

**Imperial College
London**

IMPERIAL COLLEGE OF SCIENCE, TECHNOLOGY
AND MEDICINE

DEPARTMENT OF MATHEMATICS

PHD IN MATHEMATICS

**Topological Prevalence of Finite Type
Maps and Density of Stability for
Interval Translation Mappings**

Author:

Leon STARESINIC

Supervisors:

Prof. Sebastian van STRIEN

Prof. Kostiantyn DRACH

Abstract

An *Interval Translation Mapping* (ITM) is a piece-wise translation $T: I \rightarrow I$ defined on a finite partition $I_1, \dots, I_r$ of an interval I into $r \geq 2$ sub-intervals. We do not assume that the images of these intervals are disjoint. These maps naturally generalize the classical *Interval Exchange Transformations* (IETs) by removing the bijectivity assumption. Let $\text{ITM}(r)$ be the space of all Interval Translation Mappings, where we fix r but not the intervals $I_1, \dots, I_r$, nor the translations on each of them. The set $X(T) := \bigcap_{n \geq 0} T^n(I)$ is then either a finite union of intervals (and T behaves like an IET on those intervals), in which case the map is called of *finite type*, or is a disjoint union of finitely many intervals and a Cantor set, in which case the map is called of *infinite type*.

In Chapter III, for an arbitrary $r \geq 2$, we show that the set of finite type maps contains an open and dense set of $\text{ITM}(r)$. This resolves in positive a topological version of a long-standing conjecture due to Boshernitzan and Kornfeld. More precisely, we show that there exists an open and dense subset $\mathcal{S}(r)$ of $\text{ITM}(r)$ consisting of *stable maps* such that each $T \in \mathcal{S}(r)$:

- is of finite type;
- the first return map to any component of $X(T)$ corresponds to a circle rotation;
- $\mathcal{S}(r) \ni T \mapsto X(T)$ is continuous in the Hausdorff topology.

In Chapters I and II we give an introduction to and an overview of the fields of IETs and ITMs, respectively.

Declarations

Declaration of Originality

I hereby declare that the work presented in this thesis is my own work, unless otherwise stated, and that I have rightfully acknowledged and referenced the work of others. The material presented has not previously been submitted for assessment at Imperial College London or elsewhere.

Copyright Declaration

The copyright of this thesis rests with the author. Unless otherwise indicated, its contents are licensed under a Creative Commons Attribution-Non Commercial 4.0 International Licence (CC BY-NC). Under this licence, you may copy and redistribute the material in any medium or format. You may also create and distribute modified versions of the work. This is on the condition that: you credit the author and do not use it, or any derivative works, for a commercial purpose. When reusing or sharing this work, ensure you make the licence terms clear to others by naming the licence and linking to the licence text. Where a work has been adapted, you should indicate that the work has been changed and describe those changes. Please seek permission from the copyright holder for uses of this work that are not included in this licence or permitted under UK Copyright Law.

Publications

The material in Chapter [III](#) is based on the preprint [\[DSS25\]](#), which is co-authored with Sebastian van Strien and Kostiantyn Drach, and at the time of writing has been submitted for publication.

Acknowledgements

I am very grateful for having been supervised by Prof. Sebastian van Strien and Prof. Kostiantyn Drach. Their guidance, patience, and numerous discussions are what enabled me to become a mathematician. In particular, I thank them for vastly improving my presentation, writing, and explanation skills during my PhD. I would also like to thank Prof. Dmitry Turaev for several important and interesting discussions during the examination steps of my PhD.

I am grateful to the Dynamical Systems group at Imperial for providing a very warm and intellectually stimulating environment. I am grateful to Amir, Akshunna, Dima, Evangelos, Hugo, Jossy, Martin, Matteo, Tima, and many of the other Imperial students for making London feel like a second home.

Sincere thanks goes to my parents Damir and Vesna for their continuous and unwavering support, and especially for taking full care of me during the difficult final parts of writing the thesis. I am also very grateful to my grandparents Josip and Ljerka, for making my stay in London possible, and for their continuous encouragement and support throughout my academic studies, and my other grandma Barica for her support, kind words, and unconditional love over the years. Gratitude and respect goes to Dea and Tea, especially for all the free-riding. Special thanks goes to Domagoj, Nikola, Nikolai, and Marko for their friendship and our numerous outdoor adventures.

Finally, I would like to thank Imperial College's Roth scholarship program for the financial support during my PhD.

Introduction

Interval Exchange Transformations (IETs) are the finite piece-wise continuous bijective isometries of compact intervals. They arise naturally as return maps to small transversals in the study of flows on surfaces, measured foliations, and billiards dynamics. The dynamical properties of IETs are very closely related to analogous properties of all of these objects. For this reason, and because they constitute an extremely rich class of dynamical systems themselves, IETs have been the subject of much-celebrated work by Sinai, Keane, Veech, Masur, Boshernitzan, Forni, Marmi, Moussa, Yoccoz, Avila, Viana, Chaika, and many other authors, since their introduction in the early 70's.

Perhaps most importantly for dynamical systems theories in general, the study of IETs demonstrates the power of analysing the dynamics of a *renormalization scheme* for the purpose of studying generic properties of a class of dynamical systems. Intuitively speaking, a renormalization is a map that assigns to a dynamical system $T : X \rightarrow X$ a dynamical system $T' : Y \rightarrow Y$, which is defined as the return map under T to a smaller set $Y \subset X$ (one can usually also rescale Y back to X by a nice map). Renormalization therefore speeds up the time at which the dynamics are happening and zooms into space. Iterating this process allows one to study small-scale and long-term properties of the starting dynamical system T . The renormalization scheme for IETs is called *Rauzy-Veech* induction, and it has been the central tool for establishing many of the deepest results in the field. Some of them include the proofs that almost every IET: is uniquely ergodic (known as the Keane conjecture) by Veech in [Vee82], admits solutions to the cohomological equation by Marmi–Moussa–Yoccoz in [MMY05], and is weakly mixing by Avila–Forni in [AF07].

Besides renormalization, the main tool in the study of IETs are their connections to

geometry, more specifically the construction of suspensions over IETs. It is often more convenient to work with suspensions because they are higher-dimensional and therefore less rigid, and therefore have a richer perturbation theory. The most important tools in their study are the associated *Teichmüller flow* and the action of the *mapping class group*, which can be seen as the geometric analogues of the Rauzy-Veech induction. One of the most important example of the usefulness of this connection is the proof of Keane’s conjecture by Masur in [Mas82], which was simultaneous to that of Veech in [Vee82].

There are several possible natural generalizations of IETs, each giving rise to a different subject of dynamical systems. Those that retain the bijectivity of the map, e.g. IETs with flips, GIETs or infinite IETs associated to special infinite translation surfaces, retain many of the good properties of IETs and are often intimately related to IETs themselves. In particular, at least one of the two main tools in the study of IETs, namely renormalization or connections to geometry, is retained.

The main topic of this thesis are the *Interval Translation Mappings* (ITMs) that are the finite piecewise isometries of a compact interval, first studied in [BK95]. As such they are obtained from IETs by dropping the bijectivity assumption. This makes the problems of finding a general renormalization scheme and associated geometric (topological, combinatorial) theories much more complicated than for IETs. At the time of writing this thesis, which is roughly 30 years since the study of ITMs was initiated, both of these problems remain unsolved.

There is a natural dichotomy for ITMs: each map is either of *finite* or of *infinite* type. The first example of an infinite type map was constructed in [BK95] by Boshernitzan and Kornfeld. They conjectured that such behavior should be rare:

Boshernitzan–Kornfeld Conjecture For all $r \geq 2$, the set of all infinite type maps is a measure zero subset of the set of all ITMs on r intervals $\text{ITM}(r)$.

As IETs are the dynamical systems naturally closest to ITMs, the historical approach to tackling the Boshernitzan–Kornfeld Conjecture was through similar methods that were used to study generic properties of IETs, namely renormalization. As mentioned, a general renormalization scheme has not been developed in generality, but it has been successfully used to study special families of ITMs and to establish the conjecture for these families

(e.g. in [BT03], [Bru07a], [SIA05], [BT03]). This approach culminated in the proof of the Boshernitzan–Kornfeld Conjecture for $r = 3$ by Volk in [Vol14]. However, already $r = 4$ has so far proven too complicated for the development of a renormalization scheme.

The main result of this thesis is the topological version of Boshernitzan–Kornfeld Conjecture, which the author established in a joint work with Kostiantyn Drach and Sebastian van Strien in [DSS25]:

Topological Prevalence of Finite Type Maps For every $r \geq 2$, the set of all finite type ITMs on r intervals contains an open and dense subset of the set of all ITMs on r intervals $ITM(r)$.

Our main novelty is that we do not use any type of renormalization, and we instead develop a theory of stability for ITMs and establish a powerful transversality result that allows for detailed control over global dynamics under perturbation. In the process, we also demonstrate that the theory of ITMs is, somewhat surprisingly, fairly similar to the theory of smooth real 1-dimensional dynamical dynamics. Our results also suggest the following strong refinement of the Boshernitzan–Kornfeld Conjecture:

Irrational Rotations Conjecture The set of all ITMs on r intervals that correspond to irrational circle rotations has full measure in $ITM(r)$.

More precisely, our results imply that if the Boshernitzan–Kornfeld Conjecture is true, then so is the Irrational Rotations Conjecture.

The thesis is separated into 3 chapters, which are as follows. In Chapter I, we survey the field of Interval Exchange Transformations, from its historical beginning to modern developments. The field of IETs, and especially its various connections to other dynamical systems, geometry and number theory, is too vast to cover completely, so we focus on the most important breakthroughs of the field and on the aspects of theory most related to that of ITMs. This means we often focus mostly on the ergodic and combinatorial properties of IETs, and we do not into the details of the connections with geometry. In Chapter II, we give an overview of the field of Interval Translation Mappings, focusing on the parallels and connections to IETs.

In Chapter III, which is based on the preprint [DSS25], we give a self-contained proof of the topological version of the Boshernitzan–Kornfeld Conjecture. To do this, we establish several results which are of independent interest. We prove the density of eventually periodic maps (analogues of Morse–Smale maps in our setting) in Theorem III.4.2, the characterization of stable maps by intrinsic dynamical properties called *ACC* and *Matching* in Theorem III.6.12, and show that stable maps approximate eventually periodic maps in Theorem III.7.1. Putting these three results together gives that the set of stable maps is an open and dense subset of the set of all ITMs, and since they are of finite type, we get the topological version of the Boshernitzan–Kornfeld Conjecture as a corollary. The crucial technical tool in the proofs of these results is a linear independence theorem of certain dynamically defined vectors, Theorem III.5.6.

The presentation in the first two chapters is mostly informal, and we do not go through the proofs of any of the results but give sketches of the most important ones. This is because the results in Chapter III are self-contained, and they do not rely on any existing results for IETs or ITMs. Thus the main purpose of the first two chapters is to give an overview of and compare the fields of IETs and ITMs, so as to put our results in Chapter III in the context of these theories.

Contents

I	Interval Exchange Transformations	2
I.1	Notation, definitions, conventions	2
I.2	Early history of IETs	4
I.3	Rauzy-Veech induction	5
I.4	Ergodic properties of IETs	8
I.5	Modern developments	11
I.6	Billiards	13
I.7	Generalizations of IETs	15
I.8	Appendix A: Ergodic Theory and Topological Dynamics	17
I.9	Appendix B: Number Theory	21
II	Interval Translation Mappings	23
II.1	Definitions and notation	23
II.2	Properties of general ITMs	25
II.3	Two special families	28
II.3.1	Bruin-Troubetzkoy family	28
II.3.2	Double rotations	33
II.4	Open problems	35
II.5	Spy billiards	36
III	Density of Stable Interval Translation Mappings	42
III.1	Overview of dynamical stability	42

III.2	Summary of results	44
III.2.1	Main idea of the proof	46
III.2.2	Organization of the chapter	47
III.3	Preliminaries	47
III.3.1	Notation and conventions	47
III.3.2	The non-wandering set of T	50
III.3.3	Properties of finite type ITMs	53
III.4	Product notation and eventually periodic maps	55
III.4.1	Product notation and perturbations	55
III.4.2	Density of eventually periodic maps	58
III.5	Linear independence of itinerary vectors	61
III.5.1	Statement and setup	61
III.5.2	Refining partitions of I	70
III.5.3	Refining partitions into $\mathcal{A}$ -subordinate intervals	82
III.5.4	Proof of Theorem III.5.6	89
III.6	Stability is equivalent to ACC and Matching	92
III.6.1	Definitions and examples	92
III.6.2	Stability, A1 and A2 imply Matching	99
III.6.3	Stability implies ACC	107
III.6.4	ACC + Matching implies Stability	111
III.7	Eventually periodic maps can be approximated by stable maps	115
III.7.1	Unstable number and correspondence	115
III.7.2	Proof of Theorem III.7.1	119
III.8	Almost every finite type map is stable	127
III.9	Parameter space and future work	134
III.9.1	The Bruin–Troubetzkoy family	134
III.9.2	Future work: geometry and dynamics in the boundary of stable regions	138

CHAPTER I

Interval Exchange Transformations

I.1 Notation, definitions, conventions

Interval Exchange Transformations (IETs) are the finite bijective piecewise isometries of a bounded interval I . Normalizing the bounded interval I to be equal to $[0, 1)$, we have the following formal definition:

Definition I.1.1 (Interval Exchange Transformation) An IET is a bijective map $T : I \rightarrow I$, for which there exists a partition of I into finitely many intervals $I_s = [\beta_{s-1}, \beta_s)$, where $0 = \beta_0 < \beta_1 < \dots < \beta_{r-1} < \beta_r = 1$ and $1 \leq s \leq r$, such that the restriction of T to each of these intervals is a translation by a factor γ_s . In other words, we have that:

$$T|_{I_s}(x) = x + \gamma_s,$$

for $1 \leq i \leq r$.

The standard convention is for the intervals for T to be continuous from the right, so the intervals of continuity of T are closed from on the left and open on the right. Because T is a bijection, the images of the intervals I_s are mutually disjoint and they cover the whole interval I . The order of these images inside I is encoded by a permutation σ of $\{1, \dots, r\}$

such that $\sigma(s)$, where $1 \leq s \leq r$, denotes the position of $T(I_s)$ inside I . If τ is the inverse of σ , then we must have that $\beta_{\tau(s)} + \gamma_{\tau(s)} = \beta_{\tau(s+1)} - 1 + \gamma_{\tau(s+1)}$ for all $1 \leq s \leq r - 1$. An example on 4 intervals is shown in figure I.1

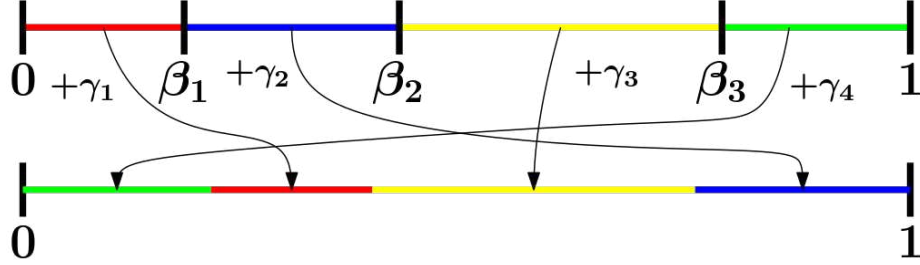


Figure I.1: Dynamical plane of a 4-IET.

The chosen notation for IETs will be more convenient for our purposes as it readily generalizes to the one used for Interval Translation Mappings (ITMs), which are the central topic of this thesis. The more standard notation is define a d -IET as a map $[0, 1) \rightarrow [0, 1)$ represented by a pair (λ, π) , where π is a permutation of $1, \dots, d$ and λ is an element of the $(d - 1)$ -dimensional standard simplex Λ . The permutation π is equivalent to σ from the preceding paragraph and $\lambda = (\lambda_1, \dots, \lambda_d)$ is a vector such that $\sum_{i=1}^d \lambda_i = 1$ and λ_i equals the length i 'th interval of continuity of T . We will occasionally use this notation when necessary, e.g. when discussing the Rauzy-Veech induction in Section I.3.

The standard assumption is that the IETs we are dealing with are *irreducible*, meaning that there is no $k < n$ such that τ permutes the first k elements, as in this case, we can reduce to the case of smaller number of intervals. The parameter space of irreducible r -IETs, denoted by $\text{IET}(r)$, is therefore defined as the disjoint union of sets $\{\sigma\} \times \Lambda$, where σ is a permutation of n elements. The natural measure on this set is the product of the counting measure and the Lebesgue measure on Λ . When we talk about a typical map, we mean with respect to this measure, and for simplicity, we will say that such a map is Lebesgue-typical. In almost all cases, our results will apply to any (or at least Lebesgue-typical) IET on any number of intervals. Thus we will often omit stating very precise information, and simply refer to T as a map, which will mean that it is an irreducible r -IET for some $r \geq 2$.

The necessary background in ergodic theory and number theory for this chapter is covered in sections [I.8](#) and [I.9](#), respectively.

I.2 Early history of IETs

IETs were first considered in print by Sinai in his 1973 book *Introduction to Ergodic Theory* [[Sin73](#)]. He proved that if the union of orbits of β 's is dense in I , then the map T has at most r ergodic probability measures. A bit later, in 1975, Keane wrote his seminal paper [[Kea75](#)] in which he introduced a condition under which an IET is minimal. In literature, this is known as either the infinite distinct orbit condition (idoc) or the *Keane* condition. We will stick to the latter name:

Definition I.2.1 (Keane condition) An IET T is said to satisfy the *Keane* condition if $T^n(\beta_i) \neq T^m(\beta_j)$ for all $0 < i, j < r$ with $i \neq j$ and $n, m \in \mathbb{Z}$.

This turned out to be the appropriate generalization of irrationality for circle rotations, as Keane proved the following:

Theorem I.2.2 An IET T that satisfies the Keane condition is minimal.

Keane also improved Sinai's result and showed that if a map T satisfies the Keane condition, it has at most r ergodic probability measures. An earlier result of Katok on surface flows [[Kat73](#)] implies a better bound of $\left\lfloor \frac{r}{2} \right\rfloor$, which was also shown by Veech in [[Vee78](#)]. However, the existence of a map that was minimal and not uniquely ergodic was not demonstrated. For $r \leq 3$, it is easy to show that minimality implies unique ergodicity, as for $r = 2$ all maps are isomorphic to circle rotations, while for $r = 3$ all maps are return maps of circle rotations. Thus the minimal number of intervals should be at least 4. The first example of such a map, for $r = 5$, was given by Keynes and Newton in 1975. Keane provided a 'minimal' possible counterexample with $r = 4$ in [[KN76](#)]. All of the constructed examples have special number-theoretic properties, so he conjectured in this paper that such maps should be rare:

Conjecture I.2.3 (Keane's conjecture) The set of all non-uniquely ergodic IETs with a given permutation π has zero Lebesgue measure.

This turned out to be a highly non-trivial problem that required the development of several new tools. The two most important ones are the connections of IETs with geometry, in particular with Riemann surfaces, Quadratic Differentials, and Teichmüller flow (see Section I.4), and the Rauzy-Veech induction, which can be seen as the combinatorial counterpart of the geometric connection.

I.3 Rauzy-Veech induction

In [Rau77] Rauzy first described an induction algorithm for IETs that generalizes the continued fraction algorithm from Diophantine approximation (see Section I.9). We will give a very minimalistic description of this algorithm, enough to state the important theorems related to it. For details see the surveys by Viana [Via06] and Yoccoz [Yoc06].

Let T be an r -IET. Let $l_0 := 1 - \beta_{r-1}$ be the length of the last interval of continuity of T . Let $l_1 := 1 - T(\beta_{r(r)-1})$ be the length of the interval whose image is last with respect to the order in I . Let T' be the return map under T to the longer of the two intervals $[0, \beta_{r-1})$, $[0, T(\beta_{r(r)-1}))$. Then it is easy to see T' is also an interval exchange transformation on r intervals (see figure I.2). By rescaling the interval I' to I , T' can be taken to again be an IET defined on $[0, 1)$. The map $\mathcal{R} : \text{IET}(r) \rightarrow \text{IET}(r)$ defined by $\mathcal{R}(T) = T'$, or $\mathcal{R}(\pi, \lambda) = (\pi', \lambda')$, is called the *Rauzy-Veech induction*, and it can be viewed as a dynamical system on $\text{IET}(r)$. It turned out to be the central tool for analyzing the properties of Lebesgue-typical IETs.

Note that in order to carry out the induction, we need that $l_1 \neq l_0$. Keane's condition turned out to be crucial in this aspect because it is equivalent to the property that the induction step can be applied infinitely many times, i.e. to the condition that $l_1 \neq l_0$ always holds. Thus the induction can be carried out for a full measure subset of IETs. For future reference let A be the matrix that describes the change of lengths: $A\lambda' = \lambda$. Thus the effect of the Rauzy induction on lengths is to apply the linear map A and then rescale the domain so that the sum is 1. For a map T that satisfies the Keane condition, let $A_1, A_2, \dots$ be the associated infinite sequence of matrices.

Interestingly, the first progress made towards the proof of Keane's conjecture did not use the Rauzy-Veech induction. In [KR80] Keane and Rauzy proved, using explicit per-

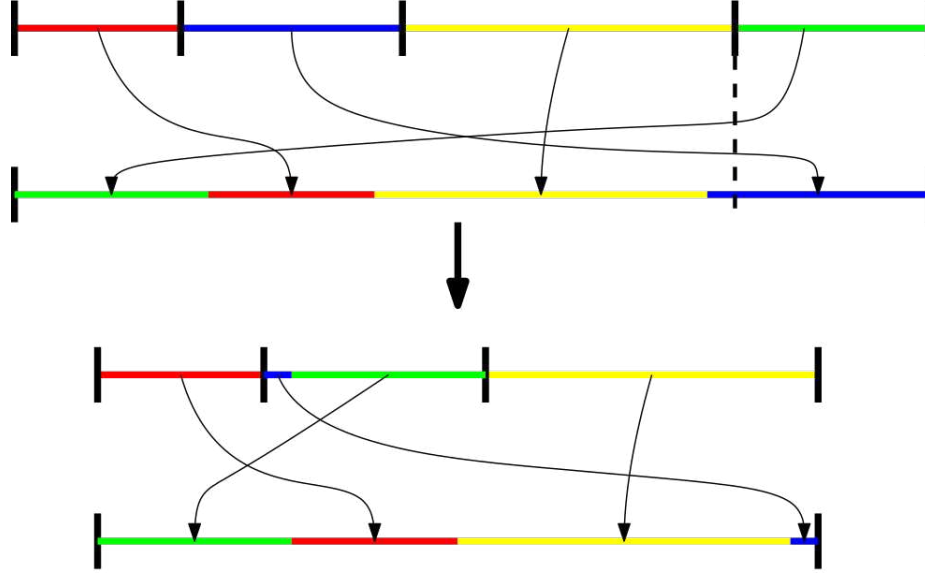


Figure I.2: Rauzy-Veech induction step without rescaling.

turbations and elementary ergodic theory, that the set of all uniquely ergodic maps is a residual subset (in the sense of Baire) of $\text{IET}(r)$. Thus they proved a weak form of the topological analogue of Keane's conjecture, but their methods proved insufficient to study the measure-theoretic question.

The important connection between Lebesgue-typical IETs and Rauzy-Veech induction was established by Veech in [Vee78] and [Vee81], building on the earlier work by Rauzy in [Rau77] and [Rau79]. He proved the following theorem:

Theorem I.3.1 If the dynamical system $\mathcal{R}$ defined by the Rauzy-Veech induction supports an absolutely continuous invariant measure on $\text{IET}(r)$, then almost every map in $\text{IET}(r)$ is uniquely ergodic.

The existence of such a measure was then established by Veech in [Vee82], completing the proof of Keane's conjecture. The proof has roughly the following steps:

1. Construction of a space of suspensions of IETs called *zippered rectangles* that is in fact a $(2r - 1)$ -dimensional manifold M ;

2. Showing that M supports a natural flow $\mathcal{S}$ and a volume element Γ invariant under $\mathcal{S}$;
3. Proving that M , $\mathcal{S}$ and Γ ‘project’ to $\text{IET}(r)$, $\mathcal{R}$ and μ , respectively, where μ is a absolutely continuous invariant measure for $\mathcal{R}$.

An issue with the constructed measure μ is that it is infinite, which complicated studying the finer ergodic properties of $\mathcal{R}$. In his seminal paper [Zor96], Zorich constructed a *finite* invariant measure μ_Z for an accelerated version $\mathcal{R}_Z$ of $\mathcal{R}$. It is defined by setting $\mathcal{R}_Z(\lambda, \pi) := \mathcal{R}^n(\lambda, \pi)$, where n is the minimal integer such that the longer of the two last two intervals $[\beta_{r-1}, 1)$ and $[\beta_{\tau(r)-1}, 1)$ becomes the shorter one. The study of the ergodic properties of this accelerated algorithm became a central topic in the field of IETs.

The most notable is perhaps the analysis of the associated *Lyapunov exponents* (of the matrix product $A_1 A_2 \dots$), which we do not cover in detail, but give a brief historical overview. The simplicity of the top exponent and various properties of the other exponents were established by Zorich in [Zor96]. The first deviation results related to the second exponent were also obtained by Zorich in [Zor97]. Forni proved in [For02] that the g ’th exponent is positive, where g is the genus of the associated translation surface (see [AM24] and [Zor06] for an introduction to these). The simplicity of the top g components (known as the Kontsevich-Zorich conjecture [Zor99], [Kon97]) was established in full generality by Avila and Viana in [AV07].

The Zorich acceleration $\mathcal{R}_Z$ and the associated measure μ_Z also played a major role in the proofs of theorem I.4.3 ([AF07]), in the study of the cohomological equations for IETs ([MMY05]) and the linearisation of GIETs ([MMY12]) (see Section I.7 for more details).

A final aspect of the Rauzy-Veech induction that we discuss is the fact that it gives rise to an equivalence relation on the set of all irreducible rotations on r symbols, by the simple criterion that two permutations π, π' are equivalent if there an IET T with permutation π that under iterates of $\mathcal{R}$ gets mapped to an IET T' with permutation π' (as stated, it is not symmetric, but can be easily shown to be so). These equivalence classes are called *reduced Rauzy classes*. The *extended Rauzy classes*, which are larger classes obtained by allowing two more operations (Rauzy-Veech induction from the left and a canonical involution) on π , were classified by Zorich and Kontsevich in [KZ03]. The classification is based on two

combinatorial properties called *hyperellipticity* and *spin parity*. A third property, called *degree of a marked singularity* was added by Bossy in [Boi12] to classify all the reduced Rauzy classes. These approaches are heavily based on geometric methods, while a completely combinatorial proof of these classifications is due to Fickenscher [Fic16]. Geometric methods were also used by Zorich in [Zor08] to find explicit representatives for all (reduced and extended) Rauzy classes and by Delecroix to give explicit formulas for the cardinalities of Rauzy classes in [Del13].

I.4 Ergodic properties of IETs

The Keane conjecture was also proved by Masur in [Mas82], at roughly the same time as Veech and by using a somewhat similar method in spirit. We state the main ingredients of the proof without defining many of the relevant objects, as this would require a longer diversion into Teichmüller theory (see [Hub06], [Hub16] for a detailed introduction):

1. The space of normalized quadratic differentials supports a natural measure μ . Its most important property for all other steps is that it is invariant under *Teichmüller flow* and the action of the *mapping class group* (but we do not explain why).
2. Each quadratic differential canonically gives rise to a vertical foliation (which is a measured foliation). Under certain assumptions, the return map to a transversal of this foliation induces an IET. Conversely, there exists a normalized quadratic differential (entire families in fact) whose vertical foliation gives rise to any chosen IET.
3. The measure μ behaves well with respect to the Lebesgue measure on the space of IETs: A set E of IETs of Lebesgue zero/non-zero Lebesgue measure gives rise to a set of quadratic differential of zero/non-zero μ -measure.
4. The set of all quadratic differentials for which the vertical foliation is not uniquely ergodic has μ -measure zero. As non-uniquely ergodic IETs give rise to non-uniquely ergodic measured foliations, this means that the Lebesgue measure of non-uniquely ergodic IETs has to be zero.

Rees gave a different proof of Keane's conjecture in [Ree81], also using geometric methods. Kerckhoff gave another one in [Ker85], using only elementary combinatorial properties of the Rauzy induction (Yoccoz further simplified this proof in [Yoc06]). The following is the sketch of the original proof:

1. For any given map T that satisfies the Keane condition, one can show that the set of T -invariant measures μ equals the set $C^\infty = \bigcap_{n=0}^\infty C_n$, where $C_n = A_1 \dots A_n C_0$, and C_0 is the infinite cone of vectors in $\mathbb{R}^{r-1}$ with non-negative entries. Thus one needs only that C^∞ is a line for a Lebesgue-typical T .
2. It is enough to show at infinitely many times n , $C^n \rightarrow C^{n+1}$ shrinks by a definite amount in both diameter and volume. This follows if the ratios of sums of entries of the matrix products $A_1 \dots A_{n+1}$ are bounded by some uniform constant C .
3. Rauzy-Veech induction $\mathcal{R}$ on the Rauzy class of given map T can be encoded by a graph G , and it can be shown that 2. follows from a combinatorial property of G called *normality*.
4. Using elementary volume and distortion estimates, one can show that normality holds for a Lebesgue-typical T , which completes the proof.

This proof in fact works in the more abstract setting of *simplicial systems* that satisfy certain combinatorial properties, and IETs with the Rauzy-Veech induction are a special case of such systems. A strong generalisation of this approach was recently developed by Fougeron in [Fou20a].

All of the previously mentioned proofs of the Keane conjecture did not establish a condition that could be used to check if a given IET is uniquely ergodic. The first such criterion was established by Boshernitzan in [Bos85]. He called this condition 'property P ' and established the following theorem:

Theorem I.4.1 (Boshernitzan) Almost every IET satisfies property P . Moreover, if an IET satisfies property P then it is uniquely ergodic.

This theorem therefore gives another proof of Keane's conjecture, one that does not rely either on connections to geometry or Rauzy induction. We do not state the definition of property P , as it is somewhat complicated. Fortunately, Veech was able to find a simpler full-measure condition in [Vee87], which he called the Boshernitzan criterion. Let T be an IET and let ϵ_n be the length of the shortest interval of continuity of T^n . Then we have the following theorem:

Theorem I.4.2 (Veech) If $\limsup_{n \rightarrow \infty} \epsilon_n n > 0$ then the T is uniquely ergodic.

We now turn to the analysis of other ergodic properties of IETs. Their definitions and basic properties are recalled in Section I.8. Katok proved in [Kat80] that IETs are *never* mixing. On the other hand, earlier work by Katok and Stepin [KS67] on general approximation results in ergodic theory implies that a typical 3 – IET is weakly mixing.

In a series of articles ([Vee84a], [Vee84b], [Vee84c]) Veech explored many ergodic and spectral properties of IETs. One of his main results (in [Vee84a]) is that for an infinite class of irreducible permutations, sometimes called Veech or W -type permutations, almost every IET is weakly mixing (a simpler proof of this result was given by Chaves and Nogueira in [CN01]).

The general question of whether a typical IET is weakly mixing remained open for the next 20 years and required major developments in the field, e.g. the results by Zorich and Forni discussed in the previous section I.3. It was finally resolved in the positive by Avila and Forni in [AF07]:

Theorem I.4.3 (Avila–Forni) A Lebesgue-typical IET whose permutation is not a rotation is weakly mixing.

At the roughly same time, Sinai and Ulcigrai also obtained in [SU05] that almost every IET of periodic type is weakly mixing and constructed explicit examples of such IETs. A bit later, in [Bos12], Boshernitzan established weakly mixing for almost all IETs in families obtained by inducing, i.e. taking return maps to smaller intervals. Nogueira and Rudolph proved in [NR97] that almost every IET is *topologically weakly mixing* (see definition I.8.15).

A convenient condition similar to Theorem I.4.2 to check if a given IET with a W -type permutation is weakly mixing was established by Boshernitzan and Nogueira in [iN04]:

Theorem I.4.4 (Boshernitzan-Nogueira) Assume that T is an r – IET with $r \geq 3$ such that it satisfies the Keane condition, Boshernitzan’s criterion and is ergodic with respect to some probability measure μ . Then T is weakly mixing with respect to μ .

I.5 Modern developments

In this section, we give a summary of recent results on IETs. As a cutoff point between classical and modern, we chose the seminal paper of Avila and Forni [AF07], the preprint of which came out roughly 20 years ago.

A natural question once the genericity in parameter space of a certain dynamical property has been established is: How large is the set of exceptional systems in the parameter space? In the setting of IETs the natural measure of largeness of such sets is the Hausdorff dimension.

The first general result in this direction was established by Masur in 1992 in [Mas92], where he proved that the set of non-uniquely ergodic r – IETs for $r \geq 4$ has Hausdorff dimension at most $r - \frac{3}{2}$. Note that the dimension of the space is $r - 1$, so the result can be rephrased to say that the Hausdorff codimension is at least $\frac{1}{2}$. Curiously, for $r = 2, 3$ minimality is equivalent to unique ergodicity, so the Hausdorff dimension in this case is equal to $r - 2$.

It was widely believed that $\frac{1}{2}$ should be the correct value for the Hausdorff codimension, but the proof for the lower bound turned out to be much harder and required almost 30 years before its completion. It was first established for $r = 4$ by Athreya and Chaika in [AC15], and in full generality by Chaika and Masur in [CM20].

For the set of non-weakly mixing IETs, the first improvement on the results of Avila and Forni was established by Avila and Leguil in [AL18], where they proved that the Hausdorff codimension is positive, without giving an explicit numerical bound. Their results were improved by Al-Saqban, Apisa, Erchenko, Khalil, Mirzadeh and Uyanik in [AS+21], where they showed that the Hausdorff codimension is at least $\frac{1}{2}$ for any $r \geq 3$. The results of Chaika and Masur in [CM20] also apply for non-weakly mixing parameters and $r \geq 4$, and give a lower bound of $\frac{1}{2}$. Thus these two results combine to show that for $r \geq 4$, the Hausdorff codimension of the set of all non-weakly mixing parameters is exactly $\frac{1}{2}$. As the

lower bound only works for $r \geq 4$, the a priori simplest case of $r = 3$ still remains open¹.

We now go through several results coming from various research directions, which are therefore not as related as the previous ones. We focus on the results concerning primarily IETs, and not the associated geometric theories and billiards (which are the topic of the next section I.6).

In [Cha08], Chaika analysed the Hausdorff dimension of the invariant ergodic measures, coming from Keane's examples in [Kea77], that are not equivalent to the Lebesgue measure. He proved that the set of possible dimensions (non-trivially) accumulates at and includes 0. He also showed that the Hausdorff dimension can be equal to 1 and that a certain subset of dimensions is contained in a special compact set. Chaika also constructed the first example of a topologically mixing IET in [Cha12b]. His example was on 4-intervals, and a bit later Boshernitzan and Chaika showed that no 3 – IET can be weakly mixing in [BC13]. They in fact prove general recurrence results about measure-preserving systems (generalizing results of Boshernitzan [Bos93]), and then they apply them to 3 – IETs. Chaika and Fickenscher then proved in [CF15] that for all $r \geq 4$, the set of topologically mixing maps is a residual subset of $\text{IET}(r)$. Robertson showed in [Rob19] that the set of all mildly mixing (see Definition I.8.9) IETs with W -type permutations has full Hausdorff dimension in $\text{IET}(r)$ for all $r \geq 1$.

A sharpening of the results by Avila–Forni [AF07] was obtained by Avila, Forni, and Safaee in [AFS23], where they described the rate of decay of Cesàro averages of Lipschitz observables for almost all IETs. Chaika showed in [Cha12a] that *every* ergodic transformation is disjoint (Definition I.8.11) from almost every IET. In [CE19] Chaika and Eskin verified the Sarnak conjecture for a full-measure of 3 – IETs that satisfy a certain Diophantine property. The same authors also proved in [CE21] that almost every 3 – IET is not 2-simple (Definition I.8.12). Athreya and Boshernitzan established in [AB13] that every finite alternating composition of iterates of an IET that of a rotation by α is uniquely ergodic for almost every α . Chaika and Robertson showed in [CR19] that the skew product of a linearly recurrent IET and almost every step function (with average equal to 0) is ergodic. Chaika generalized a theorem of Kurzweil (see [Kur55]) for circle rotations to IETs in [Cha11], and quantitative versions of this result were obtained by Chaika and Constantine

¹We would like to thank John Chaika for confirming this in written communication.

in [CC19]. Finally, Chaika and Hensel showed in [CH19] that the set of uniquely ergodic maps is a path-connected subset of $\text{IET}(r)$ for all $r \geq 4$.

I.6 Billiards

One of the main reasons IETs gathered a lot of attention is because of their connection to polygonal billiards. Roughly speaking, billiards are dynamical systems defined by two sets of data: a polygon P in the plane and a direction $\theta \in \mathbb{S}^1$. The dynamical system on the interior of P starts as the straight line flow in direction θ , with the continuation of the orbit that hits the boundary of P defined by the standard reflection law (see figure I.3). Note that this is well-defined only for points that do not eventually land on the corners of P under this flow. Rational polygons are those for which every angle is a rational multiple of π . In this special case, one can show that the return maps under the flow to appropriately chosen arcs in P have the structure of IETs (see Section II.5 for the general construction for ITMs that also applies for IETs). One can use this construction to ‘pull back’ the desired results about billiards from those for IETs. A good introduction to billiards is the survey [MT02b] by Masur and Tabachnikov.

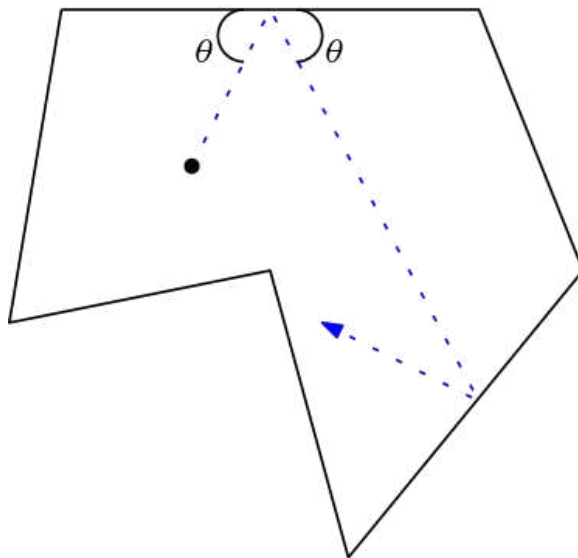


Figure I.3: Polygonal billiards

For the ergodic theory and recurrence properties of billiards, one is interested in the same set of questions as for IETs. The first generic results were obtained by Katok and Zemljakov in [ZK75a], where they established that for all but countably many directions, the billiards flow in a given rational polygon is transitive and that for an everywhere dense G_δ set of general polygons, the billiard flow is transitive in all directions. An analogous result was obtained by Boldrighini, Keane, and Marchetti in [BKM78], where they relied on the corresponding result of Keane for IET (Theorem I.2.2), and the fact that IETs arise as return maps under the billiard flow.

The analogous result to the theorems of Masur and Veech about the genericity of uniquely ergodic IETs was established by Kerckhoff, Masur, and Smillie in [KMS86]. The set of all flows on a given rational polygon is naturally identified with a one-parameter family of translation surfaces M_θ , parametrized by the angle flow θ . The billiard flow restricts to a flow of angle θ on M_θ , and the unique ergodicity of this flow is equivalent to the unique ergodicity of the billiard flow. Thus the following theorem also implies that almost every direction of a billiard flow is uniquely ergodic:

Theorem I.6.1 (Kerckhoff–Masur–Smillie) For almost every θ the flow on M_θ is uniquely ergodic.

Similar to the related questions about Hausdorff dimension for exceptional IETs from Section I.5 can be asked for the billiard flow as well. The work of Masur from [Mas92] also implies that the set of all non-uniquely ergodic directions of a billiard flow on a given polygon has Hausdorff dimension at most $\frac{1}{2}$.

The bound was shown to be strict by Cheung in [Che03] for all rectangular polygons P_λ with barriers of length $0 < \lambda < 1$ for which λ is a Diophantine number (see figure I.4). On the other hand, in the appendix of [Che03], Boshernitzan showed that for an uncountable set of λ , the Hausdorff dimension of the non-uniquely ergodic directions for the flow P_λ is zero. These were shown to be the only possible dimensions by Cheung, Hubert, and Masur in [CHM11] through the following theorem:

Theorem I.6.2 (Cheung–Hubert–Masur) The Hausdorff dimension of the non-uniquely ergodic directions for the flow on P_λ is equal to $\frac{1}{2}$ if and only if λ satisfies the Pérez-Marco condition (see Definition I.9.3). Otherwise, the Hausdorff dimension is 0.

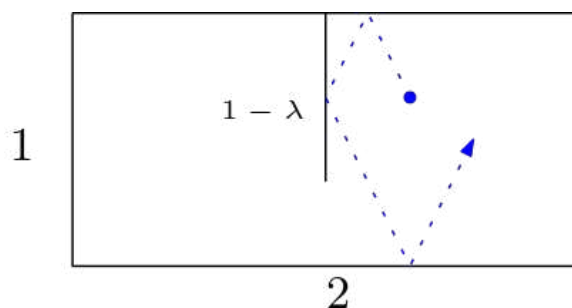


Figure I.4: Rectangular billiard with a vertical barrier

It is a general fact the translation surfaces obtained by unfolding a polygon constitute only a zero measure subset of all translation surfaces. Thus the results of Avila and Forni on weak mixing for IETs and translation surfaces (Theorem I.4.3) do not give results about generic weak mixing properties of billiards. Avila and Delecroix made the first progress on this problem in [AD16] by showing that for every regular polygon, the set of weakly mixing directions has full measure. A similar result was obtained by Málaga and Troubetzkoy in [MST17], where they showed that almost every direction is weakly mixing for a dense G_δ subset of polygons that have only horizontal and vertical sides. Chaika and Forni proved in [CF20] that the full billiard flow on the unit tangent bundle of the polygon is weak mixing (with respect to the Liouville measure) for a dense G_δ subset of polygons (not necessarily rational). In a recent preprint ([AHCF24], at the time of writing published only on arXiv), a breakthrough was obtained by Arana-Herrera, Chaika, and Forni: they gave a complete characterization of rational polygons on which the billiard flow is weakly mixing in almost every direction.

I.7 Generalizations of IETs

There are at least 4 natural generalizations of IETs, each of which is its own area of research. For a more comprehensive list of various dynamical systems related to IETs, see the survey by Skripchenko [Skr23]. Here is the list of the 4 main ones, with informal descriptions:

1. Interval Exchange Transformations with flips: The orientations of the images of the continuity intervals may be reversed.
2. Generalized Interval Exchange Transformations (GIETs): Intervals are still permuted, but they are mapped by diffeomorphisms instead of just translation.
3. Infinite Interval Exchange Transformations: Infinitely many intervals are permuted.
4. Interval Translation Mappings (ITMs): Intervals are translated, but their images may overlap.

IETs with flips display drastically different generic behaviour from IETs. Nogueira proved in [Nog89] that almost every IET with flips has a persistent periodic interval. Nogueira, Pires, and Troubetzkoy gave a detailed description of the possible dynamics of IETs with flips in [NPT13]. They showed that for the number M_T of minimal components of T and P_T , the number of periodic components of T , it holds that $2M_T + P_T \leq n$, and they characterized all possible (M_T, P_T) . Finally, Skripchenko and Troubetzkoy showed in [ST18] that the Hausdorff codimension on the set of minimal IETs with flips is in $(0, 1]$ and that the Hausdorff codimension of minimal non-uniquely ergodic IETs with flips is at most 2.

GIETs are a natural generalization of IETs from a geometric point of view, as they arise as return maps of minimal foliations of higher genus surfaces. They are interesting dynamical objects in their own right, but we are going to focus on certain aspects of their relationship with IETs. A natural question, in parallel to KAM theory and the theory of circle diffeomorphisms (see [Eli+02]), is under what condition is a GIET conjugate to an IET? Such conjugacy problems often boil down to solving linearized versions of the problem that are called *cohomological equations*. Generally speaking, for a map T , cohomological equations look for solutions ψ (which are regular and satisfy some necessary conditions) of the equation $\psi \circ T - \psi = \Phi$, where Φ is a function also satisfying some necessary regularity conditions. A first major breakthrough on cohomological equations related to IETs was obtained by Forni in [For97], where he showed that on a surface of genus $g \geq 2$, for almost every vector field X , the equation $Xu = f$ has a distributional solution u for f satisfying certain necessary conditions and that if f additionally satisfies

a finite set of linear conditions, the solution u is smooth. The IET version of this result, namely that for almost every IET T , the equation $\psi \circ T - \psi = \Phi$ has a solution, was proven by Marmi, Moussa, and Yoccoz in [MMY05]. The full-measure Diophantine condition on T is called the *Roth type* condition. A similar full-measure condition on T called the *restricted Roth type* condition was used in [MMY12] by the same authors to show that the set of GIETs that are C^r -conjugate to T forms a C^1 -submanifold of finite codimension. To finish our discussion on GIETs, we survey some of the major recent advances in a similar direction to [MMY12]: the regularity of solutions from [MMY05] was improved by Marmi and Yoccoz in [MY16], the hyperbolicity at the fixed points of the Rauzy-Veech operator on the space of GIETs was shown by Ghazouani in [Gha21], and C^1 -rigidity of d -GIETs, where $d \in \{4, 5\}$, was established by Ghazouani and Ulcigrai in [GU23].

Infinite IETs in full generality form too large of a space to be systematically studied similar to IETs. This is made clear by the result of Arnoux, Ornstein, and Weiss from [AOW85], which states that *every* aperiodic measurable transformation is isomorphic to some infinite IET. Moreover, the analogue of the fundamental Keane's Theorem I.2.2 does not hold for infinite IETs. Thus the infinite IETs that are mostly studied arise as return maps of special infinite translation surfaces. A good reference for these surfaces is the book by Delecroix, Hubert, and Valdez [DHV24].

ITMs are the main topic of this thesis, and they will be discussed in detail in the next chapter.

I.8 Appendix A: Ergodic Theory and Topological Dynamics

In this section, we define and present the basics of the ergodic and recurrence properties of general topological dynamical systems. Some good references for these are [EW11] and [Vee77], respectively.

Let X be a topological space, $\mathcal{B}$ be the Borel σ -algebra of X , and let $T : X \rightarrow X$ be a map.

Definition I.8.1 (Invariant Measure) A Borel measure μ is said to be *invariant* under T if T is a measurable function with respect to μ and:

$$\mu(T^{-1}(A)) = \mu(A), \quad \forall A \in \mathcal{B}.$$

A natural notion of the irreducibility of an invariant measure is the following:

Definition I.8.2 (Ergodic measure) An invariant measure μ is said to be *ergodic* if every T -invariant μ -measurable set has either full or zero measure.

We also say that T is ergodic with respect to μ or that T satisfies ergodicity. A map is *uniquely ergodic* if it supports only one ergodic measure μ , normalized so that $\mu(X) = 1$. The following is a crucial characterization of ergodicity:

Lemma I.8.3 A map T is ergodic if and only if for all $A, B \in \mathcal{B}$:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

The following two properties are therefore natural strengthenings of ergodicity:

Definition I.8.4 (Weakly mixing) We say that a map T is weakly mixing with respect to an invariant measure μ if

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |\mu(T^{-n}(A) \cap B) - \mu(A)\mu(B)| = 0.$$

One can show that weak mixing is equivalent to the product system $(X \times X, T \times T)$ being ergodic with respect to the product measure $\mu \times \mu$.

Definition I.8.5 (Mixing) We say that a map T is mixing if for all $A, B \in \mathcal{B}$,

$$\lim_{n \rightarrow \infty} \mu(T^{-n}(A) \cap B) = \mu(A)\mu(B).$$

All of these properties can be characterized in terms of convergence properties of the Koopman operator U_T acting on the space $L^2(T, \mu)$ by precomposition with T : $U_T(f) := f \circ T$.

Lemma I.8.6 We have the following three equivalences:

- T ergodic $\iff \forall f, g \in L^2(T, \mu)$:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \int_X U_T^n(f) g = \int_X f d\mu \int_X g d\mu.$$

- T weakly mixing $\iff \forall f, g \in L^2(T, \mu)$:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \left| \int_X U_T^n(f) g d\mu - \int_X f d\mu \int_X g d\mu \right| = 0.$$

- T mixing $\iff \forall f, g \in L^2(T, \mu)$:

$$\lim_{n \rightarrow \infty} \int_X U_T^n(f) g d\mu = \int_X f d\mu \int_X g d\mu.$$

Using the previous lemma, ergodicity and weak mixing also have the following spectral characterizations:

Lemma I.8.7 (Spectral characterizations) The following equivalences hold:

- T ergodic $\iff$ Every U_T -invariant function f is constant μ -a.e.
- T weakly mixing $\iff$ Every eigenfunction f of U_T is constant μ -a.e.

Mixing can be characterized by the existence of ‘almost’ eigenvalues:

Lemma I.8.8 (Mixing characterization) T is mixing if and only if there does not exist an $f \in L^2(X, \mu)$ with zero average and a sequence $k_n \rightarrow \infty$ such that:

$$\lim_{n \rightarrow \infty} \int_X U_T^{k_n}(f) f d\mu = 1.$$

This characterization motivates the following definition for a property that sits between weak mixing and mixing:

Definition I.8.9 (Mild mixing) We say that a map T is *mildly mixing* if $f \in L^2(X, \mu)$ is such that there exists a sequence $k_n \rightarrow \infty$ for which:

$$\lim_{n \rightarrow \infty} \left\| U_T^{k_n}(f) - f \right\|_{L^2(X, \mu)} = 0,$$

then f must be constant μ -almost everywhere.

We would like to have a natural notion of independence of two measurable systems. This property is called *disjointness*, and in order to do this we first have to define:

Definition I.8.10 (Joinings) A *joining* of systems (X, μ, T) and (Y, ν, S) is a $T \times S$ -invariant measure λ on $X \times Y$ with marginals (pushforwards of the measure under the canonical projection maps) μ and ν .

Definition I.8.11 (Disjointness) A pair of systems (X, μ, T) and (Y, ν, S) is said to be *disjoint* if their only joining is the product measure $\mu \otimes \nu$.

A relaxation of this condition is to allow for a certain class of simple joinings:

Definition I.8.12 (2-simple) An ergodic system (X, μ, T) is said to be *2-simple* if any self-joining is either the product measure $\mu \times \mu$ or its support is equal to a graph of a μ -preserving map $S : X \rightarrow X$ that commutes with T .

We now turn to strictly topological versions of our previous definitions: Let X be a topological and let $T : X \rightarrow X$ be a map.

Definition I.8.13 (Transitivity) We say that a map T is *transitive* if for all open sets U, V there exists an integer $n \geq 0$ such that $T^n(U) \cap V \neq \emptyset$.

Definition I.8.14 (Topological mixing) We say that a map T is *topologically mixing* if for all open sets U, V there exists an integer $N \geq 0$ such that for all $n \geq N$ we have that $T^n(U) \cap V \neq \emptyset$.

Considering the remark after Definition I.8.4, it makes sense to define the following:

Definition I.8.15 (Topological weak mixing) We say that a map T is *topologically weakly mixing* if $T \times T$ is transitive.

Similarly as before, one can associate to T an operator acting on the space of continuous functions $f : X \rightarrow X$ by precomposition: $U_T(f) = f \circ T$. We then have the following characterization of topological weak mixing (the assumptions are not best possible, see [Fur81] and [KR69] for stronger version of the characterization):

Lemma I.8.16 Assume that the space X is additionally compact and metrizable, and that T is minimal. Then T is topologically weakly mixing if and only if the only continuous eigenfunctions of U_T are the constant functions.

Finally, we give the definition of a type of recurrence for IETs. Let ϵ_n be the shortest interval of the partition of I by points $T^{-k}(\beta_i)$ for $0 \leq k \leq n$ and $0 < i < r$.

Definition I.8.17 (Linearly recurrent IETs) We say that an IET T is *linearly recurrent* if $\liminf_{n \rightarrow \infty} n\epsilon_n > 0$.

I.9 Appendix B: Number Theory

In this section, we are going to review some basic definitions and results coming from number theory and Diophantine approximation. A good reference for these is [Lan95].

Definition I.9.1 (Gauss map) The *Gauss map* is a map $G : (0, 1) \rightarrow (0, 1)$, defined by

$$G(x) = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor.$$

It is clear that $x \in \mathbb{Q} \cap (0, 1)$ there exists an integer $n \geq 0$ such that $G^n(x) = 0$. In the case when x is irrational, the Gauss map generates an infinite sequence of integers $a_n = \left\lfloor \frac{1}{x_{n-1}} \right\rfloor$, where $x_n = G^n(x)$.

The sequence of integers a_n constitutes the *continued fraction* of x , we write: $x = [0 : a_1 a_2 a_3 \dots]$ and it holds that:

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}},$$

where we interpret the right-hand side as a limit as $n \rightarrow \infty$ of the fractions $\frac{p_n}{q_n}$ defined inductively in the following way:

$$\frac{p_n}{q_n} := \frac{p_{n-1}a_n + p_{n-2}}{q_{n-1}a_n + q_{n-2}}.$$

The analysis of the sequences a_n and $\frac{p_n}{q_n}$ is crucial to the study of various Diophantine properties of irrational numbers. One such crucial property is the following:

Definition I.9.2 (Diophantine number) An irrational number x is said to be *Diophantine* if there exists $c, s > 0$ such that for all rationals $\frac{p}{q}$, we have that:

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^s}.$$

It is simple to show that almost all irrational numbers are Diophantine. A more sophisticated property, first introduced in [PM93]:

Definition I.9.3 (Pérez–Marco condition) An irrational number x is said to have the *Pérez–Marco condition* if the sequence of rationals $\frac{p_n}{q_n}$ generated by the Gauss map and x satisfies:

$$\sum_{k=1}^{\infty} \frac{\log \log q_{k+1}}{q_k} < \infty.$$

CHAPTER II

Interval Translation Mappings

II.1 Definitions and notation

Interval Translation mappings (ITMs) are the finite piecewise isometries of a bounded interval I . Normalizing the bounded interval I to be equal to $[0, 1)$, we have the following definition:

Definition II.1.1 (Interval Translation Mapping) An ITM is a map $T : I \rightarrow I$, for which there exists a partition of I into finitely many intervals $I_s = [\beta_{s-1}, \beta_s)$, where $0 = \beta_0 < \beta_1 < \dots < \beta_{r-1} < \beta_r = 1$ and $1 \leq s \leq r$, such that the restriction of T to each of these intervals is a translation by a factor γ_s . In other words, we have that:

$$T|_{I_s}(x) = x + \gamma_s,$$

for $1 \leq i \leq r$.

An example for $r = 4$ is shown in figure [II.1](#). Compared to the definition of Interval Exchange Transformations (Definition [I.1.1](#)), it is exactly the same except that we do not require the map to be bijective. The graphs of an ITM (left) and IET (right) on 4 intervals are displayed in figures [II.2](#) and [II.3](#), respectively.

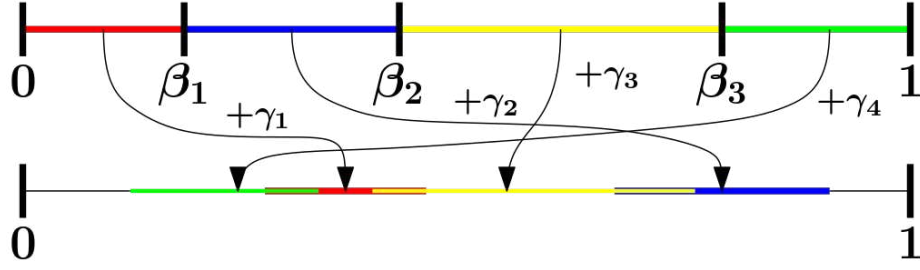


Figure II.1: The dynamical plane of a 4-ITM.

Note that we must have that $\gamma_i \in [-\beta_{i-1}, 1 - \beta_i]$ for all $1 \leq i \leq r$, since the image of T is contained in $[0, 1)$ by definition. These conditions mean that the parameter space $\text{ITM}(r)$ for ITMs on r intervals is a convex polytope in $\mathbb{R}^{2r-1}$ and we endow $\text{ITM}(r)$ with the corresponding subspace topology and restricted Lebesgue measure. In the special case when T is bijective, we get the well-known Interval Exchange Transformations (IETs). These maps were first introduced by Boshernitzan and Kornfeld in [BK95], as a natural generalization of IETs.

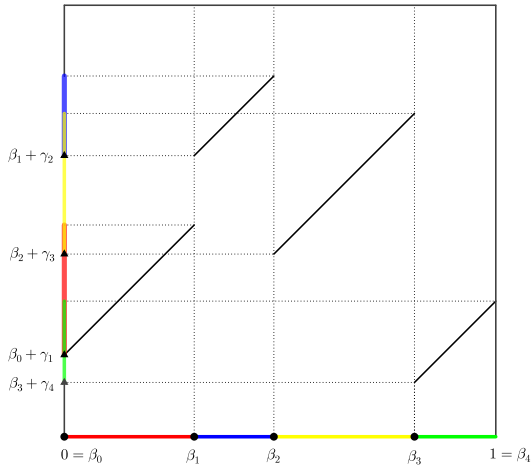


Figure II.2: ITM on 4 intervals

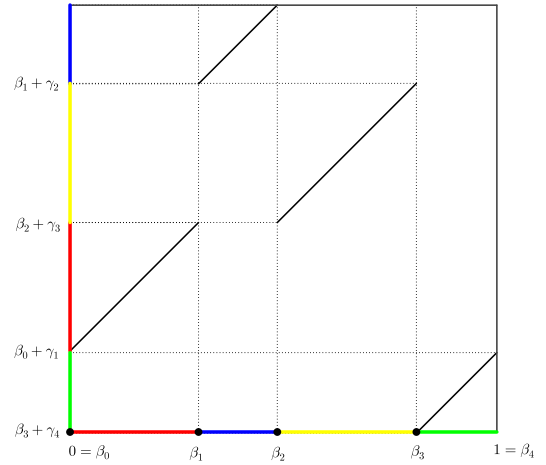


Figure II.3: IET on 4 intervals

Since the map is not surjective, it makes sense to define $X_n(T) := T^n(I)$, for $n \geq 1$,

and:

$$X := \bigcap_{n=0}^{\infty} X_n.$$

Note that $X_{n+1} \subset X_n$ for all $n \geq 0$ and that each X_n is a union of finitely many intervals, from which it is easy to show that X is non-empty. An interval translation mapping is said to be of *finite type* if $X_{n+1} = X_n$ for some n and of *infinite type* when $X_{n+1} \subsetneq X_n$ for all n . It is easy to see that the induced map $T : \overline{X} \rightarrow \overline{X}$ is surjective, and also injective except at countably many points. It is relatively simple to show that $\overline{X}$ is equal to the non-wandering set of T (see Lemma III.3.2).

Similar to IETs, a crucial role is played by return maps to intervals:

Definition II.1.2 (Return map) Let J be an interval contained in X . By R_J , we denote the first return map to J .

It is easy to see that the domain of R_J is the entire interval J , that it is bijective, and that J is partitioned into finitely many maximal half-open subintervals such that every point in the interior of these intervals does not land on a discontinuity of T before returning to J . In other words, R_J is an IET. For simplicity, we will refer to these subintervals as ‘the continuity intervals of R_J ’.

II.2 Properties of general ITMs

In this section, we chronologically go through all of the known results for general ITMs, mostly for an arbitrary number of intervals r , but sometimes focusing on $r = 3$.

The first result obtained for general ITMs was obtained Boshernitzan and Kornfeld in [BK95], and it states that if the parameters defining a map T lie in a quadratic field, then the map is of finite type. If we consider $\mathbb{R}$ as a vector field over $\mathbb{Q}$, then the rational rank of a set of real numbers is equal to the dimension of the subspace spanned by these numbers. By $R(T)$, we denote the rational rank of the set of parameters defining T .

Theorem II.2.1 If $R(T) \leq 2$, then T is of finite type.

The authors of that paper also observed that the topological entropy of every ITM must be zero. This is implied by a more general theory of symbolic flows developed by Boshernitzan in [Bos84].

The same result, but for general piecewise isometries defined on polytopes was obtained by Buzzi in [Buz01]. The setting of this paper is a straightforward generalization of ITMs to arbitrary dimension: we partition a convex polytope into finitely many convex polytopes and consider the maps which are isometries on each of the smaller polytopes.

Theorem II.2.2 Piecewise isometries on polytopes have zero topological entropy.

The first detailed analysis of the properties of general ITMs was carried out in [ST00] by Schmeling and Troubetzkoy. Their first result was the characterization of finite type maps:

Theorem II.2.3 An ITM T is of finite type if and only if X is a finite union of intervals.

Note that only the ‘if’ part of the theorem is interesting, the other follows by definition. They also show that described the non-wandering set of infinite type maps transitive on it:

Theorem II.2.4 The non-wandering set $\overline{X}$ is a Cantor set for an infinite type map T that is transitive on $\overline{X}$.

Theorems II.2.3 and II.2.4 can straightforwardly be combined to show the following that in general, $\overline{X}$ is equal to the union of a Cantor set and a finite union of intervals (see Lemma III.3.2).

The main theorem of [ST00] is an upper bound on the number of minimal components for an aperiodic ITM. Recall that a minimal set is a compact invariant subset in which every orbit is dense and that a map with no periodic points is said to be aperiodic.

Theorem II.2.5 (Number of minimal sets for ITMs) Let T be an aperiodic ITM. Then the number of minimal set of T is $\lfloor \frac{r}{2} \rfloor$.

The authors of this paper also provide examples of open subsets of the parameter space in which every map is of finite type. This is a consequence of the following theorem:

Theorem II.2.6 Let T be a finite type map such that $R(T) = 2r - 1$. Then every map in a sufficiently small neighbourhood of T is of finite type.

They also observed that the maps are conjugate on their non-wandering sets, which implies local stability (see Definition III.6.1).

In [BT03], Bruin and Troubetzkoy proved that the following well-known property of IETs generalizes to ITMs:

Theorem II.2.7 An ITM T that is transitive on $\overline{X}$ is minimal on $\overline{X}$.

In [BH04] Buzzi and Hubert generalized the well-known Milnor-Thurston theorem to piecewise monotone maps without periodic points. Recall the statement of the original theorem (see [MT88]):

Theorem II.2.8 A piecewise monotone map T with positive topological entropy is topologically semi-conjugated to a map with piece-wise constant slope $\pm e^{h_{\text{top}}(T)}$.

As maps without periodic points have zero topological entropy, the following is a generalization of Theorem II.2.8:

Theorem II.2.9 A piecewise monotone map T without periodic points is topologically conjugate to an ITM (with flips) under the necessary assumptions of no non-wandering intervals and irreducibility at infinity.

Note that in this case, we get an actual conjugacy. Thus ITMs with flips serve as rigid models for piecewise monotone maps without periodic points. Another important result shown in [BH04] is an upper bound on the number of ergodic measures of an ITM:

Theorem II.2.10 An ITM T has at most r ergodic measures that are not supported on periodic intervals.

This was shown to be sharp by Bruin and Troubetzkoy for $r = 3$ in [BT03] and for arbitrary r by Bruin in [Bru07a], by examples coming from special families. In particular, this shows that ITMs are not in general factors of IETs, because IETs have at most $\lfloor \frac{r}{2} \rfloor$ ergodic measures.

Let us now turn to the central question in the field of ITMs: the Boshernitzan–Kornfeld conjecture (first stated in [BK95]):

Conjecture II.2.11 The set of all infinite type maps in $\text{ITM}(r)$ has measure zero.

Currently, it has been established for the full parameter space $\text{ITM}(3)$ only for $r = 3$. This was shown by Volk in [Vol14] by proving that almost every ITM on three intervals can be reduced to a rotation or a double rotation (discussed in more detail in Section II.3). Since double rotations of infinite type are a zero measure subset of all double rotations (see Theorem II.3.6), we get the desired result. The main method of proof is renormalization, which is also the case for the two important families discussed in Section II.3.

The main difficulty in analysing the generic properties of ITMs for $r \geq 4$ is that there are currently no tools analogous to those of IETs. There are no associated geometric theories. There are in fact no reasonable topological or combinatorial models. Perhaps most importantly, there is no renormalization scheme analogous to Rauzy-Veech induction. In the next section we describe how, when such a scheme is available, we can prove analogous results as for IETs.

II.3 Two special families

In this chapter we are going to analyse two important special families of ITMs: the Bruin-Troubetzkoy family ([BT03]) and double rotations ([SIA05]), which are the main examples and sources of motivation in the field. The maps in the BT family are in fact special cases of double rotations, but they are better understood and sharper results are available for them.

II.3.1 Bruin-Troubetzkoy family

Our presentation in this subsection closely follows [BT03]. The Bruin-Troubetzkoy (BT) family is a 2-parameter family of maps defined as follows. Consider $U := \{(\alpha, \beta) : 0 \leq \beta \leq \alpha \leq 1\}$, $L := \{(\alpha, \beta) : 0 \leq \alpha \leq \beta + 1 \leq 1\}$ and $R := U \cup L$. For (α, β) in the interior U° of U consider the ITM $T = T_{\alpha, \beta} : [0, 1) \rightarrow [0, 1)$ defined by:

$$T_{\alpha,\beta}(x) := \begin{cases} x + \alpha & \text{for } x \in [0, 1 - \alpha) \\ x + \beta & \text{for } x \in [1 - \alpha, 1 - \beta) \\ x + \beta - 1 & \text{for } x \in [1 - \beta, 1) \end{cases}$$

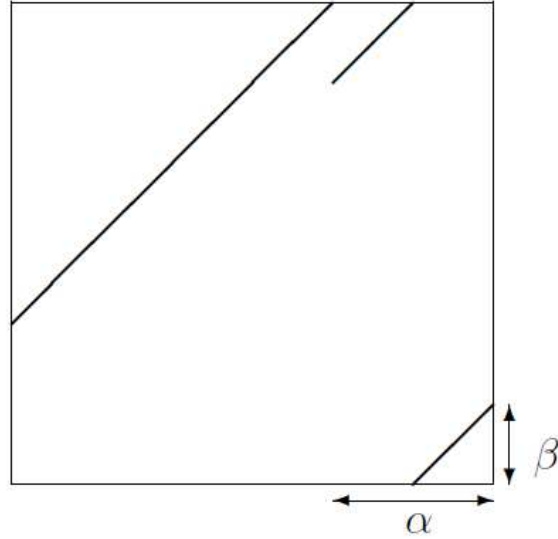


Figure II.4: Graph of a BT-map

The graph of one such map is displayed in the Figure II.4 (the picture is from [BT03]). The most important feature of this family is that there is a well-defined renormalization operator acting on the parameter space. The idea is to look at the interval $J = [1 - \alpha, 1)$ and the first return map R_J to J . We clearly have that R_J restricted to $[1 - \alpha, 1 - \beta)$ is equal to T . There are two possibilities for the interval $[1 - \beta, 1)$:

1. The whole interval $[1 - \beta, 1)$ returns at the same time. Thus the return map R_J is an ITM on 2 intervals and thus stabilizes on a rotation. This easily implies that T is of finite type.
2. There are two different return times, that is, there is a positive integer k such that the left part of the interval returns in k steps and the right part of the interval returns in

$k - 1$ steps. In this case, it is easy to check that the map R_J is, up to affine rescaling, a map $T_{\alpha', \beta'}$ in the BT-family, where:

$$(\alpha', \beta') := G(\alpha, \beta) := \left(\frac{\beta}{\alpha}, \frac{\beta - 1}{\alpha} + \left\lfloor \frac{1}{\alpha} \right\rfloor \right),$$

where $k = \left\lfloor \frac{1}{\alpha} \right\rfloor$.

The map G defined in this way is called the Gauss map, by analogy with continuous fraction algorithms (see Section I.8). It is the renormalization operator acting on parameter space of all BT-maps. One can now again consider the same interval $[1 - \alpha', 1)$ and the return map to it. If we are never in case 1. above, then the map is clearly of infinite type, while it is of finite type in the other case. In other words, the infinite type maps are the infinitely renormalizable ones. In figures II.5 and II.6, you can see the set of all finite and infinite type maps, respectively. Each closed colored triangle in the left figure II.5¹ corresponds to the set of maps that have the same orbit under the renormalization operator up to the time of renormalization to an ITM on two intervals. The right figure II.6² shows the set of infinite type maps.

Let us note that the first example of an infinite type map, constructed by Boshertizan and Kornfeld in [BK95], was a special member T_{α, α^2} of this family. The parameter α is chosen as the unique root of the polynomial $P(x) = x^3 - x^2 - 3x + 1$ in $[0, 1]$. This map is a fixed point of G , and hence of infinite type. It is therefore self-similar, in the sense that the return map defined on $[1 - \alpha, 1)$ is isomorphic to T_{α, α^2} . This example provides a counterpart to Theorem II.2.1, as it shows that parameters coming from a cubic number field can give an infinite type map.

It can be shown that the second iterate G^2 is uniformly expanding away from the parabolic fixed point $(1, 1)$. From this, it follows by standard arguments that the set of all infinite type maps A has zero measure in the parameter space of all BT maps. This gives a proof of Conjecture II.2.11 for this special family of maps.

The authors of [BT03] further study the dynamical properties of infinite type maps and

¹The picture was created by Björn Winckler.

²The picture is taken from [BT03].

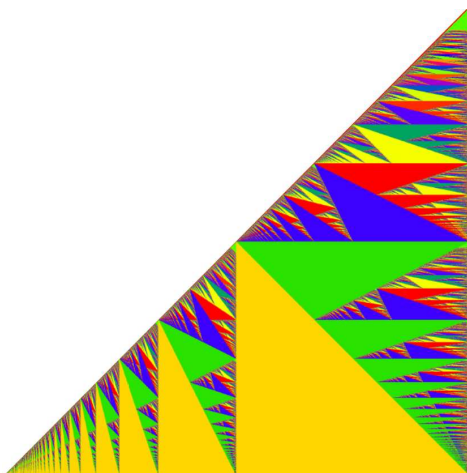


Figure II.5: Finite type maps from the family in [BT03]

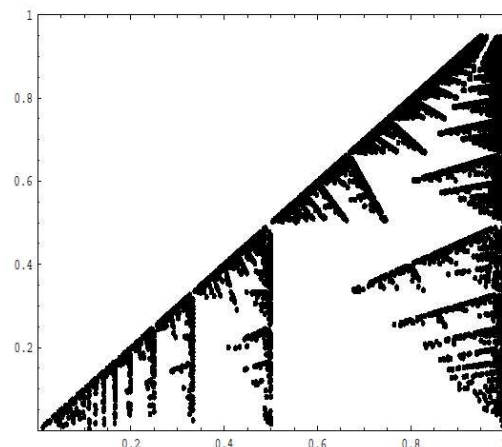


Figure II.6: Infinite type maps from the family in [BT03]

the properties of the set of all infinite type maps A in the parameter space of BT maps. They give sufficient conditions both for when an infinite type map is uniquely ergodic and when it is not. With this, they show that the set of uniquely ergodic maps is a dense G_δ subset of A . As mentioned in the previous section, they also construct examples of maps that have two ergodic measures on $\overline{X}$. They further prove that the set of all G -periodic infinite type maps forms a countable dense subset of A and that all of the Hausdorff dimensions for the non-wandering sets of infinite type maps have a uniform upper bound $r = 0.84955 \dots$.

Most of these results obtained for infinite type BT maps come from studying the subshifts associated with these maps. The following theorem from [BT03] describes this relationship:

Theorem II.3.1 (Infinite type BT maps are isomorphic to subshifts) The set A is naturally indexed by $\mathcal{K} \stackrel{\text{def}}{=} \{k = k_0 k_1 \dots; k_i \in \{1, 2, 3\}, k_{2i} \neq 1 \text{ inf. often and } k_{2i+1} \neq 1 \text{ inf. often}\}$. for any sequence k , the map $T_{\alpha, \beta}|_{\overline{X}}$ is isomorphic to the shift space $(\Sigma_{\alpha, \beta}, \sigma)$ generated by the chain of substitutions $\chi_{k_0} \circ \chi_{k_1} \circ \chi_{k_2} \circ \dots$ where

$$\chi_k := \begin{cases} 1 & \rightarrow 2 \\ 2 & \rightarrow 31^k \\ 3 & \rightarrow 31^{k-1} \end{cases}$$

The sequence $\chi_{k_0} \circ \chi_{k_1} \circ \chi_{k_2} \circ \cdots (3)$ is equal to the itinerary of the point 1^- , and since the infinite type BT maps are minimal, the itinerary of every other point is contained in the orbit closure of this sequence under the shift map σ , from which the correspondence follows.

The BT family was generalized by Bruin in [Bru07a] to an arbitrary number of intervals. He defined an analogous renormalization operator and obtained analogous results to those for the BT-family. In particular, he showed the existence of maps with $r - 1$ ergodic measures and uniform upper bound on Hausdroff dimensions of $\overline{X}$ for infinite type maps.

Finer ergodic properties of the infinite type BT-maps were recently studied by Bruin and Radinger in [BR23]. Again, this was done by studying the dynamics of the associated subshift systems coming from Theorem II.3.1. Most importantly, they establish a criterion for the systems to be linearly recurrent:

Theorem II.3.2 (Linear recurrence criterion for infinite type BT maps) The subshift (X, σ) associated to an infinite type BT map $T_{\alpha, \beta}$ is linearly recurrent if and only if $(k_i)_{i \in \mathbb{N}}$ is bounded and the sets $\{i : k_{2i} > 1\}$ and $\{i : k_{2i-1} > 1\}$ have bounded gaps.

They also show, through the study of Lyapunov exponents for the infinite matrix products associated with the renormalization sequence of an infinite type map (analogous to the case of IETs), that the weakly mixing maps are generic among the infinite type BT-maps:

Theorem II.3.3 (Weak mixing is generic among infinite type BT maps) The weakly mixing maps form a dense G_δ subset of A .

In another recent paper [AHS24], Artigiani, Hubert, and Skripchenko refined some of the results from [BT03]. They developed a new renormalization scheme, which is a Markovian multidimensional continued fraction (MCF) algorithm. This means that the results of Fougeron in [Fou20b] are applicable, which immediately gives their first main theorem:

Theorem II.3.4 (Almost every infinite type BT map is uniquely ergodic) There exists a natural measure μ , whose support set coincides with A , such that for μ -almost all $(\alpha, \beta) \in A$, the transformation $T_{\alpha, \beta}$ is uniquely ergodic.

The Gauss map from [BT03] can be obtained as an appropriate acceleration of the this new renormalization scheme. Another improvement obtained in [AHS24] are sharper bounds on the Hausdorff dimension of A :

Theorem II.3.5 (On the Hausdorff dimension of set of infinite type BT maps) The Hausdorff dimension of A can be bounded by:

$$1.5 \leq \dim_H(A) < 2$$

Moreover, the Hausdorff dimension of the set A is equal to its affinity dimension.

The upper bound is also an immediate consequence of the results on general MCF algorithms from [Fou20b], while the lower bound is obtained using methods of Jiao, Li, Pan, and Xu from [Jia+24] and of Jurga from [Jur23].

II.3.2 Double rotations

Our presentation in this subsection closely follows [BC12]. Double rotations are a natural generalization of the BT-family on which one can still define a renormalization operator and study its dynamical properties. These maps were first introduced by Aihara, Ito, and Suzuki in [SIA05]. A double rotation $T_{(\alpha, \beta, c)} : [0, 1) \rightarrow [0, 1)$ is defined by:

$$T_{(\alpha, \beta, c)}(x) = \begin{cases} x + \alpha \pmod{1} & \text{if } x \in [0, c) \\ x + \beta \pmod{1} & \text{if } x \in [c, 1) \end{cases}$$

with $\alpha, \beta, c \in [0, 1)$. These three values define the parameter space, $P = [0, 1) \times [0, 1) \times [0, 1)$, a three-dimensional cube. Thus a double rotation is an ITM with up to four subintervals of continuity. The renormalization operator S defined on this space is called Suzuki induction (in [BC12]) or the SIA induction (in [Art+21]). It is again defined as a first return map to an appropriate subset of $[0, 1)$. The formula depends on the relationship between the parameters, so we partition the parameter space P as follows

1. $P_{(1)} := \{(\alpha, \beta, c) : \alpha < \beta, c \leq 1 - \beta\};$
2. $P_{(2)} := \{(\alpha, \beta, c) : \alpha < \beta, 1 - \beta < c < 1 - \alpha\};$
3. $P_{(3)} := \{(\alpha, \beta, c) : \alpha < \beta, 1 - \alpha \leq c\};$
4. $P_{(4)} := \{(\alpha, \beta, c) : \beta < \alpha, c \leq \beta\};$
5. $P_{(5)} := \{(\alpha, \beta, c) : \beta < \alpha, \beta < c < \alpha\};$
6. $P_{(6)} := \{(\alpha, \beta, c) : \beta < \alpha, \alpha \leq c\}.$

Any parameter that is in $P_{(2)}$ or $P_{(5)}$ reduced to a rotation under renormalization. On the remaining elements of the partition, we define the renormalization S :

$$S(\alpha, \beta, c) = \begin{cases} \left(\left\{ \frac{\alpha}{1-\beta} \right\}, \left\{ \frac{\beta}{1-\beta} \right\}, \frac{c}{1-\beta} \right) & \text{if } (\alpha, \beta, c) \in P_{(1)} \\ \left(\left\{ \frac{\alpha-1}{\alpha} \right\}, \left\{ \frac{\beta-1}{\alpha} \right\}, \frac{c+\alpha-1}{\alpha} \right) & \text{if } (\alpha, \beta, c) \in P_{(3)} \\ \left(\left\{ \frac{\alpha-1}{\beta} \right\}, \left\{ \frac{\beta-1}{\beta} \right\}, \frac{c}{\beta} \right) & \text{if } (\alpha, \beta, c) \in P_{(4)} \\ \left(\left\{ \frac{\alpha}{1-\alpha} \right\}, \left\{ \frac{\beta}{1-\alpha} \right\}, \frac{c-\alpha}{1-\alpha} \right) & \text{if } (\alpha, \beta, c) \in P_{(6)} \end{cases},$$

with the renormalization defined on the sets $[0, 1 - \beta)$, $[1 - \alpha, 1)$, $[0, c) \cup [1 - \beta + c, 1)$, and $[0, c - \alpha) \cup [c, 1)$ for $P_{(1)}$, $P_{(3)}$, $P_{(4)}$, and $P_{(6)}$ respectively. If a point (α, β, c) maps to $P_{(2)}$ or $P_{(5)}$ under iteration of S , then the corresponding ITM is of finite type. Thus we can define the set B of all $(\alpha, \beta, c) \in P$ such that there exists $n \in \mathbb{N}_0$ with $S^n(\alpha, \beta, c) \in P_{(2)} \cup P_{(5)}$. The set B is equal to the set of all infinite type double rotations. Similar to the BT case, one can prove that an acceleration of S is uniformly expanding and thus get the following theorem:

Theorem II.3.6 The set B of all infinite type double rotations has zero Lebesgue measure in P .

In the original paper [SIA05], this statement was proven only for parameter slices with fixed (α, β) , where it was shown that the set of parameters c which give rise to an infinite type map is a measure zero Cantor set. The full Theorem II.3.6 was proven later by Bruin

and Clack in [BC12]. Another interesting result of [BC12] is that for any S -invariant measure defined on the B , the set of all non-uniquely ergodic maps has measure zero, but no example of such a measure is provided. It was suggested that thermodynamic formalism may be used to construct such a measure, but it was only constructed later by Artigiani, Fougeron, Hubert, and Skripchenko in [Art+21], using the results of Fougeron [Fou20b]. Similar to [AHS24] for BT maps, the authors of [Art+21] use an alternative renormalization compared to [SIA05] and [BC12], which puts them in the context of Markovian multidimensional continued fraction (MCF) algorithms. This means that the results of [Fou20b] apply, from which they recover Theorem II.3.6 via a different method and show that the Hausdorff dimension of B is strictly less than 3 (confirming a numerical result from [BC12] which gives the bound $2.88\dots$).

II.4 Open problems

In this section, we summarize the main open problems, some of which were mentioned in earlier sections. For more open questions related specifically to the stability theory of ITMs, see Section III.9 of Chapter III. The main one, as mentioned before is the Boshernitzan–Kornfeld conjecture:

Conjecture II.4.1 The set of all infinite type maps in $\text{ITM}(r)$ has measure zero.

There are two main problems for ITMs that need to be solved in order for the theory of IETs to have an analogue for ITMs. The first and, we believe, the most important one is to define a renormalization scheme for almost all ITMs:

Problem II.4.2 Define a renormalization scheme for ITMs.

The second is to find an associated geometric theory and a flow. The non-injectivity of ITMs becomes a problem here, as the flow would presumably have to be something non-injective as well.

Problem II.4.3 Associate to ITMs a geometric theory.

A softer and perhaps more traceable problem would be to find topological (combinatorial) models instead:

Problem II.4.4 Construct topological models for ITMs.

Except for these three major problems, let us also mention two other interesting ones. The first concerns the measure of Cantor sets for infinite type maps. For all known examples, they always have measure zero and Hausdorff dimension uniformly bounded away from 1. A general proof of this seems hard, because ITMs have no hyperbolic properties, so contraction results are difficult to obtain.

Question II.4.5 Does there exist an infinite type map such that the measure of the Cantor set $\overline{X}$ is non-zero?

The examples in earlier sections show that infinite type maps can have more ergodic measures than is possible for IETs. Infinite type maps seem to be rare even among the infinite type maps for these special examples, so the following is a natural question:

Question II.4.6 Which uniquely ergodic infinite type ITMs are factors of IETs?

II.5 Spy billiards

Interval Translation Maps naturally arise in the context of billiards with spy mirrors. This was first observed in [BK95] and later studied in more detail in [ST15a; ST15b]. In this section, we elaborate on an example of this construction. This section is equal to Appendix A of [DSS25].

Let P be a Euclidean polygon. A billiard path in P is the trajectory of a particle moving inside P by straight lines and bouncing off the sides of the polygon according to the physical law of reflection: the angle of incidence equals the angle of reflection. Such trajectories terminate when they meet the angles of the polygon (see [Tab05] for a standard introduction to billiards). The polygon P is called *rational* if all its angles are rational multiples of π . For such polygons, one can associate a *canonical* surface S with flat Euclidean metric in such a way that every billiard trajectory in P corresponds to a straight line (geodesic) in S . The flat metric on S is defined everywhere except at finitely many cone points, called *singularities*, for which the cone angles are multiples of 2π . The surface S is constructed by *unfolding* of P (see [ZK75b; MT02a; Sch11] for more details of the construction below).

For each side e_i of P , let $\ell_i \subset \mathbb{R}^2$ be the line passing through the origin parallel to e_i . Consider the group $\mathcal{A}_P$ generated by reflections R_{e_i} in the line ℓ_i , where e_i runs over the complete list of edges of P . Since the angles of P are rational, that is, have the form $2\pi m_i/n_i$ for some coprime integers $m_i < n_i$, the group $\mathcal{A}_P$ will contain precisely $2N$ elements, where N is the least common multiple of the n_i 's. The direction of a billiard trajectory in P is constant between bounces and changes after each bounce by the action of an element in $\mathcal{A}_P$. Thus, every billiard trajectory in a rational billiard can have only finitely many directions.

To construct S , consider $2N$ polygons of the form $g(P) + v_g$, where g runs over the elements in $\mathcal{A}_P$ and the translations v_g are chosen so that the resulting polygons are disjoint; denote them $P_1, \dots, P_{2N}$. We now glue the edges of these polygons pairwise to form a surface: for every edge e_i in P and an element $g \in \mathcal{A}_P$, consider the element $g \circ R_{e_i} \in \mathcal{A}_P$. For these two elements, define the edges

$$e' := g(e_i) + v_g \quad \text{and} \quad e'' := g \circ R_{e_i}(e_i) + v_{g \circ R_{e_i}}. \quad (\text{II.5.1})$$

It is easy to check that these edges are parallel and belong to a pair of distinct polygons in the collection $P_1, \dots, P_{2N}$. In this way, all edges of the polygons $P_1, \dots, P_{2N}$ split in such pairs e', e'' . We identify these edges by translation (explicitly given by $v_{g \circ R_{e_i}} - v_g$). After all the identifications, we obtain a closed orientable surface with a flat metric except at finitely many conical singularities (corresponding to the identified vertices of the polygons) where the angles are multiples of 2π ([Sch11, Lemma 17.3]). This is the desired surface S ; such surfaces are called *translation surfaces* to indicate that they were obtained by identifying parallel pairs of sides in a collection of polygons by Euclidean translations.

For a given direction θ , the billiard trajectories in P in direction θ correspond to a parallel geodesic flow T_θ on S in direction θ . Each trajectory, instead of reflecting off a side e of P , continues as a straight line into the reflected copy of P attached to P through e . This flow is called the *billiard flow*. If I is a segment transverse to this flow, then the first return map of T_θ to I , properly rescaled, is an *interval exchange transformation* [MT02a, Section 1.7]. This connection between rational billiards and IETs has been explored in depth in many publications, as mentioned in the introduction.

Interval translation mappings appear in a similar way once we add *spy mirrors* to our billiard table.

A *spy mirror* in P is a two-sided vertical segment, with one *transparent* side and one *reflective* side. The trajectory γ of a particle reflects according to the standard rules if it approaches the polygon sides or a spy mirror from its reflective side, and γ passes through unobstructed if it approaches a spy mirror from the mirror's transparent side. A *rational billiard with spy mirrors* is a rational polygon P together with finitely many spy mirror segments inside P and the reflection law just described; the spy mirrors are assumed to make rational angles with the sides of P . We denote such a billiard table by $\hat{P}$ ³. Unlike classical polygonal billiards, billiards in polygons with spy mirrors are non-invertible dynamical systems.

We can associate to $\hat{P}$ a surface $\hat{S}$ called a *translation surface with spy mirrors*. We utilize a similar construction as above, where instead of $\mathcal{A}_P$, we use the group $\mathcal{A}_{\hat{P}}$ generated by reflections in lines passing through the origin and parallel to all sides of P as well as to all the mirrors in P . By the same reasoning, $\mathcal{A}_{\hat{P}}$ is a finite group with $2\hat{N}$ elements for some $\hat{N} \in \mathbb{N}$. The surface $\hat{S}$ is obtained by gluing together $2\hat{N}$ rational polygons $\hat{P}_1, \dots, \hat{P}_{2\hat{N}}$ with spy mirrors, each being a reflected copy of $\hat{P}$. Each polygon in this collection is of the form $g(\hat{P}) + v_g$, and we reflect and translate not only P but also all the spy mirrors. In the images, the reflective and transparent sides of the spy mirror are swapped, and $\hat{S}$ is obtained by gluing pairs e', e'' of parallel sides in $\hat{P}_1, \dots, \hat{P}_{2\hat{N}}$ using the relation (II.5.1). Note that we do not glue along spy mirrors. In this way, we obtain a translation surface with two-sided slits. Every forward billiard trajectory $\gamma_\theta \subset \hat{P}$ in direction θ unfolds into a straight geodesic ray $t_\theta \subset \hat{S}$ with the property that if it approaches the reflective side of a spy mirror z' in a copy $\hat{P}_i = g_i(\hat{P}) + v_{g_i}$, with $z' = g_i(z) + v_{g_i}$ for some spy mirror $z \subset P$, then t_θ continues from the same position on the reflective side of the parallel mirror $z'' := g_i \circ R_z(z) + v_{g_i \circ R_z}$ in the copy of $\hat{P}$ generated by $g_i \circ R_z$ and $v_{g_i \circ R_z}$ (here, R_z is the reflection in the line through the origin parallel to z). Thus, if $I \subset \hat{P}$ is a segment in the complement of the spy mirrors, and θ' is a direction transverse to I , then all billiard trajectories in $\hat{P}$ starting on I in the direction θ' correspond to a flow T_θ of parallel lines in $\hat{S}$ starting in some segment J in some transverse direction θ (that depends on θ' and the

³The polygon P "wears" a hat $\hat{}$, as all typical spies do.

starting interval I). If this flow returns to J , then the *properly rescaled first return map* is naturally an *interval translation map* (the rescaling is done in the same way as for rational billiards).

Figure II.7 illustrates the construction in the simplest possible case when P is a square, and $\hat{P}$ is P together with two vertical spy mirrors starting at the horizontal side. In this case, $\hat{N} = 2$, the group $\mathcal{A}_{\hat{P}}$ is generated by reflections in the coordinate axes, and the corresponding translation surface $\hat{S}$ is a torus obtained by gluing four copies of the square P . The first return map of the flow T_θ on $\hat{S}$ to the global vertical transversal is shown in Figure II.8, left; this map is naturally an ITM on four intervals (same picture, right).

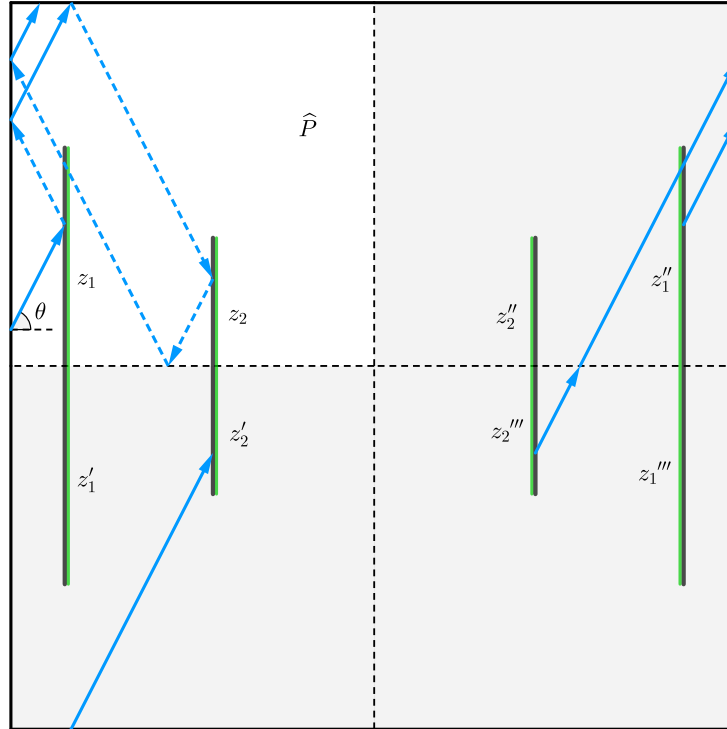


Figure II.7: Spy billiard

The rational polygon $\hat{P}$ is the white square P with two vertical spy mirrors z_1, z_2 starting on the bottom side of P . The transparent sides of the spy mirrors are indicated in green,

while all the reflective sides and the reflective edges are marked in black. The translation surface $\widehat{S}$ is the torus obtained by gluing four reflected copies of $\widehat{P}$ (one in white and three in gray). In the picture, the opposite sides of the large square are identified. The copies z_1, z'_1 and z_2, z'_2 of reflected spy mirrors are identified with the two copies z''_1, z'''_1 and z''_2, z'''_2 respectively, the reflective sides are flipped. The piece of the billiard trajectory in direction θ is shown in P (solid and dashed segments in P). This trajectory unfolds into the straight line trajectory in $\widehat{S}$ shown with solid blue lines.

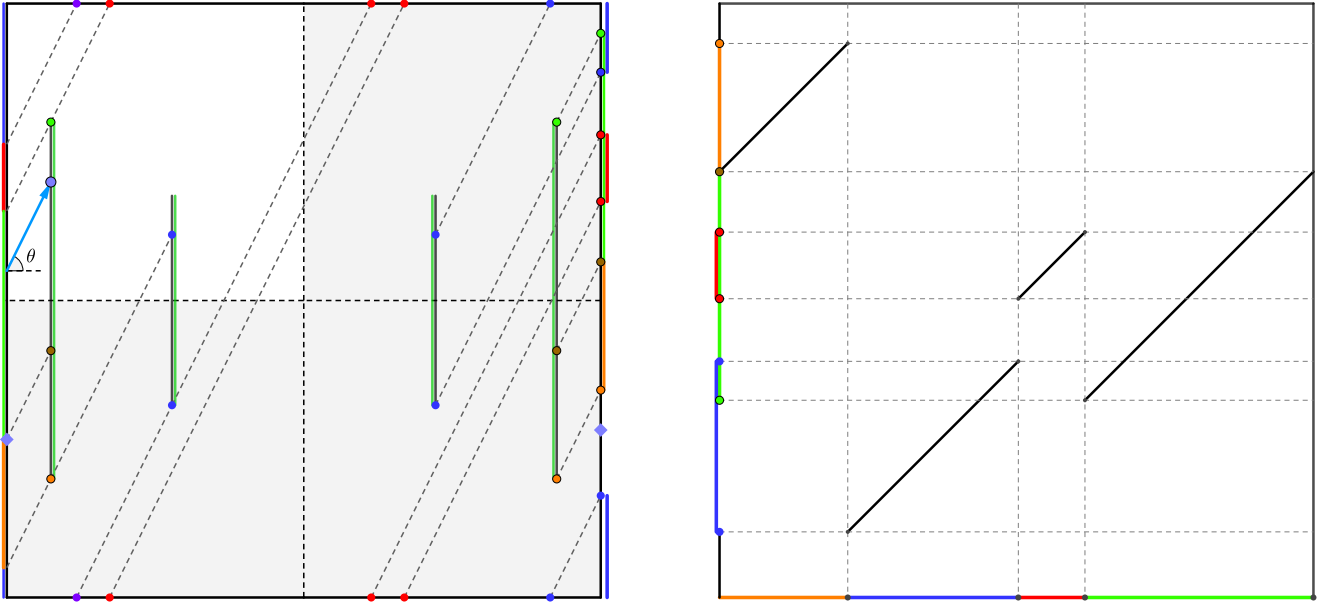


Figure II.8: (Left) ITM from a spy billiard

The flow T_θ starts on the entire vertical side on the left. Under this flow (with the corresponding jumps due to the presence of spy mirrors), the colored segments on the left (blue, red, green, and orange) are mapped isometrically to the colored segments on the right. In this way, we obtain an ITM (shown on the right as the graph of the map) as the first return map of the flow to the transversal, properly cut open from the circle to an interval; the cutting is done at the point marked with the rhombus.

The relationship between billiards and ITMs has not been fully explained, and the following fundamental problem is still open:

Question II.5.1 For which spy billiards can ITMs be obtained as return map to transverse arcs?

The main problem is that it is not clear that the domain of the return map under the flow of an arc J is the entire arc. If this is the case, then we get an ITM.

CHAPTER III

Density of Stable Interval Translation Mappings

III.1 Overview of dynamical stability

One of the main aims in the field of dynamical systems is to determine which dynamical systems are stable, and if possible, to prove that such systems are ‘typical’. It turns out that ITMs are never structurally stable (see Corollary III.2.1). For this reason, we define a map to be *stable* in this context if the next best thing holds: if -loosely speaking- its non-wandering set depends continuously on the system. This notion of stability is weaker than the usual ones in 1-dimensional dynamics because it does not require that the system and its perturbation are conjugate on their non-wandering sets (see [MS93], [McM94]). Note that for IET’s, the non-wandering set is always the entire interval, so the main result of this chapter becomes trivial.

That density of structural stability holds for smooth real 1D systems is proven in [KSS07]; that it holds for complex holomorphic maps in 1D was proven by McMullen and Sullivan [MS98]; it does not hold in higher dimension due to examples by Newhouse (and others). That it holds for ITM’s is proved in this Chapter.

stability notion	appropriate context	density holds for
structural stability	smooth systems	real 1D: $\checkmark$, $\mathbb{C}$ 1D: $\checkmark$, real > 1 D: $\times$
stability	piecewise isometry	ITM: $\checkmark$

Whether or not a map in ITM is stable is equivalent to two conditions: Absence of Critical Connections (ACC) and the Matching condition (M) (see Section III.6 and Theorem III.6.12 for more details). Here ACC means that no iterate of a discontinuity (critical point) is mapped to another discontinuity. As we have seen in Chapter I, in IET literature this is known as the Keane condition. The Matching condition means that the return map to a component of the non-wandering set has at most one discontinuity. This condition has no clear analogue for invertible systems. It rules out that one connected component of the non-wandering set consists of two pieces with independent dynamics which join up for ‘coincidental’ reasons. This characterization of stability implies in particular that stable maps are simple: their dynamics on their non-wandering set corresponds to (a union of) circle rotations.

stability notion	context	condition	density
structural stability	smooth systems > 1 D	hyperbolicity and transversality	$\times$
structural stability	smooth 1D systems	hyperbolicity and ACC	$\mathbb{R}$: $\checkmark$, $\mathbb{C}$: ?
stability	ITM	ACC and M	$\checkmark$

So in this sense, the results for ITMs of this Chapter are the analogue of the density of hyperbolicity in the real one-dimensional dynamics (proved in [KSS07]). The corresponding question for systems on the complex plane (such as quadratic maps) is the Fatou conjecture and is still wide open. Hence the ACC and M conditions can be considered as the analogues of hyperbolicity in our setting.

Definition III.1.1 (Stable maps) We say that a map T is *stable* if there a neighbourhood $\mathcal{U}$ of T in $\text{ITM}(r)$ such that:

1. The mapping $\overline{X}$ assigning to each map $\tilde{T}$ in $\mathcal{U}$ the corresponding set $\overline{X}(\tilde{T})$ is continuous with respect to the Hausdorff topology on compact subsets of $[0, 1]$. Moreover, for each $\tilde{T} \in \mathcal{U}$, $\overline{X}(\tilde{T})$ is homeomorphic to $\overline{X}(T)$.
2. The number of discontinuities in $I \setminus \overline{X}(\tilde{T})$ is constant in $\mathcal{U}$.

It is easy to see that stable maps are of finite type (see Corollary III.6.2). Moreover, the number of components of $\overline{X}(T)$ is constant in $\mathcal{U}$. In general, the map $T \mapsto \overline{X}(T)$ is neither upper nor lower semi-continuous. For example, it is possible to have that the set $\overline{X}(T)$ can jump up or down as is shown in Example III.6.6.

Let $\mathcal{S}(r)$ be the set of all stable maps in $\text{ITM}(r)$. For a stable map T , denote by $[T]$ the connected component containing T in $\mathcal{S}(r)$. We call $[T]$ the *stability component* of T . When we do not need to emphasize a specific map T , we call a connected component of $\mathcal{S}(r)$ a *stable region*. Note that the minimal number n so that $T^n(I) = X(T)$ does not need to be constant within a stable region. It is clear that $[T]$ is the maximal neighbourhood $\mathcal{U}$ for which the conditions of stability hold. It is easy to prove that for $r = 2$ the entire parameter space $\text{ITM}(2)$ is equal to the closure of a single stability component. A very interesting open problem is to describe the geometry of stable regions (see subsection III.9.2).

In the space of IETs, the definition of stability is vacuous, as the non-wandering set is always equal to $[0, 1)$, so all of the conditions for stability are trivially satisfied. On the other hand, by the Characterization of Stability Theorem III.6.12, every IET is not stable when considered as a map in the space of ITMs.

III.2 Summary of results

The results and pictures discussed in Chapter II suggest that the finite type maps should be prevalent in the parameter space for an arbitrary number of intervals. As mentioned at the beginning, this was already conjectured in the measure-theoretic sense by Boshernitzan and Kornfeld in the first paper on ITMs ([BK95]).

Our main result shows that this conjecture is true in the topological sense, i.e. that the set of all finite type maps contains an open and dense subset of the full parameter space $\text{ITM}(r)$. This is achieved by developing the theory of stability of ITMs and proving the following:

Main Theorem I (Density of Stable Interval Translation Maps) The set $\mathcal{S}(r)$ of all stable maps interval translations maps on r intervals is dense in the parameter space $\text{ITM}(r)$.

It is easy to show that the stable maps are of finite type (see Corollary III.6.2), from

which we obtain the main theorem of this Chapter:

Main Theorem II (Topological Prevalence of Finite Type Maps) Finite type maps contain an open and dense set of the parameter space $\text{ITM}(r)$, while the set of infinite type maps has empty interior in $\text{ITM}(r)$.

Proof. By Main Theorem I, the stable maps form a dense subset of $\text{ITM}(r)$. Since the set $\mathcal{S}(r)$ is open by definition and stable maps are of finite type by Corollary III.6.2, the result follows. $\square$

The proof of Main Theorem I is divided into the following three steps.

Theorem A (Eventually periodic maps are dense) The set $EP(r)$ of all eventually periodic maps is dense in $\text{ITM}(r)$.

An eventually periodic map is the one for which every point in I lands on a periodic point.

Theorem B (Stability $\iff$ ACC + Matching) A finite type map is stable if and only if it satisfies the Absence of Critical Connections and Matching conditions.

The ACC and Matching conditions will be defined in Section III.6.

Theorem C (Stable maps approximate Eventually Periodic maps) For any eventually periodic map T , there exists an arbitrarily small perturbation $\tilde{T}$ of T that is of finite type and satisfies the ACC and Matching conditions, and in particular, is stable by Theorem B.

Proof of the Main Theorem. By the theorems above we have that:

$$\begin{aligned} & \overline{\mathcal{S}(r)} \\ & \stackrel{\text{Thm B}}{=} \overline{\{\text{Maps satisfying ACC and Matching}\}} \\ & \stackrel{\text{Thm C}}{\supseteq} \overline{EP(r)} \\ & \stackrel{\text{Thm A}}{=} \text{ITM}(r). \end{aligned}$$

$\square$

Corollary III.2.1 No map in $\text{ITM}(r)$ is structurally stable or Ω -stable.

Proof. Near each map there is an eventually periodic map, by Theorem A, and one with rationally independent parameters (which implies that there are no periodic points). $\square$

In Section III.4, we prove that the eventually periodic maps are of finite type and that the set of all eventually periodic maps contains a countable dense subset of $\text{ITM}(r)$. By the Characterization of Stability Theorem III.6.12, we also know that the return maps to intervals of X must correspond to circle rotations. The stable maps which are also eventually periodic therefore correspond to rational rotations, in the sense that the return map to any component interval of X is a rational rotation, where by ‘component interval’ we mean a connected component of X , which in the finite type case is a half-open interval. In Section III.8 we show that the stable maps form a full measure subset of the set of all finite type maps.

In Subsection III.9.1, we apply our results to the special family from [BT03] to show that the connected components of stability are exactly the coloured triangle regions in Figure II.5. Moreover, any point not contained in a closure of a triangle is of infinite type. In Subsection III.9.2, we will discuss the full parameter space $\text{ITM}(r)$ of ITMs on r intervals, with a focus on the geometry of stable regions.

III.2.1 Main idea of the proof

The basic problem we deal with everywhere is: how to perturb a map T so that some of its dynamical properties change, while some others do not? In analogy to smooth one-dimensional dynamics (e.g. [MS93]), the dynamical properties of a map depend on the behaviour of critical points. For us, the discontinuities of the map serve as the critical points. Thus the problem is about controlling critical itineraries: we want some of them to change, and some of them to stay the same up to certain return times. Because our maps are piecewise linear, this can be reformulated in terms of the linear independence of certain vectors defined by the itineraries of critical points (see Section III.5 for details). The main difficulty is the following: how does the dynamics of a map T prevent the linear dependence of a set of vectors? We deal with this in the proof of Theorem III.5.6. This is

a transversality result that is crucial in the proofs of Theorem B and Theorem C. The proof of this linear independence relies on a rather detailed and subtle dynamical analysis.

III.2.2 Organization of the chapter

The structure of the chapter is as follows. In Section III.3 we prove that $\overline{X}$ (possibly without the point 1) is equal to the non-wandering set of and a few other auxiliary results used throughout the chapter. In Section III.4 we introduce the product notation, define eventually periodic maps, and prove Theorem A. In Section III.5 we prove the crucial Theorem III.5.6 about the linear dependence of critical itinerary vectors. In Section III.6 we define the ACC and Matching conditions and prove Theorem B, using a special case of Theorem III.5.6, while in Section III.7 we prove Theorem C, using the full version of Theorem III.5.6. In Section III.8 we show that the set of stable maps has full measure in the set of finite type maps, which strengthens the Boshernitzan–Kornfeld Conjecture to the Irrational Rotations Conjecture. In Section III.9 we discuss the parameter space and possible future research. The connection between ITMs and billiards is contained Appendix II.5.

III.3 Preliminaries

III.3.1 Notation and conventions

Since the map T is discontinuous, it will be useful to introduce the notion of *signed points*. Consider the relation $\sim$ defined on all half-open intervals of contained I , such that $I_1 \sim I_2$ if and only if the left endpoint of I_1 is equal to the left endpoint of I_2 . For a point $x \in [0, 1]$, we define x^+ as the equivalence class of all intervals $[x, x + \epsilon)$, for $\epsilon > 0$ sufficiently small so that $[x, x + \epsilon) \in I$. We refer to points of this form as the $+$ -type points. Analogously, we define x^- as the equivalence class (under the relation of having the same right endpoint) of all intervals $[x - \epsilon, x)$, where ϵ is sufficiently small so that $[x - \epsilon, x) \in I$, and we refer to these points as the $-$ -type points. We will also say that x^+ is the ‘ $+$ -part’ of x and that x^- is the ‘ $-$ -part’ of x . We use the term ‘geometric point’ to refer to points contained in the geometric interval $I = [0, 1)$ and the term ‘signed point’ to refer to the equivalence class

x^+ or x^- of a point $x \in I$. We will simply use the term ‘point’ if it does not matter what type we are considering.

We adopt the convention that the signed point x^+ is immediately to the right of x , while x^- is immediately to the left of x . With this convention, the definition of intervals contained in I can be extended to signed points in the following way. By $[a, b]$, where a and b can be signed points or geometric points, we will denote the set of all points (signed or geometric) between a and b , including a and b . For example, the set $[x^-, y^-]$ contains x^-, x, x^+ and y^- , but not y and y^+ , and we will write $x^-, x, x^+, y^- \in [x^-, y^-]$ and $y, y^+ \notin [x^-, y^-]$. Analogously, (a, b) will denote the set of all points between a and b , excluding a and b . We also define the sets $[a, b)$ and $(a, b]$ in the obvious way. Note with this notation we have that $0^+, 1^-$ are elements of $[0, 1)$, but $0^-, 1$, and 1^+ are not. We will refer to all of these sets as *intervals*. Thus the term ‘intervals’ will refer to these more general sets if we are dealing with signed points, and to subsets of I if we are not. Moreover, we will use the convention that the intervals of I are half-open and of the form $[a, b)$, unless stated otherwise.

We will write $a \sim b$ if the set $\{a, b\}$ is equal to $\{x^+, x^-\}$ for some point $x \in I$. We will also say that such a pair of signed points *touches* (or *is touching*).

The definition of any map T can be extended to signed points in the following way. For a point $x \in I$, let $z_1 := \lim_{y \downarrow x} T^m(y)$ and $z_2 := \lim_{y \uparrow x} T^m(y)$. Since T is discontinuous, $z_1^+ \sim z_2^-$ does not necessarily hold. Thus it makes sense to define:

$$\begin{aligned} T^m(x^+) &:= z_1^+ \\ T^m(x^-) &:= z_2^-. \end{aligned}$$

The definition of limits of signed points is also straightforward. If $\lim_{i \rightarrow \infty} z_i = z$ for some sequence $(z_i) \in I$, then we can define $\lim_{i \rightarrow \infty} z_i^\pm := z^\pm$. Finally, the distance $d(a, b)$ between two signed points a, b is defined as the distance between the geometric points in I corresponding to a and b .

We now define the *critical set* $\mathcal{C}$ of T as the following set of signed points:

$$\mathcal{C} := \{\beta_1^-, \beta_1^+, \dots, \beta_{r-1}^-, \beta_{r-1}^+\}.$$

We will refer to the elements of $\mathcal{C}$ as either the critical points or the discontinuities, depending on the context. We will sometimes use the labels $\beta_0^+ := 0^+$ and $\beta_r^- := 1^-$, but we do not consider them as critical points. We denote by $\mathcal{C}^+$ the set of all $+$ -type points in $\mathcal{C}$, and by $\mathcal{C}^-$ the set of all $-$ -type points in $\mathcal{C}$. We will also continue to use the term ‘discontinuity’ to refer to geometric points in I at which the map $T : I \rightarrow I$ is discontinuous. If we want to highlight that we are not considering a signed point, we will use the term ‘geometric discontinuity’.

If a point $x \in I$ does not land on a discontinuity of T up to some time $n > 0$, then for our purposes there is no difference between iterates $T^n(x^-)$, $T^n(x)$ and $T^n(x^+)$. If a geometric point x does land on a discontinuity of T at some time n , then we will not consider iterates of x for any time larger than n , but iterates of x^+ and x^- instead. In particular, we do not consider iterates of any geometric discontinuity β , but only of β^+ and β^- . Thus whenever we are considering iterates of some discontinuity, it is assumed that it is a signed point.

Most of the time, we do not need to know the index of a discontinuity with respect to the order in I nor whether the discontinuity is of $+$ -type or $-$ -type. That is why we will often use labels β , β_* and β_{**} to denote the discontinuities we are dealing with. If we care about the sign of a discontinuity, we will use the labels β^+ , β_-^+ , etc. In that case, we will use the same label without a sign to denote the geometric discontinuity in I corresponding to the signed discontinuity, e.g. we denote by β the geometric discontinuity such that β^+ is the $+$ -part of β . We will use the notation $\text{ind}(\beta) \in \{0, 1, \dots, r\}$ to mean the index of β with respect to this order inside I .

The perturbation of (or a map sufficiently close to) some starting map T will be denoted by $\tilde{T}$. Many of the most important objects associated to a map T , e.g. critical points, the set X and boundary points of intervals contained in X , will have well-defined continuations for sufficiently small perturbations of T . For any such object Z , we will denote its continuation by $\tilde{Z}$, e.g. $\tilde{\beta}^+$, $\tilde{X}$, $\tilde{J}$ and similar.

We associate to each point $x \in I$ its *itinerary*. Here it does not matter if a point is signed or geometric. The itinerary of x is an infinite sequence of integers $(i_0(x), i_1(x), \dots, i_n(x), \dots)$, with $1 \leq i_n(x) \leq r$, where $i_n(x) = s$ means that $T^n(x) \in I_s$. We will often use ‘itinerary up to time n ’ to refer to the first n elements of this sequence.

Finally, for a point x (can be either signed or geometric) we will refer to the set $O(x) := \{x, T(x), T^2(x), \dots\}$ as the T -orbit of x . Moreover, if S is a subset of I , we will refer to the set $O(S) := \bigcup_{x \in S} O(x)$ as the T -orbit of S . If the map T is clear from the context, we will omit it and simply use the term ‘orbit’. We will refer to the set $O(x, n) := \{x, \dots, T^{n-1}x\}$, where $n \geq 0$, as the orbit up to time n of x , and to the set $O(S, n) := \bigcup_{x \in S} O(x, n)$ as the orbit up to time n of S .

III.3.2 The non-wandering set of T

As usual, define the non-wandering set $NW(T)$ of T to be the set of $x \in I$ so that for each neighbourhood U of x there exists $n > 0$ so that $T^n(U) \cap U \neq \emptyset$. The set $X = \bigcap_{n=1}^{\infty} X_n$ is by definition the *attractor* of T . In special cases, it is possible for X to accumulate on 1, which is not contained in $I = [0, 1)$. Because of this, we assume the closure $\overline{X}$ of X is defined as the closure of X in $\mathbb{R}$, and not I , so that $\overline{X}$ is always compact. Because of this, we also consider other topological notions as happening in $\mathbb{R}$, e.g. the interior of a set or a limit of sequence,

Our main result of this subsection is that the set $\overline{X} \setminus \{1\}$ is equal to the non-wandering set of T . We first prove a lemma about the structure of $\overline{X}$ that is mentioned in several places in the literature ([BK95], [BT03], [Bru07b]), but no explicit proof is available.

Lemma III.3.1 For any map T , we have that $\overline{X} = A_1 \cup A_2$, where A_1 is a finite union of closed intervals and A_2 is a Cantor set. The union is disjoint, except possibly at the right endpoints of the intervals in A_1 . Moreover, one of the sets A_1, A_2 is allowed to be empty.

Recall that all of the intervals we consider are half-open and of the form $[a, b)$, unless stated otherwise, and that by ‘component interval’ we mean a connected component of X that is an interval.

Proof of Lemma III.3.1. Let A'_1 be the union of all maximal intervals contained in X . We first prove that A'_1 always contains at most finitely many intervals. Indeed, we claim that A'_1 is equal to the union of orbits of intervals of X that contain discontinuities of T . More precisely, $A'_1 = \bigcup_{\beta \in X} O(J_\beta)$, where J_β is the component interval of X containing $\beta \in X$. Since the return map R_J to any interval J of X always has finitely many branches, this

proves the claim. Let J be some interval of X . As a boundary point of any continuity interval of R_J must at some point land onto a discontinuity in R_J , we have that each such interval eventually lands into an interval J_β . Thus every continuity interval must be contained in the orbit of some J_β , as T is a bijection on any $O(J_\beta)$. Thus the entire interval J must be in A'_1 , which proves the claim. We set $A_1 := \overline{A'_1}$.

Let $A_2 := \{\overline{X} \setminus A_1\}$, and assume that it is non-empty. Recall the well-known characterization of a topological Cantor set: it is a compact, metrizable and totally disconnected set with no isolated points. We now show A_2 satisfies all of these properties. Compactness follows by definition, while metrizability follows from it being a subset of $[0, 1]$. It is totally disconnected because the image of any path between two different points in A_2 must be an interval in A_2 , which is impossible as $\overline{X} \setminus A_1$ is nowhere dense. Indeed, if it were dense in some interval, then this entire interval must be contained in X_n for all n , as X_n is a finite union of intervals. This gives that this interval is in X as well, so we get a contradiction with the definition of A_2 .

Finally, we prove that there are no isolated points by showing that for any point $x \in A_2$, there is a sequence in A_2 accumulating on x . First of all, we know that x is not periodic, as it would then be contained in a maximal periodic interval. As it is contained in X , the set of all of its preimages must therefore be infinite and contained in A_2 . Thus its set of preimages must have an accumulation point y in A_2 . If the orbit of y accumulates on x , we are done, so assume that this is not the case. This means that there exists a strictly monotone sequence $x_{n_k} \rightarrow y$ of n_k -preimages, with $n_1 < n_2 < \dots$, such that the itinerary of x_{n_k} and y up to time n_k is different, i.e. the iterates of the interval $[x_{n_k}, y)$ before time n_k contain a discontinuity. Since there are finitely many discontinuities, there is a discontinuity β that occurs for infinitely many k . Thus the orbit of y accumulates on β , so β is contained in A_2 and the orbit of β accumulates on x . Indeed, since y^- is not periodic (preimages of x would then be contained in a periodic interval), we know that in any finite time, the iterates of y must have bounded non-zero distance between discontinuities. Thus the times at which the interval $[x_{n_k}, y)$ lands on β must be unbounded, so x is not isolated. $\square$

Lemma III.3.2 $\overline{X} \setminus \{1\}$ is equal to the non-wandering set $NW(T)$.

Proof. We first prove that $\overline{X} \setminus \{1\} \supseteq NW(T)$. Let x be a point such that $x \notin \overline{X}$. This

means that $\overline{\mathcal{U}} \cap \overline{X} = \emptyset$ for any sufficiently small neighbourhood $\mathcal{U}$ of x . Take one such sufficiently small open neighbourhood $\mathcal{U}_0$ of x . Then there exists $n_0 \geq 1$ such that X_n has definite distance from $\mathcal{U}_0$ for all $n \geq n_0$, and thus we have that $T^n(\mathcal{U}_0) \cap \mathcal{U}_0 = \emptyset$ for all $n \geq n_0$, since $T^n(\mathcal{U}_0) \subset X_n$.

Next, we know that neither x^+ nor x^- is periodic, since $X = \bigcap_{n=1}^{\infty} T^n(I)$. Because of this, for every $n \geq 1$, there exists a sufficiently small neighbourhood $\mathcal{U}_n$ of x such that $T^i(\mathcal{U}_n) \cap \mathcal{U}_n = \emptyset$ for all $1 \leq i \leq n$. Let $\mathcal{U}_{n_0} \subset \mathcal{U}_0$ be such a sufficiently small neighbourhood. By our choice of $\mathcal{U}_0$, this means that $T^n(\mathcal{U}_{n_0}) \cap \mathcal{U}_{n_0} = \emptyset$ for all $n \geq 1$. Thus the point x is not in the non-wandering set.

We now prove that $\overline{X} \setminus \{1\} \subseteq NW(T)$. Let x be a point in $\overline{X} \setminus \{1\}$. By Lemma III.3.1, we have that $\overline{X} = A_1 \cup A_2$, where A_1 is a finite union of closed intervals and A_2 is a Cantor set (we may assume both are non-empty). Thus for any sufficiently small *closed* neighbourhood $\mathcal{U}$ of x we have that exactly one of the following holds:

1. $\mathcal{U} \cap \overline{X}$ contains exactly one closed interval J of definite length, and $x \in J$;
2. $\mathcal{U} \cap \overline{X}$ contains no interval.

In the first case, we know that there must exist $n > m \geq 1$ such that $T^m(J) \cap T^n(J)$ contains an interval of definite length since J has definite length and T is a piecewise isometry. Since X is T -invariant, $\text{int}(J) \subset X$ and $T|_X$ is a bijection, we know that $T^{m-1}(J) \cap T^{n-1}(J)$ contains an interval of definite length as well. Thus we also have that $J \cap T^{n-m}(J) \neq \emptyset$, so $x \in NW(T)$.

In the second case, we show that $x \in \omega(\beta)$ for some discontinuity $\beta \in \mathcal{C}$. This is sufficient, since all ω -limit sets are contained in the non-wandering set, and thus $x \in NW(T)$ as well. Let $\mathcal{U}$ be a neighbourhood of x as above. For any $n \geq 0$, we know that the boundary points of X_n consist of iterates of discontinuities. If $x \notin \omega(\beta)$ for every $\beta \in \mathcal{C}$, then there is a definite distance between the boundary of $\overline{X_n}$ and x for all sufficiently large n . Since x is contained in every $\overline{X_n}$, an interval of definite length containing x must also be contained in every X_n . Thus $\overline{X} \cap \mathcal{U}$ must contain an interval, which is a contradiction with our choice of $\mathcal{U}$. Thus $x \in \omega(\beta)$ for some discontinuity $\beta \in \mathcal{C}$, which finishes the proof. $\square$

III.3.3 Properties of finite type ITMs

In this subsection, we state or prove several simple results that will be continuously used throughout the chapter.

Lemma III.3.3 (Orbit Classification Lemma) For each point $x \in I$, either singed or geometric, at least one of the following three possibilities holds:

1. (Precritical) x lands on a critical point of T ;
2. (Preperiodic) x lands on a periodic point of T ;
3. (Accumulation) There exists a critical point $\beta \in \mathcal{C}$ such that $\beta \in \omega(x)$.

Proof. Assume that none of the above holds for a point $x \in I$. Since the first and the third conditions do not hold, the orbit of x stays a positive distance δ away from every critical point of T . This and the second condition give that the orbit of x also stays at least δ away from every periodic point of T . Indeed, if the point $T^n(x)$, for some $n \geq 0$, is less than δ away from some periodic point y of minimal period p , then the closed interval between $T^n(x)$ and y cannot map forward continuously until time p , because this would mean that $T^n(x)$ is periodic. Thus some iterate of this interval must contain a discontinuity, which means that the orbit of x gets less than δ close to $\mathcal{C}$, which is a contradiction.

Thus there exists a maximal interval J_0 of length at least δ and containing x in its interior such that no point in this interval lands on a critical or a periodic point. Thus the interval J_0 maps forward continuously for all time, and every interval $T^i(J_0)$ has the property that none of its points land on critical or periodic points. Let J_i be the maximal interval containing $T^i(x)$ such that no point in it lands on a critical or periodic point, and note that $T^i(J_0) \subset J_i$. Since all of the intervals J_i have length at least δ , there exists $n > m \geq 0$ such that $J_n \cap J_m \neq \emptyset$. By maximality, $J_n = J_m$ and thus $T^{n-m}(J_m) = J_m$. Since $T|_{J_m}^{n-m}$ is continuous and a translation, it must be the identity, so $T^{n-m}(T^m(x)) = T^m(x)$, which is a contradiction. $\square$

For a finite type map T , every component J of X is of the form $[x^+, y^-]$. We will refer to x^+, y^- as the ‘boundary points of X ’ (or as the points in the boundary of X). The set of all

boundary points of X will be denoted by ∂X . The points in the set $\text{int}(X) := X \setminus \partial X$ will be referred to as the ‘internal points of X ’. The next simple result describes the boundary and internal dynamics of an interval J of X .

Lemma III.3.4 Let T be a finite type ITM. Then the following two properties hold:

- (a) Every component of X is of the form $[T^{k_1}(\beta^+), T^{k_2}(\beta_*^-)]$, respectively, for some $\beta^+, \beta_*^- \in X \cap \mathcal{C}$ and $k_1, k_2 \geq 0$;
- (b) If $T^{l_1}(\beta)$ is an interior point of X for some $\beta \in X \cap \mathcal{C}$ and $l_1 \geq 0$, then there exist $\beta_* \in X \cap \mathcal{C}$ and $l_2 \geq 0$ such that $T^{l_1}(\beta) \sim T^{l_2}(\beta_*)$.

Recall that $T|_X$ is a bijection, so the inverse $T|_X^{-1}$ is well defined.

Proof. Since T is of finite type, the preimage of every boundary point of X is either a boundary point of X or a discontinuity of T . Fix x in the boundary of X and assume that $T^{-n}(x) \notin \mathcal{C}$ for every $n \geq 0$. Since there are finitely many boundary points, this means that $T^{-n}(x)$ is eventually periodic. Because $T|_X$ is a bijection, this means that x is also periodic. By assumption, it does not land on a discontinuity. Thus there is an interval of points around x that are all periodic with the same period. This contradicts x being a boundary point of X , as all periodic points are contained in X . Thus $x = T^n(\beta)$ for some discontinuity $\beta \in \mathcal{C}$, which proves (a).

For (b), let x be a point that is either a discontinuity contained in X or a boundary point in X . Since T of finite type and bijective on X , if the image of x is contained in the interior of X , then there is point y which is either a discontinuity or a boundary point of X such that $T(x) \sim T(y)$. By part (a), we have that $T(y)$ is of the form $T^{l_2}(\beta_*)$ in both cases.

Assume that $T^{l_1}(\beta) \in \text{int}(X)$ for some $n > 0$, $\beta \in X \cap \mathcal{C}$. We may assume β is the last discontinuity in $O(\beta, l_1)$. Thus $T^i(\beta) \notin \mathcal{C}$ for $0 < i < l_1$. We may further assume that $T^i(\beta) \notin \text{int}(X)$ for $0 < i < l_1$. Indeed, if some iterate $T^i(\beta)$, for $0 < i < l_1$, was in $\text{int}(X)$ and satisfied (b), then every iterate $T^j(\beta)$, for $i < j \leq l_1$, is in $\text{int}(X)$ and also satisfies (b). This is because β does not land on a discontinuity before time l_1 . This means that $T^{l_1-1}(\beta)$ is in the boundary of X , so (b) once again follows from the paragraph above. $\square$

III.4 Product notation and eventually periodic maps

III.4.1 Product notation and perturbations

In this subsection, we introduce the product notation, which will be used throughout the chapter. It is a very convenient way to represent and control iterates of critical points.

For each T , $x \in I$, $s = 1, \dots, r$ and $n \geq 1$, we define:

$$k_s(x, n, T) := \#\{T^i(x) \in I_s; 0 \leq i < n\}.$$

Thus $k_s(x, n, T)$ represents the number of entries of x into I_s up to time n . This is an analogue of the first term in the continued fraction expansion for circle rotations, see [MS93, Section 1.1]. When the map T is clear from the context, we will usually simply write $k_s(x, n)$.

Let $W(r) := \mathbb{R}^r \oplus \mathbb{R}^{r-1}$ be the $(2r - 1)$ -dimensional real vector space that contains the *coefficient vectors* (introduced below), and let $(e_1, \dots, e_r)$ and $(f_1, \dots, f_{r-1})$ be the canonical bases for $\mathbb{R}^r$ and $\mathbb{R}^{r-1}$, respectively. Recall that $\text{ITM}(r)$ is the parameter space of ITMs on r intervals, and that it is a convex polytope contained in $\mathbb{R}^{2r-1}$. We call the elements of this space *parameter vectors*. These vectors also have canonical coordinates coming from the ambient space $\mathbb{R}^{2r-1}$. We will use the shorthand $(\gamma \beta)$ for a parameter vector $(\gamma_1 \dots \gamma_r \beta_1 \dots \beta_{r-1})$.

Let $\langle \cdot, \cdot \rangle$ be the standard scalar product on $\mathbb{R}^{2r-1}$. Since $W(r) = \mathbb{R}^r \oplus \mathbb{R}^{r-1}$ and $\text{ITM}(r)$ is a subset of $\mathbb{R}^{2r-1}$, it makes sense to write $\langle v, (\gamma \beta) \rangle = \sum_{s=1}^r v_s \gamma_s + \sum_{s=1}^{r-1} v_{s+r} \beta_s$ for a coefficient vector $v = \sum_{s=1}^r v_s e_s + \sum_{s=1}^{r-1} v_{s+r} f_s \in W(r)$ and a parameter vector $(\gamma \beta) \in \text{ITM}(r)$. We call $\langle v, (\gamma \beta) \rangle$ the *product* of v and $(\gamma \beta)$.

The first reason for introducing this product is to obtain a formula for any iterate $T^n(\beta_*)$ of some discontinuity β_* . Define the vector of coefficients in $v(\beta_*, n, T) \in W(r)$ of this iterate as:

$$v(\beta_*, n, T) := \left(\sum_{1 \leq s \leq r} k_s(\beta_*, n, T) e_s, f_{\text{ind}(\beta_*)} \right).$$

Then the following holds:

$$\begin{aligned} T^n(\beta_*) &= \sum_{s=1}^r k_s(\beta_*, n, T) \gamma_s + \beta_* \\ &= \langle v(\beta_*, n, T), (\gamma \beta) \rangle, \end{aligned}$$

where the vector $(\gamma \beta)$ corresponds to the defining coefficients of T . With this notation, we can associate to any landing of some discontinuity β_* on some other discontinuity β_{**} a coefficient vector in the following way. If $T^n(\beta_*) = \beta_{**}$, then the landing vector L is defined as:

$$L := \left(\sum_{s=1}^r k_s(\beta_*, n) \mathbf{e}_s, \mathbf{f}_{\text{ind}(\beta_*)} - \mathbf{f}_{\text{ind}(\beta_{**})} \right).$$

By the above formula for the iterate $T^n(\beta_*)$, we have that:

$$\begin{aligned} 0 &= T^n(\beta_*) - \beta_{**} \\ &= \sum_{s=1}^r k_s(\beta_*, n, T) \gamma_s + \beta_* - \beta_{**} \\ &= \langle L, (\gamma \beta) \rangle. \end{aligned}$$

The second reason why the scalar product notation is useful is that it gives explicit formulas for how the T -iterates of points change after a perturbation δ of T . Let $\{\mathbf{g}_1, \dots, \mathbf{g}_r, \mathbf{h}_1, \dots, \mathbf{h}_{r-1}\}$ be the canonical basis of the tangent space $\mathbf{T}_T \text{ITM}(r)$ at point $T = (\gamma \beta)$ in $\text{ITM}(r)$. Each perturbation $\tilde{T}$ of a map T in $\text{ITM}(r)$ is given by a linear functional $\delta : \mathbf{T}_T \text{ITM}(r) \rightarrow \mathbb{R}$, so that $\tilde{\gamma}_s = \gamma_s + \delta(\mathbf{g}_s)$ and $\tilde{\beta}_s = \beta_s + \delta(\mathbf{h}_s)$. Thus $\tilde{T}$ is obtained from T by perturbing in the direction $\delta := (\delta(\mathbf{g}_1) \dots \delta(\mathbf{g}_r) \delta(\mathbf{h}_1) \dots \delta(\mathbf{h}_{r-1})) \in \mathbf{T}_T \text{ITM}(r)$. Hence, after identifying $\mathbf{T}_T \text{ITM}(r)$ and $\mathbb{R}^{2r-1}$, it makes sense to write $(\gamma \beta) + \delta = (\tilde{\gamma} \tilde{\beta})$.

If the itineraries of β under T and $\tilde{\beta}$ under $\tilde{T}$ are the same up to some finite time $n \geq 1$, then we have the following formula:

$$\begin{aligned}
\tilde{T}^n(\tilde{\beta}) - T^n(\beta) &= \sum_{s=1}^r k_s(\beta, n) \tilde{\gamma}_s + \tilde{\beta} - \sum_{s=1}^r k_s(\beta, n) \gamma_s - \beta \\
&= \sum_{s=1}^r k_s(\beta, n) (\tilde{\gamma}_s - \gamma_s) + \tilde{\beta} - \beta \\
&= \sum_{s=1}^r k_s(\beta, n) \delta(\mathbf{g}_s) + \delta(\mathbf{h}_{\text{ind}(\beta)}) \\
&= \langle v(\beta, n, T), \delta \rangle.
\end{aligned}$$

Thus if we want $T^n(\beta_*) = \beta_{**}$ to hold after perturbation as well, we must have that (assuming the itinerary of β_* up to time n does not change):

$$\langle L, \delta \rangle = 0.$$

Product notation can also be used to define parameter families of ITMs. Assume that the parameter vector $(\gamma \beta)$ of T satisfies some equation:

$$\sum_{i=1}^r a_i \gamma_i + \sum_{i=1}^{r-1} b_i \beta_i = 0,$$

where $a_i, b_i \in \mathbb{R}$. This is equivalent to saying:

$$\langle v, (\gamma \beta) \rangle = 0 \tag{III.4.1}$$

for the associated vector of coefficients $v := (a_1, \dots, a_r, b_1, \dots, b_{r-1})$. By a *parameter family*, we will refer to the subspace of $\text{ITM}(r)$ consisting of all parameter vectors for which equation (III.4.1) holds for all vectors v in some set $\mathcal{E}$. For a map T and a set of vectors $\mathcal{E}$, we will write $\langle \mathcal{E}, T \rangle = 0$ if the parameter vector corresponding to the map T satisfies (III.4.1) for all vectors $v \in \mathcal{E}$.

We will say that a family is *rational* if all of the vectors in $\mathcal{E}$ have rational coefficients. They are important because of the following simple lemma:

Lemma III.4.1 Parameters with rational coefficients are dense in rational families.

Proof. Recall that the set $IMT(r)$ of all ITMs on r intervals can be viewed as a closed convex polytope in $\mathbb{R}^{2r-1}$. Therefore, a family $\mathcal{T}$ given by a set of vectors $\mathcal{E}$ can be viewed as a convex polytope of co-dimension $t = \dim(\text{span}(\mathcal{E}))$ as the intersection of $IMT(r)$ and the linear sub-space E spanned by $\mathcal{E}$. If $\mathcal{T}$ is now a rational family, then there exists an invertible matrix A with rational coefficients that maps E to a coordinate sub-space, say, $x_1 = \dots = x_{2r-1-t} = 0$ (where we denoted by x_i some Cartesian coordinates in $\mathbb{R}^{2r-1}$). It is clear that the points with rational coordinates are dense in the image $A(\mathcal{T})$. Since the matrix A has rational coefficients, so does A^{-1} , and thus the points with rational coordinates are dense in $A^{-1} \cdot A(\mathcal{T}) = \mathcal{T}$. Hence, parameters with rational coefficients are dense in the rational family $\mathcal{T}$. $\square$

III.4.2 Density of eventually periodic maps

We say that T is *eventually periodic* if every point in $\mathcal{C}$ (recall that this is a set of signed points) is eventually periodic. Eventually periodic maps are the simplest non-trivial maps, an analogue of Morse-Smale systems from smooth interval dynamics (see [MS93]). We now prove the parametric version of Theorem A:

Theorem III.4.2 (Eventually periodic maps are dense - Parametric version) Let T be a map contained in a rational family $\mathcal{T}$ defined by a set of vectors $\mathcal{E}$. Then arbitrarily close to T in $\mathcal{T}$, there is an eventually periodic map $\tilde{T}$.

This theorem is a straightforward consequence of the fact that the maps with rational parameters are dense. Recall that the distance between signed points is defined as the distance of geometric points in I that the signed points correspond to.

Proof. Because of Lemma III.4.1, we only need to prove that maps T for which the parameter vector $(\gamma\beta)$ has rational coefficients are eventually periodic. In fact, it is enough that only the coefficients $\gamma_1, \dots, \gamma_r$ are rational. Let $\mathbb{N}(\gamma)$ be the set of all linear combinations of $\gamma_1, \dots, \gamma_r$ with integer coefficients. This set is discrete, because the minimal distance between any of its elements, if non-zero, is at least $\frac{1}{q^r}$, where q is the largest denominator of

the numbers $\gamma_1, \dots, \gamma_r$. Thus for any $\beta \in \mathcal{C}$, we have that the set $\beta + \mathbb{N}(\gamma) \cap [0, 1)$ is finite. As any iterate of β is contained in this set, we must have that β is eventually periodic. $\square$

Definition III.4.3 For $T \in \text{ITM}(r)$, we define the following sets:

- $\mathcal{C}_1$ is the set of all $\beta \in \mathcal{C}$ that eventually land on a discontinuity;
- $\mathcal{C}_2$ is the set of all $\beta \in \mathcal{C}$ that never land on a discontinuity, but are eventually periodic;
- $\mathcal{C}_0 = \mathcal{C}_1 \cup \mathcal{C}_2$;
- $\mathcal{C}_i^\pm = \mathcal{C}^\pm \cap \mathcal{C}_i$, for $i \in \{0, 1, 2\}$.

It is easy to see $\mathcal{C} = \mathcal{C}_0$ if and only if T is eventually periodic. The next proposition shows that a stronger property holds: if the $+$ -type discontinuities are eventually periodic, then so is every other point in I . In particular, all of the $-$ -type discontinuities are eventually periodic as well. Analogously, it is also sufficient that the $-$ -type discontinuities are eventually periodic.

Proposition III.4.4 If $\mathcal{C}^+ = \mathcal{C}_0^+$, then T is eventually periodic. Moreover, every point in I is eventually periodic.

Thus this proposition shows that the name ‘eventually periodic’ is justified.

Proof. We first prove that there exists an $\epsilon > 0$ such that for each $\beta^+ \in \mathcal{C}^+$, the interval $[\beta, \beta + \epsilon)$ is eventually periodic and maps forward continuously for all times. Let N be the smallest time such that $T^N(\beta^+)$ is a periodic point for all β^+ , and let p be the maximal period of these points. Let $\epsilon > 0$ be the closest positive distance the orbit of any β^+ up to time $N + p$ gets to the left of a point in $\mathcal{C}^+$. Note that β^+ can land on another critical point, but in our definition of ϵ we ignore these times and look only at the positive distances. We claim that this is our required ϵ . Indeed, if the iterates of any interval $[\beta, \beta + \epsilon)$ contain a discontinuity before time $N + p$, it would mean that the distance between a point in the orbit of β^+ and another critical point would be less than ϵ , contradicting the definition of ϵ . Thus, every such interval maps forward continuously up to time $N + p$. Since $T^N(\beta^+)$ is periodic and has period $\leq p$, we see that the entire interval $[T^N(\beta), T^N(\beta) + \epsilon)$ consists of points of the same period, which shows that ϵ has the required property.

Now assume, for the sake of contradiction, that there is some $x \in I_s$ which is not eventually periodic. By assumption, β_{s-1}^+ is eventually periodic. We define a finite sequence of eventually points $x_0 = \beta_{s-1}^+ < x_1 < \dots < x_t < x$ and times $n_0 = 0 < n_1 < \dots < n_t$ such that $T^{n_i}(x_i) \in \mathcal{C}$ and $d(T^{n_i}(x_i), T^{n_{i-1}}(x_{i-1})) \geq \epsilon$ for all $1 \leq i \leq t$ by the following inductive procedure.

If x_i is already ϵ -away from x , we stop the procedure. Otherwise, define n_{i+1} be the minimal time the orbit of the interval $[x_i, x)$ contains a critical point. Such a time exists because x and x_i must have different itineraries, as x_i is eventually periodic, while x is not. Let x_{i+1} be the rightmost point in $[x_i, x)$ which lands on a critical point at time n_{i+1} . We have that $n_{i+1} > n_i$, because x_i is the rightmost point in $[x_{i-1}, x)$ which lands on a critical point by time n_i , which means that the interval $[x_i, x)$ maps forward continuously up to time n_i . Moreover, we have that $n_{i+1} - n_i \leq N + p$, as x_i becomes periodic with period at most p , after at most N iterations. We claim that $x_{i+1} - x_i \geq \epsilon$. Indeed, by construction, we have that $T^{n_i}([x_i, x)) = [\beta, y)$, for some $\beta^+ \in \mathcal{C}^+$. The interval $[\beta, \beta + \epsilon)$ maps forward continuously for all times, which means that only the points in $[\beta + \epsilon, y)$ can land on critical points. Thus $x_{i+1} - x_i \geq \epsilon$, which means that the inductive procedure must terminate at some finite time. Since $d(x_t, x) \leq \epsilon$ and this interval maps forward continuously up to time n_t when x_t lands on some critical point β^+ , we see that all points in this interval must be eventually periodic, contradicting our choice of x . $\square$

An important corollary of the proof of this proposition is that eventually periodic maps are of finite type. Note that this fact, alongside Theorem III.4.2, shows that the finite type maps are dense in $\text{ITM}(r)$.

Corollary III.4.5 Eventually periodic maps are of finite type.

Proof. Eventually periodic maps clearly satisfy $\mathcal{C}^+ = \mathcal{C}_0^+$. The second part of the proof of Proposition III.4.4 shows that under iterates of map that satisfies $\mathcal{C}^+ = \mathcal{C}_0^+$, each point in I eventually lands into one of the finitely many periodic intervals. Thus X must be equal to the union of orbits of these periodic intervals, which shows that it is of finite type by Theorem II.2.3. Thus every eventually periodic map is of finite type. $\square$

III.5 Linear independence of itinerary vectors

III.5.1 Statement and setup

The crucial property that allows us to change one part of the dynamics while keeping the other parts as they are is the linear independence of itinerary vectors of discontinuities. This is the content of Theorem III.5.6 and Corollary III.5.8. To state these results precisely, we need some preliminary results and notation. When introducing objects that depend on some interval J , the dependence will be explicit in the notation if the object in question will be used in the context where there are several J -intervals, and it will be implicit otherwise.

Let $J = [x^{J,+}, y^{J,-}]$ be an interval contained in X . Recall that the return map R_J to J is well-defined (see II.1.2) and that there are finitely many points $a_1^J, \dots, a_{N_J-1}^J$ in the interior of J that land on discontinuities before returning to J . Let $J_1^J, \dots, J_{N_J-1}^J$ be the maximal intervals in J such that no point in their interior lands on a discontinuity, and let r_j^J be the return time of J_j^J to J . Let $a_0^{J,+} := x^+$ and $a_{N_J}^{J,-} := y^-$ be the boundary points of J . For each $1 \leq j \leq N_J - 1$, let $m_j^{J,+}$ be the number of discontinuities that $a_j^{J,+}$ lands on before returning to J and, for $1 \leq k \leq m_j^{J,+}$, let $\beta^{J,+}(j, k)$ be the k -th discontinuity along the orbit up to return time to J of $a_j^{J,+}$. Define $m_j^{J,-}$ and $\beta^{J,-}(j, k)$ similarly, for $1 \leq j \leq N_J - 1$ and $1 \leq k \leq m_j^{J,-}$. Note that for each $1 \leq j \leq N_J$, we have that $\beta^{J,+}(j, 1)$ and $\beta^{J,-}(j, 1)$ are the $+$ and $-$ part of a single discontinuity, which we denote by $\beta^J(j)$. Figure III.1 illustrates these points for an interval J for which R_J has three branches and $\sigma = (321)$.

Recall that $W(r) = \mathbb{R}^r \oplus \mathbb{R}^{r-1}$ is the space of coefficient vectors, and that $(e_1, \dots, e_r)$ and $(f_1, \dots, f_{r-1})$ are the canonical bases for $\mathbb{R}^r$ and $\mathbb{R}^{r-1}$, respectively.

We now define three types of dynamically defined vectors associated to the orbit of J . The first ones are the *L-vectors*, that correspond to the first landings of points $a_1^J, a_2^J, \dots, a_{N_J-1}^J$ to discontinuities:

Definition III.5.1 (Landing vectors) For $1 \leq j \leq N_J - 1$, let l_j^J be the landing time of a_j^J to $\beta^J(j)$ and let $L_j^J \in W(r)$ be the associated landing vector:

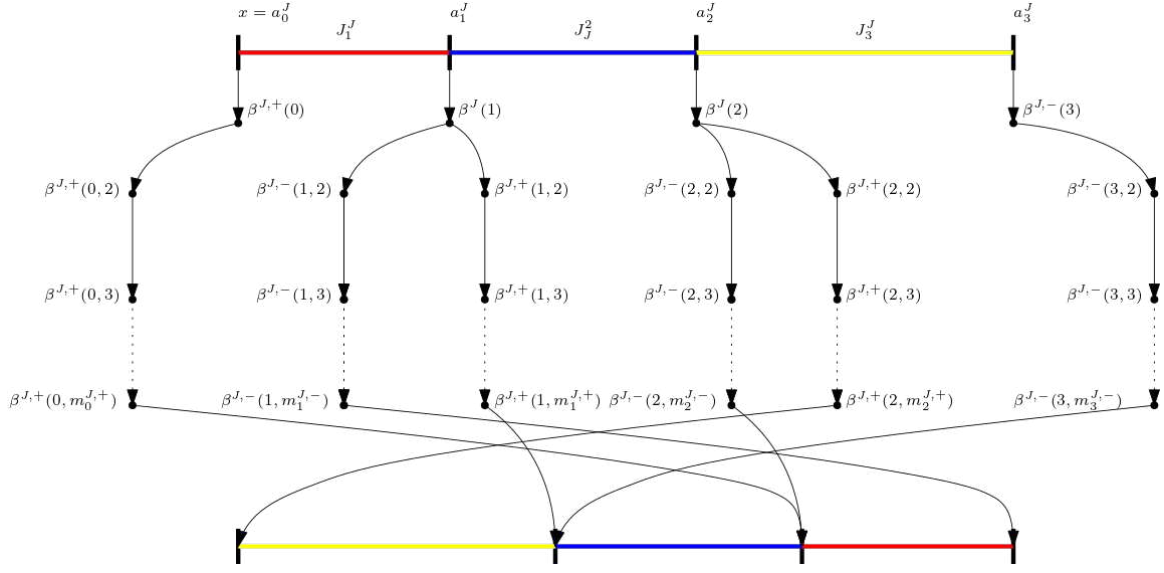


Figure III.1: All dynamically important points in the orbit of an interval J for which the return map R_J has three branches.

$$L_j^J := \left(\sum_{s=1}^r k_s(a_j^J, l_j^J) e_s, -\mathbf{f}_{\text{ind}(\beta^J(j))} \right).$$

Recall that $\text{ind}(\beta)$ is the index of β with respect to the order of critical points inside of I . Note that if a_j is a discontinuity of T , i.e. if the landing time l_j^J is zero, then $L_j^J = (0, -\mathbf{f}_{\text{ind}(\beta^J(j))})$ by definition. We call L_j^J the landing vector of a_j^J to $\beta^J(j)$ because the following holds:

$$\begin{aligned} 0 &= a_j^J + \sum_{s=1}^r k_s(a_j^J, l_j^J) \gamma_s - \beta^J(j) \\ &= \left(\sum_{s=1}^r k_s(a_j^J, l_j^J) \gamma_s - \beta^J(j) \right) + a_j^J \\ &= \langle L_j^J, (\gamma \beta) \rangle + a_j^J. \end{aligned}$$

The second type of vectors we need to consider are the *C-vectors*, which correspond to landings of discontinuities in the orbit of J onto other discontinuities (which are thus also in the orbit of J).

Definition III.5.2 (Critical connection vectors) For each $1 \leq j \leq N_J - 1$ and $1 \leq k < m_j^{J,+}$, let $q^{J,+}(j, k)$ be the landing time of $\beta^{J,+}(j, k)$ to $\beta^{J,+}(j, k+1)$ and let $C^{J,+}(j, k) \in W(r)$ be the associated landing vector:

$$C^{J,+}(j, k) := \left(\sum_{s=1}^r k_s(\beta^{J,+}(j, k), q^{J,+}(j, k)) \mathbf{e}_s, \mathbf{f}_{\text{ind}(\beta^{J,+}(j, k))} - \mathbf{f}_{\text{ind}(\beta^{J,+}(j, k+1))} \right).$$

The following holds by construction:

$$\begin{aligned} 0 &= \beta^{J,+}(j, k) + \sum_{s=1}^r k_s(\beta^{J,+}(j, k), q^{J,+}(j, k)) \gamma_s - \beta^{J,+}(j, k+1) \\ &= \sum_{s=1}^r k_s(\beta^{J,+}(j, k), q^{J,+}(j, k)) \gamma_s + \beta^{J,+}(j, k) - \beta^{J,+}(j, k+1) \\ &= \langle C^{J,+}(j, k), (\gamma \beta) \rangle. \end{aligned}$$

Define $C^{J,-}(j, k) \in W(r)$ analogously for $1 \leq j \leq N_J - 1$ and $1 \leq k < m_j^{J,-}$.

The third type of dynamical vector we need to consider are the *R-vectors*, which correspond to the return of points $a_1^{J,-}, a_1^{J,+}, \dots, a_{N_J-1}^{J,+}, a_{N_J-1}^{J,-}$ to the interval J .

Definition III.5.3 (Return vectors) For each $1 \leq j \leq N_J - 1$, let $r_j^{J,+}$ be the time at which $\beta^+(j, m_j^{J,+})$ lands into J and let $R_j^{J,+} \in W(r)$ be the landing vector:

$$R_j^{J,+} := \left(\sum_{s=1}^r k_s(\beta^{J,+}(j, m_j^{J,+}), r_j^{J,+}) \mathbf{e}_s, \mathbf{f}_{\text{ind}(\beta^{J,+}(j, m_j^{J,+}))} \right).$$

Thus we have that:

$$\langle R_j^{J,+}, (\gamma \beta) \rangle = \sum_{s=1}^r k_s(\beta^{J,+}(j, m_j^{J,+}), r_j^{J,+}) \gamma_s + \beta^+(j, m_j^{J,+}) \in J.$$

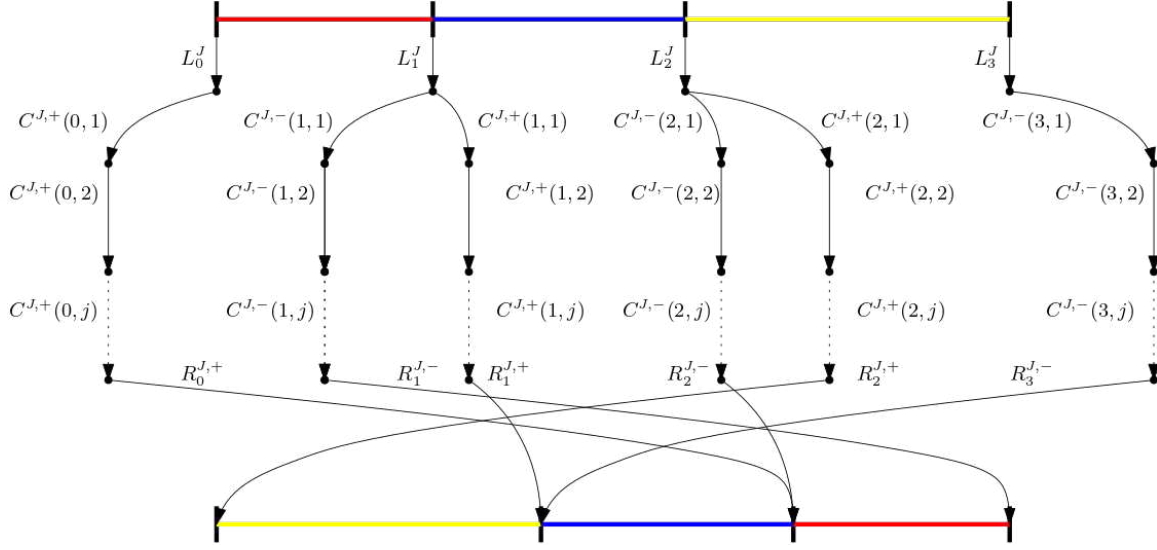


Figure III.2: Vectors associated to the orbit of an interval J for which the return map R_J has three branches.

Define $r_j^{J,-}$ and $R_j^{J,-} \in W(r)$ analogously for $1 \leq j \leq N_J - 1$.

For the boundary points $x^{J,+} = a_0^{J,+}$ and $y^- = a_{N_J}^{J,-}$, the definitions of the corresponding dynamical vectors depend on whether they land on discontinuities of T before returning to J or not. If $a_0^{J,+}$ lands on a discontinuity of returning to J , then we may analogously as for other $a_j^{J,+}$, where $1 \leq j \leq N_J - 1$, define the vectors L_0^J , $C^{J,+}(0, k)$ and $R_0^{J,+}$. In the case when $a_0^{J,+}$ does not land on a discontinuity of T before returning to J , we only define the return vector $R_0^{J,+}$ to J :

$$R_0^{J,+} := \left(\sum_{s=1}^r k_s(a_0^{J,+}, r_0^{J,+}) e_s, 0 \right),$$

where r_0^+ is the return time of a_0^+ to J . Analogously, if $a_{N_J}^{J,-}$ lands on a discontinuity of T , we may define the vectors $L_{N_J}^J$, $C^{J,-}(N_J, k)$ and $R_{N_J}^{J,-}$. If $a_{N_J}^{J,-}$ does not land on a discontinuity of T before returning to J , then define $R_{N_J}^{J,-}$ analogously as for $a_0^{J,+}$.

We would also like to have control over the dynamics of the discontinuities not contained in X . For this purpose, we also introduce the following more general notation for a

critical connection vector. Recall from Section III.4 that we say that $\beta \in \mathcal{C}_1$ if β eventually lands on a discontinuity. Let β' denote the first such discontinuity in the orbit of β , and let $q(\beta)$ be the landing time of β to β' . Note that β and β' are signed discontinuities, by the definition of $\mathcal{C}_1$ (III.4.3).

Definition III.5.4 (General critical connection vectors) Let β be a discontinuity of T which eventually lands on another discontinuity. Define the vector $v_\beta \in W$ as:

$$C_\beta := \left(\sum_{s=1}^r k_s(\beta, q(\beta)) e_s, \mathbf{f}_{\text{ind}(\beta)} - \mathbf{f}_{\text{ind}(\beta')} \right),$$

By definition, we have that:

$$\begin{aligned} 0 &= \beta + \sum_{s=1}^r k_s(\beta, q(\beta)) \gamma_s - \beta' \\ &= \langle C_\beta, (\gamma \beta) \rangle \end{aligned}$$

Recall also that $\mathcal{C}_2$ denotes the set of all discontinuities that never land on other discontinuities, but are eventually periodic. The dynamically important vector related to these discontinuities is the landing vector of such a discontinuity into X . We do not include these vectors in Theorem III.5.6. The reason is that they land into interiors of intervals which we can control with Theorem III.5.6. Thus we can already control the dynamics of these discontinuities with this theorem, without the need to include them into the statement.

In the statement of Theorem III.5.6, there are two types of intervals contained in X . There is a distinguished interval J_0 , which can be any interval contained in X , and there are in intervals $J_1, \dots, J_n$ that are *maximal periodic intervals*. Here is the definition:

Definition III.5.5 A maximal interval $J \subset I$ consisting of periodic points with the same itinerary is called a *maximal periodic interval*. It is clear that J is a half-open interval and that its boundary points have the property that they land on discontinuities of T or on the boundary points of I . Moreover, no point in the interior of P lands on a discontinuity.

To ease the notation, we will not explicitly state the ranges for all indices that appear in the statement, as they are the same as in the discussion above. Moreover, we will abbreviate

the dependence of vectors, numbers and points related to the interval J_i , for $0 \leq i \leq n$, to the dependence on the number i , e.g. we will write $C^{i,+}(j, k)$ instead of $C^{J_i,+}(j, k)$.

Because the definitions of the dynamical vectors associated to the boundary points the interval J_0 from Theorem III.5.6 are different depending on whether they land on discontinuities of T or not, this leads to different statements of Theorem III.5.6, depending on if these landings happen or not. For simplicity, we will assume that these boundary points land on discontinuities of T , because this case is more complicated, and we state Theorem III.5.6 with this assumption. The proof of Theorem III.5.6 in the case when at least one of the boundary points does not land on a discontinuity is analogous. Moreover, we also assume that all of the points $a_j^{0,\pm}$, for $0 \leq j \leq N_0$, land on at least two discontinuities before returning to J_0 . If this is not true for some $a_j^{0,\pm}$, then there are no critical connection vectors associated to this point, leading to a different statement of Theorem III.5.6. As this case is also simpler, we will assume that there are always critical connection vectors associated to every $a_j^{0,\pm}$. All of these vectors are illustrated in Figure III.2, for the same interval J as in Figure III.1.

For the maximal periodic intervals $J_1, \dots, J_n$, the only points that land on discontinuities are the boundary points, so $N_i = 1$ for $1 \leq i \leq n$. For each J_i , we will consider the vector $C^{i,+}(0, m_0^{i,+}) := R_0^{i,+} + L_0^i$, instead of the landing vector L_0^i and the return vector $R_0^{i,+}$. The reason for this is that the vector $C^{i,+}(0, m_0^{i,+})$ is equal to the critical connection vector corresponding to the landing of $\beta^{i,+}(0, m_0^{i,+} - 1)$ to $\beta^{i,+}(0, 1)$. This is the special property of maximal periodic intervals: their return and landing vectors combine into a critical connection vector, which is not true in general. This will be crucially used in the proof of Proposition III.5.14, which is one of the main steps in the proof of Theorem III.5.6. We define $C^{i,-}(1, m_1^{i,-})$ analogously.

Theorem III.5.6 (Coefficients of Linear Dependence) Let T be an ITM. Let $J_0, J_1, J_2, \dots, J_n$ be intervals contained in X that have pairwise disjoint orbits, and let $\mathcal{C}_{-X}$ be the set of all discontinuities in $\mathcal{C}_1$ that are not contained in X . Assume that J_0 is a component interval of X and that $J_1, \dots, J_n$ are maximal periodic intervals. As above, let the following be the vectors related to their dynamics:

- $L_j^0, R_j^{0,+}, R_j^{0,-}, C^{i,+}(j, k), C^{i,-}(j, k);$
 - C_β , for $\beta \in \mathcal{C}_{\neg X}$.
- (*)

Assume that there exist real coefficients:

- $\alpha_j^0, \alpha_j^{0,+}, \alpha_j^{0,-}, \alpha^{i,+}(j, k), \alpha^{i,-}(j, k);$
 - α_β , for $\beta \in \mathcal{C}_{\neg X}$.
- (**)

such that the vectors in (*) satisfy:

$$\begin{aligned}
 & \sum_{i=1}^n \left(\sum_{k=1}^{m_0^{i,+}} \alpha^{i,+}(0, k) C^{i,+}(0, k) \right) + \sum_{i=1}^n \left(\sum_{k=1}^{m_1^{i,-}} \alpha^{i,-}(1, k) C^{i,-}(1, k) \right) \\
 & + \sum_{j=0}^{N_0-1} \left(\sum_{k=1}^{m_j^{0,+}-1} \alpha^{0,+}(j, k) C^{0,+}(j, k) + \alpha_j^{0,+} R_j^{0,+} \right) \\
 & + \sum_{j=1}^{N_0} \left(\sum_{k=1}^{m_j^{0,-}-1} \alpha^{0,-}(j, k) C^{0,-}(j, k) + \alpha_j^{0,-} R_j^{0,-} \right) \\
 & + \left(\sum_{j=0}^{N_0} \alpha_j^0 L_j^0 \right) + \sum_{\beta \in \mathcal{C}_{\neg X}} \alpha_\beta C_\beta = 0.
 \end{aligned}
 \tag{III.5.1}$$

Then it must hold that:

$$\begin{aligned}
 \alpha^{i,+}(0, 1) &= \dots = \alpha^{i,+}(0, m_0^{i,+}) = -\alpha^{i,-}(1, 1) = \dots = -\alpha^{i,-}(1, m_1^{i,-}); \\
 \alpha^{0,+}(j, 1) &= \dots = \alpha^{0,+}(j, m_j^{0,+}-1) = a_j^{0,+} \\
 &= -\alpha^{0,-}(j+1, 1) = \dots = -\alpha^{0,-}(j+1, m_{j+1}^{0,-}-1) = -\alpha_{j+1}^{0,-},
 \end{aligned}
 \tag{III.5.2}$$

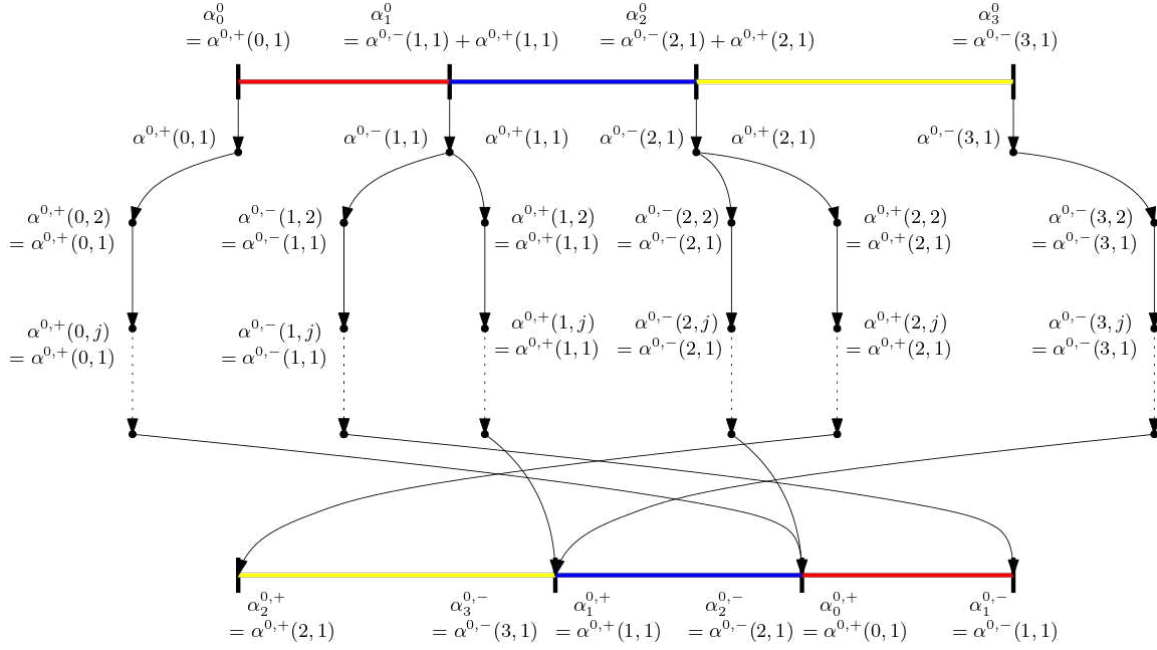


Figure III.3: The coefficients for the landing vectors and critical connection vectors are at the start of their corresponding arrows, while the coefficients for the return vectors are at the end. The indicated equalities between the coefficients come from the conclusion of Theorem III.5.6.

for all $1 \leq i \leq n$ and $0 \leq j \leq N_0 - 1$, respectively. Moreover, we have that:

$$\begin{aligned}
 \alpha_0^0 &= \alpha^{0,+}(0, 1); \\
 \alpha_{N_0}^0 &= \alpha^{0,-}(N_0, 1); \\
 \alpha_j^0 &= \alpha^{0,+}(j, 1) + \alpha^{0,-}(j, 1); \\
 \alpha_\beta &= 0,
 \end{aligned} \tag{III.5.3}$$

for all $1 \leq j \leq N_0 - 1$ and for all $\beta \in \mathcal{C}_{-X}$, respectively.

The conclusion of Theorem III.5.6 for J_0 (which is in fact the same interval as in Figures III.1 and III.2) with three branches of the return map is demonstrated in Figure III.3. The assumption $J_1, \dots, J_n$ are maximal periodic intervals is necessary for our proof to work. A natural question to ask is whether this is necessary.

Question III.5.7 Does Theorem III.5.6 hold if we assume that $J_1, \dots, J_n$ are component intervals of X ?

The proof of this theorem is split into the next three subsections. The important consequence of Theorem III.5.6 is the following corollary:

Corollary III.5.8 (Linear independence of dynamical vectors) Let T be an ITM. Let $J_0, J_1, J_2, \dots, J_n$ be intervals contained in X with pairwise disjoint orbits. Assume that J_0 is a component interval of X and that $J_1, \dots, J_n$ are maximal periodic intervals. Then the vectors:

- $L_j^0, R_j^{0,+}, C^{0,+}(j, k), C^{0,-}(j, k);$
- $C^{i,+}(0, k)$ for $1 \leq k \leq m_0^{i,+}$ and $C^{i,-}(1, k)$ for $1 \leq k < m_0^{i,-};$
- C_β , for $\beta \in \mathcal{C}_{\neg X},$

form a linearly independent set.

Proof. For the sake of contradiction, assume that they are linearly dependent with coefficients:

- $\alpha_j^0, \alpha_j^{0,+}, \alpha^{0,+}(j, k), \alpha^{0,-}(j, k);$
- $\alpha^{i,+}(0, k)$ for $1 \leq k \leq m_0^{i,+}$ and $\alpha^{i,-}(1, k)$ for $1 \leq k < m_0^{i,-};$
- α_β , for $\beta \in \mathcal{C}_{\neg X}.$

This linear dependence can be extended to include the vectors $R_j^{0,-}$ and $C^{i,-}(1, m_0^{i,-})$ by setting:

- $\alpha_j^{0,-} = 0;$
- $C^{i,-}(1, m_0^{i,-}) = 0,$

for all $1 \leq j \leq N_0$ and $1 \leq i \leq n$, respectively. Then by equation (III.5.2) from Theorem III.5.6, we must have that:

$$\begin{aligned} \alpha^{i,+}(0, 1) &= \cdots = \alpha^{i,+}(0, m_0^{i,+}) = -\alpha^{i,-}(1, 1) = \cdots = -\alpha^{i,-}(1, m_1^{i,-}) = 0; \\ \alpha^{0,+}(j, 1) &= \cdots = \alpha^{0,+}(j, m_j^{0,+} - 1) = \alpha_j^{0,+} \\ &= -\alpha^{0,-}(j+1, 1) = \cdots = -\alpha^{0,-}(j+1, m_{j+1}^{0,-} - 1) = -\alpha_{j+1}^{0,-} = 0, \end{aligned}$$

for all $1 \leq i \leq n$ and $0 \leq j \leq N_0 - 1$, respectively. Thus by equation (III.5.3) from Theorem III.5.6, we also have that:

$$\begin{aligned} \alpha_0^0 &= \alpha^{0,+}(0, 1) = 0; \\ \alpha_{N_0}^0 &= \alpha^{0,-}(N_0, 1) = 0; \\ \alpha_j^0 &= \alpha^{0,+}(j, 1) + \alpha^{0,-}(j, 1) = 0 + 0 = 0; \\ \alpha_\beta &= 0, \end{aligned}$$

for all $1 \leq j \leq N_0 - 1$ and for all $\beta \in \mathcal{C}_{-X}$, respectively. Thus all of the coefficients of the linear dependence are equal to zero, which is what we wanted to prove. $\square$

III.5.2 Refining partitions of I

For the purpose of the proof, we will simplify the notation used in the statement of Theorem III.5.6. Denote by $\mathcal{V}$ the set of all vectors from (*) and by $\mathcal{A}$ the set of all coefficients from (**).

A critical connection vector $v \in \mathcal{V}$ corresponds to the orbit of a discontinuity of T up to the time of landing on another discontinuity. For such v , let $P(v)$ be the ordered set of points along the orbit corresponding to v . The landing vectors correspond to the orbit of a discontinuity of the return map R_J up to the time of landing onto discontinuity of T , critical connection vectors corresponding to the orbit of a discontinuity of T up to the landing time to another discontinuity of T , and the return vectors correspond to the orbit of a discontinuity up to the time of landing into J . We adopt the convention that the first point in the orbit corresponding to the vector is included in $P(v)$, while the last one is not. For

example, this means that the $P(v)$ associated to the critical connection vector $C^{0,+}(0, 1)$ is equal to $\{\beta^{0,+}(0, 1), T(\beta^{0,+}(0, 1)), \dots, T^{q^{0,+}(0,2)-1}(\beta^{0,+}(0, 1))\}$. If a discontinuity of T is already contained in J_0 , then the set $P(v)$ for the associated landing vector is empty. We will exclude such vectors from the set $\mathcal{V}$. For a vector $v \in \mathcal{V}$, let us denote by v_{first} the first element of $P(v)$ and by v_{last} the last element of $P(v)$.

Definition III.5.9 Let $\mathcal{V}$ be the set of all vectors as defined above. Then we define the set of all *distinguished points* (with respect to $\mathcal{V}$) as the union of all points in $P(v)$:

$$\mathcal{P} := \bigcup_{v \in \mathcal{V}} P(v).$$

Note that $\mathcal{P}$ is by definition a set that includes both signed (from critical connection and return vectors) and geometric points (from landing vectors) points. For a vector $v \in \mathcal{V}$, we will denote by $\alpha(v)$ its corresponding coefficient. The assumption (III.5.1) of Theorem III.5.6 then reads as:

$$\sum_{v \in \mathcal{V}} \alpha(v)v = 0. \quad (\text{III.5.4})$$

Let v_s be the s -th e -coefficient of v . Equation (III.5.4) implies:

$$\sum_{v \in \mathcal{V}} \alpha(v)v_s = 0, \text{ for all } 1 \leq s \leq r.$$

Note that v_s also equals the number of points from $P(v)$ contained in I_s . This means that we have the initial partition $\mathcal{I}^0$ of I into intervals $I_s^0 := I_s$, for $1 \leq s \leq r$, such that a linear equation on the number of points from each $P(v)$ holds on each interval of this partition. This motivates the following two definitions:

Definition III.5.10 Let I^* be any interval in I . Then we define the number of points from $P(v)$ in I^* as:

$$N(I^*, v) := |P(v) \cap I^*|.$$

Next, we introduce intervals for which the number of points from each $P(v)$ satisfy a linear equation with respect to the coefficients in $\mathcal{A}$:

Definition III.5.11 ($\mathcal{A}$ -subordinate interval) Let I^* be an interval in I . We say that I^* is an $\mathcal{A}$ -subordinate interval if the following equation holds:

$$\sum_{v \in \mathcal{V}} \alpha(v) N(I^*, v) = 0.$$

Let $\mathcal{I}^* = \bigcup_{s=1}^k I_s^*$ be a partition (not necessarily dynamical) of I into intervals I_s^* , i.e. $I = \bigsqcup_{s=1}^k I_s^*$. We say $\mathcal{I}$ is an $\mathcal{A}$ -subordinate partition if every interval I_s^* is an $\mathcal{A}$ -subordinate interval. By the discussion above, we know that $\mathcal{I}^0$ is an $\mathcal{A}$ -subordinate partition. Our goal will be to refine the level-0 partition $\mathcal{I}^0$ into $\mathcal{A}$ -subordinate partitions which consist of more intervals.

More precisely, we will define a sequence of partitions $\mathcal{I}^m = \bigcup_{s=1}^{r+m} I_s^m$ of I into intervals I_s^m so that:

1. $\mathcal{I}^m$ is a refinement of $\mathcal{I}^{m-1}$;
2. $\mathcal{I}^m$ is an $\mathcal{A}$ -subordinate partition.

This procedure will terminate at some maximal level M , described in Definition III.5.16. Our first goal is to find a criterion for when the partition $\mathcal{I}^m$ can be refined into $\mathcal{I}^{m+1}$ by splitting some interval I_s^m of $\mathcal{I}^m$ into two intervals. We will show that this can be done by splitting along the gap interval between a certain pair of two neighbouring, but not touching, distinguished points. Recall that two signed points x and y touch if $\{x, y\} = \{z^+, z^-\}$ for some point z . The following lemma shows how to find such a pair:

Lemma III.5.12 (Distinguished pair) Let $\mathcal{I}^m$ be an $\mathcal{A}$ -subordinate partition. Assume that there exists a triple (I_s^m, p_1, p_2) , where I_s^m is an element of the partition, and p_1, p_2 is a pair of signed and distinct distinguished points in I_s^m , with p_1 to the left of p_2 , such that they are:

- (a) Neighbouring, but not touching, in I_s^m ;
- (b) There exist no $1 \leq j \leq N_0 - 1$ and $0 \leq k < r_j^0$ such that p_1 and p_2 are contained in $T^k(J_j^0)$, and no $i \geq 1$ and $k \geq 0$ such that p_1 and p_2 are contained in $T^k(J_i)$.

Then one can find a triple, still denoted by (I_s^m, p_1, p_2) , that additionally satisfies:

- (c) The gap interval $G = [p_1, p_2]$ between the points has maximum length among all triples satisfying (a) and (b);
- (d) Either at least one of the points p_1, p_2 has no preimage in $\mathcal{P}$ or no pair of preimages $q_1, q_2 \in \mathcal{P}$ of p_1, p_2 is contained in the same element of the partition $\mathcal{I}^m$.

We call a pair of points from the conclusion of Lemma III.5.12 a *distinguished pair*.

Proof. By assumption, we can always choose p_1 and p_2 such that they satisfy (a) and (b), and by choosing the pair with maximal distance, we can assume they also satisfy (c). Assume that this pair does not satisfy assumption (d). We will show that such a pair can be ‘pulled back’ to find a different pair of points that also satisfies (d).

By assumption, the pair of points p_1, p_2 has at least one pair of T -preimages $q_1, q_2 \in \mathcal{P}$ that are contained in a single interval of $\mathcal{I}^m$. Let us show that this pair q_1, q_2 also satisfies properties (a), (b) and (c). Indeed, if they are contained in a single element of the partition, then T maps $[q_1, q_2]$ continuously to G , as $\mathcal{I}^m$ is a refinement of $\mathcal{I}^0$. Thus q_1 and q_2 are not touching and not contained in an iterate of the form $T^k(J_j^0)$ or $T^k(J_i)$, so (b) follows. Moreover, if they are neighbouring, i.e. if there is no point from $\mathcal{P}$ between them, then they also satisfy (a) and (c).

Assume the contrary, that there is some point $q \in \mathcal{P} \cap [q_1, q_2]$ between points q_1 and q_2 . Since this interval maps forward continuously, $T(q) \in G$ is not contained in $\mathcal{P}$. This is only possible if $q = v_{last}$, where v is either a critical connection vector corresponding to landing on a point in $\mathcal{C}_2$, or $q = v_{R,last}$, where v is a return vector. In the first case, $T(q)$ must be a discontinuity, which contradicts p_1 and p_2 being in the same partition element of $\mathcal{I}^m$. In the second case, we know that $T(q) \in J_0$, for some $1 \leq i \leq n$, so the interval (p_1, p_2) must intersect J_0 . We will show that this means that p_1 and p_2 are not neighbouring in $\mathcal{P}$. Indeed, if neither of them is contained in $J_0 = [x^{0,+}, y^{0,-}]$, then the boundary points of J_0 are between them. If only p_1 is contained in J_0 , then $y^{0,-}$ is between them, since it must be contained in (p_1, p_2) . If only p_2 is in J_0 , then $x^{0,+}$ is between them. Finally, if they are both contained in J_0 , then by (b) they are not contained in the same interval J_j^0 , so they are not neighbouring in this case either. Thus the pair q_1, q_2 satisfies (c) as well.

Thus if the points p_1, p_2 satisfy (a), (b) and (c), but not (d), we can always find their pullback to another pair of points q_1, q_2 that also satisfies (a), (b) and (c). If the pair q_1, q_2 does not satisfy (d), then we can pull back this pair as well. If this can be performed indefinitely, then some pair of points p_1, p_2 must appear twice. Then the interval $G = [p_1, p_2]$ must be periodic, as the gap interval between the pullback of points maps continuously over the gap interval between the points themselves. Moreover, no point in $[p_1, p_2]$ lands on a discontinuity. Thus p_1 and p_2 are contained in X and have the same itineraries. This is only possible if they are both contained in a single iterate of the form $T^k(J_j^0)$ or $T^k(J_i)$, as points from different iterates of this kind must have different itineraries. This is a contradiction with (b), so the pullback cannot be performed indefinitely. Thus by pulling back, at some point we arrive at a pair of points that also satisfies (d). $\square$

Let $\mathcal{P}_s^m := I_s^m \cap \mathcal{P}$ and $G^\circ := G \setminus \{p_1, p_2\}$. Define $\frac{p_1+p_2}{2}$ as the average of geometric points in I corresponding to p_1 and p_2 . Note that $\frac{p_1+p_2}{2}$ is the midpoint of G . Let I^l be the interval in I to the left of $\frac{p_1+p_2}{2}$ and let I^r be the interval in I to the right of $\frac{p_1+p_2}{2}$.

Lemma III.5.13 (Left or right) Let $\mathcal{I}^m$ be an $\mathcal{A}$ -subordinate partition as in Lemma III.5.12, and let $G = [p_1, p_2]$ be a gap interval. Then for any element I_s^m of this partition, we have that the set $T(\mathcal{P}_s^m)$ is either fully to the left or fully to the right of $\frac{p_1+p_2}{2}$. More precisely, we have that either $T(\mathcal{P}_s^m) \cap I^l = \emptyset$ or $T(\mathcal{P}_s^m) \cap I^r = \emptyset$.

Proof. Assume the contrary, that there exists an interval I_s^m for which we have that $T(\mathcal{P}_s^m)$ intersects both I^l and I^r . Let $q_1, q_2 \in \mathcal{P}_s^m$ be two neighbouring (not necessarily signed) points in I_s^m such that their images are contained in $\mathcal{P}$, and q_1 gets mapped to I^l and q_2 gets mapped to I^r . Assume first that the image $T(q_1)$ is contained in G° , with the case for $T(q_2)$ being analogous. Then $T(q_1)$ is not a point in $\mathcal{P}$, since p_1 and p_2 are neighbouring. This is only possible if $q_1 = v_{last}$, where v is either a critical connection vector corresponding to landing on a point in $\mathcal{C}_2$ or a return vector. The second case was already shown to be impossible in the proof of Lemma III.5.12, while the first case is impossible since p_1 and p_2 are by assumption contained in a single interval of the partition $\mathcal{I}^m$. Thus, neither of the points $T(q_1), T(q_2)$ is contained in G° . As this pair gets mapped forward continuously, this means that the image of the gap between these intervals is at least as large as G . By property (d) of Lemma III.5.12, it is impossible that q_1 and q_2 get mapped over p_1 and p_2 ,

respectively. If q_1 and q_2 are both signed, then this is a contradiction with maximality of G . We may without loss of generality assume that q_1 is not signed, with the other case being analogous. This means that $q_1 \in P(v)$ for a landing vector v , so it is contained in the left boundary of an iterate of the form $T^k(J_j^0)$. As there is no other points in $\mathcal{P}$ contained in the interior $T^k(J_j^0)$, this means that p_1 and p_2 must both be contained in $T^{k+1}(J_j^0)$, which is a contradiction. $\square$

We now show that we can get an $\mathcal{A}$ -subordinate partition of level $m + 1$ if we split the interval $I_{s'}^m$ containing the pair p_1, p_2 along the gap interval G .

Proposition III.5.14 (Refining a partition) Let $\mathcal{I}^m$ be an $\mathcal{A}$ -subordinate partition of level m and assume that there exists at least one interval $I_{s'}^m$ of the partition as in Lemma III.5.12. Then the partition $\mathcal{I}^{m+1}$ obtained from $\mathcal{I}^m$ by splitting $I_{s'}^m$ along $\frac{p_1+p_2}{2}$ is an $\mathcal{A}$ -subordinate partition of level $m + 1$.

Proof. Let $I_{s'}^{m,l}$ be the maximal (half-open) interval to the left of $\frac{p_1+p_2}{2}$ inside $I_{s'}^m$, and let $I_{s'}^{m,r}$ be the maximal (half-open) interval to the right of $\frac{p_1+p_2}{2}$ inside $I_{s'}^m$. Then the intervals of the partition $\mathcal{I}^{m+1} = \bigcup_{1 \leq s \leq r+m+1} I_s^{m+1}$ are defined as:

$$I_s^{m+1} := \begin{cases} I_s^m, & \text{For } s < s' \\ I_{s'}^{m,l}, & \text{For } s = s' \\ I_{s'}^{m,r}, & \text{For } s = s' + 1 \\ I_{s-1}^m, & \text{For } s > s' + 1. \end{cases}$$

By construction, $\mathcal{I}^{m+1}$ is a refinement of $\mathcal{I}^0$. As only one of the intervals was modified, we know that:

$$\begin{aligned} \sum_{v \in \mathcal{V}} \alpha(v) N(I_s^{m+1}, v) &= \sum_{v \in \mathcal{V}} \alpha(v) N(I_s^m, v) = 0 \text{ for } s < s'; \\ \sum_{v \in \mathcal{V}} \alpha(v) N(I_s^{m+1}, v) &= \sum_{v \in \mathcal{V}} \alpha(v) N(I_{s-1}^m, v) = 0 \text{ for } s > s' + 1. \end{aligned} \tag{III.5.5}$$

Thus, in order to show that $\mathcal{I}^{m+1}$ is an $\mathcal{A}$ -subordinate partition, we only need to prove:

$$\begin{aligned}
\sum_{v \in \mathcal{V}} \alpha(v) N(I_{s'}^{m+1}, v) &= 0 \\
\sum_{v \in \mathcal{V}} \alpha(v) N(I_{s'+1}^{m+1}, v) &= 0.
\end{aligned} \tag{III.5.6}$$

Note that it is clearly enough to only prove one of the above equations holds, as the other follows from:

$$\begin{aligned}
&\sum_{v \in \mathcal{V}} \alpha(v) N(I_{s'}^{m+1}, v) + \sum_{v \in \mathcal{V}} \alpha(v) N(I_{s'+1}^{m+1}, v) \\
&= \sum_{v \in \mathcal{V}} \alpha(v) N(I_{s'}^m, v) = 0.
\end{aligned}$$

Because of this, we may without loss of generality assume that the interval J_0 is contained in I^r , because we can otherwise just do the computation that follows for I^r instead of I^l . This assumption will be important when we compute Δ_v (defined in (III.5.7)) for landing and return vectors. Let $\mathcal{P}^l := \mathcal{P} \cap I^l$ be the set of all distinguished points to the left of the midpoint of G . Let Ind^l be the set of all $s \in \{1, \dots, m+r\}$ such that $T(\mathcal{P}_s^m) \subset I^l$. Note that this definition makes sense because of Lemma III.5.13. Let $\mathcal{P}^{l,-1} := \bigcup_{s \in \text{Ind}^l} \mathcal{P}_s^m = T^{-1}(\mathcal{P}^l) \cap \mathcal{P}$ be the set of all distinguished points that T maps into $\mathcal{P}^l$. Consider the following equation:

$$\Delta_v := \sum_{s \in \text{Ind}^l} N(I_s^m, v) - \sum_{s \leq s'} N(I_s^{m+1}, v). \tag{III.5.7}$$

This equation states that the number of elements of $P(v)$ contained in I^l equals the number of elements of $P(v)$ that get mapped under T into I^l up to an error term Δ_v . From (III.5.7), we get:

$$\sum_{v \in \mathcal{V}} \alpha(v) N(I_{s'}^{m+1}, v)$$

$$\begin{aligned}
&= \sum_{v \in \mathcal{V}} \alpha(v) \left(\sum_{s \in \text{Ind}_L} N(I_s^m, v) - \sum_{s < s'} N(I_s^{m+1}, v) - \Delta_v \right) \\
&= \sum_{s \in \text{Ind}_L} \left(\sum_{v \in \mathcal{V}} \alpha(v) N(I_s^m, v) \right) \\
&\quad - \sum_{s < s'} \left(\sum_{v \in \mathcal{V}} \alpha(v) N(I_s^{m+1}, v) \right) - \sum_{v \in \mathcal{V}} \alpha(v) \Delta_v \\
&= - \sum_{v \in \mathcal{V}} \alpha(v) \Delta_v.
\end{aligned}$$

where the sum in the third line is zero because of the inductive assumption that each interval I_s^m is $\mathcal{A}$ -subordinate, and the sum in the fourth line is zero because of equation (III.5.5). Thus we only need to prove that:

$$\sum_{v \in \mathcal{V}} \alpha(v) \Delta_v = 0. \quad (\text{III.5.8})$$

We now compute the error term Δ_v for all vectors $v \in \mathcal{V}$. For this purpose, we define the following sets:

$$\begin{aligned}
P(v)^{l,-1} &:= P(v) \cap \left(\bigcup_{s \in \text{Ind}^l} I_s^m \right) \\
P(v)^l &:= P(v) \cap \left(\bigcup_{s \leq s'} I_s^m \right).
\end{aligned}$$

Note that the following holds:

$$\begin{aligned}
|P(v)^{l,-1}| &= \sum_{s \in \text{Ind}^l} N(I_s^m, v) \\
|P(v)^l| &= \sum_{s \leq s'} N(I_s^{m+1}, v).
\end{aligned}$$

We first compute Δ_v for a critical connection vector $v \in \mathcal{V}$. Note that this includes the vectors in $\mathcal{C}_{-X}$. For any point p in $P(v)^l$ that is not equal to v_{first} (we do not assume it is contained in $P(v)^l$), we have that there exists exactly one point q in $P(v)^{l,-1}$ such that $T(q) = p$. Indeed, this is because each element of $P(v)$ except the first one has a preimage in $P(v)$. Moreover, for each point q in $P(v)^{l,-1}$ that is not equal to v_{last} , we have that $T(q) \in P(v)^l$. Thus we get the following formula for Δ_v for a critical connection vector $v \in \mathcal{V}$:

$$\Delta_v = \begin{cases} 0, & \text{If } v_{first} \in \mathcal{P}^l \text{ and } v_{last} \in \mathcal{P}^{l,-1} \\ -1, & \text{If } v_{first} \in \mathcal{P}^l \text{ and } v_{last} \notin \mathcal{P}^{l,-1} \\ 1, & \text{If } v_{first} \notin \mathcal{P}^l \text{ and } v_{last} \in \mathcal{P}^{l,-1} \\ 0, & \text{If } v_{first} \notin \mathcal{P}^l \text{ and } v_{last} \notin \mathcal{P}^{l,-1}. \end{cases} \quad (\text{III.5.9})$$

Now we compute Δ_v for a return vector $v \in \mathcal{V}$. For every point q in $P(v)^l$ that is not equal to v_{first} , we again have that there again exists exactly one point q in $P(v)^{l,-1}$ such that $T(q) = p$. Moreover, for each point q in $P(v)^{l,-1}$ that is not equal to v_{last} , we have that $T(q) \in P(v)^l$. Recall that we have assumed that J_0 is contained in I^r . This means that $T(v_{last}) \notin \mathcal{P}^l$, and so $v_{last} \notin \mathcal{P}^{l,-1}$. Thus Δ_v in this case only depends on whether $v_{first} \in \mathcal{P}^l$, so we get the following formula:

$$\Delta_v = \begin{cases} -1, & \text{If } v_{first} \in \mathcal{P}^l \\ 0, & \text{If } v_{first} \notin \mathcal{P}^l. \end{cases} \quad (\text{III.5.10})$$

Analogously, for a landing vector v , we know that $v_{first} \notin \mathcal{P}^l$, so the value of Δ_v depends only on whether $v_{last} \in \mathcal{P}^{l,-1}$:

$$\Delta_v = \begin{cases} 1, & \text{If } v_{last} \in \mathcal{P}^{l,-1} \\ 0, & \text{If } v_{last} \notin \mathcal{P}^{l,-1}. \end{cases} \quad (\text{III.5.11})$$

Let w_β , for a (signed or geometric) discontinuity $\beta \in \mathcal{P}$, be the unique vector in $\mathcal{V}$ such that

$w_{first} = \beta$, and let $w_\beta = 0, \alpha(w_\beta) = 0$ for all $\beta \notin \mathcal{P}$. We claim that the sum in (III.5.8) is equal to:

$$\sum_{\beta \in I^l} \left(-\alpha(w_{\beta^+}) - \alpha(w_{\beta^-}) + \sum_{\substack{w \in \mathcal{V}, \\ T(w_{last}) \in \{\beta, \beta^+, \beta^-\}}} \alpha(w) \right). \quad (\text{III.5.12})$$

Here, we sum over all geometric discontinuities $\beta \in I^l$. Note for every $v \in \mathcal{V}$, the coefficient $\alpha(v)$ is equal to at most one $\alpha(w_{\beta^\pm})$ in (III.5.12). Such a $\alpha(w_\beta)$ exists if and only if v_{first} is a signed discontinuity in I^l . Moreover, $\alpha(v)$ is also equal to at most one $\alpha(w)$ in the inner sum of (III.5.12), and such $\alpha(w)$ exists if and only if $T(v_{last})$ is a (signed or geometric) discontinuity in I^l . We now prove that the following two claims hold:

1. $\Delta_v = -1$ in (III.5.8) if and only if $\alpha(v)$ is equal to some $\alpha(w_\beta)$ and not equal to any $\alpha(w)$ in (III.5.12);
2. $\Delta_v = 1$ in (III.5.8) if and only if $\alpha(v)$ is not equal to any $\alpha(w_\beta)$ and equal to some $\alpha(w)$ in (III.5.12).

The equality of (III.5.8) and (III.5.12) clearly follows from these two claims, because then the contribution of $\alpha(v)$ to equations (III.5.8) and (III.5.12) is equal for all vectors $v \in \mathcal{V}$. We prove the first claim, while the proof of the second one is analogous.

We know that if $\Delta_v = -1$, then $v_{first} \in \mathcal{P}^l$ and $v_{last} \notin \mathcal{P}^{l,-1}$. In particular, v is not a landing vector, by (III.5.10). Thus v_{first} is a signed discontinuity in I^l , while $T(v_{last})$ is not a discontinuity in I^l , so $\alpha(v)$ is equal to some $\alpha(w_\beta)$ and not equal to any $\alpha(w)$. Conversely if $\alpha(v)$ is equal to some $\alpha(w_\beta)$ and not equal any $\alpha(w)$ then the v_{first} is a signed discontinuity in I^l , so $v_{first} \in \mathcal{P}^l$, and $T(v_{last})$ is not a discontinuity in I^l , so $v_{last} \notin \mathcal{P}^{l,-1}$. Thus $\Delta_v = -1$, so the claim follows.

We now show that the sum in (III.5.12) is equal to zero, which finishes the proof. Indeed, we know that the $\mathbf{f}$ -coefficient at $\text{ind}(\beta)$ for some $\beta \in I^l$ for the resulting vector in (III.5.4) has to be zero. The only vectors in this sum for which this $\mathbf{f}$ -coefficient is non-zero are those vectors v for which $v_{first} = \beta^\pm$ or those for which $T(v_{last}) \in \{\beta, \beta^+, \beta^-\}$. Thus taking the $\mathbf{f}$ -coefficient at $\text{ind}(\beta)$ in equation (III.5.4), we get that:

$$\left(-\alpha(w_{\beta^+}) - \alpha(w_{\beta^-}) + \sum_{\substack{w \in \mathcal{V}, \\ T(w_{last}) \in \{\beta, \beta^+, \beta^-\}}} \alpha(w) \right) = 0$$

As we split along an interval G° that does not contain a discontinuity in its interior, we know that $\beta^+ \in I^l$ if and only if $\beta^- \in I^l$. Thus we must have that:

$$\sum_{\beta \in I^l} \left(-\alpha(w_{\beta^+}) - \alpha(w_{\beta^-}) + \sum_{\substack{w \in \mathcal{V}, \\ T(w_{last}) \in \{\beta, \beta^+, \beta^-\}}} \alpha(w) \right) = 0,$$

which finishes the proof. $\square$

Let us now show that Proposition III.5.14 can be applied only finitely many times.

Lemma III.5.15 We can apply Proposition III.5.14 finitely many times if we start with the initial partition $\mathcal{I}^0$ given by the continuity intervals.

Proof. We will show that applying Proposition III.5.14 always increases the number of elements of the partition that have a non-empty intersection with $\mathcal{P}$. Since $\mathcal{P}$ is a finite set, this clearly shows that Proposition III.5.14 can only be applied finitely many times. By construction, we can apply Proposition III.5.14 to a partition $\mathcal{I}^m$ if and only if we find a pair of distinguished points p_1, p_2 as in Lemma III.5.12. These points are contained in the same element $I_{s'}^m$ of the partition, so after splitting the partition along the gap interval G between them, we get two intervals $I_{s'}^{m+1}$ and $I_{s'}^{m+1}$ that contains p_1 and p_2 , respectively. Thus we have increased the number of elements of the partition that have a non-empty intersection with $\mathcal{P}$, so the proof follows. $\square$

Thus this refinement procedure must stop at some maximal level:

Definition III.5.16 (Maximal partition) Let M be the level at which the procedure of inductively applying Proposition III.5.14 stops. We call the partition $\mathcal{I}^M$ obtained by this procedure the *maximal partition*.

It is of maximal level in the sense that for the partition $\mathcal{I}^M$ of this level, there is no pair of points satisfying (a) and (b) from Lemma III.5.12. Recall that $\mathcal{C}_{-X}$ is the set of all discontinuities in $\mathcal{C}_1$ not contained in X . The maximality of $\mathcal{I}^M$ gives the following corollary:

Corollary III.5.17 All of the coefficient $\alpha(v_\beta)$, for $\beta \in \mathcal{C}_{-X}$, are equal to zero.

Proof. We will prove this for a $+$ -type discontinuity β , with the proof for a $-$ -type discontinuity being analogous. Let $\mathcal{I}^M$ be the maximal partition as in Definition III.5.16. Assume that there is another point $p \in \mathcal{P}$ contained in the element of the partition I_s^M containing β . As β is a discontinuity, it must be the leftmost point I_s^M , as $\mathcal{I}^M$ is a refinement of $\mathcal{I}^0$. Thus we may assume that β and p are neighbouring, but not touching. As $\beta \notin X$, we know that this pair of points is not contained in a periodic interval. Thus this pair of points satisfies properties (a) and (b) from Lemma III.5.12, so the partition $\mathcal{I}^M$ can in fact be refined, which is a contradiction with its maximality. Thus there is no other distinguished point in I_s^M . As $\mathcal{I}^M$ is an $\mathcal{A}$ -subordinate partition, we must have that $\alpha(v_\beta) = 0$, as this is exactly the equation which I_s^M needs to satisfy to be $\mathcal{A}$ -subordinate. $\square$

As all of these coefficients are zero, they do not contribute to the equation (III.5.4). Thus we may exclude their corresponding vectors and points from $\mathcal{V}$ and $\mathcal{P}$, respectively. This means that all points in $\mathcal{P}$ are now contained in X . This immediately gives the following corollary, which proves a part of Theorem III.5.6 for the landing vectors corresponding to points in the interior of J_0 :

Corollary III.5.18 We have that $\alpha_j^0 = \alpha^{0,+}(j, 1) + \alpha^{0,-}(j, 1)$ for all $1 \leq j \leq N_0 - 1$ in conclusion (III.5.3) of Theorem III.5.6.

Proof. By Corollary III.5.17, we know that for all $1 \leq j \leq N_0 - 1$, there exist only three vectors $v \in \mathcal{V}$ such that the $\mathbf{f}$ -coefficient at $\text{ind}(\beta^0(j))$ is non-zero. These are exactly L_j^0 , $C^{0,+}(j, 1)$ and $C^{0,-}(j, 1)$. As the $\mathbf{f}$ -coefficients at $\text{ind}(\beta^0(j))$ in equation (III.5.4) have to cancel out, we get that:

$$\alpha_j^0 = \alpha^{0,+}(j, 1) + \alpha^{0,-}(j, 1)$$

for all $1 \leq j \leq N_0 - 1$. $\square$

We may now assume that the points and vectors associated to the landing vectors L_j^0 , for all $1 \leq j \leq N_0 - 1$, are not contained in $\mathcal{P}$ and $\mathcal{V}$, respectively. Thus $\mathcal{P}$ now contains only signed points. We can make this assumption because we will be able to deduce the remaining equalities from the conclusions (III.5.2) and (III.5.3) of Theorem III.5.6 without using these landing vectors.

The maximality of $\mathcal{I}^M$ also gives the following corollary:

Corollary III.5.19 All maximal intervals contained in X that do not contain discontinuities in their interiors are $\mathcal{A}$ -subordinate intervals.

Proof. We claim that every element I_s^M of the partition, $I_s^M \cap X$ contains at most one interval that contains points in $\mathcal{P}$. Indeed, if there were at least two such intervals in this intersection, then we could choose a pair of points from two such neighbouring intervals as the pair of points from Lemma III.5.12, which would mean we can further refine $\mathcal{I}^M$, and this is a contradiction with maximality of $\mathcal{I}_s^M$. Since $\mathcal{I}^M$ is a refinement of $\mathcal{I}^0$ and since in Proposition III.5.14 we always split along a gap interval G not containing any points in $\mathcal{P}$ in its interior, a maximal interval H contained in X that does not contain discontinuities in its interior and has non-empty intersection with $\mathcal{P}$ is contained in a single element I_s^M of the partition $\mathcal{I}^M$. By Corollary III.5.17, all of the points in $\mathcal{P} \cap I_s^M$ are contained in a single such interval H . Since I_s^M is an $\mathcal{A}$ -subordinate interval, this gives that H is also an $\mathcal{A}$ -subordinate interval. Since intervals contained in X that do not contain points in $\mathcal{P}$ are trivially $\mathcal{A}$ -subordinate, the proof follows. $\square$

III.5.3 Refining partitions into $\mathcal{A}$ -subordinate intervals

Our next goal is to partition the intervals of X into smaller $\mathcal{A}$ -subordinate intervals. These smaller intervals can then be further partitioned, and we will show that we may continue with this procedure until all intervals of the form $T^k(J_j^0)$, for $1 \leq j \leq N_0 - 1$ and $l_j^0 \leq k < r_j^0$, and $T^k(J^i)$, for $1 \leq i \leq n$ and $k \geq 0$, are elements of this partition. In order to do this, we will show how a partition of an interval into smaller $\mathcal{A}$ -subordinate intervals may be further refined by ‘pushing forward’ the partition of the preimage of this interval.

Let $H = [a, b) \neq J_0$ be a component interval of X . Assume that H satisfies the following two properties:

1. There is a finite number of points $a = h_0 < h_1 < \dots < h_k < h_{k+1} = b$ such that the intervals $H_0 = [h_0, h_1), \dots, H_k = [h_k, h_{k+1})$ are $\mathcal{A}$ -subordinate, and such the intervals (h_q^+, h_{q+1}^-) contain no discontinuities for all $0 \leq q \leq k$;
2. The preimage of H in X can be partitioned into $\mathcal{A}$ -subordinate intervals $G_0 = [g_0, g_1), G_1 = [g_2, g_3), \dots, G_l = [g_{2l}, g_{2l+1})$, such that the intervals (g_{2r}^+, g_{2r+1}^-) contain no discontinuities for all $0 \leq r \leq l$. We allow for $g_{2r+1} = g_{2r+2}$, i.e. the intervals could touch at their boundary points.

We will refer to these intervals as H -intervals and G -intervals, respectively. We distinguish between the following two types of points in H :

- $h_0^+, h_1^\pm, \dots, h_{k+1}^-$ will be called *Type-1 points*;
- $T(g_0^+), T(g_1^-), \dots, T(g_{2l+1}^-)$ will be called *Type-2 points*.

We will refer to any point that is either a Type-1 or a Type-2 point as *special*. Note that a point can be both a Type-1 and a Type-2 point. For example, this is the case for boundary points a^+ and b^- .

Let $H'_0 = [h'_0, h'_1), H'_1 = [h'_1, h'_2), \dots, H'_m = [h'_m, h'_{m+1})$ be the maximal intervals in H , such that the intervals (h'_t, h'_{t+1}) contain no special points for all $0 \leq t \leq m$. In other words, the boundary of each H'_t consists of two neighboring, but not touching, special points. We will refer to these intervals as H' -intervals. These intervals form a partition of H , which is a refinement of the partition into $H_0, H_1, \dots, H_k$.

Recall that because of Corollary III.5.17 all of the points in $\mathcal{P}$ are contained in X . Since $H \neq J_0$, every point $p \in \mathcal{P} \cap H$ not equal to some $\beta^{0,\pm}(j, 1)$, for $1 \leq j \leq N_0 - 1$, has a unique preimage in $X \cap \mathcal{P}$. Moreover, p is contained in the same $P(v)$ as its unique preimage in $X \cap \mathcal{P}$ if and only if p is not a discontinuity of T . This fact will be repeatedly used in the proof below. It will also be convenient to introduce the following notation: for a $\beta \in X$ that is not equal to some $\beta^{0,\pm}(j, 1)$, for $1 \leq j \leq N_0 - 1$, denote by v_β^{-1} the unique vector in $\mathcal{V}$ such that $T(v_{last}) = \beta$. Note that this vector is now unique because of Corollary III.5.17. For a discontinuity $\beta^{0,\pm}(j, 1)$, with $1 \leq j \leq N_0 - 1$, we define $v_{\beta^{0,\pm}(j,1)}^{-1} = 0$. Recall also that v_β is the unique vector in $\mathcal{V}$ such that $v_{first} = \beta$.

Proposition III.5.20 (Push-forward of a partition) Let H satisfy the two assumptions in III.5.3. Then the intervals $H'_0, \dots, H'_m$ are also $\mathcal{A}$ -subordinate intervals. Moreover, for every discontinuity contained in H that is also a Type-2 point, we have that:

$$-\alpha(v_\beta^{-1}) + \alpha(v_\beta) = 0.$$

Proof. The proof goes by induction, starting with the leftmost interval H'_0 and going to the right. The induction has two hypotheses:

- H'_z is an $\mathcal{A}$ -subordinate interval for all $z \leq t$;
- For each discontinuity β of T contained in these intervals that is also a Type-2 point, we have that:

$$-\alpha(v_\beta^{-1}) + \alpha(v_\beta) = 0.$$

We now prove the base of the induction for $t = 0$. Note that by assumption, none of the H -intervals contain discontinuities in their interior. The proof depends on the type of the boundary point h'_1 of H'_0 . There are 3 different cases.

Case 1: h'_1 is only a Type-1 point

Then $H'_0 = H_0$, so the first hypothesis follows. Assume first that a is not a discontinuity of T . In this case, even if h'_1 is a discontinuity of T , it is not of Type-2, so the second hypothesis follows. If $a^+ = \beta^+$, then since H is assumed to be a component of X , we know that $\beta^- \notin X$. Thus the only vectors in $\mathcal{V}$ for which the $\mathbf{f}$ -coefficient at $\text{ind}(\beta^+)$ is non-zero are v_{β^+} and $v_{\beta^+}^{-1}$. Thus we have that:

$$-\alpha(v_{\beta^+}^{-1}) + \alpha(v_{\beta^+}) = 0,$$

so the second hypothesis follows in this case as well.

Case 2: h'_1 is only a Type-2 point

Note that this means that h'_1 is not a discontinuity of T , as discontinuities are Type-1 points by the assumption that H -intervals do not contain discontinuities in their interiors. Moreover, we have that $H'_0 = T(G_r)$, for some $0 \leq r \leq l$. Assume that a is not a discontinuity of T . Then the second hypothesis follows, as there are no discontinuities in

H'_0 . Moreover, all of the points in $H'_0 \cap \mathcal{P}$ belong to the same $P(v)$ as their preimages in G_r and thus $N(H'_0, v) = N(G_r, v)$ for all $v \in \mathcal{V}$. Thus H'_0 is an $\mathcal{A}$ -subordinate interval, because G_r is. Same as in the previous case, if $a^+ = \beta^+$, then:

$$\alpha(v_{\beta^+}^{-1}) - \alpha(v_{\beta^+}) = 0,$$

so the second hypothesis follows.

Case 3: h'_1 is both a Type-1 and a Type-2 point

This means that $H'_0 = H_0 = T(G_r)$, for some $0 \leq r \leq l$, so the first hypothesis follows. Assume that a is not a discontinuity of T . If the right boundary point h'_1 of H'_0 is not a discontinuity of T , the second hypothesis trivially follows. If $h_1'^- = \beta_*^-$ is a discontinuity of T , then we have that $N(H'_0, v) = N(G_r, v)$ for all $v \in \mathcal{V}$, except for v_{β_*} and $v_{\beta_*}^{-1}$. Thus we have that:

$$0 = \sum_{v \in \mathcal{V}} \alpha(v)N(G_r, v) - \sum_{v \in \mathcal{V}} \alpha(v)N(H'_0, v) = \alpha(v_{\beta_*}^{-1}) - \alpha(v_{\beta_*}),$$

so the second hypothesis follows. If we assume $a^+ = \beta^+$, then as in Case 1, we get that:

$$\alpha(v_{\beta^+}^{-1}) - \alpha(v_{\beta^+}) = 0.$$

If h'_1 is not a discontinuity of T , the second hypothesis follows, so we may assume $h_1'^- = \beta_*^-$. Then we have that:

$$0 = \sum_{v \in \mathcal{V}} \alpha(v)N(G_r, v) - \sum_{v \in \mathcal{V}} \alpha(v)N(H'_0, v) = \alpha(v_{\beta^+}^{-1}) - \alpha(v_{\beta^+}) + \alpha(v_{\beta_*}^{-1}) - \alpha(v_{\beta_*}),$$

which implies:

$$\alpha(v_{\beta_*}^{-1}) - \alpha(v_{\beta_*}) = 0,$$

so the second hypothesis follows.

Assuming the two induction hypothesis, we now want to prove that $H'_{t+1} = [h'_{t+1}, h'_{t+2})$ is also an $\mathcal{A}$ -subordinate interval, for some $t \geq 0$. The proof now depends on the type of both h'_{t+1} and h'_{t+2} , so there are 4 cases.

Case 1: Both are only Type-1 points

Then $H'_{t+1} = H_q$ for some q , so we are done, because in this case even if the boundary points are discontinuities, they are only of Type-1, so we need not check the second inductive hypothesis.

Case 2: Both are only Type-2 points

Then $H'_{t+1} = T(G_r)$ for some r , and neither of the boundary points is a discontinuity of T . Thus the first inductive hypothesis follows from the fact that $N(H'_0, v) = N(G_r, v)$ for all $v \in \mathcal{V}$, and the second because there are no discontinuities in H'_{t+1} .

Case 3: h'_{t+1}^+ is a Type-1 point and h'_{t+2}^- is a Type-2 point

Then H'_{t+1} is contained in the image of the interval G_r for which $T(g_{2r+1}^-) = h'_{t+2}^-$. Let $H'_u, \dots, H'_t$ be all of the other H' -intervals contained in $T(G_r)$ and note that they form a partition of $T(G_r)$. Thus we have that:

$$\sum_{z=u}^{t+1} N(H'_z, v) = N(G_r, v)$$

for all v except those for which either $v_{first} = \beta$ or $T(v_{last}) = \beta$, where β is a discontinuity contained in $T(G_r)$. Thus we have that:

$$\begin{aligned} \sum_{v \in \mathcal{V}} \alpha(v) N(H'_{t+1}, v) &= \sum_{v \in \mathcal{V}} \alpha(v) N(G_r, v) - \sum_{z=u}^t \left(\sum_{v \in \mathcal{V}} \alpha(v) N(H'_z, v) \right) \\ &\quad + \sum_{\beta \in T(G_r)} \left(-\alpha(v_\beta) + \alpha(v_\beta^{-1}) \right) \\ &= \sum_{\beta \in T(G_r)} \left(-\alpha(v_\beta) + \alpha(v_\beta^{-1}) \right). \end{aligned} \tag{III.5.13}$$

Here we sum over all signed discontinuities $\beta \in T(G_r)$. We now need to consider all of the discontinuities contained in $T(G_r)$. The analysis is different for discontinuities on the boundary and those in the interior. For a discontinuity β contained in the interior of $T(G_r)$, we know that:

$$-\alpha(v_{\beta+}) + \alpha(v_{\beta+}^{-1}) - \alpha(v_{\beta-}) + \alpha(v_{\beta-}^{-1}) = 0, \quad (\text{III.5.14})$$

by the assumption of Theorem III.5.6 and by Corollary III.5.17. If the left boundary point h_{t+1}^+ of $T(G_r)$ is a discontinuity β_*^+ , then we know that:

$$-\alpha(v_{\beta_*-}) + \alpha(v_{\beta_*-}^{-1}) = 0, \quad (\text{III.5.15})$$

by the second inductive hypothesis. If the right endpoint h_{t+2}^+ of $T(G_r)$ is a discontinuity β_{**}^+ , then it is a Type-1 point as well, so we know that H'_{t+1} is an H -interval by the first assumption about the H -partition, and thus an $\mathcal{A}$ -subordinate interval. Putting equations (III.5.13), (III.5.14) and (III.5.15) together gives:

$$\begin{aligned} 0 &= \sum_{v \in \mathcal{V}} \alpha(v) N(H'_{t+1}, v) \\ &\stackrel{(\text{III.5.13})}{=} \sum_{\beta \in T(G_r)} \left(-\alpha(v_{\beta}) + \alpha(v_{\beta}^{-1}) \right) \\ &\stackrel{(\text{III.5.14}) \& (\text{III.5.15})}{=} -\alpha(v_{\beta_*+}) + \alpha(v_{\beta_*+}^{-1}), \end{aligned}$$

This proves the second inductive hypothesis, so the inductive step is done in this case. Thus we may assume that h_{t+2}^+ is not a discontinuity. Then by equations (III.5.13), (III.5.14) and (III.5.15) we have that:

$$\begin{aligned} 0 &\stackrel{(\text{III.5.14}) \& (\text{III.5.15})}{=} \sum_{\beta \in T(G_r)} \left(-\alpha(v_{\beta}) + \alpha(v_{\beta}^{-1}) \right) \\ &\stackrel{(\text{III.5.13})}{=} \sum_{v \in \mathcal{V}} \alpha(v) N(H'_{t+1}, v), \end{aligned}$$

so H'_{t+1} is an $\mathcal{A}$ -subordinate interval in this case as well. Finally, if the left boundary point h_{t+1}^+ of $T(G_r)$ is not a discontinuity, then by equations (III.5.13) and (III.5.14) we have

that:

$$\begin{aligned} 0 &\stackrel{\text{III.5.14}}{=} \sum_{\beta \in T(G_r)} \left(-\alpha(v_\beta) + \alpha(v_\beta^{-1}) \right) \\ &\stackrel{\text{III.5.13}}{=} \sum_{v \in \mathcal{V}} \alpha(v) N(H'_{t+1}, v), \end{aligned}$$

which finishes the proof.

Case 4: h'_{t+1} is a Type-2 point and h'_{t+2} is a Type-1 point

Then similarly as in the previous case, we know that H'_{t+1} is contained in a larger H -interval H_q , which is partitioned into H' -intervals $H'_u, \dots, H'_t$, which are $\mathcal{A}$ -subordinate by the inductive hypothesis, and H'_{t+1} . Similarly as in the previous case, we have that:

$$\begin{aligned} \sum_{v \in \mathcal{V}} \alpha(v) N(H'_{t+1}, v) &= \sum_{v \in \mathcal{V}} \alpha(v) N(H_q, v) - \sum_{z=u}^t \left(\sum_{v \in \mathcal{V}} \alpha(v) N(H'_z, v) \right) \\ &\quad + \sum_{\beta \in H_q} \left(-\alpha(v_\beta) + \alpha(v_\beta^{-1}) \right) \\ &= \sum_{\beta \in H_q} \left(-\alpha(v_\beta) + \alpha(v_\beta^{-1}) \right). \end{aligned}$$

From here we simply need to consider whether the boundary points of H_q are discontinuities and the proof follows analogously as in the previous case. $\square$

It is interesting to note that the analogous statement for the ‘pull back’ of a partition does not hold. A simple counterexample occurs when the interval J_0 does not contain a discontinuity, but its image $T(J_0)$ contains a single discontinuity β in its interior. In this case $T(J_0)$ is partitioned into two $\mathcal{A}$ -subordinate intervals $T(J_1)$ and $T(J_2)$, where $T(J_1)$ and $T(J_2)$ touch at β . Thus if we consider $H = T(J_0)$, the pull-back version of Proposition III.5.20 would give us that J_1 and J_2 are $\mathcal{A}$ -subordinate. This would mean that $\alpha_j^0 = \alpha^{0,+}(j, 1)$ and $\alpha^{0,-}(j, 1) = 0$, where a_j is the discontinuity in $J - 0$ that lands on β . By Theorem III.5.6, we know that this does not necessarily hold, so in general we can not ‘pull back’ a partition.

III.5.4 Proof of Theorem III.5.6

In this subsection, we finish the proof of Theorem III.5.6. We do this by continuing with the process of refining partitions started in Subsection III.5.2. This will be done by refining the maximal partition $\mathcal{I}^M$ from Definition III.5.16. This refinement is obtained by further partitioning the maximal intervals in X that do not contain discontinuities in their interiors into smaller intervals.

As we will now care only about the partitions of the component intervals of X , we introduce a new label for such partitions. We will say that a partition $\mathcal{I}_*$ of the components of X into $\mathcal{A}$ -subordinate intervals is an $\mathcal{AX}$ -partition. Such a partition is defined by a finite set of points that are contained in the interiors of intervals of X , which we call *the break points* of $\mathcal{I}_*$. By Corollary III.5.19, we know that the partition of X into maximal intervals that do not contain discontinuities in their interiors is an $\mathcal{AX}$ -partition. As we do not need to consider $\mathcal{I}^M$ as a partition of I anymore, we will denote this $\mathcal{AX}$ -partition by $\mathcal{I}^M$ as well. Thus the set of break points $\mathcal{B}_0$ of $\mathcal{I}^M$ is the union of the boundary points of all maximal intervals contained in X that do not contain a discontinuity in their interiors. This set is clearly the union of all discontinuities of T and all the boundary points of X . We will show how to inductively refine this partition by applying Proposition III.5.20.

More precisely, we will show by induction that for each $m \geq 0$, there is an $\mathcal{AX}$ -partition $\mathcal{I}^{M+m}$ such that its set of break points $\mathcal{B}_m$ is equal to the union of the following two types of points:

1. Points $z \in X$ such that there exists a $q \leq m$ and a discontinuity $\beta \in X$ such that $z = T^q(\beta)$ and $T^r(\beta) \notin J_0$ for all $0 < r \leq q$;
2. Points $z \in X$ such that there exists a $q \leq m$ and a boundary point x of X such that $z = T^q(x)$, and $T^r(x) \notin J_0$ for all $0 < r \leq q$.

These two properties clearly imply that $\mathcal{B}_{m+1} \supset \mathcal{B}_m$, so $\mathcal{I}^{M+m+1}$ is a refinement of $\mathcal{I}^{M+m}$.

Proof of Theorem III.5.6. The base of induction is given by Lemma III.5.12 and Proposition III.5.14. Assuming we have a level m partition $\mathcal{I}^{M+m}$ such that its set of break points $\mathcal{B}_m$ is equal to the union of the two types of points described above, we show how to refine

it into a level $m + 1$ partition $\mathcal{I}^{M+m+1}$. We describe this refinement in terms of the new break points that get added to $\mathcal{B}_m$ by applying Proposition III.5.20.

Let H be any component of X not equal to J_0 . By the inductive assumption, $\mathcal{I}^{M+m}$ induces a partition of H into $\mathcal{A}$ -subordinate intervals. We may take these $\mathcal{A}$ -subordinate intervals to be the H -intervals from Proposition III.5.20. As H is a component interval of X , we know that its T -preimage in X consists of maximal intervals that do not contain discontinuities in their interiors, which are $\mathcal{A}$ -subordinate by Corollary III.5.19. As $\mathcal{I}^{M+m}$ is a refinement of $\mathcal{I}^M$, these preimage intervals are further partitioned into $\mathcal{A}$ -subordinate intervals, and we may take these intervals to be the G -intervals from Proposition III.5.20. Thus by Proposition III.5.20 we know that the partition into H -intervals can be refined into a partition into $\mathcal{A}$ -subordinate H' -intervals. The set of boundary points of these H' -intervals is equal to the union of the images of break points in $\mathcal{B}_m$ contained in the G -intervals and the break points in $\mathcal{B}_m \cap H$. This holds for any $H \neq J_0$, and thus for each break point $p \in \mathcal{B}_m$, we get that $T(p)$ is a break point of $\mathcal{B}_{m+1}$, provided $T(p) \notin J_0$. Thus every point of the first and second type described above is contained in $\mathcal{B}_{m+1}$. Thus the partition $\mathcal{I}^{M+m+1}$ obtained by applying Proposition III.5.20 has the desired properties, which completes the induction step.

Thus for every sufficiently large m every point of either of these two types is a break point of $\mathcal{B}_m$:

1. Points $z \in X$ such that there exists a q and a discontinuity $\beta \in X$ such that $z = T^q(\beta)$ and $T^r(\beta) \notin J_0$ for all $0 < r \leq q$;
2. Points $z \in X$ such that there exists a q and a boundary point x of X such that $z = T^q(x)$ and $T^r(x) \notin J_0$ for all $0 < r \leq q$;

Let $\beta^{0,+}(j, k)$, for $0 \leq j \leq N_0 - 1$ and $k < m_j^{0,+}$, be a discontinuity in the orbit of J_0 , and let $P(v)$ be the orbit associate to the vector v such that $v_{first} = \beta^{0,+}(j, k)$. Note that because of $k < m_j^{0,+}$, we know that $T(v_{last}) \in \mathcal{P}$. For m large enough, we therefore know that v_{last} is a break point of $\mathcal{B}_m$. Thus if we apply Proposition III.5.20 to the component interval H of X containing $T(v_{last})$, we get that:

$$\alpha^{0,+}(j, k) = \alpha^{0,+}(j, k + 1), \quad (\text{III.5.16})$$

if $k + 1 < m_j^{0,+}$, or

$$\alpha^{0,+}(j, m_j^{0,+} - 1) = \alpha^{0,+}(j, m_j^{0,+}), \quad (\text{III.5.17})$$

if $k + 1 = m_j^{0,+}$. This is because $T(v_{last}) = \beta^{0,+}(j, k + 1)$ is a Type-2 point. It is clear that an analogous argument applies to discontinuities of the form $\beta^{0,-}(j, k)$, for $1 \leq j \leq N_0$ and $k < m_j^{0,+}$, $\beta^{i,+}(0, k)$, for $1 \leq i \leq n$ and $k \leq m_0^{i,+}$, and $\beta^{i,-}(1, k)$, for $1 \leq i \leq n$ and $k \leq m_1^{i,-}$. Thus combining equations (III.5.16) and (III.5.17) for all of these discontinuities gives:

$$\begin{aligned} \alpha^{i,+}(0, 1) &= \dots = \alpha^{i,+}(0, m_0^{i,+}) \\ \alpha^{i,-}(1, 1) &= \dots = \alpha^{i,-}(1, m_1^{i,-}); \\ \alpha^{0,+}(j, 1) &= \dots = \alpha^{0,+}(j, m_j^{0,+} - 1) = a_j^{0,+}; \\ \alpha^{0,-}(j + 1, 1) &= \dots = \alpha^{0,-}(j + 1, m_{j+1}^{0,-} - 1) = \alpha_{j+1}^{0,-}; \\ \alpha_0^0 &= \alpha^{0,+}(0, 1); \\ \alpha_{N_0}^0 &= \alpha^{0,-}(N_0, 1), \end{aligned} \quad (\text{III.5.18})$$

for all $1 \leq i \leq n$ and $0 \leq j \leq N_0 - 1$, respectively.

Next, for every $0 \leq j \leq N_0$, we know that both $a_j^{0,+}$ and $a_{j+1}^{0,-}$ have to land on discontinuities before the interval $[a_j^{0,+}, a_{j+1}^{0,-}]$ return to J_0 . Thus there is a time q , such that the points $T^q(a_j^{0,+})$ and $T^q(a_{j+1}^{0,-})$ are both break points of $\mathcal{B}_m$, for a sufficiently large m . Thus the interval $[T^q(a_j^{0,+}), T^q(a_{j+1}^{0,-})]$ is $\mathcal{A}$ -subordinate, so we get that

$$\alpha(v) = -\alpha(w), \quad (\text{III.5.19})$$

where v and w are the vectors such that $T^q(a_j^{0,+}) \in P(v)$ and $T^q(a_{j+1}^{0,-}) \in P(w)$, respectively. Moreover, v is either a critical connection vector or a return vector related to the orbit of $a_j^{0,+}$, and w is either a critical connection vector or a return vector related to the orbit of $a_{j+1}^{0,-}$. Again, an analogous argument applies to discontinuities of the form $\beta^{0,-}(j, k)$, for $1 \leq j \leq N_0$ and $k < m_j^{0,+}$, $\beta^{i,+}(0, k)$, for $1 \leq i \leq n$ and $k \leq m_0^{i,+}$, and $\beta^{i,-}(1, k)$, for $1 \leq i \leq n$ and $k \leq m_1^{i,-}$. Thus by combining (III.5.18) and (III.5.19) for all of these discontinuities we get both of the conclusions (III.5.2) and (III.5.3) from the statement of

Theorem III.5.6. □

III.6 Stability is equivalent to ACC and Matching

III.6.1 Definitions and examples

Let us recall the definition of a stable ITM from the introduction:

Definition III.6.1 (Stable maps) We say that a map T is *stable* if there a neighbourhood $\mathcal{U}$ of T in $\text{ITM}(r)$ such that:

1. The mapping $\overline{X}$ assigning to each map $\tilde{T}$ in $\mathcal{U}$ the corresponding set $\overline{X}(\tilde{T})$ is continuous with respect to the Hausdorff topology on compact subsets of $[0, 1]$. Moreover, for each $\tilde{T} \in \mathcal{U}$, $\overline{X}(\tilde{T})$ is homeomorphic to $\overline{X}(T)$.
2. The number of discontinuities in $I \setminus \overline{X}(\tilde{T})$ is constant in $\mathcal{U}$.

A corollary of Theorem III.4.2 is that stable maps are of finite type:

Corollary III.6.2 Stable maps are of finite type.

Proof. By Theorem III.4.2 we know that eventually periodic maps are dense in $\text{ITM}(r)$. Thus they are also dense in the neighbourhood $\mathcal{U}$ of a stable map T on which the definition of stability III.6.1 holds. This means that there is at least one map in $\mathcal{U}$ for which the non-wandering set is equal to a finite union of intervals since eventually periodic maps are of finite type by Corollary III.4.5. Thus an infinite map can not be contained in $\mathcal{U}$, since the non-wandering set of an infinite type map must contain a Cantor set by Lemma III.3.2, which is not homeomorphic to a finite union of intervals. □

Note that this definition is softer than the usual notions of stability in 1-dimensional dynamics, like Ω -stability (see [MS93]) and J -stability (see [McM94]). The difference is that we do not require any dynamical conjugacy between the maps in the neighbourhood $\mathcal{U}$, just that several properties remain the same: finite type, number of discontinuities and the shape of the non-wandering set.

Recall that we use the shorthand $\tilde{X}$ for the non-wandering set of a perturbation $\tilde{T}$ of T . The next two lemmas follow immediately from the fact that the non-wandering set $\tilde{X}$ of a sufficiently small perturbation $\tilde{T}$ of T is close to and homeomorphic to the non-wandering set X of T .

Lemma III.6.3 For a sufficiently small perturbation $\tilde{T}$ of a stable map T , we have that $\overline{X}(\tilde{T})$ consists of the same number of intervals as X and these intervals are pairwise close to each other in sense of Hausdorff. This means that each component J of X has a well-defined continuation $\tilde{J} \subset \tilde{X}$, and in particular that its boundary points x and y also have continuations $\tilde{x}$ and $\tilde{y}$. Moreover, the continuation $\tilde{J}$ of an interval J contains the continuations of the discontinuities contained in J . $\square$

Lemma III.6.4 Each sufficiently small perturbation $\tilde{T}$ of a stable map T has the same number of discontinuities in each of the following sets:

1. Interior of the non-wandering set: $\text{int}(X)$.
2. Interior in I of the complement of the non-wandering set: $I \setminus \overline{X}$.
3. Boundary of the non-wandering set: ∂X .

Moreover, each continuation $\tilde{J}$ of an interval J of X contains the continuations of discontinuities in J . $\square$

Let us now give a non-trivial example of a stable map, shown in figure III.6.1 below.

Example III.6.5 (Stable map) Assume that $r = 3$ and that the interval $J = [T^2(\beta_2^+), T(\beta_2^-)]$ contains β_2 in its interior. Let $J^- = [T^2(\beta_2^+), \beta_2^-]$ and $J^+ = [\beta_2^+, T(\beta_2^-)]$. Assume the following about the orbits of β_2^+ and β_2^- :

- $T(\beta_2^+) \in I_1$ and $T^2(\beta_2^+) \in I_2$;
- $T(\beta_2^-) \in I_3$ and $T^2(\beta_2^-) \in I_1$;

Then:

- $T^3(\beta_2^+) = \beta_2^+ + \gamma_3 + \gamma_1 + \gamma_2$;

- $T^3(\beta_2^-) = \beta_2^- + \gamma_2 + \gamma_3 + \gamma_1,$

so $T^3(\beta_2^+) \sim T^3(\beta_2^-)$. If we additionally assume that $T^3(\beta_2^+), T^3(\beta_2^-) \in J$, then, for a suitable choice of parameters as in the figure below, T is stable and we have that $X(T)$ is equal to the orbit of J . Indeed, the itineraries of β_2^+ and β_2^- remain the same for all nearby maps $\tilde{T}$, which means that the interval $\tilde{J} = [\tilde{T}^2(\tilde{\beta}_2^+), \tilde{T}(\tilde{\beta}_2^-)]$ is a continuation of J .

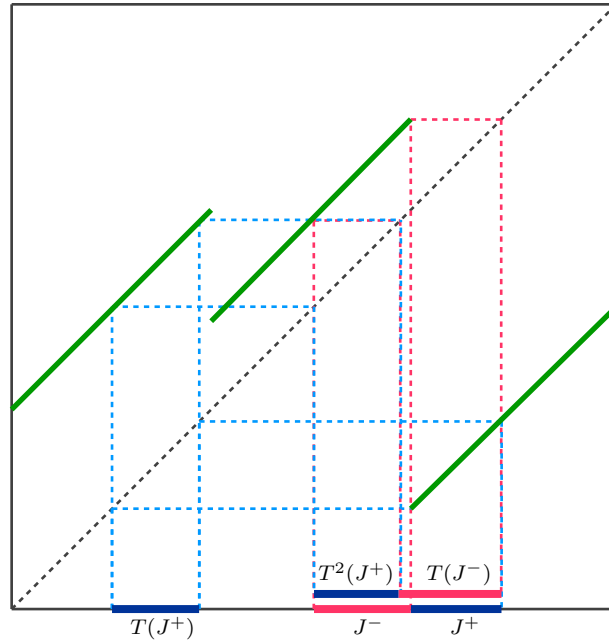


Figure III.4: A particular map as in Example III.6.5 with $\beta_0 = 0$, $\beta_1 = 1/3$, $\beta_2 = 2/3$, $\beta_3 = 1$, $\gamma_1 = 1/3$, $\gamma_2 = 1/7$, $\gamma_3 = -1/2$. The intervals J^- and $T(J^-)$ are shown in pink and the intervals J^+ , $T(J^+)$, $T^2(J^+)$ are drawn in blue.

The definition of stability mainly concerns the topology and the location of the non-wandering set. Thus the dynamical meaning of stability is unclear from the definition, and the dynamical information retained by making a small perturbation of a stable map is relatively soft. We would therefore like to characterize stability in terms of more concrete

dynamical properties. Intuitively speaking, three things can happen for an arbitrarily small perturbation $\tilde{T}$ of a map T that can break stability:

1. A definite part of X gets removed, i.e. $\tilde{X}$ is smaller (either by size or by the number of intervals) than X by a definite amount, so the mapping $\bar{X}$ is not lower semi-continuous.
2. A definite new part gets added to X , i.e. $\tilde{X}$ is bigger (either by size or by the number of intervals) than X by a definite amount, so the mapping $\bar{X}$ is not upper semi-continuous.
3. Discontinuities that were in X are not in $\tilde{X}$.

We are therefore looking for dynamical properties that guarantee that none of these three things can happen for an arbitrarily small perturbation of T . To guarantee that the first thing does not happen, we can ask for the dynamics of the full orbit of every interval J of X to remain the same after perturbation (as in the example above). For the third, we can simply require that there are no discontinuities in the boundary of X (assuming the first and the second things do not happen). The second one is the most elusive, and we will need to discuss it a bit before giving the definitions of the required dynamical properties.

The following example illustrates the simplest way in which the set X can become larger.

Example III.6.6 Assume that there are two discontinuities β_* and β_{**} for which there exist $k_1 > 0$ and $k_2 > 0$ such that:

$$T^{k_1}(\beta_*^+) = \beta_{**}^+ \text{ and } T^{k_2}(\beta_{**}^-) = \beta_*^-.$$

Moreover, assume that none of the discontinuities $\beta_*^+, \beta_*^-, \beta_{**}^+$ and β_{**}^- are in X and that there exists a perturbation $\tilde{T}$ of T such that:

- The itinerary of β_*^+ remains the same up to time k_1 and at time k_1 it lands ϵ -distance to the left of β_{**}^- ;

- The itinerary of β_{**}^- remains the same up to time k_2 and at time k_2 it lands ϵ -distance to the right of β_*^+ .

Note that there exists an interval of points to the right of β_*^+ all of which have the same itinerary as β_*^+ up to time k_1 . The same holds for β_{**}^- and its itinerary up to time k_2 . In particular, these intervals of definite size map forward continuously up to times k_1 and k_2 , respectively. By our assumption on the perturbation $\tilde{T}$, for a sufficiently small ϵ the interval $[\beta_*^+, \beta_*^+ + \epsilon)$ maps forward continuously up to time $k_1 + k_2$. Thus it is periodic with period $k_1 + k_2$ and is thus contained in $\tilde{X}$. As $\beta_*^+, \beta_*^-, \beta_{**}^+$ and β_{**}^- were by assumption not in X and therefore a definite distance away from it, we see that $\overline{X}(\tilde{T})$ and $\overline{X}(T)$ cannot be close. Thus X can not be a stable map.

This example is the simplest case of a more general phenomenon called *ghost preimages*. We give the definition for a $+$ -type discontinuity, with the one for $-$ -type being analogous.

Definition III.6.7 (Ghost preimage) Let β_*^+ a discontinuity of T . A discontinuity β_{**}^- that lands on β_*^+ is called a *ghost preimage* of β_*^+ .

Thus a ghost preimage of a $+$ -type discontinuity β_*^+ is a $-$ -type discontinuity β_{**}^- that lands on β_*^+ . The definition can easily be extended to a $-$ -type discontinuity. The motivation for the name is the following: β_{**}^- is *almost* a preimage of β_*^+ , i.e. by an arbitrarily small perturbation we can make the iterate $T^k(\beta_{**}^-)$ land to the right of β_*^+ , thus creating an actual preimage of β_*^+ .

In Example III.6.6 we have that β_*^+ is a ghost preimage of β_{**}^- and that β_{**}^- is a ghost preimage of β_*^+ as well. This is the property that allows for the enlargement of X , and it motivates the following definition:

Definition III.6.8 (Ghost tree) Let β be a discontinuity of T . The *ghost tree* $\mathcal{GT}(\beta)$ of β is defined inductively in the following way. Set β to be the root of the tree, i.e. the set of level 0 vertices of the tree. Assume that we have defined all of the vertices of level $\leq n$ and all of the edges between them. Then the level $n + 1$ vertices of the tree correspond to the set of all ghost preimages (if such exist) of the discontinuities corresponding to level n vertices of the tree. The new edges are those between discontinuities and their ghost preimages, i.e.

a new direct edge $\beta_* \rightarrow \beta_{**}$ is added if and only if β_* is a level $n + 1$ vertex, β_{**} is a level n vertex and β_* is a ghost preimage of β_{**} .

Thus $\mathcal{GT}(\beta)$ is a directed tree, with well-defined levels, and edges exist only between vertices of consecutive levels. If $\beta_* \rightarrow \beta_{**}$, then we say that β_{**} is a *parent* of β_* and that β_* is a *child* of β_{**} . The types of discontinuities alternate between consecutive levels and the tree can have finitely or infinitely many levels. The following simple lemma gives a criterion for when a ghost tree is infinite:

Lemma III.6.9 (Infinite ghost trees) The ghost tree $\mathcal{GT}(\beta)$ of some discontinuity β is infinite if β appears as a vertex in $\mathcal{GT}(\beta)$ at some level $n > 0$.

Note that this is exactly the case in Example III.6.6.

Proof. If β appears again as a vertex on some level $n > 0$, then all of the levels that appeared before the second appearance of β have to repeat. There is at least one level between these appearances, as the discontinuity type has to change from level to level. Thus the tree must be infinite. $\square$

We are now ready to state the two dynamical properties that characterize stability. All of our statements work for all components J of X , except in the special case when no point in the interior of J lands on a discontinuity and the return map R_J to this component is the identity. We will call such components of X *dynamically trivial*, and components not of this form *dynamically non-trivial*.

Definition III.6.10 (Absence of Critical Connections (ACC)) We say that a finite type T satisfies the ACC condition if the following three conditions hold:

1. (A1) For every component J of X and each point $a \in J$, we have that the orbit of a up to and including the return time to J contains at most one critical point of T ;
2. (A2) For every dynamically non-trivial component J of X , we have that none of the boundary points of J land on discontinuities up to and not including the return time to J ;
3. (A3) For every discontinuity $\beta \notin X$, its ghost tree $\mathcal{GT}(\beta)$ does not contain β .

The reason for the name is that if any of these properties is violated, then the defining parameters (γ, β) of T satisfy a non-trivial linear equation which we call a *critical connection*. Note that these are particular critical connections that we want to avoid and that we do not need to avoid all of them. An easy consequence of A1 and A2 is that there are no discontinuities in the boundary of X . Indeed, the preimage in X such a discontinuity is either a discontinuity, which violates A1, or a boundary point of a dynamically non-trivial interval of X , which violates A2.

Note that if A1 and A2 hold, then every vertex of the ghost tree $\mathcal{GT}(\beta)$ for a $\beta \notin X$ corresponds to a discontinuity that is not contained in X . Indeed, by the above, we have that there are no discontinuities in the boundary. Thus if β_2^- is a ghost preimage of β_1^+ , and β_2^- is contained in X , this means that β_1^+ is also contained in X . Thus every ghost preimage of a $\beta \notin X$ must also not be in X . Thus the claim follows by induction.

Definition III.6.11 (Matching) We say that a finite type T satisfies the *Matching* condition if for every dynamically non-trivial interval J of X , we have that exactly one point $a \in J$ lands on a critical point before returning to J , and this point a is in the interior of J .

The reason we call this property Matching is that if a is the single point from above, then we must have that:

1. $J = [R_J(a^+), R_J(a^-)]$;
2. $R_J^2(a^+) \sim R_J^2(a^-)$,

and thus the second iterates under R_J of a^+ and a^- must ‘match’. An analogous property also called Matching has been discussed in slightly different contexts (see [Bru+19], [BCK17], [NN08]). Interestingly, the authors of [DSS25] were unaware of this at the time of defining Matching in the context of that paper.

In the case when no point in the interior of J lands on a discontinuity, we know that J is a maximal periodic interval and thus both boundary points of J land on discontinuities. If A1 and A2 hold for T , then the boundary points of such intervals only land on discontinuities and are not actually themselves discontinuities of T , since there are no discontinuities in the boundary of X , by the discussion below Definition III.6.10.

With the definitions out of the way, we can now recall the statement of Theorem B:

Theorem III.6.12 (Characterization of Stability) A finite-type interval translation map T is stable if and only if it satisfies the ACC and Matching properties.

The proof of Theorem III.6.12 is spread over the next three subsections. The outline of the proof is as follows. For the ‘only if’ direction, we will show that a stable map T that already satisfies A1 and A2 must also satisfy Matching. We will then show that any nearby map must also satisfy A1 and A2, which shows that any stable map must satisfy A1 and A2. If a stable map does not satisfy A3, we will be able to produce an explicit perturbation of the map that increases the non-wandering set by a definite size, which gives a contradiction. Putting all of this together shows that stability implies ACC and Matching. For the other direction, we will show that the ACC and Matching properties are very rigid, and that therefore any map close to such a map must also satisfy these properties, giving stability.

III.6.2 Stability, A1 and A2 imply Matching

Recall that $k_s(x)$ is the number of entries of the orbit of x to the interval I_s up to, but not including, time k :

$$k_s(x) := \#\{T^j(x) \in I_s \text{ for } 0 \leq j < n\}.$$

With this definition, we have the following explicit formula for any iterate of x :

$$T^k(x) = x + \sum_{s=1}^r k_s(x) \gamma_s.$$

We now recall the notation for the dynamics of a return map R_J to J , from the beginning of Section 4. The notation is slightly changed, so that it better fits the notation of this section. Let $J = [x, y)$ be a dynamically non-trivial interval of X and let $a_1, a_2, \dots, a_{N-1}$ be the points in J which land on discontinuities before returning to J and let $l_1, l_2, \dots, l_N$ be the corresponding landing times on critical points. More precisely, let $l_i \geq 0$ be the smallest l_i such that $T^{l_i}(a_i) = \beta_{J,i} \in \mathcal{C}$. By the A1 assumption, l_i is unique. Let $J_1 = [x, a_1^-)$, $J_2 = [a_1^+, a_2^-)$, $\dots$, $J_N = [a_{N-1}^+, y)$, and let $k_1, k_2, \dots, k_N$ be the return times of these intervals, i.e. $R_J(J_i) = T^{k_i}(J_i)$. Note that we must have $l_i < k_i$ and $l_i < k_{i+1}$. We will also use the labels $a_0^+ := x$ and $a_{N+1}^- = y$, but these points do not land on discontinuities by A2.

Let us note that we consider all a 's in J which land on critical points. For simplicity, we choose to use the term ‘discontinuity of R_J ’ as a shorthand for ‘the points in J which land on discontinuities of T before returning to J ’, even though the return map might not actually be discontinuous at these points. Likewise, we will refer to the intervals $J_1, \dots, J_N$ as the ‘continuity intervals of R_J ’.

It is convenient to associate to each J a permutation σ_J which corresponds to the order on J in which the subintervals return to J . Thus if $d = 2$ and the order in which the intervals return is: $R_J(J_3) R_J(J_2) R_J(J_1)$, then $\sigma_J = (321)$.

Let $\tau_J = (i_1, i_2, \dots, i_N)$ be the inverse of σ_J , so that the order of the images under the return map is $R_J(J_{i_1}) R_J(J_{i_2}) \dots R_J(J_{i_N})$. Then the following equations must be satisfied:

$$\begin{aligned} R_J(a_{i_1}^-) &\sim R_J(a_{i_2-1}^+) \\ R_J(a_{i_2}^-) &\sim R_J(a_{i_3-1}^+) \\ &\dots \\ R_J(a_{i_N-1}^-) &\sim R_J(a_{i_N-1}^+). \end{aligned}$$

These equations are central to the stability of the interval J , i.e. to the property that J always has a continuation $\tilde{J}$ for any sufficiently small perturbation. To show this, we need to analyse what happens when we perturb a map T that is stable and satisfies A1 and A2. By $\tilde{T}$ we will denote a perturbation of T . As before, by $\tilde{Z}$ we will denote the continuation of any object of interest Z , e.g. $\tilde{\beta}^+$, $\tilde{X}$, $\tilde{T}$ and similar.

The main thing that properties A1 and A2 allow is control over finite time itineraries for any sufficiently small perturbation. To make this precise, consider any point $x \in I$ that does not land on a critical point before time n . Let $s(x, i)$, for $0 < i < n$, be index such that the iterate $T^i(x)$ is contained in $I_{s(x, i)}$. Associated to each iterate $T^i(x)$ we therefore have the following two vectors:

$$v^{i,+} := \left(\sum_{1 \leq s \leq r} i_s(x) e_s, -f_{s(x, i)-1} \right);$$

$$v^{i,-} := \left(\sum_{1 \leq s \leq r} i_s(x) e_s, -f_{s(x,i)} \right).$$

Thus we have the following formula for the distance between an iterate of x and the critical points in $I_{s(x,i)}$:

$$\begin{aligned} T^i(x) - \beta_{s(x,i)-1}^+ &= x + \langle v^{i,+}, (\gamma \beta) \rangle; \\ \beta_{s(x,i)}^- - T^i(x) &= -\langle v^{i,+}, (\gamma \beta) \rangle - x. \end{aligned}$$

By the assumption on time n , all of the quantities above are positive for $0 < i < n$. Thus they remain positive for all sufficiently small perturbation $\tilde{T}$ of T , because they clearly depend continuously on $(\gamma \beta)$. This means that the itinerary of x up to time n remains the same for all sufficiently small perturbations. Note that this in fact also holds for a small interval of definite size around this point. This follows immediately from the fact that if some orbit up to time n of some point x does not land on a discontinuity, then the same holds true for an interval of definite size (depending on n) around x .

We would like to have the same conclusions for itineraries of critical points. For this we have to slightly change the argument because the points themselves change under perturbation. Assume that some discontinuity β does not land on a discontinuity before time n . Then analogously as above we define $s(\beta, i)$, for $0 < i < n$, to be index such that the iterate $T^i(\beta)$ is contained in $I_{s(\beta,i)}$. Again, we have the following vectors:

$$\begin{aligned} v^{i,+}(\beta) &:= \left(\sum_{1 \leq s \leq r} i_s(\beta) e_s, f_{\text{ind}(\beta)} - f_{s(x,i)-1} \right); \\ v^{i,-}(\beta) &:= \left(\sum_{1 \leq s \leq r} i_s(\beta) e_s, f_{\text{ind}(\beta)} - f_{s(x,i)} \right), \end{aligned}$$

and the following formula for the distance from the critical points in $I_{s(x,i)}$:

$$\begin{aligned} T^i(\beta) - \beta_{s(x,i)-1}^+ &= \langle v^{i,+}, (\gamma \beta) \rangle; \\ \beta_{s(x,i)}^- - T^i(\beta) &= -\langle v^{i,+}, (\gamma \beta) \rangle. \end{aligned}$$

Thus by the same argument as before, we have that the itinerary of β up to time n does not change for all sufficiently small perturbations.

We can analogously do this for the backward itinerary of any point x or discontinuity β , so we get that they also persist under sufficiently small perturbations. Moreover, we can use ACC2 in the same way to prove that the itinerary of every boundary point of X also remains the same after a sufficiently small perturbation and that this also holds on a neighbourhood of definite size around these points.

We will now put together these observations to prove that the structure of the entire return map R_J to any interval J of X remains stable for a sufficiently small perturbation. More precisely, we will prove that the number of points that land on discontinuities remains the same, that the images of $J_1, \dots, J_N$ under R_J remain in the same order and that their return times remain the same. We first give a lemma that eliminates some annoying cases in the proof of Proposition III.6.14.

Lemma III.6.13 Let T be a stable map that also satisfies property A2. Then we must have that have that $R_J(J_1) \neq J_1$ and $R_J(J_N) \neq J_N$, for any dynamically non-trivial interval J of X . In other words, the return map cannot fix the boundary sub-intervals.

Proof. We give the proof for J_1 , with the one for J_N being analogous. Assume the opposite, that $R_J(J_1) = J_1$. Then we have that $R_J(x) = x$, i.e. x is periodic. Since $x = T^m(\beta^+)$, where $\beta \in X$ is a discontinuity of the starting map T , and $T|_X$ is a bijection, by pulling back with $(T|_X)^{-1}$, we see that x must land on β^+ before it returns to J . This is a contradiction with A2. $\square$

Proposition III.6.14 (Stability of the return map) Let T be a stable map that also satisfies properties A1 and A2. Let $\tilde{T}$ be a sufficiently small perturbation of T . Let $J = [x, y)$, where $x = T^{m_1}(\beta_{J, \tau(1)-1}^+)$ and $y = T^{m_2}(\beta_{J, \tau(N)}^-)$, be a dynamically non-trivial interval of X and let $\tilde{J}$ be its continuation. Then the following properties hold:

1. $\tilde{J} = [\tilde{T}^{m_1}(\tilde{\beta}_{J,\tau(1)-1}^+), \tilde{T}^{m_2}(\tilde{\beta}_{J,\tau(N)}^-)]$;
2. The only points in $\tilde{J}$ that land on critical points before they return are $\tilde{a}_1, \dots, \tilde{a}_{N-1}$, where $\tilde{a}_i := \tilde{T}^{-l_i}(\tilde{\beta}_{J,i})$ for $1 \leq i \leq N-1$;
3. Each $\tilde{a}_i^+$, for $0 \leq i \leq N-1$, and $\tilde{a}_i^-$, for $1 \leq N$, has the same itinerary up to time k_i as before the perturbation;
4. The return time of each interval $[\tilde{a}_i^+, \tilde{a}_{i+1}^-]$ is k_{i+1} (where we use the convention that $\tilde{a}_0^+ = \tilde{x}$ and $\tilde{a}_{d+1}^- = \tilde{y}$);
5. The order in $\tilde{J}$ in which the intervals return is given by the same permutation $\tau = (j_1, j_2, \dots, j_{d+1})$ as for J .

Proof. We know that the itinerary of $\tilde{a}_i^\pm$ contained in the interior of $\tilde{J}$ remains unchanged until time k_i , when it lands back into $\tilde{J}$. This comes from the stability of the backward itinerary of $\beta_{J,i}$ until time l_i and the stability of the forward orbit of $\beta_{J,i}$ up to time k_i . That the same holds for $\tilde{a}_0^+$ and $\tilde{a}_N^-$ follows from the stability of the forward orbit of the boundary points of X . Thus the third property follows.

For the second property, assume the contrary: there exists some other point $z \in \tilde{J}$ that lands on a discontinuity before returning to $\tilde{J}$. Let $\tilde{J}_i = [\tilde{a}_{i-1}^+, \tilde{a}_i^+)$ be the interval that contains z . In the case when both $\tilde{a}_{i-1}^+$ and $\tilde{a}_i^+$ return to $\tilde{J}$ at time k_i , we know that z must land on a discontinuity before time k_i . This is clearly impossible, as the itineraries of $\tilde{a}_{i-1}^+$ and $\tilde{a}_i^+$ must therefore be different, which contradicts the third property. Thus we know that $i = \tau(1)$ or $i = \tau(N)$. We may assume the former, with the proof in the other case being analogous. This means that $\tilde{a}_{i-1}^+$ and z land outside of $\tilde{J}$ at time k_i . The entire orbit of $\tilde{J}$ must be contained in $\tilde{X}$, so in particular the interval $[T^{k_i}(\tilde{a}_{i-1}^+), \tilde{a}_0^+]$ must also be contained in X . This contradicts the maximality of $\tilde{J}$, so the second property follows.

In fact, the proof of the second property also implies the fourth. Indeed, if we assume that some interval $[\tilde{a}_{i-1}^+, \tilde{a}_i^+)$ does not return to $\tilde{J}$ at time k_i , we get a contradiction with the maximality of $\tilde{J}$. The fifth property follows from this as well.

The first property follows from Lemma III.6.13: $\tilde{a}_0^+$ must be the image under $\tilde{R}_J$ of some $\tilde{a}_i^+$ with $i > 0$. This must be the same i as for T . The proof for $\tilde{a}_N^-$ is analogous, so the first property follows.

□

Let us state as a corollary the fact that this lemma immediately implies that the equations for the return map still hold after perturbation:

Corollary III.6.15 Let T and J be as in Proposition III.6.14. Then for a sufficiently small perturbation of T , all of the following equations still hold:

$$\begin{aligned}\tilde{R}_J(\tilde{a}_{i_1}^-) &\sim \tilde{R}_J(\tilde{a}_{i_2-1}^+) \\ \tilde{R}_J(\tilde{a}_{i_2}^-) &\sim \tilde{R}_J(\tilde{a}_{i_3-1}^+) \\ &\dots \\ \tilde{R}_J(\tilde{a}_{i_{N-1}}^-) &\sim \tilde{R}_J(\tilde{a}_{i_N-1}^+).\end{aligned}$$

With this proposition, we are ready to prove that any dynamically non-trivial interval J of X contains exactly one discontinuity.

Proposition III.6.16 Let T be a stable map that satisfies the ACC condition and let J be a dynamically non-trivial interval of X . Then the return map R_J to J contains exactly one discontinuity.

Proof. Assume first that the return map has 3 or more discontinuities. Then there exists an equation:

$$\tilde{R}_J(\tilde{a}_{i_j}^-) \sim \tilde{R}_J(\tilde{a}_{i_{j+1}-1}^+)$$

such that neither $\tilde{a}_{i_j}^-$ nor $\tilde{a}_{i_{j+1}-1}^+$ are boundary points of $\tilde{J}$. Thus we have that :

$$\tilde{a}_{i_j}^- + \sum_{s=1}^r (k_{i_j})_s (a_{i_j}^-) \tilde{\gamma}_s = \tilde{a}_{i_{j+1}-1}^+ + \sum_{s=1}^r (k_{i_{j+1}})_s (a_{i_{j+1}-1}^+) \tilde{\gamma}_s$$

for any sufficiently small perturbation δ . Let us first assume that the discontinuities $\tilde{a}_{i_j}^-$ and $\tilde{a}_{i_{j+1}-1}^+$ are different, i.e. that $i_j \neq i_{j+1} - 1$.

Now, let the perturbation $\tilde{T}$ be such that we only change β_{J,i_j} to $\tilde{\beta}_{J,i_j} := \beta_{i_j} + \epsilon$, where ϵ is sufficiently small so that Proposition III.6.14 applies. By Corollary III.6.15, we have that:

$$\begin{aligned} \tilde{a}_{i_j}^- + \sum_{s=1}^r (k_{i_j})_s (a_{i_j}^-) \tilde{\gamma}_s &= \tilde{a}_{i_{j+1}-1}^+ + \sum_s (k_{i_{j+1}})_s (a_{i_{j+1}-1}^+) \tilde{\gamma}_s \\ \implies \epsilon &= 0, \end{aligned}$$

as the only value that changes is $a_{i_j}^-$, since $\tilde{a}_{i_j}^- := \tilde{T}^{-l_{i_j}}(\tilde{\beta}_{J,i_j}^-)$ by definition. This is clearly a contradiction, so we may assume $i_j = i_{j+1} - 1$. Then if we make a perturbation that only changes some γ_t to $\tilde{\gamma}_t := \gamma_t + \epsilon$, we get:

$$\begin{aligned} &\epsilon \left(-(l_{i_j})_t (a_{i_j}^-) + (k_{i_j} - l_{i_j})_t (\beta_{J,i_j}^-) \right) = \\ &\epsilon \left(-(l_{i_{j+1}-1})_t (a_{i_{j+1}-1}^+) + (k_{i_{j+1}} - l_{i_{j+1}-1})_t (\beta_{J,i_{j+1}-1}^+) \right), \end{aligned}$$

by Corollary III.6.15. Since the above holds for all ϵ and t , we get:

$$(k_{i_j} - l_{i_j})_t (\beta_{J,i_j}^-) = (k_{i_{j+1}} - l_{i_{j+1}-1})_t (\beta_{J,i_{j+1}-1}^+),$$

since $(l_{i_j})_t (a_{i_j}^-) = (l_{i_{j+1}-1})_t (a_{i_{j+1}-1}^+)$, as these points have the same itinerary up to time landing time $l_{i_j} = l_{i_{j+1}-1}$ to β_{J,i_j} . Since this holds for all t , we must in fact have that the return vectors v_{i_j} and $v_{i_{j+1}}$, defined as:

$$\begin{aligned} v_{i_j} &:= \left(\sum_{1 \leq s \leq r} (k_{i_j})_s (a_{i_j-1}^+) e_s, 0 \right) \\ v_{i_{j+1}} &:= \left(\sum_{1 \leq s \leq r} (k_{i_j} + 1)_s (a_{i_j}^+) e_s, 0 \right) \end{aligned}$$

of J_{i_j} and $J_{i_{j+1}}$ are equal. This is a contradiction with Corollary III.5.8. Indeed, we have that v_{i_j} is equal to the sum of all vectors in Corollary III.5.8 related to the orbit up to return time of $a_{i_j-1}^+$ and that $v_{i_{j+1}}$ is equal to the sum of all vector related to the orbit of $a_{i_j}^+$. Thus

if they are equal, we get a non-trivial dependence between the vectors in Corollary III.5.8 by putting the coefficient 1 in front of vector in Corollary III.5.8 related to the orbit up to return time of $a_{i_j-1}^+$, coefficient -1 in front of every vector related to the orbit of $a_{i_j}^+$ and 0 in front of every other vector. This is a contradiction, so this case is impossible.

Assume now that J contains exactly two discontinuities. By Lemma III.6.13, we know that we must have $\sigma_J = (321)$. The above equations now reduce to:

$$\begin{aligned} a_1^+ + \sum_{s=1}^r (k_2)_s (a_1^+) \gamma_s &= y + \sum_{s=1}^r (k_3)_s (y) \gamma_s \\ a_2^- + \sum_{s=1}^r (k_2)_s (a_2^-) \gamma_s &= x + \sum_{s=1}^r (k_1)_s (x) \gamma_s, \end{aligned}$$

where $x = T^{k_3-l_2}(\beta_{J,2}^+)$ and $y = T^{k_1-l_1}(\beta_{J,1}^-)$. Thus by doing the same perturbations as above, we get that:

$$\begin{aligned} -(l_1)_t(a_1^+) + (k_2 - l_1)_t(\beta_{J,1}^+) &= (k_3 + k_1 - l_1)_t(\beta_{J,1}^-) \\ -(l_2)_t(a_2^-) + (k_2 - l_2)_t(\beta_{J,2}^-) &= (k_1 + k_3 - l_2)_t(\beta_{J,2}^+), \end{aligned}$$

for all $1 \leq t \leq r$. If we define the vectors v_1, v_2 and v_3 as above, this implies $v_1 + v_3 = v_2$. In the same way as above, this contradicts Corollary III.5.8, as we again get a non-trivial linear dependence between vectors from this corollary.

Thus we get that it is impossible for J to contain more than one discontinuity. As it is by assumption dynamically non-trivial, it must therefore contain exactly one discontinuity. $\square$

Proposition III.6.16 tells us that every dynamically non-trivial interval J must contain exactly one point that lands on a discontinuity before returning J , which implies Matching. Thus we have the following result:

Theorem III.6.17 (Stability + A1 + A2 $\implies$ Matching) A stable map that satisfies properties A1 and A2 must also satisfy the Matching property. $\square$

III.6.3 Stability implies ACC

Because of Theorem III.6.17, we now only need to prove that stability implies ACC to complete the first implication of Theorem III.6.12. Our proof will be indirect: we will first perturb the map so that it satisfies ACC, and therefore Matching, and use this to conclude the same thing about the starting map.

We now recall the definition of rational independence:

Definition III.6.18 (Rational independence) Let $\{s_1, s_2, \dots, s_n\}$ be a finite set of real numbers. These numbers are said to be rationally independent if the following holds:

$$\left(\sum_i q_i s_i = 0, \text{ with } q_i \in \mathbb{Q} \right) \implies (q_i = 0 \forall i).$$

This condition on the set $\{\beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_r\}$ is stronger than ACC. Indeed, if ACC does not hold, i.e. if one of A1, A2 or A3 is violated, then we have an equation of the following form:

$$\beta + \sum_{s=1}^r n_s \gamma_s = \beta_*,$$

which implies that the the numbers $\{\beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_r\}$ are rationally dependent.

Rational independence, and therefore ACC, can be achieved by an arbitrarily small perturbation. This follows from the next lemma applied to the set $\{\beta_1, \dots, \beta_r, \gamma_1, \dots, \gamma_r\}$.

Lemma III.6.19 Let $\{s_1, s_2, \dots, s_n\}$ be a finite set of real numbers. Then we can make an arbitrarily small perturbation to each of the numbers so that they become rationally independent.

Proof. The proof is by induction. Assume that the numbers $s_1, \dots, s_k$ are rationally independent. We need to show that we can add s_{k+1} do this set by an arbitrarily small perturbation. Indeed, the set of all numbers which are rationally dependent with $s_1, \dots, s_k$ is equal to the set of all linear combinations with rational coefficients of these numbers. This set is only countably infinite, so there is a real number arbitrarily close to s_{k+1} which is rationally independent from $s_1, \dots, s_k$. $\square$

Before going into the proof of Theorem III.6.22, we give two lemmas about the maps that satisfy properties A1, A2 and Matching.

Lemma III.6.20 Let T be a finite type map that satisfies properties A1, A2 and Matching. Then the orbit of any point $z \in X$ enters at most one interval of X that contains a discontinuity of T .

Proof. Assume the contrary, that for some point z we have that $T^{k_1}(z) \in J_1 = [x_1, y_1) \ni \beta$ and $T^{k_2}(z) \in J_2 = [x_2, y_2) \ni \beta_*$, with $k_1 < k_2$. Without loss of generality, we may assume that $T^{k_1}(z)$ lands into J_2 before returning to J_1 . Indeed, we can choose k_1 to be the last time z is in J_1 before landing to J_2 . Moreover, we may assume that $T^{k_2}(z)$ lands to J_1 before returning to J_2 , by choosing k_2 as the last time $T^{k_1}(z)$ is in J_2 before returning to J_1 . We may assume $T^{k_1}(z) \in [\beta^+, y_1)$, with the other case being analogous. By Matching, no point except β^+ in the interior of this interval lands on a discontinuity before returning to J_1 , β^+ does not land because of A1 and y does not land because of A2. Thus it must at time $k_2 - k_1$, when it is contained in J_2 , be compactly contained in one of the intervals $[x_2, \beta_*^-)$, $[\beta_*, y_2)$, which we call J' . As J_2 is dynamically non-trivial, by Matching and ACC1 no point in J' can land on discontinuity before returning to J_2 . By our choice of k_2 , the interval $T^{k_2-k_1}([\beta^+, y_1))$ must land to J_1 before it returns to J_2 . At this time β^+ must land on x . But the points from J' to the left of $T^{k_2-k_1}(\beta^+)$ must land outside of J . Since they must be contained in X , this contradicts the maximality of J . $\square$

Lemma III.6.21 Assume that a finite type map T satisfies properties A1, A2 and Matching. Then every trivial interval $J = [x, y)$ of X has the property that x lands only on the discontinuity β^+ and y lands only on the discontinuity β^- , for some $\beta \in X$.

Proof. Assume first that x lands on β_*^+ at the same time y lands on β_{**}^- . This means that β_*^+ and β_{**}^- are contained in the same interval of X . Since by A1 and A2 there are no discontinuities in the boundary of X , we know that the return map to this interval of X has at least two discontinuities of its return map. This is a contradiction with Matching, so this case is impossible.

Let $T^{k_1}(x) = \beta_*^+$ and $T^{k_2}(y) = \beta_{**}^-$, and assume without loss of generality that $k_1 < k_2$. By A1 and A2, the interval J_1 of X containing $T^{k_1}(J)$ must be dynamically non-trivial,

which means by Matching that no point in J_1 , except β_* , lands on a discontinuity before returning to J_1 . But β_*^+ and β_*^- also do not land on any other discontinuities because of A1. As we know that y must land on a discontinuity, this forces $\beta_{**}^- = \beta_*^-$. $\square$

We are now ready to prove the following theorem, which alongside Theorem III.6.17 proves the first implication of Theorem III.6.12:

Theorem III.6.22 A stable map T satisfies the ACC condition.

Proof. Let us first derive some simpler properties of T . Let $\tilde{T}$ be a sufficiently small perturbation of T which satisfies properties A1 and A2, and for which Lemma III.6.4. Then $\tilde{T}$ also satisfies the Matching condition, by Theorem III.6.17. This means that there are no discontinuities in the boundary of $\tilde{X}$, so the same must be true for X , by Lemma III.6.4. Moreover, each dynamically non-trivial interval $\tilde{J}$ of $\tilde{X}$ contains at most one discontinuity in its interior. By Lemma III.6.4, the same must be true for each corresponding interval J of X . For a trivial interval J , Lemma III.6.21 shows that the boundary points of J land only on the $+$ and $-$ of a single discontinuity $\beta \in X$.

Assume that T does not satisfy A1: some interval $J = [x, y)$ contains a point a which lands on a discontinuity more than once before returning to J . By the last part of the paragraph above, we know that J is dynamically non-trivial. By the first part, we know that a is in the interior of J and that a^+ and a^- are the only points in J that land on a discontinuity. We may without loss of generality assume that a^+ lands on two discontinuities, with the other case being analogous. Thus there are times $k_1 < k_2$ such that $T^{k_1}(a^+) = \beta_*^+ \in J_1$ and $T^{k_2}(a^+) = \beta_{**}^+ \in J_2$. Let z be a point in the interval $[a^+, y)$ which does not land on a critical point before return time and assume that the above perturbation is small enough so that z is still contained in $\tilde{J}$ return time and has the same itinerary up to return time to $\tilde{J}$. By Lemma III.6.20, this means that the $\tilde{T}$ -orbit of z up to return time to $\tilde{J}$ lands into at most one interval of $\tilde{X}$ that contains a discontinuity. Thus we have the same conclusion for the T -orbit of z up to return time, as the T and $\tilde{T}$ orbits of z pass through corresponding intervals of X and $\tilde{X}$, respectively. Moreover, this has to also hold for every point in the continuity interval $[a, y)$ of R_J containing z . Thus a^+ does not land in both J_1 and J_2 , which is a contradiction. Thus T must satisfy A1.

Next, assume that T violates A2: there is a point x in the boundary of a dynamically non-trivial interval $J = [x, y)$ of X that lands on a discontinuity before return time (the case for y is analogous). Let z be a point in the interior of J , to the left of a , the only discontinuity of R_J . In particular, the orbit of z up to return time does not land on any critical point. We may thus assume that the perturbation is small enough so that z has the same itinerary up to return time with respect to the intervals of X and Lemma III.6.20 holds for z and $\tilde{T}$. Thus the T -orbit of z has the same properties: it enters at most one interval of X that contains a discontinuity of T . Therefore this has to hold until return time for the continuity interval $[x, a)$ of R_J that contains z . Thus it is impossible that both x and a land on a discontinuity, which is a contradiction. Thus T must satisfy A2 as well.

Finally, assume that T violates A3. We will show that we can perturb T in such a way that the X increases by definite size, contradicting the stability of T . Indeed, if A3 does not hold, then there is some β that appears in its ghost tree $\mathcal{GT}(\beta)$. Without loss of generality, we may assume that it is of $+$ -type and equal to $\beta_{i_1}^+$. Thus we have a finite path in the ghost tree: $\beta_{i_1}^+ \leftarrow \beta_{i_2}^- \leftarrow \beta_{i_3}^+ \leftarrow \cdots \leftarrow \beta_{i_{2k+1}}^+ = \beta_{i_1}^+$. By A1 and A2, none of the discontinuities in this path are contained in X . For even j with $2 \leq j \leq 2k$, let k_j be the time at which $\beta_{i_j}^+$ lands at $\beta_{i_{j-1}}^+$ and let v_j be the landing vector:

$$v_j := \left(\sum_{1 \leq s \leq r} (k_j)_s(\beta) \mathbf{e}_s, \mathbf{f}_{\text{ind}(\beta_{i_j}^+)} - \mathbf{f}_{\text{ind}(\beta_{i_{j-1}}^+)} \right).$$

Define k_j and v_j analogously for odd i with $3 \leq j \leq 2k+1$. If the orbit of $\beta_{i_j}^+$ lands on $n_i > 0$ other discontinuities before time k_j , we split v_j into $n_j + 1$ vectors that correspond to consecutive landings in this orbit. By Corollary III.5.8, all of these vectors are linearly independent, as they correspond to discontinuities in $\mathcal{C}_{\neg X}$. Thus for sufficiently small ϵ , there exist a sufficiently small perturbation δ (in the sense that $\tilde{T}$ is still in $\mathcal{U}$) so that we have:

$$\begin{aligned} \langle v_j, \delta \rangle &= -\epsilon \text{ for odd } i ; \\ \langle v_j, \delta \rangle &= +\epsilon \text{ for even } i , \end{aligned}$$

and so that $\beta_{i_j}^+$ lands on the same discontinuities before time k_j as before, for all $1 \leq j \leq 2k + 1$.

This makes $\beta_{i_1}^+$ periodic with period $\sum_{j=1}^{2k+1} k_j$, and is therefore contained in a periodic interval contained in $\tilde{X}$. As $\beta_{i_1}^+$ was a definite distance away from X , since there are no discontinuities in the boundary of X , this means that for a sufficiently small perturbation the set $\tilde{X}$ contains more discontinuities than X . This contradicts the stability of T , so A3 follows. $\square$

III.6.4 ACC + Matching implies Stability

We are now going to prove the other direction in Theorem III.6.12, which finishes the proof.

Theorem III.6.23 Let T be a finite type map that satisfies the ACC and Matching properties. Then T is a stable map.

Proof. We will show that there for a sufficiently small perturbation of T , $\tilde{X}$ is close to and homeomorphic to X , and contains the same discontinuities as X . By definition, this shows that T is stable.

By A1 and A2 we know that there are no discontinuities in the boundary of X . Moreover, for any T that satisfies A1, A2 and Matching, every dynamically non-trivial interval J has the property that only a single point a in the interior of J lands on a discontinuity and that the return map to J is a rotation. This gives $X = \bigsqcup_{\beta \in X} O(J_\beta)$, where J_β is the dynamically non-trivial interval of X containing β . These orbits are disjoint and there is a definite distance between them because of Lemma III.6.20.

We have that every dynamically non-trivial interval J is of the form $[T^{k_2}(a^+), T^{k_1}(a^-))$, where $R_J = T^{k_1}$ on $J_1 = [x, a^-)$ and $R_J = T^{k_2}$ on $J_2 = [a^+, y)$. Moreover, for every such interval J of X , we have that there is an interval $J' = [x', y')$ containing J such that:

- No point in $[x', \beta^-)$ lands on discontinuity up to time k_1 and no point in $[\beta^+, y')$ lands on a discontinuity up to time k_2 .
- The intervals $[x', x)$ and $[y, y')$ have definite size.

Indeed, this follows from A2. We may assume that J' is sufficiently small so that for a sufficiently small perturbation of T , the itineraries of points in $[x', \tilde{a}^-)$ up to time k_1 and $[\tilde{a}^+, y')$ remain the same as for T . Moreover, we may assume that the perturbation is small enough so that the itineraries of a^+ and a^- up to time $k_1 + k_2$ remain the same as for T . Thus the interval $[\tilde{T}^{k_2}(\tilde{a}^+), \tilde{T}^{k_1}(\tilde{a}^-)]$ has a well-defined return map on it that is a rotation, so it is in particular contained in $\tilde{X}$. We will show that it must be an interval of $\tilde{X}$. For a sufficiently small perturbation, it is compactly contained in J' . This means that the points in $[x', \tilde{T}^{k_2}(\tilde{a}^+))$ land into $[\tilde{T}^{k_2}(\tilde{a}^+), \tilde{T}^{k_1}(\tilde{a}^-)]$ at time k_1 and points in $[\tilde{T}^{k_1}(\tilde{a}^-), y')$ land into $[\tilde{T}^{k_2}(\tilde{a}^+), \tilde{T}^{k_1}(\tilde{a}^-)]$ after time k_2 , which means that they are not contained in $\tilde{X}$. Thus $[\tilde{T}^{k_2}(\tilde{a}^+), \tilde{T}^{k_1}(\tilde{a}^-)]$ is an interval of $\tilde{X}$, so it is in fact equal to $\tilde{J}$.

For every trivial interval J of X , we know that there is a single dynamically non-trivial interval J_β of X that contains a discontinuity β of T into which J lands, by Lemma III.6.20 and Lemma III.6.21. Thus it must be contained in the orbit of J_β . Let J_1 and J_2 be the two continuity intervals of J_β . As J is trivial, it must in fact be equal to an iterate of either J_1 or J_2 . We may assume the former, with the other case being analogous. Assume $J = T^{k_1}(J_1)$. By the discussion for dynamically non-trivial intervals, this iterate moves continuously for sufficiently small perturbations. Thus $\tilde{T}^{k_1}(\tilde{J}_1)$ is an interval contained in $\tilde{X}$. We now show that it must be maximal. Indeed, there is again an interval $J' \supset J$ that is of a definite size larger than J that has the same itinerary as J up to the time J lands into J_β . We may assume J' is sufficiently small so that it lands into the interior of J'_β and that this continues to hold for all sufficiently small perturbations. As above, this means that the points in $J' \setminus \tilde{T}^{k_1}(\tilde{J}_1)$ are not contained in $\tilde{X}$, which show that $\tilde{T}^{k_1}(\tilde{J}_1)$ is an interval of $\tilde{X}$.

Thus we know that every interval J of X has a well-defined continuation $\tilde{J}$ for all sufficiently small perturbations. We will show that $\tilde{X}$ must in equal $\bigsqcup_{\beta \in X} \tilde{O}(\tilde{J}_\beta)$, which completes the proof. Let X' be the union of all intervals J' .

Let us first show that this is the case if the itineraries of critical points outside of X do not change up to their landing time into the interior J' . Indeed, since for any sufficiently small perturbation the points in $J' \setminus \tilde{J}$ eventually land into $\tilde{J}$, and thus into $\tilde{X}$, if none of the itineraries change, then all of the critical points outside of X still land into the interior of X' . Moreover, if these itineraries do not change, then they also do not change for intervals of definite size containing these discontinuities. We may assume these intervals are small

enough so that they also land into the interior of X' after a sufficiently small perturbation. Thus all of these intervals are not contained in $\tilde{X}$, so all of the discontinuities that were not in X remain outside of $\tilde{X}$. By the Orbit Classification Lemma III.3.3, we know that every point in $z \in I \setminus X'$ that is not eventually periodic must either land on a discontinuity or accumulate on a discontinuity. If it lands on a discontinuity, it is clearly not in $\tilde{X}$. It either accumulates onto a discontinuity inside of X' , or onto a discontinuity outside of X' , and in this case, it must eventually land into the interval of definite size that lands into the interior of X' . Thus in both cases, it eventually lands into $\bigsqcup_{\beta \in X} \tilde{O}(\tilde{J}_\beta)$, which means that it is not in $\tilde{X}$. If $z \in I \setminus X'$ is eventually periodic, it must land into a periodic interval of $\tilde{T}$. The boundary points of such an interval must be the landings of discontinuities contained in $\tilde{X}$, so z must also land into $\bigsqcup_{\beta \in X} \tilde{O}(\tilde{J}_\beta)$. Thus every point must eventually land into $\bigsqcup_{\beta \in X} \tilde{O}(\tilde{J}_\beta)$, which means $\tilde{X} = \bigsqcup_{\beta \in X} \tilde{O}(\tilde{J}_\beta)$. By the characterization of finite type maps II.2.3, we know that $\tilde{T}$ is of finite type, so the claim follows.

Thus it must hold that for any sufficiently small perturbation, there is a critical point outside of X for which the itinerary changes. By the paragraph above, we may assume that the perturbation is small enough so that if the itinerary of a critical point does not change, then there is an interval of definite size that contains it and lands into the interior of X' at the same time as before the perturbation. Moreover, we may assume that the perturbation is small enough so that the itinerary of every discontinuity outside of X up to some finite time in which it does not land on another discontinuity remains the same as before the perturbation. Thus if some $\beta \notin X$ does not land on a discontinuity before the time it lands into the interior of X' , we may assume it still does so.

Now, let $\tilde{T}$ be a sufficiently small perturbation as in the preceding paragraph. Let $\tilde{\beta}_{i_1}^+$ (we may assume it is of $+$ -type, with the other case being analogous) be a discontinuity that:

- It changes itinerary before landing into the interior of X' ;
- It does not eventually land into X' .

Such a discontinuity exists because if every discontinuity that changes itinerary still lands into X' , then there are again intervals of definite size containing such discontinuities that

also land into X' , so in the same way as in the paragraph above we must have that every point $z \in I \setminus X'$ eventually lands into X' , which gives $\tilde{X} = \bigsqcup_{\beta \in X} \tilde{O}(\tilde{J}_\beta)$.

We will show now that there is a finite path $\beta_{i_1}^+ \leftarrow \beta_{i_2}^- \leftarrow \cdots \leftarrow \beta_*$ in the ghost tree of β_1^+ such that β_* already appears earlier in the path. This means that β_* must be in its ghost tree $\mathcal{GT}(\beta_*)$, which contradicts A3.

Let n_1 be the first time at which the itinerary of $\beta_{i_1}^+$ changes. Since the perturbation is small enough, we know that $\beta_{i_1}^+$ must have landed on a discontinuity $\beta_{i_2}^+$ at time n_1 , and now it lands to the left of it. We now claim that there is a discontinuity $\beta_{i_3}^-$ such that $\beta_{i_2}^-$ lands on $\beta_{i_3}^-$ at some time n_2 and such that $\tilde{T}^{n_1+n_2}(\tilde{\beta}_{i_1}^+)$ lands to the right of $\tilde{\beta}_{i_3}^+$.

Indeed, if $\beta_{i_2}^-$ does not change itinerary before landing into X' , the same holds for a definite size interval to the left of $\tilde{\beta}_{i_2}^-$ and thus for $\tilde{T}^{n_1}(\tilde{\beta}_{i_1}^+)$, which is a contradiction with our choice of $\beta_{i_1}^+$. Let $m_{2,1}$ be the first time at which $\beta_{i_2}^-$ changes itinerary. In the same way above, we know that we must have $T^{m_{2,1}}(\beta_{i_2}^-) = \beta_{i_{2,1}}^-$ for some discontinuity $\beta_{i_{2,1}}^-$ not contained in X and that $\beta_{i_2}^-$ now lands to the right of $\beta_{i_{2,1}}^+$ after perturbation. If $\tilde{T}^{n_1+m_{2,1}}(\tilde{\beta}_{i_1}^+)$ is to the left of $\tilde{\beta}_{i_{2,1}}^-$, then we may repeat this argument for $\beta_{i_{2,1}}^-$, and get that it must land at some $\beta_{i_{2,2}}^-$ at time $m_{2,2}$, and it has to land to the right of $\tilde{\beta}_{i_{2,2}}^+$ after the perturbation. Continuing by induction, we get that there is a minimal finite t_2 such that $\tilde{T}^{n_1+m_{2,1}+\cdots+m_{2,t_2}}(\beta_{i_1}^+)$ lands to the right of $\beta_{i_{2,t_2}}^+$ and that we have $\beta_{i_{2,1}}^- \rightarrow \beta_{i_{2,2}}^- \rightarrow \cdots \rightarrow \beta_{i_{2,t_2}}^-$. There has to exist such a finite t_2 because there are finitely many discontinuities, so each iterate $\tilde{T}^{m_{2,i}}(\tilde{\beta}_{i_{2,i}}^-)$ moves to the right by at least $\epsilon > 0$. Thus at each step i of the induction $\beta_{i_{2,i}}^- - \tilde{T}^{n_1+m_{2,1}+\cdots+m_{2,i}}(\beta_{i_1}^+)$ has to be smaller than $\beta_{i_{2,i-1}}^- - \tilde{T}^{n_1+m_{2,1}+\cdots+m_{2,i-1}}(\beta_{i_1}^+)$ by at least ϵ . We can therefore set $n_2 := m_{2,1} + \cdots + m_{2,t_2}$ and $\beta_{i_3}^+ := \beta_{i_{2,t_2}}^+$.

This argument can now be repeated for $\beta_{i_3}^+$, so by induction we get an infinite path $\beta_{i_1}^+ \leftarrow \beta_{i_2}^- \leftarrow \beta_{i_3}^+ \leftarrow \cdots$ in the ghost tree of $\beta_{i_1}^+$. Since there are only finitely many discontinuities, one of them has to repeat, so we get the required finite sequence.

□

III.7 Eventually periodic maps can be approximated by stable maps

III.7.1 Unstable number and correspondence

Recall the statement of Theorem C:

Theorem III.7.1 Let T be an eventually periodic map. Then arbitrarily close to T in the parameter space, there is a stable map $\tilde{T}$.

The idea of the proof is to inductively remove the critical connections that prevent the map T from being stable. Our main tools will be Theorem III.5.6 and Corollary III.5.8. In this subsection, we formalise this idea and prove a few preliminary lemmas, before going into the proof of Theorem III.7.1 in the next subsection. We first introduce a few concepts related to the dynamics of the discontinuities of an eventually periodic map T .

Definition III.7.2 (Cycle of discontinuities) Let β be a discontinuity of an eventually periodic map T contained in X . Then the cycle of β is defined as:

$$C(\beta) = O(\beta) \cap \mathcal{C},$$

i.e. the set of all discontinuities in the orbit of β .

We say that a cycle is non-trivial if $C(\beta) \neq \{\beta\}$. In this case β lands on at least one other discontinuity of T .

Assuming T is eventually periodic, we know that for all $\beta_*, \beta_{**} \in X$ we have that $\beta_* \in C(\beta_{**})$ if and only if $\beta_{**} \in C(\beta_*)$. Thus the set of all discontinuities contained in X is partitioned into cycles, which we consider as equivalence classes of the relation $\beta_* \simeq \beta_{**}$ if and only if $\beta_* \in O(\beta_{**})$. Let $\mathcal{Z}$ be the set of all equivalence classes, i.e. the set of all cycles.

A stable eventually periodic map has the property that the size of each cycle is exactly one. Additionally, it has no discontinuities in the boundary of X . With this in mind, we define a number which tells us how far an eventually periodic map is from having this property:

Definition III.7.3 (Unstable number of T) Let T be an eventually periodic map. Then the *unstable number* of T , denote by $U(T)$ is defined as:

$$U(T) := \left(\sum_{C \in \mathcal{Z}} (|C| - 1) \right) + |\partial X \cap \mathcal{C}|$$

For a periodic discontinuity β , we will denote by $P(\beta)$ the maximal periodic interval containing β , in the sense of Definition III.5.5. For brevity, we will usually omit stating maximality when discussing periodic intervals, but it is always assumed.

Definition III.7.4 (Correspondence) Let T be an eventually periodic map. We say that T has the *correspondence property* if for each discontinuity $\beta^+ \in X$, we have that β^- eventually lands into the maximal periodic interval $P(\beta^+)$ and if analogously for every discontinuity $\beta_*^- \in X$, β_*^+ eventually lands into $P(\beta_*^-)$.

Since the orbits of β^- and β^+ both end up in the same periodic cycle of intervals, these points have ‘corresponding dynamics’. This is a fairly strong property. One of the things it implies is the following restriction on the structure of every interval J of X :

Lemma III.7.5 Let T be an eventually periodic map that has the correspondence property. Then for every interval J of X , there exists a discontinuity $\beta^+ \in X$ such that J is contained in the union of iterates of $P(\beta^+)$.

Note that since T is eventually periodic, each interval of X already has the property of being ‘tiled’ by periodic intervals. If we additionally assume correspondence, the lemma above says that there exists a $\beta^+ \in X$ such that all of these periodic intervals are iterates of $P(\beta^+)$.

Proof. Assume the contrary, that there is an interval J of X containing iterates of two periodic intervals with disjoint orbits. Let P_1 and P_2 be two touching periodic intervals in J with this property. By maximality, the point at which they touch must eventually land on a discontinuity β . Thus β^- and β^+ belong to periodic intervals with disjoint orbits, which contradicts the correspondence property. $\square$

A map that has unstable number zero and satisfies the correspondence property is almost stable:

Lemma III.7.6 Let T be an eventually periodic map. Assume that $U(T) = 0$ and that T satisfies the correspondence property. Then T also satisfies the A1, A2 and Matching properties.

Proof. Since $U(T) = 0$, no point in X can land on two discontinuities, so A1 follows.

By the Lemma III.7.5, we know that any interval J of X is tiled by intervals that are all iterates of a single maximal periodic interval $P(\beta^+)$. Since $U(T) = 0$, β^+ does not land on any other discontinuity. Since $P(\beta^+)$ is a maximal periodic interval, there exists a discontinuity $\beta_*^- \in X$ such that the right boundary point of $P(\beta^+)$ lands on β_*^- at some time k . Additionally, we have that $P(\beta_*^-) = T^k(P(\beta^+))$.

Assume that the left boundary point of a dynamically non-trivial interval J lands on a discontinuity before returning to J . The case for the right boundary point is analogous. We know that this discontinuity must be β^+ . Since the interval is dynamically non-trivial, a point in the interior of J must also land on a discontinuity of T before returning to J . This discontinuity has to be either β or β_* . In the first case, we get a contradiction with the fact that orbits of the components of a return map must be disjoint up to return time. In the second case, we get that β^+ lands on β_*^+ , which is a contradiction with $U(T) = 0$. Thus both cases are impossible, so A2 follows.

If a dynamically non-trivial interval J does not satisfy the Matching property, then there are two points in the interior of J that land on discontinuity before returning to J . These discontinuities can again be only β or β_* , and we get a contradiction in the same way as for A2. Thus Matching follows as well. $\square$

Our goal now is to produce a finite sequence of sufficiently small perturbations each of which decreases the unstable number of T , and also makes T satisfy the correspondence property. By induction, this will result in a map that has unstable number zero and satisfies the correspondence property. We then do a final perturbation to get a map that also satisfies A3, which by Lemma III.7.6 means it satisfies ACC and Matching.

The main idea of the proof is to remove the discontinuities that contribute to the unstable number of T , i.e. are in a cycle inside of X or in the boundary of X , or violate the

correspondence property, while preserving the dynamical properties of points not in the orbit of these discontinuities.

This result allows us to show that an arbitrarily small perturbation of T satisfies the correspondence property and does not increase the unstable number:

Lemma III.7.7 (Perturbation to correspondence) Let T be an eventually periodic map. There exists an arbitrarily small perturbation $\tilde{T}$ of T that satisfies the correspondence property and $U(\tilde{T}) \leq U(T)$.

Similar to the statements of Theorem III.5.6 and Corollary III.5.8, for brevity, we will omit stating the ranges of all indices explicitly.

Proof. Assume that T does not satisfy the correspondence property at some discontinuity β . Without loss of generality, assume that $\beta^+ \in X$, with the other case being analogous. Let $n = |\mathcal{Z}|$ be the number of critical cycles in X . Let $J = J_1$ be the periodic interval $P(\beta^+)$. For each of the remaining $n - 1$ cycles, let β' be any discontinuity from this cycle and let $JP(\beta')$ be the corresponding maximal periodic interval. We label these intervals as $J_2, \dots, J_n$, in any order. By construction, these are maximal intervals with disjoint orbits, so they satisfy the assumption of Corollary III.5.8 (note that we may clearly omit the interval J_0 from the statement of Corollary III.5.8 and still get the same conclusion for the vectors related to $J_1, \dots, J_n$). By linear independence, there exists an arbitrarily small perturbation δ of parameters (γ, β) such that:

$$\begin{aligned} \langle C^{i,+}(0, k), \delta \rangle &= \langle C^{1,-}(1, k), \delta \rangle = 0 \text{ for } i > 1, \\ \langle C_\beta, \delta \rangle &= 0 \text{ for } \beta \in \mathcal{C}_{-X}, \\ \langle C^{1,+}(0, k), \delta \rangle &= 0 \text{ for } k < m_0^{1,+}, \\ \langle C^{1,-}(1, k), \delta \rangle &= 0 \text{ for } k < m_1^{1,-}, \\ \langle C^{1,+}(0, m_i^+), \delta \rangle &= -\epsilon, \end{aligned}$$

where $\epsilon > 0$ is arbitrarily small. In the first line, we are using $C^{i,-}(1, m_1^{i,-}) = \sum_{k=1}^{m_0^{i,+}} C^{i,+}(0, k) - \sum_{k=1}^{m_1^{i,-}-1} C^{i,+}(1, k)$. If ϵ is sufficiently small, the first two equations above imply that the dynamics of any critical point whose orbit is disjoint from the orbit of J_1 remain unchanged.

Indeed, the first equation gives that every such point in X is still periodic and that its periodic interval moves continuously with δ . The second shows that any such point in $\mathcal{C}_1$ still lands on a periodic discontinuity. Finally, we may assume that δ is small enough so that every such point in $\mathcal{C}_2$ still lands into the interior of a periodic interval, as these intervals move continuously with δ . In particular, for any such point, it is in $\tilde{X}$ after perturbation if and only if it was in X before the perturbation.

Note that the points in $\mathcal{C}_1$ that are not contained in X still land on discontinuities in $O(J_1)$ and that the points in $\mathcal{C}_2$ still land into the interior of the orbit of J_1 for a sufficiently small perturbation.

By the third and fourth equations, the dynamics of the interval J_1 also do not change up until the time it returns to itself, when, by the fifth equation, the interval $I_\epsilon(\beta^+) = [\beta^+, \beta^+ + \epsilon)$ now lands to the left of β^- . By assumption, β^- does not land into J_1 . Since T is eventually periodic, β^- must eventually land into a periodic interval, whose orbit must be disjoint from the orbit of J_1 . By the above, the dynamics of this periodic interval and of β^- remain unchanged after the perturbation. Thus, for ϵ sufficiently small, the interval $I_\epsilon(\beta^+)$ eventually lands into a periodic interval whose orbit does not intersect $I_\epsilon(\beta^+)$. Thus $I_\epsilon(\beta^+)$ is not contained in $\tilde{X}$ anymore. By our choice of δ , every point in J_1 eventually lands into $I_\epsilon(\beta^+)$, the entire orbit of J_1 is also not in $\tilde{X}$. Thus we have that the entire critical cycle $C(\beta^+)$ is not in $\tilde{X}$ anymore. Thus every discontinuity that was not contained in X and landed into J_1 is also not contained in $\tilde{X}$. As the dynamics of all other critical points do not change, we have that $U(\tilde{T}) \leq U(T)$. We also know that $\tilde{T}$ is eventually periodic, since all critical points that land into the orbit of J_1 eventually land into a periodic interval, while all other critical points remain eventually periodic.

By induction, after a finite number of arbitrarily small perturbations, we have that $\tilde{T}$ satisfies the correspondence property and that $U(\tilde{T}) \leq U(T)$. $\square$

III.7.2 Proof of Theorem III.7.1

As mentioned before, our strategy is to remove from X the discontinuities that are contained in non-trivial critical cycles. The perturbation we make depends on the location of the critical value corresponding to the critical point we want to remove. More precisely,

we want the critical value to not be the first or the last one in an interval J of X . If the number of continuity of J is at least 4 such a critical value can easily be found. The cases with less continuity intervals need to be treated separately. In all cases, the perturbation we choose is simple and explicit, but a lot of things need to be checked to see that it works. Similar to the statements of Theorem III.5.6 and Corollary III.5.8, we will assume that the boundary points of J_0 land on discontinuities of T , and do the proof under this assumption. The analysis in the other case is simpler.

Proof of Theorem III.7.1. By Lemma III.7.7, we may assume that T satisfies the correspondence property. Assume first that there is a $\beta^+ \in X$ that lands on at least one other discontinuity, with the proof of β^- being analogous. Let J_0 be the interval of X containing such a β^+ with the maximal number of branches, and let N_0 be the number of branches of the return map to J_0 . If $N_0 \geq 2$, we may assume that β^+ is in the interior of J_0 . Indeed, if it is in the boundary, we can replace β^+ by the first critical point in the orbit of J_0 that is contained in the interior of an iterate of J_0 . Recall that σ denotes the permutation corresponding to the order in which the intervals of J_0 return to J , and that $\tau := \sigma^{-1}$.

Case 1 : $N_0 > 3$

Let v_2 be the second critical value of the return map R_{J_0} to J , with respect to the order inside of J_0 and let $J_{\tau(2)}^0$ and $J_{\tau(3)}^0$ be the two continuity intervals of the return map such that the images of their boundary points touch at v_2 . Let a_p^0 , with $1 \leq p \leq N_0 - 1$, be the first critical point of R_{J_0} in the orbit of v_2 , which exists because J_0 is, by Lemma III.7.5, equal to the union of iterates of a single periodic interval. Let $P \geq 0$ be minimal time such that $R_{J_0}^P(v_2) = a_p^0$. Let a and b be the integers:

- $a := \#\{R_{J_0}^t(v_2) \in J_{\tau(2)}^0; 0 \leq t < P\};$
- $b := \#\{R_{J_0}^t(v_2) \in J_{\tau(3)}^0; 0 \leq t < P\}.$

Let $\epsilon_1, \epsilon_2 > 0$ be sufficiently small such that for $b > 0$ we have that:

$$\frac{a}{b+1} < \frac{\epsilon_2}{\epsilon_1} < \frac{a+1}{b},$$

or if $b = 0$, such that:

$$a\epsilon_1 < \epsilon_2.$$

In both cases, ϵ_1 and ϵ_2 satisfy:

$$-\epsilon_2 < -a\epsilon_1 + b\epsilon_2 < \epsilon_1.$$

Note that ϵ_1 and ϵ_2 can clearly be chosen arbitrarily small. Moreover, we may choose ϵ_1 and ϵ_2 to be of the form $r_1|P(\beta^+)|, r_2|P(\beta^+)|$, where r_1 and r_2 are small rational numbers.

Let n be the number of critical cycles in X not contained in the orbit of J_0 . For each such cycle, let β' be one of the discontinuities contained in it and let $P(\beta')$ be the corresponding maximal periodic interval. We label these intervals as $J_1, \dots, J_n$. The intervals $J_0, J_1, \dots, J_n$ therefore satisfy the assumptions of Corollary III.5.8. Thus we may make a perturbation δ of the parameters (γ, β) such that:

$$\begin{aligned} \langle C^{i,+}(0, k), \delta \rangle &= \langle C^{i,-}(1, k), \delta \rangle = 0 \text{ for } i \geq 1, \\ \langle C_\beta, \delta \rangle &= 0 \text{ for } \beta \in \mathcal{C}_{-X}, \\ \langle L_j^0, \delta \rangle &= \langle C^{0,+}(j, k), \delta \rangle = \langle C^{0,-}(j, k), \delta \rangle = 0, \\ \langle R_{\tau(2)-1}^{0,+}, \delta \rangle &= -\epsilon_1, \\ \langle R_{\tau(3)-1}^{0,+}, \delta \rangle &= \epsilon_2, \\ \langle R_j^{0,+}, \delta \rangle &= 0, \text{ for } j \neq \tau(2) - 1, \tau(3) - 1. \end{aligned}$$

The analysis is now similar as in Lemma III.7.7: the only dynamical change is that the image of the interval $J_{\tau(2)}^0$ at the time of returning to J_0 shifts to the left by ϵ_1 (fourth equation) and the image of $J_{\tau(3)}^0$ shifts to the right by ϵ_2 (fifth equation). Indeed, the first two equations guarantee that the dynamics of critical points with orbits disjoint from J_0 have the same dynamics as before. The third equation says that for each $0 \leq j \leq N_0$, the same point a_j^0 in J_0 lands on the discontinuity $\tilde{\beta}_0(j)$. Moreover, the $\tilde{T}$ -orbit of $a_j^{0,\pm}$ up to the time $r_j^{0,\pm}$ contains same set of discontinuities as before the perturbation. Finally, the last three equations mean that each interval $[a_j^{0,+}, a_{j+1}^{0,-})$, for $0 \leq j < N_0$, returns to J_0 at the same time r_j^+ as before. Thus the return map $\tilde{R}_{J_0}$ to J_0 is well-defined and has the same intervals of continuity. The images under this return map of every interval except $J_{\tau(2)}^0$ and $J_{\tau(3)}^0$ are also the same. In particular, their itineraries remain the same for sufficiently

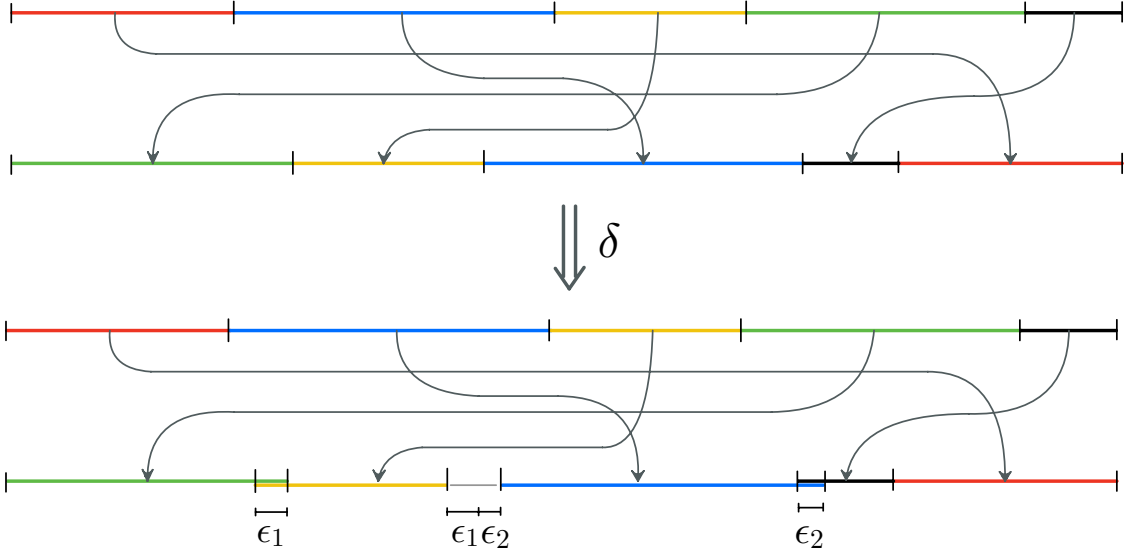


Figure III.5: The result of applying δ to the particular J_0 for which the return map has 5 branches.

small ϵ . The resulting perturbation for the return map with 5 branches and the associated permutation $\sigma = (53214)$ is shown in Figure III.5.

Because $\tilde{R}_{J_0}$ is well-defined, we have that $\tilde{X} \cap \tilde{O}(J_0) = \tilde{O}(\tilde{X} \cap J_0)$ and $\tilde{X} \cap J_0 = \bigcap_{n=1}^{\infty} \tilde{R}_{J_0}^n(J_0)$. The interval $[v_2 - \epsilon_1, v_2 + \epsilon_2)$ is not in $\tilde{X}$, since it is not in the image of $\tilde{R}_{J_0}$ and $\tilde{X} \cap J_0 = \bigcap_{n=1}^{\infty} \tilde{R}_{J_0}^n(J_0)$. On the other hand, the intervals $[v_1 - \epsilon_1, v_1)$ and $[v_3, v_3 + \epsilon_2)$ now both have two $\tilde{R}_{J_0}$ -preimages, and every other point in J_0 only has a single $\tilde{R}_{J_0}$ -preimage. As a_p is by assumption the first critical point in the orbit of v_2 , we know that v_2 must land on it before it lands on v_1 or v_3 , as they are critical values. Thus for sufficiently small ϵ_1 and ϵ_2 , we know that $[v_1 - \epsilon_1, v_1)$ and $[v_3, v_3 + \epsilon_2)$ are not contained in the $\tilde{T}$ -iterates up to and including time P of $[v_2 - \epsilon_1, v_2 + \epsilon_2)$. Thus all of these iterates of $[v_2 - \epsilon_1, v_2 + \epsilon_2)$ are also not contained in $\tilde{X}$.

The image $\tilde{R}_{J_0}^P([v_2 - \epsilon_1, v_2 + \epsilon_2))$ is equal to the image $R_{J_0}([v_2 - \epsilon_1, v_2 + \epsilon_2))$ shifted by $-a\epsilon_1 + b\epsilon_2$, by the definition of a and b . Thus by our choice of ϵ_1 and ϵ_2 , we have that a_p^0 is still contained in this image. Thus a_p^0 is also not contained in $\tilde{X}$. Consider the set of

$\tilde{T}$ iterates of the intervals J_j^0 up to the times when they return to J_0 . By our choice of δ , we know all of these intervals are mutually disjoint, with the only possible exceptions being:

1. Iterates of $J_{\tau(2)}$ for the times after $a_{\tau(2)-1}^{0,+}$ lands on $\beta^{0,+}(\tau(2)-1, m_{\tau(2)-1}^{0,+})$ and iterates of $J_{\tau(1)}$ for the times after $a_{\tau(1)}^{0,-}$ lands on $\beta^{0,-}(\tau(1), m_{\tau(1)}^{0,-})$;
2. Iterates of $J_{\tau(3)}$ after the time $a_{\tau(3)}^{0,-}$ for the times after $\beta^{0,-}(\tau(3), m_{\tau(3)}^{0,-})$ and iterates of $J_{\tau(4)}$ for the times after $a_{\tau(4)-1}^{0,+}$ lands on $\beta^{0,+}(\tau(4)-1, m_{\tau(4)-1}^{0,+})$.

Since there are only two possible exceptions to $\tilde{R}_{J_0}$ iterates up to return time of the J_i 's to not be disjoint, we know that we have that every point in the orbit of a_p^0 up to the landing on $\beta_0(p)$, as well as all of the discontinuities in the orbit up to return time of $\beta_0^+(p)$ and $\beta_0^-(p)$ to J_0 do not have any additional iterated $\tilde{T}$ -preimages contained in J_0 . Since $\tilde{X} \cap \tilde{O}(J) = \tilde{O}(\tilde{X} \cap J)$, this means that all of these points are not contained in $\tilde{X}$.

Assuming that $\tilde{T}$ is eventually periodic, this shows that $U(\tilde{T}) < U(T)$. The proof is the same as in Lemma III.7.7. The discontinuities in $\mathcal{C}_1$ and $\mathcal{C}_2$ that landed into J_0 still do so for a sufficiently small δ , so they are still not contained in $\tilde{X}$, as we can choose δ small enough so that $\tilde{O}(J_0)$ remains a positive distance away from these discontinuities. Moreover, every critical cycle not contained in the T -orbit of J_0 remains unchanged, and all of the discontinuities that land into intervals corresponding to these cycles are still not in $\tilde{X}$. Thus the contribution of all of these critical points to $U(\tilde{T})$ remains the same. The set of discontinuities contained in $\tilde{O}(J_0) \cap \tilde{X}$ is smaller by at least one and they possibly split into several cycles instead of forming a single one. Thus the contribution of these discontinuities to $U(\tilde{T})$ is strictly smaller than before, so $U(\tilde{T}) < U(T)$ follows.

Finally, we see that $\tilde{R}_{J_0}$ is conjugate, via the affine map that sends J_0 to $[0, 1)$, to an ITM on N_0 intervals for which all parameters are rational. Indeed, we know by Lemma III.7.5 that J_0 is equal to the union of iterates of $P(\beta^+)$, so the length of every J_j^0 is an integer multiple of $|P(\beta^+)|$, and by our choice of ϵ_1, ϵ_2 , every translation factor associated to every branch of $\tilde{R}_{J_0}$ is also a rational multiple of $|P(\beta^+)|$. This means that every point in J_0 is eventually periodic. Moreover, every critical point that lands into $\tilde{O}(J_0)$ is also eventually periodic. Since the dynamics of other critical points do not change, we see that $\tilde{T}$ remains eventually periodic.

Case 2 : $N_0 = 3$

This case is different because we only have two critical values of R_{J_0} and thus we are unable to make the same perturbation as in Case 1 and still have that $\tilde{R}_{J_0} \subset J_0$ (with $\tilde{R}_{J_0}$ defined as in Case 1). The perturbation δ in this case depends on the permutation σ associated to R_{J_0} . We give the proof for $\sigma = (321)$, with the proof in every other case being analogous.

Without loss of generality, we may assume that the first critical value of R_{J_0} in the backward R_{J_0} -orbit of β is v_1 , with the other case being analogous. Let P be such that $R_{J_0}^P(v_1) = \beta$. Thus we have that $v_1^- = R_{J_0}(y^{0,-})$ and $v_1^+ = R_{J_0}(a_1^{0,+})$. In this case, let:

- $a := \#\{R_J^t(v_1) \in J_3; 0 \leq t < P\}$;
- $b := \#\{R_J^t(v_1) \in J_2; 0 \leq t < P\}$.

Once again, let $\epsilon_1, \epsilon_2 > 0$ be sufficiently small and of the same form as before, so that for $b > 0$:

$$\frac{a}{b+1} < \frac{\epsilon_2}{\epsilon_1} < \frac{a+1}{b},$$

or if $b = 0$:

$$a\epsilon_1 < \epsilon_2.$$

In both cases, we again have that:

$$-\epsilon_2 < -a\epsilon_1 + b\epsilon_2 < \epsilon_1.$$

We can now choose the analogous intervals $J_1, \dots, J_n$ as in Case 1, and thus find a perturbation δ such that:

$$\begin{aligned} \langle C^{i,+}(0, k), \delta \rangle &= \langle C^{i,-}(1, k), \delta \rangle = 0 \text{ for } i \geq 1, \\ \langle C_\beta, \delta \rangle &= 0 \text{ for } \beta \in \mathcal{C}_{\neg X}, \\ \langle L_j^0, \delta \rangle &= \langle C^{0,+}(j, k), \delta \rangle = \langle C^{0,-}(j, k), \delta \rangle = 0, \\ \langle R_0^{0,+}, \delta \rangle &= -\epsilon_1, \langle R_1^{0,+}, \delta \rangle = \epsilon_2, \langle R_2^{0,+}, \delta \rangle = 0. \end{aligned}$$

Thus we once again have that $\tilde{R}_{J_0}(J_0) \subset J_0$, and the rest of the analysis is the same as in Case 1. The dynamics outside of the orbit of J_0 do not change and $\tilde{R}_{J_0}$ is conjugate to an

ITM with rational parameters, so $\tilde{T}$ is eventually periodic. To prove that $U(\tilde{T}) < U(T)$, it is enough to show that $\tilde{\beta} \notin \tilde{X}$. The intervals $[v_1, v_1 + \epsilon_2)$ and $[a_3^{0,-} - \epsilon_1, a_3^{0,-})$ are not in the image $\tilde{R}_{J_0}(J_0)$ and therefore not in $\tilde{X}$. Since $\langle R_2^{0,+}, \delta \rangle = 0$, we thus have that the only $\tilde{R}_{J_0}$ -preimage of $[v_1 - \epsilon_1, v_1)$ is $[a_3^{0,-} - \epsilon_1, a_3^{0,-})$, and therefore $[v_1 - \epsilon_1, v_1)$ is also not in $\tilde{X}$. Thus the image $\tilde{R}_{J_0}^P([v_1 - \epsilon_1, v_1 + \epsilon_2])$ is also not in $\tilde{X}$. It is equal to the image $R_{J_0}^P([v_1 - \epsilon_1, v_1 + \epsilon_2])$ shifted by exactly $-a\epsilon_1 + b\epsilon_2$, and therefore still contains β . Thus $\beta \notin \tilde{X}$.

Case 3 : $N_0 = 2$

In the first two cases, we were able to make a perturbation that removed from X a discontinuity that was the first one in the orbit of some $a_j^0 \in J_0$. Such a perturbation is not possible in this case, so we will instead remove the discontinuities that come later in the orbits of $a_0^{0,+}$, $a_1^{0,\pm}$ and $a_2^{0,-}$.

Again, we may choose δ so that the dynamics do not change outside the orbit of J_0 , but for J_0 we have:

$$\begin{aligned} \langle L_0^0, \delta \rangle &= \epsilon, \langle L_1^0, \delta \rangle = 0, \langle L_2^0, \delta \rangle = -\epsilon, \\ \langle C^{0,+}(1, 1), \delta \rangle &= \epsilon, \langle C^{0,-}(1, 1), \delta \rangle = -\epsilon, \\ \langle C_J^{0,+}(1, k), \delta \rangle &= 0, \langle C_J^{0,-}(1, k), \delta \rangle = 0, \text{ for all } k > 1, \\ \langle R_0^{0,+}, \delta \rangle &= \epsilon, \langle R_1^{0,+}, \delta \rangle = -\epsilon, \end{aligned}$$

where $\epsilon > 0$ is arbitrarily small. The first equation means that the $\tilde{T}$ -iterates on $a_0^{0,+}$ and $a_2^{0,-}$ not land on the first critical points in their T -orbits. The second and third equation give that all of the critical connections in the T -orbit of J_0 , except the landing of $\beta^{0,-}(1, 1)$ to $\beta^{0,-}(1, 2)$ and of $\beta^{0,+}(1, 1)$ to $\beta^{0,+}(1, 2)$, are preserved under the iterates of $\tilde{T}$. The fourth equation means that the return map $\tilde{R}_{J_0}$ to J_0 remains unchanged. This means that the interval $I_\epsilon^+ = [\beta^{0,+}(1, 2), \beta^{0,+}(1, 2) + \epsilon)$ maps forward continuously with $\tilde{T}$ -iterates of J_2^0 and lands to the left of J_1^0 at the time when J_2^0 returns to J_0 under the iterates of $\tilde{T}$. Analogously, the interval $I_\epsilon^- = [\beta^{0,-}(1, 2) - \epsilon, \beta^{0,-}(1, 2))$ lands to the right of J_2^0 at the time J_1^0 returns to J_0 under the iterates of $\tilde{T}$. Thus I_ϵ^+ follows J_1^0 continuously under the iterates of $\tilde{T}$ until the time J_1^0 returns to J_0 , and at this time I_ϵ^+ lands into J_0 . As the $\tilde{T}$ -orbit of J_0 is disjoint from I_ϵ^+ , we conclude that I_ϵ^+ is not in $\tilde{X}$. Analogously, we have that I_ϵ^-

is also not in $\tilde{X}$. Thus every discontinuity in the orbit of $a_0^{0,+}$, $a_2^{0,-}$, $\beta^{0,+}(1)$ and $\beta^{0,-}(1)$ is not contained in $\tilde{X}$ anymore, except for β . As $\tilde{R}_{J_0}$ is the same as before perturbation, all of these discontinuities are eventually periodic and β is still periodic. Thus $\tilde{T}$ is still eventually periodic. Moreover, the critical cycles of β^+ and β^- now have size one, so $U(\tilde{T}) < U(T)$ as well.

Case 4 : $N_0 = 1$

In this case β^+ is in the boundary of $J_0 = P(\beta^+)$. Since T satisfies the correspondence property, we know that β^- must eventually land into J . Thus we make a similar perturbation as in Case 3: we keep all of the dynamics outside of the orbit of J_0 as they are, and for J_0 we choose a δ such that:

$$\begin{aligned} \langle L_0, \delta \rangle &= 0, \langle L_1, \delta \rangle = -\epsilon, \\ \langle C^{0,+}(0, 1), \delta \rangle &= \epsilon, \\ \langle C^{0,+}(0, k), \delta \rangle &= 0, \langle C^{0,-}(1, k), \delta \rangle = 0, \text{ for all } k > 1, \\ \langle R_0^{0,+}, \delta \rangle &= -\epsilon, \end{aligned}$$

where $\epsilon > 0$ is arbitrarily small. Thus we avoid the first critical connection, for both $a_0^{0,+} = \beta^+$ and $a_1^{0,-}$, and keep all of the others in the orbit of $a_0^{0,+}$. The return map to J_0 again does not change. Similarly as in Case 3, we again have that the interval $I_\epsilon^+ = [\beta^{0,+}(0, 2), \beta^{0,+}(0, 2) + \epsilon)$ follows J_0 continuously under the iterates of $\tilde{T}$ until J_0 returns to itself, when it lands to the left of β^- . By assumption, β^- eventually lands into J_0 , and thus, for sufficiently small ϵ , J_ϵ^+ eventually lands into J as well. Since the $\tilde{T}$ -orbit of J is disjoint from it, we know that J_ϵ^+ is not in X . Same as in Case 3, we thus know that $\tilde{T}$ is eventually periodic and $U(\tilde{T}) < U(T)$.

Since we have now covered all cases, we may assume by induction that we have a map that satisfies the correspondence property and for which no critical point in X lands on a different critical point. Assume now that there exists some discontinuity β^+ in the boundary of some interval J of X , with the case for β^- being analogous. Since T satisfies the correspondence property, we know that $J_0 = P(\beta^+)$, because otherwise β^+ would have to land on some other discontinuity. We also know that β^- must eventually land into J . Let t_1 be the time at which β^- lands into J_0 and let l be the distance between β^+ and $T^{t_1}(\beta^-)$.

Let t_2 be the minimal period of β^+ . We then make a perturbation δ that keeps all dynamics outside of the orbit of J as they are and shifts the return map to J_0 to the left by ϵ , i.e. $\tilde{T}^{t_2}(\beta^+) = -\epsilon$. Note that this means that $\tilde{T}^{t_1}(\beta^-) = T^{t_1}(\beta^-)$. For a sufficiently small ϵ , the interval $[\beta - \epsilon, \beta)$ does not contain any $+$ -type discontinuity before it lands into J_0 , even after perturbation. Thus the return map to the interval $J' = [\beta - \epsilon, \beta + l)$ is now well defined and equal $\tilde{T}^{t_1}$ on $J'_1 = [\beta - \epsilon, \beta)$ and $\tilde{T}^{t_2}$ on $J'_2 = [\beta, \beta + l)$. Taking ϵ to be a rational multiple of l means that β^+ remains eventually periodic. Thus, $\tilde{T}$ is still eventually periodic and there is one less discontinuity in the boundary of $\tilde{X}$. We have added β^- to $\tilde{X}$, but by construction, it is the only discontinuity in its cycle, and therefore $U(\tilde{T}) < U(T)$.

Thus by induction, we get a map that satisfies the correspondence property and has unstable number zero. By Lemma III.7.6 we know that it also satisfies properties A1, A2 and Matching. Assume now that it does not satisfy A3. This in particular means that $C_{-X} \neq \emptyset$. Then we can make a perturbation δ that does not change the dynamics of all critical points contained in X , and for which:

$$\begin{aligned}\langle C_\beta, \delta \rangle &= -\epsilon, \text{ for all } \beta \in C_{-X} \cap \mathcal{C}^-, \\ \langle C_\beta, \delta \rangle &= \epsilon, \text{ for all } \beta \in C_{-X} \cap \mathcal{C}^+, \end{aligned}$$

where $\epsilon > 0$ is small enough so that the itineraries up to the landing time into X of discontinuities in C_{-X} do not change. Thus there are no critical connections outside of X anymore, so A3 must hold. Since the dynamics of all other critical points do not change, A1, A2 and Matching still hold. Thus the resulting map is stable. $\square$

III.8 Almost every finite type map is stable

In this section, we show that the Boshernitzan–Kornfeld Conjecture implies the Irrational Rotations Conjecture. In fact, this results follows immediately the Characterization of Stability Theorem III.6.12 and the following theorem:

Theorem III.8.1 The set of all infinite type maps has full measure in $ITM(r) \setminus S(r)$.

Proof of Boshernitzan–Kornfeld Conjecture $\implies$ Irrational Rotations Conjecture. Since infinite type maps have zero measure in $\text{ITM}(r)$, by Theorem III.8.1 the stable maps have full measure in $\text{ITM}(r)$. By Theorem III.6.12, we know that stable maps correspond to maps for which the return map to every interval is either a rotation or the identity. The set of stable maps with rationally independent coefficients forms a full measure subset of stable maps, and for these maps, every return map corresponds to an irrational circle rotation. Thus the maps corresponding to irrational circle rotations form a full measure subset of $\text{ITM}(r)$. $\square$

We now turn to the proof of Theorem III.8.1. Until now, we only needed to consider first return maps on intervals contained in X , for which it is known that they are bijective and have finitely many branches, and thus correspond to IETs. It is clear that this notion can be generalized to an arbitrary interval $J \subset X$.

Definition III.8.2 Let T be an ITM and assume that there is an interval $J \subset I$ such that each point of I returns to J under iterates of T . Then J is partitioned into maximal intervals on which the itinerary up to the return time to J is constant. The map R_J defined piecewise on these intervals as the iterate of T for which this interval returns to J is called the return map to J .

Compared to the definition of the return map to a component interval of X , the generalized return map does not need to be bijective. Thus the images of its branches are not necessarily disjoint. The next lemma tells us when a generalized return map has a continuation for all sufficiently small perturbations.

Lemma III.8.3 Let T be an ITM and let $J \subset I$ be an interval such that the generalized return map R_J is well-defined. Assume that:

- For each point a in the interior of J , we have that the orbit of a contains at most one critical point of T ;
- The boundary points x, y of J do not land on discontinuities for all time.

Then for any sufficiently small perturbation of T , there exists an interval $\tilde{J}$ close to J such that the generalized return map $R_{\tilde{J}}$ to $\tilde{J}$ is well-defined.

Proof. Let $J_1, \dots, J_n$ be the component intervals of R_J , let $r_1, \dots, r_n$ be their return times to J , and let $a_1, \dots, a_{n-1}$ be the points in the interior of J that land on discontinuities. By the second assumption, we know that J_1 does not land back on itself when it returns to J . Indeed, this would mean that x is a periodic point. Since the points to the left of x are not contained in X , this means that x must be the left boundary point of a maximal periodic interval, so it must eventually land on discontinuity, contradicting the second assumption. By an analogous argument, we know that J_n also does not land on itself when it returns to J , so both J_1 and J_n land into the interior of J when they return to J . This means that there is an interval $\hat{J}$ containing J , which is larger by a definite size, and such that a generalized return map $R_{\hat{J}}$ on $\hat{J}$ is well-defined. By the first assumption, for every sufficiently small perturbation of T , the points $a_1, \dots, a_{n-1}$ have continuations $\tilde{a}_1, \dots, \tilde{a}_{n-1}$ which land on the same discontinuities as before and have the same landing times. This means that every interval $\tilde{J}_i = [\tilde{a}_{i-1}, \tilde{a}_i]$ has the same itineraries up to time r_i . Let τ be the permutation associated to J , so that the intervals $J_{\tau(1)}$ and $J_{\tau(n)}$ get mapped to the front and back of J , respectively. For a sufficiently small perturbation $\tilde{T}^{r_{\tau(1)}}(\tilde{J}_{\tau(1)})$ and $\tilde{T}^{r_{\tau(n)}}(\tilde{J}_{\tau(n)})$ are both contained in $\hat{J}$. Thus the interval $[\tilde{T}^{r_{\tau(1)}}(a_{\tau(1)-1}^+), \tilde{T}^{r_{\tau(n)}}(a_{\tau(n)}^-)]$ has a well-defined generalized return map and is close to J . $\square$

We now show a lemma about the global stability of the critical set and critical orbits in $[T_0]$. Recall that $\mathcal{C}_X(\tilde{T})$ denotes the set of all critical points of $\tilde{T}$ contained in $X(\tilde{T})$. For a $\tilde{\beta} \in \mathcal{C}_X(\tilde{T})$, let $J(\tilde{\beta})$ be the component interval of $X(\tilde{T})$ containing $\tilde{\beta}$. Let $r(\tilde{\beta}^-)$ and $r(\tilde{\beta}^+)$ be the return times of $\tilde{\beta}^-$ and $\tilde{\beta}^+$ to $J(\tilde{\beta})$, respectively.

Lemma III.8.4 Let $[T_0]$ be the stable region. Then:

- For any $\tilde{T} \in [T_0]$ the set $\mathcal{C}_X(\tilde{T})$ is equal to the set of continuations of the discontinuities contained in $\mathcal{C}_X(T_0)$.
- If the $\tilde{\beta}$ is the continuation of a discontinuity $\beta \in \mathcal{C}_X(T_0)$, then $r(\tilde{\beta}^-) = r(\beta^-)$ and $r(\tilde{\beta}^+) = r(\beta^+)$.
- The $\tilde{T}$ -itineraries of $\tilde{\beta}^-$ and β^+ are equal to the T_0 -itineraries of β^- and β^+ up to and including the time $r(\beta^-) + r(\beta^+)$.

More briefly, we will say that $\mathcal{C}_X(T_0)$, $r(\beta^-)$, $r(\beta^+)$ and the itineraries of β^+ , β^- up to and including the time $r(\beta^-) + r(\beta^+)$ are constant in $[T_0]$. Note that for a stable map $\tilde{T}$, all of these are locally constant in a small neighbourhood of $\tilde{T}$ in parameter space. The proof is in the first part of the proof of Theorem III.6.23. Lemma III.8.4 tells us that this is true globally inside the stable region as well.

Proof. Since $[T_0]$ is connected and open, it is path-connected. Let $\eta : [0, 1] \rightarrow [T_0]$ be a path starting at T_0 and ending at $\tilde{T}$. Since $\eta(t)$ is a stable map, for every $t \in [0, 1]$, there is a small neighbourhood $\mathcal{U}(t)$ of $\eta(t)$ such that the critical set, return times and itineraries of critical points up to the times described above are constant. By compactness, the cover of $\eta([0, 1])$ by the neighbourhoods $\mathcal{U}(t)$ has a finite subcover $\mathcal{U}(t_1), \dots, \mathcal{U}(t_p)$, where $t_1 < \dots < t_p$. Since these neighbourhoods are open, $\mathcal{U}(t_i) \cap \mathcal{U}(t_{i+1})$ is non-empty, for all $1 \leq i < p$. Thus the critical set, return times, and itineraries are constant on the union of these sets as well. By induction, it must be constant on the entire path $\eta([0, 1])$, and thus constant on the entire stable region $[T_0]$. $\square$

The crucial reason why Theorem III.8.1 holds is the following theorem:

Theorem III.8.5 Let T be a finite map in the boundary of stable region. Then T has a critical connection.

This means that the parameters defining such a map must satisfy a linear equation with integer coefficients. Thus such maps live on a countable union of codimension-1 subspaces and are therefore of zero measure.

Proof. Let $[T_0]$ be the stable region such that $T \in \partial[T_0]$. Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of maps in $[T_0]$ converging to T . We may assume that T satisfies the ACC property because otherwise T has a critical connection. We will show that this means that the sequence of sets $\overline{X(T_n)}$ converges in the Hausdorff topology and that the resulting set is a subset of $\overline{X}(T)$. This immediately follows if we prove that the constancy from Lemma III.8.4 extends to T in the following sense: the continuations of discontinuities in $\mathcal{C}_X(T_0)$ are contained in $\mathcal{C}_X(T)$ and their return times and itineraries are the same as for T_0 .

For the sake of contradiction, assume that there is a discontinuity $\beta_{T_0} \in \mathcal{C}_X(T_0)$ such that its continuation β_T has a different itinerary up to the return time of β_{T_0} to its component

interval of $X(T_0)$. We may without loss of generality assume that β_T is of $+$ -type, with the other case being analogous. Let k be the smallest time for which the itineraries are different. This means that $T_0^k(\beta_{T_0}) \in I_s$ and $T^k(\beta_T) \in I_{s'}$, where by continuity $s' \in \{s-1, s+1\}$. Since β_T is of $+$ -type, we must in fact have that $s' = s+1$. Indeed, by ACC, none of the images $T_n^k(\beta_{T_n})$ land on $\beta_s^+(T_n)$, where β_{T_n} are the continuations of β_{T_0} . In particular, they are all to the right of β_s^+ . Thus passing to the limit, we have that $T^n(\beta_T)$ can't be to the left of $\beta_s(T)$, so $s' = s+1$. By a similar argument as before, we have that all of the iterates (β_{T_n}) must be strictly to the left of $\beta_{s+1}^-(T_n)$. Thus if the itinerary changes, in the limit we must have that $T_n^k(\beta_{T_n})$ lands on $\beta_{s+1}^+(T)$. This is a contradiction with our assumption that there are no critical connections, so β_T has the same itinerary up to the return time of β_{T_0} to its component interval of $X(T_0)$.

A similar argument then shows that each iterates $T^{r(\beta_T^+)}$ has the same itinerary up to time $r(\beta_T^-)$ as β_T^- . Indeed, again by looking at the smallest time when the itinerary changes, we get a critical connection for T . This clearly shows that the required constancy extends to T in a continuous way, meaning that the orbits of intervals J_β , where J_β is the interval containing $\beta \in \mathcal{C}_X(T_0)$, move continuously from $[T_0]$ to T . Finally, we will show that the orbits of different intervals J_β must remain separated, i.e. they do not touch. Indeed, if two components of different orbits touched, we would have a critical connection. This is because these different components must have different itineraries. Thus they must at some point land on different sides of some discontinuity β_s . By considering the first such time, we must have that the point at which these components touch lands on β_s , thus creating a critical connection, since the boundary points of these components are iterates of critical points. Thus $X(T_0)$ has a homeomorphic continuation inside $X(T)$ with the same dynamics as $X(T_0)$.

Since T is not stable, there exists a component interval J of $X(T)$ for which the return map violates Matching, and thus has at least three branches. By our previous discussion, the orbit of J has a definite distance from the continuation of $X(T_0)$. Indeed, it is clear that is disjoint from it and does not touch the the continuation of $X(T_0)$ because the touching point would have to land on discontinuity to separate the orbits of the touching components of $X(T_0)$ and $O(J)$, since they must have different itineraries. Since the ACC condition holds for T , the interval J satisfies all of the assumptions of Lemma III.8.3. Thus by

Lemma III.8.3, for any sufficiently small perturbation $\tilde{T}$ of T there is an interval $\tilde{J}$ close to J which has a well-defined generalized return map $R_{\tilde{J}}$. Since T is in the boundary of J , we may assume that $\tilde{T} \in [T_0]$. But then $\tilde{J}$ contain points in $\tilde{X}$ which are definite distance from the continuation of $X(T_0)$. This is clearly a contradiction, so T does not satisfy ACC and must therefore have a critical connection. $\square$

As a corollary, we get a sufficient condition for a finite type map to not be in the closure of any stable region:

Corollary III.8.6 Let T be a finite type map that does not satisfy Matching and has no critical connections. Then T is not in the closure of any stable region.

Proof. Since T does not satisfy Matching, it is not contained in any stable region. Since it does not have any critical connection, by Theorem III.8.5 it is also not contained in the boundary in any stable region. Thus it is not contained in the closure of any stable region. $\square$

Theorem III.8.5 allows us to establish the following criterion for a map to be of infinite type:

Lemma III.8.7 Let T be a map that is not a finite type stable map and assume that the defining parameters $(\gamma_1, \dots, \gamma_r, \beta_1, \dots, \beta_{r-1})$ are rationally independent. Then T is an infinite type map.

To prove this criterion, we need the following lemma about the number of equations a map needs to satisfy in order for a return map to an interval of X to have $n \geq 2$ branches.

Lemma III.8.8 Let T be a map such that there is an interval J contained in X on for which the return map R_J has exactly $n \geq 2$ continuity intervals. Then there are at least $n - 2$ linearly independent vectors $v_1, \dots, v_{n-2}$ with integer coefficients such that defining parameters (γ, β) of T satisfy $\langle v_i, (\gamma, \beta) \rangle = 0$, for all $1 \leq i \leq n - 2$.

Proof. Let σ be the permutation associated to R_J and let τ be its inverse. We have the following string of $n - 1$ equalities that must hold:

$$R_J(a_{\tau(1)}^-) = R_J(a_{\tau(2)-1}^+)$$

$$\begin{aligned}
R_J(a_{\tau(2)}^-) &= R_J(a_{\tau(3)-1}^+) \\
&\dots \\
R_J(a_{\tau(n-1)}^-) &= R_J(a_{\tau(n)-1}^+),
\end{aligned}$$

where the a 's are the discontinuities of R_J and the boundary points of J . There are two cases. The first is when the boundary points of J land on discontinuities. Then we have that $R_J(a_i^\pm) = \langle R_i^{J,\pm}, (\gamma\beta) \rangle$, where $R_i^{J,\pm}$ is the corresponding return vector, so all the equalities above are equivalent to:

$$\begin{aligned}
\langle R_{\tau(1)}^{J,-} - R_{\tau(2)-1}^{J,+}, (\gamma\beta) \rangle &= 0 \\
\langle R_{\tau(2)}^{J,-} - R_{\tau(3)-1}^{J,+}, (\gamma\beta) \rangle &= 0 \\
&\dots \\
\langle R_{\tau(n-1)}^{J,-} - R_{\tau(n)-1}^{J,+}, (\gamma\beta) \rangle &= 0
\end{aligned} \tag{III.8.1}$$

If there is a linear dependence between the $n - 1$ vectors $R_{\tau(1)}^{J,-} - R_{\tau(2)-1}^{J,+}, \dots, R_{\tau(n-1)}^{J,-} - R_{\tau(n)-1}^{J,+}$ with coefficients $\alpha_1, \dots, \alpha_{n-1}$, we can extend it to the linear dependence of all vectors associated to the return map J , by setting the coefficients for all other vectors to 0. Then it easily follows from Theorem III.5.6 that all of the coefficients of this linear dependence must be zero, so $\alpha_1 = \dots = \alpha_{n-1} = 0$. This is due to the simple fact that return vector $R_{\tau(1)-1}^{J,+}$ does not appear among the vectors in (III.8.1), so the coefficient next to it is 0, and this propagates in the conclusion of Theorem III.5.6 to all coefficients being equal to zero. Thus in this case we get $n - 1$ linearly independent vectors.

If either a_0^+ or a_n^- does not land on a discontinuity, then we must replace in (III.8.1) the vector $R_0^{J,+}$ by $R_{\tau(1)-1}^{J,+} + R_0^{J,+}$, or the vector $R_n^{J,-}$ by $R_{\tau(n)}^{J,-} + R_n^{J,-}$. This is because in this case $R_J(a_0^+) = a_0^+ + \langle R_0^{J,+}, (\gamma\beta) \rangle$ and $a_0^+ = \langle R_{\tau(1)-1}^{J,+}, (\gamma\beta) \rangle$, and similarly for a_n^- . If either of these vectors lands on a discontinuity, then the same argument as above shows that we get $n - 1$ linearly independent vectors, because either $R_{\tau(1)-1}^{J,+}$ or $R_{\tau(n)}^{J,-}$ will have coefficient zero. Thus we may assume that both boundary points do not land on discontinuities. But then we may simply take the subset of $n - 2$ vectors that does not include the vector related to the return of a_0^+ , and the same argument as before shows that these vectors are linearly

independent. □

The last part of the proof where we reduce the set from $n - 1$ to $n - 2$ vectors might appear unnecessary, but it is simple to check that for cases $n = 3$ and $n = 2$, the set of $n - 1$ vectors we get in the last part of the proof is linearly dependent.

Proof of Lemma III.8.7. Since we have assumed that the defining parameters are rationally independent, we know that T has no critical connections so it satisfies ACC. If T did not satisfy Matching, there would have to be a component interval of X with at least three branches of the return map. By Lemma III.8.8 this would mean that there is at least one non-zero vector v with integer coefficients such that $\langle v, (\gamma \beta) \rangle$ is equal to 0. This contradicts our assumption on rational independence, so T satisfies Matching. Thus if T were of finite type, it would have to be stable, but we have assumed that this is not the case. Thus T has to be of infinite type. □

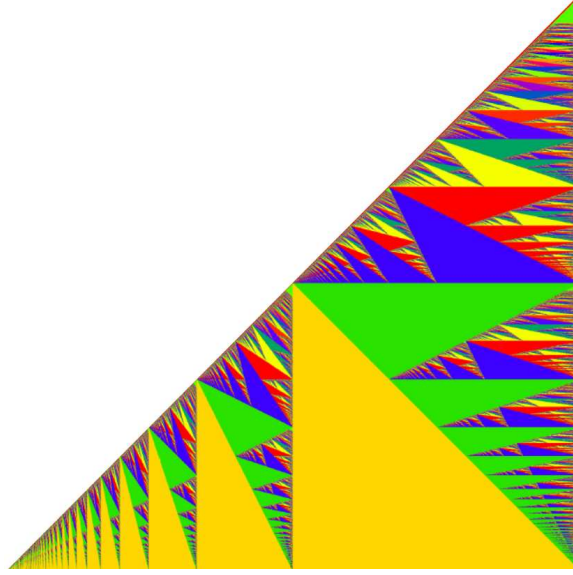
Finally, Lemma III.8.7 allows us to prove Theorem III.8.1:

Proof of Theorem III.8.1. Let $\text{INF}(r)$ be the set of all infinite type ITMs on r intervals. The set of all rationally independent parameters $\mathcal{R}(r)$ has full measure in $\text{ITM}(r)$. By Lemma III.8.8, we know that any map associated with such parameters has to be either a stable finite type map or an infinite type map. Thus $S(r) \cup \text{INF}(r)$ has full measure in $\text{ITM}(r)$, so $\text{INF}(r)$ has full measure in $\text{ITM}(r) \setminus S(r)$. □

III.9 Parameter space and future work

III.9.1 The Bruin–Troubetzkoy family

In this subsection, we are going to apply our results to the parameter space of the special family from [BT03] and show that the open coloured triangles in Figure II.5 (duplicated below for easier reference) form an open and dense subset of the parameter space.



In [BT03], the authors consider a two-parameter family $\mathcal{T}_{a,b}$, with $0 \leq b \leq a \leq 1$, consisting of maps $T_{a,b}$ defined as follows:

$$T_{a,b}(x) = \begin{cases} x + a & \text{for } x \in [0, 1 - a) \\ x + b & \text{for } x \in [1 - a, 1 - b) \\ x + b - 1 & \text{for } x \in [1 - b, 1) \end{cases}$$

In [BT03] it was shown that the set of all finite type maps in $\mathcal{T}_{a,b}$ is equal to the set of all maps that under the iterates of a certain renormalization operator get mapped into a certain region. The set of closed yellow triangles touching the bottom side of the big triangle in the figure above corresponds to the set of all maps that get mapped into this region after one step. It is easy to show that the maps on the boundaries of these triangles are not stable (in this family) because they violate ACC and that the maps in the interiors of these triangles are stable. Moreover, every other coloured triangle in the picture gets mapped homeomorphically onto one of these yellow triangles under the iterates of the renormalization operator. Thus the same conclusion holds for any triangle: the maps in the interior are stable and the maps on the boundary are not. Thus the set of all stable finite type maps is equal to the union of interiors of these triangles. In the remainder of this subsection, we are going to prove that the stable maps form a dense subset of this family,

which therefore shows that the set of (open) triangles forms a dense and open subset of the parameter space, as suggested by the figure. The authors of [BT03] do not explicitly state this result, but it can probably be derived from their results.

To apply our results to the family $\mathcal{T}_{a,b}$, we will use the following trick. We may assume that β_0^+ and β_r^- are parameters, and not the fixed points 0^+ and 1^- . The boundary points of the interval are parameters of this family but they are not discontinuities of a map. Thus we allow for the interval of the definition of an ITM to change, which enlarges the space $\text{ITM}(r)$ to $\widehat{\text{ITM}}(r)$ of $2r + 1$ parameters. The boundary conditions for this space are:

$$\begin{aligned}\beta_0^+ &\leq \beta_{s-1}^+ + \gamma_s; \\ \beta_r^- &\geq \beta_s^- + \gamma_s, \\ \beta_{s+1} &\geq \beta_s\end{aligned}$$

for all $0 \leq s \leq r - 1$. We will in fact only need to consider a small neighbourhood of $\text{ITM}(r)$ in this space, so all of the parameters we consider are uniformly bounded. The reason for making this extension is that when we make perturbations in the proofs of Theorems III.6.12 and III.7.1, we need to be able to change all of the parameters defining a map T , and this is impossible if the parameters 0 and 1 are fixed.

Thus we can consider an extension $\widehat{\mathcal{T}}_{a,b}$ of the family above in this extended space $\widehat{\text{ITM}}(3)$. The defining equations of this family are:

$$\begin{aligned}\beta_1^- + \gamma_1 &= \beta_3^- \\ \beta_2^- + \gamma_2 &= \beta_3^- \\ \beta_2^+ + \gamma_3 &= \beta_0^+.\end{aligned}$$

Theorem III.9.1 (Density of stable maps in the Bruin–Troubetzkoy family) Stable maps are dense in $\mathcal{T}_{a,b}$.

Proof. The following is the list of changes in the proof of our 4 main Theorems: III.4.2, III.5.6, III.6.12 and III.7.1.

1. Theorem III.4.2 - Density of eventually periodic maps in $\mathcal{T}_{a,b}$: We already know that

eventually periodic maps are dense in parametric families because Theorem III.4.2 was proven for rational families, and $\mathcal{T}_{a,b}$ is clearly a rational family. Thus every map in the interior of this parameter space can be approximated by an eventually periodic map in this space, so the eventually periodic maps are dense in $\mathcal{T}_{a,b}$.

2. Theorem III.5.6 - Linear Dependence of itinerary vectors for maps in $\widehat{\text{ITM}}(3)$: We need to add the itinerary vectors for β_0^+ and β_r^- to the statement and proof. This is where the trick of considering the extended parameter space $\widehat{\text{ITM}}(r)$ becomes relevant: we need 0 and 1 to be parameters so that the calculations in the proof still apply to them. The proof remains unchanged otherwise.
3. Theorem III.6.12 - Matching and ACC is equivalent to Stability in $\widehat{\text{ITM}}(3)$: The left and right boundary points β_0^+ and β_r^- are not discontinuities of a map in $\widehat{\text{ITM}}(3)$, so they are not critical points (but they are parameters of the map, which is the important point for Theorem III.5.6). Thus these points do not need to be included in the definition of ACC, Matching or stability, and so the proof that ACC and Matching characterizes stability does not change. Note that a map T which is stable in $\widehat{\text{ITM}}(3)$ and contained in $\mathcal{T}_{a,b}$ is stable within this parameter family as well, in the sense that there is neighbourhood of T in $\mathcal{T}_{a,b}$ which satisfies all of the properties of stability as in Definition III.6.1,
4. Theorem III.7.1 - Approximation of eventually periodic maps in $\mathcal{T}_{a,b}$ by stable maps in $\widehat{\mathcal{T}}_{a,b}$: By item 2, the vectors corresponding to the defining critical connections of the family $\widehat{\mathcal{T}}_{a,b}$ are included in Theorem III.5.6. Thus these critical connections may be preserved every time we make a perturbation in the proof Theorem III.7.1. Thus the same proof as before shows that we may make a finite sequence of perturbations to an eventually periodic map that makes it stable.

By item 1, if we start with a map T in $\mathcal{T}_{a,b}$, we can perturb it to an eventually periodic map that is still in $\mathcal{T}_{a,b}$. Because of item 4, we may then perturb it to a map in the extended parametric family $\widehat{\mathcal{T}}_{a,b}$ that satisfies ACC and Matching and is thus stable by item 3. This map $\tilde{T}$ is possibly not contained in $\text{ITM}(3)$, but because the perturbations we make are arbitrarily small, there is an arbitrarily small rescaling of it that is contained in $\text{ITM}(3)$. Indeed, we

may first translate all of the β -parameters by $-\beta_0$, so that the left boundary of the interval is 0^+ . Note that this preserves the itineraries of all points and the (dynamically relevant) equations satisfied by the parameters of the map, so it does not change the dynamics. Moreover, this change in parameters can be chosen arbitrarily small, since $\tilde{T}$ is arbitrarily close to a map in $\mathcal{T}_{a,b}$ for which $\beta_0^+ = 0$. Next, we multiply all of the parameters by $1/\beta_1^-$. This also does not change the dynamics, is arbitrarily small, and gives that the left boundary point is now 1^+ . Thus by this rescaling, arbitrarily close to the stable map in $\widehat{\mathcal{T}}_{a,b}$ we get a map in $\text{ITM}(3)$ with the same dynamical properties, which must therefore be stable in $\widehat{\text{ITM}}(3)$ and contained in $\mathcal{T}_{a,b}$. This map is therefore stable in $\mathcal{T}_{a,b}$, as described in item 3. Thus the stable maps are dense in $\mathcal{T}_{a,b}$. $\square$

III.9.2 Future work: geometry and dynamics in the boundary of stable regions

In this final subsection, we discuss the properties of the full parameter space $\text{ITM}(r)$ of all ITMs on r intervals, and we present a few conjectures. Recall that a connected component of the set of stable maps $\mathcal{S}(r)$ corresponds to the maximal neighbourhood $[T]$ of a stable map T that satisfies the three properties from the definition of stability (Definition III.6.1). Such sets $[T]$ are called *stable regions*. A very interesting problem is to describe the geometry of these regions.

Question III.9.2 What are the geometric properties of stable regions and their boundaries?

Ghost preimages (see Definition III.6.8 and Example III.6.6) can be used to show that stable regions are not convex in general. Since ghost preimages are codimension 1 phenomena, it seems plausible that by taking the closure of a stable region, we can make it convex.

Question III.9.3 Is the closure of a stable region always convex?

Because the stable regions are generally not convex, the following also appears to be a non-trivial question:

Question III.9.4 Are the stable regions simply connected?

Because of ghost preimages, the boundary of stable regions can contain countably many hyperplanes. The accumulation properties of these hyperplanes seem difficult to describe, and we are led to the following question:

Question III.9.5 Is the boundary of every stable region locally connected?

Besides their geometry, a step towards the proof of Boshernitzan–Kornfeld Conjecture is to understand how the stable regions are arranged in the parameter $\text{ITM}(r)$. Are they necessarily separated from each other, or can they ‘tile’ the parameter space?

Question III.9.6 Are the stable regions (partially) separated from each other? In other words, does for every stable region $[T]$ exist an open subset of the (outer) boundary on which the set of points that are contained in the boundary of some other stable component is small (e.g. meager or countable)?

Figure III.9.1 suggests that this is true: every closed coloured triangle has at least one side on which the set of points where it touches other triangles is countable.

Another interesting problem is describing the dynamics in the boundary of stable regions:

Question III.9.7 What are the dynamical properties of a map in the boundary of a stable region?

The boundary of a stable set is where the topology of a non-wandering set changes, so it is contained in *bifurcation locus* of $\text{ITM}(r)$. Thus the answer to this question would constitute the start of the bifurcation theory of ITMs. Intuitively, the bifurcations should be associated with the changes of critical itineraries, and they should in turn be associated with critical connections. This is confirmed by Theorem III.8.5 for finite type maps. It is not clear that this should also hold for infinite-type maps:

Question III.9.8 Do infinite type maps in the boundary of a stable region have critical connections?

Since a necessary condition for a finite type map to be in the boundary of a stable region is that it has a critical connection, a natural question to ask is whether the maximal possible

number of critical connections is sufficient for a map to be in the closure of a stable region. The eventually periodic maps are exactly such maps, so we conjecture the following:

Conjecture III.9.9 Each eventually periodic map is contained in the closure of some stable region.

The proof of Theorem [III.7.1](#) does not give the proof of this conjecture, even though the perturbation is arbitrarily small. The reason for this is that the perturbations we inductively make do not give us full control of the dynamical properties of the resulting map, just that the number of critical connections decreases and that the map remains eventually periodic. Thus it is possible that by making the perturbation smaller, we end up in different stable regions.

Bibliography

- [AB13] Jayadev S. Athreya and Michael Boshernitzan. “Ergodic properties of compositions of interval exchange maps and rotations”. In: *Nonlinearity* 26.2 (2013), pp. 417–423.
- [AC15] Jayadev S. Athreya and Jon Chaika. “The Hausdorff dimension of non-uniquely ergodic directions in $H(2)$ is almost everywhere $\frac{1}{2}$ ”. In: *Geom. Topol.* 19.6 (2015), pp. 3537–3563.
- [AD16] Artur Avila and Vincent Delecroix. “Weak mixing directions in non-arithmetic Veech surfaces”. In: *J. Amer. Math. Soc.* 29.4 (2016), pp. 1167–1208.
- [AF07] Artur Avila and Giovanni Forni. “Weak mixing for interval exchange transformations and translation flows”. In: *Ann. of Math. (2)* 165.2 (2007), pp. 637–664.
- [AFS23] Artur Avila, Giovanni Forni, and Pedram Safaee. “Quantitative weak mixing for interval exchange transformations”. In: *Geom. Funct. Anal.* 33.1 (2023), pp. 1–56.
- [AHCF24] Francisco Arana-Herrera, Jon Chaika, and Giovanni Forni. *Weak mixing in rational billiards*. 2024.
- [AHS24] Mauro Artigiani, Pascal Hubert, and Alexandra Skripchenko. “Renormalization for Bruin-Troubetzkoy ITMs”. In: *arXiv preprint arXiv:2412.07928* (2024).

- [AL18] Artur Avila and Martin Leguil. “Weak mixing properties of interval exchange transformations & translation flows”. In: *Bull. Soc. Math. France* 146.2 (2018), pp. 391–426.
- [AM24] Jayadev S. Athreya and Howard Masur. *Translation surfaces*. Vol. 242. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, [2024] ©2024, pp. xi+179.
- [AOW85] Pierre Arnoux, Donald S. Ornstein, and Benjamin Weiss. “Cutting and stacking, interval exchanges and geometric models”. In: *Israel J. Math.* 50.1-2 (1985), pp. 160–168.
- [Art+21] M. Artigiani, C. Fougeron, P. Hubert, and A. Skripchenko. “A note on double rotations of infinite type”. In: *Trans. Moscow Math. Soc.* 82 (2021), pp. 157–172.
- [AS+21] Hamid Al-Saqban, Paul Apisa, Alena Erchenko, Osama Khalil, Shahriar Mirzadeh, and Caglar Uyanik. “Exceptional directions for the Teichmüller geodesic flow and Hausdorff dimension”. In: *J. Eur. Math. Soc. (JEMS)* 23.5 (2021), pp. 1423–1476.
- [AV07] Artur Avila and Marcelo Viana. “Simplicity of Lyapunov spectra: proof of the Zorich-Kontsevich conjecture”. In: *Acta Math.* 198.1 (2007), pp. 1–56.
- [BC12] Henk Bruin and Gregory Clack. “Inducing and unique ergodicity of double rotations”. In: *Discrete Contin. Dyn. Syst.* 32.12 (2012), pp. 4133–4147.
- [BC13] Michael Boshernitzan and Jon Chaika. “Diophantine properties of IETs and general systems: quantitative proximality and connectivity”. In: *Invent. Math.* 192.2 (2013), pp. 375–412.
- [BCK17] Henk Bruin, Carlo Carminati, and Charlene Kalle. “Matching for generalised β -transformations”. In: *Indag. Math. (N.S.)* 28.1 (2017), pp. 55–73.
- [BH04] Jérôme Buzzi and Pascal Hubert. “Piecewise monotone maps without periodic points: rigidity, measures and complexity”. In: *Ergodic Theory Dynam. Systems* 24.2 (2004), pp. 383–405.

- [BK95] Michael Boshernitzan and Isaac Kornfeld. “Interval translation mappings”. In: *Ergodic Theory Dynam. Systems* 15.5 (1995), pp. 821–832.
- [BKM78] Carlo Boldrighini, Michael Keane, and Federico Marchetti. “Billiards in polygons”. In: *Ann. Probab.* 6.4 (1978), pp. 532–540.
- [Boi12] Corentin Boissy. “Classification of Rauzy classes in the moduli space of abelian and quadratic differentials”. In: *Discrete Contin. Dyn. Syst.* 32.10 (2012), pp. 3433–3457.
- [Bos12] Michael Boshernitzan. “A condition for weak mixing of induced IETs”. In: *Dynamical systems and group actions*. Vol. 567. Contemp. Math. Amer. Math. Soc., Providence, RI, 2012, pp. 53–65.
- [Bos84] Michael Boshernitzan. “A unique ergodicity of minimal symbolic flows with linear block growth”. In: *J. Analyse Math.* 44 (1984/85), pp. 77–96.
- [Bos85] Michael Boshernitzan. “A condition for minimal interval exchange maps to be uniquely ergodic”. In: *Duke Math. J.* 52.3 (1985), pp. 723–752.
- [Bos93] Michael D. Boshernitzan. “Quantitative recurrence results”. In: *Invent. Math.* 113.3 (1993), pp. 617–631.
- [BR23] Henk Bruin and Silvia Radinger. “Interval Translation Maps with Weakly Mixing Attractors”. In: *arXiv preprint arXiv:2312.10533* (2023).
- [Bru07a] Henk Bruin. “Renormalization in a class of interval translation maps of d branches”. In: *Dynamical Systems* 22 (2007), pp. 11–24.
- [Bru07b] Henk Bruin. “Renormalization in a class of interval translation maps of d branches”. In: *Dyn. Syst.* 22.1 (2007), pp. 11–24.
- [Bru+19] Henk Bruin, Carlo Carminati, Stefano Marmi, and Alessandro Profeti. “Matching in a family of piecewise affine maps”. In: *Nonlinearity* 32.1 (2019), pp. 172–208.
- [BT03] H. Bruin and S. Troubetzkoy. “The Gauss map on a class of interval translation mappings”. In: *Israel J. Math.* 137 (2003), pp. 125–148.

- [Buz01] Jérôme Buzzi. “Piecewise isometries have zero topological entropy”. In: *Ergodic Theory Dynam. Systems* 21.5 (2001), pp. 1371–1377.
- [CC19] Jon Chaika and David Constantine. “Quantitative shrinking target properties for rotations and interval exchanges”. In: *Israel J. Math.* 230.1 (2019), pp. 275–334.
- [CE19] Jon Chaika and Alex Eskin. “Möbius disjointness for interval exchange transformations on three intervals”. In: *J. Mod. Dyn.* 14 (2019), pp. 55–86.
- [CE21] Jon Chaika and Alex Eskin. “Self-joinings for 3-IETs”. In: *J. Eur. Math. Soc. (JEMS)* 23.8 (2021), pp. 2707–2731.
- [CF15] Jon Chaika and Jon Fickenscher. “Topological mixing for some residual sets of interval exchange transformations”. In: *Comm. Math. Phys.* 333.1 (2015), pp. 483–503.
- [CF20] Jon Chaika and Giovanni Forni. *Weakly Mixing Polygonal Billiards*. 2020.
- [CH19] Jon Chaika and Sebastian Hensel. “The set of uniquely ergodic interval exchange transformations is path-connected”. In: *Ergodic Theory Dynam. Systems* 39.2 (2019), pp. 311–356.
- [Cha08] Jon Chaika. “Hausdorff dimension for ergodic measures of interval exchange transformations”. In: *J. Mod. Dyn.* 2.3 (2008), pp. 457–464.
- [Cha11] Jon Chaika. “Shrinking targets for IETs: extending a theorem of Kurzweil”. In: *Geom. Funct. Anal.* 21.5 (2011), pp. 1020–1042.
- [Cha12a] Jon Chaika. “Every ergodic transformation is disjoint from almost every interval exchange transformation”. In: *Ann. of Math. (2)* 175.1 (2012), pp. 237–253.
- [Cha12b] Jon Chaika. “There exists a topologically mixing interval exchange transformation”. In: *Ergodic Theory Dynam. Systems* 32.3 (2012), pp. 869–875.
- [Che03] Yitwah Cheung. “Hausdorff dimension of the set of nonergodic directions”. In: *Ann. of Math. (2)* 158.2 (2003). With an appendix by M. Boshernitzan, pp. 661–678.

- [CHM11] Yitwah Cheung, Pascal Hubert, and Howard Masur. “Dichotomy for the Hausdorff dimension of the set of nonergodic directions”. In: *Invent. Math.* 183.2 (2011), pp. 337–383.
- [CM20] Jon Chaika and Howard Masur. “The set of non-uniquely ergodic d -IETs has Hausdorff codimension $1/2$ ”. In: *Invent. Math.* 222.3 (2020), pp. 749–832.
- [CN01] Juliana Chaves and Arnaldo Nogueira. “Spectral properties of interval exchange maps”. In: *Monatsh. Math.* 134.2 (2001), pp. 89–102.
- [CR19] Jon Chaika and Donald Robertson. “Ergodicity of skew products over linearly recurrent IETs”. In: *J. Lond. Math. Soc. (2)* 100.1 (2019), pp. 223–248.
- [Del13] Vincent Delecroix. “Cardinality of Rauzy classes”. In: *Ann. Inst. Fourier (Grenoble)* 63.5 (2013), pp. 1651–1715.
- [DHV24] Vincent Delecroix, Pascal Hubert, and Ferrán Valdez. *Infinite Translation Surfaces in the Wild*. 2024.
- [DSS25] Kostiantyn Drach, Leon Staresinic, and Sebastian van Strien. *Density of Stable Interval Translation Maps*. 2025.
- [Eli+02] L. H. Eliasson, S. B. Kuksin, S. Marmi, and J.-C. Yoccoz. *Dynamical systems and small divisors*. Vol. 1784. Lecture Notes in Mathematics. Lectures from the C.I.M.E. Summer School held in Cetraro, June 13–20, 1998, Edited by Marmi and Yoccoz, Fondazione CIME/CIME Foundation Subseries. Springer-Verlag, Berlin; Centro Internazionale Matematico Estivo (C.I.M.E.), Florence, 2002, pp. viii+191.
- [EW11] Manfred Einsiedler and Thomas Ward. *Ergodic theory with a view towards number theory*. Vol. 259. Graduate Texts in Mathematics. Springer-Verlag London, Ltd., London, 2011, pp. xviii+481.
- [Fic16] Jon Fickenscher. “A combinatorial proof of the Kontsevich-Zorich-Boissy classification of Rauzy classes”. In: *Discrete Contin. Dyn. Syst.* 36.4 (2016), pp. 1983–2025.
- [For02] Giovanni Forni. “Deviation of ergodic averages for area-preserving flows on surfaces of higher genus”. In: *Ann. of Math. (2)* 155.1 (2002), pp. 1–103.

- [For97] Giovanni Forni. “Solutions of the cohomological equation for area-preserving flows on compact surfaces of higher genus”. In: *Ann. of Math.* (2) 146.2 (1997), pp. 295–344.
- [Fou20a] Charles Fougeron. *Dynamical properties of simplicial systems and continued fraction algorithms*. 2020.
- [Fou20b] Charles Fougeron. “Dynamical properties of simplicial systems and continued fraction algorithms”. In: *arXiv preprint arXiv:2001.01367* (2020).
- [Fur81] H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory*. M. B. Porter Lectures. Princeton University Press, Princeton, NJ, 1981, pp. xi+203.
- [Gha21] Selim Ghazouani. “Local rigidity for periodic generalised interval exchange transformations”. In: *Invent. Math.* 226.2 (2021), pp. 467–520.
- [GU23] Selim Ghazouani and Corinna Ulcigrai. “A priori bounds for GIETs, affine shadows and rigidity of foliations in genus two”. In: *Publ. Math. Inst. Hautes Études Sci.* 138 (2023), pp. 229–366.
- [Hub06] John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 1*. Teichmüller theory, With contributions by Adrien Douady, William Dunbar, Roland Roeder, Sylvain Bonnot, David Brown, Allen Hatcher, Chris Hruska and Sudeb Mitra, With forewords by William Thurston and Clifford Earle. Matrix Editions, Ithaca, NY, 2006, pp. xx+459.
- [Hub16] John Hamal Hubbard. *Teichmüller theory and applications to geometry, topology, and dynamics. Vol. 2*. Surface homeomorphisms and rational functions. Matrix Editions, Ithaca, NY, 2016, pp. x+262.
- [iN04] Michael itzan and Arnaldo Nogueira. “Generalized eigenfunctions of interval exchange maps”. In: *Ergodic Theory Dynam. Systems* 24.3 (2004), pp. 697–705.

- [Jia+24] Yuxiang Jiao, Jialun Li, Wenyu Pan, and Disheng Xu. “On the Dimension of Limit Sets on $P(R^3)$ via Stationary Measures: Variational Principles and Applications”. In: *International Mathematics Research Notices* 2024.19 (2024), pp. 13015–13045.
- [Jur23] Natalia Jurga. “Hausdorff dimension of the Rauzy gasket”. In: *arXiv preprint arXiv:2312.04999* (2023).
- [Kat73] A. B. Katok. “Invariant measures of flows on orientable surfaces”. In: *Dokl. Akad. Nauk SSSR* 211 (1973), pp. 775–778.
- [Kat80] Anatole Katok. “Interval exchange transformations and some special flows are not mixing”. In: *Israel J. Math.* 35.4 (1980), pp. 301–310.
- [Kea75] Michael Keane. “Interval exchange transformations”. In: *Math. Z.* 141 (1975), pp. 25–31.
- [Kea77] Michael Keane. “Non-ergodic interval exchange transformations”. In: *Israel J. Math.* 26.2 (1977), pp. 188–196.
- [Ker85] S. P. Kerckhoff. “Simplicial systems for interval exchange maps and measured foliations”. In: *Ergodic Theory Dynam. Systems* 5.2 (1985), pp. 257–271.
- [KMS86] Steven Kerckhoff, Howard Masur, and John Smillie. “Ergodicity of billiard flows and quadratic differentials”. In: *Ann. of Math. (2)* 124.2 (1986), pp. 293–311.
- [KN76] Harvey B. Keynes and Dan Newton. “A “minimal”, non-uniquely ergodic interval exchange transformation”. In: *Math. Z.* 148.2 (1976), pp. 101–105.
- [Kon97] M. Kontsevich. “Lyapunov exponents and Hodge theory”. In: *The mathematical beauty of physics (Saclay, 1996)*. Vol. 24. Adv. Ser. Math. Phys. World Sci. Publ., River Edge, NJ, 1997, pp. 318–332.
- [KR69] Harvey B. Keynes and James B. Robertson. “Eigenvalue theorems in topological transformation groups”. In: *Trans. Amer. Math. Soc.* 139 (1969), pp. 359–369.

- [KR80] Michael S. Keane and Gérard Rauzy. “Stricte ergodicité des échanges d’intervalles”. In: *Math. Z.* 174.3 (1980), pp. 203–212.
- [KS67] A. B. Katok and A. M. Stepin. “Approximations in ergodic theory”. In: *Uspehi Mat. Nauk* 22.5(137) (1967), pp. 81–106.
- [KSS07] Oleg Kozlovski, Weixiao Shen, and Sebastian van Strien. “Density of hyperbolicity in dimension one”. In: *Ann. of Math. (2)* 166.1 (2007), pp. 145–182.
- [Kur55] J. Kurzweil. “On the metric theory of inhomogeneous diophantine approximations”. In: *Studia Math.* 15 (1955), pp. 84–112.
- [KZ03] Maxim Kontsevich and Anton Zorich. “Connected components of the moduli spaces of Abelian differentials with prescribed singularities”. In: *Invent. Math.* 153.3 (2003), pp. 631–678.
- [Lan95] Serge Lang. *Introduction to Diophantine approximations*. Second. Springer-Verlag, New York, 1995, pp. x+130.
- [Mas82] Howard Masur. “Interval exchange transformations and measured foliations”. In: *Ann. of Math. (2)* 115.1 (1982), pp. 169–200.
- [Mas92] Howard Masur. “Hausdorff dimension of the set of nonergodic foliations of a quadratic differential”. In: *Duke Math. J.* 66.3 (1992), pp. 387–442.
- [McM94] Curtis T. McMullen. *Complex dynamics and renormalization*. Vol. 135. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1994, pp. x+214.
- [MMY05] S. Marmi, P. Moussa, and J.-C. Yoccoz. “The cohomological equation for Roth-type interval exchange maps”. In: *J. Amer. Math. Soc.* 18.4 (2005), pp. 823–872.
- [MMY12] Stefano Marmi, Pierre Moussa, and Jean-Christophe Yoccoz. “Linearization of generalized interval exchange maps”. In: *Ann. of Math. (2)* 176.3 (2012), pp. 1583–1646.

- [MS93] Welington de Melo and Sebastian van Strien. *One-dimensional dynamics*. Vol. 25. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]. Springer-Verlag, Berlin, 1993, pp. xiv+605.
- [MS98] Curtis T. McMullen and Dennis P. Sullivan. “Quasiconformal homeomorphisms and dynamics. III. The Teichmüller space of a holomorphic dynamical system”. In: *Adv. Math.* 135.2 (1998), pp. 351–395.
- [MST17] Alba Málaga Sabogal and Serge Troubetzkoy. “Weakly mixing polygonal billiards”. In: *Bull. Lond. Math. Soc.* 49.1 (2017), pp. 141–147.
- [MT02a] Howard Masur and Serge Tabachnikov. “Chapter 13 Rational billiards and flat structures”. In: ed. by Boris Hasselblatt and Anatole Katok. Vol. 1. *Handbook of Dynamical Systems*. Elsevier Science, 2002, pp. 1015–1089.
- [MT02b] Howard Masur and Serge Tabachnikov. “Rational billiards and flat structures”. In: *Handbook of dynamical systems, Vol. 1A*. North-Holland, Amsterdam, 2002, pp. 1015–1089.
- [MT88] John Milnor and William Thurston. “On iterated maps of the interval”. In: *Dynamical systems (College Park, MD, 1986–87)*. Vol. 1342. *Lecture Notes in Math*. Springer, Berlin, 1988, pp. 465–563.
- [MY16] Stefano Marmi and Jean-Christophe Yoccoz. “Hölder regularity of the solutions of the cohomological equation for Roth type interval exchange maps”. In: *Comm. Math. Phys.* 344.1 (2016), pp. 117–139.
- [NN08] Hitoshi Nakada and Rie Natsui. “The non-monotonicity of the entropy of α -continued fraction transformations”. In: *Nonlinearity* 21.6 (2008), pp. 1207–1225.
- [Nog89] Arnaldo Nogueira. “Almost all interval exchange transformations with flips are nonergodic”. In: *Ergodic Theory Dynam. Systems* 9.3 (1989), pp. 515–525.
- [NPT13] Arnaldo Nogueira, Benito Pires, and Serge Troubetzkoy. “Orbit structure of interval exchange transformations with flip”. In: *Nonlinearity* 26.2 (2013), pp. 525–537.

- [NR97] A. Nogueira and D. Rudolph. “Topological weak-mixing of interval exchange maps”. In: *Ergodic Theory Dynam. Systems* 17.5 (1997), pp. 1183–1209.
- [PM93] Ricardo Pérez Marco. “Sur les dynamiques holomorphes non linéarisables et une conjecture de V. I. Arnol’d”. In: *Ann. Sci. École Norm. Sup. (4)* 26.5 (1993), pp. 565–644.
- [Rau77] Gérard Rauzy. “Une généralisation du développement en fraction continue”. In: *Séminaire Delange-Pisot-Poitou, 18e année: 1976/77, Théorie des nombres, Fasc. I*. Secrétariat Math., Paris, 1977, Exp. No. 15, 16.
- [Rau79] Gérard Rauzy. “Échanges d’intervalles et transformations induites”. In: *Acta Arith.* 34.4 (1979), pp. 315–328.
- [Ree81] Mary Rees. “An alternative approach to the ergodic theory of measured foliations on surfaces”. In: *Ergodic Theory Dynam. Systems* 1.4 (1981), pp. 461–488.
- [Rob19] Donald Robertson. “Mild mixing of certain interval-exchange transformations”. In: *Ergodic Theory Dynam. Systems* 39.1 (2019), pp. 248–256.
- [Sch11] Richard Evan Schwartz. *Mostly surfaces*. Vol. 60. Student Mathematical Library. American Mathematical Society, Providence, RI, 2011, pp. xiv+314.
- [SIA05] Hideyuki Suzuki, Shunji Ito, and Kazuyuki Aihara. “Double rotations”. In: *Discrete Contin. Dyn. Syst.* 13.2 (2005), pp. 515–532.
- [Sin73] Ya.Ġ. Sinai. *Introduction to ergodic theory*. Erevan Gos. Univ, 1973.
- [Skr23] A. S. Skripchenko. “Renormalization in one-dimensional dynamics”. In: *Uspekhi Mat. Nauk* 78.6(474) (2023), pp. 3–46.
- [ST00] J. Schmeling and S. Troubetzkoy. “Interval translation mappings”. In: *Dynamical systems (Luminy-Marseille, 1998)*. World Sci. Publ., River Edge, NJ, 2000, pp. 291–302.
- [ST15a] Alexandra Skripchenko and Serge Troubetzkoy. “Entropy and complexity of polygonal billiards with spy mirrors”. In: *Nonlinearity* 28.9 (2015), pp. 3443–3456.

- [ST15b] Alexandra Skripchenko and Serge Troubetzkoy. “Polygonal billiards with one sided scattering”. In: *Ann. Inst. Fourier (Grenoble)* 65.5 (2015), pp. 1881–1896.
- [ST18] Alexandra Skripchenko and Serge Troubetzkoy. “On the Hausdorff dimension of minimal interval exchange transformations with flips”. In: *J. Lond. Math. Soc. (2)* 97.2 (2018), pp. 149–169.
- [SU05] Ya.Ġ. Sinai and C. Ulcigrai. “Weak mixing in interval exchange transformations of periodic type”. In: *Lett. Math. Phys.* 74.2 (2005), pp. 111–133.
- [Tab05] Serge Tabachnikov. *Geometry and billiards*. Vol. 30. Student Mathematical Library. American Mathematical Society, Providence, 2005, pp. xii+176.
- [Vee77] William A. Veech. “Topological dynamics”. In: *Bull. Amer. Math. Soc.* 83.5 (1977), pp. 775–830.
- [Vee78] William A. Veech. “Interval exchange transformations”. In: *J. Analyse Math.* 33 (1978), pp. 222–272.
- [Vee81] William A. Veech. “Projective Swiss cheeses and uniquely ergodic interval exchange transformations”. In: *Ergodic theory and dynamical systems, I (College Park, Md., 1979–80)*. Vol. 10. Progr. Math. Birkhäuser, Boston, MA, 1981, pp. 113–193.
- [Vee82] William A. Veech. “Gauss measures for transformations on the space of interval exchange maps”. In: *Ann. of Math. (2)* 115.1 (1982), pp. 201–242.
- [Vee84a] William A. Veech. “The metric theory of interval exchange transformations. I. Generic spectral properties”. In: *Amer. J. Math.* 106.6 (1984), pp. 1331–1359.
- [Vee84b] William A. Veech. “The metric theory of interval exchange transformations. II. Approximation by primitive interval exchanges”. In: *Amer. J. Math.* 106.6 (1984), pp. 1361–1387.
- [Vee84c] William A. Veech. “The metric theory of interval exchange transformations. III. The Sah-Arnoux-Fathi invariant”. In: *Amer. J. Math.* 106.6 (1984), pp. 1389–1422.

- [Vee87] William A. Veech. “Boshernitzan’s criterion for unique ergodicity of an interval exchange transformation”. In: *Ergodic Theory Dynam. Systems* 7.1 (1987), pp. 149–153.
- [Via06] Marcelo Viana. “Ergodic theory of interval exchange maps”. In: *Rev. Mat. Complut.* 19.1 (2006), pp. 7–100.
- [Vol14] Denis Volk. “Almost every interval translation map of three intervals is finite type”. In: *Discrete Contin. Dyn. Syst.* 34.5 (2014), pp. 2307–2314.
- [Yoc06] Jean-Christophe Yoccoz. “Continued fraction algorithms for interval exchange maps: an introduction”. In: *Frontiers in number theory, physics, and geometry. I.* Springer, Berlin, 2006, pp. 401–435.
- [ZK75a] A. N. Zemljakov and A. B. Katok. “Topological transitivity of billiards in polygons”. In: *Mat. Zametki* 18.2 (1975), pp. 291–300.
- [ZK75b] Alexander N. Zemljakov and Anatole B. Katok. “Topological transitivity of billiards in polygons”. In: *Mat. Zametki* 18.2 (1975), pp. 291–300.
- [Zor06] Anton Zorich. “Flat surfaces”. In: *Frontiers in number theory, physics, and geometry. I.* Springer, Berlin, 2006, pp. 437–583.
- [Zor08] Anton Zorich. “Explicit Jenkins-Strebel representatives of all strata of abelian and quadratic differentials”. In: *J. Mod. Dyn.* 2.1 (2008), pp. 139–185.
- [Zor96] Anton Zorich. “Finite Gauss measure on the space of interval exchange transformations. Lyapunov exponents”. In: *Ann. Inst. Fourier (Grenoble)* 46.2 (1996), pp. 325–370.
- [Zor97] Anton Zorich. “Deviation for interval exchange transformations”. In: *Ergodic Theory Dynam. Systems* 17.6 (1997), pp. 1477–1499.
- [Zor99] Anton Zorich. “How do the leaves of a closed 1-form wind around a surface?”. In: *Pseudoperiodic topology*. Vol. 197. Amer. Math. Soc. Transl. Ser. 2. Amer. Math. Soc., Providence, RI, 1999, pp. 135–178.