

THE PREDICTION OF FREE TURBULENT
BOUNDARY LAYERS BY USE OF A TWO-EQUATION
MODEL OF TURBULENCE

by

Wolfgang Rodi

Thesis submitted for the degree of
Doctor of Philosophy
in the Faculty of Engineering
University of London
and
for the Diploma of Membership
of Imperial College

ABSTRACT

A model of turbulence applicable to free boundary layers is described which calculates the Reynolds stress - $\overline{uv}$ from the expression $\overline{uv} = -c_u \sqrt{k} L \partial U / \partial y$; it determines the kinetic energy, k , and the length scale, L , of the turbulence from differential transport equations for k and kL , and obtains the parameter c_u from an empirical function of $\overline{P}/\epsilon$, where the latter is the ratio of the overall production to dissipation of k at each cross section of the layer. Two simpler versions of the model are also described: one assumes that c_u is a constant, and the other takes L as uniform over the cross section.

The empirical constants and functions involved are adjusted to give best agreement between predictions and experiments for a few selected self-similar flows. Predictions are then compared with experiments for a wide variety of situations, including: plane mixing layers between streams of various velocity ratios, plane and axisymmetric jets issuing into still air and into a moving stream, plane and axisymmetric wakes, and flows behind self-propelled bodies (far field only).

The main two-equation model predicts most of these flows with a single set of constants and functions. The agreement is generally within the experimental accuracy. For free boundary layers, this model is found to be superior to simpler as well as to the existing more complex models. In common with all other models, it requires adjustment of some of the constants for the round jet. A function of the ratio of L to the radius r is introduced to extend the universality of the model to this flow.

The two simpler versions of the model work well for more restricted circumstances.

A further, more complex model is described which employs transport equations for all the Reynolds stresses and for the rate of energy dissipation. It is applied to one-dimensional flows and to some of the free flows quoted above. Its constants are found not to be universal, a feature which is attributed mainly to the inadequacy of the pressure-strain simulation used.

A comprehensive account is given of the features of free turbulent boundary layers. In a critical review of experiments, an attempt is made to identify the most reliable data for each class of flow considered.

Finally, an experimental investigation is reported in which a new technique of analysing hot-wire signals in highly turbulent flows was developed and successfully tested in a round jet.

PREFACE

This thesis represents the main result of the research which I carried out in the Mechanical Engineering Department of Imperial College during the period 1968-1972. The major part describes a two-equation model of turbulence, and its application to free boundary layers. This subject of research was suggested to me by my supervisor, Professor D.B. Spalding, who recognised that a turbulence model employing transport equations for both the intensity and the scale of turbulence should be considerably more universal for free flows than are simpler models (Spalding 1967).

My first task was to incorporate the two-equation model suggested by Spalding into the numerical procedure of Patankar and Spalding (1967), and to obtain solutions for a few self-similar free flows having strong strain (jets, mixing layers). This work, which is reported in Rodi and Spalding (1970), revealed that the plane mixing layer, the plane jet, and the radial jet could be predicted with a single set of constants, but that a different set was needed for the round jet.

I then applied the model to weak-strain flows (e.g. wakes) and found that yet another set of constants was needed for these flows. In an attempt to find a more universal model, I simultaneously investigated two lines of approach: one was to replace some of the constants in the two-equation model by functions of suitable parameters; and the other was to study the performance of more complex models which solve transport equations for the individual Reynolds stresses. At the time, Daly and Harlow (1970) had just proposed such a stress-equation model. I applied this, in slightly simplified form, to a few free flows, and found its performance to be unsatisfactory. I then constructed a new version which embodied alternative treatments of certain of the terms in the stress equations. This model performed better in free flows, but it still required the use of different constants for free flows with strong and weak strain, and for round jets.

During the last stage of my research I therefore concentrated on the construction of suitable functions for the two-equation model, and on its application to a large variety of free boundary layers.

From October 1970 to July 1971, I interrupted the theoretical work described above to carry out an experimental study. After lengthy considerations, I had concluded that, in the very limited time available, I could not make a worthwhile study of a hitherto unmeasured free flow situation. I therefore took up a suggestion of my colleague F. Durst to develop an improved method of evaluating hot-wire signals in highly turbulent flows; and I decided to test this method in the well-documented round jet.

I should now like to acknowledge the help which many individuals have offered me throughout the research for, and the writing of, this thesis. I have mentioned already the important role which Professor Spalding played in initiating the research. His active interest and guidance thereafter are also gratefully acknowledged. Furthermore, he has taught me much on the subject of technical-report writing. If the presentation of my work in this thesis is still inadequate, it would have been much worse but for Spalding's constructive criticism.

I should also like to thank Dr.M.M. Gibson and Dr.B.E. Launder for the advice which they gave me as members of my thesis committee. In addition, Dr. Launder provided many helpful discussions on turbulence models and many valuable comments on the draft of this thesis.

The development of the two-equation model was carried out in close cooperation with Dr. K.H. Ng. His influence contributed much to the progress of my work. The research on turbulence models has benefited also from discussions and correspondence with many individuals, particularly Mr. P. Bradshaw, Dr. K. Hanjalić, Dr. W.P. Jones, and Dr. J.C. Rotta.

I am grateful to Dr. F.Durst for suggesting the experimental study and for providing the original idea on which it is based. I should also like to acknowledge his expert help at all stages of this study. I am further indebted to Mr. R. Church for helping to design and for making various pieces of equipment, and to Mr. A. Robinson for his assistance in constructing the rig.

Dr. A.D. Gosman and Dr. J.H. Whitelaw made many suggestions to improve the clarity and style of the thesis. I thank them both for their help.

Space does not allow me to list the names of all the people who contributed to my work. However, two names cannot be passed over: I should like to thank Miss M.P. Steele for her help in administrative matters and Miss E.M. Archer for obtaining even the most obscure of references.

London, December, 1972

W. Rodi

CONTENTS

	<u>Page</u>
<u>ABSTRACT</u>	2
<u>PREFACE</u>	3
<u>CONTENTS</u>	6
1 <u>INTRODUCTION</u>	13
1.1 Background and objective of the investigation	13
1.2 The present contribution	16
a) Development and testing of turbulence models	16
b) Study of flow characteristics	18
c) Experimental study	19
d) Lay-out of the thesis	19
2 <u>THE GOVERNING EQUATIONS AND THE PROBLEM OF CLOSURE</u>	21
2.1 Introduction	21
2.2 Time-dependent equations	21
2.3 The time-averaged continuity and momentum equations	22
2.4 Equations for the Reynolds stresses	23
2.5 The problem of closure	26
2.6 The equations for plane and axisymmetric boundary layers at high Reynolds numbers	27
3 <u>A TWO-EQUATION MODEL FOR FREE TURBULENT BOUNDARY LAYERS</u>	29
3.1 Introduction	29
3.2 Choice of model type	29
3.2.1 The performance of various model classes	29
a) Zero-equation models	30
b) One-equation models	32
c) Two-equation models	32
d) Three-equation models	33
e) Models employing equations for all Reynolds stresses and for a length scale	33
f) Other models	33

	<u>Page</u>
3.2.2 The choice	34
3.3 The shear stress expression	34
3.4 The equations for k and k_L	38
a) The k -equation	38
b) The k_L -equation	40
3.5 The final form of model I	41
Exceptions:	42
a) The round jet	42
b) Shear-free layers	43
3.6 Two simpler versions of the model	44
a) Model II ($c_\mu = \text{const}$)	44
b) Model III ($L = L(x)$)	44
3.7 A semi-empirical derivation of the dependence $c_\mu = f(P/\epsilon)$	45
3.8 Relations with other turbulence models	47
a) Other length scale equations	47
b) The mixing length model as limiting case	48
3.9 Summarising remarks	49
 4 <u>CHARACTERISTIC FEATURES OF FREE TURBULENT BOUNDARY</u> <u>LAYERS</u>	 50
4.1 Introduction	50
4.2 Free boundaries	50
4.3 The momentum behaviour	51
4.4 Self-similar flows	54
4.4.1 Definition of self-similarity	54
4.4.2 Conditions for the existence of self-similarity	55
4.4.3 Potentially self-similar flows	57
a) Exact self-similarity	57
b) Approximate self-similarity	58
4.4.4 The flow examples	59
a) The flow situations	59
b) Decay laws	61
c) Influence of initial conditions	62
d) Momentum and energy relations	62

	<u>Page</u>
4.5 The non-similar flows considered	64
4.5.1 Transition between self-similar flows of different types	64
a) Transition mixing layer-jet	64
b) Transition strong jet - weak jet	65
4.5.2 The near field of free flows	66
5 <u>A CRITICAL REVIEW OF EXPERIMENTAL DATA</u>	68
5.1 Introduction	68
5.2 The data required	68
5.3 Measurement techniques	70
a) Impact tubes	70
b) Hot-wire anemometers	71
5.4 Criteria for the data assessment	73
5.5 The review of experimental data	74
5.5.1 Shear free flows (uniform mean velocity)	74
a) Turbulence behind grids	74
b) Flows behind self-propelled bodies	75
5.5.2 Plane mixing layer	77
5.5.3 Jets issuing into stagnant surroundings	82
a) Plane jet	82
b) Round jet	85
c) Radial jet	86
5.5.4 Self-similar jets and wakes in appropriately tailored pressure-gradients	86
5.5.5 Wakes in zero pressure gradient	87
a) Plane wakes	88
b) Round wakes	90
5.5.6 Jets issuing into a moving stream	93
5.6 Closure to the chapter	95
6 <u>APPLICATION OF THE TWO-EQUATION MODEL TO FREE TURBULENT BOUNDARY LAYERS</u>	96
6.1 Introduction	96
6.2 The equations to be solved	96
6.3 Solution procedure	97

	<u>Page</u>
6.4 Boundary and initial conditions	98
a) Boundary conditions	98
b) Initial conditions	99
6.5 Determination of the constants and functions	100
a) Choice of c_μ	100
b) Range of the constants/functions	100
c) Computer optimisation	101
d) Best-fit constants/functions	103
e) Shear-free regions	105
6.6 Presentation and discussion of results	106
6.6.1 Self-similar flows	107
6.6.1-1 Rate of spread	107
6.6.1-2 Profiles of U , k , $\overline{uv}$, and energy balance	109
a) Plane mixing layer	110
b) Plane jet with $U_E = 0$	111
c) Plane jet with $U_o/U_E = .59 = \text{const}$	111
d) Radial jet	112
e) Round jet	112
f) Plane wake	112
g) Round wake behind a sphere	113
h) Plane shear-free layer	114
6.6.1-3 L-profiles	114
6.6.1-4 Concluding remarks on the self-similar flows predictions	115
6.6.2 Non-similar flows	115
6.6.2-1 Jets issuing into a moving stream	116
6.6.2-2 The near field of jets	117
a) Idealized initial conditions	118
b) Influence of initial conditions	118
c) Comparison with other methods	120
6.6.2-3 Plane wake behind a flat plate	121
6.6.2-4 Plane wake in a pressure gradient	121
6.6.2-5 Concluding remarks on the non-similar flows predictions	121
6.7 Universality of the constants and functions	122

	<u>Page</u>
7 <u>CLOSURE</u>	125
7.1 Conclusions	125
a) Prediction methods	125
b) Flow phenomena	127
c) Accuracy of measurements	128
d) Experimental study	129
7.2 Suggestions for future research	129
7.2.1 Theoretical work	129
a) The round-jet problem	129
b) c_μ -function for wall flows	130
c) General turbulence models	130
d) Study of heat and mass transfer	131
7.2.2 Experimental work	132
 <u>REFERENCES</u>	 133
 <u>NOMENCLATURE</u>	 143
 <u>APPENDIX A</u> Derivation of the equation for kL	 146
A.1 Introduction	146
A.2 Definitions	146
A.3 The exact equation for kL	147
A.4 Simplifying assumptions	149
A.5 The spectral kL - equation	153
 <u>APPENDIX B</u> Conditions for the existence of self-similarity	 155
 <u>APPENDIX C</u> The variation of U_o and δ in self-similar flows	 158
 <u>APPENDIX D</u> The self-similar plane mixing layer	 160
 <u>APPENDIX E</u> The self-similar radial jet	 163
 <u>APPENDIX F</u> Determination of the initial value of L in free stream turbulence	 166

		<u>Page</u>
<u>APPENDIX G</u>	A general stress-equation model $(\overline{u_i u_j \epsilon})$	167
G.1	Introduction	167
G.2	Modelling of the Reynolds-stress equations	167
G.3	The final form of the stress-equation model	171
G.4	Application of the model to one-dimensional flows	172
G.5	Application of the model to free boundary layers	176
G.6	Conclusions	181
<u>APPENDIX H</u>	Predictions with the three-equation model of Hanjalic and Launder ($\tau k \epsilon$)	182
<u>APPENDIX I</u>	Algebraic stress model	185
<u>APPENDIX J</u>	Predictions with the five-equation model of Daly and Harlow	188
<u>APPENDIX K</u>	Review of integral methods	191
<u>APPENDIX L</u>	<u>Experimental study:</u>	194
	A new method of analysing hot-wire signals in highly turbulent flow, and its evaluation in a round jet	
L.1	Introduction	196
L.2	Analysis	197
	L.2.1 Basic relations in hot-wire anemometry	198
	L.2.2 Conventional methods	201
	L.2.3 The new method	204
L.3	Round jet experiments	211
	L.3.1 Experimental set-up	211
	L.3.2 Calibration of hot-wires	213
	L.3.3 Measurement procedure	215
	L.3.4 Data processing	215
	L.3.5 Results and discussion	217
L.4	Conclusions	220

	<u>Page</u>
<u>TABLES</u>	222
<u>FIGURES</u>	231

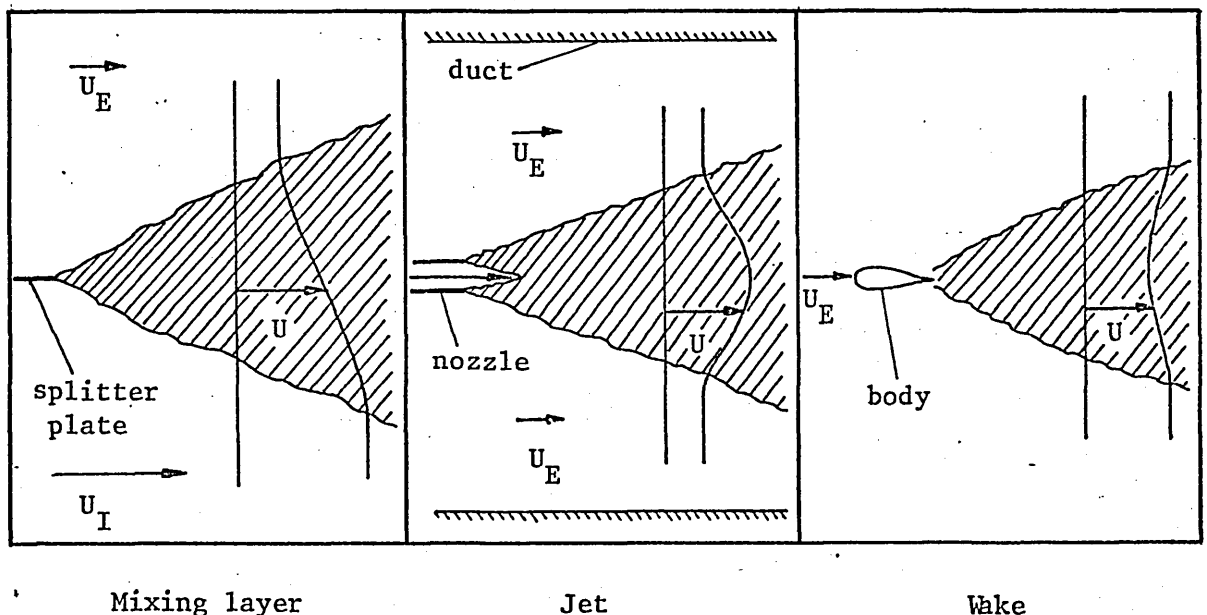
CHAPTER 1

INTRODUCTION

1.1 Background and objective of the investigation

The flows considered. The present investigation is concerned with a class of flows called "free turbulent boundary layers". The label "free" implies that the flows are remote from walls; and the label "boundary layer" is attached to those flow regions in which there is a single predominant direction of flow, and in which shear stresses, heat fluxes and diffusion fluxes are significant only in directions perpendicular to the predominant direction. Three typical examples of free turbulent boundary layers are sketched in fig.1.1: the mixing layer, the jet, and the wake. All the flows considered in this thesis are variations or combinations of these three basic examples. Both plane and axisymmetric situations are included.

Fig.1.1 The three basic examples of free turbulent boundary layers.



The shaded areas in fig.1.1 define the free turbulent boundary layer regions. The flow outside these regions is called free stream. Normally, this flow can be considered as uniform and non-turbulent. Fig.1.1 shows that walls often have an indirect influence on free flows: in all 3 examples, walls (splitter plate, nozzle lips, or wake body) create the free boundary layer; and in the example of the jet, walls govern the free stream.

Practical importance. Free turbulent boundary layers play an important role in many kinds of engineering equipment, in rivers, and in the atmosphere. Jet engines, jet ejectors, wakes behind aeroplanes and submarines, cooling water dispersal in rivers, chimney plumes, and jet streams in the atmosphere are a few examples.

The flows of practical importance may not always be purely "free" and of the plane or axisymmetric boundary-layer type. However, the process of free turbulent mixing is prominent in all these flows, whether it is a mixing of fast and slow moving streams, of hot and cold fluid, or of fuel and oxidiser. And, as a first step to study the real problem, we often study an idealised flow situation (i.e. a free boundary layer). For example, researchers concerned with the combustion process in a furnace often investigate this process with the walls absent; that is they consider the furnace flame as a free boundary layer.

The prediction problem. Because of the practical importance of free turbulent boundary layers, there is a great need for both engineers and meteorologists to be able to predict these flows.

The exact equations governing turbulent flows are well known: they are the unsteady Navier-Stokes equations. Procedures exist to solve these equations numerically; however, because the energy-dissipating eddies are very small, the computational mesh required is so fine that realistic flows cannot be calculated with present-day computers (see § 2.2).

It has therefore become customary to consider only the statistical properties of turbulence. These properties are governed by an infinite number of correlation equations. The usual way to obtain a closed system of equations is to employ a "turbulence model" which approximates the turbulence correlations in terms of knowable quantities. This turbulence-model approach has proved to be the only practicable one so far.

Because they are approximations, turbulence models involve empirical constants and functions. Predictions of unknown flows can be relied upon only when these constants and functions are universal, at least for the classes of flows considered.

The problem of predicting free turbulent boundary layers comprises a physical and a mathematical part: the physical problem is to construct a turbulence model whose constants and functions are universal for these flows. The mathematical problem is to solve the equations appearing in the model.

Previous work. Prandtl's (1925) mixing-length hypothesis was the first turbulence model which was applied with some success to free turbulent boundary layers. The model relates the turbulent shear stress uniquely to local mean flow quantities, which can then be determined by solving the time-averaged equations for momentum and continuity. These are partial differential equations which could not be solved until recently, when efficient numerical methods became available. For self-similar flows, however, the equations can be transformed into more tractable ordinary differential equations, and analytical solutions for such flows were obtained soon after Prandtl had proposed his model (Tollmien 1926, Schlichting 1930). These solutions revealed that the empirical constant associated with the mixing length model varies considerably from one boundary layer to another (see table 3.1). Other simple models (e.g. Prandtl 1942, Reichardt 1942) show a similar lack of universality.

Kolmogorov (1942), Prandtl (1945), Chou (1945), and Rotta (1951) laid the foundations of more complex models which employ differential transport equations for turbulence quantities. However, the resulting

systems of equations were even less amenable to analytical solution than in the case of the mixing-length model; so, at the time this study was started, these models had not yet been applied to free boundary layers.

Because partial differential equations were so difficult to solve, engineers took recourse to integral methods. These methods involve assumptions about the shape of the mean velocity profile and about the global behaviour of turbulence (a review is given in appendix K). The introduction of these assumptions allows the partial differential equations to be reduced to ordinary ones, which are relatively easy to solve. However, integral methods lack flexibility. For example, they are very cumbersome to use when the mean velocity profile changes its shape. Moreover, they are very difficult to extend to high Mach number flows, and to flows with non-uniform temperature and species concentration.

Recent advances in numerical methods (e.g. Patankar and Spalding 1970) have overcome the mathematical difficulties of solving partial differential equations. So engineers need no longer be content with integral methods. However, they still lack an adequate physical model to describe free turbulent boundary layers.

Objective of the investigation. The advances in numerical methods have made it possible to apply nearly any turbulence model to any free boundary layer. Thus, turbulence models can now be tested efficiently; and it is the objective of this investigation to find an economical model, and its constants and functions, which gives good agreement with experimental results for all major free turbulent boundary layers having uniform density.

1.2 The present contribution

a) Development and testing of turbulence models.

Because the time-averaged boundary layer equations can now be solved efficiently with the method of Patankar and Spalding (1970), the

attention of the investigation is deliberately confined to the physical problem, i.e. the modelling of turbulence.

We may classify turbulence models according to the number of equations they employ for turbulence quantities. Altogether seven different models are considered in the thesis which range from one-equation to five-equation models.

The development and testing of a two-equation model for free boundary layers constitutes the main contribution of this thesis. The model is introduced in three versions (I, II, III). It is constructed with the aid of the work of Kolmogorov (1942), Prandtl (1945), and Rotta (1951). It employs the eddy-viscosity concept* and relates the eddy viscosity to the intensity and scale of turbulence. The latter two quantities are determined from differential transport equations; hence the name "two-equation" has been appended to the model.

The empirical constants and functions contained in the equations are determined by reference to experimental data, mainly for self-similar flows. As a test of validity, comparisons between predictions and experiments are made for the following free turbulent boundary layers:

- (i) plane mixing layers between streams of various velocity ratios,
- (ii) plane and axisymmetric jets issuing into still air and into a uniformly moving stream,
- (iii) plane and axisymmetric wakes in zero pressure gradient,
- (iv) plane jets and wakes in a pressure gradient,
- (v) shear-free layers (occur in the far field of flows behind self-propelled bodies).

As a further contribution to turbulence modelling, a general model is developed which employs equations for all Reynolds stresses and for a length scale. This research is undertaken to allow judgement on the relative merits of models of various levels of complexity. The

* shear stress = eddy viscosity $\times$ mean velocity gradient

stress-equation model is applied to one-dimensional flows to determine certain of the constants involved. It is then simplified to yield a two-equation model for general flows, called "algebraic stress model". This model also allows all the Reynolds stresses to be calculated; it may be of use when numerical solutions with a stress-equation model are difficult, or uneconomical, to obtain.

For boundary layers, the above mentioned stress-equation model employs five equations. This five-equation model, Daly and Harlow's (1970) five-equation model, Hanjalić and Launder's (1972) three-equation model, and a one-equation model obtained by simplifying the two-equation model I, are applied to a few free boundary layers. This contributes to the knowledge about the performance of these models. A comparison shows that, for free boundary layers, the present two-equation model appears to be superior to any of the others.

Limitations. The present investigation is concerned only with the hydrodynamic properties of two-dimensional free boundary layers having steady mean quantities, high Reynolds number, uniform density, and no swirl. The investigation is complemented by the researches of Ng (1971) and Jones (1971): Ng applied a two-equation model similar to the present one to wall boundary layers, and Jones developed a two-equation model for low Reynolds number flows.

b) Study of flow characteristics

For a thorough study of a turbulence model, it is essential to have detailed knowledge about the characteristics of the flows for which the testing is to be carried out. As a further contribution, the thesis therefore assembles the available knowledge about free turbulent boundary layers, and in particular about the flows quoted in a).

Simple momentum and similarity considerations yield qualitative information about the behaviour of these flows. Quantitative information is compiled by reviewing critically a large amount of experimental data. The review recommends, for each flow case, the set of data that appears most reliable.

Apart from its importance for the testing of the models considered, this contribution may be of use to future researchers in this area.

c) Experimental study.

The accuracy of many data on turbulence quantities is rather poor, particularly when the turbulence intensity is high. This is due mainly to the inadequacy of existing measurement techniques. To develop an improved method of analysing hot-wire signals in highly turbulent flows, an experimental study is carried out. The test of this method in a round jet provides additional data for this flow case. The results are more consistent, and thus probably more reliable than those obtained with conventional methods.

d) Lay-out of the thesis.

Main body. The main body of the thesis consists of seven chapters of which the introduction is the first one.

Chapter 2 introduces exact forms of the equations for momentum, continuity, and the Reynolds stresses. The closure problem of these equations is discussed, and it is shown that the turbulence-model approach is the only practicable one at present. This approach is described in detail in § 2.5.

Chapter 3 is concerned with the development of the present two-equation model. First, in § 3.2, the performance and economy of models of various levels of complexity are compared, and it is demonstrated that a model employing two equations for turbulence quantities is the most suitable for free turbulent boundary layers. The remainder of the chapter elaborates on the details of the present two-equation model and shows its relations to other models.

Before the model is applied to free boundary layers, the features of these flows are compiled and discussed in chapters 4 and 5. Chapter 4 provides qualitative information mainly about the momentum and similarity behaviour of the flows. Chapter 5 reviews the experimental data and selects "target values" against which the predictions will be compared.

Chapter 6 concerns the application of the two-equation model. The boundary conditions are specified, and the empirical constants and functions are determined by comparing predictions and experiments for a few selected flows. These constants and functions are then used to predict all the flows quoted in a). The results are compared with experimental data and with some published predictions obtained with other methods. The chapter closes with an evaluation of the universality of the present model.

The last chapter, chapter 7, summarises the main conclusions of the thesis and makes some suggestions for future research.

Appendices. The thesis has 12 appendices to which the letters A to L are ascribed. Appendices A to F give derivations which are too long to be included in the main body. Appendix G introduces a new stress-equation model and shows its application to one-dimensional flows and to a few free boundary layers. The application of Hanjalic and Launder's (1972) three-equation model to free boundary layers is discussed in appendix H. In appendix I, a two-equation model is developed for general flows which employs algebraic modelling of all the stresses. Appendix J presents applications of Daly and Harlow's (1970) stress-equation model to the plane mixing layer and the plane jet. Integral methods are reviewed in appendix K. Finally, appendix L reports on the experimental investigation.

CHAPTER 2

THE GOVERNING EQUATIONS AND THE

PROBLEM OF CLOSURE

2.1 Introduction

The purpose of this chapter is to present the basic equations which govern turbulent flows, and to introduce the problems which their solution poses.

§ 2.2 lists a closed set of exact, time-dependent equations and explains why these equations cannot, at present, be solved for turbulent flows. We then turn to time-averaged equations: §§ 2.3 and 2.4 present the equations of continuity, momentum, and the Reynolds stresses in exact form. These equations do not constitute a closed set. The need for additional physical hypotheses to bring about closure is discussed in § 2.5. Finally, in § 2.6, the time-averaged equations are presented in the simplified forms applicable to the boundary layers considered in this thesis.

2.2 Time-dependent equations

Laminar and turbulent flows alike are governed by the laws of mass and momentum conservation. For a Newtonian fluid with constant density and viscosity, these laws are expressed by the following equations:

Mass conservation: continuity equation

$$\frac{\partial U_i}{\partial x_i} = 0, \quad (2.1)$$

Momentum conservation: Navier-Stokes equation

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho} \frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_i \partial x_i}. \quad (2.2)$$

Here, and in all the equations to follow, the summation convention applies where repeated indices appear. The symbols are defined in

the nomenclature, p. 143.

Equations (2.1) and (2.2) constitute a closed set; so all we need to predict a flow, it seems, are boundary conditions and a method of solution. For most laminar flows the equations can indeed be solved by currently available methods, but this is not the case for turbulent flows of engineering interest. The reasons are that (i) turbulence is always three-dimensional, and (ii) many of the important processes take place on very small scales (~ 0.1 mm for the flow in a furnace). The equations can be solved only by means of numerical techniques which calculate the dependent variables at discrete points in the flow. So, with a mesh size of 0.1 mm in all three directions, we would need 10^6 grid points for a volume of one cubic centimetre. However, the fast-access storage of the largest existing computers (e.g. CDC7600) has a capacity of "only" about 150 000 words. This illustrates the impossibility of solving equations (2.1) and (2.2) for realistic turbulent flows.

Fortunately however, the design engineer is not interested in every detail of the turbulent motion, but mainly in the time-averaged velocity and pressure. The next section will present equations governing these quantities.

2.3 The time-averaged continuity and momentum equations

Introduction of time-averaged quantities. For steady mean flow, the time-averaged (or mean) values of U and P are defined by

$$\bar{U}_i \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T U_i dt, \quad \bar{P} \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T P dt. \quad (2.3)$$

Following common practice, we separate the instantaneous velocity and pressure into mean and fluctuating components:

$$U_i = \bar{U}_i + u_i, \quad P = \bar{P} + p. \quad (2.4)$$

For brevity, the overbars indicating averaged values will be dropped from U and P from hereon.

The equations of continuity and momentum. Introduction of (2.4) into (2.1) and (2.2), and subsequent time-averaging yields (Hinze 1959): continuity equation:

$$\frac{\partial \overline{u_i}}{\partial x_i} = 0, \quad (2.5)$$

Momentum equation:

$$u_j \frac{\partial \overline{u_i}}{\partial x_j} = -\frac{1}{\rho} \frac{\partial \overline{P}}{\partial x_i} + \frac{\partial}{\partial x_j} \left(\nu \frac{\partial \overline{u_i}}{\partial x_j} - \overline{u_i u_j} \right). \quad (2.6)$$

These are two very important equations; they govern the mean motion and will therefore, in one form or another, be part of any prediction method* which does not solve the time-dependent equations.

The new unknown $\overline{u_i u_j}$. The momentum equation (2.6) contains a product of the time-averaging process: the Reynolds stress $-\overline{u_i u_j}$ ** This stress represents the transfer of momentum by the turbulent motion. Because of its appearance, equations (2.5) and (2.6) do not constitute a closed set. Our basic task is therefore to find a means for evaluating $\overline{u_i u_j}$.

2.4 Equation for the Reynolds stresses

If we had a transport equation for $\overline{u_i u_j}$, the most direct way to determine $\overline{u_i u_j}$ would be to solve this equation. Indeed, such an equation for $\overline{u_i u_j}$ does exist. It can be derived in the following way. The momentum equation (2.6) is subtracted from the Navier-Stokes equations (2.2) for both the components i and j . The resulting equation for the component i is multiplied with the fluctuating velocity u_j , and the equation for the component j is multiplied with u_i .

* for steady mean flows having uniform density and viscosity.

** Actually, $-\rho \overline{u_i u_j}$ is the stress; but when the density is uniform, ρ can be put equal to unity without loss of generality.

Summation of the two equations, and subsequent time-averaging yields the desired equation for $\overline{u_i u_j}$ (the details of the derivation can be found in Hinze 1959). The equation can be written in the following form:

$$\begin{aligned}
 & \underbrace{U_k \frac{\partial \overline{u_i u_j}}{\partial x_k}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial x_k} (\overline{u_k u_i u_j}) - \frac{1}{\rho} \left(\overline{\frac{\partial u_j p}{\partial x_i}} + \overline{\frac{\partial u_i p}{\partial x_j}} \right)}_{\text{diffusion}} \\
 & \quad - \underbrace{\overline{u_i u_k} \frac{\partial U_j}{\partial x_k} - \overline{u_j u_k} \frac{\partial U_i}{\partial x_k}}_{\text{production}} + \underbrace{\frac{p}{\rho} \left(\overline{\frac{\partial u_i}{\partial x_j}} + \overline{\frac{\partial u_j}{\partial x_i}} \right)}_{\text{pressure-strain}} \\
 & \quad + \underbrace{\nu \frac{\partial^2 \overline{u_i u_j}}{\partial x_k \partial x_k}}_{\text{viscous diffusion}} - \underbrace{2\nu \overline{\frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k}}}_{\text{viscous dissipation}}. \tag{2.7}
 \end{aligned}$$

Before the prospects of solving (2.7) for $\overline{u_i u_j}$ are discussed, the physical meaning of the terms in this equation will be explained briefly. The explanation is facilitated by looking at the contraction of equation (2.7), the equation for the turbulent kinetic energy, $k \equiv \frac{1}{2} \overline{u_i u_i}$:

$$\begin{aligned}
 \underbrace{U_i \frac{\partial k}{\partial x_i}}_{\text{I=convection}} &= \underbrace{\frac{\partial}{\partial x_i} \left[u_i \left(\frac{u_j u_j}{2} + \frac{p}{\rho} \right) \right]}_{\text{II=diffusion}} - \underbrace{\overline{u_i u_j} \frac{\partial U_i}{\partial x_j}}_{\text{production}} \\
 &+ \underbrace{\nu \frac{\partial^2 k}{\partial x_i \partial x_i}}_{\text{viscous diffusion}^*} - \underbrace{\nu \frac{\partial u_l}{\partial x_i} \frac{\partial u_l}{\partial x_i}}_{\text{viscous dissipation}^*} \quad (2.8)
 \end{aligned}$$

Term I represents the convective transport of energy. Term II is zero when it is integrated over the whole turbulent flow field (which is bounded by either non-turbulent flow or walls). This term is therefore a diffusion term; it expresses the diffusive transport of k by the turbulent motion. The "production" - term represents the exchange of energy between the mean and the turbulent motion. It appears with opposite sign in the equation for the energy of the mean motion, $\frac{1}{2} U_i U_i$. The production term is normally positive because, in most cases, energy is transferred from the mean to the turbulent motion. There are two viscous terms: one represents the molecular energy diffusion and the other the transfer of kinetic energy to internal energy*. This latter dissipation term is negative by definition. The molecular diffusion term is negligible when the local Reynolds number of turbulence ($\equiv \sqrt{k}L/\nu$, L = length scale) is high.

The Reynolds-stress equation (2.7) contains terms corresponding to the ones just discussed. In addition it contains the pressure-strain

* Corrsin (1953) pointed out that, to be strictly correct, the viscous terms should be split up as

$$\nu \left[\frac{\partial^2 k}{\partial x_i \partial x_i} + \frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j} \right] - \nu \frac{\partial \overline{u_i}}{\partial x_j} \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right),$$

where the first term represents diffusion and the second term represents dissipation. However, when the turbulence is either locally isotropic or homogeneous, the two ways of writing become

identical (because then $\frac{\partial^2 \overline{u_i u_j}}{\partial x_i \partial x_j} = 0$). Local isotropy means that

the small-scale turbulence, which is responsible for the dissipation, is isotropic. It prevails when the Reynolds number is high.

term which redistributes energy among its components (when $i = j$) and reduces shear stresses (when $i \neq j$). This term therefore tends to make turbulence more isotropic.

The turbulence diffusion, pressure-strain, and dissipation terms in equation (2.7) introduce new unknowns in the form of higher order correlations; so the use of equation (2.7), together with the momentum and continuity equations, does again not lead to a closed system.

2.5 The problem of closure

Exact transport equations can be derived from the Navier-Stokes equations also for the higher-order correlations appearing in equation (2.7). However, correlations of the next higher order are thereby introduced; and the example of equation (2.7) shows that the number of unknowns grows faster than the number of equations. Thus, we may conclude, closure cannot be achieved by resorting to equations for correlations of higher and higher order.

The turbulence model approach. In order to close the system, some of the correlations are usually approximated in terms of quantities that can be calculated. "Model" assumptions about turbulence are thereby introduced which may not always be entirely realistic. Instead of the real turbulent fluid we thus consider an idealised "model" - fluid which is governed by the laws prescribed by a "turbulence model". These laws relate the Reynolds stresses to other time-averaged quantities; they are expressed in differential and/or algebraic equations which, together with the momentum and continuity equations, form a closed set.

This turbulence-model approach is the only practicable one at present. It avoids the difficulty of the Navier-Stokes-equation approach, i.e. the requirement of an extremely large number of grid points, because the time-averaged quantities vary much more gradually than do the time-dependent ones.

The next chapter, § 3, will describe a turbulence model suitable for free turbulent boundary layers. For these flows, the

equations given above can be simplified considerably. The most important of these simplified equations are listed in the following section which concludes this chapter.

2.6 The equations for plane and axi-symmetric boundary layers at high Reynolds numbers.

The flows considered in this thesis are

a) plane or axisymmetric (without swirl),
i.e. $W = 0$, $\frac{\partial}{\partial z} (\text{time-averaged quantities}) = 0$, (2.9)

b) of the boundary-layer type,
i.e. $U \gg V$, $\frac{\partial}{\partial y} \gg \frac{\partial}{\partial x}$, (2.10)

c) high-Reynolds-number flows,
i.e. $v \frac{\partial U_i}{\partial x_j} \ll \overline{u_i u_j}$. (2.11)

Here, and whenever boundary layers are considered in this thesis, we adopt the notation commonly used for these flows: x , y , z for the coordinates, and U , V , W for the velocity components.

With (2.9) to (2.11), the equations governing the mean flow reduce to (Townsend 1956):

Continuity equation:

$$\frac{\partial U}{\partial x} + \frac{1}{y^j} \frac{\partial}{\partial y} (y^j V) = 0, \quad (2.12)$$

Equation for momentum in x - direction:

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = U_E \frac{dU_E}{dx} - \frac{1}{y^j} \frac{\partial}{\partial y} (y^j \overline{uv}) - \frac{\partial}{\partial x} (\overline{u^2} - \overline{v^2}), \quad (2.13)$$

where U_E is the velocity in the free stream, and

$j = 0$ for plane flows,

$j = 1$ for axisymmetric flows.

The kinetic energy equation (2.8) reduces to (Rodi 1970)*:

$$\underbrace{U \frac{\partial k}{\partial x} + V \frac{\partial k}{\partial y}}_{\text{convection}} = - \underbrace{\frac{1}{y^j} \frac{\partial}{\partial y} \left[y^j \overline{v \left(\frac{u_i u_i}{2} + \frac{p}{\rho} \right)} \right]}_{\text{diffusion}} - \underbrace{\overline{uv} \frac{\partial U}{\partial y} - (\overline{u^2 - v^2}) \frac{\partial U}{\partial x}}_{\text{production}} - \underbrace{\overline{v} \frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}}_{\text{dissipation}}.$$

(2.14)

Neglect of normal-stress terms. As a further consequence of the boundary layer assumption (2.10), we may expect the terms involving the normal stresses $(\overline{u^2} - \overline{v^2})$ to be small compared with the other terms in equations (2.13) and (2.14). These normal-stress terms will therefore be neglected in the predictions to follow; so the turbulence model has to determine only the shear stress $-\overline{uv}$. The normal stress terms have however been retained in equations (2.13) and (2.14) so that, by reference to experimental data, their importance can be evaluated and their neglect can be justified (see § 6.2).

* In Rodi (1970), the k -equation is transformed to cylindrical coordinates.

CHAPTER 3

A TWO-EQUATION MODEL FOR FREE TURBULENT BOUNDARY LAYERS

3.1 Introduction

Our objective, in this chapter, is to develop a general and economical turbulence model for free boundary layers.

Our first task is to select the type of model which is to be employed. Thus, in § 3.2, we review models of various levels of complexity and we find that the most suitable model employs transport equations for both the energy and the scale of turbulence.

In § 3.3, the shear stress is related to the energy and length scale of turbulence by way of the eddy viscosity concept; and § 3.4 discusses the assumptions necessary to model the exact equations for energy and length scale. The complete model is then presented in § 3.6. To give further support to the shear-stress expression employed, this expression is derived semi-empirically in § 3.7. Finally, the relation of the present model to other available ones is discussed in § 3.8.

A two-equation model for general flows, and a general stress-equation model are developed in appendices I and G respectively.

3.2 Choice of model type

Turbulence models have been proposed which differ greatly in complexity. For example, Prandtl's (1925) mixing length hypothesis relates the shear stress directly to the mean velocity gradient, while Daly and Harlow (1970) employ differential transport equations for all the Reynolds stresses. To enable a choice of a suitable level of complexity for the present purposes, the performance of various model classes is reviewed briefly. The classes are distinguished by the number of turbulence quantities which appear as dependent variables of differential equations.

3.2.1 The performance of various model classes

The writer has given a fairly detailed review on turbulence models in an earlier publication (Rodi and Spalding 1970). It will

therefore suffice here to repeat the main conclusions of that review, together with some further conclusions drawn from the present study. The conclusions will be supported quantitatively by table 3.1, which compares predictions due to various models with measured rates of spread for the plane jet, the radial jet, and the plane wake.

The eddy-viscosity concept. Before we turn to the individual model classes, we must first familiarize ourselves with the eddy-viscosity concept; for it is an essential component of the majority of those models which do not solve a shear stress equation. For boundary layers, this concept may be expressed as (Boussinesq 1877):

$$\overline{uv} = - \nu_t \frac{\partial U}{\partial y} , \quad (3.1)$$

where ν_t is the "eddy viscosity". ν_t is not a fluid property, but depends on the state of the turbulence; it can vary considerably from one flow to another and also across the flow.

Equation (3.1) expresses the assumption that the shear stress, $-\overline{uv}$, has the same sign as the velocity gradient, $\partial U/\partial y$. Experiments have shown that this appears to be a valid assumption in all free boundary layers.

a) Zero-equation models

The models of this group do not employ equations for turbulence quantities. They relate the shear stress uniquely to the local mean flow conditions. Probably the most familiar example is Prandtl's (1925) mixing-length hypothesis which reads:

$$\nu_t = \ell^2 \left| \frac{\partial U}{\partial y} \right| . \quad (3.2)$$

The mixing length, ℓ , is normally taken as proportional to the width of the shear layer.

Zero-equation models work well for boundary layers where the turbulence structure changes but slowly in the main flow direction. However, the empirical constants involved vary from one flow situation to another and are thus valid for restricted flow regions only. This

lack of universality is illustrated for the example of the mixing-length model in table 3.1. When the mixing length ℓ is taken as a fixed proportion of the flow width (see table 3.1 for details), agreement with experiments can be achieved for only one of the three flows considered in table 3.1.

Table 3.1 Spreading parameters: Comparison of predictions obtained with various models.*

FLOW		plane jet	radial jet	plane wake
Spreading parameter		$\frac{dy_{\frac{1}{2}}}{dx}$	$\frac{dy_{\frac{1}{2}}}{dx}$	$\frac{U_E}{U_O} \frac{dy_{\frac{1}{2}}}{dx}$
PREDICTIONS	Zero-equation model mixing length hyp. $\ell = .254 y_{\frac{1}{2}}^*$	.114	.0568	.0503
	One-equation model $L = .125 y_{\frac{1}{2}}^*$ (see § 6.6)	.114	.0557	.069
	Two-equation model (model I, § 3.5)	.114	.13	.09
	Three-equation model ($\tau k \epsilon$) (appendix H)	.114		.069
	Five-equation model ($\overline{u_i u_j} \epsilon$) (appendix G)	.114		.069
EXPERIMENTS (for references see table 6.5)		.11	.11	.098

* The symbols are defined in fig. 4.2 (P.60). The constants in the zero-and one-equation model were adjusted to give the same rate of spread for the plane jet as the two-equation model.

b) One-equation models

The members of this class employ either a transport equation for the kinetic energy of turbulence, k , or an equation for the eddy viscosity ν_t . Those which employ an equation for k assume either that (Kolmogorov 1942, Prandtl 1945)

$$\nu_t \propto \sqrt{k} L, \quad (3.3)$$

where L is a length scale of turbulence, or that $\overline{uv}$ is proportional to k (Bradshaw et al 1967). All the models require empirical specification of the length scale L .

One-equation models take better account of changes in turbulence structure than do zero-equation models^{*}. However, because they require empirical length-scale specifications, they still lack universality. This is illustrated in table 3.1 which includes predictions with two models which are distinguished only by the way in which the length scale is determined: the two-equation model, which solves a length-scale equation, gives much better general agreement than does the one-equation model, which assumes that L is a fixed proportion of the flow width.

Finally, the requirement of a prescription for the length-scale distribution makes it very difficult to apply one-equation models to more complex flows; for it is only for well documented boundary layers that these distributions are known.

c) Two-equation models

These models determine both the intensity (k) and the scale (L) of turbulence from transport equations. All the models proposed so far employ the eddy viscosity concept and the relation (3.3). All the length-scale equations proposed have $Z = k^m L^n$ as dependent variable, where m and n vary from one author to another (for further details, see § 3.8).

^{*} Note the improved prediction for the plane wake, table 3.1.

Spalding (1967a) suggested that two-equation models should predict, with a single set of constants, the spreading of the free flows for which the mixing-length hypothesis requires different constants. For these flows, the superiority of two-equation models over simpler models is made manifest by table 3.1.

The present study, complemented by Ng's (1971) study on wall flows, will show that two-equation models appear to be sufficiently universal for most boundary layers. Also, they are the simplest models which promise some success in more complex flows.

d) Three-equation models

Three-equation models do not employ the eddy-viscosity concept. Instead, they solve a transport equation for the shear stress, in addition to equations for kinetic energy and length scale (Rotta 1969, 1971, Hanjalić and Launder 1972). For the calculation of boundary layers with significant regions in which shear stress and velocity gradients have opposite signs, employment of such a model seems necessary. For other boundary layers, the existing three equation models are not superior to two-equation ones (see table 3.1, and for more details Appendix H).

e) Models employing equations for all Reynolds stresses and for a length scale

(for boundary layers: five-equation models)

In flows where all the Reynolds stresses need to be calculated, and which are highly anisotropic (e.g. strongly swirling jets), the models reviewed so far will fail. For these flows, models employing transport equations for all the Reynolds stresses are likely to be needed. For boundary layers, however, these models are not, at present, superior to two-equation models. This is exemplified in table 3.1 and, more extensively, in appendices G and J.

f) Other models

Even more complex models employing differential equations for up to 16 correlations have been proposed (e.g. Kolovandin and Vatutin 1969).

However, these models have not as yet reached the stage of application.

Deardorff's (1970) approach does not fit into the present scheme of classification: he solved numerically the time-dependent Navier-Stokes equations for the three-dimensional large scale motion in a plane channel. He modelled the motion of scale smaller than his computational mesh with a "turbulent viscosity" assumption. This method is very expensive of computer storage and time; and, in its present form, it predicts the mean flow features no better than much simpler methods. Although it is of great interest to the turbulence researcher, it is unlikely to become a practical tool for engineers for many years to come.

Economical aspects. The computation time increases with the number of equations to be solved. Thus, the fewer equations a model employs, the more economical it is. Also, it should be mentioned here, multi-equation models need more experiments and computer optimisation to determine the empirical constants than simpler models.

3.2.2 The choice

From the foregoing considerations we may conclude that a two-equation model will meet best our requirements of generality and economy. The two equations to be employed are for the kinetic energy, k , and for the product of energy times length scale, kL . The choice of kL as dependent variable for the length-scale equation will be discussed in § 3.8a.

The next sections are devoted to a detailed description of the model.

3.3 The shear stress expression

The original Kolmogorov-Prandtl formula. When we specify the state of turbulence by the intensity $\sqrt{k}$ and the length scale L^* , dimensional arguments require that the eddy viscosity ν_t be proportional to the

* L may be defined by equation (3.12).

product of $\sqrt{k}$ and L ; thus:

$$v_t = c_\mu \sqrt{k} L. \quad (3.4)$$

This equation was proposed by Kolmogorov (1942), Prandtl (1945), and others. These authors assumed that the proportionality factor c_μ is a universal constant: here we do not. Later however we shall consider a simpler version of the present model, called model II, which employs (3.4) with c_μ as a constant.

The behaviour of c_μ . The use of a constant c_μ has proved successful for many boundary layers, notably those near walls (Ng 1971). However, the predictions with model II will show (table 6.5) that flows with weak strain (e.g. wakes) require a different value of c_μ from flows with strong strain (e.g. jets, wall flows).

In order to investigate the behaviour of c_μ , we make reference to experimental data. Here it is necessary to anticipate the following assumption made in § 3.4:

$$\epsilon = c_D \frac{k^{3/2}}{L},$$

where c_D is a constant. With this assumption, equation (3.4) can be rewritten as

$$v_t = c_\mu c_D \frac{k^2}{\epsilon}, \quad (3.5)$$

and $c_\mu c_D$ can now be determined from measurements of v_t , k , and ϵ . In fig. 3.1, the results are shown for various self-similar boundary layers. The local values of $c_\mu c_D$ are plotted versus the ratio of production to dissipation of kinetic energy, P/ϵ . The importance of this ratio will be explained later. Flows are included only in fig. 3.1 for which all the data were found to be consistent*. Fig. 3.1 shows that $c_\mu c_D$ can vary considerably from one flow to another, even if we allow for a large experimental scatter.

* For consistency checks see §5.

In an attempt to understand the behaviour of $c_\mu c_D$, we turn to the transport equation governing the shear stress. When the convection and diffusion terms in the shear-stress equation (3.20) derived in appendix G are neglected, there results

$$v_t = \frac{1 - \alpha}{\omega} \frac{\overline{v^2}}{k} \frac{k^2}{\epsilon}, \quad (3.6)$$

where α and ω are constants in the pressure-strain simulation of the shear-stress equation (3.20). Comparison of (3.5) and (3.6) reveals that (3.6) becomes the Kolmogorov-Prandtl relation (3.5) when

$$c_\mu c_D = \frac{1 - \alpha}{\omega} \frac{\overline{v^2}}{k}. \quad (3.7)$$

We may therefore conclude that $c_\mu c_D$ can be a constant only when:

- a) convection and diffusion of $\overline{uv}$ are negligible,
- b) $\overline{v^2}/k$ is everywhere a constant,
- c) α and ω are everywhere constants.

We will now examine briefly whether, in the flows of interest, these conditions are satisfied. Figs. 3.3 to 3.6 display measured shear-stress balances of various boundary layers. They show that convection and diffusion of $\overline{uv}$ are indeed negligible in flows with strong strain (jets, wall flows), but that the convection of $\overline{uv}$ is very important in a round wake (weak strain). Thus, condition a) is not satisfied for flows with weak strain.

In fig. 3.7 is plotted the measured distribution of $\overline{v^2}/k$ across various self-preserving boundary layers. The figure indicates that $\overline{v^2}/k$ can vary from about .4 to .8 in these flows. Condition b) is therefore also not generally satisfied.

The pressure-strain simulation in which the constants α and ω appear is an important component of the five-equation model described in appendix G. Because this model solves transport equations for the individual stresses, it accounts for the influence of convection and diffusion of $\overline{uv}$ and for the variation of $\overline{v^2}/k$. However, it still predicts rather poorly flows with weak strain (see table 3.1). This indicates that the pressure-strain constants α and ω are not universal.

We may conclude therefore that the observed variation of $c_\mu c_D$ is a consequence of finite convection and diffusion of $\overline{uv}$ and of the lack of constancy of $\overline{v^2}/k$, α and ω .

c_μ as a function of P/ϵ . Evidently, if we want to predict flows with weak strain, our model must take account of the variation of $c_\mu c_D$. One way to do this, within the framework of a two-equation model, is to make c_μ a function of a suitable turbulence parameter.* The parameter P/ϵ appears suitable because it is a measure of the degree of strain: it represents the ratio of production of turbulence energy (by the straining action of the mean flow) to the dissipation of energy into heat. P/ϵ can also be thought of as a measure of the importance of convection and diffusion; for it indicates how close turbulence is to a state of local equilibrium (for which $P/\epsilon = 1$).

c_μ is therefore chosen to be a function of P/ϵ in the present model. When we bear in mind the experimental scatter, fig. 3.1 suggests that an empirical function of this kind may exist which is effectively universal in large regions of many flows. This function may be thought of as accounting for the influence of convection and diffusion of $\overline{uv}$, and for the variation of $\overline{v^2}/k$, α and ω .

It is clear however from fig. 3.1 that this function is certainly inadequate near symmetry lines. This was confirmed by preliminary predictions employing this function $c_\mu = f(P/\epsilon)$. We eliminate this problem by replacing the local parameter P/ϵ by the integral parameter $\overline{P/\epsilon}$ given by:

$$\overline{P/\epsilon} = \frac{\int_{y_I}^{y_E} |\overline{uv}| P/\epsilon y^j dy}{\int_{y_I}^{y_E} |\overline{uv}| y^j dy}, \quad (3.8)$$

* Actually, $c_\mu c_D$ should be made a function, but we can keep c_D constant without loss of generality. It would be equally possible to fix c_μ and vary c_D .

where y_I and y_E represent respectively the inner and outer boundary of the layer. P/ϵ is weighted with $|\overline{uv}|$ in (3.8) to ensure that regions with large shear stress have a dominating influence on the determination of c_μ .

The use of the integral parameter $\overline{P/\epsilon}$ implies the assumption that c_μ is constant over any cross section. This practice is a departure from the general practice of using only local quantities. It was suggested by the finding that the model using a single constant c_μ (model II) predicts correctly the profile shapes for all flows considered.

The principal model of this investigation, called model I, determines the shear stress from the relation (3.4), with $c_\mu = f(\overline{P/\epsilon})$. The actual form of the function employed will be determined in § 6.

It is evident from fig. 3.1 that a function suitable for most flows is not suitable for the round jet. The special treatment which this case requires is discussed in § 3.5a.

3.4 The equations for k and kL

The use of the Kolmogorov-Prandtl expression (3.4) requires knowledge of the quantities k and L over the whole flow field. We acquire this knowledge by solving differential transport equations for the quantities k and kL.

a) The k- equation

The exact equation for k as derived from the Navier-Stokes equations is given by equation (2.8). For boundary layers at high Reynolds numbers, this equation reduces to (see § 2.6):

$$\underbrace{U \frac{\partial k}{\partial x} + V \frac{\partial k}{\partial y}}_{\text{convection}} = \underbrace{- \frac{1}{y^j} \frac{\partial}{\partial y} \left[y^j v \left(\frac{\overline{u_i u_i}}{2} + \frac{p}{\rho} \right) \right]}_{\text{diffusion}} - \underbrace{\overline{uv} \frac{\partial U}{\partial y}}_{P = \text{production}} - \underbrace{\frac{\partial u_i}{\partial x_j} \frac{\partial u_i}{\partial x_j}}_{\epsilon = \text{dissipation}} \quad (3.9)$$

The meaning of the individual terms was discussed already in § 2.4. We must now make assumptions about the correlations in the diffusion and dissipation terms in order that the equation contains only knowable quantities.

Diffusion of k. We assume that k diffuses down its gradient; thus we write for the diffusion flux:

$$\overline{v \left(\frac{u_i u_i}{2} + \frac{p}{\rho} \right)} = \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial y}, \quad (3.10)$$

where σ_k is a diffusion constant.* This expression was suggested by Kolmogorov (1942), Prandtl (1945), and many others.

Dissipation of k. When the local Reynolds number of turbulence ($\equiv \sqrt{k}L/\nu$) is large, the dissipation of turbulence energy by viscous action occurs mainly in the smallest eddies. However, the rate at which energy is fed into these eddies (which is equal to the rate of dissipation) is governed by the larger energy containing eddies. At large Reynolds numbers, the rate of dissipation, ϵ , is therefore independent of the molecular viscosity, and Kolmogorov (1941) suggested that it can be expressed by

$$\epsilon = c_D \frac{k^{3/2}}{L}, \quad (3.11)$$

where c_D is a constant.*

Introduction of the assumptions (3.10) and (3.11) into (3.9) yields the final k- equation (3.14).

* σ_k and c_D are not universal constants in the present model. They will be allowed to vary in special cases (see § 3.5a, b).

b) The k_L - equation

k_L is defined here as

$$k_L = \frac{1}{2} \int_V Q_{i,i} \frac{dV}{r^2}, \quad (3.12)$$

where $Q_{i,i}$ is the correlation between the velocity fluctuations u_i at two points separated by the distance r , and V is the whole domain of interest (for an illustration, see appendix A).

The exact equation for k_L is derived in appendix A in the following way*: Hinze's (1959) equation for $Q_{i,i}$ is divided by r^2 and then integrated over V . The resulting equation (A.8) takes the form

$$\underbrace{U \frac{\partial k_L}{\partial x} + V \frac{\partial k_L}{\partial y}}_{\text{convection}} = \underbrace{DF_{k_L}}_{\text{diffusion}} + \underbrace{PR_{k_L}}_{\text{production}} + \underbrace{T}_{\text{transfer}} + \underbrace{DIS_{k_L}}_{\text{dissip.}}.$$

The actual expressions for the individual terms are very cumbersome and are therefore not listed here (they are given in appendix A). Like the k -equation, the k_L -equation contains terms representing convection, diffusion, production, and dissipation. In addition it contains a term, T , which originates from the energy transfer between eddies of various sizes. This term acts as a destruction term, while the dissipation term, DIS_{k_L} , is negligible at high Reynolds numbers (see § A.5).

The diffusion, production, and destruction terms contain new unknowns which are related to knowable quantities by model assumptions in § A.4. The following key assumptions are made:

* Rotta (1951) was the first one to derive an equation for k_L . He did this by way of the spectral energy balance (see alternative definition of k_L : equation (A.3))

- 1) All macro scales* are proportional to one another,
- 2) The two-point correlations are symmetric (this is exactly true only in homogeneous turbulence),
- 3) The diffusion of k_L is of the gradient type.

Equation (A.15) presents the resulting modelled k_L -equation. It is identical to Rotta's k_L -equation and contains the four constants c_B , c_S , c_{kL} , and c_{kL} . The diffusion term consists of two parts: the first one involves the gradient of k_L and the second one the gradient of k (and the constant c_{kL}). This form was suggested by Rotta and was employed in the earlier work of Rodi and Spalding (1970). The second part is neglected in the model described here (i.e. $c_{kL} = 0$) because, for the flows considered, the writer found that it has no appreciable influence on the predictions.

The final form of the k_L -equation, applicable to free boundary layers, is given by equation (3.15).

3.5 The final form of model I

The following equations constitute the present two-equation model I of turbulence

$$\overline{uv} = -v_t \frac{\partial U}{\partial y}, \quad v_t = c_\mu \sqrt{k_L}, \quad (3.13)$$

convection	diffusion	production	destruction
------------	-----------	------------	-------------

$$U \frac{\partial k}{\partial x} + V \frac{\partial k}{\partial y} = \frac{1}{y^j} \frac{\partial}{\partial y} \left(y^j \frac{v_t}{\sigma_k} \frac{\partial k}{\partial y} \right) + \underbrace{v_t \left(\frac{\partial U}{\partial y} \right)^2}_{= P} - \underbrace{c_D \frac{k}{L}^{3/2}}_{= \epsilon} \quad (3.14)$$

$$U \frac{\partial k_L}{\partial x} + V \frac{\partial k_L}{\partial y} = \frac{1}{y^j} \frac{\partial}{\partial y} \left(y^j \frac{v_t}{\sigma_{kL}} \frac{\partial k_L}{\partial y} \right) + c_B v_t L \left(\frac{\partial U}{\partial y} \right)^2 - c_S k^{3/2}, \quad (3.15)$$

* Different scales can be ascribed to the processes of convection, diffusion, and production. They are called "macro scales" because these processes are associated with the larger, energy containing eddies.

where c_D , c_B , c_S , σ_k , and σ_{kL} are constants except for the flows discussed below, and c_μ is a function of the parameter P/ϵ .

This parameter was defined in equation (3.8).

Exceptions

We will see in chapter 6 that the model presented above is applicable to most free boundary layers with a single set of constants and a single function for c_μ . There are, however, the following two exceptions which need special treatment:

a) The round jet

Fig. 3.1 indicates that a function $c_\mu c_D = f(P/\epsilon)$ suitable for most flows, including some axisymmetric flows (pipe flow, round wake), is not suitable for the round jet. Preliminary test predictions have confirmed this finding, and they have shown that c_S must also take a different value in order to obtain the measured level of kinetic energy. The reasons for this lack of universality are not yet fully understood; the reader will find a discussion on this matter in § 6.7.

In order to make the model applicable also to the round jet, we replace the constants c_D^* and c_S by functions of the parameters r/L and $\frac{y_E}{U_0} \frac{dU_t}{dx}$. The symbols appearing in these parameters are explained in fig. 4.2 (p. 60). The parameter r/L accounts for the presence of the axis of symmetry in round jets and distinguishes these flows from plane flows. The second parameter, $\frac{y_E}{U_0} \frac{dU_t}{dx}$, distinguishes the round jet from axisymmetric flows where the change of U with x is zero or very small (pipe flows, round wake).

The following empirical functions are proposed:

* Actually, the function $c_\mu c_D = f(P/\epsilon)$ should be altered. However, to separate the influences of P/ϵ and r/L , the function $c_\mu = f(P/\epsilon)$ is retained and c_D is altered.

$$c_D = c_{D\infty} - \frac{c_{D1}}{(r/L)^2 + c_{D2}} f, \quad (3.16)$$

$$c_S = c_{S\infty} - \frac{c_{S1}}{(r/L)^2 + c_{S2}} f, \quad (3.17)$$

with

$$f = \left| \frac{y_E}{2U_0} \left(\frac{dU_t}{dx} - \left| \frac{dU_t}{dx} \right| \right) \right|^q,$$

where c_{D1} , c_{D2} , c_{S1} , c_{S2} , and q are new constants, and $c_{S\infty}$ and $c_{D\infty}$ are the values of c_S and c_D suitable for all other flows. The form of the functions was obtained by trial and error.

The reader will notice that, for plane flows (where r/L is very large) and for axisymmetric flows with zero change of longitudinal velocity, the above functions provide the values of c_D and c_S suitable for these flows. Thus, the universality of the model is preserved.

b) Shear-free layers

The model presented so far does not predict satisfactorily the kinetic energy profile in shear-free layers (see fig. 6.13). We effect remedy by keeping v_t constant over any cross section of such layers. This practice, which is entirely empirical, is equivalent to taking the local values of v_t but allowing the σ 's to vary accordingly (so that v_t/σ remains constant). The details of this practice are discussed in § 6.5e).

Constants and functions. The constants and the function for c_μ which are suited best for free flows can be determined only by comparison of predictions with experimental results. Discussion on their determination is therefore deferred to the chapter on predictions, § 6. The constants and the function employed are presented in § 6.5.

3.6 Two simpler versions of the model

a) Model II ($c_\mu = \text{const}$)

Model II employs a universal constant for c_μ in place of the function $c_\mu = f(\overline{P/\epsilon})$ used in model I. Model II is identical to Ng's (1971) model for wall boundary layers, except for the omission of Ng's additional term near walls in the kL - equation. Model II was also used in the earlier study of Rodi and Spalding (1970).

The discussion on the behaviour of c_μ (§ 3.3) has shown that model II can be expected to perform well only in near-equilibrium flows ($\overline{P/\epsilon} \approx 1$) which do not include round jets.

b) Model III ($L = L(x)$)

The predictions with model I presented in § 6.6 indicate that L varies but moderately over any cross section of most free turbulent boundary layers. Based on this finding, model III assumes L to be a function of x only. This assumption implies that the diffusion of L is very large; the assumption is used here to transform the partial differential equation for kL into an ordinary differential equation for L : equations (3.14) and (3.15) are first integrated over the whole cross-section and then combined to yield:

$$\frac{dL}{dx} = L \frac{I_p(c_B - 1) + I_\epsilon(1 - c_S/c_D)}{I_{Uk}}, \quad (3.18)$$

where

$$I_p = \int_{y_I}^{y_E} P y^j dy, \quad I_\epsilon = \int_{y_I}^{y_E} \epsilon y^j dy, \quad I_{Uk} = \int_{y_I}^{y_E} U k y^j dy. \quad (3.19)$$

Model III employs (3.18) instead of (3.15) to determine L . In all other respects, model III is identical to model I.

Because model III solves an ordinary differential equation for the length scale instead of a partial one, it is superior to model I as regards computational economy. As for the predictive power, judgement must await the comparison of results presented in § 6.

A model employing a length scale equation similar to (3.18) has earlier been proposed by Spalding (1967a).

3.7 A semi-empirical derivation of the dependence $c_\mu = f(P/\epsilon)$

The writer had hoped initially that the function $c_\mu = f(P/\epsilon)$ could be obtained by simplifying the five-equation model of appendix G. Indeed, a function $c_\mu = f(P/\epsilon)$ can be derived in this way which reflects qualitatively the experimental behaviour of $c_\mu c_D$ shown in fig. 3.1. However, because the pressure-strain term in the five-equation model lacks universality, this function is quantitatively inadequate. Nevertheless, its derivation will be presented in this section to give some theoretical support to the entirely empirical introduction of $c_\mu = f(P/\epsilon)$ in § 3.3.

The derivation proceeds by modelling the convection and diffusion terms in the shear-stress equation given in appendix G.

The shear-stress equation of the five-equation model of appendix G reads (equation G.5-4):

$$\underbrace{U \frac{\partial \overline{uv}}{\partial x} + V \frac{\partial \overline{uv}}{\partial y}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial y} \left(\beta \frac{k \overline{v^2}}{\epsilon} \frac{\partial \overline{uv}}{\partial y} \right)}_{\text{diffusion}} - \underbrace{\omega \frac{\epsilon}{k} \overline{uv}}_{\text{pressure-strain}} + \underbrace{\alpha \overline{v^2} \frac{\partial U}{\partial y}}_{\text{product}}, \quad (3.20)$$

where β , α , and ω are assumed to be constants.

Algebraic stress modelling. The convection and diffusion terms of (3.20) make this equation a differential equation. Because, in a two-equation model, we need an algebraic shear-stress expression, we have to model these terms. This "algebraic stress modelling" is discussed for general stresses in appendix I. There, the following relation is proposed:

$$(\text{Conv. -Diff.})_{\text{of } \overline{uv}} = \frac{\overline{uv}}{k} (\text{Conv. -Diff.})_{\text{of } k} = \frac{\overline{uv}}{k} (P - \epsilon). \quad (3.21)$$

Equation (3.21) is based on the assumption that $\overline{uv}/k$ varies but slowly across the flows.

When (3.21) is introduced into (3.20), there results the following algebraic expression for $\overline{uv}$:

$$\overline{uv} = - \frac{1 - \alpha}{\omega \left[1 + \frac{1}{\omega} (P/\epsilon - 1) \right]} \frac{\overline{v^2}}{k} \frac{k^2}{\epsilon} \frac{\partial U}{\partial y}. \quad (3.22)$$

In a similar fashion, we obtain by simplifying the $\overline{v^2}$ -equation (see appendix I):

$$\overline{v^2} = \frac{2}{3} k \frac{1 - \frac{1}{\omega} (1 - \alpha P/\epsilon)}{1 + \frac{1}{\omega} (P/\epsilon - 1)}. \quad (3.23)$$

Elimination of $\overline{v^2}$ in (3.21) by means of (3.22) yields:

$$\overline{uv} = - \frac{2}{3} \frac{1 - \alpha}{\omega} \frac{1 - \frac{1}{\omega} (1 - \alpha P/\epsilon)}{\left[1 + \frac{1}{\omega} (P/\epsilon - 1) \right]^2} \frac{k^2}{\epsilon} \frac{\partial U}{\partial y}. \quad (3.24)$$

The reader will notice that (3.24) is equivalent to a combination of the eddy-viscosity relation (3.1) and the Kolmogorov-Prandtl formula (3.5) when the following equality holds:

$$c_{\mu} c_D = \frac{2}{3} \frac{1-\alpha}{\omega} \frac{1 - \frac{1}{\omega}(1-\alpha P/\epsilon)}{\left[1 + \frac{1}{\omega}(P/\epsilon - 1)\right]^2} \quad (3.25)$$

Thus, we have derived $c_{\mu} c_D$ as a function of P/ϵ . This function is shown for various sets of ω and α in fig. 3.2. Comparison with the experimental results of fig. 3.1 reveals that the general trend is predicted correctly. However, no single set of ω and α produces a function which is in quantitative agreement with the bulk of the experimental data. This, it seems, is due mainly to the lack of universality of the constants ω and α in the pressure-strain simulation.

The neglect of convection and diffusion of $\overline{uv}$ has led to $c_{\mu} c_D = \text{const}$ (§3.3). In the above derivation we have included these terms and have obtained $c_{\mu} c_D$ as a function of P/ϵ . This derivation has therefore emphasized the importance of the parameter P/ϵ as a measure of the effects of the transport of $\overline{uv}$.

3.8 Relations with other turbulence models.

a) Other length scale equations

Length scale equations were proposed in the literature for the following quantities as dependent variable:

L	(Rotta 1951, Harlow and Nakayama 1967)
kL	(Rotta 1951)
$\epsilon \propto \frac{k^{3/2}}{L}$	(Chou 1945, Davydov 1961, Daly and Harlow 1970)
$k^{1/2}/L$	(Kolmogorov 1942)
k/L^2	(Spalding 1970, Saffman 1970).

All the length-scale equations proposed are thus for a variable $Z = k^m L^n$; but they differ in respect of the choice for m and n . They all possess a common form which reads:

$$\underbrace{U \frac{\partial Z}{\partial x} + V \frac{\partial Z}{\partial y}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial y} \left(\frac{\sqrt{k} L}{\sigma_Z} \frac{\partial Z}{\partial y} \right)}_{\text{diffusion}} - \underbrace{c_{Z1} Z \frac{\overline{uv}}{k} \frac{\partial U}{\partial y}}_{\text{production}} - \underbrace{c_{Z2} Z \frac{\sqrt{k}}{L}}_{\text{destruction}} + S,$$

where S represents a secondary source term which differs according to the choice of Z and also from one author to another. With the exception of this secondary term, which is important mainly near walls*, and of the diffusion term, the terms in the equations for different Z 's may be shown to be equivalent with the aid of the k -equation (3.14). Thus, the main difference between the equations rests with the diffusion term.

When the length scale L is uniform with y , then the diffusion terms are also equivalent, and models using different Z 's are identical. In most free boundary layers, L varies but moderately with y ; hence the choice of Z has little influence on predictions of these flows. This was confirmed quantitatively by comparison of predictions obtained with the kL -equation (56), the ϵ -equation (some unreported test runs), and the k/L^2 -equation (Spalding 1971). The choice of $Z = kL$ is therefore as good as any other possible choice.

b) The mixing-length model as a limiting case

The Kolmogorov-Prandtl formula (3.4) can be converted into the mixing-length expression in the following way:

with $P = \nu_t (\partial U / \partial y)^2$, equation (3.11) can be rewritten as

$$k^{3/2} = \epsilon \frac{L}{c_D} = \frac{\epsilon}{P} \frac{L}{c_D} \nu_t \left(\frac{\partial U}{\partial y} \right)^2. \quad (3.26)$$

With the aid of (3.26), we can eliminate $\sqrt{k}$ in the Kolmogorov-Prandtl formula (3.4) and obtain

$$\nu_t = \left(\frac{c_\mu}{c_D} \frac{\epsilon}{P} \right)^{1/2} L^2 \left| \frac{\partial U}{\partial y} \right|.$$

This is the mixing-length formula already introduced in equation (3.2). The mixing length ℓ is related to the length scale L by

$$\ell = \left(\frac{c_\mu}{c_D} \frac{\epsilon}{P} \right)^{1/4} L. \quad (3.27)$$

* Thus, for example, Ng (1971) used the additional term $S = \text{const}(L/y)^6 k^{3/2}$ near walls in his kL -equation. y is the distance from the wall.

With the usual assumption that l is a fixed proportion of the shear layer width δ , the mixing length hypothesis implies that

$\left(\frac{c_\mu}{c_D} \frac{\epsilon}{P}\right)^{1/4} \frac{L}{\delta}$ is a constant everywhere*.

3.9 Summarising remarks

A two-equation model, called model I, has been developed in this chapter which possesses the following main features: the shear stress is calculated from the expression (3.13) $\overline{uv} = -c_\mu \sqrt{k} L \partial U / \partial y$, which is a combination of the eddy-viscosity concept (3.1) and the Kolmogorov-Prandtl formula (3.4). The kinetic energy k and the length scale L in this formula are determined from transport equations for k (3.14) and kL (3.15). The parameter c_μ is allowed to vary with x . It is calculated from an empirical function $c_\mu = f(\overline{P/\epsilon})$ where $\overline{P/\epsilon}$ is the ratio of the overall production to dissipation of k at each cross section.

Two simpler versions of model I have been introduced in § 3.6: model II uses $c_\mu = \text{const}$ and model III assumes that the length scale L is uniform over any cross section.

This concludes the introduction of two-equation models. § 6 will show how well they work.

* It has been stated frequently in the literature that the mixing length hypothesis implies $P = \epsilon$. Equation (3.27) shows that this need not be the case. When L is uniform with y and c_μ and c_D are constants, then the mixing length hypothesis implies that P/ϵ is constant; but P/ϵ need not equal unity.

CHAPTER 4

CHARACTERISTIC FEATURES OF FREE TURBULENT BOUNDARY LAYERS

4.1 Introduction

It may be helpful to describe the features of free turbulent boundary layers before we attempt, in § 6, to reproduce these features with the turbulence model of § 3. The present chapter describes qualitatively the behaviour of free turbulent boundary layers; and the next chapter on experimental data will provide quantitative information.

The introductory chapter, § 1, has presented already the basic features which distinguish free boundary layers from other flows. § 4.2 discusses further aspects of the presence of a free boundary; and in § 4.3 we look at the overall momentum behaviour of free flows.

§ 4.4 deals with self-similar flows at some length. These flows deserve special attention because they will provide the main empirical input for the turbulence model of § 3. They are suited particularly for a detailed comparison between predictions and experimental data: for, in general, they are independent of the exact conditions of flow generation, and they facilitate consistency checks on the experimental data.

The features of the non-similar flows considered in this thesis are discussed in § 4.5.

4.2 Free boundaries

The interface. The actual free boundary between turbulent and non-turbulent flow is sharp, highly unsteady, and contorted. This boundary is called the interface. Its essential action is to propagate vorticity, that is to make irrotational fluid rotational. This is the basic mechanism of entrainment due to which the turbulent layer width grows in the downstream direction. The transmission of vorticity occurs due to viscous shearing forces in a very thin layer, called the viscous superlayer.

Owing to the random movement of the interface, the fluid at a fixed point near the edge of free flows is alternately turbulent and non-turbulent. This phenomenon is called intermittency, and the intermittency factor γ is defined as the fraction of the time the fluid is turbulent.

Boundary conditions. We are concerned here mainly with the overall behaviour of the flow. We therefore time-average over all events and do not distinguish between turbulent and non-turbulent periods. Consequently, we must redefine the "free" boundary; simply to take the time-average position of the interface would not be appropriate; for we would cut off a region in which significant changes in mean quantities occur. The free boundary is therefore defined here as the position where the mean velocity starts to deviate from the velocity which prevails far away from the turbulent region. Thus, the mean velocity gradient is zero at the boundary; and so is the shear stress. This absence of a shearing force exerted on the flow by the boundary is the main difference in boundary conditions from wall flows.

The local Reynolds number. Another basic difference between free flows and wall flows concerns the local Reynolds number of turbulence. This Reynolds number may be defined as $\sqrt{k}L/\nu$; it is a measure of the relative importance of molecular and turbulent transport. Near walls, there is always a thin but important layer where this Reynolds number is low and the molecular transport is predominant. Although the viscous forces are essential for the mechanism of entrainment, the local Reynolds number in free flows is high and the turbulent transport dominates everywhere, provided the overall Reynolds number is high. For this reason, we could neglect the molecular diffusion terms in the transport equations of § 3; and we could assume that the constants in the turbulence model of § 3 are independent of the Reynolds number.

4.3 The momentum behaviour

The general behaviour of a free flow depends primarily on the force applied to its fluid. This force may be the momentum flow through an orifice, the drag force of a body or a combination of both. This last, most general case is pictured in fig.4.1 (symmetrical flows only).

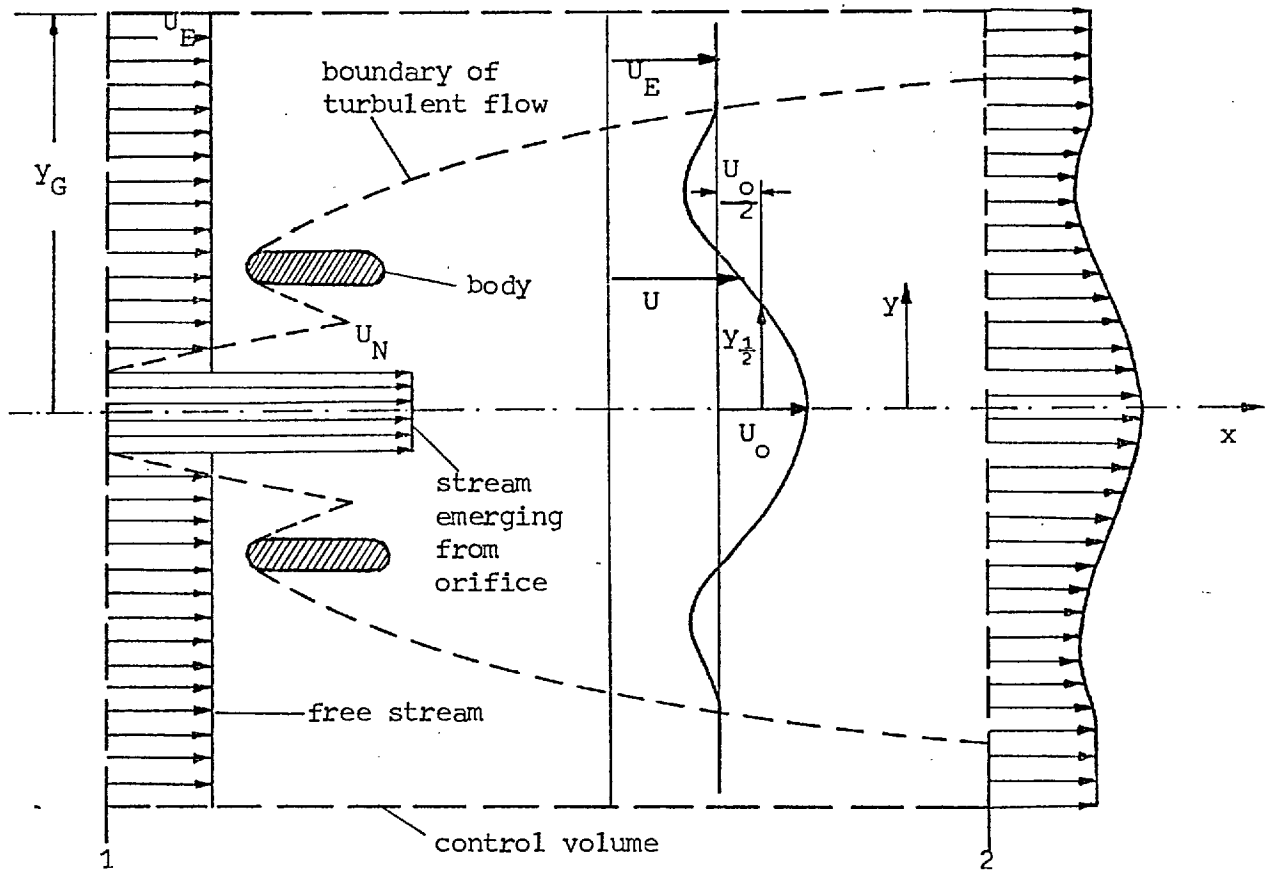


Fig. 4.1

In order to study the effect of the force on the flow, we construct the momentum balance for the fluid in the control volume drawn in fig. 4.1. This control volume is drawn to include all bodies submerged in the fluid, and the whole free turbulent flow upstream of cross section 2. Cross section 2 may be placed at any x -station downstream of the bodies (if present).

The integral momentum equation. We start our momentum considerations from the momentum equation (2.13). Elimination of V by means of the continuity equation (2.12) and integration from $y = 0$ to $y = y_G$ yields the integral momentum equation:

$$(2\pi)^j \frac{\partial}{\partial x} \int_0^{y_G} [U(U-U_E) + (\overline{u^2 - v^2})] y^j dy + (2\pi)^j \frac{dU_E}{dx} \int_0^{y_G} (U-U_E) y^j dy = - \frac{B}{\rho} . \quad (4.1)$$

B is the total body force per unit length at the cross section considered. In the free-flow region of interest (cross section 2), B is zero. There, equation (4.1) implies that the change with x of the excess momentum flux

$$M = (2\pi)^j \rho \int_0^{y_G} [U(U-U_E) + (\overline{u^2 - v^2})] y^j dy \Big|_{x=x_2} \quad (4.2)$$

is equal to the excess mass flux, $(2\pi)^j \rho \int_0^{y_G} (U-U_E) y^j dy$, times the free-stream velocity gradient.

The momentum balance for $U_E = \text{const}$. When the free-stream velocity U_E is constant, (4.1) implies that the excess momentum flux M is independent of x and equal to the force applied to the fluid:

$$M = \text{const} = (2\pi)^j \rho \int_0^{y_G} U(U-U_E) y^j dy \Big|_{x=x_1} - F . \quad (4.3)$$

This is the momentum balance, which can be derived formally by integrating (4.1) from $x = x_1$ to $x = x_2$. The first term on the right hand side is the excess momentum flux through the orifice (the normal stresses at $x = x_1$ are usually negligible), and F is the total drag force of the bodies.

The momentum balance (4.3) is applicable to all symmetrical free flows except for the radial jet. This case, and the asymmetric mixing layer, are dealt with in appendices E and D respectively.

Three groups of free flows. From (4.3) we see that M can be positive, negative, or zero. Accordingly, symmetrical free flows can be divided into the following three groups:

- a) $M > 0$. These flows are jets. The velocity is, in general, larger than U_E .
- b) $M < 0$. These flows are wakes or velocity deficit jets with wake character. The velocity is smaller than U_E .
- c) $M = 0$. These are flows behind self-propelled bodies. They are also called shear-free layers because, some way downstream of the body, the shear stress is zero (the mean velocity is uniform = U_E).

4.4 Self-similar flows

An important feature of free turbulent boundary layers is their tendency to become self-similar after a certain development region. For a study of this feature, self-similarity must be defined first.

4.4.1 Definition of self-similarity

The present definition of self-similarity is as follows:

A flow is considered self-similar when one velocity and one length scale are sufficient to render its time-averaged quantities dimensionless functions of one geometrical variable only*.

The velocity scale chosen here is the maximum excess (or deficit) velocity U_0 defined in fig. 4.1. For shear-free layers, where U_0 is zero, the square root of the maximum kinetic energy, $\sqrt{k_0}$, is taken instead. A characteristic width of the flow, δ , is employed as length scale. This width may be defined in various ways; for example we could take the half width, $y_{1/2}$, as defined in fig. 4.1. In order to keep the analysis general, δ is not specifically defined here.

* Naudascher (1967, 1968) proposed a different definition of self-similarity. He introduced two additional scales k_c and uv_c which are characteristic values of k and uv at each cross-section. Unlike the above definition, his definition allows flows to be called self-similar in which the relative intensity ($\sqrt{k}/U_0$) and shear stress (uv/U_0^2) vary from one cross-section to another.

The above definition of self-similarity implies that the following quantities of importance can be expressed as

$$\begin{aligned}
 U &= U_E + U_0 f(\eta) \\
 V &= U_0 g(\eta) \\
 k &= U_0^2 e(\eta) \\
 \overline{u^2} - \overline{v^2} &= U_0^2 q(\eta) \\
 \overline{uv} &= U_0^2 h(\eta) \\
 v_t &= U_0 \delta \tilde{v}_t(\eta) \\
 \epsilon &= \frac{U_0^3}{\delta} \tilde{\epsilon}(\eta),
 \end{aligned} \tag{4.4}$$

where $\eta = \frac{y}{\delta}$,

and f, g, e , etc are functions of η only. The free-stream velocity U_E is prescribed by constraints on the flow outside the boundary-layer region.

4.42 Conditions for the existence of self-similarity

Free turbulent boundary layers can be self-similar only when the governing momentum and energy equations can be written in self-similar form. The conditions under which this is possible are derived in appendix B. The derivation involves the assumption that the boundary layer approximations are applicable and that the viscous diffusion terms are negligible (high Reynolds number).

Exact self-similarity is possible when

$$\frac{d\delta}{dx} = \text{const}, \tag{4.5}$$

$$\frac{U_E}{U_0} = \text{const} , \quad (4.6)$$

$$\frac{\delta}{U_0} \frac{dU_0}{dx} = \text{const} . \quad (4.7)$$

The implications of these conditions are as follows. U_E must either be zero or vary in the same way as U_0 . This variation is $U_0 \propto x^m$, where m is a constant to be determined in the next section. The width of exactly self-similar flows increases linearly, irrespective of whether the flow is plane or axisymmetric.

Approximate self-similarity when $|U_E/U_0| \gg 1$.

When $|U_E/U_0| \gg 1$, certain terms are negligible in the momentum and energy equations, and approximate self-similarity is possible when (appendix B):

$$\frac{U_E}{U_0} \frac{d\delta}{dx} = \text{const} , \quad (4.8)$$

$$\frac{U_E \delta}{U_0^2} \frac{dU_0}{dx} = \text{const} , \quad (4.9)$$

$$\frac{\delta}{U_0} \frac{dU_E}{dx} = \text{const.} \quad (4.10)$$

These conditions imply that U_E must vary as $U_E \propto x^m$. Gartshore and Newman (1969) give the limits for m as

$$-\frac{j+1}{3} \leq m \leq -\frac{j}{4-j} . \quad (4.11)$$

When m is smaller than the lower limit, $|\frac{U_E}{U_0}|$ decreases with x and the assumption $|U_E/U_0| \gg 1$ is invalidated. When m exceeds the

upper limit, the Reynolds number $U_0 \delta / \nu$ decreases with x (see table 4.1), and the assumption of large Reynolds number is invalid. However, if this happens but very far downstream, there may still exist a limited similarity region. The round wake in zero pressure gradient is a prominent example for a flow of this type.

4.4.3 Potentially self-similar flows

a) Exact self-similarity

Because of the requirement $U_E/U_0 = \text{const}$, only the following free boundary layers can be exactly self-similar:

(i) Mixing layers between two constant streams

In this case U_E/U_0 is constant by definition.

(ii) Jets issuing into stagnant surroundings.

In this case $U_E/U_0 = 0 = \text{const}$.

(iii) Jets and wakes in an appropriately tailored pressure gradient.

When U_E can be made to vary in such a way that U_E/U_0 remains constant, jets and wakes can be exactly self-similar for the following ranges of U_E/U_0 :

$$-\infty < U_E/U_0 \leq -1 \quad \text{and} \quad 0 \leq U_E/U_0 < +\infty.$$

Variation of U_E , U_0 , and δ . The width of all these flows grows

linearly; and U_0 in cases (ii) and (iii), and U_E in case (iii), vary with x to the power m . In appendix C, the value of m is determined from the integral momentum equation (4.1). The result is:

$$m = -\frac{j+1}{2} I, \quad (4.12)$$

where

$$I = \frac{U_E/U_o + I_R}{3/2 U_E/U_o + I_R}$$

and

$$I_R = \frac{\int_0^{\eta_E} (f^2 + q) \eta^j d\eta}{\int_0^{\eta_E} f \eta^j d\eta} \quad (4.13)$$

For $U_E = 0$, equation (4.12) reduces to the well known relation

$$m = -\frac{j+1}{2} \quad (U_E = 0) \quad (4.14)$$

b) Approximately self-similar flows when $|U_E/U_o| \gg 1$

From conditions (4.8) to (4.11) we infer that approximate self-similarity is possible in the following flows or regions of these flows:

a) Jets and wakes with $|U_E/U_o| \gg 1$

The free stream velocity must vary as $U_E \propto x^m$, and $m \geq -\frac{j+1}{3}$. When $m > -\frac{j}{4-j}$, only a limited similarity region can exist.

b) Shear-free layers

Examples are the flow behind a self-propelled body and the turbulence front moving into non-turbulent fluid (shear-free mixing layer). The maximum kinetic energy decays as $k_o \propto x^{-n}$. The similarity region is limited when $m > 1 - n$.

Variation of U_E , U_o , and δ . In contrast to exactly self-similar flows, U_E/U_o and $d\delta/dx$ are not constant; and the power m with which U_E can vary has no constraints other than those given by equation (4.11).

For a given m , the variation of U_o and δ is derived in appendix C from the integral momentum equation (4.1) and the relations (4.8) to (4.10). The result is (for shear layers):

$$U_E \propto x^m \quad (4.15)$$

$$U_o \propto x^{\frac{m(j-1)-(j+1)}{j+2}} \quad (4.16)$$

$$\delta \propto x^{\frac{1-3m}{j+2}} \quad (4.17)$$

For shear-free layers, the momentum equation provides no information, and the variation of k_o and δ depends not only on m but also on the profiles of e , q , and $\hat{\epsilon}$. Furthermore, the limits for m given by equation (4.11) do not apply in this case.

Reality. The flows listed above are potentially self-similar because they satisfy the similarity conditions stated in § 4.4.2. Only experiments can tell whether these flows are self-similar in reality. We anticipate here the finding of the review on experimental data (§5): self-similarity was observed experimentally in all flows listed above with the exception of the shear-free mixing layer; this flow has not as yet been realised.

4.4.4 The flow examples

The present section introduces the individual flow examples; and it presents information on these flows that can be drawn from the foregoing similarity analysis. All potentially self-similar flows are included with the exception of the shear-free mixing layer.

a) The flow situations.

The flow situations are sketched in fig. 4.2. This figure explains how the flows are generated and how the free stream is controlled. The figure also defines the coordinate systems and the

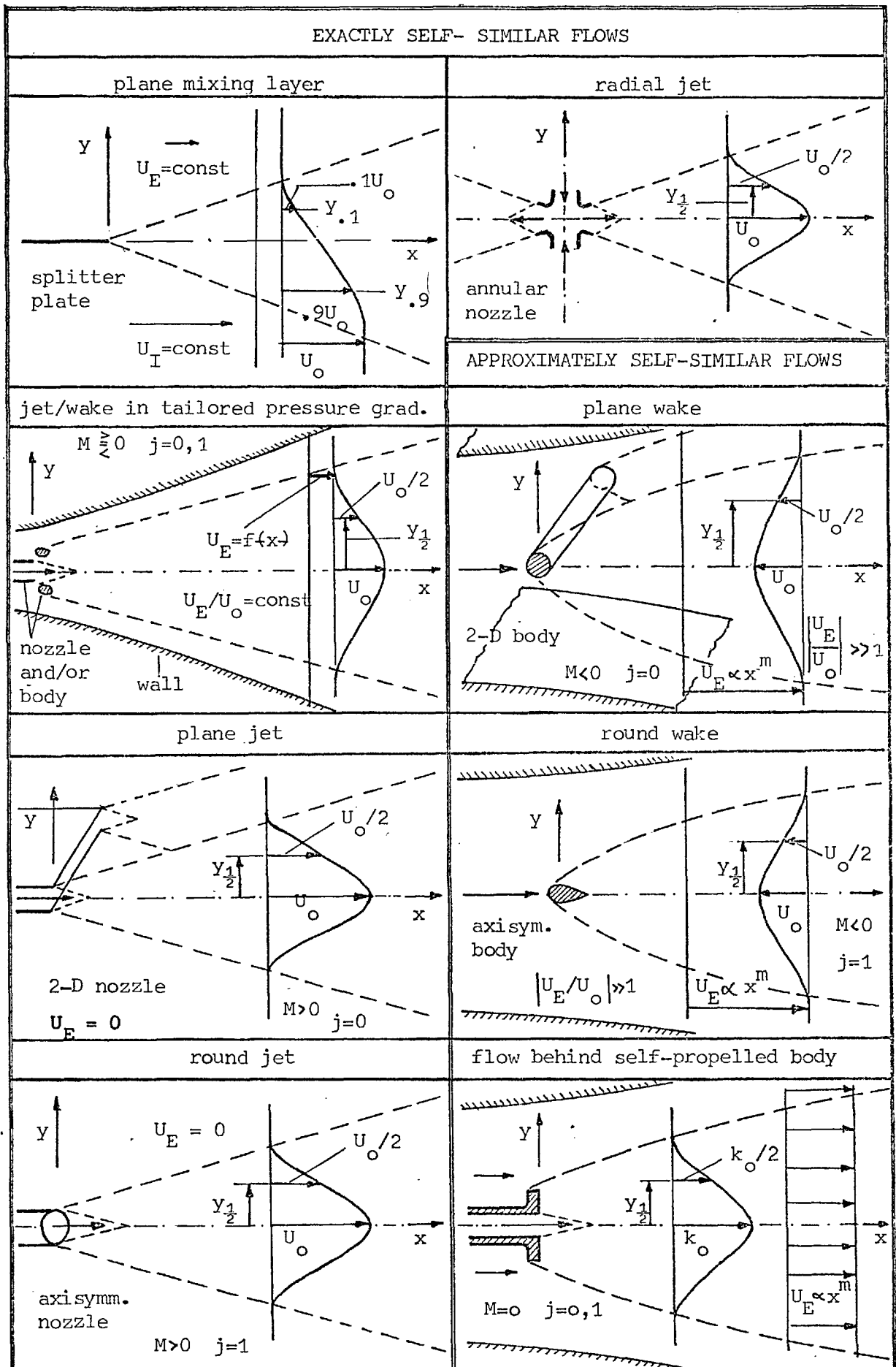


Figure 4.2 Examples of self- similar free turbulent boundary layers

Table 4.1 Decay laws of self-similar free turbulent boundary layers

FLOW		U_E	$U_O (\sqrt{k_O})$	δ	Re
exactly self-similar	plane mixing layer	const	const	$\propto X$	$\propto X$
	jet/wake, $\frac{U_E}{U_O} = \text{const}$	$\propto X^{-\frac{j+1}{2} I}$	$\propto X^{-\frac{j+1}{2} I}$	$\propto X$	$\propto X^{1-\frac{j+1}{2} I}$
	plane jet, $U_E = 0$	0	$\propto X^{-\frac{1}{2}}$	$\propto X$	$\propto X^{\frac{1}{2}}$
	round jet, $U_E = 0$	0	$\propto X^{-1}$	$\propto X$	const
	radial jet		$\propto X^{-1}$	$\propto X$	const
approx. self-similar	jet/wake, $\left \frac{U_E}{U_O} \right \gg 1$	$\propto X^m$	$\propto X^{\frac{m(j-1)-(j+1)}{j+2}}$	$\propto X^{\frac{1-3m}{j+2}}$	$\propto X^{\frac{m(j-4)-j}{j+2}}$
	plane wake, $m = 0$	const	$\propto X^{-\frac{1}{2}}$	$\propto X^{\frac{1}{2}}$	const
	round wake, $m = 0$	const	$\propto X^{-\frac{2}{3}}$	$\propto X^{\frac{1}{3}}$	$\propto X^{-\frac{1}{3}}$
	shear free layers	$\propto X^m$	$\propto X^{-\frac{n}{2}}$	$\propto X^{1-\frac{n}{2}-m}$	$\propto X^{1-n-m}$

quantities characterising the flows. The regions near the flow generation are not self-similar and are excluded from the present discussion.

b) Decay laws

Table 4.1 shows the variation with x of the following quantities: the free stream velocity U_E , the excess (or deficit) velocity U_O , the flow width δ , and the Reynolds number ($U_O \delta / \nu$).

Variation of the Reynolds number. Experiments indicate that the exponent for the decay of k_0 in shear-free layers is $n \approx 1.3$. From table 4.1 we then see that shear-free layers and round wakes in zero pressure gradient ($m = 0$) have the same decay behaviour; in particular, the Reynolds number decreases with x in both flows. The plane wake in a favourable pressure gradient ($m > 0$) also exhibits this feature. Thus, all these flows can have only a limited similarity region. For the other flows of table 4.1, the Reynolds number either increases with x or is constant. These flows have therefore unlimited similarity regions.

c) The influence of initial conditions

To be strictly correct, the power laws of table 4.1 should have $(x + x_0)$ as basis, where x_0 is the virtual origin of the self-similar flow. This origin depends on the exact conditions of flow generation, such as the velocity profile at the orifice or the shape of the wake-generating body. In all other respects, however, self-similar flows are independent of the initial conditions when the similarity region is unlimited. In particular, the constants $\frac{d\delta}{dx}$ (exact self-similarity) and $\frac{U_E}{U_0} \frac{d\delta}{dx}$ (approximate self-similarity), and the profile shapes, are independent of the exact conditions of flow generation. This feature is of great importance for the comparison of predictions with experimental results (see § 6).

For flows where the similarity region is limited, experiments indicate that the state of approximate self-similarity depends on the initial conditions (see §5.5.5b).

d) Momentum and energy relations

Tables 4.2 to 4.6 on pp. 222-230 list, for each flow, the similarity equations for momentum and turbulence energy, the latter ones in both differential and integral form. They also provide formulae to determine the lateral velocity, g , and the shear stress, h , from the profile of the longitudinal velocity, $f(\eta)$.

The equations presented draw attention to the features which distinguish the individual self-similar flows; and they are a useful tool for the evaluation of experimental data. They will be needed in § 5 for the execution of consistency checks.

Importance of convection. The momentum and energy equations reveal that, when we consider the normal stress terms as small, the differences between the flows stem mainly from the differences in the convection of momentum, energy, etc.

From an examination of the integral energy equations of table 4.5 we find:

- (i) the mixing layer is the only flow for which the convection withdraws energy from a control volume stretching across the layer.
- (ii) For symmetrical flows with zero pressure gradient, the ratio of convection to production is

$$e_o \int_0^{\infty} f \frac{e}{e_o} \eta^j d\eta / \frac{1}{2} \int_0^{\infty} f^3 \eta^j d\eta \quad \text{for jets,}$$

$$e_o \int_0^{\infty} \frac{e}{e_o} \eta^j d\eta / \frac{1}{2} \int_0^{\infty} f^2 \eta^j d\eta \quad \text{for wakes,}$$

where e_o is the energy level representative for each flow, say the value of e at $\eta = 0$. Experiments show that the velocity profiles $f(\eta)$ and the energy profiles $e(\eta)/e_o$ are roughly the same for all symmetrical flows. This leads to the following important finding:

The relative magnitude of convection increases with the relative energy level, e_o .

As a consequence, the ratio of production to dissipation, which is an important parameter in the model of § 3, decreases with increasing

e. Experiments confirm this finding.

The shear stress in plane jet, radial jet, plane wake. Experiments show that the plane and radial jet and the plane wake have not only the same velocity profile, $f(\eta)$, but also roughly the same rate of spread ($d\delta/dx$ for jets, U_E/U_O $d\delta/dx$ for the wake). The equations of table 4.4 then imply that the shear stress, h , in the radial jet is twice that in the plane jet. Also, the level of h in the plane wake is twice the level in the plane jet, but the distribution of h near the edges is different for these two cases.

4.5 The non-similar flows considered

4.5.1 Transition between self-similar flows of different types

Two flows, or rather flow regions, will be considered here which cannot be self-similar because they are transitions between self-similar flows of different types: the transition from a mixing layer to a jet, and the transition from a strong jet ($U_O > U_E$) to a weak jet ($U_O \ll U_E$). Both types of transition can be found in a jet issuing into a constant-velocity stream. Fig. 4.3 sketches the flow situation and defines the various flow regions. Both plane and axisymmetric jets are included.

a) Transition mixing layer-jet.

The lips of a jet nozzle generate mixing layers which spread until their inner boundaries meet at the jet axis. In the following transition region, the flow adjusts to the presence of a symmetry plane or line until developed jet flow is established. The shape of the velocity profile does not change much, but the turbulence energy distribution undergoes drastic changes. In the transition region, the turbulence intensities at the jet axis rise from values near zero in the potential core to fairly high values in the developed jet.

When the jet issues from a circular nozzle, the mixing layer is quasi-plane until its width is no longer small compared with the distance of the inner boundary from the axis. Then the axis of symmetry starts to exert its influence, and the transition to axisymmetric

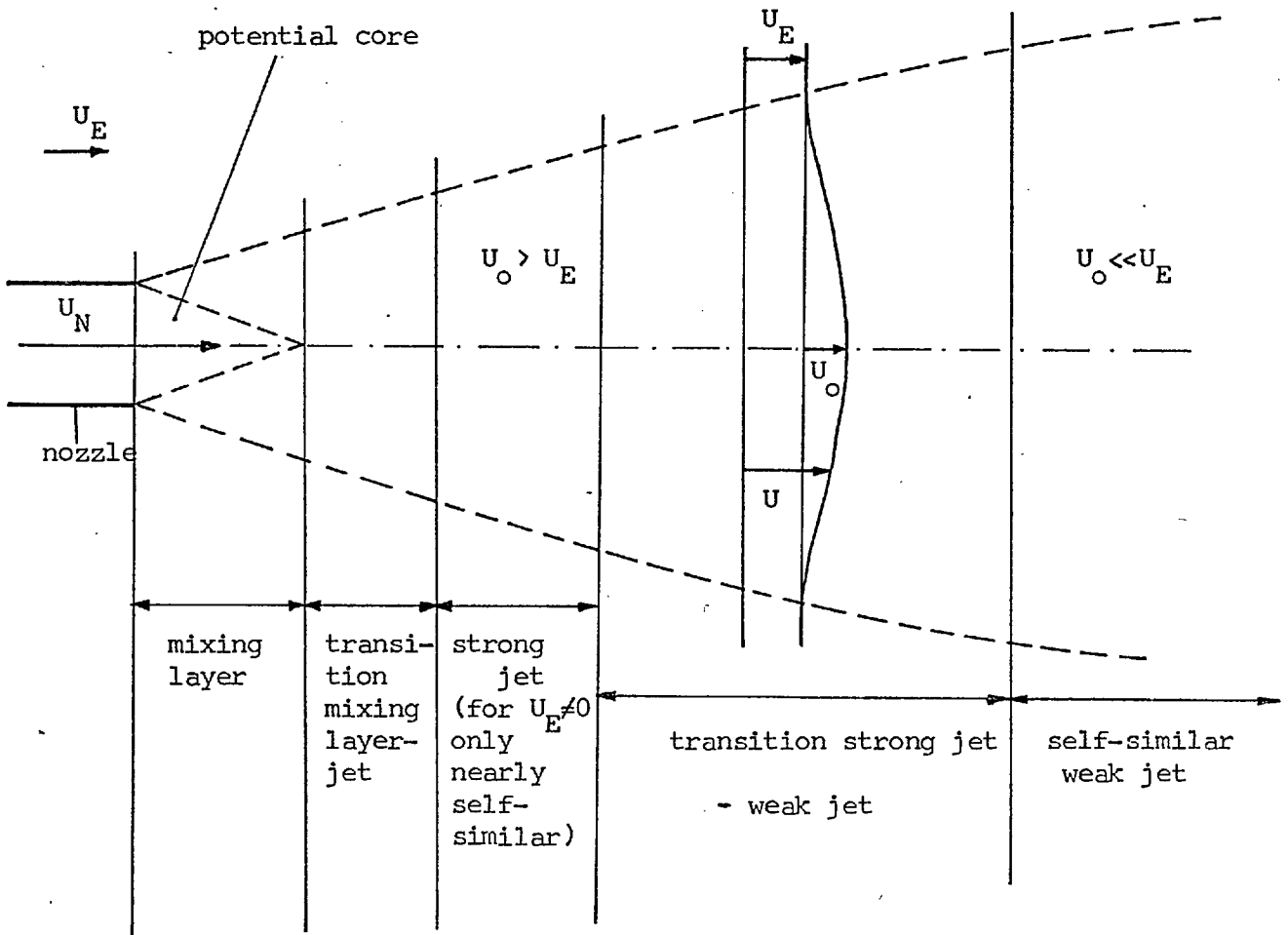


Fig. 4.3 Jet flow regions

flow occurs. This transition from a quasi-plane to an axisymmetric flow is of special interest because it puts the universality of prediction methods to a severe test.

The development of the initial mixing layer is influenced strongly by the flow conditions at the nozzle exit. This will be discussed in § 4.5.2.

b) Transition strong jet - weak jet

When U_N is much larger than U_E , the transition region discussed

under a) is followed by a "strong jet" region in which the excess velocity, U_o , is relatively large compared with U_E . Although the flow in this region cannot be exactly self-similar, (because U_o/U_E varies), it shares many features with a jet in still air.

The ratio U_o/U_E falls steadily, and, from a certain position, the flow starts to change its character. This transition continues until U_o is so small that a self-similar weak jet flow (of wake character) is expected to prevail. As was discussed in § 4.4, the difference between the two limiting flows, the strong jet and the weak jet, are quite substantial. The turbulence structure undergoes significant changes in the transition region, which makes this flow too an important test case for turbulence models.

Spalding (1958) and Bradbury and Riley (1967) deduced from dimensional arguments that, some way downstream of the flow origin, the flow depends only on the excess momentum M and not on the nozzle geometry or the velocity ratio U_N/U_E . M is defined by equation (4.2). The momentum thickness

$$\theta = \left(\frac{M}{\rho U_E^2} \right)^{\frac{1}{j+1}} \quad (4.18)^*$$

will therefore be used to non-dimensionalise all lengths of this flow situation.

For axisymmetric flow, the Reynolds number, $U_o \delta / \nu$, decreases with x in the transition and far region. It is therefore uncertain whether the development towards a self-similar weak jet can occur in this case. This matter will be discussed further in § 5.5.6.

4.5.2 The near field of free flows

It was mentioned earlier that similarity prevails only some way downstream of the flow origin. The present section is concerned with the initial region of flow development.

* Equations (4.19) to (4.58) can be found in tables 4.2 to 4.6, PP. 222-230.

Ideally, the streams separated by the nozzle lips may be thought of as uniform. In reality, there are always boundary layers of finite thickness present adjacent to the lip. The development of the initial mixing layer, and thus the potential core length, depend strongly on the properties of the boundary layers, such as velocity profiles and turbulence quantities. Turbulence carried by the two free streams also influences greatly the initial flow development.

When the two streams have the same velocity, a wake is generated. The flow past the very thin trailing edge of a flat plate has no recirculation zones and is therefore amenable to the present prediction method. This flow is ideal to study the transition from a wall boundary layer to a free boundary layer.

CHAPTER 5

A CRITICAL REVIEW OF EXPERIMENTAL DATA

5.1 Introduction

Experimental data play an important role in the development of a turbulence model: they are needed for the determination of the empirical constants and functions of the model; and for the test of its general validity. Thus, it is essential to have at hand reliable experimental "target values" against which the predictions can be compared.

Free turbulent boundary layers have been measured fairly extensively; so there are plenty of data available for most flows considered in this thesis. It is the purpose of this chapter to review these data, to check their reliability, and to select target values for a comparison with the predictions in § 6.

In § 5.2, the quantities for which we require data are stated first. § 5.3 then discusses measurement techniques and the possible errors to which they are susceptible. § 5.4 states the criteria for the assessment of the data. The data are then reviewed critically in § 5.5; and the target values are selected for the flows already introduced in the previous chapter. Finally, some suggestions for future experimental work are made in § 5.6.

5.2 The data required

Engineers require of a turbulence model primarily that it predicts the mean-flow quantities correctly. Hence, these quantities are the main target values we want to extract from experiments. However, experimental data on turbulence quantities can aid greatly the construction of turbulence models above the zero-equation level; so they are essential too.

The terms in the exact equations. It seems desirable to measure as many terms in the unmodelled equations as is possible. We are concerned in this study mainly with the transport equations for momentum (2.13), shear stress (boundary layer form of (2.7)), kinetic energy (2.14),

and length scale. The quantities appearing in the first three equations are listed in table 5.1*. The terms in the kL - equation (or any other length-scale equation) cannot as yet be measured, except in homogeneous isotropic turbulence (see § A.5). They are therefore not included in table 5.1.

Table 5.1 Quantities whose measurement is desirable

Group	1	2				3							
	Mean flow	Reynolds stresses				Other correlations							
Quantity	U	$\overline{uv}$	$\overbrace{\overline{u^2} \quad \overline{v^2} \quad \overline{w^2}}^k$			ϵ	$\overline{u^2v}$	$\overline{v^3}$	$\overline{vw^2}$	$\overline{vp}$	$\overline{uv^2}$	$\overline{up}$	$p(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})$
Equations in which it appears**	Mom	Mom	Mom	Mom									
	k	k	k	k	k	k	k	k	k	k			
	$\overline{uv}$	$\overline{uv}$		$\overline{uv}$							$\overline{uv}$	$\overline{uv}$	$\overline{uv}$

Three groups of data. The quantities of table 5.1 are divided into three groups according to their importance as target values. Classification according to the reliability with which they can be measured would lead to the same grouping.

1) Mean flow quantities. The longitudinal mean velocity U , and the rate of spread deduced from it, are the quantities of prime experimental interest.

2) Reynolds stresses. In the model of § 3, the level of turbulence

* The lateral velocity V in the momentum equation (2.13) can be eliminated with the aid of the continuity equation (2.12).

** The momentum equation is given by (2.13), the k -equation by (2.14) and the $\overline{uv}$ -equation by the boundary layer form of (2.7) (with $i=1$ and $j=2$).

is represented by the kinetic energy, k . To check whether the model describes the turbulence field correctly, we thus need measurements of k . These measurements also help to determine some of the constants. The shear stress, $-\overline{uv}$, can be calculated from the momentum equation when the other variables in this equation (mainly U) are known. However, for a consistency check on the data, direct measurement of $\overline{uv}$ is very desirable.

3) Other correlations. The remaining quantities in table 5.1 are necessary to complete the k - and $\overline{uv}$ - balances. The rate of dissipation, ϵ , is of particular interest because it is closely connected with the second turbulence quantity we solve an equation for (because $\epsilon \propto k^{3/2}/L$). Also, ϵ is needed to deduce $c_{\mu}c_D$ from experiments (equation 3.5). Unfortunately, it cannot as yet be measured reliably. Until now, attempts to measure the terms involving pressure fluctuations in laboratory flows have not been very successful.

Further measurements. There are, of course, many more quantities which give valuable information about the structure of turbulence, e.g. the intermittency factor, spectra, two-point and auto correlations, to name only a few. These data will certainly be of increasing importance when one moves on to more complex turbulence models.

5.3 Measurement techniques

In order to enable judgements of the quality of the measurements, we examine briefly the merits and shortcomings of those measurement techniques which were used to obtain the data reviewed in this chapter.

a) Impact tubes

In most of the free flows considered, the static pressure varies but very little across the flow. Thus, measurement of the total pressure by means of an impact tube is sufficient to determine the mean velocity U . The fluctuating velocities cannot be measured with an impact tube because of the slow response of the probe-manometer system.

Sources of error. Impact tubes are subject to three main sources of error: viscosity effects, displacement effects due to velocity gradients, and turbulence effects. In the turbulent flows considered here, the Reynolds number based on the probe diameter is large enough for the viscous effects to be negligible, even when the probe is small. And when the probe is small, the displacement effect becomes negligible too.

The turbulent fluctuations are the most serious source of error. When the turbulence intensity is high, say above 20%, the instantaneous yaw angle can be large enough to affect the probe reading. According to Hinze (1959), there is no reliable way to correct for this; and the errors are of unknown magnitude. However, when the intensity is low and the probe is small, the impact tube is a reliable tool for measuring the mean velocity in free boundary layers.

b) Hot-wire anemometers

The hot-wire anemometer has a sufficiently fast response to follow the turbulent fluctuations up to high frequencies (~ 20 kHz); it has therefore become the main instrument for measuring turbulent flows. Since all the data on turbulence quantities to be reviewed were obtained with this instrument, it deserves special attention. Appendix L is devoted to hot-wire measurements. There, the difficulties of obtaining reliable results, particularly when the intensity is high, are discussed at length. It may therefore suffice here to summarize briefly the limitations of the hot-wire anemometer:

Wires. The sensing wire elements are of small but finite dimensions. Their spatial resolution is therefore limited; and the smallest eddies, which are responsible for the dissipation, can be monitored reliably only in large scale flows. This appears to be the reason for the inadequacy of most of the reported dissipation measurements. On the other hand, the probe supports disturb the flow around the sensing part because the wire is not infinitely long. When multi-sensor probes are used, the sensors are likely to interfere with each other when they are placed close together; and when they are placed further apart, they do not monitor the same point as they are assumed to do. Jerome et al (1971) report that the thermal wake interference between closely spaced X-wires can cause v^2 , w^2 , and uv to be in

error by up to 20%. We believe therefore that results obtained with single-wire probes are more reliable.

Electronic equipment. Two methods can be applied in hot-wire anemometry: the constant-current method and the constant-temperature method. The constant-current method uses simpler electronics but is troubled by the thermal inertia of the wire. At high frequencies, this inertia must be compensated for; and at high intensities, the method is not suitable at all. It is now superseded almost completely by the constant-temperature method because advances in electronics have made the latter method the clearly superior one.

Most RMS - meters have a poor frequency response at very low frequencies (below ~ 3 Hz). In flows where large eddies with low frequency contain a substantial part of the fluctuating energy, these RMS-meters measure the fluctuating signal too low. It is important therefore to check, in each case, whether the RMS-meter's frequency response is adequate. Unfortunately, very few investigators have reported doing this.

Calibration. Usually, hot-wires are calibrated statically by measuring their sensitivity in a flow of very low turbulence intensity. Perry and Morrison (1971) claim that dynamic calibration (by shaking the wire) can give up to 20% different sensitivities. The turbulence measurements would be seriously affected by such an error.

Most experimenters did not calibrate their wires for yaw and pitch influence. They either assumed ideal wire response (cosine law), or they corrected their results by simply using Champagne and Sleicher's (1967) correction factors. These practices introduce further uncertainties.

Evaluation methods. The signal evaluation is troubled by two facts: the non-linear relationship between hot-wire signal and velocity, and the vectorial nature of the velocity fluctuations.

The first problem is to determine the effective cooling velocity from the electrical signal. With the electronic linearisers available

today, this is an easy task; but the turbulence data measured in highly turbulent flows without lineariser are somewhat suspect.

The second, more serious problem is to determine the mean and fluctuating velocity components from the effective cooling velocity. The standard method involves series expansions and neglects third and higher order correlations. These neglects are questionable in highly turbulent flows (above 25%). An alternative method, which is more suitable for highly turbulent flows, was developed as part of this work. It is described in appendix L.

Concluding remarks on hot-wire anemometers

The list of possible errors to which the hot-wire anemometer is susceptible seems endless; and yet it is the only fully developed and widely used instrument for measuring turbulence quantities. Many of the errors quoted can be avoided when great care is taken; and the rest need not be cumulative. Examination of the equipment and the evaluation methods employed gives some idea about the reliability of the data; but a final judgement can be based only on consistency checks.

In general, the reliability of the data decreases with increasing turbulence level.

5.4 Criteria for the data assessment

Criterion 1: The experimental flow situation must correspond closely to the one we want to predict.

We are concerned with two-dimensional (including axisymmetric) flows only; so the experimental flow situations should have been closely two-dimensional. Most predictions are for self-similar flows. For these flows it is important that the measurements should have been taken at cross sections where similarity of both mean and fluctuating quantities prevails. Lastly, in most predictions, the free stream is assumed to carry no turbulence. Thus, the experimental free stream turbulence should have been as low as possible.

Criterion 2: The data must be consistent.

The data must pass three consistency tests: firstly, the overall momentum must be conserved. This we check by means of the integral momentum equation (4.1). Secondly, the measured shear stress must agree with $\overline{uv}$ as determined from the momentum equation. This we check for self-similar flows by means of the equations given in table 4.3. Thirdly, the diffusion of kinetic energy must integrate to zero across the flow.

Criterion 3: The instruments and evaluation methods employed must have been adequate.

This criterion is difficult to apply; it will be used in this review as an additional guide only.

5.5 The review of experimental data

The experimental data on free turbulent boundary layers are too numerous for all to be included in this review. Thus, we focus attention on the more recent data. Readers interested in reviews of earlier data are referred to Abramovich (1963), Halleen (1964), Spalding (1967b), Newman (1967), and Harsha (1971).

5.5.1 Shear free flows (uniform mean velocity)

a) Turbulence behind grids

A grid placed in a uniform stream acts as a plane source of turbulence. The flow field so created is the best approximation we can devise to homogeneous turbulence; and the equations of our turbulence model reduce to a form where c_S/c_D is the only constant to appear (see § G.4a). This flow is therefore especially important for the determination of c_S/c_D .

Comte-Bellot and Corrsin (1966) and Tucker (1970) give comprehensive reviews on the experimental data. Their main findings are as follows. The turbulent kinetic energy decays as

$$k \propto (x - x_0)^{-n},$$

where x is the distance from the grid and x_0 is the hypothetical origin of the plane source of turbulence. Values of the exponent n ranging from 1 to 1.35 were measured, the majority of the results being close to 1.2. The ratio u^2/v^2 was found to be about 1.4; it decreases only slightly in the downstream direction. The approach towards isotropy is therefore very slow.

b) Flow behind self-propelled bodies (Shear-free layers)

The flow behind plane and axisymmetric self-propelled bodies represents respectively a line and a point source of turbulence in a uniform stream. The flow situation is sketched in fig. 4.2. (p.60). We are interested here only in the far region where the mean velocity is uniform and the shear stress is zero. The plane case was investigated by Townsend (1956) and Mobbs (1968), the axisymmetric case by Naudascher (1965).

Nature of the flow. Because this flow may not be so well known to the reader, a short account is given first of Mobbs' findings on the nature of the flow:

A sharp, irregular boundary separates the turbulent from the non-turbulent part of the flow. Within the turbulent part, the intensity is laterally homogeneous, so that the dependence on y of the overall intensity is caused solely by the intermittency. The profiles of k/k_0 and the intermittency factor γ/γ_t are therefore identical. γ_t decreases linearly with x . The apparent outward spreading of the turbulence is due entirely to the growing amplitude of the turbulent bulges. The volume of the turbulent fluid decreases with increasing x .

The flow seems therefore to be unique in that its overall behaviour is that of spreading inhomogeneous turbulence, while the turbulent part contracts and is homogeneous.

Self-similarity. From § 4.4 we expect the flow to be self-similar, i.e. the spreading parameter

$$S = \frac{U_E}{\sqrt{k_0}} \frac{dy_{1/2}}{dx}$$

should be independent of x , and k/k_0 and $\epsilon y_{1/2}/k_0^{3/2}$ should be functions of $y/y_{1/2}$ only (for definitions see fig. 4.2). All three investigators found the k -profile to be indeed self-similar; but there is little consensus about the parameter S . For the plane case, Townsend's results give $S = .75$, but Mobbs' results indicate a significant variation of S in the region investigated. For the axi-symmetric case, the writer has evaluated Naudascher's data and found that S is approximately constant over a very limited range only ($15 < x/D < 25$), where its value is about 0.4.

Reynolds number effects. In § 4.4 we have seen that the Reynolds number $\sqrt{k_0} y_{1/2}/\nu$ decreases with x when k_0 decays as $(x - x_0)^{-n}$ and n is larger than 1. In this case, the similarity region is restricted if existent at all. Townsend reports $n \approx 1.2$, but both Mobbs and Naudascher report $n \approx 1.7$. Because of the homogeneity of the turbulent part of the flow, we would expect the axial intensity decay to be similar to that behind grids when the Reynolds number is still high. This notion is supported by Townsend's value of $n \approx 1.2$. The high value of 1.7 reported by the other two authors indicates that their flows may have been subject to low Reynolds number effects. This is consistent with the lack of constancy of the parameter S and thus of true self-similarity.

Only Townsend's value of $S = .75$ is therefore used for comparison with the predictions.

Profile of k . The profile of k/k_0 is shown in fig. 6.13. The results of Townsend and Naudascher are virtually identical.

5.5.2 Plane mixing layer

Flow realization. The ideal mixing layer is formed by the mixing between two semi-infinite streams (see fig. 4.2). This ideal flow can be only approximated by experiments. The best approximations are the mixing between two-dimensional half-jets, and the initial regions of plane and round jets. However, in the latter case, the extent of the mixing layer is rather limited.

Tables 5.2 and 5.3 list details about the experimental conditions of the investigations to be reviewed. Table 5.2 covers the mixing layers with one stream at rest ($U_E = 0$), and table 5.3 the ones between two moving streams.

Self-similarity. From the similarity analysis of § 4.4 we expect that, downstream of a certain development region, mixing layers are self-similar for any velocity ratio U_E/U_I . This expectation is confirmed by all the experimental data on mean and, when available, turbulence quantities.

Velocity profiles. The various experimenters' similarity profiles for U are shown in fig. 5.1 for $U_E/U_I = 0$ and in fig. 5.2 for finite values of U_E/U_I . The agreement is good; the somewhat larger scatter in fig. 5.1 at the zero velocity edge indicates the difficulties in measuring accurately in this region. However, the velocity profiles agree well only when y is non-dimensionalized by a characteristic flow width, here $\delta = (y_{.1} - y_{.9})^*$. When the data are plotted versus y/x , as is often done in the literature, the agreement is poor; for there is little consensus about the rate of spread, $d\delta/dx$.

Rate of spread. The rates of spread of various experimenters are listed in tables 5.2 and 5.3. The spreading parameter σ is also listed because it appears frequently in the literature. We define σ in such a way that the use of $\eta = \sigma y/x$ as abscissa brings the velocity profiles closest to the error function.

Tables 5.2 and 5.3 show rather poor agreement about the rate of spread. This may be due to the following three reasons. Firstly, the distance to attain full development of the mixing layer is about

* $y_{.9}$ and $y_{.1}$ are defined in fig. 4.2 (p.60).

a thousand times the momentum deficit thickness of the boundary layer at the separation point (Bradshaw 1966). The data obtained at low values of $Re_x = U_I x / \nu$ are therefore likely to depend on the initial conditions, even though the velocity profile seems to be developed. Secondly, the mixing layer seems to be particularly sensitive to the flow field outside the layer, possibly because it is an asymmetrical flow. Thus, the layout of the room into which the layer spreads and room draughts may influence the rate of spread (Reichardt 1942). Thirdly, the influence of free stream turbulence appears to be quite dramatic, as is evidenced in fig. 5.3 for $U_E/U_I = 0$.

When the extreme values are discarded in table 5.2, $d\delta/dx = .16$ emerges as a fairly well supported target value for $U_E/U_I = 0$ and $Re_x > 7 \times 10^5$.

The influence of U_E/U_I on the rate of spread is illustrated in fig. 6.2. Similar figures reported in the literature were obtained by taking σ_0 , the spreading parameter at $U_E/U_I = 0$, from Liepmann and Laufer (1947). This resulted in considerable scatter. Because σ_0 can vary markedly from one experimental set-up to another (see table 5.2), it seems more reasonable to obtain σ_0 by extrapolating each experimenter's results for σ , when enough points are available. This was done in fig. 6.2; the scatter was thereby reduced considerably. We expect the influence of free stream turbulence to grow with increasing U_E/U_I . Since this influence is to decrease σ , the lower points in fig. 6.2 should be a better representation of the ideal flows.

k- profiles. The similarity profiles of k as measured by various experimenters are shown in fig. 5.4. Except for the shape of the profile, there is no agreement. Liepmann and Laufer's (1947) results are probably too low because of inadequate electronics; and Sunyach and Mathieu's (1969) results are too high because their Re_x was rather low; for Laurence (1955) and Lassiter (1957) found that $\overline{u^2}$ decreases markedly as Re_x increases.

Consistency check. To allow further judgement on the turbulence measurements, tables 5.2 and 5.3 compare the measured $\overline{uv}_{\max}$ with $\overline{uv}_{\max}$

Table 5.2 Mixing layers with $U_E = 0$

Flow situation	Experimenter	Half jet dimens.	range	max Re_x	$\frac{1}{u^2} / U$ at exit	$\frac{d(y \cdot I - y \cdot g)}{dx}$	σ	meas. $\frac{\overline{uv}}{U_I^2}$	calc. $\frac{\overline{uv}}{U_I^2}$	k_m / U_I^2	Remarks
Two- dimensional half-jet with 3 solid boundaries	Reichardt (1942)	32 x 32cm	<160cm	3.2×10^6	?	$\frac{.131}{.15}$	$\frac{13.5}{11.8}$				most runs one run
	Liepmann & Laufer (1947)	152 x 19cm	< 90cm	1.1×10^6	<1%	.16	11.	.008	.0106	.021	$\overline{w^2}$ was guessed as $w^2 = v^2$
	Wyganski & Fiedler (1970)	51 x 18cm	< 58cm	4.65×10^5	<0.1%	.2	9.	.0091	.0138	.035	wall at $x = 0$ and trip wire used
	Patel (1970)	76 x 43cm	<100cm	1.8×10^6	0.5%	.165	10.7	.0103	.0112	.0275	
initial region of a plane jet	Albertson et al (1948)		$1 < \frac{x}{D} < 4$	?	?	.155	11.4				data taken from Abramovich
	Mills (1968)	5 x 15cm	$2 < \frac{x}{D} < 4$	1.8×10^5	?	.178	10.4				
	Sunyach & Mathieu (1970)	4 x 48cm	$1.5 < \frac{x}{D} < 3$	1.6×10^5	.25%	.18	9.9	.0137	.0125	.05	
initial region of a round jet	Maydew & Reed (1963)		$.5 < \frac{x}{D} < 3.9$			.16	11.				Exit Mach number = 0.7
	Bradshaw et al (1964)	D = 5 cm	$x/D = 2$ $2.3 < \frac{x}{r} < 7$	7×10^5	≈ 0 1% at x/D = 2	.165	10.7	.010	.011	.027	
	Sami et al (1967)	D = 30 cm	$x/D = 3$ $3 < \frac{x}{r} < 15$	6.6×10^5	< .5%	.163	10.8	.0109	.0108		

Table 5.3 Mixing layer between two moving streams

Flow sit.	Experimenter	Dimens. of nozzles	range	max Re_x	$\sqrt{u^2}/U$ at exit	$\frac{U_E}{U_I}$	$\frac{d(y_{.1}-y_{.9})}{dx}$	δ	$\frac{\overline{uv}_m}{U_0^2}$ meas	$\frac{\overline{uv}_m}{U_0^2}$ calc	k_m/U_0^2	Remarks
Mixing of 2 free plane jets	Zhestkov et al see Abram. 1963	4x12.5 cm 4x12.5 cm	< 10 cm	7.1×10^5	?	0 -.64						only ratios δ/δ_0 are given
	Miles and Shih (1968)	?	< 105 cm	2.6×10^6	3%	.14 .22 .36 .46 .55 .65 .76 .83		11.6 13.2 19.8 24.3 34.4 34.8 47.0 49.0				The δ 's were taken from Brown and Roshko's (1971) paper
	Mills (1968)	5x15 cm 5x15 cm	≤ 20 cm	1.8×10^5	?	0 .3 .6	.178 .098 .07	10.4 18.3 25.0				
Mixing of 2 ducted rectangular jets	Seban and Back (1962)	.64x30 cm 11.5x30 cm	< 6 cm	1.1×10^5	?	.2 .36 .6		14.7 16.7 27.9				Influence of walls noticeable
	Sabin (1965)	?	50 cm < x < 115 cm	1.8×10^5	?	.35 .46 .61 .66		24.5 31.6 47.6 53.7				water tunnel experiments
	Spencer (1970)	19x38 cm 19x38 cm	< 127 cm	2.6×10^6	.1%	.15 .22 .305 .54 .61 .76	.078 .0328	13.3 15.7 20.4 41.4 50.4 77.0	.0132 .011	.011 .0104	.035 .032	Detailed infor- mation only for $\frac{U_E}{U_I} = .3, .6$
	Brown and Roshko (1971)	2.5x10 cm 2.5x10 cm	≤ 10 cm	5×10^5	.1% to .5%	.143 .378	.123 .084	12.8 18.7				nitrogen at 7 atm.

as calculated from the mean velocity via equation (4.27). The normal-stress terms were neglected in (4.27).

It should be mentioned here that there is considerable confusion in the literature as to how the shear stress should be calculated from the mean velocity profile. Normally, an arbitrary assumption concerning the lateral velocity V is made which is in fact unnecessary; for equation (4.27) shows that $\overline{uv}$ depends only on the rate of spread and on the similarity profile of U when the normal stress terms are neglected.

Liepmann and Laufer, Wygnanski and Fiedler, and Spencer (for $U_E/U_I = .3$ only) claim agreement between their measured and calculated shear stress. The writer has re-evaluated their data and found unsatisfactory agreement (table 5.2). For the mixing layer with $U_E/U_I = 0$, this leaves us with Bradshaw et al's (1964) and Patel's (1970) results, which agree remarkably well considering the differences in the geometry of their apparatuses (see table 5.2). They are supported by Sami et al's (1967) data which are also consistent. Unfortunately, these authors report only the longitudinal energy component.

The only consistent turbulence data for $U_E/U_I \neq 0$ appear to be those of Spencer for $U_E/U_I = 0.6$.

Energy balance. Bradshaw and Ferriss's (1965) energy balance is shown in fig. 6.4. It is consistent because the diffusion integrates to zero across the layer*. Sami's (1967) balance is in close agreement. For $U_E/U_I = 0.6$, Spencer's energy balance was modified by multiplying his measured dissipation by a factor of 3.6 to make the diffusion integrate to zero (the diffusion is determined as the difference of the other terms). This improved balance is shown in fig. 6.5.

* The measured convection and production terms in the shear-stress balance are shown in fig. H.1.

Effect of U_E/U_I on the turbulence structure. It is interesting to note that, within the experimental accuracy, the k -profile and the energy balance are the same for the cases $U_E/U_I = 0$ and $U_E/U_I = .6$. This seems to indicate that the turbulence structure is independent of the velocity ratio U_E/U_I .

5.5.3 Jets issuing into stagnant surroundings

Self-similarity. The experiments confirm our expectation in § 4.4 that jets issuing into stagnant surroundings are self-similar in the far region. The profiles of mean and turbulence quantities become similar (the former ones first), and the centre-line velocity U_0 and the jet width $y_{1/2}$ vary as predicted by table 4.1. Except for a shift in the virtual origin of the self-similar flow, the far jet is independent of the conditions at the nozzle exit. This is evidenced below by the general agreement about the rate of spread and the velocity profile.

Virtual origin. Flora and Goldschmidt (1969) found that the location of the virtual origin depends, above all, on the turbulence level at the nozzle exit. The same holds true for the potential core length, x_c . Vagt's (1970) results confirm this finding (see fig. 6.26).

Table 5.4 presents the important flow parameters for the more recent experiments on plane, round and radial jets.

a) Plane jet

Rate of spread. From Newman's (1967) review on experiments up to 1965, we find an average rate of spread of $dy_{1/2}/dx \approx .11$. Table 5.4 supports this value.

For low turbulence intensities at the exit, the potential core length is $x_c/D \approx 6$.

Velocity profile. Fig. 5.5 shows various experimenters' similarity profiles of U . Near the edge, there is considerable disagreement. Since hot-wire anemometers measure too high in this region (see appendix L), the lower values are probably more accurate. Robins' (1971)

curve is taken as target profile because Bradbury's (1965) profile is probably influenced near the edge by the slow but finite free stream he used.

Start of self-similarity. True self-similarity starts where the turbulence quantities become similar (the mean velocity does so much earlier). Table 5.4 shows that there is no consensus about the position of this start. These differences may be attributed, in part, to the difficulties in maintaining the flow two-dimensional over long distances. Also, the initial conditions may influence the turbulence quantities farther downstream than is commonly thought.

k- profiles. Fig. 5.6 compares the k- profiles of various experimenters. The agreement is rather poor. Heskestad's results are suspect because his $\overline{u^2}/U_0^2$ still rises at $x/D = 160$, and because he found a strong influence of the Reynolds number ($\equiv DU_N/\nu$). Gutmarks's results lose credibility because he reports some oddities: his velocity decay changes abruptly at $x/D = 65$, and the ratio of dissipation to production in his energy balance is of the order of 0.2, which is not plausible. Robins found $\overline{v^2}$ and $\overline{w^2}$ to be very much smaller than $\overline{u^2}$ near the edge, which is contrary to all other measurements.

Consistency check. For a consistency check, the measured and calculated values of $\overline{uv}_{\max}$ are compared in table 5.4. Two calculated values are listed: one was obtained from the full equation (4.29), and one with the normal stress neglected in (4.29). It is not surprising that Bradbury's data are the most consistent ones because his jet issued into a slow moving stream ($U_E/U_N = .16$). This stream reduced the relative intensities and thus the measurement errors.

Bradbury's data. Bradbury's jet could not be exactly self-similar, but the departures from self-similarity were insignificant in the region investigated. Also, reducing U_E/U_N from 0.16 to 0.07 had no effect on the results, nor had halving the Reynolds number. Despite the fact that Bradbury is the only one to report the somewhat implausible result that $\overline{v^2} > \overline{u^2}$ at the centre, we will take his turbulence data as target values.

Table 5.4 Jets issuing into still air

	Experimenter	Nozzle dimens.	range (x/D)	Re_D	Start of self-pres. region	$\frac{dy_1}{dx}$	meas. $\frac{\overline{uv}_m}{U_o^2}$	calc.** $\frac{\overline{uv}_m}{U_o^2}$	$\frac{k_d}{U_o^2}$	Remarks
plane jet	Bradbury (1965)	D = .95 cm L = 46 cm	14-70	3×10^4	$x/D \approx 30$	not constant	.0242	a) .024 b) 0.0249	.067	$U_E/U_N = .16$
	Heskestad (1965)	D = 1.25 cm L = 150 cm	47-155	$.47 \times 10^4$ -3.7×10^4	$x/D \approx 65$ but $\overline{u^2}_d/U_o$ keeps rising	.11	.021	a) 0.0232 b) 0.0277	.07	strong influence of Re_D on $\overline{u^2}_d/U_o$
	Patel (1970)	D = .7 cm L = 80 cm	12-152	3.5×10^4	For U and $\overline{u^2}: x/D \approx 30$	.103	.021	a) 0.0225 b) 0.0247	.064	
	Gutmark (1970)	D = 1.3 cm L = 50 cm	10-150	3×10^4	$x/D \approx 120$	.102	.024	a) 0.0238 b) 0.0287	.077	
	Robins (1971)	aspect ratios 21 - 138	5-100	1×10^4 -6×10^4	$x/D \approx 60$	.103	.02	a) 0.023 b) 0.025	.064	
	Wynanski and Fiedler (1969)	D = 2.6 cm	20-98	10^5	$x/D \approx 70$	.086	.0165	a) 0.0168 b) 0.0190	.101	strange change in slope of U/U_o versus x/D^N at $x/D=57$
round jet	Present investigation	D = 1.29 cm	62-75	8.7×10^4	< 62	.086	.0186	a) 0.0168 b) 0.0188		
radial jet	Tuve (1953)	D = 2.54 cm D = .635 cm	6-95	$.6 \times 10^4$ -3×10^4	?	total angle 23°				
	Heskestad (1966)	width of annular slot D = 0.71 cm	20-112	2.5×10^4	only U self-pres. from $x/D \approx 46$	0.111	.0254	a) 0.046 * b) 0.0508	.096 at x/D=65	for $x/D > 100$, influence of walls noticeable

* Heskestad (1966) reports a value of 0.045

 $\overline{uv}_m$ = maximum shear stress

** a) normal-stress term neglected in (4.29)

b) normal-stress term included in (4.29)

k- and uv- balances. Bradbury's energy balance is shown in fig. 6.6. It is consistent. Robins and Gutmark measured all the terms necessary to construct the shear stress balance except for the terms involving pressure fluctuations. Unfortunately, the general consistency of their measurements is not very good.

Thus, for the construction of the shear-stress balance, only the diffusion term was taken from Robins, but the other terms from Bradbury and Patel. This shear-stress balance is shown in fig. 3.4.

b) Round jet

The mean flow field of round jets was studied by numerous experimenters, but the turbulence field in the truly self-similar region only by few. Since Gibson's (1963) jet was not really symmetrical, only Wygnanski and Fiedler's (1969) and the present turbulence measurements can provide target values. A detailed discussion of the results is given in appendix L.

Rate of spread. Newman (1967) reports a rate of spread of $dy_1/dx = .086$ as average over earlier experiments. The more recent data listed in table 5.4 support this value.

For low exit turbulence level, the potential core length is $x_c/D \approx 5$ (Vagt 1970).

Profiles of U and k. The present velocity profile is shown in fig. 6.10. It agrees well with the earlier measurements. Wygnanski and Fiedler's and the present k-profiles are presented in fig. 5.7. Compared with the situation in other flows, the agreement is reasonable. The present results are more consistent, as can be seen from the measured and calculated $\overline{uv}_{\max}$ - values of table 5.4.

k- and uv- balances. Wygnanski and Fiedler's energy balance has two defects: the normal-stress production is too large by a factor of two, and the diffusion does not quite integrate to zero. An improved version is shown in fig. 6.10. The shear stress balance constructed from Wygnanski and Fiedler's measurements is shown in fig. 3.5.

c) Radial jet.

The radial jet emerging from an annular nozzle was investigated by two experimenters only. Tuve (1953) measured the mean velocity, and Heskestad (1966) both mean and turbulence quantities. The mean velocity became self-similar in both experiments. The velocity profiles are shown in fig. 5.8. Heskestad's hot-wire data are probably too high at the edge. His rate of spread is $dy_{\frac{1}{2}}/dx = .11$. This is compatible with $dy_G/dx = .42$ as measured by Tuve, where y_G is the total width of the shear layer.

Lack of self-similarity. Heskestad's turbulence quantities are not self-similar in the region investigated. $\overline{u_{\frac{1}{2}}}^2/U_o^2$ keeps rising, and the intermittency profile does not even approach a self-similar form. Beyond $x/D = 90$, $\gamma_{\frac{1}{2}}$ decreases rapidly. Heskestad quotes that the jet flow breaks down at this point because of room disturbances. Owing to this lack of true self-similarity, the turbulence quantities are not used as target values; and the mean flow results are used with some reservation only.

5.5.4 Self-similar jets and wakes in appropriately tailored pressure gradients

The similarity analysis of § 4.4 revealed that jets and wakes in a streamwise pressure gradient can be exactly self-similar when U_E/U_o is kept constant. Experimental verification of this result may be found in the work of Fekete (1970) for jets and of Gartshore (1967) for wakes. Fekete investigated jets with U_E/U_o ranging from 1 to 4, and both his mean and turbulence data confirm well the similarity behaviour described in § 4.4*. Gartshore studied two wakes with U_E/U_o equal to - 4.2 and - 5.2 respectively. The first wake showed quite good similarity behaviour, the second one did so only to a lesser degree. Both Fekete and Gartshore report that their measured shear stress agrees with $\overline{uv}$ as determined from the momentum equation.

The variation of the rate of spread, $dy_{\frac{1}{2}}/dx$, with U_o/U_E is shown in fig. 6.3.

* Unfortunately, only some of Fekete's data given in Gartshore (1971) were available to the writer.

5.5.5 Wakes in zero pressure gradient

Conditions for self-similarity. We are interested here mainly in the far wake region in which approximate self-similarity as introduced in § 4.4 is expected to prevail. Such a region can exist only when certain terms in the momentum equation are negligible. To elucidate this condition, we write the total deficit momentum flux as

$$M = M_w \left(1 + \frac{U_o}{U_E} I_R\right) = \text{const}, \quad (5.1)$$

where

$$M_w = (2\pi)^j \rho U_E U_o y_{\frac{1}{2}}^{j+1} \int_0^{\infty} f \eta^j d\eta,$$

and I_R is a shape parameter defined by equation (4.13). Experiments have shown that I_R is about .72 for $j = 0$ and .54 for $j = 1$.

The similarity analysis for wakes (§4.4) assumes the term $U_o/U_E I_R$ to be negligible compared with unity. If this term is to contribute less than, say, 3% to the total deficit momentum flux M , then $|U_o/U_E|$ must be smaller than $\approx .06$. Tables 5.5 and 5.6 show that most wake experiments met this condition; the data should therefore evidence the expected similarity behaviour.

Determination of S. With the aid of equation (C.6), the spreading parameter $S = \frac{U_E}{U_o} \frac{dy_{\frac{1}{2}}}{dx}$ can be expressed as

$$S = \frac{1}{j+2} \frac{d}{dx} \left(\frac{U_E}{U_o} y_{\frac{1}{2}} \right) \quad (5.2)$$

For the present review, S was determined from experiments by plotting $U_E/U_o y_{\frac{1}{2}}$ versus x ; the slope then yields S by means of (5.2).

S and other important wake parameters are listed in table 5.5 for plane wakes and in table 5.6 for round wakes.

a) Plane wakes

Self-similarity. Far downstream of the body, plane wakes behave as predicted by the similarity analysis: U_o varies as $x^{-\frac{1}{2}}$ and $y_{\frac{1}{2}}$ as $x^{\frac{1}{2}}$; and similarity of both mean and turbulence quantities prevails. According to Townsend (1956), the similarity region starts at $x/D \approx 500$ in the wake of a cylinder ($D \equiv$ diameter of cylinder). The approach towards similarity of the turbulence quantities is shown in fig. 5.9. It is noteworthy that the asymptotic values are approached from below in the wake of a flat plate, while they are approached from above in other wakes. Starting from a wall boundary layer, the wake has to build up turbulence energy; in contrast, the vortex shedding behind blunt bodies generates a surplus of energy which has to decay before an equilibrium can exist.

Influence of initial conditions. We expect that self-similar far wakes should be independent of the precise conditions of their generation. Thus, the body shape and size, and the free stream velocity, should have no direct influence, but only the momentum thickness θ . To test this presumption, $y_{\frac{1}{2}}/(Dc_D)$ is plotted versus $x/(Dc_D)$ in fig. 5.10. θ is related to the drag coefficient c_D by $\theta = Dc_D/2$. Only at very large values of $x/(Dc_D)$ is there some agreement (although primarily between cylinder-wake data). Thus, the initial conditions appear to exert their influence very far downstream.

Spreading parameter. There is, however, some encouraging agreement about the spreading parameter S . When the extreme values in table 5.5 are excluded, Townsend's (1956) value of $S = .098$ is supported fairly well by the rest of the data.

The spreading parameter $y_{\frac{1}{2}}/[\theta(x-x_o)]^{\frac{1}{2}}$ is also listed in table 5.5 because it is more popular in the literature.* Its calculation requires knowledge about c_D and the somewhat arbitrary x_o . We therefore prefer the parameter S , which can be determined from U_o/U_E and $y_{\frac{1}{2}}$ alone.

* The conversion formula is
$$S = \frac{y_{\frac{1}{2}}^2}{\theta(x-x_o)} \int_0^{\infty} f d\eta$$

Table 5.5 Plane wakes

Body	Experimenter	c_D	Re_D	Re_θ	$\frac{x}{D}$	$\frac{x}{\theta}$	$\frac{ U_E }{ U_o }$	S	$\frac{y_1}{[\theta(x-x_o)]^{1/2}}$	$\frac{\sqrt{u^2}}{U_o}$	$\frac{meas.}{uv_m}{U_o^2}$	$\frac{calc}{uv_m}{U_o^2}$	Remarks
circular cylinder	Reichardt (1942) Schlichting(1930)	1.32	2.38×10^4	1570		60-3400	$>.025$	.1	.328				similarity from $x/D = 400$
	Townsend (1949, 1956) **	?	1360		80-950		.093-.031	.098	.313	.27	.0505	.051	self-similarity from $x/D = 600$
	Ermshaus (1970)	.94	2500	1170	100-575	200-1230	.095-.044	.089	.303 [†]				
		1.09	6200	3370	80-240	140-440	.103-.066			.345 [*]	.035 ^{**}		
	Everitt (1971)				120-800	300-1600	.07-.038	.096	.334	.259	.0364	.050	$ U_o y_1 $ starts to rise beyond $x/D = 450$
	Alexopoulos and Keffer (1968)	?	5000		40-228		.159-.072	.107	.41 [†]	.3	.033	.055	wake not yet developed
	Uberoi and Freymuth (1969)		540 & 4320		25-800		not reported			.305			only $\bar{u}^2/U_E^2$ reported. Townsend's U_o/U_E used to determine $\bar{u}^2/U_o^2$
plate $\rightarrow$	Ermshaus(1970)	1.88	2340	2200	40-612	44-650	.206-.068	.805	.3 [†]				
aero-foil	Ermshaus(1970)	.35	3000	525	68-463	400-2600	.066-.029	.098	.349				
	Everitt (1971)					50-800	.18-.047	.172	.44	.259			
flat plate	Chevray and Kovasznay(1969)			3300		0-210	.31-.134	.062 [*]					wake far from being developed
"Jet in mov. str."	Everitt (1971)				<314		>158	.0866 [*]		.224 [*]			velocity excess

* quantity continues to change with x
[†] exponent of y_1 is not $\frac{1}{2}$

** The following quantities are not correct in Townsend's(1956) book: $\bar{uv}/u^2$ in fig. 7.10 , $\bar{u}^2/U_E$ in fig. 7.1

Profiles of U and k. The velocity profiles of various experimenters are shown in fig. 5.11. They agree very well. The k- profiles are compared in fig. 5.12. They are all somewhat suspect for the following reasons. Table 5.5 reveals that Everitt's (1971) and Alexopoulos and Keffer's (1968) measured values of $\overline{uv}_{\max}$ are 30% to 40% lower than the ones calculated by means of equation (4.33). Their kinetic energy is likely to be too low also. Townsend's (1949) measurements are consistent. However, the data of Uberoi and Freymuth (1969) and of Chevray and Kovasznay (1969), which appear to be of high quality, suggest that $\overline{u^2}$ is higher than that found by Townsend. This must be kept in mind when, for lack of something better, Townsend's k- profile is compared with the predictions in § 6.

Energy balance. Townsend's (1949) energy balance for the far wake is not consistent. His balance quoted in the books of Townsend (1956) and Hinze (1959) was measured at $x/D = 160$, which was too close to the body for the wake to be self-similar. This balance cannot therefore be used for quantitative comparison with the predictions of the self-similar wake.

b) Round wakes

Existence of self-similarity. Round wakes, together with round jets in a moving stream, are unique among free shear flows in that their local Reynolds number ($\equiv U_0 y_{1/2} / \nu$) decreases with x . As a consequence, only a limited similarity region can exist; and the question arises whether it exists at all. The answer of all the measurements beyond $x/D = 60$ is YES. Both U_0 and $u_{1/2}^2$ vary as $x^{-2/3}$ and $y_{1/2}$ as $x^{1/3}$; and the profiles of both mean and fluctuating velocities are similar. The parameter $S = U_E / U_0 (dy_{1/2} / dx)$ is constant even for $x/D < 60$.

Influence of body shape. The findings presented so far were expected, but it comes as a surprise that S should depend strongly on the shape of the wake-generating body. Table 5.6 reveals that S varies from $\approx .1$ for slender bodies to $\approx .9$ for very blunt bodies. Along with this goes a drastic increase in the turbulence level relative to the defect velocity (see fig. 5.13). It is interesting to note that, in contrast to plane wakes, the asymptotic value of $\overline{u^2}$ is approached from below. Fig. 5.14 shows that even the velocity profile depends on the

body shape. All the profiles presented are similarity profiles.

Turbulence field. There is some evidence that the turbulence field in the far wake depends much less on the body shape than does the mean field. Fig. 5.15 gives a plot of $\sqrt{u_\ell^2}/U_E$ versus x/D . The agreement of most of the data is reasonable. Table 5.6 also lists the spreading parameter of the turbulence field,

$$S_t = \frac{U_E}{\sqrt{u_\ell^2}} \frac{dy_{\frac{1}{2}t}}{dx},$$

where $y_{\frac{1}{2}t}$ is the radius at which $\sqrt{u^2}$ is half of $\sqrt{u_\ell^2}$. Unfortunately, not enough data were available to determine S_t for slender bodies, but the S_t 's for sphere and disk differ much less than do the corresponding mean field parameters S . It is interesting to note that the value of $S_t = 1.21$ for disks is close to the value of 1.26 for Townsend's (1956) shear-free layer discussed in § 5.5.1b. All these findings indicate that the turbulence field is practically de-coupled from the mean field and behaves therefore as in shear-free layers. Examination of the energy balance shown in fig. 6.12 offers an explanation for this: the production, by means of which the mean field exerts its influence, is much smaller than the other terms; in shear-free layers it is zero.

Figs. 6.12 and 3.6 show, respectively, that the convection of energy and shear stress is very important in round wakes. This appears to be the reason for the large "upstream effects" in these flows.

Reynolds number influence. A further common feature of round wakes and shear-free layers is the linear decay with x of γ_ℓ (Baldwin and Sandborn 1968, Mobbs 1968). This behaviour may be attributed to the decreasing Reynolds number; it indicates that the observed self-similarity can only be an approximate one. This makes it easier to accept the existence of different states of similarity behind different bodies. The decreasing Reynolds number may therefore be the basic reason for this apparent oddity. It would be intriguing to conduct a round-wake experiment with the free-stream velocity U_E varying as x^m , where $m > \frac{1}{3}$.

Table 5.6 Round wakes

Body	Experimenter	c_D	Re_D	x/D	$\frac{U_o}{ U_E }$	S	S_t	$\frac{\sqrt{u_z^2}}{U_o}$	$\frac{meas.}{uv_{max}} \frac{U_o}{U_o2}$	$\frac{calc.}{uv_{max}} \frac{U_o}{U_o2}$	Remarks
"round jet in moving stream"	Reichardt (1965)		$>10^4$	<160	$>.053$	$.105^*$					velocity excess
spheroid 1 : 6	Chevray (1968)	.06	2.75×10^6	0 -18	.28 -.091	$.105^*$		$.3^*$	$.041^*$	.054	flow not developed at $x/D = 18$
cone 1:6	Reichardt and Ermshaus (1962)	.68(?)	?	20 -60	?	.14					
cone 1:4	Reichardt and Ermshaus (1962)	.75(?)	?	10- 60	?	.21					
	Ermshaus (1970)	.75(?)	2.5×10^4	10- 60	.167- .041	.23		$.46^*$	$.072^*$	.12	
cone 1:2	Reichardt and Ermshaus (1962)	?	?	30- 50	?	.57					
sphere	Uberoi and Freymuth (1970)	.4(?)	8600	10- 300	.012 at $x/D=100$	.51	1.	.85	.27	.27	values quoted are for $x/D = 100$
cone 1:1, disk	Reichardt and Ermshaus (1962)	?	?	20- 50	?	.86					
disk	Ermshaus (1970)	1.1	2.5×10^4	10- 60	.078- .017	.93		.98	.5	.56	
	Carmody (1964)	1.16	$.7 \times 10^4$	<15	$>.055$	$.8^*$		.94	.5	.48	flow not fully developed at $x/D = 15$
	Cooper and Lutzky (1955)		3600 8300	23- 680			.92 1.22				mean velocity not reported
	Hwang and Baldwin (1966)		7.7×10^3 -3.1×10^4	4- 800			1.21				mean velocity not reported
square plate	Cooper and Lutzky (1955)	?	1.45×10^4	80- 680	.0153 -.0037	.92	1.23	.95		.525	body not axisymmetric

* values still change with increasing x

According to table 4.1, the Reynolds number then rises with x , and the far wake should be independent of the body shape if the above-stated hypothesis were true.

5.5.6 Jets issuing into a uniformly moving stream

Jet forgetfulness. Jets issuing into a uniformly moving stream cannot be self-similar; in contrast to the previous sections, we are therefore interested here particularly in the streamwise flow development. The principle of "jet forgetfulness" (§ 4.5.1) suggests that, some way downstream of the nozzle, this development depends on the variable $(x - x_0)/\theta$ only and not on the velocity ratio U_E/U_N . Both plane and round jet experiments confirm this. An example is given in fig. 6.18, where Reichardt's (1965) and Ortega's (1968) actual data points of $y_{1/2}/\theta$ and U_E/U_0 are plotted. The reader is reminded that the location of the virtual origin, x_0 , does depend on the flow conditions at the nozzle.

Variation of U_0 and $y_{1/2}$. With the aid of the momentum equation (4.1), the non-dimensional width, $y_{1/2}/\theta$ can be expressed as

$$\frac{y_{1/2}}{\theta} = \frac{1}{\left[2\pi^j \left(\frac{U_0}{U_E} I_1 + \left(\frac{U_0}{U_E} \right)^2 I_2 \right) \right]^{1/(j+1)}}, \quad (5.3)$$

$$\text{where } I_1 = \int_0^\infty f \eta^j d\eta \quad \text{and} \quad I_2 = \int_0^\infty f^2 \eta^j d\eta.$$

Equation (5.3) was used for this review to check the consistency of the measurements.

For the plane jet, measured values of $y_{1/2}/\theta$ and $(U_E/U_0)^2$ are plotted versus $(x - x_0)/\theta$ in fig. 6.16. The agreement between different experimenters is good only at small values of $(x - x_0)/\theta$. At larger values, Everitt's (1971) data are probably closest to the truth because he conducted the most extensive investigation.

The corresponding results for the round jet are plotted in fig. 6.18. The consensus of different experimenters is fairly good.

At high values of $(x - x_0)/\theta$, the measured values of $y_{1/2}/\theta$ do not agree with $y_{1/2}/\theta$ as calculated from equation (5.3). The measurements of U_0 are probably more accurate than those of $y_{1/2}$. The results of Forstall and Shapiro (1950) and Curtet and Ricou (1964) are not included because, for the higher values of U_E/U_N , their flows were influenced noticeably by the duct which guided the outer stream.

In both plane and round jet, the shape of the velocity profile is the same and does not vary significantly with x .

The asymptotic behaviour. The ratio U_0/U_E falls steadily with x ; and, at large distances, where U_0/U_E is very small, the flow is expected to assume wake-like character. For $|U_0/U_E| \ll 1$, the similarity analysis predicts that the flow width should vary as $x^{1/(j+2)}$ and the excess velocity U_0 as $x^{-(j+1)/(j+2)}$. The parameter

$$S = \frac{U_E}{U_0} \frac{dy_{1/2}}{dx}$$

should thus be a constant. The question is whether the measurements extended far enough downstream to satisfy the condition that $|U_0/U_E| < 0.06$ (see previous section). The lowest values of U_0/U_E included here are 0.158 and 0.053 for the plane and round jet respectively. They are probably not low enough for the establishment of truly self-similar wake-type flow. The measurements in wakes extended to much lower values (see tables 5.5 and 5.6).

Nevertheless, Everitt's data testify that the plane jet approaches the expected asymptotic state. At the furthest station, S still continues to rise, but its value of 0.087 is already close to the value of 0.098 found in wakes (table 5.5).

The round jet, on the other hand, exhibits no tendency to approach the expected asymptotic state. For $x/\theta > 50$, U_0/U_E varies approximately as x^{-1} , and $y_{1/2}$ as $x^{1/2}$. The expected behaviour is $U_0/U_E \propto x^{-2/3}$ and $y_{1/2} \propto x^{1/3}$. This surprising finding may possibly be explained by the fact that the Reynolds number decreases in this flow. The assumption of high Reynolds number, which is inherent in the similarity analysis, may thus be invalid. This explanation is

supported by the finding that the observed variation of U_0 and $y_{1/2}$ is identical to that expected in a laminar self-similar round wake (Schlichting 1960).

Turbulence measurements. Turbulence quantities were measured by Everitt in the plane jet and by Curtet and Ricou and by Ortega in the round jet. In all the cases, $u_{\frac{1}{2}}^2 / U_0^2$ rose monotonically with x . Everitt's energy balances show that the ratio of production to dissipation of energy, P/ϵ , decreases markedly as U_0/U_E decreases.

5.6 Closure to the chapter

From the foregoing sections we may conclude that the mean velocity can be measured quite accurately in free turbulent boundary layers, except in some regions where the turbulence intensity is very high. However, we may also conclude that the turbulent stresses could, in general, be obtained with an accuracy no better than 20%. Although this is not surprising in view of the many possible error sources (see § 5.3), it handicaps the development of advanced turbulence models; for these models rely greatly on experimental turbulence data. Hopefully, the development of new instruments, such as the Laser-Doppler anemometer, will improve the situation.

Suggestions for future experiments. In spite of the great number of existing data, some basic flow phenomena still lack a proper explanation; and some flows have been rather neglected. The chapter therefore closes with a few suggestions for future experimental work. The influence of the apparatus layout and the intensity and scale of free stream turbulence on the mixing layer should be studied systematically. Another attempt should be made to establish a truly self-similar radial jet. A systematic study should be carried out on the effect of the body shape on plane and round wakes. Finally, the possible effects of the Reynolds number on round wakes and round jets in a moving stream should be investigated.

CHAPTER 6

APPLICATION OF THE TWO-EQUATION MODEL TO FREE

TURBULENT BOUNDARY LAYERS

6.1 Introduction

We have now available a turbulence model and a great many of experimental target values for free turbulent boundary layers. The present chapter will show how well the model reproduces these target values.

To apply the model, we need a solution procedure and boundary conditions. § 6.3 deals briefly with the solution procedure employed; and § 6.4 specifies the boundary conditions. The empirical constants of the model are determined by comparing predicted and measured results for a few selected flows. This is discussed in § 6.5. The constants and functions are then used to predict all the flows introduced already in the previous two chapters. The results are presented and discussed in § 6.6, first for self-similar flows and then for non-similar flows. Whenever possible, the results are compared with experiments and with calculations of other prediction methods. The chapter closes with an assessment of the universality of the two-equation model.

Three higher-level models are applied to free flows in the following appendices: Daly and Harlow's (1970) five-equation model in appendix J, a new five-equation model $(\overline{u_i u_j}, \epsilon)$ in appendix G, and Hanjalić and Launder's (1972) three equation model (τ_{ke}) in appendix H.

6.2 The equations to be solved

At this point, the reader may welcome a brief reminder of what constitutes the prediction method applied in this chapter.

The method employs the momentum equation (2.13) together with the turbulence model expressed by equations (3.13) to (3.15). The normal-stress terms are neglected in the momentum equation. This neglect may not be wholly justified in some flows: table 5.4 shows that, in jets issuing into still air, the shear stress determined from the momentum equation is about 10% higher when the normal-stress terms

are included than when they are not. In principle, the normal stresses could be calculated with the algebraic stress model described in appendix I. However, because of the many assumptions made already, the inclusion of these stresses is not warranted.

The turbulence model requires the equations for k (3.14) and kL (3.15) to be solved in order to determine the eddy viscosity ν_t from the Kolmogorov-Prandtl expression (3.13). Apart from the basic version of the model, called model I, two simpler versions are considered: model II uses $c_\mu = \text{const}$ instead of $c_\mu = f(\overline{P/\epsilon})$, and model III assumes that L is uniform over any cross section.

The equations to be solved are thus three in number: they are for the three dependent variables U , k , and kL .

6.3 Solution procedure.

The equations for U , k , and kL are solved simultaneously with the implicit finite-difference method which Patankar and Spalding (1970) have developed for boundary-layer equations. Since these equations are parabolic in nature, the method integrates by marching forward (i.e. downstream), starting at an initial cross section where the profiles of U , k and kL must be specified. All the necessary information about the method is documented in the book of Patankar and Spalding and need not be repeated here.

The numerical accuracy of the solution was checked in the following way. Some computations were repeated with various sizes of the forward step and lateral grid interval, so as to verify that the solutions were unaffected by these quantities. A typical number of lateral grid points used was 20, and a typical forward step was 0.05 of the layer width. The computing time for one of the predictions presented in § 6.6 was typically 20 seconds on a CDC 6600 computer.

In some calculations, the solution tended to become unstable when c_μ was determined at each step from the function $c_\mu = f(\overline{P/\epsilon})$ given in § 6.5. To stabilise the solution, c_μ was therefore re-calculated only every n^{th} step. Various values of n were tested. As long as

n was kept small (of the order of 10), its value had no detectable influence on the results because, in boundary layers, the turbulence structure changes but slowly with x .

Otherwise, numerical instability occurred only when the forward step was chosen excessively large, or when unrealistic diffusion constants (σ_k , σ_{kL}) were employed. Some problems encountered in solving the equations of higher-level models are discussed in appendix G.

6.4 Boundary and initial conditions

a) Boundary conditions

The free boundary which separates the region of interest from the free stream was defined in § 4.2. With one exception (see §6.6.2-2b), all the flows predicted were assumed to carry no turbulence in the free stream. Thus, the conditions at a free boundary were taken as

$$U = U_E (U_I) ,$$

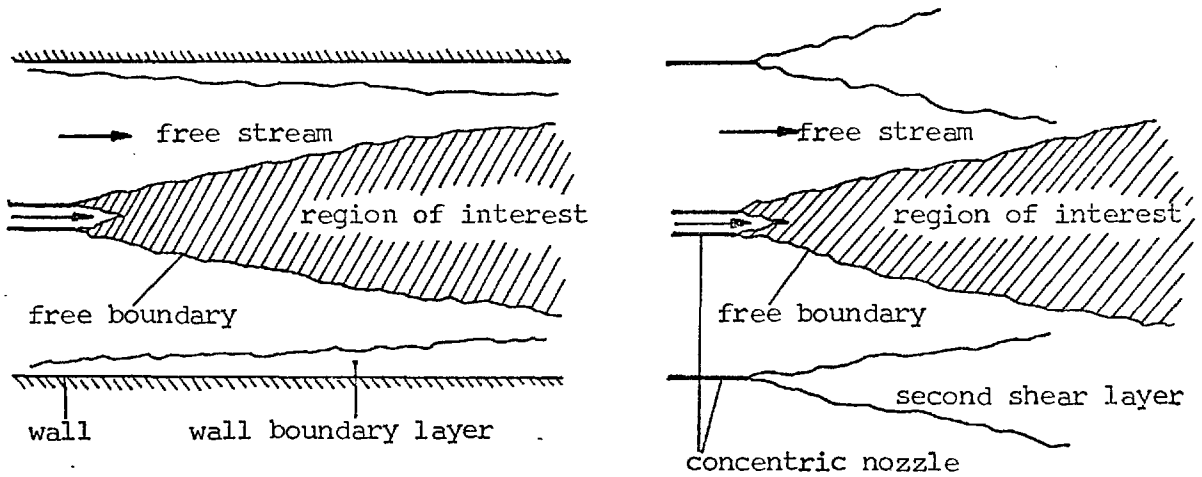
$$k = 0,$$

$$kL (= v_t \sqrt{k}/c_\mu) = 0.$$

For flows with a symmetry plane or line, this plane or line was taken as a boundary in the predictions. At this boundary, the derivatives of U , k , and kL with respect to y are zero.

Free stream turbulence. When U_E (or U_I) is finite, the free stream may carry turbulence. The predictions using the above boundary conditions can then be compared only with experiments in which the free stream turbulence level was low. Most experimenters have striven to attain this condition.

Two typical laboratory flow situation in which the free stream may carry turbulence are sketched below:



When the diffusion of k and k_L across the free boundaries is zero, the free stream turbulence is laterally homogeneous and decays like grid generated turbulence. However, in experiments like those sketched above, this decay law is seldom observed. This indicates that turbulence quantities diffuse into the free stream from the free shear layer on one side and from the wall boundary layer, or a second free shear layer, on the other side. Since the diffusion fluxes across the free boundaries are unknown, the present method can account for this kind of free stream turbulence in a realistic way only when the whole flow field (including the free stream) is solved for. Such a study would fall beyond the scope of this work; so the influence of free stream turbulence was studied for one case only, for which this turbulence was assumed to decay as behind a grid (see § 6.6.2-2b).

b) Initial conditions

Self-similar flows. For self-similar flows, arbitrary initial profiles for U , k , and k_L were chosen. The forward-marching integration was then carried out until the profiles had reached similarity. This similarity solution is independent of the initial profiles.

Non-similar flows. The downstream development of most non-similar flows is influenced by the initial conditions. These conditions must either be known from experiments, or they must be guessed. For some flows, measured initial profiles of U , k , and $\overline{uv}$ were available. The initial length scale distribution was calculated from these data via (equation 3.13)

$$L = \frac{\overline{uv}}{c_\mu \sqrt{k} \partial U / \partial y}$$

The initial value of c_μ had to be estimated^{*}. This was done with the aid of fig. 3.1, which gives experimental values of c_μ for most of the basic flows.

The initial conditions for the individual flows will be given in the sections where these flows are discussed.

6.5 Determination of the constants and functions

a) Choice of c_μ

It is the product $c_\mu c_D$ which is important in the present model, and not the values of c_μ and c_D themselves. This can be seen from equation (3.5). For a given $c_\mu c_D$, the choice of c_μ merely determines the absolute magnitude of the length scale L . Since L as defined by equation (3.12) cannot as yet be measured, the absolute magnitude of L is not of interest here. Without loss of generality, we may thus put c_μ equal to unity in model II, and in model I we may construct the function $c_\mu = f(\overline{P}/\epsilon)$ in such a way that $c_\mu = 1$ where $\overline{P}/\epsilon = 1$. This practice influences the values of c_D and c_S .

b) Range of the constants/functions

c_S/c_D from turbulence behind grids. In appendix G, the equations for k and kL^{**} are applied to homogeneous turbulence behind a grid.

* The initial value of c_μ , $c_{\mu I}$, is quoted in the relevant figures.

** Actually, the equation for ϵ is used; but the two equations are equivalent in this case.

In this flow case, c_S/c_D is the only constant appearing in the model. Comparison with experiments reveals that c_S/c_D should fall within the range 0.5 to 0.8.

c_D and $c_\mu = f(\overline{P/\epsilon})$ from experiments. Fig. 3.1, which was discussed already in § 3.3, shows $c_\mu c_D$ as determined directly from experiments. The figure includes self-similar boundary layers both near to and remote from walls, and nearly homogeneous shear flow. The figure indicates that, with $c_\mu = 1$ at $P/\epsilon = 1$, c_D is about 0.09. The figure indicates also how the function $c_\mu = f(\overline{P/\epsilon})$ must run: it must represent the mean curve to the experimental data for flow regions not too close to symmetry lines.

c) Computer optimisation

The preceding section gives constraints for some of the constants and for the c_μ -function. The exact values of the constants, and the final form of the function, were determined in such a way that predictions performed with the model give best agreement with experiments for some selected flows. This procedure is called optimisation.

Choice of flows. The optimisation of the constants c_D , c_B , c_S , σ_k , and σ_{kL} , and of the c_μ -function was carried out for the self-similar flows listed in table 6.1. Self-similar flows were chosen for the following reasons:

- (i) they constitute the limiting cases for most boundary layers,
- (ii) they facilitate the comparison of predictions with experiments because the profile shapes and spreading parameters are independent of x ,
- (iii) they facilitate consistency checks on the experimental data (see § 5).

The constants and functions of model I were optimized for free flows only. In cooperation with Ng(1971), the constants of model II were optimized also for wall flows. The constants of model III were taken from model I.

Table 6.1 Flows used to optimise c_D , c_B , c_S ,

$$\sigma_k, \sigma_{kL}, \text{ and } c_\mu = f(\overline{P}/\epsilon)$$

Model FLOW	model I $c_\mu = f(\overline{P}/\epsilon)$	model II ($c_\mu = 1$)	
		present	Rodi & Spalding
plane mixing layer	✓	✓	✓
plane jet	✓	✓	✓
radial jet	✓	✓	✓
plane wake	✓		
pl. shear-free layer	✓		
Ng's (1971) wall flows*		✓	

For model I only, the constants c_{D1} , c_{D2} , c_{S1} , c_{S2} , and q appearing in (3.16) and (3.17) were adjusted to give good agreement for round jets in both stagnant and moving surroundings.

Optimisation procedure. A great many calculations were carried out in which the constants and the c_μ -function were varied systematically. The influences of the constants and the function on the predicted rates of spread and on the profiles of U , k , $\overline{uv}$, and L were thereby studied. The constants c_D , c_B , and c_S have a strong influence on the rate of spread and on the level of k and $\overline{uv}$, while a variation of σ_k and σ_{kL} mainly changes the shape of the profiles of U , k , $\overline{uv}$, and L . The

* These flows were: flat plate boundary layer, flow in a pipe of uniform circular cross section, flow between wide parallel plates, plane wall jet in stagnant surroundings.

effect of a 5% change of the constants on the rate of spread is shown in table 6.2.

Table 6.2 Influence of a 5% increase of the individual constants on the rates of spread (% variation).

FLOW, Spreading parameter \ Constant	c_D	c_S	c_B	σ_k	σ_{kL}
plane mixing layer, $\frac{d(y_{.1}-y_{.9})}{dx}$	+ 2.5	- 7.1	+ 20.0	+ .1	+ .7
plane jet, $dy_{\frac{1}{2}}/dx$	+ 2.5	- 7.3	+ 13.2	+ .2	+ .4
radial jet, $dy_{\frac{1}{2}}/dx$	+ 2.5	- 7.3	+ 13.2	+ .3	+ .4
plane wake, $\frac{U_E}{U_O} \frac{dy_{\frac{1}{2}}}{dx}$	+ 2.5	- 9.7	+ 14.7	+ .3	0

As was to be expected, no set of constants together with a single c_μ -function produces exact agreement in respect of all the main experimental data. Since a compromise was therefore necessary, the optimum constants and the function were finally chosen in such a way that the agreement improves with the degree of reliability of the target quantity. The rate of spread has therefore priority, the shape of the velocity profile comes second, and the largest discrepancies were tolerated for the profiles of k and $\overline{uv}$.

d) The best-fit constants/function

The optimized constants are shown in table 6.3 for the three versions of the present model. The constants employed by Rodi and Spalding (1970) are also listed.

Table 6.3 Best-fit constants.

Constant, appears in Model	c_μ (3.13)	c_D (3.14)	c_B (3.15)	c_S (3.15)	σ_k (3.14)	σ_{kL} (3.15)	c_{kL} (A.15)
Rodi & Spalding	1.	.09	1.	.057	1.	.3	-.7
Model I	$f(\overline{P/\epsilon})$	.09	.95	.054	1.	1.	/
Model II	1.	.09	.98	.058	1.	1.	/
Model III	$f(\overline{P/\epsilon})$	.09	.95	.054	1.	/	/

Round jet constants for model I (equations 3.16 and 3.17)

c_{D1}	c_{D2}	c_{S1}	c_{S2}	q
14.6	229.	4.05	136.	0.2

The best-fit function $c_\mu c_D = f(\overline{P/\epsilon})$ is shown in fig. 6.1. It can be calculated from equation (3.25) when α and ω are taken as

$$\begin{aligned}\omega &= 2.8 \\ \alpha &= .55\end{aligned}\quad \text{for } \overline{P/\epsilon} > 1$$

$$\begin{aligned}\omega &= 2.8 + .525 \left\{ \sin \left[\left(\overline{P/\epsilon} - \frac{1}{2} \right) \pi \right] - 1 \right\} \\ \alpha &= .55 + .213 \left\{ \sin \left[\left(\overline{P/\epsilon} - \frac{1}{2} \right) \pi \right] - 1 \right\}\end{aligned}\quad \text{for } \overline{P/\epsilon} \leq 1$$

These expressions for α and ω were introduced for analytical convenience only; they are not appropriate for use in the pressure-strain term of the model of appendix G.

Fig. 6.1 also shows the function $c_\mu c_D = f(P/\epsilon)$ which results from Bradshaw et al's (1967) model with $\overline{uv}/k = .3 = \text{const.}$ * This function is very similar to the present one, however it approaches infinity as P/ϵ approaches zero while the present one remains finite.

From fig. 6.1 we see that the selected function represents the experimental data fairly well for flow regions away from symmetry lines or planes. Also, the values chosen for c_D and c_S/c_D agree with the constraints given in b).

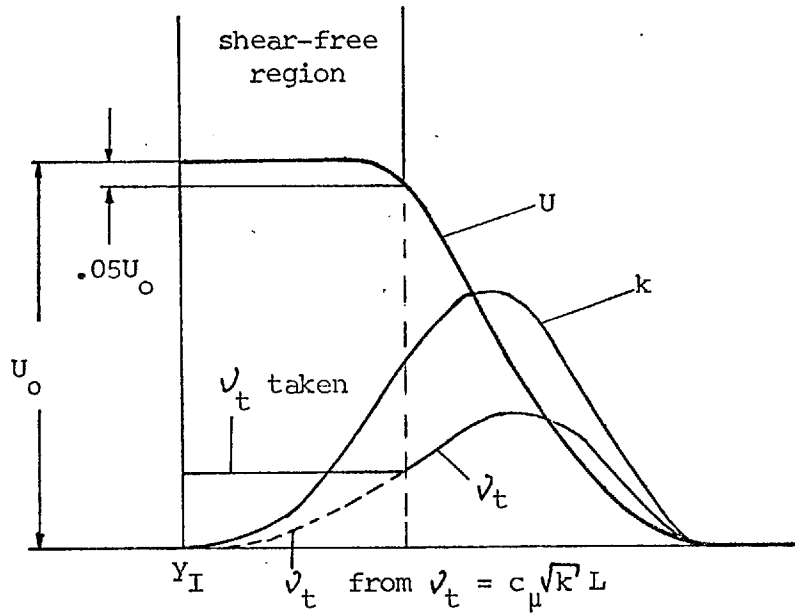
The final predictions of the flows for which the constants and the c_μ -function were optimised are presented and discussed in § 6.6.

e) Shear-free regions.

The reader is reminded here that shear-free layers which occur in the far field of flows behind self-propelled bodies, require special treatment (see § 3.5b). For these flows, v_t as calculated at the centre-line or plane is taken for the whole cross section. This practice is equivalent to taking the local value of v_t but allowing the σ 's to vary so that $v_t/\sigma = (v_t/\sigma)_d = \text{const.}$

Experiments have shown that at the edges of shear layers with an adjacent finite free-stream velocity there also exists a shear-free region. As is illustrated below, this means that k spreads out faster than does momentum (and $\overline{uv}$). Experiments indicate also that v_t does not fall to zero at these edges. Thus, and in accordance with the above practice for shear-free layers, we keep v_t constant in these regions. The details of this treatment are given in the following sketch.

* The function is $c_\mu c_D \equiv \frac{(\overline{uv}/k)^2}{P/\epsilon} = \frac{.09}{P/\epsilon}$.



The above definition of the free-stream region is somewhat arbitrary; it was taken here for the purpose of demonstration. The practice of keeping v_t constant in this region mainly serves to improve the energy balance at the edge; it has little effect on the main region of the flow, in particular it does not influence the predicted rate of spread.

6.6 Presentation and discussion of results

All the components of the present prediction method have now been introduced. In what follows, we examine how well the method works. To this end, we compare predictions obtained with the best-fit constants/functions with experimental data and with calculations of other methods. The experimental data most suitable for this comparison were selected already in chapter 5.

The principal turbulence model of this investigation is model I; this model was therefore applied to the widest range of flows. The model versions II and III were applied mainly to self-similar flows. When the type of model is not denoted specifically, the results were obtained with model I.

The results for self-similar flows will be presented and discussed first, and the results for non-similar flows second.

6.6.1 Self-similar flows

The self-similar flows considered here are sketched in fig. 4.2 (p. 60). The main features of these flows were discussed already in chapter 4; so we can concentrate here on the present method's performance in predicting these features.

6.6.1-1 Rate of spread

The spreading parameter S is defined for the various self-similar flows in table 6.4. This parameter is independent of x . The relevant symbols are defined in fig. 4.2.

Table 6.4 Definition of the spreading parameters

FLOW	mixing layer	exactly self-similar jets and wakes	small-deficit wakes	shear-free layers
$S \equiv$	$\frac{d(y_{.1}-y_{.9})}{dx}$	$\frac{dy_{\frac{1}{2}}}{dx}$	$\frac{U_E}{U_0} \frac{dy_{\frac{1}{2}}}{dx}$	$\frac{U_E}{\sqrt{k_0}} \frac{dy_{\frac{1}{2}}}{dx}$

The predictions of S are compared with measured values in table 6.5 and in figs. 6.2 and 6.3. Table 6.5 lists the results for most of the flows considered; and it includes predictions with a variety of models. Figures 6.2 and 6.3 show respectively the influence of the velocity ratio U_I/U_E on the rate of spread of mixing layers, and of U_0/U_E on the rate of spread of exactly self-similar jets and wakes. The experimental scatter of the mixing layer data in fig. 6.2 is considerable. As was discussed in § 5.5.2, the lower envelope of the data is likely to represent the experiments with the lowest free-stream turbulence. In the predictions, the free stream was assumed to carry no turbulence; so it is this lower envelope that should be compared with the predicted curve.

Table 6.5 Spreading parameter S

Comparison of predictions and experiments

FLOW Model		mix. layer $\frac{U_E}{U_I} = 0$	mix. layer $\frac{U_E}{U_I} = .6$	plane jet $U_E = 0$	radial jet	round jet $U_E = 0$	plane wake $U_E = \text{const}$	plane shear- free layer
Predictions	Rodi & Spalding	.16		.108	.114			
	Model I	.152	.0325	.114	.13	.087	.09	.75
	Model II	.159	.0338	.106	.11		.07	.246
	Model III	.124	.0294	.105	.127		.093	
	Model I, $L = .125y_{\frac{1}{2}}$			.114	.0557		.0687	
	$\tau k \epsilon$ - model	.156		.114			.069	
	$\overline{u_i u_j} \epsilon$ - model	.156	.0359	.114		.119	.069	
EXPERIMENTS		.16	.033	.11	.11	.086	.098	.75
References for experiments		Liepm. & Laufer 1947	Spencer 1970	Newman 1967	Heskestad 1966	Present study	Townsend 1956	Townsend 1956

Model I predicts reasonably well the spreading of all self-similar flows included in table 6.5 and figs. 6.2 and 6.3. Table 6.5 shows that, for flows with strong strain (jets, mixing layers), the predictions are not quite as good as those of model II. This may be ascribed to the fact that the constants of model II were optimised only for flows with strong strain, while the constants of model I were optimised to give agreement also for flows with weak strain (see table 6.1). Moreover, the function $c_{\mu} = f(\overline{P}/\epsilon)$, which is used in model I but not in model II, may not be truly universal for all flows. However, the introduction of this function is beneficial for the wake predictions; and it appears to be a necessity for flows with negligible turbulence production (e.g. shear-free layers).

Model II predicts well the spreading of flows with strong strain. In these flows, the overall production and dissipation of turbulence energy are similar in magnitude ($\overline{P/\epsilon} \approx 1$). The spreading parameter of the plane wake is underpredicted by 30%, and model II is not applicable to flows with negligible turbulence production.

Model III, which assumes that L is uniform over any cross-section, performs very similarly to model I. This was expected already in § 3.6b. The mixing layer is the only case in which the lateral distribution of L seems to play a significant role; for in this case the rate of spread is underpredicted by 22%. Indeed, the profiles of L predicted with model I (see fig. 6.14) show that among all self-similar flows the mixing layer has the strongest lateral variation of L .

One-equation model. Table 6.5 lists a few predictions obtained with a one-equation model. This model is a degenerate form of model I: the length scale L was not determined from an equation but was assumed to be a constant proportion of y_1 . The proportionality factor was so chosen as to give the same S for the plane jet as does model I ($S = .114$). Table 6.5 shows that this model gives poor predictions for the radial jet and for the plane wake. From this finding we may conclude that the employment of a length-scale equation is essential for a universal model.

Higher level models. The three-equation model ($\overline{\rho k \epsilon}$) and the five-equation model ($\overline{u_i u_j \epsilon}$) discussed respectively in appendices H and G predict well the spreading of the mixing layer and the plane jet. Both models underpredict the parameter S for the plane wake by about 30%. Also, the five-equation model does not predict correctly the rate of spread for the round jet when the same constants are used as for the plane jet.

6.6.1-2 Profiles of U , k , $\overline{uv}$, and turbulence energy balance

For the self-similar flows sketched in fig. 4.2, predicted and, where available, measured profiles of U , k , and $\overline{uv}$, as well as energy balances are presented in figs. 6.4 to 6.13. With the exception of fig. 6.7, all the predictions shown were obtained with model I. The

profile shapes predicted by model III are somewhat different, particularly near the flow edges, because, in this model, L was assumed to be uniform over any cross section. This difference can be recognized, for the example of the plane jet, by comparing figs. 6.6 and 6.7. In view of the experimental uncertainties, the difference may be considered insignificant. Because the difference is of similar magnitude for the other flows, the profiles predicted with model III are not presented for these flows.

Velocity profiles. Not much need be said about the mean velocity profiles because they agree for all cases within the experimental accuracy with the measurements.

The profiles of k and $\overline{uv}$, and the k -balance will now be discussed separately for each flow. The measured $\overline{uv}$ -profiles are included for comparison only when they were found to be consistent with the mean velocity measurements (for consistency checks see § 5).

a) Plane mixing layer (fig. 6.4 for $U_E = 0$, fig. 6.5 for $U_E/U_I = .6$)

Profiles of k and $\overline{uv}$. The overall agreement of the profiles of k and $\overline{uv}$ may be considered fairly good for both the mixing layer with one stream at rest ($U_E = 0$) and for the layer with $U_E/U_I = .6$. However, near the inner edge (where $U = U_I$) of the former layer, the agreement is rather poor. There, the experimental free stream turbulence energy was $k_I/U_I^2 \approx 3.5 \times 10^{-4}$, while k_I was taken as zero in the predictions. This difference in the boundary conditions may explain the lack of agreement at the inner edge; for the agreement is much better in the case of the layer with $U_E/U_I = .6$; and in that case the experimental k_I/U_I^2 was much lower ($\approx 10^{-5}$).

The k -balances are in fairly good agreement, considering the difficulties in measuring the dissipation accurately. In a preliminary study, Rodi and Spalding (1970) predicted an unrealistically abrupt drop to zero of convection and diffusion at the inner edge. That study employed everywhere the local value of v_t . The present practice described in § 6.5e, namely to keep v_t/σ constant near edges with finite free-stream velocity, removes this defect.

Figs. 6.4 and 6.5 show that the convection of k is negative throughout mixing layers, a result inferred already from the integral energy equation in § 4.4. As a consequence, $\overline{P}/\epsilon$ is larger than unity in mixing layers. The value obtained is $\overline{P}/\epsilon \approx 1.3$.

By comparing fig. 6.4 with fig. 6.5 we find that the profiles of k/U_o^2 and $\overline{uv}/U_o^2$, and the energy balance are affected little by a change of U_E/U_I from 0 to 0.6. Predictions and experiments agree in this respect.

b) Plane jet with $U_E = 0$ (figs. 6.6. and 6.7)

For the plane jet issuing into stagnant surroundings ($U_E = 0$), predictions are presented of both model I (fig. 6.6) and model III (fig. 6.7). In view of the experimental uncertainties near the edge of the jet, the agreement may be considered good for all the predictions.

Model III, which assumes that L is invariant with y , gives somewhat better agreement near the edge than does model I. This indicates that a value of α_{KL} smaller than unity would be more appropriate for the plane jet because it would lead to a more uniform distribution of L . However, such a value would worsen some of the other results.

The plane jet has a characteristic minimum of k at the centre. This minimum is caused by the absence of shear-stress production and by the relatively low convection at the centre.

The overall ratio of production to dissipation is about 1 in the plane jet.

c) Plane jet with $U_o/U_E = 0.59 = \text{const}$ (fig. 6.8)

The only experimental data available to the writer, the shear stress data, agree well with the predictions.

The minimum of k at the centre is even more pronounced in this case. The reason is that the convection at the centre relative to the maximum production is considerably lower than in the above jet with $U_E = 0$.

d) Radial jet (fig. 6.9)

We have learnt in § 5.5.3c that Heskestad's (1966) turbulence measurements, the only ones reported for the radial jet, did not attain similarity. Hence, they are not suitable for a comparison with the predictions.

In § 4.4 we have seen that the shear stress in the radial jet should be about twice that in the plane jet. The predictions confirm this. Since the eddy viscosity ($\approx c_\mu \sqrt{k}L$) must also be twice as high, we expect that k should also be larger. The predictions confirm this expectation.

The energy balances of radial and plane jet are markedly different in the centre portion. The convection is relatively large in the radial jet, which is explained by the factor 2 on the L.H.S. of equation (4.45). As a consequence, the k -profile has no minimum at the centre.

e) Round jet (fig. 6.10)

The profiles of k and $\overline{uv}$ agree fairly well; and so do the convection and production terms in the energy balance. There is, however, no agreement about the diffusion and dissipation terms. Since these terms cannot be measured accurately, it is difficult to say which values are closer to the truth.

As in the case of the radial jet, the convection of k is relatively large in the centre portion (note the factor 2 on the L.H.S. of equation (4.44)). For this reason, the k -profile has no minimum in the round jet either.

f) Plane wake (fig. 6.11)

It was argued in § 5.5.5a that the measured k -values are probably too low for the plane wake. With this remark in mind, we may consider the overall agreement of the k - and $\overline{uv}$ -profiles as satisfactory. At the edge, the predicted k -values approach zero more rapidly than do the measured ones. As in the case of the mixing layer, this discrepancy may be due to the presence of free stream turbulence in the experiments.

It is noteworthy that the levels of k/U_o^2 and $\overline{uv}/U_o^2$ are about twice as high as in the plane jet (fig. 6.6). Also, the k -profile has only a slight minimum at the centre because the convection is relatively large in the wake.

As was explained in § 5.5.a, Townsend's (1956) energy balance cannot be included for a quantitative comparison. However, the qualitative agreement is good. The ratio of overall production to dissipation of k is about 0.8 in the wake.

g) Round wake behind a sphere (fig. 6.12a, b)

Similarity behaviour. From the review of experimental data (§ 5) we know that the state of approximate self-similarity of a round wake depends on the geometry of the wake-generating body. However, our usual practice of predicting self-similar flows, namely to integrate until the profile shapes cease to change, can lead to one similarity solution only. A different practice was therefore adopted to calculate the round wake behind a sphere. The integration was started at $x/D = 50$ from measured profiles. The experimental wake was already self-similar at that station. So, if the measurements were exact, a perfect model would predict S and k_t/U_o^2 to remain constant as the integration proceeds downstream. Fig. 6.12a shows that the predictions with model I adjust to a state of approximate self-similarity which is only slightly different from the experimental one. This state can be seen to depend on the initial value of c_μ . The predictions agree therefore with the experimental finding that, for the round wake, different states of approximate self-similarity can exist which depend on the initial conditions. The adjective "approximate" is essential here: the predicted values of S and k_t/U_o^2 can be seen to rise slowly over the range of x/D considered in fig. 6.12a, but this variation is well within the experimental accuracy.

When the integration is carried out very far downstream, a state of similarity is reached eventually which is independent of the initial conditions. However, because the local Reynolds number decays in a round wake, such a state cannot be realised experimentally.

Fig. 6.12a also displays predictions with model II. They show the correct behaviour of S for $x/D < 100$, but a sharp rise in k_t/U_o^2 which is in contrast to the experimental observation. Thus, we may conclude, model II cannot maintain the measured state of similarity and is

therefore not applicable to round wakes.

The failure of model II in this case is easy to explain: the predictions with model I show that $c_\mu \approx 5$ is required to maintain similarity; however, model II uses $c_\mu = 1$. This example brings to light the importance of the function $c_\mu = f(\overline{P}/\epsilon)$; for it is this function which provides the required high value of c_μ (see fig. 6.1, with $\overline{P}/\epsilon = .17$ for this flow).

Profiles at $x/D = 100$. In fig. 6.12 b, we compare predicted (model I) and measured profiles of U , k , and $\overline{uv}$ and energy balances at $x/D = 100$. All the results can be seen to agree fairly well. The existence of a slight experimental minimum of k at the centre is somewhat surprising in view of the large convection. This large convection, a small production ($\overline{P}/\epsilon \approx .17$), and a very high relative kinetic energy ($k/U_o^2 \approx 1$) are characteristic features of the round wake.

h) Plane shear-free layer (fig. 6.13)

The use of constant σ 's gives poor predictions of the k -profile. When the σ 's are varied so that v_t/σ is constant across the layer (see § 6.5e), the agreement can be seen to be excellent.

The convection term of the energy balance agrees also well; the agreement of the dissipation and diffusion terms is only fair. Owing to the difficulties in measuring these terms accurately, it is hard to say which values are closer to the truth.

6.6.1-3 L - Profiles

The predicted L -profiles are shown for some of the flows in fig. 6.14. These profiles cannot as yet be compared quantitatively with experiments; for, to determine L as defined by equation (3.12), the two-point correlations of all three fluctuating velocities have to be measured for the whole flow field; and no-one has yet reported such measurements. However, from some correlation measurements of mainly the u -component, we may conclude that the predicted L -profiles are qualitatively plausible:

For the mixing layer and the jets, L rises from the inner edge, or the centre-line, to the outer boundary where its gradient with

respect to y is zero. Experimental integral scales show the same behaviour. They were determined from two-point correlation measurements for the mixing layer by Wygnanski and Fiedler (1970), for the plane jet by Tailland et al (1967), and for the round jet by Wygnanski and Fiedler (1969).

In the case of the wake, L decreases from the centre-line to the outer boundary. Townsend's (1956) integral scale for the u -fluctuations shows the same behaviour. The outer part of a wall boundary layer shares many features with a wake. In this context it is interesting to note that the length scales which Bradshaw et al (1967) and Glushko (1965) employed in their one-equation models fall to zero at the free-stream edge of wall boundary layers.

Because Rodi and Spalding (1970) employed a diffusion constant of $\sigma_{kL} = 0.3$ instead of the present value of 1, their L - profiles are more uniform than the present ones. For lack of quantitative experimental information, we cannot at present say which profiles are more realistic.

6.6.1-4 Concluding remarks on the self-similar flows predictions

In the preceding sections, we have looked at predictions of all the self-similar flows sketched in fig. 4.2. With the exception of the round wake, these predictions were independent of the initial conditions from which the integration was started. We have found that model I predicts reasonably well the main features of all the self-similar flows considered. Model II, which uses $c_\mu = \text{const}$, and the existing three- and five- equation models predict well only the flows with strong strain, excluding the round jet. The greater universality of model I (for boundary layers) is thought to be due mainly to the introduction of the function $c_\mu = f(\overline{P/\epsilon})$.

Some of the self-similar flows were used to optimise the constants and functions of the model. For these flows, the agreement is perhaps not so surprising. The application to non-similar flows, which is the subject of the next section, will be a more stringent test.

6.6.2 Non-similar flows

The non-similar flows considered here were characterised briefly in § 4.5. Most of these flows are influenced by the conditions of flow

generation; so, in most cases, we can no longer use arbitrary initial conditions for our predictions. The initial conditions employed will be described for each flow in the relevant subsection.

6.6.2-1 Jets issuing into a moving stream

In this section, we consider only the far field of non-similar jets, that is the regions termed "strong jet" and "transition strong jet - weak jet" in fig. 4.3 (p.65). In the latter region, the strain rate falls from relatively large to very small values. This region is therefore an important test case for turbulence.

Apart from a possible shift in the virtual origin, the development of the far jet depends only on the momentum thickness θ^* of the jet and not on the exact flow conditions at the nozzle (see § 4.5 and 5.5.6). In particular, the ratio of free-stream to exit velocity, U_E/U_N , has no direct influence. In this case, the integration could therefore be started from arbitrary initial conditions at the nozzle exit. Calculations were carried out for various ratios U_E/U_N . When plotted versus x/θ , all the results did indeed coincide.

Plane jet. In fig. 6.15, the quantities $(U_E/U_O)^2$, y_1/θ , k_ϵ/U_O^2 , and L/y_1 as predicted with model I are plotted versus x/θ . For $(U_E/U_O)^2$, predictions with model II, the τk_ϵ -model (appendix H) and the $u_1 u_j \epsilon$ -model (appendix G) are also included. The predictions are compared with measurements of $(U_E/U_O)^2$ and y_1/θ . Calculations obtained with various integral methods are shown in fig. 6.16 (details on the integral methods are given in appendix K).

The predictions with model I agree well with Bradbury and Riley's (1967) data. The agreement with Everitt's (1971) more comprehensive data is satisfactory only for $x/\theta < 60$. The considerable discrepancy at higher values of x/θ is hard to explain; for the spreading parameter S at extremely high values of x/θ , where the flow is self-similar, is predicted only 8% low (see wake entry, table 6.5). In view of the lack of experimental consensus at the higher values of x/θ , the performance of model I may be considered reasonable.

* θ is defined by equation (4.18).

Model II, the $k\epsilon$ - model, and to a lesser degree the $\overline{u_i u_j} \epsilon$ - model give too low predictions for the higher values of x/θ . The better performance of model I may be attributed to the use of the function $c_\mu = f(\overline{P}/\epsilon)$; for this function causes c_μ to rise with x , that is with decreasing strain rate. The development of c_μ is also shown in fig. 6.15.

By comparing fig. 6.15 with fig. 6.16 we may conclude that model I gives better predictions than any of the integral methods.

Round jet. The analogous results for the round jet are shown in figs. 6.17 (model I) and 6.18 (integral methods). Model I gives good agreement because the data of fig. 6.17 were used to adjust the constant q in the functions (3.16) and (3.17). At high values of x/θ , the experimental data are doubtful; and the local Reynolds number may have fallen to a value low enough to invalidate the high-Reynolds-number assumption inherent in the present model. Discrepancies were therefore tolerated at high values of x/θ .

Figs. 6.17 and 6.18 reveal that none of the integral methods predicts this flow as well as does model I.

Finally, it is interesting to note that k_t/U_o^2 rises much more strongly with x/θ in the round jet than in the plane jet.

The development of L . Figs. 6.15 and 6.17 show that L/y_1 increases as we move from regions with strong strain ($x/\theta \approx 0$) to regions with weak strain (large x/θ).

This draws attention to the importance of the length-scale equation; for models which do not solve a length-scale equation assume that L/y_1 is constant.

6.6.2-2 The near field of jets

The near field comprises the potential-core region and the region of the transition from mixing layer to jet. These regions are defined in fig. 4.3 (p.65). In the near field, the flow conditions at the nozzle exit such as the velocity ratio U_E/U_N , the lip boundary

layers, and the core turbulence, exert an influence on the flow development.

a) Idealised initial conditions

The experimental data available for the near field are shown in fig. 6.19 for plane jets and in fig. 6.23 for round jets. Unfortunately, the profiles necessary to start the integration at the exit were not measured in any of these experiments. So, if we want to predict these flows, we have to guess the initial conditions. The following idealized conditions were employed: the velocity profiles were taken as uniform at the exit, and the streams were assumed to carry no turbulence. The calculations were started from estimated distributions of U , k , and L in a very thin mixing layer at the edge of the nozzle lip.*

Plane jet. Fig. 6.19 compares the predicted velocity decay and jet growth for various velocity ratios U_E/U_N with experimental data.

Fig. 6.22 compares results for the development of k along the centre of the jet. Both figures reveal that the predicted transition from mixing layer to jet is too gradual. The disagreement worsens with increasing U_E/U_N . As we will see shortly, this disagreement may be due to the choice of unrealistic initial conditions.

Round jet. The velocity decay and the growth of round jets are shown in fig. 6.23 for the ratios $U_E/U_N = 0, .25, \text{ and } 0.46$. The growth is predicted well for all velocity ratios, but the velocity decay only for the two lower ones. Again the disagreement may stem from the choice of unrealistic initial conditions.

b) Influence of initial conditions

In figs. 6.19 and 6.22, results are shown of a plane jet calculation which was started at $x/D = 2$ from profiles measured by

* In the predictions, a self-similar mixing layer evolved; so the initial distributions of U , k , and L in the very thin layer had no influence on the downstream results.

Sunyach and Mathieu (1969). The agreement is now fairly good; in particular the centre-line kinetic energy starts to rise much earlier than in the predictions discussed above. This is due mainly to the presence of a small but finite amount of k in the potential core at $x/D = 2$.

Fig. 6.26(a) shows how the initial conditions influence the predictions of a round jet issuing into still air. The range which Wille (1963) measured for various experimental initial conditions is also included. Wille's upper limit represents experiments using a strongly convergent nozzle which nearly produced the idealised conditions of a) at the exit. The predictions starting from these conditions coincide approximately with this upper experimental limit. Those starting from Bradshaw et al's (1964) data at $x/D = 2$ are only slightly lower. This indicates that, in these experiments, the lip boundary layer was thin and the core turbulence intensity was low. When the flow at the nozzle exit is taken as developed pipe flow (Laufer 1954), the predictions coincide with the lower limit of Wille's range. Wille obtained these lowest data points in a jet emerging from a long pipe.

Fig. 6.26(b) shows the influence of core turbulence on the jet development. For the predictions, the turbulence was assumed to be generated upstream of the exit by a grid of mesh size M . As in a), calculations were started from a very thin mixing layer at the lip edge. The values of k and k_L at the inner boundary of the mixing layer are now finite; they are governed by the equations for k and k_L appropriate for homogeneous grid turbulence (see § G.4). With the initial energy, k_N , and the mesh size M specified, the initial length scale was determined as shown in appendix F. Calculations were carried out for two values of the parameters k_N/U_N^2 and M/D respectively. Fig. 6.26(b) shows that both parameters have a strong influence on the jet development. We note that the potential core is shortened drastically by the presence of core turbulence. The experimental data of Vagt (1970) are also included in fig. 6.26(b). Unfortunately, we cannot use these data for a quantitative comparison because Vagt has measured only the intensity but not the scale of his core turbulence. However, we find that the qualitative agreement is good.

We may conclude from the above discussion that the present turbulence model is capable of predicting the influence of various initial conditions on the jet development. The strongest influence was found to be due to core turbulence. By interpretation of their measurements, Flora and Goldschmidt (1969) arrived at the same conclusion.

We have examined the influence of the initial conditions only for the case $U_E = 0$. An even stronger influence may be expected as U_E/U_N increases; so the above lack of agreement for predictions starting from idealised conditions does not discredit the present model.

c) Comparison with other methods.

Integral methods. Predictions of the near field with the integral methods reviewed in appendix K are presented for plane jets in figs. 6.20 and 6.21, and for round jets in figs. 6.24 and 6.25. The same flow cases are considered as in a). A comparison with figs. 6.19 and 6.23 reveals that, when the exact initial conditions are not given, Bradbury's (1967) and Naudascher's (1967) simple methods predict the near field better than does the present method. However, these simple methods can in no way account for the influence of core turbulence.

Zero- and one- equation models. Fig. 6.27 compares the present predictions for a round jet in still air with predictions using the mixing length hypothesis, Prandtl's (1942) constant-eddy-viscosity model ($\nu_t = c U_0 \delta$), and the one-equation model of Harsha (1971)*. The latter predictions were all carried out by Harsha. For information on the constants used, the reader is referred to Harsha's report. All the predictions shown in fig. 6.27 were started from Bradshaw et al's (1964) measured profiles at $x/D = 2$. The experimental ranges given by Wille (1963) and Harsha are also included in fig. 6.27. When we accept Wille's range for the near field, the present model appears to give the best agreement with experiments. Also, it is the only model which predicts correctly the velocity decay in the far jet. This, it must be repeated for fairness, was accomplished by introducing the empirical functions (3.16) and (3.17).

* Harsha employs the k - equation, assumes that $\overline{uv} = -a_1 k \partial U / \partial y / |\partial U / \partial y|$, and near symmetry lines $uv = -a_1 k \partial U / \partial y / |\partial U / \partial y|_{\max}$, where a_1 is a constant.

6.6.2-3 Plane wake behind a flat plate

We now turn to the prediction of wakes. In this section we consider a wake generated by a flat plate. The wake development was predicted with model I, starting from Chevray and Kovasznay's (1969) measured profiles at the trailing edge.*

Fig. 6.28 compares the predicted velocity along the centre-line with measurements. The predictions of Harsha with the various models introduced already in the preceding section are also presented. Model I gives the best agreement; Harsha's one-equation model is very close. The zero-equation models predict the wrong asymptotic behaviour.

Fig. 6.29 compares predicted and measured velocity profiles at various stations in the wake. The agreement may be considered as very good.

6.6.2-4 Plane wake in a pressure gradient

Fig. 6.30 compares predictions of the development of a plane wake in a pressure gradient with measurements of Gartshore (1967). The integration was started at $x/D = 66$ using the measured velocity profile. The profiles of k and $\overline{uv}$ at the initial station were estimated by reference to turbulence measurements at downstream stations. The variation of the free-stream velocity U_E with x was prescribed by the experiments.

Fig. 6.30 shows good agreement of predictions and experiments. The mean flow is found to be nearly self-similar in the range covered by Gartshore's measurements, but the turbulence field is not.

6.6.2-5 Concluding remarks on the non-similar flows predictions.

The non-similar jets and wakes discussed above include three transition regions in which the flow character changes significantly, and which are therefore important test cases for turbulence models. These regions are:

*Chevray and Kovasznay did not measure the component w^2 ; this component was estimated by reference to Klebanoff's (1955) measurements in the boundary layer of a flat plate.

- (i) transition strong jet - weak jet (6.6.2-1),
- (ii) transition quasi-plane mixing layer-axisymmetric jet (6.6.2-2),
- (iii) transition wall boundary layer-free boundary layer (wake behind a flat plate, 6.6.2-3).

The comparisons with experiments have shown that, in all three cases, model I is capable of predicting the observed changes of the flow character.

Apart from the far field of jets in a moving stream, all the non-similar flows studied depended on the initial conditions. For all cases for which measured initial profiles were available, the agreement was shown to be satisfactory. In particular, model I proved to be capable of predicting the influence of various initial conditions on the flow development, such as the influence of lip boundary layers and of core turbulence.

6.7 Universality of the constants and functions

The constants and functions employed to obtain the foregoing results were the same for all flows considered. The quality of the agreement achieved reflects therefore the degree of universality of the constants and functions.

Model II ($c_{\mu} = \text{const}$). The constants of model II were adjusted to give agreement for the majority of both free and wall boundary layers with strong strain. In these flows, the overall production and dissipation of k are of similar magnitude ($\overline{P/\varepsilon} \approx 1$). The present results for free flows, together with those of Ng (1971) for wall flows, indicate that the constants can be considered universal for these flows when the round jet is excluded.

For flows with weak strain, and in particular when the production of k is very small, a constant c_{μ} is not suitable. We know from § 3.3 that the use of $c_{\mu} = \text{const}$ implies that convection and diffusion of $\overline{uv}$ are negligible, the ratio $\overline{v^2}/k$ is constant, and the parameters ω and α in the pressure strain simulation (G. 2-7) are constant. So it appears that these assumptions hold true approximately only in flows with strong strain.

Model I ($c_\mu = f(\overline{P/\epsilon})$) The empirical function

$c_\mu = f(\overline{P/\epsilon})$ was introduced to account for the effect of convection and diffusion of $\overline{uv}$, and for the variation of $\overline{v^2}/k$, ω and α .

It could hardly be expected that a single function would account for all these effects in all free boundary layers. However, the function chosen, together with the constants, is universal enough to give fairly good predictions for nearly all the flows considered. The round jet is an exception. To obtain agreement also for this case, empirical correction functions had to be introduced. The functions (3.16) and (3.17) proved to work well.

With these functions included, model I can be considered reasonably universal for free boundary layers.

The round-jet problem. It was clear already from fig. 6.1 that the function chosen for c_μ would not ^{be} suitable for the round jet. Because this function can be derived semi-empirically by simplifying the shear-stress equation (see § 3.7), examination of measured shear-stress balances may help to explain the behaviour of c_μ . We consider here the shear-stress balances for the plane jet (fig. 3.3), the round jet (fig. 3.4) and for developed pipe flow (fig. 3.6). In these balances, the pressure-strain term was determined from data of U , k , $\overline{uv}$, and ϵ via the simulation (G.2-7) using $\alpha = .4$ and $\omega = 2.5$ as determined from one-dimensional flows (§ G.4). For the plane jet and the flow in a pipe, the pressure-strain term so obtained balances quite well the other terms, for the round jet it does not. This indicates that the pressure-strain term behaves differently in the round jet than it does in other flows, which may be the basic reason for the need to use correction functions for this flow. The writer cannot offer an explanation for this behaviour of the pressure-strain term. Also, there is no evidence that any of the other pressure-strain simulations proposed can predict this behaviour.

Because of the apparent lack of universality of existing pressure-strain simulations, the round-jet problem cannot at present be resolved by employing the shear-stress equation. Predictions with the five-equation model of appendix G confirm this: when the round jet is calculated with the constants suitable for the plane jet and the mixing layer, the rate of spread is predicted 35% too high. Thus, it seems, the only way out at present is to use empirical correction functions for the round jet.

Concluding remarks. The prediction method employing model I could certainly be refined further by including the normal-stress terms^{*} and by replacing the constants c_S and c_B by functions of characteristic parameters such as $\overline{P/\varepsilon}$. However, the present state of experimental knowledge does not warrant this further complication; and it is well for us to remember that no model short of the time-dependent Navier-Stokes equations can be truly universal.

^{*} with the aid of the algebraic stress model described in appendix I.

CHAPTER 7CLOSURE7.1 Conclusions

The main conclusions that can be drawn from the present study are as follows:

a) Prediction methodsAssessment of relative merits

(i) For sufficient universality, models employing two or more equations for turbulence quantities were found to be necessary (§ 3.2).

(ii) For free turbulent boundary layers, existing models involving more than two equations were shown to be inferior in accuracy, and less economical, than two-equation models (§ 3.2).

Performance of the principal two-equation model (I) and its variants (II and III)

(iii) The two-equation model I developed in § 3 has the following main features:

- The shear stress is determined from the eddy viscosity formula, with $\nu_t = c_\mu \sqrt{k}L$.
- The kinetic energy, k , and the length scale, L , of the turbulence are obtained from differential transport equations for k and kL .
- c_μ is calculated from an empirically-determined function of $\overline{P/\epsilon}$, where $\overline{P/\epsilon}$ is the ratio of the overall production to dissipation of k at each cross section.

(iv) The present prediction method incorporating model I was found to perform as follows (§ 6):

- The method predicts the main features of a large variety of free turbulent boundary layers with a single set of constants

and functions. For both mean flow and turbulence quantities, the agreement is in general within the experimental accuracy.

- The ability of the method to predict flows with both strong and weak strain is due to the use of the function $c_\mu = f(\overline{P}/\epsilon)$.
- The predictions of most non-similar flows depended on the choice of the initial profiles of U , k , and $\overline{uv}$ (§ 6.6.2). Sometimes, a realistic choice is difficult to make; but it is one of the merits of the method that it can predict the influence of the upstream conditions.
- The present predictions compare favourably with those of other methods.
- The present method exhibits greater universality than other methods.
- In common with all other models, some of the constants of model I must be adjusted for the round jet.
- In shear-free layers, the σ 's need to be varied such that v_t/σ is constant across the layer.

(v) Two simpler variants of model I were described in § 3.6. They were found to perform adequately for more restricted circumstances (§ 6.6):

- Model II, which assumes c_μ to be a constant, is suitable for flows with strong strain (where $\overline{P}/\epsilon \geq 1$). This model, with an additional near-wall term in the kL -equation, was successfully applied also to many wall boundary layers (Ng 1971).
- Model III, which assumes L to be uniform over any cross section, performs equally well as model I in symmetrical flows.

(vi) The necessity to adjust the constants for the round jet was attributed to the different behaviour of the pressure-strain term in this flow (§ 6.7). None of the existing pressure-strain simulations can predict this behaviour. Thus, the currently-available stress-equation models also require adjustment of the constants to procure agreement for the round jet.

To extend the universality of model I to the round jet, the constants c_D and c_S were replaced by empirical functions of the parameter r/L (§ 3.5a).

More complex models

(vii) In appendix G, a model employing equations for all the Reynolds stresses and for the dissipation rate was developed and applied to a number of one-dimensional flows and to a few free boundary layers. The constants involved were found not to be universal: the constants suitable for the plane jet and the plane mixing layer yielded rates of spread which were 30% too low for the plane wake and 35% too high for the round jet.

(viii) The stress equations in the above model were simplified in appendix I to yield algebraic expressions for the Reynolds stresses. These expressions, together with equations for the kinetic energy and the scale of turbulence, form an "algebraic stress model" which may be used as a first approximation to a full stress-equation model. The model was found to predict reasonably well the distribution of v^2/k in a few free boundary layers (fig. I.1).

(ix) The weakest component of both the stress-equation and the algebraic-stress model emerged to be the pressure-strain simulation. The currently-available simulations all lack universality.

b) Flow phenomena

(i) Chapter 5 has shown that the existence of similarity has been verified experimentally for nearly all potentially self-similar flows. The similarity analysis of § 4.4 revealed that, for round wakes and shear-free layers in zero pressure gradient, the similarity region is limited because the local Reynolds number ($U_0 \delta/\nu$) decreases with x . For the other flows, the Reynolds number is constant or increases with x . For these latter flows, the state of self-similarity was found to be independent of the exact conditions of flow generation.

The decay laws, and the similarity forms of the differential and integral momentum and energy equations were given in § 4.4 for all known self-similar free boundary layers.

(ii) The examination of round-wake experiments in § 5.5.4b has revealed that, for this flow, various states of approximate self-similarity exist which depend on the shape of the wake-generating body. Round wakes behind blunt bodies spread much faster than wakes behind slender bodies.

(iii) The principle of "jet forgetfulness" was confirmed by the experiments reviewed in § 5.5.6: the far development of jets in a moving stream is independent of the exact flow conditions at the nozzle, except for a possible shift in the virtual origin; the only quantity which influences strongly the development is the momentum excess.

(iv) When the excess velocity is small compared with the free stream velocity, plane jets decay in the wake-like manner predicted by the similarity analysis. No tendency for such behaviour could be detected in round jets; rather these were found to follow the decay laws for self-similar laminar wakes when the excess velocity is relatively small. This behaviour was attributed to the axial decay of the local Reynolds number (§ 5.5.6).

(v) The experiments reviewed in § 5.5.2 indicate that for mixing layers the effect of turbulence in one or both of the free streams is to increase significantly the rate of spread. The initial region of jets is effected correspondingly: experiments have shown (fig. 6.26b) that the potential core length can be reduced considerably by the presence of turbulence in the core.

c) Accuracy of measurements

(i) It was found in § 5 that the mean velocity can be measured reliably except in regions with very high turbulence intensity. For most flows, the agreement of various experimenters about this quantity was found to be fairly good. The mixing layer is an exception because it appears to be very sensitive to the experimental lay-out and to free stream turbulence.

(ii) The review of experimental data has shown that turbulent shear and normal stresses could not, in general, be measured with an

accuracy better than 20%. Accordingly, the agreement of various experimenters was found to be rather poor. The measured dissipation had to be multiplied by factors ranging from 2 to 4 to obtain a reasonable energy balance.

(iii) It was argued in § 5.3 that, in view of the many possible sources of error in hot-wire anemometry, a valid judgement on the quality of the measurements can be based on consistency checks only. One of the most important checks is the comparison of $\overline{uv}$ as measured directly and as determined from U via the momentum equation.

For each class of flows considered, an attempt was made in § 5.5 to identify the most reliable data.

d) Experimental study

In appendix L, a new technique of analysing hot-wire signals in highly turbulent flows was developed and tested in the self-similar region of a round jet. The new technique involves less restrictive assumptions than the conventional techniques. It was argued that the round-jet data obtained with this technique are more accurate than previous ones on the basis that they are more consistent.

7.2 Suggestions for future research

7.2.1 Theoretical work

The present model I has been shown to predict satisfactorily a large variety of free turbulent boundary layers. Further refinement of this model seems not opportune at present because of the lack of more accurate and more detailed experimental information. However, there still remain some questions that require an answer.

a) The round-jet problem

Agreement for the round jet could be achieved only by introducing correction functions for this flow. Although this practice will probably meet the engineer's needs, it is unsatisfactory from a scientific point of view. The search for a model should therefore be continued which explains the different behaviour of the round jet.

Perhaps, future advances in simulating the pressure-strain term will bring the answer.

b) c_μ -function for wall flows

Model I has so far been tested for free boundary layers only. Although fig. 6.1 indicates that the chosen c_μ -function is suitable also for wall boundary layers, its exact performance in these flows is not known and should be studied.

c) General turbulence models

The ultimate goal of turbulence research is a turbulence model for general flows. The present study, and the companion study of Ng(1971), were restricted to flows with one direction of significant mean velocity. The next logical step seems to be the development of models for flows with at least two directions of significant mean velocity, such as two-dimensional re-circulating flows and swirling jets.

Shortcomings of scalar viscosity two-equation models. Already, two-equation models employing a scalar eddy viscosity and a constant c_μ have been applied to various flows of the type quoted above (e.g. Roberts 1972, Mathews 1972, Date 1972). The success was only partial, which may be ascribed to the following shortcomings of these models: (i) the assumption of a scalar eddy viscosity seems not to be generally valid and (ii) the present study has shown that c_μ is by no means a universal constant.

It should be mentioned here that the present practice, namely to make c_μ a function of a cross-sectional parameter, appears to be unsuitable for re-circulating flows.

Development of stress-equation models. To cope with more complex flows, it seems therefore necessary to employ stress-equation models or, for preliminary studies, algebraic stress models. The present study has shown that the existing models of this type are not yet satisfactory, mainly because the pressure-strain simulations employed lack universality. So, the progress in developing general turbulence models is linked

strongly with the improvement of the pressure-strain simulation; and great efforts should be taken to find a more universal simulation. Perhaps, as a first step, the constants in the existing simulations should be replaced by functions of characteristic turbulence parameters.

d) Study of heat and mass transfer

The present investigation was concerned solely with the hydrodynamics of uniform-density flows. Engineers, however, would like to be able to predict also heat- and mass-transfer processes, and flows with variable density, such as flames and supersonic jets and wakes. The prediction method described in this thesis should therefore be extended to cope with these problems.

Suggestions for the extension of the method. To account for heat- and mass-transfer phenomena, the solution of the transport equations for total enthalpy and species concentration is required. It seems consistent with the use of a two-equation model to relate the turbulent diffusion fluxes appearing in the exact forms of these equations to the respective gradients of the diffused quantity; thus

$$\text{Turbulent diffusion flux of } \Phi \equiv \overline{v\Phi} = \frac{\nu_t}{\sigma_\Phi} \frac{\partial \Phi}{\partial y},$$

where Φ stands for either enthalpy or concentration and σ_Φ is the turbulent Prandtl or Schmidt number.

When the density is non-uniform, the equations of the present model contain additional correlations involving the fluctuating density. As a first approximation, one may neglect these correlations (Morkovin's 1964 hypothesis) and simply use the local mean density, as calculated from the equation of state, where it appears in the equations.

Some predictions have already been carried out with a method as outlined above (Launder et al 1972).^{*} The majority of the results were encouraging, but for some flows the agreement with experiments was poor, especially for the case of the supersonic mixing layer. Future research should therefore give special attention to the effect of variable density.

^{*} The turbulence model used was similar to model I, but it employed an equation for ϵ instead of the kL-equation.

7.2.2 Experimental work

Improvement of measuring techniques. The review of experiments, § 5, has pointed to the need for more accurate data on the turbulent stresses; and it has shown that such data are very difficult to obtain with hot-wire anemometers, particularly when the turbulence intensity is high. New techniques should therefore be developed which are more suitable for measurements in highly turbulent flows. The Laser-Doppler anemometer has great potential in this respect (Durst 1972); so, when this instrument will be more fully developed, it should be used to repeat some of the basic measurements which were found doubtful in § 5.

Pressure-strain measurements. Another problem is the direct measurement of the pressure-strain term. Data on this term would aid greatly the development of a more universal pressure-strain simulation, which is necessary to construct a general stress-equation model. Such data may also help to resolve the round-jet problem. So far, correlations involving the fluctuating pressure could be measured reliably only in the atmospheric boundary layer (Elliott 1970). Methods should be developed which allow these correlations to be measured also in laboratory flows.

Future experiments. Suggestions for future experiments on free turbulent boundary layers have been made already in § 5.6. One further suggestion remains to be added: in order to remove the uncertainties about the choice of initial conditions for predictions, future experimenters should always measure and report the distributions of both mean and turbulence quantities at the upstream boundaries of their flows.

REFERENCES

1. ABRAMOVICH, G.N. (1963): The theory of turbulent jets.
The M.I.T. Press.
2. ALBERTSON, M.L., DAI, Y.B., JENSEN, R.A. and ROUSE, H. (1948):
"Diffusion of submerged jets". Proc.Am.Soc.Civil Engrs.,
74, p.1751.
3. ALEXOPOULOS, C.C. and KEFFER, J.F. (1968): "Extended
measurements of the two-dimensional turbulent wake".
University of Toronto, Dept. of Mech. Eng. Report UTME-TP 6811.
4. BALDWIN, L.V. and SANDBORN, V.A. (1968): "Intermittency of
far wake turbulence". AIAA J., 6, p.1163.
5. BOUSSINESQ, T.V. (1877): Mem. pres.Acad.Sci. Third Edition,
Paris XXIII, p.46.
6. BRADBURY, L.J.S. (1963): "An investigation into the structure
of a turbulent plane jet". Ph.D. Thesis, University of
London.
7. BRADBURY, L.J.S. (1965): "The structure of self-preserving
turbulent plane jet". J.Fluid Mech., 23, pp.31-64.
8. BRADBURY, L.J.S. (1967): "Simple expressions for the spread
of turbulent jets". The Aeronautical Quarterly, May 1967,
pp.133-142.
9. BRADBURY, L.J.S. (1971): "A review of theoretical and experimental
work on the development of turbulent jets". Lecture given
at the Von Karman Institute.
10. BRADBURY, L.J.S. and RILEY, J. (1967): "The spread of a
turbulent plane jet issuing into a parallel moving airstream".
J.Fluid Mech., 27, pp.381-394.
11. BRADSHAW, P. (1966): "The effect of initial conditions on
the development of a free shear layer". J.Fluid Mech., 26,
pp.225-236.
12. BRADSHAW, P., FERRISS, D.H. and JOHNSON, R.F. (1964):
"Turbulence in the noise-producing region of a circular jet".
J.Fluid Mech., 19, pp.591-624.
13. BRADSHAW, P. and FERRISS, D.H. (1965): "The spectral energy
balance in a turbulent mixing layer". NPL Aero Report 1144.
14. BRADSHAW, P., FERRISS, D.H. and ATWELL, N.P. (1967): "Calculation
of boundary-layer development using the turbulent energy
equation". J.Fluid Mech., 28, pp.593-616.
15. BROWN, G. and ROSHKO, A. (1971): "The effect of density difference
on the turbulent mixing layer". Paper delivered at AGARD
Meeting, London, Sept. 1971.

16. CARMODY, T. (1964): "Establishment of the wake behind a disk".
Trans. ASME J. of Basic Engineering, Dec.1964, pp.869-882.
17. CHAMPAGNE, F.H., SLEICHER, C.A. and WEHRMANN, O.H. (1967):
"Turbulence measurements with inclined hot-wires. Part I:
Heat transfer measurements with inclined hot-wires". J.Fluid Mech.,
28, pp.153-157.
18. CHAMPAGNE, F.H. and SLEICHER, C.A. (1967): "Turbulence measurements
with inclined hot-wires. Part 2: Hot-wire response
equations". J.Fluid Mech., 28, pp.77-182.
19. CHAMPAGNE, F.H., HARRIS, V.G. and CORRSIN, S. (1970): "Experiments
on nearly homogeneous turbulent shear flow". J.Fluid Mech., 41;
pp.81-139.
20. CHEVRAY, R. (1968): "The turbulent wake of a body of revolution".
Trans. ASME J. of Basic Engineering, 90, Series D, p.275.
21. CHEVRAY, R. and KOVASZNY, L.S.G. (1969): "Turbulence measurements
in the wake of thin flat plate". AIAA J., 7, pp.1641-1642.
22. CHOU, P.Y. (1945): "On velocity correlations and the solution
of the equations of turbulent fluctuation". Quart. Appl. Math.,
3, p.38.
23. COMTE-BELLOT, G. and CORRSIN, S. (1966): "The use of a contraction
to improve the anisotropy of grid-generated turbulence".
J.Fluid Mech., 25, pp.657-682.
24. COOPER, R.D. and LUTZKY, M. (1955): "Exploratory investigation
of turbulent wakes behind bluff bodies". David Taylor Model
Basin Report 963.
25. CORRSIN, S. (1953): "Interpretation of viscous terms in the
turbulent energy equation". J.Aero Sci., 20, p.853.
26. CORRSIN, S. and UBEROI, M.S. (1949): "Further experiments on
the flow and heat transfer in a heated turbulent air jet".
NACA TN 1865.
27. CURTET, R. and RICO, F.P. (1964): "On the tendency to self-
preservation in axisymmetric ducted jets". Trans. ASME J.
of Basic Engineering, Dec.1964, pp.765-775.
28. DALY, B.J. and HARLOW, F.H. (1970): "Transport equations of
turbulence". Phys. Fluids, 13, p.2634.
29. DATE, A.W. (1972): "Prediction of friction and heat transfer
characteristics of flow in a tube containing a twisted
tape". Ph.D. Thesis, University of London.
30. DAVYDOV, B.I. (1961): "On the statistical dynamics of an
incompressible turbulent fluid". Dok. Akad. Nauk SSSR, 136,
pp.47-50 (Soviet Physics - Doklady, 6, p.10(1961)).
31. DEARDORFF, J.W. (1970): "A numerical study of three-dimensional
turbulent channel flow at large Reynolds numbers". J.Fluid
Mech., 41, pp.453-480.

32. DONALDSON, C. du P. (1968): "A computer study of an analytical model of boundary layer transition". AIAA Paper No. 68-38.
33. DURST, F. (1971): "Evaluation of hot-wire anemometer measurements in turbulent flows". Imperial College, Dept. of Mech. Eng. Report ET/TN/A/9.
34. DURST, F. (1972): "Development and application of optical anemometers". Ph.D. Thesis, University of London.
35. DURST, F. and WHITELOW, J.H. (1970): "Evaluation of turbulence characteristics from amplitude probability density measurements of hot-wire anemometer output signals". Imperial College, Dept. of Mech. Eng. Report ET/TN/A/3.
36. ELLIOTT, J.A. (1970): "Microscale pressure fluctuations measured within the lower atmospheric boundary layer". Ph.D. Thesis, University of British Columbia.
37. ERMSHAUS, R. (1970): "Eigentümlichkeiten turbulenter Nachlaufströmungen". Mitteilungen aus dem Max-Planck-Institut für Strömungsforschung Nr. 46.
38. ESCUDIER, M.P. (1967): "The turbulent incompressible hydrodynamic boundary layer". Imperial College, Dept. of Mech. Eng. Report TWF/R/6.
39. EVERITT, K. (1971): Ph.D. Thesis, University of London.
40. FEKETE, G.I. (1970): "Two-dimensional self-preserving turbulent jets in streaming flow". McGill University Report M.E.R.L. 70-11.
41. FLORA, J.J, JR. and GOLDSCHMIDT, V.W. (1969): "Virtual origins of a free plane turbulent jet". AIAA J., 7, pp. 2344-2346.
42. FORSTALL, W., JR. (1949): "Material and momentum transfer in coaxial gas streams". Ph.D. Thesis, Massachusetts Institute of Technology.
43. FORSTALL, W., JR. and SHAPIRO, A.H. (1950): "Momentum and mass transfer in coaxial gas jets". J. of Applied Mechanics, 17, pp.399-408.
44. FUJITA, H. and KOVASZNAY, L.S.G. (1968): "Measurements of Reynolds stress by a single rotational hot-wire anemometer". The Review Scientific Instruments, 39, pp.1351-1355.
45. GARTSHORE, I.S. (1964): "Jets and wall jets in uniform streaming flow". McGill University, Mech. Eng. Report 64-4.
46. GARTSHORE, I.S. (1967): "Two-dimensional turbulent wakes". J.Fluid Mech., 30, pp.547-560.
47. GARTSHORE, I.S. (1971): "Self-preserving flow". Lecture given at the Von Karman Institute.

48. GARTSHORE, I.S. and NEWMAN, B.G. (1969): "Small -perturbation jets and wakes which are approximately self-preserving in a pressure gradient". C.A.S.I. Transaction, 2, p.101.
49. GIBSON, M.M. (1963): "Spectra of turbulence in a round jet". J.Fluid Mech., 15, pp.161-173.
50. GIBSON, C.H., CHEN, C.C. and LIN, S.C. (1968): "Measurements of turbulent velocity and temperature fluctuation in the wake of a sphere". AIAA J., 6, No. 4, pp.642-649.
51. GLUSHKO, G.S. (1965): "Turbulent boundary layer on a flat plate in an incompressible fluid". Izv.Akad.Nauk SSSR, Mekh. No.4, p.13.
52. GUITTON, D.E. (1968): "Correction of hot-wire data for high intensity turbulence, longitudinal cooling, and probe interference". McGill University, Mech.Eng.Report No. 68-6.
53. GUTMARK, E. (1970): "The two-dimensional turbulent jet". M.Sc. Thesis, Technion-Israel Institute of Technology.
54. HALLEEN, R.M. (1964): "Literature review on subsonic free turbulent shear flow". Stanford University, Mech.Eng.Report MD-11.
55. HANJALIĆ, K. and LAUNDER, B.E. (1972): "A Reynolds-stress model of turbulence and its application to thin shear flows". J. Fluid Mech., 52, p.609.
56. HARLOW, F.H. and NAKAYAMA, P.I. (1967): "Turbulent transport equations ". The Physics of Fluids, 10, 11, p.2323.
57. HARSHA, P.T. (1971): "Free turbulent mixing: a critical evaluation of theory and experiment". Arnold Engineering Development Center, Tennessee, Report AEDC - TR - 71 - 36.
58. VAN DER HEGGE ZIJNEN, B.G. (1958): "Measurements of the velocity distribution in a plane turbulent jet of air". Appl.Sci. Res., Section A, 7, pp.256-276.
59. HESKESTAD, G. (1965): "Hot-wire measurements in a plane turbulent jet". J. of Appl.Mech., Dec.1965, p.1.
60. HESKESTAD, G. (1966): "Hot-wire measurements in a radial turbulent jet". J. of Appl.Mech., June 1966, p.417.
61. HILL, P.G. (1965): "Turbulent jets in ducted streams". J.Fluid Mech., 22, pp.161-186.
62. HINZE, J.O. (1959): Turbulence, McGraw-Hill, New York.
63. HINZE, J.O. and VAN DER HEGGE ZIJNEN, B.G. (1949): "Transfer of heat and matter in the turbulent mixing zone of an axially symmetric jet". Appl.Sci.Res., Section A, 1, pp.435-461.
64. HWANG, N.H.C. and BALDWIN, L.V. (1966): "Decay of turbulence in axisymmetric wakes". Trans.ASME, J. of Basic Engineering, 88, Series D, pp.261-268.

65. JEROME, F.E. GUITTON, D.E. and PATEL, R.P. (1971):
"Experimental study of the thermal wake interference between closely spaced wires of a x-type hot-wire probe". The Aeronautical Quarterly, May 1971, pp.119-126.
66. JONES, W.P. (1971): "Laminarization in strongly accelerated boundary layers". Ph.D. Thesis, University of London.
67. KLEBANOFF, P.S. (1955): "Characteristics of turbulence in a boundary layer with zero pressure gradient". NACA Report 1247.
68. KOLMOGOROV, A.N. (1941): C.R. Acad.Sci. USSR, 30, p.301, and 32, p.16.
69. KOLMOGOROV, A.N. (1942)" "Equations of turbulent motion in an incompressible fluid". Izv.Akad.Nauk.SSSR, Seria fizicheskaya VI, No. 1-2, pp.56-58 (English translation: Imperial College, Mech.Eng.Report ON/6, (1968)).
70. KOLOVANDIN, B.A. and VATUTIN, I.A. (1969): "On statistical theory of non-uniform turbulence". Paper presented at the International Seminar on Heat and Mass Transfer, Herzeg Novi, Yugoslavia, Sept. 1969.
71. KUO, Y.H. and BALDWIN, L.V. (1966): "Diffusion and decay of turbulent elliptic wakes". AIAA J., 9, p.1566.
72. LANDIS, F. and SHAPIRO, A.H. (1951): "The turbulent mixing of coaxial jets". Proceedings of the Heat Transfer and Fluid Mechanics Institute, pp. 133-146.
73. LASSITER, L.W. (1957): "Turbulence in small air jets at exit velocities up to 705 feet per second". J.Appl.Mech., 24, pp.349-354.
74. LAUFER, J. (1951): "Investigation of turbulent flow in a two-dimensional channel". NACA Report 1053.
75. LAUFER, J. (1954): "The structure of turbulence in fully developed pipe flow". NACA Report 1174.
76. LAUNDER, B.E., MORSE, A., RODI, W. and SPALDING, D.B. (1972): "The prediction of free shear flows - a comparison of the performance of six turbulence models". Paper presented at the NASA conference on Free Shear Flows, Langley, Hampton, Virginia, July 1972 (also : Imperial College, Dept. of Mech. Eng. Report TM/TN/A/19).
77. LAURENCE, J.C. (1956): "Intensity, scale, and spectra of turbulence in mixing region of free subsonic jet". NACA Report 1292.
78. LAWN, C.J. (1971): "The determination of the rate of dissipation in turbulent pipe flow". J.Fluid Mech., 48, pp.477-505.

79. LIEPMANN, H.P. and LAUFER, J. (1947): "Investigations of free turbulent mixing". NACA TN.1257.
80. MACZYNSKI, J.F.J. (1962): "A round jet in an ambient coaxial stream". J.Fluid Mech., 13, pp.597-608.
81. MATHEWS, L. (1972): "Performance of some two-dimensional film cooling geometries". Ph.D. Thesis, University of London.
82. MAYDEW, R.C. and REED, J.F. (1963): "Turbulent mixing of axisymmetric compressible jets (in the half-jet region) with quiescent air". SANDIA Corp. Aerothermodynamics, SC - 4764.
83. MORKOVIN, M.V. (1964): "Effects of compressibility on turbulent flows". The Mechanics of Turbulence (Ed. A. Favre), Gordon and Breach, New York.
84. MILES, J.B. and SHIH, J.S. (1968): "Similarity parameter for two-stream turbulent jet-mixing region". AIAA J., 6, pp.1429-1430.
85. MILLS, R.D. (1968): "Numerical and experimental investigations of the shear layer between two parallel streams". J.Fluid Mech., 33, pp. 591-616.
86. MOBBS, F.R. (1968): "Spreading and contraction at the boundaries of free turbulent flows". J.Fluid Mech., 33, pp.227-239.
87. NAOT, D., SHAVIT, A. and WOLFSHTEIN, M. (1970): "Interactions between components of the turbulent velocity correlation tensor due to pressure fluctuations". Isreal Journal of Technology, 8, pp.259-269.
88. NAUDASCHER, E. (1965): "Flow in the wake of self-propelled bodies and related sources of turbulence". J.Fluid Mech., 22, pp.625-656.
89. NAUDASCHER, E. (1967): "Once general similarity analysis for turbulent jet and wake flows". The University of Iowa, IIHR Report No.106.
90. NAUDASCHER, E. (1968): "On the distribution and development of mean-flow and turbulence characteristics in jet and wake flows". The University of Iowa IIHR Report No. 110.
91. NEWMAN, B.G. (1967): "Turbulent jets and wakes in a pressure gradient". Fluid Mechanics of Internal Flow (Ed. G.Sovran), Elsevier, p. 170.
92. NEWMAN, B.G. (1968): "The growth of self-preserving jets and wakes". McGill University, Mech.Eng. Report 68-10.
93. NG, K.H. (1971): "Predictions of turbulent boundary-layer developments using a two-equation model of turbulence". Ph.D. Thesis, University of London (partly published in :Ng, K.H. and Spalding, D.B., Physics of Fluids, 15, p.20, 1972).

94. ORTEGA, J.J. (1968): "Characteristics of a turbulent round jet in a coaxial stream". M.S. Thesis, University of Iowa.
95. PATANKAR, S.V. and SPALDING, D.B. (1967, 1970): Heat and mass transfer in boundary layers, 1st. ed. Morgan Grampian, London, 1967. 2nd ed. Intertext, London, 1970.
96. PATEL, R.P. (1970): "A study of two-dimensional symmetric and asymmetric turbulent shear flows". Ph.D. Thesis, McGill University.
97. PATEL R.P. (1971): "Turbulent jets and wall jets in uniform streaming flow". Aeronautical Quarterly, 22, p.311.
98. PERRY, A.E. and MORRISON, G.L. (1971): "Static and dynamic calibrations of constant-temperature hot-wires systems". J.Fluid Mech., 47, pp. 765-777.
99. PRANDTL, L. (1925): "Über die ausgebildete Turbulenz". Zamm, 5, p. 136.
100. PRANDTL, L. (1942): "Bemerkungen zur Theorie der freien Turbulenz". ZAMM, 22 , pp.241-243.
101. PRANDTL, L. (1945): "Über ein neues Formel-system für die ausgebildete Turbulenz". Nachr.Akad.Wiss., Göttingen, Math.-phys. Kl.1945, p.6 (with an appendix by K.Wieghardt).
102. REICHARDT, H. (1942): "Gesetzmässigkeiten der freien Turbulenz". VDI - Forschungsheft 414.
103. REICHARDT, H. (1965): "Zur Problematik der turbulenten Strahlausbreitung in einer Grundströmung". Mitteilungen aus dem Max-Planck-Institut für Strömungsforschung Nr. 35.
104. REICHARDT, H. and ERMSHAUS, R. (1962): "Impuls - und Wärmeübertragung in turbulenten Windschatten hinter Rotationskörpern". Int.J.Heat Mass Transfer, 5, pp.251-265.
105. REYNOLDS, W.C. (1970): "Computation of turbulent flows - state-of-the-art, 1970". Stanford University, Dept. of Mech.Eng. Report MD-27.
106. ROBERTS, L.W. (1972): "Turbulent swirling flows with recirculation". Ph.D. Thesis, University of London.
107. ROBINS, A. (1971): "The structure and development of a plane turbulent free jet". Ph.D. Thesis, University of London.
108. RODI, W. (1970): "Basic equations for turbulent flow in cartesian and cylindrical coordinates". Imperial College, Dept. of Mech. Eng. Report BL/TN/A/36.
109. RODI, W. (1971): "On the equation governing the rate of turbulent energy dissipation". Imperial College, Dept. of Mech.Eng. Report TM/TN/A/14.

110. RODI, W. and SPALDING, D.B. (1970): "A two-parameter model of turbulence, and its application to free jets". Wärme- und Stoffübertragung, 3, p.85.
111. ROTTA, J.C. (1951): "Statistische Theorie nichthomogener Turbulenz". Zeitsch. f. Physik, Bd. 129, pp. 547-572, and Bd. 131, pp.51-77 (English translation: Imperial College, Dept. of Mech. Eng. Reports TWF/TN/38,39, (1968)).
112. ROTTA, J.C. (1969): "Über eine Methode zur Berechnung turbulenter Scherströmungen". Aerodynamische Versuchsanstalt Göttingen Report 69 A 14.
113. ROTTA, J.C. (1971): "Recent attempts to develop a generally applicable calculation method for turbulent shear flow layers". AGARD Conference proceedings No. 93.
114. SABIN, C.M. (1965): "An analytical and experimental study of the plane, incompressible turbulent free-shear layer with arbitrary velocity ratio and pressure gradient". ASME paper No. 64 - WA/FE 19.
115. SAFFMAN, P.G. (1970): "A model for inhomogeneous turbulent flow". Proc. of the Roy. Soc., Ser.A, 317, p.417.
116. SAMI, S. (1967): "Balance of turbulent energy in the region of jet-flow establishment". J.Fluid Mech., 29, pp.81-92.
117. SAMI, S., CARMODY, T. and ROUSE, H. (1967): "Jet diffusion in the region of flow establishment". J.Fluid Mech., 27, pp.231-252.
118. SCHLICHTING, H. (1930): "Über das ebene Windschattenproblem". Ingr. - Arch., 5, p.533.
119. SCHLICHTING, H. (1960), Boundary Layer Theory, McGraw-Hill, New York.
120. SEBAN, R.A. and BACK, L.H. (1962): "Velocity and temperature profiles in turbulent boundary layers with tangential injection". Trans. ASME, J.Heat Trans., 84, Series C, pp.45-54.
121. SPALDING, D.B. (1958): "Spread of confined turbulent pre-mixed flames". Seventh Symposium (International) on Combustion, Butterworth, London.
122. SPALDING, D.B. (1967a): "The calculation of the length scale of turbulence in some boundary layers remote from walls". Imperial College, Dept. of Mech.Eng. Report TWF/TN/31 (also in : Progress in Heat and Mass Transfer, Vol 2, (Ed. T.F.Irvine et al) Pergamon Press, 1969, p.255).
123. SPALDING, D.B. (1967b): "Monograph on turbulent boundary layers". Imperial College, Dept. of Mech.Eng. Report TWF/TN/24.

124. SPALDING, D.B. (1970): "Concentration fluctuations in a round turbulent free jet". Imperial College, Dept. of Mech. Eng. Report BL/TN/A/30 (also, Chem.Eng.Sci. 26, pp. 95-107, 1971).
125. SPALDING, D.B. (1971): "The kW model of turbulence". Imperial College, Dept. of Mech.Eng.Report TM/TN/A/16.
126. SPENCER, B.W. (1970): "Statistical investigation of turbulent velocity and pressure fields in a two-stream mixing layer". Ph.D. Thesis, University of Illinois.
127. SQUIRE, H.B. and TROUNCER, J. (1944): "Round jets in a general stream". Aero.Res.Counc.Tech.Rep.R. and M. No.1974.
128. SUNYACH, M. and MATHIEU, J. (1969): "Zone de mélange d'un jet plan; fluctuations induites dans le cône a potentiel - intermittence". Int.J.Heat Mass Transfer,12, pp. 1679-1697.
129. TAILLARD, A., SUNYACH, M. and MATHIEU, J. (1967): "Étude d'un jet plan". C.R. Acad.Sc.Paris, 264, pp.527-530.
130. TENNEKES, H. and LUMLEY, J.L. (1972): A first course in Turbulence, The M.I.T. Press.
131. TING, L. (1959): "On the mixing of two parallel streams". J.Math.Phys., 38, pp.153-165.
132. TOLLMIEH, W. (1926): "Berechnung turbulenter Ausbreitungsvorgänge". ZAMM, 22, p.5.
133. TOWNSEND, A.A. (1949): "The fully developed turbulent wake of a circular cylinder". Australian I.Sci.Research, 2A, pp.451-468.
134. TOWNSEND, A.A. (1956), The Structure of Turbulent Shear Flow, Cambridge University Press.
135. TUCKER, H.J. (1970): "The distortion of turbulence by irrotational strain". McGill University, Mech.Eng.Report No.70-7.
136. TUCKER, H.J. and REYNOLDS, A.J. (1968): "The distortion of turbulence by irrotational plane strain". J.Fluid Mech., 32, pp.657-673.
137. TUVE, G.L. (1953): "Air velocity in ventilating jets". Heating, Piping and Air Conditioning, Jan.1953.
138. UBEROI, M.S. (1956): "Effect of wind-tunnel contraction on free-stream turbulence". J.Aerospace Sci, 23, pp.754-764.
139. UBEROI, M.S. (1957): "Equipartition of energy and local isotropy in turbulent flows". J.Appl.Physics, 28, pp.1165-1170.
140. UBEROI, M.S. (1963): "Energy transfer in isotropic turbulence". Physics of Fluids, 6, 8, pp.1048-1056.

141. UBEROI, M.S. and FREYMUTH, P. (1969): "Spectra of turbulence in wakes behind circular cylinders". *Physics of Fluids*, 12, 7, pp.1359-1363.
142. UBEROI, M.S. and FREYMUTH, P. (1970): "Turbulent energy balance and spectra of the axisymmetric wake". *Physics of Fluids*, 13, 9, pp.2205-2210.
143. VAGT, J.D. (1970): "Untersuchungen zur Turbulenzstruktur runder Freistrahlen". Interner Bericht am Hermann-Föttinger-Institut für Strömungsmechanik, TU Berlin, (to be published).
144. VAGT, J.D. (1971): "Hot-wire measurement technique - some basic problems". Lecture given at the Von Karman Institute.
145. WATT, W.E. (1967): "The velocity-temperature mixing layer". University of Toronto, Mech.Eng. Report TP-6705.
146. WEINSTEN, A.S., OSTERLE, J.F. and FORSTALL, W. (1956): "Momentum diffusion from a slot jet into a moving stream". *J.Appl.Mech.*, 78, pp. 437-443.
147. WILLE, R. (1963): "Beiträge zur Phänomenologie der Freistrahlen". *Z.Flugwiss.*, 6, pp. 222-233.
148. WOLFSHTEIN, M. (1970): "On the length-scale-of-turbulence equation". *Israel Journal of Technology*, 8, pp.87-99.
149. WYGNANSKI, I. and FIEDLER, H.E. (1969): "Some measurements in the self-preserving jet". *J.Fluid Mech.*, 38, pp.577-612.
150. WYGNANSKI, I. and FIEDLER, H.E. (1970): "The two-dimensional mixing region". *J.Fluid Mech.*, 41, pp.327-363.
151. YEN, K.T. (1960): "On the indeterminateness of the boundary conditions for the mixing of two parallel streams". *J. of Appl.Mech.*, Sept. 1960, pp.390-392.
152. ZARIĆ, Z. (1969): "Statistical interpretation of the velocity measurements in high-intensity turbulence". International Summer School, Herzeg-Novli, Yugoslavia.

NOMENCLATURE

<u>Symbol</u>	<u>Meaning</u>	<u>Equation of first occurrence</u>
B	body force per unit length	(4.1)
c's	constants	
D	width/diameter of jet nozzle or wake-generating body.	
E	Energy spectrum	(A.2)
e	dimensionless kinetic energy ($= k/U_o^2$)	(4.4)
F	total body force	(4.3)
f	dimensionless longitudinal velocity, ($= (U - U_E)/U_o$)	(4.4)
	correction function	(3.16)
g	dimensionless lateral velocity ($= V/U_o$)	(4.4)
h	dimensionless shear stress ($= \overline{uv}/U_o^2$)	(4.4)
j	exponent ($= 0$ for plane flow, $= 1$ for axisymmetric flow)	(2.12)
k	kinetic energy of turbulence ($= \frac{1}{2} \overline{u_i u_i}$)	(2.8)
K_1	anisotropy parameter	
L	length scale of turbulence	(3.3)
ℓ	mixing length	(3.2)
M	excess momentum flux,	(4.2)
	mesh size of grid	
n	wave number	(A.2)
P	mean pressure,	(2.6)
	total production of k	(2.8)
p	fluctuating pressure	(2.4)
P_{ij}	production of $\overline{u_i u_j}$	(2.7)
$Q_{i,j}$	two-point velocity correlation	fig. A.1
q	$(\overline{u^2} - \overline{v^2})/U_o^2$,	(4.4)
	constant	(3.16)

Re	Reynolds number		
r	radius		(3.16)
S	spreading parameter		table 6.4
T	transfer of energy in wave number space		(A.16)
t	time		(2.2)
U_i	mean velocity components	for general flows	(2.5)
u_i	fluctuating velocity components		(2.4)
U, V, W	mean velocity components	for boundary layers	(2.9, 2.10)
U_o	velocity excess/deficit		fig.4.1
u,v,w	fluctuating velocity components		(2.13)
x_i	general cartesian coordinates		(2.1)
x,y,z	coordinates for boundary layers		(2.9, 2.10)

Greek symbols

α	constant in pressure-strain simulation	(G.2-6)
β	diffusion constant	(G.2-3)
δ	characteristic flow width	
δ_{ij}	Kronecker-delta (=1 for $i=j$, = 0 for $i \neq j$)	(G.2-2)
ϵ	rate of dissipation	(3.11)
η	dimensionless distance (= y/δ)	(4.4)
θ	momentum thickness	(4.18)
ν	kinematic viscosity	(2.2)
ρ	fluid density	(2.2)
σ	Prandtl/Schmidt number,	(3.10)
	spreading parameter	table 5.2

τ	shear stress	
Φ_{ij}	pressure-strain term	(G.2-4)
ω	constant in pressure-strain simulation	(G.2-5)

Subscripts

E	external, outer stream
I	internal, inner stream
max	maximum
min	minimum
N	nozzle
t	turbulent
.1, $\frac{1}{2}$, .9	where velocity excess/deficit is .1, half, .9 of the maximum velocity excess/deficit
o, ϕ	value at centre-line

Appendix A

Derivation of the equation for k_L

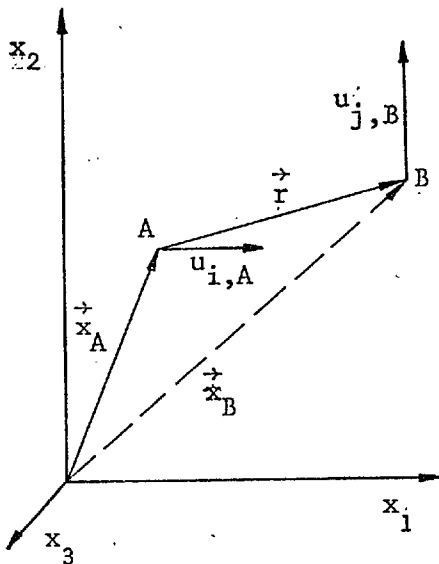
A.1 Introduction

The purpose of this appendix is to derive the equation for k_L which is employed in the turbulence models of §3.

In 1951, Rotta derived an equation for k_L from the spectral energy balance. Recently, Rotta (1969, 1971) rederived this equation in a refined form by integrating the equation for the two-point velocity correlation $Q_{i,i}$. The latter approach was also taken by Wolfshtein (1970), and it will be adopted in this appendix because it shows more clearly the assumptions which lead to the final form of the equation. However, the k_L -equation as derived from the spectral energy balance is suited better for the study of Reynolds number effects; so, to justify the neglect of certain terms, this form of the equation will be considered in §A.5.

A.2 Definitions

The two-point correlations $Q_{i,j}$, $S_{i,jk}$, and $K_{i,p}$ are defined in fig. A.1:



$$Q_{i,j} = \overline{u_{i,A} u_{j,B}}$$

$$S_{i,jk} = \overline{u_{i,A} u_{j,B} u_{k,B}}$$

$$K_{i,p} = \overline{u_{i,A} p_B}$$

For $Q_{i,i}$ the summation convention applies

x_k = components of $\vec{x}$
 r_k = components of $\vec{r}$

Fig.A.1 The two-point correlations

Geometrical relations:

$$x_k = x_{k,A} \quad (A.1)$$

$$r_k = x_{k,B} - x_{k,A}.$$

The three-dimensional energy spectrum, E , is defined in such a way that

$$k \equiv \frac{1}{2} \overline{u_i u_i} = \int_0^\infty E(n) dn, \quad (A.2)$$

where n is the wave number.

The length scale L is defined here by

$$2kL = 4\pi^2 \int_0^\infty \frac{E}{n} dn, \quad (A.3)$$

or, what is equivalent, by

$$2kL = \int_V Q_{i,i} \frac{dV}{r^2}, \quad (A.4)$$

where r is the length of the vector $\vec{r}$ and V is the whole domain of interest.

A.3 The exact equation for kL

The starting point of the derivation is Hinze's (1959) equation for $Q_{i,i}$ (Hinze, p.256):

$$\begin{aligned} & \frac{\partial Q_{i,i}}{\partial t} + U_{j,A} \left(\frac{\partial Q_{i,i}}{\partial x_j} \right)_A + U_{j,B} \left(\frac{\partial Q_{i,i}}{\partial x_j} \right)_B \\ & + Q_{j,i} \left(\frac{\partial U_i}{\partial x_j} \right)_A + Q_{i,j} \left(\frac{\partial U_i}{\partial x_j} \right)_B \\ & + \left(\frac{\partial S_{i,ij}}{\partial x_j} \right)_B + \left(\frac{\partial S_{ji,i}}{\partial x_j} \right)_A + \left(\frac{\partial K_{p,i}}{\partial x_i} \right)_A + \left(\frac{\partial K_{i,p}}{\partial x_i} \right)_B \\ & - \nu \left[\left(\frac{\partial^2 Q_{i,i}}{\partial x_j \partial x_j} \right)_A + \left(\frac{\partial^2 Q_{i,i}}{\partial x_j \partial x_j} \right)_B \right] = 0. \end{aligned} \quad (A.5)$$

In order to differentiate between the effect of the distance between the points A and B and the effect of their location in the flow field, the relations (A.1) are introduced. The differential operators then read:

$$\left(\frac{\partial}{\partial x_j} \right)_A = \frac{\partial}{\partial x_j} - \frac{\partial}{\partial r_j} ,$$

(A.6)

$$\left(\frac{\partial}{\partial x_j} \right)_B = \frac{\partial}{\partial r_j} .$$

With (A.6)*, equat.(A.5) transforms to

$$\begin{aligned} & \underbrace{\frac{\partial Q_{i,i}}{\partial t}}_{\text{rate of change with time}} + \underbrace{U_{j,A} \frac{\partial Q_{i,i}}{\partial x_j}}_{\text{convection}} + \underbrace{\left(\frac{\partial U_i}{\partial x_j} \right)_A (Q_{j,i} + Q_{i,j})}_{\text{production}} \\ & + \frac{\partial Q_{i,i}}{\partial r_j} (U_{j,B} - U_{j,A}) + Q_{i,j} \left[\left(\frac{\partial U_i}{\partial x_j} \right)_B - \left(\frac{\partial U_i}{\partial x_j} \right)_A \right] \\ & + \underbrace{\frac{\partial}{\partial r_j} (S_{i,ij} - S_{ji,i}) + \frac{\partial}{\partial r_i} (K_{i,p} - K_{p,i})}_{\text{transfer of energy between eddies of different sizes}} \\ & + \underbrace{\frac{\partial}{\partial x_j} S_{ji,i} + \frac{\partial K_{p,i}}{\partial x_i}}_{\text{diffusion}} - \underbrace{\nu \frac{\partial^2 Q_{i,i}}{\partial x_j^2} - 2\nu \left[\frac{\partial^2 Q_{i,i}}{\partial r_j \partial r_j} - \frac{\partial^2 Q_{i,i}}{\partial x_j \partial r_j} \right]}_{\text{dissipation}} = 0 \end{aligned} \quad (\text{A.7})$$

* (A.6) cannot be applied to the mean velocity, U_i , as was done by Wolfshtein (1970).

When the points A and B are brought together, the transfer term is zero and (A.7) reduces to the kinetic energy equation (2.8) multiplied by 2. To obtain an equation for kL , we divide (A.7) by r^2 and subsequently integrate it over the whole domain of interest, V . Introduction of (A.4) then yields:

$$\begin{aligned}
 & \underbrace{\frac{\partial kL}{\partial t}}_{\text{rate of change with time}} + \underbrace{U_j \frac{\partial kL}{\partial x_j}}_{\text{convection}} + \underbrace{\frac{\partial U_i}{\partial x_j} \frac{1}{2} \int_V (Q_{j,i} + Q_{i,j}) \frac{dV}{r^2}}_{\text{production}} \\
 & + \underbrace{\frac{1}{2} \int_V \frac{\partial Q_{i,i}}{\partial r_j} (U_{j,B} - U_{j,A}) \frac{dV}{r^2}}_{\text{I}} + \underbrace{\frac{1}{2} \int_V Q_{i,j} \left[\left(\frac{\partial U_i}{\partial x_j} \right)_B - \left(\frac{\partial U_i}{\partial x_j} \right)_A \right] \frac{dV}{r^2}}_{\text{II}} \\
 & + \underbrace{\frac{1}{2} \int_V \frac{\partial}{\partial r_j} (S_{i,ij} - S_{ji,i}) \frac{dV}{r^2}}_{\text{III}} + \underbrace{\frac{1}{2} \int_V \frac{\partial}{\partial r_i} (K_{i,p} - K_{p,i}) \frac{dV}{r^2}}_{\text{IV}} \\
 & T_L \equiv \text{term originating from inter - eddy energy transfer} \\
 & + \underbrace{\frac{1}{2} \frac{\partial}{\partial x_j} \int_V S_{ji,i} \frac{dV}{r^2} + \frac{1}{2} \frac{\partial}{\partial x_i} \int_V K_{p,i} \frac{dV}{r^2} - v \frac{\partial^2 kL}{\partial x_j^2}}_{\text{Diffusion}} \\
 & - v \underbrace{\left[\int_V \frac{\partial^2 Q_{i,i}}{\partial r_j \partial r_j} \frac{dV}{r^2} - \frac{\partial}{\partial x_j} \int_V \frac{\partial Q_{i,i}}{\partial r_j} \frac{dV}{r^2} \right]}_{\text{Dissipation}} = 0 \quad (A.8)
 \end{aligned}$$

This is the exact equation for kL .

A.4 Simplifying assumptions

The following two key assumptions are made:

- (i) All macro scales^{**} are proportional to one another
(ii) The two point double velocity correlations $Q_{i,j}$ are symmetrical:

$$Q_{i,j}(r_1, r_2, r_3) = Q_{i,j}(-r_1, -r_2, -r_3) \quad (A.9)$$

(A.9) holds true exactly in homogeneous turbulence. Experiments show that (A.9) is a good approximation also in most non-homogeneous flows. Only when the turbulence is strongly non-homogeneous, for example very close to walls, (A.9) appears to be inapplicable.

a) Production term.

In analogy to (A.4), a "production" length scale L_{ij} can be defined by

$$2 \overline{u_i u_j} L_{ij}^* = \int_V (Q_{j,i} + Q_{i,j}) \frac{dV}{r^2} \quad (A.10)$$

L_{ij} is a macro scale; so it can be related to L by

$$L_{ij} = c_B L, \quad (A.11)$$

where c_B is a constant which may be different for different indices i and j .

b) Dissipation term.

The second dissipation term is negligible because (A.9) implies that $\int_V \frac{\partial Q_{i,i}}{\partial r_j} \frac{dV}{r} = 0$. At high Reynolds numbers, the factor v reduces the importance of this term even further. The first dissipation term is also negligible at high Reynolds numbers because then $v \frac{\partial^2 Q_{i,i}}{\partial r_j \partial r_j}$ is significant only at very small values of r . The neglect of this term will be justified further in §A.5.

c) Term T_L

The term T_L originating from the inter-eddy energy transfer

* Here the summation convention does not apply.

** For definition of "macro scales" see footnote on p. 41.

consists of four parts (I to IV) which are defined in equat. (A.8).

Term I. Expansion of $(U_{j,B} - U_{j,A})$ by Taylor series and retention of first order terms only yields:

$$\frac{1}{2} \int_V \frac{\partial Q_{i,i}}{\partial r_j} (U_{j,B} - U_{j,A}) \frac{dV}{r^2} \approx \frac{1}{2} \frac{\partial U_j}{\partial x_k} \int_V \frac{\partial Q_{i,i}}{\partial r_j} r_k \frac{dV}{r^2}.$$

In boundary layers, the term involving the dominant velocity gradient $\partial U_1 / \partial x_2$, reads

$$\frac{1}{2} \frac{\partial U_1}{\partial x_2} \int \frac{\partial Q_{i,i}}{\partial r_1} r_2 \frac{dV}{r^2}.$$

With the symmetry assumption (A.9), the integral is zero; so term I can be neglected, at least in boundary layers.

Term II Again, by Taylor series expansion, term II can be expressed as

$$\begin{aligned} & \frac{1}{2} \int_V Q_{i,j} \left[\left(\frac{\partial U_i}{\partial x_j} \right)_B - \left(\frac{\partial U_i}{\partial x_j} \right)_A \right] \frac{dV}{r^2} \\ &= \frac{1}{2} \sum_{s=1}^{\infty} \frac{1}{s!} \frac{\partial^{s+1} U_i}{\partial x_j \partial x_{k_1} \partial x_{k_2} \dots \partial x_{k_s}} \int_V Q_{i,j} r_{k_1} r_{k_2} \dots r_{k_s} \frac{dV}{r^2}. \end{aligned}$$

When the symmetry assumption (A.9) holds, only even values of s contribute to the series. For boundary layers, the first term in the series therefore reads:

$$\text{Term II} \approx \frac{1}{4} \frac{\partial^3 U_1}{\partial x_2^3} \int_V Q_{1,2} r_2^2 \frac{dV}{r^2}. \quad (\text{A.12})$$

Rotta introduced an additional length scale to express the integral in (A.12) in terms of $\overline{u_1 u_2}$ and obtained

$$\text{Term II} \approx c_3 L^3 \frac{\partial^3 u_1}{\partial x^3}, \quad (\text{A.13})$$

where c_3 is a constant.

He states that the retention of this term is necessary to make the kL - equation, with fixed constants, applicable to flows both near to and remote from walls. The free flows considered here exhibit only mild inhomogeneity of the mean flow; so term II may be considered negligible compared with the production term. Ng (1971) did not employ (A.13) directly but he had to incorporate an additional term near walls which is close in form to (A.13) in the constant shear region*. This indicates that term II seems to be important near walls.

Term IV. Hinze (1959) showed that, for homogeneous turbulence, there follows from the continuity equation:

$$\frac{\partial}{\partial x_i} (K_{i,p} - K_{p,i}) = 0.$$

Hence term IV will be neglected.

Term III is the dominant part of T_L . While the other parts are exactly zero in homogeneous turbulence, term III balances the rate of change term when the Reynolds number is high. This will be shown in §A.5 Term III is a destruction term whose magnitude is closely related to the rate of dissipation, ϵ . The introduction of a length scale L_D representative for this destruction mechanism then leads to:

$$\frac{1}{2} \int_V \frac{\partial}{\partial x_j} (S_{i,ij} - S_{ji,i}) \frac{dV}{r^2} = -L_D \epsilon = -L_D c_D \frac{k^{3/2}}{L} = -c_S k^{3/2} \quad (\text{A.14})$$

* Ng's additional term reads: $\text{const} \left(\frac{L}{y}\right)^6 k^{3/2}$. In the constant shear layer near walls (A.13) becomes: $\text{const} \left(\frac{L}{y}\right)^2 k^{3/2}$.

† Wolfshtein (1970) wrongly neglected term III but put the dissipation term equal to the R.H.S. of (A.14).

c_S is assumed to be a constant at high Reynolds numbers. This, again, implies proportionality of all macro scales.

d) Diffusion term.

At high Reynolds numbers, the viscous diffusion, $\nu \frac{\partial^2 k_L}{\partial x_j^2}$, can be neglected. The following gradient - type approximation suggested by Rotta (1951, 1971) is used for the turbulent diffusion:

$$\frac{1}{2} \frac{\partial}{\partial x_j} \int_V S_{ji,i} \frac{dV}{r^2} + \frac{1}{2} \frac{\partial}{\partial x_j} \int_V K_{p,i} \frac{dV}{r^2} = \frac{\partial}{\partial x_j} \left[\frac{\sqrt{k} L}{\sigma_{kL}} \left(\frac{\partial k_L}{\partial x_j} + c_{kL} L \frac{\partial k}{\partial x_j} \right) \right],$$

where σ_{kL} and c_{kL} are constants.

e) The modelled equation.

Incorporation into (A.8) of all the above assumptions yields the final equation for k_L :

$$\underbrace{\frac{\partial k_L}{\partial t}}_{\text{rate of change with time}} + \underbrace{U_j \frac{\partial k_L}{\partial x_j}}_{\text{convection}} = \underbrace{\frac{\partial}{\partial x_j} \left[\frac{\sqrt{k} L}{\sigma_{kL}} \left(\frac{\partial k_L}{\partial x_j} + c_{kL} L \frac{\partial k}{\partial x_j} \right) \right]}_{\text{diffusion}} - \underbrace{c_B \overline{u_i u_j} L \frac{\partial U_i}{\partial x_j}}_{\text{production}} - \underbrace{c_S k^{3/2}}_{\text{destruction}}.$$

(A.15)

A.5 The spectral k_L - equation

When the spectral energy equation is divided by the wave number n , the spectral k_L - equation results, which upon integration from $n = 0$ to $n = \infty$ yields also the k_L - equation. For homogeneous isotropic turbulence, the spectral k_L - equation reads

$$\frac{1}{n} \frac{dE}{dt} = \frac{T}{n} - 2\nu n E.$$

rate of change term originating from inter-eddy energy transfer dissipation

(A.16)

E is the three-dimensional energy spectrum defined by (A.2), and T is the transfer spectrum which satisfies $\int_0^\infty Tdn = 0$. The spectral kL - balance (A.16) as determined from Uberoi's (1963) measurements in grid - generated turbulence is shown in fig. A.2. It is evident from this figure that the dissipation term is negligible compared with the other terms. This is a consequence of the fact that, at high Reynolds numbers, the energy is contained in eddies large relative to the eddies responsible for the dissipation. Thus, dividing the spectral energy balance by n weights the terms which are concentrated at low wave numbers heavily against the dissipation. When the Reynolds number is increased further, this effect becomes even more pronounced. Hence, term III in equation (A.8), which is equal to $\int_0^\infty \frac{T}{n} dn$, is an important one at high Reynolds numbers, while the dissipation term is indeed negligible.

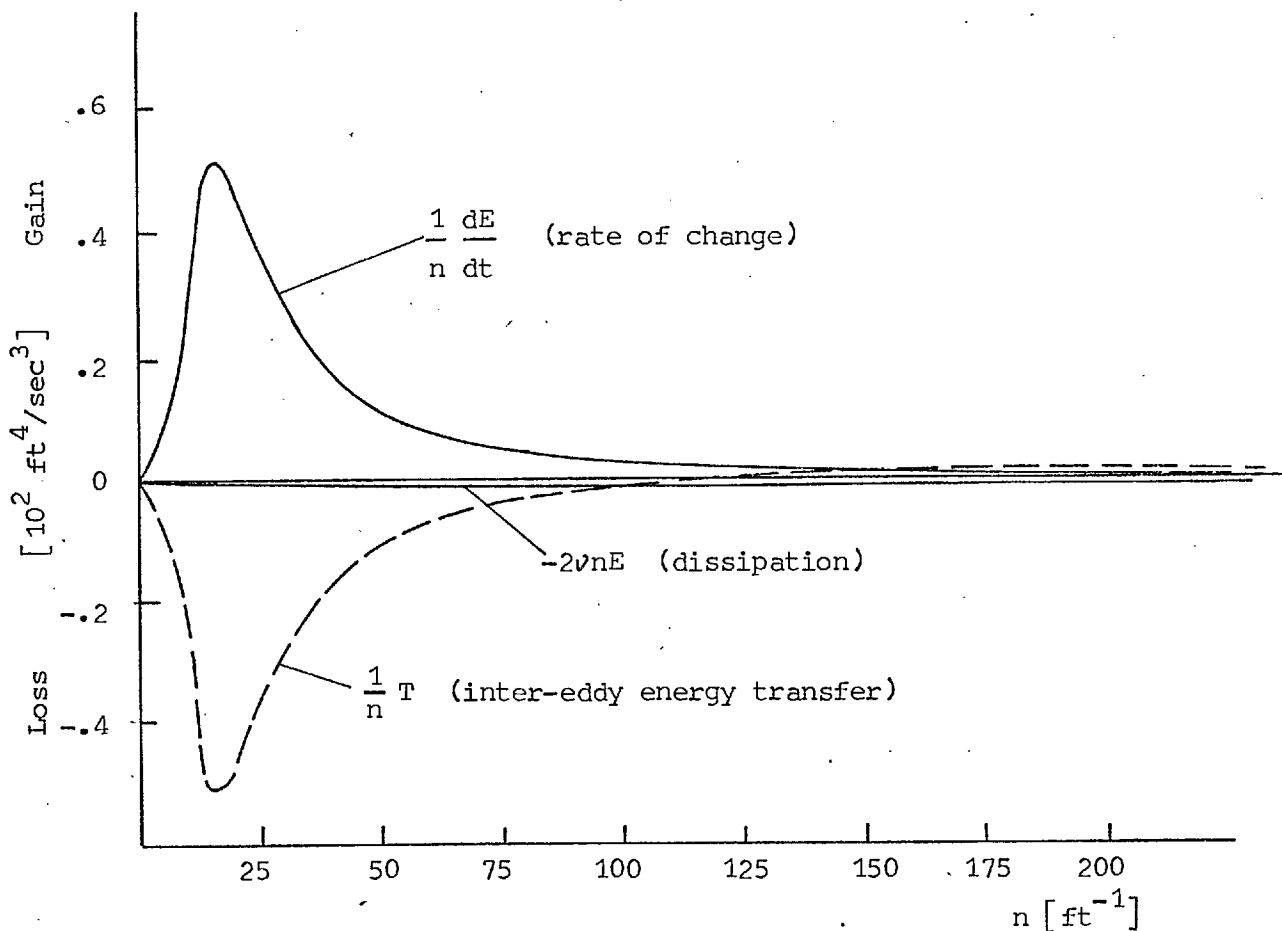


Fig. A.2 Spectral distribution of the terms in the kL -equation (from Uberoi's 1963 measurements)

Appendix B

Conditions for the existence of self-similarity

Self-similarity, as defined in §4.4.1, can exist only for those flows for which the momentum and energy equations can be written in self-similar form. To find the conditions under which this is possible, the relations (4.4) are introduced into the momentum equation (2.13) and into the turbulence energy equation (2.14)*. The analysis is valid for general free boundary layers with the exception of the radial jet, which is considered separately in appendix D.

Lateral Velocity. First, (4.4) is inserted into the continuity equation (2.12). Subsequent integration yields the dimensionless lateral velocity g :

$$g = g_0 - \left[\frac{\delta}{U_0} \frac{dU_0}{dx} + (j+1) \frac{d\delta}{dx} \frac{1}{\eta^j} \right] \int_0^\eta f \eta^j d\eta, \quad (B.1)$$

where g_0 is the value of g at $\eta = 0$. For the mixing layer, g_0 is finite; but when $\eta = 0$ is a symmetry line or plane, g_0 is zero.

Momentum equation. With (B.1) and (4.4), the momentum equation (2.13) becomes:

$$\begin{aligned} & \left\{ \frac{\delta}{U_0} \frac{dU_E}{dx} + \frac{U_E}{U_0} \frac{\delta}{U_0} \frac{dU_0}{dx} \right\} [f] + \left\{ \frac{\delta}{U_0} \frac{dU_0}{dx} \right\} [f^2 + 2q] \\ & - \left\{ \frac{U_E}{U_0} \frac{d\delta}{dx} + \frac{1}{j+1} \frac{\delta}{U_0} \frac{dU_E}{dx} \right\} \left[\frac{df}{d\eta} \right] - \left\{ \frac{\delta}{U_0} \frac{dU_0}{dx} + (j+1) \frac{d\delta}{dx} \right\} \left[\frac{df/d\eta}{\eta^j} \int_0^\eta f \eta^j d\eta \right] \\ & + \left[\frac{df}{d\eta} g_0 \right] - \left\{ \frac{d\delta}{dx} \right\} \left[\frac{dq}{d\eta} \right] = - \left[\frac{1}{\eta^j} \frac{d}{d\eta} (\eta^j h) \right] \end{aligned} \quad (B.2)$$

* Both equation (2.13) and (2.14) involve the assumptions that the boundary layer approximations are valid and that the viscous diffusion terms are negligible (high Reynolds number). Also, the diffusion term in (2.14) will be assumed to have the modelled form shown in (3.14).

For symmetrical flows, this equation was given by Newman (1967). The terms in the square brackets are functions of η only, and the terms in the curly brackets are functions of x only. There is no curly bracket in the term on the right hand side. Thus, self-similarity is possible only when all the curly-bracket terms are constant. This is the case when

$$\frac{d\delta}{dx} = \text{const}, \quad (\text{B.3})$$

$$\frac{U_E}{U_0} = \text{const}, \quad (\text{B.4})$$

$$\frac{\delta}{U_0} \frac{dU_0}{dx} = \text{const}. \quad (\text{B.5})$$

Turbulence energy equation. Insertion of (B.1) and (4.4) into the energy equation (2.14) yields:

$$\begin{aligned} & 2 \left\{ \frac{U_E}{U_0^2} \frac{dU_0}{dx} \right\} [e] + 2 \left\{ \frac{\delta}{U_0} \frac{dU_0}{dx} \right\} [ef] - \left\{ \frac{U_E}{U_0} \frac{d\delta}{dx} + \frac{1}{j+1} \frac{\delta}{U_0} \frac{dU_E}{dx} \right\} \left[\frac{de}{d\eta} \right] \\ & - \left\{ \frac{\delta}{U_0} \frac{dU_0}{dx} + (j+1) \frac{d\delta}{dx} \right\} \left[\frac{1}{\eta^j} \frac{de}{d\eta} \int_0^\eta f \eta^j d\eta \right] + \left[\frac{de}{d\eta} g_0 \right] \\ & = \left[\frac{1}{\eta^j} \frac{d}{d\eta} \left(\tilde{\eta}^j \frac{\tilde{v}_t}{\sigma_k} \frac{de}{d\eta} \right) \right] - \left[h \frac{df}{d\eta} \right] - \left\{ \frac{\delta}{U_0} \frac{dU_E}{dx} \right\} [q] \\ & - \left\{ \frac{\delta}{U_0} \frac{dU_0}{dx} \right\} [qf] + \left\{ \frac{d\delta}{dx} \right\} \left[\eta q \frac{df}{d\eta} \right] - \left[\tilde{\epsilon} \right]. \end{aligned} \quad (\text{B.6})$$

Again, self-similarity can exist only when all the curly-bracket terms are independent of x . No conditions other than (B.3) to (B.5) are necessary to procure this.

Approximate self-similarity when $|U_E/U_O| \gg 1$. When $|U_E/U_O| \gg 1$,

all terms in the curly brackets of (B.2) and (B.6) which do not contain U_E can be neglected. The momentum equation (B.2) then reduces to

$$\left\{ \frac{\delta}{U_O^2} \frac{d}{dx} (U_E U_O) \right\} \left[f \right] - \left\{ \frac{1}{j+1} \frac{1}{U_O \delta^j} \frac{d}{dx} (U_E \delta^{j+1}) \right\} \left[\frac{df}{d\eta} \right] = - \left[\frac{1}{\eta^j} \frac{d}{d\eta} (\eta^j h) \right]. \quad (B.7)$$

The kinetic energy equation (B.6) simplifies to

$$\begin{aligned} & 2 \left\{ \frac{U_E}{U_O^2} \delta \frac{dU_O}{dx} \right\} [e] - \left\{ \frac{1}{j+1} \frac{1}{U_O \delta^j} \frac{d}{dx} (U_E \delta^{j+1}) \right\} \left[\frac{de}{d\eta} \right] \\ & = \left[\frac{1}{\eta^j} \frac{d}{d\eta} \left(\eta^j \frac{\tilde{v}_t}{\sigma_k} \frac{de}{d\eta} \right) \right] - \left[h \frac{df}{d\eta} \right] - \left\{ \frac{\delta}{U_O} \frac{dU_E}{dx} \right\} [q] - \left[\tilde{\varepsilon} \right]. \end{aligned} \quad (B.8)$$

The curly brackets in (B.7) and (B.8) are constant when

$$\frac{U_E}{U_O} \frac{d\delta}{dx} = \text{const}, \quad (B.9)$$

$$\frac{U_E \delta}{U_O^2} \frac{dU_O}{dx} = \text{const}, \quad (B.10)$$

$$\frac{\delta}{U_O} \frac{dU_E}{dx} = \text{const}. \quad (B.11)$$

These are the conditions for approximate self-similarity for flows with

$$\left| \frac{U_E}{U_O} \right| \gg 1.$$

Equations (B.7) to (B.11) were first given by Gartshore and Newman (1969).

Appendix C

The variation of U_o and δ in self-similar flows

Exactly self-similar flows. δ grows linearly and U_E/U_o is constant (equations B.3 and B.4). With (4.4) and $U_E/U_o = \text{const}$, the integral momentum equation (4.1) yields

$$\frac{\delta}{U_o} \frac{dU_o}{dx} = - \frac{j+1}{2} I \frac{d\delta}{dx}, \quad (C.1)$$

where

$$I = \frac{U_E/U_o + I_R}{3/2 U_E/U_o + I_R} \quad (C.2)$$

and

$$I_R = \frac{\int_0^{\eta_E} (f^2 + q) \eta^j d\eta}{\int_0^{\eta_E} f \eta^j d\eta}. \quad (C.3)$$

Since $d\delta/dx = \text{const}$, the solution of (C.1) is

$$U_o \propto (x + x_o)^m,$$

where

$$m = - \frac{j+1}{2} I, \quad (C.4)$$

and x_o is the hypothetical origin of the flow.

Approximately self-similar flows at $|U_E/U_o| \gg 1$. Introduction of (4.4) into the integral momentum equation, and retention of terms with U_E only yields:

$$\left[\frac{d}{dx} (U_o U_E \delta^{j+1}) + U_o \delta^{j+1} \frac{dU_E}{dx} \right] \int_0^{\eta_E} f \eta^j d\eta = 0 \quad (C.5)$$

For shear flows, the square bracket must be zero. We therefore get

$$2 \frac{\delta}{U_o} \frac{dU_E}{dx} + \frac{\delta U_E}{U_o^2} \frac{dU_o}{dx} + (j+1) \frac{U_E}{U_o} \frac{d\delta}{dx} = 0 \quad (C.6)$$

Combination of (B.9) to (B.11) and (C.6) yields

$$U_E \propto (x + x_o)^m, \quad (C.7)$$

$$U_o \propto (x + x_o)^{\frac{m(j-1) - (j+1)}{j+2}} \quad (C.8)$$

$$\delta \propto (x + x_o)^{\frac{1 - 3m}{j+2}}, \quad (C.9)$$

where m may be prescribed. These relations were given by Gartshore and Newman (1969).

For shear-free layers, the momentum equation provides no information. The integral turbulence energy equation (4.58) reveals that the variation of k_o and δ depends on the distribution of e , q , and ϵ .

Appendix D

The self-similar plane mixing layer

Fig. D.1 sketches the flow situation and explains the symbols employed. The inner boundary I and the outer boundary E are drawn where

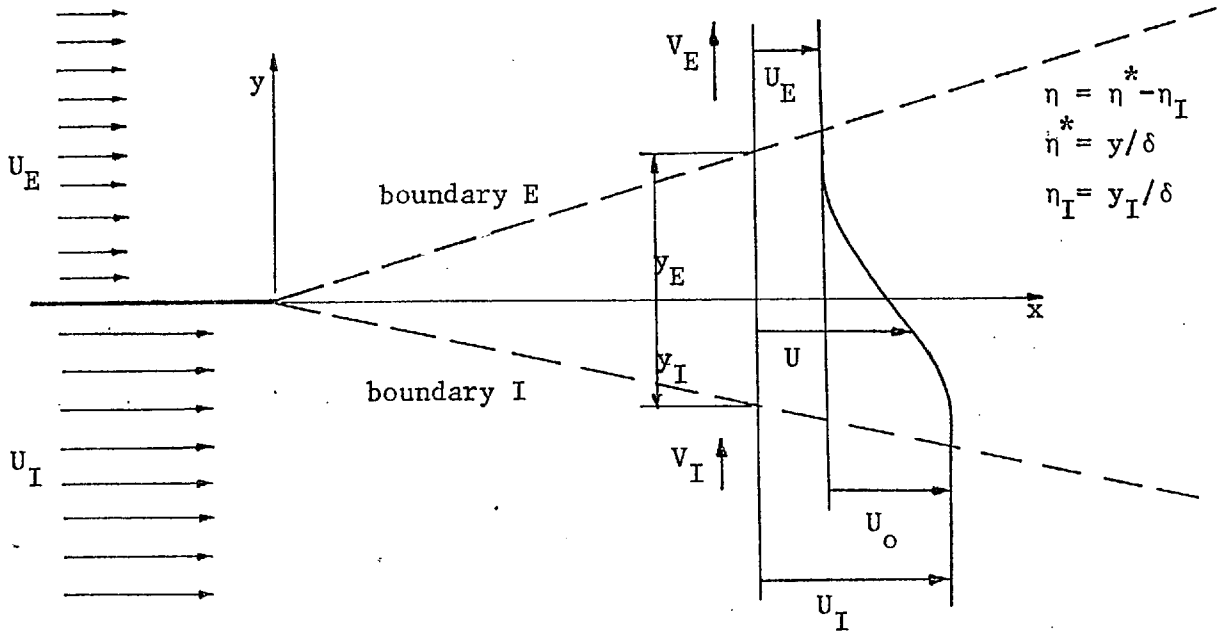


Fig. D.1 The plane mixing layer

the velocity U starts to deviate significantly from U_I and U_E respectively. The choice of what is a "significant" deviation is arbitrary, but the equations to follow remain valid when the boundaries are moved further out into the free streams.

The third boundary condition. In contrast to the other self-similar flows shown in fig. 4.2, the mixing layer has no symmetry plane or line where the lateral velocity is known to be zero. This causes mathematical problems because there is no third boundary condition apparent for the velocities. The two obvious boundary conditions are: $U = U_I$ for $y < y_I$ and $U = U_E$ for $y > y_E$. As will be shown in the next

section, there exists an infinite number of solutions of the momentum equation in the $U - \eta^*$ plane when no third boundary condition is employed. The difference between the solutions is equivalent to a shift of the velocity profile as a whole in the η^* direction. The third boundary condition is therefore necessary to determine the actual location of the mixing layer in space. Most workers in the field have avoided this question by fixing $\eta^* = 0$ at the position where $U = \frac{1}{2}(U_E + U_I)$. The correct third boundary condition was derived by Ting (1959), and independently by Yen (1960), from the balance of the momentum in the y - direction. It reads

$$U_E V_E + U_I V_I = 0 \quad . \quad (D.1)$$

Strictly speaking, this boundary condition is valid only when the two streams extend to infinity. The presence of walls or symmetry lines in the free streams has an influence on the third boundary condition and thus on the flow.

The momentum equation. Equat. (4.4) is used to transform the momentum equation (2.13) and the continuity equation (2.12) into similarity form. For convenience, the dimensionless coordinate η is now defined as (see fig. D.1)

$$\eta = \eta^* - \eta_I \quad , \quad (D.2)$$

where $\eta^* = y/\delta$ and $\eta_I = y_I/\delta$.

With (D.2.) and (4.4), the continuity equation (2.12) yields, after integration from $\eta = 0$ to η :

$$g = g_I + \frac{d\delta}{dx} \left\{ \eta_I (f - 1) + \eta f - \int_0^\eta f d\eta \right\}, \quad (D.3)$$

where g_I is the value of g at the inner boundary ($\eta = 0$).

With (D.3) and (4.4), the momentum equation transforms to:

$$\frac{dh}{d\eta} = \left\{ \frac{d\delta}{dx} \left[\frac{U_E}{U_o} (\eta + \eta_I) + \eta_I + \int_0^\eta f d\eta \right] - g_I \right\} \frac{df}{d\eta} + \frac{d\delta}{dx} (\eta + \eta_I) \frac{dq}{d\eta} . \quad (D.4)$$

The shear stress, h , is zero at both the inner and the outer boundary.

Therefore, integration of (D.4) from $\eta = 0$ to $\eta = \eta_E$ gives the following relation for g_I :

$$g_I = \frac{d\delta}{dx} \left\{ \eta_I \left(\frac{U_E}{U_o} + 1 \right) + \frac{U_E}{U_o} \int_0^{\eta_E} f d\eta + \int_0^{\eta_E} f^2 d\eta + \int_0^{\eta_E} q d\eta \right\} . \quad (D.5)$$

With g_I from (D.5), the momentum equation (D.4) becomes:

$$\frac{dh}{d\eta} = \frac{d\delta}{dx} \left\{ \frac{U_E}{U_o} \eta - \frac{U_E}{U_o} \int_0^{\eta_E} f d\eta - \int_0^{\eta_E} f^2 d\eta - \int_0^{\eta_E} q d\eta + \int_0^\eta f d\eta + (\eta + \eta_I) \frac{dq}{d\eta} \right\} . \quad (D.6)$$

This is the equation presented in table 4.2. When the normal stress terms are neglected, the solutions $f(\eta)$ and $h(\eta)$ which satisfy the equation are independent of η_I . Thus, the location of the profiles of f and h is not fixed in space, i.e. there is an infinite number of solutions when expressed as functions of the independent variable $\eta^* = y/\delta = \eta + \eta_I$. To find the location of the profiles amounts to determining η_I . This can only be done by means of the third boundary condition. With $\lambda = U_E/U_I$ and equats. (D.1), (D.3), and (D.5), there results:

$$\eta_I = - \frac{1 - \lambda^2}{1 + \lambda^2} \left(\frac{2\lambda^2}{1 - \lambda^2} \int_0^{\eta_E} f d\eta + \int_0^{\eta_E} f^2 d\eta + \int_0^{\eta_E} q d\eta \right) . \quad (D.7)$$

Turbulence energy equation. When equats. (4.4), (D.2), (D.3), and (D.5) are substituted in (2.14), the turbulence energy equation results as presented in table 4.5.

Appendix E

The self-similar radial jet

The radial jet pictured in fig. E.1 is an axisymmetric flow which is also symmetrical about the plane $y = 0$. In contrast to the other axisymmetric flows shown in fig. 4.2, the radial jet does not have the axis of rotational symmetry within the flow region of interest.

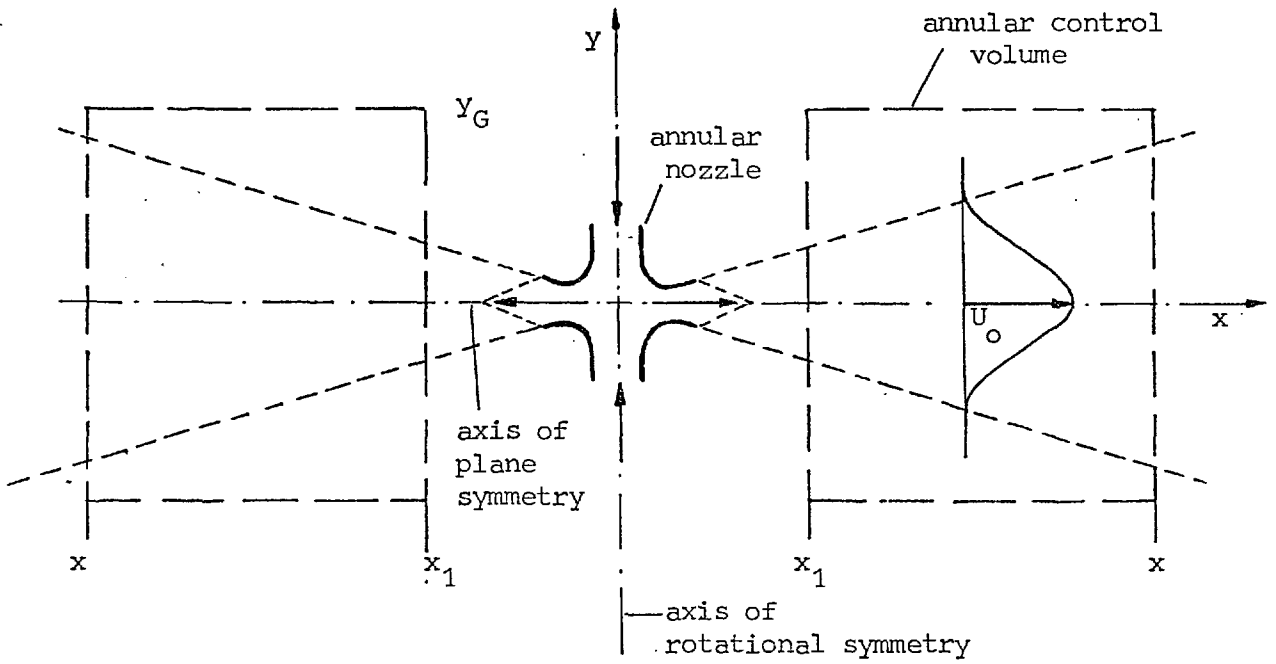


Fig. E.1 The radial jet

The basic equations for axisymmetric flows are taken from Rodi(1970):

Momentum equation. (boundary layer form)

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = - \frac{\partial \overline{uv}}{\partial y} - \frac{1}{x} \frac{\partial}{\partial x} \left[x(\overline{u^2} - \overline{v^2}) \right] - \frac{\overline{v^2} - \overline{w^2}}{x} \quad (E.1)$$

Continuity equation.

$$\frac{\partial V}{\partial y} + \frac{\partial U}{\partial x} + \frac{U}{x} = 0 \quad (E.2)$$

Turbulence energy equation.

$$U \frac{\partial k}{\partial x} + v \frac{\partial k}{\partial y} = \frac{\partial}{\partial y} \left(\frac{v_t}{\sigma_k} \frac{\partial k}{\partial y} \right) - \overline{uv} \frac{\partial U}{\partial y} - (\overline{u^2} - \overline{v^2}) \frac{\partial U}{\partial x} + (\overline{v^2} - \overline{w^2}) \frac{U}{x} - \varepsilon. \quad (\text{E.3})$$

The normal stress terms with $(\overline{v^2} - \overline{w^2})$ will be neglected because $(\overline{v^2} - \overline{w^2})$ is, in general, considerably smaller than $(\overline{u^2} - \overline{v^2})$.

Momentum balance. When V is eliminated from (E.1) by means of (E.2), and when (E.1) is then integrated from $y = 0$ to $y = y_E$ and from $x = x_1$ to x , the following momentum balance is obtained:

$$M = x \int_0^{y_E} (U^2 + \overline{u^2} - \overline{v^2}) dy \Big|_x = x_1 \int_0^{y_E} (U^2 + \overline{u^2} - \overline{v^2}) dy \Big|_{x_1} = \text{const.} \quad (\text{E.4}).$$

Similarity equations. Substitution of the similarity relations (4.4) in (E.2) yields

$$g = - \left(\frac{\delta}{U_0} \frac{dU_0}{dx} + \frac{\delta}{x} + \frac{d\delta}{dx} \right) \int_0^\eta f d\eta + \frac{d\delta}{dx} \eta f. \quad (\text{E.5})$$

With (4.4) and (E.5), the momentum equation (E.1) becomes:

$$\begin{aligned} \left\{ \frac{\delta}{U_0} \frac{dU_0}{dx} \right\} [f^2] - \left\{ \frac{\delta}{U_0} \frac{dU_0}{dx} + \frac{\delta}{x} + \frac{d\delta}{dx} \right\} \left[\frac{df}{d\eta} \int_0^\eta f d\eta \right] + \left\{ \frac{\delta}{2U_0} \frac{dU_0}{dx} + \frac{\delta}{x} \right\} [q] \\ - \left\{ \frac{d\delta}{dx} \right\} \left[\frac{dq}{d\eta} \right] = - \left[\frac{dh}{d\eta} \right]. \end{aligned} \quad (\text{E.6})$$

Self-similarity is possible only when all the curly-bracket terms are constant. This is the case when

$$\frac{\delta}{U_0} \frac{dU_0}{dx}, \quad \frac{d\delta}{dx}, \quad \text{and} \quad \frac{\delta}{x} \quad \text{are constant.}$$

Therefore $\frac{d\delta}{dx} = \frac{\delta}{x} = \text{const.} \quad (\text{E.7})$

Integration of (E.6) from $\eta = 0$ to $\eta = \eta_E$ yields the integral momentum equation

$$\left(2 \frac{\delta}{U_0} \frac{dU_0}{dx} + \frac{\delta}{x} + \frac{d\delta}{dx} \right) \int_0^{\eta_E} (f^2 + q) d\eta = 0 \quad . \quad (E.8)$$

The integral in (E.8) is not zero. Therefore we get, with (E.7):

$$\frac{\delta}{U_0} \frac{dU_0}{dx} = - \frac{d\delta}{dx} \quad . \quad (E.9)$$

From (E.2) and (E.9) we learn that δ increases linearly with x and U_0 varies as $U_0 \propto x^{-1}$.

Insertion of (E.7) and (E.9) in (E.6) gives the momentum equation presented in table 4.2. The turbulence energy equation given in table 4.5 is obtained by substituting (4.4), (E.5), (E.7) and (E.9) in (E.3).

Appendix FDetermination of the initial value of L
in free stream turbulence

The turbulence behaves like turbulence behind a grid:

$$\frac{k}{U^2} = B \left(\frac{x}{M} - \frac{x_0}{M} \right)^{-n}, \quad (F.1)$$

$$\frac{L}{M} = c \left(\frac{x}{M} - \frac{x_0}{M} \right)^{1-\frac{n}{2}}. \quad (F.2)$$

B, c, x_0 , and n are constants which depend somewhat on the grid geometry. x is the distance from the grid, and M is the grid mesh size.

The problem here is to determine $\frac{L}{D}$ when k/U^2 and M/D are given. Insertion of (F.1) and (F.2) into the k - equation

$$U \frac{dk}{dx} = - c \frac{k^{3/2}}{D_L}$$

yields

$$c = \frac{c_D \sqrt{B}}{n}. \quad (F.3)$$

(F.3) into (F.2), and elimination of $\left(\frac{x}{M} - \frac{x_0}{M} \right)$ with the aid of (F.1), yields

$$\frac{L}{D} = \frac{M}{D} \frac{c_D \sqrt{B}}{n} \left(\frac{1}{B} \frac{k}{U^2} \right)^{\frac{n-2}{2n}} \quad (F.4)$$

Average values for B and n are given by Comte-Bellot and Corrsin(1966):

$$B \approx .08$$

$$n \approx 1.25$$

APPENDIX G

A GENERAL STRESS-EQUATION MODEL ($\overline{u_i u_j}, \epsilon$)

G.1 Introduction

The present appendix presents and tests a general "stress-equation" model. The label "stress-equation" implies that the model uses transport equations for all Reynolds stresses. The model described below employs further an equation for the rate of energy dissipation. It is restricted to flows with high Reynolds number.

Stress-equation models were studied as part of this work because two-equation models employing fixed constants (e.g. model II) were found to lack universality (see § 6.6). In particular, model II needed different constants to predict flows with strong and weak strain, and again different ones for round jets. Thus, the main objective of the study reported here was to examine whether the more complex stress-equation models can predict all these flows with a single set of constants.

At the time this study was started, stress-equation models had been proposed by Rotta (1951), Donaldson (1968), and Daly and Harlow (1970). The former two proposals appeared not to be general enough;^{*} and the performance of Daly and Harlow's model was found to be unsatisfactory (see appendix J). A new stress-equation model was therefore constructed from components proposed by various authors.

In § G.2, the exact Reynolds-stress equations of § 2.4 are modelled to yield a closed system. The final form of the present stress-equation model is presented in § G.3. It is then applied to one-dimensional flows in § G.4, and to free boundary layers in § G.5.

G.2 Modelling of the Reynolds-stress equations

a) The exact equations

The exact equation for the Reynolds stress $-\overline{u_i u_j}$ was derived in § 2.4 (equation 2.7). For high Reynolds numbers, this equation

^{*} Rotta did not provide a general form for the second part of his pressure-strain simulation. Donaldson neglected this part altogether, and he did not employ a length scale equation.

reduces to

$$\begin{aligned}
 \underbrace{U_k \frac{\partial \overline{u_i u_j}}{\partial x_k}}_{\text{convection}} &= \underbrace{- \frac{\partial}{\partial x_k} (\overline{u_k u_i u_j}) - \frac{1}{\rho} \left(\overline{\frac{\partial u_j p}{\partial x_i}} + \overline{\frac{\partial u_i p}{\partial x_j}} \right)}_{\text{diffusion}} \\
 &\quad - \underbrace{\overline{u_i u_k} \frac{\partial U_j}{\partial x_k} - \overline{u_j u_k} \frac{\partial U_i}{\partial x_k}}_{P_{ij} \equiv \text{production}} + \underbrace{\frac{p}{\rho} \left(\overline{\frac{\partial u_i}{\partial x_j}} + \overline{\frac{\partial u_j}{\partial x_i}} \right)}_{\text{pressure-strain}} \\
 &\quad - \underbrace{2\nu \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_j}}{\partial x_k}}_{\text{dissipation}} . \tag{G.2-1}
 \end{aligned}$$

To obtain a closed system of equations, the following sections approximate the dissipation, diffusion, and pressure-strain expressions in terms of U_i , $\overline{u_i u_j}$, and the energy dissipation ϵ .

b) Dissipation

When the turbulent Reynolds number ($\equiv \sqrt{k}L/\nu$) is high, the small-scale structure of turbulence, which is responsible for the dissipation, may be considered isotropic (concept of local isotropy, Kolmogorov 1941). Rotta (1951) has shown that, for this condition, the dissipation of $\overline{u_i u_j}$ can be expressed as

$$2\nu \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_j}}{\partial x_k} = \frac{2}{3} \epsilon \delta_{ij} , \tag{G.2-2}$$

where

$$\epsilon = \nu \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_i}}{\partial x_k} .$$

The energy dissipation ϵ will be determined from a transport equation for this quantity (equation (G.3-2)).

c) Diffusion

The triple-correlation $\overline{u_k u_i u_j}$ represents a diffusion flux of $\overline{u_i u_j}$ which is assumed here to be proportional to the gradient of $\overline{u_i u_j}$. The simulation of Daly and Harlow (1970) is adopted which reads:

$$\overline{u_k u_i u_j} = - \beta \frac{k}{\epsilon} \overline{u_k u_\ell} \frac{\partial \overline{u_i u_j}}{\partial x_\ell}, \quad (\text{G.2-3})^*$$

where β is a diffusion constant.

Pressure diffusion. Bradshaw and Ferriss (1965), and Lawn (1971), have found for mixing layer and pipe flow respectively that the measured energy balance closes quite well when the pressure-diffusion term in the k - equation is neglected. For lack of further evidence, the pressure-diffusion term in (G.2-1), $\frac{1}{\rho} \left(\frac{\partial u_j p}{\partial x_i} + \frac{\partial u_i p}{\partial x_j} \right)$, is therefore neglected.

d) Pressure-strain

The pressure-strain term, $\overline{p/\rho (\partial u_i / \partial x_j + \partial u_j / \partial x_i)}$, is of particular importance because, in most boundary layers, it is roughly equal and opposite to the production term in the shear-stress equation; and in the normal-stress equations it redistributes the energy generated mainly in $\overline{u^2}$. Unfortunately, it is also the term which could not as yet be measured in laboratory flows.

Chou (1945) has shown that, away from the immediate vicinity of a wall, the correlation $\frac{p}{\rho} \frac{\partial u_i}{\partial x_j}$ at the point $\vec{x}$ can be expressed in terms of the velocity field by

* This relation is not tensor-invariant, which is no detriment for the flows considered here. Hanjalić and Launder (1972) simplified the exact transport equation for $\overline{u_k u_i u_j}$ to obtain the following, more complex, tensor-invariant relation:

$$\overline{u_k u_i u_j} = - \beta \frac{k}{\epsilon} \left[\overline{u_i u_\ell} \frac{\partial \overline{u_j u_k}}{\partial x_\ell} + \overline{u_j u_\ell} \frac{\partial \overline{u_k u_i}}{\partial x_\ell} + \overline{u_k u_\ell} \frac{\partial \overline{u_i u_j}}{\partial x_\ell} \right].$$

$$\begin{aligned}
 \overline{\rho \frac{\partial u_i}{\partial x_j}} &= \underbrace{\frac{1}{4\pi} \int_V \frac{\partial^2 u_\ell u_m}{\partial x_\ell \partial x_m} \langle \vec{x} + \vec{r} \rangle \frac{\partial u_i}{\partial x_j} \langle \vec{x} \rangle \frac{dV}{|\vec{r}|}}_{\Phi_{ijI}} \\
 &+ \underbrace{\frac{1}{2\pi} \int_V \frac{\partial u_\ell}{\partial x_m} \langle \vec{x} + \vec{r} \rangle \frac{\partial u_m}{\partial x_\ell} \langle \vec{x} + \vec{r} \rangle \frac{\partial u_i}{\partial x_j} \langle \vec{r} \rangle \frac{dV}{|\vec{r}|}}_{\Phi_{ijII}}. \quad (G.2-4)
 \end{aligned}$$

This equation reveals that the pressure-strain term consists of two parts: the first one is due to the interaction of the various fluctuating velocities, and the second one originates from the interaction of mean and fluctuating flow.

Φ_{ijI} . Rotta (1951) proposed to take the first part as proportional to the anisotropy of the turbulence. Accordingly he postulated:

$$(\Phi_{ij} + \Phi_{ji})_I = -\omega \frac{\varepsilon}{k} (\overline{u_i u_j} - \delta_{ij} \frac{2}{3} k), \quad (G.2-5)$$

where ω is a constant. This expression has been used in most of the subsequent stress-equation models proposed (sometimes to represent the total pressure-strain term); it will also be adopted here.

Φ_{ijII} . Naot et al (1970), and independently Reynolds (1970), proposed the following simulation for the second part of the pressure-strain term:

$$(\Phi_{ij} + \Phi_{ji})_{II} = -\alpha (P_{ij} - \frac{2}{3} \delta_{ij} P), \quad (G.2-6)$$

where P_{ij} is the production of $\overline{u_i u_j}$ as defined in equation (G.2-1), P is the production of k ($\equiv -\overline{u_k u_\ell} \partial U_k / \partial x_\ell$), and α is a constant. This simulation assumes that the second part of the pressure-strain term is proportional to the anisotropy of the production of turbulence. It is

therefore a consistent counterpart to (G.2-5) which assumes that the first part is proportional to the anisotropy of the stresses. The simulation (G.2-6) is adopted in the present model.

Alternative simulations for Φ_{ijII} have been proposed by Daly and Harlow (1970) and by Hanjalić and Launder (1972). When applied to one-dimensional flows, these simulations performed less favourably than (G.2-6).*

The complete pressure-strain model. Combination of (G.2-5) and (G.2-6) yields the following expression for the total pressure-strain term:

$$\frac{P}{\rho} \left(\frac{\partial \overline{u_i u_j}}{\partial x_j} + \frac{\partial \overline{u_j u_i}}{\partial x_i} \right) = - \omega \frac{\epsilon}{k} \left(\overline{u_i u_j} - \delta_{ij} \frac{2}{3} k \right) - \alpha \left(P_{ij} - \delta_{ij} \frac{2}{3} P \right). \quad (G.2-7)$$

G.3 The final form of the stress-equation model

Incorporation of (G.2-2), (G.2-3), and (G.2-7) into (G.2-1) yields the following modelled equation for the Reynolds stress $-\overline{u_i u_j}$:

$$\underbrace{U_k \frac{\partial \overline{u_i u_j}}{\partial x_k}}_{\text{convection}} = \underbrace{\beta \frac{\partial}{\partial x_k} \left(\frac{k}{\epsilon} \overline{u_k u_\ell} \frac{\partial \overline{u_i u_j}}{\partial x_\ell} \right)}_{\text{diffusion}} + \underbrace{P_{ij}}_{\text{production}} - \underbrace{\omega \frac{\epsilon}{k} \left(\overline{u_i u_j} - \delta_{ij} \frac{2}{3} k \right) - \alpha \left(P_{ij} - \delta_{ij} \frac{2}{3} P \right)}_{\text{pressure-strain}} - \underbrace{\frac{2}{3} \epsilon \delta_{ij}}_{\text{diss.}}, \quad (G.3-1)$$

where

$$P_{ij} = - \overline{u_i u_k} \frac{\partial U_j}{\partial x_k} - \overline{u_j u_k} \frac{\partial U_i}{\partial x_k},$$

and

$$P = \frac{1}{2} P_{ii} = - \overline{u_k u_\ell} \frac{\partial U_k}{\partial x_\ell}.$$

* Hanjalić and Launder's proposal was tested by the present writer, and Daly and Harlow's proposal by Reynolds (1970).

The rate of dissipation, ϵ , is determined from the following equation, which is taken from Hanjalić and Launder (1972):

$$\underbrace{U_k \frac{\partial \epsilon}{\partial x_k}}_{\text{convection}} = \underbrace{\beta_\epsilon \frac{\partial}{\partial x_k} \left(\frac{k}{\epsilon} \overline{u_k u_l} \frac{\partial \epsilon}{\partial x_l} \right)}_{\text{diffusion}} + \underbrace{c_1 \frac{\epsilon}{k} P - c_2 \frac{\epsilon^2}{k}}_{\text{source terms}}, \quad (\text{G.3-2})$$

where β_ϵ , c_1 , and c_2 are constants. For a physical interpretation of the source terms, the reader is referred to Hanjalić and Launder's paper.

The ϵ - equation (G.3-2) is employed here rather than the kL- equation (3.15) because, as part of a general model, it has the practical advantage over the kL- equation of not requiring an additional term near walls (see §G.5). In any case, all terms but the diffusion terms in the two equations are equivalent when

$$\begin{aligned} c_B &= 2.5 - c_1, \\ c_S/c_D &= 2.5 - c_2; \end{aligned} \quad (\text{G.3-3})$$

and in free boundary layers, the diffusion terms are also roughly equivalent (see § 3.8a).

Equations (G.3-1) and (G.3-2) constitute the present stress-equation model. Together with the momentum equation (2.6), they form a closed set. They contain the six constants ω , α , c_1 , c_2 , β , and β_ϵ which will be determined from experiments in §§G.4 and G.5.

G.4 Application of the model to one-dimensional flows

For a preliminary determination of some of the constants, the model is applied in this section to a number of one-dimensional flows. In the flows considered, x_1 is the only independent variable; and the turbulence is nearly homogeneous so that the diffusion terms can be neglected.*

* In this case, the ϵ - and kL- equations are exactly equivalent.

The present study moves from the simplest to the more complex flows. This way, the constants can be determined one after another. The constants once established are kept fixed as the study proceeds.

a) Decay of isotropic turbulence: determination of c_2

The simplest possible turbulent flow is homogeneous isotropic turbulence. The closest experimental approximation to this flow is turbulence generated by a grid in a stream of uniform velocity U_1 . In this case, the kinetic energy k is determined by the equations

$$U_1 \frac{dk}{dx_1} = -\epsilon \quad (G.4-1)$$

$$U_1 \frac{d\epsilon}{dx_1} = -c_2 \frac{\epsilon^2}{k} \quad (G.4-2)$$

The solutions are of the form

$$k \propto x_1^n, \quad \epsilon \propto x_1^m,$$

and it is required that

$$m = n - 1 \quad \text{and} \quad c_2 = 1 - \frac{1}{n}.$$

The measured values of n range from -1 to -1.35 (see § 5) so that c_2 can take values from 2 to 1.7. A value of $c_2 = 1.9$ is adopted here because it was found most suitable for the free-flow predictions of § G.5.

b) Return to isotropy in the absence of mean velocity gradients: determination of ω .

Turbulence made anisotropic by mean velocity gradients tends to return to isotropy when these gradients are removed. Equations (G.4-1) and (G.4-2) are valid also in this case, and the equation for the energy component $u_{1,2}$ (put $j = i$ in (G.3-1)), together with (G.4-1), yields:

$$\omega = 1 + \frac{\frac{k}{dk/dx_1} \frac{d\overline{u_i^2}/k}{dx_1}}{\frac{\overline{u_i^2}}{k} - \frac{2}{3}} \quad (G.4-3)$$

This relation was used to determine ω from the measured energy components in turbulence relaxing from

- (i) plane strain distortion (Tucker and Reynolds 1968),
- (ii) axisymmetric contraction (Uberoi 1957).

The results are shown in fig. G.1, together with the anisotropy parameter

$$K_1 = \frac{\left| \frac{\overline{u_{\max}^2} - \overline{u_{\min}^2}}{\overline{u_{\max}^2} + \overline{u_{\min}^2}} \right|}{},$$

where $\overline{u_{\max}^2}$ is the largest and $\overline{u_{\min}^2}$ the smallest of the components $\overline{u_i^2}$.

Fig.G.1 reveals that $\omega \approx 2.8$ for strong anisotropy (large K_1). For weak anisotropy there is no agreement between the two sets of data. Tucker and Reynolds' results agree with the general observation that the tendency towards isotropy weakens as isotropy is approached. For example, in grid-generated turbulence K_1 was found to decrease hardly at all (see fig. G.1). A constant value of ω seems therefore not appropriate for weak anisotropy. Nevertheless, $\omega = 2.5$ was chosen as average value for this study.

c) Turbulence subjected to constant strain

In this subsection, the influence of mean velocity gradients is studied by application of the model to

- (i) Turbulence distorted by plane strain (Tucker and Reynolds 1968),
- (ii) Turbulence passing through a 16:1 axisymmetric contraction (Uberoi 1956),
- (iii) Nearly homogeneous shear flow (Champagne et al 1970).

These flows were studied experimentally by the authors given in parentheses. In the former two flows, turbulence is subjected to irrotational strain, and in the latter flow to purely rotational strain.

The rates of strain were nearly constant in all three experiments.

Following Reynolds (1970), the study described here proceeded in two steps:

- 1) Determination of c_1 . For the flows considered here, the equations for k and ϵ read:

$$U_1 \frac{dk}{dx_1} = P - \epsilon,$$

$$U_1 \frac{d\epsilon}{dx} = c_1 \frac{\epsilon}{k} P - c_2 \frac{\epsilon^2}{k}.$$

With $U_1(x_1)$ and $P(x_1)$ taken from the experiments, these equations were solved numerically by the Runge - Kutta method. The integration was started from measured initial values of k and ϵ .

In figs. G.2 to G.4, the solutions for the three flows considered are compared with experimental data. Predictions with various values of c_1 are included. Flows (i) and (iii) are predicted quite well with $c_1 = 1.2$. Had Champagne et al's turbulence (flow (iii)) been truly homogeneous, $P = \epsilon$ and $c_1 = c_2 = 1.9$ would have followed. The predictions with $c_1 = 1.2$ confirm the experimental finding that the length scales* grow without bounds, which indicates that truly homogeneous turbulence cannot exist.

The 16:1 contraction distorts the turbulence so rapidly that convection and production of k nearly balance each other. Thus, the value of c_1 has little influence on the predictions (see fig. G.4). In the parallel duct following the contraction, however, the decay of k is controlled by the dissipation ϵ . The value of ϵ at the beginning of this duct must therefore be predicted correctly to get the right decay of k . Fig. G.4 shows that $c_1 = 1.2$ gives too low a value of ϵ (and thus hardly any decay of k). $c_1 = 2$ seems more appropriate in this case. It is therefore doubtful whether c_1 is a universal constant**.

* The length scales in figs. G.2 to G.4 were calculated from $L = k^{3/2}/\epsilon$.

** Tennekes and Lumley (1972) pointed out that c_1 is zero when the Reynolds number is very large.

2) Determination of α . The second step was to solve the equations for $\overline{u_1^2}$, $\overline{u_2^2}$, $\overline{u_3^2}$ (G.3-1), and ϵ (G.3-2) with only the initial conditions and the mean velocity distribution taken from the experiments. The results for various values of α are compared with experimental data in figs. G.5 to G.8. The general behaviour of the energy components is predicted well; and $\alpha = .4$ gives the best overall agreement. A few shortcomings of the present model need to be mentioned: it cannot predict the disparity between $\overline{u_2^2}$ and $\overline{u_3^2}$ measured by Champagne et al (fig. G.5); although it predicts well the anisotropy parameter for flow (i) (fig. G.7), it predicts poorly the actual energy distribution among the components (fig. G.6); finally to obtain agreement with the full range of Uberoi's measurements, c_1 and α must be changed to $c_1 = 2$ and $\alpha = .3$ (fig. G.8).

d) Best-fit constants

Table G.1 lists the best-fit constants for the majority of the one-dimensional flows considered. The constants c_B and c_S/c_D appearing in the two-equation model of § 3 are also listed; they were calculated from c_1 and c_2 via equation (G.3-3). Furthermore, Table G.1 includes values of the constants as determined by other researchers.

Table G.1 List of best-fit constants

Constant	c_1	c_2	ω	α	c_B	c_S/c_D
Appears in equation	G.3-2	G.3-2	G.3-1	G.3-1	see § 3	
Present study (1-D flows)	1.2	1.9	2.5	.4	1.3	.6
Reynolds (1970)	1.	2.	2.5	.5	1.5	.5
Hanjalić & Launder (1972)	1.45	2.	2.8	-	1.05	.5

G.5 Application of the model to free boundary layers

In this section, the stress-equation model presented in § G.3 is applied to the following free boundary layers: the self-similar plane mixing layer, plane jet, round jet, and plane wake; and the non-similar plane jet in a moving stream.

a) The boundary layer form of the model

For two-dimensional (including non-swirling axisymmetric) boundary layers, the non-zero Reynolds stresses are $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, and $\overline{uv}$, and the present model is therefore a five-equation model. The equations for these stresses, and for the dissipation ϵ , are given in boundary layer form in table G.2 (equations (G.5-1) to (G.5-5)). The normal -stress-production terms were neglected in these equations, and the form for axisymmetric flows was taken from Rodi (1970). The appearance of an additional diffusion term (II) in the $\overline{uv}$ - equation (G.5-4) for axisymmetric flows is a consequence of the transformation to cylindrical coordinates of the diffusion term in equation (G.3-1). This term is important near the axis of symmetry: when $y \rightarrow 0$, the diffusion term I approaches $+\infty$ and the term II approaches $-\infty$, but the sum of both terms goes to zero.

b) Solution procedure

Equations (G.5-1) to (G.5-5) and the momentum equation (2.13^{*}) were solved simultaneously with the finite-difference method of Patankar and Spalding (1970)^{**}.

A numerical difficulty was encountered in solving the above equations for the round jet: fig. G.12 shows that the actual values of $\overline{U}$ and $\overline{uv}$ calculated at the grid points (crosses) do not lie on a smooth curve near the symmetry-line. This difficulty is not yet resolved and needs further investigation. However, the solution was self-similar, and the gross features of the prediction are probably not distorted significantly by the above anomaly.

* The normal-stress terms in (2.13) were neglected.

**The standard Patankar-Spalding program was modified as follows:

- (i) the effective viscosity appearing in the program was ascribed a very small value, and the shear-stress term in the momentum equation was treated as a "source" term.
- (ii) for plane flows, the "slip" value of $\overline{uv}$ at symmetry-lines was taken as zero; this implies that $\overline{uv}$ was assumed to vary as $\overline{uv} \propto y$ near symmetry-lines.

Table G.2 The $\overline{u_i u_j} \epsilon$ - model for boundary layers

Stress - equations					
Convection	Diffusion	Prod.	Pressure-strain	Diss.	Equat. No.
$U \frac{\partial \overline{u^2}}{\partial x} + V \frac{\partial \overline{u^2}}{\partial y}$	$= \beta \frac{1}{y^j} \frac{\partial}{\partial y} (y^j \frac{\overline{kv^2}}{\epsilon} \frac{\partial \overline{u^2}}{\partial y})$	$- 2 \overline{uv} \frac{\partial U}{\partial y}$	$- \omega \frac{\epsilon}{k} (\overline{u^2} - \frac{2}{3} k) + \frac{4}{3} \alpha \overline{uv} \frac{\partial U}{\partial y}$	$- \frac{2}{3} \epsilon$	(G.5-1)
$U \frac{\partial \overline{v^2}}{\partial x} + V \frac{\partial \overline{v^2}}{\partial y}$	$= \beta \frac{1}{y^j} \frac{\partial}{\partial y} (y^j \frac{\overline{kv^2}}{\epsilon} \frac{\partial \overline{v^2}}{\partial y})$		$- \omega \frac{\epsilon}{k} (\overline{v^2} - \frac{2}{3} k) - \frac{2}{3} \alpha \overline{uv} \frac{\partial U}{\partial y}$	$- \frac{2}{3} \epsilon$	(G.5-2)
$U \frac{\partial \overline{w^2}}{\partial x} + V \frac{\partial \overline{w^2}}{\partial y}$	$= \beta \frac{1}{y^j} \frac{\partial}{\partial y} (y^j \frac{\overline{kw^2}}{\epsilon} \frac{\partial \overline{w^2}}{\partial y})$		$- \omega \frac{\epsilon}{k} (\overline{w^2} - \frac{2}{3} k) - \frac{2}{3} \alpha \overline{uv} \frac{\partial U}{\partial y}$	$- \frac{2}{3} \epsilon$	(G.5-3)
$U \frac{\partial \overline{uv}}{\partial x} + V \frac{\partial \overline{uv}}{\partial y}$	$= \beta \frac{1}{y^j} \frac{\partial}{\partial y} (y^j \frac{\overline{kv^2}}{\epsilon} \frac{\partial \overline{uv}}{\partial y}) - j \beta \frac{\overline{kw^2} \overline{uv}}{\epsilon y^j}$ I II	$- \overline{v^2} \frac{\partial U}{\partial y}$	$- \omega \frac{\epsilon}{k} \overline{uv} + \alpha \overline{v^2} \frac{\partial U}{\partial y}$		(G.5-4)
Dissipation equation					
$U \frac{\partial \epsilon}{\partial x} + V \frac{\partial \epsilon}{\partial y} = \beta \frac{1}{y^j} \frac{\partial}{\partial y} (y^j \frac{\overline{kv^2}}{\epsilon} \frac{\partial \epsilon}{\partial y}) - c_1 \frac{\epsilon}{k} \overline{uv} \frac{\partial U}{\partial y} - c_2 \frac{\epsilon^2}{k}$					(G.5-5)

$j = 0$ for plane flows

$j = 1$ for axisymmetric flows

c) Boundary conditions

At free boundaries, $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, $\overline{uv}$, and ϵ were put equal to zero, and $U = U_E$ (or U_I). At symmetry-line-boundaries, the gradients of all quantities except $\overline{uv}$ are zero, and $\overline{uv}$ itself is zero.

The integration was started from arbitrary initial profiles (see §6 for a justification).

d) Determination of the constants

Constants from § G.4. The constants α , ω , and c_2 were taken from the one-dimensional flows predictions (table G.1). The value of $c_1 = 1.2$ listed in table G.1 was found not to be suitable for free boundary layers.

Constant-stress layer near walls. Although we are concerned here with flows remote from walls only, it seems appropriate, for the construction of a general model, to take account of the constraints imposed on the constants by the presence of a wall. Hanjalić and Launder (1972) have shown that, for the constant-stress layer near a wall, the equations for k and ϵ , together with the log-law $\partial U / \partial y = \frac{1}{\kappa} \frac{U}{y}$, yield:

$$c_1 = c_2 - \beta_\epsilon \kappa^2 \frac{\overline{v^2}}{U_\tau^2} \frac{k^2}{U_\tau^4}, \quad (G.5-6)$$

where U_τ is the "friction velocity" and κ is the von Karman constant. With experimental values of $\overline{v^2}/U_\tau^2 = .81$ and $k/U_\tau^2 = 4.25$ from Laufer (1954), and $\kappa = .41$, there results the following constraint:

$$c_1 = c_2 - 2.47 \beta_\epsilon. \quad (G.5-7)$$

Computer-optimisation. The constants c_1 , β , and β_ϵ were determined in such a way that the predictions give best agreement with experiments for the plane mixing layer and the plane jet. The final set of constants used for the free boundary layer predictions is shown in table G.3. These constants satisfy the constraint (G.5-7).

Table G.3 Best-fit constants for free boundary layers

c_1	c_2	ω	α	β	β_ϵ
1.43	1.9	2.5	.4	.25	.19

e) Presentation and discussion of results

Rates of spread. Table G.4 compares the predicted rates of spread for some self-similar free boundary layers with experiments. It reveals that the

Table G.4 Spreading parameter S^*

Flow	mix. layer $U_E = 0$	mix. layer $U_E/U_I = .6$	plane jet $U_E = 0$	round jet $U_E = 0$	plane wake
$\overline{u_i u_j} \epsilon$ - predictions	.158	.036	.114	.119	.069
Experiments	.16	.033	.11	.086	.098

plane wake (weak strain) and the round jet are not predicted well with the constants suitable for the plane mixing layer and the plane jet (strong strain). Accordingly, as the strain becomes weaker in a jet issuing into a moving stream (increasing x/θ), the predictions shown in fig. 6.15 deviate more and more from the experiments.

From the experimental shear-stress balances discussed in § 6.7, and from the relative success of model I in predicting flows with both strong and weak strain (§ 6.6), one may infer that the lack of universality of the $\overline{u_i u_j} \epsilon$ - model is due mainly to the inadequacy of the present pressure-strain simulation. There is, however, no evidence that any of the other simulations proposed is more adequate.

* The definition of S and the references for the experimental values are given in tables 6.4 and 6.5 respectively.

Profiles. Figs. G.9 to G.12 compare predicted and measured profiles of U , k , $\overline{uv}$, $\overline{u^2/k}$, $\overline{v^2/k}$, and $\overline{w^2/k}$. The shapes of the profiles of U , k , and $\overline{uv}$ are quite well predicted, but not significantly better than with model I (§ 6.6). Of course, when the agreement with the measured rate of spread is poor, the levels of $\overline{uv}$ and k also show poor agreement.

In view of the experimental uncertainty, the agreement about the distribution of energy among its components may, in general, be considered reasonable. The inner edge of the mixing layer is an exception. The agreement may be improved by

- (i) Use of a pressure-strain simulation which allows $\overline{w^2}$ to be different from $\overline{v^2}$,
- (ii) Use of a better diffusion model,
- (iii) Inclusion of the normal-stress-production terms (see fig. I.1).

G.6 Conclusions

A stress-equation model was developed and shown to predict qualitatively well the main features of a number of one-dimensional and free boundary layer flows. However, to achieve quantitative agreement with experiments, some of the constants had to be changed from one flow to another. In particular, the constants needed to be different for flows with strong and weak strain, and again different for round jets; and table 6.5 shows that, for free boundary layers, the model is no more universal than model II (two-equation, fixed constants).

The lack of universality was attributed mainly to the inadequacy of the pressure-strain simulation used. Thus, α and ω appear not to be universal constants. The constant c_1 in the ϵ -equation was also found not to be universal.

APPENDIX H

PREDICTIONS WITH THE THREE-EQUATION

MODEL OF HANJALIĆ AND LAUNDER ($\tau k \epsilon$)

The $\tau k \epsilon$ -model. For two-dimensional boundary layers, the five-equation model of table G.2 can be reduced to a three-equation model by incorporating the assumption that $\overline{v^2}$ is proportional to k . With this assumption, the equations for $\overline{uv}$, k^* , and ϵ , given in table G.2, together with the momentum equation (2.13)** , constitute a closed set and can be written in the following form (for plane flows) :

$$\begin{array}{ccc} \text{Convection} & \text{Diffusion} & \text{Source terms} \\ U \frac{\partial \overline{uv}}{\partial x} + V \frac{\partial \overline{uv}}{\partial y} & = \frac{c_D}{\sigma_\tau} \frac{\partial}{\partial y} \left(\frac{k^2}{\epsilon} \frac{\partial \overline{uv}}{\partial y} \right) & - \omega (c_D k \frac{\partial U}{\partial y} + \epsilon \frac{\overline{uv}}{k}) \end{array} \quad (H.1)$$

$$U \frac{\partial k}{\partial x} + V \frac{\partial k}{\partial y} = \frac{c_D}{\sigma_k} \frac{\partial}{\partial y} \left(\frac{k^2}{\epsilon} \frac{\partial k}{\partial y} \right) - \overline{uv} \frac{\partial U}{\partial y} - \epsilon \quad (H.2)$$

$$U \frac{\partial \epsilon}{\partial x} + V \frac{\partial \epsilon}{\partial y} = \frac{c_D}{\sigma_\epsilon} \frac{\partial}{\partial y} \left(\frac{k^2}{\epsilon} \frac{\partial \epsilon}{\partial y} \right) - c_1 \frac{\epsilon}{k} \overline{uv} \frac{\partial U}{\partial y} - c_2 \frac{\epsilon^2}{k} \quad (H.3)$$

This is the three-equation model proposed by Hanjalić and Launder (1972). It should be mentioned, however, that these authors arrived at (H.1) and (H.2) by simplifying a stress-equation model which is different from the one discussed in appendix.G.

The σ 's in the above equations are Prandtl / Schmidt numbers, and c_D is a constant already introduced in § 3: the reader will notice that the $\overline{uv}$ - equation reduces to the Kolmogorov - Prandtl formula $\overline{uv} = -c_D (k^2/\epsilon) \partial U / \partial y$ when the convection and diffusion terms are neglected.

* The equation for k is obtained by taking half the sum of the equations for u^2 , v^2 , and w^2 .

** The normal-stress terms in (2.13) are neglected.

Constants. The constants appearing in equations (H.1) to (H.3) which were used in the predictions discussed below are as follows.

c_D	ω	c_1	c_2	σ_τ	σ_k	σ_ϵ
.09	2.8	1.45	2.	.9	1.	1.1

These values were taken from Hanjalić and Launder except for c_D , for which these authors proposed $c_D = .07$. A value of .09 is taken here because it is more suitable for flows with $P/\epsilon \approx 1$ (see fig. 3.1, with $c_\mu = 1$).

Prediction of free boundary layers. The present writer has applied the above $\tau k \epsilon$ - model to the self-similar plane jet, plane mixing layer, and plane wake, and to the non-similar plane jet issuing into a moving stream.* Table H.1 compares predicted and measured rates of spread for the self-similar flows. The results for the non-similar jet are shown in fig. 6.15.

Table H.1 Spreading parameter S^{**}

Flow	mix.layer $U_E = 0$	plane jet $U_E = 0$	plane wake
$\tau k \epsilon$ -predictions	.156	.114	.069
Experiments	.16	.11	.098

The agreement is similar to that obtained with the $\overline{u_i u_j} \epsilon$ - model. The same holds true for the profiles of U , k , and $\overline{uv}$ shown in figs. H.1 to H.3. So the comments made in § G.5 apply here too; in particular, the model cannot predict both flows with strong and weak strain with the same constants, and the reason for this fact seems to be the inadequacy of the pressure-strain simulation used in the $\overline{uv}$ -equation.

* For details on the solution procedure, see § G.5b.

** The definitions of S and the references for the experimental values are given in tables G.4 and G.5 respectively.

The lower halves of figs. H.1 to H.3 show, for the first time, predicted and, where available, measured terms of the $\overline{uv}$ - equation (shear stress balance). The general agreement may be considered fairly good.

APPENDIX I

ALGEBRAIC STRESS MODEL

Introduction. The present appendix describes a two-equation model which enables the calculation of the stresses - $\overline{u_i u_j}$ in general flows. The model employs an equation for the contraction of $\overline{u_i u_j}$, $k \equiv \frac{1}{2} \overline{u_i u_i}$, together with a length scale equation. It relates the stresses - $\overline{u_i u_j}$ to the knowable quantities U_i , k , and L by algebraic expressions which are obtained by simplifying the stress-equation (G.3-1)

Algebraic stress relations. The convection and diffusion terms in the $\overline{u_i u_j}$ - equation (G.3-1) make this equation a differential equation. In order to obtain an algebraic expression for $\overline{u_i u_j}$, these terms must therefore be simulated.

The simplest way to simulate the terms is, of course, to neglect them altogether. However, because the convection and diffusion of k in equation (I.3) are not neglected, this simulation introduces an inconsistency: the normal stresses obtained from the so truncated $\overline{u_i u_j}$ - equation do not sum up to $2k$ (except in the special case of $P = \epsilon$).

The writer therefore proposes the use of the following consistent simulation:

$$(\text{Conv.} - \text{Diff.})_{\text{of } \overline{u_i u_j}} = \frac{\overline{u_i u_j}}{k} (\text{Conv.} - \text{Diff.})_{\text{of } k} = \frac{\overline{u_i u_j}}{k} (P - \epsilon). \quad (\text{I.1})$$

This relation can be deduced from the $\overline{u_i u_j}$ - equation (G.3-1) in the following way: the convection and diffusion of $\overline{u_i u_j}$ in (G.3-1) can be rearranged to read:

Convection:

$$U \frac{\partial \overline{u_i u_j}}{\partial x} + V \frac{\partial \overline{u_i u_j}}{\partial y} = \frac{\overline{u_i u_j}}{k} \underbrace{\left(U \frac{\partial k}{\partial x} + V \frac{\partial k}{\partial y} \right)}_{\text{Convection of } k} + k \underbrace{\left(U \frac{\partial \overline{u_i u_j}/k}{\partial x} + V \frac{\partial \overline{u_i u_j}/k}{\partial y} \right)}_{\text{CII}}$$

Diffusion:

$$\beta \frac{\partial}{\partial x_k} \left(\frac{k}{\epsilon} \overline{u_k u_l} \frac{\partial \overline{u_i u_j}}{\partial x_l} \right) = \frac{\overline{u_i u_j}}{k} \underbrace{\beta \frac{\partial}{\partial x_k} \left(\frac{k}{\epsilon} \overline{u_k u_l} \frac{\partial k}{\partial x_l} \right)}_{\text{Diffusion of } k} + \beta \frac{\partial}{\partial x_k} \left(\frac{k^2}{\epsilon} \overline{u_k u_l} \frac{\partial \overline{u_i u_j}/k}{\partial x_l} \right) + \underbrace{\frac{k}{\epsilon} \overline{u_k u_l} \frac{\partial k}{\partial x_l} \frac{\partial \overline{u_i u_j}/k}{\partial x_k}}_{\text{D II}} ;$$

when the terms C_{II} and D_{II} , which contain gradients of $\overline{u_i u_j}/k$, are neglected, there results (I.1). This relation therefore implies that $\overline{u_i u_j}/k$ varies but slowly across the flow.

Incorporation of (I.1) into the $\overline{u_i u_j}$ - equation (G.3-1) yields the desired algebraic expression for $\overline{u_i u_j}$:

$$\overline{u_i u_j} = k \left[\frac{2}{3} \delta_{ij} + \frac{1-\alpha}{\omega \epsilon} \frac{P_{ij} - \frac{2}{3} \delta_{ij} P}{1 + \frac{1}{\omega} (P/\epsilon - 1)} \right] , \quad (\text{I.2})^*$$

where P_{ij} and P are defined in equation (G.3-1). The reader may verify easily that the summation over the three normal stresses yields $2k$. It should be mentioned here that the realism of (I.2) depends strongly on the realism of the pressure-strain simulation it includes.

The complete model. The algebraic stress model proposed here employs (I.2) together with the k - equation

* P_{ij} and P contain various stresses $\overline{u_i u_j}$; so, in general, a system of algebraic equations (I.2) must be solved to obtain $\overline{u_i u_j}$.

$$\underbrace{U \frac{\partial k}{\partial x} + V \frac{\partial k}{\partial y}}_{\text{convection}} = \underbrace{\beta \frac{\partial}{\partial x_k} \left(\frac{k}{\epsilon} \overline{u_k u_l} \frac{\partial k}{\partial y} \right)}_{\text{diffusion}} - \underbrace{\overline{u_k u_l} \frac{\partial U_k}{\partial x_l}}_{P \equiv \text{production}} - \underbrace{\epsilon}_{\text{diss.}} \quad (\text{I.3})^*$$

and the ϵ - equation (G.3-2) (or any other length-scale equation, with $\epsilon \propto k^{3/2}/L$).

Prediction of $\overline{v^2}/k$ in free boundary layers. To test its performance, relation (I.2) is applied to predict the distribution of $\overline{v^2}/k$ in the plane mixing layer, the plane jet, and the plane wake. For these flows, (I.2) yields (with $i = j = 2$):

$$\overline{v^2}/k = \frac{2}{3} \frac{1 - \frac{1}{\omega} (1 - \alpha P/\epsilon)}{1 + \frac{1}{\omega} (P/\epsilon - 1) - \frac{2}{\omega} (1 - \alpha) \frac{k}{\epsilon} \frac{\partial U}{\partial x}} \quad (\text{I.4})$$

The normal-stress-production terms are retained in this relation.

The results obtained with (I.4) are shown in fig. I.1. P/ϵ and $k/\epsilon (\partial U/\partial x)$ were taken from the predictions with model I (§ 6.6), the values $\omega = 2.5$ and $\alpha = .4$ were employed (taken from table G.1). Experimental data are also shown in fig. I.1. The agreement is fairly good for the jet and the wake, and also at the outer edge of the mixing layer. The agreement is better than that achieved with the stress-equation model of appendix G (see figs. G.9 to G.11). This fact may be attributed to the inclusion of the normal-stress-production terms in (I.4).

Concluding remarks. A two-equation model was developed which enables the calculation of all the Reynolds stresses in general flows. It is based on the assumption that $\overline{u_i u_j}/k$ varies but slowly across the flow. It may be of use when numerical solutions with a stress-equation model are difficult, or uneconomical, to obtain.

* (I.3) is half the contraction of (G.3-1).

APPENDIX JPREDICTIONS WITH THE FIVE-EQUATIONMODEL OF DALY AND HARLOW

The model. Daly and Harlow (1970) proposed a stress-equation model which differs from the model of appendix G in the following respects:

- (i) The dissipation of $\overline{u_i u_j}$ is calculated from

$$2\nu \frac{\partial \overline{u_i}}{\partial x_k} \frac{\partial \overline{u_j}}{\partial x_k} = \frac{\overline{u_i u_j}}{k} \epsilon$$

- (ii) The pressure-strain simulation employed is

$$\frac{p}{\rho} \left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i} \right) = \omega \frac{\epsilon}{k} \left(\overline{u_i u_j} - \frac{2}{3} k \delta_{ij} \right),$$

where ω is the following function:

$$\omega = \omega_0 + \omega_1 P/\epsilon.$$

- (iii) In the ϵ -equation, the following additional source term is employed:

$$\beta \frac{\epsilon}{k} \frac{\partial}{\partial x_k} \left(\frac{k}{\epsilon} \overline{u_l u_k} \frac{\partial k}{\partial x_l} \right).$$

Rodi (1971) has shown that the term in the exact ϵ -equation which this additional term was to represent is accounted for already by one of the other source terms. The above term was therefore not included in the predictions discussed below.

Constants. The constants suggested by Daly and Harlow are as follows

c_1	c_2	ω_0	ω_1	β	β_e
1.	2.	.5	.5	2.	2.

These constants do not satisfy the near-wall constraint (G.5-7), a fact which may explain Daly and Harlow's rather poor predictions of the log-law region.

The diffusion constants β and β_e were found to be far too large: with these constants, the predicted profiles were entirely unrealistic. A value of .5 was found to be more suitable for both β and β_e ; this value is used in the present predictions.

Prediction of free boundary layers. For boundary-layers, Daly and Harlow's model is a five-equation model. It was applied, with the modifications quoted above, to the plane mixing layer and the plane jet*. The predicted rates of spread are compared with experiments in table J.1.

Table J.1 Rate of spread S
(for definition see table 6.4)

Flow	mix.layer $U_E = 0$	pl. jet $U_E = 0$
Predictions (Daly-Harlow model)	.478	.24
Experiments (references see table 6.5)	.16	.11

There is no agreement. The profiles of U , k , and $\overline{uv}$, and the energy components are shown in fig. J.1 for the plane jet. The shapes of the profiles of U and $\overline{uv}$ agree quite well with experiments. The energy

* The normal-stress terms were neglected. For details on the solution procedure see § G.5b.

components are separated too much, a fact which indicates that the pressure-strain term in the normal-stress equations is too large.

We may conclude that the model of Daly and Harlow, with the constants given above, is not applicable to free boundary layers.

Appendix K

Review of integral methods

Integral methods are based on the experimental finding that, in most free flows, the mean velocity profiles are closely similar. These methods therefore assume a fixed shape of the velocity profile. The task is then to determine the variation with x of the velocity scale U_o and the length scale (flow width) δ . Hence, two equations are required. The integral momentum equation provides one equation. The various methods are distinguished by the second equation employed.

Table K.1 lists the particulars of the various integral methods proposed in the literature. The methods were developed to predict jets in a moving stream. The potential core region is excluded. However, the methods provide information about the length of the potential core. Table K.1 quotes the numbers of the figures which show predictions obtained with some of the methods.

None of the methods reviewed performs satisfactorily over the whole range of U_o/U_E . Naudascher's method covers the largest range. A more adequate method would be simply to take as second equation the measured relation $U_o/U_E = f(x/\theta)$, which is shown in figs. 6.16 and 6.18 for the plane jet and the round jet respectively.

Table K.1 Integral methods

	Author	Second equation	Turbulence assumption	Constants	Pred. in fig.	Remarks
1	Squire & Truncer (1949)	Momentum equat. (2.12) integrated from $y = 0$ to $y_{\frac{1}{2}}$	τ at $y_{\frac{1}{2}}$ from mix. length hyp.			Cannot be better than mixing length hypothesis
2	Spalding (1958) Bradbury (1967)	$U_o/U_E = c \left(\frac{x}{\theta}\right)^{-\frac{j+1}{2}}$ where $\theta/D = \left(\frac{\pi}{4}\right)^j \frac{U_N}{U_E} \left(\frac{U_N}{U_E} - 1\right)^{\frac{1}{j+1}}$ for uniform flow at nozzle	implicit	$j = 0:$ $c = 2.5$ $j = 1:$ $c = 7.7$	$j = 0:$ 6.16 6.20 $j = 1:$ 6.18 6.24	good only for $U_o/U_E > .5 (j=0)$, $U_o/U_E > .125 (j=1)$ good predictions of potential core length: $x_c/\theta = \left[\frac{1}{c} \left(\frac{U_N}{U_E} - 1 \right) \right]^{-\frac{2}{j+1}}$
3	Abramovich (1963)	$\frac{dy_{\frac{1}{2}}}{dx} = \frac{U_o}{c 2U_E + U_o}$	implicit	jets: $c = .097$ wakes: $c = .25$	$j = 0:$ 6.16 6.21 $j = 1:$ 6.18 6.25	Strong jets not well predicted with $c = .097$
4	Hill (1965)	Momentum equat. multiplied with y^{j+1} and integrated	Integral over τ from jets in still air			For plane jet, resulting relations identical to those of 1, for round jet only slightly different
5	Naudascher (1967)	$dy_{\frac{1}{2}}^*/dx = \frac{U_o}{c U_E + U_o}$ * $y_{\frac{1}{2}}^*$ where $U(U - U_E)$ is half center line value	implicit	$j = 0:$ $c = .073$ $j = 1:$ $c = .06$	$j = 0:$ 6.16 6.20 $j = 1:$ 6.18 6.24	profiles of momentum flux $U(U - U_E)$ assumed similar. Works well for $U_o/U_E > .3 (j=0)$, $U_o/U_E > .18 (j=1)$

All these methods have the right asymptotic behaviour for $U_o/U_E \rightarrow \infty$ but the wrong one for $U_o/U_E \rightarrow 0$.

Table K.1 continued

	Author	Second equation	Turbulence assumption	Constants	Pred. in fig.	Remarks
6	Bradbury (1963)	Integral energy equation	Townsend's large eddy hypothesis: relates		$j = 0:$ 6.16	For plane jet, both limiting cases $U_o/U_E \rightarrow \infty, U_o/U_E \rightarrow 0$ are predicted correctly, but the transition is not.
7	Gartshore (1964)	Half-momentum equation (see 1)	$R_t = \frac{U_o y_{1/2}}{v_t}$ to $\frac{\partial U}{\partial x} / \frac{\partial U}{\partial y}$ at $y_{1/2}$.			
8	Patel (1971)	$\frac{dy_{1/2}}{dx} = c \frac{U_o}{U_E + U_o}$	implicit	$j = 0:$ $c = .1$ $j = 1:$ $c = .085$	$j = 0:$ 6.16 $j = 1:$ 6.21 6.18 6.25	
9	Newman (1968)	Half momentum equation (see 1) and integral equation for total kinetic energy	$\tau \propto \sqrt{kL} \frac{\partial U}{\partial y}$ or $\tau \propto k$ k-profile assumed		6.3	Applied to predict $dy_{1/2}/dx$ of exactly self-similar jets and wakes (with $U_E/U_o = \text{const}$) as a function of U_E/U_o .

Appendix LExperimental study:

A new method of analysing hot-wire
signals in highly turbulent flow,
and its evaluation in a round jet

Nomenclature for appendix L

a	}	coefficients in equation (L.2-23)
b		
c		
d		$= \overline{E_3^2} / S^2$
D		nozzle diameter
E		linearised hot-wire signal = $\bar{E} + e$
e		fluctuating hot-wire signal
h		pitch factor
k		yaw factor
p		pressure
S		$= E/U_{\text{eff}}$ calibration factor
U, V		mean velocities in the x-y-z system
$\bar{U}$		mean velocity in the x_1 - x_2 - x_3 system
u, v, w		fluctuating velocities in the x-y-z- system
u_1, u_2, u_3		fluctuating velocities in the x_1 - x_2 - x_3 - system
x, y, z		coordinates of fixed system
x_1, x_2, x_3		coordinates of streamline-oriented system
α		yaw angle
β		pitch angle
γ		shift angle (fig 4)
ρ		density
θ		mean flow direction = angle between x-axis and x_1 - axis

Subscripts

dyn	dynamic
eff	effective cooling
m	maximum
N	normal probe, nozzle exit
R	right angle probe
o	$\alpha = 0$

L.1 Introduction

The basic problem in hot-wire analysis is to relate the velocity components to the measured hot-wire signal. For a linearised system, this signal is proportional to the effective cooling velocity which is vectorially composed of the components to be determined. In terms of mathematics this means that the components appear under a square root in the equation relating them to the effective cooling velocity (see equation (L.2-14)).

The conventional evaluation methods as discussed by Hinze (1959) and Champagne and Sleicher (1967) start from the time-averaged form of this equation. Time averaging the square root expression, however, makes it impossible to solve for the velocity components directly. The conventional methods therefore expand the square root into a series and neglect the then appearing third and higher order correlations. In highly turbulent flows these correlations are not negligible, and in extreme cases the series expansion is not permissible. Heskestad (1965), Guitton (1968), and Vagt (1971) have attempted to include the correlations by measuring some of them and guessing others. The assumptions they had to make were, however, so plentiful that their correction formulae can hardly be of universal validity.

When a 3-sensor probe is employed together with digital data analysis, the instantaneous velocity components can, in principle, be determined without assumptions. This method is very expensive, and the 3 sensors placed very close together are likely to interfere with each other.

A method specially developed for highly turbulent flow was proposed by Zaric (1969). This method makes assumptions about the probability distributions of the fluctuating velocity components. Durst and Whitelaw (1970) have shown that this method is incorrect in certain flow regions and insensitive in others.

Since all the trouble is caused by a square root, a possible remedy is to square the response equation before taking the time average. This corresponds to evaluating the squared signal, $\overline{E^2}$, rather than $\overline{E}$. Although this idea seems very obvious, it was not put forward until recently. The reasons for this may be found in the limitations of

evaluating the squared signal. Escudier (1967) presented the response equation for this signal, but he did not solve it for the velocity components because he realised that the mean and fluctuating components cannot be separated from each other. Durst (1971) was the first one to actually suggest the use of the squared signal for evaluation.

The present investigation was undertaken to develop a method which possesses most of the merits of evaluating the squared signal but is also able to separate the mean from the fluctuating components. The result is a hybrid method which determines the fluctuating velocities from the squared signal, $\overline{E^2}$, and the mean velocities from the signal $\overline{E}$. The test of validity was carried out in a round jet, and the reinvestigation of this flow was the second objective of the present study.

In §L.2, the relations for obtaining the mean and fluctuating velocities from hot-wire signals are given for both the new and the conventional methods. §L.3 discusses equipment and procedures of the round jet experiment and compares the results for U , $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, and $\overline{uv}$ for both methods with those of Wygnanski and Fiedler (1969). §L.4 closes the appendix with an assessment of the new method and the data obtained with it.

L.2 Analysis

The present analysis is restricted to two-dimensional mean flows; it can, however, be extended easily to three-dimensional flows.

L.2.1 Basic relations in hot wire anemometry

a) Coordinate Systems

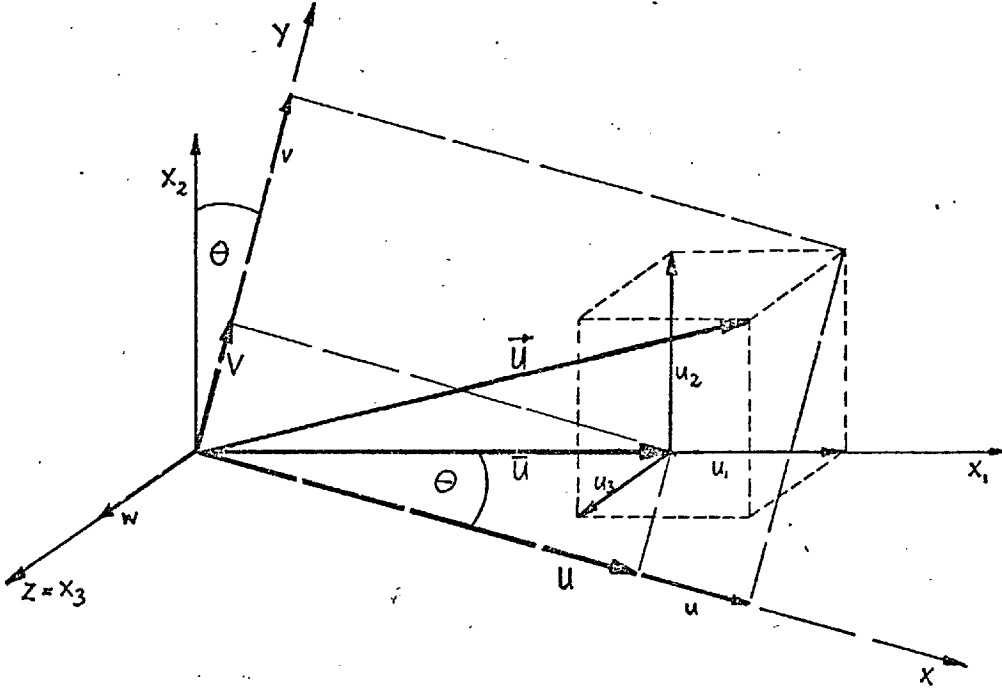


Fig. L.1 Coordinate systems

Fig.L.1 shows the fixed coordinate system x - y - z and the streamline-oriented system x_1 - x_2 - x_3 for two dimensional flow. The angle θ determines the mean flow direction. Fig. L.1 also shows the decomposition of the instantaneous velocity vector, $\vec{U}$, into the mean and fluctuating components for both systems. The analysis to follow will be carried out in the streamline-oriented system because then only one mean velocity component is present. Simple trigonometry yields the following transformation equations

$$U = \bar{U} \cos \theta \quad (\text{L.2-1})$$

$$V = \bar{U} \sin \theta \quad (\text{L.2-2})$$

$$\overline{u^2} = \overline{u_1^2} \cos^2 \theta + \overline{u_2^2} \sin^2 \theta - \overline{u_1 u_2} \sin 2\theta \quad (\text{L.2-3})$$

$$\overline{v^2} = \overline{u_1^2} \sin^2 \theta + \overline{u_2^2} \cos^2 \theta + \overline{u_1 u_2} \sin 2\theta \quad (\text{L.2-4})$$

$$\overline{w^2} = \overline{u_3^2} \quad (\text{L.2-5})$$

$$\overline{uv} = \frac{1}{2} \sin 2\theta (\overline{u_1^2} - \overline{u_2^2}) + \cos 2\theta \overline{u_1 u_2} \quad (\text{L.2-6})$$

b) The effective cooling velocity

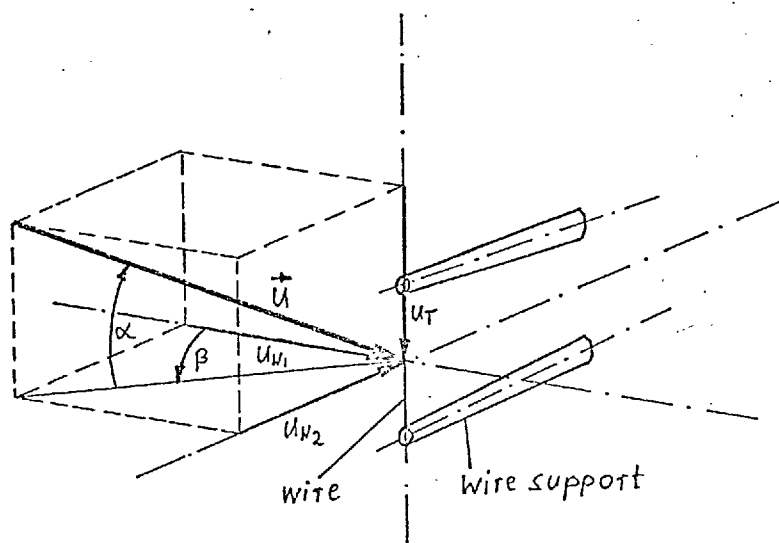


Fig. L.2 Hot-wire in a general velocity field

The velocities acting on a hot-wire are shown in fig. L.2. α is the yaw angle and β the pitch angle. If the wire were infinitely long, only the normal components U_{N1} and U_{N2} would contribute to the heat transfer. Also, the cooling would be independent of the pitch angle β . In reality, the wire is 200 to 300 diameters long, and the wire supports exercise a disturbing influence. As a consequence, U_T contributes to the cooling, and U_{N2} acts differently from U_{N1} . These facts can be expressed by

$$U_{\text{eff}}^2 = U_{N1}^2 + k^2 U_T^2 + h^2 U_{N2}^2, \quad (\text{L.2-7})$$

where k is the yaw factor and h the pitch factor. U_{eff} is the effective cooling velocity which is proportional to the electrical signal for a linearised system:

$$E = S U_{\text{eff}}, \quad (\text{L.2-8})$$

where S is the sensitivity factor to be determined from the calibration of the hot-wire.

With the trigonometric relations from fig. L. 2, equation (L.2-7) can be rewritten as

$$U_{\text{eff}}^2 = \frac{E^2}{S^2} = |\vec{U}|^2 (\cos^2 \alpha (\cos^2 \beta + h^2 \sin^2 \beta) + k^2 \sin^2 \alpha). \quad (\text{L.2-9})$$

In order to separate the influence of yaw and pitch, we can write

$$U_{\text{eff}}^2(\alpha) = \frac{\hat{E}^2(\alpha)}{S^2} = |\vec{U}|^2 (\cos^2 \alpha + k^2 \sin^2 \alpha) \text{ for } \beta = 0 \quad (\text{L.2-10})$$

$$U_{\text{eff}}^2(\beta) = \frac{E^2(\beta)}{S^2} = |\vec{U}|^2 (\cos^2 \beta + h^2 \sin^2 \beta) \text{ for } \alpha = 0. \quad (\text{L.2-11})$$

Equation (L.2-10) was proposed by Hinze (1959) to account for the effect of tangential cooling. Equation (L.2-11) seems a natural analogy to that. In general, the factors k and h are functions of α , β , and $|\vec{U}|$, but in the following analysis they will be assumed to be independent of α and β . The experimental determination of k and h , which is discussed in §L.3.2, supports this assumption.

The response equation. Fig. L.3 shows a wire placed in the x_1 - x_2 plane. α is now the angle between the mean flow direction and the normal to the wire. Decomposition of $(\bar{U} + u_1)$ and u_2 into

components normal and tangential to the wire yields:

$$U_{N1} = (\bar{U} + u_1) \cos \alpha + u_2 \sin \alpha, \quad (\text{L.2-12})$$

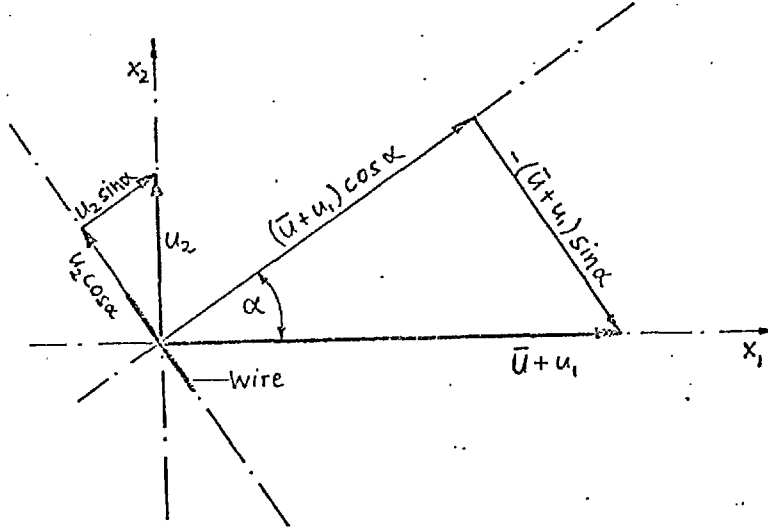


Fig. L.3 Wire in the x_1 - x_2 plane

$$\text{and } U_T = -(\bar{U} + u_1) \sin \alpha + u_2 \cos \alpha. \quad (\text{L.2-13})$$

U_{N2} is, of course, equal to u_3 .

Insertion of these components into (L.2.7) yields the response equation

$$U_{\text{eff}} = \frac{E}{S} = \left\{ [(\bar{U} + u_1) \cos \alpha + u_2 \sin \alpha]^2 + h^2 u_3^2 + k^2 [u_2 \cos \alpha - (\bar{U} + u_1) \sin \alpha]^2 \right\}^{\frac{1}{2}}. \quad (\text{L.2-14})$$

A similar equation can be derived for a wire placed in the x_1 - x_3 -plane.

L.2.2 Conventional methods

The methods labelled "conventional" in this appendix obtain

the mean velocities from the signal $\bar{E}$ and the fluctuating velocities from the signal $\bar{e^2} = \overline{(E - \bar{E})^2}$. Soluble response equations for these two signals can be deduced from equation (L.2-14) only when the square root in (L.2-14) is expanded into a series.

a) Series expansions

Equation (L.2-14) can be rearranged to yield

$$U_{\text{eff}} = \bar{U} \cos \alpha \left(1 + \frac{u_1}{\bar{U}} + \frac{u_2}{\bar{U}} \tan \alpha\right) \times \quad (\text{L.2-15})$$

$$\left[1 + \frac{\frac{h^2}{\cos^2 \alpha} \frac{u_3^2}{\bar{U}^2} + k^2 \left(\frac{u_2}{\bar{U}} - \tan \alpha - \frac{u_1}{\bar{U}} \tan \alpha \right)^2}{\left(1 + \frac{u_1}{\bar{U}} + \frac{u_2}{\bar{U}} \tan \alpha \right)^2} \right]^{\frac{1}{2}}$$

We now apply the binomial theorem to the square root and retain only terms to 4th order. When multiplied by k^2 , terms will only be retained to second order, and terms with k^4 will be neglected. This seems justified because k^2 is of the order of 0.1 or smaller. The expression for U_{eff} reads:

$$\begin{aligned} U_{\text{eff}} = \frac{E}{S} = \bar{U} \cos \alpha & \left\{ \left(1 + \frac{k^2}{2} \tan^2 \alpha \right) + \frac{u_1}{\bar{U}} \left(1 + \frac{k^2}{2} \tan^2 \alpha \right) \right. \\ & + \frac{u_2}{\bar{U}} \tan \alpha \left(1 - k^2 \left(1 + \frac{\tan^2 \alpha}{2} \right) \right) + \frac{k^2}{2} \tan^2 \alpha \left(2 + \frac{1}{\tan \alpha} + \tan^2 \alpha \right) \frac{u_2^2}{\bar{U}^2} \\ & + \frac{h^2}{2 \cos^2 \alpha} \left[\frac{u_3^2}{\bar{U}^2} - \frac{u_1 u_3^2}{\bar{U}^3} - \frac{u_2 u_3^2}{\bar{U}^3} \tan \alpha + \frac{u_1^2 u_3^2}{\bar{U}^4} \right. \\ & \left. \left. + 2 \frac{u_1 u_2 u_3^2}{\bar{U}^4} \tan \alpha + \frac{u_2^2 u_3^2}{\bar{U}^4} \tan^2 \alpha \right] \right\} \quad (\text{L.2-16}) \end{aligned}$$

Time averaging gives the desired response equation for $\bar{E}$:

$$\begin{aligned}
\frac{\bar{E}}{S} = \bar{U} \cos \alpha \left\{ 1 + \frac{k^2}{2} \tan^2 \alpha \left[1 + \left(2 + \frac{1}{\tan \alpha} + \tan^2 \alpha \right) \frac{\overline{u_2^2}}{\bar{U}^2} \right] \right. \\
+ \frac{h^2}{2 \cos^2 \alpha} \left[\frac{\overline{u_3^2}}{\bar{U}^2} - \frac{\overline{u_1 u_3^2}}{\bar{U}^3} - \frac{\overline{u_2 u_3^2}}{\bar{U}^3} \tan \alpha + \frac{\overline{u_1^2 u_2^2}}{\bar{U}^4} \right. \\
\left. \left. + 2 \frac{\overline{u_1 u_2 u_3^2}}{\bar{U}^4} \tan \alpha + \frac{\overline{u_2^2 u_3^2}}{\bar{U}^4} \tan^2 \alpha \right] \right\}. \quad (L.2-17)
\end{aligned}$$

An equation for $e = E - \bar{E}$ can be obtained by subtracting (L.2-17) from (L.2-16). Subsequent squaring and time-averaging yields the response equation for $\bar{e^2}$:

$$\begin{aligned}
\frac{\bar{e^2}}{S^2} = \cos^2 \alpha \left\{ \overline{u_1^2} (1 + k^2 \tan^2 \alpha) + 2 \overline{u_1 u_2} \tan \alpha (1 - k^2) \right. \\
+ \overline{u_2^2} \tan^2 \alpha \left[1 - 2k^2 \left(1 + \frac{\tan^2 \alpha}{2} \right) \right] \\
+ \frac{h^2}{\cos^2 \alpha} \left[\frac{\overline{u_1 u_3^2}}{\bar{U}} + \tan \alpha \frac{\overline{u_2 u_3^2}}{\bar{U}} - \frac{\overline{u_1^2 u_3^2}}{\bar{U}^2} - 2 \frac{\overline{u_1 u_2 u_3^2}}{\bar{U}^2} \tan \alpha \right. \\
\left. \left. - \tan^2 \alpha \frac{\overline{u_2^2 u_3^2}}{\bar{U}^2} \right] + \frac{h^4}{4 \cos^4 \alpha} \left[\frac{\overline{u_3^4}}{\bar{U}^2} - \frac{\overline{u_3^2}^2}{\bar{U}^2} \right] \right\}. \quad (L.2-18)
\end{aligned}$$

Equations (L.2-17) and (L.2-18) are similar to the expressions derived by Heskestad (1965) and Champagne and Sleicher (1967). It should be mentioned here that these equations do not hold when α approaches 90° .

b) Conventional relations for single wires

The conventional methods neglect all fluctuating terms in equation (L.2-17), and also the third and fourth order terms in equation (L.2-18). $\bar{U}$ can then be obtained directly from $\bar{E}$ for a given angle α . In order to obtain a system soluble for $\overline{u_1^2}$, $\overline{u_2^2}$, and $\overline{u_1 u_2}$, the signal $\bar{e^2}$ must

be measured at three different angles α . The usual choice for the wire position is $\alpha = -45^\circ, 0^\circ, +45^\circ$. The resulting relations read:

$$\overline{U} = \frac{\overline{E}(\alpha=0)}{S} \quad (\text{L.2-19})$$

$$\overline{u_1^2} = \frac{1}{S^2} \overline{e^2}(\alpha=0^\circ) \quad (\text{L.2-20})$$

$$\overline{u_2^2} = \frac{\overline{e^2}(\alpha=+45^\circ) + \overline{e^2}(\alpha=-45^\circ) - (1+k^2)\overline{e^2}(\alpha=0^\circ)}{S^2(1 - 3k^2)} \quad (\text{L.2-21})$$

$$\overline{u_1 u_2} = \frac{\overline{e^2}(\alpha=+45^\circ) - \overline{e^2}(\alpha=-45^\circ)}{2S^2(1 - k^2)} \quad (\text{L.2-22})$$

The round jet results of § L3 labelled "conventional methods" were obtained by use of these equations. The component $\overline{u_3^2}$ can be extracted from the equivalent equations for a wire in the x_1 - x_3 plane. However, in the present experimental study the wires were placed only in the x_1 - x_2 plane and perpendicular to it. Thus, $\overline{u_3^2}$ could not be obtained with the conventional method.

Before introducing the new method, it is in place here to sum up the restrictions of the conventional methods:

Equation (L.2-19) is valid only when double and higher order correlations are negligible compared with the mean velocity.

Equations (L.2-20) to (L.2-22) are valid only when triple and higher order correlations are negligible compared with double correlations.

In general, these neglects are justified only when the turbulence intensity is low, say below 25%.

L.2.3 The new method

The heart of the new method is the use of the squared signal $\overline{E^2}$. The characteristics of this signal will therefore be discussed first.

a) Characteristics of the squared signal $\overline{E^2}$.

When equation (L.2-14) is squared and then time averaged, the terms can be rearranged to yield

$$\begin{aligned}
 \frac{\overline{E^2}}{S^2} &= \underbrace{\overline{u_2^2} + h^2 \overline{u_3^2} + k^2 (\overline{U^2} + \overline{u_1^2})}_{\equiv a} && \text{independent of } \alpha \\
 &+ \underbrace{\cos^2 \alpha (\overline{U^2} + \overline{u_1^2} - \overline{u_2^2}) (1-k^2)}_{\equiv b} && \text{symmetrical about } \alpha = 0 \\
 &+ \underbrace{\sin 2\alpha \overline{u_1 u_2} (1-k^2)}_{\equiv c} && \text{antisymmetrical about } \alpha = 0
 \end{aligned}
 \tag{L.2-23}$$

The behaviour of the various terms is illustrated in fig. L.4 for one set of realistically chosen coefficients a, b, and c. Figs. L.13 to L.15* provide additional information by showing the measured distribution of $\overline{E^2}(\alpha) / S^2$ for three radial positions in a round jet. As can be seen from these figures, the measured $\overline{E^2} / S^2$ follows closely equation (L.2-23). The figures also display how a change in the coefficients a, b, and c influences the curve. At the centre of the jet, $\overline{E^2} / S^2$ is, of course, symmetrical about α because there $\overline{u_1 u_2} = 0$. With increasing distance from the centre, the ratio c/b, and thus the asymmetry of $\overline{E^2} / S^2$, increases.

*Figs. L.7 to L.22 can be found on pp. 295-310.

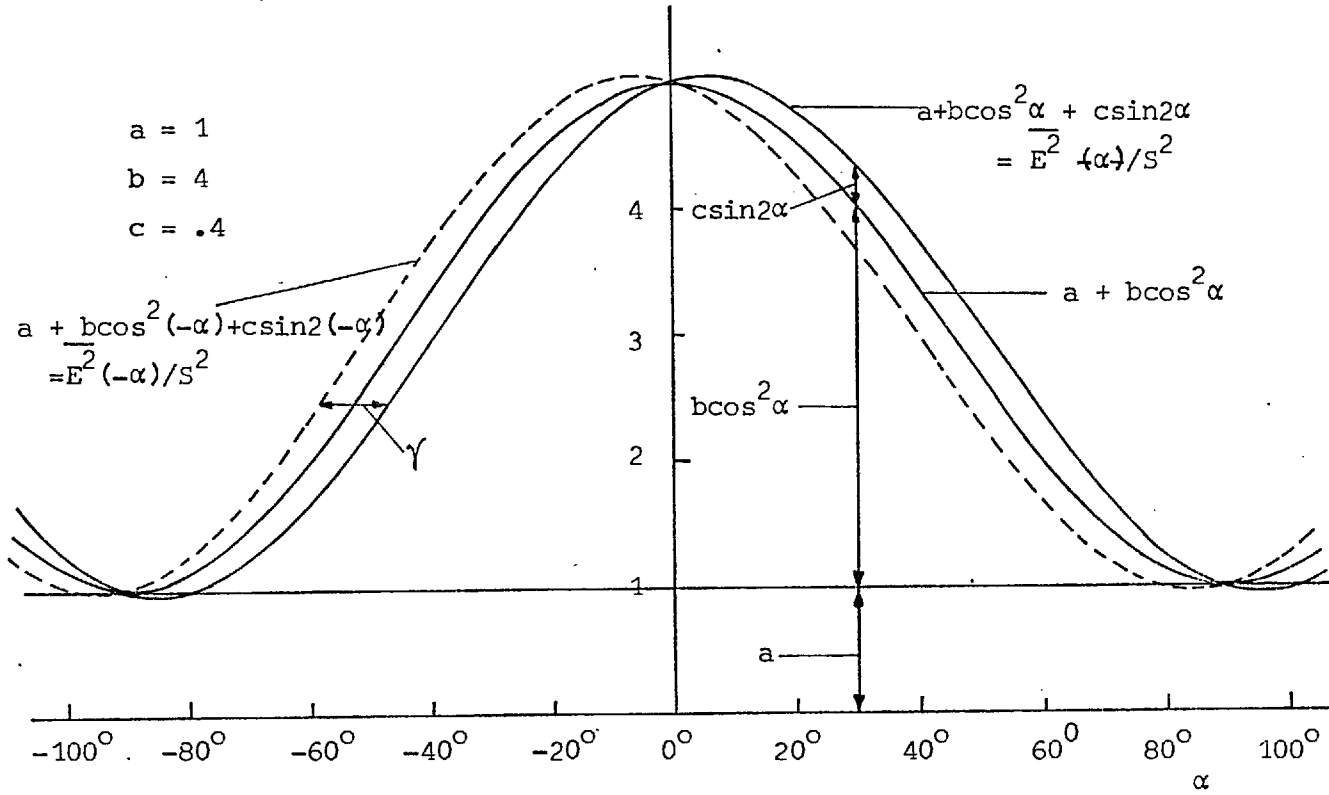


Fig. L.4 Variation of $\overline{E^2} / S^2$ with α .

Fig. L.4 shows that the curve $\overline{E^2}(-\alpha) / S^2$ can be made to coincide with $\overline{E^2}(\alpha) / S^2$ when it is shifted by the angle γ . This fact can be expressed by the equation

$$a + b \cos^2 \alpha + c \sin 2\alpha = a + b \cos^2(\gamma - \alpha) + c \sin 2(\gamma - \alpha). \quad (\text{L.2-24})$$

Trigonometric manipulations show that (L.2-24) can hold true only when

$$c = \frac{b}{2} \tan \gamma. \quad (\text{L.2-25})$$

Equation (L.2-25) offers a convenient means of determining c and thus the shear stress from the signal $\overline{E^2}(\alpha) / S^2$.

Equation (L.2-23) is written for a hot-wire sensor in the x_1 - x_2 -plane. The response equation for a sensor in the x_1 - x_3 -plane, which will also be needed, is equivalent. In both equations, $\overline{U}^2$ and u_1^2 appear only in the combination

$$U_N^2 = \overline{U}^2 + u_1^2 \quad (\text{L.2-26})$$

and can therefore not be solved for individually. Also, the mean flow direction cannot be extracted from the squared signal. Nevertheless, the fact that $\overline{u_N^2}$, $\overline{u_2^2}$, $\overline{u_3^2}$, and $\overline{u_1 u_2}$ can be obtained from the squared signal without restrictive assumptions is attractive enough to serve as basis for the new method.

b) Determination of the mean velocity vector

As discussed above, the evaluation of $\overline{E^2}$ alone does not allow a distinction between mean and fluctuating components. In order to obtain the mean velocity vector, the signal $\overline{E}$ must also be evaluated. This means a return to equation (L.2.-17), i.e. a return to series expansion and neglect of higher order correlations. There is, however, an important difference to the conventional methods: the new method considers triple and higher order correlations in (L.2-17) to be small compared with the mean velocity whereas the conventional methods must assume them to be small compared with double correlations. The new method, although again an approximate one, therefore involves much less restrictive assumptions than the conventional methods.

Mean flow direction θ . When terms of 4th order are neglected in equation (L.2-17), and k^2 is considered small, the only term which changes sign with α is

$$\tan \alpha \frac{\overline{u_2 u_3^2}}{\overline{U}^3} .$$

Wyganski and Fiedler's (1969) measurements indicate that this term is very small throughout the round jet. Neglect of this term makes $\overline{E}(\alpha)$ symmetrical about $\alpha = 0$. Determination of the flow direction, θ , thus amounts to determining the symmetry line of the function $\overline{E}(\vartheta)$, where $\vartheta = \theta + \alpha$.

Magnitude of $\overline{U}$. Neglect of third and fourth order correlations in equation (L.2-17) yields for $\alpha = 0$:

$$\overline{E_2}/S = \overline{U} \left(1 + \frac{1}{2} h^2 \frac{\overline{u_3^2}}{\overline{U}^2} \right) . \quad (L.2-27)$$

The subscript 2 attached to $\bar{E}$ indicates that the wire is placed in the direction of the x_2 -axis. When the wire coincides with the x_3 -axis, the corresponding equation reads:

$$\bar{E}_3/S = \bar{U}(1 + \frac{1}{2} h^2 \frac{\overline{u_2^2}}{\bar{U}^2}) \quad . \quad (L.2-28)$$

c) Description of the new method

The present investigation aims to determine the following six quantities from hot-wire measurements: U , θ , $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, and $\overline{uv}$. A set of six response equations, and consequently six electrical signals, are therefore required. Sufficient information could be obtained from the signals $\bar{E}\{-45^\circ\}$, $\bar{E}\{+45^\circ\}$, $\bar{E}^2\{-45^\circ\}$, $\bar{E}^2\{0^\circ\}$, $\bar{E}^2\{+45^\circ\}$, all from wires in the x_1 - x_2 -plane, and $\bar{E}_3^2$ from a wire perpendicular to this plane.

A different route is, however, taken here. In order to improve the accuracy of the measurements, the signals $\bar{E}$ and $\bar{E}^2$ for a wire in the x_1 - x_2 -plane are determined for the whole range of the angle α .^{*} From the symmetry behaviour of these curves, the flow direction, θ , and the shift angle γ can be found as discussed above.

The coefficients a , b , c of equation (L.2-23) can now be extracted from $\bar{E}^2\{\alpha\}$. For $\alpha = 0$, (L.2-23) reads

$$\bar{E}_0^2/S^2 = a + b \quad . \quad (L.2-29)$$

With (L.2-25) and (L.2-29), (L.2-23) becomes:

$$\bar{E}_0^2/S^2 - \bar{E}^2\{\alpha\}/S^2 = b(1 - \cos^2\alpha - \frac{\tan \gamma}{2} \sin 2\alpha) \quad . \quad (L.2-30)$$

The left hand side of (L.2-30) plotted against the parenthesis expression on the right yields b . Equation (L.2-29) then gives a , and (L.2-25) given c .

* For low intensity flows, Fujita and Kovasznay(1968) proposed to record $\bar{E}$ and $\bar{e}^2$ over virtually the whole range of α .

a, b and c were defined in equation (L.2-23) as

$$a = \overline{u_2^2} + h_N^2 \overline{u_3^2} + k_N^2 (\overline{U} + \overline{u_1^2}) \quad (\text{L.2-31})$$

$$b = (\overline{U} + \overline{u_1^2} - \overline{u_2^2}) (1 - k_N^2) \quad (\text{L.2-32})$$

$$c = \overline{u_1 u_2} (1 - k_N^2) \quad (\text{L.2-33})$$

$\overline{u_1 u_2}$ can be obtained directly from (L.2-33). The other two equations are not sufficient to solve for the remaining three unknowns $\overline{U} = \overline{U} + \overline{u_1^2}$, $\overline{u_2^2}$, and $\overline{u_3^2}$. Only a wire not placed in the x_1 - x_2 -plane can provide additional information. Here a wire coinciding with the x_3 -axis is chosen. Its response equation reads:

$$d \equiv \frac{\overline{E_3^2}}{S^2} = \overline{U_N^2} + h_R^2 \overline{u_2^2} + k_R^2 \overline{u_3^2} \quad (\text{L.2-34})$$

Subscripts were introduced for h and k in equations (L.2-31) and (L.2-34) because the signals $E^2(\alpha)$ and E_3^2 are, in general, measured with different types of probes; and h and k are not necessarily the same for both probes.

Equations (L.2-31), (L.2-32), and (L.2-34) can now be solved for $\overline{U_N^2}$, $\overline{u_2^2}$, and $\overline{u_3^2}$:

$$\overline{U_N^2} = \frac{b \left(\frac{h_R^2 - (k_R/h_N)^2}{1 - k_N^2} \right) + d - \left(\frac{k_R}{h_N} \right)^2 a}{h_R^2 - (k_R/h_N)^2 + 1 - (k_R k_N / h_N)^2} \quad (\text{L.2-35})$$

$$\overline{u_2^2} = \frac{d - \left(\frac{k_R}{h_N}\right)^2 a - b \frac{1 - (k_R k_N / h_N)^2}{1 - k_N^2}}{h_R^2 - (k_R / h_N)^2 + 1 - (k_R k_N / h_N)^2} \quad (\text{L.2-36})$$

$$\overline{u_3^2} = \frac{a - \overline{u_2^2} + k_N^2 \overline{u_N^2}}{h_N^2}, \quad (\text{L.2-37})$$

with $\overline{u_N^2}$ from (L.3-35) and $\overline{u_2^2}$ from (L.2-36).

With $\overline{u_2^2}$ and $\overline{u_3^2}$ known, the mean velocity $\overline{U}$ can now be calculated from (L.2-27) and, for a check, also from (L.2-28). $\overline{u_1^2}$ can then be obtained via (L.2-26) from $\overline{u_N^2}$ and $\overline{U^2}$.

The last step is to transform the now known quantities in the streamline-oriented system (x_1 - x_2 - x_3) into the desired quantities in the fixed system (x - y - z). This transformation is performed by equations (L.2-1) to (L.2-6).

d) Error analysis

The error introduced into the normal stresses by a 1% error in either $E^2\{\alpha\} / S^2$ or E_3^2 / S^2 is approximately

$$\frac{\Delta \overline{u^2}}{\overline{U^2}} = 1 + \frac{1}{2T_u^2} \quad (\text{L.2-38})$$

$T_u = \sqrt{\overline{u^2}} / \overline{U}$ is the turbulence intensity, and it was assumed that $\overline{u_1^2} = \overline{u_2^2} = \overline{u_3^2}$. For 20% turbulence intensity the error would be 13.5% and for 70% intensity it would be 2%. The new method is therefore not applicable to low intensity flows.

An error of $\Delta\gamma$ in the determination of γ causes an error in the shear stress of

$$\Delta \overline{u_1 u_2} \approx \frac{\overline{U}^2}{2} \tan \Delta\gamma. \quad (\text{L.2-39})$$

For $\Delta\gamma = 1^\circ$, the error is $\Delta \overline{u_1 u_2} = .0087 \overline{U^2}$, which is of the order of the shear stress itself in the inner region of a jet.

Fortunately, γ can be determined more accurately in this region.

L.3 Round jet experiments

The round jet was chosen as testing ground for the new method because

- a) it offers regions with medium and high turbulence intensity,
- b) it is easy to establish,
- c) it has been thoroughly investigated by previous authors,
- d) reliable knowledge about the round jet is important for the development of turbulence models.

The present measurements are restricted to the self-preserving region ($X/D > 50$).

L.3.1 Experimental equipment

Experimental set-up. Fig. L.5 shows a schematical picture of the experimental set-up. All dimensions of interest are given in fig.L.5. The jet emerged with a velocity of 101 m/s, which was maintained constant to within 1%. The Reynolds number was 8.7×10^4 . Jet and room temperature were the same, and neither one varied by more than 2°C during any test run. The mass flow through the jet nozzle was much lower than the specified lower limit of the electrostatic air cleaner. Part of the cleaned air had therefore to be spilled into the laboratory. In order to reduce the effect of room draughts, the jet was enclosed in a cage over most of its length. The traverse gears could be read to 0.05 mm and the yaw gear to 0.1° .

The exit velocity of the calibration jet could be varied continuously from 0 to 45 m/s by a motor speed controller. The turbulence level at the exit was below 2%. Hot-wire and pitot tube were placed near the centre of the potential core, one nozzle diameter downstream of the exit. A special yaw gear was designed to allow the hot-wire to be rotated around all three of its axes.

Instrumentation. The instruments used and the signal processing are shown in fig. L.6. Only gold-plated single-wire probes were employed because they cause especially little disturbance of the flow around the sensitive part of the wire.

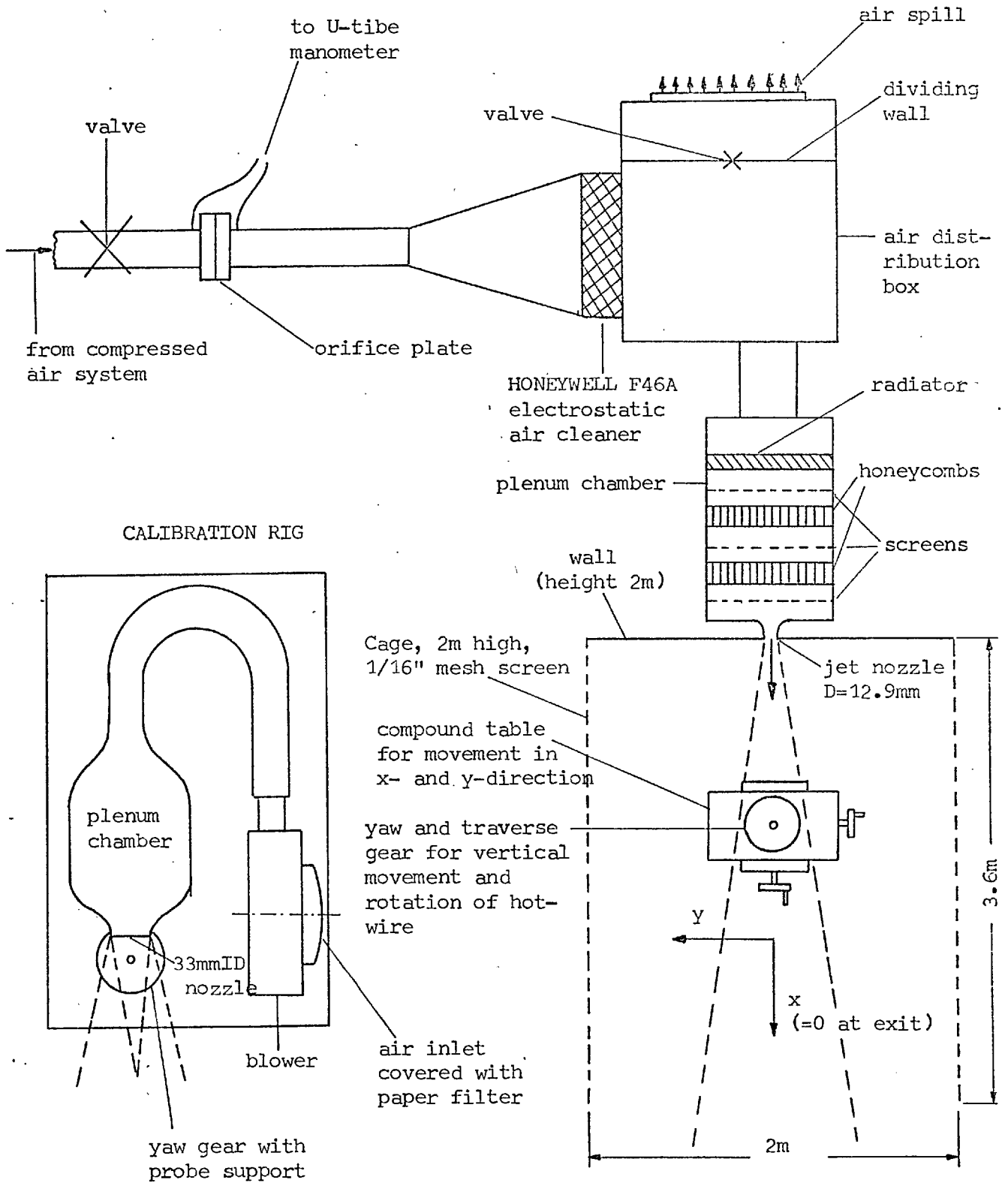


Fig. L.5 Experimental set-up

The DISA 55D10 lineariser clips all output signals larger than 10 V. In order to make sure that no turbulence peaks were clipped, the signal E was monitored on an oscilloscope.

L.3.2 Calibration of hot-wires

In this investigation, E^2 is the signal of prime interest. When linearised properly, E^2 is proportional to the velocity squared and thus to the dynamic pressure. This means that E^2 plotted versus P is a straight line. The calibration can therefore be easily performed by connecting the pressure signal, P , to the x-input and E^2 to the y-input of an x-y plotter. Variation of the calibration jet velocity from zero to the desired maximum then produces the calibration curve. This procedure took about one minute. The exponent and the zero adjustment setting of the lineariser which give the required straight line were found by trial and error. An example is given in fig. L.7. The curve E versus P was also recorded as check on the performance of the squaring device.

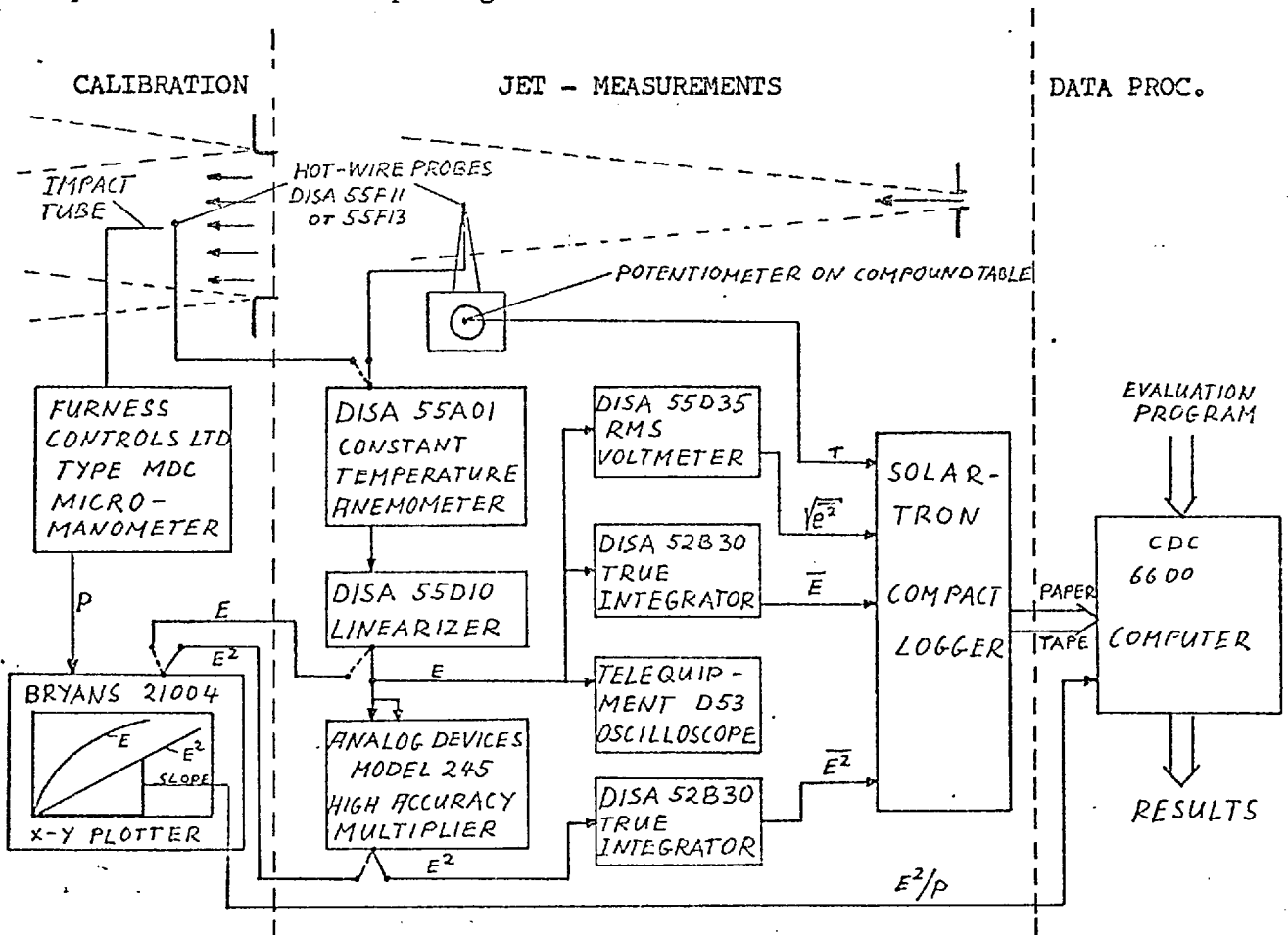


Fig. L.6 Signal processing

Determination of the factors k and h. Equations (L.2-35) to (L.2-37) show that the normal stresses are quite sensitive to the values of the factors k and h, particularly so at lower intensities. The calibration at zero yaw and pitch angle discussed above had therefore to be extended to general values of these angles. k and h are defined by equations (L.2-10) and (L.2-11). When k is independent of α , and h is independent of β , the curves $E^2(\alpha)/E^2(\alpha=0)$ versus $\sin^2\alpha$ and $E^2(\beta)/E^2(\beta=0)$ versus $\sin^2\beta$ are straight lines which intersect $\sin^2\alpha = 1$ at k^2 , and $\sin^2\beta = 1$ at h^2 . This is illustrated in fig. L.11.

In practice, E^2 was plotted versus P_{dyn} for fixed angles α and β over the whole velocity range of interest. Figs. L.8 to L.10 present examples of x-y plotter recordings for various types of probes and velocity ranges. This method is particularly suited for a quick determination of the velocity dependence of k and h.

$E^2(\alpha, \beta)/E_0^2$ as extracted from these graphs is plotted in fig. L.11 versus $\sin^2(\alpha, \beta)$ for various velocities and probes. h was found to be independent of velocity and takes the values

$$h_N = .94 \quad \text{for the DISA 55F11 normal probe,}$$

$$h_R = .96 \quad \text{for the DISA 55F13 right angle probe.}$$

Straight lines approximate the measured points in fig. L.11 well only in the range $\alpha = 0$ to 70° . k is therefore constant only in this range, but in actual flow situations the yaw angle is seldom larger than 70° . Because k is very sensitive to small measurement errors in $E^2(\alpha)$, it could only be determined to an accuracy of ± 0.06 . Champagne et al (1967) report similar scatter. For velocities larger than 7.5 m/s, no clear deviation of k from the value zero could be discerned. The mean value of k used in the evaluation is for both probes:

$$k = 0.007 (7.5 - U)^2 \quad \text{for} \quad U \leq 7.5 \text{ m/s,}$$

$$k = 0 \quad \text{for} \quad U > 7.5 \text{ m/s.}$$

Champagne et al (1967) found $k = .2$ for standard probes in the velocity range $7 \text{ m/s} < U < 35 \text{ m/s}$. From § L.2.1 we know that tangential cooling

is a consequence of the finite wire length and the flow disturbance caused by the wire supports. For the gold-plated probes employed, the sensitive part of the wire is 175 diameters away from the supports. The vanishing value of k found here for these probes seems therefore plausible.

L.3.3 Measurement procedure

First, the symmetry behaviour of the jet was checked by pitot-tube traverses in both horizontal and vertical direction at 2 x-stations. The results are shown in fig. L.12. They indicate that the jet is indeed axisymmetric.

Hot-wire traverses were then made in one half of the jet in the horizontal direction only. At the two axial stations $x/D = 62$ and $x/D = 75$, readings were taken at ten radial positions. First, the signals $E^2(\vartheta)$ and $\bar{E}(\vartheta)$ were obtained from a DISA 55F11 normal probe with the wire in the x-y-plane. At each position in the jet, this probe was rotated from $\vartheta = -75^\circ$ to $+75^\circ$ in 15° steps. Subsequently the signals E_3^2 and E_3 were measured with a DISA 55F13 right angle probe with the wire aligned in the z-direction. Each probe was calibrated and rechecked right before and after its use. An example is given in fig. L.7. The calibration curves taken before and after a 12 hour run can hardly be distinguished. This excellent wire behaviour is attributed to the efficiency of the air cleaner and to the constancy of the room temperature.

The signals E and E^2 were integrated over 200 seconds in the inner region and over 300 seconds in the outer region of the jet. These long integration times brought the total time for one traverse to about 15 hours. In order to detect changes in the experimental system and to allow to correct for them, reference values of all signals were taken every hour at exactly the same position.

L.3.4 Data processing

The data processing was carried out in three stages:

a) Evaluation of $\overline{E^2/S^2}$, $\overline{E}/S$ and $\overline{e^2/S^2}$ by computer

Input was the recorded data and the factor S^2 from the calibration curve. The reference values were used to correct the data by linear interpolation. The combined influence of changes in jet velocity, temperature, wire condition, and of lineariser drift could thereby be largely eliminated.

$\overline{e^2}$ was calculated from $\overline{E^2}$ and $\overline{E}$ via the relation

$$\overline{e^2} = \overline{E^2} - \overline{E}^2. \quad (\text{L.3-1})$$

Thus, it is important to note, the electrical information processed by both the new and the conventional method was the same.

$\sqrt{\overline{e^2}}$ as measured with the RMS-meter was not used for further evaluation but only to study the effects of the RMS-meter's poor frequency response at very low frequencies. All instruments used to measure $\overline{E^2}$ and $\overline{E}$ have good response from DC to 20 kHz.

b) Determination by hand of θ , a , b , c , $\overline{E_2/S}$, $\overline{e^2\{-45^\circ\}}$, $\overline{e^2\{0^\circ\}}$, and $\overline{e^2\{+45^\circ\}}$.

The values of $\overline{E^2/S^2}$, $\overline{E}/S$, and $\sqrt{\overline{e^2}}/S$ were plotted versus $\psi = \theta + \alpha$ and then processed as discussed in § L.2.3. The mean curves were drawn so as to make $\overline{E^2\{\alpha\}}/S^2$ follow equation (L.2-23) and $\overline{E\{\alpha\}}/S$ to be symmetrical about $\alpha = 0$. Examples are shown for $y/x = 0$, $y/x = 0.0693$, and $y/x = 0.147$ in figs. L.13 to L.15. They indicate that the experimental data behave as required by the theory. The scatter increases towards the edge of the jet, in spite of the increased integration time.

c) Computer processing of the results from b) and of $\overline{E_3^2/S^2}$ and $\overline{E_3}/S$ from a).

The calibration factors S_N^2 and S_R^2 for the normal and the right angle probe could not be matched with an accuracy much better than 1%. As was shown in § L.2.3d, such an error can distort the turbulence results when the intensity is low. It came therefore as no surprise that $\overline{v^2}$ and $\overline{w^2}$ at the jet axis were found not to be the same, as they should be. This shortcoming was remedied by a slight

correction of S_R . The effect of this correction on the results for the outer region of the jet was negligible.

L.3.5 Results and Discussion

In this section the results obtained with both the new and the conventional method are presented and compared with each other and with the results of Wygnanski and Fiedler (1969). Since these authors used conventional methods, their findings should be similar to the present results obtained with the conventional method. Differences may be attributed to the fact that Wygnanski and Fiedler (1969) used X-wires, which are subject to thermal wake interference as shown by Jerome et al (1971). Also, they did not measure k but simply adopted Champagne et al 's (1967) value of 0.2 for all velocities.

In the following presentation, y/x is used as dimensionless distance from the centre-line. This is preferred here to dividing y by $y_{1/2}$, the radius at which the velocity has half its maximum value, because deviations from similarity and differences between the methods become more apparent.

The RMS-meter was found to measure the $\sqrt{e^2}$ -value at the centre about 15% too low.* Wygnanski and Fiedler (1969) quoted a similar figure. Since most experimenters have used RMS-meters of this or of a similar type, their turbulence measurements may be unreliable.

a) Mean velocities

U/U_m versus y/x . Fig. L.16 shows the distribution of U/U_m across the jet. U_m is the value of U at the centre line. The profiles for both new and conventional method show very good similarity behaviour. The conventional method gives too high velocity values, particularly near the edge, because it neglects the $u_3^2/\bar{U}$ term in equation (L.2-27). Wygnanski and Fiedler's profile agrees well with the results of the new method. Since these authors do not state whether they included the $u_3^2/\bar{U}$ term or not, this agreement cannot be commented upon.

* because of the poor frequency response of the RMS-meter at very low frequencies.

The rate of spread can be deduced from fig. L.16 as

$$y_{\frac{1}{2}}/x = 0.086 \quad \text{for the new method,}$$

$$y_{\frac{1}{2}}/x = 0.090 \quad \text{for the conventional method.}$$

Newman (1967) reports $y_{\frac{1}{2}}/x = 0.085$ as the average of various experimenters' results.

The values of U were obtained from both normal probe and right angle probe signals via equations (L.2-27) and (L.2-28). The agreement was generally within $\frac{1}{2}\%$, which confirms the consistency of the results.

U_N/U_m versus x/D The mean velocity decay along the jet axis is plotted in fig. L.17. U_N is the jet exit velocity. The data points follow the linear distribution required by the theory for self-preserving flow. The virtual origin is at $x/D \approx 4$. Except for a small shift in the virtual origin, Wagnanski and Fiedler's (1969) results obey the same decay law down to $x/D = 57$, then the slope of their decay curve changes. This is difficult to understand.

Mean flow direction θ . The radial distribution of θ is plotted in fig. L.18. The measured θ is compared with θ as determined from the mean velocity profile via the continuity equation

$$\tan \theta = \frac{V}{U} = \frac{y}{x} - \frac{U_m/U}{y/x} \int_0^{y/x} \frac{U}{U_m} \frac{y}{x} d(y/x) . \quad (\text{L.3-2})^*$$

Towards the edge of the jet, the agreement is getting poorer because the signal $\bar{E}$ becomes less and less sensitive to α , and the turbulence intensity reaches values which invalidate the assumptions made in § L.2.3b.

b) Normal stresses

$\sqrt{u^2/U_m}$. The distribution of the longitudinal fluctuations is

* Equivalent to equation (4.38).

shown in fig. L.19. The profiles are similar and the scatter is modest. The conventional method gives slightly lower values near the axis but virtually the same values as the new method in the outer region. The agreement with Wygnanski and Fiedler's (1969) results is also good. It therefore seems that the neglects inherent in the conventional methods are justified for determining $\overline{u^2}$.

$\sqrt{\overline{v^2}/U_m}$. The radial fluctuations are presented in fig. L.20. The new evaluation method produces a considerable scatter in the inner region. This is a consequence of the magnification of measurement errors described by equation (L.2-39). Nevertheless, the value of $\overline{v^2}$ (and of $\overline{w^2}$) at the axis is fairly reliable: at the axis we have $\overline{v^2} = \overline{w^2}$ and $k \approx 0$; equation (L.2-31) then reduces to

$$\overline{v^2} = \frac{a}{1 + h^2},$$

and a can be determined quite accurately, as can be seen from fig. L.13.

The results obtained with the conventional method are, on the average, about 15% lower than those obtained with the new method. For the determination of $\overline{v^2}$ the assumptions inherent in the conventional methods are therefore not fully justified.

It was mentioned already that Wygnanski and Fiedler's results were expected to agree with the present conventional results. Fig. L.20 proves this expectation wrong. This is particularly strange because the longitudinal fluctuations in fig. L.19 agree fairly well. Jerome et al. (1971) report that the thermal wake interference of X-wires as used by Wygnanski and Fiedler can cause an error of 25% in $\overline{v^2}$, $\overline{w^2}$, and $\overline{uv}$. Moreover, $\overline{v^2}$ is quite sensitive to the value of k when it is determined from X-wire signals. Since Wygnanski and Fiedler simply took $k = 0.2$ independent of velocity, additional errors may have been introduced. The present single wire results are therefore believed to be more reliable.

$\sqrt{\overline{w^2}/U_m}$ is presented in fig. L.21. This quantity could not be obtained with the conventional method for the present wire arrangement. Nevertheless, the main arguments of the discussion on $\sqrt{\overline{v^2}}$ may be considered applicable also to $\sqrt{\overline{w^2}}$.

c) Shear stress

In fig. L.22, the measured shear stress is compared with $\overline{uv}$ as determined from the integrated momentum equation

$$\frac{\overline{uv}}{U_m^2} = \frac{1}{y/x} \frac{U}{U_m} \int_0^{y/x} \frac{U}{U_m} \frac{y}{x} d(y/x) + y/x \left(\frac{\overline{u^2}}{U_m^2} - \frac{\overline{v^2}}{U_m^2} \right) . \quad (L.3-3)^*$$

U/U_m was taken from fig. L.16 and $\frac{\overline{u^2}}{U_m^2} - \frac{\overline{v^2}}{U_m^2}$ from figs. L.19 and L.20.

The normal stress term is usually neglected for boundary layer flows. Fig. L.22 shows that it contributes about 10% to the shear stress. Bradbury (1971) reports a similar figure for the plane jet.

The measured profiles are similar, and the results obtained with the new method agree with the total shear stress from (3.3)**. Both the present, conventionally obtained, and Wygnanski and Fiedler's results agree with $\overline{uv}$ from (L.3-3) when the normal-stress part is neglected. The conventional methods therefore give too small a value of $\overline{uv}$.

L.4 Conclusions

The new method determines the fluctuating velocities from the squared signal E^2 and the mean velocity vector from the signal $\bar{E}$. The first operation does not involve any assumption, the second one assumes that third and higher order correlations are negligible compared with the mean velocity. In contrast, conventional methods consider triple correlations negligible compared with double correlations. With increasing turbulence intensity, the latter assumption becomes invalid much more rapidly than does the former one. The new method is therefore superior to the conventional methods in highly turbulent flow. It was successfully tested in a round jet, and the results indicate that, for this flow, the conventional method produces the same $\sqrt{\overline{u^2}}$ but 10% to 15% lower values of $\sqrt{\overline{v^2}}$, $\sqrt{\overline{w^2}}$, and $\overline{uv}$ than the

* Equivalent to equation (4.30).

** with θ determined from equation (L.3-2), larger γ 's and thus $\overline{uv}$ -values would have been attained in the outer region.

new method.

The new method is not applicable to low intensity flows; and it asks for very time-consuming and accurate measurements. This is the price to be paid for obtaining reliable results in highly turbulent flows.

In agreement with Wygnanski and Fiedler (1969) it was found that the widely used RMS-meters measure the r.m.s.-value too low, at least in a round jet. The agreement extends to the profiles of U and $\sqrt{u^2}$ and to the rate of spread. Somewhat different results for $\sqrt{v^2}$, $\sqrt{w^2}$, and $\overline{uv}$ were obtained. The measured shear stress agrees, for the first time, well with $\overline{uv}$ as determined from the complete momentum equation. For boundary layers, the normal-stress terms in this equation are usually neglected. This gives about 10% lower values of $\overline{uv}$.

The present results are believed to be more reliable than those of Wygnanski and Fiedler for four reasons:

- a) only single wires were used,
- b) all wires were calibrated for yaw and pitch influence,
- c) the wire inclination was varied over its whole range,
- d) the new method is more suitable for high intensity flows.

Table 4.2 Momentum equations

Flow	Equation	Equat. No.
plane mixing layer	$\frac{dh}{d\eta} = \frac{d\delta}{dx} \left\{ \frac{df}{d\eta} \left(\frac{U_E}{U_o} \eta - \frac{U_E}{U_o} \int_0^{\eta_E} f d\eta - \int_0^{\eta_E} f^2 d\eta \right. \right.$ $\left. \left. - \int_0^{\eta_E} q d\eta + \int_0^{\eta} f d\eta \right) + (\eta + \eta_I) \frac{dq}{d\eta} \right\}$ with η_I from equation (D.7)	(4.19)
jets and wakes with $\frac{U_E}{U_o} = \text{const}$	$\frac{1}{\eta^j} \frac{d}{d\eta} (\eta^j h) = \frac{j+1}{2} \frac{d\delta}{dx} \left\{ 2 \frac{U_E}{U_o} f + f^2 + \right.$ $\left. + \left(\frac{2}{I} - 1 \right) \frac{df}{d\eta} \left(\frac{U_E/U_o}{j+1} \eta + \frac{1}{\eta^j} \int_0^{\eta} f \eta^j d\eta \right) \right.$ $\left. + 2q + \frac{2}{(j+1)I} \eta \frac{dq}{d\eta} \right\}$	(4.20)
plane jet $U_E = 0$	$\frac{dh}{d\eta} = \frac{d\delta}{dx} \left\{ \frac{1}{2} f^2 + \frac{1}{2} \frac{df}{d\eta} \int_0^{\eta} f d\eta + q + \eta \frac{dq}{d\eta} \right\}$	(4.21)
round jet $U_E = 0$	$\frac{1}{\eta} \frac{d(h\eta)}{d\eta} = \frac{d\delta}{dx} \left\{ f^2 + \frac{1}{\eta} \frac{df}{d\eta} \int_0^{\eta} f \eta d\eta + 2q + \eta \frac{dq}{d\eta} \right\}$	(4.22)
radial jet	$\frac{dh}{d\eta} = \frac{d\delta}{dx} \left\{ f^2 + \frac{df}{d\eta} \int_0^{\eta} f d\eta + q + \eta \frac{dq}{d\eta} \right\}$	(4.23)
jets and wakes with $ U_E/U_o \gg 1$, $U_E \propto x^m$	$\frac{1}{\eta^j} \frac{d}{d\eta} (\eta^j h) = \frac{U_E}{U_o} \frac{d\delta}{dx} \left\{ \frac{j+1 - m(2j+1)}{1-3m} f \right.$ $\left. + \left(1 + \frac{j+2}{j+1} \frac{m}{1-3m} \right) \eta \frac{df}{d\eta} \right\}$	(4.24)

Table 4.2 Momentum equations (contd)

Flow	Equation	Equat. No.
plane wake $U_E = \text{const}$	$\frac{dh}{d\eta} = \frac{U_E}{U_o} \frac{d\delta}{dx} \left(f + \eta \frac{df}{d\eta} \right)$	(4.25)
round wake $U_E = \text{const}$	$\frac{1}{\eta} \frac{d}{d\eta} (\eta h) = \frac{U_E}{U_o} \frac{d\delta}{dx} \left(2f + \eta \frac{df}{d\eta} \right)$	(4.26)

Table 4.3 Shear stress relations

Flow	Shear stress relation	Equat. No.
plane mixing layer	$h = \frac{d\delta}{dx} \left\{ \frac{U_E}{U_o} (\eta f - \int_0^\eta f d\eta) + f \int_0^\eta f d\eta - \int_0^\eta f^2 d\eta \right.$ $- (f-1) \left(\frac{U_E}{U_o} \int_0^{\eta_E} f d\eta + \int_0^{\eta_E} f^2 d\eta + \int_0^{\eta_E} q d\eta \right)$ $\left. + (\eta + \eta_I) q - \int_0^\eta q d\eta \right\}$	(4.27)
jets and wakes with $\frac{U_E}{U_o} = \text{const}$	$h = \frac{j+1}{2} I \frac{d\delta}{dx} \left\{ \frac{1}{\eta^j} \int_0^\eta f \eta^j d\eta \left[\frac{U_E}{U_o} \left(3 - \frac{2}{I} \right) + f \left(\frac{2}{I} - 1 \right) \right] \right.$ $+ \frac{2}{\eta^j} \int_0^\eta f^2 \eta^j d\eta \left(1 - \frac{1}{I} \right) + \frac{U_E/U_o}{j+1} \left(\frac{2}{I} - 1 \right) \eta f$ $\left. + \frac{2}{\eta^j} \int_0^\eta q \eta^j d\eta \left(1 - \frac{1}{I} \right) + \frac{2}{(j+1)I} \eta q \right\}$	(4.28)
plane jet $U_E = 0$	$h = \frac{d\delta}{dx} \left(\frac{1}{2} f \int_0^\eta f d\eta + \eta q \right)$	(4.29)
round jet $U_E = 0$	$h = \frac{d\delta}{dx} \left(\frac{f}{\eta} \int_0^\eta f \eta d\eta + \eta q \right)$	(4.30)
radial jet	$h = \frac{d\delta}{dx} \left(f \int_0^\eta f d\eta + \eta q \right)$	(4.31)
jets/wakes $ U_E/U_o \gg 1$, $U_E \propto x^m$	$h = \frac{U_E}{U_o} \frac{d\delta}{dx} \left(1 + \frac{j+2}{j+1} \frac{m}{1-3m} \right) \eta f$	(4.32)

Table 4.3 Shear stress relation (contd)

Flow	Shear stress relation	Equat. No.
plane wake $U_E = \text{const}$	$h = \frac{U_E}{U_o} \frac{d\delta}{dx} \eta f$	(4.33)
round wake $U_E = \text{const}$	$h = \frac{U_E}{U_o} \frac{d\delta}{dx} \eta f$	(4.34)

Table 4.4 Lateral velocity

Flow	Lateral velocity	Equat. No.
plane mixing layer	$g = \frac{d\delta}{dx} \left\{ \eta_I \left(\frac{U_E}{U_0} + f \right) + \eta f - \int_0^\eta f d\eta \right.$ $\left. + \frac{U_E}{U_0} \int_0^{\eta_E} f d\eta + \int_0^{\eta_E} f^2 d\eta + \int_0^{\eta_E} q d\eta \right\}$	(4.35)
jets and wakes with $U_E/U_0 = \text{const}$	$g = \frac{d\delta}{dx} \left\{ \eta \left(f - \frac{U_E}{U_0} \frac{I}{2} \right) - (j+1) \left(1 - \frac{I}{2} \right) \frac{1}{\eta^j} \int_0^\eta f \eta^j d\eta \right\}$	(4.36)
plane jet $U_E = 0$	$g = \frac{d\delta}{dx} \left\{ \eta f - \frac{1}{2} \int_0^\eta f d\eta \right\}$	(4.37)
round jet $U_E = 0$	$g = \frac{d\delta}{dx} \left(\eta f - \frac{1}{\eta} \int_0^\eta f \eta d\eta \right)$	(4.38)
radial jet	$g = \frac{d\delta}{dx} \left(\eta f - \int_0^\eta f d\eta \right)$	(4.39)
jets/wakes $\left \frac{U_E}{U_0} \right \gg 1$ $U_E \propto x^m$	$g = - \frac{j+2}{j+1} \frac{m}{1-3m} \frac{U_E}{U_0} \frac{d\delta}{dx}$	(4.40)
plane wake $U_E = \text{const}$	negligibly small	
round wake $U_E = \text{const}$	negligibly small	

Table 4.5 Kinetic energy equations

Flow	Kinetic energy equation					Equat. No.
	Convection	Diffusion	sh. prod.	normal str. prod.	Diss.	
mixing layer	$-\frac{d\delta}{dx} \left(\frac{U_E}{U_0} \eta - \frac{U_E}{U_0} \int_0^{\eta_E} f d\eta - \int_0^{\eta_E} f^2 d\eta - \int_0^{\eta_E} q d\eta + \int_0^{\eta} f d\eta \right) \frac{de}{d\eta} =$	$\frac{d}{d\eta} \left(\frac{\tilde{v}_t}{\sigma_k} \frac{de}{d\eta} \right)$	$-h \frac{df}{d\eta}$	$+ \frac{d\delta}{dx} (\eta + \eta_I) q \frac{df}{d\eta}$	$- \epsilon$	(4.41)
jets/wakes with $U_E/U_0 = \text{const}$	$-\frac{d\delta}{dx} \left\{ (j+1) I \left(\frac{U_E}{U_0} + f \right) e + \left(1 - \frac{I}{2} \right) \left(\frac{U_E}{U_0} \eta + (j+1) \frac{1}{\eta j} \int_0^{\eta} f \eta^j d\eta \right) \frac{de}{d\eta} \right\} =$	$\frac{1}{\eta j} \frac{d}{d\eta} \left(\eta^j \frac{\tilde{v}_t}{\sigma_k} \frac{de}{d\eta} \right)$	$-h \frac{df}{d\eta}$	$+ \frac{d\delta}{dx} q \left\{ \frac{j+1}{2} I \left(\frac{U_E}{U_0} + f \right) + \eta \frac{df}{d\eta} \right\}$	$- \epsilon$	(4.42)
plane jet $U_E = 0$	$-\frac{d\delta}{dx} \left(fe + \frac{1}{2} \int_0^{\eta} f d\eta \frac{de}{d\eta} \right) =$	$\frac{d}{d\eta} \left(\frac{\tilde{v}_t}{\sigma_k} \frac{de}{d\eta} \right)$	$-h \frac{df}{d\eta}$	$+ \frac{d\delta}{dx} q \left(\frac{1}{2} f + \eta \frac{df}{d\eta} \right)$	$- \epsilon$	(4.43)
round jet $U_E = 0$	$-\frac{d\delta}{dx} \left(2fe + \frac{1}{\eta} \int_0^{\eta} f \eta d\eta \frac{de}{d\eta} \right) =$	$\frac{1}{\eta} \frac{d}{d\eta} \left(\frac{\tilde{v}_t}{\sigma_k} \eta \frac{de}{d\eta} \right)$	$-h \frac{df}{d\eta}$	$+ \frac{d\delta}{dx} q \left(f + \eta \frac{df}{d\eta} \right)$	$- \epsilon$	(4.44)
radial jet	$-\frac{d\delta}{dx} \left(2fe + \int_0^{\eta} f d\eta \frac{de}{d\eta} \right) =$	$\frac{d}{d\eta} \left(\frac{\tilde{v}_t}{\sigma_k} \frac{de}{d\eta} \right)$	$-h \frac{df}{d\eta}$	$+ \frac{d\delta}{dx} q \left(f + \eta \frac{df}{d\eta} \right)$	$- \epsilon$	(4.45)

Table 4.5 (contd.)

Flow	Kinetic energy equation					Equat. No.
	Convection	Diffusion	sh. prod.	normal str. prod.	Diss.	
jets/wakes $ U_E/U_o \gg 1$, $U_E \propto x^m$	$-\frac{U_E}{U_o} \frac{d\delta}{dx} \left\{ 2 \frac{(j+1) - m(j-1)}{1-3m} e + (1 + \frac{j+2}{j+1} \frac{m}{1-3m}) \eta \frac{de}{d\eta} \right\}$	$\frac{1}{\eta^j} \frac{d}{d\eta} \left(\eta^j \frac{\tilde{v}_t}{\sigma_k} \frac{de}{d\eta} \right)$	$-h \frac{df}{d\eta}$	$-\frac{U_E}{U_o} \frac{d\delta}{dx} \frac{m(j+2)}{1-3m} q$	$-\tilde{\epsilon}$	(4.46)
plane wake $U_E = \text{const}$	$-\frac{U_E}{U_o} \frac{d\delta}{dx} (2e + \eta \frac{de}{d\eta})$	$\frac{d}{d\eta} \left(\frac{\tilde{v}_t}{\sigma_k} \frac{de}{d\eta} \right)$	$-h \frac{df}{d\eta}$		$-\tilde{\epsilon}$	(4.47)
round wake $U_E = \text{const}$	$-\frac{U_E}{U_o} \frac{d\delta}{dx} (4e + \eta \frac{de}{d\eta})$	$\frac{1}{\eta} \frac{d}{d\eta} \left(\eta \frac{\tilde{v}_t}{\sigma_k} \frac{de}{d\eta} \right)$	$-h \frac{df}{d\eta}$		$-\tilde{\epsilon}$	(4.48)
shear-free layers	$\frac{U_E \delta}{k_o^{3/2}} \frac{dk_o}{dx} e - \left(\frac{U_E}{k_o^{1/2}} \frac{d\delta}{dx} + \frac{1}{j+1} \frac{\delta}{k_o^{1/2}} \frac{dU_E}{dx} \right) \eta \frac{de}{d\eta}$	$\frac{1}{\eta^j} \frac{d}{d\eta} \left(\eta^j \frac{\tilde{v}_t}{\sigma_k} \frac{de}{d\eta} \right)$		$-\frac{\delta}{k_o^{1/2}} \frac{dU_E}{dx} q$	$-\tilde{\epsilon}$	(4.49)

Table 4.6 Integral turbulence energy equations

Flow	Integral turbulence energy equation			Equat. No.
	Convection	Production	Diss.	
mixing layer	$\frac{d\delta}{dx} \int_0^\infty \left(\frac{U_E}{U_0} + f\right) e \, d\eta$	$= \frac{1}{2} \frac{d\delta}{dx} \int_0^\infty \left[(1-f) \left(\frac{U_E}{U_0} f + f^2 + q\right) - fq \right] d\eta$	$- \int_0^\infty \varepsilon \, d\eta$	(4.50)
jets/wakes with $U_E/U_0 = \text{const}$	$- (j+1) \left(\frac{3}{2}I-1\right) \frac{d\delta}{dx} \int_0^\infty \left(\frac{U_E}{U_0} + f\right) e \eta^j d\eta$	$= (j+1) \left(\frac{3}{2}I-1\right) \frac{d\delta}{dx} \int_0^\infty \left(\frac{U_E}{U_0} + f\right) \left(\frac{U_E}{U_0} f + \frac{f^2}{2} + q\right) \eta^j d\eta$	$- \int_0^\infty \varepsilon \eta^j d\eta$	(4.51)
plane jet $U_E = 0$	$- \frac{1}{2} \frac{d\delta}{dx} \int_0^\infty f e d\eta$	$= \frac{1}{2} \frac{d\delta}{dx} \int_0^\infty \left(\frac{f^3}{2} + fq\right) d\eta$	$- \int_0^\infty \varepsilon \, d\eta$	(4.52)
round jet $U_E = 0$	$- \frac{d\delta}{dx} \int_0^\infty f e \eta d\eta$	$= \frac{d\delta}{dx} \int_0^\infty \left(\frac{f^3}{2} + fq\right) \eta d\eta$	$- \int_0^\infty \varepsilon \eta d\eta$	(4.53)
radial jet	$- \frac{d\delta}{dx} \int_0^\infty f e d\eta$	$= \frac{d\delta}{dx} \int_0^\infty \left(\frac{f^3}{2} + fq\right) d\eta$	$- \int_0^\infty \varepsilon \, d\eta$	(4.54)

Table 4.6 (contd.)

Flow	Integral turbulence energy equation			Equat. No.
jets/wakes $ U_E/U_o \gg 1$ $U_E \propto x^m$	Convection	Production	Diss.	
	$-\frac{U_E}{U_o} \frac{d\delta}{dx} \frac{(j+1)+3m}{1-3m} \int_0^\infty \epsilon \eta^j d\eta$	$= \frac{1}{2} \frac{U_E}{U_o} \frac{d\delta}{dx} \left\{ \frac{(j+1)-m(1+2j)}{1-3m} \int_0^\infty f^2 \eta^j d\eta - 2 \frac{m(j+2)}{1-3m} \int_0^\infty q \eta^j d\eta \right\}$	$- \int_0^\infty \epsilon \eta^j d\eta$	(4.55)
plane wake $U_E = \text{const}$	$-\frac{U_E}{U_o} \frac{d\delta}{dx} \int_0^\infty \epsilon d\eta$	$= \frac{1}{2} \frac{U_E}{U_o} \frac{d\delta}{dx} \int_0^\infty f^2 d\eta$	$- \int_0^\infty \epsilon d\eta$	(4.56)
round wake $U_E = \text{const}$	$- 2 \frac{U_E}{U_o} \frac{d\delta}{dx} \int_0^\infty \epsilon \eta d\eta$	$= \frac{U_E}{U_o} \frac{d\delta}{dx} \int_0^\infty f^2 \eta d\eta$	$- \int_0^\infty \epsilon \eta d\eta$	(4.57)
shear-free layers	$\left(\frac{U_E}{k_o^{3/2}} \frac{dk_o}{dx} + (j+1) \frac{U_E}{k_o^{1/2}} \frac{d\delta}{dx} - \frac{\delta}{k_o^{1/2}} \frac{dU_E}{dx} \right) \int_0^\infty \epsilon \eta^j d\eta$	$= \frac{\delta}{k_o^{1/2}} \frac{dU_E}{dx} \int_0^\infty q \eta^j d\eta$	$- \int_0^\infty \epsilon \eta^j d\eta$	(4.58)

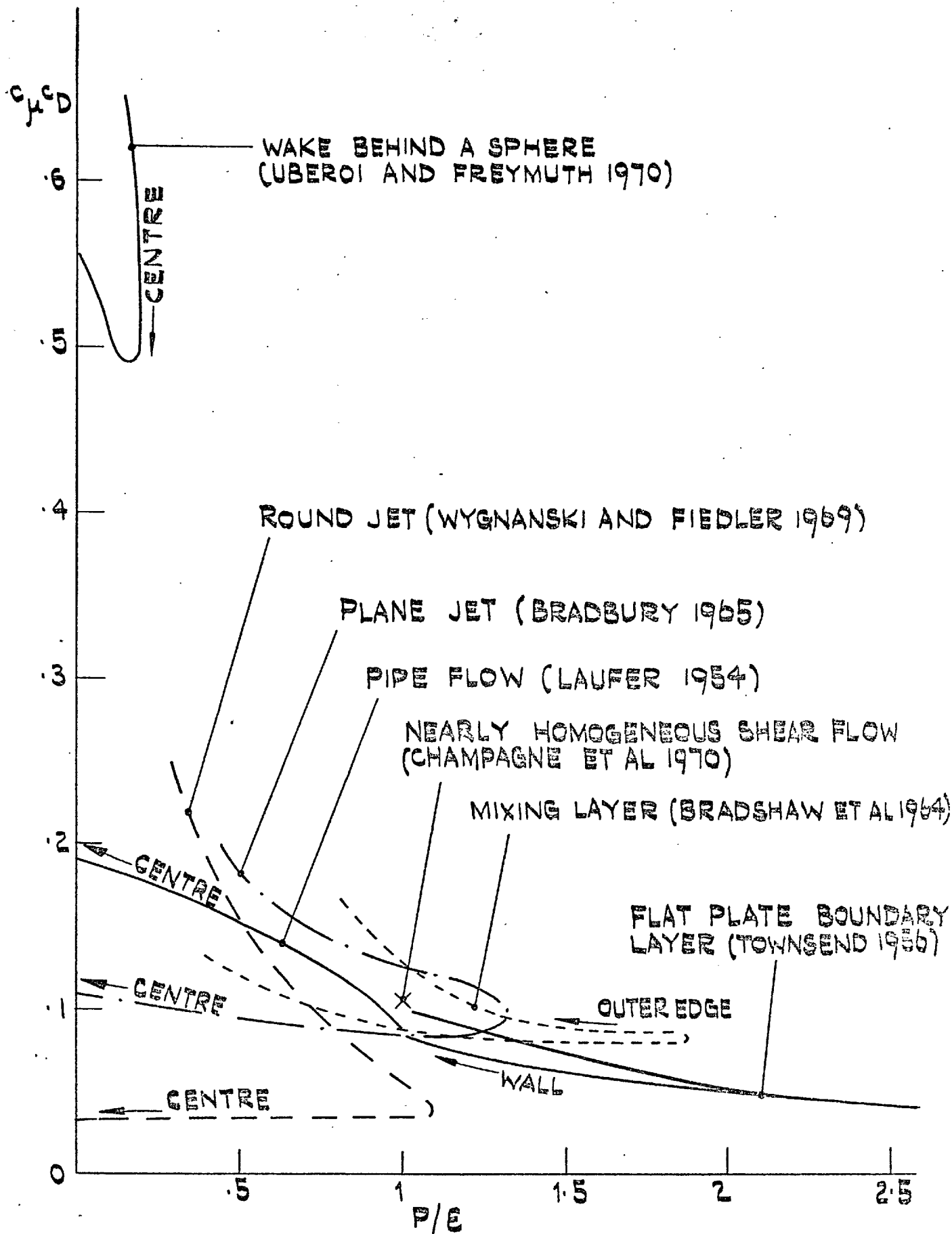


Fig. 3.1

 $c_\mu c_D$ determined from experiments

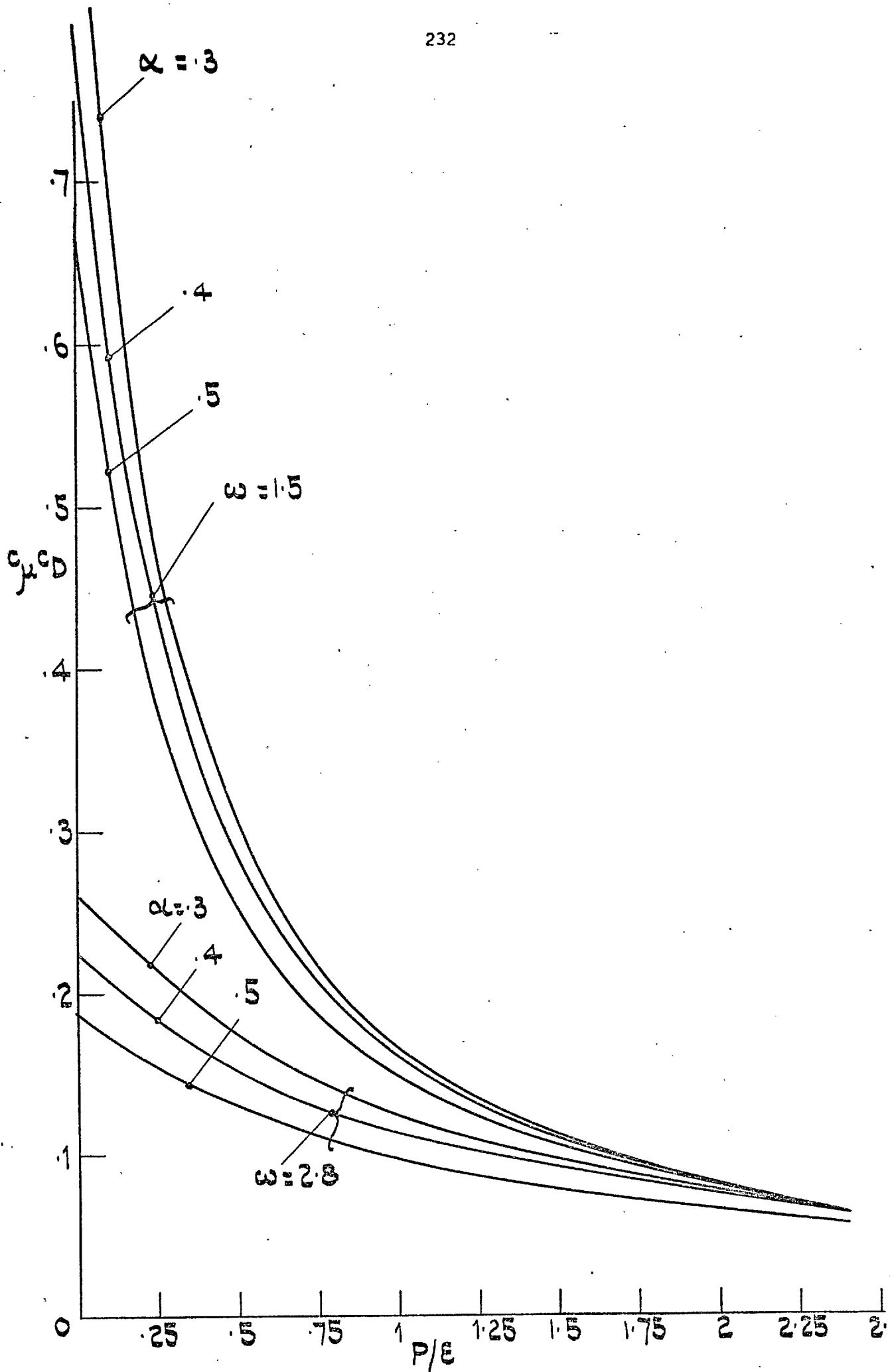


Fig. 3.2

 $c_\mu c_D$ determined from equation (4.19)

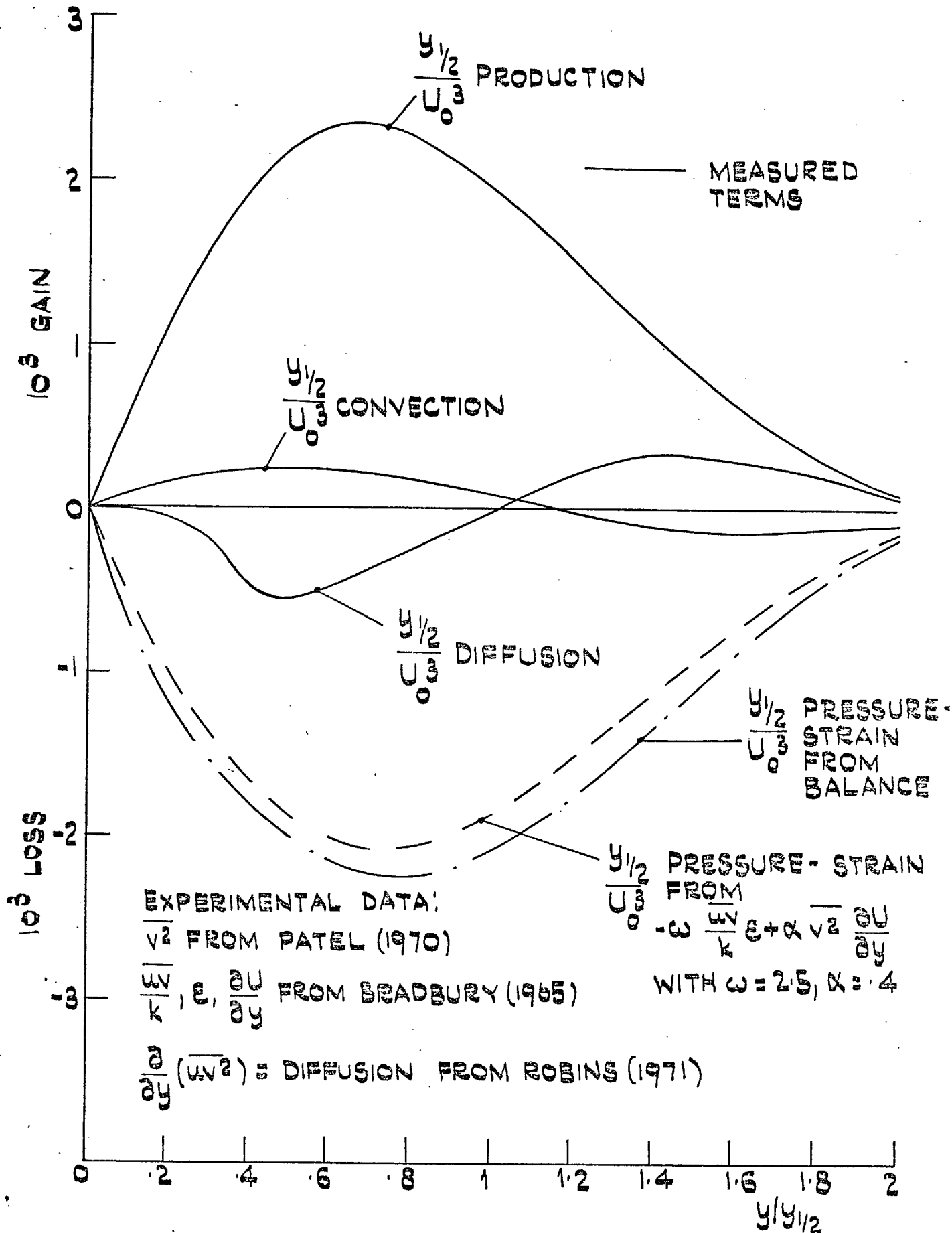


Fig. 3.3 Shear stress balance for the plane jet

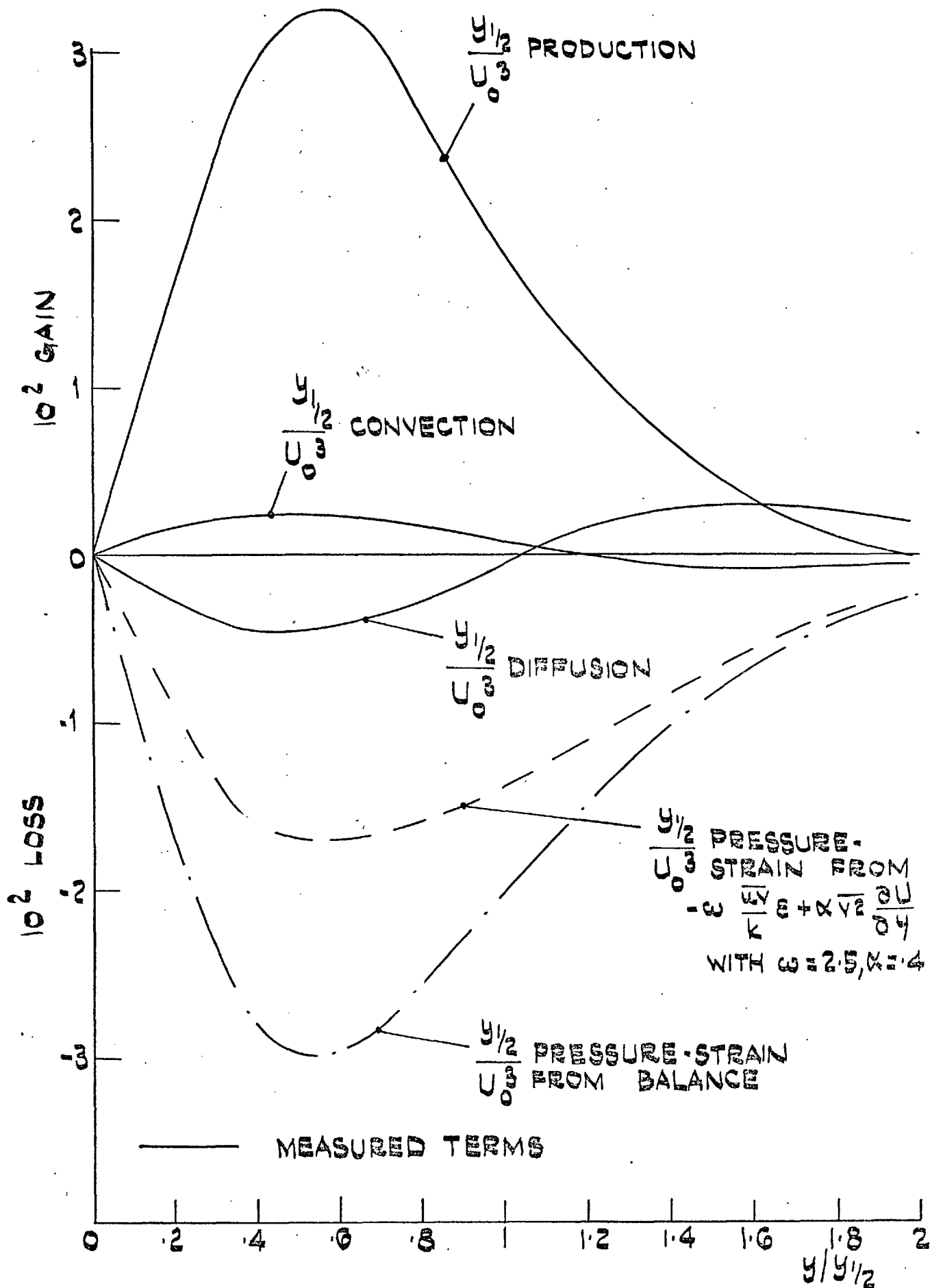


Fig.3.4 Shear stress balance for the round jet

(evaluated from data of Wagnanski and Fiedler 1969)

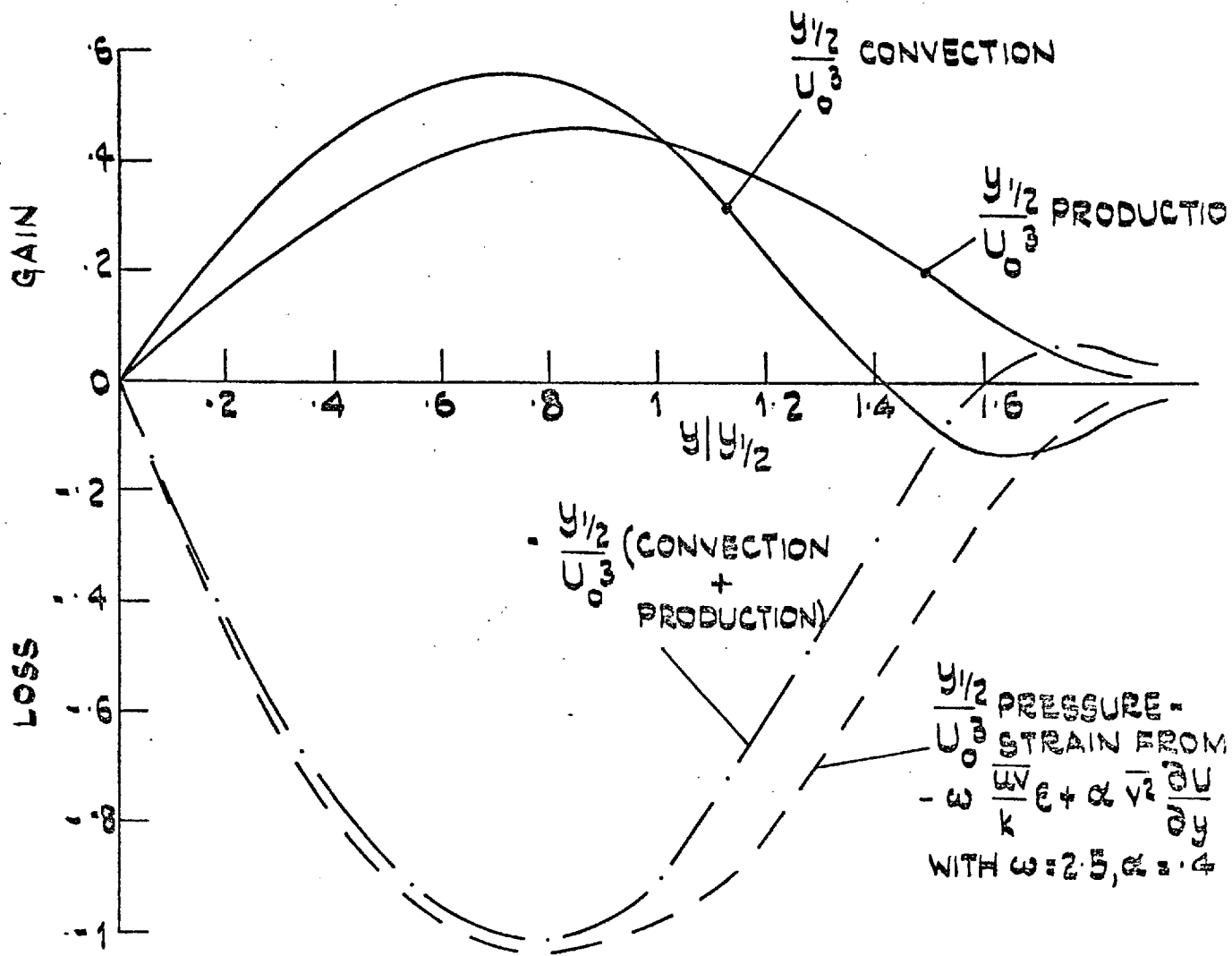


Fig. 3.5 Shear stress balance for the round wake behind a sphere (from measurements of Uberoi and Freymuth 1970)

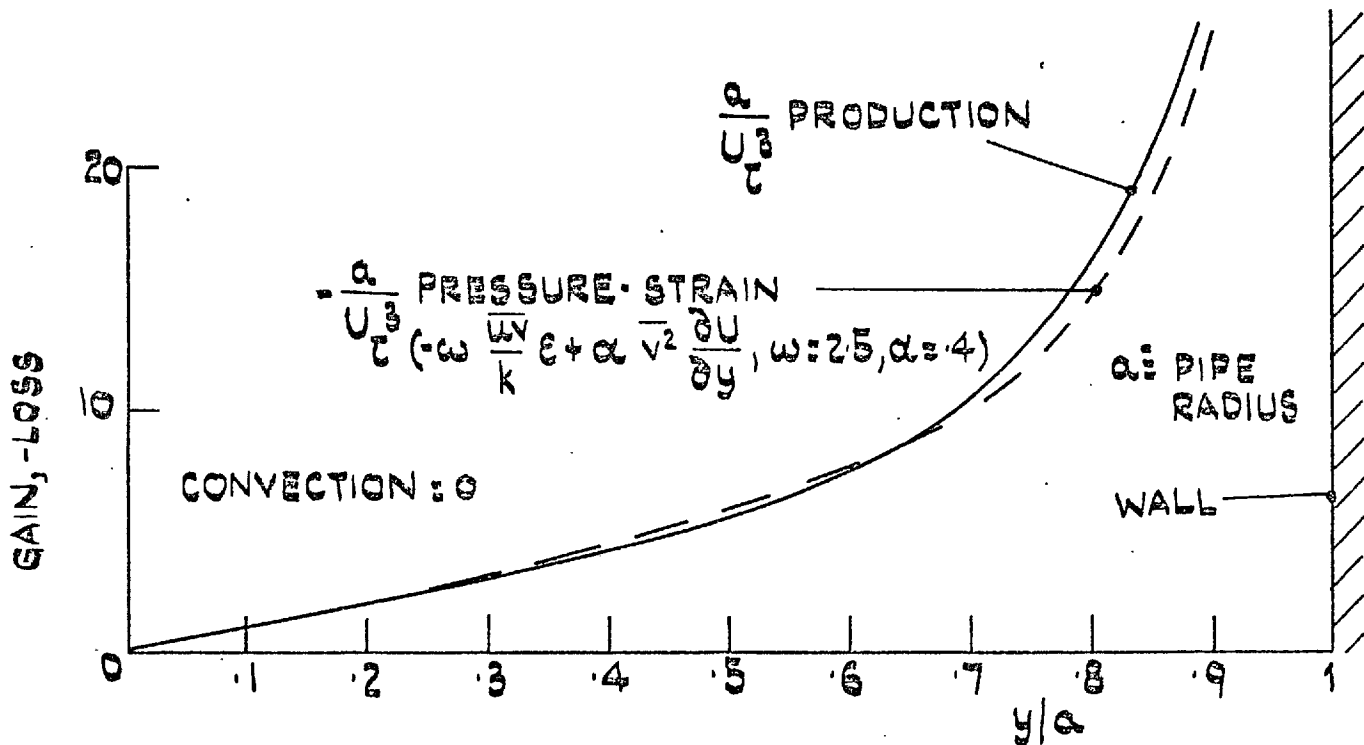


Fig. 3.6 Shear stress balance for developed pipe flow (Laufer 1954)

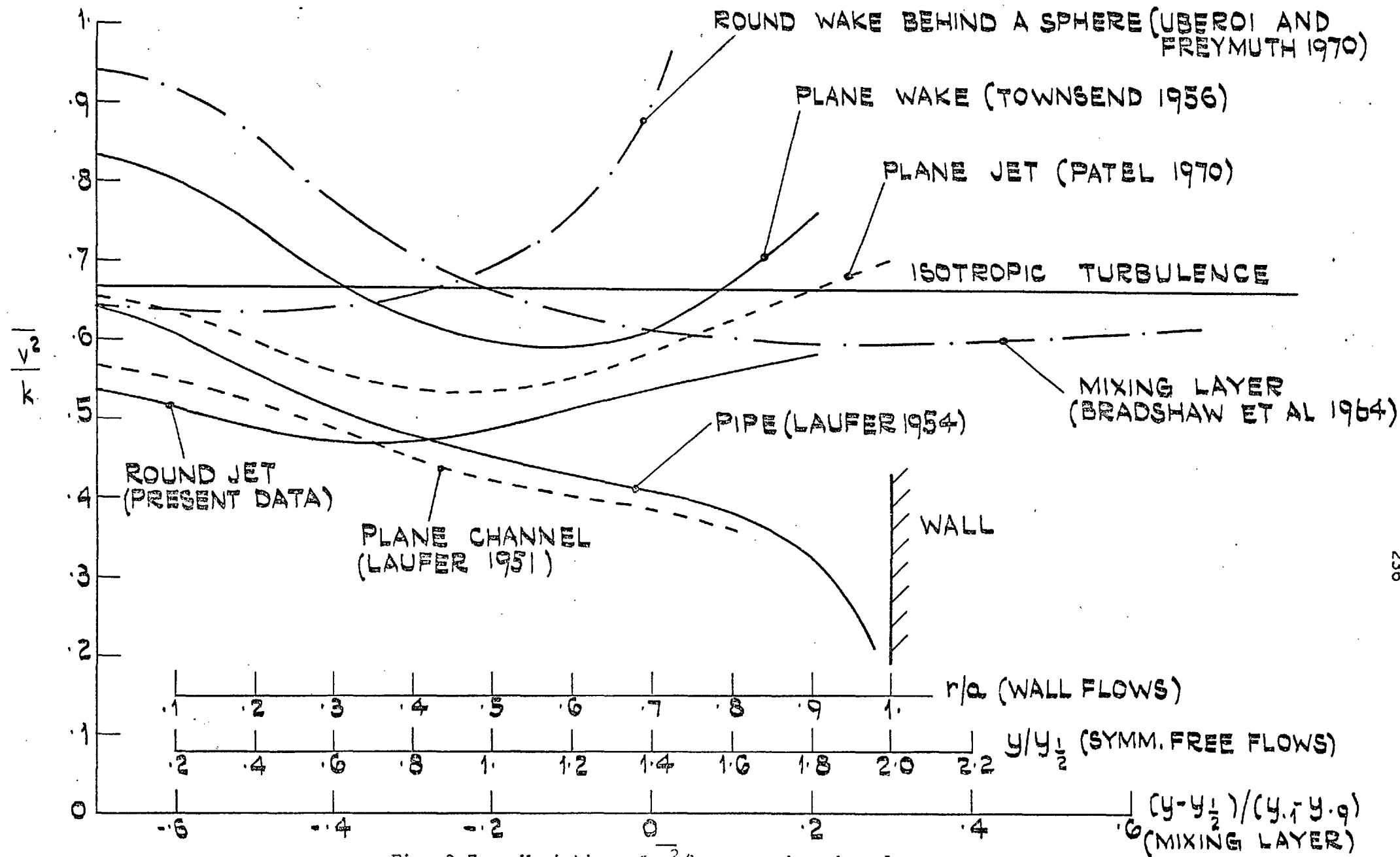


Fig. 3.7 Variation of v^2/k across boundary layers

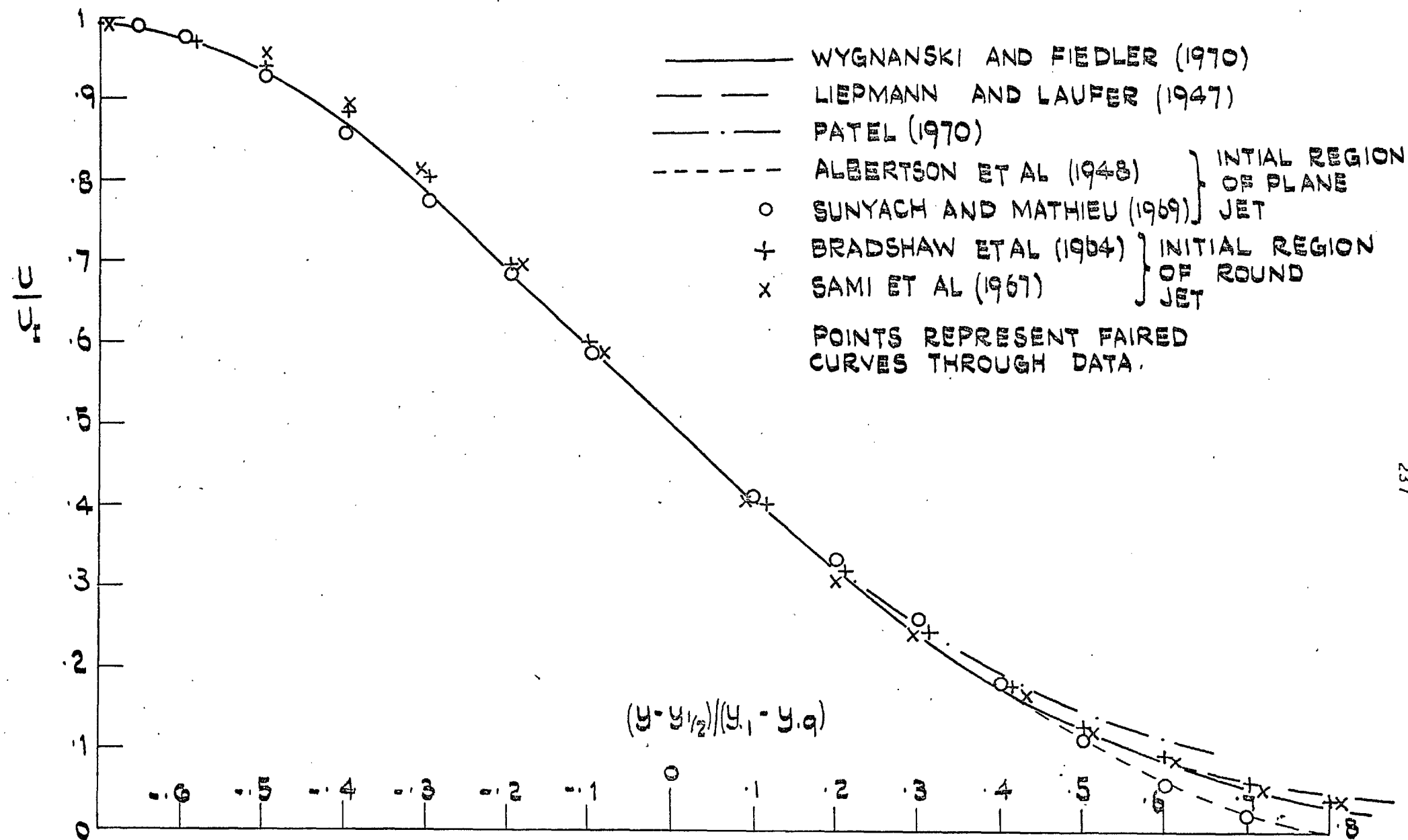


Fig.5.1 Measured velocity profiles in self-similar mixing layers with $U_E = 0$

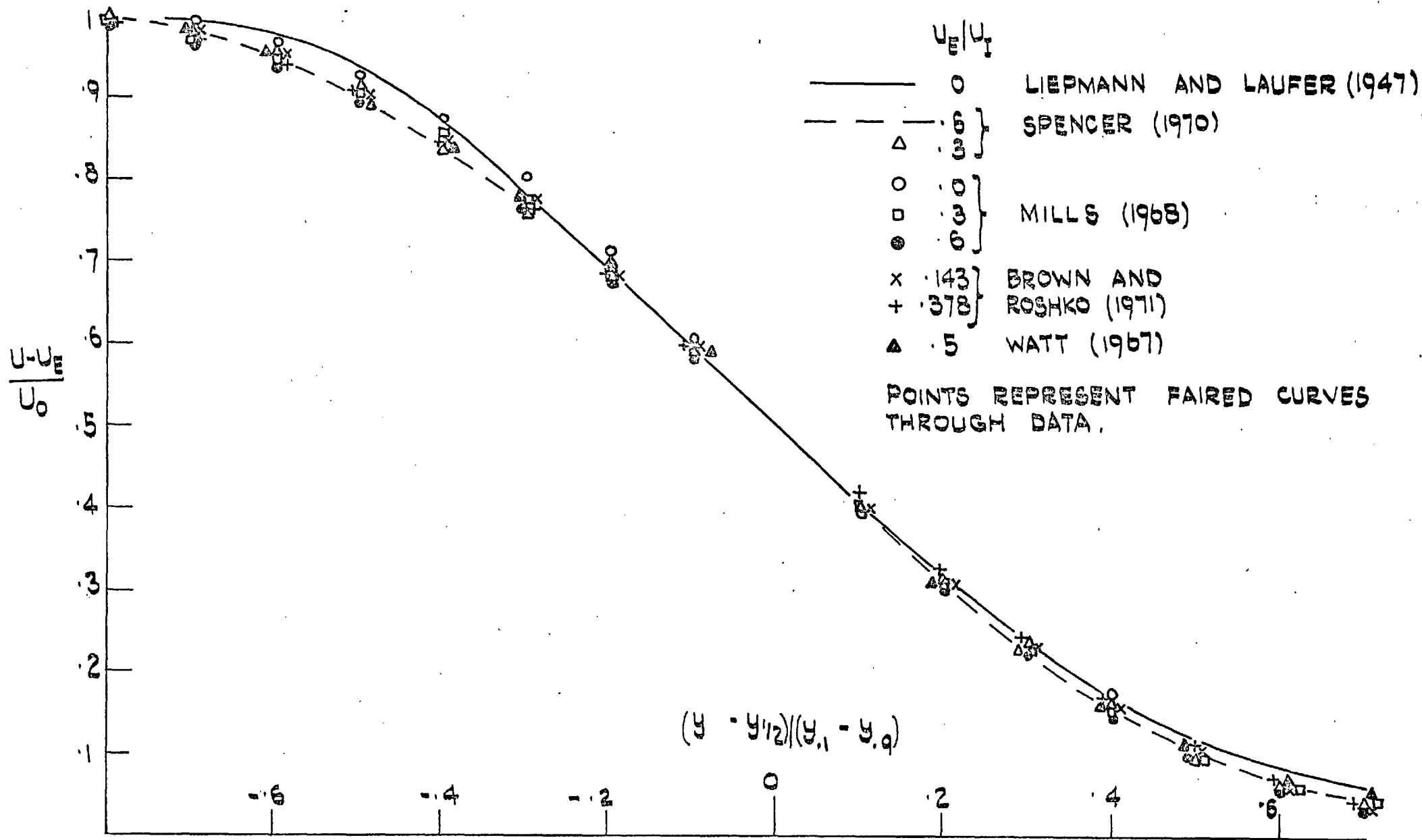


Fig. 5.2 Measured velocity profiles in mixing layers with $\frac{U_E}{U_I} = 0$

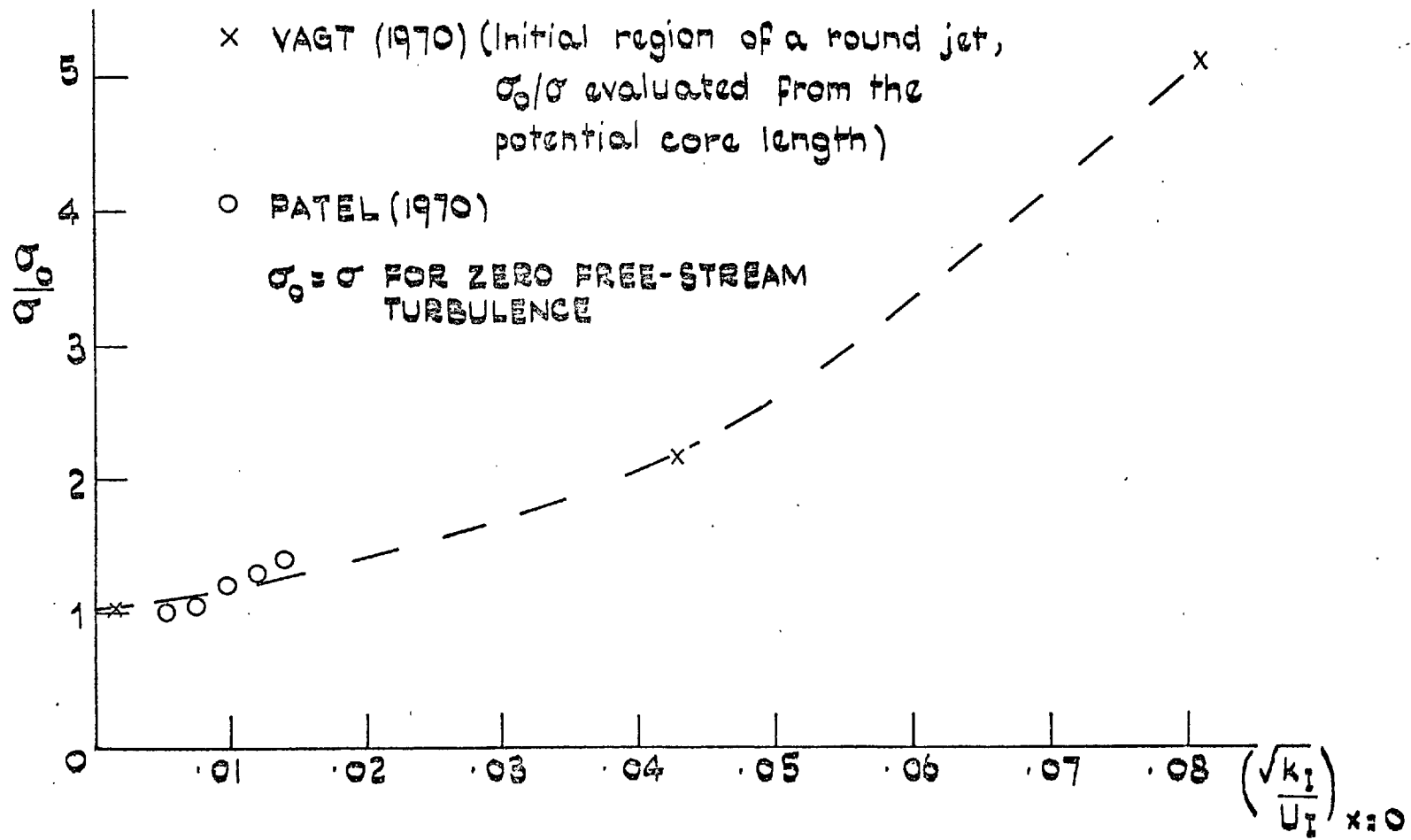


Fig. 5.3 Influence of free-stream turbulence on the rate of
spread of a mixing layer with $U_E = 0$ (measurements)

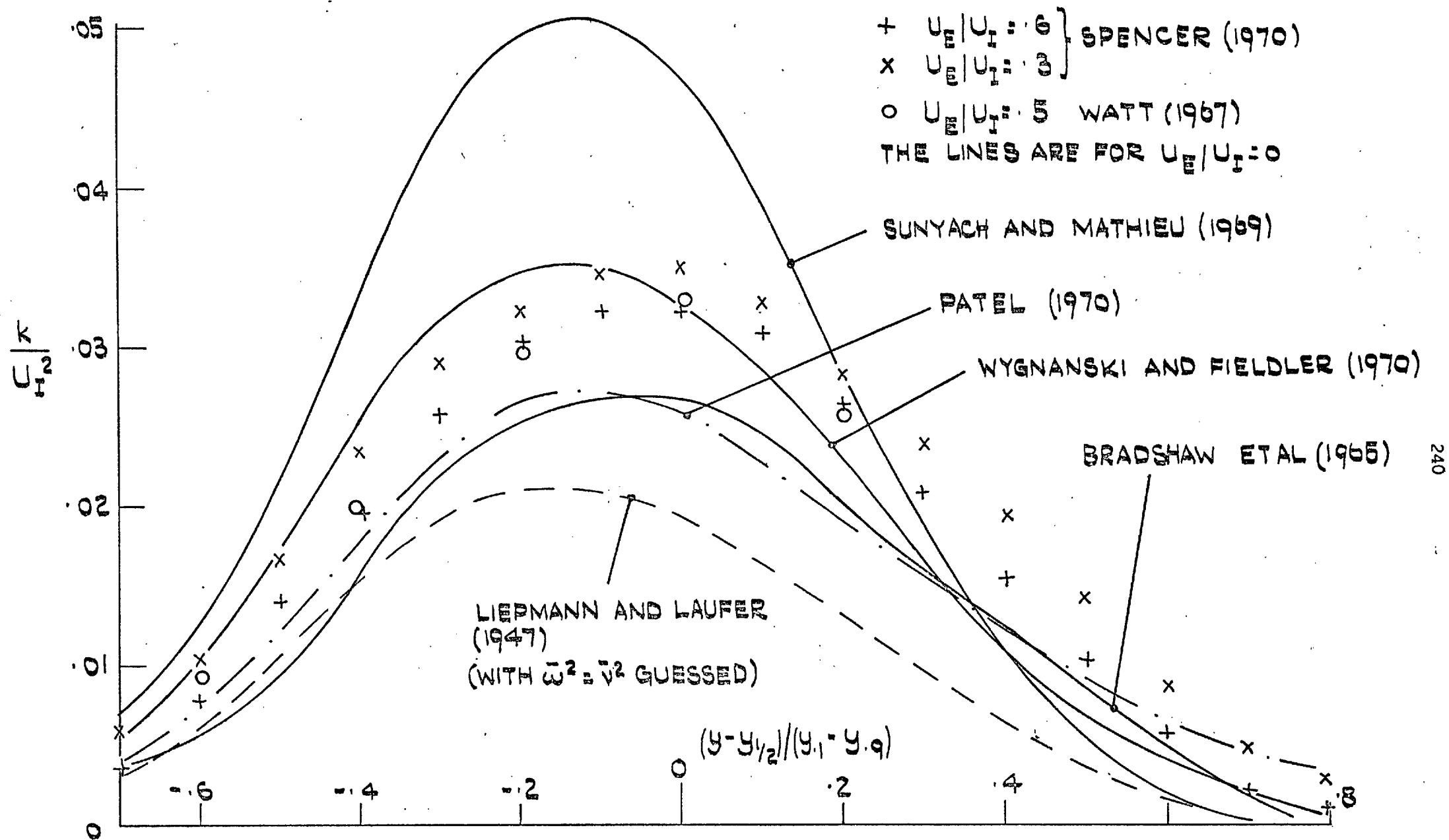


Fig. 5.4

Measured k-profiles in self-similar mixing layers

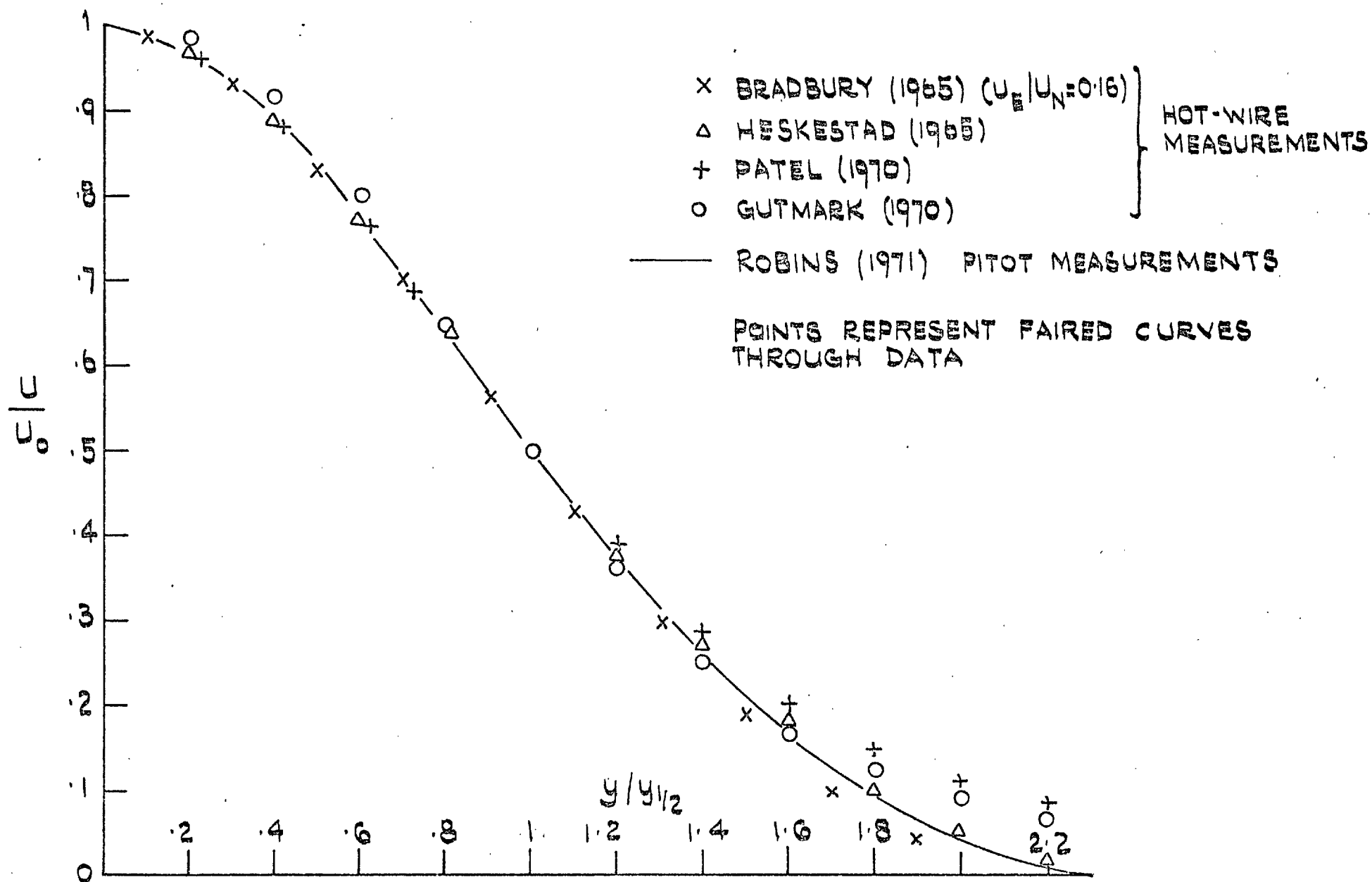


Fig. 5.5 Measured velocity profiles in the self-similar plane jet issuing into still air

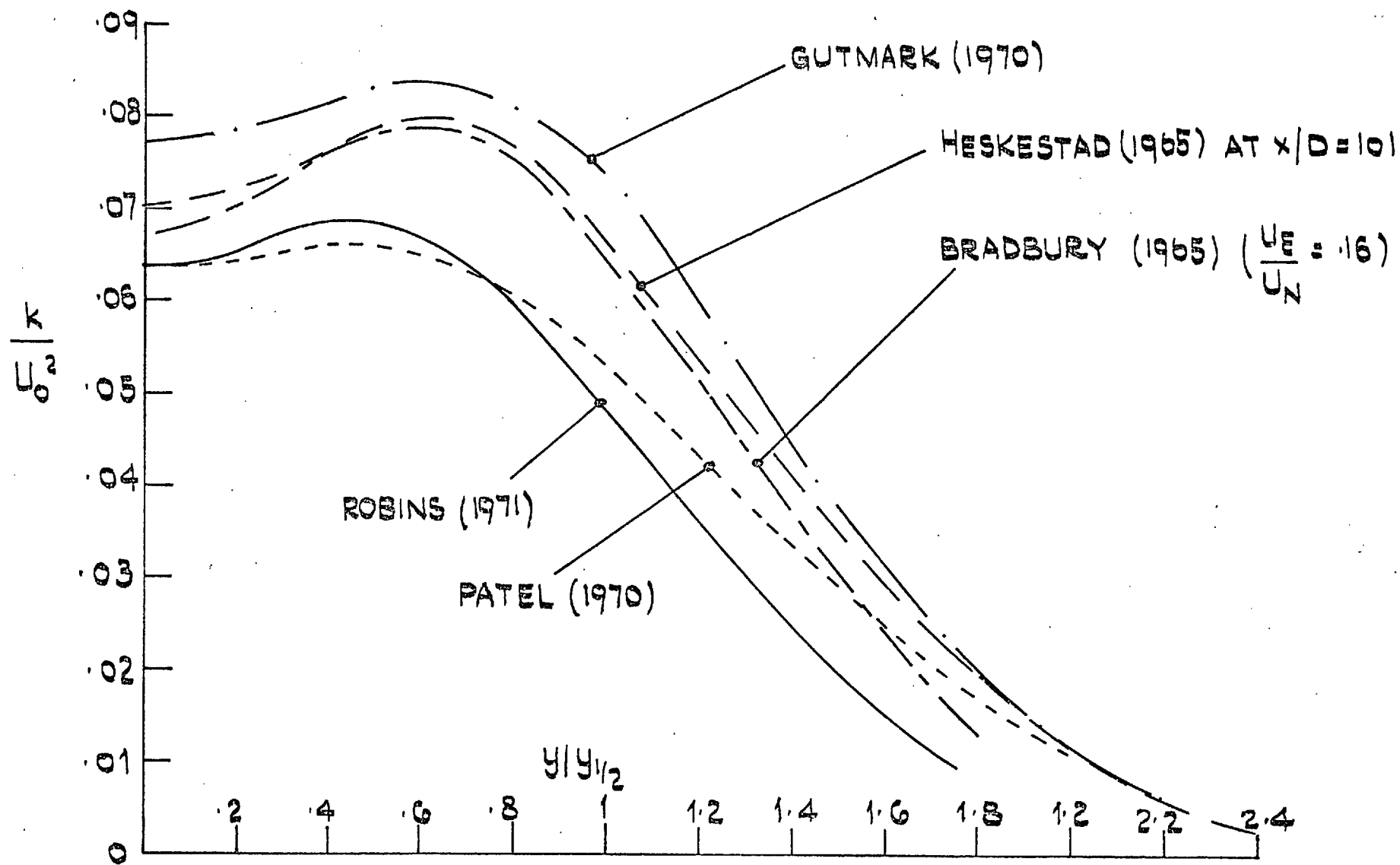


Fig. 5.6 Measured k -profiles in the self-similar plane jet issuing into still air

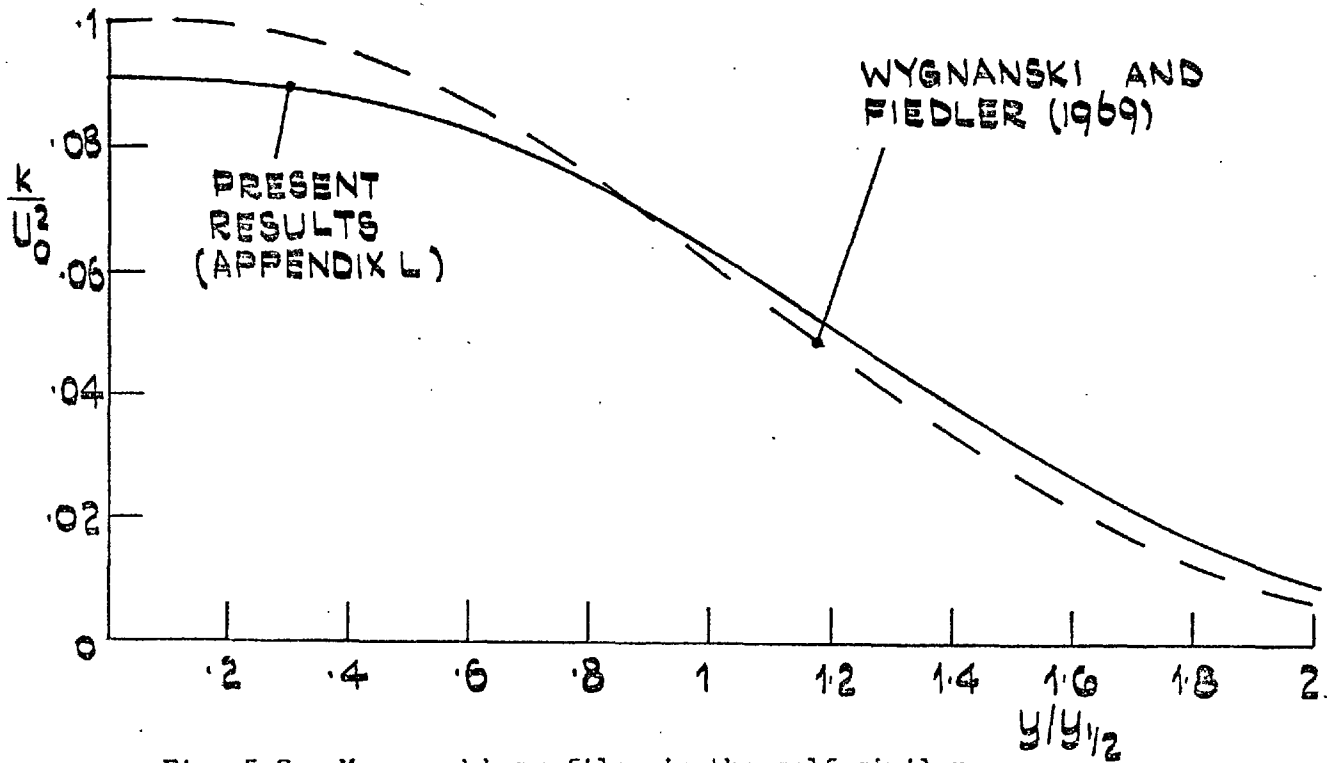


Fig. 5.7 Measured k -profiles in the self-similar round jet issuing into still air

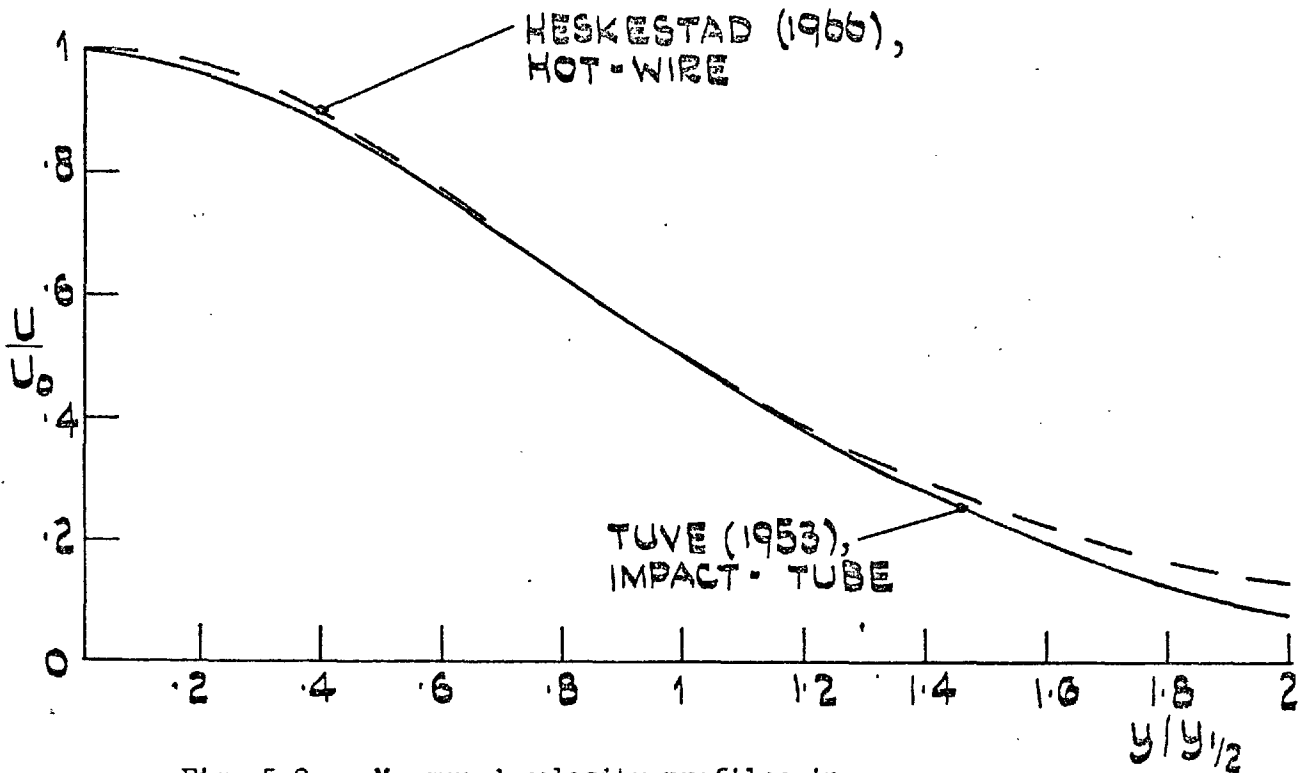


Fig. 5.8 Measured velocity profiles in a self-similar radial jet

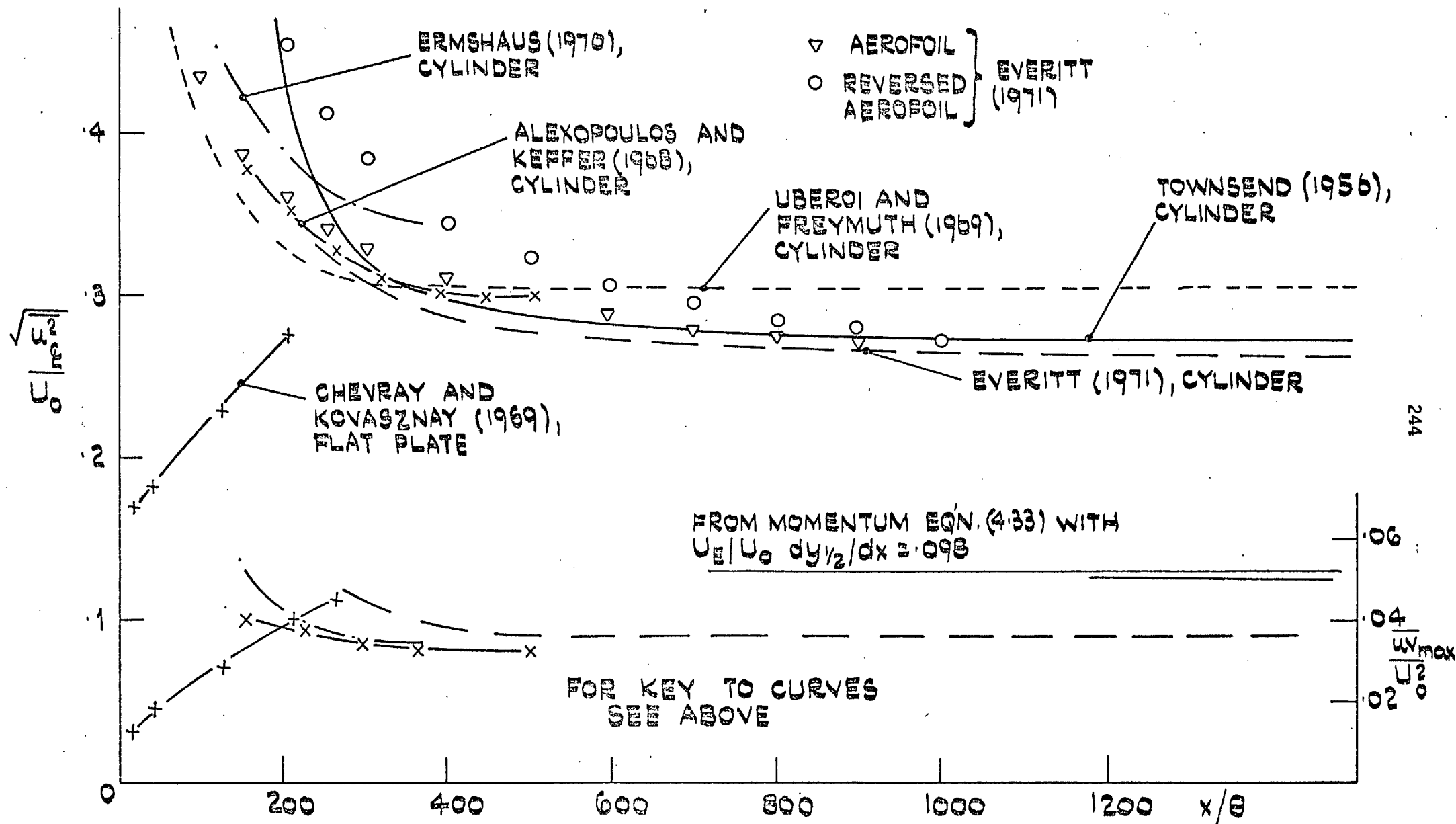
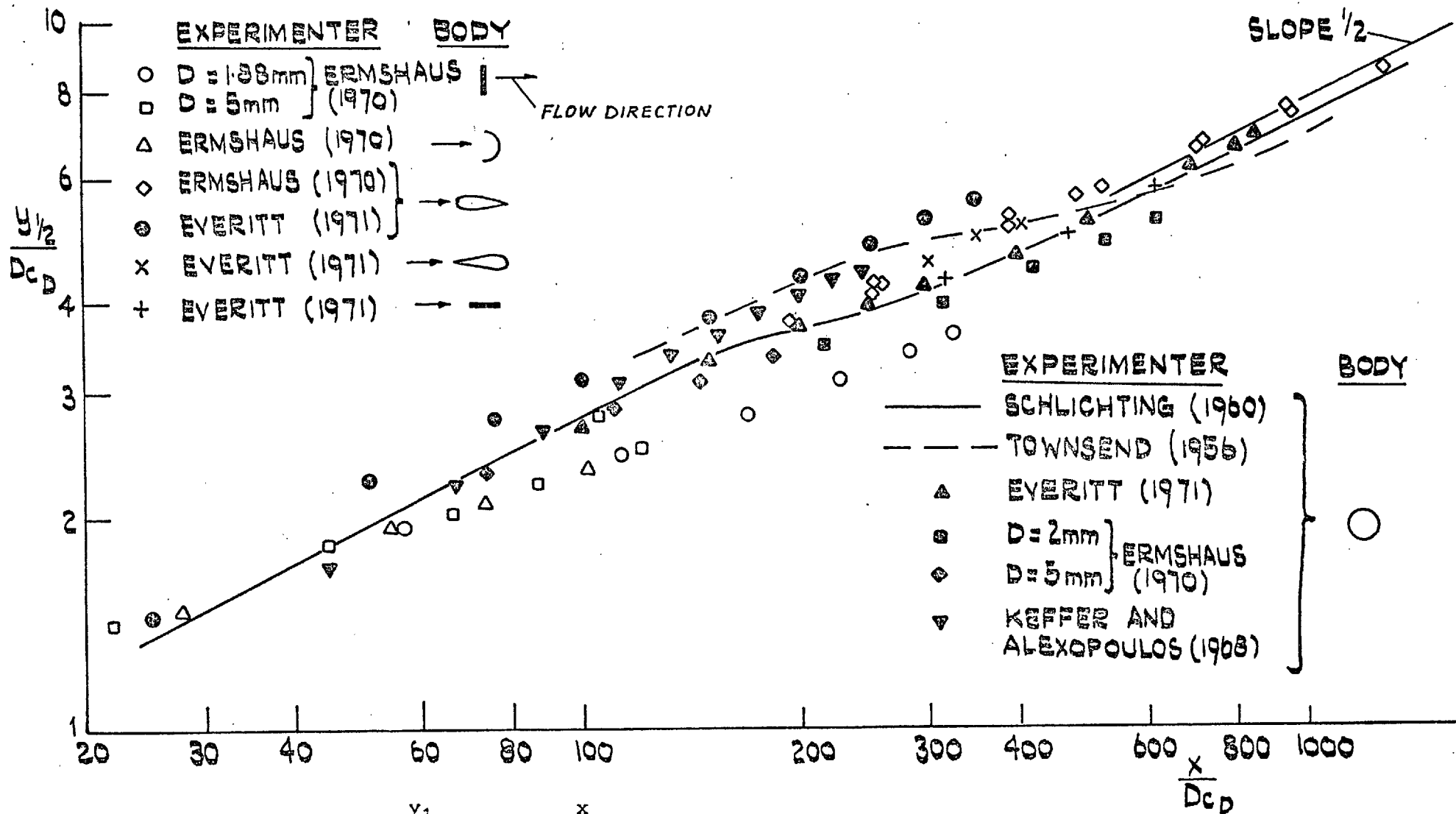


Fig. 5.9

Centr-plane intensity and maximum shear stress in plane wakes



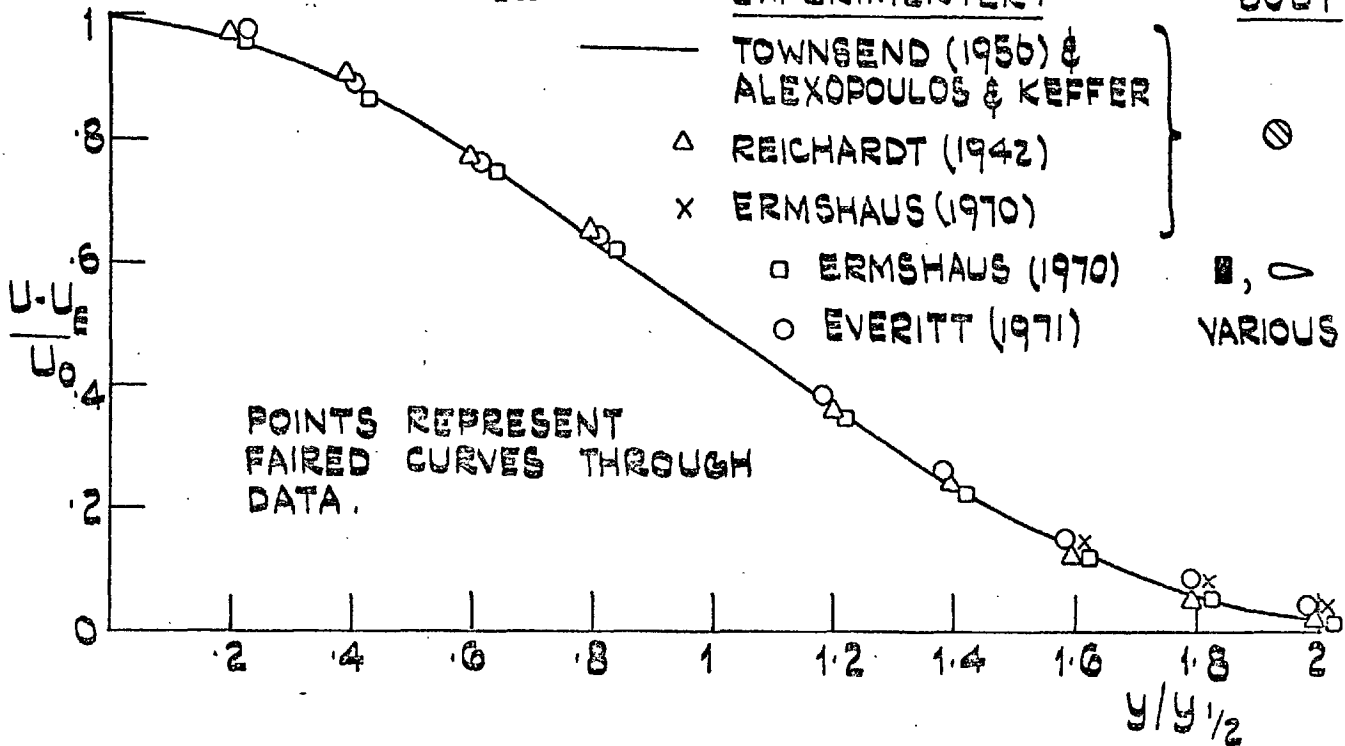


Fig. 5.11 Measured velocity profiles in self-similar plane wakes

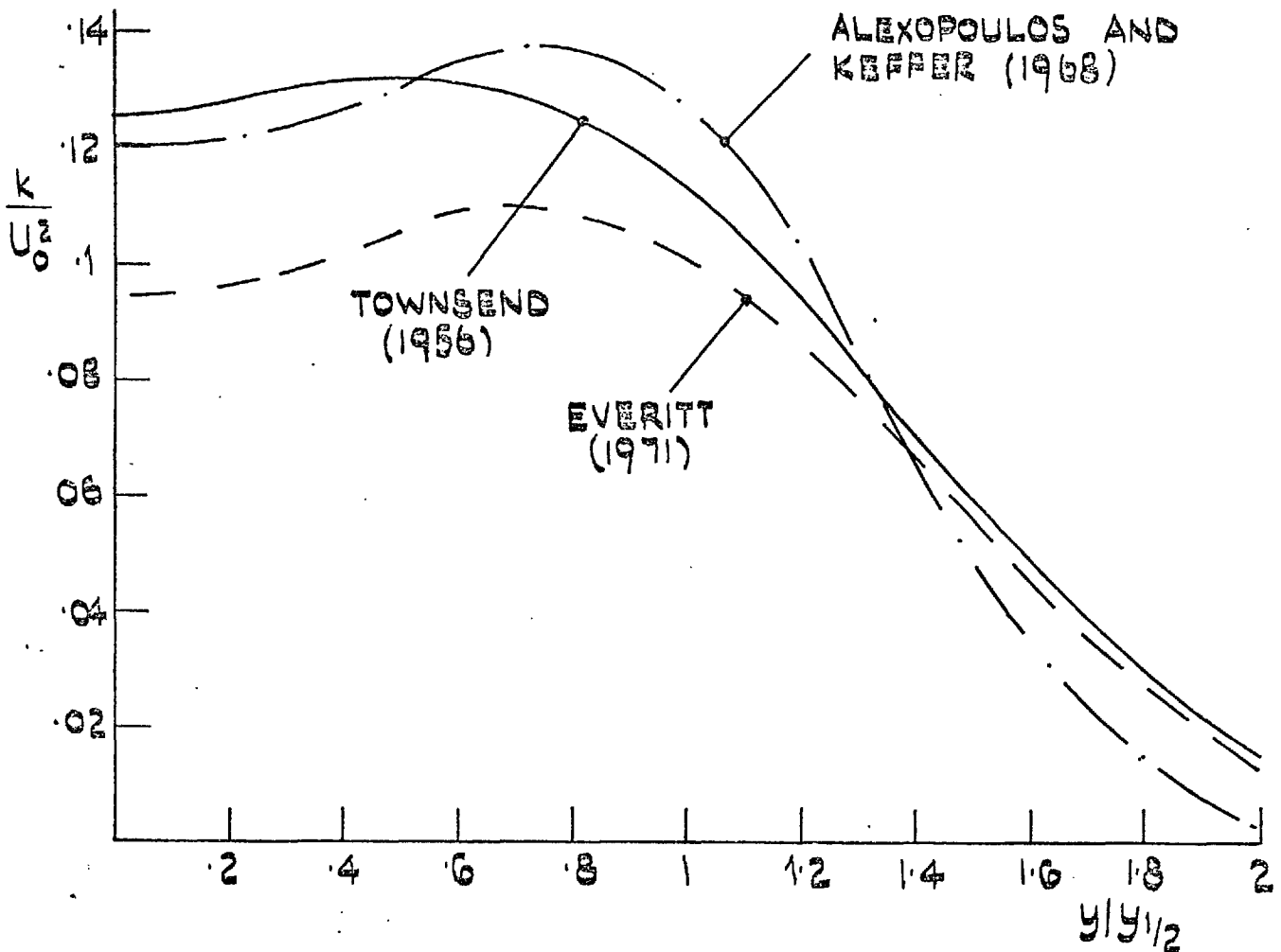


Fig. 5.12 Measured k-profiles in self-similar plane wakes (behind cylinders)

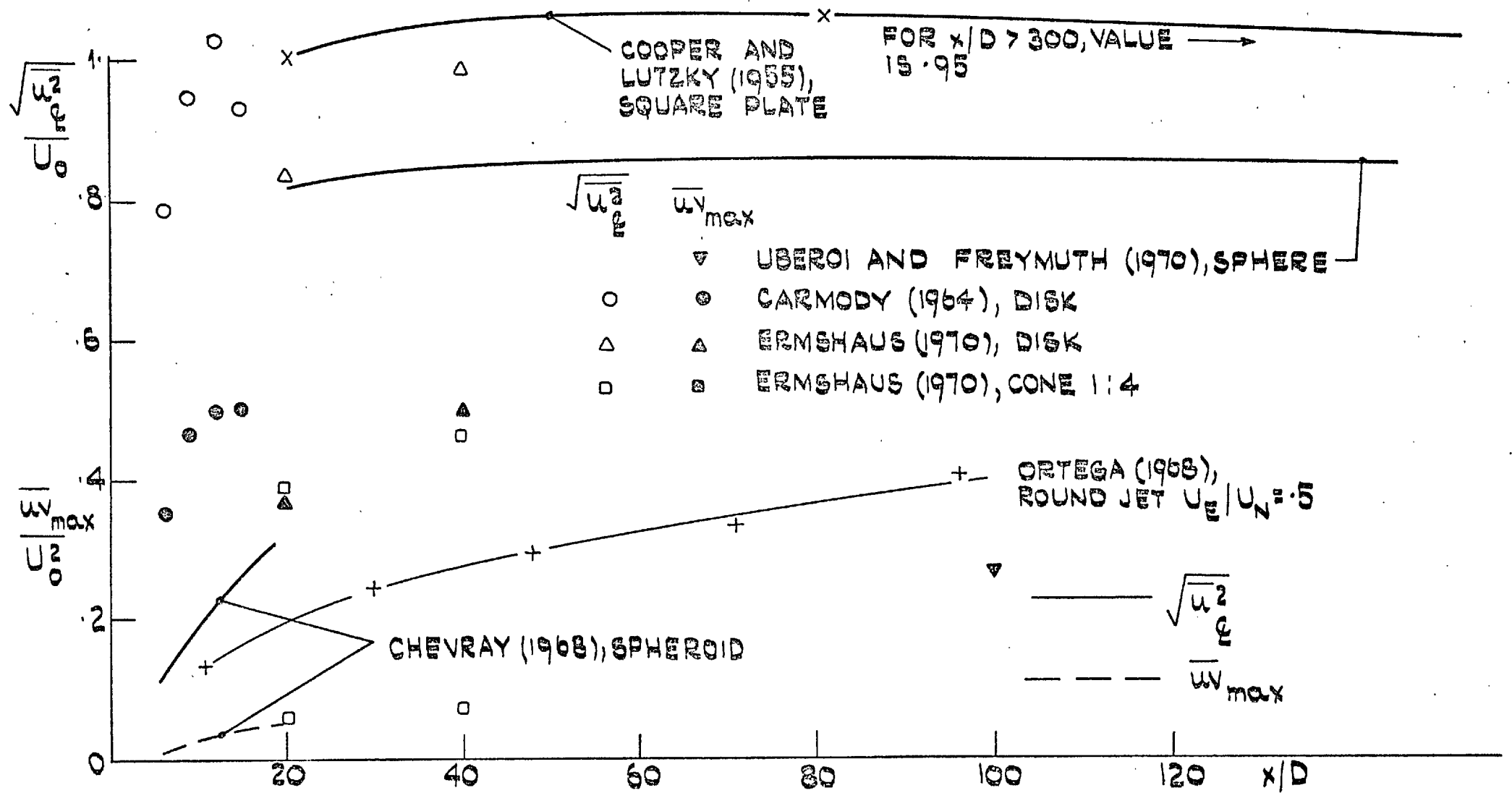


Fig. 5.13

Centre-line intensity and maximum shear stress in round wakes

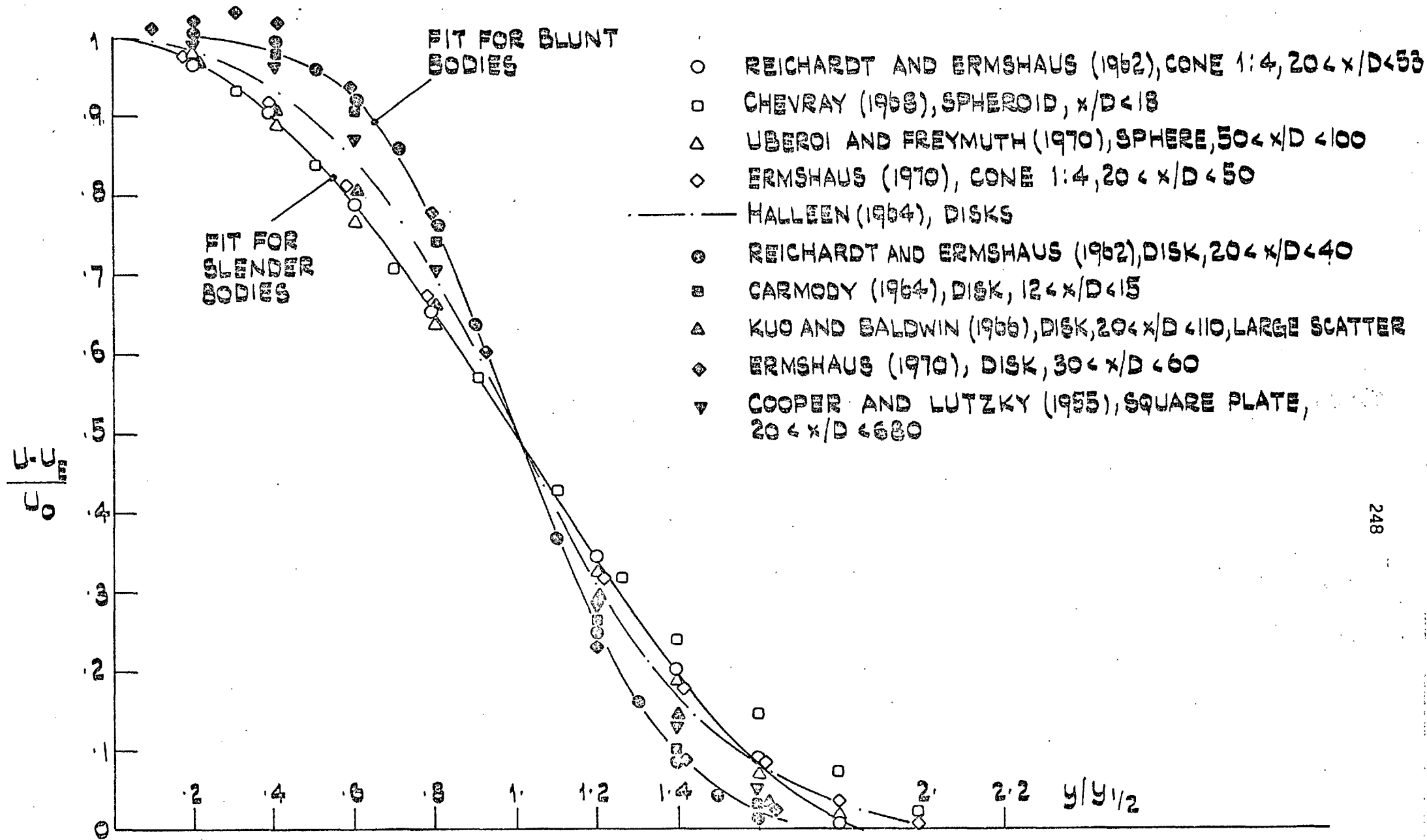


Fig. 5.14

Measured velocity profiles in self-similar round wakes

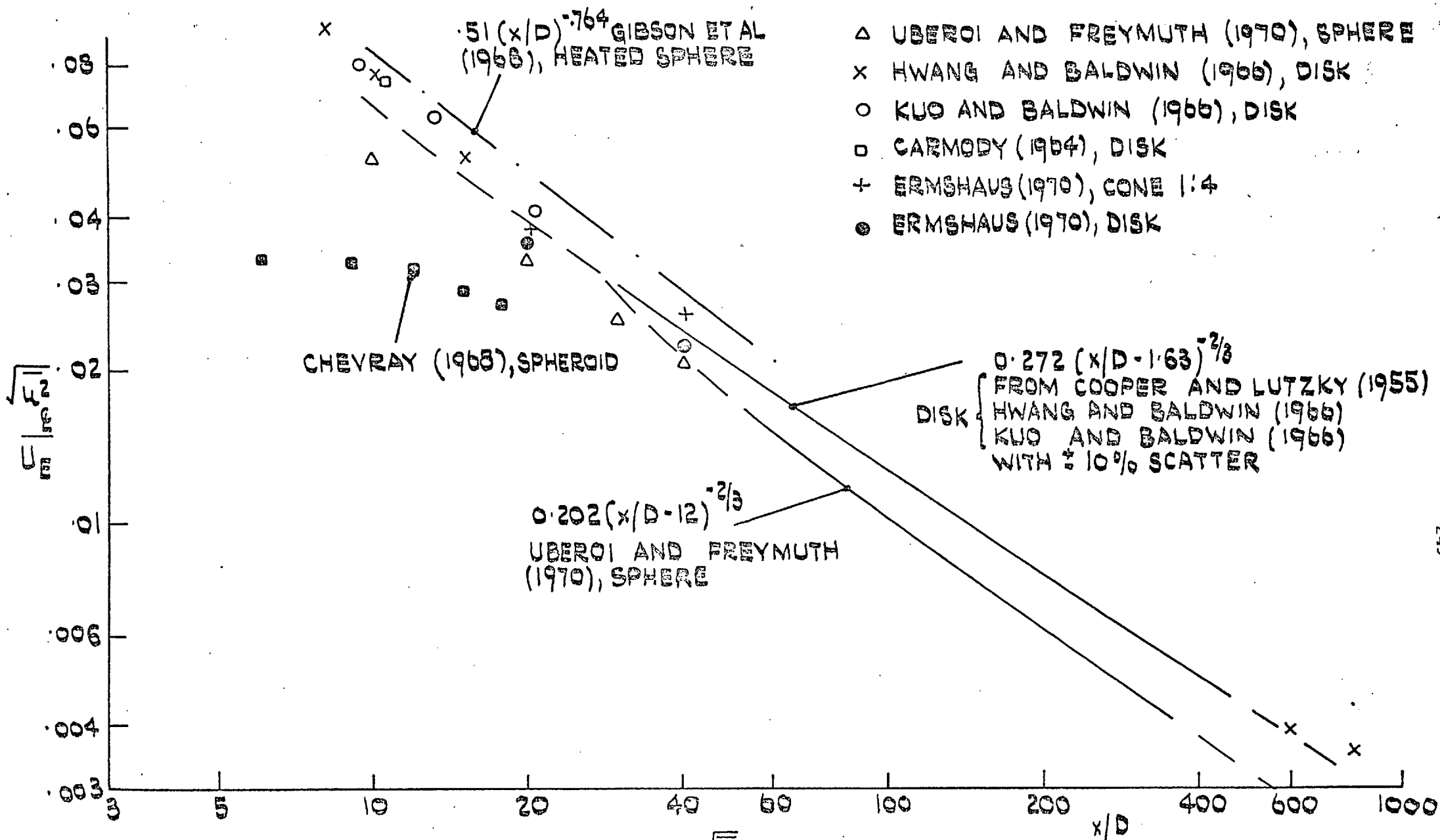


Fig. 5.15 Axial decay of $\sqrt{u'^2}$ in round wakes

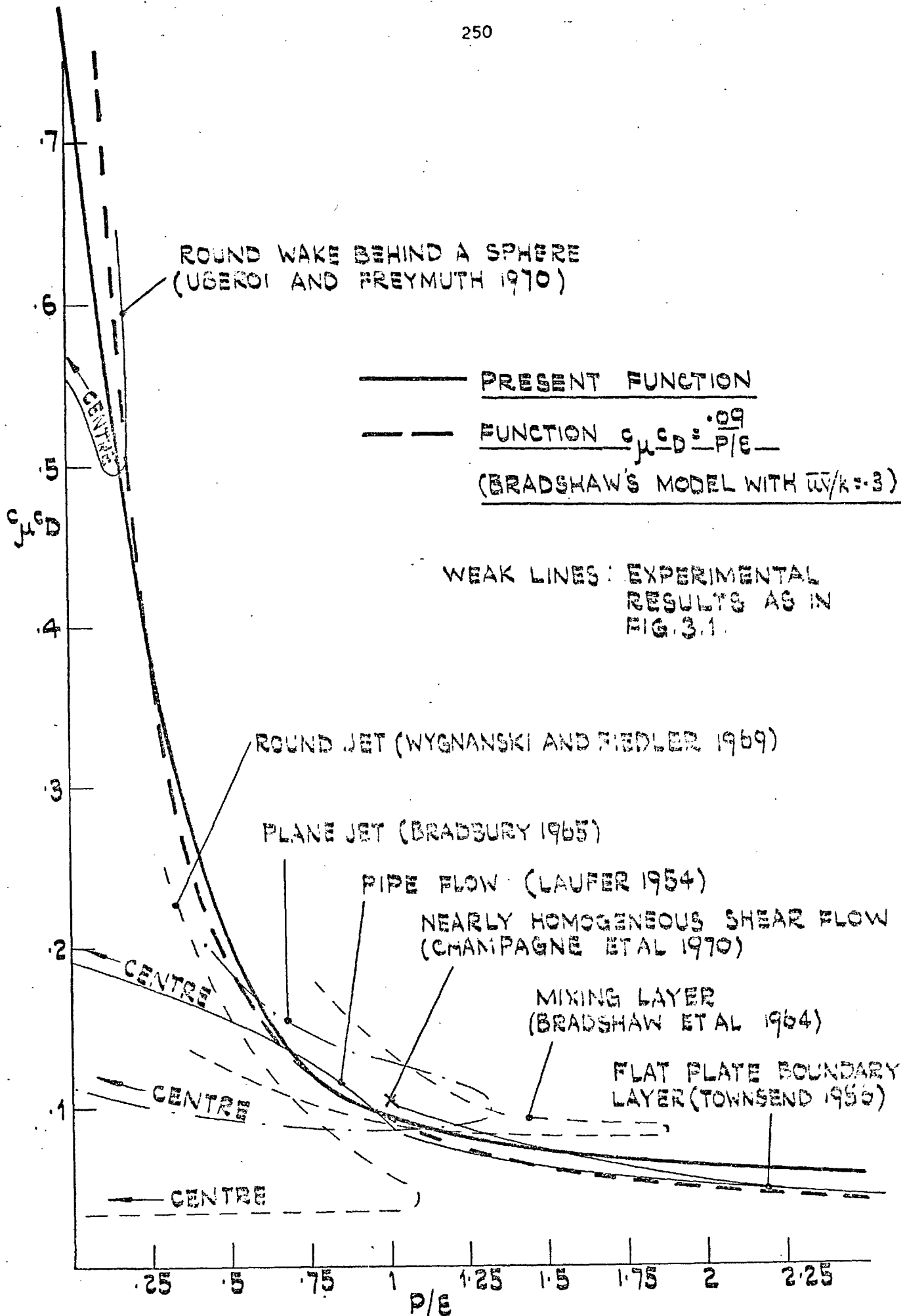
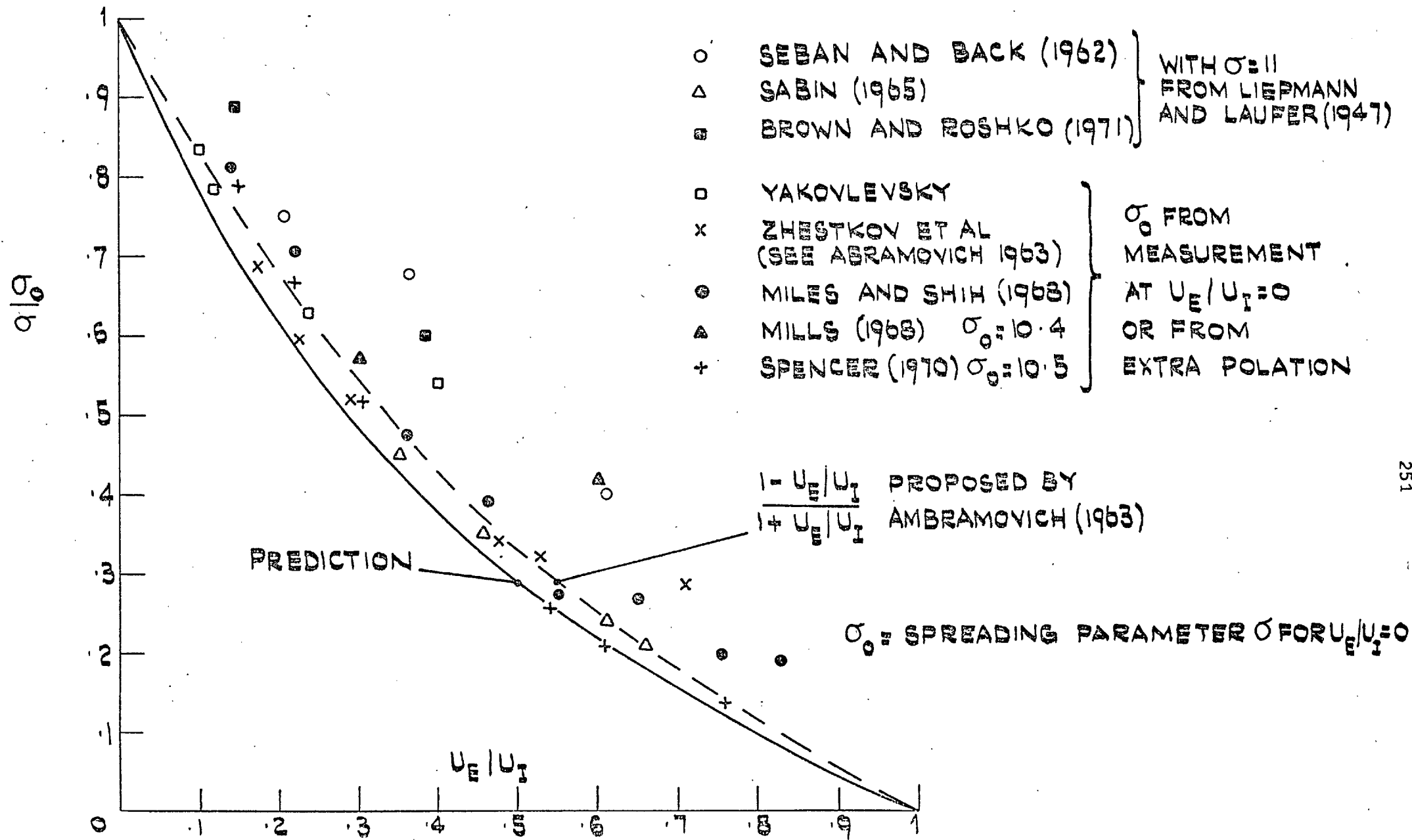


Fig. 6.1 The function $c_{\mu} c_D = f(P/\epsilon)$ employed


 Fig. 6.2 Influence of the velocity ratio U_E/U_I on the rate of spread of a mixing layer

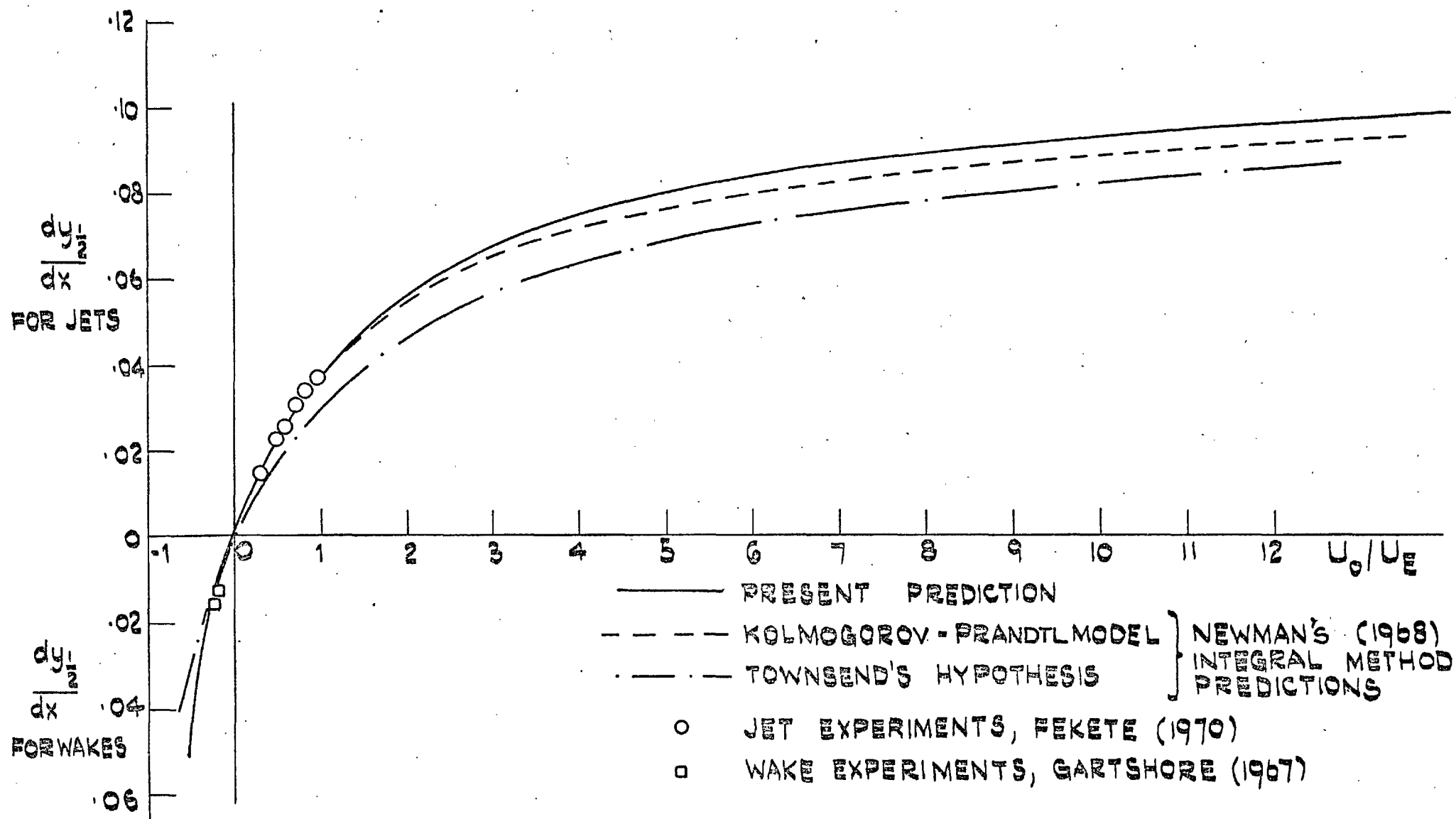


Fig. 6.3 Influence of the velocity ratio U_0/U_E on the rate of spread of exactly self-similar jets and wakes

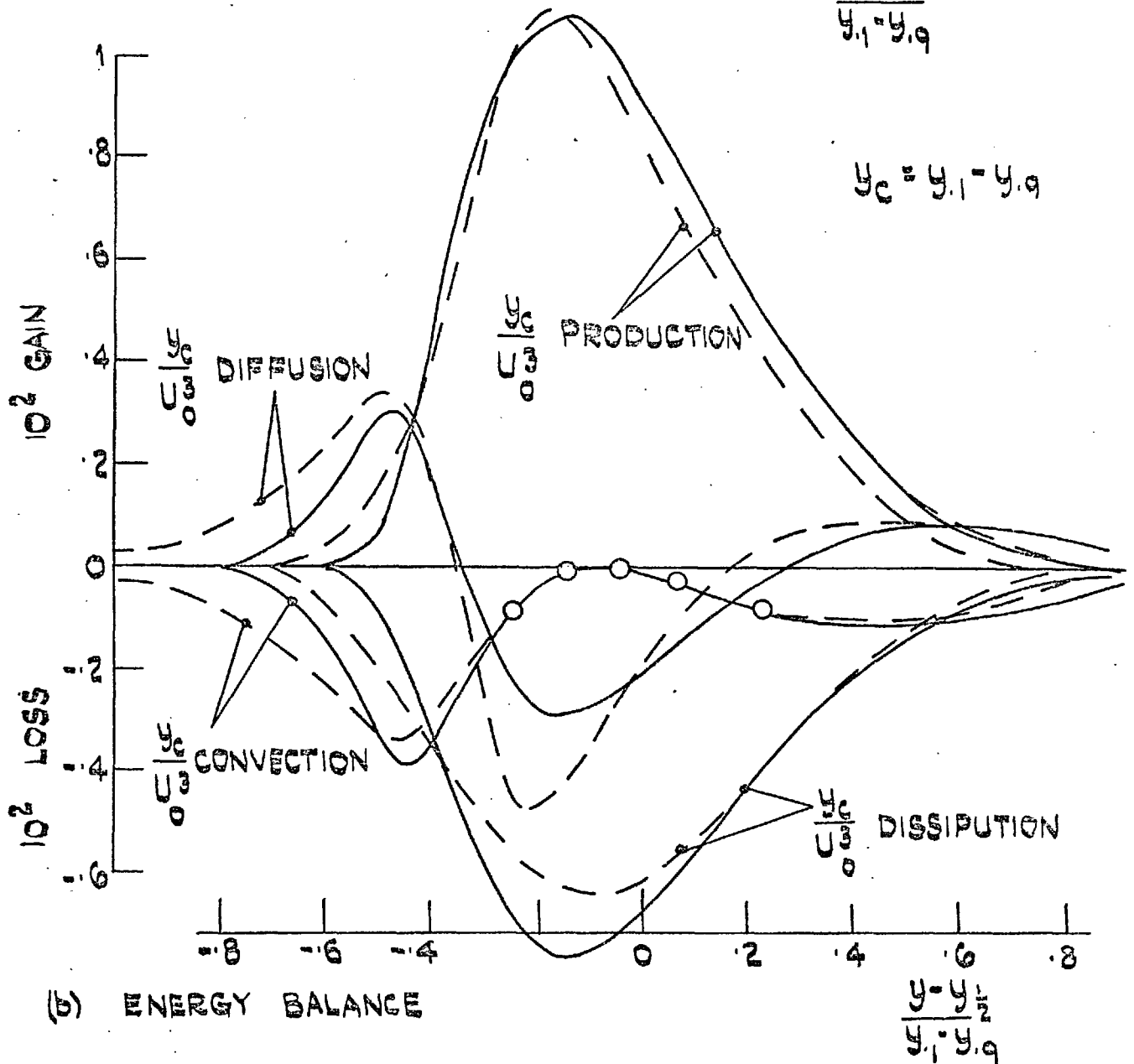
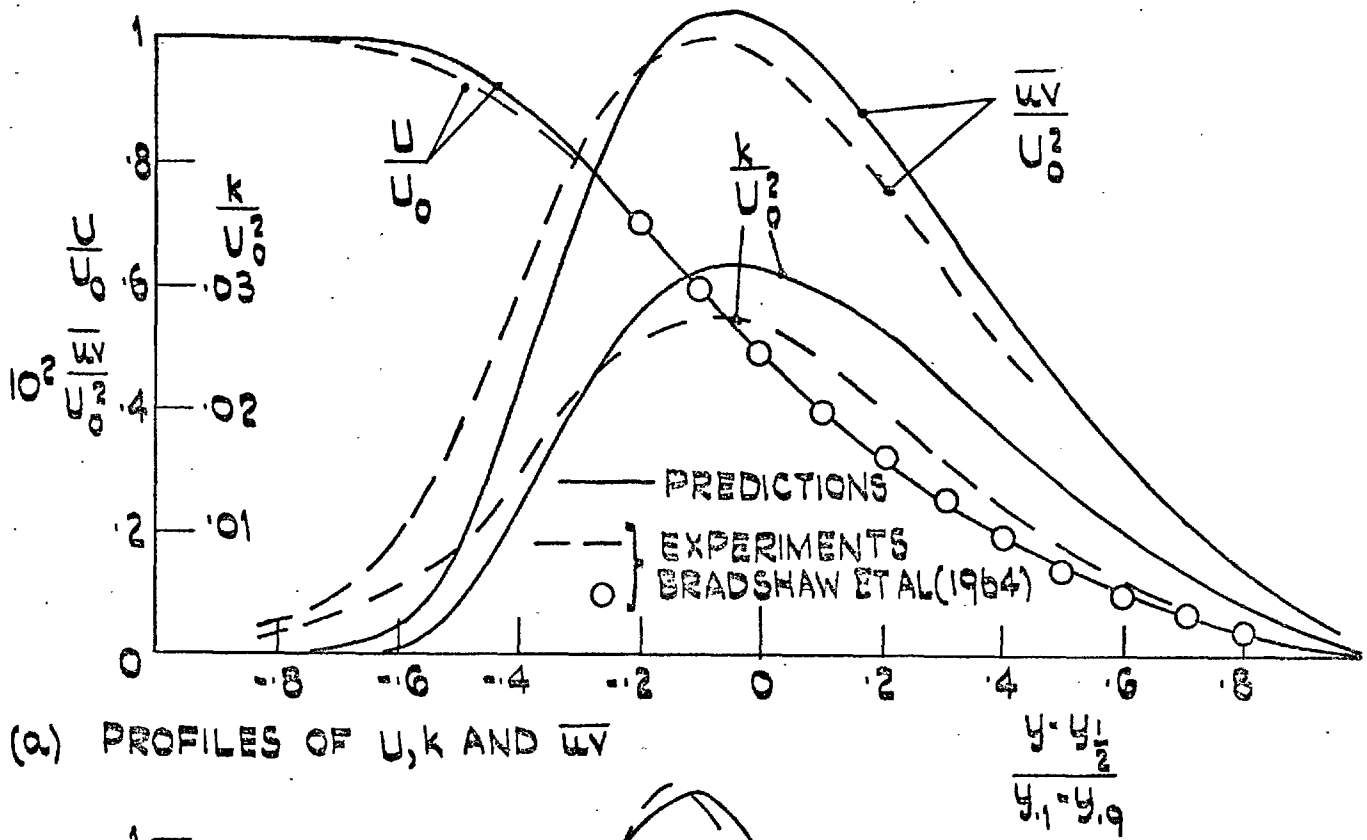


Fig. 6.4

Mixing layer with $U_E = 0$

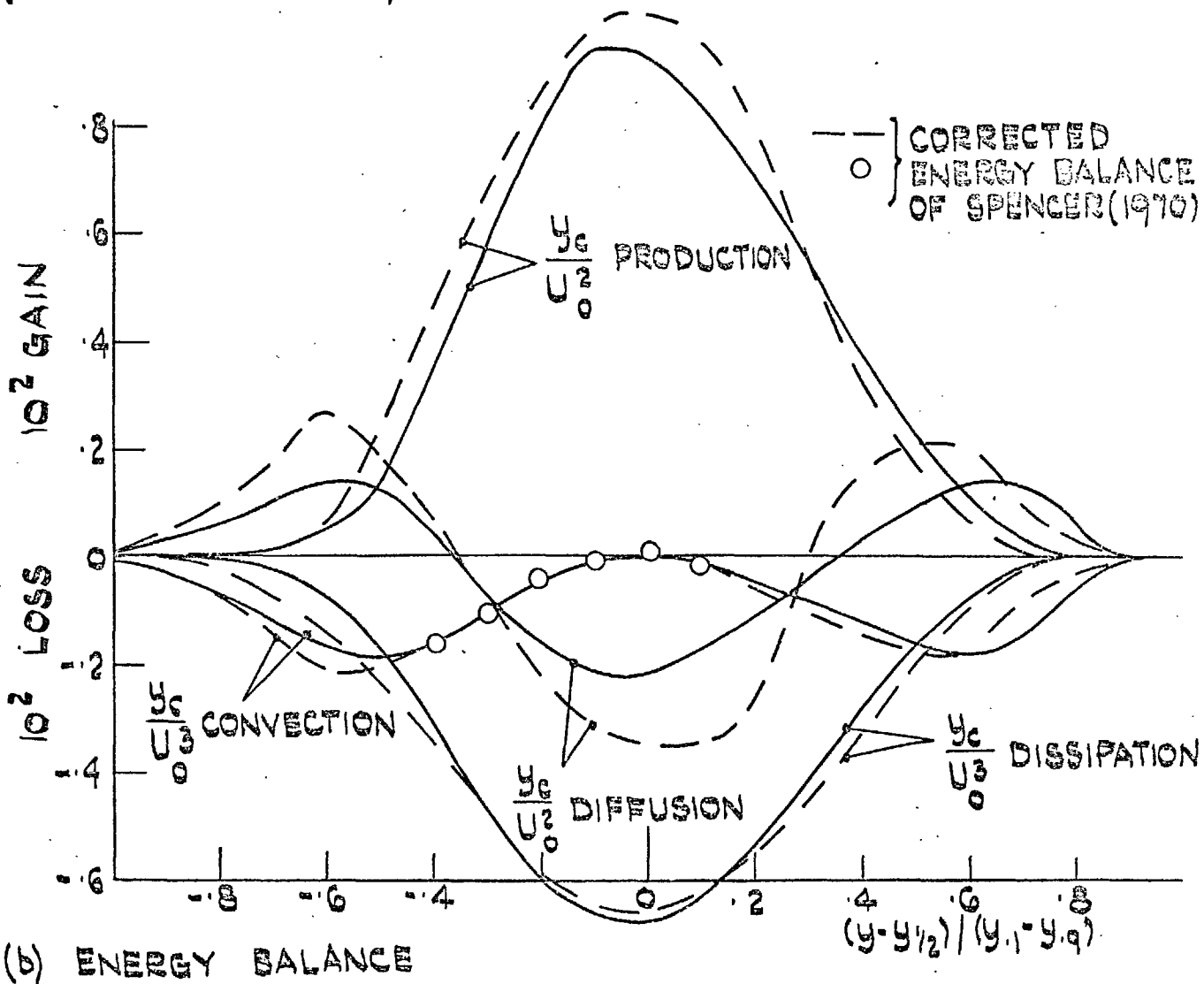
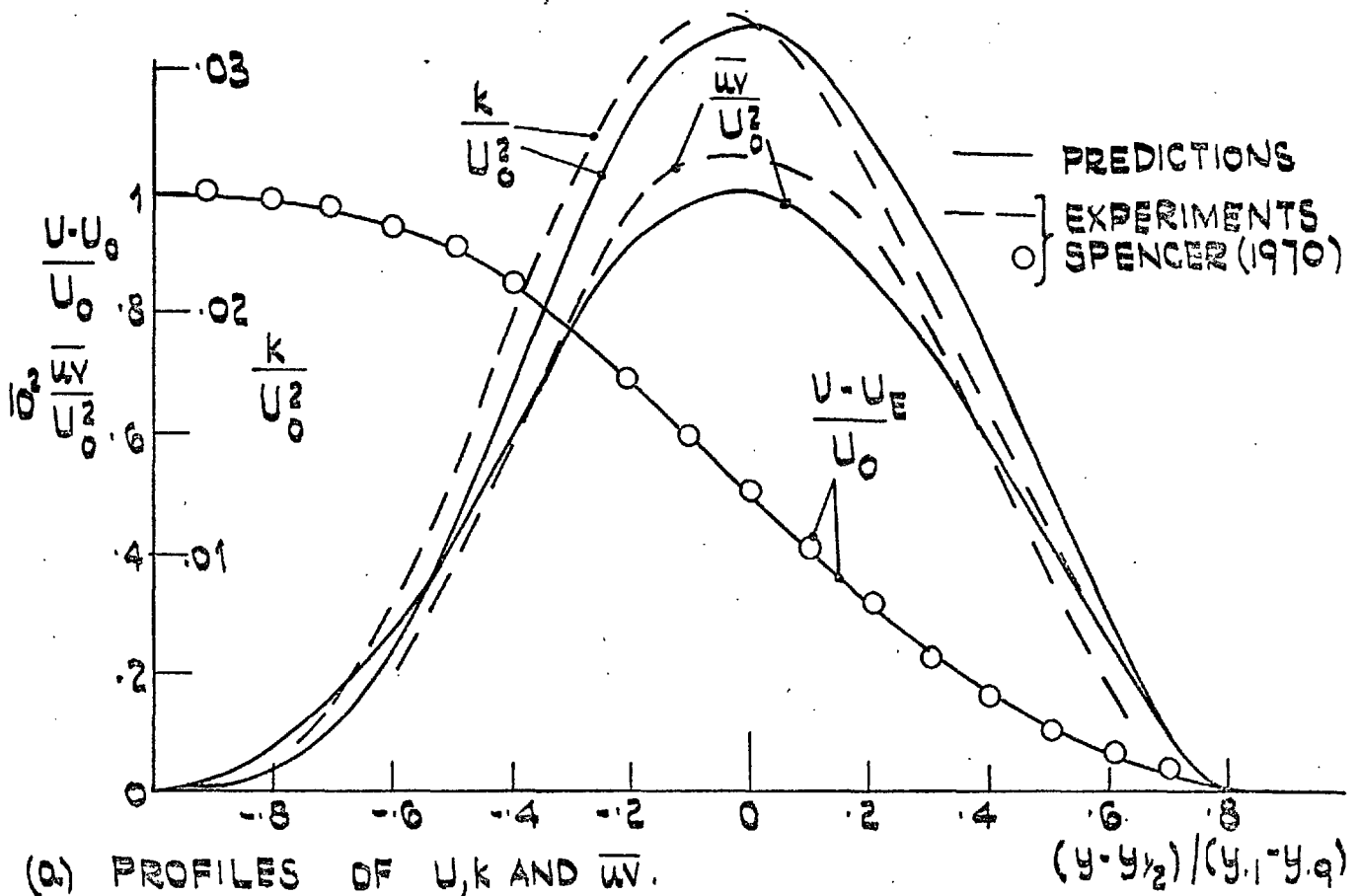


Fig 6.5 Plane mixing layer with $U_E/U_I = 0.6$

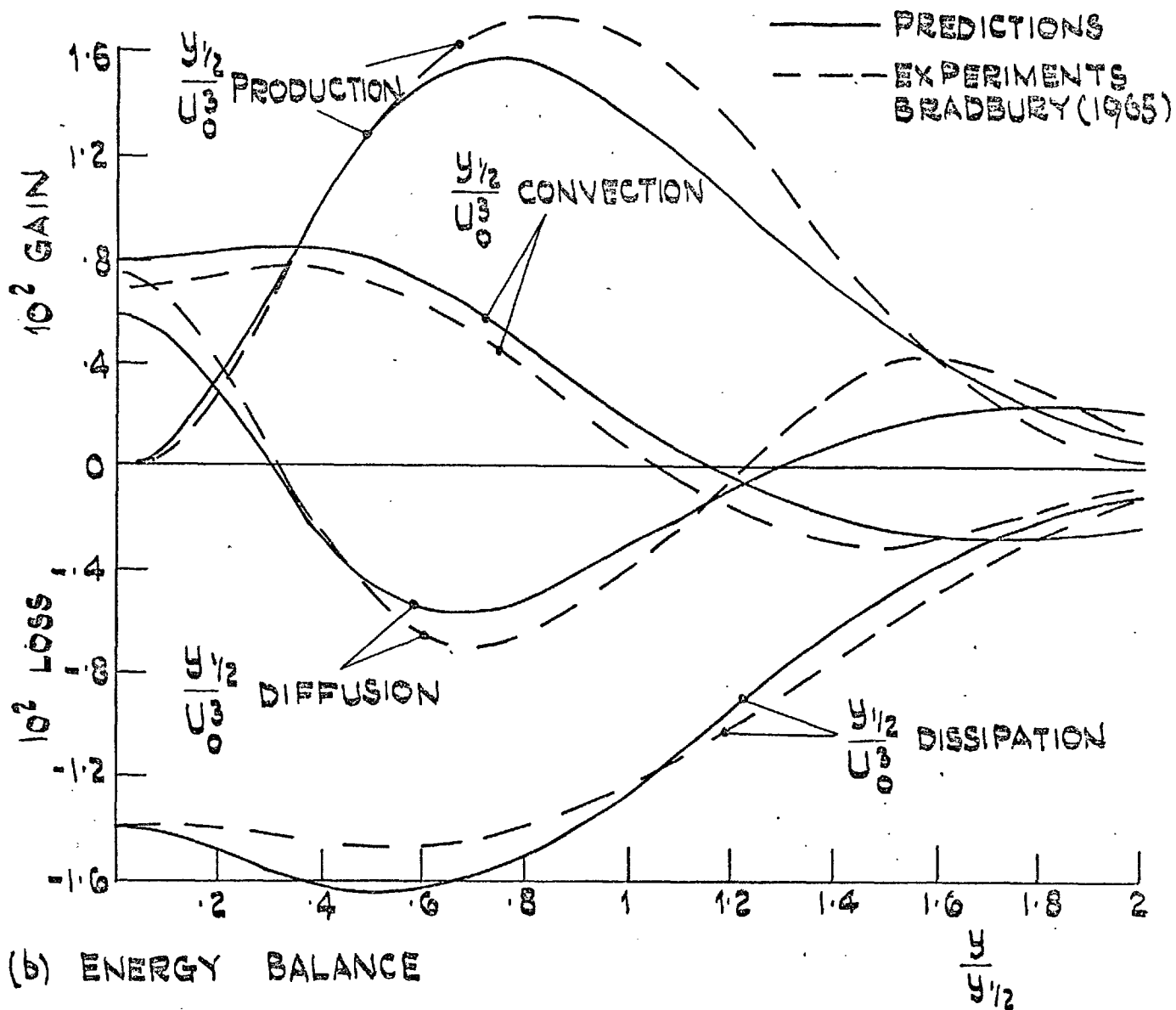
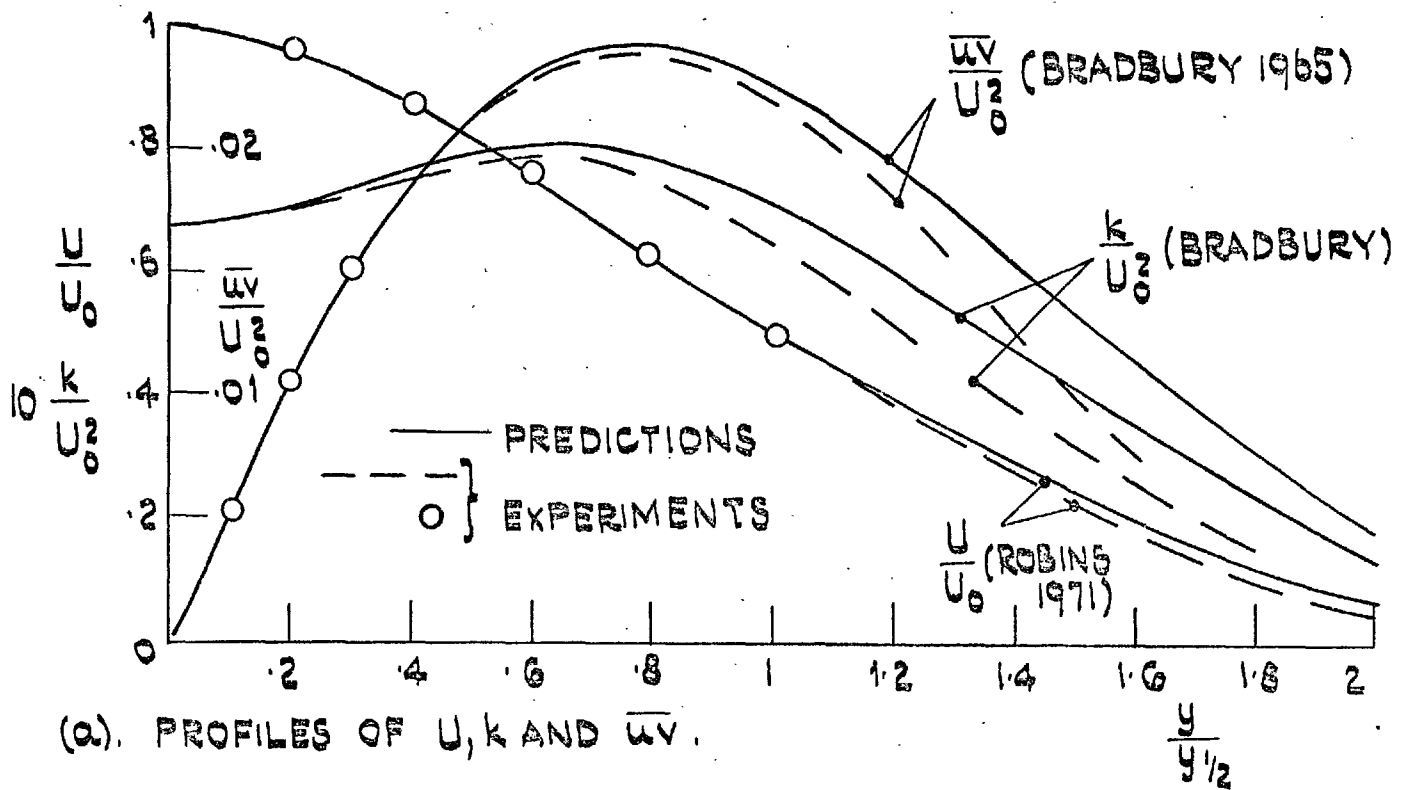


Fig. 6.6 Plane jet in stagnant surroundings (model I)

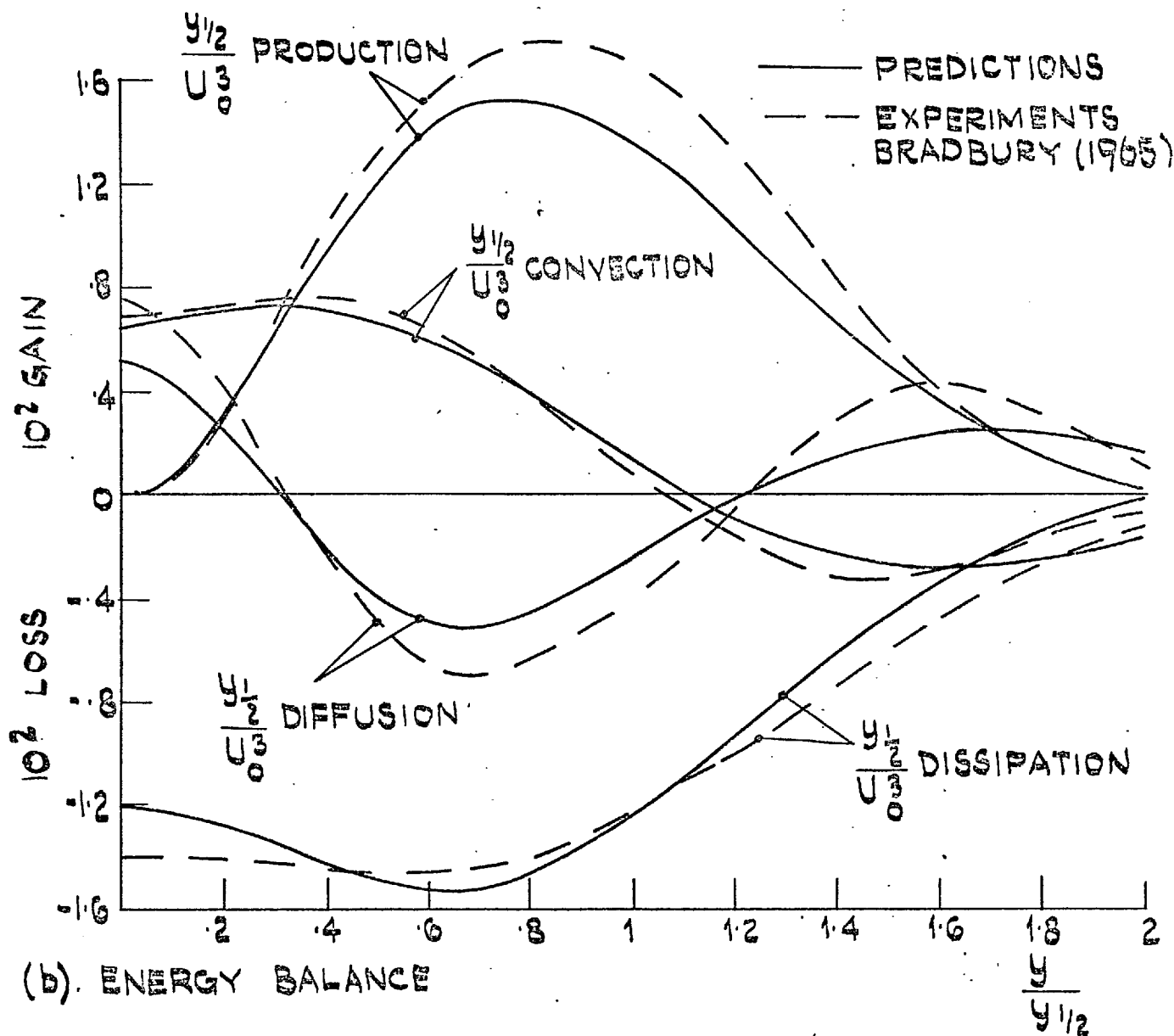
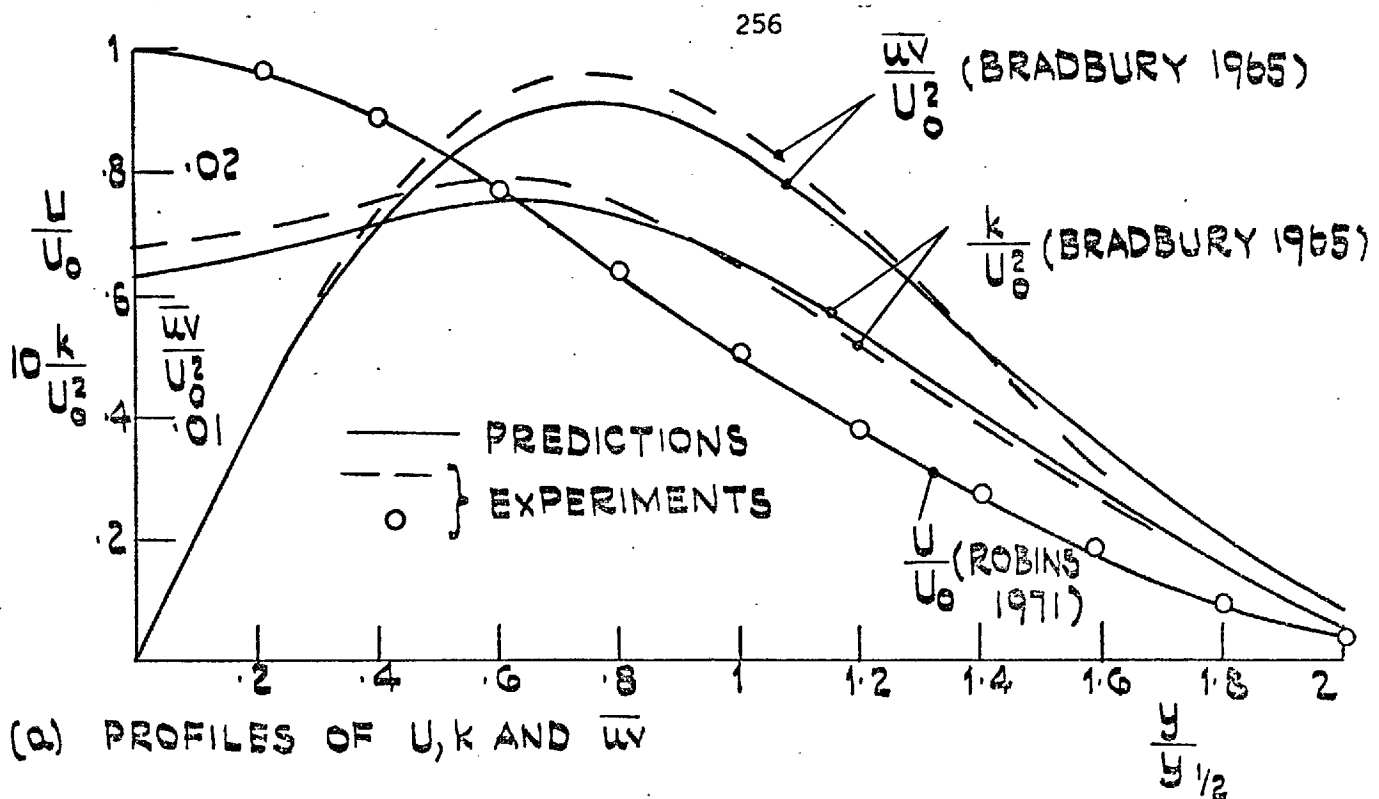


Fig. 6.7 Plane jet in stagnant surroundings (model III)

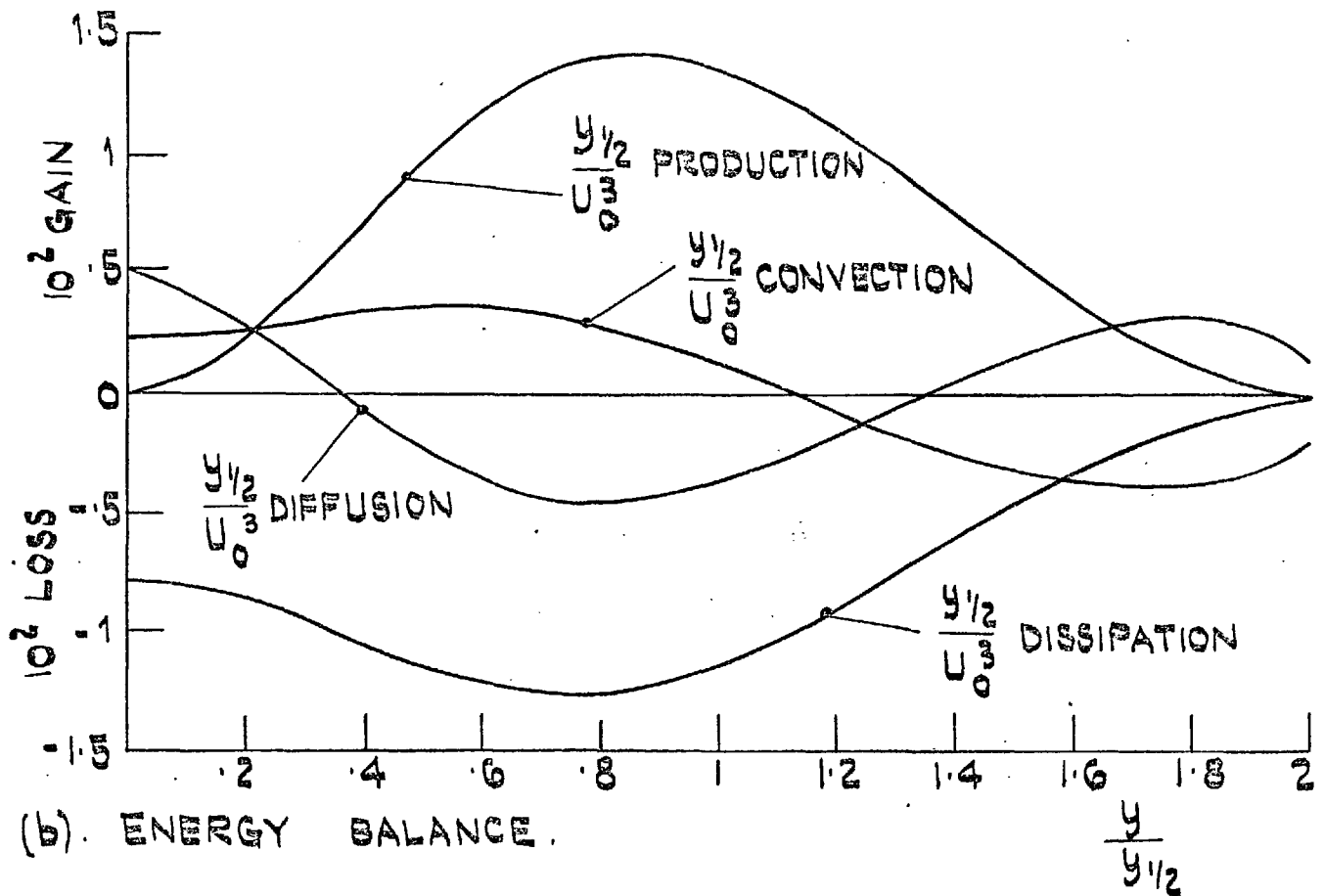
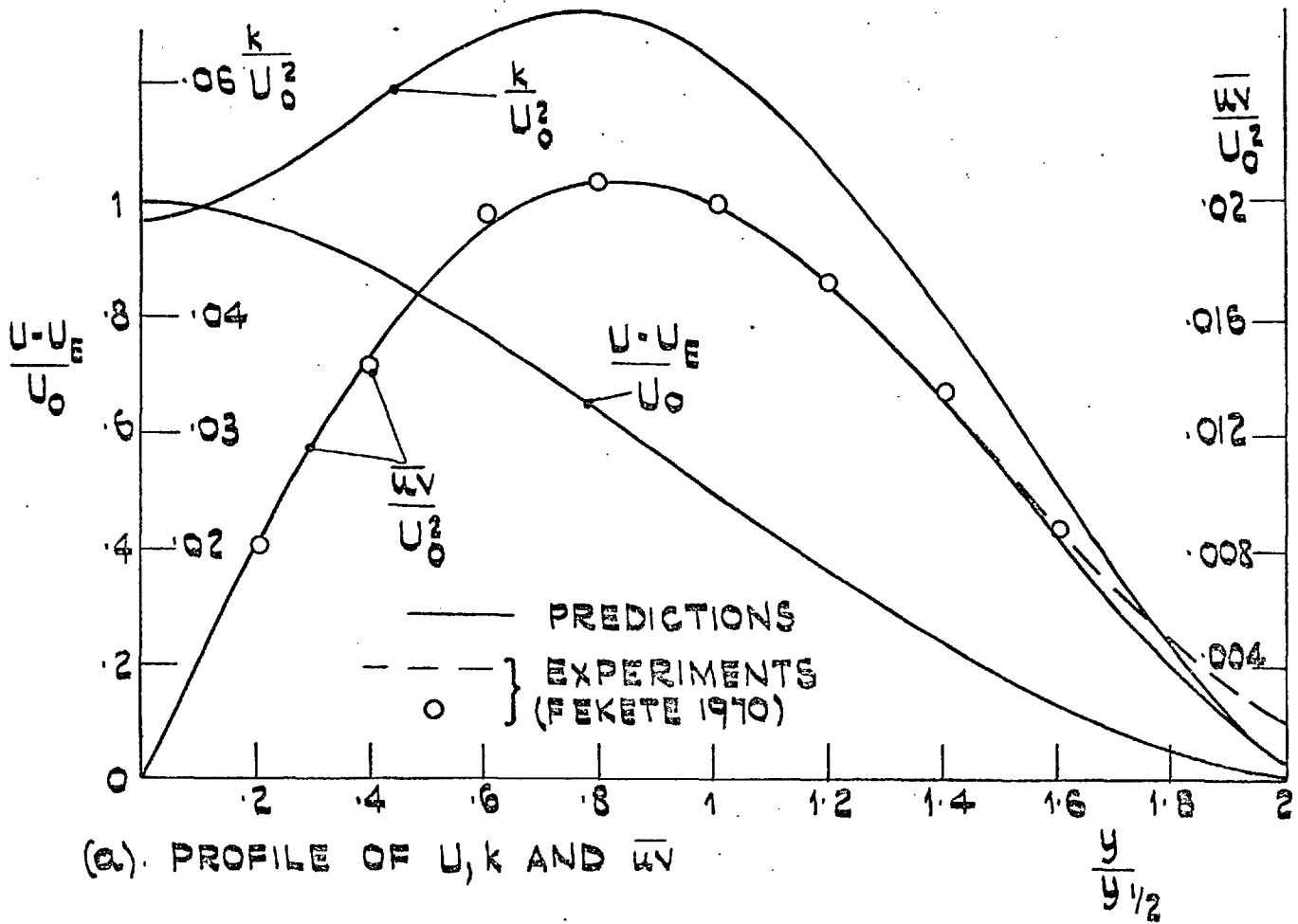
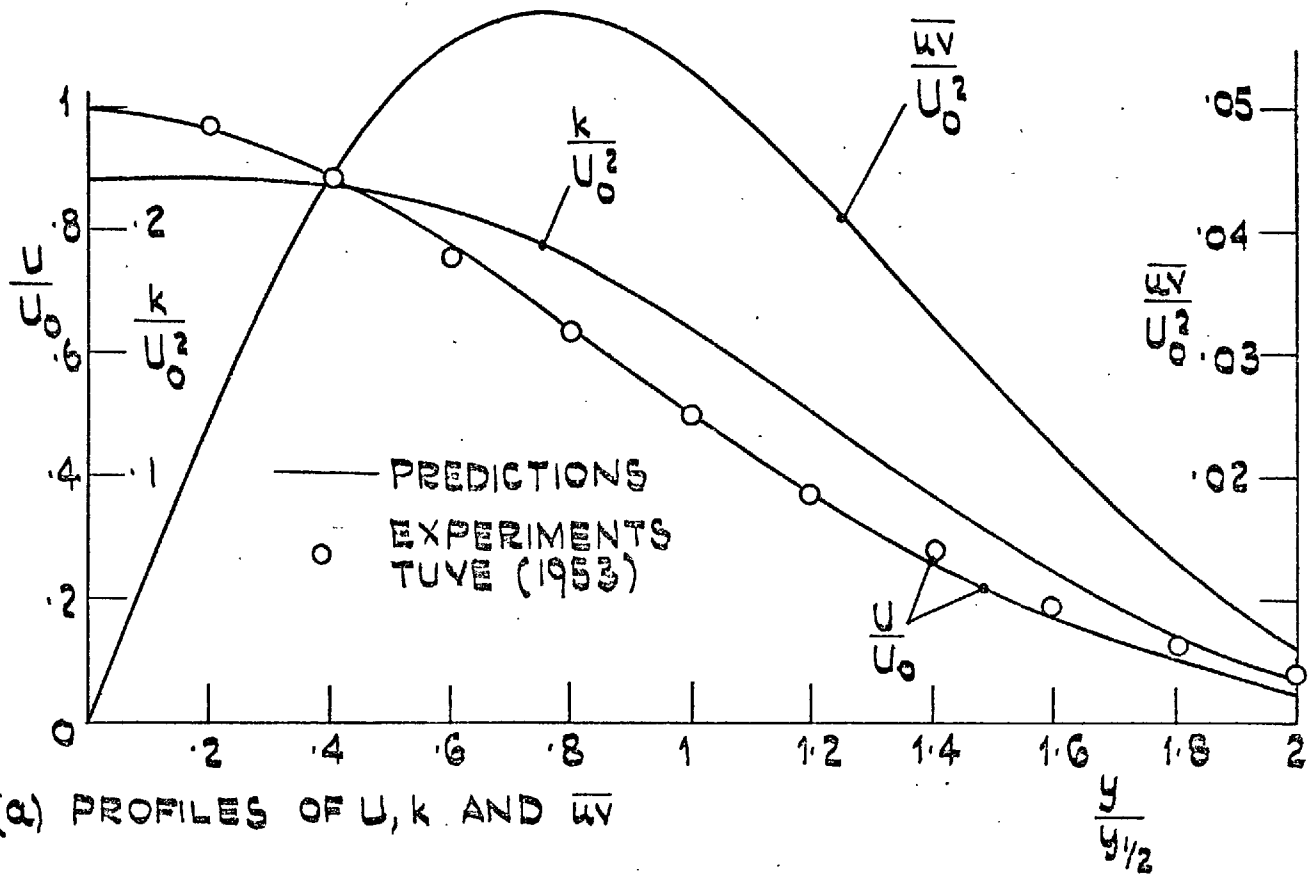
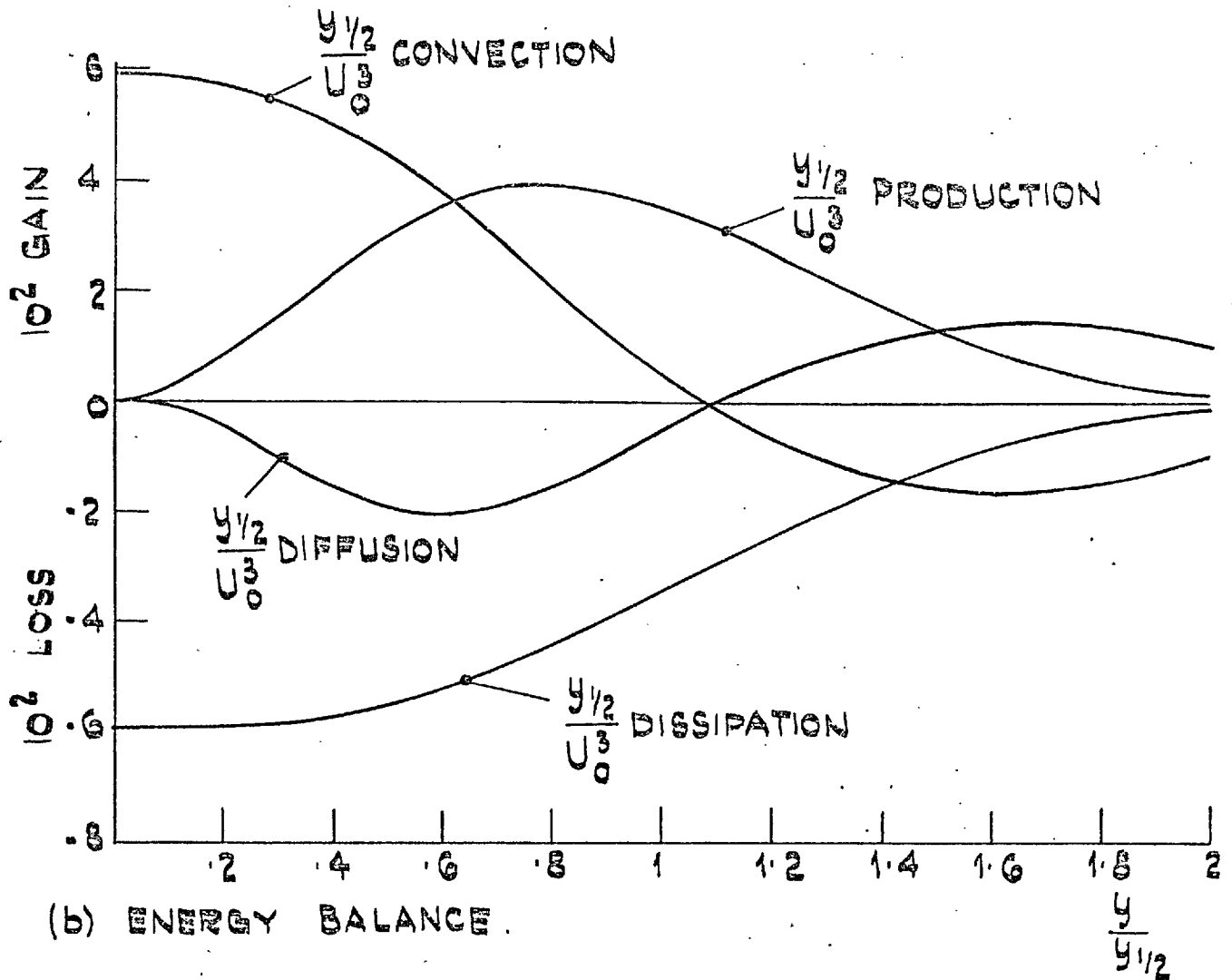


Fig. 6.8

Plane jet with $U_0/U_E = .59 = \text{const}$

(a) PROFILES OF U , k AND $\overline{uv}$ 

(b) ENERGY BALANCE.

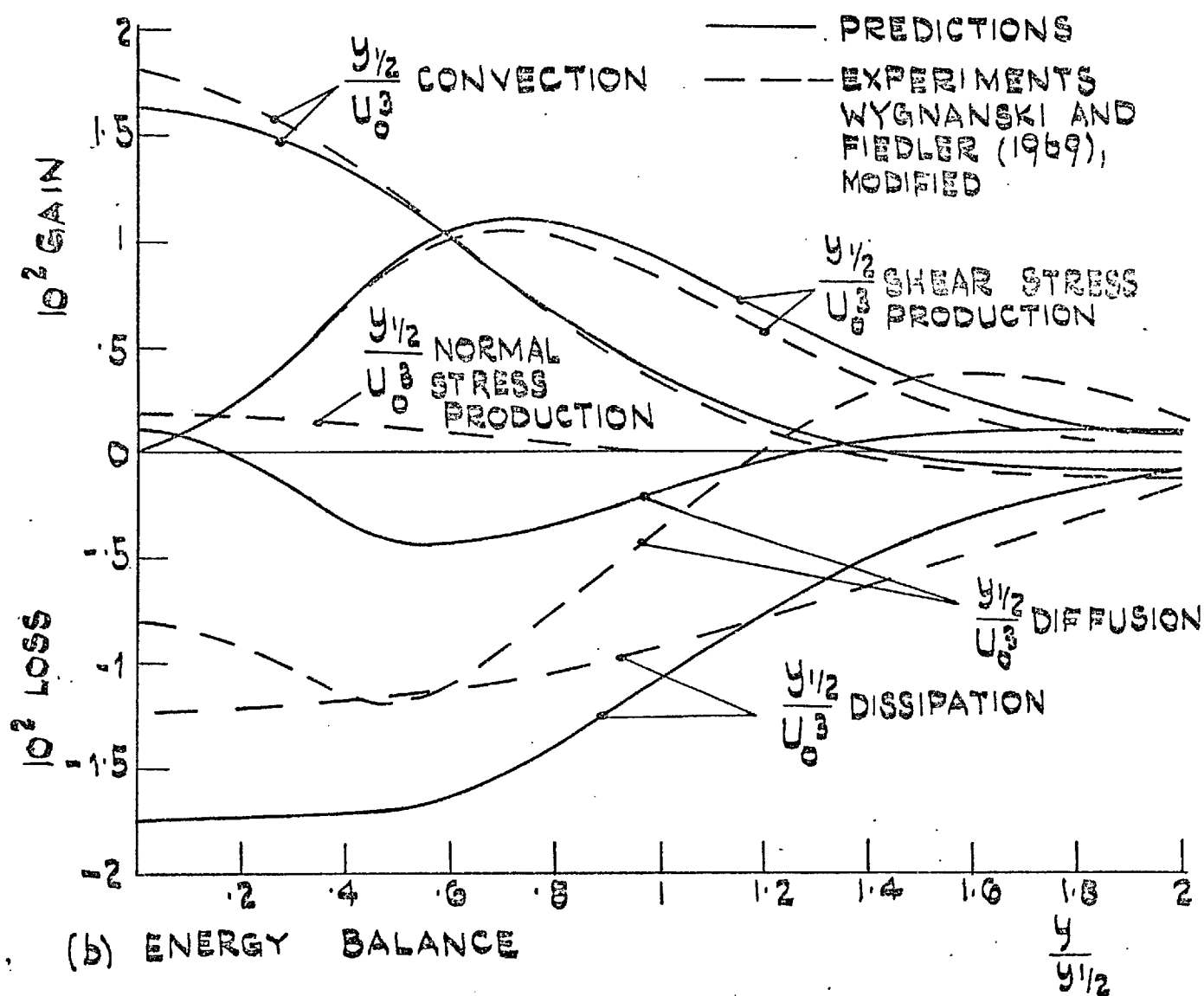
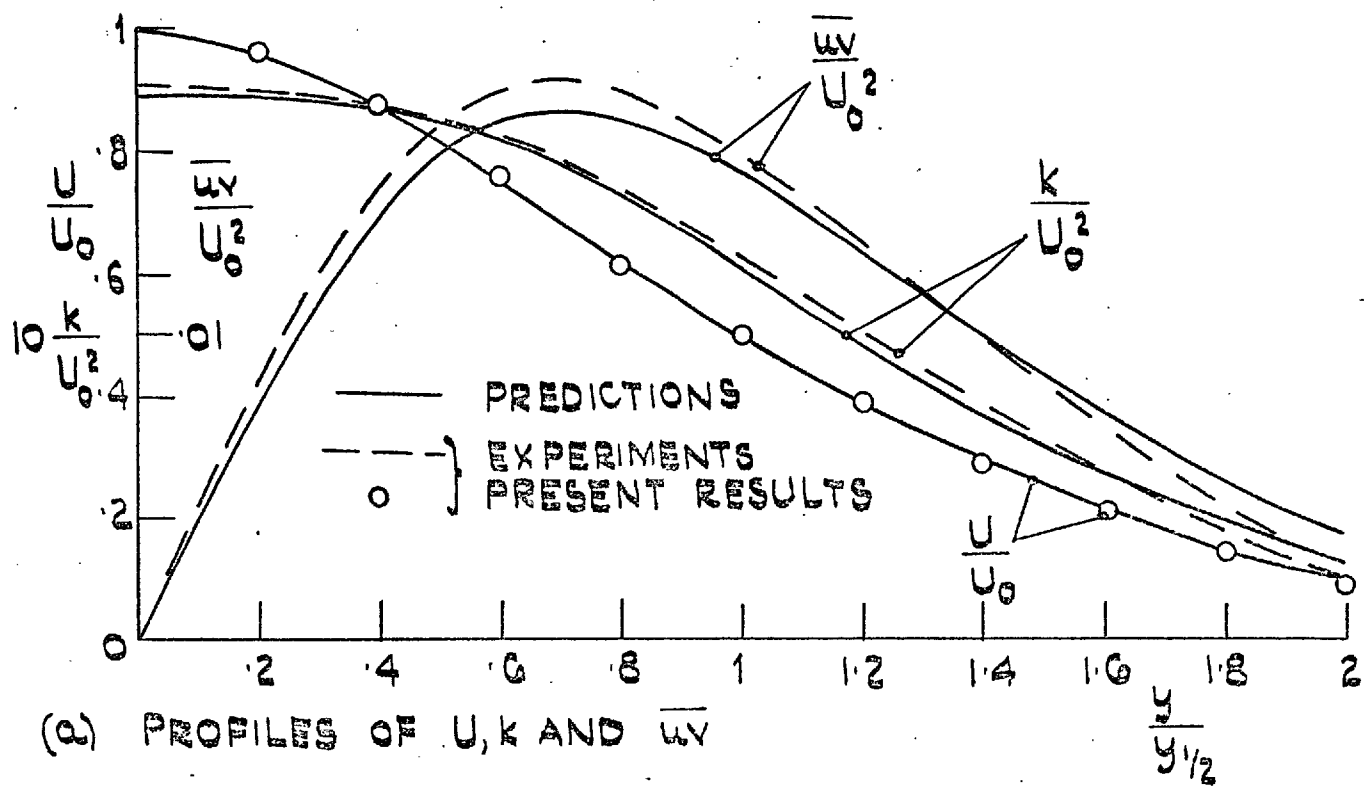
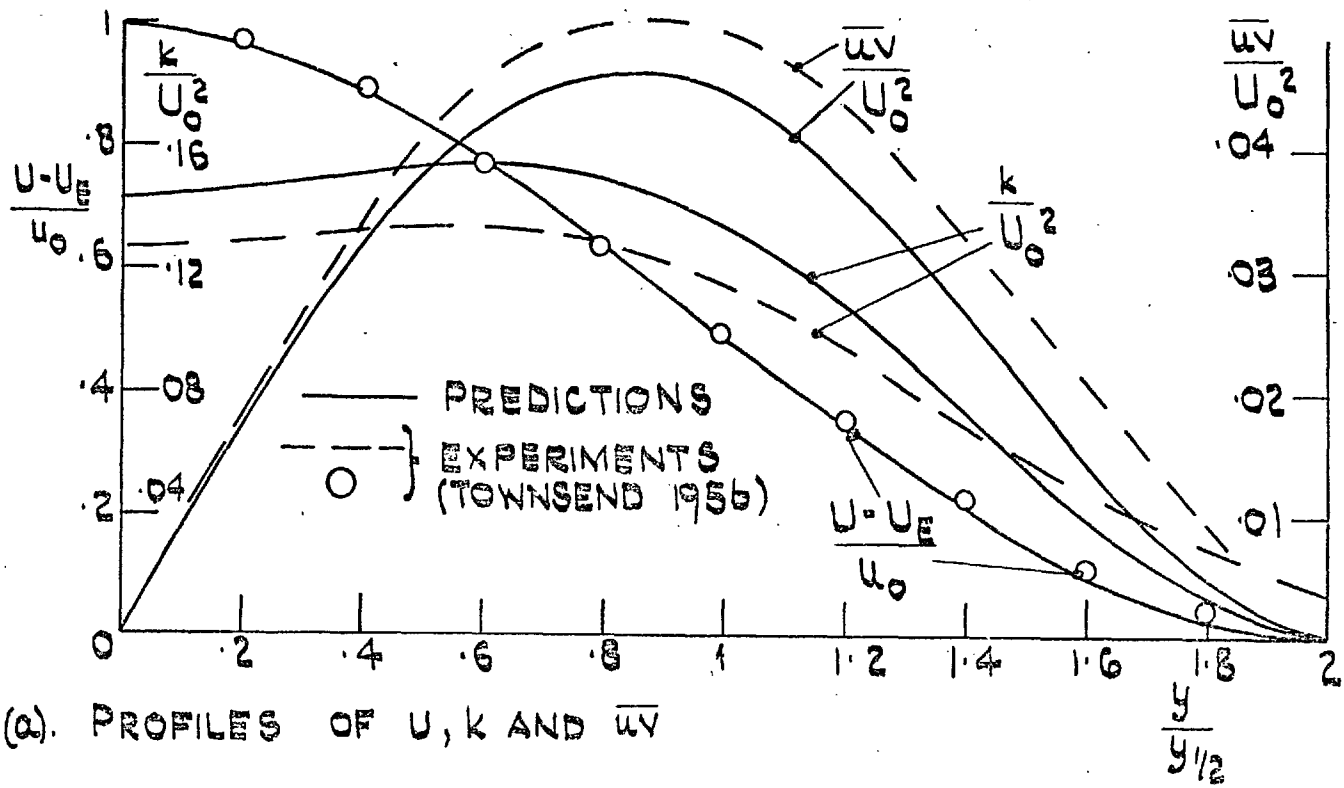
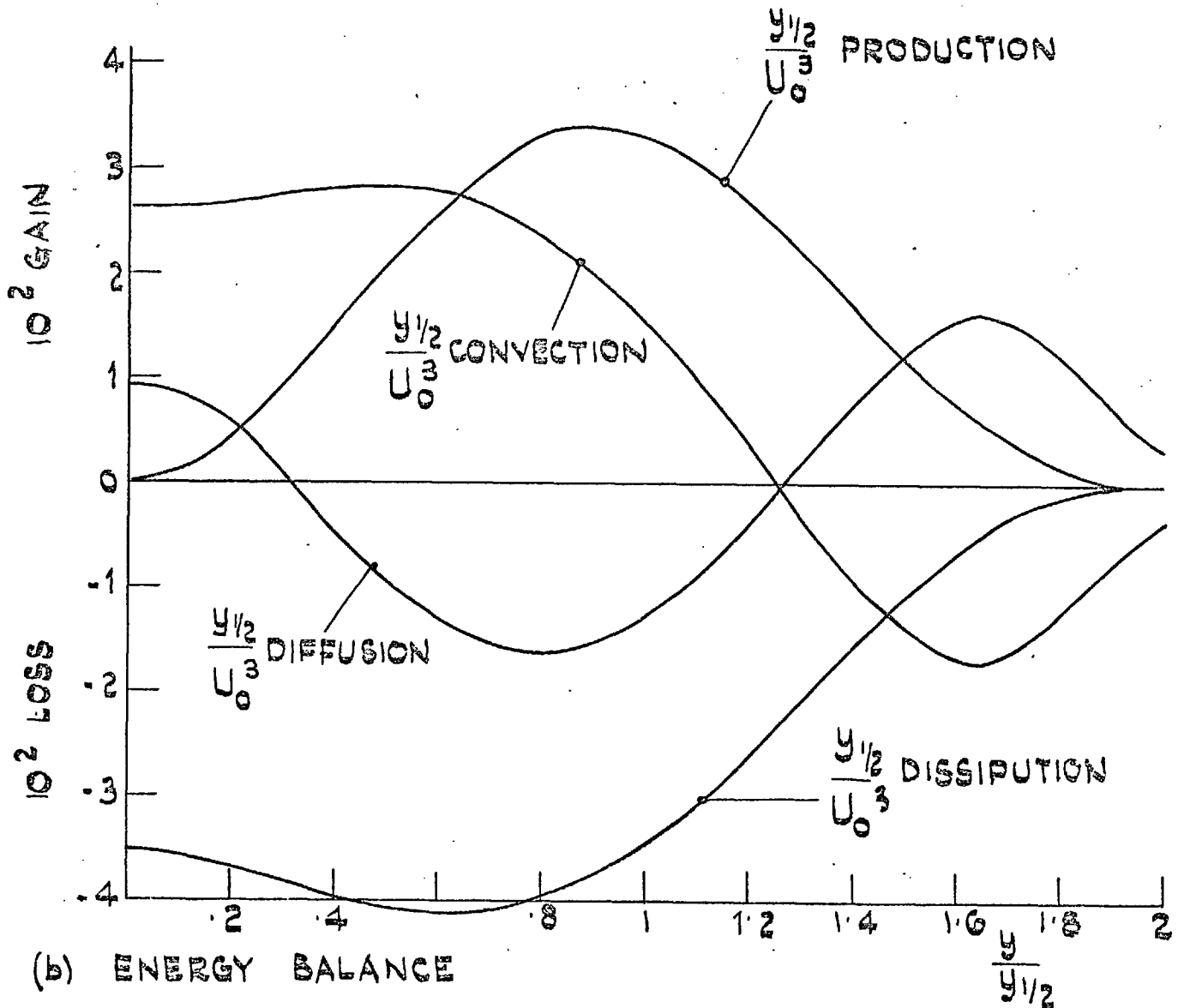


Fig. 6.10 Round jet in stagnant surroundings

(a). PROFILES OF U , k AND $\overline{uv}$ 

(b) ENERGY BALANCE

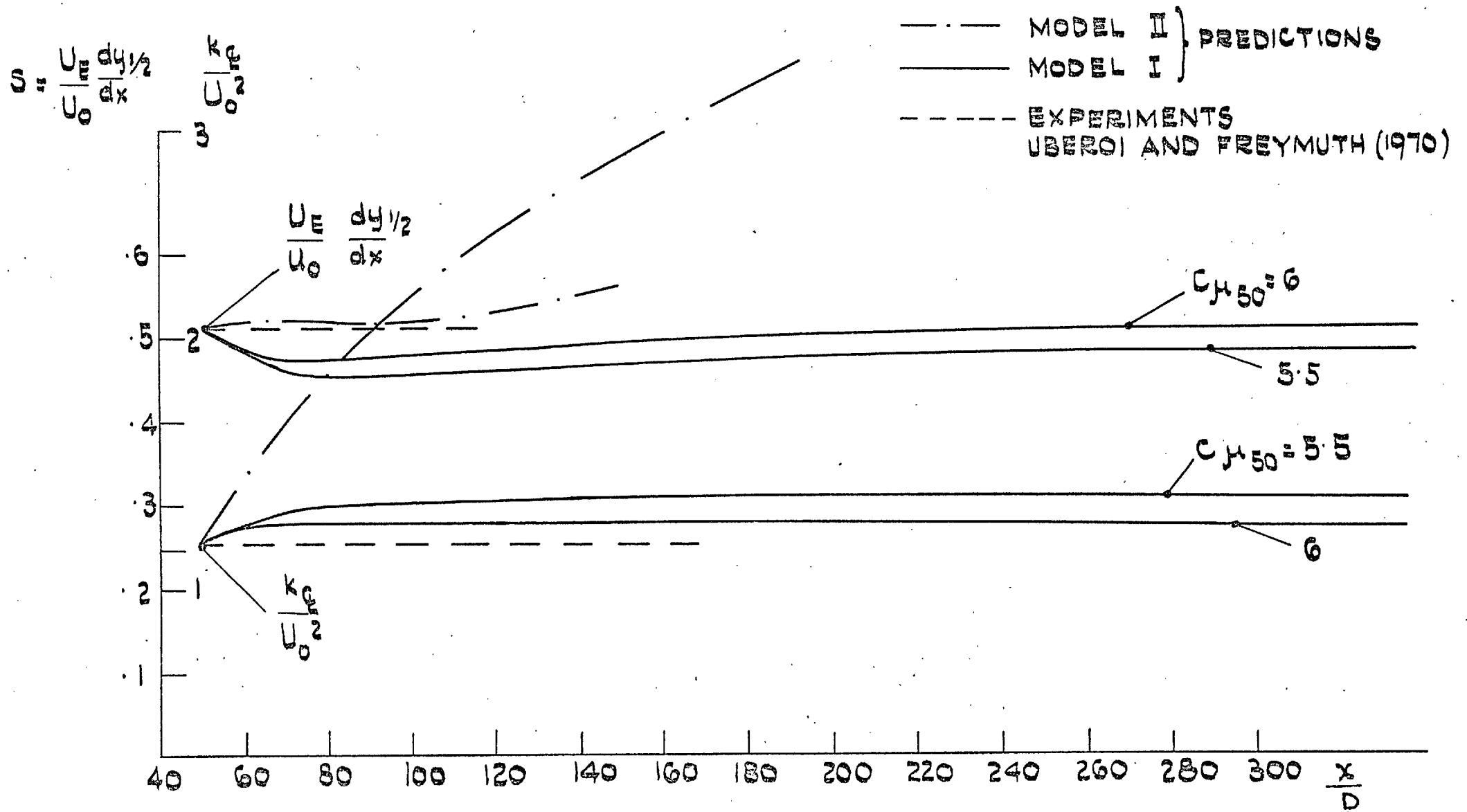


Fig. 6.12

Round wake behind a sphere

(a) Flow development

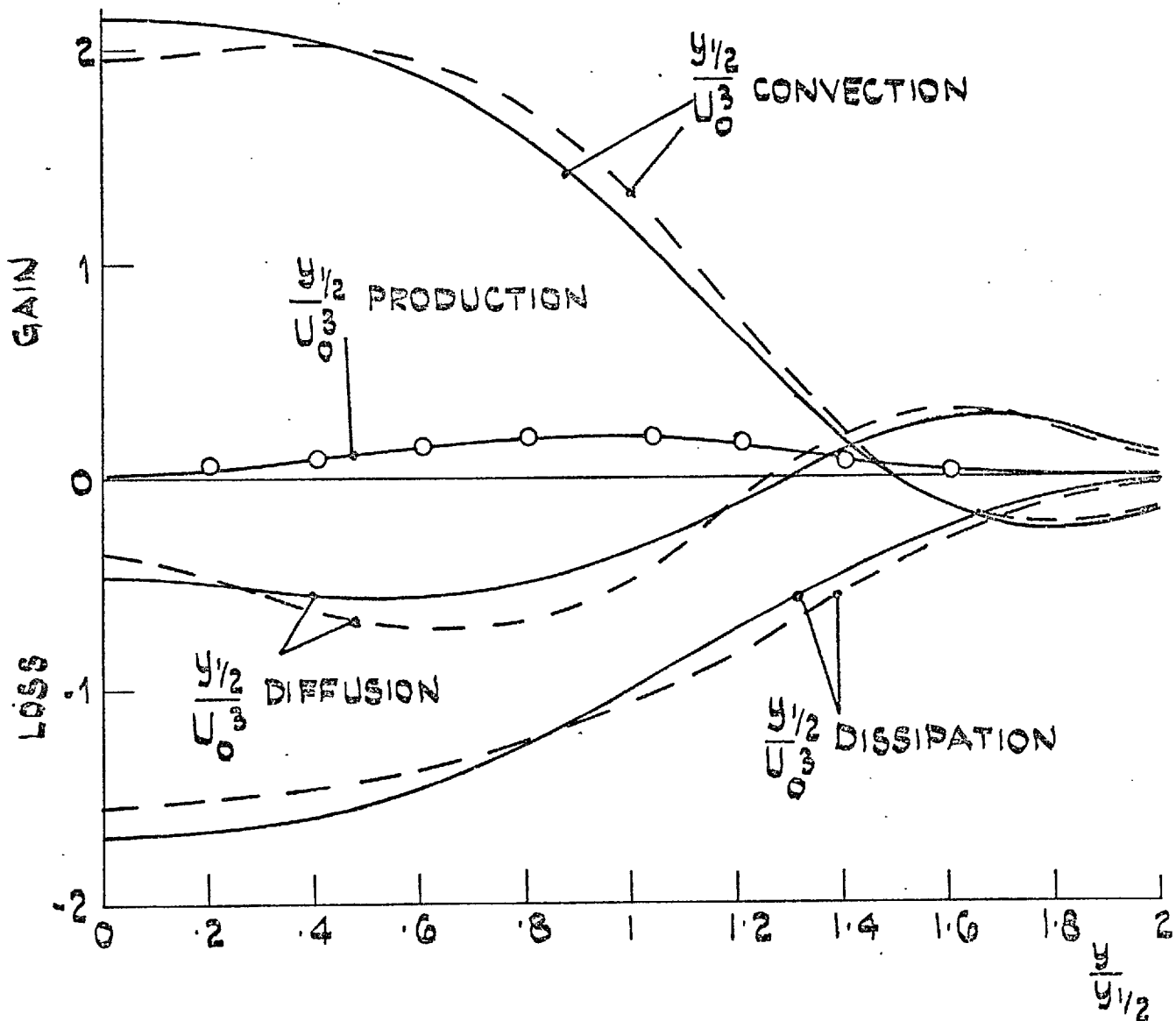
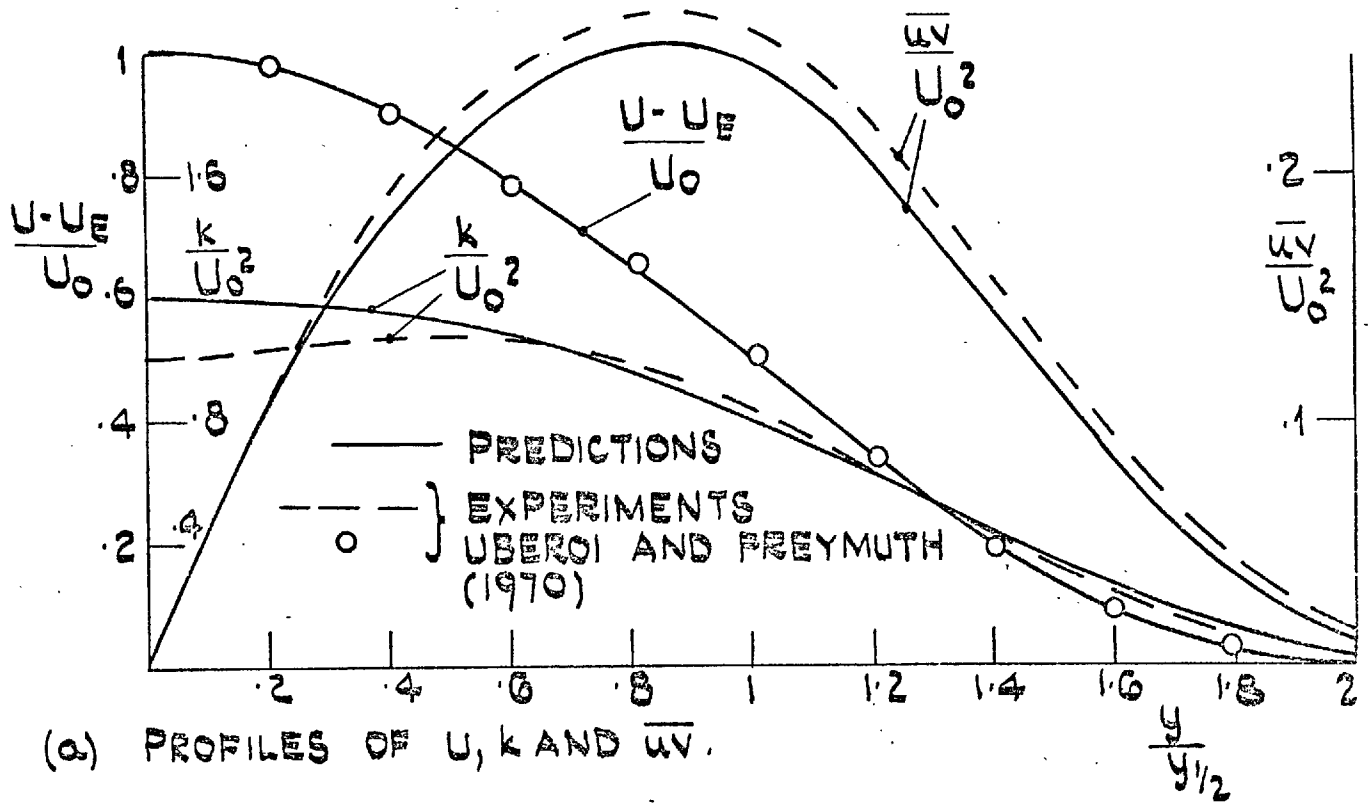
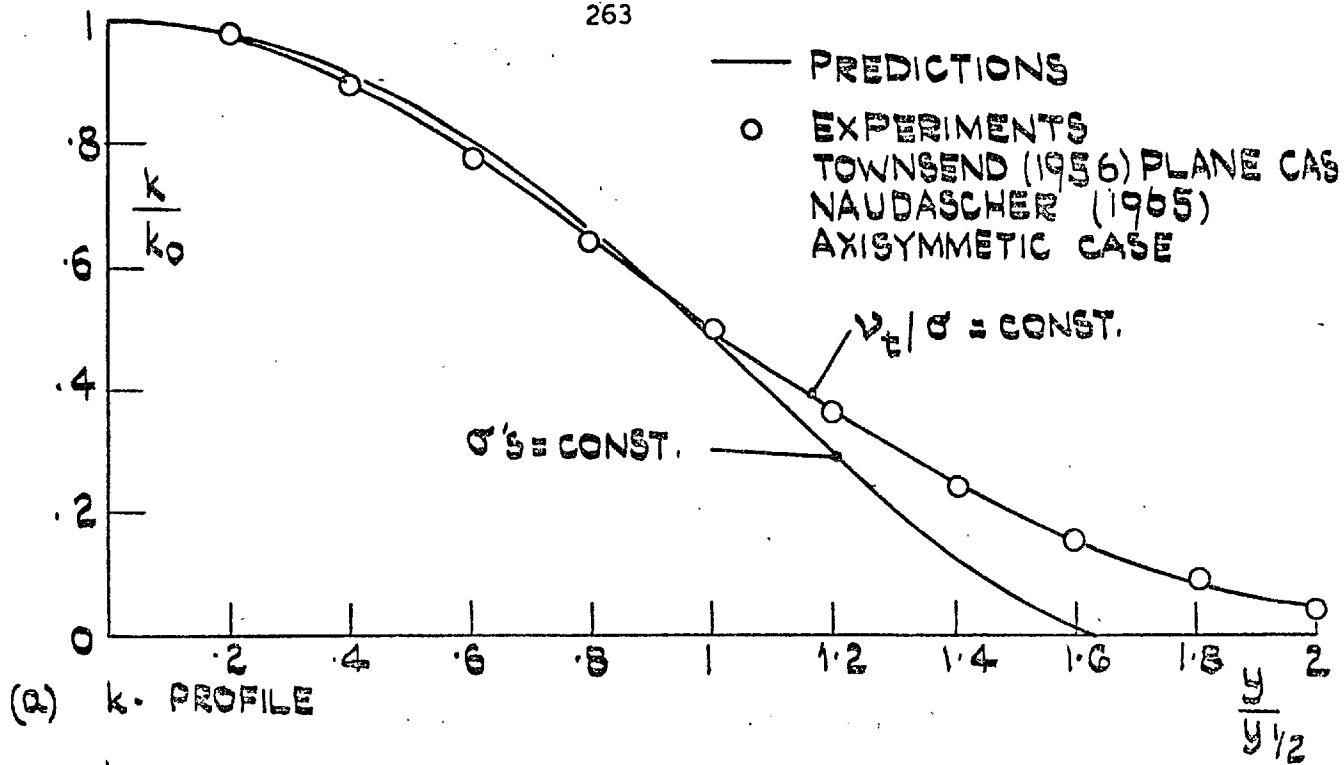
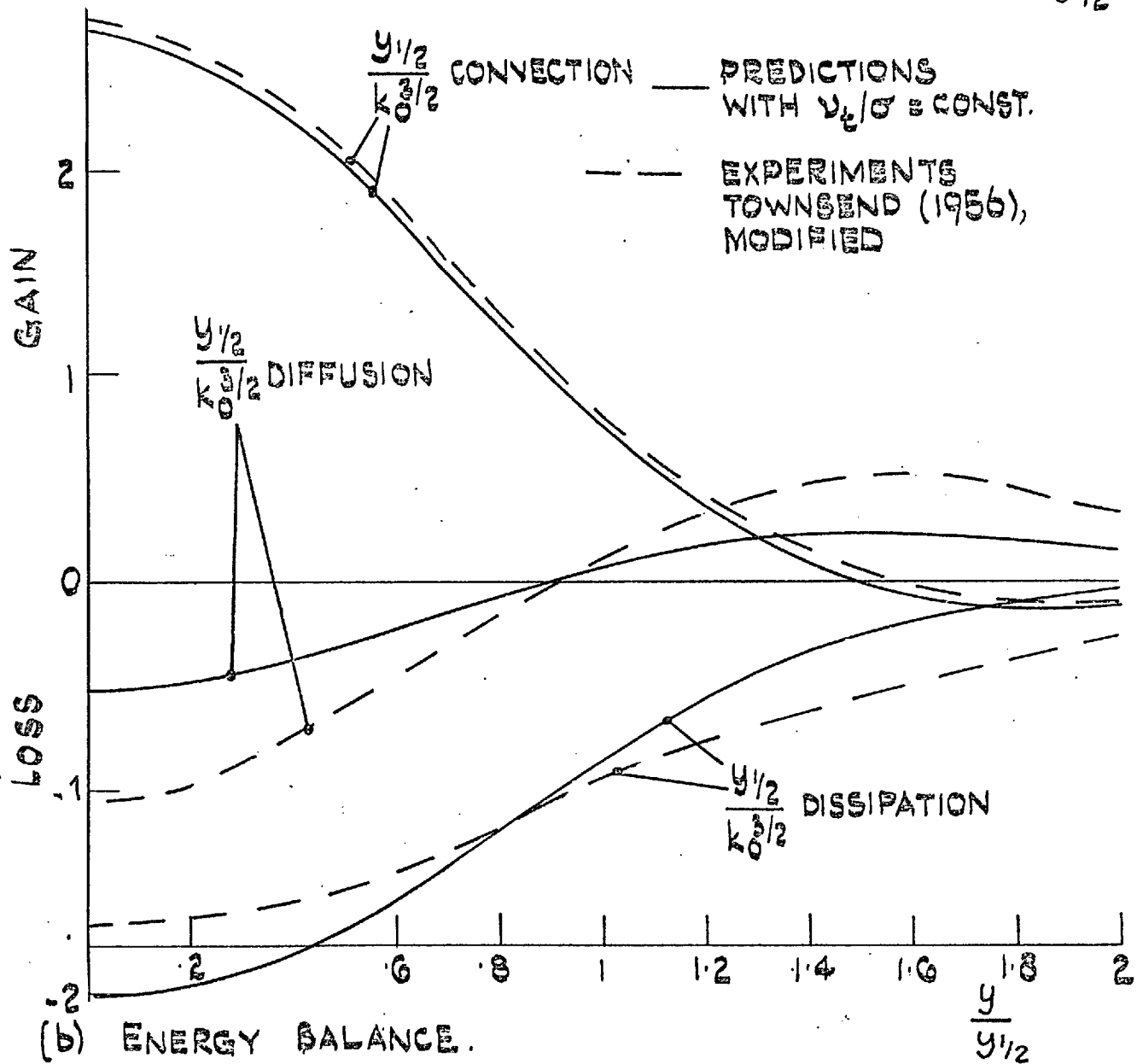


Fig. 6.12 Round wake behind a sphere

(a) k -PROFILE

(b) ENERGY BALANCE.

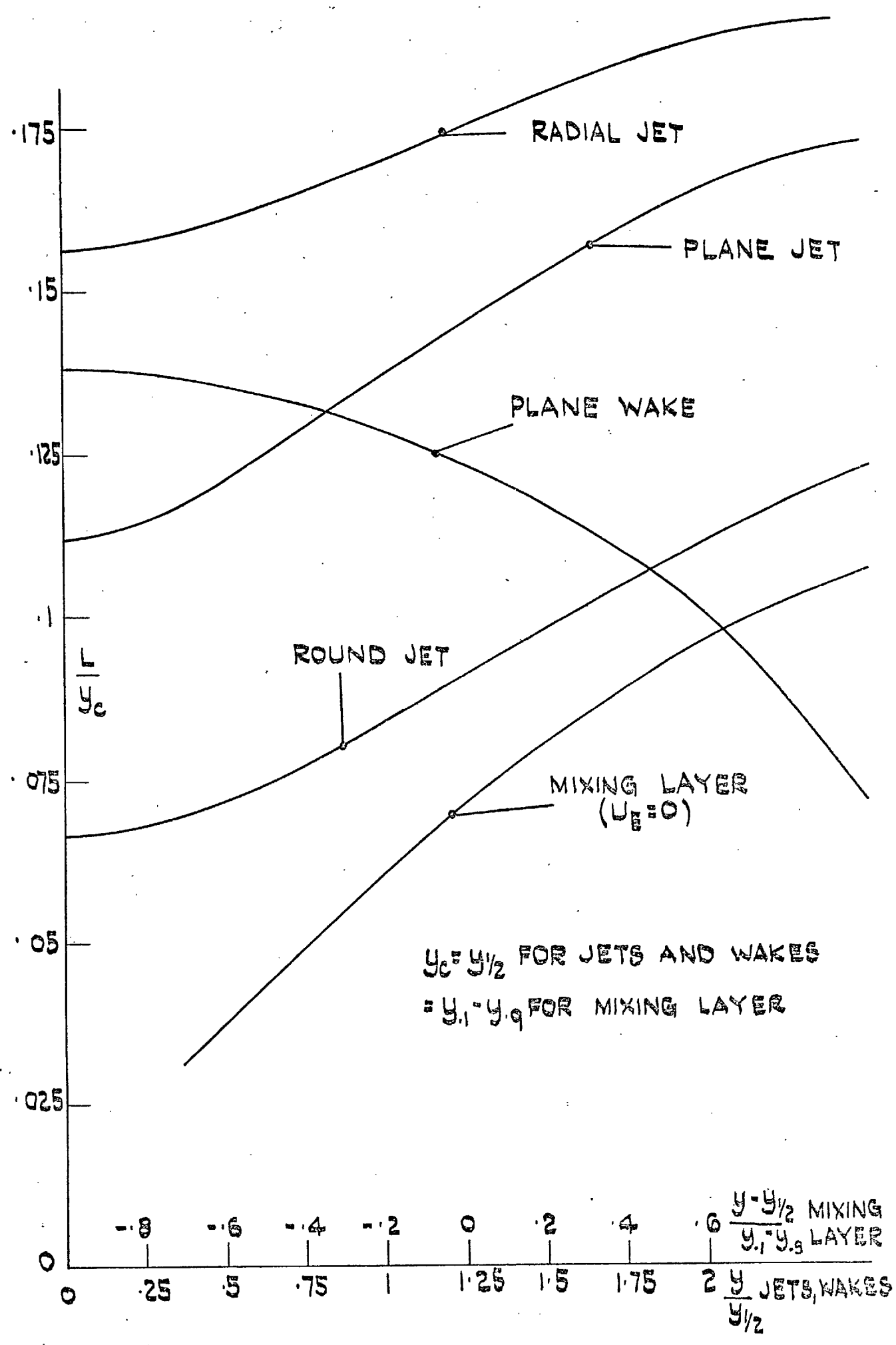


Fig. 6.14 L - profiles

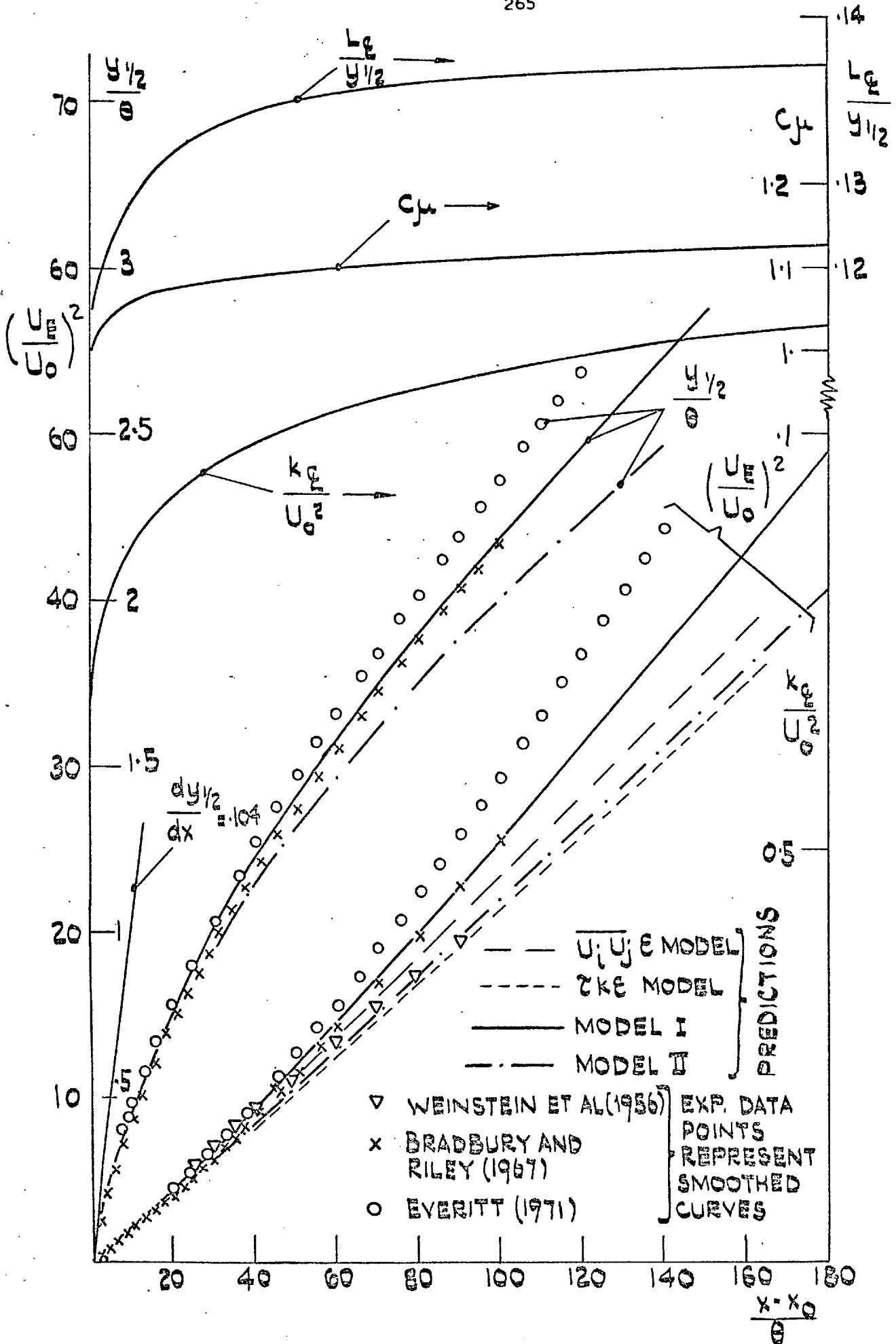


Fig. 6.15 Plane jet in a moving stream

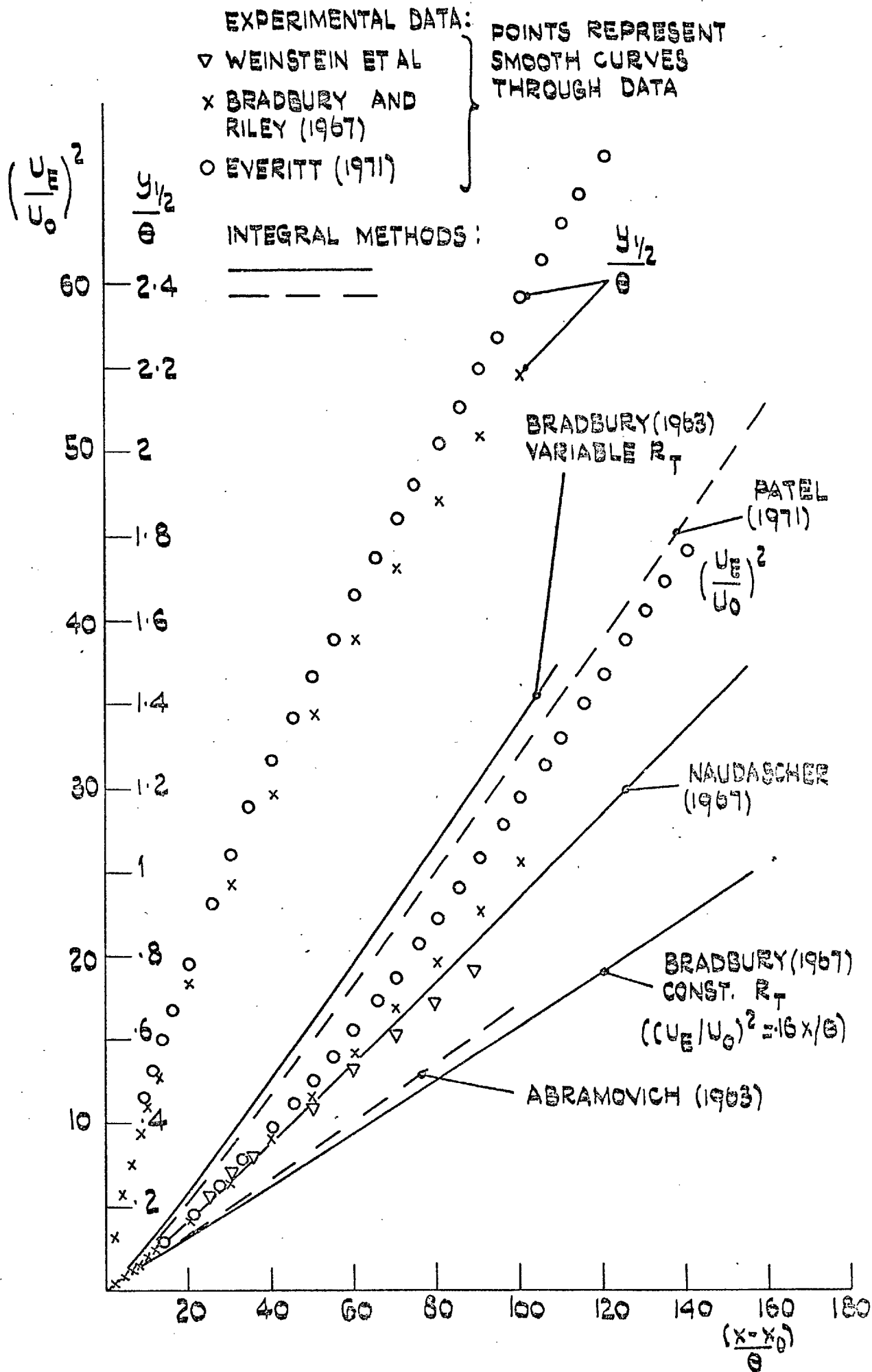


Fig. 6.16 Plane jet in a moving stream:
 Experimental data and integral methods

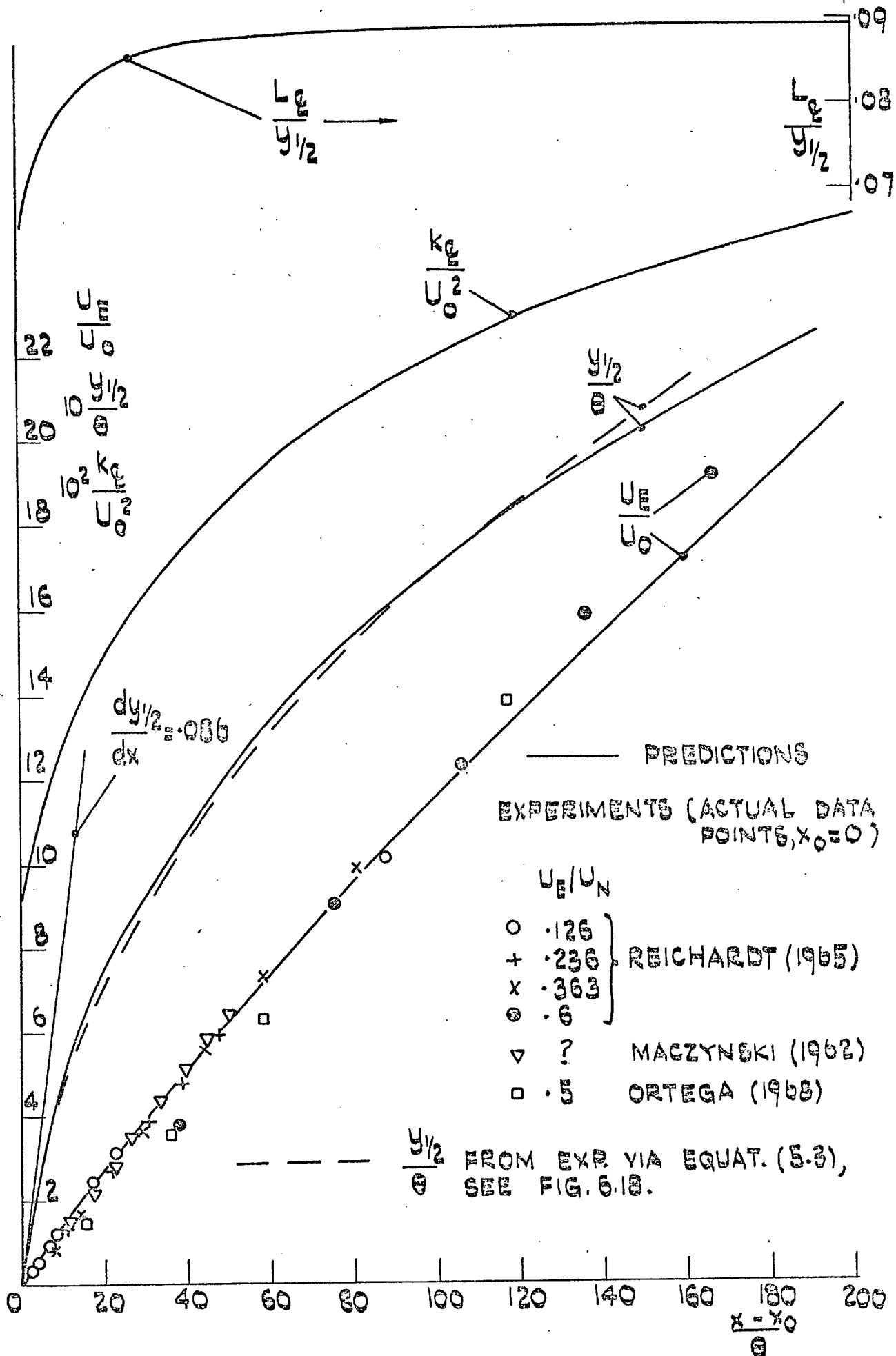


Fig. 6.17

Round jet in a moving stream

EXPERIMENTS (ACTUAL DATA POINTS, $x_0 = 0$):

U_E/U_N	
○	.126
+	.236
x	.363
● ●	.6
} REICHARDT (1965)	
▽	? MACZYNSKI (1962)
□	.5 ORTEGA (1968)

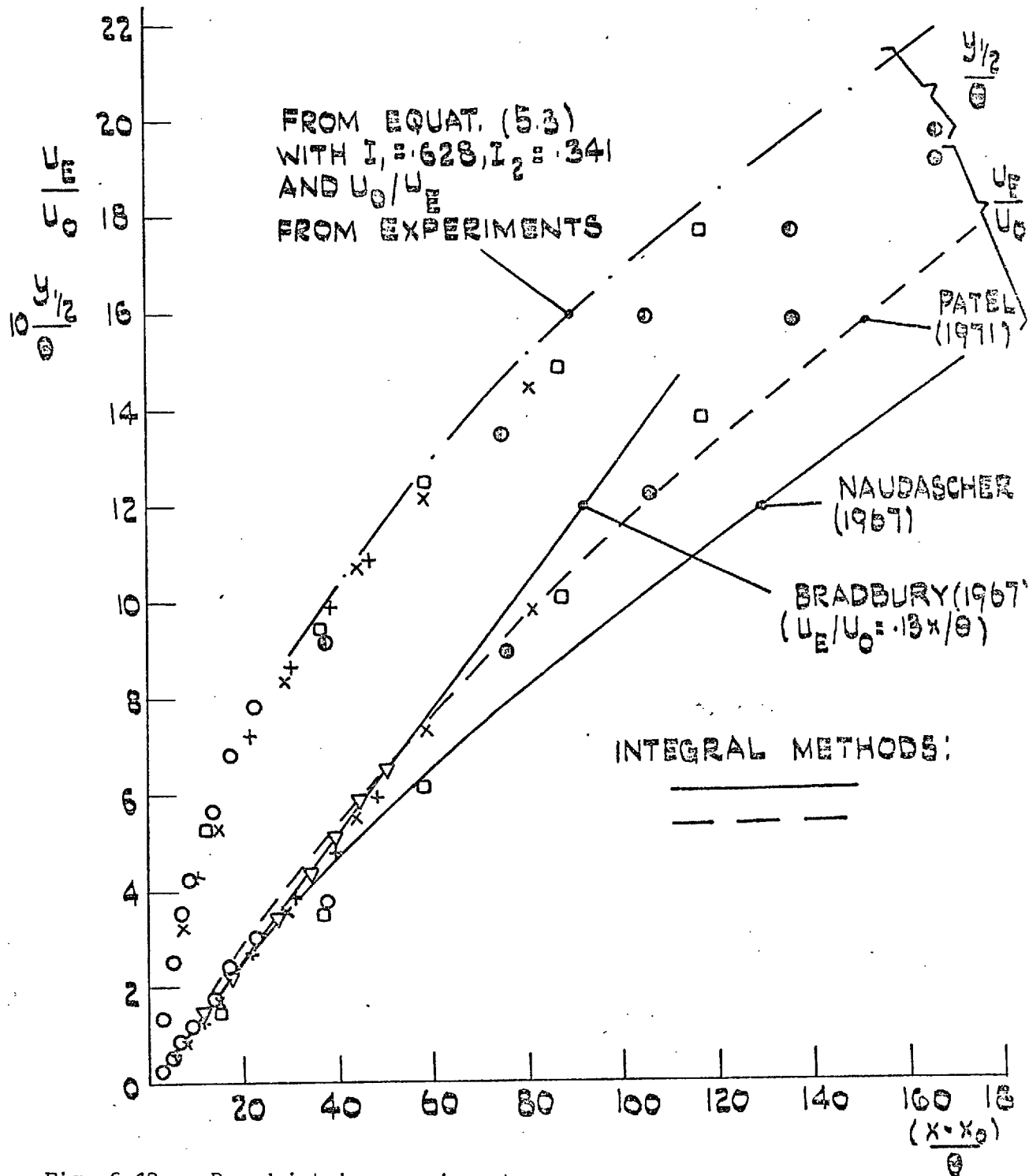


Fig. 6.18

Round jet in a moving stream:

Experimental data and integral methods

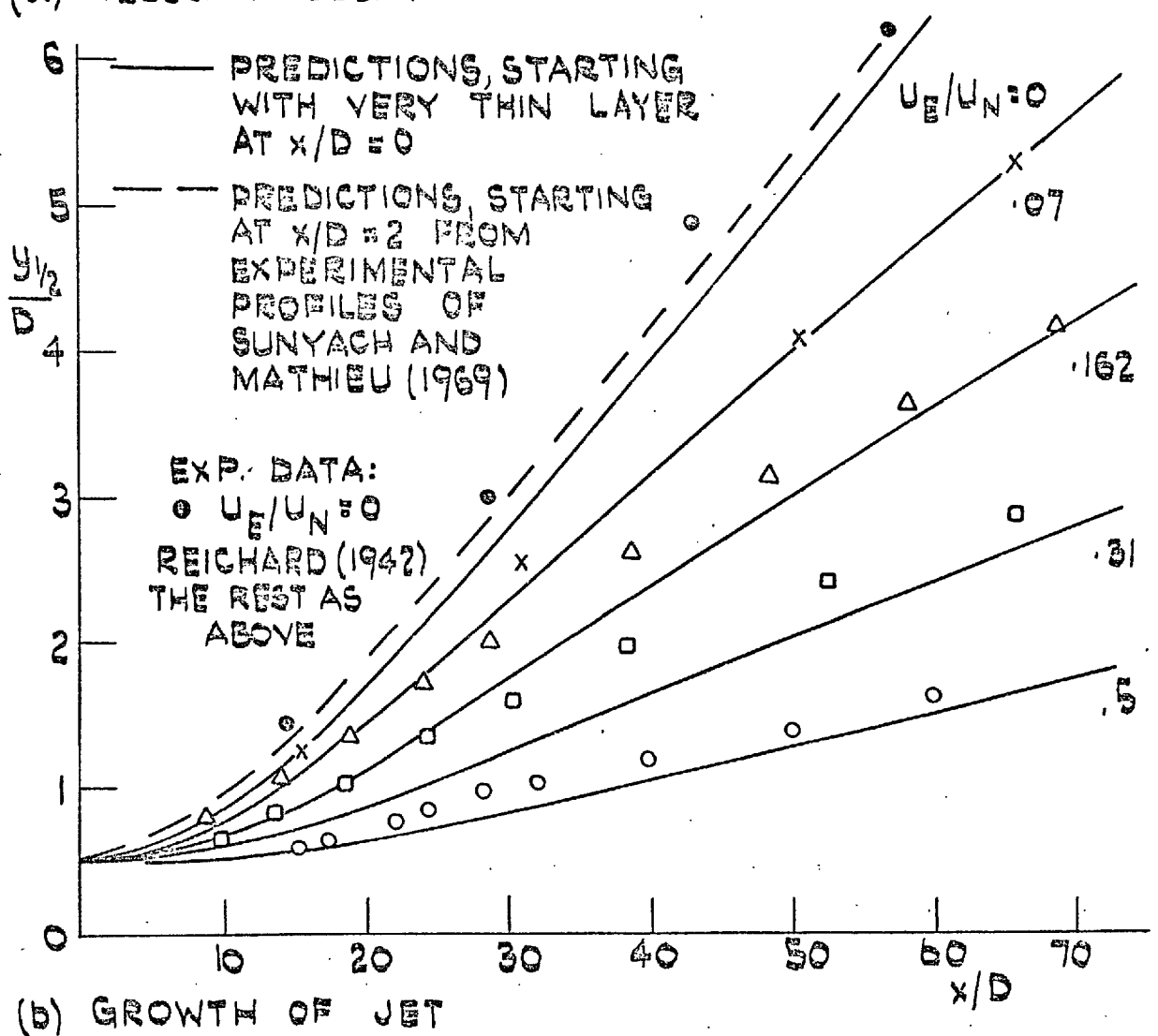
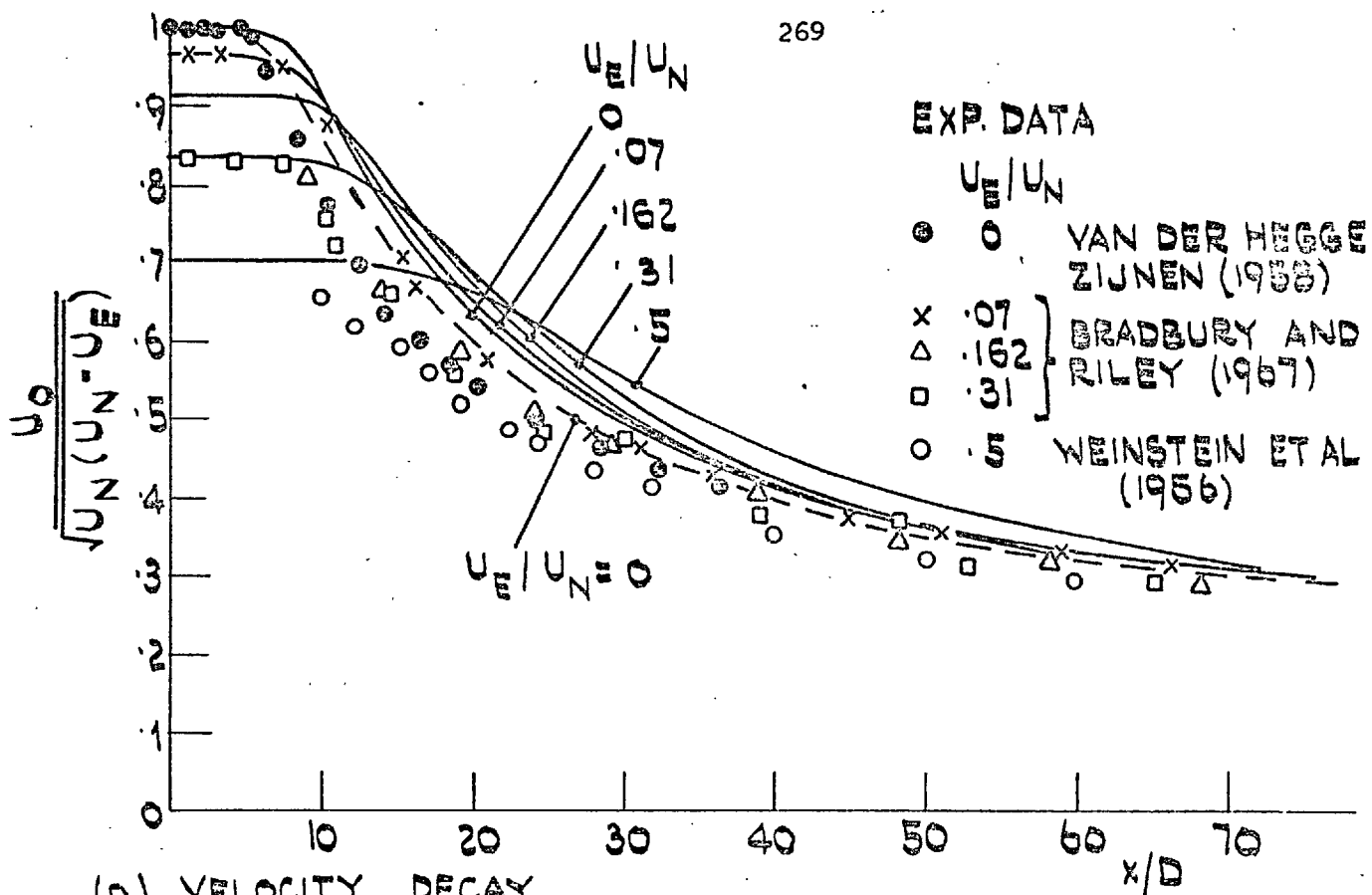
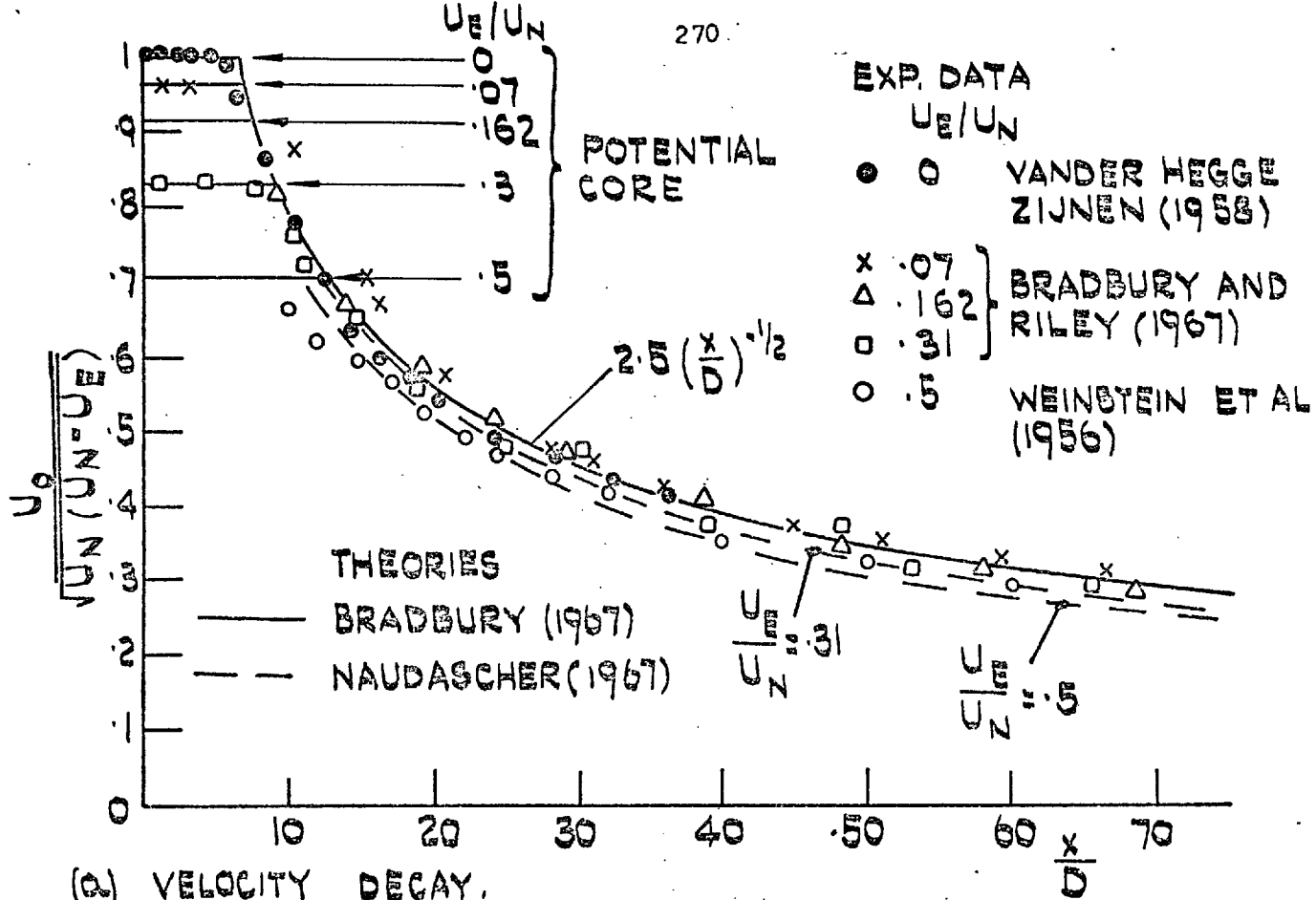
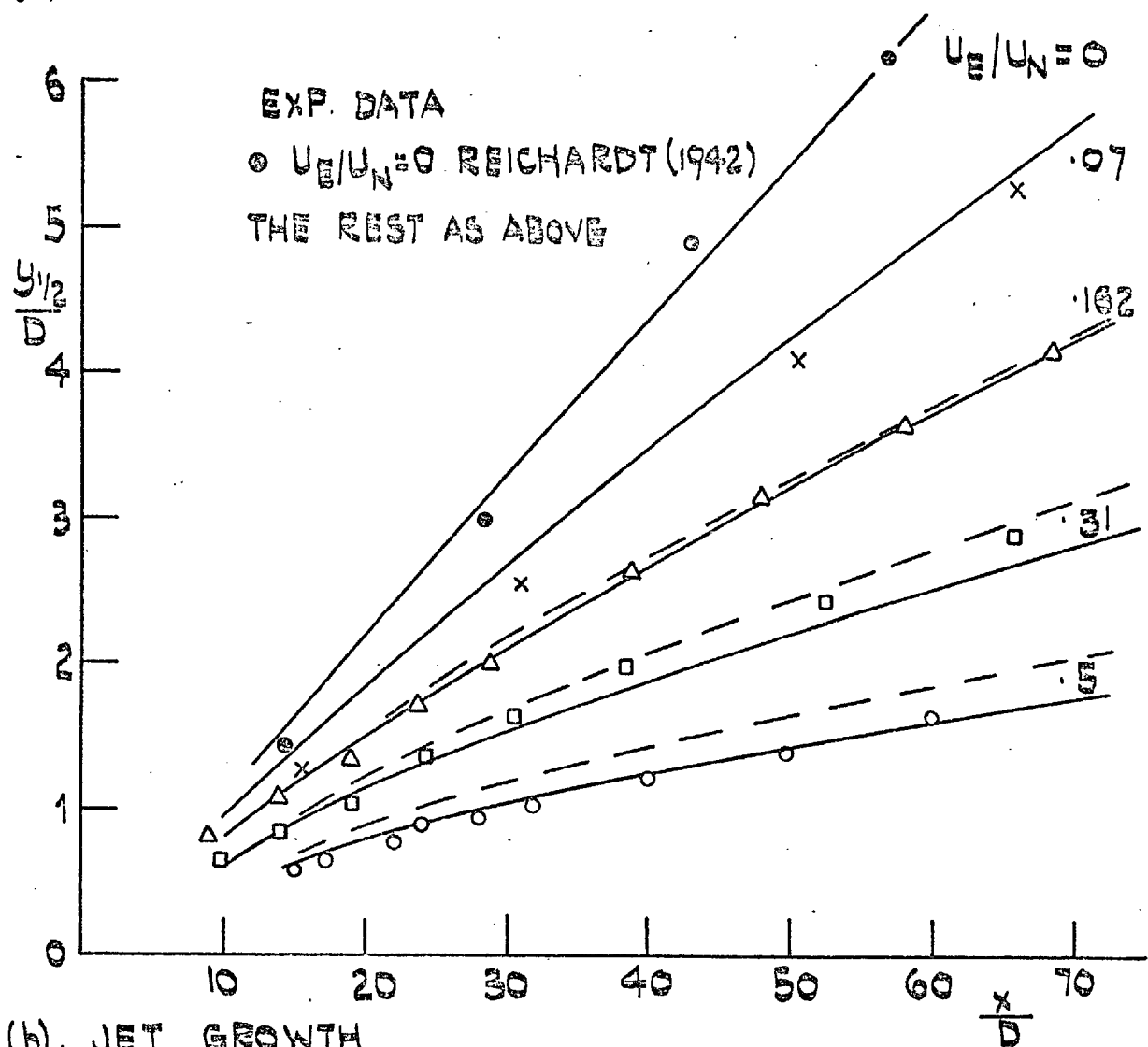


Fig. 6.19. Plane jet: transition mixing layer - far jet



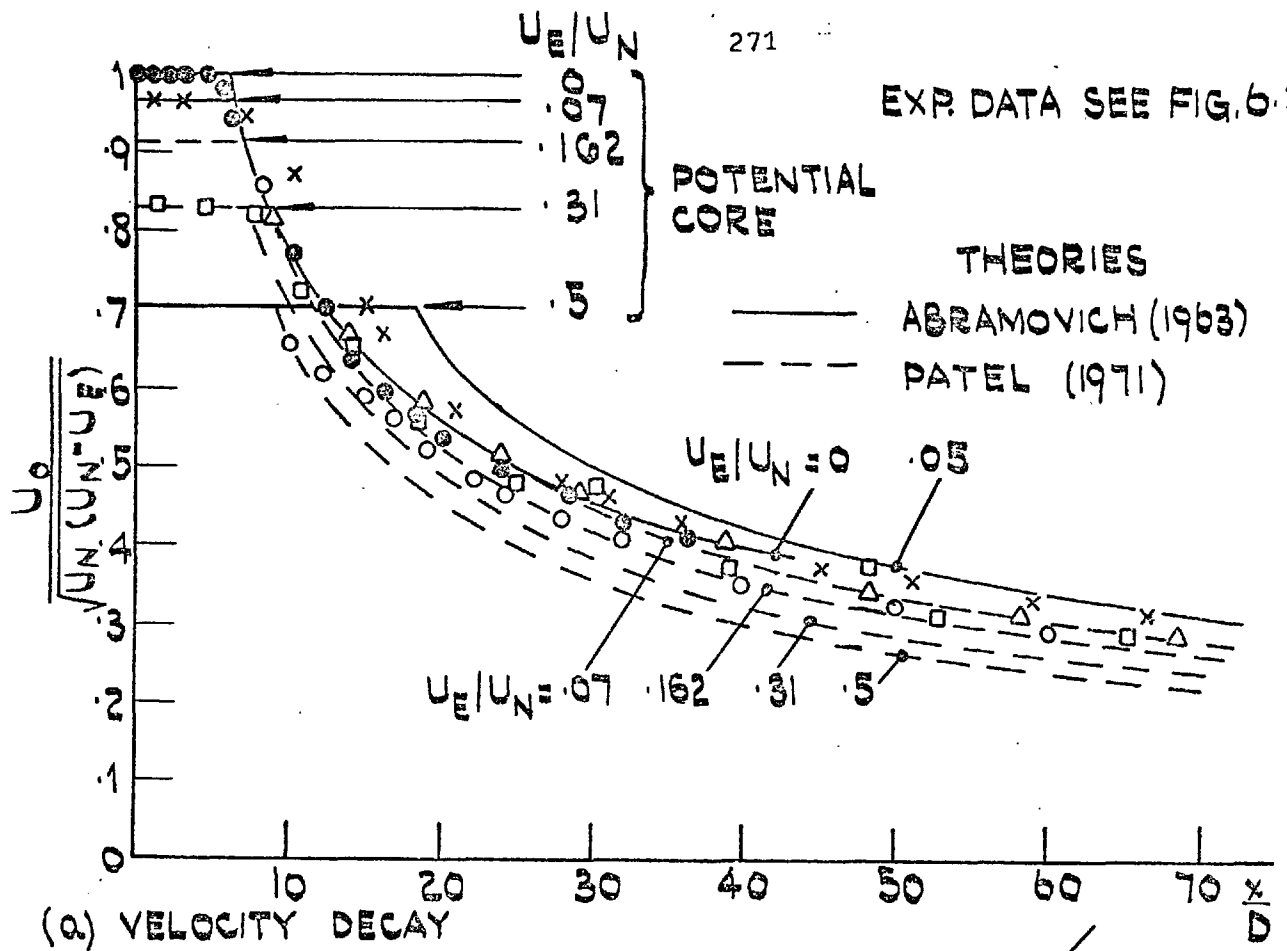
(a) VELOCITY DECAY.



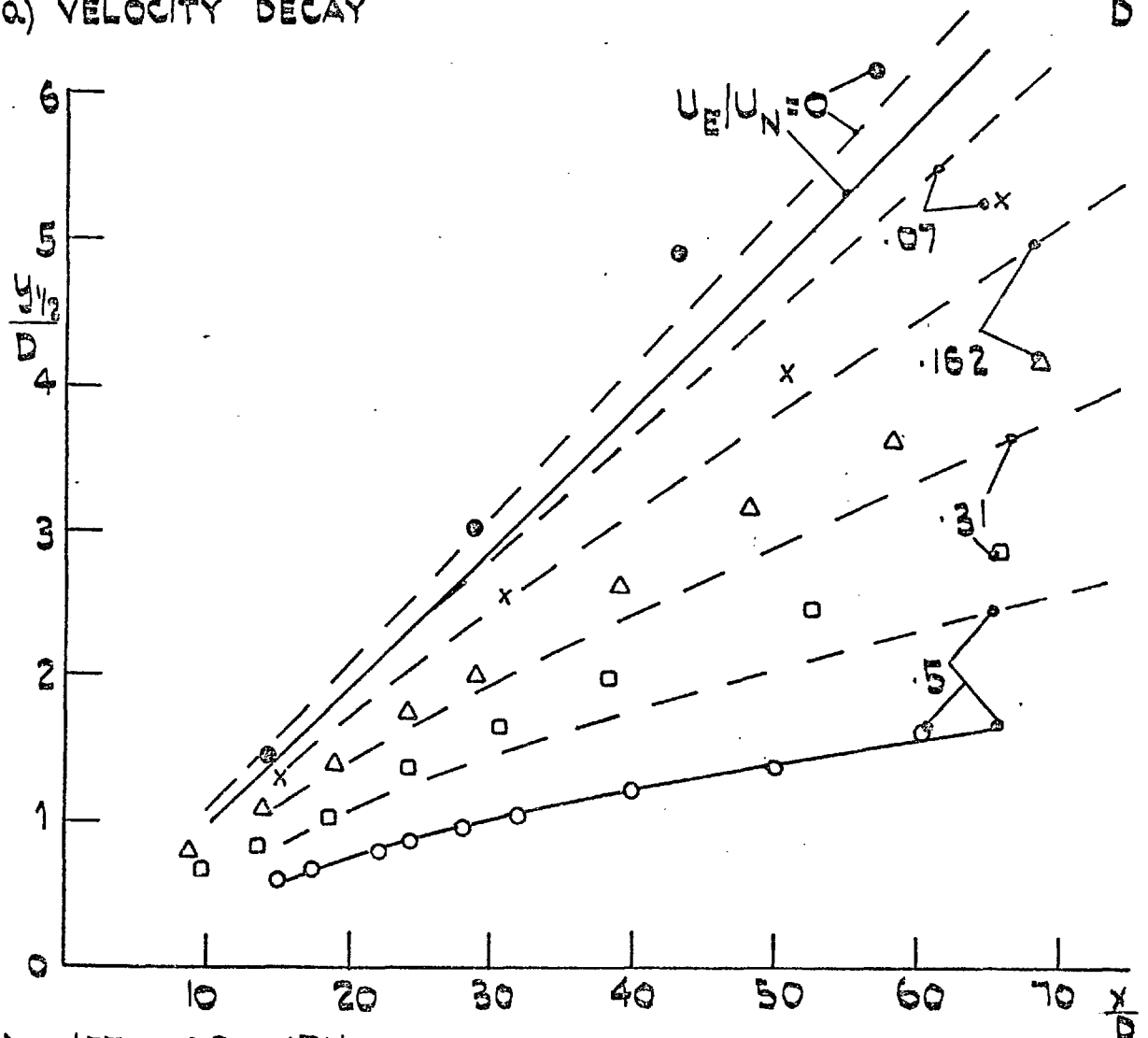
(b) JET GROWTH

Fig. 6.20 Plane jet: Transition mixing layer - far jet
Theories of Bradbury and Naudascher

EXP. DATA SEE FIG. 6.20.



(a) VELOCITY DECAY



(b) JET GROWTH

Fig. 6.21 Plane jet: Transition mixing layer - far jet
Theories of Abramovich and Patel

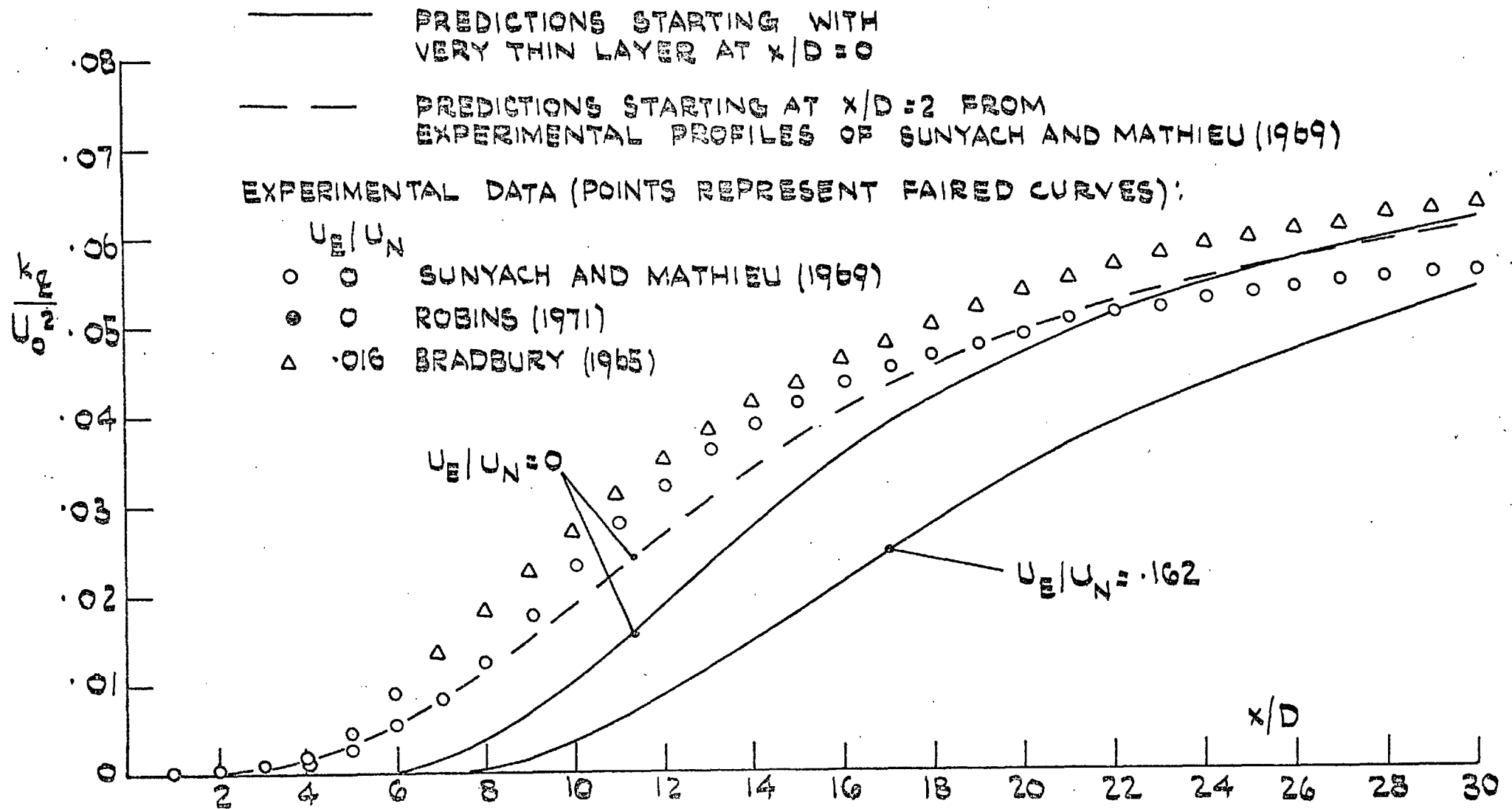


Fig. 6.22 Development of k along the axis in plane jets

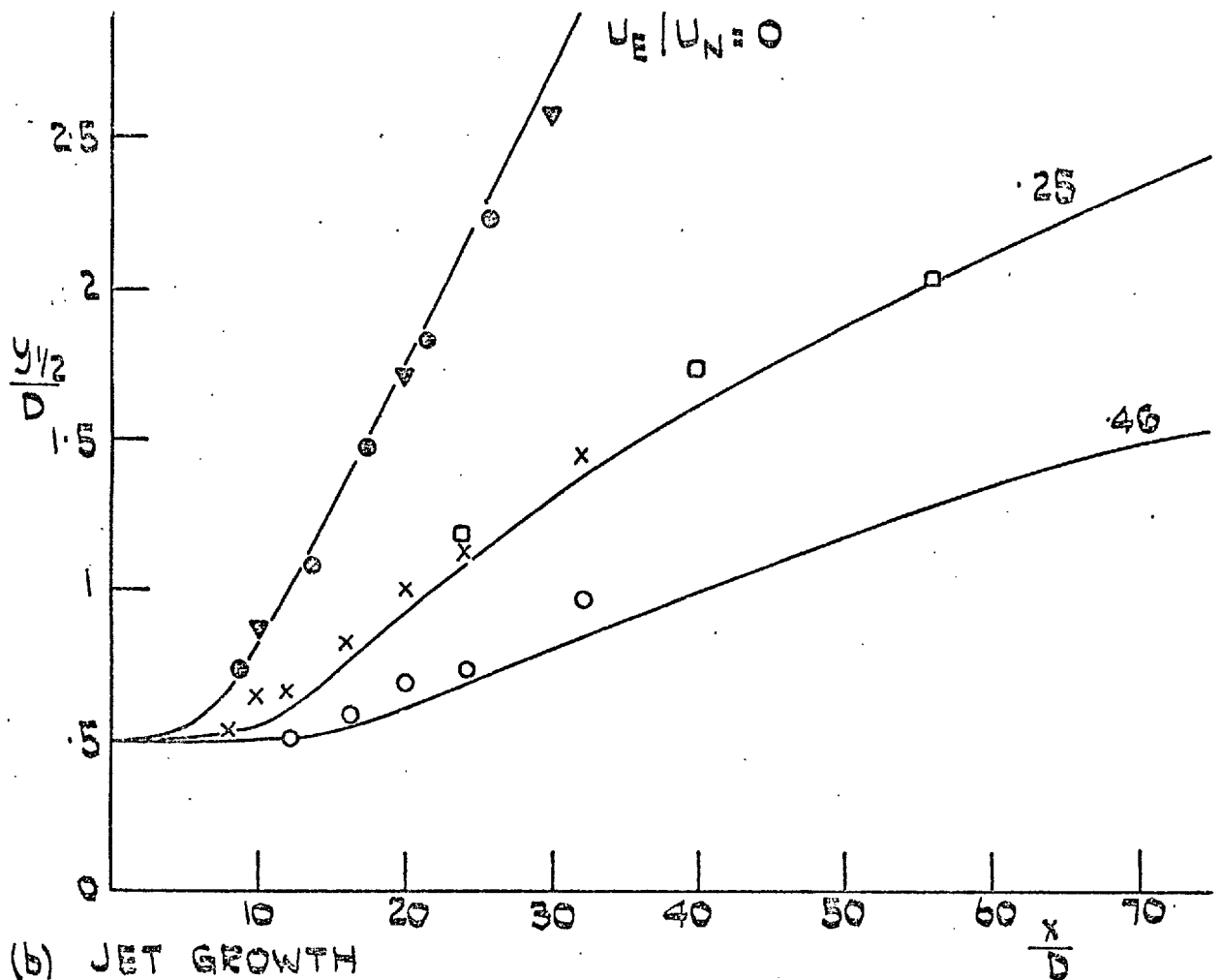
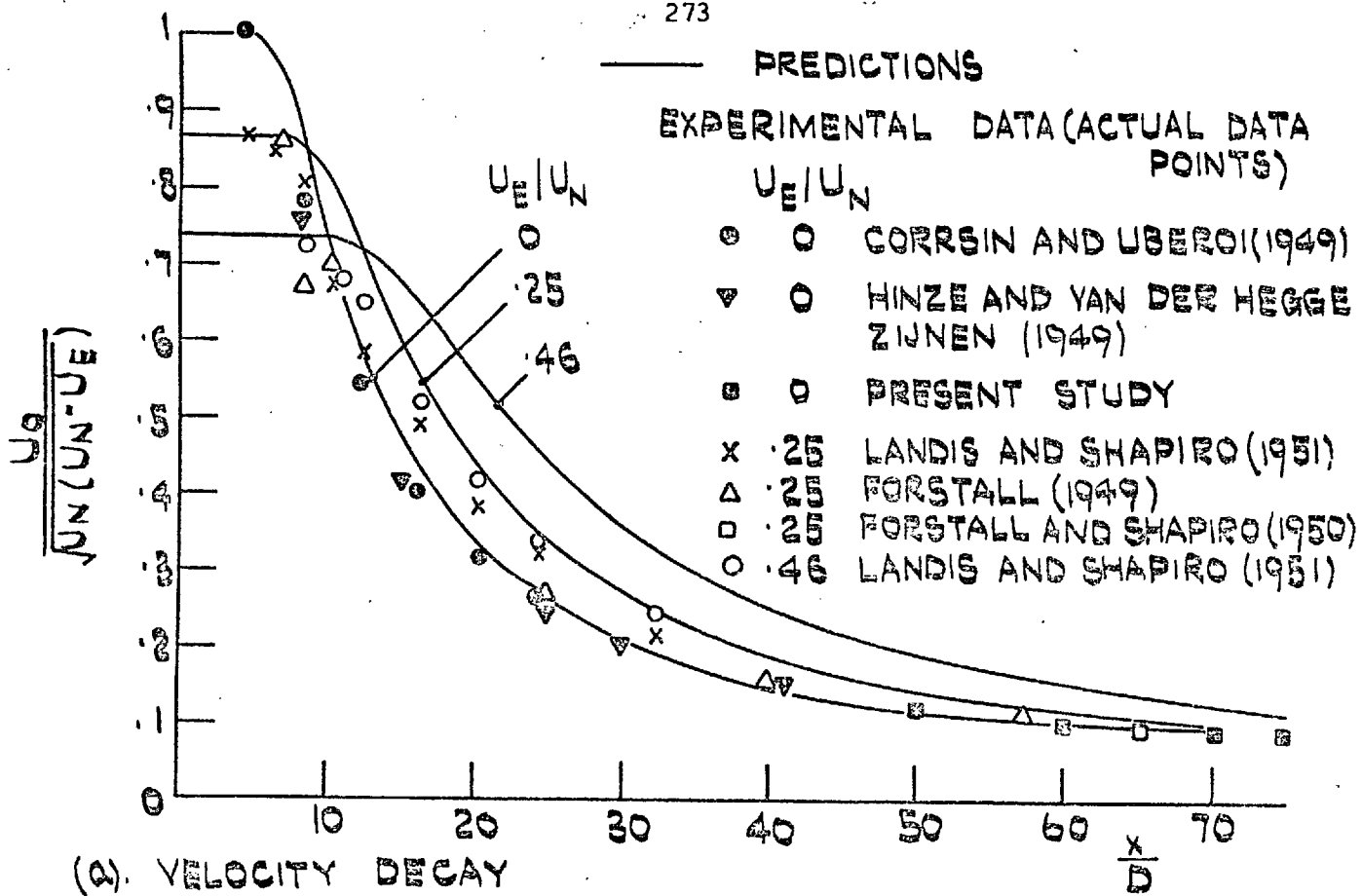
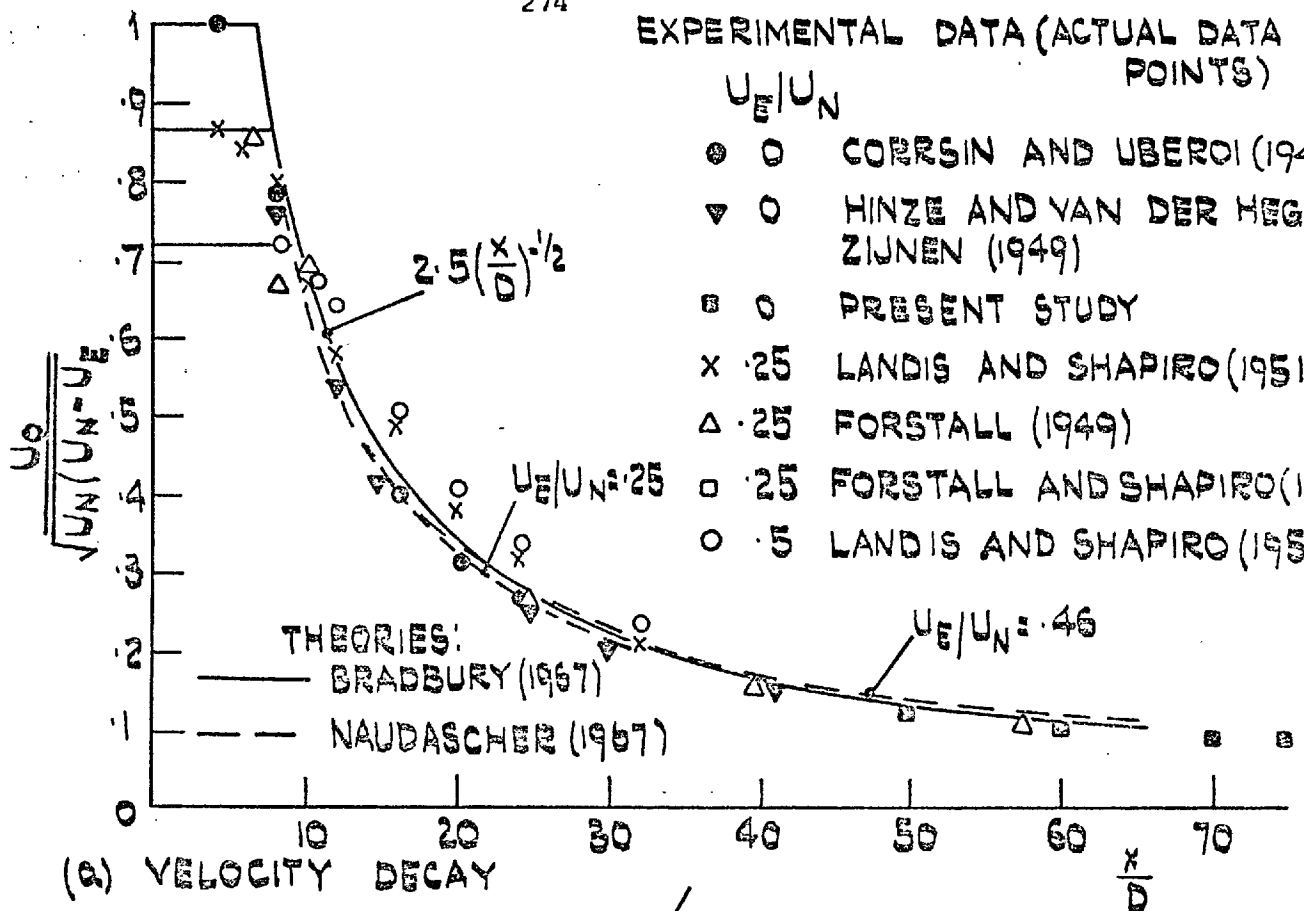


Fig. 6.23 Round jet: Transition mixing layer - far jet

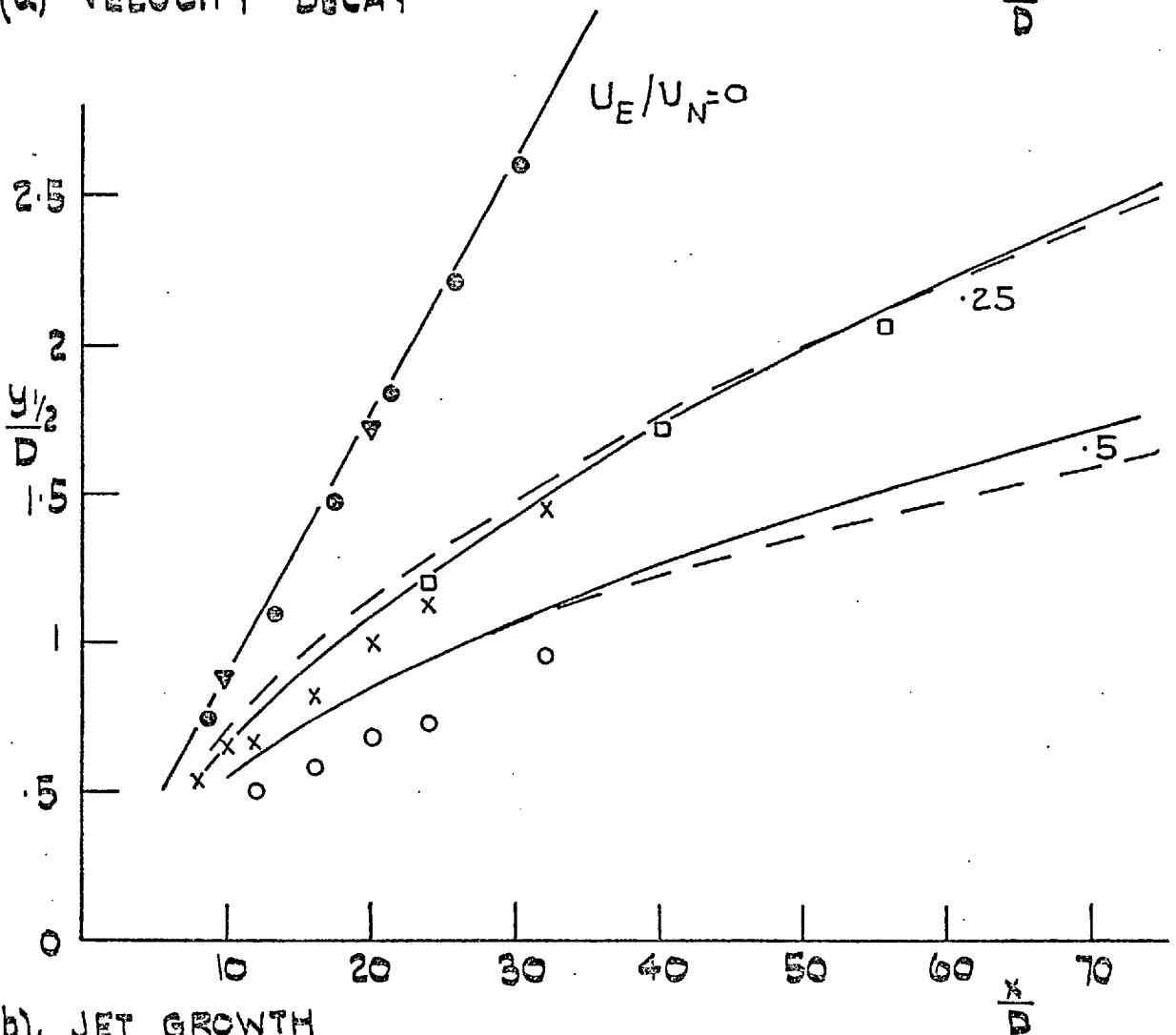
EXPERIMENTAL DATA (ACTUAL DATA POINTS)

 U_E/U_N

- ○ CORRSIN AND UBEROI (1949)
- ▼ ○ HINZE AND VAN DER HEGGE ZIJNEN (1949)
- ○ PRESENT STUDY
- x .25 LANDIS AND SHAPIRO (1951)
- △ .25 FORSTALL (1949)
- .25 FORSTALL AND SHAPIRO (1950)
- .5 LANDIS AND SHAPIRO (1951)



(a) VELOCITY DECAY



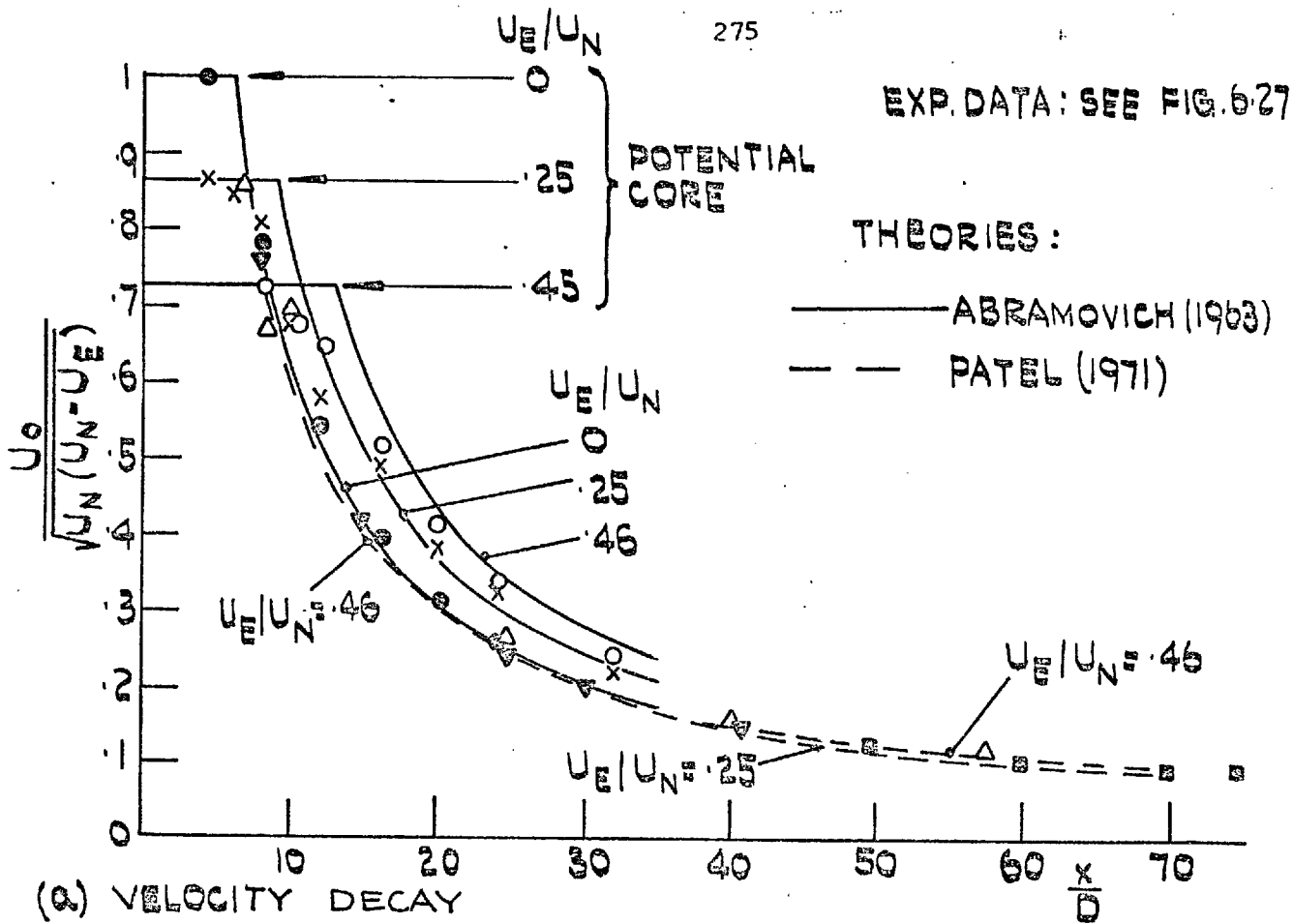
(b) JET GROWTH

Fig. 6.24

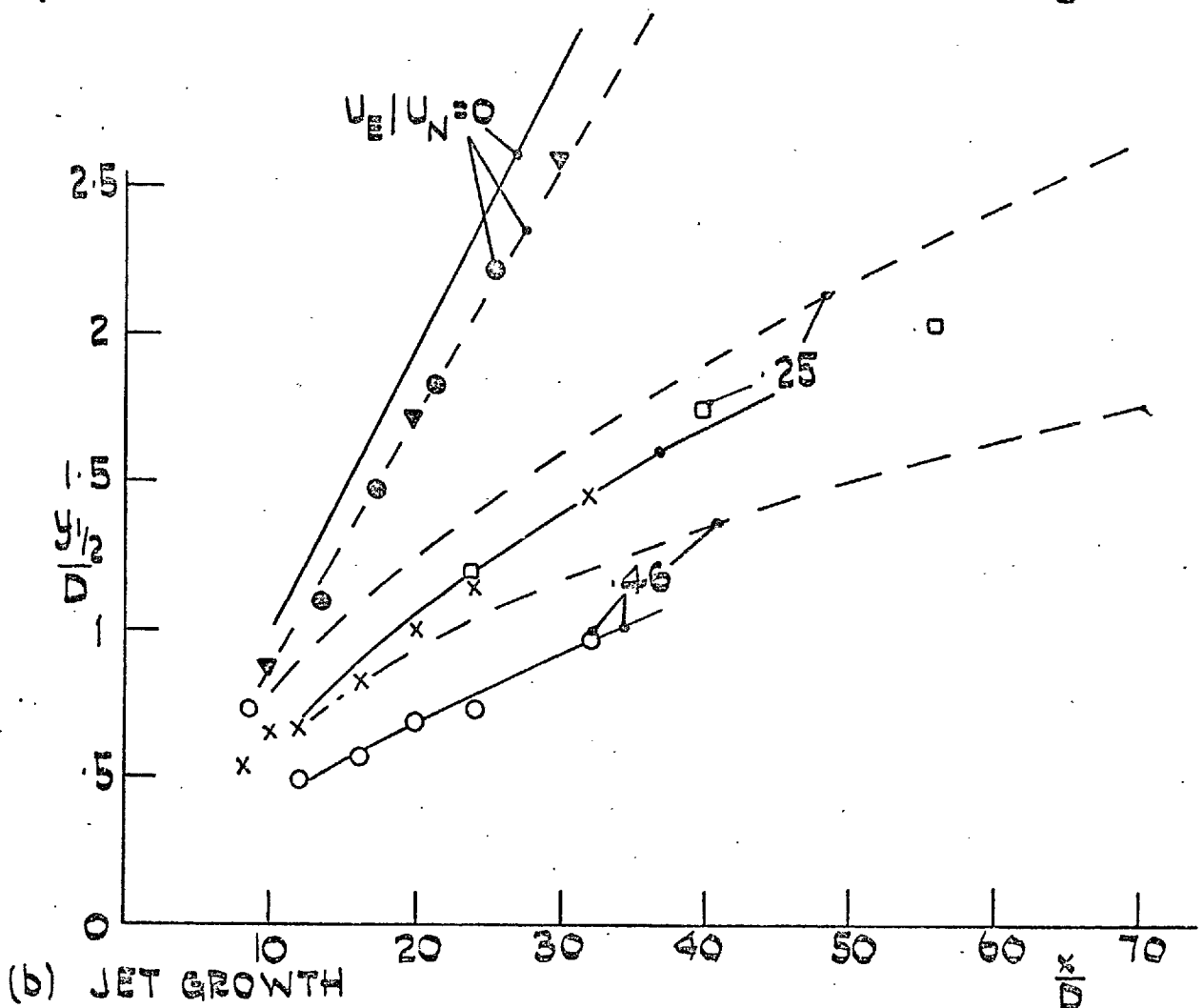
Round jet: Transition mixing layer - far jet

Theories of Bradbury and Naudascher

EXP. DATA: SEE FIG. 6.27



(a) VELOCITY DECAY

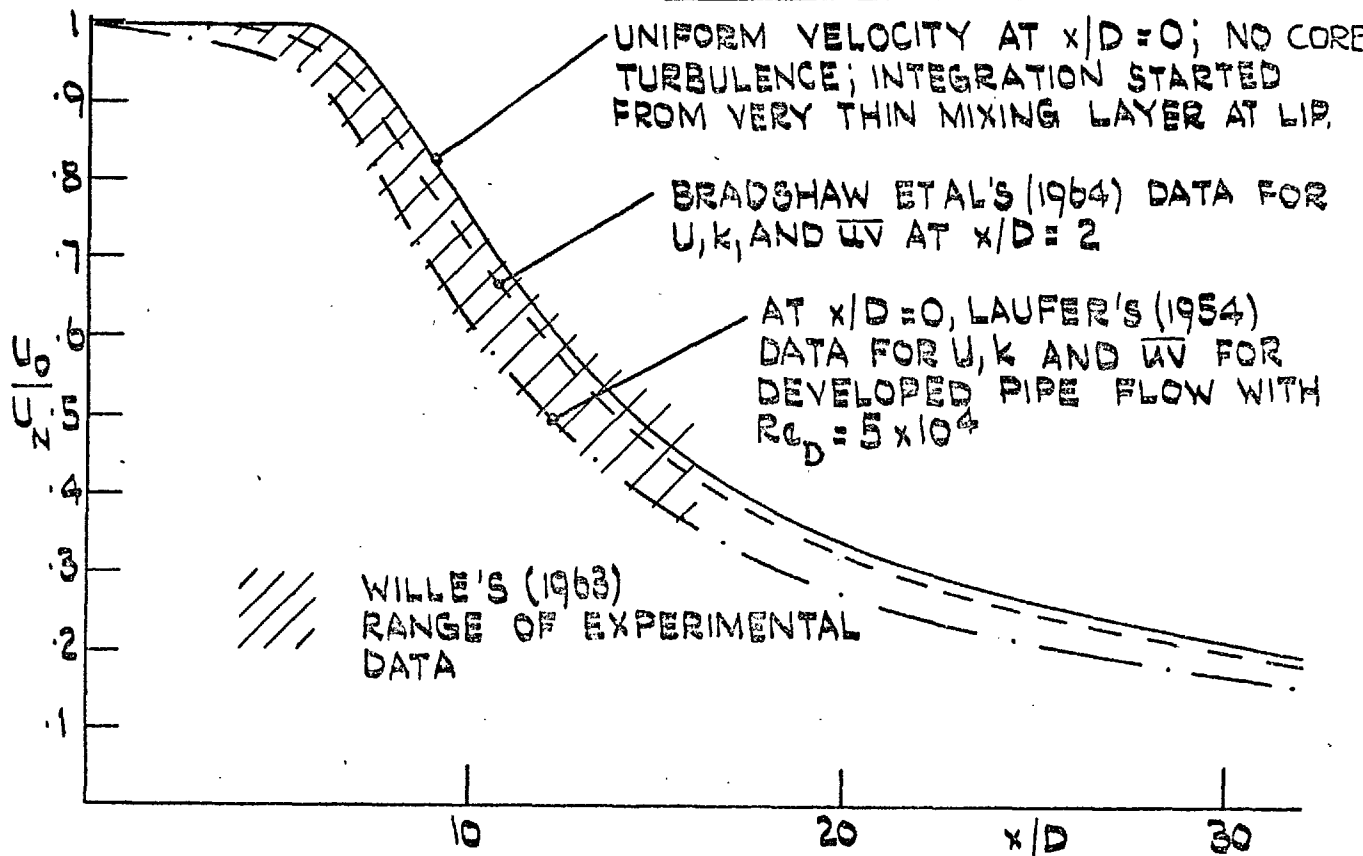


(b) JET GROWTH

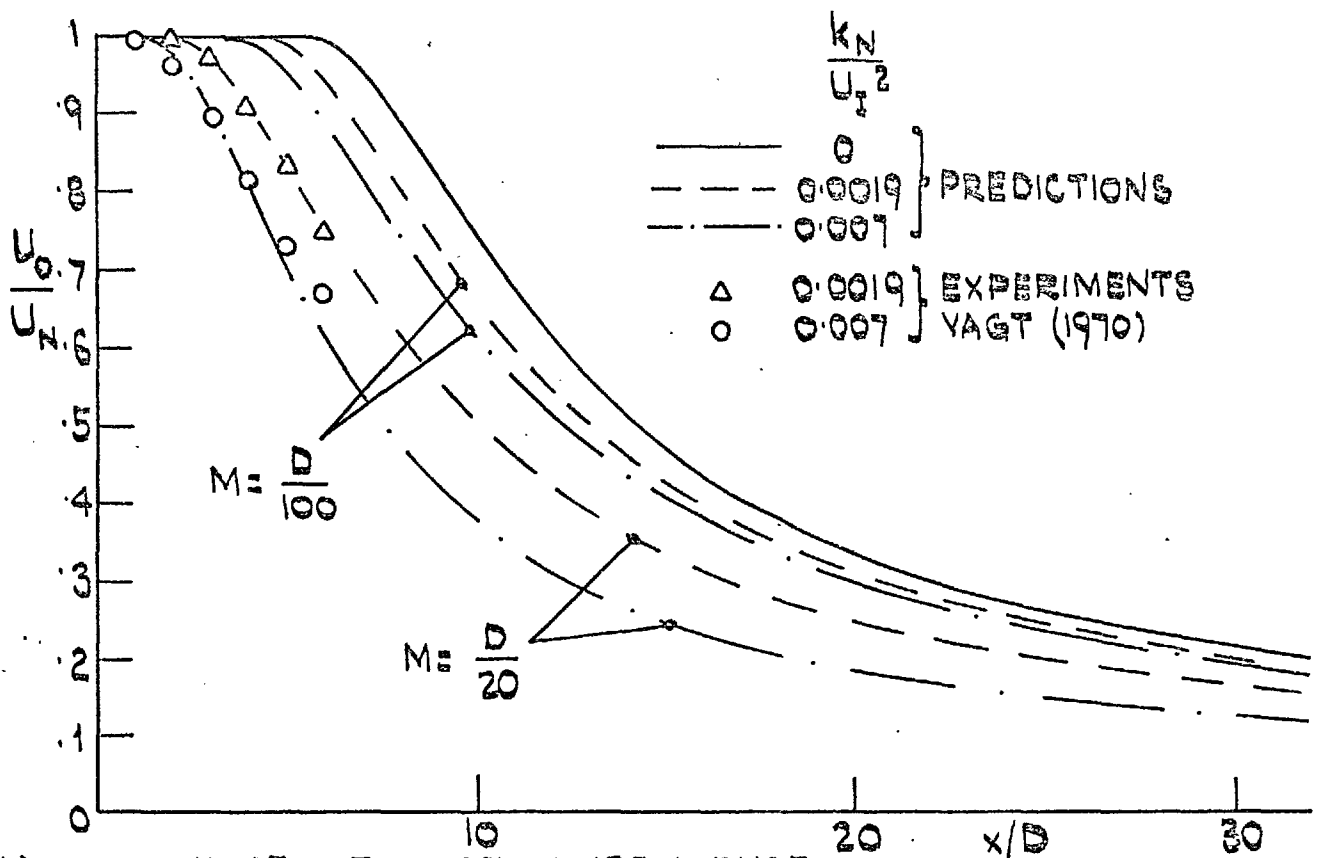
Fig. 6.25

Round jet: Transition mixing layer - far jet
Theories of Abramovich and Patel

INITIAL CONDITIONS:



(a). VARIOUS EXPERIMENTAL INITIAL CONDITIONS



(b) INFLUENCE OF CORE TURBULENCE

Fig. 6.25

Influence of initial conditions on the development of a round jet in still air

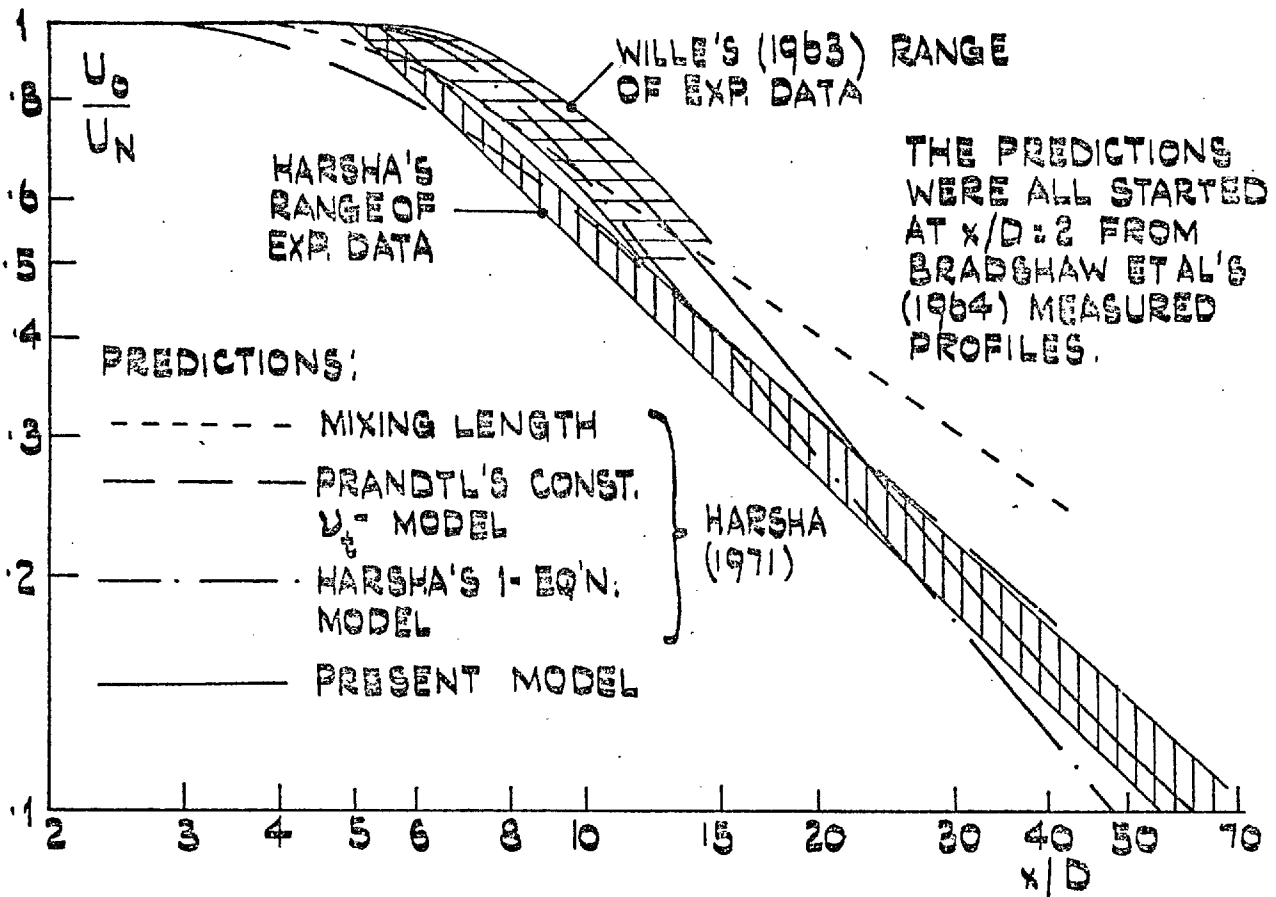


Fig. 6.27 Round jet issuing into stagnant surroundings: predictions with various models

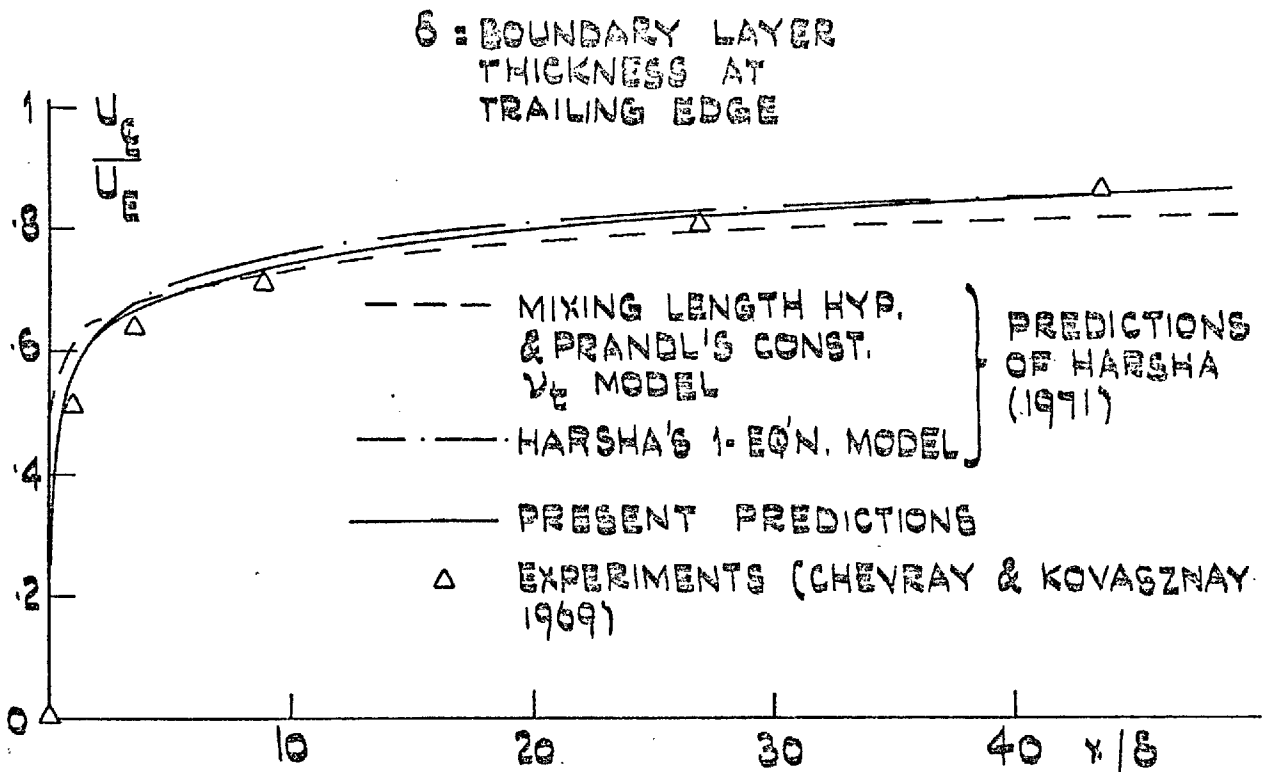


Fig. 6.28 Centr-line velocity in the wake behind a flat plate: predictions with various models

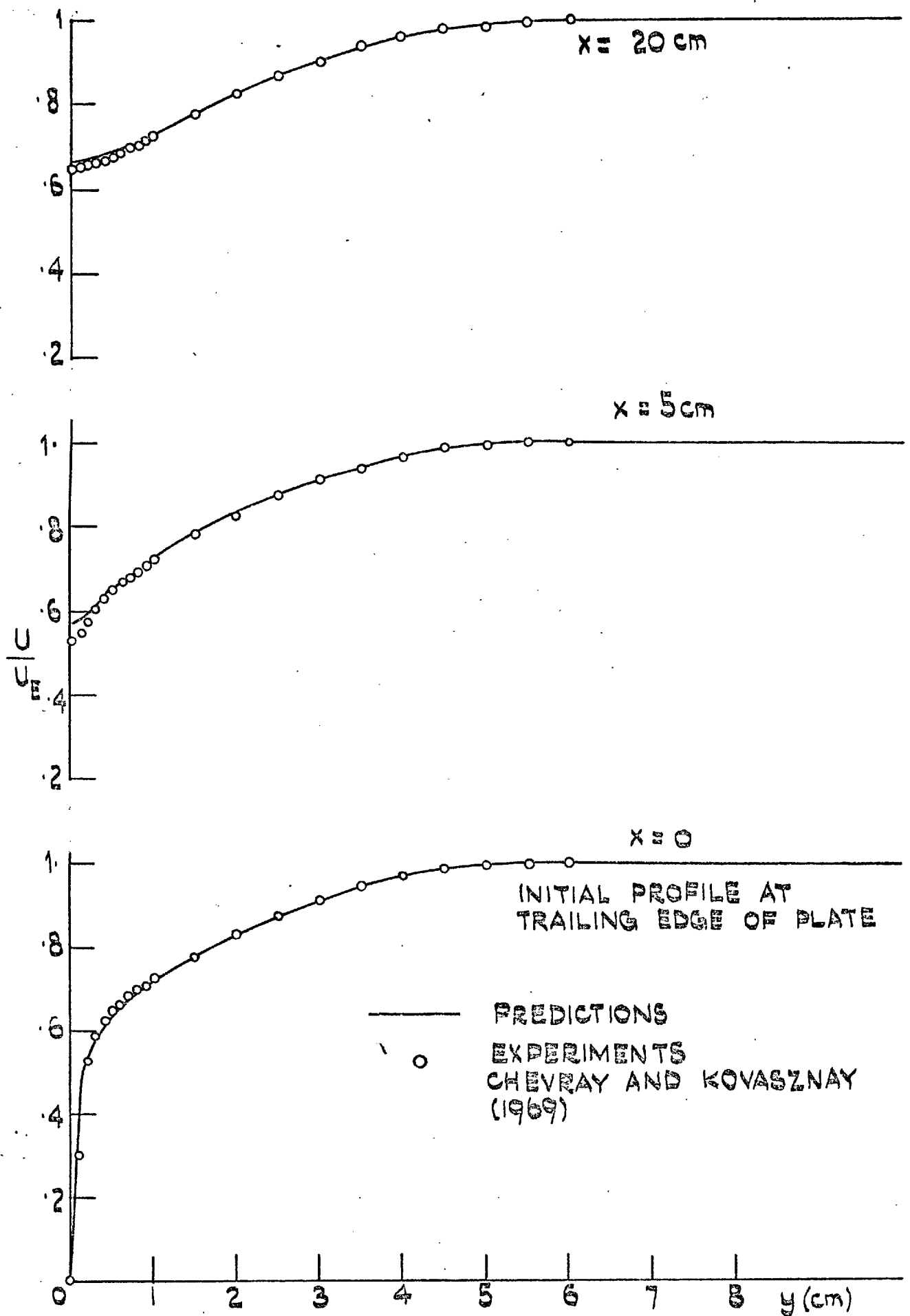


Fig. 6.29

Velocity profiles in the wake behind a flat plate

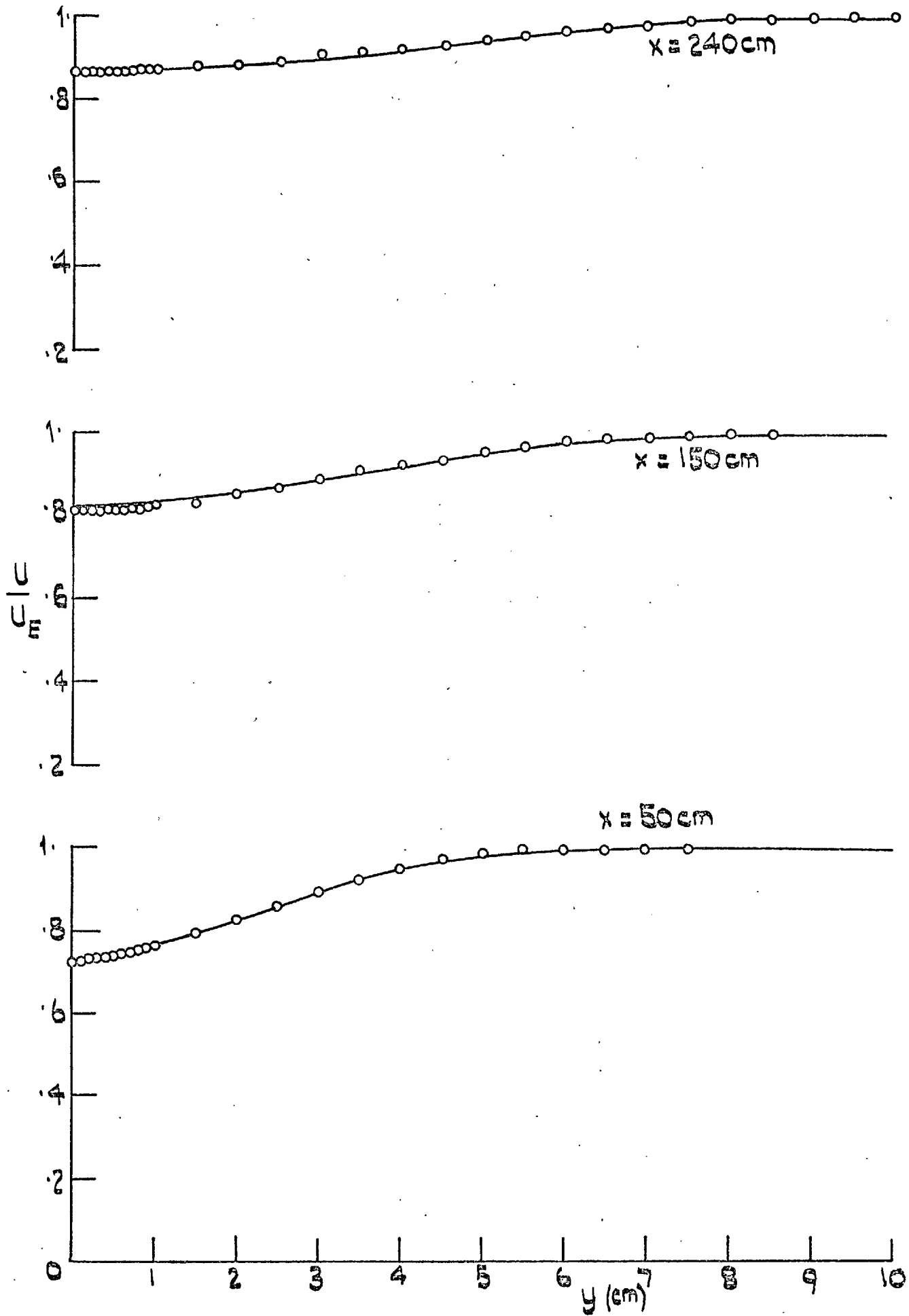


Fig. 6.29 continued

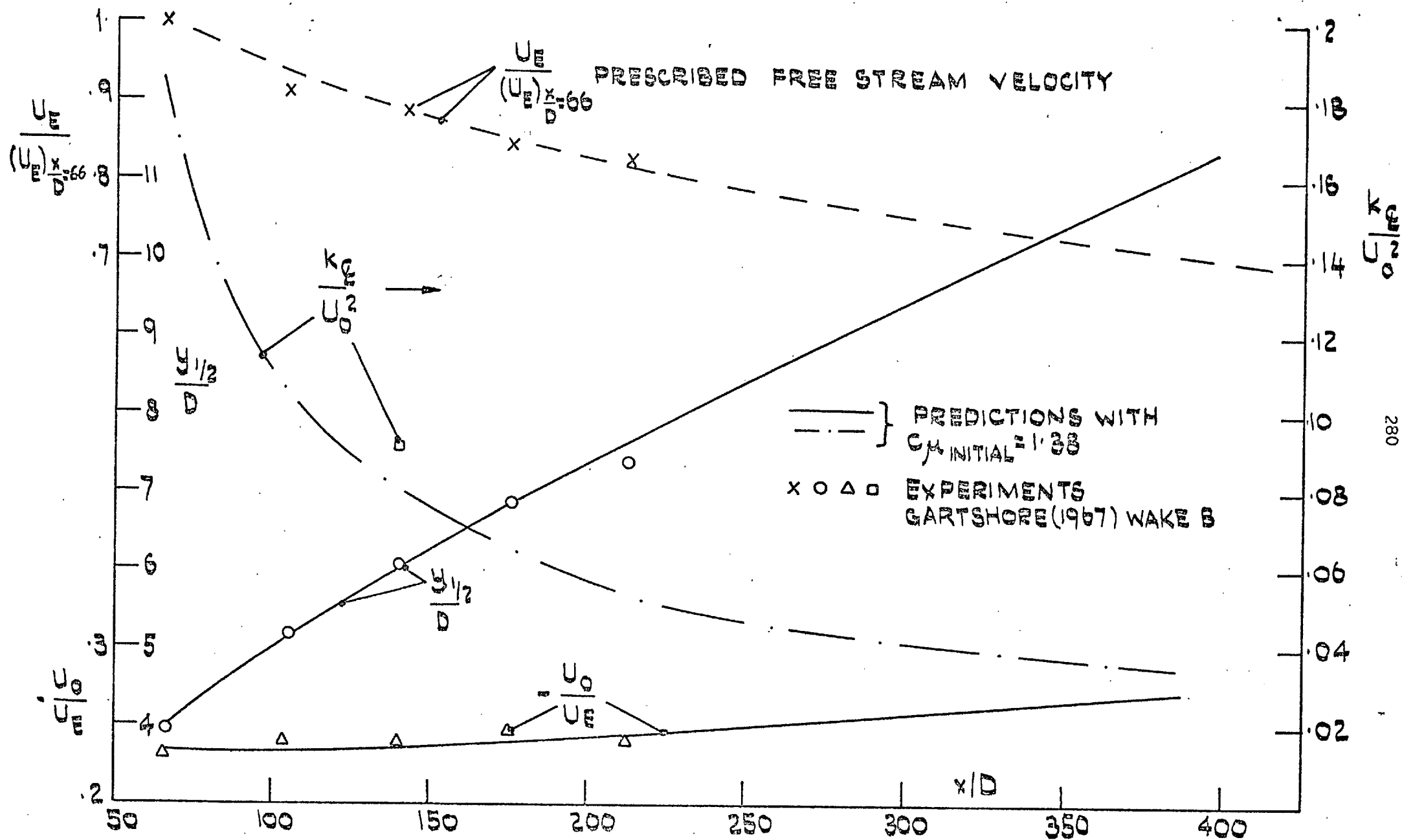


Fig. 6.30 Plane wake in a pressure gradient

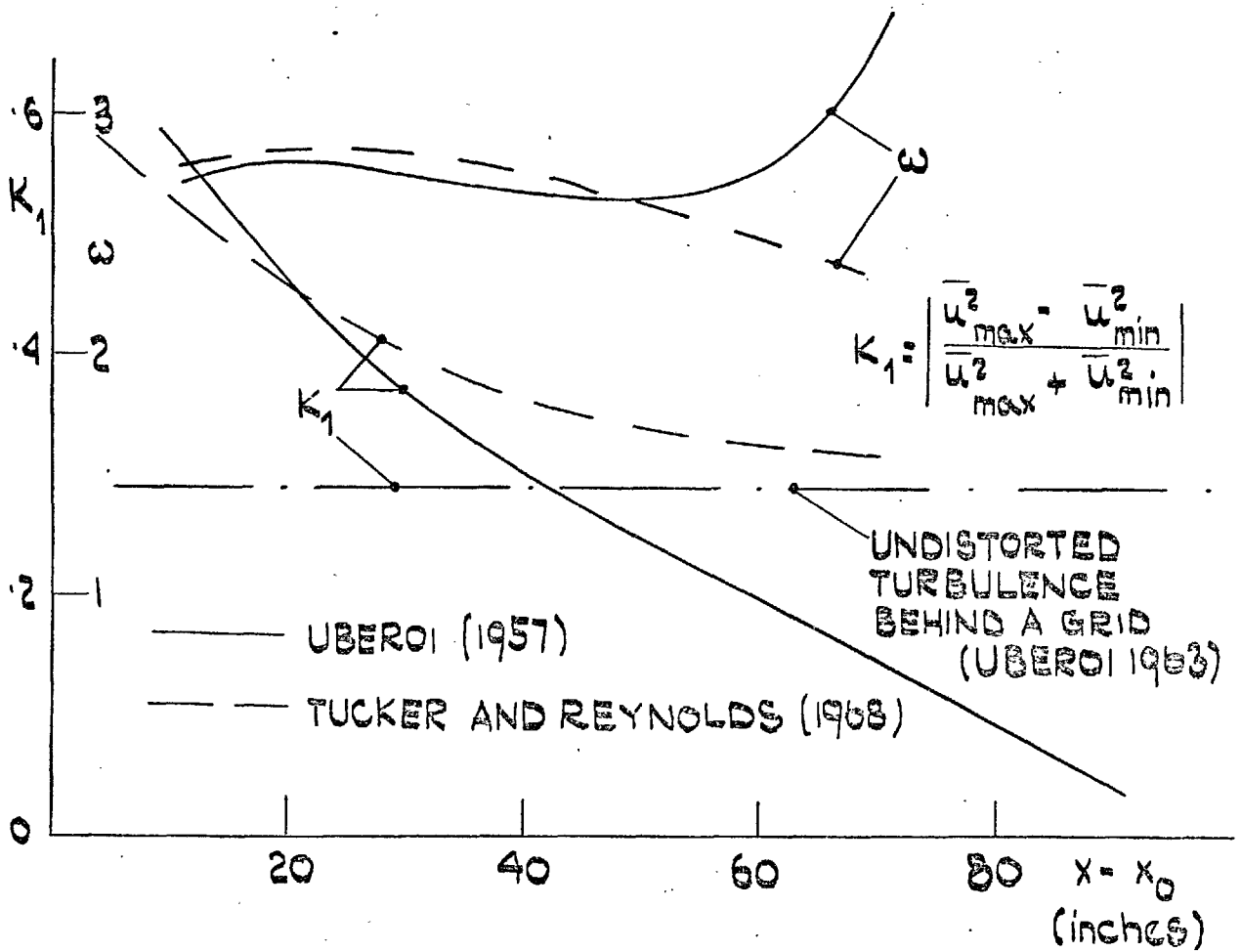


Fig. G.1 Return to isotropy in the absence of mean velocity gradients

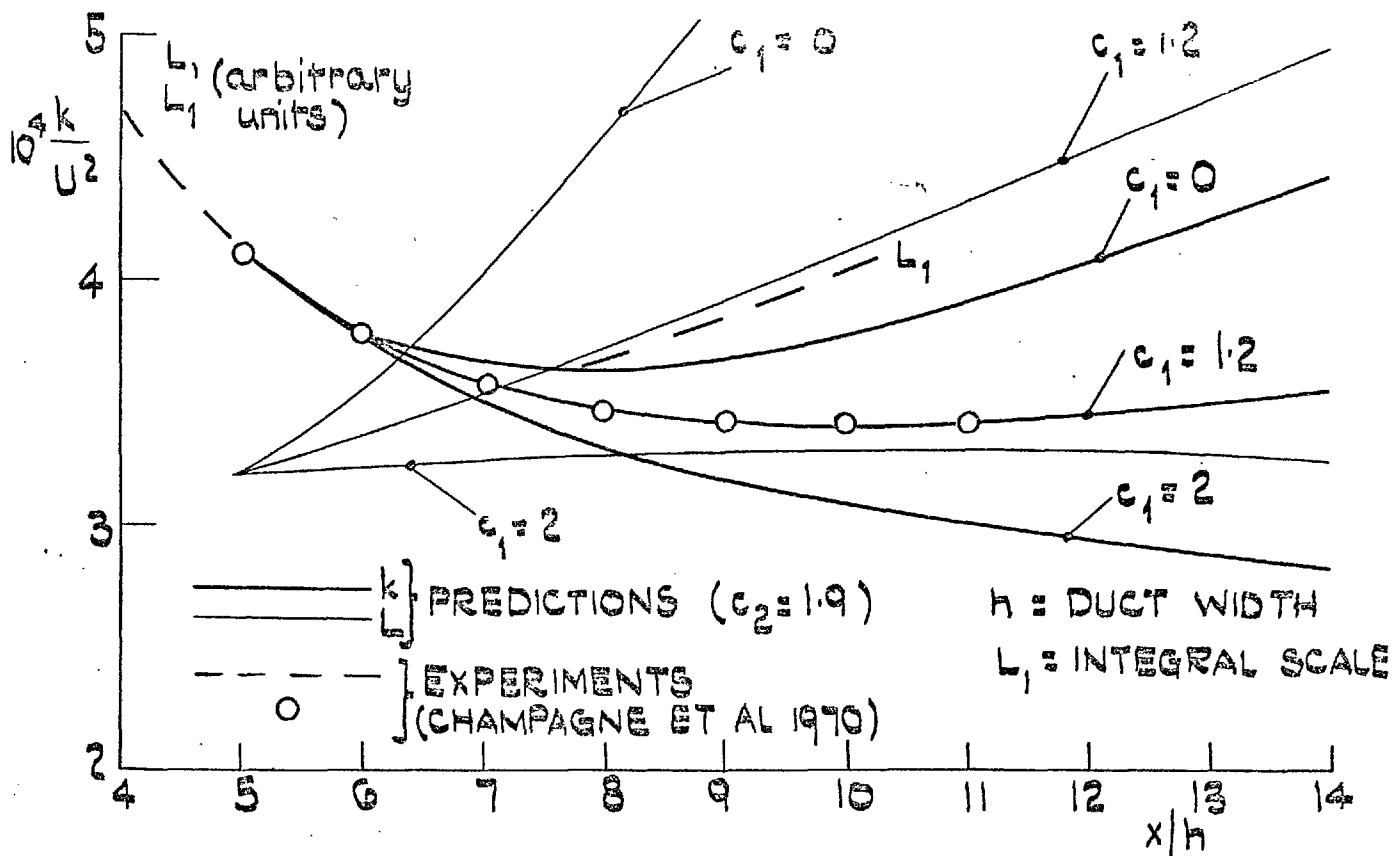


Fig. G.2 Nearly homogeneous shear flow: predictions with P from experiments

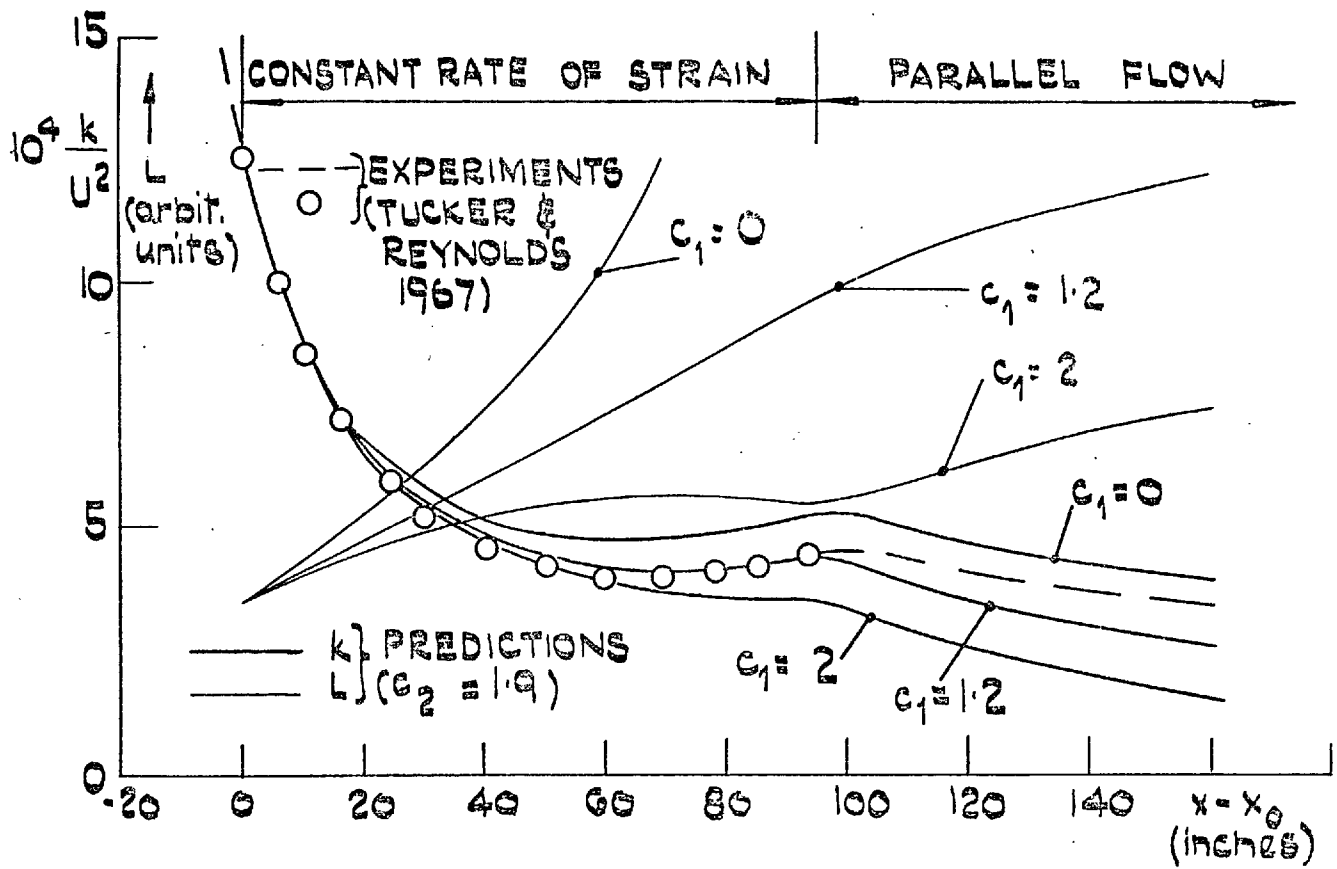


Fig. G.3 Turbulence subjected to plane strain:
predictions with P from experiments

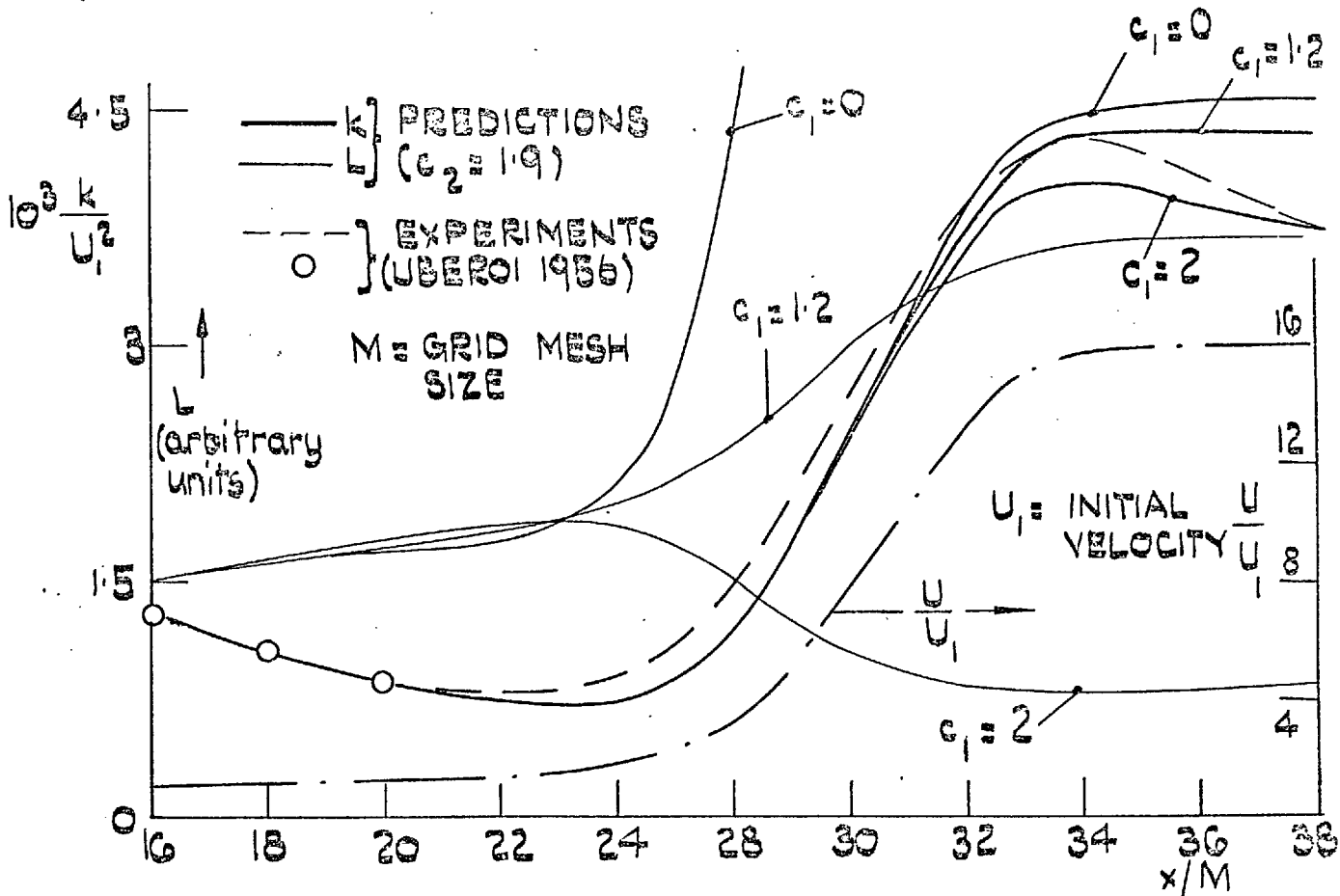


Fig. G.4 Flow through a 16:1 contraction:
predictions with P from experiments

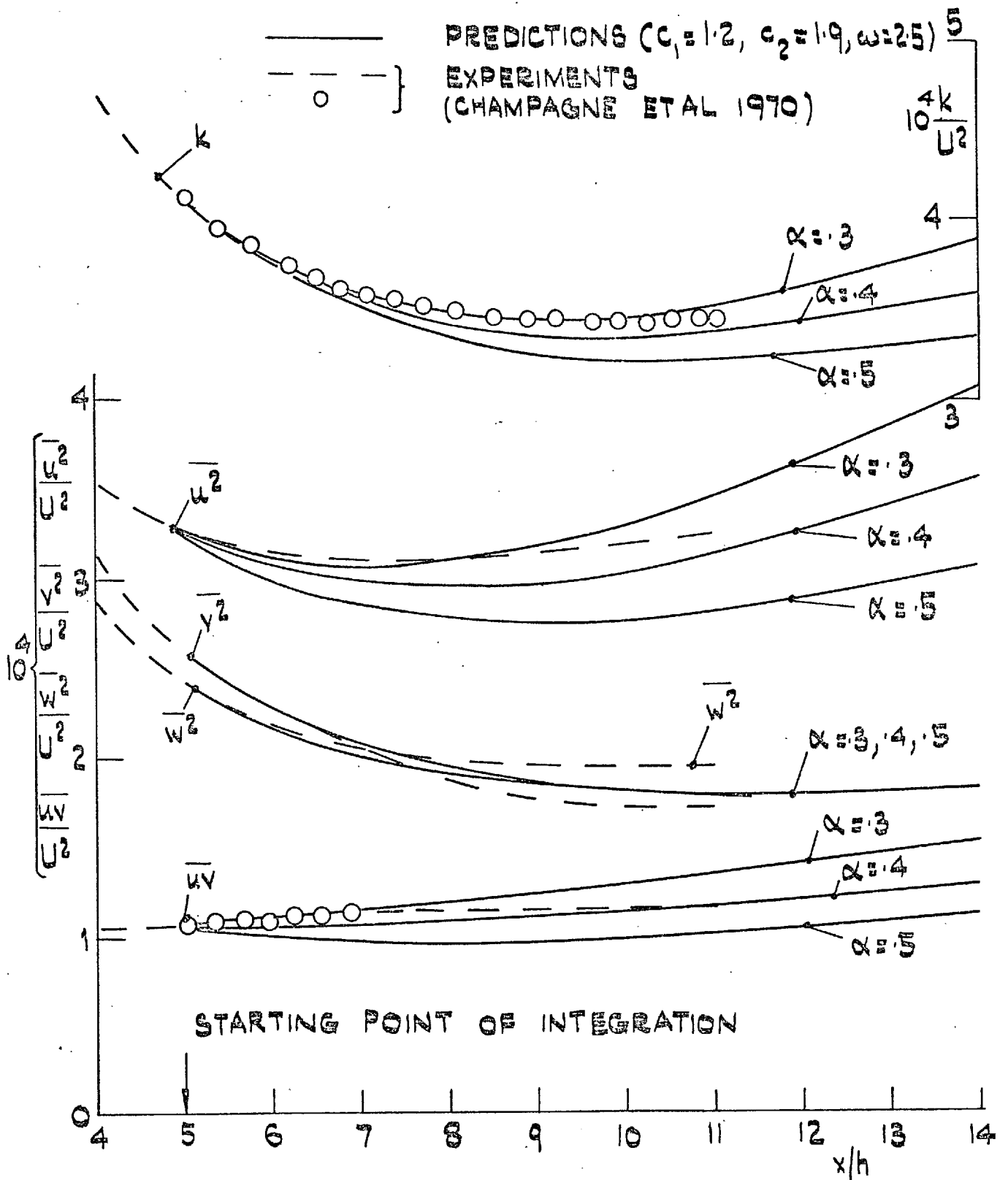


Fig. G.5

Distributions of k , $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, and $\overline{uv}$ in
 nearly homogeneous shear flow

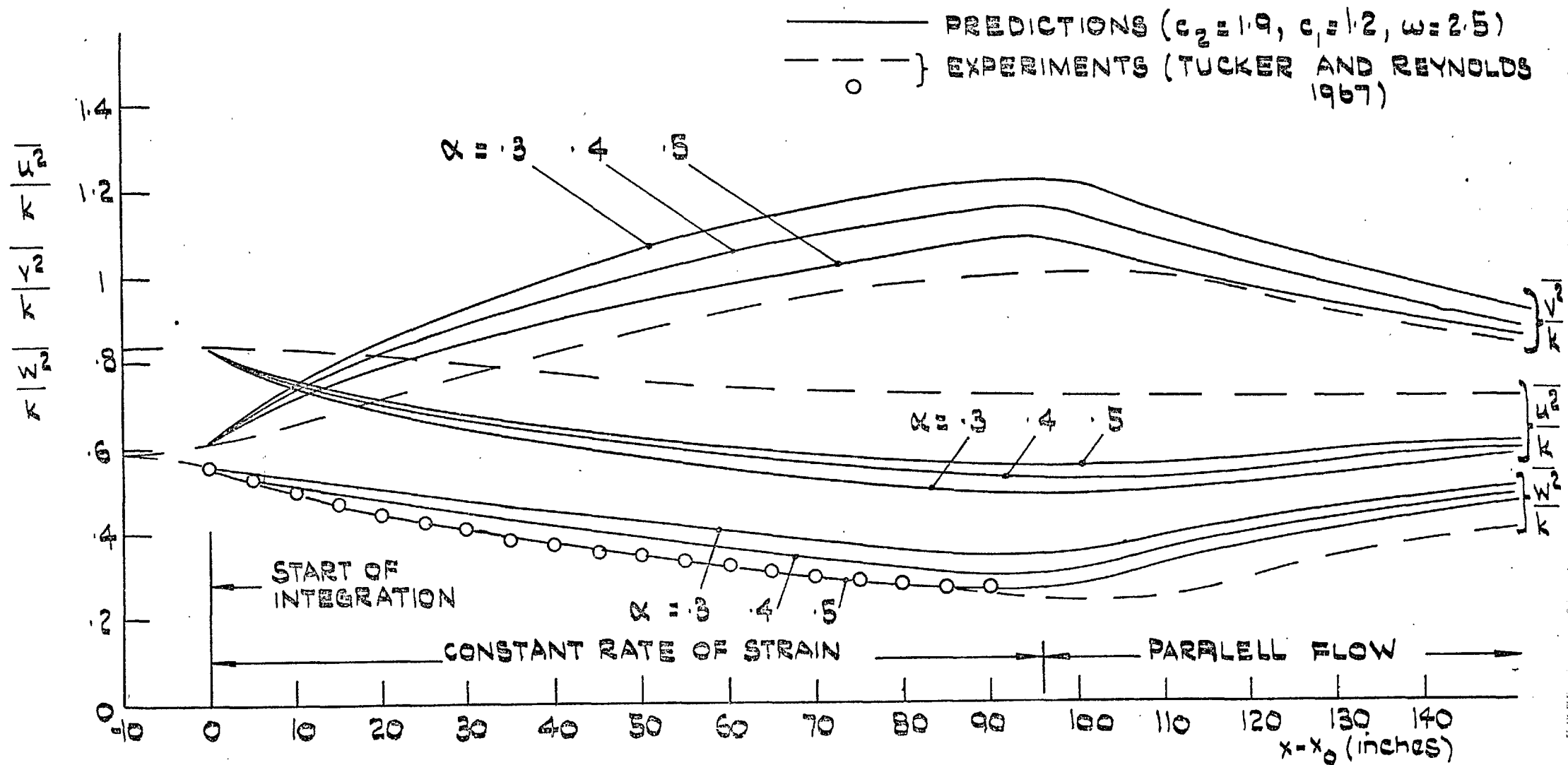


Fig. G.6 Development of $\overline{u^2}/k$, $\overline{v^2}/k$, and $\overline{w^2}/k$ in a "plane-strain duct" and in the following parallel duct

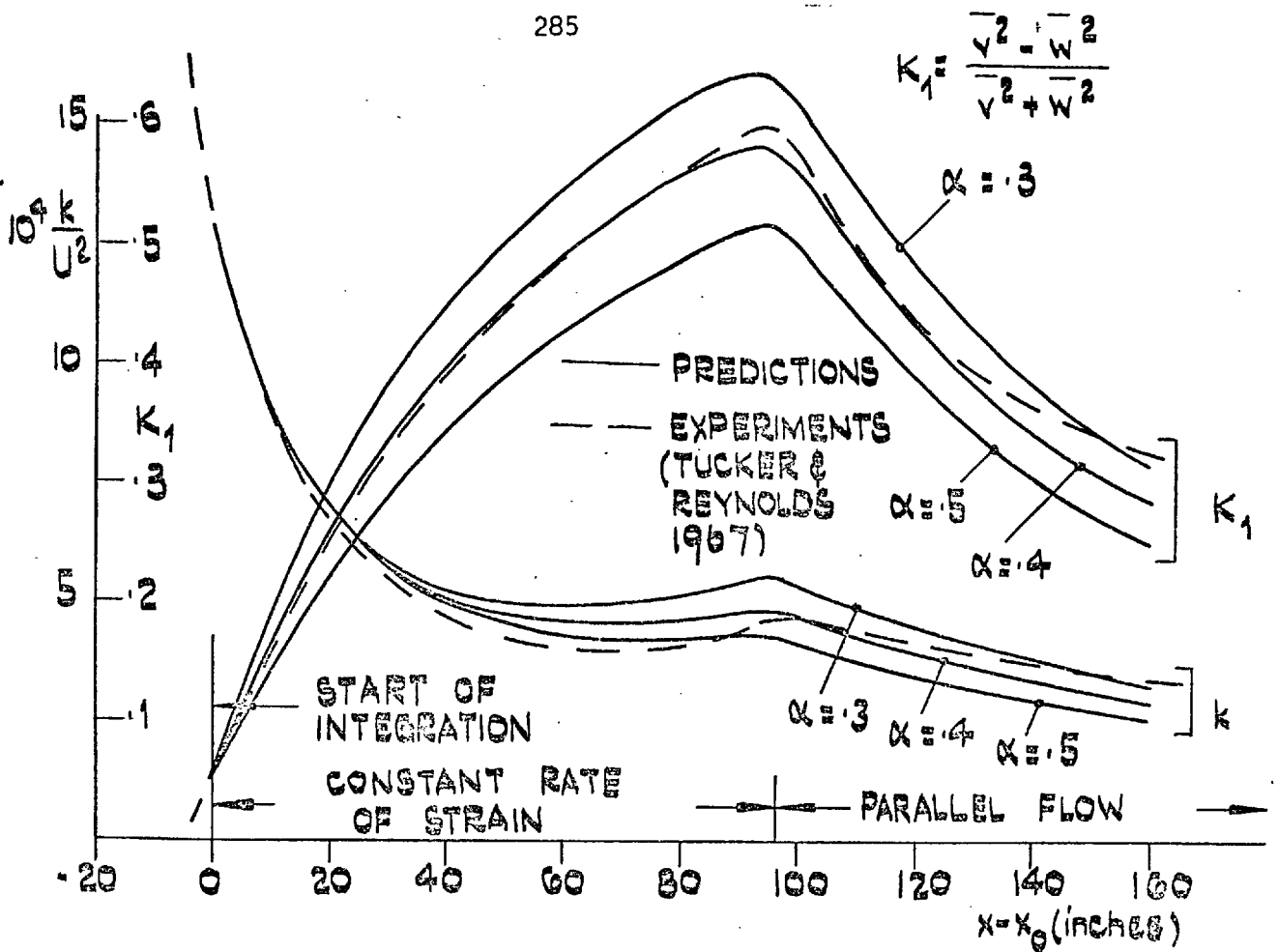


Fig. G.7 Development of k and K_1 in a "plane-strain duct" and in the following parallel duct

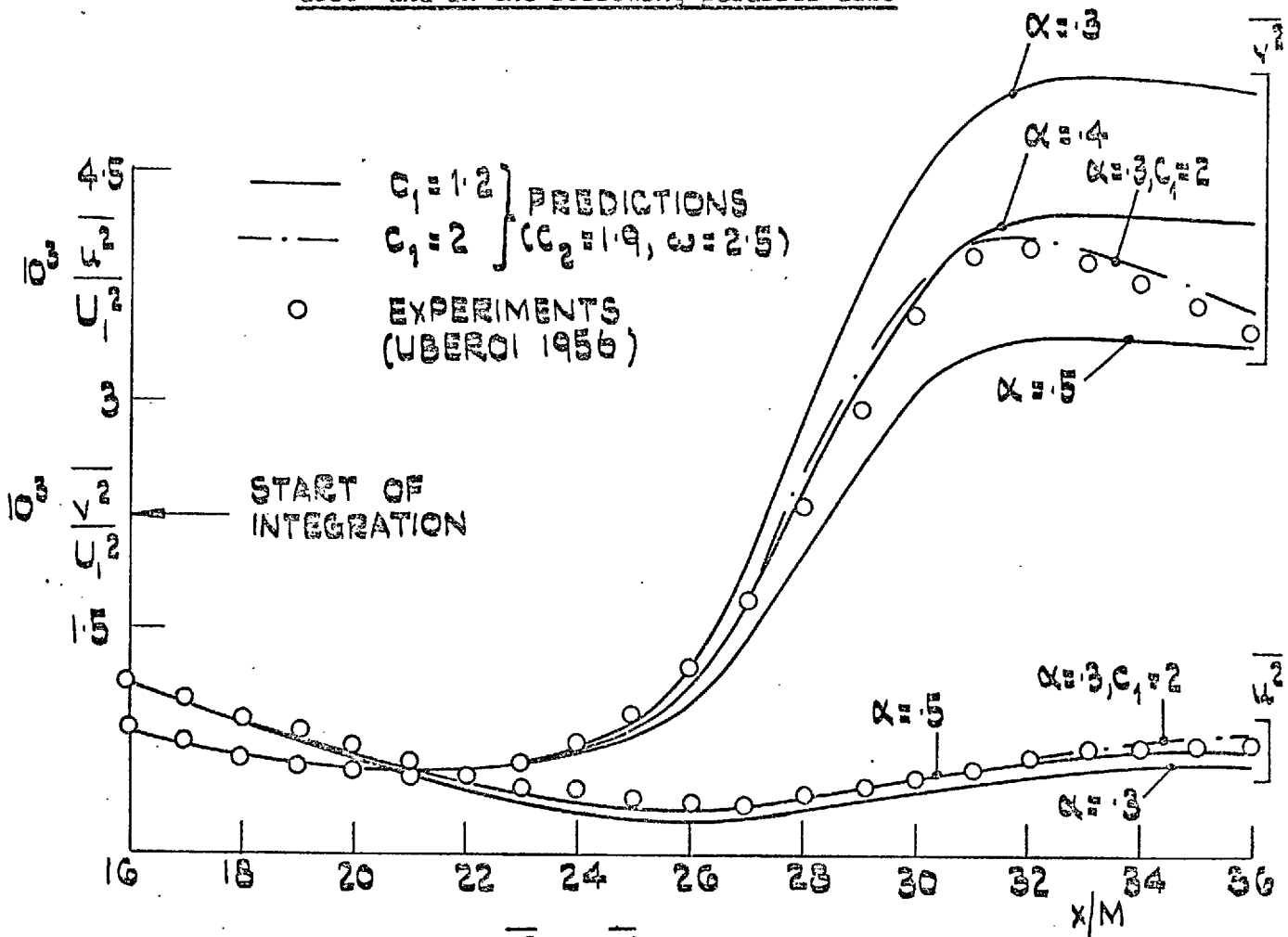


Fig. G.8 Development of $\overline{u^2}$ and $\overline{v^2}$ in a 16:1 contraction and in the following parallel duct

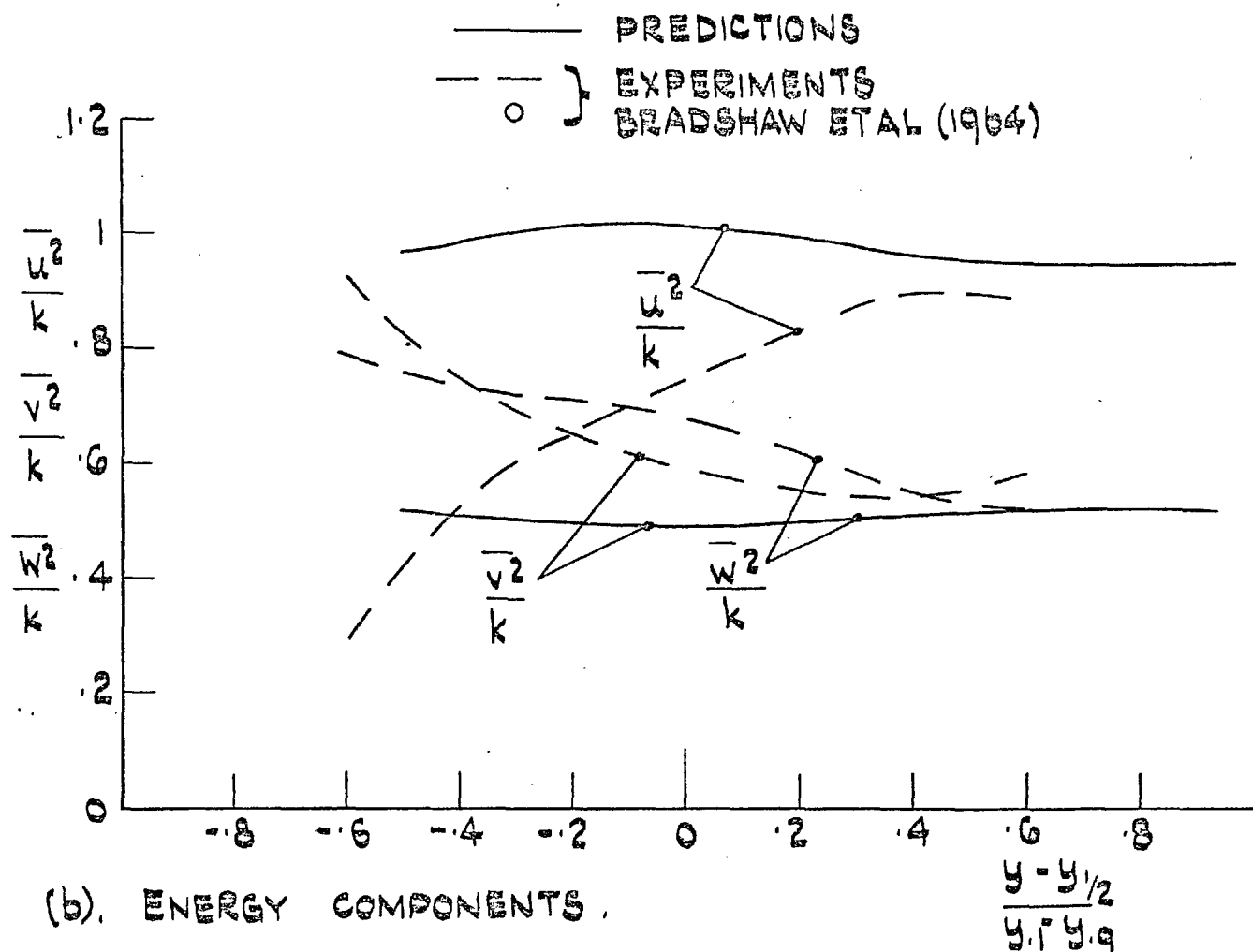
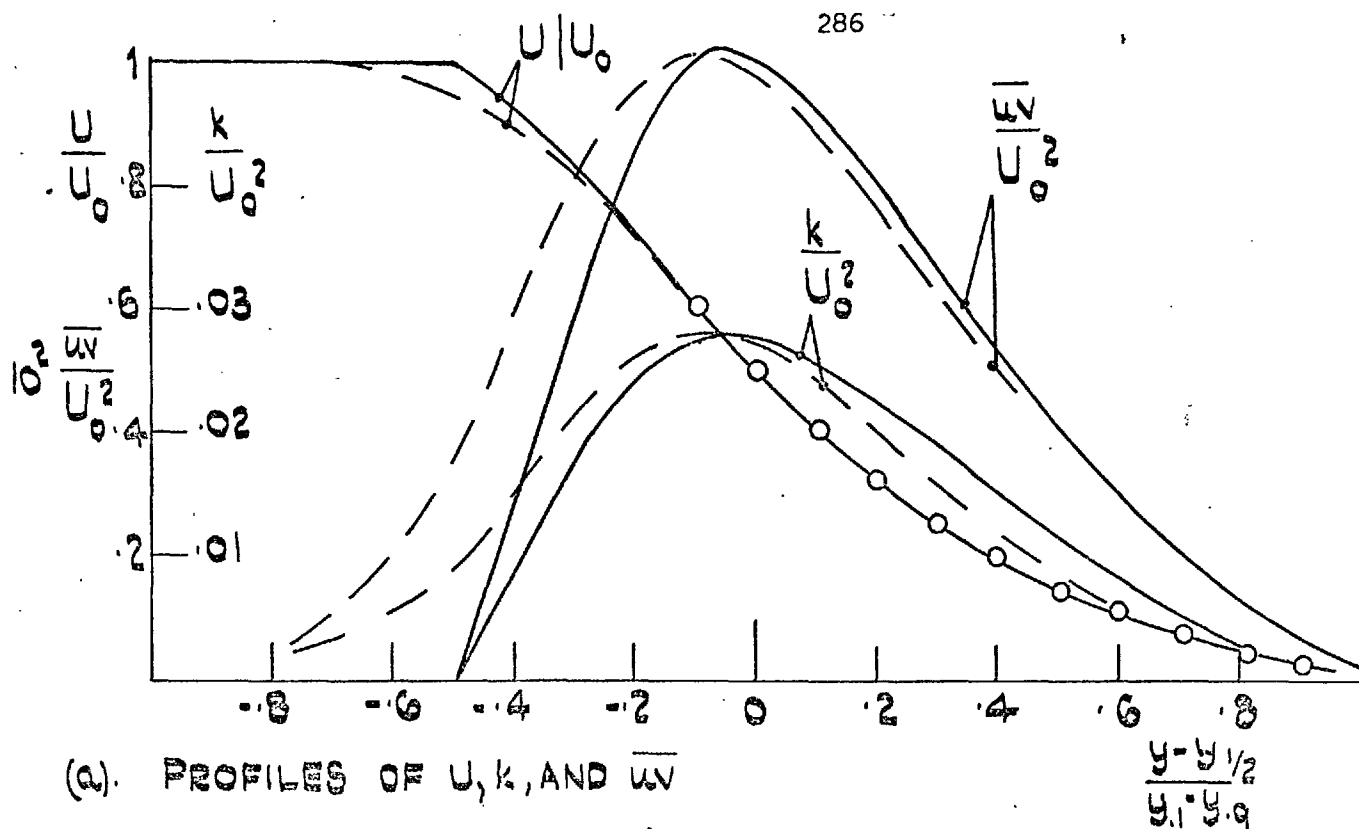


Fig. G.9

Plane mixing layer: predictions with the $\overline{u_1 u_1} \epsilon$ -model

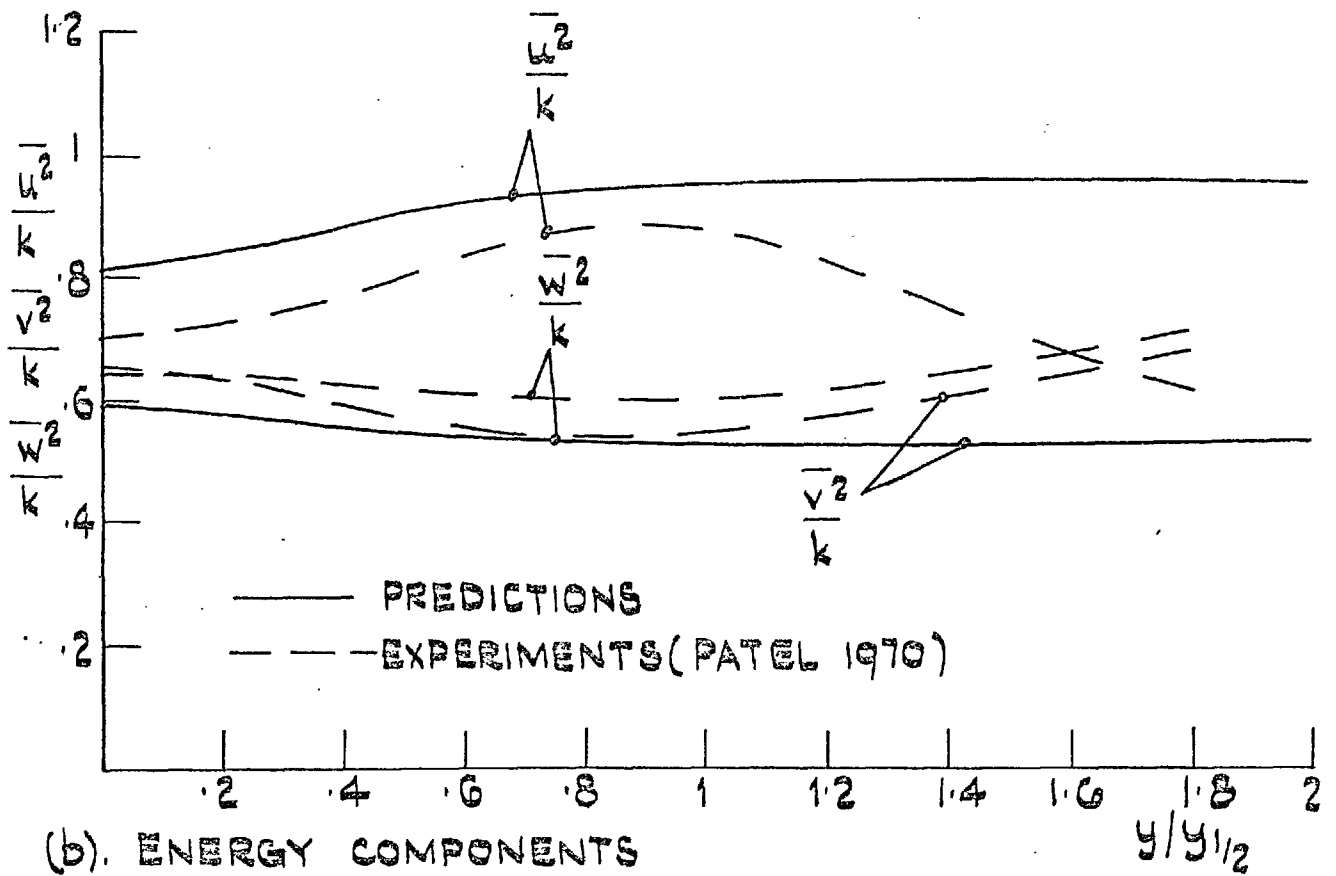
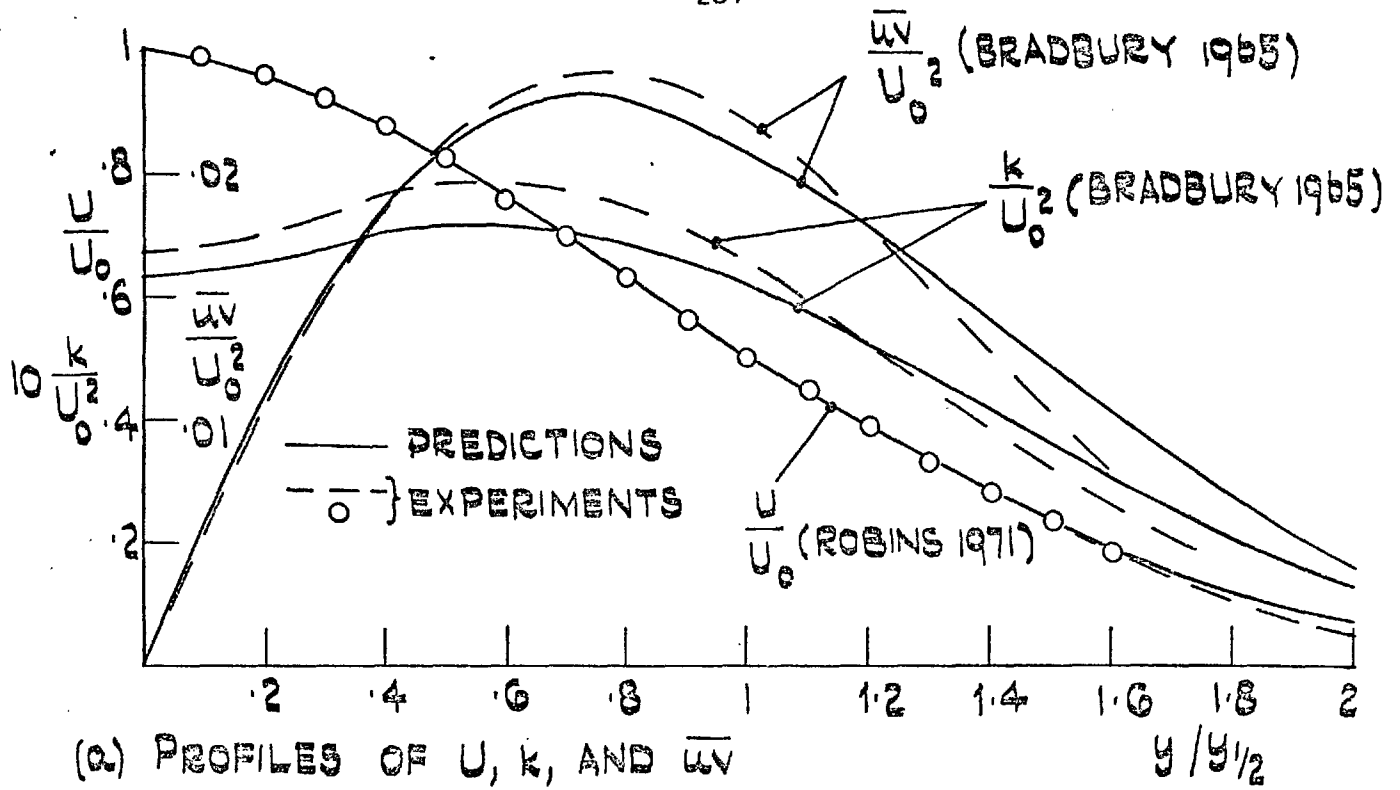
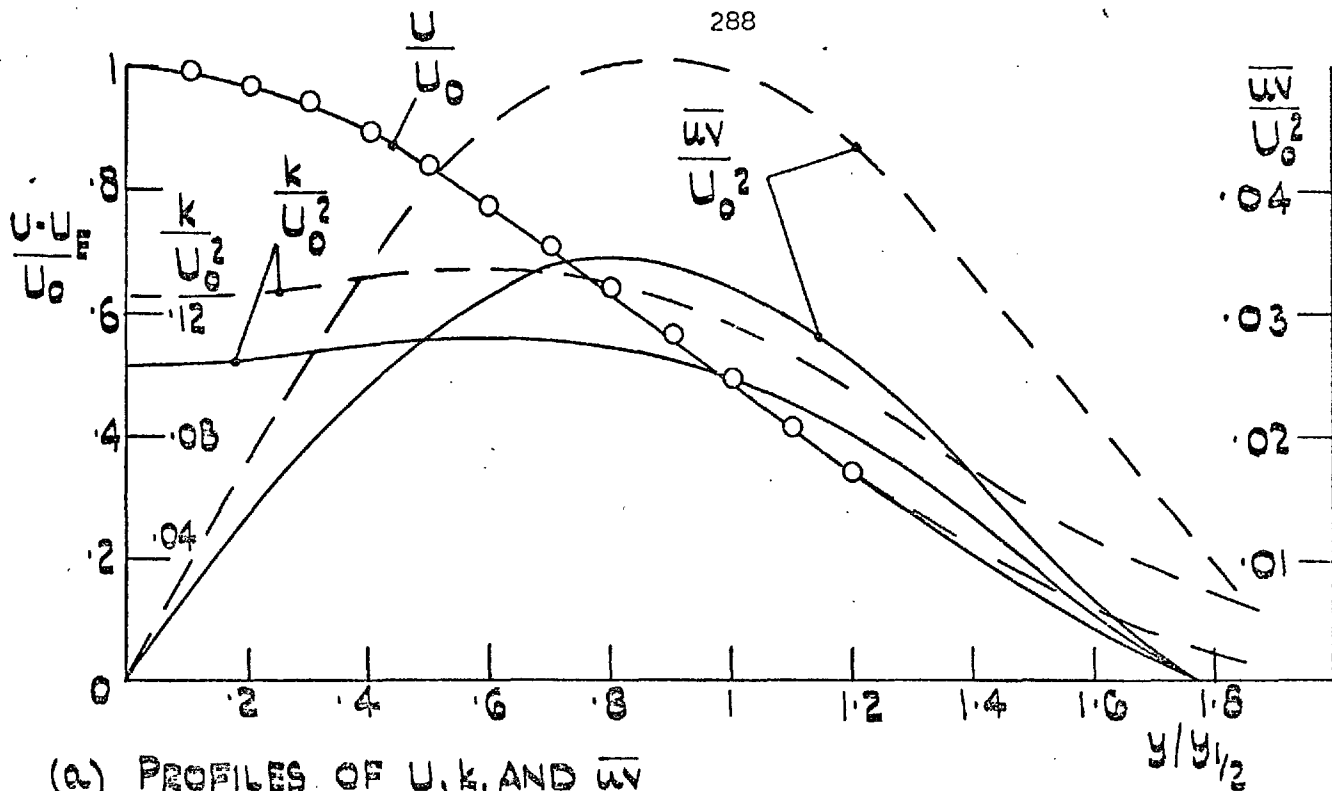
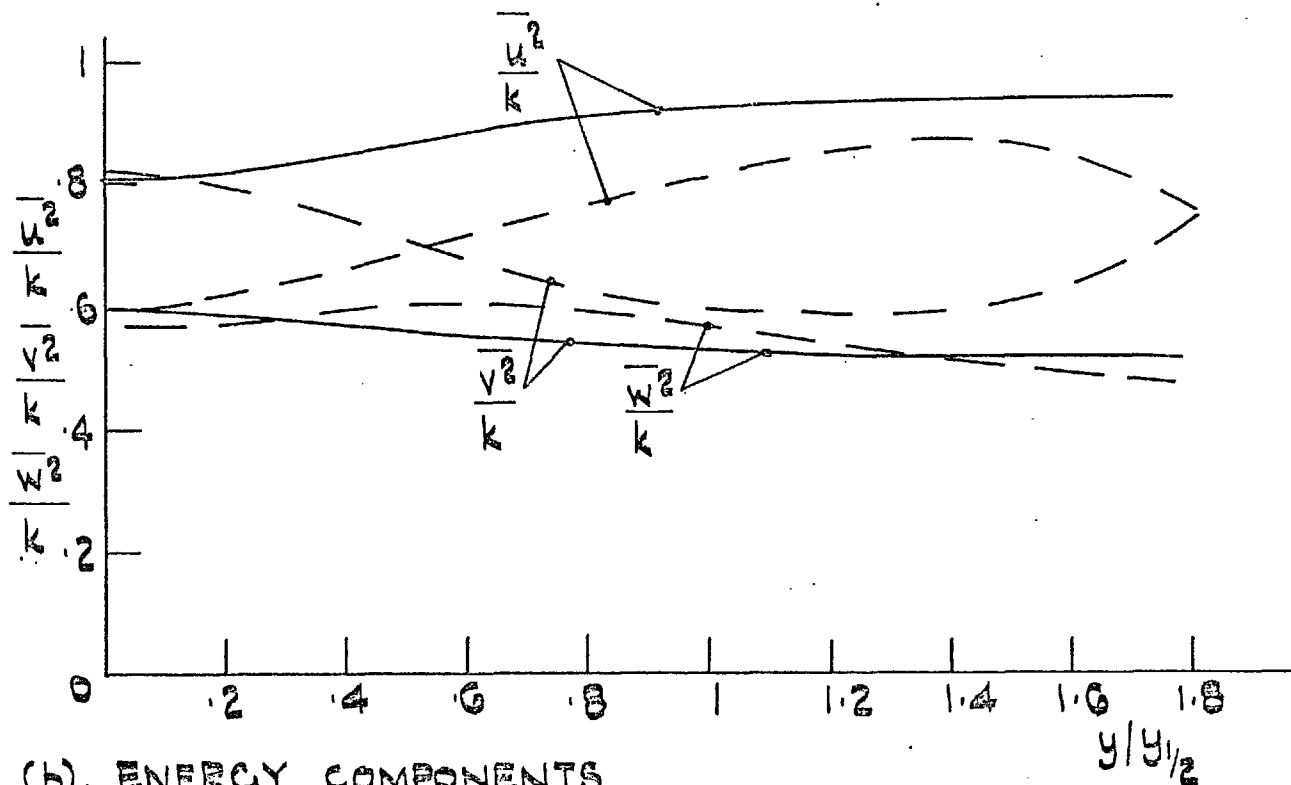


Fig. G.10 Plane jet: Predictions with the $\overline{u_i u_j} \epsilon$ -model

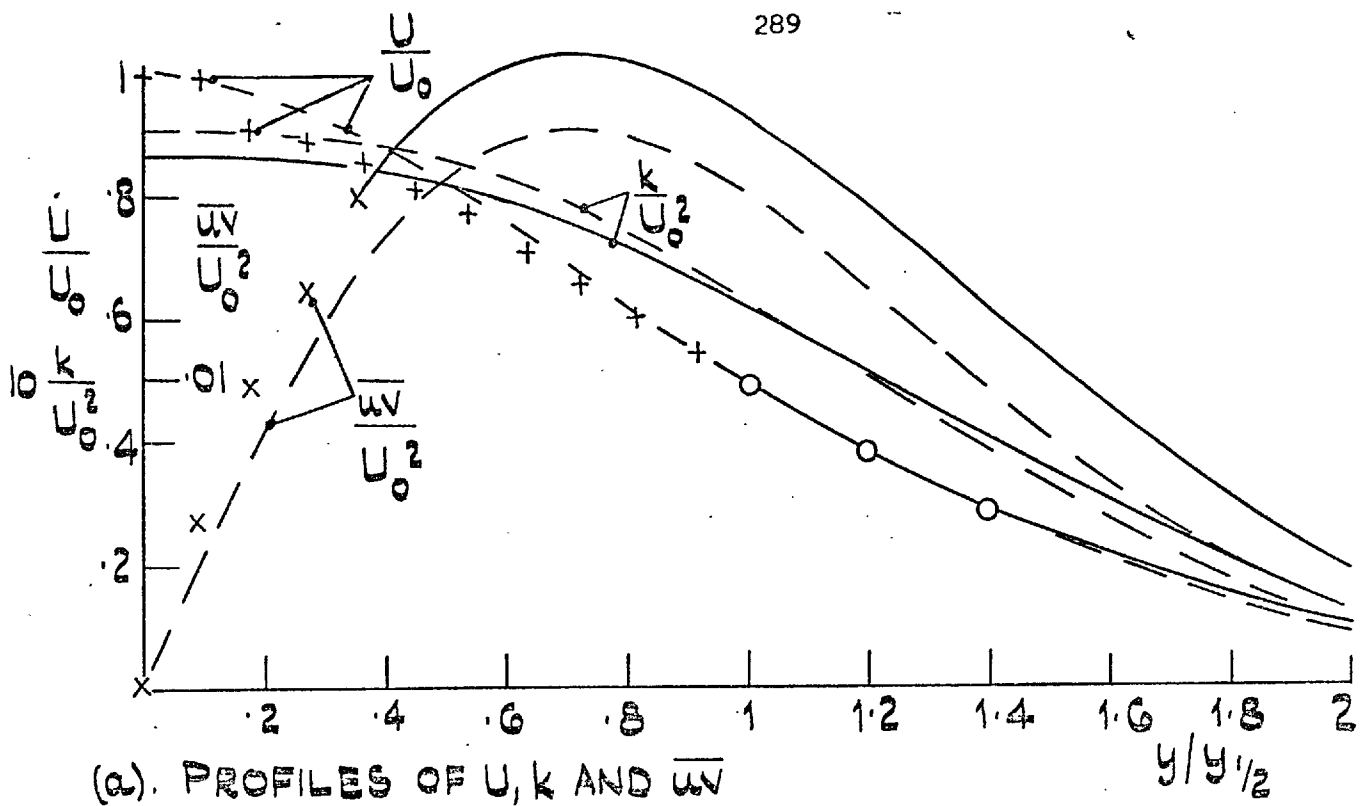


(a) PROFILES OF U , k , AND $\overline{uv}$

— PREDICTIONS
 - - - EXPERIMENTAL
 ○ (TOWNSEND 1956)



(b). ENERGY COMPONENTS



$\begin{array}{c} \text{---} \\ + \\ x \end{array} \left. \vphantom{\begin{array}{c} \text{---} \\ + \\ x \end{array}} \right\} \text{PREDICTIONS}$
 $\begin{array}{c} \text{---} \\ \circ \end{array} \left. \vphantom{\begin{array}{c} \text{---} \\ \circ \end{array}} \right\} \text{EXPERIMENTS (PRESENT DATA)}$

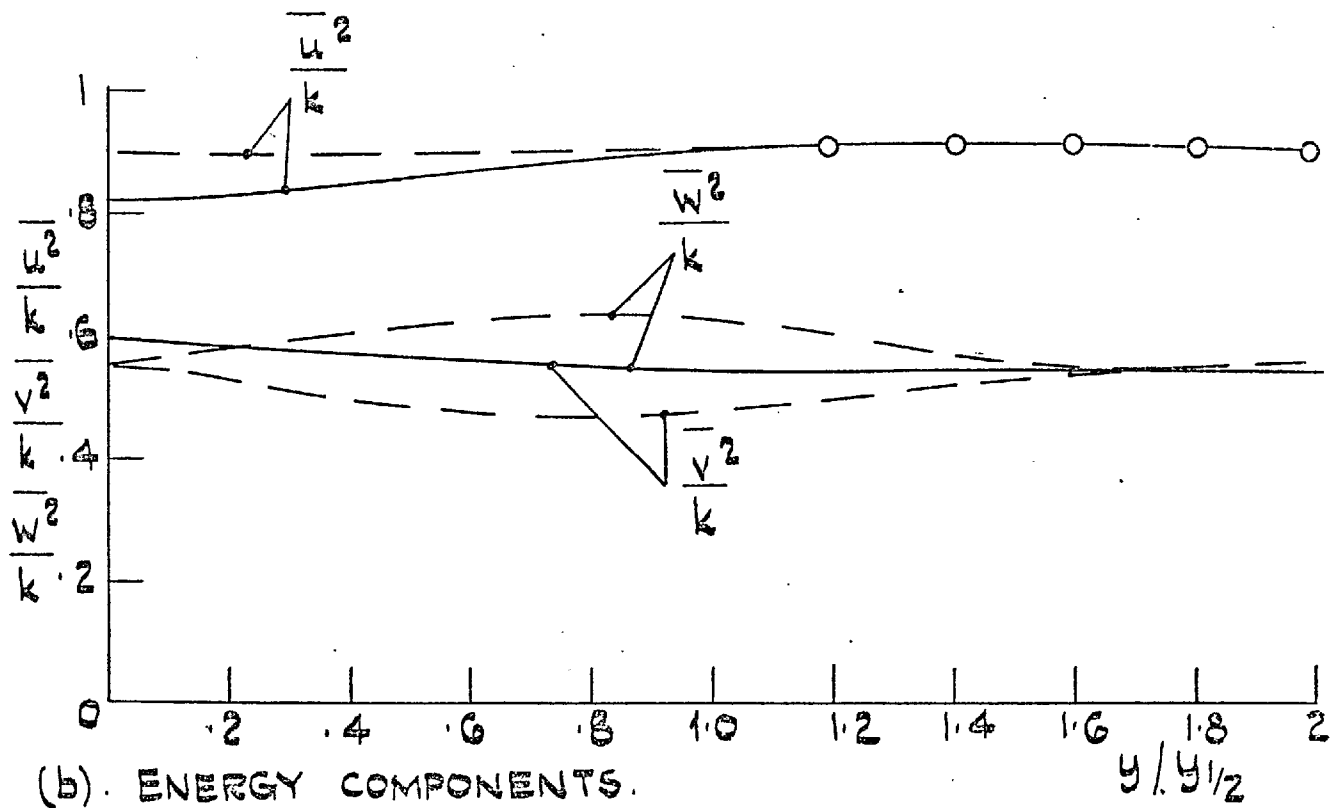


Fig. G.12 Round jet: Predictions with the $\overline{u_i u_j} \epsilon$ -model

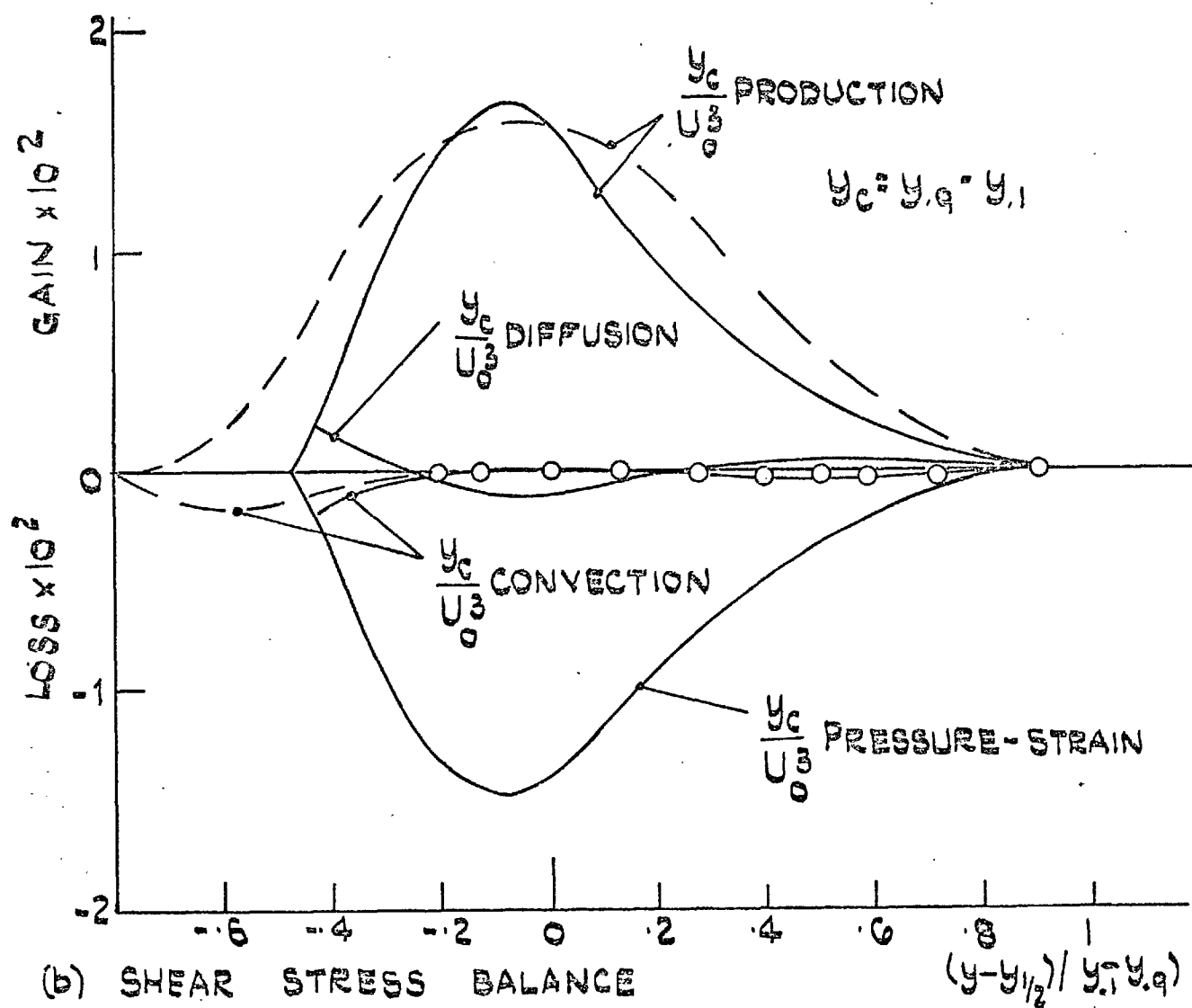
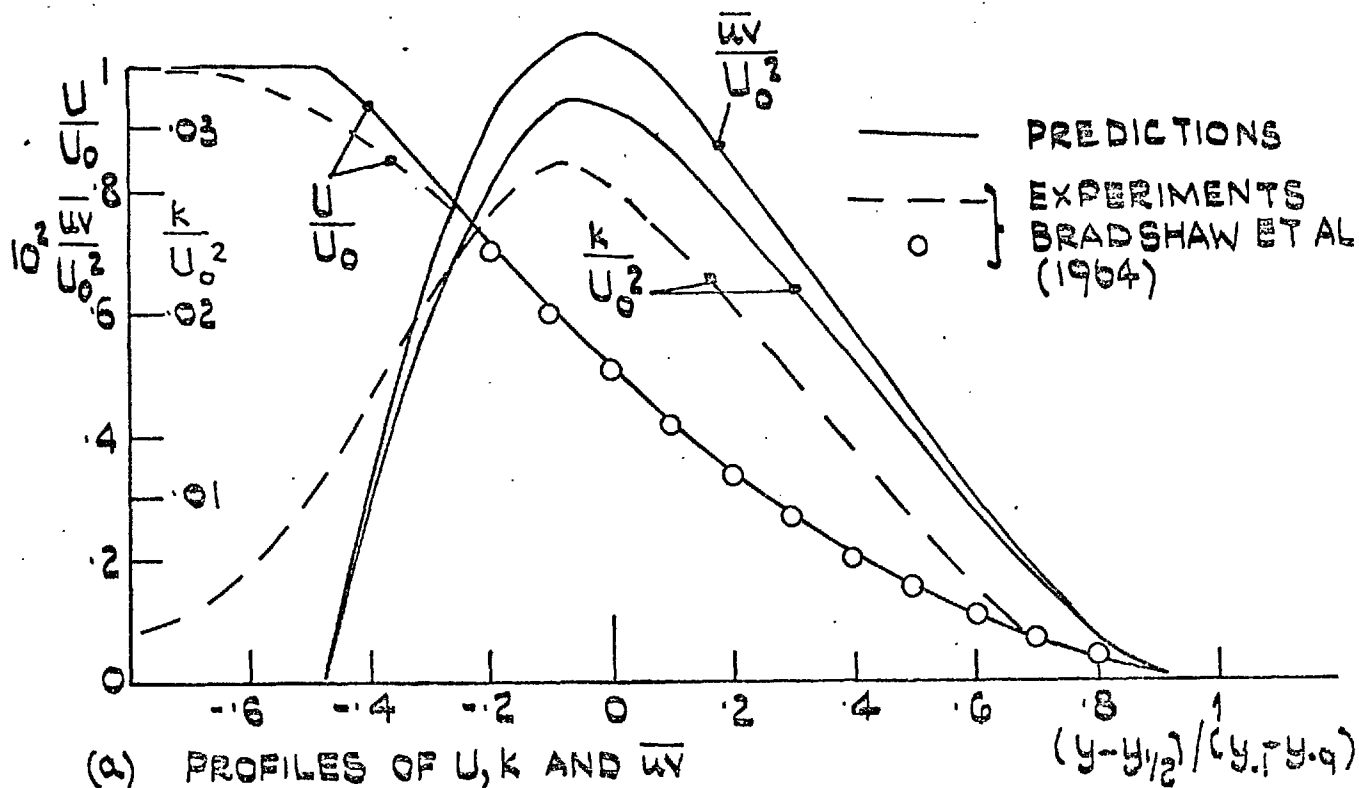


Fig. H.1 Prediction of the mixing layer ($U_E=0$) with the τ_{KE} -model

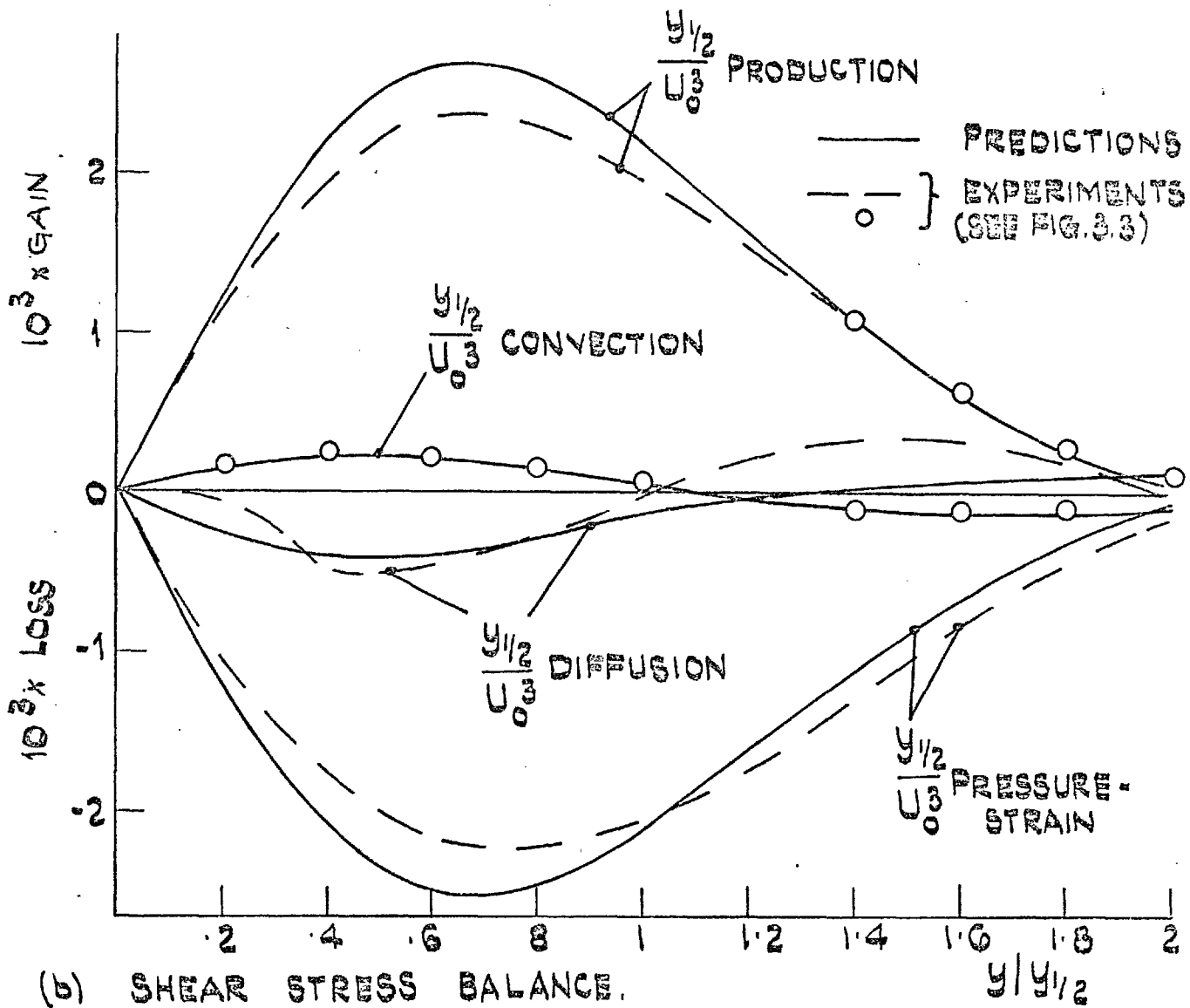
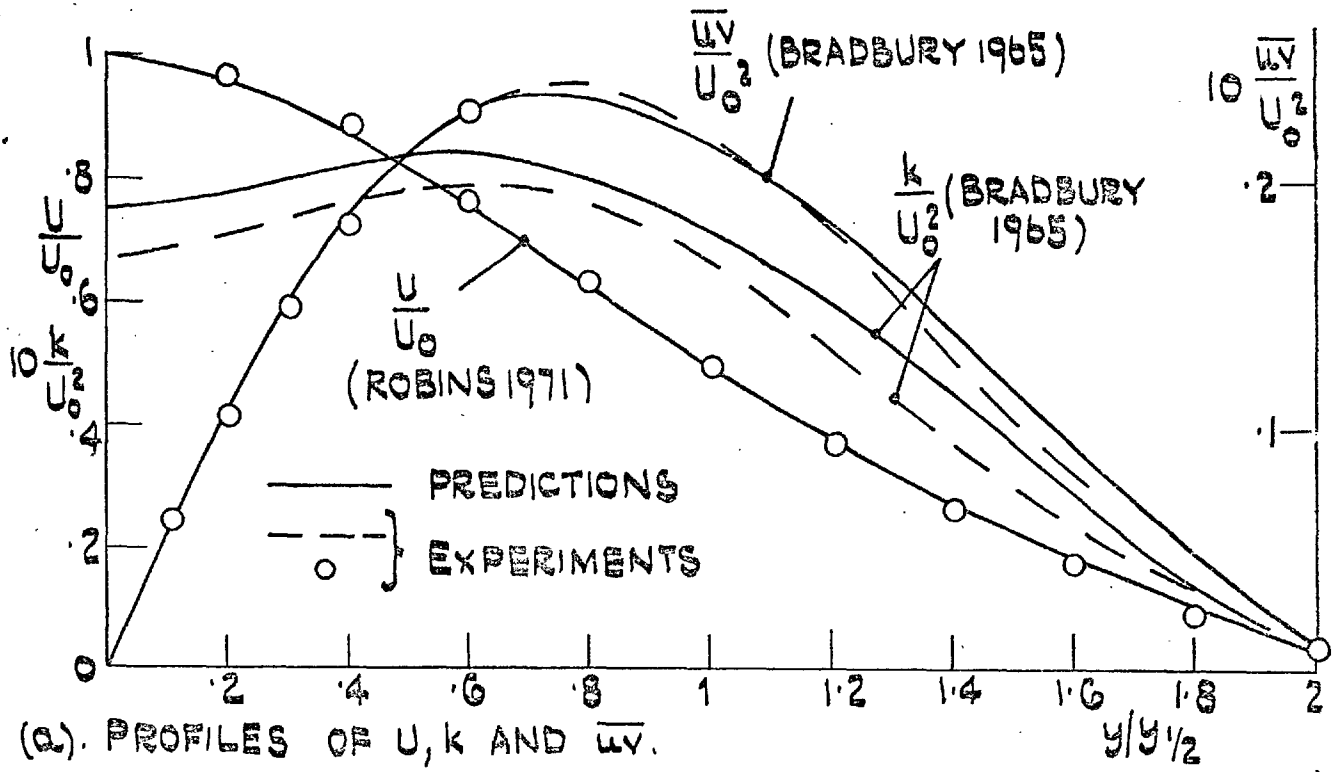


Fig. H.2 Prediction of the plane jet ($U_E=0$) with the $\eta k \epsilon$ - model

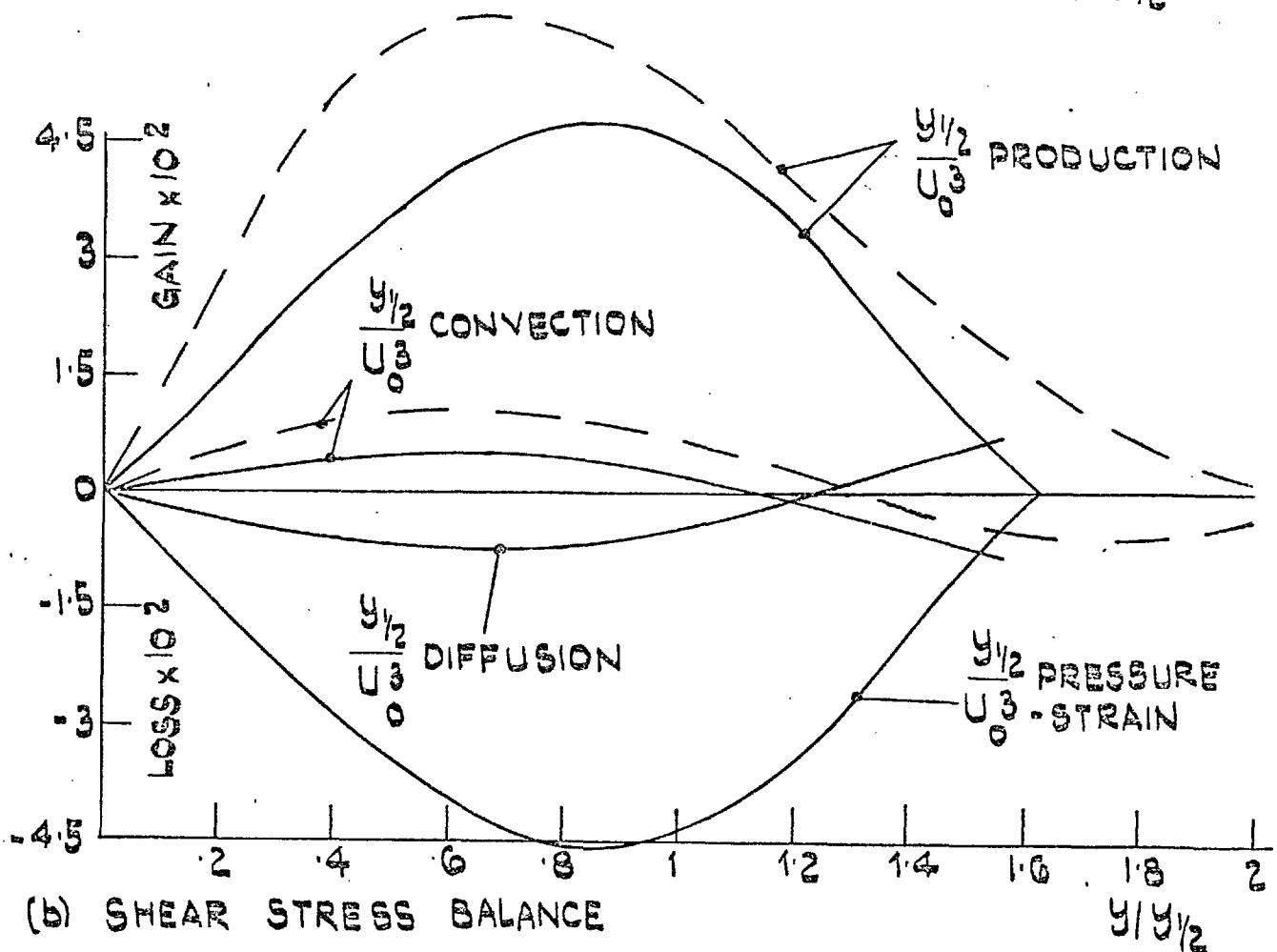
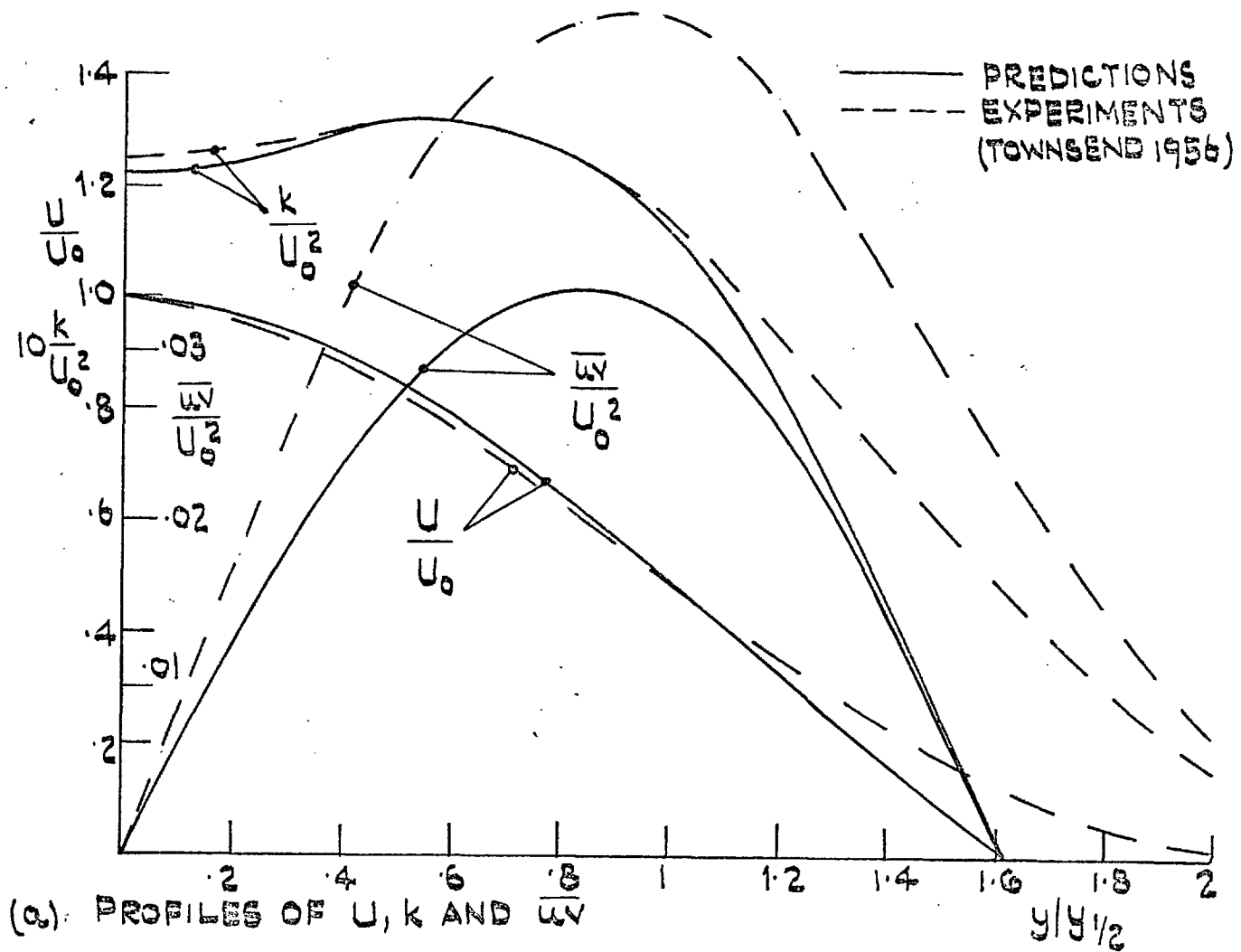


Fig. H.3 Prediction of the plane wake with the τ_{ke} - model

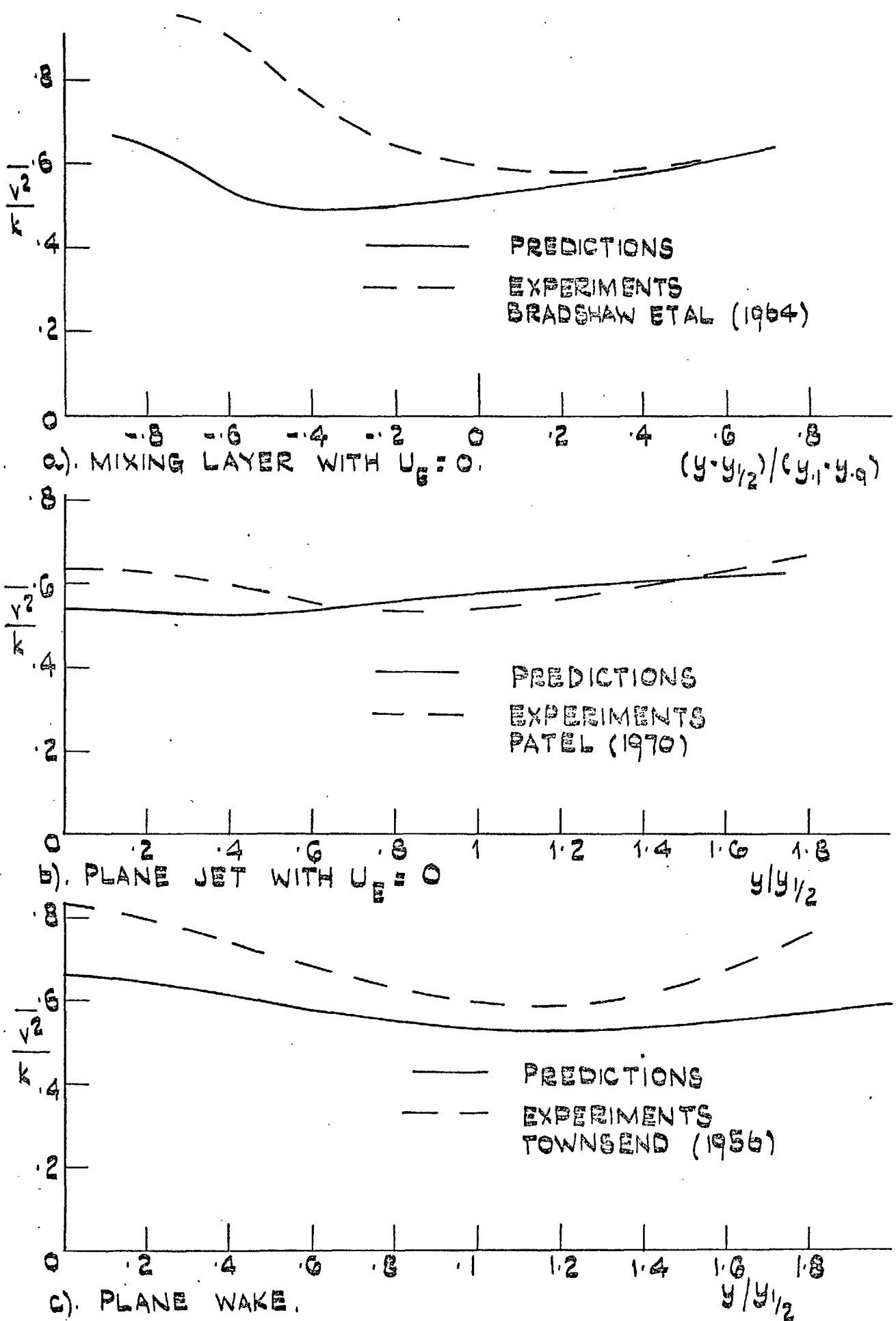
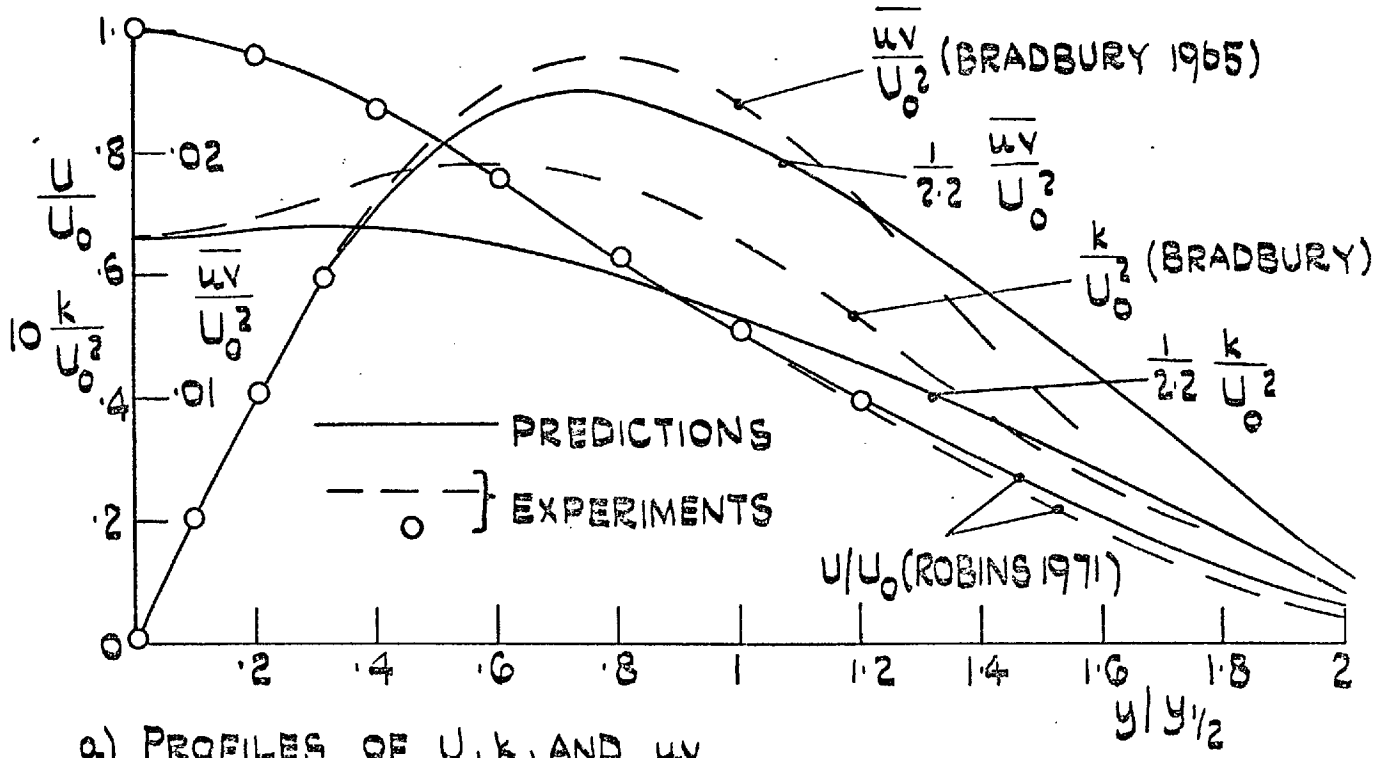
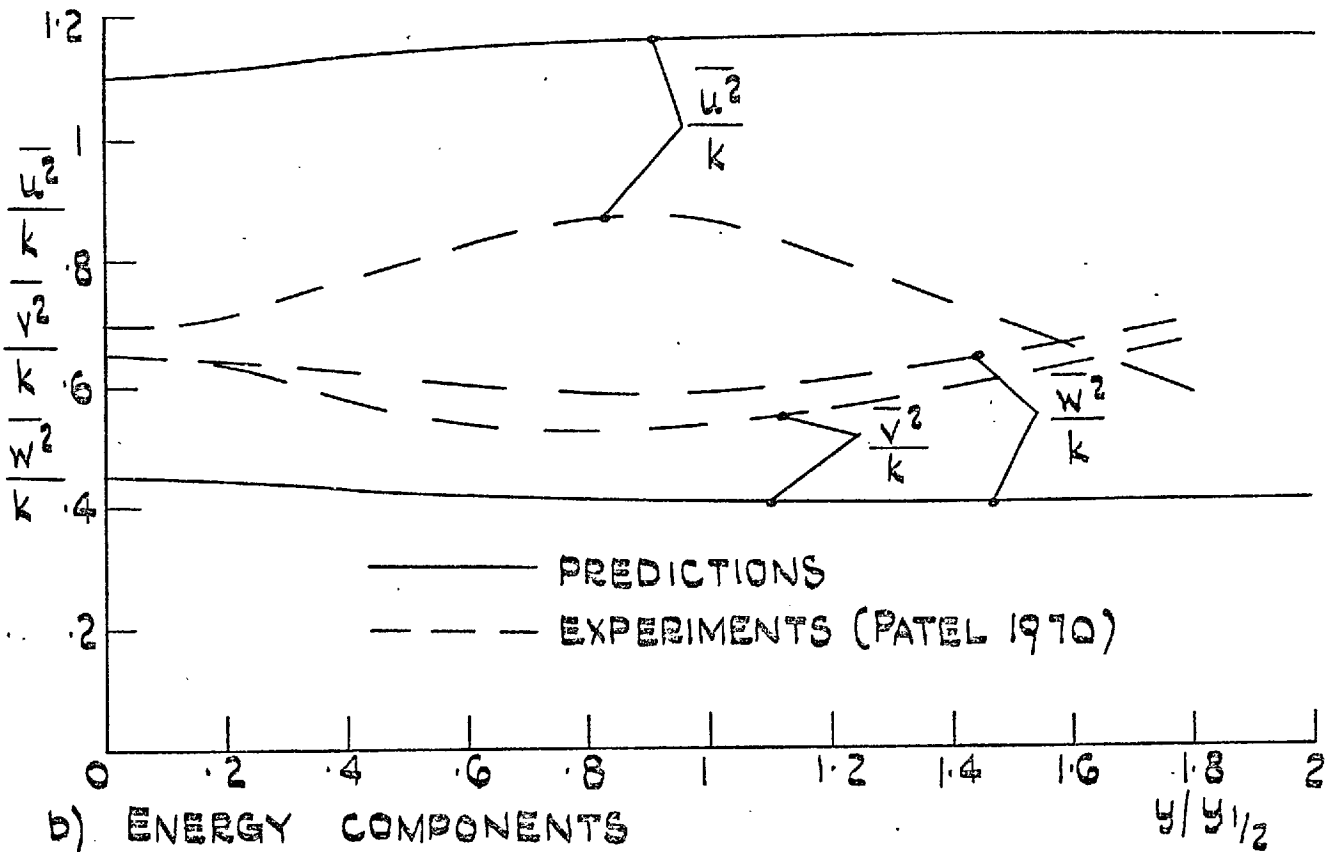


Fig. I.1 Profiles of $\frac{v^2}{k}$ predicted with relation (I.4)

a) PROFILES OF U , k , AND uv 

b) ENERGY COMPONENTS

Fig. J.1

Plane jet: predictions with the model
of Daly and Harlow

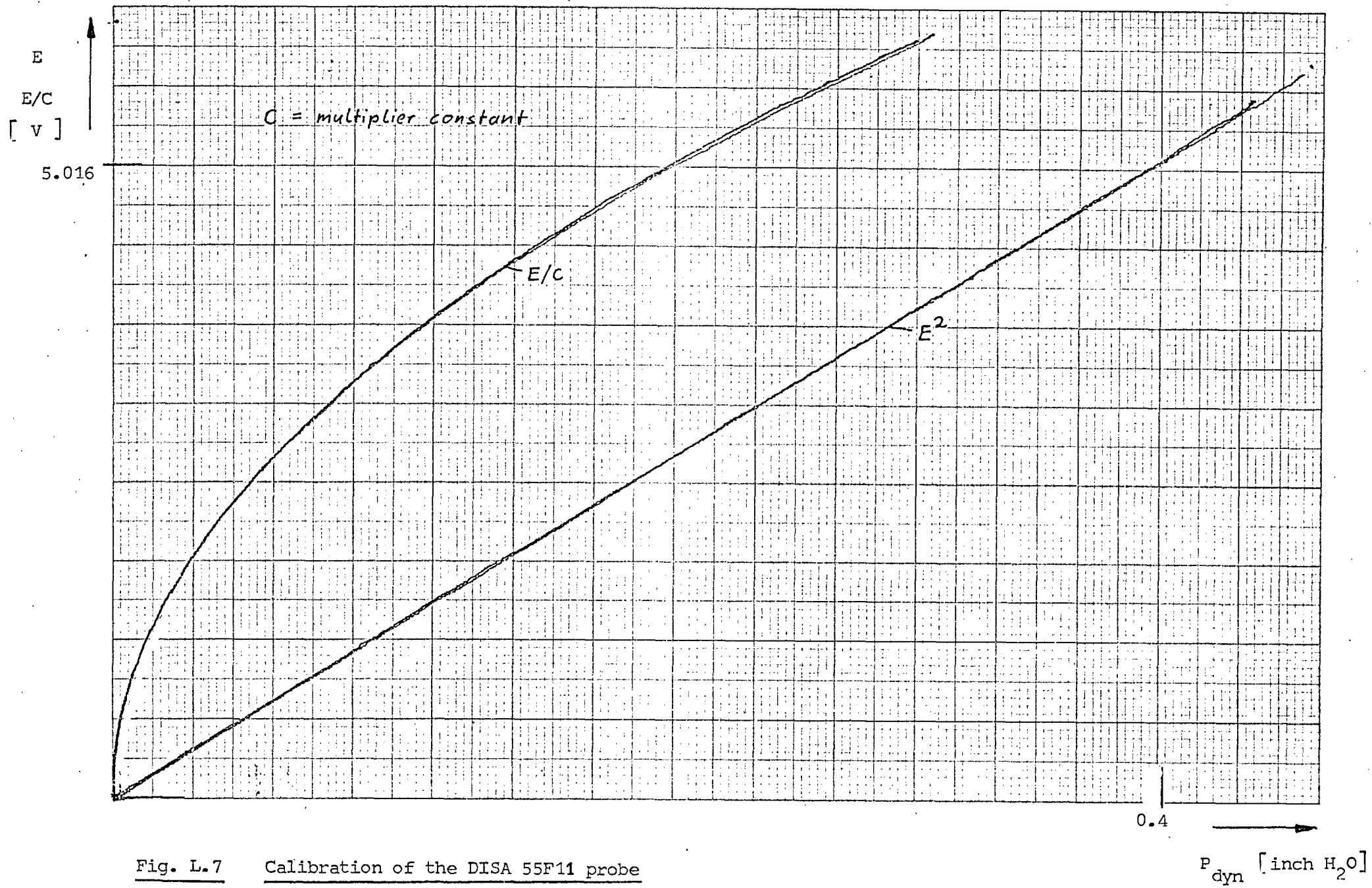


Fig. L.7 Calibration of the DISA 55F11 probe

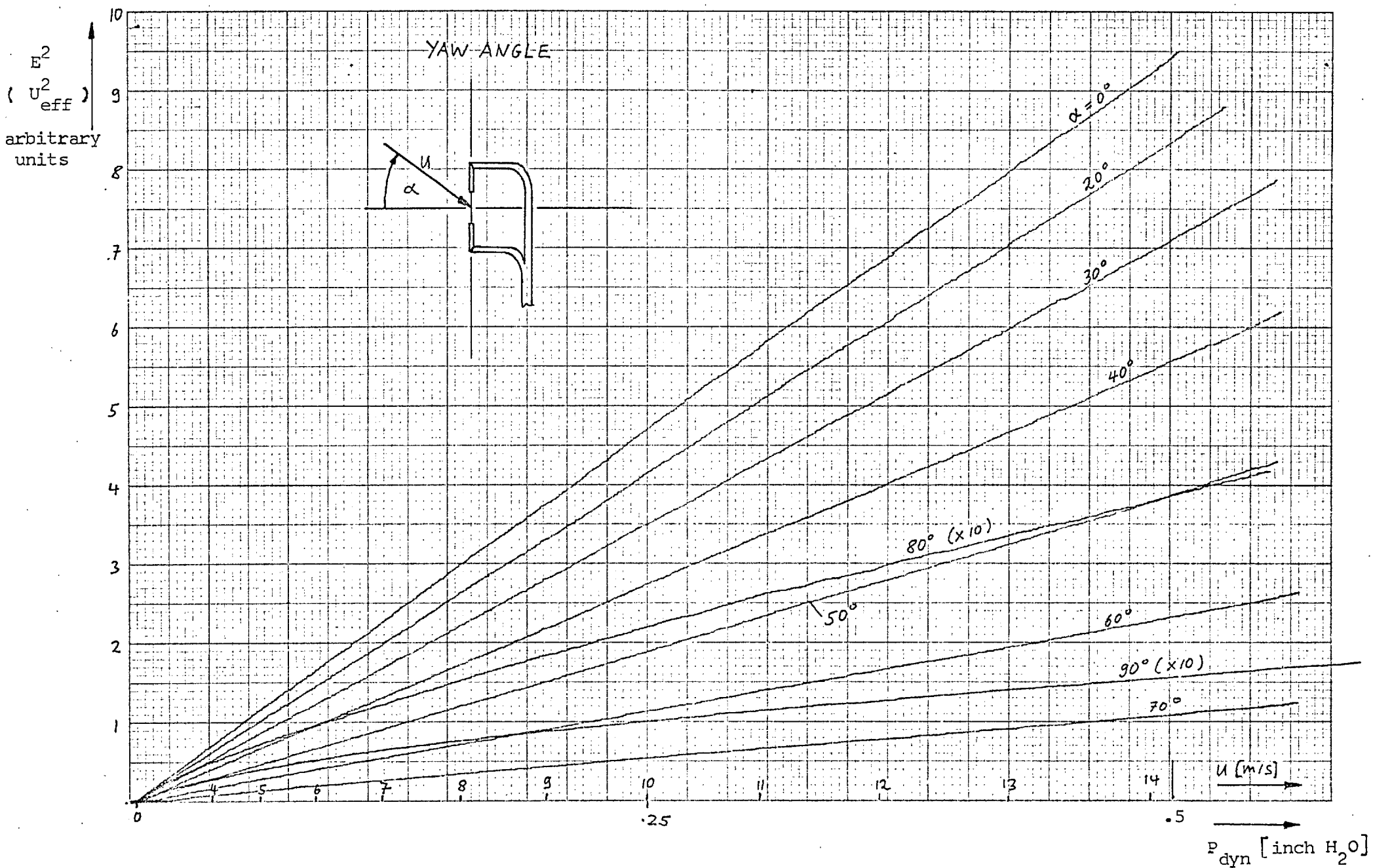


Fig. L.8 Yaw - calibration of the DISA 55F13 probe

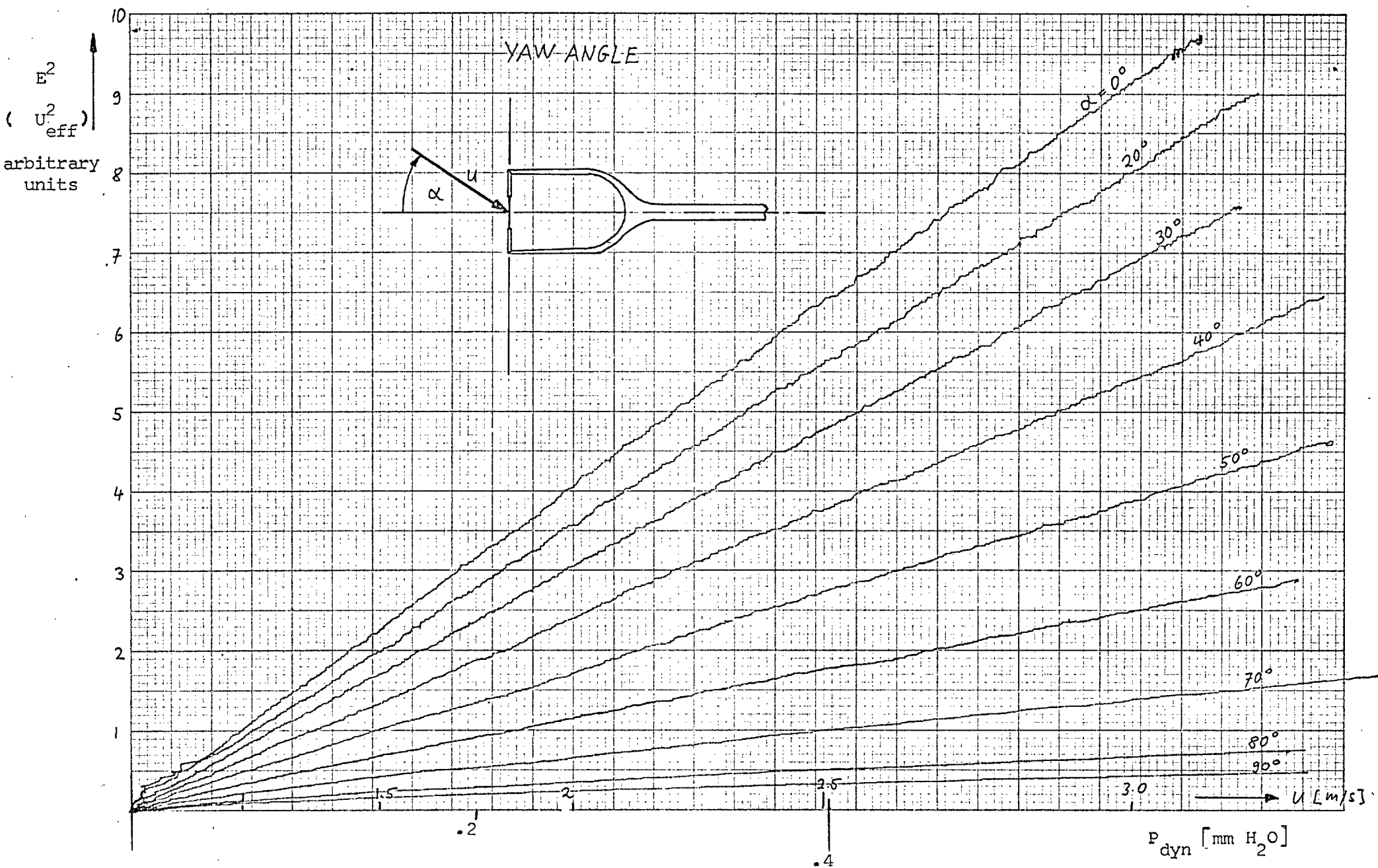


Fig. L.9 Yaw - calibration of the DISA 55F11 probe at low velocities

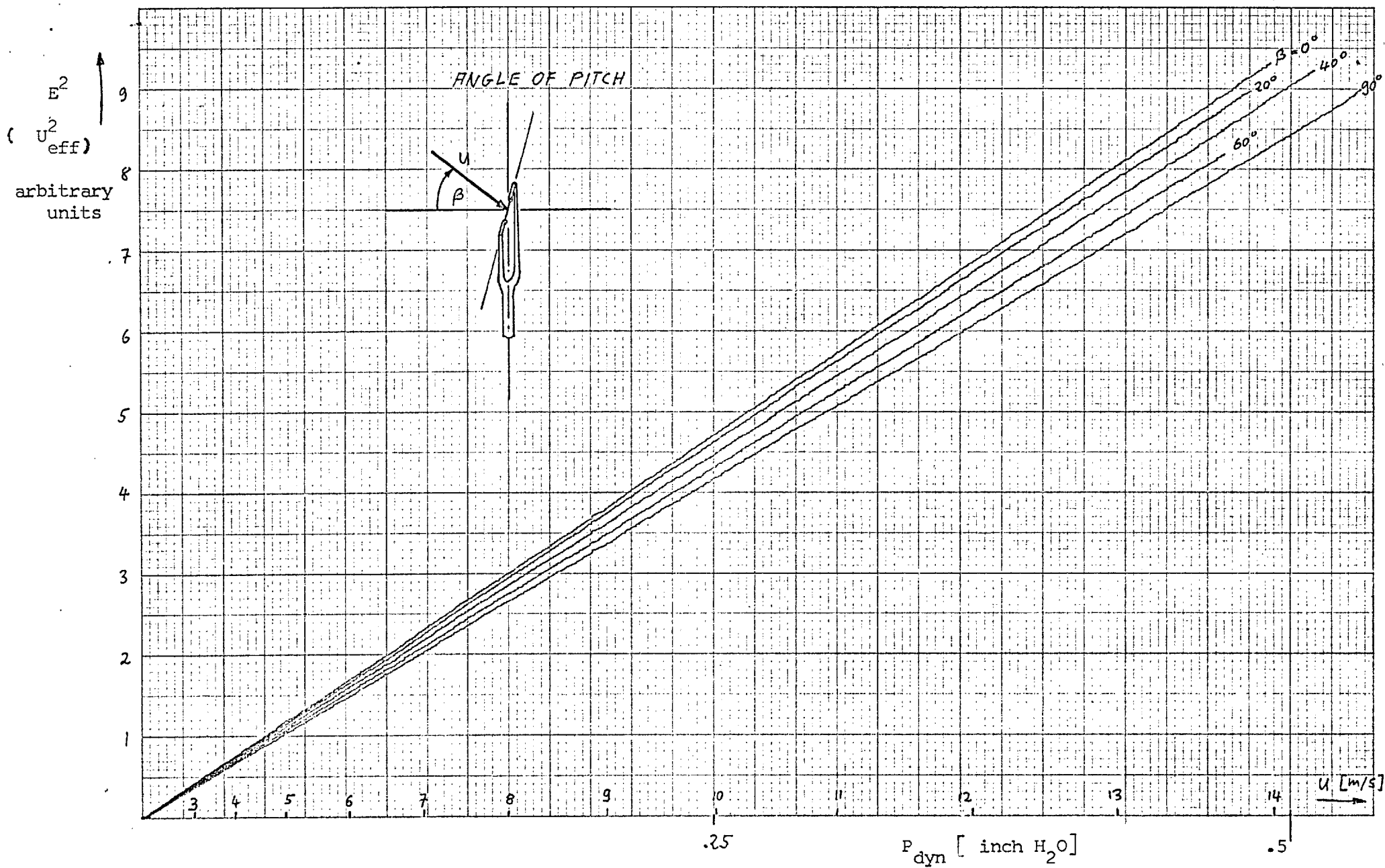


Fig. L.10 Pitch - calibration of the DISA 55F11 probe

INFLUENCE OF PITCH ANGLE β
 $\left(\frac{E\beta}{E_0}\right)^2 = 1 - \sin^2\beta + h^2 \sin^2\beta$

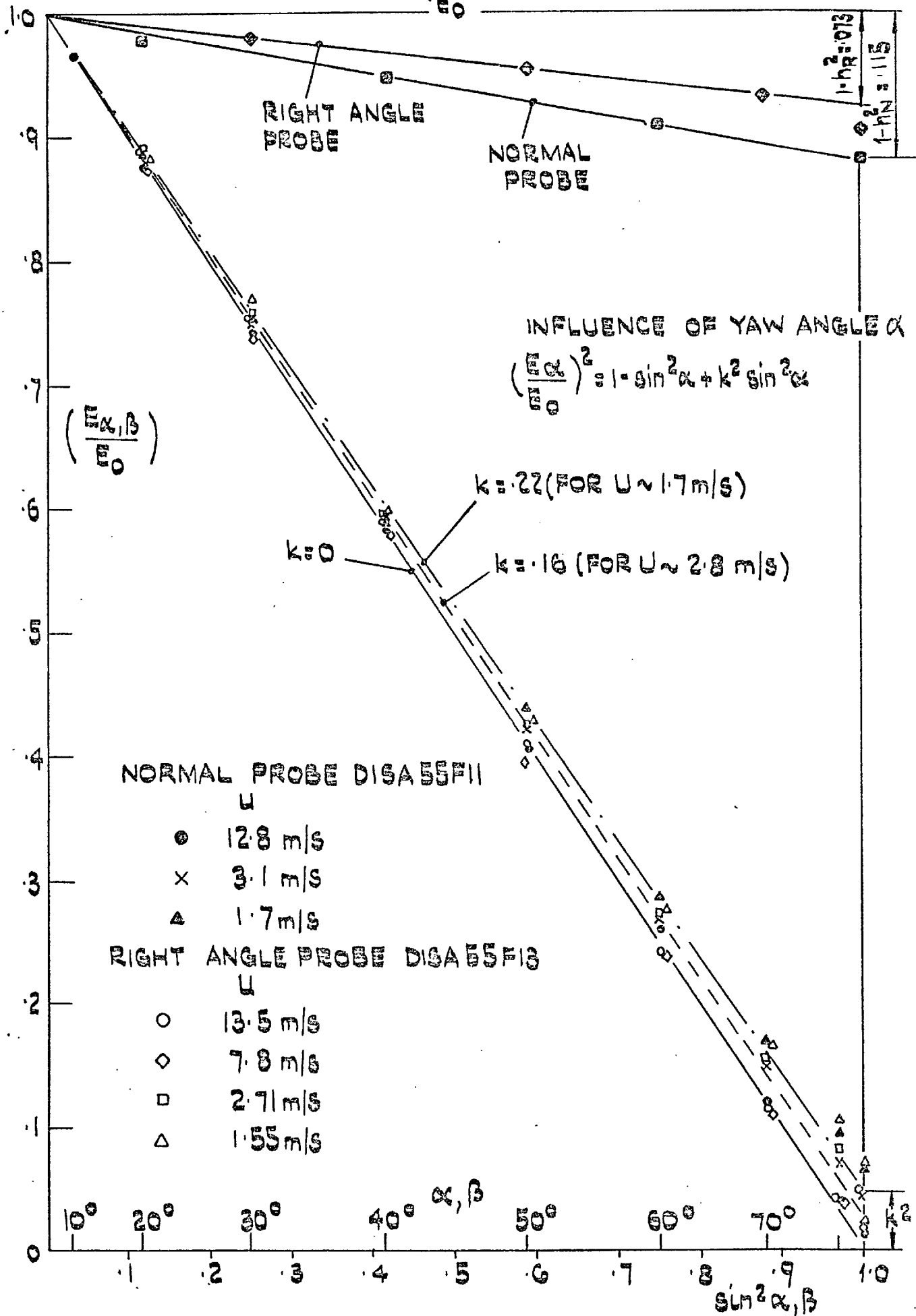


Fig. L.11

Determination of the k - and h -factors

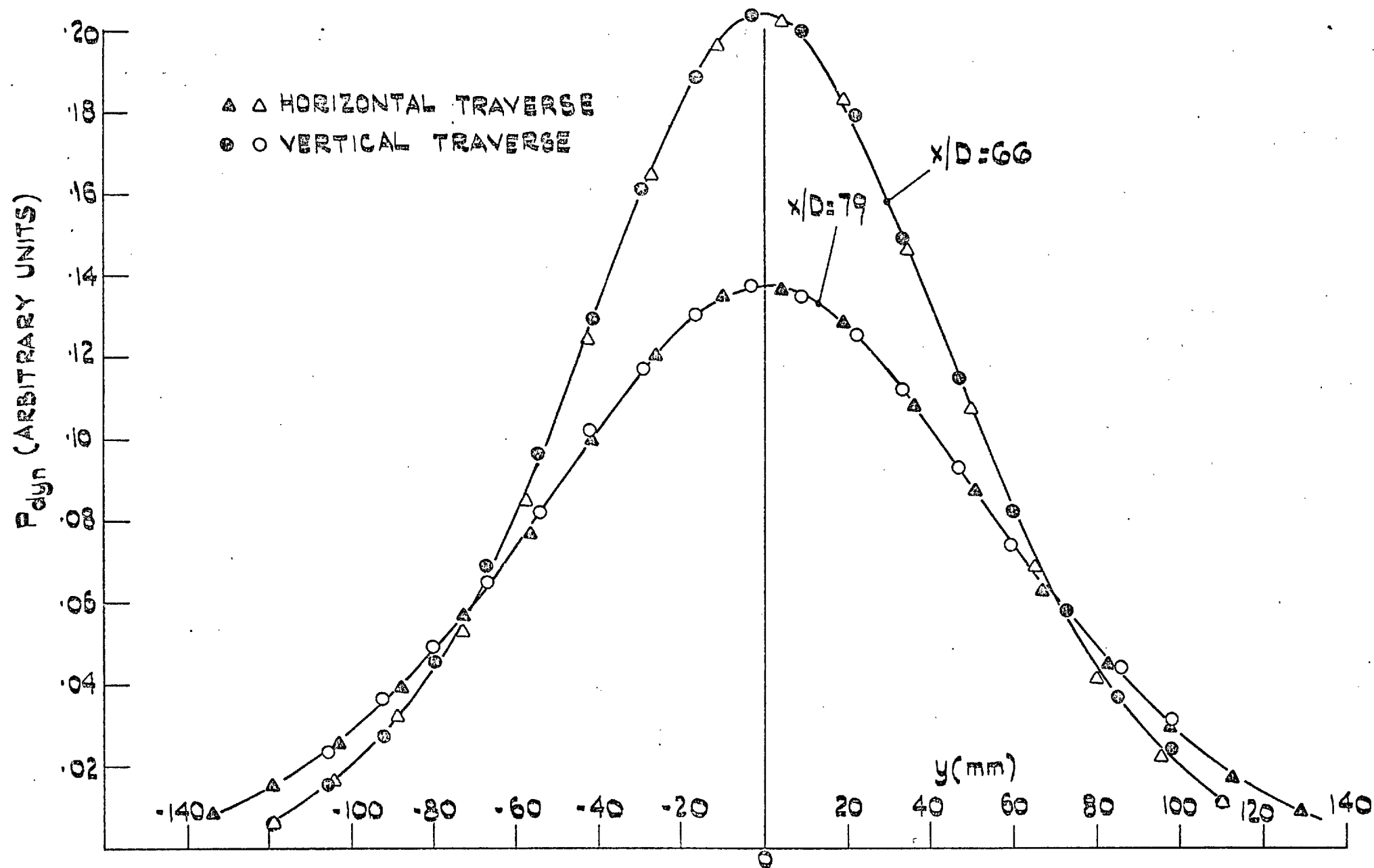


Fig. L.12 Symmetry check of the jet with a pitot tube

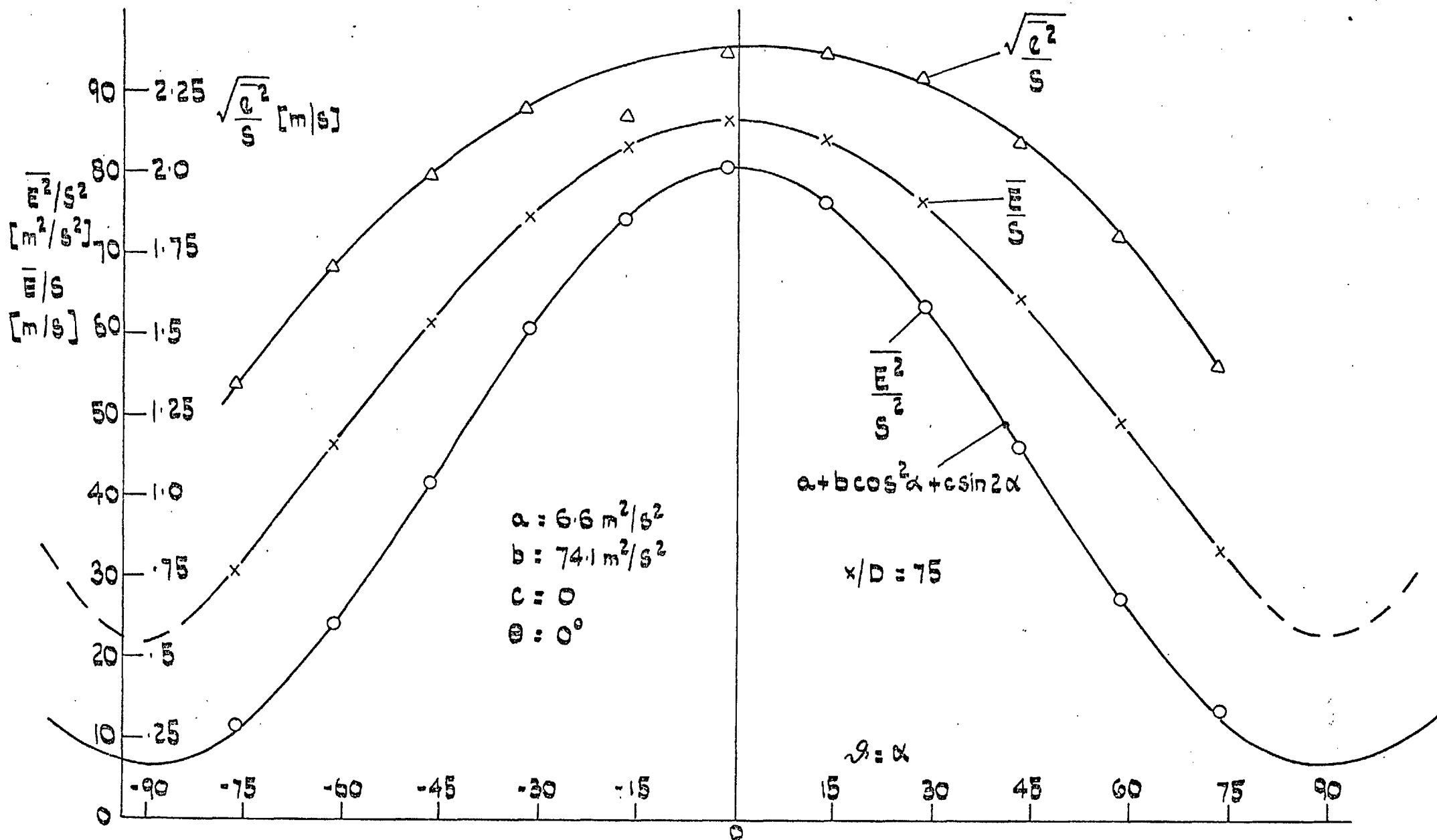


Fig. L.13 Distributions of $\frac{E^2(\alpha)}{S^2}$, $\frac{E(\alpha)}{S}$, and $\frac{\sqrt{e^2}(\alpha)}{S}$ at $y/x = 0$

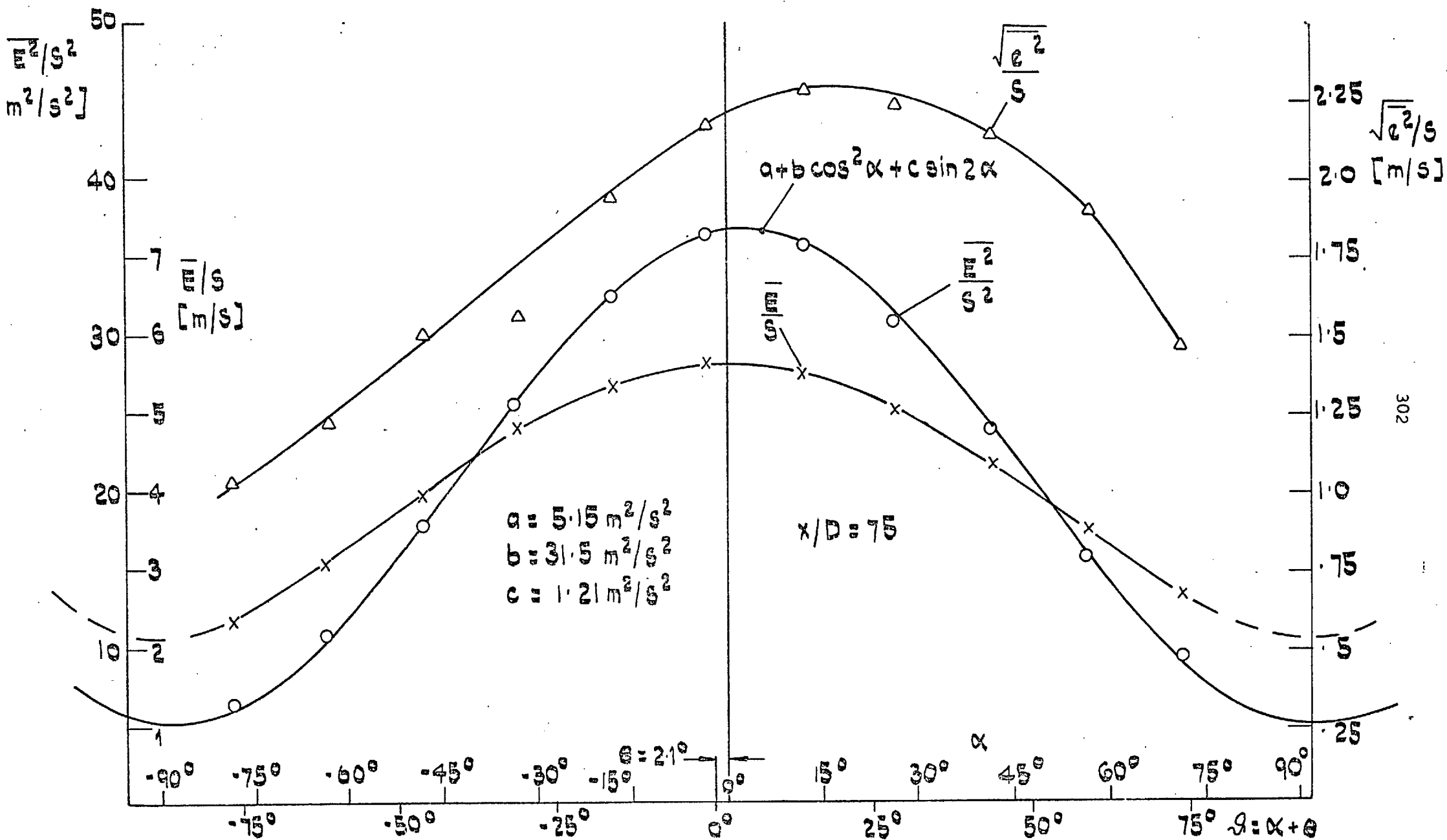


Fig. L.14

Distributions of $\frac{E^2(\alpha)}{S^2}$, $\frac{E(\alpha)}{S}$, and $\frac{\sqrt{e^2}(\alpha)}{S}$ at $y/x = 0.0693$

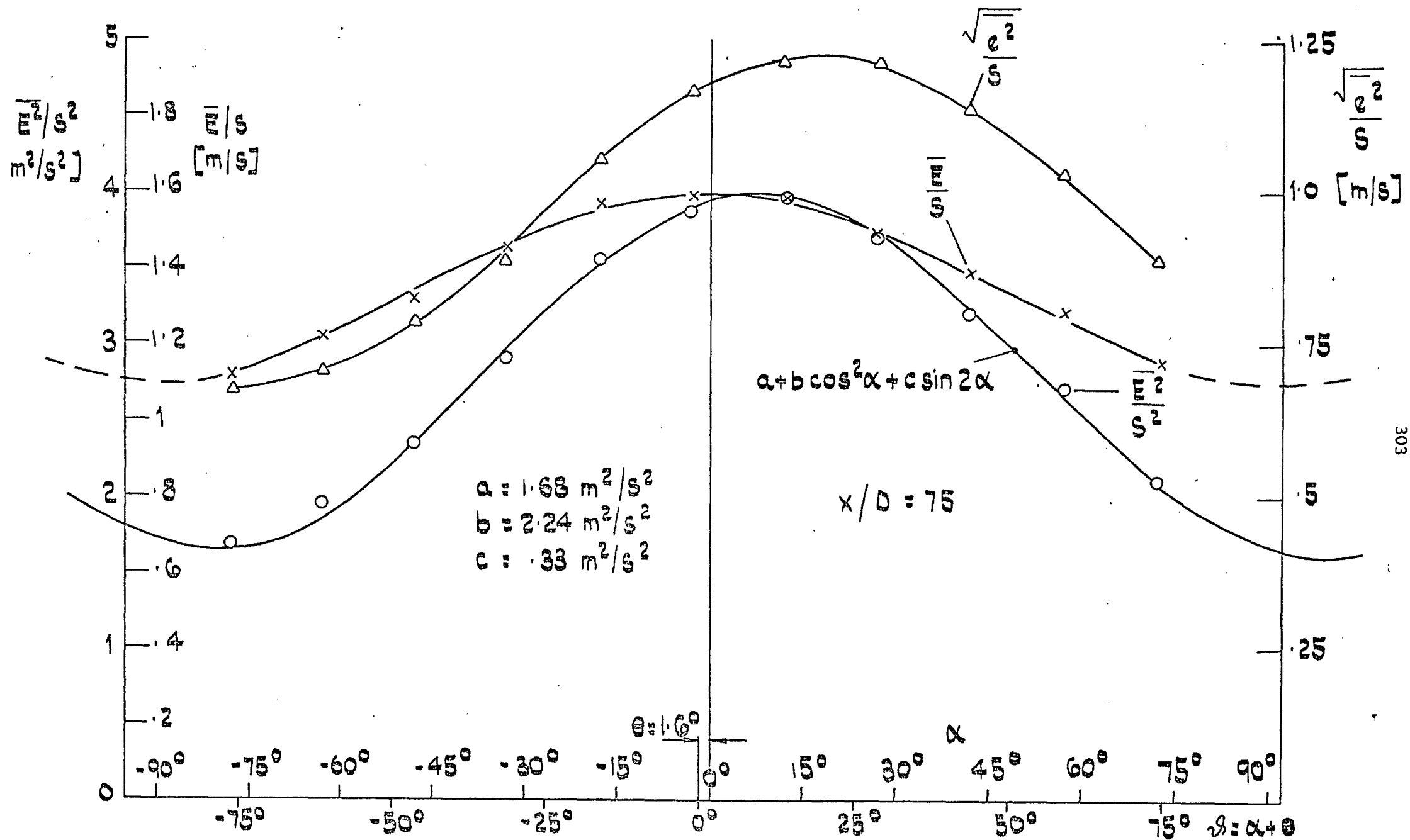


Fig. L.15

Distributions of $\frac{E^2}{S^2}$, $\frac{E}{S}$, and $\frac{\sqrt{e^2}}{S}$ at $y/x = .147$

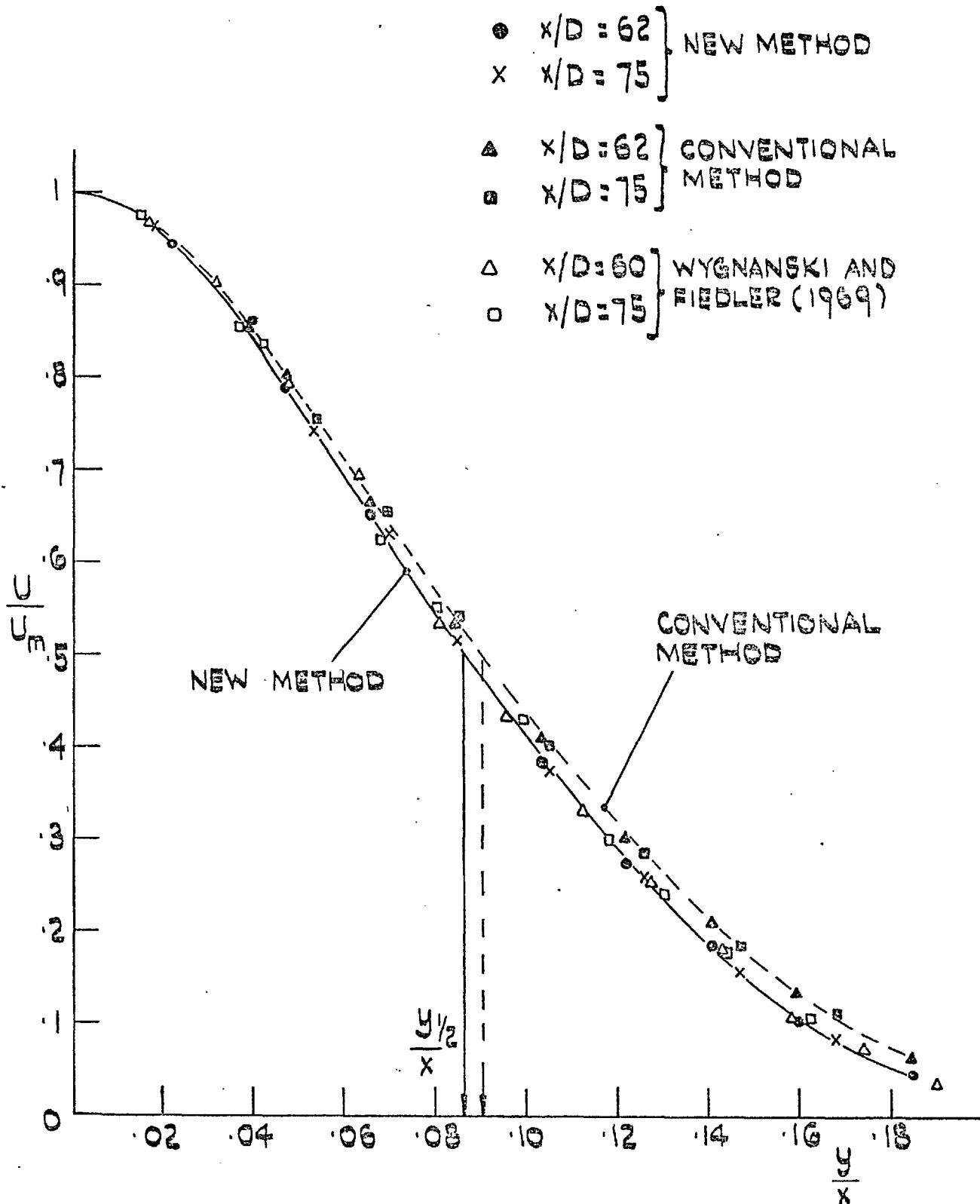


Fig. L.16

Mean velocity profile

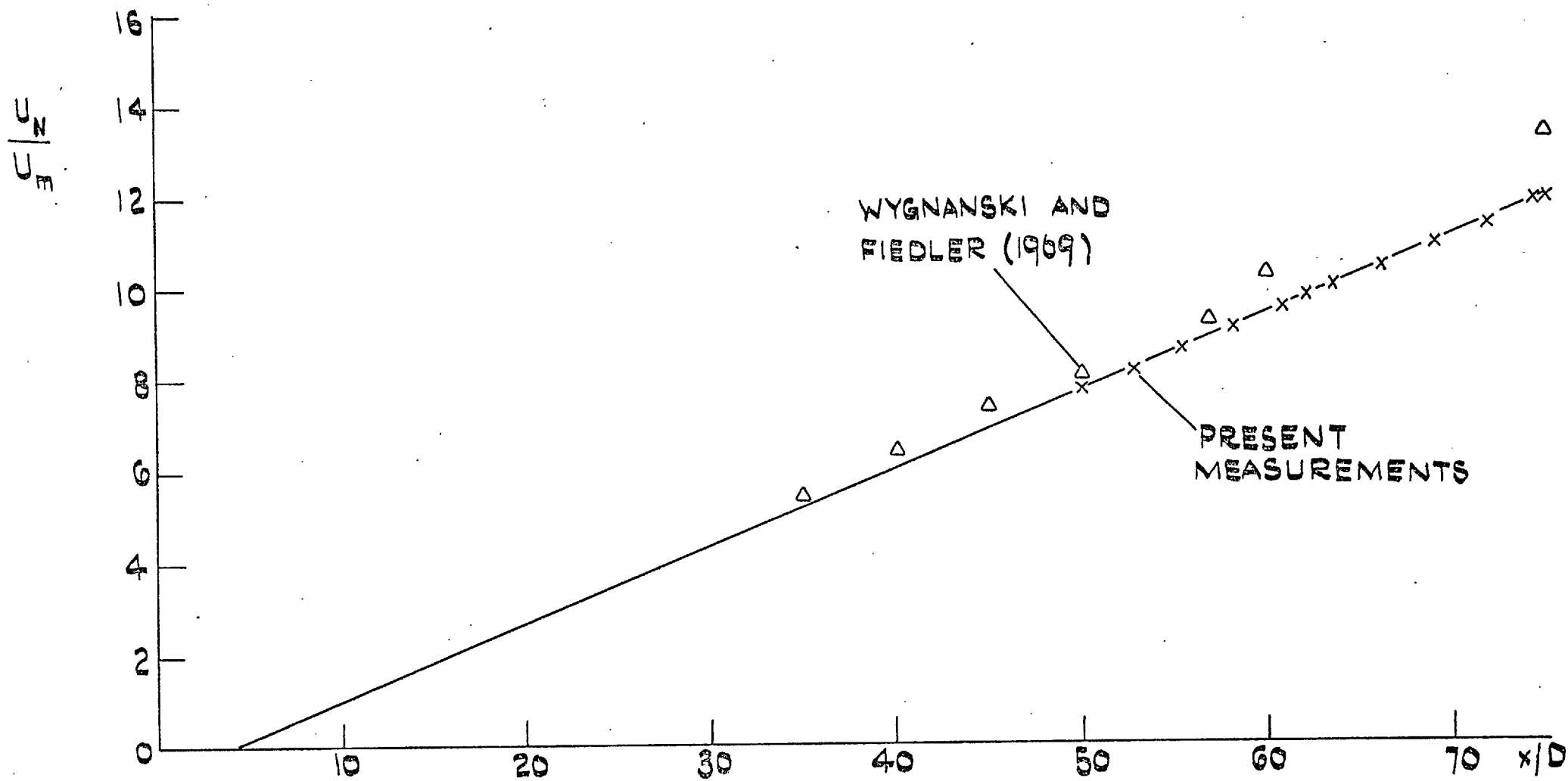


Fig. L.17 Mean velocity along the center-line

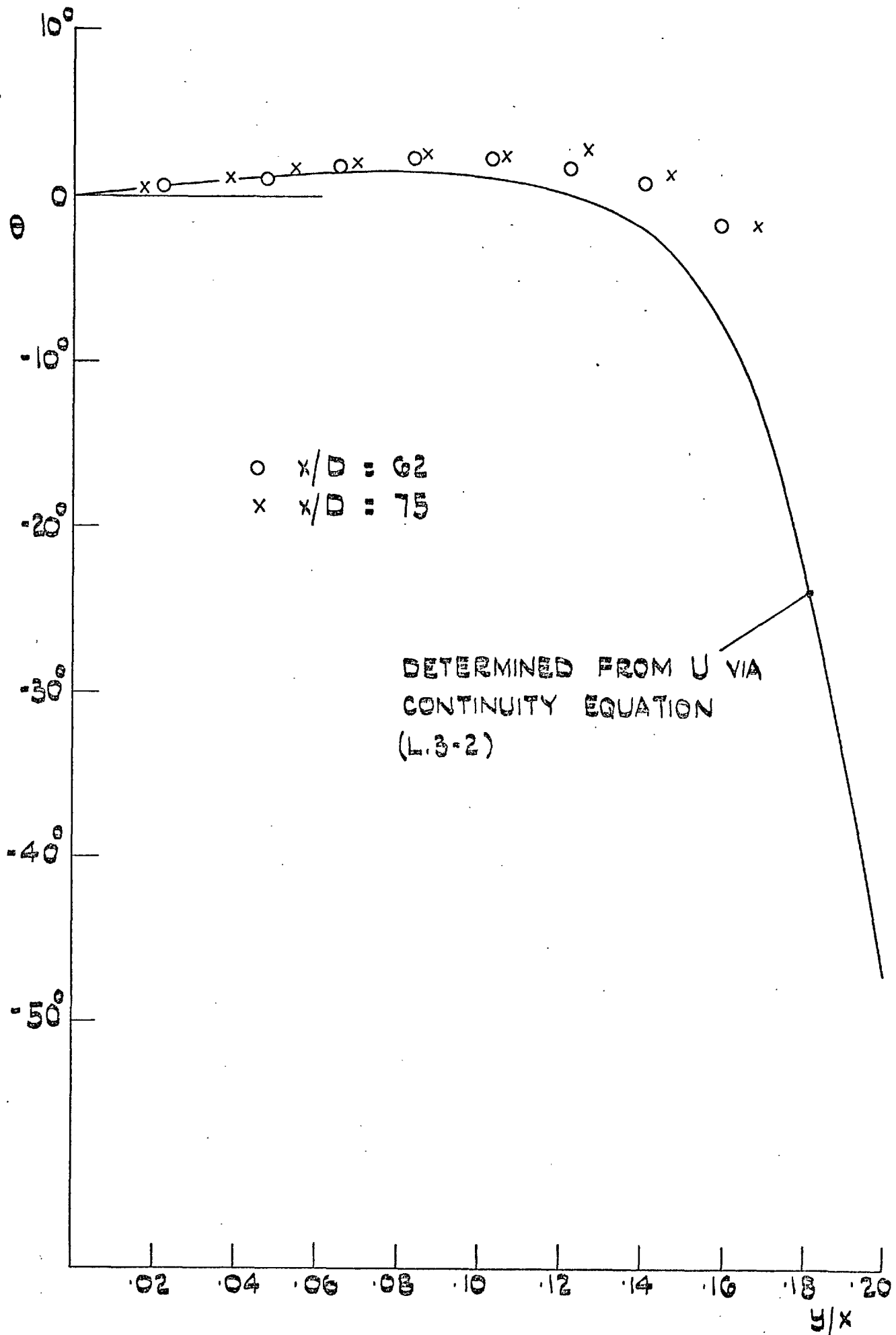


Fig. L.18

Mean flow direction

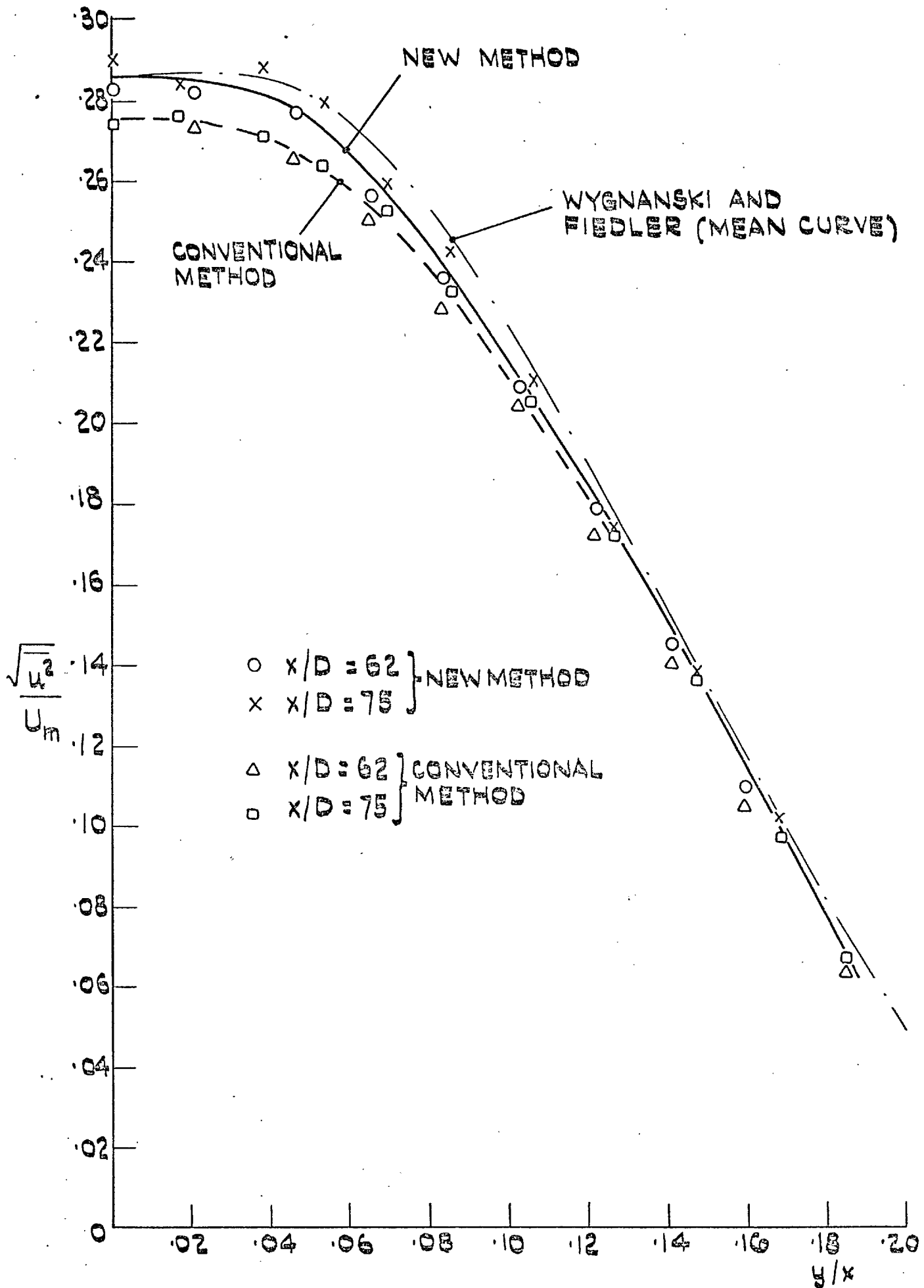


Fig. L.19 $\sqrt{u^2}/U_m$ - distribution

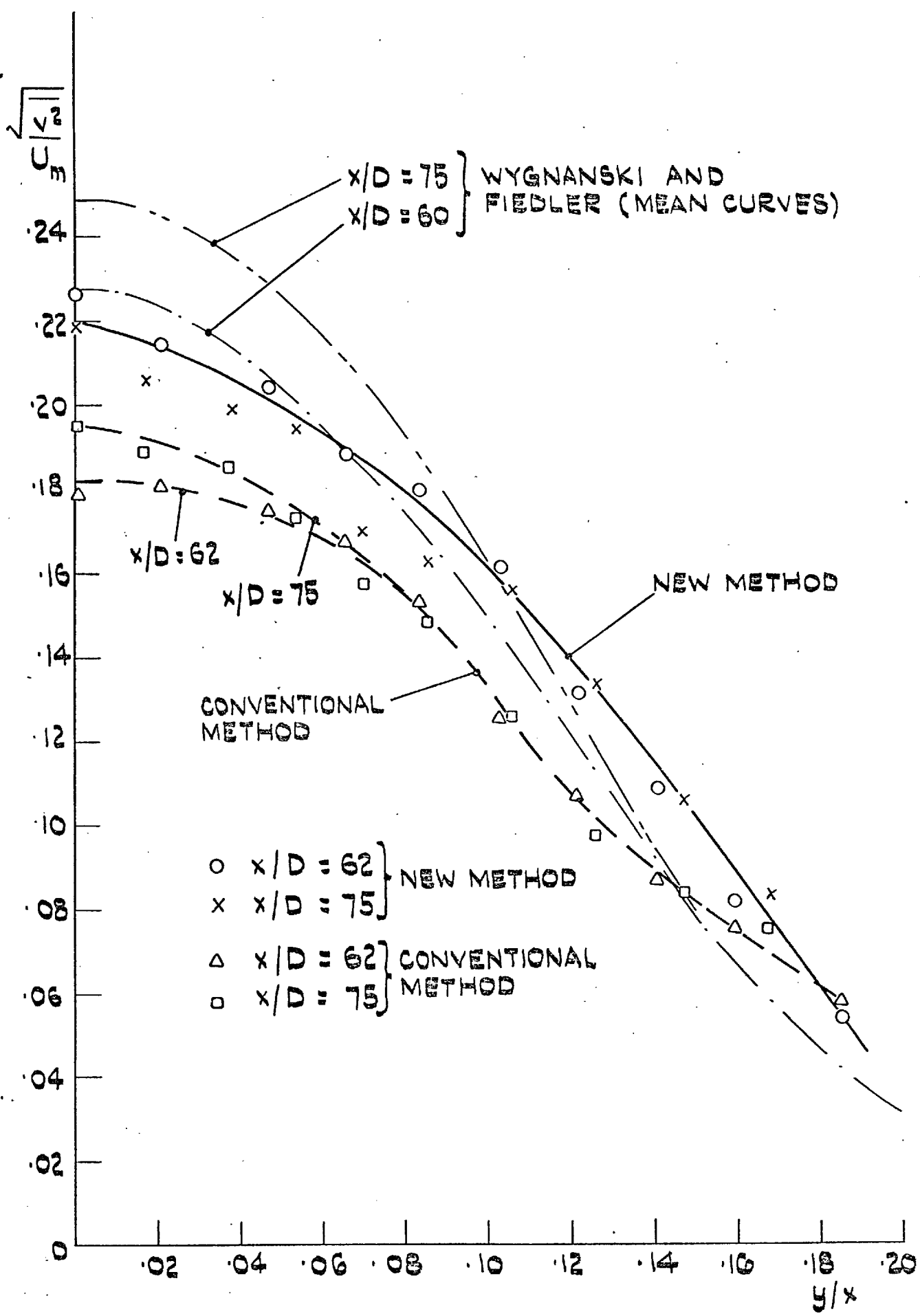


Fig. L.20 $\sqrt{v^2}/U_m$ - distribution

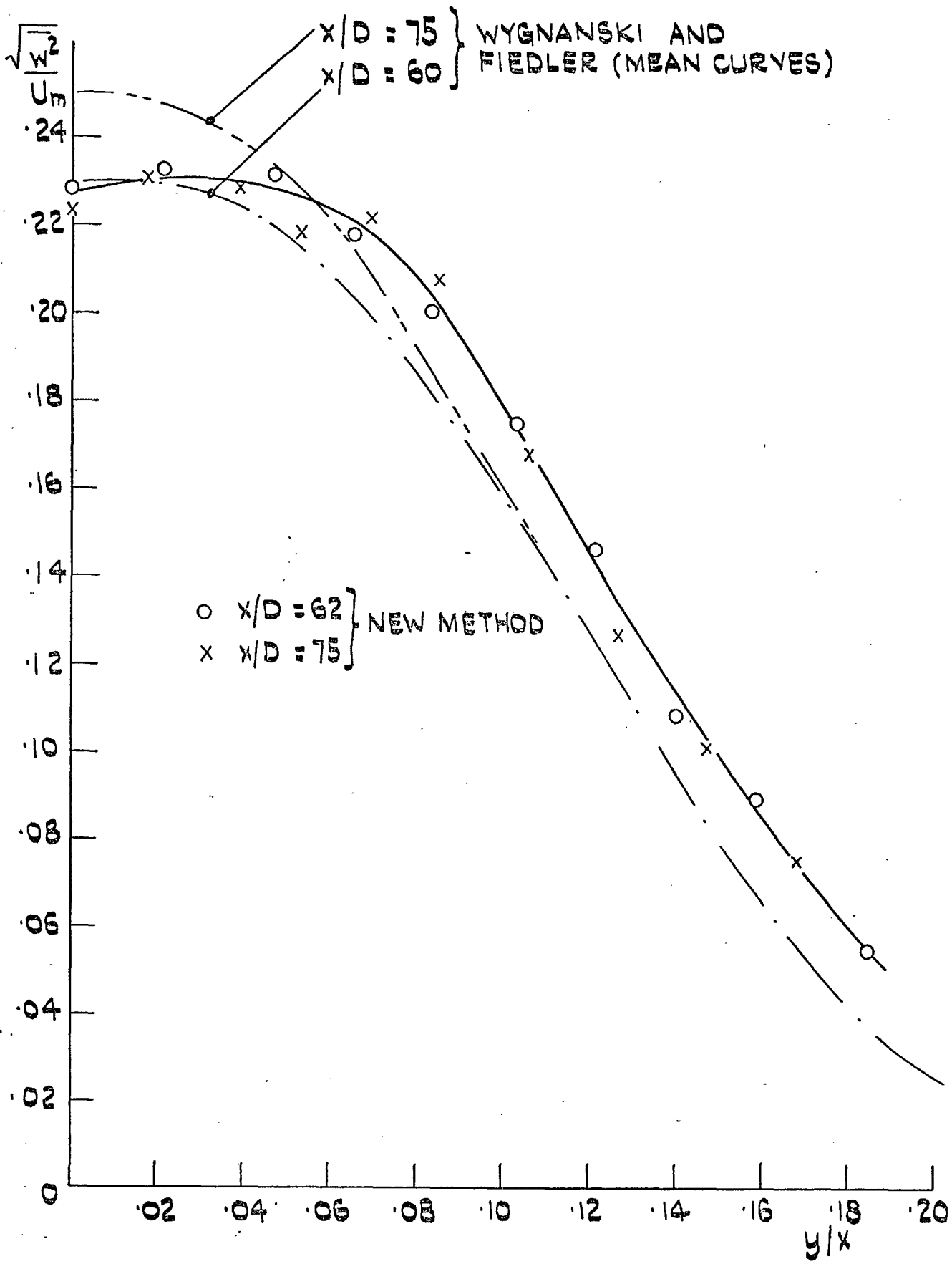


Fig. L.21 $\frac{\sqrt{w^2}}{U_m}$ - distribution

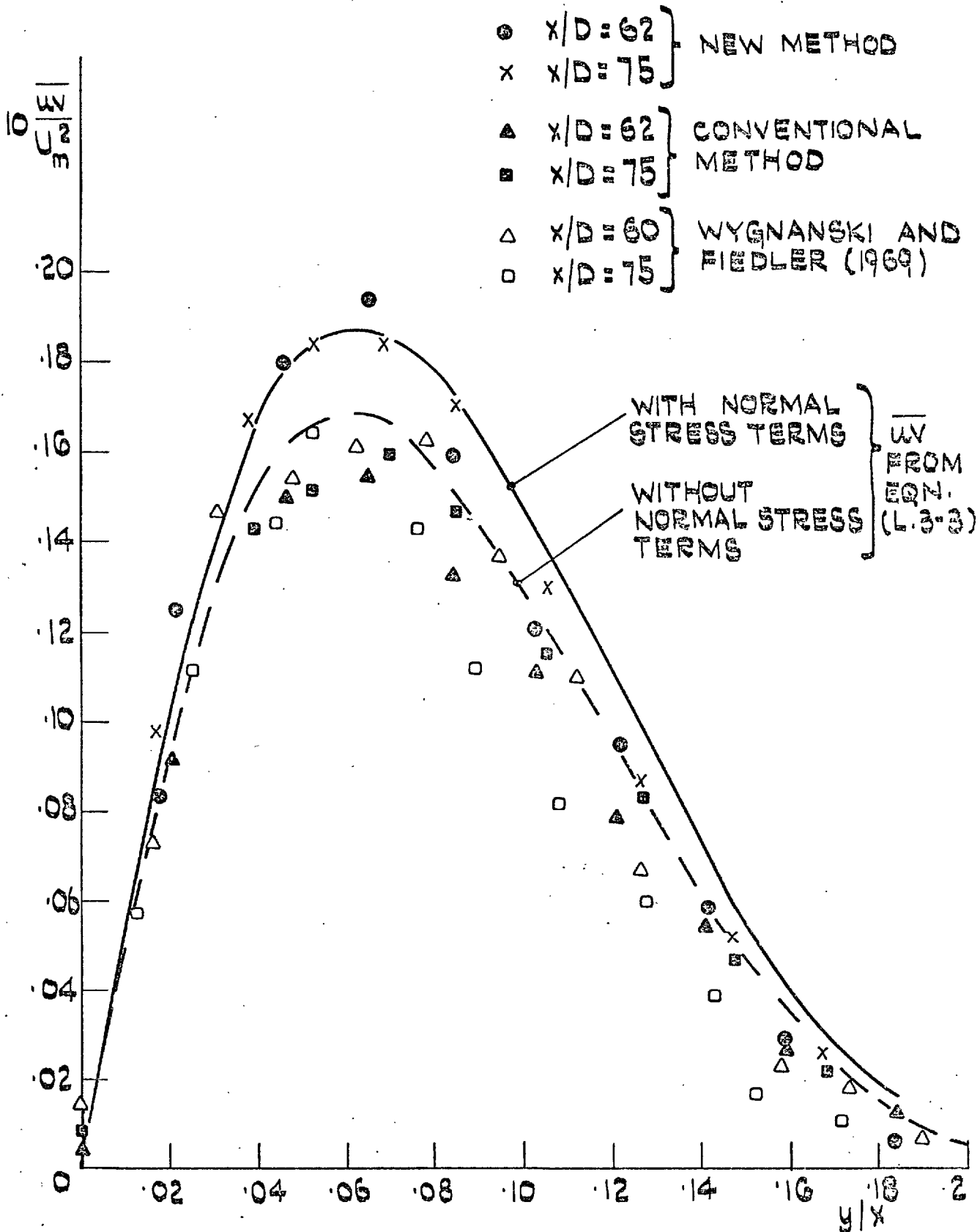


Fig. L.22 $\overline{uv}/U_m^2$ - distribution