

Nonlinear Instability of Stokes-layers

by

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To My Wife, Tiemei

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Abstract

The Stokes-layer generated by a sinusoidally oscillating flat plate in an infinite fluid is an important prototype of unsteady flows, and is known to be susceptible to high-frequency instability. According to linear theory (Tromans, 1977; Cowley, 1987), small disturbances can grow exponentially over part of a period and become neutral at some time. The present study considers nonlinear effects on the instability under a high-Reynolds-number assumption, i.e. $R \gg 1$. It is shown that in the vicinity of the neutral time, the small disturbances evolve from a linear stage to a nonlinear stage due to enhanced nonlinear interactions inside the critical layers – the neighbourhood of a level at which the phase velocity of the disturbance is equal to the basic flow velocity. Specifically, we have studied the evolution of a two-dimensional wave, a pair of oblique waves and a resonant triad of waves. The temporal-spatial modulation of near-planar waves is also considered.

For a two-dimensional disturbance, it is shown that the evolution is controlled by nonlinear effects once the linear growth rate has decreased from order one to $O(\epsilon^{\frac{2}{5}})$, where ϵ is the magnitude of the disturbance. The amplitude equation turns out to be an integro-differential one of Hickernell type (Hickernell 1984). We study the amplitude equation by series-extension and numerical methods. For the inviscid case, i.e. $\epsilon \gg R^{-\frac{5}{6}}$, the solutions always develop a finite-time singularity. When viscosity is included, i.e. $\epsilon \sim R^{-\frac{5}{6}}$, it is found that (a) viscosity generally delays the finite-time singularity, and (b) if viscosity is sufficiently large the singularity can be completely eliminated, with the result that the wave evolves to an equilibrium state, confirming the findings of Goldstein & Leib (1989). In particular, for the present problem it is found that for disturbances with wavenumbers in the range $[0.6, 1.3]$, a finite-time singularity always occurs, no matter how large the scaled viscosity parameter is. If $\epsilon \ll R^{-\frac{5}{6}}$, i.e. the disturbance is relatively small, the amplitude equation can be reduced to a classical Stuart-Landau equation by taking a limit of the integro-differential equation. The stability of the equilibria is also studied.

For the case where the disturbance consists of a pair of oblique waves, the non-linear evolution stage comes at a much earlier time, namely when linear growth rate is $O(\epsilon^{\frac{1}{3}})$, and it then evolves over a much faster time scale $O(\epsilon^{-\frac{1}{3}})$, as in Goldstein & Choi (1989). Mathematically this is due to a pole type of singularity in the outer solution for the streamwise and spanwise velocities, which is much stronger than the branch-point singularity associated with $\bar{U}_{yy}(y_c) \neq 0$. We show that for the disturbances with order-one spanwise wavenumbers, the amplitude equation is the same as that of Goldstein & Choi (1989), although $\bar{U}_{yy}(y_c) = 0$ in their case. However, we point out that when three-dimensionality is relatively weak, the fact that $\bar{U}_{yy}(y_c) \neq 0$ in our problem does bring in a difference. Moreover, we obtain an amplitude equation for the case when viscosity effects are important in the leading order equations of the critical layers, thus extend the analysis of Goldstein & Choi (1989). At least under the inviscid limit, the solution of the amplitude equation can develop a finite-time singularity, corresponding to the finite-distance singularity of

Goldstein & Choi (1989). In addition, it is noted that the nonlinear interaction of waves can generate strong vortices of the same magnitude as the waves. We suggest that these vortices may be identified as those observed in experiments, and may have relevance to the streaky structure in a turbulent boundary layer.

For the case when three waves form a resonant-triad, we show that when the amplitude of the planar and the oblique waves are $O(\epsilon^{\frac{1}{3}})$ and $O(\epsilon)$ respectively, a mutual interaction takes place and the disturbances are governed by a coupled integro-differential equation system. These amplitude equations are significantly different from those of Raetz (1959), Craik (1971), Smith & Stewart (1987) in two important aspects. Firstly the local growth rate depends on the whole history of the evolution, unlike a conventional resonant triad, where the local growth rate depends only on the instantaneous amplitudes of the disturbances (see Goldstein & Lee, 1990). Secondly the back reaction of the oblique waves on the 2-D wave is accounted for by two cubic terms and one quartic term rather than by one quadratic term. The amplitudes of the two-dimensional and the three-dimensional waves can exhibit a finite-time singularity, the structure for which is proposed. The evolution equations obtained in our study are shown to be valid until the magnitude of the disturbance becomes order one; thus they provide a full description of the development of resonant-triad waves from their linear small amplitude stage towards the fully-nonlinear order-one-amplitude stage.

For the temporal-spatial modulation case, i.e. the case when the amplitude of the disturbance is a slowly-varying function of spanwise position as well as of time, it is shown that the development of the disturbance is controlled by critical-layer nonlinearity when its linear growth rate decreases to $O(\epsilon^{\frac{2}{3}})$. Nonlinear interactions influence the evolution by producing a streamwise vortex and cross-flow distortion. The modulation equation turns out to be an integro-*partial*-differential one containing history-dependent nonlinear terms. A novel feature of the amplitude equation is that the derivatives with respect to space, including the highest derivative, appear in the nonlinear terms. These terms are associated with three-dimensionality, and hence represent vortex stretching and tilting effects. The possible properties of the amplitude equation are discussed. It is shown that a localized singularity may occur in a finite time. This provides a possible explanation for the focusing of streamwise vorticity and the formation of streaks.

The finite-time singularity, which can occur in all the four cases studied, suggests that explosive growth is induced by critical-layer nonlinear effects. Hence an initially small disturbance is further amplified by nonlinear effects; moreover this nonlinear growth can prevent the disturbance evolving into an equilibrium state as implied by linear theory. We suggest that this nonlinear blow-up of high-frequency disturbances is related to the bursting phenomena observed in experiments, and can lead to transition to turbulence.

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Chapter 1

Introduction

1.1 Stability of Steady Flows

The transition from laminar to turbulent flow is one of the central problems in fluid mechanics. It is generally attributed to some form of instability of the underlying flow. One basic step to understand the transition is to study, either experimentally or mathematically, the development of small disturbances on the basic flow.

Experimentally, many detailed investigations have concentrated on steady boundary-layer flows, e.g. Blasius flow and free shear flow, in which small disturbances are usually introduced by vibrating ribbons or loud speakers (cf. experiments for Taylor-Couette flow, where no artificial disturbance is introduced). The resulting waves or wave-packets initially are two-dimensional and evolve linearly (e.g. see Schubauer & Skramstad, 1947), then enter a nonlinear stage, at which super- and/or sub-harmonics are generated. But eventually three-dimensional disturbances grow rapidly and become appreciable (e.g. see Klebanoff *et al.*, 1962; Kachanov & Levchenko, 1984; Saric *et al.*, 1984; Cohen *et al.*, 1991), and the flow undergoes laminar breakdown locally, manifested by the onset of turbulent spots, and ultimate transition to turbulence.

Mathematically, the instability problem poses an initial-value problem of non-linear partial differential equations (PDEs), which is difficult in general. For some simple basic flows, such a problem became solvable recently by using supercomputers (see e.g. Kleiser & Zang, 1991). However when disturbances are sufficiently small, one may expect that it is reasonable to linearize the disturbance equations. The resultant linear system usually has space- and/or time-periodic solutions according to the symmetry of the flow. By seeking periodic solutions of the linear equations, the original *initial-value problem* is converted into a *linear eigenvalue problem*¹, solutions of which are usually called normal modes. The formulation and the solution of such eigenvalue problems are one of the main aims of linear instability theory. Its mathematical treatments have been summarized by, e.g. Lin (1955)

¹In contrast, Hall (1983, 1988) has shown that Gortler instability is essentially an initial-value problem, and an eigenvalue approach yields misleading results.

and Drazin & Reid (1981). By solving eigenvalue problems, the critical parameter for the onset of instability, e.g. a critical Reynolds number, may be predicted. Also at a given parameter, the most dangerous mode, i.e. the mode with largest growth rate, may be identified. Moreover, the case where the eigenvalue problem is degenerate may have a significance in the instability of shear flows, as shown by Benney & Gustavsson (1981). From a dynamical-system point of view, this linear instability theory can be regarded as a *local bifurcation analysis*. This type of analysis has proved to be highly successful for Taylor-Couette flow as illustrated beautifully by G. I. Taylor (1923). The theoretical prediction of Tollmien-Schlichting (T-S) waves for Blasius flow (Tollmien, 1929; Schlichting, 1933) and plane Poiseuille flow (Heisenberg, 1924; Lin 1944), was confirmed by landmark experiments of Schubauer & Skramstad (1947) and Nishioka *et al.* (1975) respectively. In particular, the predicted critical Reynolds number for Blasius flow is in reasonable agreement with experiments, even though the approach based on Orr-Sommerfeld (O-S) equation is strictly not mathematically justified (see e.g. Smith, 1979).

To appreciate further the usefulness of the eigenvalue approach, we should recall that the eigenfunctions form a natural function base, from which the solution of the linear initial-value problem can be constructed. This was shown by Schensted (1960) for two-dimensional disturbances in plane Poiseuille flow, where the set of eigenfunctions is complete. As demonstrated by Henningson (1991), a similar conclusion holds for three-dimensional disturbances, but to represent the solution of the initial-value problem, the so-called Squire modes, in addition to the O-S modes, have to be invoked. For Blasius flow, the eigenfunction set is not complete and a continuous spectrum exists. As a result, the representation of the solution is more involved (Gustavsson, 1971), but Gaster & Grant (1975) have demonstrated that the evolution of disturbances can be effectively treated as a linear superposition of discrete modes provided nonlinear effects have not become significant. In this sense, these normal modes are basic “building-blocks” underlying the dynamics of the evolution.

However, there are many situations where linear theory is insufficient and nonlinear effects have to be included. For instance:

(a) For plane Poiseuille flow, transition can occur at Reynolds numbers of the order of 1000, much lower than the linear-theory result 5772 (Orszag, 1971).

(b) For a growing disturbance, no matter how small it is initially, at sufficiently large times its magnitude will become so large that nonlinearity can no longer be ignored.

(c) Three-dimensional disturbances are observed to experience very fast growth, even though the initial disturbances are predominantly two dimensional.

To explain (a), Stuart (1960) and Watson (1960) developed weakly nonlinear theory. The theory is formally valid in an asymptotically small vicinity of a bifurcation point, and is able to predict whether the bifurcation is subcritical or supercritical. The subcritical case indicates that provided the amplitude is larger than a *threshold*, a *nonlinear* bifurcation is possible at a lower Reynolds number; this provides a qualitative explanation for (a). A quantitative explanation can only be given by direct

numerical simulation, but discrepancies with experiments still exist. The concept of threshold instability has been confirmed experimentally by Nishioka *et al.* (1975). Moreover, by an amplitude-expansion procedure (now referred as Stuart-Watson expansion), Stuart (1960) and Watson (1960) formally derived the Landau amplitude equations, which governs the evolution of two-dimensional modes near the neutral curve², though as a self-consistent asymptotic theory, the range of its validity may be small (Herbert, 1983).

In the context of Poiseuille flow instability, an extension of Landau's equation to take account of spatial modulation was given by Stewartson and Stuart (1971) (see also Hocking, Stewartson & Stuart, 1972; Davey, Hocking & Stewartson, 1974). They obtained an amplitude equation governing the subsequent nonlinear evolution of an initially linearly growing disturbance, and thus somewhat resolved problem (b).

Problem (c) has initiated much theoretical interest. Raetz (1959), followed by Craik (1971), introduced a *resonant-triad* model, and showed that the resonance between a two-dimensional TS wave and two subharmonic oblique waves could be the mechanism for the rapid amplification of three-dimensional disturbances. On the other hand, Herbert put forward a *linear secondary instability theory*, in which the rapid growth of 3D disturbance is attributed to the parametric instability of the two-dimensional primary wave after it has reached a certain threshold (e.g. see Herbert, 1991; Bayly, Orszag & Herbert, 1991 and references therein). See also Kelly (1967) for a discussion of a 2-D subharmonic parametric instability mechanism; he first suggested this idea.

One feature of transition is that it often, if not always, occurs at high Reynolds numbers. This suggests the application of asymptotic-expansion techniques; these have played an important role in the development of hydrodynamic instability theory. For example, using these techniques, Heisenberg (1924) and Tollmien (1929) predicted the instability for plane Poiseuille flow and Blasius boundary flow respectively. Moreover, Smith (1979a,b) has pointed out that a high-Reynolds-number-approach is the only mathematically self-consistent account of the instability of Blasius boundary layer. It does not, however, treat the minimum critical Reynolds number. Based upon the scaling yielded by earlier heuristic approximations, Smith (1979a,b) showed that TS instability in the vicinity of the lower branch of the neutral curve is described by a triple-deck structure. The fully nonlinear version of triple-deck problem is the counterpart of the fully Navier-Stokes (N-S) equations; the linearized version is equivalent to a limiting form of the O-S equation. The extended interactive version of Smith, Papageorgiou & Elliott (1984) provides the application to flows at moderate Reynolds number of practical interest. Weakly nonlinear effects, in the Stuart-Watson vein, were studied in the framework of triple-deck theory by for example, Smith (1979b, 1980) and Hall & Smith (1982, 1984). More recently,

²Generally speaking, "neutral curve" refers to the one yielded by linear analysis, and hence weakly nonlinear theory depends crucially on the existence of such a neutral curve. But Henningson (1991) argued that from the point of view of center manifold theory, the existence of a finite amplitude (nonlinear) neutral curve is sufficient.

nonlinear interactions between waves and vortices have been extensively studied, e.g. see Benney (1984), Hall & Smith (1989, 1990), Walton & Smith (1990), Hall & Smith (1991). In addition, by seeking high-frequency solutions of the triple-deck equations, Smith & Burggraf (1985) revealed some novel instability features such as eruption of vorticity from viscous sublayer and nonlinear Euler type of interactions. Along this line, Smith & Stewart (1987) studied resonant-triad interactions along the lower branch of the neutral curve. Recently, Mankbadi (1991), on the other hand considered resonant-triad interactions in the upper-branch-scaling regime; the agreement between his theoretical predictions and experiments is very impressive.

It should be noted that all the nonlinear theories so far mentioned are developed for slowly-modulated neutral modes, i.e. for modes with *vanishing small* growth rates. This possibly deviates from many practical cases, where the growth rates of disturbances, though they may be small, are *finite*. Consequently, quantitative predictions from these theories are limited, though many valuable qualitative ones have been provided. Some efforts have been made to remedy this deficiency. For instance, Zhou (1982) devised an approximation method, and calculated the threshold amplitude. The agreement with experiment is fairly good. Because of the limitation of nonlinear theories, direct numerical computations were developed, of which there are two types. The first type is to solve the initial-value problem. The second type is a *global bifurcation analysis* in the parameter space (e.g. see Ehrenstein & Koch, 1991). In this approach, instead of tracing the temporal/spatial development of an initial disturbance, one seeks nonlinear (stable or unstable) equilibrium periodic solutions directly; these solutions bifurcate at certain critical parameters. Such an approach has generally been applied to closed systems such as Taylor-Couette flow, where some bifurcated intermediate states are stable, and can be observed experimentally. In open flows, however, the intermediate states are often unstable, and therefore are not experimentally observable. Nevertheless, there is some evidence that they may be related to coherent structures observed in transitional and turbulent flows. An appealing speculation is that these equilibrium solutions are basic “building blocks” constructing a lower-dimensional manifold; the solution of a *nonlinear* initial-value problem spends a substantial fraction of time “wandering around” the neighbourhood of this manifold. If this occurs, the mean quantities might be determined accurately enough by these equilibrium solutions. The above speculation seems to be supported by an interesting finding of Rozhdestvensky & Simakin (1984). They found that a low-resolution simulation, which they believe to be adequate to take account of large-scale coherent structures, closely approximated some overall quantities of engineering interest, such as the mean-flow profile and the skin friction factor.

Most of ideas so far mentioned have mainly been developed for Tollmien-Schlichting instability, however they also apply to other instabilities. In this thesis, we will be concerned with the evolution of Rayleigh instability waves in Stokes layers. We will explore aspects like nonlinearity, three-dimensionality, in particular, resonant-triad interactions and wave-vortex interactions.

1.2 Stability of Unsteady Flows

Compared with steady flows, the instability of unsteady flows is not as well understood. On the other hand, unsteady flows, in particular time-periodic flows, occur abundantly in nature, and a knowledge of their instability is important in many applications, e.g. the transport of sediment along the ocean bed (Li 1954), physiological investigation of large blood vessels (Pedley 1980), and various pipe flows in engineering. For surveys on studies of instabilities of unsteady flows, see Davis (1974), von Kerczek *et al.* (1974, 1987), and Hall (1987). Stuart *et al.* (1990) recently studied the instability of a squeeze lubrication film, where the flow varied in both time and space.

In this study, we will consider the instability of the Stokes layer generated by a flat plate oscillating sinusoidally with speed $(U_0 \cos \omega t^*, 0)$ in an infinite fluid of kinematic viscosity ν . The resulting flow has a boundary-layer thickness of order $\delta^* = (2\nu/\omega)^{1/2}$, and the Reynolds number R based on δ^* and U_0 is $U_0(2/\omega\nu)^{1/2}$. It is an important prototype of unsteady flows as well as an *exact* solution of the Navier-Stokes equations. Experiments show that a typical phenomenon occurring in the laminar-turbulent transition process is bursts of high-frequency disturbances followed by relaminarization, see e.g. Merkli & Thomann (1975), Clarion & Pelissier (1975), Hino *et al.* (1976, 1983) and Akhavan *et al.* (1991a). Hino *et al.* (1983) found that the bursts, and the associated streamwise vortex structure were similar to those in the wall region of turbulent boundary layers.

Previous theoretical work on instability problems of time-periodic flows has so far been restricted to the study of infinitesimal disturbances. Three types of theories have been suggested. The first type is the global theory based on Floquet theory. Von Kerczek & Davis (1974) considered the instability of a finite Stokes layer, i.e. the flow between an oscillatory plate and a stationary plate, with a distance $m\delta^*$ apart. For $m = 8$, they found that the flow is stable up to $R = 800$. The numerical method they adopted only applies to finite Stokes layers. Hall (1978) presented an improved version of Floquet analysis, which can be applied to infinite Stokes layers as well as to finite Stokes layers, but he too found that the flow was stable for all Reynolds numbers investigated, i.e. $R \leq 320$. It was also shown by Hall that the flow has a continuous spectrum of damped modes at all values of Reynolds numbers, and that a set of discrete modes bifurcate from the continuous spectrum for certain values of the Reynolds number. However it appears that the least stable mode of the finite Stokes layer studied by von Kerczek & Davis does not correspond to the least stable discrete mode of the infinite Stokes layer when $m = \infty$. It is generally believed that according to Floquet theory, the flow is stable at low Reynolds numbers and possibly stable at all Reynolds numbers. Interestingly, while failing to predict instability for a flat Stokes layer, the global theory gives successful predictions for periodic flows with the presence of centrifugal force or stratification effects (Seminara & Hall, 1976, Hall 1984, von Kerczek & Davis, 1976). A mathematical explanation for this is given by Cowley (1987). Based on a large-Taylor-number assumption, he could seek the high-frequency mode to the governing equation in Seminara &

Hall (1976). At this limit, the basic-flow can be regarded as quasi-steady, i.e. the mode can be found by “freezing” the basic-flow profile at each instant. Cowley found that the most unstable mode at each instant remains to be most unstable throughout the oscillation cycle. Hence the concept of the “net growth” represented by a Floquet exponent is appropriate. For the Stokes layer over a flat plate, however, there exists no mode which remains to be most unstable over the whole period. Physically, the instabilities of the flows studied by Seminara & Hall (1976), and von Kerczek & Davis (1976), are associated with the centrifugal and buoyancy instability mechanisms respectively, while for a flat Stokes layer, the instability is associated with shear of the basic flow. As a result of the difference in instability mechanism, the instability in the stratified Stokes layer (Seminara & Hall, 1976) leads to the formation of stationary regular vortex patterns, while in the flat Stokes layer the instability is manifested by rapidly growing high-frequency disturbances followed by relaminarization. The fact that disturbances are observed to grow only over part of a period suggests that Floquet theory is too restrictive and may not be most relevant to the practical instability of Stokes layers; this is because Floquet theory concentrates on the net linear decay or growth of a disturbance over a whole period rather than on the instantaneous “growth” to larger amplitude at any part in a period. The latter appears to be more relevant.

The second approach can be classified as an Orr-Sommerfeld “quasi-steady” type, e.g. Collins (1963), Obremski & Morkovin (1969) and Monkewitz (1983). As emphasised by Cowley (1987) (see also Smith, 1979a,b), these theories are more or less heuristic approximations and inconsistent mathematically in the sense that the Reynolds number is assumed to be sufficiently large so that a multiple scale analysis can be performed in time but then the Reynolds number is also taken to be order one so that viscosity can be retained in the Orr-Sommerfeld (O-S) equation. In particular, this type of analysis predicts that the instantaneous profile has a lower critical Reynolds number at the start of the acceleration phase than at the end of it (see e.g. von Kerczek & Davis 1974). Akhavan *et al.* (1991a,b) argued that this is in conflict with the experimental observation that bursts occur at the end of the acceleration phase. This disputable prediction may be due to the aforementioned inconsistency, and thus can not be accepted. Similar inconsistencies in some approaches made to analyze the effect of nonparallelism on instability were noted by Smith (1979a,b). Also, it is difficult to take account of nonlinear effects systematically in these theories.

The third type of approach is the self-consistent theory adopted by Tromans (1977) and Cowley (1987). By assuming that the Reynolds number was large *throughout*, the earlier ad-hoc quasi-steady assumption was put on a self-consistent base. They assumed that the most relevant disturbances had frequencies of order $R\omega$, i.e. frequencies much higher than those of the basic flow. The disturbance thus evolves over a very fast time scale $(R\omega)^{-1}$. This is required for mathematical self-consistency but is also in good agreement with experimental observation. It is found that to the leading order, the disturbance is governed by the Rayleigh equation. A Rayleigh instability of the Stokes layer to high-frequency disturbances was identified

and the time interval over which such disturbances can grow was calculated.

1.3 Critical-Layer Nonlinearity

According to linear theory (Cowley, 1987), the disturbance undergoes a transient growth, and then decays. A possible transition mechanism for the Stokes layer is that before the disturbance decays, it can be further amplified by nonlinear effects and cause transition to turbulence. The present study will be concerned with the influence of nonlinearity on the disturbances considered by Cowley (1987). In particular, suppose that an instability wave with wavenumber α is excited at time τ_i on the left branch of the neutral curve "A" (see Fig.1, which arises in the paper of Cowley, 1987), and then grows exponentially according to linearized theory until the neutral time, say τ_0 , is approached. In the vicinity of the neutral time, the growth rate becomes small, and *critical layers* emerge. Mathematically, because of the singular nature of the Rayleigh equation in the neighbourhood of critical layers, some neglected terms have to be reintroduced to remove the singularity. Physically, some ignored effects become important and have to be taken into account. These effects could be viscosity (Lin 1955, Drazin & Reid 1981), nonlinearity (Benney & Bergeron 1969, Davis 1969) or the so-called translation effect (Cowley 1985, Haynes & Cowley 1986). In addition, Merkin (1984) showed that nonparallel effects representing the "meridional advection" are significant in the critical layer of Rayleigh instability waves in a nearly parallel rotating barotropic flow. The theory of critical layers have been reviewed by Maslowe (1986) and Stewartson (1981). In this study, we are particularly concerned with nonlinear effects within the critical layers.

In the vicinity of a neutral curve, nonlinear effects can be taken into account by weakly nonlinear theory. The original version of weakly nonlinear theory (Stuart 1960, Watson 1960) was developed for the case where the Reynolds number is of order one. There the disturbance is governed by the O-S equation at leading order, and the solution is regular in space; thus the nonlinear interaction takes place in the whole flow field. In high-Reynolds-number flows, the possibility that nonlinearity could first become important inside critical layers was first highlighted by Lin (1957). Later, it was shown independently by Benney & Bergeron (1969) and Davis (1969) that if the amplitude of the disturbance is sufficiently large, the singularity difficulty should be resolved by nonlinearity rather than viscosity. Cases where nonlinear and viscous effects are equally important in critical layers were considered by Haberman (1972), Smith & Bodonyi (1981), and Bodonyi & Smith (1983). It should be emphasized that in these studies, the aim was to seek nonlinear neutral modes directly, rather than to follow the development of a linearly unstable mode into a nonlinear stage. Consequently, all critical layers in these studies are in equilibrium. It is not yet clear whether such equilibrium solutions can evolve from some initial linear state; thus the relevance of this type of mode to instability remains unknown.

The critical-layer nonlinear effects in a *time-evolution problem*, were first investigated by Warn & Warn (1978), Stewartson (1978), and Brown & Stewartson

(1978). They specifically studied the evolution of a *forced* Rossby wave, and showed that when time is sufficient long, nonlinearity inside the critical layers is significant at leading order, and as a consequence the initially linear Rossby wave enters a nonlinear evolution stage. The resulting critical-layer equation is similar to that of Benney & Bergeron (1969), but differs in one important respect; this is that a time-dependent term appears in the leading order equation, i.e. the critical layer is not in equilibrium. Note that in these studies, it was assumed that free instability modes were suppressed. On the other hand, Hickernell (1984) considered the critical-layer nonlinear effects on the evolution of a *free* Rossby-wave mode, i.e. the wave satisfies homogeneous boundary conditions. By imposing an appropriate radiation condition at infinity, the eigen-mode is "singular" in the sense that the critical level does not locate at the inflexional point of the basic flow profile (see below). Unlike the forced problem stated above, the critical-layer dynamics is now governed by linear equations at leading order, and nonlinearity is weak in the sense that it enters through inhomogeneous terms. Nevertheless, it is shown that this weak nonlinearity inside the critical layer controls the growth rate of the instability wave. In particular, because the time-varying term was retained in the leading-order equations of the critical layers, he obtained an amplitude equation which contains a history-dependent nonlinear term.

More recently, Goldstein & Leib (1988) and Goldstein & Hultgren (1988) have considered the effects of critical-layer nonlinearity on the development of a two-dimensional instability mode in a free shear layer. The critical-layer structure obtained by them turns out to be the same as that in the forced Rossby wave problem studied by Stewartson (1978) and Warn & Warn (1978). However, it's worth pointing out that although the critical layer is strongly nonlinear, as in Benney and Bergeron (1969), the eigenrelation of the instability wave is not affected by nonlinearity at leading order. This is due to the fact that the critical layers evolved are regular here (see below). As will be shown in the next section, it is only *incidental* that the nonlinear terms are retained in the leading-order critical-layer equations. The numerical solutions of Goldstein & Leib (1988), and Goldstein & Hultgren (1989) show that vorticity roll up is due to critical-layer nonlinear effects. This process is initially inviscid, but sufficiently downstream, viscous effects eventually assert themselves, and cause the vorticity distribution to diffuse into a much simpler pattern with the generation of small lengthscale eddies being less pronounced compared with the inviscid case (Goldstein & Leib, 1988). The instability wave was found to grow algebraically, and consequently the critical layer evolves into a quasi-equilibrium one, which is similar to that of Benney & Bergeron (1969). But the detailed flow is different from theirs in that the vorticity in the closed streamline, i.e. within the cat's-eye boundary, is non-uniform. In both studies of Goldstein & Leib (1988) and Goldstein & Hultgren (1989), a specific mean-flow profile was used. A extension of their analysis into a general profile was made by Hultgren (1991). When an experimentally measured profile was used, his theoretical prediction for the development of the wave was quantitatively in agreement with experiments. A strongly critical layer structure was shown to describe the vorticity roll-up process in boundary layers

with weak adverse pressure gradient (Goldstein, Durbin & Leib, 1987). Goldstein & Wundrow studied the evolution of so-called acoustic mode instability (Cowley & Hall, 1990) on hypersonic boundary layers. They showed that the dynamics of the critical layer was strongly nonlinear, and was governed by a nonlinear critical-layer vorticity equation coupled with a nonlinear energy equation. Leib & Goldstein (1989) investigated the nonlinear interaction between the marginal unstable sinuous and varicose modes in the “Bickley jet”, Both modes are regular with the critical levels locating at the inflexional points (the Bickley jet has two inflexional points symmetrically off the center). The varicose mode is the subharmonic of the sinuous mode, and thus they form subharmonic resonance (Kelly, 1968), i.e. the nonlinear interaction occurs at quadratic order rather than at cubic order. The critical-layer was found to be strongly nonlinear, i.e. analogous to that of Goldstein & Leib (1988), but the equations governing the evolution of the amplitudes and the distribution of the vorticity of each mode were strongly coupled.

Goldstein & Leib (1989) considered the evolution of a *single* oblique mode in a compressible shear layer. They were concerned with the oblique modes whose critical level locates at the so-called generalized inflexional point, and hence the eigensolutions for the pressure and vertical velocity of the disturbance were regular at the critical layer. The outer solutions for the streamwise and spanwise velocity do exhibit a pole type of singularity, but in the single oblique wave case, it is appropriate to assume that the wave grows in its propagation direction, and thus this singularity as well as the spanwise velocity effects can be eliminated by an appropriate “Squire transformation” (see also Goldstein & Choi, 1989). Nevertheless, they showed that due to a pole type of singularity associated with the *temperature* fluctuation, the critical layer nonlinearity controlled the evolution of the disturbance in a way similar to that discovered by Hickernell (1984); the amplitude equation was found to be the same as that of Hickernell (1984). The numerical investigation of the amplitude equation show that the solution can either blow up at a finite distance downstream, or evolve into an equilibrium state, depending on the relative size of the disturbance and the Reynolds number as well as on the sign of the real part of the coefficient of the nonlinear terms. Leib (1991) further extended the analysis of Goldstein & Leib (1989) to the case where the critical level does not coincide with the generalized inflexional point. He also removed “restrictions of unit Prandtl number and linear viscosity-temperature relation”. In this case, the critical level is a logarithmic branch point of the leading-order solutions for the pressure and the vertical velocity; the pole type of singularity associated with three-dimensionality is again eliminated, as in Goldstein & Leib (1989). The scaling of Goldstein & Leib (1989) still applies because it is just the scaling for the case with a logarithmic-branch-point singularity (cf. Hickernell, 1984). The amplitude equation was again found to be of Hickernell type, but the “nonlinear kernel function” differs from, however is an extension of, that of Goldstein & Leib (1989). We further notice that this difference is mainly due to the additional nonlinear effect contributed by the logarithmic branch point. The removal of other restrictions also results in modifications to the kernel; however these modifications disappear at the inviscid limit, and are not dominant effects in

the final formation of the finite-distance singularity. Leib (1991) showed that the amplitude equation could have either an equilibrium solution or a solution which blows up at a finite distance downstream, depending on the physical parameters of the flow.

The evolution of interacting oblique modes, i.e. *a pair of* oblique modes with the same streamwise wavenumber but opposite spanwise wavenumber was studied by Goldstein & Choi (1989) in an incompressible free shear layer. There the critical layer dynamics is significantly different from the corresponding 2-D case, “primarily because the streamwise velocity has a simple pole type of singularity” (Goldstein & Choi, 1989). The weakly nonlinear critical layer effects of Hickernell type were identified, in contrast to the strongly nonlinear one in the corresponding 2-D case (e.g. Goldstein & Leib, 1988). Note that the case of the interacting oblique waves is also significantly different from that of a single-oblique-wave (Goldstein & Leib, 1989; Leib, 1991) in that in the latter case the nonlinear effects associated with the pole of the streamwise velocity are suppressed by an appropriate Squire transformation. ~~As a result, nonlinear effects asserts their themselves much earlier.~~

The critical-layer nonlinear effects on the development of a resonant triad of Rayleigh instability waves in adverse-pressure-gradient boundary layers ~~were~~ investigated by Goldstein & Lee (1990). On the other hand, Mankbadi (1990) studied the resonant-triad evolution of T-S waves in Blasius boundary in the upper-branch-scaling regime, in which a critical layer is distinct from the viscous wall layer (see, e.g. Drazin & Reed, 1981; Bodonyi & Smith, 1981). More comments on this aspect will be made in Chapter four. In addition, Churilov & Shukhman (1988) studied an unsteady, weakly nonlinear critical layer in a stratified flow and obtained amplitude equations evolving history effects. In particular, they found that in certain circumstances, the cubic nonlinear term disappeared due to the symmetry property of the flow, and ²quintic nonlinear term arose.

This study is designed to address the effects of the critical-layer nonlinearity on the development of initially small disturbances, in the hope of shedding some light on the transition mechanism of the Stokes layer of interest here. We are concerned with the temporal evolution since this appear to be relevant to the instability of the Stokes layer.

The overall structure of this thesis is as follows. In the next section of this chapter, which forms the cornerstone of this thesis, the underlying scalings are derived by considering nonlinear interactions inside critical layers. We show that, in the vicinity of, but before the neutral time, these nonlinear interactions control the overall development of initially linear instability waves. Consequently, disturbances enter a nonlinear evolution stage. The scalings are shown to be directly determined by the singularity properties of the disturbance vorticity near the critical levels. In Chapter two, we consider the simplest case where the disturbance is a two-dimensional wave. In Chapter three, we study the evolution of the disturbance consisting of a pair of oblique waves. A disturbance that is composed of a resonant-triad waves is studied in Chapter four. In Chapters two to four, the disturbance is assumed to be periodic in space and to evolve with time. However in Chapter five,

we consider the nonlinear *temporal-spatial* modulation of a near-planar wave, where the spanwise distribution of the disturbance is allowed to be of general form.

Each chapter has a brief summary of the main results and an introduction. The analysis in each chapter proceeds in an outer-inner expansion style. In the “outer expansion” procedure, the asymptotic solutions near a critical levels are constructed; as usual, they are found to contain unknown jumps across the critical layers. In order for the inhomogeneous Rayleigh equations to have solutions, an appropriate solvability condition has to be imposed. The “inner expansion” procedure seeks the solutions within the critical layers which match onto the corresponding outer expansions. The matching determines the jumps. It is noted that nonlinear effects control the development of the disturbance by altering these jumps. Using the solvability condition and the jumps, we obtain amplitude equations. The properties of the amplitude equations are discussed and interpreted.

Except in Chapter four, viscous effects have been included in the leading-order critical-layer equations (Chapters two, three and five), i.e. the critical-layers evolved are both unsteady and viscous in nature. However, in Chapter four, a consideration of viscosity would greatly complicate the analysis. Instead, we assume that the Reynolds number is sufficiently large so that viscous effects do not play a leading-order role in the evolution of the disturbance, and hence can be neglected.

For the two-dimensional case (Chapter two), detailed numerical as well as analytical investigations of the amplitude equation are performed. The studies of the remaining cases are mainly devoted to derivation of the amplitude equations.

1.4 General Scaling Arguments

We take the flow to be described by Cartesian coordinates (x, y, z) , where x is parallel to the direction of oscillation of the plate, y is normal to the plate and z is the spanwise direction. If (x, y, z) is non-dimensionalised by δ^* , velocity (U, V, W) by U_0 , and time τ by ω^{-1} , i.e. $\tau = \omega t^*$ (see §1.2 for the definitions for δ^* , U_0 , ω and t^*), then the basic flow field can be written as

$$(\bar{U}, \bar{V}, \bar{W}) = (\cos(\tau - y)e^{-y}, 0, 0) .$$

We denote the perturbed flow by

$$(\bar{U} + u, \bar{V} + v, \bar{W} + w) .$$

Substitution into the Navier-Stokes equations yields equations for (u, v, w) .

Following Cowley (1987), we study high-frequency instability waves, and introduce the fast time scale

$$t = R\tau , \tag{1.1}$$

to take account of the frequencies of the waves.

The normal velocity component v of a disturbance of $O(\epsilon)$, to the leading order, has the form

$$\epsilon A_0 \bar{v}(y, \tau) E \cos \beta z + c.c. ,$$

where $\epsilon \ll 1$, and β is the spanwise wavenumber and is assumed to be order one; $\beta = 0$ corresponds to the case of 2-D disturbances. The function $\bar{v}$ satisfies the Rayleigh equation, and the order-one complex constant A_0 is a measure of the scaled amplitude of the disturbance. Hereafter *c.c* represents the complex conjugate of the part written out. For convenience, we have defined

$$E = \exp(i\alpha x - i\hat{\theta}(t)) ,$$

where α is the wavenumber, and is assumed to order one in this study, and

$$\frac{d\hat{\theta}}{dt} = \frac{1}{2}\alpha c(\tau) + \frac{1}{2}R^{-\frac{1}{2}}\Omega_1(\tau) + \dots \quad (1.2)$$

with c being the complex phase velocity, i.e. $c = c_r + ic_i$. For the fast time scale t defined by (1.1), it is convenient to introduce the factor $\frac{1}{2}$ in (1.2) (Cowley, 1987). The $O(R^{-\frac{1}{2}})$ term in (1.2) represents the effect of the viscous sub-Stokes layer (near the wall) with a thickness of order $R^{-\frac{1}{2}}$; this thin layer needs to be introduced to reduce the slip velocity of the disturbance to zero at the wall (Cowley, 1987). However compared with nonlinear effects to be addressed in this study, this viscous modification to the dispersion relation is at a higher order (see Chapter two), and thus can be ignored.

As discussed earlier, we concentrate on the times near to the neutral time, say

$$\tau = \tau_0 + \epsilon^l \tau_1 ,$$

where index l is a measure of how “quickly” the nonlinear evolution stage occurs. At this stage, the linear growth rate αc_i is of order ϵ^l , as can be deduced from a Taylor expansion of $\alpha c(\tau)$ at τ_0 , i.e. $\alpha c(\tau) = \alpha c(\tau_0) + \epsilon^l c'(\tau_0)\tau_1 + \dots$. We will show that the value of the index l depends on the nature of the singularity in outer expansion of the disturbance vorticity at the critical layers.

Before proceeding to a formal asymptotic expansion, we need to set down the underlying scaling, i.e. to specify the evolution time scale and the thickness of the critical layers in terms of the magnitude of the disturbances ϵ . For this purpose, we introduce an intermediate time scale³ $t_1 = \alpha c_i t$, which is much “slower” than the time scale of the instability waves, but much “faster” than the time scale of the underlying Stokes flow. The disturbances are then described by time variables t , t_1 , and τ . In terms of these new variables, the x -momentum equation can be written as follows

$$\left\{ \alpha c_i \frac{\partial}{\partial t_1} + (\bar{U} - c_r) \frac{\partial}{\partial x} + 2R^{-1} \frac{\partial}{\partial \tau} \right\} u + v \frac{\partial \bar{U}}{\partial y} - R^{-1} \Delta u = -\frac{\partial p}{\partial x} - \frac{\partial}{\partial x} uu - \frac{\partial}{\partial y} uv - \frac{\partial}{\partial z} uw ,$$

where we have replaced $\frac{\partial}{\partial t}$ by $-c_r \frac{\partial}{\partial x}$ after using (1.2) to the order required, and

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

³For inferring underlying scaling, this time scale is loosely defined at present, but will be precisely defined in each chapter.

We are particularly interested in the case where the nonlinear evolution occurs on the *fastest* possible time scale, i.e. the time-variation term appears at leading order in the critical-layer equations. Suppose that the thickness of the critical layer is $O(\mu)$, then near a critical level, $(\bar{U} - c_r)$ is $O(\mu)$ if $\bar{U}_y(y_c) = O(1)$, where y_c is the critical level defined by $\bar{U}(y_c) = c_r$. Balancing $\alpha c_i \frac{\partial u}{\partial t_1}$ and $(\bar{U} - c_r) \frac{\partial u}{\partial x}$ yields

$$\mu \sim O(c_i) . \quad (1.3)$$

This scaling brings in a new non-equilibrium effect that does not occur in the analysis of Stuart (1960) and Hall & Smith (1982, 1984). As pointed out by Hickernell (1984), it is this effect that leads to an integro-differential type of amplitude equation. Such a non-equilibrium effect was included in the analysis of Goldstein & Leib (1988, 1989), Goldstein & Hultgren (1989), Goldstein & Choi (1989), and Goldstein & Lee (1990), Churilov & Shukhman (1988).

From the asymptotic property of the Rayleigh equation near a critical level, we know that $v \sim O(\epsilon)$ as $y \rightarrow y_c$.

For a regular critical layer of two-dimensional disturbances, i.e. $\bar{U}_{yy}(y_c) = 0$, both v and streamwise velocity u are regular, and in particular $u \sim \epsilon(y - y_c)$ as $y \rightarrow y_c$. For a singular critical layer of 2-D disturbances, i.e. $\bar{U}_{yy}(y_c) \neq 0$, u exhibits a logarithmic singularity, i.e. $u \sim \epsilon \log(y - y_c)$ as $y \rightarrow y_c$. For three-dimensional disturbances, the streamwise velocity u exhibits a pole type of singularity, i.e. $u \sim \epsilon(y - y_c)^{-1}$ as $y \rightarrow y_c$, as noted by Benney (1961). In summary, for all three cases the vorticity of the disturbances can be expressed in the following general form

$$\omega \sim u_y \sim O\left(\frac{\epsilon}{(y - y_c)^k}\right) \sim O\left(\frac{\epsilon}{\mu^k}\right) , \quad (1.4)$$

where the value of k represents the singular property of vorticity, i.e.

$$k = \begin{cases} 0 & \text{for a regular critical layer of a 2-D disturbance, i.e. } \bar{U}_{yy}(y_c) = 0, \\ 1 & \text{for a singular critical layer of a 2-D disturbance, i.e. } \bar{U}_{yy}(y_c) \neq 0, \\ 2 & \text{for a 3-D disturbance.} \end{cases}$$

Following the idea of Hickernell (1984), we now turn to analyze the nonlinear interactions inside the critical layers. The quadratic interaction through vu_y produces a forcing of $O(\frac{\epsilon^2}{\mu^k})$. Balancing this forcing term with $\alpha c_i \frac{\partial u}{\partial t_1}$ in the x -momentum equation, we know that it generates a second harmonic and a mean-flow distortion, say $u^{(2)}$,

$$u^{(2)} \sim O\left(\frac{\epsilon^2}{\mu^{k+1}}\right) .$$

Similarly, the cubic interaction through $vu_y^{(2)}$ produces a forcing of $O(\frac{\epsilon^3}{\mu^{k+2}})$. Balancing this forcing term with $\alpha c_i \frac{\partial u}{\partial t_1}$, we know that it drives a fundamental $u^{(3)}$:

$$u^{(3)} \sim O\left(\frac{\epsilon^3}{\mu^{k+3}}\right) .$$

In order for the evolution of the disturbance to be affected by nonlinear interactions inside the critical layers, this fundamental is required to match with the unsteadiness of $O(\alpha c_i \epsilon) \sim O(\mu \epsilon)$ in the outer region, i.e.

$$O\left(\frac{\epsilon^3}{\mu^{k+3}}\right) \sim O(\epsilon \mu) .$$

This rather delicate balance was first discovered by Hickernell (1984). Its physical interpretation is that the nonlinearly generated streamwise velocity is required to smooth out the $O(\epsilon \mu)$ discontinuity of the outer solution across the critical layer. The above balancing yields

$$\mu \sim O(\epsilon^{\frac{2}{k+4}}) , \quad (1.5)$$

or equivalently

$$l = \frac{2}{k+4} . \quad (1.6)$$

Clearly the scaling depends on the singular property of the critical layers. For regular critical layers, $k = 0$, so $\mu \sim O(\epsilon^{\frac{1}{2}})$. This is the scaling in Goldstein & Leib (1988) and Goldstein & Hultgren (1988). Under this scaling, the critical layers turn out to be strongly nonlinear. But we emphasise that this is only a coincidence because, unlike Benney & Bergeron (1969), we did not deliberately require nonlinear terms to be retained at leading order in critical-layer equations.

For 2-D singular critical layers, $k = 1$ so

$$\mu \sim O(\epsilon^{\frac{2}{3}}) . \quad (1.7)$$

This is the scaling obtained by Hickernell (1984). The same scaling applies to evolution of a single oblique mode in a compressible shear layer (Goldstein & Leib 1989), where the cross-flow effects and the effects associated with the pole type of singularity can be eliminated by use of an appropriate ‘‘Squire transformation’’.

In the problem concerned here, it turns out that along the right branch of the neutral curve (see Fig.1), there exist two critical layers and both are singular. Equation (1.7) thus sets the scaling for the two-dimensional disturbance, which will be studied in chapter two.

For 3-dimensional disturbances, $k = 2$, so

$$\mu \sim O(\epsilon^{\frac{1}{3}}) . \quad (1.8)$$

This scaling was obtained by Goldstein & Choi (1989) when studying three dimensional disturbances in a free shear layer, where $\bar{U}_{yy}(y_c) = 0$. Note that in the present problem, $\bar{U}_{yy}(y_c) \neq 0$. We shall examine the nonlinear evolution of a pair of oblique waves in such a flow in chapter three. It appears that (1.5) unifies the scalings for a variety of cases, and provides certain insight into the critical-layer nonlinearity.

Following a similar analysis as above, we can obtain the scalings for the resonant-triad case and temporal-spatial modulation case. But we prefer to present these

scaling arguments in chapter four and chapter five respectively, which are devoted to the study of these two case.

Finally, we check for the influence of viscosity. A typical case is that viscous diffusion terms appear at leading order in the critical-layer equations. This occurs when

$$R^{-\frac{1}{3}} \sim O(\mu) ,$$

i.e.

$$R \sim O(\epsilon^{-\frac{6}{k+4}}) . \tag{1.9}$$

Chapter 2

Nonlinear Evolution of a Two-Dimensional Wave

Abstract

The Stokes-layer generated by a sinusoidally oscillating flat plate in an infinite fluid is an important prototype of unsteady flows, and is known to be susceptible to high-frequency instability (Tromans, 1977; Cowley, 1987). According to their linear theory, a small disturbance can grow exponentially over part of a period and become neutral at some time. The present study considers nonlinear effects on the instability. We assume that the Reynolds number R is asymptotically large, i.e. $R \gg 1$. It is shown that in the vicinity of, but before the neutral time, the small disturbance evolves from a linear stage to a nonlinear stage due to enhanced nonlinear interactions inside the critical layers, i.e. thin regions centered on levels at which the phase velocity of the disturbance is equal to the basic flow velocity.

For a two-dimensional wave, it is shown that the evolution is controlled by nonlinear effects when the linear growth rate decreases to $O(\epsilon^{\frac{2}{3}})$, where ϵ is the magnitude of the disturbance. The amplitude equation turns out to be an integro-differential one of Hickernell type (Hickernell 1984). The solutions develop a finite-time singularity if viscosity is not too large, and viscosity generally delays the finite-time singularity, in agreement with the findings of Goldstein & Leib (1989). For the Stokes layer of interest in this study, it is found that for disturbances with wavenumbers in the range $[0, 0.6]$ or $[1.3, 1.43]$, the singularity can be completely eliminated if viscosity is sufficiently large; as a result the wave evolves to an equilibrium state. The stability of this equilibria is also studied. However for the disturbances with wavenumbers in the range $[0.6, 1.3]$, a finite-time singularity occurs, no matter how large the scaled viscosity parameter is. We suggest that this nonlinear explosive growth of high-frequency disturbances may be related to the bursts observed in experiments.

2.1 Introduction

As a first step in our theoretical study of nonlinear effects on the stability of Stokes layers, we consider the simplest case where the disturbance is a two-dimensional wave.

As stated in Chapter one, critical-layer nonlinearity exerts its influence on the development of a two-dimensional wave at the times near to:

$$\tau = \tau_0 + \epsilon^{\frac{2}{5}} \tau_1, \quad (2.1)$$

i.e. the time at which the linear growth rate is $O(\epsilon^{\frac{2}{5}})$. It is at this stage that nonlinear effects control the development of the disturbance.

For a two-dimensional disturbance, it is convenient to work with the stream-function equation, which is written as follows:

$$(\bar{U} \frac{\partial}{\partial x} + 2 \frac{1}{R} \frac{\partial}{\partial \tau}) \nabla^2 \psi - \bar{U}_{yy} \frac{\partial \psi}{\partial x} = \frac{1}{R} \nabla^4 \psi + \psi_x \psi_{yyy} - \psi_y \psi_{xyy}, \quad (2.2)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

For convenience, we rewrite the nonlinear evolution time scale as follows

$$t_1 = \frac{1}{2} \epsilon^{\frac{2}{5}} t. \quad (2.3)$$

The evolution of the disturbance is then described by four time variables, namely t , t_1 , τ_1 and τ_0 ; the dependence on the slow time variables τ_1 and τ_0 is parametric to the order of approximation in this study. With the multiple time scales so introduced, the time derivative in equation (2.2) should be interpreted as follows

$$\frac{\partial}{\partial \tau} \rightarrow R \frac{\partial}{\partial t} + \frac{1}{2} R \epsilon^{\frac{2}{5}} \frac{\partial}{\partial t_1} + \epsilon^{-\frac{2}{5}} \frac{\partial}{\partial \tau_1} + \frac{\partial}{\partial \tau_0}. \quad (2.4)$$

Since the basic flow $\bar{U}$ evolves on the slow time variable τ , it is sufficient to expand its profile at time τ in a Taylor series about the neutral time τ_0

$$\bar{U}(y, \tau) = \bar{U}(y, \tau_0) + \epsilon^{\frac{2}{5}} \bar{U}_\tau(y, \tau_0) \tau_1 + \dots$$

Hereafter all basic flow quantities will be evaluated at τ_0 unless otherwise mentioned.

The whole evolution process can be summarized in Fig2. As illustrated, the disturbances are initially well described by linear theory (Tromans, 1977; Cowley, 1987). As the neutral time τ_0 is approached, their growth rates become small, and linear critical layers emerge. Once the growth rates decrease further to $O(\epsilon^{\frac{2}{5}})$, i.e. at times $\tau_1 = O(1)$, nonlinear interactions take place and control the overall development. The earlier linear and the subsequent nonlinear solutions are required to match in an asymptotic sense in the mutual overlapping domain. The analysis is

largely similar to that of Hickernell (1984) because of the the same type of singularity being treated. However one different point is that instead of imposing an initial condition at some specific time, we insist, following the idea of Goldstein & Leib (1989), that the nonlinear stage should match onto the earlier linear one (see also Goldstein, Durbin & Leib, 1987; Churilov & Shukhman, 1988). Note that besides a different flow being considered, the present study also differs from Goldstein & Leib (1989) in two aspects: firstly we consider the temporal evolution, and secondly our critical-layer singularity is associated with the logarithmic branch point in the disturbance velocity rather than the simple pole associated with temperature.

2.2 Outer Expansion

In this and the next sections, we assume that the Reynolds number is large enough so that viscosity does not play a leading-order role in the evolution of the disturbance.

2.2.1 Asymptotic Solutions Near Critical Layers

Outside the critical layers, the stream function of the disturbance is expanded in the following series:

$$\psi = \epsilon\psi_1 + \epsilon^{\frac{6}{5}}\psi_2 + \epsilon^{\frac{7}{5}}\psi_3 + \dots, \quad (2.5)$$

where $\psi_1 = A(t_1, \tau_1, \tau_0)\phi_1(y, \tau_0)E + c.c$, $E = \exp(i\alpha x - i\hat{\theta}(t))$, and

$$\frac{d\hat{\theta}}{dt} = \frac{1}{2}\alpha c(\tau_0)\Omega_0(\tau_0) + \frac{1}{2}R^{-\frac{1}{2}}\Omega_1(\tau_0) + \dots \quad (2.6)$$

Here the dependence on τ_0 , and τ_1 is due to the time variation of the basic flow. They can be treated as parameters to the order of interest in this study, and thus hereafter they will not be written out explicitly.

The function ϕ_1 satisfies the Rayleigh equation:

$$(\bar{U} - c)(D^2 - \alpha^2)\phi_1 - \bar{U}_{yy}\phi_1 = 0. \quad (2.7)$$

The boundary condition is that ϕ_1 vanishes at the wall and at the infinity.

Let $\eta = y - y_c^j$, where y_c^j is the j -th critical level defined by $\bar{U}(y_c^j) = c$; then as $\eta \rightarrow \pm 0$, ϕ_1 has the following asymptotic behaviour:

$$\phi_1 \sim a_j^\pm \phi_a + b_j^\pm [\phi_b + p_j \phi_a \log |\eta|], \quad (2.8)$$

where

$$\phi_a = \eta + \frac{1}{2}p_j\eta^2 + \dots, \quad (2.9)$$

$$\phi_b = 1 + q_j\eta^2 + \dots \quad (2.10)$$

The $O(\epsilon^{\frac{6}{5}})$ term is introduced to take account of the nonlinear interaction inside the critical layer, and has the following form

$$\psi_2 = \phi_2(y, t_1)E^2 + c.c. .$$

The function ϕ_2 satisfies the same equation and hence has the same asymptotic behaviour as ϕ_1 provided α is replaced by 2α . Later, however, ϕ_2 will be found to be identically zero.

The function ψ_3 takes account of distortion of fundamental in the vicinity of the neutral time, and has the following form

$$\psi_3 = \phi_3(y, t_1)E + c.c. \quad (2.11)$$

The function ϕ_3 is the deviation of the eigenfunction from its neutral state, and is found to satisfy the following inhomogeneous Rayleigh equation:

$$[D^2 - (\alpha^2 + \frac{\bar{U}_{yy}}{\bar{U} - c})]\phi_3 = (i\alpha)^{-1} \left\{ \left[-\frac{dA}{dt_1} - (i\alpha\bar{U}_\tau\tau_1)A \right] \frac{\bar{U}_{yy}}{(\bar{U} - c)^2} + \frac{i\alpha\bar{U}_{yy\tau}\tau_1 A}{\bar{U} - c} \right\} \phi_1. \quad (2.12)$$

The asymptotic behaviour of ϕ_3 is as follows:

$$\begin{aligned} \phi_3 \sim & -b_j^\pm r_j \log |\eta| + (a_j^\pm r_j + b_j^\pm s_j) \eta \log |\eta| + \dots \\ & + c_j^\pm \phi_a + d_j^\pm [\phi_b + p_j \phi_a \log |\eta|], \end{aligned} \quad (2.13)$$

where

$$p_j = \frac{\bar{U}_{yy}}{\bar{U}_y}, \quad (2.14)$$

$$q_j = \frac{1}{2}\alpha^2 + \frac{1}{2}\frac{\bar{U}_{yyy}}{\bar{U}_y} - \frac{\bar{U}_{yy}^2}{\bar{U}_y^2}, \quad (2.15)$$

$$r_j = (i\alpha)^{-1} \frac{\bar{U}_{yy}}{\bar{U}_y^2} \left[-\frac{dA}{dt_1} - (i\alpha\bar{U}_\tau\tau_1)A \right], \quad (2.16)$$

$$\begin{aligned} s_j = & (i\alpha)^{-1} \left\{ (-i\alpha\bar{U}_{y\tau}\tau_1 A) \frac{\bar{U}_{yy}}{\bar{U}_y^2} + (i\alpha\tau_1 A) \frac{\bar{U}_{yy\tau}}{\bar{U}_y} \right. \\ & \left. + \left(-\frac{dA}{dt_1} - i\alpha\bar{U}_\tau\tau_1 A \right) \frac{\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y^3} \right\}, \end{aligned} \quad (2.17)$$

where all basic flow quantities are evaluated at time τ_0 and at the critical layer y_c^j . The jumps $(a_j^+ - a_j^-)$ etc. will be determined by analyzing the critical layers.

We now introduce an inner variable:

$$Y = \frac{\eta}{\epsilon^{\frac{2}{5}}} = \frac{y - y_c^j}{\epsilon^{\frac{2}{5}}}. \quad (2.18)$$

The outer expansion written in terms of the inner variable is as follows:

$$\begin{aligned} \psi \sim & \epsilon b_j^\pm A E + \epsilon^{\frac{7}{5}} \log \epsilon^{\frac{2}{5}} (-b_j^\pm r_j + b_j^\pm p_j A Y) E \\ & + \epsilon^{\frac{7}{5}} [(-b_j^\pm r_j \log |Y| + d_j^\pm) + A(a_j^\pm Y + b_j^\pm Y \log |Y|)] E \\ & + \epsilon^{\frac{9}{5}} \log \epsilon^{\frac{2}{5}} [(a_j^\pm r_j + b_j^\pm s_j + d_j^\pm p_j) Y + \frac{1}{2} A p_j b_j^\pm Y^2] E \\ & + \epsilon^{\frac{9}{5}} [c_j^\pm Y + (a_j^\pm r_j + b_j^\pm s_j + d_j^\pm p_j) Y \log |Y| + \dots] E \\ & + c.c. + \dots \end{aligned} \quad (2.19)$$

2.2.2 Solvability Condition

Multiplying both sides of (2.12) by ϕ_1 and integrating from 0 to $+\infty$, we obtain the solvability condition for (2.12):

$$-\sum_j FP(\Delta_j) = i\alpha^{-1} J_1 \frac{dA}{dt_1} + J_2 \tau_1 A, \quad (2.20)$$

where the summation is made over all critical layers, and FP denote the finite part of Δ_j :

$$\Delta_j = (\phi_{3y}\phi_1 - \phi_3\phi_{1y})|_{y \rightarrow y_c^+} - (\phi_{3y}\phi_1 - \phi_3\phi_{1y})|_{y \rightarrow y_c^-}. \quad (2.21)$$

The constants J_1 and J_2 are defined by the following integrals

$$J_1 = \int_0^{+\infty} \frac{\bar{U}_{yy}}{(\bar{U} - c)^2} \phi_1^2 dy, \quad (2.22)$$

$$J_2 = \int_0^{+\infty} \left[-\frac{\bar{U}_{yy}\bar{U}_\tau}{(\bar{U} - c)^2} + \frac{\bar{U}_{yy\tau}}{(\bar{U} - c)} \right] \phi_1^2 dy. \quad (2.23)$$

Note that J_1 and J_2 are singular and should be interpreted in the sense of Hadamard. By using the asymptotic expressions (2.8), (2.13) together with (2.9) and (2.10), it is found

$$\begin{aligned} FP(\Delta_j) = & (b_j^+ c_j^+ - b_j^- c_j^-) - r_j(b_j^+ a_j^+ - b_j^{-1} a_j^-) - p_j(b_j^+ d_j^+ - b_j^- d_j^-) \\ & - d_j^+(a_j^+ - a_j^-) - a_j^+(d_j^+ - d_j^-) + (a_j^+ - a_j^-)(d_j^+ - d_j^{-1}). \end{aligned} \quad (2.24)$$

We shall obtain the amplitude equation from (2.20) and (2.24) after the jumps $(a_j^+ - a_j^-)$ etc. are determined by analysing the critical layers in the next section. We emphasise that nonlinearity influences the evolution of disturbance by altering these jumps; thus in the analysis of the critical layers, it is sufficient to concentrate on these parts of the solution contributing to these jumps.

2.3 Inner Expansion

2.3.1 Inner Solutions and Jumps

The expression (2.19) suggests that the inner expansion within the j -th critical layer takes the following form:

$$\begin{aligned} \Psi = & \epsilon \Psi_1 + \epsilon^{\frac{6}{5}} \Psi_2 + \epsilon^{\frac{7}{5}} \log \epsilon^{\frac{2}{5}} \Psi_{1,l} + \epsilon^{\frac{7}{5}} \Psi_3 + \epsilon^{\frac{8}{5}} \log \epsilon^{\frac{2}{5}} \Psi_{2,l} \\ & + \epsilon^{\frac{8}{5}} \Psi_4 + \epsilon^{\frac{9}{5}} \log \epsilon^{\frac{2}{5}} \Psi_{3,l} + \epsilon^{\frac{9}{5}} \Psi_5 + \dots \end{aligned} \quad (2.25)$$

The function Ψ_1 satisfies the equation:

$$L_0 \Psi_1 = 0, \quad (2.26)$$

where

$$L_0 = \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x} \right] \frac{\partial^2}{\partial Y^2} . \quad (2.27)$$

The solution which matches the outer expansion is:

$$\Psi_1 = \hat{A}(t_1)E + c.c. , \quad (2.28)$$

where $\hat{A} = b_j A$ and $b_j^+ = b_j^- = b_j$, i.e. the jump $(b_j^+ - b_j^-)$ is zero.

The functions Ψ_2 and $\Psi_{1,l}$ both satisfy the same equation as Ψ_1 . The solutions that match the outer expansion are:

$$\Psi_2 = \hat{B}(t_1) + \hat{C}(t_1)E^2 + c.c. ,$$

$$\Psi_{1,l} = b_j(-r_j + Ap_j Y)E + c.c. .$$

The function Ψ_3 satisfies:

$$L_0 \Psi_3 = L_1 \Psi_1 , \quad (2.29)$$

where

$$L_1 = -\left(\frac{1}{2} \bar{U}_{yy} Y^2 + \bar{U}_{y\tau} \tau_1 Y + \frac{1}{2} \bar{U}_{\tau\tau} \tau_1^2 \right) \frac{\partial^3}{\partial x \partial Y^2} + \bar{U}_{yy} \frac{\partial}{\partial x} . \quad (2.30)$$

Let

$$\Psi_3 = \Phi_3(Y, t_1)E + c.c. ,$$

then Φ_3 satisfies the following equation:

$$\bar{L}_0 \Phi_3 = i\alpha \hat{A}(t_1) \bar{U}_{yy} ,$$

where

$$\bar{L}_0 = \left[\frac{\partial}{\partial t_1} + i\alpha (\bar{U}_y Y + \bar{U}_\tau \tau_1) \right] \frac{\partial^2}{\partial Y^2} .$$

It can be shown that:

$$\Phi_{3Y} = i\alpha \bar{U}_{yy} b_j F_0(Y, t_1) , \quad (2.31)$$

$$\Phi_{3Y} = \Phi_{3Y}(0, t_1) + p_j b_j F_1(Y, t_1) , \quad (2.32)$$

$$\Phi_3 = \Phi_3(0, t_1) + \Phi_{3Y}(0, t_1)Y + (i\alpha)^{-1} \frac{\bar{U}_{yy}}{\bar{U}_y^2} b_j F_2(Y, t_1) , \quad (2.33)$$

where

$$F_0(Y, t_1) = \int_{-\infty}^{t_1} e^{i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\xi - t_1)} A(\xi) d\xi , \quad (2.34)$$

$$F_1(Y, t_1) = \int_{-\infty}^{t_1} \frac{e^{i\alpha \bar{U}_y Y(\xi - t_1)} - 1}{\xi - t_1} A(\xi) e^{i\alpha \bar{U}_\tau \tau_1(\xi - t_1)} d\xi , \quad (2.35)$$

$$F_2(Y, t_1) = \int_{-\infty}^{t_1} \frac{e^{i\alpha \bar{U}_y Y(\xi - t_1)} - i\alpha \bar{U}_y Y(\xi - t_1) - 1}{(\xi - t_1)^2} A(\xi) e^{i\alpha \bar{U}_\tau \tau_1(\xi - t_1)} d\xi . \quad (2.36)$$

Using a Mellin transformation technique (Bleistein et al., 1990), we find that as $Y \rightarrow \pm\infty$, Φ_{3Y} and Φ_3 have the following asymptotic form

$$\Phi_{3Y} \sim \pm \frac{\pi}{2} i p_j \text{Sgn}(\bar{U}_y) \hat{A} + [\Phi_{3Y}(0, t_1) + p_j(\gamma + \log |\alpha \bar{U}_y Y|) \hat{A}] + O(Y^{-1}) , \quad (2.37)$$

$$\begin{aligned} \Phi_3 \sim & \{ \pm \frac{\pi}{2} i \text{Sgn}(\bar{U}_y) p_j \hat{A} + [\Phi_{3Y}(0, t_1) + p_j(\gamma + \log |\alpha \bar{U}_y Y|) \hat{A}] \} Y \\ & \mp \frac{\pi}{2} i r_j b_j \text{Sgn}(\bar{U}_y) - r_j b_j \text{Sgn}(\bar{U}_y) (\gamma + 1 + \log |\alpha \bar{U}_y Y|) + \Phi_3(0, t_1) + O(Y^{-2}) , \end{aligned} \quad (2.38)$$

where γ is the Euler constant. Matching the asymptotes of Φ_{3Y} and Φ_3 with the their outer solutions respectively gives the following jumps:

$$a_j^+ - a_j^- = \pi i p_j b_j \text{Sgn}(\bar{U}_y) , \quad (2.39)$$

$$d_j^+ - d_j^- = -\pi i r_j b_j \text{Sgn}(\bar{U}_y) . \quad (2.40)$$

In this procedure, $\Phi_3(0, t_1)$ and $\Phi_{3Y}(0, t_1)$ are determined as follows:

$$\Phi_3(0, t_1) = d_j^+ + r_j b_j \text{Sgn}(\bar{U}_y) \left(\frac{1}{2} \pi i + \gamma + 1 + \log |\alpha \bar{U}_y| \right) , \quad (2.41)$$

$$\Phi_{3Y}(0, t_1) = Q_0 A , \quad (2.42)$$

where

$$Q_0 = a_j^+ - p_j b_j \left[\frac{\pi}{2} i \text{Sgn}(\bar{U}_y) + \gamma - 1 + \log |\alpha \bar{U}_y| \right] . \quad (2.43)$$

The function Ψ_4 satisfies the following equation:

$$L_0 \Psi_4 = L_1 \Psi_2 + \Psi_{1,x} \Psi_{3,YYY} + \dots . \quad (2.44)$$

The first non-homogeneous term in (2.44) is analogous to that in (2.29) and thus it produces an analogous velocity jump. The solution driven by the second term $(\Psi_{1,x} \Psi_{3,YYY})$, i.e. by the nonlinear interaction of the fundamental, produces no velocity jump. Because $(2\alpha, c)$ is not an eigen-mode of Rayleigh's equation (Cowley, 1987), ψ_2 , and hence Ψ_2 , should be identically zero.

The solution of (2.44) has the following form:

$$\Psi_4 = \Phi_4^{(0)} + \Phi_4^{(2)} E^2 + c.c. , \quad (2.45)$$

where the mean-flow distortion, $\Phi_4^{(0)}$, and the second harmonic, $\Phi_4^{(2)}$, satisfy the following equations respectively

$$\frac{\partial}{\partial t_1} \Phi_4^{(0)}{}_{YY} = -i \alpha^3 \bar{U}_{yy} \bar{U}_y \hat{A}^*(t_1) \int_0^\infty e^{-i \alpha (\bar{U}_y Y + \bar{U}_\tau \tau_1) \xi} \hat{A}(t_1 - \xi) d\xi , \quad (2.46)$$

$$\left[\frac{\partial}{\partial t_1} + 2i \alpha (\bar{U}_y Y + \bar{U}_\tau \tau_1) \right] \Phi_4^{(2)}{}_{YY} = i \alpha^3 \bar{U}_{yy} \bar{U}_y \hat{A}(t_1) \int_0^\infty e^{-i \alpha (\bar{U}_y Y + \bar{U}_\tau \tau_1) \xi} \hat{A}(t_1 - \xi) d\xi . \quad (2.47)$$

We find that

$$\Phi_4^{(0)}{}_{YY} = -i\alpha^3 \bar{U}_{yy} \bar{U}_y \int_0^\infty \int_0^\infty e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \xi - \eta) d\xi d\eta, \quad (2.48)$$

$$\Phi_4^{(0)}{}_{YYY} = -\alpha^4 \bar{U}_{yy} \bar{U}_y^2 \int_0^\infty \int_0^\infty e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} \xi^2 \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \xi - \eta) d\xi d\eta, \quad (2.49)$$

$$\Phi_4^{(2)}{}_{YY} = i\alpha^3 \bar{U}_{yy} \bar{U}_y \int_0^\infty \int_0^\infty e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\xi+2\eta)} \xi \hat{A}(t_1 - \eta) \hat{A}(t_1 - \xi - \eta) d\xi d\eta, \quad (2.50)$$

$$\Phi_4^{(2)}{}_{YYY} = \alpha^4 \bar{U}_{yy} \bar{U}_y^2 \int_0^\infty \int_0^\infty e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\xi+2\eta)} \xi(\xi+2\eta) \hat{A}(t_1 - \eta) \hat{A}(t_1 - \xi - \eta) d\xi d\eta. \quad (2.51)$$

The function Ψ_5 satisfies the following equation:

$$L_0 \Psi_5 = L_1 \Psi_3 + L_2 \Psi_1 + \Psi_{1,x} \Psi_{4,YYY} + \dots, \quad (2.52)$$

where

$$\begin{aligned} L_2 = & -\left[\frac{1}{6} \bar{U}_{yyy} Y^3 + \bar{U}_{yy\tau} \tau_1 Y^2 + \bar{U}_{y\tau\tau} \tau_1^2 Y + \frac{1}{6} \bar{U}_{\tau\tau\tau} \tau_1^3\right] \frac{\partial^3}{\partial x \partial Y^2} \\ & + [\bar{U}_{yyy} Y + \bar{U}_{yy\tau} \tau_1] \frac{\partial}{\partial x} \\ & - \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x}\right] \frac{\partial^2}{\partial x^2}. \end{aligned} \quad (2.53)$$

We shall first seek the solution induced by the linear forcing terms, i.e. the first and the second terms in (2.52). By using (2.28), (2.30), (2.33) and (2.53), these two forcing terms can be expressed in the form of $F^{(l)}(Y, t_1)E$, where

$$\begin{aligned} F^{(l)}(Y, t_1) = & \alpha^2 \left[\frac{\partial \hat{A}}{\partial t_1} + i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1) \hat{A} \right] + i\alpha(\bar{U}_{yyy} Y + \bar{U}_{yy\tau} \tau_1) \hat{A} \\ & + \alpha^2 \bar{U}_{yy} b_j \left(\frac{1}{2} \bar{U}_{yy} Y^2 + \bar{U}_{y\tau} \tau_1 Y + \frac{1}{2} \bar{U}_{\tau\tau} \tau_1^2 \right) F_0(Y, t_1) \\ & + i\alpha \bar{U}_{yy} [\Phi_3(0, t_1) + \Phi_{3Y}(0, t_1) Y + (i\alpha)^{-1} \frac{\bar{U}_{yy}}{\bar{U}_y^2} b_j F_2(Y, t_1)]. \end{aligned} \quad (2.54)$$

Let $\Phi_{5Y}^{(l)} E$ denote the solution driven by $F^{(l)}(Y, t_1)E$; then it is found that

$$\begin{aligned} \Phi_{5Y}^{(l)} = & p_j^2 b_j G_2(Y, t_1) + \alpha^2 \bar{U}_{yy} b_j \left(\frac{1}{2} \bar{U}_{yy} Y^2 + \bar{U}_{y\tau} \tau_1 Y + \frac{1}{2} \bar{U}_{\tau\tau} \tau_1^2 \right) G_1(Y, t_1) \\ & + i\alpha [(\bar{U}_{yyy} Y + \bar{U}_{yy\tau} \tau_1) b_j + \bar{U}_{yy} Q_0 Y] F_0(Y, t_1) \\ & + \alpha^2 b_j A(t_1) + i\alpha \bar{U}_{yy} \int_{-\infty}^{t_1} \Phi_3(0, \xi) e^{i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\xi - t_1)} d\xi, \end{aligned} \quad (2.55)$$

where

$$\begin{aligned} G_1(Y, t_1) &= \int_{-\infty}^{t_1} F_0(Y, \xi) e^{i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\xi - t_1)} d\xi \\ &= \int_0^{+\infty} \xi A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} d\xi, \end{aligned} \quad (2.56)$$

$$\begin{aligned} G_2(Y, t_1) &= \int_{-\infty}^{t_1} F_2(Y, \xi) e^{i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\xi - t_1)} d\xi \\ &= \int_0^{+\infty} \frac{e^{i\alpha\bar{U}_y Y \xi} - i\alpha\bar{U}_y Y \xi - 1}{\xi} A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} d\xi. \end{aligned} \quad (2.57)$$

Integrating with respect to Y , we have:

$$\begin{aligned} \Phi_{5Y}^{(l)} &= (i\alpha\bar{U}_y)^{-1} p_j^2 b_j H_2(Y, t_1) + \frac{1}{2} \frac{i}{\alpha\bar{U}_y} p_j^2 b_j H_1(Y, t_1) - \frac{\bar{U}_{y\tau}}{\bar{U}_y} \tau_1 p_j b_j G_2(Y, t_1) \\ &\quad + i\alpha(\bar{U}_{yy} b_j + \bar{U}_{yy} Q_0) \frac{1}{(\alpha\bar{U}_y)^2} H_0(Y, t_1) + \frac{\bar{U}_{yy\tau}}{\bar{U}_y} \tau_1 b_j F_1(Y, t_1) \\ &\quad - \frac{1}{2} i\alpha\bar{U}_{\tau\tau} \tau_1^2 p_j b_j \int_0^{+\infty} (e^{i\alpha\bar{U}_y Y \xi} - 1) A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} d\xi \\ &\quad - \alpha^2 b_j A(t_1) Y + p_j \int_0^{+\infty} \frac{e^{i\alpha\bar{U}_y Y \xi} - 1}{\xi} \Phi_3(0, t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} d\xi \\ &\quad + \Phi_{5Y}(0, t_1), \end{aligned} \quad (2.58)$$

where

$$H_0(Y, t_1) = \int_0^{+\infty} \frac{1 + \alpha\bar{U}_y Y \xi - e^{i\alpha\bar{U}_y Y \xi}}{\xi^2} A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} d\xi, \quad (2.59)$$

$$H_1(Y, t_1) = \int_0^{+\infty} \frac{(\alpha\bar{U}_y Y \xi)^2 - 2i\alpha\bar{U}_y Y \xi - 2 + 2e^{i\alpha\bar{U}_y Y \xi}}{\xi^2} A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} d\xi, \quad (2.60)$$

$$H_2(Y, t_1) = \int_0^{+\infty} \frac{i\alpha\bar{U}_y Y \xi e^{i\alpha\bar{U}_y Y \xi} - 2e^{i\alpha\bar{U}_y Y \xi} + i\alpha\bar{U}_y Y \xi + 2}{\xi^2} A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} d\xi. \quad (2.61)$$

The asymptotic behaviour of $\Phi_{5Y}^{(l)}$ again can be obtained by a Mellin-transformation technique. After an extensive calculation we have:

$$\begin{aligned} \Phi_{5Y}^{(l)} &\sim (a_j^+ p_j + 2q_j b_j + \frac{1}{2} p_j^2 b_j) Y + (a_j^+ r_j + p_j d_j^+ + s_j b_j) \log |Y| \\ &\quad + \{\pm \frac{1}{2} \pi i \text{Sgn}(\bar{U}_y) (a_j^+ r_j + p_j d_j^+ + s_j b_j) + \dots\} + O(Y^{-1}). \end{aligned} \quad (2.62)$$

We now consider the solution of (2.52) induced by the nonlinear forcing term $(\Psi_{1,x}\Psi_{4,YYY})$, which, after some substitution, can be written as follows:

$$F^{(n)}(Y, t_1) = i\alpha(\hat{A}\Phi_{4YYY}^{(0)} + \hat{A}\Phi_{4YYY}^{*(0)} - \hat{A}^*\Phi_{4YYY}^{(2)})E + \dots + c.c, \quad (2.63)$$

where only the terms proportional to E need to be written out, since we are only interested in the fundamental wave. The solution is:

$$\Phi_{5Y}^{(n)} = -i\alpha^5 \bar{U}_{yy} \bar{U}_y^2 (\Pi_1 + \Pi_2 + \Pi_3), \quad (2.64)$$

where

$$\Pi_1 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^2 \hat{A}(t_1 - \zeta) \hat{A}^*(t_1 - \zeta - \rho) \hat{A}(t_1 - \xi - \zeta - \rho) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\xi + \zeta)} d\xi d\zeta d\rho, \quad (2.65)$$

$$\Pi_2 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} (\xi + 2\rho) \xi \hat{A}^*(t_1 - \zeta) \hat{A}(t_1 - \zeta - \rho) \hat{A}(t_1 - \xi - \zeta - \rho) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\xi + \zeta + 2\rho)} d\xi d\zeta d\rho, \quad (2.66)$$

$$\Pi_3 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^2 \hat{A}(t_1 - \zeta) \hat{A}(t_1 - \zeta - \rho) \hat{A}^*(t_1 - \xi - \zeta - \rho) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\zeta - \xi)} d\xi d\zeta d\rho. \quad (2.67)$$

Integrating (2.64) with respect to Y , we obtain:

$$\Phi_{5Y}^{(n)}(Y, t_1) = -i\alpha^5 \bar{U}_{yy} \bar{U}_y^2 \int_0^Y (\Pi_1 + \Pi_2 - \Pi_3) dY + \Phi_{5Y}^{(n)}(0, t_1).$$

As $Y \rightarrow \pm\infty$, $\Phi_{5Y}^{(n)}(Y, t_1)$ has the following asymptotic behaviour:

$$\Phi_{5Y}^{(n)}(Y, t_1) \sim \mp \pi i \alpha^4 \bar{U}_{yy} |\bar{U}_y| \int_0^{+\infty} \int_0^{+\infty} \xi^2 \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \zeta) \hat{A}^*(t_1 - 2\xi - \zeta) d\xi d\zeta + \dots \quad (2.68)$$

Matching $(\Phi_{5Y}^{(l)} + \Phi_{5Y}^{(n)})$ with outer expansion gives the following jump:

$$\begin{aligned} c_j^+ - c_j^- &= \pi i \text{Sgn}(\bar{U}_y) (a_j^+ r_j + p_j d_j^+ + s_j b_j) \\ &\quad - 2\pi i \alpha^4 \bar{U}_{yy} |\bar{U}_y| \int_0^{+\infty} \int_0^{+\infty} \xi^2 \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \zeta) \hat{A}^*(t_1 - 2\xi - \zeta) d\xi d\zeta. \end{aligned} \quad (2.69)$$

It is noted that $(a_j^+ - a_j^-)$ and $(d_j^+ - d_j^-)$ correspond to the classic $\pm\pi$ phase shift at the logarithmic branch point, while $(c_j^+ - c_j^-)$ is modified by nonlinearity, which as will be shown, has a significant effect on the evolution of the disturbance.

2.3.2 Evolution Equation of the Amplitude

Inserting the jumps (2.39), (2.40) and (2.69) into (2.24), and using (2.20), we obtain the crucial evolution equation as follows:

$$\frac{dA}{dt_1} = g_0 \tau_1 A + \alpha^4 g \int_0^{+\infty} \int_0^{+\infty} \xi^2 A(t_1 - \xi) A(t_1 - \xi - \zeta) A^*(t_1 - 2\xi - \zeta) d\xi d\zeta, \quad (2.70)$$

where

$$g_0 = f_0/f, \quad (2.71)$$

$$g = f_1/f, \quad (2.72)$$

$$f = i\alpha^{-1} \left\{ \sum_j \pi b_j \left[2i \frac{\bar{U}_{yy}}{|\bar{U}_y|^2} a_j^+ + ib_j \frac{\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y |\bar{U}_y|^2} + b_j \pi \frac{\bar{U}_{yy}^2}{\bar{U}_y^3} \right] + J_1 \right\}, \quad (2.73)$$

$$f_0 = \sum_j \left\{ 2\pi i b_j \frac{\bar{U}_{yy} \bar{U}_\tau}{|\bar{U}_y|^2} a_j^+ - \pi i b_j^2 \left[\frac{\bar{U}_{yy\tau}}{|\bar{U}_y|} - \frac{\bar{U}_{yy} \bar{U}_{y\tau}}{|\bar{U}_y|^2} - \frac{(\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2) \bar{U}_\tau}{\bar{U}_y |\bar{U}_y|^2} \right] \right. \\ \left. + \pi^2 b_j^2 \frac{\bar{U}_{yy} \bar{U}_\tau}{\bar{U}_y^3} \right\} - J_2, \quad (2.74)$$

$$f_1 = 2\pi i \sum_j (b_j^2 |b_j|^2 \bar{U}_{yy} |\bar{U}_y|). \quad (2.75)$$

In order to match its earlier linear evolution stage, A has the following asymptotic behaviour (see e.g. Goldstein & Leib, 1989):

$$A \rightarrow A_0 e^{g_0 \tau_1 t_1} \quad \text{as} \quad t_1 \rightarrow -\infty. \quad (2.76)$$

Equations (2.70) and (2.76) describe the evolution of disturbances.

Following the idea of Goldstein & Leib (1989), we scale out A_0 and τ_1 by introducing the rescaled variables:

$$\bar{t} = -\tau_1 t_1 - x_0, \quad (2.77)$$

$$\bar{A}(\bar{t}) = A(t_1) e^{i\bar{X}_0} / (-\tau_1)^{\frac{5}{2}}, \quad (2.78)$$

where $\bar{X}_0$ and x_0 are chosen so that:

$$g_0 x_0 - i(\bar{X}_0 - \varphi_0) = \log[A_0 / (-\tau_1)^{\frac{5}{2}}]. \quad (2.79)$$

It follows from (2.77) and (2.78) that

$$\bar{A}(\bar{t}) = A \left(\frac{\bar{t} + x_0}{-\tau_1} \right) e^{i\bar{X}_0} / (-\tau_1)^{\frac{5}{2}}.$$

Noticing this relation, we found that the evolution equation and the asymptotic behaviour then become:

$$\frac{d\bar{A}}{d\bar{t}} = -g_0 \bar{A} + \alpha^4 g \int_0^{+\infty} \int_0^{+\infty} \xi^2 \bar{A}(\bar{t} - \xi) \bar{A}(\bar{t} - \xi - \zeta) \bar{A}^*(\bar{t} - 2\xi - \zeta) d\xi d\zeta, \quad (2.80)$$

$$\bar{A} \rightarrow e^{-g_0 \bar{t} + i\varphi_0} \quad \text{as} \quad \bar{t} \rightarrow -\infty. \quad (2.81)$$

The evaluation of the coefficients g_0 and g will be discussed in §2.5.

Furthermore to scale out φ_0 , we introduce:

$$\tilde{A} = \alpha^2 \bar{A} |g|^{\frac{1}{2}} |g_0|^{-\frac{5}{2}} , \quad (2.82)$$

$$\tilde{t} = |g_0| \bar{t} - \tilde{X}_0 , \quad (2.83)$$

where φ_0 and $\tilde{X}_0$ satisfy the following equation:

$$-\frac{g_0}{|g_0|} \tilde{X}_0 + i\varphi_0 = \log\{\alpha^2 |g|^{\frac{1}{2}} |g_0|^{-\frac{5}{2}}\} . \quad (2.84)$$

Then we obtain:

$$\frac{d\tilde{A}}{d\tilde{t}} = \nu_0 \tilde{A} + \nu \int_0^{+\infty} \int_0^{+\infty} \xi^2 \tilde{A}(\tilde{t}-\xi) \tilde{A}(\tilde{t}-\xi-\zeta) \tilde{A}^*(\tilde{t}-2\xi-\zeta) d\xi d\zeta , \quad (2.85)$$

$$\tilde{A} \rightarrow e^{\nu_0 \tilde{t}} \quad \text{as} \quad \tilde{t} \rightarrow -\infty , \quad (2.86)$$

where $\nu_0 = -g_0/|g_0|$, and $\nu = g/|g|$.

2.4 Viscous Effects

In this section, the effect of viscosity on the evolution will be examined. We shall consider a typical case where the viscous effects are as important as time-variation effects in the critical layers. This happens when

$$R^{-1} = \lambda \epsilon^{\frac{6}{5}} , \quad (2.87)$$

where λ is introduced to reflect the viscosity effect, and is assumed to be $O(1)$.

The equation (2.6) now can be written as follows:

$$\frac{d\hat{\theta}}{dt} = \frac{1}{2} \alpha c(\tau_0) + \frac{1}{2} \lambda^{\frac{1}{2}} \epsilon^{\frac{3}{5}} \Omega_1 + \frac{1}{2} \lambda \epsilon^{\frac{6}{5}} \Omega_2 + \dots , \quad (2.88)$$

where the corrections to the complex frequency at $O(\epsilon^{\frac{3}{5}})$ and $O(\epsilon^{\frac{6}{5}})$, i.e. $O(R^{-\frac{1}{2}})$ and $O(R^{-1})$ respectively, are due to the viscous effects. The outer expansion is modified correspondingly as follows

$$\psi = \epsilon \psi_1 + \epsilon^{\frac{6}{5}} \psi_2 + \epsilon^{\frac{7}{5}} \psi_3 + \epsilon^{\frac{8}{5}} \psi_4 \dots ,$$

where the introduction of $\psi_4 = A(t_1) \phi_4 E + c.c.$ at $O(\epsilon^{\frac{8}{5}})$ is to take account of the viscous correction. The functions ψ_1 , ψ_2 and ψ_3 satisfy the same equations respectively, and hence have the same solutions as in the inviscid case. The function ϕ_4 satisfies the following equation:

$$[(\bar{U}-c)(D^2-\alpha^2)-\bar{U}_{yy}]\phi_4 = \lambda^{\frac{1}{2}} \Omega_1 \frac{\bar{U}_{yy}}{(\bar{U}-c_0)} \phi_1 .$$

When written in the inner variable, we find that as $y \rightarrow y_c^j$,

$$\epsilon^{\frac{8}{5}} \phi_4 \sim \epsilon^{\frac{8}{5}} \log \epsilon^{\frac{2}{5}} [-\lambda^{\frac{1}{2}} \Omega_1 i \alpha^{-1} b_j^{\pm} \bar{r}_j] + \epsilon^{\frac{8}{5}} [f_j^{\pm} - \lambda^{\frac{1}{2}} \Omega_1 i \alpha^{-1} b_j^{\pm} \bar{r}_j \log Y] + \dots ,$$

where $\bar{r}_j = \frac{\bar{U}_{yy}}{\bar{U}_y^2}$.

The inner expansion still takes the form of (2.25) and we repeat it here:

$$\begin{aligned} \Psi = & \epsilon \Psi_1 + \epsilon^{\frac{6}{5}} \Psi_2 + \epsilon^{\frac{7}{5}} \log \epsilon^{\frac{2}{5}} \Psi_{1,l} + \epsilon^{\frac{7}{5}} \Psi_3 + \epsilon^{\frac{8}{5}} \log \epsilon^{\frac{2}{5}} \Psi_{2,l} \\ & + \epsilon^{\frac{8}{5}} \Psi_4 + \epsilon^{\frac{9}{5}} \log \epsilon^{\frac{2}{5}} \Psi_{3,l} + \epsilon^{\frac{9}{5}} \Psi_5 + \dots . \end{aligned} \quad (2.89)$$

The function Ψ_1 satisfies the equation:

$$\tilde{L}_0 \Psi_1 = 0 , \quad (2.90)$$

where

$$\tilde{L}_0 = \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x} \right] \frac{\partial^2}{\partial Y^2} - \lambda \frac{\partial^4}{\partial Y^4} . \quad (2.91)$$

Note that due to viscous diffusion, the critical-layer equation becomes fourth order. The solution which matches the outer expansion is:

$$\Psi_1 = \hat{A}(t_1) E + c.c. , \quad (2.92)$$

where $\hat{A} = b_j A$ and $b_j^+ = b_j^- = b_j$.

The functions Ψ_2 and $\Psi_{1,l}$ satisfy the following equations respectively:

$$\tilde{L}_0 \Psi_2 = -\lambda^{\frac{1}{2}} \Omega_1 \frac{\partial^2 \Psi_1}{\partial Y^2} + \Psi_{1,x} \Psi_{1,YYY} - \Psi_{1,Y} \Psi_{1,xYY} ,$$

$$\tilde{L}_0 \Psi_{1,l} = 0 .$$

The solutions that match the outer expansion are:

$$\Psi_2 = \hat{B}(t_1) + \hat{C}(t_1) E^2 + c.c. .$$

$$\Psi_{1,l} = b_j (-r_j + A p_j Y) E + c.c. .$$

The function Ψ_3 satisfies:

$$\tilde{L}_0 \Psi_3 = \tilde{L}_1 \Psi_1 , \quad (2.93)$$

where

$$\tilde{L}_1 = -\left(\frac{1}{2} \bar{U}_{yy} Y^2 + \bar{U}_{y\tau} \tau_1 Y + \frac{1}{2} \bar{U}_{\tau\tau} \tau_1^2 + 2\lambda t_1 \bar{U}_\tau \right) \frac{\partial^3}{\partial x \partial Y^2} + \bar{U}_{yy} \frac{\partial}{\partial x} . \quad (2.94)$$

Let

$$\Psi_3 = \Phi_3(Y, t_1) E + c.c. ,$$

and then Φ_3 satisfies the following equation:

$$\bar{L}_0 \Phi_3 = i\alpha \hat{A}(t_1) \bar{U}_{yy} , \quad (2.95)$$

where

$$\bar{L}_0 = \left[\frac{\partial}{\partial t_1} + i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1) \right] \frac{\partial^2}{\partial Y^2} - \lambda \frac{\partial^4}{\partial Y^4} .$$

Let

$$\begin{aligned} \Gamma_3 &= \frac{\partial^2 \Phi_3}{\partial Y^2} , \\ \bar{Y} &= Y + \frac{\bar{U}_\tau}{\bar{U}_y} \tau_1 , \end{aligned}$$

then (2.95) can be written as follows

$$\left[\frac{\partial}{\partial t_1} + i\alpha \bar{U}_y \bar{Y} - \lambda \frac{\partial^2}{\partial \bar{Y}^2} \right] \Gamma_3 = i\alpha \bar{U}_{yy} \hat{A}(t_1) . \quad (2.96)$$

Following Hickernell (1984), we take a Fourier transform to equation (2.96) with respect to $\bar{Y}$, and obtain:

$$\bar{\Gamma}_{3t_1} - \alpha \bar{U}_y \bar{\Gamma}_{3k} + \lambda k^2 \bar{\Gamma}_3 = 2\pi i \alpha \bar{U}_{yy} \hat{A}(t_1) \delta(k) , \quad (2.97)$$

where $\bar{\Gamma}_3$ is the Fourier transformation of Γ_3 defined by

$$\bar{\Gamma}_3 = \int_{-\infty}^{+\infty} \Gamma_3 e^{ik\bar{Y}} d\bar{Y} .$$

On writing $T = t_1 + \frac{k}{\alpha \bar{U}_y}$ and regarding Γ_3 as a function of T and k , we have:

$$\bar{\Gamma}_{3k} - \frac{\lambda}{\alpha \bar{U}_y} k^2 \bar{\Gamma}_3 = -2\pi i p_j \hat{A}\left(T - \frac{k}{\alpha \bar{U}_y}\right) \delta(k) , \quad (2.98)$$

where $\delta(k)$ is a Dirac delta function. The solution is as follows:

$$\bar{\Gamma}_3 = 2\pi i \frac{\bar{U}_{yy}}{|\bar{U}_y|} e^{\frac{1}{3}\lambda \frac{k^3}{\alpha \bar{U}_y}} \hat{A}\left(t_1 + \frac{k}{\alpha \bar{U}_y}\right) H(-\bar{U}_y k) , \quad (2.99)$$

where $H(-\bar{U}_y k)$ is a Heaviside step function. The inverse Fourier transformation of $\bar{\Gamma}_3$ yields

$$\Gamma_3 = i\alpha \bar{U}_{yy} b_j \tilde{F}_0(Y, t_1) , \quad (2.100)$$

where

$$\tilde{F}_0(Y, t_1) = \int_0^{+\infty} A(t_1 - \xi) e^{-s_1 \xi^3 - i\alpha \bar{U}_y \bar{Y} \xi} d\xi . \quad (2.101)$$

Note that we have put

$$s_1 = \frac{1}{3} \lambda \alpha^2 \bar{U}_y^2 .$$

Integrating Γ_3 with respect to Y , we obtain:

$$\Phi_{3Y} = \Phi_{3Y}(0, t_1) + p_j b_j \tilde{F}_1(Y, t_1) , \quad (2.102)$$

$$\Phi_3 = \Phi_3(0, t_1) + \Phi_{3Y}(0, t_1)Y + (i\alpha)^{-1} \frac{\bar{U}_{yy}}{\bar{U}_y^2} b_j \tilde{F}_2(Y, t_1) , \quad (2.103)$$

where

$$\tilde{F}_1(Y, t_1) = \int_0^{+\infty} \frac{e^{i\alpha \bar{U}_y Y \xi} - 1}{\xi} A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi - s_1 \xi^3} d\xi , \quad (2.104)$$

$$\tilde{F}_2(Y, t_1) = \int_0^{+\infty} \frac{1 - i\alpha \bar{U}_y Y \xi e^{i\alpha \bar{U}_y Y \xi} - e^{i\alpha \bar{U}_y Y \xi}}{\xi^2} A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi - s_1 \xi^3} d\xi . \quad (2.105)$$

Following the same procedure as in the inviscid case, we find that the jumps across the j -th critical layer are as follows:

$$a_j^+ - a_j^- = \pi i p_j b_j \text{Sgn}(\bar{U}_y) , \quad (2.106)$$

$$d_j^+ - d_j^- = -\pi i r_j b_j \text{Sgn}(\bar{U}_y) . \quad (2.107)$$

Note that they are exactly the same as in the inviscid case, i.e. viscosity does not alter the jumps $(a_j^+ - a_j^-)$ and $(d_j^+ - d_j^-)$.

The function Ψ_4 satisfies the following equation:

$$\tilde{L}_0 \Psi_4 = \tilde{L}_1 \Psi_2 - \lambda^{\frac{1}{2}} \Omega_1 \frac{\partial^2}{\partial Y^2} \Psi_3 + \Psi_{1,x} \Psi_{3,YYY} . \quad (2.108)$$

By the same argument as in inviscid case, we conclude that ψ_2 , and Ψ_2 , should be identically zero. The solution of (2.108) has the following form:

$$\Psi_4 = \Phi_4^{(1)} E + \Phi_4^{(0)} + \Phi_4^{(2)} E^2 + c.c. . \quad (2.109)$$

In particular, $\Phi_4^{(1)}$ satisfies the following equation:

$$\bar{L}_0 \Phi_4^{(1)} = -i\alpha \bar{U}_{yy} b_j \lambda^{\frac{1}{2}} \Omega_1 \tilde{F}_0(Y, t_1) . \quad (2.110)$$

This equation can be solved in a similar way of solving for Φ_3 . It is found that

$$\Phi_{4YY}^{(1)} = -i\alpha \bar{U}_{yy} b_j \lambda^{\frac{1}{2}} \Omega_1 \tilde{G}_1(Y, t_1) , \quad (2.111)$$

$$\begin{aligned} \Phi_{4Y}^{(1)} &= p_j b_j \lambda^{\frac{1}{2}} \Omega_1 \int_0^{+\infty} (1 - e^{i\alpha \bar{U}_y Y \xi}) A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi - s_1 \xi^3} d\xi \\ &\quad + \Phi_{4Y}^{(1)}(0, t_1) , \end{aligned} \quad (2.112)$$

$$\Phi_4^{(1)} = -i\alpha^{-1} \frac{\bar{U}_{yy}}{\bar{U}_y^2} \lambda^{\frac{1}{2}} \Omega_1 \tilde{G}_2(Y, t_1) + \Phi_{4Y}^{(1)}(0, t_1)Y + \Phi_4^{(1)}(0, t_1) , \quad (2.113)$$

where

$$\tilde{G}_1(Y, t_1) = \int_0^{+\infty} \xi A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi - s_1 \xi^3} d\xi, \quad (2.114)$$

$$\tilde{G}_2(Y, t_1) = \int_0^{+\infty} \frac{e^{i\alpha\bar{U}_y Y \xi} - i\alpha\bar{U}_y Y \xi - 1}{\xi} A(t_1 - \xi) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi - s_1 \xi^3} d\xi. \quad (2.115)$$

Matching with the outer expansion gives the jump:

$$f_j^+ - f_j^- = \pi \alpha^{-1} b_j \bar{r}_j \lambda^{\frac{1}{2}} \Omega_1 \text{Sgn}(\bar{U}_y).$$

We shall see that this jump condition will not be used in deriving the evolution equation. This is not unexpected because it only provides a correction of the viscous sub-Stokes layer to the growth rate at $O(R^{-\frac{1}{2}})$ i.e. $O(\epsilon^{\frac{3}{5}})$, which is much smaller than the nonlinear growth rate of $O(\epsilon^{\frac{2}{5}})$ that we are considering.

The mean-flow distortion $\Phi_4^{(0)}$ and the second harmonic $\Phi_4^{(2)}$ satisfy the following equations respectively:

$$[\frac{\partial}{\partial t_1} - \lambda \frac{\partial^2}{\partial Y^2}] \Phi_{4Y}^{(0)} = -i\alpha^3 \bar{U}_{yy} \bar{U}_y b_j^2 A^* \tilde{G}_1(Y, t_1), \quad (2.116)$$

$$[\frac{\partial}{\partial t_1} + 2i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1) \lambda \frac{\partial^2}{\partial Y^2}] \Phi_{4Y}^{(2)} = i\alpha^3 \bar{U}_{yy} \bar{U}_y b_j^2 A \tilde{G}_1(Y, t_1). \quad (2.117)$$

The solutions are:

$$\Phi_{4Y}^{(0)} = -i\alpha^3 \bar{U}_{yy} \bar{U}_y b_j^2 \int_0^{+\infty} \int_0^{+\infty} \xi A^*(t_1 - \zeta) A(t_1 - \zeta - \xi) K_4^{(0)}(\xi, \zeta) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} d\zeta d\xi, \quad (2.118)$$

$$\Phi_{4Y}^{(2)} = i\alpha^3 \bar{U}_{yy} \bar{U}_y b_j^2 \int_0^{+\infty} \int_0^{+\infty} \xi A(t_1 - \zeta) A(t_1 - \zeta - \xi) K_4^{(2)}(\xi, \zeta) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\xi + 2\zeta)} d\zeta d\xi, \quad (2.119)$$

and so

$$\Phi_{4YY}^{(0)} = -\alpha^4 \bar{U}_{yy} \bar{U}_y^2 b_j^2 \int_0^{+\infty} \int_0^{+\infty} \xi^2 A^*(t_1 - \zeta) A(t_1 - \zeta - \xi) K_4^{(0)}(\xi, \zeta) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} d\zeta d\xi, \quad (2.120)$$

$$\Phi_{4YY}^{(2)} = \alpha^4 \bar{U}_{yy} \bar{U}_y^2 b_j^2 \int_0^{+\infty} \int_0^{+\infty} \xi(\xi + 2\zeta) A(t_1 - \zeta) A(t_1 - \zeta - \xi) K_4^{(2)}(\xi, \zeta) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)(\xi + 2\zeta)} d\zeta d\xi, \quad (2.121)$$

where

$$K_4^{(0)}(\xi, \zeta) = e^{-s_1(\xi^3 + 3\xi^2\zeta)}, \quad (2.122)$$

$$K_4^{(2)}(\xi, \zeta) = e^{-s_1(\xi^3 + 3\xi^2\zeta + 6\xi\zeta^2 + 4\zeta^3)}. \quad (2.123)$$

The function Ψ_5 satisfies the following equation:

$$\tilde{L}_0 \Psi_5 = \tilde{L}_2 \Psi_1 + \tilde{L}_1 \Psi_3 - \lambda^{\frac{1}{2}} \Omega_1 \frac{\partial^2}{\partial Y^2} \Psi_4 + \Psi_{1,x} \Psi_{4,YYY} , \quad (2.124)$$

where

$$\begin{aligned} \tilde{L}_2 = -\{ & [\frac{1}{6} \bar{U}_{yyy} Y^3 + \bar{U}_{yy\tau} \tau_1 Y^2 + \bar{U}_{y\tau\tau} \tau_1^2 Y + \frac{1}{6} \bar{U}_{\tau\tau\tau} \tau_1^3 + 2\lambda t_1 (\bar{U}_{\tau\tau} \tau_1 + \bar{U}_{y\tau} Y)] \frac{\partial}{\partial x} - \lambda \Omega_2 \} \frac{\partial^2}{\partial Y^2} \\ & + [\bar{U}_{yyy} Y + \bar{U}_{yy\tau} \tau_1] \frac{\partial}{\partial x} - [\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x}] \frac{\partial^2}{\partial x^2} . \end{aligned} \quad (2.125)$$

Due to viscous effects, the operators $\tilde{L}_1$ and $\tilde{L}_2$ are modified and additional terms appear on the right-hand side of equation (2.124). The solution driven by the first three terms on the right-hand side of (2.124) is denoted by $\Phi_5^{(l)} E$, and it is found that, to the order of interest in this study, $\Phi_{5Y}^{(l)}$ has the same asymptotic behaviour as represented by (2.68). So we shall only consider the solution forced by the non-linear interaction $(\Psi_{1,x} \Psi_{4,YYY})$. This forcing term can be written as follows:

$$F^{(n)}(Y, t_1) = i\alpha \hat{A} \Phi_{4YYY}^{*(0)} E + i\alpha (\hat{A} \Phi_{4YYY}^{(0)} - \hat{A}^* \Phi_{4YYY}^{(2)}) E + \dots + c.c , \quad (2.126)$$

where again only the terms proportional to E are written out. We find that only the solution induced by the first term contributes to the jump $c_j^+ - c_j^-$. Let us denote this solution by $\Phi_5^{(n)} E$, then $\Phi_5^{(n)}$ satisfies the following equation:

$$\bar{L}_0 \Phi_5^{(n)} = -i\alpha^5 \bar{U}_{yy} \bar{U}_y^2 b_j^3 A \int_0^{+\infty} \int_0^{+\infty} \xi^2 A(t_1 - \zeta) A^*(t_1 - \zeta - \xi) K_4^{(0)}(\xi, \zeta) e^{i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\xi} d\zeta d\xi . \quad (2.127)$$

The solution is found as follows:

$$\Phi_{5Y}^{(n)} = -i\alpha^5 \bar{U}_{yy} \bar{U}_y^2 b_j^3 \tilde{\Pi} , \quad (2.128)$$

where

$$\tilde{\Pi} = \int_0^{+\infty} \int_0^{+\infty} \int_{-\xi}^{+\infty} \xi^2 A(t_1 - \zeta - \xi) A(t_1 - \zeta - \xi - \rho) A^*(t_1 - \zeta - 2\xi - \rho) K_j(\xi, \rho) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\zeta - s_1 \zeta^3} d\zeta d\xi d\rho . \quad (2.129)$$

Integrating with respect to Y , we have:

$$\begin{aligned} \Phi_{5Y}^{(n)} = \alpha^4 \bar{U}_{yy} \bar{U}_y b_j^3 \int_0^{+\infty} \int_0^{+\infty} \int_{-\xi}^{+\infty} \frac{1 - e^{i\alpha \bar{U}_y Y \zeta}}{\zeta} \xi^2 A(t_1 - \zeta - \xi) A(t_1 - \zeta - \xi - \rho) A^*(t_1 - \zeta - 2\xi - \rho) \\ K_j(\xi, \rho) e^{-i\alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1)\zeta - s_1 \zeta^3} d\zeta d\xi d\rho . \end{aligned} \quad (2.130)$$

Matching $(\Phi_{5Y}^{(l)} + \Phi_{5Y}^{(n)})$ with the outer expansion determines the jump:

$$c_j^+ - c_j^- = \pi i \text{Sgn}(\bar{U}_y)(a_j^+ r_j + p_j d_j^+ + s_j b_j) - 2\pi i \alpha^4 \bar{U}_{yy} |\bar{U}_y| \int_0^{+\infty} \int_0^{+\infty} \xi^2 K_j(\xi, \zeta) \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \zeta) \hat{A}^*(t_1 - 2\xi - \zeta) d\xi d\zeta , \quad (2.131)$$

where

$$K_j(\xi, \zeta) = e^{-\frac{1}{3}\lambda\beta_j(2\xi^3 + 3\xi^2\zeta)} , \text{ and } \beta_j = \alpha^2 \bar{U}_y^2 . \quad (2.132)$$

Here the suffix j refers to the j -th critical layer.

By the same procedure as in the previous section, we obtain the evolution equation and the appropriate asymptotic behaviour for the viscous case as follows:

$$\frac{dA}{dt_1} = g_0 \tau_1 A + \alpha^4 \int_0^{+\infty} \int_0^{+\infty} \sum_j g_j K_j(\xi, \zeta) \xi^2 A(t_1 - \xi) A(t_1 - \xi - \zeta) A^*(t_1 - 2\xi - \zeta) d\xi d\zeta , \quad (2.133)$$

$$A \rightarrow A_0 e^{g_0 \tau_1 t_1} \quad \text{as } t_1 \rightarrow -\infty , \quad (2.134)$$

where the summation is made over all critical layers; there are two for the Stokes layer under consideration in this study. The coefficients g_0 and g_j are defined as follows

$$g_0 = f_0/f , \quad (2.135)$$

$$g_j = f_j/f , \quad (2.136)$$

where the constants f_0 and f have the same expressions as in inviscid case respectively, and

$$f_j = 2\pi i b_j^2 |b_j|^2 \bar{U}_{yy} |\bar{U}_y| . \quad (2.137)$$

Notice that f_j , hence g_j have different values at each critical level. The kernel $K_j(\xi, \zeta)$ represents the effects of viscosity. For inviscid case, $\lambda = 0$, $K_j(\xi, \zeta) = 1$, and so equation (2.133) reduces to (2.70).

As for the inviscid case, we introduce the new variables:

$$\bar{A} = A e^{i\bar{X}_0}/(-\tau_1)^{\frac{5}{2}} , \quad (2.138)$$

$$\bar{t} = -\tau_1 t_1 - x_0 , \quad (2.139)$$

$$\bar{\lambda} = \lambda/(-\tau_1)^3 , \quad (2.140)$$

to scale out A_0 and τ_1 , where $\bar{X}_0$ and x_0 are chosen so that:

$$g_0 x_0 - i(\bar{X}_0 - \varphi_0) = \log[A_0/(-\tau_1)^{\frac{5}{2}}] . \quad (2.141)$$

The evolution equation and the asymptotic behaviour then become:

$$\frac{d\bar{A}}{d\bar{t}} = -g_0\bar{A} + \alpha^4 \int_0^{+\infty} \int_0^{+\infty} \sum_j g_j K_j(\xi, \zeta) \xi^2 \bar{A}(\bar{t}-\xi) \bar{A}(\bar{t}-\xi-\zeta) \bar{A}^*(\bar{t}-2\xi-\zeta) d\xi d\zeta, \quad (2.142)$$

$$\bar{A} \rightarrow e^{-g_0\bar{t}+i\varphi_0} \quad \text{as} \quad \bar{t} \rightarrow -\infty. \quad (2.143)$$

The evaluation of the coefficients g_0 and g_j will be discussed in §2.5.

Furthermore, to scale out φ_0 we introduce:

$$\tilde{A} = \alpha^2 \bar{A} \left| \sum g_j \right|^{\frac{1}{2}} |g_0|^{-\frac{5}{2}}, \quad (2.144)$$

$$\tilde{t} = |g_0| \bar{t} - \tilde{X}_0, \quad (2.145)$$

$$\tilde{\lambda} = \lambda/3|g_0|^3, \quad (2.146)$$

where φ_0 and $\tilde{X}_0$ satisfy the following equation:

$$-\frac{g_0}{|g_0|} \tilde{X}_0 + i\varphi_0 = \log \left\{ \alpha^2 \left| \sum g_j \right|^{\frac{1}{2}} |g_0|^{-\frac{5}{2}} \right\}. \quad (2.147)$$

Then we obtain:

$$\frac{d\tilde{A}}{d\tilde{t}} = \nu_0 \tilde{A} + \int_0^{+\infty} \int_0^{+\infty} \sum_j \nu_j \tilde{K}_j(\xi, \zeta) \xi^2 \tilde{A}(\tilde{t}-\xi) \tilde{A}(\tilde{t}-\xi-\zeta) \tilde{A}^*(\tilde{t}-2\xi-\zeta) d\xi d\zeta, \quad (2.148)$$

$$\tilde{A} \rightarrow e^{\nu_0 \tilde{t}} \quad \text{as} \quad \tilde{t} \rightarrow -\infty, \quad (2.149)$$

where

$$\tilde{K}_j(\xi, \zeta) = e^{-\tilde{\lambda} \beta_j (2\xi^3 + 3\xi^2 \zeta)}, \quad (2.150)$$

and

$$\nu_0 = -g_0/|g_0|, \quad (2.151)$$

$$\nu_j = g_j / \left| \sum g_j \right|. \quad (2.152)$$

The amplitude equations (2.85) and (2.148) bear some resemblance to the Stuart-Landau equation in that they evolve cubic nonlinear terms, but they differ from the latter in one important aspect; this is that the local growth rate depends on the whole history of the evolution rather than only on the instantaneous amplitude. This form of amplitude equation was first derived by Hickernell (1984) in his study of singular modes of Rossby wave. Goldstein and Leib (1989) showed that the same amplitude equations governed the spatial evolution of an oblique wave in a compressible shear layer, although there the critical-layer nonlinearity is associated with a pole type of singularity in temperature fluctuation rather than the logarithmic branch point as in the case of Hickernell (1984) and the present case. Amplitude equations containing non-local nonlinear terms were also obtained by Churilov & Shukhman (1988) in their study of a two-dimensional disturbance in a stratified flow.

Finally note that the above theory is formally valid when ϵ is much smaller than $O(1)$ but much larger than or at the order of $O(R^{-\frac{5}{6}})$, i.e. the disturbance is relatively large. Next we turn to the limit

$$\epsilon \ll R^{-\frac{5}{6}} , \quad (2.153)$$

which corresponds to the limit $\lambda \rightarrow +\infty$. We shall examine equation (2.133) under this limit by the following variable changes:

$$\bar{\xi} = \lambda^{\frac{1}{3}} \xi , \quad \bar{\zeta} = \lambda^{\frac{1}{3}} \zeta .$$

Inserting them into (2.133) and taking the limit $\lambda \rightarrow +\infty$ yields:

$$\frac{dA}{dt_1} = g_0 \tau_1 A + \alpha^4 \lambda^{-\frac{4}{3}} \bar{g} I_0 |A|^2 A + o(\lambda^{-\frac{4}{3}}) . \quad (2.154)$$

This is a Stuart-Landau equation; here

$$I_0 = \frac{1}{9 \times 2^{\frac{1}{3}}} \Gamma\left(\frac{1}{3}\right) , \quad \bar{g} = 3^{\frac{4}{3}} \sum g_j \beta_j^{-\frac{4}{3}} .$$

That equation (2.133) reverts to a Stuart-Landau equation (2.154) was first shown by Hickernell (1984). For the problem at hand, a little more discussion seems to be appropriate. Note that all the terms in the equation are at the same order if $\tau_1 \sim O(\lambda^{-\frac{4}{3}})$. Recalling that

$$\tau_1 = \epsilon^{-\frac{2}{3}} (\tau - \tau_0) , \quad \lambda^{-\frac{4}{3}} = R^{\frac{4}{3}} \epsilon^{\frac{8}{3}} ,$$

we obtain

$$\tau - \tau_0 \sim O(R^{\frac{4}{3}} \epsilon^2) . \quad (2.155)$$

Obviously when (2.153) holds, $\tau - \tau_0 \ll R^{-\frac{1}{3}}$. This implies that the critical layers are now conventional steady-viscous ones. Churilov & Shukhman (1987) analysed such a critical layer of a stratified shear flow, and obtained an amplitude equation of Stuart-Landau type. See also Churilov & Shukhman (1987) and Huerre (1987) for studies of steady-viscous critical layer in free shear-layers.

Recalling that the sub-Stokes layer has a correction of $O(R^{-\frac{1}{2}})$ to the linear growth rate, we have an additional restriction for the validity of (2.154):

$$R^{\frac{4}{3}} \epsilon^2 \gg O(R^{-\frac{1}{2}}) , \quad \text{i.e.} \quad \epsilon \gg O(R^{-\frac{11}{12}}) .$$

We therefore conclude that when

$$R^{-\frac{11}{12}} \ll \epsilon \ll R^{-\frac{19}{12}} , \quad (2.156)$$

or equivalently

$$1 \ll \lambda \ll R^{\frac{1}{10}} , \quad (2.157)$$

the evolution can be described by (2.154). The amplitude equation (2.133), however, has a larger range of validity, namely $\lambda \ll R^{\frac{1}{10}}$. If the first of the restrictions (2.157) is violated, the evolution equation could still be of Stuart-Landau form, but its coefficients must be modified due to the sub-Stokes layer¹. When $\epsilon \sim R^{-\frac{11}{12}}$, or equivalently $\lambda \sim R^{\frac{1}{10}}$, we should concentrate on times near to $\tau_0 + R^{-\frac{1}{2}}\tau_1$. Correspondingly, $\epsilon^{\frac{2}{5}}$ and $\epsilon^{-\frac{2}{5}}$ in the second and third terms on the right-hand side of (2.4) should be replaced by $R^{-\frac{1}{2}}$ and $R^{\frac{1}{2}}$ respectively. Consequently the time scales t_1 and τ_1 are not distinct any more.

2.5 Evaluation of the Coefficients in the Amplitude Equation

The form of the amplitude equation (2.85) and (2.148) is dictated by the singularity property of critical layers, and appear to be generic for disturbances in a variety of flows. However the coefficients in these amplitude equations depend on the detailed profile of the basic flow, and on how the disturbance deviates from its linearly neutral state.

For both the inviscid and viscous cases, evaluation of the coefficients needs:

(1) A calculation of the eigenvalues and eigenfunctions. This is done by using a shooting method supplemented by a high order Frobenius solution near the critical layers, see, e.g. Benney & Bergeron (1969), Haberman (1972), and Bodonyi & Smith (1983). The sizes of $a_j^\pm$, b_j are also specified in this procedure.

(2) A suitable method for dealing with the Hadamard finite part of singular integrands of the form

$$J = \int_0^{+\infty} f(y) \phi_1^2 dy .$$

For this purpose, we write J as follows:

$$J = \sum_k J_k^s + \sum_k J_k^r + J_0 + J_\infty ,$$

where

$$J_k^s = \int_{y_c^k - h}^{y_c^k + h} f(y) \phi_1^2 dy .$$

$$J_k^r = \int_{y_c^k + h}^{y_c^{k+1} - h} f(y) \phi_1^2 dy ,$$

$$J_0 = \int_0^{y_c^1 - h} f(y) \phi_1^2 dy ,$$

$$J_\infty = \int_{y_c^N + h}^{+\infty} f(y) \phi_1^2 dy .$$

¹The author believes that Dr. J.S.B. Gajjar may have investigated such a problem.

The interval $[y_c^k - h, y_c^k + h]$ accommodates the critical level y_c^k , and hence $f(y)$ and ϕ_1 are singular, but they can be approximated by their asymptotic expansions respectively, so the finite part of J_k^s can be determined analytically. The integrals J_k^r are regular and can be evaluated numerically. By using the asymptotic expressions of $f(y)$ and ϕ_1 at large y , J_∞ can be easily calculated.

The coefficients in equation (2.80) and (2.142) are plotted against wavenumbers in Fig3 and Fig(4a-4b).

2.6 Study of the Amplitude Equation

2.6.1 Finite-Time Singularity

Equations (2.85) and (2.148) were first studied by Goldstein & Leib (1989). They found that the solutions could terminate a finite distance downstream; note that they considered spatial evolution. In this study, we consider temporal evolution. Our finite difference solutions together with the series analysis, which will be discussed in the subsequent two subsections, also suggest that solutions to (2.85) and (2.148) blow up within a finite time, say t_s . For both inviscid and viscous cases, the singularity structure is as follows

$$\tilde{A} = \frac{a_0}{(t_s - \tilde{t})^{\frac{5}{2} + i\tilde{\sigma}}} \quad (2.158)$$

Following Goldstein & Leib (1989), we insert (2.158) into (2.85) and (2.148), and find that

$$(\frac{5}{2} + i\tilde{\sigma}) = \sum_j \nu_j |a_0|^2 J_0, \quad (2.159)$$

where

$$J_0 = \int_0^{+\infty} d\xi \int_0^{+\infty} d\zeta \xi^2 [(1 + \xi)(1 + \xi + \zeta)]^{-(\frac{5}{2} + i\tilde{\sigma})} (1 + 2\xi + \zeta)^{-(\frac{5}{2} - i\tilde{\sigma})}.$$

The singularity parameters $|a_0|$ and $\tilde{\sigma}$ can be calculated from (2.159) (Goldstein & Leib, 1989). The blow-up time t_s is determined by fitting numerically calculated results to (2.158). It is noted that $|a_0|$ and t_s have been scaled by the wavenumber α through g_0 and g . The singularity-structure parameters corresponding to (2.80) and (2.142), say $|\bar{A}_0|$ and $\bar{t}_s$, are related to $|a_0|$ and t_s through the following relations:

$$\begin{aligned} \bar{A}_0 &= \alpha^{-2} |a_0| |g|^{-\frac{1}{2}}, \\ \bar{t}_s &= |g_0|^{-1} t_s + g_{0r}^{-1} \log \{ \alpha^{-2} |g_0|^{\frac{5}{2}} |g|^{-\frac{1}{2}} \}, \end{aligned}$$

where g_{0r} denotes the real part of g_0 . If disturbances of different wavenumbers have the same magnitude, then $\bar{t}_s$ reflects which disturbance blows up first and $\bar{A}_0$ reflects the "strength" of the blow-up. The $|\bar{A}_0|$, $\tilde{\sigma}$ and $\bar{t}_s$ so obtained at different wave numbers are plotted in Fig5.

2.6.2 Series Analysis and Extension

As $\tilde{t} \rightarrow -\infty$, $\tilde{A}$ can be expressed in the following form:

$$\tilde{A} = B e^{\nu_0 \tilde{t}}, \quad (2.160)$$

$$B = \sum_{n=0}^{+\infty} \bar{b}_n e^{n \nu \tilde{t}}, \quad (2.161)$$

where $\nu = \nu_0 + \nu_0^*$, and the $\bar{b}_n$ satisfy the following recursion relation:

$$\begin{aligned} \bar{b}_0 &= 1, \\ \bar{b}_{n+1} &= \frac{1}{(n+1)\nu} \sum_{l=0}^n \sum_{k=0}^l \bar{b}_{n-l}^* \bar{b}_k \bar{b}_{l-k} \sum_j \nu_j R_j(k, l, n), \\ &\quad n = 0, 1, 2, \dots, \end{aligned} \quad (2.162)$$

where

$$R_j(k, l, n) = \int_0^{+\infty} \frac{\xi^2}{\nu(k+n-l+1) + 3\lambda\beta_j\xi^2} \exp[-2\lambda\beta_j\xi^3 - \nu(2n-l+2)\xi] d\xi. \quad (2.163)$$

For the viscous case $R_j(k, l, n)$ has to be evaluated numerically, but for the inviscid case, i.e. $\lambda = 0$, it has the following expression:

$$R_j(k, l, n) = \frac{2}{\nu^4(k+n-l+1)(2n-l+2)^3}. \quad (2.164)$$

Let $\Xi = \exp(\nu \tilde{t})$, then B is a power series of the form:

$$B = \sum_{n=0} \bar{b}_n \Xi^n. \quad (2.165)$$

Such series can be analyzed using the Domb-Sykes Plot method and the method of Pade Approximants (Van Dyke, 1975, 1984).

For some wavenumbers, $\{Re(\bar{b}_n)\}, \{Im(\bar{b}_n)\}$ have fixed sign patterns. This indicate that the nearest singularity Ξ_c of B is on real positive axis, i.e. $\Xi_c \geq 0$. A Domb-Sykes Plot (Van Dyke, 1975, 1984) gives a good estimation of Ξ_c and therefore of the singularity time t_s by

$$t_s = \nu^{-1} \log(\Xi_c). \quad (2.166)$$

For $\alpha = 1.0$, the result of this procedure is shown in Fig6 and Fig7 for $\tilde{\lambda} = 0, 2.0$ respectively. The singularity time t_s so obtained has four accurate digits. However for other wavenumbers, $Re(\bar{b}_n), Im(\bar{b}_n)$ have an alternating sign pattern. This indicates that the nearest singularity $\Xi_c \leq 0$, i.e. the corresponding singularity time t_s is off real axis and hence has no physical meaning.

As shown in Fig8, for certain parameters the power-series solution (2.161) provides a good approximation up to the singularity time. The series (2.161) can

be analytically continued by recasting it into a continuous fraction. Although the method of Pade Approximants (Van Dyke, 1975, 1984) fails to give an estimation of the real singularity time, nevertheless as an approximation to $\tilde{A}$, the resultant continuous fraction does have a much larger valid range than the corresponding power series, see Fig9.

2.6.3 Finite Difference Method

The scaled evolution equations (2.85) and (2.148) were solved numerically by using a predictor-corrector scheme with fourth-order accuracy. We assumed that when $\tilde{t} \leq -T_0$, $\tilde{A}$ was approximated by its power series, where the size of T_0 can only be determined by trial. The integral domain $[0, +\infty) \times [0, +\infty]$ is divided into three domains, namely $D_0 = [0, T_0] \times [0, T_0]$, $D_1 = [T_0, +\infty] \times [0, +\infty]$, $D_2 = [0, T_0] \times [T_0, +\infty]$. The double integrals over D_0 were evaluated by Simpson rule. Tails on D_1 and D_2 were expressed by using the expressions for $\tilde{A}(\tilde{t}-\xi)$, $\tilde{A}(\tilde{t}-\xi-\zeta)$ and $\tilde{A}^*(\tilde{t}-2\xi-\zeta)$ respectively.

The amplitude evolution obtained using the finite difference method is shown in Fig5-8 for different wavenumbers. As illustrated, the solutions develop a finite time singularity in the inviscid case. Note that the instability wave amplitude does not exhibit oscillation upstream of the singularity for all wave numbers (cf. Goldstein & Leib, 1989). Viscosity generally delays the occurrence of the finite-time singularity provided the scaled viscosity parameter $\tilde{\lambda}$ does not exceed a critical value, say $\tilde{\lambda}_c^{(e)}$, i.e. viscosity is not too large. But for wavenumbers in the range of $[0.6, 1.3]$, the finite-time singularity always occurs, as shown in Fig10 and Fig11, no matter how large $\tilde{\lambda}$ is. Note that for the Stokes layer of interest here, this is a substantial range of wavenumbers. For wavenumbers beyond this range, the singularity is eliminated and consequently the disturbance evolves into an equilibrium state when $\tilde{\lambda}$ exceeds some $\tilde{\lambda}_c^{(e)}$. Results for two typical wavenumbers $\alpha = 0.5$ and $\alpha = 1.4$ are presented in Fig12 and Fig13 respectively. It turns out to be difficult to determine the size of $\tilde{\lambda}_c^{(e)}$. One may expect that $\tilde{\lambda}_c^{(e)}$ corresponds to the critical value for the equilibrium state to be stable. For this purpose, we study the stability of the equilibrium solution in the next section. However, our calculation shows that this is not the case.

2.7 The Instability of Equilibrium Solutions

2.7.1 Equilibrium Solution

For the viscous case, the evolution equation (2.148) possesses a solution of the form:

$$\tilde{A} = A_0 e^{ik\tilde{t}}, \quad (2.167)$$

as noted by Goldstein and Leib (1989). Here

$$|A_0|^2 = -\frac{Re\nu_0}{Re\tilde{\nu}_0} I_0^{-1} \tilde{\lambda}^{\frac{4}{3}}, \quad (2.168)$$

$$k = \frac{Im\nu_0 Re\bar{\nu}_0 - Re\nu_0 Im\bar{\nu}_0}{Re\bar{\nu}_0}, \quad (2.169)$$

$$\bar{\nu}_0 = \sum \nu_j \beta_j^{-\frac{4}{3}}, \quad (2.170)$$

$$I_0 = \frac{1}{9 \times 2^{\frac{1}{3}}} \Gamma\left(\frac{1}{3}\right).$$

Because $Re(-g_0)$, and hence $Re(\nu_0)$ are always positive (see equation (2.151)), it follows from (2.168) that the condition for the existence of this equilibrium solution is:

$$Re\bar{\nu}_0 \leq 0. \quad (2.171)$$

The condition (2.171) is a generalization of that of Goldstein & Leib (1989) to the multiple-critical-layer case. For the present problem, it is found that the above condition is satisfied for the wave numbers $\alpha \leq 0.6003$ or $\alpha \geq 1.293$. For $0.6003 \leq \alpha \leq 1.293$, condition (2.171) is violated (see Fig.14), and hence (2.148) has no equilibrium solution. Our study shows that in this range the solution always develops a finite-time singularity, no matter how large $\tilde{\lambda}$ is. The scaled equilibrium-amplitude $|A_0|\tilde{\lambda}^{-\frac{2}{3}}$ and k are plotted in Fig15. The next task of this section is to examine the instability of the equilibrium solutions.

2.7.2 The Instability Problem

We perturb the equilibrium solution in the following way:

$$\tilde{A} = [A_0 + B(\tilde{t})]e^{ik\tilde{t}}, \quad (2.172)$$

then B satisfies the following equation after linearization:

$$\frac{dB}{d\tilde{t}} = (\nu_0 - ik)B + A_0^2 \int_0^{+\infty} \int_0^{+\infty} \sum \tilde{\nu}_j \tilde{K}_j(\xi, \zeta) \xi^2 [B(\tilde{t} - \xi - \zeta) + B(\tilde{t} - \xi) + B^*(\tilde{t} - 2\xi - \zeta)]. \quad (2.173)$$

Following Stuart & DiPrima (1978), we shall seek a solution of the form:

$$B = C_1 e^{\sigma \tilde{t}} + C_2 e^{\sigma^* \tilde{t}}, \quad (2.174)$$

where C_1, C_2, σ generally are complex numbers. Inserting into (2.173) we get the following relations:

$$\begin{aligned} \sigma C_1 &= (\nu_0 - ik)C_1 + A_0^2 \sum \nu_j K_j^{(1)} C_1 + A_0^2 \sum \nu_j K_j^{(2)} C_2^*, \\ \sigma^* C_2 &= (\nu_0 - ik)C_2 + A_0^2 \sum \nu_j K_j^{*(1)} C_2 + A_0^2 \sum \nu_j K_j^{*(2)} C_1^*. \end{aligned}$$

The requirement that B is non-zero yields the eigenrelation as follows

$$\begin{vmatrix} \sigma - \nu_0 + ik - A_0^2 \sum \nu_j K_j^{(1)} & -A_0^2 \sum \nu_j K_j^{(2)} \\ -A_0^2 \sum \nu_j^* K_j^{(2)} & \sigma - \nu_0^* - ik - A_0^2 \sum \nu_j^* K_j^{(1)} \end{vmatrix} = 0, \quad (2.175)$$

where

$$\begin{aligned} K_j^{(1)} &= \int_0^{+\infty} d\xi \int_0^{+\infty} d\zeta \xi^2 \exp[-\tilde{\lambda}\beta_j(2\xi^3 + 3\xi^2\zeta)] [e^{-\sigma(\xi+\zeta)} + e^{-\sigma\xi}] , \\ K_j^{(2)} &= \int_0^{+\infty} d\xi \int_0^{+\infty} d\zeta \xi^2 \exp[-\tilde{\lambda}\beta_j(2\xi^3 + 3\xi^2\zeta)] e^{-\sigma(2\xi+\zeta)} . \end{aligned}$$

After integration with respect to the ζ , we have

$$K_j^{(1)} = \int_0^{+\infty} d\xi \left[\frac{\xi^2}{\sigma + 3\tilde{\lambda}\beta_j\xi^2} + \frac{1}{3}(\tilde{\lambda}\beta_j)^{-1} \right] e^{-(2\tilde{\lambda}\beta_j\xi^3 + \sigma\xi)} , \quad (2.176)$$

$$K_j^{(2)} = \int_0^{+\infty} d\xi \left[\frac{\xi^2}{\sigma + 3\tilde{\lambda}\beta_j\xi^2} \right] e^{-(2\tilde{\lambda}\beta_j\xi^3 + 2\sigma\xi)} . \quad (2.177)$$

The above eigenvalue problem generally must be solved numerically. But when $\tilde{\lambda} \rightarrow +\infty$, we can get asymptotic expressions for σ .

As $\tilde{\lambda} \rightarrow +\infty$, it is found that $K_j^{(1)}$ and $K_j^{(2)}$ have the following expansions:

$$K_j^{(1)} \sim 2I_0(\tilde{\lambda}\beta_j)^{-\frac{8}{6}} + \frac{1}{2 \times 3^{\frac{3}{2}}} s(-\sigma)^{\frac{1}{2}} \pi i (\tilde{\lambda}\beta_j)^{-\frac{9}{6}} + (-3^{-\frac{5}{3}} J_1 - \frac{2}{3} I_1) \sigma (\tilde{\lambda}\beta_j)^{-\frac{10}{6}} + o[(\tilde{\lambda}\beta_j)^{-\frac{10}{6}}] , \quad (2.178)$$

$$K_j^{(2)} \sim I_0(\tilde{\lambda}\beta_j)^{-\frac{8}{6}} + \frac{1}{2 \times 3^{\frac{3}{2}}} s(-\sigma)^{\frac{1}{2}} \pi i (\tilde{\lambda}\beta_j)^{-\frac{9}{6}} + (-3^{-\frac{5}{3}} J_1 - \frac{2}{3} I_1) \sigma (\tilde{\lambda}\beta_j)^{-\frac{10}{6}} + o[(\tilde{\lambda}\beta_j)^{-\frac{10}{6}}] , \quad (2.179)$$

where $s = \pm 1$ according to whether $Im\sqrt{-\sigma} \geq 0$ or ≤ 0 , and

$$I_1 = \int_0^{+\infty} \xi e^{-2\xi^3} d\xi , \quad J_1 = \int_0^{+\infty} \frac{e^{-\frac{2}{3}\xi^3} - 1}{\xi^2} d\xi .$$

The $O[(\tilde{\lambda}\beta_j)^{-\frac{10}{6}}]$ term is in fact identically zero if we notice that $J_1 = -6 \times 3^{-\frac{1}{3}} I_1$.

Therefore σ should be expanded as follows:

$$\sigma = \sigma_0 + \tilde{\lambda}^{-\frac{1}{6}} \sigma_1 + \tilde{\lambda}^{-\frac{2}{6}} \sigma_2 + \dots . \quad (2.180)$$

Inserting the asymptotic expressions of $K_j^{(1)}$, $K_j^{(2)}$ and σ into (2.175), we have:

$$\sigma_0 = -2Re\nu_0 , \quad (2.181)$$

$$\sigma_1 = 2^{\frac{1}{2}} \times (Re\nu_0)^{\frac{1}{2}} Re\bar{\nu}_2 A_{00} \pi i , \quad (2.182)$$

$$\sigma_2 = \frac{1}{2} (Re\bar{\nu}_2)^2 A_{00}^2 \pi^2 , \quad (2.183)$$

where

$$A_{00} = -\frac{Re\nu_0}{Re\bar{\nu}_0} I_0^{-1} , \quad \bar{\nu}_2 = \sum \nu_j \beta_j^{-\frac{3}{2}} .$$

Obviously, $\sigma_0 \leq 0$, suggesting that the equilibrium solution is stable when the viscosity parameter $\tilde{\lambda}$ is sufficiently large. However σ_1 is purely imaginary, and $\sigma_2 \geq 0$ and one may thus expect that there exists a finite critical value $\tilde{\lambda}_c$, below which the equilibria is unstable. The existence and the size of $\tilde{\lambda}_c$ are determined by numerical solution of the eigenvalue problem (2.175). The results for two wave numbers $\alpha = 0.5, 1.4$ are presented in Fig16 and Fig17 respectively. Note that for $\alpha = 0.5$, $\tilde{\lambda}_c = 2.0$, but the solution corresponding to $\tilde{\lambda} = 5.0$ still develops a singularity rather than approaching to the stable equilibrium, as can be seen from Fig12. This suggests that $\tilde{\lambda}_c < \tilde{\lambda}_c^{(e)}$, i.e. stable equilibrium state is *not* necessary a attractor. The same conclusion holds for $\alpha = 1.4$; see Fig13 and Fig17.

2.8 Conclusions and Discussion

In this study, we have shown that an initially small linear two-dimensional wave will eventually enter a nonlinear evolution stage due to enhanced nonlinearity inside the critical layers. The nonlinear evolution is governed by an integro-differential equation of Hickernell type. For relatively small amplitude waves, i.e. $\epsilon \ll R^{-\frac{2}{5}}$, this amplitude equation reverts to a classical Stuart-Landau equation.

The nonlinearity is found to have a profound effect on the development of the waves, namely it causes a blow-up in a finite time. Viscosity generally postpones the blow-up. For the disturbances with wavenumbers in the range of $[0.6, 1.3]$, nonlinearity has a robust effect in the sense that a finite-time singularity always occurs, no matter how large the scaled viscosity parameter $\tilde{\lambda}$ is. This seems to suggest that most dangerous modes should have wave lengths in the range $[5.1\delta^*, 10.5\delta^*]$, where δ^* is the thickness of Stokes layers. Monkewitz and Bunster (1987) observed that the wave length of the disturbance was about $10\delta^*$. But this apparent agreement can only be regarded to be tentative because the disturbance in experiments is three-dimensional.

For those wavenumbers beyond the range $[0.6, 1.3]$, our calculations show that if the scaled viscosity parameter $\tilde{\lambda}$ is larger than a finite critical value $\tilde{\lambda}_c^{(e)}$, the singularity is eliminated and consequently the wave evolves into an equilibrium state. The stability of this equilibrium is also studied. It is shown that when $\tilde{\lambda}$ exceeds a critical value $\tilde{\lambda}_c$, the equilibrium solution is stable. Interestingly, we find that $\tilde{\lambda}_c < \tilde{\lambda}_c^{(e)}$, i.e. the critical viscosity parameter for the equilibrium solution to be stable is smaller than that for equilibrium solution to be ^{the} long-time limit of the solution of initial-value problem. This indicates that the equilibrium state, though it is stable, does not necessarily attract the solution of the initial-value problem.

The equilibrium state, however is not observed in experiments. Such a disagreement is not uncommon in hydrodynamic instability theory. For example, for plane Poiseuille flow, an equilibrium is predicted both by theoretical and computational work (Zahn *et al.* 1974, Orszag & Kells 1980), but not verified by experiments (Nishioka, 1975). There it has been argued that such an equilibrium is destroyed by a radical secondary instability in the quasi-equilibrium stage (Herbert 1981, 1983).

For the present problem, note that when the growth rate has decayed to $O(R^{-\frac{1}{2}})$ in the process of approaching the equilibrium stage, the sub-Stokes layer will affect the evolution.

Nevertheless the blow-up is the general feature of the evolution. We suggest that this nonlinear effect may cause transition to turbulence. In particular, the explosive growth of the high-frequency wave may be related to bursts, a significant phenomena of Stokes layer instability observed in experiments.

The present nonlinear theory shows that initially small disturbances are substantially amplified by nonlinear effects in the neighbourhood of the right branch of ^{the} neutral curve, and hence can attain a finite amplitude by this nonlinear amplification. Referring to the neutral diagram of Cowley (1987), we may conclude that disturbances are most likely to be observed shortly or around the phase $\frac{\pi}{2}$ of the basic flow. This conclusion is in agreement with the observations of Monkewitz and Bunster (1987), but contradicts others, e.g. Akhavan *et al.* (1991a) and references therein. They find that explosive growth occurs at the end of the acceleration phase, i.e. phase 0 in the notation of the present study. A possible explanation is that all experiments were actually conducted for finite Stokes layers, where an additional parameter $\Gamma = h/\delta^*$ sets in, with h being the half width of a channel or the radius of a pipe. The value of Γ in most experiments is in the range of 3 to 10, but is about 44 in that of Monkewitz and Bunster (1987), closest to the infinite Stokes layer with $\Gamma = \infty$ studied in this work. Though the basic flow profile at $\Gamma = 10$ is very close to that at $\Gamma = \infty$, it is possible that at low Γ , some new mode which does not exist at high Γ , may bifurcate and cause transition. It seems that the effects of Γ on the instability of finite Stokes layers deserve further study.

Chapter 3

Nonlinear Evolution of a pair of Oblique Waves

Abstract

The nonlinear evolution of a pair of initially linear oblique waves ($\alpha, \pm\beta$) is studied under a high-Reynolds-Number assumption, where α, β are the streamwise and spanwise wavenumbers respectively. It is shown that when the growth rate decreases to $O(\epsilon^{1/3})$, where ϵ is the magnitude of the disturbance, the disturbance comes into a new evolution stage because of enhanced nonlinear effects in the critical layers. The nonlinear evolution occurs much earlier and evolves at a much faster time scale than in the corresponding two-dimensional case studied in Chapter two. As pointed out by Goldstein & Choi (1989), this is due to a pole-type singularity in the streamwise velocity, which is stronger than the logarithmic branch-point singularity of the two-dimensional case.

The amplitude equation is shown to be the same as that of Goldstein & Choi (1989), though in our case the critical levels do not locate at the inflexional points of the basic-flow profile. However it is note that when three-dimensionality is sufficiently weak, the logarithmic-branch-point singularity will assert its influence. Also the incorporation of viscosity in our analysis consists of an extension of the study of Goldstein & Choi (1989). Viscous effects are found to be represented by a complicated modification to the structure of the nonlinear term. ~~As a result, equilibrium solutions may be found.~~ The solution of the amplitude equation can develop, at least at inviscid limit, a finite-time singularity, suggesting that a substantial growth can be induced by nonlinear effects. This result gives further support to the instability mechanism suggested in Chapter two, that the subsequent nonlinear development of initially small linear disturbances may lead the flow into turbulence. The nonlinear blow-up of three-dimensional high-frequency disturbances in a finite time may be related to the bursts observed in experiments.

In addition, it is noted that nonlinear interactions between waves drives a vortex flow of the same magnitude as the fundamental waves. A vortex structure was observed by Hino *et al.* (1983), who also suggested that the bursting process, streaky

structure (Kline *et al.*, 1967; Kim *et al.*, 1971) in a turbulent boundary layer and in a Stokes layer are essentially identical. We thus believe that the results from the present study may be helpful in understanding of turbulence in a boundary layer.

3.1 Introduction

The self-consistent high-Reynolds-number approach to the instability of a Stokes layer of Tromans (1977) and Cowley (1987) makes it possible to take nonlinear effects into account systematically. Such an attempt was made for two-dimensional disturbances in Chapter two, where it was shown that nonlinear interactions inside the critical layers will inevitably affect the evolution of disturbances, causing a blow-up in a finite time. It is suggested that this subsequent nonlinear growth is responsible for the transition from laminar to turbulent states.

However in reality, the disturbances from the environment are generally three dimensional. For Tollmien-Schlichting instability, it is well recognized that only in carefully controlled experiments, are the disturbances mainly two dimensional, e.g. Schubauer & Skramstad (1947), Nishioka *et al.* (1975). Even so the two-dimensional disturbances can dominate only in the very early stage of development. Three-dimensional disturbances inevitably overtake two-dimensional ones when transition occurs, e.g. Klebanoff *et al.* (1962), Kachanov *et al.* (1984,1987), Saric *et al.* (1984). For the Stokes layer, three-dimensionality is particularly important because the disturbances grow rapidly, which makes it difficult for an artificial two-dimensional disturbance to be introduced and observed. So far there are no experiments for controlled disturbances. The present study is to extend the theory of Chapter two to the three-dimensional case. In particular, we consider the disturbance consisting of a pair of oblique waves $(\alpha, \pm\beta)$, where α and β are streamwise and spanwise wavenumbers respectively. Such a form of disturbance is important because the “quadratic interaction” produces a vortex flow. On the other hand, significant vortex structure was observed by Hino *et al.* (1983) in a finite Stokes layer.

The nonlinear *spatial* evolution of a pair of oblique waves in a free shear layer was studied by Goldstein & Choi (1989). They showed that because of the pole type of singularity in the streamwise velocity, critical-layer nonlinearity exerted its influence when the linear growth rate decreased to $O(\epsilon^{\frac{1}{3}})$, much earlier than in the corresponding two-dimensional case (Goldstein & Leib, 1988). Consequently, the critical-layer dynamics became weakly nonlinear in the three-dimensional case, in contrast to the two-dimensional case, where the critical layer is strongly nonlinear. Goldstein & Choi (1989) also point out the critical-layer dynamics of a pair of oblique modes differs from that of a single oblique mode (Goldstein & Leib, 1989; Leib, 1991): in the former case, the dynamics is dictated by the pole type of singularity associated with the three-dimensionality, while in the latter case, the effects of this singularity do not assert themselves when the wave grows in its propagation direction.

In this Chapter, we are concerned with the *temporal* development of a pair of oblique waves. Note also that $\bar{U}_{yy}(y_c) \neq 0$ for the Stokes layer of interest here.

Nevertheless, the analysis in this Chapter can be regarded as a temporal equivalent of the analysis of Goldstein & Choi (1989) when $\beta \gg \epsilon^{\frac{1}{3}}$, though the basic flow is different. When $\beta \sim \epsilon^{\frac{1}{3}}$, however, the nonlinear term associated with logarithmic singularity will come into the final amplitude equation; this term is absent for a free shear layer because there $\bar{U}_{yy}(y_c) = 0$. We shall address this issue in Chapter five in a more general situation where spanwise space-modulation will be considered. Moreover, in our analysis viscous effects will be included, and thus the critical layers evolved are thus non-equilibrium and viscous in nature. This extends the study of Goldstein & Choi (1989). Leib (1991) investigated the evolution of a *single* oblique wave whose critical level does not locate at the (generalized) inflexional point. However, it should be remarked that the present analysis is different from his. This is because in his work the effects of the pole type of singularity in the streamwise and spanwise velocities were suppressed by an appropriate Squire transformation, and nonlinear effects were associated with the logarithmic branch-point singularity and the (pole type of) singularity in the temperature.

With the scale set in Chapter one, we shall concentrate on the times specified by

$$\tau = \tau_0 + \epsilon^{\frac{1}{3}} \tau_1 ,$$

namely the times at which the linear growth rate decreases to $O(\epsilon^{\frac{1}{3}})$. It is at this stage that nonlinear effects begin to affect the evolution of the disturbances. Correspondingly we introduce the nonlinear evolution time scale

$$t_1 = \frac{1}{2} \epsilon^{\frac{1}{3}} t , \quad (3.1)$$

where $t = R\tau$ is the fast time scale introduced to take account of the frequency of the waves.

The basic flow $\bar{U}$ evolves on the slow time scale τ , it is sufficient to express its profile at time τ in a Taylor series about the neutral τ_0

$$\bar{U}(y, \tau) = \bar{U}(y, \tau_0) + \epsilon^{\frac{1}{3}} \bar{U}_\tau(y, \tau_0) \tau_1 + \dots$$

Hereafter all quantities associated with the basic flow will be evaluated at τ_0 unless otherwise mentioned.

The overall evolution of the three-dimensional disturbance is summarized in Fig2. As illustrated, the disturbance initially grows exponentially as described by linear theory (Cowley, 1987) until its growth rate decrease to $O(\epsilon^{\frac{1}{3}})$, i.e. at $\tau_1 = O(1)$. At this stage, nonlinear interactions inside the critical layers control the evolution. Following Goldstein & Choi (1989), we require this subsequent nonlinear stage to match with the earlier linear stage in the mutual overlapping domain.

For simplicity, throughout §2 to §4 of this chapter, we shall assume that the Reynolds number R is sufficiently large so that the viscosity does not play a leading-order role in the development of the disturbance. Viscous effects will be considered in §5, where the Reynolds number is scaled in such a way that the viscous diffusion term appears at leading-order in the critical-layer equations.

3.2 Outer Expansion

3.2.1 Asymptotic Solutions Near Critical Levels

Outside the critical layers, the flow is basically linear and is governed, to the order of approximation in this study, by the following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 , \quad (3.2)$$

$$2R^{-1} \frac{\partial u}{\partial \tau} + \bar{U} \frac{\partial u}{\partial x} + v \frac{\partial \bar{U}}{\partial y} = - \frac{\partial p}{\partial x} , \quad (3.3)$$

$$2R^{-1} \frac{\partial v}{\partial \tau} + \bar{U} \frac{\partial v}{\partial x} = - \frac{\partial p}{\partial y} , \quad (3.4)$$

$$2R^{-1} \frac{\partial w}{\partial \tau} + \bar{U} \frac{\partial w}{\partial x} = - \frac{\partial p}{\partial z} . \quad (3.5)$$

Elimination of pressure yields:

$$(2R^{-1} \frac{\partial}{\partial \tau} + \bar{U} \frac{\partial}{\partial x}) (\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}) - \frac{\partial \bar{U}}{\partial y} \frac{\partial w}{\partial x} = 0 , \quad (3.6)$$

and

$$(2R^{-1} \frac{\partial}{\partial \tau} + \bar{U} \frac{\partial}{\partial x}) \nabla^2 v - \bar{U}_{yy} \frac{\partial v}{\partial x} = 0 . \quad (3.7)$$

With the multiple time scale introduced, we have the following relation

$$\frac{\partial}{\partial \tau} \rightarrow R \frac{\partial}{\partial t} + R \frac{1}{2} \epsilon^{\frac{1}{3}} \frac{\partial}{\partial t_1} + \epsilon^{-\frac{1}{3}} \frac{\partial}{\partial \tau_1} + \frac{\partial}{\partial \tau_0} .$$

The velocity (u, v, w) and the pressure p of the disturbances are expanded as follows:

$$u = \epsilon u_1 + \epsilon^{\frac{4}{3}} u_2 + \dots , \quad (3.8)$$

$$v = \epsilon v_1 + \epsilon^{\frac{4}{3}} v_2 + \dots , \quad (3.9)$$

$$w = \epsilon w_1 + \epsilon^{\frac{4}{3}} w_2 + \dots , \quad (3.10)$$

$$p = \epsilon p_1 + \epsilon^{\frac{4}{3}} p_2 + \dots . \quad (3.11)$$

The “early time” linear solution suggests that v_1 can be written as

$$v_1 = A(t_1) \bar{v}_1(y) \cos \beta z E + c.c. , \quad (3.12)$$

where the function $A(t_1)$ is the amplitude of the disturbance, $E = \exp(i\alpha x - i\hat{\theta}(t))$, and

$$\frac{d\hat{\theta}}{dt} = \frac{1}{2} \alpha c(\tau_0) + \frac{1}{2} R^{-\frac{1}{2}} \Omega_1(\tau_0) + \dots . \quad (3.13)$$

Here for simplicity, we have assumed that the two oblique waves are of equal amplitude. We note, however that it is straightforward to extend the analysis to the un-equal-amplitude case. As in Chapter two, the dependence on the slow time scale τ_1 and τ_0 is parametric and will not be written out explicitly. The function $\bar{v}_1$ satisfies the Rayleigh equation:

$$(\bar{U} - c)(D^2 - \bar{\alpha}^2)\bar{v}_1 - \bar{U}_{yy}\bar{v}_1 = 0 , \quad (3.14)$$

where we have written

$$2\Omega_0 = \alpha c(\tau_0) , \quad \bar{\alpha} = (\alpha^2 + \beta^2)^{\frac{1}{2}} .$$

Let $\eta = y - y_c^j$, where y_c^j is the j -th critical level, then as $\eta \rightarrow \pm 0$, $\bar{v}_1$ has the following asymptotic behaviour:

$$\bar{v}_1 \sim a_j^\pm \phi_a + b_j^\pm [\phi_b + p_j \phi_a \log |\eta|] , \quad (3.15)$$

where

$$\begin{aligned} \phi_a &= \eta + \frac{1}{2}p_j\eta^2 + \dots , \\ \phi_b &= 1 + q_j\eta^2 + \dots . \end{aligned}$$

The function v_2 takes the following form:

$$v_2 = \bar{v}_2(y, t_1, \tau)E \cos \beta z + \bar{v}_2^{(0,2)} \cos 2\beta z + c.c + \dots , \quad (3.16)$$

where a vortex component $\bar{v}_2^{(0,2)} \cos 2\beta z$ has to be included in order to match the inner solution. The function $\bar{v}_2$ is the deviation of the eigenfunction from its neutral state, and satisfies the inhomogeneous Rayleigh equation:

$$[D^2 - (\bar{\alpha}^2 + \frac{\bar{U}_{yy}}{\bar{U} - c})]\bar{v}_2 = (i\alpha)^{-1} \{ [-\frac{dA}{dt_1} - (i\alpha\bar{U}_\tau\tau_1)A] \frac{\bar{U}_{yy}}{(\bar{U} - c)^2} + \frac{i\alpha\bar{U}_{yy\tau}\tau_1 A}{\bar{U} - c} \} \bar{v}_1 . \quad (3.17)$$

The asymptotic behaviour of $\bar{v}_2$ is as follows:

$$\begin{aligned} \bar{v}_2 \sim & -b_j^\pm r_j \log |\eta| + (a_j^\pm r_j + b_j^\pm s_j) \eta \log |\eta| + \dots \\ & + c_j^\pm \phi_a + d_j^\pm [\phi_b + p_j \phi_a \log |\eta|] , \end{aligned} \quad (3.18)$$

where

$$p_j = \frac{\bar{U}_{yy}}{\bar{U}_y} , \quad (3.19)$$

$$q_j = \frac{1}{2}\bar{\alpha}^2 + \frac{1}{2}\frac{\bar{U}_{yyy}}{\bar{U}_y} - \frac{\bar{U}_{yy}^2}{\bar{U}_y^2} , \quad (3.20)$$

$$\tau_j = (i\alpha)^{-1} \frac{\bar{U}_{yy}}{\bar{U}_y^2} [-\frac{dA}{dt_1} - (i\alpha\bar{U}_\tau\tau_1)A] , \quad (3.21)$$

$$\begin{aligned} s_j = & (i\alpha)^{-1} \{ (-i\alpha\bar{U}_{y\tau}\tau_1 A) \frac{\bar{U}_{yy}}{\bar{U}_y^2} + (i\alpha\tau_1 A) \frac{\bar{U}_{yy\tau}}{\bar{U}_y} \\ & + (-\frac{dA}{dt_1} - i\alpha\bar{U}_\tau\tau_1 A) \frac{\bar{U}_y\bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y^3} \} ; \end{aligned} \quad (3.22)$$

all the basic-flow quantities are evaluated at time τ_0 and at the critical level y_c^j . The jumps $(a_j^+ - a_j^-)$ etc. will be determined by analyzing the critical layers.

According to the continuity equation, we can write $w_1 = A\bar{w}_1 E \sin \beta z + c.c.$, then $\bar{w}_1$ satisfies the equation:

$$\frac{\partial \bar{w}_1}{\partial y} + \frac{\bar{U}_y}{\bar{U} - c} \bar{w}_1 = -\beta \bar{v}_1 ,$$

the solution of which is

$$\bar{w}_1 = \bar{\alpha}^{-1} \sin \theta \left[\frac{\bar{U}_y}{\bar{U} - c} \bar{v}_1 - \bar{v}_{1,y} \right] . \quad (3.23)$$

where

$$\sin \theta = \beta / \bar{\alpha} .$$

The velocity u_1 takes the form:

$$u_1 = A\bar{u}_1 E \cos \beta z + \bar{u}_1^{(0,2)}(y, t_1) \cos 2\beta z + c.c. ,$$

where $\bar{u}_1$ is obtained from the continuity equation as follows:

$$\bar{u}_1 = -(i\alpha)^{-1} \left\{ \left[\frac{\bar{U}_y}{\bar{U} - c} \bar{v}_1 - \bar{v}_{1,y} \right] \sin^2 \theta + \bar{v}_{1,y} \right\} . \quad (3.24)$$

It is noted that in order for u_1 to match with the inner solution, a vortex component $\bar{u}_1^{(0,2)}(y, t_1) \cos 2\beta z$ has to be included at this order. A similar vortex component was encountered at the leading-order expansion in the analysis of Goldstein & Choi (1989). As will be shown later, This vortex is driven by a streamwise slip velocity across the critical layer, which itself is generated by a nonlinear interaction inside critical layer. Note that this vortex is unexpected large in the sense that it has the same magnitude as the fundamental waves.

Similarly, we write the pressure $p_1 = A(t_1)\bar{p}_1 E \cos \beta z + c.c.$, and

$$\bar{p}_1 = i\bar{\alpha}^{-1} \cos \theta [\bar{U}_y \bar{v}_1 - (\bar{U} - c) \bar{v}_{1,y}] . \quad (3.25)$$

As $y \rightarrow y_c^j$, the asymptotic solutions of $\bar{p}_1, \bar{u}_1, \bar{w}_1$ are worked out as follows

$$\bar{p}_1 \sim i\bar{\alpha}^{-1} \bar{U}_y \cos \theta b_j^\pm + \dots , \quad (3.26)$$

$$\bar{u}_1 \sim -(i\alpha)^{-1} \sin^2 \theta b_j^\pm \eta^{-1} + \dots , \quad (3.27)$$

$$\bar{w}_1 \sim \bar{\alpha}^{-1} \sin \theta b_j^\pm \eta^{-1} + \dots , \quad (3.28)$$

where we notice that the singularity in $\bar{u}_1$ appears as a simple pole rather than a logarithmic branch of the two-dimensional case. It is this difference that results in the faster nonlinear evolution time scale in the present case, see Chapter one and Goldstein & Choi (1989).

We now introduce an inner variable:

$$Y = \frac{\eta}{\epsilon^{\frac{1}{3}}} . \quad (3.29)$$

The outer expansions written in inner variable are as follows:

$$\begin{aligned} v \sim & \epsilon b_j^\pm A E \cos \beta z + \epsilon^{\frac{4}{3}} \log \epsilon^{\frac{1}{3}} (-b_j^\pm r_j + b_j^\pm p_j A Y) E \cos \beta z \\ & + \epsilon^{\frac{4}{3}} [(-b_j^\pm r_j \log |Y| + d_j^\pm) + A(a_j^\pm Y + b_j^\pm Y \log |Y|)] E \cos \beta z \\ & + \epsilon^{\frac{5}{3}} \log \epsilon^{\frac{1}{3}} [(a_j^\pm r_j + b_j^\pm s_j + d_j^\pm p_j) Y + \frac{1}{2} A p_j b_j^\pm Y^2] E \cos \beta z \\ & + \epsilon^{\frac{5}{3}} [c_j^\pm Y + (a_j^\pm r_j + b_j^\pm s_j + d_j^\pm p_j) Y \log |Y|] E \cos \beta z + c.c. + \dots , \end{aligned} \quad (3.30)$$

$$u \sim \epsilon^{\frac{2}{3}} (-i\alpha)^{-1} \sin^2 \theta A b_j^\pm Y^{-1} E \cos \beta z + c.c. + \dots , \quad (3.31)$$

$$w \sim \epsilon^{\frac{2}{3}} \bar{\alpha}^{-1} \sin \theta b_j^\pm A Y^{-1} E \cos \beta z + c.c. + \dots , \quad (3.32)$$

$$p \sim \epsilon i \bar{\alpha}^{-1} \bar{U}_y \cos \theta A b_j^\pm E \cos \beta z + c.c. + \dots . \quad (3.33)$$

3.2.2 Solvability Condition

Multiplying both sides of (3.17) by $\bar{v}_1$, integrating from 0 to $+\infty$, and utilising the asymptotic solutions (3.15) and (3.18), we obtain the solvability condition for (3.17):

$$\begin{aligned} i\alpha^{-1} J_1 \frac{dA}{dt_1} + J_2 \tau_1 A = & - \sum_j \{ (b_j^+ c_j^+ - b_j^- c_j^-) - r_j (b_j^+ a_j^+ - b_j^- a_j^-) \\ & - p_j (b_j^+ d_j^+ - b_j^- d_j^-) - (a_j^+ d_j^+ - a_j^- d_j^-) \} , \end{aligned} \quad (3.34)$$

where the summation is made over all critical layers, and J_1 and J_2 are constants defined by the following integrals respectively

$$J_1 = \int_0^{+\infty} \frac{\bar{U}_{yy}}{(\bar{U} - c)^2} \bar{v}_1^2 dy , \quad (3.35)$$

$$J_2 = \int_0^{+\infty} \left[-\frac{\bar{U}_{yy} \bar{U}_\tau}{(\bar{U} - c)^2} + \frac{\bar{U}_{yy\tau}}{(\bar{U} - c)} \right] \bar{v}_1^2 dy . \quad (3.36)$$

Note that these integrals are singular and should be interpreted in the sense of Hadamard.

As in the previous chapter, after the jumps $(a_j^+ - a_j^-)$, etc. are determined in the next section, the amplitude equation can be derived from (3.34). Note again that nonlinear effects influence the evolution by altering these jumps; thus for the purpose of deriving the amplitude equation, we only need to consider those parts of the inner solutions contributing to these jumps (see below).

3.3 Inner Expansion

Equations (3.30)-(3.33) suggest that the inner expansions within the j -th critical layer take the following form:

$$u = \epsilon^{\frac{2}{3}} U_1 + \epsilon^{\frac{3}{3}} U_2 + \epsilon^{\frac{4}{3}} U_3 + \dots , \quad (3.37)$$

$$v = \epsilon^{\frac{3}{3}} V_1 + \epsilon^{\frac{4}{3}} V_2 + \epsilon^{\frac{5}{3}} V_3 + \dots , \quad (3.38)$$

$$w = \epsilon^{\frac{2}{3}} W_1 + \epsilon^{\frac{3}{3}} W_2 + \epsilon^{\frac{4}{3}} W_3 + \dots , \quad (3.39)$$

$$p = \epsilon^{\frac{3}{3}} P_1 + \epsilon^{\frac{4}{3}} P_2 + \epsilon^{\frac{5}{3}} P_3 + \dots . \quad (3.40)$$

where so far as deriving amplitude equation is concerned, the $\epsilon^n \log \epsilon^{\frac{1}{3}}$ terms do not need to be explicitly included since they are passive in the sense that they match onto the outer solutions automatically whenever the solutions at ϵ^n match with their corresponding outer expansions, i.e. the matching at $\epsilon^n \log \epsilon^{\frac{1}{3}}$ order does not yield additional jump condition. These terms, however, must be included if a comparison between theory and experiments is to be made.

The function V_1 satisfies the equation:

$$L_0 \frac{\partial^2 V_1}{\partial Y^2} = 0 , \quad (3.41)$$

where

$$L_0 = \frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x} . \quad (3.42)$$

The solution which matches the outer expansion is:

$$V_1 = \hat{A}(t_1) E \cos \beta z + c.c. , \quad (3.43)$$

where $\hat{A} = b_j A$, and $b_j^+ = b_j^- = b_j$, i.e. the jump $(b_j^+ - b_j^-)$ is zero.

The expansion of the y -momentum equation gives:

$$\frac{\partial P_1}{\partial Y} = 0 , \quad (3.44)$$

and so $P_1 = i\bar{\alpha}^{-1} \bar{U}_y \cos \theta \hat{A} E \cos \beta z + c.c. .$

The function W_1 satisfies the following equation

$$L_0 W_1 = -\frac{\partial P_1}{\partial z} . \quad (3.45)$$

Let $W_1 = \hat{W}_1 E \sin \beta z + c.c.$, then $\hat{W}$ satisfies:

$$\hat{L}_0^{(1)} \hat{W}_1 = i \bar{U}_y \sin \theta \cos \theta \hat{A} ,$$

where

$$\hat{L}_0^{(n)} = \frac{\partial}{\partial t_1} + n i \alpha (\bar{U}_y Y + \bar{U}_\tau \tau_1) .$$

The solution that matches with the outer solution is

$$\hat{W}_1 = i\bar{U}_y \sin \theta \cos \theta \hat{W}_0^{(0)} ,$$

where

$$\hat{W}_0^{(n)} = \int_0^{+\infty} \xi^n \hat{A}(t_1 - \xi) e^{-i\Omega \xi} d\xi , \quad \Omega = \alpha(\bar{U}_y Y + \bar{U}_\tau \tau_1) . \quad (3.46)$$

Similarly, we can write $U_1 = \hat{U}_1 E \cos \beta z + c.c.$, and then the continuity equation yields

$$\hat{U}_1 = -\bar{U}_y \sin^2 \theta \hat{W}_0^{(0)} .$$

At $O(\epsilon^{\frac{4}{3}})$, V_2 satisfies

$$L_0 V_{2,YY} = L_1 V_1 + \frac{\partial}{\partial Y} \left[\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{31}}{\partial z} \right] , \quad (3.47)$$

where

$$L_1 = -\left(\frac{1}{2} \bar{U}_{yy} Y^2 + \bar{U}_{y\tau} \tau_1 Y + \frac{1}{2} \bar{U}_{\tau\tau} \tau_1^2 \right) \frac{\partial^3}{\partial x \partial Y^2} + \bar{U}_{yy} \frac{\partial}{\partial x} , \quad (3.48)$$

and S_{11} , S_{31} are Reynolds stresses defined as follows

$$S_{11} = \frac{\partial U_1^2}{\partial x} + \frac{\partial U_1 V_1}{\partial Y} + \frac{\partial U_1 W_1}{\partial z} , \quad (3.49)$$

$$S_{31} = \frac{\partial U_1 W_1}{\partial x} + \frac{\partial V_1 W_1}{\partial Y} + \frac{\partial W_1^2}{\partial z} . \quad (3.50)$$

Furthermore, S_{11} and S_{31} can be expressed as

$$\begin{aligned} S_{11} &= S_{11}^{(0,0)} + S_{11}^{(0,2)} \cos 2\beta z + S_{11}^{(2,0)} E^2 + S_{11}^{(2,2)} E^2 \cos 2\beta z + c.c. , \\ S_{31} &= S_{31}^{(0,2)} \sin 2\beta z + S_{31}^{(2,2)} E^2 \sin 2\beta z + c.c. . \end{aligned}$$

After some calculation, we find that

$$S_{11}^{(0,0)} = \frac{1}{2} i \alpha \bar{U}_y^2 \sin^2 \theta \hat{A}^* \hat{W}_0^{(1)} , \quad (3.51)$$

$$S_{11}^{(0,2)} = \frac{1}{2} i \alpha \bar{U}_y^2 \sin^2 \theta \hat{A}^* \hat{W}_0^{(1)} , \quad (3.52)$$

$$S_{11}^{(2,0)} = \frac{1}{2} i \alpha \bar{U}_y^2 \sin^2 \theta [\hat{A} \hat{W}_0^{(1)} + 2 \sin^2 \theta \hat{W}_0^{(0)} \hat{W}_0^{(0)}] , \quad (3.53)$$

$$S_{11}^{(2,2)} = \frac{1}{2} i \alpha \bar{U}_y^2 \sin^2 \theta \hat{A} \hat{W}_0^{(1)} , \quad (3.54)$$

$$S_{31}^{(0,2)} = \frac{1}{2} \beta \bar{U}_y^2 \cos^2 \theta [\hat{A}^* \hat{W}_0^{(1)} + 2 \sin^2 \theta \hat{W}_0^{(0)} \hat{W}_0^{*(0)}] , \quad (3.55)$$

$$S_{31}^{(2,2)} = \frac{1}{2} \beta \bar{U}_y^2 \cos^2 \theta \hat{A} \hat{W}_0^{(1)} . \quad (3.56)$$

Inspecting the right-hand side of (3.47), we know V_2 has the solution of the following form

$$V_2 = \hat{V}_2^{(1)} E \cos \beta z + \hat{V}_2^{(0,2)} \cos 2\beta z + \hat{V}_2^{(2,0)} E^2 + c.c. \quad (3.57)$$

The fundamental component $\hat{V}_2^{(1)}$ is only driven by the linear forcing term, i.e. $L_1 V_1 = i\alpha \bar{U}_{yy} \hat{A}$, which is exactly the same as in the two-dimensional case studied in Chapter two. By this analogy, we can obtain the following jump conditions:

$$a_j^+ - a_j^- = \pi i p_j b_j \text{Sgn}(\bar{U}_y) , \quad (3.58)$$

$$d_j^+ - d_j^- = -\pi i r_j b_j \text{Sgn}(\bar{U}_y) . \quad (3.59)$$

Substituting (3.51) to (3.56) into (3.47), we find that

$$\frac{\partial}{\partial t_1} \hat{V}_{2,Y}^{(0,2)} = -i\bar{S}^3 \sin^2 \theta [\hat{A}^* \hat{W}_0^{(2)} + 4 \sin^2 \theta \hat{W}_0^{*(0)} \hat{W}_0^{(1)}] , \quad (3.60)$$

$$\hat{L}_0^{(2)} \hat{V}_{2,Y}^{(2,0)} = i\bar{S}^3 \sin^2 \theta [\hat{A} \hat{W}_0^{(2)} + 4 \sin^2 \theta \hat{W}_0^{(0)} \hat{W}_0^{(1)}] , \quad (3.61)$$

where

$$\bar{S} = \alpha \bar{U}_y .$$

After inserting (3.46) into (3.60), and integrating (3.60), we find the solution

$$\hat{V}_{2,Y}^{(0,2)} = -i\bar{S}^3 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta \xi (\xi + 4 \sin^2 \theta \eta) \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega \xi} . \quad (3.62)$$

In simplifying the solution into the present compact form, we have effectively dropped a purely imaginary part, which, according to (3.57) does not alter the physical velocity. Similar operation will be made in solving for $\hat{W}^{(0,2)}$ (see below).

Similarly, we substitute (3.61) into (3.61). Integration of (3.61) yields

$$\hat{V}_{2,Y}^{(2,0)} = i\bar{S}^3 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta [\xi^2 + 4 \sin^2 \theta (\xi + \eta) \eta] \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega(\xi + 2\eta)} . \quad (3.63)$$

Integrating $\hat{V}_{2,Y}^{(2,0)}$ once with respect to Y yields

$$\begin{aligned} \hat{V}_{2,Y}^{(2,0)} = & -\bar{S}^2 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta [\xi^2 + 4 \sin^2 \theta (\xi + \eta) \eta] (\xi + 2\eta)^{-1} [e^{-i\bar{\Omega}(\xi + 2\eta)} - 1] \\ & \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\bar{\omega}(\xi + 2\eta)} , \end{aligned} \quad (3.64)$$

where we have put

$$\bar{\Omega} = \alpha \bar{U}_y Y , \quad \bar{\omega} = \alpha \bar{U}_\tau \tau_1 ,$$

and the integral constant (or more precisely a function of t_1) is taken to be zero by matching with the corresponding outer solution.

Instead of integrating $\hat{V}_{2,Y}^{(2,0)}$ to get $\hat{V}_2^{(2,0)}$, we obtain it from the x -momentum equation together with the continuity equation in the following way.

The continuity equation is

$$2i\alpha\hat{U}_2^{(2,0)} + \hat{V}_{2,Y}^{(2,0)} = 0 , \quad (3.65)$$

and the x -momentum equation is

$$\hat{L}_0^{(2)}\hat{U}_2^{(2,0)} + \bar{U}_y\hat{V}_2^{(2,0)} = -S_{11}^{(2,0)} . \quad (3.66)$$

From the above two equations, we obtain

$$\hat{V}_2^{(2,0)} = -\frac{1}{2}i\bar{S}^{-1}\hat{L}_0^{(2)}\hat{V}_{2,Y}^{(2,0)} - \bar{U}_y^{-1}S_{11} . \quad (3.67)$$

It turns out that the above expression for $\hat{V}_2^{(2,0)}$ is very helpful in evaluating the asymptotic behaviour of the related solution of the next order.

The function W_2 satisfies the following equation

$$L_0W_2 = -\frac{\partial P_2}{\partial z} - F_2(Y)\frac{\partial W_1}{\partial x} - S_{31} , \quad (3.68)$$

where

$$F_2(Y) = \frac{1}{2}\bar{U}_{yy}Y^2 + \bar{U}_{y\tau}\tau_1Y + \frac{1}{2}\bar{U}_{\tau\tau}\tau_1^2 ,$$

The velocity W_2 has the form

$$W_2 = \hat{W}_2^{(1)}E \sin \beta z + \hat{W}_2^{(0,2)} \sin 2\beta z + \hat{W}_2^{(2,2)}E^2 \sin 2\beta z + c.c. . \quad (3.69)$$

Since at next order, we are only interested in the interaction that can generate the fundamental, $\hat{W}_2^{(1)}$ does not need to be worked out. The solutions forced by the nonlinear interaction are $\hat{W}_2^{(0,2)}$ and $\hat{W}_2^{(2,2)}$, which are found to satisfy the following equations respectively

$$\frac{\partial}{\partial t_1}\hat{W}_2^{(0,2)} = -S_{31}^{(0,2)} , \quad (3.70)$$

$$\hat{L}_0^{(2)}\hat{W}_2^{(2,2)} = -S_{31}^{(2,2)} . \quad (3.71)$$

The solutions are

$$\hat{W}_2^{(0,2)} = -\frac{1}{2}\beta^{-1}\bar{S}^2 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta (\xi + 4 \sin^2 \theta \eta) \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega\xi} , \quad (3.72)$$

$$\hat{W}_2^{(2,2)} = -\frac{1}{2}\beta^{-1}\bar{S}^2 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta \xi \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega(\xi+2\eta)} . \quad (3.73)$$

From the continuity equation, $\hat{V}_{2,Y}^{(0,2)} + 2\beta\hat{W}_2^{(0,2)} = 0$, we have

$$\hat{V}_{2,Y}^{(0,2)} = \bar{S}^2 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta (\xi + 4 \sin^2 \theta \eta) \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega\xi} . \quad (3.74)$$

Integration with respect to Y yields

$$\begin{aligned} \hat{V}_2^{(0,2)} = & i\bar{S} \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta \xi^{-1} (\xi + 4 \sin^2 \theta \eta) [e^{-i\bar{\Omega}\xi} - 1] \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\bar{\omega}\xi} \\ & + \hat{V}_2^{(0,2)}(0, t_1) . \end{aligned} \quad (3.75)$$

The function $\hat{V}_2^{(0,2)}(0, t_1)$ is unimportant as far as deriving the amplitude equation is concerned.

As $Y \rightarrow \pm\infty$,

$$\begin{aligned} (\hat{V}_2^{(0,2)} + c.c.) \sim & \pm 8\bar{S} \sin^4 \theta \pi \int_0^{+\infty} \eta |\hat{A}(t_1 - \eta)|^2 d\eta \\ & + \{\text{order-one continuous terms}\} + o(1) . \end{aligned}$$

It is seen here that a vortex component must be included in the $O(\epsilon^{\frac{4}{3}})$ outer expansion of v so that it matches onto $\hat{V}_2^{(0,2)}$.

The function U_2 satisfies

$$L_0 U_{2,Y} = -\bar{U}_{yy} V_1 + F_1(Y) \frac{\partial W_1}{\partial z} - F_2(Y) \frac{\partial^2 U_1}{\partial x \partial Y} - \frac{\partial}{\partial Y} S_{11} + \bar{U}_y \frac{\partial W_2}{\partial z} , \quad (3.76)$$

where

$$F_1(Y) = \bar{U}_{yy} Y + \bar{U}_{Y\tau} \tau_1 .$$

The solution takes the following form

$$U_2 = \hat{U}_2^{(1)} E \cos \beta z + \hat{U}_2^{(0,0)} + \hat{U}_2^{(0,2)} \cos 2\beta z + \hat{U}_2^{(2,0)} E^2 + \hat{U}_2^{(2,2)} E^2 \cos 2\beta z + c.c. . \quad (3.77)$$

Again only the second harmonic and mean flow distortion/vortex induced by non-linearity need to be worked out. It follows from the continuity equation that

$$\hat{U}_2^{(2,0)} = -(2i\alpha)^{-1} \hat{V}_{2,Y}^{(2,0)} , \quad \hat{U}_2^{(2,2)} = i\beta\alpha^{-1} \hat{W}_2^{(2,2)} .$$

The mean-flow distortion and vortex however satisfy the following equations respectively

$$\frac{\partial}{\partial t_1} \hat{U}_{2,Y}^{(0,0)} = -\frac{1}{2} \alpha^{-1} \bar{S}^3 \sin^2 \theta \hat{A}^* \hat{W}_0^{(2)} , \quad (3.78)$$

$$\frac{\partial}{\partial t_1} \hat{U}_{2,Y}^{(0,2)} = -\frac{\partial}{\partial Y} S_{11}^{(0,2)} + 2\beta \bar{U}_y \hat{W}_2^{(0,2)} . \quad (3.79)$$

The solutions are

$$\hat{U}_{2,Y}^{(0,0)} = -\frac{1}{2} \alpha^{-1} \bar{S}^3 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta \xi^2 \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\bar{\Omega}\xi} , \quad (3.80)$$

$$\hat{U}_{2,Y}^{(0,2)} = -\frac{1}{2} \alpha^{-1} \bar{S}^3 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta [\xi^2 + 2\xi\eta + 4 \sin^2 \theta \eta^2] \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\bar{\Omega}\xi} . \quad (3.81)$$

Integration once with respect to Y yields

$$\begin{aligned} \hat{U}_2^{(0,2)} = & -\frac{1}{2}\alpha^{-1}\bar{S}^3\sin^2\theta\int_0^{+\infty}d\xi\int_0^{+\infty}d\eta \quad \xi^{-1}[\xi^2+2\xi\eta+4\sin^2\theta\eta^2][e^{-i\bar{\Omega}\xi}-1] \\ & \hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)e^{-\bar{\omega}\xi} + \hat{U}_2^{(0,2)}(0, t_1) . \end{aligned} \quad (3.82)$$

As $Y \rightarrow \pm\infty$,

$$\begin{aligned} (\hat{U}_2^{(0,2)} + c.c) \sim & \mp 4\alpha^{-1}\bar{S}^2\sin^4\theta\pi\int_0^{+\infty}\eta^2|\hat{A}(t_1-\eta)|^2d\eta \\ & + \{\text{order-one continuous terms}\} + o(1) . \end{aligned}$$

This is in fact a streamwise slip velocity generated by the nonlinear interaction inside the critical layers, to which the outer expansion of u has to match. This is the reason that the leading-order outer expansion of u should contain a vortex component. In order to make our points, we now turn to examine the asymptotic behaviour of $(\hat{W}^{(0,2)} + c.c)$ as $Y \rightarrow \pm\infty$. It is found from (3.72) that

$$\hat{W}^{(0,2)} + c.c = 4\sin^4\theta\beta^{-1}Y^{-2}\int_0^{+\infty}\eta^2|\hat{A}(t_1-\eta)|^2d\eta + o(Y^{-2}) . \quad (3.83)$$

We notice that all vortex components generated by nonlinear interactions are bounded as $Y \rightarrow \pm\infty$, in contrast to the case of Hall & Smith (1991), where unbounded growth of these vortices was found. This is because the critical layers in the present study are unsteady, and the balance is between the inertial term and the nonlinear forcing terms. In their case, however, the critical layer is a steady-viscous one, and the balance is between the viscous diffusion term and the nonlinear forcing terms, e.g. for $\hat{W}^{(0,2)}$, the balancing is between $\hat{W}_{YY}^{(0,2)}$ (in our notation) and a nonlinear forcing, which, as $Y \rightarrow \pm\infty$ decays like Y^{-2} , as in the present case. Therefore $\hat{W}^{(0,2)}$ grows like “ $\text{Log}Y + Y$ ”. In Hall & Smith (1991), this unbounded growth of vortices was shown to be a crucial reason for the small-amplitude three-dimensional disturbances to generate an order-one mean-flow distortion. Unbounded growth of mean-flow distortion was also noted by Churilov & Shukhman (1987) in their study of a two-dimensional disturbance in a stratified shear flow. There they found that a *linear* diffusion layer, situated between the critical layer and the outer layer, had to be introduced to achieve the matching for the mean-flow distortion. It appears to us that the theory of Hall & Smith (1991) (we refer to the part on Rayleigh wave), which did not originate from, but can be related to that of Churilov & Shukhman as follows: for three-dimensional disturbances there exists a scaling such that the diffusion layer spreads into the whole flow field, and becomes *fully nonlinear*; Hall & Smith (1991) showed that, under this scaling an order-one streamwise mean-flow velocity was generated. Note that the nonlinear process addressed by Hall & Smith (1991) occurs at a much slower time/spatial scale compared with that in our study. A close examination of their analysis indicates that their evolution occurs at the time/spatial scale of $O(\epsilon^{-2}R^{-\frac{1}{3}})$, where ϵ is the amplitude of the waves, and R is

the *local* Reynolds number, i.e. the square root of their global Reynolds number. As will be shown in section five of this chapter, when viscosity effects has included, our final amplitude equation describes the evolution occurring at the time scale of $O(\epsilon^{-\frac{2}{3}}R^{-\frac{1}{3}})$, much faster than that studied by Hall & Smith (1991). They argue that their theory describes the process taking place at the final fully-fledged transition stage, at which the instability waves sature under nonlinear effects. We feel that it would be nice to be able to embed their theory into the whole evolution picture of disturbance, i.e. to examine whether an earlier linear or weakly nonlinear disturbance eventually enters the stage discussed by them (see below).

For the purpose of deriving the evolution equation, it is sufficient to seek a solution for V_3 only. We find that V_3 satisfies the following equation:

$$L_0 V_{3,Y} = L_1 V_2 + L_2 V_1 + \frac{\partial}{\partial Y} \left[\frac{\partial}{\partial x} S_{12} + \frac{\partial}{\partial z} S_{32} \right], \quad (3.84)$$

where

$$\begin{aligned} L_2 = & -\left[\frac{1}{6} \bar{U}_{yyy} Y^3 + \bar{U}_{yy\tau} \tau_1 Y^2 + \bar{U}_{y\tau\tau} \tau_1^2 Y + \frac{1}{6} \bar{U}_{\tau\tau\tau} \tau_1^3 \right] \frac{\partial^3}{\partial x \partial Y^2} \\ & + [\bar{U}_{yyY} Y + \bar{U}_{yy\tau} \tau_1] \frac{\partial}{\partial x} \\ & - \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x} \right] \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right], \end{aligned} \quad (3.85)$$

and the Reynolds stresses S_{12} and S_{32} are defined as follows

$$S_{12} = \frac{\partial}{\partial x} (2U_1 U_2) + \frac{\partial}{\partial Y} (U_1 V_2 + U_2 V_1) + \frac{\partial}{\partial z} (U_1 W_2 + U_2 W_1), \quad (3.86)$$

$$S_{32} = \frac{\partial}{\partial x} (U_1 W_2 + U_2 W_1) + \frac{\partial}{\partial Y} (W_1 V_2 + W_2 V_1) + \frac{\partial}{\partial z} (2W_1 W_2). \quad (3.87)$$

We only need to find the fundamental component at this order, i.e.

$$V_3 = \hat{V}_3 E \cos \beta z + c.c. + \dots,$$

and correspondingly, we write

$$\frac{\partial}{\partial x} S_{12} + \frac{\partial}{\partial z} S_{32} = \bar{M} E \cos \beta z + c.c. + \dots,$$

with non-fundamental components being neglected.

It is noted that the relevant part of linear forcing term, i.e. $L_1 V_2 + L_2 V_1$, is the same as $F^{(l)}(Y, t_1)$ of Chapter two, although $L_1 V_2$ contains non-fundamental components. So the solution (denoted here by $\hat{V}_3^{(l)}$) driven by it has the following asymptotic behaviour

$$\begin{aligned} \hat{V}_{3,Y}^{(l)} \sim & (a_j^+ p_j + 2q_j b_j + \frac{1}{2} p_j^2 b_j) Y + (a_j^+ r_j + p_j d_j^+ + s_j b_j) \log |Y| \\ & + \{ \pm \frac{1}{2} \pi i \text{Sng}(\bar{U}_y) (a_j^+ r_j + p_j d_j^+ + s_j b_j) + \dots \}. \end{aligned} \quad (3.88)$$

It is found to be convenient to write the nonlinear forcing term $\bar{M}$ as a sum of the following four terms, namely

$$\bar{M} = \bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_0 ,$$

where

$$\bar{M}_1 = i\alpha \hat{A} \hat{U}_{2,Y}^{*(0,0)} + \frac{1}{2} i\alpha \hat{A} \hat{U}_{2,Y}^{*(0,2)} + \frac{1}{2} \beta \hat{A} \hat{W}_{2,Y}^{*(0,2)} , \quad (3.89)$$

$$\bar{M}_2 = i\alpha \hat{U}_{1,Y} \hat{V}_2^{*(0,2)} - 2\alpha^2 \hat{U}_1 \hat{U}_2^{*(0,2)} , \quad (3.90)$$

$$\bar{M}_3 = 2i\alpha \hat{U}_{1,Y}^* \hat{V}_2^{(2,0)} , \quad (3.91)$$

$$\begin{aligned} \bar{M}_0 = & i\alpha \hat{A} \hat{U}_{2,Y}^{(0,0)} + \frac{1}{2} i\alpha \hat{A} \hat{U}_{2,Y}^{(0,2)} + \frac{1}{2} \beta \hat{A} \hat{W}_{2,Y}^{(0,2)} + i\alpha \hat{A}^* \hat{U}_2^{(2,0)} \\ & + \frac{1}{2} \alpha \hat{A}^* \hat{U}_2^{(2,2)} + \frac{1}{2} \beta \hat{A}^* \hat{W}_2^{(2,2)} + i\alpha \hat{U}_{1,Y} \hat{V}_2^{(0,2)} - 2\alpha^2 \hat{U}_1 \hat{U}_2^{(0,2)} . \end{aligned} \quad (3.92)$$

The nonlinear forcing terms are deliberately split-up as above so that the solutions and their asymptotic behaviour can be worked out easily. The solutions driven by $\bar{M}_j$ are denoted by $\hat{V}_3^{(j)}$ ($j = 0, 1, 2, 3$) respectively. It is found that $\hat{V}_3^{(0)}$ makes no contribution to the jump.

The forcing $\bar{M}_1$ represents the Reynolds stress generated by the interaction between the vertical velocity of fundamental wave and the induced vortex. It can be shown that

$$\hat{V}_{3,Y}^{(1)} = \frac{1}{2} \bar{S}^4 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \bar{K}_v^{(1)}(\xi, \eta) \hat{A}(t_1 - \zeta) \hat{A}(t_1 - \eta - \zeta) \hat{A}^*(t_1 - \xi - \eta - \zeta) e^{-i\Omega(\zeta - \xi)} d\xi d\eta d\zeta , \quad (3.93)$$

where

$$\bar{K}_v^{(1)}(\xi, \eta) = 2\xi^3 + \xi^2 \eta + 2 \sin^2 \theta (\xi^2 \eta + \xi \eta^2) . \quad (3.94)$$

Integrating with respect to Y from $-\infty$ to $+\infty$, we get the jump

$$\hat{V}_{3,Y}^{(1)}(+\infty) - \hat{V}_{3,Y}^{(1)}(-\infty) = \bar{S}^4 j_0 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \bar{K}_v^{(1)}(\xi, \eta) \hat{A}(t_1 - \xi) \hat{A}(t_1 - \eta - \xi) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta , \quad (3.95)$$

where we have put

$$j_0 = \pi |\bar{S}|^{-1} .$$

The forcing $\bar{M}_2$ represents the interactions between the streamwise velocity of the fundamental wave and the induced vortex flow. The solution driven by $\frac{\partial \bar{M}_2}{\partial Y}$ needs more effort. After some algebraic manipulation, we find

$$\begin{aligned} \frac{\partial \bar{M}_2}{\partial Y} = & \bar{S}^4 \sin^4 \theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} [\xi^2 + \xi \zeta + 2\xi \eta + 4 \sin^2 \theta (\eta + \zeta) \eta] [(\zeta - \xi) e^{-i\bar{\Omega}(\zeta - \xi)} - \zeta e^{-i\bar{\Omega}\zeta}] \\ & \hat{A}(t_1 - \zeta) \hat{A}(t_1 - \eta) \hat{A}^*(t_1 - \eta - \xi) e^{-i\bar{\omega}(\zeta - \xi)} d\xi d\eta d\zeta . \end{aligned} \quad (3.96)$$

The solution induced by it can be expressed as

$$\hat{V}_{3,Y}^{(2)} = \bar{S}^4 \sin^4 \theta J , \quad (3.97)$$

where

$$J = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} [\xi^2 + \xi\zeta + 2\xi\eta + 4\sin^2\theta(\eta + \zeta)\eta] [(\zeta - \xi)e^{-i\bar{\Omega}(\zeta + \nu - \xi)} - \zeta e^{-i\bar{\Omega}(\zeta + \nu)}] \\ \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \eta) \hat{A}^*(t_1 - \nu - \eta - \xi) e^{-i\bar{\omega}(\zeta + \nu - \xi)} d\xi d\eta d\zeta d\nu .$$

The “singular-like” factor ξ^{-1} in the kernel makes it difficult to derive the asymptotic behaviour of $\hat{V}_{3,Y}^{(2)}$. To overcome this difficulty, we split J into three parts, namely J_1, J_ν, J_0 as follows

$$J = J_1 + 4\sin^2\theta J_\nu - J_0 , \quad (3.98)$$

where

$$J_1 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [\zeta^2 - \xi^2 + 2\eta\zeta - 2\xi\eta - 4\sin^2\theta(\eta + \zeta)\zeta] e^{-i\bar{\Omega}(\zeta + \nu - \xi)} \\ \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \eta) \hat{A}^*(t_1 - \nu - \eta - \xi) d\xi d\eta d\zeta d\nu , \quad (3.99)$$

$$J_\nu = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} (\eta + \zeta) \eta \zeta [e^{-i\bar{\Omega}(\zeta + \nu - \xi)} - e^{-i\bar{\Omega}(\zeta + \nu)}] \\ \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \eta) \hat{A}^*(t_1 - \nu - \eta - \xi) e^{-i\bar{\omega}(\zeta + \nu - \xi)} d\xi d\eta d\zeta d\nu \quad (3.100)$$

$$J_0 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [\xi + 2\eta + \zeta] \zeta e^{-i\bar{\Omega}(\zeta + \nu)} \\ \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \eta) \hat{A}^*(t_1 - \nu - \eta - \xi) e^{i\bar{\omega}\xi} d\xi d\eta d\zeta d\nu .$$

The integral J_0 in fact does not contribute to the jump we need and hereafter will not be discussed.

Integration of J_1 with respect to Y from $-\infty$ to ∞ yields

$$\int_{-\infty}^{+\infty} J_1(Y) dY = 2j_0 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [\zeta^2 - (\zeta + \nu)^2 + 2\eta\zeta - 2(\zeta + \nu)\eta - 4\sin^2\theta(\eta + \zeta)\eta] \\ \hat{A}(t_1 - \nu - \eta) \hat{A}(t_1 - \nu - \zeta) \hat{A}^*(t_1 - 2\nu - \eta - \zeta) d\eta d\zeta d\nu$$

With the right-hand side simplified, the above equation can be written as follows

$$\int_{-\infty}^{+\infty} J_1(Y) dY = -2j_0 \int_0^{+\infty} \int_0^{+\infty} [2\xi^2(\xi + \eta) + 4\sin^2\theta(\frac{4}{3}\xi^3 + 2\xi^2\eta + \eta^2\xi)] \\ \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta \quad (3.101)$$

The integral J_ν can be reduced to a triple-integral

$$J_\nu = \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} \eta^2 \zeta^2 [e^{-i\bar{\Omega}(\eta - \xi)} - e^{-i\bar{\Omega}\eta}] \\ \hat{A}(t_1 - \zeta) \hat{A}(t_1 - \eta) \hat{A}^*(t_1 - \zeta - \xi) e^{-i\bar{\omega}(\eta - \xi)} d\xi d\eta d\zeta . \quad (3.102)$$

It will be seen that the factor η in the kernel brought in by combining two Reynolds stress terms is crucial for the success of the following procedure of analyzing the

asymptote of $\int_{-Y}^{+Y} J_v Y dY$ as $Y \rightarrow +\infty$. Integrating with respect to Y from $-Y$ to Y , we have

$$\int_{-Y}^{+Y} J_v(Y) dY = \bar{S}^{-1} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} \eta^2 \zeta^2 \left[\frac{\sin \bar{\Omega}(\eta - \xi)}{\eta - \xi} - \frac{\sin \bar{\Omega}(\eta)}{\eta} \right] \hat{A}(t_1 - \zeta) \hat{A}(t_1 - \eta) \hat{A}^*(t_1 - \zeta - \xi) e^{-i\bar{\omega}(\eta - \xi)} d\xi d\eta d\zeta .$$

In order to identify the main contribution to the above integral as $\bar{\Omega} \rightarrow \pm\infty$, we write the the kernel into the following form

$$\begin{aligned} & \xi^{-1} \eta^2 \zeta^2 \left[\frac{\sin \bar{\Omega}(\eta - \xi)}{\eta - \xi} - \frac{\sin \bar{\Omega}(\eta)}{\eta} \right] \\ &= \eta \zeta^2 \frac{\sin \bar{\Omega}(\eta - \xi)}{\eta - \xi} - \eta \zeta^2 \cos \bar{\Omega} \eta \frac{\sin \bar{\Omega} \xi}{\xi} - 2\eta \zeta^2 \sin \bar{\Omega} \eta \frac{\sin^2 \frac{1}{2} \bar{\Omega} \xi}{\xi} . \end{aligned} \quad (3.103)$$

The first term represents the contribution from $\xi = \eta$, the second and the third represent the contribution from $\xi = \eta = 0$. After the kernel is so split and factorized, the asymptote of the multiple integral becomes essentially the asymptote of several single integrals. By utilising a Mellin transformation method (Bleistein & Handelsman, 1990), we find that only the first term contributes an $O(1)$ jump, namely

$$\int_{-\infty}^{+\infty} J_v(Y) dY = j_0 \int_0^{+\infty} \int_0^{+\infty} \xi \zeta^2 \hat{A}(t_1 - \xi) \hat{A}(t_1 - \zeta) \hat{A}^*(t_1 - \xi - \zeta) d\xi d\zeta .$$

After some transformations, this can be rewritten as follows

$$\begin{aligned} \int_{-\infty}^{+\infty} J_v(Y) dY &= 2j_0 \int_0^{+\infty} \int_0^{+\infty} [\xi^3 + \frac{3}{2}\xi^2\eta + \frac{1}{2}\xi\eta^2] \\ &\quad \hat{A}(t_1 - \xi) \hat{A}(t_1 - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta . \end{aligned} \quad (3.104)$$

The forcing $\frac{\partial \bar{M}_3}{\partial Y}$ is the Reynolds stress generated by the interaction between streamwise velocity of the 3-D fundamental and 2-D harmonic which itself is generated by the interaction between fundamentals. Utilising (3.67), we obtain

$$\frac{\partial \bar{M}_3}{\partial Y} = \alpha \bar{S}^{-1} \hat{U}_{1,Y}^* \bar{L}_0^{(2)} \hat{V}_{2,Y}^{(2,0)} + 2i\alpha \hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} - 2i\alpha \bar{U}_y^{-1} \hat{U}_{1,Y}^* S_{11}^{(2,0)} . \quad (3.105)$$

The solution driven by the first term is

$$\alpha \bar{S}^{-1} \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1 - \tilde{t})} \hat{U}_{1,Y}^*(\tilde{t}) \left[\frac{\partial}{\partial \tilde{t}} + 2i\Omega \right] \hat{V}_{2,Y}^{(2,0)}(\tilde{t}) ,$$

which, after integration by parts, can be simplified as follows:

$$\alpha \bar{S}^{-1} \hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} - 2i\alpha \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1 - \tilde{t})} \hat{U}_{1,Y}^*(\tilde{t}) \hat{V}_{2,Y}^{(2,0)}(\tilde{t}) .$$

The solution driven by last two terms in (3.105) is

$$2i\alpha \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1-\tilde{t})} \hat{U}_{1,Y}^*(\tilde{t}) \hat{V}_{2,Y}^{(2,0)}(\tilde{t}) - 2i\alpha \bar{U}_y^{-1} \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1-\tilde{t})} \hat{U}_{1,Y}^*(\tilde{t}) S_{11}^{(2,0)}(\tilde{t}) .$$

So the solution driven by $\bar{M}_{3,Y}$ is

$$\hat{V}_{3,Y}^{(3)} = -\frac{1}{2} i \bar{S}^{-1} \hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} - 2i\alpha \bar{U}_y^{-1} \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1-\tilde{t})} \hat{U}_{1,Y}^*(\tilde{t}) S_{11}^{(2,0)}(\tilde{t}) . \quad (3.106)$$

For brevity, we write $\hat{V}_{3,Y}^{(3)}$ as

$$\hat{V}_{3,Y}^{(3)} = \bar{S}^4 \sin^4 \theta I_v + \bar{S}^4 \sin^4 \theta I_1 + 2\bar{S}^4 \sin^6 \theta I_2 ,$$

where

$$I_v = \alpha \bar{S}^{-5} \sin^{-4} \theta \hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} , \quad (3.107)$$

$$I_1 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi \zeta^2 e^{-i\Omega(\xi+\nu-\zeta)} \hat{A}(t_1-\nu) \hat{A}(t_1-\nu-\xi) \hat{A}^*(t_1-\nu-\zeta) d\xi d\zeta d\nu , \quad (3.108)$$

$$I_2 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \zeta^2 e^{-i\Omega(\xi+\eta+\nu-\zeta)} \hat{A}(t_1-\nu-\eta) \hat{A}(t_1-\nu-\xi) \hat{A}^*(t_1-\nu-\zeta) d\xi d\eta d\zeta d\nu . \quad (3.109)$$

Substituting of the expressions for $\hat{U}_{1,Y}^*$ and $\hat{V}_{2,Y}^{(2,0)}$ into I_v , we find

$$I_v = - \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \zeta^2 [\xi^2 + 4 \sin^2 \theta (\xi + \eta)(\xi + \eta)] (\xi + 2\eta)^{-1} [e^{-i\bar{\Omega}(\xi+2\eta-\zeta)} - e^{i\bar{\Omega}\zeta}] \hat{A}(t_1-\eta) \hat{A}(t_1-\xi-\eta) \hat{A}^*(t_1-\zeta) e^{-i\bar{\omega}(\xi+2\eta)} d\xi d\eta d\zeta .$$

Then integration with respect to Y yields

$$\begin{aligned} & \int_{-Y}^{+Y} I_v(Y) dY \\ &= -2\bar{S}^{-1} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \zeta^2 [\xi^2 + 4 \sin^2 \theta (\xi + \eta)(\xi + \eta)] (\xi + 2\eta)^{-1} \left[\frac{\sin \bar{\Omega}(\zeta - \xi - 2\eta)}{\zeta - \xi - 2\eta} - \frac{\sin \bar{\Omega}\zeta}{\zeta} \right] \\ & \quad \hat{A}(t_1-\eta) \hat{A}(t_1-\xi-\eta) \hat{A}^*(t_1-\zeta) e^{-i\bar{\omega}(\xi+2\eta)} d\xi d\eta d\zeta . \end{aligned}$$

Here again the kernel is split as follows

$$\begin{aligned} & \zeta^2 [\xi^2 + 4 \sin^2 \theta (\xi + \eta)(\xi + \eta)] (\xi + 2\eta)^{-1} \left[\frac{\sin \bar{\Omega}(\zeta - \xi - 2\eta)}{\zeta - \xi - 2\eta} - \frac{\sin \bar{\Omega}\zeta}{\zeta} \right] \\ &= \zeta [\xi^2 + 4 \sin^2 \theta (\xi + \eta)(\xi + \eta)] \frac{\sin \bar{\Omega}(\zeta - \xi - 2\eta)}{\zeta - \xi - 2\eta} \\ & \quad - \zeta [\xi^2 + 4 \sin^2 \theta (\xi + \eta)(\xi + \eta)] \cos \bar{\Omega}\zeta \frac{\sin \bar{\Omega}(\xi + 2\eta)}{\xi + 2\eta} \\ & \quad - 2\zeta [\xi^2 + 4 \sin^2 \theta (\xi + \eta)(\xi + \eta)] \sin \bar{\Omega}\zeta \frac{\sin^2 \frac{1}{2} \bar{\Omega}(\xi + 2\eta)}{\xi + 2\eta} . \end{aligned} \quad (3.110)$$

After a similar procedure as in treating J_v , we can obtain

$$\int_{-\infty}^{+\infty} I_v(Y) dY = -2j_0 \int_0^{+\infty} \int_0^{+\infty} [\eta^2 + 4 \sin^2 \theta (\xi + \eta) \xi] (2\xi + \eta) \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta \quad (3.111)$$

The jumps related to integrals I_1 and I_2 are straightforward, they are

$$\int_{-\infty}^{+\infty} I_1(Y) dY = 2j_0 \int_0^{+\infty} \int_0^{+\infty} (\xi + \eta)^2 \eta \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta, \quad (3.112)$$

and

$$\int_{-\infty}^{+\infty} I_2(Y) dY = 4j_0 \int_0^{+\infty} \int_0^{+\infty} \left[\frac{7}{3} \xi^3 + 3\xi^2 \eta + \xi \eta^2 \right] \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta. \quad (3.113)$$

Matching $\hat{V}_{3,Y}$ with the outer expansion requires

$$c_j^+ - c_j^- = \hat{V}_{3,Y}(+\infty) - \hat{V}_{3,Y}(-\infty), \quad (3.114)$$

With the jumps so obtained, we have

$$\begin{aligned} c_j^+ - c_j^- = & \pi i \text{Sgn}(\bar{U}_y) (a_j^+ r_j + p_j d_j^+ + s_j b_j) \\ & + \pi \alpha^3 |\bar{U}_y|^3 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} K(\xi, \eta) \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta, \end{aligned} \quad (3.115)$$

where

$$K(\xi, \eta) = (2\xi^3 + \xi^2 \eta) - 2 \sin^2 \theta (2\xi^3 - \xi \eta^2) - 4 \sin^4 \theta (\xi^2 \eta + \xi \eta^2). \quad (3.116)$$

It is noted that $(a_j^+ - a_j^-)$ and $(d_j^+ - d_j^-)$ correspond to the classic $\pm\pi$ phase shift in the outer expansion, while $(c_j^+ - c_j^-)$ is modified by nonlinearity. As will be shown, it is through this modification that nonlinear effects control the evolution of the disturbance.

3.4 Evolution Equation of the Amplitude

3.4.1 Amplitude Equation

Inserting the jumps (3.58), (3.59) and (3.115) into (3.34), we obtained the evolution equation as follows:

$$\frac{dA}{dt_1} = g_0 \tau_1 A + g_1 \int_0^{+\infty} \int_0^{+\infty} K(\xi, \eta) A(t_1 - \xi) A(t_1 - \xi - \eta) A^*(t_1 - 2\xi - \eta) d\xi d\eta, \quad (3.117)$$

where the kernel $K(\xi, \eta)$ is defined by (3.116), and

$$g_0 = f_0/f, \quad (3.118)$$

$$g_j = f_1/f. \quad (3.119)$$

The constants f_0 and f are defined by (2.73) and (2.74) respectively, and

$$f_1 = -\pi\alpha^3 \sin^2\theta \sum_j b_j^2 |b_j|^2 |\bar{U}_y|^3, \quad (3.120)$$

where the summation is made over all critical layers. To match with the earlier linear evolution stage (Goldstein & Choi, 1989), the appropriate asymptotic behaviour is

$$A \rightarrow A_0 e^{g_0 \tau_1 t_1} \quad \text{as} \quad t_1 \rightarrow -\infty, \quad (3.121)$$

Note that, although in the present problem $\bar{U}_{yy}(y_c) \neq 0$, the nonlinear kernel $K(\xi, \eta)$ is the same as that in Goldstein & Choi (1990)¹ where $\bar{U}_{yy}(y_c) = 0$. This is because nonlinear interactions inside the critical layers are only associated with the pole type of singularity in u and w , compared with which, the logarithmic branch-point singularity associated with $\bar{U}_{yy}(y_c) \neq 0$ is weaker.

In order to scale out the A_0 and τ_1 , we follow the idea of Goldstein & Choi (1989), and introduce the rescaled variables:

$$\bar{A} = A e^{i\bar{X}_0} / (-\tau_1)^3, \quad (3.122)$$

$$\bar{t} = -\tau_1 t_1 - x_0, \quad (3.123)$$

where $\bar{X}_0$ and x_0 are chosen so that:

$$g_0 x_0 - i(\bar{X}_0 - \varphi_0) = \log[A_0 / (-\tau_1)^3]. \quad (3.124)$$

Then the evolution equation and the asymptotic behaviour become:

$$\frac{d\bar{A}}{d\bar{t}} = -g_0 \bar{A} + g_1 \int_0^{+\infty} \int_0^{+\infty} K(\xi, \eta) \bar{A}(\bar{t}-\xi) \bar{A}(\bar{t}-\xi-\zeta) \bar{A}^*(\bar{t}-2\xi-\zeta) d\xi d\zeta, \quad (3.125)$$

$$\bar{A} \rightarrow e^{-g_0 \bar{t} + i\varphi_0} \quad \text{as} \quad \bar{t} \rightarrow -\infty. \quad (3.126)$$

Further to scale out φ_0 , we introduce:

$$\tilde{A} = \bar{A} |g_1|^{\frac{1}{2}} |g_0|^{-3}, \quad (3.127)$$

$$\tilde{t} = |g_0| \bar{t} - \tilde{X}_0, \quad (3.128)$$

where φ_0 and $\tilde{X}_0$ are chosen to satisfy the following equation:

$$-\frac{g_0}{|g_0|} \tilde{X}_0 + i\varphi_0 = \log\{|g_1|^{\frac{1}{2}} |g_0|^{-3}\}. \quad (3.129)$$

¹Actually, we obtained the present nonlinear kernel $K(\xi, \eta)$ before noticing the correction of Goldstein & Choi (1990). In checking the difference between (3.116) and that in Goldstein & Choi (1989), we spotted the same errors as corrected by Goldstein & Choi (1990).

Then we obtain :

$$\frac{d\tilde{A}}{d\tilde{t}} = \nu_0 \tilde{A} + \nu_1 \int_0^{+\infty} \int_0^{+\infty} K(\xi, \eta) \tilde{A}(\tilde{t}-\xi) \tilde{A}(\tilde{t}-\xi-\zeta) \tilde{A}^*(\tilde{t}-2\xi-\zeta) d\xi d\zeta , \quad (3.130)$$

$$\tilde{A} \rightarrow e^{\nu_0 \tilde{t}} \quad \text{as} \quad \tilde{t} \rightarrow -\infty , \quad (3.131)$$

where $\nu_0 = -g_0/|g_0|$, and $\nu_1 = g_1/|g_1|$.

3.4.2 Finite-Time Singularity Structure

The singularity structure of the amplitude equation (3.125) was proposed by Goldstein & Choi (1989), where the spatial evolution of disturbances in a free shear layer was studied and solutions were shown to blow up at a *finite distance* downstream. Corresponding to the temporal evolution studied in this Chapter, the solutions blow up within a *finite time*, and the singularity structure is as follows

$$\bar{A} \sim \frac{a_0}{(t_s - \bar{t})^{3+i\sigma}} \quad \text{as} \quad \bar{t} \rightarrow t_s , \quad (3.132)$$

where σ and a_0 are real and complex constants respectively. It is noted that the structure depends only on the order of the polynomial kernel $K(\xi, \eta)$.

As in Goldstein & Choi (1989), substitution of (3.132) into the amplitude equation (3.125) yields

$$(3 + i\sigma) = \bar{g}_1 D_0 |a_0|^2 , \quad (3.133)$$

where

$$D_0 = \int_0^{+\infty} \int_0^{+\infty} K(\xi, \eta) [(1+\xi)(1+\xi+\eta)]^{-(3+i\sigma)} (1+2\xi+\eta)^{-(3-i\sigma)} d\xi d\eta . \quad (3.134)$$

From (3.133), $|a_0|$ and σ can be solved. The singularity time t_s can be determined by the same way as described in Chapter two.

3.5 Viscous Effects

In this section, we consider viscosity effects on the evolution. A typical case is that the viscous diffusion terms appear in the leading-order critical-layer equations. According to the scaling arguments in Chapter one, this happens when $R \sim O(\epsilon^{-1})$. So we write

$$R^{-1} = \lambda \epsilon , \quad (3.135)$$

where the parameter λ is introduced to reflect viscosity effects, and is assumed to be of order one.

3.5.1 Amplitude Equation

In this subsection, we derive the amplitude equation for the viscous case. The outer expansions and solutions are the same as for the inviscid case. The inner expansions take the form of (3.37) to (3.40). The function V_1 satisfies the equation

$$\tilde{L}_0 V_{1YY} = 0 ,$$

where, due to viscous diffusion, the operator $\tilde{L}_0$ is of second order:

$$\tilde{L}_0 = \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x} \right] - \lambda \frac{\partial^2}{\partial Y^2} , \quad (3.136)$$

in contrast to the first-order operator of the inviscid case. This modification is the main difference that viscosity brings in. The solution which matches the outer expansion is:

$$V_1 = \hat{A}(t_1) E_{\wedge}^{c\omega\beta z} + c.c. , \quad (3.137)$$

where $\hat{A} = b_j A$ and $b_j^+ = b_j^- = b_j$, i.e. the jump $(b_j^+ - b_j^-)$ is zero.

The leading-order solution for pressure P_1 is same as in the inviscid case, i.e. $P_1 = i\bar{\alpha}^{-1} \bar{U}_y \cos \theta \hat{A} E \cos \beta z + c.c.$

The function W_1 satisfies the following equation

$$\tilde{L}_0 W_1 = -\frac{\partial P_1}{\partial z} .$$

Let $W_1 = \hat{W}_1 E \sin \beta z + c.c.$, then $\hat{W}$ satisfies:

$$\hat{L}_\lambda^{(1)} \hat{W}_1 = i \bar{U}_y \sin \theta \cos \theta \hat{A} , \quad (3.138)$$

where

$$\hat{L}_\lambda^{(n)} = \frac{\partial}{\partial t_1} + n i \alpha (\bar{U}_y Y + \bar{U}_\tau \tau_1) - \lambda \frac{\partial^2}{\partial Y^2} .$$

As in Chapter two, the equation (3.138) can be solved by a Fourier transformation method, which yields the solution

$$\hat{W}_1 = i \bar{U}_y \sin \theta \cos \theta \hat{W}_\lambda^{(0)} ,$$

where for convenience, we have introduced

$$\hat{W}_\lambda^{(n)} = \int_0^{+\infty} \xi^n \hat{A}(t_1 - \xi) e^{-s_1 \xi^3 - i \Omega \xi} d\xi , \quad s_1 = \frac{1}{3} \lambda \alpha^2 \bar{U}_y^2 . \quad (3.139)$$

Similarly, the leading-order streamwise velocity $\hat{U}_1$ can be written as $U_1 = \hat{U}_1 E \cos \beta z + c.c.$, where

$$\hat{U}_1 = -\bar{U}_y \sin^2 \theta \hat{W}_\lambda^{(0)} .$$

At $O(\epsilon^{\frac{4}{3}})$, V_2 satisfies

$$\tilde{L}_0 V_{2,YY} = L_1 V_1 + \frac{\partial}{\partial Y} \left[\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{31}}{\partial z} \right] + \dots, \quad (3.140)$$

where the forcing terms which do not contribute to the jumps are not written out. The operator L_1 and Reynolds stress S_{11} , S_{31} are defined in the same way as in the inviscid case. The solution to equation (3.140) has the following form:

$$V_2 = \hat{V}_2^{(1)} E \cos \beta z + \hat{V}_2^{(0,2)} \cos 2\beta z + \hat{V}_2^{(2,0)} E^2 + c.c. \quad (3.141)$$

The fundamental component $\hat{V}_2^{(1)}$ is only driven by the linear forcing term, i.e. $L_1 V_1 = i\alpha \bar{U}_{yy} \hat{A}$, which is exactly the same as in the two-dimensional case. Hence by this analogy, we obtain the following jump conditions:

$$a_j^+ - a_j^- = \pi i p_j b_j \text{Sgn}(\bar{U}_y), \quad (3.142)$$

$$d_j^+ - d_j^- = -\pi i r_j b_j \text{Sgn}(\bar{U}_y). \quad (3.143)$$

The functions $\hat{V}_2^{(0,2)}$ and $\hat{V}_2^{(2,0)}$ are found to satisfy the following equations respectively

$$\left\{ \frac{\partial}{\partial t_1} - \lambda \frac{\partial^2}{\partial Y^2} \right\} \hat{V}_{2,YY}^{(0,2)} = -i\bar{S}^3 \sin^2 \theta [\hat{A}^* \hat{W}_\lambda^{(2)} + 4 \sin^2 \theta \hat{W}_\lambda^{*(0)} \hat{W}_\lambda^{(1)}], \quad (3.144)$$

$$\hat{L}_\lambda^{(2)} \hat{V}_{2,YY}^{(2,0)} = i\bar{S}^3 \sin^2 \theta [\hat{A} \hat{W}_\lambda^{(2)} + 4 \sin^2 \theta \hat{W}_\lambda^{(0)} \hat{W}_\lambda^{(1)}]. \quad (3.145)$$

The solutions are found as follows

$$\hat{V}_{2,YY}^{(0,2)} = -i\bar{S}^3 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \xi I_v^{(0,2)}(\xi, \eta) \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega \xi} d\xi d\eta, \quad (3.146)$$

$$\hat{V}_{2,YY}^{(2,0)} = i\bar{S}^3 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} I_v^{(2,0)}(\xi, \eta) \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega(\xi + 2\eta)} d\xi d\eta, \quad (3.147)$$

where we have defined

$$I^{(0)}(\xi, \eta) = e^{-s_1(\xi^3 + 3\xi^2 \eta)}, \quad (3.148)$$

$$I^{(1)}(\xi, \eta) = e^{-s_1(\xi^3 + 3\xi^2 \eta + 6\xi \eta^2 + 4\eta^3)}, \quad (3.149)$$

$$I_v^{(0,2)}(\xi, \eta) = I^{(0)}(\xi, \eta) [\xi + 4 \sin^2 \theta \int_0^\eta e^{-2s_1 \zeta^3 - 3s_1 \xi \zeta^2} d\zeta], \quad (3.150)$$

$$I_v^{(2,0)}(\xi, \eta) = I^{(1)}(\xi, \eta) [\xi^2 + 4 \sin^2 \theta \int_0^\eta (\xi + 2\zeta) e^{2s_1 \zeta^3 + 3s_1 \xi \zeta^2} d\zeta]. \quad (3.151)$$

The function U_2 satisfies

$$\tilde{L}_0 U_{2,Y} = -\bar{U}_{yy} V_1 + F_1(Y) \frac{\partial W_1}{\partial z} - F_2(Y) \frac{\partial^2 U_1}{\partial x \partial Y} - \frac{\partial}{\partial Y} S_{11} + \bar{U}_y \frac{\partial W_2}{\partial z}, \quad (3.152)$$

where $F_1(Y)$ and $F_2(Y)$ are the same as in the inviscid case.

The solution takes the following form

$$U_2 = \hat{U}_2^{(1)} E \cos \beta z + \hat{U}_2^{(0,0)} + \hat{U}_2^{(0,2)} \cos 2\beta z + \hat{U}_2^{(2,0)} E^2 + \hat{U}_2^{(2,2)} E^2 \cos 2\beta z + c.c. \quad (3.153)$$

It is found that we only need to solve for the mean-flow distortion $\hat{U}_2^{(0,0)}$ and vortex flow $\hat{U}_2^{(0,2)}$, which satisfy the following equations respectively

$$\left[\frac{\partial}{\partial t_1} - \lambda \frac{\partial^2}{\partial Y^2} \right] \hat{U}_{2,Y}^{(0,0)} = -\frac{1}{2} \alpha^{-1} \bar{S}^3 \sin^2 \theta \hat{A}^* \hat{W}_\lambda^{(2)}, \quad (3.154)$$

$$\left[\frac{\partial}{\partial t_1} - \lambda \frac{\partial^2}{\partial Y^2} \right] \hat{U}_{2,Y}^{(0,2)} = -\frac{\partial}{\partial Y} \tilde{S}_{11}^{(0,2)} - \bar{U}_y \hat{V}_{2,Y}^{(0,2)}, \quad (3.155)$$

where

$$\tilde{S}_{11}^{(0,2)} = \frac{1}{2} i \alpha \bar{U}_y^2 \sin^2 \theta \hat{A}^* \hat{W}_\lambda^{(1)}. \quad (3.156)$$

The solutions are

$$\hat{U}_{2,Y}^{(0,0)} = -\frac{1}{2} \alpha^{-1} \bar{S}^3 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \xi^2 I^{(0)}(\xi, \eta) \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega\xi} d\xi d\eta, \quad (3.157)$$

$$\hat{U}_{2,Y}^{(0,2)} = -\frac{1}{2} \alpha^{-1} \bar{S}^3 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} I_u^{(0,2)}(\xi, \eta) \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega\xi} d\xi d\eta, \quad (3.158)$$

where we have put

$$I_u^{(0,2)}(\xi, \eta) = I^{(0)}(\xi, \eta) [\xi^2 + 2\xi\eta + 4 \sin^2 \theta \int_0^\eta 2(\eta - \zeta) e^{-2s_1 \zeta^3 - 3s_1 \xi \zeta^2} d\zeta]. \quad (3.159)$$

Integrating once with respect to Y , we find

$$(\hat{U}_2^{(0,2)} + c.c.) \sim -8\alpha^{-1} \bar{S}^2 \sin^4 \theta \pi \int_0^{+\infty} \int_0^\eta (\eta - \zeta) e^{-2s_1 \zeta^3} |\hat{A}(t_1 - \eta)|^2 d\zeta d\eta \\ + \{\text{order-one continuous terms}\} + o(1)$$

as $Y \rightarrow \pm\infty$. As in the inviscid case, the nonlinear interaction inside the critical layers generates a streamwise slip velocity across the critical layers. It is noted that this slip velocity is modified by viscosity.

The function W_2 satisfies the following equation

$$L_0 W_2 = -\frac{\partial P_2}{\partial z} - F_2(Y) \frac{\partial W_1}{\partial x} - S_{31}, \quad (3.160)$$

and W_2 has the form

$$W_2 = \hat{W}_2^{(1)} E \sin \beta z + \hat{W}_2^{(0,2)} \sin 2\beta z + \hat{W}_2^{(2,2)} E^2 \sin 2\beta z + c.c. . \quad (3.161)$$

But as far as deriving the amplitude equation is concerned, it is sufficient to solve for the vortex component $\hat{W}_2^{(0,2)}$ only. It follows from the continuity equation that

$$\hat{W}_2^{(0,2)} = -(2\beta)^{-1} \hat{V}_{2,Y}^{(0,2)} . \quad (3.162)$$

As in the inviscid case, the jump $(c_j^+ - c_j^-)$ has to be determined by solving for V_3 , which is found to satisfy the following equation:

$$\tilde{L}_0 V_{3,Y} = L_1 V_2 + L_2 V_1 + \frac{\partial}{\partial Y} \left[\frac{\partial}{\partial x} S_{12} + \frac{\partial}{\partial z} S_{32} \right] , \quad (3.163)$$

where L_1 , L_2 , and the Reynolds stresses S_{12} and S_{32} are defined in the same way as in the inviscid case. For the purpose of deriving amplitude equation, we only need to find the fundamental component at this order, i.e.

$$V_3 = \hat{V}_3 E \cos \beta z + c.c. + \dots ,$$

and correspondingly, we write

$$\frac{\partial}{\partial x} S_{12} + \frac{\partial}{\partial z} S_{32} = \tilde{M} E \cos \beta z + c.c. + \dots ,$$

with non-fundamental components being neglected.

It is noted that the relevant part of linear forcing term, i.e. $(L_1 V_2 + L_2 V_1)$, is the same as $F^{(l)}(Y, t_1)$ of Chapter two, although $L_1 V_2$ contains non-fundamental components. So the solution (denoted here by $\hat{V}_3^{(l)} E$) driven by it has the following asymptotic behaviour

$$\begin{aligned} \hat{V}_{3,Y}^{(l)} \sim & (a_j^+ p_j + 2q_j b_j + \frac{1}{2} p_j^2 b_j) Y + (a_j^+ r_j + p_j d_j^+ + s_j b_j) \log |Y| \\ & + \{ \pm \frac{1}{2} \pi i \text{Sng}(\bar{U}_y) (a_j^+ r_j + p_j d_j^+ + s_j b_j) + \dots \} . \end{aligned} \quad (3.164)$$

We now write the nonlinear forcing term $\tilde{M}(Y, t_1)$ as follows

$$\tilde{M} = \tilde{M}_1 + \tilde{M}_2 + \tilde{M}_3 ,$$

where $\tilde{M}_j$ ($j = 1, 2, 3$) have the same expressions as $\bar{M}_j$ of the inviscid case respectively, but the solutions evolved in the expressions, e.g. $\hat{U}_{2,Y}^{(0,0)}$ have been modified by viscous effects. We now assume that $\hat{V}_3^{(j)} E$ denotes the solution driven by $\tilde{M}_j$ and $\Gamma_j(k)$ denotes the Fourier transformation of $\hat{V}_{3,Y}^{(j)}$:

$$\Gamma_j(k) = \int_{-\infty}^{+\infty} \hat{V}_{3,Y}^{(j)} e^{-ikY} dY .$$

To obtain the jump, we need to evaluate $\{\hat{V}_{3,Y}^{(j)}(+\infty) - \hat{V}_{3,Y}^{(j)}(-\infty)\}$. It can be shown that

$$\hat{V}_{3,Y}^{(j)}(+\infty) - \hat{V}_{3,Y}^{(j)}(-\infty) = \Gamma_j(0) .$$

Using this relation, we find

$$\begin{aligned} & \hat{V}_{3,Y}^{(1)}(+\infty) - \hat{V}_{3,Y}^{(1)}(-\infty) \\ &= \bar{S}^4 j_0 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \tilde{K}_v^{(1)}(\xi, \eta) \hat{A}(t_1 - \xi) \hat{A}(t_1 - \eta - \xi) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta , \end{aligned} \quad (3.165)$$

where

$$\tilde{K}_v^{(1)}(\xi, \eta) = \tilde{K}^{(0)}(\xi, \eta) \{ 2\xi^3 + \xi^2 \eta + 2 \sin^2 \theta \int_0^\eta [\xi^2 + 2\xi(\eta - \zeta)] e^{-2s_1 \zeta^3 - 3s_1 \xi \zeta^2} d\zeta \} , \quad (3.166)$$

and

$$\tilde{K}^{(0)}(\xi, \eta) = e^{-s_1(2\xi^3 + 3\xi^2 \eta)} . \quad (3.167)$$

We now consider $\hat{V}_{3,Y}^{(2)}$, i.e. the solution driven by $\tilde{M}_2$. As in the inviscid case, there is a difficulty in evaluating the asymptotic form of $\hat{V}_{3,Y}^{(2)}$. Inspired by the experience there, we write

$$\hat{V}_{3,Y}^{(2)} = Q_2(Y, t_1) - i\alpha \bar{U}_y^{-1} \hat{U}_{1,Y} \hat{U}_2^{*(0,2)} ,$$

then $Q_2(Y, t_1)$ satisfies

$$\hat{L}_\lambda^{(1)} Q_2 = \tilde{N}_2(Y, t_1) , \quad (3.168)$$

where

$$\tilde{N}_2 = \tilde{M}_{2,Y} + i\alpha \bar{U}_y^{-1} \hat{L}_\lambda^{(1)} (\hat{U}_{1,Y} \hat{U}_2^{(0,2)}) .$$

Notice that

$$\hat{L}_\lambda^{(1)} \hat{U}_{1,Y} = -i\alpha \bar{U}_y \hat{U}_1 . \quad (3.169)$$

Differentiating with respect to Y , we obtain

$$\hat{L}_\lambda^{(1)} \hat{U}_{1,Y} = -2i\alpha \bar{U}_y \hat{U}_{1,Y} . \quad (3.170)$$

The complex conjugate of (3.155) yields

$$\left[\frac{\partial}{\partial t_1} - \lambda \frac{\partial^2}{\partial Y^2} \right] \hat{U}_{2,Y}^{*(0,2)} = -\frac{\partial}{\partial Y} \tilde{S}_{11}^{*(0,2)} - \bar{U}_y \hat{V}_{2,Y}^{*(0,2)} , \quad (3.171)$$

On making use of (3.170) and (3.171), we find

$$\tilde{N}_2 = i\alpha \hat{U}_{1,Y} \hat{V}_{2,Y}^{*(0,2)} - 2\alpha^2 \hat{U}_1 \hat{U}_{2,Y}^{*(0,2)} - 2i\lambda \alpha \bar{U}_y^{-1} \hat{U}_{1,Y} \hat{U}_{2,Y}^{*(0,2)} - i\alpha \bar{U}_y^{-1} \hat{U}_{1,Y} \tilde{S}_{11}^{*(0,2)} . \quad (3.172)$$

After solving for $Q_2(Y, t_1)$, we obtain

$$\begin{aligned} & \hat{V}_{3,Y}^{(2)}(+\infty) - \hat{V}_{3,Y}^{(2)}(-\infty) \\ &= 2\bar{S}^4 j_0 \sin^4 \theta \int_0^{+\infty} \int_0^{+\infty} \tilde{K}_v^{(2)}(\xi, \eta) \hat{A}(t_1 - \xi) \hat{A}(t_1 - \eta - \xi) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta , \end{aligned} \quad (3.173)$$

where we have put

$$\tilde{K}^{(1)}(\xi, \eta) = e^{-s_1[\xi^3 + \eta^3 + (\xi + \eta)^3]} , \quad (3.174)$$

and

$$\begin{aligned} & \tilde{K}_v^{(2)}(\xi, \eta) \\ &= \left\{ \tilde{K}^{(0)}(\xi, \eta) \int_0^\xi [\zeta(2\eta + 3\zeta) - \xi(\xi + 2\eta + 2\zeta)] e^{-3s_1\xi\zeta^2} d\zeta \right. \\ & \quad + \tilde{K}^{(1)}(\xi, \eta) \int_0^\xi [(\eta + \zeta)(\eta + 3\zeta) - (\xi + \eta)(\xi + \eta + 2\zeta)] e^{-3s_1(\xi + \eta)(2\eta + \zeta)\zeta} d\zeta \left. \right\} \\ & \quad + 4 \sin^2 \theta \left\{ \tilde{K}^{(0)}(\xi, \eta) \int_0^\xi d\zeta e^{-3s_1\xi\zeta^2} \int_0^{\eta + \zeta} (v - \eta - \zeta) [1 + 2\lambda \bar{S}^2(\xi - \zeta)\zeta^2] e^{-s_1(2v^3 + 3\xi v^2)} dv \right. \\ & \quad \left. + \tilde{K}^{(1)}(\xi, \eta) \int_0^\xi d\zeta e^{-3s_1(\xi + \eta)(2\eta + \zeta)\zeta} \int_0^\zeta (v - \eta) [1 + 2\lambda \bar{S}^2(\xi - \zeta)(\eta + \zeta)^2] e^{-s_1[2v^3 + 3(\xi + \eta)v^2]} dv \right\} . \end{aligned} \quad (3.175)$$

Finally, we solve for $\hat{V}_{3,Y}^{(3)}$, i.e. the solution driven by $\tilde{M}_3$. On writing

$$\hat{V}_{3,Y}^{(3)} = Q_3(Y, t_1) + \bar{U}_y^{-1} \hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} , \quad (3.176)$$

then Q_3 satisfies

$$\hat{L}_\lambda^{(1)} Q_3 = \tilde{N}_3(Y, t_1) , \quad (3.177)$$

where

$$\tilde{N}_3 = \tilde{M}_{3,Y} - \bar{U}_y^{-1} \hat{L}_\lambda^{(1)} (\hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)}) .$$

Noticing

$$\hat{L}_\lambda^{(2)} \hat{V}_{2,Y}^{(2,0)} = 2i\alpha \bar{U}_y \hat{V}_2^{(2,0)} + 2i\alpha \tilde{S}_{11}^{(2,0)} ,$$

and using the complex conjugate of equation (3.170), we find

$$\tilde{N}_3 = -2i\alpha \bar{U}_y^{-1} \hat{U}_{1,Y}^* \tilde{S}_{11}^{*(2,0)} + 2\lambda \bar{U}_y^{-1} \hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} , \quad (3.178)$$

where

$$\tilde{S}_{11}^{(2,0)} = \frac{1}{2} i\alpha \bar{U}_y^2 \sin^2 \theta [\hat{A} \hat{W}_\lambda^{(1)} + 2 \sin^2 \theta \hat{W}_\lambda^{(0)} \hat{W}_\lambda^{(0)}] .$$

After finding the solution for Q_3 , it can be shown

$$\begin{aligned} & \hat{V}_{3,Y}^{(3)}(+\infty) - \hat{V}_{3,Y}^{(3)}(-\infty) \\ &= 4\bar{S}^4 j_0 \sin^4 \theta \int_0^{+\infty} \int_0^{+\infty} \tilde{K}_v^{(3)}(\xi, \eta) \hat{A}(t_1 - \xi) \hat{A}(t_1 - \eta - \xi) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta , \end{aligned} \quad (3.179)$$

where

$$\begin{aligned} & \tilde{K}_v^{(3)}(\xi, \eta) \\ &= \tilde{K}^{(1)}(\xi, \eta) \left\{ \int_0^\xi \eta \zeta [1 + 2\lambda \bar{S}^2(\xi + \eta + \zeta)^2(\xi - \zeta)] \Pi_0(\xi, \eta, \zeta) d\zeta \right. \\ & \quad \left. + 4 \sin^2 \theta \int_0^\xi d\zeta \Pi_0(\xi, \eta, \zeta) \int_0^\zeta (\zeta - v) [1 + 2\lambda \bar{S}^2(\xi + \eta + \zeta)^2(\xi - \zeta)] e^{s_1(2v^3 + 3\eta v^2)} dv \right\} , \end{aligned} \quad (3.180)$$

and

$$\Pi_0(\xi, \eta, \zeta) = e^{-s_1(4\zeta^3+6\xi\zeta^2+9\eta\zeta^2+6\xi\eta\zeta+6\eta^2\zeta)}.$$

Matching onto the outer solution yields

$$\begin{aligned} c_j^+ - c_j^- &= \pi i \text{Sgn}(\bar{U}_y)(a_j^+ r_j + p_j d_j^+ + s_j b_j) \\ &\quad + \pi \alpha^3 |\bar{U}_y|^3 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \tilde{K}_j(\xi, \eta|s_1) \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta, \end{aligned} \quad (3.181)$$

where we have written

$$\tilde{K}_j(\xi, \eta|s_1) = K_s(\xi, \eta|s_1) + 4\lambda \bar{S}^2 \sin^2 \theta K_\lambda(\xi, \eta|s_1). \quad (3.182)$$

The suffix j refers to the j -th critical layer, and the argument s_1 refers to the parametric dependence on s_1 . The functions K_s and K_λ are expressed as follows

$$\begin{aligned} K_s(\xi, \eta|s_1) &= \tilde{K}^{(0)}(\xi, \eta)(2\xi^3 + \xi^2\eta) \\ &\quad + 2\sin^2 \theta \left\{ \tilde{K}^{(0)}(\xi, \eta) \int_0^\eta [\xi^2 + 2\xi(\eta - \zeta)] e^{-2s_1\zeta^3 - 3s_1\xi\zeta^2} d\zeta \right. \\ &\quad + \tilde{K}^{(0)}(\xi, \eta) \int_0^\xi [\zeta(2\eta + 3\zeta) - \xi(\xi + 2\eta + 2\zeta)] e^{-3s_1\xi\zeta^2} d\zeta \\ &\quad + 2\tilde{K}^{(1)}(\xi, \eta) \int_0^\xi \eta\zeta e^{-s_1(4\zeta^3+6\xi\zeta^2+9\eta\zeta^2+6\xi\eta\zeta+6\eta^2\zeta)} d\zeta \\ &\quad + \tilde{K}^{(1)}(\xi, \eta) \int_0^\xi [(\eta + \zeta)(\eta + 3\zeta) - (\xi + \eta)(\xi + \eta + 2\zeta)] e^{-3s_1(\xi + \eta)(2\eta + \zeta)\zeta} d\zeta \Big\} \\ &\quad + 8\sin^4 \theta \left\{ \tilde{K}^{(0)}(\xi, \eta) \int_0^\xi d\zeta e^{-3s_1\xi\zeta^2} \int_0^{\eta + \zeta} (v - \eta - \zeta) e^{-s_1(2v^3 + 3\xi v^2)} dv \right. \\ &\quad + 2\tilde{K}^{(1)}(\xi, \eta) \int_0^\xi d\zeta e^{-s_1(4\zeta^3+6\xi\zeta^2+9\eta\zeta^2+6\xi\eta\zeta+6\eta^2\zeta)} \int_0^\zeta (\zeta - v) e^{s_1(2v^3 + 3\eta v^2)} dv \\ &\quad + \tilde{K}^{(1)}(\xi, \eta) \int_0^\xi d\zeta e^{-3s_1(\xi + \eta)(2\eta + \zeta)\zeta} \int_0^\zeta (v - \zeta) e^{-s_1[2v^3 + 3(\xi + \eta)v^2]} dv \Big\}, \end{aligned} \quad (3.183)$$

and

$$\begin{aligned} K_\lambda(\xi, \eta|s_1) &= 2\tilde{K}^{(1)}(\xi, \eta) \int_0^\xi \eta\zeta(\xi + \eta + \zeta)^2(\xi - \zeta) e^{-s_1(4\zeta^3+6\xi\zeta^2+9\eta\zeta^2+6\xi\eta\zeta+6\eta^2\zeta)} d\zeta \\ &\quad + 4\sin^2 \theta \left\{ \tilde{K}^{(0)}(\xi, \eta) \int_0^\xi d\zeta e^{-3s_1\xi\zeta^2} \int_0^{\eta + \zeta} (v - \eta - \zeta)(\xi - \zeta)\zeta^2 e^{-s_1(2v^3 + 3\xi v^2)} dv \right. \\ &\quad + \tilde{K}^{(1)}(\xi, \eta) \int_0^\xi d\zeta e^{-3s_1(\xi + \eta)(2\eta + \zeta)\zeta} \int_0^\zeta (v - \zeta)(\xi - \zeta)(\eta + \zeta)^2 e^{-s_1[2v^3 + 3(\xi + \eta)v^2]} dv \end{aligned}$$

$$+ 2\tilde{K}^{(1)}(\xi, \eta) \int_0^\xi d\zeta e^{-s_1(4\zeta^3+6\xi\zeta^2+9\eta\zeta^2+6\xi\eta\zeta+6\eta^2\zeta)} \int_0^\zeta (\zeta-v)(\xi-\zeta)(\xi+\eta+\zeta)^2 e^{s_1(2v^3+3\eta v^2)} dv \Big\} . \quad (3.184)$$

Note that the structure of $K_\lambda(\xi, \eta)$ is somewhat different from $K_s(\xi, \eta)$, and its appearance is rather unexpected. Nevertheless the kernel $\tilde{K}(\xi, \eta)$ reduces to (3.116) when $\lambda = 0$, as expected.

Using the modified jump, we obtain the amplitude equation for the viscous case as follows

$$\frac{dA}{dt_1} = g_0 \tau_1 A + \int_0^{+\infty} \int_0^{+\infty} \sum g_j \tilde{K}_j(\xi, \eta) A(t_1 - \xi) A(t_1 - \xi - \eta) A^*(t_1 - 2\xi - \eta) d\xi d\eta , \quad (3.185)$$

where the summation is made over all critical layers; in the present problem there are two. The kernel $\tilde{K}_j(\xi, \eta)$ is defined by (3.182), and

$$g_0 = f_0/f , \quad (3.186)$$

$$g_j = f_j/f . \quad (3.187)$$

The constants f_0 and f have the same expressions as in the inviscid case respectively, and

$$f_j = -\pi \alpha^3 \sin^2 \theta b_j^2 |b_j|^2 |\bar{U}_y|^3 . \quad (3.188)$$

The coefficients g_j against wavenumber $\bar{\alpha} = (\alpha^2 + \beta^2)^{\frac{1}{2}}$, are plotted in Fig18a and Fig18b in a convenient scaled manner. The coefficient g_0 against $\bar{\alpha}$ is the same as shown in Fig3 except the α there is replaced by $\bar{\alpha}$.

3.5.2 The Finite-Time Singularity, Equilibrium Solution, and Viscous Limit

We note that the finite-time singularity identified for the inviscid case applies to the viscous case, i.e. it is unaltered by viscosity. However, as in the two-dimensional case (Chapter two), we expect that viscosity may delay the occurrence of the singularity, or eliminate it if the viscosity is sufficiently large.

Surprisingly, the amplitude equation (3.185) does not admit an equilibrium solution. This is a remarkable difference from the two-dimensional viscous case.

We now turn to examining the amplitude equation under the limit $\lambda \rightarrow +\infty$. This corresponds to the situation where the viscosity is relatively large or the disturbance is relatively weak (see (3.135)). It can be shown that the amplitude equation (3.185) is reduced to the following equation:

$$\lambda^{-\frac{1}{3}} \frac{dA}{dt_1} = g_0 \tau_1 A + \lambda^{-1} \bar{g} A \int_0^{+\infty} |A(t_1 - \eta)|^2 d\eta + O(\lambda^{-\frac{4}{3}}) . \quad (3.189)$$

where we have performed the variable change $t_1 \rightarrow \lambda^{\frac{1}{3}} t_1$. The constant $\bar{g}$ is defined by

$$\bar{g} = \frac{2^{\frac{2}{3}}}{9} \sin^2 \theta [1 - 2 \sin^2 \theta] \Gamma(\frac{1}{3}) \sum_j g_j \beta_j^{-1}, \quad (3.190)$$

where the summation again is made over all critical layers. Remarkably, the non-linear term is still non-local, and the amplitude equation does not appear to be of Landau-Stuart type². Note also that at the viscous limit, the coefficient $\bar{g}$ is zero, i.e. nonlinear term vanishes, when $\theta = \frac{\pi}{4}$. The same behaviour occurs in the inviscid case, as noticed by Goldstein & Choi (1990). But generally, $\tilde{K}_j(\xi, \eta|s_1)$ is non-zero when $\theta = \frac{\pi}{4}$.

As in Chapter two, we examine the validity of the amplitude equation (3.189). To keep equation (3.189) balanced, we require $\tau_1 \sim \lambda^{-\frac{1}{3}}$, and $\lambda^{-\frac{1}{3}} A = \bar{A}$ with $\bar{A} = O(1)$. The rescaled amplitude equation is then

$$\frac{dA}{dt_1} = g_0 \bar{\tau}_1 A + \bar{g} A \int_0^{+\infty} |A(t_1 - \eta)|^2 d\eta, \quad (3.191)$$

where we have written $\bar{A}$ as A , and $\tau_1 = \lambda^{-\frac{1}{3}} \bar{\tau}_1$.

Recalling that $\tau_1 = \epsilon^{-\frac{1}{3}}(\tau - \tau_0)$, and $\lambda^{-\frac{1}{3}} = (\epsilon R)^{\frac{1}{3}}$, we obtain

$$\tau - \tau_0 \sim O(\epsilon^{\frac{2}{3}} R^{\frac{1}{3}}), \quad (3.192)$$

i.e. the evolution time scale is $O(\epsilon^{-\frac{2}{3}} R^{-\frac{1}{3}})$. Note that this time scale is much faster than that in Hall & Smith (1991), where the time/spatial scale is found to be $O(\epsilon^{-2} R^{-\frac{4}{3}})$. Since $\epsilon \ll R^{-1}$, we have $(\tau - \tau_0) \ll R^{-\frac{1}{3}}$. This implies that inside the critical layers, the unsteady effects are less important than viscous effects, and the critical layers thus become steady and viscous. Recalling that the sub-Stokes layer has a correction of $O(R^{-\frac{1}{2}})$ to the growth rate, and hence in order for the amplitude equation (3.189) to be valid, we further require

$$\epsilon^{\frac{2}{3}} R^{\frac{1}{3}} \gg R^{-\frac{1}{2}}, \quad \text{i.e.} \quad \epsilon \gg O(R^{-\frac{5}{4}}).$$

We therefore can conclude that when

$$R^{-\frac{5}{4}} \ll \epsilon \ll R^{-1}, \quad (3.193)$$

or equivalently when

$$1 \ll \lambda \ll R^{\frac{1}{4}}, \quad (3.194)$$

the evolution can be described by (3.191). The validity of the *fully* integro-differential form of amplitude equation (3.185) is restricted only by $\lambda \ll R^{\frac{1}{4}}$, i.e. it has a much larger validity range than (3.191). When the first restriction of (3.193) is violated, the amplitude equation could still be of form (3.191), but its coefficients must be modified by the sub-Stokes layer near the wall. Notice also that, when $\lambda = O(R^{\frac{1}{4}})$, the unscaled amplitude is of order $(\lambda^{-\frac{1}{3}} \epsilon)$, i.e. $R^{\frac{7}{8}}$; this is just the amplitude scale in Hall & Smith (1991).

²The author is very grateful to Prof. S.N. Brown for telling him her recent result, without knowing which the present author would not have realized this point.

3.6 Conclusions and Discussions

In this chapter, we have shown that nonlinearity affects the evolution of a pair of oblique waves with $O(1)$ spanwise wave number much earlier than that of a two-dimensional wave, and the nonlinear evolution time scale is much faster. This is due to the pole type of singularity in the streamwise velocity as well as in the spanwise velocity (Goldstein & Choi, 1989), cf. the logarithmic branch-point singularity in the corresponding 2-D case.

It is shown that although $\bar{U}_{yy}(y_c) \neq 0$ in the present case, the nonlinear interaction, and as a result, the amplitude equation is the same as that in the case where $\bar{U}_{yy}(y_c) = 0$, e.g. free shear layers considered by Goldstein & Choi (1989). The amplitude equation is an integro-differential one. Explosive growth in the form of a finite-time singularity is possible and may be related to bursting phenomena observed in experiments. The result gives further support to the point of view that subsequent nonlinear growth of initially small disturbances causes transition to turbulence.

Our analysis has included viscosity, thus has extended that of Goldstein & Choi (1989). It is shown that viscous effects influence the evolution of the disturbance by modifying the structure of the nonlinear terms. The resulting amplitude equation does not seem to admit equilibrium solutions. This suggests that nonlinear effects do not render the disturbance neutral; hence it will not enter the regime described by Hall & Smith (1991) from the stage considered in this chapter.

One interesting feature is that the nonlinear interaction inside the critical layers generates a vortex of the same size as the fundamental wave in the main part of the flow. This strong vortex may be identified as that observed by Hino *et al.* (1983) in a finite Stokes layer between two plates. It may also be possible to relate this vortex to the streaky structure in a turbulent boundary layer, where high-frequency bursting is a significant phenomenon; this is because large scale coherent structures in the outer region of a boundary layer may generate a flow similar to a Stokes layer in the wall region. One may expect that the dynamics there, especially the instability mechanism, is analogous to that of the present problem; thus the results obtained here may be helpful in understanding the wall-layer phenomenon in a turbulent boundary layer. Such a point of view is supported by the experiments of Hino *et al.* (1983), who gave evidence that the generation of vortex structure and bursting processes (like ejection, sweeping) are similar in these two flows.

In this Chapter, we have assumed that the spanwise wavenumber is of order one, i.e. $\beta \sim O(1)$. It is under this condition that the nonlinearity associated with the logarithmic branch point is weaker than that associated with the pole. However, from the amplitude equation it is seen the coefficient of the nonlinear term is proportional to $\sin^2\theta = \beta^2/\bar{\alpha}^2$, and therefore it will be small if β is small, i.e. these nonlinear effects are weak for weakly three-dimensional disturbances. A typical case of special interest is when nonlinearity from the simple-pole singularity is as strong as that from the branch-point singularity. A scaling argument, similar

to that in Chapter one reveals that this occurs when

$$\beta \sim O(\epsilon^{\frac{1}{5}}) .$$

This problem will be considered in Chapter five, where the amplitude is allowed to be a function of spanwise space $Z = \epsilon^{\frac{1}{5}}z$ as well as time $t_1 = \epsilon^{\frac{2}{5}}t$.

Chapter 4

The Nonlinear Evolution of Resonant-Triad Waves

Abstract

The nonlinear evolution of initially linear high-frequency disturbances in a Stokes layer is studied under a high-Reynolds-Number assumption. The disturbances are composed of a two-dimensional wave $(2\alpha, 0)$ of magnitude δ , and a pair of oblique waves $(\alpha, \pm\beta)$ of magnitude ϵ , where α, β are the streamwise and spanwise wavenumbers respectively. We assume $\beta = \sqrt{3}\alpha$ so that resonant conditions are satisfied when disturbances become nearly neutral in the process of evolution. It is shown that the growth rate of the disturbance is controlled by nonlinear interactions inside critical layers, i.e. thin regions centered on levels at which the phase velocity of the disturbance is equal to the basic flow velocity. In order for there to be a nonlinear feedback mechanism between the two-dimensional and the three-dimensional waves, the latter are required to have a larger magnitude than the former, namely $\delta \sim O(\epsilon^{4/3})$. The nonlinear evolution time scale is $O(\epsilon^{-\frac{1}{3}})$, i.e. $O(\delta^{-\frac{1}{4}})$.

The amplitude equations turn out to be significantly different from those of Raetz (1959), Craik (1971) and Smith & Stewart (1987) in several important respects. Firstly, as in Goldstein & Lee (1990) they are integro-differential equations, i.e. the local growth rate depends on the whole history of the evolution, unlike a conventional resonant triad, where it depends only on the instantaneous amplitude of the disturbance. Secondly the back reaction of oblique waves on the 2-D wave is accounted for by two cubic terms and one quartic term in the wave amplitude rather than one quadratic term. Thirdly the amplitudes of the two- and three-dimensional waves can theoretically always exhibit a finite-time singularity, while in a conventional resonant-triad equation, the occurrence of a finite-time singularity depends delicately on the coefficients (Craik 1975). The structure of the finite-time singularity is identified, and it is found that the 2-D wave has a “more singular” structure than the 3-D waves. The finite-time singularity implies that explosive growth is induced by nonlinear effects. We argue that this nonlinear blow-up of high-frequency disturbances is related to the bursting phenomena observed in experiments, and can

lead to transition to turbulence.

4.1 Introduction

In Chapters two and three, nonlinear effects were investigated for the cases of a two-dimensional wave, and a pair of oblique waves respectively. It was shown that before the disturbance became neutral, a nonlinear interaction could take place inside the critical layers, and that this controlled the growth rate. As a result, a finite-time singularity was induced by this nonlinear interaction. Consequently, instead of evolving into an equilibrium state, as implied by linear theory (Cowley 1987), the disturbance is further amplified by nonlinear effects. This result suggests that instability and transition of the Stokes layer may be explained in terms of this *nonlinear growth* over part of a period rather than in terms of the net growth over a whole period. The explosive growth of the high-frequency disturbance may be related to the bursts observed in experiments.

In this chapter, we will explore an even stronger nonlinear interaction, namely a resonant-triad interaction. This type of interaction takes place in many kinds of wave flows, e.g. see Craik (1985). The possibility of a resonant-triad interaction between Tollmien-Schlichting (T-S) waves was put forward theoretically by Raetz (1959) and Craik (1971). The mechanism they proposed is the following. Suppose there is a pair of three-dimensional modes, say $(\alpha, \pm\beta, c_1)$, and a two-dimensional mode, say $(2\alpha, 0, c_2)$, where $\pm\beta$ are the spanwise wavenumbers of the 3-D modes, α and 2α are the streamwise wavenumbers, and c_1 and c_2 are the phase velocities of the 3-D and 2-D modes respectively. If

$$c_1 = c_2 , \quad (4.1)$$

then a nonlinear interaction can occur at (amplitude)² order, i.e. “sooner” than the (amplitude)³ order of the non-resonant case (Stuart, 1960; Watson, 1960).

It is worthwhile noting that the relevant modes involved may have more general forms and do not necessarily have to be associated with the same instability. For instance, Hall & Smith (1988) considered an interesting resonant-triad interaction in a concave channel flow. This flow can support T-S waves as well as Taylor-Gortler vortices. The triad they considered was composed of a pair of oblique T-S waves and a Gortler vortex.

The resonant-triad interaction mechanism has been used to explain the appearance of three-dimensional disturbances. For example, it is believed that the staggered arrangement of T-S wave revealed by Saric *et al.* (1984) and Kachanov *et al.* (1984) is the result of a resonant-triad interaction between a pair of oblique sub-harmonic modes with the basic fundamental two-dimensional mode.

As pointed by Stuart (1962a,b), for T-S instability, condition (4.1) is not easily satisfied, i.e. it is not always allowed by the dispersion relation of T-S waves. Craik (1971) found that in a boundary layer, (4.1) is valid only for some specific values of α and β , and he referred to this as the selective amplification mechanism. Recently,

Smith & Stewart (1987) put Craik's analysis on an asymptotic basis by using a triple-deck style analysis; they showed that the resonant triad-interaction took place for "high-frequency" T-S waves.

In the case of Rayleigh waves, condition (4.1) can be easily satisfied. Suppose $(2\alpha, 0, c)$ is a neutral mode of Rayleigh's equation, then from a consideration of Squire's transformation, we conclude that if

$$\beta = \sqrt{3}\alpha ,$$

the $(\alpha, \pm\beta, c)$ modes are also neutral, and moreover the resonant-triad condition is satisfied. Using this idea, Goldstein & Lee (1990) studied a resonant-triad interaction in a boundary layer. They assumed that an adverse pressure gradient was present, which although weak, supported long wavelength Rayleigh instability waves (Goldstein *et al.*, 1987). They first considered the case where the 2-D and 3-D waves have equal order of amplitude. It was shown that the 3-D waves are enhanced by 2-D wave, but they had *no* back reaction on the 2-D wave. As a result, the 3-D waves exhibit faster than exponential growth, and the 2-D wave grows exponentially, unaffected by the 3-D waves. Eventually, the 3-D waves will have grown sufficiently large to affect the 2-D wave and then the evolution will enter a fully-interactive stage. This fully-interactive stage was then considered by them. Importantly, they pointed out that the feedback effect of the 3-D waves on the 2-D wave was through a quartic rather than a quadratic interaction; thus the amplitude for the 2-D wave contained a quartic term. But they did not carry out a detailed critical-layer analysis to give the exact form of this term. Moreover, it seems that they overlooked the cubic interaction terms.

A fully-interactive resonant-triad of *T-S waves* in the Blasius boundary layer was studied by Mankbadi (1991). He was concerned with the upper-branch scaling regime (cf. Smith & Stewart, 1987). Along the upper branch, the viscous critical layer is distinct from the viscous wall layer and hence the flow is not described by a triple-deck structure (Drazin & Reid 1981, Bodonyi & Smith 1981). He showed that the nonlinearity dominating the evolution of the amplitude was from the critical layer, and demonstrated good agreement with experiments.

This chapter is concerned with the fully-interactive resonant-triad composed of Rayleigh instability waves. The purpose of the study in this chapter is two fold. Firstly, we shall address the role of resonant-triad interactions in the instability and transition of the Stokes layer. Secondly in a broad sense, we are concerned with the evolution of a resonant-triad of Rayleigh instability waves under nonlinear effects from the critical layers, which, by assuming that the Reynolds number is sufficiently large, are predominately *inviscid*, but *unsteady*, in contrast to the viscous and steady one in the work of Mankbadi (1991). The nonlinear terms in the amplitude equations, as will be shown, are fully determined by nonlinear interactions inside the critical layers and do not depend on the basic flow as a whole. Thus the results obtained are valid for Rayleigh instability in a broad class of flows.

4.2 Scaling Argument and Formulation

The flow is described by Cartesian coordinates (x, y, z) , where x is parallel to the oscillating direction of the plate, y is normal to the plate, and z is the spanwise direction. If we take δ^* , ω^{-1} , and U_0 as characteristic length, time and velocity scales respectively, then the basic flow is given by

$$(\bar{U}, \bar{V}, \bar{W}) = (\cos(\tau - y)e^{-y}, 0, 0) .$$

We denote the perturbed flow by

$$(\bar{U} + u, v, w) .$$

Following Cowley (1987), we study high-frequency instability waves, and introduce the fast time scale

$$t = R\tau \tag{4.2}$$

to take account of the frequency of waves, where R is the Reynolds number defined in Chapter one.

We assume that the velocity component v of a two-dimensional disturbance of magnitude $O(\delta)$ has the leading-order form

$$\delta B_0 \phi_2(y, \tau) E^2 + c.c. ,$$

where $\delta \ll 1$, and the order-one complex constant number B_0 is a measure of the scaled amplitude of the 2-D wave. Hereafter *c.c.* represents the complex conjugate of the part written out. For convenience, we have defined

$$E = \exp(i\alpha x - i\hat{\theta}(t)) ,$$

where

$$\frac{d\hat{\theta}}{dt} = \frac{1}{2}\alpha c(\tau) + \frac{1}{2}R^{-\frac{1}{2}}\Omega_1(\tau) + \dots$$

with c being the complex phase velocity, i.e. $c = c_r + ic_i$. In this study, we assume that the wavenumber α is of order one.

The three-dimensional disturbance is composed of a pair of oblique Rayleigh waves with equal magnitude ϵ and streamwise wavenumber α , but opposite spanwise wavenumbers $\pm\beta$. We assume that $\beta = \sqrt{3}\alpha$ so that they have the same phase velocity as the two-dimensional wave, i.e. the resonant condition is satisfied. The velocity component v of the 3-D disturbance can be expressed, to the leading order, in the form

$$\epsilon A_0 \bar{v}_1(y, \tau) E \cos \beta z + c.c. ,$$

where A_0 is a measure of the scaled amplitude of the 3-D waves.

According to linear theory (Cowley 1987), both the two- and three-dimensional disturbances start to grow from the same time, say τ_i , until a time, say τ_0 , at which they both become neutral (see figure 1). In the vicinity of the neutral time τ_0 , the

growth rates are small, and critical layers emerge. Due to the singular property of the Rayleigh equation near critical layers, nonlinearity may first become important there. For surveys of the importance of nonlinear effects in critical-layers, readers are referred to the papers by Stewartson (1981), Maslowe (1986), Hickernell (1984), Goldstein & Leib (1988, 1989), Goldstein & Hultgren (1989), Goldstein & Choi (1989) and Goldstein & Lee (1990). Our main concern is how this nonlinearity affects the evolution of the resonant-triad waves.

Before proceeding to a formal asymptotic expansion, we need to set down the underlying scaling, i.e. the relation between c_i , ϵ and δ . For this purpose, we introduce an intermediate time scale $t_1 = \alpha c_i t$, which is much "slower" than the time scale of the instability waves, but much "faster" than the time scale of the underlying Stokes flow; the disturbances are then described by time scales t , t_1 and τ . In terms of these new variables, the x -momentum equation can be written as follows

$$\{\alpha c_i \frac{\partial}{\partial t_1} + (\bar{U} - c_r) \frac{\partial}{\partial x} + 2R^{-1} \frac{\partial}{\partial \tau}\} u + v \frac{\partial \bar{U}}{\partial y} = -\frac{\partial p}{\partial x} - \frac{\partial}{\partial x} uu - \frac{\partial}{\partial y} uv - \frac{\partial}{\partial z} uw ,$$

where $\frac{\partial}{\partial t}$ has been replaced by $-c_r \frac{\partial}{\partial x}$.

We are particularly interested in the case where the nonlinear evolution occurs on the *fastest* possible time scale, i.e. in which the time-variation term appears at leading order in the critical-layer equations. Suppose the thickness of the critical layers is $O(\mu)$, then near a critical layer $\bar{U} - c_r$ is $O(\mu)$ if $\bar{U}_y(y_c) = O(1)$. Balancing $\alpha c_i \frac{\partial u}{\partial t_1}$ and $(\bar{U} - c_r) \frac{\partial u}{\partial x}$ yields

$$\mu \sim O(c_i) . \quad (4.3)$$

This scaling brings in a new non-equilibrium effect that does not occur in the analysis of Raetz (1959), Craik (1971), Smith & Stewart (1987) and Mankbadi (1991). Such a non-equilibrium effect was included in the analysis of Goldstein & Lee (1990), see also, e.g. Goldstein & Leib (1988, 1989), Churilov & Shukhman (1988), and Goldstein & Choi (1989). As pointed out by Hickernell (1984), such an effect leads to a different type of amplitude equation that involves history effects and is therefore an integro-differential equation.

As a critical layer is approached, the streamwise velocity of the 3-D waves exhibits a pole type of singularity, i.e.

$$u_{3D} \sim \frac{\epsilon}{(y - y_c)} ,$$

as noted by Benney (1961), where y_c is a critical level defined by $\bar{U}(y_c) = c_r$. Then the magnitude of $u_{3D,y}$ is $O(\epsilon\mu^{-2})$. According to the asymptotic property of Rayleigh's equation, the normal velocities of the two- and three-dimensional waves are of order δ and ϵ respectively. The nonlinear interaction inside the critical layer between the 2-D and the 3-D waves, say through $\frac{\partial}{\partial y}(v_{2D}u_{3D})$, produces a forcing term of $O(\epsilon\delta\mu^{-2})$. By balancing this forcing term with $\alpha c_i \frac{\partial u}{\partial t_1}$, we know that it drives a 3-D wave velocity of $O(\epsilon\delta\mu^{-3})$. We require it to balance the unsteadiness

of $O(\epsilon\alpha c_i)$, i.e. $O(\epsilon\mu)$ by (4.3), in the 3-D waves so that the evolution is affected by this resonant interaction. This balance yields

$$\mu \sim O(\delta^{\frac{1}{4}}) . \quad (4.4)$$

We will concentrate on the situation where there is energy feedback between the 2-D and 3-D waves. Conventionally, the quadratic interaction of oblique waves feeds energy back to the 2-D wave. In the present case, this interaction takes place inside the critical layers, and produces a forcing of $O(\epsilon^2\mu^{-2})$, say through $\frac{\partial}{\partial y}(u_{3D}v_{3D})$. Balancing this forcing term with $\alpha c_i \frac{\partial u}{\partial t_1}$, we know that it produce a 2-D wave component as well as a mean-flow distortion of $O(\epsilon^2\mu^{-3})$. However, the 2-D wave component generated by this interaction does not produce a velocity jump across the critical layers. As a result, the quadratic interaction between the 3-D waves does not affect the growth of the 2-D fundamental wave (see Goldstein & Lee, 1990, and also §4.2 below). It is found that the *cubic* interaction between the 2-D wave and the mean-flow distortion generates a forcing of $O(\epsilon^2\delta\mu^{-4})$. Balancing this forcing with $\alpha c_i \frac{\partial u}{\partial t_1}$, we find that it drives a 2-D wave component of $O(\epsilon^2\delta\mu^{-5})$. This 2-D wave component contributes a velocity jump of the same order (see below). It balances the unsteadiness of $O(\delta\alpha c_i) \sim O(\delta\mu)$ in the 2-D wave when $\epsilon^2\delta\mu^{-5} \sim \delta\mu$, i.e.

$$\mu \sim O(\epsilon^{\frac{1}{3}}) . \quad (4.5)$$

Notice that from (4.4), we have

$$\delta \sim O(\epsilon^{\frac{4}{3}}) . \quad (4.6)$$

Equations (4.5) and (4.6) set the underlying scaling for the analysis in this study. Note that the magnitude of the 3-D waves is required to exceed that of 2-D wave. This is the same as in the work of Goldstein & Lee (1990) and Mankbadi (1991), but in contrast to much previous work on three-dimensional instability, where it is generally assumed that the magnitudes of 3-D waves do not exceed that of 2-D wave. However, in favour of the present scaling is the fact that for Stokes layers, it has not yet been possible to identify a dominant 2-D instability stage by reducing the environmental disturbances and introducing artificial 2-D disturbances. The present scaling implies that the disturbance is dominated by the three-dimensional waves, and exhibits a “peak-valley” splitting pattern.

With the scale set above, we shall concentrate on the times near to

$$\tau = \tau_0 + \epsilon^{\frac{1}{3}}\tau_1 , \quad (4.7)$$

namely the time at which the growth rate is reduced to $O(\epsilon^{\frac{1}{3}})$; it is at this stage that nonlinearity begins to affect the evolution. The disturbance evolves over time scale t_1 , while the linear growth rate and the basic flow depend parametrically on the slow time scales τ_0 and τ_1 . Since the basic flow $\bar{U}$ changes only on the slow time scale τ , it is sufficient to expand its profile at τ in a Taylor series about τ_0

$$\bar{U}(y, \tau) = \bar{U}(y, \tau_0) + \epsilon^{\frac{1}{3}}\bar{U}_\tau(y, \tau_0)\tau_1 + \dots$$

Hereafter, all basic flow quantities will be evaluated at τ_0 unless otherwise mentioned. For convenience, we rewrite the nonlinear evolution time scale t_1 in the form

$$t_1 = \frac{1}{2}\epsilon^{\frac{1}{3}}t. \quad (4.8)$$

The evolution picture can be summarised in Fig.2. As illustrated, we assume that the disturbances are initially described by linear theory, and grow exponentially. Then as they approach the neutral time τ_0 , the growth rates become small, and linear critical layers emerge. As the growth rate decreases further to $O(\epsilon^{\frac{1}{3}})$, i.e. at times $\tau_1 = O(1)$, nonlinear interactions take place and control the overall development. The earlier linear and the subsequent nonlinear solutions should be required to match in some mutual overlapping domain.

Finally for simplicity, we shall assume that the Reynolds number is large enough so that the viscosity does not play a leading order role in the evolution of the instability waves.

4.3 Outer Expansion

4.3.1 Outer Expansions and Asymptotic Solutions Near Critical Layers

Outside the critical layers, the flow is basically linear and is governed, to the required order of approximation, i.e. $O(\epsilon^{\frac{5}{3}})$, by the following equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (4.9)$$

$$2R^{-1}\frac{\partial u}{\partial \tau} + \bar{U}\frac{\partial u}{\partial x} + v\frac{\partial \bar{U}}{\partial y} = -\frac{\partial p}{\partial x}, \quad (4.10)$$

$$2R^{-1}\frac{\partial v}{\partial \tau} + \bar{U}\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y}, \quad (4.11)$$

$$2R^{-1}\frac{\partial w}{\partial \tau} + \bar{U}\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z}. \quad (4.12)$$

Eliminating the pressure terms yields:

$$(2R^{-1}\frac{\partial}{\partial \tau} + \bar{U}\frac{\partial}{\partial x})(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}) - \frac{\partial \bar{U}}{\partial y}\frac{\partial w}{\partial x} = 0, \quad (4.13)$$

and

$$(2R^{-1}\frac{\partial}{\partial \tau} + \bar{U}\frac{\partial}{\partial x})\nabla^2 v - \bar{U}_{yy}\frac{\partial v}{\partial x} = 0. \quad (4.14)$$

With the multiple time scales introduced in §2.2, the time derivative $\frac{\partial}{\partial \tau}$ should be interpreted as follows

$$\frac{\partial}{\partial \tau} \rightarrow R\frac{\partial}{\partial t} + \frac{1}{2}R\epsilon^{\frac{1}{3}}\frac{\partial}{\partial t_1} + \epsilon^{-\frac{1}{3}}\frac{\partial}{\partial \tau_1} + \frac{\partial}{\partial \tau_0}. \quad (4.15)$$

The velocity (u, v, w) and the pressure p of the disturbances are expanded as follows:

$$u = \epsilon u_1 + \epsilon^{\frac{4}{3}} u_2 + \epsilon^{\frac{5}{3}} u_3 + \dots, \quad (4.16)$$

$$v = \epsilon v_1 + \epsilon^{\frac{4}{3}} v_2 + \epsilon^{\frac{5}{3}} v_3 + \dots, \quad (4.17)$$

$$w = \epsilon w_1 + \epsilon^{\frac{4}{3}} w_2 + \epsilon^{\frac{5}{3}} w_3 + \dots, \quad (4.18)$$

$$p = \epsilon p_1 + \epsilon^{\frac{4}{3}} p_2 + \dots. \quad (4.19)$$

The presence of the two-dimensional disturbance requires that the expansions must be taken to one order higher than is necessary for a pair of oblique waves (Chapter three). However we shall see that substantial parts of the analysis are the same.

The earlier linear solution suggests that v_1 can be written as

$$v_1 = A(t_1, \tau_1, \tau_0) \bar{v}_1(y, \tau_0) \cos \beta z E + c.c., \quad (4.20)$$

where $A(t_1)$ is the amplitude of the three-dimensional waves, $E = \exp[i\alpha x - i\hat{\theta}(t)]$, and

$$\frac{d\hat{\theta}}{dt} = \Omega_0(\tau_0) + \frac{1}{2} R^{-\frac{1}{2}} \Omega_1(\tau_0) + \dots. \quad (4.21)$$

Here the dependence on τ_0 is due to the time variation of the basic flow, and the dependence of A on τ_1 is due to the time variation of the linear growth rate. Because both τ_0 and τ_1 only appear as parameters to the order that we work, they will not be written out explicitly hereafter.

The function $\bar{v}_1$ satisfies the Rayleigh equation:

$$(\bar{U} - c)(D^2 - \bar{\alpha}^2)\bar{v}_1 - \bar{U}_{yy}\bar{v}_1 = 0, \quad (4.22)$$

where we have written

$$2\Omega_0 = \alpha c, \quad \bar{\alpha} = (\alpha^2 + \beta^2)^{\frac{1}{2}}.$$

Let $\eta = y - y_c^j$, where y_c^j is the j -th critical level, then as $\eta \rightarrow \pm 0$, $\bar{v}_1$ has the following asymptotic solution:

$$\bar{v}_1 \sim a_j^\pm \phi_a + b_j^\pm [\phi_b + p_j \phi_a \log |\eta|], \quad (4.23)$$

where

$$\phi_a = \eta + \frac{1}{2} p_j \eta^2 + \dots,$$

$$\phi_b = 1 + q_j \eta^2 + \dots.$$

The function v_2 takes the following form:

$$v_2 = \bar{v}_2(y, t_1) \cos \beta z E + B(t_1) \phi_2(y) E^2 + v^{(0,2)}(y, t_1) \cos \beta z + c.c. + \dots, \quad (4.24)$$

where $B\phi_2 E^2$ is a two-dimensional wave of scaled amplitude $B(t_1)$, and $\bar{v}_2$ is the deviation of the eigenfunction of 3-D waves from its neutral state. The function ϕ_2 satisfies Rayleigh's equation, and we have assumed $2\alpha = \bar{\alpha}$,

$$\phi_2 = \bar{v}_1. \quad (4.25)$$

As for a pair of oblique waves (Chapter three), $\bar{v}_2$ satisfies the inhomogeneous Rayleigh equation:

$$[D^2 - (\bar{\alpha}^2 + \frac{\bar{U}_{yy}}{\bar{U} - c})]\bar{v}_2 = (i\alpha)^{-1} \left\{ \left[-\frac{dA}{dt_1} - (i\alpha\bar{U}_\tau\tau_1)A \right] \frac{\bar{U}_{yy}}{(\bar{U} - c)^2} + \frac{i\alpha\bar{U}_{yy\tau}\tau_1 A}{\bar{U} - c} \right\} \bar{v}_1, \quad (4.26)$$

The asymptotic form of $\bar{v}_2$ is as follows:

$$\begin{aligned} \bar{v}_2 \sim & -b_j^\pm r_j \log |\eta| + (a_j^\pm r_j + b_j^\pm s_j) \eta \log |\eta| + \dots \\ & + c_j^\pm \phi_a + d_j^\pm [\phi_b + p_j \phi_a \log |\eta|], \end{aligned} \quad (4.27)$$

where

$$p_j = \frac{\bar{U}_{yy}}{\bar{U}_y}, \quad (4.28)$$

$$q_j = \frac{1}{2}\bar{\alpha}^2 + \frac{1}{2}\frac{\bar{U}_{yyy}}{\bar{U}_y} - \frac{\bar{U}_{yy}^2}{\bar{U}_y^2}, \quad (4.29)$$

$$r_j = (i\alpha)^{-1} \frac{\bar{U}_{yy}}{\bar{U}_y^2} \left[-\frac{dA}{dt_1} - (i\alpha\bar{U}_\tau\tau_1)A \right], \quad (4.30)$$

$$\begin{aligned} s_j = & (i\alpha)^{-1} \left\{ (-i\alpha\bar{U}_{y\tau}\tau_1 A) \frac{\bar{U}_{yy}}{\bar{U}_y^2} + (i\alpha\tau_1 A) \frac{\bar{U}_{yy\tau}}{\bar{U}_y} \right. \\ & \left. + \left(-\frac{dA}{dt_1} - i\alpha\bar{U}_\tau\tau_1 A \right) \frac{\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y^3} \right\}. \end{aligned} \quad (4.31)$$

All basic flow quantities are evaluated at time τ_0 and at the critical level y_c^j .

At $O(\epsilon^{\frac{5}{3}})$, it is sufficient to solve the deviation of the two-dimensional wave eigenfunction from its neutral state; for this purpose we write

$$\bar{v}_3 = \phi_3(y, t_1)E^2 + c.c + \dots \quad (4.32)$$

The function ϕ_3 satisfies an inhomogeneous Rayleigh equation similar to that for $\bar{v}_2$, namely

$$[D^2 - (\bar{\alpha}^2 + \frac{\bar{U}_{yy}}{\bar{U} - c})]\phi_3 = (2i\alpha)^{-1} \left\{ \left[-\frac{dB}{dt_1} - (2i\alpha\bar{U}_\tau\tau_1)B \right] \frac{\bar{U}_{yy}}{(\bar{U} - c)^2} + \frac{2i\alpha\bar{U}_{yy\tau}\tau_1 B}{\bar{U} - c} \right\} \phi_2. \quad (4.33)$$

The asymptotic behaviour of ϕ_3 is as follows:

$$\begin{aligned} \phi_3 \sim & -b_j^\pm R_j \log |\eta| + (a_j^\pm R_j + b_j^\pm S_j) \eta \log |\eta| + \dots \\ & + C_j^\pm \phi_a + D_j^\pm [\phi_b + p_j \phi_a \log |\eta|], \end{aligned} \quad (4.34)$$

where

$$R_j = (2i\alpha)^{-1} \frac{\bar{U}_{yy}}{\bar{U}_y^2} \left[-\frac{dB}{dt_1} - (2i\alpha \bar{U}_\tau \tau_1) B \right], \quad (4.35)$$

$$S_j = (2i\alpha)^{-1} \left\{ (-2i\alpha \bar{U}_{y\tau} \tau_1 B) \frac{\bar{U}_{yy}}{\bar{U}_y^2} + (2i\alpha \tau_1 B) \frac{\bar{U}_{yy\tau}}{\bar{U}_y} \right. \\ \left. + \left(-\frac{dB}{dt_1} - 2i\alpha \bar{U}_\tau \tau_1 B \right) \frac{\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y^3} \right\}. \quad (4.36)$$

We note that $C_j^\pm$ and $D_j^\pm$ are generally different from $c_j^\pm$ and $d_j^\pm$ respectively, since the right hand side of (4.26) differs from that of (4.33). The jumps $(a_j^+ - a_j^-)$ etc. will be determined by analyzing the critical layers.

According to the continuity equation, we can write $w_1 = A(t_1) \bar{w}_1(y) E \sin \beta z + c.c.$, and then $\bar{w}_1$ satisfies the following equation:

$$\frac{\partial \bar{w}_1}{\partial y} + \frac{\bar{U}_y}{\bar{U} - c} \bar{w}_1 = -\beta \bar{v}_1,$$

the solution of which is

$$\bar{w}_1 = \bar{\alpha}^{-1} \sin \theta \left[\frac{\bar{U}_y}{\bar{U} - c} \bar{v}_1 - \bar{v}_{1,y} \right], \quad (4.37)$$

where

$$\sin \theta = \beta / \bar{\alpha} = \frac{\sqrt{3}}{2}.$$

The velocity u_1 takes the following form:

$$u_1 = A(t_1) \bar{u}_1(y) E \cos \beta z + \bar{u}_1^{(0,2)}(y, t_1) \cos 2\beta z + c.c.,$$

where $\bar{u}_1$ can be obtained from the continuity equation as follows:

$$\bar{u}_1 = -(i\alpha)^{-1} \left\{ \left[\frac{\bar{U}_y}{\bar{U} - c} \bar{v}_1 - \bar{v}_{1,y} \right] \sin^2 \theta + \bar{v}_{1,y} \right\}. \quad (4.38)$$

Note that a vortex component $\bar{u}_1^{(0,2)}(y, t_1) \cos 2\beta z$ has to be included in order to match with the inner solution, and as will be shown later, this is driven by the slip velocity generated by nonlinear interactions inside the critical layers.

Similarly $p_1 = A(t_1) \bar{p}_1 E \cos \beta z + c.c.$, and

$$\bar{p}_1 = i\bar{\alpha}^{-1} \cos \theta [\bar{U}_y \bar{v}_1 - (\bar{U} - c) \bar{v}_{1,y}]. \quad (4.39)$$

As $y \rightarrow y_c^j$, the asymptotic behaviours of $\bar{p}_1, \bar{u}_1, \bar{w}_1$ can be shown to be

$$\bar{p}_1 \sim i\bar{\alpha}^{-1} \bar{U}_y \cos \theta b_j^\pm + \dots, \quad (4.40)$$

$$\bar{u}_1 \sim -(i\alpha)^{-1} \sin^2 \theta b_j^\pm \eta^{-1} + \dots, \quad (4.41)$$

$$\bar{w}_1 \sim \bar{\alpha}^{-1} \sin \theta b_j^\pm \eta^{-1} + \dots, \quad (4.42)$$

where we notice that the singularity in $\bar{u}_1$ appears as a simple pole, rather than the logarithmic branch point which is characteristic of the two-dimensional case. It is this difference that results in the faster nonlinear evolution time scale in the three-dimensional case. For free shear layers, Goldstein & Choi (1989) have shown that because of the pole type of singularity in $\bar{u}_1$, the streamwise velocity of a three-dimensional disturbance, the critical-layer dynamics of 2-D and 3-D disturbances is significantly different: for the former it is strongly nonlinear (Goldstein & Leib, 1988; Goldstein & Hultgren, 1989), but for the latter it is weakly nonlinear.

At the next order solution for pressure, we only need to solve for the 2-D wave component; therefore we write

$$p_2 = B(t_1)\bar{p}_2(y)E^2 + c.c. + \dots \quad (4.43)$$

It is found that

$$\bar{p}_2 = \frac{1}{2}i\alpha^{-1}[\bar{U}_y\phi_2 - (\bar{U} - c)\phi_{2,y}] , \quad (4.44)$$

where as $y \rightarrow y_c^j$,

$$\bar{p}_2 \sim \frac{1}{2}i\alpha^{-1}\bar{U}_y b_j^\pm + \dots \quad (4.45)$$

We now introduce an inner variable:

$$Y = \frac{\eta}{\epsilon^{\frac{1}{3}}} . \quad (4.46)$$

The outer expansions written in terms of the inner variable are as follows:

$$\begin{aligned} v \sim & \epsilon b_j^\pm A E \cos \beta z + \epsilon^{\frac{4}{3}} \log \epsilon^{\frac{1}{3}} (-b_j^\pm r_j + b_j^\pm p_j A Y) E \cos \beta z \\ & + \epsilon^{\frac{4}{3}} \{ [(-b_j^\pm r_j \log |Y| + d_j^\pm) + A(a_j^\pm Y + b_j^\pm Y \log |Y|)] E \cos \beta z + b_j^\pm B E^2 \} \\ & + \epsilon^{\frac{5}{3}} \log \epsilon^{\frac{1}{3}} \{ [(a_j^\pm r_j + b_j^\pm s_j + d_j^\pm p_j) Y + \frac{1}{2} A p_j b_j^\pm Y^2] E \cos \beta z \\ & + (-b_j^\pm R_j + b_j^\pm p_j B Y) E^2 \} \\ & + \epsilon^{\frac{5}{3}} \{ [c_j^\pm Y + (a_j^\pm r_j + b_j^\pm s_j + d_j^\pm p_j) Y \log |Y|] E \cos \beta z \\ & + [(-b_j^\pm R_j \log |Y| + D_j^\pm) + B(a_j^\pm Y + b_j^\pm Y \log |Y|)] E^2 \} \\ & + \epsilon^{\frac{6}{3}} \log \epsilon^{\frac{1}{3}} \{ [(a_j^\pm R_j + b_j^\pm S_j + D_j^\pm p_j) Y + \frac{1}{2} B p_j b_j^\pm Y^2] E^2 + \dots \} \\ & + \epsilon^{\frac{6}{3}} \{ [C_j^\pm Y + (a_j^\pm R_j + b_j^\pm S_j + D_j^\pm p_j) Y \log |Y|] E^2 + \dots \} + c.c. + \dots \end{aligned} \quad (4.47)$$

$$u \sim \epsilon^{\frac{2}{3}} (-i\alpha)^{-1} \sin^2 \theta A b_j^\pm Y^{-1} E \cos \beta z + c.c. + \dots , \quad (4.48)$$

$$w \sim \epsilon^{\frac{2}{3}} \bar{\alpha}^{-1} \sin \theta b_j^\pm A Y^{-1} E \cos \beta z + c.c. + \dots , \quad (4.49)$$

$$p \sim \epsilon i \bar{\alpha}^{-1} \bar{U}_y \cos \theta A b_j^\pm E \cos \beta z + \epsilon^{\frac{4}{3}} \frac{1}{2} i \alpha^{-1} \bar{U}_y b_j^\pm B E^2 + c.c. + \dots \quad (4.50)$$

4.3.2 Solvability Conditions

Multiplying both sides of (4.26) by $\bar{v}_1$, integrating from 0 to $+\infty$, and utilising the asymptotic solutions (4.23) and (4.27), we obtain the solvability condition for (4.26):

$$i\alpha^{-1}J_1\frac{dA}{dt_1} + J_2\tau_1A = -\sum_j\{(b_j^+c_j^+ - b_j^-c_j^-) - r_j(b_j^+a_j^+ - b_j^-a_j^-) - p_j(b_j^+d_j^+ - b_j^-d_j^-) - (a_j^+d_j^+ - a_j^-d_j^-)\}, \quad (4.51)$$

where the summation is made over all critical layers, and J_1 and J_2 are constants defined by the following integrals respectively

$$J_1 = \int_0^{+\infty} \frac{\bar{U}_{yy}}{(\bar{U} - c)^2} \bar{v}_1^2 dy, \quad (4.52)$$

$$J_2 = \int_0^{+\infty} \left[-\frac{\bar{U}_{yy}\bar{U}_\tau}{(\bar{U} - c)^2} + \frac{\bar{U}_{yy\tau}}{(\bar{U} - c)} \right] \bar{v}_1^2 dy. \quad (4.53)$$

Note that these integrals are singular and should be interpreted in the sense of Hadamard.

By the same procedure, the solvability condition for (4.33) is derived as follows:

$$i(2\alpha)^{-1}J_1\frac{dB}{dt_1} + J_2\tau_1B = -\sum_j\{(b_j^+C_j^+ - b_j^-C_j^-) - R_j(b_j^+a_j^+ - b_j^-a_j^-) - p_j(b_j^+D_j^+ - b_j^-D_j^-) - (a_j^+D_j^+ - a_j^-D_j^-)\} \quad (4.54)$$

Equations (4.51) and (4.54) are crucial in deriving ^{the}amplitude equations. The jumps $(a_j^+ - a_j^-)$ etc. will be determined by analysing the critical layers. It is important to note that the left-hand sides of (4.51) and (4.54) are linear and that nonlinear terms come from ^{the}critical layers through these jumps. Thus in the analysis of the critical layers, it is sufficient to concentrate only on the solutions contributing to these jumps. Since we find that the solution driven by the quadratic interaction of the 3-D waves does not contribute to any jump, the quadratic term in 3-D amplitude will not appear in the amplitude equation of the 2-D wave.

4.4 Inner Expansion

(4.47)-(4.50) suggest that the inner expansions within the j -th critical layer take the following form:

$$u = \epsilon^{\frac{2}{3}}U_1 + \epsilon^{\frac{3}{3}}U_2 + \epsilon^{\frac{4}{3}}U_3 + \epsilon^{\frac{5}{3}}U_4 + \dots, \quad (4.55)$$

$$v = \epsilon V_1 + \epsilon^{\frac{4}{3}}V_2 + \epsilon^{\frac{5}{3}}V_3 + \epsilon^{\frac{6}{3}}V_4 + \dots, \quad (4.56)$$

$$w = \epsilon^{\frac{2}{3}}W_1 + \epsilon^{\frac{3}{3}}W_2 + \epsilon^{\frac{4}{3}}W_3 + \epsilon^{\frac{5}{3}}W_4 + \dots, \quad (4.57)$$

$$p = \epsilon P_1 + \epsilon^{\frac{4}{3}}P_2 + \epsilon^{\frac{5}{3}}P_3 + \dots, \quad (4.58)$$

where the $\epsilon^n \log \epsilon^{\frac{1}{3}}$ terms are not written out since they match onto the outer solutions automatically whenever the solutions at ϵ^n match with their corresponding outer expansions.

We shall see that the solutions at the first two orders are available from the analysis for a pair of oblique waves (Chapter three), but with some modification due to the presence of the two-dimensional disturbance.

The function V_1 satisfies the equation:

$$L_0 \frac{\partial^2 V_1}{\partial Y^2} = 0 , \quad (4.59)$$

where

$$L_0 = \frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x} . \quad (4.60)$$

The solution which matches the outer expansion is:

$$V_1 = \hat{A}(t_1) E \cos \beta z + c.c. . \quad (4.61)$$

Matching with the outer solution yields $b_j^+ = b_j^- = b_j$, i.e. the jump $(b_j^+ - b_j^-)$ is zero, and $\hat{A}(t_1) = b_j A(t_1)$.

Expansion of the y -momentum equation gives:

$$\frac{\partial P_1}{\partial Y} = 0 , \quad (4.62)$$

and so

$$P_1 = i\bar{\alpha}^{-1} \bar{U}_y \cos \theta \hat{A} E \cos \beta z + c.c. .$$

The function W_1 satisfies the following equation

$$L_0 W_1 = -\frac{\partial P_1}{\partial z} . \quad (4.63)$$

Let $W_1 = \hat{W}_1(Y, t_1) E \sin \beta z + c.c.$, and then $\hat{W}_1$ satisfies:

$$\hat{L}_0^{(1)} \hat{W}_1 = i \bar{U}_y \sin \theta \cos \theta \hat{A} ,$$

where

$$\hat{L}_0^{(n)} = \frac{\partial}{\partial t_1} + n i \alpha (\bar{U}_y Y + \bar{U}_\tau \tau_1) .$$

The solution that matches with the outer solution is

$$\hat{W}_1 = i \bar{U}_y \sin \theta \cos \theta \hat{W}_0^{(0)} ,$$

where we have put

$$\hat{W}_0^{(n)} = \int_0^{+\infty} \xi^n \hat{A}(t_1 - \xi) e^{-i\Omega \xi} d\xi , \quad \Omega = \alpha (\bar{U}_y Y + \bar{U}_\tau \tau_1) . \quad (4.64)$$

Similarly, we write $U_1 = \hat{U}_1(Y, t_1)E \cos \beta z + c.c.$, and then

$$\hat{U}_1 = -\bar{U}_y \sin^2 \theta \hat{W}_0^{(0)} .$$

At $O(\epsilon^{\frac{4}{3}})$, the pressure is written as

$$P_2 = \hat{P}_2^{(2,0)}(Y, t_1)E^2 + \hat{P}_2^{(1,1)}E \cos \beta z + c.c. ,$$

and $\hat{P}_2^{(2,0)}$ satisfies

$$\frac{\partial \hat{P}_2^{(2,0)}}{\partial Y} = 0 . \quad (4.65)$$

The solution matching with the outer expansion is

$$\hat{P}_2^{(2,0)} = \frac{1}{2}i\alpha^{-1}\bar{U}_y\hat{B} . \quad (4.66)$$

The three-dimensional pressure component $\hat{P}_2^{(1,1)}$ is not needed in the following analysis, though it can be found.

The velocity V_2 satisfies

$$L_0 V_{2,YY} = L_1 V_1 + \frac{\partial}{\partial Y} \left[\frac{\partial S_{11}}{\partial x} + \frac{\partial S_{31}}{\partial z} \right] , \quad (4.67)$$

where

$$L_1 = -\left(\frac{1}{2}\bar{U}_{yy}Y^2 + \bar{U}_{y\tau}\tau_1 Y + \frac{1}{2}\bar{U}_{\tau\tau}\tau_1^2\right)\frac{\partial^3}{\partial x \partial Y^2} + \bar{U}_{yy}\frac{\partial}{\partial x} , \quad (4.68)$$

and S_{11}, S_{31} are Reynolds stresses defined as follows

$$S_{11} = \frac{\partial U_1^2}{\partial x} + \frac{\partial U_1 V_1}{\partial Y} + \frac{\partial U_1 W_1}{\partial z} , \quad (4.69)$$

$$S_{31} = \frac{\partial U_1 W_1}{\partial x} + \frac{\partial V_1 W_1}{\partial Y} + \frac{\partial W_1^2}{\partial z} . \quad (4.70)$$

Furthermore, S_{11} and S_{31} can be expressed as

$$S_{11} = S_{11}^{(0,0)} + S_{11}^{(0,2)} \cos 2\beta z + S_{11}^{(2,0)} E^2 + S_{11}^{(2,2)} E^2 \cos 2\beta z + c.c. ,$$

$$S_{31} = S_{31}^{(0,2)} \cos 2\beta z + S_{31}^{(2,2)} E^2 \cos 2\beta z + c.c. .$$

After some calculation, it is can be shown that

$$S_{11}^{(0,0)} = \frac{1}{2}i\alpha\bar{U}_y^2 \sin^2 \theta \hat{A}^* \hat{W}_0^{(1)} , \quad (4.71)$$

$$S_{11}^{(0,2)} = \frac{1}{2}i\alpha\bar{U}_y^2 \sin^2 \theta \hat{A}^* \hat{W}_0^{(1)} , \quad (4.72)$$

$$S_{11}^{(2,0)} = \frac{1}{2}i\alpha\bar{U}_y^2 \sin^2 \theta [\hat{A} \hat{W}_0^{(1)} + 2 \sin^2 \theta \hat{W}_0^{(0)} \hat{W}_0^{(0)}] , \quad (4.73)$$

$$S_{11}^{(2,2)} = \frac{1}{2}i\alpha\bar{U}_y^2 \sin^2 \theta \hat{A} \hat{W}_0^{(1)} , \quad (4.74)$$

$$S_{31}^{(0,2)} = \frac{1}{2}\beta\bar{U}_y^2 \cos^2 \theta [\hat{A}^* \hat{W}_0^{(1)} + 2 \sin^2 \theta \hat{W}_0^{(0)} \hat{W}_0^{*(0)}] , \quad (4.75)$$

$$S_{31}^{(2,2)} = \frac{1}{2}\beta\bar{U}_y^2 \cos^2 \theta \hat{A} \hat{W}_0^{(1)} . \quad (4.76)$$

Inspecting the right-hand-side of (4.67), we know V_2 has the solution of the following form

$$V_2 = \hat{V}_2^{(1)} E \cos \beta z + \hat{V}_2^{(0,2)} \cos 2\beta z + \hat{V}_2^{(2,0)} E^2 + c.c. . \quad (4.77)$$

The 3-D fundamental component $\hat{V}_2^{(1)}$ is only driven by the linear forcing term, i.e. $L_1 V_1 = i\alpha \bar{U}_{yy} \hat{A}$, which is exactly the same as in the two-dimensional case studied in Chapter two. By the same procedure, we can obtain the following jump conditions:

$$a_j^+ - a_j^- = \pi i p_j b_j \text{Sgn}(\bar{U}_y) , \quad (4.78)$$

$$d_j^+ - d_j^- = -\pi i r_j b_j \text{Sgn}(\bar{U}_y) . \quad (4.79)$$

Substituting (4.71)-(4.76) into (4.67), we find that

$$\frac{\partial}{\partial t_1} \hat{V}_{2,YY}^{(0,2)} = -i\bar{S}^3 \sin^2 \theta [\hat{A}^* \hat{W}_0^{(2)} + 4 \sin^2 \theta \hat{W}_0^{*(0)} \hat{W}_0^{(1)}] , \quad (4.80)$$

$$\hat{L}_0^{(2)} \hat{V}_{2,YY}^{(2,0)} = i\bar{S}^3 \sin^2 \theta [\hat{A} \hat{W}_0^{(2)} + 4 \sin^2 \theta \hat{W}_0^{(0)} \hat{W}_0^{(1)}] , \quad (4.81)$$

where we have put

$$\bar{S} = \alpha \bar{U}_y . \quad (4.82)$$

The solutions are found to be as follows

$$\hat{V}_{2,YY}^{(0,2)} = -i\bar{S}^3 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta \xi (\xi + 4 \sin^2 \theta \eta) \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega \xi} , \quad (4.83)$$

$$\hat{V}_{2,YY}^{(2,0)} = i\bar{S}^3 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta [\xi^2 + 4 \sin^2 \theta (\xi + \eta) \eta] \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega(\xi + 2\eta)} . \quad (4.84)$$

In writing $\hat{V}_{2,YY}^{(0,2)}$ into the compact form (4.83), we have dropped a purely imaginary part, which, according to (4.77) does not alter the physical velocity. The same operation will be made in simplifying the solution for $\hat{W}^{(0,2)}$. Integrating $\hat{V}_{2,YY}^{(2,0)}$ once with respect to Y yields

$$\hat{V}_{2,Y}^{(2,0)} = -\bar{S}^2 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta [\xi^2 + 4 \sin^2 \theta (\xi + \eta) \eta] (\xi + 2\eta)^{-1} [e^{-i\bar{\Omega}(\xi + 2\eta)} - 1] \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\bar{\omega}(\xi + 2\eta)} , \quad (4.85)$$

where we have put

$$\bar{\Omega} = \alpha \bar{U}_y Y , \quad \bar{\omega} = \alpha \bar{U}_\tau \tau_1 , \quad (4.86)$$

and an integral constant (or more precisely a function of t_1) is taken to be zero by matching with the corresponding outer solution. Instead of integrating $\hat{V}_{2,Y}^{(2,0)}$ to get $\hat{V}_2^{(2,0)}$, we obtain it from the x -momentum equation together with the continuity equation in the following way.

The continuity equation is

$$2i\alpha\hat{U}_2^{(2,0)} + \hat{V}_{2,Y}^{(2,0)} = 0 , \quad (4.87)$$

and the x -momentum equation is

$$\hat{L}_0^{(2)}\hat{U}_2^{(2,0)} + \bar{U}_y\hat{V}_2^{(2,0)} = -(2i\alpha)\hat{P}_2^{(2,0)} - S_{11}^{(2,0)} . \quad (4.88)$$

The first term of the right hand side, which is absent in the case of a pair of oblique waves (Chapter three), is the pressure related to the 2-D wave, and the second term is the forcing from the nonlinear interaction. From the above two equations we obtain

$$\hat{V}_2^{(2,0)} = \hat{B} - \frac{1}{2}i\bar{S}^{-1}\hat{L}_0^{(2)}\hat{V}_{2,Y}^{(2,0)} - \bar{U}_y^{-1}S_{11}^{(2,0)} . \quad (4.89)$$

It turns out that the above expression for $\hat{V}_2^{(2,0)}$ is very helpful in evaluating the asymptotic behaviours of the related solutions at higher order.

The function W_2 satisfies the following equation

$$L_0W_2 = -\frac{\partial P_2}{\partial z} - F_2(Y)\frac{\partial W_1}{\partial x} - S_{31} , \quad (4.90)$$

where

$$F_2(Y) = \frac{1}{2}\bar{U}_{yy}Y^2 + \bar{U}_{y\tau}\tau_1Y + \frac{1}{2}\bar{U}_{\tau\tau}\tau_1^2 .$$

The velocity W_2 has the form

$$W_2 = \hat{W}_2^{(1)}E \sin \beta z + \hat{W}_2^{(0,2)} \sin 2\beta z + \hat{W}_2^{(2,2)}E^2 \sin 2\beta z + c.c. , \quad (4.91)$$

where $\hat{W}_2^{(1)}$ is the component driven by linear forcing through the first terms on the right-hand side of (4.91); it does not need to be worked out because it does not contribute to the jumps. The solutions forced by the nonlinear interaction are $\hat{W}_2^{(0,2)}$ and $\hat{W}_2^{(2,2)}$, which satisfy the following equations respectively

$$\frac{\partial}{\partial t_1}\hat{W}_2^{(0,2)} = -S_{31}^{(0,2)} , \quad (4.92)$$

$$\hat{L}_0^{(2)}\hat{W}_2^{(2,2)} = -S_{31}^{(2,2)} . \quad (4.93)$$

The solutions are

$$\hat{W}_2^{(0,2)} = -\frac{1}{2}\beta^{-1}\bar{S}^2 \sin^2\theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta (\xi + 4 \sin^2\theta\eta) \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega\xi} , \quad (4.94)$$

$$\hat{W}_2^{(2,2)} = -\frac{1}{2}\beta^{-1}\bar{S}^2 \sin^2\theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta \xi \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega(\xi + 2\eta)} . \quad (4.95)$$

According to the continuity equation $\hat{V}_{2,Y}^{(0,2)} + 2\beta\hat{W}_2^{(0,2)} = 0$, we have

$$\hat{V}_{2,Y}^{(0,2)} = \bar{S}^2 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta (\xi + 4 \sin^2 \theta \eta) \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega\xi} . \quad (4.96)$$

Integration with respect to Y yields

$$\begin{aligned} \hat{V}_2^{(0,2)} &= i\bar{S} \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta \xi^{-1} (\xi + 4 \sin^2 \theta \eta) [e^{-i\bar{\Omega}\xi} - 1] \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\omega\xi} \\ &\quad + \hat{V}_2^{(0,2)}(0, t_1) . \end{aligned} \quad (4.97)$$

As $\bar{\Omega} \rightarrow \pm\infty$, i.e. $Y \rightarrow \pm\infty$ (see (4.86)),

$$\begin{aligned} (\hat{V}_2^{(0,2)} + c.c.) &\sim \pm 4\bar{S} \sin^4 \theta \pi \int_0^{+\infty} \eta |\hat{A}(t_1 - \eta)|^2 d\eta \\ &\quad + \{ \text{order-one continuous terms} \} + o(1) . \end{aligned}$$

It is seen here that a vortex component must be included in the $O(\epsilon^{\frac{4}{3}})$ outer expansion of v so that it matches to $\hat{V}_2^{(0,2)}$.

The function U_2 satisfies

$$L_0 U_{2,Y} = -\bar{U}_{yy} V_1 + F_1(Y) \frac{\partial W_1}{\partial z} - F_2(Y) \frac{\partial^2 U_1}{\partial x \partial Y} - \frac{\partial S_{11}}{\partial Y} + \bar{U}_y \frac{\partial W_2}{\partial z} , \quad (4.98)$$

where

$$F_1(Y) = \bar{U}_{yy} Y + \bar{U}_{Y\tau} \tau_1 .$$

The solution takes the following form

$$U_2 = \hat{U}_2^{(1)} E \cos \beta z + \hat{U}_2^{(0,0)} + \hat{U}_2^{(0,2)} \cos 2\beta z + \hat{U}_2^{(2,0)} E^2 + \hat{\mathcal{U}}_2^{(2,2)} E^2 \cos 2\beta z + c.c. . \quad (4.99)$$

Again only the second harmonic and mean-flow distortion/vortex induced by non-linearity need to be worked out. From the continuity equation, we obtain

$$\hat{U}_2^{(2,0)} = -(2i\alpha)^{-1} \hat{V}_{2,Y}^{(2,0)} , \quad \hat{U}_2^{(2,2)} = i\beta\alpha^{-1} \hat{W}_2^{(2,2)} .$$

The mean-flow distortion and vortex satisfy the following equations respectively

$$\frac{\partial}{\partial t_1} \hat{U}_{2,Y}^{(0,0)} = -\frac{1}{2} \alpha^{-1} \bar{S}^3 \sin^2 \theta \hat{A}^* \hat{W}_0^{(2)} , \quad (4.100)$$

$$\frac{\partial}{\partial t_1} \hat{U}_{2,Y}^{(0,2)} = -\frac{\partial}{\partial Y} S_{11}^{(0,2)} + 2\beta \bar{U}_y \hat{W}_2^{(0,2)} . \quad (4.101)$$

The solutions are

$$\hat{U}_{2,Y}^{(0,0)} = -\frac{1}{2} \alpha^{-1} \bar{S}^3 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta \xi^2 \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega\xi} , \quad (4.102)$$

$$\hat{U}_{2,Y}^{(0,2)} = -\frac{1}{2} \alpha^{-1} \bar{S}^3 \sin^2 \theta \int_0^{+\infty} d\xi \int_0^{+\infty} d\eta [\xi^2 + 2\xi\eta + 4 \sin^2 \theta \eta^2] \hat{A}^*(t_1 - \eta) \hat{A}(t_1 - \eta - \xi) e^{-i\Omega\xi} . \quad (4.103)$$

Integration once with respect to Y yields

$$\begin{aligned} \hat{U}_2^{(0,2)} = & -\frac{1}{2}i\alpha^{-1}\bar{S}^2\sin^2\theta\int_0^{+\infty}d\xi\int_0^{+\infty}d\eta \quad \xi^{-1}[\xi^2+2\xi\eta+4\sin^2\theta\eta^2][e^{-i\bar{\Omega}\xi}-1] \\ & \hat{A}^*(t_1-\eta)\hat{A}(t_1-\eta-\xi)e^{-\bar{\omega}\xi} + \hat{U}_2^{(0,2)}(0,t_1) . \end{aligned} \quad (4.104)$$

As $Y \rightarrow \pm\infty$,

$$\begin{aligned} (\hat{U}_2^{(0,2)} + c.c) \sim & \mp 4i\alpha^{-1}\bar{S}\sin^4\theta\pi \int_0^{+\infty} \eta^2 |\hat{A}(t_1 - \eta)|^2 d\eta \\ & + \{\text{order-one continuous terms}\} + o(1) , \end{aligned} \quad (4.105)$$

which is in fact the streamwise slip velocity generated by the nonlinear interaction inside the critical layers, to which the outer expansion of u has to match. This is the reason that the leading order outer expansion of u should contain a vortex component. Note that this vortex is unexpectedly strong in the sense that it has the same magnitude as the 3-D waves. The interaction of the fundamental 3-D waves with this vortex, as will be shown, affects the evolution of the waves. The significance of vortex/wave interactions has been studied extensively, e.g. see Benney (1984), Hall & Smith (1989, 1990), Smith & Walton (1990), Hall & Smith (1991). In particular, Hall & Smith (1991) have shown that the nonlinear interaction of small amplitude waves may produce an order-one vortex structure, thus completely alter the mean-flow profile from its original laminar state. Compared with the present study, their critical layers are predominately steady and viscous. This is because their nonlinear evolution time scale is much slower, due to which only very small amplitude waves are needed to generate a strong vortex. See comments in Chapter three on this issue.

We first seek a solution for V_3 ; this is found to satisfy the following equation:

$$L_0 V_{3,YY} = L_1 V_2 + L_2 V_1 + \frac{\partial}{\partial Y} \left[\frac{\partial}{\partial x} S_{12} + \frac{\partial}{\partial z} S_{32} \right] , \quad (4.106)$$

where L_1 is defined by (4.68),

$$\begin{aligned} L_2 = & -\left[\frac{1}{6}\bar{U}_{yyy}Y^3 + \bar{U}_{yy\tau}\tau_1 Y^2 + \bar{U}_{y\tau\tau}\tau_1^2 Y + \frac{1}{6}\bar{U}_{\tau\tau\tau}\tau_1^3 \right] \frac{\partial^3}{\partial x \partial Y^2} \\ & + [\bar{U}_{yyy}Y + \bar{U}_{yy\tau}\tau_1] \frac{\partial}{\partial x} \\ & - \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x} \right] \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right] , \end{aligned} \quad (4.107)$$

and the Reynolds stresses S_{12} and S_{32} are defined as follows

$$S_{12} = \frac{\partial}{\partial x}(2U_1 U_2) + \frac{\partial}{\partial Y}(U_1 V_2 + U_2 V_1) + \frac{\partial}{\partial z}(U_1 W_2 + U_2 W_1) , \quad (4.108)$$

$$S_{32} = \frac{\partial}{\partial x}(U_1 W_2 + U_2 W_1) + \frac{\partial}{\partial Y}(W_1 V_2 + W_2 V_1) + \frac{\partial}{\partial z}(2W_1 W_2) . \quad (4.109)$$

The Reynolds stresses contain different components, among which only those proportional to $E \cos \beta z$ and $E^3 \cos \beta z$ will be used later. Therefore we write

$$S_{12} = S_{12}^{(1,1)} E \cos \beta z + S_{12}^{(3,1)} E^3 \cos \beta z + c.c. + \dots, \quad (4.110)$$

$$S_{32} = S_{32}^{(1,1)} E \sin \beta z + S_{32}^{(3,1)} E^3 \sin \beta z + c.c. + \dots. \quad (4.111)$$

After some calculation, we have

$$\begin{aligned} S_{12}^{(1,1)} = & \hat{A} \hat{U}_{2,Y}^{*(0,0)} + \frac{1}{2} \hat{A} \hat{U}_{2,Y}^{*(0,2)} - \beta \hat{W}_1 \hat{U}_2^{*(0,0)} + \frac{1}{4} \hat{U}_1 \hat{V}_{2,Y}^{*(0,2)} \\ & + [\frac{3}{2} i \alpha \hat{U}_1 \hat{U}_2^{*(0,2)} + \frac{1}{2} \hat{U}_{1,Y} \hat{V}_2^{*(0,2)}] + [\hat{U}_{1,Y}^* \hat{V}_2^{(2,0)} - \frac{1}{2} \hat{U}_1^* \hat{V}_{2,Y}^{(2,0)}], \quad (4.112) \end{aligned}$$

$$S_{12}^{(3,1)} = \hat{A} \hat{U}_{2,Y}^{(2,0)} + \frac{1}{2} \hat{A} \hat{U}_{2,Y}^{(2,2)} - 3\beta \hat{U}_1 \hat{W}_2^{(2,2)} + [\hat{U}_{1,Y} \hat{V}_2^{(2,0)} - \frac{3}{2} \hat{U}_1 \hat{V}_{2,Y}^{(2,0)}] \quad (4.113)$$

$$\begin{aligned} S_{32}^{(1,1)} = & [\frac{1}{2} \hat{A} \hat{W}_{2,Y}^{*(0,2)} + i \alpha \hat{W}_1 \hat{U}_2^{*(0,0)} - \frac{1}{2} \beta \hat{W}_1 \hat{W}_2^{*(0,2)}] \\ & + [-\frac{1}{2} i \alpha \hat{W}_1 \hat{U}_2^{*(0,2)} - \frac{1}{2} \hat{W}_{1,Y} \hat{V}_2^{*(0,2)}] + [\frac{1}{2} \hat{W}_1^* \hat{V}_{2,Y}^{(2,0)} + \hat{W}_{1,Y}^* \hat{V}_2^{(2,0)}], \quad (4.114) \end{aligned}$$

$$S_{32}^{(3,1)} = \frac{1}{2} \hat{A} \hat{W}_{2,Y}^{(2,2)} - \beta \hat{W}_1 \hat{W}_2^{(2,2)} - \frac{1}{2} \hat{W}_1 \hat{V}_{2,Y}^{(2,0)} + \hat{W}_{1,Y} \hat{V}_2^{(2,0)}. \quad (4.115)$$

Inspecting the right hand side of (4.106), we know that V_3 should take the following form

$$V_3 = \hat{V}_3 E \cos \beta z + \tilde{V}_3 E^3 \cos \beta z + \hat{V}_3^{(2,0)} E^2 + c.c. + \dots,$$

where the component proportional to $E \cos 3\beta z$ is omitted since it will not be used in deriving the evolution equations. However, to derive the amplitude equation for the two-dimensional wave, two more components, i.e. $\tilde{V}_3 E^3 \cos \beta z$ and $\hat{V}_3^{(2,0)} E^2$, have to be solved. Correspondingly, we write the derivatives of the Reynolds stresses as follows

$$\frac{\partial}{\partial x} S_{12} + \frac{\partial}{\partial z} S_{32} = \bar{M} E \cos \beta z + \hat{M} E^3 \cos \beta z + \hat{M}^{(2,0)} E^2 + c.c. + \dots.$$

It is noted that the relevant part of the linear forcing term, i.e. $(L_1 V_2 + L_2 V_1)$, is the same as $F^{(l)}(Y, t_1)$ in the two-dimensional case (Chapter two). So the solution (denoted here by $\hat{V}_3^{(l)} E$) driven by it has the following asymptotic behaviour as $Y \rightarrow \pm\infty$,

$$\begin{aligned} \hat{V}_{3,Y}^{(l)} \sim & (a_j^+ p_j + 2q_j b_j + \frac{1}{2} p_j^2 b_j) Y + (a_j^+ r_j + p_j d_j^+ + s_j b_j) \log |Y| \\ & + \{\pm \frac{1}{2} \pi i \text{Sgn}(\bar{U}_y) (a_j^+ r_j + p_j d_j^+ + s_j b_j) + \dots\}. \quad (4.116) \end{aligned}$$

After some calculation, we write

$$\bar{M} = \bar{M}_1 + \bar{M}_2 + \bar{M}_3 + \bar{M}_0 ,$$

where

$$\bar{M}_1 = i\alpha \hat{A} \hat{U}_{2,Y}^{*(0,0)} + \frac{1}{2} i\alpha \hat{A} \hat{U}_{2,Y}^{*(0,2)} + \frac{1}{2} \beta \hat{A} \hat{W}_{2,Y}^{*(0,2)} , \quad (4.117)$$

$$\bar{M}_2 = i\alpha \hat{U}_{1,Y} \hat{V}_2^{*(0,2)} - 2\alpha^2 \hat{U}_1 \hat{U}_2^{*(0,2)} , \quad (4.118)$$

$$\bar{M}_3 = 2i\alpha \hat{U}_{1,Y}^* \hat{V}_2^{(2,0)} , \quad (4.119)$$

$$\begin{aligned} \bar{M}_0 = & i\alpha \hat{A} \hat{U}_{2,Y}^{(0,0)} + \frac{1}{2} i\alpha \hat{A} \hat{U}_{2,Y}^{(0,2)} + \frac{1}{2} \beta \hat{A} \hat{W}_{2,Y}^{(0,2)} + i\alpha \hat{A}^* \hat{U}_2^{(2,0)} \\ & + \frac{1}{2} \alpha \hat{A}^* \hat{U}_2^{(2,2)} + \frac{1}{2} \beta \hat{A}^* \hat{W}_2^{(2,2)} + i\alpha \hat{U}_{1,Y} \hat{V}_2^{(0,2)} - 2\alpha^2 \hat{U}_1 \hat{U}_2^{(0,2)} . \end{aligned} \quad (4.120)$$

The nonlinear forcing terms are deliberately split as above so that the solutions and their asymptotic behaviours can be worked out more easily. The solutions driven by $\bar{M}_{j,Y}$ are denoted by $\hat{V}_{3,Y}^{(j)}$ ($j = 0, 1, 2, 3$) respectively, i.e.

$$\hat{L}_0^{(1)} \hat{V}_{3,Y}^{(j)} = \bar{M}_{j,Y} , \quad (j = 0, 1, 2, 3) .$$

It is found that $\hat{V}_3^{(0)}$ makes no contribution to the jump.

It can be shown that

$$\hat{V}_{3,Y}^{(1)} = \frac{1}{2} \bar{S}^4 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \bar{K}_v^{(1)}(\xi, \eta) \hat{A}(t_1 - \zeta) \hat{A}(t_1 - \eta - \zeta) \hat{A}^*(t_1 - \xi - \eta - \zeta) e^{-i\Omega(\zeta - \xi)} d\xi d\eta d\zeta , \quad (4.121)$$

where

$$\bar{K}_v^{(1)}(\xi, \eta) = 2\xi^3 + \xi^2 \eta + 2 \sin^2 \theta (\xi^2 \eta + \xi \eta^2) . \quad (4.122)$$

Integrating with respect to Y from $-\infty$ to $+\infty$, we get the jump

$$\hat{V}_{3,Y}^{(1)}(+\infty) - \hat{V}_{3,Y}^{(1)}(-\infty) = \bar{S}^4 j_0 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \bar{K}_v^{(1)}(\xi, \eta) \hat{A}(t_1 - \xi) \hat{A}(t_1 - \eta - \xi) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta , \quad (4.123)$$

where we have put

$$j_0 = \pi |\bar{S}|^{-1} . \quad (4.124)$$

The solution driven by $\bar{M}_{2,Y}$ needs still more effort. After some algebraic manipulation, we find

$$\begin{aligned} \frac{\partial \bar{M}_2}{\partial Y} = & \bar{S}^4 \sin^4 \theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} [\xi^2 + \xi \zeta + 2\xi \eta + 4 \sin^2 \theta (\eta + \zeta) \eta] [(\zeta - \xi) e^{-i\bar{\Omega}(\zeta - \xi)} - \zeta e^{-i\bar{\Omega}\zeta}] \\ & \hat{A}(t_1 - \zeta) \hat{A}(t_1 - \eta) \hat{A}^*(t_1 - \eta - \xi) e^{-i\bar{\omega}(\zeta - \xi)} d\xi d\eta d\zeta . \end{aligned} \quad (4.125)$$

The solution induced by it can be expressed as

$$\hat{V}_{3,Y}^{(2)} = \bar{S}^4 \sin^4 \theta J , \quad (4.126)$$

where

$$J = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} [\xi^2 + \xi\zeta + 2\xi\eta + 4\sin^2\theta(\eta + \zeta)\eta] [(\zeta - \xi)e^{-i\bar{\Omega}(\zeta + \nu - \xi)} - \zeta e^{-i\bar{\Omega}(\zeta + \nu)}] , \\ \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \eta) \hat{A}^*(t_1 - \nu - \eta - \xi) e^{-i\bar{\omega}(\zeta + \nu - \xi)} d\xi d\eta d\zeta d\nu$$

The factor ξ^{-1} in the kernel creates difficulties in deriving the asymptotic behaviour of $\int_{-Y}^Y J(Y) dY$, which is needed to obtain the jump $(c_j^+ - c_j^-)$. To resolve this problem, we split J into three parts, namely J_1, J_ν, J_0 as follows

$$J = J_1 + 4\sin^2\theta J_\nu - J_0 , \quad (4.127)$$

where

$$J_1 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [\zeta^2 - \xi^2 + 2\eta\zeta - 2\xi\eta - 4\sin^2\theta(\eta + \zeta)\eta] e^{-i\bar{\Omega}(\zeta + \nu - \xi)} \\ \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \eta) \hat{A}^*(t_1 - \nu - \eta - \xi) d\xi d\eta d\zeta d\nu , \quad (4.128)$$

$$J_\nu = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} (\eta + \zeta) \eta \zeta [e^{-i\bar{\Omega}(\zeta + \nu - \xi)} - e^{-i\bar{\Omega}(\zeta + \nu)}] \\ \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \eta) \hat{A}^*(t_1 - \nu - \eta - \xi) e^{-i\bar{\omega}(\zeta + \nu - \xi)} d\xi d\eta d\zeta d\nu , \quad (4.129)$$

$$J_0 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [\xi + 2\eta + \zeta] \zeta e^{-i\bar{\Omega}(\zeta + \nu)} \\ \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \eta) \hat{A}^*(t_1 - \nu - \eta - \xi) e^{i\bar{\omega}\xi} d\xi d\eta d\zeta d\nu .$$

The integral J_0 in fact does not contribute to the jump that we need and hereafter will be ignored.

Integration of J_1 with respect to Y from $-\infty$ to $+\infty$ yields

$$\int_{-\infty}^{+\infty} J_1(Y) dY = -2j_0 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [\zeta^2 - (\zeta + \nu)^2 + 2\eta\zeta - 2(\zeta + \nu)\eta - 4\sin^2\theta(\eta + \zeta)\eta] \\ \hat{A}(t_1 - \nu - \eta) \hat{A}(t_1 - \nu - \zeta) \hat{A}^*(t_1 - 2\nu - \eta - \zeta) d\eta d\zeta d\nu .$$

With the right-hand side simplified, the above equation can be written as follows

$$\int_{-\infty}^{+\infty} J_1(Y) dY = -2j_0 \int_0^{+\infty} \int_0^{+\infty} [2\xi^2(\xi + \eta) + 4\sin^2\theta(\frac{4}{3}\xi^3 + 2\xi^2\eta + \eta^2\xi)] \\ \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta \quad (4.130)$$

The integral J_ν can be reduced to a triple-integral form

$$J_\nu = \frac{1}{2} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} \eta^2 \zeta^2 [e^{-i\bar{\Omega}(\eta - \xi)} - e^{-i\bar{\Omega}\eta}] \\ \hat{A}(t_1 - \zeta) \hat{A}(t_1 - \eta) \hat{A}^*(t_1 - \zeta - \xi) e^{-i\bar{\omega}(\eta - \xi)} d\xi d\eta d\zeta , \quad (4.131)$$

or an alternative form

$$J_v = \frac{1}{2} \hat{W}_0^{(2)} \Pi_0 , \quad (4.132)$$

where

$$\Pi_0 = \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} \zeta^2 [e^{i\bar{\Omega}\xi} - 1] e^{i\bar{\omega}\xi} \hat{A}(t_1 - \zeta) \hat{A}^*(t_1 - \zeta - \xi) d\xi d\zeta . \quad (4.133)$$

It will be seen that the factor η in the kernel of J_v (see equation (4.131)) brought in by combining two Reynolds stress terms is essential for the success of the following procedure of analyzing the asymptote of J_v . Integrating J_v with respect to Y from $-Y$ to Y , we have

$$\begin{aligned} \int_{-Y}^{+Y} J_v(Y) dY &= \bar{S}^{-1} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^{-1} \eta^2 \zeta^2 \left[\frac{\sin \bar{\Omega}(\eta - \xi)}{\eta - \xi} - \frac{\sin \bar{\Omega}\eta}{\eta} \right] \\ &\quad \hat{A}(t_1 - \zeta) \hat{A}(t_1 - \eta) \hat{A}^*(t_1 - \zeta - \xi) e^{-i\bar{\omega}(\eta - \xi)} d\xi d\eta d\zeta . \end{aligned}$$

In order to calculate the main contribution to the above integral, as $\bar{\Omega} \rightarrow \pm\infty$, we split the the kernel into the following form

$$\begin{aligned} &\xi^{-1} \eta^2 \zeta^2 \left[\frac{\sin \bar{\Omega}(\eta - \xi)}{\eta - \xi} - \frac{\sin \bar{\Omega}\eta}{\eta} \right] \\ &= \eta \zeta^2 \frac{\sin \bar{\Omega}(\eta - \xi)}{\eta - \xi} - \eta \zeta^2 \cos \bar{\Omega}\eta \frac{\sin \bar{\Omega}\xi}{\xi} - 2\eta \zeta^2 \sin \bar{\Omega}\eta \frac{\sin^2 \frac{1}{2} \bar{\Omega}\xi}{\xi} . \end{aligned} \quad (4.134)$$

The first term represents the contribution from $\xi = \eta$, the second and the third represent the contribution from $\xi = \eta = 0$. After the kernel is so split and factorized, the contributions from different stationary points are thus identified. To obtain the asymptote of the multiple integral, we essentially have to deal with the asymptotes of several single integrals. By utilizing a Mellin transformation method (Bleistein & Handelsman, 1990), we find that only the first term contributes an $O(1)$ jump, i.e.

$$\int_{-\infty}^{+\infty} J_v(Y) dY = j_0 \int_0^{+\infty} \int_0^{+\infty} \xi \zeta^2 \hat{A}(t_1 - \xi) \hat{A}(t_1 - \zeta) \hat{A}^*(t_1 - \xi - \zeta) d\xi d\zeta .$$

After some transformations, this can be rewritten as follows

$$\begin{aligned} \int_{-\infty}^{+\infty} J_v(Y) dY &= 2j_0 \int_0^{+\infty} \int_0^{+\infty} [\xi^3 + \frac{3}{2}\xi^2\eta + \frac{1}{2}\xi\eta^2] \\ &\quad \hat{A}(t_1 - \xi) \hat{A}(t_1 - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta . \end{aligned} \quad (4.135)$$

The forcing $\bar{M}_{3,YY}$ is the Reynolds stress generated by the interaction between the 3-D fundamental and the 2-D wave component. Utilizing (4.89), we can show

$$\frac{\partial \bar{M}_3}{\partial Y} = 2i\alpha \hat{B} \hat{U}_{1,Y}^* + \alpha \bar{S}^{-1} \hat{U}_{1,Y}^* \bar{L}_0^{(2)} \hat{V}_{2,Y}^{(2,0)} + 2i\alpha \hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} - 2i\alpha \bar{U}_Y^{-1} \hat{U}_{1,Y}^* S_{11}^{(2,0)} . \quad (4.136)$$

The solution driven by the first term is

$$2i\alpha \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1-\tilde{t})} \hat{B}(\tilde{t}) \hat{U}_{1,Y}^*(\tilde{t}) ,$$

while that driven by the second term is

$$\alpha \bar{S}^{-1} \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1-\tilde{t})} \hat{U}_{1,Y}^*(\tilde{t}) \left[\frac{\partial}{\partial \tilde{t}} + 2i\Omega \right] \hat{V}_{2,Y}^{(2,0)}(\tilde{t}) .$$

By integration by parts this is simplified as follows

$$\alpha \bar{S}^{-1} \hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} - 2i\alpha \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1-\tilde{t})} \hat{U}_{1,Y}^*(\tilde{t}) \hat{V}_{2,Y}^{(2,0)}(\tilde{t}) .$$

The solution driven by the last two terms on the right-hand side of (4.136) is

$$2i\alpha \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1-\tilde{t})} \hat{U}_{1,Y}^*(\tilde{t}) \hat{V}_{2,Y}^{(2,0)}(\tilde{t}) - 2i\alpha \bar{U}_y^{-1} \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1-\tilde{t})} \hat{U}_{1,Y}^*(\tilde{t}) S_{11}^{(2,0)}(\tilde{t}) .$$

So we have

$$\begin{aligned} \hat{V}_{3,Y}^{(3)} &= 2i\alpha \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1-\tilde{t})} \hat{B}(\tilde{t}) \hat{U}_{1,Y}^*(\tilde{t}) \\ &\quad + \alpha \bar{S}^{-1} \hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} - 2i\alpha \bar{U}_y^{-1} \int_{-\infty}^{t_1} d\tilde{t} e^{-i\Omega(t_1-\tilde{t})} \hat{U}_{1,Y}^*(\tilde{t}) S_{11}^{(2,0)}(\tilde{t}) . \end{aligned}$$

For brevity, we write $\hat{V}_{3,Y}^{(3)}$ as

$$\hat{V}_{3,Y}^{(3)} = 2i\bar{S}^3 \sin^2 \theta I_v^{(b)} + \bar{S}^4 \sin^4 \theta I_v + \bar{S}^4 \sin^4 \theta I_1 + 2\bar{S}^4 \sin^6 \theta I_2 ,$$

where

$$I_v^{(b)} = \int_0^{+\infty} \int_0^{+\infty} \xi^2 \hat{A}^*(t_1 - \xi - \eta) \hat{B}(t_1 - \eta) e^{-i\Omega(\eta - \xi)} d\xi d\eta , \quad (4.137)$$

$$I_v = \alpha \bar{S}^{-5} \sin^{-4} \theta \hat{U}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} , \quad (4.138)$$

$$I_1 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi \zeta^2 e^{-i\Omega(\xi + \nu - \zeta)} \hat{A}(t_1 - \nu) \hat{A}(t_1 - \nu - \xi) \hat{A}^*(t_1 - \nu - \zeta) d\xi d\zeta d\nu , \quad (4.139)$$

$$I_2 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \zeta^2 e^{-i\Omega(\xi + \eta + \nu - \zeta)} \hat{A}(t_1 - \nu - \eta) \hat{A}(t_1 - \nu - \xi) \hat{A}^*(t_1 - \nu - \zeta) d\xi d\eta d\zeta d\nu . \quad (4.140)$$

Substituting the expression for $\hat{U}_{1,Y}^*$ and $\hat{V}_{2,Y}^{(2,0)}$ into I_v , we have

$$\begin{aligned} I_v &= - \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \zeta^2 [\xi^2 + 4 \sin^2 \theta (\xi + \eta)(\xi + \eta)] (\xi + 2\eta)^{-1} [e^{-i\bar{\Omega}(\xi + 2\eta - \zeta)} - e^{i\bar{\Omega}\zeta}] \\ &\quad \hat{A}(t_1 - \eta) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - \zeta) e^{-i\bar{\omega}(\xi + 2\eta)} d\xi d\eta d\zeta . \end{aligned}$$

Integration with respect to Y yields

$$\begin{aligned} \int_{-Y}^{+Y} I_v(Y) dY \\ = -2\bar{S}^{-1} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \zeta^2 [\xi^2 + 4 \sin^2 \theta (\xi + \eta) \eta] (\xi + 2\eta)^{-1} \left[\frac{\sin \bar{\Omega}(\zeta - \xi - 2\eta)}{\zeta - \xi - 2\eta} - \frac{\sin \bar{\Omega} \zeta}{\zeta} \right] \\ \hat{A}(t_1 - \eta) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - \zeta) e^{-i\bar{\omega}(\xi + 2\eta)} d\xi d\eta d\zeta . \end{aligned}$$

Here again the kernel is split up as follows

$$\begin{aligned} & \zeta^2 [\xi^2 + 4 \sin^2 \theta (\xi + \eta) \eta] (\xi + 2\eta)^{-1} \left[\frac{\sin \bar{\Omega}(\zeta - \xi - 2\eta)}{\zeta - \xi - 2\eta} - \frac{\sin \bar{\Omega} \zeta}{\zeta} \right] \\ &= \zeta [\xi^2 + 4 \sin^2 \theta (\xi + \eta) \eta] \frac{\sin \bar{\Omega}(\zeta - \xi - 2\eta)}{\zeta - \xi - 2\eta} \\ & \quad - \zeta [\xi^2 + 4 \sin^2 \theta (\xi + \eta) \eta] \cos \bar{\Omega} \zeta \frac{\sin \bar{\Omega}(\xi + 2\eta)}{\xi + 2\eta} \\ & \quad - 2\zeta [\xi^2 + 4 \sin^2 \theta (\xi + \eta) \eta] \sin \bar{\Omega} \zeta \frac{\sin^2 \frac{1}{2} \bar{\Omega}(\xi + 2\eta)}{\xi + 2\eta} . \end{aligned} \quad (4.141)$$

Following a similar procedure as in treating J_v , we can obtain

$$\begin{aligned} \int_{-\infty}^{+\infty} I_v(Y) dY = -2j_0 \int_0^{+\infty} \int_0^{+\infty} [\eta^2 + 4 \sin^2 \theta (\xi + \eta) \xi] (2\xi + \eta) \\ \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta . \end{aligned} \quad (4.142)$$

The jumps related to the integrals $I_v^{(b)}$, I_1 and I_2 are straightforward, and are

$$\int_{-\infty}^{+\infty} I_v^{(b)}(Y) dY = 2j_0 \int_0^{+\infty} \xi^2 \hat{A}^*(t_1 - 2\xi) \hat{B}(t_1 - \xi) d\xi , \quad (4.143)$$

$$\int_{-\infty}^{+\infty} I_1(Y) dY = 2j_0 \int_0^{+\infty} \int_0^{+\infty} (\xi + \eta)^2 \eta \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta , \quad (4.144)$$

and

$$\begin{aligned} \int_{-\infty}^{+\infty} I_2(Y) dY = 4j_0 \int_0^{+\infty} \int_0^{+\infty} \left[\frac{7}{3} \xi^3 + 3\xi^2 \eta + \xi \eta^2 \right] \\ \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta . \end{aligned} \quad (4.145)$$

To match $\hat{V}_{3,Y}$ onto the outer expansion, we require

$$c_j^+ - c_j^- = \hat{V}_{3,Y}(+\infty) - \hat{V}_{3,Y}(-\infty) . \quad (4.146)$$

With the jumps so obtained, we have

$$\begin{aligned} c_j^+ - c_j^- = \pi i \text{Sgn}(\bar{U}_y) (a_j^+ r_j + p_j d_j^+ + s_j b_j) \\ + \pi |\bar{S}|^3 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} K(\xi, \eta) \hat{A}(t_1 - \xi) \hat{A}(t_1 - \xi - \eta) \hat{A}^*(t_1 - 2\xi - \eta) d\xi d\eta \\ + 4\pi i \bar{S}^2 \sin^2 \theta \int_0^{+\infty} \xi^2 \hat{A}^*(t_1 - 2\xi) \hat{B}(t_1 - \xi) d\xi , \end{aligned} \quad (4.147)$$

where

$$K(\xi, \eta) = (2\xi^3 + \xi^2\eta) - 2\sin^2\theta(2\xi^3 - \xi\eta^2) - 4\sin^4\theta(\xi^2\eta + \xi\eta^2). \quad (4.148)$$

It is noted that $(a_j^+ - a_j^-)$ and $(d_j^+ - d_j^-)$ correspond to the classic $\pm\pi$ phase shift in the outer expansion, while $(c_j^+ - c_j^-)$ is modified by nonlinearity, which, as will be shown, has a significant effect on the evolution of the disturbance.

So far the jumps obtained are sufficient for deriving the amplitude equation for the oblique waves. The next task is to seek the jumps needed for deriving the amplitude equation for the two-dimensional wave.

In order to make following algebraic manipulation easier, we shall simplify $\hat{V}_{3,YY}$ before proceeding to seek solutions for $\tilde{V}_3$, $\hat{V}_3^{(2,0)}$ etc.

After making the transformation $(\eta - \zeta) \rightarrow \eta$, we write the integral J_1 as follows

$$\begin{aligned} J_1 = & \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [3\zeta^2 - \xi^2 + 2\eta\zeta - 2\xi(\eta + \zeta) - 4\sin^2\theta(\eta + \zeta)(\eta + 2\zeta)] e^{-i\Omega(\zeta + \nu - \xi)} \\ & \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \zeta - \eta) \hat{A}^*(t_1 - \nu - \zeta - \eta - \xi) d\xi d\eta d\zeta d\nu \\ & + \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [(\eta + \zeta)^2 - \xi^2 + 2\zeta(\eta + \zeta) - 2\xi\zeta - 4\sin^2\theta\zeta(\eta + 2\zeta)] e^{-i\Omega(\eta + \zeta + \nu - \xi)} \\ & \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \zeta - \eta) \hat{A}^*(t_1 - \nu - \zeta - \xi) d\xi d\eta d\zeta d\nu. \end{aligned}$$

We further perform the transformation $(\xi - \eta) \rightarrow \xi$, and obtain

$$\begin{aligned} J_1 = & \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [6\zeta^2 + 4(\eta - \xi)\zeta - 2\xi(2\xi + \eta) - 4\sin^2\theta(\eta + 2\zeta)^2] e^{-i\Omega(\zeta + \nu - \xi)} \\ & \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \zeta - \eta) \hat{A}^*(t_1 - \nu - \zeta - \eta - \xi) d\xi d\eta d\zeta d\nu \\ & + \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [(\eta + \zeta)(\xi + 2\eta + 3\zeta) - 4\sin^2\theta\zeta(\xi + \eta + 2\zeta)] e^{-i\Omega(\zeta + \nu + \xi)} \\ & \hat{A}(t_1 - \nu - \zeta) \hat{A}(t_1 - \nu - \zeta - \eta - \xi) \hat{A}^*(t_1 - \nu - \zeta - \eta) d\xi d\eta d\zeta d\nu. \end{aligned}$$

The second integral can be dropped because it is irrelevant to the jump needed. After performing transformation $(\nu + \zeta) \rightarrow \zeta$ and integrating, we can compress the first (quartic) integral into a triple integral

$$\begin{aligned} J_1 = & \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [2\zeta^3 + 2(\eta - \xi)\zeta^2 - 2\xi(2\xi + \eta)\zeta - 4\sin^2\theta(\frac{4}{3}\zeta^3 + 2\eta\zeta^2 + \eta^2\zeta)] e^{-i\Omega(\zeta - \xi)} \\ & \hat{A}(t_1 - \zeta) \hat{A}(t_1 - \zeta - \eta) \hat{A}^*(t_1 - \zeta - \eta - \xi) d\xi d\eta d\zeta. \end{aligned}$$

Applying the similar procedure to I_1 and I_2 , we finally have the result written as

$$\hat{V}_{3,YY} = \hat{V}_{3,YY}^{(\tau)} + \hat{V}_{3,YY}^{(s)} + \hat{V}_{3,YY}^{(t)} + \hat{V}_{3,YY}^{(b)}, \quad (4.149)$$

where

$$\hat{V}_{3,YY}^{(s)} = \bar{S}^4 \sin^4\theta I_v = \alpha \bar{S}^{-1} \hat{U}_{1,YY}^* \hat{V}_{2,Y}^{(2,0)}, \quad (4.150)$$

$$\hat{V}_{3,YY}^{(t)} = 4\bar{S}^4 \sin^6\theta J_v = 2\bar{S}^4 \sin^6\theta \hat{W}_0^{(2)} \Pi_0, \quad (4.151)$$

$$\hat{V}_{3,YY}^{(b)} = 2i\bar{S}^3 \sin^2\theta I_v^{(b)}, \quad (4.152)$$

and

$$\hat{V}_{3,YY}^{(r)} = \bar{S}^4 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \hat{K}_v(\xi, \eta, \zeta) e^{-i\Omega(\zeta-\xi)} \hat{A}(t_1-\zeta) \hat{A}(t_1-\zeta-\eta) \hat{A}^*(t_1-\zeta-\eta-\xi) d\xi d\eta d\zeta dv, \quad (4.153)$$

with the kernel $\hat{K}_v(\xi, \eta, \zeta)$

$$\begin{aligned} \hat{K}_v(\xi, \eta, \zeta) = & \frac{1}{2} \xi^2 (2\xi + \eta) \\ & + 4 \sin^2 \theta \left[\frac{1}{2} \zeta^3 + \frac{1}{2} (\eta - \xi) \zeta^2 - \frac{1}{2} \xi (\xi + 2\eta) \zeta + \frac{1}{4} \eta (\xi + \eta) (2\xi + \eta) \right] \\ & + 4 \sin^4 \theta [-\zeta^3 + (\xi - \eta) \zeta^2 + (\xi^2 + 2\xi\eta) \zeta]. \end{aligned} \quad (4.154)$$

The function $\hat{V}_3^{(2,0)}$ satisfies the following equation

$$\hat{L}_0^{(2)} \hat{V}_{3,YY}^{(2,0)} = \hat{L}_1^{(2)} \hat{V}_3^{(2,0)} + \hat{M}_Y^{(2,0)}, \quad (4.155)$$

where

$$\hat{M}^{(2,0)} = 2i\alpha \hat{U}_1 \hat{U}_2^{(1)} + \frac{\partial}{\partial Y} \left[\frac{1}{2} \hat{U}_1 \hat{V}_2^{(1)} + \frac{1}{2} \hat{A} \hat{U}_2^{(1)} \right].$$

Substitution of (4.89) into (4.155) yields

$$\hat{L}_0^{(2)} \hat{V}_{3,YY}^{(2,0)} = \hat{L}_1^{(2)} \hat{B} - 2i\alpha F_2(Y) \hat{V}_{2,YY} + 2i\alpha \bar{U}_{yy} \left[-\frac{1}{2} i \bar{S}^{-1} \hat{L}_0^{(2)} \hat{V}_{2,Y}^{(2,0)} - \bar{U}_y^{-1} S_{11} \right] + \hat{M}_Y^{(2,0)}. \quad (4.156)$$

The last three terms of the right hand side are produced by the interaction between the oblique waves. After tedious calculation it is found that the solution driven by them makes no contribution to the jump needed. The first forcing term is analogous to that in the two-dimensional case, and the solution driven by it therefore produces analogous jumps (Chapter two), namely

$$\begin{aligned} a_j^+ - a_j^- &= \pi i p_j b_j \text{Sgn}(\bar{U}_y), \\ D_j^+ - D_j^- &= -\pi i R_j b_j \text{Sgn}(\bar{U}_y). \end{aligned} \quad (4.157)$$

Note that the first jump simply confirms the result obtained already.

The function $\tilde{V}_3$ satisfies the equation

$$\hat{L}_0^{(3)} \tilde{V}_{3,YY} = \tilde{M}_Y, \quad (4.158)$$

where we write

$$\tilde{M}_Y = \tilde{M}_Y^{(1)} + \tilde{M}_Y^{(2)} + \tilde{M}_Y^{(3)},$$

and

$$\begin{aligned}\tilde{M}_Y^{(1)} &= -\frac{3}{2}\hat{A}\hat{V}_{2,YY}^{(2,0)} - \beta\hat{A}\hat{W}_{2,Y}^{(2,2)}, \\ \tilde{M}_Y^{(2)} &= 8\beta^2(\hat{W}_1\hat{W}_2^{(2,2)})_Y + 4\beta\hat{W}_1\hat{V}_{2,Y}^{(2,0)}, \\ \tilde{M}_Y^{(3)} &= 2\beta\hat{W}_{1,Y}\hat{V}_{2,Y}^{(2,0)} - 2\beta\hat{W}_{1,Y}\hat{V}_2^{(2,0)}.\end{aligned}$$

The solution driven by $\tilde{M}_Y^{(1)}$ is

$$\bar{S}^4 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \tilde{K}_v^{(1)}(\xi, \eta, \zeta) e^{-i\Omega(\xi+2\zeta+3\eta)} \hat{A}(t_1-\eta) \hat{A}(t_1-\eta-\zeta) \hat{A}(t_1-\eta-\zeta-\xi) d\xi d\eta d\zeta, \quad (4.159)$$

where the kernel is

$$\tilde{K}_v^{(1)}(\xi, \eta, \zeta) = -\xi(\xi+2\zeta)(2\xi+\zeta) + 4\sin^2\theta\left[-\frac{3}{2}\zeta(\xi+\zeta)(\xi+2\zeta)\right]. \quad (4.160)$$

The solution driven by the $\tilde{M}_Y^{(2)}$ is

$$\begin{aligned}-4\bar{S}^4 \sin^4 \theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \{[2\xi^2 + 2\xi\eta + \xi\zeta] + 4\sin^2\theta(\xi+\eta)\eta\} e^{-i\Omega(\xi+\zeta+2\eta+3v)} \\ \hat{A}(t_1-v-\eta) \hat{A}(t_1-v-\zeta) \hat{A}(t_1-v-\eta-\zeta) d\xi d\eta d\zeta dv.\end{aligned}$$

After some simplification it can be written as follows

$$-4\bar{S}^4 \sin^4 \theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \tilde{K}_v^{(2)}(\xi, \eta, \zeta) e^{-i\Omega(\xi+2\zeta+3\eta)} \hat{A}(t_1-\eta) \hat{A}(t_1-\eta-\zeta) \hat{A}(t_1-\eta-\zeta-\xi) d\xi d\eta d\zeta, \quad (4.161)$$

where

$$\begin{aligned}\tilde{K}_v^{(2)}(\xi, \eta, \zeta) &= 3(\xi+\zeta)\eta^2 + 2(2\xi^2 + 4\xi\zeta + 3\zeta^2)\eta \\ &\quad + 4\sin^2\theta[\eta^3 + (\xi+2\zeta)\eta^2 + \zeta(\xi+\zeta)\eta].\end{aligned} \quad (4.162)$$

Before seeking the solution driven by $\tilde{M}_Y^{(3)}$, we write $\tilde{M}_Y^{(3)}$ as follows after utilizing (4.89):

$$\tilde{M}_Y^{(3)} = -2\beta\hat{W}_{1,Y}\hat{B} + [i\beta\bar{S}^{-1}\hat{W}_{1,Y}\hat{L}_0^{(2)}\hat{V}_{2,Y}^{(2,0)} + 2\beta\hat{W}_{1,Y}\hat{V}_{2,Y}^{(2,0)}] + 2\beta\bar{U}_y^{-1}\hat{W}_{1,Y}S_{11}^{(2,0)}.$$

The solution driven by $\tilde{M}_Y^{(3)}$ is then

$$\tilde{V}_{3,Y}^{(3)} = \bar{S}^4 \sin^4 \theta [\tilde{I}_v + \tilde{I}_1 + 2\sin^2\theta\tilde{I}_2] + 2i\bar{S}^3 \sin^2\theta\tilde{I}_v^{(b)},$$

where

$$\tilde{I}_v = i\beta\bar{S}^{-5} \sin^{-4}\theta\hat{W}_{1,Y}\hat{V}_{2,Y}^{(2,0)}, \quad (4.163)$$

$$\tilde{I}_1 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi\zeta^2 e^{-i\Omega(\xi+3v+\zeta)} \hat{A}(t_1-v) \hat{A}(t_1-v-\xi) \hat{A}(t_1-v-\zeta) d\xi d\zeta dv,$$

$$\tilde{I}_2 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \zeta^2 e^{-i\Omega(\xi+\eta+3v+\zeta)} \hat{A}(t_1-v-\eta) \hat{A}(t_1-v-\xi) \hat{A}(t_1-v-\zeta) d\xi d\eta d\zeta dv,$$

$$\tilde{I}_v^{(b)} = \int_0^{+\infty} \int_0^{+\infty} \xi^2 \hat{A}(t_1-\xi-\eta) \hat{B}(t_1-\eta) e^{-i\Omega(\xi+3\eta)} d\xi d\eta. \quad (4.164)$$

The integrals $\tilde{I}_1$ and $\tilde{I}_2$ can be simplified as follows

$$\tilde{I}_1 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \zeta(\xi+\zeta)(\xi+2\zeta)e^{-i\Omega(\xi+2\zeta+3\eta)} \hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\zeta)\hat{A}(t_1-\eta-\zeta-\xi)d\xi d\eta d\zeta, \quad (4.165)$$

$$\tilde{I}_2 = 2 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} [\eta^3 + (\xi+2\zeta)\eta^2 + (\xi^2+2\xi\zeta+2\zeta^2)\eta]e^{-i\Omega(\xi+2\zeta+3\eta)} \hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\zeta)\hat{A}(t_1-\eta-\zeta-\xi)d\xi d\eta d\zeta. \quad (4.166)$$

The solution driven by all three terms can be written as

$$\tilde{V}_{3,YY} = \tilde{V}_{3,YY}^{(r)} + \tilde{V}_{3,YY}^{(s)} + \tilde{V}_{3,YY}^{(b)}, \quad (4.167)$$

where

$$\tilde{V}_{3,YY}^{(s)} = \bar{S}^4 \sin^4 \theta \tilde{I}_v = i\beta \bar{S}^{-1} \hat{W}_{1,YY} \hat{V}_{2,Y}^{(2,0)}, \quad (4.168)$$

$$\tilde{V}_{3,YY}^{(b)} = 2i\bar{S}^3 \sin^2 \theta \tilde{I}_v^{(b)}, \quad (4.169)$$

and

$$\tilde{V}_{3,YY}^{(r)} = \bar{S}^4 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \tilde{K}_v(\xi, \eta, \zeta) e^{-i\Omega(\xi+2\zeta+3\eta)} \hat{A}(t_1-\eta)\hat{A}(t_1-\eta-\zeta)\hat{A}(t_1-\eta-\zeta-\xi)d\xi d\eta d\zeta, \quad (4.170)$$

with the kernel $\tilde{K}_v(\xi, \eta, \zeta)$:

$$\begin{aligned} \tilde{K}_v(\xi, \eta, \zeta) = & -\xi(2\xi+\zeta)(\xi+2\zeta) \\ & +4\sin^2 \theta \left[-\frac{5}{4}\zeta(\xi+\zeta)(\xi+2\zeta) - 3(\xi+\zeta)\eta^2 - (4\xi^2+8\xi\zeta+6\zeta^2)\eta \right] \\ & +4\sin^4 \theta \left[-3\eta^3 - 3(\xi+2\zeta)\eta^2 + (\xi^2-2\xi\zeta-2\zeta^2)\eta \right]. \end{aligned} \quad (4.171)$$

Now we shall seek the solution for W_3 , which satisfies the following equation

$$L_0 W_3 = -\frac{\partial P_3}{\partial z} - F_2(Y) \frac{\partial W_2}{\partial x} - F_3(Y) \frac{\partial W_1}{\partial x} - S_{32}, \quad (4.172)$$

where

$$F_3(Y) = \frac{1}{6} \bar{U}_{yyy} Y^3 + \bar{U}_{yy\tau} \tau_1 Y^2 + \bar{U}_{y\tau\tau} \tau_1^2 Y + \frac{1}{6} \bar{U}_{\tau\tau\tau} \tau_1^3.$$

The function W_3 takes the following form

$$W_3 = \hat{W}_3 E \sin \beta z + \tilde{W}_3 E^3 \sin \beta z + c.c. + \dots \quad (4.173)$$

Here again the components irrelevant to the generation of the 2-D fundamental at the next order have not been written out. It is clear from (4.172) that $\hat{W}_3$ should contain the solutions driven by first three terms on the right hand side of (4.172). However these solutions do not contribute to the jump ($C_j^+ - C_j^-$). So we shall only work out the solutions driven by $-S_{32}$, i.e. it is sufficient to solve for $\hat{W}_3$ satisfying the following equation

$$\hat{L}_0^{(1)} \hat{W}_3 = -S_{32}^{(1,1)} . \quad (4.174)$$

Following a similar procedure to that for solving and simplifying $\hat{V}_3$, we have

$$\hat{W}_3 = \hat{W}_3^{(r)} + \hat{W}_3^{(s)} + \hat{W}_3^{(t)} + \hat{W}_3^{(b)} , \quad (4.175)$$

where

$$\hat{W}_3^{(s)} = \frac{1}{2} i \bar{S}^{-1} \hat{W}_{1,Y}^* \hat{V}_{2,Y}^{(2,0)} , \quad (4.176)$$

$$\hat{W}_3^{(t)} = -i \beta^{-1} \bar{S}^3 \sin^6 \theta \hat{W}_0^{(1)} \Pi_0 , \quad (4.177)$$

$$\hat{W}_3^{(b)} = -\beta^{-1} \bar{S}^2 \sin^2 \theta I_w^{(b)} , \quad (4.178)$$

and

$$\begin{aligned} \hat{W}_3^{(r)} = & -i \beta^{-1} \bar{S}^3 \sin^2 \theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \hat{K}_w(\xi, \eta, \zeta) e^{-i\Omega(\zeta-\xi)} \\ & \hat{A}(t_1-\zeta) \hat{A}(t_1-\zeta-\eta) \hat{A}^*(t_1-\zeta-\eta-\xi) d\xi d\eta d\zeta , \end{aligned} \quad (4.179)$$

with the kernel $\hat{K}_w(\xi, \eta, \zeta)$

$$\hat{K}_w = -\frac{1}{4} \xi^2 + \sin^2 \theta [\zeta^2 + \eta \zeta - \frac{1}{2} \eta(\eta + 3\xi)] + \sin^4 \theta [-(2\xi + \eta)\zeta] . \quad (4.180)$$

The function $I_w^{(b)}$ is defined as follows

$$I_w^{(b)} = \int_0^{+\infty} \int_0^{+\infty} \xi \hat{A}^*(t_1-\xi-\eta) \hat{B}(t_1-\eta) e^{-i\Omega(\eta-\xi)} d\xi d\eta . \quad (4.181)$$

The function $\tilde{W}_3$ satisfies the following equation

$$\hat{L}_0^{(3)} \tilde{W}_3 = -S_{32}^{(3,1)} . \quad (4.182)$$

The solution can be written as follows after simplification

$$\tilde{W}_3 = \tilde{W}_3^{(r)} + \tilde{W}_3^{(s)} + \tilde{W}_3^{(b)} , \quad (4.183)$$

where

$$\tilde{W}_3^{(s)} = \frac{1}{2} i \bar{S}^{-1} \hat{W}_{1,Y} \hat{V}_{2,Y}^{(2,0)} , \quad (4.184)$$

$$\tilde{W}_3^{(b)} = -\beta^{-1} \bar{S}^2 \sin^2 \theta \tilde{I}_w^{(b)} , \quad (4.185)$$

and

$$\tilde{W}_3^{(r)} = -i\beta^{-1}\bar{S}^3 \sin^2\theta \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \tilde{K}_w(\xi, \eta, \zeta) e^{-i\Omega(\xi+2\zeta+3\eta)} \hat{A}(t_1-\eta) \hat{A}(t_1-\eta-\zeta) \hat{A}(t_1-\eta-\zeta-\xi) d\xi d\eta d\zeta, \quad (4.186)$$

with the kernel $\tilde{K}_w(\xi, \eta, \zeta)$

$$\tilde{K}_w(\xi, \eta, \zeta) = \frac{1}{4} \xi(\xi + 2\zeta) + \sin^2\theta(\xi + \zeta)(\eta - \zeta) - \sin^4\theta[3\eta^2 + 2(\xi + 2\zeta)\eta]. \quad (4.187)$$

The function $\tilde{I}_w^{(b)}$ is defined as follows

$$\tilde{I}_w^{(b)} = \int_0^{+\infty} \int_0^{+\infty} \xi \hat{A}(t_1-\xi-\eta) \hat{B}(t_1-\eta) e^{-i\Omega(\xi+3\eta)} d\xi d\eta. \quad (4.188)$$

The streamwise velocity at $O(\epsilon^{\frac{4}{3}})$ takes the form

$$U_3 = \hat{U}_3 E \cos \beta z + \tilde{U}_3 E^3 \cos \beta z + c.c. + \dots \quad (4.189)$$

From the continuity equation it follows that

$$\hat{U}_3 = -(i\alpha)^{-1}(\hat{V}_{3,Y} + \beta \hat{W}_3), \quad (4.190)$$

$$\tilde{U}_3 = -(3i\alpha)^{-1}(\tilde{V}_{3,Y} + \beta \tilde{W}_3). \quad (4.191)$$

For the purpose of deriving the amplitude equation, it is sufficient to seek the 2-D fundamental component in V_4 only, i.e.

$$V_4 = \hat{V}_4^{(2,0)} E^2 + c.c. + \dots \quad (4.192)$$

The function $\hat{V}_4^{(2,0)}$ satisfies the following equation

$$\hat{L}_0^{(2)} \hat{V}_{4,Y}^{(2,0)} = \hat{L}_1^{(2)} \hat{V}_3^{(2,0)} + \hat{L}_2^{(2)} \hat{V}_2^{(2,0)} + 2i\alpha \frac{\partial \hat{S}_{13}^{(2,0)}}{\partial Y} + 4\alpha^2 S_{21}^{(2,0)}, \quad (4.193)$$

where $S_{13}^{(2,0)}$ and $S_{21}^{(2,0)}$ are the components proportional to E^2 in Reynolds stresses S_{13} and S_{21} respectively, i.e.

$$S_{13} = S_{13}^{(2,0)} E^2 + c.c. + \dots, \quad S_{21} = S_{21}^{(2,0)} E^2 + c.c. + \dots,$$

and the Reynolds stresses are defined as follows

$$\begin{aligned} S_{13} &= \frac{\partial}{\partial x}(2U_1 U_3 + U_2^2) + \frac{\partial}{\partial Y}(U_1 V_3 + V_1 U_3 + U_2 V_2) + \frac{\partial}{\partial z}(U_1 W_3 + W_1 U_3 + U_2 W_2), \\ S_{21} &= \frac{\partial}{\partial x} V_1 U_1 + \frac{\partial}{\partial Y} V_1^2 + \frac{\partial}{\partial z} V_1 W_1. \end{aligned} \quad (4.194)$$

It is found that the solution driven by $S_{21}^{(2,0)}$ does not contribute to the jump ($C_j^+ - C_j^-$), and therefore $S_{21}^{(2,0)}$ will be discarded hereafter.

We denote the solution driven by $(\hat{L}_1^{(2)}\hat{V}_3^{(2,0)} + \hat{L}_2^{(2)}\hat{V}_2^{(2,0)})$ by $\hat{V}_4^{(l)}$. Following a similar procedure to that for the two-dimensional case (Chapter two) we find

$$\hat{V}_{4,Y}^{(l)}(+\infty) - \hat{V}_{4,Y}^{(l)}(-\infty) = \pi i \text{Sgn}(\bar{U}_y)(a_j^+ R_j + p_j D_j^+ + b_j S_j) . \quad (4.195)$$

The nonlinear driving term $2i\alpha \frac{\partial S_{13}^{(2,0)}}{\partial Y}$ is complicated. After some calculation we have

$$\begin{aligned} 2i\alpha \frac{\partial S_{13}^{(2,0)}}{\partial Y} = & -2i\alpha \hat{U}_{1,Y} \hat{V}_{3,Y} + i\alpha \hat{U}_{1,YY} \hat{V}_3 - 4i\alpha \beta \hat{U}_{1,Y} \hat{W}_3 \\ & -4i\alpha \beta \hat{U}_1 \hat{W}_{3,Y} - 3i\alpha \hat{U}_1 \hat{V}_{3,YY} - \hat{A}(\hat{V}_{3,YYY} + \beta \hat{W}_{3,YY}) \\ & + \frac{2}{3}i\alpha \hat{U}_{1,Y}^* \tilde{V}_{3,Y} + i\alpha \hat{U}_{1,YY}^* \tilde{V}_3 - \frac{4}{3}i\alpha \beta \hat{U}_{1,Y}^* \tilde{W}_3 \\ & - \frac{4}{3}i\alpha \beta \hat{U}_1^* \tilde{W}_{3,Y} - \frac{1}{3}i\alpha \hat{U}_1^* \tilde{V}_{3,YY} \\ & + 2i\alpha \hat{U}_{2,Y}^{*(0,0)} \hat{V}_2^{(2,0)} + [-4i\alpha \beta \hat{U}_2^{*(0,2)} \hat{W}_{2,Y}^{(2,2)} - \beta \hat{V}_2^{*(0,2)} \hat{W}_{2,Y}^{(2,2)}] \\ & + [-2i\alpha \hat{U}_2^{*(0,0)} \hat{V}_{2,Y}^{(2,0)} - 4i\alpha \beta \hat{U}_{2,Y}^{*(0,2)} \hat{W}_2^{(2,2)} \\ & - 2\beta \hat{V}_{2,Y}^{*(0,2)} \hat{W}_{2,Y}^{(2,2)} - \beta \hat{V}_{2,YY}^{*(0,2)} \hat{W}_2^{(2,2)}] . \end{aligned} \quad (4.196)$$

As already remarked, the $\hat{V}_{3,Y}^{(s)}, \hat{V}_{3,Y}^{(t)}$ terms in $\hat{V}_3$, and the $\hat{W}_3^{(s)}, \hat{W}_3^{(t)}$ terms in $\hat{W}_3$, etc., make the calculation of asymptotic form of the forced solutions difficult. To overcome this difficulty, we notice that from the x -momentum equation, as well as the continuity equation, that at $O(\epsilon^{\frac{3}{2}})$ we have

$$\hat{V}_3 = (i\alpha \bar{U}_y)^{-1} \hat{L}_0^{(1)}(\hat{V}_{3,Y} + \beta \hat{W}_3) - \bar{U}_y^{-1} S_{1,2}^{(1,1)} , \quad (4.197)$$

$$\tilde{V}_3 = (3i\alpha \bar{U}_y)^{-1} \hat{L}_0^{(3)}(\tilde{V}_{3,Y} + \beta \tilde{W}_3) - \bar{U}_y^{-1} S_{1,2}^{(3,1)} . \quad (4.198)$$

Utilizing the above relations and (4.89), we split $2i\alpha S_{13}^{(2,0)}$ into four parts, i.e.

$$2i\alpha S_{13}^{(2,0)} = N^{(r)} + N^{(s)} + N^{(t)} + N^{(b)} , \quad (4.199)$$

where

$$\begin{aligned} N^{(r)} = & -i\alpha \bar{U}_y^{-1} \hat{U}_{1,YY}^* [\hat{A} \hat{U}_{2,Y}^{(2,0)} + \frac{1}{2} \hat{A} \hat{U}_{2,Y}^{(2,2)} - 3\beta \hat{U}_1 \hat{W}_2^{(2,2)} - \bar{U}_y^{-1} \hat{U}_{1,Y} S_{11}^{(2,0)}] \\ & -i\alpha \bar{U}_y^{-1} \hat{U}_{1,YY}^* [\hat{A} \hat{U}_{2,Y}^{*(0,0)} + \frac{1}{2} \hat{A} \hat{U}_{2,Y}^{*(0,2)} - \beta \hat{W}_1 \hat{U}_2^{*(0,0)} + \frac{1}{4} \hat{U}_1 \hat{V}_{2,Y}^{*(0,2)} - \bar{U}_y^{-1} \hat{U}_{1,Y}^* S_{11}^{(2,0)}] \\ & -2i\alpha \beta \hat{U}_{1,Y} \hat{W}_3^{(r)} - 4i\alpha \beta \hat{U}_1 \hat{W}_{3,Y}^{(r)} - 3i\alpha \hat{U}_1 \hat{V}_{3,Y}^{(r)} - \hat{A}(\hat{V}_{3,YYY}^{(r)} + \beta \hat{W}_{3,YY}^{(r)}) \\ & -2i\alpha \beta \hat{U}_{1,Y}^* \tilde{W}_3^{(r)} - \frac{4}{3}i\alpha \beta \hat{U}_1^* [\tilde{W}_{3,Y}^{(r)} + \frac{1}{2} i \bar{S}^{-1} \hat{W}_{1,Y} \hat{V}_{2,Y}^{(2,0)}] - \frac{1}{3}i\alpha \hat{U}_1^* \tilde{V}_{3,Y}^{(r)} \end{aligned}$$

$$\begin{aligned}
& + [-2i\alpha\hat{U}_2^{*(0,0)}\hat{V}_{2,Y}^{(2,0)} - 4i\alpha\beta\hat{U}_{2,Y}^{*(0,2)}\hat{W}_2^{(2,2)} \\
& - 2\beta\hat{V}_{2,Y}^{*(0,2)}\hat{W}_{2,Y}^{(2,2)} - \beta\hat{V}_{2,Y}^{*(0,2)}\hat{W}_2^{(2,2)} - 2i\alpha\bar{U}_y^{-1}\hat{U}_{2,Y}^{*(0,0)}S_{11}^{(2,0)}] \\
& - 4i\alpha\beta\hat{U}_1[\frac{1}{2}i\bar{S}^{-1}\hat{W}_{1,Y}^*\hat{V}_{2,Y}^{(2,0)} - i\beta^{-1}\bar{S}^3\sin^6\theta\hat{W}_0^{(1)}\Pi_{0,Y}] \\
& - \hat{A}[\frac{1}{2}i\beta\bar{S}^{-1}\hat{W}_{1,Y}^*\hat{V}_{2,Y}^{(2,0)} - i\bar{S}^3\sin^6\theta\hat{W}_0^{(1)}\Pi_{0,Y}] ,
\end{aligned} \tag{4.200}$$

$$\begin{aligned}
N^{(s)} = & \bar{U}_y^{-1}\hat{U}_{1,Y}\hat{L}_0^{(1)}(\hat{V}_{3,Y} + \beta\hat{W}_3) - 2i\alpha\hat{U}_{1,Y}(\hat{V}_{3,Y} + \beta\hat{W}_3) \\
& + \frac{1}{3}\bar{U}_y^{-1}\hat{U}_{1,Y}^*\hat{L}_0^{(3)}(\tilde{V}_{3,Y} + \beta\tilde{W}_3) + \frac{2}{3}i\alpha\hat{U}_{1,Y}^*(\tilde{V}_{3,Y} + \beta\tilde{W}_3) \\
& - i\alpha\bar{U}_y^{-1}\hat{U}_{1,Y}[(2i\alpha\bar{U}_y)^{-1}\hat{U}_{1,Y}^*\hat{L}_0^{(2)}\hat{V}_{2,Y}^{(2,0)} + \frac{1}{2}\hat{U}_1^*\hat{V}_{2,Y}^{(2,0)}] \\
& - i\alpha\bar{U}_y^{-1}\hat{U}_{1,Y}^*[(2i\alpha\bar{U}_y)^{-1}\hat{U}_{1,Y}\hat{L}_0^{(2)}\hat{V}_{2,Y}^{(2,0)} - \frac{1}{2}\hat{U}_1\hat{V}_{2,Y}^{(2,0)}] \\
& + \bar{U}_y^{-1}[\hat{U}_{2,Y}^{*(0,0)}\hat{L}_0^{(2)}\hat{V}_{2,Y}^{(2,0)} - \frac{1}{2}\hat{A}\hat{U}_{1,Y}^*\hat{V}_{2,Y}^{(2,0)}] ,
\end{aligned} \tag{4.201}$$

$$\begin{aligned}
N^{(t)} = & -2i\alpha\beta\hat{U}_{1,Y}\hat{W}_3^{(t)} - 2i\alpha\hat{U}_1[\bar{S}^4\sin^6\theta\hat{W}_0^{(2)}\Pi_0] - \hat{A}[-i\bar{S}^5\sin^6\theta\hat{W}_0^{(3)}\Pi_0] \\
& - i\alpha\bar{U}_y^{-1}\hat{U}_{1,Y}[\frac{3}{2}i\alpha\hat{U}_1\hat{U}_2^{*(0,2)} + \frac{1}{2}\hat{U}_{1,Y}\hat{V}_2^{*(0,2)}] \\
& + [-4i\alpha\beta\hat{U}_2^{*(0,2)}\hat{W}_{2,Y}^{(2,2)} - \beta\hat{V}_2^{*(0,2)}\hat{W}_{2,Y}^{(2,2)}] ,
\end{aligned} \tag{4.202}$$

$$\begin{aligned}
N^{(b)} = & -2i\alpha\beta\hat{U}_{1,Y}\hat{W}_3^{(b)} - 4i\alpha\beta\hat{U}_1\hat{W}_{3,Y}^{(b)} - 3i\alpha\hat{U}_1\hat{V}_{3,Y}^{(b)} - \hat{A}(\hat{V}_{3,Y}^{(b)} + \beta\hat{W}_{3,Y}^{(b)}) \\
& - 2i\alpha\beta\hat{U}_{1,Y}^*\tilde{W}_3^{(b)} - \frac{4}{3}i\alpha\beta\hat{U}_1^*\tilde{W}_{3,Y}^{(b)} - \frac{1}{3}i\alpha\hat{U}_1^*\tilde{V}_{3,Y}^{(b)} \\
& - i\alpha\bar{U}_y^{-1}\hat{U}_{1,Y}\hat{U}_{1,Y}^*\hat{B} - i\alpha\bar{U}_y^{-1}\hat{U}_{1,Y}^*\hat{U}_{1,Y}\hat{B} + 2i\alpha\hat{U}_{2,Y}^{*(0,0)}\hat{B} .
\end{aligned} \tag{4.203}$$

The solutions driven by $N^{(r)}$, $N^{(s)}$, $N^{(t)}$, $N^{(b)}$ are denoted by $\hat{V}_4^{(r)}$, $\hat{V}_4^{(s)}$, $\hat{V}_4^{(t)}$, $\hat{V}_4^{(b)}$ respectively.

After integration by parts, we find the solution

$$\begin{aligned}
\hat{V}_{4,Y}^{(s)} = & \frac{1}{3}\bar{U}_y^{-1}\hat{U}_{1,Y}^*[\tilde{V}_{3,Y} + \beta\tilde{W}_3] + \bar{U}_y^{-1}\hat{U}_{1,Y}[\hat{V}_{3,Y} + \beta\hat{W}_3] \cdot \\
& - \frac{1}{2}\bar{U}_y^{-2}(\hat{U}_{1,Y}\hat{U}_{1,Y}^*)_Y\hat{V}_{2,Y}^{(2,0)} + \bar{U}_y^{-1}\hat{U}_{2,Y}^{*(0,0)}\hat{V}_{2,Y}^{(2,0)} .
\end{aligned} \tag{4.204}$$

After further integration by parts, we find that

$$\begin{aligned}
& \hat{V}_{4,Y}^{(s)}(+\infty) - \hat{V}_{4,Y}^{(s)}(-\infty) \\
& = 2i\bar{S}^5\sin^4\theta j_0 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} K_s(\xi, \eta, \zeta) \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \zeta) \\
& \quad \hat{A}(t_1 - \eta - \zeta - \xi) \hat{A}^*(t_1 - 3\eta - 2\zeta - \xi) d\xi d\eta d\zeta \\
& + 2\bar{S}^4\sin^4\theta j_0 \int_0^{+\infty} \int_0^{+\infty} \{ 2\zeta(\eta + \zeta)(\eta + 3\zeta) \hat{B}(t_1 - \zeta) \hat{A}(t_1 - \zeta - \eta) \hat{A}^*(t_1 - 3\zeta - \eta)
\end{aligned}$$

$$+\zeta(\eta+2\zeta)(2\eta+3\zeta)\hat{B}(t_1-\zeta-\eta)\hat{A}(t_1-\zeta)\hat{A}^*(t_1-3\zeta-2\eta)\}d\eta d\zeta, \quad (4.205)$$

where

$$\begin{aligned} K_s(\xi, \eta, \zeta) = & \frac{21}{2}\eta^4 + (3\xi + 17\zeta)\eta^3 + \left(-\frac{47}{8}\xi^2 - \frac{41}{8}\xi\zeta + \frac{113}{16}\zeta^2\right)\eta^2 \\ & + \left(-\frac{13}{4}\xi^3 - \frac{23}{2}\xi^2\zeta - \frac{45}{4}\xi\zeta^2 - \frac{7}{4}\zeta^3\right)\eta \\ & + \left(-\frac{1}{2}\xi^4 - \frac{21}{8}\xi^3\zeta - \frac{45}{8}\xi^2\zeta^2 - 5\xi\zeta^3 - \frac{3}{2}\zeta^4\right). \end{aligned} \quad (4.206)$$

After some manipulation, $\hat{V}_4^{(t)}$ can be written as

$$\hat{V}_{4,YY}^{(t)} = i\bar{S}^5 \sin^4\theta [J_t^{(0)} + \sin^2\theta(J_t^{(1)} + J_t^{(2)}) - \sin^4\theta J_t^{(3)}], \quad (4.207)$$

where

$$\begin{aligned} J_t^{(1)} = & \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \eta \zeta^2 (\eta + 2\zeta + 2v)^2 \xi^{-1} (e^{-i\bar{\Omega}(\eta+2\zeta+2v-\xi)} - e^{-i\bar{\Omega}(\eta+2\zeta+2v)}) e^{-i\bar{\omega}(\eta+2\zeta+2v-\xi)} \\ & \hat{A}(t_1-\zeta-v)\hat{A}(t_1-\zeta)\hat{A}(t_1-\eta-\zeta-v)\hat{A}^*(t_1-\zeta-\xi) d\xi d\eta d\zeta dv, \end{aligned} \quad (4.208)$$

$$\begin{aligned} J_t^{(2)} = & \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \eta (\zeta + v)^2 (\eta + 2v)^2 \xi^{-1} (e^{-i\bar{\Omega}(\eta+2v-\xi)} - e^{-i\bar{\Omega}(\eta+2v)}) e^{-i\bar{\omega}(\eta+2v-\xi)} \\ & \hat{A}(t_1-v)\hat{A}(t_1-\zeta-v)\hat{A}(t_1-\eta-v)\hat{A}^*(t_1-\zeta-v-\xi) d\xi d\eta d\zeta dv, \end{aligned} \quad (4.209)$$

$$\begin{aligned} J_t^{(3)} = & \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \eta \zeta^2 (2\eta + v)(\eta + v) \xi^{-1} (e^{-i\bar{\Omega}(2\eta+v-\xi)} - e^{-i\bar{\Omega}(2\eta+v)}) e^{-i\bar{\omega}(2\eta+v)} \\ & \hat{A}(t_1-\zeta)\hat{A}(t_1-\eta)\hat{A}(t_1-\eta-v)\hat{A}^*(t_1-\zeta-\xi) d\xi d\eta d\zeta dv, \end{aligned} \quad (4.210)$$

$$\begin{aligned} J_t^{(0)} = & \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} K_t^{(0)}(\xi, \eta, \zeta, v) e^{-i\bar{\Omega}(\zeta+2v+2\mu-\xi)} \hat{A}(t_1-\mu-\eta) \\ & \hat{A}(t_1-\mu-v)\hat{A}(t_1-\mu-v-\zeta)\hat{A}^*(t_1-\mu-\eta-\xi) d\xi d\eta d\zeta dv d\mu, \end{aligned} \quad (4.211)$$

and

$$\begin{aligned} K_t^{(0)}(\xi, \eta, \zeta, v) = & \frac{1}{2}\zeta(\zeta + 2v)(2\xi + 4\eta + \zeta + 2v) \\ & + \sin^2\theta \left[-\frac{3}{4}(\xi + 2\eta)(\zeta^2 + 2\zeta v + 2v^2) - \frac{1}{2}v(\zeta + v)(\zeta + 2v)\right]. \end{aligned} \quad (4.212)$$

The factors $(\eta + 2\zeta + 2v)$ in $J_t^{(1)}$, and $(\eta + 2v)$ in $J_t^{(2)}$ and $J_t^{(3)}$, are brought in after the forcing terms are so grouped. The appearance of these factors makes the contributions from different stationary points isolated and therefore there is no essential difficulty in obtaining the asymptotic forms of the related solutions.

After some manipulation, it is found that

$$\begin{aligned} & \hat{V}_{4,Y}^{(t)}(+\infty) - \hat{V}_{4,Y}^{(t)}(-\infty) \\ &= 2i\bar{S}^5 \sin^4 \theta j_0 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} K_t(\xi, \eta, \zeta) \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \zeta) \\ & \quad \hat{A}(t_1 - \eta - \zeta - \xi) \hat{A}^*(t_1 - 3\eta - 2\zeta - \xi) d\xi d\eta d\zeta, \end{aligned} \quad (4.213)$$

where

$$\begin{aligned} K_t(\xi, \eta, \zeta) &= -\frac{99}{16}\eta^4 + \frac{11}{4}(\xi - 2\zeta)\eta^3 + \frac{1}{16}(97\xi^2 + 243\xi\zeta + 59\zeta^2)\eta^2 \\ &+ \left(\frac{15}{8}\xi^3 + \frac{19}{2}\xi^2\zeta + \frac{57}{4}\xi\zeta^2 + \frac{11}{2}\zeta^3\right)\eta \\ &+ \left(\frac{3}{2}\xi^2\zeta^2 + 3\xi\zeta^3 + \frac{3}{2}\zeta^4\right). \end{aligned} \quad (4.214)$$

The solution $\hat{V}_{4,Y}^{(b)}$ and the asymptotic of $\hat{V}_{4,Y}^{(b)}$ as $Y \rightarrow +\infty$ are straightforward, and the final result is as follows:

$$\begin{aligned} & \hat{V}_{4,Y}^{(b)}(+\infty) - \hat{V}_{4,Y}^{(b)}(-\infty) \\ &= 2\bar{S}^4 \sin^2 \theta j_0 \int_0^{+\infty} \int_0^{+\infty} \left\{ 8\zeta^3 + \sin^2 \theta [-2\zeta(\eta^2 + 4\eta\zeta + 7\zeta^2)] \right\} \\ & \quad \hat{B}(t_1 - \zeta) \hat{A}(t_1 - \zeta - \eta) \hat{A}^*(t_1 - 3\zeta - \eta) \\ & \quad + \{ 4\zeta(\eta + \zeta)(\eta + 2\zeta) - \sin^2 \theta \zeta(\eta + 2\zeta)(6\eta + 7\zeta) \} \\ & \quad \hat{B}(t_1 - \zeta - \eta) \hat{A}(t_1 - \zeta) \hat{A}^*(t_1 - 3\zeta - 2\eta) \} d\eta d\zeta. \end{aligned} \quad (4.215)$$

The solution driven by $N^{(r)}$ can be readily written out and there is no essential difficulty in obtaining the asymptotes of $\hat{V}_{4,Y}^{(r)}$ as $Y \rightarrow \pm\infty$. However because the solution consists of many terms, it needs tremendous algebraic manipulation to simplify it. Part of the algebraic manipulation was done by *Mathematica* on a computer. The final result is

$$\begin{aligned} & \hat{V}_{4,Y}^{(r)}(+\infty) - \hat{V}_{4,Y}^{(r)}(-\infty) \\ &= 2i\bar{S}^5 \sin^2 \theta j_0 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} K^{(r)}(\xi, \eta, \zeta) \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \zeta) \\ & \quad \hat{A}(t_1 - \eta - \zeta - \xi) \hat{A}^*(t_1 - 3\eta - 2\zeta - \xi) d\xi d\eta d\zeta \end{aligned} \quad (4.216)$$

where the kernel $K^{(r)}(\xi, \eta, \zeta)$ is

$$\begin{aligned} K^{(r)}(\xi, \eta, \zeta) &= \frac{183}{16}\eta^4 + \left(-\frac{3}{4}\xi + \frac{29}{2}\zeta\right)\eta^3 + \left(-\frac{7}{16}\xi^2 - \frac{77}{16}\xi\zeta + \frac{15}{4}\zeta^2\right)\eta^2 \\ &+ \left(\frac{11}{8}\xi^3 + \frac{9}{4}\xi^2\zeta - \frac{5}{4}\xi\zeta^2 - \frac{3}{4}\zeta^3\right)\eta \\ &+ \left(\frac{1}{2}\xi^4 + \frac{21}{8}\xi^3\zeta + \frac{33}{8}\xi^2\zeta^2 + 2\xi\zeta^3\right). \end{aligned} \quad (4.217)$$

Matching $\hat{V}_{4,Y}$ with outer expansion of corresponding component requires

$$C_j^+ - C_j^- = \hat{V}_{4,Y}(+\infty) - \hat{V}_{4,Y}(-\infty) . \quad (4.218)$$

With the jumps so obtained, we have

$$\begin{aligned} C_j^+ - C_j^- &= \pi i \text{Sgn}(\bar{U}_y)(a_j^+ R_j + p_j D_j^+ + b_j S_j) \\ &+ 3\bar{S}^4 j_0 \int_0^{+\infty} \int_0^{+\infty} \zeta^3 \hat{B}(t_1 - \zeta) \hat{A}(t_1 - \zeta - \eta) \hat{A}^*(t_1 - 3\zeta - \eta) d\eta d\zeta \\ &+ \frac{3}{2} \bar{S}^4 j_0 \int_0^{+\infty} \int_0^{+\infty} \zeta(\eta + \zeta)(\eta + 2\zeta) \hat{B}(t_1 - \zeta - \eta) \hat{A}(t_1 - \zeta) \hat{A}^*(t_1 - 3\zeta - 2\eta) d\eta d\zeta \\ &+ \frac{1}{8} i \bar{S}^5 \sin^2 \theta j_0 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} K_b(\xi, \eta, \zeta) \hat{A}(t_1 - \eta) \hat{A}(t_1 - \eta - \zeta) \\ &\quad \hat{A}(t_1 - \eta - \zeta - \xi) \hat{A}^*(t_1 - 3\eta - 2\zeta - \xi) d\xi d\eta d\zeta , \end{aligned}$$

where the kernel $K_b(\xi, \eta, \zeta)$ is

$$\begin{aligned} K_b(\xi, \eta, \zeta) &= -3\eta^4 - 4(\xi + 2\zeta)\eta^3 - (3\xi^2 + \xi\zeta + 2\zeta^2)\eta^2 \\ &\quad + (3\xi^2 + 5\xi\zeta + 4\zeta^2)\zeta\eta \end{aligned} \quad (4.220)$$

4.5 Evolution Equations of the Amplitudes

4.5.1 Coupled Amplitude Equations

Substituting jumps (4.78), (4.79) and (4.147) into (4.51), we obtain the amplitude equation of the oblique waves as follows:

$$\begin{aligned} \frac{dA}{dt_1} &= g_0 \tau_1 A + g_{11} \int_0^{+\infty} \xi^2 A^*(t_1 - 2\xi) B(t_1 - \xi) d\xi \\ &\quad + g_{12} \int_0^{+\infty} \int_0^{+\infty} K_a(\xi, \eta) A(t_1 - \xi) A(t_1 - \xi - \eta) A^*(t_1 - 2\xi - \eta) d\xi d\eta , \end{aligned} \quad (4.221)$$

where we have written $g_0 = f_0/f$, and $g_{1k} = f_{1k}/f$ ($k = 1, 2$). The constants f and f_0 are the same as for a single two-dimensional wave (Chapter two), i.e.

$$\begin{aligned} f &= i\alpha^{-1} \left\{ \sum_j \pi b_j \left[2i \frac{\bar{U}_{yy}}{|\bar{U}_y|^2} a_j^+ + i b_j \frac{\bar{U}_y \bar{U}_{yyy} - \bar{U}_y^2}{\bar{U}_y \bar{U}_y^2} + b_j \pi \frac{\bar{U}_{yy}^2}{\bar{U}_y^3} \right] + J_1 \right\} , \\ f_0 &= \sum_j \left\{ 2\pi i b_j \frac{\bar{U}_{yy} \bar{U}_\tau}{\bar{U}_y^2} a_j^+ - \pi i b_j^2 \left[\frac{\bar{U}_{yy\tau}}{|\bar{U}_y|} - \frac{\bar{U}_{yy} \bar{U}_{y\tau}}{\bar{U}_y^2} - \frac{(\bar{U}_y \bar{U}_{yyy} - \bar{U}_y^2) \bar{U}_\tau}{\bar{U}_y^3} \right] \right. \\ &\quad \left. + \pi^2 b_j^2 \frac{\bar{U}_{yy} \bar{U}_\tau}{\bar{U}_y^3} \right\} - J_2 , \end{aligned}$$

where the summation is made over all critical layers. Recall that J_1 and J_2 are defined by (4.52) and (4.53) respectively. The constants f_{11} and f_{12} are found to be as follows

$$f_{11} = -3\pi i \alpha^2 \sum_j b_j |b_j|^2 |\bar{U}_y|^2, \quad (4.222)$$

$$f_{12} = \frac{3}{16} \pi \alpha^3 \sum_j b_j^2 |b_j|^2 |\bar{U}_y|^3. \quad (4.223)$$

The kernel $K_a(\xi, \eta)$ is given by

$$K_a(\xi, \eta) = \xi(4\xi^2 + 5\xi\eta + 3\eta^2). \quad (4.224)$$

Essentially the same amplitude equation for the oblique waves was obtained by Goldstein & Lee (1990) in their study of resonant-triad waves in adverse-pressure boundary layers.

Note that the first nonlinear term on the right-hand side of (4.221) represents the resonant interaction between the 2-D and the 3-D waves, and the second represents the self-interaction of the 3-D waves.

Similarly the amplitude equation of the two-dimensional wave is

$$\begin{aligned} \frac{dB}{dt_1} = & 2g_0\tau_1 B + g_{21} \int_0^{+\infty} \int_0^{+\infty} \zeta^3 B(t_1 - \zeta) A(t_1 - \zeta - \eta) A^*(t_1 - 3\zeta - \eta) d\eta d\zeta \\ & + g_{22} \int_0^{+\infty} \int_0^{+\infty} \zeta(\eta + \zeta)(\eta + 2\zeta) B(t_1 - \zeta - \eta) A(t_1 - \zeta) A^*(t_1 - 3\zeta - 2\eta) d\eta d\zeta \\ & + g_{23} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} K_b(\xi, \eta, \zeta) A(t_1 - \eta) A(t_1 - \eta - \zeta) \\ & \quad A(t_1 - \eta - \zeta - \xi) A^*(t_1 - 3\eta - 2\zeta - \xi) d\xi d\eta d\zeta, \end{aligned} \quad (4.225)$$

where we have written $g_{2k} = 2f_{2k}/f$ ($k = 1, 2, 3$), the kernel $K_b(\xi, \eta, \zeta)$ is defined by equation (4.220), and

$$f_{21} = -3\pi \alpha^3 \sum_j b_j^2 |b_j|^2 |\bar{U}_y|^3, \quad (4.226)$$

$$f_{22} = -\frac{3}{2} \pi \alpha^3 \sum_j b_j^2 |b_j|^2 |\bar{U}_y|^3, \quad (4.227)$$

$$f_{23} = -\frac{3}{32} \alpha^4 \pi i \sum_j b_j^3 |b_j|^2 |\bar{U}_y|^4. \quad (4.228)$$

To evaluate these coefficients, we need to solve the Rayleigh equation. This is done by the same method as described in Chapter two. The coefficients g_{11} , g_{21} , and g_{23} are plotted against the wave number α in Fig.19a and Fig.19b. Notice that $g_{12} = -\frac{1}{16}g_{21}$, and $g_{22} = \frac{1}{2}g_{21}$. The plot of g_0 against the wavenumber is the same as in Fig3, provided α is replaced by 2α .

Note that the 3-D waves feed energy back to the 2-D wave through the mutual cubic interaction between the 2-D wave, as well as through the quartic self-interaction

of the 3-D waves. This is very different from conventional resonant-triad amplitude equations, where the 3-D waves feed back energy through a quadratic interaction. As pointed out by Goldstein & Lee (1990), the absence of quadratic feedback from the 3-D waves is due to the fact that the 2-D wave component generated by the quadratic interaction between oblique waves causes no jump across critical layers and therefore does not appear in the amplitude equation.

The existence of the quartic term in amplitude equation was first pointed out by Goldstein & Lee (1990), but they neither gave the exact form of the kernel nor the coefficient of this quartic term. Also, the cubic feedback terms seemed to be overlooked by them¹.

In order to match to the earlier linear stage in an asymptotic sense, the amplitudes A and B should have the following asymptotic behaviours (see e.g. Goldstein & Leib, 1989)

$$A \rightarrow A_0 e^{g_0 \tau_1 t_1}, \quad B \rightarrow B_0 e^{2g_0 \tau_1 t_1} \quad \text{as } t_1 \rightarrow -\infty. \quad (4.229)$$

Thus the evolution of the resonant-triad waves is described by the above coupled amplitude equations (4.221) and (4.225) together with the appropriate "initial" conditions (4.229).

Note also that our amplitude equations bear some resemblance to those of Mankbadi (1991) in that cubic and quartic terms appear in the amplitude equation for the 2-D wave, but with quadratic term being absent. However, the nonlinearity in our equations is non-local. This is because the critical layers are unsteady in our work, in contrast to the steady-viscous one in the case of Mankbadi(1991).

As demonstrated in §4.2, in order for there to be a feedback mechanism, the magnitude of the oblique waves is required to larger than that of the planar wave. To see this point more clearly, we rescale the amplitude equations by introducing the following variables

$$\begin{aligned} \bar{t} &= -\tau_1 t_1 - x_0, \\ \bar{B} &= B e^{iX_B} / (-\tau_1)^4, \\ \bar{A} &= A e^{iX_A} / \lambda, \end{aligned}$$

where the real constants x_0 , X_A , X_B and λ are chosen so that

$$\begin{aligned} e^{-iX_B} (-\tau_1)^4 &= B_0 e^{-2g_0 x_0}, \\ e^{-iX_A} \lambda &= A_0 e^{-g_0 x_0}. \end{aligned}$$

The rescaled amplitude equations and asymptotic conditions are as follows

$$\begin{aligned} \frac{d\bar{A}}{d\bar{t}} &= -g_0 \bar{A} + g_{11} e^{-i\varphi_0} \int_0^{+\infty} \xi^2 \bar{A}^*(\bar{t} - 2\xi) \bar{B}(\bar{t} - \xi) d\xi \\ &\quad + \lambda_0 g_{12} \int_0^{+\infty} \int_0^{+\infty} K_a(\xi, \eta) \bar{A}(\bar{t} - \xi) \bar{A}(\bar{t} - \xi - \eta) \bar{A}^*(\bar{t} - 2\xi - \eta) d\xi d\eta, \end{aligned} \quad (4.230)$$

¹In a later version (1991), Goldstein & Lee obtained independently the quartic feedback terms and the cubic terms for their problem from detailed asymptotic expansions.

$$\begin{aligned}
\frac{d\bar{B}}{d\bar{t}} = & -2g_0\bar{B} + \lambda_0 g_{21} \int_0^{+\infty} \int_0^{+\infty} \zeta^3 \bar{B}(\bar{t}-\zeta) \bar{A}(\bar{t}-\zeta-\eta) \bar{A}^*(\bar{t}-3\zeta-\eta) d\eta d\zeta \\
& + \lambda_0 g_{22} \int_0^{+\infty} \int_0^{+\infty} \zeta(\eta+\zeta)(\eta+2\zeta) \bar{B}(\bar{t}-\zeta-\eta) \bar{A}(\bar{t}-\zeta) \bar{A}^*(\bar{t}-3\zeta-2\eta) d\eta d\zeta \\
& + \lambda_0^2 e^{i\varphi_0} g_{23} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} K_b(\xi, \eta, \zeta) \bar{A}(\bar{t}-\eta) \bar{A}(\bar{t}-\eta-\zeta) \\
& \quad \bar{A}(\bar{t}-\eta-\zeta-\xi) \bar{A}^*(\bar{t}-3\eta-2\zeta-\xi) d\xi d\eta d\zeta, \tag{4.231}
\end{aligned}$$

and

$$\bar{A} \rightarrow e^{-g_0 \bar{t}}, \quad \bar{B} \rightarrow e^{-2g_0 \bar{t}} \quad \text{as } \bar{t} \rightarrow -\infty, \tag{4.232}$$

where $\lambda_0 = |A_0^2/(B_0\tau_1^2)|$, and $\varphi_0 = \arg[A_0^2/(B_0\tau_1^2)]$. The parameter λ_0 represents the importance of the initial amplitude of the three-dimensional waves, and φ_0 is the initial phase difference between the three- and two-dimensional waves.

If $|\lambda_0| \ll 1$, i.e. the initial scaled amplitude of the three-dimensional waves is weak, the equations (4.230) and (4.231) become

$$\frac{d\bar{A}}{d\bar{t}} = -g_0\bar{A} + g_{11}e^{i\varphi_0} \int_0^{+\infty} \xi^2 \bar{A}^*(\bar{t}-2\xi) \bar{B}(\bar{t}-\xi) d\xi, \tag{4.233}$$

$$\frac{d\bar{B}}{d\bar{t}} = -2g_0\bar{B}. \tag{4.234}$$

This recovers the result for the equal-amplitude case considered by Goldstein & Lee (1990). It is clear that the three-dimensional waves have no back reaction on the two-dimensional wave.

Note that in our study we assume that the three-dimensional waves initially are stronger than the two-dimensional wave. If on the other hand, we assume that they are of equal magnitude initially (in fact if $\epsilon \ll \delta^{\frac{3}{4}}$), then the first stage of the evolution will be described by equations (4.233) and (4.234). In this regime, as shown by Goldstein & Lee (1990), the three-dimensional waves experience a faster than exponential growth, while the two-dimensional wave grows exponentially. Ultimately the magnitude of the three-dimensional waves will become equal to $\delta^{\frac{3}{4}}$. At this stage, the double and triple integrals in (4.230) and (4.231) become so large that the cubic and quartic terms can no longer be neglected, no matter how small λ_0 is. The evolution then enters the regime described by the fully coupled equations (4.230) and (4.231). Although an asymptotic matching of the two regimes could be constructed with some careful shift of time origin, it seems unnecessary since the fully coupled equations keep all the terms of (4.233) and (4.234), and are uniformly valid in the two regimes.

4.5.2 Finite-Time Singularity Structure

It seems that solutions of (4.221) and (4.225) develop a finite-time singularity. The structure for it is identified as follows

$$A \sim \frac{a_0}{(t_s - t_1)^{3+i\sigma}}, \quad B \sim \frac{b_0}{(t_s - t_1)^{4+2i\sigma}}, \tag{4.235}$$

as $t_1 \rightarrow t_s$, where a_0, b_0 are generally complex and σ is a real. In fact the structure depends only on the order of the polynomial kernels in the amplitude equations.

Substitution of the above expressions for A and B into (4.221) and (4.225) yields

$$(3 + i\sigma)a_0 = e^{i\varphi_0} g_{11} D_{11} a_0^* b_0 + g_{12} D_{12} a_0 |a_0|^2, \quad (4.236)$$

$$(4 + 2i\sigma)b_0 = (g_{21} D_{21} + g_{22} D_{22}) b_0 |a_0|^2 + e^{i\varphi_0} g_{23} D_{23} a_0^2 |a_0|^2, \quad (4.237)$$

where the constants D_{11} etc. are expressed by the following integrands, which are convergent:

$$\begin{aligned} D_{11} &= \int_0^{+\infty} \xi^2 (1 + \xi)^{-(4+2i\sigma)} (1 + 2\xi)^{-(3-i\sigma)} d\xi, \\ D_{12} &= \int_0^{+\infty} \int_0^{+\infty} K_a(\xi, \eta) [(1 + \xi)(1 + \xi + \eta)]^{-(3+i\sigma)} (1 + 2\xi + \eta)^{-(3-i\sigma)} d\xi d\eta, \\ D_{21} &= \int_0^{+\infty} \int_0^{+\infty} \zeta^3 (1 + \zeta)^{-(4+2i\sigma)} (1 + \zeta + \eta)^{-(3+i\sigma)} (1 + 3\zeta + \eta)^{-(3-i\sigma)} d\eta d\zeta, \\ D_{22} &= \int_0^{+\infty} \int_0^{+\infty} \zeta(\eta + \zeta)(\eta + 2\zeta)(1 + \zeta + \eta)^{-(4+2i\sigma)} \\ &\quad (1 + \zeta)^{-(3+i\sigma)} (1 + 3\zeta + 2\eta)^{-(3-i\sigma)} d\eta d\zeta, \\ D_{23} &= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} K_b(\xi, \eta, \zeta) (1 + 3\eta + 2\zeta + \xi)^{-(3-i\sigma)} \\ &\quad [(1 + \eta)(1 + \eta + \zeta)(1 + \eta + \zeta + \xi)]^{-(3+i\sigma)} d\xi d\eta d\zeta. \end{aligned}$$

Solving b_0 from (4.237) and substituting into (4.236), we have

$$(3 + i\sigma) = \frac{g_{11} g_{23} D_{11} D_{23}}{(4 + 2i\sigma) - (g_{21} D_{21} + g_{22} D_{22}) |a_0|^2} |a_0|^4 + g_{12} D_{12} |a_0|^2. \quad (4.238)$$

From (4.238), $|a_0|$ and σ can be determined. After integrating equations (4.221) and (4.225) numerically, the singularity time t_s can be determined in principle, as described in chapter two, i.e. by fitting the numerically calculated functions A and B to (4.235). However, numerical integration of (4.221) and (4.225) involves evaluation of a triple integral at each time step, and is therefore time consuming. We leave this task to the future. In the later version (1991) of Goldstein & Lee (1990), they integrated numerically the same form of amplitude equations, and confirmed the singularity structure proposed. However, the coefficients in their equations are different from those in ours.

Surprisingly, the 2-D wave has a “more singular” structure than 3-D the waves. This implies that at the late stage the 2-D wave will grow faster than the 3-D waves and attain a comparable magnitude with that of the 3-D wave; thus the initial “non-staggered” arrangement will evolve into a “staggered” pattern. A close examination reveals that the “more singular” structure is associated with the kernel K_b .

To examine the validity of the amplitude equations (4.230) and (4.231), following the idea of Goldstein & Leib (1989), we examine the asymptotic growth rates in the vicinity of the singularity. It follows from the singularity structure (4.235)

that asymptotic growth rates $|A^\dagger|_t/|A^\dagger|$ and $|B^\dagger|_t/|B^\dagger|$ behave like $(|A^\dagger|/|a_0|)^{\frac{1}{3}}$ and $(|B^\dagger|/|b_0|)^{\frac{1}{4}}$ respectively, where $A^\dagger = \epsilon A$ and $B^\dagger = \delta B$ are the unscaled amplitudes of the three- and two-dimensional waves respectively. Therefore the initial scaling, i.e. growth rate $\sim (A^\dagger)^{\frac{1}{3}} \sim (B^\dagger)^{\frac{1}{4}}$, is unchanged by the singularity. To quote from Goldstein & Leib (1989), “This suggests that the basic asymptotic structure will remain intact, and the present solution will not break down until the amplitude of the disturbance becomes order one. The flow will then be fully nonlinear and unsteady in the whole field, i.e. it will be governed by the Euler equations there”. Note that equations (4.233) and (4.234) are valid only up to when the amplitude is of order $\epsilon \log \epsilon \ll 1$.

4.6 Conclusions and Discussions

In this chapter, we have discussed a resonant-triad interaction where all three waves interplay and reinforce each other. The resulting amplitude equations are different from those of Raetz (1959), Craik (1971) and Smith & Stewart (1987) in several important aspects. Firstly, the local growth rates depend on the whole history of evolution, unlike a conventional resonant triad, where the local growth rates depend only on the instantaneous amplitudes of waves. Secondly the back reaction of the oblique waves on the 2-D wave is accounted by two cubic terms, and one quartic term, rather than only one quadratic term. Thirdly, there seems to be no reason why the solution of our amplitude equations cannot develop a finite-time singularity, whereas the occurrence of a finite-time singularity in conventional resonant triad equations depends delicately on the coefficients (Craik 1975); Smith & Stewart (1987) found that there was no finite-time singularity in the resonant-triad equations for lower-branch Tollmien-Schlichting waves in a Blasius boundary layer. It will be interesting to see whether such a singularity can be removed by viscosity effects. However, including them will greatly complicate an already complex analysis. In the case of a two-dimensional wave, it was found that viscosity effects generally delay the occurrence of the finite-time singularity and in certain cases, it can be eliminated completely if a appropriate scaled viscosity parameter exceeds a critical value (Goldstein & Leib 1989). Our study of the two-dimensional case in chapter two has shown that in a substantial range of wavenumbers, nonlinearity has a robust effect in the sense that a singularity always occurs, no matter how large the scaled viscosity parameter is.

The kernels in the amplitude equations are fully determined by nonlinear interactions inside the critical layers and do not depend on the detailed profile of the basic flow. Therefore, the amplitude equations are universally applicable to any basic flow which can support Rayleigh instability waves. The coefficients are expressed in terms of the basic flow profile and the eigenfunction of the Rayleigh equation. However in the present case, since nonlinear effects are dictated by the kernels, these coefficients lose their significance compared with the Landau coefficient in the classical weakly nonlinear theory.

It seems appropriate to compare our work with that of Goldstein & Lee (1990). By detailed asymptotic expansions and analysis of critical-layer dynamics, we have derived the quartic feedback term, while Goldstein & Lee (1990) only wrote out a general form for it by inspection. Also, we have shown that amplitude for the 2-D wave should contain cubic terms, which were overlooked by Goldstein & Lee (1990), (but in a later version (1991), they resolved this problem). By including feedback effects, the evolution equations we obtain are valid until the amplitude of the disturbance becomes order one and thus provide a full description of the evolution of the resonant-triad waves from their linear small-amplitude stage towards the fully-nonlinear order-one-amplitude stage. The equal-amplitude case, where the 2-D and 3-D waves have the same order of magnitude can be treated as a special case of the fully-interactive case.

The discovery that an initially small disturbance can be further amplified by non-linearity provides a possible explanation for instability and transition of the Stokes layer on a flat plate. It appears that instability and transition may be caused by this nonlinear instantaneous growth rather than the net growth of disturbances as studied in Floquet theory. Since the disturbances we considered have high frequencies, we believe that the bursting phenomenon observed in experiments is related to the explosive growth predicted here.

The nonlinear interaction between waves drives a vortex flow, which is unexpected strong in the sense that it has the same magnitude as the oblique waves (cf. Hall & Smith (1991)). The growth of this vortex is enhanced by a resonance mechanism as well. Indeed, both the measurement and visualisation of Hino *et al.* (1976, 1983) revealed the existence of a vortex structure in Stokes layers.

Besides the above overall qualitative agreement of our theoretical predictions with experimental observations. There are some quantitative conclusions that may be drawn by combining the present nonlinear results and the linear results of Cowley (1987). Firstly, it was shown by Cowley that the largest possible wavenumber for a growing mode was $\alpha_m = 1.43$, and we have shown that for a pair of three-dimensional modes $(\alpha, \pm\beta, c)$, resonance with a two-dimensional mode $(2\alpha, 0, c)$ occurs if $\beta = \sqrt{3}\alpha$. Obviously it is required that $2\alpha \leq \alpha_m$, i.e. $\alpha \leq 0.72$. Then $\beta = \sqrt{3}\alpha \leq \sqrt{3} \times 0.72 = 1.25$. Suppose that the disturbances are basically three dimensional and amplified by resonance, then the streamwise wavelength, say λ_x , of the observed disturbance should be larger than $8.73\delta^*$, i.e.

$$\lambda_x \geq 8.73\delta^* ,$$

and the spacing, say λ_z , between two neighbour streaks should be larger than $2.52\delta^*$, i.e.

$$\lambda_z \geq 2.52\delta^* ,$$

where δ^* is thickness of Stokes layer. Thus our theory gives a "lower bound" of space scales of observable disturbances. Monkewitz and Bunster (1987) estimated from their experiment that λ_x is about $10\delta^*$. Hino *et al.* (1983) measured that λ_z is 1.5cm, which if measured by δ^* is $3.7\delta^*$. Both seem to be in the predicted

range. Secondly our nonlinear theory shows that disturbances are substantially amplified by nonlinear effects in the neighbourhood of right branch of neutral curve, which means that initially small disturbances can attain a finite amplitude by this nonlinear amplification. Referring to Cowley's neutral diagram, we can conclude that disturbances are most likely to be observed shortly before or around the phase $\frac{\pi}{2}$ of the basic flow. This conclusion is in agreement with the observation of Monkewitz and Bunster (1987). But others, e.g. Akhavan *et al.* (1991a) and references herein, showed that explosive growth occurs at the end of the acceleration phase, i.e. phase 0 in the notation of the present study. However one should note that all experiments were actually conducted for finite Stokes layers, where an additional parameter $\Lambda = h/\delta^*$ sets in, with h being the half width of a channel or the radius of a pipe. The value of Λ in most experiments is in the range of 3 to 10, but is about 44 in that of Monkewitz and Bunster. Though this difference does not seem alter the basic flow significantly, it is possible that at low Λ , some new mode which does not exist at high Λ may bifurcate and cause transition.

In the present case, the disturbance initially has an "un-staggered" arrangement. The singular structure implies that in the later stage it will evolve to "staggered" arrangement. So far there are yet no visualisation experiments to elucidate this wave structure and its change in pattern.

Chapter 5

Nonlinear Temporal-Spatial Modulation of the Near-Planar Rayleigh Instability waves

Abstract

The nonlinear temporal-spatial modulation of an initially linear near-planar disturbance is studied under a high-Reynolds-number assumption. The disturbance consists of a Rayleigh wave whose amplitude is allowed to be a slowly varying function of spanwise position as well as of time. It is shown that the development of the disturbance is controlled by nonlinear effects from the critical layers when the linear growth rate decreases to $O(\epsilon^{\frac{2}{3}})$ in the course of development, where ϵ is the magnitude of the disturbance. Nonlinear interactions influence the evolution by producing a streamwise vortex and cross-flow distortion. The amplitude equation turns out to be an integro-*partial*-differential one containing historically-dependent nonlinear terms. In this sense it is similar to the amplitude equation of Hickernell (1984) type. But a novel feature of our amplitude equation is that the highest derivative with respect to spanwise position appears in nonlinear terms. These terms are associated with three-dimensionality, and hence represent vortex stretching and tilting effects. The possible property of the amplitude equation is discussed. It is shown that due to history accumulated nonlinear effects, a *localized singularity* may occur in a finite time. This provides a possible explanation for the focusing of streamwise vorticity and the eventual formation of streaks.

5.1 Introduction

Experiments on boundary-layer transition have revealed that after an initial two-dimensional development stage, three-dimensional disturbances grow quickly and significant streamwise streaks are observed to form. Consequently, the “mean flow” is distorted into a “peak-valley” splitting pattern, and a high shear layer appearing in the valley, where secondary small-scale instability occurs in accompaniment with

the eruption of vorticity from the wall layer, e.g. see Klebanoff *et al.* (1962). Direct numerical simulations have confirmed these observations (Kleiser & Zang, 1991). A comprehensive list of literature of experimental and computational studies can be found in Hall & Smith (1989, 1991). Similar events have also been observed in fully-developed turbulent boundary layers (e.g. Kline *et al.*, 1968; Kim *et al.*, 1971).

Such phenomena have been studied theoretically by Stuart (1965, 1984, 1987) from the viewpoint of an initial value problem. In this theory, the streamwise vortices were modeled by a predominately inviscid secondary flow embedded in an otherwise parallel shear flow. The secondary flow is assumed to be symmetric in spanwise direction and have a stagnation point. It is shown that a singularity may occur on the symmetric plane in a finite time, primarily because the flow converges towards the symmetric plane. Recently, Stuart (1991) assessed this problem using a Lagrangian description, which has been proved to be advantageous over an Eulerian description and showed that the singularity was of unsteady separation type (see e.g. Cowley, Van Dommelen & Lam, 1990). It seems that the singularity may not be eliminated by viscosity. If this is true, it is a singularity of the Navier-Stokes equations. It may be possible that near, but before, the singularity time, the stagnation-point flow becomes unstable and breaks down to small scale motions. This speculation seems to be in consistent with the observation of small scale bursts accompanying vorticity concentration and eruption, but has yet to be established. Note that in the theory of Stuart, the secondary flow can be of any kind, does not have to be associated with any instability waves. His work addressed the fundamental aspects of vorticity concentration and eruption, i.e. three-dimensionality¹ and nonlinearity. They are believed to be prerequisites for the flow to undergo stretching and tilting, and thus are essential for the formation of vorticity concentration and eruption.

The significance of three-dimensionality in the *linear* development regime was recognized by Landahl (1975), among others. He showed that the response of parallel shear flow to a three-dimensional perturbation contains a non-decaying *transient* component, in contrast to the two-dimensional case, where only a decaying transient response is excited (Gustavsson, 1978). In certain circumstances, e.g. in so-called "by-pass" transition (Breuer & Landahl, 1990), such a three-dimensional, non-vanishing, transient response may be more important than the instability mode, and may play an crucial role. The analysis of Landahl was based on inviscid and linearization assumptions. Experimental investigations of Breuer & Haritonidis (1990) have shown that the transient portion ultimately decays due to viscous effects. Nevertheless, for low-amplitude disturbances, the theoretical predictions of the early stage development and the spatial features of the flow were confirmed by the experiments for low-amplitude perturbations. A numerical investigation including viscous and nonlinear effects (Breuer & Landahl, 1990) demonstrated that after an initial linear stage, nonlinear effects soon began to enhance the growth of the perturbation.

¹Childress *et al.* (1989) gave an example of a two-dimensional stagnation point flow that developed a finite-time singularity, but this flow this was not physically realistic.

The mean flow was distorted into the form of streaks of low- and high-speed fluid, and a vertical high shear layer was formed followed by a small-scale secondary instability. In the studies of Breuer & Haritonidis (1990) and Breuer & Landahl (1990), the initial flow structure is assumed to consist of a pair of counter-rotating streamwise vortices, but not exactly the stagnation-point type of Stuart (1987, 1991). While the former flow structure appears to be more general, it requires a numerical integration of the full Navier-Stokes equations. The advantage of the stagnation-point flow assumption is that it makes analytical insight possible (Stuart 1991).

An alternative theoretical account, from the point of view as hydrodynamic instability, has been provided by Hall & Smith (1989) and Smith & Walton (1990) for channel flow and Blasius boundary layer respectively. There they considered the nonlinear interaction between planar or near-planar *Tollmien-Schlichting* (TS) waves and longitudinal vortices, induced or input. It was shown that a small three-dimensional "warping" could be amplified by nonlinear effects, and can lead to the localized concentration of streamwise vorticity. These theories are relevant to the situations where the transient response is not strong enough to overtake T-S instability waves, and thus are complementary to those developed by, e.g. Breuer & Landahl (1990). The underlying mechanism is again the combined effects of three-dimensionality and nonlinearity. Smith & Burggraf (1985) in their study of the development of "large amplitude, high-frequency" T-S waves, found that under certain conditions, the disturbance satisfied the classic unsteady boundary-layer equations. By an analogy with unsteady separation of a boundary layer, they suggested that instability might be able to initiate vorticity eruption from the wall layer. There is no doubt that this is a very interesting suggestion. However the disturbance in their study is two-dimensional, while according to experiments, eruption occurs in the three-dimensional stage, and appears to be associated with the spanwise convergence of the flow, as discussed by Stuart (1991). A three-dimensional extension of Smith & Burggraf (1985) may be more relevant. A possible connection between the vorticity eruption and unsteady separation also has been suggested by Ersoy & Walker (1985, 1987), and Hon & Walker (1988). They specifically studied the separation of the boundary-layer flows induced by the so-called hairpin vortex and discrete loop vortex; these vortices are proposed to be the basic elements of wall-layer turbulence. Comments on this point of view can be found, e.g. in Cowley, Van Dommelen & Lam (1990).

This study is to consider vorticity concentration of *Rayleigh instability waves*. We shall follow the idea of Hall & Smith (1989, 1990) and Smith & Walton (1990) in their studies of T-S waves, to examine whether a slight three-dimensional "imperfection" present in a near-planar wave can be further amplified by nonlinear interactions. However, we emphasise that this study is concerned with the Rayleigh wave, in contrast to the T-S wave considered by them. The analysis is thus significantly different from theirs.

The basic flow under consideration is a Stokes layer generated by an sinusoidally oscillating flat plate. In previous chapters, it was shown that before linear disturbances become neutral, a nonlinear interaction can take place inside the critical

layers, and that this interaction controls the growth rate. A finite-time singularity was induced by the interaction. However, so far we have restricted the amplitude of the disturbances to be slowly varying functions of time only. The present study will extend the analysis to the case where the amplitude is a slowly varying function of spanwise position as well as time, is thus allowing for a theoretical study of the vorticity concentration in space. There is evidence in the experiments of Hino *et al.* (1983) that suggests that such a process occurs in Stokes layers. However, compared with steady flows, experimental investigations on this aspect are inadequate, and much less detailed. The present study is partly initiated by this observation, and partly by theoretical interests.

On the side of theoretical studies, many theories assume that disturbances are purely periodic in spanwise direction, and thus only evolve with time or downstream. Such an assumption precludes the possibility of a study of energy and/or vorticity re-distribution in space under nonlinear effects, which as stated above is an important phenomena in transition process. On the other hand, it is impossible to keep disturbances perfectly periodic in ^{the} spanwise direction; some "imperfection" is always present. Thus an extension to the more general case of temporal-spatial modulation is desirable. Temporal-spatial modulation of TS instability waves was studied by Stewartson & Stuart (1971). They considered the development of a two-dimensional disturbance which initially centered on a streamwise location, and derived an amplitude equation of nonlinear Schrodinger type, which is a generic equation governing the envelop of a two-dimensional wave packet. An analytical and numerical investigation by Hocking, Stewartson & Stuart (1972) showed that a localized singularity could occur in a finite time, implying a focusing of vorticity in space (see also Hocking & Stewartson, 1972). Hocking, Stewartson & Stuart (1972), and Davey, Hocking & Stewartson (1974) considered a disturbance centered on a point. There in addition to a slow streamwise variable, they introduced a slow spanwise variable to take account of the spanwise modulation. In this study, we are primarily interested in spanwise focusing, so we shall assume that the disturbance is a near-planar wave with an amplitude dependent on the spanwise coordinate and time. The dependence of amplitude on Z brings in a three-dimensional "warping". Following the idea of Hall & Smith (1989, 1990) and Smith & Walton (1990) in their studies of T-S waves, we shall investigate whether such a small warping in a Rayleigh wave can be amplified by nonlinear effects. We emphasise that the theory presented in this study also applies to free shear layers, where the formation of streamwise vortices is a significant phenomena; see e.g. Jimenez (1983), Lasheras & Choi (1988), Lasheras, Choi & Maxworthy (1989), and Nygaard & Glezer (1991). Moreover, the formation of the streamwise vortices appear to be "facility-induced" (e.g. Jimenez, 1983; Nygaard & Glezer, 1991); thus the present approach seems to be closely relevant.

As in previous chapters, suppose that a stability wave with wavenumber α is excited . . . at time τ_i on the left branch of the neutral curve (see Fig.1). We assume that the dependence on the spanwise location is *weak* to ensure that the dispersion relation is not affected at leading order by this weak three-dimensionality. According

to linear theory (Tromans, 1977; Cowley, 1987), the disturbance grows exponentially until the neutral time, say τ_0 , is approached. In the vicinity of the neutral time, the linear growth rate is small, and critical layers emerge. Due to the singular nature of the Rayleigh equation near a critical layer, nonlinear interactions, among other effects may become important and have to be taken into account. The main concern of this study is how *critical-layer nonlinearity* affects the temporal-spatial evolution of such a small disturbance, in particular how the disturbance vorticity is re-distributed by this nonlinearity. For general surveys of critical-layer theories, readers are referred to Maslowe (1986) and Stewartson (1978). References closely relevant to the present study are Hickernell (1984), Goldstein & Leib (1989), Goldstein & Choi (1989), and Goldstein & Lee (1990).

5.2 General Scaling Arguments

The flow is described by Cartesian coordinates (x, y, z) , where x is parallel to the oscillating direction of the plate, y is normal to the plate and z is the spanwise direction. If (x, y, z) are nondimensionalised by δ^* , time τ by ω^{-1} , and velocity (U, V, W) by U_0 , then the basic flow field can be written as

$$(\bar{U}, \bar{V}, \bar{W}) = (\cos(\tau - y)e^{-y}, 0, 0) .$$

We denote the perturbed flow by

$$(\bar{U} + u, \bar{V} + v, \bar{W} + w) .$$

Substitution into the Navier-Stokes equations yields equations for (u, v, w) .

Following Tromans (1977) and Cowley (1987), we study high-frequency instability waves, and introduce the fast time scale,

$$t = R\tau , \tag{5.1}$$

to take account of the frequencies of the waves.

The normal velocity component v of a disturbance of $O(\epsilon)$, to the leading order, has the form

$$\epsilon A_0 \bar{v}(y, \tau) E \cos \beta z + c.c. ,$$

where $\epsilon \ll 1$, and β is the spanwise wavenumber and we assume that $\beta \ll 1$. We emphasise that for the purpose of deriving the scaling, it is sufficient to write the dependence on z in this special form. But from the next section, a general dependence on slow spatial variable $Z = \beta z$ is allowed for. The function $\bar{v}$ satisfies the Rayleigh equation, and the order-one complex constant number A_0 is a measure of the scaled amplitude of the disturbance. Hereafter *c.c* represents the complex conjugate of the part written out. For convenience, we have defined

$$E = \exp(i\alpha x - i\hat{\theta}(t)) ,$$

where

$$\frac{d\hat{\theta}}{dt} = \frac{1}{2}\alpha c(\tau) + \frac{1}{2}R^{-\frac{1}{2}}\Omega_1(\tau) + \dots$$

with c being the complex phase velocity, i.e. $c = c_r + ic_i$. In this study, we assume that the wavenumber α is of order one.

As discussed earlier, to facility critical-layer nonlinear effects, we should concentrate on the times near to the neutral time, say

$$\tau = \tau_0 + \epsilon^l \tau_1 ,$$

where the index l is a measure of how “quickly” the nonlinear evolution stage occurs.

Before proceeding to a formal asymptotic expansion, we need to set down the underlying scaling, i.e. to specify the evolution time scale, the thickness of the critical layers, and the spanwise scale β in terms of the magnitude of the disturbances ϵ . For this purpose, we introduce an intermediate time scale $t_1 = \alpha c_i t$, which is much “slower” than the time scale of the instability waves, but much “faster” than the time scale of the underlying Stokes flow; the disturbances are then described by the time variables t , t_1 , and τ . In terms of these new variables, the x -momentum equation can be written as follows

$$\left\{ \alpha c_i \frac{\partial}{\partial t_1} + (\bar{U} - c_r) \frac{\partial}{\partial x} + 2R^{-1} \frac{\partial}{\partial \tau} \right\} u + v \frac{\partial \bar{U}}{\partial y} - R^{-1} \Delta u = -\frac{\partial p}{\partial x} - \frac{\partial}{\partial x} uu - \frac{\partial}{\partial y} uv - \frac{\partial}{\partial z} uw ,$$

where $\frac{\partial}{\partial t}$ has been replaced by $-c_r \frac{\partial}{\partial x}$ to the order required, and

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} .$$

We are particularly interested in the case where the nonlinear evolution occurs on the *fastest* possible time scale, i.e. the time-variation term appears at leading order in the critical-layer equations. Suppose that the thickness of the critical layer is $O(\mu)$, then near a critical layer $(\bar{U} - c_r)$ is $O(\mu)$ if $\bar{U}_y(y_c) = O(1)$. Balancing $\alpha c_i \frac{\partial u}{\partial t_1}$ and $(\bar{U}_y - c_r) \frac{\partial u}{\partial x}$ yields

$$\mu \sim O(c_i) . \quad (5.2)$$

This scaling brings in a new non-equilibrium effect that does not occur in the analysis of, e.g. Davey, Hocking & Stewartson (1974), Hall & Smith (1989). It leads to a different type of amplitude equation that contains historically-dependent nonlinear terms of Hickernell (1984) type.

To ensure that a spatial modulation term $\frac{\partial^2}{\partial z^2}$ appears in the final amplitude equation, we require that

$$\beta^2 \sim O(\alpha c_i) . \quad (5.3)$$

From the asymptotic property of the Rayleigh equation, we know that $v \sim O(\epsilon)$ as $y \rightarrow y_c$, where y_c is the critical level defined by $\bar{U}(y_c) = c_r$. The asymptotic behaviour of the cross-flow component w is:

$$w \sim \epsilon \beta (y - y_c)^{-1} , \quad (5.4)$$

where the pole type of singularity is associated with three-dimensionality as pointed out by Benney (1961). From the continuity equation, we know that

$$u \sim \epsilon \beta^2 (y - y_c)^{-1} + \epsilon \log(y - y_c) , \quad (5.5)$$

where the logarithmic singularity is due to the fact that $\bar{U}_{yy}(y_c) \neq 0$ for the Stokes layer. It turns out that for deriving scaling, we can concentrate only on the pole type of singularity without losing generality, because the resultant scaling happens to take the logarithmic singularity into account as well (see below).

To fix the scaling, we now turn to analyze the nonlinear interaction within the critical layers, following the ideas of Hickernell (1984). The quadratic interaction, say through vu_y produces a forcing of $O(\epsilon^2 \beta^2 \mu^{-2})$. Balancing this forcing term with $\alpha c_i \frac{\partial u}{\partial t_1}$ in the x -momentum equation, we know that it generates a second harmonic and a mean-flow distortion, say $u^{(2)}$,

$$u^{(2)} \sim O(\epsilon^2 \beta^2 \mu^{-3}) .$$

The cubic interaction through $vu_y^{(2)}$ produces a forcing of $O(\epsilon^3 \beta^2 \mu^{-4})$. Balancing this forcing term with $\alpha c_i \frac{\partial u}{\partial t_1}$, we know that it drives a fundamental $u^{(3)}$:

$$u^{(3)} \sim O(\epsilon^3 \beta^2 \mu^{-5}) .$$

This fundamental is required to match with the unsteadiness of $O(\alpha c_i \epsilon) \sim O(\mu \epsilon)$ in the outer region, i.e.

$$O(\epsilon^3 \beta^2 \mu^{-5}) \sim O(\epsilon \mu) , \quad (5.6)$$

so that the evolution of the disturbance is affected by nonlinear interactions inside the critical layers. Noting that $\alpha \sim O(1)$, from (5.2), (5.3) and (5.6), we have

$$\beta \sim O(\epsilon^{\frac{1}{5}}) , \quad (5.7)$$

$$\mu \sim O(\epsilon^{\frac{2}{5}}) . \quad (5.8)$$

Finally, we check for the influence of viscosity. We consider the typical case where the viscous diffusion terms appear in the leading-order critical-layer equations. This happens when

$$R^{-\frac{1}{3}} \sim O(\mu) ,$$

i.e.

$$R \sim O(\epsilon^{-\frac{6}{5}}) . \quad (5.9)$$

So we write

$$R^{-1} = \lambda \epsilon^{\frac{6}{5}} , \quad (5.10)$$

where the parameter λ is introduced to reflect viscosity effects (cf. Haberman, 1972), and is assumed to be order one.

Equations (5.7), (5.8) and (5.10) set the underlying scalings for this study. Note that the thickness of the critical layer, and thus the nonlinear evolution time scale

happen to be the same as those in the case where the disturbance is a purely two-dimensional wave (Chapter two). However it should be emphasised that this is only a coincidence because the present scaling is obtained from the consideration of the pole type of singularity, irrespective of the presence of the logarithmic branch point. Nevertheless, under the present scaling, nonlinear effects associated with the latter singularity are also incorporated. As a result, it is expected that a corresponding nonlinear term with coefficient proportional to $\bar{U}_{yy}(y_c)$ will appear in the final amplitude equation (see §5).

With the scaling specified above, we concentrate on times near to:

$$\tau = \tau_0 + \epsilon^{\frac{2}{5}} \tau_1, \quad (5.11)$$

i.e. the time at which the linear growth rate is $O(\epsilon^{\frac{2}{5}})$. It is at this stage that nonlinear effects control the development of the disturbance, and possibly redistribute the disturbance vorticity. For convenience, the nonlinear evolution time scale is rewritten as follows

$$t_1 = \frac{1}{2} \epsilon^{\frac{2}{5}} t. \quad (5.12)$$

The evolution of the disturbance is thus described by four time variables, namely t , t_1 , τ_1 and τ_0 ; the dependence on the slow time variables τ_1 and τ_0 is parametric to the order of approximation in this study. With the multiple time scales so introduced, the time derivative $\frac{\partial}{\partial \tau}$ should be interpreted as follows

$$\frac{\partial}{\partial \tau} \rightarrow R \frac{\partial}{\partial t} + \frac{1}{2} R \epsilon^{\frac{2}{5}} \frac{\partial}{\partial t_1} + \epsilon^{-\frac{2}{5}} \frac{\partial}{\partial \tau_1} + \frac{\partial}{\partial \tau_0}. \quad (5.13)$$

Since the basic flow $\bar{U}$ evolves on the slow time variable τ , it is sufficient to expand its profile at time τ in a Taylor series about the neutral time τ_0

$$\bar{U}(y, \tau) = \bar{U}(y, \tau_0) + \epsilon^{\frac{2}{5}} \bar{U}_\tau(y, \tau_0) \tau_1 + \dots$$

Hereafter all basic flow quantities will be evaluated at τ_0 unless otherwise mentioned.

To take account of spanwise modulation, we introduce a slow spanwise spatial variable Z of the following form

$$Z = \epsilon^{\frac{1}{5}} z, \quad (5.14)$$

and thus

$$\frac{\partial}{\partial z} \rightarrow \epsilon^{\frac{1}{5}} \frac{\partial}{\partial Z}. \quad (5.15)$$

The overall evolution process can be summarized in Fig.2. As illustrated, the disturbance is initially well described by linear theory (Tromans, 1977; Cowley, 1987). As the neutral time τ_0 is approached, the growth rate becomes small, and linear critical layers emerge. As the growth rate decreases further to $O(\epsilon^{\frac{2}{5}})$, i.e. at times $\tau_1 = O(1)$, nonlinear interactions take place and control the development. In particular, the initial spatial distribution of the vorticity may be rearranged by nonlinear effects. The earlier linear and the subsequent nonlinear solutions are required to match in an asymptotic sense in the mutual overlapping domain, as insisted in, e.g. Goldstein & Leib (1988, 1989) and Goldstein & Choi (1989).

5.3 Outer Expansion

5.3.1 Asymptotic Solutions Near the Critical Layers

Outside the critical layers, the flow is basically linear and inviscid, to the order of approximation in this study. The velocity components and pressure of the disturbance are expanded as follows

$$u = \epsilon u_1 + \epsilon^{\frac{6}{5}} u_2 + \epsilon^{\frac{7}{5}} u_3 + \dots, \quad (5.16)$$

$$v = \epsilon v_1 + \epsilon^{\frac{6}{5}} v_2 + \epsilon^{\frac{7}{5}} v_3 + \dots, \quad (5.17)$$

$$w = \epsilon^{\frac{6}{5}} w_1 + \epsilon^{\frac{7}{5}} w_2 + \epsilon^{\frac{8}{5}} w_3 + \dots, \quad (5.18)$$

$$p = \epsilon p_1 + \epsilon^{\frac{6}{5}} p_2 + \epsilon^{\frac{7}{5}} p_3 + \dots. \quad (5.19)$$

The earlier linear solutions suggest that v_1 can be written as

$$v_1 = A(Z, t_1, \tau_1, \tau_0) \bar{v}_1(y, \tau_0) E + c.c. \quad (5.20)$$

where $A(Z, t_1, \tau_1, \tau_0)$ is the amplitude of the disturbance. Note that A is allowed to be an arbitrary function of Z . Here we have defined that $E = \exp(i\alpha x - i\hat{\theta})$, and

$$\frac{d\hat{\theta}}{dt} = \Omega_0(\tau_0) + \frac{1}{2} \lambda^{\frac{1}{2}} \epsilon^{\frac{3}{5}} \Omega_1(\tau_0) + \dots, \quad (5.21)$$

where the correction to the complex frequency at $O(\epsilon^{\frac{3}{5}})$, i.e. $O(R^{-\frac{1}{2}})$, is due to viscous effects. The dependence on τ_0 and τ_1 , is due to the time variation of the basic flow. Because they can be treated as parameters to the order of interest in this study, hereafter they will not be written out explicitly.

The function $\bar{v}_1$ satisfies the Rayleigh equation:

$$(\bar{U} - c)(D^2 - \alpha^2) \bar{v}_1 - \bar{U}_{yy} \bar{v}_1 = 0, \quad (5.22)$$

where we have written

$$2\Omega_0 = \alpha c.$$

Let $\eta = y - y_c^j$, where y_c^j is the j -th critical level; then as $\eta \rightarrow \pm 0$, $\bar{v}_1$ has the following asymptotic behaviour:

$$\bar{v}_1 \sim a_j^\pm \phi_a + b_j^\pm [\phi_b + p_j \phi_a \log |\eta|], \quad (5.23)$$

where

$$\phi_a = \eta + \frac{1}{2} p_j \eta^2 + \dots, \quad (5.24)$$

$$\phi_b = 1 + q_j \eta^2 + \dots. \quad (5.25)$$

The $O(\epsilon^{\frac{6}{5}})$ term is introduced to take account of nonlinear interactions inside the critical layer, and has the following form

$$v_2 = \bar{v}_2(Z, y, t_1) E^2 + c.c..$$

The function $\bar{v}_2$ satisfies the same equation and hence has the same asymptotic behaviour as $\bar{v}_1$ provided α is replaced by 2α . Later, however, $\bar{v}_2$ will be found to be identically zero.

The function v_3 takes account of unsteady effects in the vicinity of the neutral time, and takes the following form

$$v_3 = \bar{v}_3(Z, y, t_1)E + c.c. \quad (5.26)$$

The function $\bar{v}_3$ is the deviation of the eigenfunction from its neutral state, and is found to satisfy the following inhomogeneous Rayleigh equation:

$$\begin{aligned} & \{D^2 - (\alpha^2 + \frac{\bar{U}_{yy}}{\bar{U} - c})\}\bar{v}_3 \\ &= (i\alpha)^{-1} \left\{ \left[-\frac{\partial A}{\partial t_1} - (i\alpha \bar{U}_\tau \tau_1)A \right] \frac{\bar{U}_{yy}}{(\bar{U} - c)^2} + \frac{i\alpha \bar{U}_{yy\tau} \tau_1 A}{\bar{U} - c} \right\} \bar{v}_1 - \frac{\partial^2 A}{\partial Z^2} \bar{v}_1, \end{aligned} \quad (5.27)$$

where the last term of the above equation represent the three-dimensional warping.

The asymptotic behaviour of $\bar{v}_3$ is as follows:

$$\begin{aligned} \bar{v}_3 \sim & -b_j^\pm r_j \log |\eta| + (a_j^\pm r_j + b_j^\pm s_j) \eta \log |\eta| + \dots \\ & + c_j^\pm \phi_a + d_j^\pm [\phi_b + p_j \phi_a \log |\eta|], \end{aligned} \quad (5.28)$$

where

$$p_j = \frac{\bar{U}_{yy}}{\bar{U}_y}, \quad (5.29)$$

$$q_j = \frac{1}{2}\alpha^2 + \frac{1}{2}\frac{\bar{U}_{yyy}}{\bar{U}_y} - \frac{\bar{U}_{yy}^2}{\bar{U}_y^2}, \quad (5.30)$$

$$r_j = (i\alpha)^{-1} \frac{\bar{U}_{yy}}{\bar{U}_y^2} \left[-\frac{\partial A}{\partial t_1} - (i\alpha \bar{U}_\tau \tau_1)A \right], \quad (5.31)$$

$$\begin{aligned} s_j = & (i\alpha)^{-1} \left\{ (-i\alpha \bar{U}_{y\tau} \tau_1 A) \frac{\bar{U}_{yy}}{\bar{U}_y^2} + (i\alpha \tau_1 A) \frac{\bar{U}_{yy\tau}}{\bar{U}_y} \right. \\ & \left. + \left(-\frac{\partial A}{\partial t_1} - i\alpha \bar{U}_\tau \tau_1 A \right) \frac{\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y^3} \right\}, \end{aligned} \quad (5.32)$$

where all basic flow quantities are evaluated at time τ_0 and the critical layer y_c^j . The jumps $(a_j^+ - a_j^-)$ etc. will be determined by analyzing the critical layers.

The leading-order cross-flow component w_1 satisfies

$$\left(2\frac{\partial}{\partial t} + \bar{U}_y \frac{\partial}{\partial x} \right) \frac{\partial w_1}{\partial x} = \left(2\frac{\partial}{\partial t} + \bar{U}_y \frac{\partial}{\partial x} \right) \frac{\partial v_1}{\partial Z}. \quad (5.33)$$

This suggests the following form for w_1 :

$$w_1 = A_Z(Z, t_1) \bar{w}_1(y) E + c.c.$$

It is found that

$$\bar{w}_1 = \alpha^{-2} [\bar{v}_{1,y} - \frac{\bar{U}_y}{\bar{U} - c} \bar{v}_1] . \quad (5.34)$$

The streamwise velocity u_1 takes the following form

$$u_1 = A(Z, t_1) \bar{u}_1(y) E + c.c .$$

From the continuity equation, we find that

$$\bar{u}_1 = i\alpha^{-1} \bar{v}_{1,y} . \quad (5.35)$$

The $O(\epsilon^{\frac{6}{5}})$ term u_2 is introduced to take account of nonlinear interactions inside the critical layers, and has the form

$$u_2 = u_2^{(2)} E^2 + U_2^{(0)} + c.c . \quad (5.36)$$

Later, the harmonic component $u_2^{(2)}$ turns out to be zero; the mean-flow distortion is non-zero, but its determination is not needed.

From the continuity equation, the solution for u_3 is found to be $u_3 = \bar{u}_3(Y, Z, t_1) E^2 + c.c$, where

$$\bar{u}_3 = i\alpha^{-1} \{A(Z, t_1) \bar{v}_{3,Y} + A_{ZZ}(Z, t_1) \bar{w}_1\} . \quad (5.37)$$

The pressure p_1 can be written as $p_1 = A(Z, t_1) \bar{p}_1(y) E + c.c$, and

$$\bar{p}_1 = i\alpha^{-1} [\bar{U}_y \bar{v}_1 - (\bar{U} - c) \bar{v}_{1,y}] . \quad (5.38)$$

We now introduce an inner variable:

$$Y = \frac{\eta}{\epsilon^{\frac{2}{5}}} = \frac{y - y_c^j}{\epsilon^{\frac{2}{5}}} . \quad (5.39)$$

The outer expansions of u , v , w , and p written in terms of the inner variable are as follows:

$$\begin{aligned} v \sim & \epsilon b_j^\pm A E + \epsilon^{\frac{7}{5}} \log \epsilon^{\frac{2}{5}} (-b_j^\pm r_j + b_j^\pm p_j A Y) E \\ & + \epsilon^{\frac{7}{5}} [(-b_j^\pm r_j \log |Y| + d_j^\pm) + A(a_j^\pm Y + b_j^\pm Y \log |Y|)] E \\ & + \epsilon^{\frac{9}{5}} \log \epsilon^{\frac{2}{5}} [(a_j^\pm r_j + b_j^\pm s_j + d_j^\pm p_j) Y + \frac{1}{2} A p_j b_j^\pm Y^2] E \\ & + \epsilon^{\frac{9}{5}} [c_j^\pm Y + (a_j^\pm r_j + b_j^\pm s_j + d_j^\pm p_j) Y \log |Y| + \dots] E \\ & + c.c. + \dots , \end{aligned} \quad (5.40)$$

$$u \sim \epsilon i \alpha^{-1} \{A(a_j^\pm - b_j^\pm r_j Y^{-1}) - \alpha^{-2} b_j^\pm A_{ZZ} Y^{-1}\} E + c.c + \dots , \quad (5.41)$$

$$w \sim -\epsilon^{\frac{4}{5}} \alpha^{-2} b_j^\pm A_Z Y^{-1} E + c.c + \dots , \quad (5.42)$$

$$p \sim \epsilon i \alpha^{-1} \bar{U}_y b_j^\pm E + c.c + \dots . \quad (5.43)$$

5.3.2 Solvability Condition

Multiplying both sides of (5.27) by $\bar{v}_1$ and integrating from 0 to $+\infty$, we obtain the solvability condition for (5.27):

$$-\sum_j FP(\Delta_j) = i\alpha^{-1}J_1 \frac{\partial A}{\partial t_1} + J_2 \tau_1 A - J_3 \frac{\partial^2 A}{\partial Z^2}, \quad (5.44)$$

where the summation is made over all critical layers, and FP denote the finite part of Δ_j :

$$\Delta_j = (\phi_{3y}\phi_1 - \phi_3\phi_{1y})|_{y \rightarrow y_c^+} - (\phi_{3y}\phi_1 - \phi_3\phi_{1y})|_{y \rightarrow y_c^-}. \quad (5.45)$$

The constants J_1 and J_2 are defined by the following integrals

$$J_1 = \int_0^{+\infty} \frac{\bar{U}_{yy}}{(\bar{U} - c)^2} \bar{v}_1^2 dy, \quad (5.46)$$

$$J_2 = \int_0^{+\infty} \left[-\frac{\bar{U}_{yy}\bar{U}_\tau}{(\bar{U} - c)^2} + \frac{\bar{U}_{yy\tau}}{(\bar{U} - c)} \right] \bar{v}_1^2 dy, \quad (5.47)$$

$$J_3 = \int_0^\infty \bar{v}_1^2(y) dy. \quad (5.48)$$

Note that J_1 and J_2 are singular and should be interpreted in the sense of Hadamard. By using the asymptotic expressions (5.23), (5.28) together with (5.24) and (5.25) of ϕ_1 and ϕ_3 , it is found

$$\begin{aligned} \Delta_j = & b_j(c_j^+ - c_j^-) - b_j r_j(a_j^+ - a_j^-) - b_j p_j(d_j^+ - d_j^-) \\ & - (a_j^+ d_j^+ - a_j^- d_j^-). \end{aligned} \quad (5.49)$$

We shall obtain the amplitude equation from (5.44) and (5.49) after the jumps $(a_j^+ - a_j^-)$ etc. are determined by analysing the critical layers in the next section. Here it is important to note that nonlinearity influences the evolution of disturbance by altering these jumps; thus in the analysis of the critical layers, it is sufficient to concentrate only on those part of the solution contributing to these jumps.

5.4 Inner Expansion

Equations (5.40) to (5.42) suggest that the inner expansions within the j -th critical layer take the following form:

$$u = \epsilon U_1 + \epsilon^{\frac{6}{5}} U_2 + \epsilon^{\frac{7}{5}} U_3 + c.c + \dots, \quad (5.50)$$

$$v = \epsilon V_0 + \epsilon^{\frac{6}{5}} V_{01} + \epsilon^{\frac{7}{5}} V_1 + \epsilon^{\frac{8}{5}} V_2 + \epsilon^{\frac{9}{5}} V_3 + \dots, \quad (5.51)$$

$$w = \epsilon^{\frac{4}{5}} W_1 + \epsilon W_2 + \epsilon^{\frac{6}{5}} W_3 + \dots, \quad (5.52)$$

$$p = \epsilon P_1 + \epsilon^{\frac{6}{5}} P_2 + \epsilon^{\frac{7}{5}} P_3 + \dots. \quad (5.53)$$

The function V_0 satisfies the equation:

$$L_0 V_{0,YY} = 0,$$

where L_0 is the unsteady-viscous linear critical-layer operator:

$$L_0 = \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x} \right] - \lambda \frac{\partial^2}{\partial Y^2} . \quad (5.54)$$

The solution which matches the outer expansion is:

$$V_0 = \hat{A}(Z, t_1)E + c.c. , \quad (5.55)$$

where $\hat{A} = b_j A$ and $b_j^+ = b_j^- = b_j$, i.e. the jump $(b_j^+ - b_j^-)$ is zero.

The functions V_{01} satisfies the same equation as V_0 . The solutions that match the outer expansion are:

$$V_{01} = \hat{B}(Z, t_1) + \hat{C}(Z, t_1)E^2 + c.c. ,$$

Expansion of the y -momentum equation yields

$$\frac{\partial P_1}{\partial Y} = 0 ,$$

and so

$$P_1 = i\alpha^{-1} \bar{U}_y \hat{A}(Z, t_1)E + c.c. .$$

The function W_1 satisfies the following equation

$$L_0 W_1 = -\frac{\partial P_1}{\partial Z} .$$

Let $W_1 = \hat{W}_1(Y, Z, t_1)E + c.c.$, then $\hat{W}_1$ satisfies:

$$\hat{L}_0^{(1)} \hat{W}_1 = -i\alpha^{-1} \bar{U}_y A_Z(Z, t_1) , \quad (5.56)$$

where we have defined

$$\hat{L}_0^{(n)} = \frac{\partial}{\partial t_1} + n i \alpha (\bar{U}_y Y + \bar{U}_\tau \tau_1) - \lambda \frac{\partial^2}{\partial Y^2} .$$

The equation (5.56) can be solved by using a Fourier transformation method as in Chapter two; see also Hickernell (1984), and Goldstein & Leib (1989). It is found that

$$\hat{W}_1 = -i\alpha^{-1} \bar{U}_y \int_0^{+\infty} \hat{A}_Z(Z, t_1 - \xi) e^{-s_1 \xi^3 - i\Omega \xi} d\xi , \quad (5.57)$$

where we have put

$$s_1 = \frac{1}{3} \lambda \alpha^2 \bar{U}_y^2 , \quad \text{and} \quad \Omega = \alpha (\bar{U}_y Y + \bar{U}_\tau \tau_1) .$$

The velocity V_1 satisfies:

$$L_0 V_{1,YY} = L_1 V_0 , \quad (5.58)$$

where

$$L_1 = -\left(\frac{1}{2}\bar{U}_{yy}Y^2 + \bar{U}_{y\tau}\tau_1Y + \frac{1}{2}\bar{U}_{\tau\tau}\tau_1^2\right)\frac{\partial^3}{\partial x\partial Y^2} + \bar{U}_{yy}\frac{\partial}{\partial x} . \quad (5.59)$$

Let

$$V_1 = \hat{V}_1(Y, Z, t_1)E + c.c. ,$$

then $\hat{V}_1$ satisfies the following equation:

$$\hat{L}_0^{(1)}\hat{V}_{1,YY} = i\alpha\hat{A}(Z, t_1)\bar{U}_{yy} .$$

It can be shown that

$$\hat{V}_{1,YY} = i\alpha\bar{U}_{yy}\int_0^{+\infty}\hat{A}(Z, t_1-\xi)e^{-s_1\xi^3-i\Omega\xi}d\xi . \quad (5.60)$$

Note that $\hat{V}_1$ is equivalent to Φ_3 in the purely two-dimensional case (Chapter two). By this analogy, we obtain the following jumps:

$$a_j^+ - a_j^- = \pi ip_j b_j \text{Sgn}(\bar{U}_y) , \quad (5.61)$$

$$d_j^+ - d_j^- = -\pi ir_j b_j \text{Sgn}(\bar{U}_y) . \quad (5.62)$$

Similarly, we write $U_1 = \hat{U}_1(Y, Z, t_1)E + c.c$, then from the continuity equation, we obtain

$$\begin{aligned} \hat{U}_{1,Y} &= -i\alpha^{-1}\bar{U}_y^2\int_0^{+\infty}\xi\hat{A}_{ZZ}(Z, t_1-\xi)e^{-s_1\xi^3-i\Omega\xi}d\xi \\ &\quad -\bar{U}_{yy}\int_0^{+\infty}\hat{A}(Z, t_1-\xi)e^{-s_1\xi^3-i\Omega\xi}d\xi . \end{aligned} \quad (5.63)$$

The function V_2 is found to satisfy the following equation:

$$L_0V_2 = L_1V_{01} + \frac{\partial^2}{\partial Y^2}\left\{\frac{\partial}{\partial x}V_0U_1 + \frac{\partial}{\partial Z}V_0W_1\right\} + \dots . \quad (5.64)$$

The first non-homogeneous term above is analogous to that in (5.58) and thus it produces an analogous velocity jump. The solution driven by the second term, i.e. by the nonlinear interaction of the fundamental, produces no velocity jump. Because $(2\alpha, c)$ is not a fundamental mode of Rayleigh's equation (Cowley, 1987), v_2 , and hence V_{01} should be identically zero. From a consideration of the continuity equation, $u_2^{(2)}$, i.e. the harmonic in the second term in the outer expansion of u , should also be zero.

The function W_2 satisfies the following equation:

$$L_0W_2 = -\frac{\partial}{\partial Y}(V_0W_1) . \quad (5.65)$$

The solution for W_2 generally should be written as

$$W_2 = \hat{W}_2^{(0)} + \hat{W}_2^{(2)}E^2 + c.c ,$$

but it is found that at the next order the interaction associated with the harmonic $\hat{W}_2^{(2)}$ does *not* contribute the jump $(c_j^+ - c_j^-)$, and hence does not affect the amplitude equation. Therefore, it is sufficient to solve for the cross-flow distortion $\hat{W}_2^{(0)}$. After some manipulation, we find

$$\hat{W}_2^{(0)} = \bar{U}_y^2 \int_0^{+\infty} \int_0^{+\infty} \xi \hat{A}^*(Z, t_1 - \xi) \hat{A}_Z(Z, t_1 - \xi - \zeta) K_0(\xi, \zeta) e^{-i\Omega} d\xi d\zeta, \quad (5.66)$$

where

$$K_0(\xi, \zeta) = e^{-s_1(\xi^3 + 3\xi^2\zeta)}.$$

We now turn to solve for U_2 , which is found to satisfy

$$L_0 U_{2,Y} = \bar{U}_y \frac{\partial W_2}{\partial Z} - \frac{\partial^2}{\partial Y^2} (V_0 U_1), \quad (5.67)$$

The solution takes the form

$$U_2 = \hat{U}_2^{(0)} + \hat{U}_2^{(2)} E^2 + c.c.,$$

but again, as for W_2 , it is sufficient to solve for the streamwise-flow distortion $\hat{U}_2^{(0)}$. It can be shown that

$$\begin{aligned} \hat{U}_{2,Y}^{(0)} = & -i\alpha \bar{U}_{yy} \bar{U}_y \int_0^{+\infty} \int_0^{+\infty} \xi \hat{A}^*(Z, t_1 - \zeta) \hat{A}(Z, t_1 - \xi - \zeta) K_0(\xi, \zeta) e^{-i\Omega\xi} d\xi d\zeta \\ & + \bar{U}_y^3 \int_0^{+\infty} \int_0^{+\infty} \xi^2 \hat{A}^*(Z, t_1 - \zeta) \hat{A}_{ZZ}(Z, t_1 - \xi - \zeta) K_0(\xi, \zeta) e^{-i\Omega\xi} d\xi d\zeta \\ & + \bar{U}_y^3 \int_0^{+\infty} \int_0^{+\infty} \xi \zeta [\hat{A}^*(Z, t_1 - \zeta) \hat{A}_Z(Z, t_1 - \xi - \zeta)]_Z K_0(\xi, \zeta) e^{-i\Omega\xi} d\xi d\zeta. \end{aligned} \quad (5.68)$$

To determine the jump $(c_j^+ - c_j^-)$, we only need to solve for V_3 . It is found that V_3 satisfies the following equation

$$L_0 V_3 = L_1 V_1 + L_2 V_0 + i\alpha \hat{A} \hat{U}_{2,Y}^{*(0)} E + \frac{\partial}{\partial Z} (\hat{A} \hat{W}_{2,Y}^{*(0)}) E + c.c. + \dots, \quad (5.69)$$

where the forcing terms which are irrelevant to the jump are not written out. The operator L_2 is defined by

$$\begin{aligned} L_2 = & -\left[\frac{1}{6} \bar{U}_{yyy} Y^3 + \bar{U}_{yy\tau} \tau_1 Y^2 + \bar{U}_{y\tau\tau} \tau_1^2 Y + \frac{1}{6} \bar{U}_{\tau\tau\tau} \tau_1^3\right] \frac{\partial^3}{\partial x \partial Y^2} \\ & + [\bar{U}_{yyy} Y + \bar{U}_{yy\tau} \tau_1] \frac{\partial}{\partial x} \\ & - \left[\frac{\partial}{\partial t_1} + (\bar{U}_y Y + \bar{U}_\tau \tau_1) \frac{\partial}{\partial x}\right] \frac{\partial^2}{\partial x^2}. \end{aligned} \quad (5.70)$$

We shall first seek the solution induced by the linear forcing term, i.e. $(L_1 V_1 + L_2 V_0)$, which we now denote by $F^{(l)}(Z, Y, t_1) E$. Note that $F^{(l)}(Z, Y, t_1)$ is the same as $F^{(l)}(Y, t_1)$ of the two-dimensional case (Chapter two) except that the former depends

on Z . By a similar procedure, the solution (here we denote it by $\hat{V}_3^{(l)}E$) driven by it has the following asymptotic behaviour as $Y \rightarrow \pm\infty$,

$$\begin{aligned} \hat{V}_{3,Y}^{(l)} \sim & (a_j^+ p_j + 2q_j b_j + \frac{1}{2} p_j^2 b_j) Y + (a_j^+ r_j + p_j d_j^+ + s_j b_j) \log |Y| \\ & + \{\pm \frac{1}{2} \pi i \text{Sgn}(\bar{U}_y)(a_j^+ r_j + p_j d_j^+ + s_j b_j) + \dots\} + O(Y^{-1}) . \end{aligned} \quad (5.71)$$

Let $\hat{V}_3^{(n)}E$ denotes the solution induced by the nonlinear forcing term. Harmonics can be ignored as far as deriving the amplitude equation is concerned. It is found that $\hat{V}_3^{(n)}$ be written as

$$\hat{V}_3^{(n)} = -i\alpha^3 \bar{U}_y^2 \bar{U}_{yy} \Pi - \alpha^2 \bar{U}_y^4 (I_1 + I_2 + I_3) , \quad (5.72)$$

where

$$\Pi = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^2 \hat{A}(Z, t_1 - \zeta) \hat{A}(Z, t_1 - \zeta - \eta) \hat{A}^*(Z, t_1 - \xi - \zeta - \eta) K_1(\xi, \eta, \zeta) d\xi d\zeta d\eta , \quad (5.73)$$

$$I_1 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^3 \hat{A}(Z, t_1 - \zeta) \hat{A}(Z, t_1 - \zeta - \eta) \hat{A}_{ZZ}^*(Z, t_1 - \xi - \zeta - \eta) K_1(\xi, \eta, \zeta) d\xi d\zeta d\eta , \quad (5.74)$$

$$I_2 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^2 \eta \hat{A}(Z, t_1 - \zeta) [\hat{A}(Z, t_1 - \zeta - \eta) \hat{A}_Z^*(Z, t_1 - \xi - \zeta - \eta)]_Z K_1(\xi, \eta, \zeta) d\xi d\zeta d\eta , \quad (5.75)$$

$$I_3 = \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^3 [\hat{A}(Z, t_1 - \zeta) \hat{A}(Z, t_1 - \zeta - \eta) \hat{A}_Z^*(Z, t_1 - \xi - \zeta - \eta)]_Z K_1(\xi, \eta, \zeta) d\xi d\zeta d\eta . \quad (5.76)$$

The kernel $K_1(\xi, \eta, \zeta)$ is defined by

$$K_1(\xi, \eta, \zeta) = e^{-s_1[2\xi^2 + 3\xi^2\eta - (\zeta - \xi)^3] - i\Omega(\zeta - \xi)} .$$

Matching $(\hat{V}_{3,Y}^{(l)} + \hat{V}_{3,Y}^{(n)})$ with outer expansion yields the following jump:

$$\begin{aligned} c_j^+ - c_j^- &= \pi i \text{Sgn}(\bar{U}_y)(a_j^+ r_j + p_j d_j^+ + s_j b_j) \\ &\quad - 2\pi i \alpha^2 \bar{U}_{yy} |\bar{U}_y| \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^2 \hat{A}(Z, t_1 - \xi) \hat{A}(Z, t_1 - \xi - \eta) \hat{A}^*(Z, t_1 - 2\xi - \eta) K(\xi, \eta) d\xi d\eta \\ &\quad - 2\pi \alpha |\bar{U}_y|^3 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^3 \hat{A}(Z, t_1 - \xi) \hat{A}(Z, t_1 - \xi - \eta) \hat{A}_{ZZ}^*(Z, t_1 - 2\xi - \eta) K(\xi, \eta) d\xi d\eta \\ &\quad - 2\pi \alpha |\bar{U}_y|^3 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^2 \eta \hat{A}(Z, t_1 - \xi) [\hat{A}(Z, t_1 - \xi - \eta) \hat{A}_Z^*(Z, t_1 - 2\xi - \eta)]_Z K(\xi, \eta) d\xi d\eta \\ &\quad - 2\pi \alpha |\bar{U}_y|^3 \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \xi^3 [\hat{A}(Z, t_1 - \xi) \hat{A}(Z, t_1 - \xi - \eta) \hat{A}_Z^*(Z, t_1 - 2\xi - \eta)]_Z K(\xi, \eta) d\xi d\eta . \end{aligned} \quad (5.77)$$

It is noted that $(a_j^+ - a_j^-)$ and $(d_j^+ - d_j^-)$ correspond to the classic $\pm\pi$ phase shift in the outer expansion, while $(c_j^+ - c_j^-)$ is modified by nonlinearity; this has a significant effect on the evolution of the disturbance. Also note that nonlinear effects influence the evolution by producing a mean-flow distortion of vortex form, i.e. streamwise and cross-flow distortions $\hat{U}_2^{(0)}$ and $\hat{W}_2^{(0)}$, while the harmonics have no significance. In this sense, the nonlinear interaction is of wave-vortex type.

5.5 Evolution Equation of the Amplitude

Inserting the jumps (5.61), (5.62) and (5.77) into (5.49), and using (5.44), we obtain the crucial evolution equation as follows:

$$\begin{aligned}
& \frac{\partial A}{\partial t_1} - p \frac{\partial^2 A}{\partial Z^2} \\
&= g_0 \tau_1 A + \alpha^2 \int_0^{+\infty} \int_0^{+\infty} \sum g_j K_j(\xi, \eta) \xi^2 A(Z, t_1 - \xi) A(Z, t_1 - \xi - \eta) A^*(Z, t_1 - 2\xi - \eta) d\xi d\eta \\
&+ \int_0^{+\infty} \int_0^{+\infty} \sum h_j K_j(\xi, \eta) \xi^3 A(Z, t_1 - \xi) A(Z, t_1 - \xi - \eta) A_{ZZ}^*(Z, t_1 - 2\xi - \eta) d\xi d\eta \\
&+ \int_0^{+\infty} \int_0^{+\infty} \sum h_j K_j(\xi, \eta) \xi^2 \eta A(Z, t_1 - \xi) [A(Z, t_1 - \xi - \eta) A_Z^*(Z, t_1 - 2\xi - \eta)]_Z d\xi d\eta \\
&+ \int_0^{+\infty} \int_0^{+\infty} \sum h_j K_j(\xi, \eta) \xi^3 [A(Z, t_1 - \xi) A(Z, t_1 - \xi - \eta) A_Z^*(Z, t_1 - 2\xi - \eta)]_Z d\xi d\eta ,
\end{aligned} \tag{5.78}$$

where the suffix j refers to the j -th critical layers, and the summations are made over all critical layers. The kernel $K_j(\xi, \eta)$ represents the effects of viscosity, and is defined by

$$K_j(\xi, \eta) = e^{-s_1(2\xi^2 + 3\xi^2\eta)} . \tag{5.79}$$

For the inviscid case, $\lambda = 0$, and so $K_j(\xi, \zeta) = 1$. The coefficients evolved are expressed in terms of the basic-flow profile and the eigenfunction as follows:

$$g_0 = f_0/f , \tag{5.80}$$

$$g_j = 2\pi i b_j^2 |b_j|^2 \bar{U}_{yy} |\bar{U}_y|/f , \tag{5.81}$$

$$h_j = 2\pi \alpha b_j^2 |b_j|^2 |\bar{U}_y|^3/f , \tag{5.82}$$

$$p = J_3/f , \tag{5.83}$$

$$f = i\alpha^{-1} \left\{ \sum_j \pi b_j [2i \frac{\bar{U}_{yy}}{|\bar{U}_y|^2} a_j^+ + i b_j \frac{\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2}{\bar{U}_y |\bar{U}_y|^2} + b_j \pi \frac{\bar{U}_{yy}^2}{\bar{U}_y^3}] + J_1 \right\} , \tag{5.84}$$

$$\begin{aligned}
f_0 = & \sum_j \left\{ 2\pi i b_j \frac{\bar{U}_{yy} \bar{U}_\tau}{|\bar{U}_y|^2} a_j^+ - \pi i b_j^2 \left[\frac{\bar{U}_{yy\tau}}{|\bar{U}_y|} - \frac{\bar{U}_{yy} \bar{U}_{y\tau}}{|\bar{U}_y|^2} - \frac{(\bar{U}_y \bar{U}_{yyy} - \bar{U}_{yy}^2) \bar{U}_\tau}{\bar{U}_y |\bar{U}_y|^2} \right] \right. \\
& \left. + \pi^2 b_j^2 \frac{\bar{U}_{yy} \bar{U}_\tau}{\bar{U}_y^3} \right\} - J_2 .
\end{aligned} \tag{5.85}$$

These coefficients can be evaluated in the same way as described in Chapter two.

Note that the first nonlinear term is associated with the logarithmic branch point singularity, and appears to be more like the traditional cubic nonlinear term. Because this term is the same as that in the amplitude of a purely two-dimensional wave (Chapter two), we shall refer it as “two-dimensional nonlinear term”. The rest of the nonlinear terms are associated with the pole type of singularity. Besides these nonlinear terms are non-local, i.e. evolving history effects, a novel feature is that the spatial derivative, including the highest derivative appears in these terms. We shall call them “three-dimensional nonlinear terms” since they represent vortex stretching and tilting effects, which are unique for three-dimensional disturbances. We believe that these effects are crucial for the formation of focusing. We emphasise that the scaling of this analysis also applies to flows with regular critical layers, e.g. a free shear layer, for which a similar amplitude equation holds, except that the “two-dimensional nonlinear term” disappears and t_1 is replaced by a nonlinear streamwise evolution variable, say x_1 . Therefore it is sufficient for us to discuss the property of (5.78) only, while treating free shear layers as a special case with g_j being zero. It is well known that streamwise vortices exist in free shear flows. There are evidences suggesting that these vortices are “facility” forced, e.g. by small geometric details (Jimenez, 1983; Nygaard & Glezer, 19991). This is very much in the line of the present study. Moreover, the scaling of the present study indicates that streamwise vortices may form when the linear growth rate decreases to $O(\epsilon^{\frac{2}{3}})$, proceeding the roll-up process, which occurs when the growth rate decreases to $O(\epsilon^{\frac{1}{2}})$ (Goldstein & Leib, 1988, Goldstein & Hultgren, 1989). This is consistent with the experimental observation that the formation of streamwise vortices may occur before the roll-up of the spanwise vortices is completed (Lasheras & Choi, 1988; Nygaard & Glezer, 1991).

Note that the above theory is formally valid for $\beta \sim O(\epsilon^{\frac{1}{3}})$. If $\beta \ll O(\epsilon^{\frac{1}{3}})$, i.e. three-dimensionality is weaker, then there exists a difference between singular and regular critical layers: for singular critical layers, the term $\frac{\partial^2 A}{\partial Z^2}$ and the “three-dimensional nonlinear terms” can be dropped, and the “two-dimensional nonlinear term” dominates, i.e. the disturbance can be effectively treated as purely two-dimensional. But for regular critical layers, the amplitude equation degenerates to a linear one. A similar scaling argument as in §2 suggests that when $\beta \sim \epsilon^{\frac{1}{2}}$, the critical-layer structure of a two-dimensional disturbance (Goldstein & Leib, 1988) remains largely intact, i.e. the critical layer is strongly nonlinear, but subject to some modification due to the coupling with the three-dimensional velocity components.

Note also that this study yields a single amplitude equation. Davey, Hocking & Stewartson (1974) found that the analysis of Hocking, Stewartson & Stuart (1972) was incomplete in that the amplitude equation should couple to an additional function related to the secular pressure gradient. However, we note that if it were assumed in their analysis, as it is in our case, that the amplitude depended on the the spanwise coordinate only, then $B = |A|^2$ would immediately follow from their (2.28); in this case their coupled equations (2.31) and (2.34) are reduced to a single

equation, with the nonlinear term proportional to q representing three-dimensional nonlinear effects.

In order to match its earlier linear evolution stage, A has the following asymptotic behaviour:

$$A \rightarrow A_0(Z, t_1) e^{g_0 \tau_1 t_1} \quad \text{as} \quad t_1 \rightarrow -\infty, \quad (5.86)$$

where $A_0(Z, t_1)$ is the initial spanwise distribution of the amplitude, and can be any function satisfying

$$\frac{\partial A_0}{\partial t_1} - p \frac{\partial^2 A_0}{\partial Z^2} = 0. \quad (5.87)$$

One typical case, which represents a initially localized disturbance is

$$A_0 = \frac{\Delta}{(-t_1)^{\frac{1}{2}}} \exp\left(-\frac{Z^2}{4pt_1}\right),$$

as discussed in, e.g. Hocking, Stewartson & Stuart (1972), see also Hocking & Stewartson (1972). Equations (5.78) and (5.86) thus describe the evolution of disturbances.

To scale out τ_1 , we introduce the new variables:

$$\bar{A} = A e^{i\bar{X}_0 / (-\tau_1)^{\frac{5}{2}}}, \quad (5.88)$$

$$\bar{t} = -\tau_1 t_1 - x_0, \quad (5.89)$$

$$\bar{Z} = (-\tau_1)^{\frac{1}{2}} Z, \quad (5.90)$$

$$\bar{\lambda} = \lambda / (-\tau_1)^3, \quad (5.91)$$

where $\bar{X}_0$ and x_0 are chosen so that:

$$g_0 x_0 - i(\bar{X}_0 - \varphi_0) = \log[(-\tau_1)^{-\frac{5}{2}}]. \quad (5.92)$$

The evolution equation and the asymptotic behaviour then become:

$$\begin{aligned} & \frac{\partial \bar{A}}{\partial \bar{t}} - p \frac{\partial^2 \bar{A}}{\partial \bar{Z}^2} \\ &= -g_0 \bar{A} + \alpha^2 \int_0^{+\infty} \int_0^{+\infty} \sum g_j K_j(\xi, \eta) \xi^2 \bar{A}(\bar{Z}, \bar{t} - \xi) \bar{A}(\bar{Z}, \bar{t} - \xi - \eta) \bar{A}^*(\bar{Z}, \bar{t} - 2\xi - \eta) d\xi d\eta \\ &+ \int_0^{+\infty} \int_0^{+\infty} \sum h_j K_j(\xi, \eta) \xi^3 \bar{A}(\bar{Z}, \bar{t} - \xi) \bar{A}(\bar{Z}, \bar{t} - \xi - \eta) \bar{A}_{\bar{Z}\bar{Z}}^*(\bar{Z}, \bar{t} - 2\xi - \eta) d\xi d\eta \\ &+ \int_0^{+\infty} \int_0^{+\infty} \sum h_j K_j(\xi, \eta) \xi^2 \eta \bar{A}(\bar{Z}, \bar{t} - \xi) [\bar{A}(\bar{Z}, \bar{t} - \xi - \eta) \bar{A}_{\bar{Z}}^*(\bar{Z}, \bar{t} - 2\xi - \eta)] d\xi d\eta \\ &+ \int_0^{+\infty} \int_0^{+\infty} \sum h_j K_j(\xi, \eta) \xi^3 [\bar{A}(\bar{Z}, \bar{t} - \xi) \bar{A}(\bar{Z}, \bar{t} - \xi - \eta) \bar{A}_{\bar{Z}}^*(\bar{Z}, \bar{t} - 2\xi - \eta)] d\xi d\eta, \end{aligned} \quad (5.93)$$

$$\bar{A} \rightarrow A_0(\bar{Z}, \bar{t}) e^{-g_0 \bar{t} + i\varphi_0} \quad \text{as} \quad \bar{t} \rightarrow -\infty, \quad (5.94)$$

where without loss of generality, we can assume that the maximum of Δ is unity.

Furthermore, to scale out φ_0 we introduce:

$$\tilde{A} = \bar{A} |\sum q_j|^{\frac{1}{2}} |g_0|^{-\frac{5}{2}} |p|^{-\frac{1}{2}} , \quad (5.95)$$

$$\tilde{t} = |g_0| \tilde{t} - \tilde{X}_0 , \quad (5.96)$$

$$\tilde{Z} = |g_0|^{\frac{1}{2}} |p|^{-\frac{1}{2}} \bar{Z} , \quad (5.97)$$

$$\tilde{\lambda} = \lambda/3 |g_0|^3 , \quad (5.98)$$

where φ_0 and $\tilde{X}_0$ satisfy the following equation:

$$-\frac{g_0}{|g_0|} \tilde{X}_0 + i\varphi_0 = \log \{ |\sum q_j|^{\frac{1}{2}} |g_0|^{-\frac{5}{2}} |p|^{-\frac{1}{2}} \} . \quad (5.99)$$

Then we obtain:

$$\begin{aligned} & \frac{\partial \tilde{A}}{\partial \tilde{t}} - \tilde{p} \frac{\partial^2 \tilde{A}}{\partial \tilde{Z}^2} \\ &= \nu_0 \tilde{A} + s_0 \int_0^{+\infty} \int_0^{+\infty} \sum \nu_j \tilde{K}_j(\xi, \eta) \xi^2 \tilde{A}(\tilde{Z}, \tilde{t}-\xi) \tilde{A}(\tilde{Z}, \tilde{t}-\xi-\eta) \tilde{A}^*(\tilde{Z}, \tilde{t}-2\xi-\eta) d\xi d\eta \\ &+ \int_0^{+\infty} \int_0^{+\infty} \sum \tilde{q}_j \tilde{K}_j(\xi, \eta) \xi^3 \tilde{A}(\tilde{Z}, \tilde{t}-\xi) \tilde{A}(\tilde{Z}, \tilde{t}-\xi-\eta) \tilde{A}_{\tilde{Z}\tilde{Z}}^*(\tilde{Z}, \tilde{t}-2\xi-\eta) d\xi d\eta \\ &+ \int_0^{+\infty} \int_0^{+\infty} \sum \tilde{q}_j \tilde{K}_j(\xi, \eta) \xi^2 \eta \tilde{A}(\tilde{Z}, \tilde{t}-\xi) [\tilde{A}(\tilde{Z}, \tilde{t}-\xi-\eta) \tilde{A}_{\tilde{Z}}^*(\tilde{Z}, \tilde{t}-2\xi-\eta)]_{\tilde{Z}} d\xi d\eta \\ &+ \int_0^{+\infty} \int_0^{+\infty} \sum \tilde{q}_j \tilde{K}_j(\xi, \eta) \xi^3 [\tilde{A}(\tilde{Z}, \tilde{t}-\xi) \tilde{A}(\tilde{Z}, \tilde{t}-\xi-\eta) \tilde{A}_{\tilde{Z}}^*(\tilde{Z}, \tilde{t}-2\xi-\eta)]_{\tilde{Z}} d\xi d\eta , \end{aligned} \quad (5.100)$$

and

$$\tilde{A} \rightarrow A_0(\tilde{Z}, \tilde{t}) e^{-\nu_0 \tilde{t}} \quad \text{as} \quad \tilde{t} \rightarrow -\infty , \quad (5.101)$$

where $\tilde{p}$, $\tilde{q}_j$, ν_0 and ν_j are constants with unit modulus, i.e.

$$\tilde{p} = -p/|p| , \quad \tilde{q}_j = h_j/|\sum h_j| ,$$

$$\nu_0 = -g_0/|g_0| , \quad \nu_j = g_j/|\sum g_j| .$$

The kernel is rewritten as

$$\tilde{K}_j(\xi, \eta) = e^{-\tilde{\lambda} \beta_j (2\xi^3 + 3\xi^2 \eta)} . \quad (5.102)$$

The real constant s_0 is defined by

$$s_0 = \alpha^2 |p| \frac{|\sum g_j|}{|\sum h_j|} .$$

Note that s_0 is zero for regular critical layers.

5.6 Some Properties of the Amplitude Equation

5.6.1 Viscous Limit

The above theory is valid for $\epsilon = O(R^{-\frac{5}{6}})$. Next we turn to consider the limit when

$$\epsilon \ll R^{-\frac{5}{6}}, \quad (5.103)$$

which corresponds to the viscous limit $\lambda \rightarrow +\infty$. It can be shown that at this limit (5.100) reverts to the following equation:

$$\begin{aligned} \lambda^{-\frac{1}{3}} \frac{\partial A}{\partial t_1} - p \frac{\partial^2 A}{\partial Z^2} &= g_0 \tau_1 A + \alpha^2 \lambda^{-\frac{4}{3}} I_0 |A|^2 A \\ &+ \lambda^{-\frac{4}{3}} J_0 A \frac{\partial}{\partial t_1} \int_0^{+\infty} \eta^{\frac{1}{2}} [A(t_1 - \eta) A_Z^*(t_1 - \eta)]_Z d\eta + O(\lambda^{-\frac{5}{3}}) \end{aligned} \quad (5.104)$$

where we have performed the variable change $t_1 \rightarrow \lambda^{\frac{1}{3}} t_1$. The constants involved are defined as follows

$$I_0 = \frac{1}{3^{\frac{2}{3}} \times 2^{\frac{1}{3}}} \Gamma\left(\frac{1}{3}\right) \sum g_j \beta_j^{-\frac{4}{3}}, \quad J_0 = \frac{\sqrt{3}}{18} \Gamma\left(\frac{1}{2}\right) \sum h_j \beta_j^{-\frac{4}{3}}.$$

Note that even under viscous limit the nonlinear term is still non-local. Moreover, the nonlinear term contains a time as well as space derivative. Amplitude equations with nonlinear terms containing space derivatives were also encountered by Weinstein (1981), where, however, these terms are local, and served as a higher-order correction. For the regular critical layers, $I_0 = 0$, so we rescale the equation as follows: $Z = \lambda^{\frac{1}{6}} \bar{Z}$, $\tau_1 = \lambda^{-\frac{1}{3}} \bar{\tau}_1$, and $A = \lambda^{\frac{2}{3}} \bar{A}$. Then the rescaled form of (5.104) is

$$\frac{\partial A}{\partial t_1} - p \frac{\partial^2 A}{\partial Z^2} = g_0 \bar{\tau}_1 A + J_0 A \frac{\partial}{\partial t_1} \int_0^{+\infty} \eta^{\frac{1}{2}} [A(t_1 - \eta) A_Z^*(t_1 - \eta)]_Z d\eta. \quad (5.105)$$

For the singular critical layers, we introduce $Z = \lambda^{\frac{1}{6}} \bar{Z}$, $\tau_1 = \lambda^{-\frac{1}{3}} \bar{\tau}_1$, and $A = \lambda^{\frac{1}{2}} \bar{A}$. The rescaled form of (5.104)

$$\frac{\partial A}{\partial t_1} - p \frac{\partial^2 A}{\partial Z^2} = g_0 \bar{\tau}_1 A + \alpha^2 I_0 |A|^2 A. \quad (5.106)$$

Note that we can not rescale the amplitude equation (5.104) such that the two nonlinear terms are retained in the equation at the same time.

5.6.2 Equilibrium Solutions and Secondary Instability

One property of the modulation equation (5.100) is that if

$$\tilde{A} = \tilde{B}(\tilde{t}) e^{i\beta \tilde{Z}}, \quad (5.107)$$

then the “three-dimensional nonlinear terms” are identically zero, and consequently $\tilde{B}$ satisfies the same equation as the amplitude of a purely two-dimensional wave (Chapter two). Note that (5.107) corresponds to the case where the disturbance is a *single* oblique wave. Thus this property somewhat explains why the pole type of singularity has no significance for the development of a single oblique wave (Goldstein & Leib, 1989).

For flows with singular critical layers, when $\tilde{\lambda} \neq 0$, the evolution equation possesses a three-dimensional equilibrium solution of the form: $\tilde{A} = A_0 e^{i\beta\tilde{Z} + ik\tilde{t}}$. But we are primarily interested in a two-dimensional equilibria, i.e. $\beta = 0$. This equilibria was expressed in Chapter two; see (2.168) to (2.170). For the Stokes layer of interest here, the study in Chapter two found that the above condition was satisfied for wave numbers $\alpha \leq 0.6003$ or $\alpha \geq 1.293$. The next task of this section is to examine the instability of the above solution to a three-dimensional disturbance. The purpose is two fold. Firstly, as pointed out by Stuart and DiPrima (1978), such a analysis yields an understanding of sideband instability. For the present problem the sideband consists of a two dimensional wave $(\alpha, 0)$, and a pair of three-dimensional waves $(\alpha, \pm \epsilon^{\frac{1}{2}}\beta)$ (see below). Thus the sideband instability indicates a transfer of energy from the two-dimensional to the three-dimensional waves. Secondly, if the two-dimensional equilibria is unstable, it will be ruled out as a terminal solution of (5.100). Moreover, if the growth rate of this secondary instability increases sufficiently fast with β , it may indicate a finite-time focusing singularity (Hall & Smith, 1989).

Hence we perturb the equilibrium solution in the following way:

$$\tilde{A} = [A_0 + B(\tilde{t}) \cos \beta \tilde{Z}] e^{ik\tilde{t}}. \quad (5.108)$$

Then B satisfies the following equation after linearization:

$$\begin{aligned} \frac{dB}{d\tilde{t}} = & (\nu_0 - ik - \tilde{p}\beta^2)B \\ & + A_0^2 \int_0^{+\infty} \int_0^{+\infty} \sum \nu_j K_j(\xi, \zeta) \xi^2 [B(\tilde{t} - \xi - \zeta) + B(\tilde{t} - \xi) + B^*(\tilde{t} - 2\xi - \zeta)] d\xi d\zeta \\ & - A_0^2 \beta^2 \int_0^{+\infty} \int_0^{+\infty} \sum \tilde{q}_j K_j(\xi, \zeta) (2\xi^3 + \xi^2 \zeta) B^*(\tilde{t} - 2\xi - \zeta) d\xi d\zeta. \end{aligned} \quad (5.109)$$

Following Stuart & DiPrima (1978), we shall seek the solution of the form:

$$B = C_1 e^{\sigma \tilde{t}} + C_2 e^{\sigma^* \tilde{t}}, \quad (5.110)$$

where C_1, C_2, σ are complex numbers. Inserting into (5.109) we get the following relations:

$$\begin{aligned} \sigma C_1 = & (\nu_0 - ik - \tilde{p}\beta^2)C_1 + A_0^2 \sum \nu_j K_j^{(1)} C_1 + A_0^2 \sum \{\nu_j K_j^{(2)} - \beta^2 \tilde{q}_j K_j^{(3)}\} C_2^*, \\ \sigma^* C_2 = & (\nu_0 - ik - \tilde{p}\beta^2)C_2 + A_0^2 \sum \nu_j K_j^{*(1)} C_2 + A_0^2 \sum \{\nu_j K_j^{*(2)} - \beta^2 \tilde{q}_j K_j^{*(3)}\} C_1^*, \end{aligned}$$

where $K_j^{(1)}$, $K_j^{(2)}$ and $K_j^{(3)}$ are defined by the following integrals:

$$K_j^{(1)} = \int_0^{+\infty} d\xi \left[\frac{\xi^2}{\sigma + 3\tilde{\lambda}\beta_j\xi^2} + \frac{1}{3}(\tilde{\lambda}\beta_j)^{-1} \right] e^{-(2\tilde{\lambda}\beta_j\xi^3 + \sigma\xi)}, \quad (5.111)$$

$$K_j^{(2)} = \int_0^{+\infty} d\xi \left[\frac{\xi^2}{\sigma + 3\tilde{\lambda}\beta_j\xi^2} \right] e^{-(2\tilde{\lambda}\beta_j\xi^3 + 2\sigma\xi)}, \quad (5.112)$$

$$K_j^{(3)} = \int_0^{+\infty} d\xi \left\{ \frac{2\xi^3}{\sigma + 3\tilde{\lambda}\beta_j\xi^2} + \frac{\xi^2}{(\sigma + 3\tilde{\lambda}\beta_j\xi^2)^2} \right\} e^{-(2\tilde{\lambda}\beta_j\xi^3 + 2\sigma\xi)}. \quad (5.113)$$

The requirement that B is non-zero yields the eigenrelation as follows

$$\begin{vmatrix} \sigma - \nu_0 + ik + \tilde{p}\beta^2 - A_0^2 \sum \nu_j K_j^{(1)} & -A_0^2 \sum \nu_j K_j^{(2)} + \beta^2 A_0^2 \sum \tilde{q}_j K_j^{(3)} \\ -A_0^2 \sum \nu_j^* K_j^{(2)} + \beta^2 A_0^2 \sum \tilde{q}_j^* K_j^{(3)} & \sigma - \nu_0^* - ik - \tilde{p}\beta^2 - A_0^2 \sum \nu_j^* K_j^{(1)} \end{vmatrix} = 0. \quad (5.114)$$

The above eigenvalue problem generally must be solved numerically. Note that for regular critical layers, no equilibrium solution exists.

5.6.3 Terminal Solutions: Finite-Time Focusing Singularity

There are several candidates for the terminal form of the solution. For singular critical layers, i.e. $s_0 \neq 0$, the first possibility is the two-dimensional equilibrium state. If this is the case, it implies that the small three-dimensional “imperfections” are not amplified by nonlinear effects. However this equilibrium may be subject to a three-dimensional secondary instability, in which case could be ruled out as a terminal solution of (5.100).

The second possibility is that the solution blows up at a finite time, say t_s , in the *whole flow field*, i.e.

$$\tilde{A} = \frac{a_0}{(t_s - \tilde{t})^{\frac{5}{2} + i\tilde{\sigma}}}, \quad (5.115)$$

where σ and a_0 are real and complex numbers respectively. Note that the above two possibilities are associated with the “two-dimensional nonlinear term” and three-dimensionality does not play any role in the formation of the terminal form.

The system (5.100) may also permit a *three-dimensional nonlinear equilibrium solution* of the form

$$\tilde{A} = H(Z)e^{ik\tilde{t}}. \quad (5.116)$$

Substitution of (5.116) into (5.100) yields the equation for $H(Z)$

$$\tilde{p}H'' + (\nu_0 - ik)H + p_0H|H|^2 + q_0(H^2H'' + H|H'|^2) = 0, \quad (5.117)$$

where

$$p_0 = \frac{1}{9 \times 2^{\frac{1}{3}}} \Gamma\left(\frac{1}{3}\right) s_0 \sum_j \nu_j \beta_j^{-\frac{4}{3}} \lambda^{-\frac{4}{3}},$$

and

$$q_0 = -\frac{2}{9 \times 2^{\frac{2}{3}}} \Gamma\left(\frac{2}{3}\right) \sum_j \tilde{q}_j \beta_j^{-\frac{5}{3}} \lambda^{-\frac{5}{3}}.$$

After appropriate boundary conditions are imposed on $H(Z)$, e.g. $H(Z)$ decays to zero when $Z \rightarrow \pm\infty$, or periodic conditions as for a standing wave, equation (5.117) poses a nonlinear eigenvalue problem. Of particular interest is the periodic solution of (5.117) because if it exists, it represents a streamwise vortex structure. But at the present, the existence of such a form of solution remains to be established.

Another particularly interesting terminal form is that the solution blows up at a particular spanwise location, say Z_s , within a finite time, say t_s . We term this singularity a “focusing singularity”. The following structure is proposed:

$$\tilde{A} = (t_s - \tilde{t})^{-\frac{5}{2} - i\sigma} F(\hat{Z}), \quad (5.118)$$

where

$$\hat{Z} = (t_s - \tilde{t})^{-\frac{1}{2}} (\tilde{Z} - Z_s). \quad (5.119)$$

A variable of the form (5.119) was introduced in, e.g. Hocking, Stewartson & Stuart (1971), and Hall & Smith (1989). In their discussion of localized singularity of non-linear Schrodinger type of amplitude equation, Hocking, Stewartson & Stuart (1972) found that they had to introduce an additional logarithmic factor like $\log(t_s - \tilde{t})$ in $\hat{Z}$. Later, Hocking & Stewartson (1972) found that this complication was necessary for certain range parameters; the singular solution in terms of similarity variable (5.119) existed. Inserting (5.118) into (5.100) yields the an integro-differential equation for $F(\hat{Z})$

$$\begin{aligned} & -\tilde{p}F''(\hat{Z}) + \frac{1}{2}\hat{Z}F'(\hat{Z}) + \left(\frac{5}{2} + i\sigma\right)F(\hat{Z}) \\ & = s_0g \int_0^{+\infty} \int_0^{+\infty} \xi^2 \Theta_0(\xi, \eta) F\left[\frac{\hat{Z}}{(1+\xi)^{\frac{1}{2}}}\right] F\left[\frac{\hat{Z}}{(1+\xi+\eta)^{\frac{1}{2}}}\right] F^*\left[\frac{\hat{Z}}{(1+2\xi+\eta)^{\frac{1}{2}}}\right] d\xi d\eta \\ & + q \int_0^{+\infty} \int_0^{+\infty} \xi^3 \Theta_0(\xi, \eta) F\left[\frac{\hat{Z}}{(1+\xi)^{\frac{1}{2}}}\right] F\left[\frac{\hat{Z}}{(1+\xi+\eta)^{\frac{1}{2}}}\right] F_{\hat{Z}\hat{Z}}^*\left[\frac{\hat{Z}}{(1+2\xi+\eta)^{\frac{1}{2}}}\right] d\xi d\eta \\ & + q \int_0^{+\infty} \int_0^{+\infty} \xi^2 \eta \Theta_0(\xi, \eta) F\left[\frac{\hat{Z}}{(1+\xi)^{\frac{1}{2}}}\right] \left\{ F\left[\frac{\hat{Z}}{(1+\xi+\eta)^{\frac{1}{2}}}\right] F_{\hat{Z}}^*\left[\frac{\hat{Z}}{(1+2\xi+\eta)^{\frac{1}{2}}}\right] \right\} \hat{Z} d\xi d\eta \\ & + q \int_0^{+\infty} \int_0^{+\infty} \xi^3 \Theta_0(\xi, \eta) \left\{ F\left[\frac{\hat{Z}}{(1+\xi)^{\frac{1}{2}}}\right] F\left[\frac{\hat{Z}}{(1+\xi+\eta)^{\frac{1}{2}}}\right] F_{\hat{Z}}^*\left[\frac{\hat{Z}}{(1+2\xi+\eta)^{\frac{1}{2}}}\right] \right\} \hat{Z} d\xi d\eta, \end{aligned} \quad (5.120)$$

where $g = \sum g_j$, $q = \sum \tilde{q}_j$ and

$$\Theta_0(\xi, \eta) = (1 + \xi)^{-\frac{1}{2} - i\sigma} (1 + \xi + \eta)^{-\frac{1}{2} - i\sigma} (1 + 2\xi + \eta)^{-\frac{1}{2} + i\sigma}. \quad (5.121)$$

The equation (5.120) has to be solved under certain boundary conditions to be imposed on $F(\hat{Z})$. A simplest one is that the flow is symmetric about $Z_s = 0$,

and decays to zero at infinite. In that case, the appropriate boundary condition are $F'(0) = 0$ and $F(+\infty) = 0$. The validity of the focusing singularity structure (5.118) and (5.119) depends on the existence of F , which remains to be established. Note that the “two-dimensional nonlinear term” also has a contribution to the formation of the singularity. But we do not think its contribution is crucial; even if $s_0 = 0$, the singularity may well occur. This suggests that the same focusing mechanism may apply to free shear layers.

Chapter 6

Summary and Discussion

6.1 Summary

In this thesis, we studied nonlinear effects on the instability of the Stokes-layer generated by a sinusoidally oscillating flat plate in an infinite fluid under a high-Reynolds-number assumption, i.e. $R \gg 1$. According to linear theory (Tromans, 1977; Cowley, 1987), the flow is susceptible to high-frequency Rayleigh instability, and small disturbances can grow exponentially over part of a period and become neutral at some time. The present study shows that in the vicinity of the neutral time, the small disturbances evolve from a linear stage to a nonlinear stage due to enhanced nonlinear interactions inside the critical layers. We have systematically studied the nonlinear evolution of a two-dimensional wave, a pair of oblique waves, a resonant triad of waves, and the temporal-spatial modulation of a near-planar wave.

For a two-dimensional disturbance, we show that the evolution is controlled by nonlinear effects when the linear growth rate decreases to $O(\epsilon^{\frac{2}{5}})$, where ϵ is the magnitude of the disturbance. The amplitude equation turns out to be an integro-differential one of Hickernell type (Hickernell 1984). We studied the amplitude equation by series-extension and numerical methods. For the inviscid case, i.e. $1 \gg \epsilon \gg R^{-\frac{5}{6}}$, the solutions always develop a finite-time singularity proposed by Goldstein & Leib (1989). When viscosity is included, i.e. $\epsilon \sim R^{-\frac{5}{6}}$, it is found that (a) viscosity generally delays the finite-time singularity, and (b) if viscosity is sufficiently large the singularity can be completely eliminated, with the result that the wave evolves to an equilibrium state, confirming the findings of Goldstein & Leib (1989). In particular, it is found that for the disturbance of wavenumber in the range $[0.6, 1.3]$, the finite-time singularity always occurs, no matter how large the scaled viscosity is. If $\epsilon \ll R^{-\frac{5}{6}}$, i.e. the disturbance is relatively small, the amplitude is reduced to a classical Stuart-Landau equation, which can be obtained directly by taking a limit of the integro-differential equation. The stability of the equilibria is also studied.

For the case where the disturbance consists of a pair of oblique waves, the nonlinear evolution stage comes at a much earlier time, namely when linear growth

rate is $O(\epsilon^{\frac{1}{3}})$, and it then evolves over a much faster time scale $O(\epsilon^{-\frac{1}{3}})$, as in Goldstein & Choi (1989). Mathematically this is due to the pole type of singularity in the outer solution for the streamwise velocity, which is much stronger than the logarithmic-branch-point singularity associated with $\bar{U}_{yy}(y_c) \neq 0$. We show that though $\bar{U}_{yy}(y_c) \neq 0$ for the Stokes layer considered in this thesis, the amplitude equation is the same as that in Goldstein & Choi (1989). We note, however, that when three-dimensionality is sufficiently weak, the fact that $\bar{U}_{yy}(y_c) \neq 0$ will bring in a difference. Moreover, we derive the amplitude equation for the case when viscosity effects are retained at leading order in the critical-layer equations, thus extends the analysis of Goldstein & Choi (1989). The amplitude evolution equation is again of Hickernell type, with viscous effects being represented by a complicated kernel. A numerical integration of the *inviscid* amplitude equation was performed by Goldstein & Choi (1990), and it was found that the solution developed a singularity at a finite distance downstream (see also Goldstein & Choi, 1989). The coefficients of the amplitude equations depend on the overall feature of the basic flow, and thus are different in the two cases. Nevertheless, it seems that there is little doubt that the solution of the amplitude equation in the present case, at least under inviscid limit, can develop a finite-time singularity, corresponding to the finite-distance one of Goldstein & Choi (1989). However, equilibrium solution may exist when viscosity is sufficiently large. In addition, it is noted that the nonlinear interaction of waves can generate a strong vortex of the same magnitude as the waves. We believe that this vortex may be identified as that observed in the experiments of Hino *et. al.* (1983), and may have relevance to the streaky structure in a turbulent boundary layer.

For the case where the disturbances form a resonant-triad, we show that when the amplitude of the planar and the oblique waves are $O(\epsilon^{\frac{4}{3}})$ and $O(\epsilon)$ respectively, a mutual interaction takes place and the disturbances are governed by a coupled integro-differential equation system. These amplitude equations are significantly different from those of Raetz (1959), Craik (1971), Smith & Stewart (1987) in two important aspects. Firstly, as in Goldstein & Lee (1990) the local growth rate depends on the whole history of the evolution, unlike a conventional resonant triad, where the local growth rate depends only on the instantaneous amplitudes of the disturbances. Secondly the back reaction of the oblique waves on the 2-D wave is accounted for by two cubic and one quartic terms rather than by one quadratic term. The amplitudes of the two-dimensional and the three-dimensional waves can exhibit a finite-time singularity, and the structure for it is proposed (see also the later version (1991) of Goldstein & Lee, 1990). The evolution equations obtained in our study are shown to be valid until the magnitude of the disturbance becomes order one and thus provide a full description of the development of the resonant-triad waves from their linear small amplitude stage towards the fully-nonlinear order-one-amplitude stage.

For the temporal-spatial modulation case, i.e. the case when the amplitude of the disturbance is a slowly varying function of spanwise position as well as of time,

it is shown that the development of the disturbance is controlled by critical-layer nonlinearity when its linear growth rate decreases to $O(\epsilon^{\frac{2}{3}})$. Nonlinear interactions influence the evolution by producing a streamwise vortex and cross-flow distortion. The modulation equation turns out to be an integro-partial-differential one containing history-dependent nonlinear terms. A novel feature of the amplitude equation is that the derivatives, including the highest derivative, with respect to space appear in the nonlinear terms. These terms are associated with three-dimensionality, and hence represent vortex stretching and tilting effects. Some possible properties of the amplitude equation are discussed. It is shown that a localized singularity may occur in a finite time. This provides a possible explanation for the focusing of streamwise vorticity and the formation of streaks.

The finite-time singularity, which can occur in all four cases studied, suggests that explosive growth is induced by critical-layer nonlinear effects, indicating that an initially small disturbance is further amplified, rather than evolving into an equilibrium state as implied by linear theory (Tromans, 1977; Cowley, 1987). We suggest that this nonlinear blow-up of high-frequency disturbances is related to the bursting phenomena observed in experiments, and can lead to transition to turbulence.

6.2 Further Work

In chapters three to five, we only derived amplitude equations and discussed their possible properties. However, some numerical computation seems to be necessary. A numerical investigation of the inviscid amplitude equation in Chapter three has been performed by Goldstein & Choi (1989) for certain parameters. Their numerical solutions have confirmed their proposed singularity structure. It seems that for the inviscid case, i.e. $\lambda = 0$, the proposed singular structures in each chapter are the only choice. However when viscosity is included, other choices arise. The question which option actually occurs has to be settled by numerical means. In particular, the localized singularity and the periodic solution of a standing-wave form proposed at the end of Chapter five remain to be verified.

A numerical investigation of the *viscous* amplitude equation in Chapter three also appears to be important. This is because if an equilibrium solution is allowed, it is possible that before this equilibrium is reached, the disturbance may enter the stage discussed by Hall & Smith (1991) once the growth rate becomes sufficiently small. Hence one may require that these two stages to match in an asymptotic sense rather than assigning an arbitrary input for the waves and vortex (Hall & Smith, 1991).

It would be also interesting to incorporate viscous effects into the analysis of Chapter four (resonant-triad case). Though such an attempt seems to be straightforward, it will not be an easy task, as we see that the analysis in chapters three and four has already been very complicated. We note that with the results of Chapter three at hand, we can easily write down the amplitude equation including viscosity effects for the *oblique waves*. But a derivation of the viscous amplitude equation for

the planar wave evolves much more complication: .

The study in this thesis suggests that following the evolution stage described in this thesis, is the so-called Euler or Navier-Stokes stages, i.e. the stages where the disturbances are governed by the fully Euler or N-S equations. We feel that to resolve the relaminarization problem, these stages must be pursued.

The Stokes layer studied in this thesis is over a half space, and is an ideal mode of the *finite* Stokes layers that occur in practice, e.g. oscillatory flows in pipes and channel. These flows have an addition length scale, i.e. the radius of the pipe or the width of the channel. The effects of this on the instability may be the reason for the disagreement between the observations of Monkewitz & Bunster (1987), and those of others, e.g. Akhavan *et al.* (1991): the former suggest that the phase at which disturbances are likely to be observed is $\frac{\pi}{2}$, but the latter find that it is phase 0. Moreover, all experiments are conducted in finite Stokes layers, and quantitative data is available. Thus a direct comparison between theory and experiments is possible.

It also seems to be important to study oscillatory flows in curved pipes or channels with a non-zero mean-flow component -such flows can be regarded as a simple model of blood flow in vessels. Because of curvature and mean shear, Taylor-Gortler instability (Seminara & Hall, 1976) and T-S instability may occur. Also, a secondary flow is set up in the transverse plane (see e.g. Berger & Talbot, 1983). The interactions among different instability waves, the secondary flow and the induced vortices (see Chapter three, four and five) pose a very challenging problem.

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Figure Captions

Fig.1 Sketch of the neutral diagram (from Cowley, 1987). τ_i is the time from which a mode with wavenumber α starts to grow and τ_0 is the time at which the mode becomes linearly neutral. We show that at time $\tau = \tau_0 + \epsilon^l \tau_1$, i.e. in the vicinity of, but before the neutral time is reached, nonlinearity controls the development of the disturbance. The index l takes different value in each chapter.

Chapter two: $l = \frac{2}{5}$, where ϵ is the magnitude of the two-dimensional instability wave.

Chapter three: $l = \frac{1}{3}$, where ϵ is the magnitude of the three-dimensional instability waves.

Chapter four: $l = \frac{1}{3}$, where ϵ is the magnitude of the three-dimensional instability waves.

Chapter five: $l = \frac{2}{5}$, where ϵ is the magnitude of the near-planar instability wave.

Fig.2 The evolution picture and critical-layer structures. The disturbances initially grow exponentially according to linear theory. As the neutral time τ_0 is approached, the growth rates become small, and linear critical layer emerge. Once the growth rates decrease further to $O(\epsilon^l)$, nonlinear interactions inside the critical layers control the overall development. The earlier linear and the subsequent nonlinear evolution stages are required to match in the overlapping domain.

Fig.3 The coefficient g_0 in amplitude equations (2.80) and (2.142), at different wave numbers.

Fig.4a The real parts of the coefficients g in amplitude equation (2.80), g_1 and g_2 in the amplitude equation (2.142).

Fig.4b The imaginary parts of the coefficients g in amplitude equation (2.80), g_1 and g_2 in the amplitude equation (2.142).

Fig.5 Plot of singularity structure parameters $\bar{A}_0$ and σ , and the singularity time ($\tilde{\lambda} = 0.0$) against wave numbers.

Fig.6 Domb-Sykes Plot for $\alpha = 1.0$ and $\tilde{\lambda} = 0$; Δ : $Re[\bar{b}_n/\bar{b}_{n-1}]$; $\times$: $Im[\bar{b}_n/\bar{b}_{n-1}]$.

Fig.7 Domb-Sykes Plot for $\alpha = 1.0$ and $\tilde{\lambda} = 2.0$; Δ : $Re[\bar{b}_n/\bar{b}_{n-1}]$; $\times$: $Im[\bar{b}_n/\bar{b}_{n-1}]$.

Fig.8 Comparison of finite-difference, power-series and Pade-approximant solutions: $\alpha = 1.0$ and $\tilde{\lambda} = 0, 2.0$. The power series is truncated at 40-th order.

Fig.9 Comparison of finite-difference, power-series and Pade-approximant solutions: $\alpha = 1.35$ and $\tilde{\lambda} = 0, 2.0$ and 20.0 . The power series is truncated at 50-th order.

Fig.10 $Ln|\tilde{A}|$ vs. the scaled time $\tilde{t}$ for $\alpha = 0.8$: solid lines represent finite difference results; dashed lines represent the asymptote computed from equation (2.158). Notice that finite-time singularity can not be eliminated by viscosity.

Fig.11 $Ln|\tilde{A}|$ vs. the scaled time $\tilde{t}$ for $\alpha = 1.2$: solid lines represent finite difference results; dashed lines represent the asymptote computed from equation (2.158). Notice that finite-time singularity can not be eliminated by viscosity.

Fig.12 $Ln|\tilde{A}|$ vs. the scaled time $\tilde{t}$ for $\alpha = 0.5$: solid lines represent finite difference results; dashed lines represent the asymptote computed from equation (2.158); dotted lines represent the equilibria predicted by (2.168). Notice that finite-time singularity is eliminated if $\tilde{\lambda}$ is sufficiently large.

Fig.13 $Ln|\tilde{A}|$ vs. the scaled time $\tilde{t}$ for $\alpha = 1.4$: solid lines represent finite difference results; dashed lines represent the asymptote computed from equation (2.158); dotted lines represent the equilibrium predicted by (2.168). Notice that finite-time singularity is eliminated if $\tilde{\lambda}$ is sufficiently large.

Fig.14 $\bar{\nu}_0$ defined by (2.170) vs. α ; for $0.6 \leq \alpha \leq 1.3$, equilibrium solution does not exist.

Fig.15 Equilibrium solution: scaled amplitude $|A_0|\tilde{\lambda}^{-\frac{2}{3}}$ and frequency k vs. α .

Fig.16 Stability of the equilibrium solution for $\alpha = 0.5$: σ vs. $\tilde{\lambda}$. Dotted lines represent the large- $\tilde{\lambda}$ asymptotic approximation.

Fig.17 Stability of the equilibrium solution for $\alpha = 1.4$: σ vs. $\tilde{\lambda}$. Dotted lines represent the large- $\tilde{\lambda}$ asymptotic approximation.

Fig.18a The real parts of the scaled coefficients in the amplitude equations (3.125) (the inviscid case) and (3.185) (the viscous case), where $G = g/[\alpha^4 \sin^2 \theta]$, and $G_j = g_j/[\alpha^4 \sin^2 \theta]$ ($j = 1, 2$).

Fig.18b The imaginary parts of the scaled coefficients in the amplitude equations (3.125) and (3.185), where $G = g/[\alpha^4 \sin^2 \theta]$, and $G_j = g_j/[\alpha^4 \sin^2 \theta]$ ($j = 1, 2$).

Fig.19a The real parts of the coefficients g_{11} , g_{21} and g_{23} in the coupled amplitude equations (4.221) and (4.225).

Fig.19b The imaginary parts of the coefficients g_{11} , g_{21} and g_{23} in the coupled amplitude equations (2.221) and (4.225).

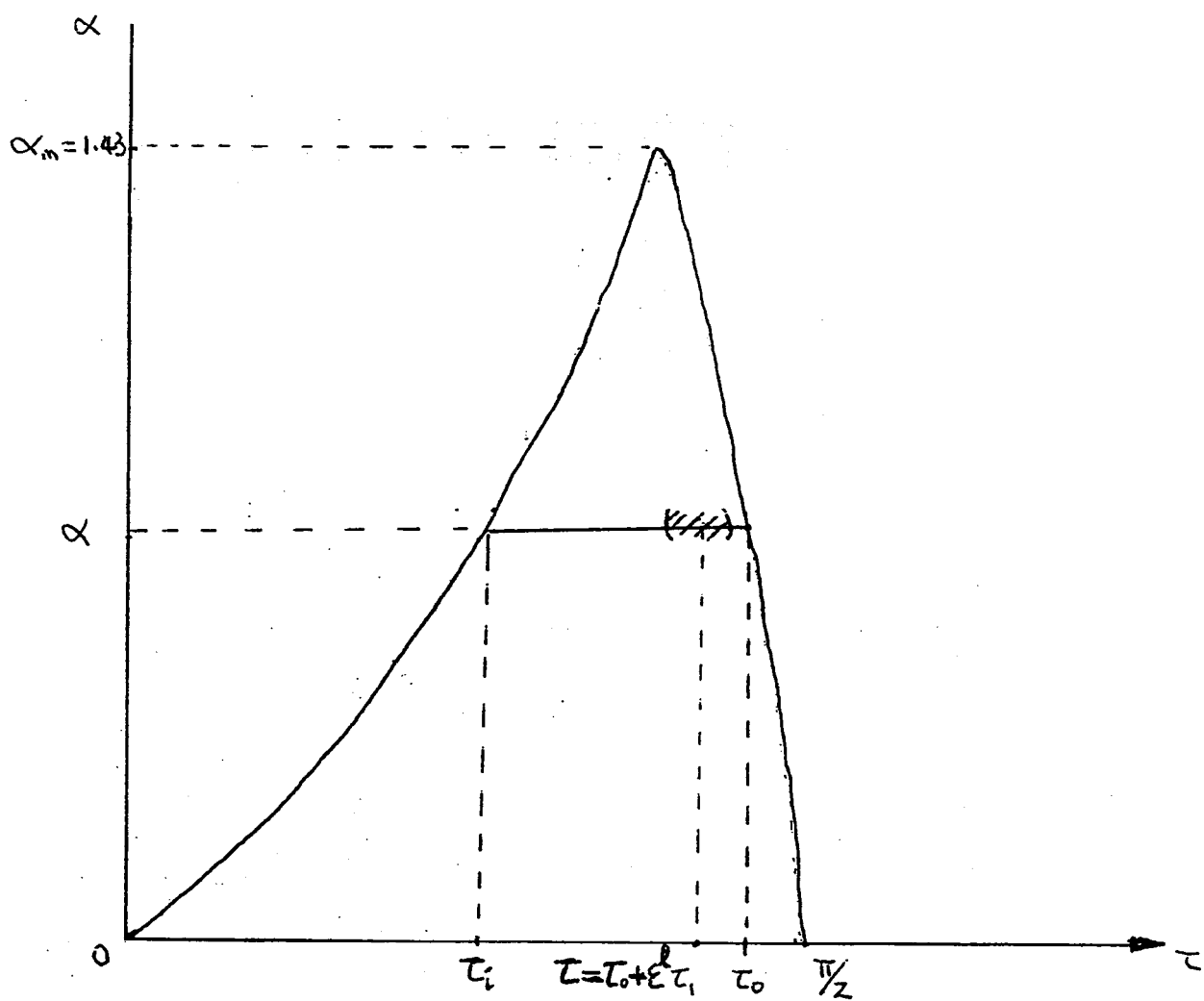


Fig 1

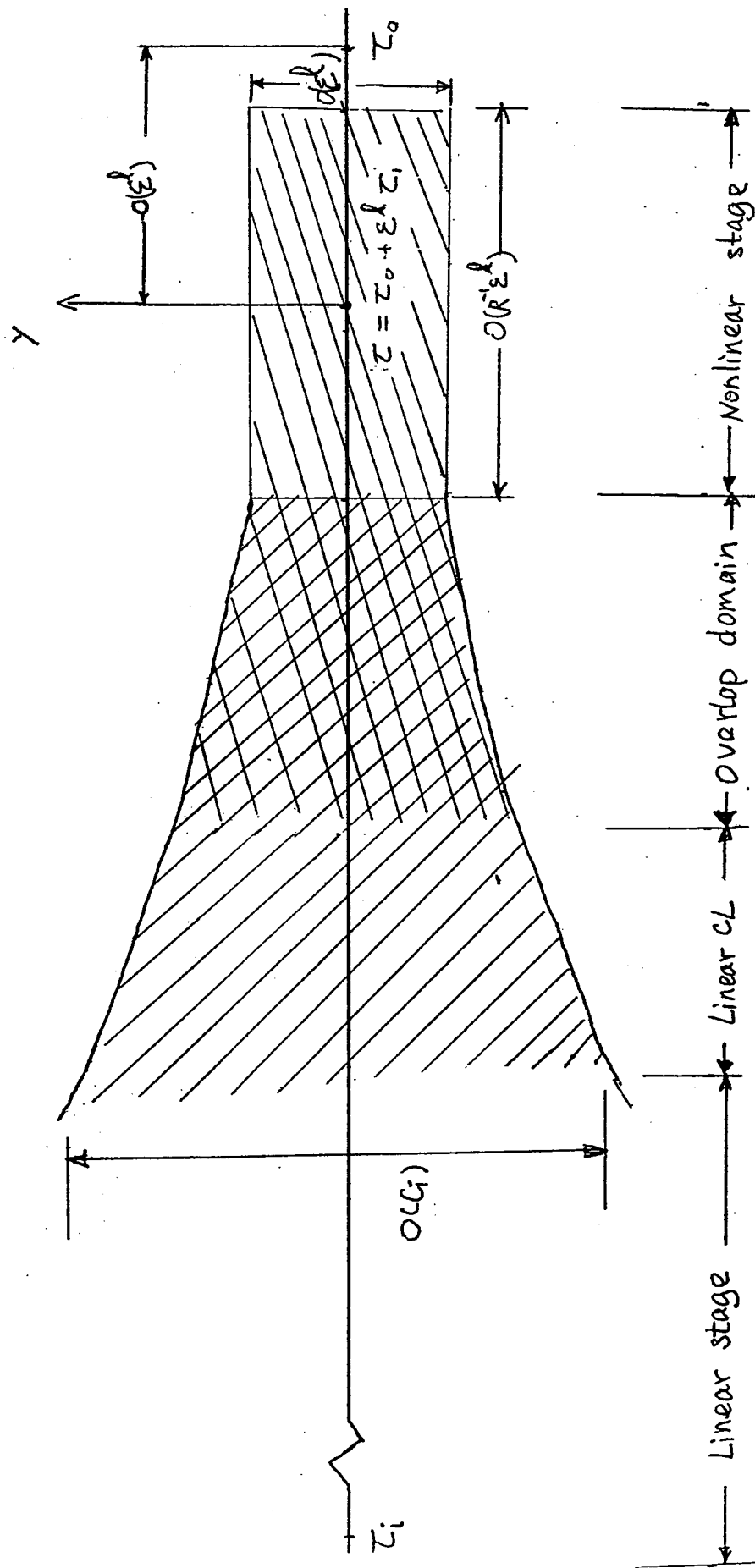


Fig. 2

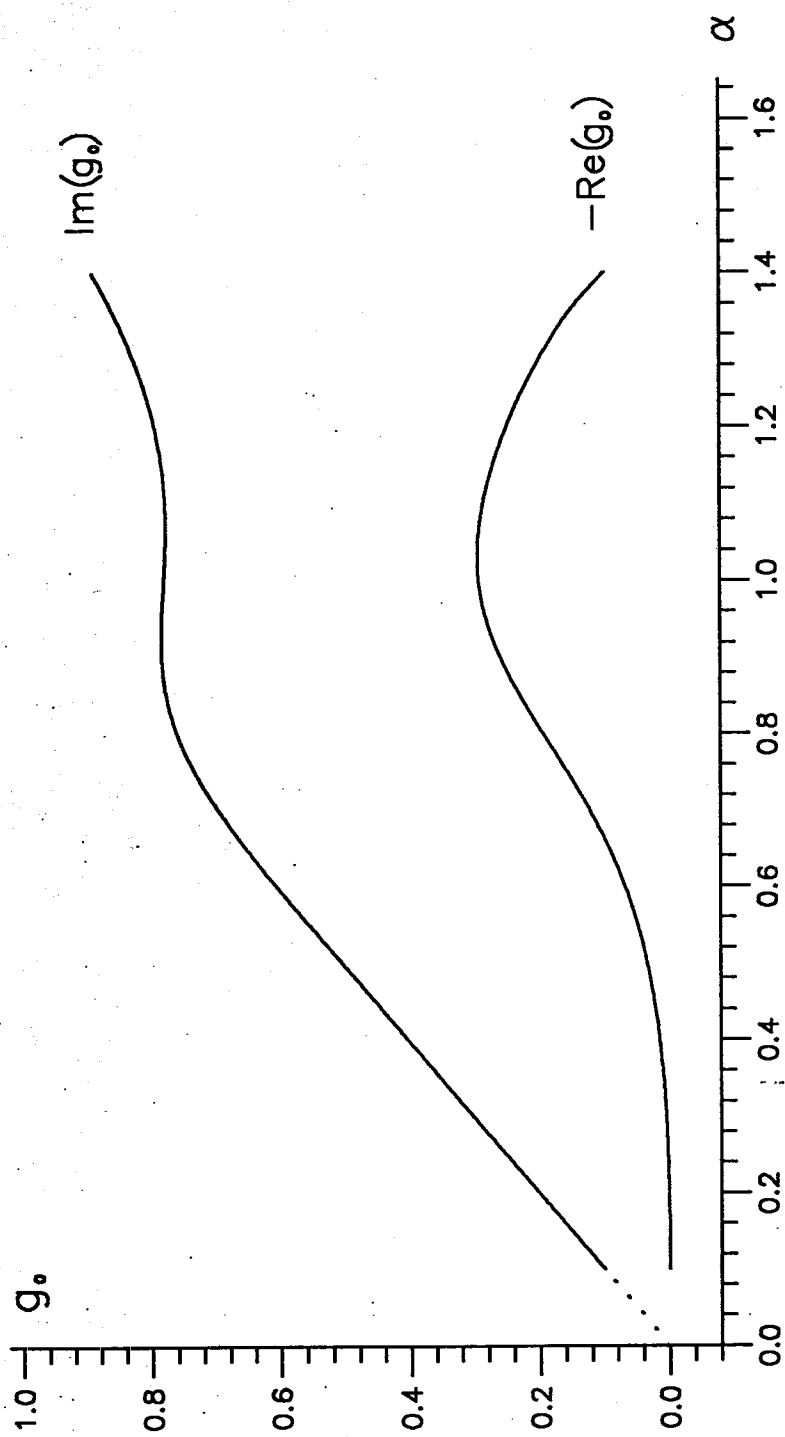


Figure 3

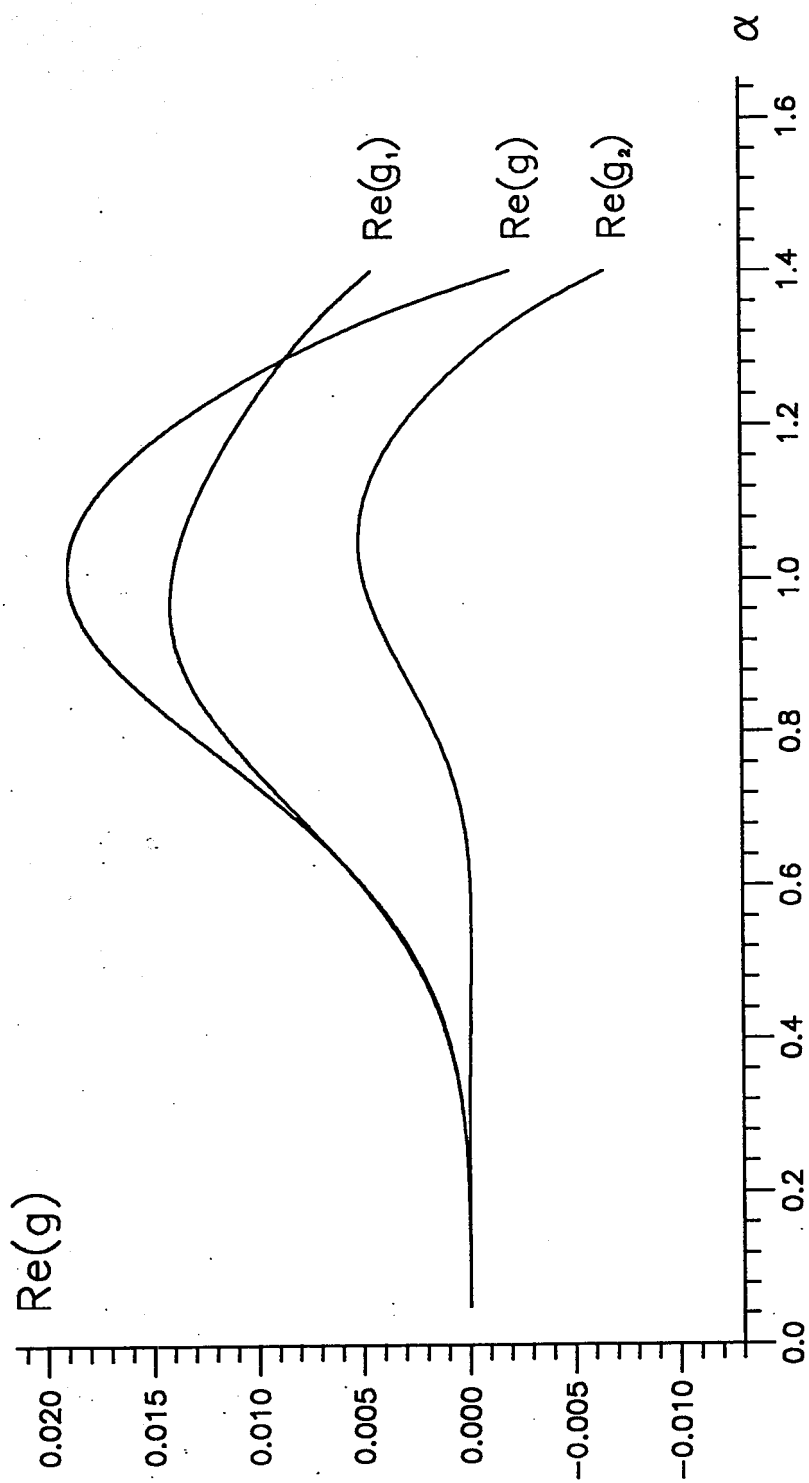


Figure 4a

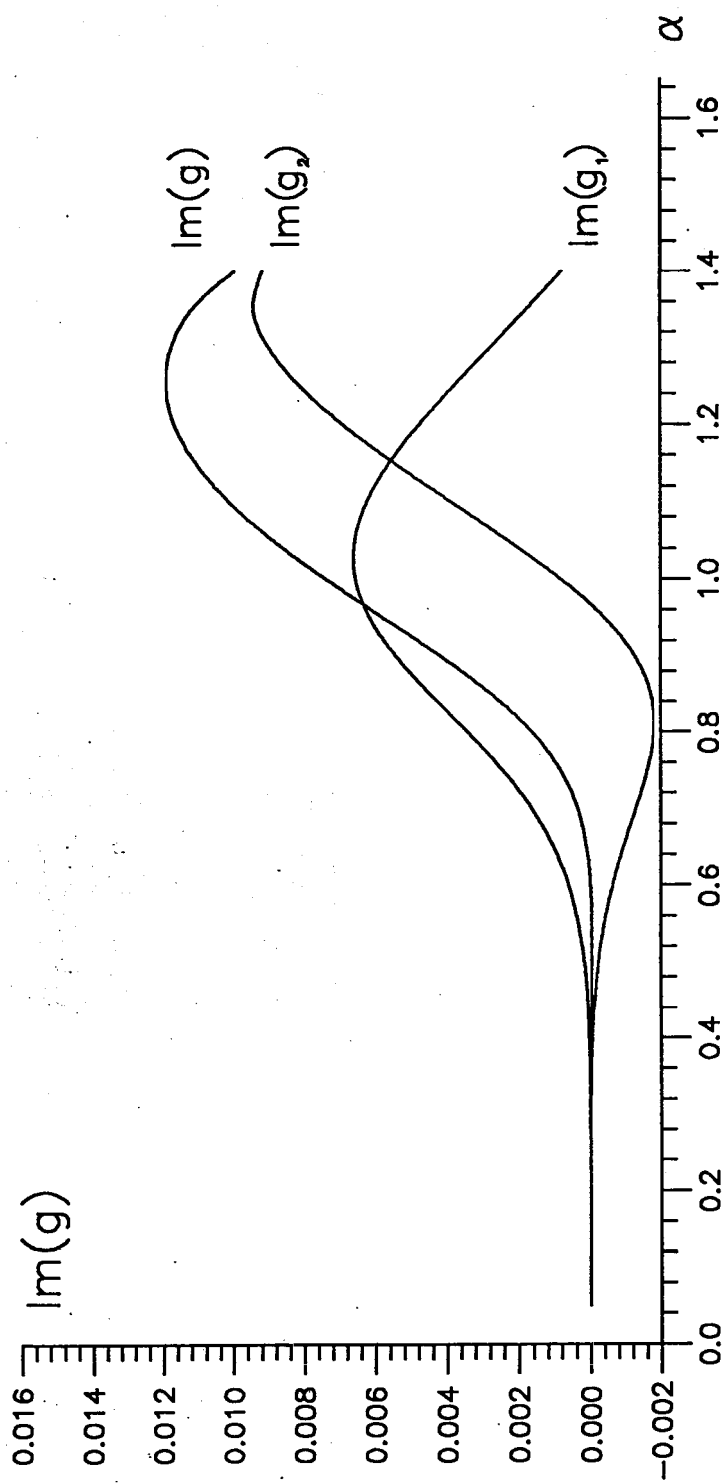


Figure 4b

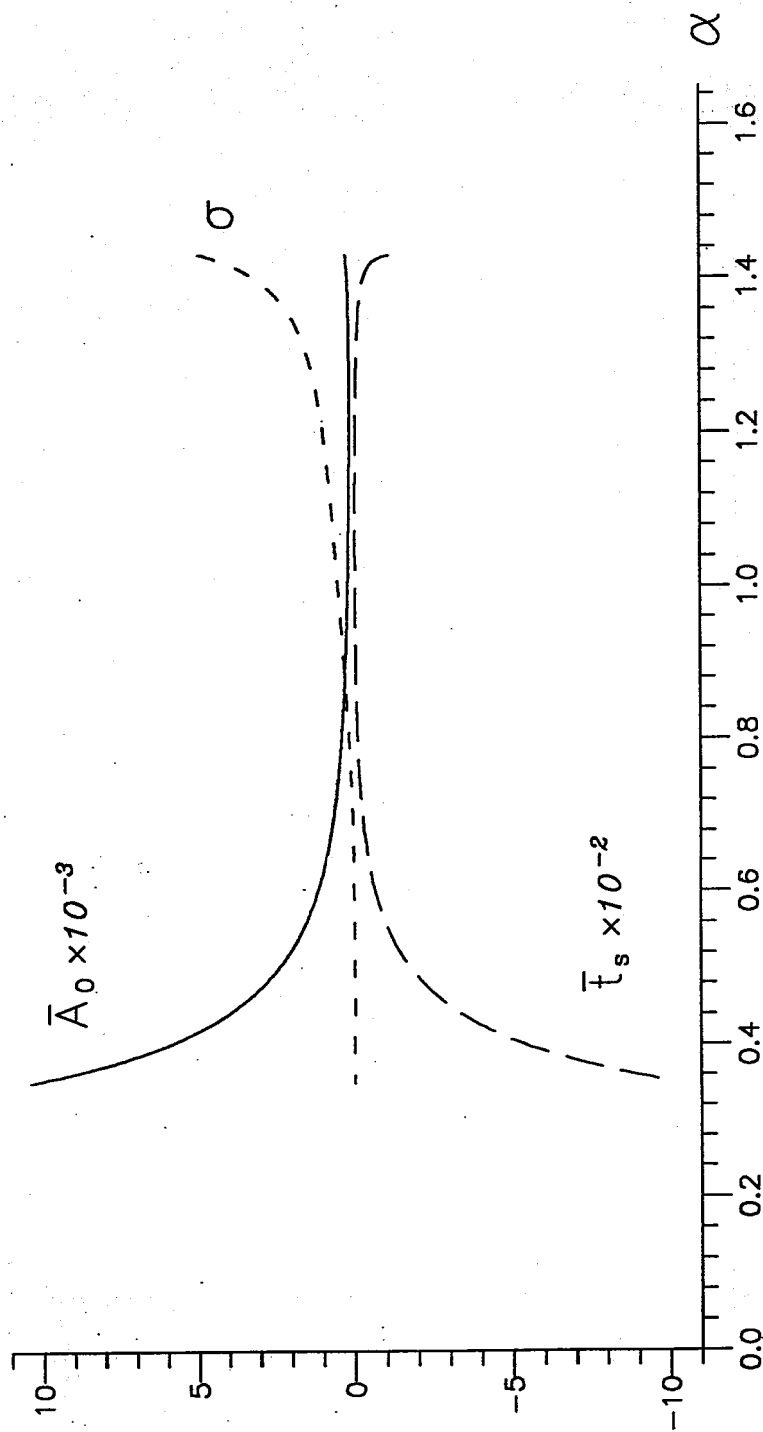


Figure 5

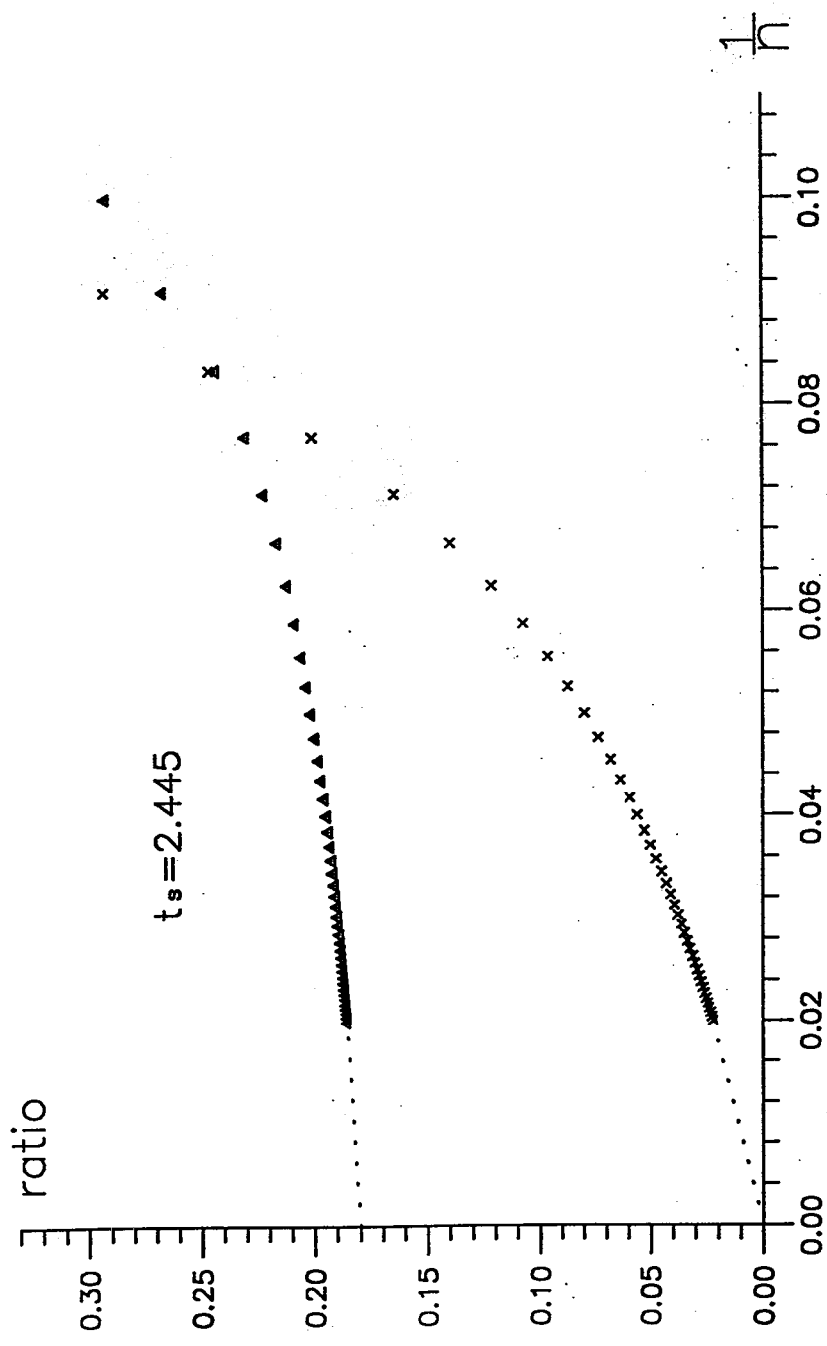


Figure 6

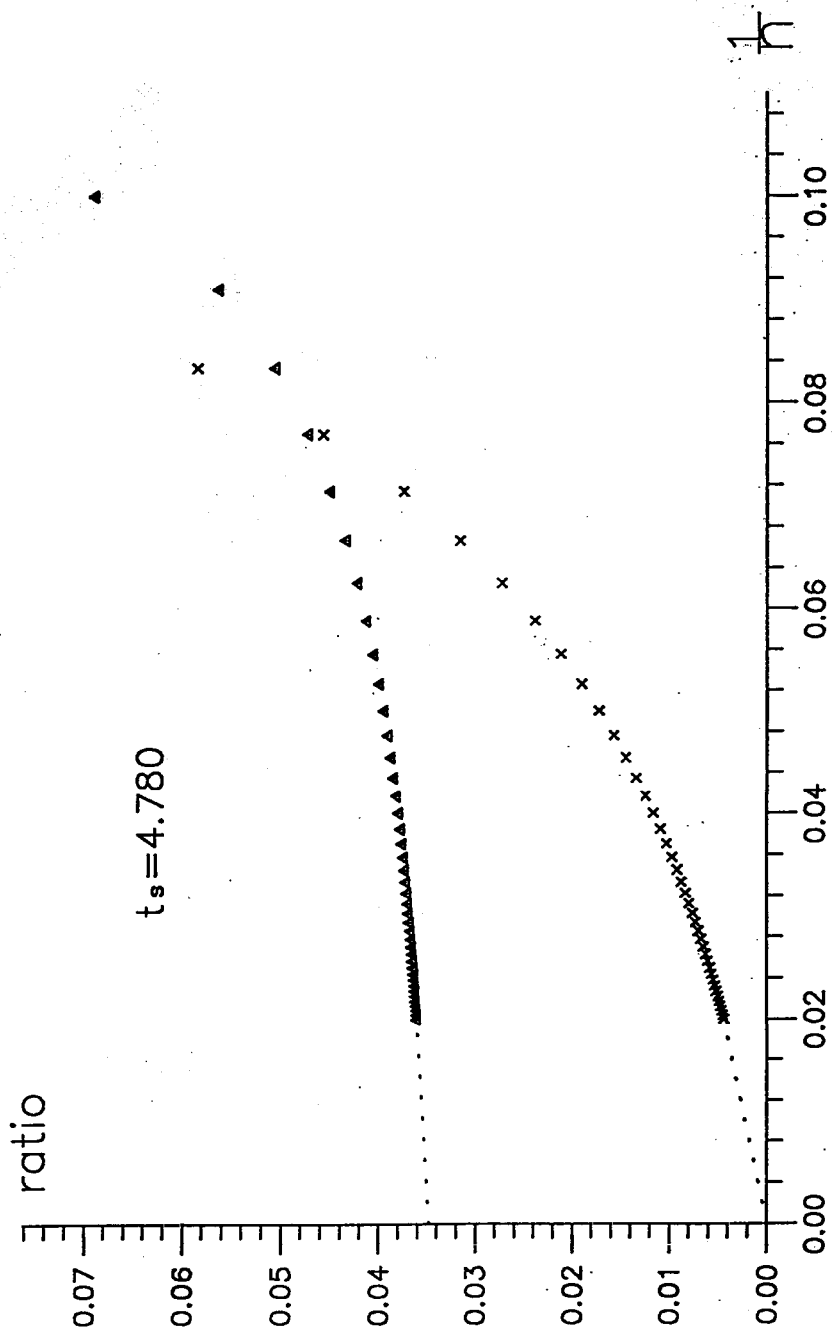


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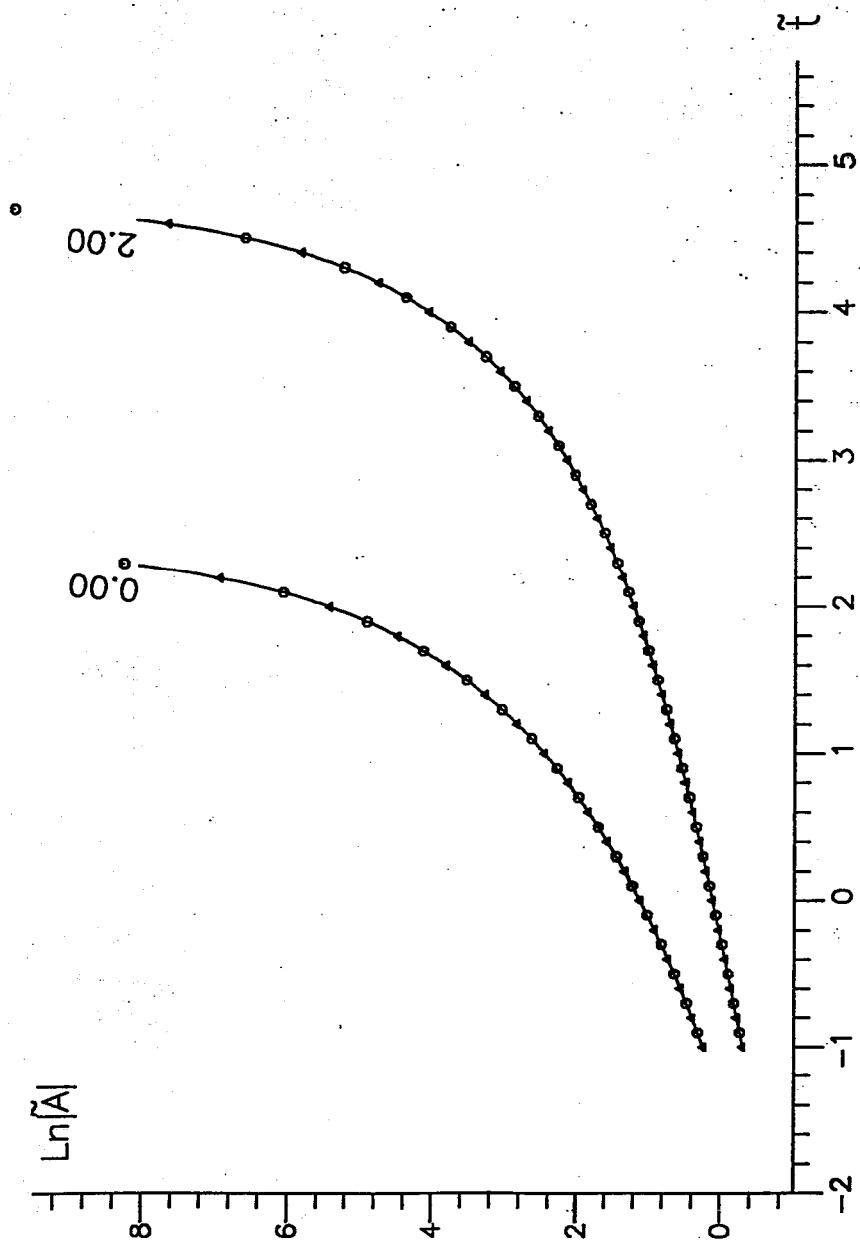


Figure 8

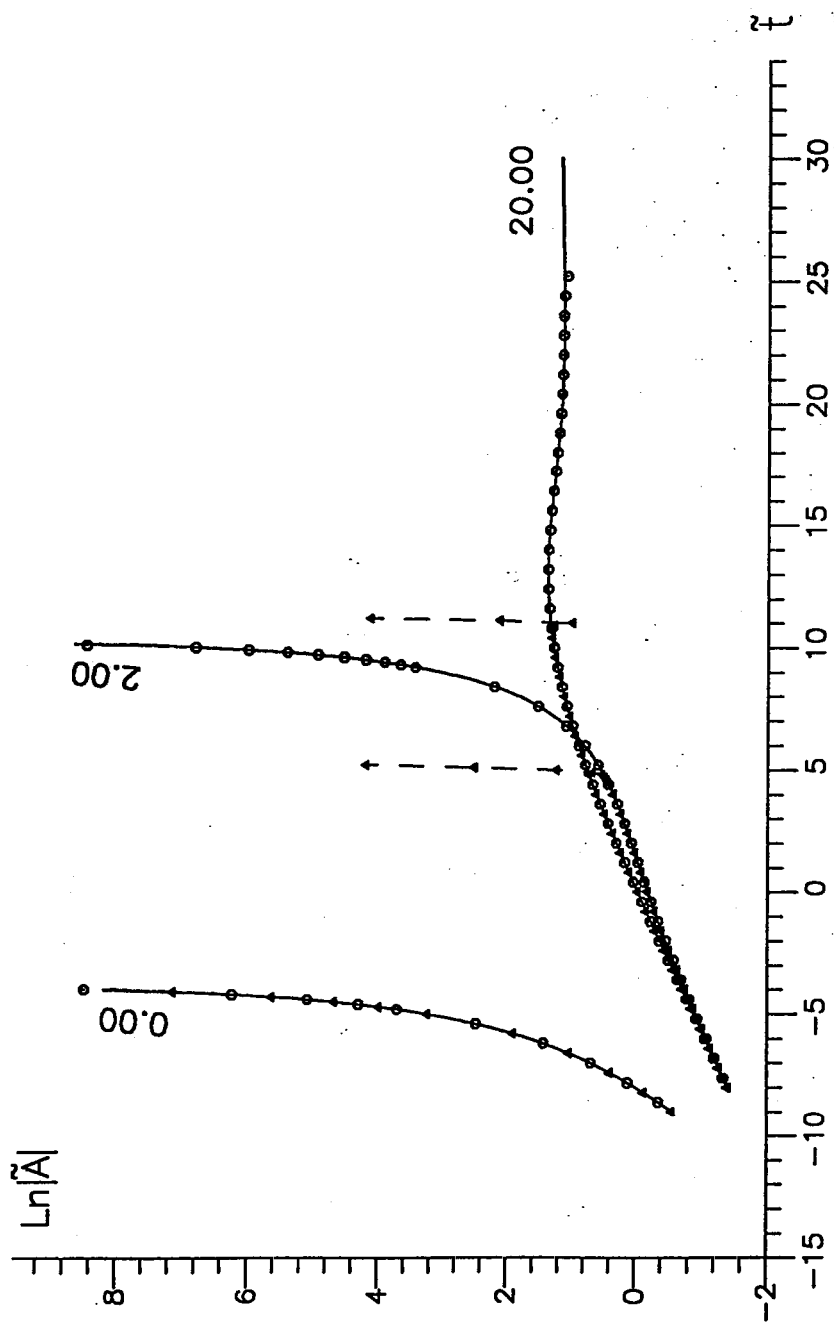


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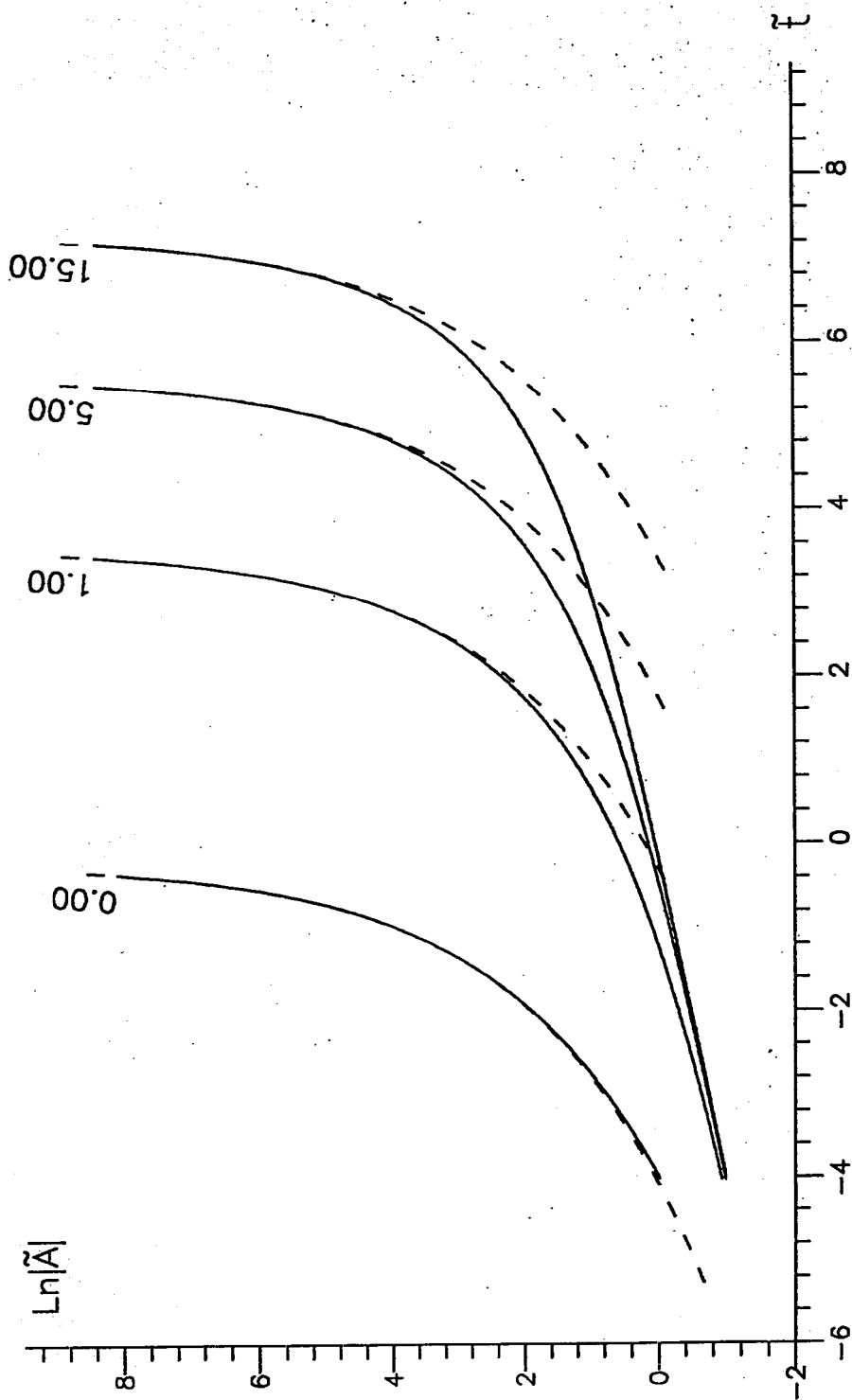


Figure 10

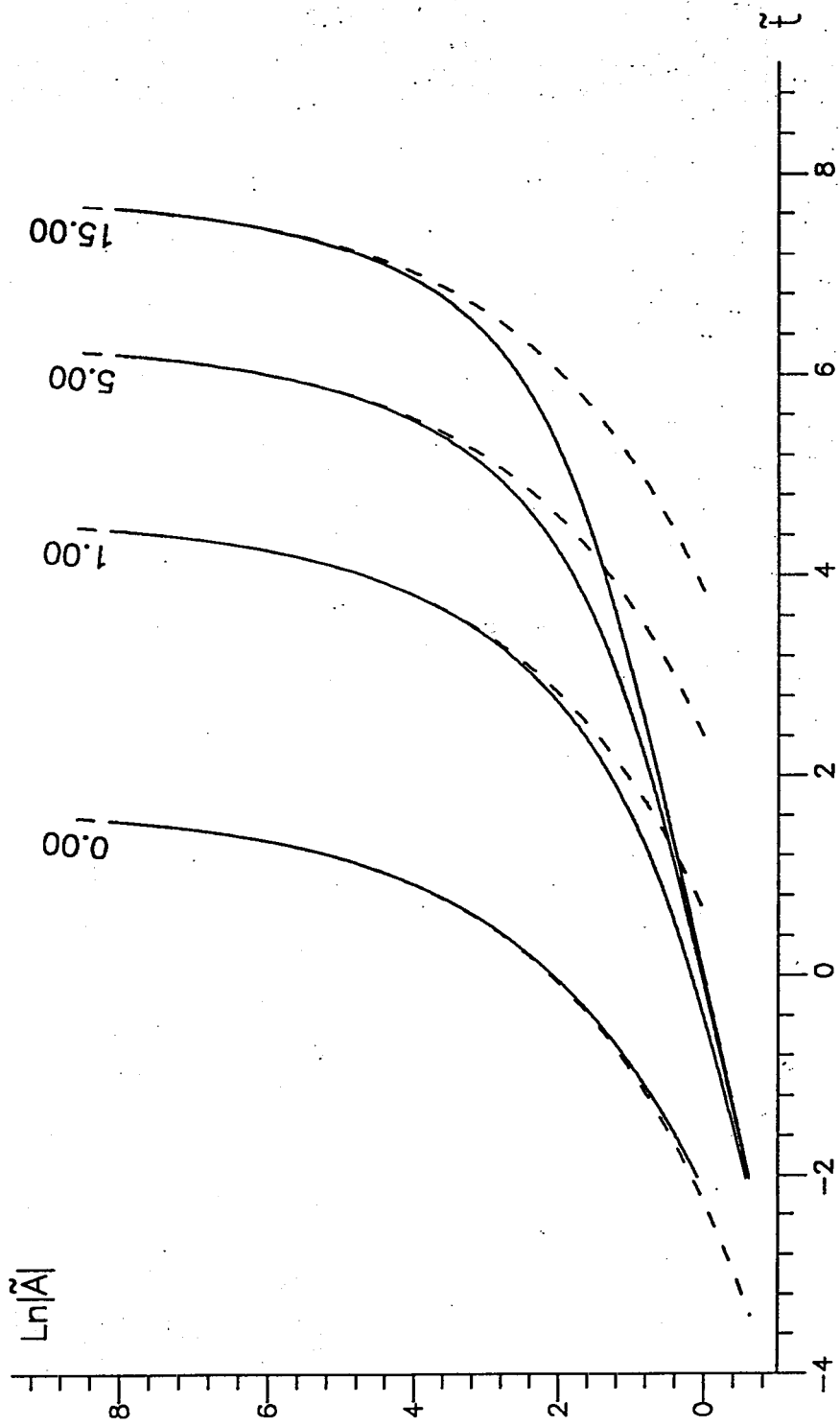


Figure 11

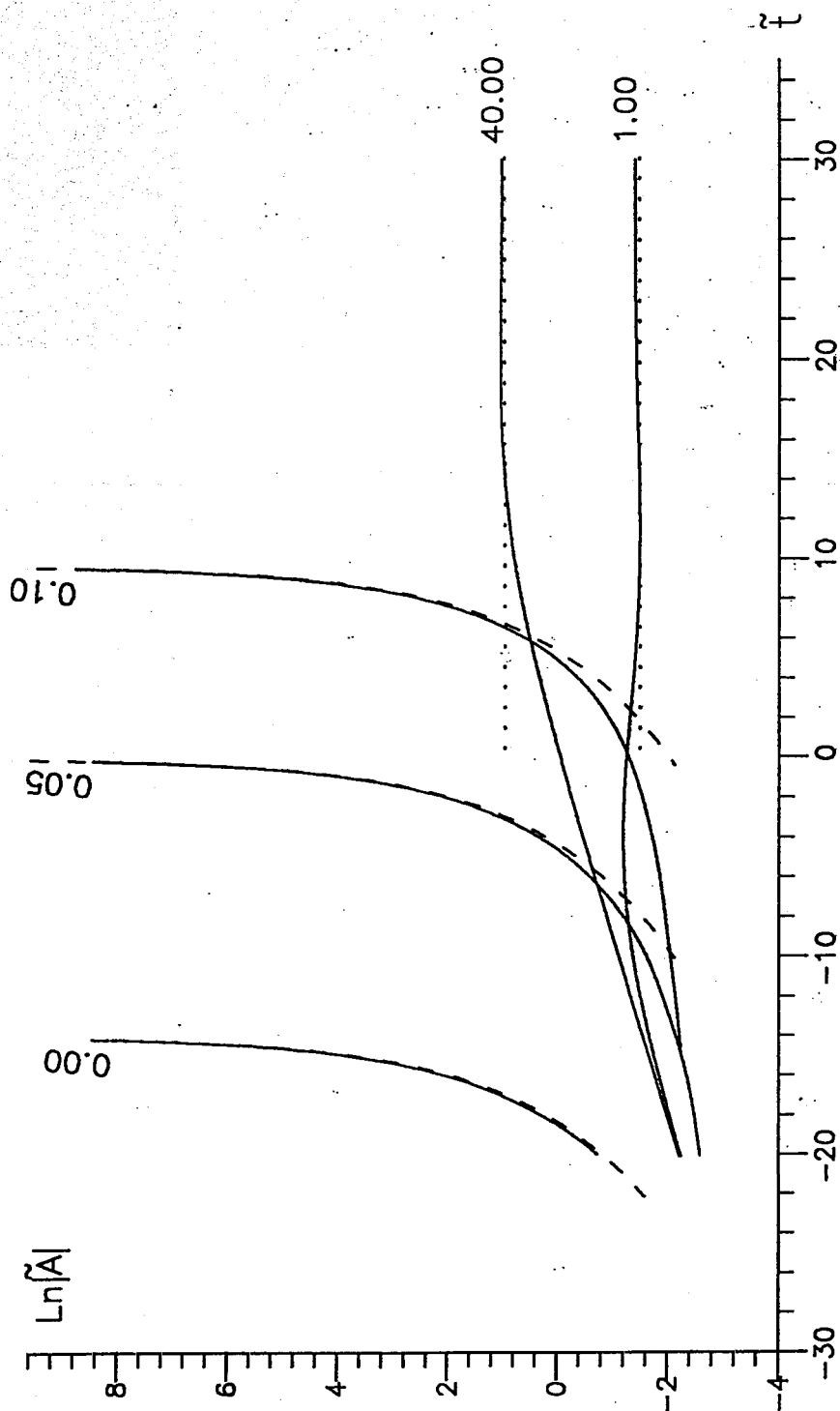


Figure 12

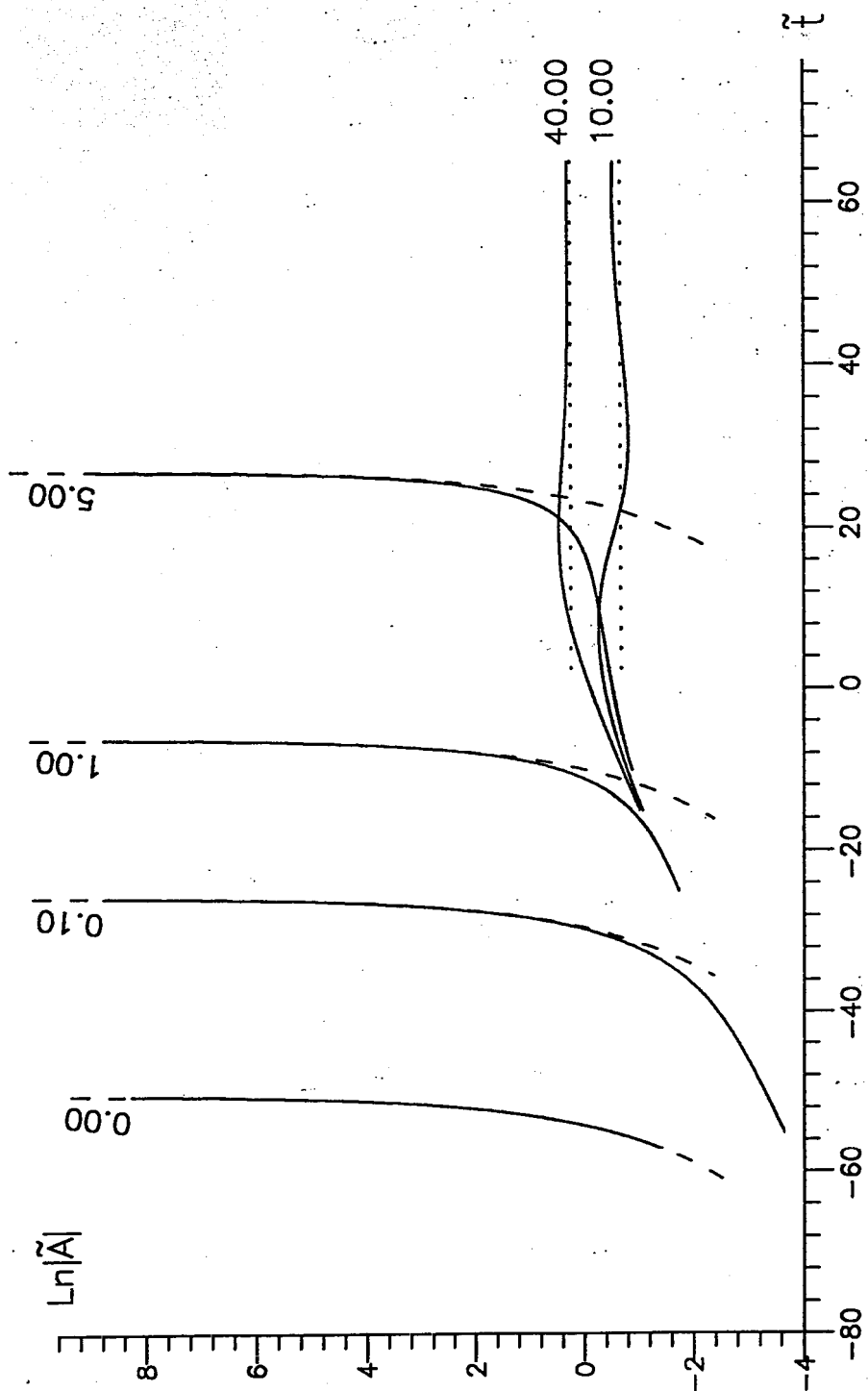


Figure 13

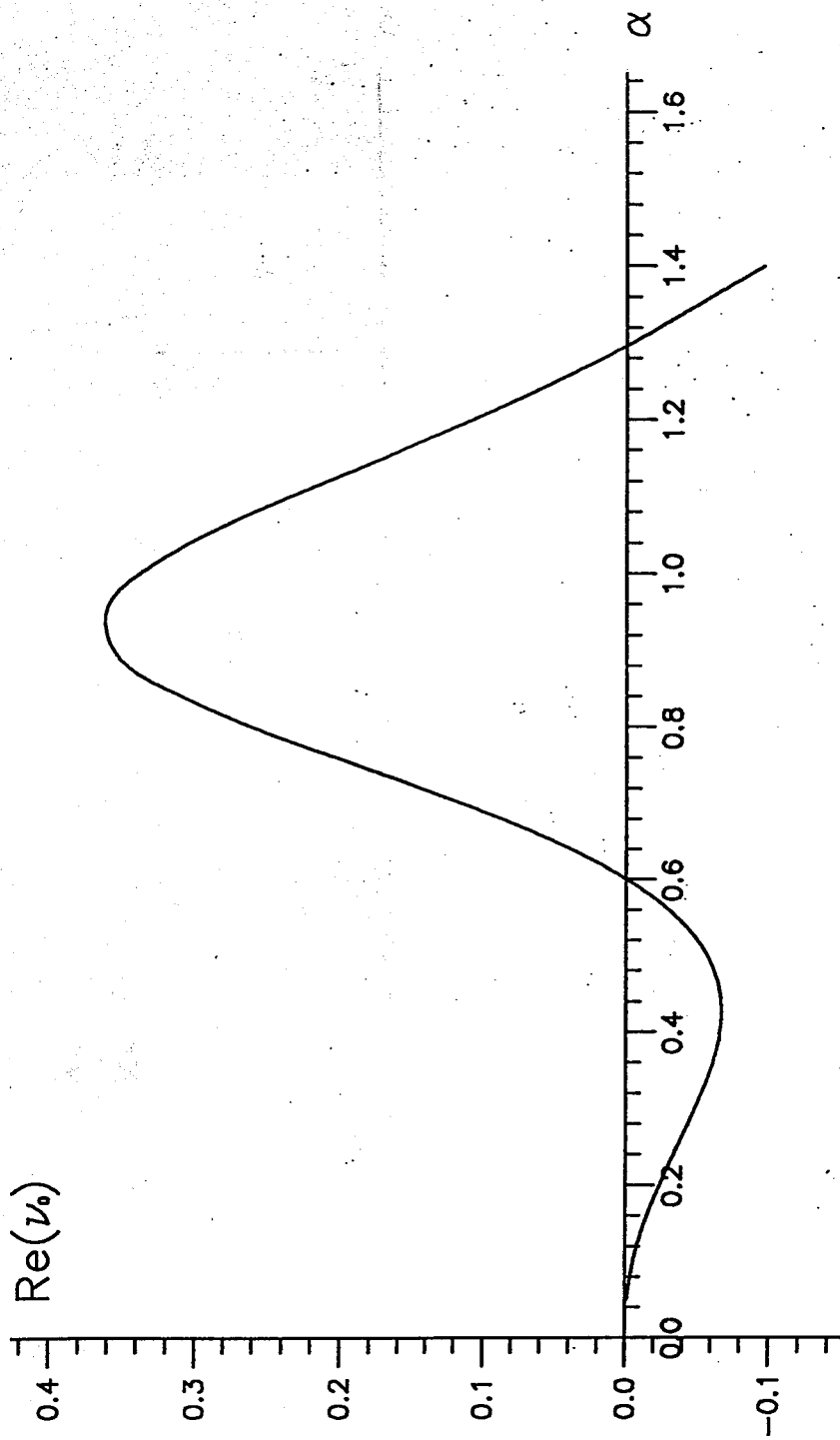


Figure 14

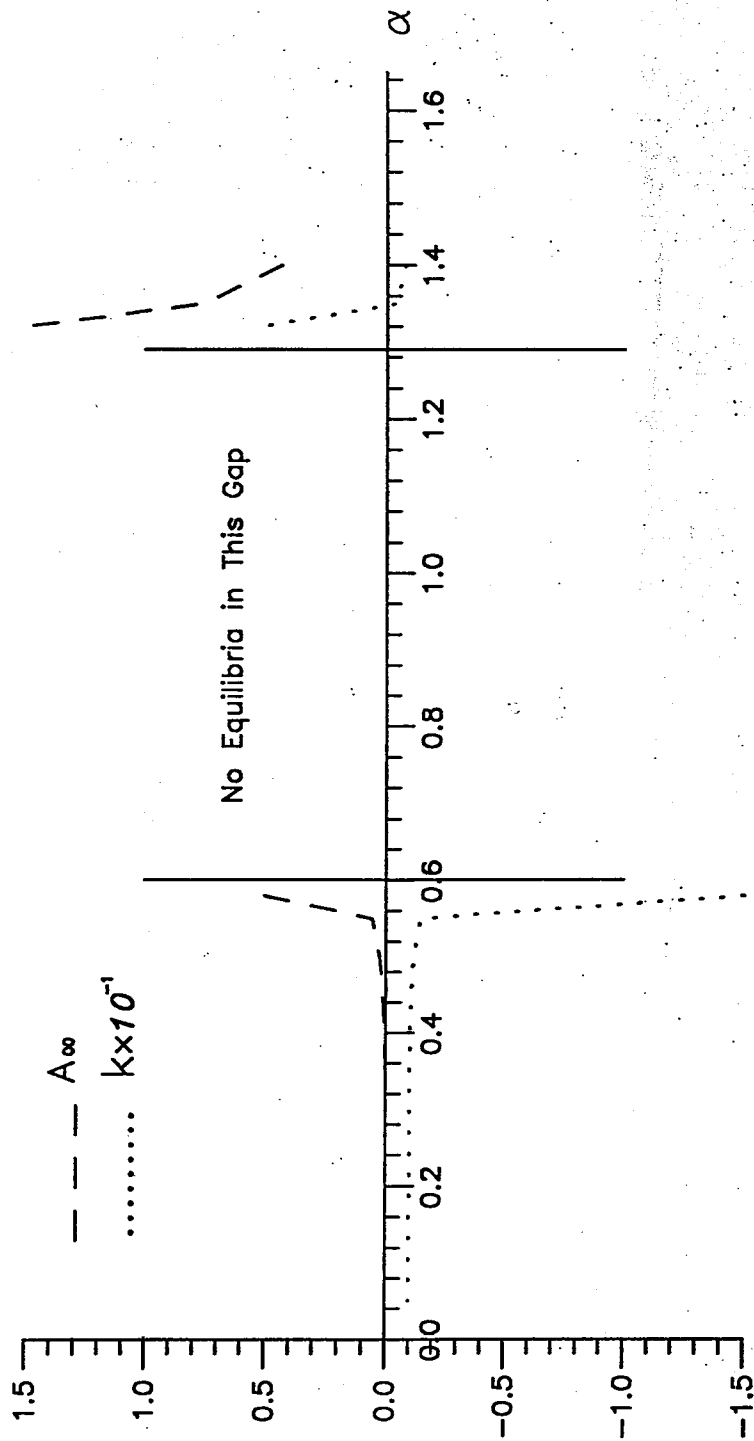


Figure 15

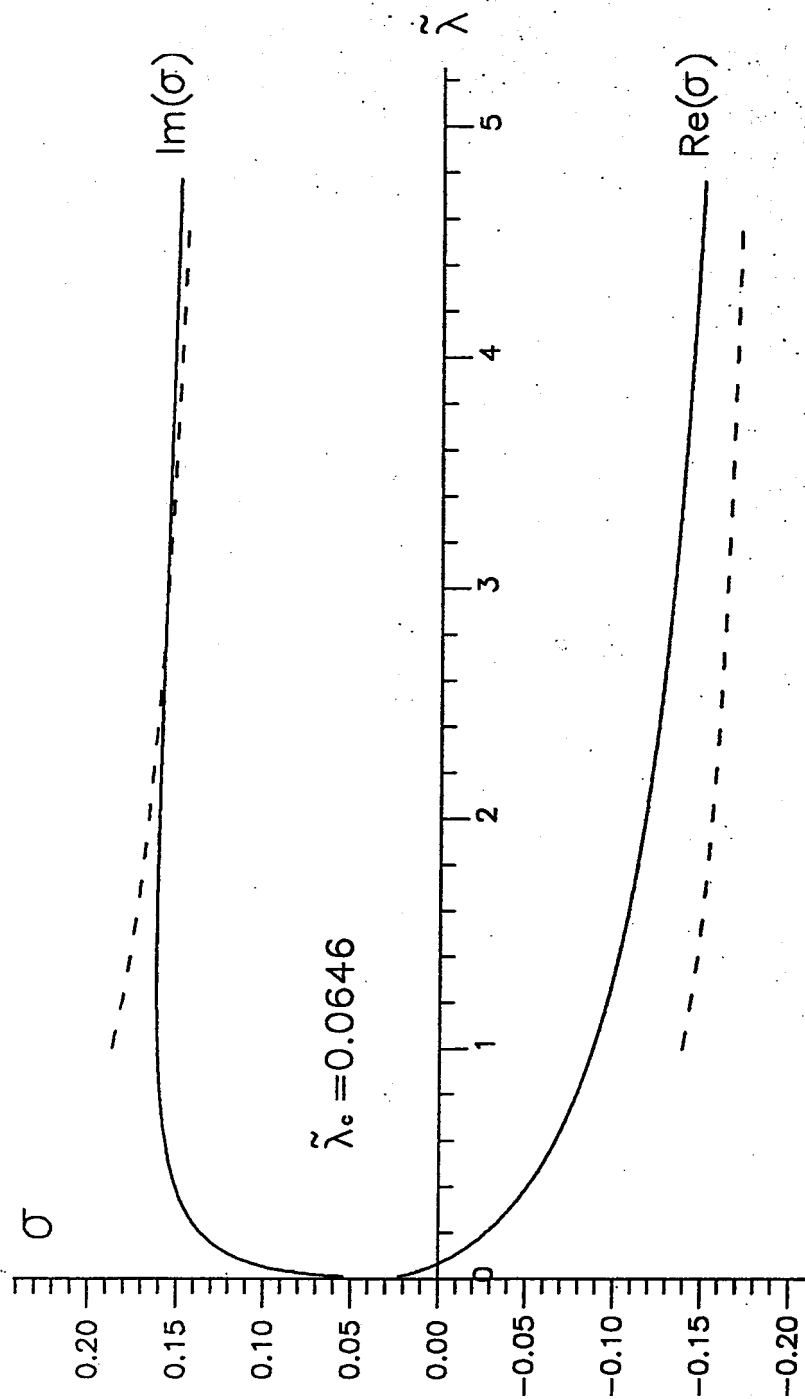


Figure 16

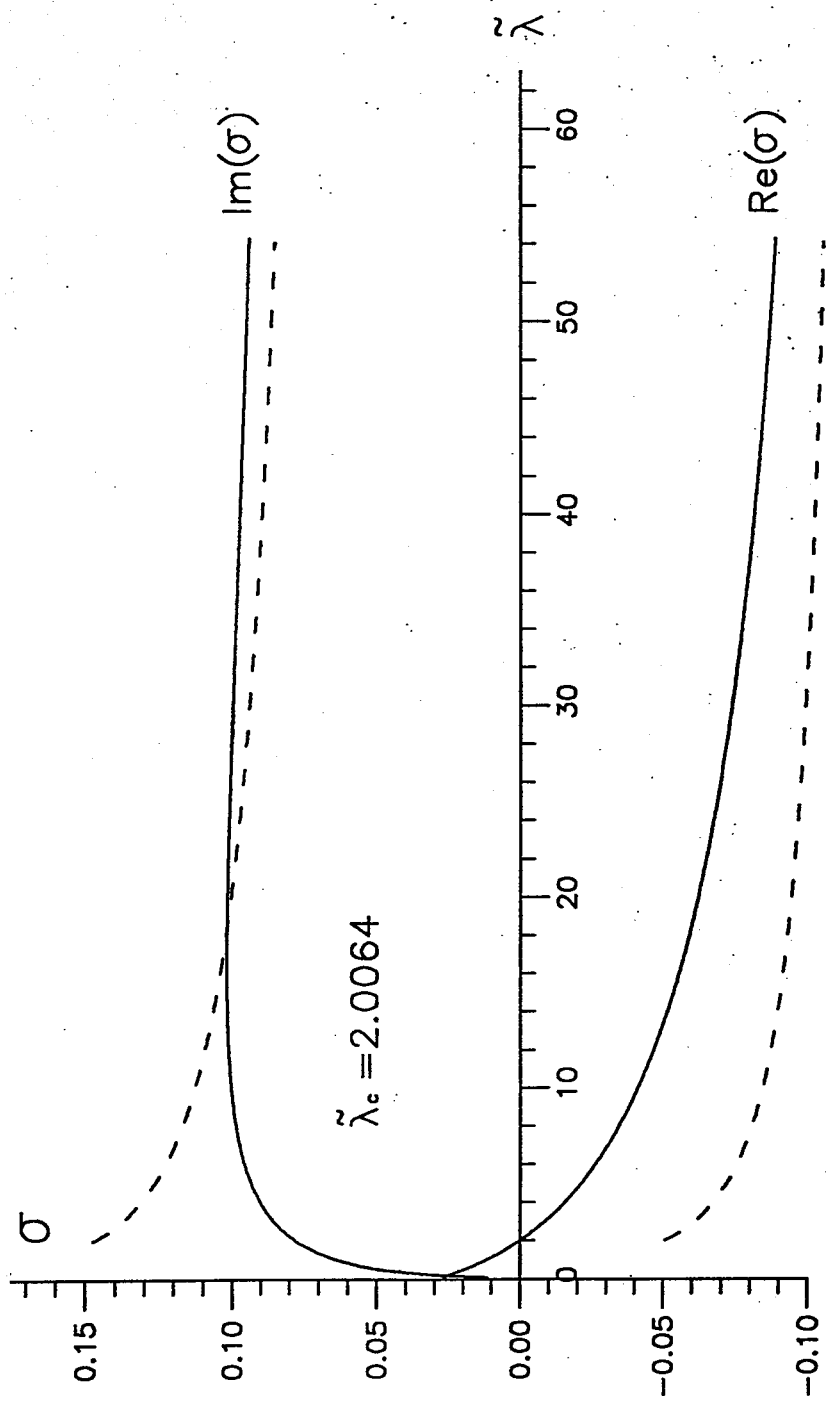


Figure 17

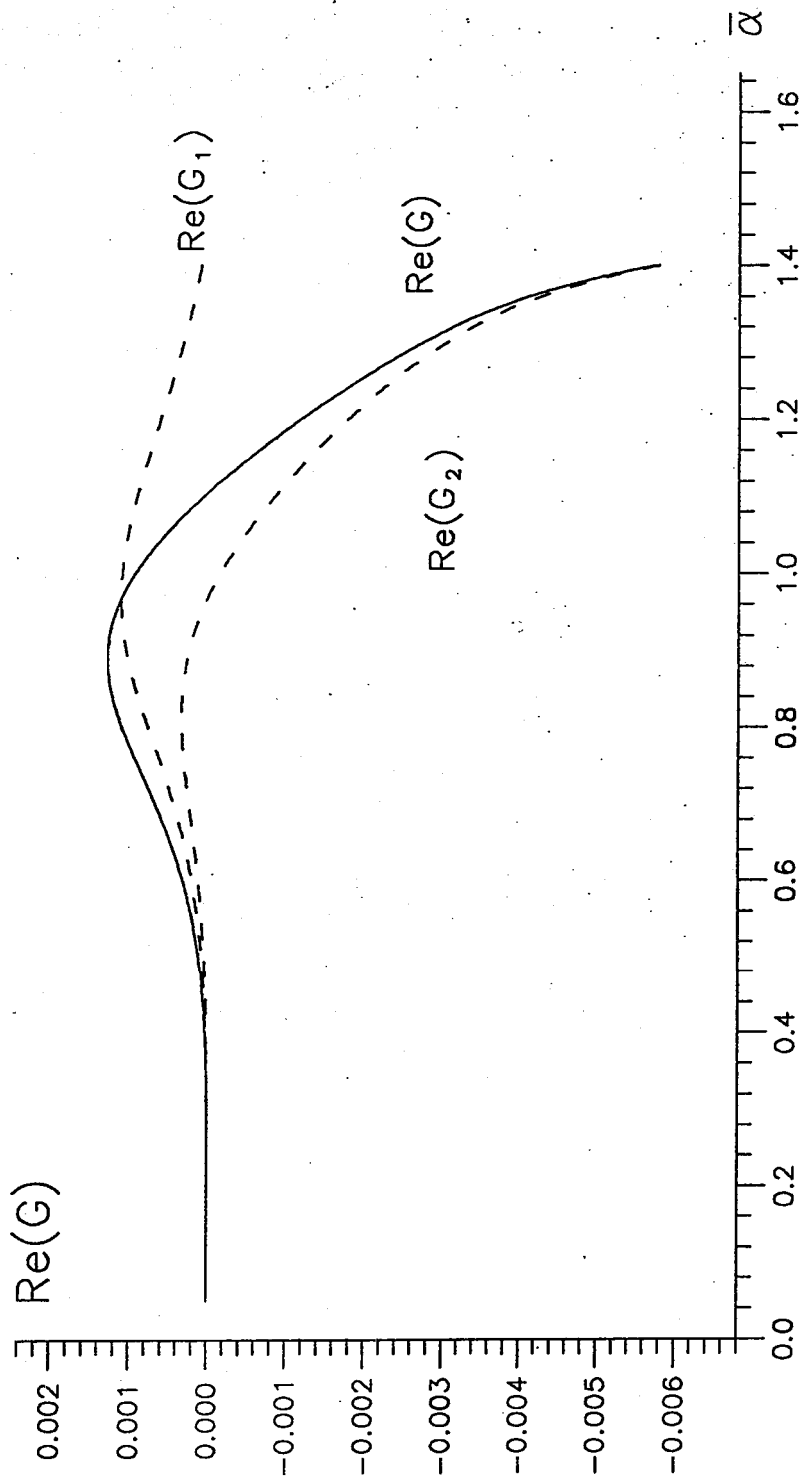


Figure 18a

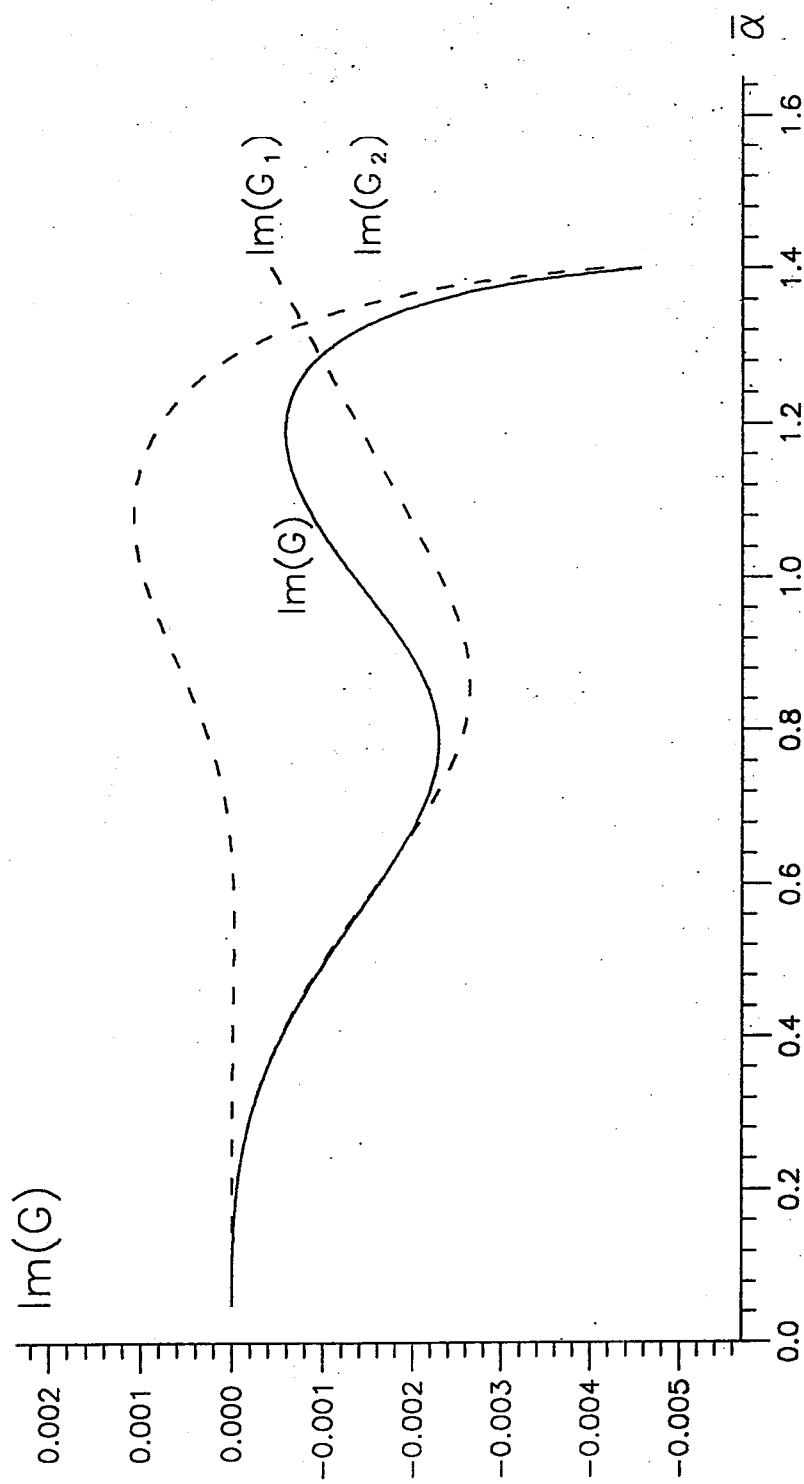


Figure 18b

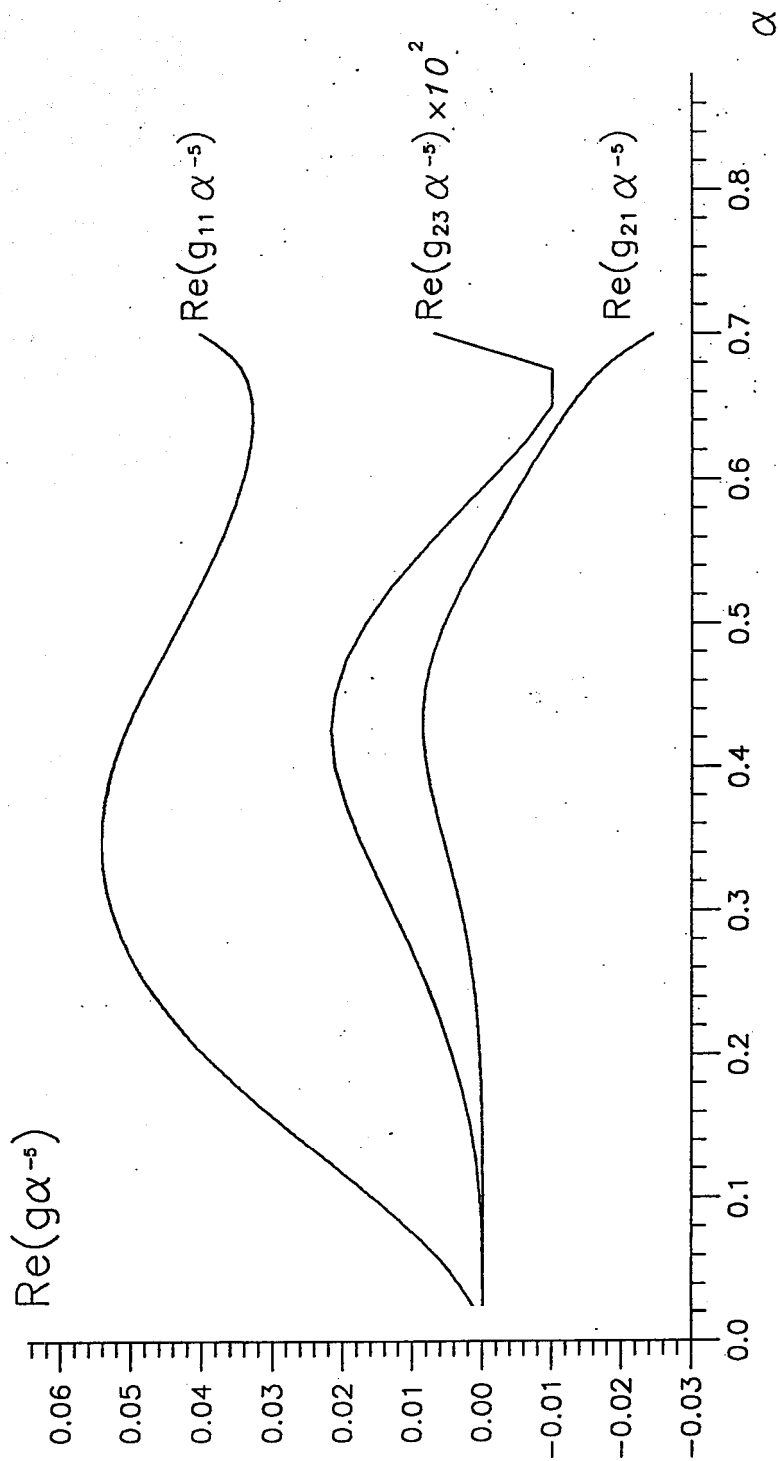


Figure 19a

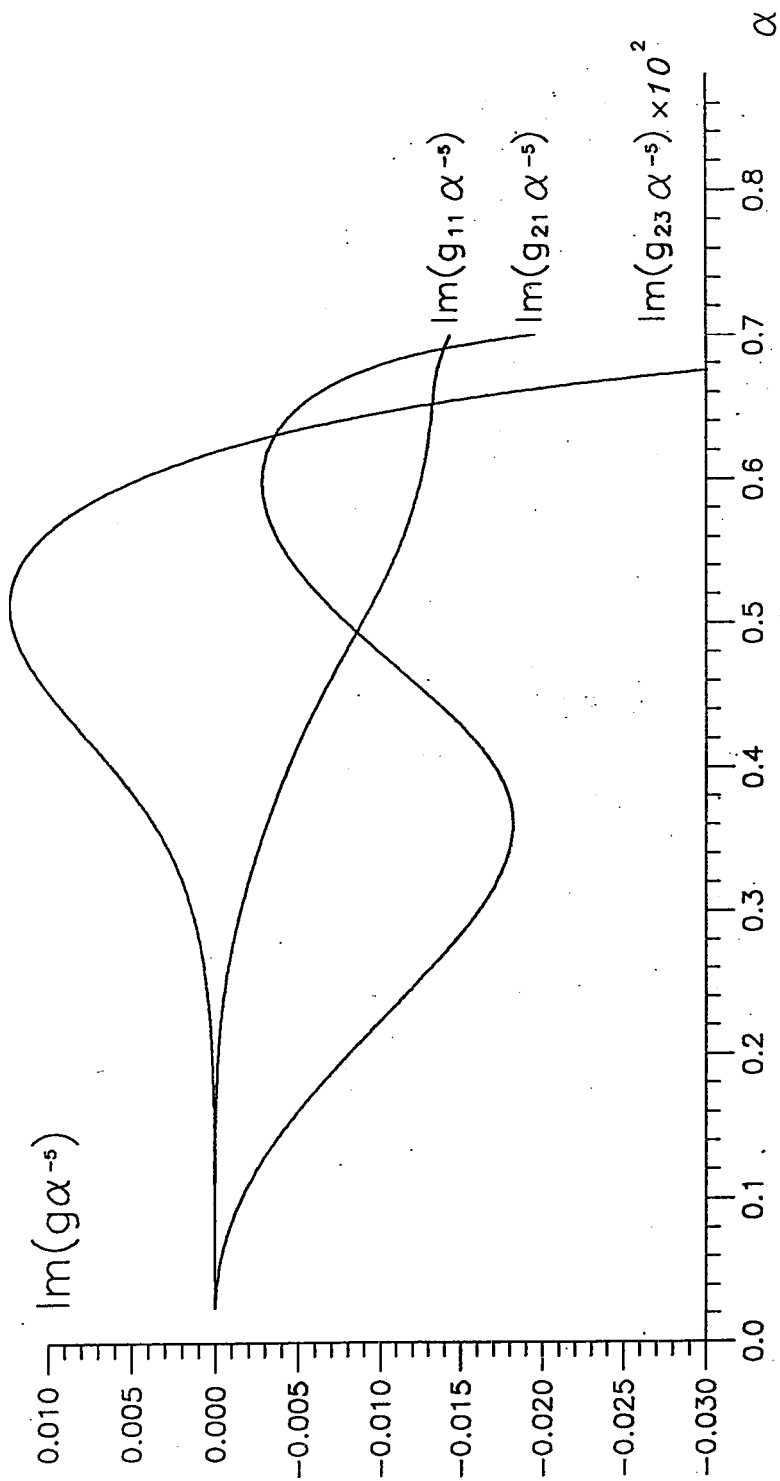


Figure 19b