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Two structural aspects in birational geometry: geography of Mori fibre spaces and Matsusaka's Theorem for surfaces in positive characteristic

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Declaration of Originality

I hereby declare that the material presented in this thesis is my own, except where otherwise acknowledged and appropriately referenced.

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There are tons of other people which deserve acknowledgements, but accuracy would produce a second thesis and concision could disappoint many. On the other hand, the risk of ending up with a cold list of people irritates me. In the end I opted for a drastic decision.

This is a maths thesis, so the first person to thank is certainly Paolo, my supervisor. I was studying on a messy desk in a tiny Parisian flat (just few blocks from here, where I am writing down these acknowledgements) when you called me for an interview. You asked me something about Riemann-Roch and it seems I made a good impression! I hope I lived up to your expectations. More or less... I learnt tons of maths under your supervision and you taught me how to discuss with colleagues about my work; at the same time you allowed me to be independent in my research. My PhD was intense, challenging and sometimes risky, but I think not many mathematicians enjoyed their PhD as I did: studying with you at Imperial was a true privilege.

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Last but not least, the backpack I brought with me from Rome: full of food to share, Zia Nina's coffee liquor and other hidden (but great) beauties.

Cheers!

*Ai miei genitori,
Clelia e Corrado.*

Abstract

“L’odore delle case dei vecchi.”

Jep Gambardella

The aim of this thesis is to investigate two questions which naturally arise in the context of the classification of algebraic varieties.

The first project concerns the structure of Mori fibre spaces: these objects naturally appear in the birational classification of higher dimensional varieties and the minimal model program. We ask which Fano varieties can appear as a fibre of a Mori fibre space and introduce the notion of *fibre-likeness* to study this property. This turns out to be a rather restrictive condition: in order to detect this property, we obtain two criteria (one sufficient and one necessary), which turn into a characterisation in the rigid case. Many applications are discussed and the basis for the classification of fibre-like Fano varieties is presented.

In the second part of the thesis, an effective version of Matsusaka’s theorem for arbitrary smooth algebraic surfaces in positive characteristic is provided: this gives an effective bound on the multiple which makes an ample line bundle D very ample. A careful study of pathological surfaces is presented here in order to bypass the classical cohomological approach. As a consequence, we obtain a Kawamata-Viehweg-type vanishing theorem for arbitrary smooth algebraic surfaces in positive characteristic.

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1 Introduction

1.1 Motivation: classification and structure

1.1.1 Some history: classification of algebraic varieties and minimal model program

Classification is a central problem in all natural sciences: sorting the objects of study according to their essential properties is fundamental not only for the full understanding of a natural phenomenon but also in order to facilitate effective communication within the scientific community. As Linnaeus started the complete classification for biological sciences during the 18th century, the Italian school of algebraic geometry (1885-1935) laid the foundations for the classification of algebraic varieties.

Although the classification of complex algebraic varieties in dimension one and two was essentially understood at that time, the problem in arbitrary dimension was first successfully attacked only in the 80's: Mori's breakthrough and the revolutionary works by Kawamata and Shokurov put the basis of modern complex birational geometry. In that period the minimal model program (or MMP) was introduced: conjecturally, one should be able to find (at least) one *good* representative in every birational class of varieties whose geometric structure is the easiest possible. Although the MMP in dimension three was completely established, this conjecture is still open in arbitrary dimension. There exist an interesting class of varieties, which are covered by rational curves: if we run an MMP for these *uniruled varieties*, there are no

chances of remaining within the same birational class. Nonetheless, we come up with a variety with a very easy geometric structure: a fibration over a lower dimensional variety, whose fibres are *Fano varieties*, i.e. varieties with ample anticanonical bundle. These fibrations are called *Mori fibre spaces* (or *MFSs*) and play a central role in the classification.

The picture for algebraic varieties over an algebraically closed field in positive characteristic is still quite mysterious and this area of research has been developed only for few years. Even for surfaces the picture is not completely clear, since many techniques in complex birational geometry heavily rely on Kodaira-type vanishing theorems, which are known to fail in positive characteristic.

1.1.2 Geography of Mori fibre spaces and Fano varieties

After the seminal work [BCHM10], it is clear that running an MMP on a uniruled variety, with reasonable singularities, always gives a MFS as end product. Furthermore, [Cor95a] and [HM13a] completely understood (for threefolds and in arbitrary dimension respectively) how distinct but birationally equivalent MFSs are related one another (the so called *Sarkisov program*).

Another fascinating research line is more focused on the explicit geometry in low dimension, particularly of threefold MFSs (cf. [BCZ04]).

Always in the stream of explicit algebraic geometry, an interesting class of algebraic variety has been extensively studied for the last decades: toric varieties. These objects have the big advantage of having a very explicit combinatorial description which makes them particularly suitable for computations, also and especially via computer algebra softwares (cf. [Cas06] and [GB⁺]). Toric Fano varieties play also an essential role in higher dimensional algebraic geometry: they are used in order to test general conjectures on arbitrary Fano varieties.

Fano varieties, which can be considered as the elementary structural pieces of uniruled varieties, are themselves object of research worldwide. In particular, several notions of stability for Fano varieties have been developed in these years both from the algebro-geometric and the differential-geometric point of view: the recent works by Chen, Donaldson and Sun and Tian (cf. [CDS15] and [Tia12]) have proven an important open conjecture relating the notion of K -stability with the existence of Kähler-Einstein metrics on smooth Fano varieties. Nonetheless, K -stability remains a rather mysterious notion from the algebro-geometric side and the picture is not even completely clear for threefolds (cf. [CS08]).

Extending the classification results and the study of Fano varieties in positive characteristic is another very active research field. The lack of many characteristic-zero technical tools makes this field extremely challenging and appealing for young algebraic geometers (cf. [HX13], [Bir13] and [GOST15]).

1.1.3 Surfaces in positive characteristic

Although the MMP for surfaces in positive characteristic has been recently established, thanks to the work of Tanaka (cf. [Tan14], [Tan12]), many interesting effectivity questions remain open in this setting, after the influential papers of Ekedahl and Shepherd-Barron (cf. [Eke88] and [SB91b]).

Conjecture 1.1 (Fujita). *Let L be an ample divisor on a smooth variety X of dimension n . Then $K_X + (n + 1)L$ is base point free and $K_X + (n + 2)L$ is very ample.*

Even in characteristic zero, this conjecture remains open for dimension bigger than two. In positive characteristic, even the two-dimensional case is not complete.

The classical proof of this conjecture for smooth surfaces relies on Reider's theorem, which is proved to hold only on non-pathological surfaces in posi-

tive characteristic (cf. Definition 3.17 and [SB91b]).

1.2 Contributions and publications

In this thesis, some of the results I obtained during my PhD studies are discussed.

Original results in Chapter 2 are marked with $\heartsuit$ and have been obtained in collaborations with Dr Giulio Codogni (Roma Tre), Dr Roberto Svaldi (MIT) and Dr Luca Tasin (Bonn). Many of them can be found in the paper [CFST14], to appear in *IMRN*.

New results in Chapter 3, marked with $\spadesuit$, have been obtained in collaborations with Dr Gabriele Di Cerbo (Columbia) and they can be found in the paper [DCF15], to appear in *Algebra and Number Theory*.

Some results which appear in this thesis are classically known and I could not find precise attribution. In this case, they are marked with $\star$.

2 Geography of Mori fibre spaces

2.1 Introduction

In this chapter we investigate the geography of Mori fibre spaces (see Definition 2.21): these objects naturally appear in the context of the classification of complex algebraic varieties in arbitrary dimension and they have been extensively studied in the last decades. It is expected that every variety with *mild* singularities which has negative Kodaira dimension is birational to a Mori fibre space: this would reduce the birational classification of these varieties to the classification of Mori fibre spaces.

All varieties involved in this chapter are defined over the complex field, with the only exceptions of Subsections 2.3.3 and 2.4.6, where varieties are defined over an algebraically closed field of arbitrary characteristic.

Ruled surfaces

Many problems concerning the classification of Mori fibre spaces already appear in the smooth two-dimensional case, so let us focus for a bit on few phenomena appearing in this setting.

It is well known that a smooth surface S with $\kappa(S) = -\infty$ is birational to a (relatively minimal) ruled surface (cf. [Băd01, Chapter 13]).

Problem 2.1. *There exist many ruled surfaces within a fixed birational class of smooth surfaces with negative Kodaira dimension. This can be easily seen*

via elementary transformations: start with a relatively minimal ruled surface $\pi_1: S_1 \rightarrow C$, blow up a point x and contract the strict transform of the fibre containing x to obtain a new ruled surface $\pi_2: S_2 \rightarrow C$ which is birational but not isomorphic to S_1 ,

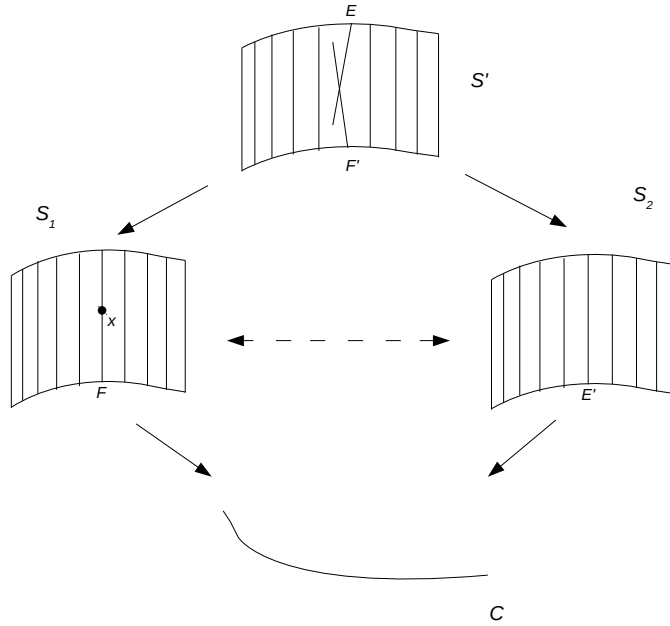


Figure 2.1: Elementary transformation between ruled surfaces

The previous example shows we can have many minimal models for a smooth surface with negative Kodaira dimension.

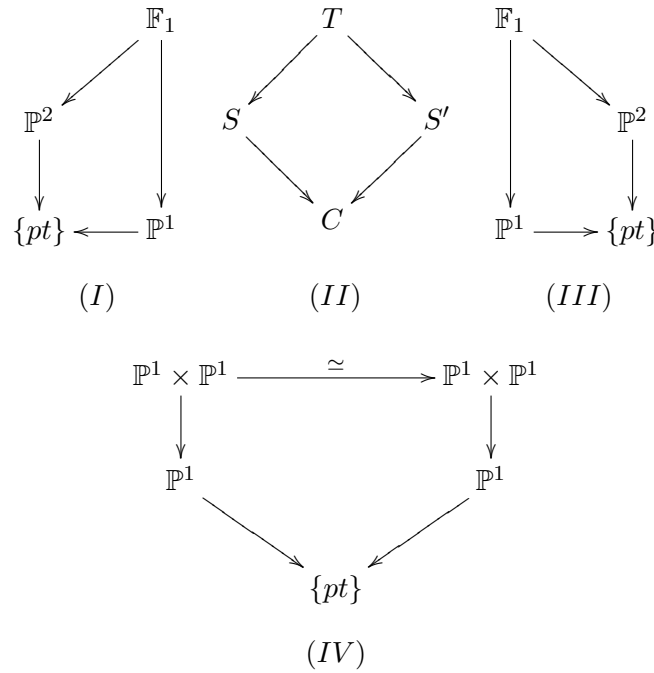
Problem 2.2. Start with $S := \mathbb{P}^1 \times C$ with C smooth curve. Then one can perform a finite number of elementary transformations to obtain a new ruled surface birational to S : this gives infinitely many relatively minimal ruled surfaces in the same birational class of S .

Actually one can prove that all ruled surfaces can be obtained this way (cf.

[Nag60]): think of the *Hirzebruch surfaces* $\mathbb{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \rightarrow \mathbb{P}^1$, which can be constructed from $\mathbb{P}^1 \times \mathbb{P}^1$ via a sequence of n elementary transformations (cf. [Băd01, Lemma 12.6]).

Let us change our point of view and focus on the minimal model program for surfaces: if our ultimate goal is to classify surfaces with negative Kodaira dimension, we can run an MMP on S and end up with a (relatively minimal) ruled surface or $\mathbb{P}^2$.

One can ask how two minimal models $S_{min,1}$ and $S_{min,2}$ within a fixed birational class are related one another. The answer is classical in this case (the so called *Castelnuovo-Noether theorem*): $S_{min,1}$ and $S_{min,2}$ are related by a composition of very easy birational maps of the following form (cf. [Mat02, Theorem 1-8-8]).



These birational maps are called *Sarkisov links*: note that the type-(II) link is simply an elementary transformation described in Problem 2.1. This gives a complete picture in the smooth two-dimensional case.

Problems 2.1 and 2.2 appear, as one can expect, also in higher dimension: they are aspects of the the so-called *Sarkisov problem*, which has been completely established by [Cor95b] in dimension three and by [HM13b] in arbitrary dimension.

The fibres of Mori fibre spaces: a preliminary discussion

Let us now focus on the fibres of Mori fibre spaces: since the only smooth curve with negative canonical divisor is $\mathbb{P}^1$, there are no structural questions regarding them for two-dimensional Mori fibre spaces. Nonetheless, moving to higher dimension (even for threefolds), it is not clear which smooth Fano varieties (i.e. varieties with anti-ample canonical divisor) can appear as fibres of a Mori fibre space.

In order to fix the ideas, let us start with a rough question which will lead our introductory discussion (it was first asked during the *Pragmatic 2013* conference in Catania).

Question 2.3. *Which smooth Fano varieties can be realised as a general fibre of a Mori fibre space?*

Let us start with some easy examples.

Example 2.4. • *Every Fano variety F with minimal Picard rank $\rho(F) = 1$ has a trivial structure of Mori fibre space over the point, so Question 2.3 is interesting only for higher Picard rank.*

- *Consider the easiest nontrivial example: $F = \mathbb{P}^1 \times \mathbb{P}^1$. There is a very nice way to obtain a Mori fibre space with generic fibre isomorphic to F : just consider, in the product $\mathbb{P}^3 \times \mathbb{P}^1$, the ample linear system $L := \mathcal{O}(2, 1)$. Let X be a general (smooth) element of L and consider*

the restriction f of the second projection π_2 to X :

$$\begin{array}{ccc} X & \longrightarrow & \mathbb{P}^3 \times \mathbb{P}^1 \\ f \downarrow & \swarrow \pi_2 & \\ \mathbb{P}^1 & & \end{array}$$

It is clear that the generic fibre of f is isomorphic to F (which degenerates to a cone $\mathbb{P}(1,1,2)$ in finitely many points). To show that $f: X \rightarrow \mathbb{P}^1$ is a Mori fibre space, we need to compute the relative Picard rank of f : this is easy, using Lefschetz hyperplane theorem (cf. [Laz04, Example 3.1.25]), which guarantees that $\rho(X) = 2$. This is enough to conclude that f is a Mori fibre space.

The second natural example to consider would be the first Hirzebruch surface $\mathbb{F}_1$, which is isomorphic to $\mathbb{P}^2$ blown up in one point. Despite $\mathbb{P}^1 \times \mathbb{P}^1$, it cannot be realised as an ample hypersurface or complete intersection in a projective space, so we are unable to perform the same construction as before. Indeed, no smooth 3-dimensional Mori fibre space over a curve has $\mathbb{F}_1$ as general fibre, as the following result due to Mori states (cf. [Mor82, Theorem 3.5]).

Theorem 2.5 (Mori (= Theorem 2.54)). *Let X be a nonsingular threefold, let R be an extremal ray which is numerically effective and $f_R: X \rightarrow Y$ the associated extremal contraction. Then Y is nonsingular and, assuming $\dim Y = 1$, the generic fibre of f is a nonsingular del Pezzo surface S , with $S \not\cong S_7, S_8$.*

The non-admissible del Pezzo fibres are precisely $\mathbb{P}^2$ blown up in one or two points. This result raises many other questions, even in the two-dimensional case. First, we would like to see some explicit examples.

Question 2.6. *Can one methodically construct examples of Mori fibre spaces with assigned general fibre, generalising the construction in Example 2.4?*

For instance, one could consider divisors of the form $\mathcal{O}(3, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^1$ to obtain, as in Example 2.4, a Mori fibre space with cubic fibres S_3 . However, cubic surfaces have nontrivial deformations and the obtained fibration will not be isotrivial. This example is relevant in order to study the geography of Mori fibre spaces, so we need to refine our notion of *general fibre*: this will be extensively discussed in Subsection 2.2.3.

Since Theorem 2.5 only deals with threefold, one could ask what happens in higher dimension.

Question 2.7. *Does a higher dimensional Mori fibre space which has S_7 or S_8 as general fibre exist?*

Furthermore, working in the context of the minimal model program, one should deal with mild singularities.

Question 2.8. *What is the picture allowing varieties to be $\mathbb{Q}$ -factorial with (log-)terminal singularities?*

The project: towards the classification of fibre-like Fano varieties

In order to give an idea of our methods, we will focus on the smooth case: so in this subsection we will limitate ourselves to Mori fibre spaces $f: X \rightarrow Y$ with X and Y smooth and the fibration f smooth too. Classical results in differential topology guarantee the existence of an Zariski open dense subset U_f^{top} over which all the fibre are smooth and the fibration is locally topologically trivial (this is Sard's Theorem combined with Ehresmann's Theorem).

Definition 2.9. Let $f: X \rightarrow Y$ be a Mori fibre space with X and Y smooth varieties. The open dense subset $U_f^{top} \subset Y$ is defined as the maximal Zariski open set for which $f|_{f^{-1}(U_f^{top})}: f^{-1}(U_f^{top}) \rightarrow U_f^{top}$ is locally topologically trivial.

The question we will address to is the following (a first refined version of Question 2.3).

Question 2.10. *Let F be a smooth Fano variety. Does it exist a Mori fibre space $f: X \rightarrow Y$ such that F is isomorphic to a fibre of f over the open dense subset $U_f^{top} \subset Y$?*

This question can be further refined allowing singularities: a new open dense subset of the base denoted by U_f will appear.

Definition 2.11 (= Definition 2.39). A Fano variety is *fibre-like* if can be realised as a fibre of a Mori fibre space $f: X \rightarrow Y$ over U_f .

The ultimate goal of this huge project would be to classify $\mathbb{Q}$ -factorial Fano variety with terminal singularities (or at least smooth). What makes this project ambitious but accessible is the restrictiveness of the fibre-like condition for a Fano varieties, which will be clear in the following sections.

The aim of this chapter is to give a precise account on the state of this project: many interesting questions have been answered via a rather general approach. After a preliminary part (cf. Section 2.2) in which the general notation of the minimal model program is briefly introduced, two criteria (one sufficient and one necessary) detecting the fibre-likeness are discussed in Section 3.3.

Theorem 2.12 ($\heartsuit$ (= Theorems 2.40, 2.42 and 2.43)).

Sufficient criterion: *A Fano variety F such that*

$$\dim \mathrm{NS}(F)_{\mathbb{Q}}^{\mathrm{Aut}(F)} = 1$$

is fibre-like.

Moreover, an explicit construction of an isotrivial MFS over a curve is provided.

Necessary criterion: A Fano variety F for which

$$\dim \mathrm{NS}(F)_{\mathbb{Q}}^{\mathrm{Mon}(F)} > 1,$$

where $\mathrm{Mon}(F)$ is the maximal subgroup of $\mathrm{GL}(\mathrm{NS}(F), \mathbb{Z})$ which preserves the birational data of F (cf. Definition 2.34), is not fibre-like.

Characterisation for rigid varieties: Assume that the Fano variety F is rigid.

Then the sufficient criterion is a characterisation.

The main idea of our approach is to look at the monodromy action in the special case of Mori fibre spaces: being a fibre of such a morphism imposes very strict constraints on its birational geometry (cf. Subsections 2.3.2). In Subsection 2.3.3 we start investigating a purely algebraic approach to the problem, which allows us to obtain some new results also in positive characteristic.

Section 2.4 is devoted to the main applications of our methods: smooth surfaces and threefolds (cf. Subsections 2.4.1 and 2.4.3), passing through a very general construction of Mori fibre spaces via complete intersections in toric varieties which provides a huge number of fibre-like Fano varieties (cf. Subsection 2.4.2).

Theorem 2.13 ($\heartsuit$ (= Theorems 2.54 and 2.67)).

Surfaces: A smooth del Pezzo surface is fibre-like if and only if it is not isomorphic to $\mathbb{P}^2$ blown up in one or two points.

Threefolds: A smooth Fano threefold F , with Picard rank bigger than one is fibre-like if and only if its deformation type is one of the eight classes appearing in Table 2.1.

Theorem 2.14 ($\heartsuit$ (= Theorem 2.62)). Let F be a smooth Fano variety and let Z be a smooth projective variety. Assume that L_i , $1 \leq i \leq k$ are

basepoint-free effective Cartier divisors on Z and that F is defined as a complete intersection of the L_i 's in Z .

Let I the incidence variety in $Z \times |L_1| \times \cdots \times |L_k|$. Assume the existence of a finite cyclic subgroup G in $\text{Aut}(Z)$ which is fix-point-free in codimension one and whose action can be lifted to I . Moreover, assume that G does not preserve F . If

$$\dim \text{NS}(Z)_{\mathbb{Q}}^G = 1,$$

then F is fibre-like.

In arbitrary dimension, we can provide a complete picture for homogeneous spaces in Subsection 2.4.4 and partial results on smooth toric varieties are presented in Subsection 2.4.5, together with the full classification of smooth toric fibre-like Fano varieties up to dimension eight (an attempt to a rigorous classification for smooth toric fibre-like Fano varieties is discussed too). Always in the toric case, an interesting connection between fibre-likeness and K -stability is proved.

Theorem 2.15 ($\heartsuit$ (= Theorem 2.85)). *Let $F(\Delta)$ be a smooth toric fibre-like Fano variety, with associated polytope Δ . Then Δ has barycentre in the origin and, as a consequence, F is K -stable.*

The last application is for singular del Pezzo in arbitrary characteristic (cf. Subsection 2.4.6).

2.2 Preliminaries

2.2.1 Some basic notation

In this subsection, we recall some standard notation which will be used in this chapter.

Notation 2.16. *By a variety X we mean an integral separated scheme of*

finite type over a field k . Unless otherwise specified we work over the field of complex numbers.

We denote by $\mathrm{NS}(X)_{\mathbb{Q}}$ the group of Cartier divisors of X modulo numerical equivalence, tensored by $\mathbb{Q}$. Dually, we denote by $N_1(X)_{\mathbb{Q}}$ the group of curves of X modulo numerical equivalence, tensored by $\mathbb{Q}$.

In the relative setting, for a projective morphism of varieties $f: X \rightarrow Y$, we denote by $N_1(X/Y)_{\mathbb{Q}}$ the subspace of $N_1(X)_{\mathbb{Q}}$ generated by curves that are mapped to a point and its dual $\mathrm{NS}(X/Y)_{\mathbb{Q}}$.

In the theory of classification in higher dimension, (mild) singularities naturally appear. We need to recall here some definitions. For the general notation about singularities, see [KM98].

Definition 2.17. A normal variety Z has terminal (resp. log-terminal) singularities if

1. K_Z is $\mathbb{Q}$ -Cartier;
2. for every resolution of singularities $r: \tilde{Z} \rightarrow Z$,

$$K_{\tilde{Z}} \simeq K_Z + \sum a_i E_i,$$

where E_i are the exceptional divisors and $a_i > 0$ (resp. $a_i > -1$) for all i .

Terminal and log-terminal singularities only require the canonical divisor to be $\mathbb{Q}$ -Cartier. This condition is usually not enough, for instance, to define intersection theory on Z . This is why one needs $\mathbb{Q}$ -factoriality.

Definition 2.18. A normal variety Z is $\mathbb{Q}$ -factorial if every Weil divisor is $\mathbb{Q}$ -Cartier.

Let us recall the definition of Fano varieties.

Definition 2.19. A normal projective variety F is said to be *Fano* if its anti-canonical divisor $-K_F$ is $\mathbb{Q}$ -Cartier and ample, i.e. there exists a sufficiently high multiple $-nK_F$ which is Cartier and the invertible sheaf $\mathcal{O}_F(-nK_F)$ is ample.

Let us recall the standard terminology for the type of steps of the minimal model program (for a beautiful introduction on the ideas of the MMP, see [Mat02]).

Definition 2.20. Let $\pi: Z \rightarrow Z'$ be an extremal contraction (associated to a negative ray R_π of the Mori cone $\overline{\text{NE}}(Z)$). Then π is a

Divisorial contraction if π is birational and the exceptional locus is an irreducible divisor and all the curves in the fibres are numerically proportional, i.e. $\rho(Z/Z') = 1$;

Flipping contraction if π is birational, the codimension of the exceptional locus is at least two and $\rho(Z/Z') = 1$;

Fibre-type contraction if π is a fibration with connected fibres and $\rho(Z/Z') = 1$.

Observe that, dually, pull-back via π identifies the nef cone of Z' with a face of the nef cone of Z .

We will be particularly interested in the fibration appearing as a fibre-type contraction.

Definition 2.21. Let $f: X \rightarrow Y$ be a dominant projective morphism of normal varieties. Then f is called a *Mori Fibre Space* (or simply *MFS*) if the following conditions are satisfied:

1. f has connected fibres, with $\dim Y < \dim X$;
2. X is $\mathbb{Q}$ -factorial with at most log-terminal singularities;

3. the relative Picard number of f is one and $-K_X$ is f -ample.

We will often refer to isotrivial fibrations. Let us recall here the definition.

Definition 2.22 (Isotrivial fibration). A morphism

$$f: X \rightarrow Y$$

is isotrivial if there exists an open dense subset U of Y and an étale cover $U' \rightarrow U$ such that $X_{U'} \cong X_U \times_U U'$ is a trivial family.

2.2.2 Monodromy for Mori fibre spaces

In this subsections we recall some classic theory of monodromy, which will form the technical core of our approach.

Notation 2.23. *Our setting for this subsection will be the following. The morphism $f: X \rightarrow Y$ will always be projective between normal varieties. Additionally, we assume that*

1. *the fibres of f are connected (i.e. that $f_*\mathcal{O}_X = \mathcal{O}_Y$);*
2. *X is $\mathbb{Q}$ -factorial with log-terminal singularities;*
3. *$-K_X$ is relatively ample over Y .*

The assumption on the ampleness of $-K_X$ implies that the general fibre is a Fano variety.

We want to consider the monodromy action associated to the fibration f , so we need to study some local systems on Y . Consider the sheaves $R^2f_*\mathbb{Q}$ and $R^1f_*\mathbb{G}_m \otimes \mathbb{Q}$ (their fibres on a point $y \in Y$ are $H^2(F_y, \mathbb{Q})$ and $\text{Pic}(F_y)_{\mathbb{Q}}$ respectively). The Fano condition on the fibres of f and the hypothesis on the singularities of X guarantees that the first Chern class's map

$$c_1: R^1f_*\mathbb{G}_m \otimes \mathbb{Q} \rightarrow R^2f_*\mathbb{Q} \tag{2.1}$$

is an isomorphism (this is a consequence of Kawamata-Viehweg vanishing theorem). The basic idea consists in finding an open dense subset of Y over which all fibres are “good enough” and, in particular, $R^2 f_* \mathbb{Q}$ is a local system.

Example 2.24 (Smooth case). *As a warm-up, we discuss the smooth case, where the core idea already appears. Assuming*

X and Y to be smooth,

Sard’s Theorem applies, producing an open dense subset U_f^s of Y over which f is a submersion (all the fibres over U_f^s are smooth). Ehresmann’s Theorem (cf. [Voi02, Proposition 9.3]) implies that f is a locally topologically trivial.

We will need (a generalised version of) a classic result by Deligne on monodromy. In the smooth case, it can be easily stated. Consider the restriction map to the fibres of the cohomology classes of the total space X :

$$H^2(X, \mathbb{Q}) \rightarrow H^0(U_f^s, R^2 f_* \mathbb{Q}).$$

One gets an isomorphism evaluating the sections at y :

$$H^0(U_f^s, R^2 f_* \mathbb{Q}) \rightarrow H^2(F_y, \mathbb{Q})^{\pi_1(U_f^s, y)}.$$

The composition gives a morphism

$$\rho: H^2(X, \mathbb{Q}) \rightarrow H^2(F_y, \mathbb{Q})^{\pi_1(U_f^s, y)}.$$

Deligne’s Theorem (cf. [Voi02, Theorem 16.24]) states that ρ is surjective for every $y \in U_f^s$.

One can find a non-empty Zariski open subset of Y over which f is a locally topologically trivial fibration also in the singular case (cf. [Ver76,

Corollary 5.1]).

Definition 2.25. The open dense subset of Y over which f is a locally topologically trivial fibration will be denoted with U_f^{top} .

Unfortunately, it is extremely hard to characterise U_f^{top} (cf. [Tei75]).

On U_f^{top} , the sheaf $R^2f_*\mathbb{Q}$ is a local system, a monodromy action of $\pi_1(U_f^{top}, y)$ on the fibre $(R^2f_*\mathbb{Q})_y$ is defined and this can be easily described (cf. [Voi02, Section 9.2.1, 15.1.1 and 15.1.2]): start with a class of a loop in $\pi_1(U_f^{top}, y)$ and pull back X to the interval $I = [0, 1]$ through this loop. More explicitly, consider γ , a representative of the class of the loop, defined by $\gamma: [0, 1] \rightarrow Y$. One can perform a fibre product

$$\begin{array}{ccc} X_\gamma & \longrightarrow & X \\ \downarrow & & \downarrow f \\ [0, 1] & \xrightarrow{\gamma} & Y \end{array}$$

After trivialising the family X_γ , a homeomorphism of F_t identifies the fibre over 0 and the fibre over 1.

In order to easily deal with terminal singularity, we need to consider another (more refined) open dense subset of the base Y (cf. [KM92] and [dFH11]).

Theorem 2.26 (Kollár-Mori, de Fernex-Hacon). *Keeping the above notation, let U_f be the open subset (possibly empty) of Y over which*

1. *Y is smooth;*
2. *f is flat;*
3. *the fibres of f are $\mathbb{Q}$ -factorial with terminal singularities.*

Then $R^1f_\mathbb{G}_m \otimes \mathbb{Q}$ is a local system on U_f with finite monodromy action.*

Proof. The assumptions on the singularity of the fibres put us in the situation described in [KM92, Conditions 12.2.1] (see also [KM92, Remark 12.2.1.4]).

We define the sheaf $\mathcal{GN}^1(X/U_f)$ as in [KM92, Definition 12.2.4]. Like for [dFH11, Proposition 6.5] one can see that $\mathcal{GN}^1(X/U_f)$ is isomorphic to $R^1 f_* \mathbb{G}_m \otimes \mathbb{Q}$ at the general point of U_f (this is a consequence of Verdier's result and the isomorphism of the first Chern class). The following step consists in showing the isomorphism at every point of U_f .

Although the paper [dFH11] deals with smooth curves as base space Y , the same argument works in arbitrary dimension. $\square$

Remark 2.27. *Assumption (3) on the singularities is needed to define the sheaf $\mathcal{GN}^1(X/U_f)$. Nonetheless, $\mathbb{Q}$ -factorial singularities and smoothness in codimension 2 is enough to prove the same result (cf. [KM92, Remark 12.2.1.4]).*

Without this assumption, U_f may be empty. Nonetheless, the assumption of terminality is enough to work in the context of the MMP, since this is the mildest singularities we are forced to allow in order to make the MMP work.

About $\mathbb{Q}$ -factoriality of the fibres, we know that in a flat family $\mathfrak{X} \rightarrow B$ of varieties with rational singularities and smooth in codimension 2 one can find an open subscheme $W \subset \mathfrak{X}$ parametrising $\mathbb{Q}$ -factorial fibres (cf. [KM92, Thm. 12.1.10]). Despite all this, even assuming the $\mathbb{Q}$ -factoriality of the space $\mathfrak{X}$, the open W may be empty. Example 2.28 gives evidence of this.

Example 2.28. *Let C be a projective curve of genus $g \geq 1$ and let $C' \rightarrow C$ be a degree-two étale cover with associated involution ι .*

Consider a rank-three quadric Q in $\mathbb{P}^4$ (this is simply a projective cone over $\mathbb{P}^1 \times \mathbb{P}^1$ via the Segre embedding). The action of the involution i switching the factors on $\mathbb{P}^1 \times \mathbb{P}^1$ lifts to an automorphism of Q (also denoted i).

Let $\mathfrak{X}$ be $Q \times C'$ mod out by the involution (i, ι) and consider the family

$\mathfrak{X} \rightarrow C$ (all fibres are isomorphic to Q). One realises that $Cl(Q \times C')$ has rank three, being isomorphic to the direct sum of the class groups of Q and C' . The monodromy acts on $\mathfrak{X}$ and the invariant part of the class group is of dimension two. On the other hand, the Picard group of $\mathfrak{X}$ has rank two, the same as $Q \times C'$. As a consequence, the morphism $\mathfrak{X} \rightarrow C$ is isotrivial of relative Picard one where $\mathfrak{X}$ is $\mathbb{Q}$ -factorial but all fibres are not $\mathbb{Q}$ -factorial.

Let us get back to the monodromy action: Theorem 2.26 produces a new open dense subset of the base Y denoted with U_f , which differs from U_f^{top} in Definition 2.25, in general. We are here considering a monodromy action of the fundamental group of different open subset of Y . However, given a normal variety W and a closed subvariety $Z \subset W$, it is known that the natural morphism $\pi_1(W \setminus Z) \rightarrow \pi_1(W)$ is always surjective (cf. [FL81, 0.7 (B)]).

This tells us that when we deal with Mori fibre spaces, restricting to open subsets of U_f^{top} will not affect the monodromy action on $R^2 f_* \mathbb{Q}$.

We are ready to generalise Deligne's Theorem to $NS_{\mathbb{Q}}$.

Theorem 2.29 (Deligne, $\heartsuit$). *Keep the notation as in Theorem 2.26. Let $U = U_f$ be the open dense subset of Theorem 2.26 and assume it contains a point y . Then the restriction map*

$$\rho: NS(X)_{\mathbb{Q}} \rightarrow NS(F_y)_{\mathbb{Q}}^{\pi_1(U_f, y)}$$

is onto for every $y \in U_f$.

Proof. First we remark that the sheaf $\mathcal{GN}^1(f^{-1}(U_f)/U_f)$ is a local system on U_f whose global sections are $NS(f^{-1}(U_f)/U_f)_{\mathbb{Q}}$: this is a consequence of Theorem 2.26 and [KM92, Corollary 12.2.9]. Furthermore, general properties of local systems (cf. [Voi02, Lemme 16.17]) guarantee that this vector space is isomorphic to $H^2(F_y, \mathbb{Q})^{\pi_1(U_f, y)}$. Finally, the Fano prop-

erty and the assumptions on the singularities of the fibres guarantee that $H^2(F_y, \mathbb{Q})^{\pi_1(U_f, y)}$ is actually isomorphic to $\mathrm{NS}(F_y)_{\mathbb{Q}}^{\pi_1(U_f, y)}$. Hence we have an isomorphism given by the restriction to F_y

$$\mathrm{NS}(f^{-1}(U_f)/U_f)_{\mathbb{Q}} \cong \mathrm{NS}(F_y)_{\mathbb{Q}}^{\pi_1(U_f, y)}.$$

By definition of $\mathrm{NS}(f^{-1}(U_f)/U_f)_{\mathbb{Q}}$, we have a surjective restriction map

$$\mathrm{NS}(f^{-1}(U_f))_{\mathbb{Q}} \twoheadrightarrow \mathrm{NS}(F_y)_{\mathbb{Q}}^{\pi_1(U_f, y)}.$$

The $\mathbb{Q}$ -factoriality of X gives the surjectivity of the required restriction map. □

2.2.3 Monodromy and MMP

This subsection provides the key application of the monodromy action for the classification of the fibres of Mori fibre spaces: the idea consists in understanding which information of the Fano fibre F_y is preserved by the monodromy. Many of the results are slight modifications or extensions of the results in [dFH12].

The following data is clearly preserved by the monodromy action:

- the intersection pairing (it can be seen as an action on the algebra $H^*(F_y, \mathbb{Z})$);
- the canonical divisor K_{F_y} ;
- the $\mathbb{Q}$ -valued bilinear form $b(A, B) := (K_{F_y})^{n-2} \cdot A \cdot B$ on $\mathrm{NS}(F_y)_{\mathbb{Q}}$, where n is the dimension of F .
- [Wis91] and [Wis09] prove that the nef cone is locally constant, assuming that the fibres are smooth;

- [dFH11, Theorem 6.8] shows that movable and pseudo effective cones are preserved, assuming terminal $\mathbb{Q}$ -factorial singularities;
- [dFH11, Theorem 6.9] shows that the Mori chambers are locally constant in the following cases: three-dimensional fibres; four-dimensional and one-canonical fibres; toric fibres (in particular the nef cone is locally constant).

Remark 2.30. In [Tot12], the Mori chambers are showed not being preserved by the monodromy action in general.

Our result in this setting is the following.

Theorem 2.31 ($\heartsuit$). *Keeping the notations of Theorem 2.26 and up to shrink U_f to a new open dense subset U'_f of Y , the monodromy action preserves $\text{Nef}(F_y)$.*

Proof. After trivialising the monodromy action via an étale cover $p: V_f \rightarrow U_f$, we consider the quotient $N_1(X_{V_f})/N_1(V_f) =: N_1(X_{V_f}/V_f)$ and remark that the restriction map (with respect to f_{V_f}) to any fibre $N_1(X_{V_f}/V_f)_{\mathbb{Q}} \rightarrow N_1(F_y)_{\mathbb{Q}}$ is an isomorphism.

The Fano hypothesis guarantees that the cone of curves $\overline{\text{NE}}(X_{V_f}/V_f)$ is rational polyhedral, generated by rays $R_1, \dots, R_k$ (corresponding to classes of a curves C_i).

Kawamata's rationality theorem implies the primitive generators of $\overline{\text{NE}}(X_{V_f}/V_f)$ are integral lattice points which lie between the hyperplanes $\mathcal{H}_0 = \{w \in N_1(X/Y) \mid (K_X \cdot w) = 0\}$ and $\mathcal{H}_{2m} = \{w \in N_1(X/Y) \mid (K_X \cdot w) = 2m\}$ ($m = \dim X$). The number of integral points in $\overline{\text{NE}}(X_{V_f}/V_f)$ laying between $\mathcal{H}_0$ and $\mathcal{H}_{2m}$ is finite and its set is denoted with $\mathcal{S}$.

Now, to any class of curves in $\mathcal{S}$ we can associate the following moduli

space (quasi-projective scheme of finite type):

$$\mathcal{M} := \bigcup_{\beta \in \mathcal{S}} \text{Mor}(\mathbb{P}^1, X_{V_f}/V_f, \beta).$$

Its elements are morphisms from $\mathbb{P}^1$ to X_{V_f} , contracted by $X_{V_f} \rightarrow V_f$ and having class belonging to $\mathcal{S}$. This moduli space comes with a map to $\pi : \mathcal{M} \rightarrow V_f$. Let $\mathcal{N}$ be the union of irreducible components not dominating V_f and let $Z := \pi(\mathcal{N})$. Let us remark that Z is a Zariski closed set, since $\mathcal{M}$ is of finite type and proper over V_f .

Let us define $W_f := V_f \setminus Z$ and $X_{W_f} := f^{-1}(W)$.

We claim that $U'_f := p(W_f)$ is the required Zariski open set: focus on the (rational polyhedral) cone $\overline{\text{NE}}(X_{W_f}/W_f)$. As before, the extremal rays are finite and their generators have bounded degree. Since every component of $\mathcal{M}$ that does not belong to $\mathcal{N}$ dominates V_f , then the classes generating the extremal rays of $\overline{\text{NE}}(X_{W_f}/W_f)$ move over W_f . This implies that the restriction $N_1(X_{W_f}/W_f)_{\mathbb{Q}} \rightarrow N_1(F_y)_{\mathbb{Q}}$ identifies $\overline{\text{NE}}(X_{W_f}/W_f)$ and $\overline{\text{NE}}(F_y)$ (the inclusion $\overline{\text{NE}}(F_y) \subset \overline{\text{NE}}(X_{W_f}/W_f)$ follows directly from the definition, while the other one is a consequence of what we just said). $\square$

Remark 2.32. *Theorem 2.31 gives no clues on how to characterise this new open set U'_f .*

We will work now on the open dense subset U'_f whose existence is guaranteed by Theorem 2.31.

Consider an element $\gamma \in \pi_1(U'_f, y)$ and assume it exchanges two maximal faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of the nef cone. These faces correspond to extremal contractions

$$\pi_i : F_y \rightarrow G_i,$$

which are associated to steps of the MMP for F_y (cf. Subsection 2.2.1).

The following is the key result which, combined with Theorem 2.42, will allow us to deal, for instance, with the low dimensional cases.

Theorem 2.33 ($\heartsuit$). *Keeping the previous notation, let $y \in U'_f$ and assume that the monodromy action identifies two faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of the nef cone of F_y . Then, the two maps*

$$\pi_1 : F_y \rightarrow G_1$$

$$\pi_2 : F_y \rightarrow G_2$$

correspond to the same kind of step in the MMP.

Assuming they are both divisorial contractions, the monodromy action exchanges the exceptional divisors.

Furthermore, the varieties G_1 and G_2 (and the corresponding morphisms) are deformation equivalent.

Proof. The proof of this result is a case-by-case analysis, depending on the type of extremal contraction. Fix the dimension of the Fano variety $\dim F_y = n$.

Divisorial contraction In this case $\pi_i : F_y \rightarrow G_i$ are birational for $i = 1, 2$, so for a divisor H_i in the relative interior of $\pi_i^* \text{Nef}(G_i)$ we have that $(H_i^n) > 0$. Let E_i be the (irreducible) exceptional divisors, which are the only effective divisor on F_y such that $(H_i^{n-1} \cdot E_i) = 0$, for every H_i in the relative interior of $\pi_i^* \text{Nef}(G_i)$. One can actually characterise the dimension of $\pi_i(E_i)$ as the maximal integer k_i for which $(H_i^{k_i} \cdot E_i) \neq 0$.

Flipping contraction Also here, as in the previous case, we know that for a given a divisor H_i in the relative interior of $\pi_i^* \text{Nef}(G_i)$, the self-intersection (H_i^n) is strictly positive. Since the two contractions π_i , with $i = 1, 2$, are small, for every effective divisor $E_i \in \mathcal{F}_i$ we have $(H_i^{n-1} \cdot E_i) > 0$. Both these conditions are preserved by monodromy.

Fibre-type contraction In this case, for every divisor H_i in the relative interior of $\pi_i^* \text{Nef}(G_i)$, we have $(H_i^n) = 0$ and $h_i := \dim G_i$ is the biggest integer such that $(H_i^{h_i}) \neq 0$. In this case, for instance, we deduce that also the dimension of the base is preserved by monodromy.

Let us produce now a flat deformation of G_1 into G_2 . After a base change $p: V'_f \rightarrow U'_f$, we obtain a new morphism with trivial monodromy $f_p: X_{V'_f} \rightarrow V'_f$

This implies that the restriction morphism $\text{NS}(X_{V'_f})_{\mathbb{Q}} \rightarrow \text{NS}(F_y)_{\mathbb{Q}}$ is surjective for every fibre of f_p . Since $\mathcal{F}_1$ and $\mathcal{F}_2$ are identified on the family $X_{U'_f} \rightarrow U'_f$, we can find two point y_1, y_2 on V'_f and a divisor H on $X_{V'_f}$ whose restrictions H_1 and H_2 to the fibres F_{y_1} and F_{y_2} lie in the relative interior of $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively.

We construct now a variety

$$\tilde{X} := \text{Proj}_{\mathcal{O}_{V'_f}} \left(\bigoplus_{m \in \mathbb{N}} f_{p*} \mathcal{O}_{X_{V'_f}}(mH) \right),$$

together with a morphism

$$\begin{array}{ccc} X_{V'_f} & \xrightarrow{g} & \tilde{X} \\ & \searrow f_p & \swarrow \pi \\ & V'_f & \end{array}$$

The fibre of π over y_i is G_i and the restriction of g over the points y_i is just the contraction associated to the face $\mathcal{F}_i$.

Since H is a Cartier divisor and f_p is flat, the sheaves $\mathcal{O}_{X_{V'_f}}(mH)$ are all flat over V'_f . Moreover all fibres have terminal singularities, so $H^i(F, \mathcal{O}_F(mH|_F)) = 0$ for all fibres F , every $i > 0$ and $m > 0$: this is a consequence of our assumptions ($H|_{F_{y_1}}$ is nef and the monodromy action on $X_{U'_f} \rightarrow U'_f$ preserves the nef cone, so we have that $H|_F$ is nef for every fibre F over V'_f).

Using some classical theory of cohomology and base-change (cf. [Mum08, Corollary 2]), the sheaves $\pi_* \mathcal{O}_{X_{V'_f}}(mH)$ are all locally free. As a consequence,

$$\bigoplus_{m \in \mathbb{N}} f_{p*} \mathcal{O}_{X_{V'_f}}(mH)$$

is a flat sheaf of algebras.

The finite generation is a consequence of Castelnuovo-Mumford regularity, up to passing to a sufficiently large multiple of H (cf. [Laz04, Example 1.8.24]).

This proves that π is flat and hence G_1 is deformation equivalent to G_2 . $\square$

We can now give one of our central definitions. Let us spend few words on motivation.

Start with an isotrivial Mori fibre space (or, more generally, with a Fano fibration). The monodromy action on $\text{NS}(F)_{\mathbb{Q}}$, where F is the fibre over U'_f , factors through the automorphism group of the fibre. This is clearly not true, for instance, for a family of cubic surfaces S_3 in $\mathbb{P}^3$. Nonetheless, all cubics are diffeomorphic and one can act on the family via diffeomorphisms preserving the canonical class K_F . These diffeomorphisms also preserve other data. We remark here that the generic cubic surface has no (holomorphic) automorphisms.

Definition 2.34 (The groups HMon and Mon , $\heartsuit$). Let F be a $\mathbb{Q}$ -factorial n -dimensional Fano variety with terminal singularities. Let $\text{Aut}(F)^0$ be the subgroup of $\text{Aut}(F)$ acting trivially on the Néron-Severi group. The group $\text{HMon}(F)$ is defined as

$$\text{HMon}(F) := \text{Aut}(F) / \text{Aut}(F)^0.$$

The group $\text{Mon}(F)$ is defined as the maximal sub-group of $\text{GL}(N^1(F), \mathbb{Z})$

which preserves the following data:

- the canonical class K_F ;
- the bilinear form $b(A, B) := (K_F)^{n-2} \cdot A \cdot B$;
- the nef cone;
- the type of step of the MMP associated to the facets of the nef cone and the exceptional divisor;
- the deformation type of the images of the contraction morphisms defined by the faces of the nef cone.

Remark 2.35. *The group $\text{HMon}(F)$ is the sub-group of $\text{Mon}(F)$ of elements induced by $\text{Aut}(F)$.*

As F has only terminal singularities, it is actually a Mori dream space.

We remark that the group $\text{Mon}(F)$ is invariant under flat deformations which preserve the nef cone. On the other hand, the group $\text{HMon}(F)$ can jump when we deform the Fano variety.

Example 2.36. *To make Definition 2.34 more transparent, we discuss the rank-two case.*

In this case $\text{Mon}(F)$ can either be trivial or of order two: in fact, if the action of $\text{Mon}(F)$ is not the trivial one, then it permutes the two primitive vectors v_1, v_2 of the nef cone (this last case can occur only if the class of the canonical divisor lies on the linear span of $v_1 + v_2$ and the extremal rays of the nef cone correspond to the same kind of step in the MMP).

Again for del Pezzo surfaces (cf. Subsection 2.4.1), it is well-known that the generic del Pezzo surface of degree 3 has no automorphisms (HMon is then trivial in this case), but Mon is far from being trivial.

Remark 2.37. *When we are dealing with isotrivial fibrations (cf. Definition 2.22), $\text{Mon}(F_y)$ factors through automorphisms of F_y (the action of a loop is induced by a (non-unique) element of $\text{Aut}(F_y)$).*

Before proceeding with our main results, we restate the main result of these subsections in a slightly different form, for future reference.

Theorem 2.38 (Deligne, ♡). *Let X and Y be normal projective varieties and let*

$$f: X \rightarrow Y$$

be a projective morphism such that

1. *the fibres of f are connected (i.e. that $f_*\mathcal{O}_X = \mathcal{O}_Y$);*
2. *X is $\mathbb{Q}$ -factorial with log-terminal singularities;*
3. *on an open dense subset of Y the fibres of f are terminal and $\mathbb{Q}$ -factorial;*
4. *$-K_X$ is relatively ample over Y ;*

Then, there exists a maximal open dense subset $U = U'_f$ of Y over which the following holds:

- *the restriction morphism $f: X_U \rightarrow U$ is a flat family of $\mathbb{Q}$ -factorial Fano varieties with terminal singularities;*
- *for every $y \in U$, the monodromy action of $\pi_1(U, y)$ on $\text{NS}(F_y)_{\mathbb{Q}}$ factors through the finite group $\text{Mon}(F_y)$. The further assumption of isotriviality for f implies that the monodromy action actually factors through $\text{HMon}(F_y)$.*

Furthermore, the restriction map

$$\rho: \text{NS}(X)_{\mathbb{Q}} \rightarrow \text{NS}(F_y)_{\mathbb{Q}}^{\pi_1(U, y)}$$

is onto for every y in U .

We close this subsection with the key definition we need for our purposes.

Definition 2.39 (Fibre-like, $\heartsuit$). A $\mathbb{Q}$ -factorial Fano variety F with terminal singularities is *fibre-like* if it can be realised as a fibre of a Mori fibre space $f: X \rightarrow Y$ over the open dense subset U'_f appearing in Theorem 2.38.

2.3 Main Results

This is the section in which the fundamental results of this project are presented. Our ultimate goal would be to classify all $\mathbb{Q}$ -factorial Fano varieties with log terminal singularities (or at least smooth) which are fibre-like.

Classification of smooth Fano varieties up to deformation have been completed only up to dimension three (cf. [MM82] and [MM03]). Classification in dimension four is in progress and many interesting results have recently appeared (cf. [CGKS14] and [CKP14]). Nonetheless, the full classification in dimension four requires a methodical (and, in some sense, “philosophical”) recourse to computer algebra.

2.3.1 Criteria of fibre-likeness

In this section we present two criteria, one sufficient and one necessary, to detect fibre-likeness of terminal Fano varieties. If the Fano variety is rigid in the sense of deformation theory, we deduce a characterisation.

The sufficient criterion gives an explicit construction which produces an isotrivial Mori fibre space with generic fibre isomorphic to the given terminal Fano variety.

The necessary criterion heavily depend on Theorem 2.38.

Theorem 2.40 (Sufficient Criterion, $\heartsuit$). *Let F be a $\mathbb{Q}$ -factorial Fano variety with terminal singularities for which*

$$\mathrm{NS}(F)_{\mathbb{Q}}^{\mathrm{Aut}(F)} = \mathbb{Q}[K_F].$$

Then F is fibre-like.

Moreover, F can be realised as the general fibre of an isotrivial Mori fibre space $f : X \rightarrow Y$ with $\dim Y = 1$.

Remark 2.41. *Before giving the proof, let us remark that K_F is always fixed by $\mathrm{Aut}(F)$. So, in general, $\dim \mathrm{NS}(F)_{\mathbb{Q}}^{\mathrm{Aut}(F)} \geq 1$. In particular, if we fix a positive integer m for which $-mK_F$ is very ample, the action of $\mathrm{Aut}(F)$ lifts faithfully to a linear action on the linear system $| -mK_F |$.*

So we are asking, for the sufficient criterion, the subspace of $\mathrm{NS}(F)_{\mathbb{Q}}$ preserved by $\mathrm{Aut}(F)$ to be minimal.

Proof. The nef cone $\mathrm{Nef}(F)$ of F is rational polyhedral and $\mathrm{HMon}(F)$ permutes the primitive vectors on the boundary. This implies that $\mathrm{HMon}(F)$ is finite. We can also assume that $\mathrm{HMon}(F)$ is not trivial: if $\mathrm{HMon}(F) = \{id\}$ the hypothesis implies that

$$\dim \mathrm{NS}(F)_{\mathbb{Q}} = 1$$

and in this case F is trivially fibre-like.

Let $[f_1], \dots, [f_g]$ ($g \geq 1$) be elements of $\mathrm{HMon}(F)$ and call G the subgroup of $\mathrm{Aut}(F)$ generated by $f_1, \dots, f_g$. Consider a genus- g curve C and denote by a_i and b_i the generators of $\pi_1(C, y)$, with base point $y \in C$. Since there is a unique relation between the a_i 's and b_i 's given by the commutator

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = 1,$$

we can define a representation

$$\begin{aligned} \rho: \quad \pi_1(C, y) &\rightarrow G. \\ a_i &\mapsto f_i \\ b_i &\mapsto f_i^{-1} \end{aligned}$$

Let $\hat{C}$ be the universal cover of C . We define our total space as

$$X := F \times \hat{C} / \pi_1(C, t),$$

where $\pi_1(C, y)$ acts on F via the map ρ . The action of $\pi_1(C, y)$ is free and properly discontinuous, hence X is an analytic space with the same singularities as F . We claim that the Mori fibre space we are looking for is

$$f : X \rightarrow C.$$

It is clearly an isotrivial fibration and the fibre is isomorphic to the given Fano variety F .

Let us prove that X is projective. Denote by $\phi_{|-mK_F|} : F \rightarrow \mathbb{P}^N$, $N = \dim |-mK_F|$, the embedding induced by the pluri-anticanonical system $|-mK_F|$, with $m > 0$ sufficiently large and divisible. Remark 2.41 implies that the action of $\pi_1(C, y)$ extends also to $\mathbb{P}^N$ and that the morphism $\phi_{|-mK_F|}$ is equivariant with respect to the action. This gives a commutative diagram

$$\begin{array}{ccc} F \times \hat{C} & \xrightarrow{\phi_{|-mK_F|} \times id_{\hat{C}}} & \mathbb{P}^N \times \hat{C} \\ \downarrow & & \downarrow \\ X = F \times \hat{C} / \pi_1(C, y) & \xrightarrow{\psi} & \mathbb{P}^N \times \hat{C} / \pi_1(C, y) = Z \\ \downarrow f & & \downarrow g \\ C & \xrightarrow{id} & C \end{array}$$

The singularities of Z are the same as $\mathbb{P}^N$ and so Z is smooth. Further-

more, since the action of $\text{Aut}(F)$ is contravariant for $\phi|_{-mK_F} \times \text{id}_{\hat{C}}$, Z maps to C and every fibre is isomorphic to the projective space $\mathbb{P}^N$. In particular, the anticanonical sheaf $\mathcal{O}_Z(-K_Z)$ is relatively ample over C . But C is projective, so Z is projective, too.

The $\mathbb{Q}$ -factoriality of X is clear from the construction (the fibration is isotrivial).

We have to prove that $f: X \rightarrow C$ has relative Picard number one. Since every curve has Picard number one, we have to prove that the Picard number of X is two. This is a consequence of the exactness of the following short sequence:

$$0 \rightarrow \text{NS}(C)_{\mathbb{Q}} \xrightarrow{f^*} \text{NS}(X)_{\mathbb{Q}} \xrightarrow{\iota^*} \text{NS}(F_y)_{\mathbb{Q}}^G \rightarrow 0.$$

The fibres of f are connected, so f^* is injective and the surjectivity of ι^* is trivial, since the vector space $\text{NS}(F_y)_{\mathbb{Q}}^G$ is one dimensional, generated by $-K_{F_y}$ (adjunction implies that $\iota^*K_X = K_{F_y}$). Clearly, $\text{im } f^* \subseteq \ker \iota^*$. The other inclusion is not trivial: it is actually very easy to produce examples for which the previous sequence is not exact (the product of two elliptic curves has three-dimensional Néron-Severi group).

To show that $\text{im } f^* \supseteq \ker \iota^*$, we pick a line bundle L on X in the kernel of ι^* and show it is actually the pullback of a line bundle on the base. Let $z \neq y$ be a point of C and let us show that $L|_{F_z}$ is trivial.

First, the restriction $L|_{F_q}$ has trivial Chern class: $c_1(L)$ is a section of the locally constant sheaf $R^2f_*\mathbb{Q}$ so if it is zero on a neighbourhood of y , it is trivial. For a contractible neighbourhood U_y of y we consider $V_{F_y} := f^{-1}(U_y)$, which is a contractible neighbourhood of the fibre F_y . This implies $c_1(L|_{V_{F_y}}) = c_1(L|_{F_y}) = 0$.

The Fano assumption on the fibres implies the triviality of the restriction line bundle (remark we are working over $\mathbb{Q}$, so we have no torsion).

In our setting, we have a flat morphism

$$f: X \rightarrow C$$

and a line bundle L on X which is trivial on every fibre. The see-saw principle (cf. [Mum08, Corollary 6, p. 54], [KM92, Proposition 12.1.4]) implies it is actually the pullback of a line bundle on the base. $\square$

We can now state the necessary criterion: this will be useful to show, for instance, that the majority of Fano varieties in low dimension with high Picard rank cannot be fibre-like.

Theorem 2.42 (Necessary Criterion, $\heartsuit$). *Let F be a $\mathbb{Q}$ -factorial Fano variety with terminal singularities. If*

$$\dim \mathrm{NS}(F)_{\mathbb{Q}}^{\mathrm{Mon}(F)} > 1,$$

F is not fibre-like.

Proof. This proof is by contradiction. Assume there exists a Mori fibre space

$$f: X \rightarrow Y$$

for which F is realised as a fibre over the open set U'_f . Furthermore, the restriction map

$$\rho: \mathrm{NS}(X)_{\mathbb{Q}} \rightarrow \mathrm{NS}(F_y)_{\mathbb{Q}}^{\pi_1(U'_f, y)}$$

is onto (F_y is a fibre over U'_f isomorphic to F).

We need to show, to get a contradiction, that $\mathrm{NS}(F_y)_{\mathbb{Q}}^{\pi_1(U'_f, y)}$ is one dimensional. For this purpose, we use the exact sequence

$$0 \rightarrow \mathrm{NS}(Y)_{\mathbb{Q}} \xrightarrow{f^*} \mathrm{NS}(X)_{\mathbb{Q}} \xrightarrow{\rho} \mathrm{NS}(F_y)_{\mathbb{Q}}^{\pi_1(U'_f, y)} \rightarrow 0$$

(exactness is a consequence of the relative Picard rank condition of MFSs).

This implies that

$$\dim \mathrm{NS}(F_y)_{\mathbb{Q}}^{\pi_1(U'_f, y)} = 1.$$

Theorem 2.38, which guarantees that the monodromy action of $\pi_1(U'_f, y)$ on $\mathrm{NS}(F_y)_{\mathbb{Q}}$ factors through the group $\mathrm{Mon}(F)$, implies that

$$\dim \mathrm{NS}(F_y)_{\mathbb{Q}}^{\mathrm{Mon}(F_y)} = 1.$$

Contradiction. □

The following is the characterisation for rigid Fano varieties.

Theorem 2.43 (Characterisation - Rigid case, ♡). *Let F be a rigid $\mathbb{Q}$ -factorial Fano variety with terminal singularities. Then F is fibre-like if and only if*

$$\mathrm{NS}(F)_{\mathbb{Q}}^{\mathrm{Aut}(F)} = \mathbb{Q}K_F.$$

Furthermore, F can be realised as the general fibre of an isotrivial MFS over a curve.

Proof. The “if” part is Theorem 2.40. and the “only if” part follows from Theorem 2.42, since the monodromy action factors through $\mathrm{HMon}(F)$. □

Remark 2.44. *If F is not rigid this characterisation does not hold (a counterexample is the del Pezzo surface of degree three, cf. Subsection 2.4.1).*

2.3.2 Necessary criterion and the MMP

In this subsection we give handy corollaries of Theorem 2.42 which allow us to deal with the applications of our criteria, since the group $\mathrm{Mon}(F)$ itself is very difficult to describe.

This group, roughly speaking, is the group of symmetries of the nef cone preserving some other birational features of F , so we want to rephrase the

necessary criterion in explicit terms of the birational geometry of the Fano variety F .

The basic idea is that, if there are faces of the nef cone of F which are “different” from the viewpoint of birational geometry, then $\mathrm{NS}(F)_{\mathbb{Q}}^{\mathrm{Mon}(F)}$ cannot be one-dimensional. As a warm-up, we start with an example assuming F with Picard number 2: $\mathrm{Nef}(F)$ has two faces, $\mathcal{F}_1$ and $\mathcal{F}_2$, each of which gives a contraction $\pi_i: F \rightarrow G_i$.

The following is a corollary of Theorem 2.42.

Corollary 2.45 ($\heartsuit$). *Let F be a terminal Fano variety with Picard rank two. Keeping the notation as before, assume that*

$$\dim G_1 \neq \dim G_2.$$

Then F is not fibre-like.

Proof. Since $\mathrm{Mon}(F)$ cannot exchange $\mathcal{F}_1$ and $\mathcal{F}_2$, it must be trivial. In particular the invariant part of $\mathrm{NS}(F)_{\mathbb{Q}}$ will be two-dimensional. $\square$

This is just an example of application of Theorem 2.42. Similar corollaries can be easily obtained as follows.

Corollary 2.46 ($\heartsuit$). *Let F be a Fano variety obtained via the blowup of another Fano variety G . Assume there are no other facets of $\mathrm{Nef}(F)$ whose associated contraction is divisorial with image a variety deformation equivalent to G . Then F is not fibre-like.*

In this statement, we are saying that F is isomorphic in a “unique way” to the blowup of another deformation type of Fano variety G . The type of an extremal contraction and the deformation type of its image are preserved under the action of $\mathrm{Mon}(F)$ because of Theorem 2.33, so this uniqueness implies that the associated morphisms must be preserved by this action. Let us give a proof.

Proof of Corollary 2.46. We know that the face of $\text{Nef}(F)$ corresponding to the pullback of $\text{Nef}(G)$ is $\text{Mon}(F)$ -invariant. We need now to provide a fixed point on this face and, consequently, a fixed one-dimensional subspace. Assume by contradiction that F is fibre-like. Then the only subspace preserved by $\text{Mon}(F)$ is the one generated by the canonical class. This class cannot lay on the pullback of $\text{Nef}(G)$. As we explained above, if $\text{Nef}(G)$ is stable by $\text{Mon}(F)$, then the class of the exceptional divisor is fixed as well. Contradiction. $\square$

We show now how one can generalise the previous results, via some examples.

Example 2.47. *Assume F has a unique fibre-type contraction to a variety G , with $\dim G = k > 0$. Then the associated face of $\text{Nef}(F)$ corresponds to the nef cone of G and is $\text{Mon}(F)$ -stable. In particular, the primitive generators of the extremal rays in the lattice $\text{NS}(F) \subset \text{NS}_{\mathbb{R}}(F)$ of this face are exchanged by $\text{Mon}(F)$ and their sum will be $\text{Mon}(F)$ -invariant. This implies that F cannot be fibre-like.*

This happens, for instance, for the projectivisation of the vector bundle associated to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1)$. Recall that this Fano variety F is isomorphic to the blow-up of the cone over a smooth quadric in $\mathbb{P}^3$ with center the vertex. F has Picard rank three and the facets of $\text{Nef}(F)$ are given by the fibre-type contraction $F \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ and the two divisorial contractions $F \rightarrow F_i$, $i = 1, 2$, given by contracting the two rulings.

To sum up, we proved the following more general result.

Corollary 2.48 ($\heartsuit$). *Let F be a Fano variety and assume that $\text{Nef}(F)$ contains a facet $\mathcal{G}$ corresponding to a certain variety G . Assume that for any other facet $\mathcal{H}$ of the nef cone, the corresponding variety H is not deformation equivalent to G . Then, F is not fibre-like.*

Let us focus now on facets that move via the $\text{Mon}(F)$ -action.

Let F be a Fano variety and $\mathcal{F}$ be a facet in the nef cone of F . Assume one obtains other faces $\mathcal{F}_1, \dots, \mathcal{F}_k$ acting with $\text{Mon}(F)$ on $\text{Nef}(F)$. Let $L, L_1, \dots, L_k$ be the sum of the primitive generators of the extremal rays spanning the faces $\mathcal{F}, \mathcal{F}_1, \dots, \mathcal{F}_k$. It is clear that the L_i 's are the orbits of L under the $\text{Mon}(F)$ -action. Hence, $L + L_1 + \dots + L_k$ is $\text{Mon}(F)$ -invariant and, in order for F to be fibre-like, it has to be a negative multiple of K_F , in particular it must be an ample class.

If the face $\mathcal{F}$ corresponds to a divisorial contraction, then the same applies to show that the sum $E + E_1 + \dots + E_k$ of the exceptional divisors must be a multiple of $-K_F$, otherwise F will not be fibre-like.

Example 2.49. *Consider the blow-up of the cone over a smooth quadric $Q \subset \mathbb{P}^3$ with center the vertex and an elliptic curve C on Q . One can see that this Fano variety F can be realised as the blow-up of $\mathbb{P}_Q(\mathcal{O}_Q \oplus \mathcal{O}_Q(1, 1))$ along an elliptic curve C' in E , where E is the negative section of $\mathbb{P}_Q(\mathcal{O}_Q \oplus \mathcal{O}_Q(1, 1))$. Let $\pi : F \rightarrow Q$ the resulting morphism.*

Clearly, the generic fibre of π is $\mathbb{P}^1$, but over $C \subset Q$ the fibres are chains of two $\mathbb{P}^1$'s intersecting in a point. F has exactly two different divisorial contractions $\psi_i : F \rightarrow \mathbb{P}_Q(\mathcal{O}_Q \oplus \mathcal{O}_Q(1, 1))$, $i = 1, 2$, given by contracting the two components of the fibres of π over C .

Assume by contradiction that F is fibre-like, then the sum of the exceptional divisors for the ψ_i 's must be ample. But this is false since such sum has intersection 0 with the generic fibre of π .

2.3.3 Another point of view

This subsection is aimed to rephrase our problem in a more algebraic setting, in order to deal, for instance, also with variety in positive characteristic and to recall some characteristic-free techniques which apply in our setting.

Varieties we are dealing with here are defined over an arbitrary algebraically closed field.

Monodromy and the Fano condition in positive characteristic. We start with monodromy. Consider a projective morphism $f: X \rightarrow Y$ between normal varieties with connected fibres and assume X to be $\mathbb{Q}$ -factorial with rational singularities. Furthermore, assume that $-K_X$ is f -ample and that there exists an open dense subset U_f^{top} over which f is locally topologically trivial (this is guaranteed in characteristic zero by the results of Verdier). The first step of our approach (cf. Theorem 2.29) consisted in working out the monodromy action on the Néron-Severi group, using the isomorphisms of the first Chern class given by the exponential sequence:

$$R^1 f_* \mathbb{G}_m \otimes \mathbb{Q} \rightarrow R^2 f_* \mathbb{Q}.$$

Let us pick a variety Z over an algebraically closed field k of arbitrary characteristic. Consider, for any prime $l \neq \text{char } k$ and n positive integer, the *Kummer exact sequence* (in the étale topology):

$$0 \rightarrow \mu_{l^n} \rightarrow \mathbb{G}_m \xrightarrow{\cdot l^n} \mathbb{G}_m \rightarrow 0. \quad (2.2)$$

Passing to cohomology, we obtain the following:

$$0 \rightarrow \text{Pic}(X)_{\mathbb{Z}/l^n \mathbb{Z}} \rightarrow H_{\text{ét}}^2(X, \mu_{l^n}) \rightarrow \text{Br}(X)_{l^n} \rightarrow 0, \quad (2.3)$$

where $\text{Br}(X) := H_{\text{ét}}^2(X, \mathbb{G}_m)$ is the (cohomological) Brauer group of X .

We can now take the inverse limit of (2.3) with respect to n and obtain

$$0 \rightarrow \text{NS}(X)_{\mathbb{Z}_l} \rightarrow H_{\text{ét}}^2(X, \mathbb{Z}_l(1)) \rightarrow T_l \text{Br}(X) \rightarrow 0, \quad (2.4)$$

where $T_l \operatorname{Br}(X) := \varprojlim \operatorname{Br}(X)_{l^n}$. Furthermore we used the following fact:

$$\operatorname{Pic}(X)_{\mathbb{Z}_l} \simeq \operatorname{NS}(X)_{\mathbb{Z}_l}.$$

We can recover basically the same result of Theorem 2.29 in this general setting, assuming, for instance, that

$$\operatorname{Br}(F_y) = 0,$$

for every fibre over U_f^{top} .

Remark 2.50. *This condition on the Brauer group is known to hold for smooth rational surfaces in arbitrary characteristic. Furthermore Bright proved in [Bri13] that the same holds for singular del Pezzo surfaces of high degree over an arbitrary algebraically closed field. This will be used to classify fibre-like singular del Pezzo surfaces in Subsection 2.4.6.*

An alternative approach to fibre-likeness. Once again, consider a projective morphism $f: X \rightarrow Y$, where X and Y are defined over an arbitrary algebraically closed field with $-K_X$ f -ample. Consider the geometric generic fibre $X_{\overline{K}}$ of f , where K is the function field $k(Y)$ of the base Y and assume it is smooth. One has the natural action of the Galois group $G := \operatorname{Gal}(\overline{K}/K)$ on $\operatorname{Pic}(X_{\overline{K}})$. Furthermore a natural exact sequence is defined (cf. [CTS87, Proposition 2.1]):

$$0 \rightarrow \operatorname{Pic}(X_K) \rightarrow \operatorname{Pic}(X_{\overline{K}})^G \xrightarrow{\star} \operatorname{Br}(K).$$

Assuming the existence of a K -point on X_K (i.e. of a section of f), the map $\star$ is the zero map and one can deduce an isomorphism

$$\operatorname{Pic}(X_K) \simeq \operatorname{Pic}(X_{\overline{K}})^G.$$

This isomorphism holds true, for instance, if Y is a smooth curve, by [GHS03] in characteristic zero (and [dJS03] in positive characteristic). Assuming the previous isomorphism, one has the following short exact sequence:

$$0 \rightarrow \mathrm{Pic}(Y)_{\mathbb{Q}} \rightarrow \mathrm{Pic}(X)_{\mathbb{Q}} \rightarrow \mathrm{Pic}(X_{\overline{K}})^G \rightarrow 0. \quad (2.5)$$

The advantage of this approach is the following: let $\mathrm{Nef}(X_{\overline{K}})$ be the nef cone of the generic fibre. Since G acts on $X_{\overline{K}}$ via automorphisms, this cone is preserved. In this setting one could work directly on the generic geometric fibre and deduce basically the same results of the previous subsections, at least for MFS over a curve.

The following is a working definition.

Definition 2.51. A $\mathbb{Q}$ -factorial Fano variety F with rational singularities over an algebraically closed field is *algebraically fibre-like over a curve* if there exists a MFS with Y a smooth curve and its generic geometric fibre $X_{\overline{K}}$, where $K := k(Y)$, is isomorphic as abstract scheme (i.e. as a scheme over $\mathrm{Spec} \mathbb{Z}$) to F .

The previous discussion gives the following result (observe that one can give the same definition of the group $\mathrm{Mon}(F)$ in this more general setting, cf. Definition 2.34).

Theorem 2.52. *A $\mathbb{Q}$ -factorial Fano variety F with rational singularities over $\mathbb{C}$ is algebraically fibre-like over a curve if and only if*

$$\dim \mathrm{NS}(F)_{\mathbb{Q}}^{\mathrm{Mon}(F)} = 1.$$

Proof. The “if” part can be obtained via the same construction of an isotrivial fibration $f: X \rightarrow C$ over a curve of Theorem 2.40. To prove that the obtained fibration is actually a MFS one considers the short exact sequence (2.5): we know by construction that $X_{\overline{K}}$, where $K := k(C)$, is isomorphic to

F as abstract scheme, by [Via13, Lemma 2.1] (this lemma is due to Totaro). In particular the Néron-Severi groups are isomorphic and the action of the Galois group on $\mathrm{NS}(F)_{\mathbb{Q}}$ factors through $\mathrm{Mon}(F)$.

The “only if” part is also a consequence of the short exact sequence (2.5).

□

2.4 Applications

2.4.1 Smooth del Pezzo surfaces

In this subsection we focus on two-dimensional smooth Fano varieties. This case was studied in [Mor82]. We provide a proof using explicit constructions of Mori fibre spaces and applying our criteria. We start with some notation.

Definition 2.53. A *del Pezzo surface* S is a smooth Fano variety of dimension two. A del Pezzo surface of degree $d := (K_S)^2$ obtained as the blow-up of $\mathbb{P}^2$ in $9 - d$ general points is denoted by S_d ($1 \leq d \leq 8$).

We refer to [Kol96, Section III.3], for the general theory of del Pezzo surfaces. The proof of the following result is obtained as a combination of Theorems 2.56, 2.59 and 2.60.

Theorem 2.54 (Mori, ♡). *A del Pezzo surface S is fibre-like if and only if it is isomorphic to $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or S_d , with $d \leq 6$.*

By comparing our result with the classification of K -stable smooth del Pezzo surfaces, we obtain the following corollary (cf. [CDS15] for the notion of K -stability for Fano varieties and the connection with the existence of Kähler-Einstein metrics).

Corollary 2.55 (♡). *A smooth del Pezzo surface is K -stable if and only if it is fibre-like.*

Before applying our criteria to surfaces, we give some explicit constructions of Mori fibre spaces whose general fibre is a del Pezzo of low degree ($1 \leq d \leq 4$).

Theorem 2.56 ($\heartsuit$). *The following del Pezzo surfaces are fibre-like:*

- $\mathbb{P}^2$;
- $\mathbb{P}^1 \times \mathbb{P}^1$;
- S_d , with $d = 3, 4$;
- the general S_d , with $d = 1, 2$.

The 2-dimensional projective space has a trivial structure of Mori fibre space: it is enough to consider the morphism $\mathbb{P}^2 \rightarrow \{pt\}$.

In the case of $\mathbb{P}^1 \times \mathbb{P}^1$, we consider a smooth hypersurface X_2 of degree $(2, m)$ with $m \geq 1$ in $\mathbb{P}^3 \times \mathbb{P}^1$. This gives a fibration in quadrics $f_2: X_2 \rightarrow \mathbb{P}^1$. Assuming $m \geq 1$, the ampleness of X_2 is guaranteed and we can apply Lefschetz hyperplane theorem to deduce that $\rho(X_2) = 2$. This implies that f_2 is a Mori fibre space, with the generic fibre isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. This MFS is isotrivial over the open set over which f_2 is a submersion.

As a second example, we consider the case of cubics in $\mathbb{P}^3$. Fix a del Pezzo S of degree three. Consider a smooth hypersurface X_3 of degree $(3, m)$, with $m \geq 1$, in $\mathbb{P}^3 \times \mathbb{P}^1$. This gives a fibration in degree-three del Pezzo surfaces $f_3: X_3 \rightarrow \mathbb{P}^1$. We can furthermore assume that one of the fibre is isomorphic to S . Arguing as before, we can show that this gives an example of MFS. Observe that this MFS is far from being isotrivial, indeed there are just finitely many fibres isomorphic to S . The general fibre will be isomorphic to S just as a topological space. Nonetheless, there will be finitely many singular fibres not even homeomorphic to S .

Let us now recall the following classical description (cf. [Kol96, Theorem III.3.5]).

Theorem 2.57 (★). *Every del Pezzo of degree d can be realised as*

- *a hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$ if $d = 1$;*
- *a hypersurface of degree 4 in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$ if $d = 2$;*
- *a hypersurface of degree 3 in $\mathbb{P}^3$, if $d = 3$;*
- *a complete intersection of degree $(2, 2)$ in $\mathbb{P}^4$ if $d = 4$.*

The degree-four del Pezzo can be treated as the degree 3. With the other 2 cases there is an issue: classical Lefschetz hyperplane theorem does not hold, because the ambient space is not smooth. We can use the following generalisation (Grothendieck-Lefschetz hyperplane theorem, cf. [RS06]).

Theorem 2.58 (Ravindra, Srinivas). *Let Z be a normal irreducible projective variety and let $|H|$ be an ample base point free linear system on Z . Then for a general $X \in |H|$, the restriction map*

$$Cl(Z) \rightarrow Cl(X)$$

is an isomorphism, if $\dim Z \geq 4$.

We can now carry out again the same construction, but we need a genericity assumption on the surface (we think we could get rid of it by careful analysis of the singularities).

This method does not help for higher dimensional Fano varieties: if a Fano variety F of dimension strictly greater than 2 is a complete intersection in an ambient space of Picard number one, then also the Picard number of F is one. This already means that F is fibre-like, so there is nothing to prove. However, this strategy could be used to study singular del Pezzo surfaces.

We apply now our criteria to the two-dimensional case.

Theorem 2.59 ($\heartsuit$). *The del Pezzo surfaces S_7 and S_8 are not fibre-like.*

Proof. In both cases the del Pezzo is rigid because a configuration of one or two points in $\mathbb{P}^2$ is unique up to projectivity, so we can use Theorem 2.43.

Let us first consider the blow up at one point. The automorphism group is given by all the projectivities which fix a marked point. This group preserves both the line bundle given by the hyperplane coming from $\mathbb{P}^2$ and the exceptional divisor.

Now we consider the blow up at two points. The automorphism group is given by all the projectivities which fix the marked points or exchange them. This group preserves the line bundle given by the hyperplane coming from $\mathbb{P}^2$ and the sum of the two exceptional divisors. $\square$

On the positive side, our sufficient criterion gives the following.

Theorem 2.60 ($\heartsuit$). *The following del Pezzo surfaces are fibre-like:*

- (rigid case) $\mathbb{P}^1 \times \mathbb{P}^1$ and S_d with $d = 5, 6$;
- (moduli case) S_d with $d = 1, 2, 4$.

Proof. Automorphisms groups of 2-dimensional Fanos have been widely studied (cf. [Koi88] and [DI09]), so we can easily apply Theorem 2.40.

$\mathbb{P}^1 \times \mathbb{P}^1$ The natural involution of the product $\mathbb{P}^1 \times \mathbb{P}^1$ guarantees that the invariant part of the Néron-Severi group has dimension 1.

$d = 1, 2$ Every del Pezzo surface obtained as the blow-up of $\mathbb{P}^2$ in 8 or 7 points comes with an involution: respectively the Bertini ($d = 1$) and Geiser ($d = 2$) involutions, denoted by ι_1 and ι_2 , which emerge when realising S_d as a $2 : 1$ cover of $\mathbb{P}(1, 1, 2)$ (resp. $\mathbb{P}^2$) if $d = 1$ (resp. $d = 2$). The automorphism groups of these del Pezzos can be much more complicated,

but these two involutions are enough to minimise the invariant Picard rank of S_i : for $i = 1, 2$ we have $\rho(S_i)^{\iota_i} = 1$.

$d = 4$ The automorphism group of all the del Pezzo surfaces in this class contains a subgroup isomorphic to $\mathbb{Z}_2^4$. The quotient is the projective plane, so the invariant part of the Picard group is one dimensional.

$d = 5, 6$ The automorphism group of every del Pezzo of degree 5 is isomorphic to $\mathbb{S}_5$; while $\text{Aut}(S_6) \simeq \mathbb{PGL}(3; P_1, P_2, P_3) \rtimes \mathbb{Z}_2$. In particular, in both cases we can permute the exceptional divisors using projectivities from $\mathbb{P}^2$. The Cremona involution centred at 3 blown-up point is a regular automorphism of the surface. This Cremona transformation does not fix the sum of exceptional divisors, so the invariant part of the Picard group is one dimensional.

□

The generic del Pezzo of degree 3 has no non-trivial automorphisms (cf. [Seg42]). Because of this, we can not apply Theorem 2.40. Nonetheless, we showed that S_3 is fibre-like.

2.4.2 Complete intersections

This is a key application of our criteria: the idea is to consider Fano varieties which can be realised as complete intersections in rigid spaces and study the action of the automorphisms of the ambient space on the system of Fano varieties.

Since the general construction is complicated and the key idea already appears in some easy examples, we start with a preliminary discussion.

Let F be a smooth divisor of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ and let σ be the involution of $\mathbb{P}^2 \times \mathbb{P}^2$. Define the projective space $\mathbb{P}^N := \mathbb{P}H^0(\mathbb{P}^2 \times$

$\mathbb{P}^2, \mathcal{O}(2, 2))^\vee$ and consider the incidence variety I in $\mathbb{P}^N \times \mathbb{P}^2 \times \mathbb{P}^2$ (it is simply a smooth divisor of degree $(1, 2, 2)$ in $\mathbb{P}^N \times \mathbb{P}^2 \times \mathbb{P}^2$). We know by Lefschetz hyperplane theorem that $\rho(I) = 3$ and we consider the fibre space $\pi: I \rightarrow \mathbb{P}^N$, whose relative Picard number is two. Acting with σ on π and defining $X := I/\sigma$ and $Y = \mathbb{P}^N/\sigma$ we obtain another fibration

$$f: X \rightarrow Y,$$

with relative Picard number one. The singularities are finite quotient singularities, they are log-terminal and $\mathbb{Q}$ -factorial by [KM98, Proposition 5.15 and Corollary 5.21]. This implies that f is a MFS. Up to acting on F with a generic element of $\mathbb{P}GL(3) \times \mathbb{P}GL(3)$, we can always assume that F is not preserved by σ ; hence the action of σ is free on a neighbourhood of F in Z . This means that F is a smooth fibre of f . In this way we have produced, for *every* smooth threefold defined by a bidegree $(2, 2)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ a MFS which makes it fibre-like.

We give now the general construction, starting with an arbitrary smooth Fano variety F , defined as a complete intersection of multi-degree $(L_1, \dots, L_k)$ in a smooth ambient variety Z , where the L_i 's are base-point-free line bundles on Z . Let I the incidence variety in $Z \times |L_1| \times \dots \times |L_k|$ (the smoothness of I is a consequence the base-point-freeness).

Lemma 2.61 ($\heartsuit$). *The restriction map*

$$\rho: \text{Pic}(Z \times |L_1| \times \dots \times |L_k|) \rightarrow \text{Pic}(I)$$

is onto.

Proof. Since the fibres of $\pi: I \rightarrow Z$ are divisors of multi-degree $(1, \dots, 1)$ in $|L_1| \times \dots \times |L_k|$, they have the same dimension. This implies that the morphism π is flat. Let us fix a point $z \in Z$ and let I_z be the corresponding

fibre for π . We claim that the following sequence is exact:

$$\mathrm{Pic}(Z) \rightarrow \mathrm{Pic}(I) \rightarrow \mathrm{Pic}(I_z).$$

This is basically the same as the last part of the proof of Theorem 2.40: let L be a line bundle on I whose restriction to I_z is trivial. Since $R^2\pi_*\mathbb{Q}$ is locally constant on Z , we have that the Chern class of L is trivial on every fibre. The Fano assumption on the fibres implies that L is actually trivial on every fibre. The map π is flat, so we conclude by the See-Saw principle.

Now, since the image of $\mathrm{Pic}(Z \times |L_1| \times \cdots \times |L_k|)$ contains the image of $\mathrm{Pic}(Z)$ and is surjective to $\mathrm{Pic}(I_z)$ we conclude. $\square$

We prove now our main result in this setting.

Theorem 2.62 ($\heartsuit$). *Let G be a finite cyclic subgroup of $\mathrm{Aut}(Z)$ which is fix-point-free in codimension one and whose action can be lifted to I . Assume that G does not preserve F and that*

$$\dim \mathrm{NS}(Z)_{\mathbb{Q}}^G = 1.$$

Then F is fibre-like.

Proof. Fix a lifting of the G -action to I . We will produce an explicit construction of MFS. Let us define $X := I/G$ and $Y := (|L_1| \times \cdots \times |L_k|)/G$. We prove that the fibration

$$f: X \rightarrow Y$$

has relative Picard number one. The total spaces of $|L_i|$'s are projective spaces, so

$$\mathrm{Pic}(Z \times |L_1| \times \cdots \times |L_k|) = \mathrm{Pic}(Z) \times \mathrm{Pic}(|L_1|) \times \cdots \times \mathrm{Pic}(|L_k|)$$

and Lemma 2.61 combined with the hypothesis implies that

$$\begin{aligned} \dim \mathrm{NS}(X)_{\mathbb{Q}} &= \dim \mathrm{NS}(I)_{\mathbb{Q}}^G \\ &\leq \dim \mathrm{NS}(|L_1| \times \cdots \times |L_k|)_{\mathbb{Q}}^G + 1 = \dim \mathrm{NS}(Y)_{\mathbb{Q}} + 1. \end{aligned}$$

The variety F is a smooth fibre of f (it is not fixed by G). Furthermore, the action of G on Z is fix-point-free in codimension one so the singularities of X and Y are $\mathbb{Q}$ -factorial and log-terminal by [KM98, Proposition 5.15 and Corollary 5.21]. $\square$

Observe that unfortunately it is not easy to check if the singularities are terminal, as the remark after [KM98, Corollary 5.21] explains.

Our result has some immediate applications, which will be useful in the following subsections.

Corollary 2.63 ($\heartsuit$). *Let r, k, d be positive integers and $n \geq 2$ such that $kd < n + 1$. Assume F to be a smooth complete intersection of k divisors of degree $(d, \dots, d)$ in $(\mathbb{P}^n)^r$. Then F is fibre-like.*

Proof. Let G be a cyclic subgroup of the symmetric group $\mathcal{S}_r$, which transitively acts on $(\mathbb{P}^n)^r$ permuting the factors. By acting with a general automorphism of $(\mathbb{P}^n)^r$ we can arrange that G does not fix F . So Theorem 2.62 applies. $\square$

Remark 2.64. *The Mori fibre space one constructs here will in general depend on the linearisation of G . For instance, in the previous example, if G acts trivially on the linear system, the base Y will be a product of projective spaces. If the lifting is not trivial, the base could be a singular variety with small Picard rank. One can also choose a smaller G (it is sufficient that its action is transitive on the copies of $\mathbb{P}^n$ and fix-point-free in codimension one on $(\mathbb{P}^n)^r$).*

Another easy corollary is the following.

Corollary 2.65 ($\heartsuit$). *Let r and $n \geq 2$ be two fixed integers and let L_i be the line bundle*

$$\mathcal{O}(1, 1, \dots, 1, 0, 1, \dots, 1)$$

on $(\mathbb{P}^n)^r$, where the 0 appears only at the i -th position. Every smooth Fano complete intersection of multi-degree $(L_1, \dots, L_r)$ in $(\mathbb{P}^n)^r$ is fibre-like.

One can prove analogous corollary basically the same way. See Subsection 2.4.3 for further applications.

2.4.3 Smooth Fano Threefolds

In this subsection we present the main application in low dimension we can obtain with our methods: the full classification of fibre-like smooth Fano threefolds.

The complete classification of smooth Fano threefolds with high Picard rank was completed by Mori and Mukai in [MM82] and [MM03]. Their main result is the following.

Theorem 2.66 (Mori-Mukai). *Smooth Fano threefold with $\rho > 1$ are classified in 88 deformation classes.*

In what follows, we will refer to [MM82, Tables 2, 3, 4, 5] where a full description of the 88 deformations types is given.

Theorem 2.67 ($\heartsuit$). *A smooth Fano threefold F with $\rho(F) > 1$ is fibre-like if and only if its deformation type is one appearing in Table 2.1*

In the second column there is the numbering used in [MM82], which we will adopt in our proof. Observe that entry 1a and 1b have the same deformation type.

No	[MM82]	$\rho(F)$	$-K_F^3$	Deformation type of F
1a	(6a)	2	12	F is a divisor of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.
1b	(6b)	2	12	F is a $2 : 1$ cover of a smooth divisor W of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ branched along a member of $ -K_W $.
2	(12)	2	20	F is the blow-up of $\mathbb{P}^3$ with center a curve of degree 6 and genus 3 which is an intersection of cubics. Alternatively, F is the intersection of three divisors of bidegree $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$.
3	(28)	2	28	F is the blow-up of $Q \subset \mathbb{P}^4$ with center a twisted quartic, a smooth rational curve of degree 4 which spans $\mathbb{P}^4$.
4	(32)	2	48	F is a divisor of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.
5	(1)	3	12	F is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ whose branch locus is a divisor of tridegree $(2, 2, 2)$.
6	(13)	3	30	F is the blowup of a smooth divisor of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ with center a curve C of bidegree $(2, 2)$ on it, such that $C \hookrightarrow W \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ is an embedding for both both projections $\mathbb{P}^2 \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$.
7	(27)	3	48	$F = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
8	(1)	4	24	F is a smooth divisor of multi degree $(1, 1, 1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Table 2.1: Deformation types of smooth fibre-like Fano threefolds

In the second column there is the numbering used in [MM82], which we will adopt in our proof. Observe that entry 1a and 1b have the same deformation type.

Remark 2.68. *These threefold have alternative descriptions, which we will use in the proof: these can be found in [Pro13] and [CCGK13].*

About the classification in [MM82], we remark that the last column in each table describes all possible ways for the given Fano threefold to be realised from another Fano threefold by blowing up a curve.

One can think of this via our approach as all the facets of the nef cone corresponding to a divisorial contraction in which the image of the exceptional divisor is a curve. It is important to underline that these contractions are listed without multiplicity: there could be more than one face giving the same contraction.

Proof of Theorem 2.67. We divide our analysis according to the Picard rank of F .

$$\rho(F) = 2$$

The nef cone of such a Fano variety is of the form

$$\mathbb{R}^+ D_1 + \mathbb{R}^+ D_2,$$

where D_1 and D_2 are two nef, semiample Cartier divisors on F (the D_i 's can be assumed to be primitive).

Remark 2.69. *Nef(F) is $\text{Mon}(F)$ -invariant. Assume that F is fibre-like; then the only invariant subspace is the line generated by the anticanonical class. As a consequence, the sum of the primitive generators of the nef cone must be a multiple of the canonical class. In particular, since $\rho(F) = 2$, the*

line of the canonical class is simply the bisector of $\text{Nef}(F)$, i.e.

$$\lambda K_F \sim D_1 + D_2$$

This implies, for instance, that a divisor of type $(1, 2)$ contained in $\mathbb{P}^2 \times \mathbb{P}^2$ cannot be fibre-like.

Consider Table 2 of [MM82]. Corollary 2.48 implies that the following classes of deformation are not fibre-like.

$$\begin{aligned} &(1), (2), (3), (4), (5), (7), (8), (9), (10), (11), (13), (14), \\ &(15), (16), (17), (18), (19), (20), (22), (23), (25), (26), \\ &(27), (28), (29), (30), (31), (33), (34), (35), (36). \end{aligned}$$

Theorem 2.62 implies that class (12), i.e. the intersection of three divisors of bidegree $(1, 1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$, is fibre-like.

Remark 2.69 implies that entry (24) is not fibre-like.

Let us work out the remaining classes: the variety corresponding to class (6a) is given by of degree- $(2, 2)$ divisor in $\mathbb{P}^2 \times \mathbb{P}^2$, while class (32) is a $(1, 1)$ -divisor in $\mathbb{P}^2 \times \mathbb{P}^2$: Corollary 2.63 implies that they are fibre-like.

Entry (6b) is a $2 : 1$ cover of a smooth divisor W of bidegree $(1, 1)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ branched along an element of $|-K_W|$. One can construct the universal family Z for this Fano variety inside $|-K_W| \times |-K_W|$ (cf. [BHPVdV04, Chapter I.17]). The variety Z is smooth, projective and it has Picard rank three. Since W has an involution ι and its action can be lifted to both $|-K_W|$ and Z , we define $Y := |-K_W|/\iota$ and $X := Z/\iota$ and obtain the required MFS which makes this variety fibre-like (cf. Theorem 2.62).

Class (28) is given by a smooth complete intersection of $L_1 := f^*\mathcal{O}_{\mathbb{P}^5}(1)$ and $L_2 := f^*\mathcal{O}_{\mathbb{P}^5}(2) - E$, where $f: Z \rightarrow \mathbb{P}^5$ is the blow-up of the Veronese surface V and E is the exceptional divisor. In order to apply Theorem 2.62

we need to construct an automorphism τ of order two of Z such that $\tau^*L_1 = L_2$. This automorphism τ is simply a special Cremona transformation: the Veronese surface is the intersection of 6 quadrics, so we have a Cremona transformation of $\mathbb{P}^5$ whose indeterminacy locus is V . By blowing-up V , we obtain a regular map from Z to $\mathbb{P}^5$ which contracts the secant variety of V , so this new map is again a blow-up. We conclude that τ lifts to a regular automorphism of Z (one can check that it acts non-trivially on the Picard group). This kind of Cremona transformations can be found in [ESB89].

$$\rho(X) = 3$$

With reference to Table 3 of [MM82], entry (1) is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched over a divisor of tridegree $(2, 2, 2)$. For this case, we use the same notation as in [CCGK13, Section 54]. This Fano variety is a member of the linear system $|2L + 2M + 2N|$ in the toric variety Z with the following weight data

x_0	x_1	y_0	y_1	z_0	z_1	w	
1	1	0	0	0	0	1	L
0	0	1	1	0	0	1	M
0	0	0	0	1	1	1	N

Also for this variety Theorem 2.62 applies, since Z carries a natural action of $\mathcal{S}_3$ exchanging the divisors L , M and N . So it lifts to the linear system $|2L + 2M + 2N|$ and this shows that class (1) is fibre-like.

Entry (13) can be described as a smooth complete intersection of three divisors of multi-degree $(0, 1, 1)$, $(1, 0, 1)$ and $(1, 1, 0)$ in $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$, so it is fibre-like (cf. Corollary 2.65).

Using Corollary 2.48, we can immediately conclude that the following classes are not fibre-like:

$$(2), (4), (5), (6), (7), (8), (11), (12), (14), (15), (16), \\ (18), (20), (21), (22), (23), (24), (26), (28), (29), (30), (31).$$

Remark 2.70. *In the case of Picard rank three, suppose that the nef cone contains two facets for which the images of the corresponding contraction morphisms are deformation equivalent. In this case, these faces may be identified by the action of $\text{Mon}(F)$. In particular, the primitive generators are exchanged and their sum is invariant. Hence, in order F to be fibre-like, this sum has to belong to the line generated by the canonical divisor.*

When the two facets correspond to divisorial contractions, the same analysis holds true for the sum of the two exceptional divisors E_i , with $i = 1, 2$. In particular, $E_1 + E_2$ has to be ample.

Using this remark, the following entries are not fibre-like:

$$(3), (9), (10), (17), (19), (25).$$

$$\rho(X) = 4$$

Here we refer to Table 4 of [MM82].

Entry (1), a divisors of multi degree $(1, 1, 1, 1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, is fibre-like (cf. Corollary 2.63).

On the other hand, Corollary 2.48 implies that following entries of the table are not fibre-like:

$$(3), (4), (5), (6), (8), (9), (10), (11), (13).$$

Generalising Remark 2.70 to Picard number four, one can exclude the

following classes:

$$(2), (7), (12).$$

$$\rho(X) \geq 5$$

Here we can easily list the possible cases.

- If F is the blow-up of Y (the blow up of a quadric $Q \subset \mathbb{P}^3$ along a conic on it with center three exceptional lines of the blowing up $Y \rightarrow Q$) then, assuming it is fibre-like, the sums of the three exceptional divisors over the lines must be a (negative) multiple of K_X . In particular it would be ample, which is clearly false, as one can see by taking an exceptional line for the map $Y \rightarrow Q$ other than those already blown up.
- Assume F to be the blow-up of $Y = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 0) \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 1))$ with center two exceptional lines l_1, l_2 of the blow-up $\phi : Y \rightarrow \mathbb{P}^3$ such that l_1 and l_2 lie on the same irreducible component of the exceptional set of ϕ . In this case, F is not fibre-like, because of Corollary 2.46.
- Products

$$\mathbb{P}^1 \times S_d, \quad d \leq 6,$$

where S_d is a del Pezzo of degree d . Since the projection onto the second factor must be $\text{Mon}(S_d)$ -invariant (as $\text{Nef}(\mathbb{P}^1 \times S_d) = \text{Nef}(\mathbb{P}^1) \times \text{Nef}(S_d)$). So none of these products can be fibre-like.

□

Our case-by-case analysis has shown the following corollary.

Corollary 2.71 ($\heartsuit$). *Let F be a smooth threefold. Then the necessary criterion (Theorem 2.42) is actually a characterisation of fibre-likeness. Moreover, fibre-likeness is preserved by smooth deformations.*

Another question concerns K -stability in this setting. It would be interesting to know whether the fibre-like Fano threefolds are K -stable or not. The following is a first piece of evidence.

Remark 2.72 (K -stability). *It is known that threefolds (4) and (7) are K -stable. In [Der15], varieties (1.b) and 5 are proved to be K -stable, being appropriate finite cover of K -stable varieties.*

2.4.4 Homogeneous Spaces

In this subsection, fibre-like rational homogeneous spaces are completely classified.

We need some notation.

Definition 2.73. A homogeneous space is a projective variety F with a transitive action of G , algebraic group.

Assuming G semi-simple, F is a rational Fano varieties and we refer to these varieties as *rational homogeneous spaces*.

Such a Fano variety can be also defined as a quotient of G by a parabolic subgroup P . General references for this circle of ideas are [Bri05] and [Dem77].

The isomorphism class of F is determined by the conjugacy class of P , which are in bijective correspondence with subsets of the nodes of the Dynkin diagram of G . We picture them by marking the corresponding nodes of the diagram; the resulting decorated diagram is called the Dynkin diagram of P .

We denote by E_P the group of symmetries of the Dynkin diagram preserving the marked nodes: it is isomorphic to $\mathrm{HMon}(F)$ (cf. Corollary 2.76).

The key technical result is the following (cf. [Dem77, Theorems 1 and 2]).

Theorem 2.74 (Demazure). *Let $F = G/P$ be a rational homogeneous space of Picard number at least 2, with G simple. Then F is rigid, that*

is $h^1(F, T_F) = 0$. Moreover, the automorphism group of F is isomorphic to the semi-direct product of G and the symmetry E_P of the Dynkin diagram of P .

Remark 2.75. The rational homogeneous spaces which are called exceptional in [Dem77] have Picard number one, so we can ignore them. The group of exterior automorphisms of G (denoted by E in [Dem77]) is known to be equal to the symmetries of the Dynkin diagram (cf. [Pro07, Section 10.6.10]). The group that Demazure calls E_π is in our notation E_P .

The following result follows directly from Theorem 2.43 and Demazure's result.

Corollary 2.76 ($\heartsuit$). *Let $F = G/P$ be a homogeneous space of Picard rank at least 2, with G simple. Then the Fano variety F is fibre-like if and only if it is isomorphic to one of the following varieties:*

1. $F = F(n, k)$ parametrises pairs of subspaces (L, H) in $\mathbb{C}^n$ such that $\dim L = k$, $\dim H = n - k$ and $L \subset H$;
2. $F = F(n)$ parametrises n dimensional isotropic subspaces of $(\mathbb{C}^{2n}, Q)$, where Q is a non-degenerate symmetric form;
3. $F = F^\tau$ parametrises pairs of isotropic subspaces (L, Π) in $(\mathbb{C}^8, Q)$, where Q is a non-degenerate symmetric form, L is a line, Π is four-dimensional and $L \subset \Pi$ (the upper-script τ stands for triality);
4. $F = F_i^\tau$ is the target of a contraction of a facet of the nef cone of F^τ ; more explicitly either $F_i^\tau = F(4)$ or Π is forced to belong to one of the two connected component of the grassmannians of isotropic 4-dimensional subspaces of $\mathbb{C}^8$.
5. $F = G/P$, where G is the exceptional group E_6 and P is associated to one of the two pairs of roots conjugated by the automorphism of the Dynkin diagram.

Before giving the proof we give a brief description of the involved varieties. Case (1) is realised as an homogeneous space with $G = SL_n$: it has Picard rank two and the faces of the nef cone are given by the projection onto grassmannians. After fixing a quadratic form Q , we obtain an automorphism of F given by $\phi_Q(L, P) = (P^\perp, L^\perp)$ exchanging the faces of $\text{Nef}(F)$.

In Case (2), the group G is SO_{2n} : the linear subspaces of an even dimensional quadric are divided into two families (the even and odd spin representations, cf. [Pro07, Section 11.7.2] or [GH78, Section 6.1]). Also this variety has Picard rank two and the action of an improper orthogonal transformation exchanges the faces of $\text{Nef}(F)$.

The third variety is homogeneous with $G = SO_8$ and has Picard rank three. The interesting automorphisms are realised via the triality (cf. [Pro07, Section 11.7.3]). We give now the proof of the classification.

Proof of Corollary 2.76. Since F is rigid, because of Theorem 2.43 we have just to study the action of

$$\text{Aut}(F) = G \rtimes E_P$$

on $\text{NS}(F)_\mathbb{Q}$.

The group G acts trivially in cohomology, so we only need to consider that action of E_P . Since $\text{NS}(F)_\mathbb{Q}$ is spanned by the line bundles associated to the simple roots of P (cf. [Bri05]), we can identify a basis of $\text{NS}(F)_\mathbb{Q}$ with the marked nodes of the diagram of P . Since this identification is equivariant with respect to E_P , $\text{HMon}(F)$ equals to E_P .

Rephrasing, $\dim \text{NS}(F)_\mathbb{Q}^{\text{Aut}(F)} = 1$ if and only if the group E_P acts transitively on the set of marked nodes.

Dynkin diagrams and their symmetries are classified (cf. [Pro07, Section 10.6.10]), so by direct inspection, we conclude that the unique F which are fibre-like are the ones listed above.

To be more explicit, the Dynkin diagrams B_n , C_n , E_7 , E_8 , F_4 and G_2 have no symmetries, so the rational homogeneous spaces for the respective groups are fibre-like if and only if the Picard number is one.

A_n has just a symmetry of order two, so for each pair of conjugated nodes one gets a fibre-like homogeneous space of Picard rank two (this is case (1)). D_n , for $n \geq 4$, has just an order-two symmetry which fixes all nodes except the two nodes associated to the spin representations (this is case (2)). The group D_4 has $\mathcal{S}_3$ as group of symmetries: so the triality gives cases (3) and (4). For the case of E_6 , one has an order-two symmetry which gives case (5). $\square$

2.4.5 The toric case

In this subsection we study the smooth toric case.

Our main result concerns the geometric structure of the polytope associated to smooth fibre-like toric Fano varieties: we prove it has barycentre in the origin (i.e. it is K -stable).

The notion of K -stability is extremely subtle and here we can provide a sufficient criterion for it, at least in the toric case. Let us remark that there are manifestly K -stable smooth toric varieties which are not fibre-like: think of $\mathbb{P}^1 \times \mathbb{P}^2$.

Preliminaries on toric geometry: primitive collections

We need to recall some basic theory of toric varieties (cf. [CLS11], [Bat91] and [Cas03] for more details).

Let N be a free abelian group of rank n and set $N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}$ and let M be the dual of N . We denote by $\Sigma \subset N_{\mathbb{Q}}$ a fan of an n -dimensional smooth toric Fano variety F and with $\Delta \subset N_{\mathbb{Q}}$ the dual polytope associated to the anti-canonical polarisation. The vertices of Δ will be denoted by $V(\Delta)$.

The following is a basic short exact sequence:

$$0 \rightarrow N_1(F) \rightarrow \mathbb{Z}^{V(\Delta)} \rightarrow N \rightarrow 0 \quad (2.6)$$

and dually

$$0 \rightarrow M \rightarrow \mathbb{Z}^{V(\Delta)} \rightarrow \text{NS}(F) \rightarrow 0. \quad (2.7)$$

Our analysis is based on techniques involving primitive collections. We need to recall them.

Definition 2.77. A subset $P \subset V(\Delta)$ is a *primitive collection* if the cone generated by P is not in Σ , while for each $x \in P$ the elements of $P \setminus \{x\}$ generate a cone in Σ .

For a primitive collection $P = \{x_1, \dots, x_k\}$ let $\sigma(P)$ be the (unique) minimal cone in Σ such that $(x_1 + \dots + x_k) \in \sigma(P)$. Let $y_1, \dots, y_h$ be generators of $\sigma(P)$. Then

$$r(P): x_1 + \dots + x_k = b_1 y_1 + \dots + b_h y_h,$$

where the b_i 's are positive integers.

Definition 2.78. The linear relation $r(P)$ is the *primitive relation of P* and the cone $\sigma(P)$ is defined as the focus of P . The integer k is the *length of $r(P)$* and the *degree of P* is defined as $\deg P = k - \sum b_i$.

Using (2.6) we identify A_1 with the group generated by relations among the elements of $V(\Delta)$:

$$A_1(X) \cong \left\{ (b_x)_{x \in V(\Delta)} \in \text{Hom}(\mathbb{Z}^m, \mathbb{Z}) \mid \sum_{x \in V(\Delta)} b_x x = 0 \right\}.$$

By abuse of notation we denote with $r(P)$ the cycle associated to $r(P)$ via the previous isomorphism.

Note that $\deg P = -(K_F \cdot r(P))$: as a consequence, in our setting, any primitive relation has strictly positive degree.

Let P a primitive collection such that $r(P)$ is *extremal* (i.e. it generates an extremal ray in $\overline{NE}(F)$). The exceptional locus of the associated contraction $\phi_P : F \rightarrow F_P$ is $Z(\sigma(P))$.

Depending on $\sigma(P)$, one obtains the different type of steps in the minimal model program.

Divisorial It appears when $\sigma(P)$ is a one-dimensional cone and the contracted divisor is precisely the one associated to the ray.

Fibre type In this case $\sigma(P)$ is the origin.

Small This appears in all the other cases.

The following results will be useful in what follows.

Lemma 2.79 (Casagrande). *Let $\gamma \in \overline{NE}(F) \cap A_1(F)$ be a cycle such that $-(K_F \cdot \gamma) = 1$. Then γ is extremal.*

Proof. This is [Cas03, Prop. 4.3]. □

Theorem 2.80 (Reid). *Let R be an extremal ray of $\overline{NE}(F)$ and $\gamma \in R \cap A_1(F)$ a primitive cycle. Then there exists a primitive collection $P = \{x_1, \dots, x_k\}$ such that*

$$\gamma = r(P) : x_1 + \dots + x_k = b_1 y_1 + \dots + b_h y_h.$$

For any cone $\nu = \langle z_1, \dots, z_t \rangle$ such that $\{z_1, \dots, z_t\} \cap \{x_1, \dots, x_k, y_1, \dots, y_h\} = \emptyset$ and $\langle y_1, \dots, y_h \rangle + \nu \in \Sigma$, we have

$$\langle x_1, \dots, \check{x}_i, \dots, x_k, y_1, \dots, y_h \rangle + \nu \in \Sigma$$

for all $i = 1, \dots, h$.

Proof. It can be found in [Rei83, Theorem 2.4] and [Cas03, Theorem 1.5].

□

Proposition 2.81 (Casagrande). *Let P a primitive extremal collection such that $\sigma(P) = \langle y_1, \dots, y_h \rangle$. Then for any primitive collection Q for which $P \cap Q \neq \emptyset$ and $Q \neq P$, the set $(Q \setminus P) \cup \{y_1, \dots, y_h\}$ contains a primitive collection.*

Proof. This is [Cas03, Prop. 3.4].

□

Remark 2.82. *Let*

$$a_1x_1 + \dots + a_kx_k = b_1y_1 + \dots + b_hy_h$$

be an (arbitrary) relation among vertices of Δ with all $\{a_i\}$ and $\{b_j\}$ positive integers. Assume that $\sum a_i \geq \sum b_j$. Then, by [Cas03, Lemma 1.4], $\langle x_1 \dots, x_k \rangle \notin \Sigma$.

We will need the following result for our main theorem (cf. [Bat91, Proposition 3.2]).

Proposition 2.83 (Batyrev). *Let F be a smooth toric Fano variety. Then there exists a primitive collection $\mathcal{P}$ such that $\sigma(\mathcal{P}) = 0$.*

Fibre-likeness implies K -stability

We present here our main result in the toric case: one expects to see a combinatorial manifestation of fibre-likeness on the polytope Δ . Moreover, one can be interested in further implications on the geometry of the smooth toric Fano variety.

As we showed in the previous sections, fibre-likeness implies some kind of “symmetry” of the Fano and it is known that the symmetry of the polytope Δ is strictly related to the K -stability (this is known to be equivalent to the

existence of Kähler-Einstein metric, cf. [WZ04] and [BB13]) of the associated Fano variety.

Mabuchi proved in [Mab87] the first result relating the K -stability with the triviality of the barycentre of Δ . This result was generalised in the singular setting in [Ber12, Corollary 1.2].

Theorem 2.84 (Berman). *Let $F(\Delta)$ be a Gorenstein toric Fano variety. Then F is K -stable if and only if the barycentre of Δ is the origin.*

The proof of the previous result is completely analytic.

In this context our main result is the following.

Theorem 2.85 ($\heartsuit$). *Every smooth toric fibre-like Fano variety $F(\Delta)$ has barycentre in the origin and, as a consequence, is K -stable.*

Remark 2.86. *Applying Theorem 2.84, we can see that there are Gorenstein terminal $\mathbb{Q}$ -factorial toric varieties which are fibre-like, but not K -stable, e.g. the weighted projective space $\mathbb{P}(1, 1, 1, 1, 2)$.*

The proof of Theorem 2.85 heavily depends on convex geometry. Let us recall some basic facts.

Remark 2.87. *The intersection of a convex polytope with an affine space is again a convex polytope.*

Lemma 2.88 ($\star$). *Let P be an n -dimensional convex polytope in an affine space $W \simeq \mathbb{Q}^n$ and let H be a k -dimensional affine subspace intersecting P in its interior. Define $P' := P \cap H$ and consider a facet $\mathcal{F}'$ of P' . Then there exists a unique face $\mathcal{F}$ of P of dimension at least $k - 1$ for which*

- $\mathcal{F}' = \mathcal{F} \cap H$;
- H meets $\mathcal{F}$ in its relative interior.

Proof. The polytope P is defined by inequalities $\{x \in W \mid a_i x \leq b_i, i = 1, \dots, t\}$, with $a_i, b_i \in \mathbb{Q}^d$, $t \geq n + 1$ and H is defined by $\{x \in W \mid c_j x = d_j, j = 1, \dots, d - k\}$, with $c_j, d_j \in \mathbb{Q}^d$.

After reordering the coefficients a_i 's and b_i 's, the facet $\mathcal{F}'$ is defined by $\{x \in W \mid a_i x = b_i, i = 1, \dots, l, a_i x \leq b_i, i = l + 1, \dots, t, c_j x = d_j, j = 1, \dots, d - k\}$, with $l \leq n - k + 1$.

The set which defines the unique face $\mathcal{F}$ of P of dimension at least $k - 1$ with the required properties is the

$$\{x \in W \mid a_i x = b_i, i = 1, \dots, l, a_i x \leq b_i, i = l + 1, \dots, t\}.$$

□

Let us pass to the action of a subgroup of automorphisms of the polytope on the vertices $V(P)$.

Lemma 2.89 ($\heartsuit$). *Let P be an n -dimensional polytope in an affine space $W \simeq \mathbb{Q}^n$ and let G be a finite subgroup of $\text{GL}(W, \mathbb{Q})$. Assume that P is invariant for the action of G on W and that W^G is k -dimensional and intersects the interior of P . Then the action of G on the vertices $V(P)$ has at least $k + 1$ orbits.*

Proof. We prove this lemma by induction on k . In the case $k = 1$, the fixed locus W^G is a line, which meets two distinct (possibly not maximal) faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of P .

The two invariant sets, which give at least two orbits are:

$$V(\mathcal{F}_1) \setminus V(\mathcal{F}_2), \quad \text{and} \quad V(\mathcal{F}_2) \setminus V(\mathcal{F}_1).$$

We prove now the inductive step. Because of Remark 2.87, we can consider the intersection polytope $P' := P \cap W^G$. Let $\mathcal{F}'$ be one of its $(k - 1)$ -dimensional facets. Lemma 2.88 produces a (unique) face $\mathcal{F}$ of P , cut in

its interior by W^G in a $k - 1$ affine space. Let H be the smallest affine subspace containing $\mathcal{F}$ (it is preserved by the action of G). By induction, we obtain at least k orbits. The extra orbit is obtained by the vertices of P not contained in $\mathcal{F}$. $\square$

We prove now the main result in the toric setting.

Proof of Theorem 2.85. Since smooth Fano toric varieties are rigid (cf. [dFH12, Corollary 4.6]), we can apply the characterisation of Theorem 2.43.

After tensoring by $\mathbb{Q}$ the short exact sequence (2.7), one gets

$$0 \rightarrow M_{\mathbb{Q}} \rightarrow \mathbb{Q}^{V(\Delta)} \rightarrow \mathrm{NS}(F(\Delta))_{\mathbb{Q}} \rightarrow 0. \quad (2.8)$$

There is a natural action of $\mathrm{Aut}(\Delta)$ on $M_{\mathbb{Q}}$ and $\mathbb{Q}^{V(\Delta)}$ and a homomorphism $\mathrm{Aut}(\Delta) \rightarrow \mathrm{Aut}(F)$, which make the sequence above equivariant. Moreover by [Cox95, Corollary 4.7] we have $\mathrm{NS}(F)_{\mathbb{Q}}^{\mathrm{Aut}(\Delta)} = \mathrm{NS}(F)_{\mathbb{Q}}^{\mathrm{Aut}(F)}$.

The following is a basic fact.

Lemma 2.90 (★). *Let G be a finite group and S a finite G -set. Call t the number of orbits of the action of G on S . Let*

$$0 \rightarrow A \rightarrow \mathbb{Q}^S \rightarrow B \rightarrow 0$$

be an equivariant exact sequence of $\mathbb{Q}$ -vector spaces with a G -action. Then

$$\dim B^G = t - \dim A^G.$$

Let us simplify the notation: $G := \mathrm{Aut}(\Delta)$, let t be the number of orbits of the action of G on $V(\Delta)$ and $k := \dim M^G$.

Lemma 2.90 and the short exact sequence (2.8) imply that F is fibre-like if and only if $t - k = 1$.

We can assume $k > 0$ and want to prove that if G has exactly $k + 1$ orbits on $V(\Delta)$, then the barycentre of Δ is the origin.

Observe that M and N are isomorphic as G -modules, so $\dim N^G = \dim M^G = k$.

Let Δ' be the intersection polytope $N^G \cap \Delta$. For every facet $\mathcal{F}'$ of Δ' one can apply Lemma 2.88 to find a unique face $\mathcal{F}$ of Δ cut by N^G in its interior.

Lemma 2.89 says that $V(\mathcal{F})$ splits in at least k orbits.

The fibre-likeness of F implies that $V(\mathcal{F})$ splits in exactly k orbits: another orbit is given by the set of vertices $V(\Delta) \setminus V(\mathcal{F})$.

Let now $\mathcal{F}'_1$ and $\mathcal{F}'_2$ be two distinct facets of Δ' , which correspond to two faces $\mathcal{F}_1$ and $\mathcal{F}_2$ of Δ and determine two sets of k orbits S_1 and S_2 in $V(\Delta)$. We claim that $S_1 \neq S_2$: otherwise

$$V(\mathcal{F}_1) = \bigcup S_1 = \bigcup S_2 = V(\mathcal{F}_2)$$

and this would imply $\mathcal{F}_1 = \mathcal{F}_2$.

Since there are exactly $k + 1$ ways to choose k elements in a set of cardinality $k + 1$ and Δ' has at least $k + 1$ facets, we conclude that any collection of k orbits is supported on a face $\mathcal{F}$ of Δ .

To conclude the proof we need to use the Fano condition, via primitive relations.

Let $\mathcal{P}$ be a primitive collection with trivial focus, whose existence is guaranteed by Proposition 2.83. Since any set of k orbits must be contained in a face, $\mathcal{P}$ must involve at least one vertex from every orbit. Acting with G on $\mathcal{P}$ one gets a family of primitive collections $\{\mathcal{P}_i\}_{1 \leq i \leq r}$ such that $\sigma(\mathcal{P}_i) = 0$ and $\bigcup \mathcal{P}_i = V(\Delta)$. Assume that $\mathcal{P}_i \cap \mathcal{P}_j \neq \emptyset$ for some i, j , i.e. $\mathcal{P}_i = \{x_1, \dots, x_k\}$ and $\mathcal{P}_j = \{x_1, \dots, x_h, y_{h+1}, \dots, y_k\}$ with $y_s \neq x_t$ for any s, t . Then

$$x_{h+1} + \dots + x_k = y_{h+1} + \dots + y_k.$$

This is impossible by Remark 2.82, since $x_{h+1}, \dots, x_k$ generate a cone in Σ .

As a consequence, all the $\mathcal{P}_i$'s are disjoint and the sum of all vertices of Δ equals the origin, i.e. the barycentre of Δ is in the origin. $\square$

The previous corollary seems to be the relative version in the toric case of the following very general conjecture (cf. [OO13, Conj. 5.1]).

Conjecture 2.91 (Odaka, Okada). *Any smooth Fano manifold X of Picard rank 1 is K -semistable.*

MAGMA computations

In the following table we collect the smooth Fano toric varieties (up to dimension 8) which are fibre-like. It has been obtained using the software MAGMA together with the Graded Ring Database [GB⁺] (for further details on the classification, cf. [Øb07]).

Remark 2.92. *In the following table, the IDs of the Fano polytopes are the ones introduced in [GB⁺].*

Remark 2.93. *The varieties V_n are known as Del Pezzo varieties (see [VK84] for more details). Varieties W_6^3, W_8^3 and $\tilde{W}$ will be described later in the subsection.*

What is interesting about Table 2.2 is the fact that all the toric varieties appearing have a very peculiar combinatorial property.

Definition 2.94 (Vertex-transitive). A polytope Δ is called *vertex-transitive* if $\text{Aut}(\Delta)$ acts transitively on the vertices of Δ . If Δ is associated to a toric Fano variety F , then F is also called vertex-transitive.

Our MAGMA computations give evidence of the following conjecture in toric geometry.

Dimension	# Vertices	Description	ID
2	6	V_2	2
2	4	$\mathbb{P}^1 \times \mathbb{P}^1$	4
2	3	$\mathbb{P}^2$	5
3	6	$(\mathbb{P}^1)^3$	21
3	4	$\mathbb{P}^3$	23
4	10	V_4	63
4	12	$V_2 \times V_2$	100
4	8	$(\mathbb{P}^1)^4$	142
4	6	$\mathbb{P}^2 \times \mathbb{P}^2$	146
4	5	$\mathbb{P}^4$	147
5	10	$(\mathbb{P}^1)^5$	1003
5	6	$\mathbb{P}^5$	1013
6	14	V_6	1930
6	12	W_6^3	5817
6	18	$(V_2)^3$	7568
6	12	$(\mathbb{P}^1)^6$	8611
6	9	$(\mathbb{P}^2)^3$	8631
6	8	$(\mathbb{P}^3)^2$	8634
6	7	$\mathbb{P}^6$	8635
7	14	$(\mathbb{P}^1)^7$	80835
7	8	$\mathbb{P}^7$	80891
8	18	V_8	106303
8	15	W_8^3	277415
8	20	$(V_4)^2$	442179
8	24	$(V_2)^4$	790981
8	12	$\tilde{W}$	830429
8	16	$(\mathbb{P}^1)^8$	830635
8	12	$(\mathbb{P}^2)^4$	830767
8	10	$(\mathbb{P}^4)^2$	830782
8	9	$\mathbb{P}^8$	830783

Table 2.2: Smooth fibre-like toric Fano varieties of dimension at most 8

Conjecture 2.95 (Characterisation in the Toric case, $\heartsuit$). *Let $F := F(\Delta)$ be a smooth toric Fano variety. Then, F is fibre-like if and only if it is vertex-transitive.*

Table 2.2 inspires also the following speculation.

Conjecture 2.96 ($\heartsuit$). *Let Δ be a smooth fibre-like polytope of dimension d . Assume that d is an odd prime number. Then either $X(\Delta) = \mathbb{P}^d$ or $X(\Delta) = (\mathbb{P}^1)^d$.*

Vertex-transitive Polytopes

In this subsection we investigate vertex-transitive Fano polytopes, which naturally appeared in the previous discussion.

Our ambitious ultimate goal would be to classify these polytopes: the transitivity condition seems rather restrictive. We can currently obtain some partial results in this direction, which are enough to classify low dimensional vertex-transitive Fanos.

From now on, Δ will denote a *smooth* Fano polytope of dimension n with m vertices and F will be the associated Fano variety.

The first interesting examples of vertex transitive Fano varieties are the so called *del Pezzo varieties*.

Definition 2.97. The n -dimensional *del Pezzo variety* V_n (with n even) is the smooth toric Fano whose associated polytope has vertices

$$V(\Delta) = \{e_1, \dots, e_n, -e_1, \dots, -e_n, (e_1 + \dots + e_n), -(e_1 + \dots + e_n)\},$$

where $e_1, \dots, e_n$ is the standard basis of $N_{\mathbb{Q}}$.

Remark 2.98. In Subsection 2.4.1, *del Pezzo surfaces* denoted all the smooth Fano varieties of dimension 2 and this could cause confusion. Nonetheless both notations are standard, so we prefer to keep it. We just remark that the

del Pezzo variety of dimension two is just S_6 , in the notation of Subsection 2.4.1.

Projective spaces and del Pezzo varieties are manifestly vertex-transitive, but they are not the only one as we will see.

The following lemma is very useful in the spirit of classification.

Lemma 2.99 ($\heartsuit$). *Let $X(\Delta)$ be a vertex-transitive Fano toric variety such that $X \cong Y \times Z$ for Y, Z toric varieties. Then there exists a vertex-transitive Fano toric variety W and positive integers r and s such that $Y \cong W^r$ and $Z \cong W^s$.*

Proof. One can assume that Y and Z are not isomorphic to products of toric varieties, since the general statement follows from this case. Let $\Phi = \text{conv}(y_1, \dots, y_{m_1})$ and $\Psi = \text{conv}(z_1, \dots, z_{m_2})$ be the polytopes of Y and Z respectively, so that $\Delta = \text{conv}(\Phi \times \{0\}, \{0\} \times \Psi)$. Consider an element $g \in \text{Aut}(\Delta)$ such that $g((y_1, 0)) = (0, z_1)$. We claim that $g(\Phi \times \{0\}) \subset \{0\} \times \Psi$, from which, by symmetry, it follows that $\Phi \cong \Psi$ and then the statement. We may assume that $g((y_i, 0)) = (0, z_i)$ for $i = 1, \dots, k$ with $k \leq m_1$ and $g((x_j, 0)) \in \Phi \times \{0\}$ for $j = k + 1, \dots, m_1$. Hence

$$g(\Phi \times \{0\}) = \text{conv}((0, z_1), \dots, (0, z_k), g(y_{k+1}, 0), \dots, g(y_{m_1}, 0)).$$

Since Y is not isomorphic to a product, we conclude that $k = m_1$ and the claim is proven. $\square$

The following is classic result in [VK84, Theorem 6].

Theorem 2.100 (Voskresenskii, Klyachko). *Let F be a smooth toric Fano such that Δ is symmetric with respect to the origin. Then F is isomorphic to the product of projective lines and del Pezzo varieties.*

The following definition is standard in convex geometry.

Definition 2.101 (K -neighbourly polytope). A polytope Δ is called k -neighbourly if every set of k vertices lies on a face of Δ .

Proposition 2.102 ($\heartsuit$). Assume that Δ is smooth and vertex-transitive. Then either

1. $F(\Delta) = (\mathbb{P}^1)^n$ or $F(\Delta) = (V_k)^r$ for positive integers r and k or
2. Δ is 2-neighbourly.

Proof. Assume that Δ is not 2-neighbourly. This implies the existence of a primitive collection with two elements.

We want to prove that there is a primitive relation of the form $x + y = 0$. Assume by contradiction that there is no such relation and consider a primitive collection $P_1 = \{x_1, x_2\}$ with relation $R_1: x_1 + x_2 = y_1$.

Acting with $\text{Aut}(\Delta)$ we obtain a family of primitive collections $\mathcal{P} = \{P_i\}_{1 \leq i \leq r}$ and relations $\mathcal{R} = \{R_i\}_{1 \leq i \leq r}$ such that $m := \#\{V(\Delta)\}$ divides r (the action is transitive). Assume that the P_i 's are all disjoint. Then $2r = m$, which is impossible. So we may assume, by symmetry, that $P_2 = \{x_1, x_3\}$ and $R_2: x_1 + x_3 = y_2$ with $x_2 \neq x_3$.

Combining P_1 and P_2 we get $x_2 + y_2 = x_3 + y_1$, from which we deduce that $\{x_2, y_2\}$ is also a primitive collection, thanks to Remark 2.82.

Write $S_1: x_3 + y_1 = z_1$ and $S_2: x_2 + y_2 = z_1$. Lemma 2.79 implies that both S_1 and S_2 are extremal.

By Theorem 2.80, $\langle y_1, y_2 \rangle \in \Sigma$, which is impossible because

$$y_1 + y_2 = x_1 + z_1.$$

This implies the existence of a primitive relation of the form $x + y = 0$.

Acting with $\text{Aut}(\Delta)$ we obtain exactly $m/2$ such relations (it is easy to check that they are disjoint). In particular, by the vertex transitive condition, for any vertex x there is a vertex y such that $x + y = 0$. This means that

Δ is symmetric. The result follows by Theorem 2.100 and Lemma 2.99. $\square$

Proposition 2.103 ($\heartsuit$). *Assume that Δ is smooth, vertex-transitive and 2-neighbourly. Then there exists an integer $k \geq 3$ and a set of primitive collections $\mathcal{P} = \{P_i\}_{i=1,\dots,r}$ for which $r = m/k$, $|P_i| = k$, $\sigma(P_i) = 0$ and $P_i \cap P_j = \emptyset$ for any $i \neq j$. Moreover, these are the only primitive relations with trivial focus.*

Moreover if just one of these relations is extremal, $F(\Delta) = (\mathbb{P}^{k-1})^r$.

By contrast, if one of these relations is not extremal then F does not admit any elementary contraction of fibre type.

Proof. By Proposition 2.83 we know that there is a primitive collection P_1 such that $\sigma(P_1) = 0$. Set $k = |P_1|$. Note that $k \geq 3$ (Δ is 2-neighbourly by hypothesis).

Acting with $\text{Aut}(\Delta)$ we obtain a family of primitive collections $\mathcal{P} = \{P_i\}_{1 \leq i \leq r}$ such that $\sigma(P_i) = 0$ and $\cup P_i = V(\Delta)$. Assume that $P_i \cap P_j \neq \emptyset$ for some i, j , that $P_i = \{x_1, \dots, x_k\}$ and $P_j = \{x_1, \dots, x_h, y_{h+1}, \dots, y_k\}$ with $y_s \neq x_t$ for any s, t . Then

$$x_{h+1} + \dots + x_k = y_{h+1} + \dots + y_k,$$

which is impossible by Remark 2.82 ($x_{h+1}, \dots, x_k$ generate a cone in Σ). It is easy to check that there are no other primitive relations with trivial focus. The first part of the proposition is hence proven.

Assume now that one of these relations (and hence all by symmetry) is extremal. We claim that there are no other primitive collections. In fact, let Q be a primitive collection such that $Q \notin \mathcal{P}$ and Q has the minimal cardinality among the primitive collections not contained in $\mathcal{P}$. We may assume that $P_1 \cap Q \neq \emptyset$. By Proposition 2.81, the set $(Q \setminus P_1)$ contains a primitive collection, which contradicts the minimality of $|Q|$.

This gives a complete description of the Mori cone: X has the same primitive relations as $(\mathbb{P}^{k-1})^r$ and hence we get the thesis.

□

Lemma 2.104 (♥). *Assume that Δ is smooth, 2-neighbourly and vertex-transitive. Then there are no extremal relations of the form*

$$x_1 + \dots + x_k = b_1 y_1.$$

In particular X does not admit any elementary divisorial contraction.

Proof. Assume, by contradiction that, acting with $\text{Aut}(\Delta)$, one gets another extremal primitive relation

$$x_1 + z_2 \dots + z_k = b_1 y_2$$

(we argue as in Proposition 2.102 to show that, moving the primitive collections with $\text{Aut}(\Delta)$, they cannot be all disjoint).

We treat the case $y_2 \notin \{x_2, \dots, x_k, y_1\}$, the other one is similar.

We have

$$b_1 y_2 + x_2 + \dots + x_k = b_1 y_1 + z_2 + \dots + z_k.$$

Since Δ is 2-neighbourly, we know that $\langle y_1, y_2 \rangle$ is a cone of Σ . We apply Theorem 2.80 to get

$$\langle y_2, x_2, \dots, x_k \rangle \in \Sigma,$$

which is impossible, again by Remark 2.82.

□

Examples

We study here a family of examples of vertex-transitive toric varieties, which are generalisations of del Pezzo varieties.

They were first introduced in [Kly84] and also appeared in [VK84], although the dual polytopes of these toric varieties naturally appeared as a spacial class of transportation polytopes (the so-called *central transportation polytopes*), which have been extensively studied in optimisation problems (cf. [EKK81]).

We will use a different notation compared to the one introduced by Klyachko (cf. Remark 2.106): our choice is more convenient for classification (cf. Remark 2.110).

Definition 2.105 (Klyachko varieties, $\heartsuit$). Let k and n be positive integers such that $n \geq 2$ and $(k-1)|n$. Fix a basis $e_1, \dots, e_n$ of a lattice $N \cong \mathbb{Z}^n$. The *Klyachko variety of order k and dimension n* is the Fano toric variety W_n^k associated to the polytope $\Delta_n^k \subset N$ with vertices

$$\begin{aligned} V(\Delta_n^k) = \{ & e_1, e_2, \dots, e_n, e_1 + \dots + e_n, \\ & -(e_1 + \dots + e_{k-1}), -(e_k + \dots + e_{2k-2}), \dots, -(e_{n-k+2} + \dots + e_n), \\ & -(e_1 + e_k + \dots + e_{n-k+2}), -(e_2 + e_{k+1} + \dots + e_{n-k+3}), \dots, \\ & -(e_{k-1} + e_{2k-2} + \dots + e_n)\}. \end{aligned}$$

Note that when n is even, W_n^2 is the del Pezzo variety V_n .

Remark 2.106. In [VK84], the varieties W_n^k are introduced with the notation $P_{g,l}$. We have the following relations between the indices:

$$n = (g-1)(l-1), \quad k = g \text{ (or } l)$$

(our definition of k is consistent because of the isomorphisms proved in Lemma 2.107).

Assume that W_n^k is smooth (cf. Proposition 2.109). Then we have the following geometric interpretation.

The one-dimensional cones of the fan of W_n^k are exactly the one-dimensional cones of the fan of the blow-up Z_n^k of $(\mathbb{P}^{k-1})^{\frac{n}{k-1}}$ in k invariant points. This implies that there exist a birational map between W_n^k and Z_n^k which is an isomorphism in codimension one. Moreover, it can be shown that this map must factor through a sequence of flips, since W_n^k is a smooth Fano: more precisely this map can be realised by a $K_{W_n^k}$ -MMP with scaling (cf. [BCHM10]).

Furthermore, the following isomorphisms hold.

Lemma 2.107 ($\heartsuit$). *For any integers n and m , we have $W_{mn}^{n+1} \cong W_{mn}^{m+1}$.*

Proof. One can assume that $m \leq n$ and write down the vertices of W_{mn}^{d+1} :

$$\begin{aligned} &\{e_1, e_2, \dots, e_{mn}, e_1 + \dots + e_{mn}, \\ &\quad -(e_1 + \dots + e_n), -(e_{n+1} + \dots + e_{2n}), \dots, -(e_{m(n-1)+1} + \dots + e_{mn}), \\ &\quad -(e_1 + e_{n+1} + \dots + e_{m(n-1)+1}), -(e_2 + e_{n+2} + \dots + e_{m(n-1)+2}), \dots, \\ &\quad -(e_n + e_{2n} + \dots + e_{mn})\}. \end{aligned}$$

It is now clear that reordering the base vectors

$$e'_{mi+j} := e_{n(j-1)+i+1}$$

with $i \in \{0, \dots, n\}$ and $j \in \{1, \dots, m-1\}$ gives the required isomorphism. $\square$

Lemma 2.108 ($\heartsuit$). *The polytope W_n^k is vertex-transitive, reflexive and terminal.*

Proof. We start proving the transitivity by induction on the dimension n . Let us fix the index k and consider the Klyachko variety with dimension W_{k-1}^k . Using Lemma 2.107 we deduce the isomorphism

$$W_{k-1}^k \cong W_{k-1}^2,$$

where the right hand side is a vertex-transitive del Pezzo variety. Assume now by induction that W_t^k with $t < n$ is vertex transitive (t has to be a multiple of $k-1$). We consider for every $i = 0, \dots, n-k+1$ the projections from the polytope Δ_n^k to the $(k-1)$ -codimensional subspaces orthogonal to

$$\langle e_{i+1}, e_{i+2}, \dots, e_{i+k-1} \rangle. \quad (2.9)$$

By inductive hypothesis, the images of the polytope Δ_n^k via the projections are all vertex transitive polytopes isomorphic to Δ_{n-k+1}^k . To deduce the transitivity of the whole polytope, we use the isomorphisms of $N_{\mathbb{Q}}$ swapping the subspaces in (2.9).

Define $W := W_n^k$ and $\Delta := \Delta_n^k$.

We prove that Δ is reflexive. It is enough to show that there are no lattice points lying between the affine hyperplane spanned by each facet of $\Delta^* \subset M_{\mathbb{Q}}$ and its parallel through the origin. Since the action of $\text{Aut}(\Delta)$ induces a transitive action on the facets of the dual Δ^* , it is sufficient to consider just one facets of Δ^* : it is obvious that $\{x_1 = -1\} \subset M_{\mathbb{Q}}$ satisfies the property we want.

We now show that W has terminal singularities, i.e. that $\Delta \cap N = V(\Delta) \cup \{0\}$. Assume by contradiction that there exists a non-zero $v \in \Delta \cap N$ such that $v \notin V(\Delta)$. We may assume that v is not in the subspace H generated by $e_1, \dots, e_{k-1}$. Let π_H the projection from H . Then $\Gamma = \pi_H(\Delta)$ is again a Klyachko polytope, $\pi_H(v) \in \Gamma$ and $\pi_H(v) \notin V(\Gamma) \cup \{0\}$. Hence we can conclude by induction, since W_d^2 is terminal for any $d \geq 2$.

□

For any positive integers n and k we denote by $\bar{n}_k$ the smallest non-negative integer r such that $n \equiv r \pmod{k}$.

Although the following result is proved in [VK84], we give a different proof using our notation. This gives a rather precise idea on how the arithmetics

of indices makes the singularity W_n^k very different.

Proposition 2.109 ($\heartsuit$). *The variety W_n^k is smooth if $\gcd(n-1, k) = 1$. If $\gcd(n-1, k) \neq 1$, then W_n^k is not $\mathbb{Q}$ -factorial.*

In particular, if k is a prime number then W_n^k is smooth, unless $n \equiv 1 \pmod k$.

Proof. For any $k \geq 2$ and $t \geq 1$ define the linear form

$$L_{k,t} = \sum_{i=0}^{k-3} (x_{t+ik} + x_{t+ik+1} + \dots + x_{t+ik+k-2} - (k-1)x_{t+ik+(k-1)}).$$

Observe that the polytope Δ_n^k for $n = (k-1)^2$ is not simplicial, since the hyperplane $\{L_{k,1} + x_n = 1\}$ supports a facet of Δ_n^k which turns out to have $k(k-1)$ vertices.

On the other hand we see that if $n = k(k-1)$, the polytope Δ_n^k is smooth. In fact it is easy to see that any facet F of Δ_n^k which contains the vertex $(1, 1, \dots, 1)$ must contain at least $f = (k-1)(k-2) + 1$ elements of the standard basis. This implies that the supporting hyperplane

$$\{a_1x_1 + \dots + a_dx_d = 1\}$$

of F has exactly f coefficients equal to 1, $k-2$ coefficients equal to $-(k-1)$ and $k-1$ coefficients equal to 0 (as for the hyperplane $\{L_{k,1} + x_k = 1\}$). One can also check that all these facets have vertices which form a basis of N . By the transitivity of $\text{Aut}(\Delta_n^k)$ we gain that W_n^k is smooth.

We now prove the general statement by induction on k and n . The case $k = 2$ (and any n) and the case $n = 2$ are immediate.

Let Δ_n^k be a Klyachko polytope with $k, n \geq 3$. If $n < (k-1)^2$ then $\Delta_n^k \cong \Delta_n^{s+1}$, where $s = k - \bar{n}_k$ and $n = s(k-1)$, so by induction on k we are done, because $\gcd(n-1, k) = \gcd(s+1, k) = \gcd(s+1, n-1)$.

We have already seen how to deal with the cases $n = (k-1)^2$ and $n = k(k-1)$.

If $n > k(k-1)$, then set $t = n - k(k-1)$ and consider the plane H spanned by $\{e_{t+1}, e_{t+2}, \dots, e_n\}$ and the associated projection π_H on the orthogonal space to H . The image $\pi_H(\Delta_n^k)$ is Δ_t^k and $\gcd(n-1, k) = \gcd(t-1, k)$. For any facet F of Δ_t^k with supporting hyperplane $\{P(x_1, \dots, x_t) = 1\}$ we get a facet $\overline{F}$ of Δ_n^k supported by

$$\{P + L_{k,(t+1)} + (x_{d-k+1} + \dots + x_{d-1} - (k-1)x_d) = 1\}.$$

We remark that $|V(\overline{F})| = |V(F)| + k(k-1)$, so if Δ_t^k is not simplicial, then Δ_n^k is not simplicial. In this way one can check that Δ_n^k is smooth if and only if Δ_t^k is smooth; the result follows now by induction on n .

□

Remark 2.110. *The 8-dimensional polytope denoted by $\tilde{W}$ in Table 2.2 is not a Klyachko variety. Nonetheless, following our notation, it can be considered as the candidate for a generalised Klyachko variety W_8^4 .*

Low-dimensional case

In this subsection, we use the previous results to give a classification of smooth vertex-transitive Fano polytopes in low dimension.

Let Δ be a transitive smooth Fano polytope of dimension n . By [Cas06, Theorem 1] we know that $|V(\Delta)| \leq 3n$ with equality if and only if n is even and $X := X(\Delta) \cong (V_2)^{n/2}$. We also note that for any extremal relation $x_1 + \dots + x_k = b_1 y_1 + \dots + b_h y_h$, we have $k + h \leq n + 1$.

Dim(X)=2. If Δ is not 2-neighbourly, then by Proposition 2.102 we get $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $X \cong V_2$.

Assume that Δ is 2-neighbourly. Then there is an extremal collection $P = \{x_1, x_2, x_3\}$ such that $\sigma(P) = 0$ and so, by Proposition 2.103, we get

$X \cong \mathbb{P}^2$.

Dim(X)=3. If Δ is not 2-neighbourly, then by Proposition 2.102 we get $X \cong (\mathbb{P}^1)^3$.

Assume that Δ is 2-neighbourly. By dimensional restriction, the extremal relations could be of the form $x_1 + x_2 + x_3 = 0$ (which is impossible by Proposition 2.103), $x_1 + x_2 + x_3 = y_1$ (which is impossible by Lemma 2.104) or $x_1 + x_2 + x_3 + x_4 = 0$, which gives $X \cong \mathbb{P}^3$ by Proposition 2.103.

In this way we can recover all the smooth vertex-transitive Fano varieties appearing in Table 2.2 (up to dimension 3).

Further analysis of primitive relations could lead to further results in this direction.

2.4.6 Singular Del Pezzo fibrations

In this subsection, we provide some further results for singular del Pezzo surfaces. These results hold in arbitrary characteristic.

Notation 2.111. *The notation here is slightly different with respect to the other subsections: we focus on the two dimensional case, but the hypothesis on the del Pezzo surfaces involved are much weaker. We only assume them to be defined over an algebraically closed field of arbitrary characteristic. In positive characteristic, we work in the étale setting: all cohomology groups, fundamental groups and Chern classes are considered with respect to the étale topology.*

Definition 2.112. A *weak del Pezzo surface* S of degree d is smooth two-dimensional variety with $-K_S$ big and nef such that $d = (K_S^2)$.

A *singular del Pezzo surface* $\tilde{S}$ of degree d is two-dimensional Fano variety with canonical singularities such that $d = (K_{\tilde{S}}^2)$.

Weak del Pezzo surface are classified: they are $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ blown up in up to 8 *almost general* points (cf. [Dem80] for a precise definition).

This condition on the points makes (-2) -curves appear on every properly (i.e non-Fano) weak del Pezzo surface and all the possible configurations of (-1) and (-2) -curves have been completely classified (cf. [Dol12]).

The aim here is to classify rigid singular del Pezzo surface which can be realised as general fibre of a Mori fibre space. In the rigid case, it is very easy to extend the definition of fibre-like.

Definition 2.113. A rigid singular del Pezzo surface is *fibre-like* if it can be realised as the general fibre (over an open dense subset U_f^{iso} , where the fibration is isotrivial) of a Mori fibre space $f: X \rightarrow Y$.

Remark 2.114. *Verdier's results in characteristic zero (cf. [Ver76]) guarantee the existence of an open dense subset over which every MFS is topologically trivial (denoted in the previous sections by U_f^{top}). In arbitrary characteristic, this is no longer clear.*

For the general theory on singular del Pezzo surfaces, see [CT88]. Here we only recall in the following theorem the fundamental properties we need for our purposes.

Theorem 2.115 (Artin, Demazure). *Every singular degree- d del Pezzo surface $\tilde{S}$ is obtained from a weak degree- d del Pezzo S via the contraction $\pi: S \rightarrow \tilde{S}$ of all (-2) -curves (π is the minimal resolution of $\tilde{S}$). Moreover this contraction morphism is associated to the (pluri)-anticanonical system $| -iK_S |$ with*

$$i = \begin{cases} 1 & \text{for } d \geq 3 \\ 2 & \text{for } d = 2 \\ 3 & \text{for } d = 1 \end{cases}$$

The main result of this section is the following.

Theorem 2.116. *Let S be a (rigid) singular del Pezzo surface of degree $d \geq 5$. Then S is fibre-like if and only if it is isomorphic to one of the*

following:

$d = 9$: $\mathbb{P}^2$;

$d = 8$: $\mathbb{P}^1 \times \mathbb{P}^1$ and Q (quadratic cone);

$d = 6$: S_6 , S_6^3 (dominated by the blow-up of $\mathbb{P}^2$ in 3 non-collinear infinitely near points) and $S_6^{3,col}$ (dominated by the blow-up of $\mathbb{P}^2$ in 3 collinear infinitely near points);

$d = 5$: S_5 and $S_5^{4,col}$ (dominated by the blow-up of $\mathbb{P}^2$ in 4 collinear infinitely near points) .

Moreover, S_6^3 is the only rigid fibre-like properly singular del Pezzo surface with Picard rank bigger than one.

Before we prove this theorem, we need to extend some of our results in the previous subsections. First, we need a slightly different version of Theorem 2.38 allowing more singularity, which holds in a general setting.

Theorem 2.117. *Let*

$$f: X \rightarrow Y$$

be a dominant morphism of projective normal varieties of arbitrary dimension, where X is $\mathbb{Q}$ -factorial with rational singularities. Assume that the anti-canonical sheaf of X is f -ample.

characteristic zero *Then the restriction map*

$$\rho: \mathrm{NS}(X)_{\mathbb{Q}} \rightarrow \mathrm{NS}(F_y)_{\mathbb{Q}}^{\pi_1(U,y)}$$

is surjective for every fibre over $U = U_f^{top}$.

positive characteristic *Assume the existence of an open dense subset U_f^{iso} over which f is isotrivial. Moreover, assume that the Brauer group of the fibre over U_f^{iso} is trivial. Then the same result holds over $U = U_f^{iso}$*

Proof. Let us focus on the characteristic zero case. Rational singularities guarantee that the first Chern class map yields an isomorphism

$$R^1 f_* \mathbb{G}_m \otimes \mathbb{Q} \rightarrow R^2 f_* \mathbb{Q} \quad (2.10)$$

of sheaves of abelian groups on U . Let us denote by $\mathcal{S}$ one of these two isomorphic sheaves, which are locally constant over U . We have a monodromy action of $\pi_1(U, y)$ on $\mathcal{S}_y = \text{NS}(F_y)_{\mathbb{Q}} = H^2(F_y, \mathbb{Q})$. This action is finite (let G be the finite group it factors through). Consider an étale Galois cover

$$p: V \rightarrow U,$$

which trivialises $p^* \mathcal{S}$. The Galois group of the cover is G and it acts on V and $V/G = U$. We can lift this action to an action on X_V , again $X_V/G = X_U$. By abuse of notation we denote again by p the map from X_V to X_U , and by f the map from X_V to V . Let $p^{-1}y = \{y_1, \dots, y_k\}$ be the pre-images of y in V . Denote by F_i the fibre over y_i . The restriction p_i of p to F_i is an isomorphism with the fibre F_y .

Let g be an element of G . Write $g(t_1) = t_i$. Pick a divisor D in $\text{NS}(F_y)$ and denote by $g(D)$ the monodromy action of g on D . Via p_1 , we can see D as a divisor on F_1 . Since $p^* \mathcal{S}$ is trivial, there exists a (non-unique) divisor $\tilde{D}$ in $\text{Pic}(X_V)$ which restricts to D on F_1 . We denote by $\tilde{D}_i$ the restriction of $\tilde{D}$ to F_i . The monodromy action on $\text{NS}(F_y)_{\mathbb{Q}}$ is given by

$$g(D) = p_i^{-1*} \tilde{D}_i = p_1^{-1*} g^* \tilde{D}_1,$$

where the second equality is due to $p_i \circ g = p_1$. We remark that if G fixes the isomorphism class of $\tilde{D}$ in $\text{Pic}(X_V)$, then G fixes D in $\text{NS}(F_y)_{\mathbb{Q}}$. The converse requires a bit of care. We want to show that if D is fixed by G , we can find a G -invariant $\tilde{D}$. Pick any $\tilde{D}$ in $\text{Pic}(X_V)$ which restricts to D on F_1 .

Since $\tilde{D}$ is possibly not G -invariant, we consider the average in $\text{Pic}(X_V)_{\mathbb{Q}}$:

$$E := \frac{1}{|G|} \sum_{g \in G} g^* \tilde{D}.$$

Since D is invariant under monodromy, the restriction of $g^* \tilde{D}$ to F_1 is isomorphic to D for any g in G . We conclude that E_1 is isomorphic to D as well. Moreover, E is G -invariant. We conclude that the restriction r_1 to the fibre F_1 composed with p_1 defines a surjective morphism $r_1: \text{Pic}(X_V)_{\mathbb{Q}}^G \rightarrow \text{Pic}(F_y)_{\mathbb{Q}}^G$. We now recall that $X_V/G = X_U$, so we have

$$\text{Pic}(X_V)_{\mathbb{Q}}^G \cong \text{Pic}(X_U)_{\mathbb{Q}}.$$

We have a surjective morphism $\text{Pic}(X_U)_{\mathbb{Q}} \rightarrow \text{Pic}(F_y)_{\mathbb{Q}}^G$ and since X is $\mathbb{Q}$ -factorial, also the restriction morphism

$$\text{Pic}(X)_{\mathbb{Q}} \rightarrow \text{Pic}(X_U)_{\mathbb{Q}}$$

is surjective (given a Cartier divisor on X_U , its Zariski closure is a Weil divisor on X). The restriction commutes with taking Chern classes, so we have the required surjection

$$\rho: \text{NS}(X)_{\mathbb{Q}} \rightarrow \text{NS}(F_y)_{\mathbb{Q}}^G.$$

In positive characteristic one can replace the first Chern class isomorphism with the one induced by Kummer exact sequence (cf. Subsection 2.3.3). The result follows repeating the same proof as in characteristic zero. $\square$

We can now extend our characterisation theorem (Theorem 2.43) in this slightly worse singular setting.

Theorem 2.118. *Let S be a (rigid) singular del Pezzo surface of degree*

$d \geq 5$. Then S is fibre-like if and only if

$$\mathrm{NS}(S)_{\mathbb{Q}}^{\mathrm{Aut}(S)} = \mathbb{Q}K_S$$

Furthermore, S can be realised as the general fibre of an isotrivial MFS over a curve.

Proof. The *if* part is precisely Theorem 2.40 (the same construction can be performed, although the singularities of the total space will be canonical instead of terminal).

Since the involved fibres are rigid and the MFS are isotrivial, the singularities are rational and $\mathbb{Q}$ -factorial and the involved MFS are isotrivial, one can apply Theorem 2.117 to conclude as in Theorem 2.42 (in characteristic p , [Bri13] proves that $\mathrm{Br}(S) = 0$ for S). $\square$

Proof of Theorem 2.116. The proof is a case-by-case analysis, looking at the automorphisms of the configurations of negative curves in the generalised del Pezzo $\tilde{S}$ dominating S (cf. [CT88, Propositions 8.1, 8.2, 8.3, 8.4]). $\square$

3 Matsusaka's Theorem for surfaces in positive characteristic

3.1 Introduction

In this chapter we present a project aimed to investigate Effective Matsusaka's Problem for surfaces in positive characteristic. In order to attach it, a new strategy has been developed, based on bend-and-break techniques combined with other positive-characteristic methods developed by Ekedahl and Shepherd-Barron to extend Bogomolov's stability for vector bundles.

Motivation and history: the moduli problem and the classical approach in characteristic zero

In order to study moduli of polarised varieties (X, D) , where D is an ample divisor on X , a smooth projective variety of dimension n , it is essential to determine a constant $m_0(D)$, only depending on the Hilbert polynomial of D , for which mD is very ample for $m \geq m_0$. This is nodal in order to determine the scheme structure of the moduli space (the so-called boundedness problem).

For curves, it is easy to show that the constant $2g+1$ work for an arbitrary ample divisor on a one-dimensional variety (i.e. there exists a constant only depending on the geometry of the curve C which works for every ample divisor on C). Unfortunately, already for smooth surfaces, one can easily

see that there is no universal constant on a fixed variety, as the following example shows (cf. [Laz04, Example 1.5.7]).

Example 3.1 (Kollár). *Let $X := E \times E$ be the self-products of an elliptic curve E and let us define, for every $n \geq 2$, the divisor*

$$A_n := nF_1 + (n^2 - n + 1)F_2 - (n - 1)\Delta,$$

where F_i are the two fibres of the projections and Δ is the diagonal. It is easy to show that A_n is ample.

Let $R := F_1 + F_2$ and consider a smooth element B of the linear system $|2R|$. After constructing a $2 : 1$ cover $f : Y \rightarrow X$ ramified over R , one can show that the following injection

$$H^0(X, \mathcal{O}_X(nA_n)) \hookrightarrow H^0(Y, \mathcal{O}_Y(nD_n))$$

*is an isomorphism, where we have defined $D_n := f^*A_n$. This implies that nD_n cannot be very ample and, as a consequence, that*

$$m_0(D_n) > n.$$

This implies that the effective value m_0 actually depends on the ample divisor we choose.

A celebrated theorem of Matsusaka (cf. [Mat72]) solves the problem in characteristic zero, providing a constant $m_0(D)$, only depending on the Hilbert polynomial of D , and a further refined result by Kollár and Matsusaka, always in characteristic zero, (cf. [KM83]) shows that this integer only depends on the intersection numbers (D^n) and $(K_X \cdot D^{n-1})$.

More generally, one can ask the following question in arbitrary characteristic.

Question 3.2. *Let X be a smooth variety over an algebraically closed field, let D and B be an ample and a nef divisor on X respectively. Then there exists an integer M depending only on (D^2) , $(K_X \cdot D)$ and $(D \cdot B)$ such that*

$$mD - B$$

is very ample for all $m \geq M$.

In the same spirit, the stronger question requiring an *explicit* integer M as a function of (D^n) and $(K_X \cdot D^{n-1})$ can be formulated: this is the so called *effective* problem. The first effective versions of Matsusaka's theorem are due to Siu (cf. [Siu02a], [Siu02b]) and Demailly (cf. [Dem96a], [Dem96b]): their methods, which only work in characteristic zero, are cohomological and rely on vanishing theorems. See also [Laz04] for a full account of this approach.

Let us state the main theorem of this chapter.

Theorem 3.3 (= Theorem 3.5) *Effective Matsusaka for surfaces in characteristic p , $\spadesuit$). Let X be a smooth surface over an algebraically closed field, let D and B be an ample and a nef divisor on X respectively. Then there exists an integer M depending only on (D^2) , $(K_X \cdot D)$ and $(D \cdot B)$ such that*

$$mD - B$$

is very ample for all $m \geq M$. Furthermore an explicit value of M is provided.

Since we will be interested in the case of surfaces, let us recall the work by Fernandez del Busto (cf. [FdB96]) where basically the optimal bounds for smooth surfaces in characteristic zero (with $B = 0$) are obtained. This approach was generalised for some classes of smooth surfaces in positive characteristic in [Bal96].

In this chapter we will develop a method which works for *arbitrary* smooth surfaces defined over an arbitrary algebraically closed field.

Pathologies for surfaces in characteristic p

This is a very rich and interesting topic with a rather long history: the seminal works by Bomberi and Mumford (cf. [Mum69], [BM77] and [BM76]) extended Enriques' classification of smooth surfaces in positive characteristic, focusing on new classes of surfaces which do not exist in characteristic zero. On them, weird phenomena appear: Raynaud in [Ray78] provided examples of quasi-elliptic surfaces on which Kodaira vanishing fails. Mukai characterised in [Muk13] all surfaces over which Ramanujam vanishing fails (cf. Theorem 3.49).

Although also the minimal model program for surfaces in positive characteristic has been recently established, thanks to the work of Tanaka (cf. [Tan14] and [Tan12]), some interesting effectivity questions remain open in this setting, after the influential papers of Ekedahl and Shepherd-Barron (cf. [Eke88] and [SB91b]), where, for instance, effective birationality, base-point freeness and very ampleness for pluricanonical linear systems on smooth surfaces in positive characteristic was studied and completely solved.

One interesting strategy to study surfaces in characteristic p is via purely inseparable covers. Actually, all new classes of surfaces which do not exist in characteristic zero basically appear because of purely inseparable fields extensions. In some sense, Frobenius morphism is the cause and the solution for these new classes. We describe here a beautiful construction (cf. [Kol96, Construction II.6.1.6]), which will be useful for our purposes. We present it here, since it is extremely explicit and gives a good idea of the techniques in this chapter.

First of all, we fix the notation about the Frobenius morphism: let X be a smooth variety defined over an algebraically closed field of positive characteristic $p > 0$. We will denote with

$$F: X \rightarrow X$$

the *absolute* Frobenius, corresponding at the rings level to the map $x \mapsto x^p$.

Construction 3.4 (Ekedahl, Kollár). *Let X be a smooth variety of dimension n defined over a field of characteristic $p > 0$ and let D be a divisor for which the following holds:*

$$H^1(X, \mathcal{O}_X(D)) \neq 0, \quad \text{but} \quad H^1(X, \mathcal{O}_X(pD)) = 0.$$

This gives an element in the kernel of the induced Frobenius map

$$F^*: H^1(X, \mathcal{O}_X(D)) \rightarrow H^1(X, \mathcal{O}_X(pD)) = 0,$$

which is the same as a (non-split) rank-two vector bundle

$$0 \rightarrow \mathcal{O}_X(D) \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X \rightarrow 0,$$

for which the Frobenius action gives a splitting

$$F^*(\mathcal{E}) \simeq \mathcal{O}_X(pD) \oplus \mathcal{O}_X.$$

Since $F^(\mathcal{E}) \subset S^p(\mathcal{E})$, the previous splitting gives a section*

$$\sigma \in H^0(X, \mathcal{O}_X) \hookrightarrow H^0(X, \mathcal{O}_X(S^p(\mathcal{E}))) \simeq H^0(Y, \mathcal{O}_Y(p)),$$

where $Y := \mathbb{P}(\mathcal{E})$ is the projective bundle associated to the rank-two bundle $\mathcal{E}$.

The associated zero section $Z \subset Y$ determines a purely inseparable cover $\pi: Z \rightarrow X$, where Z is integral (cf. [Kol96, Proposition II.6.1.2]), but often non normal (even if X is a smooth curve).

This construction can be very useful, for instance, if we want to produce rational curves on X (cf. Section 3.2). We will also use it in order to deduce effective Kawamata-Viehweg-type theorems for pathological surfaces

(cf. Definition 3.17 and Section 3.5).

A new approach to Effective Matsusaka for surfaces in positive characteristic

The main purpose of this chapter is to prove the following result.

Theorem 3.5 ($\spadesuit$). *Let D and B be respectively an ample divisor and a nef divisor on a smooth surface X over an algebraically closed field k , with $\text{char } k = p > 0$. Then $mD - B$ is very ample for any*

$$m > \frac{2D \cdot (H + B)}{D^2} ((K_X + 2D) \cdot D + 1),$$

where

- $H := K_X + 4D$, if X is not-pathological (see Definition 3.17);
- $H := K_X + 19D$, if X is pathological with $\kappa(X) = 1$;
- $H := 2K_X + 19D$, if X is pathological of general type.

Remark 3.6. *The effective bound obtained with $H = K_X + 4D$ is expected to hold for all surfaces. Note that the bound in this case is not far from being sharp even in characteristic zero (cf. [FdB96]).*

The proof of Theorem 3.5 does not rely directly on vanishing theorems, but rather on Fujita's conjectures, which are known to hold for smooth surfaces in characteristic zero (cf. [Rei88]) and for non-pathological smooth surfaces in positive characteristic (cf. [SB91b] and [Ter99]).

Conjecture 3.7 (Fujita). *Let X be a smooth n -dimensional projective variety and let D be an ample divisor on it. Then $K_X + kD$ is base point free for $k \geq n + 1$ and very ample for $k \geq n + 2$.*

Another new result of this chapter is an effective very ampleness result *à la Fujita* for surfaces which are quasi-elliptic with $\kappa(X) = 1$ or of general type

we can prove the following effective result in the spirit of Fujita's conjecture (cf. Section 3.4).

Theorem 3.8 (= Theorem 3.33, ♠). *Let X a smooth surface over an algebraically closed field of characteristic $p > 0$, D an ample Cartier divisor on X and let $L(a, b) := aK_X + bD$ for positive integers a and b . Then $L(a, b)$ is very ample for the following values of a and b :*

1. *if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 3$, $a = 1$ and $b \geq 8$;*
2. *if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 2$, $a = 1$ and $b \geq 19$;*
3. *if X is of general type with $p \geq 3$, $a = 2$ and $b \geq 4$;*
4. *if X is of general type with $p = 2$, $a = 2$ and $b \geq 19$.*

The key ingredient of this result is a combination of a Reider-type result due to Shepherd-Barron and bend-and-break techniques.

In the last part of this chapter (cf. Section 3.5), a Kawamata-Viehweg-type vanishing theorem which holds for surfaces which are quasi-elliptic with $\kappa(X) = 1$ or of general type is proved (cf. Theorem 3.55 and Corollary 3.57): this generalises the vanishing result in [Ter99].

3.2 Preliminary results

In this section we recall some results and techniques developed in the last twenty years, some of them quite recently, in order to extend classical results in characteristic zero to algebraically closed fields in characteristic p . In some cases, we will start from characteristic-zero results in order to make our discussion more clear.

Volume of divisors

When deforming a variety X of general type in flat families, one preserves the Hilbert polynomial and in particular the top coefficient: this is, up to a

constant depending on the dimension, the volume of X .

More generally, let D be a Cartier divisor on a normal variety X . The volume of D measures the asymptotic growth of the space of global sections of multiples of D . We need to recall some basic properties of $\text{vol}(D)$ (cf. [Laz04] for a detailed discussion).

Definition 3.9. Let D be a Cartier divisor on X with $\dim(X) = n$. The volume of D is

$$\text{vol}(D) := \limsup_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

The volume of X is defined as $\text{vol}(X) := \text{vol}(K_X)$.

Assuming the divisor D big and nef, one can show that $\text{vol}(D) = D^n$. Although, in general, computing this invariant is incredibly hard, Fujita's approximation theorem allows us to deduce some properties if D is ample. A complete proof of this result can be found in [Laz04] assuming the characteristic to be zero. Just very recently, Takagi proved of the same theorem in positive characteristic (cf. [Tak07]). As a consequence, we can deduce the log-concavity of the volume function in characteristic $p > 0$. The proof of this implication, which is characteristic-free, is in [Laz04, Theorem 11.4.9].

Theorem 3.10 (Fujita, Takagi). *Let D and D' be big Cartier divisors on a normal variety X defined over an algebraically closed field. Then*

$$\text{vol}(D + D')^{1/n} \geq \text{vol}(D)^{1/n} + \text{vol}(D')^{1/n}.$$

Bogomolov's inequality and Sakai's theorems

Here we present the notion of semi-stability for rank-two vector bundles on surfaces. In this subsection, X is a smooth surface defined over an algebraically closed field.

Definition 3.11. A rank-two vector bundle $\mathcal{E}$ on X is *unstable* if it fits in a short exact sequence

$$0 \rightarrow \mathcal{O}_X(D_1) \rightarrow \mathcal{E} \rightarrow \mathcal{I}_Z \cdot \mathcal{O}_X(D_2) \rightarrow 0,$$

where D_1 and D_2 are Cartier divisors such that $D' := D_1 - D_2$ is big with $(D'^2) > 0$ and Z is an effective 0-cycle on X .

The vector bundle $\mathcal{E}$ is *semi-stable* if it is not unstable.

In characteristic zero, Bogomolov proved the following classical result (cf. [Bog78, Main Theorem]).

Theorem 3.12 (Bogomolov). *Let X be defined over a field of characteristic zero. Then every rank-two vector bundle $\mathcal{E}$ for which $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$ is unstable.*

As a consequence, one can deduce the following theorem, due to Sakai (cf. [Sak90, Proposition 1]): the idea is that non-vanishing of the H^1 of adjoint line bundles $K_X + D$ with some weak positivity hypothesis on D allows us to deduce something on the “negative part” of D .

In [DC13], the two statements are proved to be equivalent in characteristic zero.

Theorem 3.13 (Sakai). *Let D be a nonzero big divisor with $D^2 > 0$ on a smooth projective surface X over a field of characteristic zero. If $H^1(X, \mathcal{O}_X(K_X + D)) \neq 0$ then there exists a non-zero effective divisor E such that*

- $D - 2E$ is big;
- $(D - E) \cdot E \leq 0$.

As a consequence, one obtains Reider’s theorem.

Theorem 3.14 (Reider). *Let D be a nef divisor with $D^2 > 4$ on a smooth projective surface X over a field of characteristic zero. Then $K_X + D$ has no base point unless there exists a non-zero effective divisor E such that $D \cdot E = 0$ and $(E^2) = -1$ or $D \cdot E = 1$ and $(E^2) = 0$.*

This is enough for surfaces in characteristic zero to prove Fujita's conjectures.

Corollary 3.15 (Fujita Conjectures for surfaces, char 0). *Let $D_1, \dots, D_k$ be ample divisors on X smooth, over a field of characteristic zero. Then $K_X + D_1 + \dots + D_k$ is base-point free if $k \geq 3$ and very ample if $k \geq 4$.*

Remark 3.16. *Theorem 3.13 is not known in general for smooth surfaces in positive characteristic, although Fujita's conjectures are expected to hold in general: classical cohomological techniques are too weak in positive characteristic.*

In order to make our presentation more clear, we introduce the following definition for surfaces in positive characteristic over which Sakai's theorem holds.

Definition 3.17 (Non-pathological surface). A smooth surface X defined over an algebraically closed field of positive characteristic is *non-pathological* if the following holds: for every $D \neq 0$ big divisor with $D^2 > 0$ on X which verifies

$$H^1(X, \mathcal{O}_X(K_X + D)) \neq 0,$$

there exists a non-zero effective divisor E such that

- $D - 2E$ is big;
- $(D - E) \cdot E \leq 0$.

A smooth surface X defined over an algebraically closed field of positive characteristic which does not verify the previous condition is said *pathological*.

Some pathologies for surfaces in positive characteristic: quasi-elliptic surfaces

Before focusing on specific techniques we survey some interesting classes appearing for smooth surfaces in positive characteristic. For a detailed account (with some proofs) on the classification in positive characteristic, see the survey paper [Lie13].

Quasi-elliptic surfaces appeared in Bombieri-Mumford classification of surfaces in positive characteristic and play an important role in constructing counterexamples to Kodaira vanishing. Starting with a surface X and a fibration $f: X \rightarrow C$ on a curve, we know that the generic fibre F is reduced and irreducible. Nonetheless, we cannot deduce the smoothness of F over $k(C)$, because of the failure of Bertini's theorem.

Definition 3.18. A surface X endowed with a fibration $f: X \rightarrow C$ on a smooth curve such that the generic fibre is a singular integral curve of arithmetic genus one is called *quasi-elliptic*.

The following fundamental theorem by Tate shows that quasi-elliptic surfaces only appear in low characteristic (cf. [Băd01, Theorem 7.18]).

Theorem 3.19 (Tate). *Quasi-elliptic surfaces only exist in characteristic 2 and 3. Furthermore, the general fibre of the fibration is a genus one curve with only one ordinary cusp.*

Another interesting aspect of quasi-elliptic surfaces is concerning their uniruledness.

Remark 3.20. *Let $f: X \dashrightarrow Y$ be a dominant, generically finite map. In characteristic 0, the assumption $\kappa(X) = -\infty$ implies $\kappa(Y) = -\infty$. By contrast, in positive characteristic the existence of purely inseparable morphisms produces examples where the pull-back of a non-zero pluricanonical form annihilates.*

For a quasi-elliptic surface $f: X \rightarrow C$, there exists a purely inseparable extension $K/k(C)$ of degree $p = \text{char}(k)$ such that $F \times_{\text{Spec } k(C)} \text{Spec } K$ is not regular and whose normalisation is $\mathbb{P}^1$. We are saying that every quasi-elliptic surface is purely inseparably uniruled.

Ekedahl's construction and Shepherd-Barron's theorem

In this subsection some classical results on smooth surfaces in positive characteristic are recalled (cf. [Eke88], [SB91b] and [SB91a] for further details).

In particular, we need to discuss here a construction which is due to Tango for the case of curves (cf. [Tan72]) and Ekedahl (cf. [Eke88]) for surfaces. There are many variations on the same theme (we saw an explicit example in Section 3.1), but we will focus on the one which is more related to stability of vector bundles. The following result is due to Bogomolov and basically generalises Theorem 3.12. For a proof, see [SB91b, Theorem 1].

Theorem 3.21 (Bogomolov, Shepherd-Barron). *Let $\mathcal{E}$ be a rank-two vector bundle on a smooth projective surface X over a field of positive characteristic such that the Bogomolov's inequality does not hold (i.e. such that $c_1^2(\mathcal{E}) > 4c_2(\mathcal{E})$). Then there exists a reduced and irreducible surface Y contained in the ruled threefold $\mathbb{P}(\mathcal{E})$ such that*

- *the restriction $\rho: Y \rightarrow X$ is p^e -purely inseparable for some $e > 0$;*
- *$(F^*)^e(\mathcal{E})$ is unstable.*

Remark 3.22. *In the proof of the previous result, an explicit construction of the purely inseparable cover is provided.*

Construction 3.23 (Shepherd-Barron). *Let $\mathcal{E}$ be a rank-two vector bundle such that the Bogomolov's inequality does not hold and let e be an integer such that $F^{e*}\mathcal{E}$ is unstable. One obtains the following commutative diagram.*

$$\begin{array}{ccc} \mathbb{P}(F^{e*}\mathcal{E}) & \xrightarrow{G} & \mathbb{P}(\mathcal{E}) \\ p' \downarrow & & \downarrow p \\ X & \xrightarrow{F^e} & X \end{array}$$

Since $F^{e}\mathcal{E}$ is unstable, we obtain a short exact sequence*

$$0 \rightarrow \mathcal{O}_X(D_1) \rightarrow F^{e*}\mathcal{E} \rightarrow \mathcal{I}_Z \cdot \mathcal{O}_X(D_2) \rightarrow 0$$

and a quasi-section X_0 of $\mathbb{P}(F^{e}\mathcal{E})$ (i.e. $p'_{|X_0}: X_0 \rightarrow X$ is birational).*

Let Y be the image of X_0 via G . One can show that the induced morphism

$$\rho: Y \rightarrow X$$

is p^e -purely inseparable. Let us define $D' := D_1 - D_2$; one can show (cf. [SB91b, Corollary 5]) that the following relation on canonical divisors holds.

$$K_Y \equiv \rho^* \left(K_X - \frac{p^e - 1}{p^e} D' \right).$$

Remark 3.24. *The case we are interested in is when the rank-two vector bundle $\mathcal{E}$ appears as a nontrivial extension*

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X(D) \rightarrow 0$$

associated to a non-zero element $\gamma \in H^1(X, \mathcal{O}_X(-D))$, where D is a big Cartier divisor such that $(D^2) > 0$.

Indeed, the instability of $F^{e}\mathcal{E}$ guarantees the existence of a diagram (keep*

the notation as in Definition 3.11:

$$\begin{array}{ccccccc}
& & & 0 & & & \\
& & & \downarrow & & & \\
& & & \mathcal{O}_X(D' + D_2) & & & \\
& & & \downarrow f_1 & & & \\
0 & \longrightarrow & \mathcal{O}_X & \xrightarrow{g_1} & F^{e*}\mathcal{E} & \xrightarrow{g_2} & \mathcal{O}_X(p^e D) \longrightarrow 0 \\
& & \searrow \sigma & & \downarrow f_2 & & \\
& & & \mathcal{I}_Z \cdot \mathcal{O}_X(D_2) & & & \\
& & & \downarrow & & & \\
& & & 0 & & &
\end{array}$$

First, we claim that the composition map σ is non-zero: assume, by contradiction that $\sigma \equiv 0$. This gives a nonzero section $\sigma': \mathcal{O}_X \rightarrow \mathcal{O}_X(D' + D_2)$. This forces the composition $\tau := g_2 \circ f_1$ to be zero. But this implies that $D' + D_2 \leq 0$. This is a contradiction (cf. the proof of [Sak90, Proposition 1] and [SB91b, Lemma 16]).

This implies that $D_2 \simeq E \geq 0$: one can then rewrite the vertical exact sequence as follows:

$$0 \rightarrow \mathcal{O}_X(p^e D - E) \rightarrow F^{e*}\mathcal{E} \rightarrow \mathcal{I}_Z \cdot \mathcal{O}_X(E) \rightarrow 0.$$

Since [SB91b, Corollary 8] implies that Corollary 3.15 holds true for smooth surfaces in positive characteristic which are neither quasi-elliptic with $\kappa(X) = 1$ nor of general type, we need to deduce effective base-point-freeness and very ampleness results only for these two classes of surfaces.

We recall here the following key result from [SB91b], based on Bombieri-Mumford classification of smooth surfaces in positive characteristic.

Theorem 3.25 (Shepherd-Barron). *Let $\mathcal{E}$ be a rank-two vector bundle on a smooth projective surface X over an algebraically closed field of positive*

characteristic such that the Bogomolov's inequality does not hold and is semi-stable. Then

- if X is not of general type, then X is quasi-elliptic with $\kappa(X) = 1$;
- if X is of general type and

$$c_1^2(\mathcal{E}) - 4c_2(\mathcal{E}) > \frac{\text{vol}(X)}{(p-1)^2},$$

then X is purely inseparably uniruled. More precisely, in the notation of Theorem 3.21, Y is uniruled.

Proof. This is [SB91b, Theorem 7], since the volume of a surface X with minimal model X' equals $(K_{X'}^2)$. $\square$

The corollary we are interested in is the following (cf. [SB91b, Corollary 8]).

Corollary 3.26 (Shepherd-Barron). *A smooth surface X in positive characteristic which is neither quasi-elliptic of Kodaira dimension one nor of general type is non-pathological and Corollary 3.15 holds true for X .*

Bend-and-break lemmas

This subsection is dedicated to the key technical result our approach is based on. It is a celebrated method due to Mori (see [Kol96] for an insight into these techniques) which allows us to find rational curves on varieties with “negative” canonical divisor with a precise control on the intersection numbers.

Mori theory deals with effective 1-cycles in a variety X ; more specifically we will consider non-constant morphisms $h: C \rightarrow X$, where C is a smooth curve. These techniques allow us to deform curves for which

$$(K_X \cdot C) := \deg_C h^* K_X < 0.$$

On the space $Z_1(X)$ of 1-cycles, many equivalence relations can be imposed: following [Kol96] we will denote with $\overset{e}{\approx}$ the *effective algebraic equivalence* defined on $Z_1(X)$ (cf. [Kol96, Definition II.4.1]).

Theorem 3.27 (Mori’s Bend-and-break). *Let X be a variety over an algebraically closed field and let C be a smooth, projective and irreducible curve with a morphism $h: C \rightarrow X$ such that X has local complete intersection singularities along $h(C)$ and $h(C)$ intersects the smooth locus of X . Assume the following numerical condition holds:*

$$(K_X \cdot C) < 0.$$

Then for every point $x \in C$, there exists a rational curve C_x in X passing through x such that

$$h_*[C] \overset{e}{\approx} k_0[C_x] + \sum_{i \neq 0} k_i[C_i] \tag{3.1}$$

(as algebraic cycles) with $k_i \geq 0$ for all i and

$$-(K_X \cdot C_x) \leq \dim X + 1.$$

Proof. See [Kol96, Theorem II.5.14 and Remark II.5.15]. The relation (3.1) can be deduced looking directly at the proofs of the bend-and-break lemmas (cf. [Kol96, Corollary II.5.6 and Theorem II.5.7]): our notation is slightly different, since in (3.1) we have isolated a rational curve with the required intersection properties. $\square$

The application we need for our purpose is the following.

Corollary 3.28 ($\spadesuit$). *Let X be a surface which fibres over a curve C via $f: X \rightarrow C$ and let F be the general fibre of f . Assume that X has only local complete intersection singularities along F and that F is a (possibly*

singular) rational curve such that

$$(K_X \cdot F) < 0.$$

Then

$$-(K_X \cdot F) \leq 3.$$

Proof. We are in the hypothesis of Theorem 3.27, so we can take a point x in the smooth locus of X and deduce the existence of a rational curve C' passing through x such that

$$-(K_X \cdot C') \leq 3 \text{ and } [F] \stackrel{e}{\approx} k_0[C'] + \sum_{i \neq 0} k_i[C_i].$$

By [Kol96, Exercise II.4.1.10], the curves appearing on the right hand side of the previous equation must be contained in the fibres of f . F is the general fibre, so the second relation implies that $k_0 = 1$ and $k_i = 0$ for all $i \neq 0$ and so that $C' = F$. $\square$

3.3 Effective Matsusaka's theorem

In this section we prove our main theorem, assuming the results on effective very ampleness that we will prove in the next section. If not specified, X will denote a smooth surface over an algebraically closed field of arbitrary characteristic.

We start recalling few results. The first is a numerical criterion for bigness, whose characteristic-free proof is based on Riemann-Roch theorem (cf. [Laz04, Theorem 2.2.15]).

Theorem 3.29 (★). *Let D and E be nef $\mathbb{Q}$ -divisors on X and assume that*

$$D^2 > 2(D \cdot E).$$

Then $D - E$ is big.

Some further lemmas are needed for the proof.

Lemma 3.30 (★). *Let D be an ample divisor on X . Then $K_X + 2D + C$ is nef for any irreducible curve $C \subset X$.*

Proof. If $X = \mathbb{P}^2$ then the lemma is trivially proved. By the Cone theorem (cf. [KM92, Theorem 1.24]) and the classification of surfaces with extremal rays of maximal length, we have that $K_X + 2D$ is always a nef divisor. This implies that $K_X + 2D + C$ may have negative intersection number only when intersected with C . On the other hand, by adjunction, $(K_X + C) \cdot C = 2g - 2 \geq -2$, where g is the arithmetic genus of C . The divisor D is ample, so we obtain the statement. $\square$

Assuming the characteristic of the base field to be arbitrary and at least in the case of surfaces, the key result to prove an effective Matsusaka's theorem (on very ampleness) is an effective nefness result (cf. [Laz04, Theorem 10.2.4]).

Theorem 3.31 (♠). *Let D be an ample divisor and let B be a nef divisor on X . Then $mD - B$ is nef for any*

$$m \geq \frac{2D \cdot B}{D^2} ((K_X + 2D) \cdot D + 1) + 1.$$

Proof. To simplify the notation in the proof let us define the following thresholds for the pair (D, B) :

$$\begin{aligned} \eta &= \eta(D, B) := \inf \{t \in \mathbb{R}_{>0} \mid tD - B \text{ is nef}\}, \\ \gamma &= \gamma(D, B) := \inf \{t \in \mathbb{R}_{>0} \mid tD - B \text{ is pseudo-effective}\}. \end{aligned}$$

In order to show the theorem, we need to find an upper bound on η (clearly, $\gamma \leq \eta$ since a nef divisor is also pseudo-effective).

By definition of the threshold, $\eta D - B$ is contained in the boundary of the nef cone $\text{Nef}(X)$. Nakai-Moishezon criterion implies that we are in one of the following two cases:

- $(\eta D - B)^2 = 0$, or
- there exists an irreducible curve C such that $(\eta D - B) \cdot C = 0$.

If $(\eta D - B)^2 = 0$, then it is easy to see that

$$\eta \leq 2 \frac{D \cdot B}{D^2}.$$

Let us assume that there is an irreducible curve C such that $\eta D \cdot C = B \cdot C$.

Let us define $G := \gamma D - B$. Then

$$G \cdot C = (\gamma - \eta) D \cdot C \leq (\gamma - \eta).$$

To simplify our computations, define $A := K_X + 2D$. By Lemma 3.30 and the definition of G , we have that $(A + C) \cdot G \geq 0$. Combining with the previous inequality we get

$$(\eta - \gamma) \leq -G \cdot C \leq A \cdot G = \gamma A \cdot D - A \cdot B.$$

In particular,

$$\eta \leq \gamma(A \cdot D + 1) - A \cdot B \leq \gamma(A \cdot D + 1).$$

The statement of our result follows from Theorem 3.29, which guarantees that $\gamma < \frac{2D \cdot B}{D^2}$.

□

Remark 3.32. *The previous proof is completely characteristic-free, although the new result is for surfaces in positive characteristic.*

We can now prove our main theorem, assuming the results in the next section.

Proof of Theorem 3.5. By Corollary 3.26, if X is non-pathological and $H = K_X + 4D$ then $H + N$ is very ample for any nef divisor N . By Theorem 3.31, $mD - (H + B)$ is nef for any m as in the statement. Then $K_X + 4D + (mD - K_X - 4D - B)$ is very ample. For pathological surfaces, we can use Theorem 3.43 and Theorem 3.46 to obtain the desired very ample divisor H and deduce the result. $\square$

3.4 Effective very ampleness in positive characteristic

In order to complete the proof of Theorem 3.5, we need an effective very ampleness theorem for pathological quasi-elliptic surfaces of Kodaira dimension one and pathological surfaces of general type. Our ultimate goal is to prove the following result via a case-by-case analysis.

Theorem 3.33 ($\spadesuit$). *Let X a smooth surface over an algebraically closed field of characteristic $p > 0$, D an ample Cartier divisor on X and let $L(a, b) := aK_X + bD$ for positive integers a and b . Then $L(a, b)$ is very ample for the following values of a and b :*

1. *if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 3$, $a = 1$ and $b \geq 8$;*
2. *if X is quasi-elliptic with $\kappa(X) = 1$ and $p = 2$, $a = 1$ and $b \geq 19$;*
3. *if X is of general type with $p \geq 3$, $a = 2$ and $b \geq 4$;*
4. *if X is of general type with $p = 2$, $a = 2$ and $b \geq 19$.*

In positive characteristic we cannot expect to obtain precisely Theorem 3.13. Nonetheless, we will deduce a (worse) effective result for pathological surfaces. Let us fix the notation.

Definition 3.34. A big divisor D on a smooth surface X with $(D^2) > 0$ is m -unstable for a positive integer m if either

- $H^1(X, \mathcal{O}_X(-D)) = 0$; or
- $H^1(X, \mathcal{O}_X(-D)) \neq 0$ and there exists a nonzero effective divisor E such that
 - $mD - 2E$ is big;
 - $(mD - E) \cdot E \leq 0$.

Remark 3.35. *Theorem 3.13 tells us that in characteristic zero every big divisor D on a smooth surface X with $(D^2) > 0$ is 1-unstable. The same holds in positive characteristic, if we assume that the surface is non-pathological. Our goal here is to clarify the picture in the remaining cases.*

We can start our analysis with quasi-elliptic surfaces of maximal Kodaira dimension.

Proposition 3.36 (♠). *Let X be a quasi-elliptic with $\kappa(X) = 1$ and let D be a big divisor on X with $(D^2) > 0$. Then*

1. *if $p = 3$, then D is 3-unstable;*
2. *if $p = 2$, then D is 4-unstable.*

Proof. Let us work out the case $p = 3$. Assume $H^1(X, \mathcal{O}_X(-D)) \neq 0$. This nonzero element gives a (non-split) extension

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X(D) \rightarrow 0.$$

Bogomolov's result (Theorem 3.21) implies that $(F^*)^e \mathcal{E}$ is unstable for e sufficiently large. We need to bound this coefficient.

To prove the proposition we need to show that $e = 1$.

Assume $e \geq 2$ and let F be the general element of the pencil of the fibration $f: X \rightarrow B$ in curves with a cusp. Let $\rho: Y \rightarrow X$ be the p^e -purely inseparable morphism appearing in Construction 3.23. Then $\{C_i := \rho^*F\}$ is a family of movable rational curves in Y . Let us define $g := f \circ \rho$ and consider its Stein factorisation:

$$\begin{array}{ccccc}
 & Y & & & \\
 & \downarrow \rho & \searrow h & & \\
 g \swarrow & X & & B' & \\
 & \downarrow f & \swarrow c & & \\
 & B & & &
 \end{array}$$

Since the C_i 's are precisely the fibres of h , we can use Corollary 3.28 on $h: Y \rightarrow B'$ (Y is defined by a quasi-section in a $\mathbb{P}^1$ -bundle over X , so it has hypersurface singularities along the general element of $\{C_i\}$) and deduce that

$$0 < -(K_Y \cdot C_i) \leq 3.$$

This gives a contradiction, since the following chain of inequality holds:

$$\begin{aligned}
 3 &\geq -(K_Y \cdot C_i) = \left(\rho^* \left(\frac{p^e - 1}{p^e} (p^e D - 2E) - K_X \right) \cdot C_i \right) \\
 &= p^e \left(\left(\frac{p^e - 1}{p^e} (p^e D - 2E) - K_X \right) \cdot F \right) = ((p^e - 1)(p^e D - 2E) \cdot F) \geq p^e - 1 \geq 8,
 \end{aligned}$$

where E is the effective divisor appearing in Remark 3.24.

The same proof for $p = 2$ gives, in that case $e \leq 2$. □

Let us move to surfaces of general type. For this case we need to use the following result by Shepherd-Barron (cf. [SB91b, Theorem 12]).

Theorem 3.37 (Shepherd-Barron). *Let D be a big Cartier divisor on a smooth surface X of general type which verifies one of the following hypoth-*

esis:

- $p \geq 3$ and $(D^2) > \text{vol}(X)$;
- $p = 2$ and $(D^2) > \max\{\text{vol}(X), \text{vol}(X) - 3\chi(\mathcal{O}_X) + 2\}$.

Then D is 1-unstable.

Since the bound of the previous theorem depends on $\chi(\mathcal{O}_X)$ if $p = 2$, we need further analysis: another result by Shepherd-Barrons useful for this (cf. [SB91a, Theorem 8]).

Theorem 3.38 (Shepherd-Barron). *Let X be a surface in characteristic $p = 2$ of general type with $\chi(\mathcal{O}_X) < 0$. Then there is a fibration $f: X \rightarrow C$ over a smooth curve C , whose generic fibre is a singular rational curve with arithmetic genus $2 \leq g \leq 4$.*

Let us prove our effectivity result for general type surfaces.

Proposition 3.39 ($\spadesuit$). *Let D be a big Cartier divisor on a surface in characteristic $p = 2$ of general type with $\chi(\mathcal{O}_X) < 0$ such that $(D^2) > \text{vol}(X)$. Then D is 4-unstable.*

Proof. As in Proposition 3.36, assume $H^1(X, \mathcal{O}_X(-D)) \neq 0$. Also here we have a non-split extension

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E} \rightarrow \mathcal{O}_X(D) \rightarrow 0.$$

Using Theorem 3.21 one deduces that $(F^*)^e \mathcal{E}$ for e sufficiently large is unstable. Let F be the general element of the pencil which gives the fibration in singular rational curves given by Theorem 3.38 and let $\rho: Y \rightarrow X$ be the p^e -purely inseparable morphism of Construction 3.23. Like in the proof of Proposition 3.36, we use Corollary 3.28 on Y and deduce that

$0 < -(K_Y \cdot C_i) \leq 3$. This implies the following relations

$$\begin{aligned} 3 &\geq -(K_Y \cdot C_i) = \left(\rho^* \left(\frac{2^e - 1}{2^e} (2^e D - 2E) - K_X \right) \cdot C_i \right) \\ &= 2^e \left(\left(\frac{2^e - 1}{2^e} (2^e D - 2E) - K_X \right) \cdot F \right) = (((2^e - 1)(2^e D - 2E) - 2^e K_X) \cdot F) \geq 1. \end{aligned}$$

So, the following holds:

$$(((2^e - 1)(2^{e-1} D - E) - 2^{e-1} K_X) \cdot F) = 1.$$

As a consequence, we apply Theorem 3.38 to bound the intersection $(K_X \cdot F)$:

$$(2^e - 1)((2^{e-1} D - E) \cdot F) = 2^e(g - 1) + 1,$$

where g is the arithmetic genus of F . Some basic arithmetic give that the only possibilities for the pair (g, e) are $(2, 1)$, $(3, 1)$, $(3, 2)$ and $(4, 1)$. □

Theorem 3.37 can be used to deduce a variant of Reider's theorem in positive characteristic. We recall a technical proposition we will need later (cf. [Sak90, Proposition 2], the proof is characteristic-free).

Proposition 3.40 (Sakai). *Let $\pi : Y \rightarrow X$ be a birational morphism between two normal surfaces. Let $\tilde{D}$ be a Cartier divisor on Y such that $\tilde{D}^2 > 0$. Assume there is a nonzero effective divisor $\tilde{E}$ such that*

- $\tilde{D} - 2\tilde{E}$ is big and
- $(\tilde{D} - \tilde{E}) \cdot \tilde{E} \leq 0$.

Set $D := \pi_ \tilde{D}$, $E := \pi_* \tilde{E}$ and $\alpha = D^2 - \tilde{D}^2$. If D is nef and E is a non-zero effective divisor, then*

- $0 \leq D \cdot E < \alpha/2$,

- $D \cdot E - \alpha/4 \leq E^2 \leq (D \cdot E)^2/D^2$.

In our case, we obtain this immediate corollary.

Corollary 3.41 ($\spadesuit$). *Let $\pi : Y \rightarrow X$ be a birational morphism between two smooth surfaces and let $\tilde{D}$ be a big Cartier divisor on Y such that $(\tilde{D}^2) > 0$. Assume that*

- $H^1(X, \mathcal{O}_X(-\tilde{D})) \neq 0$;
- $\tilde{D}$ is m -unstable for some $m > 0$.

Set $D := \pi_\tilde{D}$ and $\alpha = D^2 - \tilde{D}^2$. Then if D is nef, there exists a nonzero effective divisor E on X such that*

- $0 \leq D \cdot E < m\alpha/2$,
- $mD \cdot E - m^2\alpha/4 \leq E^2 \leq (D \cdot E)^2/D^2$.

We prove our main theorem in this section case-by case, starting with quasi-elliptic surfaces. For them, we apply Theorem 3.36 and the previous corollary.

Proposition 3.42 ($\spadesuit$). *Let X be a quasi-elliptic surface with maximal Kodaira dimension. Let D be a big and nef divisor on X . Then the following holds.*

- $p = 3$:
 - if $D^2 > 4$ and $|K_X + D|$ has a base point at $x \in X$, there exists a curve C such that $D \cdot C \leq 5$;
 - if $D^2 > 9$ and $|K_X + D|$ does not separate any two points $x, y \in X$, there exists a curve C such that $D \cdot C \leq 13$;
- $p = 2$:
 - if $D^2 > 4$ and $|K_X + D|$ has a base point at $x \in X$, there exists a curve C such that $D \cdot C \leq 7$;

- if $D^2 > 9$ and $|K_X + D|$ does not separate any two points $x, y \in X$,
there exists a curve C such that $D \cdot C \leq 17$;

Proof. We will give explicit computations for the case $p = 3$. Assume that $|K_X + D|$ has a base point at $x \in X$. Let $\pi : Y \rightarrow X$ be the blow up at x . Since x is a base point we have that $H^1(Y, \mathcal{O}_Y(K_Y + \pi^*D - 2F)) \neq 0$, where F is the exceptional divisor of π . Let $\tilde{D} := \pi^*D - 2F$.

By our assumption we have that $\tilde{D}^2 > 0$ and so, by Theorem 3.36, we can find an effective divisor $\tilde{E}$ such that $p\tilde{D} - 2\tilde{E}$ is big and $(p\tilde{D} - \tilde{E}) \cdot \tilde{E} \leq 0$. The previous inequality easily implies that $\tilde{E}$ is not a positive multiple of the exceptional divisor and in particular $E := \pi_*\tilde{E}$ is a non-zero effective divisor. Moreover, $D = \pi_*\tilde{D}$ is nef by assumption, thus we can apply Corollary 3.41 and since $\alpha = (D^2 - \tilde{D}^2) = 4$, the first inequality of the corollary implies that $D \cdot E \leq 5$.

The statement on separation of points follows exactly in the same way. Note that we allow the case $x = y$.

The bounds for the case $p = 2$ can be obtained the same way ($\tilde{D}$ is p^2 -unstable in this case). $\square$

Effective very ampleness for adjoint divisors for quasi elliptic surfaces can now be proved.

Theorem 3.43 ($\spadesuit$). *Let D be an ample Cartier divisor on a smooth quasi-elliptic surface X with $\kappa(X) = 1$. Then*

- if $p = 3$, the divisor $K_X + kD$ is base-point free for any $k \geq 4$ and it is very ample for any $k \geq 8$;
- if $p = 2$, the divisor $K_X + kD$ is base-point free for any $k \geq 5$ and it is very ample for any $k \geq 19$.

In particular, if N is any nef divisor, $K_X + kD + N$ is always very ample for any $k \geq 8$ (resp. $k \geq 19$) in characteristic 3 (resp. 2).

Proof. Also here we show one of the computations.

We compute here the minimal multiple of D which contradicts the second inequality of Corollary 3.41 to prove the theorem.

Let us start with base-point-freeness for $p = 3$. Assume that $k \geq 5$, $K_X + kD$ has a base point and define $D' := kD$. Then, by Proposition 3.42, we know that there exists an effective divisor E such that $(D' \cdot E) \leq 5$.

This implies

$$(D \cdot E) \leq 1.$$

Now use the second inequality of Corollary 3.41 on D' to deduce that

$$15 - 9 \leq 3(D' \cdot E) - 9 \leq \frac{(D' \cdot E)^2}{(D'^2)} \leq 1.$$

Contradiction.

Similar computations give the other bounds. $\square$

For surfaces of general type, the problem is more subtle, since we need some extra positivity from the canonical divisor itself to guarantee basepoint-freeness and very ampleness. The analogous of Proposition 3.42 is the following.

Proposition 3.44 ($\spadesuit$). *Let X be a surface of general type and let D be a big and nef divisor on X . Then the following holds.*

- $p \geq 3$:
 - if $D^2 > \text{vol}(X) + 4$ and $|K_X + D|$ has a base point at $x \in X$, there exists a curve C such that $D \cdot C \leq 1$;
 - if $D^2 > \text{vol}(X) + 9$ and $|K_X + D|$ does not separate any two points $x, y \in X$, there exists a curve C such that $D \cdot C \leq 2$;
- $p = 2$:

- if $D^2 > \text{vol}(X) + 6$ and $|K_X + D|$ has a base point at $x \in X$, there exists a curve C such that $D \cdot C \leq 7$;
- if $D^2 > \text{vol}(X) + 11$ and $|K_X + D|$ does not separate any two points $x, y \in X$, there exists a curve C such that $D \cdot C \leq 17$;

Proof. The proof is basically the same as Proposition 3.42. Let $p \geq 3$ and assume that $|K_X + D|$ has a base point at $x \in X$. Using the same notation as Proposition 3.42, we can blow up x and deduce the existence of an effective divisor $\tilde{E}$ such that $\tilde{D} - 2\tilde{E}$ is big and $(\tilde{D} - \tilde{E}) \cdot \tilde{E} \leq 0$ (in order to deduce 1-instability we use Theorem 3.37). Also here, the first inequality of Corollary 3.41 implies that $(D \cdot E) \leq 1$.

The statement on separation of points follow the same way.

The bounds for the case $p = 2$ we use the same strategy, via a combination of Theorem 3.38 and Proposition 3.39. $\square$

The following effective very ampleness statement can be deduced for this class of surfaces.

Remark 3.45. *If we directly apply Proposition 3.44, this would provide bounds that depend on the volume. Nonetheless, it is possible to get a uniform bound if we work with linear systems of the type $|2K_X + mD|$. Note that we get sharp statements for those linear systems.*

Theorem 3.46 ($\spadesuit$). *Let D be an ample Cartier divisor on a smooth surface X of general type. Then*

- *if $p \geq 3$, the divisor $2K_X + kD$ is base-point free for any $k \geq 3$ and it is very ample for any $k \geq 4$.*
- *if $p = 2$ the divisor $2K_X + kD$ is base-point free for any $k \geq 5$ and it is very ample for any $k \geq 19$.*

In particular, if N is any nef divisor, $2K_X + kD + N$ is always very ample for any $k \geq 4$ (resp. $k \geq 19$) in characteristic $p \geq 3$ (resp. $p = 2$).

Proof. Since negative extremal rays of general type surfaces have length one, if $m \geq 3$, we know that $L := K_X + mD$ is an ample divisor and $L \cdot C \geq 2$ for any irreducible curve $C \subset X$. Moreover, Theorem 3.10 gives

$$L^2 = \text{vol}(L) \geq \text{vol}(K_X) + 9D^2 > \text{vol}(X) + 4$$

and Proposition 3.44 implies that $K_X + L = 2K_X + kD$ is base-point free for any $k \geq 4$. A similar computation allows us to derive very ampleness.

The same strategy gives the result for $p = 2$. $\square$

Proof of Theorem 3.33. This is a combination of Theorem 3.43 and Theorem 3.46. $\square$

Remark 3.47. In [Ter99], similar results can be found. Nonetheless our approach allows us to work out effective base-point freeness and very ampleness also on quasi-elliptic surfaces and arbitrary surfaces of general type.

3.5 A Kawamata-Viehweg-type Vanishing Theorem in positive characteristic

Our techniques can be used also in order to deduce effective vanishing theorems for adjoint divisors: in this section we extend the results in [Ter99], where the author uses [SB91b] to deduce a Kawamata-Viehweg-type theorem for non-pathological smooth surfaces. Our techniques apply in general and we can deduce an effective Kawamata-Viehweg type theorem in positive characteristic for surfaces on which even Kodaira vanishing fails.

Let us first recall the classical Kawamata-Viehweg vanishing theorem in its general version (cf. [KM92] for the general notation).

Theorem 3.48 (Kawamata, Viehweg). *Let (X, B) be a klt pair over an algebraically closed field of characteristic zero and let D be a Cartier divisor*

on X such that $D - (K_X + B)$ is big and nef. Then

$$H^i(X, \mathcal{O}_X(D)) = 0$$

for any $i > 0$.

In positive characteristic, even for non-pathological smooth surfaces, there exist counterexamples to Theorem 3.48: Xie has interesting examples in [Xie10] of relatively minimal irregular ruled surfaces in every characteristic where Theorem 3.48 fails.

In the case when the boundary B is trivial, Mukai deduced the following structural result (cf. [Muk13]).

Theorem 3.49 (Mukai). *Let X be a smooth surface in positive characteristic. Assume there exists a big and nef Cartier divisor D on X such that*

$$H^1(X, \mathcal{O}_X(K_X + D)) \neq 0.$$

Then:

- X is either quasi-elliptic of Kodaira dimension one or of general type;
- up to a sequence of blow-ups, X has the structure of a fibered surface over a smooth curve such that every fibre is connected and singular.

In weaker hypothesis, Terakawa deduced the following vanishing theorem (cf. [Ter99]), using the techniques in [SB91b].

Theorem 3.50 (Terakawa). *Let X be a smooth projective surface over an algebraically closed field of characteristic $p > 0$ and let D be a big and nef Cartier divisor on X . Assume that either*

1. $\kappa(X) \neq 2$ and X is not quasi-elliptic with $\kappa(X) = 1$; or
2. X is of general type with

- $p \geq 3$ and $(D^2) > \text{vol}(X)$; or
- $p = 2$ and $(D^2) > \max\{\text{vol}(X), \text{vol}(X) - 3\chi(\mathcal{O}_X) + 2\}$.

Then

$$H^i(X, \mathcal{O}_X(K_X + D)) = 0$$

for all $i > 0$.

Our aim is to improve this theorem for arbitrary surfaces, via bend-and-break techniques.

More generally, we will deduce some results on the injectivity of cohomological maps

$$H^1(X, \mathcal{O}_X(-D)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-pD))$$

for a big divisor on X .

The following result by Kollár, based on Construction 3.4 is an application of bend-and-break lemmas (cf. Theorem 3.27), specialised in our two-dimensional setting.

Theorem 3.51 (Kollár). *Let X be a smooth projective variety over a field of positive characteristic and let D be a Cartier divisor on X such that:*

1. $H^1(X, \mathcal{O}_X(-mD)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-pmD))$ is not injective for some integer $m > 0$;
2. there exists a curve C on X such that

$$(p-1)(D \cdot C) - (K_X \cdot C) > 0.$$

Then through every point x of C there is a rational curve C_x such that

$$[C] \stackrel{e}{\approx} k_0[C_x] + \sum_{i \neq 0} k_i[C_i] \tag{3.2}$$

(as algebraic cycles), with $k_i \geq 0$ for all i and

$$(p-1)(D \cdot C_x) - (K_X \cdot C_x) \leq \dim(X) + 1.$$

Proof. We can prove this result via a slight modification of [Kol96, Theorem II.6.2]. Assumption (1) allows us to construct (cf. Construction 3.4) a finite morphism

$$\pi: Y \rightarrow X,$$

where Y is defined as a Cartier divisor in the projectivisation of a non-split rank-two bundle over X . Furthermore, the following property holds:

$$K_Y = \pi^*(K_X + (k(1-p)D)),$$

where k is the largest integer for which $H^1(X, -kD) \neq 0$.

Consider the curve given in (2) and define $C' := \text{red } \pi^{-1}(C)$. The hypothesis on the intersection numbers and the formula for the canonical divisor of Y guarantee that $(K_Y \cdot C') < 0$. Let $y \in C'$ be a pre-image of x in Y . One can apply Theorem 3.27 and deduce the existence of a rational curve C'_y passing through y . Via the projection formula, we obtain a curve C_x on X for which:

$$(p-1)(D \cdot C_x) - (K_X \cdot C_x) \leq \dim(X) + 1.$$

□

Remark 3.52. *If we assume $\dim X = 2$ and the divisor D to be big and nef, the asymptotic condition*

$$H^1(X, \mathcal{O}_X(-mD)) = 0,$$

for m sufficiently large is guaranteed by a result of Szpiro (cf. [Szp79]).

As a consequence, the following holds.

Corollary 3.53 (Kollár). *Let X be a smooth projective surface over a field of positive characteristic and let D be a big and nef Cartier divisor on X such that $H^1(X, \mathcal{O}_X(-D)) \neq 0$. Assume there exists a curve C on X such that*

$$(p-1)(D \cdot C) - (K_X \cdot C) > 0.$$

Then through every point x of C there is a rational curve C_x such that

$$(p-1)(D \cdot C_x) - (K_X \cdot C_x) \leq 3.$$

We will show later how to use Corollary 3.53 to deduce our effective version of Kawamata-Viehweg vanishing.

In what follows, we will also need the following lemma on fibered surfaces, which explicitly gives a bound on the genus of the fibre with respect to the volume of the surface.

Lemma 3.54 ($\spadesuit$). *Let $f: X \rightarrow C$ be a fibered surface of general type and let g be the arithmetic genus of the general fibre F . Then*

$$\text{vol}(X) \geq g - 4.$$

Proof. We divide our analysis according to the genus b of the base curve, after having assumed the fibration is relatively minimal (i.e. that $K_{X/C}$ is nef).

$b \geq 2$ In this case we can deduce a better estimate. Indeed,

$$\text{vol}(X) \geq (K_X^2) = (K_{X/C}^2) + 8(g-1)(b-1) \geq 8g - 8.$$

$b = 1$ In this case we need a more careful analysis, since in positive characteristic we cannot assume the semi-positivity of $f_*K_{X/C}$. Nonetheless

the following general formula holds:

$$\deg(f_*K_{X/C}) = \chi(\mathcal{O}_X) - (g-1)(b-1), \quad (3.3)$$

which specialises to

$$\deg(f_*K_{X/C}) = \chi(\mathcal{O}_X) \geq 0.$$

Formula (3.3) can be obtained via Riemann-Roch, since we know that $R^1f_*K_{X/C} = \mathcal{O}_C$ and that $R^1f_*nK_{X/C} = 0$ for $n \geq 2$ by relative minimality. This inequality can be assumed by [SB91a, Theorem 8]. Furthermore, one can apply the following formula

$$\deg(f_*(nK_{X/C})) = \deg(f_*K_{X/C}) + \frac{n(n-1)}{2}(K_{X/C}^2).$$

Having $K_{X/C}$ big, we deduce that

$$\deg(f_*(2K_{X/C})) \geq 1.$$

As a consequence, we can apply the results of [Ati57] and deduce a decomposition of $f_*(2K_{X/C})$ into indecomposable vector bundles

$$f_*(2K_{X/C}) = \bigoplus_i E_i,$$

where we can assume that $\deg(E_1) \geq 1$. This implies that all quotient bundles of E_1 have positive degree. We want to show now that there exists a degree-one divisor L_1 on C for which $h^0(C, f_*(2K_{X/C}) \otimes \mathcal{O}_C(-L_1)) \neq 0$.

But this is clear, since, for every degree-one divisor L on C , one has that all quotient bundles of $f_*(2K_{X/C}) \otimes \mathcal{O}_C(-L)$ have degree zero and, up to a twisting by a degree-zero divisor on C , one can assume

there exists a quotient

$$f_*(2K_{X/C} \otimes \mathcal{O}_C(-L_1)) \rightarrow \mathcal{O}_C \rightarrow 0.$$

This implies that $h^0(X, \mathcal{O}_X(2K_{X/C} - F)) (= h^0(C, f_*(2K_{X/C}) \otimes \mathcal{O}_C(-L_1))) \neq 0$, where F is the general fibre of f and, since $K_X = K_{X/C}$ is nef, that

$$(K_X \cdot (2K_X - F)) \geq 0.$$

We have obtained in this case the bound

$$\text{vol}(X) \geq (K_X^2) \geq g - 1.$$

$b = 0$ Also in this case we can assume that $\chi(\mathcal{O}_X) \geq 0$ and, as a consequence, that

$$\deg(f_*K_{X/\mathbb{P}^1}) = \chi(\mathcal{O}_X) + g - 1 \geq g - 1.$$

If $g \geq 6$,

$$\deg(f_*K_{X/\mathbb{P}^1}) \geq 5.$$

This implies that $\deg(f_*K_X \otimes \mathcal{O}_{\mathbb{P}^1}(3)) \geq 0$ and, as a consequence of Grothendieck's classification of vector bundles on $\mathbb{P}^1$,

$$h^0(X, \mathcal{O}_X(K_X - f^*\mathcal{O}_C(-3))) \neq 0.$$

As before, we have assumed that $K_{X/\mathbb{P}^1} = K_X + f^*\mathcal{O}_C(2)$ is nef, so

$$((K_X + f^*\mathcal{O}_C(2)) \cdot (K_X - f^*\mathcal{O}_C(-3))) \geq 0.$$

As a consequence,

$$\text{vol}(X) \geq (K_X^2) \geq 2g - 2.$$

If $g \leq 5$, we simply use the trivial inequality $\text{vol}(X) \geq 1$ to deduce

$$\text{vol}(X) \geq g - 4.$$

□

The effectiveness of our result only depend on the birational geometry of X . Here is the statement.

Theorem 3.55 (♠). *Let X be a smooth surface in characteristic $p > 0$ and let D be a big Cartier divisor D on X . Then, for all integers*

$$m > m_0 = \frac{2 \text{vol}(X) + 9}{p - 1},$$

the induced Frobenius map

$$H^1(X, \mathcal{O}_X(-mD)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-pmD))$$

is injective.

(If $\kappa(X) \neq 2$, the volume $\text{vol}(X) = 0$).

Remark 3.56. *The previous result becomes trivial if $H^1(X, \mathcal{O}_X(-D)) = 0$. Moreover, combined with Corollary 3.53, gives an effective version of Kawamata-Viehweg theorem (cf. Corollary 3.57) in the case of big and nef divisors.*

Proof. We prove the result by contradiction. Assume that

$$H^1(X, \mathcal{O}_X(-\lceil m_0 \rceil D)) \xrightarrow{F^*} H^1(X, \mathcal{O}_X(-p\lceil m_0 \rceil D))$$

has a nontrivial kernel. Then, after a sequence of blow-ups $f: X' \rightarrow X$, we can assume the existence of a (relatively minimal) fibration (possibly with

singular general fibre) of arithmetic genus g

$$\pi: X' \rightarrow C.$$

We remark that we can reduce to prove our result on X' : since $D' := f^*D$ is a big divisor we have the following commutative diagram

$$\begin{array}{ccc} H^1(X', \mathcal{O}_{X'}(-[m_0]D')) & \xrightarrow{F^*} & H^1(X', \mathcal{O}_{X'}(-p[m_0]D')) \\ \simeq \downarrow & & \downarrow \simeq \\ H^1(X, \mathcal{O}_X(-[m_0]D)) & \xrightarrow{F^*} & H^1(X, \mathcal{O}_X(-p[m_0]D)) \end{array}$$

where the vertical isomorphisms holds because of $R^1f_*\mathcal{O}_{X'} = 0$.

We can than apply Theorem 3.51 to $[m_0]D'$: choose C to be a general fibre F of π , which certainly intersects positively D' , and we can use Lemma 3.54 and obtain

$$(p-1)[m_0](D' \cdot F) - (K_{X'} \cdot F) \geq (p-1)[m_0] - (2g-2) > 3. \quad (3.4)$$

So we can apply Theorem 3.51: fix a point $x \in F$ and find a rational curve C_x such that

$$(p-1)m_0(D \cdot C_x) - (K_X \cdot C_x) \leq 3.$$

By construction, $F = C_x$ because of (3.2) in Theorem 3.51. But this is a contradiction, because of (3.4). $\square$

We finally obtain our effective vanishing theorem as a corollary.

Corollary 3.57 ($\spadesuit$). *Let X be a smooth surface in characteristic $p > 0$ and let D be a big and nef Cartier divisor D on X . Then,*

$$H^1(X, \mathcal{O}_X(K_X + mD)) = 0$$

for all integers $m > m_0$, where

- $m_0 = \frac{3}{p-1}$ if X is quasi-elliptic with $\kappa(X) = 1$;
- $m_0 = \frac{2\text{vol}(X)+9}{p-1}$ if X is of general type.

Proof. For surfaces of general type, the result is trivial. For quasi-elliptic surfaces, a better bound can be obtained, since $(K_X \cdot F) = 0$, for the general fibre F . □

4 Conclusion: Future Work

In this chapter, I present some research projects I plan to develop in a near future.

This project is deeply related to many different research lines in algebraic geometry (such as classification, MMP, Fano varieties, toric geometry, K -stability, positive characteristic methods, etc), interesting for complex differential geometers (because of the connection with Kähler-Einstein metrics) and potentially interdisciplinary (thanks to the unexplored connection between fibre-like toric polytopes and optimisation problems).

Investigation of smooth four-dimensional fibre-like Fano varieties

In the results in Chapter 2, we analyse the three-dimensional picture, using the classification (up to deformation) of smooth Fano varieties in dimension three due to Mori and Mukai: our strategy consists in applying our general criteria to deduce information on the structure of the nef cone (and in particular on the induced monodromy action of the MFS on it) for three-dimensional Fano varieties.

Furthermore, our approach can be applied also in higher dimension and we can easily provide examples of Fano varieties in arbitrary dimensions which are not fibre-like. This study of threefolds is based on explicit constructions of MFS's.

Classification of smooth Fano varieties is a rather active field of research: *Fanosearch* group at Imperial College London has an ongoing classification

project via quantum periods for dimension four and some partial results have been already obtained.

Classification of fibre-like Fano varieties in dimension four, contextually to the advance of classification in dimension four, seems reasonable, since the majority of Fano fourfold can be realised as complete intersections in toric varieties and Theorem 2.62 could work to prove that *very symmetric* Fano fourfold can be realised as a general fibre of a MFS. On the other hand, one can use the necessary criterion of Theorem 2.42 to show that the majority of smooth Fano fourfolds cannot be fibre-like.

Complete classification of fibre-like smooth Fano toric varieties

For this project, we want to consider Conjecture 2.95, proved up to dimension eight, to give an explicit characterisation of smooth toric fibre-like Fanos via their polytopes.

Some classes of fibre-like polytopes have been already studied, but the full classification is far from being completely understood. In particular, it seems that (the dual of) fibre-like polytopes also emerge in classical optimisation problems: this connection between fibre-like toric Fano varieties and transportation polytopes is almost completely unexplored (cf. [VK84]). In this research line, we would act in two directions:

1. use of computer algebra softwares, such as MAGMA, in order to explicitly study the low dimensional fibre-like polytopes (a very detailed database of smooth toric Fano varieties is available up to dimension eight) with special care for *sporadic* fibre-like toric Fano varieties in dimension larger than seven and algorithmically construct new fibre-like Fano polytopes (the fibre-like condition can be verified with MAGMA in arbitrary dimension);
2. investigate the connection between fibre-like and transportation poly-

topes (cf. [YKK84]).

In this way, new classes of fibre-like polytopes would emerge and the general picture would be sensibly clarified.

Connection between fibre-likeness and K -semistability for smooth Fano varieties

Being realised as a general fibre of a MFS seems to be a tricky condition and it is natural to understand its connection with other algebro-geometric notions for Fano varieties which have been developed in these years.

Conjecture 4.1. *Every smooth fibre-like Fano variety is K -semistable.*

This conjecture on K -semistability, is far from being verified. It is a relative version of Conjecture 2.91.

In this context, the notion of fibre-likeness would give a completely new prospective to verify K -stability for Fano varieties in a purely algebro-geometric way, bypassing birational rigidity, i.e. the notion of being the unique end product of an MMP, or the explicit computations of alpha-invariants. In this context, our point of view is radically different: fibre-likeness does not deal with the structure of the total space of a MFS, but rather with the geometric constraints on the fibres which appear in MFS's. This line of research is now at a preliminary status.

The plan is to:

- work out the K -stability of the 8 classes of smooth Fano threefold appearing in Theorem 2.67, by explicit computations of α_G -invariant in the style of [CS08] for very symmetric elements in each class and deduce the K -stability at least for the general element of the family via the methods developed in [Oda12];

- extend the method to smooth Fano varieties in higher dimension which can be realised as complete intersections in products of projective spaces (or more generally toric varieties), using the automorphisms of the ambient space induced on the linear system defined by the complete intersection.

Fano varieties and divisorial contractions

Always in the context of the MMP, one can modify the setting and discuss the following question.

Question 4.2. *Which Fano varieties can be realised as an exceptional divisor of a divisorial contraction to a point?*

In [Mor82], the fibres of a divisorial contraction to a point from a smooth threefold have been classified and Beltrametti extended Mori's analysis to smooth fourfolds in [Bel87]. Furthermore, the extensive work by Kawakita gives an exhaustive answer for threefolds in the local analytic setting (cf. [Kaw01], [Kaw02] and [Kaw05]).

Nonetheless, we are interested in understanding Question 4.2 in the algebraic category, rather than in the analytic one.

The reason why this question could be related to the one on MFS's is that, given a divisorial contraction $\pi: X \rightarrow Y$ which contracts a divisor E to a point, one can see that Y is analytically $\mathbb{Q}$ -factorial if and only if $\rho(E) = 1$. In this sense, this question becomes *trivial* for Fano varieties with Picard number one, in analogy with the question for Mori fibre spaces.

The following example is interesting in this setting: it shows that a cubic del Pezzo surface S_3 can be realised as an exceptional divisor over a point in a crepant (instead of extremal) divisorial contraction.

Example 4.3. Let $f: Z \rightarrow \mathbb{P}^4$ the blow-up of a point P with exceptional divisor $E \simeq \mathbb{P}^3$. Take Y to be a cubic in $\mathbb{P}^4$ with a triple point in P and

define $X := f^*(Y) - 3E$. The restriction $f: X \rightarrow Y$ gives a crepant divisorial contraction of a del Pezzo S_3 to a point.

One could try to construct other explicit examples of divisorial contractions in the algebraic category in dimension three, in the spirit of Example 4.3 in order to clarify the three-dimensional picture.

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