Towards Constructive Nonlinear Control Systems Analysis and Design

by
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Abstract

This work presents a novel method to solve analysis and design problems for nonlinear control systems, the classical solutions of which rely on the solvability, or on the solution itself, of partial differential equations or inequalities. The first part of the thesis is dedicated to the analysis of nonlinear systems. The notion of Dynamic Lyapunov function is introduced. These functions allow to study stability properties of equilibrium points, similarly to standard Lyapunov functions. In the former, however, a positive definite function is combined with a dynamical system that render Dynamic Lyapunov functions easier to construct than Lyapunov functions. These ideas are then extended to characterize Dynamic Controllability and Observability functions, which are exploited in the model reduction problem for nonlinear systems. Constructive solutions to the $\mathcal{L}_2$-disturbance attenuation and the optimal control problems are proposed in the second part of the thesis. The key aspect of these solutions is the definition of Dynamic Value functions that, generalizing Dynamic Lyapunov functions, consist of a dynamical feedback and a positive definite function. In the last part of the thesis a similar approach is utilized to simplify the observer design problem via the Immersion and Invariance technique. Finally, the effectiveness of the methodologies is illustrated by means of several applications, including range estimation and the optimal robust control of mechanical systems, combustion engine test benches and the air path of a diesel engine.

Declaration

As required by the College, I hereby declare that this thesis is the result of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are appropriately referenced.

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To my parents

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Chapter 1

Introduction

1.1 Motivations

The theory of control, as the name entails, has been created with the aim of providing a set of tools to identify, characterize and, possibly, adjust the behavior of a dynamical system, be it the mathematical model of a physical plant or the abstraction of processes in the fields for example of economics, sociology or biology. Since the origins of this theory, mathematicians and engineers have been usually more interested in defining concepts that would allow them to *describe* properties of a dynamical system rather than to *construct*, *e.g.* by means of feedback control laws, systems possessing such properties.

It appears useful at this preliminary stage to clarify the meaning of *constructive* solutions to problems of control systems analysis. Towards this end, note that in several circumstances properties of dynamical systems may be related to the existence of specific functions. This is the case for instance of (asymptotic) stability in the sense of Lyapunov. Therefore, despite the fact that a Lyapunov function may be unknown for the mere purpose of analysis, an explicit expression of the function itself may be of interest for different reasons, *e.g.* control design methodologies based on the knowledge of a Lyapunov function. Hence, we refer to constructive solutions to analysis problems meaning that the solution not only allows to assess the desired property but provides simultaneously an explicit expression of the required function.

In recent years, mainly due to the more and more demanding performances expected from engineering applications, the need for constructive solutions has become of paramount importance. Thus, the focus of research activities has moved from the mere theoretical characterization of control systems properties to the actual design of implementable - both in simulation and in practice - solutions to analysis and design control problems. The impact of this shift in the point of view is attested by the definition of several concepts or techniques that, strictly speaking, may not be classified solely as neither design nor analysis tools. Control Lyapunov functions and feedback passivation, for instance, entail the attempt of extending a well-established analysis tool, namely Lyapunov functions, and a dynamical systems property, namely passivity, to the control design framework.

In this light a significative distinction must be introduced between linear and nonlinear control systems. In the case of linear models, in fact, constructive solutions are provided for the vast majority of control problems in terms of the solutions of linear matrix equations or inequalities. These solutions may then be computed (or approximated) employing widely available and reliable numerical algorithms. The extension of these approaches to the nonlinear setting is usually conceptually straightforward, leading to partial differential equations or inequalities, but computationally impracticable. In fact, due to the intrinsic nonlinear nature of the arising partial differential equations or inequalities, the closed-form solutions can be determined solely in the most favorable cases, which usually do not include practical applications. In addition, to complicate the picture boundary conditions are often imposed in terms of positivity conditions (*e.g.* in stabilization problems), of rank conditions (*e.g.* in the feedback linearization problem) or of algebraic constraints (*e.g.* in the tracking and regulation problem). As a result, very few computational tools are available, and these often rely on non-constructive arguments.

The objective of this thesis is to provide a first step towards the definition of constructive methods for the solutions of some important nonlinear control problems. These include analysis problems, such as assessing the stability properties of equilibrium points or characterizing the input/output behavior of dynamical systems, as well as design problems, ranging from disturbance attenuation and optimal control to observer design.

A complete discussion on the aforementioned research areas is beyond the scope of this section, since the literature concerning these problems is incredibly vast. A brief introduction to each topic is available at the beginning of the chapter in which the specific problem is covered.

1.2 Contribution of the Thesis

Due to the variety of different topics dealt with in the present work, encompassing Lyapunov stability analysis, input/output behavior characterization, disturbance attenuation, optimal control and observer design, it appears that a detailed and comprehensive discussion concerning the contributions of the thesis in each field is neither viable nor useful at this stage. Therefore, in this section, we limit our attention to few prominent features that are shared by the approaches in the different research areas, before concluding the section with a brief outline of each chapter.

In the present work we are mainly concerned with nonlinear control systems analysis and design problems, the solutions of which are given in terms of partial differential equations or inequalities. The key underlying idea may be stated as follows. To begin with, a different notion of solution of the partial differential equation or inequality – arising in the problem under examination – is defined. In fact, having in mind the objective of avoiding the integrability and, partly, the positivity (in the cases in which the latter condition is imposed) constraints, we seek for a smooth mapping, the so-called *algebraic solution*, that satisfies an algebraic *equivalent* of the corresponding partial differential equation or inequality. In particular, in the expression of the former condition the unknowns of the latter, *i.e.* the components of the gradient vector of the scalar function sought for, are replaced by the elements of a row vector. Then we consider a dynamic extension driven by the state of the original system. We anticipate that, throughout the entire thesis, the state of the dynamic extension is denoted by ξ . Chapter 6, however, which deals with the observer design problem, represents an exception. In fact, since therein the dynamic extension possesses an obvious physical meaning, namely it describes a filtered version of the measured output, we prefer to adopt a notation, *i.e.* $\hat{y}$, that identifies the nature of the dynamic extension.

Finally, the immersion of the underlying ordinary differential equation into the system composed of the above equation and the ξ-dynamics allows to construct, in the extended state-space, a class of functions such that each element of this family satisfies a "modified" version of the original partial differential equation or inequality, together with the corresponding boundary conditions. The implications of the above approximation are commented on in details in the following chapters. The contributions and the results of each chapter are summarized below.

In Chapter 2 the notion of Dynamic Lyapunov function is introduced. These functions allow to assess stability properties of equilibrium points, similarly to standard Lyapunov functions. In the former, however, a positive definite function is combined with additional dynamics that render Dynamic Lyapunov functions easier to construct than Lyapunov functions. The result is achieved exploiting the *time-varying* part of the function the behavior of which is autonomously adjusted and defined in terms of the solution of a dynamical system driven by the state of the system.

The balancing and model reduction problems for nonlinear systems are investigated in Chapter 3 introducing the notion of Dynamic Generalized Controllability and Observability functions. These functions are called dynamic and generalized since a dynamic extension is introduced and partial differential inequalities are solved in place of partial differential equations. It is shown that a class of Dynamic Generalized Controllability and Observability functions can be explicitly constructed, which is a crucial advantage with respect to Controllability and Observability functions. This aspect, in fact, allows to define a specific change of coordinates such that, in the new coordinates, the functions are in the so-called balanced form.

Chapter 4 introduces the notion of Dynamic Storage function and relates the existence of a Dynamic Storage function to the solution of the $\mathcal{L}_2$-disturbance attenuation problem. In particular the knowledge of a Dynamic Storage function allows to derive a dynamic control law that solves a disturbance attenuation problem. Interestingly the control law provided by the method reduces to the standard solution of the H_∞ disturbance attenuation problem in the case of linear time-invariant systems.

The finite-horizon and infinite-horizon optimal control problems are discussed in Chapter 5. In particular, similarly to Chapter 4, we define the notion of Dynamic Value function. These functions may be obtained from Dynamic Storage functions when the system is not affected by disturbance. Mimicking the above construction, it is shown that a value function is combined with a dynamical system, interpreted as a dynamic control law that approximates the solution of the optimal control problem. The approximation is twofold: a modified optimal control problem defined in an extended state-space is considered, on one hand, and partial differential inequalities are solved in place of equations, on the other hand. The resulting approximation errors may be reduced initializing the dynamic extension and shaping the additional running cost, respectively.

Chapter 6 extends the approach of the previous chapters to the reduced-order observer design problem via the Immersion and Invariance technique. This methodology hinges upon the solution of a system of partial differential equations which ensures attractivity of a desired invariant subset of the extended state-space of the plant and the observer. The main result of this chapter consists in showing that the issue of determining a closed-form solution of the pde may be avoided making use of an output filter and a dynamic scaling factor. The approach is then specialized to the challenging problem of estimating the range and the orientation of an object observed through a single pin-hole camera.

The thesis is concluded by Chapter 7 in which the performances of the proposed methodology are validated by means of several examples, including the optimal and robust control of fully actuated mechanical systems, the optimal control of a combustion engine speed and torque at a test bench and of the air path in a diesel engine.

1.3 Notation and Preliminaries

The purpose of this section is to provide the reader with the basic notation and mathematical background that are employed throughout the thesis. Some of the definitions or the preliminary results may already be familiar to the reader. Nevertheless they are reported in this section for completeness and to set a consistent framework for the remaining of the thesis.

Let $M \in \mathbb{R}^{n\times m}$ be a matrix. $M^\top$ denotes the transpose of M while $\|M\|$ denotes the induced 2-norm of M, namely $\|M\| = \bar{\sigma}(M)$, where $\bar{\sigma}(M)$ is the maximum singular value of M. Similarly, $\underline{\sigma}(M)$ is the minimum singular value of M.

Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix, namely such that $M = M^\top$. M is positive definite (positive semi-definite, resp.), denoted $M > 0$ ($M \geq 0$, resp.), if $x^\top Mx > 0$

($x^\top M x \geq 0$, resp.) for all $x \in \mathbb{R}^n \setminus \{0\}$. M is negative definite (negative semi-definite, resp.) if $-M$ is positive definite (positive semi-definite, resp.). Let v be a vector in $\mathbb{R}^n$ and $M = M^\top > 0$, then $\|v\|_M$ denotes the Euclidean norm of v weighted by the matrix M, that is $\|v\|_M = (v^\top M v)^{1/2}$.

Lemma 1.1. [3] Let M be an $n \times n$ symmetric matrix and C an $m \times n$ matrix with $rank(C) = m$, where $m < n$. Let Z denote a basis for the null space of C.

(i) If $Z^T M Z$ is positive semidefinite and singular, then there exists a finite $\bar{k} \geq 0$ such that $M + kC^T C$ is positive semidefinite for all $k \geq \bar{k}$, if and only if $Ker(Z^T M Z) = Ker(MZ)$. Moreover, $M + kC^T C$ is singular for all k.

(ii) $Z^T M Z$ is positive definite if and only if there exists a finite $\bar{k} \geq 0$ such that $M + kC^T C$ is positive definite for all $k > \bar{k}$. ◇

Definition 1.1. Let $V : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function, with $\mathcal{D}$ containing the origin. V is positive definite in $\mathcal{D}$ around the origin if $V(0) = 0$ and $V(x) > 0$ for all $x \in \mathcal{D} \setminus \{0\}$. The function V is negative definite in $\mathcal{D}$ if $-V$ is positive definite in $\mathcal{D}$.

The notation $\dfrac{\partial V}{\partial x}$ (V_x when there is no danger of confusion) represents the gradient of the continuously differentiable function V with respect to the vector $x \in \mathbb{R}^n$. Similarly, V_{xx} denotes the Hessian matrix of the scalar function V with respect to x. Let $P(x) = [P_1(x), ..., P_n(x)]$, with $P_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, ..., n$, be a continuously differentiable mapping. Then there exists $V : \mathbb{R}^n \to \mathbb{R}$ such that $V_x(x) = P(x)$ if and only if (see for instance Lemma 2.22 of [10])

$$\frac{\partial P_i}{\partial x_j}(x) = \frac{\partial P_j}{\partial x_i}(x)\,, \tag{1.1}$$

for all x and $i, j = 1, 2, ..., n$.

Consider the autonomous nonlinear system described by equations of the form

$$\dot{x} = f(x), \tag{1.2}$$

where $x(t) \in \mathbb{R}^n$ and $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, with $\mathcal{D}$ containing the origin of $\mathbb{R}^n$. The time derivative of the function V along the trajectories of the system (1.2) is defined as $\dot{V}(x(t)) = V_x f(x)\Big|_{x=x(t)}$. Moreover f_x denotes the Jacobian matrix of the mapping $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$.

Definition 1.2. A continuous function $\alpha : [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is said to belong to class $\mathcal{K}_\infty$ if in addition it is unbounded, *i.e.* $\alpha(s) \to \infty$, as $s \to \infty$.

Definition 1.3. A continuous function $\beta[0,\infty)\times[0,\infty)\to[0,\infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the mapping $\beta(r,s)$ belongs to class $\mathcal{K}$ with respect to r and, for each fixed r, the mapping $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s)\to 0$ as $s\to\infty$.

Consider the autonomous nonlinear system (1.2). Suppose that $x_e = 0$ is an equilibrium point for the system (1.2), *i.e.* $f(0) = 0$. The stability properties of the zero equilibrium of (1.2) can be characterized in the sense of Lyapunov as detailed in the following definition.

Definition 1.4. [47, 56] The equilibrium point $x_e = 0$ of (1.2) is

(i) stable if, for each $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|x(0)\| < \delta \Longrightarrow \|x(t)\| < \varepsilon$ for all $t \geq 0$;

(ii) unstable if it is not stable;

(iii) asymptotically stable if it is stable and δ can be chosen such that $\|x(0)\| < \delta \Longrightarrow \lim_{t\to\infty} x(t) = 0$;

(iv) exponentially stable if $\|x(t)\| \leq k\|x(0)\|e^{-\gamma t}$, for all $t \geq 0$ and for some $k > 0$, $\gamma > 0$.

Theorem 1.1. [47] Let $x_e = 0$ be an equilibrium point of (1.2) and $\mathcal{D} \subseteq \mathbb{R}^n$ be a domain containing the origin. Let $V : \mathcal{D} \to \mathbb{R}$ be a $\mathcal{C}^1$ positive definite function around the origin. Then, $x_e = 0$ is stable if $\dot{V} \leq 0$ in $\mathcal{D}$. Moreover, if $\dot{V} < 0$ in $\mathcal{D} \setminus \{0\}$ then $x_e = 0$ is asymptotically stable. $\diamond$

A function V satisfying the conditions of Theorem 1.1 is called a *Lyapunov function* if $\dot{V}$ is negative definite and a *weak Lyapunov function* if $\dot{V}$ is negative semi-definite.

Definition 1.5. A subset $\mathcal{F} \subset \mathbb{R}^n$ is said to be *invariant* with respect to the system (1.2) if $x(0) \in \mathcal{F}$ implies $x(t) \in \mathcal{F}$ for all $t \geq 0$, where $x(t)$ denotes the solution of the system (1.2) with initial condition $x(0)$. Moreover, a subset $\mathcal{F}$ is said to be *almost-invariant* with respect to the system (1.2) if, for any given $\varepsilon > 0$, $\mathrm{dist}(x(0),\mathcal{F}) \leq \varepsilon$ implies $\mathrm{dist}(x(t),\mathcal{F}) \leq \varepsilon$ for all $t \geq 0$, where $\mathrm{dist}(x(t),\mathcal{F})$ denotes the distance of $x(t)$ from the subset $\mathcal{F}$.

Definition 1.6. Let $P : \mathbb{R}^n \to \mathbb{R}^{1\times n}$ be a continuously differentiable mapping. Then the function

$$V(x,\xi) \triangleq P(\xi)x + \frac{1}{2}\|x-\xi\|_R^2, \tag{1.3}$$

with $\xi \in \mathbb{R}^n$ and $R = R^\top > 0$, is called the *extension* of P.

Lemma 1.2. Let $P : \mathbb{R}^n \to \mathbb{R}^{1\times n}$ be a continuously differentiable mapping and suppose that $P_x(x)\Big|_{x=0} = P_x(x)^\top\Big|_{x=0} > 0$. Then there exist a non-empty neighborhood $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$, containing the origin, and a matrix $\bar{R} = \bar{R}^\top > 0$ such that the extension of P is positive definite for all $(x, \xi) \in \Omega$ and for all $R > \bar{R}$. $\diamond$

Proof. Since the mapping P is such that $P_x(x)\Big|_{x=0} = \bar{P} > 0$, then the function $P(x)x : \mathbb{R}^n \to \mathbb{R}$, is quadratic locally around the origin and moreover has a local minimum for $x = 0$. To show this claim note that the first-order derivative of the function is zero at $x = 0$ and $(P(x)x)_{xx}\Big|_{x=0} = 2\bar{P} > 0$. Hence, the existence of $\bar{R}$ can be proved noting that the function $P(\xi)x$ is (locally) quadratic and, restricted to the set

$$\mathcal{F} \triangleq \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi = x\},$$

is positive definite for all $x \neq 0$ in a non-empty neighborhood Ω. $\square$

Finally, consider the definition of Lebesgue function spaces, *i.e.* $\mathcal{L}_p$-spaces, together with the corresponding norms.

Definition 1.7. The set $\mathcal{L}_q$, $q \in \{1, 2, ...\}$, $q < \infty$, consists of all the measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ that satisfy

$$\int_0^\infty \|f(t)\|^q dt < \infty\,.$$

Moreover

$$\left(\int_0^\infty \|f(t)\|^q dt\right)^{\frac{1}{q}}$$

is the $\mathcal{L}_q$-norm of the function $f \in \mathcal{L}_q$. If $q = \infty$, the set $\mathcal{L}_\infty$ consists of all measurable functions $f : \mathbb{R}_+ \to \mathbb{R}$ which are bounded, namely

$$\sup_{t\in\mathbb{R}_+} \|f(t)\| < \infty\,,$$

with norm $\|f\|_\infty = \sup_{t\in\mathbb{R}_+} \|f(t)\|$.

1.4 Published Material

The results presented in Chapter 2 are included in the papers [84, 89, 91]. Chapter 3 is partly contained in [90]. The work of Chapter 4 and Section 5.4 is included in [83]. Preliminary versions are contained in the papers [85] and [86], respectively. The results of Section 5.3 are discussed in [82] and [87]. The results of Section 6.2 are contained in [44] while the papers [16, 92, 93] encompass the topic of Section 6.3. The application of Section 7.2 is dealt with in [83, 88]. The experimental results of Sections 7.3 and 7.4 are included in the papers [72] and [94], respectively.

Chapter 2

Dynamic Lyapunov Functions

2.1 Introduction

In practical applications a system initialized at an equilibrium rarely remain in the same configuration indefinitely. In fact, in the vast majority of the situations noise or external disturbances acting on the system may perturb the state of the system driving it away from the equilibrium point. Therefore it is not surprising that the problem of studying the behavior of the trajectories of a system initialized in a neighborhood of an equilibrium point dates back to the early years of the theory of ordinary differential equations. Even before the introduction of the formal definition of stability of equilibrium points several results showed that, without external disturbances, if an equilibrium point of a conservative mechanical system is in addition a minimum of the potential energy then it is a stable (in the sense of Definition 1.4) equilibrium point. Over the past century this challenging topic encouraged a number of researchers to develop theories to tackle the aforementioned problem. Clearly, such a theory should allow to characterize the behavior of the trajectories of the system near the equilibrium point without actually determining the trajectories of the system, namely without knowledge of the explicit solution of the underlying ordinary differential equation. A first step towards this end is the definition of the notion of stability of equilibrium points, usually characterized in the sense of Lyapunov, see [56, 79] for more detailed discussions.

Lyapunov's theory provides a mathematical tool to assess stability, instability, asymptotic stability or exponential stability of equilibrium points of linear and nonlinear systems, avoiding the explicit computation of the solution of the underlying ordinary differential equation. In what follows the attention is focused on the time-invariant case. It is well-known, see *e.g.* [6, 47], that the existence of a scalar function, positive definite around an equilibrium point, the time derivative of which along the trajectories of the system is negative definite, is sufficient to guarantee asymptotic stability of the equilibrium point. Such a function is generally referred to as a Lyapunov function. Moreover if the scalar function is positive definite and its time derivative is only negative semi-definite then stability of the equilibrium point can be proved and the function is called a *weak* Lyapunov function.

Conversely, several theorems, the so-called *converse Lyapunov theorems*, implying the existence of a Lyapunov function defined in a neighborhood of an asymptotically stable equilibrium point, have been established, see for instance [47, 51, 52, 61]. However the mere knowledge of the existence of a Lyapunov function is often not satisfactory since the actual computation of the analytic expression of the function may be extremely difficult. This construction usually relies upon the knowledge of the explicit solution of the ordinary differential equation which is somewhat contradictory to the *spirit* of Lyapunov's theory, the aim of which is mainly to avoid the computation of all the trajectories of the system to assess the stability properties of an equilibrium point. From a practical point of view this is the main drawback of Lyapunov's methods [7].

A different approach consists in determining, if it exists, a *weak* Lyapunov function, *i.e.* a function the time derivative of which is only negative semi-definite along the trajectories of the system, and then prove asymptotic properties by means of *Invariance Principles* [47]. Finally, a somewhat more flexible approach is pursued in [59], where the authors give sufficient conditions under which *weak* Lyapunov functions, which may be more easily available, can be employed to construct Lyapunov functions.

In recent years the development of control techniques requiring the explicit knowledge of a Lyapunov function, such as Control Lyapunov functions, *backstepping* and *forwarding*, see *e.g.* [39,47,48,63,76,99], have conferred a crucial role to the construction of Lyapunov functions as design tools. In addition Lyapunov functions are useful to characterize and to estimate the region of attraction of locally asymptotically stable equilibrium points, see for instance [19] and [102], where the notion of *maximal* Lyapunov function has been introduced.

The rest of the chapter is organized as follows. The notion of Dynamic Lyapunov function is introduced in Section 2.2 together with basic results relating the stability properties of an equilibrium point to the existence of a Dynamic Lyapunov function. The topic of Section 2.3 is the construction of Dynamic Lyapunov functions for linear time-invariant systems. The extension to nonlinear systems is dealt with in Section 2.4 providing the proof of Theorem 2.2, which states a converse result for Dynamic Lyapunov functions, presented in Section 2.2 for the general nonlinear case. Section 2.5 deals with the problem of constructing Lyapunov functions from the knowledge of a Dynamic Lyapunov function for linear and nonlinear systems. The chapter is concluded by several examples and further results in Section 2.6.

2.2 Dynamic Lyapunov Functions

Consider the nonlinear autonomous system described by equations of the form

$$\dot{x} = f(x)\,, \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system and $f : \mathcal{D} \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, with $\mathcal{D}$ containing the origin of $\mathbb{R}^n$.

Definition 2.1. Consider the nonlinear autonomous system (2.1) and suppose that the origin of the state-space is an equilibrium point of (2.1). A (weak) *Dynamic Lyapunov function* $\mathcal{V}$ is a pair (D_α, V) defined as follows

- D_α is the ordinary differential equation $\dot{\xi} = \alpha(x, \xi)$, with $\xi(t) \in \mathbb{R}^n$, $\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz and $\alpha(0, 0) = 0$.
- $V : \Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is positive definite around $(x, \xi) = (0, 0)$ and it is such that

  $$\dot{V}(x, \xi) = V_x f(x) + V_\xi \alpha(x, \xi) < 0,$$

  for all $(x, \xi) \in \Omega \setminus \{0\}$ ($\dot{V}(x, \xi) \leq 0$ for a weak Dynamic Lyapunov function).

Theorem 2.1. Consider the nonlinear autonomous system (2.1) and suppose that the origin of the state-space is an equilibrium point of (2.1). Suppose that there exists a Dynamic Lyapunov function for system (2.1). Then $x = 0$ is an asymptotically stable equilibrium point of the system (2.1). $\diamond$

Proof. Suppose that $\mathcal{V}$ is a Dynamic Lyapunov function for (2.1). Then the positive definite function V is a Lyapunov function for the *augmented* system

$$\begin{aligned} \dot{x} &= f(x)\,, \\ \dot{\xi} &= \alpha(x, \xi)\,, \end{aligned} \tag{2.2}$$

implying, by Lyapunov's Theorem 1.1, asymptotic stability of the equilibrium point $(x, \xi) = (0, 0)$ of the system (2.2). By Lemma 4.5 of [47], the latter is equivalent to the existence of a class $\mathcal{KL}$ function β such that $\|(x(t),\, \xi(t))\| \leq \beta(\|(x(0),\, \xi(0))\|, t)$ for all $t \geq 0$ and for any $(x(0),\, \xi(0)) \in \Omega$. Therefore $\|x(t)\| \leq \beta(\|(x(0),\, 0)\|, t) \triangleq \bar{\beta}(\|x(0)\|, t)$ proving asymptotic stability of the origin of the system (2.1). $\square$

Theorem 2.1 states that Dynamic Lyapunov functions represent a mathematical tool to investigate Lyapunov stability properties of equilibrium points, different from standard Lyapunov functions.

Remark 2.1. In the following sections it is explained how to select α to enforce negativity of the time derivative of the function V as in Definition 2.1. The key aspect consists in introducing a class of positive definite functions and then designing α such that each element of this class becomes a Lyapunov function for the augmented system (2.2). Therefore, the importance of the following Theorem lies in the fact that its proof is not carried out by augmenting system (2.1) with an asymptotically stable autonomous system and then resorting to arguments similar to those in the proofs of standard converse Lyapunov theorems. On the contrary, the proof provides a systematic methodology to construct Dynamic Lyapunov functions without assuming knowledge of the solution of the underlying differential equation and without involving any partial differential equation. In practical situations this aspect represents an advantage of Dynamic Lyapunov functions over Lyapunov functions. ▲

The following theorem establishes a *converse* result that guarantees the existence of a Dynamic Lyapunov function in a neighborhood of a locally exponentially stable equilibrium point.

Theorem 2.2. Consider the nonlinear autonomous system (2.1) and suppose that the origin of the state-space is a locally exponentially stable equilibrium point. Then there exists a Dynamic Lyapunov function for system (2.1). ◇

The proof of Theorem 2.2 is constructive and it is given in Sections 2.3 and 2.4 for linear and nonlinear systems, respectively. In particular, the statement of Theorem 2.2 guarantees the existence of a Dynamic Lyapunov function in a neighborhood of an exponentially stable equilibrium point, while the proof provides explicitly a Dynamic Lyapunov function $\mathcal{V} = (D_\alpha, V)$.

The result is achieved noting that a Dynamic Lyapunov function implicitly includes a *time-varying* term the behavior of which is autonomously adjusted and defined in terms of the solution of a differential equation. It is important to stress that Dynamic Lyapunov functions can be determined from the knowledge of a local quadratic Lyapunov function and, as demonstrated by the examples, provide a tool to study stability properties of nonlinear systems which compares favorably against the use of quadratic Lyapunov functions, for instance yielding sharper estimates of domains of attraction.

We conclude this section showing that the knowledge of Dynamic Lyapunov functions can be exploited to *construct* Lyapunov functions for the system (2.1).

Theorem 2.3. Consider the nonlinear autonomous system (2.1). Suppose that $\mathcal{V} = (D_\alpha, V)$ is a Dynamic Lyapunov function for (2.1) and that there exists a $\mathcal{C}^1$ mapping $h : \mathbb{R}^n \to \mathbb{R}^n$, $h(0) = 0$, such that

$$h_x(x) f(x) = \alpha(x, h(x)) \,. \tag{2.3}$$

Then $V_{\mathcal{M}}(x) \triangleq V(x, h(x))$ is a Lyapunov function for the system (2.1). $\diamond$

Proof. The condition (2.3) implies that the set $\mathcal{M} = \{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n : \xi = h(x)\}$ is invariant for the dynamics of the *augmented* system (2.2). The restriction of the system (2.2) to the invariant set is a copy of the dynamics of the system (2.1). Note that, by definition of Dynamic Lyapunov function, $V(x, \xi) > 0$ and $\dot{V}(x, \xi) < 0$ for all $(x, \xi) \in \Omega \subset \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$. Moreover

$$\begin{aligned}\dot{V}_{\mathcal{M}} &= V_x(x, \lambda)\Big|_{\lambda=h(x)} f(x) + V_\lambda(x, \lambda)\Big|_{\lambda=h(x)} h_x(x) f(x) \\ &= V_x(x, \lambda)\Big|_{\lambda=h(x)} f(x) + V_\lambda(x, \lambda)\Big|_{\lambda=h(x)} \alpha(x, h(x)) = \dot{V}(x, h(x)) < 0\,,\end{aligned}$$

where the second equality is obtained considering the equation (2.3). The function $V_{\mathcal{M}}$ depends only on x, is positive definite around $x = 0$ and its time derivative is negative definite, which proves the claim. $\square$

Corollary 2.1. Suppose that $h : \mathbb{R}^n \to \mathbb{R}^n$, $h(0) = 0$, is a solution of (2.3) such that

(i) $rank \left(\dfrac{\partial h(x)}{\partial x}\Big|_{x=0} \right) = n$,

(ii) $h(x)^\top \alpha(x, h(x)) < 0$ for all x in a neighborhood $\Omega \subseteq \mathbb{R}^n$ of the origin.

Then there exists a set $\hat{\Omega} \subseteq \Omega$ such that $V(x) = \frac{1}{2} h(x)^\top h(x)$ is a Lyapunov function for the system (2.1) in $\hat{\Omega}$. $\diamond$

Proof. By condition (i) the function $V(x) = \frac{1}{2} h(x)^\top h(x)$ is positive definite around the origin. Moreover the time derivative of the function V along the trajectories of the system (2.1) is

$$\dot{V} = h(x)^\top h_x(x) f(x) = h(x)^\top \alpha(x, h(x)) < 0\,,$$

where the last equality and inequality are obtained by the conditions (2.3) and (ii), respectively. $\square$

Remark 2.2. If condition (i) in Corollary 2.1 is replaced by the assumptions that $h : \mathbb{R}^n \to \mathbb{R}^n$ is a global diffeomorphism and the function $h^\top h$ is radially unbounded and (ii) holds with $\Omega = \mathbb{R}^n$, then $V(x) = \frac{1}{2} h(x)^\top h(x)$ is a global Lyapunov function for the system (2.1) $\blacktriangle$

In the following sections it is shown how Lyapunov functions can be obtained from Dynamic Lyapunov functions, namely guaranteeing the invariance, with respect to the system (2.2), of a subset of the *extended* state-space parameterized in x, as detailed in

Sections 2.3 and 2.4 for linear and nonlinear systems, respectively. It is interesting to note that for nonlinear systems the condition to achieve the invariance of the desired subset is given in terms of a system of first-order partial differential equations similar to the Lyapunov pde, namely $V_x f(x) = -\nu(V)$ where ν is a class $\mathcal{K}$ function. However, as explained in Section 2.4, differently from the Lyapunov pde the solution of which must be positive definite around the origin, no sign constraint is imposed on the solution h or on the mapping α, which is an advantage of the latter approach with respect to the former.

Finally, the invariance condition (2.3) is given in terms of a system of first-order nonlinear partial differential equations, which may be hard or impossible to solve. In such cases, the closed-form solution h of (2.3) can be replaced by an approximation as clarified in the following statement.

Theorem 2.4. Consider the nonlinear autonomous system (2.1). Suppose that $\mathcal{V} = (D_\alpha, V)$ is a Dynamic Lyapunov function for (2.1) and that there exists a $\mathcal{C}^1$ mapping $\hat{h} : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\|\hat{h}_x(x) f(x) - \alpha(x, \hat{h}(x))\| < \frac{\|\dot{V}(x, \hat{h}(x))\|}{\|\kappa(x, \hat{h}(x))\|}\,, \tag{2.4}$$

for all $x \in \Omega \setminus \{0\}$, where $\kappa(x, \hat{h}(x)) = V_\lambda(x, \lambda)\Big|_{\lambda = \hat{h}(x)}$. Then $\hat{V}_{\mathcal{M}}(x) \triangleq V(x, \hat{h}(x))$ is a Lyapunov function for the system (2.1). $\diamond$

Proof. To begin with note that $\hat{V}_{\mathcal{M}}$ is positive definite around the origin. The time derivative of $\hat{V}_{\mathcal{M}}$ along the trajectories of the system (2.1) is

$$\begin{aligned}
\dot{\hat{V}}_{\mathcal{M}} &= V_x(x, \lambda)\Big|_{\lambda = \hat{h}(x)} f(x) + V_\lambda(x, \lambda)\Big|_{\lambda = \hat{h}(x)} \hat{h}_x(x) f(x) \\
&= V_x(x, \lambda)\Big|_{\lambda = \hat{h}(x)} f(x) + V_\lambda(x, \lambda)\Big|_{\lambda = \hat{h}(x)} \alpha(x, \hat{h}(x)) \\
&\quad + V_\lambda(x, \lambda)\Big|_{\lambda = \hat{h}(x)} \left[\hat{h}_x(x) f(x) - \alpha(x, \hat{h}(x))\right] \\
&\le \dot{V}(x, \hat{h}(x)) + \|\kappa(x, \hat{h}(x))\|\, \|\hat{h}_x(x) f(x) - \alpha(x, \hat{h}(x))\| < 0\,,
\end{aligned}$$

for all $x \in \Omega \setminus \{0\}$, where the last strict inequality is derived considering the condition (2.4). $\square$

Remark 2.3. Every mapping h that solves the partial differential equation (2.3) is also a solution of (2.4) since in this case the left-hand side of (2.4) is equal to zero for all $x \in \mathbb{R}^n$. $\blacktriangle$

2.3 Construction of Dynamic Lyapunov Functions for Linear Systems

Consider a linear, time-invariant, autonomous system described by equations of the form

$$\dot{x} = Ax\,, \tag{2.5}$$

with $x(t) \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Consider the system (2.5) and suppose that there exists a mapping $x^\top P$, with $P = P^\top > 0$ such that

$$\frac{1}{2}x^\top PAx + \frac{1}{2}x^\top A^\top Px = -x^\top Qx\,, \tag{2.6}$$

for some given $Q = Q^\top > 0$ and for all $x \in \mathbb{R}^n$. Note that the mapping $x^\top P$ is an exact differential, however to present the main ideas of the proposed approach and to prove Theorem 2.2 for linear systems suppose that, instead of integrating the mapping $x^\top P$ obtaining the quadratic function

$$V_\ell(x) = \frac{1}{2}x^\top Px = \int_0^1 (\zeta(\sigma)^\top P)d\sigma\,, \tag{2.7}$$

for any state trajectory such that $\zeta(0) = 0$ and $\zeta(1) = x$, we exploit the mapping $P(x) = x^\top P$ to construct an *auxiliary* function defined in an *extended* space, namely

$$V(x,\xi) = \xi^\top Px + \frac{1}{2}\|x-\xi\|_R^2\,, \tag{2.8}$$

with $\xi \in \mathbb{R}^n$ and $R = R^\top > 0$ to be determined. A Schur complement argument shows that the function V is globally positive definite provided $R > \frac{1}{2}P$. Define now the *augmented* linear system in triangular form described by the equations

$$\begin{aligned} \dot{x} &= Ax\,, \\ \dot{\xi} &= F\xi + Gx\,, \end{aligned} \tag{2.9}$$

with F and G to be determined, and consider the problem of studying the stability properties of the origin of the system (2.9) using the function V, defined in (2.8), as a *candidate* Lyapunov function.

Lemma 2.1. Consider the linear, time-invariant, system (2.5) and suppose that the origin is an asymptotically stable equilibrium point. Let $P = P^\top > 0$ be the solution of (2.6) for some positive definite matrix Q. Let the matrices F and G be defined as

$$F = -kR \tag{2.10}$$

and

$$G = k(R - P)\,. \tag{2.11}$$

Suppose that

$$\underline{\sigma}(R) > \frac{1}{2}\bar{\sigma}(P)\left[\frac{\bar{\sigma}(PA)}{\underline{\sigma}(Q)}\right]. \tag{2.12}$$

Then the function V, defined in (2.8), is positive definite and there exists $\bar{k} \geq 0$ such that for all $k > \bar{k}$ the time derivative of V along the trajectories of the system (2.9) is negative definite. ◇

Proof. To prove that V is globally positive definite it is sufficient to show that the condition (2.12) implies $R > \frac{1}{2}P$. The latter follows immediately noting that, by (2.6) and the inequality $\bar{\sigma}(B_1 + B_2) \leq \bar{\sigma}(B_1) + \bar{\sigma}(B_2)$, $\bar{\sigma}(PA) \geq \underline{\sigma}(Q)$.

Note that the partial derivatives of the function V in (2.8) are given by

$$\begin{aligned} V_x &= x^\top P + (x - \xi)^\top (R - P)\,, \\ V_\xi &= x^\top P - (x - \xi)^\top R\,. \end{aligned} \tag{2.13}$$

Therefore, the time derivative of the function V along the trajectories of the augmented system (2.9) is $\dot{V} = V_x Ax + V_\xi(F\xi + Gx)$. Setting the matrices F and G as in (2.10) and (2.11), respectively, yields $\dot{\xi} = -kV_\xi^\top$. Consequently,

$$\begin{aligned} \dot{V}(x, \xi) &= x^\top PAx + x^\top A^\top (R - P)(x - \xi) - k(x^\top P - (x - \xi)^\top R)(Px - R(x - \xi)) \\ &= -x^\top Qx + x^\top A^\top (R - P)(x - \xi) - k(x^\top P - (x - \xi)^\top R)(Px - R(x - \xi)) \\ &= -x^\top Qx + x^\top A^\top (R - P)(x - \xi) - k[x^\top\ (x - \xi)^\top]C^\top C[x^\top\ (x - \xi)^\top]^\top\,, \end{aligned} \tag{2.14}$$

with $C = [P\ \ -R]$, where the second equality is obtained using the condition (2.6). Note that the time derivative (2.14) can be rewritten as a quadratic form in x and $(x - \xi)$, *i.e.*

$$\dot{V}(x, \xi) = -[x^\top\ (x - \xi)^\top][M + kC^\top C][x^\top\ (x - \xi)^\top]^\top\,,$$

where the matrix M is defined as

$$M = \begin{bmatrix} Q & -\frac{1}{2}A^\top (R - P) \\ -\frac{1}{2}(R - P)A & 0_n \end{bmatrix}.$$

The kernel of C is spanned by the columns of the matrix $Z = [I\ PR^{-1}]^\top$. As a result, the condition of positive definiteness of the matrix M restricted to Z, required by Lemma 1.1, reduces to the condition

$$\frac{1}{2}PR^{-1}(R-P)A + \frac{1}{2}A^\top(R-P)R^{-1}P < Q\,. \tag{2.15}$$

The left-hand side of the inequality (2.15) can be rewritten as

$$\begin{aligned}\frac{1}{2}PR^{-1}(R-P)A + \frac{1}{2}A^\top(R-P)R^{-1}P &= \frac{1}{2}(PA + A^\top P) - \frac{1}{2}PR^{-1}PA - \frac{1}{2}A^\top PR^{-1}P \\ &= -Q - \frac{1}{2}PR^{-1}PA - \frac{1}{2}A^\top PR^{-1}P\,.\end{aligned}$$

Therefore, the condition (2.15) is equivalent to

$$-\frac{1}{2}PR^{-1}PA - \frac{1}{2}A^\top PR^{-1}P < 2Q\,. \tag{2.16}$$

Moreover, recalling that $\bar{\sigma}(B^{-1}) = 1/\underline{\sigma}(B)$ yields

$$\begin{aligned}&-\frac{1}{2}PR^{-1}PA - \frac{1}{2}A^\top PR^{-1}P \leq \frac{1}{2}\|P\|\|R^{-1}\|\|PA\| + \frac{1}{2}\|P\|\|R^{-1}\|\|A^TP\| \\ &= \frac{1}{2}\bar{\sigma}(R^{-1})\bar{\sigma}(P)(\|PA\| + \|A^\top P\|) = \frac{1}{2}\frac{\bar{\sigma}(P)}{\underline{\sigma}(R)}(\|PA\| + \|A^\top P\|)\,.\end{aligned}$$

Hence, by condition (2.12), the inequality (2.16) holds. Therefore, by Lemma 1.1, there exists a value $\bar{k} \geq 0$ such that for all $k > \bar{k}$ the time derivative of V in (2.8) is negative definite along the trajectories of the augmented system (2.9) for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. □

Remark 2.4. The condition (2.12) can be satisfied selecting the matrix R *sufficiently large*. ▲

As a consequence of Lemma 2.1 consider the following statement, which provides a constructive proof of Theorem 2.2 for linear time-invariant systems.

Proposition 2.1. Consider the system (2.5) and suppose that the origin is an asymptotically stable equilibrium point. Let P and Q be such that (2.6) holds. Let $R = R^\top > 0$ be such that (2.12) holds. Then there exists $\bar{k} \geq 0$ such that $\mathcal{V} = (D_\alpha, V)$, where D_α is the differential equation

$$\dot{\xi} = -kR\xi + k(R-P)x \tag{2.17}$$

and V is defined in (2.8), is a Dynamic Lyapunov function for the system (2.5) for all $k > \bar{k}$. ◇

Proof. The claim follows immediately from Lemma 2.1, since V in (2.8) is positive definite around the origin, provided R satisfies condition (2.12), and moreover $V_xAx + V_\xi\alpha(x, \xi)$ is negative definite with the choice of α given in (2.17) and k sufficiently large. □

2.4 Construction of Dynamic Lyapunov Functions for Nonlinear Systems

Consider the nonlinear autonomous system (2.1) and suppose that the origin of the state-space is an equilibrium point, *i.e.* $f(0) = 0$. Hence, there exists a continuous (possibly non-unique) matrix-valued function $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $f(x) = F(x)x$ for all $x \in \mathbb{R}^n$.

Assumption 2.1. The equilibrium point $x = 0$ of the system (2.1) is locally exponentially stable, *i.e.* there exists a matrix $\bar{P} = \bar{P}^\top > 0$ such that

$$\frac{1}{2}\bar{P}A + \frac{1}{2}A^\top \bar{P} = -Q\,, \tag{2.18}$$

where $Q = Q^\top > 0$ and $A = \left.\dfrac{\partial f}{\partial x}\right|_{x=0} = F(0)$.

In this section - mimicking the results and the construction in Section 2.3 for linear systems - we present a constructive methodology to obtain a Dynamic Lyapunov function for the system (2.1) thus providing a constructive proof of Theorem 2.2. Clearly, by equation (2.18), the quadratic function

$$V_\ell(x) = \frac{1}{2}x^\top \bar{P}x\,, \tag{2.19}$$

is a local (around the origin) Lyapunov function for the nonlinear system (2.1).

Consider the Lyapunov partial differential inequality

$$V_x f(x) < 0\,, \tag{2.20}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$ and the following notion of solution.

Definition 2.2. Let $\Gamma(x) = \Gamma(x)^\top > 0$ for all $x \in \mathbb{R}^n$. A $\mathcal{X}$-*algebraic* $\bar{P}$ *solution* of the inequality (2.20) is a continuously differentiable mapping $P : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, with $P(0) = 0$, such that

$$P(x)f(x) \leq -x^\top \Gamma(x)x\,, \tag{2.21}$$

for all $x \in \mathcal{X} \subset \mathbb{R}^n$, with $\mathcal{X}$ containing the origin, and such that P is *tangent* at the origin to $\bar{P}$, namely

$$\left. P_x(x)\right|_{x=0} = \bar{P}.$$

If $\mathcal{X} = \mathbb{R}^n$ then P is called an *algebraic* $\bar{P}$ *solution.*

In what follows we assume the existence of an *algebraic* $\bar{P}$ *solution*, *i.e.* we assume $\mathcal{X} = \mathbb{R}^n$. Note that all the statements can be modified accordingly if $\mathcal{X} \subset \mathbb{R}^n$. Note that (2.21) implies that $\Gamma(0) \leq Q$. The mapping P does not need to be the *gradient vector* of any scalar function. Hence the condition (2.21) may be interpreted as the algebraic equivalent of (2.20), since in the former the *integrability* and (partly) the *positivity* constraints are relaxed. Similarly to (2.8), define the *extension* of the mapping P, namely the function

$$V(x,\xi) = P(\xi)x + \frac{1}{2}\|x-\xi\|_R^2\,, \tag{2.22}$$

with $\xi \in \mathbb{R}^n$ and $R = R^\top \in \mathbb{R}^{n\times n}$ positive definite.

Since the mapping P is *tangent* at the origin to $\bar{P}$, then the hypotheses of Lemma 1.2 are satisfied, hence there exists $\bar{R} = \bar{R}^\top > 0$ such that the *extension* of P, namely V as in (2.22), is positive definite on a neighborhood of the origin for all $R > \bar{R}$.

The partial derivatives of the function V defined in (2.22) are given by

$$\begin{aligned} V_x &= P(x) + (x-\xi)^\top (R - \Phi(x,\xi))^\top\,, \\ V_\xi &= x^\top P_\xi(\xi) - (x-\xi)^\top R\,, \end{aligned} \tag{2.23}$$

where $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a continuous matrix-valued function such that $P(x)-P(\xi) = (x-\xi)^\top \Phi(x,\xi)^\top$. As in the linear setting, define an *augmented* nonlinear system described by equations of the form

$$\begin{aligned} \dot{x} &= f(x)\,, \\ \dot{\xi} &= -k(P_\xi(\xi) - R)^\top x - kR\xi \triangleq g(\xi)x - kR\xi\,, \end{aligned} \tag{2.24}$$

and let the function (2.22) be a *candidate* Lyapunov function to investigate the stability properties of the equilibrium point $(x,\xi) = (0,0)$ of the system (2.24). To streamline the presentation of the following result - providing conditions on the choice of the parameter k such that V in (2.22) is indeed a Lyapunov function for the augmented system (2.24) - define the continuous matrix-valued function $\Delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ as

$$\Delta(x,\xi) = (R - \Phi(x,\xi))R^{-1}P_\xi(\xi)^\top\,. \tag{2.25}$$

Lemma 2.2. Consider the system (2.1). Suppose Assumption 2.1 holds. There exist a set $\Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ and a constant $\bar{k} \geq 0$ such that V, defined in (2.22), is positive definite in Ω and its time derivative along the trajectories of the system (2.24) is negative definite for all $k > \bar{k}$ if and only if

$$\frac{1}{2}F(x)^\top \Delta(x,\xi) + \frac{1}{2}\Delta(x,\xi)^\top F(x) < \Gamma(x)\,, \tag{2.26}$$

for all $(x,\xi) \in \Omega \setminus \{0\}$. $\diamond$

Proof. The time derivative of the function V defined in (2.22) is

$$\begin{aligned}\dot{V} =& V_x f(x) + V_\xi \left(g(\xi)x - kR\xi\right) = P(x)f(x) + x^\top F(x)^\top (R - \Phi(x,\xi))(x-\xi)\\ &- k[x^\top\ (x-\xi)^\top]C(\xi)^\top C(\xi)[x^\top\ (x-\xi)^\top]^\top ,\end{aligned}$$

with $C(\xi) = [P_\xi(\xi)^\top \ \ -R]$. Note that the matrix $C : \mathbb{R}^n \to \mathbb{R}^{n\times 2n}$ has constant rank n for all $\xi \in \mathbb{R}^n$, since R is non-singular. The columns of the matrix

$$Z(\xi) \triangleq \begin{bmatrix} I \\ R^{-1}P_\xi(\xi)^\top \end{bmatrix},$$

which has constant rank, span the kernel of the matrix $C(\xi)$ for all $\xi \in \mathbb{R}^n$. Consider now the restriction of the matrix

$$M(x,\xi) \triangleq \begin{bmatrix} \Gamma(x) & -\frac{1}{2}F(x)^\top (R - \Phi(x,\xi)) \\ -\frac{1}{2}(R - \Phi(x,\xi))^\top F(x) & 0_n \end{bmatrix}$$

to the set $\mathcal{P} = \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : P_\xi(\xi)^\top x - R(x-\xi) = 0\}$, namely $Z(\xi)^\top M(x,\xi)Z(\xi)$. Condition (2.26) implies that the matrix $Z(\xi)^\top M(x,\xi)Z(\xi)$ is positive definite for all $(x,\xi) \in \Omega$. Therefore, by Lemma 1.1, condition (2.26) guarantees the existence of a constant $\bar{k} \geq 0$ and of a non-empty subset $\Omega \subset \mathbb{R}^n$ such that, for all $k > \bar{k}$, $\dot{V}(x,\xi) < 0$ for all $(x,\xi) \in \Omega \subset \mathbb{R}^n \times \mathbb{R}^n$ and $(x,\xi) \neq (0,0)$. □

Remark 2.5. If the *algebraic $\bar{P}$ solution* of the inequality (2.20) is linear in x, *i.e.* $P(x) = x^\top \bar{P}$, then $\Phi(x,\xi) = \bar{P}$. Moreover the choice $R = \bar{P}$ is such that in equation (2.24) $g \equiv 0$ and the condition (2.26) is satisfied for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$. ▲

Remark 2.6. The *gain* k in (2.24) may be defined as a function of x and ξ. ▲

The following result provides the proof of Theorem 2.2 for nonlinear systems.

Proposition 2.2. Consider the nonlinear system (2.1) and suppose that the origin is a locally exponentially stable equilibrium point of (2.1). Let P be an *algebraic $\bar{P}$ solution* of the inequality (2.20). Let $R = \Phi(0,0) = \bar{P}$. Then there exist a constant $\bar{k} \geq 0$ and a non-empty set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that $\mathcal{V} = (D_\alpha, V)$, where D_α is the differential equation $\dot{\xi} = g(\xi)x - kR\xi$ and V is defined in (2.22), is a Dynamic Lyapunov function for the system (2.1) in Ω for all $k > \bar{k}$. ◇

Proof. Since $\Phi(0,0) = \bar{P}$ and recalling Lemma 1.2, there exists a set $\Omega_1 \subseteq \mathbb{R}^n \times \mathbb{R}^n$ containing the origin in which the function V defined in (2.22) is positive definite. Therefore, to prove the claim it is sufficient to show that the condition (2.26) of Proposition 2.2 is, at least locally, satisfied. Note that the choice $R = \Phi(0, 0)$ implies that the left-hand side of the inequality (2.26) is zero at the origin, whereas the right-hand side, *i.e.* $\Gamma(0)$, is positive definite. Hence, by continuity, there exists a non-empty set $\Omega_2 \subseteq \mathbb{R}^n \times \mathbb{R}^n$ containing the origin in which the condition (2.26) holds, which proves the claim with $\Omega = \Omega_1 \cap \Omega_2$. □

2.5 From Dynamic Lyapunov functions to Lyapunov Functions

2.5.1 Linear Systems

The result in Lemma 2.1 can be exploited to *construct* a Lyapunov function for the linear system (2.5), as detailed in the following statement, which is an application of Theorem 2.3 to linear time-invariant systems.

Corollary 2.2. Consider the linear time-invariant system (2.5). Suppose that the conditions (2.6) and (2.12) are satisfied, fix $k > \bar{k}$ and let $Y \in \mathbb{R}^{n\times n}$ be the solution of the Sylvester equation

$$k(R - P) - kRY = YA. \tag{2.27}$$

Then the subspace $\mathcal{L} = \{(x, \xi) \in \mathbb{R}^{2n} : \xi = Yx\}$ is invariant and the restriction of the function V in (2.8) to $\mathcal{L}$, defined as

$$V_{\mathcal{L}}(x) = V(x, Yx) = \frac{1}{2}x^\top[Y^T P + PY + (I - Y)^\top R(I - Y)]x, \tag{2.28}$$

is positive for all $x \in \mathbb{R}^n \setminus \{0\}$ and its time derivative along the trajectories of the system (2.5) is negative definite, hence $V_{\mathcal{L}}$ is a Lyapunov function for the system (2.5). ⋄

Proof. The condition $\sigma(A) \cap \sigma(-kR) = \emptyset$, which holds for almost all k, guarantees existence and unicity of the matrix Y and therefore the existence of the invariant subspace $\mathcal{L}$. The claim is proved showing that the assumptions of Theorem 2.3 are satisfied. To begin with, by Proposition 2.1, $\mathcal{V} = (D_\alpha, V)$, where D_α is the differential equation (2.17) and V is defined in (2.8), is a Dynamic Lyapunov function for the system (2.5) for all $k > \bar{k}$. Moreover, by (2.27), the mapping $h(x) = Yx$ is a solution of the partial differential equation (2.3), which reduces in the linear case to the equation

$$\begin{bmatrix} A & 0_n \\ k(R-P) & -kR \end{bmatrix} \begin{bmatrix} I_n \\ Y \end{bmatrix} = \begin{bmatrix} I_n \\ Y \end{bmatrix} A\,.$$

□

Remark 2.7. The condition (2.27) implies the existence of the linear subspace $\mathcal{L}$ parameterized in x, which is invariant with respect to the dynamics of the augmented system (2.9) and such that the flow of the system (2.9) restricted to $\mathcal{L}$ is a copy of the flow of the system (2.5). Moreover, $V_{\mathcal{L}}$ describes a family of Lyapunov functions for the system parameterized by the matrix $R > \frac{1}{2}P$ and $k > \bar{k}$. ▲

Corollary 2.3. Suppose that Y is a common solution of the Sylvester equation (2.27) and of the algebraic Riccati equation

$$Y^\top(P-R) + (P-R)Y + Y^\top RY - (P-R) = 0\,, \tag{2.29}$$

for some $k > \bar{k}$. Then $V_{\mathcal{L}}$ coincides with the *original* quadratic Lyapunov function (2.7), *i.e.* $V(x, Yx) = \frac{1}{2}x^\top Px$. ◇

Remark 2.8. The Lyapunov function V_ℓ defined as in (2.7) does not necessarily belong to the family parameterized by $V_{\mathcal{L}}$, hence the need of condition (2.29). Recall in fact that the matrix P is defined together with the matrix Q, *i.e.* the pair (P, Q) is such that V_ℓ in (2.7) is a quadratic positive definite function and $\dot{V}_\ell = -x^\top Qx$ along the trajectories of the linear system (2.5). Therefore the function V_ℓ in (2.7) belongs to the family of Lyapunov functions $V_{\mathcal{L}}$ if and only there exists $k > \bar{k}$ and R such that (2.12) holds and such that $\dot{V}_{\mathcal{L}} = -x^\top Qx$. ▲

Remark 2.9. If (2.12) is satisfied and $\mathcal{L} \neq Ker(C)$, larger values of the constant k yield a *more negative* $\dot{V}(x, \xi)$ for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Thus, if we let k be such that $\dot{V}(x, Yx) < -x^\top Qx$ for some $x \in \mathbb{R}^n$ then it is obvious that the quadratic function V_ℓ defined in (2.7) cannot belong to the family $V_{\mathcal{L}}$. On the contrary, if the parameter k is selected such that the solution Y of the Sylvester equation is in the set of solutions of the algebraic Riccati equation (2.29), then the function (2.7) belongs to the family of Lyapunov functions $V_{\mathcal{L}}$. ▲

Remark 2.10. In the case $\mathcal{L} = Ker(C)$, $\dot{V}(x, Yx)$ is not affected by the choice of the parameter k. To see this let $Ker(C) = \{(x, \xi) : \xi = -R^{-1}(P-R)x\}$. Obviously, if $Ker(C)$ coincides with $\mathcal{L}$, then the matrix $K = -R^{-1}(P-R)$ must satisfy the Sylvester equation that defines Y. As a matter of fact $Ker(C) = \mathcal{L}$ if and only if $R = P$ and in this case not only the family of Lyapunov functions automatically contains the *original* Lyapunov function but actually the family *reduces* to the function V_ℓ defined in (2.7). Since the solution of the Sylvester equation is $Y = 0$, the invariant subspace is defined by $\xi = 0$, hence, with $R = P$, $V(x, 0) = \frac{1}{2}x^\top Px$ and moreover the time derivative $\dot{V}(x, \xi)$ in (2.14) is equal to $-x^\top Qx$ for any value of the parameter k. ▲

2.5.2 Nonlinear Systems

As in the linear case a Dynamic Lyapunov function can be employed to construct a family of Lyapunov functions for the system (2.1). The following proposition represents a particularization of Theorem 2.3 to the dynamics of the *augmented* system defined as in (2.24).

Corollary 2.4. Consider the system (2.1). Suppose Assumption 2.1 holds. Let $k > \bar{k}$. Suppose that the condition (2.26) is satisfied and that there exists a smooth mapping $h : \mathbb{R}^n \to \mathbb{R}^{n\times 1}$ such that the set $\mathcal{M} = \{(x, \xi) \in \mathbb{R}^{2n} : \xi = h(x)\}$ is invariant with respect to the dynamics of the *augmented* system (2.24), *i.e.*

$$g(h(x))x - kRh(x) = \frac{\partial h}{\partial x} f(x)\,. \tag{2.30}$$

Then the restriction of the function V in (2.22) to the set $\mathcal{M}$, namely

$$V_{\mathcal{M}}(x) = P(h(x))x + \frac{1}{2}\|x - h(x)\|_R^2\,, \tag{2.31}$$

yields a family of Lyapunov functions for the nonlinear system (2.1). $\diamond$

Remark 2.11. The family of Lyapunov functions (2.31) is parameterized by R and $k > \bar{k}$. $\blacktriangle$

Note that, by (2.30), $\mathcal{M}$ is invariant under the flow of the system (2.24) and moreover the restriction of the flow of the *augmented* system (2.24) to the set $\mathcal{M}$ is a *copy* of the flow of the nonlinear system (2.1). The condition (2.30) is a system of partial differential equations similar to (2.20), without any sign constraint on the solution, *i.e.* the mapping h, or on the sign of the term on the left-hand side of the equation.

2.5.3 Approximate Solution of the Invariance Pde

An explicit solution of the partial differential equation (2.30) may still be difficult to determine even without the sign constraints. Therefore, consider the following *algebraic* condition which allows to approximate, with an arbitrary degree of accuracy, the closed-form solution of the partial differential equation (2.30). Suppose that there exists a matrix valued function $H_{k,R} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that

$$H_{k,R}(x)f(x) + kRH_{k,R}(x)x - g(H_{k,R}(x)x)x = 0. \tag{2.32}$$

Note that the solution of the equation (2.32) is parameterized by k and R. Let now $\hat{h}(x) = H_{k,R}(x)x$ and consider the subset $\mathcal{M}_\eta \triangleq \{(x, \xi) \in \mathbb{R}^{2n} : \xi = \hat{h}(x)\}$.

Lemma 2.3. Let $\mathcal{W} \subset \mathbb{R}^n \times \mathbb{R}^n$ be a compact set containing the origin. Suppose that the condition (2.32) is satisfied and that

(i) there exists a function[1] $\phi_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|H_{k,R}(x)\| < \phi_R(\|x\|)\,, \tag{2.33}$$

for all $k > \bar{k}$, with $\bar{k}$ defined in Proposition 2.2;

(ii) there exists $R = R^\top > \bar{R}$ such that

$$\underline{\sigma}(R) > \frac{\|G(x,\xi)\|}{1-\mu}\,,$$

for some $\mu \in (0,1)$, for all $(x,\xi) \in \mathcal{W}$, where $G : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is such that

$$P_\xi(H_{k,R}(x)x)^\top x - P_\xi(\xi)^\top x = G(x,\xi)(\xi - H_{k,R}(x)x)\,. \tag{2.34}$$

Then there exists $k > \bar{k}$ such that the subset $\mathcal{M}_\eta \cap \mathcal{W}$ is *almost*-invariant for the system (2.24). $\diamond$

Proof. Define the error variable $\eta = \xi - H_{k,R}(x)x$. The dynamics of η are given by

$$\dot{\eta} = -kR\xi + g(\xi)x - H_{k,R}(x)f(x) - \theta(x)f(x)\,, \tag{2.35}$$

where

$$\theta(x) = \frac{\partial(H_{k,R}(x)\lambda)}{\partial x}\bigg|_{\lambda=x}\,.$$

Letting $\xi = \eta + H_{k,R}(x)x$, adding to the equation (2.35) the term

$$H_{k,R}(x)f(x) + kRH_{k,R}(x)x - g(H_{k,R}(x)x)x\,,$$

which is equal to zero by the condition (2.32) and recalling the definition of the mapping g in (2.24), the equation (2.35) yields

$$\begin{aligned} \dot{\eta} &= -\,kR\eta - \theta(x)f(x) + k[P_\xi(H_{k,R}(x)x) - P_\xi(\eta + H_{k,R}(x)x)]^\top x \\ &\triangleq -\,kR\eta - \theta(x)f(x) + kG(x,\eta + H_{k,R}(x))\eta\,. \end{aligned} \tag{2.36}$$

Therefore the system (2.36) can be rewritten as

$$\dot{\eta} = -k(R - G(x,\eta + H_{k,R}(x)))\eta - \theta(x)f(x)\,. \tag{2.37}$$

Consider now the Lyapunov function $V = \eta^\top R^{-1}\eta$ the time derivative of which along the

[1] The analytic expression of the function ϕ_R may be unknown.

trajectories of system (2.37) is

$$\begin{aligned}\dot{V} &= -2k\eta^\top\eta + 2k\eta^\top R^{-1}G(x,\xi)\eta - 2\eta^\top R^{-1}\theta(x)f(x)\\ &\leq -2k\|\eta\|^2 + 2k\bar{\sigma}(R^{-1})\|G(x,\xi)\|\|\eta\|^2 + 2\bar{\sigma}(R^{-1})\|\theta(x)f(x)\|\|\eta\|\\ &= -2\|\eta\|\left[k\left(1-\frac{\|G(x,\xi)\|}{\underline{\sigma}(R)}\right)\|\eta\| - \frac{\|\theta(x)f(x)\|}{\underline{\sigma}(R)}\right],\end{aligned}$$

which is negative definite in the set $\{(x,\xi)\in\mathcal{W} : \|\eta\| > \|\theta(x)f(x)\|/\left(k\mu\underline{\sigma}(R)\right)\}$. Let

$$\tilde{k} = \sup_{x\in\mathcal{W}} \frac{\|\theta(x)f(x)\|}{\bar{\varepsilon}\mu\underline{\sigma}(R)}\,.$$

Then the Lyapunov function V is such that $\dot{V} < 0$ when $\|\eta\| > \bar{\varepsilon}$ for all $k \geq \tilde{k}$. For any $\varepsilon > 0$ let $\bar{\varepsilon} = \sqrt{\underline{\sigma}(R)/\bar{\sigma}(R)}\varepsilon$, then the set $\{\eta\in\mathbb{R}^n : \|\eta\| < \varepsilon\}$ is attractive and positively invariant hence the subset $\mathcal{M}_\eta \cap \mathcal{W}$ is almost-invariant □

Remark 2.12. If the *algebraic* $\bar{P}$ *solution* of the inequality (2.20) is linear in x, *i.e.* $P(x) = x^\top\bar{P}$, then the condition (ii) of Lemma 2.3 is satisfied for any constant $\mu \in (0,1)$ since $G(x) = 0$ for all $x\in\mathbb{R}^n$. ▲

Note that the result of Lemma 2.3 implies the existence of a continuously differentiable mapping $\pi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\xi(t) = \hat{h}(x(t)) + \pi(x(t))$ and $\|\pi(x(t))\| \leq \varepsilon$ for all $t \geq 0$.

Corollary 2.5. Suppose that the conditions of Lemma 2.3 are satisfied. Suppose that

$$\left\|\frac{\partial\pi}{\partial x}\right\| < \frac{\mu\|\dot{V}(x,\hat{h}(x))\|}{\|\kappa(x,\hat{h}(x))f(x)\|} \tag{2.38}$$

with $\mu\in(0,1)$ and $\kappa(x,\hat{h}(x)) = V_\lambda(x,\lambda)\Big|_{\lambda=\hat{h}(x)}$. Then there exist a matrix $R > \bar{R}$ and a constant $k > \bar{k}$ such that each element of the family of functions

$$V_{\mathcal{M}_\eta}(x) = V(x,\hat{h}(x)) = P(H_{k,R}(x))x)x + \frac{1}{2}\|x - H_{k,R}(x))x\|_R^2\,, \tag{2.39}$$

parameterized by R and k, is a Lyapunov function for the nonlinear system (2.1). ◇

Proof. Since the subset $\mathcal{M}_\eta \cap \mathcal{W}$ is almost-invariant, the time derivative of the function

$V_{\mathcal{M}_\eta}$ as in (2.39) yields

$$\begin{aligned}
\dot V_{\mathcal{M}_\eta} &= V_x(x,\lambda)\Big|_{\lambda=\hat h(x)+\pi(x)} f(x) + V_\lambda(x,\lambda)\Big|_{\lambda=\hat h(x)+\pi(x)} \left(\hat h_x(x)f(x) + \pi_x(x)f(x)\right)\\
&= V_x(x,\lambda)\Big|_{\lambda=\hat h(x)} f(x) + \left(V_x(x,\lambda)\Big|_{\lambda=\hat h(x)+\pi(x)} - V_x(x,\lambda)\Big|_{\lambda=\hat h(x)}\right) f(x)\\
&\quad + V_\lambda(x,\lambda)\Big|_{\lambda=\hat h(x)} \hat h_x(x)f(x) + \left(V_\lambda(x,\lambda)\Big|_{\lambda=\hat h(x)+\pi(x)} - V_\lambda(x,\lambda)\Big|_{\lambda=\hat h(x)}\right) \hat h_x(x)f(x)\\
&\quad + V_\lambda(x,\lambda)\Big|_{\lambda=\hat h(x)} \pi_x(x)f(x) + \left(V_\lambda(x,\lambda)\Big|_{\lambda=\hat h(x)+\pi(x)} - V_\lambda(x,\lambda)\Big|_{\lambda=\hat h(x)}\right) \pi_x(x)f(x)\\
&\le \dot V(x,\hat h(x)) + \|\pi(x)\| \left(L_x + L_\lambda\|\hat h_x\| + L_\lambda\|\pi_x\|\right)\|f(x)\| + \|\pi_x(x)\|\|\kappa(x,\hat h(x))f(x)\|\\
&\le \mu\dot V(x,\hat h(x)) + \|\pi_x(x)\|\|\kappa(x,\hat h(x))f(x)\| < 0
\end{aligned}$$

with $\mu \in (0,1)$, $L_x > 0$ and $L_\lambda > 0$, where the last three inequalities are obtained considering that V_x and V_λ are continuous functions and $x \in \mathcal{W}$, recalling that $\|\pi(x)\| \le \varepsilon$, with $\varepsilon > 0$ arbitrarily small, and by the condition (2.38), respectively. □

It is interesting to note that *almost-invariance* of the subset $\mathcal{M}_\eta \cap \mathcal{W}$ is not enough to ensure that the restriction of the function $V(x,\xi)$ to the subset $\mathcal{M}_\eta \cap \mathcal{W}$ is a Lyapunov function for the system (2.1), hence the need for the condition (2.38). The latter condition entails the fact that the time derivative of the distance of the trajectory $(x(t),\xi(t))$ from the subset must be *sufficiently* small.

2.6 Examples and Further Results

In this section several numerical examples and further results on control design using Dynamic Lyapunov functions are presented.

2.6.1 An Academic Example

Consider the nonlinear system described by the equations

$$\begin{aligned}
\dot x_1 &= -x_1\,,\\
\dot x_2 &= x_1^2 - x_2\,,
\end{aligned} \tag{2.40}$$

with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$. Note that the zero equilibrium of the system (2.40) is globally asymptotically stable and locally exponentially stable. A choice of a Lyapunov function for the linearization around the origin of the system (2.40) is provided by $V_\ell(x_1,x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, *i.e.* $\bar P = I$ which is a solution of the equation (2.18) with $Q = I$. The quadratic function V_ℓ may be employed to estimate the region of attraction,

$\mathcal{R}_0$, of the zero equilibrium of the nonlinear system (2.40). The estimate is given by the largest connected component, containing the origin of the state-space, of the level set of the considered Lyapunov function entirely contained in the set $\mathcal{N} \triangleq \{x \in \mathbb{R}^2 : \dot{V} < 0\}$. Figure 2.1 displays the zero level line (solid line) of the time derivative of V_ℓ along the

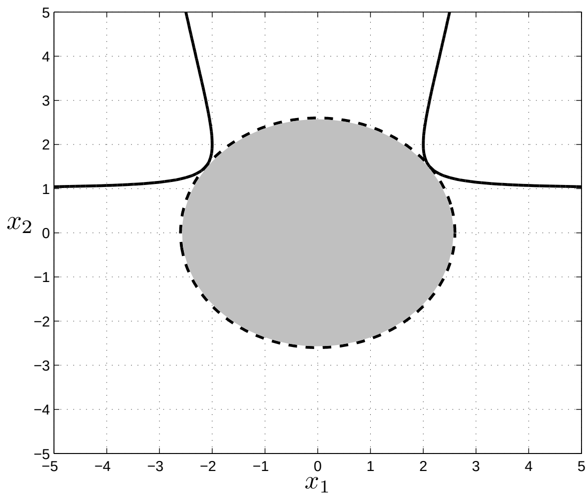

Figure 2.1: The gray region describes the estimate of the region of attraction of the zero equilibrium of system (2.40) obtained using the quadratic function V_ℓ. The solid line is the zero level line of the time derivative of V_ℓ along the trajectories of system (2.40).

trajectories of system (2.40) together with the largest level set of V_ℓ entirely contained in the set $\mathcal{N}$ (gray area). Note that $\mathcal{N} \subset \mathbb{R}^2$ and consequently $\mathcal{R}_0 \subset \mathbb{R}^2$. The use of any quadratic function $V_q(x) = \frac{1}{2}x^\top \bar{P}x$, with

$$\bar{P} = \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix},$$

and $p_2 \neq 0$, does not allow to obtain $\mathcal{N} = \mathbb{R}^2$. In fact, the time derivative of V_q along the trajectories of the system (2.40) yields $\dot{V}_q = -p_1x_1^2 - 2p_2x_1x_2 - p_3x_2^2 + p_2x_1^3 + p_3x_2x_1^2$, which, if evaluated along $x_2 = 0$, is equal to

$$\dot{V}_q\Big|_{x_2=0} = -x_1^2(p_1 - p_2x_1).$$

Therefore, $\dot{V}_q > 0$ for $x_2 = 0$ and $sign(p_2)x_1 > \frac{p_1}{|p_2|}$, hence $\mathcal{R}_0 \neq \mathbb{R}^2$. In what follows we show that the notion of Dynamic Lyapunov function allows to construct a Lyapunov function proving global asymptotic stability of the zero equilibrium of system (2.40). Interestingly a (global) Lyapunov function for system (2.40) can be constructed noting that the system has a cascaded structure and exploiting *forwarding* arguments.

The mapping $P : \mathbb{R}^2 \to \mathbb{R}^2$ defined as the gradient vector of the quadratic function V_ℓ

is a $\mathcal{X}$-*algebraic* $\bar{P}$ *solution* of the inequality (2.20) for the nonlinear system (2.40). The choice $R = \bar{P}$ guarantees that the mapping g is identically equal to zero and that the condition (2.26) is trivially satisfied for all $(x, \xi) \in \mathbb{R}^4 \setminus \{0\}$.

In the following three different approaches to construct a Lyapunov function for the system (2.40) are proposed.

To construct the Lyapunov function V_d defined in Corollary 2.4 it is required to determine mappings $h_1 : \mathbb{R}^2 \to \mathbb{R}$ and $h_2 : \mathbb{R}^2 \to \mathbb{R}$ such that the set

$$\{(x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4 : \xi_1 = h_1(x_1, x_2), \xi_2 = h_2(x_1, x_2)\}$$

is invariant for the dynamics of the *augmented* system (2.24). Since $R = \bar{P} = I$, the system of partial differential equations (2.30) reduces to two identical (decoupled) partial differential equations given by

$$-\frac{\partial h_i}{\partial x_1}(x_1, x_2)x_1 + \frac{\partial h_i}{\partial x_2}(x_1, x_2)(x_1^2 - x_2) + kh_i(x_1, x_2) = 0\,, \tag{2.41}$$

for $i = 1, 2$. The solutions h_1 and h_2 are defined on $\mathbb{R}^2 \setminus (\mathbb{R} \times 0)$, and by continuous extension on all $\mathbb{R}^2$, according to the formula

$$h_1(x) = h_2(x) = L\left(\frac{x_2 + x_1^2}{x_1}\right) x_1^k\,,$$

$k \geq 1$, where L is any differentiable function. Let, for example, $L(a) = a$ and construct the family of Lyapunov functions

$$V_d^k(x) = h(x)^\top x + \frac{1}{2}\|x - h(x)\|^2 = \frac{1}{2}(x_1^2 + x_2^2) + (x_2 + x_1^2)^2 \left(x_1^{k-1}\right)^2\,, \tag{2.42}$$

with $h(x) = [h_1(x),\, h_2(x)]^\top$, and $k \geq 1$. Letting $k = 1$ yields

$$V_d^1(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + (x_2 + x_1^2)^2\,,$$

the time derivative of which along the trajectories of the system (2.40), namely $\dot{V}_d^1(x_1, x_2) = -x_1^2 - 3x_2^2 - 3x_2x_1^2 - 2x_1^4$, is negative definite for all $(x_1, x_2) \in \mathbb{R}^2$, hence $\mathcal{N} = \mathbb{R}^2$, which proves global asymptotic stability of the zero equilibrium. Finally, note that $\dot{V}_d^k(x_1, x_2)$ is negative definite for all $(x_1, x_2) \in \mathbb{R}^2$ and for all $k \geq 1$. Interestingly, the partial differential equation (2.41) has a structure similar to the equation (2.20), but the solution h_i is not positive definite, hence it does not qualify as a Lyapunov function. Figure 2.2 shows the phase portraits of the trajectories of the system (2.40) together with the level lines of the Lyapunov function V_d^1.

To illustrate a second approach for the construction of a Lyapunov function, let the function L be identically equal to one in the solution h_1 while let L be the identity function

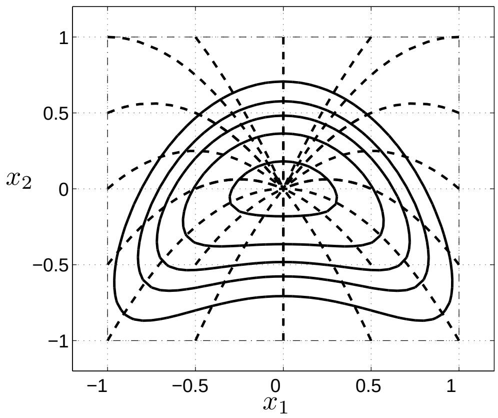

Figure 2.2: Phase portraits (dashed) of the system (2.40) together with the level lines (solid) of the Lyapunov function V_d^1.

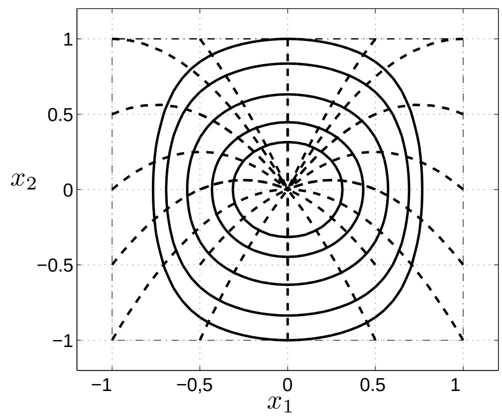

Figure 2.3: Phase portraits (dashed) of the system (2.40) together with the level lines (solid) of the Lyapunov function V_{d_a}.

in the definition of h_2, with $k = 1$. The mapping $h(x) = [h_1(x),\, h_2(x)] = [x_1,\, x_2 + x_1^2]$ satisfies the assumptions of Corollary 2.1, since h is a global diffeomorphism, $h^\top h$ is radially unbounded and moreover $\alpha(x, h(x)) = -kh(x)$. Therefore by Corollary 2.1 the function $V_f(x) = \frac{1}{2}h(x)^\top h(x) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1^2)^2$ is a Lyapunov function for the system (2.40). Interestingly the function V_f is the Lyapunov function obtained also using *forwarding* arguments [63] for the planar system (2.40).

Finally we show how Corollary 2.5 can be used to construct a Lyapunov function. Let

$$H(x) = \begin{bmatrix} \bar{h}_1(x) & \bar{h}_2(x) \\ \bar{h}_3(x) & \bar{h}_4(x) \end{bmatrix}, \tag{2.43}$$

and consider the condition (2.32), which reduces to two identical conditions on $\bar{h}_1$, $\bar{h}_2$ and $\bar{h}_3$, $\bar{h}_4$, namely

$$\begin{aligned} -x_1\bar{h}_1 + \bar{h}_2(x_1^2 - x_2) + k\bar{h}_1 x_1 + k\bar{h}_2 x_2 = 0\,, \\ -x_1\bar{h}_3 + \bar{h}_4(x_1^2 - x_2) + k\bar{h}_3 x_1 + k\bar{h}_4 x_2 = 0\,. \end{aligned} \tag{2.44}$$

A solution of (2.44) is given by $\bar{h}_1(x) = \bar{h}_3(x) = -x_2 - x_1^2(k-1)^{-1}$ and $\bar{h}_2(x) = \bar{h}_4(x) = x_1$ and, since the condition (2.33) holds together with the condition (*ii*) of Corollary 2.5, by Remark 2.12, the set $\{(x,\xi) \in \mathbb{R}^4 : \xi_1 = \xi_2 = -x_1^3(k-1)^{-1}\}$ is *almost invariant* for $k > 1$. Moreover note that the mapping $\delta(x) \triangleq [\bar{h}_1(x),\, \bar{h}_2(x)]$ is not the gradient of a scalar function, since the Jacobian $\delta_x(x)$ is not a symmetric matrix for all $x \in \mathbb{R}^2$.

Setting $k = 2$ and using Corollary 2.5 yield the Lyapunov function

$$V_{d_a}(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + x_1^6\,,$$

the time derivative of which along the trajectories of the system (2.40) is $\dot{V}_{d_a}(x_1, x_2) =$

$x_2x_1^2 - x_1^2 - x_2^2 - 6x_1^6$ which is negative for all $(x_1, x_2) \in \mathbb{R}^2$. Figure 2.3 displays the phase portraits of the trajectories of the system (2.40) together with the level lines of the Lyapunov function V_{d_a}.

2.6.2 A Polynomial System without Polynomial Lyapunov Function

Consider the nonlinear system described by the equations

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_1x_2 \,, \\ \dot{x}_2 &= -x_2 \,, \end{aligned} \tag{2.45}$$

with $x_1(t) \in \mathbb{R}$ and $x_2(t) \in \mathbb{R}$, and note that the zero equilibrium of the system (2.45) is globally asymptotically stable. In [1] it has been shown that the system (2.45) does not possess a polynomial Lyapunov function. In the same paper the authors have shown that $V(x_1, x_2) = ln(1 + x_1^2) + x_2^2$ is a Lyapunov function for the system (2.45), *i.e.* it is such that $\dot{V} < 0$ for all $(x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$. The quadratic function $V_\ell(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, *i.e.* $V_\ell = \frac{1}{2}x^\top \bar{P}x$ with $\bar{P} = I$, is a Lyapunov function for the linearized system. Note that the mapping $[x_1,\, x_2]^\top \bar{P}$ is a $\mathcal{X}-$*algebraic* $\bar{P}$ *solution*, with $\mathcal{X} = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 > x_1^2x_2\}$. Let $R = \bar{P}$, then $\mathcal{V} = (D_\alpha, V)$ with

$$\dot{\xi}_i = -k\xi_i \,,$$

$i = 1, 2$, with $k > 0$, and

$$V(x_1, x_2, \xi_1, \xi_2) = \frac{1}{2}(x_1^2 + x_2^2 + \xi_1^2 + \xi_2^2) \,,$$

is a Dynamic Lyapunov function, the time derivative of which is negative definite in a neighborhood of the origin of the *extended* state space.

The *invariance* partial differential equation reduces to two identical equations for h_1 and h_2, namely

$$\frac{\partial h_i}{\partial x_1}(x_1, x_2)(x_1x_2 - x_1) - \frac{\partial h_i}{\partial x_2}x_2 + kh_i(x_1, x_2) = 0 \,, \tag{2.46}$$

for $i = 1, 2$. The function

$$h_1(x_1, x_2) = h_2(x_1, x_2) \triangleq x_1x_2^{k-1}e^{x_2} \,,$$

with $k \geq 1$ is a solution of the partial differential equation (2.46). Note that the function h is not positive definite, hence it is not a Lyapunov function. Letting, for instance, $k = 1$ the restriction of the Dynamic Lyapunov function to the invariant subset $\mathcal{M} \triangleq$

$\{(x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^4 : \xi_1 = h_1(x_1, x_2), \xi_2 = h_2(x_1, x_2)\}$ is

$$V_{\mathcal{M}}(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2) + x_1^2 e^{2x_2}\,,$$

which is positive definite in $\mathbb{R}^2$. Finally note that the time derivative of the function $V_{\mathcal{M}}$ along the trajectories of the system (2.45) is

$$\dot{V}_{\mathcal{M}}(x_1, x_2) = -x_1^2 - x_2^2 + x_1^2 x_2 - 2x_1^2 e^{2x_2}\,,$$

which is negative definite in $\mathbb{R}^2$.

2.6.3 Synchronous Generator

The problem of estimating, as precisely as possible, the region of attraction of an equilibrium point is crucial in power system analysis [31]. In fact, when a fault occurs in the system, the operating condition is pushed to a different point in the state-space. The possibility of assessing whether the equilibrium is recovered after the fault critically depends on the estimate of the region of attraction of the equilibrium point itself, hence enlargements of the estimate of the basin of attraction may be of paramount importance in case of failure occurrence.

Consider the model of a synchronous generator described by the equation [31]

$$\frac{d^2\delta}{dt^2} + d\frac{d\delta}{dt} + \sin(\delta + \delta_0) - \sin(\delta_0)\,, \tag{2.47}$$

where $d > 0$ is the damping factor, δ_0 is the power angle and $\delta(t) \in \mathbb{R}$ is the power angle variation. Setting $x_1 = \delta$ and $x_2 = \dot{\delta}$, yields

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= -dx_2 - \sin(x_1 + \delta_0) + \sin(\delta_0)\,, \end{aligned} \tag{2.48}$$

with $x(t) = (x_1(t), x_2(t)) \in \mathbb{R}^2$. The origin of the state-space is a locally exponentially stable equilibrium point of the system (2.48) provided $|\delta_0| < \frac{\pi}{2}$. Consider the linearization of the system (2.48) around the equilibrium point $(x_1,\, x_2) = (0,\, 0)$, namely

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\cos(\delta_0) & -d \end{bmatrix} x\,. \tag{2.49}$$

Select as in [31] $d = 0.5$ and $\delta_0 = 0.412$ and define the quadratic Lyapunov function $V_\ell(x) = \frac{1}{2}x^\top \bar{P} x$, where $\bar{P} = \bar{P}^\top > 0$ is the solution of the Lyapunov equation (2.18) with $Q = I$, namely

$$V_\ell(x) = \frac{1}{2}x^\top \begin{bmatrix} 4.3783 & 1.0913 \\ 1.0913 & 4.1826 \end{bmatrix} x\,.$$

To construct a Lyapunov function note that the mapping $P(x) = x^\top \bar{P}$ is an *algebraic* $\bar{P}$ *solution* of the equation (2.21). Letting $R = \bar{P}$ and noting that the partial differential equation (2.30) does not admit a closed-form solution, consider the algebraic equivalent of the invariance partial differential equation, namely the equation (2.32). Partitioning the mapping H as in (2.43) yields

$$\begin{aligned} 0 &= \bar{h}_1 x_2 + \bar{h}_2(-0.5x_2 - (\sin(x_1+\delta_0) - sin(\delta_0))) + 4.3783k(\bar{h}_1 x_1 + \bar{h}_2 x_2) \\ &\quad +1.0913k(\bar{h}_3 x_1 + \bar{h}_4 x_2)\,, \\ 0 &= \bar{h}_3 x_2 + \bar{h}_4(-0.5x_2 - (\sin(x_1+\delta_0) - sin(\delta_0))) + 1.0913k(\bar{h}_1 x_1 + \bar{h}_2 x_2) \\ &\quad +4.1826k(\bar{h}_3 x_1 + \bar{h}_4 x_2)\,. \end{aligned} \tag{2.50}$$

The solution H of (2.50) is exploited to construct the function $V_m(x) = x^\top H(x)^\top \bar{P} x + \frac{1}{2}\|x - H(x)x\|^2$. The estimates of the basin of attraction obtained with the quadratic function V_ℓ and with the function V_m are displayed in Figure 2.4, together with the level line corresponding to $\dot{V}_m = 0$. Figure 2.5 shows the phase-portrait of the system (2.48), with $\delta_0 = 0.412$ and $d = 0.5$ together with the estimate of the basin of attraction given by the function V_m.

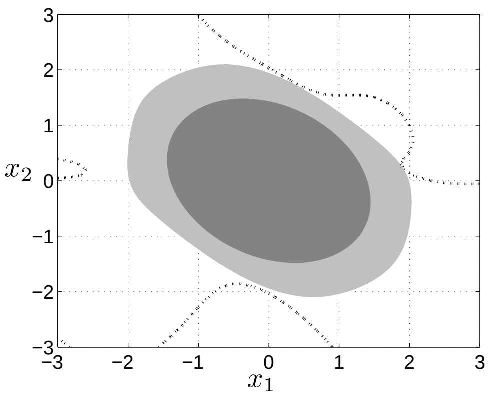

Figure 2.4: estimates of the basin of attraction given by the quadratic Lyapunov function V_ℓ (dark-gray region) and by the dynamic Lyapunov function V_m (light-gray region), together with the level line corresponding to $\dot{V}_m = 0$, dash-dotted line.

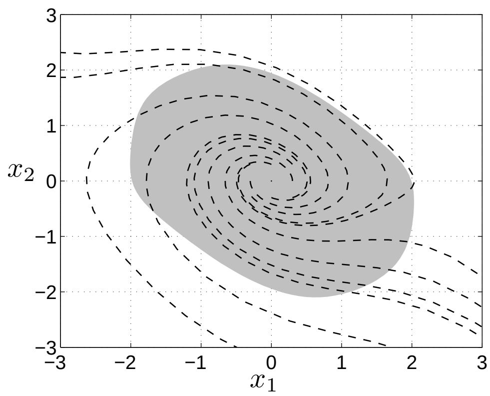

Figure 2.5: Phase-portrait of the system (2.48), with $\delta_0 = 0.412$, together with the estimate of the basin of attraction obtained using the function V_m, gray region.

2.6.4 Back-Stepping with Dynamic Lyapunov Functions

Consider a nonlinear system in feedback form described by equations of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_2)\,, \\ \dot{x}_2 &= u\,, \end{aligned} \tag{2.51}$$

with $f(0,0) = 0$, $x_1(t) \in \mathbb{R}^n$, $x_2(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$. Suppose that there exists a smooth function ϕ such that the zero equilibrium of the system

$$\dot{x}_1 = f(x_1, \phi(x_1)) \tag{2.52}$$

is locally exponentially stable. It is well-known, see [47], that if there exists a Lyapunov function for the system $\dot{x}_1 = f(x_1, \phi(x_1))$, then the zero equilibrium of the system (2.51) can be stabilized by a static state feedback via *back-stepping*. This result requires that an explicit expression of the Lyapunov function be known. In this section we show that this requirement can be relaxed assuming only local exponential stabilizability of the equilibrium point, provided other assumptions are satisfied. In fact, if the former assumption holds a Dynamic Lyapunov function can be constructed for the system $\dot{x}_1 = f(x_1, \phi(x_1))$ and moreover it can be shown that the Dynamic Lyapunov function for the system (2.51) in closed-loop with a specific stabilizing control u has the same *structure* of the Dynamic Lyapunov function of the system $\dot{x}_1 = f(x_1, \phi(x_1))$. A straightforward consequence of the previous comment is that the procedure can be iterated for higher-dimensional systems.

By Theorem 2.2, there exists a neighborhood $\Omega \subseteq \mathbb{R}^n$, containing the origin, and a Dynamic Lyapunov function $\mathcal{V}_1 = (D_{\alpha_1}, V_1)$, with $V_1 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of the form

$$V_1(x_1, \xi_1) = P(\xi_1)x_1 + \frac{1}{2}(x_1 - \xi_1)^\top R(x_1 - \xi_1)\,, \tag{2.53}$$

which is positive definite around the origin in Ω, where $P : \mathbb{R}^n \to \mathbb{R}^n$ is an *algebraic* $\bar{P}$ *solution* of the inequality (2.20) for the system (2.52). The differential equation D_{α_1} is defined as

$$\dot{\xi}_1 = -k(P_{\xi_1}(\xi_1)^\top x_1 - R(x_1 - \xi_1))\,. \tag{2.54}$$

Moreover there exist a constant $\bar{k} \geq 0$ and a positive definite function $\tau : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ such that

$$\begin{aligned} &- k(x_1^\top P_{\xi_1}(\xi_1) - (x_1 - \xi_1)^\top R)(P_{\xi_1}(\xi_1)^\top x_1 - R(x_1 - \xi_1)) \\ &+ (P(\xi_1) + (x_1 - \xi_1)^\top R) f(x_1, \phi(x_1)) < -\tau(x_1, \xi_1)\,, \end{aligned}$$

for all $k > \bar{k}$. The following result shows that a Dynamic Lyapunov function of the *form*

$$
\begin{aligned}
V_2 &= P(\xi_1)x_1 + p(\xi_2)(x_2 - \phi(x_1)) + \frac{1}{2}(x_1 - \xi_1)^\top R(x_1 - \xi_1) + \frac{r}{2}((x_2 - \phi(x_1)) - \xi_2)^2 \\
&\triangleq \tilde{P}(\xi)z + \frac{1}{2}(z - \xi)^\top \tilde{R}(z - \xi)\,,
\end{aligned}
\tag{2.55}
$$

where $z = [x_1^\top, x_2 - \phi(x_1)]^\top$, $\xi = [\xi_1^\top, \xi_2]^\top$ and $r > 0$, can be constructed for the system (2.51) with a suitable choice of the control u and of the dynamics D_{α_2}, where $\tilde{P}(x) = [P(x_1)^\top, p(x_2)]^\top$ and

$$
\tilde{R} = \begin{bmatrix} R & 0 \\ 0 & r \end{bmatrix}.
$$

To provide a concise statement of the following result let $M(x_1, x_2)$ be such that $f(x_1, x_2) - f(x_1, \phi(x_1)) = M(x_1, x_2)(x_2 - \phi(x_1))$.

Proposition 2.3. Consider the nonlinear system (2.51) and suppose that the zero equilibrium of the system (2.52) is locally exponentially stable. Let $p(\xi_2) = r\xi_2$ and

$$
u = -\frac{\gamma_1}{r}(x_2 - \phi(x_1)) - \frac{1}{r}\left(P(\xi_1) + (x_1 - \xi_1)^\top R\right) M(x_1, x_2) + \frac{\partial \phi}{\partial x_1} f(x_1, x_2)\,, \tag{2.56}
$$

with $\gamma_1 > 0$. Then $\mathcal{V}_2 = (D_{\alpha_2}, V_2)$, with

$$
D_{\alpha_2} : \begin{cases} \dot{\xi}_1 &= -k(P_{\xi_1}(\xi_1)^\top x_1 - R(x_1 - \xi_1))\,, \\ \dot{\xi}_2 &= -\dfrac{\gamma_2}{r}\xi_2\,, \end{cases} \tag{2.57}
$$

$\gamma_2 > 0$, and V_2 defined in (2.55), is a (local) Dynamic Lyapunov function for the system (2.51). ◇

Proof. Local exponential stability of the zero equilibrium of the system (2.52) implies the existence of a Dynamic Lyapunov function $\mathcal{V}_1 = (D_{\alpha_1}, V_1)$ with V_1 defined as in (2.53) and D_{α_1} described by (2.54). With the given choice of $p(\xi_2)$, there exists a non-empty open set $\tilde{\Omega} \subseteq \mathbb{R}^{2n+2}$, containing the origin, such that the function V_2 is positive definite around the origin in $\tilde{\Omega}$. Define the change of coordinates $z_1 = x_1$ and $z_2 = x_2 - \phi(x_1)$. In the new coordinates the dynamics are described by equations of the form

$$
\begin{aligned}
\dot{z}_1 &= f(z_1, \phi(z_1)) + M(z_1, z_2 + \phi(z_1))z_2\,, \\
\dot{z}_2 &= u - \frac{\partial \phi}{\partial z_1} f(z_1, z_2 + \phi(z_1))\,.
\end{aligned}
\tag{2.58}
$$

The time derivative of the function V_2 in (2.55) along the trajectories of the system (2.58) is

$$
\begin{aligned}
\dot{V}_2 = & -k(z_1^\top P_{\xi_1}(\xi_1) - (z_1-\xi_1)^\top R)(P_{\xi_1}(\xi_1)^\top z_1 - R(z_1-\xi_1)) \\
& + (P(\xi_1) + (z_1-\xi_1)^\top R) f(z_1, z_2+\phi(z_1)) + r z_2 \left[u - \frac{\partial \phi}{\partial z_1} f(z_1, z_2+\phi(z_1)) \right] + r\xi_2\dot{\xi}_2 \\
\leq & -\tau(z_1,\xi_1) + (P(\xi_1) + (z_1-\xi_1)^\top R)\left[f(z_1, z_2+\phi(z_1)) - f(z_1,\phi(z_1)) \right] \\
& + r z_2 \left[u - \frac{\partial \phi}{\partial z_1} f(z_1, z_2+\phi(z_1)) \right] + r\xi_2\dot{\xi}_2 \\
= & -\tau(z_1,\xi_1) + (P(\xi_1) + (z_1-\xi_1)^\top R) M(z_1, z_2+\phi(z_1)) z_2 \\
& + r z_2 \left[u - \frac{\partial \phi}{\partial z_1} f(z_1, z_2+\phi(z_1)) \right] + r\xi_2\dot{\xi}_2 \\
\leq & -\tau(z_1,\xi_1) - \gamma_1 z_2^2 - \gamma_2 \xi_2^2 < 0 \,,
\end{aligned}
$$

for all (z_1, z_2, ξ_1, ξ_2) different from zero, where the last inequality is obtained considering, in the new coordinates, the definition of the control law u and of the dynamics of the variable ξ_2, which proves the claim. □

Note that, letting $\xi_2(0) = 0$, only the dynamic extension ξ_1 is required for the control design.

Remark 2.13. If the first equation of system (2.51) is affine in the state x_2, *i.e.* $\dot{x}_1 = f(x_1) + g(x_1)x_2$, then $M(x_1, x_2) = g(x_1)$ and the control law (2.56)-(2.57) is

$$
\begin{aligned}
\dot{\xi}_1 &= -k(P_{\xi_1}(\xi_1)^\top x_1 - R(x_1-\xi_1)) \,, \\
\dot{\xi}_2 &= -\frac{\gamma_2}{r}\xi_2 \,, \\
u &= -\frac{\gamma_1}{r}(x_2 - \phi(x_1)) - \frac{1}{r}\left(P(\xi_1) + (x_1-\xi_1)^\top R \right) g(x_1) + \frac{\partial \phi}{\partial x_1}\left[f(x_1) + g(x_1)x_2 \right] .
\end{aligned}
$$

▲

Example 2.1. Consider the nonlinear system described by the equations

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_{11} \\ \dot{x}_{12} \end{bmatrix} &= \begin{bmatrix} -x_2 \\ x_{11}x_2 - x_{12} \end{bmatrix} , \\
\dot{x}_2 &= x_{11}^2 + x_{12}x_2 + u \,,
\end{aligned}
\tag{2.59}
$$

and note that $P(x_1) = [x_{11} + x_{11}x_{12},\ x_{12}]$ is an *algebraic* $\bar{P}$ *solution* of the inequality (2.20) for the system

$$
\begin{aligned}
\dot{x}_{11} &= -x_{11} \,, \\
\dot{x}_{12} &= x_{11}^2 - x_{12} \,,
\end{aligned}
$$

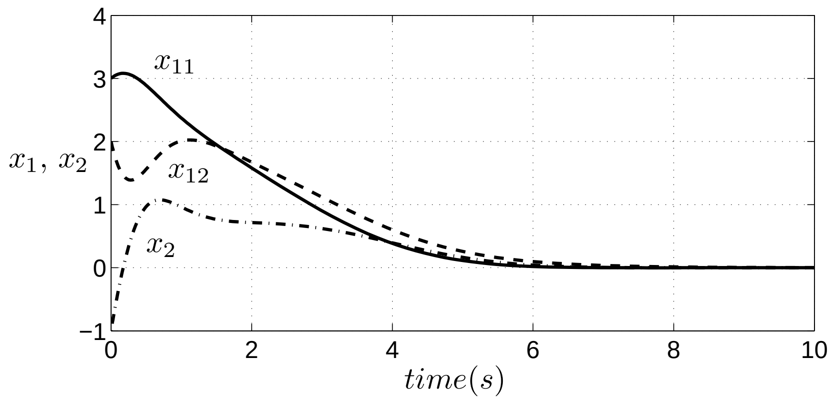

Figure 2.6: Time histories of the state of the system (2.59) from the initial condition $(x_{11}(0), x_{12}(0), x_2(0)) = (3, 2, -1)$ in closed-loop with the dynamic control law (2.60)-(2.62).

with $\bar{P} = I$. Define the Dynamic Lyapunov Function $\mathcal{V}$ for the system $\dot{x}_1 = f_1(x_1) + g_1(x_1)x_{11}$, with the differential equation

$$\begin{aligned} \dot{\xi}_{11} &= -k(x_{11}\xi_{12} + \xi_{11}), \\ \dot{\xi}_{12} &= -k(x_{11}\xi_{11} + \xi_{12}), \end{aligned} \tag{2.60}$$

$k > 0$, and the locally positive definite function

$$V(x_1, \xi_1) = \frac{1}{2}x_1^\top x_1 + \frac{1}{2}\xi_1^\top \xi_1 + \xi_{11}\xi_{12}x_{11}. \tag{2.61}$$

The control law that asymptotically stabilizes the zero equilibrium of the system (2.59) is given by (see (2.13))

$$u = -\gamma(x_2 - x_{11}) - (x_{11}x_{12} - \xi_{11}\xi_{12} - x_{11}) - x_2 - x_{11}^2 - x_{12}x_2, \tag{2.62}$$

with the dynamics of ξ defined as in (2.60). Figure 2.6 shows the time histories of the trajectories of system (2.59) from the initial condition $(x_{11}(0), x_{12}(0), x_2(0)) = (3, 2, -1)$ driven by the dynamic control law (2.60),(2.62), with $(\xi_{11}(0), \xi_{12}(0)) = (1,\ 1)$.

Finally, since standard back-stepping techniques may be considered to stabilize the zero equilibrium of system (2.59), a comparison between the two approaches is performed. In particular, letting as above $\phi(x_1) = x_{11}$ and considering the Lyapunov function

$$V(x_1) = \frac{1}{2}\left(x_{11}^2 + x_{11}^4\right) + \frac{1}{2}x_{12}^2,$$

classical back-stepping arguments yield the static control law

$$u = -\gamma(x_2 - x_{11}) + (x_{11} + 2x_{11}^3 - x_{11}x_{12}) - x_2 - x_{11}^2 - x_{12}x_2. \tag{2.63}$$

Figure 2.7 shows the phase portrait of the state of system (2.59) in closed-loop with the dynamic control law (2.60)-(2.62) and with the control law (2.63).

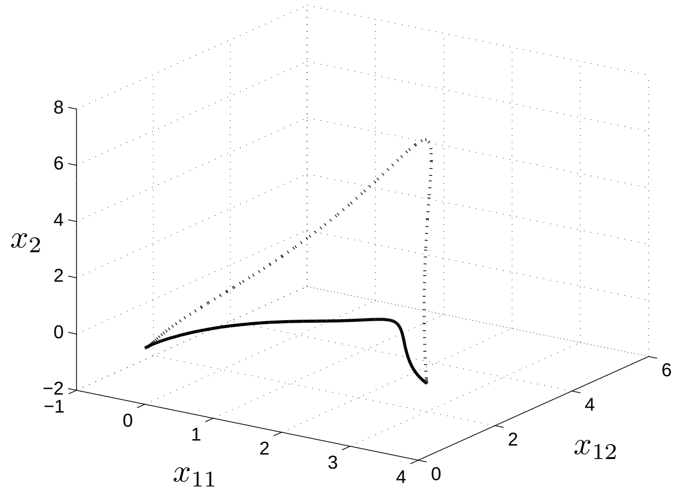

Figure 2.7: Phase portrait of the state of system (2.59) in closed-loop with the dynamic control law (2.60)-(2.62) and with the control law (2.63), solid and dashed lines, respectively.

■

2.6.5 Forwarding with Dynamic Lyapunov Functions

Consider a nonlinear system in feedforward form described by equations of the form [76]

$$\begin{aligned} \dot{y} &= h(x)\,, \\ \dot{x} &= f(x) + g(x)u\,, \end{aligned} \tag{2.64}$$

with $y(t) \in \mathbb{R}$, $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^p$. Suppose that the linearization of the system (2.64) around the origin is controllable. Suppose that the zero equilibrium of the system $\dot{x} = f(x)$ is globally asymptotically stable and locally exponentially stable.

It is important to stress that the knowledge of a Lyapunov function for the system $\dot{x} = f(x)$ is not assumed, differently from standard *forwarding* technique [63, 76, 99]. The exponential stability assumption implies the existence of a smooth function $M : \mathbb{R}^n \to \mathbb{R}$ such that

$$h(x) = \frac{\partial M(x)}{\partial x} f(x)\,,$$

together with the boundary condition $M(0) = 0$. Consider the change of coordinates

$(x, z) = (x, y - M(x))$ yielding

$$\begin{aligned} \dot{z} &= -\frac{\partial M(x)}{\partial x} g(x) u \,, \\ \dot{x} &= f(x) + g(x) u \,. \end{aligned} \tag{2.65}$$

Then, by the exponential stability assumption, there exists a Dynamic Lyapunov function $\mathcal{V}_1 = (D_{\alpha_1}, V_1)$ for the x-subsystem, defined in a neighborhood of the origin, with V_1 as in (2.53) and D_{α_1} described by (2.54). The following result shows that a Dynamic Lyapunov function of the same form as $\mathcal{V}_1$ can be constructed for the system (2.65), mimicking the construction of the Dynamic Lyapunov function (2.55), for a suitable choice of the input u.

Proposition 2.4. Consider the nonlinear system (2.65). Let

$$u = -g(x)^\top \left[P(\xi_1) + (x - \xi_1)^\top R - rz \frac{\partial M(x)}{\partial x} \right]^\top \,, \tag{2.66}$$

with $r > 0$. Then $\mathcal{V}_2 = (D_{\alpha_2}, V_2)$, with

$$D_{\alpha_2} : \begin{cases} \dot{\xi}_1 &= -k(P_{\xi_1}(\xi_1)^\top x - R(x - \xi_1)) \,, \\ \dot{\xi}_2 &= -\dfrac{\gamma_2}{r} \xi_2 \,, \end{cases} \tag{2.67}$$

$\gamma_2 > 0$, and V_2 defined in (2.55) with $p(\xi_2) = r\xi_2$, is a Dynamic Lyapunov function for the system (2.65). ◇

Proof. The time derivative of the function V_2, with $p(\xi_2) = r\xi_2$, along the trajectories of the system (2.65)-(2.67) is, provided $k > \bar{k}$,

$$\dot{V}_2 < -\tau(x, \xi_1) - \gamma_2 \xi_2^2 - \left\| \left[P(\xi_1) + (x - \xi_1)^\top R - rz \frac{\partial M(x)}{\partial x} \right] g(x) \right\|^2 \leq 0 \,,$$

where the last inequality is obtained from the definition of u in (2.66). Finally, controllabilty of the linearization of the system (2.64) around the origin guarantees that

$$\left. \frac{\partial M(x)}{\partial x} \right|_{x=0} g(0) \neq 0$$

and consequently, by LaSalle's invariance principle, the equilibrium $(x, z) = (0, 0)$ is asymptotically stable. □

2.7 Conclusions

The notion of Dynamic Lyapunov function has been introduced. Similarly to the classical notion of Lyapunov function, this notion allows to study stability properties of equilibrium points of linear and nonlinear systems. Unlike Lyapunov functions, Dynamic Lyapunov functions may be constructed without the knowledge of the explicit solution of the underlying ordinary differential equation and involving the solution of any partial differential equation or inequality. In addition Dynamic Lyapunov functions allow to construct families of Lyapunov functions. Implications of Dynamic Lyapunov functions have been discussed and examples have been used to illustrate the advantages of Dynamic Lyapunov functions.

Chapter 3

Observability and Controllability Functions

3.1 Introduction

Minimal realization theory for linear systems provides the tool to determine a state-space description of minimal size, the impulse response of which matches the impulse response of the original system. However, in situations in which this minimal realization cannot be obtained for practical or theoretical reasons, one may still be interested in finding a lower-order representation, *i.e.* a reduced order model, that *approximates* the behavior of the original model, see [66, 73, 97, 98].

The first step towards the solution of the model reduction problem is the characterization of a *measure* of importance of the state components according to some criterion. In the literature the latter is typically defined in terms of the contribution of each state component to the input/output behavior of the system. In the case of locally asymptotically stable systems, this information is contained in the controllability and observability functions, which provide the minimum energy necessary to steer the state from the origin to a state $\bar{x}$ in infinite time and the output energy generated from the system initialized at $\bar{x}$, respectively. Once these functions have been determined, there exists a (nonlinear) change of coordinates such that, in the new coordinates, the functions are in the so-called *balanced* form [98]. After the application of the coordinates transformation the components of the state may be ordered according to the energy required to steer the state and the output energy released by the corresponding initial condition. Therefore, if a state demands a large amount of energy to be controlled, on one hand, and it is *hardly* observable (in terms of output energy), on the other hand, then the contribution of the aforementioned state to the input/output behavior of the system could be ignored in a lower-order approximation.

In the linear case, the controllability and observability functions are determined as the solutions of Lyapunov matrix equations and are related to the controllability and observability Gramians of the system. On the other hand, the controllability and observability functions of nonlinear systems are given in terms of the solutions of nonlinear first-order partial differential equations. This represents a serious drawback for the application of model reduction by balancing (of the controllability and the observability functions) to practical cases.

In the case of unstable nonlinear systems several techniques have been proposed, such as LQG, HJB or H_∞ balancing. The latter has been introduced for linear systems in [68] and subsequently extended to the nonlinear case in [96]. Moreover, it is shown in [97] that the HJB singular value functions, obtained from the past and future energy functions, are strongly interconnected with the graph Hankel singular value functions, derived balancing the controllability and observability functions of the normalized coprime factorizations of the nonlinear system. As a result, the reduced order models obtained with the two different approaches coincide. Finally, the assumption of zero-state observabilty is relaxed in [32] allowing for non-zero inputs in the definition of the Observability function.

The rest of the chapter is organized as follows. Section 3.2 introduces the notion of Dynamic Generalized controllability and observability functions. They are said to be generalized and dynamic since partial differential inequalities are solved, in place of equations, in an extended state-space. While the notion of generalized Gramians has been introduced in the literature, see for instance [74, 80, 81], the idea of considering *dynamic* Gramians is entirely new. A class of Dynamic Generalized controllability and observability functions is constructed in Section 3.3. The application of the Dynamic Generalized controllability and observability functions to the problem of balancing and model reduction for nonlinear systems is discussed in Section 3.4. It is shown that, once the functions have been obtained, a nonlinear system can be transformed into a *dynamically balanced form* by means of a change of coordinates.

3.2 Dynamic Generalized Controllability and Observability Functions

Consider a nonlinear system described by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u\,, \\ y &= h(x)\,, \end{aligned} \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system, $u(t) \in \mathbb{R}^m$ denotes the input, and $y(t) \in \mathbb{R}^p$ is the output. The mappings $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ are assumed to be sufficiently smooth and, without loss of generality, such that $f(0) = 0$ and $h(0) = 0$. In what follows it is assumed that the linearization of system (3.1) around the origin is controllable and observable.

Assumption 3.1. The system (3.1) is zero-state detectable, *i.e.* $u(t) = 0$ and $y(t) = 0$ for all $t \geq 0$ imply $\lim\limits_{t \to \infty} x(t) = 0$.

Note that zero-state detectability holds locally by observability of the pair (A, C).

Assumption 3.2. The zero equilibrium of the system (3.1) is locally asymptotically stable.

Definition 3.1. Consider the nonlinear system (3.1). The functions, if they exist,

$$L_c(\bar{x}) = \min_{\substack{u \in \mathcal{L}_2(-\infty,\, 0) \\ x(-\infty) = 0,\, x(0) = \bar{x}}} \frac{1}{2} \int_{-\infty}^{0} \|u(t)\|^2 dt \tag{3.2}$$

and

$$L_o(\bar{x}) = \frac{1}{2} \int_{0}^{\infty} \|y(t)\|^2 dt, \;\; x(0) = \bar{x},\, u(t) \equiv 0, \tag{3.3}$$

are the controllability and observability functions, respectively, of system (3.1).

The function L_o may take on infinity as a value if the zero equilibrium of system (3.1) is unstable whereas the function L_c takes on infinity by convention if the state $\bar{x}$ cannot be reached from zero. For simplicity, we suppose throughout the paper that the controllability and observability functions are defined, namely finite, for all $\bar{x}$ in a neighborhood of the origin. If the system (3.1) is linear, *i.e.*

$$\begin{aligned} \dot{x} &= Ax + Bu\,, \\ y &= Cx\,, \end{aligned} \tag{3.4}$$

the controllability and observability functions are defined, with the assumptions of asymptotic stability of the zero equilibrium point and observability of the pair (A, C), as $L_c(\bar{x}) = \frac{1}{2}\bar{x}^\top \bar{P}^{-1}\bar{x}$ and $L_o(\bar{x}) = \frac{1}{2}\bar{x}^\top \bar{Q}\bar{x}$, respectively, where the matrices $\bar{P} = \bar{P}^\top > 0$ and $\bar{Q} = \bar{Q}^\top > 0$ are solutions of the Lyapunov equations

$$A\bar{P} + \bar{P}A^\top + BB^\top = 0, \tag{3.5}$$

$$A^\top \bar{Q} + \bar{Q}A + C^\top C = 0. \tag{3.6}$$

The notion of Dynamic Generalized Controllability and Observability functions is introduced in the two following definitions, respectively.

Definition 3.2. Consider the nonlinear system (3.1). A *Dynamic Generalized controllability function* $\mathcal{V}_c$ is a pair $(\mathcal{D}_c, \mathcal{L}_c)$ defined as follows.

- $\mathcal{D}_c$ is the ordinary differential equation

$$\dot{\xi}_c = \phi_c(x,\, \xi_c), \tag{3.7}$$

  with $\xi_c(t) \in \mathbb{R}^n$, $\phi_c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz and $\phi_c(0,\, 0) = 0$.

- $\mathcal{L}_c : \Omega_c \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\mathcal{L}_c(0,\, 0) = 0$ and $\mathcal{L}_c(x,\, \xi_c) > 0$ for all $(x,\, \xi_c) \in \Omega_c \setminus \{0\}$, is such that

$$\frac{\partial \mathcal{L}_c}{\partial x} f(x) + \frac{\partial \mathcal{L}_c}{\partial \xi_c} \phi_c(x,\, \xi_c) + \frac{1}{2} \frac{\partial \mathcal{L}_c}{\partial x} g(x) g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top \leq 0, \tag{3.8}$$

  for all $(x, \xi_c) \in \Omega_c$. Moreover $(0,\, 0)$ is an asymptotically stable equilibrium point of the system

$$\begin{aligned} \dot{x} &= -f(x) - g(x)g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top, \\ \dot{\xi}_c &= -\phi_c(x,\, \xi_c)\,. \end{aligned} \tag{3.9}$$

Definition 3.3. Consider the nonlinear system (3.1). A *Dynamic Generalized observability function* $\mathcal{V}_o$ is a pair $(\mathcal{D}_o, \mathcal{L}_o)$ defined as follows.

- $\mathcal{D}_o$ is the ordinary differential equation

$$\dot{\xi}_o = \phi_o(x,\, \xi_o)\,, \tag{3.10}$$

  with $\xi_o(t) \in \mathbb{R}^n$, $\phi_o : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz, $\phi_o(0,\, 0) = 0$ and such that $\xi_o = 0$ is an asymptotically stable equilibrium point of $\phi_o(0,\, \xi_o)$.

- $\mathcal{L}_o : \Omega_o \subseteq \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, $\mathcal{L}_o(0,\, 0) = 0$ and $\mathcal{L}_o(x,\, \xi_o) > 0$ for all $(x,\, \xi_o) \in \Omega_o \setminus \{0\}$, is such that

$$\frac{\partial \mathcal{L}_o}{\partial x} f(x) + \frac{\partial \mathcal{L}_o}{\partial \xi_o} \phi(x,\, \xi_o) + \frac{1}{2} h(x)^\top h(x) \leq 0, \tag{3.11}$$

  for all $(x,\, \xi_o) \in \Omega_o$.

Note that the Dynamic Generalized controllability and observability functions are not unique.

The relation between the Dynamic Generalized controllability and observability functions and the controllability and observability functions as in Definition 3.1 is now investigated. In particular the following proposition shows that the function $\mathcal{L}_c$ in a Dynamic Generalized controllability function $\mathcal{V}_c$ of the system (3.1) is a lower-bound for the minimum energy necessary to steer the state of the system from the initial condition $x = 0$ at time $t = -\infty$ to the final condition $x = \bar{x}$ at time $t = 0$.

Proposition 3.1. Suppose that $\mathcal{V}_c = (\mathcal{D}_c, \mathcal{L}_c)$ is a Dynamic Generalized controllability function for the system (3.1). Then

$$\mathcal{L}_c(\bar{x}, \bar{\xi}_c) \leq L_c(\bar{x}), \tag{3.12}$$

for any $(\bar{x}, \bar{\xi}_c) \in \Omega_c \subseteq \mathbb{R}^n \times \mathbb{R}^n$. ◇

Proof. By the condition (3.8)

$$\begin{aligned}\dot{\mathcal{L}}_c &= \frac{\partial \mathcal{L}_c}{\partial x}(f(x) + g(x)u) + \frac{\partial \mathcal{L}_c}{\partial \xi_c}\phi_c(x, \xi_c) \leq \frac{\partial \mathcal{L}_c}{\partial x}g(x)u - \frac{1}{2}\frac{\partial \mathcal{L}_c}{\partial x}g(x)g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top \\ &= \frac{1}{2}\|u\|^2 - \frac{1}{2}\left\|u - g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top\right\|^2,\end{aligned}$$

where the last equality is obtained completing the squares. Therefore, by definition of controllability function of the system (3.1)-(3.7) and letting

$$u = g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top, \tag{3.13}$$

it follows that

$$\mathcal{L}_c(\bar{x}, \bar{\xi}_c) = \int_{-\infty}^{0} \dot{\mathcal{L}}_c(x(t), \xi_c(t))dt \leq \frac{1}{2}\int_{-\infty}^{0} \|u(t)\|^2 dt = L_c(\bar{x}),$$

with $(x(0), \xi_c(0)) = (\bar{x}, \bar{\xi}_c)$, where the first equality is obtained by asymptotic stability of the system (3.9). □

A similar result can be established for the Dynamic Generalized observability function.

Proposition 3.2. Suppose that $\mathcal{V}_o = (\mathcal{D}_o, \mathcal{L}_o)$ is a Dynamic Generalized observability function for the system (3.1). Then

$$L_o(\bar{x}) \leq \mathcal{L}_o(\bar{x}, \bar{\xi}_o), \tag{3.14}$$

for any $(\bar{x}, \bar{\xi}_o) \in \Omega_o \subseteq \mathbb{R}^n \times \mathbb{R}^n$. ◇

Proof. Since $\mathcal{V}_o$ is a Dynamic Generalized observability function, by the condition (3.11),

$$\frac{1}{2}h(x)^\top h(x) \leq -\dot{\mathcal{L}}_o(x, \xi_o).$$

Then, by definition of observability function,

$$\begin{aligned}L_o(\bar{x}) &= \frac{1}{2}\int_0^\infty \|y(t)\|^2 dt = \frac{1}{2}\int_0^\infty h(x)^\top h(x)dt \leq -\int_0^\infty \dot{\mathcal{L}}_o dt = -\mathcal{L}_o(x(\infty), \xi_o(\infty)) \\ &\quad + \mathcal{L}_o(x(0), \xi_o(0)) = \mathcal{L}_o(\bar{x}, \bar{\xi}_o),\end{aligned}$$

where the last equality is obtained recalling that, by assumption, $x = 0$ is an asymptotically stable equilibrium point of f and noting that $\xi_o(t)$ converges to zero as time tends to infinity. To show the last claim consider the triangular system

$$\begin{aligned} \dot{x} &= f(x)\,, \\ \dot{\xi}_o &= \phi_o(x,\,\xi_o)\,, \end{aligned} \tag{3.15}$$

and note that, since $\mathcal{L}_o$ is positive definite by assumption and $\dot{\mathcal{L}}_o \leq 0$ by (3.11), all the trajectories of the system (3.15) are bounded. Moreover, since $x = 0$ and $\xi_o = 0$ are asymptotically stable equilibrium points of f and of $\phi_o(0,\,\xi_o)$, respectively, standard arguments on *cascaded* systems prove the claim. □

Proposition 3.2 states that the amount of output energy generated by the initial condition $\bar{x}$ is upper-bounded by the *estimate* provided by the Dynamic Generalized observability function.

Remark 3.1. The inequalities (3.12) and (3.14) guarantee that the analysis, in view of the model reduction problem, of the input/output behavior of the nonlinear system (3.1) is not *modified* by the dynamic extension $(\xi_o,\,\xi_c)$. In fact the output energy released by the system (3.1) from the initial condition $x = \bar{x}$ is not greater than the output energy released by the system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u, \\ \dot{\xi}_o &= \phi_o(x,\,\xi_o), \\ \dot{\xi}_c &= \phi_c(x,\,\xi_c), \\ y &= h(x), \end{aligned} \tag{3.16}$$

with $u(t) = 0$, for all $t \geq 0$, initialized at $(\bar{x},\,\bar{\xi}_o,\,\bar{\xi}_c)$, for any $(\bar{\xi}_o,\,\bar{\xi}_c)$. Similarly, the minimum energy necessary to steer the state of the system (3.1) from $x(-\infty) = 0$ to $x(0) = \bar{x}$ is not smaller than the energy necessary to drive the system (3.16) from $(x(-\infty),\,\xi_o(-\infty),\,\xi_c(-\infty)) = (0,0,0)$ to $(\bar{x},\,\bar{\xi}_o,\,\bar{\xi}_c)$, for any $(\bar{\xi}_o,\,\bar{\xi}_c)$. ▲

Remark 3.2. The difference between the functions $\mathcal{L}_c(\bar{x},\,\bar{\xi}_c)$ and $L_c(\bar{x})$ and between the functions $\mathcal{L}_o(\bar{x},\,\bar{\xi}_o)$ and $L_o(\bar{x})$ can be minimized, for a given initial condition $x(0) = \bar{x}$, initializing the dynamic extension such that

$$\xi_c(0) = \xi_c^* \triangleq \arg\max_{\xi_c} \mathcal{L}_c(\bar{x},\xi_c), \tag{3.17}$$

$$\xi_o(0) = \xi_o^* \triangleq \arg\min_{\xi_o} \mathcal{L}_o(\bar{x},\xi_o), \tag{3.18}$$

respectively. ▲

An alternative minimization of the difference between L_c and $\mathcal{L}_c$ is now considered. Note that, since $\mathcal{L}_c(0,0)=0$,

$$\mathcal{L}_c(x,\,\xi_c) = \int_{-\infty}^{0} \dot{\mathcal{L}}_c(x(t),\,\xi_c(t))dt\,, \tag{3.19}$$

where $x(t)$ is the solution of the system (3.1), with $x(0)=x$, u as in (3.13), and $\xi_c(t)$ any continuously differentiable function of time such that $\xi_c(0)=\xi_c$ and $\lim_{t\to-\infty}(x(t),\,\xi_c(t)) = (0,0)$. Therefore, one may be interested in determining a *point-wise*, namely with respect to time, maximizer of (3.19), denoted $\hat{\xi}_c(t)$, for fixed $x(0)$, which is a continuously differentiable scalar function that maximizes the cost functional

$$\begin{aligned} J(\xi_c(t)) = \int_{-\infty}^{0} \Big(&\frac{\partial \mathcal{L}_c}{\partial x}(x(t),\xi_c(t))f(x(t)) + \frac{\partial \mathcal{L}_c}{\partial \xi_c}(x(t),\xi_c(t))\dot{\xi}_c(t) \\ &+ \frac{\partial \mathcal{L}_c}{\partial x}(x(t),\xi_c(t))g(x(t))g(x(t))^\top \frac{\partial \mathcal{L}_c}{\partial x}(x(t),\xi_c(t))^\top \Big) dt, \end{aligned} \tag{3.20}$$

subject to

$$\begin{cases} \xi_c(0) = \xi_c, \\ \lim_{t\to-\infty}(x(t),\,\xi_c(t)) = (0,0), \\ \dot{\mathcal{L}}_c(x(t),\,\xi_c(t)) \le \dfrac{1}{2}\dfrac{\partial \mathcal{L}_c}{\partial x} g(x)g(x)^\top \dfrac{\partial \mathcal{L}_c}{\partial x}^\top, \quad \forall t \ge 0. \end{cases} \tag{3.21}$$

Note that the last constraint is needed in order to guarantee that $\mathcal{L}_c(x,\,\xi_c) \le L_c(x)$, for all $(x,\,\xi_c) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proposition 3.3. For any initial condition $x(0)=x_0$, let $(x^*(t),\,\xi_c^*(t))$ denote the solution of

$$\begin{aligned} \dot{x} &= f(x) + g(x)g(x)\frac{\partial \mathcal{L}_c}{\partial x}^\top, \\ \dot{\xi}_c(t) &= \phi_c(x,\xi_c), \\ x(0) &= x_0, \\ \xi_c(0) &= \xi_c^*, \end{aligned} \tag{3.22}$$

where ξ_c^* is defined in (3.17). The solution $\xi_c^*(t)$ is a maximizer $\hat{\xi}_c(t)$ of (3.20) for all $t \in \mathbb{R}$. $\diamond$

Proof. The claim follows immediately noting that, by definition of Dynamic Generalized controllability function, $\xi_c^*(t)$ is admissible for the problem (3.20)-(3.21). Moreover notice that the integral of an exact differential is independent of the path. Then, by definition of the maximizer $\hat{\xi}_c(t)$,

$$\mathcal{L}_c(x,\,\hat{\xi}_c) = \int_{-\infty}^{0} \dot{\mathcal{L}}_c(x(t),\,\hat{\xi}_c(t))dt \ge \int_{-\infty}^{0} \dot{\mathcal{L}}_c(x(t),\,\xi_c^*(t))dt = \mathcal{L}_c(x,\,\xi_c^*).$$

On the other hand, by (3.17), $\mathcal{L}_c(x,\, \xi_c^*) \geq \mathcal{L}_c(x,\, \xi_c)$, for any $\xi_c \in \mathbb{R}^n$. Hence

$$\int_{-\infty}^{0} \dot{\mathcal{L}}_c(x(t),\, \xi_c^*(t))dt = \mathcal{L}_c(x,\, \xi_c^*) \geq \mathcal{L}_c(x,\, \xi_c) = \int_{-\infty}^{0} \dot{\mathcal{L}}_c(x(t),\, \xi_c(t))dt\,,$$

for any function $\xi_c(t)$ admissible for (3.21). Since the maximizer $\hat{\xi}_c(t)$ must be admissible, it follows that

$$\int_{-\infty}^{0} \dot{\mathcal{L}}_c(x(t),\, \hat{\xi}_c(t))dt \geq \int_{-\infty}^{0} \dot{\mathcal{L}}_c(x(t),\, \xi_c^*(t))dt \geq \int_{-\infty}^{0} \dot{\mathcal{L}}_c(x(t),\, \hat{\xi}_c(t))dt\,,$$

proving the claim. □

The result can be extended to the Dynamic Generalized observability function. In particular a point-wise minimizer, denoted $\hat{\xi}_o(t)$, for fixed $x(0)$ is a continuously differentiable scalar function that minimizes the cost functional

$$J(\xi_o(t)) = \int_{0}^{\infty} -\Big(\frac{\partial \mathcal{L}_o}{\partial x}(x(t), \xi_o(t))f(x(t)) + \frac{\partial \mathcal{L}_o}{\partial \xi_o}(x(t), \xi_o(t))\dot{\xi}_o(t)\Big)dt, \tag{3.23}$$

subject to

$$\begin{cases} \xi_o(0) = \xi_o, \\ \lim\limits_{t\to\infty} \xi_o(t) = 0, \\ \dot{\mathcal{L}}_o(x(t),\, \xi_o(t)) \leq -\frac{1}{2}h(x(t))^\top h(x(t)), \quad \forall t \geq 0. \end{cases} \tag{3.24}$$

Note that the last constraint is needed in order to guarantee that $\mathcal{L}_o(x,\, \xi_o) \geq L_o(x)$, for all $(x,\, \xi_o) \in \mathbb{R}^n \times \mathbb{R}^n$.

Proposition 3.4. For any initial condition $x(0) = x_0$, let $(x^*(t),\, \xi_o^*(t))$ denote the solution of

$$\begin{aligned} \dot{x} &= f(x), \\ \dot{\xi}_o(t) &= \phi_o(x, \xi_o), \\ x(0) &= x_0, \\ \xi_o(0) &= \xi_o^*, \end{aligned} \tag{3.25}$$

where ξ_o^* is defined in (3.18). The solution $\xi_o^*(t)$ is a minimizer $\hat{\xi}_o(t)$ of (3.23) for all $t \geq 0$. ◇

3.3 A Class of Dynamic Generalized Controllability and Observability Functions

The notion of Dynamic Generalized controllability and observability functions is motivated by the fact that the differential equation $\mathcal{D}_c$ ($\mathcal{D}_o$, respectively) and the function

$\mathcal{L}_c$ ($\mathcal{L}_o$, respectively) can be explicitly constructed in a neighborhood of the origin. This construction hinges upon the solution of algebraic equations and it does not involve the solution of any partial differential equation or inequality.

3.3.1 Controllability

The following result provides a class of Dynamic Generalized controllability functions.

Proposition 3.5. Consider the nonlinear system (3.1) and let $\Sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, with $\Sigma(0) > 0$, $x^\top \Sigma(x) x \geq 0$ for all $x \in \mathbb{R}^n$. Let $P : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, $P(0) = 0$, be such that

$$P(x)f(x) + \frac{1}{2}P(x)g(x)g(x)^\top P(x) + \frac{1}{2}x^\top \Sigma(x)x \leq 0, \tag{3.26}$$

with

$$\frac{\partial P}{\partial x}(x)^\top \Big|_{x=0} = \bar{P}\,, \tag{3.27}$$

where $\bar{P} = \bar{P}^\top > 0$ is the solution of $\bar{P}A + A^\top \bar{P} + \bar{P}BB^\top \bar{P} + \Sigma(0) = 0$. Consider the function

$$\mathcal{L}_c(x,\, \xi_c) = P(\xi_c)x + \frac{1}{2}\|x - \xi_c\|^2_{R_c}, \tag{3.28}$$

and let

$$\dot{\xi}_c = -k\frac{\partial \mathcal{L}_c}{\partial \xi_c}^\top = -k\left(\frac{\partial P}{\partial \xi_c}(\xi_c)^\top x - R_c(x - \xi_c)\right). \tag{3.29}$$

Then there exist $k_l \geq 0$ and a non-empty open set $\Omega_c \subset \mathbb{R}^n \times \mathbb{R}^n$ such that $\mathcal{L}_c$ satisfies the partial differential inequality (3.8) for all $k > k_l$ and for all $(x,\, \xi_c) \in \Omega_c$. If, additionally, there exists $k_u > k_l$ such that

$$\frac{\partial \mathcal{L}_c}{\partial x}f(x) + (1-\epsilon)\frac{\partial \mathcal{L}_c}{\partial x}g(x)g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top - k\frac{\partial \mathcal{L}_c}{\partial \xi_c}\frac{\partial \mathcal{L}_c}{\partial \xi_c}^\top \geq 0, \tag{3.30}$$

for some $\epsilon \in (0, \frac{1}{2})$ and $k_l < k \leq k_u$, then the pair $\mathcal{V}_c = (\mathcal{D}_c,\, \mathcal{L}_c)$ with $\mathcal{L}_c$ as in (3.28) and $\mathcal{D}_c$ as in (3.29) is a Dynamic Generalized controllability function for system (3.1) for all $k \in (k_l,\, k_u]$ and all $(x,\, \xi_c) \in \Omega_c$. $\diamond$

Proof. To begin with note that the selection

$$\phi_c(x,\, \xi_c) = -k\frac{\partial \mathcal{L}_c}{\partial \xi_c}(x,\, \xi_c)^\top \tag{3.31}$$

is such that the inequality (3.8) yields

$$\frac{\partial \mathcal{L}_c}{\partial x}f(x) + \frac{1}{2}\frac{\partial \mathcal{L}_c}{\partial x}g(x)g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top - k\frac{1}{2}\frac{\partial \mathcal{L}_c}{\partial \xi_c}\frac{\partial V_c}{\partial \xi_c}^\top \leq 0.$$

By arguments similar to those in the proofs of Lemma 2.2 and Proposition 2.2, there exist

$k_l \geq 0$ and a set $\Omega_c \subset \mathbb{R}^n \times \mathbb{R}^n$ such that $\mathcal{L}_c$ satisfies the partial differential inequality (3.8) for all $k > k_l$ and for all $(x,\, \xi_c) \in \Omega_c$. To prove the second claim it remains to show that the point $(x,\, \xi_c) = (0, 0)$ is an asymptotically stable equilibrium point of the system (3.9), with ϕ_c as in (3.31). Towards this end consider $\mathcal{L}_c$ as in (3.28) and note that, by Lemma 1.2, there exist a compact set $\mathcal{W} \subseteq \mathbb{R}^n \times \mathbb{R}^n$ containing the origin and a positive definite matrix $\bar{R}$ such that for all $R_c > \bar{R}$ the function $\mathcal{L}_c$ is positive definite for all $(x, \xi_c) \in \mathcal{W}$. Then suppose that $\mathcal{L}_c$ is a *candidate* Lyapunov function for the system (3.9). The time derivative of $\mathcal{L}_c$ along the trajectories of (3.9) is

$$\dot{\mathcal{L}}_c = -\frac{\partial \mathcal{L}_c}{\partial x} f(x) + k \frac{\partial \mathcal{L}_c}{\partial \xi_c} \frac{\partial \mathcal{L}_c}{\partial \xi_c}^\top - \frac{\partial \mathcal{L}_c}{\partial x} g(x) g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top \leq -\epsilon \frac{\partial \mathcal{L}_c}{\partial x} g(x) g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top \leq 0, \tag{3.32}$$

where the first inequality is obtained by equation (3.30). Consider now the set

$$U_1 \triangleq \{(x,\, \xi_c) \in \Omega_c : g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top = 0\},$$

and the restriction of (3.9) to U_1. Then finding the maximal invariant set of (3.9) in U_1 is equivalent, by (3.32), to determining the maximal invariant set in which the function $\mathcal{L}_c(x(t),\, \xi_c(t))$ is non-decreasing for all $t \geq 0$ along the trajectories of the system

$$\begin{aligned} \dot{x} &= f(x)\,, \\ \dot{\xi}_c &= -k \frac{\partial \mathcal{L}_c}{\partial \xi_c}\,. \end{aligned} \tag{3.33}$$

Since the function $\mathcal{L}_c$ is positive definite and the zero equilibrium of system (3.33) is asymptotically stable – as can be shown using arguments similar to those in the proof of Proposition 3.2 – the set coincides with the point $(x,\, \xi_c) = (0, 0)$. Then, since the maximal invariant set in U_1 reduces to the origin and U_1 contains the set $U_2 \triangleq \{(x,\, \xi_c) \in \Omega_c : \dot{\mathcal{L}}_c = 0\}$, asymptotic stability of the zero equilibrium of the system (3.9) can be concluded by LaSalle's invariance principle. □

The following remark provides an interpretation of the condition (3.30) required to conclude asymptotic stability of the zero equilibrium of the system (3.9).

Remark 3.3. Suppose that $\mathcal{L}_c$ is a solution of the partial differential inequality (3.8). Then there exists some non-negative smooth function $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ such that

$$\frac{\partial \mathcal{L}_c}{\partial x} f(x) + \frac{1}{2} \frac{\partial \mathcal{L}_c}{\partial x} g(x) g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top - k \frac{\partial \mathcal{L}_c}{\partial \xi_c} \frac{\partial \mathcal{L}_c}{\partial \xi_c}^\top + c(x,\, \xi_c) = 0. \tag{3.34}$$

The condition (3.30) is equivalent to requiring that the *additional negativity* of the partial differential inequality (3.8) with respect to the corresponding equation, namely the function c, is bounded as

$$c(x, \xi_c) < \frac{1}{2}\frac{\partial \mathcal{L}_c}{\partial x} g(x)g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top ,$$

for all $(x, \xi_c) \in \Omega_c \setminus \{0\}$. ▲

Remark 3.4. The gain k in the ordinary differential equation (3.29) may be a function of x and ξ_c. ▲

As detailed in Proposition 3.3, all admissible trajectories of system (3.1) that satisfy the end-point constraints – obtained for instance by varying the gain k of the dynamics of ξ_c – yield the same value of the integral (3.20), hence of the minimum energy to steer the state from $x = 0$ to $x = \bar{x}$. Nevertheless we may *select* one of these trajectories according to some desired criterion. In the following we show how to minimize the term c, the interpretation of which is given in (3.34).

To streamline the presentation of the result define, for $\varepsilon > 0$, the open set

$$\mathcal{M}_\varepsilon \triangleq \{(x, \xi_c) \in \mathbb{R}^n \times \mathbb{R}^n : \left\| \frac{\partial P}{\partial \xi_c}(\xi_c)^\top x - R_c(x - \xi_c) \right\| < \varepsilon\},$$

and consider the continuous function

$$sat_\delta(x) = \begin{cases} x, & |x| \le \delta \\ sign(x)\delta, & |x| > \delta \end{cases} \tag{3.35}$$

with $x \in \mathbb{R}$ and $\delta > 0$.

Proposition 3.6. Consider the nonlinear system (3.1). Suppose that P is a solution of (3.26) such that (3.27) holds and that (3.30) holds for all $k_l < k \le k_u$. Consider the function $\mathcal{L}_c$ as in (3.28) with ξ_c obtained as the solution of (3.29) for some constant $k_l < \hat{k} \le k_u$. Let $\varepsilon > 0$ and $k(x, \xi_c)$ in (3.29) be defined as

$$k(x, \xi_c) = sat_{\hat{k}}\left(m_c(x, \xi_c)\right) , \tag{3.36}$$

with

$$m_c(x, \xi_c) \triangleq \left(\frac{\partial \mathcal{L}_c}{\partial \xi_c} \frac{\partial \mathcal{L}_c}{\partial \xi_c}^\top \right)^{-1} \left(\frac{\partial \mathcal{L}_c}{\partial x} f(x) + \frac{1}{2}\frac{\partial \mathcal{L}_c}{\partial x} g(x)g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top \right). \tag{3.37}$$

Then the Dynamic Generalized controllability function[1] $\mathcal{V}_c^\varepsilon = (\mathcal{D}_c^\varepsilon, \mathcal{L}_c^\varepsilon)$ is such that

$$\begin{cases} c^\varepsilon(x, \xi_c) &= 0 & (x, \xi_c) \in \Omega_c \setminus \mathcal{M}_\varepsilon, \\ c^\varepsilon(x, \xi_c) &\leq c(x, \xi_c) & (x, \xi_c) \in \Omega_c \cap \mathcal{M}_\varepsilon. \end{cases} \tag{3.38}$$

◇

Proof. The claim is proved in two steps. First it is shown that the function m_c is not saturated for all $(x, \xi_c) \in \Omega_c \setminus \mathcal{M}_\varepsilon$, then that (3.38) holds.

To begin with, rewrite the function (3.36) as

$$k(x, \xi_c) = \begin{cases} m_c(x, \xi_c), & (x, \xi_c) \in \Omega_c \setminus \mathcal{M}_\varepsilon \\ \\ sat_{\hat{k}}\left(m(x, \xi_c)\right), & (x, \xi_c) \in \Omega_c \cap \mathcal{M}_\varepsilon \end{cases} \tag{3.39}$$

and note that the function m_c is smaller than or equal to $\hat{k}$ for all $(x, \xi_c) \in \Omega_c \setminus \mathcal{M}_\varepsilon$. In fact in the latter set $\mathcal{L}_c$ is such that

$$\left(\frac{\partial \mathcal{L}_c}{\partial \xi_c} \frac{\partial \mathcal{L}_c}{\partial \xi_c}^\top\right) \geq \varepsilon^2,$$

and

$$\frac{\partial \mathcal{L}_c}{\partial x} f(x) - m_c \frac{\partial \mathcal{L}_c}{\partial \xi_c} \frac{\partial \mathcal{L}_c}{\partial \xi_c}^\top + \frac{1}{2} \frac{\partial \mathcal{L}_c}{\partial x} g(x) g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top = 0, \tag{3.40}$$

where the equality is obtained by substitution of the function m_c as in (3.37). Moreover, by Proposition 3.5,

$$\frac{\partial \mathcal{L}_c}{\partial x} f(x) - \hat{k} \frac{\partial \mathcal{L}_c}{\partial \xi_c} \frac{\partial \mathcal{L}_c}{\partial \xi_c}^\top + \frac{1}{2} \frac{\partial \mathcal{L}_c}{\partial x} g(x) g(x)^\top \frac{\partial \mathcal{L}_c}{\partial x}^\top \leq 0, \tag{3.41}$$

for all $(x, \xi_c) \in \Omega_c$. Therefore, subtracting (3.41) and (3.40) yields in $\Omega_c \setminus \mathcal{M}_\varepsilon$

$$-(\hat{k} - m_c(x, \xi_c)) \frac{\partial \mathcal{L}_c}{\partial \xi_c} \frac{\partial \mathcal{L}_c}{\partial \xi_c}^\top \leq -(\hat{k} - m_c(x, \xi_c))\varepsilon^2 \leq 0\,,$$

hence $m(x, \xi_c) \leq \hat{k}$ and $k(x, \xi_c)$ is not saturated outside the set $\Omega_c \cap \mathcal{M}_\varepsilon$.

To show that (3.38) holds note that, by (3.40), the function $\mathcal{L}_c$ satisfies the partial differential inequality (3.8) with the equality sign for all $(x, \xi_c) \in \Omega_c \setminus \mathcal{M}_\varepsilon$, hence $c^\varepsilon = 0$. Moreover, by the definition of $k(x, \xi_c)$ as in (3.36), $k(x, \xi_c) \leq \hat{k}$ for all $(x, \xi) \in \Omega_c \cap \mathcal{M}_\varepsilon$. Finally, the proof is concluded noting that since $c^\varepsilon(x, \xi_c) \leq c(x, \xi_c)$ for all $(x, \xi_c) \in \Omega_c$, then the inequality (3.30) is satisfied also by $\mathcal{L}_c^\varepsilon$, hence $\mathcal{V}_c^\varepsilon = (\mathcal{D}_c^\varepsilon, \mathcal{L}_c^\varepsilon)$ is indeed a Dynamic

[1]The notation $(\mathcal{D}_c^\varepsilon, \mathcal{L}_c^\varepsilon)$ and $c^\varepsilon(x, \xi_c)$ describes the vector field of the differential equation (3.31) and the function $\mathcal{L}_c(x, \xi_c)$ in (3.28) with ξ_c obtained as the solution of (3.29)-(3.36) and the corresponding additional negativity defined as in (3.34), respectively.

Generalized controllability function. $\square$

3.3.2 Observability

In this section the results presented above for Dynamic Generalized controllability functions are developed also for Dynamic Generalized observability functions.

Proposition 3.7. Consider the nonlinear system (3.1) and let $\Gamma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, with $\Gamma(0) > 0$, $x^\top \Gamma(x)x \geq 0$, for all $x \in \mathbb{R}^n$. Let $Q : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, $Q(0) = 0$, be such that

$$Q(x)f(x) + \frac{1}{2}h(x)^\top h(x) + x^\top \Gamma(x)x \leq 0, \tag{3.42}$$

with

$$\frac{\partial Q}{\partial x}(x)\Big|_{x=0} = \bar{Q}\,, \tag{3.43}$$

where $\bar{Q} = \bar{Q}^\top$ is the solution of $A^\top \bar{Q} + \bar{Q}A + C^\top C + \Gamma(0) = 0$. Consider the function

$$\mathcal{L}_o(x,\, \xi_o) = Q(\xi_o)x + \frac{1}{2}\|x - \xi_o\|^2_{R_o}. \tag{3.44}$$

Then there exist $k_o \geq 0$ and a set $\Omega_o \subset \mathbb{R}^n \times \mathbb{R}^n$ such that the pair $\mathcal{V}_o = (\mathcal{D}_o,\, \mathcal{L}_o)$ with $\mathcal{L}_o$ as in (3.44) and $\mathcal{D}_o$ defined by

$$\dot{\xi}_o = -k\frac{\partial V_o}{\partial \xi_o}^\top = -k\left(\frac{\partial Q}{\partial \xi_o}(\xi_o)^\top x - R_o(x - \xi_o)\right), \tag{3.45}$$

is a Dynamic Generalized Observability function of system (3.1) for all $k > k_o$ and all $(x,\, \xi_o) \in \Omega_o$. $\diamond$

Proof. To begin with note that the partial derivatives of the function $\mathcal{L}_o$ with respect to x and ξ_o are

$$\begin{aligned} \frac{\partial \mathcal{L}_o}{\partial x} &= Q(x) + (x - \xi_o)^\top (R_o - \Phi(x,\, \xi_o))^\top\,, \\ \frac{\partial \mathcal{L}_o}{\partial \xi_o} &= x^\top Q_{\xi_o}(\xi_o) - (x - \xi_o)^\top R_o\,, \end{aligned} \tag{3.46}$$

where the matrix-valued function $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is such that

$$Q(x) - Q(\xi_o) = (x - \xi_o)^\top \Phi(x,\, \xi_o)^\top.$$

Therefore the partial differential inequality (3.11) reduces to

$$
\begin{aligned}
&Q(x)f(x) + (x-\xi_o)^\top (R_o - \Phi(x,\,\xi_o))^\top f(x) + \frac{\partial \mathcal{L}_o}{\partial \xi_o}\phi_o(x,\,\xi_o) + \frac{1}{2}h(x)^\top h(x) \\
&\leq -x^\top \Gamma(x) x + x^\top F(x)(R_o - \Phi(x,\,\xi_o))(x-\xi_o) - k\frac{\partial \mathcal{L}_o}{\partial \xi_o}\frac{\partial \mathcal{L}_o}{\partial \xi_o}^\top \\
&= -[x^\top\ (x-\xi)^\top]\left[M + kCC^\top\right][x^\top\ (x-\xi)^\top]^\top \leq 0
\end{aligned}
$$

with

$$
M = \begin{bmatrix} \Gamma & -\frac{1}{2}F^\top(R_o - \Phi) \\ -\frac{1}{2}(R_o-\Phi)^\top F & 0 \end{bmatrix}, \tag{3.47}
$$

and

$$
C = \begin{bmatrix} \frac{\partial Q}{\partial \xi_o}^\top & -R_o \end{bmatrix}.
$$

Arguments similar to those in the proofs of Lemma 2.2 and Proposition 2.2 imply that there exist $k_o \geq 0$ and a set $\Omega_o \subseteq \mathbb{R}^n \times \mathbb{R}^n$ such that the function $\mathcal{L}_o$ as in (3.3) satisfies the partial differential inequality (3.11) for all $(x,\,\xi_o) \in \Omega_o$ and for all $k > k_o$. Finally to conclude the proof note that $\phi_o(0,\,\xi_o) = -kR_o\xi_o$ hence the asymptotic stability assumption is satisfied.

□

Similarly to the controllability case, the following statement provides conditions that allow to minimize the *additional output*, denoted by $d(x,\,\xi_o) \geq 0$, for all $(x,\,\xi_o) \in \Omega_o$, implicitly introduced in (3.11), namely

$$
d(x,\,\xi_o) \triangleq -\frac{\partial \mathcal{L}_o}{\partial x}f(x) + k\frac{\partial \mathcal{L}_o}{\partial \xi_o}\frac{\partial \mathcal{L}_o}{\partial \xi_o}^\top - \frac{1}{2}h(x)^\top h(x)\,. \tag{3.48}
$$

To streamline the presentation of the result define, for $\varepsilon > 0$, the open set

$$
\mathcal{N}_\varepsilon \triangleq \{(x,\,\xi_o) \in \mathbb{R}^n \times \mathbb{R}^n : \left\| \frac{\partial Q}{\partial \xi_o}(\xi_o)^\top x - R_o(x-\xi_o) \right\| < \varepsilon\}.
$$

Proposition 3.8. Consider the nonlinear system (3.1). Suppose that Q is a solution of (3.42) such that (3.43) holds and consider the function $\mathcal{L}_o$ as in (3.44) with ξ_o obtained as the solution of (3.45) for some constant $\hat{k} > k_o$. Let $\varepsilon > 0$ and $k(x,\,\xi_o)$ in (3.45) be defined as

$$
k(x,\,\xi_o) = sat_{\hat{k}}\left(m_o(x,\,\xi_o)\right)\,, \tag{3.49}
$$

with

$$
m_o(x,\,\xi_o) \triangleq \left(\frac{\partial \mathcal{L}_o}{\partial \xi_o}\frac{\partial \mathcal{L}_o}{\partial \xi_o}^\top\right)^{-1}\left(\frac{\partial \mathcal{L}_o}{\partial x}f(x) + \frac{1}{2}h(x)^\top h(x)\right).
$$

Then the Dynamic Generalized observability function $\mathcal{V}_o^\varepsilon = (\mathcal{D}_o^\varepsilon,\, \mathcal{L}_o^\varepsilon)$ is such that

$$\begin{cases} d^\varepsilon(x,\,\xi_o) &= 0 & (x,\,\xi_o) \in \Omega_o \setminus \mathcal{N}_\varepsilon, \\ d^\varepsilon(x,\,\xi_o) &\leq d(x,\,\xi_o) & (x,\,\xi_o) \in \Omega_o \cap \mathcal{N}_\varepsilon. \end{cases} \tag{3.50}$$

◇

3.3.3 Example

Consider the scalar, single-input, single-output, nonlinear system described by the equations

$$\begin{aligned} \dot{x} &= -\tfrac{1}{2}x + (1+x^2)u, \\ y &= \sin(x), \end{aligned} \tag{3.51}$$

with $x(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$. The controllability and observability functions of the system (3.51) can be explicitly computed allowing for a direct comparison with the lower and upper bounds provided by (3.28) and (3.44), respectively. Alternative bounds are also given in [33]. To begin with, let $x^\top \Sigma(x) x = \mu_c x^2$ and $x^\top \Gamma(x) x = \mu_o x^2$, $0 < \mu_c < 0.25$, $\mu_o > 0$ and note that

$$\bar{P} = \frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\mu_c}$$

and $\bar{Q} = 1 + \mu_o$. Then, let

$$\begin{aligned} P(x) &= \frac{\bar{P}x}{1+\gamma x^2}, \\ Q(x) &= \mu_o x + \arctan(x), \end{aligned} \tag{3.52}$$

be algebraic solutions of the inequality (3.26) and (3.42), respectively, for all $x \in [-1, 1]$. Define then the functions

$$\begin{aligned} \mathcal{L}_c &= \frac{\bar{P}\xi x}{1+\gamma \xi^2} + \tfrac{1}{2} r_c (x - \xi_c)^2 \\ \mathcal{L}_o &= \left(\mu_o \xi_o + \arctan(\xi_o)\right) x + \tfrac{1}{2} r_o (x - \xi_o)^2. \end{aligned} \tag{3.53}$$

To obtain a tighter bound on the observability function the initial condition ξ_o is selected, for each state $x \in \mathbb{R}$, according to (3.18), since the minimizer (3.18) can be computed analytically for the function $\mathcal{L}_o$. In the case of the controllability function the initial condition ξ_c is selected, for each state $x \in \mathbb{R}$, as $\xi_c = \lambda x$, $\lambda > 0$, since the analytic expression of the maximizer (3.17) is not available.

Let $\mu_c = \mu_o = 0.01$, $r_c = 0.6$, $r_o = 1$ and $\gamma = 10$. Figure 3.1 displays the comparison between the controllability function of the system (3.51), the bound proposed in [33] and the function $\mathcal{L}_c(x, \lambda x)$, with $\lambda = 0.44$. Note that the function $\mathcal{L}_c(x, \lambda x)$ provides a tighter lower-bound than the function in [33] and the zero equilibrium of the system (3.9) is,

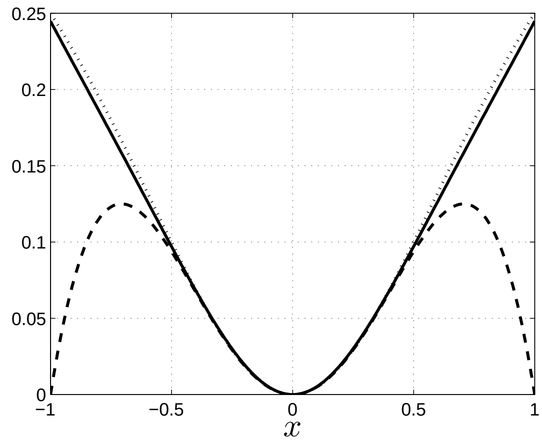

Figure 3.1: Comparison between the controllability function of the system (3.51) (dotted line), the bound proposed in [33] (dashed line) and the function $\mathcal{L}_c(x, \xi_c^*(x))$ (solid line).

similarly to the corresponding system in [33], locally asymptotically stable. Figure 3.2 shows the comparison between the observability function of the system (3.51), the bound proposed in [33] and the function $\mathcal{L}_o(x, \xi_o^*(x))$. The latter provides a tighter upper-bound than the approximation in [33].

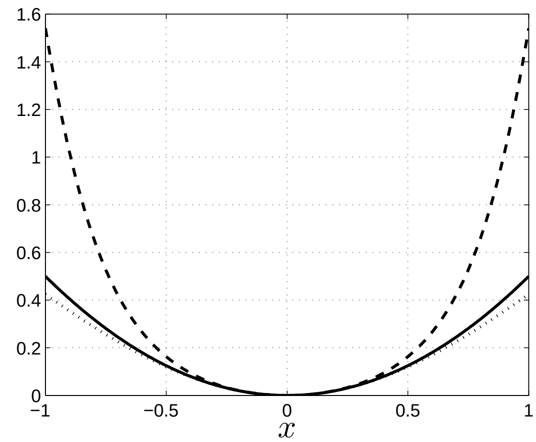

Figure 3.2: Comparison between the observability function of the system (3.51) (dotted line), the bound proposed in [33] (dashed line) and the function $\mathcal{L}_o(x, \xi_o^*(x))$ (solid line).

3.4 Application to Model Reduction

Once the controllability and the observability functions have been obtained, standard model reduction techniques for nonlinear systems rely upon the definition of a sequence of changes of coordinates such that, in the new coordinates, the functions are in the so-called *balanced form.* Often these transformations are not constructive but exploit, for instance, Morse's Lemma.

The main advantage of the notion of Dynamic Generalized controllability and observability functions over controllability and observability functions in balancing for nonlinear systems consists in the fact that these permit the construction of a class of Dynamic Generalized functions. In fact – differently from the balancing technique proposed in [98], in which the existence of a change of coordinates is implied by Morse's Lemma while no constructive result is provided – using Dynamic Generalized functions a specific change of coordinates is given which transforms the functions into the desired structure, as detailed in the following[2].

Proposition 3.9. Suppose that $\mathcal{V}_c = (\mathcal{D}_c, \mathcal{L}_c)$ and $\mathcal{V}_o = (\mathcal{D}_o, \mathcal{L}_o)$ are the Dynamic Generalized controllability and observability functions as in Propositions 3.5 and 3.7, respectively, of the system (3.1). Let

$$H_o(\xi_o) \triangleq N_o(\xi_o) + N_o(\xi_o)^\top - N_o(\xi_o) R_o^{-1} N_o(\xi_o)^\top, \tag{3.54}$$

with $N_o : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $Q(\xi_o)^\top = N_o(\xi_o)\ \xi_o$. Then there exist a matrix $\hat{R} = \hat{R}^\top > 0$ and a neighborhood $\mathcal{W}$ of the origin of $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ in which the change of coordinates $\zeta \triangleq (\hat{x}, \hat{\xi}_o, \hat{\xi}_c) = \Lambda(x, \xi_o, \xi_c)$ defined by

$$\begin{aligned} \hat{x} &= H_o(\xi_o)^{\frac{1}{2}} x, \\ \hat{\xi}_o &= R_o^{\frac{1}{2}} \left(\xi_o - (I - R_o^{-1} N_o(\xi_o))^\top x\right), \\ \hat{\xi}_c &= \xi_c, \end{aligned} \tag{3.55}$$

is such that

$$\mathcal{L}_o(\Lambda^{-1}(\zeta)) = \frac{1}{2}\hat{x}^\top \hat{x} + \frac{1}{2}\hat{\xi}_o^\top \hat{\xi}_o \triangleq \frac{1}{2}\zeta^\top J \zeta, \tag{3.56}$$

$$\mathcal{L}_c(\Lambda^{-1}(\zeta)) = \frac{1}{2}\zeta^\top M_c(\zeta)\zeta\,, \tag{3.57}$$

for all $R_o \geq \hat{R}$, with

$$J = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and $M_c : \mathbb{R}^{n\times n\times n} \to \mathbb{R}^{3n\times 3n}$ a positive semi-definite matrix-valued function. ⋄

Proof. By definition of Q, $N(0)$ is equal to $\bar{Q}$, hence the constant terms of the matrix $H_o(\xi_o)$ are defined by

$$\bar{H} \triangleq \bar{Q} + \bar{Q}^\top - \bar{Q} R_o^{-1} \bar{Q}.$$

[2] The results that follow can be similarly obtained exchanging the roles of the Dynamic Generalized controllability and observability functions.

Therefore, selecting R_o *sufficiently* large, there exists a neighborhood of the origin in which the matrix $H(\xi_o)$ is positive definite, hence the matrix $H(\xi_o)^{\frac{1}{2}}$ is locally well-defined. To show that Λ is a diffeomorphism note that $x = H_o(\xi_o)^{-\frac{1}{2}}\hat{x}$ and consider the equation

$$\hat{\xi}_o - R_o^{-\frac{1}{2}}\pi(\hat{x},\, \xi_o) = 0, \tag{3.58}$$

which corresponds to the second equation of (3.55), with $\pi(\hat{x},\, \xi_o) = R_o\xi_o + \left(N(\xi_o)^\top - R_o\right) H_o(\xi_o)^{-\frac{1}{2}}\hat{x}$. The partial derivative of π with respect to ξ_o is given by

$$\frac{\partial \pi}{\partial \xi_o}(\hat{x},\, \xi_o) = R_o + \frac{\partial\left[\left(N(\xi_o)^\top - R_o\right) H_o(\xi_o)^{-\frac{1}{2}}\hat{x}\right]}{\partial \xi_o}. \tag{3.59}$$

Thus, since by definition of the mapping N

$$\left.\frac{\partial\left[\left(N(\xi_o)^\top - R_o\right) H_o(\xi_o)^{-\frac{1}{2}}\hat{x}\right]}{\partial \xi_o}\right|_{(\hat{x},\, \xi_o)=(0,0)} = 0,$$

there exists a neighborhood $\mathcal{W}$ of the origin in which the rank of the partial derivative (3.59) is equal to n. Then, by the Implicit Function theorem, there exists a unique continuously differentiable mapping $\chi\ :\ \mathbb{R}^{n\times n}\ \to\ \mathbb{R}^n$ such that

$$\hat{\xi}_o - R_o^{-\frac{1}{2}}\pi(\hat{x},\, \chi(\hat{x},\, \hat{\xi}_o)) = 0\,,$$

for all $(\hat{x},\, \hat{\xi}_o) \in \mathcal{W}$. Therefore the inverse change of coordinates Λ^{-1} is defined as

$$\begin{aligned} x &= H_o(\chi(\hat{x},\, \hat{\xi}_o))^{-\frac{1}{2}}\hat{x}, \\ \xi_o &= \chi(\hat{x},\, \hat{\xi}_o), \\ \xi_c &= \hat{\xi}_c\,. \end{aligned} \tag{3.60}$$

Finally note that, once we have shown that Λ is indeed a change of coordinates, $\mathcal{L}_o(\Lambda^{-1}(\zeta))$ can be obtained by direct substitution of

$$\begin{aligned} x &= H_o(\xi_o)^{-\frac{1}{2}}\hat{x}, \\ \xi_o &= R_o^{-\frac{1}{2}}\hat{\xi}_o + R_o^{-1}(R_o - N_o(\xi_o))H_o(\xi_o)^{-\frac{1}{2}}\hat{x}, \\ \xi_c &= \hat{\xi}_c, \end{aligned} \tag{3.61}$$

which is obtained from (3.55), into (3.44), yielding the function (3.56), while the expression of the function (3.57) is obtained noting that in the new coordinates the function $\mathcal{L}_c(\Lambda^{-1}(\zeta))$ is positive semi-definite. □

Interestingly the equations (3.61) entail that the explicit expression of the function χ is not required to compute $\mathcal{L}_o(\Lambda^{-1}(\zeta))$.

Remark 3.5. By the definition of the inverse coordinate transformation Λ^{-1}, the function $\mathcal{L}_c(\Lambda^{-1}(\zeta))$ depends explicitly also on $\hat{\xi}_o$, differently from $\mathcal{L}_c(x, \xi_c)$. ▲

In the following the notion of *dynamically balanced* nonlinear system is introduced. Towards this end consider the following assumption together with some preliminary results. (Note that the assumptions and the notation are partly borrowed from [98].)

Assumption 3.3. There exists a neighborhood of the origin where the number of distinct eigenvalues of $M_c(\zeta)$ is constant.

Assumption 3.3 implies that the eigenvalues of $M_c(\zeta)$, denoted by $\lambda_i(\zeta)$ for $i = 1, ..., 3n$, are smooth functions of ζ. Note that, since $M_c(\zeta)$ is only positive semi-definite, n eigenvalues are identically equal to zero. Moreover, since the matrix $M_c(0) \geq 0$ is diagonalizable, provided $R_c > \frac{1}{2}\bar{P}$, we can find an orthogonal matrix $T : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{3n\times 3n}$, *i.e.* such that $T(\zeta)^\top T(\zeta) = I$, such that $M_c(\zeta) = T(\zeta)^\top \Delta(\zeta) T(\zeta)$ with $\Delta(\zeta) = \mathrm{diag}(\lambda_1(\zeta), ..., \lambda_{2n}(\zeta), 0, ..., 0)$ and

$$\lambda_1(\zeta) \geq ... \geq \lambda_{2n}(\zeta) > 0\,.$$

Finally, define the new variables $z = \nu(\zeta) \triangleq T(\zeta)\zeta$, $z \in \mathbb{R}^{3n}$, and

$$\bar{z}_i = \begin{cases} \eta_i(z_i) \triangleq \tau_i(0, ..., 0, z_i, 0, ..., 0)^{\frac{1}{4}} z_i, & i = 1, ..., 2n\,, \\ \eta_i(z_i) \triangleq z_i, & i = 2n+1, ..., 3n\,, \end{cases}$$

where the smooth functions $\tau_i(z) \triangleq \lambda(\nu^{-1}(z)) > 0$, $i = 1, ..., 2n$ are called the *dynamic singular value functions.*

Definition 3.4. The nonlinear system (3.1) is dynamically balanced if the functions $\mathcal{L}_o$ and $\mathcal{L}_c$ in the Dynamic Generalized observability and controllability functions, respectively, are such that

$$\mathcal{L}_o(\rho) \triangleq \frac{1}{2}\rho^\top \begin{bmatrix} \mathrm{diag}(\sigma_i(\rho_i)^{-1}) & 0 \\ 0 & 0 \end{bmatrix} \rho, \tag{3.62}$$

and

$$\mathcal{L}_c(\rho) \triangleq \frac{1}{2}\rho^\top \begin{bmatrix} \mathrm{diag}(\sigma_i(\rho_i)^{-1}\tau_i(\rho)) & 0 \\ 0 & 0 \end{bmatrix} \rho, \tag{3.63}$$

with $\rho \in \mathbb{R}^{3n}$ and $\sigma_i(\rho) \triangleq \tau_i(0, ..., \rho_i, 0, ..., 0)^{\frac{1}{2}}$, $i = 1, ..., 2n$, namely such that

$$\mathcal{L}_o(\rho_i) = \begin{cases} \frac{1}{2}\rho_i^2 \sigma_i(\rho_i)^{-1} & if\ \ i \leq 2n\,, \\ 0 & if\ \ i > 2n\,, \end{cases}$$

and

$$\mathcal{L}_c(\rho_i) = \begin{cases} \frac{1}{2}\rho_i^2\sigma_i(\rho_i) & if \;\; i \leq 2n\,, \\ 0 & if \;\; i > 2n\,. \end{cases}$$

The following result summarizes the previous considerations stating that the nonlinear system (3.1) can be dynamically balanced by means of a coordinate transformation.

Proposition 3.10. Consider the nonlinear system (3.1). Let $\mathcal{V}_c = (\mathcal{D}_c, \mathcal{L}_c)$ and $\mathcal{V}_o = (\mathcal{D}_o, \mathcal{L}_o)$ be the Dynamic Generalized controllability and observability functions defined in Propositions 3.5 and 3.7, respectively, and define the change of coordinates $\bar{z} = \eta(\nu(\Lambda(x, \xi_o, \xi_c)))$. Then, in the new coordinates, the system (3.1) is *dynamically balanced.* $\diamond$

3.5 Conclusions

The balancing and model reduction problems for nonlinear systems are considered introducing the notion of Dynamic Generalized Controllability and Observability functions. The latter are called dynamic and generalized since a dynamic extension to the state of the state is considered and a partial differential inequality is solved in place of a partial differential equation. As a matter of fact a class of Dynamic Generalized Controllability and Observability functions can be constructively defined, which is a crucial advantage with respect to Controllability and Observability functions. In fact, exploiting the specific structure of the class of functions, an explicit change of coordinates can be given that transforms the functions into the dynamically balanced form.

Chapter 4

$\mathcal{L}_2$-disturbance Attenuation

4.1 Introduction

In most control applications the internal state or the output of a dynamical system may be affected by disturbances, which are in general unknown. An ideal achievement of the control law would be to render (the output of) the system *insensitive* to the disturbance. However, it has been shown that the conditions under which this goal can be achieved are seldom satisfied, see [39]. To avoid this issue a different approach has been pursued, namely the design of a control law that guarantees that the *effect* of the disturbance on (the output of) the system is kept under a desired level and that an equilibrium of the closed-loop system is (locally) asymptotically stable in the absence of disturbances. Obviously, in order to tackle this problem, the first issue to clarify is how the effect of an unknown disturbance on (the output of) a dynamical system can be measured. A possible choice is to consider the $\mathcal{L}_2$-induced norm of the nonlinear system, see for instance [22, 101]. For linear systems the $\mathcal{L}_2$-disturbance attenuation problem reduces to the time-domain formulation of the H_∞ control problem. The classic solution of the $\mathcal{L}_2$-disturbance attenuation problem is defined in terms of the solution of an Hamilton-Jacobi (HJ) partial differential inequality, which is a nonlinear first-order partial differential inequality.

In this chapter we introduce the notion of Dynamic Storage function and we investigate the relation between these functions and the solution of the $\mathcal{L}_2$-disturbance attenuation problem. Moreover it is shown how to construct dynamically, *i.e.* by means of a dynamic extension, an exact solution of a (modified) HJ inequality, for input-affine nonlinear systems, without solving any partial differential equation or inequality. The conditions are given in terms of algebraic equations or inequalities, that can be shown to be solvable for specific classes of nonlinear systems including feedback linearizable, feedforward and fully actuated mechanical systems.

The rest of the chapter is organized as follows. In Section 4.2 the $\mathcal{L}_2$-disturbance attenuation problem and the concept of Dynamic Storage function are formally introduced and their relation is discussed in details. Section 4.3 is dedicated to the definition of the notion of *algebraic $\bar{P}$ solution*, which is instrumental for the construction of a class of Dynamic Storage functions for input-affine nonlinear systems. The proposed dynamic control law is defined in Section 4.4. In Section 4.4.1 the particularization of these results to linear time-invariant systems is proposed and the relation with the standard solution of the H_∞ disturbance attenuation problem is investigated. A procedure to systematically determine an *algebraic $\bar{P}$ solution* for classes of nonlinear systems in the presence of matched disturbance is proposed in Section 4.5.

4.2 Definition of the Problem

Consider a nonlinear dynamical system affected by unknown disturbances described by equations of the form

$$\begin{aligned} \dot{x} &= f(x) + g(x)u + p(x)d\,, \\ z &= h(x) + l(x)u\,, \end{aligned} \tag{4.1}$$

where the first equation describes a plant with state $x(t) \in \mathbb{R}^n$, control input $u(t) \in \mathbb{R}^m$ and exogenous input $d(t) \in \mathbb{R}^p$, while the second equation defines a regulated output $z(t) \in \mathbb{R}^q$. In addition $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $p : \mathbb{R}^n \to \mathbb{R}^{n\times p}$, $l : \mathbb{R}^n \to \mathbb{R}^{q\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ are smooth mappings.

Assumption 4.1. The mappings h and l in the system (4.1) are such that $h(x)^\top l(x) = 0$ and $l(x)^\top l(x) = I$, for all $x \in \mathbb{R}^n$.

The second condition in the Assumption 4.1 is needed to avoid a so-called *singular problem*, see for instance [4, 57].

Assumption 4.2. The mappings f and h are such that $f(0) = 0$ and $h(0) = 0$. Moreover, the nonlinear system (4.1), with $d(t) = 0$ for all $t \geq 0$ and output $y = h(x)$, is zero-state detectable, *i.e.* $u(t) = 0$ and $y(t) = 0$ for all $t \geq 0$ imply $\lim_{t\to\infty} x(t) = 0$.

As a consequence of Assumption 4.2 there exists some, possibly not unique, continuous matrix-valued function $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $f(x) = F(x)x$, for all x.

Problem 4.1. Consider the system (4.1) and let $\gamma > 0$. The *regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation* problem with stability consists in determining an integer $\tilde{n} \geq 0$, a dynamic control law of the form

$$\begin{aligned} \dot{\xi} &= \alpha(t, x, \xi)\,, \\ u &= \beta(t, x, \xi)\,, \end{aligned} \tag{4.2}$$

with $\xi(t) \in \mathbb{R}^{\tilde{n}}$, $\alpha : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{n}}$, $\beta : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^m$ smooth mappings, $\alpha(t,0,0) = 0$, $\beta(t,0,0) = 0$ for all $t \geq 0$, and a set $\bar{\Omega} \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$, containing the origin of $\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$, such that the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\beta(t,x,\xi) + p(x)d\,, \\ \dot{\xi} &= \alpha(t,x,\xi)\,, \\ z &= h(x) + l(x)\beta(t,x,\xi)\,, \end{aligned} \tag{4.3}$$

has the following properties.

(i) The zero equilibrium of the system (4.3) with $d(t) = 0$, for all $t \geq 0$, is uniformly asymptotically stable with region of attraction containing $\bar{\Omega}$.

(ii) For every $d \in \mathcal{L}_2(0,T)$ such that the trajectories of the system remain in $\bar{\Omega}$, the $\mathcal{L}_2$-gain of the system (4.3) from d to z is less than or equal to γ, *i.e.*

$$\int_0^T \|z(t)\|^2 dt \leq \gamma^2 \int_0^T \|d(t)\|^2 dt\,, \tag{4.4}$$

for all $T \geq 0$.

Note that if $\tilde{n} = 0$ and the control law is time-invariant then the problem is the standard static state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability.

It is well-known, see [101], that there exists a solution to the static state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability, in some neighborhood of the origin, if there exists a smooth positive definite solution $V : \mathbb{R}^n \to \mathbb{R}_+$ of the Hamilton-Jacobi inequality

$$V_x f(x) + \frac{1}{2} V_x \left[\frac{1}{\gamma^2} p(x)p(x)^\top - g(x)g(x)^T \right] V_x^\top + \frac{1}{2} h(x)^\top h(x) \leq 0\,, \tag{4.5}$$

with $V(0) = 0$, for the given $\gamma > 0$. Moreover the control law that solves the problem is given by $u = -g(x)^\top V_x^\top$.

In the linear case the solution of the disturbance attenuation problem is given by a linear static state feedback of the form $u = -B_1^\top \bar{P} x$, where $\bar{P}$ is the symmetric positive definite solution of the algebraic Riccati equation

$$\bar{P}A + A^\top \bar{P} + \bar{P} \left[\frac{1}{\gamma^2} B_2 B_2^\top - B_1 B_1^\top \right] \bar{P} + H^\top H = 0\,, \tag{4.6}$$

where

$$A \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=0} = F(0)\,, \quad H \triangleq \left.\frac{\partial h}{\partial x}\right|_{x=0}\,, \quad B_1 \triangleq g(0)\,, \quad B_2 \triangleq p(0)\,. \tag{4.7}$$

Remark 4.1. As noted in [101] if the pair (H, A) is detectable, the solvability of the equation (4.6) guarantees the existence of a neighborhood $\mathcal{W}$ of the origin and of a smooth positive definite function V defined on $\mathcal{W}$ such that V is a solution of the Hamilton-Jacobi inequality (4.5). However, in most practical cases there is no *a priori* knowledge about the size of the neighborhood $\mathcal{W}$ in which the solution is defined. ▲

The following definition introduces the notion of Dynamic Storage function, which is instrumental in designing a dynamic control law that solves the regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability.

Definition 4.1. (Dynamic Storage Function). Consider the system (4.1) and let $\gamma > 0$. A time-varying *Dynamic Storage function* $\mathcal{V}$ is a pair (D_α, V) defined as follows.

- D_α is the dynamical system

$$\begin{aligned} \dot{\xi} &= \alpha(t, x, \xi)\,, \\ u &= -g(x)^\top V_x(t, x,\, \xi)^\top\,, \end{aligned} \tag{4.8}$$

  with $\xi(t) \in \mathbb{R}^n$, $\alpha : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz, $\alpha(t, 0, 0) = 0$ for all $t \geq 0$. Moreover α is such that the zero equilibrium of $\dot{\xi} = \alpha(t, 0, \xi)$ is asymptotically stable, uniformly in t.

- $V : \mathbb{R} \times \Omega \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is positive definite around $(x, \xi) = (0, 0)$, for all $t \geq 0$, and it is such that

$$V_t + V_x f(x) + V_\xi \alpha(t, x, \xi) + \frac{1}{2} V_x \left[\frac{1}{\gamma^2} p(x)(x)^\top - g(x)g(x)^\top \right] V_x^\top + \frac{1}{2} h(x)^\top h(x) \leq 0\,, \tag{4.9}$$

  for all $(x, \xi) \in \Omega$ and all $t \geq 0$.

The first term of (4.9) *disappears* if the function V does not depend explicitly on time. In the time-invariant case, namely with $V_t = 0$ and $\alpha = \alpha(x, \xi)$, we refer to $\mathcal{V}$ as a Dynamic Storage function.

Remark 4.2. Dynamic Storage functions generalize Dynamic Lyapunov functions introduced in Chapter 2, since the former reduce to the latter for autonomous systems without inputs and disturbances. ▲

The following result relates the notion of Dynamic Storage function to the solution of the *regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation* problem with stability, namely Problem 4.1.

Lemma 4.1. Consider the system (4.1) and let $\gamma > 0$. Suppose that Assumptions 4.1 and 4.2 hold. Let $\mathcal{V} = (D_\alpha, V)$ be a Dynamic Storage function for the system (4.1) for all $(x, \xi) \in \Omega$. Then the dynamical system (4.8), with α and V independent of time, solves the *regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation* problem with stability for all $(x, \xi) \in \bar{\Omega}$, where $\bar{\Omega}$ is the largest level set of V contained in Ω. ⋄

Proof. The claim follows directly noting that, by (4.9), V is a continuously differentiable storage function for the closed-loop system (4.1)-(4.8), with α and V independent of time, with respect to the *supply rate* $s(z, d) = \gamma^{-2}\|d\|^2 - \|z\|^2$. Therefore the $\mathcal{L}_2$-gain from d to z is smaller than γ. Let now $d = 0$ and V be a candidate Lyapunov function for the closed-loop system (4.1)-(4.8). By inequality (4.9) all the trajectories are bounded and, by Assumption 4.2, $x(t)$ converges to zero as time tends to infinity. Moreover, by definition, the zero equilibrium of the system $\dot{\xi} = \alpha(0, \xi)$ is uniformly asymptotically stable. □

As it appears considering the arguments in the proof of Lemma 4.1, the requirement of asymptotic stability for the system $\dot{\xi} = \alpha(t, 0, \xi)$ introduced in the definition of Dynamic Storage function may be replaced by zero-state detectability of the extended closed-loop system

$$
\begin{aligned}
\dot{x} &= f(x) + g(x)\beta(t, x, \xi)\,, \\
\dot{\xi} &= \alpha(t, x, \xi)\,,
\end{aligned}
$$

with respect to the output $y = h(x)$.

4.3 Algebraic $\bar{P}$ Solution of the HJ Partial Differential Inequality

In the following we show how to construct a class of Dynamic Storage functions without involving the solution of any partial differential equation or inequality, which is a significative advantage of Dynamic Storage functions over standard storage functions. Consider the HJ inequality (4.5) and suppose that it can be solved *algebraically*, as detailed in the following definition.

Definition 4.2. Consider the system (4.1), with Assumption 4.1, and let $\gamma > 0$. Let $\sigma(x) \triangleq x^\top \Sigma(x) x \geq 0$, for all $x \in \mathbb{R}^n$, with $\Sigma : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\Sigma(0) = 0$. A $\mathcal{C}^1$ mapping $P : \mathbb{R}^n \to \mathbb{R}^{1\times n}$, $P(0) = 0$, is said to be a *$\mathcal{X}$-algebraic $\bar{P}$ solution* of inequality (4.5) if the following holds.

(i) For all $x \in \mathcal{X} \subseteq \mathbb{R}^n$

$$
P(x)f(x) + \frac{1}{2}P(x)\left[\frac{1}{\gamma^2}p(x)p(x)^\top - g(x)g(x)^\top\right]P(x)^\top + \frac{1}{2}h(x)^\top h(x) + \sigma(x) \leq 0. \tag{4.10}
$$

(ii) P is tangent at $x = 0$ to the symmetric positive definite solution of (4.6), *i.e.* $P_x^\top(x)\Big|_{x=0} = \bar{P}$.

If condition (i) holds for all $x \in \mathbb{R}^n$, *i.e.* $\mathcal{X} = \mathbb{R}^n$, then P is an *algebraic $\bar{P}$ solution.*

Note that any $\mathcal{C}^2$ positive function $\sigma : \mathbb{R}^n \to \mathbb{R}$ can be written as $x^\top \Sigma(x) x$ for some continuous non-unique matrix-valued function Σ. Moreover note that P is not assumed to be a gradient vector. A similar approach is proposed in [54] where the solutions of the Hamilton-Jacobi partial differential inequality are characterized in terms of nonlinear matrix inequalities (NLMI). Therein, however, it is assumed that the solution of the NLMI is a gradient vector.

Remark 4.3. For any given σ the condition (4.10) is an algebraic inequality in n unknowns, namely the components P_i, $i = 1, ..., n$, of the mapping P. Moreover, any solution V of the HJ inequality is such that V_x is a solution of the inequality (4.10). Finally, since $\Sigma(0) = 0$, the solvability of the algebraic Riccati equation (4.6) implies the existence of a $\mathcal{X}$*-algebraic $\bar{P}$ solution* for some non-empty set $\mathcal{X} \subseteq \mathbb{R}^n$. ▲

In what follows we assume the existence of an *algebraic $\bar{P}$ solution*, *i.e.* we assume $\mathcal{X} = \mathbb{R}^n$. Note that all the statements can be modified accordingly if $\mathcal{X} \subset \mathbb{R}^n$.

Using the *algebraic $\bar{P}$ solution* P, define the *extension* $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ as

$$V(x, \xi) = P(\xi)x + \frac{1}{2}\|x - \xi\|_R^2 , \tag{4.11}$$

with $\xi \in \mathbb{R}^n$ and $R = R^\top \in \mathbb{R}^{n \times n}$ positive definite. Note that since P satisfies the assumptions of Lemma 1.2 then there exists $\bar{R} = \bar{R}^\top > 0$ such that V is locally positive definite for all $R > \bar{R}$.

To provide concise statements of the main results, define the continuous matrix-valued function $\Delta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$, with

$$\Delta(x, \xi) = (R - \Phi(x, \xi))\Lambda(\xi)^\top , \tag{4.12}$$

$\Lambda(\xi) = P_\xi(\xi)R^{-1}$, and where $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n}$ is a continuous matrix-valued function such that

$$P(x) - P(\xi) = (x - \xi)^\top \Phi(x, \xi)^\top , \tag{4.13}$$

$A_{cl}(x) = F(x) + \Pi(x)N(x)$ with $N : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ such that $P(x) = x^\top N(x)^\top$. Note that the functions Φ and N always exist since $P(x) - P(\xi) = 0$ for $x = \xi$ and $P(x) = 0$ for $x = 0$, respectively.

4.4 $\mathcal{L}_2$-disturbance Attenuation Problem

The following statement provides a solution to the regional dynamic state feedback $\mathcal{L}_2$ - disturbance attenuation problem with stability, namely Problem 4.1, in terms of the existence of a Dynamic Storage function for system (4.1), the construction of which relies upon an *algebraic $\bar{P}$ solution* of inequality (4.5).

Theorem 4.1. Consider system (4.1), with Assumptions 4.1 and 4.2, and let $\gamma > 0$. Let P be an *algebraic $\bar{P}$ solution* of (4.5). Let the matrix $R = R^\top > 0$ be such that the function V defined in (4.11) is positive definite in a set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ containing the origin and such that

$$\frac{1}{2}A_{cl}(x)^\top \Delta + \frac{1}{2}\Delta^\top A_{cl}(x) + \frac{1}{2}\Delta^\top \Pi(x)\Delta < \Sigma(x)\,, \tag{4.14}$$

for all $(x,\xi) \in \Omega \setminus \{0\}$, with $\Pi(x) = \gamma^{-2}p(x)p(x)^\top - g(x)g(x)^\top$. Then there exists $\bar{k} \geq 0$ such that for all $k > \bar{k}$ the function V satisfies the Hamilton-Jacobi inequality

$$\mathcal{HJ}(x,\xi) \triangleq V_x f(x) + V_\xi \dot{\xi} + \frac{1}{2}h(x)^\top h(x) + \frac{1}{2}V_x\left[\frac{1}{\gamma^2}p(x)p(x)^\top - g(x)g(x)^\top\right]V_x^\top \leq 0\,, \tag{4.15}$$

for all $(x,\xi) \in \Omega$, with $\dot{\xi} = -kV_\xi^\top = -k(P_\xi(\xi)^\top x - R(x-\xi))$. Hence $\mathcal{V} = (D_\alpha,\, V)$, with D_α defined as

$$\begin{aligned} \dot{\xi} &= -k(P_\xi(\xi)^\top x - R(x-\xi))\,, \\ u &= -g(x)^\top\left[P(x)^\top + (R - \Phi(x,\xi))(x-\xi)\right]\,, \end{aligned} \tag{4.16}$$

and V as in (4.11), is a Dynamic Storage function for the system (4.1) and, by Lemma 4.1, (4.16) solves the regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability for all $(x,\xi) \in \bar{\Omega}$, where $\bar{\Omega}$ is the largest level set of V contained in Ω. ◇

Remark 4.4. Interestingly, the result entails that, differently from classical Storage functions, Dynamic Storage functions can be constructed without involving the solution of any partial differential equation or inequality, since they rely on the computation of an *algebraic $\bar{P}$ solution* of (4.5). Moreover the control input u defined in (4.16) contains the *algebraic input*, *i.e.* the term $-g(x)^\top P(x)$, obtained from the solution of the equation (4.10), and a *dynamic compensation* term, *i.e.* $-g(x)^\top(R - \Phi(x,\xi))(x-\xi)$. ▲

Proof. To begin with note that the partial derivatives of the function V defined in (4.11) are

$$\begin{aligned} V_x &= P(x) + P(\xi) - P(x) + (x-\xi)^\top R = P(x) + (x-\xi)^\top(R - \Phi(x,\xi))^\top\,, \\ V_\xi &= x^\top P_\xi(\xi) - (x-\xi)^\top R\,. \end{aligned} \tag{4.17}$$

The Hamilton-Jacobi inequality (4.15), considering the partial derivatives of V as in (4.17), the dynamics of the controller as in (4.16) and recalling that the mapping P is an *algebraic $\bar{P}$ solution* of the inequality (4.10), reads

$$\begin{aligned}&-x^\top\Sigma(x)x+(x-\xi)^\top(R-\Phi)^\top F(x)x+\frac{1}{2}(x-\xi)^\top(R-\Phi)^\top\Pi(x)(R-\Phi)(x-\xi)\\&+x^\top N(x)^\top\Pi(x)(R-\Phi)(x-\xi)-k(x^\top P_\xi(\xi)-(x-\xi)^\top R)(P_\xi(\xi)^\top x-R(x-\xi))\leq 0\,.\end{aligned}\tag{4.18}$$

Rewriting the inequality (4.18) as a *quadratic* form in the variables x and $(x-\xi)$ yields

$$-[x^\top\;(x-\xi)^\top][M(x,\xi)+kC(\xi)^\top C(\xi)]\begin{bmatrix}x\\(x-\xi)\end{bmatrix}\leq 0\,,\tag{4.19}$$

where $C(\xi)=[P_\xi(\xi)^\top\;\;-R]$ is a $n\times 2n$ matrix with constant rank n for all $\xi\in\mathbb{R}^n$, since R is positive definite, and

$$M(x,\xi)=\begin{bmatrix}\Sigma(x) & \Gamma_1(x,\xi)\\ \Gamma_1(x,\xi)^\top & \Gamma_2(x,\xi)\end{bmatrix},$$

where $\Gamma_1(x,\xi)=-\frac{1}{2}A_{cl}(x)^\top(R-\Phi(x,\xi))$ and $\Gamma_2(x,\xi)=-\frac{1}{2}(R-\Phi(x,\xi))^\top\Pi(x)(R-\Phi(x,\xi))$. Note that the null space of $C(\xi)$ is spanned by the columns of the matrix

$$Z(\xi)=\begin{bmatrix}I\\R^{-1}P_\xi(\xi)^\top\end{bmatrix}=\begin{bmatrix}I\\\Lambda(\xi)^\top\end{bmatrix},\tag{4.20}$$

for all $\xi\in\mathbb{R}^n$. Exploiting the results in [3], consider the restriction of the matrix $M(x,\xi)$ to the subspace defined by the columns of the matrix $Z(\xi)\in\mathbb{R}^{2n\times n}$ and note that the inertia of the matrix $Z(\xi)^\top M(x,\xi)Z(\xi)$ does not depend on the choice of the basis for the null space of $C(\xi)$. Finally, condition (4.14) implies that the matrix $Z(\xi)^\top M(x,\xi)Z(\xi)$ is positive definite. This proves that there exists $\bar{k}\geq 0$ such that the Hamilton-Jacobi inequality (4.15) is satisfied for all $(x,\xi)\in\Omega\setminus\{0\}$ and for all $k>\bar{k}$. Furthermore $\mathcal{HJ}(0,0)=0$, hence, by continuity of the mappings in system (4.1), $\mathcal{HJ}(x,\xi)$ is continuous and smaller than or equal to zero for all $(x,\xi)\in\Omega$. Moreover, note that, by the condition (4.15), V is a weak Lyapunov function for the closed-loop system (4.3) with $d(t)=0$ for all $t\geq 0$. In fact, $V(x,\xi)>0$ for all $(x,\xi)\in\Omega\setminus\{0\}$ and $\dot{V}\leq 0$. Moreover the system (4.3), with $d(t)=0$ for all t and α and β defined in (4.16), is zero-state detectable with respect to the output $y=h(x)$. To prove this claim consider system (4.3) and note that $d(t)=0$, $\beta(x(t),\xi(t))=0$, $h(x(t))=0$, for all $t\geq 0$, imply, by Assumption 4.2, that $x(t)$ asymptotically converges to zero while $\xi(t)$ belongs, by (4.15), to the compact set $\{(x,\xi):V(x,\xi)\leq V(x(0),\xi(0))\}$ for all $t\geq 0$. Therefore, since $\xi=0$ is a globally asymptotically stable equilibrium point of the system $\dot{\xi} = \alpha(0, \xi)$, standard arguments on interconnected systems allow to conclude that also $\xi(t)$ tends to zero for t that goes to infinity.

Hence by LaSalle's invariance principle and zero-state detectability, the feedback (4.16) asymptotically stabilizes the zero equilibrium of the closed-loop system. Thus the dynamic control law $\dot{\xi} = -kV_\xi(x, \xi)^\top$, $u = -g(x)^\top V_x(x, \xi)^\top$ solves the regional dynamic $\mathcal{L}_2$-disturbance attenuation problem with stability for the system (4.1). Finally, since the condition (4.4) is satisfied for all the trajectories of system (4.1) driven by the dynamic control law (4.16) that remain in the set Ω, then it is straightforward to conclude that $u(t), y(t) \in \mathcal{L}_2(0, \infty)$ for all such disturbances $d(t) \in \mathcal{L}_2(0, \infty)$. $\square$

Remark 4.5. The vector field $A_{cl}(x)x = (F(x) + \Pi(x)N(x))x$ describes the closed-loop nonlinear system when the *algebraic* feedback control law $u = -g(x)^\top P(x)^\top$ and the *algebraic* worst case disturbance $d = \frac{1}{\gamma^2}p(x)^\top P(x)^\top$ are *implemented.* ▲

Remark 4.6. The conditions (4.10) and (4.14) imply

$$\mathcal{P}(x, \xi)^\top f(x) + \frac{1}{2}h(x)^\top h(x) + \frac{1}{2}\mathcal{P}(x, \xi)^\top \Pi(x)\mathcal{P}(x, \xi) < 0\,, \tag{4.21}$$

for all $x \in \Omega \setminus \{0\}$, where $\mathcal{P}(x, \xi) = P(x) + x^\top \Delta(x, \xi)$, which highlights that inequality (4.10) has to be satisfied *robustly.* Note that (4.21) guarantees that the conditions (4.10) and (4.14) are independent of the choice of the matrices $N(x)$ and $F(x)$ for all $x \in \mathbb{R}^n$. ▲

The following statement provides an alternative solution to the regional $\mathcal{L}_2$ - disturbance attenuation problem with stability obtained by considering a time-varying Dynamic Storage function.

Theorem 4.2. Consider system (4.1), with Assumptions 4.1 and 4.2, and let $\gamma > 0$. Let P be an *algebraic* $\bar{P}$ *solution* of (4.5) and suppose that $\Phi(x, \xi) = \Phi(x, \xi)^\top > \bar{R}$, for all $(x, \xi) \in \Omega$, where $\bar{R}$ is such that the function $\bar{V}(x, \xi) = P(\xi)x + \frac{1}{2}\|x - \xi\|^2_{\bar{R}}$ is positive definite in Ω. Assume additionally that

$$\Lambda(\xi)\dot{\Phi}(x, \xi)\Lambda(\xi)^\top < \Sigma(x)\,, \tag{4.22}$$

for all $(x, \xi) \in \Omega \setminus \{0\}$. Then the control law $u(x) = -g(x)^\top P(x)$ solves the regional dynamic $\mathcal{L}_2$-disturbance attenuation problem with stability. $\diamond$

Remark 4.7. The closed-loop system $\dot{x} = f(x) - g(x)g(x)^\top P(x)$ is independent of the dynamic extension ξ, *i.e.* Theorem 4.2 provides a *static* control law that solves the regional dynamic $\mathcal{L}_2$-disturbance attenuation problem with stability. ▲

Proof. To begin with, define

$$V(t,x,\xi) = P(\xi)x + \frac{1}{2}(x-\xi)^\top R(t)(x-\xi)\,, \tag{4.23}$$

where $R(t) = \Phi(x(t), \xi(t))$, for all $t \geq 0$. Note that, by the assumptions on the mapping Φ, $R(t)$ is such that the function V is uniformly positive definite for all $(t,x,\xi) \in \mathbb{R} \times \Omega$. Consider now the Hamilton-Jacobi partial differential inequality

$$V_t + \mathcal{HJ}(x,\xi) \leq 0\,. \tag{4.24}$$

Following the same steps of the proof of Theorem 4.1, it can be shown that the condition (4.22), similarly to the condition (4.14) of Theorem 4.1, implies that the function V satisfies the HJ inequality (4.24) for all $(t,x,\xi) \in \mathbb{R} \times \Omega$.

Hence the control law

$$\begin{aligned} \dot{\xi} &= -k(P_\xi(\xi)^\top x - R(t)(x-\xi))\,, \\ u &= -g(x)^\top P(x)^\top\,, \end{aligned} \tag{4.25}$$

solves the regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability for all $(x,\xi) \in \Omega$. Finally note that $\mathcal{V} = (D_\alpha, V)$, with $\alpha = -kV_\xi(t,x,\xi)^\top$ and V as in (4.23), is a time-varying Dynamic Storage function for the system (4.1). □

The conditions of Theorem 4.2 can be further relaxed under additional assumptions. To this end, consider the class of *algebraic $\bar{P}$ solutions* yielding a matrix $\Phi(x,\xi)$ such that[1]

$$(x-\xi)^\top(\Phi(x,\xi) - \bar{R})(x-\xi) \geq 0\,, \tag{4.26}$$

for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Condition (4.26) is sufficient to guarantee that the function V in (4.23) is uniformly positive definite in the set Ω and for all $t \geq 0$.

Lemma 4.2. Suppose that $\Phi(x,\xi) = \Phi(x,\xi)^\top$, then

$$\left(\frac{\partial \mathcal{Q}^\top}{\partial x}\right) + \left(\frac{\partial \mathcal{Q}^\top}{\partial x}\right)^\top \geq \bar{R} - \bar{P}\,, \tag{4.27}$$

for all $x \in \mathbb{R}^n$, where $\mathcal{Q}(x) = P(x) - x^\top \bar{P}$, if and only if condition (4.26) holds for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. ◇

Proof. Since $\bar{P} > \bar{R} = \frac{1}{2}\bar{P}$, the condition (4.27) is equivalent to requiring that the mapping $P(x) - x^\top \bar{R}$ is monotone, *i.e.*

$$(P(x) - x^\top \bar{R} - P(\xi) + \xi^\top \bar{R})(x-\xi) \geq 0\,, \tag{4.28}$$

[1] Condition (4.26) is milder than requiring $(\Phi(x,\xi) - \bar{R}) \geq 0$, *i.e.* $v^\top(\Phi(x,\xi) - \bar{R})v \geq 0$ for all $v \in \mathbb{R}^n$.

for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Recalling (4.13),

$$(P(x) - x^\top \bar{R} - P(\xi) + \xi^\top \bar{R}) = (x - \xi)^\top (\Phi(x, \xi) - \bar{R}) \,. \tag{4.29}$$

Necessity follows immediately right multiplying both sides of equation (4.29) by $(x - \xi)$. To show sufficiency, suppose, by contradiction, that there exists $(\bar{x}, \bar{\xi})$ such that (4.26) does not hold. Then

$$0 > (\bar{x} - \bar{\xi})^\top (\Phi(\bar{x}, \bar{\xi}) - \bar{R})(\bar{x} - \bar{\xi}) = (P(\bar{x}) - \bar{x}^\top \bar{R} - P(\bar{\xi}) + \bar{\xi}^\top \bar{R})(\bar{x} - \bar{\xi}) \geq 0 \,,$$

where the last inequality is obtained by monotonicity of $P(x) - x^\top \bar{R}$, hence the hypothesis of existence of $(\bar{x}, \bar{\xi})$ is contradicted. $\square$

We summarize the above discussion in the following result.

Corollary 4.1. Consider system (4.1), with Assumptions 4.1 and 4.2, and let $\gamma > 0$. Let P be an *algebraic $\bar{P}$ solution* of (4.5) and suppose that condition (4.27) holds. Assume that the condition (4.22) is satisfied for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$. If $\bar{V}$ is positive definite and radially unbounded for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$, then the control law $u(x) = -g(x)^\top P(x)$ globally solves the state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability. $\diamond$

Remark 4.8. Selecting $R = \Phi(0, 0) > \bar{R}$ the matrix-valued function Δ is such that $\Delta(0, 0) = 0$, hence, since it is continuous, it is sufficiently small in a neighborhood of the origin. Moreover, assume additionally that $\Sigma(0) = \bar{\Sigma} > 0$ in the definition of *algebraic $\bar{P}$ solution*. Then, by continuity of the left-hand side of inequality (4.14), there exists a non-empty subset $\Omega_2 \subset \mathbb{R}^n \times \mathbb{R}^n$ containing the origin such that the condition (4.14) is satisfied for all $(x, \xi) \in \Omega_2$. Therefore, the *algebraic $\bar{P}$ solution* of (4.10), with $\Sigma(0) = \bar{\Sigma}$, allows to construct a Dynamic Storage function that solves the regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability for all $(x, \xi) \in \Omega_1 \cap \Omega_2$, with Ω_1 defined in Lemma 1.2. $\blacktriangle$

Corollary 4.2. Consider system (4.1), with Assumptions 4.1 and 4.2, and let $\gamma > 0$. Let P be an *algebraic $\bar{P}$ solution* of (4.10) with $\Sigma(0) > 0$. Then, letting $R = \Phi(0, 0)$, there exist a neighborhood of the origin $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ and $\bar{k} \geq 0$ such that for all $k > \bar{k}$ the function V defined in (4.11) is positive definite and satisfies the partial differential inequality (4.15) for all $(x, \xi) \in \Omega$. $\diamond$

4.4.1 Linear Systems

In this section we consider linear time-invariant systems affected by an unknown disturbance described by equations of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 u + B_2 d\,, \\ z &= Hx + Lu\,, \end{aligned} \tag{4.30}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$, $B_2 \in \mathbb{R}^{n\times p}$, $H \in \mathbb{R}^{q\times n}$ are defined in (4.7) and $L = l(0) \in \mathbb{R}^{q\times m}$. The following statement represents a particularization of the results presented in Theorem 4.1 to linear systems. It is interesting to note that the methodology proposed therein yields the standard solution to the $\mathcal{L}_2$-disturbance attenuation for linear time-invariant systems.

Assumption 4.3. The coefficient matrices of system (4.30) are such that $H^\top L = 0$ and $L^\top L \; = I$.

Proposition 4.1. Consider system (4.30), with Assumption 4.3, and let $\gamma > 0$. Suppose that there exists a positive definite matrix $\bar{P} = \bar{P}^\top \in \mathbb{R}^{n\times n}$ such that[2]

$$\bar{P}A + A^\top \bar{P} + \bar{P}\left[\frac{1}{\gamma^2}B_2B_2^\top - B_1B_1^\top\right]\bar{P} + H^\top H \leq 0\,. \tag{4.31}$$

Then there exists R such that the conditions in Theorem 4.1 hold for all $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Moreover the control law (4.16) reduces to

$$\begin{aligned} \dot{\xi} &= -k\bar{P}\xi\,, \\ u &= -B_1^\top \bar{P}x\,. \end{aligned} \tag{4.32}$$

i.e. the standard solution of the $\mathcal{L}_2$-disturbance attenuation problem for linear systems is recovered. ◇

Proof. Select $R = \bar{P}$ and note that in the linear case the function V is quadratic and is given by

$$V(x,\xi) = \frac{1}{2}\left[\begin{array}{cc} x^\top & \xi^\top \end{array}\right]\left(\begin{array}{cc} \bar{P} & 0 \\ 0 & \bar{P} \end{array}\right)\left[\begin{array}{c} x \\ \xi \end{array}\right].$$

To complete the proof it remains to show that the linear inequality equivalent to the condition (4.14) is satisfied. To this end, it is enough to note that, with the selection $R = \bar{P}$, $\Delta(x,\xi) = [I - \bar{P}R^{-1}]\bar{P} = 0$. In addition the left-hand side of the condition (4.14) is

[2] Since $\Sigma(0) = 0$, the equation is the linear version of condition (4.10) in the definition of the *algebraic $\bar{P}$ solution.*

identically equal to zero, hence the matrix M is only positive semi-definite in the null space of $C = [\bar{P} \ \ -R]$. However, $M = 0$ and this guarantees that the condition $Ker(Z^\top MZ) = Ker(MZ) = \mathbb{R}^n$ holds. Thus, by [3], there exists a positive constant $\bar{k}$ such that the left-hand side of the inequality (4.19) is negative semi-definite and singular for all $k \geq \bar{k}$. Finally note that the dynamic control law (4.16) *reduces* to (4.32) yielding the standard static state feedback solution of the disturbance attenuation problem for linear systems. □

4.5 Algebraic $\bar{P}$ Solution for Classes of Nonlinear Systems

In this section a systematic procedure to compute *algebraic $\bar{P}$ solutions* for classes of nonlinear systems with matched disturbance is presented.

4.5.1 Feedback Linearizable Systems with Matched Disturbance

Consider the class of nonlinear systems described by $\dot{x} = f(x) + g(x)(u + d)$, with

$$\begin{aligned} \dot{x}_1 &= x_2 \,, \\ \dot{x}_2 &= x_3 + f_2(x_2) \,, \\ &\ \vdots \\ \dot{x}_{n-1} &= x_n + f_{n-1}(x_2, ..., x_{n-1}) \,, \\ \dot{x}_n &= f_n(x_1, x_2, ..., x_n) + b(x_1, x_2, ..., x_n)(u + d) \,, \end{aligned} \tag{4.33}$$

where $x(t) = (x_1(t), ..., x_n(t)) \in \mathbb{R}^n$, $d(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$. The equations (4.33) describe a class of nonlinear systems in *feedback form* in which the functions $f_2, ..., f_{n-1}$ do not depend on the variable x_1. Suppose that the system (4.33) has an equilibrium at $x = 0$, hence $f_2(0) = f_3(0, 0) = ... = f_n(0, ..., 0) = 0$, and moreover that $b(x) \neq 0$, for all $x \in \mathbb{R}^n$ and let $b(0) = b_1$. Hence $b(x) = b_1 + b_2(x)$, for some continuous function $b_2(x)$ with $b_2(0) = 0$. Since the origin of the state space is an equilibrium point there exist continuous functions $\phi_{i,j}$ and constants $a_{i,j}$, $i = 2, ..., n$, $j = 2, ..., i$ such that the functions $f_2, ..., f_n$ of the system (4.33) can be expressed as

$$\begin{aligned} f_2(x) &= a_{2,2}x_2 + \phi_{2,2}(x)x_2 \,, \\ f_3(x) &= a_{3,2}x_2 + a_{3,3}x_3 + \phi_{3,2}(x)x_2 + \phi_{3,3}(x)x_3 \,, \\ &\ \vdots \\ f_n(x) &= a_{n,1}x_1 + ... + a_{n,n}x_n + \phi_{n,1}(x)x_1 + \phi_{n,2}(x)x_2 + ... + \phi_{n,n}(x)x_n \,. \end{aligned}$$

In particular, note that $\partial f_2(x)/\partial x_2(0) = a_{2,2}$, ..., $\partial f_n(x)/\partial x(0) = [a_{n,2}, ..., a_{n,n}]$.

Let $\epsilon^2 = -(1/\gamma^2 - 1)$, where $\gamma \in (1, \infty]$ is the desired disturbance attenuation level.

Note that $\epsilon \in (0,1)$. Let

$$\bar{P} = \begin{bmatrix} p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\ p_{1,2} & p_{2,2} & \cdots & p_{2,n} \\ \vdots & \ddots & \ddots & \vdots \\ p_{1,n} & p_{2,n} & \cdots & p_{n,n} \end{bmatrix}$$

be the symmetric solution of the algebraic Riccati equation for the linearized problem, *i.e.* $\bar{P}A + A^\top \bar{P} - \epsilon^2 \bar{P}BB^\top \bar{P} + H^\top H \leq 0$ where

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & a_{2,2} & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & a_{n-1,2} & a_{n-1,3} & \dots & 1 \\ a_{n,1} & a_{n,2} & a_{n,3} & \dots & a_{n,n} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b_1 \end{bmatrix}$$

and H defines a penalty variable. To determine an *algebraic* $\bar{P}$ *solution* let $P(x) = x^\top \bar{P} + \mathcal{Q}(x)$, $\bar{f}(x) = f(x) - Ax$ and consider the *algebraic* inequality in the unknown $\mathcal{Q}(x) = [\mathcal{Q}_1(x), ..., \mathcal{Q}_n(x)]$

$$\begin{aligned} &2(x^\top \bar{P} + \mathcal{Q}(x))(Ax + \bar{f}(x)) + x^\top H^\top H x + 2x^\top \Sigma(x) x \\ &- \epsilon^2 (x^\top \bar{P} + \mathcal{Q}(x)) g(x) g(x)^\top (x^\top \bar{P} + \mathcal{Q}(x))^\top \leq 0\,, \end{aligned} \tag{4.34}$$

where $\Sigma(x) = diag\{\sigma_i(x)\}_{i=1,\dots,n}$, with $\sigma_i(x) > 0$, for all $x \neq 0$.

Proposition 4.2. Consider a nonlinear system of the form (4.33) and the HJ inequality (4.34) and suppose that

$$a_{n,1} + \phi_{n,1}(x) - \epsilon^2 p_{1,n} b(x)^2 \neq 0\,, \tag{4.35}$$

for all $x \in \mathbb{R}^n$. Then the mapping $P(x) = x^\top \bar{P} + \mathcal{Q}(x)$, with

$$\begin{aligned} \mathcal{Q}_n(x) = &\frac{1}{a_{n,1} + \phi_{n,1}(x) - \epsilon^2 p_{1,n} b(x)^2} \Big[- x_1 \sigma_1(x) - \phi_{n,1}(x_1 p_{1,n} + ... + x_n p_{n,n}) \\ &- \epsilon^2 p_{1,n} b_1 (x_1 p_{1,n} + ... + x_n p_{n,n}) b_2(x) \Big]\,, \end{aligned} \tag{4.36}$$

$$\begin{aligned} \mathcal{Q}_i(x) = &\epsilon^2 p_{i+1,n} b_1 (x_1 p_{1,n} + ... + x_n p_{n,n}) b_2(x) + \epsilon^2 p_{i+1,n} b(x)^2 \mathcal{Q}_n(x) - x_{i+1}\sigma_{i+1} \\ &- \sum_{j=i+1}^{n} \Big[(a_{j,i+1} + \phi_{j,i+1}(x)) \mathcal{Q}_j(x) + \phi_{j,i+1}(x)(x_1 p_{1,j} + ... + x_n p_{j,n}) \Big]\,, \end{aligned} \tag{4.37}$$

$i = 1, ..., n-1$, is an *algebraic* $\bar{P}$ *solution* of the equation (4.33). $\diamond$

Remark 4.9. The condition (4.35) is trivially satisfied if the function f_n does not depend on the variable x_1, *i.e.* $\partial f_n(x)/\partial x_1 = 0$ for all $x \in \mathbb{R}^n$ and $p_{1,n} \neq 0$. Note that, since the i-th element of $\mathcal{Q}$ depends on the components $\mathcal{Q}_j$ with $j > i$, the solution can be determined iteratively from $\mathcal{Q}_{n-1}$ to $\mathcal{Q}_1$. ▲

Proof. The proof is by direct substitution. Considering the definition of the matrices A, Σ and H and of the vectors $\bar{f}$ and g, straightforward computations show that the mapping $P(x) = x^\top \bar{P} + \mathcal{Q}(x)$, where $\bar{P}$ is the solution of the algebraic Riccati equation and $\mathcal{Q}$ is defined in (4.36)-(4.37), is a solution of the algebraic inequality (4.34). □

4.5.2 Strict Feedforward Form

Consider a nonlinear system in *strict feedforward form* described by equations of the form

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_2, ..., x_n)\,, \\
\dot{x}_2 &= f_2(x_3, ..., x_n)\,, \\
&\vdots \\
\dot{x}_{n-1} &= f_{n-1}(x_n)\,, \\
\dot{x}_n &= u + d\,,
\end{aligned}
\tag{4.38}
$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$ and $d(t) \in \mathbb{R}$ and $f = [f_1, f_2, ..., f_{n-1}, 0]^\top$. Suppose that the origin is an equilibrium point for the system (4.38), *i.e.* $f_i(0) = 0$ $i = 1, ..., n-1$, hence there exist continuous functions $\phi_{i,j}$ and constants $a_{i,j}$, $i = 2, ..., n$, $j = 2, ..., i$ such that

$$
f(x) = \begin{pmatrix}
a_{1,2}x_2 + ... + a_{1,n}x_n + \phi_{1,2}(x)x_2 + ... + \phi_{1,n}(x)x_n \\
a_{2,3}x_3 + ... + a_{2,n}x_n + \phi_{2,3}(x)x_3 + ... + \phi_{2,n}(x)x_n \\
\vdots \\
a_{n-1,n}x_n + \phi_{n-1,n}x_n \\
0
\end{pmatrix}.
$$

Proposition 4.3. Consider a nonlinear system of the form (4.38) and suppose that

$$
a_{i,i+1} + \phi_{i,i+1}(x) \neq 0\,, \tag{4.39}
$$

for all $x \in \mathbb{R}^n$, for $i = 1, ..., n-1$. Then the mapping $P(x) = x^\top \bar{P} + \mathcal{Q}(x)$, with $\mathcal{Q}_n(x) = \frac{x_1\sigma_1(x)}{\epsilon^2 p_{1,n}}$ and

$$
\begin{aligned}
\mathcal{Q}_i(x) = &-\frac{1}{a_{i,i+1} + \phi_{i,i+1}(x)}\Big[-\epsilon^2 p_{i+1,n}\mathcal{Q}_n(x) + \sum_{j=1}^{i-1}\Big(\phi_{j,i}(x)(x_1 p_{1,j} + ... + x_n p_{j,n}) \\
&- x_{i+1}\sigma_{i+1}(x) + \mathcal{Q}_j(x)(a_{j,i+1} + \phi_{j,i+1}(x))\Big)\Big]\,,
\end{aligned}
\tag{4.40}
$$

$i = 1, ..., n-1$, solves the inequality (4.34) and is an *algebraic* $\bar{P}$ *solution* for the nonlinear system (4.38). ◇

Proof. As in the proof of Proposition 4.2 the claim is proved by direct substitution. □

Remark 4.10. The condition (4.39) is trivially satisfied if $f_i(x) = x_{i+1} + \bar{f}_i(x_{i+2}, ..., x_n)$, for $i = 1, ..., n-1$. ▲

4.6 Conclusions

In this chapter we have introduced the notion of (time-varying) Dynamic Storage function and we have related the existence of a Dynamic Storage function to the solution of the $\mathcal{L}_2$-disturbance attenuation problem. In particular the knowledge of Dynamic Storage function allows to construct a dynamic control law that solves the attenuation problem. Moreover, it is shown that a specific choice of time-varying Dynamic Storage function yields a static feedback. The key aspect of novelty lies in the fact that, differently from classical storage functions, partial differential equations or inequalities are not involved in the construction of Dynamic Storage functions and a constructive methodology is provided. In particular this approach relies upon the notion of *algebraic* $\bar{P}$ *solution*, which is defined in terms of the solution of an algebraic scalar inequality. In the last two sections of the chapter it is shown that the control law provided by the method reduces to the standard solution of the H_∞ attenuation problem in the case of linear time invariant systems and that *algebraic* $\bar{P}$ *solutions* can be systematically computed for classes of input-affine nonlinear systems in the presence of matched disturbance.

Chapter 5

Optimal Control

5.1 Introduction

The main objective in optimal control problems consists in determining control signals to steer the trajectories of an ordinary differential equation in such a way that a desired criterion of optimality is minimized, or maximized. Two different – and, in some respect, antagonistic – approaches to solve optimal control problems have been introduced in the literature, namely Pontryagin's Minimum Principle [103] and Dynamic Programming [11]. The present chapter deals solely with the latter, which relies upon the solution of the well-known Hamilton-Jacobi-Bellman (HJB) partial differential equation. In the last decades, the definition of several constraints and requirements has given rise to a wide variety of problems that may be formalized within the framework of optimal control theory. Among such problems, for instance, the Calculus of Variations – the origins of which are independent from optimal control and date back long before the birth of the latter theory – may nonetheless be considered as a special case of optimal control in which the dynamical constraint is replaced by an integrator.

A significant distinction is also emphasized between finite-horizon and infinite-horizon optimal control problems, which are the topics of Sections 5.3 and 5.4, respectively. In the latter the purpose of the control law is to steer the state of the system in such a way that a desired criterion of optimality is minimized while ensuring asymptotic stability of an equilibrium point of the underlying differential equation. In the finite-horizon case the time interval is finite and decided *a priori*, as it is obviously the initial condition. Furthermore, a penalty term imposed on the value of the state at the terminal time may additionally be considered.

In either case the *value function*, namely the cost-to-go function, is defined in terms of the solution of a first-order nonlinear partial differential equation, the HJB partial differential equation, which may be hard or impossible to determine in closed-form.

Therefore, in recent years, several approaches to approximate, with a desired degree of accuracy, the solution of the Hamilton-Jacobi-Bellman partial differential equation or inequality in a neighborhood possibly large of an equilibrium point have been proposed, see [8, 9, 21, 37, 49, 55, 64, 78, 104].

Many of these results rely either on the iterative computation of the coefficients of a local expansion of the solution, provided that all the functions of the nonlinear system are analytic (or at least smooth) or on the solution of the HJ inequality along the trajectories of the system. In particular, in [55], under the assumption of stabilizability of the zero equilibrium of the nonlinear system, which is supposed to be real analytic about the origin, an optimal stabilizing control law defined as the sum of a linear control law, that solves the linearized problem, and a convergent power series about the origin, starting with terms of order two, is proposed. Sufficient conditions that guarantee the convergence of the Galerkin approximation of the HJB equation over a compact set containing the origin are given in [9]. The local solution proposed in [104] hinges upon the technique of *apparent linearization* and the repeated computation of the steady-state solution of the Riccati equation. The basic idea of the state dependent Riccati equation (SDRE) is to solve the Riccati equation pointwise, using a state-dependent linear representations (SDLR), see [36]. The SDLR is not unique and for particular choices the SDRE not only provides a performance different from the optimal, but it may even fail to yield stability of the equilibrium point of the system. In [49] conditions for the existence of a local solution of the HJB equation, for a parameterized family of infinite horizon optimal control problems are given. In [64] it is shown that the solution of the HJB equation is the eigenfunction, relative to the zero eigenvalue, of the semigroup, which is max-plus linear, corresponding to the HJB equation, and a discrete-time approximation of the semigroup guarantees the convergence of the approximate solution to the actual one.

Finally, a large effort has been devoted to avoid the hypothesis of differentiability of the storage function, interpreting the HJB equation in the viscosity sense, see [8,21,78]. In particular the issue of existence, or uniqueness, of viscosity solutions has been extensively addressed and explored, see for instance [27]. A method to approximate viscosity solutions by means of a discretization in time and in the state variable has been proposed in [28] and using a domain decomposition without *overlapping* in [15].

The rest of the chapter is organized as follows. The notion of Dynamic Value function is introduced and discussed in Section 5.2. In both finite and infinite-horizon optimal control problems these functions generalize standard value functions since the former consist of a positive definite function combined with a dynamical system. As a matter of fact, this additional system may be interpreted as a dynamic control law that approximates the solution of the underlying optimal control problem. Sections 5.3 and 5.4 deal with the finite-horizon and the infinite-horizon optimal control problems, respectively. Each section is concluded by a numerical example.

5.2 Dynamic Value Function

In this section we lay the foundations for a common framework that encompasses finite-horizon as well as infinite-horizon optimal control problems. These problems are then discussed separately in the two following sections. Consider a nonlinear system described by equations of the form

$$\dot{x} = f(x) + g(x)u\,, \tag{5.1}$$

where $x(t) \in \mathbb{R}^n$ is the state of the system, while $u(t) \in \mathbb{R}^m$ denotes the control input. The mappings $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are assumed to be sufficiently smooth. The (finite-horizon) optimal control problem consists in determining a control input u that minimizes the cost functional[1]

$$J(u) \triangleq \frac{1}{2}\int_0^T (q(x(t)) + u(t)^\top u(t))dt + \frac{1}{2}x(T)^\top Kx(T)\,, \tag{5.2}$$

where $T > 0$ is the fixed terminal time, $q : \mathbb{R}^n \to \mathbb{R}_+$ is the running cost imposed on the state of the system, $K = K^\top \geq 0$ penalizes the value of the state of the system at the terminal time, subject to the dynamic constraint (5.1) and the initial condition $x(0) = x_0$. In the infinite-horizon case, namely when T tends to infinity, the terminal cost is replaced with the requirement that the zero equilibrium of the closed-loop system be locally asymptotically stable. Consider the following standing assumptions.

Assumption 5.1. The vector field f is such that $f(0) = 0$, *i.e.* $x = 0$ is an equilibrium point for the system (5.1) when $u(t) = 0$ for all $t \geq 0$.

Assumption 5.2. The nonlinear system (5.1) with output $y = q(x)$ is zero-state detectable.

Note that, by Assumption 5.1, there exists a non-unique continuous matrix-valued function $F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $f(x) = F(x)x$, for all $x \in \mathbb{R}^n$.

Given a time instant $t \geq 0$ and a state x, the *value function* yields the minimum value of J, *i.e.* the cost of the optimal solution, from (t, x). As is well-known, a value function is defined in terms of the solution of the HJB partial differential equation, together with the corresponding boundary condition. This aspect may somewhat restrict the scope of applicability of the Dynamic Programming approach, since the analytic solution of the HJB equation may be seldom determined. Extending the definition of a Dynamic Lyapunov function and mimicking the construction of a Dynamic Storage function, in this section we introduce the notion of Dynamic Value function that generalizes the concept of value function, as detailed in the following definition.

[1] For simplicity we consider only a quadratic terminal cost and quadratic penalty on the input u.

Definition 5.1. (Dynamic Value Function). Consider the system (5.1) and suppose that Assumptions 5.1 and 5.2 hold. A *Dynamic Value function* $\mathcal{V}$ is a pair (D_α, V) defined as follows.

- D_α is the dynamical system

$$\begin{aligned} \dot{\zeta} &= \alpha(t, x, \zeta)\,, \\ u &= -g(x)^\top V_x(t, x, \zeta)^\top\,, \end{aligned} \tag{5.3}$$

  with $\zeta(t) \in \mathbb{R}^{\tilde{n}}$, $\alpha : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{n}}$, $\alpha(t, 0, 0) = 0$ for all $t \geq 0$.

- $V : \mathbb{R} \times \Omega \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}$ is non-negative, for all $t \geq 0$, and it is such that

$$\mathcal{HJB}(t, x, \zeta) \triangleq V_t + V_x f(x) + V_\zeta \alpha(t, x, \zeta) - \frac{1}{2} V_x g(x) g(x)^\top V_x^\top + \frac{1}{2} q(x) \leq 0\,, \tag{5.4}$$

  for all $(x, \zeta) \in \Omega$, $t \geq 0$, and $V(T, x, \zeta(T)) = \frac{1}{2} x^\top K_d x$, with $K_d = K_d^\top \geq K$.

Note that the function V, similarly to standard value functions, and the vector field α do not depend explicitly on time in the infinite-horizon case, hence the first term of (5.4) disappears. Moreover, the boundary condition reduces to $V(0, 0) = 0$. In the following sections, Dynamic Value functions are employed to characterize – and, in some respect, to give a novel interpretation to – the solutions of optimal control problems.

5.3 Finite-Horizon Optimal Control Problem

The Dynamic Programming solution of the fixed terminal time, free-endpoint, optimal control problem hinges upon the solution, $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, of the Hamilton-Jacobi-Bellman partial differential equation

$$V_t(t, x) + V_x(t, x) f(x) + \frac{1}{2} q(x) - \frac{1}{2} V_x(t, x) g(x) g(x)^\top V_x(t, x)^\top = 0\,, \tag{5.5}$$

together with the boundary condition

$$V(T, x) = \frac{1}{2} x^\top K x\,, \tag{5.6}$$

for all $x \in \mathbb{R}^n$. The solution of the HJB equation (5.5), if it exists, is the *value function* of the optimal control problem, *i.e.* it is a function that associates to every initial state x_0 the optimal cost, namely

$$V(0, x_0) = \min_u \frac{1}{2} \left(\int_0^T (q(x(t)) + u(t)^\top u(t)) dt + x(T)^\top K x(T) \right). \tag{5.7}$$

Problem 5.1. Consider system (5.1) and the cost (5.2). The *regional dynamic finite-horizon optimal control* problem consists in determining an integer $\tilde{n} \geq 0$, a dynamic control law described by the equations

$$\begin{aligned} \dot{\zeta} &= \alpha(t, x, \zeta)\,, \\ u &= \beta(t, x, \zeta)\,, \end{aligned} \tag{5.8}$$

with $\alpha : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^{\tilde{n}}$ and $\beta : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}^m$ smooth mappings, $\alpha(t, 0, 0) = 0$, $\beta(t, 0, 0) = 0$ for all $t \geq 0$, and a set $\bar{\Omega} \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ containing the origin of $\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ such that the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\beta(t, x, \zeta)\,, \\ \dot{\zeta} &= \alpha(t, x, \zeta)\,, \end{aligned} \tag{5.9}$$

is such that the inequality $J(\beta) \leq J(\bar{u})$ holds for any $\bar{u}$ and any (x_0, ζ_0) such that the trajectory of the system (5.9) remains in $\bar{\Omega}$ for all $t \in [0, T]$.

Herein an approximate solution of the *regional dynamic finite-horizon optimal control* problem is determined, in the sense specified by the following definition.

Problem 5.2. Consider system (5.1) and the cost (5.2). The *approximate regional dynamic finite-horizon optimal control* problem consists in determining an integer $\tilde{n} \geq 0$, a dynamic control law described by the equations (5.8), a set $\bar{\Omega} \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ containing the origin of $\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ and non-negative functions $\rho_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}_+$ and $\rho_2 : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}_+$ such that the regional dynamic finite-horizon optimal control problem is solved with respect to the running cost

$$\mathcal{L}(x, \zeta, u) \triangleq q(x) + \rho_1(t, x, \zeta) + u^\top u\,, \tag{5.10}$$

and the terminal cost

$$\mathcal{T}(x(T), \zeta(T)) \triangleq \frac{1}{2} x(T)^\top K x(T) + \rho_2(x(T), \zeta(T))\,. \tag{5.11}$$

In the linear-quadratic case, *i.e.* in the case in which the underlying system is

$$\dot{x} = Ax + Bu\,, \tag{5.12}$$

and the running cost on the state is $q(x) = x^\top Q x$, the solution of the finite-horizon optimal control problem is a linear time-varying state feedback of the form $u = -B^\top \bar{P}(t) x$, where

$\bar{P}(t)$ is the (symmetric, positive semidefinite) solution of the differential Riccati equation

$$\dot{\bar{P}}(t) + \bar{P}(t)A + A^\top \bar{P}(t) - \bar{P}(t)BB^\top \bar{P}(t) + Q = 0\,, \tag{5.13}$$

with the boundary condition $\bar{P}(T) = K$.

We conclude this section discussing the relation between a Dynamic Value function and the solution of the approximate regional dynamic finite-horizon optimal control problem, namely Problem 5.2.

Lemma 5.1. Consider system (5.1) and the cost (5.2). Let $\mathcal{V} = (D_\alpha, V)$ be a Dynamic Value function for the system (5.1). Then the system (5.3) is a dynamic time-varying control law that solves the approximate regional dynamic finite-horizon optimal control problem with $\rho_1 = -2\,\mathcal{HJB}(t, x, \zeta)$ and $\rho_2 = \frac{1}{2}x^\top(K_d - K)x$. $\diamond$

Proof. The claim follows immediately noting that, by the partial differential inequality (5.4), the control law u in (5.3) solves a finite-horizon optimal control problem associated with the extended nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u\,, \\ \dot{\zeta} &= \alpha(t, x, \zeta)\,. \end{aligned}$$

In particular, an additional terminal cost ρ_2 is induced by the difference between the boundary condition on the Dynamic Value function and the condition (5.6). Finally, the motivation behind the additional running cost ρ_1 lies in the fact that a partial differential inequality is solved in place of an equation, hence the *gap* between the former and the latter contributes to the running cost imposed on the state of the extended system. $\square$

Rephrasing the result of Lemma 5.1 it appears that the existence of a Dynamic Value function implies solvability of Problem 5.2. Indeed, the previous statement entails something more than the mere solvability since an explicit dynamic control law that solves Problem 5.2 is obtained, similarly to standard value functions, from the knowledge of a Dynamic Value function. The advantage of the latter over the former will become evident in the following section in which it is shown that a class of Dynamic Value functions can be constructively defined without resorting to the solution of any partial differential equation or inequality.

5.3.1 Algebraic $\bar{P}$ Solution and Dynamic Value Function

In this section a notion of solution of the Hamilton-Jacobi-Bellman equation (5.5) is presented. This notion is instrumental in providing a constructive methodology to define a class of Dynamic Value functions, mimicking the ideas of the previous chapter. To this end, consider the augmented system

$$\dot{z} = \mathcal{F}(z) + \mathcal{G}(z)u\,, \tag{5.14}$$

with $z(t) = (x(t)^\top, \tau(t))^\top \in \mathbb{R}^{n+1}$, $\mathcal{F}(z) = (f(x)^\top, 1)^\top$ and $\mathcal{G}(z) = (g(x)^\top, 0)^\top$. Note that the partial differential equation (5.5) can be rewritten as

$$V_z(z)\mathcal{F}(z) + \frac{1}{2}q(x) - \frac{1}{2}V_z(z)\mathcal{G}(z)\mathcal{G}(z)^\top V_z(z)^\top = 0\,, \tag{5.15}$$

with the boundary condition (5.6). Consider the HJB equation (5.15) and suppose that it can be solved *algebraically*, as detailed in the following definition.

Definition 5.2. Let $\Sigma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, with $x^\top \Sigma(\tau, x)x \geq 0$ for all $(\tau, x) \in \mathbb{R} \times \mathbb{R}^n$, and $\sigma : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}_+$. A continuously differentiable mapping $P(\tau, x) = [p(\tau, x),\, r(\tau, x)]$, $p : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{1\times n}$, $r : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, is a $\mathcal{X}$*-algebraic* $\bar{P}$ *solution* of (5.5) if

$$p(\tau, x)f(x) + r(\tau, x) - \frac{1}{2}p(\tau, x)g(x)g(x)^\top p(\tau, x)^\top + \frac{1}{2}q(x) + x^\top \Sigma(\tau, x)x + \tau^2\sigma(\tau, x) = 0\,, \tag{5.16}$$

for all $(\tau, x) \in \mathcal{X} \subseteq \mathbb{R} \times \mathbb{R}^n$, $p(\tau, 0) = 0$, $r(\tau, 0) = 0$,

$$\begin{array}{ll} \left.\dfrac{\partial p^\top}{\partial \tau}\right|_{(\tau,0)} = 0\,, & \left.\dfrac{\partial r}{\partial \tau}\right|_{(\tau,0)} = 0\,, \\ \left.\dfrac{\partial p^\top}{\partial x}\right|_{(\tau,0)} = \bar{P}(\tau)\,, & \left.\dfrac{\partial^2 r}{\partial x^2}\right|_{(\tau,0)} = \dot{\bar{P}}(\tau)\,, \end{array} \tag{5.17}$$

where $\bar{P}(\tau)$ is the solution of the differential Riccati equation (5.13) together with the boundary condition $\bar{P}(T) = K$.

Obviously, since an arbitrary mapping that solves the equation (5.16) is selected, the mapping P may not be a *gradient vector*. In the following we assume $\mathcal{X} = \mathbb{R} \times \mathbb{R}^n$ without loss of generality, as explained in the previous chapter.

Using an *algebraic* $\bar{P}$ *solution* of the equation (5.16), define the *extension* of P, namely

$$V(\tau, x, s, \xi) = p(s, \xi)x + r(s, \xi)\tau + \frac{1}{2}\|x - \xi\|_R^2 + \frac{1}{2}b\|\tau - s\|^2\,, \tag{5.18}$$

where $\xi \in \mathbb{R}^n$, $s \in \mathbb{R}$, $R = R^\top > 0$ and $b > 0$. To streamline the presentation and provide a concise statement of the main result let

$$\Delta(x, s, \xi) = (R - \Phi(x, s, \xi))\Lambda(s, \xi)^\top\,, \tag{5.19}$$

$$\delta(x, s, \xi) = (R - \Phi(x, s, \xi))\lambda(s, \xi)^\top\,, \tag{5.20}$$

with $\Lambda(s, \xi) = p_\xi(s, \xi)R^{-1}$, $\lambda(s, \xi) = r_\xi(s, \xi)R^{-1}$, where $\Phi : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is a continuous matrix-valued function such that $p(s,x) - p(s,\xi) = (x-\xi)^\top \Phi(x,s,\xi)^\top$ for all $(x,s,\xi) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$.

Moreover, $A_{cl}(\tau,x) = F(x) - g(x)g(x)^\top N(\tau,x)$, with $N : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $p(\tau,x) = x^\top N(\tau,x)^\top$ for all $(\tau,x) \in \mathbb{R} \times \mathbb{R}^n$. Finally let ℓ, H, Π, W_1, W_2, D_1 and D_2 be such that

$$\begin{aligned}
r(s,x) - r(\tau,x) &= \ell(\tau,x,s)(s-\tau)\,,\\
r(s,\xi) - r(s,x) &= (x-\xi)^\top H(x,s,\xi)(x-\xi)\,,\\
p(s,x) - p(\tau,x) &= x^\top \Pi(\tau,x,s)\,,\\
p_s(s,\xi) - p_s(s,x) &= W_1(x,s,\xi)(x-\xi)\,,\\
p_s(x,s) &= W_2(s,x)x\,,\\
r_s(s,\xi) - r_s(s,x) &= D_1(x,s,\xi)(x-\xi)\,,\\
r_s(s,x) &= D_2(x,s)x\,.
\end{aligned}$$

Remark 5.1. The vector field $A_{cl}(\tau,x)x$ describes the closed-loop nonlinear system when only the *algebraic input*, namely $u = -g(x)^\top p(\tau,x)^\top$, is applied. ▲

The following statement provides a solution to the *approximate regional dynamic finite-horizon optimal control problem* for the cost (5.2) subject to the dynamical constraint (5.1), in terms of the existence of a Dynamic Value function. Moreover, we show that the function (5.18) is indeed a *value function* for the system (5.1) and the cost (5.2), *i.e.* solves the Hamilton-Jacobi-Bellman partial differential inequality associated to the *extended* system (5.9), namely

$$\mathcal{HJB} \triangleq V_z\mathcal{F}(z) + V_\xi\dot{\xi} + V_s\dot{s} + \frac{1}{2}q(x) - \frac{1}{2}V_z\mathcal{G}(z)\mathcal{G}(z)^\top V_z^\top \le 0\,, \tag{5.21}$$

and satisfies a boundary condition of the form

$$V(T,x,s(T),\xi(T)) = \frac{1}{2}x^\top K_d x\,, \tag{5.22}$$

with $K_d = K_d^\top \ge K$.

Proposition 5.1. Consider the system (5.1) and the cost (5.2). Let P be an *algebraic* $\bar{P}$ *solution* of (5.15).

Let $R = R^\top \ge \sup_{\tau\in[0,T]} \bar{P}(\tau)$ and $b > 0$ be such that

$$\begin{bmatrix} L_1 & L_2 \\ L_2^\top & L_3 \end{bmatrix} < \begin{bmatrix} \Sigma & 0 \\ 0 & \sigma \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \Delta^\top \\ \delta^\top \end{bmatrix} gg^\top \begin{bmatrix} \Delta & \delta \end{bmatrix}, \tag{5.23}$$

for all $(\tau, x, s, \xi) \in \Omega \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, where Ω is a non-empty set containing the origin, with

$$\begin{aligned} L_1 &= A_{cl}^\top(\tau, x)\Pi^\top + \gamma W_2 + \Lambda Y + Y^\top \Lambda^\top + \Lambda H \Lambda^\top , \\ L_2 &= \frac{\gamma}{2} D_2^\top + Y^\top \lambda^\top + \Lambda H \lambda^\top + \frac{\gamma}{2}\Lambda D_1^\top , \\ L_3 &= \lambda H \lambda^\top + \frac{\gamma}{2} D_1 \lambda^\top + \frac{\gamma}{2}\lambda D_1^\top , \\ Y &= \frac{1}{2}(R - \Phi)^\top A_{cl}(s, x) + \frac{\gamma}{2} W_1 , \end{aligned}$$

and $\gamma(\tau, x, s) = 1 - \dfrac{\ell(\tau, x, s)}{b}$.

Then there exists $\bar{k} \geq 0$ such that for all $k > \bar{k}$ the function V defined in (5.18) satisfies the Hamilton-Jacobi-Bellman inequality (5.21) for all $(\tau, x, s, \xi) \in \Omega$, with $\dot{\xi} = -kV_\xi^\top$ and $\dot{s} = \gamma(\tau, x, s)$. Furthermore, selecting $s(T) = T$ and $\xi(T) = 0$, yields $V(T, x, T, 0) = \frac{1}{2}x^\top Rx$. Hence $\mathcal{V} = (D_\alpha, V)$, with D_α defined as

$$\begin{aligned} \dot{s} &= \gamma(\tau, x, s) , \\ \dot{\xi} &= -k(p_\xi(s, \xi)^\top x - R(x - \xi) + r_\xi(s, \xi)^\top \tau) , \\ u &= -g(x)^\top \left[p(\tau, x)^\top + (R - \Phi(x, s, \xi))(x - \xi) + x^\top \Pi(\tau, x, s)\right] \end{aligned} \tag{5.24}$$

and V as in (5.18), is a Dynamic Value function for the system (5.1), and, by Lemma 5.1, (5.24) solves the approximate regional dynamic finite-horizon optimal control problem with $\rho_1 = -2\mathcal{HJB} \geq 0$ and $\rho_2 = \frac{1}{2}x(T)^\top(R - K)x(T)$. Finally, suppose that condition (5.23) is satisfied with $R = K$ then the additional cost on the final state is zero. $\diamond$

Remark 5.2. Since the variable τ, which *describes* time, belongs to the set Ω it may not be possible to solve the finite-horizon optimal control problem for the cost (5.2) subject to the dynamical constraint (5.1) for all terminal time $T > 0$. ▲

Proof. Consider the partial derivatives of the function V defined in (5.18), namely

$$\begin{aligned} V_x &= p(\tau, x) + (x - \xi)^\top (R - \Phi)^\top + x^\top \Pi , \\ V_\tau &= r(\tau, x) + b(\tau - s) + (x - \xi)^\top H(x - \xi) + r(s, x) - r(\tau, x) , \\ V_\xi &= x^\top p_\xi + \tau r_\xi - (x - \xi)^\top R , \\ V_s &= x^\top p_s(s, \xi) + \tau r_s(s, \xi) - b(\tau - s) , \end{aligned} \tag{5.25}$$

and the HJB partial differential inequality (5.21) with the partial derivatives of V defined as in (5.25). In particular, letting $\dot{\xi} = -kV_\xi^\top$, the left-hand side of the inequality (5.21) can be rewritten as the sum of two terms, *i.e.*

$$\mathcal{HJB} = \mu_1(\tau, x, s, \xi) + \mu_2(\tau, x, s) \leq 0 , \tag{5.26}$$

where μ_1 is a *quadratic form* in the variables x, $(x-\xi)$ and τ, *i.e.*

$$\mu_1 = -\begin{bmatrix} x^\top & (x-\xi)^\top & \tau \end{bmatrix} [M + kC^\top C] \begin{bmatrix} x \\ (x-\xi) \\ \tau \end{bmatrix}, \tag{5.27}$$

with $C(s,\xi) = [p_\xi^\top(s,\xi) \quad -R \quad r_\xi^\top(s,\xi)]$ for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and

$$M = \begin{bmatrix} M_{11} & -Y^\top & -\frac{\gamma}{2}D_2^\top \\ -Y & M_{22} & -\frac{\gamma}{2}D_1^\top \\ -\frac{\gamma}{2}D_2 & -\frac{\gamma}{2}D_1 & \sigma \end{bmatrix},$$

with

$$M_{11} = \Sigma - A_{cl}^\top(\tau,x)\Pi^\top - \gamma W_2 + \frac{1}{2}\Pi g g^\top \Pi^\top$$

and $M_{22} = -H + \frac{1}{2}(R-\Phi)^\top g g^\top (R-\Phi)$. On the other hand, the term μ_2 is defined as $\mu_2(\tau,x,s) \triangleq r(s,x) - r(x,\tau) + b(\tau - s) - b(\tau - s)\dot{s}$. Note that the definition of μ_2 stems from the fact that, since $z=0$ is not an equilibrium point for $\mathcal{F}$ defined in (5.14), there are terms, *e.g.* in V_τ, that cannot be written in the form of (5.27). As a first step in showing that the inequality (5.26) is satisfied, note that the choice of the dynamics of s as in (5.24) guarantees that the term μ_2 is identically equal to zero for all $(\tau,x,s,\xi) \in \Omega$.

Moreover, the matrix $C(s,\xi) \in \mathbb{R}^{n\times(2n+1)}$ has constant rank n for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$ and the columns of the matrix

$$Z = \begin{bmatrix} I & 0 \\ R^{-1}p_\xi^\top & R^{-1}r_\xi^\top \\ 0 & 1 \end{bmatrix}$$

span the kernel of the matrix $C(s,\xi)$ for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Exploiting the results in [3], consider the restriction of the matrix-valued function M to the subspace defined by the columns of the matrix $Z(\xi,s) \in \mathbb{R}^{2n+1\times n+1}$ for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^n$. Finally, tedious but straightforward computations show that condition (5.23) implies that the matrix $Z^\top M Z$ is positive definite, hence there exists $\bar{k} \geq 0$ such that $\mu_1 \leq 0$ for all $k > \bar{k}$. This proves that the Hamilton-Jacobi-Bellman inequality (5.21) is satisfied for all $(\tau,x,s,\xi) \in \Omega \setminus \{0\}$. Furthermore $\mathcal{HJB}(0,0,0,0) = 0$, hence, by continuity of the mappings in system (5.1), $\mathcal{HJB}(\tau,x,s,\xi)$ is continuous and smaller than or equal to zero for all $(\tau,x,s,\xi) \in \Omega$. To conclude the proof, note that the boundary conditions on the dynamic extension yield $V(T,x,T,0) = \frac{1}{2}x^\top R x$ and the boundary condition on V is satisfied considering the additional cost $\rho_2 = \frac{1}{2}x(T)^\top(R-K)x(T)$. □

The following statement shows that there exists a neighborhood of the origin of $\mathbb{R} \times$

$\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ in which the condition (5.23) holds provided that Σ and σ in the definition of an *algebraic $\bar{P}$ solution* are strictly positive at the origin.

Corollary 5.1. Consider the system (5.1) and the cost (5.2). Let P be an *algebraic $\bar{P}$ solution* of (5.15) with $\Sigma(0,0) = \bar{\Sigma} > 0$ and $\sigma(0,0) = \bar{\sigma} > 0$. Let $R = R^\top \geq \sup_{\tau\in[0,T]} \bar{P}(\tau)$ and $b > 0$. Then there exist $\bar{k} \geq 0$ and a set $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ such that for all $k > \bar{k}$ the function V defined in (5.18) satisfies the Hamilton-Jacobi-Bellman inequality (5.21) for all $(\tau, x, s, \xi) \in \Omega$, with $\dot{\xi} = -kV_\xi^\top$ and $\dot{s} = \gamma(x, \tau, s)$. ⋄

Proof. Assume that $\Sigma(0,0) = \bar{\Sigma} > 0$ and $\sigma(0,0) = \bar{\sigma} > 0$ in the definition of the *algebraic $\bar{P}$ solution*. Then, since the left-hand side of the condition (5.23) is zero at the origin, by continuity of the left-hand side of inequality (5.23), there exists a non-empty subset $\Omega \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$, containing the origin, such that the condition (5.23) is satisfied for all $(\tau, x, s, \xi) \in \Omega$, proving the claim. □

Remark 5.3. The gain k in the dynamics of ξ, namely the second equation of (5.24), may be defined as a function of (τ, x, s, ξ) in order to reduce the instantaneous additional cost ρ_1 (see also the numerical example for more details). ▲

To streamline the presentation of the following result, providing conditions that allow to guarantee that the *extended* Hamilton-Jacobi-Bellman partial differential inequality (5.21) can be solved with the equality sign, define the set

$$\mathcal{N}_\varepsilon \triangleq \{(\tau, x, s, \xi) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n : \|x^\top p_\xi(\xi, s) + \tau r_\xi(s, \xi) - (x - \xi)^\top R\| < \varepsilon\}, \quad (5.28)$$

parameterized by the constant $\varepsilon > 0$.

Corollary 5.2. Consider the system (5.1) and the cost (5.2). Let P be an *algebraic $\bar{P}$ solution* of (5.15). Let $R = R^\top \geq \sup_{\tau\in[0,T]} \bar{P}(\tau)$, $b > 0$ and $\varepsilon > 0$. Then there exists a set $\Omega \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ such that the function V defined in (5.18) satisfies the Hamilton-Jacobi-Bellman inequality (5.21) with the equality sign for all $(\tau, x, s, \xi) \in \Omega \setminus \mathcal{N}_\varepsilon$, with $\dot{\xi} = -k(x, \tau, \xi, s)V_\xi^\top$,

$$k(\tau, x, s, \xi) = \left(V_\xi V_\xi^\top\right)^{-1} [x^\top \; (x - \xi)^\top \; \tau] M [x^\top \; (x - \xi)^\top \; \tau]^\top, \quad (5.29)$$

and $\dot{s} = \gamma(\tau, x, s)$. ⋄

Proof. Note that by definition of the set $\mathcal{N}_\varepsilon$, the gain k is well-defined for all $(\tau, x, s, \xi) \in \Omega \setminus \mathcal{N}_\varepsilon$, since $V_\xi V_\xi^\top > \varepsilon^2$ for all $(\tau, x, s, \xi) \notin \mathcal{N}_\varepsilon$. Then the choice of k as in (5.29) implies that $\mu_1 \equiv 0$ for all $(\tau, x, s, \xi) \in \Omega$. The proof is concluded recalling that the choice of $\dot{s}$ guarantees also that $\mu_2 \equiv 0$. □

Remark 5.4. Under the assumptions of Corollary 5.2, the additional running cost ρ_1 is identically equal to zero along all the trajectories of the closed-loop system (5.1)-(5.24), with k as in (5.29), such that $(\tau(t), x(t), s(t), \xi(t)) \notin \mathcal{N}_\varepsilon$ for all $t \in [0, T]$, namely trajectories that stay *sufficiently* away from the set in which $V_\xi = 0$. If additionally the selection $R = K$ is such that V as in (5.18) is non-negative in Ω then $\rho_2 = 0$. Therefore, for any $T > 0$ and for any initial condition $(0, x(0), s(0), \xi(0)) \in \Omega$ such that the closed-loop system (5.1)-(5.24) satisfies $\xi(T) = 0$, $s(T) = T$ and $(\tau(t), x(t), s(t), \xi(t)) \in \Omega \setminus \mathcal{N}_\varepsilon$ for all $t \in [0, T]$ the dynamic time-varying control law (5.24), with k as in (5.29), is the optimal solution of the finite horizon optimal control problem with the (original) cost (5.2). ▲

Remark 5.5. Suppose that $P(z)z > 0$ for all $z \neq 0$ and consider the limit of (5.2) as T tends to infinity. Let $\mathcal{A} = \{(\tau, x, s, \xi) \in \Omega : x = \xi = 0, s = \tau\}$. Then V as in (5.18) is zero in $\mathcal{A}$, *i.e.* $V|_{\mathcal{A}} = 0$, and there exist $R = R^\top > 0$, $b > 0$ and a set $\tilde{\Omega} \supset \Omega$ such that $V|_{\tilde{\Omega}\setminus\mathcal{A}} > 0$. If R is such that (5.21) is satisfied then $\dot{V} \leq 0$, along the trajectories of the closed-loop system (5.14)-(5.24), and the set $\mathcal{A}$ is stable for the system (5.14)-(5.24). ▲

Remark 5.6. The closed-loop system (5.14)-(5.24) is defined by a system of $2(n+1)$ ordinary differential equations with *two-point boundary conditions*, similar to the problem obtained exploiting the arguments of the Minimum Principle. However in the approach proposed herein, if the solution of the *two-point boundary value* problem is not precisely computed, *i.e.* $s(T) = \varepsilon_1 \neq T$ and $\xi(T) = \varepsilon_2 \neq 0$, then, by Remark 5.5, the resulting trajectories are not *far away* from the optimal evolutions and the additional cost resulting from this *mismatch* is given by

$$
\begin{aligned}
\rho_2(x(T), \varepsilon_1, \varepsilon_2) &= p(\varepsilon_1, \varepsilon_2)x(T) + r(\varepsilon_1, \varepsilon_2)T \\
&+ \frac{1}{2}(x(T) - \varepsilon_2)^\top (R - K)(x(T) - \varepsilon_2) + \frac{1}{2}b(T - \varepsilon_1)^2 .
\end{aligned}
$$

As a result, if an approximate solution of the problem is sought, then the solution of the *two-point boundary value* problem can be avoided selecting the initial condition of ξ and s such that the quantity ρ_2 is minimized. ▲

Remark 5.7. The choice $\dot{s} = \bar{\gamma}$, with $\bar{\gamma}$ constant, is sufficient to guarantee that the term μ_2 is smaller than zero. In fact, let $\dot{s} = \bar{\gamma}$, $s(0) = (1 - \bar{\gamma})T$ and note that $s(t) \geq \tau(t)$ and $s(T) = T$. Let $\bar{\ell} > 0$ be such that $|r(s, x) - r(\tau, x)| < \bar{\ell}(s - \tau)$ for all $(\tau, x, s) \in \Omega$, then $\bar{\gamma} \leq 1 - \frac{\bar{\ell}}{b}$ guarantees that $\mu_2 \leq 0$ for all $(\tau, x, s) \in \Omega$. In this case, the state variable s can be substituted in the dynamic control law (5.24) by the function of time $s(t) = T + \bar{\gamma}(t - T)$. ▲

5.3.2 Linear Systems

Consider the system (5.12) and the cost (5.2) with $q(x) = x^\top Qx$. From the definition of *algebraic* $\bar{P}$ *solution* of the equation (5.15), we expect p to *approximate* (in the sense defined in (5.16)) the partial derivative of the value function with respect to the state x, whereas r *represents* the partial derivative of V with respect to time. Therefore, in the linear case an *algebraic* $\bar{P}$ *solution* is given by

$$P(x,\tau) = [x^\top \bar{P}(\tau),\ \frac{1}{2}x^\top \dot{\bar{P}}(\tau)x]\,, \tag{5.30}$$

which satisfies the equation (5.16) together with the conditions (5.17), since $\bar{P}(\tau)$ satisfies the differential Riccati equation (5.13). Then, define the function (5.18), *i.e.*

$$V(\tau,x,s,\xi) = \xi^\top \bar{P}(s)x + \frac{1}{2}\xi^\top \dot{\bar{P}}(s)\xi\tau + \frac{1}{2}\|x-\xi\|^2_{R(s)} + \frac{1}{2}b\|\tau - s\|^2\,, \tag{5.31}$$

with $R(s) = R(s)^\top > 0$.

Proposition 5.2. Consider the linear system (5.12) and the cost (5.2) with $q(x) = x^\top Qx$. Let $\bar{P}(\tau) = \bar{P}(\tau)^\top > 0$ be the solution of the differential Riccati equation (5.13) with the boundary condition $\bar{P}(T) = K$. Let $R(\tau) = \bar{P}(\tau)$ for all $\tau \in [0,T]$. Then there exist $\bar{T} > 0$ and $\bar{k} \geq 0$ such that, for all $T \leq \bar{T}$ and all $k > \bar{k}$, V as in (5.31) satisfies the partial differential inequality (5.21) with $\dot{s} = 1$ and $\dot{\xi} = -kV_\xi^\top$. Furthermore, selecting $s(0) = 0$ and $\xi(0) = 0$, yields $V(T,x,T,0) = \frac{1}{2}x^\top Kx$ and the boundary condition (5.22) is satisfied. Hence

$$\begin{aligned}
\dot{s} &= 1\,,\\
\dot{\xi} &= -kV_\xi^\top = -k\left(\dot{\bar{P}}(s)\tau + \bar{P}(s)\right)\xi\,,\\
u &= -B^\top V_x^\top = -B^\top \bar{P}(\tau)x + B^\top(\bar{P}(\tau) - \bar{P}(s))x\,,
\end{aligned} \tag{5.32}$$

with $s(0) = 0$ and $\xi(0) = 0$, solves the regional dynamic finite-horizon optimal control problem. ◇

Proof. The proof is performed in two steps. First we show that the function V in (5.31) satisfies the partial differential inequality (5.21). Then we prove that the boundary condition (5.22) is satisfied with the initialization of the state of the dynamic extension given in the statement.

Consider the partial derivatives of the function V defined in (5.31), namely

$$\begin{aligned}
V_x &= \xi^\top \bar{P}(s) + (x-\xi)^\top R(s)\,,\\
V_\tau &= \tfrac{1}{2}\xi^\top \dot{\bar{P}}(s)\xi + b(\tau - s)\,,\\
V_\xi &= x^\top \bar{P}(s) + \xi^\top \dot{\bar{P}}(s)\tau - (x-\xi)^\top R(s)\,,\\
V_s &= \xi^\top \frac{\partial \bar{P}}{\partial s}(s)x + \frac{1}{2}\xi^\top \frac{\partial \dot{\bar{P}}}{\partial s}(s)\xi\tau + \frac{1}{2}(x-\xi)^\top \frac{\partial R}{\partial s}(s)(x-\xi) - b(\tau - s)\,.
\end{aligned} \tag{5.33}$$

Note that (the same consideration applies also to the partial derivative of $\dot{\bar{P}}$ and R with respect to s)

$$\frac{\partial \bar{P}(s(t))}{\partial s} = \dot{\bar{P}}(s(t))\,,$$

since $\dot{s} = 1$. Recalling that $\bar{P}(\tau)$ is positive definite by assumption and that $R(\tau) = \bar{P}(\tau)$ for all $\tau \in [0,\, T]$, by direct substitution of (5.33) and (5.32) in (5.21), the latter reduces to

$$\xi^\top \left[-k\left(\dot{\bar{P}}(s)\tau + \bar{P}(s)\right)\left(\dot{\bar{P}}(s)\tau + \bar{P}(s)\right) + \dot{\bar{P}}(s) + \frac{1}{2}\ddot{\bar{P}}(s)\tau\right]\xi \le 0\,, \tag{5.34}$$

hence there exist $\bar{T} > 0$ and $\bar{k} \ge 0$, which in general depends on $\bar{T}$, such that the partial differential inequality (5.21) is satisfied for all $k > \bar{k}$ and all $\tau \le \bar{T}$.

Finally note that, by the definition of the dynamics of ξ and s and setting $\xi(0) = 0$ and $s(0) = 0$, $\xi(t) = 0$ for all $t \ge 0$ and $s(T) = T$, hence

$$V(T, x, T, 0) = \frac{1}{2}x^\top R(T)x = \frac{1}{2}x^\top \bar{P}(T)x\,,$$

and the boundary condition (5.22) is satisfied. □

Corollary 5.3. Consider the linear system (5.12) and the cost (5.2) with $q(x) = x^\top Qx$. Let $\bar{P}(\tau)$ be the solution of the differential Riccati equation (5.13) with the boundary condition $\bar{P}(T) = K$. Let $R(\tau) = \bar{P}(\tau)$ for all $\tau \in [0, T]$. Suppose that

$$rank\left(\bar{P}(s(t)) + \tau(t)\dot{\bar{P}}(s(t))\right) = n\,, \tag{5.35}$$

for all $t \ge 0$.

Then the time-varying control law (5.32), with $s(0) = 0$ and $\xi(0) = 0$, solves the regional dynamic finite-horizon optimal control problem for all $T > 0$. ⋄

Proof. The proof follows immediately from the proof of Proposition 5.2 noting that, by the condition (5.35), for any $T > 0$ there exists $\bar{k}$ such that the inequality (5.34) is satisfied for all $k > \bar{k}$. □

Remark 5.8. The control law (5.32), in the linear case, is consistent with the control law (5.24) given in Proposition 5.1. In fact, (5.32) can be obtained from (5.24) letting $\gamma = \bar{\gamma} = 1$ and noting that, by the assumptions in Proposition 5.2,

$$\dot{\xi} = -k\left[\dot{\bar{P}}(s)\tau + \bar{P}(\tau)\right]\xi\,,$$

with an equilibrium point in $\xi = 0$, hence $\xi(0) = 0$, $s(0) = 0$ provides a solution to the *two-point boundary value* problem defined in Proposition 5.1. ▲

5.3.3 Example

To illustrate the results of Section 5.3.1 consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= x_1x_2 + u\,, \end{aligned} \tag{5.36}$$

with $x(t) = (x_1(t), x_2(t))^\top \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$ and the cost

$$J(u) = \frac{1}{2}\int_0^T u(t)^2 dt + \frac{1}{2}\left[x_1(T)^2 + x_2(T)^2\right]\,. \tag{5.37}$$

Note that no running cost is imposed on the state of the system, hence only the value of the state at the terminal time is penalized, together with the control effort. Let

$$\bar{P}(\tau) = \begin{bmatrix} \bar{p}_{11}(\tau) & \bar{p}_{12}(\tau) \\ \bar{p}_{12}(\tau) & \bar{p}_{22}(\tau) \end{bmatrix}$$

be the solution of the differential Riccati equation (5.13) with the boundary condition $\bar{P}(T) = I$. Note that the solution $\bar{P}(\tau)$ is such that the condition (5.35) holds for all $t \geq 0$. Letting $\Sigma = \vartheta I$, where I is the identity matrix and $\vartheta > 0$, consider

$$\begin{aligned} p(\tau, x) &= x^\top \bar{P}(\tau) + \mathcal{Q}(\tau, x)\,, \\ r(\tau, x) &= \tfrac{1}{2}x^\top \dot{\bar{P}}(\tau)x\,, \end{aligned} \tag{5.38}$$

with $\mathcal{Q} = [\mathcal{Q}_1,\ \mathcal{Q}_2] \in \mathbb{R}^{1\times 2}$ and

$$\mathcal{Q}_1(\tau, x) = -\bar{p}_{12}(\tau)x_1^2 - \bar{p}_{22}(\tau)x_1x_2\,,$$

$$\mathcal{Q}_2(\tau, x) = -\bar{p}_{12}(\tau)x_1 - \bar{p}_{22}(\tau)x_2 + x_1x_2 + \sqrt{\chi(x_1, x_2, \tau)}\,, \tag{5.39}$$

with

$$\begin{aligned} \chi(x_1, x_2, \tau) \triangleq{} & \bar{p}_{12}(\tau)^2x_1^2 + 2\bar{p}_{12}(\tau)\bar{p}_{22}(\tau)x_1x_2 + \bar{p}_{22}(\tau)^2x_2^2 \\ & - 2\bar{p}_{12}(\tau)x_1^2x_2 - 2\bar{p}_{22}(\tau)x_1x_2^2 + x_1^2x_2^2 + 2\vartheta x_1^2 + 2\vartheta x_2^2\,. \end{aligned}$$

It can be shown that $P(\tau, x) = [p(\tau, x), r(\tau, x)]$, with p and r defined in (5.38), is an *algebraic* $\bar{P}$ *solution* for the system (5.36) as defined in (5.16).

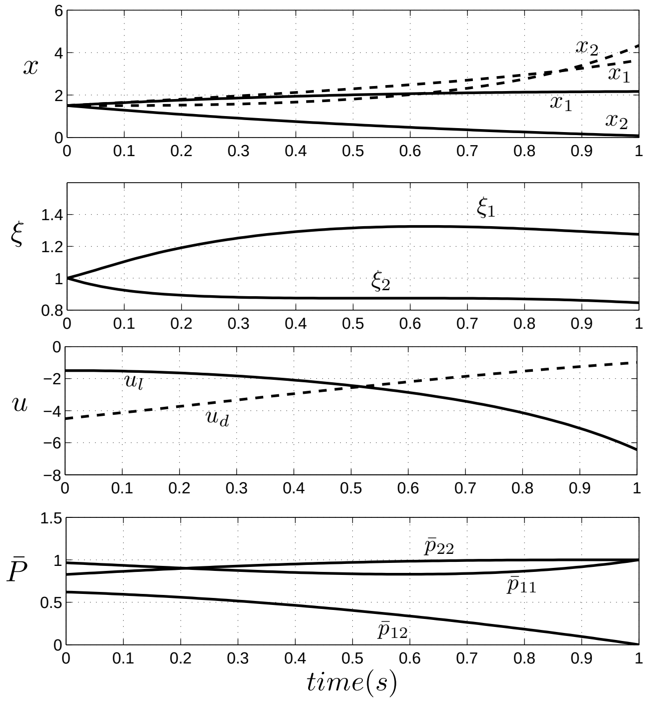

Figure 5.1: Top graph: Time histories of the state of the system (5.36) in closed-loop with the linear control law (dashed line) and with the dynamic control law (5.24) (solid line). Upper Middle graph: Time histories of the state of the dynamic extension ξ, with $\xi(0)$ such that ρ_2 is minimized. Lower Middle graph: Time histories of the control action u_l (dashed line) and of the dynamic control law (5.24) (solid line). Bottom graph: Time histories of the entries of the matrix $\bar{P}(t)$, solution of the differential Riccati equation (5.13).

The dynamic solution (5.24) proposed herein is compared with the optimal solution of the linearized problem, namely $u_l = -B^\top \bar{P}(\tau)x$. Note that the control law u_l is designed for the linear part of the system (5.36), which is described by a double integrator, and may perform poorly on the nonlinear system (5.36).

The simulations show that the performances of the dynamic control (5.24), obtained determining a solution of the algebraic equation (5.16) and constructing the *augmented value function* V as in (5.18), compares favorably to the performances of the control law u_l.

To show the latter claim we consider the ratio

$$\eta = \frac{J(u_d)}{J(u_l)},$$

for the same initial condition, where J is defined in (5.37) and u_d denotes the dynamic control law (5.24). A ratio smaller than one implies that the cost *yielded* by the dynamic

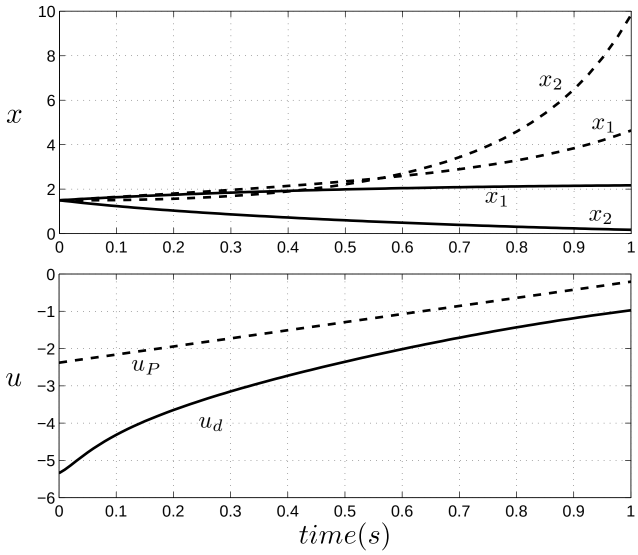

Figure 5.2: Top graph: Time histories of the state of the system (5.36) in closed-loop with the control law provided by the Minimum Principle (dashed line) and with the dynamic control law (5.24) (solid line). Bottom graph: Time histories of the control action u_P (dashed line) and of the dynamic control law (5.24) (solid line).

control law is smaller than the cost yielded by the linear control law u_l.

In the first simulation we select $(x_1(0), x_2(0)) = (\frac{3}{2}, \frac{3}{2})$, $R = \kappa I_2$, with $\kappa > 1$, $\bar{\gamma} = 1$ and the terminal time $T = 1$. The gain $k(\tau, x, s, \xi)$ is selected as in Corollary 5.2 and it is such that the partial differential equation (5.21) and consequently the additional cost ρ_1 are identically zero. The top graph of Figure 5.1 displays the time histories of the state of the system (5.36) in closed-loop with the linear control law u_l (solution of the linearized problem) and with the dynamic control law (5.24). The time histories of the control signals u_l and (5.24) are depicted in the lower middle graph of Figure 5.1. Note that, considering ρ_1 as defined in Proposition 5.1 and ρ_2 obtained as in Remark 5.6, the ratio η is equal to 0.3083. The upper middle graph of Figure 5.1 shows the time histories of the dynamic extension ξ with $\xi(0)$ selected such that ρ_2, namely the additional cost on the terminal *augmented* state, is minimized. Finally, the time histories of the elements of the matrix $\bar{P}(\tau)$, solution of the differential Riccati equation (5.13) are displayed in the bottom graph of Figure 5.1. The top graph of Figure 5.2 displays the time histories of the state of the system (5.36) driven by the open-loop control resulting from the Minimum Principle applied to the linearized system, namely $u_P = -\lambda_2$ with $\dot{\lambda} = -A^\top \lambda$ and $\lambda(0) = \bar{P}(0)x(0)$, and by the dynamic control law (5.24). The bottom graph shows the time histories of the control action u_P and of the dynamic control law (5.24). Note that the open-loop control u_P does not compensate for the nonlinear dynamics neglected in the design, see Figure 5.2. Figure 5.3 displays the phase portraits of the system (5.36) in closed-loop with

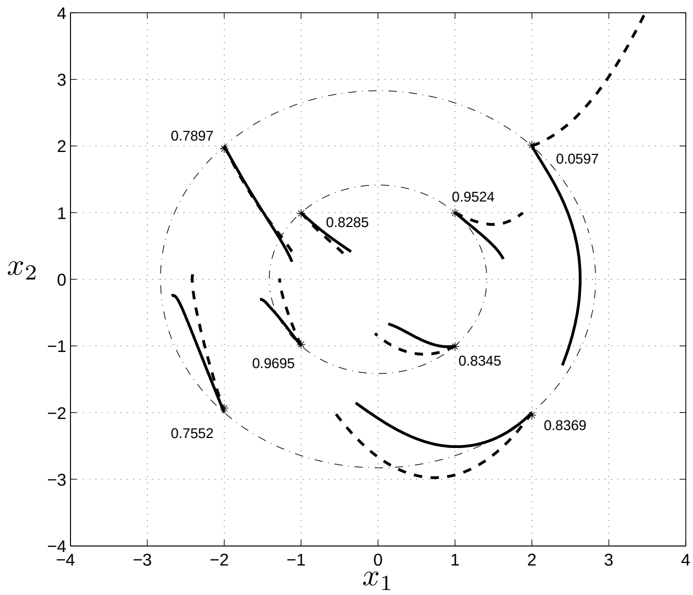

Figure 5.3: Phase portraits of the system (5.36) in closed-loop with the linear control law u_l (dashed line) and with the dynamic control law (5.24) (solid line). The values denote the ratio η.

the linear control law u_l and with the dynamic control law (5.24).

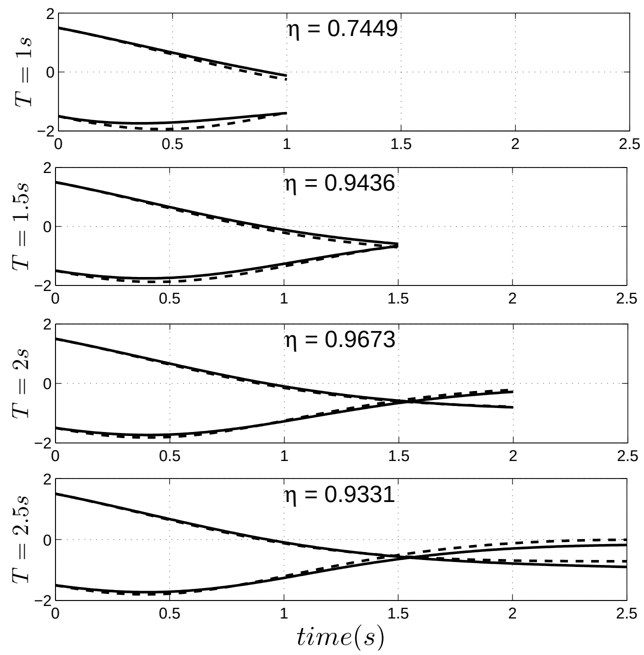

Figure 5.4: Time histories of the states of the system (5.36) in closed-loop with the linear control law u_l (dashed line) and with the dynamic control law (5.24) (solid line) for different terminal times.

Finally, Figure 5.4 shows the time histories of the system (5.36) in closed-loop with the linear control law u_l and with the dynamic control law (5.24) for different terminal times, namely $T = 1, 1.5, 2, 2.5$, and the initial condition $(x_1(0),\, x_2(0)) = (2,\, -2)$.

5.4 Infinite-Horizon Optimal Control Problem

As discussed in Section 5.2, the *infinite-horizon* optimal control problem with stability consists in finding a control input u that minimizes the cost functional

$$J(u(t)) = \frac{1}{2}\int_0^\infty (q(x(t)) + u(t)^\top u(t))dt\,, \tag{5.40}$$

where $q : \mathbb{R}^n \to \mathbb{R}_+$ is positive semi-definite, subject to the dynamical constraint (5.1), the initial condition $x(0) = x_0$ and the requirement that the zero equilibrium of the closed-loop system be locally asymptotically stable.

The classical optimal control design methodology relies upon the solution of the HJB equation [11, 13, 103]

$$\min_u \left\{ V_x(f(x) + g(x)u) + \frac{1}{2}q(x) + \frac{1}{2}u^\top u \right\} = 0\,. \tag{5.41}$$

The solution of the HJB equation (5.41), if it exists, is the *value function* of the optimal control problem, *i.e.* it is a function which associates to every point in the state space, x_0, the optimal cost of the trajectory of system (5.1) with $x(0) = x_0$, *i.e.*

$$V(x_0) = \min_u \frac{1}{2}\int_0^\infty (q(x(t)) + u(t)^\top u(t))dt\,. \tag{5.42}$$

The knowledge of the value function on the entire state space allows to determine the minimizing input for all initial conditions. It is easy to check that the minimum of equation (5.41) with respect to u is attained for

$$u_o = -g(x)^\top V_x^\top\,. \tag{5.43}$$

Thus, if we are able to solve analytically the partial differential equation

$$V_x f(x) - \frac{1}{2}V_x g(x)g(x)^\top V_x^\top + \frac{1}{2}q(x) = 0\,, \tag{5.44}$$

we can design the optimal control law given by (5.43). We formalize the above discussion in the following definition.

Problem 5.3. Consider the system (5.1), with Assumptions 5.1 and 5.2, and the cost functional (5.40). The *infinite-horizon regional dynamic optimal control* problem with stability consists in determining an integer $\tilde{n} \geq 0$, a dynamic control law described by equations (4.2) and a set $\bar{\Omega} \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ containing the origin of $\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ such that the closed-loop system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\beta(x,\xi)\,, \\ \dot{\xi} &= \alpha(x,\xi)\,, \end{aligned} \tag{5.45}$$

has the following properties.

(i) The zero equilibrium of the system (5.45) is asymptotically stable with region of attraction containing $\bar{\Omega}$.

(ii) For any $\bar{u}$ and any (x_0, ξ_0) such that the trajectory of the system (5.45) remains in $\bar{\Omega}$ the inequality $J(\beta) \leq J(\bar{u})$ holds.

In this context we consider also an approximate problem in the sense specified by the following statement.

Problem 5.4. Consider the system (5.1), with Assumptions 5.1 and 5.2, and the cost functional (5.40). The *approximate regional dynamic optimal control* problem with stability consists in determining an integer $\tilde{n} \geq 0$, a dynamic control law described by equations (4.2), a set $\bar{\Omega} \subset \mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ containing the origin of $\mathbb{R}^n \times \mathbb{R}^{\tilde{n}}$ and a non-negative function $c : \mathbb{R}^n \times \mathbb{R}^{\tilde{n}} \to \mathbb{R}_+$ such that the regional dynamic optimal control problem is solved with respect to the running cost $q(x) + c(x,\xi) + u^\top u$.

It is worth recalling that in the linearized case the solution of the optimal control problem is a linear static state feedback of the form $u = -B_1^\top \bar{P}x$, where $\bar{P}$ is the symmetric positive definite solution of the algebraic Riccati equation

$$\bar{P}A + A^\top \bar{P} - \bar{P}B_1B_1^\top \bar{P} + Q = 0\,, \tag{5.46}$$

the matrices A and B_1 are defined in (4.7) and

$$Q \triangleq \frac{1}{2}\frac{\partial^2 q}{\partial x^2}\bigg|_{x=0}\,.$$

Finally, mimicking the discussion in the previous chapter, we investigate the relation between a Dynamic Value function and the solution of the approximate regional dynamic optimal control problem.

Lemma 5.2. Consider the system (5.1), with Assumptions 5.1 and 5.2, and the cost functional (5.40). Let $\mathcal{V} = (D_\alpha, V)$ be a Dynamic Value function for the system (5.1) for all $(x,\, \xi) \in \Omega$ such that V is positive definite around the origin and the zero-equilibrium of $\dot{\xi} = \alpha(0, \xi)$ is asymptotically stable[2].

Then the dynamical system (5.3), which reduces to

$$\begin{aligned} \dot{\xi} &= \alpha(x, \xi)\,, \\ u &= -g(x)^\top V_x(x, \xi)^\top\,, \end{aligned} \tag{5.47}$$

in the infinite-horizon case, solves the approximate regional dynamic optimal control problem with stability for all $(x,\, \xi) \in \bar{\Omega}$, where $\bar{\Omega}$ is the largest level set of V contained in Ω. $\diamond$

Proof. The claim follows directly noting that, by (5.4) which reduces to

$$\mathcal{HJB}_i(x, \xi) \triangleq V_x f(x) + V_\xi \alpha(x,\, \xi) + \frac{1}{2} q(x) - \frac{1}{2} V_x g(x) g(x)^\top V_x^T \leq 0\,, \tag{5.48}$$

in the infinite-horizon case, V is a continuously differentiable value function for the closed-loop system (5.1)-(5.47), with respect to a cost functional of the form (5.40) and a running cost $q(x) + c(x,\, \xi) + u^\top u$, where the additional cost $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ is defined as $c(x,\, \xi) = -2\,\mathcal{HJB}_i(x,\, \xi)$. Moreover, asymptotic stability of the origin is proved, letting V be a candidate Lyapunov function, combining (5.48) with the Assumption 5.2 and asymptotic stability of the zero-equilibrium of $\dot{\xi} = \alpha(0, \xi)$. $\square$

5.4.1 Algebraic $\bar{P}$ Solution and Dynamic Value Function

In the previous section we have shown that a Dynamic Value functions is strongly related to the solution of the approximate optimal control Problem 5.2. In the following we present a constructive methodology to systematically define a class of Dynamic Value function for input-affine nonlinear systems without requiring the solution of any partial differential equation. Towards this end, similarly to Definition 5.2, consider the HJB equation (5.44) and suppose that it can be solved *algebraically*, as detailed in the following definition.

Definition 5.3. Consider the system (5.1) and the cost functional (5.40). Let $\sigma(x) \triangleq x^\top \Sigma(x) x \geq 0$, for all $x \in \mathbb{R}^n$, with $\Sigma : \mathbb{R}^n \to \mathbb{R}^{n \times n}$, $\Sigma(0) = 0$. A $\mathcal{C}^1$ mapping $P : \mathbb{R}^n \to \mathbb{R}^{1 \times n}$, $P(0) = 0$, is said to be a *$\mathcal{X}$-algebraic $\bar{P}$ solution* of equation (5.44) if the following holds.

[2] As pointed out in the previous chapter, this requirement may be replaced by zero-state detectability of the closed-loop extended system

$$\begin{aligned} \dot{x} &= f(x) + g(x)\beta(x, \xi)\,, \\ \dot{\xi} &= \alpha(x, \xi)\,, \end{aligned}$$

with respect to the output $y = q(x)$.

(i) For all $x \in \mathcal{X} \subseteq \mathbb{R}^n$

$$P(x)f(x) - \frac{1}{2}P(x)g(x)g(x)^\top P(x)^\top + \frac{1}{2}q(x) + \sigma(x) = 0. \tag{5.49}$$

(ii) P is tangent at $x = 0$ to the symmetric positive definite solution of (5.46), *i.e.* $P_x^\top(x)\Big|_{x=0} = \bar{P}$.

If condition (i) holds for all $x \in \mathbb{R}^n$, *i.e.* $\mathcal{X} = \mathbb{R}^n$, then P is an *algebraic* $\bar{P}$ *solution.*

Finally, similarly to Chapter 4, define the *extension* of P, namely the function V in (4.11).

In what follows a systematic construction of a class of Dynamic Value function is proposed and the approximate regional dynamic nonlinear optimal control problem is solved.

Theorem 5.1. Consider the system (5.1), with Assumptions 5.1 and 5.2 and the cost defined in (5.40). Let P be an *algebraic* $\bar{P}$ *solution* of (5.44). Let the matrix $R = R^\top > 0$ be such that V defined in (4.11) is positive definite in a set $\Omega \subseteq \mathbb{R}^{2n}$ containing the origin and such that

$$\frac{1}{2}A_{cl}(x)^\top \Delta + \frac{1}{2}\Delta^\top A_{cl}(x) < \Sigma(x) + \frac{1}{2}\Delta^\top g(x)g(x)^\top \Delta\,, \tag{5.50}$$

for all $(x, \xi) \in \Omega \setminus \{0\}$, with Δ defined in (4.12). Then there exists $\bar{k} \geq 0$ such that for all $k > \bar{k}$ the function V satisfies the Hamilton-Jacobi-Bellman inequality (5.48) for all $(x, \xi) \in \Omega$, with $\dot{\xi} = -kV_\xi^\top = -k(P_\xi(\xi)^\top x - R(x - \xi))$. Hence $\mathcal{V} = (D_\alpha, V)$, with D_α defined as

$$\begin{aligned} \dot{\xi} &= -k(P_\xi(\xi)^\top x - R(x - \xi))\,, \\ u &= -g(x)^\top \left[P(x)^\top + (R - \Phi(x, \xi))(x - \xi)\right]\,, \end{aligned} \tag{5.51}$$

and V as in (4.11), is a Dynamic Value function for the system (5.1) and, by Lemma 5.2, (5.51) solves the approximate regional dynamic optimal control problem with $c(x, \xi) \geq 0$ such that $\mathcal{HJB}_i(x, \xi) + \frac{1}{2}c(x, \xi) = 0$ for all $(x,\, \xi) \in \bar{\Omega}$, where $\bar{\Omega}$ is the largest level set of V contained in Ω. $\diamond$

Proof. The proof follows the same lines of the proof of Theorem 4.1. The Hamilton-Jacobi-Bellman inequality for the *extended* system (5.45), *i.e.* the inequality (5.48), considering the partial derivatives of V as in (4.17), the controller as in (5.51) and recalling that the function P is an *algebraic $\bar{P}$ solution* of the equation (5.49), can be written as

$$\begin{aligned}\mathcal{HJB}_i(x,\xi) =& - x^\top \Sigma(x) x + (x-\xi)^\top (R-\Phi)^\top F(x) x \\ & - \frac{1}{2}(x-\xi)^\top (R-\Phi)^\top g(x) g(x)^\top (R-\Phi)(x-\xi) \\ & - x^\top N(x)^\top g(x) g(x)^\top (R-\Phi)(x-\xi) \\ & - k(P_\xi(\xi)^\top x - R(x-\xi))^\top (P_\xi(\xi)^\top x - R(x-\xi)) \leq 0 \,.\end{aligned} \tag{5.52}$$

Rewriting (5.52) as a *quadratic* form in x and $(x-\xi)$ yields (4.19) where $C(\xi)$ and $M(x,\xi)$ are defined in the proof of Theorem 4.1, with $\Pi(x) = -g(x)g(x)^\top$. Let the space spanned by the columns of the matrix $Z(\xi)$ be the null space of $C(\xi)$. Consider now the restriction of the matrix $M(x,\xi)$ to the subspace defined by the columns of the matrix $Z(\xi) \in \mathbb{R}^{2n\times n}$. The condition (5.50) guarantees that the matrix $Z(\xi)^\top M(x,\xi) Z(\xi)$ is positive definite. This proves, by [3], that there exists $\bar{k} \geq 0$ such that the Hamilton-Jacobi-Bellman inequality (5.48) is satisfied for all $(x,\xi) \in \Omega \setminus \{0\}$ and for all $k > \bar{k}$. Furthermore $\mathcal{HJB}_i(0,0) = 0$, hence, by continuity of the mappings in system (5.1) and in the control (5.51), the function $\mathcal{HJB}_i(x,\xi)$ is continuous and smaller than or equal to zero for all $(x,\xi) \in \Omega$. Hence the dynamic control law $\dot{\xi} = -kV_\xi(x,\xi)^\top$, $u = -g(x)^\top V_x(x,\xi)^\top$ is a dynamic optimal control for the system (5.1) and the running cost $q(x)+c(x,\xi)+u^\top u$. Finally, $V > 0$, by assumption, and $\dot{V} \leq 0$, by the condition (5.48), for all $(x,\xi) \in \Omega$. Hence, by LaSalle's invariance principle and zero-state detectability, the feedback (5.51) asymptotically stabilizes the zero equilibrium of the closed-loop system. □

Remark 5.9. The problem solved is intrinsically different from the so-called *inverse optimal control problem* [29], where it is shown that optimality with respect some cost functional guarantees several robustness properties of the closed-loop system. In fact, herein optimality of the control law is ensured with respect to a cost functional which upperbounds the original one, *i.e.* the approximate optimal control is determined with respect to the original cost and an *additional cost.* ▲

In what follows two different methodologies to reduce the *approximation error* of the solution of Problem 5.2 are proposed. The running cost imposed on the state of the extended system, namely $q(x) + c(x,\xi)$, can be *shaped* to approximate the original cost, on one hand, and the initial condition of the dynamic extension $\xi(0)$ can be selected to obtain the minimum value of the cost *paid* by the solution on the other hand, as detailed in the following results.

Theorem 5.2. Consider the system (5.1), with Assumptions 5.1 and 5.2 and the cost defined in (5.40). Let P be an *algebraic $\bar{P}$ solution* of (5.44). Let $\mathcal{M}_C \triangleq \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n : P_\xi(\xi)x - R(x-\xi) = 0\}$ and the matrix $R = R^\top > 0$ be such that V defined in (4.11) is positive definite in a set $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^n$ containing the origin and such that the condition

$$0 < \Sigma(x) + \frac{1}{2}\Delta^\top g(x)g(x)^\top \Delta - \frac{1}{2}A_{cl}(x)^\top \Delta - \frac{1}{2}\Delta^\top A_{cl}(x) \leq \varepsilon I \tag{5.53}$$

is satisfied for all $(x, \xi) \in \Omega \backslash \{0\}$ and for some $\varepsilon \in \mathbb{R}_+$. Let $K : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$ be defined as $K(x, \xi) = (CC^\top)^{-1}\mathcal{G}(x, \xi)(CC^\top)^{-1}$, where $\mathcal{G} = CMZ\left(Z^\top MZ\right)^{-1} Z^\top MC^\top - CMC^\top$ and suppose that K is continuous. Then the control law

$$\begin{aligned} \dot{\xi} \ &= -K(x, \xi)(P_\xi(\xi)^\top x - R(x - \xi)) , \\ u \ &= -g(x)^\top \left[P(x)^\top + (R - \Phi(x, \xi))(x - \xi)\right] , \end{aligned} \tag{5.54}$$

solves the approximate regional dynamic optimal control problem with $c(x, \xi) = 0$ for all $(x, \xi) \in \Omega \setminus \mathcal{M}_C$ and $0 \leq c(x, \xi) \leq \varepsilon y_1^\top y_1$, for all $(x, \xi) \in \mathcal{M}_C$, with

$$y_1 = \begin{bmatrix} I_n & (Z^\top MZ)^{-1}Z^\top MC^\top \end{bmatrix} \begin{bmatrix} Z^\top \\ C \end{bmatrix}^{-1} \begin{bmatrix} x \\ x - \xi \end{bmatrix} .$$

◇

Proof. Let $T = [Z\ C^\top]$ and note that the matrix T is full rank. Multiply, to the left and to the right, the matrix $M(x) + C(\xi)^\top K(x, \xi)C(\xi)$ by $T^\top$ and T, respectively, yielding

$$\begin{bmatrix} Z^\top MZ & Z^\top MC^\top \\ CMZ & CMC^\top + CC^\top KCC^\top \end{bmatrix} , \tag{5.55}$$

where we have used the fact that $C(\xi)Z(\xi) = 0$ and that $C(\xi)C(\xi)^\top \in \mathbb{R}^{n\times n}$ is non-singular, for all $\xi \in \mathbb{R}^n$. Note that the quadratic forms associated to the matrices (5.55) and $M + C^\top KC$ are congruent. By Sylvester's law of inertia, the inertia of the matrix (5.55) is equal to the inertia of the matrix $M + C^\top KC$. The matrix valued function K is selected to maximize the number of zero eigenvalues of the *quadratic* form c. In particular the matrix (5.55) loses rank if and only if the Schur complement of the element $Z^\top MZ$, *i.e.* $CMC^\top + CC^\top KCC^\top - CMZ(Z^\top MZ)^{-1}Z^\top MC^\top$, is zero. Note that $Z^\top MZ$ is positive definite by condition (5.53), as shown in the proof of Theorem 5.1, hence the Schur complement is well-defined. Thus the matrix $K(x, \xi)$ is selected to zero the Schur complement of the term $Z^\top MZ$ and it is such that the additional cost $c(x, \xi)$ is identically zero for all $(x, \xi) \in \Omega \setminus \mathcal{M}_C$ whereas the condition (5.53) implies that $c(x, \xi)$ is upper-bounded by $\varepsilon y_1^\top y_1$ for all $(x, \xi) \in \mathcal{M}_C$, proving the claim. □

Remark 5.10. If the closed-loop system (5.45), with α and β as in (5.54), is such that $(x(t), \xi(t)) \in \Omega \setminus \mathcal{M}_C$ for all $t \geq 0$, then the control law (5.54) is the optimal solution with respect to the original cost (5.40). ▲

We conclude the section noting that, by definition of value function, $V(x(0), \xi(0))$ is the cost yielded by the dynamic control law defined in (5.54) for the system (5.1) initialized in $(x(0), \xi(0))$. Therefore to minimize the cost, for a given initial condition $\bar{x}(0)$ of the system (5.1), it is possible to select the initial condition of the dynamic extension $\xi(0)$ such that

$$\xi(0) = \arg\min_{\xi} V(\bar{x}(0), \xi) \,. \tag{5.56}$$

5.4.2 Linear systems

In this section we consider linear systems described by equations of the form

$$\dot{x} = Ax + B_1 u \,, \tag{5.57}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$ are defined in (4.7). The following statement represents a particularization of the results presented in Theorem 5.1 to linear time invariant systems. It is interesting to note that the methodology proposed yields the standard solution of the infinite horizon quadratic optimal control problems.

Proposition 5.3. Consider system (5.57) and let $q(x) = x^\top Q x$ in (5.40) for all $x \in \mathbb{R}^n$. Suppose that there exists a positive definite matrix $\bar{P} = \bar{P}^\top \in \mathbb{R}^{n\times n}$ such that[3]

$$\bar{P}A + A^\top \bar{P} - \bar{P}B_1 B_1^\top \bar{P} + Q = 0 \,. \tag{5.58}$$

Then there exists $R = R^\top > 0$ such that the conditions in Theorem 5.1 hold for all $(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n$. Moreover the control law (5.51) *reduces* to (4.32), hence the standard solution to the infinite horizon optimal control problem for linear systems with quadratic cost is recovered. ◇

Proof. The proof of Proposition 5.3 follows the same steps of the proof of Proposition 4.1. □

Remark 5.11. The choice of the initial condition of the dynamic extension ξ is determined as the solution of the minimization problem (5.56). Note that in the linear case the minimizer can be determined explicitly as a function of the initial condition of the system, namely

$$\xi(0) = R^{-1}(R - \bar{P})\bar{x}(0) = 0 \,, \tag{5.59}$$

which yields $\xi(t) = 0$ for all $t \geq 0$. ▲

[3] Since $\Sigma(0) = 0$, the equation is the linear version of the condition (5.49) in the definition of the algebraic $\bar{P}$ solution.

5.4.3 Example: Controlled Van Der Pol Oscillator

Consider the single-input, single-output, nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= -x_1 - \mu(1 - x_1^2)x_2 + x_1 u\,, \end{aligned} \tag{5.60}$$

known as the controlled Van der Pol oscillator, with $x(t) = (x_1(t)\,, x_2(t))^\top \in \mathbb{R}^2$ and $u(t) \in \mathbb{R}$. The parameter μ describes the strength of the damping effect and in the following is selected as $\mu = 0.5$, hence the oscillator has a stable but linearly uncontrollable equilibrium at the origin surrounded by an unstable limit cycle. Define the positive definite cost integrand $q(x) = x_2^2$, *i.e.* the control action minimizes the *speed* of the oscillator together with the control effort.

Note that the solution of the HJB equation for system (5.60) with the given cost can be explicitly determined, namely $V_o(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$, and the resulting optimal control is $u_o = -x_1 x_2$. The interesting aspect of the existence of the optimal solution is in allowing the explicit comparison between the optimal control feedback, the dynamic solution and the optimal solution of the linearized problem, as detailed in the following.

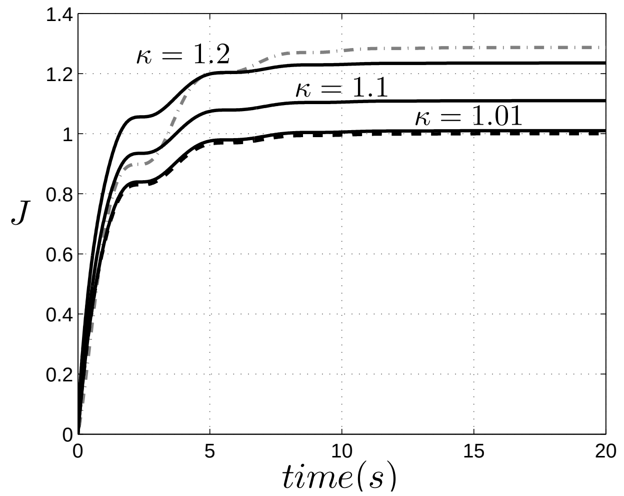

Figure 5.5: Time histories of the integral of the running cost applying the control law u_o from the initial condition $x(0) = (1, -1)$ (dashed line) and the control law $u = 0$, *i.e.* the optimal solution of the linearized problem, (dash-dotted line) and the dynamic control from the initial condition $(x(0), \xi(0)) = (1, -1, 0, 0)$ (solid lines), for different values of κ.

The linearization of the nonlinear system around the origin is given by $\dot{x} = Ax$, *i.e.* it is not affected by u, with A Hurwitz, and the solution of the algebraic Riccati equation (5.46) corresponding to the linearized problem is $\bar{P} = I$. The selection $\Sigma(x) = \frac{\lambda}{2}\ diag\{x_2^2,\, x_1^2\}$ with $\lambda > 0$, is such that $P(x) = [x_1(1 - \lambda x_1 x_2)\,, x_2]$ is an *algebraic* $\bar{P}$ *solution* of the Hamilton-Jacobi-Bellman equation. Note that for this solution the condition (1.1) is not

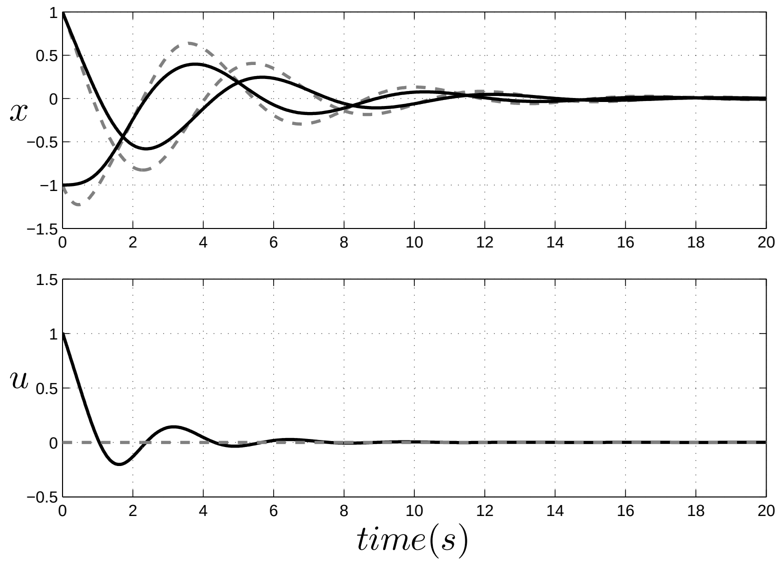

Figure 5.6: Top graph: time histories of the state of the system (5.60) driven by the control law $u(x,\xi)$, with $\kappa = 1.01$, and by the optimal solution of the linearized problem, *i.e.* $u = 0$, solid and dashed lines, respectively. Bottom graph: time histories of the control input $u(x(t),\xi(t))$ (solid line) and the input $u = 0$ (dashed line).

satisfied, hence P is not an exact differential. Let $R = diag(\kappa,\kappa)$ with $\kappa > 1$, then the function

$$V(x,\xi) = x_1\xi_1(1-\xi_1\xi_2) + x_2\xi_2 + \frac{\kappa}{2}(x_1-\xi_1)^2 + \frac{\kappa}{2}(x_2-\xi_2)^2 \tag{5.61}$$

is locally positive definite in the set $\Omega_1 = \{\ (x,\xi) \in \mathbb{R}^4 : \xi_1\xi_2 \le 1, \xi_1\xi_2 > 1-2\kappa\}$ and moreover the condition (5.50) is strictly satisfied for all $(x,\xi) \in \Omega\setminus\{0\}$, where $\Omega = \Omega_1\cap\Omega_2 \triangleq \{(x,\xi)\in\mathbb{R}^4 : \zeta_\kappa(x,\xi) > 0\}$, ζ_κ being a known continuous function parameterized by κ. Note that the set Ω_2 is non-empty for any $\kappa > 1$ and it is empty for $\kappa = 1$. Thus it is possible to determine a value $\bar{k} \ge 0$ such that the *extended* Hamilton-Jacobi-Bellman inequality (5.48) holds for all $k > \bar{k}$. Since $R - \Phi(x,\xi)$ is not zero in $Im(g(x))$ then the dynamic control law

$$\begin{aligned}
\dot{\xi}_1 &= -k\left(x_1 - 2\lambda x_1\xi_1\xi_2 - \lambda\xi_1^2 x_2 - \kappa(x_1-\xi_1)\right)\,,\\
\dot{\xi}_2 &= -k\left(x_2 - \kappa(x_2-\xi_2)\right)\,,\\
u &= -\kappa x_1x_2 + (\kappa-1)x_1\xi_2
\end{aligned}$$

solves the dynamic optimal control problem with running cost $x_2^2 + u^2 + c(x,\xi)$, where $c(x,\xi) \ \ge\ 0$ is defined as in Theorem 5.1.

The control law $u(x,\xi)$ is such that for any $\epsilon > 0$ there exists $\kappa > 1$ such that the ratio $\rho = V(x(0),0)/V_o(x(0))$ is smaller than ϵ. Figure 5.5 shows the time histories of the integral of the running cost applying the control law u_o from the initial condition $x(0) = (1, -1)$ (dashed line), the control law $u = 0$, *i.e.* the optimal solution of the linearized problem, (dash-dotted line) and the dynamic control from the initial condition $(x(0), \xi(0)) = (1, -1, 0, 0)$ (solid line), for different values of κ. The top graph of Figure 5.6 displays the trajectories of the system (5.60) driven by the control law $u(x, \xi)$ and by the optimal solution of the linearized problem, *i.e.* $u = 0$, solid and dashed lines, respectively. The bottom graph shows the time histories of the control input $u(x(t), \xi(t))$, with $\kappa = 1.01$, (solid line) and the input $u = 0$ (dashed line). It is important to stress that the value function V_o is determined in principle as the solution of a partial differential equation while simple algebraic conditions are involved in the computation of V in (5.61).

5.5 Conclusions

In this chapter we have discussed the finite-horizon and infinite-horizon optimal control problems. The key novelty lies in the definition of the notion of Dynamic Value function. These functions are derived from Dynamic Storage functions when the system is not affected by disturbance. Mimicking the construction of Chapter 4, it is shown that a value function is combined with a dynamical system, interpreted as a dynamic control law that approximates the solution of the optimal control problem. The approximation is twofold: a modified optimal control problem defined in an extended state-space is considered, on one hand, and partial differential inequalities are solved in place of equations, on the other hand. The resulting approximation errors may be reduced initializing appropriately the dynamic extension and/or shaping the additional running cost.

Chapter 6

Nonlinear Observer Design

6.1 Introduction

The problem of constructing observers for nonlinear systems has received increasing attention due to its importance in practical applications in which some of the states may not be measurable. Classic approaches to nonlinear observer design consist in finding a transformation that linearizes the plant up to an output injection term and then applying standard linear observer design techniques [30]. On the other hand, *high-gain* techniques have been developed that make it possible to robustly estimate the derivatives of the output [24]. A somewhat alternative methodology is to define linear observer dynamics with a nonlinear output mapping [2, 46]. The latter approach represents the natural extension of Luenberger's original ideas to the nonlinear case. Finally, large effort has been devoted to the definition of optimization-based observers, see for instance [65, 67] which explore the possibility of reconstructing the state of the system by minimizing a cost function over the preceding interval (moving horizon). Herein the observer design problem is formulated as a problem of rendering attractive an appropriately selected invariant set in the extended state-space of the plant and the observer, see also [45,70] for more detailed discussions.

We consider nonlinear time-varying systems described by equations of the form

$$\begin{aligned} \dot{y} &= f_1(y, x, t)\,, \\ \dot{x} &= f_2(y, x, t)\,, \end{aligned} \tag{6.1}$$

where $y(t) \in \mathbb{R}^m$ is the measured part of the state and $x(t) \in \mathbb{R}^n$ is the unmeasured part of the state. As proposed in [45] an asymptotically convergent observer for $x(t)$ can be obtained by rendering attractive an appropriately selected subset of the extended state-space of the plant and the observer. A parameterized description of the latter is given and the observer dynamics are selected to render the set invariant. The crucial issue is therefore the attractivity of the set, which has to be achieved by solving a partial differential equation. If the vector fields f_1 and f_2 of system (6.1) are linear in the unmeasured states then such pde takes the form

$$\beta_y = B(y), \tag{6.2}$$

where $\beta : \mathbb{R}^m \to \mathbb{R}^n$ is a mapping of dimension equal to the number of unmeasured states, and $B : \mathbb{R}^m \to \mathbb{R}^{n\times m}$ is a given matrix that depends on the system. When the dimension of y is larger than one, the pde (6.2) is solvable only if B is a Jacobian matrix, which occurs only under certain (restrictive) conditions. The difficulty in finding a solution for these equations has proven to be a major drawback for the original methodology proposed in [45].

In this chapter it is shown how to circumvent the issue of determining an explicit solution of the partial differential equation (6.2) still ensuring attractivity of the selected invariant set. The rest of the chapter is organized as follows. The construction of an Immersion and Invariance observer without pde's is described in Section 6.2 for nonlinear systems that are affine in the unmeasured state. Moreover, a nonlinear observer to estimate the rotor fluxes and the unknown (constant) load torque is designed for a voltage-fed induction motor. In Section 6.2.2 the techniques developed in Section 6.2 are specialized to the case in which the vector of unmeasured states consists solely of unknown constant parameters, which is the case for instance in adaptive control. These results allow to solve the aircraft longitudinal control problem in the presence of uncertain parameters. In the following section the observer design method is tested on the practical problem of 3-dimensional *structure* estimation. It is shown that a globally asymptotically convergent observer for the depth of a moving object observed through a single pin-hole camera can be constructed. Similarly, a locally convergent observer can be defined to estimate simultaneously the depth and the orientation of the observed object.

6.2 Nonlinear Observers for Systems Affine in the Unmeasured State

Consider a class of nonlinear systems described by equations of the form

$$\begin{aligned} \dot{y} &= f(y,u) + \Phi(y)x, \\ \dot{x} &= h(y,u) + A(y)x, \end{aligned} \tag{6.3}$$

where $y(t) \in \mathbb{R}^m$ is the measured part of the state and $x(t) \in \mathbb{R}^n$ is the unmeasured part of the state. Necessary and sufficient conditions for existence of a change of coordinates transforming a nonlinear system into the form of (6.3) are provided in [12]. Note that a similar observer design problem is approached in [75], where a full-order observer is proposed.

Define the observation error

$$z = \hat{x} - x = \zeta + \beta(y) - x, \tag{6.4}$$

where $\zeta(t) \in \mathbb{R}^n$ is the observer state and $\beta : \mathbb{R}^m \to \mathbb{R}^n$ is a mapping to be determined.

Defining the observer dynamics as

$$\dot{\zeta} = h(y,u) + A(y)\hat{x} - \beta_y \left(f(y,u) + \Phi(y)\hat{x} \right) \tag{6.5}$$

yields the error dynamics

$$\dot{z} = \left[A(y) - \beta_y \Phi(y) \right] z. \tag{6.6}$$

To complete the design it is necessary to select the function β so that the system (6.6) has a uniformly in y (asymptotically, if convergence of the estimation error is required) stable equilibrium at zero.

Note that, if the system (6.3) is detectable, we expect to be able to find an output injection matrix $B(y)$ such that the system $\dot{z} = \left[A(y) - B(y)\Phi(y) \right] z$ has a uniformly (asymptotically) stable equilibrium at zero. However, if the dimension of y is larger than one, it may not be possible to find a mapping β such that (6.2) holds, *i.e.* B may not be a Jacobian matrix. Obviously, if y has dimension one, we can simply select β to be the first integral of B. To overcome this obstacle we propose a dynamic extension to the reduced-order observer which consists of an output filter, of order m, and a dynamic scaling factor, *i.e.* the proposed observer is of order $n + m + 1$. The idea is to employ the output filter to ensure that

$$\beta_y = \Psi(y, \hat{y}), \tag{6.7}$$

where $\hat{y}$ is the filtered output and $\Psi : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{n \times m}$ is such that $\Psi(y,y) = B(y)$, and then use dynamic scaling to compensate for the mismatch between y and $\hat{y}$.The obvious gain from this modification is that Ψ can be chosen such that (6.7), in contrast with (6.2), admits a closed-form solution.

Remark 6.1. Since in the observer design problem the dynamic extension has a physical meaning, we adopt the more suggestive notation $\hat{y}$ in place of ξ, differently from the previous chapters. ▲

To begin with consider the following detectability-like assumption.

Assumption 6.1. There exist a continuously differentiable $n \times m$ matrix-valued function B, a non-negative function $\rho : \mathbb{R}^m \to \mathbb{R}_+$ and a constant $\gamma > 0$ such that

$$\frac{1}{2} \left(\left[A(y) - B(y)\Phi(y) \right]^\top + \left[A(y) - B(y)\Phi(y) \right] \right) \leq -\rho(y) I - \gamma \Phi(y)^\top \Phi(y).$$

Remark 6.2. Assumption 6.1 implies that the system $\dot{z} = [A(y) - B(y)\Phi(y)]\, z$ has a uniformly globally stable equilibrium at zero and $\Phi(y(t))z(t)$ is square-integrable. If, in addition, ρ is strictly positive, then the observer error z converges to zero, provided $y(t)$ exists for all $t \geq 0$. ▲

In fact, while the stability property of the system (6.6) is uniform with respect to the trajectories of (6.3), the rate of convergence to the origin depends in general on the measured state y. Thus, for particular trajectories of (6.3), namely such that the functions ρ and Φ evaluated along these trajectories decay to zero *faster* than z, $z(t)$ converges to a constant value different from zero. If ρ is strictly positive, then a minimum rate of convergence of the estimation error to zero can be guaranteed. Similarly, convergence results can be proved assuming *persistency of excitation* conditions [60, 95].

Note that, when $A(y)+A(y)^\top \leq 0$, Assumption 6.1 is trivially satisfied (with $\rho(y) \equiv 0$) by letting $B(y) = \gamma\Phi(y)^\top$. This is the case, for instance, in adaptive control, where η is a vector of unknown parameters and $A(y) = 0$. Thus, in general, the injection matrix $B(y)$ can be selected in the form $\bar{B}(y) + \gamma\Phi(y)^\top$, where $\bar{B}(y)$ is not identically zero and appropriately defined if $A(y) + A(y)^\top > 0$. To conclude, even if $\bar{B}(y) \equiv 0$, there may be other possible selections of $\rho(y)$, strictly positive, to guarantee a desired rate of convergence.

Consider now the scaled observer error

$$z = \frac{\hat{x} - x}{r} = \frac{\zeta + \beta(y, \hat{y}) - x}{r}, \tag{6.8}$$

with $r(t) \in \mathbb{R}$, where $\beta(y, \hat{y}) = [\beta_1(y, \hat{y}), \ldots, \beta_n(y, \hat{y})]^\top$ are functions to be specified and the auxiliary state $\hat{y}$ is obtained from the filter

$$\dot{\hat{y}} = f(y, u) + \Phi(y)\hat{x} - K(y, r, \hat{y} - y)(\hat{y} - y), \tag{6.9}$$

where $K : \mathbb{R}^m \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is a positive definite matrix-valued function.

Defining the observer dynamics as

$$\dot{\zeta} = h(y, u) + A(y)\hat{x} - \beta_y\left(f(y, u) + \Phi(y)\hat{x}\right) - \beta_{\hat{y}}\dot{\hat{y}} \tag{6.10}$$

yields the error dynamics

$$\dot{z} = [A(y) - \beta_y\Phi(y)]\, z - \frac{\dot{r}}{r}z. \tag{6.11}$$

The observer design problem is now reduced to the problem of finding a function $\beta(y, \hat{y})$ and the dynamics $\dot{r}$ such that the system (6.11) has a uniformly globally (asymptotically) stable equilibrium at the origin.

Let the desired output injection matrix, satisfying Assumption 6.1, be given by

$$B(y) = \left[\begin{array}{ccc} B_1(y) & \cdots & B_m(y) \end{array} \right] = \left[\begin{array}{ccc} b_{11}(y) & \cdots & b_{1m}(y) \\ \vdots & & \vdots \\ b_{n1}(y) & \cdots & b_{nm}(y) \end{array} \right]$$

and consider the function

$$\beta(y, \hat{y}) = \int_0^{y_1} B_1(\sigma, \hat{y}_2, \ldots, \hat{y}_m) d\sigma + \cdots + \int_0^{y_m} B_m(\hat{y}_1, \ldots, \hat{y}_{m-1}, \sigma) d\sigma, \tag{6.12}$$

which is such that

$$\beta_y = \left[\begin{array}{ccc} B_1(y_1, \hat{y}_2, \ldots, \hat{y}_m) & \cdots & B_m(\hat{y}_1, \ldots, \hat{y}_{m-1}, y_m) \end{array} \right].$$

Let $\eta = \hat{y} - y$ and note that, since B is continuously differentiable, we can write

$$\begin{array}{rcl} B_1(y_1, \hat{y}_2, \ldots, \hat{y}_m) & = & B_1(y) - \sum_{j=1}^{m} \eta_j \delta_{1j}(y, \eta), \\ & \vdots & \\ B_m(\hat{y}_1, \ldots, \hat{y}_{m-1}, y_m) & = & B_m(y) - \sum_{j=1}^{m} \eta_j \delta_{mj}(y, \eta), \end{array}$$

for some functions $\delta_{ij} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ with $\delta_{ii}(y, \eta) \equiv 0$. Substituting the above equations into (6.11) yields

$$\dot{z} = [A(y) - B(y)\Phi(y)]\, z + \sum_{j=1}^{m} \eta_j \Delta_j(y, \eta)\Phi(y) z - \frac{\dot{r}}{r} z, \tag{6.13}$$

where $\Delta_j(y, \eta) = [\delta_{1j}(y, \eta), \ldots, \delta_{mj}(y, \eta)]$, while the dynamics of η are given by

$$\dot{\eta} = -K(y, r, \eta)\eta + r\Phi(y) z. \tag{6.14}$$

Remark 6.3. A different definition of the mapping β can be obtained letting, similarly to the previous chapters, $\beta(y, \hat{y}) = B(\hat{y})y$, namely substituting in B the entire measured state y with the corresponding components of the dynamic extension $\hat{y}$. Clearly the mapping β is such that

$$\beta_y(y, \hat{y}) = B(y) + [B(\hat{y}) - B(y)] \triangleq B(y) + \Lambda(y, \eta) = B(y) + \sum_{j=1}^{m} \eta_j \hat{\Lambda}_j(y, \eta)\,,$$

with $\hat{\Lambda}_i : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{n \times m}$, $i = 1, ..., m$. Differently from Δ_j in the mapping β defined in (6.12), $\hat{\lambda}_{jj} : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$, which denotes the j-th column of the matrix $\hat{\Lambda}_j$, is not identically equal to zero. ▲

The system (6.13)-(6.14) has an equilibrium at zero and this can be rendered uniformly globally stable by selecting the dynamics of the scaling factor r and the matrix K appropriately, as described in the following theorem.

Theorem 6.1. Consider the system (6.3). Suppose that the solutions of system (6.3) exist for all $t \geq 0$ and that Assumption 6.1 holds. Let

$$\dot{r} = -\frac{\rho(y)}{2}(r-1) + cr\sum_{j=1}^{m}\eta_j^2\|\Delta_j(y,\eta)\|^2, \quad r(0) \geq 1, \tag{6.15}$$

with $c \geq m/(2\gamma)$ and

$$K(y,r,\eta) = kr^2 I + \epsilon c r^2 diag(\|\Delta_j(y,\eta)\|^2), \tag{6.16}$$

with $k > 0$ and $\epsilon > 0$ constants. Then the system (6.13)-(6.14)-(6.15) has a globally stable, uniformly in y, set of equilibria defined by

$$\Omega = \{\,(z,r,\eta) \mid z = \eta = 0\,\}.$$

Moreover, $z, r, \eta \in \mathcal{L}_\infty$, $\eta, \Phi(y)z \in \mathcal{L}_2$ and $r(t) \geq 1$ for all $t \geq 0$. If, in addition, $\rho > 0$, then $z(t)$ converges to zero. $\diamond$

Proof. To begin with note that for $r = 1$, $\dot{r} \geq 0$, hence $r(t) \geq 1$ for all t. Consider the positive-definite and proper function $V(z) = \frac{1}{2}\|z\|^2$, whose time-derivative along the trajectories of (6.13) satisfies

$$\begin{aligned}
\dot{V} \;\leq\;& -\frac{\rho(y)}{2}\|z\|^2 - \gamma\|\Phi(y)z\|^2 + \frac{1}{2}\sum_{j=1}^{m}\eta_j z^\top\left[\Delta_j(y,\eta)\Phi(y) + \Phi(y)^\top\Delta_j(y,\eta)^\top\right]z \\
& -c\sum_{j=1}^{m}\eta_j^2\|\Delta_j(y,\eta)\|^2\|z\|^2 \\
\leq\;& -\frac{\rho(y)}{2}\|z\|^2 - \frac{\gamma}{2}\|\Phi(y)z\|^2 + \frac{m}{2\gamma}\sum_{j=1}^{m}\eta_j^2\|\Delta_j(y,\eta)^\top z\|^2 - c\sum_{j=1}^{m}\eta_j^2\|\Delta_j(y,\eta)\|^2\|z\|^2 \\
\leq\;& -\frac{\rho(y)}{2}\|z\|^2 - \frac{\gamma}{2}\|\Phi(y)z\|^2.
\end{aligned}$$

As a result, the system (6.13) has a uniformly globally stable equilibrium at the origin, $z \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\Phi(y)z \in \mathcal{L}_2$ (the latter is obtained by integrating the last inequality). Moreover, the above holds true independently of the behavior of $y(t)$ and $\eta(t)$.

Consider now the function $W(\eta, z) = \frac{1}{2}\|\eta\|^2 + \frac{1}{k\gamma}V(z)$, whose time-derivative along the trajectories of (6.13)-(6.14) satisfies

$$\dot{W} \leq -kr^2\|\eta\|^2 + r\eta^\top\Phi(y)z - \frac{1}{2k}\|\Phi(y)z\|^2 \leq -\frac{k}{2}r^2\|\eta\|^2,$$

from which we conclude that the system (6.13)-(6.14) has a uniformly globally stable equilibrium at $(z, \eta) = (0, 0)$, hence the set Ω is uniformly globally stable, and $r\eta \in \mathcal{L}_2$. It remains to show that r is bounded. To this end, consider the combined Lyapunov function $U(\eta, z, r) = W(\eta, z) + \frac{\epsilon}{2} r^2$, whose time-derivative along the trajectories of (6.13)-(6.14)-(6.15) satisfies

$$\dot{U} \leq -\frac{k}{2} r^2 \|\eta\|^2 - \epsilon c r^2 \eta^\top diag(\|\Delta_j(y, \eta)\|^2)\eta + \epsilon c r^2 \sum_{j=1}^{m} \eta_j^2 \|\Delta_j(y, \eta)\|^2 = -\frac{k}{2} r^2 \|\eta\|^2.$$

Note that the last two terms cancel out, hence $r \in \mathcal{L}_\infty$. Thus $\eta \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and $\lim_{t \to \infty} \eta(t) = 0$, which concludes the proof. □

Note that the square-integrability of the function $\Phi(y(t))z(t)$, implied by Assumption 6.1 for (6.6), is guaranteed also for trajectories of (6.13). The assumption of existence of the trajectories of (6.3) for all $t \geq 0$ can be relaxed. In fact, we can show that the estimation errors z and η are bounded for all time for which the solution of (6.3) is defined and converge to zero if the solution exists for all $t \geq 0$, see a similar consideration in [100].

Remark 6.4. The term $-\frac{\rho(y)}{2}(r - 1)$ appearing in (6.15) is not needed to prove stability of the system (6.13)-(6.14)-(6.15) (note that $\rho(y)$ may be zero), but it has been introduced to ensure that, when $\rho(y) > 0$, r stays bounded in the presence of noise. ▲

In summary the observer is described by the equations (6.9)-(6.10)-(6.12)-(6.15) and (6.16) with $\hat{x} = \zeta + \beta(y, \hat{y})$.

6.2.1 Example: Induction Motor

The two-phase equivalent model of a voltage-fed induction motor in the stator reference frame is described by the equations [40]

$$\begin{aligned} \dot{y} &= \begin{bmatrix} -a_0 y_1 + a_2 u_1 \\ -a_0 y_2 + a_2 u_2 \\ 0 \end{bmatrix} + \begin{bmatrix} a_1 \mu & a_1 y_3 & 0 \\ -a_1 y_3 & a_1 \mu & 0 \\ a_3 y_2 & -a_3 y_1 & -1 \end{bmatrix} x, \\ \dot{x} &= \begin{bmatrix} a_4 y_1 \\ a_4 y_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -\mu & -y_3 & 0 \\ y_3 & -\mu & 0 \\ 0 & 0 & 0 \end{bmatrix} x, \end{aligned} \tag{6.17}$$

where $y = [i_a, i_b, n_p \omega]^\top$, $x = [\lambda_a, \lambda_b, n_p \tau_L / J_m]^\top$, i_a, i_b are the stator currents, λ_a, λ_b describe the rotor fluxes, ω describes the rotor speed, u_1, u_2 describe the stator voltages, n_p is the number of pole pairs, J_m is the rotor moment of inertia, and τ_L is the (unknown) load torque. The parameters a_0, a_1, a_2, a_3, a_4 and μ are positive constants. The system (6.17) is of the form (6.3) with

$$\Phi(y) = \begin{bmatrix} a_1\mu & a_1 y_3 & 0 \\ -a_1 y_3 & a_1\mu & 0 \\ a_3 y_2 & -a_3 y_1 & -1 \end{bmatrix}, \quad A(y) = \begin{bmatrix} -\mu & -y_3 & 0 \\ y_3 & -\mu & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $A(y) + A(y)^\top \leq 0$, Assumption 6.1 is satisfied by selecting the output injection

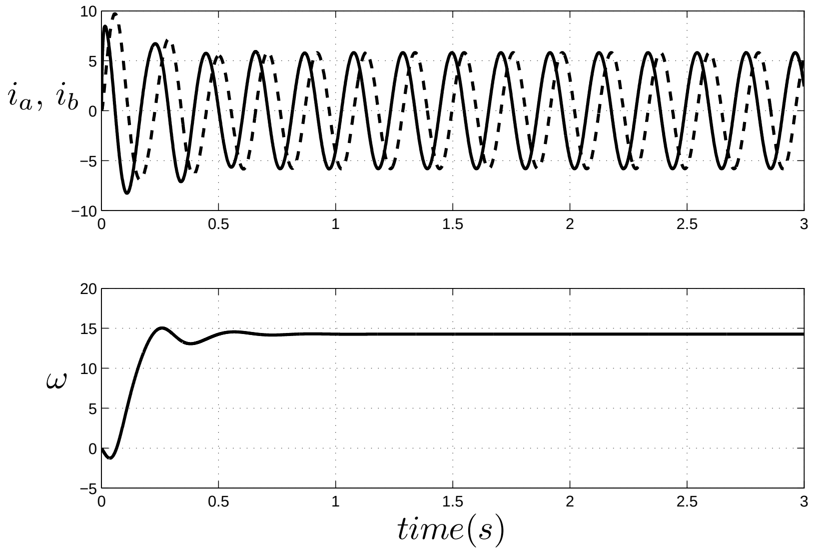

Figure 6.1: Top graph: time histories of the stator currents $i_a(t)$ (solid line) and $i_b(t)$ (dashed line) of the voltage-fed induction motor (6.17) with $u_1(t) = 100\cos(30t)$ and $u_2(t) = 100\sin(30t)$. Bottom graph: time history of the rotor speed $\omega(t)$ of the voltage-fed induction motor (6.17).

matrix

$$B(y) = \gamma\Phi(y)^\top = \gamma \begin{bmatrix} a_1\mu & -a_1 y_3 & a_3 y_2 \\ a_1 y_3 & a_1\mu & -a_3 y_1 \\ 0 & 0 & -1 \end{bmatrix}.$$

Note that, as a result of the specific structure of the matrices $A(y)$ and $\Phi(y)$, the Assumption 6.1 holds for some strictly positive function ρ, upper-bounded by a positive constant that depends on the parameter γ and on the constant μ. In the following we let $\gamma = 5$ for which the Assumption 6.1 is satisfied with some constant $\rho(y) \equiv \bar{\rho} \in (0, \frac{1}{2}]$. Consider now the function

$$\beta(y, \hat{y}) = \gamma \begin{bmatrix} a_1\mu y_1 - a_1\hat{y}_3 y_2 + a_3\hat{y}_2 y_3 \\ a_1\hat{y}_3 y_1 + a_1\mu y_2 - a_3\hat{y}_1 y_3 \\ -y_3 \end{bmatrix},$$

which is such that

$$\beta_y = \begin{bmatrix} a_1\mu & -a_1\hat{y}_3 & a_3\hat{y}_2 \\ a_1\hat{y}_3 & a_1\mu & -a_3\hat{y}_1 \\ 0 & 0 & -1 \end{bmatrix} = B(y) - \begin{bmatrix} 0 & a_1\eta_3 & -a_3\eta_2 \\ -a_1\eta_3 & 0 & a_3\eta_1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the last matrix we have

$$\Delta_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a_3 \\ 0 & 0 & 0 \end{bmatrix}, \Delta_2 = \begin{bmatrix} 0 & 0 & -a_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \Delta_3 = \begin{bmatrix} 0 & a_1 & 0 \\ -a_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Interestingly, in the case of the voltage-fed induction motor (6.17), the definitions of

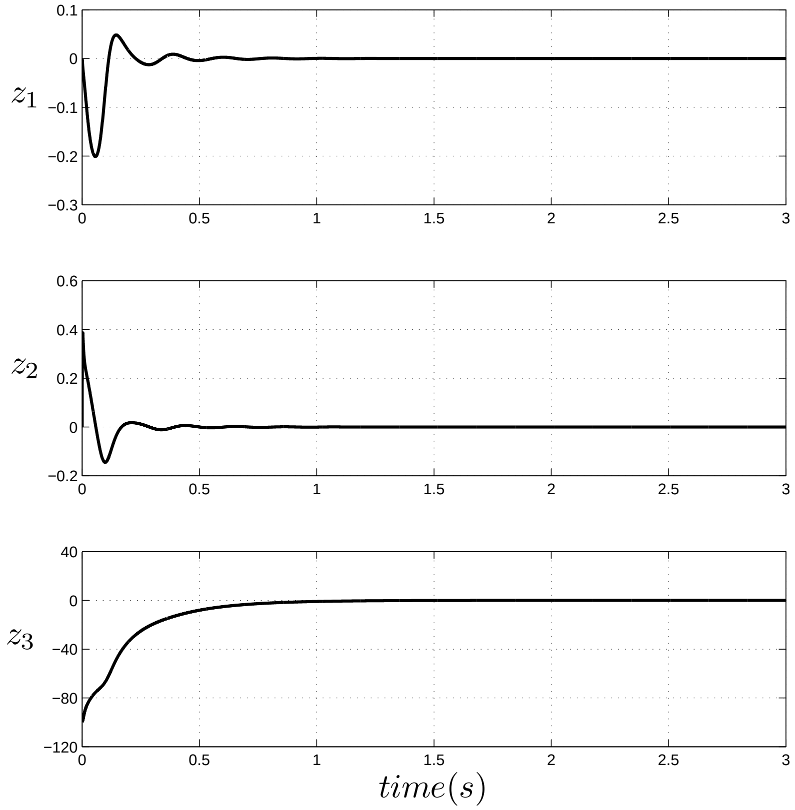

Figure 6.2: graph: time history of the estimation error of the rotor flux $\lambda_a(t)$ in the voltage-fed induction motor (6.17). Middle graph: time history of the estimation error of the rotor flux $\lambda_b(t)$ in the voltage-fed induction motor (6.17). Bottom graph: time history of the estimation error of the unknown load torque τ_L acting on the voltage-fed induction motor (6.17).

the mapping β as in (6.12) and as discussed in Remark 6.3 coincide since $\dfrac{\partial B_i}{\partial y_i} = 0$ for $i = 1, 2, 3$. Thus, from (6.15) and (6.16) the dynamic scaling and the gain matrix K are selected, respectively, as

$$\dot{r} = \frac{3}{2\gamma} r \left(a_3^2 \eta_1^2 + a_3^2 \eta_2^2 + a_1^2 \eta_3^2 \right) - \frac{\bar{\rho}}{2}(r-1)$$

and

$$K(r) = kr^2 I + \frac{3\epsilon}{2\gamma} r^2 diag(a_3^2, a_3^2, a_1^2).$$

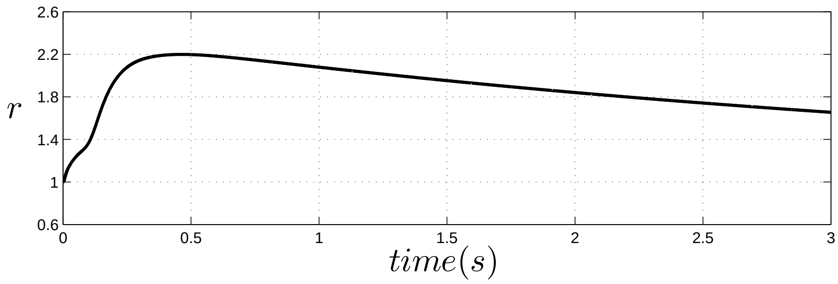

Figure 6.3: Time history of the dynamic scaling $r(t)$.

In the simulation we have selected $y(0) = [0, 0, 0]$ and supposed that the rotor fluxes at time $t = 0$ and the constant load torque τ_L are equal to zero and $3Nm$, respectively. The stator voltages are enforced by the control actions $u_1(t) = 100\cos(30t)$ and $u_2(t) = 100\sin(30t)$. Finally, since y describes the measured part of the state, the dynamic extension $\hat{y}$ is initialized such that $\eta(0) = y(0) - \hat{y}(0) = 0$. Moreover, let $r(0) = 1$ and $\hat{x}(0) = 0$. Figure 6.1 shows the time histories of the stator currents $i_a(t)$ (solid line) and $i_b(t)$ (dashed line) of the voltage-fed induction motor (6.17) when the inputs $u_1(t)$ and $u_2(t)$ are applied (top graph). The bottom graph displays the time histories of the rotor speed $\omega(t)$ of the voltage-fed induction motor (6.17) when the inputs u_1 and $u_2(t)$ are applied. Letting $k = 2$ and $\epsilon = 1$, the time histories of the estimation errors of the rotor fluxes $\lambda_a(t)$ and $\lambda_b(t)$, namely $z_1 = (\hat{\lambda}_a - \lambda_a)/r$ and $z_2 = (\hat{\lambda}_b - \lambda_b)/r$, are displayed in the top and middle graphs, respectively, of Figure 6.2. The bottom graph of Figure 6.2 shows the time history of the estimation error of the unknown load torque $\tau_L(t)$, namely $z_3 = \frac{n_p}{J_m r}(\hat{\tau}_L - \tau_L)$, acting on the voltage-fed induction motor (6.17), Finally the time history of the dynamic scaling $r(t)$ is displayed in Figure 6.3.

6.2.2 Adaptive Control Design

In this section we consider a special case of the class of systems (6.3), where the unmeasured state consists solely of unknown constant parameters, and we provide conditions for constructing an adaptive control law using the observer of Section 6.2. The algorithm is then applied to the longitudinal control problem for an aircraft with unknown aerodynamic coefficients. Consider linearly parameterized systems of the form

$$\dot{x} = f(x,u) + \Phi(x)\theta, \tag{6.18}$$

with state $x(t) \in \mathbb{R}^n$ and input $u(t) \in \mathbb{R}^m$, where $\theta \in \mathbb{R}^p$ is an unknown constant vector and each element of the vector $\Phi(x)\theta$ has the form $\varphi_i(x)^\top \theta_i$, with $\theta_i \in \mathbb{R}^{p_i}$. The control problem is to find an adaptive state feedback control law such that all trajectories of the closed-loop system are bounded and

$$\lim_{t\to\infty} x(t) = x^*, \tag{6.19}$$

where x^* is a desired set point. The same class of systems is considered in [105]. Therein the function $f(x,u)$ is linear in x, *i.e.* $f(x,u) = A(u,y)x$ and may include a nonlinear term depending on the measured part of the state and on the input u, hence the two formulations coincide when the entire state is directly measurable.

Remark 6.5. A special subclass of the class of systems described by equations (6.18) is the so-called parametric strict feedback form which is given by

$$\dot{x}_i = x_{i+1} + \varphi_i(x_1,\ldots,x_i)^\top \theta_i, \tag{6.20}$$

with states $x_i(t) \in \mathbb{R}$, $i = 1,\ldots,n$ and control input $u \triangleq x_{n+1}$. This class of systems can be stabilized using adaptive back-stepping, see [50] and related works. The drawback of the latter approach is that the resulting closed-loop system dynamics depend strongly on the estimation error which is only guaranteed to be bounded. This can have a detrimental effect on performance.

To counteract this problem, an alternative method has been developed in [42], see also [43], which is based on the reduced-order observer design of Section 6.2 and which effectively recovers the performance of the known-parameters controller by imposing a closed-loop cascaded structure. However, the application of this method relies on a rather restrictive structural assumption (see Assumption 1 in [42]). The result in this section allows to remove this assumption. ▲

The result in Theorem 6.1 is directly applicable to the system (6.18). However, in this section we design a separate observer for each vector θ_i to facilitate the control design (for example, to enable a step-by-step construction of the control law, which is necessary when dealing with systems in feedback form).

Consider the system (6.18) and let

$$z_i = \frac{\hat{\theta}_i - \theta_i}{r_i} = \frac{\zeta_i + \beta_i(x_i,\hat{x}) - \theta_i}{r_i},$$

for $i = 1, \ldots, n$, where $\zeta_i(t) \in \mathbb{R}^{p_i}$ are the estimator states, $r_i(t) \in \mathbb{R}$ are scaling factors, β_i are functions to be specified, and the auxiliary states $\hat{x}_i$ are obtained from the filters

$$\dot{\hat{x}}_i = f_i(x, u) + \varphi_i(x)^\top \hat{\theta}_i - k_i(x, r, \hat{x} - x)(\hat{x}_i - x_i), \tag{6.21}$$

for $i = 1, \ldots, n$, where $k_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ are positive functions. Using the above definitions and the update laws

$$\dot{\zeta}_i = -\frac{\partial \beta_i}{\partial x_i}\big(f_i(x, u) + \varphi_i(x)^\top \hat{\theta}_i\big) - \sum_{j=1}^{n} \frac{\partial \beta_i}{\partial \hat{x}_j}\dot{\hat{x}}_j\,, \tag{6.22}$$

yields the error dynamics

$$\dot{z}_i = -\frac{\partial \beta_i}{\partial x_i}\varphi_i(x)^\top z_i - \frac{\dot{r}_i}{r_i} z_i. \tag{6.23}$$

Note that the system (6.23), for $i = 1, \ldots, n$, can be regarded as a linear time-varying system with a block diagonal dynamic matrix. In order to render the diagonal blocks negative-semidefinite, we select the functions β_i as

$$\beta_i(x_i, \hat{x}) = \gamma_i \int_0^{x_i} \varphi_i(\hat{x}_1, \ldots, \hat{x}_{i-1}, \sigma, \hat{x}_{i+1}, \ldots, \hat{x}_n) d\sigma, \tag{6.24}$$

where γ_i are positive constants.

Let $\eta_i = \hat{x}_i - x_i$ and note that, since φ_i is continuously differentiable, we can write

$$\varphi_i(\hat{x}_1, \ldots, \hat{x}_{i-1}, x_i, \hat{x}_{i+1}, \ldots, \hat{x}_n) = \varphi_i(x) - \sum_{j=1}^{n} \eta_j \delta_{ij}(x, \eta),$$

for some mappings δ_{ij}, with $\delta_{ii}(x, \eta) \equiv 0$. Using the above equation and substituting (6.24) into (6.23) yields the error dynamics

$$\dot{z}_i = -\gamma_i \varphi_i(x)\varphi_i(x)^\top z_i + \gamma_i \sum_{j=1}^{n} \eta_j \delta_{ij}(x, \eta)\varphi_i(x)^\top z_i - \frac{\dot{r}_i}{r_i} z_i, \tag{6.25}$$

while, from (6.18) and (6.21), the dynamics of η_i are given by

$$\dot{\eta}_i = -k_i(x, r, \eta)\eta_i + r_i \varphi_i(x)^\top z_i. \tag{6.26}$$

The system (6.25)-(6.26) has an equilibrium at zero and this can be rendered uniformly globally stable by selecting the dynamics of the scaling factors r_i and the functions k_i as described in the following lemma.

Lemma 6.1. Consider the system (6.18). Suppose that the solutions of (6.18) exist for all $t \geq 0$ and let

$$\dot{r}_i = c_i r_i \sum_{j=1}^{n} \eta_j^2 \|\delta_{ij}(x,\eta)\|^2, \qquad r_i(0) = 1, \tag{6.27}$$

for $i = 1, \ldots, n$, with $c_i \geq \gamma_i n/2$ and

$$k_i(x,r,\eta) = \lambda_i r_i^2 + \epsilon \sum_{\ell=1}^{n} c_\ell r_\ell^2 \|\delta_{\ell i}(x,\eta)\|^2, \tag{6.28}$$

where $\lambda_i > 0$ and $\epsilon > 0$ are constants.

Then the system (6.25)-(6.26)-(6.27) has a globally stable, uniformly in x, set of equilibria defined by

$$\Omega = \{\, (z,r,\eta) \mid z = \eta = 0 \,\}.$$

Moreover, $z_i \in \mathcal{L}_\infty$, $r_i \in \mathcal{L}_\infty$, $\eta_i \in \mathcal{L}_2 \cap \mathcal{L}_\infty$, and $\varphi_i(x)^\top z_i \in \mathcal{L}_2$, for all $i = 1, \ldots, n$. If, in addition, $\varphi_i(x(t))$ and its time-derivative are bounded, then the signals $\varphi_i(x(t))^\top z_i(t)$ converge to zero. $\diamond$

Proof. Consider the positive-definite and proper function $V_i(z_i) = \frac{1}{2\gamma_i}\|z_i\|^2$, whose time-derivative along the trajectories of (6.25)-(6.27) satisfies

$$\dot{V}_i \leq -\left(\varphi_i(x)^\top z_i\right)^2 + \sum_{j=1}^{n}\left[\frac{1}{2n}\left(\varphi_i(x)^\top z_i\right)^2 + \frac{n}{2}\eta_j^2\left(\delta_{ij}^\top z_i\right)^2\right] - \frac{\dot{r}_i}{\gamma_i r_i}\|z_i\|^2 \leq -\frac{1}{2}\left(\varphi_i(x)^\top z_i\right)^2.$$

As a result, the system (6.25) has a globally stable, uniformly in x, equilibrium at the origin, $z_i \in \mathcal{L}_\infty$ and $\varphi_i(x)^\top z_i \in \mathcal{L}_2$, for all $i = 1, \ldots, n$. Consider now the function $W_i(\eta_i, z_i) = \frac{1}{2}\eta_i^2 + \frac{1}{\lambda_i}V_i(z_i)$, whose time-derivative along the trajectories of (6.25)-(6.26) satisfies

$$\dot{W}_i \leq -k_i(x,r,\eta)\eta_i^2 + \frac{\lambda_i}{2}r_i^2\eta_i^2 \leq -\frac{\lambda_i}{2}r_i^2\eta_i^2,$$

for any $\lambda_i > 0$, from which we conclude that the system (6.25)-(6.26) has a globally stable, uniformly in x, equilibrium at $(z_i, \eta_i) = (0,0)$, hence the set Ω is uniformly globally stable, and $r_i\eta_i \in \mathcal{L}_2$. To show that the r_i's are bounded consider the combined Lyapunov function $U(\eta, z, r) = \sum_{i=1}^{n}\left[W_i(\eta_i, z_i) + \frac{\epsilon}{2}r_i^2\right]$, whose time-derivative along the trajectories of (6.25)-(6.26)-(6.27) satisfies

$$\dot{U} \leq -\sum_{i=1}^{n}\left(k_i(x,r,\eta) - \frac{\lambda_i}{2}r_i^2\right)\eta_i^2 + \epsilon\sum_{i=1}^{n}\left[c_i r_i^2 \sum_{j=1}^{n}\eta_j^2\|\delta_{ij}(x,\eta)\|^2\right].$$

Note now that the last term is equal to

$$\epsilon\sum_{i=1}^{n}\sum_{\ell=1}^{n} c_\ell r_\ell^2 \|\delta_{\ell i}(x,\eta)\|^2 \eta_i^2,$$

hence selecting k_i from (6.28) ensures that

$$\dot{U} \le -\sum_{i=1}^{n} \frac{\lambda_i}{2}\eta_i^2,$$

which proves that $r_i \in \mathcal{L}_\infty$, hence we conclude that $\eta_i \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ and that $\lim_{t\to\infty} \eta_i(t) = 0$. Finally, when $\varphi_i(x(t))$ and its time-derivative are bounded, it follows from Barbalat's Lemma that $\varphi_i(x(t))^\top z_i(t)$ converge to zero. □

Remark 6.6. Mimicking the arguments in Remark 6.3 a different definition of the mappings $\beta_i(x_i,\, \hat{x})$ can be given. In particular let $\beta_i(x_i,\, \hat{x}) = \gamma_i\varphi_i(\hat{x})x_i$ and note that

$$\frac{\partial\beta_i}{\partial x_i} = \gamma_i\varphi_i(x) - \gamma_i\left[\varphi_i(x) - \varphi_i(\hat{x})\right] .$$

Define $\dot{r}_i = c_i r_i\|\varphi_i(x) - \varphi_i(\hat{x})\|^2$ and $k_i(x, r, \eta) = \lambda_i r_i^2 + \epsilon\sum_{\ell=1}^{n} c_\ell r_\ell^2\|\pi_{\ell i}(x, \eta)\|^2$, where c_i, λ_i and ϵ, $i = 1, \ldots, n$, are defined in Lemma 6.1 and $\pi_{\ell i} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{p_i}$ are such that $\varphi_i(x) - \varphi_i(\hat{x}) = \Pi_i(x, \eta)\eta = \sum_{j=1}^{n} \pi_{ij}\eta_j$. Then the results of Lemma 6.1 can be proved following exactly the same arguments as those in the above proof. ▲

Remark 6.7. In the special case in which φ_i is a function of x_i only, the auxiliary states (6.21) are not used in the adaptive law and $\delta_{ij}(x, \eta) \equiv 0$ which implies that $\dot{r}_i = 0$, hence we can simply fix the scaling factors r_i to be equal to one. The same simplification occurs when only one of the functions φ_i is nonzero. ▲

A possible way of exploiting the estimator properties given in Lemma 6.1 to design an adaptive control law based on the certainty equivalence principle is presented in the following theorem.

Theorem 6.2. Consider the system (6.18) and suppose that there exists a control law $u = v(x, \theta + z)$ such that, for all trajectories of the closed-loop system,

(*a*) $\varphi_i(x(t))^\top z_i(t) \in \mathcal{L}_2 \Longrightarrow x(t) \in \mathcal{L}_\infty$

(*b*) $\lim_{t\to\infty} \varphi_i(x(t))^\top z_i(t) = 0 \Longrightarrow \lim_{t\to\infty} x(t) = x^*$.

Then there exists an adaptive state feedback control law such that all closed-loop signals are bounded and (6.19) holds. ◇

The proof of the statement follows directly from Lemma 6.1 and conditions (*a*) and (*b*), hence it is omitted.

Finally, it is possible to derive a counterpart of the result in [42] with the dynamic scaling-based estimator replacing the estimator in [42], as the following corollary shows.

Corollary 6.1. Consider the system (6.20) with input $u \triangleq x_{n+1}$, where $\phi_i : \mathbb{R}^n \to \mathbb{R}^{p_i}$ are $\mathcal{C}^{n-i}$ mappings. Then there exists an adaptive state feedback control law such that all trajectories of the closed-loop system are bounded and $\lim_{t\to\infty}[x_1(t) - x_1^*(t)] = 0$, where $x_1^*(t)$ is a bounded $\mathcal{C}^n$ reference signal. ◇

6.2.3 Example: Aircraft Longitudinal Control

The longitudinal motion of an aircraft can be described by the equations [14, 26]

$$\begin{aligned}
\dot{V} &= -g\sin(\vartheta - \alpha) + \frac{T_x}{m}\cos(\alpha) - \frac{D}{m}, &(6.29)\\
\dot{\alpha} &= q + \frac{1}{V}\Big(g\cos(\alpha - \vartheta) - \frac{T_x}{m}\sin(\alpha) - \frac{L}{m}\Big) &(6.30)\\
\dot{\vartheta} &= q, &(6.31)\\
\dot{q} &= \frac{M}{I_y}, &(6.32)
\end{aligned}$$

where V describes the total airspeed, α describes the incidence angle, ϑ describes the pitch angle, q describes the pitch rate, T_x describes the thrust (along the x body axis), m describes the aircraft mass, g describes the gravitational acceleration, I_y describes the moment of inertia, M describes the pitching moment, and D and L describe the aerodynamic forces corresponding to drag and lift, respectively, see [26] for more details. For the purposes of this example we consider a simple model for the drag and lift given by the parameterized functions

$$D = \frac{1}{2}\rho V^2 S\left(C_{D0} + C_{D\alpha}\alpha^2\right), \quad L = \frac{1}{2}\rho V^2 S C_{L\alpha}\alpha,$$

where ρ is the air density, S is the wing area and $C_{D0}, C_{D\alpha}$, $C_{L\alpha}$ are constant coefficients. To simplify the control design we also assume that the pitch rate can be directly controlled and focus on the first two equations. Define the states $x_1 = V$, $x_2 = \alpha$, the control inputs $u_1 = T_x/m$, $u_2 = q$, and the unknown parameters

$$\theta_1 = -\frac{\rho S}{2m}\begin{bmatrix} C_{D0} \\ C_{D\alpha} \end{bmatrix}, \quad \theta_2 = -\frac{\rho S}{2m} C_{L\alpha},$$

and note that the system can be rewritten in the form (6.18), namely

$$\begin{aligned}
\dot{x}_1 &= -g\sin(\vartheta - x_2) + u_1\cos(x_2) + \varphi_1(x)^\top\theta_1,\\
\dot{x}_2 &= u_2 + \frac{g}{x_1}\cos(x_2 - \vartheta) - \frac{u_1}{x_1}\sin(x_2) + \varphi_2(x)^\top\theta_2,
\end{aligned} \qquad (6.33)$$

where $\varphi_1(x) = \begin{bmatrix} x_1^2, & x_1^2 x_2^2 \end{bmatrix}^\top$ and $\varphi_2(x) = x_1 x_2$.

Note that in the set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, \ |x_2| < \pi/2 \ \}$ the system (6.33) is well-defined and controllable. The control objective is to drive the airspeed x_1 and the incidence angle x_2 to their respective set-points x_1^* and x_2^*, despite the lack of information on the drag and lift coefficients. The output filter is defined as

$$\begin{aligned}
\dot{\hat{x}}_1 &= -g\sin(\vartheta - x_2) + u_1\cos(x_2) + \varphi_1(x)^\top\hat{\theta}_1 - k_1 e_1, \\
\dot{\hat{x}}_2 &= u_2 + \frac{g}{x_1}\cos(x_2 - \vartheta) - \frac{u_1}{x_1}\sin(x_2) + \varphi_2(x)^\top\hat{\theta}_2 - k_2 e_2,
\end{aligned}$$

where

$$k_1 = \lambda_1 r_1^2 + \epsilon\frac{\gamma_2}{2}r_2^2 x_2^2, \quad k_2 = \lambda_2 r_2^2 + \epsilon\frac{\gamma_1}{2}r_1^2 x_1^4(2x_2 + \eta_2)^2,$$

and the estimator dynamics are given by the equations

$$\begin{aligned}
\dot{\zeta}_1 &= -\frac{\partial\beta_1}{\partial x_1}\left(\dot{\hat{x}}_1 + k_1\eta_1\right) - \frac{\partial\beta_1}{\partial\hat{x}_2}\dot{\hat{x}}_2, \\
\dot{\zeta}_2 &= -\frac{\partial\beta_2}{\partial x_2}\left(\dot{\hat{x}}_2 + k_2\eta_2\right) - \frac{\partial\beta_2}{\partial\hat{x}_1}\dot{\hat{x}}_1,
\end{aligned}$$

where $\lambda_1 > 0$, $\lambda_2 > 0$ and $\epsilon > 0$ are constants and the functions β_i are obtained from (6.24), namely

$$\beta_1(x_1, \hat{x}_2) = \gamma_1\frac{x_1^3}{3}\begin{bmatrix} 1 \\ \hat{x}_2^2 \end{bmatrix}, \quad \beta_2(x_2, \hat{x}_1) = \gamma_2\hat{x}_1\frac{x_2^2}{2},$$

with $\gamma_1 > 0$ and $\gamma_2 > 0$. Note now that

$$\begin{aligned}
\frac{\partial\beta_1}{\partial x_1} &= \gamma_1\varphi_1(x) - \gamma_1\begin{bmatrix} 0 \\ -x_1^2(2x_2 + \eta_2) \end{bmatrix}\eta_2, \\
\frac{\partial\beta_2}{\partial x_2} &= \gamma_2\varphi_2(x) - \gamma_2\left(-x_2\right)\eta_1,
\end{aligned}$$

where the terms in brackets correspond to the functions δ_{12} and δ_{21}, repsectively. Hence, from (6.27) and (6.28), the dynamic scaling parameters r_i and the gains k_i are given by

$$\dot{r}_1 = \gamma_1 x_1^4(2x_2 + \eta_2)^2\eta_2^2 r_1, \quad \dot{r}_2 = \gamma_2 x_2^2\eta_1^2 r_2.$$

Finally, a certainty equivalence control law for the system (6.33) is given by

$$u_1 = \frac{1}{\cos(x_2)}\left(g\sin(\vartheta - x_2) - \varphi_1(x)^\top\hat{\theta}_1 - \mu_1(x_1 - x_1^*)\right), \tag{6.34}$$

$$u_2 = -\frac{g}{x_1}\cos(x_2 - \vartheta) + \frac{u_1}{x_1}\sin(x_2) - \varphi_2(x)^\top\hat{\theta}_2 - \mu_2(x_2 - x_2^*), \tag{6.35}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are constants. The closed-loop system has been simulated using the parameters $\theta_1 = [-0.00063, -0.0358]^\top$ and $\theta_2 = -0.092$, which correspond to an Eclipse-type unmanned aerial vehicle [26]. We consider two set-point step changes: at time $t = 1$ s the airspeed reference x_1^* changes from 30 m/s to 35 m/s, and at time $t = 3$ s, the incidence angle reference x_2^* changes from 0.1 rad to 0.2 rad. The definition of β_i as in (6.24) and as in Remark 6.6 are compared, with respect to the same set of parameters, namely $\gamma_1 = 10^{-3}$, $\gamma_2 = 0.5$ and $\lambda_1 = \lambda_2 = \epsilon = 1$. In the latter case, $\dot{r}_i$ and k_i, $i = 1, 2$, are defined according to Remark 6.6 with

$$\pi_{11} = \begin{bmatrix} -(2x_1 + \eta_1) \\ -2x_1x_2^2 - \eta_1x_2^2 - 4x_1x_2\eta_2 - 2x_1\eta_2^2 \end{bmatrix}, \quad \pi_{12} = \begin{bmatrix} 0 \\ -2x_1^2x_2 - x_1^2\eta_2 - 2x_2\eta_1^2 - \eta_1^2\eta_2 \end{bmatrix}$$

and $\pi_{21} = -x_2$, $\pi_{22} = -\eta_1 - x_1$. The top graph of Figure 6.4 shows the time histories of the total airspeed for the system (6.29) in closed-loop with the known parameters control (dash-dotted line) and with the adaptive control (6.34), where β_i is defined as in (6.24) and as in Remark 6.6, solid and dashed line, respectively. The bottom graph displays the time histories of the incidence angle for the system (6.29) in closed-loop with the known parameters control (dash-dotted line) and with the adaptive control (6.34), where β_i is defined as in (6.24) and as in Remark 6.6, solid and dashed line, respectively.

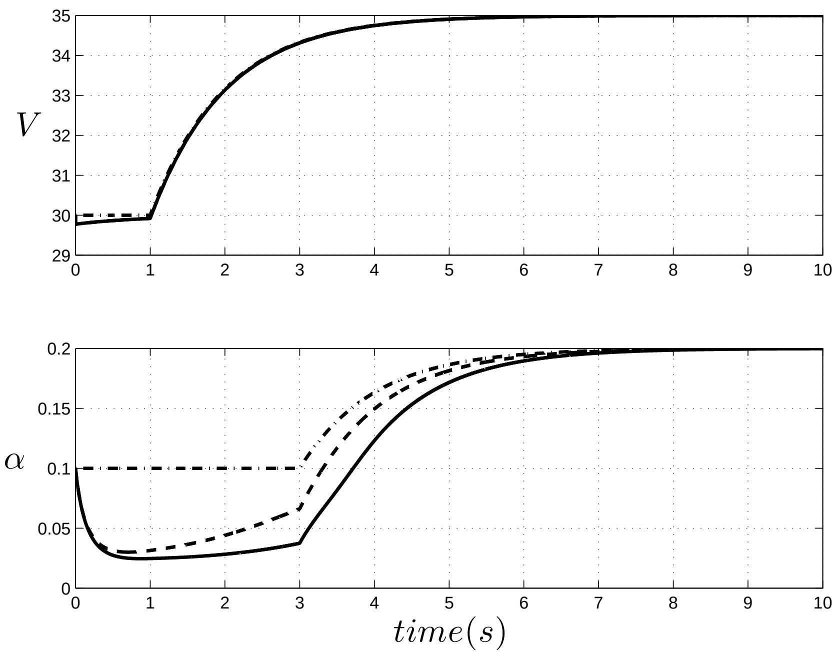

Figure 6.4: Top graph: time histories of the total airspeed for the system (6.29) in closed-loop with the known parameters control (dash-dotted line) and with the adaptive control (6.34), where β_i is defined as in (6.24) and as in Remark 6.6, solid and dashed line, respectively. Bottom graph: time histories of the incidence angle for the system (6.29) in closed-loop with the known parameters control (dash-dotted line) and with the adaptive control (6.34), where β_i is defined as in (6.24) and as in Remark 6.6, solid and dashed line, respectively.

6.3 Observer Design for Range and Orientation

Vision guided robotics has been one of the major research issues in recent years, since the applications of visually guided systems are numerous, *e.g.* intelligent agents, robotic surgery, exploration rovers, home automation and the SLAM (Simultaneous Localization And Mapping) problem see *e.g.* [20, 25, 69]. The use of direct visual information, which is acquired from measurable characteristics of the environment, to provide feedback about the state of the environment and to control a robot is commonly termed *visual servoing*. The observed characteristics are usually referred to as *features*. Features are extracted from the image and their motion is mapped to the velocity *twist* of the camera via an *interaction matrix* [25]. The image of the target is a function of the relative pose between the camera and the target and the distance between them is frequently referred to as *depth* or range.

Two different approaches for visual servoing have been defined [18, 20, 38]. Image-based visual servoing uses features extracted from the image to directly provide a command to the robot. Position-based visual servoing is a model based technique, since the 3-dimensional pose, with respect to the camera frame, of the object of interest is estimated from the extracted features, and then the robot is controlled by zeroing, in the 3-dimensional space, the error between the current configuration of the object and the final one.

The challenge in developing these types of controllers is that a camera provides a 2-dimensional image information of the 3-dimensional Euclidean space through a perspective, *i.e.* range dependant, projection, whereas in general the interaction matrix depends on 3-dimensional information. In the case of point features this information consists exclusively of the *depth* of the object with respect to the camera, while to exploit image moments as observed features more general information is needed, *e.g.* the object *depth* and the orientation of the plane containing the object, [77].

The projection on the image plane causes loss of information of the *depth* and orientation of the plane containing the object. In some applications the *depth* can be considered constant and approximated by the distance of the object in the desired pose, but, with this assumption, it is possible to achieve only local results, since the stability region for Image-based visual servoing in the presence of errors on the *depth* distribution is not large [58]. Thus, the distance of the object from the camera has to be estimated to exploit the visual information to control the robot. For this purpose, the dynamics of the features, *e.g.* geometric moments, and of the 3-dimensional information of the object must be expressed in terms of the known translational and rotational velocities of the camera.

The problem of designing an observer to estimate the *depth* of an object has received significant attention, see [5, 23, 41, 62, 77]. In [77] a local observer that exploits the basic formulation of the *persistency of excitation* lemma [60] is presented. Therein the estimation error system consists of an exponentially stable slowly varying perturbed linear system. Thus, stability of the zero equilibrium of the error system can be guaranteed if the initial estimation error is sufficiently small. In [62] a Kalman filter allows to consider at the same time new measurements and old estimates of the *depth* to determine the current uncertainty. The method integrates new measurements with existing *depth* estimates to reduce the uncertainty over time. In [23] the range identification problem is studied considering object features evolving with affine motion dynamics, with known motion parameters. In [41] a solution to the range identification problem is proposed based on a nonlinear observer design which is inspired by the immersion and invariance methodology [70] and the reduced order observer in [5].

The first step towards the solution of the estimation problem consists in determining the equation describing the time derivative $\dot{m}_{ij}^{\mathcal{R}}$ of the moment $m_{ij}^{\mathcal{R}}$, *i.e.* the geometric moment of order $i+j$ relative to the region $\mathcal{R}$ which defines the observed object, as a function of the twist $\nu = (v, \omega)$ between the camera and the object, where v and ω represent the relative translational and rotational velocity components, respectively. Note that the geometric moments may be computed exploiting the technique described in [16]. Without loss of generality it is possible to suppose that the observed object is fixed and so ν represents exactly the imposed camera velocities. Following [17] the equation are given by

$$\dot{m}_{ij}^{\mathcal{R}} = L_{m_{ij}^{\mathcal{R}}} \nu, \tag{6.36}$$

where $L_{m_{ij}^{\mathcal{R}}}$ is the interaction matrix associated to the geometric moment $m_{ij}^{\mathcal{R}}$, depending in general on the 3-dimensional parameters. Consider the following standing assumption, which is standard in this context [77].

Assumption 6.2. The observed object $\mathcal{O}$ is planar or has a planar surface with plane equation

$$n \cdot p + d = 0, \tag{6.37}$$

where $n = [\ n_x\ n_y\ n_z\ ]^\top$ is the plane unit normal, d is the distance to the origin of the moving frame relative to the camera and p is the vector from the origin to a generic point, P, lying on the plane (6.37).

Note that if Assumption 6.2 is satisfied then the *depth* Z of any point lying on the plane containing the object can be expressed in terms of the image coordinates as

$$\frac{1}{Z} = A_{00} + A_{10}\pi_1 + A_{01}\pi_2, \tag{6.38}$$

where

$$\begin{bmatrix} A_{10} \\ A_{01} \\ A_{00} \end{bmatrix} = -n/d.$$

With these assumptions the time variation of a moment of order $i+j$ can be expressed from the moments of order up to $i+j+1$, and from the parameters A_{00}, A_{10} and A_{01}. If additionally the unit plane normal n is known the only 3-dimensional information to be estimated reduces to the distance d.

6.3.1 Local Range and Orientation Identification

Let $y(t) = [m^{\mathcal{R}}_{i_1 j_1}(t), ..., m^{\mathcal{R}}_{i_n j_n}(t)]^\top \in \mathbb{R}^n$ be a collection of n geometric moments extracted from the object, represented by the region $\mathcal{R}$, and let $x(t) = [x_1(t)\, x_2(t)\, x_3(t)]^\top \in \mathbb{R}^3$ describe the 3-dimensional information of the object, namely $x_1 = A_{10} = -n_x/d$, $x_2 = A_{01} = -n_y/d$ and $x_3 = A_{00} = -n_z/d$, where d is the *depth* of the object with respect to the camera frame. The dynamics of the evaluated geometric moments and of the 3-dimensional parameters of the object are described by equations of the form [17, 77]

$$\begin{aligned} \dot{y}(t) &= F(y(t), v(t))x(t) + G(y(t), \omega(t))\,, \\ \dot{x}(t) &= x(t)x(t)^\top v(t) + x(t) \times \omega(t)\,, \end{aligned} \tag{6.39}$$

where $\times$ denotes the vector product and the matrix functions $F : \mathbb{R}^n \times \mathbb{R}^3 \to \mathbb{R}^{n\times 3}$ and $G : \mathbb{R}^n \times \mathbb{R}^3 \to \mathbb{R}^n$ are obtained from equation (6.36). Moreover, due to the linearity in v and ω of the relation (6.36), F and G are identically zero when the translational velocity, $v(t)$, and the angular velocity, $\omega(t)$, are zero, respectively.

Assumption 6.3. F and G are known continuous mappings.

In what follows we consider the problem of designing a reduced-order observer for the unmeasured part of the state of the system (6.39), *i.e.* $x(t)$, under Assumptions 6.2 and 6.3. To begin with, only translational velocities are allowed, *i.e.* $\omega(t) = 0$ for all $t \geq 0$, hence $G(y(t), \omega(t)) \equiv 0$.

Proposition 6.1. Consider system (6.39). Suppose that the solutions of system (6.39) exist for all $t \geq 0$ and that $x_i(t) > 0$ for all $t \geq 0$, $i = 1, 2, 3$. Moreover suppose that Assumptions 6.2 and 6.3 hold. Suppose that there exist scalar functions $\beta_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$, and non-negative scalar functions $\gamma_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$, not in $\mathcal{L}_1$, *i.e.* $\int_0^\infty \gamma_i(\tau)d\tau = \infty$, such that

$$\frac{\partial \beta_i}{\partial y}(y, t)F(y, v) - (v_x\ v_y\ v_z) = \gamma_i(t)e_i\,, \tag{6.40}$$

where e_i is the i-th vector of the canonical basis in $\mathbb{R}^n$. Let

$$\dot{\zeta}_i = -\gamma_i(t)\left(e^{\zeta_i} + \beta_i(y, t)\right) - \frac{\partial \beta_i}{\partial t}\,, \tag{6.41}$$

where $\zeta_i(t) \in \mathbb{R}$. Then all trajectories of the interconnected system (6.39)-(6.41) are such that

$$\lim_{t\to\infty} [x(t) - \hat{x}(t)] = 0\,,$$

where

$$\hat{x}_i = e^{\zeta_i + \beta_i(y,t)}\,, \tag{6.42}$$

$i = 1, 2, 3$. That is (6.41)-(6.42) is a locally convergent observer for the system (6.39). Moreover

$$\lim_{t\to\infty} \left[\frac{1}{d(t)^2} - (\hat{x}_1(t)^2 + \hat{x}_2(t)^2 + \hat{x}_3(t)^2) \right] = 0\,.$$

◇

Proof. Let the estimation errors be $z_i = \zeta_i + \beta_i(y,t) - \ln(x_i)$, $i = 1, 2, 3$, and note that they are defined for all $t \geq 0$, and that

$$\dot{z} = Az = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} z\,, \tag{6.43}$$

where

$$a_{ij} = \hat{x}_j \left[\frac{\partial \beta_i}{\partial y}(y,t) F_j(y,t) - v_j(t) \right] \rho(z_j)\,,$$

with $F_i \in \mathbb{R}^{n\times 1}$ the i-th column of the matrix function F. For the sake of notational simplicity $v_j(t)$ denotes $v_x(t), v_y(t)$ and $v_z(t)$ for $j = 1, 2, 3$, respectively. Moreover, the function ρ, which is such that $z\rho(z) = e^{-z} - 1$, is continuous and negative for all values of its argument, by monotonicity of $e^{-z} - 1$.

Considering the condition (6.40) it is straightforward to see that the matrix A in the estimation error dynamics 6.43 is equal to

$$A = \operatorname{diag}\{\hat{x}_1\rho(z_1)\gamma_1(t),\, \hat{x}_2\rho(z_2)\gamma_2(t),\, \hat{x}_3\rho(z_3)\gamma_3(t)\}\,.$$

Since the signals $\gamma_i(t)$, $t \geq 0$, $i = 1, 2, 3$, are *sufficiently exciting* by assumption, all trajectories of the system (6.43) asymptotically converge to $z = 0$. Hence the trajectories of system (6.39)-(6.41)-(6.42) are such that $\hat{x}_i(t) = e^{\zeta_i(t)+\beta_i(y,t)}$ is an asymptotically convergent estimate of the unmeasured state x_i. The second claim in the statement follows immediately recalling that the vector $n(t)$ is the unit plane normal thus the relation $n_x(t)^2 + n_y(t)^2 + n_z(t)^2 = 1$ holds for all $t \geq 0$. □

Remark 6.8. The assumption that x_i is always positive, $i = 1, 2, 3$, is mild since, if the angular velocities of the camera are zero, the set $\{x \in \mathbb{R} : x \geq 0\}$ is invariant for the dynamics of the system (6.39). ▲

Remark 6.9. Suppose that the matrix function F satisfies the following *instantaneous observability* and boundedness condition[1]

$$\epsilon_1 I < F(y,t)^\top F(y,t) < \epsilon_2 I\,, \tag{6.44}$$

where ϵ_1 and ϵ_2 are two positive constants, for all $t \geq 0$ and all $y \in \mathbb{R}^n$, then all the conditions of Proposition 6.1 are satisfied selecting $\gamma_i(t) \equiv \bar{\gamma}_i > 0$, for $i = 1, 2, 3$ and selecting β as the solution, if it exists, of the partial differential equations

$$\frac{\partial \beta_i}{\partial y}(y,t) = [(v_x \; v_y \; v_z) + \bar{\gamma}_i e_i]\,(F^\top F)^{-1} F^\top\,, \tag{6.45}$$

for $i = 1, 2, 3$, which is such that

$$\frac{\partial \beta_i}{\partial y}(F_1 \; F_2 \; F_3) = (v_x \; v_y \; v_z) + \bar{\gamma}_i e_i\,. \tag{6.46}$$

Note that (6.45) is equivalent to (6.46), since the *instantaneous observability* condition (6.44) implies that the matrix $F(y,t)$ is full-rank for all $t \geq 0$ and all possible values of the measured moments. Exploiting the condition (6.44) uniform asymptotic stability of the zero equilibrium of system (6.43) can be proven using the Lyapunov function $V(z) = 1/2 z^\top z$. ▲

6.3.2 Global Range Identification

Let $y(t) \in \mathbb{R}^n$ be as in (6.39) and let $\chi(t) \in \mathbb{R}_+$ be equal to $1/d(t)$, *i.e.* the inverse of the *depth* of the center of mass of the observed object, with respect to the camera. The nonlinear system, describing the dynamics of the evaluated geometric moments and of the inverse of the distance, is described by equations of the form [17, 77]

$$\begin{aligned} \dot{y}(t) &= F_s(y(t), v(t))\chi(t) + G_s(y(t), \omega(t))\,, \\ \dot{\chi}(t) &= -c(t)\chi(t)^2\,, \end{aligned} \tag{6.47}$$

where the mapping F_s is obtained as a linear combination of the columns of the matrix function F in (6.39), with coefficients given by $[-n_x, -n_y, -n_z]$.

Assumption 6.4. $F_s : \mathbb{R}^n \times \mathbb{R}^3 \to \mathbb{R}^{n\times 1}$, $G_s : \mathbb{R}^n \times \mathbb{R}^3 \to \mathbb{R}^{n\times 1}$ and $c(t) = n \cdot v$ are known continuous mappings. Moreover, there exists $c_0 > 0$ such that, for all $t \geq 0$, $\|c(t)\| \leq c_0$.

In what follows we consider the problem of design a reduced-order observer for the unmeasured part of the state of the systems (6.47), *i.e.* $\chi(t)$, under Assumptions 6.2 and 6.4.

[1] With a minor abuse of notation $F(y,t)$ is used to denote the function $F(y(t), v(t))$ in (6.39).

Proposition 6.2. Consider system (6.47). Suppose that the solutions of system (6.47) exist for all $t \geq 0$, that $\chi(t) > 0$ for all $t \geq 0$ and that Assumptions 6.2 and 6.4 hold. Moreover suppose that there exist a scalar function $\beta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and a non-negative scalar function $\gamma : \mathbb{R} \to \mathbb{R}$, not in $\mathcal{L}_1$, such that

$$\frac{\partial \beta}{\partial y}(y,t)F_s(y,t) + c(t) \geq \gamma(t)\,, \tag{6.48}$$

for all $t \geq 0$ and all $y \in \mathbb{R}^n$. Let

$$\dot{\zeta} = -\frac{\partial \beta}{\partial y}G_s(y,t) - \left[\frac{\partial \beta}{\partial y}(y,t)F_s(y,t) + c(t)\right]\left(e^{\zeta} + \beta(y,t)\right) - \frac{\partial \beta}{\partial t}\,, \tag{6.49}$$

with $\zeta(t) \in \mathbb{R}$. Then all trajectories of the interconnected system (6.47), (6.49) are such that

$$\lim_{t\to\infty}[\chi(t) - \hat{\chi}(t)] = 0\,,$$

where

$$\hat{\chi} = e^{\zeta + \beta(y,t)}\,. \tag{6.50}$$

◇

Remark 6.10. The assumption on the positivity of $\chi(t)$ for all $t \geq 0$ reduces to requiring that $\chi(t)$ never reaches zero, *i.e.* that the distance of the object never diverges to infinity. In fact from the physics of the problem, *i.e.* the distance of the object from the image plane is always non-negative, and from the second equation of system (6.47), it is possible to deduce that $\chi(t) \geq 0$, for all $t > 0$, provided $\chi(0) \geq 0$. ▲

Proof. Let the estimation error be $z = \zeta + \beta(y,t) - \ln(\chi)$, and note that the estimation error is defined for all $t \geq 0$. Note now that

$$\dot{z} = e^{\zeta} + \beta(y,t)\left[\frac{\partial \beta}{\partial y}(y,t)F_s(y,t) + c(t)\right](e^{-z}-1) = xe^{z}\left[\frac{\partial \beta}{\partial y}(y,t)F_s(y,t) + c(t)\right](e^{-z}-1), \tag{6.51}$$

which has an equilibrium in $z = 0$, and that the term $e^{-z} - 1$ can be written as $z\rho(z)$ where $\rho(0) = -1$ and $\rho(z) = \sum_{i=0}^{\infty} \frac{(-z)^i}{(i+1)!}$ is, by monotonicity of $e^{-z} - 1$, sign definite, namely negative.

System (6.51) can be considered as a time-varying system with state z. Moreover, since the signal $\gamma(t)$ is *sufficiently exciting*, by (6.48) all the trajectories of system (6.51) uniformly asymptotically converge to $z = 0$, which implies that the trajectories of system (6.47)-(6.49) are such that $\hat{\chi} = e^{\zeta + \beta(y,t)}$ is an asymptotically converging estimate of the unmeasured state χ. □

Remark 6.11. If there exists a constant ϵ such that the *instantaneous observability* condition

$$\|F_s(y,t)\|^2 \geq \epsilon > 0 \tag{6.52}$$

holds for all $t \geq 0$ and all $y \in \mathbb{R}^n$, then the assumptions of Proposition 6.2 are satisfied selecting $\gamma(t) \equiv c_0\bar{\gamma}$, with $\bar{\gamma} > 1$ a positive constant, provided β is such that

$$\frac{\partial \beta}{\partial y} = \frac{2\bar{\gamma}c_0}{\epsilon} F_s(y,t)^\top \,. \tag{6.53}$$

In this case global uniform asymptotic stability of the zero equilibrium of system (6.51) can be proved using the Lyapunov function $V(z) = \frac{1}{2}\, z^2$. In fact, the time derivative of the Lyapunov function along the trajectories of the system (6.51) satisfies

$$\dot{V} \leq e^{\zeta + \beta(y,t)} \rho(z)\bar{\gamma}c_0\|F_s(y,t)\|^2 z^2 < 0 \,,$$

for all $z \neq 0$. ▲

Remark 6.12. The solution of the partial differential equations (6.53) when $y \in \mathbb{R}^n$ with $n > 1$ can be hard to find, and the equation may not even have a solution. (Obviously, if y has dimension one the solution of (6.53) is the integral of the right-hand side of the equation). At the end of this section it is shown that even if $n > 1$ an approximate solution is sufficient to solve the considered estimation problem. ▲

Remark 6.13. The choice of the estimation error z is natural, since the logarithm of χ allows to compensate, in the error dynamics, the quadratic terms of the dynamics of the unmeasured part of the state. ▲

Remark 6.14. In [45] the problem of estimating the range of an object moving in the 3-dimensional space by observing the motion of its *point feature* is studied. The system considered is similar to the system (6.47), *i.e.* quadratic in the unmeasured state. The result therein heavily relies on the particular structure of the functions of the perspective system considered, which are such that

$$F(y,t) = b(t)I_n y + b_0(t) \,,$$

for some known scalar function $b(t)$ and some known vector $b_0(t)$. Note that this is not the case for the system (6.47). ▲

6.3.3 Approximate Solution with Dynamic Scaling

The solution of the problem of estimating simultaneously the depth and the orientation of the object hinges upon the solvability of the partial differential equations (6.45)[2]. Obviously, the pde (6.45) do not always admit a solution, and even if there exists a solution this can be not easy to determine. This issue can be resolved by introducing a dynamic scaling factor, $r(t) \in \mathbb{R}$, and an output filter, with state $\hat{y}(t) \in \mathbb{R}^n$, and proving that an approximate solution of (6.45) allows to guarantee asymptotic stability of the zero equilibrium of the estimation error dynamics under condition (6.44). Define a scaled observer error

$$z_i = \frac{\zeta_i + \beta_i(y, \hat{y}, t) - \ln(x_i)}{\ln(r(t) - \mu)}\,, \tag{6.54}$$

$i = 1, 2, 3$, where $\mu \in (0, 1)$ and let the mapping $\beta_i : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ be defined as

$$\beta_i(y, \hat{y}, t) = \left[v^\top + \bar{\gamma}_i e_i\right] (F(\hat{y}, v)^\top F(\hat{y}, v))^{-1} F(\hat{y}, v)^\top y\,, \tag{6.55}$$

i.e. such that

$$\frac{\partial \beta_i}{\partial y}(y, \hat{y}, t) = \left[v^\top + \bar{\gamma}_i e_i\right] (F(y, v)^\top F(y, v))^{-1} F(y, v)^\top + \Delta_i(y, \eta)\,, \tag{6.56}$$

where $\eta = \hat{y} - y$ and $\Delta_i : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, 3$, is a mapping such that $\Delta_i(y, 0) \equiv 0$. Define then the matrix $\Delta(y, \eta) = \left[\Delta_1(y, \eta)^\top \; \Delta_2(y, \eta)^\top \; \Delta_3(y, \eta)^\top\right]^\top \in \mathbb{R}^{3\times n}$ and note that there exist matrices $\tilde{\Delta}_j \in \mathbb{R}^{3\times n}$, for $j = 1, ..., n$, such that $\Delta(y, \eta) = \sum_{j=1}^n \tilde{\Delta}_j \eta_j$. Moreover, the auxiliary state $\hat{y}$ is obtained from the filter

$$\dot{\hat{y}} = F(y, v)\hat{x} - K(r, y, \eta)(\hat{y} - y)\,, \tag{6.57}$$

where K is a positive definite matrix-valued function.

To streamline the presentation of the following results, let $\Psi : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^{3\times 3}$ be defined as $\Psi(\hat{x}, z, r) = \mathrm{diag}\{\hat{x}_i \rho(z_i \ln(r - \mu))\}$, namely such that $x - \hat{x} = \ln(r - \mu)\Psi(\hat{x}, z, r)z$. Considering (6.57), the dynamics of the error η are

$$\dot{\eta}(t) = -\ln(r - \mu)F(y, v)\Psi(\hat{x}, z, r)z - K(r, y, \eta)\eta\,. \tag{6.58}$$

Recalling the equations of the time derivative of the estimation error z in (6.54) and the definition of the mapping β as in (6.55) and letting

$$\dot{\zeta}_i = -\frac{\partial \beta_i}{\partial t} - \frac{\partial \beta_i}{\partial \hat{y}}\dot{\hat{y}} - \left[\frac{\partial \beta_i}{\partial y}F(y, v) - v^\top\right]\hat{x}$$

[2]Similar considerations apply to depth estimation problem.

yields the estimation error dynamics

$$\dot{z} = A(\hat{x}, z, r)z + \Delta(y, \eta)F(y, v)\Psi(\hat{x}, z, r)z - \frac{\dot{r}}{(r-\mu)\ln(r-\mu)}z\,, \tag{6.59}$$

with $A(\hat{x}, z, r) = \mathrm{diag}\{\hat{x}_i \rho(z_i \ln(r-\mu))\bar{\gamma}_i\}$.

Assumption 6.5. There exist positive constants $\bar{\gamma}_i$, $i = 1, 2, 3$, and a non-negative constant $\bar{\rho} \in \mathbb{R}$ such that

$$A(\hat{x}, z, r) \leq -\bar{\rho}I - \lambda\Psi(\hat{x}, z, r)F(y, v)^\top F(y, v)\Psi(\hat{x}, z, r)\,, \tag{6.60}$$

for some positive constant $\lambda \in \mathbb{R}$.

By (6.44) and the definitions of ρ and $\hat{x}$, it follows that the Assumption 6.5 holds in any compact set $(x, z, r) \in \Omega \subset \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ for arbitrary $\lambda > 0$ and $\bar{\rho} > 0$, selecting

$$\bar{\gamma}_i \geq \lambda\epsilon_2\alpha_i^2 + \frac{\bar{\rho}}{\alpha_i}\,,$$

where $\alpha_i = \max\limits_{(x,z,r)\in\Omega}\{e^{z_i \ln(r-\mu)}x_i\|\rho(z_i \ln(r-\mu))\|\}$, $i = 1, 2, 3$.

Proposition 6.3. Consider the system (6.59). Suppose that its solutions exist for all $t \geq 0$, that $x_i(t) \in (0, \infty)$, $i = 1, 2, 3$, for all $t \geq 0$, and that Assumptions 6.2 and 6.5 and the condition (6.44) hold with $\lambda > 0$. Let

$$\dot{r}(t) = c(r(t) - \mu)\ln(r(t) - \mu)\|\Delta(y, \eta)\|^2\,, \tag{6.61}$$

with $r(0) > e + \mu$, $c > \frac{1}{2\lambda}$ and

$$K(y, \eta, r) = \left(\frac{\ln(r-\mu)^2}{2\lambda} + k\right)I + cr(r-\mu)\ln(r-\mu)\mathrm{diag}\{\|\tilde{\Delta}_i(y, \eta)\|^2\}\,, \tag{6.62}$$

where k is a positive constant. Then the system (6.58)-(6.59)-(6.61) has a globally stable set of equilibria defined by $\{(z, \eta, r) : z = 0, \eta = 0\}$. Moreover z, $\dot{z}$, $r \in \mathcal{L}_\infty$, $\eta \in \mathcal{L}_\infty \cap \mathcal{L}_2$. If additionally Assumption 6.5 holds with $\bar{\rho} > 0$, then $z \in \mathcal{L}_2$ and $z(t)$ converges to zero. $\diamond$

Proof. To begin with note that $\dot{r}(t) \geq 0$, for all $t \geq 0$, hence $r(t) \geq r(0) > \mu + e$. This, in turn, guarantees that $\ln(r-\mu) > 1$. Consider now the positive-definite and proper function $V(z) = \frac{1}{2}z^\top z$, whose time derivatives along the trajectories of system (6.59) satisfies, by

Assumption 6.5 and replacing the dynamics of the scaling as in (6.61),

$$\begin{aligned}
\dot{V} &\leq -\bar{\rho}\|z\|^2 - \lambda\|F(y,v)\Psi(\hat{x},z,r)z\|^2 + \|z^\top\Delta(y,\eta)\|\|F(y,v)\Psi(\hat{x},z,r)z\| \\
&\quad - \frac{\dot{r}}{(r-\mu)\ln(r-\mu)}\|z\|^2 \\
&\leq -\bar{\rho}\|z\|^2 - \frac{\lambda}{2}\|F(y,v)\Psi(\hat{x},z,r)z\|^2 + \frac{1}{2\lambda}\|\Delta(y,\eta)\|^2\|z\|^2 - \frac{\dot{r}}{(r-\mu)\ln(r-\mu)}\|z\|^2 \\
&\leq -\bar{\rho}\|z\|^2 - \frac{\lambda}{2}\|F(y,v)\Psi(\hat{x},z,r)z\|^2 .
\end{aligned} \tag{6.63}$$

Hence the system (6.59) has a uniformly globally stable equilibrium at the origin and $z \in \mathcal{L}_\infty \cap \mathcal{L}_2$, provided $\bar{\rho} > 0$.

Consider now the candidate Lyapunov function $W(z,\eta) = V(z) + \frac{1}{2}\eta^\top\eta$ the time derivative of which satisfies

$$\begin{aligned}
\dot{W} &\leq -\eta^\top K\eta - \eta^\top\ln(r-\mu)F(y,v)\Psi(\hat{x},z,r)z - \frac{\lambda}{2}\|F(y,v)\Psi(\hat{x},z,r)z\|^2 \\
&\leq -\eta^\top K\eta + \ln(r-\mu)\|\eta\|\|F(y,v)\Psi(\hat{x},z,r)z\| - \frac{\lambda}{2}\|F(y,v)\Psi(\hat{x},z,r)z\|^2 \\
&\leq -\eta^\top K\eta + \frac{1}{2\lambda}\ln(r-\mu)^2\|\eta\|^2 \\
&\leq -k\|\eta\|^2 ,
\end{aligned} \tag{6.64}$$

where the last inequality is obtained from the definition of the matrix $K(r,y,\eta)$ in (6.62). The equation (6.64) implies that $\eta \in \mathcal{L}_\infty \cap \mathcal{L}_2$. To conclude the proof it remains to show that the scaling $r(t)$ is bounded. To show this consider the Lyapunov function $U(\eta,z,r) = W(z,\eta) + \frac{1}{2}r^2$, which is such that

$$\begin{aligned}
\dot{U} &\leq cr(r-\mu)\ln(r-\mu)\|\Delta(y,\eta)\|^2 - c(r-\mu)\ln(r-\mu)\eta^\top\mathrm{diag}\{\|\tilde{\Delta}_i(y,\eta)\|^2\}\eta \\
&\leq cr(r-\mu)\ln(r-\mu)\sum_{j=1}^{n}\|\tilde{\Delta}(y,\eta)\|^2\eta_j^2 - c(r-\mu)\ln(r-\mu)\eta^\top\mathrm{diag}\{\|\tilde{\Delta}_i(y,\eta)\|^2\}\eta \leq 0 ,
\end{aligned} \tag{6.65}$$

which proves the claim. Note that, by continuity of F and by equation (6.44), $y \in \mathcal{L}_\infty$ and moreover $\hat{x}_i = x_i e^{z_i\ln(r-\mu)} \in \mathcal{L}_\infty$, since $x, z, r \in \mathcal{L}_\infty$ by assumption, by (6.63) and by (6.65), respectively. Hence, by continuity of the mappings on the right-hand side of equation (6.59), $\dot{z} \in \mathcal{L}_\infty$. Finally, by Barbalat's lemma, provided $\bar{\rho} > 0$, $z(t)$ converges to zero. □

6.3.4 Examples

The first case study considers the situation in which the orientation of the plane containing the observed object is known for all $t \geq 0$. Suppose that the object is a disk, *e.g.* a coin. The motion profiles of the camera are selected to show the importance of the use of several image geometric moments to recover the *depth* of the object.

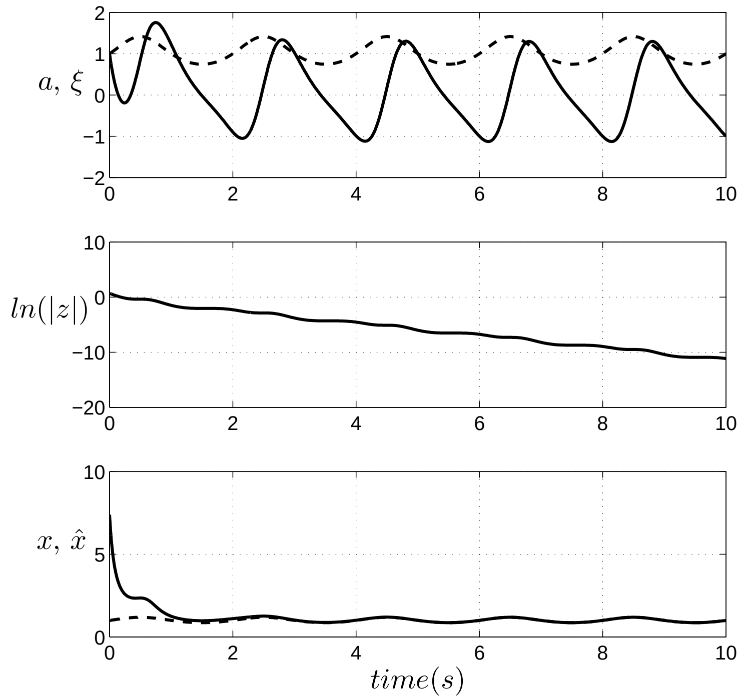

Figure 6.5: Top graph: time histories of the area $a(t)$ of the object (dashed line) and of the internal state of the observer $\xi(t)$ (solid line). Middle graph: time history of the logarithm of the modulo of the estimation error. Bottom graph: time histories of the estimate of the distance (solid line) and of the actual distance (dashed line).

In the first simulation let $y(t) = a(t) \in \mathbb{R}$. Suppose that the plane containing the object observed is parallel to the image plane of the camera, hence $n_x = n_y = 0$ and $n_z = -1$. The camera is commanded with a periodic predefined motion according to the velocity profiles $v_x(t) = v_y(t) = 0$ and $v_z(t) = 0.5\cos(\pi t)$, hence Assumption 6.4 is satisfied with $c_0 = 0.5$. In this situation an observer designed for point feature, such as the one in [41], cannot recover the *depth* of the object, *i.e.* the distance of the object from the camera. In the simulation we set $x(0) = 1$, $\hat{x}(0) = 7.5$ and $\gamma = 2$. Figure 6.5 shows (bottom graph) that the estimate $\hat{x}(t)$ approaches $x(t)$ after about 3 seconds of motion of the camera. The top graph shows the evolution of the area of the observed object due to the motion imposed to the camera and the time history of the internal state

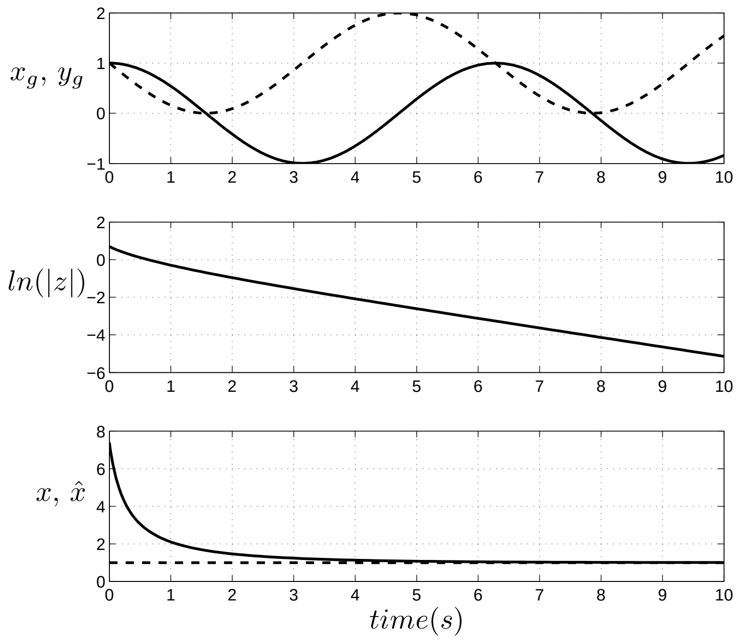

Figure 6.6: Top graph: time histories of x_g (dashed line) and of y_g (solid line). Middle graph: time history of the logarithm of the estimation error. Bottom graph: time histories of the estimation of the distance (solid line) and of the actual distance (dashed line).

of the observer, *i.e.* $\xi(t)$. The middle graph shows the time history of the logarithm of the modulo of the estimation error, highlighting the exponential convergence rate. Note that with the considered choice of camera velocities there are isolated time instants in which the instantaneous observability condition (6.52) is violated, namely the instants of inversion of the motion direction of the camera. Nevertheless, exponential convergence of the estimation error is guaranteed.

In the second case study suppose that the motion of the camera describes a circle on the (x_1, x_2)-plane, *i.e.* $v_x(t) = R\cos(lt)$, $v_y(t) = R\sin(lt)$ and $v_z(t) = 0$, where l and R are constants. In this situation the area of the observed coin does not change over time and so the geometric moment of order zero cannot be used to recover the *depth*, which is constant. However, the two geometric moments of order one, *i.e.* the coordinates of the center of mass of the object, can be exploited to guarantee convergence of the observer. In this case the solution $\beta(x_g, y_g, t)$ of the partial differential equation defined in Remark 6.11 is given by $\beta(x_g, y_g, t) = -\gamma Rl(v_x(t)x_g(t) + v_y(t)y_g(t))$, yielding the estimation error dynamics $\dot{z} = e^{\xi+\beta(y,t)}\gamma R^3 l(e^{-z} - 1)$ which have a globally asymptotically stable equilibrium at $z = 0$. Figure 6.6 (top graph) shows the periodic behavior of the coordinates of the center of mass of the object, due to the imposed camera velocities, resulting in a circular motion of the center of mass in the (x_1, x_2)-plane. The bottom graph shows the convergence of the estimate of the *depth* of the coin to the real value, which is constant. Note that, differently from the first case study, the condition (6.52) is satisfied for every $t \geq 0$ and hence stability of the equilibrium $z = 0$ is guaranteed for the error dynamics. Finally, the middle graph displays the history of the modulo of the logarithm of the estimation error. Note, again, the exponential convergence rate.

Finally, the problem of estimating the *depth* of the object is studied in the case in which the orientation of the plane containing the object is unknown, using the results in Section 6.3.1. Suppose that the information about the initial orthant of the unit normal is available and that the camera is moving away from the observed object with a normal velocity decreasing as $v_z(t) = 1/(t+1)^2$. For this motion profile of the camera condition (6.44) is satisfied for all possible values of the moments and for all $t \geq 0$. Assume that the initial configuration, which is unknown, of the unit normal over the distance, that has to be estimated, is $x(0) = [1, -0.5, -1]^\top$.

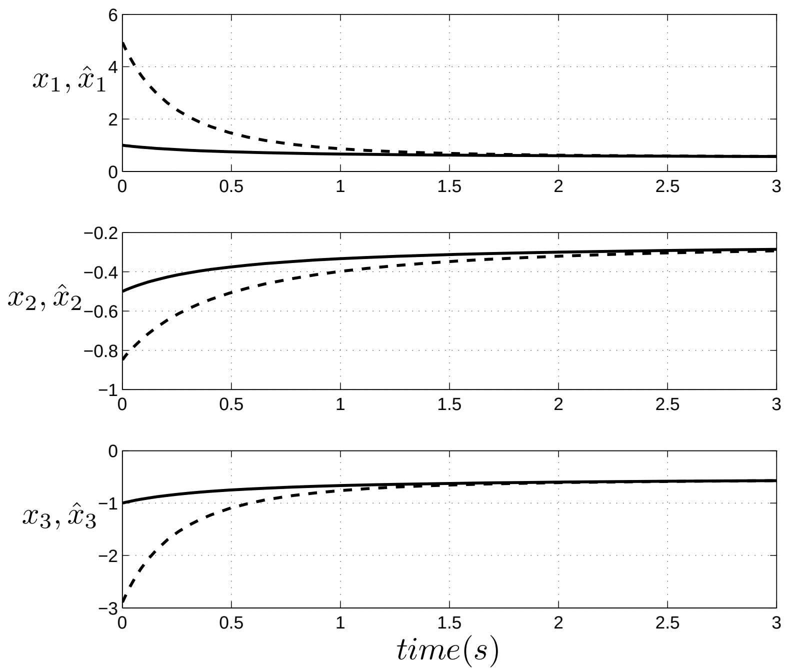

Figure 6.7: Time histories of the actual orientation, $(x_1(t),\ x_2(t),\ x_3(t))$, of the plane containing a planar face of the observed object (solid lines) together with the estimated orientation $\hat{x}(t)$ (dashed lines).

Note that the particular motion profile imposed to the camera yields not only a variation of the area of the object but also of the observed position of the center of mass. With simple computations it is possible to determine the structure of the matrix function

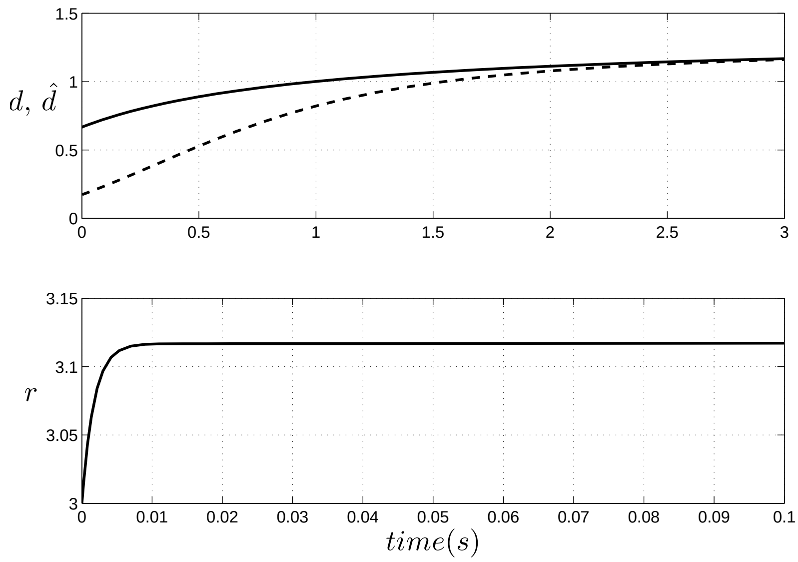

Figure 6.8: Top graph: time histories of the actual distance of the object (solid line) and the estimated distance (dashed line). Bottom graph: time history of the dynamic scaling $r(t)$, with $r(0)$.

$F(y, v)$, *i.e.*

$$F(y,v) = \begin{pmatrix} 3ax_g v_z & 3ay_g v_z & 2av_z \\ x_g^2 v_z + 4n_{20} v_z & x_g y_g v_z & x_g v_z \\ x_g y_g v_z & y_g^2 v_z + 4n_{02} v_z & y_g v_z \end{pmatrix},$$

where n_{20} and n_{02} denote the central moments, *i.e.* geometric moments of order two computed around the center of mass and normalized with respect to the zero order moment, namely the area of the object. Solving the partial differential equation (6.45), with the matrix $F(y, v)$ defined as above, is an hard task. However, as shown in Section 6.3.1 it is possible to design an observer introducing an output filter and a dynamic scaling factor, $r(t)$. In Figure 6.7 the time histories of the actual orientation, $x(t)$, of the plane containing a planar face of the observed object (solid lines) together with the estimated orientation $\hat{x}(t)$ (dashed lines) are displayed. The time histories of the actual distance of the object $d(t)$ (solid line) and the estimated distance $\hat{d}(t)$ (dashed line) are in the top graph of Figure 6.8. The bottom graph of Figure 6.8 shows the time history of the single dynamic scaling factor $r(t)$ with $r(0) = 3$. Note the different time scale.

6.4 Conclusions

In this chapter we have extended the approach based on the dynamic extension to encompass the reduced-order observer design problem via the Immersion and Invariance technique. The latter methodology, in fact, hinges upon the solution of a system of partial differential equations which ensures attractivity of a desired invariant subset of the augmented state-space of the plant and the observer. The main result of this chapter consists in showing that the computation of a closed-form solution of the pde arising may be avoided making use of an output filter and a dynamic scaling factor. The design is then specialized to tackle the problem of estimating the range and the orientation of an object observed through a single pin-hole camera.

Chapter 7

Applications

7.1 Introduction

In this chapter several examples are illustrated to demonstrate the applicability of the approach, highlighting advantages and drawbacks of the proposed methodology. The applications range from optimal and robust control of fully actuated mechanical systems in Section 7.2 to optimal control problems in internal combustion engines discussed in Sections 7.3 and 7.4.

7.2 Fully Actuated Mechanical Systems

Consider fully actuated mechanical systems described by the Euler-Lagrange equations [71], namely

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tilde{\tau}\,, \tag{7.1}$$

where $G(q)$ contains the potential forces and the matrix $C(q,\dot{q})$, which is linear in the second argument, describes the Coriolis and centripetal forces and is computed using the Christoffel symbols of the first kind.

Defining the variables $x_1 = q$ and $x_2 = \dot{q}$, with $x_1(t) \in \mathbb{R}^n$, $x_2(t) \in \mathbb{R}^n$ yields

$$\begin{aligned} \dot{x}_1 &= x_2\,, \\ \dot{x}_2 &= M(x_1)^{-1}\left[\tilde{\tau} - C(x_1,x_2)x_2 - G(x_1)\right]. \end{aligned} \tag{7.2}$$

Assume that the preliminary feedback $\tilde{\tau} = \hat{\tau} + G(0)$, $\hat{\tau} \in \mathbb{R}^n$, is applied to compensate for the effect of gravity at the origin. Under this assumption, the gravitational term becomes $\tilde{G}(x_1) = G(x_1) - G(0)$. Therefore there exists a continuous matrix valued function $\bar{G} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ such that $\tilde{G}(x_1) = \bar{G}(x_1)x_1$, for all $x_1 \in \mathbb{R}^n$.

Suppose that the variables of interest – to minimize in the optimal control problem or for which a desired attenuation level needs to be guaranteed in the disturbance attenuation problem – are the positions x_1 and the velocities x_2 of the joints. Finally let $\hat{\tau} = u + d$ in the disturbance attenuation problem, whereas let $\hat{\tau} = u$ in the optimal control problem. Note that, as in to Chapter 4, $\epsilon^2 = -(1/\gamma^2 - 1)$, where $\gamma \in (1, \infty]$ is the desired disturbance attenuation level.

The definition of an *algebraic $\bar{P}$ solution* for the system (7.2) requires the computation of the positive definite matrix $\bar{P}$. To this end consider the first-order approximation of the nonlinear system (7.2), namely

$$\dot{x} = \begin{bmatrix} 0 & I_n \\ -M(0)^{-1}D & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ M(0)^{-1} \end{bmatrix} u \,,$$

where $D = D^\top \in \mathbb{R}^{n \times n}$ is a constant matrix defined as $D = \partial G/\partial x_1(0)$. Note that D is the Hessian matrix of the potential energy, denoted by $\mathcal{U}(x_1)$, evaluated in $x_1 = 0$, therefore if the potential energy has a strict local minimum at the origin then D is positive definite. The algebraic Riccati equation arising is given by

$$\bar{P}A + A^\top \bar{P} - \epsilon^2 \bar{P}BB^\top \bar{P} + H^\top H = 0 \,, \tag{7.3}$$

where $H = diag\{\mu_1 I_n,\ \mu_2 I_n\}$. The positive constants μ_1 and μ_2 are weighting factors for the positions and the velocities of the joints, respectively. Partition the matrix $\bar{P}$ as

$$\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \\ \bar{P}_2^\top & \bar{P}_3 \end{bmatrix} ,$$

and let the matrices $\bar{P}_1, \bar{P}_2$ and $\bar{P}_3 \in \mathbb{R}^{n \times n}$ be defined as the solutions of the system of quadratic matrix equations

$$\begin{aligned} \mu_1^2 I_n &= \bar{P}_2 M(0)^{-1} D + D M(0)^{-1} \bar{P}_2^\top + \epsilon^2 \bar{P}_2 M(0)^{-2} \bar{P}_2^\top \,, \\ \bar{P}_1 &= D^\top M(0)^{-1} \bar{P}_3 + \epsilon^2 \bar{P}_2 M(0)^{-2} \bar{P}_3 \,, \\ \bar{P}_3 &= \frac{1}{\epsilon} M(0) [\bar{P}_2 + \bar{P}_2^\top + \mu_2^2 I_n]^{1/2} \,. \end{aligned} \tag{7.4}$$

Note that the positive definite solution $\bar{P}$ of the matrix equations (7.4) exists and is unique, by observability and controllability of the linearized system.

The following result provides the solutions to Problems 4.1 and 5.4 in Sections 4.2 and 5.4 ,respectively, for fully actuated mechanical systems described by the equations (7.2).

Proposition 7.1. Consider the mechanical system (7.2). Suppose that $\Sigma_i(0) > 0$, $i = 1, 2$, let $\gamma \in (1, \infty]$. Let $x_i^\top \Upsilon_i^\top(x) \Upsilon_i(x) x_i = x_i^\top (\mu_i^2 I_n + \Sigma_i(x)) x_i \ > \ 0$, $i = 1, 2$, let $W_1(x_1, x_2)$ be such that $\Upsilon_1^\top(x)\Upsilon_1(x) = W_1 M(x_1)^{-1} \bar{G}(x_1) + \bar{G}(x_1)^\top M(x_1)^{-1} W_1 + \epsilon^2 W_1 M(x_1)^{-2} W_1$

and let

$$V_1(x) = W_1 M(x_1)^{-1} C(x) + \bar{G}(x_1)^\top M(x_1)^{-1} W_2 + \epsilon^2 W_1 M(x_1)^{-2} W_2\,,$$

$$V_2(x) = W_2 M(x_1)^{-1} C(x) + \frac{\epsilon^2}{2} W_2 M(x_1)^{-2} W_2 - \frac{1}{2} \Upsilon_2(x)^\top \Upsilon_2(x)\,,$$

with $W_2(x_1, x_2)$ such that $W_2(0,0) = \bar{P}_3$. Then there exist a matrix $R = R^\top > 0$, a neighborhood of the origin $\Omega \subseteq \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ and $\bar{k} \geq 0$ such that the function $V > 0$ as in (4.11), with

$$P(x) = [x_1^\top V_1 + x_2^\top V_2, x_1^\top W_1 + x_2^\top W_2]\,, \tag{7.5}$$

satisfies the Hamilton-Jacobi partial differential inequality (4.15) for all $k > \bar{k}$ and $(x, \xi) \in \Omega$, hence the dynamic control law (4.16) solves Problem 4.1 if $\epsilon \in (0,1)$ and Problem 5.4 if $\epsilon = 1$. ◇

Proof. Let $\Sigma_i(0) > 0$, $i = 1, 2$,

$$f(x) \triangleq \begin{bmatrix} x_2 \\ -M(x_1)^{-1}\left(C(x_1, x_2)x_2 + \tilde{G}(x_1)\right) \end{bmatrix}, \tag{7.6}$$

and

$$\Pi(x) \triangleq -\epsilon^2 \begin{bmatrix} 0 & 0 \\ 0 & M(x_1)^{-2} \end{bmatrix}, \tag{7.7}$$

$\epsilon \in (0,1)$, and note that P as in (7.5) is an *algebraic* $\bar{P}$ *solution* of the equation (4.10), *i.e.*

$$\begin{aligned} &2x_1^\top V_1 x_2 + 2x_2^\top V_2 x_2 - 2x_1^\top W_1 M^{-1} C x_2 - 2x_2^\top W_2 M^{-1} C x_2 + x_1^\top \Upsilon_1^\top \Upsilon_1 x_1 + x_2^\top \Upsilon_2^\top \Upsilon_2 x_2 \\ &- 2x_1^\top W_1 M^{-1} \tilde{G} - 2x_2^\top W_2 M^{-1} \tilde{G} - \epsilon^2 \left(x_1^\top W_1 + x_2^\top W_2\right) M^{-2} \left(W_1 x_1 + W_2 x_2\right) = 0. \end{aligned} \tag{7.8}$$

Then, by Theorem 4.1 and Corollary 4.2 there exist $\bar{k}$, R and a set $\Omega \subseteq \mathbb{R}^{4n}$ such that the dynamic control law (4.16) solves the regional dynamic state feedback $\mathcal{L}_2$-disturbance attenuation problem with stability. If $\epsilon = 1$ in (7.7) then the dynamic control law (4.16) solves the approximate regional dynamic optimal control problem. □

7.2.1 Planar Mechanical Systems

In the case of planar mechanical systems, *i.e.* systems with $G(x_1) = 0$ for all $x_1 \in \mathbb{R}^n$, the solution of the equation (7.8) can be given in closed-form. Towards this end note that the solution of the algebraic Riccati equation (7.3) is

$$\bar{P}_1 = \mu_1 \left[\frac{2\mu_1}{\epsilon} M(0) + \mu_2^2 I_n \right]^{1/2}, \ \bar{P}_2 = \frac{\mu_1}{\epsilon} M(0), \ \bar{P}_3 = \frac{1}{2}[S + S^\top], \tag{7.9}$$

with $S = \epsilon^{-1} M(0) \left[2\mu_1 \epsilon^{-1} M(0) + \mu_2^2 I_n \right]^{1/2}$.

Corollary 7.1. Consider the mechanical system (7.2) with $G(x_1) = 0$ for all $x_1 \in \mathbb{R}^n$. Suppose that $\Sigma_i(0) > 0$, $i = 1, 2$, let $\gamma \in (1, \infty]$. Let $W_1(x_1, x_2) = \frac{1}{\epsilon} \Upsilon(x)^\top M(x_1)$ and

$$V_1(x_1, x_2) = \Upsilon(x) \left[\frac{1}{\epsilon} C(x_1, x_2) + \epsilon M(x_1)^{-1} W_2 \right],$$

$$V_2(x_1, x_2) = W_2 M(x_1)^{-1} \left[C(x_1, x_2) + \frac{\epsilon^2}{2} M(x_1)^{-1} W_2 \right] - \frac{1}{2} \Upsilon_2(x)^\top \Upsilon_2(x),$$

with $W_2(x_1, x_2)$ such that $W_2(0, 0) = \bar{P}_3$. Then there exist a matrix $R = R^\top > 0$, a neighborhood $\Omega \subset \mathbb{R}^{2n} \times \mathbb{R}^{2n}$ containing the origin and $\bar{k} \geq 0$ such that for all $k > \bar{k}$ the dynamic control law 4.16), with P as in (7.5), solves Problem 4.1 if $\epsilon \in (0, 1)$ and Problem 5.4 if $\epsilon = 1$. $\diamond$

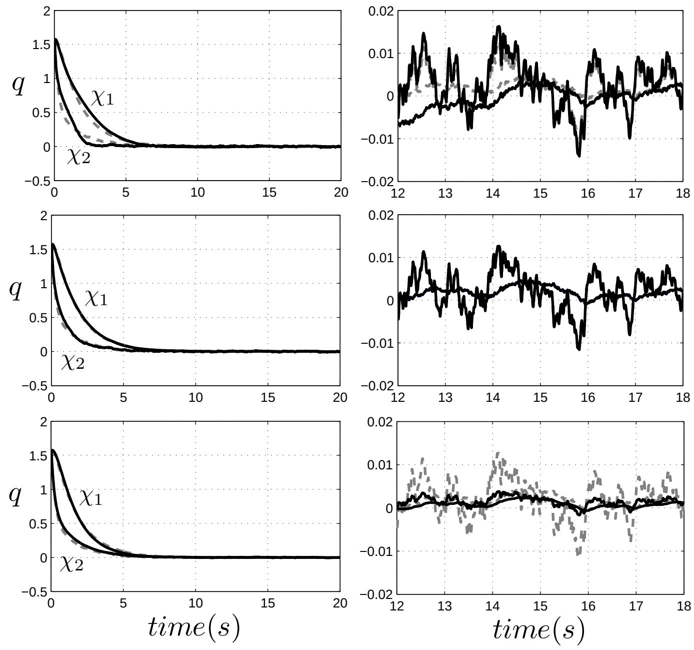

Figure 7.1: Left graphs: time histories of the angular positions of the joints for different values of the parameter κ (Top graph: $\kappa = 0.8$. Middle graph: $\kappa = 1$. Bottom graph: $\kappa = 5$) when the linearized control law u_o and the dynamic control law are applied, dashed and solid lines, respectively. Right graphs: zoom of the time histories in the left graphs in the interval $t \in [12, 18]$.

7.2.2 2-dof Planar Robot

Consider a planar fully actuated robot with two rotational joints and let $x_1 = [\chi_1, \chi_2] \in \mathbb{R}^2$ be the relative positions of the joints and $x_2 = [\chi_3, \chi_4] \in \mathbb{R}^2$ be the corresponding velocities. The dynamics of the mechanical system can be described by equations of the form (7.2) with

$$M = \begin{bmatrix} a_1 + 2a_2\cos(\chi_2) & a_2\cos(\chi_2) + a_3 \\ a_2\cos(\chi_2) + a_3 & a_3 \end{bmatrix}, C = \begin{bmatrix} 0 & -a_2\sin(\chi_2)(\chi_4 + 2\chi_3) \\ a_2\sin(\chi_2)\chi_3 & 0 \end{bmatrix}$$

where $a_1 = I_1 + m_1 d_1^2 + I_2 + m_2 d_2^2 + m_2 l_1^2$, $a_2 = m_2 l_1 d_2$ and $a_3 = I_2 + m_2 d_2^2$, with I_i, m_i, l_i and d_i the moment of inertia, the mass, the length and the distance between the center of mass and the tip of the i-th joint, $i = 1, 2$, respectively. In the simulations we let $m_1 = m_2 = 0.5Kg$, $l_1 = l_2 = 0.3m$, $d_1 = d_2 = 0.15m$ and $I_1 = I_2 = 0.0037Kg \cdot m^2$. To begin with, suppose that the action of the actuators is corrupted by white noise and consider a desired attenuation level on the position of the joints close to $\gamma = 1$, *e.g.* define $\epsilon = 0.1$. Let $\Sigma_1(\chi_1, \chi_2) = diag\{10^{-3}(1 + \chi_1^2), 10^{-3}(1 + \chi_2^2)\}$ and determine the *algebraic* $\bar{P}$ *solution* as described in Corollary 7.1. Since the Hamilton-Jacobi partial differential equation or inequality that yields the solution of the disturbance attenuation problem for the considered planar robot does not admit a closed-form solution, the performance of the dynamic control law defined in (4.16) is compared with the performance of the linearized problem, *i.e.* the static state feedback given by $u_o = -B_1^\top \bar{P} x$. In the dynamic control law, the matrix R is selected as $R = \kappa\Phi(0, 0)$, with $\kappa \in \mathbb{R}_+$.

Let the initial condition of the planar robot be $[\chi_1(0), \chi_2(0), \chi_3(0), \chi_4(0)] = [\pi/2, \pi/2, 0, 0]$. Figure 7.1 displays the time histories of the angular positions of the joints for different values of the parameter κ when the linearized control law u_o and the dynamic control law (4.16) are applied, dashed and solid lines, respectively. In all the plots the same disturbance affects the actuators. The behavior of the joints with $\kappa = 0.8$ is displayed in the top graph and it can be noted that the dynamic control law guarantees a *worse* rejection of the matched disturbances than the control law u_o. Increasing the value of the parameter κ improves the performance of the dynamic control law. In particular, the choice $R = \Phi(0, 0)$ (middle graph) yields a performance almost identical to that of u_o, whereas selecting $\kappa = 5$ the disturbance attenuation is significantly improved (bottom graph).

Consider the ideal case of absence of disturbances and let $\epsilon = 1$, *i.e.* $\gamma = \infty$. Figure 7.2 displays, for different initial conditions, the ratio between the costs yielded by the dynamic control law - considering the optimal value of $[\xi_1(0), \xi_2(0)]$ for each $\chi(0)$, letting $[\xi_3(0), \xi_4(0)] = [0, 0]$ - and by the optimal static state feedback for the linearized system, *i.e.* $\eta = V_d(\chi(0), \xi(0))/V_o(\chi(0))$, where V_d is the value function defined in (4.11), with P as in (7.5), and V_o is the quadratic value function for the linearized problem. Obviously,

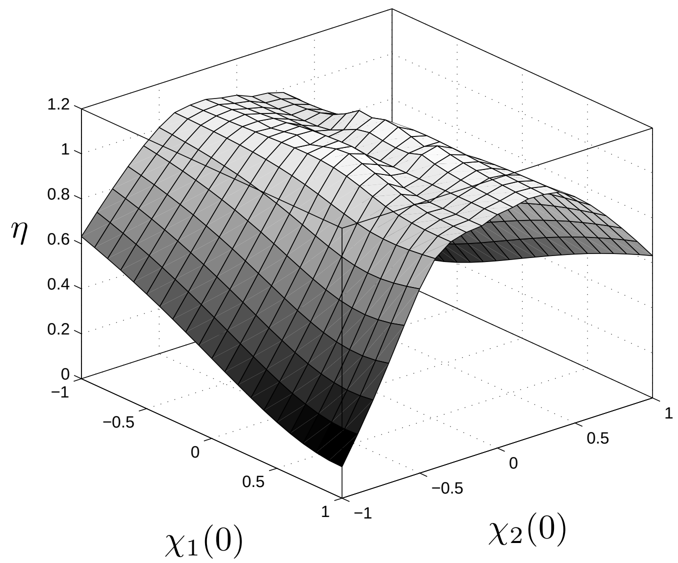

Figure 7.2: The ratio η between the costs yielded by the dynamic control law - considering the optimal value of $[\xi_1(0),\, \xi_2(0)]$ for each $\chi(0)$ and letting $[\xi_3(0),\, \xi_4(0)] = [0, 0]$ - and by the optimal static state feedback for the linearized system.

$\eta < 1$ implies that the cost paid by the dynamic control law is smaller than the cost of the optimal static state feedback for the linearized system. In Figure 7.2, the ratio η is greater than 1 in a neighborhood of the origin since $\Sigma_1(0) > 0$.

Let $\kappa = 1.2$ and $\chi(0) = [\pi/4, \pi/4, 0, 0]$. As above, the dynamic control law is compared with the solution of the linearized problem obtained with $\epsilon = 1$, the optimal cost of which is $V(\chi(0)) = \frac{1}{2}\chi(0)^\top \bar{P}\chi(0) = 0.2606$. Set $[\xi_3(0), \xi_4(0)] = [0, 0]$. The optimization of the value function $V(\chi(0), \xi(0))$, which gives the optimal cost *paid* by the solution, with respect to $\xi_1(0)$ and $\xi_2(0)$, yields the values $\xi_1^o(0) = 0.3$, $\xi_2^o(0) = -0.9$ and the corresponding value attained by the function is $V(\chi(0), \xi_1^o(0), \xi_2^o(0), 0, 0) = 0.2081$. The top graph of Figure 7.3 shows the time histories of the angular positions of the two joints when the dynamic control law, with $\kappa = 1.2$, and the optimal state feedback u_o are applied, left and right graph, respectively. The time histories of the dynamic control law and the optimal local state feedback are displayed in the bottom graph of Figure 7.3, left and right graph, respectively.

7.3 Combustion Engine Test Bench

Combustion engines are operated at test benches in the same way as in a passenger car or a heavy-duty truck, since the former allow to generate the same load a combustion engine would undergo in normal operations. The crucial advantage in the use of a test bench resides in the possibility of reproducing desired conditions in terms of temperature

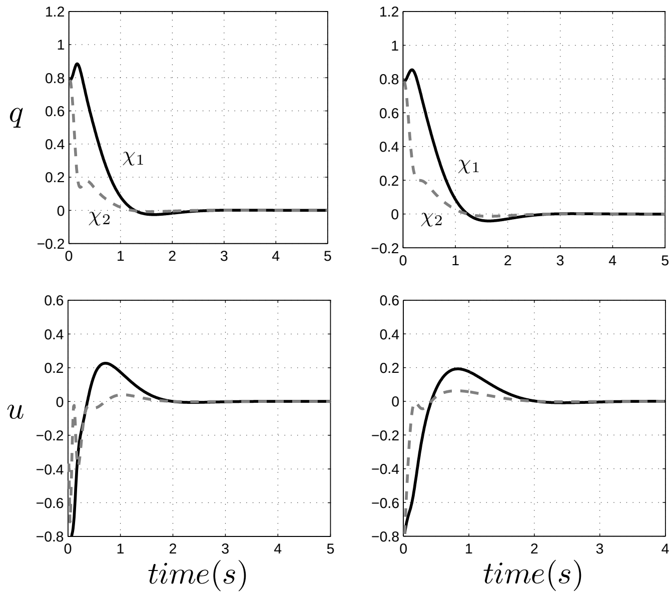

Figure 7.3: Top graphs: time histories of the positions of the two joints when the dynamic control law and the optimal state feedback for the linearized problem are applied, left and right graph, respectively. Bottom graphs: time histories of the dynamic control law and of the optimal state feedback for the linearized problem, left and right graph, respectively.

and pressure, and, consequently, of drastically reducing the cost and time required for development and configuration.

In a vehicle the velocity and, by means of the transmission, the rotational speed ω_E result from the engine and the load torques. For this reason the engine torque T_E as well as the engine speed ω_E need to be controlled to operate a combustion engine on a test bench. In industrial practice a test bench is usually controlled by means of two separate control loops: the torque is often influenced by the accelerator pedal position α of the engine under test, while the speed is controlled by the loading machine. This section deals with the design of a multi-input/multi-output dynamic control law for the torque and the speed of the combustion engine on a test bench.

7.3.1 Simplified Model of the Test Bench

In the standard setting of a test bench, the combustion engine is connected via a single shaft to a different main power unit. The latter may either be a purely passive brake or an electric machine, which offers the possibility of an active operation. In this situation, the accelerator pedal position α of the combustion engine and the set value $T_{D,\,set}$ of the dynamometer torque provide the inputs to the test bench. In the following let T_x denote the measured torque, while ω_x represents the measured rotational speed, where x may be equal to E or D, denoting the combustion engine and the dynamometer, respectively (see [34] and the references therein for more details). To streamline the presentation of the main results only the equations necessary for the control design are summarized. To this end the entire mechanical system – a two-mass oscillator – can be described by the equations

$$\Delta\dot{\varphi} = \omega_E - \omega_D, \tag{7.10}$$

$$\theta_E \dot{\omega}_E = T_E - c\Delta\varphi - d\left(\omega_E - \omega_D\right), \tag{7.11}$$

$$\theta_D \dot{\omega}_D = c\Delta\varphi + d\left(\omega_E - \omega_D\right) - T_D, \tag{7.12}$$

where $\Delta\varphi$ is the torsion of the connection shaft while θ_E and θ_D denotes the inertias of the combustion engine and the dynamometer, respectively. The contributions due to the inertias of the adapter flanges, the damping element, the shaft torque measurement device and the flywheel are already included in these values. The parameter c characterizes the stiffness of the connection shaft, whereas d describes its damping.

The behavior of a combustion engine is complex. However, a *simplified* model of the torque generated by a diesel engine can be described by the equation [34]

$$\dot{T}_E = -\left(c_0 + c_1\omega_E + c_2{\omega_E}^2\right) T_E + m\left(\omega_E, T_E, \alpha\right), \tag{7.13}$$

where $c_i > 0$, $i = 1, \ldots, 3$ are constant parameters and $m : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a nonlinear static function.

The employed electric dynamometer is modeled as a second order low-pass filter, with dynamics significantly faster than those of the other components of the test bench, hence they can be neglected in the design. Within the range of maximum torque and maximum rate of change, the torque of the dynamometer can be described by $T_D = T_{D,\,set}$. Letting $v = m\left(\omega_E, T_E, \alpha\right)$, the system (7.10)-(7.13) can be rewritten as

$$\dot{x} = Ax + f\left(x\right) + Bu \tag{7.14}$$

with

$$A = \begin{pmatrix} -c_0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ \frac{1}{\theta_E} & -\frac{c}{\theta_E} & -\frac{d}{\theta_E} & \frac{d}{\theta_E} \\ 0 & \frac{c}{\theta_D} & \frac{d}{\theta_D} & -\frac{d}{\theta_D} \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{1}{\theta_D} \end{pmatrix}$$

$$f\left(x\right) = \begin{pmatrix} -\left(c_1\, x_3 + c_2\, {x_3}^2\right) x_1 & 0 & 0 & 0 \end{pmatrix}^\top,$$

where $x(t) \in \mathbb{R}^4$, $x = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^\top = \begin{pmatrix} T_E & \Delta\varphi & \omega_E & \omega_D \end{pmatrix}^\top$, denotes the state of the system and $u(t) \in \mathbb{R}^2$, $u = \begin{pmatrix} v & T_{D,\,set} \end{pmatrix}^\top$, the input. The actual control input to apply in order to generate the desired signal v, namely α, is then obtained in practice by an approximate inversion of the nonlinear function m.

7.3.2 MIMO Control Design

To begin with, let e_i denote the regulation error of the component x_i with respect to the corresponding reference value, namely $e_i = x_i - x_i^*$, $i = 1, \dots, 4$. Let $q(e) = e^\top Q e$ in the cost functional (5.40), where the positive definite matrix $Q \in \mathbb{R}^{4\times 4}$ weights the relative regulation errors for the different components of the state. Without loss of generality, let the *algebraic $\bar{P}$ solution* be of the form $P(e) = e^\top \bar{P} + \mathcal{Q}(e)$, where $\mathcal{Q} : \mathbb{R}^4 \to \mathbb{R}^{1\times 4}$ contains higher-order polynomials of the error variable e. The matrix $\bar{P}$ is the symmetric and positive definite solution of the algebraic Riccati equation (5.46) corresponding to the linearized error system and the quadratic cost $q(e)$. Note that, depending on the specific operating point, some of the terms in f may contribute to the linearized system. The proposed structure of the *algebraic $\bar{P}$ solution* P intuitively suggests that the linear solution $e^\top \bar{P}$ is modified, by the addition of the term $\mathcal{Q}$, in order to compensate for the nonlinear terms of the algebraic equation (5.49) associated with the system (7.14).

In the following, exploiting the specific structure of the vector field f in system (7.14), we let $\mathcal{Q}$ be defined as $\mathcal{Q} = (\mathcal{Q}_1, 0, 0, 0)$. It is worth noting that this specific choice is arbitrary and alternative choices can be made. Then, the algebraic Hamilton-Jacobi-Bellman inequality (5.49) is solved with respect to the unknown $\mathcal{Q}_1(e)$ obtaining a solution of the form $\mathcal{Q}_1(e) = N(e)D(e)^{-1}$, with the function D strictly positive for all the values of *interest*, namely the desired operating range, of the state variable x. Once an *algebraic $\bar{P}$ solution* has been obtained we define the dynamic control law

$$\begin{aligned} \dot{\xi} &= -kV_\xi^\top(e,\xi)\,, \\ \nu &= -B^\top V_x^\top(e,\xi)\,, \end{aligned} \tag{7.15}$$

where the function V is defined as in (4.11) with P as given above. The control law (7.15) approximates the solution of the regional dynamic optimal control problem for the system (7.14) with respect to the running cost $q(e) = e^\top Q e$.

The dynamic control law (7.15) is designed considering a *simplified* model of the combustion engine test bench. Therefore the control action needs to be modified, as explained in the following, in order to cope with some of the nonlinearities of the combustion engine not included in the model.

In particular, an integral action is added to the dynamic control law (7.15). More specifically, letting ν_i, $i = 1, 2$, denote the i-th component of the dynamic control law (7.15) we define the actual control inputs implemented on the combustion engine test bench as

$$u_i = \nu_i(e,\xi) + k_i \int_0^t \nu_i(e(\tau),\xi(\tau))d\tau\,, \tag{7.16}$$

for $i = 1, 2$. Additionally we let the gain k_2 be a function of the derivative – which is implemented as $\frac{s}{\delta s+1}$, with $\delta \gg 0$ – of the regulation error for the speed of the engine, namely $k_2(\dot{e}_3)$. In particular, the gain is defined such that, when the regulation error of the engine speed changes *too fast*, the integral action is negligible compared to the other components of the control signal. This choice is reasonable, as shown in the simulations, and needed in order to avoid an excessively aggressive reference profile for the torque of the dynamometer. Large errors between the desired and the measured engine speed and torque, due to changes in the references, are mainly, and relatively rapidly, compensated by the action provided by the dynamic control law (7.15).

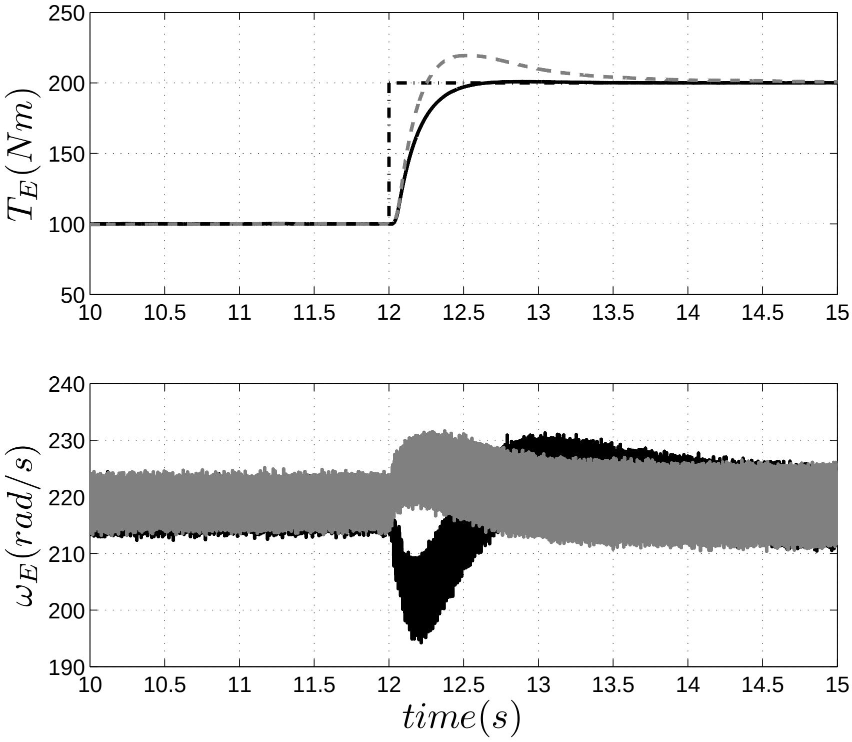

Figure 7.4: Top graph: time histories of the engine torque T_E determined by an accurate model of a combustion engine test bench in closed-loop with the control law (7.15)-(7.16) (solid line) and with the control law in [34] (dashed line), together with the desired reference value (dash-dotted line). Bottom graph: time histories of the engine speed ω_E determined by an accurate model of a combustion engine test bench in closed-loop with the control law (7.15)-(7.16) (dark line) and with the control law in [34] (grey line).

Note that, locally around the origin, the linear part of the dynamic control law is dominant with respect to higher-order terms, hence the integrals provide a *standard* integral action.

Finally the proposed control law (7.15)-(7.16) is tested and validated on a high quality test bench simulator developed by the Institute for Design and Control of Mechatronical

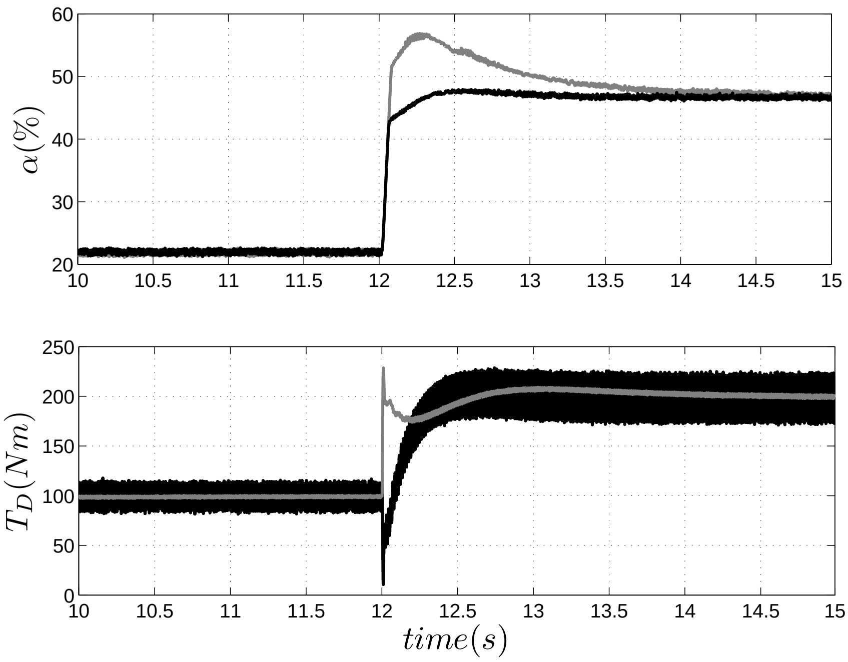

Figure 7.5: Top graph: time histories of the accelerator pedal position α determined by the control law (7.15)-(7.16) (solid line) and by the control law in [34] (dashed line). Bottom graph: time histories of the torque of the dynamometer T_D determined by the control law (7.15)-(7.16) (dark line) and by the control law in [34] (grey line).

Systems, Kepler University, Linz. In addition to the dynamics of the entire mechanical description the simulator also includes a more accurate, genuinely nonlinear, data-based model of the combustion engine, limitations of the dynamometer as well as disturbance effects. The combustion engine model takes for instance the dynamics of the accelerator pedal and combustion oscillations into account. Measurement noise similar to the one observed on an actual combustion engine test bench is superimposed to all relevant values.

In the simulation results the control law (7.15)-(7.16) is compared with the control law developed in [34], which already has a significantly improved performance with respect to existing standard implementations based on two separate control loops. Let the cost $q(e)$ be defined as $e^\top e$. The top graph of Figure 7.4 shows the time histories of the engine torque T_E determined by an accurate model of a combustion engine test bench in closed-loop with the control law (7.15)-(7.16) (solid line) and with the control law in [34] (dashed line). The bottom graph of Figure 7.4 displays the time histories of the engine speed ω_E determined by an accurate model of a combustion engine test bench in closed-loop with the control law (7.15)-(7.16) (solid line) and with the control law in [34] (dashed line). The ratio between the cost yielded by the dynamic control law u and the control law in [34] is equal to 0.984. The use of the controller developed herein leads to a significant reduced overshooting of the engine torque T_E when a change of the operation point is required. Although the engine torque T_E shows a slightly increased rise time, the final value is reached approximately four times faster. The coupling effect is also improved by the dynamic control law proposed, as it can be appreciated at $t = 12s$, when an increase in the engine torque from $100Nm$ to $200Nm$ occurs at constant speed, namely 220 radians per second. Note that the speed is corrupted by additive disturbance caused by the resolution of the shaft encoders and by combustion oscillations.

Significative differences are also visible in the input signals determined by the control laws, see Figure 7.5. In fact at time $t = 12s$, the control law (7.15)-(7.16) avoids an overshoot in the accelerator pedal position α, which is an advantage from the polluting emissions point of view, while requiring, on the other side, a more demanding behavior to the electric dynamometer.

7.4 Diesel Engine Air Path

The principal control objective for an internal combustion engine consists in designing a control action such that the engine itself generates the torque requested by the driver via the acceleration pedal position and minimizes on the other hand the fuel consumption while keeping specific quantities, such as the cumulated emissions or the exhaust temperatures, below a given threshold. Since the aforementioned problem has gained increasing attention in recent years, Diesel engines have evolved to complex systems to cope with these requirements in terms of economy, pollution emissions and driveability. Several degrees of freedom, such as a variable geometry turbocharger (VGT) or an exhaust gas recirculation (EGR) valve, are available to achieve a satisfactory trade-off between conflicting goals.

In internal combustion engines, pollution emissions are mainly generated as a result of imperfect combustion. However, the production is strongly affected by the operating conditions, in particular the rate of injection, the gas composition and the gas distribution. Several traditional approaches are based on a limitation of the injected fuel amount. Smoke maps in the Engine Control Unit (ECU) of a Diesel engine or control based on the air-fuel ratio are possible alternatives. On the other hand, one can try to improve the dynamics of the air path using, for instance, a variable geometry turbocharger or *assisted* turbochargers.

The topic of this section is the design of a multi-input multi-output controller for the air path of a turbocharged Diesel engine such that the the fresh mass airflow MAF and the absolute pressure in the intake manifold MAP are regulated, by means of a variable geometry turbocharger and an exhaust gas recirculation, to desired reference values.

7.4.1 Model of the Air Path

The air path is equipped with a variable geometry turbocharger and a high pressure exhaust gas recirculation. Fresh air from the ambient enters the engine via the air filter, compressor, intercooler and intake manifold. The pressure of the fresh air is increased by the compressor, while the density is raised at (almost) constant pressure by the intercooler. When the EGR valve is open the fresh air in the intake manifold is mixed with a portion of the exhaust gases. These are then recirculated from the exhaust manifold by the EGR cooler and the EGR valve. The gas mixture enters the combustion chambers by the intake ports whereas, after closing the intake ports, the fuel is injected around the Top Dead Center (TDC) of the piston movement. After the combustion the major fraction of the hot exhaust gases leave the internal combustion engine through the turbine and the tailpipe to the ambient. A part of the exhaust energy content is recovered by an expansion process in the turbine.

The vane position X_{VGT} of the turbocharger and the EGR valve opening level X_{EGR} provide the control inputs of the air path. The regulated variables are the fresh mass airflow MAF and the absolute pressure in the intake manifold MAP. Typically, this control loop is decoupled from the control of the fuel injection which affects the injected fuel amount W_f and the engine speed n_{eng}. However, the latter two determine the reference values for MAF and MAP, which are additionally strongly coupled to each other.

The air path system of a turbocharged Diesel engine is approximated by two coupled equations each with five inputs and one output. Each of these equations is a polynomial Nonlinear AutoRegressive model with eXogenous input (NARX) described by equations of the form

$$y\,(k) = \theta_0 + \sum_{m=1}^{l} \sum_{\substack{i_1\ \ldots\ i_m=1 \\ i_1 \leq i_2 \leq \ldots \leq i_m}}^{n} \theta_{i_1\ \ldots\ i_m} \cdot \prod_{p=1}^{m} x_{i_p}\,(k) + \varepsilon\,(k)\,, \tag{7.17}$$

where the output $y\,(k)$ consists of the fresh mass airflow MAF and the absolute pressure in the intake MAP. The vector θ describes the parameters of the polynomial estimated during an identification procedure of the model. The error term $\varepsilon\,(k)$ is assumed to be white noise. The variable $x\,(k)$ is composed by delayed outputs (up to maximal lag of n_y) and the five inputs (up to maximal lag of n_{u_i}, $i = 1, ..., 5$), namely

$$x(k) = [y(k-1), ..., y(k-n_y), u_1(k-1), ..., u_1(k-n_{u_1}), ..., u_5(k-1), ..., u_5(k-n_{u_5})]$$

The values of the parameters can be estimated from input/output data. Measurements are taken on an internal combustion engine test bench varying the input variables X_{VGT}, X_{EGR}, W_f and n_{eng} according to an optimal input design procedure [35]. Since the model is linear in the parameters, the latter can be computed using standard least squares algorithms, namely minimizing the sum of the squared prediction error over N measured samples (see, *e.g.*, [53] for more details)

$$\hat{\theta} = \arg\min_{\theta} \sum_{k=1}^{N} \Big(y\left(k\right) - f^{l}\left(x\left(k\right), \theta\right) \Big)^{2}. \tag{7.18}$$

Letting y_1 and y_2 correspond to the MAP and the MAF respectively, the discrete-time model of the air path of a turbocharged Diesel engine is then described by difference equations of the form

$$\begin{aligned} y_1(k) = & f_1 \triangleq \theta_0 + \theta_1 y_1(k-1) + \theta_2 y_1(k-2) + \theta_3^\top u(k-1) + \theta_4^\top u(k-2) + \theta_5 y_2(k-1) \\ & + \theta_6 y_2(k-2) + \theta_7 y_1(k-1)^2 + \theta_8^\top u(k-1) y_1(k-1) + u(k-1)^\top \Theta_1 u(k-1) \\ & + \theta_9 y_1(k-1) y_2(k-1) + \theta_{10}^\top u(k-1) y_2(k-1) + \theta_{11} y_1(k-2)^2 \\ & + u(k-2)^\top \Theta_2 u(k-2) + \theta_{12}^\top u(k-2) y_1(k-2) + \theta_{13} y_1(k-2) y_2(k-2) \\ & + \theta_{14}^\top u(k-2) y_2(k-2) + \theta_{15} y_2(k-1)^2 + \theta_{16} y_2(k-2)^2 , \end{aligned} \tag{7.19}$$

$$\begin{aligned} y_2(k) = & f_2 \triangleq \eta_1 y_2(k-1) + \eta_2 y_2(k-2) + \eta_3^\top u(k-1) + \eta_4^\top u(k-2) + \eta_5 y_1(k-1) \\ & \eta_0 + \eta_6 y_1(k-2) + \eta_7 y_2(k-1)^2 + \eta_8^\top u(k-1) y_2(k-1) + u(k-1)^\top \Delta_1 u(k-1) \\ & + \eta_9 y_1(k-1) y_2(k-1) + \eta_{10}^\top u(k-1) y_1(k-1) + \eta_{11} y_2(k-2)^2 \\ & + u(k-2)^\top \Delta_2 u(k-2) + \eta_{12}^\top u(k-2) y_2(k-2) + \eta_{13} y_1(k-2) y_2(k-2) \\ & + \eta_{14}^\top u(k-2) y_1(k-2) + \eta_{15} y_1(k-1)^2 + \eta_{16} y_1(k-2)^2 , \end{aligned} \tag{7.20}$$

with $n_y = n_{u_i} = 2$ for all i in the model (7.17), where f_1 and f_2 are polynomials of degree two and $u = [W_f,\, n_{eng},\, X_{EGR},\, X_{VGT}]$. The numerical values of the parameters θ_i, η_i, for $i = 1, ..., 14$ and the matrices Θ_j, Δ_j, with $j = 1, 2$, are determined as described above.

A continuous-time description of the model (7.19)-(7.20) is then obtained. Towards this end, the first step consists in determining an equivalent state-space discrete-time description of the difference equations (7.19)-(7.20). In particular this is obtained introducing the change of coordinates

$$\begin{aligned} x_1(k) &\triangleq y_1(k-2) , \\ x_2(k) &\triangleq y_1(k-1) - \phi_1(k) , \\ x_3(k) &\triangleq y_2(k-2) , \\ x_4(k) &\triangleq y_2(k-2) - \phi_2(k) , \end{aligned}$$

where the functions $\phi_1(k)$ and $\phi_2(k)$ contain all the terms depending on the variable

$u(k-1)$ of f_1 and f_2, respectively, evaluated at the step $k-2$, namely

$$\phi_1(k) \triangleq \theta_3^\top u(k-2) + \theta_8^\top u(k-2)y_1(k-2) + \theta_{10}^\top u(k-2)y_2(k-2) + u(k-2)^\top \Theta_1 u(k-2)\,,$$

$$\phi_2(k) \triangleq \eta_3^\top u(k-2) + \eta_8^\top u(k-2)y_2(k-2) + \eta_{10}^\top u(k-2)y_1(k-2) + u(k-2)^\top \Delta_1 u(k-2)\,.$$

Introducing the notation $x(k) \triangleq [x_1(k),\, x_2(k),\, x_3(k),\, x_4(k)]^\top$ and letting $v = u(k-2)$, the discrete-time state-space model of the air path is described by equations of the form

$$x(k+1) = F(x(k), v)\,, \tag{7.21}$$

where $F : \mathbb{R}^4 \times \mathbb{R}^2 \to \mathbb{R}^4$ is directly obtained from the functions f_1 and f_2 and the proposed change of coordinates.

A continuous-time description can be now derived using an *inverse Euler transformation*, namely

$$\dot{x}(t) = \frac{1}{T}\left(-x(t) + F(x(t), v)\right) \tag{7.22}$$

where T is the sampling time and the vector-field F is defined as in (7.21). Finally note that in the continuous-time model (7.22) the variables MAP and MAF are described by the components x_1 and x_3 of the state, respectively.

7.4.2 Regulation Problem

The continuous-time model (7.22) is modified to fit the class of nonlinear systems considered in Chapter 5, namely input-affine systems. Specifically, we add continuous-time asymptotically stable (sufficiently *fast*) filters, the state of which is denoted by $z(t) \in \mathbb{R}^2$, for the two inputs $v_1 = X_{EGR}$ and $v_2 = X_{VGT}$. Note then that the injected fuel amount W_f and the engine speed n_{eng} can not be directly controlled but are determined by the driver's demand and the vehicle resistance. Consequently W_f and n_{eng} are assumed to be constant known parameters of the model (7.22), namely $p_1 = W_f$ and $p_2 = n_{eng}$. Letting $v = [v_1,\, v_2]^\top$, the resulting system is then defined as

$$\begin{aligned} \dot{x} &= \frac{1}{T}\left(-x + F(x, p_1, p_2, z)\right) \\ \dot{z} &= Az + Hv \end{aligned} \tag{7.23}$$

where $A \in \mathbb{R}^{2\times 2}$ is a Hurwitz matrix and $H \in \mathbb{R}^{2\times 2}$ is a diagonal non-singular matrix. Note that the system (7.23) has the same structure as the systems dealt with in Section 5. Consider now the regulation problem.

Problem 7.1. Let $x_1^* \in \mathbb{R}$ and $x_3^* \in \mathbb{R}$. The *air path regulation problem* consists in determining a control law

$$\begin{aligned} \dot{\xi} &= \alpha(x, \xi, x_1^*, x_3^*)\,, \\ u &= \beta(x, \xi, x_1^*, x_3^*)\,, \end{aligned} \tag{7.24}$$

such that all the trajectories of the closed-loop system (7.23)-(7.24) are such that $x(t) \in \mathcal{L}_\infty$, $z(t) \in \mathcal{L}_\infty$,

$$\lim_{t\to\infty} \|x_1(t) - x_1^*\| = 0\,, \quad \lim_{t\to\infty} \|x_3(t) - x_3^*\| = 0\,,$$

and moreover the cost functional

$$J(u) = \frac{1}{2}\int_0^\infty (q(x(t) - x^*) + (v - z^*)^\top (v - z^*))dt\,, \tag{7.25}$$

is minimized along the trajectories of the closed-loop system.

The following result provides a control law that approximates, in the sense specified in Chapter 5, the solution of Problem 7.1. To streamline the presentation of the result, define the *error variables* $e_x = [e_1,\, e_2,\, e_3,\, e_4]^\top$ with $e_1 = x_1 - x_1^*$, $e_2 = x_2 - x_2^*$, $e_3 = x_3 - x_3^*$, $e_4 = x_4 - x_4^*$ and $e_z = [e_5,\, e_6]^T$ with $e_5 = z_1 - z_1^*$ and $e_6 = z_2 - z_2^*$.

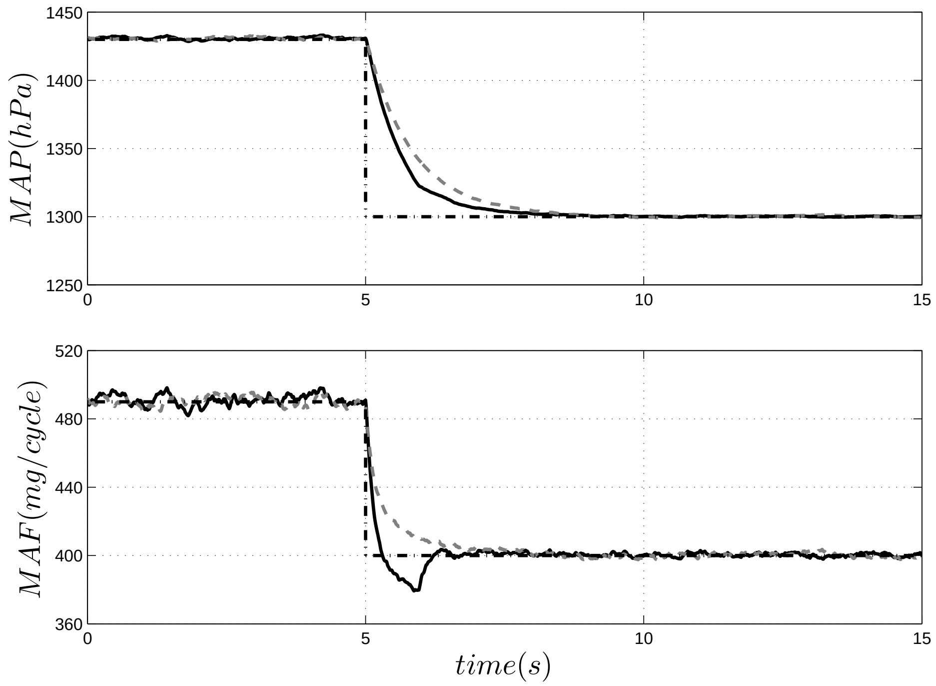

Figure 7.6: Top graph: time histories of the absolute pressure in the intake manifold MAP determined by the dynamic control law (7.26) (solid line) and the linear control law u_l (dashed line), together with the desired reference value (dash-dotted line). Bottom graph: time histories of the fresh mass airflow MAF determined by the dynamic control law (7.26) (solid line) and the linear control law u_l (dashed line), together with the desired reference value (dash-dotted line).

Proposition 7.2. Consider the nonlinear system (7.23) and the cost functional (7.25). Let x_2^*, x_4^*, z_1^* and z_2^* be such that[1]

$$x^* = F(x_1^*, x_2, x_3^*, x_4, p_1, p_2, z_1, z_2)\,.$$

Let $P : \mathbb{R}^6 \to \mathbb{R}^{1\times 6}$ be an *algebraic* $\bar{P}$ *solution* of the inequality (5.49) with $\Sigma(0) > 0$ and with the mappings f and g defined as $f(e_x + x^*, e_z + z^*)$ and $B \triangleq [0_{2\times 4}, H]^\top$, respectively.

Then there exist a non-empty set $\Omega \subseteq \mathbb{R}^6 \times \mathbb{R}^6$, containing the origin, $\bar{k} \geq 0$ and $R = R^\top > 0$ such that

$$\begin{aligned} \dot{\xi} &= -k\left(\Psi(\xi)^\top e - R(e-\xi)\right)\,, \\ v &= v^* - B^\top P(\xi) - B^\top R(e-\xi)\,, \end{aligned} \tag{7.26}$$

with $e = [e_x^\top, e_z^\top]^\top$ and $v^* = -H^{-1}Az^*$, approximates the solution of Problem 7.1, for all $k > \bar{k}$ and for all $(e, \xi) \in \Omega$. $\diamond$

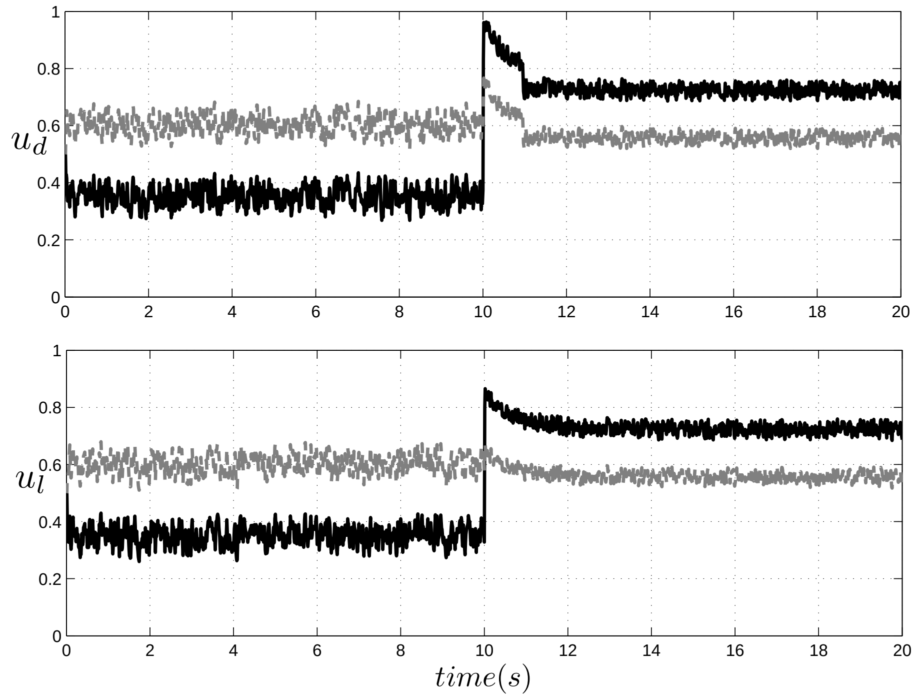

Figure 7.7: Time histories of the *EGR* valve position (black line) and the *VGT* actuator signal (gray line) determined by the dynamic control law (7.26) and the linear control law u_l, top and bottom graph, respectively.

Remark 7.1. Since no constraint, *e.g.* integrability or positivity, is imposed on the algebraic solution, a closed-form expression is determined for the latter, as a function of

[1] This is a system of four quadratic equations in four unknowns.

the data of the problem, namely $P(x, z, f, B, q, k)$. Therefore, to implement the dynamic control law (7.26), at each step of integration of the ordinary differential equations (7.23)-(7.26), given the current value of the state ($\hat{x}$, $\hat{z}$), it is enough to compute the vector $\hat{P} = P(\hat{x}, \hat{z}, f, g, q, k)$, which is such that

$$\hat{P}f(\hat{x}, \hat{z}) + \frac{1}{2}q(\hat{x}, \hat{z}) - \frac{1}{2}\hat{P}BB^\top\hat{P}^\top + \sigma(\hat{x}, \hat{z}) = 0\,.$$

▲

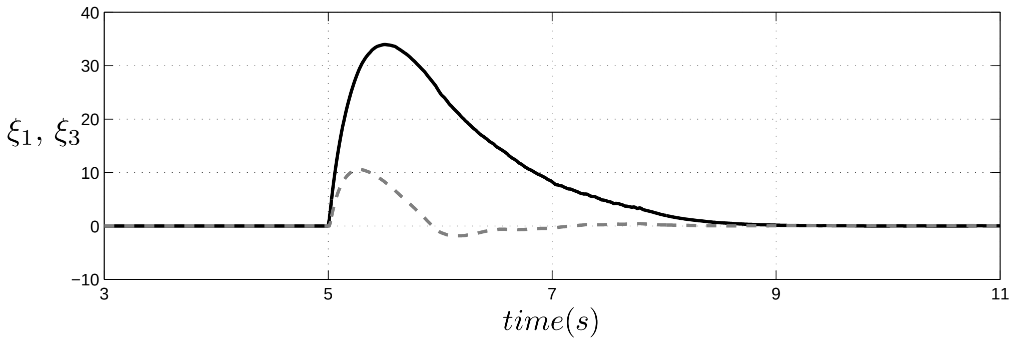

Figure 7.8: Time histories of the dynamic extension $\xi_1(t)$ (solid line) and $\xi_3(t)$ (dashed line) determined as the solution of the first equation in (7.26).

The performances of the dynamic control law (7.26) are tested in simulation and compared with the optimal solution of the linearized problem $u_l = -B^\top\bar{P}e$ where $\bar{P}$ is the solution of the corresponding algebraic Riccati equation. The running cost q is selected as $q(e) = 20(x_1 - x_1^*)^2 + (x_2 - x_2^*)^2 + 20(x_3 - x_3^*)^2 + (x_4 - x_4^*)^2$. In the considered setting the injected fuel amount W_f and the the engine speed n_{eng} are kept constantly equal to $20mg/inj$ and $2000rpm$, respectively, whereas the EGR valve position and the VGT actuator signal are determined in closed-loop according to the dynamic control law (7.26). In the present setting perfect knowledge of the state is assumed. Moreover the actuators are corrupted by additive white noise with an amplitude of 10% with respect to the admissible range of values. The two graphs of Figure 7.6 show the time histories of the absolute pressure in the intake manifold MAP and the fresh mass airflow MAF, top and bottom graph respectively, in closed-loop with the control (7.26) (solid lines) and in closed-loop with the linear feedback u_l (dashed lines) together with the reference signals (dash-dotted lines). At time $t = 5s$ the desired reference value for the MAP is decreased from $1430hPa$ to $1300hPa$ while the reference value for the MAF is decreased from $490mg/cycle$ to $400mg/cycle$ and an increase in the EGR valve position can be observed. The ratio between the cost yielded by the dynamic control law u_d and by the linear control law u_l, namely $J(u_d)/J(u_l)$ is equal to 0.854. Figure 7.7 displays the time histories of the EGR valve position (black line) and the VGT actuator signal (gray line), *i.e.* the components of the control input v, determined by the dynamic control law (7.26) and the linear control law u_l, top and bottom graph, respectively. Finally, the evolutions of the components of the dynamic extension $\xi(t) \in \mathbb{R}^6$, $\xi(0) = 0$, corresponding to the MAP and the MAF are depicted in Figure 7.8.

7.5 Conclusions

In this chapter we have presented several examples to validate the performances of the proposed dynamic control law in practical applications. These examples allow us to point out advantages and drawbacks of the approach and to demonstrate the applicability of the latter under realistic and practice-motivated assumptions, including the optimal and robust control of fully actuated mechanical systems in Section 7.2 to optimal control problems in internal combustion engines discussed in Sections 7.3 and 7.4.

Chapter 8

Conclusions

8.1 Main Contribution

Several analysis and design control problems for linear and nonlinear systems have been addressed introducing the notion of *dynamic function*. These functions are defined as pairs consisting of a dynamical system combined with a positive definite function taking values in a subset of the extended state-space of the original plant and the dynamic extension. Such a dynamic function implicitly contains a *time-varying* part the behavior of which is defined in terms of the solution of an ordinary differential equation driven by the state of the system. In particular, the evolution of the dynamic extension is autonomously adjusted, along the trajectories of the plant, to ensure that a (modified) version of the partial differential equation arising in the nonlinear control problem under examination is satisfied. The main idea is then specialized to various problems in nonlinear control theory such as stability and input/output analysis, robust and optimal control and observer design.

The key aspect of novelty lies in the definition of an *algebraic solution* of the underlying partial differential equation. More precisely, algebraic solutions allow to avoid two structural and restrictive assumptions of the classical approaches proposed in the literature to tackle the aforementioned control problems, namely integrability and (partly) positivity of the solutions. Moreover, these solutions are instrumental in defining classes of dynamic functions. The latter then yield alternative solutions to challenging nonlinear control problems, including the construction of Lyapunov, Controllability and Observability functions, the design of dynamic control laws to approximate the solution of robust and optimal control problems and the observer design problem formalized within the framework of the Immersion and Invariance technique.

It must be noted that, exploiting the knowledge of algebraic solutions, dynamic functions may be constructed without requiring the knowledge of the solution of the underlying ordinary differential equation or of the solution of any partial differential equation or inequality, thus providing constructive solutions to the control problems mentioned above.

The potential of the proposed methodology in solving various nonlinear control problems has been highlighted by means of several applications and examples, encompassing Lyapunov functions construction for power systems, observer design to estimate the range of objects observed through a single camera as well as the optimal and robust control of mechanical systems and combustion engine test benches.

8.2 Future Research Directions

This section aims at suggesting possible developments of the methodology proposed in the thesis. Two main directions may be identified, which may be pursued in parallel: advances of the theory and advances of the design techniques. The objective of the former is to extend the approach to different control problems the solution of which is defined in terms of partial differential equations, such as output regulation and feedback linearization. On the other hand, as far as the latter is concerned, the focus lies in the development of a set of computational tools which allow to construct solutions to practical problems, similarly to the applications of the previous chapter, without requiring restrictive assumptions. This set must necessarily contain techniques to systematically obtain algebraic solutions for classes of nonlinear systems, thus extending the results of Section 4.5.

Once these guidelines have been defined, extensions of the results in each chapter may be discussed. In particular, it may prove interesting to define Dynamic Lyapunov functions for time-varying nonlinear systems as well as systems described by stochastic differential equations. Moreover, the limitation of the notions introduced in Chapters 2 and 3, namely Dynamic Lyapunov functions and Dynamic Generalized Controllability and Observability functions, of being merely analysis tools may be removed. This would lead, on one hand, to the definition of Dynamic Control Lyapunov functions and to the design of observers and control law with specific properties with respect to the input/output energy point of view, on the other hand.

Finally, it appears challenging to approach additional problems that can be framed within the Dynamic Programming methodology, hence exploiting Dynamic Value functions to solve minimum time problems or optimal control in the presence of bounded inputs.

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