Optimal control of systems with a unitary semigroup and with colocated control and observation

George Weiss
Dept. of Electrical & Electronic Eng.
Imperial College of Sci., Technol. & Medicine
Exhibition Road, London SW7 2BT
United Kingdom
e-mail: G.Weiss@ic.ac.uk

Abstract. We solve the quadratic optimal control problem on an infinite time interval for a class of linear systems whose state space is a Hilbert space and whose operator semigroup is unitary. The difficulty is that the systems in this class, having unbounded control and observation operators, may be ill-posed. We show that there is a surprizingly simple solution to the problem (the optimal feedback turns out to be output feedback). Our approach is to use a change of variables which transforms the system into a one which, according to recent research, is known to be conservative. We show that, under a mild assumption, the transfer function of this conservative system is inner, and then it follows that the optimal control of this conservative system is trivial. We give an example with the wave equation on an n-dimensional domain, with Neumann control and Dirichlet observation of the velocity.

Key words. Semigroup of unitary operators, colocated control and observation, conservative linear system, quadratic optimal control, unbounded control and observation operators, output feedback, Riccati equation.

1. Problem formulation and main result

In this paper we investigate the standard quadratic optimal control problem for a class of systems described by a second order differential equation in a Hilbert space. Such a differential equation is rather common in describing undamped oscillatory systems, such as waves, beams or plates, where the control and the sensing are colocated, for example, acting through the same part of the boundary. The original second order equation can of course be rewritten as a first order differential equation in a product Hilbert space $\mathcal{X}$ called the state space. The operator semigroup

associated with the first order equation is unitary, so that the system is not strongly stable. The system may have unbounded control and observation operators, and these are adjoint to each other (this is the formal meaning of "colocated").

The main difficulty is that this oscillatory system may be ill-posed. Actually, it may violate up to three out of the four conditions for well-posedness listed in Curtain and Weiss [4] (the one it does not violate is semigroup generation). Thus, it does not fit into any framework established for the treatment of the quadratic optimal control problem. For example, in Lasiecka and Triggiani [7, 8], either the semigroup is assumed to be analytic, or the control operator is assumed to be admissible, which is not necessarily the case here. In Callier and Winkin [2], Curtain [3], Staffans [10, 11] as well as in Weiss[2] [17], the system to be controlled is assumed to be well-posed. (All these references make also various other assumptions.)

In spite of the difficulty explained above, we provide a surprisingly simple solution to our optimal control problem, by a transformation which leads to a conservative linear system of a special kind, studied in Tucsnak and Weiss [12, 16].

Let H be a Hilbert space, and let $A_0 : \mathcal{D}(A_0) \to H$ be a self-adjoint, positive and boundedly invertible operator. We introduce the scale of Hilbert spaces H_α, $\alpha \in \mathbb{R}$, as follows: for every $\alpha \geq 0$, $H_\alpha = \mathcal{D}(A_0^\alpha)$, with the norm $\|z\|_\alpha = \|A_0^\alpha z\|_H$. The space $H_{-\alpha}$ is defined by duality with respect to the pivot space H as follows: $H_{-\alpha} = H_\alpha^*$ for $\alpha > 0$. Equivalently, $H_{-\alpha}$ is the completion of H with respect to the norm $\|z\|_{-\alpha} = \left\|A_0^{-\alpha} z\right\|_H$. The operator A_0 can be extended (or restricted) to each H_α, such that it becomes a bounded operator

$$A_0 : H_\alpha \to H_{\alpha-1} \qquad \forall\, \alpha \in \mathbb{R}.$$

Let C_1 be a bounded linear operator from $H_{\frac{1}{2}}$ to $\mathcal{U}$, where $\mathcal{U}$ is another Hilbert space. We identify $\mathcal{U}$ with its dual, so that $\mathcal{U} = \mathcal{U}^*$. We denote $B_1 = C_1^*$, so that $B_1 \in \mathcal{L}(\mathcal{U}, H_{-\frac{1}{2}})$. The system Σ^u studied here is described by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} z(t) + A_0 z(t) = B_1 u(t), \tag{1.1}$$

$$z(0) = z_0, \qquad \dot{z}(0) = w_0, \tag{1.2}$$

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t} C_1 z(t), \tag{1.3}$$

where $t \in [0, \infty)$ is the time. The equation (1.1) is understood as an equation in $H_{-\frac{1}{2}}$, i.e., all the terms are in $H_{-\frac{1}{2}}$. The signal u is the input function, with values in $\mathcal{U}$, and the signal y is the output function, with values in $\mathcal{U}$ as well. The state $x(t)$ of this system, its initial state x_0 and its state space $\mathcal{X}$ are defined by

$$x(t) = \begin{bmatrix} z(t) \\ \dot{z}(t) \end{bmatrix}, \qquad x_0 = \begin{bmatrix} z_0 \\ w_0 \end{bmatrix}, \qquad \mathcal{X} = H_{\frac{1}{2}} \times H. \tag{1.4}$$

As mentioned earlier, systems in the class just described may be not well-posed, so we have to be careful about the meaning of state trajectories and output functions. To discuss this, we rewrite (1.1) as a first order differential equation:

$$\dot{x}(t) = A^u x(t) + B^u u(t), \tag{1.5}$$

where

$$A^u = \begin{bmatrix} 0 & I \\ -A_0 & 0 \end{bmatrix}, \qquad B^u = \begin{bmatrix} 0 \\ B_1 \end{bmatrix}, \tag{1.6}$$

$$\mathcal{D}(A^u) = H_1 \times H_{\frac{1}{2}}.$$

It is easy to check that A^u is skew-adjoint on $\mathcal{X}$ and hence, it generates a strongly continuous group of unitary operators on $\mathcal{X}$, denoted by $\mathbb{T}^u$. Such a group may describe, for example, oscillations of an undamped flexible structure. The superscript "u" used above and in Σ^u may stand for "unitary" or for "unstable". It will be useful to note that for every $s \in \mathbb{C}$ with $-s^2 \in \rho(A_0)$,

$$(sI - A^u)^{-1} = (s^2 I + A_0)^{-1} \begin{bmatrix} sI & I \\ -A_0 & sI \end{bmatrix}. \tag{1.7}$$

The operators $\mathbb{T}^u_t$ (with $t \in \mathbb{R}$) have a natural bounded extension to the Hilbert space $\mathcal{X}^u_{-1}$ defined by

$$\mathcal{X}^u_{-1} = H \times H_{-\frac{1}{2}},$$

the generator of this extended semigroup is an extension of A^u whose domain is $\mathcal{X}$, and we have $B^u \in \mathcal{L}(\mathcal{U}, \mathcal{X}^u_{-1})$. The resolvent $(sI - A^u)^{-1}$ (whenever it exists) has a bounded extension to X^u_{-1}. We use the same notation for the original operators and the extended ones. Now it it clear that, if $x_0 \in \mathcal{X}$ and $u \in L^2([0,\infty), \mathcal{U})$, then the state trajectory x is a continuous function with values in $\mathcal{X}^u_{-1}$, given by

$$x(t) = \mathbb{T}^u_t x_0 + \int_0^t \mathbb{T}^u_{t-\sigma} B^u u(\sigma) \mathrm{d}\sigma. \tag{1.8}$$

If B^u were an admissible control operator for $\mathbb{T}^u$, in the sense of [13] or [4], then x from (1.8) would be a continuous $\mathcal{X}$-valued function of t. However, under the given assumptions, this may be not true. Note that we have

$$\hat{x}(s) = (sI - A^u)^{-1} x_0 + (sI - A^u)^{-1} B^u \hat{u}(s),$$

where a hat denotes the Laplace transformation and $s \in \mathbb{C}_0$, where $\mathbb{C}_0$ is the open right half-plane. In particular, looking at z, the first component of x, we obtain, using the formula (1.7), that for all $s \in \mathbb{C}_0$,

$$\hat{z}(s) = s(s^2 I + A_0)^{-1} z_0 + (s^2 I + A_0)^{-1} w_0 + (s^2 I + A_0)^{-1} B_1 \hat{u}(s), \tag{1.9}$$

where z_0 and w_0 are the components of x_0, as in (1.4).

Now we discuss the interpretation of the output equation (1.3). If x is only known to be a continuous function with values in $\mathcal{X}^u_{-1}$, then (1.3) makes no sense, because $z(t)$ (the first component of $x(t)$) is not in the domain of C_1 (which is $H_{\frac{1}{2}}$). Even if it happens that $z(t)$ is in the domain of C_1, it is still unclear if we can differentiate $C_1 z(t)$ with respect to t. We shall overcome these difficulties in defining y by using the Laplace transformation. Indeed, if C_1 were bounded, i.e., if $C_1 \in \mathcal{L}(H, \mathcal{U})$, then (1.9) and (1.3) would imply that for every $u \in L^2([0,\infty), \mathcal{U})$,

$$\begin{aligned} \hat{y}(s) = \; & -C_1 A_0 (s^2 I + A_0)^{-1} z_0 + C_1 s (s^2 I + A_0)^{-1} w_0 \\ & + C_1 s (s^2 I + A_0)^{-1} B_1 \hat{u}(s) \qquad \forall\, s \in \mathbb{C}_0\,. \end{aligned} \tag{1.10}$$

This expression for $\hat{y}(s)$ is well defined for every $x_0 \in \mathcal{X}$ (i.e., for every $z_0 \in H_{\frac{1}{2}}$ and $w_0 \in H$) and for every $u \in L^2([0,\infty), \mathcal{U})$, even if we remove the boundedness assumption on C_1. In this paper we are only interested in the situation when the output signal y is in $L^2([0,\infty), \mathcal{U})$. Recall that, according to a well known theorem of Paley and Wiener, the Laplace transformation is an isomorphism from $L^2([0,\infty), \mathcal{U})$ to the Hardy space $\mathcal{H}^2(\mathcal{U})$ of $\mathcal{U}$-valued analytic functions on the right half-plane $\mathbb{C}_0$. These facts may serve as an intuitive justification for the following definition.

Definition 1.1. We use the standing assumptions on A_0, B_1, C_1, as stated before (1.1). For every $x_0 \in \mathcal{X}$, we define the set $\mathcal{D}_{x_0}$ by

$$\mathcal{D}_{x_0} = \left\{ u \in L^2([0,\infty), \mathcal{U}) \;\middle|\; \hat{y} \text{ given by (1.10) is in } \mathcal{H}^2(\mathcal{U}) \right\}. \tag{1.11}$$

If $x_0 \in \mathcal{X}$ and $u \in \mathcal{D}_{x_0}$, then we define the corresponding output function y of the system Σ^u as the inverse Laplace transform of $\hat{y}$ defined in (1.10).

It is easy to see that $\mathcal{D}_0$ is a vector space and, if $u \in \mathcal{D}_{x_0}$ then $\mathcal{D}_{x_0} = u + \mathcal{D}_0$. Hence, $\mathcal{D}_{x_0}$ is either empty or it is a linear manifold whose supporting vector space is $\mathcal{D}_0$. If u, x_0 and y are as in the definition, then (1.10) can be rewritten (using (1.7)) in the form

$$\hat{y}(s) = B^{u*}(sI - A^u)^{-1} x_0 + \mathbf{G}^u(s)\hat{u}(s), \tag{1.12}$$

where $B^{u*} = \begin{bmatrix} 0 & C_1 \end{bmatrix}$ and, for all $s \in \mathbb{C}_0$,

$$\mathbf{G}^u(s) = C_1 s (s^2 I + A_0)^{-1} B_1\,.$$

We call $\mathbf{G}^u$ the *transfer function* of Σ^u, because (1.12) looks like the formula for the Laplace transform of the output function of a well-posed linear system with transfer function $\mathbf{G}^u$ (see [10, 14]), even though Σ^u may be not well-posed.

The above definition immediately raises the following questions: (1) Is the set $\mathcal{D}_{x_0}$ rich enough (in particular, not empty)? (2) If u, x_0 and y are as in the definition and z is the first component of the state trajectory x from (1.8) (equivalently, z is given by (1.9)), does y satisfy (1.3) in some reasonable sense? Both answers are positive, and they are contained in the following proposition.

Proposition 1.2. *With the above notation, for every $x_0 \in \mathcal{X}$, $\mathcal{D}_{x_0}$ is an infinite-dimensional linear manifold. For $u \in \mathcal{D}_{x_0}$, the state trajectory x from (1.8) is a continuous function with values in $\mathcal{X}$ (so that its first component z is continuous with values in $H_{\frac{1}{2}}$). Moreover, the function $C_1 z$ is in the Sobolev space $\mathcal{H}^1(0,\infty;\mathcal{U})$, and its distributional derivative is the output function $y \in L^2([0,\infty),\mathcal{U})$.*

The proof of Proposition 1.2 will be given in Section 3.

Remark 1.3. The output signal y of Σ^u could be defined via (1.10) for every $x_0 \in \mathcal{X}$ and for every $u \in L^2([0,\infty),\mathcal{U})$. To see this, first we factor

$$C_1 = C_b A_0^{\frac{1}{2}}, \qquad B_1 = A_0^{\frac{1}{2}} B_b,$$

where $C_b \in \mathcal{L}(H,\mathcal{U})$ and $B_b \in \mathcal{L}(\mathcal{U},H)$ (the subscript "b" stands for "bounded"). Then we have from (1.10), via a short computation

$$\hat{y}(s) = C_b \left[I - s^2(s^2 I + A_0)^{-1}\right] \left[-A_0^{\frac{1}{2}} z_0 + s A_0^{-\frac{1}{2}} w_0 + s B_b \hat{u}(s)\right].$$

Since $(s^2 I + A_0)^{-1}$ and $\hat{u}(s)$ are uniformly bounded on the right half-plane where $\operatorname{Re} s > 1$, it follows that $|\hat{y}(s)| \le K|s|^3$ on this half-plane. Hence, the function

$$\hat{q}(s) = \frac{1}{(s+1)^4}\, \hat{y}(s+1)$$

is in the Hardy space $\mathcal{H}^2(\mathcal{U})$, so that it is the Laplace transform of $q \in L^2([0,\infty),\mathcal{U})$. From here, after extending q to be zero for $t < 0$, we can define

$$y(t) = \frac{\mathrm{d}^4}{\mathrm{d}t^4}\left(e^t q(t)\right),$$

in the sense of distributions in $\mathcal{D}'(\mathbb{R})$. However, such a definition of the output signal is not needed in this paper, because for the quadratic optimal control problem we only consider those inputs which produce an output in $L^2([0,\infty),\mathcal{U})$.

We associate to the system Σ^u from (1.1)–(1.3) the following *cost function*:

$$J(x_0,u) = \int_0^\infty \left[\|y(t)\|^2 + r^2\|u(t)\|^2\right] \mathrm{d}t, \tag{1.13}$$

where $r > 0$. Clearly, $J(x_0,u)$ is finite for every $u \in \mathcal{D}_{x_0}$. The *optimal control problem* is to find, for each $x_0 \in \mathcal{X}$, the function $u \in \mathcal{D}_{x_0}$ which minimizes $J(x_0,u)$. Moreover, it is desireable to express this optimal input function in feedback form, i.e., to express $u(t)$ as a function of $x(t)$.

We introduce the operator $A : \mathcal{D}(A) \to \mathcal{X}$ by

$$A = \begin{bmatrix} 0 & I \\ -A_0 & -\frac{1}{r} B_1 C_1 \end{bmatrix},$$

$$\mathcal{D}(A) = \left\{ \begin{bmatrix} z \\ w \end{bmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}} \;\middle|\; A_0 z + \frac{1}{r} B_1 C_1 w \in H \right\}.$$

It is easy to verify (see, for example, [16, Section 5]) that A is dissipative and onto, and hence it generates a contraction semigroup on X, denoted by $\mathbb{T}$.

Theorem 1.4. (1) *With the above notation, for every $x_0 \in \mathcal{X}$, the function φ defined for $t \geq 0$ by $\varphi(t) = \begin{bmatrix} C_1 & 0 \end{bmatrix} \mathbb{T}_t x_0$ is in $\mathcal{H}^1(0,\infty;\mathcal{U})$. We define the function $u_0 : [0,\infty) \to \mathcal{U}$ by $u_0 = -\frac{1}{r}\dot{\varphi}(t)$, i.e., for almost every $t \geq 0$,*

$$u_0(t) = -\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} C_1 & 0 \end{bmatrix} \mathbb{T}_t x_0 \,. \tag{1.14}$$

Then $u_0 \in \mathcal{D}_{x_0} \subset L^2([0,\infty),U)$ and $J(x_0,u_0) \leq r\|x_0\|^2$.

(2) *For every $x_0 \in \mathcal{X}$, there exists a unique $u^{\mathrm{opt}} \in \mathcal{D}_{x_0}$, called the* optimal input function*, which minimizes $J(x_0,u)$ over all $u \in \mathcal{D}_{x_0}$.*

(3) *If $\sigma(A_0)$ has measure zero in $\mathbb{R}$, then the optimal input function u^{opt} is the function u_0 defined in* (1.14)*.*

(4) *If $x_0 \in \mathcal{X}$ is such that $\lim_{t\to\infty} \mathbb{T}_t x_0 = 0$, then (again) the optimal input function u^{opt} is u_0 defined in* (1.14)*, and moreover $J(x_0,u_0) = r\|x_0\|^2$.*

(5) *The input function u_0 can be obtained by closing the output-feedback loop*

$$u(t) = -\frac{1}{r}y(t) \tag{1.15}$$

around the original system Σ^u described by (1.1)–(1.3)*. The closed-loop semigroup corresponding to this feedback is $\mathbb{T}$.*

Note that if A_0^{-1} is compact (which is usually the case in applications), then $\sigma(A_0)$ is countable, so that $\sigma(A_0)$ has measure zero. Then, according to (3) above, we have solved the optimal control problem for Σ^u. Moreover, according to (5) above, we have expressed the optimal input in feedback form. If the condition in (3) is not satisfied, then the solution of the optimal control problem (which exists according to (2)) may be much more difficult to express. The proof is provided in Section 3.

Note that for any $x_0 \in \mathcal{X}$, the first component of $x(t) = \mathbb{T}_t x_0$ is continuously differentiable as an H-valued function of t and its derivative is the second component of $x(t)$. (The semigroup $\mathbb{T}^u$ also has this property.) Similarly, if $x_0 \in \mathcal{D}(A)$ then the first component of $x(t)$ is continuously differentiable as an $H_{\frac{1}{2}}$-valued function of t and its derivative is the second component of $x(t)$. From this property of $\mathbb{T}$ it follows that if $x_0 \in \mathcal{D}(A)$, then u_0 from (1.14) can also be expressed as

$$u_0(t) = -\frac{1}{r}B^{u*}\mathbb{T}_t x_0 = -\frac{1}{r}\begin{bmatrix} 0 & C_1 \end{bmatrix} \mathbb{T}_t x_0 \,. \tag{1.16}$$

In this form, the formula for u_0 looks like what we would expect, based on finite-dimensional optimal control theory (i.e., considering A_0, B_1 and C_1 to be matrices).

Remark 1.5. The semigroup $\mathbb{T}$ is called *strongly stable* if $\lim_{t\to\infty} \mathbb{T}_t x_0 = 0$ for all $x_0 \in \mathcal{X}$. Several equivalent conditions for the strong stability of semigroups with this structure are given in [12]. Suppose that $\mathbb{T}$ is strongly stable, so that (by point (4) above) for every $x_0 \in \mathcal{X}$ we have $u^{\mathrm{opt}} = u_0$. Then the formula $J(x_0,u_0) = r\|x_0\|^2$

means that the *optimal cost operator* corresponding to our optimal control problem is $P = rI$, so that $J(x_0, u^{\rm opt}) = \langle Px_0, x_0\rangle$. It is easy to see that this P satisfies

$$A^{u*}P + PA^u + B^uB^{u*} = \frac{1}{r^2}PB^uB^{u*}P \qquad (A^{u*} = -A^u).$$

This is the algebraic Riccati equation that we would expect to hold based on the theory with bounded control and observation operators in Curtain and Zwart [5], or on the theory that allows unbounded operators in Lasiecka and Triggiani [7, 8], even though the assumptions in [7, 8] are not satisfied. However, from this fact we cannot conclude directly that the feedback

$$u(t) = -\frac{1}{r^2}B^{u*}Px(t) = -\frac{1}{r}y(t)$$

leads to an optimal input function, because (as already mentioned) there is no Riccati equation theory that covers our ill-posed system Σ^u. Besides, the above Riccati equation with $P = rI$ holds regardless if $\mathbb{T}$ is strongly stable (and regardless if $\sigma(A_0)$ has measure zero), so that it holds also for systems in our class where $u^{\rm opt} \neq u_0$ and/or where P is not the optimal cost operator. It is trivial to find examples where $P = rI$ is not the optimal cost operator: take $C_1 = 0$.

2. Reduction to another optimal control problem

In this section we introduce a conservative linear system Σ using the operators A_0, B_1 and C_1 from the description (1.1)–(1.3) of the original unstable system Σ^u. The input, state and output spaces remain $\mathcal{U}$, $\mathcal{X}$ and $\mathcal{U}$. We show that the optimal control problem for Σ^u is equivalent to an optimal control problem for Σ.

First we rewrite the cost $J(x_0, u)$ from (1.13) using the parallelogram identity:

$$\|y(t)\|^2 + r^2\|u(t)\|^2 = \frac{1}{2}\left[\|ru(t) - y(t)\|^2 + \|ru(t) + y(t)\|^2\right].$$

Thus, if we denote

$$y_1(t) = ru(t) - y(t), \qquad u_1(t) = ru(t) + y(t), \tag{2.1}$$

and if we regard y_1 as the new output function and u_1 as the new input function, then

$$J(x_0, u) = J_1(x_0, u_1) = \frac{1}{2}\int_0^\infty \left[\|y_1(t)\|^2 + \|u_1(t)\|^2\right] {\rm d}t.$$

In terms of the new signals y_1 and u_1, the equations (1.1) and (1.3) become

$$\frac{{\rm d}^2}{{\rm d}t^2}z(t) + A_0z(t) + \frac{1}{r}B_1\frac{\rm d}{{\rm d}t}C_1z(t) = \frac{1}{r}B_1u_1(t), \tag{2.2}$$

$$y_1(t) = -2\frac{\mathrm{d}}{\mathrm{d}t}C_1 z(t) + u_1(t). \tag{2.3}$$

Now we introduce the scaled versions of C_1 and B_1 defined by

$$C_0 = \sqrt{\frac{2}{r}}C_1\,, \qquad B_0 = \sqrt{\frac{2}{r}}B_1\,,$$

so that $B_0 = C_0^*$. Then (2.2) and (2.3) can be rewritten as

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) + A_0 z(t) + \frac{1}{2}B_0\frac{\mathrm{d}}{\mathrm{d}t}C_0 z(t) = B_0\frac{1}{\sqrt{2r}}u_1(t),$$

$$\frac{1}{\sqrt{2r}}y_1(t) = -\frac{\mathrm{d}}{\mathrm{d}t}C_0 z(t) + \frac{1}{\sqrt{2r}}u_1(t).$$

Finally, we introduce the scaled versions of y_1 and u_1 by

$$\tilde{y}(t) = \frac{1}{\sqrt{2r}}y_1(t), \qquad \tilde{u}(t) = \frac{1}{\sqrt{2r}}u_1(t). \tag{2.4}$$

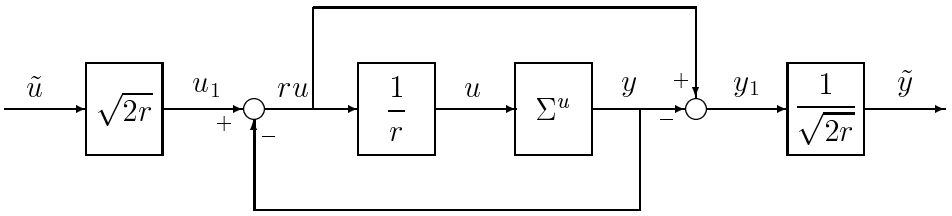


Figure 1: The conservative system Σ with input $\tilde{u}$ and output $\tilde{y}$, as obtained from the possibly ill-posed system Σ^u (with input u and output y).

———————

Then the last two equations become

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}z(t) + A_0 z(t) + \frac{1}{2}B_0\frac{\mathrm{d}}{\mathrm{d}t}C_0 z(t) = B_0\tilde{u}(t), \tag{2.5}$$

$$\tilde{y}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}C_0 z(t) + \tilde{u}(t), \tag{2.6}$$

and the cost function becomes

$$J(x_0, u) = \tilde{J}(x_0, \tilde{u}) = r\int_0^\infty \left[\|\tilde{y}(t)\|^2 + \|\tilde{u}(t)\|^2\right]\mathrm{d}t. \tag{2.7}$$

The transformations (2.1) and (2.4) are shown as a block diagram in Figure 1. If we regard $\tilde{u}$ as the new input signal and $\tilde{y}$ as the new output signal, then this is a

new system Σ, with the same state and the same state space as for Σ^u. However, Σ is much "nicer" because it is well-posed, as we shall see.

It is important to note that the transformations (2.1) and (2.4) are reversible, as expressed in matrix form:

$$\begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix} = \frac{1}{\sqrt{2r}} \begin{bmatrix} r & 1 \\ r & -1 \end{bmatrix} \begin{bmatrix} u \\ y \end{bmatrix}, \qquad \begin{bmatrix} u \\ y \end{bmatrix} = \frac{1}{\sqrt{2r}} \begin{bmatrix} 1 & 1 \\ r & -r \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix}. \tag{2.8}$$

Thus, $\tilde{u}$, z and $\tilde{y}$ satisfy (2.5) and (2.6) (in the general sense of the equality of the Laplace transforms of the sides) if and only if the corresponding u, z and y satisfy (1.1) and (1.3) (again in the sense of Laplace transforms).

Some properties of the system Σ. The system Σ described by (2.5) and (2.6) fits into the framework of the papers [16] and [12] by M. Tucsnak and the author. We know from [16, Theorem 1.1] that (2.5) and (2.6), together with (1.2) define a conservative linear system Σ with input and output space $\mathcal{U}$ and state space $\mathcal{X}$. For the concept of a conservative linear system we refer to Arov and Nudelman [1], Weiss, Staffans and Tucsnak [15] or [16]. The fact that Σ is conservative implies, in particular, that Σ is a well-posed linear system and for every $\tau > 0$ we have the balance equation

$$\|x(\tau)\|^2 + \int_0^\tau \|\tilde{y}(t)\|^2 \mathrm{d}t = \|x(0)\|^2 + \int_0^\tau \|\tilde{u}(t)\|^2 \mathrm{d}t. \tag{2.9}$$

Moreover, a similar balance equation holds for the dual system of Σ.

The semigroup generator of Σ is A as defined before Theorem 1.4, see [16, Theorem 1.3], so that its semigroup is the contraction semigroup $\mathbb{T}$ appearing in Theorem 1.4. For every $s \in \mathbb{C}_0$, the operator $s^2 I + A_0 + \frac{s}{2} B_0 C_0 \in \mathcal{L}(H_{\frac{1}{2}}, H_{-\frac{1}{2}})$ is invertible, see [16, Proposition 5.3], and we denote

$$V(s) = \left(s^2 I + A_0 + \frac{s}{2} B_0 C_0 \right)^{-1} \in \mathcal{L}(H_{-\frac{1}{2}}, H_{\frac{1}{2}}). \tag{2.10}$$

We denote by $\mathcal{X}_{-1}$ the completion of $\mathcal{X}$ with respect to the norm

$$\|x_0\|_{-1} = \|(I - A)^{-1} x_0\|.$$

The semigroup $\mathbb{T}$ has a continuous extension to $\mathcal{X}_{-1}$, whose generator is an extension of A, with domain $\mathcal{X}$. Hence, for every $s \in \rho(A)$, $(sI - A)^{-1}$ can be extended to a bounded operator from $\mathcal{X}_{-1}$ to $\mathcal{X}$. We use the same notation for the original operators and the extended ones. It has been proved in [16, Section 5] that $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$ (which obviously contains $\mathcal{X}$) is a subspace of $\mathcal{X}_{-1}$ and on this subspace we have

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s}[I - V(s)A_0] & V(s) \\ -V(s)A_0 & sV(s) \end{bmatrix}, \tag{2.11}$$

for every $s \in \mathbb{C}_0$. The control and observation operators of Σ are given by

$$B = \begin{bmatrix} 0 \\ B_0 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & -C_0 \end{bmatrix}.$$

Note that B maps into $H_{\frac{1}{2}} \times H_{-\frac{1}{2}}$ and hence, into $\mathcal{X}_{-1}$. The domain of C is $\mathcal{D}(A)$. The *input maps* of Σ are defined, as usual, by

$$\Phi_\tau \tilde{u} = \int_0^\tau \mathbb{T}_{\tau-\sigma} B\tilde{u}(\sigma)\mathrm{d}\sigma\,, \tag{2.12}$$

for all $\tau \geq 0$. We have $\Phi_\tau \in \mathcal{L}(L^2([0,\tau],\mathcal{U}),\mathcal{X})$ with $\|\Phi_\tau\| \leq 1$ for all $\tau \geq 0$, see [16, Proposition 6.1]. The *state trajectory* of Σ corresponding to the initial state $x_0 \in \mathcal{X}$ and the input function $\tilde{u} \in L^2([0,\infty),\mathcal{U})$ is given by

$$x(t) = \mathbb{T}_t\, x_0 + \Phi_t \tilde{u} \qquad \forall\, t \geq 0\,.$$

This is a continuous and bounded $\mathcal{X}$-valued function of t. If we denote its first component by $z(t)$, then its second component is $\dot{z}(t)$ and z satisfies (2.5) for almost every $t \geq 0$ (as an equation in $H_{-\frac{1}{2}}$), see [16, Theorem 1.1] for details.

The *extended output map* of Σ is defined, as usual, by

$$(\Psi\, x_0)(t) = C\mathbb{T}_t\, x_0 \qquad \forall\, x_0 \in \mathcal{D}(A),\ t \geq 0\,,$$

and this operator has a unique continuous extension to $\mathcal{X}$, denoted by the same symbol, so that $\Psi \in \mathcal{L}(\mathcal{X}, L^2([0,\infty),\mathcal{U}))$. Moreover, we have $\|\Psi\| \leq 1$, see [16, Proposition 6.2]. We have $y = \Psi\, x_0$ if and only if $\hat{y}(s) = C(sI - A)^{-1}x_0$. ■

We have seen in Section 1 that the (possibly ill-posed) system Σ^u from (1.1)–(1.3) has the following transfer function:

$$\mathbf{G}^u(s) = C_1 s(s^2 I + A_0)^{-1}B_1 = \frac{r}{2} C_0 s(s^2 I + A_0)^{-1}B_0\,, \tag{2.13}$$

which is analytic on $\mathbb{C}_0$, the open right half-plane. This means that if $x_0 = 0$ and $u \in \mathcal{D}_0$, then $\hat{u}$ and $\hat{y}$, the Laplace transforms of u and y, are related by

$$\hat{y}(s) = \mathbf{G}^u(s)\hat{u}(s) \qquad \forall\, s \in \mathbb{C}_0\,.$$

The following proposition lists some properties of the transfer function of Σ, denoted by $\mathbf{G}$, which is related to $\mathbf{G}^u$.

Proposition 2.1. *The transfer function of Σ is given by*

$$\mathbf{G}(s) = I - C_0 sV(s)B_0 \qquad \forall\, s \in \mathbb{C}_0\,, \tag{2.14}$$

where $V(s)$ is the operator defined in (2.10)*. We have*

$$\mathbf{G}(s) = (rI - \mathbf{G}^u(s))(rI + \mathbf{G}^u(s))^{-1} \qquad \forall\, s \in \mathbb{C}_0\,. \tag{2.15}$$

The function $\mathbf{G}$ satisfies $\|\mathbf{G}(s)\| \leq 1$ for all $s \in \mathbb{C}_0$. If $\omega \in \mathbb{R}$ is such that $\omega^2 \in \rho(A_0)$, then $\mathbf{G}$ has an analytic extension to a neighborhood of $i\omega$ and

$$\mathbf{G}^*(i\omega)\mathbf{G}(i\omega) = \mathbf{G}(i\omega)\mathbf{G}^*(i\omega) = I\,. \tag{2.16}$$

In particular, if $\sigma(A_0)$ has measure zero, then (2.16) *holds for almost every $\omega \in \mathbb{R}$.*

Proof. The formula (2.14) and the fact that $\|\mathbf{G}(s)\| \leq 1$ are contained in [16, Theorem 1.3], along with other properties of $\mathbf{G}$. It is easy to check, using (2.1) and (2.4) (or using the block diagram in Figure 1) that the transfer function of Σ is also given by (2.15). For any $\omega \in \mathbb{R}$ such that $\omega^2 \in \rho(A_0)$, it follows from (2.13) that $\mathbf{G}^u$ has an analytic continuation to a neighborhood of $i\omega$. For such ω, we can factor $\mathbf{G}^u(i\omega) = iT(\omega)$, where $T(\omega)$ is a self-adjoint operator in $\mathcal{L}(\mathcal{U})$. We have $(rI - \mathbf{G}(i\omega))^* = rI + iT(\omega)$, $(rI + \mathbf{G}(i\omega))^* = rI - iT(\omega)$, so that using (2.15),

$$\mathbf{G}^*(i\omega)\mathbf{G}(i\omega) = (rI - iT(\omega))^{-1}(rI + iT(\omega))(rI - iT(\omega))(rI + iT(\omega))^{-1}.$$

Since the factors on the right-hand side commute, we obtain $\mathbf{G}^*(i\omega)\mathbf{G}(i\omega) = I$. The proof of $\mathbf{G}(i\omega)\mathbf{G}^*(i\omega) = I$ is similar. ■

Remark 2.2. A bounded analytic $\mathcal{L}(\mathcal{U})$-valued function defined on $\mathbb{C}_0$ which satisfies $\mathbf{G}^*(i\omega)\mathbf{G}(i\omega) = I$ for almost every $\omega \in \mathbb{R}$ is called *inner*. The values $\mathbf{G}(i\omega)$ are defined for almost every $\omega \in \mathbb{R}$ by non-tangential strong limits, see [9, Theorem 4.5]. If the order of the factors $\mathbf{G}^*(i\omega)$ and $\mathbf{G}(i\omega)$ is reversed, then $\mathbf{G}$ is called *co-inner*. Suppose that Σ is a conservative linear system with semigroup $\mathbb{T}$ and transfer function $\mathbf{G}$. It is not difficult to prove that if $\mathbb{T}$ is strongly stable, then $\mathbf{G}$ is inner. Similarly, if $\mathbb{T}^*$ is strongly stable, then $\mathbf{G}$ is co-inner.

3. Proof of the main results

We continue to use the notation from Sections 1 and 2. The following proposition shows that u_0, our candidate optimal input function from Theorem 1.4, can be expressed using Ψ, the extended output map of Σ.

Proposition 3.1. *For every $x_0 \in \mathcal{X}$, the function φ from Theorem 1.4 is in $\mathcal{H}^1(0,\infty;\mathcal{U})$, so that $u_0 = -\frac{1}{r}\dot{\varphi} \in L^2([0,\infty),\mathcal{U})$. This function u_0 from (1.14) is also given by*

$$u_0 = \frac{1}{\sqrt{2r}}\Psi x_0. \tag{3.1}$$

If the input function of Σ^u is u_0 and its initial state is x_0, then the corresponding output function of Σ^u (see (1.10) or (1.12)) is $-ru_0$.

Proof. Denoting $z(t) = \begin{bmatrix} I & 0 \end{bmatrix} \mathbb{T}_t x_0$, it follows from [16, Theorem 1.3] that z is a solution of (2.5) corresponding to $\tilde{u} = 0$. Now it follows from [16, Theorem 1.1] that $C_0 z \in \mathcal{H}^1(0,\infty;\mathcal{U})$, and hence the same is true for φ. We see from (2.6) (with $\tilde{u} = 0$) that $(\Psi x_0)(t) = -\frac{\mathrm{d}}{\mathrm{d}t} C_0 z(t)$. Using that $C_1 = \sqrt{\frac{r}{2}} C_0$, we obtain (3.1).

If the system Σ has input function $\tilde{u} = 0$ and initial state x_0, then its output function is $\tilde{y} = \Psi x_0$. According to (2.8), the corresponding signals u and y are

$$u = \frac{1}{\sqrt{2r}}\tilde{y} = u_0, \qquad y = -\frac{1}{\sqrt{2r}} r\tilde{y} = -ru_0.$$

This proves the last statement in the proposition. ■

We denote by $\mathbb{F}$ the *extended input-output operator of* Σ. Thus, $\mathbb{F}$ is a bounded shift-invariant operator on $L^2([0,\infty),\mathcal{U})$ and $y=\mathbb{F}u$ if and only if $\hat{y}=\mathbf{G}\hat{u}$ (see [16, Section 3]). If the input function of Σ is $\tilde{u}\in L^2([0,\infty),\mathcal{U})$ and its initial state is $x_0\in\mathcal{X}$, then its output function is (as for any well-posed system)

$$\tilde{y}=\Psi x_0+\mathbb{F}\tilde{u}\,, \tag{3.2}$$

and $\tilde{y}\in L^2([0,\infty),\mathcal{U})$. In the following proposition, we use $\mathbb{F}$ to describe $\mathcal{D}_{x_0}$.

Proposition 3.2. *For every $x_0\in\mathcal{X}$, the set $\mathcal{D}_{x_0}$ defined in* (1.11) *is described by*

$$\mathcal{D}_{x_0}=u_0+(I+\mathbb{F})L^2([0,\infty),\mathcal{U}),$$

where u_0 is the function defined in (1.14).

Proof. The last part of Proposition 3.1 together with (1.12) implies that

$$-r\hat{u}_0(s)=B^{u*}(sI-A^u)^{-1}x_0+\mathbf{G}^u(s)\hat{u}_0(s). \tag{3.3}$$

Let $x_0\in\mathcal{X}$ and suppose that $u\in L^2([0,\infty),\mathcal{U})$ is of the form given in the proposition, i.e., $u=u_0+(I+\mathbb{F})v$, with $v\in L^2([0,\infty),\mathcal{U})$. Then $\hat{u}=\hat{u}_0+(I+\mathbf{G})\hat{v}$. Substituting this into (1.12) and using (3.3), we obtain that y, the corresponding output function of Σ^u, is given by

$$\hat{y}=\ -r\hat{u}_0+\mathbf{G}^u(I+\mathbf{G})\hat{v}.$$

Note that (2.15) implies $\mathbf{G}^u(I+\mathbf{G})=r(I-\mathbf{G})$, so that

$$\hat{y}=\ -r\hat{u}_0+r(I-\mathbf{G})\hat{v},$$

which shows that $y\in L^2([0,\infty),\mathcal{U})$ (because $u_0\in L^2([0,\infty),\mathcal{U})$). Hence, $u\in\mathcal{D}_{x_0}$.

Conversely, let $x_0\in\mathcal{X}$ and suppose that $u\in\mathcal{D}_{x_0}$. Then, by the definition of $\mathcal{D}_{x_0}$, the corresponding output function y of Σ^u is also in $L^2([0,\infty),\mathcal{U})$. We see from (2.8) that $\tilde{u}$, the corresponding input function of Σ, is also in $L^2([0,\infty),\mathcal{U})$. The corresponding output function of Σ, $\tilde{y}\in L^2([0,\infty),\mathcal{U})$ is given by (3.2). Using (2.8), then (3.2) and finally (3.1), we have

$$u=\frac{1}{\sqrt{2r}}(\tilde{u}+\tilde{y})=\frac{1}{\sqrt{2r}}\left[\Psi x_0+(I+\mathbb{F})\tilde{u}\right]=u_0+(I+\mathbb{F})\frac{1}{\sqrt{2r}}\tilde{u}\,.$$

Denoting $v=\frac{1}{\sqrt{2r}}\tilde{u}$, we see that u has the structure claimed in the proposition. ■

Proof of Proposition 1.2. The fact that $\mathcal{D}_{x_0}$ is an infinite-dimensional linear manifold follows from Proposition 3.2. Let $x_0\in\mathcal{X}$ be the initial state of Σ^u and let $u\in\mathcal{D}_{x_0}$ be its input function. Thus, the corresponding output function of Σ^u, y is in $L^2([0,\infty),\mathcal{U})$. From (2.8) we see that $\tilde{u}$, the corresponding input function of Σ,

is in $L^2([0,\infty),\mathcal{U})$. The state trajectory x of Σ^u corresponding to x_0 and u is the same as the state trajectory of Σ corresonding to x_0 and $\tilde{u}$. Now the properties of x and $C_1 z$ claimed in Proposition 1.2 follow from [16, Theorem 1.1]. ■

The following proposition is a general result about conservative linear systems. It is simple and probably well-known to specialists in conservative systems, but we do not know a good reference for it.

Proposition 3.3. *Let Σ be a conservative linear system with input space $\mathcal{U}$, state space $\mathcal{X}$, semigroup $\mathbb{T}$, extended output map Ψ and extended input-output map $\mathbb{F}$.*

(a) *If $u \in L^2([0,\infty),\mathcal{U})$ is such that $\|\mathbb{F}u\| = \|u\|$, then $\Psi^*\mathbb{F}u = 0$.*

(b) *If $x_0 \in \mathcal{X}$ is such that $\lim_{t\to\infty} \mathbb{T}_t x_0 = 0$, then $\mathbb{F}^*\Psi x_0 = 0$.*

Proof. We will need the input maps of Σ, denoted (as usual) by Φ_τ, see (2.12). Let $\mathcal{Y}$ denote the output space of Σ. Let $\mathbf{P}_\tau$ denote the truncation operator which maps $L^2([0,\infty),\mathcal{U})$ onto $L^2([0,\tau],\mathcal{U})$, and similarly for $\mathcal{Y}$ in place of $\mathcal{U}$. We introduce

$$\Psi_\tau = \mathbf{P}_\tau \Psi, \qquad \mathbb{F}_\tau = \mathbf{P}_\tau \mathbb{F} \mathbf{P}_\tau \qquad \forall\, \tau \geq 0,$$

which are the usual operators appearing in the definition of a well-posed linear system, see for example [14]. It is clear that we have, for any $z_0 \in \mathcal{X}$ and any $v \in L^2([0,\infty),\mathcal{U})$,

$$\lim_{\tau\to\infty} \Psi_\tau z_0 = \Psi z_0, \qquad \lim_{\tau\to\infty} \mathbb{F}_\tau v = \mathbb{F}v. \tag{3.4}$$

The fact that Σ is conservative means that the operators

$$\Sigma_\tau = \begin{bmatrix} \mathbb{T}_\tau & \Phi_\tau \\ \Psi_\tau & \mathbb{F}_\tau \end{bmatrix}$$

are unitary, see [12, 16, 15] for details. From $\Sigma_\tau^*\Sigma_\tau = I$ we see that

$$\mathbb{T}_\tau^*\Phi_\tau + \Psi_\tau^*\mathbb{F}_\tau = 0, \quad \text{or equivalently,} \quad \Phi_\tau^*\mathbb{T}_\tau + \mathbb{F}_\tau^*\Psi_\tau = 0. \tag{3.5}$$

Now we prove point (a). If $\|\mathbb{F}u\| = \|u\|$, then $\Phi_\tau u$ converges to zero, because of the balance equation (2.9) rewritten for the initial state zero:

$$\|\Phi_\tau u\|^2 + \|\mathbb{F}_\tau u\|^2 = \|\mathbf{P}_\tau u\|^2.$$

Now we see from the first equation in (3.5) and from the uniform boundedness of the operators $\mathbb{T}_\tau^*$ that

$$\lim_{t\to\infty} \Psi_\tau^*\mathbb{F}_\tau u = 0.$$

This implies that for any $z_0 \in \mathcal{X}$ we have $\lim_{t\to\infty}\langle \mathbb{F}_\tau u, \Psi_\tau z_0\rangle = 0$. From here, using (3.4) we see that $\langle \mathbb{F}u, \Psi z_0\rangle = 0$, which implies that $\Psi^*\mathbb{F}u = 0$.

We proceed to the proof of (b). If $\mathbb{T}_\tau x_0$ converges to zero as $t\to\infty$, then we see from the second equation in (3.5) and from the uniform boundedness of the operators Φ_τ^* that

$$\lim_{\tau\to\infty} \mathbb{F}_\tau^*\Psi_\tau x_0 = 0.$$

This implies that for any $v \in L^2([0,\infty),\mathcal{U})$ we have $\lim_{\tau\to\infty}\langle \Psi_\tau x_0, \mathbb{F}_\tau v\rangle = 0$. From here, using (3.4) we see that $\langle \Psi x_0, \mathbb{F}v\rangle = 0$, so that $\mathbb{F}^*\Psi x_0 = 0$. ■

Proof of Theorem 1.4. (1) We know from Proposition 3.1 that indeed $\varphi \in \mathcal{H}^1(0,\infty;\mathcal{U})$, so that $u_0 \in L^2([0,\infty),\mathcal{U})$. From the same proposition we know that the output function of Σ^u corresponding to the input function u_0 is $-ru_0$, which implies that $u_0 \in \mathcal{D}_{x_0}$. Substituting into (2.8) we see that $\tilde{u} = 0$, so that according to (2.7) we obtain $J(x_0,u) = r\|\tilde{y}\|^2$. Since (according to (3.2)) we have $\tilde{y} = \Psi x_0$, we get $J(x_0,u) = r\|\Psi x_0\|^2$. Since $\|\Psi\| \le 1$, we obtain that $J(x_0,u) \le r\|x_0\|^2$.

(2) The optimal control problem for Σ^u has been reduced, via the transformations (2.8), to the optimal control problem for Σ. This can be addressed using the techniques in Staffans [10] or Weiss[2] [17]. The paper [17] usually assumes that the system to be controlled is weakly regular, but in [17, Section 7] it is pointed out that for the results in that section, the regularity assumption is not needed. For subjective reasons, we will now use the terminology and a result from that section. The Popov function corresponding to Σ with the cost (2.7) is

$$\Pi(i\omega) = r\left[\mathbf{G}^*(i\omega)\mathbf{G}(i\omega) + I\right], \tag{3.6}$$

which is positive and bounded from below. Hence, by [17, Proposition 7.2] there is a unique optimal input function corresponding to every initial state of Σ, and this can be translated via (2.8) into a unique optimal input function for Σ^u.

(3) If $\sigma(A_0)$ has measure zero, then (according to Proposition 2.1) $\mathbf{G}$ is inner. The Popov function from (3.6) becomes $\Pi(i\omega) = 2rI$, which implies that the Toeplitz operator with symbol Π is also $2rI$. Using the formula from [17, Proposition 7.2], we see that the optimal input function for Σ is

$$\tilde{u}^{\rm opt} \;=\; -\frac{1}{2}\mathbb{F}^*\,\Psi x_0\,.$$

The fact that $\mathbf{G}$ is inner implies that we have $\|\mathbb{F}u\| = \|u\|$ for all $u \in L^2([0,\infty),\mathcal{U})$. According to point (a) of Proposition 3.3 we have $\Psi^*\mathbb{F} = 0$, whence $\mathbb{F}^*\Psi = 0$. Thus, the above formula for $\tilde{u}^{\rm opt}$ shows that in fact $\tilde{u}^{\rm opt} = 0$. The corresponding output function of Σ is of course Ψx_0. Using the transformation (2.8) to compute the corresponding input of Σ^u, we obtain that the optimal input function of Σ^u is $u^{\rm opt} = \frac{1}{\sqrt{2r}}\Psi x_0$. According to Proposition 3.1, this is the same as u_0 from (1.14).

(4) If x_0 is such that $\mathbb{T}_t x_0$ converges to zero, then according to point (b) of Proposition 3.3 we have $\mathbb{F}^*\Psi x_0 = 0$. Using again the formula from [17, Proposition 7.2], we see that the optimal input function for Σ is $\tilde{u}^{\rm opt} = 0$. By the same argument as in the proof of (3), we obtain that the optimal input function of Σ^u is $u^{\rm opt} = u_0$. Since $\mathbb{T}_t x_0$ converges to zero, from the balance equation (2.9) with $\tilde{u} = 0$ we see that $\|\Psi x_0\| = \|x_0\|$. We have seen in the proof of (1) that $J(x_0,u) = r\|\Psi x_0\|^2$. Combining this with our earlier conclusion, we obtain that $J(x_0,u) = r\|x_0\|^2$.

(5) Closing the feedback (1.15) around Σ^u (i.e., imposing the relation (1.15) on u and y) is equivalent, according to (2.8), to imposing the restriction $\tilde{u} = 0$ on Σ. It is

clear that this leads to a unique input function, state trajectory and output function for Σ^u. We know from the last part of Proposition 3.1 that the corresponding input function is u_0. It is clear that the state trajectory of Σ corresponding to $\tilde{u} = 0$ is $x(t) = \mathbb{T}_t x_0$, and this is the same as the state trajectory of Σ^u with the feedback (1.15). Thus, the closed-loop semigroup is $\mathbb{T}$. ■

4. An example based on the wave equation

We describe a challanging example of an unstable system Σ^u of the type introduced in Section 1. We assume that $\Omega \subset \mathbb{R}^n$ is a (possibly unbounded) domain with Lipschitz boundary Γ (such a boundary admits corners and edges). Γ_0 and Γ_1 are nonempty open subsets of Γ such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $\overline{\Gamma_0 \cup \Gamma_1} = \Gamma$. We denote by x the space variable ($x \in \overline{\Omega}$). We assume that the Poincaré inequality holds for Ω and Γ_0. This means that there exists a $c > 0$ such that for every $f \in \mathcal{H}^1(\Omega)$ with $f|_{\Gamma_0} = 0$,

$$\int_\Omega \|(\nabla f)(x)\|^2 \mathrm{d}x \geq c \int_\Omega |f(x)|^2 \mathrm{d}x.$$

This holds, in particular, if Ω is bounded. A function $b \in L^\infty(\Gamma_1)$ is given, with $b(x) \neq 0$ for almost every $x \in \Gamma_1$. The equations of the system Σ^u are

$$\begin{cases} \ddot{z}(x,t) = \Delta z(x,t) & \text{on } \Omega \times [0,\infty), \\ z(x,t) = 0 & \text{on } \Gamma_0 \times [0,\infty), \\ \frac{\partial}{\partial \nu} z(x,t) = b(x) u(x,t) & \text{on } \Gamma_1 \times [0,\infty), \\ y(x,t) = \overline{b(x)}\, \dot{z}(x,t) & \text{on } \Gamma_1 \times [0,\infty), \\ z(x,0) = z_0(x), \quad \dot{z}(x,0) = w_0(x) & \text{on } \Omega, \end{cases} \tag{4.1}$$

where u is the input function and y is the output function. The functions z_0 and w_0 are the initial state of the system. We shall often write $z(t)$ to denote a function of x, meaning that $z(t)(x) = z(x,t)$, and similarly for other functions.

To put the equations (4.1) into the framework (1.1)–(1.3) studied in this paper, we introduce the Hilbert spaces $H = L^2(\Omega)$ and $\mathcal{U} = L^2(\Gamma_1)$. The *Dirichlet trace* operator γ is initially defined for any function $g \in C^1(\overline{\Omega})$ by

$$\gamma g = g|_\Gamma.$$

If we regard γg as an element of $L^2(\Gamma)$, then the operator γ has a continuous extension to $\mathcal{H}^1(\Omega)$. We denote by $\mathcal{R}$ the usual restriction operator mapping $L^2(\Gamma)$ onto $L^2(\Gamma_1)$ and for all $g \in \mathcal{H}^1(\Omega)$ we put

$$\gamma_0 g = \mathcal{R}\gamma g.$$

We call $\gamma_0 g$ the *Dirichlet trace* of g on Γ_1. If we regard $L^2(\Gamma_1)$ as a subspace of $L^2(\Gamma)$, then $I - \mathcal{R}$ is the restriction from $L^2(\Gamma)$ onto $L^2(\Gamma_0)$ and we define the Hilbert space

$$\mathcal{H}^1_{\Gamma_0}(\Omega) = \{g \in \mathcal{H}^1(\Omega) \mid (I - \mathcal{R})\gamma g = 0\}, \qquad \|g\|_{\mathcal{H}^1} = \|\nabla g\|_{H^n}.$$

The *Neumann trace* γ_1 is an operator originally defined on $C^1(\overline{\Omega})$ by

$$\gamma_1 f = \frac{\partial}{\partial \nu} f|_{\Gamma_1} = \langle \nabla f, \nu \rangle|_{\Gamma_1},$$

where ν is the unit vector in the outward normal direction to Γ_1, which is defined almost everywhere on Γ_1. Thus, γ_1 is the outward normal derivative restricted to Γ_1. Using Green's formula, it is possible to extend γ_1 to all those $f \in \mathcal{H}^1_{\Gamma_0}(\Omega)$ for which $\Delta f \in L^2(\Omega)$ (Δf is computed in the sense of distributions on Ω). For the details we refer to [16, Section 7] (without any claim of originality). We put

$$Z_0 = \left\{ f \in \mathcal{H}^1_{\Gamma_0}(\Omega) \;\middle|\; \Delta f \in L^2(\Omega),\ \gamma_1 f \in bL^2(\Gamma_1) \right\}.$$

We define the operator $A_0 : \mathcal{D}(A_0) \subset L^2(\Omega) \to L^2(\Omega)$ by

$$A_0 z = -\Delta z, \quad \mathcal{D}(A_0) = \{ z \in Z_0 \mid \gamma_1 z = 0 \}.$$

Then A_0 is self-adjoint, positive and boundedly invertible (the bounded invertibility of A_0 follows from the Poincaré inequality).

The norms $\|z\|_\alpha$ and the spaces H_α, with $\alpha \in \mathbb{R}$, are defined as in the Section 1. In particular, it can be checked (see [16, Section 7]) that

$$H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{H}^1_{\Gamma_0}(\Omega)$$

and

$$\|z\|^2_{\frac{1}{2}} = \left\| A_0^{\frac{1}{2}} z \right\|^2_H = \int_\Omega \|\nabla z(x)\|^2 \mathrm{d}x.$$

We define the operator $C_1 \in \mathcal{L}(H_{\frac{1}{2}}, U)$ by

$$C_1 = \overline{b} \gamma_0.$$

Here, $\overline{b}$ is the operator of pointwise multiplication with the complex conjugate of the function b introduced earlier. We put $B_1 = C_1^*$, as in Section 1. An explicit description of B_1 can be found in [16, Section 7]. We will also need

$$C_0 = \sqrt{2} C_1, \qquad B_0 = \sqrt{2} B_1,$$

to make it easier to follow [16], which is written in terms of C_0 and B_0.

It can be checked (see again [16, Section 7]) that we have $Z_0 = H_1 + A_0^{-1} B_0 \mathcal{U}$. We define the operators $G_0, G_1 : Z_0 \to \mathcal{U}$ by

$$G_1 = b^{-1} \gamma_1, \qquad G_0 = \frac{1}{\sqrt{2}} G_1.$$

Note that $b^{-1}\gamma_1$ cannot be defined on the larger space of those $f \in \mathcal{H}^1_{\Gamma_0}(\Omega)$ for which $\Delta f \in L^2(\Omega)$, but on Z_0, its definition makes sense because $\gamma_1 f \in bL^2(\Gamma_1)$. Clearly we have $G_0 H_1 = \{0\}$ and it can be checked (see [16, Section 7]) that

$$G_0 A_0^{-1} B_0 = I, \quad \text{or equivalently,} \quad G_1 A_0^{-1} B_1 = I. \tag{4.2}$$

We need one more operator: $L_0 : Z_0 \to H$ is defined by $L_0 = A_0 - B_0G_0 = A_0 - B_1G_1$. The fact that L_0 maps indeed into H follows from (4.2), see [16, Section 6]. It is not difficult to check that in fact $L_0 = -\Delta$, see [16, Section 7]. Now assuming that $z(t) \in Z_0$, $\dot{z}(t) \in H_{\frac{1}{2}}$ and $\ddot{z}(t) \in H$, we can rewrite (4.1) in the form

$$\begin{cases} \ddot{z}(t) + L_0 z(t) &= 0, \qquad G_1 z(t) = u(t), \\ z(0) &= z_0, \qquad \dot{z}(0) = w_0, \\ y(t) &= C_1 \dot{z}(t). \end{cases} \tag{4.3}$$

Using the formulas (4.2) and $L_0 = A_0 - B_1G_1$, it is easy to transform these into the equations (1.1)–(1.3). The transformations from (1.1)–(1.3) to (4.1) work also in the opposite way, if we assume again that $z(t) \in Z_0$, $\dot{z}(t) \in H_{\frac{1}{2}}$ and $\ddot{z}(t) \in H$.

The state space $\mathcal{X}$ is defined, as in Section 1, by $\mathcal{X} = H_{\frac{1}{2}} \times H$, so that

$$\mathcal{X} = \mathcal{H}^1_{\Gamma_0}(\Omega) \times L^2(\Omega).$$

It is known that for $n > 1$, this system is ill-posed. In fact, using the notation from (1.6), B^u is not admissible for the unitary group generated by A^u, see Lasiecka and Triggiani [6]. We define the cost function

$$J(x_0, u) = \int_0^\infty \left[\|y(t)\|^2 + \|u(t)\|^2 \right] \mathrm{d}t,$$

which corresponds to (1.13) with $r = 1$. Now we see that C_0 and B_0 defined above are the same as those defined in Section 2. The semigroup $\mathbb{T}$ is defined as in Section 1. It is proved in [12] that $\mathbb{T}$ is always strongly stable. Hence, we can apply point (4) of Theorem 1.4 to conclude that the optimal input function is generated by the feedback $u = -y$. Moreover, the optimal cost operator corresponding to this system with this cost function is $P = I$, and the closed-loop semigroup is $\mathbb{T}$.

References

[1] D.Z. Arov and M.A. Nudelman. Passive linear stationary dynamical scattering systems with continuous time. *Integral Eq. & Operator Theory*, 24:1–45, 1996.

[2] F. Callier and J. Winkin. LQ-optimal control of infinite-dimensional systems by spectral factorization. *Automatica*, **28**, pp. 757–770, 1992.

[3] R.F. Curtain. Linear operator inequalities for strongly stable weakly regular linear systems. *Math. Control, Signals and Systems*, 14:299–337, 2001.

[4] R.F. Curtain and G. Weiss. Well–posedness of triples of operators (in the sense of linear systems theory). *Control and Estimation of Distributed Parameter Systems*, F. Kappel, K. Kunisch, and W. Schappacher, Eds., Birkhäuser Verlag, Basel, 1989, pp. 41–59.

[5] R.F. Curtain and H.J. Zwart. *An Introduction to Infinite-Dimensional Linear Systems Theory*. Springer-Verlag, New York, 1995.

[6] I. Lasiecka and R. Triggiani. A cosine operator approach to modeling $L_2(0,T;L_2(\Gamma))$–boundary input hyperbolic equations. *Applied Mathematics and Optimization*, 7:35–93, 1981.

[7] I. Lasiecka and R. Triggiani. *Differential and Algebraic Riccati Equations with Applications to Boundary/Point Control Problems: Continuous Theory and Approximation Theory*. Lecture Notes in Control and Information Sciences, **164**, Springer-Verlag, Berlin, 1991.

[8] I. Lasiecka and R. Triggiani. *Control Theory for Partial Differential Equations: Continuous and Approximation Theories*. Cambridge University Press, Cambridge, UK, 2000.

[9] M. Rosenblum and J. Rovnyak. *Hardy Classes and Operator Theory*. Oxford University Press, New York, 1985.

[10] O.J. Staffans. Quadratic optimal control of stable well-posed linear systems. *Trans. Amer. Math. Soc.*, 349:3679–3715, 1997.

[11] O.J. Staffans. Quadratic optimal control of well-posed linear systems. *SIAM J. Control and Optim.*, 37:131–164, 1998.

[12] M. Tucsnak and G. Weiss. How to get a conservative well-posed linear system out of thin air, Part II: Controllability and stability. Submitted in 2001.

[13] G. Weiss. Admissibility of unbounded control operators. *SIAM J. Control Optim.*, 27:527–545, 1989.

[14] G. Weiss. Transfer functions of regular linear systems. Part I: characterizations of regularity. *Trans. Amer. Math. Soc.*, 342:827–854, 1994.

[15] G. Weiss, O.J. Staffans and M. Tucsnak. Well-posed linear systems -a survey with emphasis on conservative systems. *Applied Mathematics and Computer Science* 11:101–127, 2001.

[16] G. Weiss and M. Tucsnak. How to get a conservative well-posed linear system out of thin air, Part I: Well-posedness and energy balance. Submitted in 2001.

[17] M. Weiss and G. Weiss. Optimal control of stable weakly regular linear systems, *Mathematics of Control, Signals and Systems* 10:287–330, 1997.