

Constriction size distributions of granular filters: a numerical study

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This is an author generated postprint of the article:

Shire, T., O'Sullivan, C. (2016). Constriction size distributions of granular filters: a numerical study. *Géotechnique* (Accepted for publication)

Abstract:

The retention capability of granular filters is controlled by the narrow constrictions connecting the voids within the filter. The theoretical justification for empirical filter rules used in practice includes consideration of an idealised soil fabric in which constrictions form between co-planar combinations of spherical filter particles. This idealised fabric has not been confirmed by experimental or numerical observations of real constrictions. This paper reports the results of direct, particle-scale measurement of the constriction size distribution (CSD) within virtual samples of granular filters created using the discrete element method (DEM). A previously proposed analytical method that predicts the full CSD using inscribed circles to estimate constriction sizes is found to poorly predict the CSD for widely graded filters due to an over-idealisation of the soil fabric. The DEM data generated are used to explore quantitatively the influence of the coefficient of uniformity, particle size distribution and relative density of the filter on the CSD. For a given relative density CSDs form a narrow band of similarly shaped curves when normalised by characteristic filter diameters. This lends support to the practical use of characteristic diameters to assess filter retention capability.

Keywords: Filters; Internal Erosion; Discrete Element Modelling; Voids

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30 Introduction

31 Granular filters are placed in zoned embankment dams and flood embankments in order to protect
 32 adjacent base soils which are vulnerable to internal erosion, in particular the watertight core material.
 33 Granular filters function by retaining base material within the narrowest constrictions in the void
 34 network between the filter particles. Particle retention within granular materials is also important in
 35 other areas such as water treatment (Yuan et al. 2012), bacterial filtration (Martin et al. 1996), sand
 36 production in wells (Penberthy and Shaughnessy 1992) and leachate collection systems (Rowe 2005).

37 Design criteria for granular filters typically take the form of a comparison between a base particle
 38 diameter and a filter particle diameter, where the filter particle diameter implicitly characterises the
 39 constrictions within the filter. For the protection of cohesionless base soils, Sherard and Dunnigan
 40 (1989) and ICOLD (2013) recommend Terzaghi's classic filter rule that $D_{15F} < 4 D_{85B}$, where D_{XF} is the
 41 filter particle diameter for which X% of the material by mass is finer and D_{XB} is the base particle diameter
 42 for which X% of the material by mass is finer. In Terzaghi's original patent application the rule is justified
 43 on the grounds that "the pore size of a broadly-graded filter comprises at maximum $1/5^{\text{th}}$ of the
 44 diameter of the biggest grain of the finest fraction of the filter materials" (Fannin, 2008). This implies
 45 that if $D_{15F} < 5 D_{85B}$, then the D_{85B} will not pass through the pores of the filter (the design rule includes
 46 a safety factor). The same basis has also been used to justify the use of characteristic diameters in other
 47 areas of engineering, for example the internal stability of soils, e.g. Kézdi (1979), permeability (Chapuis
 48 2012) and leachate collection systems (Yu and Rowe 2012).

49 Kenney et al. (1985) defined the "controlling constriction size", D_c^* , of a granular filter as the diameter
 50 of the smallest constriction which a base particle is likely to encounter on a given flow path. They
 51 considered this to be the governing property of a granular filter. Kenney et al. carried out an extensive
 52 series of filter tests using uniform cohesionless base materials and linearly graded filters of varying
 53 thickness with coefficients of uniformity, $C_u = D_{60F}/D_{10F} = 1.2$ to 12. Extreme hydraulic conditions and
 54 light vibration were applied to ensure that the largest possible base particles were transported through

the filter. The largest particle which was transported through each filter was taken to be equal to D_c^* . Kenney et al. related D_c^* to the characteristic diameters D_{0F} , D_{5F} and D_{15F} . For the most uniform filters with $C_u = 1.2$, they found $D_c^* \approx 0.18D_{0F}$, whereas for filters with $C_u \geq 3$, they found $D_c^* \approx 0.25 D_{0F}$. When normalised by D_{5F} and D_{15F} it was found that D_c^* was similar for all filters regardless of C_u value, with upper limits to the values given by $D_c^* \leq 0.25 D_{5F}$ and $D_c^* \leq 0.20 D_{15F}$. The results using D_{15F} were found to give slightly more scatter than D_{5F} , but were still sufficiently good to recommend the continued use of Terzaghi's $D_{15F} / D_{85B} \leq 4$ ratio for design for filters with $C_u \leq 6$. Limited experimental work has been carried out for filters with $C_u > 6$, notable exceptions being Sherard et al. (1984) and Lafleur et al. (1989), both of whom found that D_{15F} remained a suitable characteristic filter particle diameter. However, based on permeability data, Lafleur & Tétreault (1986) found that relative density was more important for filters with $C_u > 6$ than for more uniformly graded filters, suggesting that more research is needed in this area.

Analytical approaches to filter design in geomechanics have focussed on the “inscribed circle method”, which is used to calculate the full constriction size distribution (CSD) and to relate this to the retention ability of the filter (e.g. Silveira et al. 1975, Indraratna et al. 2007). The CSD is the cumulative distribution of constriction diameters (by number) within a filter. The inscribed circle method pioneered by Silveira et al. (1975) calculates the CSD by considering the largest ‘constriction’ circle which can be inscribed in a highly idealised 2D (planar) representation where both the filter particles and the constrictions are represented as circles, as shown in Figures 1 (a) and (b). The “dense” state constriction diameter, D_{cD} , is given by the largest circle which can be inscribed between three mutually touching filter particles (Figure 1(a)). The “loose” state constriction diameter, D_{cL} , is given by the equivalent diameter of the maximum area, $S_{c,max}$, between four touching filter particles, as shown in Figure 1(b). The probability of a filter particle forming a constriction (i.e. acting as a vertex in a combination of contacting spheres) is related to the probability of its occurrence in the particle size distribution (either by number, surface area or mass). Silveira et al. (1975) related the probability of a particle of a given diameter forming a constriction to the PSD by number. The Silveira et al. method was further developed by Locke et al.

(2001), who proposed that the CSD should be related to the PSD by surface area and should vary linearly with relative density according to the relationship:

$$D_{cP} = D_{cD} + P (1 - R_d)(D_{cL} - D_{cD}) \quad 1$$

where: P : fraction smaller of the CSD ($P = 0$ at the smallest constriction and $P = 1$ at the largest constriction); D_{cP} : constriction size at P ; R_d : relative density; D_{cD} , D_{cL} : dense (3-particle) and loose (4-particle) constriction diameters for P .

Locke et al. (2001) also proposed using a regraded PSD to calculate the CSD so that loose fines which do not contribute to the constriction-forming skeleton, and coarse particles which are “enmeshed in a matrix of fines” are not considered. These coarse particles are removed from the PSD as any constrictions between them will be filled with finer particles, so will not be true constrictions.

Kenney et al. (1985) used the Silveira et al. inscribed circle method in their study into the controlling constriction size for granular filters. They used a 1D infiltration model of a void network to determine the controlling constriction size (i.e. the maximum size particle that can be transported through a filter of a given thickness). For filters of at least 10 layers with the same D_{0F} and with coefficients of uniformity ranging from $C_u = 2$ to 12, Kenney et al. found that the controlling constrictions were similar for all filters, in agreement with their experimental work. Noting that the method used takes the PSD by number to calculate the CSD, they attributed the similarity to the relatively large number of fine particles dominating the CSD and therefore the controlling constrictions. The analytical and experimental work of Kenney et al. (1985) formed the basis for the design rule developed by Kenney and Lau (1985).

Locke et al. (2001) extended the work of Kenney et al. (1985) by replacing the 1D model of Kenney et al. (1985) with a 3D cubic model allowing greater degrees of freedom for the permeating fines, as proposed by Schuler (1996). Based on this, Indraratna et al. (2007) proposed that the controlling constriction size, D_c^* , should be equal to D_{c35} , the 35th percentile from the Locke CSD calculation method. Indraratna et al. (2007) argued that the advantage of calculating D_c^* from the CSD, as opposed to taking a characteristic value from the filter PSD, is that the complete PSD shape and relative density are taken into account. Overall, this method predicts that as filters become more widely graded, the CSD will also become more widely graded. Note that the regrading rule proposed by Locke et al. (2001) is apparently not applied in the Indraratna et al. (2007) method.

Typical permeameter data can only reliably give values for D_c^* and cannot be used to define a CSD. The above analytical approach considers idealised combinations of spherical particles and hitherto the validity of the underlying assumptions has not been directly assessed. Advances in numerical modelling (DEM) and image analysis (micro computed tomography) mean that constriction topology and be examined at the particle scale and a CSD can be determined (Ketcham and Carlson 2001; O’Sullivan 2011). The resultant data enable a scientific re-examination of the geometrical hypotheses that date from Terzaghi’s work and underlie modern filter design (e.g. ICOLD, 2015).

Here the discrete element method (DEM) is used to generate numerical samples for which constriction size distributions (CSDs) can be determined to enable a re-examination of the conclusions developed from the inscribed-circle assumption. The algorithm proposed by Reboul (2008) to generate constriction size distributions (CSDs) is introduced. The inscribed circle assumption is then critically analysed by comparing CSDs obtained from the method proposed by Locke et al. (2001) and Indraratna et al. (2007) with CSDs obtained using Reboul’s algorithm. The variation in the shape of the CSD with particle size distribution (PSD) and density is then considered. Finally, the idea of using a characteristic diameter as a means to relate particle sizes and constriction sizes is quantitatively assessed.

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132 **Analysis Approach**

133 DEM is a numerical approach that was developed by Cundall & Strack (1979) to enable particle scale
 134 simulation of the mechanical behavior of granular materials. In a DEM model each particle and each
 135 particle-particle contact are explicitly considered. Ideal particle geometries (spheres) are used to
 136 control the computational cost of the DEM simulations. The effect of particle shape on filtration is
 137 unclear and it is typically not considered in design criteria, although Wu et al. (2012) found that filters
 138 comprised of glass beads gave a similar experimentally derived CSD to rounded sands, whereas angular
 139 sands gave a wider range of constriction sizes. The data directly generated from a DEM simulation
 140 include the particle positions, the contact locations and the contact forces. Shire et al. (2014) showed
 141 that these data can be used to assess the internal stability of granular filters. The current study extends
 142 this earlier contribution by using DEM data to quantify constriction sizes.

143 The DEM simulations were carried out on cubic samples using a modified version of the open-source
 144 DEM code Granular LAMMPS (Plimpton, 1995). Periodic boundary conditions were used to create a
 145 sample which is effectively of infinite size and is free from the boundary effects associated with rigid
 146 boundaries, allowing particle numbers to be kept to a reasonable level. A Hertz-Mindlin contact model
 147 was used and the simulation input parameters were Poisson's ratio $\nu = 0.3$, shear modulus $G = 27.0$ GPa
 148 and particle density $\rho = 2670$ kg/m³, which are approximately equal to experimentally derived values
 149 for spherical glass beads used by Barreto (2009). Particles are initially placed in random, non-touching
 150 positions within the periodic cell using an in-house placement code, creating a homogeneous, high
 151 porosity sample, and then applying isotropic compression with a uniform strain field following the
 152 algorithm proposed by Cundall (1988). The target isotropic stress level was $p' = (\sigma'_1 + \sigma'_2 + \sigma'_3) / 3 = 50$
 153 kPa where $\sigma'_1, \sigma'_2, \sigma'_3$ are the three principal stresses. Samples created in this way are homogeneous
 154 and isotropic. Simulations were terminated when the mean normal stress reached the target level and

the coordination number (the number of contacts per particle) remained constant for 20,000 simulation cycles.

Referring to Table 1 and Figure 2, 18 of the DEM samples created had linearly graded particle size distributions (PSDs) with $C_u = 1.2, 1.5, 2, 3, 4.5$ or 6 , a further 10 of the samples had bi-linear PSDs with $C_u = 1.5$ to 2.6 . Each of the PSDs is presented normalised by the smallest filter particle diameter (D_{OF}). The PSDs of each of the bi-linear samples follow a linear PSD with $C_u = 3$ to a diameter D_X , where X is specified by the number in the sample name (D_5 for BL5, D_{10} for BL10, etc.). Up to three coefficients of friction (μ) were used for each grading considered (i.e. $\mu = 0.0, 0.1$, and 0.3) to give a range of void ratios, and the resultant samples are termed “dense”, “medium” and “loose” respectively. $\mu = 0.3$ was selected for the loose samples as this is approximately equal to the value reported for physical glass beads by Barreto (2009). The number of particles considered increased with C_u and varied between 8,262 and 59,183. As detailed by Shire (2014) DEM simulations with PSDs with $C_u = 1.5, 3$ and 6 were repeated using samples with fewer particles to confirm that a representative element volume (REV) had been achieved. Referring to Table 1 it is clear that for a given μ value, the void ratio at $p' = 50$ kPa falls as C_u increases. The maximum coefficient of uniformity considered was $C_u = 6$ to limit the computational time required for simulations, as discussed above.

The approach used here to determine the constriction sizes and hence the constriction size distributions (CSDs) was the weighted Delaunay tessellation-based algorithm originally proposed by Al-Raoush et al. (2003) and Reboul (2008). A 2D schematic of the algorithm is included as Figure 3. A 3D weighted Delaunay tessellation was formed with the tetrahedra vertices being located at the particle centroids; during the creation of the tessellation the vertices were weighted by their corresponding particle radii (Edelsbrunner and Shah 1996). The weighted triangulation was achieved using the CGAL Regular Delaunay algorithm (CGAL, 2013). Referring to Figure 3(b), inter-void constrictions are identified using the faces of the tetrahedra. On each face the constriction diameter is taken to be the

diameter of the smallest circle that can be inscribed between the three particles that form that face or between the particles forming the face and any other non-vertex particles which cross over the tetrahedron face (Reboul et al. 2010).

In order to avoid over segmentation of the void space, adjacent tetrahedra were merged using a criterion proposed by Al-Raoush et al. (2003) and Reboul et al. (2008). This can be explained by reference to a simple 2D analogy of uniform particles on a square grid, as illustrated in Figure 3. Figure 3(a) illustrates the constriction locations. The weighted Delaunay triangulation to identify the voids is presented in Figure 3(b). As illustrated in Figure 3(c) if constrictions are located at each of the triangulation edges (equivalent to tetrahedron faces in 3D), the voids are essentially over segmented, i.e. a space that would intuitively be considered a single void is subdivided. As illustrated in Figure 3(d) circles (spheres in 3D) that are tangential to all the particles forming a Delaunay cell are identified. Then the overlap between this tangent sphere and each of the tangent spheres in adjacent Delaunay cells is calculated and if this overlap exceeds a user-specified value, the cells are merged to form a single void cell, as illustrated in Figure 3(e).

The choice of user-defined merging overlap value is subjective; a smaller value results in the formation of fewer, larger voids. While this subjectivity is indeed a limitation of the algorithm, it is important to realise that as soils have a continuous network of interconnected voids the exact definition of boundaries between voids in the system is itself inherently subjective. The variation in the calculated CSD with the merging overlap selected can be appreciated by reference to Figure 4, which illustrates CSDs for loose samples with a linear gradings of $C_u = 1.2$ and $C_u = 6.0$. Merging overlap values of 0%, 50% and 100% were used to generate three CSDs for each PSD. For both C_u values, as the merging overlap increases fewer cells merge and more constrictions are measured; the constriction sizes also increase with merging overlap. The effect of changing the merging overlap is less pronounced for the smaller than the larger constrictions. Typically a merging overlap of 50% has been used in the analyses

presented here, however, where appropriate more than one merging overlap value has been considered to account for the subjectivity induced through this parameter. As discussed by O'Sullivan et al. (2015) and Taylor et al. (2015) the weighted Delaunay method gives comparable results to other available particle-scale algorithms for measuring constrictions. Reboul et al. (2010) compared CSDs generated with the triangulation approach to those calculated using the inscribed circle method. They considered loose and dense DEM samples of soils with uniformly to moderately graded PSDs ($C_u = 1.67$ and 3.75) and they also found that for both filter gradings that the inscribed circle CSDs for $R_d = 100\%$ gave poor agreement and were much finer than the weighted Delaunay CSDs for their dense DEM samples ($\mu = 0$). They did find that the inscribed circle method for $R_d = 0\%$ gave a reasonable match to the loose DEM samples, and the deviation from the results here is probably because they used a higher interparticle friction coefficient for their loose samples ($\mu = 0.7$). In addition to the study of filtration, the weighted Delaunay method has also been used for the prediction of permeability in porous media (Gao et al. 2012).

Evaluation of idealized constriction configurations

The data that can be generated using this approach enable a re-examination of the hypothetical constriction configurations considered in prior studies and illustrated in Figures 1(a) and (b). These assumptions are embedded in the inscribed circle method proposed by Silveira et al. (1975) and Locke et al. (2001). Both the inscribed circle method and the weighted Delaunay method give data on the CSD as well as the particles bounding each constriction. Direct comparison of data generated using these approaches is therefore well suited to a re-evaluation of the use of idealized configurations when generating a CSD. The inscribed circle method was implemented using Matlab as described in Shire (2014). The implementation was successfully validated against an example from Raut (2006).

Figure 5 compares the CSDs obtained using the inscribed circle method with those obtained using the weighted Delaunay method for $C_u=1.5$ (Figure 5(a)) and $C_u=6$ (Figure 5(b)). The weighted Delaunay

method clearly generates a much narrower range of CSDs than the inscribed circle method in both cases. For the uniformly graded material with $C_u = 1.5$ shown in Figure 5(a), the weighted Delaunay CSDs plot approximately between the inscribed circle CSDs with $R_d = 30\%$ and $R_d = 60\%$. Within these limits, the shape of the CSDs is in agreement. The densest possible arrangement of the inscribed circle method ($R_d = 100\%$) which assumes all constriction configurations are 3-particle (Figure 1(a)) gives constriction diameters that are much smaller than those obtained using the weighted Delaunay method for the “dense” sample. The “dense” weighted Delaunay CSD was created considering frictionless, perfectly spherical particles and so it represents an upper bound that is denser than any sample that could be physically created. Thus, it can be concluded that a fabric with constrictions consisting entirely of three mutually touching spheres is unlikely to occur in reality. Also, it is unclear whether it is possible for spheres to form a stable fabric which would be loose enough to match the loosest possible arrangement ($R_d = 0\%$), as the “loose” DEM sample was created using an interparticle friction of $\mu = 0.3$, which prior research suggests is close to the upper limit of mineral friction as discussed by Huang et al. (2014). Huang et al also showed that using a value of $\mu \geq 0.5$ can give a material response that is not representative of soil behaviour. The $R_d = 0\%$ data for the inscribed circle method takes the constriction formed by the 4-particle configuration using the largest particles (Figure 1(b)) to be D_{c100} ; the data presented here indicate that that assumption significantly overestimates the constriction sizes.

The narrower range of CSDs obtained using the weighted Delaunay method may be due in part to the use of perfect spheres, which tend to give a smaller void ratio range than real soils. However, even taking this into account, the weighted Delaunay result does not support the use of Equation 1 to model the variation of the CSD with relative density. Figure 1(c) to (f) illustrate representative particle-constriction configurations obtained in the DEM models. These are presented adjacent to the assumed three-particle and four-particle configurations (Figure 1(a) and (b)) to illustrate that while these idealized configurations occur in some cases e.g. Figure 1 (c) and (d), other constriction configurations do not resemble these idealised combinations e.g. Figure 1(e) and (f).

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260 Figure 5(b) shows that the agreement between the weighted Delaunay and inscribed circle methods
 261 reduces as the C_u increases to $C_u = 6$. Although both methods give the same $D_{c0} \approx 0.155D_{0F}$, the
 262 inscribed circle method results in a more widely-graded CSD than the weighted Delaunay method. The
 263 divergence between the weighted Delaunay method and the inscribed circle method for intermediate
 264 relative densities ($R_d = 50\%$ and 70%) occurs for the largest 50-60% of constrictions in the CSD, where
 265 the inscribed circle method results in much larger diameter constrictions. The densest and loosest CSDs
 266 from the inscribed circle method diverge greatly from the weighted Delaunay result. The densest CSD
 267 generated using the inscribed circle method includes constrictions that are larger than those from any
 268 of the weighted Delaunay CSDs.

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270 This reduction in the agreement between the two methods as C_u increases can be explained by
 271 examining the soil fabric. Figure 6 gives the cumulative distributions of the largest, intermediate and
 272 smallest particles bounding constrictions for the inscribed circle and weighted Delaunay methods for
 273 the material with $C_u = 1.5$. The inscribed circle distribution is based on the assumption that the
 274 probability is proportional to a particle's surface area (Locke et al., 2001) whereas the weighted
 275 Delaunay distribution is directly measured from the numerical model. For this uniform filter, the surface
 276 area assumption is reasonably accurate, as shown in Figure 6(a). The agreement decreases with
 277 increasing C_u and the cumulative distributions for the filter with $C_u = 6$ are presented in Figure 7. It is
 278 clear that the particles assumed to form constrictions according to the inscribed circle method are
 279 consistently coarser than those from the weighted Delaunay. These data show that the assumption that
 280 surface area is a measure of how likely a particle is to form a constriction does not hold for the more
 281 broadly-graded filters as the coarse particles are separated by finer particles, as shown schematically in
 282 Figure 7(b). This shortcoming was recognised by Locke et al. (2001), who suggested regrading well
 283 graded filters by removing particles with $D_{nF} < D_{2nF}/4$, where D_{nF} is the filter diameter for $n\%$ smaller and

D_{2nF} is the filter diameter for 2n% smaller. For linear gradings this equates to regrading the PSD to $C_u = 4$. It can be seen in Figure 5(b) that this gives only marginally better agreement.

These comparison data indicate that the idealized configurations in Figures 1(a) and (b) are not representative of constriction topologies. This lack of agreement motivated a systematic re-examination of the influence of C_u and density on constriction sizes as well as a re-assessment of the characteristic diameters used in common design rules.

Effect of Coefficient of Uniformity and Density on CSD

Figure 8(a) shows the CSDs obtained using the weighted Delaunay method with merging overlap = 50% for the loose, linearly graded samples, normalised by D_{0F} . All the CSDs for the linear gradings considered have generally similar shapes; referring to Table 1 and considering all packing densities, the CSD C_u varies between 1.48 and 1.64. As the filter grading increases from $C_u = 1.2$ to $C_u = 3$, the CSDs coarsen. However, for samples with $C_u \geq 3$ the CSDs are very similar, with those for $C_u = 4.5$ and $C_u = 6$ being virtually indistinguishable as shown by comparing the D_{c50} and C_{uCSD} values in Table 1. The simulations carried out with the bi-linear gradings are also summarised in Table 1. Figure 8(b) shows the CSDs for the bi-linear PSDs given in Figure 2(b). As with the CSDs from linear PSDs, the CSDs for the bi-linear PSDs maintain a relatively similar shape. For BL5, BL10 and BL15 the CSDs lie between the limits set by $C_u = 1.5$ and $C_u = 3$, the CSDs becoming more similar to $C_u = 3$ as the volume fraction with $C_u = 3$ increases. However, as the proportion of $C_u = 3$ material moves to 25% and above the CSDs become very similar to, but slightly coarser than, the CSD of the linear filter with $C_u = 3$. The trends observed for the linear and bi-linear gradings were also observed in additional analyses using merging overlaps of 0% and 100% and for the other density values considered.

Relative density is an important factor in filter effectiveness (Lafleur and T  treault 1986), although it is not considered by empirical rules such as Terzaghi (Fannin 2008) or Sherard and Dunnigan (1989). The

effect of density is illustrated by comparing the loose, medium and dense CSDs for samples with $C_u = 1.5$ and $C_u = 6$ in Figure 9. As shown in Table 2, the smallest constrictions are the same for loose and dense samples ($\approx 0.155 D_{0F}$), and the remaining fractile diameters (e.g. D_{c25} , D_{c50} , etc.) are consistently approximately 10% smaller in the dense sample than the loose, indicating that while the constriction diameters show some sensitivity to density, the shape of the CSD curve remains similar.

Comparison with characteristic diameters proposed by Kenney et al. (1985)

Kenney et al. (1985) proposed normalising the CSDs by D_{5F} and D_{15F} to relate the controlling constriction size to the PSD. Figure 10(a) illustrates the CSDs normalised by D_{5F} of the corresponding filter PSDs and it is clear that the CSDs from the samples with $C_u \geq 2$ form a narrow band of curves, with the CSDs from the samples with $C_u = 1.2$ and 1.5 lying to the left of this (i.e. having smaller normalized constrictions). The controlling constriction size $D_c^* \approx 0.25 D_{5F}$ proposed by Kenney et al. (1985) is included on Figure 10(a). Although the CSD alone does not allow the controlling constriction size to be calculated, qualitatively it can be considered that similar normalised CSDs would translate into similar values of D_c^* for these filters. Figure 10(b) considers the CSDs normalised by D_{15F} and in this case the samples with $C_u \leq 3$ form a narrow band, with more widely graded filters having progressively finer normalised CSDs. The controlling constriction size $D_c^* \approx 0.2 D_{15F}$ proposed by Kenney et al. (1985) is also marked on Figure 10(b). Figure 11 shows the CSDs normalised by D_{15F} for the bi-linear PSDs. As was the case with the linear PSDs with $C_u \leq 3$, the normalised bi-linear filter CSDs maintain a similar shape and form a narrow band of curves.

Figure 12(a) shows the variation of the normalised median constriction size D_{c50}/D_{5F} and the experimental controlling constriction size D_c^*/D_{5F} with C_u . Note that the y-axes have different scales in order to allow comparison between the trends. The experimental values are taken from Kenney et al. (1985), who tested filters with $C_u = 1.2, 3, 6$ and 12 . For the numerical data loose samples with merging overlaps of 0%, 50% and 100% and dense and medium samples with a merging overlap of 50% are

shown. The same overall trend is seen irrespective of the merging overlap and density: D_{c50}/D_{5F} increases sharply as sample C_u increases from $C_u = 1.2$ to $C_u = 2$, then falls slightly between $C_u = 4.5$ and $C_u = 6$. The experimental values of D_c^*/D_{5F} show a similar pattern, initially increasing between $C_u = 1.2$ and 3, before falling gently as C_u increases to 12. Referring to Figure 12(b) a similar trend is observed when D_{15F} is used to normalise the results: both numerical D_{c50}/D_{15F} and experimental D_c^*/D_{15F} increase as sample C_u initially increases with C_u then decreases again when $C_u > 3$.

The fact that the median constriction diameter depends on the chosen merging overlap highlights that the definition of the actual CSD is ambiguous. However, although a unique CSD cannot be defined there are similar trends in CSD variation with C_u irrespective of the chosen merging overlap used to define the CSD. This lends support to the approach of Kenney et al. (1985), who proposed using an experimentally determined proportion of a characteristic filter diameter to get D_c^* , rather than attempting to explicitly calculate the full CSD for design. However, the sensitivity of the CSD to relative density shows that the PSD alone is insufficient to describe the controlling constriction at all density levels.

A comparison between the normalised median constriction sizes for each normalising parameter (D_{0F} , D_{5F} and D_{15F}) is given in Table 3. This confirms that D_{5F} gives the narrowest band of normalised values for more widely graded filters with $C_u \geq 2$, whereas D_{15F} gives the narrowest band of normalised values for more uniform filters with $C_u \leq 3$. Both have a similar overall scatter for the samples analysed, measured by the coefficient of variation (standard deviation divided by mean). For linearly graded materials with $C_u \leq 3$ it was noted above that D_{15F} was the best normalising parameter. As shown in Table 3, the same pattern is found for the bi-linearly graded materials, with the CSDs normalised by D_{15F} giving a narrower band of CSDs than the normalised CSDs of either D_{5F} or D_{0F} .

Conclusions

Inter-void constrictions are a key characteristic of granular filters and can be described by a constriction size distribution (CSD). Here, an extensive set of CSDs was calculated from data of 28 DEM simulations of varying coefficient of uniformity up to $C_u = 6$, PSD shape and relative density, using the numerical weighted Delaunay method proposed by Al-Raoush et al. (2003) and Reboul (2008). The user-defined merging overlap parameter used to join adjacent Delaunay tetrahedra was varied from 0% to 100%. As the merging overlap decreases more tetrahedra merge to form larger voids and therefore fewer constrictions with a larger average diameter are identified. This highlights the fact that for a granular material, void boundaries are subjective and no unique CSD can be identified. However, the trends observed for a single merging overlap value were the same regardless of whether the value was 0%, 50% or 100% and therefore general conclusions about the relationship between PSD, relative density and CSD can be drawn. The calculated CSDs were compared to two practical methods of filter design, (i) the inscribed circle method to analytically define a full CSD based on the largest circle which can be inscribed between combinations of three or four touching spheres (Locke et al., 2001); (ii) characteristic filter particle diameters to represent the controlling constriction diameter, D_c^* (Kenney et al. 1985). Based on this analysis, the following conclusions can be drawn.

- a) For a given merging overlap value, the CSDs are similarly shaped regardless of the filter uniformity. For linearly graded filters with the same smallest particle, D_{OF} , the CSDs become coarser as filter C_u increases from $C_u = 1.2$ to $C_u = 3$. For samples with $C_u \geq 3$ the CSD curves form a narrow band. This shows very good qualitative agreement with the experimental work of Kenney et al. (1985), who found that when normalised by D_{OF} the controlling constriction size increased as filter C_u increased from $C_u = 1.2$ to $C_u = 3$, and then remained approximately constant up to at least $C_u = 12$.

- b) Constriction sizes reduce with increasing relative density. However, the CSD shape remains similar. A difference in constriction diameter of around 10% was found between the CSDs from samples in the loosest and densest states analysed.
- c) A comparison between CSDs from the numerical weighted Delaunay and the analytical inscribed circle methods gave reasonable agreement for filters of low C_u values ($C_u \leq 2$) and moderate relative densities in the inscribed circle method ($R_d = 30\text{-}60\%$). However, the large range of possible inscribed circle CSDs, depending on R_d , was not replicated in the weighted Delaunay CSDs. Agreement between the methods became progressively worse as C_u increased to $C_u = 6$. In particular the inscribed circle method resulted in much larger diameter constrictions. This poor agreement was due to an over-idealisation of soil fabric in the inscribed circle method. The assumption that the probability of a filter particle forming a constriction is proportional to surface area was found to be poor for more widely graded filters. This was due to coarse particles being separated from one another by smaller particles in these materials.
- d) Narrow bands of CSD curves are formed when normalising PSDs by D_{5F} and D_{15F} , which Kenney et al. (1985) suggested were useful for practical purposes. In agreement with Kenney et al., D_{5F} was found to give narrowest band of CSD curves for filters with $C_u \geq 2$, whereas the more commonly used D_{15F} was found to give the narrowest band for filters with $C_u \leq 3$, regardless of whether the PSD was linear or bi-linear. These findings give fundamental support to the use of characteristic diameters in filter design.

The CSDs presented here were calculated for homogeneous DEM samples of spherical particles. Future work should consider the fabric effects resulting from changes in particle shape, anisotropy and inhomogeneity (i.e. segregation) in real filters. In particular, micro-CT based techniques (e.g. Taylor et al. 2015) allow the effect of particle morphology and soil fabric to be assessed at the microscale from 3D images of real soils.

411 Acknowledgements

412 Computing resources from Research Councils UK's HECToR HPC (via EPSRC grant EP/I006761/1), the
 413 European PRACE Program and the CX1 HPC at Imperial College are gratefully acknowledged. The first
 414 author was funded by an EPSRC Doctoral Training Account.

415

416 Notation

417 The following symbols are used in this paper:

418	C_u	Coefficient of uniformity
419	$C_{u,CSD}$	Coefficient of uniformity of the constriction size distribution
420	D_{XB}	Base particle diameter for which X% of the material by mass is finer
421	D_{XF}	Filter particle diameter for which X% of the material by mass is finer
422	D_c^*	Controlling constriction diameter
423	D_{cX}	Constriction diameter for which X% of the constrictions are smaller
424	D_{cD}	Largest circle which can be inscribed between three mutually touching filter particles
425	D_{cL}	Equivalent diameter of the maximum area between four touching filter particles
426	P	Fraction smaller of the CSD
427	R_d	Relative density
428	$S_{c,max}$	Maximum area between four touching filter particles
429	p'	Mean effective stress
430	μ	Coefficient of interparticle friction
431	σ'	Effective stress

432

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PSD shape (Linear or Non-linear)	Coefficient of uniformity, Cu	Number of particles	Interparticle friction coefficient, μ	Void ratio, e	Coefficient of uniformity of the CSD, Cu_{CSD}	Median constriction diameter, D_{c50} / D_{of}
Linear	1.2	8262	0.0	0.558	1.48	0.25
			0.1	0.651	1.58	0.27
			0.3	0.714	1.64	0.29
Linear	1.5	9313	0.0	0.531	1.52	0.29
			0.1	0.608	1.60	0.31
			0.3	0.658	1.63	0.33
Linear	2	12115	0.0	0.467	1.55	0.32
			0.1	0.523	1.60	0.35
			0.3	0.555	1.62	0.37
Linear	3	22600	0.0	0.382	1.57	0.35
			0.1	0.426	1.60	0.38
			0.3	0.455	1.60	0.39
Linear	4.5	44821	0.0	0.320	1.58	0.36
			0.1	0.356	1.60	0.38
			0.3	0.384	1.59	0.40
Linear	6	59183	0.0	0.265	1.58	0.37
			0.1	0.292	1.60	0.38
			0.3	0.314	1.58	0.40
Bi-Linear (BL5)	1.5	18632	0.0	0.448	1.63	0.31
			0.3	0.519	1.52	0.35
Bi-Linear (BL10)	1.5	19915	0.0	0.442	1.63	0.32
			0.3	0.506	1.52	0.37
Bi-Linear (BL15)	1.6	24757	0.0	0.440	1.63	0.34
			0.3	0.500	1.53	0.38
Bi-Linear (BL25)	1.9	30011	0.0	0.498	1.61	0.36
			0.3	0.584	1.55	0.40
Bi-Linear (BL50)	2.6	39990	0.0	0.431	1.60	0.37
			0.3	0.503	1.57	0.41

Table 1: Summary of DEM Simulations

Coefficient of uniformity, Cu	$D_{c0,dense} /$ $D_{c0,loose}$	$D_{c25,dense} /$ $D_{c25,loose}$	$D_{c50,dense} /$ $D_{c50,loose}$	$D_{c75,dense} /$ $D_{c75,loose}$	$D_{c99,dense} /$ $D_{c99,loose}$
1.5	1.0	0.91	0.88	0.89	0.88
6	1.0	0.91	0.91	0.92	0.93

Table 2. Comparison of CSD diameters for dense and loose samples with Cu = 1.5 and 6, overlap = 50%.

PSD shape (Linear or Non-linear)	Normalising particle diameter (D_{norm})	Cu (linear), Sample name (bi-linear)	D_{c50} / D_{norm}	Coefficient of variation of D_{c50} / D_{norm}
Linear	D_{0F}	1.2	0.29	0.13
		1.5	0.33	
		2	0.37	
		3	0.39	
		4.5	0.40	
		6	0.40	
	D_{5F}	1.2	0.28	0.08
		1.5	0.32	
		2	0.34	
		3	0.35	
		4.5	0.35	
		6	0.33	
	D_{15F}	1.2	0.28	0.09
		1.5	0.29	
		2	0.30	
		3	0.28	
		4.5	0.26	
		6	0.23	
Bi-linear	D_{0F}	BL5	0.31	0.07
		BL10	0.32	
		BL15	0.34	
		BL25	0.36	
		BL50	0.37	
	D_{5F}	BL5	0.28	0.07
		BL10	0.29	
		BL15	0.3	
		BL25	0.32	
		BL50	0.33	
	D_{15F}	BL5	0.25	0.03
		BL10	0.25	
		BL15	0.24	
		BL25	0.26	
		BL50	0.27	

Table 3 Linear gradings: Comparison of CSD characteristics obtained using normalizing diameters D_{0F} , D_{5F} and D_{15F}

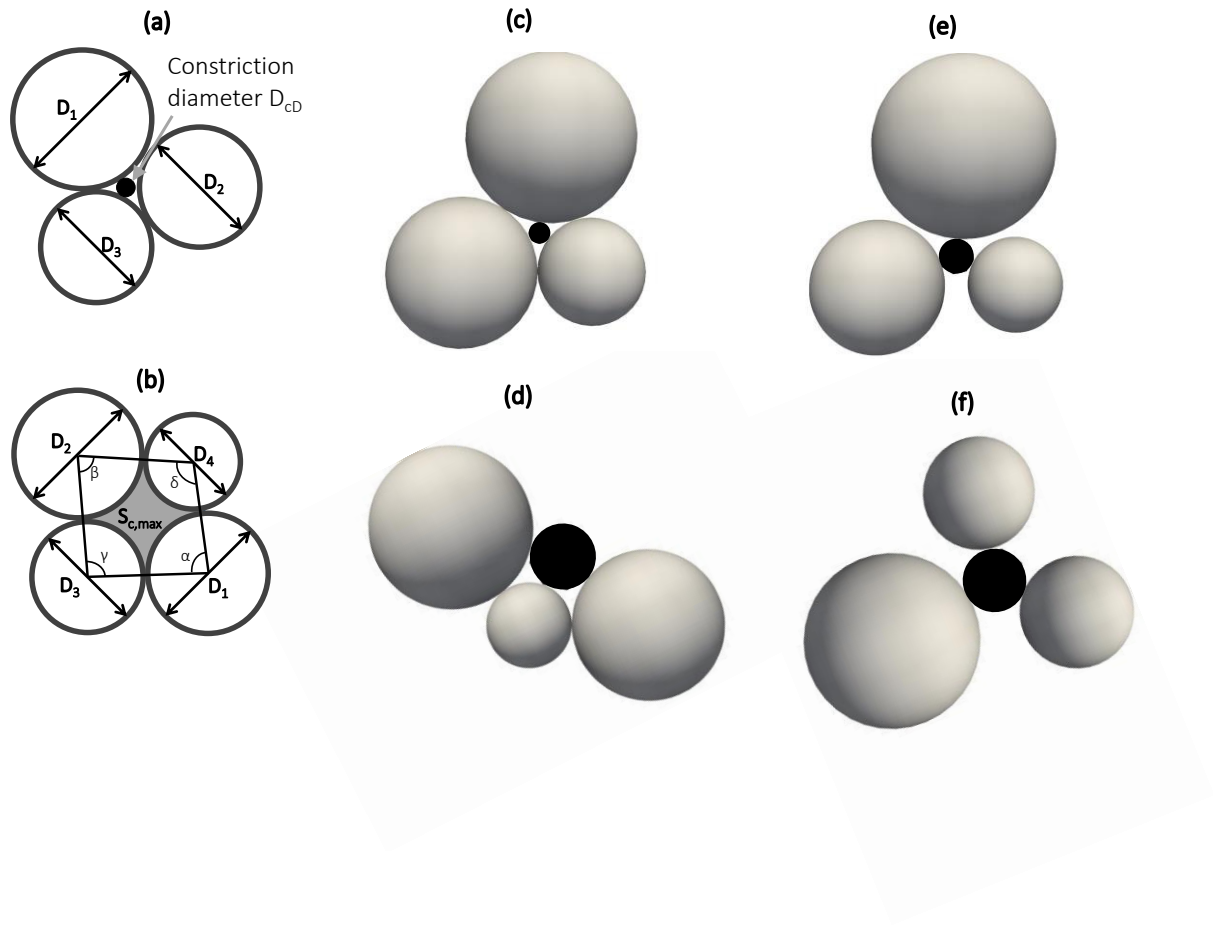
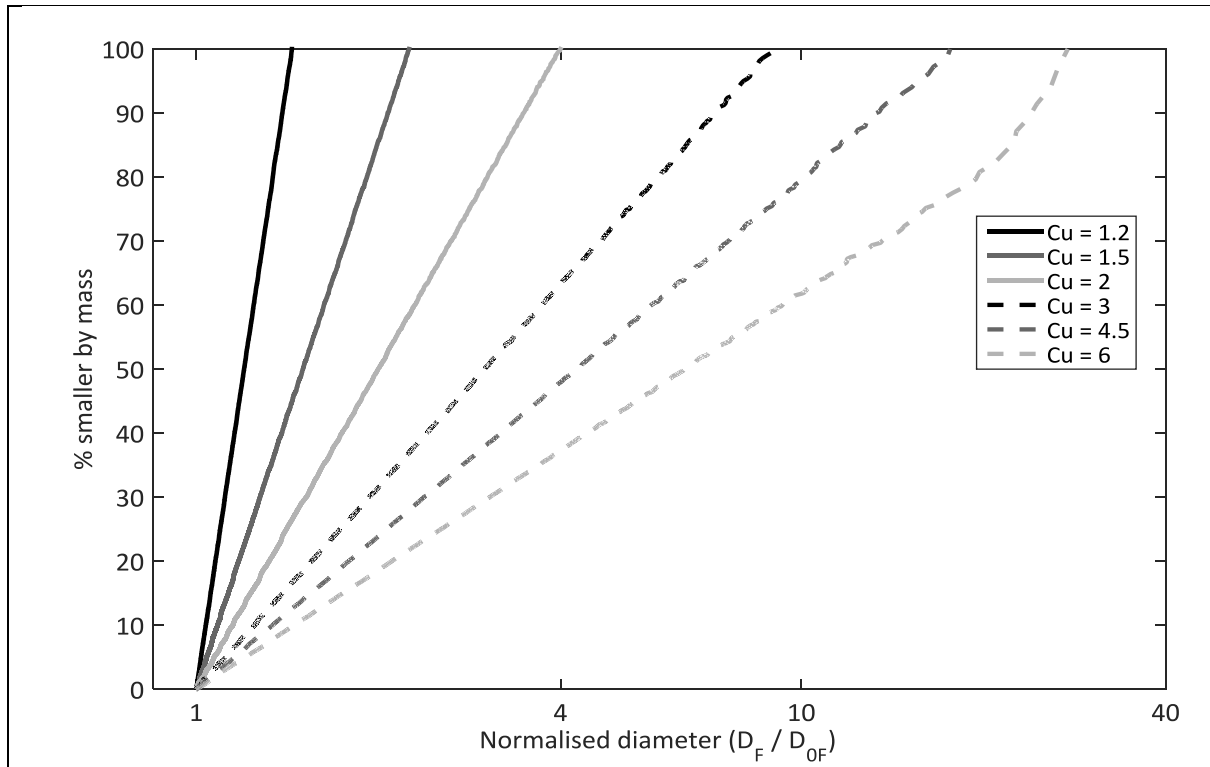
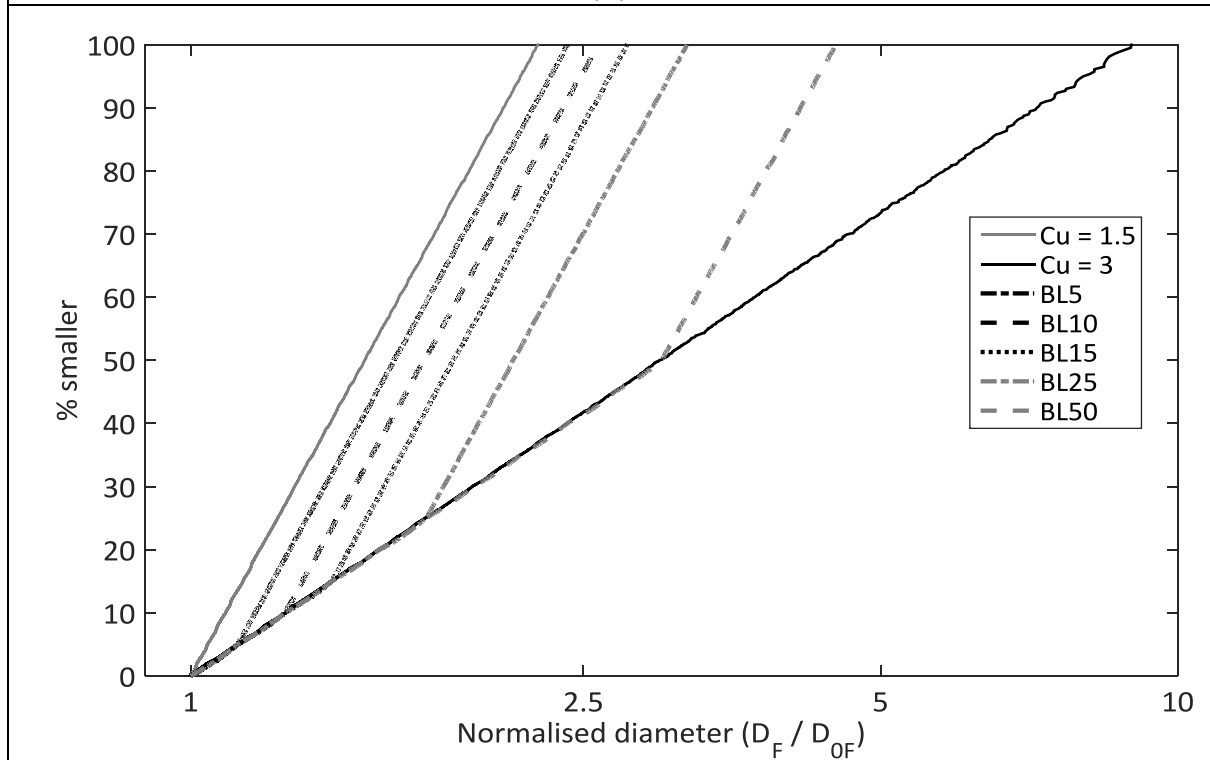


Figure 1. (a) idealised 2D three-particle arrangement used in the inscribed circle method (Silveira et al., 1975); (b) idealised 2D four-particle arrangement used in the inscribed circle method (Silveira et al., 1975); (c) to (f) images of weighted Delaunay constrictions inscribed between 3D DEM particles.



(a)



(b)

Figure 2. Normalised particle size distributions (PSDs) for samples considered (a) Linear PSDs (b) Bi-linear PSDs.

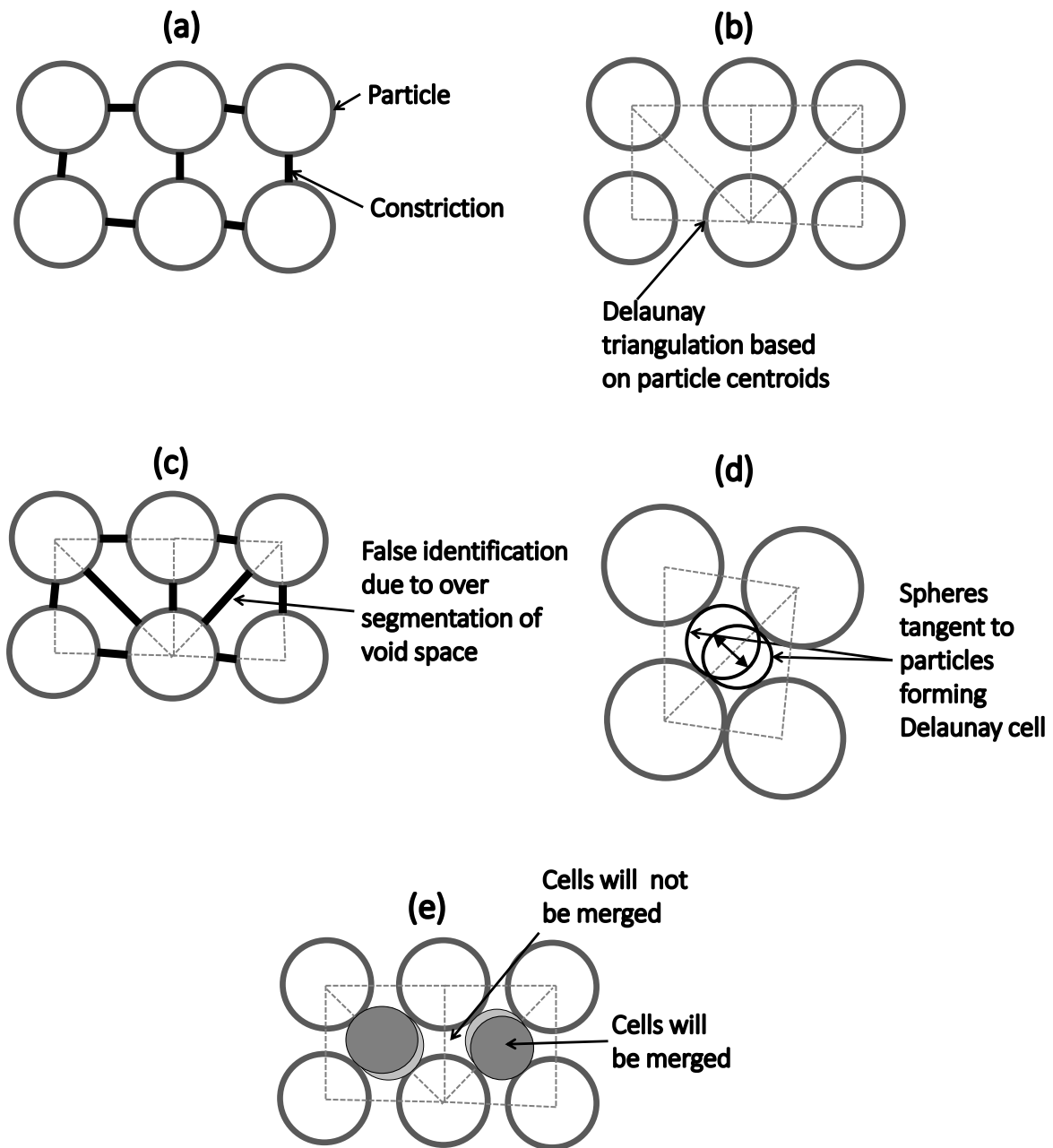


Figure 3. Two dimensional analogy of three-dimensional merging of Delaunay cells to form voids (Adapted from Al-Raoush et al., 2003). (a) correct constriction locations; (b) Delaunay triangulation of the void space; (c) false identification of constrictions based on Delaunay cell edges; (d) tangent sphere overlap used for Delaunay cell merging; (e) merging of Delaunay cells to form larger voids. Note that 2D triangles represent 3D tetrahedra, triangle sides represent tetrahedra faces and tangent circles represent tangent spheres.

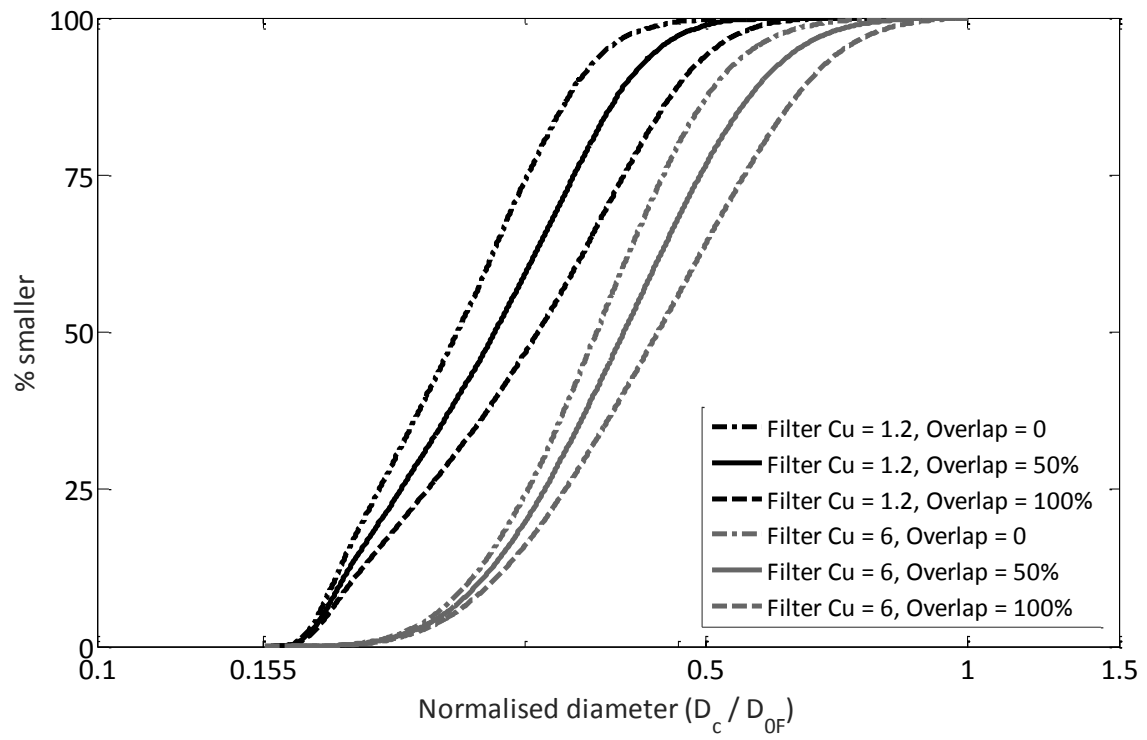


Figure 4. Comparison of CSDs generated by weighted Delaunay method with varying merging overlap values from 0% to 100% for filter PSDs with $Cu = 1.2$ and $Cu = 6$.

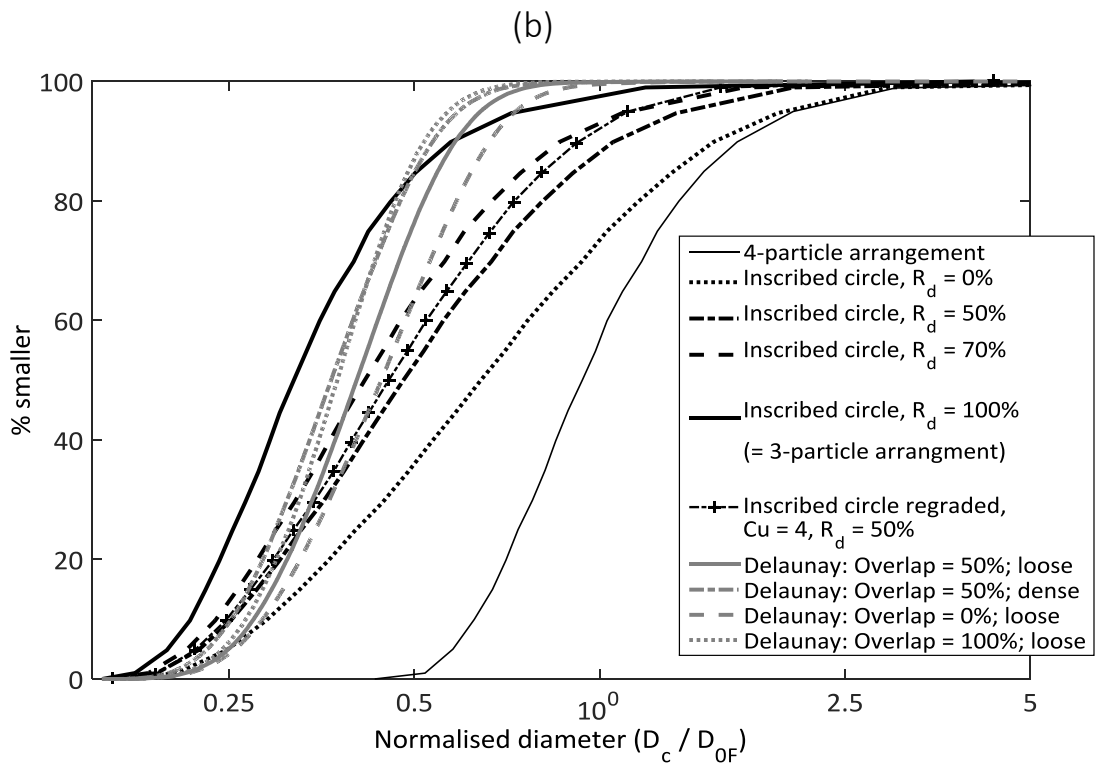
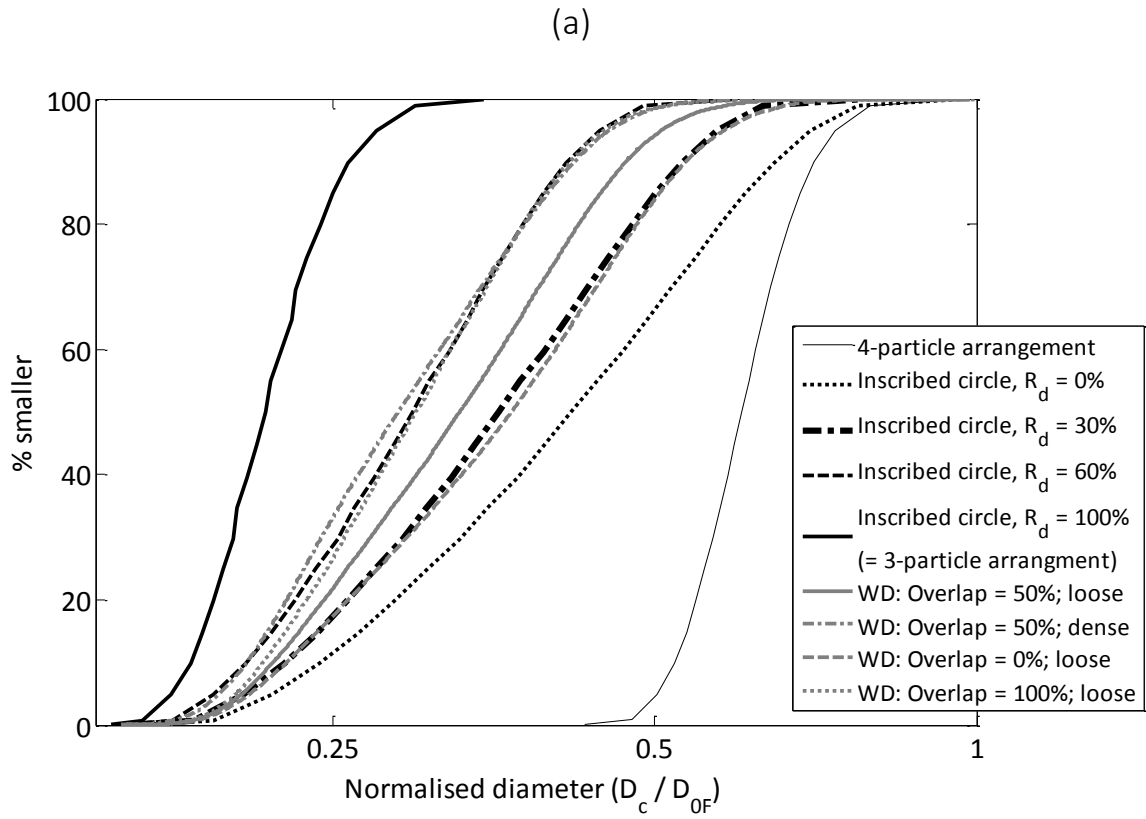
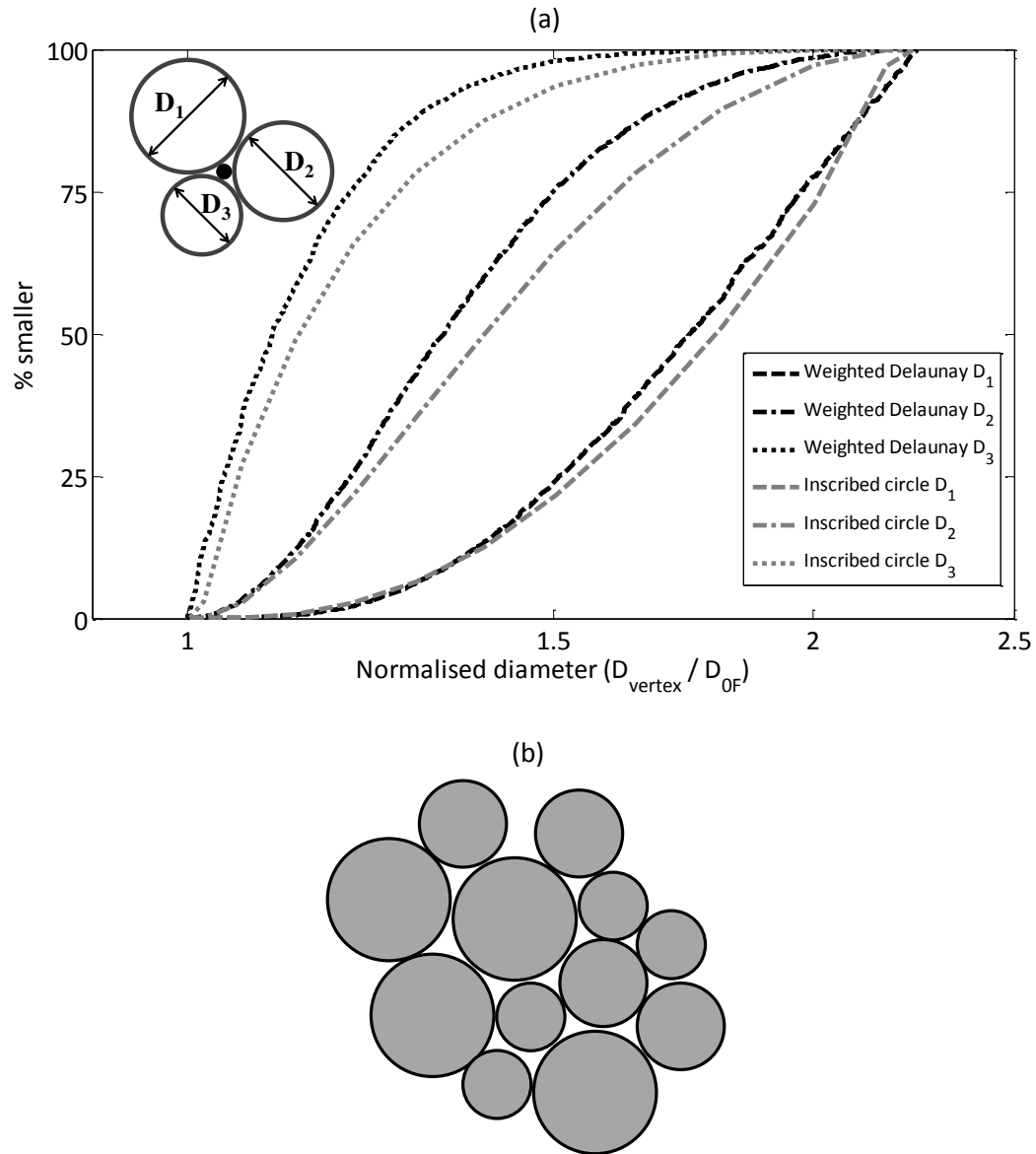


Figure 5 Comparison of CSDs calculated with the inscribed circle method and the weighted Delaunay method (a) $Cu=1.5$, (b) $Cu=6$. The three and four particle arrangements are shown in Figure 1(a) and (b).



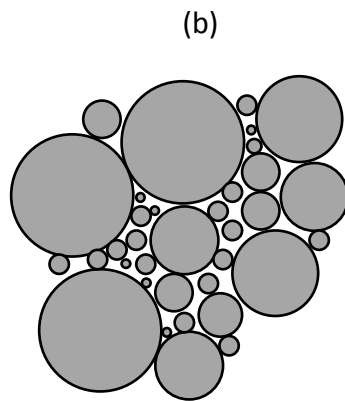
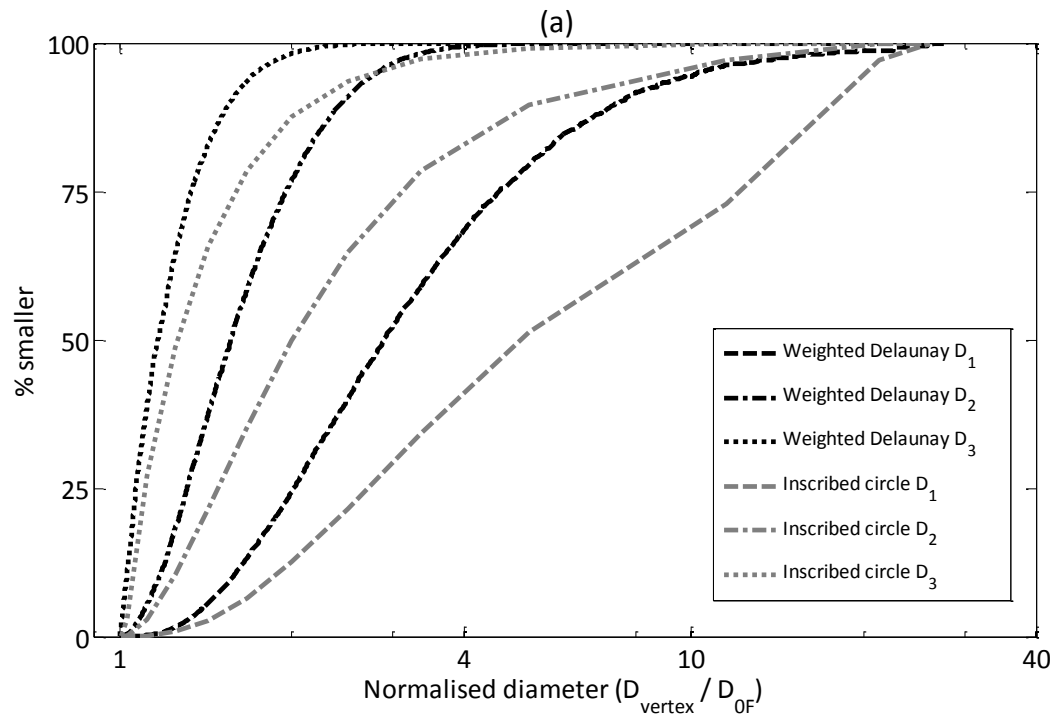


Figure 7 (a) $C_u = 6$: Comparison of diameters of filter particles making up vertices of constrictions according to inscribed circle and weighted Delaunay methods ; (b) schematic of soil fabric for widely graded soils in which larger particles are separated by smaller particles.

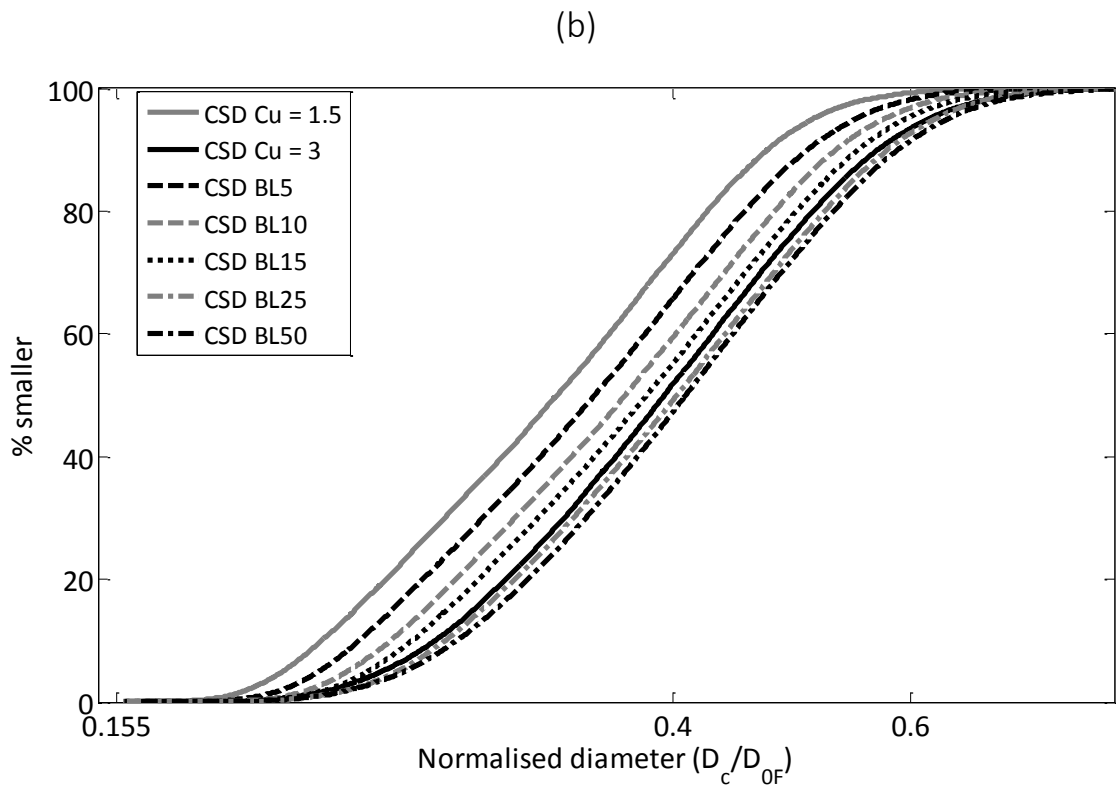
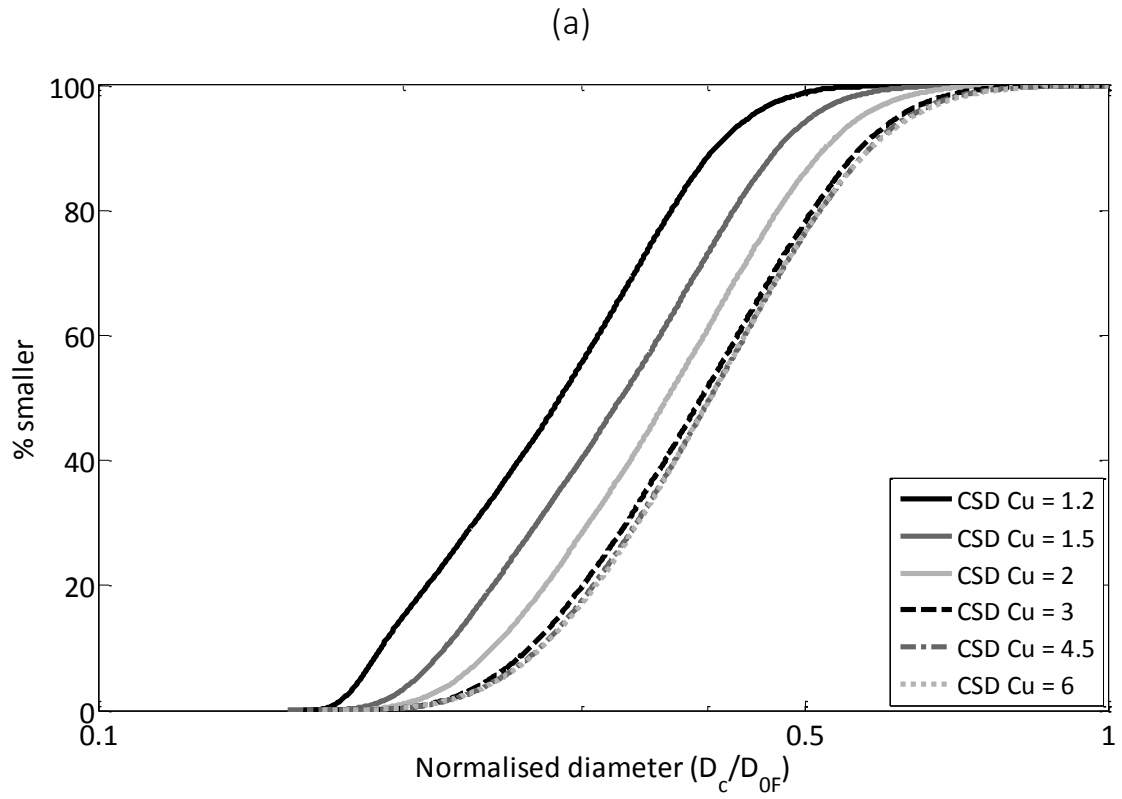


Figure 8. Normalised CSDs for linear gradings obtained with weighted Delaunay method using a 50% overlap: (a) linear gradings; (b) bi-linear gradings

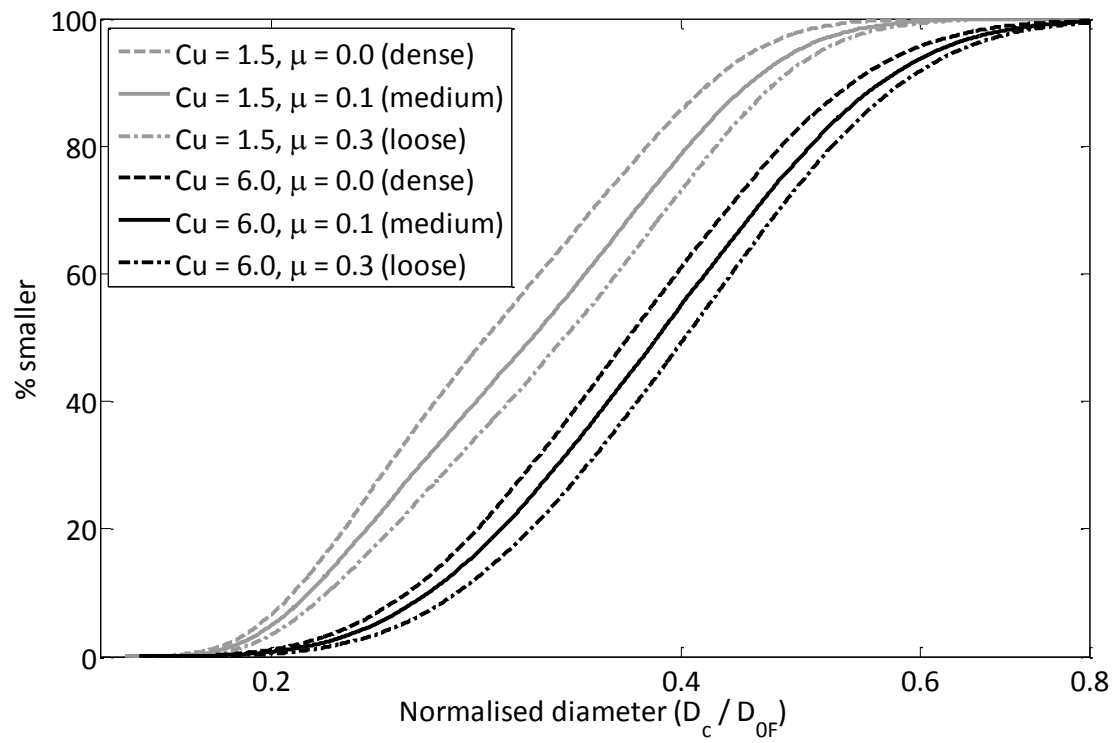


Figure 9. Normalised CSDs obtained with weighted Delaunay method using a 50% overlap for loose, medium dense and dense and DEM samples with $Cu = 1.5$ and $Cu = 6$.

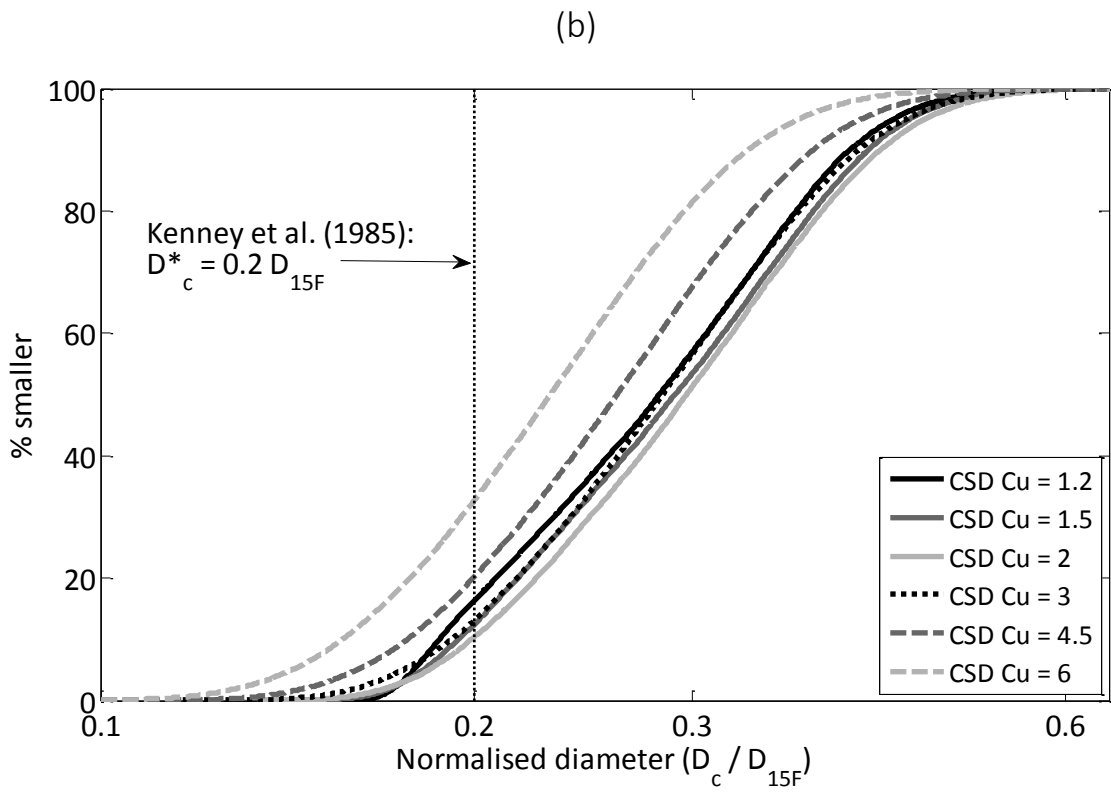
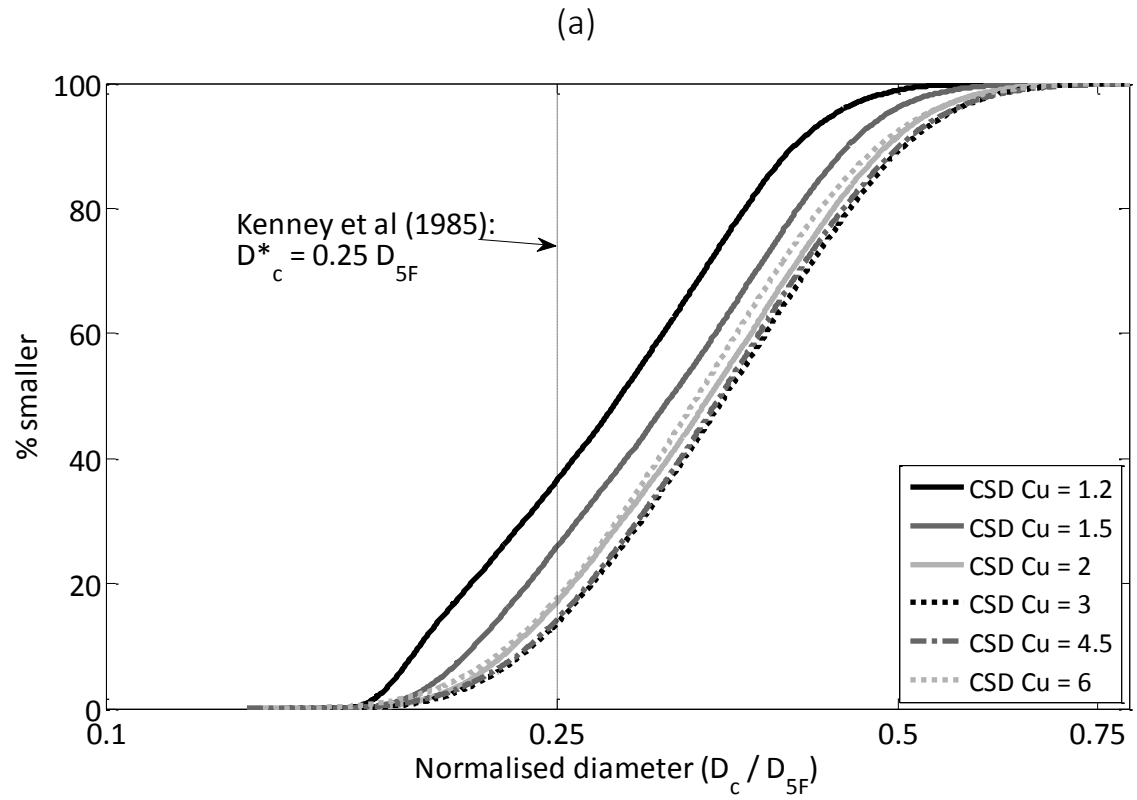


Figure 10. CSDs obtained with weighted Delaunay method using a 50% overlap for loose linearly graded DEM samples normalised by (a) D_{5F} and (b) D_{15F}

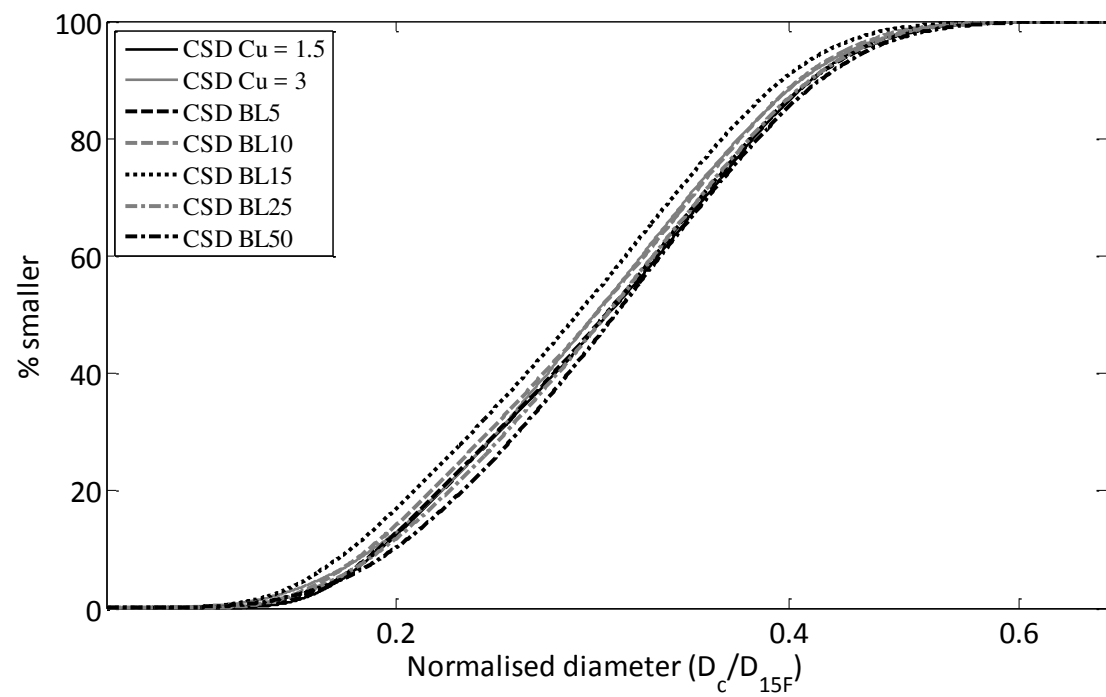
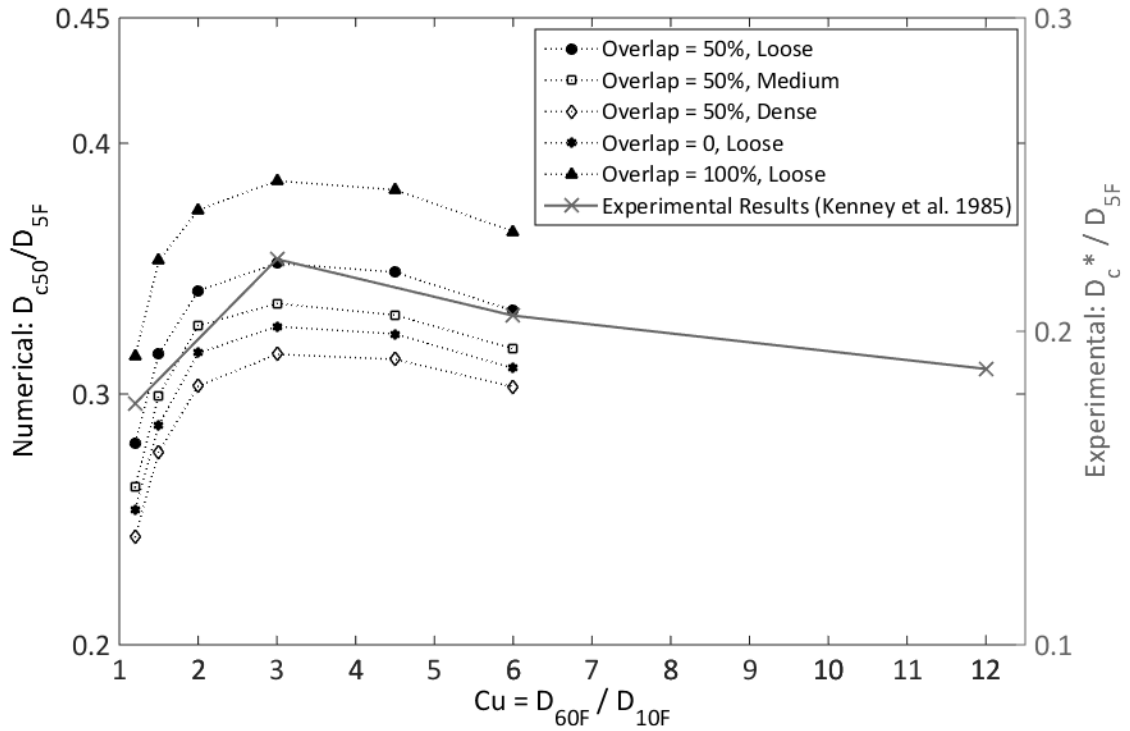


Figure 11. CSDs obtained with weighted Delaunay method using a 50% overlap for loose bi-linearly graded DEM samples normalised by D_{15F}



(a)

(b)

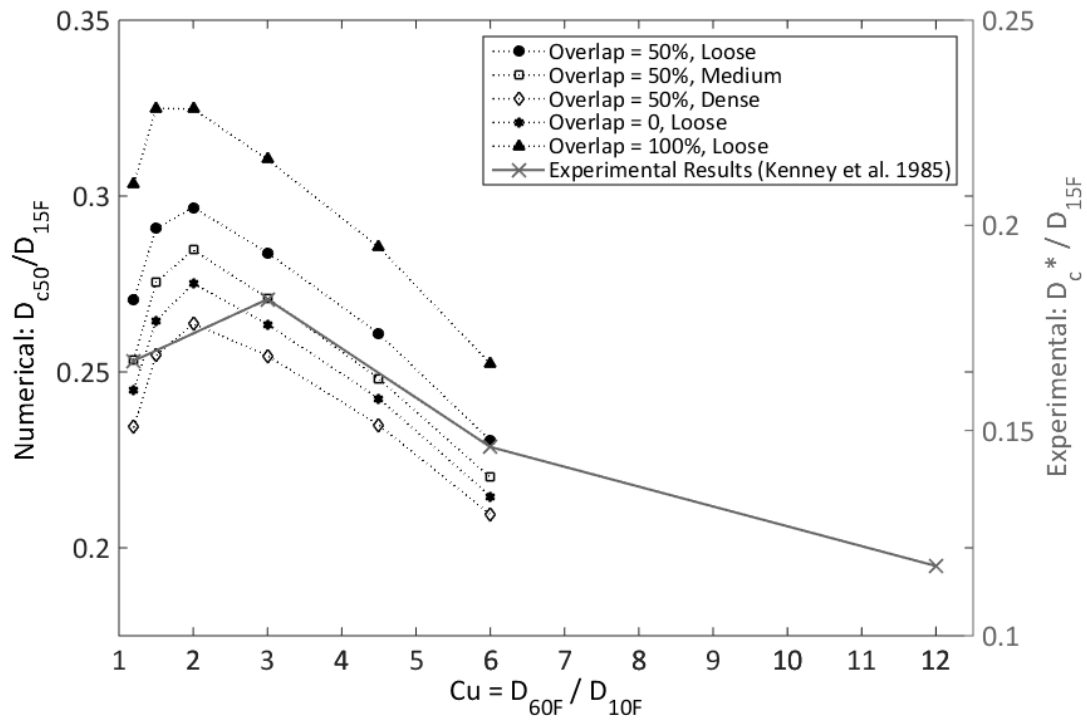


Figure 12 Variation of median constriction size D_{c50} and experimental controlling constriction size with coefficient of uniformity, Cu : (a) normalised by D_{5F} ; (b) normalised by D_{15F} . D_c^* values taken from Kenney et al. (1985):