INTERPOLATION INEQUALITIES IN PRESENCE OF A MAGNETIC FIELD AND APPLICATION TO SCHRÖDINGER OPERATORS

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Let us consider the Schrödinger operator with exterior magnetic field $\mathbf{B}$. It is defined via a magnetic potential $\mathbf{A}$ satisfying $\mathrm{curl}\,\mathbf{A} = \mathbf{B}$ as follows:

$$-\Delta_{\mathbf{A}} = -\Delta - 2\,i\,\mathbf{A}\cdot\nabla + |\mathbf{A}|^2 - 2\,i\,\mathrm{div}\mathbf{A},$$

and the associated quadratic form is given by

$$\int_{\mathbb{R}^d} |\nabla_{\mathbf{A}} u|^2\,dx$$

where

$$\nabla_{\mathbf{A}} := \nabla + i\,\mathbf{A}$$

is the covariant derivative. The quadratic form is well defined for all functions

$$u \in \mathrm{H}^1_{\mathbf{A}} := \left\{ u \in \mathrm{L}^2(\mathbb{R}^d) \,:\, \nabla_{\mathbf{A}} u \in \mathrm{L}^2(\mathbb{R}^d) \right\}.$$

In this paper we are mainly interested in the case of a constant magnetic field $\mathbf{B}$ of strength B in dimension $d = 2$, with $\mathbf{A} = \frac{B}{2}(-x_2, x_1)$. It is well-known that the infimum of the spectrum of $-\Delta_{\mathbf{A}}$ is equal to B. Moreover the spectrum of $-\Delta_{\mathbf{A}}$ is a the countable set $\{(2j+1)\,B \,:\, j \in \mathbb{N}\}$. The eigenspaces corresponding to these elements of the spectrum are called the Landau levels and they are all infinity dimensional. In particular the eigenspace corresponding to the lowest level B is called the LLL (lowest Landau level). The case of a constant magnetic field $\mathbf{B} = (0,0,B)$ with magnetic potential $\mathbf{A} = \frac{B}{2}(-x_2, x_1, 0)$ in dimension $d = 3$ is similar, but is not studied in this paper.

1. Some functional inequalities in presence of a constant magnetic field

1.1. Some preliminary results in case $p \in (2,\infty)$.

Proposition 1. *Consider a constant magnetic field with field strength B in two dimensions and denote the covariant derivative by $\nabla_{\mathbf{A}}$. Then given any $2 \le p < \infty$, there exists a constant S_p independent of B such that*

$$\|\nabla_{\mathbf{A}}\psi\|_2 \ge S_p\, B^{1/p}\, \|\psi\|_p\,.$$

Proof. Recall the Gagliardo-Nirenberg inequality

$$\|\nabla|\psi|\|_2^{1-2/p}\, \|\psi\|_2^{2/p} \ge S_p\, \|\psi\|_p \quad \forall\, \psi \in \mathrm{H}^1(\mathbb{R}^2)\,. \tag{1}$$

Now observe that for a constant magnetic field

$$B\,\|\psi\|_2^2 \le \|\nabla_{\mathbf{A}}\psi\|_2^2$$

and moreover by the diamagnetic inequality

$$\|\nabla|\psi|\|_2 \le \|\nabla_{\mathbf{A}}\psi\|_2$$

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and hence

$$S_p \|\psi\|_p \le \|\nabla|\psi|\|_2^{1-2/p} \|\psi\|_2^{2/p} \le \|\nabla_{\mathbf{A}}\psi\|_2^{1-2/p} \|\psi\|_2^{2/p} \le B^{-1/p} \|\nabla_{\mathbf{A}}\psi\|_2 .$$

□

The following result follows from the proof of [23, Theorem 3.1].

Lemma 2. *With the notations of the Proposition 1, for every $c \in [0,1]$, we have*

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\psi|^2 \, dx \ge \left(1-c^2\right) \int_{\mathbb{R}^2} |\nabla\psi|^2 \, dx + c\,B \int_{\mathbb{R}^2} \psi^2 \, dx , \tag{2}$$

and equality holds with $\psi = u\, e^{iS}$ and $u > 0$ if and only if

$$\left(-\partial_2 u^2, \partial_1 u^2\right) = \frac{2\,u^2}{c} \left(\mathbf{A} + \nabla S\right) . \tag{3}$$

Proof. For every $c \in [0,1]$,

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\psi|^2 \, dx = \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} |\mathbf{A}+\nabla S|^2 \, u^2 \, dx$$
$$= \left(1-c^2\right) \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} \left(c^2 |\nabla u|^2 + |\mathbf{A}+\nabla S|^2 \, u^2\right) dx .$$

Now,

$$\int_{\mathbb{R}^2} \left(c^2 |\nabla u|^2 + |\mathbf{A}+\nabla S|^2 \, u^2\right) dx \ge \int_{\mathbb{R}^2} 2\,c\,|\nabla u|\,|\mathbf{A}+\nabla S|\,u \, dx .$$

with equality only if $c|\nabla u| = |\mathbf{A}+\nabla S|\,u$. Since we have chosen the gauge

$$\mathbf{A}_1 = \tfrac{B}{2} x_2 , \quad \mathbf{A}_2 = -\tfrac{B}{2} x_1$$

so that $\partial_2 \mathbf{A}_1 - \partial_1 \mathbf{A}_2 = B$. Next,

$$2\,|\nabla u|\,|\mathbf{A}+\nabla S|\,u = |\nabla u^2|\,|\mathbf{A}+\nabla S| \ge \left(\nabla u^2\right)^\perp \cdot (\mathbf{A}+\nabla S) ,$$

where $\left(\nabla u^2\right)^\perp := \left(-\partial_2 u^2, \partial_1 u^2\right)$, and there is equality only if

$$\left(-\partial_2 u^2, \partial_1 u^2\right) = \gamma \left(\mathbf{A}+\nabla S\right)$$

for some γ. Since $c|\nabla u| = |\mathbf{A}+\nabla S|\,u$, we have $\gamma = 2\,u^2/c$. Integration by parts yields

$$\int_{\mathbb{R}^2} \left(c^2 |\nabla u|^2 + |\mathbf{A}+\nabla S|^2 \, u^2\right) dx \ge B\,c \int_{\mathbb{R}^2} u^2 \, dx .$$

□

Let C_p denote the optimal constant of the embedding of $\mathrm{H}^1(\mathbb{R}^2)$ into $\mathrm{L}^p(\mathbb{R}^d)$, that is, the best constant in the inequality

$$\|\nabla u\|_2^2 + \|u\|_2^2 \ge C_p^2 \, \|u\|_p^2 \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d) . \tag{4}$$

By considering the scaling $u_\lambda(x) = u\big(x/\lambda\big)$, we find that

$$\|\nabla u\|_2^2 + \lambda^2 \, \|u\|_2^2 \ge C_p^2 \, \lambda^{2-d\left(1-\frac{2}{p}\right)} \, \|u\|_p^2 \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d) . \tag{5}$$

From now on, we assume that $d = 2$. An optimization on $\lambda > 0$ shows that the optimal constant in (1) is given by

$$S_p^2 = \tfrac{2}{p} \left(1 - \tfrac{2}{p}\right)^{1-\frac{2}{p}} C_p^2 .$$

1.2. **Case $p \in (2,\infty)$.**

Proposition 3. *Consider a constant magnetic field with field strength B in two dimensions and denote the covariant derivative by $\nabla_{\mathbf{A}}$. Given any $p \in (2,+\infty)$, and any $\alpha > -B$, we have*

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 \geq C_p^2\left(1-c^2\right)^{1-\frac{2}{p}}(\alpha + c\,B)^{\frac{2}{p}}\,\|\psi\|_p^2 \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^2) \tag{6}$$

with $1/c = \eta + \sqrt{\eta^2+p-1} > 1$ and $\eta = \alpha\,(p-2)/(2\,B)$. Moreover, when either $\alpha \to +\infty$ for a fixed given $B>0$ or $B \to 0_+$ for a given $\alpha > 0$, we have

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 \geq C_p^2\,\alpha^{\frac{2}{p}}\,\|\psi\|_p^2\,(1+o(1))\,.$$

Proof. For any $\alpha > -B$ and $c \in [0,1]$ such that $\alpha + c\,B \geq 0$, we use Lemma 2 to write

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 \geq \left(1-c^2\right)\int_{\mathbb{R}^2}|\nabla u|^2\,dx + (\alpha + c\,B)\int_{\mathbb{R}^2}u^2\,dx\,.$$

By applying (5) with $\lambda^2 = (\alpha + c\,B)/\left(1-c^2\right)$, we get

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 \geq C_p^2\left(1-c^2\right)^{1-\frac{2}{p}}(\alpha + c\,B)^{\frac{2}{p}}\,\|\psi\|_p^2\,.$$

Next we optimize the function $c \mapsto \left(1-c^2\right)^{1-\frac{2}{p}}(\alpha + c\,B)^{\frac{2}{p}}$ in the interval $[0,1]$. This function reaches its maximum at c such that

$$(p-2)\,c\,(\alpha + c\,B) = B\,(1-c^2)\,.$$

Notice that $\alpha + c\,B$ is nonnegative. With

$$\eta = \frac{\alpha\,(p-2)}{2\,B}\,,$$

the equation for c becomes

$$(p-1)\,c^2 + 2\,\eta\,c - 1 = 0\,.$$

An elementary computation shows that

$$c = c(p,\eta) = \frac{\sqrt{\eta^2+p-1}-\eta}{p-1} = \frac{1}{\eta + \sqrt{\eta^2+p-1}} \in (0,1)\,.$$

In the asymptotic regimes as either $\alpha \to +\infty$ or $B \to 0_+$, c converges to 0. □

1.3. **Logarithmic Sobolev inequality.** By passing to the limit as $p \to 2_+$ in (6), we obtain a logarithmic Sobolev inequality.

Corollary 4. *Consider a constant magnetic field with field strength $B>0$ in two dimensions. Then for any $\eta > 0$ we have*

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 \geq \eta\,B\int_{\mathbb{R}^2}\psi^2\log\left(\frac{\psi^2}{\|\psi\|_2^2}\right)dx + B\left[c(2,\eta) + \eta\log\left(\tfrac{2\,\pi\,e^2\,c(2,\eta)}{B}\right)\right]\|\psi\|_2^2 \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^2)\,, \tag{7}$$

where $c(2,\eta) = \sqrt{\eta^2+1}-\eta$.

Proof. First let us consider the Gagliardo-Nirenberg inequality

$$\|\nabla\psi\|_2^2 + \|\psi\|_2^2 \geq C_p^2\,\|\psi\|_p^2 \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^d)$$

written with optimal constant C_p^2 for $p>2$. Let us apply (5) with $\lambda^2 = \frac{1}{p-2}$, so that our inequality now reads

$$\|\nabla\psi\|_2^2 - \tfrac{1}{p-2}\left(\|\psi\|_p^2 - \|\psi\|_2^2\right) \geq \left[C_p^2\left(\tfrac{1}{p-2}\right)^{1-d\,\frac{p-2}{2p}} - \tfrac{1}{p-2}\right]\|\psi\|_p^2\,.$$

By passing to the limit as $p \to 2$, we recover the Euclidean logarithmic Sobolev inequality with optimal constant, that is

$$\|\nabla\psi\|_2^2 \geq \tfrac{1}{2}\int_{\mathbb{R}^d} \psi^2 \log\left(\frac{\psi^2}{\|\psi\|_2^2}\right) dx + \tfrac{d}{4}\log\left(2\pi e^2\right)\|\psi\|_2^2 \quad \forall\,\psi\in \mathrm{H}^1(\mathbb{R}^d)\,. \tag{8}$$

As a consequence, we obtain that $\lim_{p\to 2_+} C_p = 1$ and

$$\lim_{p\to 2_+}\left[C_p^2\left(\tfrac{1}{p-2}\right)^{1-d\,\frac{p-2}{2p}} - \tfrac{1}{p-2}\right] = \tfrac{d}{4}\log\left(2\pi e^2\right). \tag{9}$$

In other words, this means that

$$C_p^2 = 1 - \tfrac{d}{2p}\,(p-2)\log(p-2) + \tfrac{d}{4}\log\left(2\pi e^2\right)(p-2) + o(p-2) \quad \text{as} \quad p\to 2_+\,.$$

Assume that $d=2$ and let $\varepsilon = p-2\to 0_+$. We have shown that

$$C_p^2 = 1 - \tfrac{\varepsilon}{2}\log\varepsilon + \tfrac{\varepsilon}{2}\log\left(2\pi e^2\right) + o(\varepsilon) \tag{10}$$

and, with $c = c(p,\eta)$,

$$\begin{aligned} &C_p^2\,(1-c^2)^{1-\frac{2}{p}}\left(\tfrac{2\eta B}{p-2}+c\,B\right)^{\frac{2}{p}} - \tfrac{2\eta B}{p-2} \\ &\quad= \tfrac{2\eta B}{\varepsilon}\left[\left(1-\tfrac{\varepsilon}{2}\log\varepsilon+\tfrac{\varepsilon}{2}\log\left(2\pi e^2\right)\right)\left(1+\tfrac{\varepsilon}{2}\log\left(1-c^2\right)\right)\left(1+\tfrac{\varepsilon}{2}\,\tfrac{c}{\eta}\right)\left(1-\tfrac{\varepsilon}{2}\log\left(\tfrac{2\eta B}{\varepsilon}\right)\right)-1\right] \\ &\qquad\qquad \to B\left[c(2,\eta)+\eta\log\left(\tfrac{2\pi e^2\,c(2,\eta)}{B}\right)\right] \end{aligned}$$

because $1-c(2,\eta)^2 = 2\eta\,c(2,\eta)$. By rewriting (6) as

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 \geq \tfrac{2\eta B}{p-2}\left(\|\psi\|_p^2 - \|\psi\|_2^2\right) + \left[C_p^2\,(1-c^2)^{1-\frac{2}{p}}\left(\tfrac{2\eta B}{p-2}+c\,B\right)^{\frac{2}{p}} - \tfrac{2\eta B}{p-2}\right]\|\psi\|_p^2$$

we can pass to the limit as $p\to 2_+$ and establish (7). □

As proved in the Proposition below, it turns out that the above magnetic logarithmic Sobolev inequality is optimal, and that the optimizers are all known.

Proposition 5. *Consider a constant magnetic field with field strength $B>0$ in two dimensions. Then the logarithmic Sobolev inequality (7) is optimal and the optimizers are all the functions*

$$\psi(x) = C\,e^{-\frac{B}{c}\left[\frac{|x-\mathbf{b}|^2}{4}+i\,\frac{c}{2}\,(b_1x_2-b_2x_1)\right]}\,,\quad C\in\mathbb{C}\,,\quad c\in\mathbb{R}\,,\quad b\in\mathbb{R}^2\,.$$

Note that the vectors $\mathbf{b}=(b_1,b_2)$ are precisely the magnetic translations. The magnetic Hamiltonian is not invariant under the standard translations, but it commutes with the magnetic translations.

Proof. In order to prove this result, let us see first that (7) can be proved also in the following way. First write ψ as $\psi = u\,e^{iS}$, with $u>0$, and note that for every $\sigma>0$ the logarithmic Sobolev inequality with $B=0$ can be written as

$$\int_{\mathbb{R}^2}|\nabla u|^2\,dx \geq \frac{\sigma^2}{2}\int_{\mathbb{R}^2} u^2\log\left(\frac{u^2}{\|u\|_2^2}\right)dx + \frac{\sigma^2}{2}\log\left(\frac{2\pi e^2}{\sigma^2}\right)\int_{\mathbb{R}^2}|u|^2\,dx\,,$$

and as proven in [4], there is equality in the above inequality if and only if, up to translation and multiplication by a constant, $u = e^{-\frac{\sigma^2}{4}|x|^2}$. Hence, combining this inequality with Lemma 2, we find that for all $\sigma>0$, for all $c\in[0,1]$,

$$\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}u|^2\,dx \geq \frac{\sigma^2\,(1-c^2)}{2}\int_{\mathbb{R}^2} u^2\log\left(\frac{u^2}{\|u\|_2^2}\right)dx + \left(Bc + \frac{\sigma^2\,(1-c^2)}{2}\log\left(\frac{2\pi e^2}{\sigma^2}\right)\right)\int_{\mathbb{R}^2}|u|^2\,dx\,,$$

and choosing

$$\sigma^2 = \frac{B}{c}\,,\quad c = \sqrt{\eta^2+1}-\eta\,,$$

we see that the above inequality is exactly the same as (7). Moreover, the only possible functions for which there is equality are the Gaussians $C\,e^{-\frac{\sigma^2}{4}|x-\mathbf{b}|^2}$ satisfying (3), with $C\in\mathbb{C}$ and $\mathbf{b}\in\mathbb{R}^2$. This means that S has to satisfy

$$\partial_1 S=\frac{B}{2}\,b_2\,,\quad \partial_2 S=-\frac{B}{2}\,b_1\,,$$

which implies $S=\frac{B}{2}\,(b_2x_1-b_1x_2)+D$. This ends the proof. □

1.4. **Case $p\in(1,2)$.** Now let us turn our attention to the case $p\in(1,2)$. With a straightforward abuse of notations, we shall use the notations $\|u\|_p=\left(\int_{\mathbb{R}^d}|u|^p\,dx\right)^{1/p}$ and $\mathrm{L}^p(\mathbb{R}^d)$ for measurable functions such that $\|u\|_p<\infty$, even when $p<1$.

Let us consider the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_2^2+\|u\|_p^2\geq C_p^2\,\|u\|_2^2\quad\forall\,u\in\mathrm{H}^1\cap\mathrm{L}^p(\mathbb{R}^d) \tag{11}$$

where, compared to the case $p>2$, the position of the norms $\|u\|_2^2$ and $\|u\|_p^2$ have been exchanged. A scaling similar to the one of (5) shows that, for any $\lambda>0$,

$$\|\nabla u\|_2^2+\lambda^{2+d\,\frac{2-p}{p}}\,\|u\|_p^2\geq C_p^2\,\lambda^2\,\|u\|_2^2\quad\forall\,u\in\mathrm{H}^1\cap\mathrm{L}^p(\mathbb{R}^d)\,. \tag{12}$$

By optimizing on $\lambda>0$ when $d=2$, we obtain the scale invariant form of the Gagliardo-Nirenberg inequality

$$\|\nabla\psi\|_2^{1-\frac{p}{2}}\,\|\psi\|_p^{\frac{p}{2}}\geq S_p\,\|\psi\|_2\quad\forall\,\psi\in\mathrm{H}^1\cap\mathrm{L}^p(\mathbb{R}^2)\,,$$

with $S_p^2=\left(\frac{p}{2}\right)^{\frac{p}{2}}\left(1-\frac{p}{2}\right)^{1-\frac{p}{2}}\,C_p^2$. The strategy of the proof of Proposition 3 applies: for any $c\in(0,1)$, for any $\beta>0$, by applying (12) with $\lambda^{4/p}=\beta/(1-c^2)$, we obtain

$$\|\nabla_{\mathbf{A}}\psi\|_2^2+\beta\,\|\psi\|_p^2\geq\left(c\,B+C_p^2\,\beta^{\frac{p}{2}}\,(1-c^2)^{1-\frac{p}{2}}\right)\|\psi\|_2^2\,.$$

The function $c\mapsto c\,B+C_p^2\,\beta^{\frac{p}{2}}\,(1-c^2)^{1-\frac{p}{2}}$ is positive in $[0,1]$ and its derivative is positive at 0, and negative in a neighborhood of 1_-. Hence it achieves its maximum, $K^p=c_*\,B+C_p^2\,\beta^{\frac{p}{2}}\,(1-c_*^2)^{1-\frac{p}{2}}$, at the unique point $c_*\in(0,1)$ given by $\frac{c}{(1-c^2)^{p/2}}=\frac{p\,B}{2\,(2-p)\,C_p\,\beta^{p/2}}$.

Proposition 6. *Consider a constant magnetic field with field strength B in two dimensions. Given any $p\in(1,2)$, and any $\beta>0$, we have*

$$\|\nabla_{\mathbf{A}}\psi\|_2^2+\beta\,\|\psi\|_p^2\geq K^p\,\|\psi\|_2^2\quad\forall\,\psi\in\mathrm{H}^1\cap\mathrm{L}^p(\mathbb{R}^2)$$

with K^p as above. Moreover, for a given fixed B and $\beta\to+\infty$ or for $B\to0$ with a fixed $\beta>0$,

$$\|\nabla_{\mathbf{A}}\psi\|_2^2+\beta\,\|\psi\|_p^2\geq C_p^2\,\beta^{\frac{p}{2}}\,\|\psi\|_2^2\,(1+o(1))\,.$$

1.5. **Convergence towards the lowest Landau levels.** We conclude this section by considerations on the limit as $\alpha\to(-B)_+$. Let $\mu_{\mathbf{B}}(\alpha)$ be the optimal constant in the inequality

$$\|\nabla_{\mathbf{A}}\psi\|_2^2+\alpha\,\|\psi\|_2^2\geq\mu_{\mathbf{B}}(\alpha)\,\|\psi\|_p^2\quad\forall\,\psi\in\mathrm{H}^1_{\mathbf{A}}\,.$$

Notice that by homogeneity, we can always consider optimal functions such that $\|\psi\|_p=1$. Optimal functions exist according to [15].

Proposition 7. *Let $d=2$ and consider a constant magnetic field with field strength B. If u_α is a minimizer for $\mu_{\mathbf{B}}(\alpha)$ such that $\|u_\alpha\|_p=1$, then there exists a non trivial $v_\alpha\in LLL$ such that*

$$\lim_{\alpha\searrow-B}\|u_\alpha-v_\alpha\|_{\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^2)}=0\,.$$

Proof. Let us consider a minimizer u_α of $\mu_{\mathbf{B}}((\alpha)$ such that $\|u_\alpha\|_p = 1$ and decompose it as $u_\alpha = v_\alpha + w_\alpha$, where $v_\alpha \in$ LLL and w_α is in the orthogonal of LLL. Then, by the orthogonality of v_α and w_α, we get

$$\mu_{\mathbf{B}}(\alpha) \ge (\alpha + B)\,\|v_\alpha\|_2^2 + (\alpha + 3B)\,\|w_\alpha\|_2^2 \ge (\alpha + 3B)\,\|w_\alpha\|_2^2 \sim 2B\,\|w_\alpha\|_2^2 \quad \text{as} \quad \alpha \searrow -B\,.$$

Since $\lim_{\alpha\searrow -B} \mu_{\mathbf{B}}(\alpha) = 0$, this implies that $\lim_{\alpha\searrow -B} \|w_\alpha\|_2 = 0$. On the other hand, we know that

$$\mu_{\mathbf{B}}(\alpha) = (\alpha + B)\,\|v_\alpha\|_2^2 + \|\nabla_A\, w_\alpha\|_2^2 + \alpha\,\|w_\alpha\|_2^2\,,$$

which concludes the proof. □

2. Other functional inequalities in presence of a magnetic field

Although the estimates below are written only for constant magnetic fields, it is possible to generalize them to non-constant magnetic fields **B** for which the spectral following gap inequality holds

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 \ge \Lambda[\mathbf{B}]\,\|\psi\|_2^2 \quad \forall\,\psi \in \mathrm{H}^1(\mathbb{R}^d, \mathbb{C})\,, \tag{13}$$

where **A** is any magnetic potential satisfying curl**A** = **B**. We recall that $\Lambda[\mathbf{B}] = B$ if **B** is a constant magnetic field, $d = 2$ and $\mathbf{A} = \frac{B}{2}(-x_2, x_1)$. From now on, unless it is explicitely specified, we assume for simplicity that $d = 2$.

2.1. Case $p \in (2,\infty)$.

Here we generalize to a magnetic field **B** satisfying (13) with $\Lambda[\mathbf{B}] > 0$ the results obtained in Section 1. The first result is an extension of Proposition 3, which is actually valid for general magnetic fields.

Proposition 8. *Let $d = 2$. Consider a magnetic field* **B** *with magnetic potential* **A** *and assume that* (13) *holds for some $\Lambda = \Lambda[\mathbf{B}] > 0$. For any $p \in (2,\infty)$, any $\alpha > -\Lambda$, and any $\psi \in \mathrm{H}^1_{\mathbf{A}}$ we have*

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 \ge K_p\,\|\psi\|_p^2 \tag{14}$$

where

$$K_p = S_p^2\,(\alpha + \Lambda)\,\Lambda^{\frac{2}{p}-1} \quad \text{if} \quad \alpha \in \left[-\Lambda, \tfrac{2\Lambda}{p-2}\right], \quad K_p = C_p^2\,\alpha^{\frac{2}{p}} \quad \text{if} \quad \alpha \ge \tfrac{2\Lambda}{p-2}\,,$$

and C_p^2 denotes the optimal constant in (4).

When $d = 2$, we recall that $S_p^2 = C_p^2\left(\frac{2}{p}\right)^{\frac{2}{p}}\left(1-\frac{2}{p}\right)^{1-\frac{2}{p}}$. If $\alpha = 0$, we observe that $K_p = S_p^2\,\Lambda^{\frac{2}{p}}$, from which we deduce the inequality

$$\|\nabla_{\mathbf{A}}\psi\|_2 \ge S_p\,\Lambda^{\frac{1}{p}}\,\|\psi\|_p \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}\,. \tag{15}$$

Proof. For any $t \in [0,1]$, we deduce from the diamagnetic inequality

$$\|\nabla|\psi|\|_2 \le \|\nabla_{\mathbf{A}}\psi\|_2 \tag{16}$$

and from (13) and (5) that

$$\begin{aligned}\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 &= t\left(\|\nabla_{\mathbf{A}}\psi\|_2^2 - \Lambda\,\|\psi\|_2^2\right) + (1-t)\left(\|\nabla_{\mathbf{A}}\psi\|_2^2 + \frac{\alpha + \Lambda\,t}{1-t}\,\|\psi\|_2^2\right)\\ &\ge (1-t)\,\|\nabla|\psi|\|_2^2 + (\alpha + t\,\Lambda)\,\|\psi\|_2^2 \ge C_p^2\,(1-t)^{1-\frac{2}{p}}\,(\alpha + t\,\Lambda)^{\frac{2}{p}}\,\|\psi\|_p^2\,.\end{aligned}$$

Finally we have to optimize the quantity

$$t \mapsto (1-t)^{1-\frac{2}{p}}\,(\alpha + t\,\Lambda)^{\frac{2}{p}}$$

on the interval $t \in [\max\{0, -\alpha/\Lambda\}, 1]$. The optimum value in the interval $(-\alpha/\Lambda, 1)$ is achieved for $t = \frac{2}{p} - \frac{\alpha\,(p-2)}{\Lambda\,p}$, which proves the first inequality. For $\alpha \ge 2\,\Lambda/(p-2)$, the maximum is achieved at $t = 0$, which proves the second inequality. □

2.2. **Further interpolation inequalities in case $p \in (2,\infty)$.** Without magnetic field, Gagliardo-Nirenberg interpolation inequalities can be put in scale-invariant form (1) by optimizing (5) on the scale parameter $\lambda > 0$. In presence of a magnetic field, one may wonder if an inequality similar to (1) exists. Let us assume that $\theta \in (1-2/p, 1)$ and let

$$q = \frac{2\,p\,\theta}{2-p+p\,\theta}\,.$$

By Hölder's inequality we obtain that

$$\left(\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2\right)^{\theta/2} \|\psi\|_2^{1-\theta} \geq \left(K_q\,\|\psi\|_q^2\right)^{\theta/2} \|\psi\|_2^{1-\theta} \geq K_q^{\theta/2}\,\|\psi\|_p\,.$$

A similar estimate, which also covers the limit case $\theta = 1-2/p$, also holds.

Corollary 9. *Under the assumptions of Proposition 8, with $\Lambda = \Lambda[\mathbf{B}]$, for any $\theta \in [1-2/p, 1)$ and any $\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$, we have*

$$\left(\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2\right)^{\theta/2} \|\psi\|_2^{1-\theta} \geq K_p^{\frac14\,(p\theta-p+2)} \left(\min\left\{1, 1+\tfrac{\alpha}{\Lambda}\right\} S_p\right)^{\frac{p}{2}\,(1-\theta)} \|\psi\|_p\,.$$

Proof. With $\theta_\star = 1-\frac{2}{p}$, we can write

$$\begin{aligned}\left(\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2\right)^{\theta/2} \|\psi\|_2^{1-\theta} &= \left(\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2\right)^{\frac12\,\frac{\theta-\theta_\star}{1-\theta_\star}} \left(\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2\right)^{\frac{1-\theta}{1-\theta_\star}} \\ &\geq \left(K_p\,\|\psi\|_p^2\right)^{\frac12\,\frac{\theta-\theta_\star}{1-\theta_\star}} \left(\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2\right)^{\frac{1-\theta}{1-\theta_\star}}\end{aligned}$$

If $\alpha \in (-\Lambda, 0]$, it follows from (13) and (16) that

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 \geq \left(1+\tfrac{\alpha}{\Lambda}\right)\|\nabla_{\mathbf{A}}\psi\|_2 \geq \left(1+\tfrac{\alpha}{\Lambda}\right)\|\nabla|\psi|\|_2\,,$$

while we can simply drop $\alpha\,\|\psi\|_2^2$ when $\alpha \geq 0$. Hence it follows from (1) that

$$\left(\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2\right)^{\theta/2} \|\psi\|_2^{1-\theta} \geq \left(K_p\,\|\psi\|_p^2\right)^{\frac12\,\frac{\theta-\theta_\star}{1-\theta_\star}} \left(\min\left\{1, 1+\tfrac{\alpha}{\Lambda}\right\} S_p\,\|\psi\|_p\right)^{\frac{1-\theta}{1-\theta_\star}}\,,$$

which concludes the proof. □

2.3. **Case $p \to +\infty$: a magnetic Onofri inequality.** From many points of view, Onofri's inequality plays in dimension $d = 2$, the role of Sobolev's inequality in higher dimensions. According to the Euclidean version of Onofri's inequality, for any $u \in \mathrm{L}^1(\mathbb{R}^2, d\mu)$ with $\nabla u \in \mathrm{L}^2(\mathbb{R}^2, dx)$, we have

$$\log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) \leq \int_{\mathbb{R}^2} u\,d\mu + \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx\,,$$

where $d\mu = \mu(x)\,dx$ denotes the probability measure defined by $\mu(x) = \frac{1}{\pi}\,(1+|x|^2)^{-2}$, $x \in \mathbb{R}^2$. On the one hand, we can apply this inequality with $u = |\psi|$, and on the other hand rely on the Cauchy-Schwarz inequality and (13) to estimate

$$\int_{\mathbb{R}^2} u\,d\mu \leq \|u\|_2\,\|\mu\|_2 = \frac{1}{\sqrt{3\pi}}\,\|\psi\|_2 \leq \frac{1}{\sqrt{3\pi\,\Lambda}}\,\|\nabla_{\mathbf{A}}\psi\|_2\,.$$

We conclude by using the diamagnetic inequality (16) with $\Lambda = \Lambda[\mathbf{B}]$ that

$$\log\left(\int_{\mathbb{R}^2} e^{|\psi|}\,d\mu\right) \leq \frac{1}{\sqrt{3\pi\,\Lambda}}\,\|\nabla_{\mathbf{A}}\psi\|_2 + \frac{1}{16\,\pi}\,\|\nabla_{\mathbf{A}}\psi\|_2^2\,.$$

This is equivalent to the following result.

Proposition 10. *Let $d = 2$. Consider a magnetic field $\mathbf{B}$ with magnetic potential $\mathbf{A}$ and assume that (13) holds for some $\Lambda = \Lambda[\mathbf{B}] > 0$. Then*

$$8\,\sqrt{\tfrac{\pi}{3\,\Lambda}}\left(-1+\sqrt{1+\tfrac34\,\Lambda\,\log\left(\textstyle\int_{\mathbb{R}^2} e^{|\psi|}\,d\mu\right)}\right) \leq \|\nabla_{\mathbf{A}}\psi\|_2 \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}\,.$$

The optimality in Onofri's inequality is achieved by considering the test functions $u_\varepsilon = 1+\varepsilon\varphi$ as $\varepsilon \to 0$, with

$$\varphi(x) = \frac{|x|^2-1}{|x|^2+1}\,, \quad x \in \mathbb{R}^2\,.$$

We recall that φ solves $-\Delta\varphi = 8\pi\,\mu\,\varphi$. It is compatible neither with the equality in the Cauchy-Schwarz inequality nor with the optimality in (13), and so the question of an optimal magnetic Onofri inequality is open.

One can alternatively consider the unique, smooth and radially symmetric solution (n_M, c_M) of

$$-\Delta c_M = M\,\frac{e^{-\frac12|x|^2+c_M}}{\int_{\mathbb{R}^2} e^{-\frac12|x|^2+c}\,dx} =: n_M\,, \quad x\in\mathbb{R}^2$$

for any $M \in (0, 8\pi)$. We refer to [8] for details. With $\mu_M := n_M/M$, we have the following inequality:

$$\log\left(\int_{\mathbb{R}^2} e^u\,d\mu_M\right) - \int_{\mathbb{R}^2} u\,d\mu_M \le \tfrac{1}{2M}\int_{\mathbb{R}^2}|\nabla u|^2\,dx\,,$$

and, with the same scheme as for the standard Onofri inequality, we have

$$\log\left(\int_{\mathbb{R}^2} e^{|\psi|}\,d\mu_M\right) \le \tfrac{1}{\sqrt{3\pi\,\Lambda}}\,\|\nabla_{\mathbf{A}}\psi\|_2 + \tfrac{1}{2M}\,\|\nabla_{\mathbf{A}}\psi\|_2^2\,,$$

which finally gives an inequality similar to the one of Proposition 10 in case $M < 8\pi$, that is,

$$\tfrac{M}{\sqrt{3\pi\,\Lambda}}\left(-1+\sqrt{1+\tfrac34\,\Lambda\log\left(\textstyle\int_{\mathbb{R}^2} e^{|\psi|}\,d\mu_M\right)}\right) \le \|\nabla_{\mathbf{A}}\psi\|_2 \quad \forall\,\psi\in \mathrm{H}^1_{\mathbf{A}}\,.$$

2.4. **Logarithmic Sobolev inequality.** The extension of Corollary 4 to non-constant magnetic fields goes as follows.

Proposition 11. *Let $d = 2$. Consider a magnetic field $\mathbf{B}$ with magnetic potential $\mathbf{A}$ and assume that (13) holds for some $\Lambda = \Lambda[\mathbf{B}] > 0$. For any $u \in \mathrm{H}^1_{\mathbf{A}}$ and any $\eta \ge 1$, we have*

$$\|\nabla_{\mathbf{A}} u\|_2^2 \ge \eta\,\Lambda\int_{\mathbb{R}^2} u^2\log\left(\frac{u^2}{\|u\|_2^2}\right)dx + \eta\,\Lambda\log\left(\tfrac{\pi\,e^2}{\eta\,\Lambda}\right)\|u\|_2^2 \quad \forall\,u\in\mathrm{H}^1(\mathbb{R}^2)\,. \tag{17}$$

Notice that in the semi-classical regime corresponding to a limit of the magnetic field $\mathbf{B}$ such that $\Lambda = \Lambda[\mathbf{B}] \to 0$, then with $\eta = \frac{1}{2\,\Lambda} \to +\infty$, we recover the classical logarithmic Sobolev inequality (8) without magnetic field.

Proof. For any $\alpha \ge \frac{2\,\Lambda}{p-2}$, Inequality (14) takes the form

$$\|\nabla_{\mathbf{A}} u\|_2^2 + \alpha\,\|u\|_2^2 \ge C_p^2\,\alpha^{\frac{2}{p}}\,\|u\|_p^2 \quad \forall\,u\in\mathrm{H}^1(\mathbb{R}^d)\,.$$

Let us choose $\alpha = \frac{2\eta\,\Lambda}{p-2}$, with $\eta \ge 1$. Hence

$$\|\nabla_{\mathbf{A}} u\|_2^2 \ge \tfrac{2\eta\,\Lambda}{p-2}\left(\|u\|_p^2 - \|u\|_2^2\right) + \left[C_p^2\left(\tfrac{2\eta\,\Lambda}{p-2}\right)^{\frac{2}{p}} - \tfrac{2\eta\,\Lambda}{p-2}\right]\|u\|_p^2\,.$$

With $d = 2$, by passing to the limit as $p \to 2_+$ and using the fact that

$$\lim_{p\to 2_+}\tfrac{2}{p-2}\left(\|u\|_p^2 - \|u\|_2^2\right) = \int_{\mathbb{R}^2} u^2\log\left(\frac{u^2}{\|u\|_2^2}\right)dx\,,$$

we obtain (17). □

2.5. **Case $p \in (1,2)$.** We have a result similar to the one of Proposition 8.

Proposition 12. *Let $d = 2$. Consider a magnetic field $\mathbf{B}$ with magnetic potential $\mathbf{A}$ and assume that (13) holds for some $\Lambda = \Lambda[\mathbf{B}] > 0$. For any $p \in (1,2)$, any $\beta > 0$, and any $\psi \in \mathrm{H}^1_{\mathbf{A}}$ we have*

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \beta\,\|\psi\|_p^2 \geq K_p\,\|\psi\|_2^2$$

where

$$K_p = C_p^2\,\beta^{\frac{p}{2}} \quad \text{if} \quad \beta \geq \beta_\star := \left(\tfrac{2\Lambda}{2-p}\right)^{\frac{2}{p}} C_p^{-\frac{4}{p}}, \quad K_p = \Lambda + \Lambda^{1-\frac{p}{2}}\,\tfrac{p}{2}\left(1-\tfrac{p}{2}\right)^{\frac{2}{p}-1}\beta\,C_p^{\frac{4}{p}} \quad \text{if} \quad \beta \in [0,\beta_\star]\,,$$

and C_p^2 denotes the optimal constant in (11).

Proof. For all $\psi \in \mathrm{H}^1_{\mathbf{A}}$, by (13) and (16), we obtain that

$$\begin{aligned}\|\nabla_{\mathbf{A}}\psi\|_2^2 + \beta\,\|\psi\|_p^2 &= t\left(\|\nabla_{\mathbf{A}}\psi\|_2^2 - \Lambda\,\|\psi\|_2^2\right) + (1-t)\,\|\nabla_{\mathbf{A}}\psi\|_2^2 + \beta\,\|\psi\|_p^2 + \Lambda\,t\,\|\psi\|_2^2\\ &\geq (1-t)\,\|\nabla|\psi|\|_2^2 + \beta\,\|\psi\|_p^2 + \Lambda\,t\,\|\psi\|_2^2\,.\end{aligned}$$

Next we apply (12) to $u = |\psi|$ with $\lambda^2 = \left(\frac{\beta}{1-t}\right)^{p/2}$. This yields

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \beta\,\|\psi\|_p^2 \geq \left[(1-t)^{1-\frac{p}{2}}\,\beta^{\frac{p}{2}}\,C_p^2 + \Lambda\,t\right]\|\psi\|_2^2\,.$$

If $\beta \leq \beta_\star$, the right hand side is maximal for some $t \in [0,1]$, otherwise the maximum on $[0,1]$ is achieved by $t = 0$, which concludes the proof. □

2.6. **Optimal constant in the magnetic interpolation inequality, in case $p \in (2,2^*)$.** Let $2^* = 2\,d/(d-2)$ if $d \geq 3$ and $2^* = +\infty$ if $d = 2$. For any $\alpha > -\Lambda[\mathbf{B}]$ and any $p \in (2,2^*)$, let us consider the optimal constant $\mu_{\mathbf{B}}(\alpha)$ in the inequality

$$\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2 \geq \mu_{\mathbf{B}}(\alpha)\,\|\psi\|_p^2 \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}\,, \tag{18}$$

that is,

$$\mu_{\mathbf{B}}(\alpha) := \inf_{\psi \in \mathrm{H}^1_{\mathbf{A}}\setminus\{0\}} \frac{\|\nabla_{\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2}{\|\psi\|_p^2}\,.$$

Proposition 13. *Let $d \geq 2$ and $p \in (2,2^*)$. Consider a magnetic field $\mathbf{B}$ with magnetic potential $\mathbf{A}$ and assume that (13) holds for some $\Lambda = \Lambda[\mathbf{B}] > 0$. The function $\alpha \mapsto \mu_{\mathbf{B}}(\alpha)$ is positive on the interval $(-\Lambda[\mathbf{B}], +\infty)$, monotone increasing, concave and such that*

$$\lim_{\alpha \to (-\Lambda)_+} \mu_{\mathbf{B}}(\alpha) = 0 \quad \text{and} \quad \lim_{\alpha \to +\infty} \mu_{\mathbf{B}}(\alpha)\,\alpha^{\frac{d-2}{2}-\frac{d}{p}} = C_p^2\,.$$

As a consequence, for any given $\alpha > 0$, we obtain

$$\lim_{\varepsilon \to 0} \mu_{\varepsilon\mathbf{B}}(\alpha) = C_p^2\,\alpha^{\frac{d}{p}-\frac{d-2}{2}}\,.$$

Proof. The positivity is a consequence of Proposition 8, while the concavity follows from the fact that $\alpha \mapsto \mu_{\mathbf{B}}(\alpha)$ is achieved as minimizing an affine function of α on a set of functions ψ. The estimate as $\alpha \to (-\Lambda)_+$ is obtained by considering test functions for (13). As $\alpha \to +\infty$, a lower estimate is obtained by arguing as in Proposition 8 for any $d \geq 2$ and using (5). To see that this estimate is optimal, we argue as in the proof of [10, Theorem 1].

The small magnetic field regime follows from the change of variables $\psi(x) = \phi(\varepsilon^{1/2}x)$. Since

$$\frac{\|\nabla_{\varepsilon\mathbf{A}}\psi\|_2^2 + \alpha\,\|\psi\|_2^2}{\|\psi\|_p^2} = \varepsilon^{\frac{d}{p}-\frac{d-2}{2}}\,\frac{\|\nabla_{\varepsilon\mathbf{A}}\phi\|_2^2 + \alpha\,\varepsilon^{-1}\,\|\phi\|_2^2}{\|\phi\|_p^2}\,,$$

we deduce that

$$\mu_{\varepsilon\mathbf{B}}(\alpha) = \varepsilon^{\frac{d}{p}-\frac{d-2}{2}}\,\mu_{\mathbf{B}}\left(\alpha\,\varepsilon^{-1}\right). \tag{19}$$

The conclusion follows by considering the asymptotic regime as $\alpha\,\varepsilon^{-1}\to+\infty$. □

2.7. **Optimal constant in the magnetic interpolation inequality, in case $p\in(1,2)$.** For any $\beta>0$, let us define

$$\nu_{\mathbf{B}}(\beta):=\inf_{\psi\in\mathrm{H}^1_{\mathbf{A}}\setminus\{0\}}\frac{\|\nabla_{\mathbf{A}}\psi\|_2^2+\beta\,\|\psi\|_p^2}{\|\psi\|_2^2}\,,$$

i.e., $\nu_{\mathbf{B}}(\beta)$ is the optimal constant in the interpolation inequality

$$\|\nabla_{\mathbf{A}}\psi\|_2^2+\beta\,\|\psi\|_p^2\ge\nu_{\mathbf{B}}(\beta)\,\|\psi\|_2^2\quad\forall\,\psi\in\mathrm{H}^1\cap\mathrm{L}^p(\mathbb{R}^2)\,. \tag{20}$$

Proposition 14. *Let $d=2$ and $p\in(1,2)$. Consider a magnetic field $\mathbf{B}$ with magnetic potential $\mathbf{A}$ and assume that (13) holds for some $\Lambda=\Lambda[\mathbf{B}]>0$. The function $\beta\mapsto\nu_{\mathbf{B}}(\beta)$ is positive on the interval $(0,+\infty)$, monotone increasing, concave and such that*

$$\lim_{\beta\to0_+}\nu_{\mathbf{B}}(\beta)=\Lambda\quad\textit{and}\quad\lim_{\beta\to+\infty}\nu_{\mathbf{B}}(\alpha)\,\beta^{-\frac{p}{2}}=C_p^2\,.$$

As a consequence, for any given $\beta>0$, we obtain

$$\lim_{\varepsilon\to0}\nu_{\varepsilon\mathbf{B}}(\beta)=C_p^2\,\beta^{\frac{p}{2}}\,.$$

The proof is very similar to that of Proposition 13. We will not detail it here. It relies on a change of variables from which we deduce that, for all $\beta>0$ and $\varepsilon>0$,

$$\nu_{\varepsilon\mathbf{B}}(\beta)=\varepsilon\,\nu_{\mathbf{B}}(\beta\,\varepsilon^{-\frac{2}{p}})\,.$$

3. Keller–Lieb–Thirring estimates for magnetic Schrödinger operators

3.1. **A Keller–Lieb–Thirring estimate for negative potentials.** Let us consider the Schrödinger energy of some function $\psi\in\mathrm{H}^1_{\mathbf{A}}$ in case of a negative potential $V\in\mathrm{L}^q(\mathbb{R}^d)$. Using Hölder's inequality with $1/q+2/p=1$, with $q>d/2$, we know that

$$\int_{\mathbb{R}^d}|\nabla_{\mathbf{A}}\psi|^2\,dx+\int_{\mathbb{R}^d}V\,|\psi|^2\,dx\ge\int_{\mathbb{R}^d}|\nabla_{\mathbf{A}}\psi|^2\,dx-\int_{\mathbb{R}^d}|V|\,|\psi|^2\,dx\ge\|\nabla_{\mathbf{A}}\psi\|_2^2-\|V\|_q\,\|\psi\|_p^2\,. \tag{21}$$

We know from Proposition 13 that there exists a unique α such that $\mu_{\mathbf{B}}(\alpha)=\|V\|_q$. Using (18), we obtain that the right hand side is bounded from below by $-\alpha\,\|\psi\|_2^2$. Let us denote by $\alpha_{\mathbf{B}}:[0,+\infty)\to[-\Lambda[\mathbf{B}],+\infty)$ the inverse of the function $\alpha\mapsto\mu_{\mathbf{B}}(\alpha)$ and by $\lambda_{\mathbf{A},V}$ the principal eigenvalue of the operator $-\Delta_{\mathbf{A}}+V$ defined as the infimum of the quotient of the Schrödinger energy of ψ by $\|\psi\|_2^2$. This can be summarized as follows.

Corollary 15. *Let $d\ge2$ and $q>d/2$. Consider a magnetic field $\mathbf{B}$ with magnetic potential $\mathbf{A}$ and assume that (13) holds for some $\Lambda=\Lambda[\mathbf{B}]>0$. With the above notations,*

$$\lambda_{\mathbf{A},V}\ge-\alpha_{\mathbf{B}}(\|V\|_q)\quad\forall\,V\in\mathrm{L}^q(\mathbb{R}^d)\quad\textit{such that}\quad V\le0\ \textit{a.e.} \tag{22}$$

This estimate is optimal. Moreover, $-\alpha_{\mathbf{B}}(\mu)$ is nonnegative if $\mu=\|V\|_q\le\mu_{\mathbf{B}}(0)$ and

$$\lim_{\mu\to+\infty}\alpha_{\mathbf{B}}(\mu)\,\mu^{-\frac{2q}{2q-d}}=C_p^{-\frac{4q}{2q-d}}$$

with $p=\frac{2q}{q-1}$. As a consequence, for any given $\mu>0$, we obtain

$$\lim_{\varepsilon\to0}\alpha_{\varepsilon\mathbf{B}}(\mu)=\left(C_p^{-2}\,\mu\right)^{\frac{2q}{2q-d}}\,.$$

Proof. The optimality case in (22) is equivalent to the optimality case in (18) because $V=-|\psi|^{p-2}$ realizes the equality case in (21). The small magnetic field regime is obtained from (19) by writing $\mu=\mu_{\varepsilon\mathbf{B}}(\alpha_{\varepsilon\mathbf{B}}(\mu))$ and composing both sides of the identity with $\alpha_{\mathbf{B}}$, hence showing that

$$\alpha_{\varepsilon\mathbf{B}}(\mu)=\varepsilon\,\alpha_{\mathbf{B}}\left(\mu\,\varepsilon^{-\frac{2q-d}{2q}}\right),$$

and using the asymptotics as $\mu\,\varepsilon^{-\frac{2q-d}{2q}}\to+\infty$. □

Explicit estimates follow from Proposition 8 when $d=2$ by inverting the inequality $\mu_{\mathbf{B}}(\alpha)\ge K_p$.

Corollary 16. *Let $d=2$. Under the same assumptions as in Proposition 15, with $p=\frac{2\,q}{q-1}$, we have*

$$\alpha_{\mathbf{B}}(\mu)\le S_p^{-2}\,\Lambda^{\frac1q}\,\mu-\Lambda\quad\text{if}\quad\mu\in\left[0,q\,\Lambda^{1-\frac1q}\,S_p^2\right],\quad\alpha_{\mathbf{B}}(\mu)\le\left(C_p^{-2}\,\mu\right)^{\frac{q}{q-1}}\quad\text{if}\quad\mu\ge q\,\Lambda^{1-\frac1q}\,S_p^2.$$

A better estimate, although non explicit, can be obtained from Proposition 3 in case of a constant magnetic field. See Fig. 1.

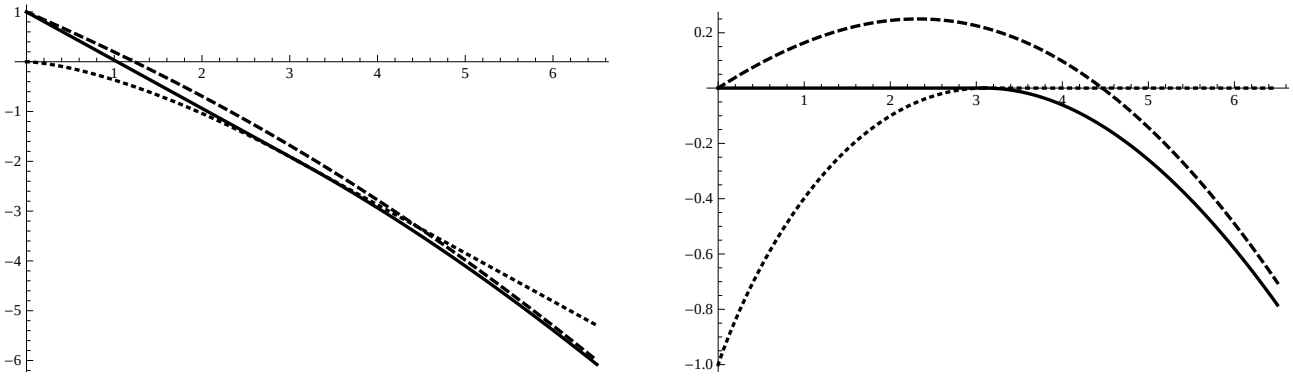


FIGURE 1. Case $p>2$: plot of lower estimates of the principale eigenvalue $\lambda_{\mathbf{A},V}$ as a function of Λ with $d=2$, $p=7$ and $\Lambda=1$ as a function of μ (left). For more readability, the same computation is represented, except that the straight line estimate $\mu\mapsto S_p^{-2}\,\Lambda^{1/q}\,\mu-\Lambda$ has been subtracted (right). The result of Corollary 16 is given by the plain curve. In case of constant magnetic field, we have $\Lambda=B$ and the improved (although purely numerical) result that can be deduced from Proposition 3 is represented by the dashed curve.

3.2. **A Keller–Lieb–Thirring estimate for positive potentials.** By Hölder's inequality with exponents $2/p$ and $2/(2-p)$, with $p\in(1,2)$, if W is a positive function, we have

$$\|\psi\|_p^2=\left(\int_{\mathbb{R}^d}W^{-\frac p2}\left(W\,|\psi|^2\right)^{\frac p2}\,dx\right)^{2/p}\le\|W^{-1}\|_q\int_{\mathbb{R}^d}W\,|\psi|^2\,dx,$$

with $q=\frac{p}{2-p}$. Using (18), we know that

$$\int_{\mathbb{R}^d}|\nabla_{\mathbf{A}}\psi|^2\,dx+\int_{\mathbb{R}^d}W\,|\psi|^2\,dx\ge\int_{\mathbb{R}^d}|\nabla_{\mathbf{A}}\psi|^2\,dx+\beta\,\|\psi\|_p^2\ge\nu_{\mathbf{B}}(\beta)\int_{\mathbb{R}^d}|\psi|^2\,dx.$$

with $\beta=1/\|W^{-1}\|_q$. The results of Proposition 14 provide us with estimates on the principal eigenvalue of the Schrödinger operator $-\Delta_{\mathbf{A}}+W$.

Corollary 17. *Let $d\ge2$ and $q\in(1,+\infty)$. Consider a magnetic field $\mathbf{B}$ with magnetic potential $\mathbf{A}$ and assume that (13) holds for some $\Lambda=\Lambda[\mathbf{B}]>0$. If W is a positive potential, then*

$$\lambda_{\mathbf{A},W}\ge\nu_{\mathbf{B}}\left(\|W^{-1}\|_q^{-1}\right)\quad\forall\,W\text{ such that }W^{-1}\in\mathrm{L}^q(\mathbb{R}^d).$$

This estimate is optimal.

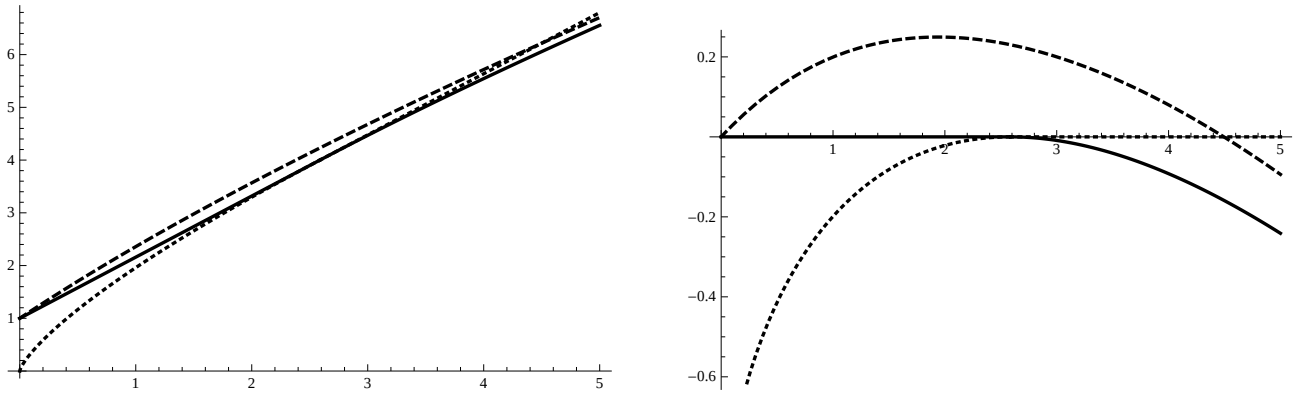


FIGURE 2. Case $p \in (1,2)$: plot of lower estimates of the principale eigenvalue $\lambda_{\mathbf{A},V}$ as a function of Λ with $d=2$, $p=3/2$ and $\Lambda=1$ as a function of β (left). The same computation is represented (right), except that the straight line estimate

$$\beta \mapsto \Lambda + \Lambda^{1-\frac{p}{2}}\,\frac{p}{2}\,\big(1-\tfrac{p}{2}\big)^{\frac{2}{p}-1}\,\beta\,C_p^{\frac{4}{p}}$$

has been subtracted. The result that can be deduced from Proposition 12 is given by the plain curve. In case of constant magnetic field, we have $\Lambda = B$ and the improved (although purely numerical) result that can be deduced from Proposition 6 is represented by the dashed curve.

If $d=2$, we recall that

$$\nu_{\mathbf{B}}(\beta) \ge \Lambda + \Lambda^{1-\frac{p}{2}}\,\frac{p}{2}\,\big(1-\tfrac{p}{2}\big)^{\frac{2}{p}-1}\,\beta\,C_p^{\frac{4}{p}} \quad \text{if} \quad \beta\in[0,\beta_\star]\,, \quad \nu_{\mathbf{B}}(\beta) \ge C_p^2\,\beta^{\frac{p}{2}} \quad \text{if} \quad \beta\ge\beta_\star = \Big(\tfrac{2\,\Lambda}{2-p}\Big)^{\frac{2}{p}}\,C_p^{-\frac{4}{p}}\,,$$

with $\Lambda = \Lambda[\mathbf{B}]$ given by (13), according to Proposition 12. If, additionally, $\mathbf{B}$ is a constant magnetic field of intensity $B>0$, then $\Lambda = B$ and we deduce from Proposition 6 the improved estimate

$$\nu_{\mathbf{B}}(\beta) \ge c_*\,B + C_p^2\,\beta^{\frac{p}{2}}\,(1-c_*^2)^{1-\frac{p}{2}} \quad \text{where } c_*\in(0,1) \text{ solves} \quad \frac{c}{(1-c^2)^{p/2}} = \frac{p\,B}{2\,(2-p)\,C_p\,\beta^{p/2}}\,.$$

See Fig. 2.

3.3. **Keller–Lieb–Thirring estimates for general potentials.** The estimates of Sections 3.1 and 3.2 are respectively limited to nonpositive and nonnegative potentials V and W, which induces some restrictions on the behavior of V as $|x|\to+\infty$ and on the behavior of W in a neighborhood of the minimum if $\operatorname{infess}_{\mathbb{R}^d} W = 0$. However, it is easy to extend our results to a general potential Φ. We can first rewrite (21) in a more general setting as

$$\int_{\mathbb{R}^d} |\nabla_{\mathbf{A}}\psi|^2\,dx + \int_{\mathbb{R}^d} \Phi\,|\psi|^2\,dx \ge \int_{\mathbb{R}^d} |\nabla_{\mathbf{A}}\psi|^2\,dx + \lambda\int_{\mathbb{R}^d} |\psi|^2\,dx - \int_{\mathbb{R}^d} (\lambda-\Phi)_+\,|\psi|^2\,dx$$
$$\ge \|\nabla_{\mathbf{A}}\psi\|_2^2 - \mu\,\|\psi\|_p^2 + \lambda\,\|\psi\|_2^2$$

with $\lambda\in\mathbb{R}$, $\mu = \|(\lambda-\Phi)_+\|_q$ and $q = \frac{p}{p-2}$. Using (18), we know that

$$\int_{\mathbb{R}^d} |\nabla_{\mathbf{A}}\psi|^2\,dx + \int_{\mathbb{R}^d} \Phi\,|\psi|^2\,dx \ge -\big(\alpha_{\mathbf{B}}(\mu)-\lambda\big)\int_{\mathbb{R}^d} |\psi|^2\,dx\,.$$

This makes sense of course if all above quantities are finite and well defined. For instance, one cannot choose $\lambda > \operatorname{supess}_{\mathbb{R}^d}\Phi$.

A similar estimate holds in the range $p\in(1,2)$. Let $\lambda\le \mathrm{infess}_{\mathbb{R}^d}\Phi$. Then we have

$$\|\psi\|_p^2=\left(\int_{\mathbb{R}^d}(\Phi-\lambda)^{-\frac{p}{2}}\left((\Phi-\lambda)\,|\psi|^2\right)^{\frac{p}{2}}dx\right)^{2/p}\le\frac{1}{\beta}\int_{\mathbb{R}^d}(\Phi-\lambda)\,|\psi|^2\,dx,$$

with $1/\beta=\|(\Phi-\lambda)^{-1}\|_q$ and $q=\frac{p}{2-p}$, using (20), we know that

$$\int_{\mathbb{R}^d}|\nabla_{\mathbf{A}}\psi|^2\,dx+\int_{\mathbb{R}^d}\Phi\,|\psi|^2\,dx\ge\int_{\mathbb{R}^d}|\nabla_{\mathbf{A}}\psi|^2\,dx+\beta\,\|\psi\|_p^2+\lambda\,\|\psi\|_2^2\ge\left(\nu_{\mathbf{B}}(\beta)+\lambda\right)\|\psi\|_2^2.$$

We can collect these estimates in the following result.

Corollary 18. *Let $d\ge2$. Consider a magnetic field $\mathbf{B}$ with magnetic potential $\mathbf{A}$ and assume that (13) holds for some $\Lambda=\Lambda[\mathbf{B}]>0$. Let $\Phi\in\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ be an arbitrary potential.*

(i) If $q>d/2$, $p=\frac{2\,q}{q-1}$ and $\alpha_{\mathbf{B}}$ is defined as in Corollary 15, we have

$$\lambda_{\mathbf{A},\Phi}\ge-\left(\alpha_{\mathbf{B}}\left(\|(\lambda-\Phi)_+\|_q\right)-\lambda\right).$$

(ii) If $q\in(1,+\infty)$, $p=\frac{2\,q}{q+1}$ and $\nu_{\mathbf{B}}$ defined as in Corollary 17, we have

$$\lambda_{\mathbf{A},\Phi}\ge\lambda+\nu_{\mathbf{B}}\left(\|(\Phi-\lambda)^{-1}\|_q^{-1}\right).$$

These estimates hold for any $\lambda\in\mathbb{R}$ such that all above norms are well defined.

4. Some numerical results

In $\mathbb{R}^2$, we consider the magnetic gradient associated with the constant magnetic field of strength B as

$$\nabla_{\mathbf{A}}u=\nabla u+i\,\frac{B}{2}\begin{pmatrix}-x_2\\x_1\end{pmatrix}u.$$

Let $r=\sqrt{x_1^2+x_2^2}=|x|$ be the radial coordinate associated to any $x=(x_1,x_2)\in\mathbb{R}^2$.

4.1. Case $p\in(2,\infty)$.

Let us consider the problem of minimizing

$$\mathcal{Q}_\alpha^{(p)}[u]:=\frac{\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}u|^2\,dx+\alpha\int_{\mathbb{R}^2}|u|^2\,dx}{\left(\int_{\mathbb{R}^2}|u|^p\,dx\right)^{2/p}}$$

for a given fixed $p\in(2,+\infty)$. Upper estimates of $\alpha\mapsto\mu_{\mathbf{B}}(\alpha)$ can be obtained by looking for the minimum of $\mathcal{Q}_\alpha^{(p)}$ on a subspace of $\mathrm{H}^1_{\mathbf{A}}\setminus\{0\}$.

For every integer $k\in\mathbb{N}$ we introduce the special symmetry class

$$(\mathscr{C}_k)\qquad u(x)=\left(\frac{x_2+i\,x_1}{|x|}\right)^k v(|x|)\quad\forall\,x\in\mathbb{R}^2.$$

With this notation, if $u\in\mathscr{C}_k$, then

$$\mathcal{Q}_\alpha^{(p)}[u]=(2\,\pi)^{1-\frac{2}{p}}\,\mathcal{Q}_{k,\alpha}^{(p)}[v]$$

where

$$\mathcal{Q}_{k,\alpha}^{(p)}[v]:=\frac{\displaystyle\int_{\mathbb{R}^2}|v'|^2\,dx+\int_{\mathbb{R}^2}\left(\left(\frac{k}{r}-\frac{B\,r}{2}\right)^2+\alpha\right)|v|^2\,dx}{\displaystyle\left(\int_{\mathbb{R}^2}|v|^p\,dx\right)^{2/p}}.$$

Simple estimates can be obtained by using test functions like $v_\vartheta(r)=e^{-\vartheta\,r^2}$ and optimizing on the value of ϑ. See Figs. 3 and 4.

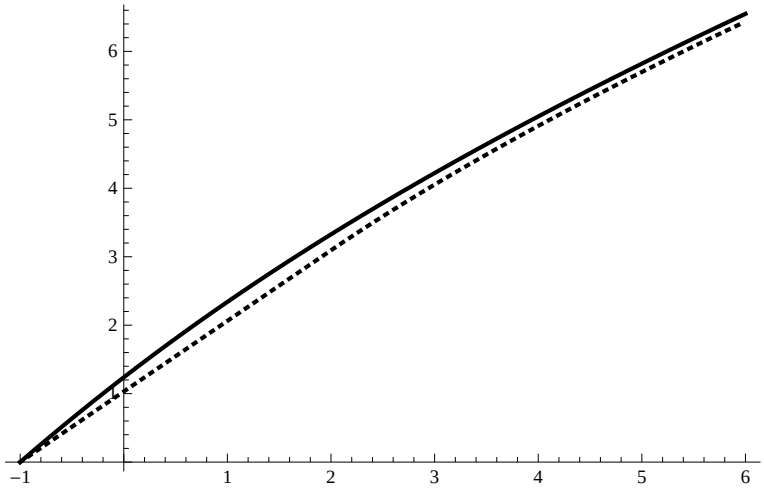


FIGURE 3. Case $p = 3$, $B = 1$: estimates of $\alpha \mapsto \mu_{\mathbf{B}}(\alpha)$. The dotted curve is the lower estimate of Proposition 6 valid for an arbitrary magnetic field while the plain curve is an upper estimate based on the optimization of $\mathscr{Q}_{0,\alpha}$ on Gaussian test functions. The horizontal axis corresponds to α while the vertical axis is measured in units of $(2\pi)^{1-2/p}$.

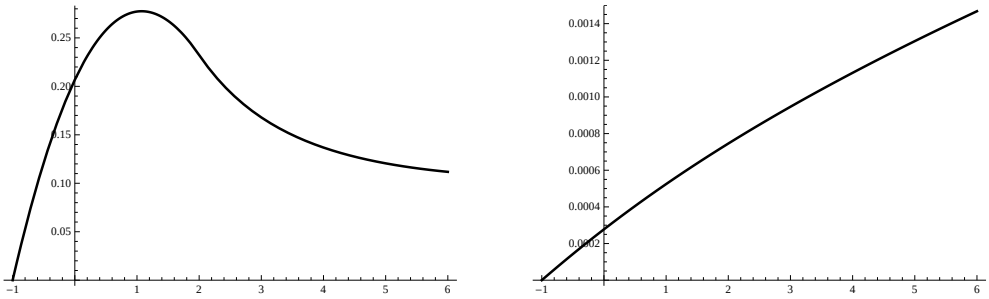


FIGURE 4. For the same parameters as in Fig. 3, the difference of the upper estimate (based on Gaussian test functions) and the lower estimate of Proposition 6 (valid for an arbitrary magnetic field: dotted curve in Fig. 3) is shown on the left. The difference of the upper estimate based on Gaussian test functions and the lower estimate of Proposition 3 is of much smaller order (right).

In principle, a better estimate can be found by numerically computing $\mu_{\mathbf{B},0}(\alpha)$ where

$$\mu_{\mathbf{B},k}(\alpha) := \inf_{0\neq v\in\mathscr{C}_k} \mathscr{Q}^{(p)}_{k,\alpha}[v]\,.$$

A minimizer of $\mu_{\mathbf{B},0}(\alpha)$ solves the Euler-Lagrange equation associated with the minimization of $\mathscr{Q}_{0,\alpha}$, that is,

$$-v'' - \frac{v'}{r} + \left(\tfrac{B^2}{4}\,r^2 + \alpha\right) v = \mu_{\mathbf{B},0}(\alpha) \left(\int_0^{+\infty} |v|^p\, r\, dr\right)^{\frac{2}{p}-1} |v|^{p-2}\, v\,. \tag{23}$$

With no restriction, we can restrict the problem to positive solutions such that

$$\mu_{\mathbf{B},0}(\alpha) = \left(\int_0^{+\infty} |v|^p\, r\, dr\right)^{1-\frac{2}{p}}$$

and have therefore to solve the reduced problem

$$-v'' - \frac{v'}{r} + \left(\tfrac{B^2}{4}\,r^2 + \alpha\right) v = |v|^{p-2}\, v$$

among positive functions in $\mathrm{H}^1((0,+\infty), r\,dr)$ such that $\int_0^{+\infty} r^2\,|v|^2\,r\,dr < +\infty$. From the existence result [15, Theorem 3.5], we know that $\mu_{\mathbf{B},k}(\alpha)$ is given by the infimum of $\left(\int_0^{+\infty}|v|^p\,r\,dr\right)^{1-2/p}$ on the set of solutions. Numerically, we solve the ODE on a finite interval $[0,R]$, which induces a numerical error: R has to be chosen large enough.

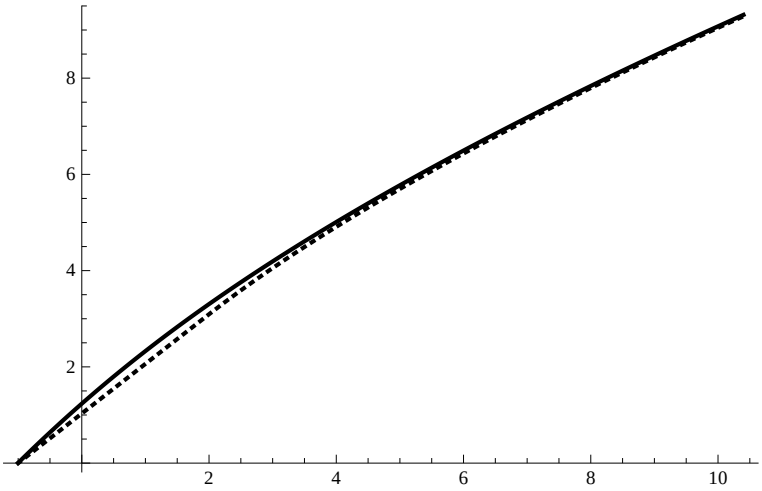


FIGURE 5. Case $p=3$, $B=1$: estimates of $\alpha \mapsto \mu_{\mathbf{B}}(\alpha)$. The dotted curve is the lower estimate valid for an arbitrary magnetic field while the plain curve is an upper estimate based on (23). The plain curve gives a slightly better upper bound for large values of α. Notice that there is a numerical error due to the fact that (23) is solved on a finite interval. The estimate based on Proposition 3 cannot be distinguished from the plain curve.

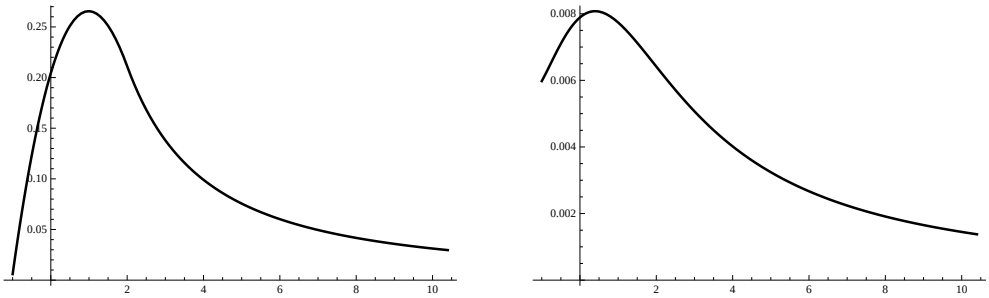


FIGURE 6. For the same parameters as in Fig. 5, the difference of the upper and the lower estimates are shown. The upper estimate is computed by numerically solving (23). Left: lower estimate of Proposition 6. Right: lower estimate of Proposition 3, which is about 3% of the estimate of Proposition 6.

4.2. **Case $p \in (1,2)$.** Let us consider the problem of minimizing

$$\mathcal{Q}_\beta^{(p)}[u] := \frac{\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} u|^2\,dx + \beta\left(\int_{\mathbb{R}^2}|u|^p\,dx\right)^{2/p}}{\int_{\mathbb{R}^2}|u|^2\,dx}$$

for a given fixed $p \in (2,+\infty)$. Upper estimates of $\beta \mapsto \nu_{\mathbf{B}}(\beta)$ can be obtained by looking for the minimum of $\mathcal{Q}_\beta^{(p)}$ on a subspace of $\mathrm{H}^1_{\mathbf{A}} \setminus \{0\}$. We assume that $\mathbf{B}$ is a constant magnetic field of intensity B. As in the case $p>2$, we can consider the special symmetry class $\mathscr{C}_k$: if $u \in \mathscr{C}_k$, then

$$\mathcal{Q}_\beta^{(p)}[u] = \mathcal{Q}_{k,\beta}^{(p)}[v]$$

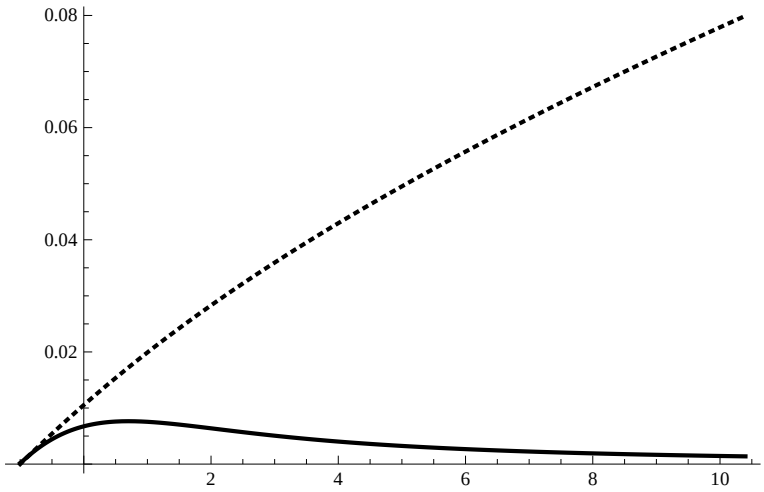


FIGURE 7. Case $p = 3$, $B = 1$: the plain curve is reproduced from Fig. 6 (right), and corresponds to the maximal error when estimating $\alpha \mapsto \mu_{\mathbf{B}}(\alpha)$, with an upper estimate given by (23). The lower estimate is based on Proposition 3. It has to be compared with the difference between the plain curve in Fig. 3, corresponding to the upper estimate based on Gaussian test functions, and the lower estimate of Proposition 3, shown as a dotted curve. This explains in which sense the upper estimate of Fig. 5 is better than the upper estimate of Fig. 3.

where

$$\mathcal{Q}^{(p)}_{k,\beta}[v] := \frac{\displaystyle\int_{\mathbb{R}^2} |v'|^2\,dx + \int_{\mathbb{R}^2} \left(\frac{k}{r} - \frac{B\,r}{2}\right)^2 |v|^2\,dx + (2\,\pi)^{\frac{2}{p}-1}\,\beta\left(\int_{\mathbb{R}^2} |v|^p\,dx\right)^{2/p}}{\displaystyle\int_{\mathbb{R}^2} |v|^2\,dx}\,.$$

Simple estimates can again be obtained by using test functions like $v_\vartheta(r) = e^{-\vartheta\,r^2}$ and optimizing on the value of ϑ. See Fig. 8.

In principle, a better estimate can be found by numerically computing $\nu_{\mathbf{B},0}(\beta)$ where

$$\nu_{\mathbf{B},k}(\beta) := \inf_{0\neq v\in\mathscr{C}_k} \mathcal{Q}^{(p)}_{k,\beta}[v]\,.$$

A minimizer of $\nu_{\mathbf{B},0}(\beta)$ solves the Euler-Lagrange equation associated with the minimization of $\mathcal{Q}_{0,\beta}$, that is,

$$-\,v'' - \frac{v'}{r} + \tfrac{B^2}{4}\,r^2\,v = \nu_{\mathbf{B},0}(\beta)\,v - \beta\left(\int_0^{+\infty} |v|^p\,r\,dr\right)^{\frac{2}{p}-1} |v|^{p-2}\,v\,.$$

With no restriction, we can restrict the problem to positive solutions such that

$$\nu_{\mathbf{B},0}(\beta) = \left(\int_0^{+\infty} |v|^p\,r\,dr\right)^{1-\frac{2}{p}}$$

and have therefore to solve the reduced problem

$$-\,v'' - \frac{v'}{r} + \tfrac{B^2}{4}\,r^2\,v = \nu\,v - |v|^{p-2}\,v \tag{24}$$

among nonnegative functions in $\mathrm{H}^1((0,+\infty),\,r\,dr)$ such that $\int_0^{+\infty} r^2\,|v|^p\,r\,dr < +\infty$. Notice that the *compact support principle* applies according to, *e.g.*, [1, 5, 6, 25], since $p-1<1$ so that the nonlinearity in the right hand side of (24) is non-Lipschitz. Numerically, we can therefore solve (24) using a shooting method, with a shooting parameter $a = v(0) > 0$ that has to be adjusted to provide a nonnegative solution with compact support, which minimizes $\int_0^{+\infty} |v|^p\,r\,dr$ and thus determines

$\beta \mapsto \nu_{\mathbf{B}}(\beta)$. Since we compute the size of the support of the approximated solution, there is no numerical error due to finite size truncation.

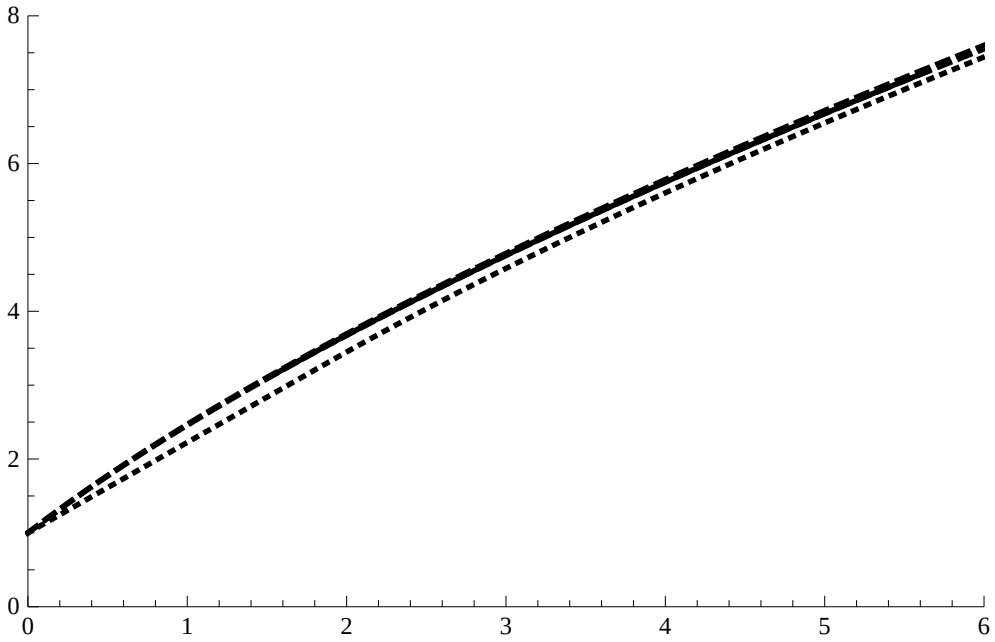


FIGURE 8. Case $p = 1.4$, $B = 1$: Estimates of $\nu_{\mathbf{B}}(\beta)$. The horizontal axis corresponds to β measured in units of $(2\pi)^{1-2/p}$. Estimates based on Gaussian functions correspond to the dashed curve, and can be distinguished only for large values of β from the numerical solutions found by solving (24), which are represented by the plain curve. The dotted curve is the lower estimate of Proposition 12.

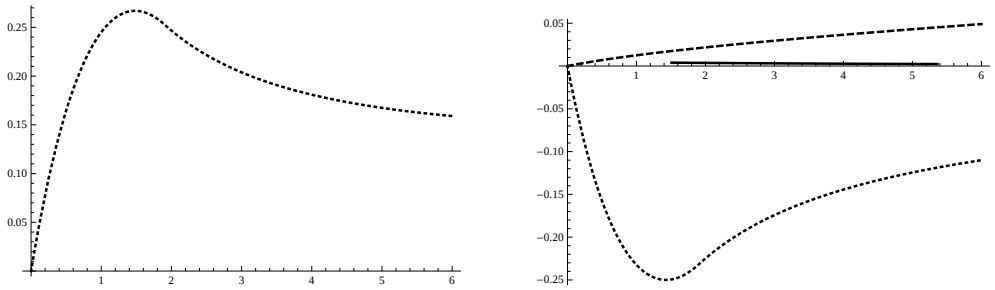


FIGURE 9. The parameters are fixed to the same values as in Fig. 8. Left: difference of the estimate based on Gaussian test functions and of the lower estimate of Proposition 12. Right: difference with respect to the lower estimate of Proposition 6 of the estimate based on Gaussian functions (dashed curve), of the estimate based on the numerical solutions found by solving (24) (plain curve) and of the lower estimate of Proposition 12.

5. INVESTIGATION OF THE STABILITY OF THE $k = 0$ COMPONENT: A PERTURBATION APPROACH

Let us consider the problem of minimizing

$$\mathcal{Q}_\alpha^{(p)}[\psi] := \frac{\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\psi|^2\,dx + \alpha \int_{\mathbb{R}^2} |\psi|^2\,dx}{\left(\int_{\mathbb{R}^2} |\psi|^p\,dx\right)^{2/p}}$$

for a given fixed $p \in (2,+\infty)$. Let us denote by u a minimizer of $\mathcal{Q}_\alpha^{(p)}$ on the class $(\mathscr{C}_0)$ of radial functions, normalized so that it solves

$$-u''-\frac{u'}{r}+\left(\tfrac{B^2}{4}\,r^2+\alpha\right)u=|u|^{p-2}\,u$$

and let us consider the test function

$$\psi_\varepsilon=u+\varepsilon\,e^{i\,\theta}\,v\,,$$

where v is a radial function, depending only on $r=|x|$, and $e^{i\,\theta}=(x_1+i\,x_2)/r$. In the asymptotic regime as $\varepsilon\to 0_+$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\psi_\varepsilon|^2\,dx+\alpha\int_{\mathbb{R}^2}|\psi_\varepsilon|^2\,dx-\left(\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}u|^2\,dx+\alpha\int_{\mathbb{R}^2}|u|^2\,dx\right)\\ =\left(\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}v|^2\,dx+\alpha\int_{\mathbb{R}^2}|v|^2\,dx\right)\varepsilon^2+o(\varepsilon^2)\\ =2\,\pi\int_0^{+\infty}\left[|v'|^2+\left(\left(\tfrac{1}{r}-\tfrac{B\,r}{2}\right)^2+\alpha\right)|v|^2\right]r\,dr\,\varepsilon^2+o(\varepsilon^2)\,,\end{aligned}$$

$$\|\psi_\varepsilon\|_p^2-\|u\|_p^2=2\,\pi\,\tfrac{p}{2}\,\|u\|_p^{2-p}\left(\int_0^{+\infty}|u|^{p-2}\,v^2\,r\,dr\right)\varepsilon^2+o(\varepsilon^2)\,,$$

and finally

$$\begin{aligned}\left(\mathcal{Q}_\alpha^{(p)}[\psi_\varepsilon]-\mu_0(\alpha)\right)\|u\|_p^2\\ =2\,\pi\left[\int_{\mathbb{R}^2}|v'|^2\,dx+\int_{\mathbb{R}^2}\left(\left(\tfrac{1}{r}-\tfrac{B\,r}{2}\right)^2+\alpha\right)|v|^2\,dx-\tfrac{p}{2}\int_0^{+\infty}|u|^{p-2}\,v^2\,r\,dr\right]\varepsilon^2+o(\varepsilon^2)\end{aligned}$$

where $\mu_0(\alpha)=\|u\|_p^{p-2}=\mathcal{Q}_\alpha^{(p)}[u]$.

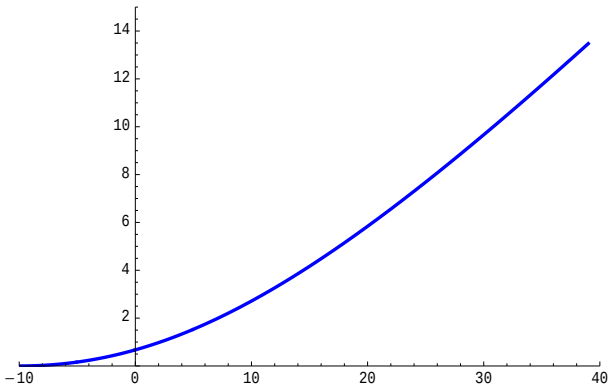


FIGURE 10. Case $p=3$ and $B=10$: plot of μ solving (25) as a function of α.

The linear stability of $\mathcal{Q}_\alpha^{(p)}$ can therefore be recast as the eigenvalue problem of finding the lowest $\mu\in\mathbb{R}$ for which

$$-v''-\frac{v'}{r}+\left(\left(\tfrac{1}{r}-\tfrac{B\,r}{2}\right)^2+\alpha\right)v-\tfrac{p}{2}\,|u|^{p-2}\,v=\mu\,v \tag{25}$$

has a solution in $u\in \mathrm{H}^1_{\mathbf{A}}$. The quotient $\mathcal{Q}_\alpha^{(p)}$ is linearly unstable at u if $\mu<0$.

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