

HEEGAARD FLOER HOMOLOGY AND LINK COBORDISMS

A THESIS PRESENTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY OF IMPERIAL COLLEGE LONDON

AND THE
DIPLOMA OF IMPERIAL COLLEGE

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MARCH 2017

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Heegaard Floer homology and link cobordisms

ABSTRACT

The purpose of this thesis is to study link cobordisms. The main tool we use to do so is given by Juhász's cobordism maps, although we sometimes make use of a recent construction by Zemke. Given a decorated link cobordism $\mathcal{X}$, that is, a link cobordism with additional structure, there is a map $F_{\mathcal{X}}$ induced at the level of link Floer homology ($\widehat{\text{HFL}}$, also denoted $\widehat{\text{HFK}}$ for knots).

When $\mathcal{X}$ is a decorated *concordance* from K_0 to K_1 , we prove that $F_{\mathcal{X}}$ preserves the natural bigrading of $\widehat{\text{HFK}}$, and that there is a morphism of spectral sequences from $\widehat{\text{HFK}}(K_i)$ to $\widehat{\text{HF}}(S^3)$ which agrees with $F_{\mathcal{X}}$ on the E^1 page, and with $\text{id}_{\widehat{\text{HF}}(S^3)}$ on the E^∞ page. We use this result to obstruct the existence of invertible concordances between given knots, and to define a non-vanishing element of $\widehat{\text{HFK}}(K)$ associated to a slice disc for the knot K .

We then give a full description of the maps induced by elementary decorated link cobordisms, which generate all decorated link cobordisms. In particular, we relate the map associated to a saddle cobordism to a skein exact triangle which generalises Ozsváth and Szabó's oriented skein exact triangle in $\widehat{\text{HFL}}$ to decorated links.

Lastly, we use this description to prove that the TQFT defined by $\widehat{\text{HFL}}$ on a category of unlinks and cobordisms between them agrees with Khovanov's reduced TQFT.

ACKNOWLEDGEMENTS

Ανδράς μοι ἔννεπε, Μοῦσα.* I am mostly indebted to my supervisor **András Juhász**, without whom this thesis would not exist. His patience and his continuous interest in my Ph.D. has been much more than usually expected from an academic mentor.

I am also very grateful for the financial help I have received during my Ph.D., in particular from the EPSRC doctoral training award, and from the Doris Chen mobility award.

A special acknowledgement is reserved for **Adam Levine**, who kindly offered to be my host during my visiting period at Princeton University, and with whom I had several interesting mathematical discussions, and to **Paolo Lisca**, who has always been interested in my Ph.D.—even after I left Pisa.

I am really grateful to the numerous people who devoted some of their time to discussing mathematics with me. In particular, I would like to thank those who, with comments or helpful conversations, contributed to this thesis: **John Baldwin, Michel Boileau, Alessio Corti, Matt Hedden, Tom Hockenhull, Jen Hom, Andrew Lobb, Ciprian Manolescu, Tom Mrowka, Peter Ozsváth, Jake Rasmussen, Daniel Ruberman, and Ian Zemke**. I would also like to thank the two anonymous referees for my joint papers with Juhász^{33,32}, whose suggestions and corrections were invaluable.

Throughout my Ph.D. years it has been an honour to share my office space (in London and/or in Princeton) with such great companions as **Alberto, Andrew, Fëdor, Irena, James, Lisa, Michael, Nurgissa, Senja, and So**. A special mention is reserved for **Dan**, for the great moments we have spent

*Sing to me, o Muse, of András.

together in the last two years[†], and for **Tom**, with whom I have shared countless adventures in these years. I am also very grateful to all the fellow Ph.D. students I met at Imperial College, in particular **Andrea, Chiara, Enrica, Fabio, Francesca, Luca, Massimiliano, Matteo, Mirko, Nick, Radu, Susana**, and **Zak**.

The Imperial bureaucracy would have probably prevented me from graduating if it was not for **Anderson**, a real load-bearing structure in the department: you're irreplaceable!³⁹ I am also grateful to **Nick Jones** for his help and support at the end of my course.

The next group I would like to thank is the one of conference buddies from Pisa (broadly speaking), whom I have had the opportunity to know better in the last years: **Agnese, Antonio, Daniele, Francesco, Marco, Paolo II**, and **Paolo III**.

The memory of Pisa leads me to thank friends whom I have not seen in a while, but whom I have not forgotten: **ABS, Ale, Alessandra**, my mum **Davide, Enrico**, my concrete brother **Fabrizio, Gena, Gio, Giove, Giulio, Ila, Irene, Maddi, Maria, Martina, Michele, Peppe, Pietro, Sara, Sasha**, and **Stefano**.

I would also like to recall my friends from my visiting period in Princeton: **Brad, Elisabetta, Federica, Giacomo, Jan**, and **Margarita**.

From my friends in London, firstly I would like to thank all my flatmates over the time: **Ale, Capra, Edoardo, Geppetto, Gio, Giovanni, Grisu, Simone, Stefano, Tommy**, and especially my brother **William**. I am very grateful to **Angelica, Christoph, Coma, Fe, Gabri, Jack, Lorenz, Maria, Paola, Robberta[‡], Scionti, Teresa**, and **Teresa** for their friendship. Warm thanks go to **Cristina**, who cooked dinner for me the night before the submission deadline.[§] Moreover, I thank a lot the **Govettos**, the **Rossis**, the **Sinisis**, the **Solomons**, and the **Youngs**. A special acknowledgement goes to **Gema**, and to her wonderful family, who, even if for just few months,

[†]A thesis is forever. A whiteboard too.

[‡]This is not a typo.

[§]The careful reader will notice that this implies that I am still writing on the night before the submission deadline.

made my time at Imperial unexpectedly enjoyable!

A separate acknowledgement, with an attached apology for disappearing in the last weeks, goes to the people involved in the scientific exhibition for the London Encounter, and especially to **Chiara, Giovanni, Lucia, Maria, Paolo, and Sara**.

I would also like to mention my cousins **Gianna, Ilaria, Letizia, Luca, and Matteo**, who are very special friends to me.

Lastly, I wish to thank those people who mostly supported me in the course of my Ph.D., namely my family, to whom this thesis is dedicated. A very warm acknowledgement goes to mamma **Silvia**, papà **Giuseppe**, my siblings **Cristina** and **Matteo**, and **Marco**, who will soon be part of the family too.

Un ringraziamento e un saluto particolare va infine ai miei nonni **Anna-maria, Luigi, Anna Afra e Bruno**, per il calore con cui mi accolgono ogni volta che torno a casa.

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To my family

*Allor mi dolsi, e ora mi ridoglio
quando drizzo la mente a ciò ch'io vidi,
e più lo 'ngegno affreno ch'i' non soglio,
perché non corra che virtù nol guidi;
sì che, se stella bona o miglior cosa
m'ha dato 'l ben, ch'io stessi nol m'invidi.*

D. Alighieri

1

INTRODUCTION

TOPOLOGY is usually divided into low-dimensional topology, which is the study of manifolds of dimension less than or equal to 4, and high-dimensional topology, which focuses on manifolds of dimension greater than or equal to 5. The historical reason of this subdivision should be sought in Smale's h -cobordism theorem⁷⁶. A key step in the proof of Smale's theorem is a technique to replace immersed discs with embedded discs, known as the *Whitney trick*, which (in the smooth category) holds true in every dimension greater than or equal to 5, where, loosely speaking, there is enough room to make discs embedded. As a consequence, the topology of smooth manifolds in higher dimensions (≥ 5) can be controlled. In low dimension, manifolds of dimensions 1 and 2 are easy to understand, and are fully classified. There has been a great advance in the comprehension of 3-manifolds in the last years, especially due to Perelman's proof of the Poincaré conjecture⁶⁵. In the same years, the development of Heegaard Floer homology⁶⁰ has been another important step towards the understanding of 3-manifolds. As for the last dimension, namely 4, the topology of 4-manifolds remains quite mysterious, and presents a surprising dichotomy between the topological category and the

smooth category. On one hand, by the work of Freedman¹⁴, we know that the homeomorphism type of a simply-connected 4-manifold is completely determined by algebraic invariants; on the other hand, Donaldson in a seminal work¹² exhibited examples of non-diffeomorphic smooth 4-manifolds which are homeomorphic. As a further remark about the lack of our understanding of 4-manifold topology, note that the 4-dimensional smooth Poincaré conjecture is the last unsettled case of the generalised Poincaré conjecture (by contrast we know that the corresponding *topological* version holds true, thanks to the work of Freedman). Moreover, if the smooth 4-dimensional Poincaré conjecture is false, and an exotic 4-sphere exists, then the latter must not be a *twisted sphere*, that is a closed manifold admitting a Morse function with two critical points. Note that, for all other dimensions, all exotic spheres are twisted spheres.

Knot theory is closely tied with the topology of 3- and 4-manifolds. A theorem by Lickorish⁴³ and Wallace⁸¹ states that every closed oriented 3-manifold is obtained from S^3 by a sequence of Dehn surgeries, and associated to each integral Dehn surgery there is a 4-dimensional cobordism, called the *trace* of the surgery. A very interesting connection between knot theory and the topology of smooth 4-manifolds is given by the study of smooth surfaces in a given 4-manifold X which are bounded by a given knot in ∂X . If X is the standard 4-ball B^4 , then this is a well-studied and absolutely non-trivial subject: remarkably, the knots in S^3 that bound the easiest possible surface, i.e. a disc, in B^4 , form a non-trivial subset of the set of knots. As we shall see, this implies that smooth surfaces could in principle be used to detect non-standard smooth structures on 4-manifolds.

The topic of link cobordisms lies in between dimensions 3 and 4. One can exploit the most popular techniques for 3-manifold topology, such as Heegaard Floer homology, and at the same time have an insight into the complicated world of embedded surfaces in 4-manifolds. The preeminent tool that we will utilise in the present thesis is the map that a link cobordism induces in Heegaard Floer homology, as defined by Juhász³¹. In the next paragraph I will give a very quick overview of Heegaard Floer homology and cobordism maps, before stating the main results of this work.

1.1 AN OVERVIEW ON HEEGAARD FLOER HOMOLOGY

Heegaard Floer theory is a package of invariants for 3-manifolds, knots and links in 3-manifolds, and, more generally, sutured manifolds. Moreover, there are maps associated to cobordisms between such objects, and an invariant for closed 4-manifolds. Since its conception by Ozsváth and Szabó⁶⁰ in the early 2000s, it has proved very effective in establishing results in low-dimensional topology, due to its power and the fact that it is often a computable invariant.

The seminal paper by Ozsváth and Szabó⁶⁰ introduced four invariants of any closed connected oriented 3-manifold Y , denoted by $\mathrm{HF}^\circ(Y)$, where $\circ$ is a symbol among ∞ , $-$, $+$, and $\wedge$. These four symbols are called the *flavours* of *Heegaard Floer homology*. Ozsváth and Szabó⁶² also defined maps F_W° associated to a cobordism W between two 3-manifolds. The invariants are natural by the work of Juhász and Thurston³⁴, and the naturality of the cobordism maps was achieved by Juhász³¹ and Zemke⁸⁴. These invariants proved very powerful: for instance, Ozsváth and Szabó⁵⁵ used them to give an alternate proof of Donaldson's diagonalisability theorem¹². Moreover, they are algorithmically computable^{73,50}.

Concerning the structure of these invariants, $\mathrm{HF}^\infty(Y)$ is a module over $\mathbb{F}_2[U, U^{-1}]$, $\mathrm{HF}^\pm(Y)$ is a module over $\mathbb{F}_2[U]$, and $\widehat{\mathrm{HF}}(Y)$ is a vector space over $\mathbb{F}_2$. Actually, these invariants are also defined over $\mathbb{Z}$, but, to the best of the author's knowledge, this construction with $\mathbb{Z}$ coefficients does not extend to cobordism maps at the moment.

Ozsváth and Szabó⁵⁸ and, independently, Rasmussen⁶⁶, defined invariants for nullhomologous knots in closed oriented 3-manifolds, called *knot Floer homology*, and denoted by HFK° . These were then generalised to invariants HFL° for links in 3-manifolds by Ozsváth and Szabó⁶³, called *link Floer homology*. These invariants contain a lot of information about the knots or links: for instance, $\widehat{\mathrm{HFK}}$ of a knot K in S^3 determines the Alexander polynomial of K ^{58,66}, its genus⁵⁷, and whether K is fibred^{17,28,29,52,53}. Moreover, they are algorithmically computable invariants^{48,49}. Juhász³¹ realised that, in order to make $\widehat{\mathrm{HFL}}$ natural and functorial, one needs

to consider *decorated links* rather than links, whose definition we recall below. Sarkar showed that decorations are necessary for functoriality⁷². Zemke achieved naturality and functoriality of HFL° for all flavours. The fact that these invariants are natural follows from the work of Juhász and Thurston³⁴.

Definition 2.16 (Juhász³¹). A *decorated link* is given by a triple (Y, L, P) , where L is a (not necessarily oriented) non-empty link in the connected oriented 3-manifold Y , and $P \subset L$ is a finite set of points. We require that for every component L_0 of L , the number $|L_0 \cap P|$ is positive and even. Furthermore, we are given a decomposition of L into compact 1-manifolds $R_+(P)$ and $R_-(P)$ such that $R_+(P) \cap R_-(P) = P$.

We say that a decorated link is a *decorated knot* if L is a knot and P consists of 2 points. Moreover, we drop the manifold Y from the notation whenever $Y = S^3$. Juhász³¹ and Zemke⁸⁵ defined maps associated to decorated link cobordisms, whose definition we now recall.

Definition 2.16 (Juhász³¹). The triple $\mathcal{X} = (X, F, \sigma)$ is a *decorated link cobordism* from (Y_0, L_0, P_0) to (Y_1, L_1, P_1) if

1. (X, F) is a link cobordism from (Y_0, L_0) to (Y_1, L_1) ;
2. (F, σ) is a compact orientable surface F , possibly with boundary, together with a properly embedded 1-manifold σ that divides F into two compact subsurfaces that meet along σ , such that the map

$$\pi_0(\partial\sigma) \rightarrow \pi_0((L_0 \setminus P_0) \cup (L_1 \setminus P_1))$$

is a bijection;

3. we can orient each component R of $F \setminus \sigma$ such that whenever $\partial\bar{R}$ crosses a point of P_0 , it goes from $R_+(P_0)$ to $R_-(P_0)$, and whenever it crosses a point of P_1 , it goes from $R_-(P_1)$ to $R_+(P_1)$; and
4. if F_0 is a closed component of F , then $\sigma \cap F_0 \neq \emptyset$.

If $X = S^3 \times I$, we drop it from the notation.

To every decorated link cobordism $\mathcal{X}$ between decorated links $\mathcal{L}_0$ and $\mathcal{L}_1$ Juhász³¹ associated a map

$$F_{\mathcal{X}}: \widehat{\text{HFL}}(\mathcal{L}_0) \rightarrow \widehat{\text{HFL}}(\mathcal{L}_1).$$

More recently, Zemke⁸⁵ constructed maps

$$F_{\mathcal{X}}^{\circ}: \text{HFL}^{\circ}(\mathcal{L}_0) \rightarrow \text{HFL}^{\circ}(\mathcal{L}_1).$$

Note that at the moment it is unknown whether the maps $F_{\mathcal{X}}$ and $\widehat{F}_{\mathcal{X}}$ coincide. It is also worth mentioning that there are other closely related constructions. Alishahi and Eftekhary¹ defined maps in a version of Heegaard Floer homology for tangles, and Sarkar⁷¹ defined a map in the combinatorial version of link Floer homology (although it is unclear if his map is invariant under isotopy of the surface F).

Heegaard Floer homology and link Floer homology come with gradings. First, $\text{HF}^{\circ}(Y)$ splits along the spin^c structures on Y :

$$\text{HF}^{\circ}(Y) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} \text{HF}^{\circ}(Y, \mathfrak{s}).$$

Recall that $\text{Spin}^c(Y)$ is an affine space over $H^2(Y; \mathbb{Z})$. For HFL° there is a similar splitting along *relative* spin^c structures. If L is nullhomologous and oriented, and S is a Seifert surface for L , one can define a *collapsed Alexander grading* $\mathcal{A}_S$, which attains values in $\frac{1}{2}\mathbb{Z}$, associated to a relative spin^c structure. When Y is a rational homology sphere, then all Seifert surfaces for L are rationally homologous, the Alexander grading does not depend on S , and therefore we can simply denote it by $\mathcal{A}$.

Each Heegaard Floer homology group $\text{HF}^{\circ}(Y, \mathfrak{s})$ also carries a homological grading, called the *Maslov grading*. In the most general case, this is a relative $\mathbb{Z}/(\mathfrak{d}(\mathfrak{s}))$ grading, where $\mathfrak{d}(\mathfrak{s})$ is the divisibility of the spin^c structure $\mathfrak{s}$. If $\mathfrak{s}$ is a *torsion* spin^c structure on Y (i.e., its first Chern class is torsion), then the Maslov grading on $\text{HF}^{\circ}(Y, \mathfrak{s})$ can be lifted to a relative $\mathbb{Z}$ grading

and absolute $\mathbb{Q}$ grading. $\mathrm{HFL}^\circ(Y, L, P)$ also carries a relative $\mathbb{Z}$ grading and absolute $\mathbb{Q}$ grading, when $Y = S^3$ and L is oriented.

1.2 OVERVIEW OF THE MAIN RESULTS

The main goal of this thesis is to improve our understanding of the maps induced by decorated link cobordisms and discuss some applications. We will focus mainly on Juhász's maps.

1.2.1 DECORATED CONCORDANCES

Decorated concordances are the simplest kind of decorated link cobordism. Given decorated knots (K_0, P_0) and (K_1, P_1) in S^3 , we say that a decorated link cobordism $\mathcal{C} = (X, F, \sigma)$ between them is a *decorated concordance* if X is a 4-manifold with boundary $S^3 \sqcup -S^3$ and with the same integral homology as $S^3 \times I$, F is an annulus, and σ consists of two parallel arcs on F joining the two different components of ∂F . If two knots K_0 and K_1 are classically concordant (i.e., they co-bound an embedded annulus in $S^3 \times I$), then the decorated knots (K_0, P_0) and (K_1, P_1) are decoratedly concordant.

What of the structure of HFK° is preserved under concordances is a question not completely understood. On one hand, there are several concordance invariants arising from Heegaard Floer homology, such as the τ invariant⁵⁶, the correction terms V_i ⁶⁷, the ε invariant²¹, and the Υ invariant⁵⁴. On the other hand, a natural conjecture about the structure of $\widehat{\mathrm{HFK}}$ of slice knots turns out to be false. Recall that if a knot K is slice, then its (symmetrised) Alexander polynomial factors as

$$\Delta_K(t) = f(t)f(t^{-1})$$

for some $f \in \mathbb{Z}[t, t^{-1}]$ (this is known as the *Fox-Milnor condition*). Ozsváth and Szabó⁵⁸ and, independently, Rasmussen⁶⁶ proved that knot Floer homology is a categorification of the Alexander polynomial, in the sense that,

if $\widehat{\text{HFK}}_i(K, j)$ denotes the summand of $\widehat{\text{HFK}}(K)$ in Maslov grading i and Alexander grading j , then

$$\Delta_K(t) = \sum_{i,j \in \mathbb{Z}} (-1)^{i+j} t^j \dim(\widehat{\text{HFK}}_i(K, j)).$$

In light of this, it is natural to ask whether $\widehat{\text{HFK}}(K)$ also factors in a Fox-Milnor fashion as the tensor product of a vector space V and its dual V^* , with gradings on V^* defined by $(V^*)_{i,j} = (V_{-i,-j})^*$. However, as pointed out to me by Lisca and Ozsváth, this is not the case, because the Kinoshita-Terasaka knot KT is slice, and $\dim(\widehat{\text{HFK}}(KT)) = 33$, which is not a perfect square in $\mathbb{Z}$. Note that KT is also ribbon, and a symmetric union, and therefore it is unlikely that a Fox-Milnor condition for $\widehat{\text{HFK}}$ holds for a meaningful subset of slice knots. Therefore, what of the structure of HFK° is preserved under concordance is not completely understood yet. In this thesis, we give some results in this direction.

Ozsváth and Szabó⁵⁸ and, independently, Rasmussen⁶⁶ proved that, given a decorated knot (K, P) , there is a spectral sequence

$$\widehat{\text{HFK}}(K, P) \Rightarrow \mathbb{F}_2,$$

where $\mathbb{F}_2$ should be thought of as $\widehat{\text{HF}}$ of the ambient manifold S^3 . In joint work with Juhász, we prove the following theorem.

Theorem 4.2 (Juhász-M.³³, Theorem 1.2). *Let $\mathcal{C} = (X, F, \sigma)$ be a decorated concordance between decorated knots (K_0, P_0) and (K_1, P_1) , such that $H_1(X) = H_2(X) = 0$. Then*

$$F_{\mathcal{C}} \left(\widehat{\text{HFK}}_j(K_0, P_0, i) \right) \leq \widehat{\text{HFK}}_j(K_1, P_1, i)$$

for every $i, j \in \mathbb{Z}$.

Furthermore, given an admissible diagram $(\Sigma_r, \alpha_r, \beta_r, w_r, z_r)$ of (K_r, P_r) for $r \in \{0, 1\}$, there is a filtered chain map

$$f_{\mathcal{C}}: \widehat{\text{CF}}(\Sigma_0, \alpha_0, \beta_0, w_0) \rightarrow \widehat{\text{CF}}(\Sigma_1, \alpha_1, \beta_1, w_1)$$

of homological degree zero such that the induced morphism of spectral sequences agrees with F_C on the E^1 page and with $\text{id}_{\mathbb{F}_2}$ on the total homology and on the E^∞ page.

As a first application of Theorem 4.2, we give a new, conceptual proof that the Ozsváth-Szabó τ invariant is a (decorated) concordance invariant (see Corollary 4.20).

Another immediate application of Theorem 4.2 is that F_C is always non-vanishing (see Corollary 4.46). This follows from the fact that F_C is the E^1 map in a morphism of spectral sequences where the E^∞ map is also non-vanishing (because it is $\text{id}_{\mathbb{F}_2}$).

More generally, for $j \in \{0, 1\}$ let $A_j(K, P)$ denote the subset of $\widehat{\text{HFK}}(K, P)$ of elements that represent a cycle in each page of the spectral sequence, and which limit to $j \in \mathbb{F}_2$ in the E^∞ page (see Definition 4.45). Then $A_j(K, P)$ is a knot invariant, it is non-empty, and

$$F_C(A_j(K_0, P_0)) \subset A_j(K_1, P_1).$$

Suppose that B is an integer homology 4-ball, and that $S \subset B$ is a slice disc for the decorated knot (K, P) in S^3 . If we remove a ball from B about a point of S , we obtain a concordance $\mathcal{C}(S)$ from the unknot U to K , on which we put arbitrary decorations. We define the element

$$t_{S,P} := F_{\mathcal{C}(S)}(1) \in \widehat{\text{HFK}}_0(K, P, 0).$$

Then, as a corollary of Theorem 4.2, $t_{S,P}$ enjoys the following properties.

Corollary 4.49 (Juhász-M.³³). *The element $t_{S,P}$ does not depend on the decorations chosen on $\mathcal{C}(S)$, it is always non-zero, and it is an invariant of the surface S up to isotopy in B fixing K .*

At the time of writing, it is still an open question whether $t_{S,P}$ can distinguish between different slice discs.

In Section 4.5.1 we use the element $t_{S,P}$ to propose a refinement of an ap-

proach of Freedman, Gompf, Morrison and Walker for disproving the 4-dimensional Poincaré conjecture¹⁵.

We also use Theorem 4.2, and specifically the fact that F_C preserves the bigrading of $\widehat{\text{HFK}}$, as an obstruction to invertible concordances. Motivated by a question of Fox¹³ asking which knots are doubly slice, Sumners⁷⁸ introduced the notion of invertible concordances: we say that a concordance $(S^3 \times I, F)$ from K_0 to K_1 is *invertible* if there is a concordance $(S^3 \times I, F')$ from K_1 to K_0 such that the composition of $(S^3 \times I, F)$ and $(S^3 \times I, F')$ from K_0 to K_0 is equivalent to the trivial cobordism. Sumners uses the terminology ‘invertible cobordism’ instead of ‘invertible concordance’. We write $K_0 \leq K_1$ if there is an invertible concordance from K_0 to K_1 . A knot K is doubly slice if and only if $U \leq K$, and the relation $\leq$ induces a partial order on the set of knots. As a corollary of Theorem 4.2, we get the following obstruction.

Theorem 4.56 (Juhász-M.³³, Theorem 1.6). *If there is an invertible concordance from K_0 to K_1 , then*

$$\dim \widehat{\text{HFK}}_j(K_0, i) \leq \dim \widehat{\text{HFK}}_j(K_1, i) \quad (1.1)$$

for every $i, j \in \mathbb{Z}$.

By using the fact that $\widehat{\text{HFK}}$ determines both the genus of a knot and if it is fibred, we deduce as a corollary the following unpublished result of Ruberman.

Corollary 4.57 (Juhász-M.³³, Corollary 1.7). *Suppose that $K_0 \leq K_1$. Then*

1. $g(K_0) \leq g(K_1)$; and
2. if $g(K_0) = g(K_1)$ and K_1 is fibred, then so is K_0 .

Given an invertible concordance from K_0 to K_1 , one can build a degree one map $E(K_1) \rightarrow E(K_0)$, where $E(K_i)$ denotes the exterior of K_i . We let $K_0 \preceq_1 K_1$ if there is a degree one map as above. The relation $\preceq_1$ also defines a partial order on the set of knots. In view of Theorem 4.56, we

can ask whether Equation (1.1) holds under the more general hypothesis $K_0 \preceq_1 K_1$. This turns out to be false, as shown by a counterexample constructed by Hom (see Example 4.62). In view of that, we ask the following, weaker question (see Question 4.63): if $K_0 \preceq_1 K_1$, is it true that $\dim \widehat{\text{HFK}}(K_1) \geq \dim \widehat{\text{HFK}}(K_0)$? Compare this question with the following conjecture.

Conjecture 4.64 (Karakurt-Lidman³⁵). *If $f: Y_1 \rightarrow Y_0$ is a non-zero degree map between integer homology spheres, then $\dim \widehat{\text{HF}}(Y_1) \geq \dim \widehat{\text{HF}}(Y_0)$.*

Following a very helpful conversation with Levine, I considered Zemke's cobordism maps F° too. Although for generic decorated link cobordisms it is not clear whether Zemke's map $\widehat{F}_\chi$ agrees with Juhász's map F_χ , they do agree on decorated concordances. Theorem 4.2 readily generalises to all flavours of knot Floer homology without modifications in the statement (see Theorem 4.75). We remark that the first part of the statement, about the fact that F_C° preserves the bigrading of HFK° , was independently proven by Zemke⁸⁷ with different methods. Theorem 4.75 can be applied to generalise the sets $A_j(K, P)$ and the element $t_{S,P}$ associated to a slice disc to all flavours of HFK. Levine also pointed out that the fact that f° actually preserves the double filtration on HFK^∞ gives an alternate proof of the invariance under concordance of many Heegaard Floer theoretical invariants, including Ozsváth-Szabó's ν invariant⁶⁴, Rasmussen's correction terms, usually denoted V_i , and Ozsváth-Stipsicz-Szabó's Υ function⁵⁴. An interesting feature of this viewpoint is that it may be possible to generalise these concordance invariants to more general settings, possibly including 3-manifolds that are not L -spaces (i.e., 3-manifolds Y whose Heegaard Floer homology is not determined by $|H_1(Y; \mathbb{Z})|$). This viewpoint is explored in a forthcoming paper of Hom, Levine, and Lidman²². The same approach was also used by Grigsby, Licata, and Wehrli¹⁹ to prove that their Υ -type invariants d_t , defined in the setting of annular Khovanov-Lee homology, are braid concordance invariants. Lastly, as an application of Theorem 4.75, we get the following result.

Corollary 4.78. *The E^∞ page of the spectral sequence induced by the Alexander filtration on $\text{CFK}^\circ(K, P)$ is a decorated concordance invariant.*

Examples of $E^\infty(\text{CFK}^\pm)$ for the trefoil knots can be found in Figure 4.4, at the end of Chapter 4.

1.2.2 A MORSE THEORETICAL DESCRIPTION

By Morse theory, a decorated link cobordism (F, σ) in $S^3 \times I$ can be broken into a composition of *elementary cobordisms*. Therefore, in order to understand the map associated to a generic decorated link cobordism, it is enough to compute it on its elementary pieces, which we define below. Note that this gives an alternative definition of the cobordism maps: one can define them on elementary cobordisms and then extend by composition. This is actually the approach taken by Zemke⁸⁵ to define his cobordism maps in HFL° . The cobordism maps defined by Alishahi and Eftekhary¹ in tangle Floer homology, and by Sarkar⁷¹ in grid homology, also use this strategy.

We will suppose that the surface F is *oriented* (as opposed to being just orientable). If (F, σ) is a decorated link cobordism from (L_0, P_0) to (L_1, P_1) , then we orient the links so that $\partial F = L_1 \sqcup -L_0$. As we mentioned in Section 1.2, a choice of orientation of the decorated link $\mathcal{L}$ induces two gradings on $\widehat{\text{HFL}}(\mathcal{L})$, the Alexander grading and the Maslov grading. An orientation of F induces a natural splitting $F \setminus \sigma = R_-(\sigma) \sqcup R_+(\sigma)$, as explained in Definition 3.15.

In an elementary cobordism (F, σ) , the height function $h: S^3 \times I \rightarrow I$ has at most one generic critical point when restricted to F and to σ , and $\text{Crit}(h|_F) \subseteq \text{Crit}(h|_\sigma)$. In a *birth* cobordism, $h|_F$ has a local minimum. In a *saddle*, $h|_F$ has index one and $h|_\sigma$ has a local maximum. We distinguish *merge* and *split* saddles depending on whether the number of link components decreases or increases by one, respectively. In a *death* cobordism, $h|_F$ has a local maximum. A *stabilisation* is an elementary cobordism where $h|_F$ has no critical points, and $h|_\sigma$ has a local minimum, and a *destabilisation* is when $h|_\sigma$ has a local maximum. A (de)stabilisation is *positive* (resp. *negative*) if the bigon component of $F \setminus \sigma$ lies in $R_+(\sigma)$ (resp. in $R_-(\sigma)$). Finally, an *isotopy* is a cobordism where $h|_F$ and $h|_\sigma$ have no critical points.

It follows from Proposition 5.6 that elementary cobordisms generate all decorated link cobordisms.

By the work of Juhász³⁰, the maps induced by isotopies agree with diffeomorphism maps (see Section 5.2). We focus on the other elementary cobordisms. Let V denote the bigraded $\mathbb{F}_2$ vector space $\mathbb{F}_2\langle B, T \rangle$, generated by homogeneous elements B in Maslov grading $-1/2$ and Alexander grading 0, and T in Maslov grading $1/2$ and Alexander grading 0. We also define W as the bigraded vector space $\mathbb{F}_2\langle b, t \rangle$, where b is in Maslov and Alexander gradings $-1/2$, and t is in Maslov and Alexander gradings $1/2$. Then, for stabilisations and destabilisations, we have the following results.

Proposition 5.12 (Juhász-M.³²). *Let $\mathcal{S} = (F, \sigma)$ be a stabilisation from (L, P_0) to (L, P_1) . Then, there is a canonical isomorphism*

$$I_{\mathcal{S}}: \widehat{\text{HFL}}(L, P_0) \otimes W \xrightarrow{\sim} \widehat{\text{HFL}}(L, P_1)$$

such that the link cobordism map

$$F_{\mathcal{S}}: \widehat{\text{HFL}}(L, P_0) \rightarrow \widehat{\text{HFL}}(L, P_1)$$

factorises as $I_{\mathcal{S}} \circ s_{\mathcal{S}}$, where

$$s_{\mathcal{S}}: \widehat{\text{HFL}}(L, P_0) \rightarrow \widehat{\text{HFL}}(L, P_0) \otimes W$$

is given by $s_{\mathcal{S}}(x) = x \otimes t$ in case of a positive stabilisation, and $s_{\mathcal{S}}(x) = x \otimes b$ in case of a negative stabilisation.

Proposition 5.14 (Juhász-M.³²). *Suppose that $\mathcal{D} = (F, \sigma)$ is a destabilisation from (L, P_0) to (L, P_1) . Then, there is a canonical isomorphism*

$$I_{\mathcal{D}}: \widehat{\text{HFL}}(L, P_0) \xrightarrow{\sim} \widehat{\text{HFL}}(L, P_1) \otimes W$$

such that the link cobordism map

$$F_{\mathcal{D}}: \widehat{\text{HFL}}(L, P_0) \rightarrow \widehat{\text{HFL}}(L, P_1)$$

factorises as $d_{\mathcal{D}} \circ I_{\mathcal{D}}$, where

$$d_{\mathcal{D}}: \widehat{\text{HFL}}(L, P_1) \otimes W \rightarrow \widehat{\text{HFL}}(L, P_1)$$

is given by $d_{\mathcal{D}}(x \otimes b) = x$ and $d_{\mathcal{D}}(x \otimes t) = 0$ in case of a positive destabilisation, and by $d_{\mathcal{D}}(x \otimes t) = x$ and $d_{\mathcal{D}}(x \otimes b) = 0$ in case of a negative destabilisation.

For births and deaths, the result is quite similar, but we ought to use the vector space V instead of W .

Theorem 5.21 (Juhász-M.³²). *Suppose that $\mathcal{B}$ is a birth cobordism from (L_0, P_0) to (L_1, P_1) . Then there is an isomorphism*

$$\widehat{\text{HFL}}(L_1, P_1) \cong \widehat{\text{HFL}}(L_0, P_0) \otimes V$$

such that $F_{\mathcal{B}}(x) = x \otimes T$ for every $x \in \widehat{\text{HFL}}(L_0, P_0)$.

Theorem 5.23 (Juhász-M.³²). *Let $\mathcal{D}$ be a death cobordism from (L_0, P_0) to (L_1, P_1) . Then there is an isomorphism*

$$\widehat{\text{HFL}}(L_0, P_0) \cong \widehat{\text{HFL}}(L_1, P_1) \otimes V$$

such that $F_{\mathcal{D}}(x \otimes T) = 0$ and $F_{\mathcal{D}}(x \otimes B) = x$ for every $x \in \widehat{\text{HFL}}(L_0, P_0)$.

For decorated saddles, we relate the map to a decorated exact triangle in knot Floer homology, which we now describe. Let (Y, L, P) be a decorated link, and let D be a disc embedded in Y such that $D \cap L = \{r_-, r_+\}$, with $r_{\pm} \in R_{\pm}(P)$. Define the decorated link (Y, L_0, P_0) as the link obtained from $L \setminus \mathcal{N}(D)$ by reconnecting the endpoints with arcs parallel to D , and by adding a decoration on each new arc. Furthermore, let (Y, L_+, P) be the decorated link obtained from (Y, L, P) by adding a full left-handed twist along D . See Figure 1.1 for a local picture of the three decorated links.

The following result is a decorated generalisation of Ozsváth and Szabó's oriented exact triangle. For a more precise and natural statement see Section 5.5.

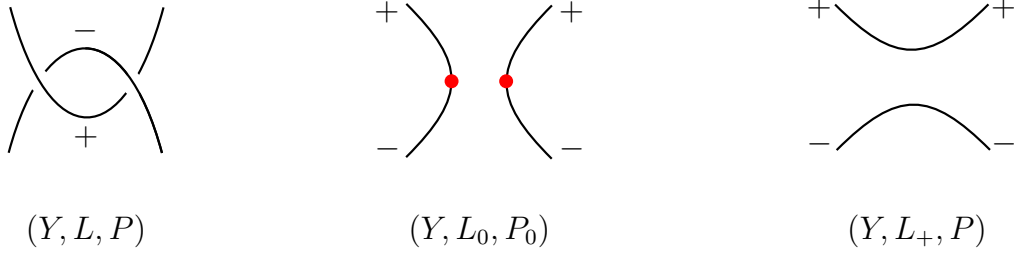


Figure 1.1: A local picture of the decorated links (Y, L_0, P_0) , (Y, L, P) , and (Y, L_+, P) . The disc D is contained in a plane perpendicular to the sheet, and it intersects both $R_+(P)$ and $R_-(P)$ in one point.

Theorem 5.15 (Juhász-M.³²). *Given decorated links (Y, L, P) , (Y, L_0, P_0) , and (Y, L_+, P) as described above, there is an exact triangle*

$$\cdots \longrightarrow \widehat{\text{HFL}}(Y, L, P) \xrightarrow{e} \widehat{\text{HFL}}(Y, L_0, P_0) \xrightarrow{f} \widehat{\text{HFL}}(Y, L_+, P) \longrightarrow \cdots \quad (1.2)$$

Note that we do not require that (Y, L, P) and (Y, L_0, P_0) can be oriented coherently (this happens if and only if $|L_0| = |L| \pm 1$). If $Y = S^3$ and the links are coherently oriented, then we recover Ozsváth and Szabó's oriented skein exact triangle in $\widehat{\text{HFK}}$ ⁵⁸. Specifically, if (L, P) has two decorations on each component and $|L_0| = |L| + 1$, then (L_0, P_0) also has two decorations on each component, and Equation (1.2) specialises to Ozsváth and Szabó's⁵⁸ exact sequence (7):

$$\cdots \longrightarrow \widehat{\text{HFK}}(L) \longrightarrow \widehat{\text{HFK}}(L_0) \longrightarrow \widehat{\text{HFK}}(L_+) \longrightarrow \cdots$$

Here we used the fact that, by definition, if a decorate link (L, P) has two decorations on each component, then $\widehat{\text{HFL}}(L, P) \cong \widehat{\text{HFK}}(L)$. Suppose now that (L, P) has two decorations on each component and $|L_0| = |L| - 1$. Then (L_0, P_0) has six decorations on one component, and two on all other components, and in this case $\widehat{\text{HFL}}(L_0, P_0) \cong \widehat{\text{HFK}}(L_0) \otimes W^{\otimes 2}$. It follows that Equation (1.2) specialises to Ozsváth and Szabó's⁵⁸ exact sequence (8):

$$\cdots \longrightarrow \widehat{\text{HFK}}(L) \longrightarrow \widehat{\text{HFK}}(L_0) \otimes W^{\otimes 2} \longrightarrow \widehat{\text{HFK}}(L_+) \longrightarrow \cdots$$

Furthermore, when L , L_0 and L_+ are coherently oriented, the Alexander and the Maslov grading shift are as in the paper by Ozsváth and Szabó⁵⁸, so all maps in Equation (1.2) preserve the Alexander grading, and e and f decrease the Maslov grading by $1/2$.

If $\mathcal{S} = (F, \sigma)$ is a decorated saddle cobordism from (L_0, P_0) to (L_1, P_1) , we give an interpretation of the saddle map $F_{\mathcal{S}}$ in terms of the decorated exact triangle described above. Therefore, with the following theorem, the Morse theoretical description of cobordism maps is completed.

Theorem 5.17 (Juhász-M.³²). *Let the triple (L, P) , (L_0, P_0) , (L_+, P) be as in Figure 5.5, and suppose that they can be oriented coherently. Then the map*

$$f: \widehat{\text{HFL}}(L_0, P_0) \longrightarrow \widehat{\text{HFL}}(L_+, P)$$

appearing in the decorated skein exact triangle of Theorem 5.15 coincides with the cobordism map induced by the saddle cobordism from (L_0, P_0) to (L_+, P) .

As an application of the description of the maps induced by elementary cobordisms, we compute the Alexander and Maslov grading shift of any decorated link cobordism map, and relate it to the topological invariants of the cobordism.

Theorem 5.27 (Juhász-M.³²). *If $\mathcal{X} = (F, \sigma)$ is a decorated link cobordism, then the map induced on $\widehat{\text{HFL}}$ is homogeneous with respect to both the Alexander and the Maslov gradings. $F_{\mathcal{X}}$ shifts the Maslov grading by*

$$\frac{\chi(F) + \chi(R_+(\sigma)) - \chi(R_-(\sigma))}{2}$$

and the Alexander grading by

$$\frac{\chi(R_+(\sigma)) - \chi(R_-(\sigma))}{2}.$$

Given a decorated link cobordism $\mathcal{X} = (X, F, \sigma)$ from $\mathcal{L}_0 = (Y_0, L_0, P_0)$ to $\mathcal{L}_1 = (Y_1, L_1, P_1)$, one can consider the upside down cobordism $\overline{\mathcal{X}} = (X, F, \sigma)$ from $\overline{\mathcal{L}}_1 = (-Y_1, L_1, P_1)$ to $\overline{\mathcal{L}}_0 = (-Y_0, L_0, P_0)$. There is a canonical duality $\widehat{\text{HFL}}(\overline{\mathcal{L}}_i) \cong \widehat{\text{HFL}}(\mathcal{L}_i)^*$, and Juhász³¹ asked whether $F_{\overline{\mathcal{X}}}$ is dual

to F_χ . Actually, Juhász's original question was formulated in terms of sutured manifold cobordisms, and is therefore more generic. From the Morse theoretical description above, it is straightforward to show that births, deaths, and (de)stabilisations enjoy the duality property. An answer to the duality question for saddles would prove duality for all decorated link cobordisms, but is at the moment out of reach.

Another open question is how the Morse theoretical description given above relates to Zemke's definition of link cobordism maps. Once again, it is quite straightforward to verify that Juhász's maps and Zemke's maps agree for births, deaths, and (de)stabilisations, while the relation between the two saddle maps is still not understood. Another interesting question is how these two descriptions relate to Alishahi and Eftekhary's tangle Floer maps¹ and to Sarkar's grid homology maps⁷¹.

1.2.3 RELATIONS TO KHOVANOV HOMOLOGY

Like link Floer homology, Khovanov homology is a homology theory for links which has been introduced, and subsequently extensively studied, in the last two decades. Khovanov³⁷ defined a link invariant which is a bigraded Abelian group $\text{Kh}(L) = \bigoplus_{i,j \in \mathbb{Z}} \text{Kh}^{i,j}(L)$, and he proved that it contains at least as much information as the Jones polynomial. Indeed, Khovanov homology is a so-called *categorification* of the Jones polynomial, in the sense that the Jones polynomial (up to a factor) is recovered as the graded Euler characteristic of $\text{Kh}(L)$. We refer to Khovanov's paper³⁷ and Bar-Natan's paper⁷ for more information.

For the purposes of this thesis, we will only consider the reduced homology³⁸ $\widetilde{\text{Kh}}(L)$, associated to a *link with a marked component* $L \subset S^3$, and we will always work with $\mathbb{F}_2$ coefficients. We now quickly recall Khovanov's constructions. Given a marked diagram D for a marked link L , we say that a *resolution* of D is a function

$$u: \{\text{crossings of } D\} \rightarrow \{0, 1\}.$$

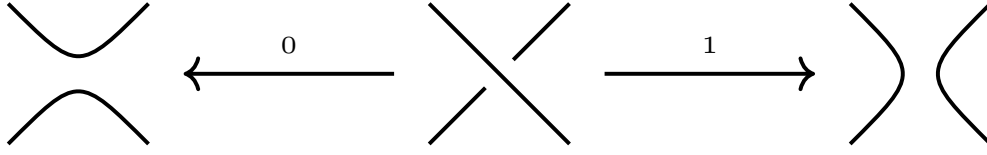


Figure 1.2: The figure shows the 0-smoothing and the 1-smoothing for a crossing of an unoriented link.

Given a resolution u , by smoothing D at each crossing according to Figure 1.2, we define a new marked diagram L_u , which represents a marked unlink. Given two resolutions u and v , we say that $u \leq v$ if v is obtained from u by changing a single 0-smoothing to a 1-smoothing. If $u \leq v$, then there is a saddle cobordism $\mathcal{G}_{u,v}$ from L_u to L_v , given by attaching a band in correspondence of the crossing which is smoothed in different ways. This construction is usually referred to as a *cube of resolutions* for the marked diagram D . Khovanov defined a complex

$$\left(\bigoplus_u \widetilde{\text{Kh}}(L_u), \sum_{u \leq v} \widetilde{\text{Kh}}(\mathcal{G}_{u,v}) \right),$$

where $\widetilde{\text{Kh}}$ here denotes a $(1+1)$ -dimensional TQFT that associates an $\mathbb{F}_2$ vector space to any closed 1-manifold with a marked component, and a map to each (marked) cobordism between two such objects. Such a TQFT is defined in Definition 6.20. The homology of the above complex is an invariant of the link L with a marked component, and is called its reduced Khovanov homology $\widetilde{\text{Kh}}(L)$. Note that, in view of the above construction, $\widetilde{\text{Kh}}(L)$ is fully determined by the choice of the $(1+1)$ -dimensional TQFT.

Rasmussen⁶⁸ pointed out some parallels between the reduced Khovanov homology and the knot Floer homology of a knot $K \subset S^3$. He raised the question of whether there exists a spectral sequence from $\widetilde{\text{Kh}}(\overline{K})$ to $\widehat{\text{HFK}}(K)$, where $\overline{K}$ denotes the mirror image of K . This is usually referred to as the Rasmussen conjecture. In Chapter 6, in joint work with Juhász³² we prove the following theorem, which, as we explain in Section 6.4, can be seen as a first step in a programme towards the proof of the Rasmussen conjecture.

Theorem 6.2 (Juhász-M.³²). *The $(1 + 1)$ -dimensional TQFT induced by $\widehat{\text{HFL}}$ on a category of decorated unlinks and decorated cobordisms between them is the same as Khovanov's reduced TQFT.*

The strategy we use to prove the above theorem is the following. We describe the $\widehat{\text{HFL}}$ TQFT on a category of unlinks and cobordisms between them, by using the Morse theoretical description and a connected sum formula. Although we only need a minimal version of the connected sum formula, we prove a general version for sutured Floer homology, which can be of independent interest (see Theorem 6.14). In order to compare $\widehat{\text{HFL}}$ with $\widetilde{\text{Kh}}$, we need a preferred basis of $\widehat{\text{HFL}}(U_n, P_{U_n})$, where (U_n, P_{U_n}) is the n -component unlink with standard decorations. This is achieved by marking a component of U_n . By using the preferred basis, we have an isomorphism $\widehat{\text{HFL}}(U_n, P_{U_n}) \cong \widetilde{\text{Kh}}(U_n)$, with respect to which the maps induced on the cobordisms of the category coincide.

Since the reduced Khovanov homology of a knot is completely determined by the $(1+1)$ -TQFT $\widetilde{\text{Kh}}$, we have the following immediate corollary of Theorem 6.2.

Corollary 6.25 (Juhász-M.³²). *The reduced Khovanov homology of a marked link L (with $\mathbb{F}_2$ coefficients) can be computed by applying the TQFT $\widehat{\text{HFL}}$ to a cube of resolutions of L .*

1.3 ORGANISATION

This thesis is organised as follows.

In Chapter 2 we review Heegaard Floer homology, from the sutured viewpoint, and recall the splitting along spin^c structures and the homological grading. We then introduce the Heegaard Floer homology of a 3-manifold and the link Floer homology of a link as particular cases of sutured Floer homology. Lastly, in order to work in a natural setting, we define decorated links and their link Floer homology.

In Chapter 3 we recall how Heegaard Floer homology is made functorial.

We focus on the case of decorated link cobordisms, and on the maps that they induce in link Floer homology defined by Juhász³¹.

Chapter 4 is devoted to the study of Juhász's maps in the case of decorated concordances. The main result here, obtained in joint work with Juhász³³, is that concordance maps preserve the bigrading of $\widehat{\text{HFK}}$ and fit into a morphism of spectral sequences (Theorem 4.2). The proof of this result will take most of the chapter. We then focus on the applications that we described in Section 1.2.1. Lastly, we conclude with some remarks about the concordance maps defined by Zemke⁸⁵.

In Chapter 5 we give a full description of the maps induced by elementary decorated link cobordisms, based on joint work with Juhász³². In the same chapter we prove the decorated exact triangle introduced in Section 1.2.2 (Theorem 5.15).

Lastly, in Chapter 6, which is based on joint work with Juhász³², we explore the relation between the link Floer TQFT and the reduced Khovanov TQFT, and prove Theorem 6.2.

2

HEEGAARD FLOER HOMOLOGY AND LINK COMPLEMENTS

THE purpose of this chapter is to give an introduction to the Heegaard Floer theory for knots and links in 3-manifolds. We will define link Floer homology in terms of sutured Floer homology (SFH), since Juhász's link cobordism maps³¹, which we will recall in the next chapter, are defined in terms of SFH. It is worth noting that we do not aim at a comprehensive explanation of sutured Floer homology (or any other version of Heegaard Floer homology), but we only aim at recalling the definitions and results which are necessary for the purposes of this thesis.

2.1 SUTURED FLOER HOMOLOGY

2.1.1 SUTURED MANIFOLDS

Sutured manifolds were first introduced by Gabai in 1983. His original definition is the following.

Definition 2.1 (Gabai¹⁶, Definition 2.6). A *sutured manifold* is a compact

oriented 3-manifold M with boundary together with a set $\gamma \subseteq \partial M$ of pairwise disjoint annuli $A(\gamma)$ and tori $T(\gamma)$. Furthermore, the interior of each component of $A(\gamma)$ contains a homologically non-trivial oriented simple closed curve, called a *suture*. We denote the set of sutures by $s(\gamma)$.

Finally, every component of $R(\gamma) = \partial M \setminus \text{Int}(\gamma)$ is oriented such that $\partial R(\gamma)$ is coherent with the sutures. Let $R_+(\gamma)$ (or $R_-(\gamma)$) denote the components of $R(\gamma)$ whose normal vectors point out of (into) M .

In order to define a version of Heegaard Floer homology for sutured manifolds, the natural setting used in Juhász's original paper is given by *balanced* sutured manifolds.

Definition 2.2 (Juhász²⁷, Definition 2.2). We say that a sutured manifold (M, γ) is *balanced* if M has no closed components, $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$, and the map $\pi_0(A(\gamma)) \rightarrow \pi_0(\partial M)$ is surjective.

From now on, we only consider balanced sutured manifolds (unless otherwise specified), and we view γ as a “thickened” oriented 1-manifold. So we often do not distinguish between γ and $s(\gamma)$; it shall be clear from the context which one we mean.

2.1.2 RELATIVE SPIN^c STRUCTURES

Definition 2.3 (Juhász³¹, Definition 3.1). Given a sutured manifold (M, γ) , we say that a vector field v defined on a subset of M containing ∂M is *admissible* if it is nowhere vanishing, it points into M along $R_-(\gamma)$, it points out of M along $R_+(\gamma)$, and $v|_\gamma$ is tangent to ∂M and either points into $R_+(\gamma)$ or is positively tangent to γ (we think of ∂M as a smooth surface, and of γ as a 1-manifold).

Let v and w be admissible vector fields on M . We say that v and w are *homologous*, and we write $v \sim w$, if there is a collection of balls $B \subseteq M$, one in each component of M , such that v and w are homotopic on $M \setminus B$ through admissible vector fields. Then, the set of *relative spin^c structures*,

denoted by $\text{Spin}^c(M, \gamma)$, is the set of homology classes of admissible vector fields on M .

If (M, γ) is balanced, $\text{Spin}^c(M, \gamma)$ is an affine space over $H^2(M, \partial M)$. We will denote relative spin^c structures by $\mathfrak{s}^\circ$, to distinguish them from ordinary spin^c structures on oriented 3-manifolds, usually denoted by $\mathfrak{s}$.

Remark 2.4. Let v_0 be a fixed vector field on ∂M arising as $v|_{\partial M}$ for some admissible vector field v on M . We define $\text{Spin}_{v_0}^c(M, \gamma)$ as the set of nowhere vanishing vector fields on M that restrict to v_0 on ∂M , up to isotopy in the complement of a collection of balls through such vector fields. Since the space of all possible v_0 is contractible, $\text{Spin}_{v_0}^c(M, \gamma)$ can be canonically identified with $\text{Spin}^c(M, \gamma)$. This was the approach taken by Juhász in his first paper on sutured Floer homology²⁷.

2.1.3 SUTURED FLOER HOMOLOGY

The sutured Floer homology²⁷ of a balanced sutured manifold (M, γ) is a finite dimensional $\mathbb{F}_2$ vector space $\text{SFH}(M, \gamma)$, which admits a splitting along relative spin^c structures:

$$\text{SFH}(M, \gamma) = \bigoplus_{\mathfrak{s}^\circ \in \text{Spin}^c(M, \gamma)} \text{SFH}(M, \gamma, \mathfrak{s}^\circ).$$

Each vector space $\text{SFH}(M, \gamma, \mathfrak{s}^\circ)$ is an invariant of the sutured manifold (M, γ) equipped with the relative spin^c structure $\mathfrak{s}^\circ$.

We now briefly recall the definition of sutured Floer homology. For a full exposition on the subject, we refer to Juhász's original paper²⁷ and to Altman's paper².

Definition 2.5 (Juhász²⁷, Definition 2.7). A (balanced) *sutured Heegaard diagram* is a triple $\mathcal{H} = (\Sigma, \alpha, \beta)$, such that Σ is a compact oriented surface with boundary with no closed components and $\alpha = \{\alpha_1, \dots, \alpha_d\}$ and $\beta = \{\beta_1, \dots, \beta_d\}$ are two sets of d pairwise disjoint simple closed curves linearly independent in $H_1(\Sigma; \mathbb{Z})$, which intersect transversely.

One can associate a balanced sutured manifold (M, γ) to each sutured Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta)$: To build M , consider the manifold $\Sigma \times I$, and attach d negative 2-handles along $\alpha \times \{0\}$ and d positive 2-handles along $\beta \times \{1\}$. The sutures γ are given by $\partial\Sigma \times I$. Every balanced sutured manifold can be expressed by a sutured Heegaard diagram.

Given a Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta)$, Juhász constructs the manifold

$$\text{Sym}^d \Sigma := \Sigma^{\times d} / S_d.$$

The sets of curves α and β define d -dimensional tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ in $\text{Sym}^d \Sigma$, from which Juhász defines the set of *generators*

$$\mathcal{G} := \mathbb{T}_\alpha \cap \mathbb{T}_\beta.$$

Every generator $\mathbf{x}$ defines a relative spin^c structure $\mathfrak{s}^\circ(\mathbf{x}) \in \text{Spin}^c(M, \gamma)$. If $\mathfrak{s}^\circ \in \text{Spin}^c(M, \gamma)$, Juhász defines

$$\begin{aligned} \text{SFH}(\mathcal{H}) &= H_* \left(\bigoplus_{\mathbf{x} \in \mathcal{G}} \mathbb{F}_2 \langle \mathbf{x} \rangle, \partial \right), \\ \text{SFH}(\mathcal{H}, \mathfrak{s}^\circ) &= H_* \left(\bigoplus_{\mathfrak{s}^\circ(\mathbf{x}) = \mathfrak{s}^\circ} \mathbb{F}_2 \langle \mathbf{x} \rangle, \partial \right), \end{aligned}$$

where the differential ∂ is defined on the set of generators as

$$\partial(\mathbf{x}) = \sum_{\mathbf{y} \in \mathcal{G}} \sum_{\substack{D \in \mathcal{D}(\mathbf{x}, \mathbf{y}) \\ \mu(D)=1}} \# \widehat{\mathcal{M}}(D) \cdot \mathbf{y},$$

and then extended by linearity. Here $\mathcal{D}(\mathbf{x}, \mathbf{y})$ denotes the set of *domains* from $\mathbf{x}$ to $\mathbf{y}$ (see Definition 2.6 below), and $\widehat{\mathcal{M}}(D)$ denotes the reduced moduli space of pseudoholomorphic discs $u: \mathbb{D}^2 \rightarrow \text{Sym}^d \Sigma$ with domain D , with respect to a proper choice of almost complex structure on $\text{Sym}^d \Sigma$.

Definition 2.6 (Juhász²⁷, Definition 3.6). Let $\mathcal{H} = (\Sigma, \alpha, \beta)$ be a sutured Heegaard diagram. Let $D_1, \dots, D_l$ denote the closures of the components

of $\Sigma \setminus (\alpha \cup \beta)$. Then the set of *domains* in $\mathcal{H}$ is

$$\mathcal{D}(\mathcal{H}) = \mathbb{Z}\langle D_1, \dots, D_l \rangle.$$

Let $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$. A domain D is said to connect $\mathbf{x}$ to $\mathbf{y}$ if for every $1 \leq i \leq d$ the equalities $\partial(\alpha_i \cap \partial D) = \mathbf{y} \cap \alpha_i - \mathbf{x} \cap \alpha_i$ and $\partial(\beta_i \cap \partial D) = \mathbf{x} \cap \beta_i - \mathbf{y} \cap \beta_i$ hold. We will denote by $\mathcal{D}(\mathbf{x}, \mathbf{y})$ the set of domains connecting $\mathbf{x}$ to $\mathbf{y}$.

By Juhász²⁷ (Corollary 4.8), $\mathcal{D}(\mathbf{x}, \mathbf{y})$ is empty if $\mathfrak{s}^\circ(\mathbf{x}) \neq \mathfrak{s}^\circ(\mathbf{y})$. Thus, the differential splits along the relative spin^c structure, and we have that

$$\text{SFH}(\mathcal{H}) = \bigoplus_{\mathfrak{s}^\circ \in \text{Spin}^c(M, \gamma)} \text{SFH}(\mathcal{H}, \mathfrak{s}^\circ).$$

We should mention here that, in order to define sutured Floer homology, one should only work with a particular kind of sutured Heegaard diagrams, known as *admissible* diagrams, which are defined below.

Definition 2.7 (Juhász²⁷, Definition 3.6). Given a sutured Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta)$, a domain $\mathcal{P} \in \mathcal{D}(\mathcal{H})$ is called *periodic* if $\partial \mathcal{P}$ is a linear combination of α and β curves.

Definition 2.8 (Juhász²⁷, Definition 3.11). A sutured Heegaard diagram $\mathcal{H}$ is called *admissible* if every periodic domain $\mathcal{P} \neq 0$ has both positive and negative coefficients.

Every sutured Heegaard diagram can be perturbed to an admissible diagram as shown by Juhász²⁷. From now on, when dealing with sutured Heegaard diagrams, we will suppose that they are admissible. It is beyond the scope of this thesis to explain why admissible diagrams are required, and to give the details of the construction of sutured Floer homology. We refer the reader to Juhász²⁷ for further details.

Given sutured Heegaard diagrams $\mathcal{H}$ and $\mathcal{H}'$ for a balanced sutured manifold (M, γ) , Juhász and Thurston³⁴ defined maps

$$F_{\mathcal{H}, \mathcal{H}'}: \text{SFH}(\mathcal{H}) \rightarrow \text{SFH}(\mathcal{H}'),$$

$$F_{\mathcal{H}, \mathcal{H}', \mathfrak{s}^\circ} : \text{SFH}(\mathcal{H}, \mathfrak{s}^\circ) \rightarrow \text{SFH}(\mathcal{H}', \mathfrak{s}^\circ),$$

that satisfy a functoriality condition, that is, given Heegaard diagrams $\mathcal{H}$, $\mathcal{H}'$ and $\mathcal{H}''$, we have that

$$F_{\mathcal{H}', \mathcal{H}'', \mathfrak{s}^\circ} \circ F_{\mathcal{H}, \mathcal{H}'} = F_{\mathcal{H}, \mathcal{H}''} \quad \text{and} \quad F_{\mathcal{H}', \mathcal{H}'', \mathfrak{s}^\circ} \circ F_{\mathcal{H}, \mathcal{H}', \mathfrak{s}^\circ} = F_{\mathcal{H}, \mathcal{H}'', \mathfrak{s}^\circ}.$$

Therefore, we can define $\text{SFH}(M, \gamma)$ and $\text{SFH}(M, \gamma, \mathfrak{s}^\circ)$ functorially. The naturality maps are defined on a chain level for each elementary Heegaard move, and to get the aforementioned maps one should take the induced maps in homology. See also Section 5.2 of Juhász's paper³¹.

In Section 8 of Juhász's paper on sutured Floer homology²⁷, a homological grading is also described: every group $\text{SFH}(M, \gamma, \mathfrak{s}^\circ)$ carries a relative $\mathbb{Z}/(\mathfrak{d}(\mathfrak{s}^\circ))$ grading, where $\mathfrak{d}(\mathfrak{s}^\circ)$ denotes the divisibility of the spin^c structure:

$$\mathfrak{d}(\mathfrak{s}^\circ) = \gcd_{\xi \in H_2(M; \mathbb{Z})} \langle c_1(\mathfrak{s}^\circ), \xi \rangle.$$

In fact, the grading is already defined on a chain level, and the differential lowers the relative $\mathbb{Z}/(\mathfrak{d}(\mathfrak{s}^\circ))$ grading by 1.

2.2 HEEGAARD FLOER HOMOLOGY FOR CLOSED 3-MANIFOLDS, KNOTS AND LINKS

Sutured Floer homology generalises at once the *hat* flavour of the Heegaard Floer homologies of 3-manifolds, knots and links. In this section I will recall how to recover these invariants, and I will explain why it is natural to consider *decorated* knots and links (introduced in Section 2.3) when studying their Heegaard Floer homologies.

2.2.1 HEEGAARD FLOER HOMOLOGY FOR CLOSED 3-MANIFOLDS

Definition 2.9 (Juhász²⁷, Example 2.3). Let Y denote a closed oriented connected 3-manifold. Then $Y(1)$ is the sutured manifold with underlying 3-manifold $Y \setminus B^3$, and with a single suture on $\partial Y = S^2$.

In the following proposition, Juhász relates SFH to the Heegaard Floer homology defined by Ozsváth and Szabó⁶⁰.

Proposition 2.10 (Juhász²⁷, Proposition 9.1). *Let Y be a closed connected oriented 3-manifold. Then*

$$\widehat{\text{HF}}(Y) \cong \text{SFH}(Y(1)). \quad (2.1)$$

We will actually use (2.1) as a definition of $\widehat{\text{HF}}$. We remark that, in order to make $\widehat{\text{HF}}$ functorial, one should actually have a *based* 3-manifold (Y, p) . The sutured manifold associated to it would be obtained by removing a ball in correspondence of the basepoint, and the functoriality of $\widehat{\text{HF}}$ would then follow from the work of Juhász and Thurston³⁴.

The splitting of SFH along relative spin^c structures is reflected in a splitting of $\widehat{\text{HF}}$ along $\text{Spin}^c(Y)$:

$$\widehat{\text{HF}}(Y) = \bigoplus_{\mathfrak{s} \in \text{Spin}^c(Y)} \widehat{\text{HF}}(Y, \mathfrak{s}).$$

If $\mathfrak{s}$ is a *torsion* spin^c structure (that is, its first Chern class $c_1(\mathfrak{s}) \in H^2(Y; \mathbb{Z})$ is torsion), then $\mathfrak{d}(\mathfrak{s}|_{Y \setminus B^3}) = 0$, and therefore $\widehat{\text{HF}}(Y, \mathfrak{s})$ admits a relative $\mathbb{Z}$ grading. Ozsváth and Szabó⁶² proved that this can actually be lifted to an absolute $\mathbb{Q}$ grading, which is usually referred to as the *Maslov grading* or *homological grading*.

2.2.2 KNOT FLOER HOMOLOGY

Suppose that $K \subset Y$ is a *knot*, that is a closed connected oriented 1-manifold in a closed connected oriented 3-manifold. Then Ozsváth and Szabó⁵⁸ and, independently, Rasmussen⁶⁶, associate to it a *knot Floer homology* $\widehat{\text{HFK}}(Y, K)$. The definition we give here is in terms of SFH.

Definition 2.11 (Juhász²⁷, Example 2.4). Let $K \subset Y$ be a knot. Then $Y(K)$ denotes the sutured manifold $Y \setminus \mathcal{N}(K)$ complementary to a regular neighbourhood $\mathcal{N}(K)$ of K , with two oppositely oriented meridians on $\partial(Y(K))$ as sutures.

Proposition 2.12 (Juhász²⁷, Proposition 9.2). *Let $K \subset Y$ be a knot. Then*

$$\widehat{\mathrm{HFK}}(Y, K) \cong \mathrm{SFH}(Y(K)).$$

As a consequence of the spin^c decomposition of SFH , we have that

$$\widehat{\mathrm{HFK}}(Y, K) = \bigoplus_{\mathfrak{s}^\circ \in \mathrm{Spin}^c(Y(K))} \widehat{\mathrm{HFK}}(Y, K, \mathfrak{s}^\circ).$$

In Section 3.3 of their paper⁵⁸, Ozsváth and Szabó explain that, if $\mathfrak{s}^\circ \in \mathrm{Spin}^c(Y(K))$ extends to a torsion spin^c structure on Y , then $\widehat{\mathrm{HFK}}(Y, K, \mathfrak{s}^\circ)$ carries a Maslov grading, which is a relative $\mathbb{Z}$ grading and an absolute $\mathbb{Q}$ grading. As a particular case, if Y is a rational homology sphere (which we will denote by $\mathbb{Q}HS^3$), then the Maslov grading is well-defined on the whole group.

If K is nullhomologous and *oriented*, and S is a Seifert surface for K , then there is an *Alexander grading*, defined on the set of generators by the formula

$$\mathcal{A}_S(\mathbf{x}) = \frac{1}{2} \langle c_1(\mathfrak{s}^\circ(\mathbf{x}), t), [S] \rangle. \quad (2.2)$$

Here $c_1(\mathfrak{s}^\circ(\mathbf{x}), t)$ is the relative Chern class associated to the trivialisation of $v_0^\perp$ given by the meridional direction (see Juhász's paper²⁸, Definition 3.7 and proof of Theorem 1.5 at page 333), and v_0 is a vector field as in Remark 2.4. When Y is a $\mathbb{Q}HS^3$, then all Seifert surfaces for K are rationally homologous, and therefore we can denote the Alexander grading simply by $\mathcal{A}$.

In the special case of $Y = S^3$, both Maslov and Alexander gradings are indexed by $\mathbb{Z}$, and by convention $\widehat{\mathrm{HFK}}_i(K, j)$ denotes the knot Floer homology in Maslov grading i and Alexander grading j .

Finally note that the sutured manifold $Y(K)$ complementary to K is not well defined, because there is a choice on where to put the sutures on $\partial Y(K)$. This will be fixed in Section 2.3.

2.2.3 LINK FLOER HOMOLOGY

Suppose now that $L \subset Y$ is a *link*, that is a closed oriented 1-manifold in a closed connected oriented 3-manifold. Then Ozsváth and Szabó⁶³ defined its *link Floer homology* $\widehat{\text{HFL}}(Y, L)$, which we define, once again, in terms of SFH.

Definition 2.13 (Juhász²⁷, Example 2.4). Let $L \subset Y$ be a link. Then $Y(L)$ denotes the sutured manifold $Y \setminus \mathcal{N}(L)$, with two oppositely oriented meridians on each component of $\partial(Y(L))$ as sutures.

Proposition 2.14 (Juhász²⁷, Proposition 9.2). *Let $L \subset Y$ be a link. Then*

$$\widehat{\text{HFL}}(Y, L) \cong \text{SFH}(Y(L)).$$

We will mainly focus on links in $Y = S^3$, in which case $\widehat{\text{HFL}}$ is endowed with a (collapsed) Alexander grading and a Maslov grading. We will give precise definitions in Section 2.3, and refer to Ozsváth and Szabó⁶³, and Zemke⁸⁷ for a full discussion on gradings in $\widehat{\text{HFL}}$.

2.3 DECORATED LINKS AND THE HFL FUNCTOR

We already mentioned that the definitions of $\widehat{\text{HFK}}$ and $\widehat{\text{HFL}}$ in Section 2.2 present some functoriality issues. First, the neighbourhood $\mathcal{N}(L)$ of a link L in Y is not chosen canonically, and therefore the manifold $Y(L)$ is not well defined. This problem was fixed by Juhász by replacing the removal of $\mathcal{N}(L)$ by a real blow-up construction, as detailed below. A second and more fundamental issue is given by the fact that the sutures on $\partial Y(L)$ are not chosen canonically. Different choices γ_1 and γ_2 of the two meridional sutures yield isomorphic groups $\text{SFH}(Y(L), \gamma_1) \cong \text{SFH}(Y(L), \gamma_2)$, but—as pointed out by Sarkar⁷² and Zemke⁸⁶—in general there is no canonical isomorphism between them. Therefore, the natural setting for a definition of link Floer homology is given by decorated links.

We start by reviewing the real blow-up construction.

Definition 2.15 (Arone and Kankaanrinta³, Juhász³¹). Suppose that M is a smooth manifold, and let $L \subset M$ be a properly embedded submanifold. For every $p \in L$, let $N_p L = T_p M / T_p L$ be the fiber of the normal bundle of L over p , and let $UN_p L = (N_p L \setminus \{0\}) / \mathbb{R}_+$ be the fiber of the unit normal bundle of L over p . Then the (spherical) blowup of M along L , denoted by $\text{Bl}_L(M)$, is a manifold with boundary obtained from M by replacing each point $p \in L$ by $UN_p L$. There is a natural projection $\text{Bl}_L(M) \rightarrow M$.

We now review decorated links.

Definition 2.16 (Juhász³¹, Definition 4.4). A *decorated link* is given by a triple (Y, L, P) , where L is a (not necessarily oriented) non-empty link in the connected oriented 3-manifold Y , and $P \subset L$ is a finite set of points. We require that for every component L_0 of L , the number $|L_0 \cap P|$ is positive and even. Furthermore, we are given a decomposition of L into compact 1-manifolds $R_+(P)$ and $R_-(P)$ such that $R_+(P) \cap R_-(P) = P$.

We can canonically assign a balanced sutured manifold $Y(L, P) = (M, \gamma)$ to every decorated link (Y, L, P) , as follows. Let $M = \text{Bl}_L(Y)$ and $\gamma = \bigcup_{p \in P} UN_p L$. Furthermore,

$$R_{\pm}(\gamma) := \bigcup_{x \in R_{\pm}(P)} UN_x L,$$

oriented as $\pm \partial M$, and we orient γ as $\partial R_+(\gamma)$.

Definition 2.17 (Juhász-M.³³). We say that a decorated link (Y, L, P) is a *decorated knot* if $|K| = 1$ and $|P| = 2$.

We will often consider decorated knots and links in $Y = S^3$, in which case we drop the 3-manifold Y from the notation.

Definition 2.18. For a decorated link (Y, L, P) , we define its *link Floer homology* as

$$\text{HFL}(Y, L, P) := \text{SFH}(Y(L, P)).$$

Definition 2.19 (Ozsváth-Szabó⁵⁸, Section 2.3). Let (Y, L, P) be a decorated link, L be nullhomologous and oriented, and let S be a Seifert surface for

L . For a sutured Heegaard diagram $\mathcal{H}$ for $Y(L, P)$, we define the *collapsed Alexander grading* of a generator $\mathbf{x}$ of $\text{SFC}(\mathcal{H})$ as

$$\mathcal{A}_S(\mathbf{x}) := \frac{1}{2} \langle c_1(\mathfrak{s}^\circ(\mathbf{x}), t), [S] \rangle,$$

where t is the trivialisation of $v_0^\perp$ given by the meridional direction. $\mathcal{A}_S(\mathbf{x})$ is preserved under the differential and naturality maps, and descends to a grading on $\text{HFL}(Y, L, P)$.

Remark 2.20. The collapsed Alexander grading is a generalisation of the Alexander grading defined in Equation 2.2, it changes sign if the orientation of each component of L is reversed, it satisfies $\mathcal{A}_S(\mathbf{x}) \in \frac{1}{2}\mathbb{Z}$, and, if Y is a $\mathbb{Q}HS^3$, it does not depend on S .

Remark 2.21. If $Y = S^3$, and L is oriented, then $\text{HFL}(L, P)$ carries a *Maslov grading*, which is a relative $\mathbb{Z}$ grading and absolute $\frac{1}{2}\mathbb{Z}$ grading. The orientation of L allows us to split $P = w \cup z$, in such a way that $R_-(L)$ is oriented from w to z . Then $\text{HFL}(L, P)$ inherits the Maslov grading from $\widehat{\text{HF}}(S^3, w)$.

Note that for the definition of $\text{HFL}(Y, L, P)$ we do not require that L is oriented. The only place where the orientation of L comes into play is in the definitions of the (collapsed) Alexander grading and of the Maslov grading. It is possible to specify an orientation of L already at the level of Heegaard diagrams. For this reason we introduce multi-pointed Heegaard diagrams.

Definition 2.22 (Ozsváth-Szabó⁶³, Definition 3.1). A *multi-pointed Heegaard diagram* is a tuple

$$\mathcal{H} = (\Sigma, \alpha, \beta, w, z),$$

where Σ is a closed connected oriented genus g surface, $\alpha = \{\alpha_1, \dots, \alpha_d\}$ (resp. $\beta = \{\beta_1, \dots, \beta_d\}$) is a set of $d \geq g$ pairwise disjoint simple closed curves, and $w = \{w_1, \dots, w_{d-g+1}\}$ (resp. $z = \{z_1, \dots, z_{d-g+1}\}$) is a set of points contained in $\Sigma \setminus (\alpha \cup \beta)$, such that α (resp. β) are linearly independent in $H_1(\Sigma \setminus w; \mathbb{Z})$ and in $H_1(\Sigma \setminus z; \mathbb{Z})$. We denote by $\ell := d - g + 1$.

The original definition by Ozsváth and Szabó is slightly different, but equivalent to ours. We outline why this is true, by showing that our definition implies that the α curves generate a g -dimensional subspace of $H_1(\Sigma; \mathbb{Z})$, and that each component of $\Sigma \setminus \alpha$ (resp. $\Sigma \setminus \beta$) contains exactly one w basepoint and one z basepoint. Note that $H_1(\Sigma \setminus w; \mathbb{Z}) \cong \mathbb{Z}^{d+g}$, $H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ and the map

$$H_1(\Sigma \setminus w; \mathbb{Z}) \rightarrow H_1(\Sigma; \mathbb{Z})$$

is surjective. Since the d α curves are linearly independent in $H_1(\Sigma \setminus w; \mathbb{Z})$, they must span a g -dimensional subspace of $H_1(\Sigma; \mathbb{Z})$. Therefore, cutting Σ along α gives a planar surface. The fact that each component of $\Sigma \setminus \alpha$ has exactly one w_i follows again from the fact that α are linearly independent in $H_1(\Sigma \setminus w; \mathbb{Z})$: if there is a component with two basepoints w_i and w_j , we could add a small circle about w_i to the α curve, and we would get a set of $d+1$ disjoint simple closed curves on Σ which are linearly independent on in $H_1(\Sigma \setminus w; \mathbb{Z})$, which is a contradiction.

A multi-pointed Heegaard diagram defines a decorated link as follows. Let U_α (resp. U_β) be the handlebody obtained by attaching handles along α (resp. β) on the negative (resp. positive) side of $\Sigma \times [-1, 0]$ (resp. $\Sigma \times [0, 1]$) and by capping off the newly obtained S^2 components of the boundary with 3-handles. We identify $\Sigma \times \{0\}$ with Σ , and we let $Y = U_\alpha \cup_\Sigma U_\beta$. In each component of $\Sigma \setminus \alpha$ (resp. $\Sigma \setminus \beta$) there is exactly one w_i and one z_j . Choose arcs connecting the w basepoints and the z basepoints in $\Sigma \setminus \alpha$ (resp. $\Sigma \setminus \beta$), and push them into U_α (resp. U_β) so that they are properly embedded: the union of all such arcs yields a link L . We also set $P = w \cup z$, $R_-(P) = L \cap U_\alpha$, and $R_+(P) = L \cap U_\beta$. The distinction between w and z basepoints also allows us to define an *orientation* of L , by declaring that $R_-(P)$ is oriented from the w basepoints to the z basepoints.

Remark 2.23. Given a multi-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ for a decorated link (Y, L, P) , then the sutured diagram $(\Sigma \setminus \mathcal{N}(w \cup z), \alpha, \beta)$ defines the balanced sutured manifold $Y(L, P)$. Here $\mathcal{N}(w \cup z)$ denotes a collection of small discs about the w and z basepoints. Conversely, given a sutured Heegaard diagram $(\Sigma^\circ, \alpha, \beta)$ for $Y(L, P)$, one can fill the bound-

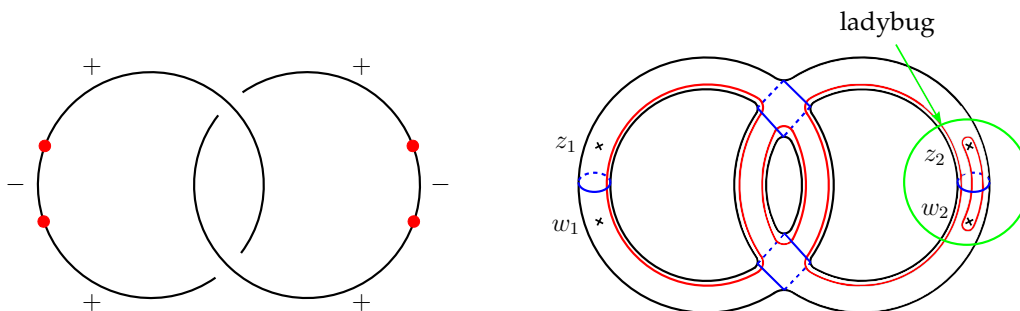


Figure 2.1: The picture on the left shows a projection of the Hopf link with two decorations on each component. From this projection, one can construct the Heegaard diagram on the right-hand side.

aries of Σ° with discs, put a basepoint in the middle of each disc, and label them w_i and z_i in an alternating fashion, to obtain a multi-pointed Heegaard diagram for (Y, L, P) .

Note that the two descriptions are equivalent except that the multi-pointed diagram also defines an *orientation* of the link. When we will talk about unoriented links, we will use the two descriptions indifferently and make no distinction between them.

2.4 THE HEEGAARD DIAGRAM ARISING FROM A DECORATED LINK PROJECTION

Ozsváth and Szabó⁵⁸ described how to construct a Heegaard diagram from a knot projection. We extend their construction to decorated links in S^3 ; see Figure 2.1 for an illustration.

Let (L, P) be an oriented decorated n -component link with $|P| = 2\ell$, and let D be a connected projection of L onto $\mathbb{R}^2 \subseteq \mathbb{R}^3$. Let Σ be the boundary of a regular neighbourhood of D in $\mathbb{R}^3$. Then Σ is a surface of some genus g , oriented as the boundary of its exterior. Place a basepoint on Σ right above each point of the decoration P . The basepoints are alternately labelled by w_i and z_i such that each component of $R_-(P)$ is oriented from w_i to z_i . We let $\mathbf{w} = \{w_1, \dots, w_\ell\}$ and $\mathbf{z} = \{z_1, \dots, z_\ell\}$.

For each crossing of D , add a β -curve on Σ according to the orientation

of the crossing. We call these curves $\beta_1, \dots, \beta_{g-1}$. Furthermore, for each component of $R_-(P)$, add a meridional β -curve that we call $\beta_g, \dots, \beta_{g+\ell-1}$. What we finally get is a set

$$\beta = \{ \beta_1, \dots, \beta_{g+\ell-1} \}$$

of attaching circles.

As for the α -curves, draw a curve around each bounded region of $\mathbb{R}^2 \setminus D$, and call these curves $\alpha_1, \dots, \alpha_g$. Furthermore, for each component of $R_-(P)$ except one, add an inessential α -curve that contains z_i and w_i as in Figure 2.1. We call these *ladybugs*, and are labelled $\alpha_{g+1}, \dots, \alpha_{g+\ell-1}$. We set

$$\alpha = \{ \alpha_1, \dots, \alpha_{g+\ell-1} \}.$$

Then, the diagram $(\Sigma, \alpha, \beta, w, z)$ is a multi-pointed Heegaard diagram representing the decorated link (L, P) .

3

HEEGAARD FLOER HOMOLOGY AND DECORATED LINK COBORDISMS

IN this chapter we review the construction of maps in HFL associated to decorated link cobordisms as defined by Juhász³¹. We already mentioned that, by the work of Sarkar⁷² and Zemke⁸⁶, for HFL to be a well defined vector space one needs to consider decorated links, otherwise what one gets is a vector space up to isomorphisms. More details of this fact will be given in Section 3.3.

In a similar way, if one considers link cobordisms without decorations, the best thing that one could hope is to get a map defined up to composition by automorphisms of the domain and of the target. Therefore, the natural setting for the definition of cobordism maps is given by decorated link cobordisms, which we recall in Section 3.2. The way they are defined by Juhász³¹ is as a particular case of sutured cobordism maps, which we review in Section 3.1.

Before reviewing these constructions, it is worth mentioning that two different constructions of maps in Heegaard Floer homology associated to link cobordisms were recently posted online, by Alishahi and Eftekhary¹, and by Zemke⁸⁵. We will say a little more in Section 3.4.

3.1 SUTURED COBORDISMS AND SUTURED MAPS

3.1.1 SUTURED COBORDISMS

We first recall the notion of cobordism of sutured manifolds.

Definition 3.1 (Juhász³¹, Definition 2.3). Let (M, γ) be a sutured manifold, and suppose that ξ_0 and ξ_1 are contact structures on M such that ∂M is a convex surface with dividing set γ with respect to both ξ_0 and ξ_1 . Then we say that ξ_0 and ξ_1 are *equivalent* if there is a 1-parameter family $\{\xi_t : t \in I\}$ of contact structures such that ∂M is convex with dividing set γ with respect to ξ_t for every $t \in I$. In this case, we write $\xi_0 \sim \xi_1$, and we denote by $[\xi]$ the equivalence class of the contact structure ξ .

Definition 3.2 (Juhász³¹, Definitions 2.4 and 2.14). Given sutured manifolds (M_0, γ_0) and (M_1, γ_1) , a *cobordism* from (M_0, γ_0) to (M_1, γ_1) is a triple $\mathcal{W} = (W, Z, [\xi])$, where

- W is a compact oriented 4-manifold with boundary;
- $Z \subseteq \partial W$ is a compact, codimension-0 submanifold with boundary (viewed within ∂W), such that $\partial W \setminus \text{Int}(Z) = -M_0 \sqcup M_1$, and we view Z as a sutured manifold with sutures $\gamma_0 \cup \gamma_1$; and
- ξ is a positive contact structure on Z such that ∂Z is a convex surface with dividing set γ_i on ∂M_i for $i \in \{0, 1\}$.

Finally, a cobordism is called *balanced* if both (M_0, γ_0) and (M_1, γ_1) are balanced.

Note that in this thesis we only consider balanced sutured manifolds and balanced cobordisms.

Definition 3.3 (Juhász³¹, Definition 2.7). Two cobordisms $\mathcal{W} = (W, Z, [\xi])$ and $\mathcal{W}' = (W', Z', [\xi'])$ from (M_0, γ_0) to (M_1, γ_1) are called *equivalent* if there is an orientation preserving diffeomorphism $\varphi : W \rightarrow W'$ such that $d(Z) = Z'$, $d_*(\xi) = \xi'$, and $d|_{M_0 \cup M_1} = \text{id}$.

Definition 3.4 (Juhász³¹, Definition 10.4). A cobordism $\mathcal{W} = (W, Z, [\xi])$ from (M_0, γ_0) to (N, γ_1) is called a *boundary cobordism* if W is balanced, N is parallel to $M_0 \cup (-Z)$, and we are also given a deformation retraction $r : W \times [0, 1] \rightarrow M_0 \cup (-Z)$ such that $r_0|_W = \text{id}_W$ and $r_1|_N$ is an orientation preserving diffeomorphism from N to $M_0 \cup (-Z)$.

Definition 3.5 (Juhász³¹, Definition 5.1). We say that a cobordism $\mathcal{W} = (W, Z, [\xi])$ from (M_0, γ_0) to (M_1, γ_1) is *special* if

1. $\mathcal{W}$ is balanced;
2. $\partial M_0 = \partial M_1$, and $Z = \partial M_0 \times I$ is the trivial cobordism between them; and
3. ξ is an I -invariant contact structure on Z such that each $\partial M_0 \times \{t\}$ is a convex surface with dividing set $\gamma_0 \times \{t\}$ for every $t \in I$ with respect to the contact vector field $\partial/\partial t$.

In particular, it follows from 3. that $\gamma_0 = \gamma_1$.

Remark 3.6. Every sutured cobordism can be seen as the composition of a boundary cobordism and a special cobordism as explained by Juhász³¹ in Definition 10.1. Let $\mathcal{W} = (W, Z, [\xi])$ be a balanced cobordism from (M_0, γ_0) to (M_1, γ_1) . Let (N, γ_1) be the sutured manifold $(M_0 \cup (-Z), \gamma_1)$. Then we can think of the cobordism $\mathcal{W}$ as a composition $\mathcal{W}^s \circ \mathcal{W}^b$, where $\mathcal{W}^b$ is a boundary cobordism from (M_0, γ_0) to (N, γ_1) and $\mathcal{W}^s$ is a special cobordism from (N, γ_1) to (M_1, γ_1) .

3.1.2 RELATIVE SPIN^c STRUCTURES

Before introducing the notion of spin^c cobordism of sutured manifolds, we need to recall the following definition.

Definition 3.7 (Juhász³¹, Definition 3.2). Let (M, γ) be a sutured manifold. We say that an oriented 2-plane field ξ defined on a subset of M containing ∂M is *admissible* if there exists a Riemannian metric g on M such that $\xi^{\perp_g}$ is an admissible vector field. If ξ is defined on the whole manifold M , we write

$$\mathfrak{s}_\xi^\circ = [\xi^{\perp_g}] \in \text{Spin}^c(M, \gamma).$$

This is independent of the choice of g since the space of metrics g for which $\xi^{\perp_g}$ is an admissible vector field is convex.

We now recall the notion of relative spin^c structures on sutured cobordisms. If J is an almost complex structure on a 4-manifold W and H is a 3-dimensional submanifold, then there is a 2-plane field induced on H called the *field of complex tangencies* along H , as explained in Lemma 3.4 of Juhász's paper³¹.

Definition 3.8 (Juhász³¹, Definition 3.5). Suppose that $\mathcal{W} = (W, Z, [\xi])$ is a cobordism from the sutured manifold (M_0, γ_0) to (M_1, γ_1) . We say that an almost complex structure J defined on a subset of W containing ∂Z is *admissible* if the field of complex tangencies on M_i (defined on a subset of M_i containing ∂M_i) is admissible in (M_i, γ_i) for $i \in \{0, 1\}$, and the field ξ_J of complex tangencies on Z (defined on a subset of Z containing ∂Z) is admissible in $(Z, \gamma_0 \cup \gamma_1)$.

A *relative spin^c structure* on $\mathcal{W}$ is a homology class of pairs (J, P) , where:

- $P \subseteq \text{Int}(W)$ is a finite collection of points;
- J is an admissible almost complex structure defined over $W \setminus P$; and
- if ξ_J is the field of complex tangencies along Z , then $\mathfrak{s}_\xi^\circ = \mathfrak{s}_{\xi_J}^\circ$.

We say that (J, P) and (J', P') are *homologous* if there exists a compact 1-manifold $C \subseteq W \setminus \partial Z$ such that $P, P' \subseteq C$; furthermore, $J|_{W \setminus C}$ and $J'|_{W \setminus C}$ are isotopic through admissible almost complex structures. We denote by $\text{Spin}^c(\mathcal{W})$ the set of relative spin^c structures over $\mathcal{W}$.

Remark 3.9. As in the case of sutured manifolds, we will denote relative spin^c structures on sutured cobordisms by $\mathfrak{s}^\circ$, in order to distinguish them from ordinary spin^c structures on oriented 4-manifolds, which we denote by $\mathfrak{s}$, in analogy with the case of oriented 3-manifolds.

Remark 3.10. $\text{Spin}^c(\mathcal{W})$ is an affine space over

$$\ker \left(H^2(W, \partial Z) \longrightarrow H^2(Z, \partial Z) \right).$$

There are restriction maps

$$\text{Spin}^c(\mathcal{W}) \longrightarrow \text{Spin}^c(M_i, \gamma_i)$$

for $i \in \{0, 1\}$.

3.1.3 COBORDISM MAPS IN SUTURED FLOER HOMOLOGY

Juhász proved³¹ that a balanced cobordism $\mathcal{W}$ from (M_0, γ_0) to (M_1, γ_1) induces a homomorphism

$$F_{\mathcal{W}}: \text{SFH}(M_0, \gamma_0) \longrightarrow \text{SFH}(M_1, \gamma_1).$$

If $\mathcal{W}$ is endowed with a relative spin^c structure $\mathfrak{s}^\circ$, then we also have a map

$$F_{\mathcal{W}, \mathfrak{s}^\circ}: \text{SFH}(M_0, \gamma_0, \mathfrak{s}^\circ|_{M_0}) \longrightarrow \text{SFH}(M_1, \gamma_1, \mathfrak{s}^\circ|_{M_1}).$$

Let **BSut** denote the category of balanced sutured manifolds and equivalence classes of cobordisms, whereas **Vect** denotes the category of vector spaces over $\mathbb{F}_2$.

Theorem 3.11 (Juhász³¹, Theorem 11.12). *SFH defines a functor*

$$\mathbf{BSut} \rightarrow \mathbf{Vect},$$

which is a $(3 + 1)$ -dimensional TQFT in the sense of Atiyah⁴, and Blanchet and Turaev⁸.

We now outline the construction of the cobordism map associated to a balanced cobordism. Let $\mathcal{W} = (W, Z, [\xi])$ be a balanced cobordism from (M_0, γ_0) to (M_1, γ_1) , and suppose that every component Z_0 of Z intersects M_1 (this last hypothesis can actually be dropped, as explained below). According to Remark 3.6, we can view $\mathcal{W}$ as the composition of a boundary cobordism $\mathcal{W}^b$ from (M_0, γ_0) to (N, γ_1) and a special cobordism $\mathcal{W}^s$ from (N, γ_1) to (M_1, γ_1) . Using the *contact gluing map* defined by Honda, Kazez, and Matić²⁴, Juhász³¹ constructed in Section 9 a map

$$F_{\mathcal{W}^b} : \text{SFH}(M_0, \gamma_0) \longrightarrow \text{SFH}(N, \gamma_1)$$

associated to the special cobordism $\mathcal{W}^b$.

The special cobordism $\mathcal{W}^s$ also induces a map: Choose a decomposition of $\mathcal{W}^s$ as $\mathcal{W}_3 \circ \mathcal{W}_2 \circ \mathcal{W}_1$, where $\mathcal{W}_i$ is the trace of i -handle attachments. Juhász³¹ defined a map $F_{\mathcal{W}_i}$ associated to each cobordism $\mathcal{W}_i$, and the map associated to $\mathcal{W}^s$ is defined as

$$F_{\mathcal{W}^s} = F_{\mathcal{W}_3} \circ F_{\mathcal{W}_2} \circ F_{\mathcal{W}_1} : \text{SFH}(N, \gamma_1) \longrightarrow \text{SFH}(M_1, \gamma_1).$$

Finally, the cobordism map $F_{\mathcal{W}}$ is the composition $F_{\mathcal{W}^s} \circ F_{\mathcal{W}^b}$, which is independent of all the choices made.

We supposed above that there are no components of $Z = N \setminus M_0$ that do not intersect M_1 (these are also called *isolated components*); this requirement was needed for the definition of the gluing map (and also because N may not be balanced). In the presence of isolated components, the map associated to a cobordism $\mathcal{W}$ is defined as follows: Remove a standard contact ball B^3 (i.e.; a ball with a single suture on the boundary) from each isolated component, and add them to M_1 . Then you get a cobordism $\mathcal{W}'$ from (M_0, γ_0) to $(M_1 \sqcup \coprod B^3, \gamma_1 \sqcup \coprod \gamma_{B^3})$ with no isolated components. By composing the already defined map

$$F_{\mathcal{W}'} : \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}\left(M_1 \sqcup \coprod B^3, \gamma_1 \sqcup \coprod \gamma_{B^3}\right)$$

with the natural isomorphism

$$\mathrm{SFH}\left(M_1 \sqcup \coprod B^3, \gamma_1 \sqcup \coprod \gamma_{B^3}\right) \xrightarrow{\cong} \mathrm{SFH}(M_1, \gamma_1),$$

one obtains a map $F_{\mathcal{W}}: \mathrm{SFH}(M_0, \gamma_0) \rightarrow \mathrm{SFH}(M_1, \gamma_1)$.

All cobordism maps above admit refinements $F_{\mathcal{W}, s^\circ}$ along relative spin^c structures. The map $F_{\mathcal{W}}$ can be recovered from the maps $F_{\mathcal{W}, s^\circ}$ for all spin^c structures (see Juhász³¹, Definition 10.9 and Proposition 10.11), and the spin^c cobordism maps satisfy a type of composition law (see Juhász³¹, Theorem 11.3).

3.2 DECORATED LINK COBORDISMS

We now recall the definition of decorated link cobordisms and of the maps that they induce, therefore making HFL functorial.

Definition 3.12 (Juhász³¹, Definition 4.2). A *surface with divides* (S, σ) is a compact orientable surface S , possibly with boundary, together with a properly embedded 1-manifold σ that divides S into two compact subsurfaces that meet along σ .

Definition 3.13 (Juhász³¹, Definition 4.1). For $i \in \{0, 1\}$, let Y_i be a connected, oriented 3-manifold, and let L_i be a non-empty link in Y_i . Then a *link cobordism* from (Y_0, L_0) to (Y_1, L_1) is a pair (X, F) , where

1. X is a connected, oriented cobordism from Y_0 to Y_1 ;
2. F is a properly embedded, compact, orientable surface in X ; and
3. $\partial F = L_0 \cup L_1$.

We are now ready to define decorated link cobordisms.

Definition 3.14 (Juhász³¹, Definition 4.5). The triple $\mathcal{X} = (X, F, \sigma)$ is a *decorated link cobordism* from (Y_0, L_0, P_0) to (Y_1, L_1, P_1) if

1. (X, F) is a link cobordism from (Y_0, L_0) to (Y_1, L_1) ;

2. (F, σ) is a surface with divides such that the map

$$\pi_0(\partial\sigma) \rightarrow \pi_0((L_0 \setminus P_0) \cup (L_1 \setminus P_1))$$

is a bijection;

3. we can orient each component R of $F \setminus \sigma$ such that whenever $\partial\bar{R}$ crosses a point of P_0 , it goes from $R_+(P_0)$ to $R_-(P_0)$, and whenever it crosses a point of P_1 , it goes from $R_-(P_1)$ to $R_+(P_1)$; and
4. if F_0 is a closed component of F , then $\sigma \cap F_0 \neq \emptyset$.

Furthermore, we say that two decorated link cobordisms $\mathcal{X} = (X, F, \sigma)$ and $\mathcal{X}' = (X', F', \sigma')$ from (Y_0, L_0, P_0) to (Y_1, L_1, P_1) are *equivalent* if there is an orientation-preserving diffeomorphism $\varphi : X \rightarrow X'$ such that $\varphi(F) = F'$, $\varphi(\sigma) = \sigma'$, $\varphi|_{Y_0} = \text{id}_{Y_0}$, and $\varphi|_{Y_1} = \text{id}_{Y_1}$.

The category of decorated links and equivalence classes of decorated link cobordisms is denoted by **DLink**.

According to Juhász³¹, there is a natural sutured manifold cobordism complementary to a decorated link cobordism whose construction we recall in the next definition. For this purpose, we first discuss S^1 -invariant contact structures on circle bundles; see also Juhász³¹ (Section 4). Let $\pi : M \rightarrow F$ be a principal circle bundle over a compact oriented surface F . An S^1 -invariant contact structure ξ on M determines a dividing set σ on the base F , by requiring that $x \in \sigma$ if and only if ξ is tangent to $\pi^{-1}(x)$, and a splitting of F as $R_+(\sigma) \cup R_-(\sigma)$. The image of any local section of π is a convex surface with dividing set projecting onto σ . In the opposite direction, according to Lutz⁴⁵ and Honda²³, given a dividing set σ on F that intersects each component of F non-trivially and divides F into subsurfaces $R_+(\sigma)$ and $R_-(\sigma)$, there is a unique S^1 -invariant contact structure ξ_σ on M , up to isotopy, such that the dividing set associated to ξ_σ is exactly σ , the coorientation of ξ_σ induces the splitting $R_\pm(\sigma)$, and the boundary ∂M is convex.

Definition 3.15 (Juhász³¹, Definition 4.9). Let (X, F, σ) be a decorated link cobordism from (Y_0, L_0, P_0) to (Y_1, L_1, P_1) . Then we define the sutured

cobordism $\mathcal{W} = \mathcal{W}(X, F, \sigma)$ from $Y_0(L_0, P_0)$ to $Y_1(L_1, P_1)$ as follows. Pick an arbitrary splitting of F into $R_+(\sigma)$ and $R_-(\sigma)$ such that $R_+(\sigma) \cap R_-(\sigma) = \sigma$, and orient F such that $\partial R_+(\sigma)$ (with $R_+(\sigma)$ oriented as a subsurface of F) crosses P_0 from $R_+(P_0)$ to $R_-(P_0)$ and P_1 from $R_-(P_1)$ to $R_+(P_1)$. Then $\mathcal{W}$ is defined to be the triple $(W, Z, [\xi])$, where $W = \text{Bl}_F(X)$ and $Z = UNF$, oriented as a submanifold of ∂W , finally $\xi = \xi_\sigma$ is an S^1 -invariant contact structure with dividing set σ on F and convex boundary ∂Z with dividing set projecting to $P_0 \cup P_1$.

The contact structure ξ is independent of the splitting of F into $R_+(\sigma)$ and $R_-(\sigma)$.

Note that, in the above definition, the contact vector fields with respect to which a local section of $UNF \rightarrow F$ and ∂Z are transverse are different, so they can project to different subsets of $L_0 \cup L_1$. Specifically, we have that the dividing set for ∂Z projects to $P_0 \cup P_1$, while $\partial\sigma$ is disjoint from $P_0 \cup P_1$.

Moreover, note that if F does not have any closed component, then it deformation retracts onto a 1-dimensional CW complex, and therefore any S^1 -bundle on it has a section, hence is trivial if the bundle is orientable. In particular, $UNF \approx F \times S^1$.

According to Juhász³¹, if (X, F, σ) and (X', F', σ') are equivalent, so are the cobordisms of sutured manifolds associated to them. Therefore, we obtain a functor

$$\mathcal{W} : \mathbf{DLink} \rightarrow \mathbf{BSut}.$$

Notice that $\mathcal{W}(Y, L, P)$ is what we called $Y(L, P)$. By composing with SFH, we obtain a functor from $\mathbf{DLink}$ to $\mathbf{Vect}$.

Definition 3.16. We define the functor

$$\mathbf{HFL} : \mathbf{DLink} \rightarrow \mathbf{Vect}$$

to be the composition $\mathbf{SFH} \circ \mathcal{W}$.

We conclude this section with two remarks concerning notation. First, given a decorated link cobordism, with a slight abuse of notation, we will

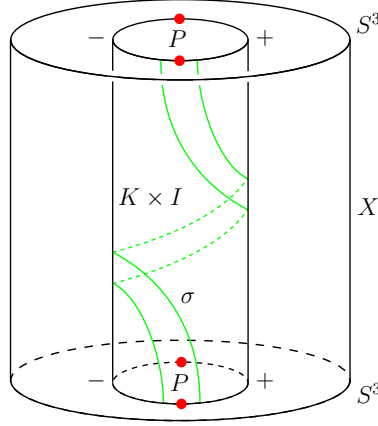


Figure 3.1: The figure shows the decorated cobordism $\mathcal{X}$ from (K, P) to itself. The knot decorations P are shown in red, and the cobordism decorations σ on the surface $K \times I$ are shown in green. The signs $-$ and $+$ respectively stand for $R_-(P)$ and $R_+(P)$.

also denote the cobordism of sutured manifolds associated to it by the same letter. Second, if we choose an orientation of F , then we always orient L_0 as $-\partial F$ and L_1 as ∂F .

3.3 DECORATIONS MATTER

We already mentioned that decorations are fundamental to the definition of cobordism maps. Here we give a concrete example of why this is the case. Given a decorated knot (K, P) in S^3 , construct a decorated link cobordism $\mathcal{X} = (S^3 \times I, K \times I, \sigma)$, where the decorations σ consist of two parallel arcs which wind once along the cylinder $K \times I$ (see Figure 3.1).

As we said in Section 3.2, $\mathcal{X}$ induces a map

$$F_{\mathcal{X}}: \text{HFL}(K, P) \rightarrow \text{HFL}(K, P).$$

An alternative definition of $\mathcal{X}$ is the following. Consider the product cobordism $(S^3 \times I, K \times I, P \times I)$, and identify both top and bottom of

it to (S^3, K, P) , using the identity diffeomorphism for the bottom

$$\text{id}: (S^3 \times \{0\}, K \times \{0\}, P \times \{0\}) \rightarrow (S^3, K, P),$$

and a *fingermove* diffeomorphism for the top

$$d: (S^3 \times \{1\}, K \times \{1\}, P \times \{1\}) \rightarrow (S^3, K, P).$$

More explicitly, d is a diffeomorphism which agrees with id away from K , and which twists once along K in a neighbourhood of K itself. Note that this diffeomorphism is not isotopic to the identity through diffeomorphisms fixing K . As explained by Juhász³¹, the map induced by $\mathcal{X}$ agrees with the action of d on $\text{HFL}(K, P)$ described by Juhász and Thurston³⁴. Given a decorated link (Y, L, P) , according to Juhász and Thurston, there is a representation of the mapping class group on $\text{HFL}(Y, L, P)$:

$$\rho: \text{MCG}(Y, K, P) \rightarrow \text{Aut}(\text{HFL}(Y, L, P)).$$

Sarkar⁷² proved that if (Y, L, P) is a decorated knot (K, P) in S^3 , then $\rho(d)$, where d is the diffeomorphism defined above, often acts non-trivially on $\text{HFL}(Y, L, P)$. As a result, the map $F_{\mathcal{X}}$ associated to the decorated link cobordism in Figure 3.1 is not the identity map, and therefore this shows that cobordism maps are sensitive to the choice of decorations.

We also mention that in the case of decorated links in S^3 , Zemke⁸⁶ gave a formula for $\rho(d_i)$, where d_i is the diffeomorphism associated to a fingermove along the i -th component of the link.

3.4 COBORDISM MAPS IN THE LITERATURE

There are some other constructions of maps in Heegaard Floer homology associated to (decorated) link cobordisms, whose description is beyond the scope of this thesis. Nevertheless, we give in this section a brief overview of the recent literature on the subject for the interested reader.

In 2010, a year after Juhász's paper³¹, Sarkar⁷¹ gave a construction of link

cobordism maps in grid Floer homology associated to a surface in $S^3 \times I$, which secretly carried decorations. However, he has not proved that such maps are invariant under isotopy of the surface in $S^3 \times I$. In his PhD thesis, Graham¹⁸, attempted to show that Sarkar's maps do not change under surface isotopy by trying to show their invariance under the so-called *movie moves*, but was able to prove it only for a subset of them. To the best of the author's knowledge, the question whether Sarkar's maps are invariant under isotopy is still unsettled.

More recently, Zemke⁸⁵ defined maps associated to decorated link cobordisms for all flavours of link Floer homology. The spirit of his definition is quite different from the one of Juhász. Rather than going through sutured Floer homology, Zemke defines a map associated to each elementary decorated link cobordism, and extends his construction to any cobordism by composing elementary cobordisms. He also shows that such a map does not depend on the choice of a splitting of the decorated link cobordisms into elementary pieces. To the best of the author's knowledge, it is unclear whether his maps coincide with Juhász's maps. Note that in Chapter 5 we give a description of Juhász's maps for each elementary decorated link cobordism. However, the set of elementary link cobordism used by Zemke is different from ours, specifically in the decorations chosen for a saddle cobordism, and therefore it is not clear whether Zemke's and Juhász's maps agree, although I should mention that the maps associated to all other elementary cobordisms agree.

Lastly, Alishahi and Eftekhary¹ defined maps associated to tangle cobordisms, which also yield maps for link cobordisms. Their construction is more similar in spirit to Zemke's one than to Juhász's one.

4

DECORATED CONCORDANCES

DECORATED concordances are the easiest kind of decorated link cobordism. A *classical concordance* from a knot $K_0 \subset S^3$ to a knot $K_1 \subset S^3$ is an annulus F , properly embedded in $S^3 \times I$, such that $\partial F = -K_0 \times \{0\} \sqcup K_1 \times \{1\}$. This notion defines an interesting equivalence relation on the set of knots: there are knots which are concordant to the unknot (they are called *slice* knots), and knots which are not. Note that, by contrast, if you allow any cobordism (that is, if you allow F to be a surface of any genus), then every knot is cobordant to every other. There is a very rich literature on concordance, for example about determining the structure of the *concordance group* (which is the set of knots modulo concordance, endowed with the connected sum operation). In this chapter, we focus on the decorated counterpart of concordances, which we imaginatively call *decorated concordances*. We will require that not only the surface is the easiest possible, but that also the decorations are in a particularly nice form.

Recall from Definition 2.17 that a decorated knot (K, P) in S^3 is a knot $K \subset S^3$, and a pair of points $P \subset K$, together with a decomposition of K into compact 1-manifolds $K = R_+(P) \cup R_-(P)$ such that $R_+(P) \cap R_-(P) = P$. We will suppose that K is oriented.

Definition 4.1 (Juhász-M.³³, Definition 3.6). A *decorated concordance* between decorated knots (K_0, P_0) and (K_1, P_1) is a decorated link cobordism $\mathcal{C} = (X, F, \sigma)$ as in Definition 3.14 such that

1. X is an integer homology $S^3 \times I$ with boundary $(-S^3) \sqcup S^3$;
2. the surface F is an oriented annulus with $\partial F = -K_0 \sqcup K_1$; and
3. σ consists of two arcs connecting the two components of ∂F .

If $X = S^3 \times I$, we drop X from the notation and only write (F, σ) .

Being a decorated link cobordism, a decorated concordance $\mathcal{C} = (X, F, \sigma)$ from (K_0, P_0) to (K_1, P_1) induces a homomorphism

$$F_{\mathcal{C}}: \widehat{\text{HFK}}(K_0, P_0) \rightarrow \widehat{\text{HFK}}(K_1, P_1),$$

as explained in Sections 3.1.3 and 3.2. By $\widehat{\text{HFK}}$ we mean the functor HFL from Section 3.2 restricted to decorated knots. Recall that the map $F_{\mathcal{C}}$ depends on the decorations, as shown in Section 3.3. Actually, the decorated link cobordism in Figure 3.1, which is discussed in that section, is also a decorated concordance.

The main result of this chapter, from joint work with Juhász³³, gives a link between the two spectral sequences associated to (decorated) concordant knots (K_0, P_0) and (K_1, P_1) . Ozsváth and Szabó⁵⁸ and, independently, Rasmussen⁶⁶ proved that for every decorated knot (K, P) in S^3 , there is a corresponding spectral sequence

$$\widehat{\text{HFK}}(K, P) \Rightarrow \widehat{\text{HF}}(S^3) \cong \mathbb{F}_2.$$

Given an admissible doubly-pointed Heegaard diagram $(\Sigma, \alpha, \beta, w, z)$ for (K, P) , the singly-pointed diagram $(\Sigma, \alpha, \beta, w)$ represents (S^3, w) , and z gives rise to the knot filtration on $\widehat{\text{CF}}(\Sigma, \alpha, \beta, w)$. The spectral sequence arises from this filtered complex. The E^0 page is the associated graded complex $\widehat{\text{CFK}}(\Sigma, \alpha, \beta, w, z)$, whose homology is $\widehat{\text{HFK}}(K, P)$, the E^1 page. The spectral sequence limits to the homology of $\widehat{\text{CF}}(\Sigma, \alpha, \beta, w)$, which is

$\widehat{\text{HF}}(S^3) \cong \mathbb{F}_2$. The filtration level of the generator of $\mathbb{F}_2$ in the E^∞ page is the τ invariant, denoted by $\tau(K)$. This was introduced by Ozsváth and Szabó⁵⁶ and, independently, Rasmussen⁶⁶ with the notation $s(K)$.

The main result of this chapter, in joint work with Juhász³³, is that a decorated concordance $\mathcal{C}$ induces a non-vanishing homomorphism $F_{\mathcal{C}}$ on knot Floer homology that preserves the Alexander and homological gradings, and also induces a morphism of the corresponding spectral sequences. Recall that the Alexander grading and the homological (or Maslov) grading were introduced in Section 2.2.2. We denote by $\widehat{\text{HFK}}_j(K, P, i)$ the summand of $\widehat{\text{HFK}}(K, P)$ in Alexander grading i and homological grading j .

Theorem 4.2 (Juhász-M.³³, Theorem 1.2). *Let $\mathcal{C} = (X, F, \sigma)$ be a decorated concordance between decorated knots (K_0, P_0) and (K_1, P_1) , such that $H_1(X) = H_2(X) = 0$. Then*

$$F_{\mathcal{C}} \left(\widehat{\text{HFK}}_j(K_0, P_0, i) \right) \leq \widehat{\text{HFK}}_j(K_1, P_1, i)$$

for every $i, j \in \mathbb{Z}$.

Furthermore, given an admissible diagram $(\Sigma_r, \alpha_r, \beta_r, w_r, z_r)$ of (K_r, P_r) for $r \in \{0, 1\}$, there is a filtered chain map

$$f_{\mathcal{C}}: \widehat{\text{CF}}(\Sigma_0, \alpha_0, \beta_0, w_0) \rightarrow \widehat{\text{CF}}(\Sigma_1, \alpha_1, \beta_1, w_1)$$

of homological degree zero such that the induced morphism of spectral sequences agrees with $F_{\mathcal{C}}$ on the E^1 page and with $\text{id}_{\mathbb{F}_2}$ on the total homology and on the E^∞ page.

The fact that the map induced by a decorated concordance preserves both the Alexander and the Maslov gradings has been shown independently in a recent pre-print by Zemke⁸⁷, who uses a different strategy. His techniques are closer in spirit to Chapter 5 of this thesis. We will explain how our work relates to Zemke's work there.

Note that the fact that the map induced by a filtered map f on the total homology is an isomorphism in general does not imply that the map f^∞

induced between the E^∞ pages is also an isomorphism. As an example, consider a complex $C \cong \mathbb{F}_2$ in filtration level one, and a complex $\overline{C} \cong \mathbb{F}_2$ in filtration level zero. If $f: C \rightarrow \overline{C}$ is an isomorphism, then $H(f)$ is an isomorphism but f^∞ is not.

In the case of the filtered map f_C induced by a decorated concordance $\mathcal{C}$, the fact that f_C^∞ is an isomorphism follows from the fact that $\tau(K_0) = \tau(K_1)$, which was shown by Ozsváth and Szabó⁵⁶ (Theorem 1.1), and of which we give an alternative proof in Corollary 4.20, following a reasoning already used by Rasmussen⁶⁹ and Sarkar⁷¹ in different settings.

From Theorem 4.2 we also deduce that the concordance map F_C is always non-vanishing, since it is the map induced on the E^1 page in a morphism of spectral sequences where the E^∞ map is non-vanishing. We use this result to define an invariant of slice discs, which is discussed in Section 4.5.

Another application of Theorem 4.2, in combination with previous results by Ozsváth and Szabó⁵⁷, Ghiggini¹⁷, and Ni^{52,53}, is a new proof of an unpublished result of Ruberman, concerning invertible concordances, which will be discussed in Section 4.6.

We now outline the structure of the proof of Theorem 4.2, which is quite involved and will take most of this chapter.

Structure of the proof of Theorem 4.2. We will break the proof of the theorem in three main steps:

1. We first prove that the concordance map F_C in $\widehat{\text{HFK}}$ preserves the Alexander grading (Proposition 4.6). More precisely, we prove that a relative spin^c structure on $\mathcal{C}$ restricts to relative spin^c structures on $S^3(K_0, P_0)$ and $S^3(K_1, P_1)$ with “same” first Chern class (see Lemma 4.5 for a precise statement). This is done in Section 4.1. An independent proof of Proposition 4.6 will also be given in Chapter 5 (see Corollary 5.29).
2. By using Lemma 4.5, we are then able to prove that the chain map f_C

preserves the Alexander filtration and therefore induces a morphism of spectral sequences satisfying the properties stated above. This is done in Section 4.3 (see Theorems 4.19 and 4.21).

3. Lastly, by using the fact that the map induced in total homology is $\text{id}_{\mathbb{F}_2}$ (which is proven in Theorem 4.19), we prove that $F_{\mathcal{C}}$ preserves also the Maslov grading (see Theorem 4.40, in Section 4.4). $\square$

This chapter is organised as follows. In Section 4.1 we study spin^c structures on the sutured cobordism complementary to $\mathcal{C}$, and we prove that $F_{\mathcal{C}}$ preserves the Alexander grading. In Section 4.2 we recall spectral sequences and morphisms between them, and we establish some lemmas concerning the τ invariant in a more general setting. In Section 4.3 we show that $f_{\mathcal{C}}$ preserves the Alexander filtration, and therefore induces a morphism of spectral sequences. By applying the results in Section 4.2, we prove that τ is a concordance invariant, and that the map induced on the E^∞ page is the identity. In Section 4.4 we prove that $F_{\mathcal{C}}$ also preserves the homological grading. In Section 4.5 we note that $F_{\mathcal{C}}$ is non-vanishing, and discuss some consequences, including the definition of a new invariant of slice discs. In Section 4.6 we study applications to invertible concordances. Section 4.7 is devoted to a side topic, that is the relation between invertible concordances and imitations as defined by Kawauchi³⁶. Lastly, in Section 4.8 we extend the construction to other flavours.

4.1 DECORATED CONCORDANCES AND RELATIVE SPIN^c STRUCTURES

In this section we study relative spin^c structures on decorated concordances, which behave more nicely than for generic link cobordisms.

First of all, we prove the following lemma concerning 4-manifolds underlying decorated concordances.

Lemma 4.3 (Juhász-M.³³, Lemma 3.7). *Let X be an oriented cobordism from S^3 to S^3 . Then X has the same homology and cohomology as $S^3 \times I$ if and only if $H_1(X) = H_2(X) = 0$.*

Proof. The “only if” part is obvious. So suppose that $H_1(X) = H_2(X) = 0$. Then let $\bar{X}$ be the closed 4-manifold obtained by gluing two 4-balls to ∂X . We denote by $B \subset X$ the union of these 4-balls. Then, for $i \in \{1, 2\}$, we have

$$0 = H_i(X) \cong H^{4-i}(X, \partial X) \cong H^{4-i}(\bar{X}, B) \cong H^{4-i}(\bar{X}).$$

Here, the first isomorphism follows from Poincaré-Lefschetz duality, the second from excision, and the third from the cohomological long exact sequence of the pair $(\bar{X}, B)$. So $H^2(\bar{X}) = H^3(\bar{X}) = 0$, hence $H_1(\bar{X}) \cong H^3(\bar{X}) = 0$, and $H^1(\bar{X}) = \text{Hom}(H_1(\bar{X}), \mathbb{Z}) = 0$. As $\bar{X}$ has the same integral cohomology as S^4 , after removing two balls, X has the same integral homology and cohomology as $S^3 \times I$. $\square$

We now turn the attention to relative spin^c structures.

Lemma 4.4 (Juhász-M.³³, Lemma 3.8). *Let $\mathcal{C} = (X, F, \sigma)$ be a decorated concordance from (K_0, P_0) to (K_1, P_1) . If $(M_i, \gamma_i) = S^3(K_i, P_i)$ is the balanced sutured manifold complementary to (K_i, P_i) for $i \in \{0, 1\}$, and $\mathcal{W} = \mathcal{W}(\mathcal{C}) = (W, Z, [\xi])$ is the sutured manifold cobordism from (M_0, γ_0) to (M_1, γ_1) complementary to $\mathcal{C}$, then*

$$F_{\mathcal{W}} = \bigoplus_{\mathfrak{s}^\circ \in \text{Spin}^c(\mathcal{W})} F_{\mathcal{W}, \mathfrak{s}^\circ}. \quad (4.1)$$

Furthermore, $\text{Spin}^c(\mathcal{W})$ is an affine space over $H^2(W, \mathbb{Z}) \cong \mathbb{Z}$, and the restriction maps

$$r_i: \text{Spin}^c(\mathcal{W}) \longrightarrow \text{Spin}^c(M_i, \gamma_i)$$

are isomorphisms for $i \in \{0, 1\}$.

Proof. As in Remark 3.6, we write $\mathcal{W} = \mathcal{W}^s \circ \mathcal{W}^b$, where $\mathcal{W}^b$ is a boundary cobordism from (M_0, γ_0) to (N, γ_1) , where $N = M_0 \cup (-Z)$, and $\mathcal{W}^s$ is a special cobordism from (N, γ_1) to (M_1, γ_1) . As Z is a product, N is diffeomorphic to the knot complement $M_0 \approx S^3 \setminus \mathcal{N}(K_0)$, and hence $H_2(N) = 0$. So, by Remark 10.10 and Proposition 10.11 of Juhász’s paper³¹,

$$F_{\mathcal{W}} = \bigoplus_{\mathfrak{s}^\circ \in \text{Spin}^c(\mathcal{W})} F_{\mathcal{W}, \mathfrak{s}^\circ}.$$

As $H^k(Z, \partial M_1) = 0$ for $k \in \{1, 2\}$, by Juhász³¹ (Lemma 3.7) we conclude that $\text{Spin}^c(\mathcal{W})$ is affine over $H^2(W, \partial M_1)$. Of course, $H^2(W, \partial M_1) \cong H^2(W, \partial M_0) \cong H^2(W, Z)$. By excision, $H^2(W, Z) \cong H^2(X, \mathcal{N}(F))$, where $\mathcal{N}(F)$ is a regular neighbourhood of F . From the long exact sequence associated to the pair $(X, \mathcal{N}(F))$, the fact that $H^1(X) = H^2(X) = 0$, and since $H^1(\mathcal{N}(F)) \cong H^1(S^1) \cong \mathbb{Z}$, we obtain that $H^2(X, \mathcal{N}(F)) \cong \mathbb{Z}$.

The restriction maps

$$r_i: \text{Spin}^c(\mathcal{W}) \rightarrow \text{Spin}^c(M_i, \gamma_i)$$

for $i \in \{0, 1\}$ are modelled on the restriction maps

$$H^2(W, \partial M_i) \rightarrow H^2(M_i, \partial M_i)$$

for $i \in \{0, 1\}$. From the long exact sequence of the triple $(W, M_i, \partial M_i)$, the sequence

$$H^2(W, M_i) \rightarrow H^2(W, \partial M_i) \rightarrow H^2(M_i, \partial M_i) \rightarrow H^3(W, M_i) \quad (4.2)$$

is exact. Now consider the relative Mayer-Vietoris sequence of the pairs (W, M_i) and $(\mathcal{N}(F), \mathcal{N}(K_i))$, whose union is $(X, \partial_i X)$, where $\partial_i X \approx S^3$ is the ingoing boundary component of X when $i = 0$, and is the outgoing boundary component when $i = 1$:

$$H^k(X, \partial_i X) \rightarrow H^k(W, M_i) \oplus H^k(\mathcal{N}(F), \mathcal{N}(K_i)) \rightarrow H^k(Z, \partial M_i).$$

Here, $H^k(X, \partial_i X) \cong H^k(S^3 \times I, S^3 \times \{0\}) = 0$, and the last term is zero as Z deformation retracts onto ∂M_i . Consequently, $H^k(W, M_i) = 0$ for every k , and by the exact sequence (4.2), this means that the restriction maps r_i are isomorphisms for $i \in \{0, 1\}$. $\square$

In the following lemma, v_0 denotes any fixed vector field on a balanced sutured manifold (M, γ) obtained by restricting an admissible vector field to ∂M (see Definition 2.3 and Remark 2.4).

Lemma 4.5 (Juhász-M.³³, Lemma 3.9). *Let $\mathcal{C} = (X, F, \sigma)$ be a knot concor-*

dance from (K_0, P_0) to (K_1, P_1) . As in Lemma 4.4, let $(M_i, \gamma_i) = S^3(K_i, P_i)$ for $i \in \{0, 1\}$, and let

$$\mathcal{W} = \mathcal{W}(\mathcal{C}) = (W, Z, [\xi]).$$

For $i \in \{0, 1\}$, let S_i be a Seifert surface for K_i , and let t_i be the trivialisation of $v_0^\perp$ given by a vector field tangent to ∂M_i in the meridional direction. Then, for any relative spin^c structure $\mathfrak{s}^\circ \in \text{Spin}^c(\mathcal{W})$,

$$\langle c_1(r_0(\mathfrak{s}^\circ), t_0), [S_0] \rangle = \langle c_1(r_1(\mathfrak{s}^\circ), t_1), [S_1] \rangle, \quad (4.3)$$

where r_0 and r_1 are the restriction maps in Lemma 4.4.

From Lemma 4.5, we can already deduce the following proposition, which is the first step towards the proof of Theorem 4.2.

Proposition 4.6 (Juhász-M.³³, Proposition 3.10). *If $\mathcal{C}$ is a decorated concordance between two knots (K_0, P_0) and (K_1, P_1) , then the map induced between the knot Floer homologies preserves the Alexander grading, that is*

$$F_{\mathcal{C}} \left(\widehat{\text{HFK}}(K_0, P_0, i) \right) \leq \widehat{\text{HFK}}(K_1, P_1, i)$$

for every $i \in \mathbb{Z}$.

Proof. We use the same notation as in Lemmas 4.4 and 4.5. It follows from Lemma 4.4 that the map $F_{\mathcal{C}} = F_{\mathcal{W}}$ splits as the sum of the maps $F_{\mathcal{W}, \mathfrak{s}^\circ}$ for $\mathfrak{s}^\circ \in \text{Spin}^c(\mathcal{W})$, see Equation (4.1). It is therefore sufficient to check that, for every relative spin^c structure $\mathfrak{s}^\circ \in \text{Spin}^c(\mathcal{W})$, the map

$$F_{\mathcal{W}, \mathfrak{s}^\circ} : \text{SFH}(M_0, \gamma_0, \mathfrak{s}^\circ|_{M_0}) \longrightarrow \text{SFH}(M_1, \gamma_1, \mathfrak{s}^\circ|_{M_1})$$

preserves the Alexander grading.

According to the proof of Theorem 1.5 on page 333 of Juhász's paper²⁸, if t_i is the trivialisation of $v_0^\perp$ given by a vector field tangent to ∂M_i in the meridional direction, then

$$\text{SFH}(M_i, \gamma_i, \mathfrak{t}^\circ) = \widehat{\text{HFK}}(K_i, P_i, -\langle c_1(\mathfrak{t}^\circ, t_i), [S_i] \rangle / 2),$$

where S_i is a Seifert surface of K_i for $i \in \{0, 1\}$. The result now follows from Lemma 4.5, which states that

$$\langle c_1(\mathfrak{s}^\circ|_{M_0}, t_0), [S_0] \rangle = \langle c_1(\mathfrak{s}^\circ|_{M_1}, t_1), [S_1] \rangle. \quad \square$$

Proof of Lemma 4.5. Choose an admissible almost complex structure J on $W \setminus P$ whose homology class is $\mathfrak{s}^\circ$, where $P \subset \text{Int}(W)$ is a finite set of points, as in Definition 3.8. Let ξ_J be the field of complex tangencies of J along Z (as defined in Section 3.1.2). Then, by definition, $\mathfrak{s}_\xi^\circ = \mathfrak{s}_{\xi_J}^\circ$. In fact, we can choose J such that $\xi_J = \xi$. Choose a trivialisation of the normal S^1 -bundle of F whose total space is Z . If we identify F with $S^1 \times I$ such that σ maps to $Q_0 \times I$ for $Q_0 = \sigma \cap K_0$, then this identification, together with the above trivialisation, induces a diffeomorphism $d: Z \rightarrow S^1 \times S^1 \times I$, where the first factor is the fiber direction, and such that ξ is mapped to an I -invariant contact structure with dividing set $S^1 \times P_0 \times \{a\}$ on $S^1 \times S^1 \times \{a\}$ for every $a \in I$, and $\{\vartheta\} \times Q_0 \times I$ on $\{\vartheta\} \times S^1 \times I$ for every $\vartheta \in S^1$. Hence, we can perturb the 2-plane field ξ such that it is always tangent to the second S^1 factor; i.e., the longitudinal direction. So we can choose J such that ξ_J is also invariant in the I direction, and it contains the longitudinal direction. If v is a nowhere zero section of ξ_J tangent to the longitudinal direction, then—under a homotopy of $\xi_J|_{\partial M_i}$ to $v_0^\perp$ through admissible 2-plane fields—the vector field $v|_{\partial M_0}$ represents a trivialisation τ_0 that corresponds to t_0 and $v|_{\partial M_1}$ represents a trivialisation τ_1 that corresponds to t_1 .

The 2-plane field ξ_J , together with the trivialisation given by v , gives a complex 1-dimensional subbundle of $(TW|_Z, J)$ together with a trivialisation. The complement of ξ_J is also trivial, canonically trivialised by its intersection with TZ , which then gives rise to a trivialisation τ of $TW|_Z$. As J is defined over the 3-skeleton of W , it makes sense to talk about the relative Chern class $c_1(TW, J, \tau) \in H^2(W, Z)$. If ξ_J^i denotes the field of complex tangencies of J along M_i , then the complement of ξ_J^i is a trivial bundle (trivialised by its intersection with TM_i), so

$$c_1(\xi_J^i, \tau_i) = c_1(TW|_{M_i}, J, \tau) = c_1(TW, J, \tau)|_{M_i},$$

where the second equality follows from the naturality of Chern classes. By construction, ξ_J^i represents $\mathfrak{s}_i^\circ = r_i(\mathfrak{s}^\circ)$.

Recall that S_i is a Seifert surface of K_i for $i \in \{0, 1\}$. Note that $H_2(W, \mathbb{Z}) \cong \mathbb{Z}$, and that there is a bilinear intersection pairing

$$H_2(W, \mathbb{Z}) \otimes H_2(W, M_0 \cup M_1) \rightarrow \mathbb{Z}.$$

Consider the cycle $m = S^1 \times \{\text{pt}\} \times I$ in $C_2(W, M_0 \cup M_1)$. As both S_0 and S_1 intersect m once positively, they both represent the generator of $H_2(W, \mathbb{Z}) \cong \mathbb{Z}$.

Hence

$$\begin{aligned} \langle c_1(\mathfrak{s}_0^\circ, \tau_0), [S_0] \rangle &= \langle c_1(TW, J, \tau), [S_0] \rangle \\ &= \langle c_1(TW, J, \tau), [S_1] \rangle \\ &= \langle c_1(\mathfrak{s}_1^\circ, \tau_1), [S_1] \rangle, \end{aligned}$$

and Equation (4.3) follows as we saw that τ_0 corresponds to t_0 and τ_1 corresponds to t_1 . $\square$

4.2 FILTERED COMPLEXES AND SPECTRAL SEQUENCES

In this section, we briefly recall the definitions and properties of spectral sequences that we need. We mainly refer to the book of McCleary⁵¹. The spectral sequences we are interested in arise from filtered chain complexes, so we focus on this case only.

Definition 4.7. A *filtered chain complex* is a chain complex

$$(C = \bigoplus_{k \in \mathbb{Z}} C_k, \partial),$$

such that $\partial C_k \subseteq C_{k-1}$, and with is a nested sequence of subcomplexes

$$\dots \subseteq \mathcal{F}_{p-1}C \subseteq \mathcal{F}_pC \subseteq \mathcal{F}_{p+1}C \subseteq \dots$$

such that $\bigcup_{p \in \mathbb{Z}} \mathcal{F}_p C = C$ and $\partial(\mathcal{F}_p C) \subseteq \mathcal{F}_p C$.

We say that the filtered chain complex is *bounded* if there are integers $a \leq b$ such that

$$\{0\} = \mathcal{F}_a C \subseteq \dots \subseteq \mathcal{F}_b C = C.$$

We obtain a spectral sequence from a filtered chain complex as follows (see also the proof of Theorem 2.6 in McCleary's book⁵¹).

Definition 4.8. For $p, q, r \in \mathbb{Z}$, we define

$$\begin{aligned} Z_{p,q}^r &= \mathcal{F}_p C_{p+q} \cap \partial^{-1}(\mathcal{F}_{p-r} C_{p+q-1}), \\ B_{p,q}^r &= \mathcal{F}_p C_{p+q} \cap \partial(\mathcal{F}_{p+r} C_{p+q+1}), \\ Z_{p,q}^\infty &= \mathcal{F}_p C_{p+q} \cap \ker \partial, \\ B_{p,q}^\infty &= \mathcal{F}_p C_{p+q} \cap \operatorname{im} \partial. \end{aligned}$$

For $0 \leq r \leq \infty$, the r -page (or r -term) is the complex $(E^r = \bigoplus_{p,q \in \mathbb{Z}} E_{p,q}^r, \partial^r)$, where

$$E_{p,q}^r = \frac{Z_{p,q}^r}{Z_{p-1,q+1}^{r-1} + B_{p,q}^{r-1}},$$

and the differential

$$\partial^r : E_{p,q}^r \rightarrow E_{p-r,q+r-1}^r$$

is induced by the differential ∂ on the complex C .

Sometimes we only focus on the p grading. In such cases, we drop q from the notation, and write $E_p^r = \bigoplus_{q \in \mathbb{Z}} E_{p,q}^r$. For the following, see the proof of Theorem 2.6 in McCleary's book⁵¹.

Theorem 4.9. The pages $\{(E^r, \partial^r)\}$ induced by a filtered chain complex form a spectral sequence; i.e.,

$$E_{p,q}^{r+1} = H_{p,q}(E_{*,*}^r, \partial^r) := \frac{\ker(\partial^r|_{E_{p,q}^r})}{\operatorname{im}(\partial^r|_{E_{p+r,q-r+1}^r})}.$$

If the filtration is bounded, then there is a canonical isomorphism

$$E_{p,q}^\infty \cong \frac{\mathcal{F}_p(H_{p+q}(C))}{\mathcal{F}_{p-1}(H_{p+q}(C))},$$

where the filtration on the total homology $H(C) = \bigoplus_{k \in \mathbb{Z}} H_k(C)$ is the one induced from C :

$$\mathcal{F}_p(H(C)) := \text{im} \left(H(\mathcal{F}_p C, \partial|_{\mathcal{F}_p C}) \rightarrow H(C, \partial) \right).$$

Remark 4.10. Notice that $E_{p,q}^0$ is the graded module

$$\frac{\mathcal{F}_p C_{p+q}}{\mathcal{F}_{p-1} C_{p+q}}$$

associated with the filtration. The page $E_{p,q}^1$ is the homology $H_q(E_{p,*}^0, \partial^0)$ of the associated graded module with the induced differential.

4.2.1 MORPHISMS OF SPECTRAL SEQUENCES

According to McCleary⁵¹, we have the following.

Definition 4.11. Let (E^r, ∂^r) and $(\overline{E}^r, \overline{\partial}^r)$ be spectral sequences. A *morphism of spectral sequences* is a sequence of module homomorphisms

$$f^r : E_{*,*}^r \rightarrow \overline{E}_{*,*}^r$$

for $r \in \mathbb{N}$, of bidegree $(0, 0)$, such that f^r commutes with the differentials; i.e., $f^r \circ \partial^r = \overline{\partial}^r \circ f^r$, and each f^{r+1} is induced by f^r on homology; i.e., f^{r+1} is the composite

$$f^{r+1} : E_{*,*}^{r+1} \cong H(E_{*,*}^r, \partial^r) \xrightarrow{H(f^r)} H(\overline{E}_{*,*}^r, \overline{\partial}^r) \cong \overline{E}_{*,*}^{r+1}.$$

Remark 4.12. Let $f : C \rightarrow \overline{C}$ be a map of filtered complexes of homological degree zero; i.e.,

- $f(C_k) \subseteq \overline{C}_k$;

- $f \circ \partial = \bar{\partial} \circ f$; and
- $f(\mathcal{F}_p C) \subseteq \mathcal{F}_p \bar{C}$.

Then f induces a morphism between the spectral sequences associated to C and $\bar{C}$.

Remark 4.13. If (E^r, ∂^r) and $(\bar{E}^r, \bar{\partial}^r)$ are bounded spectral sequences,

$$\{f^r : E^r \rightarrow \bar{E}^r\}$$

is a morphism of spectral sequences, and f^∞ is non-zero on $E_{p,q}^\infty$, then f^r is non-zero on $E_{p,q}^r$ for all $r \in \mathbb{N}$.

4.2.2 THE τ INVARIANT

In this subsection, we recall the definition and few properties of the τ invariant^{56,66}, and we discuss it in a slightly more general setting.

Definition 4.14. If C is a non-acyclic bounded filtered complex over $\mathbb{F}_2$, we define

$$\tau(C) := \min \{p \in \mathbb{Z} : H(\mathcal{F}_p C) \rightarrow H(C) \text{ is non-trivial}\}.$$

Definition 4.14 generalises the Ozsváth-Szabó τ invariant in the sense that, if $C = \widehat{\text{CF}}(\mathcal{H})$ for some Heegaard diagram for a decorated knot (K, P) , then $\tau(C) = \tau(K)$.

Remark 4.15. An alternative definition of $\tau(C)$ is given by the following property:

$$E_p^\infty(C) \begin{cases} = 0 & \text{if } p < \tau(C) \\ \neq 0 & \text{if } p = \tau(C). \end{cases}$$

Furthermore, if the total homology $H(C) = \mathbb{F}_2$, then

$$E_p^\infty(C) \begin{cases} = 0 & \text{if } p \neq \tau(C) \\ \neq 0 & \text{if } p = \tau(C). \end{cases}$$

The proof of the following lemma is already contained in the proofs of Theorem 1 in Rasmussen's paper⁶⁹ and of Theorem 3.4 of Sarkar's paper⁷¹. We decide to make it explicit here since it has appeared in several settings.

Lemma 4.16. *Let $f: C \rightarrow \overline{C}$ be a filtered map of degree zero between bounded filtered complexes over $\mathbb{F}_2$ such that*

1. *the total homologies $H(C) = H(\overline{C}) = \mathbb{F}_2$; and*
2. *$H(f)$ is the identity map in total homology.*

Then $\tau(C) \geq \tau(\overline{C})$.

Proof. Let $c \in \mathcal{F}_{\tau(C)}C$ be a cycle that represents $1 \in H(C)$. Then $f(c) \in \mathcal{F}_{\tau(C)}\overline{C}$ is also a cycle and also represents $1 \in H(\overline{C})$ (because the map $H(f)$ is the identity). It follows that $\tau(\overline{C}) \leq \tau(C)$. $\square$

We conclude the section with a technical lemma that we will use to prove that a decorated concordance induces a non-trivial map between the E^∞ pages of the spectral sequences arising from the knot filtrations.

Lemma 4.17 (Juhász-M.³³, Lemma 4.10). *Let $f: C \rightarrow \overline{C}$ be a filtered map of degree zero between bounded filtered complexes over $\mathbb{F}_2$ such that*

1. *$H(C) \cong \mathbb{F}_2$ and $H(\overline{C}) \cong \mathbb{F}_2$;*
2. *$\tau(C) = \tau(\overline{C})$; and*
3. *$H(f): H(C) \rightarrow H(\overline{C})$ is an isomorphism.*

Then $E_\tau^\infty(C) \cong \mathbb{F}_2$, $E_\tau^\infty(\overline{C}) \cong \mathbb{F}_2$, and the map $f^\infty: E_\tau^\infty(C) \rightarrow E_\tau^\infty(\overline{C})$ is also an isomorphism.

Proof. Since 1. and 2. hold, by Theorem 4.9 and Definition 4.14, there are canonical isomorphisms

$$E_\tau^\infty(C) \cong H(C) \cong \mathbb{F}_2 \quad \text{and} \quad E_\tau^\infty(\overline{C}) \cong H(\overline{C}) \cong \mathbb{F}_2.$$

The commutativity of the following diagram concludes the proof.

$$\begin{array}{ccc} E_\tau^\infty(C) & \xrightarrow{f^\infty} & E_\tau^\infty(\overline{C}) \\ \downarrow \text{\tiny 15} & & \downarrow \text{\tiny 15} \\ H(C) & \xrightarrow[H(f)]{\simeq} & H(\overline{C}) \end{array}$$

□

4.3 CONCORDANCE MAPS PRESERVE THE KNOT FILTRATION

4.3.1 THE KNOT FILTRATION

Let (K, P) denote a null-homologous decorated knot in a closed oriented 3-manifold Y . Ozsváth and Szabó⁵⁸ and, independently, Rasmussen⁶⁶ proved that (K, P) gives rise to a filtration of the Heegaard Floer chain complex $\widehat{\text{CF}}(Y)$, well-defined up to filtered chain homotopy equivalence, called the *knot filtration*, or the *Alexander filtration*, since it can be defined in terms of the Alexander grading.

Let S be a Seifert surface for the knot K , and let $(\Sigma, \alpha, \beta, w, z)$ be a doubly pointed Heegaard diagram for (K, P) . Given a generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we already defined its S -Alexander grading in (2.2) as

$$\mathcal{A}_S(\mathbf{x}) = \frac{1}{2} \langle c_1(\mathfrak{s}^\circ(\mathbf{x}), t), [S] \rangle.$$

Recall that if Y is a $\mathbb{Q}HS^3$, the Alexander grading does not depend on S , and we simply denote it by $\mathcal{A}(\mathbf{x})$, and the filtration by $\mathcal{F}(\mathbf{x})$.

The following lemma describes how the relative Alexander grading can be read off the Heegaard diagram; see Ozsváth and Szabó⁵⁸ (Lemma 2.5), and Rasmussen⁶⁶ (page 25).

Lemma 4.18. *Let $(\Sigma, \alpha, \beta, w, z)$ be a doubly-pointed Heegaard diagram for a null-homologous decorated knot (K, P) in a 3-manifold Y , and let S be a Seifert surface for K . If $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, then*

$$n_z(\phi) - n_w(\phi) = \mathcal{A}_S(\mathbf{x}) - \mathcal{A}_S(\mathbf{y}).$$

Here $n_z(\phi)$ and $n_w(\phi)$ denote the multiplicities of the domain associated to ϕ at z and w .

As explained in Section 4.2, the Alexander filtration defines a spectral sequence whose first page is $\widehat{\text{HFK}}(K, P)$ and whose abutment is $\widehat{\text{HF}}(S^3)$:

$$\widehat{\text{HFK}}(K, P) \Rightarrow \widehat{\text{HF}}(S^3).$$

4.3.2 KNOT FILTRATION AND CONCORDANCES

In the following theorem we prove that the knot filtration is preserved by the chain maps induced by concordances.

Theorem 4.19 (Juhász-M.³³, Theorem 5.4). *Let $\mathcal{C}$ be a decorated concordance from (K_0, P_0) to (K_1, P_1) , and let $(\Sigma_i, \alpha_i, \beta_i, w_i, z_i)$ be a doubly pointed diagram representing (K_i, P_i) for $i \in \{0, 1\}$. Then there is a chain map*

$$f_{\mathcal{C}}: \widehat{\text{CF}}(\Sigma_0, \alpha_0, \beta_0, w_0, z_0) \rightarrow \widehat{\text{CF}}(\Sigma_1, \alpha_1, \beta_1, w_1, z_1)$$

preserving the knot filtration; i.e., for every generator $\mathbf{x} \in \mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0}$,

$$\mathcal{A}(f_{\mathcal{C}}(\mathbf{x})) \leq \mathcal{A}(\mathbf{x}),$$

such that $f_{\mathcal{C}}$ induces the identity of $\widehat{\text{HF}}(S^3)$ on the total homology, and $F_{\mathcal{C}}$ on the homology of the associated graded complexes.

As a first application of Theorem 4.19 we obtain a new conceptual proof of the fact that τ is a concordance invariant.

Corollary 4.20 (Juhász-M.³³). *τ is a (decorated) concordance invariant.*

Proof. Given concordant knots K_0 and K_1 , choose arbitrary decorations on them and on a concordance between them. By Theorem 4.19 we have a filtered map $f_{\mathcal{C}}$ between filtered chain complexes associated to (Heegaard diagrams of) (K_0, P_0) and (K_1, P_1) , which is the identity in total homology. Lemma 4.16 then implies that $\tau(K_1) \leq \tau(K_0)$. By swapping the roles of K_0 and K_1 we get the other inequality. $\square$

Note that the proof of Corollary 4.20 works for concordances in a homology $S^3 \times I$ with boundary $(-S^3) \sqcup S^3$.

Theorem 4.19 yields a morphism of spectral sequences in the sense of Definition 4.11, hence we have the following corollary.

Theorem 4.21 (Juhász-M.³³, Theorem 5.5). *Let $\mathcal{C}$ be a decorated concordance from (K_0, P_0) to (K_1, P_1) . Then the map $f_{\mathcal{C}}$ from Theorem 4.19 induces a morphism of spectral sequences from $\widehat{\text{HFK}}(K_0, P_0) \Rightarrow \widehat{\text{HF}}(S^3)$ to $\widehat{\text{HFK}}(K_1, P_1) \Rightarrow \widehat{\text{HF}}(S^3)$ such that the map induced on the E^1 page is $F_{\mathcal{C}}$, and the map induced on the E^∞ page is $\text{id}_{\widehat{\text{HF}}(S^3)}$.*

Proof. Suppose that $\mathcal{C} = (X, F, \sigma)$. Since $H_1(X) = H_2(X) = 0$, it follows from the work of Ozsváth and Szabó⁵⁶ (Theorem 1.1), or from Corollary 4.20, that $\tau(K_0) = \tau(K_1)$. By Theorem 4.19, we have a filtered map $f_{\mathcal{C}}$ that induces an isomorphism on the total homology. We can therefore apply Lemma 4.17 to conclude that the map induced on the E^∞ page is also an isomorphism. $\square$

We now turn to the proof of Theorem 4.19, which will take the rest of Section 4.3.

4.3.3 TRIVIALITY OF THE GLUING MAP

As explained in Section 3.1.3, given a sutured manifold cobordism $\mathcal{W} = (W, Z, [\xi])$ from (M_0, γ_0) to (M_1, γ_1) , the map

$$F_{\mathcal{W}}: \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(M_1, \gamma_1)$$

is the composition $F_{\mathcal{W}^s} \circ \Phi_{-\xi}$, where

$$\Phi_{-\xi}: \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(N, \gamma_1)$$

is the gluing map of Honda, Kazez, and Matić²⁴ for the sutured submanifold $(-M_0, -\gamma_0)$ of $(-N, -\gamma_1)$ with $N = M_0 \cup (-Z)$, and $F_{\mathcal{W}^s}$ is the surgery map corresponding to handles attached along the *interior* of N .

If $\mathcal{C} = (X, F, \sigma)$ is a decorated concordance from (K_0, P_0) to (K_1, P_1) , let

$$\mathcal{W} = \mathcal{W}(\mathcal{C}): (M_0, \gamma_0) \rightarrow (M_1, \gamma_1)$$

be the complementary sutured manifold cobordism, where

$$(M_0, \gamma_0) = S^3(K_0, P_0) \text{ and } (M_1, \gamma_1) = S^3(K_1, P_1).$$

Let $T^2 \times I$ be a collar neighbourhood of ∂M_0 such that $T^2 \times \{1\}$ is identified with ∂M_0 . Since the dividing set on F consists of two arcs connecting the two components of ∂F , there is a diffeomorphism $d: T^2 \times I \rightarrow Z$ such that $\xi' = d^*(\xi)$ is an I -invariant contact structure on $T^2 \times I$, and hence induces the trivial gluing map as proven by Honda, Kazez, and Matić²⁴ (Theorem 6.1). More precisely, if we write $M'_0 = \overline{M_0 \setminus (T^2 \times I)}$ and γ'_0 for the projection of γ_0 to $T^2 \times \{0\}$, then there is a diffeomorphism $\varphi: (M'_0, \gamma'_0) \rightarrow (M_0, \gamma_0)$ supported in a neighbourhood of $T^2 \times \{0\}$ such that

$$\Phi_{-\xi'} = \varphi_*: \text{SFH}(M'_0, \gamma'_0) \rightarrow \text{SFH}(M_0, \gamma_0).$$

Let $D: M_0 \rightarrow N$ be the diffeomorphism that agrees with φ on M'_0 and with d on $T^2 \times I$, smoothed along $T^2 \times \{0\}$. By the diffeomorphism invariance of the gluing construction, the diagram

$$\begin{array}{ccc} \text{SFH}(M'_0, \gamma'_0) & \xrightarrow{\varphi_*} & \text{SFH}(M_0, \gamma_0) \\ \downarrow \Phi_{-\xi'} & & \downarrow \Phi_{-\xi} \\ \text{SFH}(M_0, \gamma_0) & \xrightarrow{D_*} & \text{SFH}(N, \gamma), \end{array}$$

is commutative, hence $\Phi_{-\xi} = D_*$.

We now show that D_* preserves the Alexander grading on the chain level. If we glue $D^2 \times S^1$ to N along ∂N such that the meridian is glued to a suture in $s(\gamma_1)$, we obtain a 3-manifold Y diffeomorphic to S^3 , and the image of $\{0\} \times S^1$ is a knot K' in Y . We can canonically extend D to a diffeomorphism from (S^3, K_0) to (Y, K') . Given a knot diagram $\mathcal{H}_0 = (\Sigma_0, \alpha_0, \beta_0, w_0, z_0)$ for (S^3, K_0) , its image $D(\mathcal{H}_0)$ is a diagram of (Y, K') .

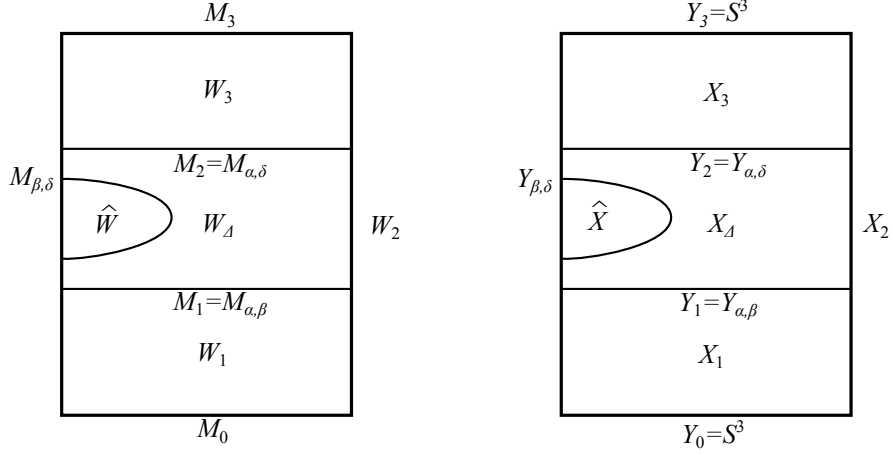


Figure 4.1: The left-hand side shows the sutured cobordism $\mathcal{W} = (W, Z, [\xi])$, and how we split it into different pieces. The picture on the right-hand side shows the cobordism of 3-manifolds X , and the corresponding decomposition into smaller cobordisms.

Given a Seifert surface S of K_0 and a generator $\mathbf{x} \in \mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0}$, the image $D(S)$ is a Seifert surface of K' , and $D(\mathbf{x})$ satisfies

$$\langle c_1(\mathfrak{s}^\circ(\mathbf{x}), t), [S] \rangle = \langle c_1(\mathfrak{s}^\circ(D(\mathbf{x})), D_*(t)), [D(S)] \rangle.$$

As $D(\gamma_0) = \gamma_1$, the trivialisation $D_*(t)$ points in the meridional direction for K' , and it follows that $\mathcal{A}(\mathbf{x}) = \mathcal{A}(D(\mathbf{x}))$. It is apparent from the above discussion that we can identify (S^3, K_0) and (Y, K') via D , so from now on we will think of $\mathcal{W}$ as a special cobordism from (S^3, K_0) to (S^3, K_1) .

4.3.4 NOTATION

In this subsection, we fix the notation for the rest of the section. Recall that (K_0, P_0) and (K_1, P_1) denote two decorated knots in S^3 , and that we have a decorated concordance $\mathcal{C} = (X, F, \sigma)$ from (K_0, P_0) to (K_1, P_1) .

We denote by $\mathcal{W} = (W, Z, [\xi])$ the sutured cobordism $\mathcal{W}(\mathcal{C})$ associated to the knot concordance $\mathcal{C}$. It follows from the discussion in Section 4.3.3 that $\mathcal{W}$ can be thought of as a special cobordism. The 4-manifold W can be obtained by attaching to $M_0 \times I$ along the interior of $M_0 \times \{1\}$ a sequence

of 4-dimensional 1-handles, followed by 2-handles, and finally 3-handles. We denote the number of i -handles by c_i for $i \in \{1, 2, 3\}$, and often write p for c_1 and ℓ for c_2 . We split the cobordism $\mathcal{W}$ into three parts $\mathcal{W}_1$, $\mathcal{W}_2$, and $\mathcal{W}_3$, in such a way that $\mathcal{W}_i = (W_i, Z_i, [\xi_i])$ is a cobordism from (M_{i-1}, γ_{i-1}) to (M_i, γ_i) , and is the trace of the i -handle attachments; see the left-hand side of Figure 4.1. Notice that, by construction, $(M_0, \gamma_0) = S^3(K_0, P_0)$ and $(M_3, \gamma_3) = S^3(K_1, P_1)$.

In order to represent sutured manifolds, we use doubly-pointed Heegaard diagrams. If $w, z \in \Sigma \setminus (\alpha \cup \beta)$, the Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ represents the complement of a knot in a 3-manifold (recall Remark 2.23).

Let $\mathcal{T} = (\Sigma, \alpha, \beta, \delta, w, z)$ be a doubly-pointed triple diagram for the cobordism $\mathcal{W}_2$ (see Section 4.3.8), where $|\alpha| = |\beta| = |\delta| = d$. Furthermore, suppose that the 2-handles are attached along an ℓ -component framed link $\mathbb{L}$. Then we further split the manifold W_2 into two pieces according to Juhász³¹ (Proposition 6.6): The piece $\mathcal{W}_{\alpha, \beta, \delta} = (W_\Delta, Z_\Delta, \xi_\Delta)$ denotes the sutured manifold cobordism obtained from the triangle construction in Juhász paper³¹ (Sections 5 and 6), while $\mathcal{W}_\beta(\mathbb{L}) = (\widehat{W}, \widehat{Z}, \widehat{\xi})$ is a sutured manifold cobordism from

$$(R_+(\gamma_1), \partial R_+(\gamma_1) \times I) \# (\#_{i=1}^{d-\ell} (S^2 \times S^1))$$

to the empty set. The horizontal boundary of $\widehat{W}$ is the sutured manifold $M_{\beta, \delta}$, defined by the diagram $(\Sigma, \beta, \delta, w, z)$. By analogy, we also use the notation $M_{\alpha, \beta} \cong (M_1, \gamma_1)$ and $M_{\alpha, \delta} \cong (M_2, \gamma_2)$.

We can fill in the vertical boundary of the sutured cobordism $\mathcal{W}$ by gluing $D^2 \times S^1 \times I$ along $S^1 \times S^1 \times I$ to Z such that $S^1 \times \{(1, 0)\}$ is glued to a meridian of K_0 to obtain cobordisms of closed 3-manifolds rather than knot complements. In terms of Heegaard diagrams, this amounts to forgetting the z basepoints. We denote the closed 3-manifolds by the letter Y rather than M . As for the cobordisms, we use the letter X instead of the letter W . See the right-hand side of Figure 4.1.

Lastly, let $S_0 \subseteq M_0$ and $S_3 \subseteq M_3$ be Seifert surfaces for K_0 and K_1 , respectively. Since (M_1, γ_1) is obtained from (M_0, γ_0) by taking connected sums

with copies of $S^1 \times S^2$, the surface S_0 also defines a surface $S_1 \subseteq M_1$, which is contained in the M_0 summand of M_1 . Analogously, the Seifert surface S_3 induces a Seifert surface $S_2 \subseteq M_2$.

4.3.5 DEFINITION OF THE CHAIN MAP $f_{\mathcal{C}}$

We now define the chain map $f_{\mathcal{C}}$. Given an admissible doubly-pointed diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ for a decorated knot (Y, K, P) , we denote by $\widehat{\text{CF}}(\mathcal{H})$ the Heegaard Floer chain complex that counts discs avoiding w and filtered by z . Its homology is $\widehat{\text{HF}}(Y, w)$, while the homology of the associated graded complex $\widehat{\text{CFK}}(\mathcal{H})$ is $\widehat{\text{HFK}}(Y, K, P)$.

Suppose that the 1-handles are attached along p framed pairs of points $\mathbb{P} \subset M_0$. Pick an admissible diagram $\mathcal{H}^0$ of (M_0, γ) subordinate to $\mathbb{P}$, and let

$$f_{\mathcal{H}^0, \mathbb{P}}: \widehat{\text{CF}}(\mathcal{H}^0) \rightarrow \widehat{\text{CF}}(\mathcal{H}_{\mathbb{P}}^0)$$

be the 1-handle map defined by Juhász³¹ (Definition 7.5). The 2-handles are attached along an ℓ -component framed link $\mathbb{L} \subset M_1$. Choose an admissible diagram $\mathcal{H}^1$ subordinate to $\mathbb{L}$, and let

$$f_{\mathcal{H}^1, \mathbb{L}}: \widehat{\text{CF}}(\mathcal{H}^1) \rightarrow \widehat{\text{CF}}(\mathcal{H}_{\mathbb{L}}^1)$$

be the 2-handle map defined by Juhász³¹ (Definition 6.8), on the chain level. This map counts triangles that avoid w but might pass through z . Finally, let $\mathcal{H}^2$ be an admissible diagram of (M_2, γ) subordinate to framed spheres $\mathbb{S} \subset M_2$ corresponding to the 3-handles. The corresponding 3-handle map

$$f_{\mathcal{H}^2, \mathbb{S}}: \widehat{\text{CF}}(\mathcal{H}^2) \rightarrow \widehat{\text{CF}}(\mathcal{H}_{\mathbb{S}}^2)$$

was introduced in Definition 7.8 of Juhász's paper³¹.

As we explained in Section 2.1.3, given admissible diagrams $\mathcal{H}$ and $\mathcal{H}'$ of a sutured manifold (M, γ) , there is a canonical isomorphism

$$F_{\mathcal{H}, \mathcal{H}'}: \text{SFH}(\mathcal{H}) \rightarrow \text{SFH}(\mathcal{H}').$$

We can obtain a chain level representative by connecting $\mathcal{H}$ and $\mathcal{H}'$ through a sequence of ambient isotopies, (de)stabilisations, and equivalences of the attaching sets. If (M, γ) is complementary to a knot (Y, K) , we can view this as a sequence of moves on knot diagrams. Each induces a chain homotopy equivalence on $\widehat{\text{CF}}$ preserving the knot filtration according to Ozsváth and Szabó⁵⁸, and Rasmussen⁶⁶, and induces an isomorphism on both the homology of the whole complex (isomorphic to $\widehat{\text{HF}}(Y)$), and the homology of the associated graded complex (isomorphic to $\widehat{\text{HFK}}(Y, K)$). Note that the triangle maps corresponding to changing the set of attaching curves do not pass over w but might cross z , so they are in fact naturality maps for the closed 3-manifold and *not* the knot. Juhász and Thurston³⁴ proved that the maps on the homology are independent of the sequence of moves connecting $\mathcal{H}$ and $\mathcal{H}'$. We write $f_{\mathcal{H}, \mathcal{H}'}$ for the chain level representative of $F_{\mathcal{H}, \mathcal{H}'}$ described above. With the above notation in place, we set

$$f_{\mathcal{C}} := f_{\mathcal{H}^2, \mathbb{S}} \circ f_{\mathcal{H}_{\mathbb{L}}^1, \mathcal{H}^2} \circ f_{\mathcal{H}^1, \mathbb{L}} \circ f_{\mathcal{H}_{\mathbb{P}}^0, \mathcal{H}^1} \circ f_{\mathcal{H}^0, \mathbb{P}}, \quad (4.4)$$

from $\widehat{\text{CF}}(\mathcal{H}^0)$ to $\widehat{\text{CF}}(\mathcal{H}_{\mathbb{S}}^2)$. Note that in each of the diagrams involved in the construction above we can distinguish z and w . If we are given diagrams $\mathcal{H}$ of (M_0, γ_0) and $\mathcal{H}'$ of (M_3, γ_3) , then we have to pre- and post-compose the above map $f_{\mathcal{C}}$ with $f_{\mathcal{H}_{\mathbb{S}}^2, \mathcal{H}'}$ and $f_{\mathcal{H}, \mathcal{H}^0}$.

We split the proof of Theorem 4.19 into a number of steps, and we prove that for each $\mathcal{W}_i$ the knot filtration is preserved.

4.3.6 1- AND 3-HANDLES

First, consider the case of the 1-handle attachments along the framed pairs of points $\mathbb{P} \subset \text{Int}(M_0)$. As in Section 4.3.4, we write $\mathcal{W}_1 := \mathcal{W}(\mathbb{P})$ for the trace of the surgery along $\mathbb{P}$; this is a cobordism from (M_0, γ_0) to (M_1, γ_1) . As explained by Juhász³¹ (Section 7), there is an isomorphism

$$\text{Spin}^c(\mathcal{W}_1) \cong \text{Spin}^c(M_0, \gamma_0).$$

Furthermore, a spin^c structure $\mathfrak{s}^\circ \in \text{Spin}^c(M_1, \gamma_1)$ extends to $\mathcal{W}_1$ if and only if $c_1(\mathfrak{s}^\circ)$ vanishes on the belt spheres of all the 1-handles. Given $\mathfrak{s}^\circ \in \text{Spin}^c(\mathcal{W})$, we write $\mathfrak{s}_0^\circ$ for its restriction to (M_0, γ_0) , and $\mathfrak{s}_1^\circ$ for its restriction to (M_1, γ_1) .

Lemma 4.22 (Juhász-M.³³, Lemma 5.8). *Let $\mathfrak{s}_0^\circ \in \text{Spin}^c(M_0, \gamma_0)$, and let $\mathfrak{s}_1^\circ \in \text{Spin}^c(M_1, \gamma_1)$ denote the corresponding spin^c structure. Then*

$$\langle c_1(\mathfrak{s}_0^\circ, t), [S_0] \rangle = \langle c_1(\mathfrak{s}_1^\circ, t), [S_1] \rangle.$$

Proof. This is a consequence of the naturality of the first Chern class and the fact that both S_0 and S_1 are actually contained in $M_0 \setminus \mathcal{N}(\mathbb{P})$. We can suppose that S_0 is properly embedded in $M_0 \setminus \mathcal{N}(\mathbb{P})$. By definition, S_1 is a surface contained in $M_0 \setminus \mathcal{N}(\mathbb{P}) \subseteq M_1$ that is isotopic to S_0 in $M_0 \setminus \mathcal{N}(\mathbb{P})$.

Since S_0 and S_1 are isotopic in $M_0 \setminus \mathcal{N}(\mathbb{P})$ and $\mathfrak{s}_1^\circ|_{M_0 \setminus \mathcal{N}(\mathbb{P})} = \mathfrak{s}_0^\circ|_{M_0 \setminus \mathcal{N}(\mathbb{P})}$, by the naturality of the first Chern class

$$\begin{aligned} \langle c_1(\mathfrak{s}_1^\circ, t), [S_1] \rangle &= \langle c_1(\mathfrak{s}_1^\circ|_{M_0 \setminus \mathcal{N}(\mathbb{P})}, t), [S_1] \rangle \\ &= \langle c_1(\mathfrak{s}_0^\circ|_{M_0 \setminus \mathcal{N}(\mathbb{P})}, t), [S_0] \rangle \\ &= \langle c_1(\mathfrak{s}_0^\circ, t), [S_0] \rangle. \end{aligned}$$

Notice that the trivialisation t of the vector field v_0 on $\partial M_0 = \partial M_1$ does not change because the boundary is left unaffected by the surgery. $\square$

Remark 4.23. Since $c_1(\mathfrak{s}_1^\circ, t)$ vanishes on the belt spheres of the 1-handles, the above result also holds for an arbitrary Seifert surface S_1 .

Corollary 4.24 (Juhász-M.³³, Corollary 5.10). *The map induced by the attachment of the 1-handles $f_{\mathcal{H}^0, \mathbb{P}} : \widehat{\text{CF}}(\mathcal{H}^0) \rightarrow \widehat{\text{CF}}(\mathcal{H}_{\mathbb{P}}^0)$ preserves the Alexander grading with respect to arbitrary Seifert surfaces S_0 and S_1 ; i.e.,*

$$\mathcal{A}_{S_1}(f_{\mathcal{H}^0, \mathbb{P}}(\mathbf{x})) = \mathcal{A}_{S_0}(\mathbf{x})$$

for any $\mathbf{x} \in \mathbb{T}_{\alpha^0} \cap \mathbb{T}_{\beta^0}$, where $\mathcal{H}^0 = (\Sigma^0, \alpha^0, \beta^0, w^0, z^0)$.

Proof. This is a straightforward consequence of Lemma 4.22, Remark 4.23,

and the fact that the relative spin^c structure induced by $\mathfrak{s}^\circ(\mathbf{x})$ on (M_1, γ_1) is exactly $\mathfrak{s}^\circ(f_{\mathcal{H}^0, \mathbb{P}}(\mathbf{x}))$. $\square$

A dual reasoning gives the following results for the map $f_{\mathcal{H}^2, \mathbb{S}}$, which are analogous to Lemma 4.22 and Corollary 4.24.

Lemma 4.25 (Juhász-M.³³, Lemma 5.11). *Let $\mathfrak{s}_3^\circ \in \text{Spin}^c(M_3, \gamma_3)$, and let $\mathfrak{s}_2^\circ \in \text{Spin}^c(M_2, \gamma_2)$ denote the corresponding spin^c structure. Then*

$$\langle c_1(\mathfrak{s}_2^\circ, t), [S_2] \rangle = \langle c_1(\mathfrak{s}_3^\circ, t), [S_3] \rangle.$$

Corollary 4.26 (Juhász-M.³³, Corollary 5.12). *The map induced by the attachment of the 3-handles $f_{\mathcal{H}^2, \mathbb{S}} : \widehat{\text{CF}}(\mathcal{H}^2) \rightarrow \widehat{\text{CF}}(\mathcal{H}_{\mathbb{S}}^2)$ preserves the Alexander grading with respect to arbitrary Seifert surfaces S_2 and S_3 ; i.e.,*

$$\mathcal{A}_{S_3}(f_{\mathcal{H}^2, \mathbb{S}}(\mathbf{x})) = \mathcal{A}_{S_2}(\mathbf{x})$$

for any $\mathbf{x} \in \mathbb{T}_{\alpha^2} \cap \mathbb{T}_{\beta^2}$ such that $f_{\mathcal{H}^2, \mathbb{S}}(\mathbf{x}) \neq 0$, where $\mathcal{H}^2 = (\Sigma^2, \alpha^2, \beta^2, w^2, z^2)$.

4.3.7 2-HANDLES

The proof that the Alexander filtration is preserved under the attachment of the 2-handles is less straightforward than in the case of 1-handles and 3-handles.

Lemma 4.27 (Juhász-M.³³, Lemma 5.13). *Let $\mathcal{C}$ be a decorated concordance from (K_0, P_0) to (K_1, P_1) . With the notation of Section 4.3.4, let $\mathcal{W}_2$ denote the 2-handle cobordism from (M_1, γ_1) to (M_2, γ_2) obtained by surgery along a framed link $\mathbb{L}$, and let S_1 and S_2 be corresponding Seifert surfaces. Then there is an admissible doubly pointed triple diagram $(\Sigma, \alpha, \beta, \delta, w, z)$ subordinate to a bouquet for $\mathbb{L}$ as follows: If $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is such that $\mathfrak{s}(\mathbf{x}) \in \text{Spin}^c(Y_{\alpha, \beta})$ extends to X_1 , then for any $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ that appears with non-zero coefficient in $f_{\mathcal{H}^1, \mathbb{L}}(\mathbf{x})$, and such that $\mathfrak{s}(\mathbf{y}) \in \text{Spin}^c(Y_{\alpha, \delta})$ extends to X_3 , we have*

$$\mathcal{F}_{S_2}(\mathbf{y}) \leq \mathcal{F}_{S_1}(\mathbf{x}).$$

Moreover, if ψ is a holomorphic triangle connecting $\mathbf{x}$, ϑ (the top-graded generator of $\widehat{\text{CF}}(\Sigma, \beta, \delta, w, z)$), and $\mathbf{y}$ that does not cross w , then

$$\mathcal{F}_{S_2}(\mathbf{y}) = \mathcal{F}_{S_1}(\mathbf{x}) - n_z(\psi). \quad (4.5)$$

Notice that, in Lemma 4.27, we consider ordinary spin^c structures rather than relative ones. Recall that relative spin^c structures are defined for sutured cobordisms, which we denote by the letter $\mathcal{W}$, while ordinary spin^c structures are defined for cobordisms of 3-manifolds, which we denote by the letter X , cf. Figure 4.1.

Idea of the proof. Consider an admissible triple diagram $(\Sigma, \alpha, \beta, \delta)$ subordinate to a bouquet for a framed link $\mathbb{L}$, as explained by Juhász³¹ (Section 6). Suppose that $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ is such that $\mathfrak{s}(\mathbf{x}) \in \text{Spin}^c(Y_{\alpha,\beta})$ extends to X_1 . Let $\vartheta \in \mathbb{T}_\beta \cap \mathbb{T}_\delta$ be the top-graded generator of $\widehat{\text{CF}}(\Sigma, \beta, \delta)$, and let $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ be such that $\mathfrak{s}(\mathbf{y}) \in \text{Spin}^c(Y_{\alpha,\delta})$ extends to X_3 . Given a Whitney triangle $\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$, let

$$c = \mathcal{A}_{S_2}(\mathbf{y}) - \mathcal{A}_{S_1}(\mathbf{x}) + n_z(\psi) - n_w(\psi).$$

First, we prove that c is independent of ψ , $\mathbf{x}$, and $\mathbf{y}$. If $\psi_1, \psi_2 \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$, then the domain $\mathcal{D}(\psi_1) - \mathcal{D}(\psi_2)$ is triply periodic. If we prove that, for every triply periodic domain D , we have

$$n_z(D) - n_w(D) = 0,$$

then c is independent of ψ . For this reason, the next subsection is devoted to the study of triply periodic domains in the setting of Lemma 4.27.

Given different intersection points $\mathbf{x}' \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{y}' \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ such that $\mathfrak{s}(\mathbf{x}') \in \text{Spin}^c(Y_{\alpha,\beta})$ extends to X_1 and $\mathfrak{s}(\mathbf{y}') \in \text{Spin}^c(Y_{\alpha,\delta})$ extends to X_3 , there are domains $D_{\mathbf{x}}$ connecting $\mathbf{x}$ with $\mathbf{x}'$ and $D_{\mathbf{y}}$ connecting $\mathbf{y}$ with $\mathbf{y}'$ that do not pass through w (but might have non-trivial multiplicities at z). Adding these domains to $D(\psi)$, we get a triangle domain connecting $\mathbf{x}'$, ϑ , and $\mathbf{y}'$ with the same c by Lemma 4.18.

Then we show that $c = 0$ by isotoping α to obtain a diagram where such

$\mathbf{x}$, $\mathbf{y}$, and ψ as above exist, and by applying Lemma 4.5. Finally, if ψ appears in the surgery map $f_{\mathcal{H}^1, \mathbb{L}}(\mathbf{x})$, then $n_w(\psi) = 0$ and it has a pseudo-holomorphic representative, so $n_z(\psi) \geq 0$. Consequently, $A_{S_2}(\mathbf{y}) \leq A_{S_1}(\mathbf{x})$, as desired. $\square$

We now explain the missing details in the above outline.

4.3.8 TRIPLY PERIODIC DOMAINS

The following argument was motivated by the work of Manolescu and Ozsváth⁴⁷.

Definition 4.28. A *doubly pointed triple Heegaard diagram* is a tuple

$$\mathcal{T} = (\Sigma, \alpha, \beta, \delta, w, z),$$

where Σ is a closed, oriented surface, and there is an integer $d \geq 0$ such that the sets α , β and δ all consist of d pairwise disjoint simple closed curves in $\Sigma \setminus \{w, z\}$ that are linearly independent in $H_1(\Sigma \setminus \{w, z\})$.

We denote by $Y_{\alpha, \beta}$, $Y_{\alpha, \delta}$, and $Y_{\beta, \delta}$ the 3-manifolds represented by the Heegaard diagrams (Σ, α, β) , (Σ, α, δ) , and (Σ, β, δ) , respectively.

Definition 4.29. Let $\mathcal{T} = (\Sigma, \alpha, \beta, \delta, w, z)$ be a doubly pointed triple Heegaard diagram. Let $D_1, \dots, D_l$ denote the closures of the components of $\Sigma \setminus (\alpha \cup \beta \cup \delta)$. Then the set of *domains* in $\mathcal{T}$ is

$$D(\mathcal{T}) = \mathbb{Z}\langle D_1, \dots, D_l \rangle.$$

We denote by $n_z(\mathcal{D})$ (respectively $n_w(\mathcal{D})$) the multiplicity of a domain $\mathcal{D} \in D(\mathcal{T})$ in the region D_i that contains z (respectively w).

A *triply periodic domain* is an element $\mathcal{P} \in D(\mathcal{T})$ such that $\partial\mathcal{P}$ is a $\mathbb{Z}$ -linear combination of curves in $\alpha \cup \beta \cup \delta$. We denote the set of triply periodic domains by $\Pi_{\alpha, \beta, \delta}$.

A *doubly periodic domain* is an element $\mathcal{P} \in D(\mathcal{T})$ such that $\partial\mathcal{P}$ is a $\mathbb{Z}$ -linear combination of curves in either $\alpha \cup \beta$, or in $\beta \cup \delta$, or in $\alpha \cup \delta$. We denote

the set of the three types of doubly periodic domains by $\Pi_{\alpha,\beta}$, $\Pi_{\alpha,\delta}$, and $\Pi_{\beta,\delta}$, respectively.

The following result states that every triply periodic domain in the diagram describing the surgery map for $\mathcal{W}_2$ can be written as a sum of doubly periodic domains.

Proposition 4.30 (Juhász-M.³³, Proposition 5.16). *Let $(\Sigma, \alpha, \beta, \delta)$ denote a Heegaard diagram associated to the cobordism X_2 . Then*

$$\Pi_{\alpha,\beta,\delta} = \Pi_{\alpha,\beta} + \Pi_{\alpha,\delta} + \Pi_{\beta,\delta}.$$

Given a triple diagram associated to a surgery on an ℓ -component link $\mathbb{L}$, one can construct a 4-manifold X_Δ as Ozsváth and Szabó do⁶² (Section 2.2). See Juhász³¹ (Section 5) for the analogous construction in the sutured setting. In ∂X_Δ naturally sit $Y_{\alpha,\beta}$, $Y_{\alpha,\delta}$, and $Y_{\beta,\delta}$, which are the 3-manifolds defined by the Heegaard diagrams (Σ, α, β) , (Σ, α, δ) , and (Σ, β, δ) , respectively. The cobordism X_2 corresponding to the attachment of the 2-handles is obtained by gluing the 4-manifold $\hat{X} = \natural_{i=1}^{d-\ell}(S^1 \times D^3)$ to X_Δ along $Y_{\beta,\delta} \cong \#_{i=1}^{d-\ell}(S^1 \times S^2)$.

Lemma 4.31 (Ozsváth-Szabó⁶⁰, Propositions 2.15 and 8.3). *Given a pointed triple Heegaard diagram $(\Sigma, \alpha, \beta, \delta, z)$, there are isomorphisms*

- $\pi_{\alpha,\beta} : \Pi_{\alpha,\beta} \xrightarrow{\cong} \mathbb{Z} \oplus H_2(Y_{\alpha,\beta})$ and
- $\pi_{\alpha,\beta,\delta} : \Pi_{\alpha,\beta,\delta} \xrightarrow{\cong} \mathbb{Z} \oplus H_2(X_\Delta)$.

In both cases, the projection onto the $\mathbb{Z}$ summand is given by n_z .

Lemma 4.32 (Juhász-M.³³, Lemma 5.18). *Given a pointed triple Heegaard diagram $(\Sigma, \alpha, \beta, \delta, z)$, the isomorphisms from Lemma 4.31 fit into the commutative diagram*

$$\begin{array}{ccc} \Pi_{\alpha,\beta} & \xrightarrow{\pi_{\alpha,\beta}} & \mathbb{Z} \oplus H_2(Y_{\alpha,\beta}) \\ \downarrow & & \downarrow \text{id}_{\mathbb{Z}} \oplus i_* \\ \Pi_{\alpha,\beta,\delta} & \xrightarrow{\pi_{\alpha,\beta,\delta}} & \mathbb{Z} \oplus H_2(X_\Delta), \end{array}$$

where $i: Y_{\alpha,\beta} \rightarrow X_\Delta$ is the embedding.

Proof. Let $\mathcal{P}$ be a doubly periodic domain in $\Pi_{\alpha,\beta}$. By construction, the 2-chain in X_Δ associated to $\mathcal{P}$ —thought of as a triply periodic domain—is homotopic, hence homologous to $i_*(H(\mathcal{P}))$, where $H(\mathcal{P})$ is the 2-chain in $Y_{\alpha,\beta}$ obtained by capping off the boundary of the doubly periodic domain $\mathcal{P}$. Therefore, the projections onto the second summand commute. The projections onto the $\mathbb{Z}$ summand commute because in both cases they are obtained by taking n_z . $\square$

Proof of Proposition 4.30. By Lemmas 4.31 and 4.32, it is sufficient to prove that the map

$$\chi: H_2(Y_{\alpha,\beta}) \oplus H_2(Y_{\alpha,\delta}) \oplus H_2(Y_{\beta,\delta}) \longrightarrow H_2(X_\Delta)$$

is surjective. From the long exact sequence of the pair $(X_\Delta, Y_{\alpha,\beta} \sqcup Y_{\alpha,\delta} \sqcup Y_{\beta,\delta})$, we see that the map χ is surjective if and only if

$$\varphi: H_2(X_\Delta, Y_{\alpha,\beta} \sqcup Y_{\alpha,\delta} \sqcup Y_{\beta,\delta}) \longrightarrow H_1(Y_{\alpha,\beta} \sqcup Y_{\alpha,\delta} \sqcup Y_{\beta,\delta})$$

is injective. From the inclusion of pairs

$$i_{\alpha,\beta,\delta}: (X_\Delta, Y_{\alpha,\beta} \sqcup Y_{\alpha,\delta} \sqcup Y_{\beta,\delta}) \hookrightarrow (X, X_1 \sqcup X_3 \sqcup \widehat{X}),$$

we obtain the commutativity of the following diagram:

$$\begin{array}{ccc} H_2(X_\Delta, Y_{\alpha,\beta} \sqcup Y_{\alpha,\delta} \sqcup Y_{\beta,\delta}) & \xrightarrow{\varphi} & H_1(Y_{\alpha,\beta}) \oplus H_1(Y_{\alpha,\delta}) \oplus H_1(Y_{\beta,\delta}) \\ (i_{\alpha,\beta,\delta})_* \downarrow \wr & & \downarrow (i_{\alpha,\beta})_* \oplus (i_{\alpha,\delta})_* \oplus (i_{\beta,\delta})_* \\ H_2(X, X_1 \sqcup X_3 \sqcup \widehat{X}) & \xrightarrow[\tilde{\varphi}]{\cong} & H_1(X_1) \oplus H_1(X_3) \oplus H_1(\widehat{X}), \end{array}$$

where $i_{\alpha,\beta}$, $i_{\alpha,\delta}$, and $i_{\beta,\delta}$ are the restrictions of $i_{\alpha,\beta,\delta}$ to $Y_{\alpha,\beta}$, $Y_{\alpha,\delta}$, and $Y_{\beta,\delta}$, respectively. The map $(i_{\alpha,\beta,\delta})_*$ is an isomorphism by excision. The fact that $\tilde{\varphi}$ is an isomorphism follows from the long exact sequence in homology associated with the pair $(X, X_1 \sqcup X_3 \sqcup \widehat{X})$, together with the fact that $H_2(X) = H_1(X) = 0$.

The commutativity of the above diagram implies that the map φ is injective, and therefore concludes the proof of the proposition. $\square$

Remark 4.33. The important condition in Proposition 4.30 is that the map

$$\rho: H_2(X_1) \oplus H_2(X_3) \oplus H_2(\widehat{X}) \longrightarrow H_2(X)$$

is surjective, which is obviously true as $H_2(X) = 0$. The surjectivity of ρ is equivalent to the injectivity of $\tilde{\varphi}$, which implies the injectivity of φ .

In Proposition 4.30, we saw that, in the case of a triple diagram describing the 2-handle attachments in the cobordism X , every triply periodic domain can be expressed as a sum of doubly periodic domains. We now analyze the doubly periodic domains.

Proposition 4.34. *Consider a null-homologous knot K in a 3-manifold Y . If $(\Sigma, \alpha, \beta, w, z)$ is a doubly pointed Heegaard diagram for (Y, K) , then for every periodic domain $\mathcal{P}$,*

$$n_z(\mathcal{P}) - n_w(\mathcal{P}) = 0.$$

Proof. Let $H(\mathcal{P}) \in C_2(Y)$ be the 2-cycle obtained by capping off the boundary of $\mathcal{P}$ with the cores of the 3-dimensional 2-handles attached to $\Sigma \times I$ along $\alpha \times \{0\}$ and $\beta \times \{1\}$. Then $n_z(\mathcal{P}) - n_w(\mathcal{P})$ is precisely the algebraic intersection number of $H(\mathcal{P})$ and K , which is zero as K is null-homologous. $\square$

4.3.9 REPRESENTING HOMOLOGY CLASSES

Let (Σ, α, β) be a Heegaard diagram for a 3-manifold Y . It is straightforward to see that any element of $H_1(Y)$ can be represented by a 1-cycle in Σ . In this subsection, we strengthen this result for the case of concordances in the following sense.

Lemma 4.35 (Juhász-M.³³, Lemma 5.21). *Choose an arbitrary handle decomposition of the cobordism X from S^3 to S^3 , and let X_2 denote the trace of the*

2-handle attachments. Suppose that $(\Sigma, \alpha, \beta, \delta, w)$ is a pointed triple Heegaard diagram subordinate to a bouquet for a link $\mathbb{L}$ that defines X_2 . Then the map

$$i: H_1(\Sigma) \longrightarrow H_1(Y_{\alpha, \beta}) \oplus H_1(Y_{\alpha, \delta}),$$

induced by the inclusions $\Sigma \hookrightarrow Y_{\alpha, \beta}$ and $\Sigma \hookrightarrow Y_{\alpha, \delta}$, is surjective.

In other words, given any two classes in the first homologies of $Y_{\alpha, \beta}$ and $Y_{\alpha, \delta}$, there is a 1-cycle in Σ that represents both simultaneously.

Proof. Consider the following short exact sequence of abelian groups:

$$0 \rightarrow \frac{H_1(\Sigma)}{\langle \alpha, \beta \rangle \cap \langle \alpha, \delta \rangle} \rightarrow \frac{H_1(\Sigma)}{\langle \alpha, \beta \rangle} \oplus \frac{H_1(\Sigma)}{\langle \alpha, \delta \rangle} \rightarrow \frac{H_1(\Sigma)}{\langle \alpha, \beta, \delta \rangle} \rightarrow 0.$$

The middle term is isomorphic to $H_1(Y_{\alpha, \beta}) \oplus H_1(Y_{\alpha, \delta})$, and the last term is isomorphic to $H_1(X_\Delta)$, where X_Δ is the 4-manifold obtained by the triangle construction, as explained by Ozsváth and Szabó⁶⁰ (Proposition 8.2). The short exact sequence above can then be rewritten as

$$0 \rightarrow \frac{H_1(\Sigma)}{\langle \alpha, \beta \rangle \cap \langle \alpha, \delta \rangle} \xrightarrow{f} H_1(Y_{\alpha, \beta}) \oplus H_1(Y_{\alpha, \delta}) \xrightarrow{g} H_1(X_\Delta) \rightarrow 0.$$

If we prove that $H_1(X_\Delta) = 0$, then by exactness we have that the map f is surjective. So the map i in the statement of the lemma is surjective too, because it is obtained by composing the following two maps:

$$H_1(\Sigma) \rightarrow \frac{H_1(\Sigma)}{\langle \alpha, \beta \rangle \cap \langle \alpha, \delta \rangle} \xrightarrow{f} H_1(Y_{\alpha, \beta}) \oplus H_1(Y_{\alpha, \delta}).$$

In order to prove that $H_1(X_\Delta) = 0$, consider the Mayer-Vietoris long exact sequence associated to the decomposition $X = A \cup B$, where $A = X_\Delta$ and $B = X_1 \sqcup X_3 \sqcup \widehat{X}$. A portion of the long exact sequence is

$$H_2(X) \rightarrow H_1(A \cap B) \xrightarrow{\iota} H_1(A) \oplus H_1(B) \rightarrow H_1(X).$$

Since X has trivial first and second homology groups, by exactness the

map ι gives an isomorphism

$$H_1(Y_{\alpha,\beta}) \oplus H_1(Y_{\alpha,\delta}) \oplus H_1(Y_{\beta,\delta}) \xrightarrow{\sim} H_1(X_\Delta) \oplus H_1(X_1) \oplus H_1(X_3) \oplus H_1(\widehat{X}). \quad (4.6)$$

If c_k denotes the number of k -handles in the decomposition of the cobordism X and $d = |\alpha|$, then it is straightforward to check that

$$\begin{aligned} H_1(Y_{\alpha,\beta}) &\cong H_1(X_1) \cong \mathbb{Z}^{c_1}, \\ H_1(Y_{\beta,\delta}) &\cong H_1(\widehat{X}) \cong \mathbb{Z}^{d-c_2}, \text{ and} \\ H_1(Y_{\alpha,\delta}) &\cong H_1(X_3) \cong \mathbb{Z}^{c_3}. \end{aligned}$$

It now follows from Equation (4.6) that $H_1(X_\Delta) = 0$, which concludes the proof. $\square$

4.3.10 PROOF OF LEMMA 4.27

The cobordism $\mathcal{W}_2$ can be represented via surgery on a framed ℓ -component link $\mathbb{L}$. Let $\mathcal{T} = (\Sigma, \alpha, \beta, \delta, w, z)$ be a doubly pointed triple Heegaard diagram subordinate to a bouquet for the framed link $\mathbb{L}$. As Juhász³¹ explains in Section 6, we suppose that $d = |\alpha| = |\beta| = |\delta|$, and that the curve δ_i is an isotopic translates of β_i for $i \in \{\ell + 1, \dots, d\}$.

Following the notation established in Section 4.3.4 and in Figure 4.1, let $Y_{\alpha,\beta}$, $Y_{\alpha,\delta}$, and $Y_{\beta,\delta}$ denote the closed manifolds associated to the Heegaard diagrams (Σ, α, β) , (Σ, α, δ) , and (Σ, β, δ) , respectively. Each of these closed manifolds contains a knot, defined by the basepoints w and z . We denote the knot exteriors—thought of as sutured manifolds—by $M_{\alpha,\beta}$, $M_{\alpha,\delta}$, and $M_{\beta,\delta}$. We let γ denote the sutures of all three sutured manifolds.

Let $\mathfrak{s}$ be the unique spin^c structure on X . By definition, $\mathfrak{s}|_{X_\Delta}$ is the unique spin^c structure on X_Δ that extends to the whole cobordism X . Suppose that $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ are such that $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}|_{Y_{\alpha,\beta}}$ and $\mathfrak{s}(\mathbf{y}) = \mathfrak{s}|_{Y_{\alpha,\delta}}$. Let $\vartheta \in \mathbb{T}_\beta \cap \mathbb{T}_\delta$ denote the top-graded generator. Consider a Whitney

triangle $\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$, possibly crossing the basepoints z and w , and let

$$c = \mathcal{A}_{S_2}(\mathbf{y}) - \mathcal{A}_{S_1}(\mathbf{x}) + n_z(\psi) - n_w(\psi). \quad (4.7)$$

Our aim is to show that $c = 0$. First, we show that c is independent of the triangle $\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$ for fixed $\mathbf{x}$ and $\mathbf{y}$. Indeed, let $\psi_1, \psi_2 \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$. The domain $\mathcal{P} = \mathcal{D}(\psi_1) - \mathcal{D}(\psi_2)$ is triply periodic. By Proposition 4.30, $\mathcal{P}$ can be expressed as the sum of three doubly periodic domains $\mathcal{P}_{\alpha,\beta}$, $\mathcal{P}_{\beta,\delta}$, and $\mathcal{P}_{\alpha,\delta}$.

Since $(\Sigma, \alpha, \beta, \delta)$ is subordinate to a bouquet for $\mathbb{L}$, each of the diagrams $(\Sigma, \alpha, \beta, w, z)$, $(\Sigma, \beta, \delta, w, z)$, and $(\Sigma, \alpha, \delta, w, z)$ defines a null-homologous knot in a connected sum of a number of copies of $S^1 \times S^2$. Hence, by Proposition 4.34, $n_z(P) = n_w(P)$ for every $P \in \{\mathcal{P}_{\alpha,\beta}, \mathcal{P}_{\beta,\delta}, \mathcal{P}_{\alpha,\delta}\}$. So $n_z(\mathcal{P}) = n_w(\mathcal{P})$, and

$$n_z(\psi_1) - n_w(\psi_1) = n_z(\psi_2) - n_w(\psi_2).$$

Therefore, c is independent of the triangle ψ for fixed $\mathbf{x}$ and $\mathbf{y}$, cf. Equation (4.7).

To check that c is independent of $\mathbf{x}$, consider another generator $\mathbf{x}'$ such that $\mathfrak{s}(\mathbf{x}') = \mathfrak{s}|_{Y_{\alpha,\beta}} = \mathfrak{s}(\mathbf{x})$. Since $\mathbf{x}$ and $\mathbf{x}'$ represent the same spin^c structure, there is a Whitney disc $\phi \in \pi_2(\mathbf{x}', \mathbf{x})$ (that possibly crosses the basepoints w and z). If $\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$, then $\phi \# \psi \in \pi_2(\mathbf{x}', \vartheta, \mathbf{y})$. By Lemma 4.18, the number c defined in Equation (4.7) is the same for ψ and $\phi \# \psi$, so c does not depend on $\mathbf{x}$. A similar reasoning also proves that c is independent of $\mathbf{y}$.

What remains to prove is that $c = 0$. We do this by constructing a Whitney triangle ψ for which $c = 0$.

By isotoping the α -curves, we can create intersection points $\tilde{\mathbf{x}} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\tilde{\mathbf{y}} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ such that there is a “small” triangle $\tilde{\psi} \in \pi_2(\tilde{\mathbf{x}}, \vartheta, \tilde{\mathbf{y}})$. The domain of $\tilde{\psi}$ is shown in Figure 4.2. For each $i \in \{\ell + 1, \dots, d\}$, we isotope α_i —pushing the other α -curves alongside—until it intersects both δ_i and β_i near ϑ_i , and consider the shaded triangle shown on the left-hand side of the figure. For each $i \in \{1, \dots, \ell\}$, after some finger moves on the

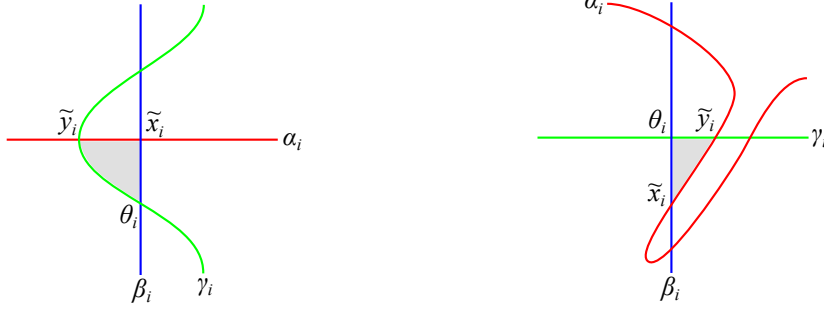


Figure 4.2: This show the domain of the Whitney triangle $\tilde{\psi}$. For $i \in \{\ell + 1, \dots, d\}$, the curves β_i and δ_i are small isotopic translates of each other, and—after isotoping α_i —we can find a “small” triangle bounded by α_i , β_i , and δ_i , shown shaded on the left. For $i \in \{1, \dots, \ell\}$, after applying finger moves to the α -curves, we can assume that there is a triangle, shown shaded on the right. The sum of all these triangles is the domain of $\tilde{\psi}$.

α -curves—again, pushing the other α -curves along—we can assume that there is a small triangle near each intersection point $\vartheta_i = \beta_i \cap \delta_i$, as shown shaded on the right-hand side of the figure. The sum of all these small triangles is the domain of the Whitney triangle $\tilde{\psi}$. We denote by $\tilde{\mathbf{x}} \in \mathbb{T}_{\alpha, \beta}$ and $\tilde{\mathbf{y}} \in \mathbb{T}_{\alpha, \delta}$ the generators connected by $\tilde{\psi}$; i.e., $\tilde{\psi} \in \pi_2(\tilde{\mathbf{x}}, \vartheta, \tilde{\mathbf{y}})$.

The Whitney triangle $\tilde{\psi}$ satisfies $n_z(\tilde{\psi}) = n_w(\tilde{\psi}) = 0$, but the constant c is not necessarily defined for it, because $\mathfrak{s}(\tilde{\mathbf{x}})$ and $\mathfrak{s}(\tilde{\mathbf{y}})$ might not coincide with $\mathfrak{s}|_{Y_{\alpha, \beta}}$ and $\mathfrak{s}|_{Y_{\alpha, \delta}}$, respectively, where $\mathfrak{s} \in \text{Spin}^c(X)$ is the unique spin^c structure, cf. Equation (4.7). The next lemma proves that we can replace $\tilde{\psi}$ with a Whitney triangle ψ for which the constant c is defined.

Lemma 4.36 (Juhász-M.³³, Lemma 5.22). *We can further isotope the α -curves such that there is a Whitney triangle $\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$ satisfying*

- $n_z(\psi) = n_w(\psi) = 0$; and
- $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}|_{Y_{\alpha, \beta}}$ and $\mathfrak{s}(\mathbf{y}) = \mathfrak{s}|_{Y_{\alpha, \delta}}$.

Proof. Given generators $\mathbf{x}' = (x'_1, \dots, x'_d)$ and $\mathbf{x}'' = (x''_1, \dots, x''_d)$ in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$, Ozsváth and Szabó⁶⁰ (Definition 2.11) associate to them a class $\varepsilon(\mathbf{x}', \mathbf{x}'') \in H_1(Y_{\alpha, \beta})$. Choose 1-chains $a \subset \alpha$ and $b \subset \beta$ such that

$$\partial a = \partial b = x''_1 + \dots + x''_d - x'_1 - \dots - x'_d.$$

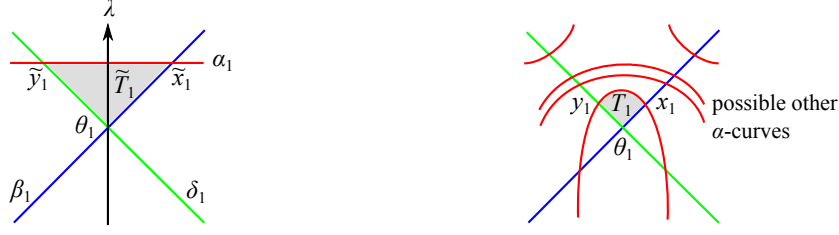


Figure 4.3: The picture above shows how to modify the Whitney triangle $\tilde{\psi}$ defined in Figure 4.2 to obtain a Whitney triangle ψ satisfying the requirements of Lemma 4.36. The picture on the left shows the loop λ near the triangle $\tilde{T}_1$. The picture on the right shows the new triangle T_1 in the triple Heegaard diagram obtained after performing a finger move on the α -curves along λ .

Then $a - b$ represents an element of $H_1(\Sigma)$ whose image in $H_1(Y_{\alpha,\beta})$ under the inclusion map is $\varepsilon(\mathbf{x}', \mathbf{x}'')$. Ozsváth and Szabó⁶⁰ (Lemma 2.19) proved that

$$\mathfrak{s}(\mathbf{x}'') - \mathfrak{s}(\mathbf{x}') = \text{PD}(\varepsilon(\mathbf{x}', \mathbf{x}'')). \quad (4.8)$$

Consider the Whitney triangle $\tilde{\psi} \in \pi_2(\tilde{\mathbf{x}}, \vartheta, \tilde{\mathbf{y}})$ defined above, and whose domain is shown in Figure 4.2. Its domain is the disjoint union of d triangles $\tilde{T}_1, \dots, \tilde{T}_d$.

We define the homology classes $h_1 \in H_1(Y_{\alpha,\beta})$ and $h_2 \in H_1(Y_{\alpha,\delta})$ as

$$h_1 = \text{PD}(\mathfrak{s}|_{Y_{\alpha,\beta}} - \mathfrak{s}(\tilde{\mathbf{x}})) \text{ and} \quad (4.9a)$$

$$h_2 = \text{PD}(\mathfrak{s}|_{Y_{\alpha,\delta}} - \mathfrak{s}(\tilde{\mathbf{y}})), \quad (4.9b)$$

where $\mathfrak{s}$ is the unique spin^c structure on X . By Lemma 4.35, there is a homology class $h \in H_1(\Sigma)$ such that $i(h) = (h_1, h_2)$; i.e., h represents h_1 in $H_1(Y_{\alpha,\beta})$ and h_2 in $H_1(Y_{\alpha,\delta})$. We can represent h as the sum of simple closed curves λ on Σ that satisfies the following conditions:

- λ intersects the triangle $\tilde{T}_1$ as on the left-hand side of Figure 4.3;
- λ is disjoint from all the triangles $\tilde{T}_2, \dots, \tilde{T}_d$; and
- λ is disjoint from the basepoints z and w .

If we perform a finger move on the α -curves along each loop λ , the re-

sult will look like the right-hand side of Figure 4.3. If x_1 and y_1 are as on the right-hand side of Figure 4.3, we define $\mathbf{x} = (x_1, \tilde{x}_2, \dots, \tilde{x}_d)$ and $\mathbf{y} = (y_1, \tilde{y}_2, \dots, \tilde{y}_d)$. Notice that, by construction,

$$\varepsilon(\tilde{\mathbf{x}}, \mathbf{x}) = h_1 \text{ and } \varepsilon(\tilde{\mathbf{y}}, \mathbf{y}) = h_2. \quad (4.10)$$

Let ψ be a Whitney triangle with domain $T_1 \sqcup \tilde{T}_2 \sqcup \dots \sqcup \tilde{T}_d$, where T_1 is the shaded triangle on the right-hand side of Figure 4.3. By construction, we have $n_z(\psi) = n_w(\psi) = 0$. Furthermore, due to Equations (4.8), (4.10), and (4.9), we have

$$\begin{aligned} \mathfrak{s}(\mathbf{x}) &= \mathfrak{s}(\tilde{\mathbf{x}}) + \text{PD}(\varepsilon(\tilde{\mathbf{x}}, \mathbf{x})) \\ &= \mathfrak{s}(\tilde{\mathbf{x}}) + \text{PD}(h_1) \\ &= \mathfrak{s}(\tilde{\mathbf{x}}) + (\mathfrak{s}|_{Y_{\alpha,\beta}} - \mathfrak{s}(\tilde{\mathbf{x}})) = \mathfrak{s}|_{Y_{\alpha,\beta}}. \end{aligned}$$

Analogously, we have $\mathfrak{s}(\mathbf{y}) = \mathfrak{s}|_{Y_{\alpha,\delta}}$. □

Before showing that $c = 0$ for the triangle $\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$ constructed above, we prove that the relative spin^c structure $\mathfrak{s}^\circ(\psi) \in \text{Spin}^c(\mathcal{W}_2)$ extends to a relative spin^c structure on $\mathcal{W}$.

Recall that $Y_1 = Y_{\alpha,\beta}$ is obtained from Y_0 by performing surgery along some framed 0-spheres. The belt circles of the 1-handles involved give rise to embedded 2-spheres $O_1, \dots, O_p \subset Y_1$. Similarly, $Y_2 = Y_{\alpha,\delta}$ is obtained from Y_3 by surgery along some framed 0-spheres, giving rise to embedded spheres $O'_1, \dots, O'_s \subset Y_2$. In Lemma 4.36, we achieved that $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}|_{Y_{\alpha,\beta}}$ and $\mathfrak{s}(\mathbf{y}) = \mathfrak{s}|_{Y_{\alpha,\delta}}$. This implies that $\mathfrak{s}(\mathbf{x})$ extends to X_1 , or equivalently, that $\langle c_1(\mathfrak{s}(\mathbf{x})), [O_i] \rangle = 0$ for every $i \in \{1, \dots, p\}$. Similarly, $\langle c_1(\mathfrak{s}(\mathbf{y})), [O'_j] \rangle = 0$ for every $j \in \{1, \dots, s\}$. However

$$\langle c_1(\mathfrak{s}(\mathbf{x})), [O_i] \rangle = \langle c_1(\mathfrak{s}^\circ(\mathbf{x}), t), [O_i] \rangle,$$

as $\mathfrak{s}^\circ(\mathbf{x})$ and $\mathfrak{s}(\mathbf{x})$ are represented by the same vector field on $M_1 \subset Y_1$. Since M_0 is obtained from M_1 by compressing the 2-spheres $O_1, \dots, O_p$,

the equality

$$\langle c_1(\mathfrak{s}^\circ(\mathbf{x}), t), [O_i] \rangle = 0$$

implies that $\mathfrak{s}^\circ(\mathbf{x})$ extends to $\mathfrak{s}_1^\circ \in \text{Spin}^c(\mathcal{W}_1)$. Similarly, $\mathfrak{s}^\circ(\mathbf{y})$ extends to $\mathfrak{s}_3^\circ \in \text{Spin}^c(\mathcal{W}_3)$. The Mayer-Vietoris sequence now implies that there is a Spin^c structure $\mathfrak{s}^\circ \in \text{Spin}^c(\mathcal{W})$ such that $\mathfrak{s}^\circ|_{\mathcal{W}_1} = \mathfrak{s}_1^\circ$, $\mathfrak{s}^\circ|_{\mathcal{W}_2} = \mathfrak{s}^\circ(\psi)$, and $\mathfrak{s}^\circ|_{\mathcal{W}_3} = \mathfrak{s}_3^\circ$.

We are now ready to prove that, for the Whitney triangle ψ constructed above, $c = 0$. Recall that, by definition,

$$c = \mathcal{A}_{S_2}(\mathbf{y}) - \mathcal{A}_{S_1}(\mathbf{x}) = \frac{1}{2} (\langle c_2(\mathfrak{s}^\circ(\mathbf{y}), t), [S_2] \rangle - \langle c_1(\mathfrak{s}^\circ(\mathbf{x}), t), [S_1] \rangle),$$

cf. Equation (4.7) and Equation (2.2). Since ψ is a Whitney triangle connecting $\mathbf{x}$, ϑ , and $\mathbf{y}$, we have that $\mathfrak{s}^\circ(\psi)|_{M_1} = \mathfrak{s}^\circ(\mathbf{x})$ and $\mathfrak{s}^\circ(\psi)|_{M_2} = \mathfrak{s}^\circ(\mathbf{y})$, and therefore

$$c = \frac{1}{2} (\langle c_1(\mathfrak{s}^\circ(\psi), t), [S_2] \rangle - \langle c_1(\mathfrak{s}^\circ(\psi), t), [S_1] \rangle).$$

Notice that we can omit the restrictions of the (relative) spin^c structures by the naturality of Chern classes.

Now the relative spin^c structure $\mathfrak{s}^\circ(\psi)$ extends to some relative spin^c structure $\mathfrak{s}^\circ \in \text{Spin}^c(\mathcal{W})$ by the above. Then, by Lemmas 4.22 and 4.25, we have

$$c = \frac{1}{2} (\langle c_1(\mathfrak{s}^\circ, t), [S_3] \rangle - \langle c_1(\mathfrak{s}^\circ, t), [S_0] \rangle).$$

From Lemma 4.5, it finally follows that $c = 0$.

We can now conclude the proof of Lemma 4.27. By Equation (4.7), for any Whitney triangle $\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$, where $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ are such that $\mathfrak{s}(\mathbf{x})$ and $\mathfrak{s}(\mathbf{y})$ extend to X_1 and X_3 , respectively, we have

$$\mathcal{A}_{S_2}(\mathbf{y}) - \mathcal{A}_{S_1}(\mathbf{x}) + n_z(\psi) - n_w(\psi) = 0.$$

If ψ contributes to the surgery map $f_{\mathcal{H}^1, \mathbb{L}}(\mathbf{x})$, then $n_w(\psi) = 0$, and it has a pseudo-holomorphic representative, so $n_z(\psi) \geq 0$. Therefore, $\mathcal{A}_{S_2}(\mathbf{y}) \leq$

$A_{S_1}(\mathbf{x})$, as desired. □

4.3.11 NATURALITY MAPS

Recall that³⁴, given two admissible Heegaard diagrams $\mathcal{H}$ and $\mathcal{H}'$ for the same 3-manifold Y , there is a naturality map

$$f_{\mathcal{H}, \mathcal{H}'}: \widehat{\text{CF}}(\mathcal{H}) \longrightarrow \widehat{\text{CF}}(\mathcal{H}'),$$

which is the composition of maps associated to isotopies of the attaching sets, handleslides, (de)stabilisations, and diffeomorphisms of the Heegaard surface isotopic to the identity in Y . On the homology, it induces an isomorphism

$$F_{\mathcal{H}, \mathcal{H}'}: \widehat{\text{HF}}(\mathcal{H}) \rightarrow \widehat{\text{HF}}(\mathcal{H}')$$

that is independent of the sequence of Heegaard moves.

In our case, $\mathcal{H}$ and $\mathcal{H}'$ are doubly pointed Heegaard diagrams, which define the same decorated knot (Y, K, P) . Juhász and Thurston³⁴ (Proposition 2.37) proved that $\mathcal{H}$ and $\mathcal{H}'$ can be connected by a sequence of Heegaard moves *that do not cross the basepoints w and z* . If we forget about the z basepoint, this sequence induces the naturality map $f_{\mathcal{H}, \mathcal{H}'}$ above. As we explained, the z basepoints on $\mathcal{H}$ and $\mathcal{H}'$ induce filtrations on $\widehat{\text{CF}}(\mathcal{H})$ and $\widehat{\text{CF}}(\mathcal{H}')$. It follows from the work of Ozsváth-Szabó⁵⁸ and Rasmussen⁶⁶ that, if $f_{\mathcal{H}, \mathcal{H}'}$ is the map associated to either an isotopy, a handleslide, a (de)stabilisation, or a diffeomorphism of the Heegaard surface isotopic to the identity in Y , then it preserves the knot filtration. If $f_{\mathcal{H}, \mathcal{H}'}$ is an isotopy map or a handleslide map, then the map induced on the E^1 page is the corresponding naturality map $F_{\mathcal{H}, \mathcal{H}'}$ on $\widehat{\text{HFK}}$; i.e., it is the map obtained by counting all holomorphic triangles that do not cross z . If $f_{\mathcal{H}, \mathcal{H}'}$ is a (de)stabilisation or diffeomorphism map, then it is an isomorphism of filtered complexes.

As the above result is only outlined in the works of Ozsváth and Szabó⁵⁸ and Rasmussen⁶⁶, we provide a bit more detail. With the techniques of this paper, we can prove the following analogue of Lemma 4.27.

Lemma 4.37 (Juhász-M.³³, Lemma 5.23). *Let K be a null-homologous knot in $Y = \#_{i=1}^p(S^1 \times S^2)$. Choose a Seifert surface S for K . Suppose that $\mathcal{H}$ and $\mathcal{H}'$ are admissible doubly pointed Heegaard diagrams for (Y, K, P) that only differ by an isotopy or a handleslide.*

Given an admissible doubly pointed triple diagram $(\Sigma, \alpha, \beta, \delta, w, z)$ for the Heegaard move $\mathcal{H} \rightarrow \mathcal{H}'$, if $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, then for any $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ that has non-trivial coefficient in the expansion of $f_{\mathcal{H}, \mathcal{H}'}(\mathbf{x})$, we have that

$$\mathcal{F}_S(\mathbf{y}) \leq \mathcal{F}_S(\mathbf{x}).$$

Furthermore, if ψ is a holomorphic triangle connecting $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $\vartheta \in \mathbb{T}_\beta \cap \mathbb{T}_\delta$ (the top-dimensional generator of $\widehat{\text{CF}}(\Sigma, \beta, \delta, w, z)$), and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ that does not cross w , then

$$\mathcal{F}_S(\mathbf{y}) = \mathcal{F}_S(\mathbf{x}) - n_z(\psi). \quad (4.11)$$

Remark 4.38. The (de)stabilisation and diffeomorphism maps do not appear in Lemma 4.37 because they are not triangle maps. They are already isomorphisms at the level of filtered chain complexes.

Idea of the proof. As in the proof of Lemma 4.27, we let

$$c = \mathcal{A}_S(\mathbf{y}) - \mathcal{A}_S(\mathbf{x}) - n_w(\psi) + n_z(\psi),$$

and prove that this is independent of ψ , $\mathbf{x}$, and $\mathbf{y}$. The main differences from the proof of Lemma 4.27 are the following:

- **Triply periodic domains:** We closely follow the proof of Proposition 4.30. In this case, $X \cong Y \times I$, the boundary of the 4-manifold X_Δ consists of $Y \sqcup Y \sqcup Y_{\beta, \delta}$, and the cobordisms X_1 and X_3 are replaced by identity cobordisms $Y \times I$. Finally, the injectivity of φ follows from the surjectivity of the map

$$\rho: H_2(Y \times I) \oplus H_2(Y \times I) \oplus H_2(\widehat{X}) \longrightarrow H_2(X),$$

as noted in Remark 4.33.

- **Doubly periodic domains:** One can use Proposition 4.34 for the two

copies of Y and for $Y_{\beta,\delta}$.

- **Proving that $c = 0$:** Note that, in contrast with Lemma 4.27, we cannot isotope the α curves in this case, because the Heegaard diagram would change. However, this is not necessary, because the homology of the associated graded vector space is $\text{HFL}(\#S^1 \times S^2, K, P)$, whose splitting into Alexander grading is preserved under naturality maps, by the functoriality of the sutured Floer homology³⁴ $\text{SFH}(M, \gamma, \mathfrak{s}^\circ)$. This forces the grading shift c to be 0. $\square$

4.3.12 PROOF OF THEOREM 4.19

We are now ready to prove Theorem 4.19. In the proof we use the notation introduced in Section 4.3.4, and we assume that the gluing map is the identity map, as explained in Section 4.3.3.

Suppose that $\mathbf{x}$ is a generator of $\widehat{\text{CF}}(\mathcal{H}^0)$ such that $f_c(\mathbf{x}) \neq 0$. Let $\mathbf{y}$ be a generator of $\widehat{\text{CF}}(\mathcal{H}_{\mathbb{S}}^2)$ that appears in the expression of $f_c(\mathbf{x})$ with non-zero coefficient. Then there exist generators $\mathbf{x}' \in \widehat{\text{CF}}(\mathcal{H}_{\mathbb{P}}^0)$, $\mathbf{x}'' \in \widehat{\text{CF}}(\mathcal{H}^1)$, $\mathbf{y}'' \in \widehat{\text{CF}}(\mathcal{H}_{\mathbb{L}}^1)$, and $\mathbf{y}' \in \widehat{\text{CF}}(\mathcal{H}^2)$ that appear with non-zero coefficient in $f_{\mathcal{H}^0, \mathbb{P}}(\mathbf{x})$, $f_{\mathcal{H}_{\mathbb{P}}^0, \mathcal{H}^1}(\mathbf{x}')$, $f_{\mathcal{H}^1, \mathbb{L}}(\mathbf{x}'')$, and $f_{\mathcal{H}_{\mathbb{L}}^1, \mathcal{H}^2}(\mathbf{y}'')$, respectively, and such that $\mathbf{y}$ appears with non-zero coefficient in $f_{\mathcal{H}^2, \mathbb{S}}(\mathbf{y}')$. Notice that, by construction, $\mathfrak{s}(\mathbf{x}'')$ extends to X_1 and $\mathfrak{s}(\mathbf{y}'')$ extends to X_3 .

By Lemma 4.37, we know that the naturality maps preserve the knot filtration, and by Corollaries 4.24 and 4.26 so do the maps $f_{\mathcal{H}^0, \mathbb{P}}$ and $f_{\mathcal{H}^2, \mathbb{S}}$. Finally, Lemma 4.27 proves that $\mathcal{F}_{S_2}(\mathbf{y}'') \leq \mathcal{F}_{S_1}(\mathbf{x}'')$. By putting all these together, we obtain that

$$\mathcal{F}_{S_3}(\mathbf{y}) = \mathcal{F}_{S_2}(\mathbf{y}') \leq \mathcal{F}_{S_2}(\mathbf{y}'') \leq \mathcal{F}_{S_1}(\mathbf{x}'') \leq \mathcal{F}_{S_1}(\mathbf{x}') = \mathcal{F}_{S_0}(\mathbf{x}). \quad (4.12)$$

Thus f_c is a map of filtered complexes and so, by Remark 4.12, it induces a morphism of spectral sequences.

Furthermore, each of the maps $f_{\mathcal{H}^0, \mathbb{P}}$, $f_{\mathcal{H}_{\mathbb{P}}^0, \mathcal{H}^1}$, $f_{\mathcal{H}^1, \mathbb{L}}$, $f_{\mathcal{H}_{\mathbb{L}}^1, \mathcal{H}^2}$, and $f_{\mathcal{H}^2, \mathbb{S}}$ is a map of filtered complexes. The map induced by f_c on the E^1 page is the

composition of the maps induced by each of the above maps on the E^1 page.

We now consider the case when the inequalities in Equation (4.12) are all equalities. Lemmas 4.22 and 4.25 imply that the maps induced by $f_{\mathcal{H}^0, \mathbb{P}}$ and $f_{\mathcal{H}^2, \mathbb{S}}$ on the E^1 page are the 1- and 3-handle maps for $\widehat{\text{HFK}}$. As for the 2-handle map $f_{\mathcal{H}^1, \mathbb{L}}$, by Equation (4.5) in Lemma 4.27, we have that $\mathcal{F}(\mathbf{y}'') = \mathcal{F}(\mathbf{x}'')$ if and only if there is a pseudo-holomorphic triangle ψ connecting $\mathbf{x}''$, ϑ , and $\mathbf{y}''$ such that $n_w(\psi) = n_z(\psi) = 0$, and in this case all such holomorphic triangles satisfy this equality. Hence, the map induced by $f_{\mathcal{H}^1, \mathbb{L}}$ on the E^1 page is the 2-handle map for $\widehat{\text{HFK}}$. Finally, it follows from the discussion in Section 4.3.11 that the maps induced on the E^1 page by the naturality maps for $\widehat{\text{CF}}$ are the naturality maps for $\widehat{\text{HFK}}$. Alternatively, one can use Equation (4.11) in Lemma 4.37 and argue in the same way as for the 2-handle maps.

This immediately implies that the map induced by $f_{\mathcal{C}}$ on the E^1 page is obtained by counting (for the naturality maps and the 2-handle map) the pseudo-holomorphic triangles that do not cross w and z , and so it is $F_{\mathcal{C}}$.

On the other hand, the map induced by $f_{\mathcal{C}}$ on the total homology is given by counting all holomorphic triangles that do not cross w but might cross z . This is precisely the map $\widehat{F}_X: \widehat{\text{HF}}(S^3) \rightarrow \widehat{\text{HF}}(S^3)$ induced by the cobordism X . Since $H_1(X) = H_2(X) = 0$, the map $\widehat{F}_X = \text{id}_{\widehat{\text{HF}}(S^3)}$, according to Ozsváth and Szabó⁵⁶ (Lemma 3.4). $\square$

4.4 CONCORDANCE MAPS PRESERVE THE HOMOLOGICAL GRADING

In this section, we show that concordance maps also behave well with respect to the homological grading.

Let $\mathcal{H}$ be an admissible pointed Heegaard diagram for the closed, connected, oriented, based 3-manifold (Y, w) , together with a spin^c structure $\mathfrak{s} \in \text{Spin}^c(Y)$ such that $c_1(\mathfrak{s}) \in H^2(Y)$ is torsion. Ozsváth and Szabó⁶⁰ (Section 4) showed that $\widehat{\text{CF}}(\mathcal{H}, \mathfrak{s})$ admits a relative $\mathbb{Z}$ -grading (see also Section 2.2.1 of this thesis). For generators $\mathbf{x}, \mathbf{y} \in \widehat{\text{CF}}(\mathcal{H}, \mathfrak{s})$ and $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we

have

$$gr(\mathbf{x}, \mathbf{y}) = \mu(\phi) - 2n_w(\phi). \quad (4.13)$$

According to Ozsváth and Szabó⁶² (Theorem 7.1) that this can be lifted to an absolute $\mathbb{Q}$ -grading $\tilde{gr}$, in the sense that $gr(\mathbf{x}, \mathbf{y}) = \tilde{gr}(\mathbf{x}) - \tilde{gr}(\mathbf{y})$. Such a grading is called the *Maslov grading* or *homological grading*.

Example 4.39. If $Y = S^3$ with its unique spin^c structure $\mathfrak{s}_0$, and if $\mathcal{H}$ is a Heegaard diagram of Y , then on $\widehat{\text{CF}}(\mathcal{H}, \mathfrak{s}_0)$ the absolute $\mathbb{Q}$ -grading is actually an absolute $\mathbb{Z}$ -grading. The generator of $\widehat{\text{HF}}(S^3, \mathfrak{s}_0) \cong \mathbb{F}_2$ is homogeneous of grading zero.

More generally, if $Y = \#_{i=1}^k (S^1 \times S^2)$ with Heegaard diagram $\mathcal{H}$, and $\mathfrak{s}_0 \in \text{Spin}^c(Y)$ is such that $c_1(\mathfrak{s}_0) = 0$, then $\tilde{gr}$ is an absolute $\frac{1}{2}\mathbb{Z}$ -grading on $\widehat{\text{CF}}(\mathcal{H}, \mathfrak{s}_0)$.

The main result of this section is the following.

Theorem 4.40 (Juhász-M.³³, Theorem 6.2). *Let $\mathcal{C}$ be a decorated concordance from (K_0, P_0) to (K_1, P_1) , and let $\mathcal{H}_i$ be an admissible doubly pointed diagram of (K_i, P_i) for $i \in \{0, 1\}$. Then, the chain map*

$$f_{\mathcal{C}}: \widehat{\text{CF}}(\mathcal{H}_0) \longrightarrow \widehat{\text{CF}}(\mathcal{H}_1)$$

preserves the absolute homological grading; that is, if $x \in \widehat{\text{CF}}(\mathcal{H}_0)$ is $\tilde{gr}$ -homogeneous, so is $f_{\mathcal{C}}(x)$, and if $f_{\mathcal{C}}(x) \neq 0$, then

$$\tilde{gr}(f_{\mathcal{C}}(x)) = \tilde{gr}(x).$$

Remark 4.41. Notice that the statement of Theorem 4.40 is stronger than the fact that $f_{\mathcal{C}}$ preserves the Maslov filtration. We actually claim that the Maslov grading is not decreased by $f_{\mathcal{C}}$.

Idea of the proof. We proceed similarly to the proof of Theorem 4.19, and use the notation from Section 4.3.4 and Figure 4.1. As the diffeomorphism D constructed in Section 4.3.3 induces a homomorphism D_* that preserves the homological grading, we can assume the gluing map is trivial and we are dealing with a special cobordism.

First, we prove that, in the right spin^c structure, each map $f_{\mathcal{H}^0, \mathbb{P}}$, $f_{\mathcal{H}^0, \mathcal{H}^1}$, $f_{\mathcal{H}^1, \mathbb{L}}$, $f_{\mathcal{H}^1, \mathcal{H}^2}$, and $f_{\mathcal{H}^2, \mathbb{S}}$ preserves the relative Maslov grading gr . This is only implicit in the work of Ozsváth and Szabó⁶², so we provide more detail. Then we show that the absolute grading shift of $f_{\mathcal{C}}$, which is the composition of all the above maps, is zero.

For the 1- and 3-handle maps $f_{\mathcal{H}^0, \mathbb{P}}$ and $f_{\mathcal{H}^2, \mathbb{S}}$, it is straightforward to check that the relative Maslov grading is preserved using Equation (4.13) above.

Now consider the 2-handle map $f_{\mathcal{H}^1, \mathbb{L}}$. Let $(\Sigma, \alpha, \beta, \delta, w, z)$ be an admissible triple Heegaard diagram subordinate to a bouquet for $\mathbb{L}$. For generators $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ such that $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}|_{Y_{\alpha, \beta}}$ and $\mathfrak{s}(\mathbf{y}) = \mathfrak{s}|_{Y_{\alpha, \delta}}$, where $\mathfrak{s}$ denotes the unique spin^c structure on X , and for every Whitney triangle $\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$, we let

$$d = \tilde{gr}(\mathbf{y}) - \tilde{gr}(\mathbf{x}) + \mu(\psi) - 2n_w(\psi).$$

We show that d is independent of ψ , $\mathbf{x}$, and $\mathbf{y}$. Since the triangles ψ contributing to $f_{\mathcal{H}^1, \mathbb{L}}$ have $\mu(\psi) = 0$ and $n_w(\psi) = 0$, it follows that the absolute grading is shifted by d , so the relative grading is preserved.

We already know from the work of Ozsváth and Szabó⁶⁰ that the natural maps $f_{\mathcal{H}^0, \mathcal{H}^1}$ and $f_{\mathcal{H}^1, \mathcal{H}^2}$ preserve the relative homological grading gr . Alternatively, this can also be shown using the techniques of Section 4.3.11.

Finally, $f_{\mathcal{C}}$, which is the composition of all the above maps, preserves the relative homological grading, or equivalently, it shifts the absolute homological grading by some constant e . This implies that, for every $r \in \mathbb{N}$, the map $E^r(f_{\mathcal{C}})$ shifts the homological grading by the same constant e independent of r . Since we know that the map in total homology is $\text{id}_{\widehat{\text{HF}}(S^3)}$ and preserves the absolute grading by the work of Ozsváth and Szabó⁵⁶ (Lemma 3.4), it immediately follows that $e = 0$. $\square$

The rest of this section is devoted to filling in the details of the above outline.

4.4.1 SPIN^c STRUCTURES

Let $\mathfrak{s}$ be the unique spin^c structure on X . Then

$$f_{\mathcal{C}} = f_{\mathcal{C}, \mathfrak{s}} = f_{\mathcal{H}^2, \mathbb{S}, \mathfrak{s}} \circ \cdots \circ f_{\mathcal{H}^0, \mathbb{P}, \mathfrak{s}},$$

where the restrictions of $\mathfrak{s}$ are omitted for the sake of clarity.

So it suffices to consider the above maps in the spin^c structure $\mathfrak{s}$. In the rest of the section, we will focus on the maps $f_{\mathcal{H}^2, \mathbb{S}, \mathfrak{s}}, \dots, f_{\mathcal{H}^0, \mathbb{P}, \mathfrak{s}}$, and for simplicity, we will denote the restrictions of $\mathfrak{s}$ by the same letter.

4.4.2 1- AND 3-HANDLES

The 1-handle map $f_{\mathcal{H}^0, \mathbb{P}, \mathfrak{s}}$ satisfies the following.

Lemma 4.42 (Juhász-M.³³, Lemma 6.4). *Let $\mathbf{x}'', \tilde{\mathbf{x}}'' \in \widehat{\text{CF}}(\mathcal{H}^0, \mathfrak{s})$ be generators. Then*

$$gr(\mathbf{x}'', \tilde{\mathbf{x}}'') = gr(f_{\mathcal{H}^0, \mathbb{P}, \mathfrak{s}}(\mathbf{x}''), f_{\mathcal{H}^0, \mathbb{P}, \mathfrak{s}}(\tilde{\mathbf{x}}''));$$

i.e., the relative homological grading is preserved under the 1-handle map.

Proof. Let $\phi \in \pi_2(\mathbf{x}'', \tilde{\mathbf{x}}'')$. Then the domain of ϕ also represents a Whitney disc between $f_{\mathcal{H}^0, \mathbb{P}}(\mathbf{x}'')$ and $f_{\mathcal{H}^0, \mathbb{P}}(\tilde{\mathbf{x}}'')$ in the Heegaard diagram $\mathcal{H}_{\mathbb{P}}^0$ that we also denote by ϕ . By Equation (4.13), we have

$$gr(\mathbf{x}'', \tilde{\mathbf{x}}'') = \mu(\phi) - 2n_w(\phi) = gr(f_{\mathcal{H}^0, \mathbb{P}, \mathfrak{s}}(\mathbf{x}''), f_{\mathcal{H}^0, \mathbb{P}, \mathfrak{s}}(\tilde{\mathbf{x}}'')). \quad \square$$

A dual argument gives the following result for the 3-handle map $f_{\mathcal{H}^2, \mathbb{S}, \mathfrak{s}}$.

Lemma 4.43 (Juhász-M.³³, Lemma 6.5). *Let $\mathbf{y}', \tilde{\mathbf{y}}' \in \widehat{\text{CF}}(\mathcal{H}^2, \mathfrak{s})$ be generators such that $f_{\mathcal{H}^2, \mathbb{S}, \mathfrak{s}}(\mathbf{y}') \neq 0$ and $f_{\mathcal{H}^2, \mathbb{S}, \mathfrak{s}}(\tilde{\mathbf{y}}') \neq 0$. Then*

$$gr(\mathbf{y}', \tilde{\mathbf{y}}') = gr(f_{\mathcal{H}^2, \mathbb{S}, \mathfrak{s}}(\mathbf{y}'), f_{\mathcal{H}^2, \mathbb{S}, \mathfrak{s}}(\tilde{\mathbf{y}}'));$$

i.e., the relative homological grading is preserved under the 3-handle map.

4.4.3 2-HANDLES

For 2-handles, we have the following.

Lemma 4.44 (Juhász-M.³³, Lemma 6.6). *Let $\mathbf{x}, \tilde{\mathbf{x}} \in \widehat{\text{CF}}(\mathcal{H}^1)$ be generators such that $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}(\tilde{\mathbf{x}}) = \mathfrak{s}$. Then $f_{\mathcal{H}^1, \mathbb{L}, \mathfrak{s}}(\mathbf{x})$ and $f_{\mathcal{H}^1, \mathbb{L}, \mathfrak{s}}(\tilde{\mathbf{x}})$ are $\tilde{gr}$ -homogeneous, and if they are non-zero, then*

$$gr(\mathbf{x}, \tilde{\mathbf{x}}) = gr(f_{\mathcal{H}^1, \mathbb{L}, \mathfrak{s}}(\mathbf{x}), f_{\mathcal{H}^1, \mathbb{L}, \mathfrak{s}}(\tilde{\mathbf{x}})).$$

Proof. For $\mathbf{x} \in \widehat{\text{CF}}(\mathcal{H}^1)$, $\mathbf{y} \in \widehat{\text{CF}}(\mathcal{H}_{\mathbb{L}}^1)$, and $\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y})$ such that $\mathfrak{s}(\psi) = \mathfrak{s}$, let

$$d = \tilde{gr}(\mathbf{y}) - \tilde{gr}(\mathbf{x}) + \mu(\psi) - 2n_w(\psi). \quad (4.14)$$

First, we check that d is independent of ψ . As in the proof of Lemma 4.27, it suffices to show that, for every triply periodic domain $\mathcal{P}$,

$$\mu(\mathcal{P}) = 2n_w(\mathcal{P}). \quad (4.15)$$

Since every triply periodic domain is the sum of doubly periodic domains by Proposition 4.30, it is sufficient to prove Equation (4.15) in the case of doubly periodic domains in Heegaard diagrams of $Y_{\alpha, \beta}$, $Y_{\alpha, \delta}$, and $Y_{\beta, \delta}$.

Consider, for example, $Y_{\alpha, \beta}$ and $\mathbf{z} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\beta}$, with a periodic domain $\mathcal{P} \in \Pi_{\alpha, \beta}$ based at $\mathbf{z}$. As $\mathfrak{s}(\mathbf{z})$ extends to the cobordism X_1 , we see that $c_1(\mathfrak{s}(\mathbf{z}))$ vanishes on the belt spheres of the 1-handles. Furthermore, since $H(\mathcal{P}) \in H_2(Y_{\alpha, \beta})$ is a linear combination of the belt spheres, we obtain that

$$\langle c_1(\mathfrak{s}(\mathbf{z})), H(\mathcal{P}) \rangle = 0.$$

By the work of Ozsváth and Szabó⁶⁰ (Theorem 4.9) and Lipshitz⁴⁴ (Corollary 4.12),

$$\mu(\mathcal{P}) = \langle c_1(\mathfrak{s}(\mathbf{z})), H(\mathcal{P}) \rangle + 2n_w(\mathcal{P}),$$

and the result follows. This proves that d is independent of ψ .

Next, we check that d is independent of $\mathbf{x}$ and $\mathbf{y}$. Let $\tilde{\mathbf{x}}$ be another generator of $\widehat{\text{CF}}(\mathcal{H}^1)$ such that $\mathfrak{s}(\tilde{\mathbf{x}}) = \mathfrak{s}$. Then there is a Whitney disc $\phi \in \pi_2(\tilde{\mathbf{x}}, \mathbf{x})$,

hence $\phi \# \psi \in \pi_2(\tilde{\mathbf{x}}, \vartheta, \mathbf{y})$. Then, by Equation (4.13),

$$\begin{aligned} d &= \tilde{g}r(\mathbf{y}) - \tilde{g}r(\mathbf{x}) + \mu(\psi) - 2n_w(\psi) \\ &= \tilde{g}r(\mathbf{y}) - \tilde{g}r(\mathbf{x}) + \mu(\psi) - 2n_w(\psi) + (\tilde{g}r(\mathbf{x}) - \tilde{g}r(\tilde{\mathbf{x}}) + \mu(\phi) - 2n_w(\phi)) \\ &= \tilde{g}r(\mathbf{y}) - \tilde{g}r(\tilde{\mathbf{x}}) + \mu(\phi \# \psi) - 2n_w(\phi \# \psi). \end{aligned}$$

Thus, d is independent of $\mathbf{x}$. An analogous argument shows independence of $\mathbf{y}$.

Finally, all the holomorphic triangles that appear in the definition of the map $f_{\mathcal{H}^1, \mathbb{L}, \mathfrak{s}}$ satisfy $\mu(\psi) = 0$ and $n_w(\psi) = 0$. Then, it follows from Equation (4.14) that $f_{\mathcal{H}^1, \mathbb{L}, \mathfrak{s}}$ increases the absolute grading $\tilde{g}r$ by d . In particular, it preserves the relative grading gr . $\square$

4.4.4 NATURALITY MAPS

We already know from the work of Ozsváth and Szabó⁶⁰ that the naturality maps preserve the Maslov grading. Alternatively, one can prove that the handleslide and isotopy maps preserve the Maslov grading using the techniques of Lemma 4.44. The (de)stabilisation maps are already isomorphisms on the chain level.

4.4.5 PROOF OF THEOREM 4.40

As explained in Section 4.4.1,

$$f_{\mathcal{C}} = f_{\mathcal{H}^2, \mathbb{S}, \mathfrak{s}} \circ f_{\mathcal{H}_{\mathbb{L}}^1, \mathcal{H}^2, \mathfrak{s}} \circ f_{\mathcal{H}^1, \mathbb{L}, \mathfrak{s}} \circ f_{\mathcal{H}_{\mathbb{P}}^0, \mathcal{H}^1, \mathfrak{s}} \circ f_{\mathcal{H}^0, \mathbb{P}, \mathfrak{s}}.$$

All the above maps preserve the relative Maslov grading by Lemmas 4.42, 4.43 and 4.44, so $f_{\mathcal{C}}$ shifts the absolute Maslov grading by some constant e . It follows that the maps induced between the spectral sequences $E^r(f_{\mathcal{C}})$ shift the absolute Maslov grading by the same constant e . On the other hand, the map in total homology is $\text{id}_{\widehat{\text{HF}}(S^3)}$, which is homogeneous of degree 0, so we obtain that $e = 0$. $\square$

4.5 A NON-VANISHING RESULT

By Theorem 4.2, the map induced by $f_{\mathcal{C}}$ in the E^∞ page is non-trivial, hence so must be the map in the first page E^1 . In this section, we make this statement more precise, by keeping in mind also the gradings.

Definition 4.45 (Juhász-M.³³, Definition 5.6). We say that an element $x \in \widehat{\text{HFK}}(K, P)$ *survives* the spectral sequence to $\widehat{\text{HF}}(S^3) \cong \mathbb{F}_2$ if there is a sequence of cycles $x_i \in E^i$ for $i \geq 1$ such that $x_1 = x$ and $x_{i+1} = [x_i]$; we denote the set of such elements by $A(K, P)$. Furthermore, we have a partition $A(K, P) = A_0(K, P) \cup A_1(K, P)$, where $A_j(K, P)$ consists of those elements for which $x_i = j \in \mathbb{F}_2$ for i sufficiently large (note that the spectral sequence is bounded).

The subset $A_0(K, P)$ is a linear subspace of $A(K, P)$, and $A_1(K, P)$ is an affine translate of $A_0(K, P)$. Each set $A(K, P)$, $A_0(K, P)$, and $A_1(K, P)$ is a knot invariant.

It follows from the definition of the τ invariant (see Section 4.2.2) that

$$A_1(K, P) \cap \widehat{\text{HFK}}(K, P, i) \begin{cases} = \emptyset & \text{if } i \neq \tau(K) \\ \neq \emptyset & \text{if } i = \tau(K). \end{cases} \quad (4.16)$$

If $a \in A_1(K)$, let a_0 denote the homogeneous component of a in homological grading zero. It is straightforward to check that a_0 survives the spectral sequence. Since the homological grading on $\widehat{\text{CFK}}$ is inherited from the one on $\widehat{\text{CF}}$, and since the homological grading of $1 \in \widehat{\text{HF}}(S^3)$ is zero, it follows that $a_0 \in A_1(K)$. Combined with Equation (4.16), this implies that

$$A'_1(K, P) := A_1(K, P) \cap \widehat{\text{HFK}}_0(K, P, \tau(K)) \neq \emptyset. \quad (4.17)$$

Note that $A'_1(K, P)$ is also a knot invariant.

Corollary 4.46 (Juhász-M.³³, Corollary 5.7). *Let $\mathcal{C} = (X, F, \sigma)$ be a decorated concordance from (K_0, P_0) to (K_1, P_1) , and let $\tau = \tau(K_0) = \tau(K_1)$. Then, for $j \in \{0, 1\}$,*

$$F_{\mathcal{C}}(A_j(K_0)) \subseteq A_j(K_1)$$

and hence F_C is non-zero from $\widehat{\text{HFK}}_0(K_0, P_0, \tau)$ to $\widehat{\text{HFK}}_0(K_1, P_1, \tau)$.

Proof. The fact that $F_C(A_j(K_0)) \subseteq A_j(K_1)$ follows from the fact that F_C is the E^1 map in a morphism of spectral sequences, where the E^∞ map is $\text{id}_{\mathbb{F}_2}$ (see Theorem 4.2). Again by Theorem 4.2, F_C preserves both the Alexander and the Maslov grading, hence it maps $\widehat{\text{HFK}}_0(K_0, P_0, \tau)$ to $\widehat{\text{HFK}}_0(K_1, P_1, \tau)$. So we only need to prove that this map is non-zero.

By Equation (4.17), we have $A'_1(K_0, P_0) \neq \emptyset$; let $x \in A'_1(K_0, P_0)$. Then, by the previous paragraph,

$$F_C(x) \in A_1(K_1, P_1) \neq 0,$$

hence $F_C(x) \neq 0$. □

4.5.1 AN INVARIANT OF SLICE DISCS

In this section we define an invariant of slice discs, and we apply it to propose a refinement of an approach of Freedman, Gompf, Morrison and Walker for disproving the 4-dimensional Poincaré conjecture. We start with a technical lemma.

Lemma 4.47 (Juhász-M.³³, Lemma 3.11). *Let $\mathcal{C} = (X, F, \sigma)$ be a decorated concordance from (K_0, P_0) to (K_1, P_1) . If K_0 is the unknot U , then the element*

$$F_C(1) \in \widehat{\text{HFK}}_0(K_1, P_1, 0),$$

where $1 \in \widehat{\text{HFK}}(K_0, P_0) \cong \mathbb{F}_2$, is independent of the decorations σ and P_0 .

Proof. Suppose that σ' is another decoration with the same endpoints as σ , let $\mathcal{C}' = (X, F, \sigma')$, and define

$$k = [\sigma' - \sigma] \in H_1(F) \cong \mathbb{Z}.$$

Consider the decorated concordance $\mathcal{C}_k = (S^3 \times I, U \times I, \sigma_k)$, where σ_k spirals around k times. Then $\mathcal{C}' = \mathcal{C} \circ \mathcal{C}_k$. As $\widehat{\text{HFK}}(U) \cong \mathbb{F}_2$, we have

$F_{\mathcal{C}_k} = \text{id}_{\mathbb{F}_2}$. By the functoriality of the knot concordance maps, we obtain that $F_{\mathcal{C}'} = F_{\mathcal{C}}$.

Since $\widehat{\text{HFK}}(U) \cong \mathbb{F}_2$ has no non-trivial automorphisms, it does not matter how we choose the markings P_0 . $\square$

Definition 4.48 (Juhász-M.³³). Let B be an integral homology 4-ball with boundary S^3 . Suppose that $S \subset B$ is a slice disc for a decorated knot (K, P) in S^3 . If we remove a ball from B about a point of S , we obtain a concordance $\mathcal{C}(S)$ from the unknot U to K , which we endow with two arbitrary arcs as decorations. We define the element

$$t_{S,P} := F_{\mathcal{C}(S)}(1) \in \widehat{\text{HFK}}_0(K, P, 0).$$

Corollary 4.49 (Juhász-M.³³). *The element $t_{S,P}$ does not depend on the decorations chosen on $\mathcal{C}(S)$, it is always non-zero, and it is an invariant of the surface S up to isotopy in B fixing K .*

Proof. This is a straightforward consequence of Lemma 4.47 and Corollary 4.46. $\square$

Question 4.50 (Juhász-M.³³, Question 1.4). Can $t_{S,P}$ distinguish different slice discs? More precisely, is there a decorated knot (K, P) in S^3 that has two different slice discs S and S' in D^4 such that $t_{S,P} \neq t_{S',P}$?

Note that, given different decorations P and P' on K , the basepoint moving map of Sarkar⁷² takes $t_{S,P}$ to $t_{S,P'}$, so the answer is independent of the choice of basepoints.

We can use the element $t_{S,P}$ to refine the approach of Freedman, Gompf, Morrison, and Walker¹⁵ for disproving the smooth 4-dimensional Poincaré conjecture.

Conjecture 4.51 (SPC4). *If a smooth 4-manifold Σ^4 is homotopy equivalent to S^4 , then it is diffeomorphic to S^4 .*

Note that, by the work of Freedman¹⁴, we know that Σ must be *homeomorphic* to S^4 .

Our proposed refined approach to disprove SPC4 goes as follows. Suppose that we are given a counterexample with no 3-handles and a single 4-handle. Removing the 4-handle, we obtain an exotic 4-ball B with boundary homeomorphic to S^3 . The belt circles of the 2-handles give a link $L \subset \partial B$, and the cocores of the 2-handles give a collection of discs $C \subset B$ with boundary L . If we band sum the components of L in some way, we obtain a knot $K \subset \partial B$, together with a disc $D \subset B$ obtained from C . Hence D induces an element $t_{D,P} \in \widehat{\text{HFK}}(K, P)$ for any decoration P . If $t_{D,P} \neq t_{S,P}$ for S an arbitrary slice disc of K , then this implies that B is indeed exotic.

The approach of Freedman et al. only works if K is not slice in the standard 4-ball, but it is in the homotopy 4-ball B . By the work of Ozsváth and Szabó⁵⁶ (Theorem 1.1), the τ invariant vanishes if K bounds a disc in a homotopy ball, and so does Rasmussen's s invariant according to Kronheimer and Mrowka⁴², so neither can be used for the above purpose. We could use any other theory equipped with knot concordance maps in manifolds homeomorphic to $S^3 \times I$. However, note that the Khovanov homology concordance maps of Jacobsson²⁶ are only defined when the ambient manifold is diffeomorphic to $S^3 \times I$.

4.6 INVERTIBLE CONCORDANCES

A knot is called *doubly slice* if it is a hyperplane cross-section of an unknotted S^2 in S^4 . Motivated by a question of Fox¹³ asking which knots are doubly slice, Sumners⁷⁸ introduced the notion of invertible knot cobordisms. In his terminology, cobordism stands for concordance; we use the latter for clarity.

Definition 4.52. Let K_0 and K_1 be knots in S^3 . We say that a concordance $(S^3 \times I, F)$ from K_0 to K_1 is *invertible* if there is a concordance $(S^3 \times I, F')$ from K_1 to K_0 such that the composition of $(S^3 \times I, F)$ and $(S^3 \times I, F')$ from K_0 to K_0 is equivalent to the trivial cobordism. We write $K_0 \leq K_1$ if there is an invertible concordance from K_0 to K_1 .

In other words, F is invertible if and only if $(S^3 \times I, F)$ has a left inverse in the cobordism category of links. A knot K is doubly slice if and only if $U \leq K$. The relation $\leq$ is a partial order on the set of knots in S^3 , as we shall explain later. There are also two other partial orders on the set of knots in S^3 which compare to the one induced by invertible concordances.

Definition 4.53 (Rong⁷⁰, Wang⁸²). Given two knots K_0 and K_1 in S^3 , we say that K_1 *1-dominates* K_0 if there is a degree one map

$$(E(K_1), \partial E(K_1)) \rightarrow (E(K_0), \partial E(K_0)),$$

where $E(K_i)$ denotes the exterior of K_i . In such a case we write $K_1 \succeq_1 K_0$.

Definition 4.54 (Silver-Whitten⁷⁵). For $i = 0, 1$, let $K_i \subset S^3$ be knots with meridian-longitude pair m_i, ℓ_i . We denote $K_1 \succeq K_0$ if there is an epimorphism $f: \pi_1(K_1) \rightarrow \pi_1(K_0)$ that *preserves the peripheral structure*, i.e., such that $f(\langle m_1, \ell_1 \rangle)$ is conjugate to a subgroup of $\langle m_0, \ell_0 \rangle$.

Silver and Whitten⁷⁵ proved that $\succeq$ is a partial order on the set of knots. By the following lemma, also $\geq$ and $\succeq_1$ are partial orders on the set of knots.

Lemma 4.55. *For K_0 and K_1 knots in S^3 , we have the following chain of implications:*

$$K_1 \geq K_0 \Rightarrow K_1 \succeq_1 K_0 \Rightarrow K_1 \succeq K_0.$$

As far as I can trace it, the first implication in the above lemma was essentially observed by Ruberman, whereas the second one is due to Silver and Whitten⁷⁵.

Proof. For the first implication, let F be an invertible concordance from K_0 to K_1 with inverse F' . Then there is a diffeomorphism $d: S^3 \times I \rightarrow S^3 \times I$ such that $d(F' \circ F) = K_0 \times I$ and $d|_{S^3 \times \partial I}$ is the identity. Let $i: S^3 \rightarrow S^3 \times I$ be the embedding $i(x) = (x, 1/2)$, and let $p: S^3 \times I \rightarrow S^3$ be the projection. Then the composition

$$f = p \circ d \circ i: S^3 \rightarrow S^3$$

restricts to a degree one map between the knot exteriors. The fact that $f|_{E(K_1)}$ is degree one follows from the fact that we can isotope d such that $d(K_1 \times \{1/2\})$ becomes transverse to the I -fibration of $K_0 \times I$, and hence $f|_{K_1}$ is an embedding with image K_0 .

For the second implication, see Section 4 of Silver and Whitten's paper⁷⁵.

□

From Theorem 4.2 we get the following obstruction to the existence of invertible concordances from K_0 to K_1 .

Theorem 4.56 (Juhász-M.³³, Theorem 1.6). *If there is an invertible concordance from K_0 to K_1 , then*

$$\dim \widehat{\text{HFK}}_j(K_0, i) \leq \dim \widehat{\text{HFK}}_j(K_1, i)$$

for every $i, j \in \mathbb{Z}$.

Proof. Suppose that F is an invertible concordance from K_0 to K_1 . Choose an arbitrary pair of points P_0 on K_0 and P_1 on K_1 , making them into decorated knots, and an arbitrary pair of arcs σ on F making F into a decorated concordance from (K_0, P_0) to (K_1, P_1) . Let F' be the inverse of F , and choose a decoration σ' on it such that (F', σ') is a decorated concordance from (K_1, P_1) to (K_0, P_0) . As the composition of F and F' is equivalent to the trivial cobordism $K_0 \times I$ from K_0 to K_0 , we can choose σ' such that the composition of $\mathcal{C} = (F, \sigma)$ and $\mathcal{C}' = (F', \sigma')$ is equivalent to the product decorated cobordism $(K_0 \times I, P \times I)$, where $P = \sigma \cap K_0$ is a pair of points. By the functoriality of $F_{\mathcal{C}}$ and the fact that a product cobordism induces the identity map,

$$F_{\mathcal{C}'} \circ F_{\mathcal{C}} = \text{id}_{\widehat{\text{HFK}}(K_0, P_0)},$$

and so $F_{\mathcal{C}}$ is injective. The injectivity of $F_{\mathcal{C}}$, together with the fact it preserves the bigrading (Theorem 4.2), implies that

$$\dim \widehat{\text{HFK}}_j(K_0, P_0, i) \leq \dim \widehat{\text{HFK}}_j(K_1, P_1, i)$$

for every $i, j \in \mathbb{Z}$. Up to isomorphism, $\widehat{\text{HFK}}_j(K_r, P_r, i)$ is independent of the choice of P_r , and the result follows. $\square$

For a knot K in S^3 , we denote its Seifert genus by $g(K)$. Ozsváth and Szabó⁵⁷ proved that knot Floer homology detects the genus of a knot, in the sense that

$$g(K) = \max \left\{ i \in \mathbb{Z} \mid \widehat{\text{HFK}}_*(K, i) \neq 0 \right\}.$$

Furthermore, knot Floer homology also detects fibredness of knots, as $\dim \widehat{\text{HFK}}_*(K, g(K)) = 1$ if and only if K is fibred. This was shown by Ghigini¹⁷ in the genus one case, and by Ni^{52,53} in the general case. A different proof was also given by Juhász^{28,29}. These two results, together with Theorem 4.56, immediately imply the following unpublished result of Ruberman.

Corollary 4.57 (Juhász-M.³³, Corollary 1.7). *Suppose that $K_0 \leq K_1$. Then*

1. $g(K_0) \leq g(K_1)$;
2. *if $g(K_0) = g(K_1)$ and K_1 is fibred, then so is K_0 .*

We now outline a more elementary proof of these results, which was communicated to me and Juhász by Ruberman. Actually, we will prove the following stronger statement. Also see the proof of Proposition 3.7 and the paragraph following it in Silver and Whitten's paper⁷⁵.

Proposition 4.58 (Ruberman). *Suppose that $K_0 \preceq_1 K_1$. Then*

1. $g(K_0) \leq g(K_1)$;
2. *if K_1 is fibred, then so is K_0 .*

Proof. In Section 4 of their paper, Silver and Whitten⁷⁵ show that, if there is a degree one map

$$f: (E(K_1), \partial E(K_1)) \rightarrow (E(K_0), \partial E(K_0)),$$

then, up to reversing orientation, $f(m_1) = m_0$ and $f(\ell_1) = \ell_0$. It follows that $f|_{\partial E(K_1)}$ is homotopic to a diffeomorphism. Therefore, we can suppose that f restricts to a diffeomorphism on the boundary. If S is a minimal genus Seifert surface for K_1 , then $f|_S$ satisfies the conditions of Gabai's generalisation of Dehn's lemma¹⁶ (Corollary 6.23), hence there exists a Seifert surface T of $K_0 = f(K_1)$ such that $g(T) \leq g(S)$. It follows that $g(K_0) \leq g(K_1)$. The generalised Dehn's lemma states that if M is a compact oriented 3-manifold, S a compact oriented surface with connected boundary, and $f: S \rightarrow M$ a map such that $f|_{\partial S}$ is an embedding and $f^{-1}(f(\partial S)) = \partial S$, then there exists an embedded surface T in M such that $\partial T = f(\partial S)$ and $g(T) \leq g(S)$.

For the proof of 2, Lemma 1.2 of Rong's paper⁷⁰ shows that f induces a surjection on the fundamental groups, and also on the commutator subgroups. If K_1 is fibred, then the commutator subgroup $\pi_1(E(K_1))'$ must be finitely generated, hence $\pi_1(E(K_0))'$ is also finitely generated, so K_0 is fibred by a result of Stallings⁷⁷. $\square$

Based on the above proof and Theorem 4.56, it is natural to ask the following questions.

Question 4.59. Does $K_1 \succeq K_0$ imply $\dim \widehat{\text{HFK}}_*(K_1, i) \geq \dim \widehat{\text{HFK}}_*(K_0, i)$ for every $i \in \mathbb{Z}$?

Question 4.60. Does $K_1 \succeq_1 K_0$ imply $\dim \widehat{\text{HFK}}_*(K_1, i) \geq \dim \widehat{\text{HFK}}_*(K_0, i)$ for every $i \in \mathbb{Z}$?

Note that, in light of Lemma 4.55, Question 4.60 is weaker than Question 4.59. A positive answer to Question 4.59 would imply the following conjecture.

Conjecture 4.61 (Silver-Whitten⁷⁵). *If $K_1 \succeq K_0$, then $g(K_1) \geq g(K_0)$.*

However, Question 4.60 (and therefore also Question 4.59) has a negative answer due to the following example constructed by Hom.

Example 4.62. Let $K_1 = (T_{2,3})_{2,3}$ be the $(2, 3)$ -cable of the right-handed trefoil $T_{2,3}$, and let $K_0 = T_{2,3}$. Then there is a degree one map

$$(E(K_1), \partial E(K_1)) \rightarrow (E(K_0), \partial E(K_0)).$$

Indeed, let $T \subset E(K_1)$ be the boundary of the solid torus used in the satellite construction for K_1 . Then the exterior of T is $E(K_0)$, hence fibred over S^1 . If we collapse the fibers to discs, we obtain a degree one map from the exterior of T to $D^2 \times S^1$, and hence from $E(K_1)$ to $E(K_0)$. But both K_0 and K_1 are determined by their Alexander polynomials, K_0 because it is alternating, and K_1 by the work of Hedden²⁰ (Theorem 1.0.6). The symmetrised Alexander polynomial of K_1 is

$$t^3 - t^2 + 1 - t^{-2} + t^{-3},$$

while the symmetrised Alexander polynomial of K_0 is $t - 1 + t^{-1}$. So $\widehat{\text{HFK}}(K_1, 1) = 0$ and $\widehat{\text{HFK}}(K_0, 1) = \mathbb{F}_2$, contradicting the desired inequality.

In light of this, we propose the following weaker question.

Question 4.63 (Juhász-M.³³, Question 1.9). Suppose that $K_1 \succeq K_0$. Then is it true that

$$\dim \widehat{\text{HFK}}(K_1) \geq \dim \widehat{\text{HFK}}(K_0)?$$

Compare this with the following conjecture.

Conjecture 4.64 (Karakurt-Lidman³⁵). *If $f: Y_1 \rightarrow Y_0$ is a non-zero degree map between integer homology spheres, then $\dim \widehat{\text{HF}}(Y_1) \geq \dim \widehat{\text{HF}}(Y_0)$.*

4.7 THE IMITATION GAME

As pointed out to me by Boileau, invertible concordances are closely related to the notion of imitation, introduced by Kawachi³⁶. In this section there are no new results, but we explain the connection and review some classical questions.

Following Kawauchi's terminology, we say that an (m, ℓ) -manifold pair (M, L) consists of a smooth m -manifold M and a properly embedded ℓ -submanifold L .

Definition 4.65 (Kawauchi³⁶). Let (M, L) be an (m, ℓ) -manifold pair, and $I = [-1, 1]$. A *reflection* in $(M, L) \times I$ is a smooth involution α on $(M, L) \times I$ such that $\alpha(M \times \{1\}) = M \times \{-1\}$, the set of fixed points $\text{Fix}(\alpha)$ is an m -manifold, and α acts non-freely on each component of $M \times I$. A smooth embedding ϕ from an (m, ℓ) -manifold pair (M_1, L_1) to $(M_0, L_0) \times I$ with $\phi(M_1, L_1) = \text{Fix}(\alpha)$ for a reflection α in $(M_0, L_0) \times I$ is called a *reflector* (of the reflection α).

Kawauchi³⁶ showed that, given a reflection on $(M, L) \times I$, $\text{Fix}(\alpha)$ is an (m, ℓ) -manifold pair.

Definition 4.66 (Kawauchi³⁶). A map $f: (M_1, L_1) \rightarrow (M_0, L_0)$ between (m, ℓ) -manifold pairs is called an *imitation* if there is a reflector

$$\phi: (M_1, L_1) \rightarrow (M_0, L_0) \times I$$

such that $f = p_1 \circ \phi$, where $p_1: (M_0, L_0) \times I \rightarrow (M_0, L_0)$ is the projection on the first factor.

The imitation f is called *pure* if $f = p_1 \circ \phi$ for a reflector ϕ of a reflection α satisfying $\alpha(x, 1) = (x, -1)$ for all $x \in M$.

Definition 4.67. If K_0 and K_1 are knots in S^3 , we say that $K_1 \succeq_i K_0$ if there is a pure imitation $(S^3, K_1) \rightarrow (S^3, K_0)$.

It follows from the work of Kawauchi that $\succeq_i$ is a partial order on the set of knots. The transitive property was proven by Kawauchi³⁶ (Proposition 2.1), and the antisymmetric property follows from

$$K_1 \succeq_i K_0 \Rightarrow K_1 \succeq_1 K_0, \tag{4.18}$$

together with the fact that $\succeq_1$ is already known to be a partial order. Equation (4.18) follows from Properties I and II of the work of Kawauchi³⁶. Also compare (4.18) with Lemma 4.55.

Remark 4.68. Assuming the smooth 4-dimensional Schönflies conjecture (SSC4), which is recalled below, it is straightforward to show that $K_1 \succeq_i K_0$ also implies $K_1 \geq K_0$.

Conjecture 4.69 (SSC4). *A smoothly embedded S^3 in S^4 separates S^4 into two standard 4-balls.*

A similar relation to $\succeq_i$ can also be defined on connected closed oriented 3-manifolds. We let $Y_1 \succeq_i Y_0$ if there is a pure imitation $f: Y_1 \rightarrow Y_0$. We also define $Y_1 \geq Y_0$ if there is an invertible cobordism X^4 from Y_0 to Y_1 , and $Y_1 \succeq_1 Y_0$ if there is a degree one map from Y_1 to Y_0 . Here, by analogy with invertible concordances (cf. Definition 4.52), we call the cobordism X^4 *invertible* if there is a left inverse in the cobordism category of 3-manifolds. It follows from Lemma 1.1 of Kawauchi's paper³⁶ that

$$Y_1 \succeq_i Y_0 \Rightarrow Y_1 \geq Y_0, \quad (4.19)$$

and there is an invertible cobordism X from Y_0 to Y_1 such that its left inverse is X turned upside down. Moreover, we can show that

$$Y_1 \geq Y_0 \Rightarrow Y_1 \succeq_1 Y_0. \quad (4.20)$$

Indeed, if we denote by X_- the invertible cobordism from Y_0 to Y_1 , and by X_+ its left inverse, we have that

$$\mathbb{Z} \cong H_3(Y_0; \mathbb{Z}) \subseteq H_3(X_{\pm}; \mathbb{Z}),$$

since $X_{\pm}$ retracts onto Y_0 . By the long exact sequence in homology associated to the splitting $Y_0 \times I = X_- \cup_{Y_1} X_+$, and the fact that $H_2(Y_1; \mathbb{Z})$ is torsion-free, we get a short exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & H_3(Y_1) & \xrightarrow{(\iota_-, \iota_+)} & H_3(X_-) \oplus H_3(X_+) & \xrightarrow{\pi_- - \pi_+} & H_3(Y_0 \times I) \longrightarrow 0 \\ & & \downarrow \text{id} & & \downarrow \text{id} & & \downarrow \text{id} \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \oplus \mathbb{Z} & \longrightarrow & \mathbb{Z} \longrightarrow 0 \end{array}$$

If we prove that ι_- and ι_+ are isomorphisms, then the maps $\pi_{\pm} \circ \iota_{\pm}$ are

isomorphisms too, so the composition map

$$Y_1 \hookrightarrow X_- \hookrightarrow Y_0 \times I \rightarrow Y_0$$

would be a degree one map. We now prove that ι_- is an isomorphism. We already know that it is injective. By Poincaré duality, we can consider it as a map

$$\iota_- : H^0(Y_1) \rightarrow H^1(X_-, -Y_0 \cup Y_1).$$

The fact that it is also surjective follows from the long exact sequences in cohomology associated to the pair $(X_-, -Y_0 \cup Y_1)$, together with the fact that $H^1(X_-)$ is torsion-free.

Rong⁷⁰ proved that $\succeq_1$ is a partial order on the set of 3-manifolds modulo homotopy equivalence. It follows that $\succeq_i$ (and analogously $\geq$) is a partial order on the set of 3-manifolds modulo some equivalence relation, which can be thinner than that of homotopy equivalence. In particular, as far as I know, it is also possible that $\succeq_i$ is a partial order on the set of 3-manifolds.

Question 4.70. Is $\succeq_i$ a partial order on the set of 3-manifolds?

Note that for a large class of 3-manifolds, namely hyperbolic 3-manifolds, this is true. However, also $\succeq_1$ is a partial order on this class of 3-manifolds, because, by Mostow rigidity, two hyperbolic 3-manifolds are homotopy equivalent if and only if they are homeomorphic.

The converses of (4.19) and (4.20) do not hold. For (4.20), every 3-manifold admits a degree one map to S^3 , but, for example, there is no invertible cobordism from S^3 to Poincaré sphere $\Sigma(2, 3, 5)$: if there was one, then one would get an embedding of $\Sigma(2, 3, 5)$ into S^4 , which is impossible because the Rokhlin invariant of $\Sigma(2, 3, 5)$ is 1. For (4.19), a cancelling pair of 4-dimensional 1- and 2-handles gives a product cobordism $S^3 \times I$, where the middle level set is $S^1 \times S^2$. Therefore, $S^1 \times S^2 \geq S^3$. However, only integer homology spheres can imitate S^3 , by Kawauchi³⁶ (Property II). Connected to this point, we ask the following question.

Question 4.71. Which integer homology spheres imitate S^3 ?

Since an imitation induces an epimorphism in the fundamental groups, it follows that the only 3-manifold that S^3 can imitate is S^3 . However, there may be reflections α on $S^3 \times I$, with $\text{Fix}(\alpha) \cong S^3$, which are not isotopic to the standard one. In light of Remark 4.73, the following question is a weaker version of SSC4.

Question 4.72. Is every reflection α on $S^3 \times I$, with $\text{Fix}(\alpha) \cong S^3$, isotopic to the standard one?

Remark 4.73. The following are equivalent:

1. Every invertible cobordism from S^3 to S^3 is the product cobordism;
2. SSC4 holds true.

In light of Remark 4.73, an analogue of the smooth Schönflies problem for knots would be whether every invertible concordance from the unknot to itself is a product concordance. When translated to a language closer to the one of SSC4, this question appears as in the following conjecture, for which we propose the evocative name of *four spheres' conjecture*.

Conjecture 4.74. *Suppose that a smoothly embedded S^3 in S^4 cuts it into two standard 4-balls and intersects an unknotted S^2 in an unknotted S^1 . Then there is a decomposition*

$$(S^4, S^2) = (B^4, B^2)_{\text{std}} \cup_{S^3} (B^4, B^2)_{\text{std}}.$$

Equivalently, if we view $S^4 = \{\mathbf{x} = (x^1, \dots, x^5) \mid \|\mathbf{x}\| = 1\}$, then, up to ambient isotopy, $S^3 = S^4 \cap \{x^5 = 0\}$ and $S^2 = S^4 \cap \{x^3 = x^4 = 0\}$.

Note that, as in the case of SSC4, the Heegaard Floer TQFT approach cannot detect a counterexample to Conjecture 4.74, since all invertible concordances from the unknot U to itself induce the only automorphism in $\widehat{\text{HFK}}(U) \cong \mathbb{F}_2$.

4.8 OTHER FLAVOURS OF HEEGAARD FLOER HOMOLOGY

In this section we generalise Theorem 4.2 to all flavours of Heegaard Floer homology. The content of this section was developed after a conversation with Levine, for whom I reserve a special acknowledgement.

Independent generalisations of Theorem 4.2 were done by Tovstopyat-Nelip⁸⁰ and Zemke⁸⁷.

First, we need to give a quick overview of all flavours of Heegaard Floer homology. We keep using $\mathbb{F}_2$ coefficients since we do not know whether Heegaard Floer homology is functorial with $\mathbb{Z}$ coefficients.

Suppose that Y^3 is a closed oriented 3-manifold, and $\mathcal{H} = (\Sigma, \alpha, \beta, w)$ is a pointed Heegaard diagram for it. Then we can define a chain complex

$$\mathrm{CF}^\infty(\mathcal{H}) = \mathbb{F}_2 \langle [\mathbf{x}, i] \mid \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, i \in \mathbb{Z} \rangle,$$

with differential

$$\partial([\mathbf{x}, i]) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{D \in \mathcal{D}(\mathbf{x}, \mathbf{y}) \\ \mu(D)=1}} \# \widehat{\mathcal{M}}(D) \cdot [\mathbf{y}, i - n_w(D)].$$

$\mathrm{CF}^\infty(\mathcal{H})$ has a structure of $\mathbb{F}_2[U]$ -module, where the U action is given by

$$U \cdot [\mathbf{x}, i] = [\mathbf{x}, i - 1].$$

Note that the differential commutes with the U action. Actually, $\mathrm{CF}^\infty(\mathcal{H})$ is also an $\mathbb{F}_2[U, U^{-1}]$ -module.

There is a subcomplex $\mathrm{CF}^-(\mathcal{H}) \subset \mathrm{CF}^\infty(\mathcal{H})$, generated by all elements $[\mathbf{x}, i]$ with $i \leq 0$, which is still an $\mathbb{F}_2[U]$ -module. The quotient complex, which is again an $\mathbb{F}_2[U]$ -module, is denoted by

$$\mathrm{CF}^+(\mathcal{H}) = \frac{\mathrm{CF}^\infty(\mathcal{H})}{U \cdot \mathrm{CF}^-(\mathcal{H})}.$$

Note that, by definition, we have that, as chain complexes,

$$\widehat{\text{CF}}(\mathcal{H}) = \frac{\text{CF}^-(\mathcal{H})}{U \cdot \text{CF}^-(\mathcal{H})}.$$

The U action on $\widehat{\text{CF}}(\mathcal{H})$ sends every element to 0, and therefore we only consider $\widehat{\text{CF}}(\mathcal{H})$ as a vector space over $\mathbb{F}_2$.

Ozsváth and Szabó⁶⁰ proved that, for each $\circ \in \{+, -, \infty, \wedge\}$, the homologies $\text{HF}^\circ(Y, p)$ of the above complexes are invariants of Y . Moreover, all above groups admit a natural splitting along spin^c structures on Y .

Given a doubly pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ for a null-homologous knot K in Y , with a Seifert surface S , one can build the complex

$$\text{CFK}_S^\infty(\mathcal{H}) = \mathbb{F}_2 \langle [\mathbf{x}, i, j] \mid \mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta, i, j \in \mathbb{Z}, j - i = \mathcal{A}_S(\mathbf{x}) \rangle,$$

with differential

$$\partial([\mathbf{x}, i, j]) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{D \in \mathcal{D}(\mathbf{x}, \mathbf{y}) \\ \mu(D)=1}} \# \widehat{\mathcal{M}}(D) \cdot [\mathbf{y}, i - n_w(D), j - n_z(D)].$$

As before, one can define $\text{CFK}_S^-(\mathcal{H})$, $\text{CFK}_S^+(\mathcal{H})$ and $\widehat{\text{CFK}}_S(\mathcal{H})$. All such complexes carry a $(\mathbb{Z} \oplus \mathbb{Z})$ -filtration, given by i and j . We call *Alexander filtration* the $\mathbb{Z}$ -filtration given by j . When Y is a $\mathbb{Q}HS^3$, or when K is a *local* knot in $\#_{i=1}^n S^1 \times S^2$, i.e., when K is contained in a ball $B^3 \subset S^1 \times S^2$ (this terminology was used by Celoria¹⁰), then the Alexander grading does not depend on the Seifert surface, and the filtered chain homotopy type of $\text{CFK}^\circ(Y, K, P)$ is an invariant of the decorated knot^{58,66}. The homology of the associated graded module is denoted by $\text{HFK}^\circ(Y, K, P)$. Note that by collapsing the j grading in $\text{CFK}^\circ(Y, K, \{w, z\})$ one obtains exactly $\text{CF}^\circ(Y, w)$. Note that, assuming Y $\mathbb{Q}HS^3$ or K local in $\#_{i=1}^n S^1 \times S^2$, the Alexander filtration on CFK° induces a spectral sequence

$$\text{HFK}^\circ(Y, K, \{w, z\}) \Rightarrow \text{HF}^\circ(Y, w).$$

If Y is a 3-manifold equipped with a torsion spin^c structure $\mathfrak{s}$, then Ozsváth and Szabó⁶⁰ showed that $\text{CF}^\circ(\mathcal{H}, \mathfrak{s})$ admits a relative $\mathbb{Z}$ -grading. For generators $[\mathbf{x}, i], [\mathbf{y}, \bar{i}] \in \text{CF}^\circ(\mathcal{H}, \mathfrak{s})$ and $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, we have

$$gr([\mathbf{x}, i], [\mathbf{y}, \bar{i}]) = 2(i - \bar{i}) + \mu(\phi) - 2n_w(\phi).$$

According to Ozsváth and Szabó⁶² (Theorem 7.1) that this can be lifted to an absolute $\mathbb{Q}$ -grading $\tilde{gr}$, in the sense that

$$gr([\mathbf{x}, i], [\mathbf{y}, \bar{i}]) = \tilde{gr}([\mathbf{x}, i]) - \tilde{gr}([\mathbf{y}, \bar{i}]).$$

Such a grading is called the *Maslov grading* or *homological grading*. It can alternatively be defined as

$$\tilde{gr}([\mathbf{x}, i]) = \tilde{gr}(\mathbf{x}) - 2i.$$

When Y is a $\mathbb{Q}HS^3$, or when K is a local knot in $\#_{i=1}^n S^1 \times S^2$, such a grading is defined in an analogous way on CFK° .

From the works of Ozsváth and Szabó⁶², Juhász³¹, and Zemke⁸⁴, there are maps induced in all flavours of HF. Given a handle decomposition of a cobordism (X, γ) from (Y_0, w_0) to (Y_1, w_1) , there are maps associated to each handle attachment, which behave functorially.

Recall that we proved in Section 4.3.3 that the gluing map in $\widehat{\text{HFK}}$ associated to a decorated concordance is trivial. One can then try to define, for all other flavours of HFK, maps associated to 1-, 2- and 3-handles attached away from the knot, just as in the case of cobordisms of 3-manifolds, and define the concordance map as the composition of such maps. This was done by Zemke⁸⁵, who constructed maps associated to every decorated link cobordism by following a Morse theoretical approach.

For each $\circ \in \{+, -, \infty, \wedge\}$, we denote by F_C° Zemke's concordance map. Note that $\widehat{F_C}$ agrees with Juhász's map F_C . Then we generalise Theorem 4.2 as follows.

Theorem 4.75. *Let $\mathcal{C} = (X, F, \sigma)$ be a decorated concordance between deco-*

rated knots (K_0, P_0) and (K_1, P_1) , such that $H_1(X) = H_2(X) = 0$, and let $\circ \in \{+, -, \infty, \wedge\}$. Then

$$F_C^\circ: \text{HFK}^\circ(K_0, P_0) \rightarrow \text{HFK}^\circ(K_1, P_1)$$

preserves both Maslov and Alexander grading.

Furthermore, given an admissible diagram $(\Sigma_r, \alpha_r, \beta_r, w_r, z_r)$ of (K_r, P_r) for $r \in \{0, 1\}$, there is a $(\mathbb{Z} \oplus \mathbb{Z})$ -filtered chain map

$$f_C^\circ: \text{CFK}^\circ(\Sigma_0, \alpha_0, \beta_0, w_0) \rightarrow \text{CFK}^\circ(\Sigma_1, \alpha_1, \beta_1, w_1)$$

of homological degree zero such that the induced morphism of spectral sequences arising from the Alexander filtration agrees with F_C° on the E^1 page and with $\text{id}_{\text{HF}^\circ(S^3)}$ on the total homology and on the E^∞ page.

Remark 4.76. Similar results were independently proven by Tovstopyat-Nelip⁸⁰ and Zemke⁸⁷. Tovstopyat-Nelip⁸⁰ gives an alternative definition of the map induced by a decorated concordance in HFK^- , and proves that it preserves the Alexander grading and the absolute $\mathbb{Z}/2$ Maslov grading. Zemke⁸⁷ gives computes the Alexander and Maslov grading shifts for all cobordism maps in all flavours of link Floer homology. As a particular case, he proves that the maps associated to decorated concordances preserve both the Alexander and the Maslov grading, which is exactly the first part of Theorem 4.75.

Proof. The chain map f_C° is defined exactly as in Equation (4.4), with the exception that all the map are now between CFK° complexes:

$$f_C^\circ := f_{\mathcal{H}^2, \mathbb{S}}^\circ \circ f_{\mathcal{H}_L^1, \mathcal{H}^2}^\circ \circ f_{\mathcal{H}^1, \mathbb{L}}^\circ \circ f_{\mathcal{H}_\mathbb{P}^0, \mathcal{H}^1}^\circ \circ f_{\mathcal{H}^0, \mathbb{P}}^\circ.$$

We use here the same notation as in Section 4.3.5. We will now check that each map preserves the $(\mathbb{Z} \oplus \mathbb{Z})$ -filtration and the relative homological grading.

The map associated to the 1-handles is defined as

$$f_{\mathcal{H}^0, \mathbb{P}}^\circ([\mathbf{x}, i, j]) = [\mathbf{x} \otimes T, i, j],$$

where T is the tuple of top-dimensional intersection points of the new α and β curves. The target of the map is CFK° because, due to Lemma 4.22, we have that

$$j - i = \mathcal{A}(\mathbf{x}) = \mathcal{A}(\mathbf{x} \otimes T).$$

The $(\mathbb{Z} \oplus \mathbb{Z})$ -filtration, and actually also the $(\mathbb{Z} \oplus \mathbb{Z})$ -grading, are preserved by $f_{\mathcal{H}^0, \mathbb{P}}^\circ$ because i and j remain unchanged under the 1-handle maps. The relative Maslov grading is also preserved, since any two intersection points $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are connected by a Whitney disc $\phi \in \pi_2(\mathbf{x}, \mathbf{y})$, and

$$gr([\mathbf{x} \otimes T, i, j], [\mathbf{y} \otimes T, \bar{i}, \bar{j}]) = 2(i - \bar{i}) + \mu(\phi) - 2n_w(\phi) = gr([\mathbf{x}, i, j], [\mathbf{y}, \bar{i}, \bar{j}]).$$

A dual reasoning shows that also 3-handle maps preserve the absolute $(\mathbb{Z} \oplus \mathbb{Z})$ -grading and the relative Maslov grading.

The 2-handle map is defined by

$$f_{\mathcal{H}^1, \mathbb{L}}^\circ([\mathbf{x}, i, j]) = \sum_{\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\gamma} \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y}) \\ \mu(\psi)=0}} \# \mathcal{M}(\psi)[\mathbf{y}, i - n_w(\psi), j - n_z(\psi)],$$

where ϑ denotes the top-dimensional intersection point of $\mathbb{T}_\beta \cap \mathbb{T}_\gamma$. Note that this is the map on the complexes for the 3-manifold, where the j grading is collapsed. To get the 2-handle map for the knot complexes, one should require that $n_z(\psi) = 0$.

The fact that the map $f_{\mathcal{H}^1, \mathbb{L}}^\circ$ is well defined follows from the fact that

$$\mathcal{A}(\mathbf{y}) + (i - n_w(\psi)) - (j - n_z(\psi)) = \mathcal{A}(\mathbf{x}) + i - j = 0,$$

where the first equality follows from the proof of Lemma 4.27 (in particular Equation (4.7) and the fact that $c = 0$).

The fact that the 2-handle map preserves the $(\mathbb{Z} \oplus \mathbb{Z})$ -filtration follows from

the fact that, if ψ has a pseudo-holomorphic representative, then $n_w(\psi) \geq 0$ and $n_z(\psi) \geq 0$. For the relative Maslov grading, consider the following quantity:

$$d = \tilde{g}r(\mathbf{y}) - \tilde{g}r(\mathbf{x}) + \mu(\psi) - 2n_w(\psi) + 2i - 2\bar{i},$$

associated to generators $[\mathbf{x}, i, j]$ and $[\mathbf{y}, \bar{i}, \bar{j}]$ in the torsion spin^c structure of Y_1 and Y_2 . Following the proof of Lemma 4.44, it is straightforward to show that d is independent of ψ , $\mathbf{x}$ and $\mathbf{y}$, and that therefore the relative Maslov grading is preserved by the 2-handle map.

The fact that naturality maps preserve the $(\mathbb{Z} \oplus \mathbb{Z})$ -filtration and the relative Maslov grading was proved by Ozsváth and Szabó⁵⁸, and Rasmussen⁶⁶. Alternatively, the strategy we used for the 2-handle maps readily applies to the naturality maps associated to isotopies and handleslides. The maps associated to (de)stabilisations are already isomorphisms at the level of chain complexes.

By composing all the maps, we obtain a map which preserves the $(\mathbb{Z} \oplus \mathbb{Z})$ -filtration and the relative Maslov grading. Therefore, it induces a morphism of spectral sequences. The map induced on the first page of the spectral sequence is obtained by setting $n_z(\psi) = 0$ in all the triangle maps. This yields exactly Zemke's cobordism maps⁸⁵ and the naturality maps for the knot chain complexes^{58,66}. We already proved in Theorem 4.2 that the map in total homology is the identity map on $\widehat{\text{HF}}(S^3)$. The fact that the map induces the identity in total homology also in the other flavours of Heegaard Floer homology follows from the commutation with the exact sequences (2) and (3) in Ozsváth and Szabó's paper⁵⁵.

We now prove that the absolute Maslov grading is preserved under F_C° . We already proved that the relative Maslov grading is preserved by each handle map and naturality map, therefore the absolute grading is shifted by a constant e . Since the maps induced in the total homology are the identity maps, $e = 0$ for all flavours.

Lastly, we need to show that the map in the E^∞ page is the identity. For a knot in S^3 , and $q \in \mathbb{Z}$, let x_q denote the unique Maslov grading q homoge-

neous non-trivial element of $\mathrm{HF}^\circ(S^3)$, if it exists. We define

$$\tau_q^\circ(K) := \min \{p \in \mathbb{Z} \mid \exists x \in \mathcal{F}_p(\mathrm{CFK}^\circ(K)) : [x] = x_q\}.$$

More generally, for a filtered complex C over $\mathbb{F}_2$ and $q \in \mathbb{Z}$ such that $H_q(C) \cong \mathbb{F}_2$, we define

$$\tau_q(C) := \min \{p \in \mathbb{Z} \mid \exists x \in \mathcal{F}_p(C) : [x] = 1 \in H_q(C)\}. \quad (4.21)$$

In a similar way as in Lemma 4.16, we prove that τ_q° is a decorated concordance invariant. Let C and $\overline{C}$ denote the complexes $\mathrm{CFK}^\circ(K_0, P_0)$ and $\mathrm{CFK}^\circ(K_1, P_1)$, $f: C \rightarrow \overline{C}$ the filtered map f_C° , and $p = \tau_q^\circ(C)$. Then we have the following commutative diagram.

$$\begin{array}{ccc} H(\mathcal{F}_p(C)) & \xrightarrow{f} & H(\mathcal{F}_p(\overline{C})) \\ \downarrow \iota_C & & \downarrow \iota_{\overline{C}} \\ \mathrm{HF}^\circ(S^3) & \xrightarrow{\mathrm{id}} & \mathrm{HF}^\circ(S^3) \end{array}$$

If $x \in \mathcal{F}_p(C)$ is an element representing x_q in $H(C)$, then by commutativity we have

$$\iota_{\overline{C}}(f([x])) = \iota_C([x]) = x_q,$$

and therefore $\tau_q^\circ(\overline{C}) \leq p = \tau_q^\circ(C)$. By exchanging the roles of C and $\overline{C}$ we get the other inequality.

The fact that the E^∞ map is the identity then follows from the Lemma 4.77, which is a refinement of Lemma 4.17. $\square$

Lemma 4.77. *Let $f: C \rightarrow \overline{C}$ be a filtered map of degree zero, homogeneous in the homological grading, of filtered complexes over $\mathbb{F}_2$, bounded in every homological grading (see Theorem 2.6 in McCleary's book⁵¹), such that*

1. $H_q(C) \cong \mathbb{F}_2$ and $H_q(\overline{C}) \cong \mathbb{F}_2$;
2. $\tau_q(C) = \tau_q(\overline{C}) = \tau$; and
3. $H_q(f): H_q(C) \rightarrow H_q(\overline{C})$ is an isomorphism.

Then $E_{\tau,q-\tau}^\infty(C) \cong \mathbb{F}_2$, $E_{\tau,q-\tau}^\infty(\overline{C}) \cong \mathbb{F}_2$, and the map

$$E_{\tau,q-\tau}^\infty(f): E_{\tau,q-\tau}^\infty(C) \rightarrow E_{\tau,q-\tau}^\infty(\overline{C})$$

is also an isomorphism.

Proof. Since 1. and 2. hold, by Theorem 4.9 and Equation (4.21), there are canonical isomorphisms

$$E_{\tau,q-\tau}^\infty(C) \cong H_q(C) \cong \mathbb{F}_2 \quad \text{and} \quad E_{\tau,q-\tau}^\infty(\overline{C}) \cong H_q(\overline{C}) \cong \mathbb{F}_2.$$

The commutativity of the following diagram concludes the proof.

$$\begin{array}{ccc} E_{\tau,q-\tau}^\infty(C) & \xrightarrow{E_{\tau,q-\tau}^\infty(f)} & E_{\tau,q-\tau}^\infty(\overline{C}) \\ \downarrow \cong & & \downarrow \cong \\ H_q(C) & \xrightarrow[H_q(f)]{\cong} & H_q(\overline{C}) \end{array}$$

□

As an immediate corollary of Theorem 4.75 we have the following result.

Corollary 4.78. *The E^∞ page of the spectral sequence induced by the Alexander filtration on $\text{CFK}^\circ(K, P)$ is a decorated concordance invariant.*

The E^∞ page is always isomorphic to $\text{HF}^\circ(S^3)$ as an $\mathbb{F}_2[U]$ -module, but the Alexander filtration carries interesting information about the knot. For instance, the information contained in $E^\infty(\widehat{\text{CFK}}(K))$ is precisely the classical τ invariant, because $E^\infty(\widehat{\text{CFK}}(K)) \cong \mathbb{F}_2$ in filtration level $\tau(K)$.

Note that the U action strictly decreases the filtration level on the E^∞ page. Suppose that $\mathcal{F}([a]) = p$. By definition, we can choose a representative a for $[a]$ in filtration level p . Then $\mathcal{F}(Ua) = p - 1$, and therefore $\mathcal{F}(U[a]) \leq p - 1$.

Remark 4.79. Note that, since the chain map f_C° is filtered also with respect to the i filtration, the E^∞ page carries also an i grading. Let $y_{\bar{j}}$ denote the non-trivial element in filtration level $\bar{j}$ in the E^∞ page, if it exists. Then,

the quantity

$$\min \{ \bar{i} \in \mathbb{Z} \mid \exists x \in C \{ j \leq \bar{j} \text{ or } (j = \bar{j} \text{ and } i \leq \bar{i}) \} : [x]_\infty = y_{\bar{j}} \}$$

is invariant under decorated concordance. Once again, this follows from the existence of filtered maps in both directions.

As an example, Figure 4.4 shows the E^∞ pages associated to $\text{CFK}^\pm$ of the two trefoil knots.

The E^∞ page associated to $\text{CFK}^-(S^3, K, P)$ looks very similar to Kratochvíč's reduced knot Floer complex⁴⁰. Kratochvíč's complex arises as a complex chain homotopy equivalent to $\text{CFK}^-(S^3, K, P)$. The E^∞ page is rather its homology.

Finally, one can apply Theorem 4.75 as in Section 4.5 to obtain a distinguished tower in $\text{CFK}^\circ(K, P)$ which is an invariant of a slice disc for K .

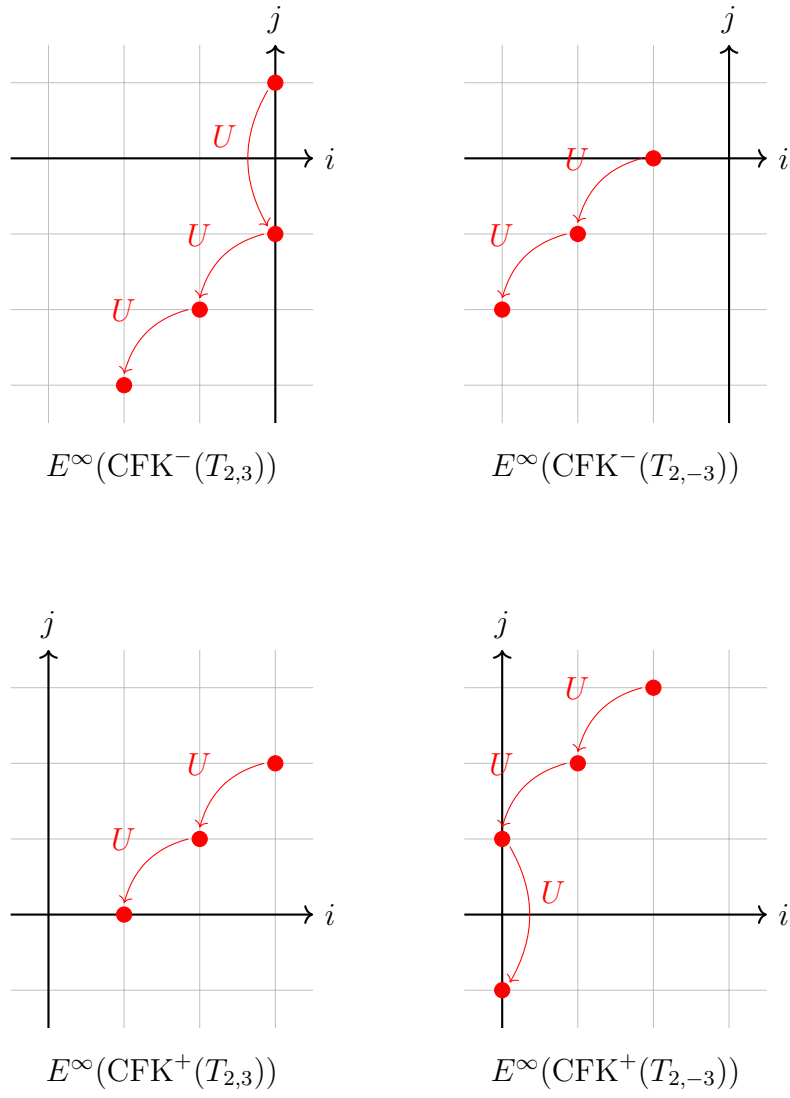


Figure 4.4: The picture shows the E^∞ pages for the complexes $\text{CFK}^\pm$ associated to the right-handed trefoil knot $T_{2,3}$ and the left-handed trefoil knot $T_{2,-3}$.

5

A MORSE THEORETICAL APPROACH TO LINK COBORDISMS

THIS chapter, whose content is based on joint work with Juhász³², is devoted to developing tools for computing Juhász's cobordism maps, by reinterpreting the maps associated to each elementary decorated link cobordism. In this chapter we only focus on decorated link cobordisms in $S^3 \times I$. Via Morse theory, we can write every decorated link cobordism as a composition of elementary link cobordisms. We define these elementary cobordisms in Section 5.1, and prove that they generate the decorated link cobordism category in Proposition 5.6. In an elementary cobordism (F, σ) , the height function $h: S^3 \times I \rightarrow I$ has at most one generic critical point when restricted to F and to σ , and $\text{Crit}(h|_F) \subseteq \text{Crit}(h|_\sigma)$. In a *birth* cobordism, $h|_F$ has a local minimum. In a *saddle*, $h|_F$ has index one and $h|_\sigma$ has a local maximum. We distinguish *merge* and *split* saddles depending on whether the number of link components decreases or increases by one, respectively. In a *death* cobordism, $h|_F$ has a local maximum. A *stabilisation* is an elementary cobordism where $h|_F$ has no critical points, and $h|_\sigma$ has a local minimum, and a *destabilisation* is when $h|_\sigma$ has a local maximum. A

(de)stabilisation is *positive* (resp. *negative*) if the bigon component of $F \setminus \sigma$ lies in $R_+(\sigma)$ (resp. in $R_-(\sigma)$). Finally, an *isotopy* is a cobordism where $h|_F$ and $h|_\sigma$ have no critical points.

In Sections 5.2-5.8 we give a description of the maps associated to the aforementioned elementary cobordisms. The most interesting one is perhaps the one associated to a saddle cobordism, which coincides with a map appearing in an oriented skein exact triangle proven by Ozsváth and Szabó, or, more precisely, a decorated version of it, which we state in Section 5.5. This description of the cobordism maps in terms of elementary pieces is more similar in spirit to the constructions of Alishahi and Eftekhary¹, Sarkar⁷¹, and Zemke⁸⁵.

In Section 5.9 we study the duality properties of the link cobordism maps. It follows from the work of Ozsváth and Szabó⁶² (Theorem 3.5) that the maps induced in Heegaard Floer homology by cobordisms of 3-manifolds enjoy a duality property. See also Juhász's paper³¹ (Theorem 11.8) for a version for special cobordisms in sutured Floer homology. It is in general an open question whether any sutured cobordism enjoys a duality property. In the case of link cobordism maps, we prove a duality statement for all elementary cobordisms except saddles, which still remain mysterious.

Lastly, in Section 5.10, we use the Morse theoretical description of decorated link cobordisms to give a formula for the Alexander and Maslov grading shifts of the map induced in link Floer homology. Zemke⁸⁷ computed the grading shifts for the maps that he defined for all flavours of link Floer homology. Note that, at the time of writing, it is not clear whether the maps that he defined for the hat flavour agree with Juhász's maps. However, the grading shifts that he finds for the hat flavour of link Floer homology agree with the ones that we find in Section 5.10.

5.1 ELEMENTARY LINK COBORDISMS

We now define the elementary decorated link cobordisms, and prove that they generate all decorated link cobordisms in $S^3 \times I$. Since we will always

consider links in S^3 and link cobordisms in $S^3 \times I$, we drop them from the notation, and we only write (L, P) for a decorated link, and (F, σ) for a decorated link cobordism. Recall that the link Floer homology and the cobordism maps are defined for orientable L and F , and a choice of orientation is not required, strictly speaking, for their definitions. However, since we will compute the Alexander and the Maslov grading shifts, and since we will use the gradings to distinguish elements in link Floer homology, we will suppose that links and surfaces are always oriented. Recall from Definition 3.15 that an orientation of F induces a splitting $F = R_+(\sigma) \cup R_-(\sigma)$.

Definition 5.1. A decorated cobordism (F, σ) from (L, P_0) to (L, P_1) is a *stabilisation* (resp. *destabilisation*) if

- $F = L \times I$;
- the only critical point of the height function $h: L \times I \rightarrow I$ on σ is a non-degenerate minimum (resp. maximum); and
- all but one component of σ is of the form $\{x\} \times I$ for some $x \in L$.

If the bigon component of $F \setminus \sigma$ lies in $R_+(\sigma)$, then we say that (F, σ) is a *positive* stabilisation (resp. destabilisation), and is *negative* otherwise. Note that this depends on the orientation of F .

Definition 5.2. A *birth* cobordism from (L_0, P_0) to (L_1, P_1) is a decorated cobordism (F, σ) such that

- (L_1, P_1) is the split union of (L_0, P_0) and (U_1, P_{U_1}) , the unknot with two decorations;
- $F = F_0 \sqcup D$, where $F_0 = L_0 \times I \subseteq S^3 \times I$ and D is a disc such that $\partial D = U_1$, and embedded in $S^3 \times I$ such that the height function $h|_D$ has a single critical point of index 0; and
- if $\sigma_0 = \sigma \cap F_0$ and $\sigma_D = \sigma \cap D$, then σ_0 consists of $|P_0|$ vertical lines, one for each component of $L_0 \setminus P_0$, and σ_D is a single arc on D such that $\text{Crit}(h|_{\sigma_D}) = \text{Crit}(h|_D)$.

Definition 5.3. A *death* cobordism from (L_0, P_0) to (L_1, P_1) is a decorated cobordism (F, σ) such that

- (L_0, P_0) is the split union of (L_1, P_1) and (U_1, P_{U_1}) , the unknot with two decorations;
- $F = F_1 \sqcup D$, where $F_1 = L_1 \times I \subseteq S^3 \times I$ and D is a disc such that $\partial D = U_1$, and embedded in $S^3 \times I$ such that the height function $h|_D$ has a single critical point of index 2; and
- if $\sigma_1 = \sigma \cap F_1$ and $\sigma_D = \sigma \cap D$, then σ_1 consists of $|P_1|$ vertical lines, one for each component of $L_1 \setminus P_1$, and σ_D is a single arc on D such that $\text{Crit}(h|_{\sigma_D}) = \text{Crit}(h|_D)$.

Notice that, by turning a birth (resp. death) cobordism upside down, one obtains a death (resp. birth) cobordism.

Definition 5.4. A *saddle* cobordism from (L_0, P_0) to (L_1, P_1) is a decorated cobordism (F, σ) such that

- $|P_1| = |P_0| - 2$;
- the diffeomorphism $f: S^3 \times \{0\} \rightarrow S^3 \times \{1\}$ defined by $f(x, 0) = (x, 1)$ maps L_0 to L_1 , except in the neighbourhood of a small square $S \approx I \times I$ in S^3 that intersects L_0 and L_1 in the arcs $I \times \partial I$ and $\partial I \times I$, respectively, as illustrated in Figure 5.1; furthermore, $f^{-1}(P_1) \subseteq P_0$;
- the surface F in $S^3 \times I$ is a product outside a neighbourhood of $S \times I$, and it interpolates between L_0 and L_1 inside $S \times I$, as shown in Figure 5.1; and
- σ consists of an arc supported on the square S shown in green in Figure 5.1, and outside the square it consists of small parallel translates of $P_0 \times I$, in such a way that

$$\pi_0(\partial\sigma) \rightarrow \pi_0((L_0 \setminus P_0) \sqcup (L_1 \setminus P_1))$$

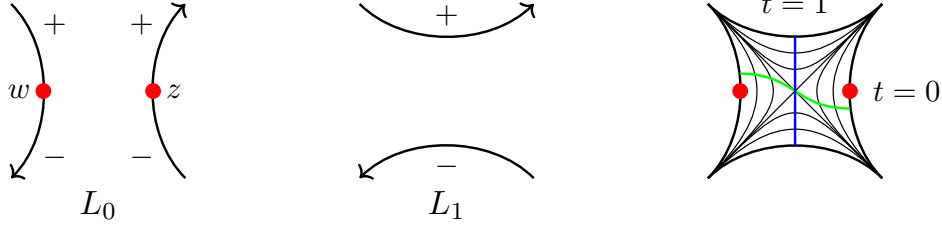


Figure 5.1: This figure illustrates a saddle cobordism. The first two pictures show how L_0 and L_1 differ in a neighbourhood of the square S . The last picture shows how the square S is embedded in $S^3 \times I$, interpolating from L_0 when $t = 0$ to L_1 when $t = 1$. The green arc in the last picture is the decoration σ on S .

is a bijection. In particular, $\text{Crit}(h|_\sigma) = \text{Crit}(h|_F)$, and $\text{Crit}(h|_\sigma)$ is a local maximum.

Note that $|L_1| = |L_0| \pm 1$. We say that (F, σ) is a *merge* saddle if $|L_1| = |L_0| - 1$, and it is a *split* saddle if $|L_1| = |L_0| + 1$.

Definition 5.5. Let (F, σ) be a decorated link cobordism from (L_0, P_0) to (L_1, P_1) , and let $h: S^3 \times I \rightarrow I$ be the height function. Then we say that (F, σ) is an *isotopy* if $h|_F$ and $h|_\sigma$ have no critical points.

Proposition 5.6 (Juhász-M.³²). *The category of decorated links in S^3 and decorated link cobordisms in $S^3 \times I$ is generated by stabilisations, destabilisations, births, deaths, saddles, and isotopies.*

Proof. By a slight perturbation of F , we can assume $h|_F$ is a generic Morse function. Then isotope σ such that it passes through the critical points of $h|_F$, and such that $h|_\sigma$ is also a generic Morse function with $\text{Crit}(h|_F) \subset \text{Crit}(h|_\sigma)$. We also require that if p is a saddle point of $h|_F$, then $h|_\sigma$ has a local maximum. Finally, if $t \in I$ is a regular value of $h|_\sigma$, then each component of $h^{-1}(t) \cap F$ contains at least two points of $h^{-1}(t) \cap \sigma$. For an illustration, see Figure 5.2. The cobordism that we obtain is equivalent to the original. Then let $c_1 < c_2 < \dots < c_n$ be an enumeration of the critical values of $h|_\sigma$, where $n = |\text{Crit}(h|_\sigma)|$. We write $c_0 = 0$ and $c_{n+1} = 1$, and let

$$\varepsilon = \min\{(c_i - c_{i-1})/3 : 1 \leq i \leq n+1\}.$$

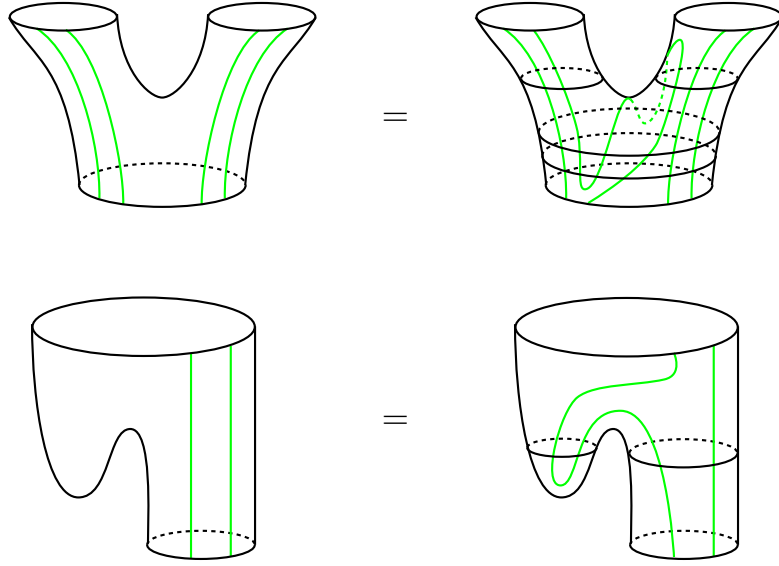


Figure 5.2: Isotoping the dividing set σ such that it passes through the critical points of the height function on F , after which it can be written as a product of elementary cobordisms.

We set $t_{2i} = c_i - \varepsilon$ and $t_{2i+1} = c_i + \varepsilon$ for $i \in \{1, \dots, n\}$, and write $t_1 = 0$ and $t_{2n+2} = 1$.

Let $L_j = F \cap h^{-1}(t_j)$, and let P_j be a decoration of L_j such that its points alternate with $\sigma \cap L_j$. Then (F, σ) is the product of the cobordisms

$$(F_j, \sigma_j) = (F \cap h^{-1}[t_j, t_{j+1}], \sigma \cap h^{-1}[t_j, t_{j+1}])$$

for $j \in \{1, \dots, 2n+1\}$ from (L_j, P_j) to (L_{j+1}, P_{j+1}) . The splitting $L_j = R_+(P_j) \cup R_-(P_j)$ is chosen such that $\partial R_+(\sigma_j)$ goes from $R_+(P_j)$ to $R_-(P_j)$ as it passes P_j , where $R_+(\sigma_j) = R_+(\sigma) \cap F_j$.

When j is even, $h|_{\sigma_j}$ has a single critical point p_j (that is possibly also a critical point of $h|_{F_j}$), and we can isotope F and σ such that F_j and σ_j are products outside a small neighbourhood of the critical point, and all of F_j is a product if $p \notin \text{Crit}(h|_F)$. Then every (F_j, σ_j) is an elementary cobordism. In particular, it is an isotopy if and only if j is odd. $\square$

5.2 ISOTOPIES

Let $\mathcal{X} = (F, \sigma)$ be an isotopy from (L_0, P_0) to (L_1, P_1) , and let $\sigma' \subset F$ be a push-off of σ in a normal direction such that $\partial\sigma' = P_0 \cup P_1$. If we set $L_t = F \cap h^{-1}(t)$ and $P_t = \sigma' \cap h^{-1}(t)$ for $t \in I$, then there is an ambient isotopy $d_t: S^3 \rightarrow S^3$ for $t \in I$ such that $L_t = d_t(L_0)$ and $P_t = d_t(P_0)$. Then d_1 is well-defined up to isotopy, and Juhász proved³⁰ (Lemma 2.22) that the link cobordism map

$$F_{\mathcal{X}}: \text{HFL}(L_0, P_0) \rightarrow \text{HFL}(L_1, P_1)$$

agrees with the map $(d_1)_*$ defined by Juhász and Thurston³⁴. In particular, it preserves both the Alexander and the Maslov gradings.

5.3 PRODUCT ANNULI

Before studying the maps induced by stabilisations and destabilisations, we review sutured decompositions along product annuli. For a full exposition on sutured manifold decompositions, we refer to Juhász's paper²⁸.

Definition 5.7 (Juhász²⁸, Definition 2.9). A *product annulus* A is a properly embedded annulus in a sutured manifold (M, γ) such that one component of A is contained in $R_-(\gamma)$ and the other one is contained in $R_+(\gamma)$.

A product annulus defines a sutured manifold decomposition

$$(M, \gamma) \xrightarrow{A} (M', \gamma'),$$

where

$$\begin{aligned} M' &= M \setminus \text{Int}(\mathcal{N}(A)), \\ \gamma' &= (\gamma \cap M') \cup \mathcal{N}(A'_+ \cap R_-(\gamma)) \cup \mathcal{N}(A'_- \cap R_+(\gamma)), \\ R_+(\gamma') &= ((R_+(\gamma) \cap M') \cup A'_+) \setminus \text{Int}(\gamma'), \\ R_-(\gamma') &= ((R_-(\gamma) \cap M') \cup A'_-) \setminus \text{Int}(\gamma'), \end{aligned}$$

where A'_+ (resp. A'_-) denotes the component of $\partial\mathcal{N}(A) \cap M'$ whose normal vector points out of (resp. into) M' .

Juhász²⁸ actually considers sutured manifold decompositions along more generic *decomposing surfaces*. For the scope of this thesis, any decomposing surface that we will consider will be a product annulus.

Definition 5.8 (Juhász²⁸, Definition 1.2). Suppose that R is a compact, oriented surface with non-empty boundary. Let C be an oriented simple closed curve in R . If $[C] = 0$ in $H_1(R; \mathbb{Z})$, then $R \setminus C$ can be written as $R_1 \sqcup R_2$, where R_1 is the component of $R \setminus C$ that is disjoint from ∂R and satisfies $\partial R_1 = C$. We call R_1 the *interior* and R_2 the *exterior* of C .

We say that the curve C is *boundary-coherent* if either $[C] \neq 0$ in $H_1(R; \mathbb{Z})$, or if $[C] = 0$ in $H_1(R; \mathbb{Z})$ and C is oriented as the boundary of its interior.

Lemma 5.9 (Juhász²⁸, Lemma 8.9). Suppose that (M, γ) is a balanced sutured manifold such that $H_2(M; \mathbb{Z}) = 0$. Let $A \subset (M, \gamma)$ be a product annulus such that at least one component of ∂A is non-zero in $H_1(R(\gamma); \mathbb{Z})$ or both components are boundary-coherent in $\partial R(\gamma)$. Then A defines a sutured manifold decomposition $(M, \gamma) \xrightarrow{A} (M', \gamma')$, and there is an isomorphism

$$\Psi_A: \text{SFH}(M', \gamma') \xrightarrow{\cong} \text{SFH}(M, \gamma).$$

Honda, Kazez, and Matić give a potentially different construction of a direct summand map associated to a sutured manifold decomposition. Suppose that $S \subset -M$ is a properly embedded surface, together with a contact structure ξ on $\mathcal{N}(S)$, such that S is convex with Legendrian boundary, and with ∂ -parallel dividing set. Then $(M, \gamma) \xrightarrow{S} (M', \gamma')$ is a sutured manifold decomposition, which induces an inclusion

$$\Phi_A: \text{SFH}(M', \gamma') \rightarrow \text{SFH}(M, \gamma).$$

The construction of such a map is given in Section 6 of their paper²⁵. It coincides with the gluing map Φ_ξ , as explained by Honda, Kazez and Matić²⁴ (Proposition 6.4). In the next lemma we study the grading shift induced by Φ_A .

Lemma 5.10 (Juhász-M.³²). *For $i \in \{0, 1\}$, let (Y_i, L_i, P_i) be decorated nullhomologous links, such that there is a product annulus decomposition between the complementary sutured manifolds*

$$(M_1, \gamma_1) \xrightarrow{A} (M_0, \gamma_0).$$

Let S_1 be a Seifert surface for L_1 that intersects A in an essential arc. Then $S_0 := S_1 \cap M_0$ is a Seifert surface for L_0 . With the orientations of L_0 and L_1 induced by the Seifert surfaces, the map

$$\Phi_A: \text{HFL}(Y_0, L_0, P_0) \rightarrow \text{HFL}(Y_1, L_1, P_1)$$

preserves the S_i -Alexander grading and the relative Maslov grading.

Proof. By the work of Juhász³¹ (Proposition 9.11), we know that

$$\Phi_A(\text{SFH}(M_0, \gamma_0, \mathfrak{s}^\circ)) \subseteq \text{SFH}(M_1, \gamma_1, f_{-\xi}(\mathfrak{s}^\circ)),$$

where $f_{-\xi}(\mathfrak{s}^\circ)$ is obtained from $\mathfrak{s}^\circ$ by gluing along the product annulus. Specifically, if v_0 is an admissible vector field on (M_0, γ_0) representing $\mathfrak{s}^\circ$, then we can glue v_0 to itself along the sutures that are identified by A , to obtain an admissible vector field v_1 on (M_1, γ_1) . Note that by construction $v_1|_{M_0} = v_0$. If t_0 is the trivialisation of $v_0^\perp$ on ∂M_0 given by the meridional direction, then it can be extended across A to a trivialisation t_1 of $(v_1)^\perp$ on $\partial M_1 \cup A$, in such a way that $t_1|_{\partial M_1}$ is again along the meridional direction (this follows from the fact that the orientations of L_0 and L_1 are compatible). Hence, the obstruction $c_1(f_{-\xi}(\mathfrak{s}^\circ), t_1)$ to extending t_1 to a trivialisation of $(v_1)^\perp$ restricts on M_0 to $c_1(\mathfrak{s}^\circ, t_0)$, and both can be represented, by Poincaré duality, by a 1-manifold $C \subset M_0$.

Then, if $\mathbf{y}$ appears with non-zero multiplicity in $\Phi_A(\mathbf{x})$, and $\mathfrak{s}^\circ$ denotes $\mathfrak{s}^\circ(\mathbf{x})$,

$$\frac{1}{2} \langle c_1(f_{-\xi}(\mathfrak{s}^\circ), t_1), [S_1] \rangle = C.[S_1] = C.[S_0] = \frac{1}{2} \langle c_1(\mathfrak{s}^\circ, t_0), [S_0] \rangle,$$

hence $\mathcal{A}_{S_1}(\mathbf{y}) = \mathcal{A}_{S_0}(\mathbf{x})$.

We now turn to the relative Maslov grading. Recall from the work of Honda, Kazez, and Matić²⁴ that the gluing map is defined as follows. One chooses a *contact compatible* Heegaard diagram $\mathcal{H}_0 = (\Sigma_0, \alpha_0, \beta_0)$ for (M_0, γ_0) , and extends it to a contact compatible Heegaard diagram $\mathcal{H}_1 = (\Sigma_1, \alpha_1, \beta_1)$, where $\Sigma_0 \subseteq \Sigma_1$, $\alpha_1 \setminus \alpha_0 \subset \Sigma_1 \setminus \Sigma_0$, and $\beta_1 \setminus \beta_0 \subset \Sigma_1 \setminus \Sigma_0$. The Heegaard diagram $\mathcal{H}_A = (\Sigma_A, \alpha_A, \beta_A) = (\Sigma_1 \setminus \Sigma_0, \alpha_1 \setminus \alpha_0, \beta_1 \setminus \beta_0)$ defines the sutured manifold corresponding to a regular neighbourhood of the annulus A . Σ_A is glued to Σ_0 along two sutures corresponding to a w and a z basepoints, which we denote by w_0 and z_0 . We also denote by $\overline{\mathcal{H}_0}$, $\overline{\mathcal{H}_1}$ and $\overline{\mathcal{H}_A}$ the multipointed Heegaard diagrams obtained by filling the sutures with discs, and placing a basepoint (w or z depending on the orientation) on each disc. We call w_A and z_A the basepoints on $\overline{\mathcal{H}_A}$. The gluing map on the chain level is defined as

$$\begin{aligned} \Phi_A: \text{SFC}(\mathcal{H}_0) &\rightarrow \text{SFC}(\mathcal{H}_1) \\ \mathbf{x} &\mapsto \mathbf{x} \otimes \text{EH}(-\xi) \end{aligned}$$

where $\text{EH}(-\xi)$ is a distinguished generator of $\text{SFC}(\mathcal{H}_A)$.

Given $\mathbf{x}, \mathbf{y}$ in $\text{SFC}(\mathcal{H}_0)$, let $\mathbf{x}'$ and $\mathbf{y}'$ denote their images through Φ_A . Suppose that D_1 is a domain on $\overline{\mathcal{H}_1}$ between $\mathbf{x}'$ and $\mathbf{y}'$. Then this induces a domain D_0 on $\overline{\mathcal{H}_0}$ from $\mathbf{x}$ to $\mathbf{y}$, and a periodic domain D_A on $\overline{\mathcal{H}_A}$, from $\text{EH}(-\xi)$ to itself. By Lemma 4.31, $D_A = m \cdot \overline{\Sigma_A}$ for some $m \in \mathbb{Z}$. Recall that $\mathcal{H}_A$ defines a neighbourhood of A as sutured manifold, which has vanishing Euler characteristic. It follows that the number of α_A curves (and analogously of β_A curves) must equal the genus g of Σ_A . It follows that the Euler characteristic of D_1 is obtained by adding $m((2-2g)-4)$ to the Euler characteristic of D_0 (where the 4 is given by the fact that we are removing four discs, about w_0, z_0, w_A and z_A). By Lipshitz's formula⁴⁴ (Corollary 4.10), we have

$$\begin{aligned} \mu(D_1) &= \mu_{\mathbf{x}'}(D_1) + \mu_{\mathbf{y}'}(D_1) + e(D_1) \\ &= (\mu_{\mathbf{x}}(D_0) + mg) + (\mu_{\mathbf{y}}(D_0) + mg) + (e(D_0) - m(2g + 2)) \\ &= \mu(D_0) - 2m = \mu(D_0) - 2n_{w_0}(D_0). \end{aligned}$$

Here $\mu_{\mathbf{x}}(D)$ denotes the *point measure* of the domain, obtained by summing the multiplicities of D in all regions adjacent to $\mathbf{x}$ and dividing by 4, while $e(D)$ denotes the *Euler measure* of D . If S is a surface with boundary and punctures, then $e(S)$ is $\frac{1}{2\pi}$ times the integral over S of the curvature of a metric on S for which ∂S is geodesic and the punctures of S are right angles.

By definition of the relative Maslov grading (see Equation (4.13)) and the above computation, we have

$$gr(\mathbf{x}', \mathbf{y}') = \mu(D_1) - 2n_{w \setminus w_0}(D_1) = \mu(D_0) - 2n_w(D_0) = gr(\mathbf{x}, \mathbf{y}). \quad \square$$

The following lemma will be useful.

Lemma 5.11 (Juhász-M.³²). *Let (Y, L_0, P_0) be a decorated link locally looking as in Figure 5.5, and suppose that (Y, L_+, P) as in Figure 5.5 is still a decorated link (i.e., every link component has at least two decorations). Let $(Y \# (S^1 \times S^2), L_0^*, P)$ be the decorated link obtained by attaching a 1-handle to Y in correspondence of the decorations shown in Figure 5.5, and by reconnecting the two arcs in $R_+(P_0)$ (and analogously $R_-(P_0)$) inside the 1-handle. Then, if (M_0, γ_0) is the sutured manifold complementary to (Y, L_0, P_0) , and (M_1, γ_1) is the sutured manifold complementary to $(Y \# (S^1 \times S^2), L_0^*, P)$, there is a product annulus decomposition $(M_1, \gamma_1) \xrightarrow{A} (M_0, \gamma_0)$, where A is given by the cocore of the 1-handle, and the map*

$$\Phi_A: \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(M_1, \gamma_1)$$

is an isomorphism.

Proof. The cocore of the 1-handle intersects the link L_0^* in two points, one of which is in $R_+(P)$ and the other one in $R_-(P)$. It follows that it defines a product annulus A in the sutured manifold (M_1, γ_1) . By the work of Honda, Kazez, and Matic²⁴, the annulus A induces an injection

$$\Phi_A: \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(M_1, \gamma_1).$$

We will now prove that the ranks of $\text{SFH}(M_0, \gamma_0)$ and $\text{SFH}(M_1, \gamma_1)$ coin-

cide, implying that Φ_A is an isomorphism.

Lemma 5.9, which is Lemma 8.9 by Juhász²⁸, proves that if $H_2(M_1) = 0$, then the product annulus decomposition induces an isomorphism between $\text{SFH}(M_1, \gamma_1)$ and $\text{SFH}(M_0, \gamma_0)$. Looking at the proof of Lemma 8.9 in Juhász's paper²⁸, we see that the technical assumption $H_2(M_1) = 0$ ensures that there is an *admissible* surface diagram $\mathcal{H}$ adapted to the product annulus A such that $\text{CF}(\mathcal{H}, \mathfrak{s}) = 0$ for all $\mathfrak{s} \notin O_A$, where O_A is the set of outer spin^c structures. Although $H_2(M_1)$ may be non-zero in our case, we will prove that there is a Heegaard diagram satisfying the condition above.

Let $P_0 \setminus P = \{p_1, p_2\}$. Given a Heegaard diagram $\mathcal{H}_0$ for (M_0, γ_0) , a Heegaard diagram $\mathcal{H}_1$ for (M_1, γ_1) is obtained from $\mathcal{H}_0$ simply by connecting the two sutures $\gamma_0(p_1)$ and $\gamma_0(p_2)$ corresponding to p_1 and p_2 with an annulus; see Figure 5.8. The core a of the connecting annulus—shown in green in the picture—spans the decomposing annulus A in M_1 .

We claim that if we glue product 2-handles to M_0 along $\gamma_0(p_1)$ and $\gamma_0(p_2)$, corresponding to filling two basepoints p_1 and p_2 in the diagram $\mathcal{H}_0$, then the resulting sutured manifold (M'_0, γ'_0) is still balanced. The only thing we need to check is that each component of $\partial M'_0$ has a suture.

First, suppose we have a merge saddle from L_0 to L_+ ; i.e., that p_1 and p_2 belong to different components of L_0 . Then, by definition, P_0 has at least two decorations on each component of L_0 . Hence gluing product 2-handles to $\gamma_0(p_1)$ and $\gamma_0(p_2)$ results in two S^2 components of $\partial M'_0$ with at least one suture on each.

Now suppose that we have a split saddle; i.e., that p_1 and p_2 lie on the same component of L_0 and $|L_+| = |L_0| + 1$. As the components of L_+ involved in the saddle move each have at least two decorations of P , the corresponding component of L_0 has at least six points of P_0 (including p_1 and p_2). Furthermore, p_1 and p_2 cannot be consecutive decorations, as that would result in a component of L_+ with no decorations. Hence $\partial M'_0$ has two S^2 components with at least one suture on each.

Lastly, suppose that we have an unorientable saddle, i.e., that p_1 and p_2 lie

on the same component of L_0 and $|L_+| = |L_0|$. Then, the component of L_0 containing p_1 and p_2 must contain at least four decorations, and p_1 and p_2 cannot be consecutive. Hence $\partial M'_0$ has two S^2 components with at least one suture on each.

Let $\mathcal{H}'_0$ be the Heegaard diagram for (M'_0, γ'_0) obtained by filling $\gamma_0(p_1)$ and $\gamma_0(p_2)$. Since (M'_0, γ'_0) is still a balanced sutured manifold by the discussion above, it follows from the work of Juhász²⁷ (Proposition 3.15) that $\mathcal{H}'_0$ can be made admissible by winding the β -curves. Without loss of generality, we can assume that $\mathcal{H}_0$ is already such that $\mathcal{H}'_0$ is admissible. Then we claim that $\mathcal{H}_1$ is also admissible. Indeed, if P is a non-zero periodic domain in $\mathcal{H}_1$, then compressing it along the green curve a , we obtain a periodic domain P' in $\mathcal{H}'_0$. As $\mathcal{H}'_0$ is admissible, P' , and hence also P , has both positive and negative multiplicities.

In particular, we can assume that both diagrams $\mathcal{H}_0$ and $\mathcal{H}_1$ are admissible. Furthermore, $\mathcal{H}_1$ is adapted to A , and the quasi-polygon P that defines the annulus A is a regular neighbourhood of the green curve a in Figure 5.8. All the spin^c structures associated to a generator of the complex $\text{SFC}(\mathcal{H}_1)$ are outer with respect to A . This is because, by construction, they are represented by a vector field which is tangent to A . Since P is disjoint from all α - and β -curves, we can apply Juhász's product annulus decomposition formula²⁸ (Lemma 8.9) to obtain that

$$\dim(\text{SFH}(M_0, \gamma_0)) = \dim(\text{SFH}(M_1, \gamma_1)),$$

and that therefore Φ_A is an isomorphism. □

5.4 STABILISATIONS AND DESTABILISATIONS

We now turn the attention to the maps associated to stabilisations and destabilisations.

Proposition 5.12 (Juhász-M.³²). *Let $\mathcal{S} = (F, \sigma)$ be a stabilisation from (L, P_0) to (L, P_1) . We denote by W the bigraded vector space $\mathbb{F}_2 \oplus \mathbb{F}_2$, where the first summand is generated by the element b in Maslov and Alexander gradings $-1/2$,*

and the second summand is generated by the element t in Maslov and Alexander gradings $1/2$. Then, given an orientation of F , there is a canonical isomorphism

$$I_S: \text{HFL}(L, P_0) \otimes W \xrightarrow{\sim} \text{HFL}(L, P_1)$$

which preserves the bigrading, such that the link cobordism map

$$F_S: \text{HFL}(L, P_0) \rightarrow \text{HFL}(L, P_1)$$

factorises as $I_S \circ s_S$, where

$$s_S: \text{HFL}(L, P_0) \rightarrow \text{HFL}(L, P_0) \otimes W$$

is given by $s_S(x) = x \otimes t$ in case of a positive stabilisation, and $s_S(x) = x \otimes b$ in case of a negative stabilisation.

Proof. Let $\mathcal{W} = \mathcal{W}(F, \sigma) = (W_F, Z_F, \xi)$ be the sutured manifold cobordism complementary to (F, σ) from the sutured manifold $(M, \gamma_0) = \mathcal{W}(L, P_0)$ complementary to (L, P_0) to the sutured manifold $(M, \gamma_1) = \mathcal{W}(L, P_1)$ complementary to (L, P_1) . Here $Z = Z_F \cong -F \times S^1$ is oriented as the boundary of W_F , and hence the orientation of F induces an orientation of the S^1 factor. This is the orientation for which S^1 has linking number one with F .

By definition, the cobordism map $F_{\mathcal{W}}$ is the composition of the gluing map

$$\Phi_{-\xi}: \text{SFH}(M, \gamma_0) \rightarrow \text{SFH}(N, \gamma_1),$$

where $N = M \cup (-Z)$, and a special cobordism map

$$F_{\mathcal{W}^s}: \text{SFH}(N, \gamma_1) \rightarrow \text{SFH}(M, \gamma_1).$$

As $\mathcal{W}^s$ is a product, it induces the identity map. So, it suffices to compute $\Phi_{-\xi}$.

We denote by σ_0 the component of σ that contains the critical point of the height function $L \times I \rightarrow I$. Let a be a properly embedded arc in $F \setminus \sigma$

that is parallel to σ_0 , is disjoint from the bigon component of $F \setminus \sigma$, and such that there are two points of P_1 in the rectangular region between a and σ_0 , on different sides of the rectangle. Then $A = a \times S^1$ is a product annulus in (N, γ_1) . If we decompose (N, γ_1) along A , we obtain the disjoint union of (M, γ_0) and the sutured manifold $(D^2 \times S^1, \gamma)$, where γ consists of four longitudinal sutures. By the work of Juhász²⁹ (Proposition 9.1), we have $\text{SFH}(D^2 \times S^1, \gamma) \cong W$, and the isomorphism is uniquely given by the orientation of the S^1 factor, which in turn gives an orientation of $H_1(D^2 \times S^1)$, and hence allows one to distinguish between the two spin^c structures in which the two $\mathbb{F}_2$ summands live. Hence, by Honda, Kazez, and Matić²⁴ (Proposition 6.4) the gluing map from the product annulus gives a canonical injection

$$I_S: \text{HFL}(L, P_0) \otimes W \xrightarrow{\sim} \text{HFL}(L, P_1), \quad (5.1)$$

which is an isomorphism because the ranks of domain and range agree (see for instance Proposition 2.3 by Manolescu, Ozsváth, and Sarkar⁴⁸). I_S preserves the Alexander grading and the relative Maslov grading by Lemma 5.10. By Equation (3) in Ozsváth and Szabó's paper⁵⁸, both sides of Equation (5.1) enjoy a symmetry property $C_{a,m} \cong C_{-a,m-2a}$, where a is the Alexander grading and m the Maslov grading. This property, with the fact that I_S is an isomorphism which preserves the Alexander grading and shifts the Maslov grading by a constant e , implies that $e = 0$. Note that I_S depends only on the choice of the product annulus, which in turn depends only on the bigon region in the stabilisation.

We write the gluing map $\Phi_{-\xi}$ as a composition of two gluing maps, which is possible according to Honda, Kazez, and Matić²⁴ (Proposition 6.2). The first gluing map s_S corresponds to the sutured submanifold $(-M, -\gamma_0)$ of $(-M, -\gamma_0) \sqcup (-D^2 \times S^1, -\gamma)$ with the contact structure $-\xi|_{D^2 \times S^1}$ on the difference. The second gluing map I_S , which is the one in Equation (5.1), corresponds to the sutured submanifold $(-M, -\gamma_0) \sqcup (-D^2 \times S^1, -\gamma)$ of $(-N, -\gamma_1)$, using the contact structure $-\xi$ on $(-N) \setminus ((-M) \sqcup (-D^2 \times S^1))$.

By construction, the gluing map s_S maps $x \in \text{HFL}(L, P_0)$ to

$$x \otimes \text{EH}(-\xi|_{D^2 \times S^1}) \in \text{HFL}(L, P_0) \otimes \text{SFH}(D^2 \times S^1, \gamma).$$

Recall that $-\xi$ is a positive contact structure on $(-D^2 \times S^1, -\gamma)$. In case of a positive stabilisation, the dividing set of $-\xi$ on $-D^2 \times \{p\}$ is given by the left-hand side of Figure 14 in the paper of Honda, Kazez, and Matić²⁴, while it is the right-hand side of it in case of a negative stabilisation. As explained there, the evaluation of the first Chern class of the spin^c structure of the contact element $\text{EH}(-\xi|_{D^2 \times S^1})$ on the surface $D^2 \times \{p\}$ is given by $\chi(R_+) - \chi(R_-)$ for the dividing set on $D^2 \times \{p\}$. Hence $\text{EH}(-\xi|_{D^2 \times S^1}) = t$ in case of a positive stabilisation and $\text{EH}(-\xi|_{D^2 \times S^1}) = b$ in case of a negative stabilisation. $\square$

Remark 5.13. If we chose a different product annulus $A' = a \times S^1$, with a as on the left hand side of Figure 5.3, with the two basepoints in the rectangular region delimited by σ_0 and a on the same side of the rectangle, then we would have obtained a different isomorphism

$$I'_S: \text{HFL}(L, P_0) \otimes W \xrightarrow{\sim} \text{HFL}(L, P_1).$$

However, the $D^2 \times S^1$ component of (N, γ_1) decomposed along A' would have induced the same contact element in $\text{SFH}(D^2 \times S^1, \gamma)$, on the grounds that the convex surface $D^2 \times \{p\}$ carries the same dividing set and decomposition into R_+ and R_- .

Proposition 5.14 (Juhász-M.³²). *Suppose that $\mathcal{D} = (F, \sigma)$ is a destabilisation from (L, P_0) to (L, P_1) . Let W be as in Proposition 5.12. Then, given an orientation of F , there is a canonical isomorphism*

$$I_{\mathcal{D}}: \text{HFL}(L, P_0) \xrightarrow{\sim} \text{HFL}(L, P_1) \otimes W$$

which preserves the bigrading, such that the link cobordism map

$$F_{\mathcal{D}}: \text{HFL}(L, P_0) \rightarrow \text{HFL}(L, P_1)$$

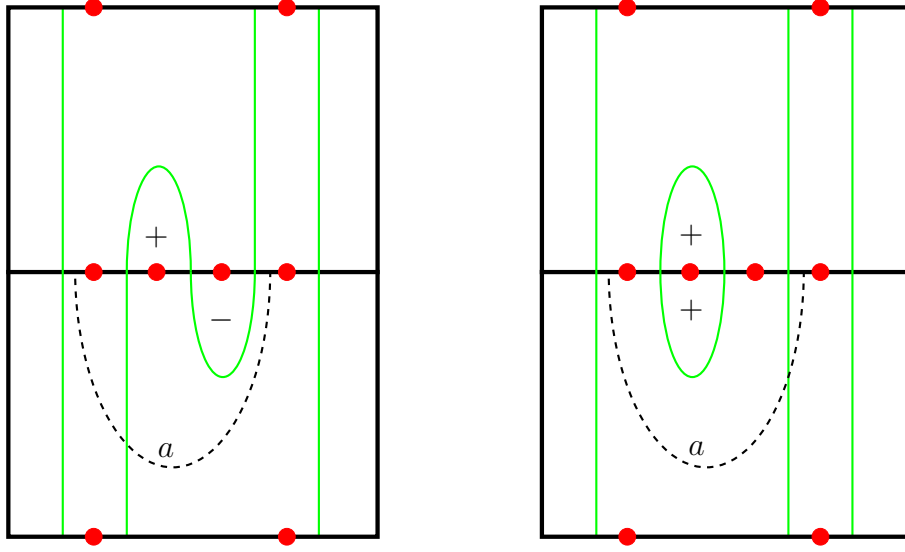


Figure 5.3: The composition of a negative stabilisation and positive destabilisation on the left, and the composition of a positive stabilisation and a positive destabilisation on the right.

factorises as $d_{\mathcal{D}} \circ I_{\mathcal{D}}$, where

$$d_{\mathcal{D}}: \text{HFL}(L, P_1) \otimes W \rightarrow \text{HFL}(L, P_1)$$

is given by $d_{\mathcal{D}}(x \otimes b) = x$ and $d_{\mathcal{D}}(x \otimes t) = 0$ in case of a positive destabilisation, and by $d_{\mathcal{D}}(x \otimes t) = x$ and $d_{\mathcal{D}}(x \otimes b) = 0$ in case of a negative destabilisation.

Proof. Given a positive destabilisation $\mathcal{D} = (F, \sigma)$, let $\mathcal{S}_- = (F, \sigma_-)$ be a negative stabilisation from (L, P_1) to (L, P_0) such that $\mathcal{D} \circ \mathcal{S}_-$ exists and is as on the left-hand side of Figure 5.3. Then $\mathcal{D} \circ \mathcal{S}_-$ is the identity cobordism from (L, P_1) to itself, and hence it induces the identity of $\text{HFL}(L, P_1)$.

Now let $\mathcal{S}_+ = (F, \sigma_+)$ be the positive stabilisation shown on the right-hand side of Figure 5.3. Then $F_{\mathcal{D}} \circ F_{\mathcal{S}_+} = 0$, since the dividing set $\sigma \cup \sigma_+$ contains a closed component that bounds a disc, so the corresponding S^1 -invariant contact structure is overtwisted, and hence the gluing map vanishes by Honda, Kazez, and Matic²⁴.

Let a be the properly embedded dashed arc in F shown in Figure 5.3. The

bigon component D of $F \setminus a$ contains the bigon components of $F \setminus \sigma_+$ and $F \setminus \sigma_-$ and the three points of P_0 involved in the destabilisation, and intersects both σ_+ and σ_- in two arcs. Then $A = a \times S^1$ is a product annulus in $(M, \gamma_0) = \mathcal{W}(L, P_0)$.

By Proposition 5.12,

$$F_{S_+} = I_{S_+} \circ s_{S_+},$$

where we use the above product annulus A to define the gluing map

$$I_{S_+}: \text{HFL}(L, P_1) \otimes W \rightarrow \text{HFL}(L, P_1).$$

Let

$$d_{\mathcal{D}} = F_{\mathcal{D}} \circ I_{S_+}.$$

If we set $I_{\mathcal{D}} := I_{S_+}^{-1}$, then

$$F_{\mathcal{D}} = d_{\mathcal{D}} \circ I_{\mathcal{D}}.$$

Given $x \in \text{HFL}(L, P_1)$, we obtain that

$$0 = F_{\mathcal{D}} \circ F_{S_+}(x) = d_{\mathcal{D}} \circ s_{S_+}(x) = d_{\mathcal{D}}(x \otimes t),$$

as claimed.

In case of S_- , we repeat the proof of Proposition 5.12 with the same product annulus A , as explained in Remark 5.13, to obtain that

$$F_{S_-} = I_{S_-} \circ s_{S_-}.$$

Indeed, the dividing set of the S^1 -invariant contact structure corresponding to σ_- on the bigon D consists of two arcs such that $\chi(R_+) - \chi(R_-) = -1$. Given $x \in \text{HFL}(L, P_1)$, we obtain that

$$x = F_{\mathcal{D}} \circ F_{S_-}(x) = d_{\mathcal{D}} \circ s_{S_-}(x) = d_{\mathcal{D}}(x \otimes b),$$

which completes the proof. □

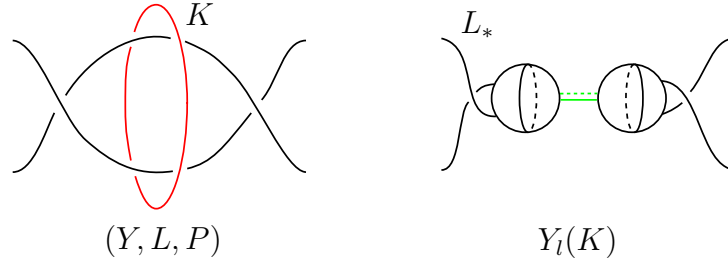


Figure 5.4: The left-hand side of the figure shows the decorated link (Y, L, P) with the small unknot $K = \partial D$ in red. In the middle, we see the result of l -surgery along K . The manifold $Y_l(K)$, which is diffeomorphic to $Y \# (S^1 \times S^2)$, is locally represented as the complement of two balls, with their boundaries identified via a reflection.

5.5 A DECORATED SKEIN EXACT TRIANGLE

Ozsváth and Szabó⁵⁸ defined an oriented skein exact triangle in knot Floer homology. Recall that $\widehat{\text{HFK}}(Y, L)$ of an n -component link L in a closed, connected, oriented 3-manifold Y is obtained as follows: Choose $2n - 2$ points $p_1, \dots, p_{n-1}$ and $q_1, \dots, q_{n-1}$ in L such that if we identify each p_i and q_i in L , then we obtain a connected graph. After attaching 3-dimensional 1-handles to Y with feet at p_i and q_i , and inside each 1-handle taking the connected sum of the link components, we get a knot $\kappa(L, \{p_i, q_i\})$ in a 3-manifold $\kappa(Y, \{p_i, q_i\})$ diffeomorphic to $Y \#^{n-1}(S^2 \times S^1)$. Then

$$\widehat{\text{HFK}}(L) := \widehat{\text{HFK}}(\kappa(Y, \{p_i, q_i\}), \kappa(L, \{p_i, q_i\})).$$

The skein exact triangle is stated in terms of this invariant $\widehat{\text{HFK}}$. However, the construction of $\widehat{\text{HFK}}(Y, L)$ is not natural, as it depends on the choice of points p_i and q_i , and is not stated in terms of decorated knots or links. In this subsection, we outline a natural version of the Ozsváth-Szabó skein exact triangle in terms of HFL that works for multi-pointed and unoriented (but decorated) links.

Let (Y, L, P) be a decorated link in a closed oriented 3-manifold. Suppose that D is a disc embedded in Y that intersects both $R_+(P)$ and $R_-(P)$ transversely in a single point. Let $K = \partial D$, we view this as a knot in the sutured manifold $(M, \gamma) := Y(L, P)$. Let m , l , and s be oriented sim-

ple closed curves on $\partial\mathcal{N}(K)$ such that m is a meridian of K , the curve $l = D \cap \partial\mathcal{N}(K)$ is a longitude, and s intersects both m and l once algebraically. For $c \in \{m, l, s\}$, let $M_c(K)$ denote the manifold obtained by Dehn filling $M \setminus \mathcal{N}(K)$ along c . Then there is a surgery exact triangle

$$\cdots \longrightarrow \mathrm{SFH}(M_m(K), \gamma) \xrightarrow{e'} \mathrm{SFH}(M_l(K), \gamma) \xrightarrow{f'} \mathrm{SFH}(M_s(K), \gamma) \longrightarrow \cdots$$

Here $M_m(K) = M$, so $\mathrm{SFH}(M_m(K), \gamma) = \mathrm{HFL}(Y, L, P)$. In $M_l(K)$ (see Figure 5.4), consider the annulus A obtained by capping off $D \setminus \mathcal{N}(K)$ with the disc $\{1\} \times D^2 \subset S^1 \times D^2$ whose boundary is glued to l . Such an annulus appears as one of the two spheres on the right hand side of Figure 5.4, which are punctured twice by the dual link L_* , which is the image of L in $Y_l(K)$. Then A is a product annulus since $|D \cap R_{\pm}(P)| = 1$. Decomposing $(M_l(K), \gamma)$ along A , we obtain a sutured manifold (M', γ') that coincides with $Y(L_0, P_0)$, where L_0 is obtained from $L \setminus \mathcal{N}(D)$ by reconnecting the endpoints parallel to D , and adding one decoration to P along each of these parallel arcs. The gluing map

$$\Phi_A: \mathrm{SFH}(Y(L_0, P_0)) \rightarrow \mathrm{SFH}(M_l(K), \gamma)$$

is an isomorphism by Lemma 5.11. By the work of Juhász³¹ (Proposition 9.11), the spin^c -grading on $\mathrm{SFH}(M_l(K), \gamma)$ is obtained by gluing spin^c structures along A_+ and A_- , the two sides of A after the decomposition. Finally, $(M_s(K), \gamma) = Y(L_+, P)$, where L_+ is a link in Y obtained by adding a full left-handed twist to L along D . For an illustration showing L , L_0 , and L_+ , see Figure 5.5. Hence, we obtain the following natural version of the Ozsváth-Szabó skein exact triangle.

Theorem 5.15 (Juhász-M.³²). *Let (Y, L, P) be a decorated link in a closed, connected, oriented 3-manifold. Suppose that D is a disc embedded in Y such that it intersects both $R_+(P)$ and $R_-(P)$ transversely in a single point, and let (Y, L_0, P_0) and (Y, L_+, P) be the decorated links defined above (see Figure 5.5). Then there is an exact triangle*

$$\cdots \longrightarrow \mathrm{HFL}(Y, L, P) \xrightarrow{e} \mathrm{HFL}(Y, L_0, P_0) \xrightarrow{f} \mathrm{HFL}(Y, L_+, P) \longrightarrow \cdots,$$

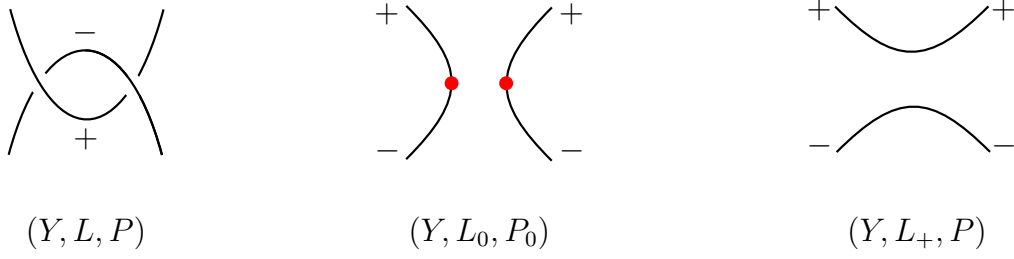


Figure 5.5: A local picture of the decorated links (Y, L_0, P_0) , (Y, L, P) , and (Y, L_+, P) .

where $e = \Phi_A^{-1} \circ e'$ and $f = f' \circ \Phi_A$.

If $Y = S^3$, $|L_0| = |L| \pm 1$, and P consists of two decorations on each component of L and L_+ , then the above exact triangle coincides with that of Ozsváth and Szabó⁵⁸, after collapsing the multi-grading onto the main diagonal. The condition $|L_0| = |L| \pm 1$ means that the links L , L_0 , and L_+ can be oriented coherently, which was an assumption made by Ozsváth and Szabó⁵⁸, but which is not necessary for Theorem 5.15. Note that Ozsváth and Szabó distinguish two cases depending on whether the two strands of L_+ meeting D belong to the same component. In both cases, we have $\widehat{\text{HFL}}(L, P) \cong \widehat{\text{HFK}}(L)$ and $\widehat{\text{HFL}}(L_+, P) \cong \widehat{\text{HFK}}(L_+)$. If the two strands belong to the same component of L_+ , then $|L_0| = |L| + 1$, and P_0 consists of exactly two decorations on each component of L_0 , hence $\widehat{\text{HFL}}(L_0, P_0) \cong \widehat{\text{HFK}}(L_0)$. So, in this case, Theorem 5.15 specialises to exact sequence (7) in Ozsváth and Szabó's paper⁵⁸:

$$\cdots \longrightarrow \widehat{\text{HFK}}(L) \longrightarrow \widehat{\text{HFK}}(L_0) \longrightarrow \widehat{\text{HFK}}(L_+) \longrightarrow \cdots$$

On the other hand, if the two strands belong to different components of L_+ , then P_0 consists of six decorations on one component of L_0 and two on all other components, hence $\widehat{\text{HFL}}(L_0, P_0) \cong \widehat{\text{HFK}}(L_0) \otimes W \otimes W$. Here $W \otimes W$ is a 4-dimensional vector space with a 1-dimensional summand generated by $b \otimes b$ in Maslov and Alexander gradings -1 , a 2-dimensional summand generated by $b \otimes t$ and $t \otimes b$ in Maslov and Alexander gradings 0 , and a 1-dimensional summand generated by $t \otimes t$ in Maslov and Alexander gradings 1 . Ozsváth and Szabó⁵⁸ denoted this vector space V , though

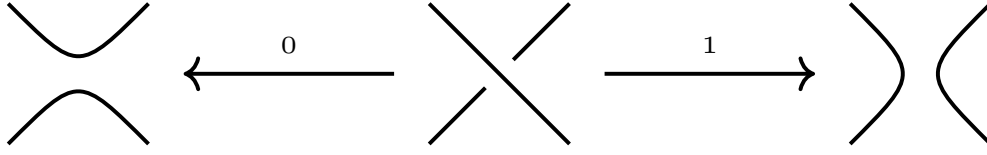


Figure 5.6: The figure shows the 0-smoothing and the 1-smoothing for a crossing of an unoriented link.

in this thesis we use that notation for a different purpose. Hence, in this case, our exact sequence specialises to exact sequence (8) in Ozsváth and Szabó's paper⁵⁸:

$$\cdots \longrightarrow \widehat{\mathrm{HFK}}(L) \longrightarrow \widehat{\mathrm{HFK}}(L_0) \otimes W \otimes W \longrightarrow \widehat{\mathrm{HFK}}(L_+) \longrightarrow \cdots$$

In particular, the Alexander and Maslov grading shifts are as in Ozsváth and Szabó's paper⁵⁸, so all the maps preserve the Alexander grading, and e and f decrease the Maslov grading by $1/2$. A completely analogous proof shows that this also holds for arbitrary decorated links in S^3 with compatible orientations.

Other unoriented skein exact triangles in link Floer homology appeared in the literature, by Manolescu⁴⁶, Manolescu and Ozsváth⁴⁷, and Wong⁸³. However, such unoriented triangles relate the link Floer homology of a link with that of its smoothings at a crossing, as in Figure 5.6. The result of Theorem 5.15 is more similar to Ozsváth and Szabó's oriented skein exact triangle⁵⁸.

5.6 SADDLES

In this section we will give a description of the map associated to a saddle cobordism, which is given by the following theorem. Recall that we work with *oriented* decorated link cobordisms. Therefore, the two basepoints that disappear with the saddle move (cf. Definition 5.4) are a w basepoint and a z basepoint.

Theorem 5.16 (Juhász-M.³²). *Let $\mathcal{S} = (F, \sigma)$ be an oriented saddle cobor-*

dism from (L_0, P_0) to (L_1, P_1) , as in Definition 5.4. Call the two basepoints that disappear in the saddle move w and z . Let $(M_0, \gamma_0) = \mathcal{W}(L_0, P_0)$ and $(M_1, \gamma_1) = \mathcal{W}(L_1, P_1)$ denote the sutured manifolds complementary to the decorated links, and let $\gamma_0(w)$ and $\gamma_0(z)$ denote the annular components of γ_0 determined by w and z . Then the cobordism map associated to $\mathcal{S}$ is

$$F_{\mathcal{S}} = F_{\mathcal{W}^s} \circ \Phi_A,$$

where the gluing map Φ_A is an isomorphism induced by the gluing of a product annulus A joining the annuli $\gamma_0(w)$ and $\gamma_0(z)$.

The special cobordism consists of a 2-handle attachment, and it induces a map $F_{\mathcal{W}^s}$ as follows. Let $\mathcal{H}_0$ denote a Heegaard diagram for (L_0, P_0) arising from a connected link projection, as explained in Section 2.4, which in a neighbourhood of the band corresponding to the saddle move appears as in Figure 5.7 on the left hand side. Let $\mathcal{H}$ be the triple Heegaard diagram obtained by locally modifying $\mathcal{H}_0$ as in Figure 5.7 on the right hand side, and by setting all the δ curves that are not shown as isotopic translates of the respective β curves. Then $F_{\mathcal{W}^s}$ is the map induced by the chain map that counts pseudo-holomorphic triangles in $\mathcal{H}$:

$$f_{\mathcal{W}^s}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\delta}} \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y}) \\ \mu(\psi) = 0}} \# \mathcal{M}(\psi) \cdot \mathbf{y}.$$

We will break the proof of Theorem 5.16 into two parts. We decompose the sutured cobordism complementary to the saddle cobordism into two cobordisms, a *boundary cobordism* $\mathcal{W}^b$ and a *special cobordism* $\mathcal{W}^s$, and we will prove that the maps that they induce are Φ_A and $F_{\mathcal{W}^s}$ as in the statement of the theorem. This will be done in Sections 5.6.1 and 5.6.2.

From the Heegaard diagrammatic description of $F_{\mathcal{W}^s}$ given in Theorem 5.16 we will also derive a relation between the 2-handle map and the decorated skein exact triangle, which we prove in Section 5.6.3.

Theorem 5.17 (Juhász-M.³²). *Let the triple (L, P) , (L_0, P_0) , (L_+, P) be as in*

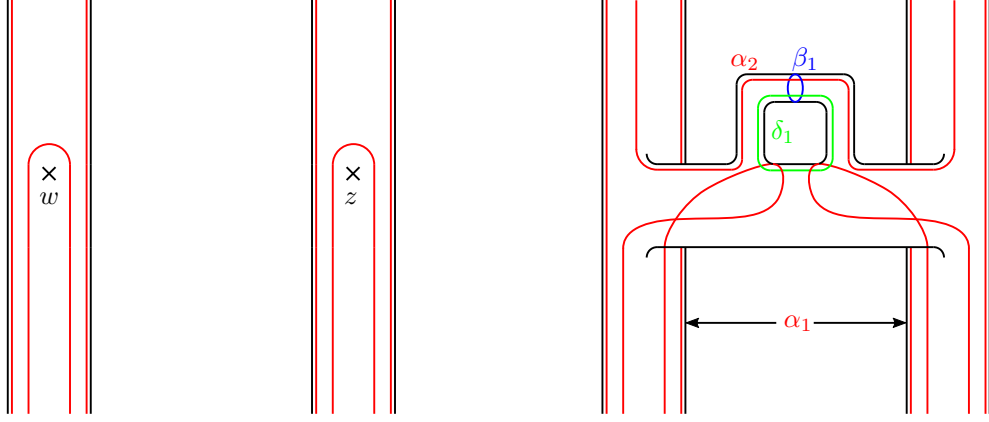


Figure 5.7: The figure on the left is a local picture of the Heegaard diagram $\mathcal{H}_0$ for (L_0, P_0) arising from a connected link projection. The triple Heegaard diagram $\mathcal{H}$ on the right hand side is the one subordinate to the 2-handle attachment of which the special cobordism associated to a saddle consists.

Figure 5.5, and suppose that they can be oriented coherently. Then the map

$$f: \text{HFL}(L_0, P_0) \longrightarrow \text{HFL}(L_+, P)$$

appearing in the decorated skein exact triangle of Theorem 5.15 coincides with the cobordism map induced by the saddle cobordism from (L_0, P_0) to (L_+, P) .

5.6.1 THE BOUNDARY COBORDISM ASSOCIATED TO A SADDLE

Let (F, σ) be a saddle cobordism from (L_0, P_0) to (L_1, P_1) . Let (M_i, γ_i) be the sutured manifold complementary to (L_i, P_i) , and let $\mathcal{W} = (W, Z, [\xi])$ be the sutured cobordism complementary to (F, σ) . Let $\mathcal{W}^b$ and $\mathcal{W}^s$ denote the boundary and the special cobordisms such that $\mathcal{W} = \mathcal{W}^s \circ \mathcal{W}^b$. Let (N, γ_1) denote the outgoing end of the boundary cobordism. Since F is oriented,

$$N \cong M_0 \cup_{(L_0 \cap S) \times S^1} (S \times S^1),$$

where S is the square shown in Figure 5.1.

Proposition 5.18 (Juhász-M.³²). *Let $\mathcal{S} = (F, \sigma)$ be an oriented saddle cobordism from (L_0, P_0) to (L_1, P_1) , as in Definition 5.4. Call the two basepoints that*

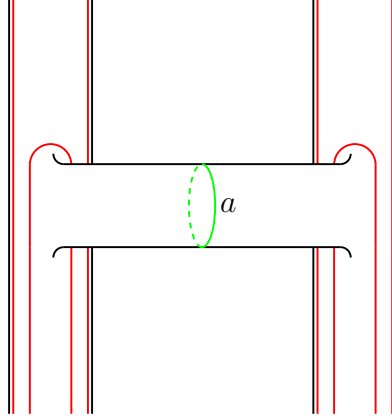


Figure 5.8: The figure shows a Heegaard diagram for the sutured manifold (N, γ_1) after the boundary cobordism, obtained by connecting w and z by a tube. The core of the tube is the green curve a .

disappear in the saddle move w and z . Let $\gamma_0(w)$ and $\gamma_0(z)$ denote the annular components of γ_0 determined by w and z . Then the gluing map Φ_A associated to $\mathcal{W}^b$ is an isomorphism induced by the gluing of a product annulus A joining the annuli $\gamma_0(w)$ and $\gamma_0(z)$.

Furthermore, the map Φ_A preserves the Alexander and the Maslov gradings.

Proof. Consider the annulus $A = b \times S^1 \subseteq N$, where $b = I \times \{1/2\}$ is the arc contained in the square S drawn in blue on the right-hand side of Figure 5.1. Such an annulus is a *product annulus*, and induces a sutured manifold decomposition

$$(N, \gamma_1) \xrightarrow{A} (M_0, \gamma_0).$$

By the work of Honda, Kazez, and Matić²⁴ (Proposition 6.4), the annulus A induces an injection

$$\Phi_A: \text{SFH}(M_0, \gamma_0) \rightarrow \text{SFH}(N, \gamma_1),$$

which coincides with the gluing map $\Phi_{-\xi}$, and which is an isomorphism by Lemma 5.11.

Φ_A preserves the Alexander grading and the relative Maslov grading by

Lemma 5.10. Together with the fact that Φ_A is an isomorphism, and that domain and range of Φ_A enjoy the symmetry property $C_{a,m} \cong C_{-a,m-2a}$ (see Equation (3) by Ozsváth and Szabó⁵⁸), this implies that Φ_A preserves also the absolute Maslov grading. $\square$

5.6.2 THE SPECIAL COBORDISM ASSOCIATED TO A SADDLE

We now turn the attention to the special cobordism $\mathcal{W}^s$ associated to a saddle. Recall that the sutured manifold (N, γ) after the boundary cobordism is obtained from (M_0, γ_0) by gluing the annuli $\gamma_0(w)$ and $\gamma_0(z)$ with a tube, which we will denote by T , whose core is a product annulus A . Figure 5.8 shows a Heegaard diagram for (N, γ_1) . The annulus A is spanned by the curve a , and T is spanned by a tubular neighbourhood of a in that diagram.

As we explained in Figure 5.1, the saddle critical point of F happens inside a square region. We denote by $B \subset F$ the band embedded in $S^3 \times I$, and by S its projection onto $S^3 \times \{0\}$. Without loss of generality, we can suppose that B projects diffeomorphically onto S through a diffeomorphism d . Note that the result of surgery on L_0 along S gives L_1 . Then we have that $T = UNB \cong B \times S^1$ inside $\text{Bl}_F(S^3 \times I)$. Let c be the core of the band S , joining the basepoints w and z of L_0 , and let c^* denotes its cocore. Furthermore, let $\tilde{c} = d^{-1}(c)$ and $\tilde{c}^* = d^{-1}(c^*)$ in B . Then $A = UN\tilde{c}^* \cong \tilde{c}^* \times S^1$. See Figure 5.9.

Proposition 5.19 (Juhász-M.³²). *The special cobordism $\mathcal{W}^s$ associated to a saddle cobordism consists of a single 2-handle attachment. With the notation set above, let $b \subset UN\tilde{c}$ be a section of $\tilde{c}$ joining $UNw \cap S$ to $UNz \cap S$, let c' be a parallel translate of c in S , disjoint from c , and b' be a parallel translate of b in UNB which is contained in $UN(d^{-1}(c'))$. Then, the attaching sphere of the 2-handle is the curve $c \cup b$, with framing $c' \cup b'$.*

Proof. Up to isotopy, we can suppose that the band S is the square centred in the origin of $\mathbb{R}^3 \subset S^3 \times \{0\}$, with edges parallel to the x and y axes and of length 2. We can further suppose that the edges parallel to the y axis

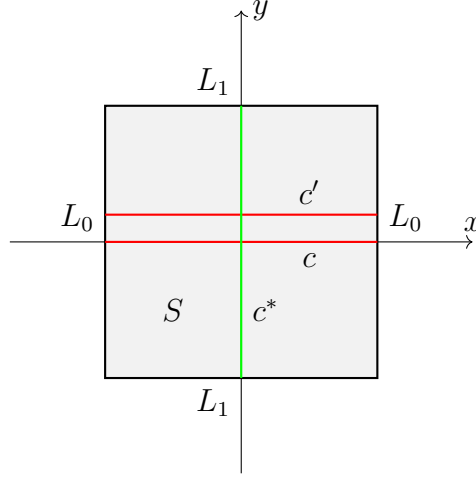


Figure 5.9: The figure shows the band $S \subset S^3$ associated to the saddle cobordism. The curves c and c^* denote the core and cocore of the band S .

belong to L_0 , and that the rest of L_0 is contained in $\mathbb{R}^3 \setminus S \times \mathbb{R}$.

We now introduce a cancelling pair of 4-dimensional 1-/2-handles, which define a product cobordism from $S^3 \times I$ to itself. The 1-handle $\mathfrak{h}^1$ is an ε -neighbourhood of the band B , where $\varepsilon > 0$ is small enough, and the 2-handle is

$$\mathfrak{h}^2 = \{(x, y, z, t) \mid \exists \bar{t} \geq t : (x, y, z, \bar{t}) \in B\}.$$

If we remove an ε -neighbourhood of F from $S^3 \times I$ we obtain the special cobordism $\mathcal{W}^s$, or, more precisely, a cobordism diffeomorphic to it. The 1-handle $\mathfrak{h}^1$ is contained in $\mathcal{N}_\varepsilon(F)$, and therefore disappears when we consider $\mathcal{W}^s$. On the other hand, $\mathfrak{h}^2$ is contained in $\mathcal{W}^s$, except a small portion near the boundary, which is contained in $\mathcal{N}_\varepsilon(F)$. An illustration of $\mathfrak{h}^2$ is available in Figure 5.10 on the left hand side.

The core of the 2-handle is

$$C = \mathfrak{h}^2 \cap \{y = z = 0\},$$

whose boundary is $c \cup b$, which is therefore the attaching sphere.

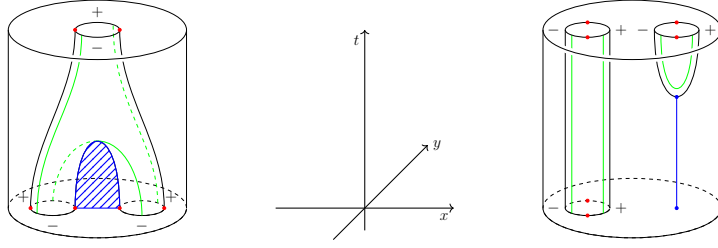


Figure 5.10: The figure above illustrates two decorated cobordisms. In both cases, the decorated links are embedded in S^3 , and the cobordism is embedded in $S^3 \times I$. The figure represents the $\{z = 0\}$ section $\mathbb{R}^2 \times \{0\} \times I$ of $\mathbb{R}^3 \times I \subseteq S^3 \times I$, with coordinates as shown above. Here, the coordinates on $\mathbb{R}^3$ are x , y , and z , and the coordinate on I is t . We use this convention every time a (decorated) cobordism in $S^3 \times I$ is represented enclosed in a cylinder, as above. The figure on the left shows a merge saddle cobordism from (U_2, P_{U_2}) —the two-component unlink with standard decorations—to (U_1, P_{U_1}) —the unknot with standard decorations. The core of the 2-handle described in Section 5.6.2 is shown in blue. The picture on the right shows a birth cobordism from (U_1, P_{U_1}) to (U_2, P_{U_2}) . The core of the 1-handle described in Section 5.7 is shown in blue.

For the framing of $\mathfrak{h}^2$, a framing disc, parallel to C , in $\mathfrak{h}^2$ is given by

$$C' = \mathfrak{h}^2 \cap \{z = 0, y = \varepsilon\},$$

whose boundary, which is the framing of $\mathfrak{h}^2$, is $c' \cup b'$.

The height function on $\mathcal{W}^s \setminus \mathfrak{h}^2$ does not have any critical points, and can be rescaled to be a Morse function. It follows that $\mathcal{W}^s$ consists of the single attachment of $\mathfrak{h}^2$. $\square$

It follows from Proposition 5.19 and from the work of Juhász³¹ (Section 6) that the cobordism map $F_{\mathcal{W}^s}$ can be computed from any triple sutured Heegaard diagram subordinate to the 2-handle attachment. We give an explicit example of such a diagram in the next proposition.

Proposition 5.20 (Juhász-M.³²). *Let $\mathcal{W}^s$ denote the special cobordism associated to a saddle cobordism (F, σ) from (L_0, P_0) to (L_1, P_1) . Let $\mathcal{H}_0$ denote a Heegaard diagram for (L_0, P_0) arising from a connected link projection, as explained in Section 2.4, which in a neighbourhood of the band corresponding to the saddle move appears as in Figure 5.7 on the left hand side. Let $\mathcal{H}$ be the triple Heegaard diagram obtained by locally modifying $\mathcal{H}_0$ as in Figure 5.7 on the right hand side,*

and by setting all the δ curves that are not shown as isotopic translates of the respective β curves. Then $\mathcal{H}$ is a triple sutured Heegaard diagram subordinate to the 2-handle described in Proposition 5.19.

Proof. Consider the Heegaard diagram on the right hand side of Figure 5.8, and suppose that it is embedded in S^3 as in the figure. Let $\Sigma \times I$ be a regular neighbourhood of the Heegaard surface Σ in S^3 . The translates $\Sigma_+ = \Sigma \times \{1\}$ and $\Sigma_- = \Sigma \times \{0\}$ of Σ appear on the inside and on the outside of $\Sigma = \Sigma \times \{0.5\}$, respectively.

Let $\tilde{C} \subset \Sigma$ denote the annulus joining the two vertical tubes. With the notation established before Proposition 5.19, $\tilde{C} = UN\tilde{c}$. We denote the translates of the annulus $\tilde{C} \subset \Sigma$ by $\tilde{C}_+ \subset \Sigma_+$ and $\tilde{C}_- \subset \Sigma_-$. By Proposition 5.19, the attaching circle $\mathbb{L} = c \cup b$ consists of two arcs, such that $b \subset \tilde{C}$, and c lies on the square $S \subset M_0$. Let $B(\mathbb{L})$ be the bouquet obtained by connecting $\mathbb{L}$ to $\tilde{C}_+$ via a straight arc entirely contained in $C \times I$.

Consider now the (α, β) -Heegaard diagram on the right hand side of Figure 5.7, which again we can think of as embedded in S^3 , and denote it by (Σ', α, β) . In order to construct the sutured manifold associated to it, one should take $\Sigma' \times I$, and attach thickened discs to Σ'_- along the α -curves and to Σ'_+ along the β -curves. We denote these discs by $D_{\alpha_i}^2$ and $D_{\beta_i}^2$. Given subsets $\alpha' \subset \alpha$ and $\beta' \subset \beta$, let $\Sigma'(\alpha', \beta')$ be the sutured manifold obtained from $\Sigma' \times I$ by attaching the thickened discs corresponding to the α - and β -curves in α' and β' . The manifold $\Sigma' \times I$ can also be viewed as a subset of S^3 , as well as the disc $D_{\alpha_1}^2$, which lies in the plane of the link projection as it arose from the Ozsváth-Szabó diagram explained in Subsection 2.4, so it follows that $\Sigma'(\alpha_1)$ is also embedded in S^3 . Note that the band S is contained within $D_{\alpha_1}^2$.

The manifold $\Sigma'(\alpha_1, \alpha_2)$ is obtained from $\Sigma'(\alpha_1)$ by attaching the thickened disc $D_{\alpha_2}^2$. The upper half of α_2 is parallel to the upper half of α_1 along the top of Figure 5.7 and outside of it, hence we can slide the upper half of $D_{\alpha_2}^2$ along $D_{\alpha_1}^2$ until the upper boundary becomes a horizontal arc across $D_{\alpha_1}^2$, and $D_{\alpha_2}^2$ becomes perpendicular to $D_{\alpha_1}^2$. After the isotopy, $D_{\alpha_2}^2$ connects the small handle in the centre of Figure 5.7 to $D_{\alpha_1}^2 \times I$. This implies that

$\Sigma'(\alpha_1, \alpha_2)$ can also be seen as embedded in S^3 , and we can isotope the small handle in the centre of Figure 5.7 onto $D_{\alpha_1}^2$. Attaching all the other handles along the α - and β -curves except β_1 , one obtains $\Sigma'(\alpha_1, \dots, \alpha_g, \beta_2, \dots, \beta_g)$, which is the same manifold as the one defined by the Heegaard diagram in Figure 5.8—that is, N —with the difference that a neighbourhood of $B(\mathbb{L})$ has been removed from it. The curve β_1 is the meridian of $\mathbb{L}$, and δ_1 is exactly the framing given by Proposition 5.19. Therefore, this is a Heegaard diagram subordinate to the 2-handle attachment. $\square$

We can now conclude the proof of Theorem 5.16.

Proof of Theorem 5.16. The proof follows by combining Propositions 5.18, 5.19, and 5.20, and by the fact that, by the work of Juhász³¹ (Section 6), $F_{\mathcal{W}^s}$ is the map induced by the chain map that counts pseudo-holomorphic triangles in $\mathcal{H}$:

$$f_{\mathcal{W}^s}(\mathbf{x}) = \sum_{\mathbf{y} \in \mathbb{T}_{\alpha} \cap \mathbb{T}_{\delta}} \sum_{\substack{\psi \in \pi_2(\mathbf{x}, \vartheta, \mathbf{y}) \\ \mu(\psi)=0}} \# \mathcal{M}(\psi) \cdot \mathbf{y}. \quad \square$$

5.6.3 SADDLE COBORDISMS AND THE DECORATED SURGERY EXACT TRIANGLE

In this subsection, we use the Heegaard diagrammatic description of the saddle cobordism to prove Theorem 5.17, which is re-stated below.

Suppose that (L_0, P_0) and (L_+, P) are links in S^3 that are related by a saddle cobordism as in Figure 5.5. Let L be the link obtained from L_+ by adding a full right-handed twist. Then (L, P) , (L_0, P_0) , and (L_+, P) have projections that differ only locally as in Figure 5.5, and hence fit into the surgery exact triangle

$$\longrightarrow \text{HFL}(L, P) \xrightarrow{e} \text{HFL}(L_0, P_0) \xrightarrow{f} \text{HFL}(L_+, P) \longrightarrow$$

of Theorem 5.15. Indeed, consider the disc D that is perpendicular to the plane of Figure 5.5 and intersects it in a horizontal arc that meets L in two

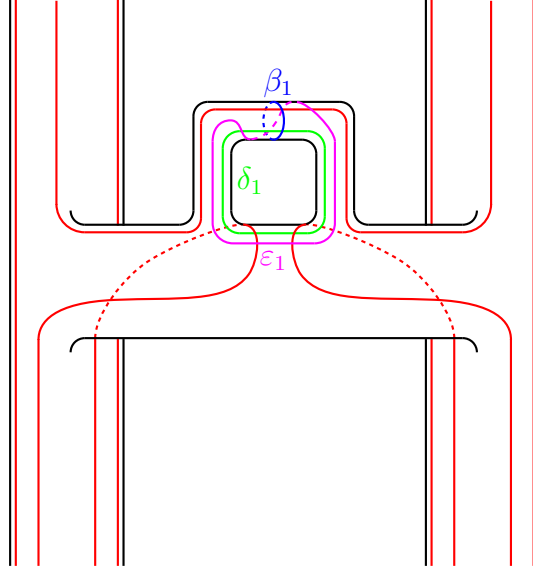


Figure 5.11: The quadruple diagram $(\Sigma, \alpha, \varepsilon, \beta, \delta)$ above has four sets of attaching curves. The triple diagram $(\Sigma, \alpha, \beta, \delta)$ is subordinate to the 2-handle attachment described in Section 5.6.2. The diagram $(\Sigma, \alpha, \varepsilon)$ is for $S^3(L, P) = (M_\infty(K), \gamma)$, while (Σ, α, β) is for $(M_0(K), \gamma)$, and (Σ, α, δ) is for $S^3(L_+, P) = (M_1(K), \gamma)$. The δ -curves and the ε -curves not shown are small isotopic translates of the β -curves.

points. Then resolving L along D gives L_0 , and doing a left twist gives L_+ .

Theorem 5.17 (Juhász-M.³²). *Let the triple (L, P) , (L_0, P_0) , (L_+, P) be as in Figure 5.5, and suppose that they can be oriented coherently. Then the map*

$$f: \text{HFL}(L_0, P_0) \longrightarrow \text{HFL}(L_+, P)$$

appearing in the decorated skein exact triangle of Theorem 5.15 coincides with the cobordism map induced by the saddle cobordism from (L_0, P_0) to (L_+, P) .

Proof. Consider the saddle cobordism $\mathcal{S}$ from (L_0, P_0) to (L_+, P) . Then the cobordism map

$$F_{\mathcal{S}}: \text{HFL}(L_0, P_0) \rightarrow \text{HFL}(L_+, P)$$

is given by the isomorphism induced by gluing along a product annulus as in Section 5.6.1, followed by the map obtained by counting holomorphic triangles in the triple diagram $(\Sigma, \alpha, \beta, \delta)$ in Figure 5.7 on the right.

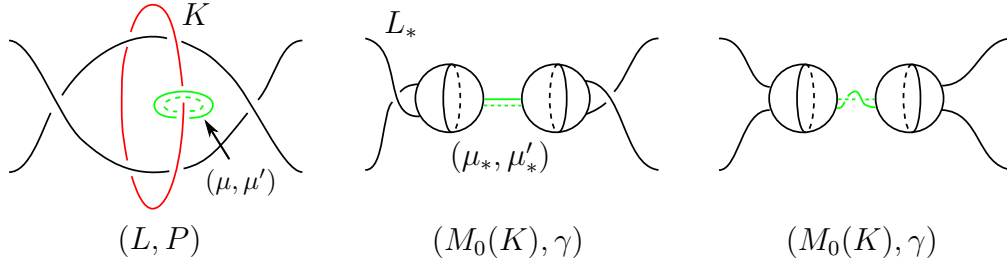


Figure 5.12: The left-hand side of the figure shows the link (L, P) with the small unknot $K = \partial D$ in red, and the two concentric meridians of K , denoted by μ and μ' , in green. In the middle, we see the result of 0-surgery along K . The manifold $S^3_0(K) = S^1 \times S^2$ is represented here as the complement of two balls, with their boundaries identified via a reflection. The framed knot (μ_*, μ'_*) is the image of (μ, μ') . After untwisting L^* , we get the figure on the right, making the solid and the dashed green arcs linked.

Consider the crossing disc D that intersects L in two points, and let $K = \partial D$. As in Section 5.5, the framings $m = \infty$, $l = 0$, and $s = 1$ of K give rise to the surgery exact triangle

$$\cdots \longrightarrow \mathrm{SFH}(M_\infty(K), \gamma) \xrightarrow{e'} \mathrm{SFH}(M_0(K), \gamma) \xrightarrow{f'} \mathrm{SFH}(M_1(K), \gamma) \longrightarrow \cdots \quad (5.2)$$

If A denotes the product annulus obtained from D in $M_0(K)$, and

$$\Phi_A: \mathrm{HFL}(L_0, P_0) \rightarrow \mathrm{SFH}(M_0(K), \gamma)$$

is the gluing map, then we obtain the exact triangle

$$\cdots \longrightarrow \mathrm{HFL}(L, P) \xrightarrow{e} \mathrm{HFL}(L_0, P_0) \xrightarrow{f} \mathrm{HFL}(L_+, P) \longrightarrow \cdots$$

by setting $e = \Phi_A^{-1} \circ e'$ and $f = f' \circ \Phi_A$.

The result follows once we show that the quadruple diagram $(\Sigma, \alpha, \varepsilon, \beta, \delta)$ in Figure 5.11 is as in the statement of Lemma 9.2 of Ozsváth and Szabó's paper⁵⁹, and counting holomorphic triangles in such a diagram yields the exact sequence (5.2).

Consider a local projection for L as on the left of Figure 5.12, and let K be the boundary of the crossing disc D . Let μ and μ' be two concentric meridians of K . We view μ as a knot with framing μ' . After performing

0-surgery along K , the framed knot (μ, μ') becomes isotopic to the core of the solid torus that we glued in during the surgery, and which we denote by (μ_*, μ'_*) in $S^1 \times S^2 = S_0^3(K)$; i.e., μ_* is $S^1 \times \{\text{pt}\}$ and μ'_* is $S^1 \times \{\text{pt}'\}$. These are shown in green in the middle of Figure 5.12. This means that performing surgery along the framed knot (μ_*, μ'_*) in $(M_0(K), \gamma)$, we recover $S^3(L, P) = (M_\infty(K), \gamma)$. In particular, $M_0(K) \setminus \mathcal{N}(\mu_*) = M_\infty(K) \setminus \mathcal{N}(K)$.

The disc D can be capped off in $M_0(K) \subset S^1 \times S^2$ to obtain a sphere $S^2 \subset M_0(K)$. If we cut $S^1 \times S^2$ along this sphere, the manifold that we obtain is the complement of the two balls in the middle of Figure 5.12. To recover $S^1 \times S^2$, glue the boundaries of the balls via an identification given by a reflection. Note that, in the middle of Figure 5.12, the framed knot (μ_*, μ'_*) appears as two parallel arcs.

Performing 0-surgery on K sends the link L to the link L_* in the middle of Figure 5.12. By twisting the left-hand sphere by π to the left and the right-hand sphere by π to the right along the horizontal axis (this is an automorphism of $S^1 \times S^2$), we remove the two crossings of L , but introduce a full right-handed twist in the green pair of arcs μ_* and μ'_* ; see the right-hand side of Figure 5.12. By the twist, the link L_* is sent to the link on the right-hand side of Figure 5.12, whose complement has a Heegaard diagram as in Figure 5.8.

By Proposition 5.20 that the triple diagram $(\Sigma, \alpha, \beta, \delta)$ is subordinate to performing surgery on $(M_0(K), \gamma)$ along the framed knot $(c \cup b, c' \cup b')$. On the right of Figure 5.12, $(c \cup b, c' \cup b')$ corresponds to μ_* with a framing f that is a parallel vertical copy of μ_* (see the green pair in Figure 5.4 on the right). Hence, surgery on $(M_0(K), \gamma)$ along (μ_*, μ'_*) gives $(M_\infty(K), \gamma)$, while surgery along (μ_*, f) gives $(M_1(K), \gamma)$.

Let $B(\mu_*)$ be a bouquet for the knot μ_* in $(M_\infty(K), \gamma)$, and let $B(K)$ be the corresponding bouquet of K in $(M_0(K), \gamma)$, as defined by Juhász³¹. The Heegaard subdiagram $(\Sigma, \alpha, \beta \setminus \beta_1)$ of the diagram on the right hand side of Figure 5.7 represents the sutured manifold $(M_0(K) \setminus B(\mu_*), \gamma)$, or, equivalently, $(M_\infty(K) \setminus B(K), \gamma)$. The curve β_1 on the right hand side of

Figure 5.7 corresponds to a meridian of μ_* , so Dehn filling the manifold described by $(\Sigma, \alpha, \beta \setminus \beta_1)$ along β_1 yields $(M_0(K), \gamma)$. The curve δ_1 corresponds to f , so Dehn filling along it yields $S^3(L_+, P) = (M_1(K), \gamma)$. The curve ε_1 in Figure 5.11 corresponds to μ'_* , and so filling along it yields $S^3(L, P) = (M_\infty(K), \gamma)$. Furthermore, it satisfies the condition $[\beta_1] + [\delta_1] = [\varepsilon_1]$ in the first homology of the boundary, and β_1 , δ_1 , and ε_1 are oriented such that their pairwise algebraic intersection numbers are -1 . It follows that the Heegaard diagram $(\Sigma, \alpha, \varepsilon, \beta, \delta)$ satisfies the hypotheses of Lemma 9.2 by Ozsváth and Szabó⁵⁹, and therefore the count of holomorphic triangles in such a diagram gives the exact triangle (5.2). $\square$

5.7 BIRTHS

We now turn to the description of the map associated to a birth cobordism. Let V denote a 2-dimensional vector space over $\mathbb{F}_2$, generated by two homogeneous elements T , in Alexander grading 0 and Maslov grading $1/2$, and B , in Alexander grading 0 and Maslov grading $-1/2$. Then we have the following result.

Theorem 5.21 (Juhász-M.³²). *Suppose that $\mathcal{B}$ is a birth cobordism from (L_0, P_0) to (L_1, P_1) . Then there is an isomorphism*

$$\mathrm{HFL}(L_1, P_1) \cong \mathrm{HFL}(L_0, P_0) \otimes V$$

such that $F_{\mathcal{B}}(x) = x \otimes T$ for every $x \in \mathrm{HFL}(L_0, P_0)$.

Proof. Let $(M_0, \gamma_0) = \mathcal{W}(L, P)$ and $(M_1, \gamma_1) = \mathcal{W}(L \sqcup U_1, P \sqcup P_{U_1})$ denote the sutured manifolds complementary to the links. As usual, we write the cobordism $\mathcal{W}(\mathcal{B})$ as the composition of a boundary cobordism $\mathcal{W}^b$ from (M_0, γ_0) to (N, γ_1) , and a special cobordism $\mathcal{W}^s$ from (N, γ_1) to (M_1, γ_1) , where

$$(N, \gamma_1) = (M_0, \gamma_0) \sqcup (S^1 \times D^2, \gamma),$$

and γ consists of two longitudes as sutures. Note that $\mathrm{SFH}(S^1 \times D^2, \gamma) \cong$

$\mathbb{F}_2$, and hence

$$\mathrm{SFH}(N, \gamma_1) \cong \mathrm{SFH}(M_0, \gamma_0) \otimes \mathbb{F}_2$$

in a canonical way by the multiplicativity of sutured Floer homology³¹ (Theorem 11.11). Since one component of the boundary cobordism is a product $(M_0, \gamma_0) \times I$, by multiplicativity of SFH the gluing map

$$\Phi_{-\xi}: \mathrm{SFH}(M_0, \gamma_0) \rightarrow \mathrm{SFH}(M_0, \gamma_0) \otimes \mathbb{F}_2$$

is either the zero map or the tautological isomorphism. Since $\mathcal{B}$ has a left inverse, which is a saddle cobordism, by functoriality the map $\Phi_{-\xi}$ must be the tautological isomorphism.

We can now focus on the special cobordism $\mathcal{W}^s$, for which we have the following lemma.

Lemma 5.22 (Juhász-M.³²). *The special cobordism $\mathcal{W}^s$ associated to a birth consists of the 1-handle attachment that connects the two components of N .*

Proof. Suppose that the birth of the new component is centred (with respect to some local coordinate system on S^3) at $O = (0, 0, 0)$. Without loss of generality, we can assume that the birth takes place in a ball $B_R(O)$ of radius R centered at the origin so small that it is disjoint from L .

We can suppose that the sections $\mathcal{W}_t^s$ of the special cobordism inside $S^3 \times I$ appear as follows:

- for $t \in [0, 1/2]$, we have $\mathcal{W}^s = M_0$;
- for $t \in (1/2, 3/4)$, we have $\mathcal{W}^s = M_0 \setminus B_R(O)$; and
- for $t \in [3/4, 1]$, since $U_1 \subseteq B_R(O)$, we can suppose that, if $t > \bar{t}$, then $\mathcal{W}_t^s \supseteq \mathcal{W}_{\bar{t}}^s$, and $\mathcal{W}_1^s = M_1$.

We define

$$\mathfrak{H}^1 := \{(x, y, z, t) \mid (x, y, z) \in B_R(O) \text{ and } t \in [0, 1/2]\}.$$

This is a 1-handle that connects the two components of N . An illustration of $\mathfrak{H}^1$ in a particular link cobordism can be found on the right hand side of Figure 5.10. The 1-handle structure is clear from the diffeomorphism

$$\mathfrak{H}^1 \cong [0, 1/2] \times B_R(O) \cong D^1 \times D^3.$$

The height function on $\mathcal{W}^s \setminus \mathfrak{H}^1$ does not have any critical points, and can be rescaled to be a Morse function. It follows that $\mathcal{W}^s$ consists of the single attachment of $\mathfrak{H}^1$. $\square$

Hence, by Juhász³¹ (Definition 7.5), the map $F_{\mathcal{W}^s}$ is

$$\begin{aligned} F_{\mathcal{W}^s} : \text{SFH}(M_0, \gamma_0) &\rightarrow \text{SFH}(M_0, \gamma_0) \otimes V. \\ x &\mapsto x \otimes T \end{aligned}$$

It follows that the composition $F_{\mathcal{B}} = F_{\mathcal{W}^s} \circ \Phi_{-\xi}$ is as in the statement of the theorem. $\square$

5.8 DEATHS

For a death cobordism $\mathcal{D} = (F, \sigma)$ from $(L \sqcup U_1, P \sqcup P_{U_1})$ to (L, P) , where (U_1, P_{U_1}) is the unknot with two decorations, we have a result dual to Theorem 5.21. As before, V denotes a 2-dimensional vector space over $\mathbb{F}_2$, generated by two homogeneous elements T , in Alexander grading 0 and Maslov grading 1/2, and B , in Alexander grading 0 and Maslov grading $-1/2$.

Theorem 5.23 (Juhász-M.³²). *Let $\mathcal{D}$ be a death cobordism from (L_0, P_0) to (L_1, P_1) . Then there is an isomorphism*

$$\text{HFL}(L_0, P_0) \cong \text{HFL}(L_1, P_1) \otimes V$$

such that $F_{\mathcal{D}}(x \otimes T) = 0$ and $F_{\mathcal{D}}(x \otimes B) = x$ for every $x \in \text{HFL}(L_0, P_0)$.

Proof. By turning the cobordism $\mathcal{W}(\mathcal{D})$ upside down, we obtain a birth

cobordism $\overline{\mathcal{W}(\mathcal{D})}$, which we can break into a boundary cobordism $\overline{\mathcal{W}^b}$ followed by a special cobordism $\overline{\mathcal{W}^s}$. By turning this decomposition upside down, we get a decomposition $\mathcal{W}(\mathcal{D}) = \mathcal{W}^b \circ \mathcal{W}^s$, where

- $\mathcal{W}^s$ is the *special* cobordism obtained by turning $\overline{\mathcal{W}^s}$ upside down;
- $\mathcal{W}^b$ is a (*not necessarily boundary*) cobordism, obtained by turning $\overline{\mathcal{W}^b}$ upside down.

$\mathcal{W}^s$ is a cobordism from $(M_0, \gamma_0) \cong (M_1, \gamma_1) \# (S^1 \times D^2, \gamma)$ to $(M_1, \gamma_1) \sqcup (S^1 \times D^2, \gamma)$. Here we are using the same notation as in Section 5.7. The map $F_{\mathcal{W}^s}$ is dual to $F_{\overline{\mathcal{W}^s}}$ by Juhász³¹ (Theorem 11.8). More explicitly, $\mathcal{W}^s$ consists of a single 3-handle dual to the 1-handle $\mathfrak{H}^1$ from Lemma 5.22. By Juhász³¹ (Definition 7.8), there is an isomorphism

$$\mathrm{SFH}(M_0, \gamma_0) \cong \mathrm{SFH}(M_1, \gamma_1) \otimes V$$

such that the map

$$F_{\mathcal{W}^s}: \mathrm{SFH}(M_1, \gamma_1) \otimes V \rightarrow \mathrm{SFH}(M_1, \gamma_1) \otimes \mathbb{F}_2$$

is given by $F_{\mathcal{W}^s}(x \otimes T) = 0$ and $F_{\mathcal{W}^s}(x \otimes B) = x \otimes 1$ for all $x \in \mathrm{SFH}(M_1, \gamma_1)$. Note that, as in Section 5.7, we are identifying $\mathrm{SFH}((M_1, \gamma_1) \sqcup (S^1 \times D^2, \gamma))$ with $\mathrm{SFH}(M_1, \gamma_1) \otimes \mathbb{F}_2$ via the canonical isomorphism given by the multiplicativity of sutured Floer homology.

We now need to understand the map

$$F_{\mathcal{W}^b}: \mathrm{SFH}(M_1, \gamma_1) \otimes \mathbb{F}_2 \rightarrow \mathrm{SFH}(M_1, \gamma_1).$$

By multiplicativity of sutured Floer homology, this is either the tautological isomorphism or the zero map. Since $\mathcal{D}$ has a right inverse, which is a saddle preceded by two stabilisations, the map $F_{\mathcal{W}^b}$ is the tautological isomorphism. By composing $F_{\mathcal{W}^b} \circ F_{\mathcal{W}^s}$ we get the map $F_{\mathcal{D}}$ as in the statement. $\square$

5.9 DUALITY

If $\mathcal{X}$ is a decorated link cobordism from (L_0, P_0) to (L_1, P_1) (or more generally, a sutured cobordism from (M_0, γ_0) to (M_1, γ_1)), then one can turn it upside down to get a decorated link cobordism from $(\overline{L_1}, \overline{P_1})$ to $(\overline{L_0}, \overline{P_0})$ (or more generally a sutured cobordism from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$).

Juhász³¹ (Theorem 11.8) proved that if you turn a *special* cobordism upside down, then the map induced on SFH is the dual map, and asked in Question 11.9 whether this is true for any sutured cobordism. A weaker question is whether this is true at least for decorated link cobordisms. An approach to answer this weaker question would be to check what happens for elementary cobordisms.

Proposition 5.24 (Juhász-M.³²). *Let $\mathcal{S} = (F, \sigma)$ be a stabilisation from (L_0, P_0) to (L_1, P_1) , and let $\mathcal{D}$ be the destabilisation obtained by turning $\mathcal{S}$ upside down. Then the maps $F_{\mathcal{S}}$ and $F_{\mathcal{D}}$ are dual to each other.*

Proof. Let $\sigma_0 \subseteq \sigma$ denote the arc with both endpoints in L_1 . Let p, q and r denote three consecutive points on P_1 such that the two points of $\partial\sigma_0$ are contained in the components of $L_1 \setminus P_1$ bounded by p and q , and by q and r . Call $\mathcal{N}$ a neighbourhood in S^3 of the component of $L_1 \setminus \{p, r\}$ containing q . Then $\partial\mathcal{N}$ is a sphere cutting L_1 in two points, which induces a product annulus A in (M_1, γ_1) . By Propositions 5.12 and 5.14, A induces an isomorphism

$$\Phi_A = I_{\mathcal{S}} = I_{\mathcal{D}}^{-1}: \text{HFL}(L_0, P_0) \otimes W \rightarrow \text{HFL}(L_1, P_1).$$

Under this identifications, the maps induced by $\mathcal{S}$ and $\mathcal{D}$ are dual to each other, by the formulas given in Propositions 5.12 and 5.14. $\square$

Proposition 5.25 (Juhász-M.³²). *Let $\mathcal{B}$ be a birth cobordism, and let $\mathcal{D}$ be the death cobordism obtained by turning $\mathcal{B}$ upside down. Then the maps $F_{\mathcal{B}}$ and $F_{\mathcal{D}}$ are dual to each other.*

Proof. Write $\mathcal{W}(\mathcal{B}) = \mathcal{W}^s \circ \mathcal{W}^b$, where $\mathcal{W}^b$ is boundary and $\mathcal{W}^s$ special.

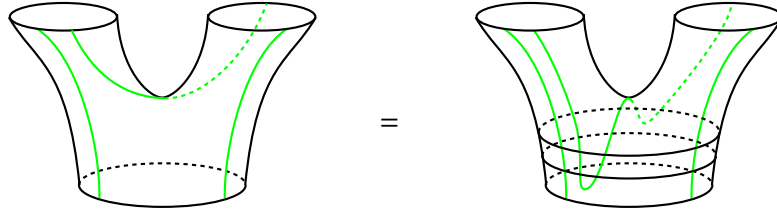


Figure 5.13: If we turn a saddle cobordism upside down, then we can write it as a composition of two stabilisations, followed by another saddle cobordism.

Then, by turning the decomposition upside down, we obtain a decomposition $\mathcal{W}(\mathcal{D}) = \overline{\mathcal{W}}^b \circ \overline{\mathcal{W}}^s$, where $\overline{\mathcal{W}}^s$ is still special. The maps $F_{\mathcal{W}^s}$ and $F_{\overline{\mathcal{W}}^s}$ are dual to each other by Juhász³¹ (Theorem 11.8), and the maps $F_{\mathcal{W}^b}$ and $F_{\overline{\mathcal{W}}^b}$ are dual to each other because they both coincide with the tautological isomorphism (see proofs of Theorems 5.21 and 5.23). $\square$

The situation is more complicated for the saddle maps. First, note that if we turn a saddle cobordism upside down, then the height function has a local minimum on σ as opposed to a maximum, so this is not an elementary saddle cobordism, according to our definition. However, as in Figure 5.13, we can write this as a composition of two stabilisations and another saddle cobordism (which is a split if the original cobordism was a merge, and vice versa).

Let $\mathcal{W}$ be the sutured cobordism associated to a saddle. As usual, we write it as $\mathcal{W}^s \circ \mathcal{W}^b$, where

- $\mathcal{W}^b$ is a boundary cobordism from (M_0, γ_0) to (N, γ_1) , given by the gluing of a product annulus;
- $\mathcal{W}^s$ is a special cobordism from (N, γ_1) to (M_1, γ_1) , consisting of a 2-handle attachment.

By turning this decomposition upside down, we obtain a decomposition for the upside down sutured cobordism $\overline{\mathcal{W}}$ as $\overline{\mathcal{W}}^b \circ \overline{\mathcal{W}}^s$, where

- $\overline{\mathcal{W}}^s$ is a special cobordism from $(-M_1, -\gamma_1)$ to $(-N, -\gamma_1)$ (consisting of the 2-handle dual to the one for $\mathcal{W}^s$);

- $\overline{\mathcal{W}}^b$ is a cobordism from $(-N, -\gamma_1)$ to $(-M_0, -\gamma_0)$ that is *not* a boundary cobordism.

The map $F_{\mathcal{W}^s}$ induced by the special cobordism is dual to the map $F_{\overline{\mathcal{W}}^s}$ by Juhász³¹ (Theorem 11.8).

The map $F_{\overline{\mathcal{W}}^b}$ is not so clear. If one could prove that this is the inverse of the isomorphism induced by the product annulus decomposition

$$(-N, -\gamma_1) \rightsquigarrow (-M_0, -\gamma_0),$$

then the upside down cobordism would induce the dual map. Therefore, the question about duality in the decorated link category boils down to the following question.

Question 5.26 (Juhász-M.³²). Let $(N, \gamma_1) \rightsquigarrow (M_0, \gamma_0)$ be a product annulus decomposition, and let $\mathcal{W}$ be the boundary cobordism from (M_0, γ_0) to (N, γ_1) given by gluing along the product annulus. Does the map induced by $\overline{\mathcal{W}}$ coincide with the inverse of the inclusion induced by the product annulus decomposition?

5.10 GRADINGS

Recall that HFL of a link L in S^3 can be naturally endowed with two gradings, the (collapsed) Alexander grading (denoted by $\mathcal{A}$) and the Maslov grading (denoted by $\tilde{gr}$). They were both defined in Section 2.3. We will also consider the δ -grading, defined by $\delta = \mathcal{A} - \tilde{gr}$, following the convention established by Manolescu and Ozsváth⁴⁷.

Theorem 5.27 (Juhász-M.³²). *If $\mathcal{X} = (F, \sigma)$ is a decorated link cobordism, then the map induced on HFL is homogeneous with respect to both the Alexander and the Maslov gradings. Furthermore, if $\mathcal{X}$ is an elementary cobordism, then the shifts are as given in Table 5.1. Consequently, $F_{\mathcal{X}}$ shifts the Maslov grading by*

$$\frac{\chi(F) + \chi(R_+(\sigma)) - \chi(R_-(\sigma))}{2} \quad (5.3)$$

	Alexander grading	Maslov grading
Positive (de)stabilisation	$+\frac{1}{2}$	$+\frac{1}{2}$
Negative (de)stabilisation	$-\frac{1}{2}$	$-\frac{1}{2}$
Birth or death	0	$+\frac{1}{2}$
Saddle	0	$-\frac{1}{2}$

Table 5.1: The table shows the shift in the Alexander grading and the Maslov grading associated to each elementary cobordism.

and the Alexander grading by

$$\frac{\chi(R_+(\sigma)) - \chi(R_-(\sigma))}{2}. \quad (5.4)$$

Remark 5.28. As $\delta = \mathcal{A} - \tilde{g}r$, an immediate consequence of Theorem 5.27 is that F_χ is also δ -homogeneous, and that the δ -grading shift is $-\chi(F)/2$ and therefore does not depend on the decorations.

Proof. Since every decorated cobordism is a product of elementary cobordisms by Proposition 5.6, and formulas (5.3) and (5.4) are additive under composition of decorated link cobordisms, it is sufficient to prove that the maps associated to elementary cobordisms are homogeneous.

Stabilisation maps (see Proposition 5.12) are defined on the chain level as tensor products with a contact element, so they are homogeneous. The Alexander and the Maslov gradings of such an element (t or b) coincides with the grading shift. Destabilisation maps (see Propositions 5.14 and 5.24) are dual to stabilisation maps, so they are also homogeneous, and the grading shifts are the same as for stabilisations.

By Theorem 5.21, the birth cobordism map is induced by the tensor product with the top-graded element of the vector space generated by two elements (called T and B) in different homological gradings, but in the same Alexander grading. Therefore the map is homogeneous and the grading shift is $(\mathcal{A}, \tilde{g}r) = (0, +\frac{1}{2})$. By Proposition 5.25, the death cobordism map is dual to the birth cobordism map, so the grading shift is the same.

By Theorem 5.17, the saddle map is given by an isomorphism Φ_A induced by a product annulus—which does not change the gradings by Proposition 5.18—followed by the map f' appearing in the exact triangle (5.2). Following Ozsváth and Szabó’s proof of Theorem 10.2⁵⁸, we can prove that f' preserves the Alexander grading and drops the Maslov grading by $\frac{1}{2}$, so the saddle map does the same.

Finally, isotopies induce diffeomorphism maps, which preserve both the Alexander and Maslov gradings. $\square$

As a straightforward corollary of Theorem 5.27, we can compute the shift in the Alexander grading and Maslov grading for decorated concordances, hence obtaining an independent proof of the first part of Theorem 4.2, when the decorated concordance is in $S^3 \times I$.

Corollary 5.29. *Let $\mathcal{C} = (S^3 \times I, F, \sigma)$ be a decorated concordance between decorated knots (K_0, P_0) and (K_1, P_1) . Then*

$$F_{\mathcal{C}} \left(\widehat{\text{HFK}}_i(K_0, P_0, j) \right) \subset \widehat{\text{HFK}}_i(K_1, P_1, j),$$

that is, $F_{\mathcal{C}}$ preserves both the Alexander and the Maslov grading.

5.11 COMPARISON TO OTHER THEORIES

In Section 3.4 we mentioned that there are other constructions of cobordism maps in various versions of link Floer homology, which are defined in a spirit similar to the one that motivates this chapter.

Zemke’s construction⁸⁵ gives maps in all flavour of HFL° . His maps are defined on elementary decorated link cobordisms, and then on arbitrary link cobordisms by composition. The set of elementary cobordisms chosen by Zemke is a bit different from the one we chose for this chapter, in the decorations chosen on the saddle cobordism. The map on $\widehat{\text{HFL}}$ that Zemke associates to a stabilisation (which he calls quasi-stabilisation), and the one he associates to a destabilisation, coincide with the ones computed in Section 5.4. Similarly, the maps that Zemke associates to birth and death

cobordisms agree with the ones computed in Section 5.7 and 5.8. For the saddle cobordism it is unclear whether the two maps are the same or not, even after correcting for the different choice of decorations. The question whether Juhász's theory and Zemke's theory for cobordism maps in $\widehat{\text{HFL}}$ agree is therefore unsettled.

A similar question concerning the relation between Juhász's maps and Sarkar's maps⁷¹, or Alishahi and Eftekhary's tangle Floer maps¹, is also unsettled, as far as I know.

Question 5.30. How does the Morse theoretical description of Juhász's maps given in this chapter relate to the constructions of decorated cobordism maps by Alishahi and Eftekhary¹, Sarkar⁷¹, and Zemke⁸⁵?

6

THE HFL FUNCTOR AND KHOVANOV'S REDUCED TQFT

K^HHOVANOV homology is a homology theory for links introduced by Khovanov³⁷, which categorifies the Jones polynomial. In a subsequent paper³⁸, Khovanov introduced a reduced version of this homology, which was shown to have strong connections with Floer theoretical invariants.

The first such connection was found by Ozsváth and Szabó⁶¹, who proved the existence of a spectral sequence

$$\widetilde{\text{Kh}}(\overline{K}) \Rightarrow \widehat{\text{HF}}(\Sigma_2(K)),$$

where $K \subset S^3$ is a knot, $\overline{K}$ is its mirror image, and $\Sigma_2(K)$ is the branched double cover of S^3 along K .

After Ozsváth and Szabó, Kronheimer and Mrowka⁴¹ constructed a spectral sequence

$$\widetilde{\text{Kh}}(\overline{K}) \Rightarrow \text{KHI}(K),$$

where $\text{KHI}(K)$ is the *sutured instanton homology* of the knot complement. Kronheimer and Mrowka used such a spectral sequence to prove that reduced Khovanov homology is an unknot-detector.

Many related results appeared in the literature^{5,9,11,74,79}, but the relationship between (reduced) Khovanov homology and knot Floer homology is still unsettled. The following conjecture, motivated by the above results, is usually attributed to Rasmussen, who first raised the question about the connection between the two theories⁶⁸.

Conjecture 6.1. *Given a knot $K \subset S^3$, there is a spectral sequence whose E^2 page is $\widetilde{\text{Kh}}(\overline{K})$ and whose E^∞ page is $\widehat{\text{HFK}}(K)$.*

The goal of this chapter is to prove the relation between the two theories expressed by the following theorem. The content of this chapter is based on joint work with Juhász³².

Theorem 6.2 (Juhász-M.³²). *The $(1 + 1)$ -dimensional TQFT induced by HFL on a category of decorated unlinks and decorated cobordisms between them is the same as Khovanov's reduced TQFT. Consequently, the homology of the complex obtained from a cube of resolutions for a knot K by applying the functor HFL is the reduced Khovanov homology of K with $\mathbb{F}_2$ coefficients.*

A more precise statement of the result is given in Theorem 6.21. In order to prove it, we will first determine the maps induced by some elementary decorated cobordisms between the unknot and the two-component unlink, specifically the ones appearing in Figure 6.1. This will be done in Section 6.1 by using the Morse theoretical description of the cobordism maps achieved in Chapter 5. In Section 6.2 we will prove a disjoint union formula for link cobordism maps. In Section 6.3 we define the categories EmbCob^* and AbsCob^* of decorated unlinks and decorated cobordisms between them, and, by using the results from Sections 6.1 and 6.2, we prove Theorem 6.2. Lastly, in Section 6.4 we explain how Theorem 6.2 fits into the literature listed above.

6.1 ELEMENTARY LINK COBORDISMS BETWEEN UNLINKS

This section is devoted to computing the maps induced on HFL by elementary decorated link cobordisms between unlinks. We will denote by (U_n, P_{U_n}) , or, with slight abuse of notation, just by U_n , the standard n -component unlink with two decorations on each component. We will often consider it as embedded in the plane $\{z = 0\} \subseteq \mathbb{R}^3 \subseteq S^3$, with the centres of the components on the x axis, and for each component we will suppose that the x -coordinate of each point of $R_-(P_{U_n})$ (respectively, $R_+(P_{U_n})$) is less than or equal (respectively greater than or equal) to the x -coordinate of the centre; roughly speaking, we suppose that the $-$ arc is on the left and the $+$ arc is on the right. We will often omit the labels $-$ and $+$ in our figures.

The figures illustrating decorated link cobordisms will follow the conventions explained in Figure 5.10: The cylinder enclosing a cobordism means that the cobordism is embedded in $S^3 \times I$ and that the figure represents the section $\{z = 0\}$ in $\mathbb{R}^3 \times I$, with framing as shown in Figure 5.10.

In this section we focus on the following four cobordisms.

Definition 6.3. Let V denote the split saddle cobordism from (U_1, P_{U_1}) to (U_2, P_{U_2}) with decorations as in Figure 6.1. Note that this is not an elementary decorated cobordism, according to Definition 5.4, because the height function restricted to the decorations has a minimum rather than a maximum.

Let Λ denote the merge saddle cobordism from (U_2, P_{U_2}) to (U_1, P_{U_1}) with decorations as in Figure 6.1.

Let Γ denote the birth cobordism from (U_1, P_{U_1}) to (U_2, P_{U_2}) with decorations as in Figure 6.1.

Let L denote the death cobordism from (U_2, P_{U_2}) to (U_1, P_{U_1}) with decorations as in Figure 6.1.

We will now determine the maps associated to the cobordisms defined above. Recall that by V we denote the vector space $\mathbb{F}_2\langle B, T \rangle$ generated

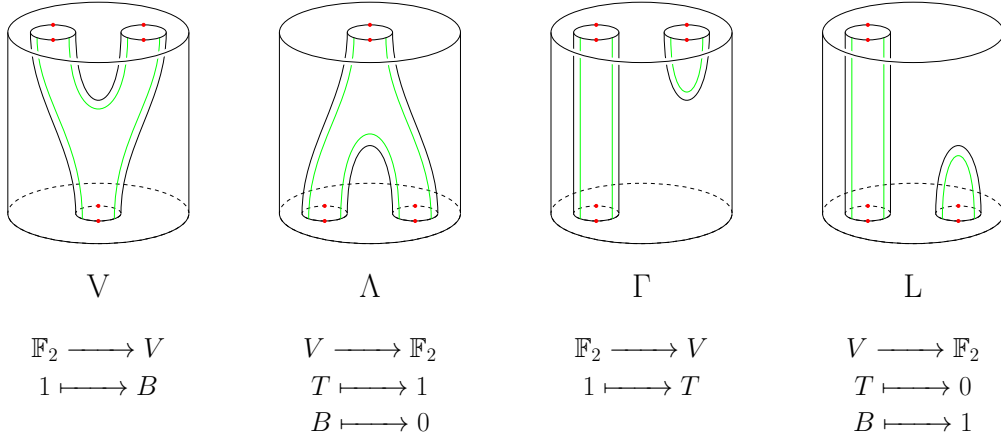


Figure 6.1: The decorated link cobordisms V , Λ , Γ , and L , and the maps induced on link Floer homology. As explained in Figure 5.10, the figures represent the section $\{z = 0\}$ in $\mathbb{R}^3 \times I$.

by two (Maslov) homogenous elements B (in grading $-1/2$) and T (in grading $1/2$).

Theorem 6.4 (Juhász-M.³²). *Under the canonical identifications*

$$\mathrm{HFL}(U_1, P_{U_1}) \cong \mathbb{F}_2 \quad \text{and} \quad \mathrm{HFL}(U_2, P_{U_2}) \cong V,$$

the maps associated to the decorated link cobordisms in Definition 6.3 are as in Figure 6.1.

Proof. Consider the cobordism V . By Theorem 5.27 we know that F_V is homogeneous with respect to the Maslov grading, and that the Maslov grading shift is $-1/2$. Therefore, $F_V(1)$ is either to 0 or to B . Since V has a left inverse, in the sense that $L \circ V = \mathrm{id}_{(U_1, P_{U_1})}$, by functoriality F_V cannot be vanishing. Therefore, $F_V(1) = B$, and the map is determined.

Similar considerations lead to the determination of the maps F_Λ , F_Γ and F_L . □

Remark 6.5. The maps in Figure 6.1 also apply to cobordisms $\tilde{V}$, $\tilde{\Lambda}$, $\tilde{\Gamma}$, and $\tilde{L}$ which have the same surfaces as V , Λ , Γ , and L , but with different decorations, provided that the following two conditions are satisfied:

- There is a partial (left or right) inverse in the category of decorated link cobordisms; and
- $\chi(R_-(\sigma)) = \chi(R_+(\sigma))$.

The proof goes in the same way as for Theorem 6.4.

6.2 A DISJOINT UNION FORMULA

In this section, we focus on a formula for the disjoint union of link cobordisms, with the aim of putting the maps associated to the cobordisms in Figure 6.1 into a TQFT. Although we actually need a minimal version of it, we give a proof in the general setting that might be of independent interest; cf. Theorem 6.14.

It is well known—see for instance Juhász²⁷ (Proposition 9.15)—that the link Floer homology of the “split” disjoint union of two links K and L satisfies

$$\mathrm{HFL}(K \sqcup L) \cong \mathrm{HFL}(K) \otimes \mathrm{HFL}(L) \otimes \mathbb{F}_2^2. \quad (6.1)$$

We would like to understand if the cobordism map associated to a disjoint union of link cobordisms splits in a similar way.

Definition 6.6 (Juhász-M.³²). Suppose that $\mathcal{X}_K = (F, \sigma_F)$ is a decorated link cobordism from (K_0, P_{K_0}) to (K_1, P_{K_1}) , and that $\mathcal{X}_L = (G, \sigma_G)$ is a decorated link cobordism from (L_0, P_{L_0}) to (L_1, P_{L_1}) . We define the *disjoint union cobordism* to be the cobordism in $S^3 \times I$ obtained by taking the split disjoint union of the two surfaces F and G with decoration $\sigma_F \sqcup \sigma_G$. We denote it by $\mathcal{X}_K \amalg \mathcal{X}_L$. Such a cobordism connects the decorated link $(K_0 \sqcup L_0, P_{K_0} \sqcup P_{L_0})$ (the “split” disjoint union of (K_0, P_{K_0}) and (L_0, P_{L_0})) to the decorated link $(K_1 \sqcup L_1, P_{K_1} \sqcup P_{L_1})$.

One would hope that the map induced by the cobordism $\mathcal{X}_K \amalg \mathcal{X}_L$ on sutured Floer homology is the tensor product of the maps induced respectively by $\mathcal{X}_K$ and $\mathcal{X}_L$ and the map $\mathrm{id}_{\mathbb{F}_2^2}$. We will show that this is true under some mild restrictions.

We will actually prove it in a generalised setting, by considering sutured manifolds. Recall that the complement of a decorated link is naturally a sutured manifold; cf. Definition 2.16. By Juhász²⁷ (Proposition 9.15), Equation (6.1) generalises as follows: If (M, γ) and (N, ν) are balanced sutured manifolds, then

$$\mathrm{SFH}((M, \gamma) \# (N, \nu)) \cong \mathrm{SFH}(M, \gamma) \otimes \mathrm{SFH}(N, \nu) \otimes \mathbb{F}_2^2. \quad (6.2)$$

Actually, we need to make the splitting in Equation (6.2) functorial, and for this purpose we use *framed pairs of points, bouquets, and sutured diagrams subordinate to bouquets* as defined by Juhász³¹ (Section 7).

Given a framed pair of points $\mathbb{P}$ for $(M, \gamma) \sqcup (N, \nu)$ such that $p_- \in M$ and $p_+ \in N$, let $M^\circ = \mathrm{Bl}_{p_-}(M)$ and $N^\circ = \mathrm{Bl}_{p_+}(N)$. We can view M° and N° as embedded in $M \#_{\mathbb{P}} N$. Denote by S^2 the sphere $M^\circ \cap N^\circ$. The relative Mayer-Vietoris sequences in cohomology for $(M, \partial M) = (M^\circ, \partial M) \cup B^3$, $(N, \partial N) = (N^\circ, \partial N) \cup B^3$, and $(M \#_{\mathbb{P}} N, \partial(M \#_{\mathbb{P}} N)) = (M^\circ, \partial M) \cup (N^\circ, \partial N)$ imply that the map

$$\begin{aligned} \phi : \mathrm{Spin}^c(M, \gamma) \times \mathrm{Spin}^c(N, \nu) &\longrightarrow \mathrm{Spin}^c(M \#_{\mathbb{P}} N, \gamma \cup \nu) \\ (\mathfrak{s}_M, \mathfrak{s}_N) &\longmapsto (\mathfrak{s}_M)|_{M^\circ} \cup (\mathfrak{s}_N)|_{N^\circ} \end{aligned} \quad (6.3)$$

is well-defined and injective. We will write $\mathfrak{s}_M \#_{\mathbb{P}} \mathfrak{s}_N = \phi(\mathfrak{s}_M, \mathfrak{s}_N)$. Furthermore, $\mathfrak{s} \in \mathrm{Spin}^c(M \#_{\mathbb{P}} N, \gamma \cup \nu)$ is in the image of ϕ if and only if $\mathfrak{s}|_{S^2} = 0$ in $\mathrm{Spin}^c(S^2)$; i.e., if and only if it is the unique Spin^c structure on S^2 with vanishing first Chern class. In this case, we call $\mathfrak{s}|_M$ and $\mathfrak{s}|_N$ the unique spin^c structures such that $\phi(\mathfrak{s}|_M, \mathfrak{s}|_N) = \mathfrak{s}$.

The following is a spin^c refinement of Proposition 9.15 by Juhász²⁷.

Lemma 6.7 (Juhász-M.³²). *Let $\mathbb{P}$ be a framed set of points in the sutured manifold $(M, \gamma) \sqcup (N, \nu)$, in such a way that $p_- \in M$ and $p_+ \in N$. Let $\mathcal{H}_M = (\Sigma_M, \alpha_M, \beta_M)$ and $\mathcal{H}_N = (\Sigma_N, \alpha_N, \beta_N)$ be admissible balanced diagrams such that $\mathcal{H}_M \sqcup \mathcal{H}_N$ is subordinate to a bouquet $B(\mathbb{P})$ for $\mathbb{P}$. If $\mathfrak{s} \in \mathrm{Spin}^c(M \#_{\mathbb{P}} N, \gamma \cup \nu)$*

is such that $\mathfrak{s}|_{S^2} \neq 0$, then

$$\mathrm{SFH}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathfrak{s}) = 0.$$

If $\mathfrak{s}|_{S^2} = 0$, then there is an isomorphism

$$\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{P}, \mathfrak{s}}: \mathrm{SFH}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathfrak{s}) \rightarrow \mathrm{SFH}(\mathcal{H}_M, \mathfrak{s}|_M) \otimes \mathrm{SFH}(\mathcal{H}_N, \mathfrak{s}|_N) \otimes V,$$

where $V = \mathbb{F}_2\langle T, B \rangle$ denotes a 2-dimensional vector space generated by two homogeneous elements T (top) and B (bottom). Moreover, there is an isomorphism

$$\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{P}}: \mathrm{SFH}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N) \rightarrow \mathrm{SFH}(\mathcal{H}_M) \otimes \mathrm{SFH}(\mathcal{H}_N) \otimes V.$$

Proof. For each generator $\mathbf{x} \in \mathrm{SFC}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N)$, we have $\mathfrak{s}(\mathbf{x})|_{S^2} = 0$. Therefore, $\mathrm{SFH}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathfrak{s}) = 0$ unless $\mathfrak{s}|_{S^2} = 0$. There is a self-evident bijection

$$\mathrm{SFC}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathfrak{s}) \rightarrow \mathrm{SFC}(\mathcal{H}_M, \mathfrak{s}|_M) \otimes \mathrm{SFC}(\mathcal{H}_N, \mathfrak{s}|_N) \otimes V.$$

Since $\mathcal{H}_M \sqcup \mathcal{H}_N$ is subordinate to the bouquet $B(\mathbb{P})$, all holomorphic discs on $\Sigma_M \# \Sigma_N$ split as the disjoint union of a holomorphic disc on Σ_M , one on Σ_N , and one on the annulus A attached to the framed pair of points $\mathbb{P}$. Each of these discs has multiplicity 0 near the boundary. It follows that the above bijection descends to a map in homology. $\square$

The following Lemma proves that the isomorphisms defined in Lemma 6.7 are natural.

Lemma 6.8 (Juhász-M.³²). *Let $\mathbb{P}$ be a framed set of points in the sutured manifold $(M, \gamma) \sqcup (N, \nu)$, in such a way that $p_- \in M$ and $p_+ \in N$. Let $\mathcal{H}_M = (\Sigma_M, \alpha_M, \beta_M)$ and $\mathcal{H}'_M = (\Sigma'_M, \alpha'_M, \beta'_M)$ be admissible balanced diagrams for (M, γ) and $\mathcal{H}_N = (\Sigma_N, \alpha_N, \beta_N)$ and $\mathcal{H}'_N = (\Sigma'_N, \alpha'_N, \beta'_N)$ be admissible balanced diagrams for (N, ν) such that $\mathcal{H}_M \sqcup \mathcal{H}_N$ and $\mathcal{H}'_M \sqcup \mathcal{H}'_N$ are subordinate to the bouquets $B(\mathbb{P})$ and $B(\mathbb{P})'$ for $\mathbb{P}$, respectively. Let $\mathfrak{s}_M \in \mathrm{Spin}^c(M, \gamma)$ and $\mathfrak{s}_N \in \mathrm{Spin}^c(N, \nu)$, and let $\mathfrak{s}_{\mathbb{P}} = \mathfrak{s}_M \#_{\mathbb{P}} \mathfrak{s}_N \in \mathrm{Spin}^c(M \#_{\mathbb{P}} N, \gamma \sqcup \nu)$ denote the connected sum of the Spin^c structures. Then, if $\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N$ and $\mathcal{H}'_M \#_{\mathbb{P}} \mathcal{H}'_N$ denote*

the Heegaard diagrams obtained from $\mathcal{H}_M \sqcup \mathcal{H}_N$ and $\mathcal{H}'_M \sqcup \mathcal{H}'_N$ by surgery on $\mathbb{P}$, we have that the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{SFH}(\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathfrak{s}_{\mathbb{P}}) & \xrightarrow{\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{P}, \mathfrak{s}_{\mathbb{P}}}} & \mathrm{SFH}(\mathcal{H}_M, \mathfrak{s}_M) \otimes \mathrm{SFH}(\mathcal{H}_N, \mathfrak{s}_N) \otimes V \\
\downarrow F_{\mathcal{H}_M \#_{\mathbb{P}} \mathcal{H}_N, \mathcal{H}'_M \#_{\mathbb{P}} \mathcal{H}'_N} & & \downarrow F_{\mathcal{H}_M, \mathcal{H}'_M} \otimes F_{\mathcal{H}_N, \mathcal{H}'_N} \otimes \mathrm{id}_V \\
\mathrm{SFH}(\mathcal{H}'_M \#_{\mathbb{P}} \mathcal{H}'_N, \mathfrak{s}_{\mathbb{P}}) & \xrightarrow{\varphi_{\mathcal{H}'_M, \mathcal{H}'_N, \mathbb{P}, \mathfrak{s}_{\mathbb{P}}}} & \mathrm{SFH}(\mathcal{H}'_M, \mathfrak{s}_M) \otimes \mathrm{SFH}(\mathcal{H}'_N, \mathfrak{s}_N) \otimes V.
\end{array}$$

Here $F_{\mathcal{H}, \mathcal{H}'}$ denotes the naturality map defined by Juhász and Thurston³⁴. An analogous statement holds for $\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{P}}$ and $\varphi_{\mathcal{H}'_M, \mathcal{H}'_N, \mathbb{P}}$.

Proof. The proof is analogous to Theorem 7.6 of Juhász's paper³¹. $\square$

Corollary 6.9 (Juhász-M.³²). *Let $\mathbb{P}$ be a framed set of points in the sutured manifold $(M, \gamma) \sqcup (N, \nu)$, in such a way that $p_- \in M$ and $p_+ \in N$. If $\mathfrak{s} \in \mathrm{Spin}^c(M \#_{\mathbb{P}} N, \gamma \sqcup \nu)$ is such that $\mathfrak{s}|_{S^2} \neq 0$, then*

$$\mathrm{SFH}(M \#_{\mathbb{P}} N, \mathfrak{s}) = 0.$$

If $\mathfrak{s}|_{S^2} = 0$, then there is a well-defined isomorphism

$$\varphi_{\mathbb{P}, \mathfrak{s}}: \mathrm{SFH}(M \#_{\mathbb{P}} N, \mathfrak{s}) \rightarrow \mathrm{SFH}(M, \mathfrak{s}|_M) \otimes \mathrm{SFH}(N, \mathfrak{s}|_N) \otimes V.$$

Moreover, there is an isomorphism

$$\varphi_{\mathbb{P}}: \mathrm{SFH}(M \#_{\mathbb{P}} N) \rightarrow \mathrm{SFH}(M) \otimes \mathrm{SFH}(N) \otimes V.$$

We now generalise the notion of framed pair of points to cobordisms, in order to define the connected sum of sutured cobordisms in a natural way.

Definition 6.10 (Juhász-M.³²). Suppose that $\mathcal{W} = (W, Z_W, [\xi_W])$ (respectively $\mathcal{U} = (U, Z_U, [\xi_U])$) is a cobordism between sutured manifolds (M_0, γ_0) and (M_1, γ_1) (respectively (N_0, ν_0) and (N_1, ν_1)). A *framed pair of arcs* $\mathbb{A}$ consists of two properly embedded arcs $a_-: I \rightarrow W$ and $a_+: I \rightarrow U$, such that

$a_-(i) \in M_i$ and $a_+(i) \in N_i$ for $i \in \{0, 1\}$, together with a negative frame $\langle v_1^-(t), v_2^-(t), v_3^-(t), v_4^-(t) \rangle$ of the tangent space $T_{a_-(t)}W$ and a positive frame $\langle v_1^+(t), v_2^+(t), v_3^+(t), v_4^+(t) \rangle$ of the tangent space $T_{a_+(t)}U$ for all $t \in I$ such that:

- the set $\mathbb{A}(i) := \{a_-(i), a_+(i)\}$ with the frames $\langle v_1^\pm(i), v_2^\pm(i), v_3^\pm(i) \rangle$ is a framed pair of points for $(M_i \sqcup N_i, \gamma_i \sqcup \nu_i)$, for $i \in \{0, 1\}$; and
- $v_4 = a'_\pm = da_\pm/dt$.

Given a framed pair of arcs, we can define the connected sum of the two cobordisms, as explained in Definition 6.11, below. We will denote by $(M, \gamma) \#_{\mathbb{P}} (N, \nu)$ the connected sum of the sutured manifolds (M, γ) and (N, ν) along the framed pair of points $\mathbb{P}$, assuming that $p_- \in M$ and $p_+ \in N$.

Definition 6.11 (Juhász-M.³²). Suppose that $\mathcal{W} = (W, Z_W, [\xi_W])$ (respectively $\mathcal{U} = (U, Z_U, [\xi_U])$) is a cobordism between sutured manifolds (M_0, γ_0) and (M_1, γ_1) (respectively (N_0, ν_0) and (N_1, ν_1)), and let $\mathbb{A}$ be a framed pair of arcs.

We define the *connected sum* of $\mathcal{W}$ and $\mathcal{U}$ as the cobordism

$$\mathcal{W} \#_{\mathbb{A}} \mathcal{U} = (V, Z_V, [\xi_V])$$

from $(M_0, \gamma_0) \#_{\mathbb{A}(0)} (N_0, \nu_0)$ to $(M_1, \gamma_1) \#_{\mathbb{A}(1)} (N_1, \nu_1)$, where

- V is the 4-manifold obtained by gluing $\text{Bl}_{a_-} W$ and $\text{Bl}_{a_+} U$ respectively along UNa_- and UNa_+ , in such a way that the frame of $a_-(t)$ is identified with the one of $a_+(t)$;
- $Z_V = Z_W \sqcup Z_U$; and
- $\xi_V = \xi_W \sqcup \xi_U$.

Remark 6.12. Let $\mathcal{X}_K = (F, \sigma_F)$ (resp. $\mathcal{X}_L = (G, \sigma_G)$) be a decorated link cobordism from (K_0, P_{K_0}) to (K_1, P_{K_1}) (resp. from (L_0, P_{L_0}) to (L_1, P_{L_1})). Let $(x_-, y_-, z_-) \in S^3$ be such that

$$a_- = \{ (x_-, y_-, z_-, t) : t \in I \}$$

is disjoint from F . This holds for generic (x_-, y_-, z_-) , after possibly isotoping F . Analogously, choose $(x_+, y_+, z_+) \in S^3$ and define a_+ to be disjoint from G . Consider the pair of arcs $\mathbb{A} = \{a_-, a_+\}$ with frames $\langle \pm \partial_x, \partial_y, \partial_z, \partial_t \rangle$. Then

$$\mathcal{W}(\mathcal{X}_K \# \mathcal{X}_L) = \mathcal{W}(\mathcal{X}_K) \#_{\mathbb{A}} \mathcal{W}(\mathcal{X}_L).$$

As in the case of the connected sum of sutured manifolds, one can define the connected sum of Spin^c structures for cobordisms. Suppose that $\mathcal{W} = (W, Z_W, [\xi_W])$ and $\mathcal{U} = (U, Z_U, [\xi_U])$ are cobordisms of sutured manifolds, and let $\mathcal{W} \#_{\mathbb{A}} \mathcal{U} = (W \#_{\mathbb{A}} U, Z, [\xi])$ denote their connected sum along some framed pair of arcs $\mathbb{A}$. Let $W^\circ = \text{Bl}_{a_-} W$ and $U^\circ = \text{Bl}_{a_+} U$. Then W° and U° can be seen as embedded in $W \#_{\mathbb{A}} U$. Denote by $S^2 \times I$ the thickened sphere obtained by intersecting $W^\circ \cap U^\circ$. Then, the relative Mayer-Vietoris sequences for $(W, Z_W) = (W^\circ, Z_W) \cup B^3 \times I$, $(U, Z_U) = (U^\circ, Z_U) \cup B^3 \times I$, and $(W \#_{\mathbb{A}} U, Z) = (W^\circ, Z_W) \cup (U^\circ, Z_U)$ imply that the map

$$\begin{aligned} \phi: \text{Spin}^c(\mathcal{W}) \times \text{Spin}^c(\mathcal{U}) &\longrightarrow \text{Spin}^c(\mathcal{W} \#_{\mathbb{A}} \mathcal{U}) \\ (\mathfrak{s}_W, \mathfrak{s}_U) &\longmapsto (\mathfrak{s}_W)|_{W^\circ} \cup (\mathfrak{s}_U)|_{U^\circ} \end{aligned}$$

is well-defined and injective. We will denote $\phi(\mathfrak{s}_W, \mathfrak{s}_U)$ by $\mathfrak{s}_W \#_{\mathbb{A}} \mathfrak{s}_U$. Moreover, $\mathfrak{s} \in \text{Spin}^c(\mathcal{W} \#_{\mathbb{A}} \mathcal{U})$ is in the image of ϕ if and only if $\mathfrak{s}|_{S^2 \times I} = 0$ in $\text{Spin}^c(S^2 \times I)$ (i.e., if and only if it is the unique Spin^c structure on $S^2 \times I$ with vanishing first Chern class). In such a case we call $\mathfrak{s}|_W$ and $\mathfrak{s}|_U$ the unique spin^c structures such that $\phi(\mathfrak{s}|_W, \mathfrak{s}|_U) = \mathfrak{s}$.

Definition 6.13 (Juhász-M.³²). Let $\mathbb{A}$ be a framed pair of arcs for $\mathcal{W} = (W, Z_W, [\xi_W])$ and $\mathcal{U} = (U, Z_U, [\xi_U])$. A *bouquet* $B(\mathbb{A})$ for $\mathbb{A}$ consists of a pair of embedded squares $d_-: I \times I \rightarrow W$ and $d_+: I \times I \rightarrow U$ and a framing of $d_\pm$ given by a normal vector field $v_\pm$ such that:

- $d_\pm(0, t) = a_\pm(t)$ and $d_\pm(1, t) \in Z_W \cup Z_U$ for all $t \in I$;
- for $i \in \{0, 1\}$, $d_\pm(s, i)$ with $v_\pm(s, i)$, which we denote by $B(\mathbb{A})_i$, is a bouquet for $\mathbb{A}(i)$ (in particular, $d_-(s, i)$ lies in M_i and $d_+(s, i)$ lies in N_i);
- $\partial_s d_\pm(0, t) = v_1^\pm$ and $\partial_s d_\pm(1, t)$ is transverse to Z ;

- $v_{\pm}(0, t) = v_2^{\pm}$;
- along $d_{\pm}(1, t)$, $\xi = \langle \partial_t d_{\pm}, v_{\pm} \rangle$; and
- $\partial_t d_{\pm}(0, t) = v_4^{\pm}$ and $\partial_t d_{\pm}(s, i)$ is transverse to $M_i \sqcup N_i$ for $i \in \{0, 1\}$.

Theorem 6.14 (Juhász-M.³²). *Suppose that $\mathcal{W} = (W, Z_W, [\xi_W])$ (respectively $\mathcal{U} = (U, Z_U, [\xi_U])$) is a cobordism between balanced sutured manifolds (M_0, γ_0) and (M_1, γ_1) (respectively (N_0, ν_0) and (N_1, ν_1)). Let $\mathbb{A}$ be a framed pair of arcs for the cobordisms $\mathcal{W}$ and $\mathcal{U}$, and suppose that there exists some bouquet $B(\mathbb{A})$ for $\mathbb{A}$. Then the following diagram commutes (the sutures have been dropped from the notations because they are clear).*

$$\begin{array}{ccc}
\mathrm{SFH}(M_0 \#_{\mathbb{A}(0)} N_0) & \xrightarrow{\varphi_{\mathbb{A}(0)}} & \mathrm{SFH}(M_0) \otimes \mathrm{SFH}(N_0) \otimes V \\
\downarrow F_{\mathcal{W} \#_{\mathbb{A}} \mathcal{U}} & & \downarrow F_{\mathcal{W}} \otimes F_{\mathcal{U}} \otimes \mathrm{id}_V \\
\mathrm{SFH}(M_1 \#_{\mathbb{A}(1)} N_1) & \xrightarrow{\varphi_{\mathbb{A}(1)}} & \mathrm{SFH}(M_1) \otimes \mathrm{SFH}(N_1) \otimes V
\end{array}$$

Example 6.15. The existence of a bouquet $B(\mathbb{A})$ in the statement of Theorem 6.14 is not just a technical condition, as we are going to show in this example. Consider the cobordism $\Gamma \circ L$ (cf. Definition 6.3). By the computations in Section 6.1 and functoriality, we know that the map

$$F_{\Gamma \circ L} : V \longrightarrow V$$

induced on $\mathrm{SFH}(U_2, P_{U_2})$ has rank 1.

This cobordism can also be seen as the disjoint union $\mathcal{X}_1 \# \mathcal{X}_2$ of the identity cobordism $\mathcal{X}_1$ from (U_1, P_{U_1}) to (U_1, P_{U_1}) and a “death and re-birth” cobordism $\mathcal{X}_2$ again from (U_1, P_{U_1}) to (U_1, P_{U_1}) ; cf. Figure 6.2. Choose $\mathbb{A}$ as in Remark 6.12. Then we have that

$$F_{\Gamma \circ L} = F_{\mathcal{W}(\mathcal{X}_1) \#_{\mathbb{A}} \mathcal{W}(\mathcal{X}_2)}.$$

We cannot apply Theorem 6.14 here because there is no bouquet for $\mathbb{A}$. If the diagram in Theorem 6.14 were commutative also in this case, then the

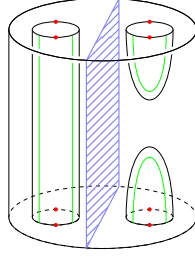


Figure 6.2: The cobordism $\Gamma \circ L$ can be seen as the disjoint union cobordism of two cobordisms from (U_1, P_{U_1}) to (U_1, P_{U_1}) , namely the identity cobordism to the left of the blue rectangle, which we denote by $\mathcal{X}_1$, and the one to its right, which we denote by $\mathcal{X}_2$.

map induced by $\mathcal{W}(\mathcal{X}_1) \parallel_{\mathbb{A}} \mathcal{W}(\mathcal{X}_2)$ would have rank either 0 or 2, contradicting our computations.

Proof of Theorem 6.14. Split $\mathcal{W}$ and $\mathcal{U}$ into a boundary cobordism and a special cobordism: $\mathcal{W} = \mathcal{W}^s \circ \mathcal{W}^b$ and $\mathcal{U} = \mathcal{U}^s \circ \mathcal{U}^b$. Then $\mathbb{A}$ determines framed pairs of arcs $\mathbb{A}^b$ for $\mathcal{W}^b$ and $\mathcal{U}^b$, and $\mathbb{A}^s$ for $\mathcal{W}^s$ and $\mathcal{U}^s$. Note that

$$\mathcal{W} \parallel_{\mathbb{A}} \mathcal{U} = (\mathcal{W}^s \parallel_{\mathbb{A}^s} \mathcal{U}^s) \circ (\mathcal{W}^b \parallel_{\mathbb{A}^b} \mathcal{U}^b)$$

is a decomposition of $\mathcal{W} \parallel_{\mathbb{A}} \mathcal{U}$ into a boundary cobordism and a special cobordism.

Fix a bouquet $B(\mathbb{A})$ for $\mathbb{A}$. Note that this induces bouquets $B(\mathbb{A}^b)$ and $B(\mathbb{A}^s)$. Therefore, we can consider the cases of boundary cobordism and special cobordism separately. We now split the proof into three cases.

CASE 1: $\mathcal{W}$ AND $\mathcal{U}$ ARE IDENTITY COBORDISMS. We first consider the case when $\mathcal{W}$ and $\mathcal{U}$ are identity cobordisms, and $\mathbb{A}$ is any framed pair of arcs that admits a bouquet $B(\mathbb{A})$. Let B_- be a regular neighbourhood of $\text{im}(d_-)$ in W , and let B_+ be a regular neighbourhood of $\text{im}(d_+)$ in U . For $i \in \{0, 1\}$, let $B_{\pm}(i) = B_{\pm} \cap (M_i \sqcup N_i)$. The fact that the contact structure ξ is parallel near the bouquet implies that $B_{\pm}$ can be turned into a special cobordism

$$\mathcal{B}_{\pm}: B_{\pm}(0) \rightarrow B_{\pm}(1)$$

by extending ξ to $\partial B_{\pm} \cap \text{Int}(W \cup U)$. If we remove $\mathcal{B}_{\pm}$ from $\mathcal{W}$ and $\mathcal{U}$, we obtain two identity cobordisms $\mathcal{W}^{\circ} = \text{id}_M$ and $\mathcal{U}^{\circ} = \text{id}_N$, where $M = M_0 \setminus B_{-}(0)$ and $N = N_0 \setminus B_{+}(0)$. Note that $\mathbb{A}$ is a framed pair of arcs and $B(\mathbb{A})$ is a bouquet in $\mathcal{B}_{\pm}$. The cobordism $\mathcal{B}_{-} \#_{\mathbb{A}} \mathcal{B}_{+}$ is diffeomorphic to an identity cobordism, therefore it gives a diffeomorphism

$$\psi: B_{-}(0) \#_{\mathbb{A}(0)} B_{+}(0) \rightarrow B_{-}(1) \#_{\mathbb{A}(1)} B_{+}(1),$$

and, without loss of generality, we can suppose that

- $\psi(a_{\pm}(s, 0)) = a_{\pm}(s, 1)$; and
- $d\psi(v_{\pm}(s, 0)) = v_{\pm}(s, 1)$.

Let $D_{\pm}(0)$ be a 2-disc embedded in $B_{\pm}(0)$ containing $a_{\pm}(\cdot, 0)$ such that $v_{\pm}(\cdot, 0)$ is tangent to $D_{\pm}(0)$, and let $D_{\pm}(1) = \psi(D_{\pm}(0))$. $D_{-}(i) \#_{\mathbb{A}(i)} D_{+}(i)$, together with a meridional α - and a β -curve on the connecting tube that intersect in two points, is a Heegaard diagram $\mathcal{H}_i$ for $B_{-}(i) \#_{\mathbb{A}(i)} B_{+}(i)$, and $\psi_{*}(\mathcal{H}_0) = \mathcal{H}_1$. The map induced by such a diffeomorphism on sutured Floer homology preserves the relative homological grading, so it must be the identity map:

$$\begin{array}{ccc} \text{SFH}(\mathcal{H}_0) & \xrightarrow{\simeq} & V \\ \downarrow \psi_{*} & & \downarrow \text{id}_V \\ \text{SFH}(\mathcal{H}_1) & \xrightarrow{\simeq} & V. \end{array}$$

We then choose Heegaard diagrams $\mathcal{H}_M$ and $\mathcal{H}_N$ for the sutured manifolds M and N such that

$$\mathcal{H}_{M \#_{\mathbb{A}(0)} N} = \mathcal{H}_M \natural \mathcal{H}_0 \natural \mathcal{H}_N$$

is a Heegaard diagram for the sutured manifold $M \#_{\mathbb{A}(0)} N$ (or, more pre-

cisely, for $M_0 \#_{\mathbb{A}(0)} N_0$). The cobordism $\mathcal{W} \parallel_{\mathbb{A}} \mathcal{U}$ induces a diffeomorphism

$$\Psi_*: \mathcal{H}_{M \#_{\mathbb{A}(0)} N} \rightarrow \mathcal{H}_{M \#_{\mathbb{A}(1)} N} = \mathcal{H}_M \natural \mathcal{H}_1 \natural \mathcal{H}_N$$

such that the following diagram of diffeomorphisms commutes:

$$\begin{array}{ccc} \mathcal{H}_{M \#_{\mathbb{A}(0)} N} & \xrightarrow{\simeq} & \mathcal{H}_M \natural \mathcal{H}_0 \natural \mathcal{H}_N \\ \downarrow \Psi_* & & \downarrow \text{id}_{\mathcal{H}_M} \natural \psi_* \natural \text{id}_{\mathcal{H}_N} \\ \mathcal{H}_{M \#_{\mathbb{A}(1)} N} & \xrightarrow{\simeq} & \mathcal{H}_M \natural \mathcal{H}_1 \natural \mathcal{H}_N. \end{array}$$

From this, it follows that the diagram below commutes:

$$\begin{array}{ccc} \text{SFH}(\mathcal{H}_{M \#_{\mathbb{A}(0)} N}) & \xrightarrow{\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{A}(0)}} & \text{SFH}(\mathcal{H}_M) \otimes \text{SFH}(\mathcal{H}_N) \otimes V \\ \downarrow F_{\mathcal{H}_{M \#_{\mathbb{A}(0)} N}, \mathcal{H}_{M \#_{\mathbb{A}(1)} N}} & & \downarrow \text{id}_{\text{SFH}(\mathcal{H}_M)} \otimes \text{id}_{\text{SFH}(\mathcal{H}_N)} \otimes \text{id}_V \\ \text{SFH}(\mathcal{H}_{M \#_{\mathbb{A}(1)} N}) & \xrightarrow{\varphi_{\mathcal{H}_M, \mathcal{H}_N, \mathbb{A}(1)}} & \text{SFH}(\mathcal{H}_M) \otimes \text{SFH}(\mathcal{H}_N) \otimes V. \end{array}$$

CASE 2: $\mathcal{W}$ AND $\mathcal{U}$ ARE BOUNDARY COBORDISMS. Since $\mathcal{W}$ and $\mathcal{U}$ are boundary cobordisms, recall that $M_1 \cong M_0 \cup -Z_W$ and $N_1 \cong N_0 \cup -Z_U$. Given a bouquet $B(\mathbb{A}) = \{d_{\pm}, v_{\pm}\}$ for $\mathbb{A}$, we define a bouquet $B'(\mathbb{A}(0))$ for $\mathbb{A}(0)$ in the sutured manifold $(M_1, \gamma_1) \sqcup (N_1, \nu_1)$. The arcs of $B'(\mathbb{A}(0))$ are $\eta_{\pm} = d_{\pm}(\cdot, 0) \cup d_{\pm}(1, \cdot)$, and the normal vector field is just the restriction of $v_{\pm}$.

Consider a sutured Heegaard diagram $\mathcal{H}_{M_0} = (\Sigma_{M_0}, \alpha_{M_0}, \beta_{M_0})$ for the sutured manifold (M_0, γ_0) , and an extension of it to a sutured Heegaard diagram $\mathcal{H}_{M_1} = (\Sigma_{M_1}, \alpha_{M_1}, \beta_{M_1})$ for (M_1, γ_1) that are compatible with the contact structure ξ_W as in the definition of the gluing map²⁴. Choose analogous sutured Heegaard diagrams for (N_0, ν_0) and (N_1, ν_1) . The gluing

maps are then defined at the complex level as the tensor product with contact classes, denoted by $\mathbf{x}''_M$ and $\mathbf{x}''_N$.

Using a notation similar to Honda, Kazez, and Matić²⁴, let $M' = M_0 \setminus \mathcal{N}(\partial M_0)$, $M'' = -Z_W \setminus \mathcal{N}(\partial M_0)$, and let $M''' = \mathcal{N}(\partial M_0)$. Let $L' = \eta_- \cap M'$, $L'' = \eta_- \cap M''$, and $L''' = \eta_- \cap M'''$. By definition of bouquet, L'' and L''' are Legendrian, and L' is Legendrian close to $\partial M'$. Let K'' be a Legendrian skeleton of M'' containing L'' as a connected component, and let K''' be a Legendrian skeleton of M''' dual to a Legendrian skeleton containing L''' as a component. Lastly, let L' be a contact compatible 1-skeleton for M' , such that a component of $L' \cap \mathcal{N}(\partial M')$ is K' . We can choose a contact cellular decomposition M' , M'' , and M''' such that the 1-skeletons are respectively L' , L'' , and L''' , and construct contact compatible Heegaard diagrams as in²⁴. It follows that the arc η_- can be isotoped onto the Heegaard surface Σ_{M_1} , and is disjoint from $\alpha_{M''}$ and $\beta_{M''}$. After performing a fingermove on the $\alpha_{M'}$ and $\beta_{M'}$ curves, η_- is disjoint from all α_{M_1} and β_{M_1} curves.

Thus, we can suppose that $\mathcal{H}_{M_0} \#_{\mathbb{A}(0)} \mathcal{H}_{N_0}$ is subordinate to the bouquet $B(\mathbb{A}(0))$ and $\mathcal{H}_{M_1} \#_{\mathbb{A}(0)} \mathcal{H}_{N_1}$ is subordinate to the bouquet $B'(\mathbb{A}(0))$. Furthermore, the sutured diagrams $\mathcal{H}_{M_0} \#_{\mathbb{A}(0)} \mathcal{H}_{N_0}$ and $\mathcal{H}_{M_1} \#_{\mathbb{A}(0)} \mathcal{H}_{N_1}$ are contact-compatible, and the contact class is just $\mathbf{x}''_M \otimes \mathbf{x}''_N$. Therefore, the diagram below commutes:

$$\begin{array}{ccc}
\mathrm{SFH}(\mathcal{H}_{M_0} \#_{\mathbb{A}(0)} \mathcal{H}_{N_0}) & \xrightarrow{\varphi_{\mathcal{H}_{M_0}, \mathcal{H}_{N_0}, \mathbb{A}(0)}} & \mathrm{SFH}(\mathcal{H}_{M_0}) \otimes \mathrm{SFH}(\mathcal{H}_{N_0}) \otimes V \\
\downarrow \Phi_{\xi_W \sqcup \xi_U} & & \downarrow \Phi_{\xi_W} \otimes \Phi_{\xi_U} \otimes \mathrm{id}_V \\
\mathrm{SFH}(\mathcal{H}_{M_1} \#_{\mathbb{A}(0)} \mathcal{H}_{N_1}) & \xrightarrow{\varphi_{\mathcal{H}_{M_1}, \mathcal{H}_{N_1}, \mathbb{A}(0)}} & \mathrm{SFH}(\mathcal{H}_{M_1}) \otimes \mathrm{SFH}(\mathcal{H}_{N_1}) \otimes V.
\end{array}$$

We are now left with the connected sum cobordism $\mathrm{id}_{M_1} \parallel_{\mathbb{A}} \mathrm{id}_{N_1}$, which still contains a bouquet, induced by $B(\mathbb{A})$. It follows from Case 1 that we have a commutative diagram as below, which concludes the proof in the case of boundary cobordisms:

$$\begin{array}{ccc}
\mathrm{SFH}(\mathcal{H}_{M_1} \#_{\mathbb{A}(0)} \mathcal{H}_{N_1}) & \xrightarrow{\varphi_{\mathcal{H}_{M_1}, \mathcal{H}_{N_1}, \mathbb{A}(0)}} & \mathrm{SFH}(\mathcal{H}_{M_1}) \otimes \mathrm{SFH}(\mathcal{H}_{N_1}) \otimes V \\
\downarrow F_{\mathcal{H}_{M_1} \#_{\mathbb{A}(0)} \mathcal{H}_{N_1}, \mathcal{H}_{M_1} \#_{\mathbb{A}(1)} \mathcal{H}_{N_1}} & & \downarrow \mathrm{id}_{\mathrm{SFH}(\mathcal{H}_{M_1})} \otimes \mathrm{id}_{\mathrm{SFH}(\mathcal{H}_{N_1})} \otimes \mathrm{id}_V \\
\mathrm{SFH}(\mathcal{H}_{M_1} \#_{\mathbb{A}(1)} \mathcal{H}_{N_1}) & \xrightarrow{\varphi_{\mathcal{H}_{M_1}, \mathcal{H}_{N_1}, \mathbb{A}(1)}} & \mathrm{SFH}(\mathcal{H}_{M_1}) \otimes \mathrm{SFH}(\mathcal{H}_{N_1}) \otimes V.
\end{array}$$

CASE 3: $\mathcal{W}$ AND $\mathcal{U}$ ARE SPECIAL COBORDISMS. Consider the function t on $B(\mathbb{A})$, and extend it to a Morse function on $\mathcal{W} \sqcup \mathcal{U}$. This gives a decomposition of the cobordisms $\mathcal{W}$ and $\mathcal{U}$ into 1-, 2-, and 3-handles that are attached *away from the bouquet*. Furthermore, $B(\mathbb{A})$ is a product with respect to this handle decomposition. In particular, $\mathbb{A}(0)$ and $B(\mathbb{A})_0$ coincide with $\mathbb{A}(1)$ and $B(\mathbb{A})_1$ after the handle attachments.

For each handle attachment, one can choose adapted Heegaard diagrams that are also subordinate to the bouquet $B(\mathbb{A})_0$ for the framed pair of points $\mathbb{A}(0)$. We therefore obtain a commutative diagram

$$\begin{array}{ccc}
\mathrm{SFH}(M_0 \#_{\mathbb{A}(0)} N_0) & \xrightarrow{\varphi_{\mathbb{A}(0)}} & \mathrm{SFH}(M_0) \otimes \mathrm{SFH}(N_0) \otimes V \\
\downarrow F_{\mathcal{W} \#_{\mathbb{A}} \mathcal{U}}^s & & \downarrow F_{\mathcal{W}}^s \otimes F_{\mathcal{U}}^s \otimes \mathrm{id}_V \\
\mathrm{SFH}(M_1 \#_{\mathbb{A}(0)} M_1) & \xrightarrow{\varphi_{\mathbb{A}(0)}} & \mathrm{SFH}(M_1) \otimes \mathrm{SFH}(N_1) \otimes V.
\end{array}$$

This concludes the proof of Theorem 6.14. □

6.3 A $(1 + 1)$ -DIMENSIONAL TQFT

The aim of this section is to determine the $(1 + 1)$ -dimensional TQFT defined by the cobordism maps computed in Section 6.1, and to compare it with Khovanov's reduced TQFT.

6.3.1 THE CATEGORY OF MARKED EMBEDDED COBORDISMS AND THE HFL TQFT

We first define the cobordism category that we are interested in.

Definition 6.16 (Juhász-M.³²). We define the category of *marked decorated embedded cobordisms* EmbCob^* as follows. The objects are the standard n -component unlinks (U_n, P_{U_n}) for every $n \in \mathbb{N}$ with two decorations on each component. As Kronheimer and Mrowka suggest⁴¹ in Section 8.2, we can take a specific model for it: We define U_n to be the union of standard circles $L_1, \dots, L_n$ in the (x, y) -plane, each of diameter 0.5, centered at the first n integer lattice points along the x -axis, and such that $R_-(P_{U_n}) \cap L_i = L_i \cap \{x \leq i\}$. Furthermore, we suppose that the component L_1 of the unlink U_n is marked (in addition to being decorated), and denote it by a dot on the component itself.

The morphisms are all cobordisms in $S^3 \times I$ generated by

- V_n^e ($n \geq 1$), the cobordism from (U_n, P_{U_n}) to $(U_{n+1}, P_{U_{n+1}})$ obtained by stacking horizontally the cobordism V in Figure 6.1 and the identity cobordism of $(U_{n-1}, P_{U_{n-1}})$. Note that the marked components are in the V part of the cobordism.
- Λ_n^e ($n \geq 2$), the cobordism from (U_n, P_{U_n}) to $(U_{n-1}, P_{U_{n-1}})$ obtained by stacking horizontally the cobordism Λ in Figure 6.1 and the identity cobordism of $(U_{n-2}, P_{U_{n-2}})$. Note that the marked components are in the Λ part of the cobordism.
- $\text{IX}_{i,n}^e$ (for $2 \leq i \leq n-1$ and $n \geq 3$), obtained by swapping the i -th and the $(i+1)$ -th component of (U_n, P_{U_n}) . Note that the marked components are on the first component of the cobordism, and that therefore the marked component cannot be swapped.
- IV_n^e ($n \geq 2$), the cobordism from (U_n, P_{U_n}) to $(U_{n+1}, P_{U_{n+1}})$ obtained by stacking horizontally an identity cobordism I between the marked components, the cobordism V and, finally, identity cobordisms on the last $n-2$ components.

- Λ_n^e ($n \geq 3$), the cobordism from (U_n, P_{U_n}) to $(U_{n-1}, P_{U_{n-1}})$ obtained by stacking horizontally an identity cobordism I between the marked components, the cobordism Λ and, finally, identity cobordisms on the last $n - 3$ components.

The above cobordisms are represented in Figure 6.3.

Our aim is to describe the TQFT

$$\text{HFL}: \mathbf{EmbCob}^* \longrightarrow \mathbf{Vect}_{\mathbb{F}_2}.$$

To do so, we need to deal with some technicalities due to the fact that link Floer homology does not naturally arise as an invariant of a marked link. For this reason, we now explain how marking a component of (U_n, P_{U_n}) gives an isomorphism

$$\psi_{U_n}: \text{HFL}(U_n, P_{U_n}) \xrightarrow{\cong} V^{\otimes(n-1)}. \quad (6.4)$$

Note that a sutured Heegaard diagram for the unknot in S^3 is given by $(A, \emptyset, \emptyset)$, where A is an annulus. We can construct a (sutured) Heegaard diagram for (U_n, P_{U_n}) by taking n copies of this Heegaard diagram (each one associated to a link component), and by connecting the first annulus (i.e., the one that corresponds to the marked component) to all the other annuli with tubes. Each tube contains a homotopically non-trivial α -curve and a homotopically non-trivial β -curve that intersect in two points, which we call B and T (for bottom-graded and top-graded). Then we have an isomorphism

$$\psi_{U_n}: \text{HFL}(U_n, P_{U_n}) \xrightarrow{\cong} V_2 \otimes \dots \otimes V_n,$$

where by V_i we mean the vector space $V = \mathbb{F}_2\langle B, T \rangle$ associated to the tube connecting the first component to the i -th component. As usual, B denotes an element in homological grading $-1/2$, and T denotes an element in homological grading $1/2$.

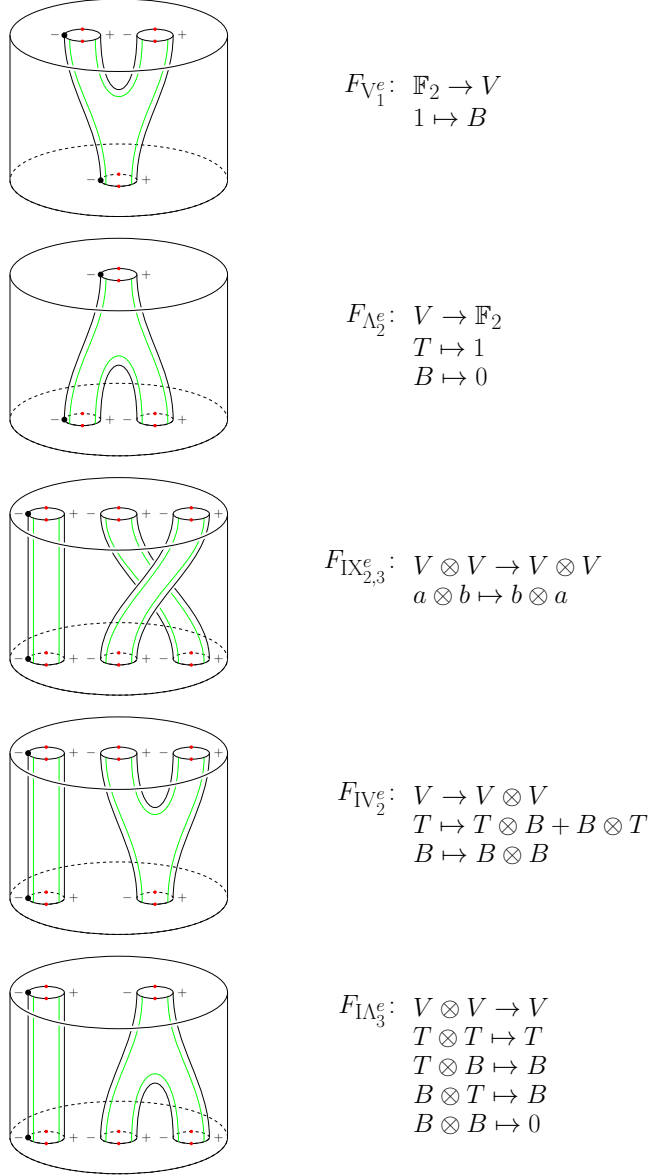


Figure 6.3: The figure shows the cobordisms V_1^e , Λ_2^e , $\text{IX}_{2,3}^e$, IV_2^e and IA_3^e in the category EmbCob^* (represented following the conventions explained in Figure 5.10), and the maps they induce via the TQFT HFL. Note that the marked component allows us to choose a canonical identification $\psi_{U_n}: \text{HFL}(U_n, P_{U_n}) \rightarrow V^{\otimes(n-1)}$.

Having set the isomorphism in Equation (6.4), our next aim is to understand the cobordism maps induced by the generators of $\mathbf{EmbCob}^*$ with respect to the standard basis consisting of elements of the form

$$v_2 \otimes \cdots \otimes v_n,$$

where $v_i \in \{B, T\}$ for every $i \in \{2, \dots, n\}$.

THE COBORDISM V_n^e

This cobordism is the disjoint union of the cobordism V and the identity cobordism $I^{\otimes(n-1)} = \text{id}_{U_{n-1}}$. Consider a framed pair of arcs $\mathbb{A}$ such that the arc in V is close to the surface and its endpoints are close to the marked component of the link. Then by, Corollary 6.9, there are isomorphisms

$$\begin{aligned} \varphi_{U_n} &: \text{HFL}(U_n, P_{U_n}) \rightarrow \mathbb{F}_2 \otimes V^{\otimes(n-2)} \otimes V, \text{ and} \\ \varphi_{U_{n+1}} &: \text{HFL}(U_{n+1}, P_{U_{n+1}}) \rightarrow V \otimes V^{\otimes(n-2)} \otimes V \end{aligned}$$

such that the cobordism map can be easily expressed with respect to these isomorphisms as in Theorem 6.14. However, φ_{U_n} and $\varphi_{U_{n+1}}$ are not the isomorphisms ψ_{U_n} and $\psi_{U_{n+1}}$ given in Equation (6.4) for decorated unlinks with a marked component.

Choose Heegaard diagrams giving the splittings φ_{U_n} and $\varphi_{U_{n+1}}$ above. Since one of the two points in $\mathbb{A}(0)$ and $\mathbb{A}(1)$ is close to the marked component, the connecting tube (to which the last V factor is associated) can be supposed to connect the first (i.e., the marked) component of the unlink to a Heegaard diagram for $(U_{n-1}, P_{U_{n-1}})$ or (U_n, P_{U_n}) . It follows that we can change the given Heegaard diagrams for U_n and U_{n+1} , which we denote by $\mathcal{H}$, to the standard ones $\mathcal{H}_{\text{std}}$ that give the isomorphisms ψ_{U_n} and $\psi_{U_{n+1}}$ in such a way that the naturality map affects only the last factors $V^{\otimes(n-2)}$ and V .

Therefore, the following diagram commutes.

$$\begin{array}{ccc}
\text{HFL}(U_n, P_{U_n}) & \xrightarrow{\psi_{U_n}} & \mathbb{F}_2 \otimes V^{\otimes(n-1)} \\
\downarrow \text{id} & & \downarrow \text{id}_{\mathbb{F}_2} \otimes F_{\mathcal{H}_{\text{std}}, \mathcal{H}} \\
\text{HFL}(U_n, P_{U_n}) & \xrightarrow{\varphi_{U_n}} & \mathbb{F}_2 \otimes (V^{\otimes(n-2)} \otimes V) \\
\downarrow F_{V_n^e} & & \downarrow F_V \otimes \text{id}_{V^{\otimes(n-2)}} \otimes \text{id}_V \\
\text{HFL}(U_{n+1}, P_{U_{n+1}}) & \xrightarrow{\varphi_{U_{n+1}}} & V \otimes (V^{\otimes(n-2)} \otimes V) \\
\downarrow \text{id} & & \downarrow \text{id}_V \otimes F_{\mathcal{H}_{\text{std}}, \mathcal{H}}^{-1} \\
\text{HFL}(U_{n+1}, P_{U_{n+1}}) & \xrightarrow{\psi_{U_{n+1}}} & V \otimes V^{\otimes(n-1)},
\end{array}$$

Then we see that the map $F_{V_n^e}$ with respect to ψ_{U_n} and $\psi_{U_{n+1}}$ is

$$\begin{array}{ccc}
F_{V_n^e} : \mathbb{F}_2 \otimes V^{\otimes(n-1)} & \longrightarrow & V \otimes V^{\otimes(n-1)}. \\
1 \otimes * & \longmapsto & B \otimes *
\end{array}$$

THE COBORDISM Λ_n^e

This cobordism is the disjoint union of the merge cobordism Λ as in Figure 6.1, from the first two components of U_n to the first component of U_{n-1} , and the identity cobordism on the last $n-2$ components. By arguing as in the case of V_n^e , we see that the map $F_{\Lambda_n^e}$ with respect to ψ_{U_n} and $\psi_{U_{n-1}}$ is

$$\begin{array}{ccc}
F_{\Lambda_n^e} : V \otimes V^{\otimes(n-2)} & \longrightarrow & \mathbb{F}_2 \otimes V^{\otimes(n-2)}. \\
T \otimes * & \longmapsto & 1 \otimes * \\
B \otimes * & \longmapsto & 0
\end{array}$$

THE COBORDISM $\text{IX}_{i,n}^e$

The effect of the cobordism $\text{IX}_{i,n}^e$ on the Heegaard diagram that induces the isomorphism ψ_{U_n} in Equation (6.4) is a swap between the two (unmarked)

components i and $i + 1$. The map on HFL is therefore

$$\begin{array}{ccc} F_{IX_{i,n}^e} : V^{\otimes(i-2)} \otimes V \otimes V \otimes V^{\otimes(n-i-1)} & \longrightarrow & V^{\otimes(i-2)} \otimes V \otimes V \otimes V^{\otimes(n-i-1)}. \\ * \otimes a \otimes b \otimes * & \longmapsto & * \otimes b \otimes a \otimes * \end{array}$$

THE COBORDISM IV_n^e

We now turn our attention to the cobordism IV_n^e . By Theorem 6.14, and arguing as in the case of V_n^e , we can restrict to the case $n = 2$. Then we can split the cobordism IV_2^e as the disjoint union of the identity cobordism I and the split cobordism V from Figure 6.1. By Theorem 6.14, we have isomorphisms

$$\begin{aligned} \varphi_{U_2} : \text{HFL}(U_2, P_{U_2}) &\rightarrow \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes V, \text{ and} \\ \varphi_{U_3} : \text{HFL}(U_3, P_{U_3}) &\rightarrow \mathbb{F}_2 \otimes V \otimes V, \end{aligned}$$

such that the map $F_{IV_2^e}$ under these identifications is as follows:

$$\begin{array}{ccc} F_{IV_2^e} : \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes V & \longrightarrow & \mathbb{F}_2 \otimes V \otimes V. \\ 1 \otimes 1 \otimes T & \longmapsto & 1 \otimes B \otimes T \\ 1 \otimes 1 \otimes B & \longmapsto & 1 \otimes B \otimes B \end{array}$$

Recall that we are however interested in writing the map with respect to the identifications in Equation (6.4). Notice that the map

$$\varphi_{U_2} \circ \psi_{U_2}^{-1} : V \rightarrow \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes V$$

is clear because it preserves the grading. We can therefore identify

$$\text{HFL}(U_2, P_{U_2}) \cong V \cong \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes V.$$

We need to understand $\psi_{U_3} \circ F_{IV_2^e}(T)$ and $\psi_{U_3} \circ F_{IV_2^e}(B)$. Notice that $\varphi_{U_3} \circ F_{IV_2^e}(B)$ is the unique bottom-graded element of $\mathbb{F}_2 \otimes V \otimes V$, therefore we deduce that

$$\psi_{U_3} \circ F_{IV_2^e}(B) = B \otimes B \in V \otimes V.$$

On the other hand, T is mapped to one of the three middle-graded homogeneous elements of $\mathbb{F}_2 \otimes V \otimes V$, so $\psi_{U_3} \circ F_{IV_2^e}(T)$ is also one of the middle-graded homogeneous elements: $T \otimes B$, $B \otimes T$, or $T \otimes B + B \otimes T$. If we consider the composition of cobordisms $IX_{2,3}^e \circ IV_2^e$, we obtain the cobordism IV_2^e with different decorations, which by Remark 6.5 and Theorem 6.14 induces the same map as IV_2^e on HFL. Therefore, we have that the map $\psi_{U_3} \circ F_{IV_2^e}$ is invariant under swap. In the end, the map associated to $F_{IV_n^e}$ with respect to the identifications in Equation (6.4) is

$$\begin{aligned} F_{IV_n^e} : V \otimes V^{\otimes(n-2)} &\longrightarrow V^{\otimes 2} \otimes V^{\otimes(n-2)}. \\ T \otimes * &\longmapsto (T \otimes B + B \otimes T) \otimes * \\ B \otimes * &\longmapsto (B \otimes B) \otimes * \end{aligned}$$

THE COBORDISM IA_n^e

The last case that we have to study is the cobordism IA_n^e . As in the case of IV_n^e , we can restrict our attention to IA_3^e .

By Theorem 6.14, we have isomorphisms φ_{U_2} and φ_{U_3} such that the map $F_{IA_3^e}$ under these identifications is as follows:

$$\begin{aligned} F_{IA_3^e} : \mathbb{F}_2 \otimes V \otimes V &\longrightarrow \mathbb{F}_2 \otimes \mathbb{F}_2 \otimes V. \\ 1 \otimes T \otimes T &\longmapsto 1 \otimes 1 \otimes T \\ 1 \otimes T \otimes B &\longmapsto 1 \otimes 1 \otimes B \\ 1 \otimes B \otimes T &\longmapsto 0 \\ 1 \otimes B \otimes B &\longmapsto 0 \end{aligned}$$

As in the case of IV_2^e , we need to understand this map with respect to the isomorphisms given by Equation (6.4). By the fact that $\text{I}\Lambda_3^e \circ \text{IX}_{2,3}^e$ is the cobordism $\text{I}\Lambda_3^e$ (with different decorations) and by Remark 6.5, we know that the map $F_{\text{I}\Lambda_3^e}$ with respect to such identifications is invariant under swaps. Since we know that there is a unique middle-graded homogeneous element of $V^{\otimes 2}$ that is in the kernel of $F_{\text{I}\Lambda_3^e}$, and since $\ker(F_{\text{I}\Lambda_3^e})$ is invariant under swaps, we deduce that such an element is $T \otimes B + B \otimes T$. Therefore, the map $F_{\text{I}\Lambda_n^e}$ with respect to the identifications in Equation (6.4) is given by

$$\begin{aligned}
F_{\text{I}\Lambda_n^e} : V^{\otimes 2} \otimes V^{\otimes(n-3)} &\longrightarrow V \otimes V^{\otimes(n-3)}. \\
(T \otimes T) \otimes * &\longmapsto T \otimes * \\
(T \otimes B) \otimes * &\longmapsto B \otimes * \\
(B \otimes T) \otimes * &\longmapsto B \otimes * \\
(B \otimes B) \otimes * &\longmapsto 0
\end{aligned}$$

Remark 6.17. The results above completely determine the TQFT HFL. This is expressed in a concise form in Figure 6.3.

6.3.2 THE CATEGORY OF MARKED ABSTRACT COBORDISMS AND THE REDUCED KHOVANOV TQFT

We now compare the TQFT HFL from Section 6.3.1 with another TQFT from a category of marked cobordisms, namely the reduced Khovanov TQFT, which gives rise to the reduced Khovanov homology of a marked link. We start with the following definition.

Definition 6.18 (Juhász-M.³²). The category of *marked abstract cobordisms* AbsCob^* is defined as follows. The objects are closed 1-manifolds with a marked component, which we will denote by a dot on the component itself. Let U_n be a particular closed 1-manifold with n components; we will suppose that each component is labelled by a number between 1 and n , and that the marked component is the first one. Notice that any object

of AbsCob^* is diffeomorphic to some U_n .

The morphisms are all abstract cobordisms between any two closed 1-manifolds generated by

- V_n^a ($n \geq 1$), the cobordism from U_n to U_{n+1} which consists of a pair-of-pants between the first component of U_n and the first two components of U_{n+1} , and, for each $i > 1$, of a cylinder between the i -th component of U_n and the $(i + 1)$ -th component of U_{n+1} . Notice that the marked components of U_n and U_{n+1} are both contained in the pair-of-pants.
- Λ_n^a ($n \geq 2$), the cobordism from U_n to U_{n-1} which consists of a pair-of-pants between the first two components of U_n and the first component of U_{n-1} , and, for each $i > 2$, of a cylinder between the i -th component of U_n and the $(i - 1)$ -th component of U_{n-1} . Notice that the marked components of U_n and U_{n-1} are both contained in the pair-of-pants.
- $\text{IX}_{i,n}^a$ (for $2 \leq i \leq n - 1$ and $n \geq 3$), obtained by swapping the i -th and the $(i + 1)$ -th component of U_n . Notice that the marked components are on the first I component of the cobordism, and that therefore the marked component cannot be swapped.
- IV_n^a ($n \geq 2$), the cobordism from U_n to U_{n+1} which consists of a pair-of-pants between the second component of U_n and the second and the third components of U_{n+1} , and of cylinders between all the other components. The marked components of U_n and U_{n+1} are connected by a cylinder.
- $\text{I}\Lambda_n^a$ ($n \geq 3$), the cobordism from U_n to U_{n-1} which consists of a pair-of-pants between the second and the third components of U_n and the second component of U_{n-1} , and of cylinders between all the other components. The marked components of U_n and U_{n-1} are connected by a cylinder.

The above cobordisms are represented in Figure 6.4.

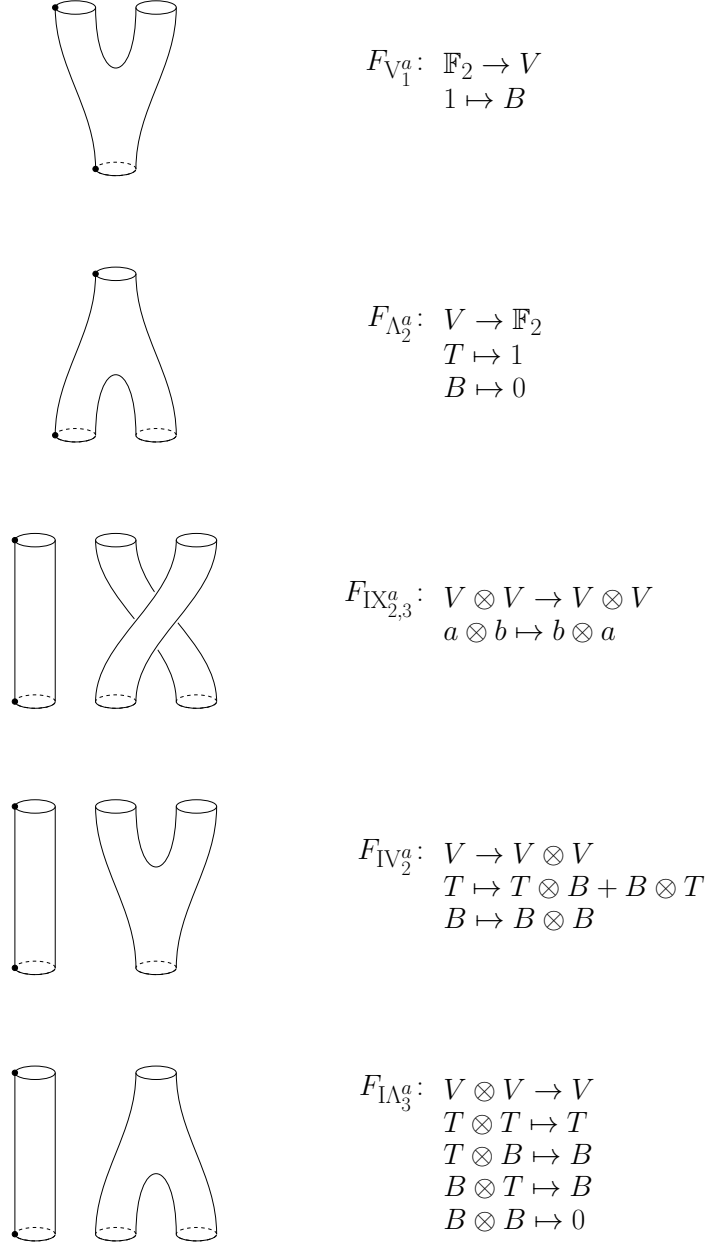


Figure 6.4: The figure shows the cobordisms V_1^a , Λ_2^a , $IX_{2,3}^a$, IV_2^a and $I\Lambda_3^a$ in the category $\mathbf{AbsCob}^*$ and the maps they induce via the reduced Khovanov TQFT $\widetilde{\mathbf{Kh}}$. Note that the marked component gives a canonical identification $\widetilde{\mathbf{Kh}}(U_n) = V^{\otimes(n-1)}$.

Definition 6.19. Let

$$\text{Ob}: \mathbf{EmbCob}^* \rightarrow \mathbf{AbsCob}^*,$$

be the forgetful functor such that on the objects $\text{Ob}(U_n, P_{U_n}) = U_n$, and on the cobordisms it simply forgets all the decorations and the embedding.

Definition 6.20. The reduced Khovanov TQFT is the functor

$$\widetilde{\text{Kh}}: \mathbf{AbsCob}^* \rightarrow \mathbf{Vect}_{\mathbb{F}_2}$$

such that on the objects $\widetilde{\text{Kh}}(U_n) = V^{\otimes(n-1)}$, where each V factor should be thought of as associated to an unmarked component, and on the morphisms it is defined as in Figure 6.4.

The main result of the section is expressed in the following theorem.

Theorem 6.21 (Juhász-M.³²). *There is a commutative triangle of functors*

$$\begin{array}{ccc} \mathbf{EmbCob}^* & \xrightarrow{\text{HFL}} & \mathbf{Vect}_{\mathbb{F}_2} \\ & \searrow \text{Ob} \quad \nearrow \widetilde{\text{Kh}} & \\ & \mathbf{AbsCob}^* & \end{array}$$

Proof. We have by definition

$$\text{HFL}(U_n, P_{U_n}) \cong V^{\otimes(n-1)} = \widetilde{\text{Kh}}(U_n) = \widetilde{\text{Kh}}(\text{Ob}(U_n, P_{U_n})).$$

The commutativity of the triangle on the morphisms can be checked just on the generators of the morphisms. This is achieved by comparing Figures 6.3 and 6.4. $\square$

As a consequence of Theorem 6.21, we note that the reduced Khovanov homology of L can be computed by using the HFL TQFT. In order to prove it, we need to quickly recall the definition of (reduced) Khovanov homology.

Definition 6.22. Let L be a marked link in S^3 , and let D be a marked diagram for L . A *resolution* of D is a map u from the set of crossings of D to $\{0, 1\}$.

Given a resolution u , we denote by L_u the marked unlink obtained by smoothing each crossing according to the conventions in Figure 5.6. We will also call L_u a *resolution*.

If u and v are two resolutions of D , we say that $u \prec v$ if v is obtained by changing the value of a single crossing of u from 0 to 1.

We denote the number of 1's in the resolution u by $|u|$.

Definition 6.23. If $u \prec v$, then L_v is obtained from L_u by a pair-of-pants cobordism. We denote it by $\mathcal{G}_{u,v}$.

We say that the family of all resolutions L_u together with the cobordisms $\mathcal{G}_{u,v}$ such that $u \prec v$ is a *cube of resolutions* for the marked diagram D .

Given a TQFT

$$\widetilde{\text{Kh}}: \text{AbsCob}^* \rightarrow \text{Vect}_{\mathbb{F}_2},$$

we define the reduced Khovanov homology of any link as follows.

Definition 6.24. Let D be a marked diagram for a marked link L in S^3 . Its *reduced Khovanov homology* $\widetilde{\text{Kh}}(L)$ is the (co-)homology of the complex

$$\left(\bigoplus_{i \in \mathbb{Z}} C^i, d \right),$$

where we set

$$C^i := \bigoplus_{|u|=i} \widetilde{\text{Kh}}(L_u)$$

and

$$d|_{\widetilde{\text{Kh}}(L_u)} := \sum_{u \prec v} \widetilde{\text{Kh}}(\mathcal{G}_{u,v}).$$

Since the definition of reduced Khovanov homology is completely determined by the TQFT $\widetilde{\text{Kh}}$, we have the following corollary of Theorem 6.21.

Corollary 6.25 (Juhász-M.³²). *The reduced Khovanov homology of a marked link L (with $\mathbb{F}_2$ coefficients) can be computed by applying the TQFT HFL to a cube of resolutions of L .*

6.4 ON THE RELATION BETWEEN $\widetilde{\text{Kh}}$ AND HFL

Although the relation between HFL and $\widetilde{\text{Kh}}$ proven in Theorem 6.21 is far from an answer to Conjecture 6.1, there is still a connection between them, which we point out in this final section.

As mentioned at the beginning of this chapter, Kronheimer and Mrowka⁴¹ proved the existence of a spectral sequence

$$\widetilde{\text{Kh}}(\overline{K}) \Rightarrow \text{KHI}(K).$$

We now outline the strategy of their proof:

1. They define a singular instanton homology for links $I^\natural(L)$, which coincides with $\text{KHI}(L)$ when L is a knot, and which restricts to a functor $\text{EmbCob}^* \rightarrow \text{Vect}_{\mathbb{F}_2}$ which fits a triangle as in Theorem 6.21;
2. Given a diagram D for a link L , they construct the cube of resolutions of D , and associate $I^\natural(L_u)$ to each vertex/resolution, and $I^\natural(\mathcal{G}_{u,v})$ to each edge/cobordism. Since $I^\natural$ restricted to EmbCob^* fits a triangle as in Theorem 6.21, the homology of the complex that they obtain from the cube of resolution is the reduced Khovanov homology of (the mirror image of) L ;
3. They introduce higher differentials in the complex, and, by a mapping cone argument, they show that the total homology of the complex is isomorphic to $I^\natural(L)$. Since for knots $I^\natural$ coincides with KHI , the result is established.

Inspired by Kronheimer and Mrowka, one could try to prove the existence of a spectral sequence from $\widetilde{\text{Kh}}(\overline{K})$ to $\widehat{\text{HFK}}(K)$ by using the same strategy as above with HFL in place of $I^\natural$. Note that HFL coincides with $\widehat{\text{HFK}}$ on

knots by definition. Theorem 6.21 implies that points 1 and 2 hold unaffected also in the case of HFL. However, point 3 is false for HFL, because the mapping cone argument breaks already in the most elementary case. This elementary case would be an unoriented exact triangle involving HFL of three links L_0 , L and L_1 that locally look as in Figure 5.6. However, there are counterexamples to such a triangle. It is noteworthy that Manolescu⁴⁶ proved a stabilised version of the aforementioned exact triangle, which is not enough to prove Conjecture 6.1.

In order to make Kronheimer and Mrowka's proof work for $\widehat{\text{HFK}}$, one would need an invariant $\widetilde{\text{HFL}}$ for links such that:

1. $\widetilde{\text{HFL}}$ agrees with HFL on $\mathbf{EmbCob}^*$;
2. for every knot K in S^3 , $\widetilde{\text{HFL}}(K) \cong \widehat{\text{HFK}}(K)$; and
3. $\widetilde{\text{HFL}}$ satisfies an unoriented exact triangle.

Lastly, I would like to mention that Baldwin, Levine, and Sarkar⁶ proposed a different strategy to tackle the question about the relation between HFL and $\widetilde{\text{Kh}}$. Instead of trying to define a suitable variation of link Floer homology, they define a multi-pointed variation of reduced Khovanov homology. The goal of their programme would not be a spectral sequence from $\widetilde{\text{Kh}}(\overline{K})$ to $\widehat{\text{HFK}}(\overline{K})$, but between equally stabilised versions, which would anyway give a rank inequality

$$\dim_{\mathbb{F}_2}(\widetilde{\text{Kh}}(\overline{K})) \geq \dim_{\mathbb{F}_2}(\widehat{\text{HFK}}(K))$$

for all knots K in S^3 .

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