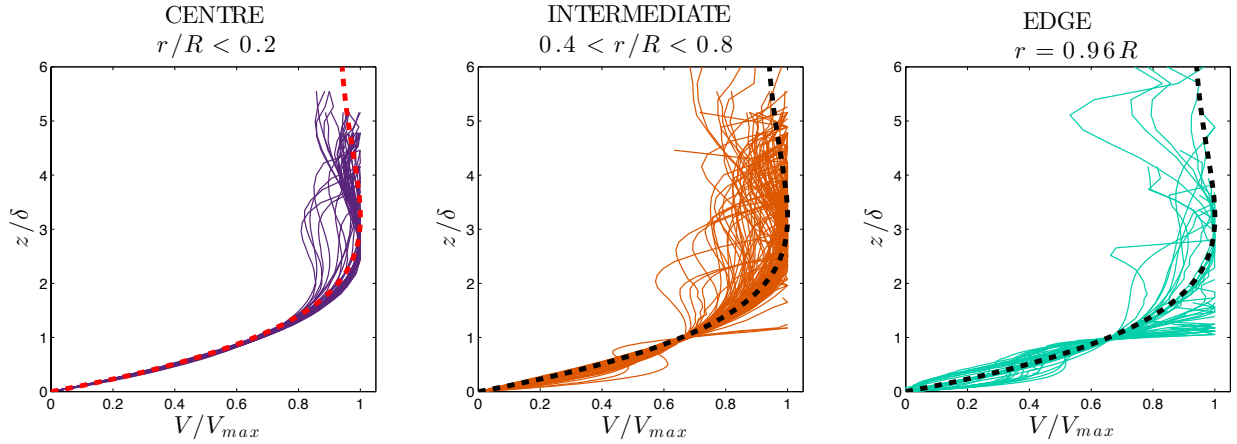


FIG. 6. Profiles of velocity magnitude profiles normalised by the boundary layer characteristic thickness δ in the vertical direction z for the different regions of the container in simulations of the L cases. The discontinuous line (red or black) is the theoretical profile (14).

that differs from the theoretical prediction. The latter is caused by the influence of the side walls, which is not considered in the theoretical boundary layer model.

The Ekman spiral (Fig. 3[c]) connects τ_w to $\mathbf{v}_s$ in magnitude and direction by $\tau_w = \rho V_s \sqrt{\nu \omega}$ (18) and $\alpha_\tau = \alpha_v + 45^\circ$ (19), respectively. Fig. 7 plots the values obtained in the simulations on the abscissa against the prediction of the theory on the ordinate. A linear

regression with the equation $x = y$ was used to quantify the agreement or lack of it. The simulation result was taken as the dependent variable¹⁸. The coefficient of determination r^2 is used as a measure of the goodness of the correlation between the simulation result and the prediction. Note that r^2 can be negative because the constant term of the fitting curve has been omitted; $r^2 < 0$ indicates a fit that is poorer than just fitting to the mean. The τ_w magnitude (Fig. 7a-c) is predicted well in the centre and intermediate regions ($r^2 = 0.97$ and 0.91 , respectively); in the edge region, the correlation decreases, again reflecting the influence of the side walls. It should in principle be possible to correct for the influence of the side boundaries. Indeed, on the side boundaries, one would expect a boundary layer of thickness $O(\sqrt{\nu/\omega})$ to form in which the velocities will smoothly converge to zero (and become parallel to the wall). However, the simulation data suggests that the reality is more complex, and further analysis beyond the scope of this study is needed to improve the correlations very close to the side walls.

The simulated τ_w angle in the centre region (Fig. 7d) remains practically constant at 125° , but the prediction varies between $+80^\circ$ and $+145^\circ$. The scatter increases in the intermediate region and even more towards the edge (Fig. 7[e][f]) ($r^2 < 0$). The discrepancy in the centre and intermediate regions can be attributed to disturbances near the free surface, where, despite the perfectly flat surface, the velocity vectors deviate from the linear solution and point in the direction of a small bifurcation line; these disturbances vanish gradually from the free surface to the deeper fluid. At the edge, the discrepancy is mainly caused by the side walls.

B. PT-Stokes wall shear stress model

The surface velocities obtained from the simulations have been compared to the Stokes solution (28) (Fig. 8a-c) and the potential theory (29) (Fig. 8d-f). The Stokes solution V_s^{St} overpredicts the surface velocity magnitude by a factor of 10. The potential theory velocity magnitude V_s^{PT} gives a markedly better prediction in the centre and intermediate regions ($r^2 = 0.92$) — V_s is underestimated (by 30% on average) due to the absence of the viscous effects in the potential theory, as discussed in Part 1¹. The surface velocity direction α_v has also been analysed (data not shown). The Stokes model predicts a constant $\alpha_v^{\text{St}} = -90^\circ$, while the simulated α_v varies from 15° to 160° along the free surface. Focusing at the centre

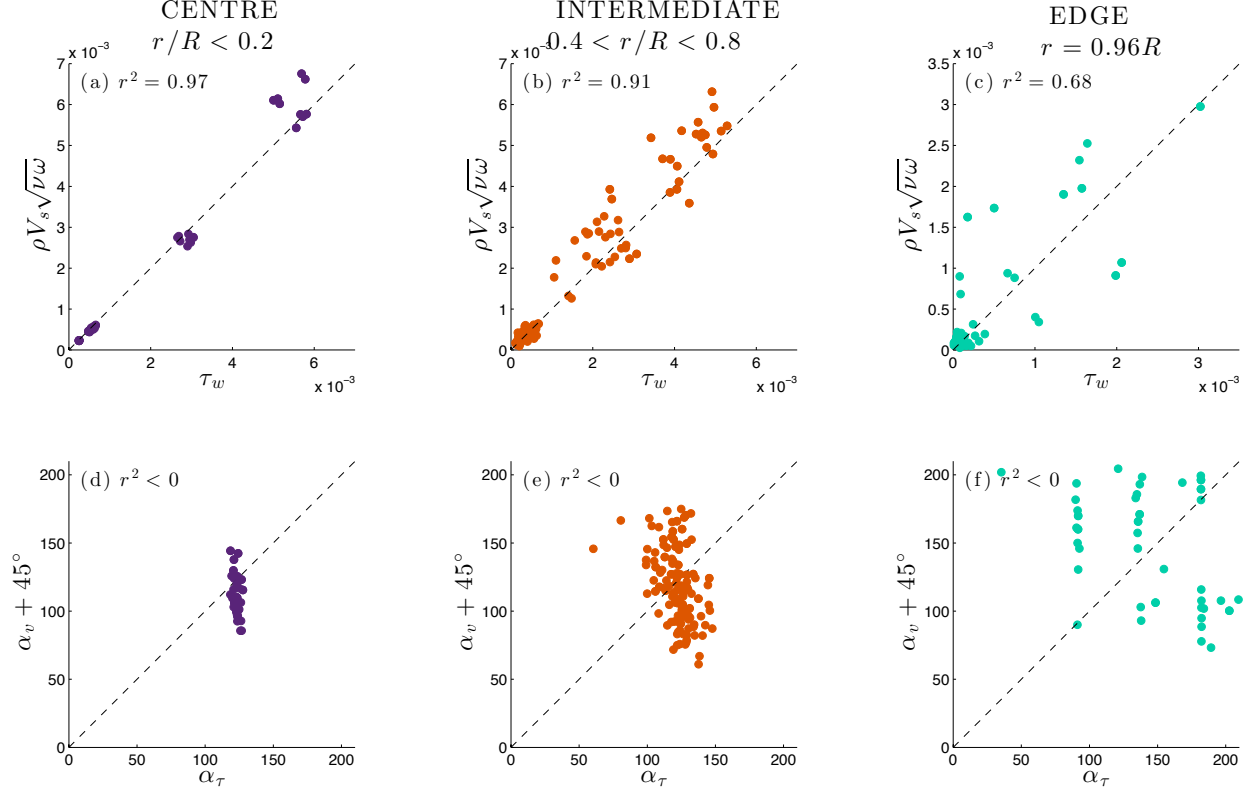


FIG. 7. Relationship of τ_w to $\mathbf{v}_s$ for L cases in the different container regions. (a)-(c) simulated τ_w against the prediction $\rho V_s \sqrt{\nu \omega}$ (values presented in Pa); (d)-(f) simulated α_τ against the prediction $\alpha_v + 45^\circ$ (values presented in degrees). r^2 is the coefficient of determination. The discontinuous line marks the linear relationship of proportionality one.

region, where the Stokes solution is expected to work better, α_v varies from 40° to 100° , an average angle being $\bar{\alpha}_s = 72^\circ$, almost the opposite direction to α_v^{St} . The potential theory predicts the angle notably better: α_v^{PT} in the centre region has an approximately constant value of 90° . The greater variability in the simulated angle results from disturbances observed near the free surface, as previously noted. In fact, the potential theory correctly predicts the velocity direction some computational cells below the free surface.

Lastly, we compared the simulated τ_w and α_τ to the PT-Stokes WSS model prediction (2)-(3). The magnitude τ_w (Fig. 9a-c) shows a very good correlation in the centre and intermediate regions ($r^2 = 0.96$). As expected from the results in the potential theory velocity prediction, τ_w is underestimated by 30% on average. Interestingly, the angle prediction (Fig. 9d-f) is better than the results given by the Ekman-spiral relationship shown in Fig. 7d-f;

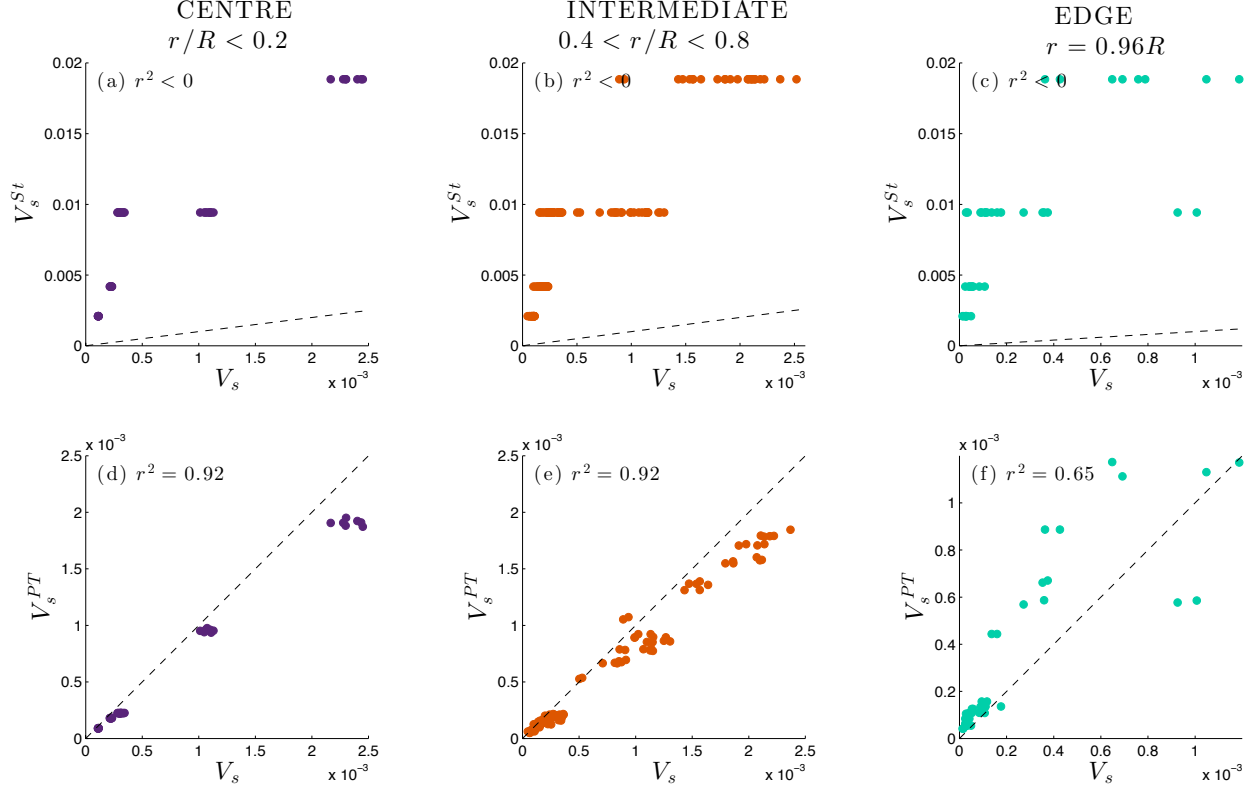


FIG. 8. Velocity predictions compared to simulation results for L cases in the different container regions (values presented in m/s). (a-c) V_s against the velocity predicted by Stokes's model V_s^{St} (28). (d-f) V_s against the velocity predicted by the potential theory V_s^{PT} (29). r^2 is the coefficient of determination. The discontinuous line marks the linear relationship of proportionality one.

in particular, the dispersion decreases substantially at the centre. This result supports the idea that disturbances of unknown origin are occurring near the free surface for the L cases, which do not follow the expected linear behaviour.

V. APPLICABILITY TO HIGHLY FORCED CASES

The aim of this section is to explore the applicability of the PT-Stokes WSS model to highly forced cases using the simulations summarised in Table II.

Firstly, a qualitative schematic overview of the spirals observed at the centre point ($x = y = 0$) for each case study in the E - F - Re parameter space is presented in Fig. 10. Each framed spiral is depicted by the velocity vectors $\mathbf{v}$ (black arrows) in the $\{x, y\}$ plane

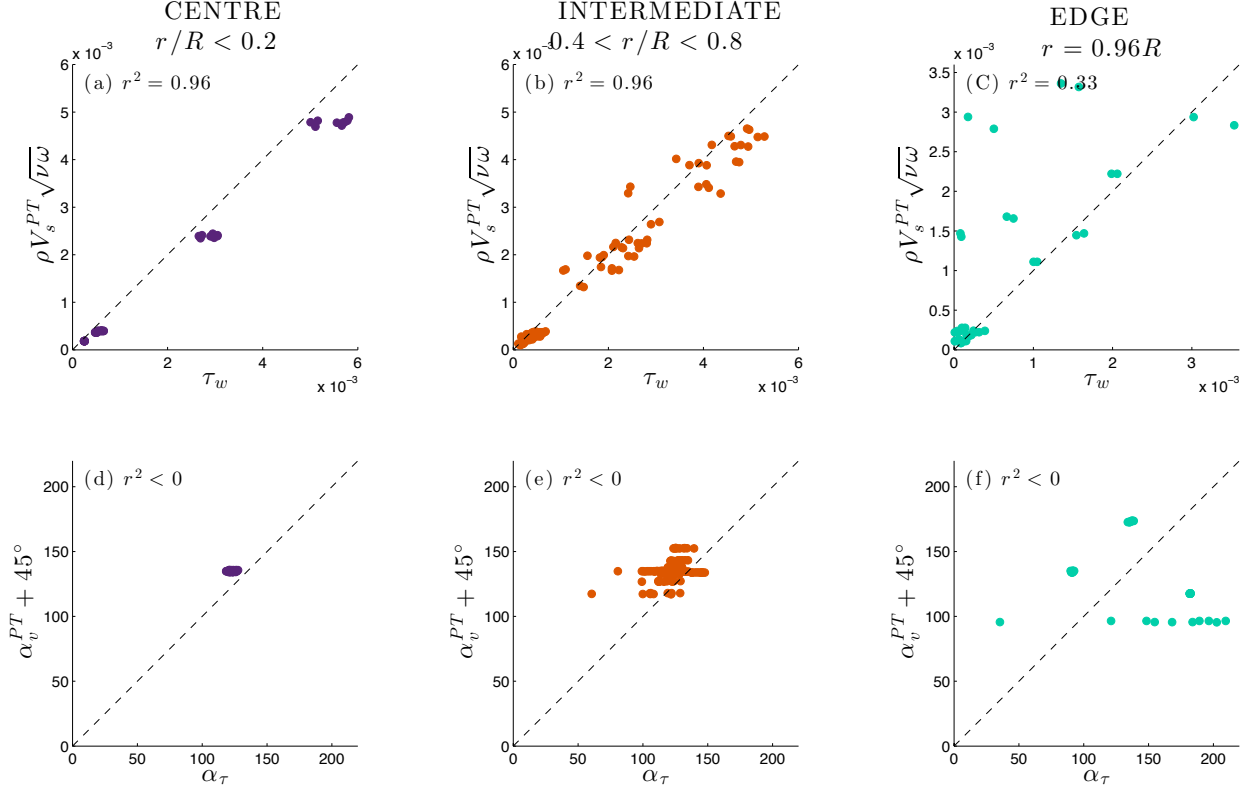


FIG. 9. τ_w prediction by the PT-Stokes WSS model compared to simulation results for L cases in the different container regions. (a-c) Simulated τ_w against the prediction $V_s^{\text{PT}} \sqrt{\rho \mu \omega}$ (values presented in Pa). (d-f) Simulated α_τ against the prediction $\alpha_v + 45^\circ$ (values presented in degrees). r^2 is the coefficient of determination. The discontinuous line marks the linear relationship of proportionality one.

projection from $z = 0$ to $z = h$, and τ_w (red arrow) at the bottom of the container (see § II A for the description of the theoretical Ekman spiral). The spirals are framed using a colour code based on the breaking classification used in Part 1¹: non-breaking (black), breaking and bottom covered (green), and breaking and bottom uncovered (blue). Furthermore, some of the cases are pinned with a star to denote that they have distinctive features, and to connect them with the points in Fig. 11; they are grouped under the following characteristics:

- Group (a) (black frame, yellow star) corresponds to low-forcing cases (L cases), where a wider spiral than the theoretical prediction is observed due to the disturbances observed in the velocity pattern near the free surface (as noted in § IV A).

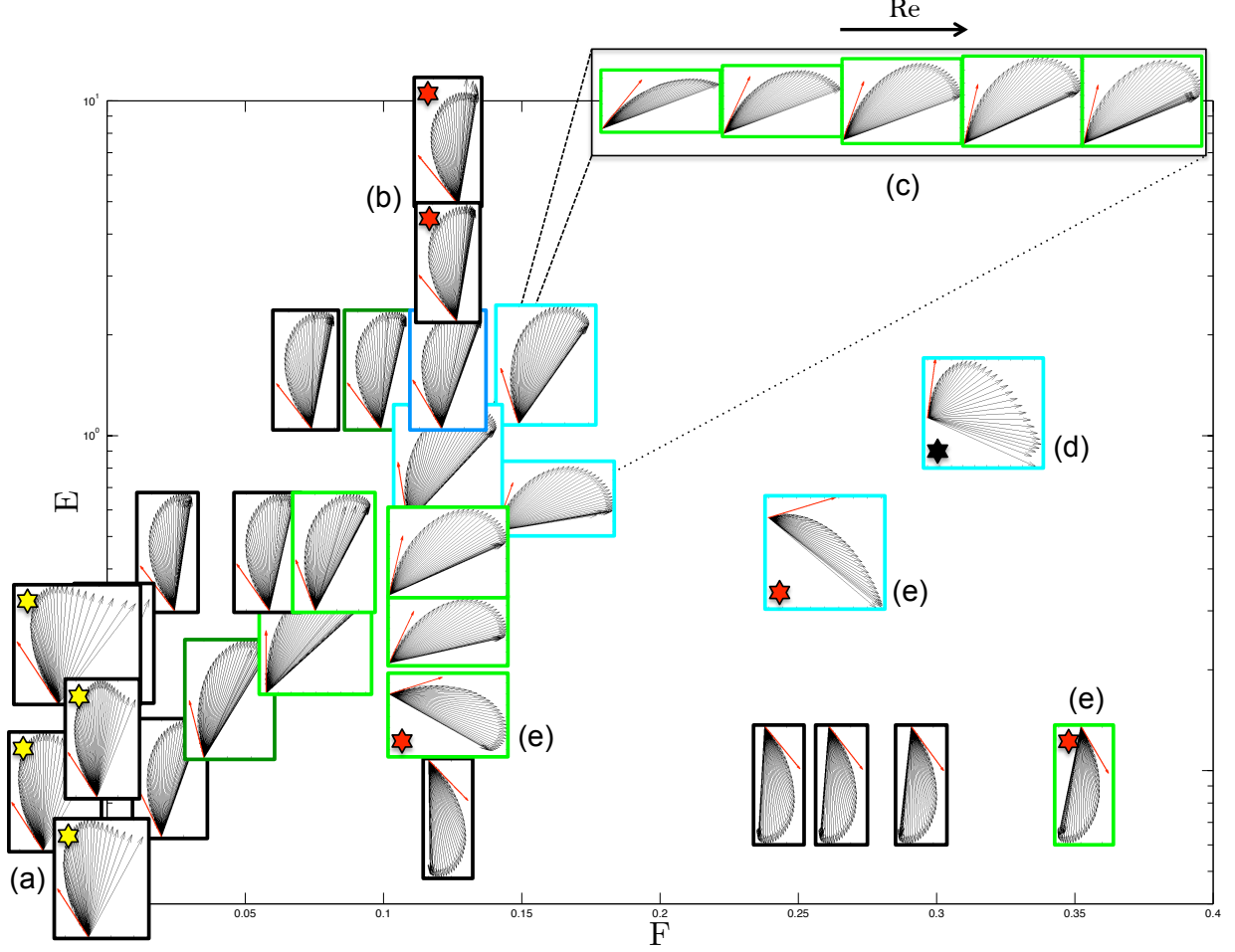


FIG. 10. Ekman-like spirals at the container centre for all case studies. Each frame represents the velocity vectors (black arrows) from $z = 0$ to $z = h$ in the $\{x, y\}$ plane projection at the centre ($x = y = 0$) for all 33 simulations in a E - F - Re parameter space; the red arrow represents τ_w . The frames are colour coded as non-breaking and covered (black), breaking and covered (green), and breaking and uncovered (blue). Some of the cases are pinned with a black, red or yellow star denoting cases with distinctive features (to connect them with Fig. 11), and they are grouped with letters (a) to (e) and described in the main text.

- Group (b) (black frame, red star) corresponds to cases with very high E , where the spiral looks similar to the theoretical Ekman spiral but the velocity counterrotates about 10° and increases in magnitude in the region close to the free surface.
- Group (c) (supplementary window, green frames) corresponds to the Re cases, where

only viscosity is varied, and the spiral is observed to unfold as the viscosity decreases.

- Group (d) (blue frame, black star) corresponds to a case with very high F , where a large breaking area is observed in the free surface.
- Group (e) (green and blue frame, red star) corresponds to cases close to a resonance line, where the potential model overpredicts the surface and velocity amplitudes due to the lack of viscosity to damp resonant frequencies.

A quantitative analysis of the boundary layer thickness δ , the relationship of $\boldsymbol{\tau}_s$ to $\mathbf{v}_s$, and the PT-Stokes WSS model magnitude is presented in Fig. 11 for in the centre, intermediate and edge regions of all cases. The focus below is on explaining the behaviour in the centre region.

The boundary layer characteristic thickness δ shows a good correlation with the viscous penetration depth $\sqrt{\nu/\omega}$ (Fig. 11a, $r^2 = 0.72$) except for the following cases: severely breaking cases (blue-black, group d); and very high E cases (black-red, group b), where points with $\delta > 10^{-3}$ have been omitted since there is an overestimation of δ due to the relative increment of the velocity magnitude near the free surface, deviating the simulated velocity profile shape from the theoretical one.

The WSS angle α_τ is well predicted from α_v (Fig. 11d, $r^2 = 0.94$) except for the following cases: low-forcing cases (black-yellow, group a), which experience a rapid change of direction in the velocity vector at the surface (see spirals in Fig. 10), opening up the spiral and leading to angles between $\mathbf{v}_s$ and $\boldsymbol{\tau}_s$ greater than 45° ; cases with high E (black-red, group b), which have a peculiar behaviour at the free surface despite the expected linear behaviour of the flat surface; and cases with large breaking areas (blue-red, blue-black), which show large scattering.

The WSS magnitude τ_w is well predicted from V_s (Fig. 11g, $r^2 = 0.89$) except for the following cases: very high E cases (black-red, group b), which show a peculiar behaviour in the free surface, as previously noted — it has been verified that the relationship of $\boldsymbol{\tau}_s$ to $\mathbf{v}_s$ improves a couple of computational cells underneath the free surface; and very highly forced cases (blue-black, group d), which show a dispersive behaviour even at the centre region.

The PT-Stokes WSS model predicts τ_w magnitude well (Fig. 11j) except for the following cases: very highly forced cases (blue-black, group d), which are in a very non-linear regime; and the cases close to the resonance lines (green-red and blue-red, group e), where the

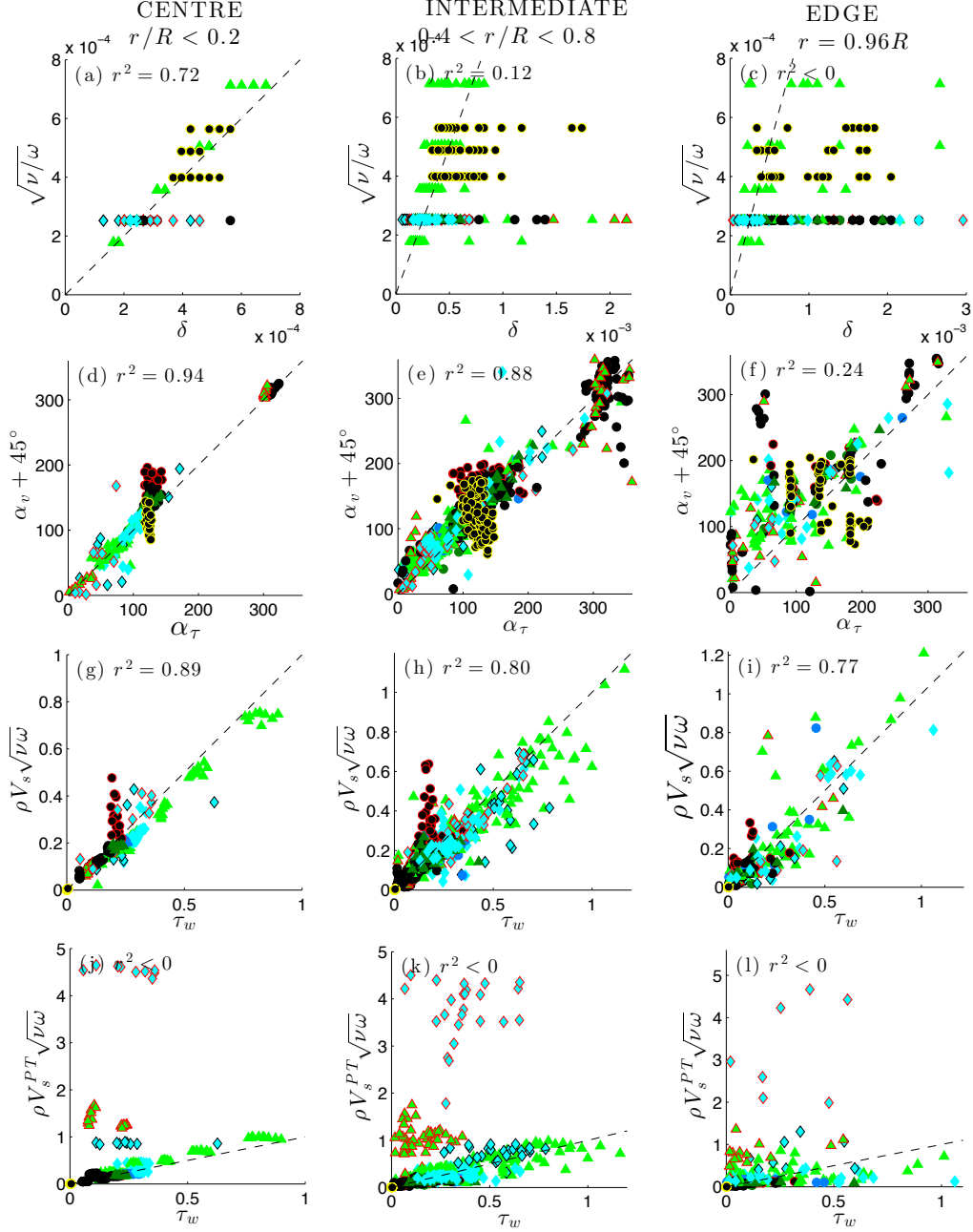


FIG. 11. Prediction versus simulations in all case studies at the centre, intermediate and edge regions. (a)-(c) Boundary layer characteristic thickness δ against the theoretical characteristic length $\sqrt{\nu/\omega}$ (units in m). (d)-(f) Simulated α_τ against the prediction $\alpha_v + 45^\circ$ (units in degrees). (g)-(i) Simulated τ_w against the prediction $\rho V_s \sqrt{\nu\omega}$ in the different regions (units in Pa). (j)-(l) Simulated τ_w against the prediction $\rho V_s^{PT} \sqrt{\nu\omega}$ (units in Pa). Marker's code: black circle (non-breaking and covered), green triangle (breaking and covered), and blue diamonds (breaking and uncovered). Marker's edge colour are used in the text to explain the cases featured.

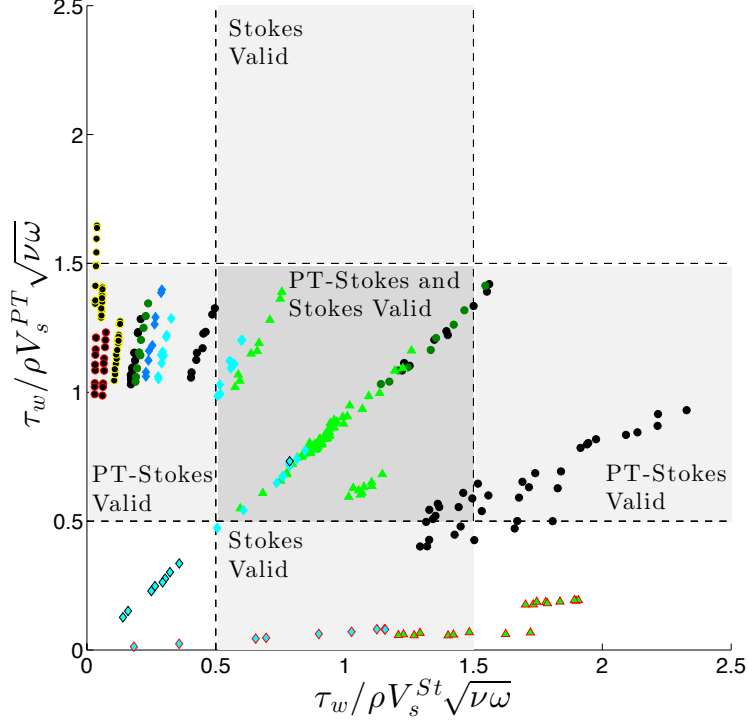


FIG. 12. Comparison of τ_w simulation to prediction ratio in the Stokes and PT-Stokes WSS models for all cases in the centre region. The valid grey shadowed areas are regions where the discrepancy between simulation results and the model specified is $< 50\%$. The marker's code is the same as in § V.

potential prediction overestimates the velocity magnitude due to the lack of viscosity to damp the resonant frequencies. The correlation is poor ($r^2 < 0$); however, if the coefficient of determination is calculated omitting the resonant cases, the correlation improves notably ($r^2 = 0.4$).

The PT-Stokes WSS model is not able to predict the α_τ angle for the highly forced cases (data not shown). The potential theory model used to predict the surface velocity is linear and inviscid, and cannot reproduce the surface velocity angles in the highly force regime where wave breaking alters the linearity of the free surface (a potential and inviscid flow can only satisfy the boundary conditions at a stationary free-surface plane¹⁶, p.366).

The PT-Stokes WSS model is a notable improvement over the Stokes WSS model, as Fig. 12 demonstrates. We have plotted the centre region points for all case studies in a

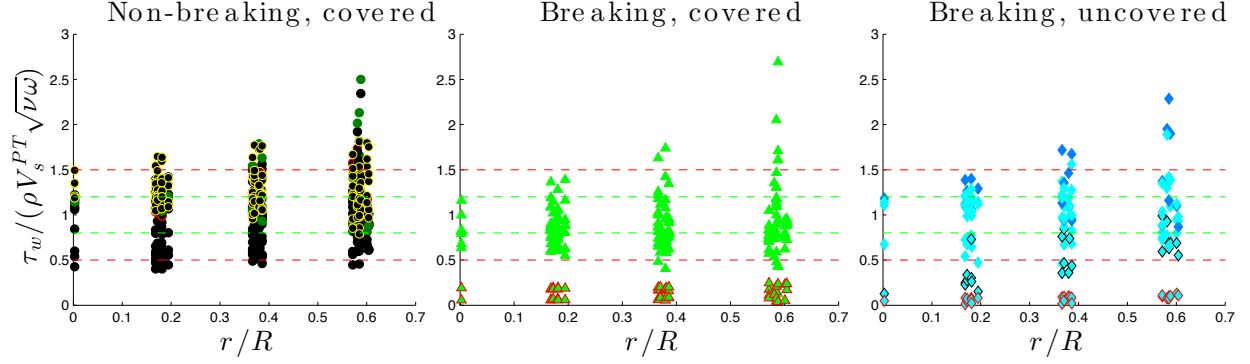


FIG. 13. τ_w simulation to prediction ratio in the PT-Stokes WSS model for the three different regimes identified in Part 1¹.

graph with the simulated τ_w magnitude over Stokes' prediction in the abscissa, and τ_w over PT-Stokes' prediction in the ordinate. The validity domain of each of the models, defined as the region where the discrepancy between simulation and model is $< 50\%$, is depicted as light grey shaded areas, and a dark grey shaded area where they overlap. The new model's validity domain covers practically all the cases covered successfully by the Stokes solution and extends its validity to all non-breaking or incipient breaking cases (black, dark green, and dark blue markers); the exceptions are the resonant cases (blue-red diamonds and green-red triangles), which are partly captured by the Stokes solution but overpredicted by the PT-Stokes WSS model due to the absence of viscosity, as previously discussed. The very highly forced case (blue-black diamonds) is overpredicted by both solutions. Overall, the PT-Stokes WSS model provides better wall shear stress predictions than the classical Stokes WSS model in 63% of cases (21 out of 33).

The PT-Stokes WSS model provides a spatial prediction of τ_w . Fig. 13 shows the simulation to prediction ratio as a function of the normalised radial distance. The PT-Stokes WSS model estimates τ_w within a 50% discrepancy up to the radial distance $r = 0.5R$, with the exception of the resonant and very highly forced cases.

VI. CONCLUSIONS

We have presented an improved Stokes wall shear stress model, termed the PT-Stokes WSS model, that predicts WSS in magnitude and direction for orbitally shaken shallow

fluid layers. The model assumes that the shaken fluid behaves as a potential flow in the bulk of the flows with a relatively thin Stokes boundary layer at the container bottom. The Stokes boundary layer has a characteristic thickness $\sqrt{\nu/\omega}$ (viscous penetration depth), and the horizontal velocity profile takes a spiral-like shape (Ekman spiral) that relates the WSS vector $\boldsymbol{\tau}_w$ to the velocity in the outer layer $\mathbf{v}_s$ (18)-(19). Beyond the boundary layer, the fluid velocity is assumed to be irrotational and described by potential flow theory $\mathbf{v}_s^{\text{PT}}$. The PT-Stokes WSS model is summarised in equations (2), (3), and (29).

The PT-Stokes model was validated by numerical simulation of low-forcing cases (L cases). The simulated vertical velocity profiles are similar to the theoretical prediction in the centre and intermediate regions, and the spiral relationship of the simulated $\boldsymbol{\tau}_w$ to simulated $\mathbf{v}_s$ correlates strongly in magnitude. The Stokes and potential theory prediction for the surface velocity were compared to the simulations. Unsurprisingly, Stokes' solution fails to predict the low-forcing cases, whereas the potential theory is in good agreement. The shear stress predictions of the PT-Stokes WSS model are in good agreement with the simulations, except very close to the side walls, which have their corresponding boundary layer which is not incorporated in the present theory.

The model's applicability to highly forced cases was examined using 33 numerical simulations that cover a wide range of flow behaviours from linear to breaking, uncovered bottom, and resonance. Our simulated results at the centre of the container reveal that all cases show an Ekman-like spiral, with distinctive features in extreme cases. In particular, the influence of viscosity in the ratio static fluid height to boundary layer thickness can be observed in the Re cases, where the spiral is observed to unfold as the viscosity decreases. The relationship of simulated $\boldsymbol{\tau}_w$ to simulated $\mathbf{v}_s$ display a remarkably strong correlation, particularly given the large forcing values that are considered. The PT-Stokes WSS model was shown to predict $\boldsymbol{\tau}_w$ adequately, with the exception of cases close to resonant frequencies, where the potential theory overpredicts the velocity magnitude due to the absence of viscous damping.

The PT-Stokes WSS model represents a significant improvement on the prediction of WSS by the classical Stokes solution for shallow orbitally shaken flows. The novel model has a broader validity than the Stokes model: it predicts τ_w within a 50% discrepancy in areas up to $0.5R$ from the container centre in all cases other than for those close to the resonant line and very highly forced cases. The most important feature of new WSS model is the relationship of $\boldsymbol{\tau}_w$ to the surface velocity $\mathbf{v}_s$ in magnitude and direction, which has

been shown to be valid up to the edge of the container and for most case studies.

The demonstration of a dependence of τ_w magnitude and directionality on the behaviour of the free surface is a significant advance. However, the non-linear nature of the flow complicates the characterisation of orbital shaker fluid dynamics, which the linear and inviscid potential theory cannot capture. The potential model can be refined by expanding on the secondary modes^{19,20}, and by adding viscosity to the irrotational flow solution to cope with free-surface curvature and viscous damping^{20–22}. Furthermore, it is important to understand the occurrence of the secondary flows (toroidal vortex)^{3,23,24} and its relation to the free surface and the wave breaking front. Lastly, further work is required to improve the performance of the PT Stokes WSS model near the shaker’s side boundaries.

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