

(Cyclo-passive) Port-Controlled Hamiltonian dynamics in LQ differential games

M. Sassano, T. Mylvaganam and A. Astolfi

Abstract—It is shown that the state/costate dynamics arising in a certain class of linear quadratic differential games can be interpreted as the interconnection of (cyclo-passive) Port-Controlled Hamiltonian systems. This property relies on the fact that the (virtual) energy functions associated to each player depend only on the interplay between the inputs of the players, as opposed to the system's matrix or the individual cost functionals. Finally, it is shown that an arbitrarily accurate approximation of an open-loop Nash equilibrium strategy, obtained from the trajectories of the state/costate system, can be *robustified* by externally stabilizing the stable eigenspace of the underlying state/costate system.

I. INTRODUCTION

Game theory plays an important role in a large variety of control engineering applications, such as for power systems, robotics and biomedical systems (see, *e.g.* [1], [2], [3], [4], [5], [6]). Differential games, in particular, provide the ideal framework to model settings in which multiple control inputs (referred to as *players* in this context) influence the evolution of a dynamical system and to study the strategic interactions between the players [7], [8], [9], [10]. Both optimal control problems and differential games are *dynamic optimisation* problems and, consequently, share many similarities (see, *e.g.* [9], [11], [12]). In fact, observing that the former can be viewed as single-player games, differential games can be considered a generalisation of optimal control problems. Despite their similarities, there are certain interesting differences between the two classes of problems and their solutions, which warrant some attention. For instance, several solution concepts exist for differential games, such as open-loop/feedback Nash/Stackelberg equilibrium strategies (see, for instance, [7] for details regarding the different solution concepts). In particular, in what follows we consider *open-loop Nash equilibrium strategies*.

It is well known that the solution of an optimal control problem is closely related to the stable, invariant manifold of its associated Hamiltonian lifted system [13], [14], [15].

M. Sassano with Dipartimento di Ingegneria Civile ed Ingegneria Informatica, Università di Roma Tor Vergata, 00133 Roma, Italy.

T. Mylvaganam is with the Department of Aeronautics, Imperial College London, London SW7 2AZ, UK.

A. Astolfi is with the Department of Electrical and Electronic Engineering, Imperial College London, London SW7 2AZ, UK, and the Dipartimento di Ingegneria Civile ed Ingegneria Informatica, Università di Roma Tor Vergata, 00133 Roma, Italy.

The work of A. Astolfi has been supported in part by the European Union's Horizon 2020 Research and Innovation Programme under grant agreement No 739551 (KIOS CoE) and by the Italian Ministry for Research in the framework of the 2017 Program for Research Projects of National Interest (PRIN), Grant no. 2017YKXYXJ. The work of M. Sassano has been (partially) supported by the Univ. "Tor Vergata", Rome, 2019 "Beyond the Borders" Program, Grant no. E89C20000600005

The Hamiltonian structure of the lifted system has revealed several useful insights related to the optimal control policy (see, *e.g.* [13]). In the context of differential games, on the other hand, despite the fact that open-loop Nash equilibrium strategies are computed by firstly associating a classical *pre-Hamiltonian* function to each player, the peculiar nature of the minimization task induced by the presence of multiple players leads to a lifted system that does not possess an overall classical Hamiltonian structure [10]. Consequently, several insights that are available for optimal control problems, *e.g.* with regards to existence of solutions, are unavailable for differential games.

Motivated by the stark contrast between the similarities and the differences between optimal control problems and differential games, we focus on the specific, yet interesting case of two-player linear quadratic (LQ) differential games and study how the behaviour of each *individual* player is modified by the presence of the other player. We demonstrate that an *individual* state/costate system with a Port-Controlled Hamiltonian (PCH) structure can be associated with each individual player. Under certain conditions, relating to the coupling of the two players, the PCH systems are cyclo-passive. We further demonstrate that the overall differential game can be viewed as a *positive* feedback interconnection of the individual PCH systems, with the solution (in terms of open-loop Nash equilibrium strategies) given by the outputs of the interconnected system. Interestingly, due to the *positive* feedback, the interconnected system does not inherit any cyclo-passivity properties from the individual PCH systems and is itself intrinsically unstable. Returning attention to the original (*common*) state/costate system associated with the two-player game, we further demonstrate that the addition of *virtual* control inputs enables determining arbitrarily accurate approximate open-loop Nash equilibrium strategies. Namely, the trajectories of the state/costate system - and hence the open-loop Nash equilibrium strategies - can be *robustified* by externally stabilizing the stable eigenspace of the state/costate system.

The remainder of the paper is organised as follows. Two-player LQ differential games and their solutions (in terms of open-loop Nash equilibrium strategies) are introduced, along with some preliminaries, in Section II. A structural interpretation of the the state/costate dynamics arising in LQ differential games, which relies on the introduction of *individual* state/costate dynamics, is then provided in Section III. A constructive result enabling robust computation of open-loop Nash equilibrium strategies, via the addition of virtual control inputs to the original state/costate dynamics,

is provided in Section IV, before some concluding remarks are given in Section V.

II. PRELIMINARIES AND PROBLEM DEFINITION

Consider two-player LQ differential games characterized by the underlying dynamics

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2, \quad (1)$$

with $x : \mathbb{R} \rightarrow \mathbb{R}^n$ and $u_i : \mathbb{R} \rightarrow \mathbb{R}^{m_i}$ describing the *shared* state of the plant and the control actions available to each player, respectively, and the individual cost functionals

$$\mathcal{J}_i(u) = \int_0^\infty (\|x(\tau)\|_{Q_i}^2 + \|u_i(\tau)\|^2) d\tau, \quad (2)$$

for $i = 1, 2$. It has been shown (see, e.g., [10] for comprehensive discussions and further insights) that *open-loop Nash equilibrium strategies* are provided by the outputs

$$u_i^*(t) = -B_i^\top \lambda_i^*(t), \quad (3)$$

of the extended state/costate system

$$\dot{x} = Ax - B_1 B_1^\top \lambda_1 - B_2 B_2^\top \lambda_2, \quad (4a)$$

$$\dot{\lambda}_1 = -Q_1 x - A^\top \lambda_1, \quad (4b)$$

$$\dot{\lambda}_2 = -Q_2 x - A^\top \lambda_2, \quad (4c)$$

initialized at $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, P_1 x_0, P_2 x_0)$, where the matrices $P_1 \in \mathbb{R}^{n \times n}$ and $P_2 \in \mathbb{R}^{n \times n}$, not necessarily symmetric, denote a pair of stabilizing solutions of the so-called (asymmetric) coupled Riccati equations [10]

$$\begin{aligned} 0 &= P_1 A + A^\top P_1 + Q_1 - P_1 S_1 P_1 - P_1 S_2 P_2, \\ 0 &= P_2 A + A^\top P_2 + Q_2 - P_2 S_2 P_2 - P_2 S_1 P_1, \end{aligned} \quad (5)$$

where $S_i = -B_i B_i^\top$, for $i = 1, 2$. The following structural assumption, borrowed from [10], characterizes the class of two-player LQ differential games that admit a unique open-loop Nash equilibrium strategies. To this end, let $H \in \mathbb{R}^{3n \times 3n}$ describe the dynamic matrix of the system (4), namely

$$H = \left[\begin{array}{c|cc} A & -S_1 & -S_2 \\ \hline -Q_1 & -A^\top & 0 \\ -Q_2 & 0 & -A^\top \end{array} \right]. \quad (6)$$

Assumption 1. The matrix $H \in \mathbb{R}^{3n \times 3n}$ in (6) possesses an n -dimensional stable graph subspace and $2n$ eigenvalues, counted with their algebraic multiplicities, in $\mathbb{C}^+$. $\circ$

Even if the proof of the claims in the preceding discussion is not reported here, it is worth mentioning that the system (4) is derived from the individual, classic *pre-Hamiltonian* functions $\mathcal{H}_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{m_1+m_2}$, $i = 1, 2$, defined for each player by

$$\mathcal{H}_i(x, \lambda_i, u) = \frac{1}{2} x^\top Q_i x + \frac{1}{2} u^\top \bar{R}_i u + \lambda_i^\top A x + \lambda_i^\top \bar{B} u, \quad (7)$$

with $u = (u_1, u_2)$, $\bar{R}_1 = \text{blkdiag}(I, 0)$, $\bar{R}_2 = \text{blkdiag}(0, I)$ and $\bar{B} = [B_1, B_2]$. It is then not surprising that the equations

(4a), (4b) and (4a), (4c) individually *resemble* the classical Hamiltonian structure underlying the solution to an open-loop optimal control problem (see, for instance, [15]). However, the peculiar nature of the minimization process associated to the computation of the Nash equilibrium strategy, namely

$$\begin{aligned} u_1^* &= \min_v \mathcal{H}_1(x, \lambda_1, (v, u_2^*)), \\ u_2^* &= \min_v \mathcal{H}_2(x, \lambda_2, (u_1^*, v)), \end{aligned} \quad (8)$$

is such that neither the classical Hamiltonian structure of the overall system (4), nor some of the particularly appealing properties resulting from such a structure (e.g. the split spectrum of its eigenvalues, see [13] for more details) are preserved. It is therefore worth establishing whether also the overall system (4) possesses a certain Hamiltonian interpretation of its dynamics.

III. NASH STRATEGIES AS THE OUTPUT OF (CYCLO-PASSIVE) PCH SYSTEMS IN FULL-AUTHORITY DIFFERENTIAL GAMES

The main objective of this section is to investigate the structure of the state/costate dynamics arising in the context of differential games with the aim - as suggested in Section II - of identifying a Hamiltonian interpretation of the dynamics (4). To this end, consider LQ differential games characterized by the underlying dynamics (1) and the cost functionals (2), for $i = 1, 2$, and suppose that the following structural assumption, which defines what we refer to as the class of *full-authority* differential games, holds.

Assumption 2. The LQ differential game described by (1) and (2) is such that $m_i = n$ and $\text{rank } B_i = n$ for $i = 1, 2$. $\circ$

To provide a concise statement of the following result, consider the (virtual) *energy function*

$$\hat{\mathcal{H}}_i(x_i, \lambda_i) = -\lambda_i^\top B_i B_i^{-1} x_i := -\lambda_i^\top \Pi_i x_i, \quad (9)$$

with $(x_i, \lambda_i) \in \mathbb{R}^n \times \mathbb{R}^n$, $i = 1, 2$. While invertibility of B_j follows immediately from the structural condition and the rank condition in Assumption 2, the latter additionally implies, by the property that $\det \Pi_i = \det B_i \det B_j^{-1}$, that also the matrices Π_i , $i = 1, 2$, are invertible.

Consider now the PCH system described by the energy function (9), together with the *interconnection matrix*

$$J_i = \frac{1}{2} \begin{bmatrix} (B_i B_j^\top - B_j B_i^\top) & -(\Pi_i^{-1} A + A \Pi_i^{-1}) \\ (A^\top \Pi_i^{-\top} + \Pi_i^{-\top} A^\top) & (Q_i \Pi_i^{-1} - \Pi_i^{-\top} Q_i) \end{bmatrix}, \quad (10)$$

and the *dissipation matrix*

$$D_i = \frac{1}{2} \begin{bmatrix} -(B_i B_j^\top + B_j B_i^\top) & (A \Pi_i^{-1} - \Pi_i^{-1} A) \\ (\Pi_i^{-\top} A^\top - A^\top \Pi_i^{-\top}) & -(Q_i \Pi_i^{-1} + \Pi_i^{-\top} Q_i) \end{bmatrix}, \quad (11)$$

$i = 1, 2$, $j = 1, 2$ and $i \neq j$, namely the PCH system

$$\Sigma_i := \begin{cases} \dot{z}_i &= (J_i - D_i) \nabla \hat{\mathcal{H}}_i(z_i) + G_i v_i \\ y_i &= G_i^\top \nabla \hat{\mathcal{H}}_i(z_i), \end{cases} \quad (12)$$

with $z_i = (x_i, \lambda_i)$ and $G_1 = [B_2^\top, 0]^\top$, $G_2 = [B_1^\top, 0]^\top$. In the following statement, the PCH systems (12) are employed to interpret the (collective and individual) behaviour of the players.

Theorem 1. Consider the feedback interconnection $\bar{\Sigma}$ of Σ_1 and Σ_2 according to $v_1 = y_2$ and $v_2 = y_1$. Then

- (i) the manifold $\mathcal{M} := \{(z_1, z_2) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} : x_1 = x_2, \lambda_1 = P_1 x_1, \lambda_2 = P_2 x_2\}$ is invariant for $\bar{\Sigma}$, for all $x \in \mathbb{R}^n$;
- (ii) the outputs y_i of the interconnected system $\bar{\Sigma}$, initialized at $(x_i(0), \lambda_i(0)) = (x_0, P_i x_0)$ are such that $(z_1(t), z_2(t)) \in \mathcal{M}$ for all $t \geq 0$ and such that $y_i(t) = u_i^*(t)$. $\circ$

Remark 1. In the case that Assumption 2 does not hold, a PCH interpretation of the system (4), similar to that of Theorem 1, is still feasible, albeit possibly with different energy function and interconnection and damping matrices. Namely, define the candidate energy function $\hat{\mathcal{H}}_i(x_i, \lambda_i) = (1/2)x_i^\top M_{i,1} x_i + \lambda_i^\top M_{i,2} x_i + (1/2)\lambda_i^\top M_{i,3} \lambda_i$, with $M_{i,j} \in \mathbb{R}^{n \times n}$, $j = 1, 2, 3$, and let

$$J_i = \begin{bmatrix} J_i^{11} & J_i^{12} \\ -(J_i^{12})^\top & J_i^{22} \end{bmatrix}, \quad D_i = \begin{bmatrix} D_i^{11} & D_i^{12} \\ (D_i^{12})^\top & D_i^{22} \end{bmatrix},$$

with $J_i^{jj} = -J_i^{jj^\top}$ and $D_i^{jj} = D_i^{jj^\top}$. For the LQ differential games that do not satisfy Assumption 2 the conditions to obtain a PCH interconnected structure are given by the following system of equations

$$0 = M_{i,1} B_j, \quad (13a)$$

$$-B_i = M_{i,2} B_j, \quad (13b)$$

$$-Q_i = -(J_i^{12} + D_i^{12})^\top M_{i,1} + (J_i^{22} - D_i^{22}) M_{i,2}, \quad (13c)$$

$$-A^\top = -(J_i^{12} + D_i^{12})^\top M_{i,2} + (J_i^{22} - D_i^{22}) M_{i,3}, \quad (13d)$$

$$A = (J_i^{11} - D_i^{11}) M_{i,1} + (J_i^{12} - D_i^{12}) M_{i,2}, \quad (13e)$$

$$-B_i B_i^\top = (J_i^{11} - D_i^{11}) M_{i,2}^\top + (J_i^{12} - D_i^{12}) M_{i,3}, \quad (13f)$$

for $i = 1, 2$. Whenever Assumption 2 holds, (13a) immediately implies that $M_{i,1} = 0$, consistently with Theorem 1. Considering instead equation (13b), the generic solution is provided by $M_{i,2} = -B_i (B_j^\top B_j)^{-1} B_j^\top + W B_j^\perp$, provided B_j is full column rank, as it is reasonable to expect to avoid trivialities, and where $B_j^\perp$ is a matrix whose rows span the space orthogonal to B_j , i.e. $B_j^\perp B_j = 0$, and W is an arbitrary matrix of appropriate dimensions. $\blacktriangle$

The notion of *cyclo-passivity* is discussed herein specifically in the case of dynamics as in (12), since a detailed discussion about this property would be beyond the scope of this paper¹.

Definition 1. Let $w_i : \mathbb{R}^{m_i} \times \mathbb{R}^{m_j} \rightarrow \mathbb{R}$, $w_i = y_i^\top v_i$, define a *supply rate* for the system (12). Then the latter is said to be *cyclo-passive* if there exists a *storage function* $\mathcal{S}_i : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ such that the dissipation inequality

$$\mathcal{S}_i(x_i(0), \lambda_i(0)) + \int_0^T w_i(y_i(t), v_i(t)) dt \geq \mathcal{S}_i(x_i(T), \lambda_i(T)), \quad (14)$$

holds along the trajectories of (12) for all $T > 0$, any input v_i , and any initial condition $(x_i(0), \lambda_i(0))$. $\circ$

Although not particularly evident from Definition 1, the key difference between the notions of passivity and cyclo-passivity consists in the property that in the latter the storage function $\mathcal{S}_i$ is not required to be *bounded from below* (see, e.g. [17], [18]). Similarly to the context of passivity, an infinitesimal version of the dissipation inequality (14) can be given, provided the storage function is continuously differentiable, as $\dot{\mathcal{S}}_i \leq y_i^\top v_i$. In particular, it allows to insightfully interpret the behaviour of several thermodynamical systems (with *entropy* typically qualifying as a storage function not necessarily bounded from below), as well as electrical networks with inductors. In fact, a dynamical system is *cyclo-passive* if and only if

$$\oint w_i(y_i(t), v_i(t)) dt \geq 0,$$

for all $T \geq 0$ and $v_i : [0, T] \rightarrow \mathbb{R}^{m_i}$ such that $x_i(0) = x_i(T)$, namely it exhibits a *passive* behaviour on closed paths.

In the remainder of this section we characterize the class of games for which the PCH systems (12), for $i = 1, 2$, are *cyclo-passive*. Towards this end, consider the following assumption, which relates to the coupling between the two players.

Assumption 3. The input matrices are such that $m_1 = m_2 = m$, namely $B_1 \in \mathbb{R}^{n \times m}$ and $B_2 \in \mathbb{R}^{n \times m}$, and such that $B_i B_j^\top + B_j B_i^\top < 0$. $\circ$

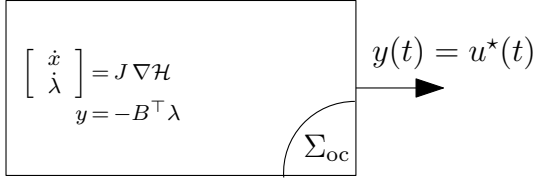
Note that the above assumption is without loss of generality if Assumption 2 holds. In fact, if the latter inequality is not verified it is always possible to consider a coordinate transformation in the input space such that $\tilde{u}_1 = -(B_1^\top B_2^\top)^{-1} u_1$ and $\tilde{u}_2 = u_2$: the resulting game satisfies the inequality and admits identical solutions P_i to the coupled AREs (5).

Proposition 1. Consider the PCH systems (12) described by the energy functions $\hat{\mathcal{H}}_i$ and the interconnection and dissipation matrices (10) and (11), respectively. Suppose that Assumption 3 holds and that

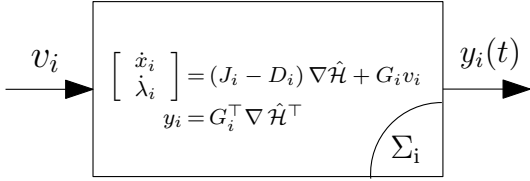
$$Q_i \Pi_i^{-1} + \Pi_i^{-T} Q_i < 0, \quad (15)$$

¹The interested reader is referred to [16], [17] for a complete discussion.

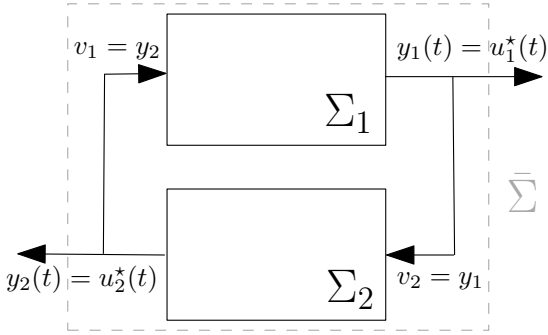
for $i = 1, 2$ and $j = 1, 2$, with $j \neq i$. Then there exists $\varepsilon_p \in \mathbb{R}_{>0}$ such that the PCH system (4) is cyclo-passive for the class of LQ differential games (1), (2) with the property that $\|A\| \leq \varepsilon_p$. $\circ$



(a) optimal control: single-player behaviour



(b) two-player LQ game: individual behaviours



(c) two-player LQ game: interconnected behaviour

Fig. 1. Block diagram representations of the individual and interconnected behaviours of the decision-making agents in a game.

Remark 2. As schematically illustrated by the block diagram (a) in Figure 1, in (single-player) optimal control problems the behaviour of the decision-making agent can be interpreted and explained as the output of an autonomous Hamiltonian system with the canonical skew-symmetric interconnection matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

together with the (minimized) Hamiltonian function $\mathcal{H}(x, \lambda) = (1/2)x^T Q x + \lambda^T A x - (1/2)\lambda^T B B^T \lambda$. On the other hand, extending similar ideas to the context of multi-player dynamic *optimization* problems, the statement of Theorem 1 entails that instead the *individual behaviour* of each player in the framework of full-authority games is governed by the output of a PCH system generated by the (virtual) energy function $\hat{\mathcal{H}}_i$ in (9) that captures the interplay between the control authorities of each player, as shown in the block diagram (b) in Figure 1. Moreover, such

a PCH is *cyclo-passive* from the input v_i to the output y_i , provided the conditions of Proposition 1 hold. Note that v_i summarizes the interaction of the player i with the other agent in the game and, in this perspective, it is not surprising that the *generator* of the most desirable behaviour for the agent i , namely Σ_i , is a *non-autonomous* system capable of reacting to the interaction induced by the opponent. Finally, by relying on invariance of the manifold $\mathcal{M}$ proved in Theorem 1, the overall interaction between the two players can be explained via a *positive* feedback interconnection of the individual PCH systems, as depicted by the block diagram (c) in Figure 1. Note that the interconnection does not preserve the (potential) cyclo-passivity of each player and it is in fact intrinsically unstable. $\blacktriangle$

IV. ROBUST COMPUTATION OF OPEN-LOOP NASH EQUILIBRIUM STRATEGIES

The statement of Theorem 1 and the remarks in the previous section immediately reveal that, similarly to the context of (single-player) optimal control problems, the open-loop Nash equilibrium strategies are generated as the output of an intrinsically unstable system: the classic Hamiltonian system - described by the Hamiltonian matrix that possesses a spectrum symmetric with respect to the imaginary axis (*split spectrum*) - in the case of optimal control problems and the state/costate dynamics (4) - which have been interpreted as the *positive* interconnection of two (cyclo-passive) PCH systems - in the case of differential games.

This aspect becomes particularly relevant in the case of games, since the effective methods and routines developed in the past decades to (numerically) solve the standard Algebraic Riccati Equation (ARE) arising in optimal control problems have not been paralleled hitherto by similar techniques in the case of the *coupled* Riccati equations (5), especially for large-scale systems. To motivate the results presented in this section consider first the following numerical example.

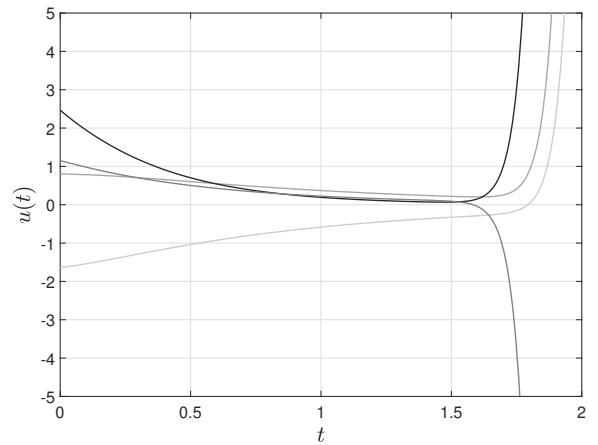


Fig. 2. Time histories of the open-loop Nash equilibrium strategies $u_1(t)$ and $u_2(t)$ computed as the output of the state/costate dynamics (4) for the LQ differential game described in Example 1.

Example 1. Consider a two-player LQ differential game described by the randomly generated matrices

$$A = \begin{bmatrix} -5.6547 & 3.4813 & -0.4920 & -2.7317 & 4.7415 \\ 3.8814 & -5.8622 & -0.3012 & 3.9582 & -5.0316 \\ -0.4297 & -0.0703 & -1.9284 & -2.5866 & 1.4709 \\ -3.9984 & 5.2990 & -1.6464 & -9.7696 & 5.1261 \\ 3.3269 & -3.9920 & 2.4495 & 6.5753 & -11.0227 \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} 0.0075 & 0 \\ 0.4829 & -0.2945 \\ -0.3514 & 0.6022 \\ 0.1208 & -0.5150 \\ 0.1461 & -0.0030 \end{bmatrix}, B_2 = \begin{bmatrix} -1.1980 & 0 \\ 0 & -0.4816 \\ -0.1661 & -0.4482 \\ 0 & 0.3315 \\ -0.3181 & 0 \end{bmatrix}$$

together with individual cost functionals described by $Q_i = I$, $i = 1, 2$. The solutions P_1 and P_2 are then obtained by computing the eigenstructure of the underlying matrix H via the routine **eig** in Matlab and by subsequently determining a basis for the stable eigenspace of the form $[I \ P_1^\top \ P_2^\top]^\top$. Despite the fact that $\max\{\|X_1(P_1, P_2)\|, \|X_2(P_1, P_2)\|\} < 10^{-14}$, where X_i denotes the right-hand side of the equations (5), respectively, the graph of Figure 2 shows that the open-loop Nash equilibrium strategies generated for each player by the state/costate system (4) initialized at $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, P_1 x_0, P_2 x_0)$ diverge rapidly to infinity. $\circ$

Lemma 1. Consider the LQ differential game described by the dynamics (1) and the individual cost functionals (2). Suppose that Assumption 1 holds. Then the “equilibrium” costate is provided by

$$\lambda_i^*(t) = P_i e^{(A - S_1 P_1 - S_2 P_2)t} x_0, \quad (16)$$

for all $t \geq 0$ and $i = 1, 2$. $\circ$

Consider instead the controlled state/costate system

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda}_1 \\ \dot{\lambda}_2 \end{bmatrix} = H \begin{bmatrix} x \\ \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} v := Hz + Lv, \quad (17)$$

where $v : \mathbb{R} \rightarrow \mathbb{R}^{2n}$, $v = (v_1, v_2)$, denotes a virtual control input that allows to “control” the costate dynamics, as discussed below.

Lemma 2. Consider the controlled state/costate system (17) in closed loop with

$$v_1 = (A^\top - P_1 S_1 + F_1)(\lambda_1 - P_1 x) - P_1 S_2(\lambda_2 - P_2 x) \quad (18a)$$

$$v_2 = (A^\top - P_2 S_2 + F_2)(\lambda_2 - P_2 x) - P_2 S_1(\lambda_1 - P_1 x) \quad (18b)$$

with $F_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$, arbitrary Hurwitz matrices. Then the resulting controlled costate is given by

$$\lambda_i^s(t) = e^{F_i t} \xi_i + P_i e^{A_{cl} t} \left(x_0 - \sum_{j=1}^2 \tilde{M}_j(t) \xi_j \right), \quad (19)$$

for $i = 1, 2$, with $\xi_j = \lambda_j(0) - P_j x_0$ and $\tilde{M}_j(t) = \int_0^t e^{-A_{cl} \tau} S_j e^{F_j \tau} d\tau$. $\circ$

Note that the control inputs v_1 and v_2 are such that the costate dynamics, in the transformed coordinates and in closed loop, are decoupled and asymptotically stable. For simplicity of presentation of the following result, suppose that the Hurwitz matrices $F_1 = F_2$ are defined as the real, diagonal matrix $F = \text{diag}\{-\sigma_F, \dots, -\sigma_F\}$, with $\sigma_F > 0$. Moreover, to provide a concise statement, let $c_1 \in \mathbb{R}_{>0}$ and $c_2 \in \mathbb{R}_{>0}$ be such that $\|e^{-A_{cl} t}\| < c_1 e^{c_2 t}$, whereas $d_1 \in \mathbb{R}_{>0}$ and $d_2 \in \mathbb{R}_{>0}$ are such that $\|e^{A_{cl} t}\| < d_1 e^{-d_2 t}$.

Proposition 2. Consider the state/costate dynamics (4), with $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, P_1 x_0, P_2 x_0)$, together with the controlled state/costate dynamics (17) in closed loop with (18). Fix any $T \in \mathbb{R}_{>0}$, $\varepsilon \in \mathbb{R}_{>0}$ and $\mu \in \mathbb{R}_{>0}$ and let

$$\sigma_F > \max \left\{ \frac{2c_1 d_1 \mu \beta (\|S_1\| + \|S_2\|)}{\varepsilon} + c_2, \frac{1}{T} \log \left(\frac{\varepsilon}{2\mu} \right) \right\},$$

where $\beta = \max\{\|P_1\|, \|P_2\|\}$. Define $e_i(t) := \lambda_i^*(t) - \lambda_i^s(t)$, $i = 1, 2$. Then

$$\|e_i(t)\| \leq \varepsilon, \quad (20)$$

for all $t > T$ and for all $(x_0, \lambda_1(0), \lambda_2(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ such that $\|(x_0, \xi_1, \xi_2)\| < \mu$. $\circ$

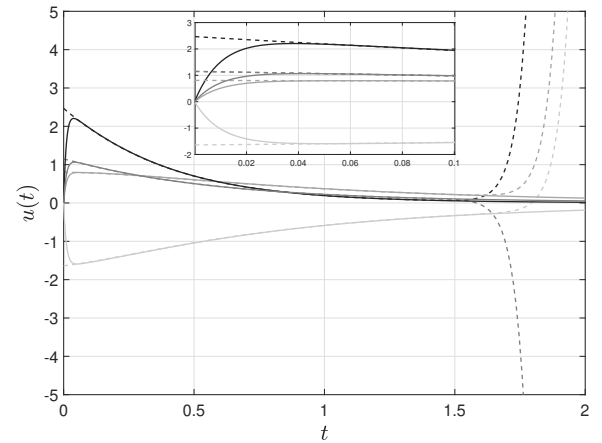


Fig. 3. Comparison between the time histories of the open-loop Nash equilibrium strategies generated by (4) and those provided by the controlled state/costate dynamics (17) in closed loop with (18) and initialized at $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, 0, 0)$.

Remark 3. Differently from the computation of a solution of the classic ARE arising in optimal control problems, for which numerically reliable techniques have been developed in the past decades, the systematic solution of the coupled

AREs arising in the context of differential games essentially remains an open issue. This may have seriously detrimental consequences whenever the state of the plant cannot be measured by the players for any time $t > 0$, namely the so-called open-loop Nash equilibrium strategies in which the game information structure consists of the knowledge of the dynamics and only of the initial configuration $x_0 \in \mathbb{R}^n$ of the plant. As recalled above, such equilibrium strategies are provided by the output of an extended dynamics that can be interpreted as the positive interconnection of two (cyclo-passive) PCH systems for each player precisely initialized at $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, P_1 x_0, P_2 x_0)$. However, since the practical computation of the matrices P_i that solve the AREs (5) are inevitably affected by numerical errors - *i.e.* the matrices $\tilde{P}_i = P_i + \Delta P_i$, $i = 1, 2$, are typically employed in the above architecture in place of P_i - the latter scheme would yield the open-loop control laws $u_i(t) = -B_i^\top \lambda_i(t)$, with

$$\lambda_i(t) = \lambda_i^*(t) + (N_i e^{A_u t} - P_i e^{A_{cl} t} M(t)) \begin{bmatrix} \Delta P_1 \\ \Delta P_2 \end{bmatrix} x_0,$$

which is such that $\lim_{t \rightarrow \infty} \|u_i(t)\| = \infty$ for any $x_0 \in \mathbb{R}^n$ since $\sigma(A_u) \subset \mathbb{C}^+$ by Assumption 1. On the other hand, by employing the solution of the controlled state/costate dynamics (17) in closed loop with $\tilde{v}_i = v_i|_{P_i = \tilde{P}_i}$, with v_i as in (18), one has that $u_i^s(t) = -B_i^\top \tilde{\lambda}_i^s(t) = -B_i^\top (\hat{z}_{i+1} + P_i \hat{z}_1(t))$, where $\hat{z}$ is provided by

$$\begin{aligned} \dot{\hat{z}} &= \left[\begin{array}{c|cc} A_{cl} & -S_1 & -S_2 \\ \hline Y_1(\Delta P_1, \Delta P_2) & F_1 - \Delta P_1 S_1 & -\Delta P_1 S_2 \\ Y_2(\Delta P_1, \Delta P_2) & -\Delta P_2 S_1 & F_2 - \Delta P_2 S_2 \end{array} \right] \hat{z} \\ &:= H_F(\Delta P_1, \Delta P_2) \hat{z}, \end{aligned}$$

with $Y_i = -(A^\top - P_i S_i + F_i - \Delta P_i S_i) \Delta P_i + P_i S_j \Delta P_j + \Delta P_i S_j \Delta P_j$, $i = 1, 2$, $j = 1, 2$, $j \neq i$. Hence, since the eigenvalues of H_F are continuous functions of the entries of the uncertain matrices ΔP_i and $\sigma(H_F(0, 0)) \subset \mathbb{C}^-$, there exists $\varepsilon > 0$ such that the open-loop control laws u_i^s remain bounded for all time for all numerical errors such that $\max\{\|\Delta P_1\|, \|\Delta P_2\|\} < \varepsilon$. $\blacktriangle$

Example 2. Consider the LQ differential game introduced in Example 1. Figure 3 depicts the comparison between the time histories of the open-loop Nash equilibrium strategies generated by (4) and those provided by the *controlled* state/costate dynamics (17) in closed loop with (18) and initialized at $(x(0), \lambda_1(0), \lambda_2(0)) = (x_0, 0, 0)$ and $\sigma_F = 10$. $\circ$

V. CONCLUSIONS AND FURTHER DISCUSSIONS

We have studied two-player LQ differential games and their solutions in terms of open-loop Nash equilibrium strategies. Focusing on how the behaviour of each individual player is modified by the presence of the other player, it has been demonstrated that each player can be associated with an *individual* state/costate system with a PCH structure and the

open-loop Nash equilibrium strategies are generated by the outputs resulting from a positive feedback interconnection of the two PCH systems (which are cyclo-passive under certain conditions). It has been further demonstrated that approximate open-loop Nash equilibrium strategies - with the level of approximation arbitrarily small - can be obtained via the addition of virtual inputs to the original state/costate system associated with the game. In addition, the latter yields a method to *robustify* open-loop Nash equilibrium strategies. As further developments of the proposed robustified architecture, it may be possible to envision a strategy in which the achieved asymptotic stability properties of the stabilized state/costate dynamics are employed to iteratively compute the open-loop Nash equilibrium strategy, thus circumventing the need for an explicit solution of the coupled Riccati equations. Furthermore, a similar approach may be envisioned also for the case of nonlinear differential games.

REFERENCES

- [1] J. R. Marden and J. S. Shamma, "Game theory and control," *Annual Review of Control, Robotics, and Autonomous Systems*, vol. 1, pp. 105–134, 2018.
- [2] J. R. Marden, S. D. Ruben, and L. Y. Pao, "A model-free approach to wind farm control using game theoretic methods," *IEEE Transactions on Control Systems Technology*, vol. 21, no. 4, pp. 1207–1214, 2013.
- [3] S. M. LaValle and S. Hutchinson, "Game theory as a unifying structure for a variety of robot tasks," in *Proceedings of 8th IEEE International Symposium on Intelligent Control*, 1993, pp. 429–434.
- [4] D. Cappello, S. Garcin, Z. Mao, M. Sassano, A. Paranjape, and T. Mylvaganam, "A hybrid controller for multi-agent collision avoidance via a differential game formulation," *IEEE Transactions on Control Systems Technology*, 2020.
- [5] T. Mylvaganam and M. Sassano, "Autonomous collision avoidance for wheeled mobile robots using a differential game approach," *European Journal of Control*, vol. 40, pp. 53–61, 2018.
- [6] K. Staňková, J. S. Brown, W. S. Dalton, and R. A. Gatenby, "Optimizing cancer treatment using game theory: A review," *JAMA oncology*, vol. 5, no. 1, pp. 96–103, 2019.
- [7] T. Basar and G. Olsder, *Dynamic Noncooperative Game Theory*. Academic Press, 1982.
- [8] A. W. Starr and Y. C. Ho, "Nonzero-sum differential games," *Journal of Optimization Theory and Applications*, vol. 3, pp. 184–206, 1969.
- [9] A. W. Starr and Y.-C. Ho, "Further properties of nonzero-sum differential games," *Journal of Optimization Theory and Applications*, vol. 3, no. 4, pp. 207–219, 1969.
- [10] J. Engwerda, *LQ dynamic optimization and differential games*. Chichester: John Wiley & Sons, 2005.
- [11] Y.-C. Ho, "Differential games, dynamic optimization, and generalized control theory," *Journal of Optimization Theory and Applications*, vol. 6, no. 3, pp. 179–209, 1970.
- [12] M. Sassano, T. Mylvaganam, and A. Astolfi, "An algebraic approach to dynamic optimisation of nonlinear systems: a survey and some new results," *Journal of Control and Decision*, vol. 6, no. 1, pp. 1–29, 2019.
- [13] K. Zhou, J. Doyle, and K. Glover, *Robust and Optimal Control*, 1st ed. Prentice Hall, 1996.
- [14] G. Blankenstein and A. van der Schaft, "Optimal control and implicit Hamiltonian systems," in *Nonlinear control in the year 2000*. Springer, 2001, pp. 185–205.
- [15] D. Liberzon, *Calculus of variations and optimal control theory: a concise introduction*. Princeton University Press, 2011.
- [16] J.C. Willems, "Qualitative behavior of interconnected systems," *Annals of Systems Research*, vol. 3, pp. 61–80, 1973.
- [17] A. van der Schaft, "Cyclo-dissipativity revisited," *IEEE Transactions on Automatic Control*, pp. 1–1, 2020.
- [18] J. C. Willems, "Dissipative dynamical systems part i: General theory," *Archive for rational mechanics and analysis*, vol. 45, no. 5, pp. 321–351, 1972.