Real-Analytic Perturbation Theory

Arnold Diffusion in Billiards and Homoclinic Geodesics on Hypersurfaces

A Thesis Submitted for the Degree of Doctor of Philosophy
of Imperial College London

by

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Declaration

I declare that this thesis and the research to which it pertains are my own work. Any ideas from the work of other people that are mentioned in this thesis are fully acknowledged in accordance with standard referencing practices.

Andrew Clarke

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Abstract

In this thesis we study two different, but related, dynamical systems on hypersurfaces of Euclidean space. First, we consider billiard dynamics in a real-analytic, strictly convex domain. In dimension 2, if a billiard trajectory begins with the velocity vector making a positive angle with the boundary, then this angle remains bounded away from zero at each collision due to the existence of a family of invariant curves called caustics. We show that in higher dimensions, trajectories drifting towards the boundary can exist. Namely, using the geometric techniques of Arnold diffusion, we show that in three or more dimensions, assuming the geodesic flow on the boundary of the domain has a hyperbolic periodic orbit and a transverse homoclinic, the existence of trajectories asymptotically approaching the billiard boundary is a generic phenomenon in the real-analytic topology. Secondly, we address the assumptions from the first problem. We show that real-analytic, strictly convex, closed surfaces in 3-dimensional Euclidean space generically have a hyperbolic closed geodesic with a transverse homoclinic. It is also shown that real-analytic, closed hypersurfaces with an elliptic closed geodesic generically have a hyperbolic closed geodesic and a transverse homoclinic geodesic, in arbitrary dimension, and without the assumption of strict convexity. These are among the first perturbation-theoretic results for real-analytic geodesic flows.

For my mother, Mona O'Duffy Clarke (1963-2017)

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Publications

The content of Chapter 3 of this thesis is based on joint work with Dmitry Turaev, and has been submitted for publication in the paper [27]. The content of Chapter 4 of this thesis has been submitted for publication in the paper [26].

Contents

1 Introduction **9**

2 Background and Preliminaries **19**

2.1 Hyperbolic Dynamics 19

2.1.1 Homoclinic Trajectories and Hyperbolic Sets 19

2.1.2 Normally Hyperbolic Invariant Manifolds 22

2.1.3 Scattering Maps 25

2.2 Real-Analytic Perturbation Theory 27

2.3 Abraham's Transversality Theorem 29

2.4 Hamiltonian and Symplectic Dynamics 30

2.4.1 Hamiltonian Flows and Symplectic Maps 30

2.4.2 KAM Theory 33

2.4.3 Arnold Diffusion 34

2.4.4 k-Jets of Symplectic Automorphisms 37

2.5 Hypersurface Geometry 39

2.6 Convex Billiards 41

2.6.1 The Billiard Map 42

2.6.2 Caustics 44

2.6.3 Trajectories Asymptotic to the Boundary 45

2.6.4 Oscillatory Motions 46

2.7 Geodesic Flows 47

2.7.1 The Geodesic Flow on Riemannian Manifolds 47

2.7.2 The Geodesic Flow on Hypersurfaces 48

2.7.3 Classification of Closed Geodesics 49

3 Arnold Diffusion in Billiards **51**

3.1 Results 51

3.2 Billiard Trajectories Near the Boundary 55

3.2.1 Estimates Near the Boundary 56
3.2.2 Normally Hyperbolic Invariant Manifold for the Billiard Map . . 59
3.3 Homoclinic Cylinders and Pseudo-Orbits 65
3.3.1 Analysis of Hyperbolic and Homoclinic Cylinders 65
3.3.2 Shadowing of Pseudo-Orbits 69
3.4 Effect of a Perturbation on the Billiard and Scattering Maps 71
3.4.1 Perturbing the Billiard Map 71
3.4.2 Perturbative Computation of the Scattering Map 73
3.5 Diffusive Orbits 78

4 Generic Properties of Geodesic Flows 89
4.1 Results 89
4.2 Geometry and Perturbations 93
4.2.1 Effect of a Local Perturbation on the Geodesic Flow 93
4.2.2 Fermi Coordinates 95
4.2.3 Perturbations in Fermi Coordinates 96
4.2.4 A Class of Metrics on a Reference Manifold 98
4.3 Generic Nondegeneracy of Closed Geodesics 98
4.3.1 Perturbation-Invariant Phase Space and Proof of Theorem 4.1.1 98
4.3.2 Proof of Lemma 4.3.1 102
4.4 Generic Properties of k-Jets of Poincaré Maps 107
4.4.1 Effect of a Perturbation on the k-Jet 108
4.4.2 k-General Position of Families of Poincaré Maps 111
4.4.3 Proof of Theorem 4.4.1 114
4.5 Transversality and the Kupka-Smale Theorem 117
4.6 Generic Existence of Transverse Homoclinic Geodesics on Surfaces . . . 123

Bibliography 127

— 1 —

Introduction

A few years after Vladimir Arnold proved the KAM theorem – probably the most famous result about *stability* of near-integrable Hamiltonian systems – he published a short note with an example of a near-integrable Hamiltonian system exhibiting a certain type of *instability*, and expressed his belief in the existence of such instability in the general case. Chirikov subsequently named this instability phenomenon *Arnold diffusion*, and now, more than 50 years later, it is a very active area of research around which there has been built a deep theoretical framework.

Arnold diffusion takes place in the realm of *perturbation theory*. Typically one begins with a dynamical system that conserves some quantity, and then it is shown that this quantity can drift arbitrarily far under arbitrarily small perturbations of the system. The difficulty usually lies in showing that this phenomenon occurs not just for a specific perturbation, but for a *generic* perturbation.

The techniques employed by Arnold are of a geometric nature. He studied dynamics near particular invariant subsets of the system's phase space; these subsets are characterised by the existence of nearby manifolds, one of which consists of orbits approaching the invariant subset, and the other of orbits departing it. He observed that there are orbits departing one invariant subset but asymptotically approaching another. Arnold constructed a chain of such invariant sets connected via departing and approaching trajectories, and noticed that nearby trajectories must behave in a similar fashion, but without being bound by the invariant sets. He concluded that there exist orbits starting at the bottom of the chain and finishing at the top, thus *diffusing* across the phase space.

The motivation to study KAM theory and Arnold diffusion is linked to the question of stability of the solar system. The *n-body problem* – a mathematical model of the solar system constructed according to Newton's law of gravitation – requires formidable computations. Indeed, this hindered the progress of KAM theory in celestial mechanics for many years until Pinzari discovered a way of dealing with a degeneracy in the problem [100]. Despite progress in recent years, the problem of Arnold diffusion in

celestial mechanics, as well as in the general setting, remains open.

The billiard problem, introduced by Birkhoff [18], concerns the motion of a point particle in a domain. The particle travels along straight lines in the interior of the domain, and experiences elastic collisions at the boundary. Katok referred to the billiard problem as a "mathematician's playground" where "various questions, conjectures, methods of solution in the theory of dynamical systems are tested" [72]. In this thesis we use the geometric techniques of Arnold diffusion to investigate generic properties of convex billiards.

Drifting Trajectories in Convex Billiards

Let M be a real-analytic, closed, and strictly convex hypersurface of $\mathbb{R}^d$. The billiard map takes a point $x_0 \in M$ and an inward-pointing velocity vector v_0 of norm 1 and follows the straight line in the direction of v_0 until the next point of intersection with M. At this point, say x_1, the velocity vector transforms according to the optical law of reflection: the angle of reflection α_1 equals the angle of incidence β_1 (see Figure 1.1).

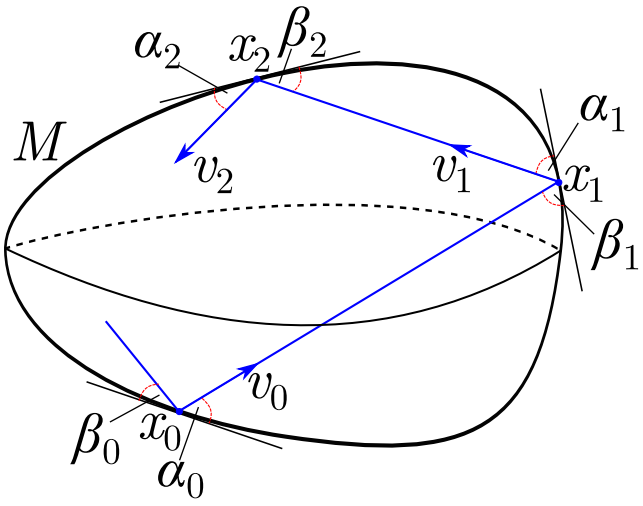


Figure 1.1: The billiard map takes a point $x_0 \in M$ and an inward-pointing velocity vector v_0 to the point x_1 and the velocity vector v_1. Iterating the billiard map gives an orbit $\{(x_n, v_n)\}_{n\in\mathbb{N}}$ with a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of angles of reflection.

Suppose first that M is a real-analytic, strictly convex, closed plane curve in $\mathbb{R}^2$. Consider another strictly convex closed plane curve C, contained inside the region bounded by M, with the following property: if one segment of a billiard trajectory is tangent to C, then so is every segment of that trajectory. A well-known example is the case when M is an ellipse; in this case, every confocal ellipse contained inside M is a caustic (see Figure 1.2). In fact, a result of Lazutkin implies the existence of a family of caustics accumulating near the boundary of any sufficiently smooth, strictly convex, closed billiard domain in $\mathbb{R}^2$ [80]. The phase space of the 2-dimensional billiard is a

2-dimensional cylinder with the angular coordinate being an arclength parameter on M, and the height corresponding to the angle of reflection. The family of caustics near the boundary of M corresponds to a family of invariant essential curves near the boundary of the phase space, where the angle of reflection is zero. Since the billiard map is close to the identity near the boundary of its phase space, these invariant curves divide the phase space into invariant regions where the angle of reflection is bounded away from zero. In the case where there is at least one flat point (i.e. a point on M where the curvature vanishes) however, it was shown by Mather that the caustics are destroyed and that there are billiard trajectories that come arbitrarily close to being tangent to the boundary [86].

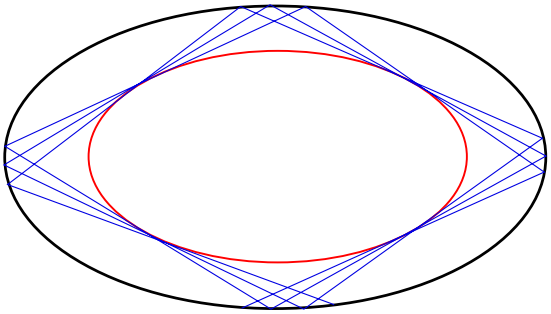


Figure 1.2: The red curve is an elliptical caustic of an elliptical billiard in dimension 2. The blue line is a few iterates of a billiard orbit tangent to this caustic.

In higher dimensions the picture is different. The notion of caustics extends to higher dimensions as follows: a caustic is a strictly convex closed hypersurface C contained inside the region bounded by the closed and strictly convex hypersurface M such that if one segment of a billiard trajectory is tangent to C then so is every segment of that trajectory. As in the 2-dimensional case, if $M \subset \mathbb{R}^d$ (where $d \geq 3$) is an ellipsoid, then there are many ellipsoidal caustics contained in the domain bounded by M. It turns out, however, that *only ellipsoidal billiard tables in* $\mathbb{R}^d$ *have caustics when* $d \geq 3$ [14, 59]. Therefore if $M \subset \mathbb{R}^d$ is a real-analytic, closed, and strictly convex billiard table where $d \geq 3$, and M is not an ellipsoid, then the sequence of angles of reflection of a billiard trajectory may not necessarily remain approximately constant as in the 2-dimensional case. In particular, there appears to be no obstacle in higher dimensions to the existence of billiard trajectories along which the angle of reflection tends to 0.

Such billiard trajectories were first studied by Gruber, who referred to them as

"trajectories terminating asymptotically on the boundary" [58]. He proved that the measure of such orbits is zero, but the question of existence remained open. That question is addressed in this thesis.

A general principle of dynamical folklore says: the dynamics of the billiard map in the limit as the angle of reflection goes to 0 is determined by the geodesic flow on M. This statement is formulated and proved rigorously in this thesis. Consider the following assumptions on the geodesic flow.

[A1] There is a hyperbolic closed geodesic γ on M; and

[A2] There is a transverse homoclinic geodesic ξ to γ on M.

The first main result of this thesis may be stated as follows.

Theorem A. For a generic real-analytic, closed, strictly convex hypersurface M of $\mathbb{R}^d$ (where $d \geq 3$) satisfying assumptions [A1, 2], there are billiard trajectories in M such that the corresponding sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of angles of reflection satisfies $\alpha_n \to 0$ as $n \to \infty$.

These billiard trajectories begin with a small positive angle of reflection α_0, and come closer and closer to being tangent with the boundary as the orbit progresses.

The word "generic" means that the phenomenon occurs on a residual set of hypersurfaces M in the real-analytic topology. This topology is very restrictive as bump functions are not real-analytic, and so they cannot be used explicitly when making perturbations. This means that any small analytic perturbation affects dynamics globally. To overcome this difficulty, we use a method introduced by Broer and Tangerman [22]: determine open conditions (in a weaker topology, e.g. C^4) to be satisfied by a family of perturbations to obtain the desired effect for arbitrarily small values of the parameter; show that these conditions are satisfied by a locally-supported family of perturbations; and approximate the family of perturbed systems sufficiently well by a real-analytic family. Since the conditions are open in a weaker topology, they are satisfied by the real-analytic family since the approximation can be made arbitrarily well in C^r. Therefore the desired effect is obtained for arbitrarily small values of the parameter of the real-analytic family. As a consequence of this approach, the theorem is also true in the C^r topology for $r = 4, 5, \ldots, \infty$.

Assumptions [A1,2] imply the existence of a normally hyperbolic invariant manifold $\bar{A}$ for the geodesic flow, the stable and unstable manifolds of which have a transverse homoclinic intersection. We show that when the initial angle of reflection α_0 is close to 0, the billiard map can be considered as a small perturbation of a time-shift of the geodesic flow, but with fluctuating speed. We adapt the theory of Fenichel [40, 41, 42] to infer that the billiard map inherits the normally hyperbolic invariant manifold, and

that the transverse homoclinic intersection persists. Therefore this is the *a priori chaotic* setting as defined in [32]. The union of a small neighbourhood of the normally hyperbolic invariant manifold with small neighbourhoods of several homoclinic manifolds is called the *homoclinic channel* [34]. We prove that there are trajectories of the billiard map that move up the channel, and that the angle of reflection tends to 0 for these trajectories. These trajectories will mostly stay in a neighbourhood of the hyperbolic closed geodesic γ, occasionally making excursions near a transverse homoclinic intersection.

A related concept is that of Fermi acceleration [19, 35, 52]. It takes place in the setting of nonautonomous Hamiltonian systems, and is characterised by unbounded growth of energy. In the context of billiards, an example of a nonautonomous system would be the billiard dynamics inside a table with moving walls, for instance. There have been several results in recent years establishing Fermi acceleration in such circumstances [37, 50, 51]. In this thesis the billiard system is autonomous, and so the billiard map conserves energy.

The geometric techniques used in this thesis rely on the scattering map of a normally hyperbolic invariant manifold, introduced in [32]. It is known that there are actual orbits (of the billiard map, in this case) that shadow orbits of the scattering map (see e.g. [53, 54]). Therefore one can prove the existence of diffusive trajectories by destroying the invariant curves of scattering maps on an invariant cylinder. This idea was first considered in [89], and developed further in [53, 82, 96]. Variational techniques pioneered by Mather [87] have also had success in proving generic existence of diffusive orbits.

The techniques used in this thesis are also suitable for describing oscillatory motions. A billiard trajectory is said to be *oscillatory* if the corresponding sequence of angles of reflection has positive limit superior, but limit inferior equal to 0.

Theorem B. For a generic real-analytic, closed, strictly convex hypersurface M of $\mathbb{R}^d$ (where $d \geq 3$) satisfying [A1, 2], there are oscillatory billiard trajectories in M.

Remarkably, as a consequence of the proof, the generic hypersurfaces described in Theorem B are the same hypersurfaces as those described in Theorem A. In other words, in generic billiard domains in $\mathbb{R}^d$ where assumptions [A1,2] are satisfied, there are both diffusive billiard trajectories *and* oscillatory billiard trajectories.

As an example of the above discussion and results, consider a triaxial ellipsoid M_0 in $\mathbb{R}^3$ defined by

$$M_0 = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \right\}$$

where $0 < a < b < c$. Then there are exactly 3 simple closed geodesics on M_0, given by $\{x = 0\}, \{y = 0\}, \{z = 0\}$. Moreover the lengths of these 3 geodesics are mutually

distinct. Denote by $\gamma = \{y = 0\}$ the simple closed geodesic of intermediate length. It was shown by Knieper and Weiss that γ is a hyperbolic closed geodesic [76]. The curve γ provides two periodic orbits of the geodesic flow, one by traversing it clockwise, and the other by traversing it anticlockwise. Denote by γ_+ one of these periodic orbits, and by γ_- the other. Since $\gamma_\pm$ are hyperbolic periodic orbits, they have stable and unstable invariant manifolds $W^{s,u}(\gamma_\pm)$. The symmetry of the ellipsoid M_0 implies that $W^s(\gamma_+) = W^u(\gamma_-)$ and $W^u(\gamma_+) = W^s(\gamma_-)$ [76]. A result proved in this thesis (Theorem 4.5.5) implies that we can find a closed hypersurface M_* of $\mathbb{R}^3$, arbitrarily close to M_0 in the real-analytic topology such that the heteroclinic cycle between γ_+ and γ_- becomes transverse. The existence of a transverse heteroclinic cycle implies the existence of a transverse homoclinic orbit ξ to γ_+ (or, equivalently, to γ_-). Therefore M_* satisfies the assumptions of Theorems A and B. It follows that, arbitrarily close to M_* in the real-analytic topology, we can find a hypersurface M_1 in which there are both billiard trajectories drifting towards the boundary and oscillatory billiard trajectories. We have started with an ellipsoid M_0 in which the billiard trajectories are tangent to caustics for all time, and by arbitrarily small real-analytic perturbation, we have found a hypersurface M_1 where the billiard dynamics has an extremely rich orbit structure near the hyperbolic closed geodesic γ_+, consisting periodic orbits, diffusive orbits, and oscillatory orbits. That is to say, we have moved from integrability to chaos.

While this example shows that there are hypersurfaces arbitrarily near ellipsoids on which assumptions [A1,2] are satisfied, we would like Theorems A and B to apply more generally. Therefore we study geodesic flows.

Doubly Asymptotic Geodesics on Hypersurfaces

Let M be a real-analytic, closed (not necessarily convex) hypersurface of $\mathbb{R}^d$ where $d \geq 3$. The standard Euclidean inner product on $\mathbb{R}^d$ induces a Riemannian metric on the hypersurface M. The *geodesics* of M are the curves that locally minimise length with respect to this Riemannian metric. The geodesic flow takes a point $x \in M$ and a tangent vector $u \in T_xM$ and follows the unique geodesic passing through x in the direction u at a constant speed $\|u\|$.

As in the case of billiards, the geodesic flow on an ellipsoid is integrable [69]. The existence of a transverse homoclinic geodesic, on the other hand, implies the existence of chaotic motions [108, 109]. The study of homoclinic geodesics goes back to Hadamard and Poincaré [61, 101, 102], and was continued by Hedlund and Morse (among others) in the next half-century [62, 90, 91, 92]. In a seminal paper, Anosov studied the geodesic flow on closed Riemannian manifolds of negative curvature [6]. In this case, the unit tangent bundle of the Riemannian manifold is a hyperbolic set for the geodesic flow, so there are symbolic dynamics, positive topological entropy, and

infinitely many homoclinic geodesics. The case of nonnegative curvature, however, is more subtle. The most general result in this direction was proved by Contreras, who showed that C^2-generic Riemannian metrics on an arbitrary sufficiently smooth closed manifold give rise to transverse homoclinic geodesics [29]. It was shown by Knieper and Weiss that C^∞-generic Riemannian metrics on a 2-sphere have a transverse homoclinic geodesic [77].

These results do not apply directly to the case considered in this thesis for the following two reasons. Firstly, if M is a real-analytic, closed hypersurface of $\mathbb{R}^d$, and g is a small perturbation of the standard metric on M, the Nash embedding theorem guarantees that (M, g) can be isometrically embedded in *some* Euclidean space $\mathbb{R}^N$, but it is not necessarily the case that $N = d$ [94, 95]. Therefore attention must be restricted to those metrics g that do not change the dimension of the embedding space. Secondly, the genericity proved in the results of Contreras and of Knieper and Weiss is in the C^2 and C^∞ topologies, respectively. To prove genericity in those topologies it is sufficient to use C^∞-small locally-supported functions. Such an approach is not valid in the real-analytic topology, as bump functions fail to be analytic. Instead, we use again the method of Broer and Tangerman [22], and prove results for parametrised families of locally supported perturbations for arbitrarily small values of the perturbation parameter, before approximating by a real-analytic family.

Using these methods, the following general result is obtained in dimension $d \geq 3$ by analysing the dynamics near an elliptic closed geodesic.

Theorem C. Let M be a real-analytic closed hypersurface of $\mathbb{R}^d$ (with $d \geq 3$) equipped with the standard metric. Suppose M has a nonhyperbolic closed geodesic. Then M generically has a hyperbolic closed geodesic with a transverse homoclinic.

In the case when $d = 3$ and when it is assumed in addition that M is strictly convex, genericity in the analytic topology is obtained.

Theorem D. Let M be a real-analytic, closed, and strictly convex surface in $\mathbb{R}^3$ equipped with the standard metric. Then M generically has a hyperbolic closed geodesic with a transverse homoclinic.

The main tool used in proving Theorems C and D is a Kupka-Smale theorem for analytic geodesic flows on hypersurfaces, which says: for generic real-analytic closed hypersurfaces of $\mathbb{R}^d$, the Poincaré map of every closed geodesic is in general position, and every homoclinic or heteroclinic connection between hyperbolic closed geodesics is transverse (Theorem 4.1.5). The proof of this theorem is as follows. It is first shown that by an arbitrarily small analytic perturbation, the Poincaré map of *a single closed geodesic* can be brought into general position. It is shown moreover that an arbitrarily

small analytic perturbation can make connections transverse. It is then shown that for each $T > 0$, there is an open and dense set of real-analytic closed hypersurfaces M of $\mathbb{R}^d$ such that M has at most finitely many closed geodesics of length at most T. Starting in this open dense set for $T = 1$, we can make finitely many real-analytic perturbations to bring the Poincaré map of each of the finitely many closed geodesics of length at most T into general position, and to make any homoclinic or heteroclinic connections between them transverse. Repeating this process for each $T \in \mathbb{N}$ gives countably many open dense sets, the intersection of which is the generic set we are seeking.

Theorem C is immediate from the Kupka-Smale theorem and an argument of Contreras that is based on the work of Zehnder [29, 118]. Theorem D follows from the Kupka-Smale theorem combined with arguments of Birkhoff, Mather, and Knieper and Weiss [17, 77, 85]. The reason that the case of surfaces is special is that a simple closed geodesic γ on a surface M in $\mathbb{R}^3$ splits M into northern and southern hemispheres. It was then shown by Birkhoff that if M is strictly convex, every geodesic on M must cross γ infinitely many times, both forward and backward in time. Consequently, γ gives rise to a global Poincaré map, which is a mapping of a 2-dimensional annulus. We can therefore use results about mappings of 2-dimensional annuli to complete the proof.

Combined Results and Future Research

Combining Theorems A and B with Theorem C we obtain the following result near elliptic closed geodesics.

Theorem E. Let M be a real-analytic, closed, and strictly convex hypersurface of $\mathbb{R}^d$ where $d \geq 3$. Suppose M has a nonhyperbolic closed geodesic. Then M generically contains both billiard trajectories that drift towards the boundary, as well as oscillatory billiard trajectories.

Finally, in dimension 3 we get the following general result upon combining Theorems A and B with Theorem D, where assumptions [A1] and [A2] have been completely removed.

Theorem F. For a generic real-analytic, closed, strictly convex surface M in $\mathbb{R}^3$, there are both billiard trajectories that drift towards the boundary, and oscillatory billiard trajectories.

It is, of course, desirable to have a version of Theorem F in dimension $d > 3$. This would follow from a higher-dimensional analogue of Theorem D, but it turns out that such a result is very difficult to prove (if indeed it is true). The closed strictly convex

hypersurfaces considered in this thesis are topological spheres, and therefore have trivial fundamental group. From a variational point of view, this means that standard minimising techniques will fail. There are also "minimax" techniques which comprise first maximising over some family of curves before minimising over homotopy classes of such families [17], but these techniques, when applied to the search for homoclinic curves, seem to require control of the first derivative of the curve. Such control is apparently not granted by standard variational methods.

An approach that could potentially work is the following. As a result of Theorem C it is only necessary to consider the case where M is a real-analytic closed hypersurface of $\mathbb{R}^d$ on which every closed geodesic is hyperbolic. It is well-known that closed geodesics are critical points of a certain action functional. Hill's formula, along with the hyperbolicity assumption, implies that every closed geodesic is a *nondegenerate critical point* of the action [20, 64, 117]. Therefore the action functional is a Morse function on the loop space of M, and so we can construct its Morse homology. Such a construction is called *Floer homology* [44, 45, 46]. It is possible that the profound theory surrounding these constructions could be used to construct homoclinic geodesics on generic M. A result of this sort would be a major coup, for two reasons: first of all, it would improve significantly on any existing results about generic existence of homoclinic geodesics; and secondly, it would be evidence of Floer homology being a very powerful analytical tool in the theory of dynamical systems.

Structure of This Thesis

In Chapter 2 we define many of the mathematical tools and objects used in this thesis, including hyperbolic sets and normally hyperbolic invariant manifolds, scattering maps, and symplectic structures. We consider the geometry of hypersurfaces, and define the billiard map and geodesic flow. A proof of Gruber's theorem regarding the measure of billiard trajectories drifting towards the boundary is given.

In Chapter 3 we study billiard dynamics. It is shown that, under assumptions [A1,2], the billiard map inherits a normally hyperbolic invariant manifold from the geodesic flow, the stable and unstable manifolds of which have a transverse homoclinic intersection. We construct an iterated function system on the invariant manifold comprising the billiard map and 8 scattering maps, and show how to destroy all invariant curves of the iterated function system by arbitrarily small analytic perturbations, thus proving Theorems A and B.

In Chapter 4 we study the geodesic flow. A Kupka-Smale theorem for geodesic flows on real-analytic closed hypersurfaces of $\mathbb{R}^d$ is proved. In order to do this we prove analogues of theorems of Anosov, Klingenberg and Takens, and Contreras and Paternain [7, 29, 30, 75]. Theorem C follows immediately from this theorem and

an argument of Contreras [29]. Finally, it is shown how to prove Theorem D by constructing a global Poincaré map and analysing its hyperbolic fixed points, using an argument of Knieper and Weiss [77].

— 2 —

Background and Preliminaries

2.1 Hyperbolic Dynamics

Hyperbolicity is characterised by the existence of expanding and contracting directions for the linearisation of a map or flow at an invariant object. This ubiquitous concept gives rise to many dynamical phenomena associated with chaos.

In this section we review some of the basic ideas and objects from hyperbolicity theory that are fundamental in the study of Arnold diffusion and instability in general: hyperbolic sets near transverse homoclinic points, normally hyperbolic invariant manifolds, and scattering maps.

2.1.1 Homoclinic Trajectories and Hyperbolic Sets

Let M be a smooth manifold, and $f : M \to M$ a diffeomorphism with a fixed point $x \in M$. The fixed point x is said to be:

- *Parabolic* if 1 is an eigenvalue of the linearisation $D_x f$ of f at x;
- *Degenerate* if $D_x f$ has an eigenvalue equal to a root of unity;
- *Hyperbolic* if $D_x f$ has no eigenvalue of absolute value 1; and
- *Elliptic* if it nondegenerate and nonhyperbolic.

Suppose x is a hyperbolic fixed point. In this case the theorem of Hadamard and Perron (see for example [73]) guarantees the existence of f-invariant submanifolds $W^s(x), W^u(x)$ of M, called the *stable* and *unstable manifolds* of x which satisfy

$$W^s(x) = \{y \in M : f^n(y) \to x \text{ as } n \to \infty\},$$

$$W^u(x) = \{y \in M : f^{-n}(y) \to x \text{ as } n \to \infty\},$$

and

$$T_x W^s(x) \oplus T_x W^u(x) = T_x M.$$

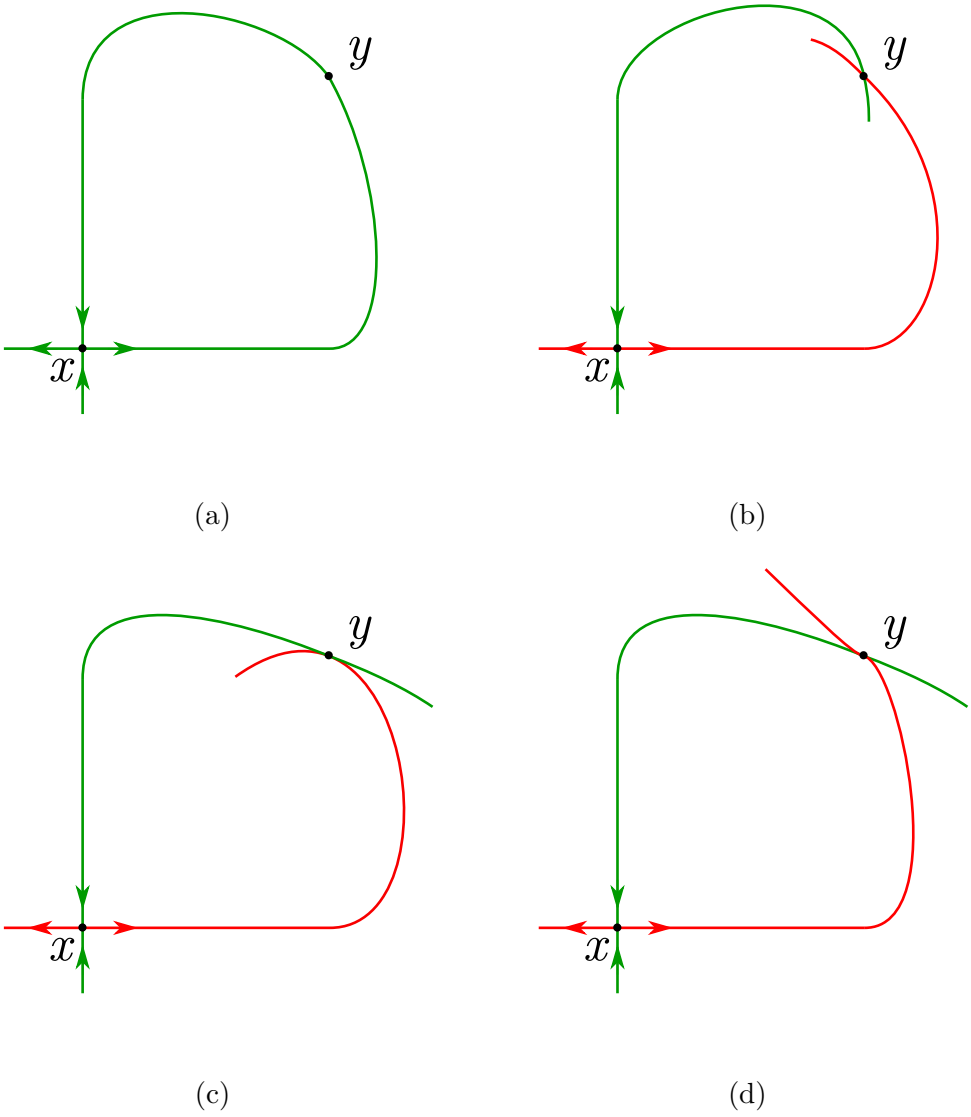


Figure 2.1: There are several ways that stable and unstable manifolds can intersect at a homoclinic point y to a hyperbolic fixed point x. In (a), the stable and unstable manifolds of x coincide to give a homoclinic loop; in (b) the intersection is transverse, so the tangents to the invariant manifolds span the tangent space of $\mathbb{R}^2$ at y; in (c) there is a homoclinic tangency; and (d) shows a nontransverse two-sided topological crossing.

Definition 2.1.1. Let $x, y \in M$ be hyperbolic fixed points of f. A point $z \in M$ is called a *heteroclinic* point if $z \in W^s(x) \cap W^u(y)$. If $x = y$ then such a point z (where $z \neq x$) is called a *homoclinic* point.

If $z \in W^s(x) \cap W^u(y)$ is a heteroclinic point then we clearly have $f^n(z) \to x$ as $n \to +\infty$ and $f^n(z) \to y$ as $n \to -\infty$. There are a number of different ways in which the stable and unstable manifolds can intersect at a homoclinic or heteroclinic point (see Figure 2.1), the most important (in this thesis) being the following.

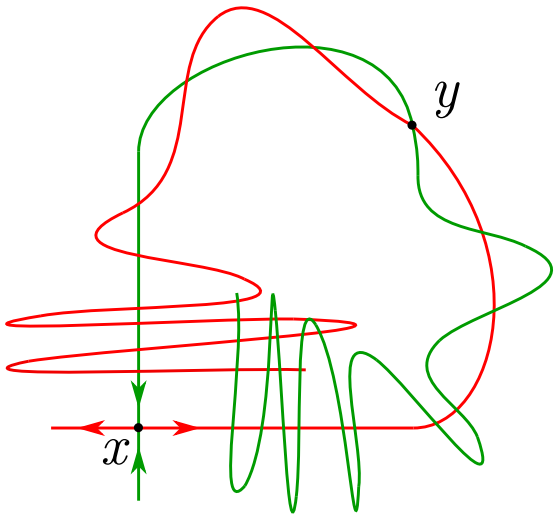


Figure 2.2: If y is a transverse homoclinic point to a hyperbolic fixed point x, the invariance of the stable and unstable manifolds imply that all forward and backward iterates of y under f are also transverse homoclinic points. Near x, the stable and unstable manifolds fold on themselves as they cannot have self-intersections. This leads to a complicated web of homoclinic intersections called the *homoclinic tangle*.

Definition 2.1.2. A heteroclinic (or homoclinic, in the case where $x = y$) point $z \in W^s(x) \cap W^u(y)$ is *transverse* if

$$T_z W^s(x) + T_z W^u(y) = T_z M.$$

The existence of a transverse homoclinic point leads to very complicated dynamics near the hyperbolic fixed point (see Figure 2.2). Indeed, Poincaré first drew this picture after initially believing that transverse homoclinic points were too complicated to occur in nature [101].

We will now be more precise about the word 'complicated': let us define hyperbolic sets.

Definition 2.1.3. A subset $\Lambda \subset M$ is called a *hyperbolic set* for f if there is $\lambda \in (0,1)$, a positive constant C, and an invariant splitting

$$T_\Lambda M = E^s \oplus E^u$$

of the tangent bundle such that for all $x \in \Lambda$ and $n \in \mathbb{N}$ we have:

- $\|D_x f^n(v)\| \leq C\lambda^n \|v\|$ for all $v \in E^s_x$; and
- $\|D_x f^{-n}(v)\| \leq C\lambda^n \|v\|$ for all $v \in E^u_x$.

A hyperbolic fixed point is an example of a hyperbolic set. In order to avoid such trivial examples we also require the following additional definitions.

Definition 2.1.4. Let $\Lambda \subset M$ be a hyperbolic set for f. The set Λ is said to be:

- *Nontrivial* if it contains more than one orbit;
- *Locally maximal* if there is a neighbourhood V of Λ in M such that

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(V);$$

- *Basic* if it is locally maximal and contains a dense orbit.

A proof of the following well-known result can be found, for example, in [73].

Theorem 2.1.5. *Suppose $\Lambda \subset M$ is a nontrivial hyperbolic basic set for f. Then $f|_\Lambda$ is topologically semiconjugate to a transitive subshift of finite type.*

Theorem 2.1.5 tells us that the dynamics on a nontrivial hyperbolic basic set can be explained symbolically. The transitivity of the symbolic dynamics implies that there are infinitely many periodic and homoclinic orbits. The following theorem, due to Smale [109, 110] (see also [108]), further establishes the relationship between hyperbolic sets and homoclinic points.

Theorem 2.1.6. *Let $x \in M$ be a hyperbolic fixed point of f, and let $y \in M$ be a transverse homoclinic point to x. Then, arbitrarily close to x, we can find a nontrivial hyperbolic basic set Λ for an iterate of f such that $f^n(y) \in \Lambda$ for some $n \in \mathbb{N}$.*

Combining Theorems 2.1.5 and 2.1.6, we obtain the following corollary.

Corollary 2.1.7. *If f has a single transverse homoclinic orbit to a hyperbolic fixed point x, then it has infinitely many.*

2.1.2 Normally Hyperbolic Invariant Manifolds

In this section, let M be a smooth manifold, and $\phi^t : M \to M$ a smooth flow with

$$\left.\frac{d}{dt}\right|_{t=0} \phi^t = X.$$

One of the central objects of study in this thesis is a normally hyperbolic invariant manifold.

Definition 2.1.8. Let $A \subset M$ be a compact invariant manifold for ϕ^t. We say that A is a *normally hyperbolic invariant manifold* if there is $0 < \lambda < \mu^{-1} < 1$, a positive constant C, and an invariant splitting of the tangent bundle

$$T_A M = TA \oplus E^s \oplus E^u$$

such that:

$$\begin{aligned} \|D\phi^t|_{E^s}\| &\leq C\lambda^t \text{ for all } t \geq 0, \\ \|D\phi^t|_{E^u}\| &\leq C\lambda^{-t} \text{ for all } t \leq 0, \\ \|D\phi^t|_{TA}\| &\leq C\mu^{|t|} \text{ for all } t \in \mathbb{R}. \end{aligned} \tag{2.1}$$

Remark 2.1.9. *This is actually stronger than the standard definition. Notice that the rate of expansion λ is equal to the rate of contraction. Such invariant manifolds are called symmetrically normally hyperbolic. Moreover, the standard definition of normally hyperbolic invariant manifolds allows the parameters λ, μ to depend on the point in A. However the definition given here is suitable for the purposes of this thesis.*

Definition 2.1.10. Let A be a normally hyperbolic invariant manifold for ϕ^t, and let λ, μ denote the hyperbolicity parameters as in the definition. If for some $r \in \mathbb{N}$ we have $0 < \lambda < \mu^{-r} < 1$, then A is called an *r-normally hyperbolic invariant manifold.*

This condition on the hyperbolicity parameters is referred to as a *large spectral gap* condition. Setting $t = 1$ in (2.1) gives the definition of a normally hyperbolic invariant manifold for the map $f = \phi^1$.

It was proved by Fenichel in the 70's that normally hyperbolic invariant manifolds are stable under small perturbations [40, 41, 42]. The result was improved by Hirsch, Pugh, and Shub to obtain the following, which is sometimes referred to as the *fundamental theorem of normally hyperbolic invariant manifolds* [65].

Theorem 2.1.11. *Let $r \in \mathbb{N}$, and suppose $A \subset M$ is an r-normally hyperbolic invariant manifold for ϕ^t. Let $T_A M = TA \oplus E^s \oplus E^u$ denote the corresponding invariant splitting of the tangent bundle.*

1. *There are ϕ^t-invariant stable and unstable submanifolds $W^s(A)$, $W^u(A) \subset M$ given by*

$$W^s(A) = \{x \in M : d(\phi^t(x), A) \leq C(x)\lambda^t \text{ for all } t \geq 0\},$$

$$W^u(A) = \{x \in M : d(\phi^t(x), A) \leq C(x)\lambda^{-t} \text{ for all } t \leq 0\},$$

   that are tangent at A to $TA \oplus E^s$, $TA \oplus E^u$ respectively;

2. *$W^s(A), W^u(A)$ are unique in the sense that any set V near A satisfying $\phi^t(V) \subseteq V$ for all $t > 0$ is contained in $W^s(A)$, and any set V near A satisfying $\phi^{-t}(V) \subseteq V$ for all $t > 0$ is contained in $W^u(A)$;*

3. *$W^s(A)$ consists of all points whose forward orbits never leave some neighbourhood of A, and $W^u(A)$ is the set of points whose backward orbits never leave some neighbourhood of A;*

4. *$W^s(A), W^u(A)$, and A are C^r-smooth submanifolds;*

5. *On $W^s(A), W^u(A)$ there are invariant foliations by submanifolds $W^s(x)$, $W^u(x)$ for each $x \in A$ where $T_x W^s(x) = E^s_x$ and $T_x W^u(x) = E^u_x$;*

6. *If Y is a vector field C^r-close to X then there is a manifold A_Y C^r-close to A that is a normally hyperbolic invariant manifold for the flow of Y. Moreover, its stable and unstable manifolds and foliations are C^r-close to those of A; and*

7. *The flow ϕ^t near A is topologically conjugate to the flow of $D\phi^t|_{E^s \oplus E^u}$.*

Remark 2.1.12. *Analogous results hold for maps. The theorem also includes similar statements about* eventually *r-normally hyperbolic flows, but these statements are not required for the purposes of this thesis.*

Let $A \subset M$ be an r-normally hyperbolic invariant manifold for ϕ^t. The stable and unstable invariant manifolds are given by

$$W^s(A) = \{x \in M : d(\phi^t(x), A) \leq C(x)\lambda^t \text{ for all } t \geq 0\},$$
$$W^u(A) = \{x \in M : d(\phi^t(x), A) \leq C(x)\lambda^{-t} \text{ for all } t \leq 0\},$$

where C is a constant that is allowed to depend on x. The foliations described in part 5 of Theorem 2.1.11 are called the *strong stable* and *strong unstable foliations*. The leaves of the foliations corresponding to $x \in A$ are

$$W^s(x) = \{y \in M : d(\phi^t(x), \phi^t(y)) \leq C(x, y)\lambda^t \text{ for all } t \geq 0\},$$
$$W^u(x) = \{y \in M : d(\phi^t(x), \phi^t(y)) \leq C(x, y)\lambda^{-t} \text{ for all } t \leq 0\}.$$

Two leaves of the strong stable or strong unstable foliation corresponding to different points of A are disjoint, and

$$W^s(A) = \bigcup_{x \in A} W^s(x), \quad W^u(A) = \bigcup_{x \in A} W^u(x).$$

The foliations are invariant in the sense that for any $x \in A$ and $t \in \mathbb{R}$ we have

$$\phi^t(W^s(x)) = W^s(\phi^t(x)), \quad \phi^t(W^u(x)) = W^u(\phi^t(x)). \tag{2.2}$$

Moreover, the foliations are C^{r-1}-regular, meaning that the fields of tangents to the leaves are C^{r-1}-smooth.

2.1.3 Scattering Maps

Let M be a smooth manifold and $\phi^t : M \to M$ a smooth flow. Suppose $A \subset M$ is an r-normally hyperbolic invariant manifold for ϕ^t.

Definition 2.1.13. The *holonomy maps* $\pi^s : W^s(A) \to A$ and $\pi^u : W^u(A) \to A$ are the projections along leaves of the strong stable and strong unstable foliations, respectively: if $y \in W^s(A)$ (resp. $W^u(A)$) then there is a unique $x \in A$ such that $y \in W^s(x)$ (resp. $y \in W^u(x)$). We define $\pi^s(y) = x$ (resp. $\pi^u(y) = x$).

Now, suppose there is a homoclinic point $x \in (W^s(A) \cap W^u(A)) \setminus A$. We call x a *transverse homoclinic* point if $W^s(A)$ and $W^u(A)$ are transverse at x, in which case we write $x \in (W^s(A) \pitchfork W^u(A)) \setminus A$. If x is a transverse homoclinic point, the implicit function theorem implies that there is an open neighbourhood V of x contained in $W^s(A) \cap W^u(A)$ such that every $y \in V$ is a transverse homoclinic point.

Since $x \in W^s(A) \cap W^u(A)$, there are unique $x_\pm \in A$ such that $x \in W^s(x_+) \cap W^u(x_-)$. The homoclinic intersection at x is said to be *strongly transverse* if

$$T_x W^s(x_+) \oplus T_x(W^s(A) \cap W^u(A)) = T_x W^s(A),$$

$$T_x W^u(x_-) \oplus T_x(W^s(A) \cap W^u(A)) = T_x W^u(A).$$

In this case the strong stable foliation is transverse to V in $W^s(A)$, and the strong unstable foliation is transverse to V in $W^u(A)$, assuming V is small enough. In this case we can define the scattering map (see Figure 2.3).

Definition 2.1.14. With the notation and assumptions as above, we define the *scattering map*

$$\begin{aligned} s : \pi^u(V) &\longrightarrow \pi^s(V) \\ x &\longmapsto \pi^s \circ (\pi^u)^{-1}(x). \end{aligned} \tag{2.3}$$

The scattering map was introduced by Delshams, de la Llave, and Seara [32, 33], and has become a widely used tool in proofs of diffusion.

Now, consider the map $f = \phi^1$. Assume that A has the structure of a cylinder, so there is a C^r-diffeomorphism $h : \mathbb{T} \times [0,1] \to A$.

Definition 2.1.15. Let $B \subset (W^s(A) \pitchfork W^u(A)) \setminus A$ be a smooth two-dimensional manifold. We say that B is a *homoclinic cylinder* if the scattering map

$$s_B : A \longrightarrow A$$

relative to B is a C^{r-1} diffeomorphism.

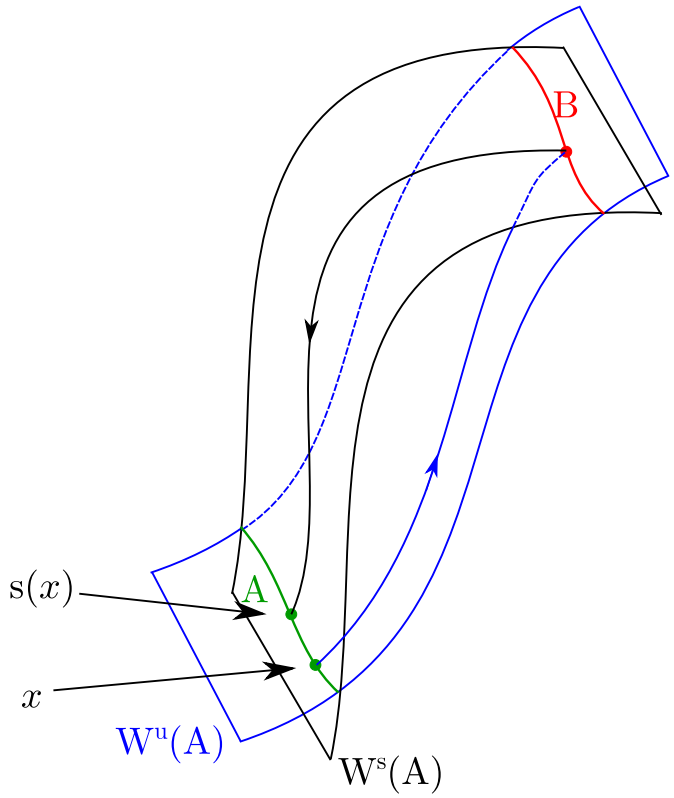


Figure 2.3: The scattering map s takes a point x on the normally hyperbolic invariant manifold A, follows a leaf of the strong unstable foliation of $W^u(A)$ until it reaches the homoclinic manifold B, and then follows a leaf of the strong stable foliation of $W^s(A)$ back to A.

A transverse homoclinic intersection remains transverse under sufficiently small perturbations, as a result of part 6 of Theorem 2.1.11. It follows that, if f has a C^r-smooth homoclinic cylinder B, and if g is sufficiently C^r-close to f, then the map g has a homoclinic cylinder $\bar{B}$ C^r-close to B. Moreover, the corresponding scattering map $\bar{s}$ is C^{r-1}-close to the unperturbed scattering map s.

It is clear that if B is a homoclinic cylinder, then so too is $f^n(B)$ for any $n \in \mathbb{Z}$. The following identities are an immediate consequence of the invariance (2.2) of the strong stable and strong unstable foliations:

$$\pi^s_{f(B)} = f \circ \pi^s_B \circ f^{-1}, \quad \pi^u_{f(B)} = f \circ \pi^u_B \circ f^{-1}. \tag{2.4}$$

We also have analogous statements for time shifts of the flow ϕ^t. Similar expressions hold for $f^n(B)$ for any $n \in \mathbb{Z}$. Furthermore, combining (2.3) and (2.4) gives an analogous expression for the scattering map $s_{f^n(B)}$.

2.2 Real-Analytic Perturbation Theory

In the theory of perturbations of dynamical systems, it is often convenient to make some small change to the dynamics in a small region of the phase space while leaving the dynamics elsewhere in the phase space unchanged. Locally-supported functions, however, are not real-analytic. This is a significant obstacle when proving perturbation-theoretic results for real-analytic systems, and specific techniques are required to overcome this obstacle.

We begin with some basic definitions.

Definition 2.2.1. Let X be a topological space. Then:

- A subset $Y \subseteq X$ is *residual* if it can be written
$$Y = \bigcap_{n\in\mathbb{N}} Y_n$$
where Y_n is an open dense subset of X for each $n \in \mathbb{N}$; and
- A property of points $x \in X$ is said to be *generic* if it occurs on a residual subset of X.

Let $d \in \mathbb{N}$ and let $U \subset \mathbb{R}^d$ be a domain.

Definition 2.2.2. A function $\psi : U \to \mathbb{R}$ is said to be *real-analytic* if it admits a holomorphic extension in a complex neighbourhood of U.

Equivalently, the function is real-analytic if it can locally be written as a convergent power series in a neighbourhood of every point of U. If $\psi : U \to \mathbb{R}$ is real-analytic then we write $\psi \in C^\omega(U)$. The following elementary lemma tells us that we cannot use bump functions (i.e. locally-supported functions) explicitly in a C^ω perturbation theory.

Lemma 2.2.3. *Bump functions are not real-analytic.*

Proof. Let $U \subset \mathbb{R}^d$ be a domain, and let $\psi : U \to \mathbb{R}$ be a smooth function. Suppose for contradiction that both the support of ψ is compactly embedded in U, and $\psi \in C^\omega(U)$.

Let $V = \mathrm{supp}(\psi)$. Since V is compactly embedded in U we can find a point $x_0 \in \partial V$ such that every open neighbourhood of x_0 in U contains an open subset of $U \setminus \overline{V}$. Therefore $\psi(x_0) = 0$ along with all of its derivatives. Since $x_0 \in \partial V$, every open neighbourhood of x_0 in U also contains an open subset of V. Since $\psi \in C^\omega(U)$, we can find an open neighbourhood W of $x_0 \in U$ and constants $a_\alpha \in \mathbb{R}$ for each $\alpha \in \mathbb{N}_0^d$ such that
$$\psi(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha (x - x_0)^\alpha$$

for all $x \in W$, where we have used multi-index notation. Since ψ vanishes at x_0 along with all of its derivatives, we find that $a_\alpha = 0$ for all $\alpha \in \mathbb{N}_0^d$, and so $\psi|_W \equiv 0$. Since W is an open neighbourhood of x_0 it contains points of V, which is a contradiction. □

Define the C^ω topology on $C^\omega(U)$ as follows. Two functions $f, g \in C^\omega(U)$ by definition admit holomorphic extensions $\hat{f}, \hat{g}$ on a compact complex neighbourhood $\hat{U}$ of U. We say that f, g are close in the C^ω topology if $\hat{f}, \hat{g}$ are uniformly close on $\hat{U}$.

Suppose now we want to show that a property of functions in $C^\omega(U)$ occurring on a set

$$Y = \bigcap_{n \in \mathbb{N}} Y_n \subseteq C^\omega(U)$$

is generic. It follows from the definitions that it is sufficient to show that each Y_n is open and dense in $C^\omega(U)$. Lemma 2.2.3 implies that the C^ω topology is finer than the C^r topology for any $r \in \mathbb{N}_0 \cup \{\infty\}$. Therefore for openness, it is sufficient to show that Y_n is open in the C^r topology for some $r \in \mathbb{N}_0$, since this in turn implies C^ω-openness. For denseness, on the other hand, this is not sufficient.

This means that if we have a function $f \in C^\omega(U)$ and some C^∞-small bump function $\psi : U \to \mathbb{R}$ such that $f + \psi$ has the desired properties, this is not enough to prove denseness for a real-analytic perturbation theory. In order to overcome this difficulty, we use a technique introduced by Broer and Tangerman [22]:

- Determine open conditions in a weaker topology (e.g. the C^r-topology for some $r \in \mathbb{N}$) to be satisfied by a family of perturbations in order to obtain the desired effect for arbitrarily small values of the parameter;
- Show that these conditions are satisfied by $f + \epsilon\psi$ for arbitrarily small values of ϵ where ψ is a bump function;
- Approximate $f + \epsilon\psi$ sufficiently well in the C^r topology by a real-analytic family f_ϵ where $f_0 \equiv f$; and
- Since the conditions are open in the C^r-topology, and since the approximation is sufficiently good in the C^r topology, conclude that we obtain the desired effect for arbitrarily small values $\epsilon \neq 0$ of the parameter for the real-analytic family f_ϵ.

As an application of this technique, Broer and Tangerman proved the following theorem in [22].

Theorem 2.2.4. *(Real-Analytic Kupka-Smale Theorem) Let (M, g) be a real-analytic Riemannian manifold. Let $\mathfrak{X}^\omega(M)$ denote the space of real-analytic vector fields on M, equipped with the C^ω topology. Then for generic $X \in \mathfrak{X}^\omega(M)$ we have:*

1. *Every periodic orbit of X is hyperbolic; and*

2. *Any homoclinic or heteroclinic connections are transverse.*

2.3 Abraham's Transversality Theorem

A theorem of Abraham regarding the transversality of a map to a submanifold of its image has turned out to be very useful when proving results about generic properties of dynamical systems. The theorem is stated using the terminology of *pseudorepresentations*, but the parallels to the case of maps are clear.

We first define pseudorepresentations and pseudotransversality. Let $\mathcal{E}$ be a Baire topological space (i.e. every residual subset of $\mathcal{E}$ is dense), and let M, N be second countable, finite-dimensional manifolds. Let D be a subset of M, and Δ a submanifold of N.

Definition 2.3.1. A C^1-*pseudorepresentation* is a map $F : \mathcal{E} \to C^1(M, N)$ such that the evaluation map

$$ev\left(F^{(1)}\right) : \mathcal{E} \times TM \longrightarrow TN$$

defined by

$$ev\left(F^{(1)}\right)(G, z) = D\left(F(G)\right)(z)$$

is continuous.

Definition 2.3.2. A C^1-pseudorepresentation $F : \mathcal{E} \to C^1(M, N)$ is C^r- *pseudotransverse* to Δ on D if there is a dense subset $K \subset \mathcal{E}$ such that for all $G \in K$ there is an open set B_G in a separable Banach space B, a continuous mapping $\psi_G : B_G \to \mathcal{E}$ and $\overline{G} \in B_G$ such that $\psi_G\left(\overline{G}\right) = G$ and the map $ev\left(F \circ \psi_G\right) : B_G \times M \to N$ is C^r-smooth and transverse to Δ on D.

Abraham's transversality theorem can now be stated as follows.

Theorem 2.3.3. *Suppose $F : \mathcal{E} \to C^1(M, N)$ is C^r-pseudotransverse to Δ on D, where r is sufficiently large. Denote by R the set of $G \in \mathcal{E}$ for which $F(G)$ is transverse to Δ at points of D.*

- *If $D = M$ then R is residual in $\mathcal{E}$; and*
- *If Δ is a closed submanifold and $D \subset M$ is compact then R is open and dense in $\mathcal{E}$.*

Various statements and proofs of this theorem can be found in the writings of Abraham [1, 4, 5]. However in Abraham's statements, the space $\mathcal{E}$ is required to

be a Banach space. This was loosened by Robinson [107] (see also [105]) to the requirement above, that it is a Baire topological space. Notice that the space $C^1(M, N)$ is not given any topology. In this thesis, the set $\mathcal{E}$ will be a space of real-analytic embeddings, equipped with the analytic topology. Theorem 2.3.3 is capable of handling this topology.

As an example of how this theorem is used, let M, N be smooth finite-dimensional manifolds without boundary, and let $f_t : M \to N$ be a one-parameter family of smooth mappings with smooth dependence on the parameter $t \in \mathbb{R}$. Let $\Delta \subset N$ be a smooth submanifold without boundary. Consider the map $F : \mathbb{R} \times M \to N$ defined by $F(t, x) = f_t(x)$. Then F is a smooth map of manifolds. Suppose F is transverse to Δ in the sense that

$$D_{(t,x)}F\left(T_{(t,x)}\left(\mathbb{R} \times M\right)\right) + T_{F(t,x)}\Delta = T_{F(t,x)}N$$

whenever $F(t, x) \in \Delta$. Then Theorem 2.3.3 guarantees that there is a residual set $R \subset \mathbb{R}$ such that f_t is transverse to Δ for all $t \in \mathbb{R}$. The latter is a version of the classical transversality theorem of Thom [115, 116].

2.4 Hamiltonian and Symplectic Dynamics

In classical mechanics, energy is determined entirely by position and momentum, and is conserved due to the principle of conservation of energy. Hamiltonian mechanics is a mathematical formulation of these concepts. From a dynamical point of view, orbits of a Hamiltonian flow evolve in such a way that an energy functional called the Hamiltonian remains constant along trajectories. Symplectic maps can be thought of as a discrete analogue of Hamiltonian flows.

Integrable Hamiltonian systems are distinguished by the existence of *action-angle coordinates* $(\varphi, I) \in T\mathbb{T}^d$ such that the Hamiltonian depends only on the action variable I. In this case, the dynamics takes place on invariant tori $\mathbb{T}^d$ on which the action variable is constant and the dynamics is smoothly conjugate to a rotation by a fixed frequency vector. The problems of stability (KAM theory) and instability (Arnold diffusion) of Hamiltonian systems relates to the perseverance of these invariant tori under small perturbations of the system, and to the consequences of their destruction. The study of these problems is motivated by the question of stability of the solar system, which is modelled by the n-body problem.

2.4.1 Hamiltonian Flows and Symplectic Maps

Let M be a smooth manifold of dimension d. If $x \in M$ and $u \in T_xM$, we write $(x, u) \in TM$. A *Hamiltonian* is a smooth function $H : TM \to \mathbb{R}$ that represents

energy[1]. The function H defines a dynamical system as follows. The orbit $(x(t), u(t))$ with initial condition $(x, u) \in TM$ should evolve so that the energy functional H remains constant, so

$$0 = \frac{d}{dt}H(x(t), u(t)) = \sum_{j=1}^{d} \left(\frac{\partial H}{\partial x_j}(x(t), u(t))\dot{x}_j(t) + \frac{\partial H}{\partial u_j}(x(t), u(t))\dot{u}_j(t) \right)$$

where the dot represents the time derivative. Notice that this equation is satisfied if

$$\begin{cases} \dot{x}_j = \dfrac{\partial H}{\partial u_j} \\ \dot{u}_j = -\dfrac{\partial H}{\partial x_j}. \end{cases} \tag{2.5}$$

The equations (2.5) are called *Hamilton's equations of motion*. The manifold M is called the *configuration space*, and the tangent bundle TM is the *phase space*.

Denote by I_d the $d \times d$ identity matrix. The *standard* $2d \times 2d$ *symplectic matrix* is

$$\Omega = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix}. \tag{2.6}$$

Then the equations (2.5) can also be written in the concise form

$$(\dot{x}, \dot{u}) = X(x, u) \coloneqq \Omega \nabla H,$$

where X is referred to as the *Hamiltonian vector field* of the Hamiltonian function H. The flow $\phi^t : TM \to TM$ defined by

$$\left.\frac{d}{dt}\right|_{t=0} \phi^t = X \tag{2.7}$$

is the *Hamiltonian flow* of the Hamiltonian function H.

This description can be generalised somewhat. A differential k-form α on TM is called *closed* if $d\alpha = 0$, and when $k = 2$ it is called *nondegenerate* if for any $(x, u) \in TM$ and $\xi \in T_{(x,u)}TM \setminus \{0\}$, there is $\eta \in T_{(x,u)}TM$ such that $\alpha(\xi, \eta) \neq 0$.

Definition 2.4.1. A differential 2-form ω is called a *symplectic form* if it is closed and nondegenerate.

The *standard symplectic form* is

$$\omega = dx \wedge du = dx_1 \wedge du_1 + \cdots + dx_d \wedge du_d. \tag{2.8}$$

[1]Typically a Hamiltonian is defined on the cotangent bundle rather than the tangent bundle. In this thesis there is always a Riemannian structure on the manifold M which allows us to associate the tangent and cotangent bundles.

Notice that, for $(x,u) \in TM$ and $\xi, \eta \in T_{(x,u)}TM$ we have

$$\omega(\xi,\eta) = \xi^T \Omega \eta$$

where Ω is the standard symplectic matrix defined by (2.26). In order to define a Hamiltonian system, it is sufficient only to have a symplectic form ω and a sufficiently smooth function H on an even-dimensional manifold N.

Suppose now that N is a $2d$-dimensional manifold and ω is a symplectic form on N. It turns out that we can always write ω (locally) in the form (2.8) (see [11], for example).

Theorem 2.4.2. *(Darboux's Theorem) The manifold N can be covered by charts with coordinates $(x,u) = (x_1,\ldots,x_d,u_1,\ldots,u_d)$ such that the symplectic form ω is written in the form* (2.8).

Now, let $H : N \to \mathbb{R}$ be a Hamiltonian function. Denote by $\phi^t : N \to N$ the Hamiltonian flow associated with H.

Lemma 2.4.3. *The Hamiltonian flow $\phi^t : N \to N$ preserves the symplectic form.*

Proof. There is a quick proof of this lemma using Cartan's formula, but the following is more instructive. By Theorem 2.4.2, we can cover N by charts with coordinates (x,u) such that ω is the standard symplectic form. Let $(x(t),u(t)) = \phi^t(x,u)$ denote the orbit with initial conditions (x,u). Then

$$\left(\phi^t\right)^* \omega = dx(t) \wedge du(t) = \left(\frac{\partial x(t)}{\partial x}\frac{\partial u(t)}{\partial u} - \frac{\partial x(t)}{\partial u}\frac{\partial u(t)}{\partial x}\right) dx \wedge du.$$

It follows from (2.5), (2.7), and the previous equation that

$$\left.\frac{d}{dt}\right|_{t=0} \left(\left(\phi^t\right)^* \omega\right) = \left(\frac{\partial^2 H}{\partial x \partial u} - \frac{\partial^2 H}{\partial u \partial x}\right) dx \wedge du = 0.$$

□

Definition 2.4.4. The *Liouville measure* on N is defined as the integral over the volume form ω^d.

The next result is an immediate corollary of Lemma 2.4.3.

Corollary 2.4.5. *The Hamiltonian flow $\phi^t : N \to N$ preserves the Liouville measure.*

We now define what it means for a map to be symplectic or exact symplectic - the discrete analogue of Hamiltonian flows.

Definition 2.4.6. A smooth map $f : N \to N$ is said to be *symplectic* if $f^*\omega = \omega$.

Suppose now the symplectic form is *exact*, so there is a 1-form λ on N such that $\omega = d\lambda$. Such a 1-form is called the *Liouville 1-form*.

Definition 2.4.7. The smooth map $f : N \to N$ is *exact symplectic* if there is a smooth function $h : N \to \mathbb{R}$ such that

$$f^*\lambda - \lambda = dh.$$

As in Corollary 2.4.5, a symplectic map preserves the Liouville measure.

2.4.2 KAM Theory

Let $(\varphi, I) \in T\mathbb{T}^d$ and consider the Hamiltonian $H = H(I)$. Such a system is called *integrable* since the equations of motion

$$\begin{cases} \dot{\varphi} = \dfrac{\partial H}{\partial I} \\ \dot{I} = -\dfrac{\partial H}{\partial \varphi} = 0 \end{cases}$$

can simply be integrated to obtain

$$\begin{cases} \varphi(t) = \varphi(0) + \nu(I(0))t \\ I(t) = I(0) \end{cases}$$

where $\nu(I) = \frac{\partial H}{\partial I}(I)$. Therefore the dynamics takes place on invariant tori $\mathbb{T}^d \times \{I(0)\}$ and is a rigid rotation with frequency $\nu(I(0))$.

Definition 2.4.8. A vector $\nu \in \mathbb{R}^d$ is said to be:

- *Resonant* if there is $k \in \mathbb{Z}^d \setminus \{0\}$ such that $\langle k, \nu \rangle = 0$;
- *Nonresonant* if it is not resonant; and
- *Strongly nonresonant* or *Diophantine* if there are $\gamma, \tau > 0$ such that

$$|\langle k, \nu \rangle| \geq \frac{\gamma}{|k|^\tau} \tag{2.9}$$

for all $k \in \mathbb{Z}^d \setminus \{0\}$, where $|k| = |k_1| + \cdots + |k_d|$.

It turns out that the persistence properties of an invariant torus under perturbation are heavily dependent on the properties of the underlying frequency vector.

Suppose now we make a perturbation

$$H \to H_\epsilon = H + \epsilon \bar{H}$$

where $\bar{H} = \bar{H}(\varphi, I)$. Poincaré showed that tori with resonant frequencies generally do not survive, even for arbitrarily small values of ϵ, and it was widely believed that the remaining tori should also be destroyed, but no (correct) proof appeared. This is because it is not true, as Kolmogorov pointed out. Let Δ_τ^γ denote the set of all vectors $\nu \in \mathbb{R}^d$ satisfying (2.9), and let

$$\Delta_\tau = \bigcup_{\gamma>0} \Delta_\tau^\gamma.$$

Let $E \subset \mathbb{R}^d$ be a bounded domain, and denote by E_γ the set of all vectors $\nu \in E \cap \Delta_\gamma$ such that $\|\nu - \partial E\| \geq \gamma$.

Theorem 2.4.9. *(KAM Theorem) Suppose there is a set $D \subset \mathbb{R}^d$ such that the frequency map $\nu : D \to E$ is a diffeomorphism, the Hamiltonian $H_\epsilon = H + \epsilon\bar{H}$ is real-analytic on $\mathbb{T}^d \times \overline{D}$, and*

$$\det \frac{\partial^2 H}{\partial I^2} \neq 0.$$

Then there is $\delta > 0$ such that for

$$|\epsilon| < \delta\gamma^2,$$

the invariant tori $\mathbb{T}^d \times \{I_0\}$ with frequency vectors $\nu(I_0) \in E_\gamma$ survive the perturbation.

Remark 2.4.10. *This is not the sharpest possible statement of the KAM theorem - rather it was chosen in [103] as a relatively simple statement to facilitate the exposition.*

The KAM theorem was first stated without proof by Kolmogorov [78] (see also an appendix of [3] for an English translation of the famous address given by Kolmogorov at the ICM in 1954). It was later proved separately by Arnold [9] and Moser [93], and is named for these three. The author found [68, 103] to be particularly useful references on KAM theory, and this exposition is based on them. The proof of the theorem, which reduces to solving a certain homological equation via Newton's method of successive approximations, is beyond the scope of this thesis.

2.4.3 Arnold Diffusion

In his original paper [10] on the subject, Arnold considered the nonautonomous Hamiltonian $H_\epsilon : T\mathbb{T}^2 \times \mathbb{R} \to \mathbb{R}$ given by

$$H_\epsilon(\varphi, I; t) = \frac{I^2}{2} + \epsilon(-1 + \cos\varphi_1)(1 + \mu(\sin\varphi_2 + \cos t)) \tag{2.10}$$

where ϵ, μ are small parameters. For $\epsilon = 0$, the system is integrable, and the dynamics takes place on invariant tori $\mathbb{T}^2 \times \{I_0\}$. KAM theory implies that many of these invariant tori survive perturbation. Arnold proved, however, that we also have the following instability phenomenon.

Theorem 2.4.11. *(Arnold's Example) For any $0 < A < B$ and $\epsilon > 0$, we can find $\mu_0 > 0$ such that for $0 < \mu < \mu_0$ the system (2.10) has a trajectory $(\varphi(t), I(t))$ such that $I_2(0) < A$ and $I_2(\tau) > B$ for some (potentially very large) $\tau > 0$.*

A sketch of the proof of this statement is as follows. Arnold calls an invariant torus T a *whiskered torus* if it is contained in the intersection of invariant manifolds Y^+, Y^-, called the arriving and departing whiskers of T. The manifolds $Y^\pm$ can be thought of as stable and unstable manifolds. A collection $T_1, \dots, T_n$ of whiskered tori is called a *transition chain* if the departing whisker Y_j^- of T_j has a transverse intersection with the arriving whisker Y_{j+1}^+ of T_j, as well as some transversality properties of $Y_j^\pm$ near T_j (see Figure 2.4). For $\mu = 0$, the equations of motion of the (autonomous) system (2.10) are

$$\begin{cases} \dot{\varphi}_1 = \dfrac{\partial H_\epsilon}{\partial I_1} = I_1, \quad \dot{\varphi}_2 = \dfrac{\partial H_\epsilon}{\partial I_2} = I_2 \\ \dot{I}_1 = -\dfrac{\partial H_\epsilon}{\partial \varphi_1} = \epsilon \sin \varphi_1, \quad \dot{I}_2 = -\left.\dfrac{\partial H_\epsilon}{\partial \varphi_2}\right|_{\mu=0} = 0. \end{cases}$$

Let $\nu \in \mathbb{R} \setminus \mathbb{Q}$ and consider the torus

$$T_\nu = \left\{(\varphi, I) \in T\mathbb{T}^2 : I_1 = \varphi_1 = 0, \quad I_2 = \nu\right\}.$$

Then T_ν is a whiskered torus of H_ϵ with $Y^+ = Y^-$ when $\mu = 0$. Arnold points out that, for small $\mu \neq 0$, the invariant torus persists, and the separatrices $Y^\pm$ split and intersect transversely [10, 88, 101]. He then shows that we can find $\nu_1, \dots, \nu_n \in \mathbb{R} \setminus \mathbb{Q}$ such that $\nu_1 < A < \nu_2 < \cdots < \nu_{n-1} < B < \nu_n$, and such that $T_{\nu_1}, \dots, T_{\nu_n}$ is a transition chain. Finally, he observes that arbitrarily small neighbourhoods of T_{ν_1} are connected to arbitrarily small neighbourhoods of T_{ν_n} by orbits of H_ϵ, thus proving Theorem 2.4.11.

When $\mu = 0$, the system (2.10) is autonomous, and the phase space is 4-dimensional. In this case the 2-dimensional invariant tori divide the phase space into regions that are invariant under the Hamiltonian flow of H_ϵ. However, when $\mu \neq 0$ the system depends periodically on time. This can be thought of as increasing the dimension of the phase space, with the fifth variable, time, defined by $\dot{t} = 1$. The 2-dimensional invariant tori do not divide the 5-dimensional phase space into invariant regions, and so diffusion is possible.

The system (2.10) is a very specific example of what Arnold believed to be a much more general phenomenon: "I believe that the mechanism of 'transition chains' which guarantees that instability in our example is also applicable to the general case". He continued, however, "The search for whiskered tori and especially the investigation of their intersections in the general problem of perturbation theory demands very complicated calculations" [10].

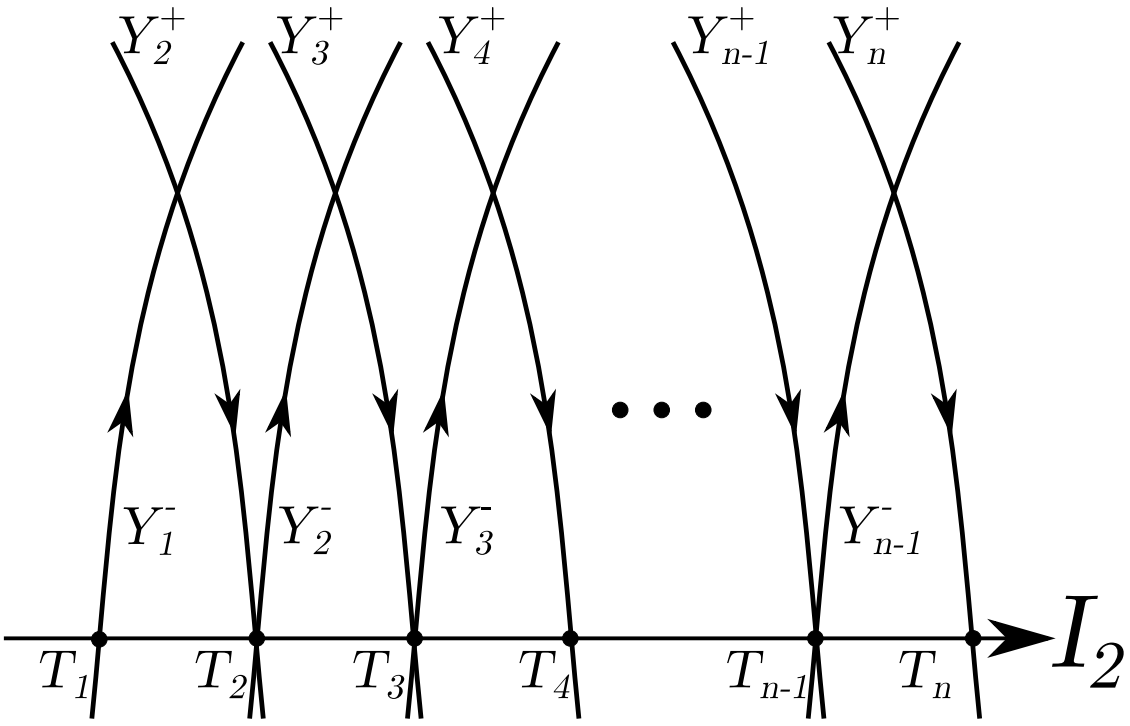


Figure 2.4: A transition chain of whiskered tori along which I_2 increases.

Modern descriptions of Arnold diffusion are similar in spirit. Typically the system has a normally hyperbolic invariant manifold A (or indeed a chain of them with heteroclinic connections between them), which for the purposes of this heuristic discussion we assume to have the structure of a cylinder. In the case where the unperturbed system does not have such an invariant manifold, it must be shown that one can be constructed by arbitrarily small perturbation. Suppose, moreover, a small perturbation can be made that makes the stable and unstable manifolds of A intersect transversely. Then we can define scattering maps $s_1, \ldots, s_n$ on subsets of A. Consider the iterated function system ("IFS") on A consisting the n scattering maps and a time-shift of the restriction of the Hamiltonian flow to the invariant manifold A. Orbits of the IFS are called "pseudo-orbits". The next step is to find (perhaps by small perturbation) diffusive pseudo-orbits – that is, pseudo-orbits which move up along the cylinder. Finally, it is shown that there are true orbits of the Hamiltonian system that shadow the diffusive pseudo-orbits.

In the real-analytic case, the Nekhoroshev estimates apply [21, 97]. These estimates give a lower bound on diffusion time that is exponentially large in the perturbation parameter ϵ. As a result, the diffusion is an extremely slow process.

There are various settings in which Arnold diffusion can take place. These settings

can be categorised as follows.

- *A priori stable:* the unperturbed system is integrable;
- *A priori unstable:* the unperturbed system has a normally hyperbolic invariant manifold; and
- *A priori chaotic:* the unperturbed system has a normally hyperbolic invariant manifold, the stable and unstable manifolds of which have a transverse homoclinic intersection.

The description given above relates to the *geometrical* methods of Arnold diffusion. Variational methods pioneered by Mather have also had success in proving diffusion [87]. These results however typically use a lemma of Cheng and Yan which requires essentially the use of bump functions which cannot be approximated, and so the method of Broer and Tangerman cannot be used [24]. This has presented an obstacle to producing results in the real-analytic category.

2.4.4 k-Jets of Symplectic Automorphisms

Let M be a smooth manifold of dimension d, and let $H : TM \to \mathbb{R}$ be a Hamiltonian function. Suppose $z \in TM$ is a periodic point of the Hamiltonian flow ϕ^t. Let Σ denote a Poincaré section to the orbit of z in the energy level $H(z)$. It follows that, in some coordinates, the corresponding Poincaré map $P : \Sigma \to \Sigma$ is a symplectic diffeomorphism from a neighbourhood of the origin in $\mathbb{R}^{2(d-1)}$ to itself, with the origin as a fixed point. Often it is easier to study this Poincaré map than it is to study the flow.

Let $\mathcal{S}(d-1)$ denote the set of symplectic diffeomorphisms from $\mathbb{R}^{2(d-1)}$ to itself with a fixed point at the origin. Let Ω denote the standard $2(d-1) \times 2(d-1)$ symplectic matrix. A $2(d-1) \times 2(d-1)$ matrix σ with real entries is said to be a *symplectic matrix* if

$$\sigma^T \Omega \sigma = \Omega. \tag{2.11}$$

Let $f \in \mathcal{S}(d-1)$. The fact that f is symplectic guarantees that $D_0 f$ is a symplectic matrix.

Lemma 2.4.12. *Let σ be a symplectic matrix. Then the eigenvalues of σ come in reciprocal pairs.*

Proof. Notice that $\det(\Omega) = 1$. This, together with equation (2.11) implies that $\det(\sigma) = \pm 1$. Therefore (2.11) imples that

$$\sigma = -\Omega \sigma^{-T} \Omega.$$

Using this formula together with the formulas $\det(\Omega) = \det(-\Omega) = 1$ and $\Omega^{-1} = -\Omega$, and the fact that the determinant of a product of matrices is the product of determinants, we see that the characteristic polynomial of σ is

$$\begin{aligned}\det(\sigma - \lambda I) &= \det(-\Omega\sigma^{-T}\Omega + \lambda\Omega^2) \\ &= \det(-\Omega)\det(\sigma^{-T} - \lambda I)\det(\Omega) = \det(\sigma^{-T} - \lambda I).\end{aligned}$$

Therefore σ and σ^{-T} have the same characteristic polynomial. It follows that σ and σ^{-1} have the same spectrum. □

Another property of symplectic matrices implies that elliptic fixed points of symplectic diffeomorphisms cannot be made hyperbolic by arbitrarily small symplectic perturbations, meaning that a direct analogue of Theorem 2.2.4 is not true in the class of Hamiltonian vector fields (or any subsets of this class). As an example, suppose $f \in \mathcal{S}(1)$ and let

$$\sigma = D_0 f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By Lemma 2.4.12 we have $\det(\sigma) = 1$. Therefore the characteristic polynomial of σ is $\lambda^2 - (a+d)\lambda + 1 = 0$, and so the eigenvalues are

$$\frac{(a+d) \pm \sqrt{(a+d)^2 - 4}}{2}.$$

The product of these eigenvalues is 1, and the eigenvalues are nonreal if and only if

$$(a+d)^2 < 4,$$

in which case they must lie on the complex unit circle. The latter is clearly a C^1-open property of f in $\mathcal{S}(1)$. Therefore a different formulation of part 1 of Theorem 2.2.4 is required in the Hamiltonian setting.

Let $f \in \mathcal{S}(d-1)$ and $k \in \mathbb{N}$. The *k-jet* $J_0^k f$ *of* f *at* 0 is the truncated Taylor series at 0,

$$J_0^k f(x) = \sum_{j=0}^{k} \frac{1}{j!} D^j f(0) x^{\otimes j}$$

considered as a polynomial in x. Denote by $J_s^k(d-1)$ the set of k-jets of symplectic C^k-diffeomorphisms acting on $\mathbb{R}^{2(d-1)}$ that fix the origin, and equip this space with the product

$$J_0^k f_1 \cdot J_0^k f_2 = J_0^k (f_2 \circ f_1).$$

Definition 2.4.13. Let $\mathcal{J} \subset J_s^k(d-1)$. The set $\mathcal{J}$ is said to be:

- *Open* if for all $J_0^k f$ in $\mathcal{J}$, for any g in a sufficiently C^k-small neighbourhood of f, we have $J_0^k g \in \mathcal{J}$;
- *Dense* if for any $f \in \mathcal{S}(d-1)$ we can find g arbitrarily C^k-close to f such that $J_0^k g \in \mathcal{J}$; and
- *Invariant* if for any $J_0^k f \in \mathcal{J}$ and any $g \in \mathcal{S}(d-1)$,

$$J_0^k g \cdot J_0^k f \cdot J_0^k g^{-1} \in \mathcal{J}.$$

Remark 2.4.14. *The invariance of a set of jets allows us to make symplectic changes of coordinates without leaving the set.*

A suitable reformulation of part 1 of Theorem 2.2.4 for Hamiltonian vector fields is the following: if $\mathcal{J} \subset J_s^k(d-1)$ is open, dense, and invariant, there is a residual set of C^k-smooth Hamiltonian vector fields X on TM such that the k-jet of the Poincaré map of every closed orbit of the flow of X is in $\mathcal{J}$. Theorems of this type were proved by Robinson for conservative systems in [105, 106], and for Mañé generic Hamiltonians by Rifford and Ruggiero [104] and by Carballo and Miranda [23].

In practice, k-jets of symplectic diffeomorphisms can be hard to compute. Instead, we will look at the $(k+1)$-jet of the Hamiltonian, which takes the form of a more familiar polynomial, and look at the effect of a generic perturbation on this function.

2.5 Hypersurface Geometry

Denote by $\mathcal{W}^d$ the set of all real-analytic functions $Q : \mathbb{R}^d \to \mathbb{R}$ such that the set

$$M = M(Q) = \{x \in \mathbb{R}^d : Q(x) = 0\} \tag{2.12}$$

is a closed hypersurface of $\mathbb{R}^d$. Equip $\mathcal{W}^d$ with the real-analytic topology (see Section 2.2). Let $Q \in \mathcal{W}^d$, and let $M = M(Q)$. Then for $x \in M$, the unit normal vector

$$n(x) = -\frac{\nabla Q(x)}{\|\nabla Q(x)\|} \tag{2.13}$$

is inward-pointing[2]. The curvature matrix $C(x)$ is the matrix of second partial derivatives of Q divided by the norm of the gradient:

$$C(x) = \|\nabla Q(x)\|^{-1} \left(\frac{\partial^2 Q}{\partial x_i \partial x_j}(x) \right)_{i,j=1,\ldots,d+2} \tag{2.14}$$

[2]The minus sign is a convention that leads to the standard definitions of convexity and concavity.

Let $u \in T_xM$. The shape operator $S(x) : T_xM \to T_xM$ is defined as

$$S(x)u = -Dn(x)u = C(x)u - \langle C(x)u, n(x)\rangle n(x).$$

This enables a definition of the normal curvature at x in the direction u via

$$\kappa(x, u) = \langle S(x)u, u\rangle = \langle C(x)u, u\rangle.$$

Definition 2.5.1. The manifold M is *strictly convex* if

$$\kappa(x, u) > 0$$

for all $x \in M$ and $0 \neq u \in T_xM$. Denote by $\mathcal{V}^d$ the set of all $Q \in \mathcal{W}^d$ such that $M = M(Q)$ is strictly convex.

It is clear from the above definition that strict convexity is an open property of a compact manifold, in the sense that all sufficiently small perturbations of the manifold preserve strict convexity. It follows that $\mathcal{V}^d$ is an open subset of $\mathcal{W}^d$.

The following lemma allows us to simplify some formulas.

Lemma 2.5.2. *Let $Q \in \mathcal{W}^d$ and $M = M(Q)$. Then we may assume without loss of generality that $\|\nabla Q(x)\| = 1$ for all $x \in M$.*

Proof. Replace Q by $\tilde{Q} = \|\nabla Q(x)\|^{-1}Q(x)$. Then $\tilde{Q}(x) = 0$ if and only if $Q(x) = 0$, and so $M = M(\tilde{Q})$.Moreover, if $x \in M$ then

$$\nabla\tilde{Q}(x) = \|\nabla Q(x)\|^{-1}\nabla Q(x) + Q(x)\nabla\|\nabla Q(x)\|^{-1} = \|\nabla Q(x)\|^{-1}\nabla Q(x)$$

since $Q(x) = 0$ for $x \in M$. It follows that $\|\nabla\tilde{Q}(x)\| = 1$ for all $x \in M$. □

For such a $Q \in \mathcal{W}^d$ with $\|\nabla Q(x)\| = 1$ whenever $x \in M$, the unit normal vector is

$$n(x) = -\nabla Q(x). \tag{2.15}$$

Therefore the shape operator takes the form

$$S(x)u = D\left(\nabla Q(x)\right)u = \left(\frac{\partial^2 Q}{\partial x^2}(x)\right)u = C(x)u. \tag{2.16}$$

Remark 2.5.3. *When we perform perturbations to the hypersurface M, we still have to use equation* (2.13) *rather than* (2.15) *to compute the perturbed normal vector. This is because the perturbed normal vector will not necessarily have norm* 1 *and so must be renormalised. However, once the perturbation is made, we can again assume that $\|\nabla Q(x)\| = 1$ for all $x \in M$ due to Lemma 2.5.2.*

Let

$$\pi : TM \longrightarrow M$$
$$T_xM \ni u \longmapsto x$$

denote the canonical projection along fibres of the tangent bundle. We use the notation $(x, u) \in TM$ to mean $x \in M$ and $u \in T_xM$, so that $\pi(x, u) = x$.

Throughout this thesis we consider Hamiltonian and symplectic dynamics on hypersurfaces M. The coordinates used are those of the ambient Euclidean space $\mathbb{R}^{2d}$, but restricted to the tangent bundle TM. It is therefore necessary that the symplectic form on $\mathbb{R}^{2d}$ remains a symplectic form when restricted to TM.

Lemma 2.5.4. *Denote by $\omega = dx \wedge du$ the standard symplectic form on $\mathbb{R}^{2d}$. Then the restriction $\omega|_{TM}$ is nondegenerate.*

Proof. Let $x \in M$ and $u \in T_xM$. Let $X \in T_{(x,u)}TM \setminus \{0\}$. Rotating M if necessary we may assume without loss of generality that

$$T_xM = \{v = (v_1, \ldots, v_d) \in \mathbb{R}^d : v_d = 0\}.$$

In that case,

$$X = (X_x, X_u) = \left(\left(X_{x,1}, \ldots, X_{x,d-1}, 0\right), \left(X_{u,1}, \ldots, X_{u,d-1}, 0\right)\right).$$

Since $X \neq 0$, there is some $j \in \{1, \ldots, d-1\}$ such that $X_{x,j} \neq 0$ or $X_{u,j} \neq 0$. Without loss of generality $X_{x,1} \neq 0$. Let

$$Y = ((0, \ldots, 0), (1, 0, \ldots, 0)).$$

Then $Y \in T_{(x,u)}TM$, and

$$\begin{aligned}\omega(X, Y) = X^T \Omega Y &= \begin{pmatrix} X_x & X_u \end{pmatrix} \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \begin{pmatrix} Y_x \\ Y_u \end{pmatrix} = \begin{pmatrix} X_x & X_u \end{pmatrix} \begin{pmatrix} Y_u \\ -Y_x \end{pmatrix} \\ &= \langle X_x, Y_u \rangle - \langle X_u, Y_x \rangle = X_{x,1} \neq 0.\end{aligned}$$

□

2.6 Convex Billiards

The billiard map takes a point on a strictly convex hypersurface M of $\mathbb{R}^d$ and an inward-pointing velocity vector of norm 1, and follows the velocity vector to the next point of collision with the boundary. At this point, the new velocity vector is obtained

via the optical law of reflection: the angle of reflection equals the angle of incidence. Billiard dynamics were first considered by Birkhoff [17, 18], who said "in this problem the formal side, usually so formidable in dynamics, almost completely disappears, and only the interesting qualitative questions need to be considered". See also e.g. [25, 113, 114] for modern expositions.

2.6.1 The Billiard Map

Let $Q \in \mathcal{V}^d$, and let $M = M(Q)$ denote the hypersurface of $\mathbb{R}^d$ defined by (2.12). The phase space $\mathcal{M}$ of the billiard map consists of the closed unit ball around the origin of each tangent space to M,

$$\mathcal{M} = \mathcal{M}(Q) = \{(x, u) \in TM : \|u\| \leq 1\}. \tag{2.17}$$

For a point $(x, u) \in \mathcal{M}$, the inward-pointing velocity vector is

$$v = v(x, u) = u + \sqrt{1 - u^2}\, n(x),$$

where $u^2 = \|u\|^2$. Note that $\|v\| = 1$ since u is a tangent vector to M at x and $n(x)$ is normal to M at x. The boundary $\partial\mathcal{M}$ of the phase space is the set of points $(x, u) \in \mathcal{M}$ for which $\|u\| = 1$, and the interior is $\mathrm{Int}(\mathcal{M}) = \mathcal{M} \setminus \partial\mathcal{M}$. Define the free flight time $\tau : \mathrm{Int}(\mathcal{M}) \to \mathbb{R}$ between collisions with the boundary as

$$\tau(x, u) = \sup\{t > 0 : x + tv(x, u) \in M\}. \tag{2.18}$$

This is well-defined on the interior of $\mathcal{M}$ by strict convexity. Extend it continuously to the boundary of $\mathcal{M}$ by setting $\tau|_{\partial\mathcal{M}} \equiv 0$. It can be shown that $\tau|_{\mathrm{Int}(\mathcal{M})}$ is real-analytic under our assumptions on M, but analyticity of τ fails on $\partial\mathcal{M}$ in these coordinates (see Lemma 3.2.1). Since $\|v\| = 1$, the flight time $\tau(x, u)$ is equal to the Euclidean distance from x to the next point of collision with the boundary. Define the billiard map $f : \mathcal{M} \to \mathcal{M}$ by $(\bar{x}, \bar{u}) = f(x, u)$ where

$$\begin{cases} \bar{x} = x + \tau(x, u)\, v \\ \bar{u} = v - \langle v, n(\bar{x}) \rangle\, n(\bar{x}). \end{cases} \tag{2.19}$$

The second equation of (2.19) is simply the optical law of reflection. Clearly $f|_{\partial\mathcal{M}} \equiv \mathrm{Id}$ since the flight time vanishes identically on the boundary. Moreover, a point $(x, u) \in \mathcal{M}$ is a fixed point of the billiard map if and only if $\|u\| = 1$.

The billiard map satisfies the following reversibility property: if $\mathcal{I} : TM \to TM$ denotes the operator $\mathcal{I}(x, u) = (x, -u)$ then $f^{-1} = \mathcal{I} \circ f \circ \mathcal{I}$. It follows that the billiard map is bijective.

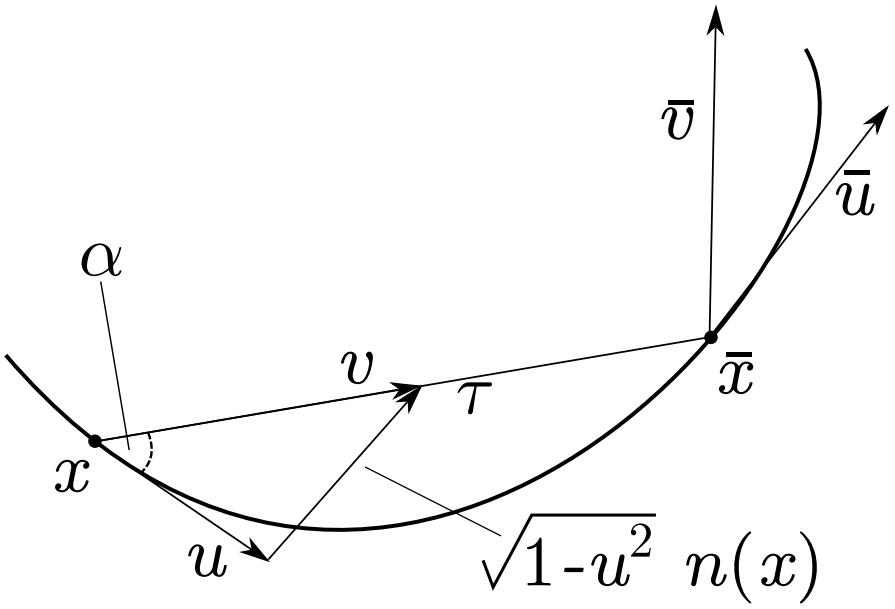


Figure 2.5: An iterate of the billiard map. The billiard map takes as input a point x, and a tangent vector u to M at x of norm at most 1. With $v = u + \sqrt{1-u^2}\, n(x)$ denoting the inward-pointing velocity vector and τ the flight time, the particle travels a distance τ in the direction of v until the next point of collision with the boundary at $\bar{x}$. If $\|u\| = 1$, the inward-pointing component of the velocity vanishes, so the velocity vector is tangent to the boundary, and the angle of reflection α is zero.

If $\lambda = -u\, dx$ is the Liouville 1-form, then the standard symplectic form on $\mathbb{R}^{2d}$ is given by $\omega = d\lambda = dx \wedge du$. The restriction of ω to $\mathcal{M}$ is a symplectic form by Lemma 2.5.4.

Lemma 2.6.1. *The restriction of the billiard map f to the interior Int($\mathcal{M}$) of the phase space is exact symplectic, with*

$$f^*\lambda - \lambda = -d\tau.$$

Proof. Applying the differential to the first equation of (2.19) we see that

$$d\bar{x} = dx + v\, d\tau + \tau\, dv.$$

Using this together with the second equation of (2.19), the fact that $n(x)\, dx = n(\bar{x})\, d\bar{x} = 0$, and the fact that $\langle v, v\rangle = 1$, we see that

$$\begin{aligned} -f^*\lambda = \bar{u}\, d\bar{x} &= v\, d\bar{x} - \langle v, n(\bar{x})\rangle n(\bar{x})\, d\bar{x} = v\, d\bar{x} \\ &= v\, dx + \langle v, v\rangle\, d\tau + \tau v\, dv \\ &= u\, dx + \sqrt{1-u^2}\, n(x)\, dx + d\tau + \tau v\, dv \\ &= -\lambda + d\tau + \tau v\, dv. \end{aligned}$$

Applying the differential to the identity $\langle v, v\rangle = 1$ we see that $v\,dv = 0$, which completes the proof of the lemma. □

Since the restriction of ω to $\mathcal{M}$ is a symplectic form, there is a corresponding Liouville measure μ on $\mathcal{M}$ that is preserved by f as a result of Lemma 2.6.1.

2.6.2 Caustics

Let $Q \in \mathcal{V}^d$, let $M = M(Q)$ denote the corresponding hypersurface of $\mathbb{R}^d$, and let $\mathcal{M}$ denote the phase space of the billiard map f. Let $(x, u) \in \mathcal{M}$. A *segment of the billiard trajectory* $(x^n, u^n) = f^n(x, u)$ is the straight line between two consecutive points of collision x^n and x^{n+1}.

Definition 2.6.2. A *caustic* is a closed and strictly convex hypersurface C of $\mathbb{R}^d$ compactly embedded in the interior of the region bounded by M that has the following property: if one segment of a billiard trajectory is tangent to C, then so is every segment of that trajectory.

Remark 2.6.3. *Some authors do not require the assumptions of closedness or strict convexity. The definition given above is suitable for the purposes of this thesis.*

A well-known example of caustics is for the billiard inside an ellipsoid M in $\mathbb{R}^d$. In this case every confocal ellipsoid contained in the domain bounded by M is a caustic. A consequence of this is that the billiard in an ellipsoid is integrable. In fact, a conjecture of Birkhoff says that in $\mathbb{R}^2$ the billiard in an ellipse is the *only* integrable billiard. Some progress has been made on this (see, for example, [12, 15, 36, 71]), but the conjecture in its full generality remains open.

We now present various results regarding existence and nonexistence of such convex caustics. The following famous result was proved in [80].

Theorem 2.6.4. *(Lazutkin) If M is a sufficiently smooth, closed, strictly convex hypersurface of $\mathbb{R}^2$, then there is a family of caustics near the boundary occupying a set of positive measure.*

Remark 2.6.5. *Sufficient smoothness in Lazutkin's theorem was C^{553} – this was motivated by the KAM techniques he employed to prove the theorem. This requirement was later reduced to C^6 by Douady [39].*

On the other hand, Halpern gave an example of a billiard boundary in $\mathbb{R}^2$ which is strictly convex but only C^2 smooth, and on which there are billiard trajectories that become tangent to the boundary in finite time, implying that there are no caustics.

It was shown by Mather that if a billiard boundary in $\mathbb{R}^2$ is at least C^2-smooth but has a flat point (i.e. a point at which the curvature is 0), then there are billiard trajectories which come arbitrarily close to being tangent with the boundary [86]. As a consequence of this and Birkhoff's theorem about invariant circles of twist maps being graphs of Lipschitz functions, he showed the absence of caustics in this case.

The following result was proved in its full generality in [59], improving results in [14] which required some additional assumptions.

Theorem 2.6.6. *(Gruber, Berger) Let $Q \in \mathcal{V}^d$ and let $M = M(Q)$ denote the corresponding hypersurface of $\mathbb{R}^d$ where $d \geq 3$. Suppose there exist caustics in M. Then M is an ellipsoid.*

Theorem 2.6.6 implies that there is an open and dense subset of $\mathcal{V}^d$ such that for every Q in this subset, the billiard map defined by the hypersurface $M = M(Q)$ of $\mathbb{R}^d$ (with $d \geq 3$) has no caustics.

2.6.3 Trajectories Asymptotic to the Boundary

Let $Q \in \mathcal{V}^d$, let $M = M(Q)$ denote the corresponding hypersurface of $\mathbb{R}^d$, and let $\mathcal{M}$ denote the phase space of the billiard map f.

Definition 2.6.7. Let $(x, u) \in \mathcal{M}$. The trajectory $(x^n, u^n) = f^n(x, u)$ is said to *terminate asymptotically on the boundary* if

$$\lim_{n\to\infty} \|u^n\| = 1.$$

Such trajectories were first considered by Gruber [58]. Notice that the angle of reflection along such a trajectory tends to zero, and so the trajectory comes closer to being tangent with the boundary as $n \to \infty$.

In the case where $d = 2$, Theorem 2.6.4 implies the existence of a family of caustics near the boundary. The phase space of the 2-dimensional billiard has the structure of a cylinder, with an angular coordinate on M and the angle the trajectory makes with the boundary taking values in an interval. The caustics divide this 2-dimensional phase space into invariant regions, rendering the existence of trajectories asymptotically terminating on the boundary impossible.

Suppose now that $d \geq 3$. Let $Q \in \mathcal{V}^d$ and suppose $M = M(Q)$ is not an ellipsoid. Theorem 2.6.6 implies that M has no caustics. Even though caustics do not exist, invariant tori with a different structure may be present (see [79] for a construction of KAM tori for the billiard map in a neighbourhood of an elliptic closed geodesic on M). In dimension 3, the known examples of invariant tori are of dimension 2 or lower,

and so do not divide the 4-dimensional phase space. This removes an obstacle to the existence of trajectories asymptotically terminating on the boundary.

The following result is due to Gruber [58], but has been adapted here to use the notation and coordinates of this thesis. Let μ denote the Liouville measure on $\mathcal{M}$ induced by the restriction of the standard symplectic form ω to $\mathcal{M}$.

Theorem 2.6.8. *(Gruber) For μ-almost every $(x,u) \in Int(\mathcal{M})$ the trajectory $f^n(x,u)$ does not terminate asymptotically on the boundary.*

Proof. Denote by $\mathcal{T}$ the set of all points $(x,u) \in \mathrm{Int}(\mathcal{M})$ such that $f^n(x,u)$ terminates asymptotically on the boundary. As it is not clear that the set $\mathcal{T}$ is measurable, we proceed using the outer measure corresponding to μ, which for convenience we also denote μ.

Suppose for contradiction $\mu(\mathcal{T}) > 0$. For $\epsilon > 0$ let

$$A_\epsilon = \left\{(x,u) \in \mathrm{Int}(\mathcal{M}) : \sqrt{1-u^2} < \epsilon\right\}.$$

Clearly $\mu(A_\epsilon) \to 0$ as $\epsilon \to 0$, and so we can find some $\epsilon > 0$ such that

$$\mu(A_\epsilon) < \mu(\mathcal{T}). \tag{2.20}$$

For each $(x,u) \in \mathcal{T}$ there is $N \in \mathbb{N}$ such that $f^n(x,u) \in A_\epsilon$ for all $n \geq N$. It follows that

$$\mathcal{T} \subseteq \bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} f^{-n}(A_\epsilon). \tag{2.21}$$

It follows from the μ-invariance of f, equations (2.20) and (2.21), and part (i) of Theorem 10.16 in [63] that

$$\mu(\mathcal{T}) \leq \mu\left(\bigcup_{N=1}^{\infty} \bigcap_{n=N}^{\infty} f^{-n}(A_\epsilon)\right) \leq \liminf_{n\to\infty} \mu\left(f^{-n}(A_\epsilon)\right) = \mu(A_\epsilon) < \mu(\mathcal{T}),$$

which is a contradiction. □

2.6.4 Oscillatory Motions

Let $Q \in \mathcal{V}^d$, let $M = M(Q)$ denote the corresponding hypersurface of $\mathbb{R}^d$, and let $\mathcal{M}$ denote the phase space of the billiard map f.

Definition 2.6.9. Let $(x,u) \in \mathcal{M}$. The trajectory $(x^n,u^n) = f^n(x,u)$ is said to be *oscillatory* if there is $\delta > 0$ such that

$$\liminf_{n\to\infty} \|u^n\| = 1-\delta, \quad \limsup_{n\to\infty} \|u^n\| = 1.$$

It is not known whether an analogue of Theorem 2.6.8 is true for oscillatory motions. The study of oscillatory motions in celestial mechanics is an active area of research: see for example [57, 60, 83]. In particular, Arnold called the question of the measure of oscillatory trajectories the "central problem of celestial mechanics" [57]. The measure of such trajectories is conjectured to be zero in the context of celestial mechanics. In this thesis the generic existence of oscillatory trajectories in billiards is proved, and we conjecture that the measure of such trajectories is zero.

2.7 Geodesic Flows

Geodesics - the equivalent of straight lines on curved surfaces - are a central object of study in Riemannian geometry, and give rise to a dynamical system of classical interest. Let M be a smooth manifold endowed with a Riemannian metric. The geodesic flow takes a tangent vector $u \in TM$ and follows the unique geodesic passing through $\pi(u)$ with first derivative equal to u. The geodesic flow is a natural form of the kinetic energy of a Hamiltonian system, and so knowledge of generic properties of these flows can be useful in that context.

2.7.1 The Geodesic Flow on Riemannian Manifolds

Let M be a smooth manifold of dimension d endowed with a Riemannian metric g. The geodesic flow is the flow of the Hamiltonian function

$$H(x,u) = \frac{1}{2}g(x)(u,u).$$

In a coordinate chart, the equations of motion take the form

$$\begin{cases} \dot{x}_k = \dfrac{\partial H}{\partial u_k}(x,u) = \displaystyle\sum_{j=1}^{d} g_{jk}(x)u_j \\ \dot{u}_k = -\dfrac{\partial H}{\partial x_k}(x,u) = -\dfrac{1}{2}\displaystyle\sum_{i,j=1}^{d} \dfrac{\partial g_{ij}}{\partial x_k}(x)u_i u_j. \end{cases}$$

The question of existence of homoclinic orbits for flows of this type is a classical problem that goes back to Poincaré [101]. We present here two results in this direction obtained relatively recently. The following theorem was proved in [77].

Theorem 2.7.1. *(Knieper and Weiss) There is a C^∞-open and dense set of strictly convex Riemannian metrics on $\mathbb{S}^2$ for which the geodesic flow has a transverse homoclinic orbit.*

The next result, proved by Contreras in [29], built on work by Contreras and Paternain in [30].

Theorem 2.7.2. *(Contreras) Let M be a smooth closed manifold of dimension $d \geq 2$. There is a C^2-open and dense set of Riemannian metrics on M for which the geodesic flow has a transverse homoclinic orbit.*

2.7.2 The Geodesic Flow on Hypersurfaces

Let $Q \in \mathcal{W}^d$, and let $M = M(Q)$ denote the closed hypersurface of $\mathbb{R}^d$ defined by Q. The geodesic flows studied in this thesis are associated with the induced metric and coordinates of the ambient Euclidean space $\mathbb{R}^d$, so a different formulation to the above is required. Consider a smooth curve $\gamma : [a, b] \to M$. The tangential component $\gamma''(t)^\top$ of its acceleration is given by

$$\gamma''(t)^\top = \gamma''(t) - \langle \gamma''(t), n(\gamma(t)) \rangle n(\gamma(t)). \tag{2.22}$$

Since $\gamma'(t) \in T_{\gamma(t)}M$ and $n(\gamma(t))$ is normal to M at $\gamma(t)$ for all $t \in [a, b]$ we have

$$0 = \frac{d}{dt}\langle \gamma'(t), n(\gamma(t)) \rangle = \langle \gamma''(t), n(\gamma(t)) \rangle - \kappa(\gamma(t), \gamma'(t)). \tag{2.23}$$

The curve γ is a geodesic if and only if $\gamma''(t)^\top = 0$, so from (2.22) and (2.23) we see that γ is a geodesic if and only if

$$\gamma''(t) = \kappa(\gamma(t), \gamma'(t))\, n(\gamma(t)).$$

It follows that the geodesic flow $\phi^t : TM \to TM$ is defined by

$$\left.\frac{d}{dt}\right|_{t=0} \phi^t(x, u) = X(x, u),$$

where the vector field $X(x, u) = (\dot{x}, \dot{u})$ is given by

$$\begin{cases} \dot{x} = u \\ \dot{u} = \kappa(x, u)\, n(x). \end{cases} \tag{2.24}$$

Consider the function $H : TM \to \mathbb{R}$ given by

$$H(x, u) = \frac{u^2}{2} + \kappa(x, u)\frac{Q(x)}{\|\nabla Q(x)\|}. \tag{2.25}$$

It is not hard to see that $X = \Omega \nabla H$ where

$$\Omega = \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} \tag{2.26}$$

is the standard symplectic matrix. This is equivalent to the statement that the geodesic flow is a Hamiltonian system with Hamiltonian H and symplectic form

$$\omega = dx \wedge du.$$

Notice that the second term of (2.25) vanishes identically on TM, and so the energy $\frac{u^2}{2}$ is conserved. Since the Hamiltonian H is homogeneous of second order in u, the dynamics of the geodesic flow is the same on every energy level.

The following lemma is geometrically clear: if the manifold is not curved in the direction of a geodesic then the geodesic goes to infinity and cannot be periodic. Since this fact is used repeatedly in later chapters, we give a proof. Similarly, it can be shown that if a geodesic segment on the closed manifold M is sufficiently long, then M must be curved in the direction of the geodesic segment.

Lemma 2.7.3. *Let $Q \in W^d$ and let $M = M(Q)$. Suppose $(x_0, u_0) \in TM \setminus \{0\}$ is a periodic point of the geodesic flow ϕ^t. Denote by $(x(t), u(t)) = \phi^t(x_0, u_0)$ its orbit. Then there is some $t > 0$ such that*

$$\kappa(x(t), u(t)) \neq 0.$$

Proof. If not, then

$$u(t) = u_0 + \int_0^t \dot{u}(s)\, ds = u_0 + \int_0^t \kappa(x(s), u(s))\, n(x(s))\, ds = u_0$$

for all $t \geq 0$, and so

$$x(t) = x_0 + \int_0^t \dot{x}(s)\, ds = x_0 + t u_0.$$

Therefore $\|x(t)\| \to \infty$ as $t \to \infty$, so (x_0, u_0) cannot be periodic, a contradiction. □

2.7.3 Classification of Closed Geodesics

Let $Q \in \mathcal{W}^d$, and let $M = M(Q)$ denote the closed hypersurface of $\mathbb{R}^d$ defined by Q. Closed geodesics on M correspond to periodic orbits of the geodesic flow. A method of construction of closed geodesics on any Riemannian manifold was proposed by Birkhoff [16, 17] (see also [31]; see [28] for a modern exposition). This implies that the geodesic flow on M has a periodic orbit γ. Consider a transverse section to γ in TM, and the corresponding Poincaré map of the periodic orbit γ in the energy level $\|u\| = c$. Similarly to the classification of fixed points in Section 2.1.1, the closed geodesic γ is said to be:

- *Parabolic* if 1 is an eigenvalue of the linearisation of the Poincaré map;
- *Degenerate* if the linearised Poincaré map has an eigenvalue equal to a root of unity;
- *Hyperbolic* if the linearised Poincaré map has no eigenvalue of absolute value 1;
- *Elliptic* if it is nondegenerate and nonhyperbolic; and

- *q-elliptic* if the linearised Poincaré map has exactly $2q$ eigenvalues of absolute value 1.

Since the geodesic flow is a Hamiltonian system, the eigenvalues of the linearised Poincaré map come in reciprocal pairs by Lemma 2.4.12. As the dynamics of the geodesic flow is the same on every energy level (see Section 2.7.2), there is a periodic orbit γ in each energy level, and moreover the above classification is independent of the level set in consideration. Furthermore, the classification is independent of the choice of Poincaré map [74].

— 3 —

Arnold Diffusion in Billiards

In this section it is shown that the billiard map on real-analytic closed hypersurfaces of $\mathbb{R}^d$ (where d is at least 3) generically has trajectories that drift asymptotically towards the boundary, assuming that the geodesic flow on the boundary has a hyperbolic periodic orbit and a transverse homoclinic. On these same manifolds, there also exist oscillatory billiard trajectories.

3.1 Results

Let $d \geq 3$, and recall from Section 2.5 that $\mathcal{V}^d$ denotes the set of all real-analytic functions $Q : \mathbb{R}^d \to \mathbb{R}$ such that the set defined as in (2.12) by

$$M = M(Q) = \{x \in \mathbb{R}^d : Q(x) = 0\}$$

is a real-analytic, closed, and strictly convex hypersurface of $\mathbb{R}^d$. Moreover, $\mathcal{V}^d$ is equipped with the real-analytic topology. Denote by $\mathcal{V}^d_*$ the set of all $Q \in \mathcal{V}^d$ such that assumptions [A1,2] hold:

[A1] There is a hyperbolic closed geodesic γ on M; and

[A2] There is a transverse homoclinic geodesic ξ to γ.

It is well-known that these conditions are C^2-open in terms of the Hamiltonian function of the geodesic flow (i.e. in terms of the Riemannian metric), and in our case the Hamiltonian (as well as the metric) depends on the curvature of the hypersurface (i.e. the second derivative). Therefore the set of hypersurfaces satisfying these assumptions is C^4-open. In particular, $\mathcal{V}^d_*$ is open in $\mathcal{V}^d$ in the real-analytic topology.

Recall that the phase space $\mathcal{M} = \mathcal{M}(Q)$ of the billiard map is defined by equation (2.17), and the billiard map f itself is defined by (2.19).

Theorem 3.1.1. *There is a residual subset $\overline{\mathcal{V}}$ of $\mathcal{V}_*^d$ such that for every $Q \in \overline{\mathcal{V}}$, there is $(x,u) \in \mathcal{M} = \mathcal{M}(Q)$ with $\|u\| < 1$ and $\|u^n\| \to 1$ as $n \to \infty$, where $(x^n, u^n) = f^n(x,u)$.*

Recall from Section 2.6.4 that a billiard trajectory $\{(x^n, u^n) = f^n(x,u)\}_{n\in\mathbb{N}_0}$ is oscillatory if the limit superior as n goes to infinity of $\|u^n\|$ is 1, and the limit inferior is less than 1.

Theorem 3.1.2. *Let $\overline{\mathcal{V}}$ denote the residual subset of $\mathcal{V}_*^d$ described in Theorem 3.1.1. Then for every $Q \in \overline{\mathcal{V}}$ there is $(x,u) \in \mathcal{M} = \mathcal{M}(Q)$ for which the trajectory $\{(x^n, u^n) = f^n(x,u)\}_{n\in\mathbb{N}_0}$ is oscillatory.*

Note that the trajectories described in Theorem 3.1.1 (and obviously those of Theorem 3.1.2) cannot occur in a finite number of iterations: since the billiard map is bijective, the preimage of a fixed point (i.e. $\|u\| = 1$) is a fixed point. In fact, the process is extremely slow. Since the proof is valid in the analytic category, the Nekhoroshev estimates apply, giving an exponential lower bound on the stability time of finite segments of diffusive trajectories [97].

Let us describe these trajectories. Assumption [A1] guarantees that the geodesic flow has a normally hyperbolic invariant manifold consisting the hyperbolic closed geodesic γ on an interval of energy levels (see (3.25)). We prove that the billiard map inherits such a normally hyperbolic invariant manifold A from the geodesic flow. The manifold A has the structure of a 2-dimensional cylinder, where the height component can be thought of as the reciprocal of the angle of reflection, and the angular component is close to the angular variable on the closed geodesic γ. The diffusive trajectories occur arbitrarily close to (but not on) this cylinder. Due to normal hyperbolicity, A has stable and unstable invariant manifolds $W^{s,u}(A)$. Moreover, these invariant manifolds have a transverse homoclinic intersection by [A2] (see (3.29) for a definition of the homoclinic cylinder B). The union of some small neighbourhoods of the normally hyperbolic cylinder and submanifolds of the transverse homoclinic intersection is called the *homoclinic channel*. Diffusive trajectories occur in the homoclinic channel, starting near the cylinder, and making excursions near a transverse homoclinic intersection before returning close to A and remaining in a neighbourhood of A for some iterations (see Figure 3.1). This process is repeated infinitely many times.

On the stable and unstable manifolds of the normally hyperbolic invariant manifold A are the strong stable and strong unstable foliations respectively (see Section 2.1.2). Each point x on the manifold A uniquely defines a leaf $W^s(x)$ of the strong stable foliation and a leaf $W^u(x)$ of the strong unstable foliation. Suppose now that y is a point of transverse homoclinic intersection of $W^{s,u}(A)$. The implicit function theorem then implies that there is a manifold B such that $y \in B \subset W^s(A) \pitchfork W^u(A)$. Suppose

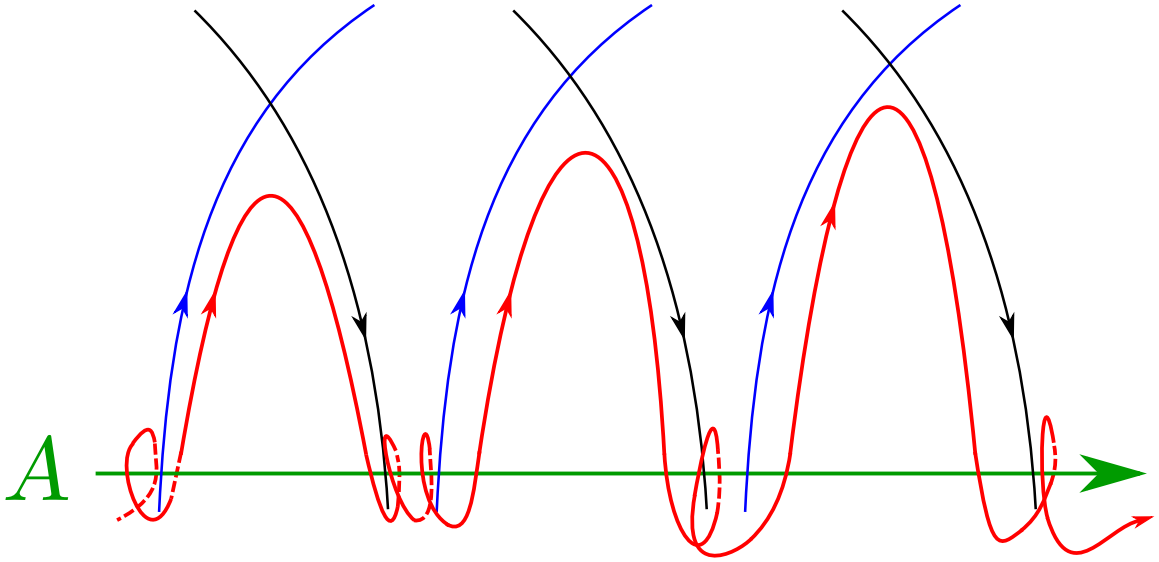


Figure 3.1: The green line represents the height component of the normally hyperbolic invariant cylinder A, which is equal to $\|u\|$ plus a small correction. The blue lines represent pieces of the unstable manifold $W^u(A)$, and the black lines pieces of the stable manifold $W^s(A)$. The red curve is a diffusive trajectory. Compare with Figure 2.4.

moreover that the intersection at z is strongly transverse for each $z \in B$. Let A' denote the subset of A consisting of points x for which the leaf $W^u(x)$ of the strong unstable foliation intersects B at exactly one point. Recall from Section 2.1.3 that the scattering map $s_B : A' \to A$ is defined as follows: from $x \in A'$, follow the leaf $W^u(x)$ of the strong unstable foliation until it reaches B. At this point there is a unique leaf of the strong stable foliation passing through B. Follow this leaf back to A. The resulting point is $s_B(x)$ (see Figure 2.3). The scattering map was introduced by Delshams, de la Llave, and Seara [32, 33], and is the key tool for analysing dynamics near a transverse homoclinic intersection.

In Section 3.2, we prove that the billiard map has a normally hyperbolic invariant cylinder A, with height component inversely proportional to the angle of reflection, and such that the stable and unstable manifolds of A have a transverse homoclinic intersection. The invariant manifold A is 2-dimensional, and the stable and unstable manifolds are each d-dimensional. Each leaf of the strong stable and strong unstable foliations is $(d-2)$-dimensional. We prove the existence of A by using the fact that,

as mentioned above, the geodesic flow under assumptions [A1,2] has such objects. Expanding the Taylor series of the billiard map in powers of the flight time τ, we find that the term of order τ in this expansion is the vector field of the geodesic flow. The problem with taking this approach directly is that the 'small parameter' τ is in fact a function, and this function has unbounded u-derivative as $\|u\| \to 1$. This implies that the approximation of the billiard map by a time shift of the geodesic flow is only good in the C^0 topology, in these coordinates. Therefore we fix some small constant value $\tau_* > 0$ of the flight time, and make a change of coordinates so that the billiard map is real-analytic in some subset of the phase space. If we expand the billiard map now in powers of τ_*, we find that the term of order τ_* is a vector field Z, whose flow traces the geodesics of M, but at a fluctuating (nonzero) speed. Moreover Z preserves an integral, and so has a normally hyperbolic invariant manifold consisting the hyperbolic closed geodesic on an interval of values of the integral. We conclude that the billiard map is $O(\tau_*^2)$ close to the time-τ_* shift of the flow of Z. Fenichel theory guarantees the persistence of the normally hyperbolic invariant manifold A for any dynamics C^1-close enough to Z [40, 41, 42]. Since we can make τ_* as small as we like by considering trajectories nearer to the boundary, we get the desired persistent manifold for the billiard map.

Notice that as $\tau \to 0$, the billiard map tends to the identity. Therefore as we move up the cylinder A, the hyperbolicity parameters become weaker. To deal with this, we look at a large (i.e. $1/\tau$) iterate of the billiard map, and compare this to the time-1 shift of the geodesic flow. Of course this means that we must exclude the case $\tau = 0$, which corresponds to the upper boundary of the normally hyperbolic invariant manifold A. As a consequence, the resulting persistent manifold A for the billiard map is a non-compact non-uniformly normally hyperbolic invariant manifold. Aside from this, the non-uniformity of hyperbolicity parameters does not explicitly impact our proof, although it does mean that diffusive trajectories become slower the further up the cylinder they move. Non-compactness, however, means that we must adapt some proofs of results we use from [53] and [54].

We conduct an analysis of the homoclinic cylinders in Section 3.3. We show in this section that it is sufficient to make perturbations along (most of) a fundamental domain of the billiard map on a homoclinic trajectory. We also give conditions for the existence of diffusive pseudo-orbits, and describe how there are orbits of the billiard map that shadow pseudo-orbits arbitrarily well.

In Section 3.4 we show how to perturb the billiard map by perturbing the function Q that defines the manifold M. We make a perturbation of size ϵ, and compute the first order expansion of the map resulting from this change of the surface. Since the billiard map is symplectic, the first order term is a Hamiltonian vector field, corresponding

to some Hamiltonian H_{pert}. Similarly, the first order term in the expansion of the scattering map in powers of ϵ is a Hamiltonian vector field. Delshams, de la Llave, and Seara proved a formula (Theorem 31 of [34]) for the Hamiltonian function of this vector field in terms of H_{pert} and the unperturbed billiard dynamics. We make use of this formula here.

In Section 3.5, we use the machinery developed in [53] and [54] to determine conditions which, when satisfied, guarantee the existence of diffusive trajectories. We then prove that these conditions are satisfied generically in the real-analytic category. Since we have a homoclinic cylinder for the billiard map, there is a nontrivial hyperbolic basic set near the normally hyperbolic manifold A. This in turn implies the existence of infinitely many independent homoclinic cylinders (see Section 2.1.1). Take 8 such cylinders, and construct an iterated function system $\{f, s_1, \ldots, s_8\}$ on A, consisting of the 8 scattering maps and the restriction of the billiard map to A. Orbits of the IFS are often called pseudo-orbits, and it can be shown that there are orbits of the map f that shadow pseudo-orbits arbitrarily well [54]. An adaptation of a result from [53] implies that if the IFS has no essential invariant curves on the cylinder A, then there exist diffusive pseudo-orbits: orbits of the IFS that drift up the cylinder A. We then prove that the absence of invariant essential curves of the IFS is a generic property. This is done by embedding the billiard map in a two-parameter family of real-analytic billiard maps. In Lemma 3.5.2, we give open conditions on this family which, if satisfied, imply that the set of parameters for which there are no invariant essential curves is of full measure. The lemma is proved by contradiction, using a combinatorial argument that requires 8 scattering maps. It is based on a result in [53], and finishes the proof of Theorem 3.1.1. The idea of using families of real-analytic mappings to approximate a system perturbed by a bump function was used in [22, 53, 56].

3.2 Billiard Trajectories Near the Boundary

The goal of this section is to show that the billiard map inherits a normally hyperbolic invariant manifold from the geodesic flow, and to analyse the homoclinic intersection of its stable and unstable manifolds. This is done by showing that, in some coordinates and for sufficiently small $\tau_* > 0$, the billiard map f is approximated up to order τ_*^2 by the time-τ_* shift of the flow of some vector field Z in the C^r-topology for any finite r. We show that Z has a normally hyperbolic invariant manifold that does not depend on τ_* as a result of assumptions [A1,2]. Therefore, shrinking τ_* if necessary, we can use the theory of Fenichel to imply the persistence of the normally hyperbolic invariant manifold for f [40, 41, 42]. This process is repeated a countable number of times, and the resulting invariant manifolds are glued together to obtain a noncompact normally

hyperbolic invariant manifold for f.

Let $Q \in \mathcal{V}_*^d$ and let $M = M(Q)$, $\mathcal{M} = \mathcal{M}(Q)$ be as in (2.12), (2.17) respectively. Lemma 2.5.2 implies that we can assume without loss of generality that $\|\nabla Q(x)\| = 1$ for all $x \in M$. Therefore $n(x), C(x), S(x)$ simplify via (2.15), (2.16).

3.2.1 Estimates Near the Boundary

We consider trajectories near the boundary by taking $\delta > 0$ small and restricting our attention to the subset

$$\mathcal{M}_\delta = \{(x, u) \in TM : 1 - \delta \leq \|u\| \leq 1\}$$

of the billiard map's phase space.

Lemma 3.2.1. *Let $(x, u) \in \mathcal{M}_\delta$. Recall $\tau = \tau(x, u)$ denotes the flight time defined by (2.18), and $v = u + \sqrt{1 - u^2}\, n(x)$ is the inward-pointing velocity vector.*

(i) We have

$$\sqrt{1 - u^2} = \frac{1}{2}\tau\kappa(x, v) + \frac{1}{3}\tau^2 R(x, v) + O(\tau^3) \tag{3.1}$$

where

$$R(x, v) = \sum_{i,j,k=1}^{d} \frac{\partial^3 Q}{\partial x_i \partial x_j \partial x_k}(x) v_i v_j v_k. \tag{3.2}$$

(ii) Write $(\bar{x}, \bar{u}) = f(x, u)$. Then

$$(\bar{x}, \bar{u}) = (x, u) + \tau X(x, u) + O(\tau^2) \tag{3.3}$$

where X is the vector field of the geodesic flow, defined by equation (2.24).

(iii) With $n(x)$ as in (2.15),

$$n(\bar{x}) = n(x) - \tau S(x) v - \frac{1}{2}\tau^2 r(x, v) + O(\tau^3) \tag{3.4}$$

where r is the vector-valued function with components

$$r_i(x, v) = \sum_{j,k=1}^{d} \frac{\partial^3 Q}{\partial x_i \partial x_j \partial x_k}(x) v_j v_k. \tag{3.5}$$

(iv) With $R(x, v)$ as in (3.2),

$$\sqrt{1 - \bar{u}^2} = \sqrt{1 - u^2} - \frac{1}{6}\tau^2 R(x, v) + O(\tau^3). \tag{3.6}$$

Proof. These statements are proved by taking Taylor expansions in powers of τ. Notice that

$$\langle n(x), v\rangle = \langle n(x), u\rangle + \sqrt{1-u^2}\langle n(x), n(x)\rangle = \sqrt{1-u^2}. \tag{3.7}$$

From (2.15), (2.16), (3.7), and the fact that $Q(x) = Q(\bar{x}) = 0$ we get

$$\begin{aligned}
0 = Q(\bar{x}) &= Q(x+\tau v)\\
&= Q(x) + \tau\langle \nabla Q(x), v\rangle + \frac{1}{2}\tau^2 \sum_{i,j=1}^{d} \frac{\partial^2 Q}{\partial x_i \partial x_j}(x) v_i v_j + \\
&\qquad + \frac{1}{3}\tau^3 \sum_{i,j,k=1}^{d} \frac{\partial^3 Q}{\partial x_i \partial x_j \partial x_k}(x) v_i v_j v_k + O(\tau^4)\\
&= -\tau\langle n(x), v\rangle + \frac{1}{2}\tau^2\langle S(x)v, v\rangle + \frac{1}{3}\tau^3 R(x,v) + O(\tau^4)\\
&= -\tau\sqrt{1-u^2} + \frac{1}{2}\tau^2\kappa(x,v) + \frac{1}{3}\tau^3 R(x,v) + O(\tau^4).
\end{aligned}$$

Dividing by τ and rearranging terms gives (i), where the terms of higher order are uniformly bounded due to the real-analyticity of Q.

For (ii) we must compute the term of order τ in the expansion of $(\bar{x}, \bar{u})$. From (3.1) we get

$$\bar{x} = x + \tau v = x + \tau u + \tau\sqrt{1-u^2}\, n(x) = x + \tau u + O(\tau^2). \tag{3.8}$$

Also by (3.7),

$$\bar{u}|_{\tau=0} = u + \sqrt{1-u^2}\, n(x) - \langle n(x), v\rangle n(x) = u$$

and moreover

$$\begin{aligned}
\left.\frac{d}{d\tau}\right|_{\tau=0} \bar{u} &= -\langle v, Dn(\bar{x})v\rangle n(\bar{x})|_{\tau=0} - \langle v, n(\bar{x})\rangle Dn(\bar{x})v|_{\tau=0}\\
&= \langle v, S(x)v\rangle n(x) + \left.\sqrt{1-u^2}\right|_{\tau=0} S(x)v\\
&= \kappa(x,u)n(x)
\end{aligned}$$

where (3.1) was used. It follows that

$$\bar{u} = u + \tau\kappa(x,u)n(x) + O(\tau^2). \tag{3.9}$$

Comparing (3.8) and (3.9) with (2.24) completes the proof of (ii).

From (2.15) we get

$$\begin{aligned}
n(\bar{x}) &= n(x) + \tau Dn(x)v + \frac{1}{2}\tau^2 D^2 n(x) v^{\otimes 2} + O(\tau^3)\\
&= n(x) - \tau S(x)v - \frac{1}{2}\tau^2 r(x,v) + O(\tau^3)
\end{aligned}$$

which is (iii).

For part (iv) notice that the second equation of (2.19) combined with basic geometry implies that

$$\bar{v} = v - 2\langle n(\bar{x}), v\rangle n(\bar{x}) = \bar{u} - \langle n(\bar{x}), v\rangle n(\bar{x})$$

which, along with (3.7), gives

$$\langle n(\bar{x}), \bar{v}\rangle = -\langle n(\bar{x}), v\rangle. \tag{3.10}$$

Comparing (3.2) and (3.5) we see that

$$\langle r(x,v), v\rangle = \sum_{i,j,k=1}^{d} \frac{\partial^3 Q}{\partial x_i \partial x_j \partial x_k}(x) v_i v_j v_k = R(x,v).$$

Using this together with (3.4),

$$\begin{aligned}\langle n(\bar{x}), v\rangle &= \langle n(x), v\rangle - \tau\langle S(x)v, v\rangle - \frac{1}{2}\tau^2\langle r(x,v), v\rangle + O(\tau^3)\\ &= \langle n(x), v\rangle - \tau\kappa(x,v) - \frac{1}{2}\tau^2 R(x,v) + O(\tau^3).\end{aligned} \tag{3.11}$$

From (3.1), (3.7), (3.10), and (3.11) we get

$$\begin{aligned}\sqrt{1-\bar{u}^2} &= \langle n(\bar{x}), \bar{v}\rangle = -\langle n(\bar{x}), v\rangle\\ &= -\langle n(x), v\rangle + \tau\kappa(x,v) + \frac{1}{2}\tau^2 R(x,v) + O(\tau^3)\\ &= -\frac{1}{2}\tau\kappa(x,v) - \frac{1}{3}\tau^2 R(x,v) + \tau\kappa(x,v) + \frac{1}{2}\tau^2 R(x,v) + O(\tau^3)\\ &= \frac{1}{2}\tau\kappa(x,v) + \frac{1}{3}\tau^2 R(x,v) - \frac{1}{6}\tau^2 R(x,v) + O(\tau^3)\\ &= \langle n(x), v\rangle - \frac{1}{6}\tau^2 R(x,v) + O(\tau^3)\\ &= \sqrt{1-u^2} - \frac{1}{6}\tau^2 R(x,v) + O(\tau^3)\end{aligned}$$

which completes the proof of the lemma. □

Remark 3.2.2. *Equation* (3.3) *implies that the billiard map is approximated up to order* τ^2 *by the time-*τ *shift of the geodesic flow. Equation* (3.1)*, however, implies that*

$$\frac{\partial \tau}{\partial u}(x,u) = \frac{-2u}{\kappa(x,u)\sqrt{1-u^2}} + O(1)$$

which is unbounded as $\|u\|$ *tends to 1. Therefore the approximation of the billiard map by a time shift of the geodesic flow is only good in the* C^0 *topology, which is not enough to use the theory of Fenichel. However, if we consider* τ *as a variable rather than a function and consider the billiard map as a function of* $x, u,$ *and* τ*, then the billiard map is real-analytic in* x, u, τ *by* (3.1) *and* (3.6)*, and the estimates contained in Lemma 3.2.1 hold in the* C^r*-topology for* r *finite.*

3.2.2 Normally Hyperbolic Invariant Manifold for the Billiard Map

By Lemma (3.1), the principal part of τ is $\frac{2\sqrt{1-u^2}}{\kappa(x,u)}$. Fix some small $\tau_* > 0$ and consider the coordinates (x, w, z) where

$$w = \frac{u}{\|u\|}, \quad z = \tau_*^{-1}\frac{2\sqrt{1-u^2}}{\kappa(x,w)}. \tag{3.12}$$

By Remark 3.2.2, the billiard map is real-analytic in these coordinates. We will show that it is also near to the identity. Fix any $b > a > 0$ and let

$$\mathcal{D}_* = \{(x, w, z) : a < z < b\} \subset \mathcal{M}. \tag{3.13}$$

Consider the vector field $Z(x, w, z) = (\dot{x}, \dot{w}, \dot{z})$ where

$$\begin{cases} \dot{x} = zw \\ \dot{w} = z\kappa(x,w)n(x) \\ \dot{z} = -\frac{4}{3}z^2\kappa(x,w)^{-1}R(x,w). \end{cases} \tag{3.14}$$

Comparing Z with X, it is clear that the orbits of Z trace the geodesics of M, but at a fluctuating speed z. The following lemma is the general principle mentioned in the introduction: the dynamics of the billiard map in the limit as the angle of reflection goes to 0 is determined by the geodesic flow. More specifically, it says that the billiard map is near to the identity. Therefore it is interpolated by the time τ_* shift of the flow of some vector field, which the lemma says is equal, up to terms of order τ_*, to Z.

Lemma 3.2.3. *On $\mathcal{D}_* \subset \mathcal{M}$ we have*

$$f = Id + \tau_* Z + O(\tau_*^2) \tag{3.15}$$

in the (x, w, z)-coordinates on $\mathcal{M}$ in the C^r-topology for any finite r.

Proof. The second equation of (3.12) implies that

$$\|u\| = \sqrt{1 - \frac{1}{4}\tau_*^2\kappa(x,w)^2z^2} = 1 + O(\tau_*^2).$$

This together with the first equation of (3.12) gives

$$w = u + O(\tau_*^2). \tag{3.16}$$

From (3.1) and (3.16) we get

$$\tau = \frac{2\sqrt{1-u^2}}{\kappa(x,v)} + O(\tau^2) = \frac{2\sqrt{1-u^2}}{\kappa(x,w)} + O(\tau^2),$$

and so

$$\tau = \tau_* z + O(\tau^2) = \tau_* z + O(\tau_*^2). \tag{3.17}$$

Write $(\bar{x}, \bar{w}, \bar{z}) = f(x, w, z)$. From (3.3) and (3.16) we get

$$\bar{x} = x + \tau v = x + \tau_* z u + O(\tau_*^2) = x + \tau_* z w + O(\tau_*^2) \tag{3.18}$$

and

$$\begin{aligned} \bar{w} &= \bar{u} + O(\tau_*^2) = u + \tau_* z \kappa(x,u) n(x) + O(\tau_*^2) \\ &= w + \tau_* z \kappa(x,w) n(x) + O(\tau_*^2). \end{aligned} \tag{3.19}$$

Let us determine $\bar{z}$. Equations (2.16), (3.18), and (3.19) imply

$$\begin{aligned} \left.\frac{d}{d\tau_*}\right|_{\tau_*=0} \kappa(\bar{x}, \bar{w}) = & \left.\frac{d}{d\tau_*}\right|_{\tau_*=0} \sum_{i,j=1}^{d} \frac{\partial^2 Q}{\partial x_i \partial x_j}(\bar{x}) \bar{w}_i \bar{w}_j \\ &= z \sum_{i,j,k=1}^{d} \frac{\partial^3 Q}{\partial x_i \partial x_j \partial x_k}(x) w_i w_j w_k + \\ &\qquad + 2z\kappa(x,w) \sum_{i,j=1}^{d} \frac{\partial^2 Q}{\partial x_i \partial x_j}(x) w_i n_j(x) \\ &= zR(x,w) + 2z\kappa(x,w)\langle S(x)w, n(x)\rangle \\ &= zR(x,w) \end{aligned} \tag{3.20}$$

since $S(x)w$ is tangent to M and $n(x)$ is normal. Therefore

$$\kappa(\bar{x}, \bar{w})^{-1} = \kappa(x,w)^{-1} - \tau_* z \kappa(x,w)^{-2} R(x,w) + O(\tau_*^2).$$

Moreover (3.1), (3.6), (3.16), and (3.17) imply

$$\sqrt{1-u^2} = \frac{1}{2}\tau_* z \kappa(x,w) + O(\tau_*^2)$$

and

$$\sqrt{1-\bar{u}^2} = \sqrt{1-u^2} - \frac{1}{6}\tau_*^2 z^2 R(x,w) + O(\tau_*^3).$$

Combining the last three equations with the second equation of (3.12),

$$\begin{aligned} \bar{z} &= \tau_*^{-1} \frac{2\sqrt{1-\bar{u}^2}}{\kappa(\bar{x}, \bar{w})} \\ &= 2\tau_*^{-1} \left(\sqrt{1-u^2} - \frac{1}{6}\tau_*^2 z^2 R(x,w) \right) \times \\ &\qquad \times \left(\kappa(x,w)^{-1} - \tau_* z \kappa(x,w)^{-2} R(x,w) \right) + O(\tau_*^2) \\ &= \tau_*^{-1} \frac{2\sqrt{1-u^2}}{\kappa(x,w)} - 2z\sqrt{1-u^2}\kappa(x,w)^{-2} R(x,w) - \\ &\qquad - \frac{1}{3}\tau_* z^2 \kappa(x,w)^{-1} R(x,w) + O(\tau_*^2) \\ &= z - \frac{4}{3}\tau_* z^2 \kappa(x,w)^{-1} R(x,w) + O(\tau_*^2). \end{aligned}$$

It follows that

$$f = Id + \tau_* Z + O(\tau_*^2)$$

where the vector field $Z(x, w, z)$ is as in (3.14). □

Assumptions [A1,2] imply that there exists a hyperbolic closed geodesic $\gamma : \mathbb{T} \to M$, and a transverse homoclinic geodesic $\xi : \mathbb{R} \to M$. Without loss of generality, the length of γ is 1, and both curves are parametrised by arclength.

There are three main points to take from the following lemma. First of all, the vector field Z preserves an integral y defined by (3.22). The integral tells us that the speed of the vector field Z varies depending on the curvature of the manifold – for example on a manifold of constant curvature (i.e. a sphere) the speed z is constant and the flow of Z is exactly the geodesic flow. The preservation of this quantity ensures that the speed z remains bounded away from 0 along orbits of the vector field Z. Therefore geodesics on M which are hyperbolic for the geodesic flow are also hyperbolic for Z. Secondly, the lemma tells us that the vector field Z has a normally hyperbolic invariant manifold $\bar{A}_*$. Since there is a hyperbolic closed geodesic γ, the flow of Z has a hyperbolic periodic orbit for each fixed value of the integral y, and the projection of this periodic orbit onto the hypersurface M is exactly γ. Finally, the existence of a transverse homoclinic geodesic ξ to γ ensures that the normally hyperbolic invariant manifold $\bar{A}_*$ has a transverse homoclinic intersection.

Lemma 3.2.4. *The vector field Z has an integral y, and the set*

$$\bar{A}_* = \left\{ \left(\gamma(q), \gamma'(q), y^{-1} \kappa(\gamma(s), \gamma'(s))^{-\frac{4}{3}} \right) : q \in \mathbb{T}, y \in \left[\frac{1}{2}, \frac{3}{2} \right] \right\} \tag{3.21}$$

is an r-normally hyperbolic invariant manifold for Z. Moreover the stable and unstable manifolds of $\bar{A}_*$ have a transverse homoclinic intersection.

Proof. Notice that, from (3.14), and by a similar computation to (3.20),

$$\frac{d}{dt} \kappa(x, w) = z R(x, w).$$

It follows that

$$\frac{d}{dt} \ln z = \frac{\dot{z}}{z} = -\frac{4}{3} z \kappa(x, w)^{-1} R(x, w) = -\frac{4}{3} \frac{d}{dt} \ln \kappa(x, w) = \frac{d}{dt} \ln \kappa(x, w)^{-\frac{4}{3}}$$

which implies that $z = C\kappa(x, w)^{-\frac{4}{3}}$ for some positive constant C. Therefore

$$y^{-1} = z \kappa(x, w)^{\frac{4}{3}} \tag{3.22}$$

is constant along trajectories. Since $z = y^{-1} \kappa(x, w)^{-\frac{4}{3}}$ it is bounded along trajectories, and bounded away from 0 for strictly positive y. It follows that the vector field

Z follows the geodesics on M at a fluctuating speed that is bounded away from 0. Therefore $\bar{A}_*$ as defined in (3.21) is a normally hyperbolic invariant manifold for Z.

Notice that every point $(q, y) \in \bar{A}_*$ is periodic with minimal period $l = l(y)$ depending only on y. We have

$$l(y) = \int_{q\in\mathbb{T}} z(q)dq = y^{-1} \int_{q\in\mathbb{T}} \kappa(\gamma(q), \gamma'(q))^{-\frac{4}{3}} dq = K^{-1}y^{-1}$$

where

$$K^{-1} = \int_{s\in\mathbb{T}} \kappa(\gamma(s), \gamma'(s))^{-\frac{4}{3}} ds$$

is a positive constant depending only on the periodic orbit γ. Then we can write the cylinder as

$$\bar{A}_* = \left\{ \phi_Z^{\theta l(y)} \left(\gamma(0), \gamma'(0), y^{-1}\kappa(\gamma(0), \gamma'(0))^{-\frac{4}{3}} \right) : \theta \in \mathbb{T}, y \in \left[\frac{1}{2}, \frac{3}{2}\right] \right\}.$$

In this equation the time t along the orbit is equal to $\theta l(y)$, and so we get $\dot{\theta} = Ky$. Therefore we have

$$Z(\theta, y) = (Ky, 0)$$

and

$$\phi_Z^t(\theta, y) = (\theta + tKy, y)$$

where ϕ_Z^t denotes the time-t shift of the flow of Z. It follows that the differential of the time-t shift of the flow of Z is

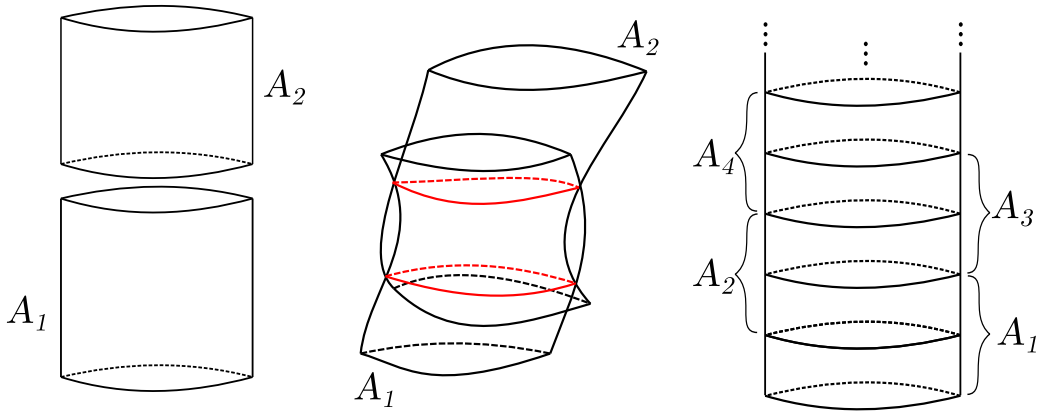
$$D\phi_Z^t = \begin{pmatrix} 1 & tK \\ 0 & 1 \end{pmatrix}$$

which has a double eigenvalue of 1. Therefore there is an infinitely large spectral gap in these coordinates, in the sense that for any fixed $r \in \mathbb{N}$ we can always find a norm such that the hyperbolicity parameters λ, μ satisfy $0 < \lambda < \mu^{-r} < 1$ (see Definition 2.1.10). It follows that $\bar{A}_*$ is an r-normally hyperbolic invariant manifold.

Moreover there is a transverse homoclinic orbit that follows the geodesic ξ. Thus the stable and unstable manifolds of $\bar{A}_*$ have a transverse homoclinic intersection. □

Remark 3.2.5. *By choosing $b > a > 0$ appropriately in equation* (3.13) *we can guarantee that $\bar{A}_* \subset \mathcal{D}_*$. Therefore combining the r-normally hyperbolic invariant manifold $\bar{A}_*$ of Lemma 3.2.4 with* (3.15) *and Fenichel theory [40, 41, 42] implies that we obtain a C^r-smooth normally hyperbolic invariant manifold A_* for f, $O(\tau_*^2)$-close to $\bar{A}_*$ in the C^r-topology for any finite r, for sufficiently small values of τ_* (where τ_* depends on r). Moreover, the results imply that when we perturb the surface M, the normally hyperbolic invariant manifold A_* of the billiard map persists. This is because the invariant manifold $\bar{A}_*$ for the geodesic flow persists in the perturbed manifold, and so we*

can apply the results again to obtain a normally hyperbolic invariant manifold for the billiard map on the perturbed manifold.



(a) (b) (c)

Figure 3.2: Three possible pictures of how the perturbed compact normally hyperbolic invariant cylinders (i.e. the compact invariant cylinders of the billiard map) meet. In (a), the two cylinders miss each other entirely; in (b) the cylinders agree transversely along the KAM curves (in red), but not between them; and (c) shows a sequence of cylinders that overlap and fit together to form a noncompact invariant cylinder. In Proposition 3.2.6 it is shown how to construct a sequence of compact invariant cylinders for the billiard map as in (c). The angular component of the cylinder is close to the angular variable on the closed geodesic γ, and the height is close to values of the integral y of the vector field Z. In particular, the angle of reflection tends to zero as we move up the cylinder.

The next result shows that we can choose a sequence $\{\bar{A}_n\}_{n\in\mathbb{N}}$ of normally hyperbolic invariant cylinders for the flow of Z such that: the cylinders overlap; as we move up the cylinders $\|u\| \to 1$; and the corresponding persistent cylinders A_n of the billiard map fit together to form a manifold, itself having the structure of a noncompact cylinder (see Figure 3.2).

Proposition 3.2.6. *We can choose a sequence of constants $\{\tau_n\}_{n\in\mathbb{N}}$ such that $\tau_n \to 0$ as $n \to \infty$, and so that the corresponding cylinders A_n for the billiard map fit together to form a noncompact nonuniformly normally hyperbolic invariant manifold. Moreover, $\|u\| \to 1$ as we move up the cylinder.*

Proof. We choose the sequence $\{\tau_n\}_{n\in\mathbb{N}}$ so that the top of the cylinder $\bar{A}_1$ which is $y_1 = \frac{3}{2}$ matches up with the middle of A_2 which is $y_2 = 1$ (see equation (3.21)). If we can choose τ_1, τ_2 so that y_1, y_2 correspond to the same value of $\|u\|$ at $(\gamma(q), \gamma'(q))$ for

each $q \in \mathbb{T}$, then the cylinders match up. We have

$$y_j^{-1} = z_j \kappa(\gamma(q), \gamma'(q))^{\frac{4}{3}} = 2\tau_j^{-1}\sqrt{1-u^2}\,\kappa(\gamma(q), \gamma'(q))^{\frac{1}{3}}, \tag{3.23}$$

so we need

$$\frac{2}{3}\tau_1 = \tau_1 y_1^{-1} = 2\sqrt{1-u^2}\,\kappa(\gamma(q), \gamma'(q))^{\frac{1}{3}} = \tau_2 y_2^{-1} = \tau_2$$

or more generally,

$$\tau_{n+1} = \frac{2}{3}\tau_n = \cdots = \left(\frac{2}{3}\right)^n \tau_1 \tag{3.24}$$

which tends to 0 as $n \to \infty$. Therefore we choose $\tau_1 > 0$ sufficiently small so that Fenichel theory guarantees the persistence of the cylinder $\bar{A}_1$ as a C^r-smooth invariant cylinder for the billiard map (where r is sufficiently large), along with its transverse homoclinic intersection. Then define τ_n according to (3.24). Since the sequence is strictly decreasing, and since $\bar{A}_n$ as defined in (3.21) does not depend explicitly on τ_n, each cylinder $\bar{A}_n$ persists for the billiard map, along with its transverse homoclinic intersection. Clearly $\|u\| \to 1$ as we move up the cylinders since, from (3.23), and for any $q \in \mathbb{T}$,

$$\sqrt{1-u^2} = \frac{1}{2}\tau_n \kappa(\gamma(q), \gamma'(q))^{-\frac{1}{3}} y^{-1} \longrightarrow 0 \text{ as } n \to \infty.$$

Now, we must prove that

$$\bigcup_{n=1}^{\infty} A_n$$

has the structure of a cylinder. It is clear that $\mathrm{Int}(\bar{A}_n) \cap \mathrm{Int}(\bar{A}_{n+1}) \neq \emptyset$. Recall that each closed essential curve on $\bar{A}_n$ corresponding to constant y is an invariant circle of the flow of Z with frequency $\rho = \rho(y)$ depending only on y. Moreover, ρ is strictly monotone in y. KAM theory ensures that curves corresponding to Diophantine frequencies survive perturbation and so are invariant circles for the billiard map. We may assume without loss of generality (as we may slightly shift the boundary curves of the cylinders $\bar{A}_n$ if necessary) that each cylinder $\bar{A}_n$ is bounded by curves with Diophantine frequencies.

Let $\bar{\gamma}_1$ be the lower boundary curve of $\bar{A}_{n+1}$ corresponding to $y = y_1$. For any $\epsilon > 0$ we can find $y_2 \in (y_1, y_1 + \epsilon)$ such that $\rho(y_2)$ is Diophantine. Let $\bar{\gamma}_2$ be the invariant circle corresponding to $y = y_2$. Then there are invariant circles γ_1, γ_2 for the billiard map that are τ_n^2 close in the C^r-topology to $\bar{\gamma}_1$, $\bar{\gamma}_2$, respectively. Clearly we must have $\gamma_1, \gamma_2 \subset A_n \cap A_{n+1}$. Let D_1 be the region in A_n bounded by γ_1 and γ_2. Then D_1 is an invariant set for the billiard map. There is a characterisation of the local stable (respectively unstable) manifold of A_{n+1} as the set of points whose forward (resp. backward) orbit never leaves a neighbourhood V of A_{n+1} (see Theorem 2.1.11). Therefore the manifold A_{n+1} is the set of points whose forward and backward orbits never leave V. By choosing ϵ small enough, we can guarantee that $D_1 \subset V$

since A_n and A_{n+1} are τ_n^2-close in the C^r-topology. It follows that $D_1 \subset A_{n+1}$, and so the cylinders A_n and A_{n+1} agree between the curves γ_1 and γ_2. Now choose curves $\bar{\gamma}_3, \ldots, \bar{\gamma}_N$ for some $N \in \mathbb{N}$ such that $\bar{\gamma}_N$ is the upper boundary of $\bar{A}_n$, and such that each $\bar{\gamma}_j \subset \bar{A}_n$ corresponds to $y = y_j$ where $\rho(y_j)$ is a Diophantine number where $0 < y_j - y_{j-1} < \epsilon$. Then this argument applies to the area between each pair of consecutive curves γ_j, γ_{j+1}, and so the cylinders A_n, A_{n+1} agree where they overlap. This clearly applies for each $n \in \mathbb{N}$. □

3.3 Homoclinic Cylinders and Pseudo-Orbits

In this section we construct an iterated function system consisting the restriction of the billiard map to the noncompact normally hyperbolic invariant manifold together with n scattering maps. An orbit of the iterated function system is called a *pseudo-orbit*. We give sufficient conditions for the existence of diffusive pseudo-orbits, and explain how there are orbits of the billiard map that shadow pseudo-orbits arbitrarily well.

3.3.1 Analysis of Hyperbolic and Homoclinic Cylinders

It was shown in Section 3.2 that in (x, w, z)-coordinates defined by (3.12), the billiard map f is approximated up to order τ_*^2 by the time τ_*-shift of the flow of the vector field Z, given by (3.14). Recall it was shown in the proof of Lemma 3.2.4 that we can write the normally hyperbolic invariant manifold $\bar{A}_*$ of the vector field Z as

$$\bar{A}_* = \left\{ \phi_Z^{\theta l(y)} \left(\gamma(0), \gamma'(0), y^{-1} \kappa(\gamma(0), \gamma'(0))^{-\frac{4}{3}} \right) : \theta \in \mathbb{T}, y \in \left[\frac{1}{2}, \frac{3}{2} \right] \right\}$$

where

$$l(y) = K^{-1} y^{-1}$$

is the minimal period of a point on $\bar{A}_*$ with height coordinate y, and

$$K^{-1} = \int_{s \in \mathbb{T}} \kappa(\gamma(s), \gamma'(s))^{-\frac{4}{3}} ds$$

is a positive constant depending only on the periodic orbit γ. Moreover it was shown that

$$Z(\theta, y) = (Ky, 0)$$

and

$$\phi_Z^t(\theta, y) = (\theta + tKy, y).$$

Since $K > 0$ depends only on γ, we can replace y by Ky and write

$$\bar{A}_* = \left\{ \phi_Z^{\theta y^{-1}} \left(\gamma(0), \gamma'(0), K y^{-1} \kappa(\gamma(0), \gamma'(0))^{-\frac{4}{3}} \right) : \theta \in \mathbb{T}, y \in \left[\frac{K}{2}, \frac{3K}{2} \right] \right\}. \quad (3.25)$$

Then the vector field and the flow take the simple form

$$Z(\theta, y) = (y, 0)$$

and

$$\phi_Z^t(\theta, y) = (\theta + ty, y). \tag{3.26}$$

Notice that the speed z of the flow of Z is now given by

$$z = Ky^{-1}\kappa(x, w)^{-\frac{4}{3}}. \tag{3.27}$$

Moreover the billiard map is given by

$$f : (\theta, y) \longmapsto (\theta + \tau_* y, y) + O(\tau_*^2), \tag{3.28}$$

since it is τ_*^2-close to the time-τ_* shift of the flow of Z.

Since $\bar{A}_*$ is a 2-dimensional normally hyperbolic invariant manifold in the $2(d-1)$-dimensional phase space $\mathcal{M}$, it has d-dimensional stable and unstable manifolds $W^{s,u}(\bar{A}_*)$. On the stable (resp. unstable) manifold is the strong stable (resp. strong unstable) foliation, each leaf of which is $(d-2)$-dimensional, and is uniquely determined by a point $(\theta, y) \in \bar{A}_*$ (see Section 2.1.2 for definitions). As shown in Lemma 3.2.4, $W^s(\bar{A}_*)$ and $W^u(\bar{A}_*)$ have a transverse homoclinic intersection along the homoclinic geodesic ξ. Recall that along a transverse homoclinic intersection we have the holonomy maps $\bar{\pi}^{s,u}$ defined as projections onto $\bar{A}_*$ along leaves of the strong stable and strong unstable foliations. Moreover, recall that a manifold $B \subset \left(W^s(\bar{A}_*) \pitchfork W^u(\bar{A}_*)\right) \setminus \bar{A}_*$ is called a homoclinic cylinder if the scattering map

$$s_B : \bar{A}_* \longrightarrow \bar{A}_*$$

relative to B is a C^r-diffeomorphism.

Lemma 3.3.1. *The set*

$$\bar{B}_* = \left\{ \phi_Z^{\theta y^{-1}} \left(\xi(0), \xi'(0), Ky^{-1}\kappa(\xi(0), \xi'(0))^{-\frac{4}{3}} \right) : \theta \in [0, 1), y \in \left[\frac{K}{2}, \frac{3K}{2} \right] \right\} \tag{3.29}$$

is a homoclinic cylinder. Moreover there are constants $a_\pm \in \mathbb{R}$ that do not depend on the point (θ, y) such that

$$\bar{\pi}^s_{\bar{B}_*}(\theta, y) = (\theta + a_+, y), \quad \bar{\pi}^u_{\bar{B}_*}(\theta, y) = (\theta + a_-, y). \tag{3.30}$$

Proof. It is clear that the holonomy maps are well-defined on $\bar{B}_*$ since every point on the homoclinic orbit ξ is the intersection of exactly one leaf of the strong stable foliation and one leaf of the strong unstable foliation. Moreover they are C^r-smooth due to the regularity of the foliations [65].

Consider the point $(0, y) \in \bar{B}_*$, and let $(a_+, y_+) = \bar{\pi}^s_{\bar{B}_*}(0, y)$. Since y is constant along trajectories of the flow, and since the holonomy map is defined via asymptotic trajectories, we have $y_+ = y$. Now let $(\theta, y) \in \bar{B}_*$ be another point with a different θ-value and the same y-value. Then there is t_* such that $0 < t_* < l(y) = y^{-1}$ and

$$(\theta, y) = \phi^{t_*}_Z(0, y) = (t_* y, y) \tag{3.31}$$

where we have used (3.26). From (2.4), (3.26), and (3.31) we get

$$\begin{aligned}\bar{\pi}^s_{\bar{B}_*}(\theta, y) &= \bar{\pi}^s_{\bar{B}_*} \circ \phi^{t_*}_Z(0, y) = \phi^{t_*}_Z \circ \bar{\pi}^s_{\bar{B}_*}(0, y) = \\ &= \phi^{t_*}_Z(a_+, y) = (a_+ + t_* y, y) \\ &= (\theta + a_+, y)\end{aligned}$$

so a_+ does not depend on θ. Moreover it does not depend on y since the dynamics of the flow (3.26) is the same on every energy level $y = const$. The second equation of (3.30) is proved analogously. □

Corollary 3.3.2. *The scattering map*

$$s_{\bar{B}_*} : \bar{A}_* \longrightarrow \bar{A}_*$$

is defined by

$$s_{\bar{B}_*}(\theta, y) = \bar{\pi}^s_{\bar{B}_*} \circ \left(\bar{\pi}^u_{\bar{B}_*}\right)^{-1}(\theta, y) = (\theta + \Delta, y) \tag{3.32}$$

where

$$\Delta = a_+ - a_-.$$

Remark 3.3.3. *In Section 3.5 we will take several homoclinic cylinders B and we will make perturbations along most of the projected set $\pi(B) \subset M$ (where $\pi : TM \to M$ is the canonical projection). However, the projections of the cylinders defined in* (3.29) *are long curves (of length ≈ 1), whose iterates accumulate asymptotically on γ. This means that any two of these cylinders $\bar{B}_1, \bar{B}_2$ are likely to have intersecting projections (or perhaps even self-intersections). Therefore any perturbation made along the entirety of $\pi(\bar{B}_1)$ is likely to interfere with a perturbation made along $\pi(\bar{B}_2)$, making computations very difficult. To avoid these complicated situations, we define an equivalence relation in a neighbourhood of the cylinder $\bar{A}_*$ that results in homoclinic cylinders with projections of length $O(\tau_*)$. Then we will only have to make perturbations along some fundamental domain of the billiard map.*

There is a neighbourhood U of $\bar{A}_*$ in $\mathcal{M}$ in which we can introduce Fenichel coordinates: these are coordinates that straighten the stable and unstable manifolds $W^{s,u}(\bar{A}_*)$ (see [70]). Let $((\theta, y), \alpha, \beta)$ denote these coordinates where $\alpha = (\alpha^1, \ldots, \alpha^{d-2})$ and $\beta = (\beta^1, \ldots, \beta^{d-2})$. Let (θ_0, y_0) denote a fixed point on $\bar{A}_*$. Then we have:

- $W^s(\bar{A}_*) = \{\alpha = 0\}$;
- $W^s(\theta_0, y_0) = \{((\theta, y), \alpha, \beta) \in U : (\theta, y) = (\theta_0, y_0), \alpha = 0\}$;
- $W^u(\bar{A}_*) = \{\beta = 0\}$; and
- $W^u(\theta_0, y_0) = \{((\theta, y), \alpha, \beta) \in U : (\theta, y) = (\theta_0, y_0), \beta = 0\}$.

Slightly shrinking τ_* if necessary, we may assume without loss of generality that $\tau_*^{-1} \in \mathbb{N}$. On the set U define the equivalence relation $\sim$ where $((\theta_1, y_1), \alpha_1, \beta_1) \sim ((\theta_2, y_2), \alpha_2, \beta_2)$ if and only if

$$\theta_1 = \theta_2 \mod \tau_*, \quad y_1 = y_2, \quad \alpha_1 = \alpha_2, \quad \beta_1 = \beta_2.$$

Define the coordinate φ on $U/\sim$ by

$$\varphi = \tau_*^{-1} (\theta \mod \tau_*) \in \mathbb{T} = \mathbb{R}/\mathbb{Z}.$$

Since τ_*^{-1} is a natural number, this equivalence relation corresponds to wrapping the torus $\mathbb{T}$ around itself exactly τ_*^{-1} times.

Notice that, without loss of generality, we may assume that $\bar{B}_* \subset U$, where $\bar{B}_*$ is defined as in (3.29). This is because $\phi_Z^t(\bar{B}_*)$ is a homoclinic cylinder for all $t \in \mathbb{R}$ and $\phi_Z^t(\bar{B}_*) \to \bar{A}_*$ as $t \to \pm\infty$. Therefore we can find some sufficiently large $|T|$ so that $\phi_Z^T(\bar{B}_*) \subset U$ and replace $\bar{B}_*$ by $\phi_Z^T(\bar{B}_*)$.

Clearly the vector field Z induces a vector field on $U/\sim$ since Z does not depend on θ. For simplicity we also denote the quotient vector field by Z. Now, the cylinders $\bar{A}_*/\sim$ and $\bar{B}_*/\sim$ can be written

$$\bar{A}_*/\sim = \left\{ \phi_Z^{\tau_* \varphi y^{-1}} \left(\gamma(0), \gamma'(0), K y^{-1} \kappa(\gamma(0), \gamma'(0))^{-\frac{4}{3}} \right) : \varphi \in \mathbb{T}, y \in \left[\frac{K}{2}, \frac{3K}{2} \right] \right\} \tag{3.33}$$

and

$$\bar{B}_*/\sim = \left\{ \phi_Z^{\tau_* \varphi y^{-1}} \left(\xi(0), \xi'(0), K y^{-1} \kappa(\xi(0), \xi'(0))^{-\frac{4}{3}} \right) : \varphi \in [0, 1), y \in \left[\frac{K}{2}, \frac{3K}{2} \right] \right\} \tag{3.34}$$

where inside the brackets we are still using (x, w, z) coordinates. By the same logic as before, we can write the restriction of Z and ϕ_Z^t to these cylinders as

$$Z(\varphi, y) = (\tau_*^{-1} y, 0)$$

and

$$\phi_Z^t(\varphi, y) = (\varphi + t\tau_*^{-1} y, y). \tag{3.35}$$

Proposition 3.3.4. *The manifold $\bar{A}_*/\sim$ as defined in* (3.33) *is an r-normally hyperbolic invariant manifold for the flow of Z for any finite r. The manifold $\bar{B}_*/\sim$ as defined in* (3.34) *is a homoclinic cylinder to $\bar{A}_*/\sim$. Moreover the holonomy maps*

$$\bar{\pi}^{s,u}_{\bar{B}_*/\sim} : \bar{B}_*/\sim \longrightarrow \bar{A}_*/\sim$$

are defined by

$$\bar{\pi}^s_{\bar{B}_*/\sim}(\varphi, y) = (\varphi + a_+, y), \quad \bar{\pi}^u_{\bar{B}_*/\sim}(\varphi, y) = (\varphi + a_-, y) \tag{3.36}$$

where $a_\pm \in \mathbb{R}$ *do not depend on the point* $(\varphi, y) \in \bar{A}_*/\sim$.

Proof. In Fenichel coordinates, normal hyperbolicity means hyperbolicity in the α, β directions. Since these directions are not affected by the equivalence relation $\sim$, the derivatives in the α, β directions still exhibit hyperbolicity with the same parameters. From (3.35) we see that on the manifold $\bar{A}_*/\sim$, the differential of the time-t shift of the flow of Z is

$$D\phi^t_Z = \begin{pmatrix} 1 & t\tau_*^{-1} \\ 0 & 1 \end{pmatrix}$$

which has a double eigenvalue of 1. Therefore the spectral gap is infinitely large, in the sense that for any $r \in \mathbb{N}$ we can find some norm for which the hyperbolicity parameters λ, μ satisfy $0 < \lambda < \mu^{-r} < 1$ (see Definition 2.1.10). It follows that $\bar{A}_*/\sim$ is an r-normally hyperbolic invariant manifold for any finite r.

The proof that $\bar{B}_*/\sim$ is a homoclinic cylinder and that the holonomy maps are defined as in (3.36) is analogous to the proof of Lemma 3.3.1. □

3.3.2 Shadowing of Pseudo-Orbits

Let $\{A_n\}_{n\in\mathbb{N}}$ denote the sequence of normally hyperbolic invariant cylinders for the billiard map f obtained in Proposition 3.2.6, and let

$$A = \bigcup_{n\in\mathbb{N}} A_n$$

denote the noncompact cylinder.

Standard results imply that the existence of a single transverse homoclinic geodesic ξ implies the existence of infinitely many (see e.g. [73]). Suppose we take N such geodesics $\xi_1, \ldots, \xi_N$. Let $B_1, \ldots, B_N$ denote the corresponding homoclinic cylinders for the billiard map, and consider the scattering maps $s_j = s_{B_j}$. We now construct an iterated function system $\{f|_A, s_1, \ldots, s_N\}$ on A, and call orbits of the IFS pseudo-orbits.

The following theorem is a generalisation of a theorem of Moeckel [89]. See also [53, 82]. The proof here relies on the global definition of the scattering maps, but recent progress has relaxed this assumption [55]. The theorem tells us that, given a prescribed sequence $\{A_{m_n}\}$ of consecutive cylinders, we can find a pseudo-orbit $\{\mu_n\}$ visiting the cylinders A_{m_n} in the prescribed order.

Theorem 3.3.5. *(Diffusive and Oscillatory Pseudo-Orbits) Suppose $f, s_1, \dots, s_N$ have no common invariant essential curves in the cylinder A. Let $\{m_n\}_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $m_{n+1} = m_n \pm 1$ for each n. Then there is a pseudo-orbit $\{\mu_n\}_{n\in\mathbb{N}} \subset A$ and $\{k_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$ with*

$$k_{n+1} > k_n, \quad \mu_{k_n} \in A_{m_n}$$

for each $n \in \mathbb{N}$, and

$$\mu_j \in A_{m_n}$$

for each $j = k_n + 1, k_n + 2, \dots, k_{n+1} - 1$.

Proof. The theorem is proved by applying the proof of Theorem 3 of [53] inductively to each compact invariant subcylinder A_{m_n}. Indeed, it follows from (3.28) that the billiard map is a twist map, and since the scattering maps are globally defined diffeomorphisms that are τ_*^2-close to the maps (3.32), they are homotopic to the identity. Therefore the assumptions of Theorem 3 of [53] are satisfied.

Let $\gamma_\pm$ denote the essential curves bounding the bottom and top of the cylinder A_{m_1}, and assume $m_2 = m_1 + 1$. Construct a curve $\tilde{\gamma}$ as follows: the curves $f(\gamma_-), s_1(\gamma_-), \dots, s_N(\gamma_-)$ divide A_{m_1} into connected components. If one of these curves intersects γ_+ then we have already found a pseudo orbit connecting A_{m_1} and A_{m_2}. If not, let $\tilde{\gamma}$ denote the lower boundary of the connected component of A_{m_1} whose upper boundary is γ_+. Write $\tilde{\gamma} = F(\gamma)$. It is shown in [53] that by iterating F on γ_-, we can obtain a curve $F^{n_1}(\gamma_-)$ that intersects γ_+, which in turn gives a pseudo-orbit connecting the bottom of A_{m_1} to the top. Since A_{m_1} and A_{m_2} overlap (see Proposition 3.2.6), we can apply the process again to show that there is some $n_2 \in \mathbb{N}$ such that $F^{n_1+n_2}(\gamma_-)$ intersects the upper boundary curve of A_{m_2}. Similarly if $m_2 = m_1 - 1$ we can find pseudo-orbits starting on γ_+ and ending on γ_-, thus connecting A_{m_1} and A_{m_2} in that case. Repeating the process inductively gives the proof of Theorem 3.3.5. □

Lemma 3.3.6. *(Shadowing Lemma) Let $\epsilon > 0$, and let $\{\mu_n\}_{n=0}^{\infty}$ be a pseudo-orbit, so for each $n = 0, 1, \dots$ we have either $\mu_{n+1} = f(\mu_n)$, or there is an $\alpha_n \in \{1, \dots, N\}$ such that*

$$\mu_{n+1} = s_{\alpha_n}(\mu_n).$$

Then there is an orbit $\{\nu_n\}_{n=0}^{\infty}$ of f in $\mathcal{M}$ such that for some $\{m_n\}_{n=0}^{\infty} \subset \mathbb{N}$ we have

$$\nu_{n+1} = f^{m_n}(\nu_n)$$

and

$$\|\mu_n - \nu_n\| \leq \epsilon.$$

The Lemma is equivalent to Lemma 3 of [53] and Lemma 3.11 of [54], so we do not include it here. In those papers it is assumed that the manifold A is compact. The

purpose of compactness is to allow application of the Poincaré recurrence theorem. In our case, the structure of A as a countable union of compact invariant subcylinders allows us to adapt the proof as we can apply the recurrence theorem in each compact invariant subcylinder.

It follows from Theorem 3.3.5 and Lemma 3.3.6 that if we can show that the property of the map $f|_A$ and the N scattering maps having no common invariant essential curves on A is a residual property of $Q \in \mathcal{V}_*^d$, then Theorems 3.1.1 and 3.1.2 are proved.

3.4 Effect of a Perturbation on the Billiard and Scattering Maps

3.4.1 Perturbing the Billiard Map

In order to perturb the billiard map, we perturb the surface M. Recall that M is given implicitly as the set of zeros of a real-analytic function $Q \in \mathcal{V}_*^d$. In this section, we will show how the billiard map is affected by compactly supported perturbations of the surface M. Since such perturbations are not real-analytic, we will later approximate the perturbed system $Q + \epsilon\psi$ by a real-analytic family $Q_\epsilon \subset \mathcal{V}_*^d$.

Fix some $(x, u) \in \mathcal{M}$ and let $(\bar{x}, \bar{u}) = f(x, u)$. Let $\epsilon > 0$ and $\psi : \mathbb{R}^d \to \mathbb{R}$ be a smooth function supported in a neighbourhood of $\bar{x}$. In particular, assume $x \notin \text{supp}(\psi)$. Make the perturbation

$$Q \longrightarrow Q_\epsilon = Q + \epsilon\psi. \tag{3.37}$$

Denote by f_ϵ the perturbed billiard map, and let $(\tilde{x}, \tilde{u}) = f_\epsilon(x, u)$. Referring to (2.19), we have

$$\begin{cases} \tilde{x} = x + \tau_\epsilon(x, u)\, v \\ \tilde{u} = v - \langle v, \tilde{n}(\tilde{x})\rangle\, \tilde{n}(\tilde{x}), \end{cases} \tag{3.38}$$

where $\tau_\epsilon(x, u) = \{t > 0 : Q_\epsilon(x + tv) = 0\}$, and

$$\tilde{n}(x) = -\frac{\nabla Q_\epsilon(x)}{\|\nabla Q_\epsilon(x)\|}. \tag{3.39}$$

Since ϵ is sufficiently small, the new surface is still strictly convex and so τ_ϵ is still well-defined. Clearly for the $\tilde{x}$-component, the perturbation is manifested only through τ_ϵ. Therefore the term of order ϵ in the Taylor series of $\tilde{x}$ is $\Theta(\bar{x}, \bar{u})v$, where $\Theta : \mathcal{M} \to \mathbb{R}$ is a smooth function. We determine Θ by computing the first term in the Taylor series

of $Q_\epsilon(\tilde{x}) = 0$. Since this term must itself vanish, we have

$$\begin{aligned} 0 = \frac{d}{d\epsilon}\Big|_{\epsilon=0} Q_\epsilon(\tilde{x}) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \big[Q(\bar{x} + \epsilon\Theta(\bar{x},\bar{u})v + O(\epsilon^2)) + \epsilon\psi(\tilde{x})\big] \\ &= [\Theta(\bar{x},\bar{u})\langle \nabla Q(\tilde{x}), v\rangle + \psi(\tilde{x}) + O(\epsilon)]\big|_{\epsilon=0} \\ &= -\Theta(\bar{x},\bar{u})\|\nabla Q(\bar{x})\|\langle n(\bar{x}), v\rangle + \psi(\bar{x}) \end{aligned}$$

Elementary geometrical considerations imply that we can write v in terms of $(\bar{x},\bar{u})$ as $v = \bar{u} - \sqrt{1-\bar{u}^2}\, n(\bar{x})$. Therefore we have

$$\Theta(\bar{x},\bar{u}) = -\frac{\psi(\bar{x})}{\|\nabla Q(\bar{x})\|\sqrt{1-\bar{u}^2}}. \tag{3.40}$$

In order to compute the effect of the perturbation on the u-component, we write $\tilde{u} = \bar{u} + \epsilon\Pi(\bar{x},\bar{u}) + O(\epsilon^2)$. This computation is more direct: we first compute the expansion of $\tilde{n}(\tilde{x})$ up to first order, using (3.39) and (3.40), and then substitute this into the expression for $\tilde{u}$ in (3.38). Write $\tilde{n}(\tilde{x}) = n(\bar{x}) + \epsilon V(\bar{x},\bar{u}) + O(\epsilon^2)$. A straightforward computation shows that

$$\begin{aligned} V(\bar{x},\bar{u}) &= \frac{d}{d\epsilon}\Big|_{\epsilon=0} \tilde{n}(\tilde{x}) \\ &= -\Theta(\bar{x},\bar{u})S(\bar{x})v - \|\nabla Q(\bar{x})\|^{-1}[\nabla\psi(\bar{x}) - \langle\nabla\psi(\bar{x}), n(\bar{x})\rangle\, n(\bar{x})]. \end{aligned}$$

Notice that $V(\bar{x},\bar{u}) \in T_{\bar{x}}M$. It follows from (3.38) that

$$\begin{aligned} \tilde{u} &= v - \langle v, n(\bar{x}) + \epsilon V(\bar{x},\bar{u})\rangle(n(\bar{x}) + \epsilon V(\bar{x},\bar{u})) + O(\epsilon^2) \\ &= v - \langle v, n(\bar{x})\rangle\, n(\bar{x}) - \epsilon[\langle v, V(\bar{x},\bar{u})\rangle\, n(\bar{x}) + \langle v, n(\bar{x})\rangle\, V(\bar{x},\bar{u})] + O(\epsilon^2) \\ &= \bar{u} + \epsilon\left[\sqrt{1-\bar{u}^2}\, V(\bar{x},\bar{u}) - \langle v, V(\bar{x},\bar{u})\rangle\, n(\bar{x})\right] + O(\epsilon^2). \end{aligned}$$

We compute

$$\begin{aligned} \langle v, V(\bar{x},\bar{u})\rangle &= \langle \bar{u}, V(\bar{x},\bar{u})\rangle \\ &= -\Theta(\bar{x},\bar{u})\kappa(\bar{x},\bar{u}) + \sqrt{1-\bar{u}^2}\,\Theta(\bar{x},\bar{u})\langle C(\bar{x})\bar{u}, n(\bar{x})\rangle \\ &\quad - \|\nabla Q(\bar{x})\|^{-1}\langle\nabla\psi(\bar{x}), \bar{u}\rangle \end{aligned}$$

We thus obtain the following expressions for Π:

$$\Pi(\bar{x},\bar{u}) = \sqrt{1-\bar{u}^2}\, V(\bar{x},\bar{u}) - \langle v, V(\bar{x},\bar{u})\rangle\, n(\bar{x}) \tag{3.41}$$

$$\begin{aligned} &= -\sqrt{1-\bar{u}^2}\,\Theta(\bar{x},\bar{u})S(\bar{x})v - \\ &\quad - \sqrt{1-\bar{u}^2}\,\|\nabla Q(\bar{x})\|^{-1}\,[\nabla\psi(\bar{x}) - \langle\nabla\psi(\bar{x}), n(\bar{x})\rangle\, n(\bar{x})] + \\ &\quad + \Big[\Theta(\bar{x},\bar{u})\kappa(\bar{x},\bar{u}) - \Theta(\bar{x},\bar{u})\sqrt{1-\bar{u}^2}\langle C(\bar{x})\bar{u}, n(\bar{x})\rangle + \\ &\quad + \|\nabla Q(\bar{x})\|^{-1}\langle\nabla\psi(\bar{x}), \bar{u}\rangle\Big] n(\bar{x}) \end{aligned} \tag{3.42}$$

Equation (3.41) will be useful later since $V(\bar{x},\bar{u}) \in T_{\bar{x}}M$, whereas the second term is normal to M at $\bar{x}$. Using equations (3.40) and (3.42), we see that the vector field $(\Theta(\bar{x},\bar{u})\,v, \Pi(\bar{x},\bar{u}))$ of the perturbation is Hamiltonian. Indeed, the function

$$H_{pert}(\bar{x},\bar{u}) = \frac{\psi(\bar{x})}{\|\nabla Q(\bar{x})\|}\left[\sqrt{1-\bar{u}^2} + \langle n(\bar{x}),\bar{u}\rangle\right] - \\ - \left[\Theta(\bar{x},\bar{u})\langle C(\bar{x})v,\bar{v}\rangle + \|\nabla Q(\bar{x})\|^{-1}\langle\nabla\psi(\bar{x}),\bar{v}\rangle\right]\frac{Q(\bar{x})}{\|\nabla Q(\bar{x})\|},$$

where $\bar{v} = \bar{u} + \sqrt{1-\bar{u}^2}\,n(\bar{x})$, is such that

$$\begin{cases} \dfrac{\partial H_{pert}}{\partial \bar{u}_j}(\bar{x},\bar{u}) = \Theta(\bar{x},\bar{u})v_j \\ -\dfrac{\partial H_{pert}}{\partial \bar{x}_j}(\bar{x},\bar{u}) = \Pi_j(\bar{x},\bar{u}). \end{cases}$$

3.4.2 Perturbative Computation of the Scattering Map

Let $\bar{A}_*$ denote a compact normally hyperbolic invariant manifold of the flow of Z as in (3.33), and $\bar{B}_*$ the homoclinic cylinder as in (3.34), where in this section we drop the modulo $\sim$ notation. Let A_* and B_* denote the corresponding normally hyperbolic invariant manifold and homoclinic cylinder for f. As we do not have a precise formula for B_*, it is difficult to perturbatively compute the scattering map of f. Instead, denote by g the time-τ_* shift of the flow of Z, make the perturbation $Q \to Q + \epsilon\psi$, and define the map

$$g_\epsilon = \phi^\epsilon_{pert} \circ g$$

where ϕ^ϵ_{pert} is the Hamiltonian flow defined by

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi^\epsilon_{pert} = \Omega\nabla H_{pert}$$

and H_{pert} is the Hamiltonian of the perturbation of the billiard map. Then we have

$$g_\epsilon = g + \epsilon\Omega\nabla H_{pert} \circ g + O(\epsilon^2). \tag{3.43}$$

This is no longer a time-shift of the geodesic flow, but rather a "billiard-like" perturbation of a time-shift of the geodesic flow. Moreover

$$f_\epsilon = g_\epsilon + O(\tau_*^2) \tag{3.44}$$

outside the equivalence relation, in the C^r-topology for any finite r. Assume now that ψ is supported near $\pi \circ g(\bar{B}_*)$. Let

$$s = \bar{\pi}^s \circ (\bar{\pi}^u)^{-1}$$

where the holonomy maps $\bar{\pi}^{s,u}$ are defined as in (3.36), and let s_ϵ denote the scattering map corresponding to g_ϵ. The scattering map s has smooth dependence on ϵ. Therefore we can write

$$s_\epsilon(\varphi, y) = (\Psi(\varphi, y, \epsilon), Y(\varphi, y, \epsilon)).$$

Proposition 3.4.1. *Let $(\bar{x}, \bar{u}) = g \circ (\bar{\pi}^s)^{-1}(\varphi, y)$. Then there is a strictly positive function $h : \bar{A}_* \to \mathbb{R}_{>0}$ such that*

$$\frac{\partial \Psi}{\partial \epsilon}(s^{-1}(\varphi, y), 0) = -\frac{1}{2}\tau_* h(\varphi, y)\psi(\bar{x})Ky^{-2}\kappa(\bar{x}, \bar{u})^{-\frac{1}{3}} + O(\tau_*^2) \tag{3.45}$$

and

$$\begin{aligned}\frac{\partial Y}{\partial \epsilon}(s^{-1}(\varphi, y), 0) = h(\varphi, y)\Bigg[&\frac{1}{6}\tau_*^2\psi(\bar{x})K^2y^{-3}\kappa(\bar{x}, \bar{u})^{-\frac{8}{3}}R(\bar{x}, \bar{u}) - \\ &- \frac{1}{2}\tau_*^2K^2y^{-3}\kappa(\bar{x}, \bar{u})^{-\frac{5}{3}}\langle\nabla\psi(\bar{x}), \bar{u}\rangle\Bigg] + O(\tau_*^4)\end{aligned} \tag{3.46}$$

where R is defined as in (3.2)*, and where the higher order terms are uniformly bounded.*

Proof. Since $\bar{A}_*$ is symmetrically normally hyperbolic, the restriction of the symplectic form $\omega|_{\bar{A}_*}$ is nondegenerate (see e.g. Proposition 4 of [53]). Moreover there is a positive function $h : \bar{A}_* \to \mathbb{R}_{>0}$ such that

$$\omega|_{\bar{A}_*} = h(\varphi, y)d\varphi \wedge dy.$$

The map g_ϵ is symplectic as it is the composition of an exact symplectic map with a time-shift of a Hamiltonian flow. It was shown in [34] that the scattering map is therefore symplectic. It follows that there is some Hamiltonian function $\tilde{H}_{pert}$ such that the perturbed scattering map is

$$s_\epsilon = s + \epsilon h \circ s\Omega\nabla\tilde{H}_{pert} \circ s + O(\epsilon^2)$$

where the unperturbed scattering map $s_0 = s$ is as in (3.32) due to Proposition 3.3.4. Notice that the assumptions of Theorem 31 of [34] are satisfied. Therefore we have

$$\begin{aligned}\tilde{H}_{pert} = &\lim_{N_\pm \to +\infty}\Bigg[\sum_{j=0}^{N_- - 1}\left(H_{pert} \circ g^{-j} \circ (\bar{\pi}^u)^{-1} \circ s^{-1} - H_{pert} \circ g^{-j} \circ s^{-1}\right) + \\ &+ \sum_{j=1}^{N_+}\left(H_{pert} \circ g^j \circ (\bar{\pi}^s)^{-1} - H_{pert} \circ g^j\right)\Bigg] \\ = &\, H_{pert} \circ g \circ (\bar{\pi}^s)^{-1}\end{aligned} \tag{3.47}$$

where all but one term has vanished due to ψ being supported only near $\pi \circ g(\bar{B}_*)$.

Recall in Lemma 2.5.2 we assumed that $\|\nabla Q(x)\| = 1$ for all $x \in M$ which gave us the simple formula (2.16) for the shape operator. Now that we have made perturbations, we may again make this assumption without loss of generality. Therefore the perturbative vector field

$$\tilde{X}(\bar{x}, \bar{u}) = \Omega \nabla H_{pert}(\bar{x}, \bar{u}) = (\Theta(\bar{x}, \bar{u})v, \Pi(\bar{x}, \bar{u})) \tag{3.48}$$

is defined by

$$\Theta(\bar{x}, \bar{u}) = -\frac{\psi(\bar{x})}{\sqrt{1-\bar{u}^2}}, \tag{3.49}$$

$$v = \bar{u} - \sqrt{1-\bar{u}^2}n(\bar{x}) \tag{3.50}$$

and

$$\Pi(\bar{x}, \bar{u}) = \sqrt{1-\bar{u}^2}V(\bar{x}, \bar{u}) - \langle \bar{u}, V(\bar{x}, \bar{u})\rangle n(\bar{x}) \tag{3.51}$$

where

$$V(\bar{x}, \bar{u}) = -\Theta(\bar{x}, \bar{u})S(\bar{x})v - (\nabla\psi(\bar{x}) - \langle \nabla\psi(\bar{x}), n(\bar{x})\rangle n(\bar{x}))\,. \tag{3.52}$$

Now, let $(\varphi, y) \in \bar{A}_*$, and let $(\bar{x}, \bar{u})$ denote the point $g \circ (\bar{\pi}^s)^{-1}(\varphi, y)$ in Euclidean coordinates. We must determine the derivatives of $(\bar{x}, \bar{u})$ with respect to (φ, y). By (3.35) we have

$$g : (\varphi, y) \longmapsto (\varphi + y, y). \tag{3.53}$$

Notice that, reparametrising the homoclinic geodesic ξ by a translation if necessary, we may assume that $a_+ = 0$ so that $\bar{\pi}^s$, defined as in (3.36), is just the identity. Combining (3.34) and (3.53), we see that the set $g(\bar{B}_*)$ is equal to

$$\left\{ \phi_Z^{\tau_*(\varphi y^{-1}+1)} \left(\xi(0), \xi'(0), Ky^{-1}\kappa(\xi(0), \xi'(0))^{-\frac{4}{3}} \right) : \varphi \in [0,1), y \in \left[\frac{K}{2}, \frac{3K}{2}\right] \right\}.$$

Let

$$\bar{\theta} = \tau_*(\varphi y^{-1} + 1)$$

denote the time along the homoclinic geodesic ξ in $g(\bar{B}_*)$ so that

$$\frac{\partial\bar{\theta}}{\partial\varphi} = \tau_* y^{-1}, \quad \frac{\partial\bar{\theta}}{\partial y} = -\tau_*\varphi y^{-2}. \tag{3.54}$$

There is an arclength parameter $\bar{q} = \bar{q}(\varphi, y)$ such that

$$\bar{x} = \xi(\bar{q}), \quad \bar{u} = \|\bar{u}\|\xi'(\bar{q}) \tag{3.55}$$

where

$$\|\bar{u}\| = \sqrt{1 - \frac{1}{4}\tau_*^2 K^2 y^{-2}\kappa(\xi(\bar{q}), \xi'(\bar{q}))^{-\frac{2}{3}}} \tag{3.56}$$

by the second equation of (3.12) and by (3.27). Since z is the speed of the vector field Z, we get from (3.27)

$$\bar{q} = \int_0^{\bar{\theta}} z(s)ds = Ky^{-1} \int_0^{\bar{\theta}} \kappa(x(s), w(s))^{-\frac{4}{3}} ds.$$

Notice that

$$\bar{q} = O(\tau_*), \tag{3.57}$$

uniformly. We have

$$\begin{aligned} \frac{\partial \bar{q}}{\partial \varphi} &= Ky^{-1}\kappa\left(x\left(\bar{\theta}\right), w\left(\bar{\theta}\right)\right)^{-\frac{4}{3}} \frac{\partial \bar{\theta}}{\partial \varphi} \\ &= \tau_* K y^{-2} \kappa\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right)^{-\frac{4}{3}} \end{aligned} \tag{3.58}$$

and

$$\begin{aligned} \frac{\partial \bar{q}}{\partial y} =& -y^{-1}\bar{q} + Ky^{-1}\kappa\left(x\left(\bar{\theta}\right), w\left(\bar{\theta}\right)\right)^{-\frac{4}{3}} \frac{\partial \bar{\theta}}{\partial y} \\ =& -y^{-1}\bar{q} - \tau_* K\varphi y^{-3}\kappa\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right)^{-\frac{4}{3}} \end{aligned} \tag{3.59}$$

where we have used (3.54). The second equation of (2.24) implies that

$$\xi''(\bar{q}) = \kappa(\xi(\bar{q}), \xi'(\bar{q}))n(\xi(\bar{q})). \tag{3.60}$$

From (2.16) and (3.60) and by a similar computation to (3.20) we get

$$\frac{\partial}{\partial \bar{q}}\kappa\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right) = R\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right) \tag{3.61}$$

where R is as in (3.2). Differentiating (3.56) with respect to φ, y and using (3.58), (3.59), (3.61) yields

$$\frac{\partial \|\bar{u}\|}{\partial \varphi} = \frac{1}{12}\tau_*^3 \|\bar{u}\|^{-1} K^3 y^{-4} \kappa\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right)^{-3} R\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right) \tag{3.62}$$

and

$$\begin{aligned} \frac{\partial \|\bar{u}\|}{\partial y} =& \frac{1}{4}\tau_*^2 \|\bar{u}\|^{-1} K^2 y^{-3} \kappa\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right)^{-\frac{2}{3}} - \\ &- \frac{1}{12}\tau_*^2 \|\bar{u}\|^{-1} K^2 y^{-3} \bar{q} \kappa\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right)^{-\frac{5}{3}} R\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right) - \\ &- \frac{1}{12}\tau_*^3 \|\bar{u}\|^{-1} K^3 \varphi y^{-5} \kappa\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right)^{-3} R\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right). \end{aligned} \tag{3.63}$$

Differentiating (3.55) with respect to φ, y and using (3.58), (3.59), and (3.60) we get

$$\frac{\partial \bar{x}}{\partial \varphi} = \xi'\left(\bar{q}\right) \frac{\partial \bar{q}}{\partial \varphi} = \tau_* K y^{-2} \kappa\left(\xi\left(\bar{q}\right), \xi'\left(\bar{q}\right)\right)^{-\frac{4}{3}} \xi'\left(\bar{q}\right), \tag{3.64}$$

$$\frac{\partial \bar{x}}{\partial y} = \xi'(\bar{q})\frac{\partial \bar{q}}{\partial y} = -y^{-1}\bar{q}\xi'(\bar{q}) - \tau_* K\varphi y^{-3}\kappa\left(\xi(\bar{q}), \xi'(\bar{q})\right)^{-\frac{4}{3}}\xi'(\bar{q}), \tag{3.65}$$

$$\begin{aligned}\frac{\partial \bar{u}}{\partial \varphi} &= \frac{\partial \|\bar{u}\|}{\partial \varphi}\xi'(\bar{q}) + \|\bar{u}\|\xi''(\bar{q})\frac{\partial \bar{q}}{\partial \varphi} = \\ &= \frac{\partial \|\bar{u}\|}{\partial \varphi}\xi'(\bar{q}) + \tau_*\|\bar{u}\|Ky^{-2}\kappa\left(\xi(\bar{q}), \xi'(\bar{q})\right)^{-\frac{1}{3}} n(\xi(\bar{q})),\end{aligned} \tag{3.66}$$

and

$$\begin{aligned}\frac{\partial \bar{u}}{\partial y} &= \frac{\partial \|\bar{u}\|}{\partial y}\xi'(\bar{q}) + \|\bar{u}\|\xi''(\bar{q})\frac{\partial \bar{q}}{\partial y} = \\ &= \frac{\partial \|\bar{u}\|}{\partial y}\xi'(\bar{q}) - \|\bar{u}\|y^{-1}\bar{q}\kappa\left(\xi(\bar{q}), \xi'(\bar{q})\right) n(\xi(\bar{q})) - \\ &\qquad - \tau_*\|\bar{u}\|K\varphi y^{-3}\kappa\left(\xi(\bar{q}), \xi'(\bar{q})\right)^{-\frac{1}{3}} n(\xi(\bar{q})).\end{aligned} \tag{3.67}$$

Moreover, (3.12) and (3.27) imply that

$$\sqrt{1-\bar{u}^2} = \frac{1}{2}\tau_* Ky^{-1}\kappa(\xi(\bar{q}), \xi'(\bar{q}))^{-\frac{1}{3}}, \tag{3.68}$$

and (3.56) implies that

$$\|\bar{u}\| = 1 + O(\tau_*^2) \tag{3.69}$$

where the higher order terms are uniformly bounded. Finally, differentiating (3.47) with respect to φ, y and using (3.48), (3.49), (3.50), (3.51), (3.52), (3.57), (3.62), (3.63), (3.64), (3.65), (3.66), (3.67), (3.68), and (3.69):

$$\begin{aligned}\frac{\partial \tilde{H}_{pert}}{\partial \varphi} &= \left\langle \frac{\partial H_{pert}}{\partial \bar{x}}(\bar{x}, \bar{u}), \frac{\partial \bar{x}}{\partial \varphi}\right\rangle + \left\langle \frac{\partial H_{pert}}{\partial \bar{u}}(\bar{x}, \bar{u}), \frac{\partial \bar{u}}{\partial \varphi}\right\rangle \\ &= -\left\langle \Pi(\bar{x}, \bar{u}), \frac{\partial \bar{x}}{\partial \varphi}\right\rangle + \Theta(\bar{x}, \bar{u})\left\langle v, \frac{\partial \bar{u}}{\partial \varphi}\right\rangle \\ &= -\tau_*\sqrt{1-\bar{u}^2}Ky^{-2}\kappa(\xi(\bar{q}), \xi'(\bar{q}))^{-\frac{4}{3}}\langle V(\bar{x}, \bar{u}), \xi'(\bar{q})\rangle + \\ &\quad + \Theta(\bar{x}, \bar{u})\|\bar{u}\|\frac{\partial \|\bar{u}\|}{\partial \varphi} - \tau_*\sqrt{1-\bar{u}^2}\Theta(\bar{x}, \bar{u})Ky^{-2}\kappa(\xi(\bar{q}), \xi'(\bar{q}))^{-\frac{1}{3}} \\ &= \tau_*\sqrt{1-\bar{u}^2}\Theta(\bar{x}, \bar{u})Ky^{-2}\kappa(\xi(\bar{q}), \xi'(\bar{q}))^{-\frac{4}{3}}\langle S(\bar{x})v, \xi'(\bar{q})\rangle + \\ &\quad + \tau_*\sqrt{1-\bar{u}^2}Ky^{-2}\kappa(\xi(\bar{q}), \xi'(\bar{q}))^{-\frac{4}{3}}\langle \nabla\psi(\bar{x}), \xi'(\bar{q})\rangle + \\ &\quad + \Theta(\bar{x}, \bar{u})\|\bar{u}\|\frac{\partial \|\bar{u}\|}{\partial \varphi} - \tau_*\sqrt{1-\bar{u}^2}\Theta(\bar{x}, \bar{u})Ky^{-2}\kappa(\xi(\bar{q}), \xi'(\bar{q}))^{-\frac{1}{3}} \\ &= \frac{1}{2}\tau_*^2K^2y^{-3}\kappa(\xi(\bar{q}), \xi'(\bar{q}))^{-\frac{5}{3}}\langle \nabla\psi(\bar{x}), \xi'(\bar{q})\rangle + \\ &\quad + \frac{1}{12}\tau_*^3\Theta(\bar{x}, \bar{u})K^3y^{-4}\kappa(\xi(\bar{q}), \xi'(\bar{q}))^{-3}R(\xi(\bar{q}), \xi'(\bar{q})) + O(\tau_*^4) \\ &= \frac{1}{2}\tau_*^2K^2y^{-3}\kappa(\bar{x}, \bar{u})^{-\frac{5}{3}}\langle \nabla\psi(\bar{x}), \bar{u}\rangle - \\ &\quad - \frac{1}{6}\tau_*^2\psi(\bar{x})K^2y^{-3}\kappa(\bar{x}, \bar{u})^{-\frac{8}{3}}R(\bar{x}, \bar{u}) + O(\tau_*^4)\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial \tilde{H}_{pert}}{\partial y} &= \left\langle \frac{\partial H_{pert}}{\partial \bar{x}}(\bar{x},\bar{u}), \frac{\partial \bar{x}}{\partial y} \right\rangle + \left\langle \frac{\partial H_{pert}}{\partial \bar{u}}(\bar{x},\bar{u}), \frac{\partial \bar{u}}{\partial y} \right\rangle \\
&= -\left\langle \Pi(\bar{x},\bar{u}), \frac{\partial \bar{x}}{\partial y} \right\rangle + \Theta(\bar{x},\bar{u}) \left\langle v, \frac{\partial \bar{u}}{\partial y} \right\rangle \\
&= \sqrt{1-\bar{u}^2}\Theta(\bar{x},\bar{u}) \left\langle S(\bar{x})v, \frac{\partial \bar{x}}{\partial y} \right\rangle + \sqrt{1-\bar{u}^2} \left\langle \nabla\psi(\bar{x}), \frac{\partial \bar{x}}{\partial y} \right\rangle + \\
&\quad + \Theta(\bar{x},\bar{u})\|\bar{u}\| \frac{\partial \|\bar{u}\|}{\partial y} + \bar{q}\sqrt{1-\bar{u}^2}\Theta(\bar{x},\bar{u})\|\bar{u}\| y^{-1}\kappa(\xi(\bar{q}),\xi'(\bar{q}))+ \\
&\quad + \tau_*\sqrt{1-\bar{u}^2}\Theta(\bar{x},\bar{u})K\varphi y^{-3}\kappa(\xi(\bar{q}),\xi'(\bar{q}))^{-\frac{1}{3}} \\
&= -\bar{q}\sqrt{1-\bar{u}^2}\Theta(\bar{x},\bar{u})y^{-1}\langle S(\bar{x})v, \xi'(\bar{q})\rangle - \\
&\quad - \tau_*\sqrt{1-\bar{u}^2}\Theta(\bar{x},\bar{u})K\varphi y^{-3}\kappa(\xi(\bar{q}),\xi'(\bar{q}))^{-\frac{4}{3}}\langle S(\bar{x})v, \xi'(\bar{q})\rangle - \\
&\quad - \bar{q}\sqrt{1-\bar{u}^2}y^{-1}\langle \nabla\psi(\bar{x}), \xi'(\bar{q})\rangle - \\
&\quad - \tau_*\sqrt{1-\bar{u}^2}K\varphi y^{-3}\kappa(\xi(\bar{q}),\xi'(\bar{q}))^{-\frac{4}{3}}\langle \nabla\psi(\bar{x}), \xi'(\bar{q})\rangle + \\
&\quad + \Theta(\bar{x},\bar{u})\|\bar{u}\| \frac{\partial \|\bar{u}\|}{\partial y} + \bar{q}\sqrt{1-\bar{u}^2}\Theta(\bar{x},\bar{u})\|\bar{u}\| y^{-1}\kappa(\xi(\bar{q}),\xi'(\bar{q}))+ \\
&\quad + \tau_*\sqrt{1-\bar{u}^2}\Theta(\bar{x},\bar{u})K\varphi y^{-3}\kappa(\xi(\bar{q}),\xi'(\bar{q}))^{-\frac{1}{3}} \\
&= \frac{1}{4}\tau_*^2\Theta(\bar{x},\bar{u})K^2 y^{-3}\kappa(\xi(\bar{q}),\xi'(\bar{q}))^{-\frac{2}{3}} + O(\tau_*^2) \\
&= -\frac{1}{2}\tau_*\psi(\bar{x})K y^{-2}\kappa(\bar{x},\bar{u})^{-\frac{1}{3}} + O(\tau_*^2).
\end{aligned}$$

These derivatives combined with the fact that the term of order ϵ in the expansion of s_ϵ is $h \circ s\Omega\nabla\tilde{H}_{pert} \circ s$ completes the proof of the proposition. □

3.5 Diffusive Orbits

As mentioned in Section 3.3.2, the existence of a single transverse homoclinic geodesic $\xi : \mathbb{R} \to M$ implies the existence of infinitely many. Choose 8 distinct transverse homoclinic geodesics $\xi_1, \ldots, \xi_8$. The reason we choose 8 homoclinics is because 8 scattering maps are needed for the combinatorial argument of Lemma 3.5.2. Each curve ξ_j is parametrised by arclength, but we may reparametrise by translation without disturbing this property. Recall it was shown in Section 3.2 that in (x, w, z)-coordinates, f is approximated up to order τ_*^2 by the time-τ_* shift of the flow of a vector field Z. Moreover, it was shown that this vector field has a normally hyperbolic invariant manifold $\bar{A}_*$. We then defined an equivalence relation to shorten the length of the homoclinic cylinders. In fact, we showed that the quotient homoclinic cylinder $\bar{B}_{*,j}/\sim$

defined by ξ_j is given by

$$\left\{\phi_Z^{\tau_*\varphi y^{-1}}\left(\xi_j(0),\xi_j'(0),Ky^{-1}\kappa(\xi_j(0),\xi_j'(0))^{-\frac{4}{3}}\right):\varphi\in[0,1),y\in\left[\frac{K}{2},\frac{3K}{2}\right]\right\}.$$

Moreover, in these coordinates the map $g=\phi_Z^{\tau_*}$ is given by $g(\varphi,y)=(\varphi+y,y)$.

The set of times $t\in\mathbb{R}$ for which $\xi_j(t)$ is either an intersection point of ξ_j and ξ_k for $j\neq k$, or a point of self-intersection for ξ_j is discrete. Translate the parametrisation of each curve ξ_j such that for all sufficiently small $\tau_*>0$ there exist tubular neighbourhoods V_j of $\pi\circ g\left(\bar{B}_{*,j}/\sim\right)$ in M such that:

- $V_j\cap\gamma=\emptyset$
- $V_j\cap V_k=\emptyset$ if $j\neq k$
- $V_j\cap\xi_j=\pi\circ g\left(\bar{B}_{*,j}/\sim\right)$

In what follows we will make perturbations inside each V_j to affect only the scattering map corresponding to $\bar{B}_{*,j}/\sim$, without interfering with the other scattering maps.

Recall that τ_*^2-close to $\bar{A}_*$ and $\bar{B}_{*,j}$ we have analogous objects A_* and $B_{*,j}$ for the billiard map f. Moreover, we have a sequence of such cylinders so that the set

$$A=\bigcup_{n\in\mathbb{N}}A_n$$

is a noncompact normally hyperbolic invariant manifold for f. Combining the following theorem with Theorem 3.3.5 and Lemma 3.3.6 completes the proof of Theorem 3.1.1.

Theorem 3.5.1. *The billiard map f and the 8 scattering maps corresponding to the homoclinic cylinders $B_{*,j}$ having no common invariant essential curves on the noncompact normally hyperbolic invariant cylinder A is a residual property of $Q\in\mathcal{V}_*^d$.*

Proof. We show that all common invariant essential curves of the maps $g_\epsilon, s_1,\ldots,s_8$ on $\bar{A}_*$ can be destroyed by arbitrarily small perturbation of the hypersurface where g_ϵ is defined by (3.43) and s_j is the corresponding scattering map relative to the homoclinic cylinder $\bar{B}_{*,j}$. Since the perturbations will be supported away from $\bar{A}_*$ it can be seen that $g_\epsilon|_{\bar{A}_*}$ is precisely the time-τ_* shift of the flow of Z:

$$g_\epsilon|_{\bar{A}_*}(\theta,y)=g|_{\bar{A}_*}(\theta,y)=(\theta+\tau_*y,y).$$

Therefore every essential curve of $\bar{A}_*$ corresponding to $y=const$ is an invariant curve of g. Since the derivative of the θ component of g with respect to y is τ_*, g is a twist map, as is every map in some C^1-neighbourhood of g. Therefore all invariant essential curves of these maps on $\bar{A}_*$ are graphs of Lipschitz functions. Let L_* be the supremum

of Lipschitz constants of every map in some C^1-neighbourhood of g. In particular, this C^1-neighbourhood is large enough to contain the billiard map since $f = g+O(\tau_*^2)$. It is sufficient to show that all L_*-Lipschitz common invariant curves of $s_1, \dots, s_8$ in $\bar{A}_*$ can be destroyed by small perturbation. Now, there is some $L > 0$ such that if the quotient scattering maps defined by $s_1, \dots, s_8$ on $\bar{A}_*/\sim$ have no L-Lipschitz common invariant curves, then the maps $s_1, \dots, s_8$ have no L_*-Lipschitz common invariant curves on $\bar{A}_*$.

Write $\tilde{A}_* = A_*/\sim$. The maps $s_j(\theta, y) = (\theta + \Delta_j, y)$ induce well-defined maps on $\tilde{A}_*$, and the perturbations we make will preserve this property. Consider the space of all L-Lipschitz functions $y' : \mathbb{T} \to \mathbb{R}$ equipped with the C^0-topology. Let $\mathcal{L}$ denote the subset consisting all such functions whose graph is contained in $\tilde{A}_*$ and is invariant under each s_j. Clearly $\mathcal{L}$ is compact. If $\mathcal{L} = \emptyset$ then there is nothing to prove. Suppose otherwise. Then for any $\mu > 0$ there are finitely many $y_1, \dots, y_q \in \mathcal{L}$ such that for any $y' \in \mathcal{L}$ we have

$$\operatorname{graph}(y') \subset A_*^{(p)} \coloneqq \{(\varphi, y) \in \tilde{A}_* : |y - y_p(\varphi)| \le \mu\} \tag{3.70}$$

for some $p \in \{1, \dots, q\}$. Inclusion (3.70) is also true for invariant curves of all maps in some C^1 neighbourhood of g_ϵ. The set $A_*^{(p)}$ containing no L_*-Lipschitz common invariant curves of $s_1, \dots, s_8$ is an open property of the hypersurface.

Let R be some positive constant such that

$$\left\| \frac{\partial s_j}{\partial(\varphi, y)} \right\| < R \tag{3.71}$$

for each $j = 1, \dots, 8$ and for the 8 scattering maps corresponding to all maps in some C^1-neighbourhood of g_ϵ.

Since the scattering maps are defined on a cylinder, perturbations must be periodic in the angular component. To this end consider arcs $I_{1,2,3,4} \subsetneq \mathbb{T}$ such that $I_1 \cup I_2 = I_3 \cup I_4 = \mathbb{T}$. Write $I_{kl} = I_k \setminus I_l$, and assume that $I_{12}, I_{34}, I_{21}, I_{43}$ are mutually disjoint and lie on $\mathbb{T}$ in this order (see Figure 3.3). Since each I_k is a proper subset of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ we have $l(I_k) < 1$.

For $y_C \in \mathcal{L}$, let $C = \operatorname{graph}(y_C)$. Then C is an L-Lipschitz s_j-invariant curve for each j. Each arc I_k defines a subset of the curve C: $\hat{I}_k = \{(\varphi, y_C(\varphi)) : \varphi \in I_k\}$. Since C is s_j-invariant, there is an arc $\bar{I}_k^j \subsetneq \mathbb{T}$ such that $s_j(\hat{I}_k) = \{(\varphi, y_C(\varphi)) : \varphi \in \bar{I}_k^j\}$. Clearly $l(\bar{I}_k^j) < 1$. Let

$$E = \max_{y_C \in \mathcal{L}} \max_{j,k} l(\bar{I}_k^j).$$

Then $E < 1$, so

$$\mu = \frac{1 - E}{R} > 0. \tag{3.72}$$

By the above argument, there are finitely many $y_1, \dots, y_q \in \mathcal{L}$ such that every L-Lipschitz common invariant curve of $s_1, \dots, s_8$ in $\tilde{A}_*$ lies in one of the cylinders $A_*^{(p)}$ as defined in (3.70) corresponding to this value of μ.

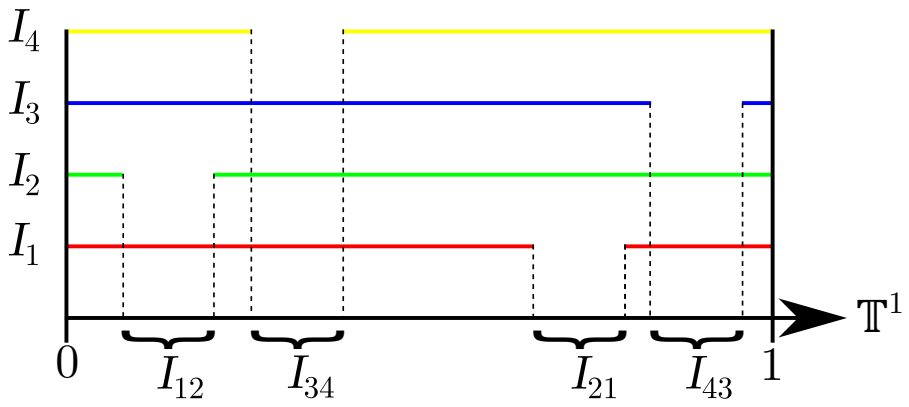


Figure 3.3: We choose the arcs $I_{1,2,3,4} \subsetneq \mathbb{T}$ so that $\mathbb{T} = I_1 \cup I_2 = I_3 \cup I_4$, and so that if $I_{kl} = I_k \setminus I_l$ then $I_{12}, I_{34}, I_{21}, I_{43}$ are mutually disjoint and lie on $\mathbb{T}$ in this order. In what follows, we make 8 perturbations (of size, say, ϵ): one perturbation per scattering map. Each interval I_k corresponds to 2 scattering maps. We give conditions on the term of order ϵ in the expansion of the scattering maps. To avoid monodromy, these conditions apply only to φ in the corresponding interval I_k. The rest of $\mathbb{T}$ is dedicated to guaranteeing a well-defined perturbation.

Consider one of these cylinders $A_*^{(p)}$ around $y_p \in \mathcal{L}$, and let $C_p = \text{graph}(y_p)$. By (3.71), $s_j(A_*^{(p)} \cap \{\varphi \in I_k\})$ is contained in the $(R\mu)$-neighbourhood of $s_j(C_p \cap \{\varphi \in I_k\})$. This curve is a subset of C_p, since C_p is s_j-invariant, and corresponds to an interval $\bar{I}_k^j$ where $l(\bar{I}_k^j) \leq E$. Therefore (3.72) implies that there is an arc $\hat{I}_{jk}^{(p)} \subsetneq \mathbb{T}$ such that

$$s_j(A_*^{(p)} \cap \{\varphi \in I_k\}) \subset \{(\varphi, y) \in A : |y - y_p(\varphi)| < R\mu,\ \varphi \in \hat{I}_{jk}^{(p)}\}.$$

The fact that $\hat{I}_{jk}^{(p)}$ is a proper sub-arc of $\mathbb{T}$ will allow us to define a perturbation however we like on $\hat{I}_{jk}^{(p)}$ and extend it to the rest of $\mathbb{T}$ without monodromy.

We will now construct a two-parameter family of functions Q_ϵ where $\epsilon = (\epsilon_1, \epsilon_2)$ and $Q_0 = Q$. Since the scattering maps s_j depend smoothly on the map g_ϵ, they depend smoothly on ϵ. Therefore, there are smooth functions Ψ_j, Y_j for each $j = 1, \ldots, 8$ such that $s_j(\varphi, y, \epsilon) = (\Psi_j(\varphi, y, \epsilon), Y_j(\varphi, y, \epsilon))$.

Consider the perturbation

$$Q_\epsilon(x) = Q(x) + \epsilon_1(\psi_1(x) + \cdots + \psi_4(x)) + \epsilon_2(\psi_5(x) + \cdots + \psi_8(x)) \tag{3.73}$$

of Q, where for now we assume that $\text{supp}(\psi_j) \cap M \subset V_j$. Later, we will approximate

Q_ϵ by a real-analytic family. Let $M_\epsilon = \{x \in \mathbb{R}^d : Q_\epsilon(x) = 0\}$. It follows that

$$\frac{\partial \Psi_{1,2,3,4}}{\partial \epsilon_2} = \frac{\partial Y_{1,2,3,4}}{\partial \epsilon_2} = \frac{\partial \Psi_{5,6,7,8}}{\partial \epsilon_1} = \frac{\partial Y_{5,6,7,8}}{\partial \epsilon_1} = 0. \tag{3.74}$$

The following lemma provides conditions on the ϵ-derivatives of the scattering maps which, if satisfied, ensure the absence of L-Lipschitz common invariant curves of $s_1, \ldots, s_8$ in $A_*^{(p)}$. The lemma plays the same role as Lemma 5 in [53]. However we have had to improve the estimates, as the assumptions of Lemma 5 of [53] do not directly apply in our case.

Lemma 3.5.2. *Suppose the scattering maps $s_1, \ldots, s_8$ corresponding to the family M_ϵ satisfy:*

(i) For $j = 1, 2$, and whenever $\Psi_j(\varphi, y, 0) \in I_j$ we have

$$\frac{\partial Y_j}{\partial \epsilon_1}(\varphi, y, \epsilon) > 2L \left| \frac{\partial \Psi_j}{\partial \epsilon_1}(\varphi, y, \epsilon) \right|, \tag{3.75}$$

and

$$\frac{\partial Y_{j+4}}{\partial \epsilon_2}(\varphi, y, \epsilon) > 2L \left| \frac{\partial \Psi_{j+4}}{\partial \epsilon_2}(\varphi, y, \epsilon) \right|. \tag{3.76}$$

(ii) For $j = 3, 4$, and whenever $\Psi_j(\varphi, y, 0) \in I_j$ we have

$$\frac{\partial Y_j}{\partial \epsilon_1}(\varphi, y, \epsilon) < -2L \left| \frac{\partial \Psi_j}{\partial \epsilon_1}(\varphi, y, \epsilon) \right|, \tag{3.77}$$

and

$$\frac{\partial Y_{j+4}}{\partial \epsilon_2}(\varphi, y, \epsilon) < -2L \left| \frac{\partial \Psi_{j+4}}{\partial \epsilon_2}(\varphi, y, \epsilon) \right|. \tag{3.78}$$

(iii) Moreover for $j = 1, 2, 3, 4$ we have

$$\left| \frac{\partial \Psi_j}{\partial \epsilon_1}(\varphi, y, \epsilon) \right| > \max\{1, (1 + L^{-1}) R \tau_*^2\} \tag{3.79}$$

whenever $\Psi_j(\varphi, y, 0) \in I_j$, and

$$\left| \frac{\partial \Psi_{j+4}}{\partial \epsilon_2}(\varphi, y, \epsilon) \right| > \max\{1, (1 + L^{-1}) R \tau_*^2\} \tag{3.80}$$

whenever $\Psi_{j+4}(\varphi, y, 0) \in I_j$.

Then the set of parameters $\epsilon = (\epsilon_1, \epsilon_2)$ for which the scattering maps $s_1, \ldots, s_8$ have an L-Lipschitz common invariant curve in $A_*^{(p)}$ has Lebesgue measure 0, and so there are arbitrarily small values of ϵ for which the scattering maps have no L-Lipschitz common invariant curve in $A_*^{(p)}$.

Proof. On the space of parameters we use the norm $\|\epsilon\| = \max\{|\epsilon_1|, |\epsilon_2|\}$. Take two values ϵ^* and ϵ^{**} of ϵ, and suppose the scattering maps $s_1, \dots, s_8$ have an L-Lipschitz common invariant curve $\mathcal{L}^* \subset A_*^{(p)}$ at ϵ^*, and $\mathcal{L}^{**} \subset A_*^{(p)}$ at ϵ^{**}. These curves can be written as $\mathcal{L}^* = \text{graph}(y^*)$ and $\mathcal{L}^{**} = \text{graph}(y^{**})$ where y^*, $y^{**} : \mathbb{T} \to \mathbb{R}$. We claim that

$$\|\epsilon^* - \epsilon^{**}\| \leq \tau_*^{-2} |y^*(0) - y^{**}(0)|. \tag{3.81}$$

The lemma follows from (3.81). Indeed, suppose it is true, and let $\mathcal{E} \subset \mathbb{R}^2$ denote the set of all ϵ for which the scattering maps have an L-Lipschitz common invariant curve in $A_*^{(p)}$. Let $\mathcal{Y}$ denote the set of all points $y_0 \in \mathbb{R}$ for which there is an L-Lipschitz common invariant curve $\mathcal{L}' = \text{graph}(y') \subset A_*^{(p)}$ of $s_1, \dots, s_8$ at any value of ϵ with $y'(0) = y_0$. By (3.81), for each $y_0 \in \mathcal{Y}$ there is exactly one $\epsilon \in \mathcal{E}$ at which the scattering maps have an L-Lipschitz common invariant curve in $A_*^{(p)}$ that intersects the y-axis at y_0. It follows that there is a well-defined surjective map from $\mathcal{Y}$ to $\mathcal{E}$ sending y_0 to ϵ. Moreover, (3.81) also implies that this map is (τ_*^{-2})-Lipschitz. Since Lipschitz maps do not increase Hausdorff dimension, it follows that $\mathcal{E}$ is a 1-dimensional set in $\mathbb{R}^2$, and so it has Lebesgue measure 0.

We must now establish estimate (3.81). Suppose, for a contradiction, that it is not true. Without loss of generality, we may assume that

$$y^*(0) \geq y^{**}(0) \quad \text{and} \quad |\epsilon_2^* - \epsilon_2^{**}| \leq |\epsilon_1^* - \epsilon_1^{**}| = \epsilon_1^* - \epsilon_1^{**}. \tag{3.82}$$

It follows that $\Delta\epsilon \coloneqq \|\epsilon^* - \epsilon^{**}\| = \epsilon_1^* - \epsilon_1^{**} > 0$. The supposition that (3.81) is not true implies that

$$0 \leq y^*(0) - y^{**}(0) < \tau_*^2 \Delta\epsilon. \tag{3.83}$$

We have that $\varphi = 0$ is in at least one of I_3 or I_4 since $\mathbb{T} = I_3 \cup I_4$. Assume $0 \in I_3$. We write

$$\begin{cases} (\bar{\varphi}^*, \bar{y}^*) = s_3(0, y^*(0), \epsilon^*) = (\Psi_3(0, y^*(0), \epsilon^*), Y_3(0, y^*(0), \epsilon^*)) \\ (\bar{\varphi}^{**}, \bar{y}^{**}) = s_3(0, y^{**}(0), \epsilon^{**}) = (\Psi_3(0, y^{**}(0), \epsilon^{**}), Y_3(0, y^{**}(0), \epsilon^{**})). \end{cases}$$

Since $\mathcal{L}^*$, $\mathcal{L}^{**}$ are invariant under $s_1, \dots, s_8$ at ϵ^*, ϵ^{**} respectively, we have $\bar{y}^* = y^*(\bar{\varphi}^*)$ and $\bar{y}^{**} = y^{**}(\bar{\varphi}^{**})$. It follows from the mean value theorem and (3.71), (3.74), (3.77), (3.82), and (3.83) that

$$|\bar{\varphi}^* - \bar{\varphi}^{**}| < \left|\frac{\partial \Psi_3}{\partial \epsilon_1}\right| \Delta\epsilon + R\tau_*^2 \Delta\epsilon, \quad \bar{y}^* - \bar{y}^{**} < -2L \left|\frac{\partial \Psi_3}{\partial \epsilon_1}\right| \Delta\epsilon + R\tau_*^2 \Delta\epsilon.$$

These inequalities together with the mean value theorem again, the L-Lipschitz property of y^{**}, and (3.79) imply that

$$y^*(\bar{\varphi}^*) - y^{**}(\bar{\varphi}^*) < \Delta\epsilon \left[R\tau_*^2 (L+1) - L \left|\frac{\partial \Psi_3}{\partial \epsilon_1}\right| \right] < 0.$$

We have assumed that $y^*(0) \geq y^{**}(0)$ and we have just shown that there is $\varphi = \bar{\varphi}^{**} \in \mathbb{T}$ such that $y^*(\varphi) < y^{**}(\varphi)$. It follows that

$$\mathcal{L}^* \cap \mathcal{L}^{**} \neq \emptyset. \tag{3.84}$$

Let us define what it means for an arc to be positive or negative. Let $I \subset \mathbb{T}$ be an arc such that $y^*(\varphi) = y^{**}(\varphi)$ for $\varphi \in \partial I$. We say that I is:

- *Positive* if $y^*(\varphi) > y^{**}(\varphi)$ for all $\varphi \in \mathrm{Int}(I)$; and
- *Negative* if $y^*(\varphi) < y^{**}(\varphi)$ for all $\varphi \in \mathrm{Int}(I)$.

We consider a single point in $\mathcal{L}^* \cap \mathcal{L}^{**}$ to be both a positive and negative arc. It follows that the only arcs in $\mathbb{T}$ that are both positive and negative are points in $\mathcal{L}^* \cap \mathcal{L}^{**}$. We will show that (3.83) implies that we can construct a positive arc and a negative arc in $\mathbb{T}$ so that the interior of their intersection is non-empty. This is absurd, and will complete the proof by contradiction.

We have already shown via (3.84) that there is at least one positive arc and one negative arc. Let $J \subset \mathbb{T}$ be a positive arc, and let

$$\mathcal{L}^*_J = \{(\varphi, y^*(\varphi)) : \varphi \in J\}, \quad \mathcal{L}^{**}_J = \{(\varphi, y^{**}(\varphi)) : \varphi \in J\},$$

and

$$\mathcal{D}_J = \{(\varphi, y) : \varphi \in J,\, y^{**}(\varphi) \leq y \leq y^*(\varphi)\}.$$

We will show that if $J \subseteq I_j$ for $j \in \{1, 2\}$ then there is a positive arc $J' \subset \mathbb{T}$ such that $s_j(\mathcal{L}^*_J, \epsilon^*) \subset \mathcal{L}^*_{J'}$ and

$$l(J') > \Delta\epsilon, \quad \mathrm{Area}(\mathcal{D}_{J'}) > \mathrm{Area}(\mathcal{D}_J). \tag{3.85}$$

Write $s^*_j(\cdot,\cdot) = s_j(\cdot,\cdot,\epsilon^*)$, and $s^{**}_j(\cdot,\cdot) = s_j(\cdot,\cdot,\epsilon^{**})$. Let $\varphi \in J$, and let $P = (\varphi, y^*(\varphi)) \in \mathcal{L}^*_J$ denote the corresponding point in $\mathcal{L}^*_J$. Write

$$\begin{cases} P^* = (\varphi^*, y^*(\varphi^*)) = s^*_j(P) = (\Psi_j(\varphi, y^*(\varphi), \epsilon^*), Y_j(\varphi, y^*(\varphi), \epsilon^*)) \\ P' = (\varphi', y') = s^{**}_j(P) = (\Psi_j(\varphi, y^*(\varphi), \epsilon^{**}), Y_j(\varphi, y^*(\varphi), \epsilon^{**}). \end{cases} \tag{3.86}$$

Clearly $P^* \in \mathcal{L}^*$ and $P' \in s^{**}_j(\mathcal{L}^*_J)$. Since J is positive we have $y^{**}(\varphi) \leq y^*(\varphi)$, and so P is either on or above $\mathcal{L}^{**}$. Moreover since $\mathcal{L}^{**}$ is s^{**}_j-invariant, P' must also lie on or above $\mathcal{L}^{**}$, and so

$$y' \geq y^{**}(\varphi'). \tag{3.87}$$

The mean value theorem together with (3.74), (3.75), and expressions (3.86) implies

$$|\varphi^* - \varphi'| \leq \left|\frac{\partial \Psi_j}{\partial \epsilon_1}\right| \Delta\epsilon, \quad y^*(\varphi^*) - y' > 2L \left|\frac{\partial \Psi_j}{\partial \epsilon_1}\right| \Delta\epsilon.$$

These inequalities combined with (3.79), (3.87) and the L-Lipschitz property of y^* and y^{**} give

$$y^*(\varphi^*) - y^{**}(\varphi^*) > L\left|\frac{\partial \Psi_j}{\partial \epsilon_1}\right|\Delta\epsilon > 0, \quad y^*(\varphi') - y' > L\left|\frac{\partial \Psi_j}{\partial \epsilon_1}\right|\Delta\epsilon > 0. \tag{3.88}$$

The first inequality of (3.88) shows that there is a positive arc $J' \subset \mathbb{T}$ such that $s_j^*(\mathcal{L}_J^*) \subset \mathcal{L}_{J'}^*$. Let $\tilde{y}^* = y^* - y^{**}$. Then $\tilde{y}^*$ is 0 on the endpoints of J' and strictly positive on $\mathrm{Int}(J')$. Moreover $\tilde{y}^*$ is $2L$-Lipschitz since y^* and y^{**} are L-Lipschitz. Let $\tilde{s}_j : \mathbb{T} \to \mathbb{T}$ denote the map $\tilde{s}_j(\varphi) = \Psi_j(\varphi, y^*(\varphi), \epsilon^*)$. Then $\tilde{s}_j(J)$ is compactly embedded in J'. Let φ_1, φ_4 denote the endpoints of J', and φ_2, φ_3 the endpoints of $\tilde{s}_j(J)$. It is possible that $\varphi_2 = \varphi_3$. Without loss of generality, we may assume (by translating our φ-coordinate on $\mathbb{T}$ if necessary) that $\varphi_1 < \varphi_2 \leq \varphi_3 < \varphi_4$. Notice that the first inequality of (3.88) holds at φ_2, φ_3 since it holds at all points of $\tilde{s}_j(J)$. These facts combined with (3.79) imply that

$$\begin{aligned}
l(J') &= \varphi_4 - \varphi_1 \geq \varphi_4 - \varphi_3 + \varphi_2 - \varphi_1 \\
&= |\varphi_4 - \varphi_3| + |\varphi_2 - \varphi_1| \\
&\geq (2L)^{-1}\left[|\tilde{y}^*(\varphi_4) - \tilde{y}^*(\varphi_3)| + |\tilde{y}^*(\varphi_2) - \tilde{y}^*(\varphi_1)|\right] \\
&= (2L)^{-1}\left(\tilde{y}^*(\varphi_2) + \tilde{y}^*(\varphi_3)\right) \\
&> \left|\frac{\partial \Psi_j}{\partial \epsilon_1}\right|\Delta\epsilon > \Delta\epsilon
\end{aligned}$$

which is the first inequality of (3.85).

Due to (3.87) and the second inequality of (3.88), the curve $\mathcal{L}' = s_j^{**}(\mathcal{L}_J^*)$ lies above $\mathcal{L}_{J'}^{**}$, and strictly below $\mathcal{L}_{J'}^*$. Since $s_j^{**}(\mathcal{D}_J)$ is the region bounded by $\mathcal{L}'$ and $s_j^{**}(\mathcal{L}_J^*)$ it follows that

$$\mathrm{Area}(D_{J'}) > \mathrm{Area}(s_j^{**}(D_J)) = \mathrm{Area}(D_J)$$

where the last equality is due to the symplecticity of s_j^{**}. This is the second inequality of (3.85).

Now, let $J \subseteq I_j$ be a positive arc where $j \in \{1, 2\}$, and construct a sequence of positive arcs J_s as follows: $J_0 = J$, and $\tilde{s}_{j_s}(J_s) \subset J_{s+1}$ where $j_s = 1$ if $J_s \subseteq I_1$ and $j_s = 2$ if $J_s \subseteq I_2$. The sequence terminates if neither $J_s \not\subset I_1$ nor $J_s \not\subset I_2$. Due to the second inequality of (3.85), $\mathrm{Area}(\mathcal{D}_{J_s})$ is strictly increasing as s increases, and so $J_{s_1} \neq J_{s_2}$ if $s_1 \neq s_2$. Since all the arcs J_s are positive, it follows that they are mutually disjoint. Therefore, the first inequality of (3.85) guarantees that there cannot be more than $\frac{1}{\Delta\epsilon}$ such arcs, and so the sequence must terminate at some positive arc J_+ such that $J_+ \cap I_{12} \neq \emptyset$ and $J_+ \cap I_{21} \neq \emptyset$.

Applying similar logic to negative arcs in the intervals I_3 and I_4, we construct a negative arc J_- such that $J_- \cap I_{34} \neq \emptyset$ and $J_- \cap I_{43} \neq \emptyset$. But since $I_{12}, I_{34}, I_{21}, I_{43}$ are

located in this order on $\mathbb{T}$, this means that $\mathrm{Int}(J_+) \cap \mathrm{Int}(J_-) \neq \emptyset$, which is impossible since one is positive and the other is negative. This contradiction implies (3.81), and completes the proof of the lemma. □

Now, suppose we make a perturbation as in (3.73) where each ψ_j is supported near $\pi \circ g(\bar{B}_{*,j}/\sim) \subset V_j$ (where V_j are the neighbourhoods described at the beginning of the section). First we will show that we can choose $\psi_1, \dots, \psi_8$ so that the scattering maps corresponding to g_ϵ satisfy the assumptions of Lemma 3.5.2. Then using (3.44) and shrinking τ_*, the scattering maps corresponding to the billiard map will also satisfy these conditions.

Notice that since g_ϵ is constant in the y-component there is some constant $\tilde{L}$ not depending on τ_* such that

$$L = \tau_*^2 \tilde{L}. \tag{3.89}$$

Now, suppose $j \in \{1,2,3,4\}$ and suppose $(\varphi, y) \in A_*^{(p)}$ such that $\Psi_j(\varphi, y, 0) \in I_j$, where now $s_j = (\Psi_j, Y_j)$ are the scattering maps corresponding to g_ϵ. Let

$$(\bar{x}, \bar{u}) = g \circ \left(\bar{\pi}_j^u\right)^{-1} (\varphi, y) = g \circ \left(\bar{\pi}_j^s\right)^{-1} \circ s_j(\varphi, y).$$

By (3.45) and (3.46) we have

$$\frac{\partial}{\partial \epsilon_1} \Psi_j(\varphi, y, \epsilon) = -\frac{1}{2} \tau_* h(s_j(\varphi, y)) \psi_j(\bar{x}) K y^{-2} \kappa(\bar{x}, \bar{u})^{-\frac{1}{3}} + O(\epsilon_1, \tau_*^2)$$

and

$$\frac{\partial}{\partial \epsilon_1} Y_j(\varphi, y, \epsilon) = h(s_j(\varphi, y)) \Bigg[\frac{1}{6} \tau_*^2 \psi_j(\bar{x}) K^2 y^{-3} \kappa(\bar{x}, \bar{u})^{-\frac{8}{3}} R(\bar{x}, \bar{u}) - \\ - \frac{1}{2} \tau_*^2 K^2 y^{-3} \kappa(\bar{x}, \bar{u})^{-\frac{5}{3}} \langle \nabla \psi_j(\bar{x}), \bar{u} \rangle \Bigg] + O(\epsilon_1, \tau_*^4).$$

Assume $\psi_j > 0$. Using these expressions, the fact that h is positive, and equation (3.89), we see that conditions (3.75) and (3.77) are of the form:

$$\langle \nabla \psi_j(\bar{x}), \bar{u} \rangle + O(\epsilon_1, \tau_*) < \frac{1}{3} \kappa(\bar{x}, \bar{u})^{-1} R(\bar{x}, \bar{u}) \psi_j(\bar{x}) \tag{3.90}$$

for $j = 1, 2$ whenever $\Psi_j(\varphi, y, 0) \in I_j$; and

$$\langle \nabla \psi_j(\bar{x}), \bar{u} \rangle + O(\epsilon_1, \tau_*) > \frac{1}{3} \kappa(\bar{x}, \bar{u})^{-1} R(\bar{x}, \bar{u}) \psi_j(\bar{x}) \tag{3.91}$$

for $j = 3, 4$ whenever $\Psi_j(\varphi, y, 0) \in I_j$. Moreover condition (3.79) is

$$\psi_j(\bar{x}) > 2\tau_*^{-1} h(s_j(\varphi, y))^{-1} K^{-1} y^2 \kappa(\bar{x}, \bar{u})^{\frac{1}{3}} \max\{1, (1 + L^{-1}) R \tau_*^2\} + \\ + O(\epsilon_1, \tau_*). \tag{3.92}$$

Notice that for all $(\bar{x}, \bar{u}) \in g(\bar{B}_{*,j}/\sim)$ we have

$$\langle \xi_j'(0), \bar{u} \rangle = 1 + O(\tau_*)$$

where the terms of order τ_* are uniformly bounded. For constants $C_{j,1} > 0$ and $C_{j,2} \in \mathbb{R}$ which are yet to be determined, let

$$\psi_j(\bar{x}) = C_{j,1} \exp\left(C_{j,2} \sum_{k=1}^{d} \xi_{j,k}'(0) \bar{x}_k \right)$$

whenever $\bar{x}$ is a point on the homoclinic curve $\xi_j = (\xi_{j,1}, \ldots, \xi_{j,d})$ corresponding to (φ, y) such that $\Psi_j(\varphi, y, 0) \in I_j$. Moreover ψ_j is 0 outside V_j, and ψ_j is at least C^4-smooth to guarantee persistence of homoclinic structures. For $j = 1, 2$ let

$$C_{j,2} = -1 + \frac{1}{3} \min_{(\bar{x},\bar{u}) \in g(\bar{B}_{*,j}/\sim)} \left(\kappa(\bar{x}, \bar{u})^{-1} R(\bar{x}, \bar{u}) \right).$$

Since $\psi_j > 0$, we may divide both sides of (3.90) by $\psi_j(\bar{x})$ to see that it is equivalent to

$$\frac{1}{3} \kappa(\bar{x}, \bar{u})^{-1} R(\bar{x}, \bar{u}) > \langle \nabla \ln \psi_j(\bar{x}), \bar{u} \rangle + O(\epsilon_1, \tau_*) = C_{j,2} + O(\epsilon_1, \tau_*)$$

which is true for sufficiently small ϵ_1, τ_*. Since the exponential term is nonzero, we may choose $C_{j,1} > 0$ large enough so that (3.92) is satisfied[3]. For $j = 3, 4$, let

$$C_{j,2} = 1 + \frac{1}{3} \max_{(\bar{x},\bar{u}) \in g(\bar{B}_{*,j}/\sim)} \left(\kappa(\bar{x}, \bar{u})^{-1} R(\bar{x}, \bar{u}) \right).$$

Then a similar computation shows that ψ_j satisfies (3.91) and (3.92) for sufficiently small ϵ_1, τ_*. Similar computations show that similar functions satisfy conditions (3.76), (3.78), and, (3.80).

Therefore, Lemma 3.5.2 implies that the scattering maps $s_1, \ldots, s_8$ corresponding to g_ϵ have no L-Lipschitz common invariant curves in the cylinder $A_*^{(p)}$. Approximating the perturbed function Q_ϵ sufficiently well by a real-analytic family $\tilde{Q}_\epsilon \subset \mathcal{V}_*^d$, we can guarantee that for arbitrarily small values of ϵ, the conditions of Lemma 3.5.2 are still satisfied for the analytic family. Now, apply this process to clear the L-Lipschitz common invariant curves in $A_*^{(1)}$. Since the property of the scattering maps having no invariant curves in $A_*^{(1)}$ is open, we may repeat the process with sufficiently small values of ϵ to kill the invariant curves in $A_*^{(2)}$ without creating new invariant curves in $A_*^{(1)}$. Repeating this process q times, we can simultaneously clear the subcylinders $A_*^{(1)}, \ldots, A_*^{(q)}$ of invariant curves. Since all L-Lipschitz common invariant curves of $s_1, \ldots, s_8$ in $\bar{A}_*/\sim$ were contained in these subcylinders, we have therefore cleared the cylinder $\bar{A}_*/\sim$ of common invariant curves of the scattering maps. This implies

[3]It does not matter that $C_{j,1}$ is very large (of order τ_*^{-1}) since we can just shrink ϵ_1 accordingly.

that the IFS $\{g, s_1, \ldots, s_8\}$ has no L_*-Lipschitz common invariant curves in $\bar{A}_*$, which in turn implies that the IFS $\{f, s_1, \ldots, s_8\}$ corresponding to the billiard map has no common invariant essential curves on A_*, for sufficiently small τ_*.

We have shown that by making arbitrarily small analytic perturbations, we can destroy the invariant essential curves on the cylinder A_* for sufficiently small τ_*. This implies that the property of the IFS having no invariant essential curves on A_* is dense in $\mathcal{V}_*^d$ for sufficiently small τ_*. As mentioned earlier, it is also open. Choosing the sequence of cylinders $\{A_n\}_{n\in\mathbb{N}}$ so that the corresponding sequence $\{\tau_n\}_{n\in\mathbb{N}}$ of constants are all sufficiently small, we therefore obtain for each $n \in \mathbb{N}$ an open dense set $\overline{\mathcal{V}}_n \subset \mathcal{V}_*^d$ with the property that for all $Q \in \overline{\mathcal{V}}_n$, the corresponding IFS comprising the billiard map and 8 scattering maps has no invariant essential curves on the cylinder A_n. It follows that

$$\overline{\mathcal{V}} = \bigcap_{n\in\mathbb{N}} \overline{\mathcal{V}}_n$$

is the residual set we are looking for.

□

— 4 —

Generic Properties of Geodesic Flows

In this chapter, the question of existence of transverse homoclinic geodesics on hypersurfaces of Euclidean space is addressed. In particular, this question is answered positively for generic strictly convex surfaces in $\mathbb{R}^3$, and generically in the presence of an elliptic closed geodesic on hypersurfaces of $\mathbb{R}^d$ with the real-analytic topology.

4.1 Results

Let $d \geq 3$, and recall from Section 2.5 that $\mathcal{W}^d$ denotes the set of all real-analytic functions $Q : \mathbb{R}^d \to \mathbb{R}$ such that the set

$$M = M(Q) = \{x \in \mathbb{R}^d : Q(x) = 0\}$$

is a closed (not necessarily convex) hypersurface of $\mathbb{R}^d$. Recall from Section 2.7.2 that the geodesic flow ϕ^t takes a point $x \in M$ and a tangent vector $u \in T_xM$ and follows the unique geodesic through x in the direction u at a constant speed $\|u\|$. We write

$$d = n + 2$$

where $n \geq 1$ for convenience throughout this chapter. Recall from Section 2.7.3 the classification of closed geodesics.

Theorem 4.1.1. *There is a residual set $\mathcal{B} \subset \mathcal{W}^{n+2}$ such that if $Q \in \mathcal{B}$ then every closed geodesic on $M(Q)$ is nondegenerate.*

Recall from Section 2.4.4, for $k \in \mathbb{N}_0$, $J^k_s(n)$ denotes the set of k-jets of symplectic autormorphisms of $\mathbb{R}^{2n}$ that fix the origin. The Poincaré map of a geodesic on M is such a symplectic automorphism of a neighbourhood of the origin after making a suitable coordinate transformation (e.g. Fermi coordinates - see Section 4.2). Moreover, the geodesic need not be closed: one can simply consider the Poincaré map between two transverse sections along a geodesic.

Theorem 4.1.2. *Let $Q \in \mathcal{W}^{n+2}$, and let $M = M(Q)$. Let $k \in \mathbb{N}$ and let $\mathcal{J} \subset J^k_s(n)$ be open, dense, and invariant. Let $\gamma : [0,1] \to M$ denote a nonconstant geodesic segment such that the normal curvature along γ is not identically zero. Assume moreover that $\gamma(0)$ is not a point of self-intersection of γ. Let $l : [0,1] \to TM$ be the corresponding orbit segment $l = (\gamma, \gamma')$ of the geodesic flow. Let Σ_0, Σ_1 be transverse sections to l at $l(0), l(1)$ respectively. Let P_Q denote the Poincaré map from Σ_0 to Σ_1. Then we can find $\tilde{Q} \in \mathcal{W}^{n+2}$ arbitrarily close to Q such that the perturbed Poincaré map $P_{\tilde{Q}}$ corresponding to the perturbed orbit segment $\tilde{l} : [0,1] \to TM(\tilde{Q})$ satisfies $J^k_{\tilde{l}(0)} P_{\tilde{Q}} \in \mathcal{J}$.*

Remark 4.1.3. *The orbits $l, \tilde{l}$ do not lie on the same manifold, so it is not immediately clear that we can compare them. However we can fix some reference manifold M_0 such that the hypersurface M in consideration admits a C^ω-diffeomorphism $G : M_0 \to M$, so we can pull back the Euclidean metric on M to M_0 via G to get a Riemannian metric on the hypersurface M_0. Therefore we can consider $l, \tilde{l}$ as orbit segments of geodesic flows corresponding to different metrics on M_0. See Section 4.2.4.*

Remark 4.1.4. *The assumption that the normal curvature along γ is not identically zero is necessary because the result of a perturbation of the hypersurface on the metric in Fermi coordinates is equal to some function times a matrix related to the curvature matrix (see Section 4.2 for definitions and details). In particular, if the curvature is zero at a point, then so is the perturbation of the metric. This assumption is not particularly restrictive, as closed geodesics necessarily have this property, and therefore so too do homoclinic and heteroclinic geodesics. Moreover, given a geodesic segment with normal curvature identically zero, it is expected that a generic C^ω-small perturbation of the hypersurface would curve this geodesic segment.*

Let γ be a hyperbolic periodic orbit of ϕ^t. Let $\theta \in \gamma$. Recall that the strong stable and strong unstable manifolds $W^{s,u}(\theta)$ are defined as

$$W^s(\theta) = \left\{\bar{\theta} \in TM : \|\phi^t(\bar{\theta}) - \phi^t(\theta)\| \to 0 \text{ as } t \to +\infty\right\},$$
$$W^u(\theta) = \left\{\bar{\theta} \in TM : \|\phi^t(\bar{\theta}) - \phi^t(\theta)\| \to 0 \text{ as } t \to -\infty\right\}.$$

The stable and unstable manifolds $W^{s,u}(\gamma)$ of the hyperbolic periodic orbit γ are

$$W^s(\gamma) = \bigcup_{\theta \in \gamma} W^s(\theta), \quad W^u(\gamma) = \bigcup_{\theta \in \gamma} W^u(\theta),$$

and we have

$$\dim W^s(\gamma) = \dim W^u(\gamma) = n + 1.$$

Now suppose η is another hyperbolic periodic orbit. A point $\bar{\theta} \in TM$ is *heteroclinic* if there are $\theta_1 \in \gamma, \theta_2 \in \eta$ such that

$$\|\phi^t(\bar{\theta}) - \phi^t(\theta_1)\| \to 0 \text{ as } t \to +\infty \quad \text{and} \quad \|\phi^t(\bar{\theta}) - \phi^t(\theta_2)\| \to 0 \text{ as } t \to -\infty.$$

Clearly then $\bar{\theta} \in W^s(\gamma) \cap W^u(\eta)$. Suppose the orbits γ, η lie in the unit tangent bundle T^1M consisting of tangent vectors with length equal to 1. Then $\bar{\theta}$ is a *transverse heteroclinic* point if $\bar{\theta} \in W^s(\gamma) \cap W^u(\eta)$ and

$$T_{\bar{\theta}}W^s(\gamma) + T_{\bar{\theta}}W^u(\eta) = T_{\bar{\theta}}T^1M.$$

If $\gamma = \eta$ then $\bar{\theta}$ is a *homoclinic* point. If a homoclinic or heteroclinic point is transverse, then so is every point in its orbit.

Theorem 4.1.5. *Let $k \in \mathbb{N}$ and let $\mathcal{J} \subset J^k_s(n)$ be open, dense, and invariant. There is a residual set $\mathcal{K} \subset \mathcal{W}^{n+2}$ such that if $Q \in \mathcal{K}$ and $M = M(Q)$, then:*

(i) The k-jet of the Poincaré map of every closed geodesic on M lies in $\mathcal{J}$; and

(ii) Any homoclinic or heteroclinic connections between hyperbolic closed geodesics are transverse.

In the proof of Theorem 4.1.5, a result (Theorem 4.5.5) that may be of independent interest is proved, which says: if two hyperbolic closed geodesics have a heteroclinic (or homoclinic, in the case where the two geodesics coincide) connection, then by an arbitrarily small real-analytic perturbation of the hypersurface, this connection can be made transverse.

Combining Theorem 4.1.5 with an argument of Contreras [29], the following result is obtained.

Theorem 4.1.6. *There is a residual set $\mathcal{W}^{n+2}_0 \subset \mathcal{W}^{n+2}$ such that for all $Q \in \mathcal{W}^{n+2}_0$, if the geodesic flow on $M(Q)$ with respect to the Euclidean metric has a nonhyperbolic periodic orbit, then it has a hyperbolic periodic orbit with a transverse homoclinic.*

Recall $\mathcal{V}^{n+2}$ denotes the subset of $\mathcal{W}^{n+2}$ consisting functions Q for which $M = M(Q)$ is strictly convex. Moreover $\mathcal{V}^{n+2}_*$ denotes the subset of $\mathcal{V}^{n+2}$ consisting functions Q for which $M = M(Q)$ has a transverse homoclinic geodesic. Then Theorems 4.1.5 and 4.1.6 together with a theorem of Mather [85] and an argument of Knieper and Weiss [77] imply the following result.

Theorem 4.1.7. *$\mathcal{V}^3_*$ is open and dense in $\mathcal{V}^3$.*

Remark 4.1.8. *It is quite possible that Theorem 4.1.7 is also true in the nonconvex case (i.e. for an open and dense set in $\mathcal{W}^3$). The proof relies on the existence of a global surface of section. In the strictly convex case, the existence of such a surface was shown by Birkhoff [17]. In the case of 3-dimensional Reeb flows, conditions were given by Hofer, Wysocki, and Zehnder which, if satisfied, guarantee the existence of a global surface of section [66, 67]. The restriction of the geodesic flow to the unit tangent*

bundle is a 3-dimensional Reeb flow. One would need to check that the conditions of those papers are C^ω-generically satisified by geodesic flows on real-analytic closed surfaces in $\mathbb{R}^3$, before applying the same reasoning as in Section 4.6.

It is well-known that the geodesic flow on manifolds of negative curvature has the Bernoulli property, and so exhibits chaotic motions [6], but the positive curvature case is more subtle. The first explicit example of a smooth, positively-curved Riemannian metric with a nontrivial hyperbolic basic set was given in [76].

In the following discussion, the setting of perturbations of Riemannian metrics on a fixed closed manifold is referred to as the *classical setting*.

In the classical setting, it was proved by Contreras that there is a C^2-open and dense set of C^∞ Riemannian metrics on an arbitrary closed manifold M for which the geodesic flow has a nontrivial hyperbolic basic set [29]. His result hinges on an application of Mañé's theory of dominated splittings [84], and an analogue of Franks' Lemma for geodesic flows, which only works for C^2-small perturbations of the metric. Knieper and Weiss proved that, whenever there exists a global surface of section, by an arbitrarily C^∞ small perturbation of the metric, a geodesic flow with a nontrivial hyperbolic basic set can be obtained [77]. Combining this with results of Hofer, Wysocki, and Zehnder [66, 67] implies C^∞-generic existence of nontrivial hyperbolic basic sets on surfaces in $\mathbb{R}^3$ in the classical setting.

Theorem 4.1.1 was originally stated (without proof) by Abraham in the classical setting [2] for C^r-smooth metrics where $5 \leq r < \infty$. Some time later, the first proof was produced by Anosov [7], which worked for $2 \leq r \leq \infty$. A C^∞ version of the theorem for hypersurfaces with the Euclidean metric was proved by Stojanov [111]. The classical result is typically referred to as the bumpy metric theorem, but we resist that nomenclature in this instance to avoid any potential confusion with essential use of bump functions, since they fail to be real-analytic.

Theorem 4.1.2 was originally proved for k-jets of C^{k+1}-smooth Poincaré maps arising in the classical setting in the C^{k+1}-topology for any $k \in \mathbb{N}$ by Klingenberg and Takens [75]. For hypersurfaces with the Euclidean metric, it was first shown by Stojanov and Takens in [112], but this result did not include the analytic case. An analogue for the case of Mañé generic Hamiltonians was proved for $k = 1$ in [104] and for $k \geq 2$ in [23].

Theorem 4.1.5 is an analogue of the Kupka-Smale theorem for geodesic flows on real-analytic, closed, and strictly convex hypersurfaces of Euclidean space. Part (i) of Theorem 4.1.5 follows from Theorems 4.1.1 and 4.1.2, as pointed out by Anosov [7]. Part (ii) was proved in the classical setting by Contreras and Paternain [30].

It was shown by Contreras that Theorem 4.1.6 is equivalent to Theorem 4.1.5 in the classical setting [29]. As his proof applies directly, once Theorem 4.1.5 is proved,

to the case of geodesic flows on real-analytic closed hypersurfaces of Euclidean space, we do not include it here. The complicated structure of orbits near elliptic periodic points was proved in the case of real-analytic symplectic diffeomorphisms of the plane with an elliptic fixed point by Zehnder in the 70's [118], and the proof in the case of geodesic flows is based on the ideas in that paper. The general idea is that the restriction of the Poincaré map of an elliptic closed geodesic to its centre manifold can be put in a Birkhoff normal form. Then, using techniques developed by Arnaud and Herman [8], it can be shown that such a map generically has a 1-elliptic periodic orbit. The restriction of the Poincaré map corresponding to the 1-elliptic periodic orbit to its 2-dimensional centre manifold is a twist map of the annulus, of Kupka-Smale type, and therefore has a transverse homoclinic orbit [81].

The reason that the proofs of these theorems from the classical setting do not apply to this case is that the class of perturbations allowed here is more restrictive. The Nash embedding theorem guarantees that any Riemannian manifold can be isometrically embedded in some Euclidean space $\mathbb{R}^N$, but small perturbations of the metric do not necessarily preserve N. In the next section we compute the effect of a perturbation of the surface on the metric, and we will see that this results only in a small subset of all possible perturbations of the metric. It follows that many perturbations of the metric are not allowed in the current setting. Moreover, the proofs presented here apply in the real-analytic topology (as well as weaker topologies).

The remainder of this chapter is structured as follows. In Section 4.2, we introduce some basic ideas regarding the geometry of the hypersurface, and show how a (local) perturbation of the manifold affects the geodesic flow. In Section 4.3, Theorem 4.1.1 is proved. In Section 4.4, Theorem 4.1.2 is proved, and Theorem 4.1.5 is proved in Section 4.5, along with a theorem that breaks homoclinic or heteroclinic connections by arbitrarily small real-analytic perturbations of the hypersurface (see [30, 38, 99] for proofs of similar results in the classical setting). In Section 4.6, it is shown how to combine results of this paper with the argument of Knieper and Weiss to prove Theorem 4.1.7.

4.2 Geometry and Perturbations

4.2.1 Effect of a Local Perturbation on the Geodesic Flow

Let $Q \in \mathcal{W}^{n+2}$ and $M = M(Q)$. Let $x \in M$, and suppose we make a local perturbation

$$Q \longrightarrow Q + \epsilon\psi \tag{4.1}$$

where ϵ is small and ψ is a C^∞ function supported near x. Then the Hamiltonian H of the geodesic flow defined as in (2.25) perturbs as

$$H \longrightarrow H_\epsilon = H + \epsilon \bar{H} + O(\epsilon^2). \tag{4.2}$$

Let us determine $\bar{H}$. For $u \in T_x M$ we have

$$H_\epsilon(x,u) = \frac{u^2}{2} + \kappa_\epsilon(x,u) \frac{Q(x) + \epsilon\psi(x)}{\|\nabla Q(x) + \epsilon \nabla\psi(x)\|}. \tag{4.3}$$

The perturbed normal curvature is

$$\kappa_\epsilon(x,u) = \langle C_\epsilon(x)u, u\rangle \tag{4.4}$$

where

$$C_\epsilon(x) = \|\nabla Q(x) + \epsilon\nabla\psi(x)\|^{-1} \left(\left(\frac{\partial^2 Q}{\partial x_i \partial x_j}(x) \right)_{i,j} + \epsilon \left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}(x) \right)_{i,j} \right) \tag{4.5}$$

is the perturbed curvature matrix. We have

$$\begin{aligned} \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \|\nabla Q(x) + \epsilon\nabla\psi(x)\| &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \left(\sum_{j=1}^{n+2} \left(\frac{\partial Q}{\partial x_j}(x) + \epsilon \frac{\partial \psi}{\partial x_j}(x) \right)^2 \right)^{\frac{1}{2}} \\ &= \|\nabla Q(x)\|^{-1} \langle \nabla Q(x), \nabla\psi(x)\rangle \\ &= -\langle n(x), \nabla\psi(x)\rangle \end{aligned}$$

where we have used (2.13). Therefore

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \|\nabla Q(x) + \epsilon\nabla\psi(x)\|^{-1} = \|\nabla Q(x)\|^{-2} \langle n(x), \nabla\psi(x)\rangle. \tag{4.6}$$

From (2.14), (4.5) and (4.6) we get

$$\begin{aligned} C_\epsilon(x) = {} & \left(\|\nabla Q(x)\|^{-1} + \epsilon \|\nabla Q(x)\|^{-2} \langle n(x), \nabla\psi(x)\rangle \right) \times \\ & \times \left(\left(\frac{\partial^2 Q}{\partial x_i \partial x_j}(x) \right)_{i,j} + \epsilon \left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}(x) \right)_{i,j} \right) + O(\epsilon^2) \\ = {} & \|\nabla Q(x)\|^{-1} \left(\frac{\partial^2 Q}{\partial x_i \partial x_j}(x) \right)_{i,j} + \epsilon \|\nabla Q(x)\|^{-1} \left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}(x) \right)_{i,j} + \\ & + \epsilon \|\nabla Q(x)\|^{-2} \langle n(x), \nabla\psi(x)\rangle \left(\frac{\partial^2 Q}{\partial x_i \partial x_j}(x) \right)_{i,j} + O(\epsilon^2) \\ = {} & C(x) + \epsilon \|\nabla Q(x)\|^{-1} \Bigg[\left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}(x) \right)_{i,j} + \\ & + \langle n(x), \nabla\psi(x)\rangle C(x) \Bigg] + O(\epsilon^2). \end{aligned} \tag{4.7}$$

Then (4.4) and (4.7) imply

$$\kappa_\epsilon(x,u) = \kappa(x,u) + \epsilon\|\nabla Q(x)\|^{-1}\Bigg[\left\langle\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}(x)\right)_{i,j}u,u\right\rangle + \\ + \langle n(x),\nabla\psi(x)\rangle\kappa(x,u)\Bigg] + O(\epsilon^2). \tag{4.8}$$

Combining (4.3), (4.6), and (4.8) yields

$$\begin{aligned} H_\epsilon(x,u) = \frac{u^2}{2} + \Bigg[\kappa(x,u) + \epsilon\|\nabla Q(x)\|^{-1}\Bigg(\left\langle\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}(x)\right)_{i,j}u,u\right\rangle + \\ + \langle n(x),\nabla\psi(x)\rangle\kappa(x,u)\Bigg)\Bigg]\Bigg[\|\nabla Q(x)\|^{-1} + \\ + \epsilon\|\nabla Q(x)\|^{-2}\langle n(x),\nabla\psi(x)\rangle\Bigg]\Bigg[Q(x)+\epsilon\psi(x)\Bigg] + O(\epsilon^2) \\ = \frac{u^2}{2} + \kappa(x,u)\frac{Q(x)}{\|\nabla Q(x)\|} + \epsilon\|\nabla Q(x)\|^{-1}\Bigg[\kappa(x,u)\psi(x) + \\ + 2\langle n(x),\nabla\psi(x)\rangle\kappa(x,u)\frac{Q(x)}{\|\nabla Q(x)\|} + \\ + \left\langle\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}(x)\right)_{i,j}u,u\right\rangle\frac{Q(x)}{\|\nabla Q(x)\|}\Bigg] + O(\epsilon^2). \end{aligned} \tag{4.9}$$

Therefore (4.2) and (4.9) imply

$$\bar{H}(x,u) = \|\nabla Q(x)\|^{-1}\Bigg[\kappa(x,u)\psi(x) + \Bigg(2\langle n(x),\nabla\psi(x)\rangle\kappa(x,u) + \\ + \left\langle\left(\frac{\partial^2\psi}{\partial x_i\partial x_j}(x)\right)_{i,j}u,u\right\rangle\Bigg)\frac{Q(x)}{\|\nabla Q(x)\|}\Bigg]. \tag{4.10}$$

4.2.2 Fermi Coordinates

Let $Q \in \mathcal{W}^{n+2}$ and $M = M(Q)$. Recall that ϕ^t denotes the time-t shift along orbits of the geodesic flow. Let $x \in M$, and define the *exponential map*

$$\begin{aligned} \exp_x : T_xM &\longrightarrow M \\ u &\longmapsto \phi^1(x,u) \end{aligned}$$

It is well-known that the exponential map is a diffeomorphism in a neighbourhood of $0 \in T_xM$.

Let $\gamma : [a,b] \to M$ be a nonconstant geodesic segment with no self-intersections, parametrised to have unit speed. Then we can choose an orthonormal basis $\gamma'(a), e_1, \ldots,$

e_n of $T_{\gamma(a)}M$. Moving this basis by parallel transport along the geodesic segment γ gives an orthonormal basis $\gamma'(t), e_1(t), \ldots, e_n(t)$ of $T_{\gamma(t)}M$ for each $t \in [a,b]$. It follows that there is a $\delta > 0$ such that if $B^n_\delta(0)$ denotes the ball of radius δ around the origin in $\mathbb{R}^n$ then the map

$$\begin{aligned} &: [a,b] \times B^n_\delta(0) \longrightarrow M \\ (y_0, y) = (y_0, y_1, \ldots, y_n) &\longmapsto \exp_{\gamma(y_0)} \sum_{j=1}^{n} y_j e_j(y_0) \end{aligned}$$

defines a chart on M (since γ has no self-intersections). We can still use such 'coordinates' if γ has self-intersections, but they will not define a chart.

Let $(v_0, v) = (v_0, v_1, \ldots, v_n)$ denote the corresponding tangent coordinates, in the sense that (v_0, v) corresponds to

$$\sum_{j=0}^{n} v_j \frac{\partial}{\partial y_j}$$

in the tangent space. Let g_{ij} denote the metric in these coordinates, and recall that $\delta_{ij} = 1$ if $i = j$ and 0 otherwise. Then along the geodesic segment γ we have

$$g_{ij} = \delta_{ij}, \quad \frac{\partial}{\partial y_k} g_{ij} = 0, \quad \frac{d}{dt} y_i = \delta_{0i}.$$

The equation of the geodesic is

$$\gamma(t) = (t, 0) = (t, 0, \ldots, 0), \quad \gamma'(t) = (1, 0) = (1, 0, \ldots, 0). \tag{4.11}$$

These coordinates were first introduced by Fermi and typically bear his name [43]. See [74] for more details.

4.2.3 Perturbations in Fermi Coordinates

Let $Q \in \mathcal{W}^{n+2}$ and $M = M(Q)$. Lemma 2.5.2 implies that we can assume without loss of generality that $\|\nabla Q(x)\| = 1$ for all $x \in M$. Therefore $n(x), C(x), S(x)$ simplify via (2.15), (2.16).

Suppose we make a perturbation as in (4.1). The Hamiltonian of the perturbation must be of the form

$$\bar{H}(x,u) = \frac{1}{2}\bar{h}(x)(u,u) \tag{4.12}$$

where $\bar{h}(x)$ is a symmetric matrix with smooth dependence on x. Comparing (4.10) with (4.12), and bearing in mind that $\|\nabla Q(x)\| = 1$, we see that

$$\begin{aligned} \bar{h}(x) = 2\psi(x)C(x) + \\ + \left(4\langle n(x), \nabla\psi(x)\rangle C(x) + 2\left(\frac{\partial^2 \psi}{\partial x_i \partial x_j}(x)\right)_{i,j} \right) Q(x). \end{aligned} \tag{4.13}$$

Let g denote the metric in Fermi coordinates (y_0, y). Then g perturbs as

$$g \longrightarrow g_\epsilon = g + \epsilon\bar{g} + O(\epsilon^2).$$

In Fermi coordinates we have

$$Q(y_0, y) \equiv 0. \tag{4.14}$$

It is well-known that the formula for $\bar{g}$ in terms of $\bar{h}$ at a point q on M is

$$\bar{g}_{ij}(q) = \sum_{k,l=1}^{n+2} \frac{\partial x_k}{\partial y_i}(q)\frac{\partial x_l}{\partial y_j}(q)\bar{h}_{kl}(q). \tag{4.15}$$

Combining (4.13), (4.14), and (4.15) gives

$$\bar{g}(y_0, y) = 2\psi(y_0, y)\tilde{C}(y_0, y) \tag{4.16}$$

where the matrix $\tilde{C}$ is defined by

$$\tilde{C}_{ij}(q) = \sum_{k,l=1}^{n+2} \frac{\partial^2 Q}{\partial x_k \partial x_l}(q)\frac{\partial x_k}{\partial y_i}(q)\frac{\partial x_l}{\partial y_j}(q).$$

Due to the substantially less complicated nature of (4.16) relative to (4.13), we will make perturbations exclusively in Fermi coordinates.

Notice that, at points $(y_0, 0)$ of γ, we have

$$\frac{\partial x_k}{\partial y_0}(y_0, 0) = \gamma_k'(y_0)$$

since y_0 is time along γ. It follows that

$$\tilde{C}_{00}(y_0, 0) = \sum_{k,l=1}^{n+2} \frac{\partial^2 Q}{\partial x_k \partial x_l}(\gamma(y_0))\gamma_k'(y_0)\gamma_l'(y_0) = \kappa(\gamma(y_0), \gamma'(y_0)). \tag{4.17}$$

Remark 4.2.1. *In the setting of generic Riemannian metrics on a smooth manifold, the perturbed metric $\bar{g}$ can be any symmetric positive-definite matrix. In the present case the perturbed metric must be the matrix $\tilde{C}$ times some function. Therefore this setting is significantly more restrictive - this (along with the more stringent topology) is why the original proofs of the classical theorems do not apply directly.*

If $\bar{H}$ denotes the Hamiltonian of the perturbation as in (4.2), then

$$\bar{H}(y_0, y, v_0, v) = \frac{1}{2}\sum_{i,j=0}^{n} \bar{g}_{ij}(y_0, y)v_i v_j = \psi(y_0, y)\sum_{i,j=0}^{n} \tilde{C}_{ij}(y_0, y)v_i v_j. \tag{4.18}$$

Let

$$X_\epsilon = X + \epsilon\bar{X} + O(\epsilon^2)$$

denote the vector field of the perturbed geodesic flow. Then

$$\bar{X} = \Omega \nabla \bar{H} = \left(\frac{\partial \bar{H}}{\partial v}, -\frac{\partial \bar{H}}{\partial y} \right) \tag{4.19}$$

where

$$\frac{\partial \bar{H}}{\partial y_k} = \sum_{i,j=0}^{n} \left(\frac{\partial \psi}{\partial y_k}(y_0, y) \tilde{C}_{ij}(y_0, y) + \psi(y_0, y) \frac{\partial \tilde{C}_{ij}}{\partial y_k}(y_0, y) \right) v_i v_j$$

and

$$\frac{\partial \bar{H}}{\partial v_k} = 2\psi(y_0, y) \sum_{j=0}^{n} \tilde{C}_{jk}(y_0, y) v_j.$$

4.2.4 A Class of Metrics on a Reference Manifold

Fix $Q_0 \in \mathcal{W}^{n+2}$ and let $M_0 = M(Q_0)$. Denote by $\mathcal{E}_{Q_0}$ the set of all C^ω-embeddings $G : M_0 \hookrightarrow \mathbb{R}^{n+2}$. Then for each $G \in \mathcal{E}_{Q_0}$, there is $Q \in \mathcal{W}^{n+2}$ such that $G(M_0) = M(Q)$. Moreover M_0 and $M(Q)$ are topologically equivalent. Equip $\mathcal{E}_{Q_0}$ with the topology induced by $\mathcal{W}^{n+2}$. Let δ denote the Euclidean metric on $M(Q)$. Then the embedding G defines a Riemannian metric

$$g = G^* \delta \tag{4.20}$$

via the pullback of δ by the map G to M_0.

This construction allows us to draw comparison between orbits on different (topologically equivalent) hypersurfaces $M(Q_j)$ by projecting them onto M_0 using the corresponding map G_j. Similar constructions were used in [111, 112].

4.3 Generic Nondegeneracy of Closed Geodesics

4.3.1 Perturbation-Invariant Phase Space and Proof of Theorem 4.1.1

Let $Q_0 \in \mathcal{W}^{n+2}$ and let $M_0 = M(Q_0)$ denote the reference manifold. Recall the definition of the space $\mathcal{E}_{Q_0}$ from Section 4.2.4. Let $G \in \mathcal{E}_{Q_0}$. Then there is $Q \in \mathcal{W}^{n+2}$ such that $G(M_0) = M$ where $M = M(Q)$. To make comparisons easier, recall that G defines a metric $g = g(G)$ on M_0 via (4.20).

Let $\phi_G^t : TM_0 \to TM_0$ denote the geodesic flow on M_0 corresponding to the metric $g = g(G)$. Since the dynamics of the geodesic flow is the same on every energy level, it is sufficient to perform analysis on one energy level, say the bundle $T^1 M_0$ of tangent vectors with norm 1. However, $T^1 M_0$ depends on the metric g, and therefore on the function $G \in \mathcal{E}_{Q_0}$, so it will change when we make perturbations. Define the sphere

bundle SM_0 of M_0 as follows: if $x \in M_0$, then $\tilde{u} \in S_x M_0$ if there is a nonzero tangent vector $u \in T_x M_0$ such that

$$\tilde{u} = \{\lambda u : \lambda \in (0, \infty)\} \tag{4.21}$$

is the ray emanating from the origin in the direction u. Let p denote the obvious projection from the set of nonzero tangent vectors in TM_0 to SM_0: if $0 \neq u \in TM_0$ then $p(u) = \tilde{u}$ where $\tilde{u}$ is defined as in (4.21). Denote by p_G the restriction of the projection p to $T^1 M_0$, and let $i_G = p_G^{-1}$. Consider the following maps.

$$\begin{array}{ccccc} TM_0 \setminus \{0\} & \xrightarrow{p} & SM_0 & \xrightarrow{i_G} & T^1 M_0 \\ \phi_G^t \downarrow & & \psi_G^t \downarrow & & \phi_G^t \downarrow \\ TM_0 \setminus \{0\} & \xrightarrow{p} & SM_0 & \xrightarrow{i_G} & T^1 M_0 \end{array}$$

Here we define $\psi_G^t = p \circ \phi_G^t \circ i_G$. Now the phase space SM_0 is independent of $G \in \mathcal{E}_{Q_0}$ (i.e. and $Q \in \mathcal{W}^{n+2}$).

Define

$$\begin{array}{rcl} F : SM_0 \times \mathbb{R}_{>0} \times \mathcal{E}_{Q_0} & \longrightarrow & SM_0 \times SM_0 \\ (z, t, G) & \longmapsto & (z, \psi_G^t(z)), \end{array}$$

and consider the diagonal set

$$\Delta = \{(z, z) : z \in SM_0\} \subset SM_0 \times SM_0.$$

Then $z \in SM_0$ is an L-periodic point of the flow ψ_G^t if and only if $F(z, L, G) \in \Delta$.

Lemma 4.3.1. *Suppose $F(z, L, G) \in \Delta$, and suppose either:*

1. *The orbit $\{\psi_G^t(z) : t \in [0, L]\}$ is nonparabolic; or*

2. *L is the minimal period of z with respect to the flow ψ_G^t.*

Then F is transverse to Δ at (z, L, G).

In the remainder of this section, it is shown how to deduce Theorem 4.1.1 from Lemma 4.3.1. Lemma 4.3.1 itself is proved in Section 4.3.2. The following arguments are adaptations of those in Section 4 of [7]. In the cases where the statements are entirely analogous to those of Anosov, the reader is asked to refer to [7] for the proof.

Lemma 4.3.2. *Let $G_0 \in \mathcal{E}_{Q_0}$. Then there is a neighbourhood $\mathcal{U}$ of G_0 in $\mathcal{E}_{Q_0}$ and $\alpha > 0$ such that the flow ψ_G^t has no periodic orbits with period less than or equal to α for all $G \in \mathcal{U}$.*

The proof of Lemma 4.3.2 is equivalent to the content of Section 4(c) in [7], so it is not included here.

For $0 < a \leq b$, let $\mathcal{B}_{Q_0}(a,b)$ denote the set of $G \in \mathcal{E}_{Q_0}$ for which every periodic orbit of ψ_G^t with period at most b and minimal period at most a is nonparabolic. Let $\mathcal{B}_{Q_0}(a) = \mathcal{B}_{Q_0}(a,a)$. Notice that if $0 < a_1 \leq b_1$ and $0 < a_2 \leq b_2$ where $a_1 \leq a_2$ and $b_1 \leq b_2$ then

$$\mathcal{B}_{Q_0}(a_2,b_2) \subseteq \mathcal{B}_{Q_0}(a_1,b_1).$$

The goal is to show that $\mathcal{B}_{Q_0}(a)$ is open and dense in $\mathcal{E}_{Q_0}$. It turns out that openness follows directly from the argument in [7], and so we get:

Lemma 4.3.3. *$\mathcal{B}_{Q_0}(a,b)$ is open in $\mathcal{E}_{Q_0}$.*

The proof of the following result is where Lemma 4.3.1 is used.

Lemma 4.3.4. *If $(z,L,G) \in SM_0 \times (0,2a) \times \mathcal{B}_{Q_0}(a,2a)$ and $F(z,L,G) \in \Delta$, then F is transverse to Δ at (z,L,G).*

Proof. If L is the minimal period then the lemma follows from part 2 of Lemma 4.3.1. If not, then the minimal period is at most $\frac{L}{2}$ which is strictly less than a since $L \in (0,2a)$. Therefore z is a periodic point of ψ_G^t with period at most $2a$ and minimal period at most a, so it is nonparabolic since $G \in \mathcal{B}_{Q_0}(a,2a)$. Then part 1 of Lemma 4.3.1 completes the proof. □

Lemma 4.3.5. *$\mathcal{B}_{Q_0}(\frac{3a}{2}) \cap \mathcal{B}_{Q_0}(a,2a)$ is dense in $\mathcal{B}_{Q_0}(a,2a)$.*

Proof. Lemma 4.3.4 implies that F restricted to $SM_0 \times (0,2a) \times \mathcal{B}_{Q_0}(a,2a)$ satisfies the assumptions of Abraham's transversality theorem. Therefore Theorem 2.3.3 implies that there is a dense set of embeddings $G \in \mathcal{B}_{Q_0}(a,2a)$ for which the map from $SM_0 \times (0,2a) \to SM_0 \times SM_0$ that sends $(z,t) \mapsto (z, \psi_G^t(z))$ is transverse to Δ. But for those G, all periodic orbits of ψ_G^t with period at most $2a$ are nonparabolic [4, 7, 47, 48, 49]. Therefore $G \in \mathcal{B}(2a) \subseteq \mathcal{B}_{Q_0}(\frac{3a}{2})$. □

Lemma 4.3.6. *Let $a \in (0,\infty)$, let $G \in \mathcal{B}_{Q_0}(a)$. Then ψ_G^t has only finitely many periodic orbits of period at most a.*

Proof. Let $Q \in \mathcal{W}^{n+2}$ such that $G(M_0) = M$ where $M = M(Q)$, and let $\alpha > 0$ denote a lower bound on the lengths of closed geodesics on M, given by Lemma 4.3.2. Recall that $\phi^t : TM \to TM$ denotes the geodesic flow on M. Suppose the statement is not true: suppose $\{z_i\}_{i\in\mathbb{N}} \subset T^1M$ and $\{t_i\}_{i\in\mathbb{N}} \subset (\alpha, a]$ such that

$$\phi^{t_i}(z_i) = z_i,$$

and if $i \neq j$ then for all $t \in \mathbb{R}$ we have

$$z_i \neq \phi^t(z_j).$$

By passing to a convergent subsequence, we may assume that there is $z_0 \in T^1M$ and $t_0 \in [\alpha, a]$ such that $z_i \to z_0$ and $t_i \to t_0$ as $i \to \infty$. Then we have

$$\phi^{t_0}(z_0) = \lim_{i\to\infty} \phi^{t_i}(z_i) = \lim_{i\to\infty} z_i = z_0$$

and so z_0 is a periodic point of ϕ^t with period $t_0 \in (\alpha, a]$. Let $\gamma_i = \pi \circ \phi^t(z_i)$ denote the closed geodesic corresponding to the periodic point z_i for each $i \in \mathbb{N}$. Notice that γ_0 is nondegenerate since $G \in \mathcal{B}_{Q_0}(a)$. Let Σ denote a transverse section to γ_0 at some point θ_0. For i sufficiently large, γ_i is transverse to Σ. Let θ_i denote the point at which γ_i meets Σ. Then $\theta_i \neq \theta_j$ whenever $i \neq j$, and if $P : \Sigma \to \Sigma$ denotes the Poincaré map, then $P(\theta_i) = \theta_i$ for each i. Let

$$Y = \lim_{i\to\infty} \frac{\theta_i - \theta_0}{\|\theta_i - \theta_0\|} = \lim_{i\to\infty} \frac{P(\theta_i) - P(\theta_0)}{\|\theta_i - \theta_0\|}.$$

The limit exists because P is smooth. Moreover $\|Y\| = 1$, so $0 \neq Y \in T_{\theta_0}\Sigma$. Furthermore we have

$$D_{\theta_0}P(Y) = Y$$

by construction, and so Y is an eigenvector of $D_{\theta_0}P$ with eigenvalue 1. This implies that γ_0 is parabolic, which is a contradiction. □

Now, let $G_0 \in \mathcal{B}_{Q_0}(a)$. Lemma 4.3.6 implies that there is $N \in \mathbb{N}$ and $z_1, \ldots, z_N \in SM_0$ such that

$$\psi^t_{G_0}(z_0), \ldots, \psi^t_{G_0}(z_N)$$

are all of the periodic orbits of $\psi^t_{G_0}$ of period at most a. Let L_i denote the minimal period of z_i. The following lemma is a consequence of the arguments of Section 4(i) in [7].

Lemma 4.3.7. *There is a neighbourhood $\mathcal{U}$ of G_0 in $\mathcal{E}_{Q_0}$ and continuous functions*

$$z_i^* : \mathcal{U} \longrightarrow SM_0, \quad L_i^* : \mathcal{U} \longrightarrow \mathbb{R}$$

for each $i = 1, \ldots, N$ such that

$$z_i^*(G_0) = z_i, \quad L_i^*(G_0) = L_i$$

and $z_i^*(G)$ is a periodic point of ψ^t_G with period $L_i^*(G)$ for each $G \in \mathcal{U}$. Moreover,

$$\psi^t_G(z_1^*(G)), \ldots, \psi^t_G(z_N^*(G)) \tag{4.22}$$

are the only periodic orbits of ψ^t_G with period at most a for all $G \in \mathcal{U}$.

Lemma 4.3.8. *$\mathcal{B}_{Q_0}(a, 2a)$ is dense in $\mathcal{B}_{Q_0}(a)$.*

Proof. Let $G_0 \in \mathcal{B}_{Q_0}(a)$ and let $\mathcal{U}^*$ be a neighbourhood of G_0 in $\mathcal{B}_{Q_0}(a)$. We will show that $\mathcal{U}^* \cap \mathcal{B}_{Q_0}(a, 2a) \neq \emptyset$. Let $\mathcal{U} \subset \mathcal{U}^*$ be a neighbourhood of G_0 as in Lemma 4.3.7. By applying Theorem 4.1.2 to each of the trajectories (4.22), we can find G arbitrarily close to G_0 in $\mathcal{E}_{Q_0}$ such that each of these orbits is nonparabolic as a periodic orbit with period at most $2a$. Since $G \in \mathcal{U}$, all periodic orbits of ψ_G^t with minimal period at most a are in (4.22). Therefore $G \in \mathcal{B}_{Q_0}(a, 2a) \cap \mathcal{U}^*$. □

Proof of Theorem 4.1.1. Lemma 4.3.5 and Lemma 4.3.8 together imply that $\mathcal{B}_{Q_0}(\frac{3a}{2})$ is dense in $\mathcal{B}_{Q_0}(a)$. It follows that $\mathcal{B}_{Q_0}\left(\left(\frac{3}{2}\right)^i a\right)$ is dense in $\mathcal{B}_{Q_0}(a)$ for each $i \in \mathbb{N}$. Let $G_0 \in \mathcal{E}_{Q_0}$. By Lemma 4.3.2 there is $\alpha \in (0, a)$ such that $\psi_{G_0}^t$ has no periodic orbits of period less than or equal to α. This trivially implies that $G_0 \in \mathcal{B}_{Q_0}(\alpha)$. Choose $i \in \mathbb{N}$ such that

$$\alpha_* \coloneqq \left(\frac{2}{3}\right)^i a \leq \alpha$$

which implies that $G_0 \in \mathcal{B}_{Q_0}(\alpha_*)$ since $\mathcal{B}_{Q_0}(\alpha) \subseteq \mathcal{B}_{Q_0}(\alpha_*)$.

Now, $\mathcal{B}_{Q_0}(a) = \mathcal{B}_{Q_0}\left(\left(\frac{3}{2}\right)^i \alpha_*\right)$ is dense in $\mathcal{B}_{Q_0}(\alpha_*)$, so we can find G arbitrarily close to G_0 in $\mathcal{W}^{n+2}$ such that $G \in \mathcal{B}_{Q_0}(a)$. Therefore $\mathcal{B}_{Q_0}(a)$ is dense. By Lemma 4.3.3, it is also open.

Let $\mathcal{B}(a)$ denote the subset of $\mathcal{W}^{n+2}$ consisting functions Q for which there is $Q_0 \in \mathcal{W}^{n+2}$ and $G \in \mathcal{B}_{Q_0}(a)$ such that $G(M(Q_0)) = M(Q)$. Then $\mathcal{B}(a)$ is open and dense in $\mathcal{W}^{n+2}$. It follows that

$$\mathcal{B} = \bigcap_{a \in \mathbb{N}} \mathcal{B}(a)$$

is the residual set we are looking for. □

4.3.2 Proof of Lemma 4.3.1

Let $x \in M_0$ and $u \in T_x M_0$. Recall the vertical subspace $T_u^V T M_0$ of $T_u T M_0$ is

$$T_u^V T M_0 = T_u T_x M_0.$$

Let $G \in \mathcal{E}_{Q_0}$ and let $g = g(G)$ denote the corresponding Riemannian metric on M_0. Recall the horizontal subspace $T_u^H T M_0$ is the orthogonal complement of $T_u^V T M_0$ in $T_u T M_0$ with respect to g. Then we have

$$T_u T M_0 = T_u^H T M_0 \oplus T_u^V T M_0 \simeq T_x M_0 \oplus T_x M_0.$$

If $u_1, u_2 \in T_x M_0$, we let $(u_1; u_2) \in T_u T M_0$ denote the vector in the second tangent space defined by this correspondence.

Define

$$\begin{array}{rcl} \Phi : TM_0 \times \mathbb{R}_{>0} \times \mathcal{E}_{Q_0} & \longrightarrow & TM_0 \\ (\theta, t, G) & \longmapsto & \phi_G^t(\theta). \end{array}$$

Fix $(\theta, L, G) \in SM_0 \times \mathbb{R}_{>0} \times \mathcal{E}_{Q_0}$, and let $\bar{G} \in T_G\mathcal{E}_{Q_0}$, so there is a real-analytic family $G_\epsilon \subset \mathcal{E}_{Q_0}$ with $G_0 \equiv G$ and $\frac{d}{d\epsilon}\big|_{\epsilon=0} G_\epsilon = \bar{G}$. Define the curve

$$\gamma_\epsilon : [0, L+1] \longrightarrow M_0$$

by

$$\gamma_\epsilon(t) = \pi \circ \phi_{G_\epsilon}^t(\theta),$$

and write

$$\gamma_\epsilon(t) = \gamma(t) + \epsilon\bar{\gamma}(t) + O(\epsilon^2).$$

Lemma 4.3.9. *Let D_G denote the derivative with respect to G, and let $\nabla_t = \nabla_{\gamma'(t)}$ denote the covariant time derivative with respect to the Levi-Civita connection of the metric $g = g(G)$. Then we have*

$$D_G\Phi(\theta, L, G)(\bar{G}) = (\bar{\gamma}(L); \nabla_t\bar{\gamma}(L)).$$

Proof. Write $\theta = (x, u)$ where $x \in TM_0$ and $u \in T_xM_0$, and let $V \in T_uTM_0$. Let (U, φ) be a coordinate chart with $x \in U$ and write

$$\varphi(x) = y, \quad D\varphi(u) = (y, v), \quad D^2\varphi(V) = (y, v, w, z). \tag{4.23}$$

Let $\Gamma_{ijk}(\cdot) = \Gamma_{ijk}(\cdot, G)$ denote the Christoffel symbols with respect to $g = g(G)$ and the chart (U, φ), and let $\Gamma(y)$ denote the $(n+1)$-dimensional vector valued function with components

$$(\Gamma(y)(v, w))_i = \sum_{j,k=0}^{n} \Gamma_{ijk}(y)v_jw_k.$$

As pointed out in [6, 7] (see also [74]) we have $V = (Y; Z)$ where

$$D\varphi(Y) = (y, w), \quad D\varphi(Z) = (y, z + \Gamma(y)(v, w)). \tag{4.24}$$

Let us show that

$$(\gamma(t), \gamma'(t), \bar{\gamma}(t), \bar{\gamma}'(t)) = (\bar{\gamma}(t); \nabla_t\bar{\gamma}(t)). \tag{4.25}$$

If

$$(\gamma, \gamma', \bar{\gamma}, \bar{\gamma}') = (Y; Z)$$

then (4.23) and the first equation of (4.24) imply that

$$D\varphi(Y) = (\gamma, \bar{\gamma}),$$

and so $Y = \bar{\gamma}$. Moreover, (4.23) and the second equation of (4.24) give

$$\begin{aligned} D\varphi(Z) &= (\gamma, \bar{\gamma}' + \Gamma(\gamma)(\gamma', \bar{\gamma})) \\ &= (\gamma, \frac{d}{dt}\bar{\gamma} + \Gamma(\gamma)(\gamma', \bar{\gamma})) \end{aligned}$$

which implies that $Z = \nabla_t \bar{\gamma}$. This proves (4.25). Using (4.25) we see that

$$\begin{aligned} D_G\Phi(\theta, L, G)(\bar{G}) &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \Phi(\theta, t, G_\epsilon) \\ &= \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} \phi^t_{G_\epsilon}(\theta) = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} (\gamma_\epsilon(t), \gamma'_\epsilon(t)) \\ &= (\gamma(t), \gamma'(t), \bar{\gamma}(t), \bar{\gamma}'(t)) \\ &= (\bar{\gamma}(t); \nabla_t \bar{\gamma}(t)) \end{aligned}$$

which completes the proof of the lemma. □

Lemma 4.3.10. *Let $G \in \mathcal{E}_{Q_0}$ and let $g = g(G)$ denote the corresponding Riemannian metric on M_0. Let $x \in M_0$, and suppose $u \in T_xM_0$ is such that $g(x)(u,u) = 1$ and (x,u) is a periodic point of ϕ^t_G with minimal period L. Let $v, w \in T_xM_0$ such that*

$$g(x)(u,v) = g(x)(u,w) = 0. \tag{4.26}$$

Then there is $\bar{G} \in T_G\mathcal{E}_{Q_0}$ such that

$$D_G\Phi((x,u), L, G)(\bar{G}) = (v; w) \tag{4.27}$$

where D_G denotes the derivative with respect to G.

Proof. Let $\gamma(t) = \pi \circ \phi^t_G(x,u)$ denote the closed geodesic corresponding to the orbit of (x,u), and let $(y_0, y) = (y_0, y_1, \ldots, y_n)$ denote Fermi coordinates in a neighbourhood of γ so that $\gamma(t) = (t, 0, \ldots, 0)$. Now if $v, w \in T_xM_0$ such that (4.26) holds, then $v = (0, v_1, \ldots, v_n)$ and $w = (0, w_1, \ldots, w_n)$.

Let $Q \in \mathcal{W}^{n+2}$ such that $G(M_0) = M(Q) \equiv M$. Consider a real-analytic family $Q_\epsilon \subset \mathcal{W}^{n+2}$ with $Q_0 = Q$. Let $\gamma_\epsilon(t)$ denote the perturbed geodesic, and write

$$\gamma_\epsilon(t) = \gamma(t) + \epsilon\bar{\gamma}(t) + O(\epsilon^2).$$

Let

$$\bar{Q} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} Q_\epsilon \in T_Q\mathcal{W}^{n+2}. \tag{4.28}$$

By Lemma 4.3.9, if we can show that, by appropriate choice of real-analytic family Q_ϵ, any vector $(\bar{\gamma}_1(L), \ldots, \bar{\gamma}_n(L), (\bar{\gamma}_1)'(L), \ldots, (\bar{\gamma}_n)'(L)) \in \mathbb{R}^{2n}$ can be obtained, then the lemma is proved.

Suppose we make a localised perturbation $Q \to Q + \epsilon\psi$, and denote by $\bar{g}$ the term of order ϵ in the Taylor expansion of the perturbed metric. Then $\bar{\gamma}(t)$ is the solution of the initial value problem

$$\frac{d^2}{dt^2}\bar{\gamma}_i(t) + \sum_{k=1}^{n} R_{i0k0}(t,0)\bar{\gamma}_k(t) + \bar{\Gamma}_{i00}(t,0) = 0, \quad \bar{\gamma}(0) = \frac{d}{dt}\bar{\gamma}(0) = 0 \tag{4.29}$$

where R_{ijkl} is the Riemann curvature tensor, and

$$\bar{\Gamma}_{ijk} = \frac{1}{2}\left(\frac{\partial \bar{g}_{ij}}{\partial y_k} + \frac{\partial \bar{g}_{ik}}{\partial y_j} - \frac{\partial \bar{g}_{jk}}{\partial y_i}\right) \tag{4.30}$$

are the Christoffel symbols corresponding to the perturbation $\bar{g}$ of the metric [7].

Since γ is a closed geodesic, Lemma 2.7.3 implies that there is a time t for which $\kappa(\gamma(t), \gamma'(t)) \neq 0$. It follows that we must have $\kappa(\gamma(t), \gamma'(t)) \neq 0$ for all t in some interval $[a, b]$. We assume that the perturbation ψ is supported only when $y_0 \in (a, b)$. It follows from (4.17) that

$$\tilde{C}_{00}(y_0, 0) = \kappa(\gamma(y_0), \gamma'(y_0)) \neq 0$$

whenever $y_0 \in (a, b)$. For these values of y_0 and for small y, define

$$\psi(y_0, y) = \tilde{C}_{00}(y_0, y)^{-1} \sum_{j=1}^{n} y_j f_j(y_0) \tag{4.31}$$

where $f_1, \ldots, f_n$ are functions of y_0 that we will choose later. Moreover, we define ψ in such a way that it goes to 0 smoothly in a small neighbourhood of $\gamma((a, b))$. Combining (4.16), (4.30), and (4.31), we see that

$$\bar{\Gamma}_{000}(y_0, 0) = \frac{1}{2}\frac{\bar{g}_{00}}{\partial y_0}(y_0, 0) = 0$$

and

$$\bar{\Gamma}_{i00}(y_0, 0) = -f_i(y_0)$$

for $i = 1, \ldots, n$.

Since $\bar{g} \equiv 0$ along γ, we have $\bar{\gamma}_0 = 0$. Therefore we need only consider $\bar{\gamma} = (\bar{\gamma}_1, \ldots, \bar{\gamma}_n)$. Write

$$f(t) = (f_1(t), \ldots, f_n(t)), \quad R(t) = (R_{i0k0}(t, 0))_{i,k=1,\ldots,n}.$$

The initial value problem (4.29) can be written as

$$\frac{d}{dt}\begin{pmatrix} \bar{\gamma}(t) \\ \bar{\gamma}'(t) \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -R(t) & 0 \end{pmatrix}\begin{pmatrix} \bar{\gamma}(t) \\ \bar{\gamma}'(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \quad \begin{pmatrix} \bar{\gamma}(0) \\ \bar{\gamma}'(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $U(t)$ denote the fundamental matrix solution of this initial value problem, so we have

$$\frac{d}{dt}U(t) = \begin{pmatrix} 0 & I_n \\ -R(t) & 0 \end{pmatrix} U(t), \quad U(0) = I_{2n}.$$

Then $U(t)$ is a $2n \times 2n$ invertible matrix, and upon differentiating the identity $I_{2n} = U^{-1}(t)U(t)$ we see that its inverse satisfies

$$\frac{d}{dt}U^{-1}(t) = -U^{-1}(t) \begin{pmatrix} 0 & I_n \\ -R(t) & 0 \end{pmatrix}. \tag{4.32}$$

Choose some $t_0 \in (a, b)$ such that $\gamma(t_0)$ is not a point of self-intersection of γ. Let $\delta(t - t_0)$ denote the Dirac delta function, a generalised function that can be thought of heuristically as taking the 'value' ∞ at $t = t_0$ and 0 elsewhere. Suppose

$$f(t) = \alpha\delta(t - t_0) + \beta\delta'(t - t_0)$$

where $\alpha, \beta \in \mathbb{R}^n$. Then the equation for variation of parameters together with properties of the Dirac delta function and equation (4.32) implies

$$\begin{aligned} \begin{pmatrix} \bar{\gamma}(L) \\ \bar{\gamma}'(L) \end{pmatrix} &= U(L) \int_0^L U^{-1}(t) \begin{pmatrix} 0 \\ f(t) \end{pmatrix} dt = U(L) \int_{t_0}^L U^{-1}(t) \begin{pmatrix} 0 \\ f(t) \end{pmatrix} dt \\ &= U(L)U^{-1}(t_0) \begin{pmatrix} 0 \\ \alpha \end{pmatrix} - U(L)\frac{d}{dt}U^{-1}(t_0) \begin{pmatrix} 0 \\ \beta \end{pmatrix} \\ &= U(L)U^{-1}(t_0) \begin{pmatrix} 0 \\ \alpha \end{pmatrix} + U(L)U^{-1}(t_0) \begin{pmatrix} 0 & I_n \\ -R(t) & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \beta \end{pmatrix} \\ &= U(L)U^{-1}(t_0) \begin{pmatrix} \beta \\ \alpha \end{pmatrix}. \end{aligned}$$

Therefore, by appropriate choice of α, β we can obtain any vectors $\bar{\gamma}(L)$, $\bar{\gamma}'(L) \in \mathbb{R}^n$. However, δ is not a function, so instead choose some bump function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\operatorname{supp}(\varphi) \subseteq (-1, 1), \quad \int_{-\infty}^{\infty} \varphi(t)dt = 1.$$

For $\epsilon_0 > 0$ define $\varphi_{\epsilon_0}(t) = \frac{1}{\epsilon_0}\varphi\left(\frac{t}{\epsilon_0}\right)$. Then we have

$$\operatorname{supp}(\varphi_{\epsilon_0}) \subseteq (-\epsilon_0, \epsilon_0), \quad \int_{-\infty}^{\infty} \varphi_{\epsilon_0}(t)dt = 1.$$

Then $\varphi_{\epsilon_0} \to \delta$ as $\epsilon_0 \to 0$. For sufficiently small $\epsilon_0 > 0$, let $f(t) = \alpha\varphi_{\epsilon_0}(t - t_0) + \beta\varphi'_{\epsilon_0}(t - t_0)$. Then we can still obtain any vectors by varying α, β. Finally, approximate $Q + \epsilon\psi$ by a one-parameter real-analytic family Q_ϵ such that the term $\bar{Q}$ of order ϵ in the

expansion of Q_ϵ (see (4.28)) is sufficiently close to ψ along γ, along with its first derivative. Now, we can find a real-analytic family $G_\epsilon \subset \mathcal{E}_{Q_0}$ with the property that $G_\epsilon(M_0) = M(Q_\epsilon)$ for all ϵ. Let

$$\bar{G} = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} G_\epsilon.$$

Then $\bar{G} \in T_G\mathcal{E}_{Q_0}$ satisfies (4.27). □

The deduction of Lemma 4.3.1 from Lemma 4.3.10 is identical to the deduction of Lemma 1 of [7] from Lemma 2 of [7], and so it is not repeated here.

4.4 Generic Properties of k-Jets of Poincaré Maps

Let $Q \in \mathcal{W}^{n+2}$, and denote by $M = M(Q)$ the corresponding closed C^ω-smooth hypersurface of $\mathbb{R}^{n+2}$ equipped with the Euclidean metric. Let $\gamma : [0,1] \to M$ denote a nonconstant geodesic segment, and denote by $l = (\gamma, \gamma')$ the corresponding orbit segment of the geodesic flow. Let Σ_0, Σ_1 denote transverse sections to l in TM at $l(0), l(1)$ respectively, restricted to the energy level of l. Let $P_{Q,1} : \Sigma_0 \to \Sigma_1$ denote the corresponding Poincaré map.

We assume moreover that the normal curvature $\kappa(\gamma(t), \gamma'(t))$ is not identically zero, and $\gamma(0)$ is not a point of self-intersection of γ. It follows from (2.24) that we can find $0 < a < b < c < 1$ and a sufficiently small cylindrical neighbourhood W of the curve segment $\gamma([0,c])$ such that:

- $\kappa(\gamma(t), \gamma'(t)) \neq 0$ for all $t \in [a,b]$;
- $\gamma([0,1]) \cap W = \gamma([0,c])$; and
- W is contained in a neighbourhood of γ in which Fermi coordinates are valid, and the Fermi coordinates define a chart in W.

Notice that the third point implies that γ has no self-intersections in W. Let $k \in \mathbb{N}$ and $\mathcal{J} \subset J^k_s(n)$ be open, dense, and invariant.

Theorem 4.4.1. *There are $n_0 \in \mathbb{N}$ and C^∞-smooth locally-supported functions $\psi_j : \mathbb{R}^{n+2} \to \mathbb{R}$ for $j = 1, \ldots, n_0$ such that:*

1. *For each j, we have $\gamma(0), \gamma(1) \notin supp(\psi_j)$;*
2. *For each j, we have $supp(\psi_j) \cap M \subset W$;*

3. γ is still a geodesic on the perturbed manifold

$$M_\epsilon = M(Q + \psi_\epsilon)$$

where

$$\psi_\epsilon = \epsilon_1\psi_1 + \cdots + \epsilon_{n_0}\psi_{n_0}$$

for all sufficiently small values of $\epsilon = (\epsilon_1, \ldots, \epsilon_{n_0})$, in the sense that $\phi^t_{M_\epsilon}(\gamma(0), \gamma'(0)) = (\gamma(t), \gamma'(t))$ where $\phi^t_{M_\epsilon}$ denotes the geodesic flow on M_ϵ with respect to the Euclidean metric; and

4. There are arbitrarily small values of the parameters $\epsilon = (\epsilon_1, \ldots, \epsilon_{n_0})$ for which the k-jet $J^k_{l(0)}P_{Q+\psi_\epsilon,1}$ of the Poincaré map of γ on the perturbed manifold M_ϵ lies in the set $\mathcal{J}$.

This section is devoted to the proof of Theorem 4.4.1. Theorem 4.1.2 follows immediately from Theorem 4.4.1, since we can approximate $Q + \psi_\epsilon$ sufficiently well by a real-analytic family $Q_\epsilon = Q_{(\epsilon_1,\ldots,\epsilon_{n_0})}$ where $Q_0 \equiv Q$. Then we can find arbitrarily small values of the n_0 parameters ϵ for which the k-jet of the Poincaré map of the perturbed geodesic $\tilde{\gamma}$ on the manifold $M(Q_\epsilon)$ lies in $\mathcal{J}$.

4.4.1 Effect of a Perturbation on the k-Jet

Let (y_0, y) denote Fermi coordinates adapted to γ, and let (v_0, v) denote the corresponding tangent coordinates, as described in Section 4.2.2. Therefore (4.11) is the equation of l. For each $t \in [0, 1]$ let

$$\Sigma(t) = \{(y_0, y, v_0, v) : y_0 = t\}.$$

Then $\Sigma(t)$ is a smooth family of transverse sections to l in TM. Suppose now we make a perturbation as in (4.1) with ψ supported near γ. Let

$$P_{Q+\epsilon\psi,t} : \Sigma(0) \longrightarrow \Sigma(t) \tag{4.33}$$

denote the Poincaré map (in the above Fermi coordinates) corresponding to the geodesic flow on $M(Q+\epsilon\psi)$ (i.e. $P_{Q,t}$ denotes the unperturbed Poincaré map from $\Sigma(0)$ to $\Sigma(t)$). Define the map

$$R_{Q,\epsilon\psi,t} : \Sigma(0) \longrightarrow \Sigma(0) \tag{4.34}$$

by

$$R_{Q,\epsilon\psi,t} = P^{-1}_{Q,t} \circ P_{Q+\epsilon\psi,t}.$$

Remark 4.4.2. *The maps $P_{Q+\epsilon\psi,t}$ and $R_{Q,\epsilon\psi,t}$ may not be defined on all of $\Sigma(0)$, so we restrict them to a neighbourhood of the point $l(0)$ in $\Sigma(0)$ where they are defined, and keep the notation as in* (4.33) *and* (4.34) *for convenience.*

Let $\bar{X}$ denote the vector field of the perturbation as in (4.19), and assume:

(i) $J^{k-1}_{l(t)}\bar{X} = 0$ for all $t \in [0,1]$;

(ii) $l(0), l(1) \notin \operatorname{supp}\left(\bar{X}\right)$; and

(iii) $\bar{X}\big|_{\Sigma(t)}$ is tangent to $\Sigma(t)$ for all $t \in [0,1]$.

Consider the nonautonomous vector field

$$\bar{X}_t = P^*_{Q,t}\left(\bar{X}|_{\Sigma(t)}\right)$$

defined as the pullback to $\Sigma(0)$ under the unperturbed Poincaré map $P_{Q,t}$ of the restriction of $\bar{X}$ to the transverse section $\Sigma(t)$. Let $\phi^t_{\epsilon\bar{X}_t}$ denote the time-t shift along orbits of the flow of $\epsilon\bar{X}_t$. The following result relating the k-jet of this flow with the k-jet of $R_{Q,\epsilon\psi,t}$ is analogous to Proposition 2.1 in [75].

Proposition 4.4.3. $J^k_{l(0)}R_{Q,\epsilon\psi,t} = J^k_{l(0)}\phi^t_{\epsilon\bar{X}_t} + O(\epsilon^2)$ *for each* $t \in [0,1]$.

For each $t \in [0,1]$ define

$$\tilde{\Sigma}(t) = \{(y_0, y, v_0, v) : y_0 = t, v_0 = 1\}. \tag{4.35}$$

Clearly $\tilde{\Sigma}(t) \subset \Sigma(t)$. However, the manifold $\tilde{\Sigma}(0)$ is not necessarily invariant under $R_{Q,\epsilon\psi,t}$. The following lemma was proved in [75] (Lemma 3.1 and Remark 2.3).

Lemma 4.4.4. $P^{-1}_{Q,t}(\tilde{\Sigma}(t))$ *and* $\Sigma(0)$ *are tangent to order k at $l(0)$. Consequently, $\bar{X}_t$ is tangent to $\tilde{\Sigma}(0)$ with respect to k-jets, and so $\tilde{\Sigma}(0)$ and $R_{Q,\epsilon\psi,t}(\tilde{\Sigma}(0))$ have a tangency of order k at $l(0)$.*

Define the perturbation space $\mathcal{P}^k$ as the space of C^∞-smooth functions $\psi : \mathbb{R}^{n+2} \to \mathbb{R}$ such that $M(Q+\epsilon\psi)$ is a C^∞-smooth closed hypersurface of $\mathbb{R}^{n+2}$ for all sufficiently smooth values of ϵ, and such that, near γ, ψ takes the form

$$\psi(y_0, y) = \alpha(y_0)\beta(y)$$

such that:

- α is a real-valued function in y_0 with

$$\operatorname{supp}(\alpha) \subset (a, b);$$

- β is a real-valued function in $y = (y_1, \ldots, y_n)$ with

$$J_0^k \beta = 0$$

for each $t \in [0,1]$, and β is supported sufficiently close to the origin so that

$$\mathrm{supp}(\psi) \cap M \subset W.$$

Notice that the restriction of the symplectic form ω to $\tilde{\Sigma}(t)$ is a symplectic form, and so it makes sense to discuss Hamiltonian functions and Hamiltonian vector fields on $\tilde{\Sigma}(t)$.

Proposition 4.4.5. *Let $\psi \in \mathcal{P}^k$ where $\psi(y_0, y) = \alpha(y_0)\beta(y)$. Then $J^k_{l(0)} R_{Q,\epsilon\psi,t}$ is equal (up to terms of order ϵ^2) to the k-jet at $l(0)$ of the time-t shift along orbits of the flow of the nonautonomous Hamiltonian function*

$$\epsilon \left[\tilde{H}_t\right] \circ P_{Q,t} : \tilde{\Sigma}(0) \longrightarrow \mathbb{R}$$

where

$$\left[\tilde{H}_t\right](y) = \kappa(\gamma(t), \gamma'(t))\alpha(t)\beta(y)$$

is a one-parameter family (with parameter $t \in [0,1]$, but supported only when $t \in [a,b]$) of homogeneous polynomials of degree $k+1$ in $y = (y_1, \ldots, y_n)$ that is entirely determined by our choice of $\psi \in \mathcal{P}^k$ and the normal curvature along γ. Moreover, any one-parameter family of homogeneous polynomials of degree $k+1$ in $y = (y_1, \ldots, y_n)$ with t-support in (a,b) occurs for some $\psi \in \mathcal{P}^k$.

Proof. Suppose we make the perturbation $Q \to Q + \epsilon\psi$ with $\psi \in \mathcal{P}^k$. Recall from equation (4.18) that the Hamiltonian of the perturbation is

$$\bar{H}(y_0, y, v_0, v) = \psi(y_0, y) \sum_{i,j=0}^{n} \tilde{C}_{ij}(y_0, y) v_i v_j$$

where

$$\tilde{C}_{ij} = \sum_{k,l=1}^{n+2} \frac{\partial^2 Q}{\partial x_k \partial x_l} \frac{\partial x_k}{\partial y_i} \frac{\partial x_l}{\partial y_j}.$$

Let

$$\tilde{H}_t = \bar{H}\big|_{\tilde{\Sigma}(t)}$$

so that

$$\tilde{H}_t(y, v) = \psi(t, y) \left[\tilde{C}_{00}(t, y) + 2 \sum_{j=1}^{n} \tilde{C}_{0j}(t, y) v_j + \sum_{i,j=1}^{n} \tilde{C}_{ij}(t, y) v_i v_j \right].$$

Recall that $l(t) = (\gamma(t), \gamma'(t))$ where

$$\gamma(t) = (t, 0, \ldots, 0), \quad \gamma'(t) = (1, 0, \ldots, 0).$$

It follows from this and the definition of $\mathcal{P}^k$ that

$$J^k_{\gamma(t)}\psi = \alpha(t) J^k_0 \beta = 0,$$

and

$$J^{k+1}_{\gamma(t)}\psi = \alpha(t) J^{k+1}_0 \beta.$$

Now, by (4.17), the $(k+1)$-jet of $\tilde{H}_t$ is

$$J^{k+1}_{l(t)}\tilde{H}_t = \kappa(\gamma(t), \gamma'(t))\alpha(t) J^{k+1}_0 \beta = J^{k+1}_{l(0)}\left[\tilde{H}_t\right].$$

We have

$$0 \neq \kappa(\gamma(t), \gamma'(t))$$

for all $t \in [a, b]$ by assumption. Since along γ, α can be any smooth function in y_0 and β can be any smooth function in $y = (y_1, \ldots, y_n)$ with vanishing k-jet, we can obtain any one-parameter family of homogeneous polynomials of degree $k+1$ in $y = (y_1, \ldots, y_n)$ with t-support in (a, b) by varying our choice of $\psi \in \mathcal{P}^k$.

Notice that perturbations $\psi \in \mathcal{P}^k$ give rise to perturbative vector fields $\bar{X}$ satisfying assumptions (i)-(iii) of Proposition 4.4.3. Therefore, combining Proposition 4.4.3 and Lemma 4.4.4, we find that

$$J^k_{l(0)} R_{Q,\epsilon\psi,t} = J^k_{l(0)} \phi^t_{\epsilon\tilde{X}_t} + O(\epsilon^2)$$

where

$$\tilde{X}_t = P^*_{Q,t}\left(\bar{X}\big|_{\tilde{\Sigma}(t)}\right).$$

Since $P^*_{Q,t}$ is determined by the 1-jet of the symplectic map $P_{Q,t}$, and since $\tilde{H}_t$ is the Hamiltonian function with Hamiltonian vector field $\bar{X}\big|_{\tilde{\Sigma}(t)}$, the k-jet of $\tilde{X}_t$ is the k-jet of the Hamiltonian vector field with Hamiltonian function $\left[\tilde{H}_t\right] \circ P_{Q,t}$. □

4.4.2 k-General Position of Families of Poincaré Maps

Recall that the set $J^1_s(n)$ of 1-jets of symplectic automorphisms of $\mathbb{R}^{2n}$ that fix the origin is just the set $\mathrm{Sp}(2n, \mathbb{R})$ of real $2n \times 2n$ symplectic matrices, that is, matrices σ satisfying

$$\sigma^T \Omega \sigma = \Omega$$

where

$$\Omega = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

is the standard symplectic matrix.

Let $\mathbb{R}_k[y,v]$ denote the set of real homogeneous polynomials of degree k in $(y,v) = (y_1,\dots,y_n,v_1,\dots,v_n)$. This is a real vector space of dimension

$$N = \dim \mathbb{R}_k[y,v] = \binom{2n-1+k}{k}.$$

Definition 4.4.6. A vector of matrices $(\sigma_1,\dots,\sigma_N) \in \mathrm{Sp}(2n,\mathbb{R})^N$ is *k-general* if there are homogeneous polynomials $f_1,\dots,f_N \in \mathbb{R}_k[y]$ such that

$$\{f_1 \circ \sigma_1, \dots f_N \circ \sigma_N\}$$

forms a basis of $\mathbb{R}_k[y,v]$. Write

$$G_k = \left\{(\sigma_1,\dots,\sigma_N) \in \mathrm{Sp}(2n,\mathbb{R})^N : (\sigma_1,\dots,\sigma_N) \text{ is } k\text{-general}\right\}.$$

Proposition 4.4.7. *G_k is open and dense in $Sp(2n,\mathbb{R})^N$ for each $k \in \mathbb{N}$.*

Proof. Let $F(y,v) = y_1^k$. It is shown in [23] (Proposition 6) that the set

$$\left\{(\sigma_1,\dots,\sigma_N) \in \mathrm{Sp}(2n,\mathbb{R})^N : \{F\circ\sigma_1,\dots,F\circ\sigma_N\} \text{ is a basis of } \mathbb{R}_k[y,v]\right\}$$

is dense in $\mathrm{Sp}(2n,\mathbb{R})^N$. This set is contained in G_k, so G_k is dense. Since a sufficiently small perturbation of a basis is a basis, G_k is open. □

The following basic linear algebra result is the key step in passing from the C^∞ to the C^ω topology.

Lemma 4.4.8. *Let $N \in \mathbb{N}$ and let V be an N-dimensional vector space. Let $u_1,\dots,u_N$, $v_1,\dots,v_N \in V$ and $\epsilon^* > 0$ such that*

$$u_1 + \epsilon^* v_1, \dots, u_N + \epsilon^* v_N \tag{4.36}$$

is a basis of V. Then

$$u_1 + \epsilon v_1, \dots, u_N + \epsilon v_N \tag{4.37}$$

is a basis of V for all but a finite number of $\epsilon \in [0,\epsilon^*]$.

Proof. Consider the matrix

$$A(\epsilon) = [u_1 + \epsilon v_1 | \cdots | u_N + \epsilon v_N]$$

and the degree N polynomial

$$f(\epsilon) = \det(A(\epsilon)).$$

Then the vectors (4.37) form a basis of V if and only if $f(\epsilon) \neq 0$. Since (4.36) is a basis of V, we have $f(\epsilon^*) \neq 0$. Therefore f has at most N zeros in $[0,\epsilon^*]$, and so there are at most N values of $\epsilon \in [0,\epsilon^*]$ for which (4.37) is not a basis of V. □

Definition 4.4.9. Let $\{\sigma_t\}_{t\in[0,1]}$ be a one-parameter family of symplectic automorphisms of $\mathbb{R}^{2n}$ that fix the origin. This family is *k-general* if there are times $t_1, \ldots, t_N \in [0,1]$ such that

$$\left(J_0^1\sigma_{t_1}, \ldots, J_0^1\sigma_{t_N}\right) \in G_k.$$

Proposition 4.4.10. *Let the assumptions and notation be as in the statement of Theorem 4.4.1. Denote by (y_0, y) Fermi coordinates adapted to γ, and by (v_0, v) the corresponding tangent coordinates. Let the transverse sections $\tilde{\Sigma}(t)$ be defined as in (4.35). Then we can find $\psi \in \mathcal{P}^1$ such that the family of linearised Poincaré maps*

$$\left\{D_{l(0)}P_{Q+\epsilon\psi,t} : T_{l(0)}\tilde{\Sigma}(0) \longrightarrow T_{l(t)}\tilde{\Sigma}(t)\right\}_{t\in[0,1]}$$

is k-general for arbitrarily small values of ϵ where the times $t_1, \ldots, t_N$ in the definition of k-generality satisfy

$$a < t_1 < \cdots < t_N < b. \tag{4.38}$$

Proof. The easiest way to prove the proposition is to start with the analogous result for the C^∞ case and use Lemma 4.4.8 to pass to the C^ω case. By Proposition 4.4.7, k-generality is an open and dense property of 1-jets. It was shown by Stojanov and Takens that we can find an arbitrarily C^∞-small function $\tilde{\psi} \in \mathcal{P}^1$ such that the family of differentials

$$\left\{D_{l(0)}P_{Q+\tilde{\psi},t} : T_{l(0)}\tilde{\Sigma}(0) \to T_{l(t)}\tilde{\Sigma}(t)\right\}_{t\in[0,1]}$$

is k-general, where the times $t_1, \ldots, t_N$ in the definition of k-generality satisfy (4.38) [112]. Choose some small $\epsilon^* > 0$ and write

$$\tilde{\psi} = \epsilon^*\psi. \tag{4.39}$$

Then there are times $a < t_1 < \cdots < t_N < b$ and homogeneous polynomials $f_1, \ldots, f_N \in \mathbb{R}_k[y]$ such that

$$f_1 \circ \sigma_{1,\epsilon^*}, \ldots, f_N \circ \sigma_{N,\epsilon^*}$$

is a basis of $\mathbb{R}_k[y, v]$, where

$$\sigma_{j,\epsilon} = D_{l(0)}P_{Q+\epsilon\psi,t_j}$$

for $\epsilon \in [0, \epsilon^*]$. Momentarily ignoring the second order terms in ϵ^*, we may assume, shrinking ϵ^* and redefining (4.39) if necessary, that

$$f_1 \circ \sigma_1 + \epsilon^*\bar{f}_1, \ldots, f_N \circ \sigma_N + \epsilon^*\bar{f}_N$$

is a basis of $\mathbb{R}_k[y, v]$, where $\sigma_j = \sigma_{j,0}$ and

$$\bar{f}_j = \left.\frac{d}{d\epsilon}\right|_{\epsilon=0} f_j \circ \sigma_{j,\epsilon} \in \mathbb{R}_k[y, v].$$

By Lemma 4.4.8, there are at most finitely many $\epsilon \in (0, \epsilon_*)$ for which

$$f_1 \circ \sigma_1 + \epsilon \bar{f}_1, \ldots, f_N \circ \sigma_N + \epsilon \bar{f}_N$$

is not a basis of $\mathbb{R}_k[y, v]$. Then, since a sufficiently small perturbation of a basis is a basis,

$$\left\{ D_{l(0)} P_{Q+\epsilon\psi,t} : T_{l(0)}\tilde{\Sigma}(0) \longrightarrow T_{l(t)}\tilde{\Sigma}(t) \right\}_{t\in[0,1]}$$

is k-general for arbitrarily small values of ϵ, where the k-generality times satisfy (4.38). □

4.4.3 Proof of Theorem 4.4.1

Let the assumptions and notation be as in the statement of Theorem 4.4.1, with (y_0, y) Fermi coordinates, (v_0, v) the corresponding tangent coordinates, and the transverse sections $\tilde{\Sigma}(t)$ as defined in (4.35). By Proposition 4.4.10 we may assume without loss of generality (in the context of Theorem 4.4.1) that the family of Poincaré maps $\{D_{l(0)} P_{Q,t}\}_{t\in[0,1]}$ is $(m+1)$-general for each $m = 1, \ldots, k$, where the times in the definition of m-generality satisfy (4.38). Recall the definition of the space $\mathcal{P}^m$ of perturbations from Section 4.4.1, and consider the map

$$\begin{aligned} S_m : \mathcal{P}^m &\longrightarrow J^m_s(n) \\ \tilde{\psi} &\longmapsto J^m_{l(0)} R_{Q,\tilde{\psi},1}. \end{aligned}$$

Denote by

$$\pi_m : J^m_s(n) \longrightarrow J^{m-1}_s(n)$$

the projection by truncation. Then the kernel of π_m is the set of $\sigma \in J^m_s(n)$ for which $\pi_m(\sigma)$ is equal to the $(m-1)$-jet of the identity map on $\mathbb{R}^{2n}$. It is clear that the image of S_m is contained in the kernel of π_m.

Lemma 4.4.11. *For each $m = 1, \ldots, k$ we can find $\psi_{m,1}, \ldots, \psi_{m,N_m} \in \mathcal{P}^m$, where*

$$N_m = \dim \mathbb{R}_{m+1}[y, v]$$

such that the map

$$\begin{aligned} : \mathbb{R}^{N_m} &\longrightarrow \ker(\pi_m) \\ (\epsilon_1, \ldots, \epsilon_{N_m}) &\longmapsto S_m(\epsilon_1 \psi_{m,1} + \cdots + \epsilon_{N_m} \psi_{m,N_m}) \end{aligned}$$

is open in a neighbourhood of $0 \in \mathbb{R}^{N_m}$.

Proof. Fix some $m \in \{1, \ldots, k\}$. Since $\{D_{l(0)}P_{Q,t}\}_{t\in[0,1]}$ is $(m+1)$-general with times in (a, b), there are $f_1, \ldots, f_N \in \mathbb{R}_{m+1}[y]$ and times $a < t_1 < \cdots < t_N < b$ such that

$$f_1 \circ D_{l(0)}P_{Q,t_1}, \ldots, f_N \circ D_{l(0)}P_{Q,t_N} \tag{4.40}$$

is a basis of $\mathbb{R}_{m+1}[y, v]$ where

$$N = \dim \mathbb{R}_{m+1}[y, v].$$

As in the proof of Lemma 4.3.10, we will first use delta functions to perform computations, and then approximate the delta functions by C^∞-smooth locally-supported functions in $\mathcal{P}^m$ to complete the proof of the Lemma.

Since the normal curvature along the geodesic segment $\gamma([a, b])$ is nonvanishing, and since the times t_i are in (a, b), we have $\kappa(\gamma(t_i), \gamma'(t_i)) \neq 0$. For each $i = 1, \ldots, N$ let

$$\tilde{\psi}_i(y_0, y) = \kappa(\gamma(y_0), \gamma'(y_0))^{-1}\delta(y_0 - t_i)f_i(y).$$

Write

$$\tilde{\psi}_\epsilon = \sum_{i=1}^{N} \epsilon_i \tilde{\psi}_i$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$ takes values in a neighbourhood of $0 \in \mathbb{R}^N$. By Proposition 4.4.5, the m-jet at $l(0)$ of $R_{Q,\tilde{\psi}_\epsilon,1}$ is equal (up to terms of order ϵ^2) to the m-jet at $l(0)$ of the time-1 shift of the flow of the nonautonomous Hamiltonian function

$$\sum_{i=1}^{N} \epsilon_i \delta(t - t_i) f_i \circ P_{Q,t}.$$

Denote by

$$Y = \sum_{i=1}^{N} Y_i$$

the corresponding Hamiltonian vector field, with the summands Y_i being given by

$$Y_i = \epsilon_i \delta(t - t_i)\Omega\nabla\left(f_i \circ P_{Q,t}\right)$$

where Ω is the standard $2d \times 2d$ symplectic matrix. Let ϕ_Y^t denote the flow of the

vector field Y. Let $(y,v) \in \tilde{\Sigma}(0)$ and write $(y(t),v(t)) = \phi_Y^t(y,v)$. We have

$$\begin{aligned}
\phi_Y^1(y,v) &= (y,v) + \sum_{i=1}^N \epsilon_i \int_0^1 Y_i(y(t),v(t);t)dt \\
&= (y,v) + \sum_{i=1}^N \epsilon_i \int_0^1 \delta(t-t_i)\Omega\nabla(f_i \circ P_{Q,t})(y(t),v(t))dt \\
&= (y,v) + \\
&\quad + \sum_{i=1}^N \epsilon_i \int_0^1 \delta(t-t_i)\Omega\left[\nabla f_i(P_{Q,t}(y(t),v(t)))\right]^T D_{(y(t),v(t))}P_{Q,t}dt \\
&= (y,v) + \sum_{i=1}^N \epsilon_i\Omega\left[\nabla f_i(P_{Q,t_i}(y(t_i),v(t_i)))\right]^T D_{(y(t_i),v(t_i))}P_{Q,t_i}.
\end{aligned}$$

Let I denote the identity map on $\mathbb{R}^{2n}$. Since $J^m_{l(0)}f_i = 0$, the $(m-1)$-jet of ∇f_i at $l(0)$ is zero. Moreover, since $l(0)$ is an equilibrium of Y we find that

$$\begin{aligned}
J^m_{l(0)}R_{Q,\tilde{\psi}_\epsilon,1} &= J^m_{l(0)}\phi_Y^1 + O\left(\epsilon^2\right) \\
&= J^1_{l(0)}I + \sum_{i=1}^N \epsilon_i\Omega\left(J^m_{l(0)}\nabla f_i\right) \circ D_{l(0)}P_{Q,t_i} + O\left(\epsilon^2\right). \qquad (4.41)
\end{aligned}$$

Since any homogeneous polynomial of degree $(m+1)$ in (y,v) can be obtained as a linear combination of the polynomials (4.40), from (4.41) we can obtain the 1-jet of the identity plus any homogeneous m-jet of a symplectic automorphism of $\mathbb{R}^{2n}$ by varying $\epsilon = (\epsilon_1,\ldots,\epsilon_N)$. Therefore the map

$$(\epsilon_1,\ldots,\epsilon_N) \longmapsto S_m(\epsilon_1\tilde{\psi}_1 + \cdots + \epsilon_N\tilde{\psi}_N)$$

is open.

Now, choose a C^∞ bump function $\varphi : \mathbb{R} \to \mathbb{R}$ such that

$$\operatorname{supp}(\varphi) \subset [-1,1], \quad \int_{-\infty}^{\infty} \varphi(t)dt = 1.$$

For small $\epsilon_0 > 0$ let $\varphi_{\epsilon_0}(t) = \frac{1}{\epsilon_0}\varphi\left(\frac{t}{\epsilon_0}\right)$. Then

$$\operatorname{supp}(\varphi_{\epsilon_0}) \subset [-\epsilon_0,\epsilon_0], \quad \int_{-\infty}^{\infty} \varphi_{\epsilon_0}(t)dt = 1.$$

It follows that $\varphi_{\epsilon_0} \to \delta$ as $\epsilon_0 \to 0$. Fix some sufficiently small $\epsilon_0 > 0$ and let

$$\psi_i(y_0,y) = \kappa(\gamma(y_0),\gamma'(y_0))^{-1}\varphi_{\epsilon_0}(y_0 - t_i)f_i(y).$$

Since ϵ_0 is sufficiently small, we have

$$\operatorname{supp}(\psi_{i_1}) \cap \operatorname{supp}(\psi_{i_2}) = \emptyset$$

if $i_1 \neq i_2$, and moreover

$$\operatorname{supp}(\psi_i) \subset (a, b).$$

Therefore $\psi_1, \ldots, \psi_N \in \mathcal{P}^m$, and since a sufficiently small perturbation of a basis is a basis, we obtain that the map

$$(\epsilon_1, \ldots, \epsilon_N) \longmapsto S_m\left(\epsilon_1\psi_1 + \cdots + \epsilon_N\psi_N\right)$$

is open in a neighbourhood of the origin, as required. □

Proof of Theorem 4.4.1. Let $\mathfrak{X}^k$ denote the space of C^∞-smooth functions $\psi : \mathbb{R}^{n+2} \to \mathbb{R}$ of the form $\psi = \psi_1 + \cdots + \psi_N$ for some $N \in \mathbb{N}$ where

$$\psi_1, \ldots, \psi_N \in \mathcal{P}^1 \cup \cdots \cup \mathcal{P}^k,$$

and define the map

$$\begin{aligned} S : \mathfrak{X}^k &\longrightarrow J^k_s(n) \\ \psi &\longmapsto J^k_{l(0)} R_{Q,\psi,1}. \end{aligned}$$

Consider the functions $\{\psi_{m,i}\}^{i=1,\ldots,N_m}_{m=1,\ldots,k}$ found in Lemma 4.4.11. Relabel these functions as $\psi_1, \ldots, \psi_N$ where

$$N = \sum_{m=1}^{k} N_m.$$

By Lemma 4.4.11, the map : $\mathbb{R}^N \to J^k_s(n)$ defined by

$$(\epsilon_1, \ldots, \epsilon_N) \longmapsto S(\epsilon_1\psi_1 + \cdots + \epsilon_N\psi_N)$$

is open in a neighbourhood of $0 \in \mathbb{R}^N$. Since $\mathcal{J}$ is open and dense, there exist arbitrarily small values of $\epsilon = (\epsilon_1, \ldots, \epsilon_N)$ for which $J^k_{l(0)} P_{Q+\psi_\epsilon,1} \in \mathcal{J}$ where $\psi_\epsilon = \epsilon_1\psi_1 + \cdots + \epsilon_N\psi_N$. □

4.5 Transversality and the Kupka-Smale Theorem

Recall that a submanifold N of a symplectic manifold (TM, ω) is called *Lagrangian* if the restriction of ω to TN vanishes identically, and the dimension of N is maximal with respect to this property (i.e. $\dim N = \frac{1}{2}\dim TM$). A proof of the following lemma can be found in Appendix A of [30].

Lemma 4.5.1. *If a Lagrangian submanifold N is contained in an energy level $H^{-1}(E)$ of a Hamiltonian function H, then the Hamiltonian vector field associated with H is tangent to N.*

The following local perturbation result (which uses Lemma 2.7.3 in its statement) allows us to make a heteroclinic intersection transverse in a neighbourhood of a heteroclinic point using locally-supported perturbations. In practice, we will make these locally-supported perturbations and then approximate by a real-analytic family to ensure we remain in the space $\mathcal{W}^{n+2}$. This lemma is an adaptation of Lemma 2.6 in [30].

Lemma 4.5.2. *Let $Q \in \mathcal{W}^{n+2}$ and $M = M(Q)$, and suppose γ, η are hyperbolic closed geodesics on M (where γ is allowed to equal η). Choose t_0 such that*

$$\kappa(\gamma(t_0), \gamma'(t_0)) \neq 0 \tag{4.42}$$

and let $\mathcal{U}$ be a sufficiently small open neighbourhood of $(\gamma(t_0), \gamma'(t_0))$ in TM. Suppose $\theta \in W^u(\gamma) \cap W^s(\eta) \cap \mathcal{U}$ is such that the projection $\pi : TM \to M$ is a diffeomorphism in a neighbourhood of θ in $W^u(\gamma)$. Assume moreover that $\theta \in T^1M$ has unit length. Take any sufficiently small neighbourhoods $U_1 \subset U_2 \subset U_3$ of θ in $W^u(\gamma)$ such that $\overline{U}_3 \subset \mathcal{U}$. Then there is a C^∞-smooth function $\psi : \mathbb{R}^{d+2} \to \mathbb{R}$ such that

$$\mathrm{supp}(\psi) \cap M \subseteq \pi(U_3)$$

and, with $M_\epsilon = M(Q + \epsilon\psi)$, for all sufficiently small values of ϵ, the connected component of $W^u(\gamma) \cap U_1$ containing θ is transverse to $W^s(\eta)$.

Proof. Since $\mathcal{U}$ is siffuciently small, and since $\theta \in \mathcal{U}$, we may pass to Fermi coordinates $(y_0, y) = (y_0, y_1, \ldots, y_n)$ adapted to γ (see Section 4.2.2). Let $(v_0, v) = (v_0, v_1, \ldots, v_n)$ denote the corresponding tangent coordinates. Then we know that the geodesic flow is the flow of the Hamiltonian function

$$H(y_0, y, v_0, v) = \frac{1}{2} g(y_0, y)((v_0, v), (v_0, v)) = \frac{1}{2} \sum_{i,j=0}^{n} g_{ij}(y_0, y) v_i v_j.$$

Since $\pi|_{W^u(\gamma)}$ is a diffeomorphism in a neighbourhood of θ, the connected component W of $W^u(\gamma) \cap U_3$ containing θ is

$$W = \mathrm{graph}(v^*) = \{((y_0, y), v^*(y_0, y)) : (y_0, y) \in \pi(W)\}$$

where $v^* = (v_0^*, v_1^*, \ldots, v_n^*)$. A result of Robinson implies that there is a C^∞ smooth potential V (that does not necessarily correspond to a geodesic flow) such that

$$\mathrm{supp}(V) \subseteq U_2$$

and if we write

$$\tilde{H}_\epsilon = H + \epsilon V$$

then the perturbed unstable manifold $\tilde{W}^u_\epsilon(\gamma)$ corresponding to the Hamiltonian flow of $\tilde{H}_\epsilon$ is transverse to the unperturbed stable manifold $W^s(\eta)$ in the neighbourhood U_1 of θ for arbitrarily small values[4] of the parameter [105]. Since $W^s(\eta)$ is unchanged up to the support of the perturbation, the new stable and unstable manifolds are transverse.

Let W_ϵ denote the connected component of $\tilde{W}^u_\epsilon(\gamma) \cap U_3$ containing θ. If ϵ is small enough then $\pi|_{W_\epsilon}$ is still a diffeomorphism, so we can write

$$W_\epsilon = \text{graph}(v^* + \epsilon\bar{v}^*) = \{((y_0, y), v^*(y_0, y) + \epsilon\bar{v}^*(y_0, y)) : (y_0, y) \in \pi(W_\epsilon)\}$$

where $\bar{v}^*$ may depend on ϵ. Now make the perturbation $Q \to Q + \epsilon\psi$ and consider the perturbed metric

$$g_\epsilon = g + \epsilon\bar{g} + O(\epsilon^2)$$

where

$$\bar{g}(y_0, y) = 2\psi(y_0, y)\tilde{C}(y_0, y)$$

and

$$\tilde{C}_{ij}(y_0, y) = \sum_{k,l=1}^{n+2} \frac{\partial^2 Q}{\partial x_k \partial x_l}(y_0, y)\frac{\partial x_k}{\partial y_i}(y_0, y)\frac{\partial x_l}{\partial y_j}(y_0, y).$$

Write

$$\tilde{\kappa}((y_0, y), (v_0, v)) = \sum_{i,j=0}^{n} \tilde{C}_{ij}(y_0, y)v_i v_j.$$

Due to (4.17) and (4.42) we have

$$\tilde{\kappa}((t_0, 0), (1, 0)) = \tilde{C}_{00}(t_0, 0) = \kappa(\gamma(t_0), \gamma'(t_0)) \neq 0.$$

Since $\mathcal{U}$ is a sufficiently small neighbourhood of $(\gamma(t_0), \gamma'(t_0))$ we have

$$\tilde{\kappa}((y_0, y), (v_0, v)) \neq 0 \tag{4.43}$$

for all $((y_0, y), (v_0, v)) \in \mathcal{U}$.

The Hamiltonian of the perturbed geodesic flow is

$$H_\epsilon(y_0, y, v_0, v) = \frac{1}{2}g(y_0, y)((v_0, v), (v_0, v)) + \frac{1}{2}\epsilon\bar{g}(y_0, y)((v_0, v), (v_0, v)) + O(\epsilon^2).$$

On the manifold W_ϵ we have

$$\begin{aligned}
&g_\epsilon(y_0, y)(v^*(y_0, y) + \epsilon\bar{v}^*(y_0, y), v^*(y_0, y) + \epsilon\bar{v}^*(y_0, y)) = \\
&= g(y_0, y)(v^*(y_0, y) + \epsilon\bar{v}^*(y_0, y), v^*(y_0, y) + \epsilon\bar{v}^*(y_0, y)) + \\
&\quad + \epsilon\bar{g}(v^*(y_0, y), v^*(y_0, y)) + O(\epsilon^2) \\
&= g(y_0, y)(v^*(y_0, y), v^*(y_0, y)) + \\
&\quad + \epsilon\left[2\psi(y_0, y)\tilde{\kappa}((y_0, y), v^*(y_0, y)) + 2g(y_0, y)(v^*(y_0, y), \bar{v}^*(y_0, y))\right] + O(\epsilon^2).
\end{aligned}$$

[4] In the proof of Claim a, Theorem 3 of [105], c is the small parameter. The claim is not proved explicitly in that paper, but it is equivalent to the proof of Claim a of Theorem 1 of [105].

Notice that the term of order ϵ vanishes if we let

$$\psi(y_0, y) = \begin{cases} -\frac{g(y_0,y)(v^*(y_0,y),\bar{v}^*(y_0,y))}{\tilde{\kappa}((y_0,y),v^*(y_0,y))} & \text{if } (y_0, y) \in \pi(U_2) \\ 0 & \text{if } (y_0, y) \notin \pi(U_3) \end{cases}$$

with a smooth transition on $\pi(U_3) \setminus \pi(U_2)$. This is well-defined due to (4.43). Consider the truncated perturbed Hamiltonian

$$H^*_\epsilon = H + \epsilon \bar{H}$$

where we have removed the terms of order ϵ^2. On the manifold W_ϵ we have

$$H^*_\epsilon((y_0, y), v^*(y_0, y) + \epsilon \bar{v}^*(y_0, y)) = H((y_0, y), v^*(y_0, y)) = \frac{1}{2}$$

since

$$W = \text{graph}(v^*) \subset T^1M.$$

It follows that

$$W_\epsilon \subset (H^*_\epsilon)^{-1}\left(\frac{1}{2}\right).$$

It is well-known that the stable and unstable manifolds of a hyperbolic periodic orbit of a Hamiltonian flow are Lagrangian submanifolds. Indeed, the Hamiltonian flow preserves the symplectic form and contracts vectors on the stable manifold, which is only possible if the symplectic form vanishes identically on the stable manifold – similarly for the unstable manifold. Therefore, by Lemma 4.5.1, the Hamiltonian vector field of H^*_ϵ is tangent to W_ϵ. It follows that W_ϵ is a piece of the unstable manifold of γ with respect to the flow of H^*_ϵ. Moreover, W_ϵ is transverse to $W^s(\eta)$. Since H_ϵ is $O(\epsilon^2)$-close to H^*_ϵ, the unstable manifold of γ is transverse to the stable manifold of η with respect to H_ϵ in the neighbourhood U_1 of θ. □

The following result is also useful. A proof can be found in [98] (Proposition 2.11).

Proposition 4.5.3. *Let $\phi^t : TM \to TM$ denote the geodesic flow, $\pi : TM \to M$ the canonical projection, and let $\theta \in TM$. Suppose $L \subset T_\theta TM$ is a Lagrangian subspace. Then the set*

$$\left\{t \in \mathbb{R} : D_\theta \phi^t(L) \cap \ker\left(D_{\phi^t(\theta)}\pi\right) \neq \{0\}\right\}$$

is discrete.

Remark 4.5.4. *In Lemma 4.5.2, it is assumed that the restriction of the projection $\pi : TM \to M$ to $W^u(\gamma)$ is a diffeomorphism in a neighbourhood of a point $\theta \in W^u(\gamma)$. Proposition 4.5.3 combined with the inverse function theorem tells us that this assumption fails to be true on an at most countable set of points on the orbit of θ. Therefore if we start at a point $\theta \in W^u(\gamma)$ where the projection is not a diffeomorphism,*

then we can find arbitrarily small t so that it is a diffeomorphism at $\phi^t(\theta)$, and so we can apply the lemma. Since transversality of stable and unstable manifolds is a property of orbits and not just points, if the perturbed invariant manifolds are transverse in a neighbourhood of $\phi^t(\theta)$, then they are transverse in a neighbourhood of θ.

Theorem 4.5.5. *Let $Q \in \mathcal{V}$ and $M = M(Q)$, and suppose γ, η are hyperbolic closed geodesics with a heteroclinic (or homoclinic in the case where $\gamma = \eta$) connection. Then there is $\tilde{Q}$ arbitrarily close to Q in $\mathcal{V}$ such that γ, η are still hyperbolic closed geodesics on $M(\tilde{Q})$, and the connection between γ and η is transverse.*

Proof. Since $\gamma : \mathbb{T} \to M$ is a closed geodesic, there exists $t_0 \in \mathbb{T}$ such that $\kappa(\gamma(t_0), \gamma'(t_0)) \neq 0$, as a result of (2.24). Let $\mathcal{U}$ be a sufficiently small neighbourhood of $(\gamma(t_0), \gamma'(t_0))$ in TM, as in Lemma 4.5.2.

Suppose $W^u(\gamma) \cap W^s(\eta) \neq \emptyset$. Recall that a fundamental domain for the geodesic flow ϕ^t on $W^u(\gamma)$ is a subset of $W^u(\gamma)$ that intersects every orbit of $\phi^t|_{W^u(\gamma)}$ exactly once. We can find arbitrarily small fundamental domains arbitrarily close to γ. Let $K \subset W^u(\gamma) \cap \mathcal{U}$ be a sufficiently small compact set, sufficiently close to γ, such that K contains a fundamental domain of $\phi^t|_{W^u(\gamma)}$.

Let $\theta_1 \in K$, and suppose $\pi|_{W^u(\gamma)}$ is a diffeomorphism in a neighbourhood K_1 of θ_1, where K_1 is sufficiently small (i.e. K_1 plays the role of the set U_1 in Lemma 4.5.2). Then, by Lemma 4.5.2, there is a locally supported function ψ_1 such that for arbitrarily small values of ϵ, the connected component of $W^u(\gamma) \cap K_1$ containing θ_1 is transverse to $W^s(\eta)$ on the manifold $M(Q + \epsilon\psi_1)$.

If θ_1 does not have the property that $\pi|_{W^u(\gamma)}$ is a diffeomorphism in a neighbourhood of θ_1, then we can find some small time t such that $\phi^t(\theta_1) \in \mathcal{U}$ does have that property, by Proposition 4.5.3. Then let $\bar{K}_1$ denote a sufficiently small neighbourhood of $\phi^t(\theta_1)$ where we can apply Lemma 4.5.2 to find the appropriate perturbation ψ_1, localised near $\phi^t(\theta_1)$. Let $K_1 = \phi^{-t}(\bar{K}_1)$. Then for arbitrarily small values of ϵ we get transversality of $W^s(\eta)$ and $W^u(\gamma)$ in the set $K_1 \cap M(Q + \epsilon\psi_1)$ (see Remark 4.5.4).

Since K is compact, there is $n_0 \in \mathbb{N}$ such that for $j = 1, \ldots, n_0$ we have a neighbourhood K_j of θ_j in $W^u(\gamma)$ and locally supported functions ψ_j such that $W^s(\eta)$ and $W^u(\gamma)$ are transverse in K_j on the manifold $M(Q + \epsilon\psi_j)$, and

$$K \subset \bigcup_{j=1}^{n_0} K_j.$$

We first make a sufficiently small perturbation $Q \to Q + \epsilon_1\psi_1$ to get transversality on K_1. Since the property of $W^s(\eta)$ and $W^u(\gamma)$ being transverse in K_1 is open, we may then find sufficiently small ϵ_2 such that if we make the perturbation $Q + \epsilon_1\psi_1 \to Q + \epsilon_1\psi_1 + \epsilon_2\psi_2$, we obtain transversality of $W^s(\eta)$ and $W^u(\gamma)$ in K_2 without destroying

the transversality in K_1. Repeating this process n_0 times, each time taking care not to destroy transversality in the previous neighbourhood K_j, we find that there are arbitrarily small values of the parameters $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ such that if

$$\tilde{Q}_\epsilon = Q + \sum_{j=1}^{n_0} \epsilon_j \psi_j,$$

then the stable manifold of η is transverse to the unstable manifold of γ on the manifold $M(\tilde{Q}_\epsilon)$. We can now approximate $\tilde{Q}_\epsilon$ arbitrarily well by an n_0-parameter real-analytic family Q_ϵ. If the approximation is sufficiently good in the C^4-topology, then η, γ are still hyperbolic closed geodesics on the manifold $M(Q_\epsilon)$, and $W^s(\eta), W^u(\gamma)$ are still transverse for arbitrarily small values of ϵ. Therefore we can take ϵ small enough to find $\tilde{Q}$ arbitrarily close to Q in $\mathcal{W}^{n+2}$ for which we get the desired transversality property. □

Proof of Theorem 4.1.5. Let $N \in \mathbb{N}$ and let $\mathcal{H}(N)$ denote the set of $Q \in \mathcal{W}^{n+2}$ for which the k-jet of the Poincaré map of every closed geodesic of length at most N in $M(Q)$ lies in $\mathcal{J}$. Intersect $\mathcal{J}$ with the set of k-jets of symplectic automorphisms of $\mathbb{R}^{2n}$ whose linearisation at the origin does not have an eigenvalue equal to 1. Then $\mathcal{J}$ is still open, dense, and invariant. On such a manifold $M(Q)$ there can be only finitely many periodic orbits of length at most N (see Lemma 4.3.6). Therefore $\mathcal{H}(N)$ is open since $\mathcal{J}$ is. By Lemma 4.3.3 and Lemma 4.3.8, there is an open and dense set $\mathcal{B}(N) \subset \mathcal{W}^{n+2}$ such that for all $Q \in \mathcal{B}(N)$, every closed geodesic of length at most N on $M(Q)$ is nondegenerate. By Lemma 4.3.6, for each Q in $\mathcal{B}(N)$, the manifold $M(Q)$ admits only finitely many closed geodesics of length at most N.

Now, let $Q_0 \in \mathcal{W}^{n+2}$. Arbitrarily close to Q_0 we can find $Q_1 \in \mathcal{B}(N)$ since $\mathcal{B}(N)$ is dense. Then there are only finitely many closed geodesics of length at most N on $M(Q_1)$. Therefore we can apply Theorem 4.1.2 to each of these closed geodesics to find some Q_2 arbitrarily close to Q_1 such that the k-jet of the Poincaré map of every closed geodesic of length at most N in $M(Q_2)$ is in $\mathcal{J}$. Therefore $Q_2 \in \mathcal{H}^k(N)$. Since Q_2 is arbitrarily close to Q_0, this proves that $\mathcal{H}(N)$ is dense. This proves part (i) of Theorem 4.1.5.

Let $\mathcal{K}(N)$ denote the set of $Q \in \mathcal{H}(N)$ such that if γ, η are hyperbolic closed geodesics on $M(Q)$ of length at most N, then $W^s(\eta), W^u(\gamma)$ are transverse. Since transversality is an open property, $\mathcal{K}(N)$ is open. It remains to prove that $\mathcal{K}(N)$ is dense.

Let $Q \in \mathcal{H}(N)$, and let γ, η be hyperbolic closed geodesics on $M(Q)$ of length at most N. If there is no heteroclinic connection (or homoclinic if $\gamma = \eta$) between γ and η, or if any such connection is transverse, there is nothing to prove. Otherwise,

there is a nontransverse intersection. We may then apply Theorem 4.5.5 to make this intersection transverse.

Repeating this process for each of the finitely many pairs of hyperbolic closed geodesics of length at most N completes the proof of density of $\mathcal{K}(N)$. It follows that

$$\mathcal{K} = \bigcap_{N \in \mathbb{N}} \mathcal{K}(N)$$

is the residual set we are looking for. □

4.6 Generic Existence of Transverse Homoclinic Geodesics on Surfaces

In this section it is shown that the geodesic flow with respect to the Euclidean metric on real-analytic, closed, and strictly convex surfaces in $\mathbb{R}^3$ generically (i.e. for a C^ω open and dense set of such surfaces) has a transverse homoclinic orbit as a result of Theorems 4.1.5 and 4.1.6, a theorem of Mather [85], and an argument of Knieper and Weiss [77], thus proving Theorem 4.1.7. Since a surface $M = M(Q)$ having a transverse homoclinic orbit for its geodesic flow is a C^4-open property of $Q \in \mathcal{V}^3$, it is required to show only that it is C^ω-dense.

Theorem 4.1.6 implies that any closed strictly convex analytic surface in $\mathbb{R}^3$ with a nonhyperbolic closed geodesic C^ω-generically has a transverse homoclinic to a hyperbolic closed geodesic. Moreover, Theorem 4.1.5 implies that, on C^ω-generic surfaces, all homoclinic and heteroclinic connections are transverse. Therefore in what follows we consider only the case where every closed geodesic is hyperbolic and every homoclinic or heteroclinic connection is transverse (the "Kupka-Smale condition").

Let $Q \in \mathcal{V}^3$, $M = M(Q)$ such that the above assumptions are satisfied, and recall that $\phi^t : TM \to TM$ denotes the geodesic flow with respect to the Euclidean metric on M. A variational construction due to Birkhoff implies the existence of a simple closed geodesic $\gamma : \mathbb{T} \to M$ which is referred to as the "minimax" geodesic [17]. Let $A = \mathbb{T} \times [0, \pi]$. Notice that the simple closed curve γ divides M into two hemispheres. Since M is strictly convex and γ is the minimax geodesic, a theorem of Birkhoff (Section VI, 10 of [17]) implies that there is a well-defined analytic function $\tau : \mathrm{Int}(A) \to \mathbb{R}$ defined as follows. If $(\varphi, y) \in \mathrm{Int}(A)$ and $x = \gamma(\varphi)$, then there is a uniquely defined unit tangent vector $u \in T^1_x M$ pointing into the northern hemisphere and making an angle y with $\gamma'(\varphi)$. Then $\tau(\varphi, y)$ is the minimum value of $t > 0$ for which $(\bar{x}, \bar{u}) = \phi^t(x, u)$ consists a point $\bar{x}$ on the curve γ and a unit tangent vector $\bar{u} \in T^1_{\bar{x}} M$ pointing again into the northern hemisphere. The function τ is then extended continuously to ∂A.

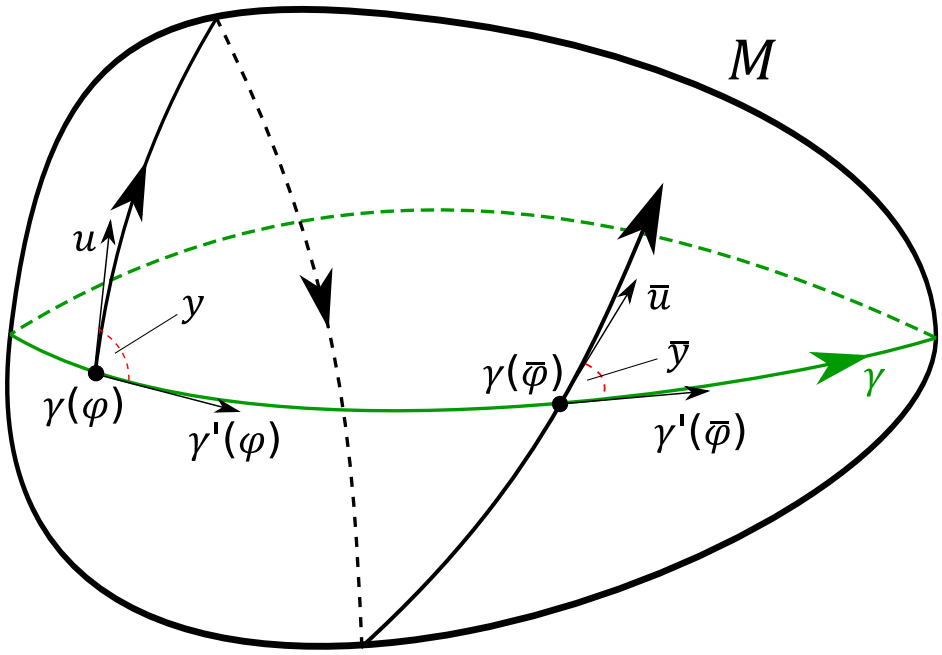


Figure 4.1: Birkhoff proved that if M is a strictly convex surface in $\mathbb{R}^3$, then a simple closed geodesic gives rise to a global Poincaré map $P : (\varphi, y) \mapsto (\bar{\varphi}, \bar{y})$.

This implies the existence of a global Poincaré map $P : A \to A$ defined as follows. If $(\varphi, y) \in A$ and (x, u) are as above, then let $(\bar{x}, \bar{u}) = \phi^{\tau(\varphi,y)}(x, u)$. Then there is $\bar{\varphi} \in \mathbb{T}$ such that $\bar{x} = \gamma(\bar{\varphi})$. Since $\bar{u}$ is pointing into the northern hemisphere, the angle $\bar{y}$ between $\bar{u}$ and $\gamma'(\bar{\varphi})$ is in $[0, \pi]$. Then $P(\varphi, y) = (\bar{\varphi}, \bar{y})$ (see Figure 4.1). Moreover P is analytic on $\mathrm{Int}(A)$ and continuous on ∂A.

Since M is topologically a 2-sphere, the theorem of the three closed geodesics implies that M has at least three geometrically distinct simple closed geodesics [13] (see also [74]). Let $\eta : \mathbb{T} \to M$ denote one of these simple closed geodesics, geometrically distinct from γ. This geodesic is hyperbolic by assumption, and therefore implies the existence of a hyperbolic fixed point $z \in \mathrm{Int}(A)$ for (an iterate of) the global Poincaré map P. The idea of the proof is to apply the following theorem of Mather [85].

Theorem 4.6.1 (Mather)**.** *Let $U \subseteq \mathbb{S}^2$ be an open set and $f : U \to f(U) \subseteq \mathbb{S}^2$ a homeomorphism. Suppose every fixed point of f is sectorial periodic or Moser stable, and suppose f has no fixed connections. Let O be a sectorial fixed point of f and let b_1, b_2 be any two stable or unstable branches of O. Then the closures of b_1 and b_2 coincide.*

Since the map must be defined on an open set, notice that we can extend P con-

tinuously to an open annulus A' containing A, for example by reflecting the dynamics of $P|_{\mathrm{Int}(A)}$ across ∂A.

The precise definitions of *sectorial periodic* and *Moser stable* fixed points can be found in [85], but sectorial periodic points can be thought of as a topological alternative to hyperbolic periodic points with one contracting and one expanding eigendirection. Indeed, Mather points out in [85] that hyperbolic fixed points of maps in dimension 2 with one expanding and one contracting eigendirection are sectorial periodic. Since all closed geodesics on M are hyperbolic, so too are all fixed points of P. Moreover, P is a symplectic map so the characteristic eigenvalues of hyperbolic fixed points are reciprocal pairs. Therefore every fixed point of P in $\mathrm{Int}(A)$ is sectorial periodic.

A *fixed connection* is a P-invariant arc ξ with fixed endpoints [85]. The existence of such an arc for P would imply the existence of a non-transverse homoclinic or heteroclinic connection, which is impossible due to the Kupka-Smale condition that we have assumed. It follows that P satisfies the assumptions of Mather's theorem, and so the closure of all 4 stable and unstable branches of the hyperbolic fixed point z coincide. Therefore the following result of Knieper and Weiss [77] applies.

Proposition 4.6.2 (Knieper and Weiss)**.** *Let A be a surface to which the Jordan curve theorem can be applied, and let $f : A \to A$ be a diffeomorphism with a hyperbolic fixed point z. Suppose the closure of a branch of $W^s(z)$ coincides with the closure of a branch of $W^u(z)$. Then the two branches have a topological crossing.*

The proposition implies that the stable and unstable manifolds of z intersect. By the Kupka-Smale condition, this intersection is transverse. The transverse homoclinic intersection of $W^s(z), W^u(z)$ for P implies the transverse homoclinic intersection of $W^s(\eta), W^u(\eta)$ for the geodesic flow, completing the proof of Theorem 4.1.7.

Bibliography

[1] R. Abraham. Transversality in manifolds of mappings. *Bull. Amer. Math. Soc*, 69(196):470–474, 1963.

[2] R. Abraham. Bumpy metrics. In *Global Analysis, Proc. Sympos. Pure Math*, volume 14, pages 1–3, 1970.

[3] R. Abraham and J. E. Marsden. *Foundations of mechanics*, volume 36. Benjamin/Cummings Publishing Company Reading, Massachusetts, 1978.

[4] R. Abraham and J. Robbin. *Transversal mappings and flows*. WA Benjamin New York, 1967.

[5] R. H. Abraham and S. Smale. *Lectures of Smale on differential topology*. Columbia University, Department of Mathematics, 1963.

[6] D. V. Anosov. Geodesic flows on closed riemannian manifolds of negative curvature. *Trudy Matematicheskogo Instituta Imeni VA Steklova*, 90:3–210, 1967.

[7] D. V. Anosov. On generic properties of closed geodesics. *Mathematics of the USSR-Izvestiya*, 21(1):1, 1983.

[8] M.-C. Arnaud. Type des points fixes des difféomorphismes symplectiques de $\mathbb{T}^n \times \mathbb{R}^n$. *Mém. Soc. Math. France (NS) No*, 48:63, 1992.

[9] V. Arnold. Proof of a theorem of A. N. Kolmogorov on the invariance of quasi-periodic motions under small perturbations of the Hamiltonian. *Russian Mathematical Surveys*, 18:9–36, 1963.

[10] V. I. Arnold. Instability of dynamical systems with many degrees of freedom. In *Dokl. Akad. Nauk SSSR*, volume 156, pages 9–12, 1964.

[11] V. I. Arnold. *Mathematical methods of classical mechanics*, volume 60. Springer, 1978.

[12] A. Avila, J. De Simoi, and V. Kaloshin. An integrable deformation of an ellipse of small eccentricity is an ellipse. *Annals of Mathematics*, pages 527–558, 2016.

[13] W. Ballmann. Der satz von Lusternik und Schnirelmann. *Mathematische Shriften*, 102:1–25, 1978.

[14] M. Berger. Seules les quadriques admettent des caustiques. *Bulletin de la Société Mathématique de France*, 123(1):107–116, 1995.

[15] M. Bialy. Convex billiards and a theorem by E. Hopf. *Mathematische Zeitschrift*, 214(1):147–154, 1993.

[16] G. D. Birkhoff. Dynamical systems with two degrees of freedom. *Proceedings of the National Academy of Sciences of the United States of America*, 3(4):314, 1917.

[17] G. D. Birkhoff. *Dynamical systems*, volume 9. American Mathematical Soc., 1927.

[18] G. D. Birkhoff. On the periodic motions of dynamical systems. *Acta Mathematica*, 50(1):359–379, 1927.

[19] S. Bolotin and D. Treschev. Unbounded growth of energy in nonautonomous Hamiltonian systems. *Nonlinearity*, 12(2):365, 1999.

[20] S. V. Bolotin and D. V. Treschev. Hill's formula. *Russian Mathematical Surveys*, 65(2):191, 2010.

[21] A. Bounemoura and J.-P. Marco. Improved exponential stability for near-integrable quasi-convex Hamiltonians. *Nonlinearity*, 24(1):97, 2010.

[22] H. Broer and F. Tangerman. From a differentiable to a real analytic perturbation theory, applications to the Kupka Smale theorems. *Ergodic Theory and Dynamical Systems*, 6(3):345–362, 1986.

[23] C. Carballo and J. Miranda. Jets of closed orbits of Mané's generic Hamiltonian flows. *Bulletin of the Brazilian Mathematical Society, New Series*, 44(2):219–232, 2013.

[24] C.-Q. Cheng and J. Yan. Existence of diffusion orbits in a priori unstable hamiltonian systems. *J. Differential Geom*, 67(3):457–517, 2004.

[25] N. Chernov and R. Markarian. *Chaotic billiards*, volume 127. American Mathematical Soc., 2006.

[26] A. Clarke. Generic properties of geodesic flows on analytic hypersurfaces of Euclidean space. *arXiv preprint arXiv:1908.04662*, 2019.

[27] A. Clarke and D. Turaev. Arnold diffusion in multi-dimensional convex billiards. *arXiv preprint arXiv:1906.07778*, 2019.

[28] T. H. Colding and W. P. Minicozzi. *A course in minimal surfaces*, volume 121. American Mathematical Soc., 2011.

[29] G. Contreras. Geodesic flows with positive topological entropy, twist maps and hyperbolicity. *Annals of mathematics*, pages 761–808, 2010.

[30] G. Contreras-Barandiarán and G. P. Paternain. Genericity of geodesic flows with positive topological entropy on $\mathbb{S}^2$. *Journal of Differential Geometry*, 61(1):1–49, 2002.

[31] C. B. Croke. Area and the length of the shortest closed geodesic. *Journal of Differential Geometry*, 27(1):1–21, 1988.

[32] A. Delshams, R. de la Llave, and T. M. Seara. A geometric approach to the existence of orbits with unbounded energy in generic periodic perturbations by a potential of generic geodesic flows of $\mathbb{T}^2$. *Communications in Mathematical Physics*, 209(2):353–392, 2000.

[33] A. Delshams, R. de la Llave, and T. M. Seara. A geometric mechanism for diffusion in hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model. *Mem. Amer. Math. Soc*, 179:844, 2003.

[34] A. Delshams, R. De La Llave, and T. M. Seara. Geometric properties of the scattering map of a normally hyperbolic invariant manifold. *Advances in Mathematics*, 217(3):1096–1153, 2008.

[35] A. Delshams, M. Gidea, R. de la Llave, and T. M. Seara. Geometric approaches to the problem of instability in hamiltonian systems. an informal presentation. In *Hamiltonian dynamical systems and applications*, pages 285–336. Springer, 2008.

[36] A. Delshams and R. Ramirez-Ros. On Birkhoff's conjecture about convex billiards. *Proceedings of the 2nd Catalan Days on Applied Mathematics*, pages 85–94, 1995.

[37] C. P. Dettmann, V. Fain, and D. Turaev. Splitting of separatrices, scattering maps, and energy growth for a billiard inside a time-dependent symmetric domain close to an ellipse. *Nonlinearity*, 31(3):667, 2018.

[38] V. J. Donnay. Transverse homoclinic connections for geodesic flows. In *Hamiltonian dynamical systems*, pages 115–125. Springer, 1995.

[39] R. Douady. *Applications du théoreme des tores invariants*. PhD thesis, Université Paris Diderot, 1982.

[40] N. Fenichel. Persistence and smoothness of invariant manifolds for flows. *Indiana University Mathematics Journal*, 21(3):193–226, 1971.

[41] N. Fenichel. Asymptotic stability with rate conditions. *Indiana University Mathematics Journal*, 23(12):1109–1137, 1974.

[42] N. Fenichel. Asymptotic stability with rate conditions, II. *Indiana University Mathematics Journal*, 26(1):81–93, 1977.

[43] E. Fermi. Sopra i fenomeni che avvengono in vicinanza di una linea oraria. *Rend. Lincei, 1922, 31 (1), pp. 21-23, 51-52, 101-103 (in Itallian)*, 31:21–23, 1922.

[44] A. Floer. An instanton-invariant for 3-manifolds. *Communications in mathematical physics*, 118(2):215–240, 1988.

[45] A. Floer. Morse theory for Lagrangian intersections. *Journal of differential geometry*, 28(3):513–547, 1988.

[46] A. Floer. The unregularized gradient flow of the symplectic action. *Communications on Pure and Applied Mathematics*, 41(6):775–813, 1988.

[47] F. B. Fuller. The treatment of periodic orbits by the methods of fixed point theory. *Bulletin of the American Mathematical Society*, 72(5):838–840, 1966.

[48] F. B. Fuller. An index of fixed point type for periodic orbits. *American Journal of Mathematics*, 89(1):133–148, 1967.

[49] F. B. Fuller. *The Existence of Periodic Orbits. Seminaire Trajectorien*. Institut de Recherche Mathématique Avancée, Strasbourg, 1968.

[50] V. Gelfreich, V. Rom-Kedar, K. Shah, and D. Turaev. Robust exponential acceleration in time-dependent billiards. *Physical review letters*, 106(7):074101, 2011.

[51] V. Gelfreich, V. Rom-Kedar, and D. Turaev. Fermi acceleration and adiabatic invariants for non-autonomous billiards. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 22(3):033116, 2012.

[52] V. Gelfreich and D. Turaev. Unbounded energy growth in hamiltonian systems with a slowly varying parameter. *Communications in Mathematical Physics*, 283(3):769, 2008.

[53] V. Gelfreich and D. Turaev. Arnold diffusion in a priori chaotic symplectic maps. *Communications in Mathematical Physics*, 353(2):507–547, 2017.

[54] M. Gidea, R. de la Llave, and T. M-Seara. A general mechanism of diffusion in hamiltonian systems: Qualitative results. *Communications on Pure and Applied Mathematics*, 2019.

[55] M. Gidea and J.-P. Marco. Diffusion along chains of normally hyperbolic cylinders. *arXiv preprint arXiv:1708.08314*, 2017.

[56] S. Gonchenko, D. Turaev, and L. Shilnikov. Homoclinic tangencies of arbitrarily high orders in conservative and dissipative two-dimensional maps. *Nonlinearity*, 20(2):241, 2007.

[57] A. Gorodetski and V. Kaloshin. On Hausdorff dimension of oscillatory motions in three body problems. *Preprint, available at http://www2.math.umd.edu/∼vkaloshi/papers/HD-Nov2013.pdf*, 2011.

[58] P. M. Gruber. Convex billiards. *Geometriae Dedicata*, 33(2):205–226, 1990.

[59] P. M. Gruber. Only ellipsoids have caustics. *Mathematische Annalen*, 303(1):185–194, 1995.

[60] M. Guardia, P. Martín, and T. M. Seara. Oscillatory motions for the restricted planar circular three body problem. *Inventiones mathematicae*, 203(2):417–492, 2016.

[61] J. Hadamard. Les surfaces à courbures opposées et leurs lignes géodésique. *Journal de Mathématiques Pures et Appliquées*, 4:27–73, 1898.

[62] G. A. Hedlund. The dynamics of geodesic flows. *Bulletin of the American Mathematical Society*, 45(4):241–260, 1939.

[63] E. Hewitt and K. Stromberg. *Real and abstract analysis: a modern treatment of the theory of functions of a real variable*. Springer-Verlag, 2013.

[64] G. W. Hill. On the part of the motion of the lunar perigee which is a function of the mean motions of the sun and moon. *Acta mathematica*, 8:1–36, 1886.

[65] M. W. Hirsch, C. Pugh, and M. Shub. Invariant manifolds (lecture notes in mathematics vol 583), 1977.

[66] H. Hofer, K. Wysocki, and E. Zehnder. Pseudoholomorphic curves and dynamics in three dimensions. In *Handbook of dynamical systems*, volume 1, pages 1129–1188. Elsevier, 2002.

[67] H. Hofer, K. Wysocki, and E. Zehnder. Finite energy foliations of tight three-spheres and hamiltonian dynamics. *Annals of Mathematics*, pages 125–255, 2003.

[68] J. Hubbard and Y. Ilyashenko. A proof of Kolmogorov's theorem. *Discrete and continuous dynamical systems*, 10(1/2):367–386, 2004.

[69] C. G. J. Jacobi. *Vorlesungen über dynamik*. G. Reimer, Berlin, 1884.

[70] C. K. Jones and S.-K. Tin. Generalized exchange lemmas and orbits heteroclinic to invariant manifolds. *Discrete & Continuous Dynamical Systems-S*, 2(4):967–1023, 2009.

[71] V. Kaloshin and A. Sorrentino. On the local Birkhoff conjecture for convex billiards. *Annals of Mathematics*, 188(1):315–380, 2018.

[72] A. Katok. Billiard table as a mathematician's playground. In *Student colloquium lecture series*, volume 2, pages 8–36, 2002.

[73] A. Katok and B. Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54. Cambridge university press, 1995.

[74] W. Klingenberg. *Lectures on closed geodesics*. Mathematisches Institut der Universitat Bonn, 1976.

[75] W. Klingenberg and F. Takens. Generic properties of geodesic flows. *Mathematische Annalen*, 197(4):323–334, 1972.

[76] G. Knieper and H. Weiss. A surface with positive curvature and positive topological entropy. *Journal of Differential geometry*, 39(2):229–249, 1994.

[77] G. Knieper and H. Weiss. C^∞ genericity of positive topological entropy for geodesic flows on $\mathbb{S}^2$. *Journal of Differential Geometry*, 62(1):127–141, 2002.

[78] A. N. Kolmogorov. On conservation of conditionally periodic motions for a small change in Hamilton's function. In *Dokl. Akad. Nauk SSSR*, volume 98, pages 527–530, 1954.

[79] V. Kovachev and G. Popov. Invariant tori for the billiard ball map. *Transactions of the American Mathematical Society*, 317(1):45–81, 1990.

[80] V. F. Lazutkin. The existence of caustics for a billiard problem in a convex domain. *Izvestiya: Mathematics*, 7(1):185–214, 1973.

[81] P. Le Calvez. *Propriétés dynamiques des difféomorphismes de l'anneau et du tore*. Société mathématique de France, 1991.

[82] P. Le Calvez. Drift orbits for families of twist maps of the annulus. *Ergodic Theory and Dynamical Systems*, 27(3):869–879, 2007.

[83] J. Llibre and C. Simó. Oscillatory solutions in the planar restricted three-body problem. *Mathematische Annalen*, 248(2):153–184, 1980.

[84] R. Mañé. Oseledec's theorem from the generic viewpoint. In *Proceedings of the international Congress of Mathematicians*, volume 1, page 2. Warsaw, 1983.

[85] J. Mather. Invariant subsets of area-preserving homeomorphisms of surfaces. *Mathematical Analysis and Applications*, 7B:531–561, 1981.

[86] J. N. Mather. Glancing billiards. *Ergodic theory and dynamical systems*, 2(3-4):397–403, 1982.

[87] J. N. Mather. Arnold diffusion by variational methods. In *Essays in mathematics and its applications*, pages 271–285. Springer, 2012.

[88] V. K. Mel'nikov. On the stability of a center for time-periodic perturbations. *Trudy moskovskogo matematicheskogo obshchestva*, 12:3–52, 1963.

[89] R. Moeckel. Generic drift on cantor sets of annuli. In *Celestial Mechanics*, page 163, 2002.

[90] H. M. Morse. Recurrent geodesics on a surface of negative curvature. *Transactions of the American Mathematical Society*, 22(1):84–100, 1921.

[91] M. Morse and G. A. Hedlund. Symbolic dynamics. *American Journal of Mathematics*, 60(4):815–866, 1938.

[92] M. Morse and G. A. Hedlund. Symbolic dynamics II. Sturmian trajectories. *American Journal of Mathematics*, 62(1):1–42, 1940.

[93] J. Moser. On invariant curves of area-preserving mappings of an annulus. *Nachr. Akad. Wiss. Göttingen, II*, pages 1–20, 1962.

[94] J. Nash. C^1 isometric imbeddings. *Annals of mathematics*, pages 383–396, 1954.

[95] J. Nash. The imbedding problem for Riemannian manifolds. *Annals of mathematics*, pages 20–63, 1956.

[96] M. Nassiri and E. R. Pujals. Robust transitivity in Hamiltonian dynamics. In *Annales scientifiques de l'École Normale Supérieure*, volume 45, pages 191–239, 2012.

[97] N. N. Nekhoroshev. An exponential estimate of the time of stability of nearly-integrable Hamiltonian systems. *Russian Mathematical Surveys*, 32(6):1–65, 1977.

[98] G. P. Paternain. *Geodesic flows*, volume 180. Springer Science & Business Media, 2012.

[99] D. Petroll. *Existenz und Transversalität von homoklinen und heteroklinen Orbits beim geodätischen Fluß*. PhD thesis, University of Freiburg, 1996.

[100] G. Pinzari. *On the Kolmogorov set for many-body problems*. PhD thesis, Università degli Studi di Roma Tre, 2009.

[101] H. Poincaré. *Les méthodes nouvelles de la mécanique celeste*. Springer, 1899.

[102] H. Poincaré. Sur les lignes géodésiques des surfaces convexes. *Transactions of the American Mathematical Society*, 6(3):237–274, 1905.

[103] J. Pöschel. A lecture on the classical KAM theorem. In *Proc. Symp. Pure Math*, pages 707–732, 2000.

[104] L. Rifford and R. O. Ruggiero. Generic properties of closed orbits of hamiltonian flows from Mañé's viewpoint. *International Mathematics Research Notices*, 2012(22):5246–5265, 2012.

[105] R. C. Robinson. Generic properties of conservative systems. *American Journal of Mathematics*, 92(3):562–603, 1970.

[106] R. C. Robinson. Generic properties of conservative systems II. *American Journal of Mathematics*, 92(4):897–906, 1970.

[107] R. C. Robinson. A global approximation theorem for Hamiltonian systems. In *Global Analysis (Proc. Sympos. Pure Math*, volume 14, pages 233–243, 1970.

[108] L. Shilnikov. A case of the existence of a countable number of periodic orbits. In *Sov. Math. Dokl*, volume 6, pages 163–166, 1965.

[109] S. Smale. Diffeomorphisms with many periodic points. In *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)*. Princeton University Press, 1965.

[110] S. Smale. Differentiable dynamical systems. *Bulletin of the American mathematical Society*, 73(6):747–817, 1967.

[111] L. Stojanov. A bumpy metric theorem and the poisson relation for generic strictly convex domains. *Mathematische Annalen*, 287(1):675–696, 1990.

[112] L. Stojanov and F. Takens. Generic properties of closed geodesics on smooth hypersurfaces. *Mathematische Annalen*, 296(1):385–402, 1993.

[113] S. Tabachnikov. *Billiards*. SMF, Panoramas et Syntheses, 1995.

[114] S. Tabachnikov. *Geometry and billiards*, volume 30. American Mathematical Society, 2005.

[115] R. Thom. Quelques propriétés globales des variétés différentiables. *Commentarii Mathematici Helvetici*, 28(1):17–86, 1954.

[116] R. Thom. Un lemme sur les applications différentiables. *Bol. Soc. Mat. Mex*, 2(1):59–71, 1956.

[117] D. Treschev and O. Zubelevich. *Introduction to the perturbation theory of Hamiltonian systems*. Springer Science & Business Media, 2009.

[118] E. Zehnder. Homoclinic points near elliptic fixed points. *Communications on Pure and Applied Mathematics*, 26(2):131–182, 1973.