Pricing of Options in Interrupted Markets Using Utility Maximisation Theory

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Abstract

In the financial industry, the standard method of pricing options is using the Black & Scholes model or one of its immediate extensions. One of the basic assumptions in this model is that the underlying of the option contract is available for continuous trading, thus, via hedging, one can completely offset the risk generated by trading in such derivatives. In this thesis an attempt is made to price options in cases where the Black & Scholes model cannot be applied. The particular restriction considered, is the a priori known unavailability of the underlying for continuous trading. The objectives of this thesis are the following:

a) investigate the effect such deterministic trading interruptions, under a generic utility maximisation framework,

b) exploit the concepts developed by [11] , in order to obtain a more tractable model, for options on one underlying asset, and

c) extend the model developed using the [11] concepts and investigate the effects of deterministic interrupts for options with two underlyings.

By investigating a parallel branch of pricing options to Black & Scholes, namely the utility maximization theory (which is popular in portfolio pricing), it is possible to treat many of the shortcomings of the Black & Scholes model, including the problem dealt with in this thesis. We will consider the Black & Scholes price, as the base case for the options price and compare the prices obtained from our models to this base case price. The advantage of using utility maximization theory is that, the results yielded from such a model converge to the Black & Scholes base case price when we assume that no restrictions

apply to our underlyings.

Three models are developed, all based on an initial portfolio composed of a bond and risky assets. Initially, we consider a general form of the utility function from the HARA class of utility functions and provide prices for both base and restricted cases under three different utility functions. Then, a negative exponential utility function is used to develop a second model. By using this utility function, we can simplify the form of our problem since we can ignore the initial level of the bond in the portfolio, which is an optimisation parameter that, this utility function is insensitive to.

The first two models developed can price options based on one underlying only. An improvement is achieved by increasing to two the number of underlyings of the option, hence widening the class of options that can be priced. Options such as the spread option belong to this class, an option very popular in the energy industry. In the end of this thesis, it is shown how the two-asset model can be applied to a real problem from the energy sector. The case of a petroleum refinery is investigated, and results are drawn on how the refinery value can be increased by rescheduling its maintenance periods.

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Contents

List of Symbols

$\|x\|$ is the Euclidean norm, defined as $\sqrt{\sum_{k=1}^{n} x_i^2}$ $\max_x g(x)$ signifies the maximisation of function $g(x)$, with respect to x

$g(x)^{-1}$ is the inverse function of $g(x)$

$\sup_A g(x)$ is defined as the least upper bound of the real-valued $g(x)$ over the subset A of its domain

$I_A(x)$ is an indicator function, such that $I_A(x) := \{1 \text{ if } x \in A, 0 \text{ otherwise}\}$

$g_x(x, y)$ is the first derivative of the function, with respect to x and $g_{xx}(x, y)$ is the second derivative, withe respect to x

x^- is defined as the negative part of x, i.e. $\max(0, -x)$

$\Rightarrow$ is the standard mathematical operation for the expression: the left-hand side equation implies the right-hand side equation

Chapter 1
Introduction

Financial derivatives have become extremely popular contracts, particularly in the last 20 years. Their underlyings and structure have become increasingly more complex and as new methods and models are created to satisfy this demand, a vicious circle is set in motion with a continuous demand in new models for new derivatives. The major part of this development can be credited to the research of Black and Scholes [5] . Most of these methodologies are based on the assumptions made and use the methodology developed by Black and Scholes [5] , almost 30 years ago. Clearly, their analysis has been ground-breaking, since it has allowed a myriad of models to stem from this single piece of research.

The assumptions of the Black and Scholes model include the following:

1) The stock pays no dividends during the option's life. Most companies pay dividends to their shareholders, so this might seem a serious limitation to the model considering the observation that higher dividend yields elicit lower call premiums. A common way of adjusting the model for this situation is to subtract the discounted value of a future dividend from the stock price.

2) European exercise terms are used

European exercise terms dictate that the option can only be exercised on the expiration date. American exercise terms allow the option to be exercised at any time during the life of the option, making American options more valuable due to their greater flexibility. This limitation is not a major concern because very few calls are ever exercised before the last few days of their life. This is true because when you exercise a call early, you forfeit the remaining time value on the call and collect the intrinsic value. Towards the end of the life of a call, the remaining time value is very small, but the intrinsic value is the same.

3) Markets are efficient

This assumption suggests that people cannot consistently predict the direction of the market or an individual stock. The market operates continuously with share prices following a continuous Itô process. To understand what a continuous Itô process is, you must first know that a Markov process is "one where the observation in any time period depends only on the preceding observation." Loosely speaking, an Itô process is a Markov process in continuous time. If you were to draw a continuous process you would do so without picking the pen up from the piece of paper.

4) No commissions are charged

Usually market participants do have to pay a commission to buy or sell options. Even floor traders pay some kind of fee, but it is usually very small. The fees that individual investors pay are more substantial and can often distort the output of the model.

5) Interest rates remain constant and known

The Black and Scholes model uses the risk-free rate to represent this constant and known rate. In reality there is no such thing as the risk-free rate, but the discount rate on U.S. Government Treasury Bills with 30 days left until maturity is usually used to represent it. During periods of rapidly changing interest rates, these 30 day rates are often subject to change, thereby violating one of the assumptions of the model.

6) Returns are lognormally distributed

This assumption suggests, returns on the underlying stock are normally distributed, which is reasonable for most assets that offer options.

The problem dealt with here is probably one of the most fundamental of these assumptions. By considering the nature of the markets, in which the underlyings of these derivatives trade in, one cannot help but notice that they are far from complete. Transaction costs are present even in the most liquid of the markets (e.g. foreign exchange markets) while trading of indexed prices (which means that the prices of the assets are directly unavailable) is quite commonplace, particularly in the commodities markets (e.g. the Dubai crude trades indexed to Brent crude or WTI).

These two problems (transaction costs and direct trading unavailability of assets) are quite closely linked. An extensive literature is available on the subject of transaction costs, while the subject of asset illiquidity is becoming an increasingly a popular subject of research. A review of this literature is presented later in this chapter.

The standard methodology of treating problems, that are associated with the assumption of market completeness, borrows concepts from portfolio optimisation, initiated by Markowitz [25] based on the mean-variance approach and suited for static portfolio decisions. The portfolio optimisation theory was further developed, primarily, by Merton for portfolio allocations in a continuous-time setting. Merton's methodologies can be extended (and certainly complemented) to solve problems of much more complex nature than the allocations in the portfolio and consumption of the portfolio holder.

As mentioned above, a vast selection of literature deals with problems in incomplete markets. Most of these methods use the definitions and methods provided in the next two sections.

1.1 Review of the continuous time market model

A set of definitions will be presented in this section, useful for building the rest of this chapter. These definitions are fairly common, but are presented here for completeness.

These continuous time market model definitions cover all aspects of the model, from asset processes to completeness of the market. We will start by defining the processes for the assets participating in our market. Two assets will be considered, one with a riskless return (such as a bank account or a zero coupon government bond) and an asset with fluctuating (or risky) return, such as a stock. The process describing the riskless asset is the following

$$dB_t = rB_t dt \tag{1.1}$$

and the process describing the changes in the price of the risky asset is

$$dS_t = S_t\left(\mu dt + \sigma dW_t\right) \tag{1.2}$$

where r is a constant rate of interest and μ is a constant drift rate for the return of the risky asset, which fluctuates with a constant volatility σ. W_t is a one-dimensional Brownian motion. This stochastic process is defined over a probability space $(\Omega, \mathbb{F}, \mathbb{P})$, where Ω is the set of all possible outcomes of the price process, $\mathbb{F}$ is a collection of subsets of Ω and $\mathbb{P}$ is the implied probability measure under which W_t is defined.

The (explicit) solutions of the two processes are provided below

$$B_t = \exp(rt)$$

for the riskless asset and

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right)$$

for the risky asset.

Definition 1 *Trading Strategy*
Trading strategies are real valued F_t-adapted processes (φ_t, ψ_t), $t \in [0, T]$, that satisfy

$$\int_0^T |\varphi_t|\, dt < \infty \ a.s.$$

$$\int_0^T (\psi_t S_t \sigma)^2\, dt < \infty \ a.s.$$

Definition 2 *Wealth Process*
A process X_t

$$X_t := \varphi_t B_t + \psi_t S_t \tag{1.3}$$

is called a wealth process if (φ_t, ψ_t) are two trading strategies.

A direct extension of the above definition gives the following very important definition.

Definition 3 *Self-Financing Strategy*
The triplet (φ_t, ψ_t, c_t), for some positive adapted process c_t (a consumption process), $t \in [0, T]$, satisfying

$$\int_0^T c_t dt < \infty, \ a.s.$$

is called a self financing trading strategy, if the wealth process X_t corresponding to (φ_t, ψ_t, c_t), satisfies

$$X_t = X_0 + \int_0^t \varphi_s dB_s + \int_0^t \psi_s dS_s - \int_0^T c_s ds \tag{1.4}$$

The self-financing condition dictates that the wealth of an investor at time $t + 1$ must be equal to his wealth at time t plus any trading activity minus any consumption that takes place between these two times (Korn [20]). This condition dictates that no "new" wealth is introduced into the portfolio of the investor at any time $t \in [0, T]$.

Definition 4 *Portfolio Process*

A quotient p_t defined as

$$p_t := \frac{\psi_t S_t}{X_t}$$

is called a portfolio process for the risky asset. The portfolio process for the riskless asset is $1 - p_t$. From this definition and 1.3 we get that

$$\frac{\varphi_t B_t}{X_t} = 1 - p_t$$

Using the differential form of 1.4 and substituting the forms of the risky and riskless assets and the portfolio processes, we get

$$\frac{dX_t}{X_t} = \left((1 - p_t) r + p_t \mu\right) dt + p_t \sigma dW_t - \frac{c_t}{X_t} dt \tag{1.5}$$

with a unique solution

$$X_t = Z_t \left(X_0 + \int_0^t \frac{c_s}{Z_s} ds \right)$$

where

$$Z_t = \exp\left(\left(r + p_s (\mu - r) - \frac{\|p_s \sigma\|}{2} \right) ds + p_s \sigma dW_s \right)$$

Definition 5 *Self-Financing Process*

A pair (p_t, c_t) is called a self-financing strategy and p_t is called a self-financed portfolio process, if c_t is a consumption process and p_t is a real valued F_t-adapted process for $t \in [0, T]$. Furthermore the pair (p_t, c_t) must satisfy the condition that the stochastic differential equation 1.5 has a unique solution X_t such that

$$\int_0^T \|p_t X_t\|^2 dt < \infty \text{ a.s.}$$

This condition is required because for the problem to make sense, the precess $p_t X_t$ must be square-integrable, semi-martingale.

Definition 6 *Set of Admissible Strategies*

If a selffinancing pair (p_t, c_t) has a corresponding wealth process with initial wealth $X_0 > 0$, the wealth process is called admissible if $X_t \geq 0 \; \forall \, t \in [0, T]$ a.s. Furthermore, $A(x)$ defined as

$$A(x) := [(p_t, c_t) \mid (p_t, c_t) \text{ selffinancing strategy with admissible } X_t \text{ with } X_0 = x]$$

is called the set of admissible strategies.

Finally the complete market condition is provided for completeness.

Theorem 1 *Complete Market (see* [20]*, section 2.3)*

For every selffinancing trading strategy $(p_t, c_t) \in A(x)$, *with corresponding wealth process* X_t, *we get*

$$I\!E\left[\int_0^t H_s c_s ds + H_t X_t\right] \leq x$$

For every F_T*-measurable random variable* $B \geq 0$ *and* c_t, $t \in [0,T]$ *with*

$$x := I\!E\left[\int_0^t H_s c_s ds + H_T B\right] \leq \infty$$

the wealth process X_t *satisfies*

$$X_T = B \text{ a.s.}$$

if there exists a portfolio process p_t, $t \in [0,T]$, $(p_t, c_t) \in A_x$, *for the* X_t.

In this theorem H_t *is defined by*

$$H_t = \exp\left(-\int_0^t \left(r + \frac{\left\|\frac{\mu-r}{\sigma}\right\|^2}{2}\right) ds - \int_0^t \frac{\mu - r}{\sigma} dW_s\right)$$

1.2 Portfolio maximisation in continuous-time

The general approach to maximising a portfolio value is to set up a portfolio, containing typically a riskless and a risky asset along with a consumption process and then obtain the maximum utility for the portfolio over all admissible trading strategies. There are two main streams in portfolio maximisation theory; the stochastic control and the martingale approach. The martingale approach uses stochastic integration and is based on martingale theory, but assumes a complete market to develop the theory required for solving a portfolio problem. The main contributors to this theory include Pliska [34] and Karatzas, Lehoczky and Shreve [19] . The stochastic control approach was initiated by Merton [29] in the late 60's and early 70's. This approach requires the solution to the Hamilton-Jacobi-Bellman equation and the theory is largely based on stochastic control theory. Although this method does not guarantee that an explicit solution is obtainable, it can be used in more general problems and in areas where the martingale method cannot be used. By contrast to the martingale approach, the stochastic control theory does not require the market completeness condition in order to solve the portfolio problem, which makes it an ideal approach for solving problems of the type considered here (the discontinuous trading of assets).

The base definitions for the two methods providing solutions to the portfolio problem are common. In order to simplify the introduction to the two models, we will assume that a complete market does exist. The first model presented, is derived using the martingale approach, while the second model is derived using the stochastic control approach.

Based on the definitions of the previous section, we will now build the optimisation problem.

Definition 7 *Utility Function*
If a strictly concave function $U : (0, +\infty) \to I\!R$ *satisfies*

$$U'(0) := \lim_{x \to 0} U'(x) = +\infty \text{ and } U'(+\infty) := \lim_{x \to +\infty} U'(x) = 0$$

it will be called a utility function.

More on the form and properties of utility functions can be found later in this chapter. A utility function is the "summary" of an investor's preferences, so it is worth looking into at different forms with different characteristics.

The starting point of building any portfolio problem is to define the portfolio wealth process, therefore we assume that a process $X_t^{x,\pi,c}$ for $t \in [0, T]$ exists, that describes the wealth of a portfolio with an initial endowment $x > 0$. The pair (π, c) is a pair of admissible strategies followed by the holder of this portfolio. By following this trading and consumption strategy, the portfolio's value can be determined by the following equation

$$J(x; \pi, c) := I\!E^{x,0} \left[\int_0^T U_1(t, c_t)\, dt + U_2\left(X_T^{x,p,c}\right) \right] \tag{1.6}$$

where U_1 and U_2 are two utility functions, x is the initial endowment and (p, c) is the pair of admissible strategies in $\Im(x)$.

It is now necessary to define a set $\Im'(x) \subseteq \Im(x)$ such that

$$\Im'(x) := \left\{ (p,c) \in \Im(x) \,|\, I\!E^{x,0} \left[\int_0^T \min\left(U_1(t,c_t)\right) dt + \min\left(U_2\left(X_T^{x,p,c}\right)\right) \right] < \infty \right\} \tag{1.7}$$

The optimisation problem is fully described by the following statement

$$\max_{(p,c)\in\Im'(x)} J(x;p,c)$$

where $J(.)$ and $\Im'(.)$ are given by the above definitions.

1.2.1 The martingale approach

The martingale approach was primarily developed by Karatzas, Lehoczky and Shreve[19] and Pliska[34] . The methodology can be separated into two parts. First, the optimal consumption and terminal wealth are determined (static optimisation problem), followed by the solution of the problem, which yields the strategy that obtains these cash flows (representation problem). For brevity, only the main points of this methodology will be presented.

As an introduction to the solution of the problem, a simpler case is considered (consumption is ignored by setting $c_t = 0, \forall\ t \in [0,T]$). We begin by decomposing the optimisation problem from the previous section (equation 1.6) into the two problems

$$\max_{B\in B(x)} I\!E\left[U(B)\right]$$

where $B(x)$ is defined by

$$B(x) := \left\{ B | B \geq 0, B \text{ is } F_T\text{-measurable}, I\!E\left[H_T B\right] \leq x, I\!E\left[U(B)^-\right] < \infty \right\}$$

which states the static optimisation problem; and

$$\begin{aligned} \text{find the process } p^* &\in A'(x) \text{ such that} \\ X_T^{x,p^*} &= B^* \text{ a.s.} \end{aligned}$$

where B^* is the solution to the static optimisation problem. This decomposition is only valid because of the assumption that the market is complete.

Via the use of Lagrangian multipliers (see Luenberger [24]), we can transform the static optimisation problem from above to the following problem

$$\max_{B \in B(x)} I\!E\left[U(B)\right]$$

subject to

$$I\!E\left[H_T B\right] = x$$

with the Lagrangian equivalent

$$L(B, y) = I\!E\left[U(B) - y\left(H_T B - x\right)\right]$$

The optimal B^* for this problem is given by

$$B^* = U^{-1}\left(D^{-1}(x)\, H_T\right)$$

where $D(y)$ is defined by

$$D(y) := I\!E\left[H_T U^{-1}\left(y H_T\right)\right]$$

A more rigorous discussion on this point can be found in [20] , section 3.4. This concludes the presentation of the simplified static optimisation problem. The solution to the problem including consumption will now be presented. Prior to the statement of the problem, a new definition for the utility function will be provided.

Definition 8 *Utility Function (extended)*
A strictly concave function $U : (0, \infty) \to I\!R$ will be called a utility function if it satisfies

$$U'(0) := \lim_{x \to 0} U'(x) > 0,\ U'(z) = 0$$

for some $z \in (0,\infty]$. *Furthermore* $U'(.)$ *has a continuous inverse function* $I^* : [0, U'(0)] \to [0, z]$ *such that*

$$I(y) := \left\{ \begin{array}{c} I^*(y),\, y \in [0, U'(0)] \\ 0,\, y \geq U'(0) \end{array} \right\}$$

Further, a function $U(t,x)$, *with the properties described above for* x, *will also be called a utility function on* x.

Now we can restate the optimisation problem as follows

$$\max_{(p,c)\in\Im'(y),\, y\leq x} J(y;p,c)$$

If $U_1(t,.)$ and $U_2(.)$ are strictly increasing functions, then the condition $y \leq x$ is not required.

Similarly to the previous case (no consumption process), we can define function D in the following way

$$D(y) := I\!E\left[H_T I_2(yH_T) + \int_0^T H_t I_1(t, yH_t)\, dt\right] \quad \forall\, y > 0$$

The solution to this problem is provided by the following theorem.

Theorem 2 *Assuming* $D(y) < \infty\, \forall\, y \in (0,\infty)$ *in the case* $U_1'(t,0) < \infty\, \forall\, t \in [0,T]$ *and* $U_2'(0) < \infty$, *then the optimal wealth* B^* *is given by*

$$B^* = \left\{ \begin{array}{c} z_2,\ \text{if } x \geq z^* \\ I_2\left(D^{-1}(x) H_T\right)\ \text{otherwise} \end{array} \right\}$$

and the optimal consumption process is given by

$$c_t^* = \left\{ \begin{array}{c} z_1,\ \text{if } x \geq z^* \\ I_1\left(t, D^{-1}(x) H_T\right)\ \text{otherwise} \end{array} \right\}$$

where z^* *is defined as*

$$z^* := \left\{ \begin{array}{c} z_1 I\!E\left[\int_0^T H_t dt\right] + z_2 I\!E[H_T]\ \text{for finite } z_1 \text{ and } z_2 \\ \infty\ \text{otherwise} \end{array} \right\}$$

and

$$\begin{aligned} D(0) &: = \lim_{y\to 0} D(y) = \left\{ \begin{array}{c} \infty,\ \text{for } \lim_{z\to\infty} U_2'(z) = 0 \text{ or } \lim_{z\to\infty} U_1'(t,z) = 0\ \forall\, t \in [0,T] \\ z_1 I\!E\left[\int_0^T H_t dt\right] + z_2 I\!E[H_T]\ \text{otherwise} \end{array} \right\} \\ D(\infty) &: = \lim_{y\to\infty} D(y) = 0 \end{aligned}$$

for any positive x. $x^* \in [0,x]$ *and* $p_t^*\ \forall\ t \in [0,T]$ *exist, such that the pair* (p_t^*, c_t^*) *solves the portfolio problem. In the case where* $U_1'(t,0)$ *or* $U_2'(0)$ *are not finite we have* $x^* = x$.

The optimal portfolio process p_t^* is given by

$$p_t^* = \frac{\nabla_x g(t, W_t)}{\sigma X_t}$$

which gives

$$\varphi_t = \frac{X_t - \psi_t S_t}{B_t}$$

and

$$\psi_t = \frac{\nabla_x g(t, W_t)}{\sigma S_t}$$

where W_t is a Brownian motion and $g(.,.)$ is a non-negative function with $g(0,0) = x^*$ which is defined by

$$g(t, W_t) = \frac{I\!E\left[\int_t^T H_s c_t^* ds + H_T B^* | F_t\right]}{H_t}$$

1.2.2 The stochastic control approach

The stochastic control approach was developed primarily by Merton [29]. As suggested by the title of the approach, the concepts for solving the portfolio problem are borrowed from stochastic control theory and involve the transformation of the wealth stochastic differential equation (equation 1.5) into a controlled process of the form

$$\frac{dX_t^k}{X_t^k} = \mu'\left(t, X_t^k, k\right) dt + \sigma'\left(t, X_t^k, k\right) dW_t \tag{1.8}$$

where $k = (k_1, k_2) := (p, c), \mu'\left(t, X_t^k, k\right) := \mu k_{1,t} + r\left(1 - k_{1,t}\right) - \frac{k_{2,t}}{X_t^k}$ and $\sigma'\left(t, X_t^k, k\right) := \sigma k_{1,t}$ which is the same expression as equation 1.5. To proceed with the presentation of the solution, the following value function is introduced

$$V(t,x) := \sup_{k \in K(t,x)} J(t,x,k) = \sup_{k \in K(t,x)} I\!E^{t,x}\left[\int_t^T U_1(s, k_{2,s}) ds + U_2\left(X_T^k\right)\right]$$

where $X_t^k = x$ and $K(t,x)$ denotes the set of admissible controls for initial conditions t

and x. U_1 and U_2 must also satisfy the following conditions

$$\begin{aligned} |U_1(t,k)| &\leq C(1+|x|^a+\|k\|^a) \\ |U_2(x)| &\leq C(1+|x|^a) \end{aligned}$$

for all $(t,x,k) \in [0,T] \times I\!R \times K$ and constants $C > 0$ and $a \in I\!N$ (as per [20] , appendix C). This value function ($V(t,x)$) denotes the maximum possible utility of the portfolio problem if the optimisation starts at t with an initial endowment (for the portfolio) x. We will now rewrite the value function as

$$V(t,x) := \sup_{k \in K([t,t'],x)} I\!E^{t,x}\left[\int_t^{t'} U_1(s,k_{2,s})\,ds + V\left(t',X_s^k\right)\right]$$

which forms part of the Bellman principle. Obviously $K([t,t'],x)$ is a subset of $K(t,x)$ with $t < t'$. The last part of the value function $V(t,x)$ is the value when an optimal strategy is assumed between t' and T. The rest of the expression represents the increase in utility for the strategy k in the period between t and t'. By applying Itô's lemma to the last part of the value function (i.e. $V\left(t',X_s^k\right)$) we get

$$\begin{aligned} V(t,x) &= V(t,x) + \sup_{k \in K([t,s],x)} I\!E^{t,x}[\int_t^s \{U_1(s,k_{2,s}) + V_t\left(s,X_s^k\right) \\ &\quad + \left[\left((\mu-r)k_{1,s}+r\right)X_s^k - k_{2,s}\right]V_x\left(s,X_s^k\right) \\ &\quad + \frac{1}{2}\left\|\sigma k_{1,s}X_s^k\right\|^2 V_{xx}\left(s,X_s^k\right)\}ds \\ &\quad + \int_t^s \sigma k_{1,s}X_s^k V_x\left(s,X_s^k\right)dW_s] \end{aligned}$$

where the notation $G_x(x,y)$ denotes the partial derivative of $G(x,y)$ with respect to x. If we make the assumption that the final integral is a martingale, divide both sides by $s-t$ and take the limit as $s \to t$, we get

$$0 = \sup_{k \in K([t,s],x)} [U_1(t,k_{2,t}) + V_t(t,x) + \left[\left((\mu-r)k_{1,t}+r\right)x - k_{2,t}\right]V_x(t,x) \quad (1.9)$$

$$+\frac{1}{2}\sigma^2 k_{1,t}^2 x^2 V_{xx}(t,x)]$$

The boundary condition is

$$V(T,x) = U_2(x) \text{ and } V(t,0) = U_2(0) \tag{1.10}$$

These two equations constitute the Hamilton-Jacobi-Bellman (HJB) equation. By differentiating equation 1.9 with respect to k_1 and k_2 (considering that $V(t,x)$ is concave, which ensures the existence of a maximum) we get the two expressions for the maximising k_1 and k_2

$$\begin{aligned} 0 &= (\mu - r) V_x(t,x) + \sigma^2 k_{1,t}^* x V_{xx}(t,x) \Rightarrow \\ k_{1,t}^* &= \frac{(r-\mu) V_x(t,x)}{\sigma^2 x V_{xx}(t,x)} \end{aligned}$$

and

$$\begin{aligned} 0 &= U_{1k}\left(t, k_{2,t}^*\right) - V_x(t,x) \Rightarrow \\ k_{2,t}^* &= U_{1k}^{-1}(V_x(t,x)) \end{aligned}$$

where $U_{1k}(.)$ denotes the first derivative of $U_1(.)$ with respect to k and $U_{1k}^{-1}(.)$ is its inverse function.

We will now substitute the pair of optimal strategies (k_1^*, k_2^*) into 1.9

$$\begin{aligned} 0 &= U_1\left(U_{1k}^{-1}(V_x(t,x))\right) + V_t(t,x) + \left[rx - U_{1k}^{-1}(V_x(t,x))\right] V_x(t,x) \\ &\quad -\frac{1}{2}\frac{(\mu - r)^2 V_x(t,x)^2}{\sigma^2 V_{xx}(t,x)} \end{aligned}$$

In order to complete the solution to the portfolio problem using the stochastic control method the verification theorem must be satisfied. The verification theorem only constitutes a sufficient condition of optimality for the problem, therefore a general solution to the problem might not always be feasible. However, if such an HJB equation is attainable the solution coincides with the value function.

Theorem 3 *The Verification Theorem (see* [20] *, App C for further references)*

Assume a solution to the Hamilton-Jacobi-Bellman equation (equations 1.9 and 1.10) $V'(t,x)$ *which is continuous on* $[0,T] \times I\!R$ *and* $V' \in C^{1,2}([0,T) \times I\!R)$*, then* $V'(t,x) \geq V(t,x)\ \forall\ (t,x) \in [0,T) \times I\!R$*. Furthermore, for an admissible control* k_t^* *such that*

$$\begin{aligned} k_t^* &= \arg\max_{p \in K} \left[U_1(t, X_t^*, c) + V_t(t, X_t^*) + \left[((\mu - r)p + rp)x - c\right] V_x(t, X_t^*)\right. \\ &\quad \left. + \frac{1}{2}\sigma^2 p^2 X_t^* V_{xx}(t, X_t^*)\right] \end{aligned}$$

where X_t^* *is the solution to the controlled stochastic differential equation 1.8, it holds that*

$$V'(t,x) = V(t,x) = J(t,x,k^*)\ \forall\ (t,x) \in [0,T) \times I\!R$$

and k_t^* *is an optimal control.*

From this theorem, we can extract an algorithm to solve the portfolio problem as defined in this section. The two steps of the algorithm are:

1. Solve the optimisation problem described by the Hamilton-Jacobi-Bellman equation for the optimal controls $k^* = (k_1^*, k_2^*)$ (which represent the pair (p, c) from the condition 1.7) with respect to the unknown value of $V(t,x)$.

2. Substitute the controls in the Hamilton-Jacobi-Bellman equation (1.9 with bounds 1.10) with the optimal solution (k_1^*, k_2^*) of the previous step. Since the controls are optimal, the supremum operator is now redundant. The result is obtained by solving the partial differential equation.

The application of this method is demonstrated in the second chapter. Next some comments on the types of the utility functions that can be used with these models are provided, followed with the set-up of an option pricing problem in a complete market. Finally, the review of the literature in the area of option pricing and hedging in incomplete markets and trading with transaction costs will be provided.

1.3 Review of utility functions

The most common class of utility functions used in portfolio theory, is the hyperbolic absolute risk aversion class (HARA for short). A utility function is classified as a HARA function if it is of the following form

$$U(x) = \frac{1-c}{c}\left(\frac{ax}{1-c}+b\right)^{c} \tag{1.11}$$

where the constants must be in the following ranges; $c < 1$, $c \neq 0$, $a > 0$, $\frac{ax}{1-c}+b > 0$.

The name of the function stems from the form of the Arrow-Pratt measure for the absolute risk-aversion

$$ARA(x) = \frac{U_{xx}(x)}{U_x(x)} = \left(\frac{x}{1-c}+\frac{b}{a}\right)^{-1}$$

The $ARA(x)$ function is of hyperbolic form.

From the basic definition of the utility function (definition 1.7) we can see that the utility functions used here must be increasing and strictly concave, therefore the $ARA(x)$ function is a positive number for any combination of a, b, c. An investor with $ARA(x) > 0$ is called risk averse. The explanation of risk aversion is the following; assume that B is the certain outcome of wealth at time T and W is the random variable describing the wealth, then

$$U(B) = \mathbb{E}[U(W)]$$

and the investor is indifferent between which amount to choose. In the case of a risk averse investor, the utility function is a strictly increasing and concave function and the certain outcome for the terminal amount is always smaller than the expectation of the random amount. This argument is proved in Merton [29] and is shown by using the Jensen inequality; since $U(x)$ is strictly concave, then

$$U(B) = I\!E[U(W)] < U(I\!E[W])$$

Another popular risk measure is the relative risk-aversion function, which is given by

$$RRA(x) = \frac{U_{xx}(x) \cdot x}{U_x(x)} = \left(\frac{x}{1-c} + \frac{a}{b}\right)^{-1} x$$

Some special cases of utility functions (in the HARA) class will be presented since they are some of the most commonly used utility functions. By setting $a = 1$, $b = 0$ and $c < 1$ we get the power utility, which has the form

$$U(x) = \frac{x^c}{c}$$

which has an ARA function equal to $\frac{1-c}{x}$ and a RRA function of $1 - c$.

By setting $b = c = 0$ we get the logarithmic utility function. Equation 1.11 is not defined for $c = 0$, but by applying the L'Hopital rule (see Ingersoll [18]), we get

$$U(x) \lim_{c \to 0} \frac{x^c - 1}{c} = \lim_{c \to 0} \frac{x^c \log(x)}{c} = \log(x)$$

which has an ARA function equal to x^{-1} and a RRA function equal to 1.

Finally, by using $c = -\infty$ and $b = 1$ we get the negative exponential utility which has the form

$$U(x) = 1 - \exp(-ax)$$

which has an ARA function equal to a, hence a RRA function equal to ax.

1.4 Pricing of options using portfolio maximisation

We will now demonstrate the pricing of options in the context discussed so far. In this example, a complete market is assumed to exist. In order to solve this problem we will consider minimising the distance of the final wealth to the claim of the option (the strike of the option). The particular example considered here is a call option on an asset, whose stochastic differential equation is given by equation 1.2.

The option's payoff function is given by

$$O(T,S,K)=\max(S_T-K,0)$$

for some positive strike price K. We start by setting up a portfolio whose wealth process is given by

$$X_t=B_t+y_tS_t \tag{1.12}$$

which is a self financing strategy for $c_t=0\,\forall\, t\in[0,T]$, and a trading strategy $(y_t,1)$. The stochastic differential equation for the wealth of the portfolio, which contains the risky underlying of the option and a riskless asset as defined in equation 1.1, is given by

$$dX_t=dB_t+y_tdS_t$$

with boundary condition

$$X_T=B_T+y_TS_T \text{ when } S_T\leq K$$

and

$$X_T-K \text{ when } S_T>K \tag{1.13}$$

The differential equation for the wealth process can be rewritten

$$\begin{aligned}
dX_t &= rB_tdt+y_tS_t(\mu dt+\sigma dW_t)\Rightarrow\\
dX_t &= rB_tdt+y_tS_t\mu dt+y_tS_t\sigma dW_t\Rightarrow\\
dX_t &= r\frac{B_t}{X_t}X_tdt+\frac{y_tS_t}{X_t}X_t\mu dt+\frac{y_tS_t}{X_t}X_t\sigma dW_t
\end{aligned}$$

From equation 1.12 we get that, for a portfolio process $p_t=\frac{y_tS_t}{X_t}$, $\frac{B_t}{X_t}=1-p_t$.

$$dX_t = r(1-p_t)X_tdt+p_tX_t\mu dt+p_tX_t\sigma dW_t\Rightarrow$$

$$dX_t = [(\mu - r) p_t + r] X_t dt + p_t X_t \sigma dW_t$$

This final version of the wealth stochastic differential equation has the same form as equation 1.8 for $\mu' (t, X_t^p, p) = (\mu - r) p_t + r$ and $\sigma' (t, X_t^p, p) = p_t X_t \sigma$. It must be noted that in this example the consumption process is zero, therefore the control $k_{2,t} = 0$. Comparing the drift and volatility from our final equation to the drift and volatility from equation 1.8, we can see that p_t is the control $k_{1,t}$, hence the stochastic control approach can be used in combination with 1.13. Since we consider a complete market, the martingale approach can be used as well, for optimally assessing the trading strategy of the risky asset.

1.5 Review of transaction costs literature

It is a well known fact that Black and Scholes's interpretation of risk in derivatives is an elegant and effective one. The assumptions underlying the methodology devised by them, are generally accepted and are widely used. It is only in specific cases that their implementation breaks down. The special case dealt with here, affects one the most fundamental assumptions of the Black and Scholes model. The effort here is to price the risk associated with the breakdown of trading, whose continuity is fundamental if the Black and Scholes model is to be used. Their model predetermines that if one were to offset the risk generated from a derivative position perfectly, he would have to trade (as in take specific positions) in the underlying asset as often as possible (given that the underlying has moved in price). Nevertheless, this is not always true in real markets, as there can be transaction costs associated with the purchasing and selling of assets (stocks, foreign exchange, etc.), as well as more critical events such as trading interrupts.

Transaction costs can either be of the form of a proportional amount to the price of the asset or a fixed amount charged at each transaction. Furthermore, there can be escalating transaction costs for purchasing different volumes of the asset, e.g. in the commodity markets, different prices are charged for different volumes of commodities. Trading interruptions can occur in many different ways and for varying reasons, but they can roughly be categorised in two types of interrupts; the asset is not trading but its price is observable and the asset is not trading and its value is unobservable.

In such a setting, one would have to find an effective way of treating these periods of unavailability (or illiquidity) of the underlying risky asset. One could consider hedging the derivative with another risky asset possibly available during the period the underlying is unavailable and incur some basis risk. Another way of treating such problems is to use the methodologies developed in the portfolio maximisation section. The idea is that we can control the terminal wealth of the underwriter of a derivative in a market dominated by interrupts, by optimally assessing his holding in the underlying during the periods of normal trading.

Quite extensive literature has been concentrated around this problem. The initiator of this stream of research was Leland [23] who addressed the problems in hedging derivative contracts. We are particularly interested in a specific branch of this stream. The argument surrounding this research is that perfect replication (in a Black and Scholes context) is no longer possible, so the optimal hedging of such contracts is considered. First, Hodges and Neuberger [16] developed a model treating such risks in an optimal replication scheme. Their model is dealing with the problem of proportional transaction costs and solves it using loss functions in a stochastic optimisation setup. The specific argument behind this implementation is that exact replication of the payoff function is impossible under the assumption of a finite cost for the strategy. A substantial contribution to this branch of methodologies was introduced by Davis, Panas and Zariphopoulou [11] where they treat a similar market setting and solve the problem via the use of unique viscosity solutions, which increases the optimality of the strategy. Both methods assume a portfolio of a cash amount (or bond) and one risky asset to reduce the dimensionality of the problem, but generally more assets can (and maybe should) be considered. As mentioned in Davis, Panas and Zariphopoulou "...in the presence of transaction costs they (the underwriters of such derivatives) might well wish to invest in other securities also.", which is a result in line with the classical portfolio diversification concepts. The technique developed by Davis, Panas and Zariphopoulou has been used in their context with success, although it has received some critique (Martellini [26]). Nevertheless, the in-built optimality of the method is highly desirable and their results constitute one of the milestone researches in their area (Martellini and Priaulet [27]).

Other research around the area of transaction costs and incomplete markets, in general, include articles such as Clewlow and Hodges's work [7] where they extend Hodges and Neuberger work to examine the problem of delta-hedging of options under fixed and proportional transaction costs, while Whalley and Wilmott [38] study fixed and proportional transaction costs in the context of Davis Panas and Zariphopoulou. Duffie and Fleming and Soner and Zariphopoulou [15] solve a problem of an investor who is endowed with a stochastic income and solve the problem using similar technology to Davis and Panas and Zariphopoulou. A different approach to the solution proposed by most of the literature examined so far can be found in Spivak and Cvitanic [36] who price an option using a complete market model, based on a duality method in a utility maximisation setting. Then, they move on to price an option in a market where only the distribution of the vector of returns is available. Similarly, Schachermayer [35] emphasises on a duality solution to the problem of optimal investment in incomplete markets after giving a review of results on expected utility maximisation.

Two extensions that have complemented the study of such problems can be found in Owen [33] and Davis [10] . Owen extends the definition of the utility function to accommodate for negative wealth and provides a solution to a utility maximisation problem, in incomplete markets, when the investor holds a short position on a contingent claim. Davis on the other hand, assumes that there a exists an asset closely related to the underlying of the option and uses this asset to hedge the risk in the derivative.

We can adapt such methodologies and transfer them to our problem (or market setting) directly, without any loss of optimality or generality. In short, we can treat our case as a special case of the transaction costs settings, where we can treat periods of normal market conditions as periods of negligible transaction costs, while periods when transactions in the risky asset are not permitted, can be though of as periods when transaction costs are infinite. This is not a realistic scenario, but it is used to demonstrate the analogy, between the work of [11] and the work in this thesis.

In the second chapter, we present two methodologies for pricing options on one illiquid asset. The illiquidity of the asset concerns the interruption in trading, meaning that the asset is unavailable for trading for certain intervals during the life of the option. Firstly, a model using a generic utility function is presented. The solution to the problem is given under three different utility functions, the negative exponential, the power and the logarithmic utility functions. A numerical example is provided, for the power utility, which is followed by the comparative results for the three functions. Then, the relative merits and disadvantages of each function are discussed. By exploiting the advantages of the negative exponential utility function a second model is developed. By using this utility function, we can ignore the one factor of the optimisation problem (the initial wealth of the portfolio) and derive a more tractable solution to the problem. The continuous case is firstly presented, independently of the option payoff function; concluded by the statement of the new optimisation problem. The problem is then solved and discretised in order to put it in a form that can be implemented in an algorithm. The presentation of the algorithm is followed by the results for this model.

In the third chapter, the second model developed in the second chapter is extended in order to accommodate a second risky asset in the portfolio, assumed by the writer of the option. This changes the class of options that can be considered, in the sense that a multitude of option payoff functions can now be considered. The type of options still have a European payoff but options such as spread and exchange options can be priced. The chapter starts with a presentation of the parts comprising the portfolio and states the optimisation problem for the two asset version of the option. Based on the conclusions of the second chapter, the problem is solved by using the negative exponential utility function. The solution is then presented in a discrete case and a payoff function is assigned to the problem. An outline of the algorithm is followed by a numerical example pricing the option with the particular payoff function. Finally the results obtained from simulating this model under different scenarios are presented.

In the fourth chapter, a realistic problem is described. The model developed in the third chapter is used to price the optionality inherent in the scheduling of the maintenance periods of a petroleum refinery. The model is slightly adjusted to accommodate for the new contracts required for the pricing of the options involved and the results obtained are discussed in detail. A discussion on the main points drawn from this thesis, along with points regarding further work and extensions of the present work, can be found in the conclusion of the thesis.

Chapter 2
Pricing an option on a single illiquid asset

We will now consider the case of a derivative with one illiquid risky underlying. In order to develop a methodology for pricing such an option, one needs to create a function describing the wealth of a portfolio which is not constructed specifically for the purpose of hedging the exposure generated by trading the derivative. We will assume that the market consists of only a single risky asset (to reduce the dimensions of the problem) and that the portfolio's value will fluctuate by the moves (in price) of its two assets, the risky asset and the cash amount. A function will be created to transform the portfolio into a cash amount by liquidating the position in the risky asset. A general outline of the model will be provided, followed by a comparison of the performance of three different utility functions. Namely, the logarithmic, power series and negative exponential utility functions will be compared. Then the negative exponential utility function will be used to develop a second, more rigorous, model for pricing such options.

The whole process is developed from the side of the underwriter and it is the price of the option, that he is happy to sell it at, that is calculated here. Obviously, this will not coincide with the price the investor would expect, but this is a common problem for pricing options in incomplete markets. It is only in the Black and Scholes "perfect" market that these two prices coincide. Furthermore, the use of a utility function makes the pricing of the option dependent on the risk aversion of the underwriter. Nevertheless, this is not a huge problem since we can assume total risk aversion (which is an argument in Black and Scholes as well) and only consider such cases. Then the only effect of the risk aversion factor is one of flexibility since it adds a degree of freedom to the pricing.

In order to define the problem, we will begin by defining the processes of the two assets involved. The payoff function will be generically defined, while the problem will be solved for a general case of a utility function first.

2.1 General definitions of the problem

We need to set-up a portfolio in order to replicate the terminal claim of the buyer of the option. In order to do that we form a portfolio containing a risky asset (such as a stock) and a riskless asset in the form of a cash amount. The price process of the risky asset is of the following form

$$dS_t = S_t \left(\mu dt + \sigma dW_t \right)$$

This price process is defined on a probability space $(\Omega, I\!F, P)$ over $[0, T]$ where, μ is a constant drift rate, σ is the constant volatility of the risky asset's return and W_t is a Brownian motion. The riskless asset is also defined on the same interval $[0, T]$ and its price follows the process

$$dB_t = rB_t dt$$

where, B_t is the cash amount in our portfolio and r is the constant interest rate.

In order to be able to replicate the final claim of the investor we need to create the cash value of our portfolio. To do this, we define a process y_t which describes the level in the holding of the risky asset. The product $y_t S_t$ (where y_t is a function defined over $[0, T]$) provides the cash value of the risky position of the portfolio. This way, the total cash value of our portfolio is

$$P_t = B_t + y_t S_t$$

When considering a self financing portfolio, we need to modify the price process of the riskless asset, as we assume that cash amounts will be deducted or induced from and to this investment, to finance the purchasing or selling of the risky asset. This changes the diffusion of the riskless asset in the following way

$$dB_t = rB_t dt - S_t dy_t \tag{2.1}$$

The aim of this formulation is to have a matching of the investor's terminal claim K (superreplication), such that

$$K \leq y_T S_T + B_T$$

where T is the time when the claim takes place (the option's expiry time).

The option payoff function does not need to be defined at this point as it is irrelevant to the formulation of the problem, thus a generic function $O(T, y, S, K)$ will be used. The payoff condition $f(T, S, K) \geq 0$ will be used to indicate whether the option is exercised or not. In these functions, T denotes the time that the rest of the variables are considered at (apart of K, which is constant).

These definitions will now be used to define and solve the problem.

2.2 General definition of the optimisation problem

The starting point of the analysis is to devise a portfolio, similar to the one defined earlier.

The idea of having a portfolio to trade on, introduces flexibility in the way the option can be described, as well as the opportunities available to the investor. Following the rational in [11] , the investor may choose not to underwrite an option at all, but instead to manage the portfolio by trading in the market for some profit. By optimally trading around his portfolio, the investor is expected to make a some profit. One can think of this situation from the markets' prospective, that is anybody wishing to enter in the game of trading should pay a premium for doing so. Therefore, the investor would (on average) end up with a zero profit.

On the other hand, by taking the risk of underwriting an option, the investor should charge a premium to cover himself from potential losses while hedging his position. Therefore, in order to underwrite an option, the investor would incur a premium (for the option), but also will be charged the market premium as well. This means that the portfolio of the underwriter will have the following amount of cash (at the time of trading the option, s)

$$p_s = B_{o,s} - B_{m,s} \tag{2.2}$$

where $B_{o,s}$ is the premium associated with the option and $B_{m,s}$ is the premium associated with the market return. At this price (p_s) the underwriter has no preference whether he charges $B_o - B_m$ and hedges the option or if he trades in the market for his own profit (for a further discussion and the proof of equation 2.2, see Davis, Panas and Zariphopoulou [11] , Theorem 1).

In the first case the final value of the portfolio is

$$P_m(T, B_m, y, S) = B_{m,T} + y_T S_T$$

where y_T is the final holding of the risky asset at the horizon time T. $B_{m,T}$ is the amount of cash left in the portfolio at T.

When considering the case of a traded option, there are two possibilities; the option will either be exercised, or it will not. To express this, the use of an indicator function is needed, such as $I_{excond}[g(x)]$ (where *excond* denotes an exercise condition, typically an inequality), which gives $g(x)$ whenever the exercise condition is satisfied and zero elsewhere. The following function describes the cash value of the portfolio at the expiry time T, based on the definitions above;

$$P_o(T, B_o, y, S) = B_{o,T} + I_{f(T,S,K)\leq 0}[y_T S_T] + I_{f(T,S,K)>0}[O(T, y, S, K)]$$

where $B_{o,T}$ is the cash amount left in the portfolio after adjusting the portfolio to hedge the position in the option. Formulating the final value of the portfolio in such a way (in cash terms) is convenient since, the underwriter might not be able to accumulate the required amount of the risky asset, but only the cash value representing the required amount.

Given a random trading strategy z in a set of trading strategies $\Im(B)$ the underwriter's maximum utility (at the expiry time of the option, T), given an initial cash amount, B, is defined by the following value function

$$V_o(T, B_o^z, y, S) = \sup_{z\in\Im(B)} I\!E\{U(P_o(T, B_o^z, y, S))\}$$

where $I\!E$ is the expectation operator and where $(s, B, y, S) \in [0, T] \times I\!R \times I\!R \times I\!R^+$. $V_o(T, B_o, y, S)$ is a continuous and monotonic increasing function and is bounded for all $B \in I\!R$.

We next consider the case when the underwriter does not sell any option but only enters the market to trade. In such a case the value function becomes

$$V_m\left(T, B_m^z, y, S\right) = \sup_{z \in \Im(B)} I\!E\left\{U\left(P_m\left(T, B_m^z, y, S\right)\right)\right\}$$

This function gives the maximum utility available to the underwriter after trading in the market. In both cases the value function is zero, assuming that the underwriter holds no stock at the starting time s and that B_o and B_m are premiums for hedging the option and trading in the market, respectively.

By exploiting the way the portfolio value is specified, we can generalise the form of the function and separate the cash amount from the final value of the asset holdings. This creates a new function $G_j\left(T, y, S\right)$ which is independent of the cash amount in the portfolio. Its form is given below

$$G_j\left(T, y, S\right) = P_j\left(T, B_j, y, S\right) - B_{j,T}$$

where the index j is used to distinguish between the case when the portfolio is traded in the market and the case when an option is written.

By introducing this definition into our value functions, we get a general description of the value function which completes the construction of the optimisation problem

$$V_j\left(T, B_j^z, y, S\right) = \sup_{z \in \Im(B)} I\!E\left\{U\left(G_j\left(T, y, S\right) + B_{j,T}^z\right)\right\} \tag{2.3}$$

2.3 Definition of the optimisation problem for a generic utility function

We will now attempt to provide a method of solving this problem for a generic utility function. This will provide a useful comparison between different types of utilities as well as valuable insight into which are the major factors affecting the optimisation problem. In the concluding remarks of [11] , it is stated that "... the form of the utility function is unimportant ... only its curvature at the origin plays any real role.". In this section, an effort will be made to demonstrate this fact in cases where the continuous trading of the underlying is not allowed. Based on other research in the area of transaction costs, there are specific utility functions that can handle such problems, but this has not been demonstrated in the case of trading interruption.

In order to solve the optimisation problem stated by equation 2.3, we will start by stating the conditions governing the utility function. $U(x)$ must be defined in $I\!R \to I\!R$ and must be a strictly increasing and concave function. Furthermore, the utility function must also be twice continuously differentiable. The utility functions used in this section will be chosen from the HARA family of utilities, which are discussed in the previous chapter.

An approach similar to [11] is followed in Andersen and Damgaard [2] in order to solve the problem of option pricing with proportional transaction costs. Andersen and Damgaard go one step further and propose an algorithm for solving such problems under a generic utility function using convex optimisation on the problem modeled by [11], which is similar to the problem posed by 2.3.

Following Andersen's and Damgaard's methodology, this problem can be shown to have a discrete time equivalent $V_j^{(n)}(.)$, with abstract discretisation parameter n, such that

$$V_j^{(n)}\left(T, B_j^z, y, S\right) \to V_j\left(T, B_j^z, y, S\right) \text{ for } n \to \infty$$

In [11] it is shown that this is true for the case of a specific utility function. This is done via the use of viscosity solutions of partial differential equations for a portfolio containing a riskless asset and the underlying of the option. By exploiting the fact that the general shape and properties of the utility functions in the HARA class are similar, it seems reasonable to argue that this fact should hold for more utility functions and not just for the negative exponential utility.

We will now borrow some concepts from [2] in order to develop a model for our optimisation problem. As with [11] , Andersen and Damgaard specify the option price as the price that makes the investor indifferent between trading his portfolio in the market and buying an option in addition to trading on his portfolio. This automatically gives an outline of the procedure that will be followed in order to solve the problem. The maximum expected utility of the investor for trading in the market (say V_m^*) must be smaller than, or if possible equal, to the maximum expected utility when he buys the option (V_o^*).

At this point, Andersen and Damgaard differentiate their specification slightly from [11] and include the option price directly in the wealth process. In order to distinguish between the two cases (V_m and V_o), the option price is set to zero at the time of the pricing and has a zero payoff at the horizon of the problem (which coincides with the expiry of the option). In the case when the option is sold, the option price is included in the wealth process at time s, while the payoff at the expiry of the option is included in the calculation of the terminal utility.

The portfolio values at time s are given by

$$P_m(s, B_m, y, S) = B_{m,s} - y_s S_s$$

in the case of trading with this portfolio in the market (the substraction of the yS denotes any initial purchasing of the risky asset), while the value of the portfolio in the case of selling the option is

$$P_o(s, B_o, y, S) = B_{o,s} - y_s S_s = B_{m,s} - y_s S_s + p_s$$

Since we are investigating the inventory of an underwriter of an option, it is apparent that he will not incur any costs for selling an option, therefore the premium of the option is added to his wealth at time 0. The only cost associated with the option is the final payoff which takes place at the expiry of the option.

In order to complete the specification of the portfolio process for the two cases, we need to specify the process for all intermediate times, t (in terms of the instantaneous previous step $t-1$). The two processes have the following form

$$P_j\left(t, y, S\right) = \exp(rdt)P_{j,t-1} - \left(y_t - y_{t-1}\right) S_t \tag{2.4}$$

which is the same for both cases.

We can now state the optimisation problem in its discrete form, as following

$$\begin{aligned} V_j^{(n)}\left(T, B_j, y, S\right) &= \max_{z \in \Im(B)} I\!E\left\{U\left(P_j\left(T, B_j^z, y, S\right)\right)\right\} \\ &\text{subject to} \\ P_j(T, B_j^z, y, S) &\leq G_j\left(T, y, S\right) + B_{j,T}^z \\ P_j(s, B_j^z, y, S) &\leq \left\{ \begin{array}{ll} B_{m,s} - y_s S_s, & \text{for } j = m \\ B_{o,s} - y_s S_s + p_s, & \text{for } j = o \end{array} \right\} \\ P_j(t, B_j^z, y, S) &\leq \exp(rdt)P_{j,t-1} - \left(y_t - y_{t-1}\right) S_t \end{aligned} \tag{2.5}$$

The procedure to solve this problem is to maximise the $V_m^{(n)}$ function to find the optimum utility V_m^* and then optimise the option premium for $V_o^{(n)}$ for the following optimisation problem

$$\begin{aligned} p_s^* &= \min\ V_o^{(n)}\left(s, B_o, y, S\right) - V_m^*\left(s, B_m, y, S\right) \\ &\text{subject to} \\ V_o^{(n)}\left(s, B_o, y, S\right) &\geq V_m^*\left(s, B_m, y, S\right) \end{aligned} \tag{2.6}$$

Obviously, the only financially viable situation will be the case when the two value functions are equal; otherwise the investor will no longer be indifferent as to which strategy to select.

In the next two sections, we will investigate different types of discretisation of the movements of the risky asset as well as ways to approximate the expected value of derivatives and give functional forms to the payoff functions specified earlier. The next section is an important part for the optimisation problem described above, as it provides the tools for calculating the expected value of the $V_j^{(n)}$ functions and will form the backbone for finding the solution to the optimisation problem.

2.4 Asset price discretisation

In this section we will present two of the methods widely used for the discretisation of the risky asset's price process. Namely, the binomial and trinomial lattices will be investigated and specific guidelines for the construction of the lattices will be given. The two lattices have very similar construction but different properties. Both can be constructed adjointly to the optimisation problem described before and therefore bare no significance in the construction of the problem as it has been described up to now. Nevertheless, they play a key role in the pricing process and therefore are an important part of the methodology described. Such implementation methods are variants of Markov chain approximations based on explicit finite differences. Such methods are weakly convergent schemes to the continuous time, requiring no lateral boundary conditions and are used for ease of implementation.

2.4.1 Binomial lattice on a single asset

The main feature that makes binomial lattices attractive to use is the simplicity of their implementation. They are based on the simple assumption that the price of the asset that is to be discretised, will either increase or decrease in the next time-step, which is generally true. That is the reason why they are so popular and are widely used for the simulation of spot prices for various financial instruments, from foreign exchange and stock prices to interest rates. The basic construction of a binomial lattice can be seen below (Figure 2.1)

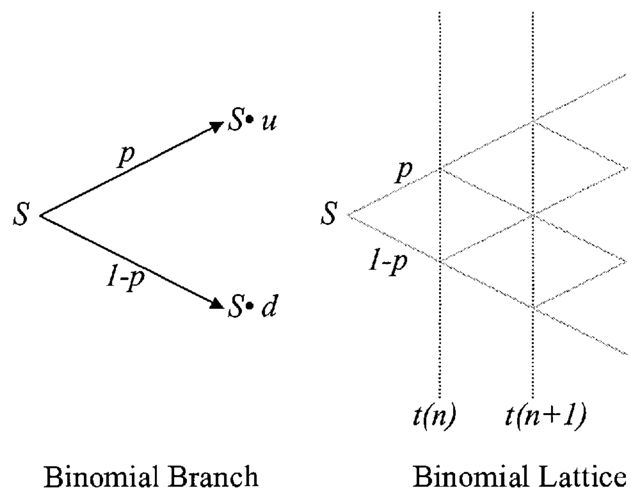


Figure 2.1: One factor Binomial Lattice

From Figure 2.1 it can be seen that the rate at which the binomial lattice is growing is $+1$. Therefore at the time-step n the number of price nodes are $n + 1$. The fairly compact size of this lattice is another attractive feature since it makes simulations of the prices fast. The binomial lattice properties were first studied by Cox, Ross and Rubinstein [9] . More on binomial tree implementation and applications can be found in [17] .

From figure Figure 2.1 we can see that the discrete changes in the price are given by

$$\delta S_m = \left\{ \begin{array}{l} uS_m - S_m, \text{ with probability } p \\ dS_m - S_m, \text{ with probability } 1-p \end{array} \right\}$$

where S_m is the current level of the price, u is and upward movement factor, d is a downward movement factor and p is the probability of the price rising. The way to solve the problem for the three parameters involved in this implementation is to follow the ideas dictated by a risk-neutral valuation and the mean and the variance of the original variable (the price of the risky asset) must be matched. Therefore we have

$$\begin{aligned} puS + (1-p)\,dS &= \exp(r\delta t)\,S \\ p\,(uS)^2 + (1-p)\,(dS)^2 - (puS + (1-p)\,dS)^2 &= \exp(2r\delta t)\left(\exp\left(\sigma^2\delta t\right) - 1\right) S^2 + \ldots \\ & \quad \text{higher order terms of } \delta t \end{aligned}$$

where σ is the volatility of the returns of the risky asset, r is the interest rate level, δt is the time-step and $exp\,(\delta t)\,S$ is the forward price of the risky asset. Below is the full parameter list used in the discretisation process for a binomial lattice if we assume that $u = d^{-1}$.

$$\begin{aligned} u &= \exp\left(\sigma\sqrt{\delta t}\right) \\ d &= \exp\left(-\sigma\sqrt{\delta t}\right) \\ p &= \frac{e^{r\delta t} - d}{u - d} \end{aligned}$$

2.4.2 Trinomial lattice on a single asset

In contradiction to the previous argument, that prices generally tend to increase or decrease, it is often common to use the trinomial lattice, which uses an extra branch for no change in price, to discretise price processes. This is the major difference between the two discretisations, which implies many differences in both their implementation and significance.

The trinomial lattice discretisation described here will follow the methodology described in Derman, Kani and Chriss [13] which is produced by using a constant volatility trinomial lattice. Extensions to this method can be made by introducing a volatility term structure and forward price tracking described in Derman, Kani and Chriss and in Clewlow and Strikland [8] respectively.

One of the reasons for using such a discretisation method is that by using the constant volatility trinomial lattice we summarise two time-steps of the equivalent binomial lattice in one time-step.. This can be seen in the diagram below.

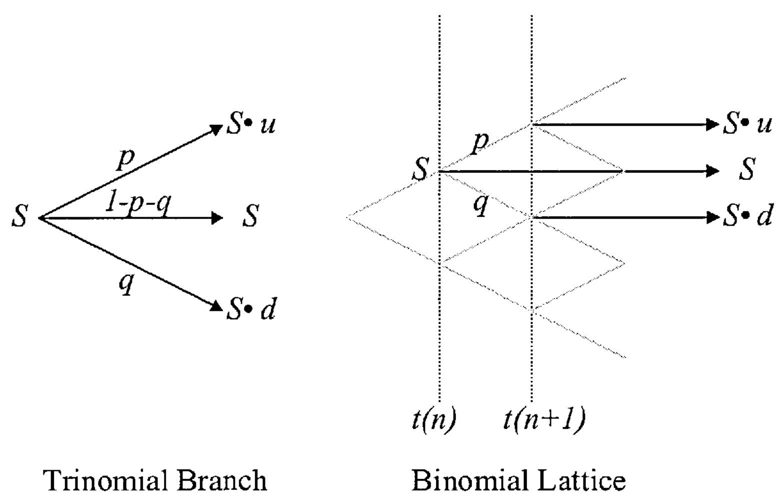


Figure 2.2: One factor Trinomial Lattice

where S is the risky asset's price at time $t(n)$, p is the transition probability for an upward movement, q is the probability for a downward movement in the asset's price, u

is the factor by which the price increases and d is the factor by which the price decreases. In order to have a risk-neutral valuation, we need to have that

$$pSu + (1-p-q)S + qSd = \exp(r\delta t)\, S$$

where $exp\,(r\delta t)\, S = F$ is the forward price of the risky asset. Also, if the volatility during the time period is σ, then the transition probabilities satisfy

$$p(Su-F)^2 + (1-p-q)(S-F)^2 + q(Sd-F)^2 = \exp(2r\delta t)\left(\exp\left(\sigma^2\delta t\right) - 1\right) S^2 + \ldots$$

$$\text{higher order terms of } \delta t$$

These two equations can give a solution for two of the four parameters that need to be determined. One possible solution to this scheme is given by the following equations

$$\begin{aligned}
u &= \exp\left(\sigma\sqrt{2\delta t}\right) \\
d &= 1/u \\
p &= \left(\frac{\exp\left(\frac{r\delta t}{2}\right) - \exp\left(-\sigma\sqrt{\frac{\delta t}{2}}\right)}{\exp\left(\sigma\sqrt{\frac{\delta t}{2}}\right) - \exp\left(-\sigma\sqrt{\frac{\delta t}{2}}\right)}\right)^2 \\
q &= \left(\frac{\exp\left(\sigma\sqrt{\frac{\delta t}{2}}\right) - \exp\left(\frac{r\delta t}{2}\right)}{\exp\left(\sigma\sqrt{\frac{\delta t}{2}}\right) - \exp\left(-\sigma\sqrt{\frac{\delta t}{2}}\right)}\right)^2 \\
m &= 1-p-q
\end{aligned}$$

As mentioned before, this is only one possible method of discretising a trinomial lattice. More methods are described in Derman, Kani and Chriss [13] , all with different properties. The main property of all these types of discretisations is that they converge to the continuous time Black and Scholes equation when the $\delta t \mapsto 0$, as is the case for the binomial lattice.

2.5 Specification of the payoff functions

By using a generic option payoff function in the development of our methodology, we have the flexibility to price any kind of single asset option as long as it has a European style payoff. Throughout the rest of the chapter we will consider a vanilla option (both calls and puts can be priced). In the next chapter results are presented for the following payoff function

$$\max\{S_T - K, 0\}$$

The function $O(T, y, S, K)$ becomes

$$O(T, y, S, K) = (y_T - 1) S_T + K$$

and the exercise condition becomes

$$f(T, S, K) = S_T - K > 0$$

2.6 Solution of the optimisation problem for a generic utility function

Having defined all the tools required for solving the optimisation problem, we will now present an outline of the solution followed by a simple numerical example. In order to solve the problem, a binomial lattice will be used to discretise the risky asset. Then the lattice will be broken up into its individual branches, in order to calculate the expected value for each path and eventually each path will be scaled by its according probability and summed up to calculate the expected value of the value function.

In more detail, in order to calculate the expected payoff in 2.5, we need to observe that by discretising the asset price on a lattice (e.g. binomial), the expected value of the final time-step can be calculated using the following function

$$\mathbb{E}\left[U\left(x\right)\right] = \sum_{I\in\mathbb{F}} p^{N(p,I)} q^{N(q,I)} U\left(x\right)$$

where I is an individual path through the lattice, p is the probability of an upward movement in the asset price, q is a downward movement and $\mathbb{F}$ is the collection of all complete paths through the binomial lattice. $N\left(x, I\right)$ is a function counting the number of occurrences of an event, such as an upward or a downward movement for the specific path I. The function for a trinomial lattice is similar, the only difference is that the final time-step value is scaled by all three probabilities.

In order to clarify the reason why we need to decompose the lattice into individual paths, we must consider the form of 2.4. For each step through each path we require the information of the level of asset holding in the previous time-step.. This is not feasible if we move along a lattice, as information about previous states is lost. Therefore, we need to decompose the lattice and calculate each path separately.

By using the counter functions we can assign the individual probability to each path and therefore scale its result. In order to get the value of the expected utility, we need to sum up all the scaled paths. Using a large number of steps (hence a large number of paths; the number of paths grows exponentially with each added step), we can ensure that we get an accurate approximation to the real expected value.

The final statement of the problem is given below

$$V_j^{(n)}\left(T, B_j, y, S\right) = \max_{z\in\Im(B)} \sum_{I\in\mathbb{F}} p^{N(p,I)} q^{N(q,I)} U\left(P_j\left(T, B_j^z, y, S\right)\right)$$

subject to

$$
\begin{aligned}
P_j(T, B_j^z, y, S) &= \exp(rdt)P_{j,T-1} + G_j\,(T, y, S) \\
P_j(s, B_j^z, y, S) &= \left\{ \begin{array}{l} B_{m,s} - y_s S_s, \text{ for } j = m \\ B_{m,s} - y_s S_s + p_s, \text{ for } j = o \end{array} \right\} \\
P_j(t, B_j^z, y, S) &= \exp(rdt)P_{j,t-1} - (y_t - y_{t-1})\, S_t
\end{aligned}
$$

In this statement we have substituted the inequalities by equalities for the following reason. The utility function is a concave function, and since the target is to maximise its value, it is optimal that we take the maximum value of the wealth at each time-step through the paths. Any point lower than the equality point for the constraints is clearly suboptimal and therefore the inequalities are replaced by equalities. Furthermore, in order to calculate the model for each path we will use the wealth functions recursively, so that the problem in reality is constrained only by the upper and lower limits of the holding values. Typically, no short selling will be allowed (therefore the lower limit for each holding is zero) and no more than the asset can be held (therefore the upper bound is one).

As mentioned before, the procedure for solving the problem is to maximise the expected value of the portfolio for trading in the market and then find the maximum option premium that makes the expected value of the portfolio with the option greater or equal to the 'market' portfolio. In order to demonstrate the procedure, a numerical example is provided below.

2.6.1 Numerical example

For this example, the initial price of the underlying, S, is set to 1, the strike K, is set to 1, the volatility of the underlying, σ is 0.3, its drift is set to the constant interest rate $r = 0.03$, and the expiry of the option (and hence the horizon of the problem) T is set to 0.25. In order to make the problem easier to follow we will use two steps ($N = 2$) hence the problem has four paths. We assume that the initial wealth $B_{m,s}$ is equal to 2 and that the initial guess for our vector of asset holdings $Y = \{y_{s,I_1}, y_{s+1,I_1}, ..., y_{T,I_1}, y_{s,I_2}, y_{s+1,I_2}, ..., y_{T,I_{2N}}\}$ is alternating between 0 and 1, starting from 0 for each path. The discrete time-step is $ds = 0.125$, while $tU = 1.1119$ is the upward price jump and $p = 0.4912$ is the probability of an upward movement. We will use a power utility of the form $U(x) = \frac{x^a}{a}$ with $a = 0.4$. The objective of the optimisation problem is to find the values $y_{s,I}$ (denoting time-step and path) of the asset holding, that maximise the expected utility. In order to do this we need to be able (for the current vector of holdings) to calculate the expected utility. This is done in the following way.

We start from the vertex of the lattice ($N = 0$) and select the path for which we will be calculating the value of the utility. At this point we evaluate the wealth using the following function.

$P_m(s, B_m, y, S) = B_{m,s} - y_{s,I_1} S_s = 2 - 0 * 1 = 2$

We then move to the next time-step and assume that we followed the node of an upward movement

$P(s + 1, B, y, S) = \exp(rds)P_s - (y_{s+1,I_1} - y_{s,I_1}) S_s * tU = 1.0038 * 2 - (1 - 0) * 1 * 1.1119 = 0.8956$

$N(p, I_1) = 1$

$N(q, I_1) = 0$

We now reach the final time-step of the lattice and calculate P_T using the same function (again we assume we followed the upward movement node)

$P(T, B, y, S) = \exp(rds)P_{s+1} - (y_{T,I_1} - y_{s+1,I_1}) S_{s+1} * tU = 1.0038 * 0.8956 - (0 -$

$1) * 1.1119 * 1.1119 = 2.1353$

$N(p, I_1) = 2$

$N(q, I_1) = 0$

We will now calculate the value of the utility for this path

$U(P_T, I_1) = \frac{P_T^{0.4}}{0.4} = 3.3863$

and the associated probability of this path is

$Q_{I_1} = 0.4912^2 = 0.2413$

We now repeat this procedure for a second path.

$P_s = B_{m,s} - y_{s,I_2} S_s = 2 - 0 * 1 = 2$

We start by following the node of an upward movement.

$P_{s+1} = \exp(rds)P_s - (y_{s+1,I_2} - y_{s,I_2}) S_s * tU = 1.0038*2-(1-0)*1*1.1119 = 0.8956$

$N(p, I_2) = 1$

$N(q, I_2) = 0$

We then follow a downward movement.

$P_T = \exp(rds)P_{s+1} - (y_{T,I_2} - y_{s+1,I_2}) S_{s+1}/tU = 1.0038 * 0.8956 - (0 - 1) * 1.1119/1.1119 = 1.8990$

$N(p, I_2) = 1$

$N(q, I_2) = 1$

$U(P_T, I_2) = \frac{P_T^{0.4}}{0.4} = 3.2311$

$Q_{I_2} = 0.4912 * (1 - 0.4912) = 0.2499$

The next path (I_3) will follow the downward movement node first and the upward movement node next.

$P_s = B_{m,s} - y_{s,I_3} S_s = 2 - 0 * 1 = 2$

$P_{s+1} = \exp(rds)P_s - (y_{s+1,I_3} - y_{s,I_3}) S_s / tU = 1.0038*2-(1-0)*1/1.1119 = 1.1082$

$N(p, I_3) = 0$

$N(q, I_3) = 1$

$P_T = \exp(rds)P_{s+1} - (y_{T,I_3} - y_{s+1,I_3}) S_{s+1} * tU = 1.0038 * 1.1082 - (0-1) * 0.8994 * 1.1119 = 2.1132$

$N(p, I_3) = 1$

$N(q, I_3) = 1$

$U(P_T, I_3) = \frac{P_T^{0.4}}{0.4} = 3.3717$

$Q_{I_3} = 0.4912 * (1 - 0.4912) = 0.2499$

The final path (I_4) follows all the downward moving nodes in the lattice and its calculations are shown below

$P_s = B_{m,s} - y_{s,I_4} S_s = 2 - 0 * 1 = 2$

$P_{s+1} = \exp(rds)P_s - (y_{s+1,I_4} - y_{s,I_4}) S_s / tU = 1.0038*2-(1-0)*1/1.1119 = 1.1082$

$N(p, I_4) = 0$

$N(q, I_4) = 1$

$P_T = \exp(rds)P_{s+1} - (y_{T,I_4} - y_{s+1,I_4}) S_{s+1} / tU = 1.0038 * 1.1082 - (0-1) * 0.8994 * 0.8994 = 1.9212$

$N(p, I_4) = 0$

$N(q, I_4) = 2$

$U(P_T, I_4) = \frac{P_T^{0.4}}{0.4} = 3.2461$

$Q_{I_4} = 0.4912 * (1 - 0.4912) = 0.2589$

The expected utility for this example is given by

$\mathbb{E}\left[U\left(P_T\right)\right]=\sum_{i=1}^{4} Q_{I_i} * U\left(P_T, I_i\right)=0.2413 * 3.3863+0.2499 * 3.2311+0.2499 * 3.3717+0.2589 * 3.2461=3.3076$

We now need to change the values in vector Y so that we achieve the maximum value for $\mathbb{E}\left[U\left(P_T\right)\right]$. Any large scale optimisation package that can handle quadratic problems can be used for this purpose, as the utility functions are convex and smooth functions. A solution to this particular example gives a value for the expected utility to be $V_m^*=3.3770$, and the optimal holding vector is $Y^*=\{1,1,0,1,0,0,0,1,0,0,0,0\}$.

We are now left with the task of maximising the premium in $V_o^{(n)}$ in the secondary optimisation problem 2.6. This is a much simpler optimisation problem as even a simple secant method algorithm can solve it, since $V_o^{(n)}\left(p_s\right)$ is a strictly increasing function of p_s. By recalculating the value function, as demonstrated above, with subtracting the payoff at each P_T, we get (for $p_s=0$) a value for the expected utility $V_o^{(n)}(0)=3.3408$. The premium that solves the problem is $p_s=0.05607$. The equivalent Black & Scholes price for this problem is 0.0634 and the Cox, Ross & Rubinstein price is 0.0570.

This method of solution is very flexible as it allows direct manipulation of the holding at each node for each path of the decomposed lattice. This can be used to introduce interruptions in the trading of the underlying, by directly assigning the value of the holding to the immediately preceding one on the path. This makes this method a very useful tool in the study of trading interruptions. Furthermore, it allows us to study this effect under different utility functions.

2.6.2 Results

We will now present some results obtained, using the procedure described above. Two sets of results will be presented. In the first set, the effects of the risk aversion factor and of the initial wealth will be investigated in a continuous trading setting. In the second set of results, the effect of interrupted trading on the price of the option will be investigated.

2.6.2.1 Continuous trading

The most important effect that we need to demonstrate in this section is the convergence of the option price of this model to the price of a Black & Scholes (BS) model. In order to make this more plausible, the price of Cox, Ross & Rubinstein (CRR) option will be provided for each set of results. The main problem of this model is the number of free variables, which grow at a rate of $2^{nSteps}\,(nSteps+1)+1$, where $nSteps$ is the number of time-steps of the lattice. Therefore, the solution of a problem with six time-steps, requires the optimisation of 448 free variables. As a consequence, a large number of time-steps is prohibited if the solution is to be attained within a reasonable time. In this study, 3, 4, 5 and 6 time-steps were used (33, 81, 193 and 449 free variables respectively).

The input parameters for this section are the following. The price of the underlying is set to 1 $(S=1)$, the strike K is 1, the yield of the underlying is set to 0.02 and its volatility σ is 0.3. The constant interest rate r is 0.03 and the option expires in four months $(T=0.25)$. The initial wealth (W_0) is set to 1. The BS price is 0.0607. The option prices were calculated using three utility functions, namely the negative exponential, power and logarithmic utility functions. Firstly, we take a look at the effect of the absolute risk aversion factor (denoted as gamma) for the different utility functions for different time-steps..

nSteps	3	crr	0.0662
gamma	NEU	PWU	LGU
0.1	0.0654	0.0636	0.0634
0.2	0.0652	0.0638	0.0634
0.3	0.0649	0.0641	0.0634
0.4	0.0646	0.0643	0.0634
0.5	0.0644	0.0645	0.0634
0.6	0.0641	0.0648	0.0634
0.7	0.0638	0.0650	0.0634
0.8	0.0636	0.0652	0.0634
0.9	0.0633	0.0655	0.0634
1	0.0630	0.0657	0.0634

Table 2.1: Effect of risk aversion (nSteps=3)

The CRR price is provided along the different time-steps..

In Table 2.1, the results obtained from a three time-steps lattice are presented. It can be observed that the risk aversion factor has no effect on the logarithmic utility function and this is due to the fact that $\log(\gamma x) = \log(\gamma) + \log(x)$. Further, it can be observed that increasing the risk aversion factor has an opposite effect on the negative exponential utility than that for the power utility function. As a final remark, all three utilities produce premiums very close to the premium of BS and CRR.

The effects observed for the previous set of results are true as the number of time-steps increases. The most important point to note, is that the option prices of the three different models (negative exponential (NEU), power (PWU) and logarithmic (LGU) utility) mimic the behaviour of the CRR price. That means that they follow a similar trend around the BS price. In Table 2.3 and Table 2.4, the results for 5 and 6 time-steps are presented.

Next, we will investigate the behaviour of the option price under the three models in the case of changing the initial wealth $B_m = W_0$. For this set of results the options prices were calculated for 3 and 5 time-steps..

In the two tables that follow (Table 2.5 and Table 2.6) the change in the option prices

nSteps	4	crr	0.0575
gamma	NEU	PWU	LGU
0.1	0.0569	0.0553	0.0551
0.2	0.0566	0.0555	0.0551
0.3	0.0564	0.0557	0.0551
0.4	0.0562	0.0559	0.0551
0.5	0.0559	0.0561	0.0551
0.6	0.0557	0.0563	0.0551
0.7	0.0555	0.0565	0.0551
0.8	0.0552	0.0567	0.0551
0.9	0.0550	0.0569	0.0551
1	0.0548	0.0571	0.0551

Table 2.2: Effect of risk aversion (nSteps=4)

nSteps	5	crr	0.0642
gamma	NEU	PWU	LGU
0.1	0.0634	0.0615	0.0613
0.2	0.0631	0.0618	0.0613
0.3	0.0628	0.0620	0.0613
0.4	0.0625	0.0622	0.0613
0.5	0.0623	0.0625	0.0613
0.6	0.0620	0.0627	0.0613
0.7	0.0617	0.0630	0.0613
0.8	0.0614	0.0632	0.0613
0.9	0.0611	0.0634	0.0613
1	0.0609	0.0637	0.0613

Table 2.3: Effect of risk aversion (nSteps=5)

nsteps	6	crr	0.0587
gamma	NEU	PWU	LGU
0.1	0.0580	0.0563	0.0561
0.2	0.0577	0.0565	0.0561
0.3	0.0575	0.0567	0.0561
0.4	0.0572	0.0569	0.0561
0.5	0.0569	0.0572	0.0561
0.6	0.0567	0.0574	0.0561
0.7	0.0564	0.0576	0.0561
0.8	0.0562	0.0578	0.0561
0.9	0.0559	0.0580	0.0561
1	0.0556	0.0582	0.0561

Table 2.4: Effect of risk aversion (nSteps=6)

nSteps	3	crr	0.0662
Wo	NEU	PWU	LGU
0.5	0.0644	0.0636	0.0615
1	0.0644	0.0645	0.0634
1.5	0.0644	0.0649	0.0641
2	0.0644	0.0651	0.0645
4	0.0644	0.0654	0.0651
6	0.0644	0.0655	0.0653
10	0.0644	0.0656	0.0654

Table 2.5: Effect of initial wealth (nSteps=3)

can be seen with increasing initial wealth, while the risk aversion parameter gamma is set to 0.5. The NEU is the only model that is insensitive to changes to this parameter. Both PWU and LGU demonstrate increase in their premiums, as the initial wealth increases. This is possibly due to the nature of the functions. The NEU is the only function that can be decomposed to two parts, the initial wealth part and the optimisation parameters part since $\exp(x+y) = \exp(x)\exp(y)$. This is a welcome effect as we could possibly drop this factor W_0 as an optimisation parameter.

As a further point, we can observe that the option prices obtained from the other two models (PWU and LGU) are comparable to the CRR price, but this factor seems to have a much bigger effect on the LGU model.

2.6.2.2 Interrupted trading

In this section we demonstrate the properties of the three models under different types of deterministic interrupted trading periods. For these results the same set of parameters is used and the initial wealth is set to $W_0 = 1$ and the risk aversion factor is set to 0.5.

We start by presenting the results for a decreasing interruption period, where initially the period is the whole option life and decreases towards the end of the expiry of the option.

In Table 2.7 the effect of the starting time can be observed. The starting time $(tStr)$

nSteps	5	crr	0.0642
Wo	NEU	PWU	LGU
0.5	0.0623	0.0616	0.0595
1	0.0623	0.0625	0.0613
1.5	0.0623	0.0628	0.0620
2	0.0623	0.0630	0.0624
4	0.0623	0.0633	0.0630
6	0.0623	0.0634	0.0632
10	0.0623	0.0635	0.0634

Table 2.6: Effect of initial wealth (nSteps=5)

denotes the start of the interruption to trading. The stopping time ($tStp$) is the period when trading resumes as normal. By observing the three columns of the table, is can be seen that all three option premiums decrease as the period of interruption becomes smaller.

Table 2.8, Table 2.9 and Table 2.10 all show the results obtained for a lattice with 4, 5 and 6 time-steps..

In Table 2.11 and Table 2.12 the option prices obtained for an increasing period of trading interruption (increasing from the start of pricing towards the expiry of the option) are displayed.

It can be observed that for an increasing period of interruption the option prices are increasing as well. It is also worth noting, that when the period of interruption is 0, the option prices revert to the prices obtained in the continuous trading section.

Next the effect of a fixed period of interruption is shifted from the start of the option towards its expiry.

In this setting the interruption is assumed to be lasting for one time period. It can be observed that the effect to the option price is small but it seems to be increasing as the period approaches the expiry of the option. In both tables (Table 2.13 and Table 2.14) the prices increase monotonically.

nSteps	3	tStp	0.25
tStr	NEU	PWU	LGU
0.000	0.0656	0.0656	0.0656
0.083	0.0656	0.0656	0.0656
0.167	0.0646	0.0647	0.0636
0.250	0.0644	0.0645	0.0634

Table 2.7: Effect of starting time (nSteps=3)

nSteps	4	tStp	0.25
tStr	NEU	PWU	LGU
0.000	0.0578	0.0579	0.0587
0.063	0.0578	0.0579	0.0587
0.125	0.0571	0.0571	0.0571
0.188	0.0562	0.0563	0.0555
0.250	0.0559	0.0561	0.0551

Table 2.8: Effect of starting time (nSteps=4)

nSteps	5	tStp	0.25
tStr	NEU	PWU	LGU
0.000	0.0646	0.0646	0.0656
0.050	0.0646	0.0646	0.0656
0.100	0.0639	0.0640	0.0643
0.150	0.0633	0.0633	0.0630
0.200	0.0625	0.0627	0.0617
0.250	0.0623	0.0625	0.0613

Table 2.9: Effect of starting time (nSteps=5)

nSteps	6	tStp	0.25
tStr	NEU	PWU	LGU
0.000	0.0595	0.0595	0.0595
0.042	0.0595	0.0596	0.0611
0.083	0.0590	0.0590	0.0599
0.125	0.0585	0.0585	0.0587
0.167	0.0579	0.0579	0.0576
0.208	0.0572	0.0574	0.0565
0.250	0.0569	0.0572	0.0561

Table 2.10: Effect of starting time (nSteps=6)

nSteps	3	tStr	0
tStp	NEU	PWU	LGU
0.000	0.0644	0.0645	0.0634
0.083	0.0644	0.0646	0.0635
0.167	0.0656	0.0656	0.0656
0.250	0.0679	0.0680	0.0704

Table 2.11: Effect of stopping time (nSteps=3)

nSteps	4	tStr	0
tStp	NEU	PWU	LGU
0.000	0.0559	0.0561	0.0551
0.063	0.0560	0.0562	0.0552
0.125	0.0569	0.0569	0.0567
0.188	0.0578	0.0579	0.0587
0.250	0.0596	0.0597	0.0624

Table 2.12: Effect of stopping time (nSteps=4)

	nSteps	5		
tStr	tStp	NEU	PWU	LGU
0.000	0.050	0.0623	0.0624	0.0612
0.050	0.100	0.0623	0.0624	0.0613
0.100	0.150	0.0624	0.0625	0.0614
0.150	0.200	0.0625	0.0626	0.0615
0.200	0.250	0.0625	0.0627	0.0617

Table 2.13: Fixed period of interruption (nSteps=5)

	nSteps	6		
tStr	tStp	NEU	PWU	LGU
0.000	0.042	0.0569	0.0571	0.0560
0.042	0.083	0.0570	0.0571	0.0560
0.083	0.125	0.0570	0.0572	0.0561
0.125	0.167	0.0571	0.0572	0.0562
0.167	0.208	0.0572	0.0573	0.0563
0.208	0.250	0.0572	0.0574	0.0565

Table 2.14: Fixed period of interruption (nSteps=6)

From the above short analysis of the behaviour of the tree models (NEU, PWU and LGU) it can be concluded that all three models exhibit similar properties. The major difference in behaviour was the insensitivity of the NEU model to the initial wealth level and the insensitivity of the LGU model to the risk aversion parameter. The PWU could be used as an intermediate model with more free parameters, but in our case, the property of the NEU model is advantageous. By dropping the initial wealth parameter, we can reduce the number of optimisation parameters. Although, the algorithm presented in this section has a lot of limitations (number of time steps must be small for a feasible solution, being the main), it provides the flexibility of a generic utility function. This is the only reason this model was presented. Nevertheless, calculation times (growing exponentially with increasing time steps) make the use of such a model restrictive, in a commercial sense.

In the next section a model exploiting the form of the negative exponential utility function is presented. This model has many advantages as opposed to the NEU, PWU and LGU models, which are limited in use, due to the large-scale optimisation problems they require to be solved in order to obtain the option price. Nevertheless, they have provided some useful insight into the behaviour of utility based models when used for pricing replicated portfolios.

2.7 Negative exponential utility - a simplified solution for the optimisation problem

The negative exponential utility function has the following form

$$U(x) = 1 - \exp(-\gamma x)$$

This function has certain attractive features; starting from the risk aversion coefficient, which is calculated in the following way

$$-U''(x)/U'(x) = \gamma$$

Since the risk aversion coefficient is independent of the underwriter's wealth, it shows that, irrespective of where the underwriter's wealth lies, the underwriter will use as much cash required to optimise his position in the form of trading in the risky asset. This applies to both the cases when the underwriter hedges the position that the written option creates and when he trades for his own profit. This is an appealing feature, as the optimality of the position in the riskless asset is not an issue and the problem is reduced by one dimension. The formulation hereafter, follows closely the derivations in [11] , where the negative exponential utility function was used in the problem of pricing European options, under proportional transaction costs.

By including the negative exponential utility in our value functions 2.3, we get

$$V_j\left(T, B_j^z, y^z, S\right) = \sup_{z \in \Im(B)} I\!E\left\{1 - \exp\left(-\gamma\left(G_j\left(T, y_T^z, S_T\right) + B_{j,T}^z\right)\right)\right\} \tag{2.7}$$

where j has been used in the place of o and m to indicate the two possibilities available to the underwriter. By taking 2.7 further, we get

$$V_j\left(T, B_j^z, y^z, S\right) = \sup_{z \in \Im(B)} \left[1 - I\!E\left\{\exp\left(-\gamma\left(G_j\left(T, y_T^z, S_T\right) + B_{j,T}^z\right)\right)\right\}\right] \Leftrightarrow$$

$$V_j\left(T, B_j^z, y^z, S\right) = 1 - \inf_{z \in \Im(B)} I\!E\left\{\exp\left(-\gamma B_{j,T}^z\right)\exp\left(-\gamma G_j\left(T, y_T^z, S_T\right)\right)\right\} \tag{2.8}$$

One thing worth mentioning at this point, is that the problem is no longer one of maximisation, but the contrary. By altering the value function 2.7 in a way to form function 2.8 we have turned the problem into one of minimisation.

By exploiting the form of equation 2.1 and integrating, we get the following expression for B_T

$$B_T = \exp\left(r\left(T - s\right)\right) B_{j,s} - \int_s^T \exp\left(r\left(T - t\right)\right) S_t dy_t$$

Now we define the bond component $F\left(s\right) = \exp\left(r\left(T - s\right)\right)$ and introduce the expression for B_T into 2.8; we get

$$V_j\left(s, T, B_j^z, y^z, S\right) = 1 - \inf_{z \in \Im(B)} I\!E \left\{ \begin{array}{c} \exp\left(-\gamma\left[F\left(s\right) B_{j,s} - \int_s^T F\left(t\right) S_t dy_t^z\right]\right) \\ \exp\left(-\gamma G_j\left(T, y_T^z, S_T\right)\right) \end{array} \right\}$$

By separating the terms to those that refer to the trading strategy adopted and to those that refer to the initial endowment $B_{j,s}$ at time s we get

$$V_j\left(s, T, B_j^z, y^z, S\right) = 1 - \inf_{z \in \Im(B)} I\!E \left\{ \begin{array}{c} \exp\left(-\gamma F\left(s\right) B_{j,s}\right) \exp\left(\gamma \int_s^T F\left(t\right) S_t dy_t^z\right) \cdot \\ \cdot \exp\left(-\gamma G_j\left(B_{j,T}^z, y_T^z, S_T\right)\right) \end{array} \right\}$$

Next we observe that the term $\exp\left(-\gamma F\left(s\right) B_{j,s}\right)$ can be taken out of the expectation as it has a know value at time s. Hence,

$$V_j\left(s, T, B_j^z, y^z, S\right) = 1 - \exp\left(-\gamma F\left(s\right) B_{j,s}\right) \inf_{z \in \Im(B)} I\!E \left\{ \exp\left(\begin{array}{c} \gamma \int_s^T F\left(t\right) S_t dy_t^z \\ -\gamma G_j\left(T, y_T^z, S_T\right) \end{array} \right) \right\}$$

Having separated the value function $V_j\left(s, T, B_j^z, y^z, S\right)$ in such a way, we can define a function $Q_j\left(s, T, y^z, S\right)$ independent of the initial endowment $B_{j,s}$, such that

$$Q_j\left(s, T, y^z, S\right) = \inf_{z \in \Im(B)} I\!E \left\{ \exp\left(\gamma \int_s^T F\left(t\right) S_t dy_t^z - \gamma G_j\left(T, y_T^z, S_T\right)\right) \right\} \tag{2.9}$$

which only depends on the trading process which will minimise the value function. We can now rewrite equation 2.8 in the following way

$$V_j\left(s, T, B_j^z, y^z, S\right) = 1 - \exp\left(-\gamma F\left(s\right) B_{j,s}\right) Q_j\left(s, T, y^z, S\right) \tag{2.10}$$

From the above and by setting the initial endowment $B_{j,s}$ to 0, it can be shown that

$$Q_j\left(s, T, y^z, S\right) = 1 - V_j\left(s, T, 0, y^z, S\right)$$

and the boundary condition for 2.9 at final time T is

$$Q_j\left(T, T, y^z, S\right) = \exp\left(-\gamma G_j\left(T, y_T^z, S_T\right)\right)$$

Solving 2.10 for $B_{j,s}$ yields

$$B_{j,s} = -\frac{1}{\gamma F(s)} \ln\left(\frac{1 - V_j\left(s, T, B_j^z, y^z, S\right)}{Q_j\left(s, T, y^z, S\right)}\right)$$

Evaluating the above function at the points $B_{o,s}$ and $B_{m,s}$ and returning to the definition of the asking price $p_s = B_{o,s} - B_{m,s}$, we get

$$\begin{aligned} p_s &= \frac{1}{\gamma F(s)}\left[-\ln\left(\frac{1 - V_o\left(s, T, B_o^z, y^z, S\right)}{Q_o\left(s, T, y^z, S\right)}\right) + \ln\left(\frac{1 - V_m\left(s, T, B_m^z, y^z, S\right)}{Q_m\left(s, T, y^z, S\right)}\right)\right] \\ p_s &= \frac{1}{\gamma F(s)} \ln\left(\frac{\left(1 - V_m\left(s, T, B_m^z, y^z, S\right)\right) Q_o\left(s, T, y^z, S\right)}{\left(1 - V_o\left(s, T, B_o^z, y^z, S\right)\right) Q_m\left(s, T, y^z, S\right)}\right) \end{aligned}$$

At time s (the time the option is traded), if the underwriter is assumed to hold none of the risky asset ($y = 0$) then the value function $V_j\left(s, T, B_o^z, 0, S\right) = 0$ by definition. This leads to the explicit definition of the asking price for the option in terms of $Q_j\left(s, T, 0, S\right)$

$$p_s = \frac{1}{\gamma F(s)} \ln\left(\frac{Q_o\left(s, T, 0, S\right)}{Q_m\left(s, T, 0, S\right)}\right)$$

which makes the calculation of the asking price totally independent of the initial cash amount $B_{j,s}$. The fact that the option price predetermines that the underwriter has no position of the risky asset (the underlying of the option), at the time of trading the option, also reverts to the definition of the asking price, where the underwriter has no preference between hedging the option or trading in the market for his own profit. If the underwriter had any position in the underlying at the time of trading the option, hedging the exposure would prove advantageous and effectively reduce the asking price for this option (with the specific underlying).

It would be useful to be able to calculate the value function $Q_j\left(s, T, y^z, S\right)$ parametrically from another value of the function, specifically from a different level of holding in the risky asset. To do this we have to consider two cases. We can either buy an amount of δy to increase our holding or sell δy (buy $-\delta y$) to decrease the position. We will consider the case of buying. Starting from our value function $V_j\left(s,T,B_o^z,y^z,S\right)$, in order to increase our holding we need to buy δy. This is shown in the following function

$$V_j\left(s,T,B_j^z,y^z,S\right) = V_j(s,B_{\delta y} - S\delta y, y_{\delta y} + \delta y, S) \tag{2.11}$$

By allowing $\delta y \longrightarrow 0$ we get

$$\frac{\partial V_j\left(s,T,B_j^z,y^z,S\right)}{\partial y} - S\frac{\partial V_j\left(s,T,B_j^z,y^z,S\right)}{\partial B} = 0$$

Now we can use 2.10 and its partial derivatives with respect to y and B to get

$$\begin{aligned}
\exp(-\gamma F\left(s\right)B_{j,s})\frac{\partial Q_j\left(s,T,y^z,S\right)}{\partial y} &= -\gamma F\left(s\right)\exp(-\gamma F\left(s\right)B_{j,s})SQ_j\left(s,T,y^z,S\right) \\
\int_{y_{\delta y}}^{y} \frac{\partial Q_j\left(s,T,y^z,S\right)}{Q_j\left(s,T,y^z,S\right)} &= -\int_{y_{\delta y}}^{y} \gamma F\left(s\right)S\partial y
\end{aligned}$$

where $y = y_{\delta y} + \delta y$. By solving the integral we get

$$Q_j\left(s,T,y^z,S\right) = \exp\left(-\gamma F\left(s\right)S\left(y - y_{\delta y}\right)\right)Q_j(s,y_{\delta y},S) \tag{2.12}$$

We get the same result when we consider selling an amount δy, as in $y = y_{\delta y} - \delta y$. Then, equation 2.11 becomes

$$\begin{gathered}
V_j\left(s,T,B_j^z,y^z,S\right) = V_j(s,T,B_{\delta y} + S\delta y, y_{\delta y} - \delta y, S) \\
-\frac{\partial V_j\left(s,T,B_j^z,y^z,S\right)}{\partial y} + S\frac{\partial V_j\left(s,T,B_j^z,y^z,S\right)}{\partial B} = 0
\end{gathered}$$

and by integrating from y to $y_{\delta y}$ we get equation 2.12.

2.7.1 Problem discretisation

The use of the negative exponential utility function significantly simplified the problem by reducing its dimensionality. Under the negative exponential utility function, we derived the price of selling the option at which it is independent of the initial endowment. In this section we describe the discretisation process for this problem and give an outline of the algorithm we use to computationally price such options.

In order to discretise the value function 2.10 we have to present the discretisation of the elements used to replicate the option payoff. We start by defining a discrete small time-step δs. We now need to define the discrete changes in the cash amount so, we define

$$\delta\Upsilon_s = -\Gamma_s \delta\upsilon_s$$

where Γ_s represents the price of the risky asset, at time s. We do not need to define an explicit discretisation scheme for the risky asset's price here, the only thing need mentioning is that the discrete changes in the price is $\delta\Gamma_s$ at time-step s. The schemes used for this purpose have been discussed earlier in the chapter. Lastly, we also define $\delta\upsilon_s$, the discrete changes in the amount held in the risky asset.

We now try to find a suitable discretisation for our value functions. The proposed discretisation scheme, using the definitions of the parameters of the underwriter's portfolio, looks like

$$V_j^{(n)}(s,T,\Upsilon,\upsilon,\Gamma) = \max\left\{ \begin{array}{l} I\!\!E\left\{V_j^{(n)}\left(s+\delta s, T, \exp\left(r\delta s\right)\left(\Upsilon - \delta\Upsilon\right), \upsilon + \delta\upsilon, \Gamma + \delta\Gamma\right)\right\}, \\ I\!\!E\left\{V_j^{(n)}\left(s+\delta s, T, \exp\left(r\delta s\right)\Upsilon, \upsilon, \Gamma + \delta\Gamma\right)\right\} \end{array} \right\}$$

where all the parameters are considered at time s and n the number of discretisation steps. In the first line, the value function is evaluated when a change in the holding of the risky asset takes place, while in the second line, no transaction takes place but the price of the risky asset changes. The boundary conditions are

$$V_j^{(n)}(T,T,\Upsilon,\upsilon,\Gamma) = U\left[P_j\left(T,\Upsilon,\upsilon,\Gamma\right)\right]$$

Equation 2.10 becomes

$$V_j^{(n)}\left(s,T,\Upsilon,\upsilon,\Gamma\right) = 1 - \exp\left(-\gamma F\left(s\right)\Upsilon\right) Q_j^{(n)}\left(s,T,\upsilon,\Gamma\right)$$

since the exponential utility function has the same effect in the discrete case as for the continuous case and $F\left(s\right) = exp\left(r\left(T-s\right)\right)$. The value function $Q_j^{(n)}\left(s,T,\upsilon,\Gamma\right)$ is defined in the same way as in the continuous case and its discretisation is given below

$$Q_j^{(n)}(s,T,v,\Gamma) = \min \left\{ \begin{array}{l} \exp\left(-\gamma F\left(s\right)\Gamma\delta v\right) I\!E\left\{Q_j^{(n)}\left(s+\delta s, T, v+\delta v, \Gamma+\delta\Gamma\right)\right\}, \\ I\!E\left\{Q_j^{(n)}\left(s+\delta s, T, v, \Gamma+\delta\Gamma\right)\right\} \end{array} \right\} \tag{2.13}$$

Again the first line assumes that some transaction has taken place. We do not need to have a separate line for buying and selling since only one form of transaction will revert the current holding to the optimal level. The way δv has been specified takes care of any sign changes in the equation. The second line is the expected value of the function when (at the same level of holding) the price of the risky asset diffuses. The boundary conditions for $Q_j^{(n)}(s,T,v,\Gamma)$ are

$$Q_j^{(n)}\left(T,T,v,\Gamma\right) = \exp\left(-\gamma G_j^{(n)}\left(T,v,\Gamma\right)\right) \tag{2.14}$$

The discrete form of equation 2.12 is the following

$$Q_j^{(n)}(s,T,v,\Gamma) = \exp\left(-\gamma F\left(s\right)\Gamma\delta v\right) Q_j^{(n)}(s,T,v-\delta v,\Gamma) \tag{2.15}$$

for the case of buying an amount δv. This leads to the definition of the discrete option writing price, which is

$$p_s = \frac{1}{\gamma F\left(s\right)} \ln\left(\frac{Q_a^{(n)}(s,T,0,\Gamma)}{Q_m^{(n)}(s,T,0,\Gamma)}\right) \tag{2.16}$$

when the underwriter holds none of the risky assets at the writing time s.

2.7.2 Pricing algorithm

The pricing algorithm shown in this section follows the concepts detailed in [11] . Equations 2.14 and 2.15 are the two equations that are used to price the option. In recursive form, we need to start from the boundary condition (equation 2.14) and by using equation 2.15 reach the starting time s. The prices for the risky asset are generated on a lattice, as described earlier and at each time-step the value function $Q_j^{(n)}(s,T,y,S)$ is evaluated for all possible holdings y. In order to make the process feasible we need to define upper and lower bounds (e.g. upper bound of 1 for holding 100% of the risky asset and lower bound of 0 for no asset holding) for y such that would make a reasonable holding of the risky asset. This is a dynamic programming algorithm and the computational algorithm is described below.

Assuming a lattice (either binomial or trinomial), we start from the expiration time T where the value function is calculated for all nodes based on the option payoff. Then, moving backwards on the lattice, we can calculate the holding in the risky asset that minimises the first case in our value function. Using function 2.15 all of the points of y (between the minimum and maximum holdings) are calculated for each node on the lattice at the particular time-step. Then the two cases in 2.13 are checked against each other and the minimising strategy is adopted for each node. In the event that trading is interrupted, only the second case is considered. This is repeated until the first time-step (representing the option writing time) is reached. This process is followed for both selling an option and then hedging it and for trading in the market for profit. The two values are stored and using equation 2.16 we can calculate the price for the option.

2.7.3 Results

In this section, the results that were obtained from simulating the single asset model implemented, will be presented. Further results evidencing the convergence of the algorithm to the Black-Scholes model [5] and the CRR model [9] can be seen in the Appendix I. For the first three sets of results the discretisation was performed using both binomial and trinomial lattices described before, while in the last section, a binomial lattice was used. Several different scenarios were applied to the algorithm yielding several interesting results. The pricing of the options were performed with an a priori knowledge of the bounds (and duration) of the illiquidity period. In all of the figures the y-axis represents the option writing price while the x-axis represents time, unless otherwise stated. Four different scenarios are tested. Namely, in the first scenario, we investigate the effect of the illiquidity period starting time (trading stopping time), in the second, the effect of the illiquidity period stopping time (trading resuming time) and in the third, we assume that a fixed period of illiquidity occurs, starting at different times. In the final set of results we investigate the effect of splitting a fixed interruption period into smaller ones, amounting to the initial period of interruption.

For the first three scenarios, some parameters are fixed. These remain constant throughout and concern the current price of the risky asset, $S = 23$, the option contract strike price, $K = 23$, the expiration of the contract, $T = 1$, the interest rate level, $r = 10\%$, the risky asset's average return, $\mu = 0\%$ and the risk aversion factor $\gamma = 1$. The risky asset's return volatility will be fluctuating and reference to the changes in its level, will be made whenever appropriate. The number of time steps used are shown next to the name of the asset price discretisation method used for the pricing and the number of holding steps is set to 100.

2.7.3.1 Effect of the starting time for the illiquidity period

The first set of results presents the case when trading is interrupted at a time after selling the option. The first test case was to explore what happens to the price of the sold option when there can be no trading, throughout the life of the contract, in the underlying asset. Afterwards, the period that illiquidity starts, is pushed further towards the expiration of the option. In the two figures that follow (Figure 2.3 and Figure 2.4), two different volatilities for the underlying's returns are assumed, 10% in the first and 20% in the second. The time the illiquidity period starts is shown on the x-axis. In this scenario, it is assumed that trading does not resume before the expiration of the option.

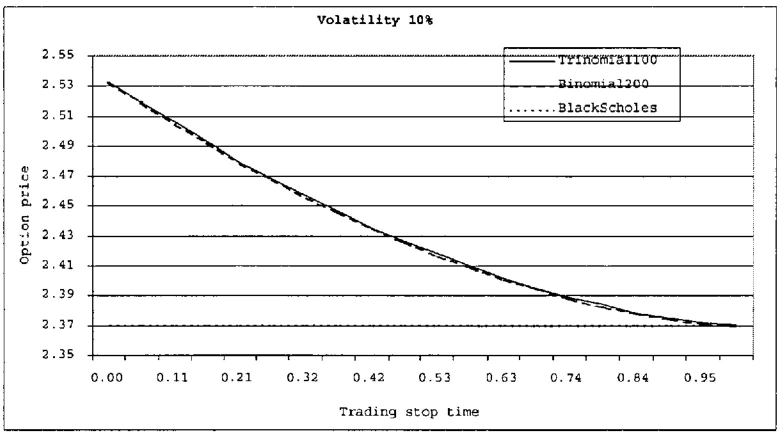


Figure 2.3: Effect of trading stopping time

Starting with the obvious results, we can see that when we assume that there is no illiquidity period (as in that the underlying can be freely traded during the lifetime of the option), the price we arrive at using the dynamic programming algorithm, coincides with the Black and Scholes price calculated for an option defined with the equivalent parameters. This is quite reassuring, because it forms a sort of a safe check for the method we are employing. Starting from this point we can now draw the intuitive results from the option's price behaviour. The decrease in the price of the option as the periods of illiquidity become smaller, agrees with the intuition, since this means that he will have increasingly more chances of hedging his exposure created by selling the option. The bigger the illiquidity period, the bigger the exposure to fluctuations of the underlying's price. To put it in other words, we can think of this scheme as a sort of hedge-and-forget scheme but with the periods of the static hedge becoming bigger and bigger.

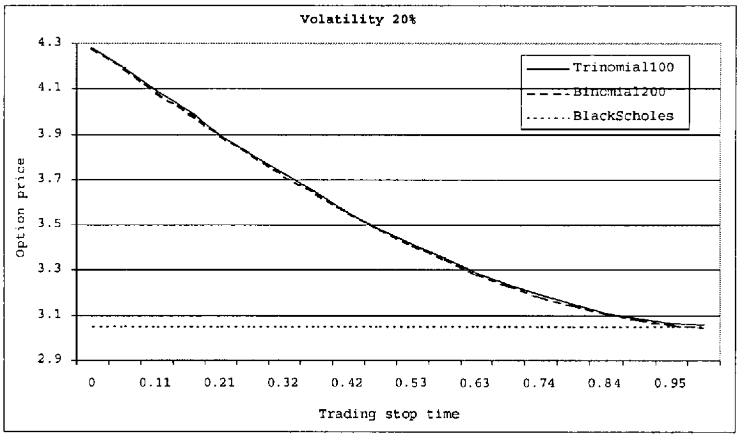


Figure 2.4: Effect of trading stopping time

Comparing the two figures (Figure 2.3 and Figure 2.4), for different levels of volatility, we can see that again the results agree with intuition, since the prices for the option with higher volatility are higher. Moving forward, we can observe that the changes in the option prices are also bigger. This has a dual explanation. Firstly, higher volatility means bigger price fluctuations, hence, the underwriter is exposed to a higher risk. Secondly, due to the same fact, the underwriter is not able to take advantage of more privileged prices during his hedging.

2.7.3.2 Effect of stopping time for the illiquidity period

Similar results to the above are observed when we reverse the experiment. In this case the starting time of the illiquidity period is fixed at the time of selling the option, while the time the illiquidity period stops, is moved further away from this starting time to the expiry of the option. In this case, the only time when the option is fully hedged is at the far left of the figure, where the starting and stopping time of the illiquidity period coincide at time 0. In this example the volatility was set to 10% and 20% respectively for Figure 2.5 and Figure 2.6.

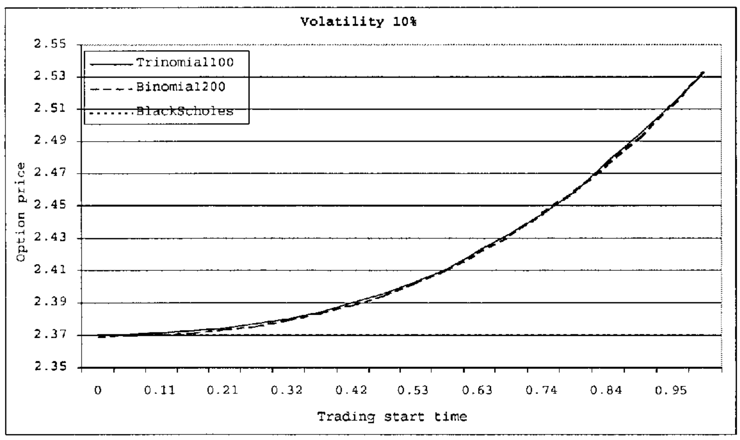


Figure 2.5: Effect of trading resuming time

By comparing the prices for the two sets of results (shifting the illiquidity period starting time and then the stopping time; set of Figure 2.3, Figure 2.5 and Figure 2.4, Figure 2.6 for volatility 10% and 20% respectively) we can observe that the prices achieved for similar periods of illiquidity are not the same, which clearly shows that apart from the effect on the size of the illiquid period, its position during the lifetime of the option must have an effect on the option's price as well. In particular, we can see that the price difference of an option with an illiquidity period of half its lifetime is almost 0.03 (for volatility 10%) and 0.1 (for volatility 20%) bigger when this illiquidity period takes place at the second half of the option's lifetime. This effect is investigated next.

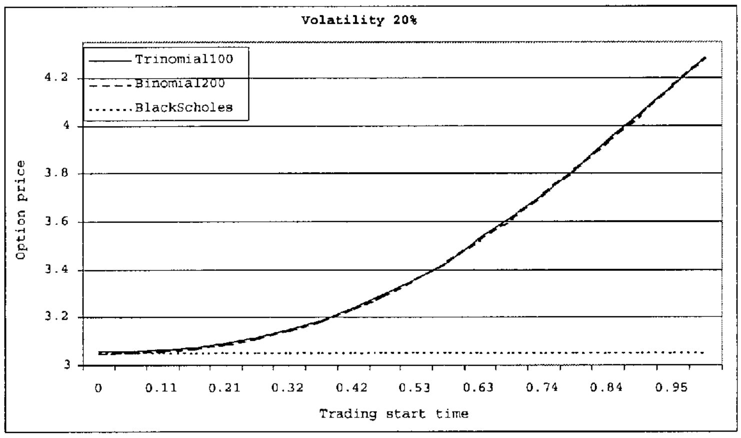


Figure 2.6: Effect of trading resuming time

2.7.3.3 Effect of a fixed length illiquidity period

Now we will explore the effect of a fixed illiquidity period over the lifetime of the option, but will investigate the effect it has on the option price if we shift it from the time the option is sold towards the time the option expires. In the following examples, the same parameters, as before, have been used but, the volatility has been set at 20% to make any features for this sort of change more pronounced. Again, the y-axis represents the option prices and the x-axis represents the beginning of the illiquidity period. In the first figure (Figure 2.7) we assume a period of illiquidity of 0.2 periods, where in the second (Figure 2.8) we assume an illiquidity period of 0.4. As in the previous set of results, time 1 is the expiration of the option.

Unfortunately, in this case we cannot compare the price of the option calculated using the dynamic programming algorithm at any point with the Black and Scholes price, since we assume that there is a fixed period of illiquidity throughout the life of the option. What we can do though, is to check the price calculated for an option whose illiquidity period

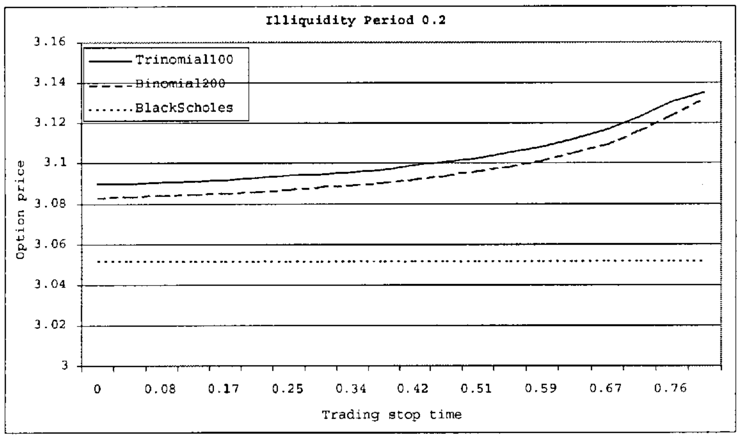


Figure 2.7: Effect of fixed illiquidity period starting time

ends at time 1 (that is similar to the illiquidity period starting at period 0.8 for the first of the two figures and at 0.6 for the second) and compare these prices with the prices on Figure 2.4 for the equivalent periods. Indeed the prices match, so we have another safe check for the prices presented here.

What we observe instantly is that the written option prices increase as the periods of illiquidity tend to end towards the expiration of the option. This can be explained by the intuitive notion, that the underwriter (assuming that a liquid period proceeds the period of illiquidity) will hedge his exposure initially, do no hedging during the period of illiquidity and eventually compensate for any misshedging toward the expiration of the option.

The curves produced are monotonically increasing, which again makes sense. The bigger the period he has left to correct any misshedging, the more accurate the replication will be. A further point we can draw by comparing the two figures is that the bigger the period of illiquidity the bigger the differences in the option prices become (note that there is a difference of scale).

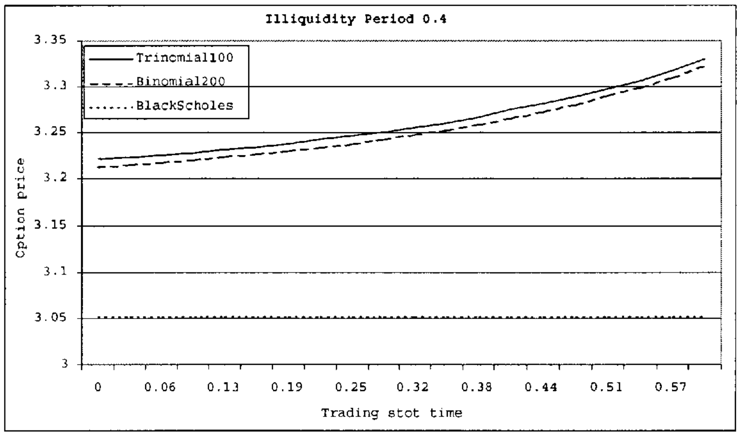


Figure 2.8: Effect of fixed illiquidity period starting time

2.7.3.4 Effect of multiple trading interruption periods

So far, we have only investigated the effect a single period of interruption has on the option price. It is very useful to be able to restrict the problem not just during one period, but over many periods during the time horizon. In the following examples, this effect is investigated. The analysis will start with a demonstration of how an increasing number of periods (of the same length) affect the option price. For all the results in this part, the spot price S is set to 20, the yield is set to 0 and the volatility is set to 30%. The strike of the option, K, is set to 20, while the option expires in one year. The constant risk free interest rate is set to 3% and the risk aversion factor, γ, is 0.5. The results were obtained using only a binomial lattice. From our previous analysis, it has been demonstrated that an increasing period of interruption has a proportional effect on the option price, that means that an increasing period of interruption increases the option price. One would expect an increasing number of interruption periods (of fixed duration) to have the same effect.

In Figure 2.9 the number of interruption periods is increased from one week to four

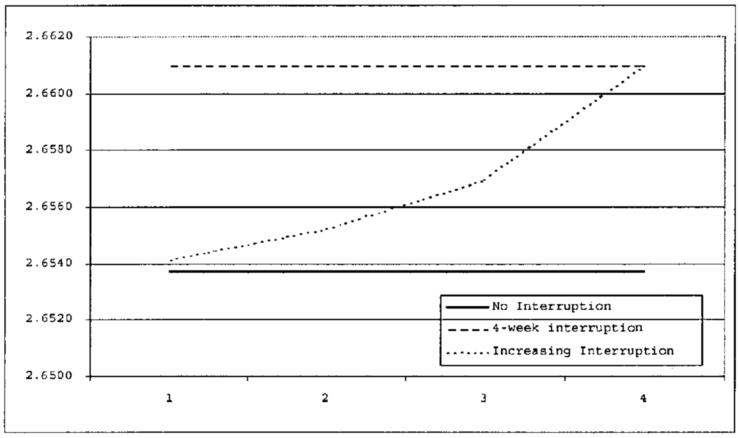


Figure 2.9: Effect of increasing number of interruption periods

weeks. The period of interruption starts at the middle of the year and increases towards the expiry of the option. The interruption periods are adjacent to each other, thus amounting to a month. The price for each additional period of interrupt is compared to the price of an option with no interrupted trading and an option price for an option with a single period of interruption four weeks long. It can be seen that as each week is added onto the initial period of interruption, the value of the option moves from the no interruption price to the four week single interruption option price.

Having established that increasing number of interrupts (when adjacent) amount to a single interrupt lasting for the sum of the individual interrupts, we will now investigate the effect of splitting an interruption period into smaller periods. The periods will not be adjacent to each other, but the sum of the individual periods will amount to the single interrupt. For example a four week period of trading interruption will be progressively split into two two-week periods of interruption, three one and a third-week periods and four one-week periods. The first example, demonstrates the effect of splitting a six-week interruption period, ending at the beginning of the sixth month, into progressively smaller periods. The additional periods are placed inside the first six months of the option's life. In the following table, the data used for this example are displayed. The first row of each box indicates the duration of each interruption period (e.g. the first box has one interruption period of six weeks, the second has two three-periods and so on) while the rest of the rows indicate the times at which the interruption periods start and end.

dur	6.0	dur	3.0	dur	2.0
start	end	start	end	start	end
0.3846	0.5000	0.1923	0.2500	0.1282	0.1667
		0.4423	0.5000	0.2949	0.3333
				0.4615	0.5000

dur	1.5	dur	1.2	dur	1.0
start	end	start	end	start	end
0.0962	0.1250	0.0769	0.1000	0.0641	0.0833
0.2212	0.2500	0.1769	0.2000	0.1474	0.1667
0.3462	0.3750	0.2769	0.3000	0.2308	0.2500
0.4712	0.5000	0.3769	0.4000	0.3141	0.3333
		0.4769	0.5000	0.3974	0.4167
				0.4808	0.5000

In Figure 2.10 the results obtained for this example can be seen. The x-axis in the figure denotes the number of interruption periods, while the y-axis gives the corresponding option price. The first thing observed from this figure is that the option price reduces as the number of interruption periods gets bigger. This change in value is probably due to the fact that increasingly smaller periods intervene between the normal trading periods, making the handling of the risk associated easier. In the next example the same effect is investigated. Starting from a six-week interruption period, we split it down to two periods of three-week interruptions, etc. In this example the periods of interruption are distributed evenly across the year. The durations, starting and ending times for this example are shown below.

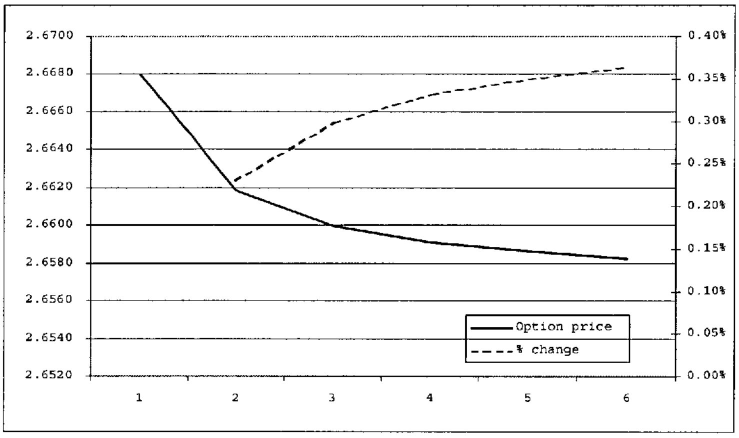


Figure 2.10: Effect of splitting the interruption period

dur	6	dur	3	dur	2
start	end	start	end	start	end
0.442308	0.557692	0.304487	0.362179	0.230769	0.269231
		0.637821	0.695513	0.480769	0.519231
				0.730769	0.769231

dur	1.5	dur	1.2	dur	1
start	end	start	end	start	end
0.185577	0.214423	0.155128	0.178205	0.133242	0.152473
0.385577	0.414423	0.321795	0.344872	0.276099	0.29533
0.585577	0.614423	0.488462	0.511538	0.418956	0.438187
0.785577	0.814423	0.655128	0.678205	0.561813	0.581044
		0.821795	0.844872	0.70467	0.723901
				0.847527	0.866758

The results obtained from this set of dates are shown in Figure 2.11. We can observe that the effect is exactly the same as before, suggesting that the positioning of the periods is less important than the number of interruption periods.

We will now investigate the sensitivity of these observations to parameters such as the volatility of the risky asset and the risk aversion factor. For this set of results the volatility of the risky, σ, is set to 20% and the risk aversion factor γ is set to 0.5 (unless otherwise stated). The rest of the parameters are the same as with the previous example. The initial period is a twelve-week period, which is situated at the middle of the year. This period is split into two six-week periods, three four-week periods, four three-week periods, six two-

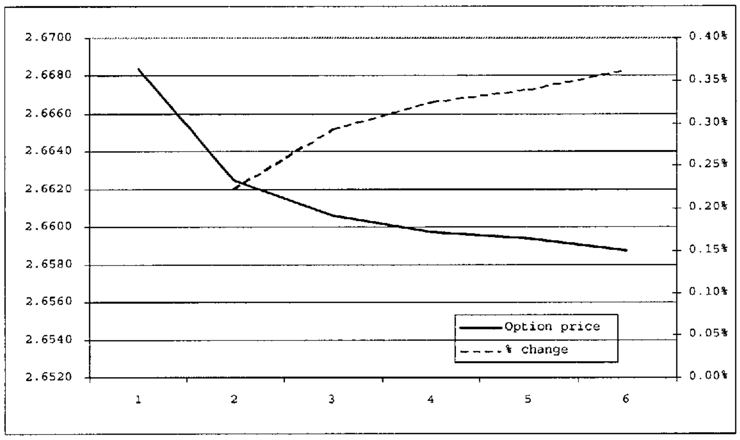


Figure 2.11: Effect of splitting the interruption period (even distribution)

week periods and twelve one-week periods progressively and are all evenly distributed across the year. The times for these interruption periods are shown in the table below

dur	12	dur	6	dur	4
start	end	start	end	start	end
0.3846	0.6154	0.2500	0.3654	0.1923	0.2692
		0.6346	0.7500	0.4615	0.5385
				0.7308	0.8077

dur	3	dur	2	dur	1
start	end	start	end	start	end
0.1548	0.2125	0.0971	0.1356	0.0769	0.0962
0.3663	0.4240	0.2510	0.2894	0.1538	0.1731
0.5779	0.6356	0.4048	0.4433	0.2308	0.2500
0.7894	0.8471	0.5587	0.5971	0.3077	0.3269
		0.7125	0.7510	0.3846	0.4038
		0.8663	0.9048	0.4615	0.4808
				0.5385	0.5577
				0.6154	0.6346
				0.6923	0.7115
				0.7692	0.7885
				0.8462	0.8654
				0.9231	0.9423

Volatility sensitivity

In the next three figures, the volatility of the risky asset is increased from 10% to 20% and finally to 30%. The results can be seen in Figure 2.12, Figure 2.13 and Figure 2.14. In these figures the solid line represents the option price (on the major axis) and the dotted line represents the change in value for each split from the single twelve-week period (secondary axis).

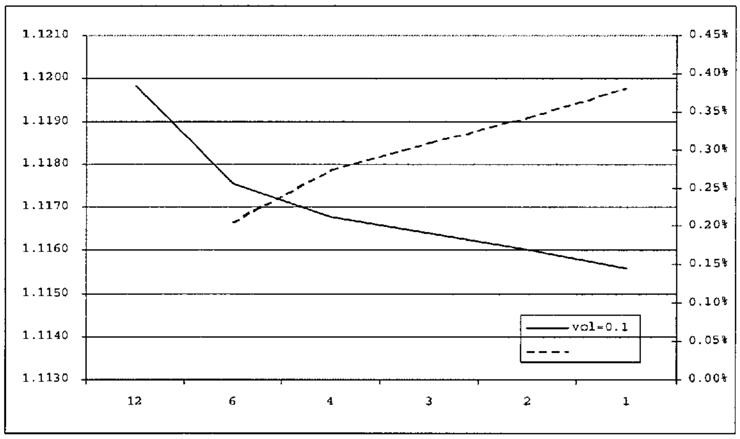


Figure 2.12: Volatility sensitivity (vol = 10%)

The three figures look very similar in shape. The option price is reduced for each split performed. Taking into account that the number of periods after each split is not linear, one may assume that there is a lower bound for the reduction of the option price. It is very difficult to establish what this lower bound is, as it requires an increasing number of steps (for the discretisation of the lattice) in order to capture the decreasing interruption periods.

The main point of that can be drawn from this sensitivity analysis is that the reduction achieved by splitting the initial interruption period is increasing as the volatility of the asset is increased. The last sensitivity demonstrated for this model is the sensitivity to the risk aversion factor (γ), provided in the next set of results.

Gamma sensitivity

In this set of results, the sensitivity of the change in option price to the risk aversion factor, γ, is demonstrated. Similarly to the volatility sensitivity, three figures are presented

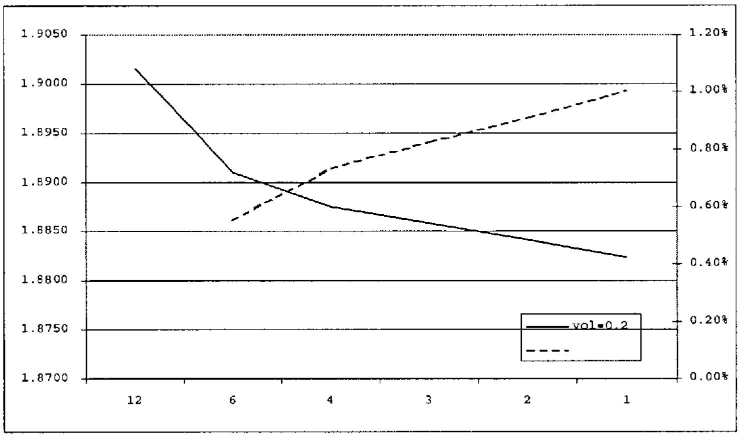


Figure 2.13: Volatility sensitivity (vol = 20%)

Figure 2.15, Figure 2.16 and Figure 2.17.

The risk aversion factor is increased from 0.2, to 0.5 and finally to 0.7. As was the case for the volatility sensitivities, the option price is reduced every time we split the interruption period. The reduction for the different levels of risk aversion is different, which suggests that a more risk averse investor (higher levels of γ) will gain more from following such a strategy (distributing the risk into many periods, rather than a single period).

This set of results concludes this chapter. In this results section we have demonstrated some intuitive results, such as the increase of the option price with increasing periods of interruption and the increase of the option price with the increase of the number of fixed length periods of interruption. Further analysis has shown some less intuitive results, such as the decrease of the option price with the increase of the number of interruption periods, when the overall interruption throughout the option life remains constant. The analysis has also shown that there is some sensitivity to the chronological placement of a fixed

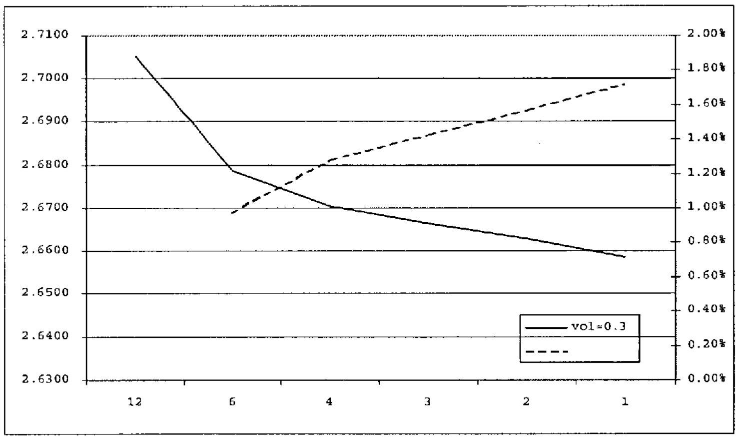


Figure 2.14: Volatility sensitivity (vol = 30%)

illiquidity period.

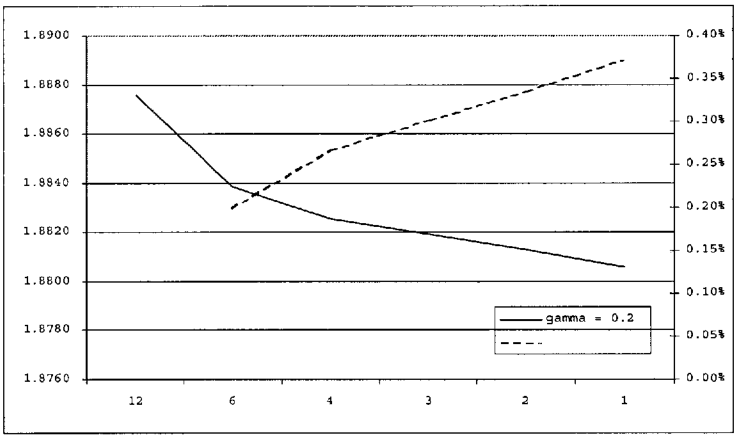


Figure 2.15: Risk aversion sensitivity (gamma = 0.2)

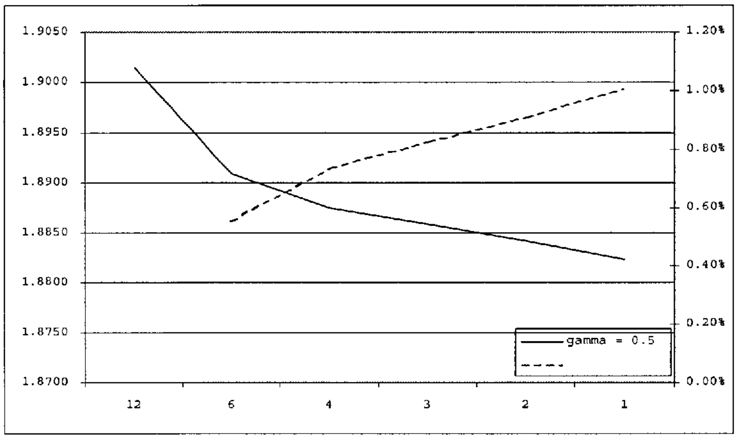


Figure 2.16: Risk aversion sensitivity (gamma = 0.5)

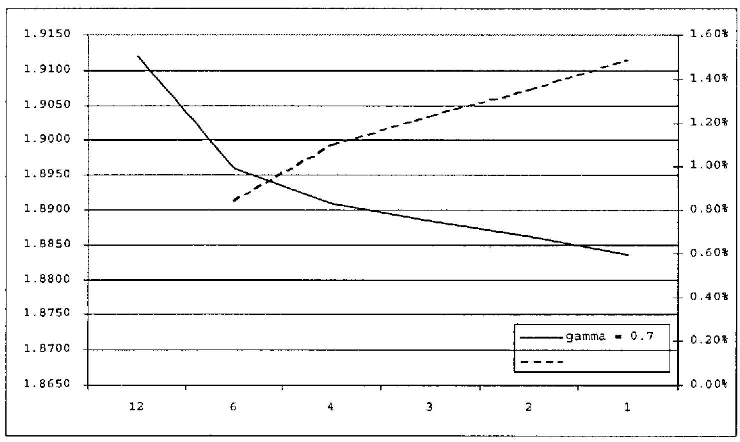


Figure 2.17: Risk aversion sensitivity (gamma = 0.7)

Chapter 3
Pricing an option on two illiquid assets

In this chapter we will investigate the same problem as before (the effect of trading interrupts of the underlying on the option price), but consider two assets as the underlying. This sort of options are particularly popular in the energy sector where a multiple number of options such as spread options on different commodities (crack and spark spreads) or different futures prices of the same commodity (calendar spreads) are traded. Other forms of two-factor options are very common across the border of trading, such as the exchange and binary options. Similarly to the previous chapter, we will attempt to price the option using dynamic programming, therefore the form of the payoff function is irrelevant to the method that will be developed, hence all these options can be considered as possible applications of the model.

The investigation in the previous chapter has provided us with a good foundation on which to build this extension, along with valuable insight in the form of the utility function. The basic concepts and methodology used in the pricing of the single factor derivative, of the previous chapter (and indeed the concepts from [11]), are directly transferable to the pricing of higher order derivatives. The basic concept on the value of the derivative is practically the same, while the methodology described before will be closely followed here as well. Furthermore, the negative exponential utility function simplifies the optimisation problem significantly, since it reduces the dimensionality of the problem by one degree, something which is highly desirable in the setting of a two-factor portfolio. This should considerably reduce computation time and could possibly allow the development of higher order portfolios.

3.1 Background of the pricing method

In this section the basic model for pricing derivatives on two assets will be developed. The concepts presented in the previous chapter will be used in order to develop this model. The modification from the previous model concerns the portfolio assumed at the start of the pricing period, which is different, in that two risky assets are assumed to be present along with the riskless investment (i.e. a cash amount).

Using the same argument as before, we can see that the form of the derivative is irrelevant, as is the number of assets involved in its pricing. Hence, the price of the option is again

$$p_s = B_{o,s} - B_{m,s}$$

where B_o and B_m are the two premiums associated with the two possibilities available to the underwriter of the option.

The prices of the two risky assets are assumed to be lognormally distributed; the processes are given below

$$\begin{aligned} dS_{1,t} &= S_{1,t}\left(\mu_1 dt + \sigma_1 dw_{1,t}\right) \\ dS_{2,t} &= S_{2,t}\left(\mu_2 dt + \sigma_2 dw_{2,t}\right) \end{aligned} \tag{3.1}$$

where μ_1 and μ_2 are the two constant drifts and σ_1 and σ_2 are the two constant volatilities of the two assets. The two processes are driven by two correlated, normally distributed Brownian motions, $w_{1,t}$ and $w_{2,t}$. The instantaneous correlation between the two assets (assumed constant) is ρ, giving

$$\mathbb{E}\left[dw_{1,t}dw_{2,t}\right] = \rho dt$$

The cash portion of the portfolio is driven by the following process, depending on the price and the holding processes (dY_1 and dY_2) for the two assets, where $y_{1,t}$ and $y_{2,t}$ are the number of units held in assets S_1 and S_2, respectively

$$dB_t = rB_t dt - S_{1,t}dY_{1,t} - S_{2,t}dY_{2,t} \tag{3.2}$$

where r is a constant interest rate. Using these definitions we will now try to devise a portfolio under the two earlier assumptions.

3.2 Constructing the value of the portfolio

As already mentioned in the previous chapter, there are two possibilities available to the portfolio holder; either trade an option and hedge away the risk undertaken by doing so, or trade in the market for some profit. For these two cases we need to start by describing the value of the portfolio at the expiry time that we are interested in. Since we are not yet concerned with the payoff of the option, we will construct a function for designating the payoff, which will depend on the prices of the two assets and the strike price agreed. Therefore, we devise a function $O\left(T, y_1, y_2, S_1, S_2, K\right)$ that replaces the option payoff at the expiry time T, for a strike price K and an exercise condition $f\left(T, S_1, S_2, K\right) > 0$. Obviously the prices S_1 and S_2 refer to the prices of the assets at time T. Both functions will be specified later.

Starting from the simpler case of the underwriter trading for profit in the market, the final value of the portfolio will be

$$P_m\left(T, B_m, y_1, y_2, S_1, S_2\right) = B_{m,T} + y_{1,T}S_{1,T} + y_{2,T}S_{2,T}$$

since there is no payoff at the problem horizon T. In this function $y_{i,T}$ is the final holding of the ith asset at expiry time T. B_m is the amount of cash left in the portfolio after the expiry.

The case where an option has been traded creates two possibilities, when the exercise condition is satisfied the option is exercised; when the exercise condition is false the option expires unexercised. To express these two cases the use of an indicator function is required again, hence the indicator $I_{excond}[g(x)]$ from the previous chapter is used. The following function describes the value of the portfolio at the expiry time T, based on the definitions above,

$$\begin{aligned} P_o\left(T, B_o, y_1, y_2, S_1, S_2\right) &= B_{o,T} + I_{f(T,S_1,S_2,K)\leq 0}\left[y_{1,T}S_{1,T} + y_{2,T}S_{2,T}\right] \\ &\quad + I_{f(T,S_1,S_2,K)>0}\left[O\left(T, y_1, y_2, S_1, S_2, K\right)\right] \end{aligned}$$

where B_o is the cash amount left in the portfolio at the expiry T after trading in S_1 and S_2.

Since the two final values of the portfolio have the above form we can generalise the form of the function and separate the cash amount from the final value of the asset holdings. This creates a similar function to the one in the previous chapter which is independent of the cash amount in the portfolio. Its form is given below

$$G_j\left(T, y_1, y_2, S_1, S_2\right) = P_j\left(T, B_j, y_1, y_2, S_1, S_2\right) - B_{j,T}$$

where the index j is used to distinguish between the two cases.

Now we are ready to develop a mathematical framework to tackle the option price.

The expected value of the portfolio must be at least as great as the claim of the option and therefore, we use the following definition for our new value function that describes this expected value of the portfolio for some trading strategy z

$$V_j\left(T, B_j, y_1, y_2, S_1, S_2\right) = \sup_{z \in \Im(B)} I\!E\left\{U\left[G_j\left(T, y_1^z, y_2^z, S_1^z, S_2^z\right) + B_{j,T}^z\right]\right\} \qquad (3.3)$$

where $U(x)$ is a utility function that has the following form

$$U(x) = 1 - \exp(-\gamma x)$$

where γ is a risk aversion factor between $(0, 1]$.

3.2.1 Calculation of the option price

By introducing the utility function equation, we can transform $V_j\left(T, B, y_1, y_2, S_1, S_2\right)$ to

$$V_j\left(T, B_j, y_1, y_2, S_1, S_2\right) = 1 - \inf_{z \in \Im(B)} I\!E\left\{\exp\left(-\gamma B_{j,T}^z - \gamma G_j\left(T, y_1^z, y_2^z, S_1^z, S_2^z\right)\right)\right\}$$

which transforms the problem into a minimisation problem. We can further simplify the problem, because we know the form of the process dB (equation 3.2). By integrating the riskless investment process, we get (a similar equation has been solved in Davis, Panas and Zariphopoulou [11] for a single asset with transaction costs)

$$B_T = \exp\left(r\left(T - s\right)\right) B - \int_s^T \exp\left(r\left(T - t\right)\right) S_{1,t} dy_{1,t} - \int_s^T \exp\left(r\left(T - t\right)\right) S_{2,t} dy_{2,t}$$

The introduction of this explicit function for B_T, allows us to calculate the value function $V_j\left(T, B_j, y_1, y_2, S_1, S_2\right)$ from any time s to the expiry time T.

$$V_j\left(s, T, B_j, y_1, y_2, S_1, S_2\right) = 1 - \exp\left(-\gamma F(s) B_j\right) Q_j\left(s, T, y_1, y_2, S_1, S_2\right) \qquad (3.4)$$

where $F(s) = \exp\left(r\left(T - s\right)\right)$. $Q_j\left(s, T, y_1, y_2, S_1, S_2\right)$ is the function that we now need to dynamically assess, in order to price the option. $Q_j\left(s, T, y_1, y_2, S_1, S_2\right)$ has the following form

$$Q_j(s,T,y_1,y_2,S_1,S_2) = \inf_{z\in\Im(B)} \{\exp(-\gamma G_j(T,y^z_{1,T},y^z_{2,T},S^z_{1,T},S^z_{2,T}) + \gamma\int_s^T F(t)\,S^z_{1,t}dy^z_{1,t} + \gamma\int_s^T F(t)\,S^z_{2,t}dy^z_{2,t})\}$$

The only thing required now is to get a solution for the premium B_j and substitute it in the definition of the option price. Solving equation 3.4 for B_j yields

$$B_{j,s} = -\frac{1}{\gamma F(s)}\ln\left(\frac{1-V_j(s,T,B_j,y_1,y_2,S_1,S_2)}{Q_j(s,T,y_1,y_2,S_1,S_2)}\right)$$

By replacing the premiums in the definition of the option price, equation 2.2, we get

$$p_s = \frac{1}{\gamma F(s)}\ln\left(\frac{(1-V_m(s,T,B_j,y_1,y_2,S_1,S_2))\,Q_o(s,T,0,0,S_1,S_2)}{(1-V_o(s,T,B_j,y_1,y_2,S_1,S_2))\,Q_m(s,T,0,0,S_1,S_2)}\right)$$

The underwriter of the option must not hold any amount of stock at the time of trading the option, otherwise hedging the option would be advantageous. As mentioned before, both the $V_j(.)$ functions are zero at the time of trading the option (s), when the amount of stock is zero. Hence, the final value of the option becomes

$$p_s = \frac{1}{\gamma F(s)}\ln\left(\frac{Q_o(s,T,0,0,S_1,S_2)}{Q_m(s,T,0,0,S_1,S_2)}\right) \tag{3.5}$$

As it can be seen, the form of the price of the option is similar to the expression from the previous chapter. Next the algorithm for pricing the option will be described followed by the discretisation of the problem, where the constraints to the pricing will be set.

3.3 Parametric form of Q_j for the dynamic program

One vital step to price this option is to construct a parametric form for Q_j. This will be used in the dynamic algorithm to price the option. There are four distinct regions we must consider (Figure 3.18). The four regions determine the combined buying and selling of the two assets (e.g. buy δy_1, sell δy_2 in the figure below), while the dotted lines represent the cases when the buying or selling of either asset is exclusive and the intersection of the lines represents the case of trading absence (neither y_1 or y_2 are adjusted, either because it is optimal to do so, or because it is not allowed).

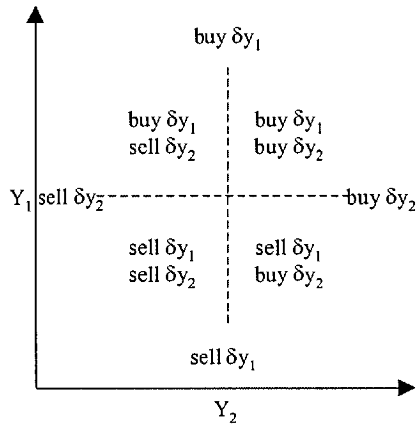


Figure 3.18: Buy and sell regions

We can distinguish between the two adjustments in the holdings of the assets, since they are independent (the holding in one asset does not affect the holding in the other) and hence can proceed in the following way. By treating the adjustment in the holding of the first asset, we can say that

$$V_j\left(s,T,B_j,y_1,y_2,S_1,S_2\right)=V_j\left(s,T,B_{j,\delta y_1}-S_1\delta y_1,y_{\delta y_1}+\delta y_1,y_2,S_1,S_2\right)$$

where δy_1 is the change in the first asset (we can treat the case of buying and selling under the same scheme, since selling an amount δy_1 is equivalent to buying an amount $-\delta y_1$). By allowing $\delta y_1 \to 0$ we get the following partial differential equation

$$\frac{\partial V_j\left(s,T,B_j,y_1,y_2,S_1,S_2\right)}{\partial y_1}-S_1\frac{\partial V_j\left(s,T,B_j,y_1,y_2,S_1,S_2\right)}{\partial B}=0$$

By performing the partial differentiation and integrating the above function between $[y_1, y_{\delta y_1}+\delta y_1]$, we get

$$\frac{\partial Q_j(s,T,y_1,y_2,S_1,S_2)}{\partial y_1} = -\gamma F(s) S_1 Q_j(s,T,y_1,y_2,S_1,S_2)$$

By solving the equation we can get the parametric solution for $Q_j(.)$ from any point $y_{\delta y_1}$ (that is δy_1 away from y_1).

$$Q_j(s,T,y_1,y_2,S_1,S_2) = \exp\left(-\gamma F(s) S_1 (y_1 - y_{\delta y_1})\right) Q_j(s,T,y_{\delta y_1},y_2,S_1,S_2)$$

This result holds for both buying and selling of a δy_1 amount of the asset. The result yielded for the second asset is the same

$$Q_j(s,T,y_1,y_2,S_1,S_2) = \exp\left(-\gamma F(s) S_2 (y_2 - y_{\delta y_2})\right) Q_j(s,T,y_1,y_{\delta y_2},S_1,S_2)$$

but, these equations are only able to deal with the case when the adjustment of the holdings are exclusive.

In the case that two moves have to take place at the same time, then another two cases have to be considered (since the buy and sell moves are equivalent). The first case is when δy_1 is bought/sold first and the second when δy_2 is bought/sold first. In both cases the resulting equation is the same. For example, lets consider the case when δy_1 is bought/sold first (we start from a point $Q_j(s,T,y_{\delta y_1},y_{\delta y_2},S_1,S_2)$)

$$Q_j(s,T,y_1,y_{\delta y_2},S_1,S_2) = \exp\left(-\gamma F(s) S_1 (y_1 - y_{\delta y_1})\right) Q_j(s,T,y_{\delta y_1},y_{\delta y_2},S_1,S_2)$$

We already know the form of the function for $Q_j(s,T,y_1,y_{\delta y_2},S_1,S_2)$, so by replacing the equation the above for a buy/sell of δy_2, we get

$$\begin{aligned} Q_j(s,T,y_1,y_2,S_1,S_2) &= \exp\left(-\gamma F(s)\left(S_1(y_1 - y_{\delta y_1}) + S_2(y_2 - y_{\delta y_2})\right)\right) \\ & \quad Q_j(s,T,y_{\delta y_1},y_{\delta y_2},S_1,S_2) \end{aligned} \tag{3.6}$$

From the form of equation 3.6 it is clear that the result is the same when δy_2 is traded first.

3.4 Discretisation of the problem

In order to price the option, we need to develop a discrete time dynamic algorithm. The discrete versions of the continuous processes for the asset prices and the amount of cash (equation 3.1 and equation 3.2, respectively) will be used to that effect. The three discretisation parameters that will be used here are the changes in the level of holding in each asset ($\delta \upsilon_1$ and $\delta \upsilon_2$) and the time increments (δs). By allowing these three parameters to go to zero, the discrete version of the value function (equation 3.3) $V_j^{(n)}(s, T, \Upsilon_j, \upsilon_1, \upsilon_2, \Gamma_1, \Gamma_2)$, must converge to its continuous time counterpart (this can be proved in a similar way as Theorem 4 in Davis, Panas and Zariphopoulou [3]). First the discrete evolution of the assets' price and the cash amount will be provided.

The changes in the amount of cash in the portfolio are described by the following equation

$$\delta \Upsilon_s = -\Gamma_{1,s}\delta \upsilon_{1,s} - \Gamma_{2,s}\delta \upsilon_{2,s}$$

The asset price updates are described later in this chapter. The only thing required here is to mention that the change in price for the two risky assets is assigned to the $\delta \Gamma_{i,s}$ variable.

The discrete value function $V_j^{(n)}(s, T, \Upsilon_j, \upsilon_1, \upsilon_2, \Gamma_1, \Gamma_2)$ is given by the following non-linear function

$$\max_{z \in \Im(B)} \left\{ \begin{array}{l} \mathbb{E}\left[V_j^{(n)} \left(\begin{array}{c} s + \delta s, T, \exp(r\delta s)\left(\Upsilon_{j,s}^z - \delta \Upsilon_{j,s}^z\right), \upsilon_{1,s}^z + \delta \upsilon_{1,s}^z, \\ \upsilon_{2,s}^z + \delta \upsilon_{2,s}^z, \Gamma_{1,s} + \delta \Gamma_{1,s}, \Gamma_{2,s} + \delta \Gamma_{2,s} \end{array} \right) \right], \\ \mathbb{E}\left[V_j^{(n)} \left(s + \delta s, T, \exp(r\delta s)\Upsilon_{j,s}^z, \upsilon_{1,s}^z, \upsilon_{2,s}^z, \Gamma_{1,s} + \delta \Gamma_{1,s}, \Gamma_{2,s} + \delta \Gamma_{2,s} \right) \right] \end{array} \right\}$$

We can use this definition to progress to the final statement of the problem, namely to use the solution of the continuous time problem (via exploiting the form of the utility function) and express the problem as one of minimisation, in terms of the discrete time form of the $Q_j(s, T, y_1, y_2, S_1, S_2)$ function. Equation 3.4 is used in combination with the parametric equation 3.6 and the discrete time problem becomes the minimisation of

$$Q_j^{(n)}(s, T, \upsilon_1, \upsilon_2, \Gamma_1, \Gamma_2)$$
$$\min_{z\in\Im(B)} \left\{ \begin{array}{l} \exp\left(-\gamma F(s)\left(\Gamma_{1,s}\delta\upsilon_{1,s} + \Gamma_{2,s}\delta\upsilon_{2,s}\right)\right) Q_j^{(n)}(s, \upsilon_{1,s}^z + \delta\upsilon_{1,s}^z, \upsilon_{2,s}^z + \delta\upsilon_{2,s}^z, \Gamma_{1,s} \\ +\delta\Gamma_{1,s}, \Gamma_{2,s} + \delta\Gamma_{2,s}),\ I\!E\left\{Q_j^{(n)}\left(s + \delta s, \upsilon_{1,s}^z, \upsilon_{2,s}^z, \Gamma_{1,s} + \delta\Gamma_{1,s}, \Gamma_{2,s} + \delta\Gamma_{2,s}\right)\right\} \end{array} \right\} \tag{3.7}$$

The boundary conditions for our problem are

$$Q_m^{(n)}(T, T, \upsilon_1, \upsilon_2, \Gamma_1, \Gamma_2) = \exp\left(-\gamma\left(\upsilon_{1,T}\Gamma_{1,T} + \upsilon_{2,T}\Gamma_{2,T}\right)\right) \tag{3.8}$$

when an option has not been traded and

$$\begin{aligned} Q_o^{(n)}(T, T, \upsilon_1, \upsilon_2, \Gamma_1, \Gamma_2) &= \exp(-\gamma I_{f(T,\Gamma_1,\Gamma_2,K)\leq 0}\left[\upsilon_{1,T}\Gamma_{1,T} + \upsilon_{2,T}\Gamma_{2,T}\right] \\ &\quad -\gamma I_{f(T,\Gamma_1,\Gamma_2,K)>0}\left[O\left(T, \upsilon_1, \upsilon_2, \Gamma_1, \Gamma_2, K\right)\right]) \end{aligned} \tag{3.9}$$

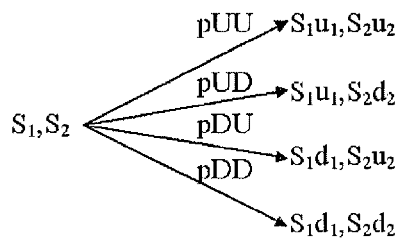
when an option has been traded.

We can now impose restrictions to these functions in the forms of non-trading periods. This is done by nominating a period, between the pricing date and the expiry date, and restricting the minimisation of equation 3.7 to the lower function, thus allowing no control over the diffusion of the process. More restrictions can be applied to the units of assets held.

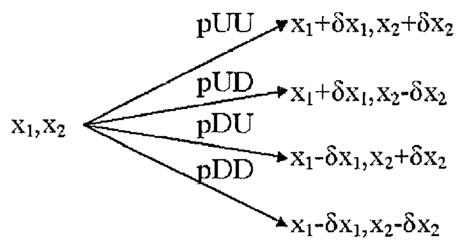
We have yet to specify a payoff function for our option (the payoff function was set to an unspecified function $O(T, \upsilon_1, \upsilon_2, \Gamma_1, \Gamma_2, K)$ with an exercise condition $f(T, \Gamma_1, \Gamma_2, K) > 0$). In order to complete the specification of the problem, we must specify the form of these two functions and replace them in the boundary conditions equation 3.9.

3.4.1 Assets' price discretisation

The two assets can be discretised using a binomial lattice for two factors. More complex schemes can be employed such as a trinomial lattice, but in sake of simplicity, a two factor binomial lattice scheme is presented below.



Price updates



Natural logarithm updates

In order to make the system of equations simpler to solve, we use the natural logarithmic forms of the two processed describing the two assets

$$\begin{aligned} dx_1 &= v_1 dt + \sigma_1 dw_1 \text{ for asset price } S_1 \text{ and} \\ dx_2 &= v_2 dt + \sigma_2 dw_2 \text{ for asset price } S_2 \end{aligned}$$

where $\nu_i = \mu_i - \frac{\sigma_i^2}{2}$. μ_i is the constant drift rate of each of the two assets' return and σ_i is their respective volatility. By matching the expected values, variances and correlation of the two assets, the following set of equations is used to derive the values of the price updates and transition probabilities for the lattice

$$\begin{aligned} I\!E\left[\delta x_1\right] &= \left(pUU + pUD\right)\delta x_1 - \left(pDU + pDD\right)\delta x_1 = v_1 \delta t \\ I\!E\left[\delta x_1^2\right] &= \left(pUU + pUD\right)\delta x_1^2 + \left(pDU + pDD\right)\delta x_1^2 = \sigma_1 \delta t + v_1 \delta t^2 \\ I\!E\left[\delta x_2\right] &= \left(pUU + pDU\right)\delta x_2 - \left(pUD + pDD\right)\delta x_2 = v_2 \delta t \end{aligned}$$

$$
\begin{aligned}
I\!E\left[\delta x_2^2\right] &= (pUU + pDU)\,\delta x_2^2 + (pUD + pDD)\,\delta x_2^2 = \sigma_2 \delta t + v_2 \delta t^2 \\
I\!E\left[\delta x_1 \delta x_2\right] &= (pUU - pUD - pDU + pDD)\,\delta x_1 \delta x_2 = \rho \sigma_1 \sigma_2 \delta t + v_1 v_2 \delta t^2
\end{aligned}
$$

and to complete the system (since we have six unknowns, we need six equations) we impose the condition

$$pUU + pUD + pDU + pDD = 1$$

This process is similar to the one described in Clewlow and Strickland [8] (chapter 2.11) and is given below.

The discrete price jumps for the two assets are

$$\delta \Gamma_{i,\tau} = \exp\left(\delta x_i\right), \text{where } \delta x_i = \sigma_i \sqrt{\delta s}$$

for an upward movement in price and

$$\delta \Gamma_{i,s} = exp\left(-\delta x_i\right)$$

for a downwards movement. The probabilities with which these transitions take place, are given below

$$
\begin{aligned}
pUU &= \frac{\delta x_1 \delta x_2 + \delta x_2 \nu_1 \delta \tau + \delta x_1 \nu_2 \delta \tau + \rho \sigma_1 \sigma_2 \delta \tau}{4 \delta x_1 \delta x_2} \\
pUD &= \frac{\delta x_1 \delta x_2 + \delta x_2 \nu_1 \delta \tau - \delta x_1 \nu_2 \delta \tau - \rho \sigma_1 \sigma_2 \delta \tau}{4 \delta x_1 \delta x_2} \\
pDU &= \frac{\delta x_1 \delta x_2 - \delta x_2 \nu_1 \delta \tau + \delta x_1 \nu_2 \delta \tau - \rho \sigma_1 \sigma_2 \delta \tau}{4 \delta x_1 \delta x_2} \\
pDD &= \frac{\delta x_1 \delta x_2 - \delta x_2 \nu_1 \delta \tau - \delta x_1 \nu_2 \delta \tau + \rho \sigma_1 \sigma_2 \delta \tau}{4 \delta x_1 \delta x_2}
\end{aligned}
$$

where σ_i and ρ are the volatilities and the correlation of the assets, pUU denotes an upwards movement in both asset prices and pUD denotes an upwards movement in the first asset and a downwards movement in the second, etc.

3.4.2 Specification of the payoff function

We can use the methodology developed in this chapter to price a spread option, that is an option on the difference in price of the two assets in our portfolio. This makes the payoff specification

$$\max\{(S_{1,T} - S_{2,T}) - K, 0\}$$

so our payoff function becomes

$$O(T, y_1, y_2, S_1, S_2, K) = (y_{1,T} - 1)\, S_{1,T} + (y_{2,T} + 1)\, S_{2,T} + K$$

and the exercise condition is

$$f(T, S_1, S_2, K) = (S_{1,T} - S_{2,T}) - K > 0$$

From this procedure it can be seen that many different payoffs can be considered, as long as the payoff is of European type. American, Asian and other exotic payoffs, that do not have an explicit payoff (dependent only on the asset prices at the expiry of the option) can be implemented with the payoff function specified a priori and included in the procedure of the solution of the option price.

3.5 Pricing algorithm and numerical examples

Having fully specified the problem, a rough description of the procedure followed to price the option will be outlined. Following that, a more detailed outline of the optimisation loops will be provided.

3.5.1 The pricing algorithm

General outline of the algorithm

As mentioned throughout this chapter, there are two legs to the pricing of an option under the scheme proposed here. First, the value function must be calculated for the case when an option has not been traded. This will affect the payoff function as there will be no exercise condition and thus, the value of the function at expiry is the value of the portfolio (cash and stock holding). In the second run through the algorithm, the effect of an option payoff is taken into account, so the terminal value of the function includes the payoff specified for the option. In both cases the procedure after valuing the portfolio at expiry is the same. One has to work backwards, using the dynamic scheme specified by equation 3.7. Working backwards through the time-steps up to the time of pricing the option, one must account for any periods of illiquidity. At those time-steps the value of the function can only drift uncontrolled, meaning that the portfolio cannot be rebalanced since, trading in the assets is not available.

At times of normal trading, the algorithm will compare the value of the function for all discrete holdings, find the holding that minimises the value function and compare it with the value of the function when no rebalancing takes place. The algorithm will register the lowest value as the optimal value. At periods of interrupted trading, the algorithm will only take into account the 'drifting' value of the portfolio, for the holding level at the last rebalancing.

In a similar way, the algorithm will progress through all the time-steps required, until it reaches the pricing time. The two prices (for no traded option and for a traded option) will be stored and will be used by equation 3.5 to price the option.

A more rigorous outline of the algorithm

This section describes a thorough run through the algorithm for pricing an option with

European style payoff. The underlyings of the option are assumed to be unavailable for hedging during some period, between the pricing and the expiry of the option inclusive. To initiate the pricing of the option, the payoff must be specified along with the exercise condition.

In the first run through the algorithm, we will assume that an option has not been traded, so the value at the expiry of the option will be given by equation 3.8. The value of the function will have to be evaluated for all possible holding increments and for every price at the expiry time. This yields a four-dimensional array, two dimensions for the assets' prices and two dimensions for the assets' holdings.

The next step is to identify the holding pair that minimises the value function. This value and the holdings are then used to calculate the value function for the previous time-step.. The dynamic program is invoked on a two-variable binomial lattice, meaning the each node belonging to time-step s is connected with four nodes in the time-step $s + \delta s$, with each node assigned a probability according to the price discretisation scheme proposed in the previous section. Having calculated the function fully for any time-step $s + \delta s$, the nodes in the previous time-step can be calculated using equation 3.7 and the associated probabilities.

If trading is not allowed, then the set of strategies available for evaluating the function, is a subset of the freely rebalanced value function strategies. This restricts the calculation of the value function just at the corresponding assets' holdings (from holding pair (y_1, y_2) to pair (y_1, y_2)), since the assets are not available for trading. In the case of free trading, the full evaluation of the function can take place, that is from any holding pair to any other, for the specific prices pair. The minimising holding for the assets' prices pair is stored and the value function is calculated for every possible holding combination. Eventually, the function is compared with the no-trading strategy and the optimal strategy is propagated. This is repeated until the final time-step, where the value of the function for the holding pair $(0, 0)$ at the assets' price pair that corresponds to the initial price set for the assets.

A second run is required in order to calculate the option price for the option. This is identical to the above description, but with a modified boundary equation. Instead of equation 3.8, equation 3.9 should be used, in order to take into account the effect of the option payoff. The rest of the procedure is identical as before. Once the value for the second run is obtained for the same holding and price points as before, the two values are used in equation 3.5 to obtain the option price.

The inputs required to price the option are the assets' initial prices (S_i), the volatility of returns and drifts (σ_i, μ_i) in annualised terms, the correlation of the two assets' returns (ρ), the constant interest rate (r), the strike price of the option (K), the expiry of the option (T), the risk aversion factor (γ), the time and holding discretisations $(\delta s, \delta v)$ and the times that the illiquidity period starts and ends (τ_{st}, τ_{sp}). The minimum and maximum holdings permissible $(\underline{y}, \bar{y})$ are set by default to -1 and 1 respectively. The justification for allowing to hold negative amounts of either asset, comes from the fact that (specifically in the spread option) we trade on the difference of the two assets. The difference between the assets is not a tradable, so we need to replicate it by trading in the two assets separately; therefore negative amounts of the second asset need to be held.

Two numerical examples will be provided now in order to demonstrate the use of the discretisation and the algorithm described above. The first example deals with a spread option and assumes that trading is allowed over all of the life of the option. The second example deals with the same option, but assumes that an interruption occurs at some point during its life.

First, some basic variables will be defined. $nFibres$ is the number of time-steps of the lattice and $nHSteps$ is the number of steps in the discretisation of the holding of the two assets. For simplicity, it is assumed that both asset holdings are discretised by the same number of steps. The minimum allowed holding $(minH)$ is -1 and the maximum $(maxH)$ is 1. The price of the option is calculated at fibre (time-step) 0. Fibre 0 has $(nHSteps\ ,\ nHSteps)$ elements for all different holdings of the assets. Fibre 1 has $(2 * nHSteps\ ,\ 2 * nHSteps)$ elements, etc. Since the option price is taken at the point of not holding any of the assets, the option is taken between the points $((nHSteps - 1)/2\ ,\ (nHSteps - 1)/2)$ and $((nHSteps + 1)/2\ ,\ (nHSteps + 1)/2)$ if $nHSteps$ is odd and at $(nHSteps/2\ ,\ nHSteps/2)$ when it is even. In order to store all the values required, the following structures are going to be used:

A matrix $mOpt$ of size $((nFibres + 1) * nHSteps\ ,\ (nFibres + 1) * nHSteps)$ which stores all the option calculations.

A vector $vHold$ of size $(nHSteps)$ which holds the discrete steps of the holdings.

The asset prices can be calculated during the algorithm so they do not need to be stored in a structure.

In order to provide some context, the pseudo-algorithm for pricing the option is provided below.

Algorithm 1 Calculation of the premium of an option on two illiquid assets

Inputs
Underlying initial prices: S_1 and S_2
Annualised yield and volatility for the two underlyings: μ_1, σ_1 and μ_2, σ_2
Instantaneous correlation between the two underlyings returns: ρ
Risk-free constant interest rate: r
Risk aversion parameter: γ
Option strike price: K
Option expiry: T
Times (in years) that the illiquidity period starts and ends: τ_{st}, τ_{sp}
Calculate all discrete up and down moves for the asset prices and transition probabilities
$\delta\Gamma_i$, pUU, pUD, pDU and pDD.
Calculate the levels of holding

```
for (hStep = 0 to nHSteps − 1)
  vHold(hStep) = (maxH − minH) * hStep / nHSteps + minH
```

Initialise $mOpt$ using the boundary condition at the final time step of lattice

```
for (idxX1 = 0 to nFibres)
| X1 = S1 * δΓ1^(nFibres−2*idxX1)
| for (idxX2 = 0 to nFibres)
| | X2 = S2 * δΓ2^(nFibres−2*idxX2)
| | for (idxH1 = 0 to nHSteps − 1)
| | | for (idxH2 = 0 to nHSteps − 1)
| | | | mOpt(idxX1 * nHSteps + idxH1, idxX2 * nHSteps + idxH2) =
| | | | exp(−γ * (vHold[idxH1] * X1 + vHold[idxH2] * X2))
| | | ⌊next idxH2
| | ⌊next idxH1
| ⌊next idxX2
⌊next idxX1
```

Update the prices in $mOpt$ by stepping backwards through the time-steps of the lattice
At each time-step find the optimal trade and calculate the new values for the current adjustment

```
for (idxF = nFibres − 1 to 0) step − 1
| Calculate the time step's interest factor
| F = exp(r * (T − idxF * dt))
| if (trading allowed) (i.e. we are not within τst, τsp) allowed to trade to optimal holding
| | for (idxX1 = 0 to idxF)
| | | X1 = S1 * δΓ1^(idxF−2*idxX1)
| | | for (idxX2 = 0 to idxF)
| | | | X2 = S2 * δΓ2^(idxF−2*idxX2)
| | | | set the minimum option to infinity, to ensure minimisation
| | | | minMOpt = +∞
| | | | for (idxH1 = 0 to nHSteps − 1)
| | | | | for (idxH2 = 0 to nHSteps − 1)
| | | | | | Find optimal trade movement
| | | | | | D = exp(−γ * ((vHold[0] − vHold[idxH1]) * X1 + (vHold[0] − vHold[idxH2])*
| | | | | |                                                                   X2) * F)
| | | | | | mOpt(idxX1 * nHSteps + idxH1, idxX2 * nHSteps + idxH2) =
| | | | | |  F * (pUU * mOpt(idxX1 * nHSteps + idxH1, idxX2 * nHSteps + idxH2)+
| | | | | |  pUD * mOpt(idxX1 * nHSteps + idxH1, idxX2 * nHSteps + idxH2 + nHSteps)+
| | | | | |  pDU * mOpt(idxX1 * nHSteps + idxH1 + nHSteps, idxX2 * nHSteps + idxH2)+
| | | | | |  pDD * mOpt(idxX1 * nHSteps + idxH1 + nHSteps, idxX2 * nHSteps + idxH2+
| | | | | |                                                                 nHSteps))
| | | | | | if (mOpt(idxX1 * nHSteps + idxH1, idxX2 * nHSteps + idxH2) < minMOpt)
| | | | | | | minMOpt = mOpt(idxX1 * nHSteps + idxH1, idxX2 * nHSteps + idxH2)
| | | | | | |                                                                  / D
| | | | | | | minH1 = vHold[idxH1], minH2 = vHold[idxH2]
| | | | | | ⌊end if
```

```
| | | | | ⌊next idxH2
| | | | ⌊next idxH1
| | | | Calculate value for adjustment
| | | | for (idxH1 = 0 to nHSteps − 1)
| | | | | for (idxH2 = 0 to nHSteps − 1)
| | | | | | D = exp(−γ ∗ ((vHold[idxH1] − minH1) ∗ X1 + (vHold[idxH2] − minH2)∗
| | | | | |                                                              X2) ∗ F)
| | | | | | mOpt(idxX1 ∗ nHSteps + idxH1, idxX2 ∗ nHSteps + idxH2) = D ∗ minMOpt
| | | | | ⌊next idxH2
| | | | ⌊next idxH1
| | | ⌊next idxX2
| | ⌊next idxX1
| ⌊else if (trading not allowed) (i.e. we are between τst, τsp) just calculate diffused value of mOpt
| | for (idxX1 = 0 to idxF)
| | | X1 = S1 ∗ δΓ1^(idxF−2∗idxX1)
| | | for (idxX2 = 0 to idxF)
| | | | X2 = S2 ∗ δΓ2^(idxF−2∗idxX2)
| | | | for (idxH1 = 0 to nHSteps − 1)
| | | | | for (idxH2 = 0 to nHSteps − 1)
| | | | | | mOpt(idxX1 ∗ nHSteps + idxH1, idxX2 ∗ nHSteps + idxH2) =
| | | | | | pUU ∗ mOpt(idxX1 ∗ nHSteps + idxH1, idxX2 ∗ nHSteps + idxH2)+
| | | | | | pUD ∗ mOpt(idxX1 ∗ nHSteps + idxH1, idxX2 ∗ nHSteps + idxH2 + nHSteps)+
| | | | | | pDU ∗ mOpt(idxX1 ∗ nHSteps + idxH1 + nHSteps, idxX2 ∗ nHSteps + idxH2)+
| | | | | | pDD ∗ mOpt(idxX1 ∗ nHSteps + idxH1 + nHSteps, idxX2 ∗ nHSteps + idxH2+
| | | | | |                                                              nHSteps)
| | | | | ⌊next idxH2
| | | | ⌊next idxH1
| | | ⌊next idxX2
| | ⌊next idxX1
| ⌊end if
⌊next idxF
Return mOpt price
  marketPrem = mOptn((HSteps − 1)/2, (nHSteps − 1)/2)
Recalculate the initialisation function (to include the option payoff)
  mOpt(idxX1 ∗ nHSteps + idxH1, idxX2 ∗ nHSteps + idxH2) = exp(−γ ∗ (vHold[idxH1]∗
    X1 + vHold[idxH2] ∗ X2 − max(f(T, S1, S2, K), 0)))
Update the prices in mOpt by stepping backwards through the time-steps of the lattice
Return mOpt price
  optionPrem = mOptn((HSteps − 1)/2, (nHSteps − 1)/2)
Calculate the option price
  optValue = exp(−r ∗ T) ∗ ln(optionPrem / marketPrem) / γ
```

3.5.2 Numerical example

In this numerical example the following set of values is going to be used. The initial price of the first asset is $S_1 = 100$, its return drift is set to the risk free interest rate ($\mu_1 = r$) (for simplicity), and its volatility is $\sigma_1 = 0.2$. The price of the second asset is set to $S_2 = 110$, with drift $\mu_2 = r$ and volatility $\sigma_2 = 0.3$. The correlation ρ between the two assets is set to 0.5. The strike of the spread option $K = 1.2$, while the option expires in one year ($T = 1$) and the risk free interest rate r is 0.06. In order to make the example more readable, a small number of fibres ($nFibres = 2$) and asset discretisation steps ($nHSteps = 7$) is selected. The payoff of a spread option is used to initialise the final fibre values. Using these numbers we get, $\mu_1 = \mu_2 = r = 0.06$, $\delta t = T \,/\, nFibres \Rightarrow \delta t = 0.5$ and $vi = \mu_i - 0.5 * \sigma_i^2 \Rightarrow v_1 = 0.0400$, $v_2 = 0.0150$.

The up and down steps for this example are given by:

$\delta\Gamma_i = \exp(\sigma_i * \sqrt{\delta t}) \Rightarrow \delta\Gamma_1 = \exp(0.2 * \sqrt{0.5}) = \exp(0.1414) = 1.1519$ for an up move and $\delta\Gamma_2 = \exp(0.2121) = 1.2363$.

The probabilities are given below

$$pUU = \frac{0.0300+(0.0085+0.0021+0.0300)0.5}{0.0800} = 0.4192$$

$$pUD = \frac{0.0300+(0.0085-0.0021-0.0300)0.5}{0.0800} = 0.1515$$

$$pDU = \frac{0.0300+(-0.0085+0.0021-0.0300)0.5}{0.0800} = 0.0985$$

$$pDD = \frac{0.0300+(-0.0085-0.0021+0.0300)0.5}{0.0800} = 0.3308$$

Market premium calculation

Fibre step 2

As an example, the first calculation of the matrix is provided (for idxX1=0, idxX2=0, idxH1=0, idxH2=0):

$$X_1 = S_1 * \delta\Gamma_1^{nFibres-2*idxX1} = 100 * 1.1519^{2-0} = 132.6896$$

$$X_2 = S_2 * \delta\Gamma_2^{nFibres-2*idxX2} = 100 * 1.2363^{2-0} = 168.1312$$

$mOpt\,(0,0) = \exp(-\gamma * (vHold[0] * X_1 + vHold[0] * X_2)) = \exp(-0.5 * (-1 * 132.6896 - 1 * 168.1312)) = 2.1009E + 65$

The full matrix for the final step is provided below

For $idxX1 = 0, idxX2 = 0$ we get

	X2	168.1312						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
132.6896	-1.00	2.1009E+65	1.4212E+53	9.6146E+40	6.5041E+28	4.4000E+16	2.9765E+04	2.0136E-08
	-0.67	5.2241E+55	3.5341E+43	2.3907E+31	1.6173E+19	1.0941E+07	7.4014E-06	5.0070E-18
	-0.33	1.2990E+46	8.7877E+33	5.9448E+21	4.0216E+09	2.7206E-03	1.8404E-15	1.2450E-27
	0.00	3.2301E+36	2.1851E+24	1.4782E+12	1.0000E+00	6.7649E-13	4.5764E-25	3.0959E-37
	0.33	8.0320E+26	5.4335E+14	3.6757E+02	2.4866E-10	1.6821E-22	1.1380E-34	7.6981E-47
	0.67	1.9972E+17	1.3511E+05	9.1400E-08	6.1831E-20	4.1828E-32	2.8296E-44	1.9142E-56
	1.00	4.9663E+07	3.3596E-05	2.2727E-17	1.5375E-29	1.0401E-41	7.0361E-54	4.7598E-66

For $idxX1 = 0, idxX2 = 1$ we get

	X2	110.0000						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
132.6896	-1.00	5.0048E+52	5.4616E+44	5.9601E+36	6.5041E+28	7.0978E+20	7.7457E+12	8.4527E+04
	-0.67	1.2445E+43	1.3581E+35	1.4820E+27	1.6173E+19	1.7649E+11	1.9260E+03	2.1018E-05
	-0.33	3.0945E+33	3.3770E+25	3.6852E+17	4.0216E+09	4.3887E+01	4.7892E-07	5.2264E-15
	0.00	7.6948E+23	8.3971E+15	9.1636E+07	1.0000E+00	1.0913E-08	1.1909E-16	1.2996E-24
	0.33	1.9134E+14	2.0880E+06	2.2786E-02	2.4866E-10	2.7135E-18	2.9612E-26	3.2315E-34
	0.67	4.7578E+04	5.1920E-04	5.6659E-12	6.1831E-20	6.7475E-28	7.3634E-36	8.0354E-44
	1.00	1.1831E-05	1.2910E-13	1.4089E-21	1.5375E-29	1.6778E-37	1.8310E-45	1.9981E-53

For $idxX1 = 0, idxX2 = 2$ we get

	X2	71.9676						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
132.6896	-1.00	2.7591E+44	1.7044E+39	1.0529E+34	6.5041E+28	4.0179E+23	2.4820E+18	1.5333E+13
	-0.67	6.8606E+34	4.2381E+29	2.6181E+24	1.6173E+19	9.9909E+13	6.1718E+08	3.8126E+03
	-0.33	1.7060E+25	1.0538E+20	6.5101E+14	4.0216E+09	2.4843E+04	1.5347E-01	9.4804E-07
	0.00	4.2420E+15	2.6205E+10	1.6188E+05	1.0000E+00	6.1775E-06	3.8161E-11	2.3574E-16
	0.33	1.0548E+06	6.5160E+00	4.0253E-05	2.4866E-10	1.5361E-15	9.4891E-21	5.8618E-26
	0.67	2.6229E-04	1.6203E-09	1.0009E-14	6.1831E-20	3.8196E-25	2.3595E-30	1.4576E-35
	1.00	6.5220E-14	4.0289E-19	2.4889E-24	1.5375E-29	9.4977E-35	5.8672E-40	3.6244E-45

For $idxX1 = 1, idxX2 = 0$ we get

	X2	168.1312						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	1.6747E+58	1.1329E+46	7.6641E+33	5.1847E+21	3.5074E+09	2.3727E-03	1.6051E-15
	-0.67	9.6761E+50	6.5458E+38	4.4282E+26	2.9956E+14	2.0265E+02	1.3709E-10	9.2739E-23
	-0.33	5.5906E+43	3.7820E+31	2.5585E+19	1.7308E+07	1.1709E-05	7.9207E-18	5.3582E-30
	0.00	3.2301E+36	2.1851E+24	1.4782E+12	1.0000E+00	6.7649E-13	4.5764E-25	3.0959E-37
	0.33	1.8663E+29	1.2625E+17	8.5408E+04	5.7777E-08	3.9086E-20	2.6441E-32	1.7887E-44
	0.67	1.0783E+22	7.2945E+09	4.9347E-03	3.3382E-15	2.2583E-27	1.5277E-39	1.0335E-51
	1.00	6.2301E+14	4.2146E+02	2.8511E-10	1.9287E-22	1.3048E-34	8.8267E-47	5.9711E-59

For $idxX1 = 1, idxX2 = 1$ we get

	X2	110.0000						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	3.9895E+45	4.3537E+37	4.7510E+29	5.1847E+21	5.6579E+13	6.1744E+05	6.7379E-03
	-0.67	2.3050E+38	2.5154E+30	2.7450E+22	2.9956E+14	3.2690E+06	3.5674E-02	3.8930E-10
	-0.33	1.3318E+31	1.4534E+23	1.5860E+15	1.7308E+07	1.8888E-01	2.0612E-09	2.2493E-17
	0.00	7.6948E+23	8.3971E+15	9.1636E+07	1.0000E+00	1.0913E-08	1.1909E-16	1.2996E-24
	0.33	4.4459E+16	4.8517E+08	5.2945E+00	5.7777E-08	6.3051E-16	6.8806E-24	7.5087E-32
	0.67	2.5687E+09	2.8032E+01	3.0590E-07	3.3382E-15	3.6429E-23	3.9754E-31	4.3383E-39
	1.00	1.4841E+02	1.6196E-06	1.7674E-14	1.9287E-22	2.1048E-30	2.2969E-38	2.5066E-46

For $idxX1 = 1$, $idxX2 = 2$ we get

	X2	71.9676						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	2.1993E+37	1.3586E+32	8.3929E+26	5.1847E+21	3.2028E+16	1.9785E+11	1.2222E+06
	-0.67	1.2707E+30	7.8499E+24	4.8492E+19	2.9956E+14	1.8505E+09	1.1431E+04	7.0618E-02
	-0.33	7.3420E+22	4.5355E+17	2.8018E+12	1.7308E+07	1.0692E+02	6.6048E-04	4.0801E-09
	0.00	4.2420E+15	2.6205E+10	1.6188E+05	1.0000E+00	6.1775E-06	3.8161E-11	2.3574E-16
	0.33	2.4509E+08	1.5140E+03	9.3530E-03	5.7777E-08	3.5692E-13	2.2048E-18	1.3620E-23
	0.67	1.4161E+01	8.7478E-05	5.4039E-10	3.3382E-15	2.0622E-20	1.2739E-25	7.8695E-31
	1.00	8.1817E-07	5.0542E-12	3.1222E-17	1.9287E-22	1.1915E-27	7.3603E-33	4.5468E-38

For $idxX1 = 2$, $idxX2 = 0$ we get

	X2	168.1312						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
75.3638	-1.00	7.4863E+52	5.0644E+40	3.4260E+28	2.3177E+16	1.5679E+04	1.0606E-08	7.1751E-21
	-0.67	2.6257E+47	1.7763E+35	1.2016E+23	8.1289E+10	5.4991E-02	3.7201E-14	2.5166E-26
	-0.33	9.2095E+41	6.2301E+29	4.2146E+17	2.8511E+05	1.9288E-07	1.3048E-19	8.8267E-32
	0.00	3.2301E+36	2.1851E+24	1.4782E+12	1.0000E+00	6.7649E-13	4.5764E-25	3.0959E-37
	0.33	1.1329E+31	7.6641E+18	5.1847E+06	3.5074E-06	2.3727E-18	1.6051E-30	1.0858E-42
	0.67	3.9736E+25	2.6881E+13	1.8185E+01	1.2302E-11	8.3220E-24	5.6297E-36	3.8085E-48
	1.00	1.3937E+20	9.4283E+07	6.3781E-05	4.3147E-17	2.9189E-29	1.9746E-41	1.3358E-53

For $idxX1 = 2$, $idxX2 = 1$ we get

	X2	110.0000						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
75.3638	-1.00	1.7834E+40	1.9462E+32	2.1238E+24	2.3177E+16	2.5292E+08	2.7601E+00	3.0120E-08
	-0.67	6.2550E+34	6.8259E+26	7.4490E+18	8.1289E+10	8.8709E+02	9.6806E-06	1.0564E-13
	-0.33	2.1939E+29	2.3941E+21	2.6127E+13	2.8511E+05	3.1114E-03	3.3954E-11	3.7053E-19
	0.00	7.6948E+23	8.3971E+15	9.1636E+07	1.0000E+00	1.0913E-08	1.1909E-16	1.2996E-24
	0.33	2.6989E+18	2.9452E+10	3.2140E+02	3.5074E-06	3.8275E-14	4.1769E-22	4.5581E-30
	0.67	9.4660E+12	1.0330E+05	1.1273E-03	1.2302E-11	1.3425E-19	1.4650E-27	1.5987E-35
	1.00	3.3201E+07	3.6231E-01	3.9538E-09	4.3147E-17	4.7085E-25	5.1383E-33	5.6073E-41

For $idxX1 = 2$, $idxX2 = 2$ we get

	X2	71.9676						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
75.3638	-1.00	9.8315E+31	6.0733E+26	3.7518E+21	2.3177E+16	1.4317E+11	8.8444E+05	5.4636E+00
	-0.67	3.4483E+26	2.1302E+21	1.3159E+16	8.1289E+10	5.0216E+05	3.1021E+00	1.9163E-05
	-0.33	1.2094E+21	7.4713E+15	4.6154E+10	2.8511E+05	1.7613E+00	1.0880E-05	6.7212E-11
	0.00	4.2420E+15	2.6205E+10	1.6188E+05	1.0000E+00	6.1775E-06	3.8161E-11	2.3574E-16
	0.33	1.4878E+10	9.1910E+04	5.6777E-01	3.5074E-06	2.1667E-11	1.3385E-16	8.2683E-22
	0.67	5.2184E+04	3.2237E-01	1.9914E-06	1.2302E-11	7.5994E-17	4.6945E-22	2.9000E-27
	1.00	1.8303E-01	1.1307E-06	6.9846E-12	4.3147E-17	2.6654E-22	1.6465E-27	1.0171E-32

Fibre step 1

In the next step, we move one fibre back on the lattice and update the matrix by finding the minimum value for $mOpt$. For this time-step we need to calculate the interest factor $F = \exp(r * (T - idxF * dt)) \Rightarrow F = \exp(0.06 * (1 - 1 * 0.5)) = \exp(0.0300) = 1.0305$

We now need to recalculate the asset prices for each node of the matrix and compute the new values for $mOpt$. From these values the minimum will be selected and used to calculate the adjustment of the portfolio.

The new prices for $idxX1 = 0$ and $idxX2 = 1$ are

$X_1 = S_1 * \delta\Gamma_1^{idxF-2*idxX1} = 100 * 1.1519^{1-0} = 115.1910$

$X_2 = S_2 * \delta\Gamma_2^{idxF-2*idxX2} = 100 * 1.2363^{1-2} = 88.9744$

We now need to calculate the factor D. We show an example for this calculation when $idxH1 = 3$ and $idxH2 = 2$

$D = \exp(-\gamma * ((vHold[0] - vHold[idxH1]) * X_1 + (vHold[0] - vHold[idxH2]) * X_2) * F) \Rightarrow$

$D = \exp(-0.5*((-1-0)*115.1910+(-1+0.33)*88.9744)*1.0305) = 1.11635E+39$

$mOpt(3,9) = D * (pUU * mOpt(3,9) + pUD * mOpt(3,16) + pDU * mOpt(10,9) + pDD * mOpt(10,16)) \Rightarrow$

$mOpt(3,9) = (1.11635E + 39) * (0.4192 * (9.1636E + 07) + 0.1515 * (1.6188E + 05) + 0.0985 * (9.1636E + 07) + 0.3308 * (1.6188E + 05))$

$mOpt(3,9) = (5.3045.E + 46)$

For $idxX1 = 0$, $idxX2 = 0$ we get

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
115.1910	-1.00	8.8069.E+64	8.2881.E+62	7.8001.E+60	9.9936.E+58	4.0285.E+60	6.1148.E+62	9.2829.E+64
minMOpt	-0.67	8.5537.E+63	8.0499.E+61	7.5758.E+59	9.7064.E+57	3.9129.E+59	5.9392.E+61	9.0164.E+63
1.0000	-0.33	8.3161.E+62	7.8263.E+60	7.3654.E+58	9.4577.E+56	3.8359.E+58	5.8224.E+60	8.8390.E+62
minH1	0.00	9.9645.E+61	9.3775.E+59	8.8256.E+57	1.6043.E+56	1.1748.E+58	1.7834.E+60	2.7074.E+62
0.00	0.33	4.3564.E+62	4.0998.E+60	3.8591.E+58	1.5632.E+57	1.8219.E+59	2.7658.E+61	4.1988.E+63
minH2	0.67	9.6552.E+63	9.0865.E+61	8.5530.E+59	3.5077.E+58	4.1035.E+60	6.2295.E+62	9.4571.E+64
0.00	1.00	2.1788.E+65	2.0504.E+63	1.9301.E+61	7.9158.E+59	9.2605.E+61	1.4058.E+64	2.1342.E+66

From this matrix we get the minimum value for holding pair $(0, 0)$. We now need to recalculate this part of the $mOpt$ matrix for trading to the minimising pair $(0, 0)$. We need to recalculate D first (at point $idxH1 = 0, idxH2 = 0$)

$D = \exp(-\gamma * ((vHold[idxH1] - minH1) * X_1 + (vHold[idxH2] - minH2) * X_2) * F) \Rightarrow$

$D = \exp(-0.5 * (-115.1910 - 135.9942) * 1.0305) = 1.6043.E + 56$

$mOpt(0, 0) = D * minMOpt = (1.6043.E + 56) * 1 = 1.6043.E + 56$

By calculating the value of trading to this point for all $idxH1$ and $idxH2$ we get the following values for $mOpt$

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
115.1910	-1.00	1.6043.E+56	1.1532.E+46	8.2899.E+35	5.9591.E+25	4.2836.E+15	3.0792.E+05	2.2134.E-05
	-0.67	4.1074.E+47	2.9525.E+37	2.1224.E+27	1.5256.E+17	1.0967.E+07	7.8833.E-04	5.6668.E-14
	-0.33	1.0516.E+39	7.5591.E+28	5.4337.E+18	3.9059.E+08	2.8077.E-02	2.0183.E-12	1.4508.E-22
	0.00	2.6923.E+30	1.9353.E+20	1.3911.E+10	1.0000.E+00	7.1883.E-11	5.1672.E-21	3.7144.E-31
	0.33	6.8927.E+21	4.9547.E+11	3.5616.E+01	2.5602.E-09	1.8404.E-19	1.3229.E-29	9.5095.E-40
	0.67	1.7647.E+13	1.2685.E+03	9.1184.E-08	6.5546.E-18	4.7117.E-28	3.3869.E-38	2.4346.E-48
	1.00	4.5179.E+04	3.2476.E-06	2.3345.E-16	1.6781.E-26	1.2063.E-36	8.6712.E-47	6.2331.E-57

For $idxX1 = 0, idxX2 = 1$ we get

	X2	88.9744						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
115.1910	-1.00	2.0980.E+52	9.9094.E+50	4.6835.E+49	3.0098.E+48	2.1470.E+49	5.7126.E+50	1.5274.E+52
minMOpt	-0.67	2.0377.E+51	9.6245.E+49	4.5489.E+48	2.9233.E+47	2.0853.E+48	5.5486.E+49	1.4835.E+51
1.0000	-0.33	1.9811.E+50	9.3572.E+48	4.4225.E+47	2.8484.E+46	2.0442.E+47	5.4394.E+48	1.4544.E+50
minH1	0.00	2.3737.E+49	1.1212.E+48	5.3045.E+46	4.8319.E+45	6.2430.E+46	1.6661.E+48	4.4547.E+49
0.00	0.33	1.0378.E+50	4.9018.E+48	2.3288.E+47	4.7079.E+46	9.6690.E+47	2.5839.E+49	6.9087.E+50
minH2	0.67	2.3001.E+51	1.0864.E+50	5.1618.E+48	1.0564.E+48	2.1778.E+49	5.8198.E+50	1.5561.E+52
0.00	1.00	5.1903.E+52	2.4516.E+51	1.1648.E+50	2.3841.E+49	4.9146.E+50	1.3134.E+52	3.5116.E+53

and the corresponding values after trading to the minimising holding

	X2	88.9744						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
115.1910	-1.00	4.8319.E+45	1.1164.E+39	2.5792.E+32	5.9591.E+25	1.3768.E+19	3.1809.E+12	7.3492.E+05
	-0.67	1.2371.E+37	2.8581.E+30	6.6033.E+23	1.5256.E+17	3.5248.E+10	8.1438.E+03	1.8815.E-03
	-0.33	3.1671.E+28	7.3173.E+21	1.6906.E+15	3.9059.E+08	9.0243.E+01	2.0850.E-05	4.8171.E-12
	0.00	8.1084.E+19	1.8734.E+13	4.3282.E+06	1.0000.E+00	2.3104.E-07	5.3380.E-14	1.2333.E-20
	0.33	2.0759.E+11	4.7962.E+04	1.1081.E-02	2.5602.E-09	5.9151.E-16	1.3666.E-22	3.1575.E-29
	0.67	5.3148.E+02	1.2279.E-04	2.8370.E-11	6.5546.E-18	1.5144.E-24	3.4988.E-31	8.0837.E-38
	1.00	1.3607.E-06	3.1437.E-13	7.2633.E-20	1.6781.E-26	3.8771.E-33	8.9577.E-40	2.0696.E-46

For $idxX1 = 1, idxX2 = 0$ we get

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	7.0204.E+57	6.6068.E+55	6.2177.E+53	7.9663.E+51	3.2113.E+53	4.8744.E+55	7.3999.E+57
minMOpt	-0.67	1.2112.E+57	1.1398.E+55	1.0727.E+53	1.3745.E+51	5.5431.E+52	8.4136.E+54	1.2773.E+57
1.0000	-0.33	2.0973.E+56	1.9738.E+54	1.8575.E+52	2.4001.E+50	9.9003.E+51	1.5027.E+54	2.2813.E+56
minH1	0.00	4.4508.E+55	4.1886.E+53	3.9421.E+51	7.1660.E+49	5.2475.E+51	7.9658.E+53	1.2093.E+56
0.00	0.33	9.4888.E+55	8.9299.E+53	8.4055.E+51	3.2921.E+50	3.7972.E+52	5.7646.E+54	8.7513.E+56
minH2	0.67	9.2965.E+56	8.7489.E+54	8.2352.E+52	3.3749.E+51	3.9472.E+53	5.9923.E+55	9.0970.E+57
0.00	1.00	9.7245.E+57	9.1516.E+55	8.6143.E+53	3.5330.E+52	4.1331.E+54	6.2745.E+56	9.5254.E+58

and

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	7.1660.E+49	5.1512.E+39	3.7028.E+29	2.6617.E+19	1.9133.E+09	1.3754.E-01	9.8866.E-12
	-0.67	2.4001.E+43	1.7252.E+33	1.2402.E+23	8.9147.E+12	6.4082.E+02	4.6064.E-08	3.3112.E-18
	-0.33	8.0384.E+36	5.7783.E+26	4.1536.E+16	2.9858.E+06	2.1463.E-04	1.5428.E-14	1.1090.E-24
	0.00	2.6923.E+30	1.9353.E+20	1.3911.E+10	1.0000.E+00	7.1883.E-11	5.1672.E-21	3.7144.E-31
	0.33	9.0170.E+23	6.4817.E+13	4.6593.E+03	3.3492.E-07	2.4075.E-17	1.7306.E-27	1.2440.E-37
	0.67	3.0200.E+17	2.1709.E+07	1.5605.E-03	1.1217.E-13	8.0634.E-24	5.7963.E-34	4.1665.E-44
	1.00	1.0115.E+11	7.2708.E+00	5.2265.E-10	3.7570.E-20	2.7006.E-30	1.9413.E-40	1.3955.E-50

For $idxX1 = 1, idxX2 = 1$ we get

	X2	88.9744						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	1.6724.E+45	7.8992.E+43	3.7334.E+42	2.3993.E+41	1.7115.E+42	4.5538.E+43	1.2176.E+45
minMOpt	-0.67	2.8852.E+44	1.3628.E+43	6.4409.E+41	4.1398.E+40	2.9541.E+41	7.8603.E+42	2.1016.E+44
1.0000	-0.33	4.9962.E+43	2.3599.E+42	1.1154.E+41	7.2285.E+39	5.2755.E+40	1.4039.E+42	3.7537.E+43
minH1	0.00	1.0603.E+43	5.0080.E+41	2.3693.E+40	2.1582.E+39	2.7886.E+40	7.4419.E+41	1.9898.E+43
0.00	0.33	2.2604.E+43	1.0677.E+42	5.0711.E+40	9.9150.E+39	2.0153.E+41	5.3854.E+42	1.4399.E+44
minH2	0.67	2.2146.E+44	1.0460.E+43	4.9700.E+41	1.0164.E+41	2.0948.E+42	5.5982.E+43	1.4968.E+45
0.00	1.00	2.3166.E+45	1.0942.E+44	5.1988.E+42	1.0641.E+42	2.1935.E+43	5.8618.E+44	1.5673.E+46

and

	X2	88.9744						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	2.1582.E+39	4.9864.E+32	1.1521.E+26	2.6617.E+19	6.1496.E+12	1.4208.E+06	3.2827.E-01
	-0.67	7.2284.E+32	1.6701.E+26	3.8585.E+19	8.9147.E+12	2.0597.E+06	4.7586.E-01	1.0994.E-07
	-0.33	2.4210.E+26	5.5934.E+19	1.2923.E+13	2.9858.E+06	6.8983.E-01	1.5938.E-07	3.6823.E-14
	0.00	8.1084.E+19	1.8734.E+13	4.3282.E+06	1.0000.E+00	2.3104.E-07	5.3380.E-14	1.2333.E-20
	0.33	2.7157.E+13	6.2744.E+06	1.4496.E+00	3.3492.E-07	7.7381.E-14	1.7878.E-20	4.1306.E-27
	0.67	9.0956.E+06	2.1014.E+00	4.8552.E-07	1.1217.E-13	2.5917.E-20	5.9878.E-27	1.3834.E-33
	1.00	3.0463.E+00	7.0382.E-07	1.6261.E-13	3.7570.E-20	8.6801.E-27	2.0055.E-33	4.6334.E-40

Fibre step 0

We now move to the final time-step (fibre 0) and perform the same calculations for idxX1=0 and idxX2 =0 only. The values returned are the following

For $idxX1 = 1, idxX2 = 1$ we get

	X2	110.0000						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	6.7253.E+55	1.3764.E+54	2.8173.E+52	7.8493.E+50	1.3720.E+52	9.0173.E+53	5.9316.E+55
minMOpt	-0.67	8.3525.E+54	1.7095.E+53	3.4990.E+51	9.7488.E+49	1.7041.E+51	1.1200.E+53	7.3677.E+54
1.0000	-0.33	1.0392.E+54	2.1268.E+52	4.3534.E+50	1.2177.E+49	2.1515.E+50	1.4141.E+52	9.3018.E+53
minH1	0.00	1.5910.E+53	3.2561.E+51	6.6661.E+49	2.6347.E+48	8.3619.E+49	5.4987.E+51	3.6171.E+53
0.00	0.33	5.0775.E+53	1.0392.E+52	2.1290.E+50	1.8563.E+49	9.3483.E+50	6.1488.E+52	4.0447.E+54
minH2	0.67	7.9915.E+54	1.6356.E+53	3.3509.E+51	2.9858.E+50	1.5136.E+52	9.9554.E+53	6.5488.E+55
0.00	1.00	1.2981.E+56	2.6567.E+54	5.4429.E+52	4.8507.E+51	2.4590.E+53	1.6174.E+55	1.0640.E+57

After adjusting for the optimal holding, we get the final form of the $mOpt$ matrix

	X2	110.0000						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	2.6347.E+48	9.2536.E+39	3.2501.E+31	1.1415.E+23	4.0094.E+14	1.4082.E+06	4.9460.E-03
	-0.67	5.4312.E+40	1.9076.E+32	6.6999.E+23	2.3532.E+15	8.2651.E+06	2.9029.E-02	1.0196.E-10
	-0.33	1.1196.E+33	3.9324.E+24	1.3812.E+16	4.8510.E+07	1.7038.E-01	5.9842.E-10	2.1018.E-18
	0.00	2.3080.E+25	8.1063.E+16	2.8472.E+08	1.0000.E+00	3.5123.E-09	1.2336.E-17	4.3327.E-26
	0.33	4.7578.E+17	1.6711.E+09	5.8693.E+00	2.0614.E-08	7.2403.E-17	2.5430.E-25	8.9317.E-34
	0.67	9.8079.E+09	3.4448.E+01	1.2099.E-07	4.2495.E-16	1.4926.E-24	5.2422.E-33	1.8412.E-41
	1.00	2.0218.E+02	7.1013.E-07	2.4942.E-15	8.7602.E-24	3.0768.E-32	1.0807.E-40	3.7956.E-49

The price for the market premium is given by $marketPrem = mOpt(3, 3) = 1$.

Option premium calculation

Fibre step 2

We now repeat the procedure and change the initialisation function to $mOpt(idxX1 * nHSteps + idxH1, idxX2 * nHSteps + idxH2) = \exp(-\gamma * (vHold[idxH1] * X_1 + vHold[idxH2] * X_2 - \max(S_{1,T} - S_{2,T} - K, 0)))$.

As an example, we consider the case of $idxX1 = 0$ and $idxX2 = 2$. Also, assume that $idxH1 = 1$ and $idxH2 = 1$. We get

$mOpt(1, 15) = \exp(-0.5*(-0.6667*132.6896-0.6667*71.9676-\max(132.6896-71.9676-1.2, 0))) \Rightarrow$

$mOpt(1, 15) = \exp(97.9801) = 3.5663E + 42$

The full span of the $mOpt$ matrix is given below

For $idxX1 = 0$, $idxX2 = 0$ we get

	X2	168.1312						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
132.6896	-1.00	2.1009E+65	1.4212E+53	9.6146E+40	6.5041E+28	4.4000E+16	2.9765E+04	2.0136E-08
	-0.67	5.2241E+55	3.5341E+43	2.3907E+31	1.6173E+19	1.0941E+07	7.4014E-06	5.0070E-18
	-0.33	1.2990E+46	8.7877E+33	5.9448E+21	4.0216E+09	2.7206E-03	1.8404E-15	1.2450E-27
	0.00	3.2301E+36	2.1851E+24	1.4782E+12	1.0000E+00	6.7649E-13	4.5764E-25	3.0959E-37
	0.33	8.0320E+26	5.4335E+14	3.6757E+02	2.4866E-10	1.6821E-22	1.1380E-34	7.6981E-47
	0.67	1.9972E+17	1.3511E+05	9.1400E-08	6.1831E-20	4.1828E-32	2.8296E-44	1.9142E-56
	1.00	4.9663E+07	3.3596E-05	2.2727E-17	1.5375E-29	1.0401E-41	7.0361E-54	4.7598E-66

For $idxX1 = 0$, $idxX2 = 1$ we get

	X2	110.0000						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
132.6896	-1.00	2.3217E+57	2.5336E+49	2.7649E+41	3.0172E+33	3.2926E+25	3.5932E+17	3.9211E+09
	-0.67	5.7731E+47	6.3000E+39	6.8750E+31	7.5026E+23	8.1874E+15	8.9347E+07	9.7502E-01
	-0.33	1.4355E+38	1.5666E+30	1.7095E+22	1.8656E+14	2.0359E+06	2.2217E-02	2.4245E-10
	0.00	3.5695E+28	3.8954E+20	4.2509E+12	4.6389E+04	5.0623E-04	5.5244E-12	6.0287E-20
	0.33	8.8760E+18	9.6862E+10	1.0570E+03	1.1535E-05	1.2588E-13	1.3737E-21	1.4991E-29
	0.67	2.2071E+09	2.4085E+01	2.6284E-07	2.8683E-15	3.1301E-23	3.4158E-31	3.7276E-39
	1.00	5.4881E-01	5.9890E-09	6.5357E-17	7.1323E-25	7.7833E-33	8.4937E-41	9.2689E-49

For $idxX1 = 0$, $idxX2 = 2$ we get

	X2	71.9676						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
132.6896	-1.00	2.3217E+57	1.4342E+52	8.8598E+46	5.4731E+41	3.3810E+36	2.0886E+31	1.2902E+26
	-0.67	5.7731E+47	3.5663E+42	2.2031E+37	1.3609E+32	8.4071E+26	5.1935E+21	3.2082E+16
	-0.33	1.4355E+38	8.8679E+32	5.4781E+27	3.3841E+22	2.0905E+17	1.2914E+12	7.9776E+06
	0.00	3.5695E+28	2.2051E+23	1.3622E+18	8.4148E+12	5.1982E+07	3.2112E+02	1.9837E-03
	0.33	8.8760E+18	5.4831E+13	3.3872E+08	2.0924E+03	1.2926E-02	7.9848E-08	4.9326E-13
	0.67	2.2071E+09	1.3634E+04	8.4225E-02	5.2030E-07	3.2141E-12	1.9855E-17	1.2265E-22
	1.00	5.4881E-01	3.3903E-06	2.0943E-11	1.2938E-16	7.9921E-22	4.9371E-27	3.0499E-32

For $idxX1 = 1$, $idxX2 = 0$ we get

	X2	168.1312						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	1.6747E+58	1.1329E+46	7.6641E+33	5.1847E+21	3.5074E+09	2.3727E-03	1.6051E-15
	-0.67	9.6761E+50	6.5458E+38	4.4282E+26	2.9956E+14	2.0265E+02	1.3709E-10	9.2739E-23
	-0.33	5.5906E+43	3.7820E+31	2.5585E+19	1.7308E+07	1.1709E-05	7.9207E-18	5.3582E-30
	0.00	3.2301E+36	2.1851E+24	1.4782E+12	1.0000E+00	6.7649E-13	4.5764E-25	3.0959E-37
	0.33	1.8663E+29	1.2625E+17	8.5408E+04	5.7777E-08	3.9086E-20	2.6441E-32	1.7887E-44
	0.67	1.0783E+22	7.2945E+09	4.9347E-03	3.3382E-15	2.2583E-27	1.5277E-39	1.0335E-51
	1.00	6.2301E+14	4.2146E+02	2.8511E-10	1.9287E-22	1.3048E-34	8.8267E-47	5.9711E-59

For $idxX1 = 1$, $idxX2 = 1$ we get

	X2	110.0000						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	3.9895E+45	4.3537E+37	4.7510E+29	5.1847E+21	5.6579E+13	6.1744E+05	6.7379E-03
	-0.67	2.3050E+38	2.5154E+30	2.7450E+22	2.9956E+14	3.2690E+06	3.5674E-02	3.8930E-10
	-0.33	1.3318E+31	1.4534E+23	1.5860E+15	1.7308E+07	1.8888E-01	2.0612E-09	2.2493E-17
	0.00	7.6948E+23	8.3971E+15	9.1636E+07	1.0000E+00	1.0913E-08	1.1909E-16	1.2996E-24
	0.33	4.4459E+16	4.8517E+08	5.2945E+00	5.7777E-08	6.3051E-16	6.8806E-24	7.5087E-32
	0.67	2.5687E+09	2.8032E+01	3.0590E-07	3.3382E-15	3.6429E-23	3.9754E-31	4.3383E-39
	1.00	1.4841E+02	1.6196E-06	1.7674E-14	1.9287E-22	2.1048E-30	2.2969E-38	2.5066E-46

For $idxX1 = 1$, $idxX2 = 2$ we get

	X2	71.9676						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	1.4753E+43	9.1134E+37	5.6298E+32	3.4778E+27	2.1484E+22	1.3272E+17	8.1984E+11
	-0.67	8.5237E+35	5.2655E+30	3.2527E+25	2.0094E+20	1.2413E+15	7.6680E+09	4.7369E+04
	-0.33	4.9248E+28	3.0423E+23	1.8794E+18	1.1610E+13	7.1718E+07	4.4304E+02	2.7368E-03
	0.00	2.8454E+21	1.7578E+16	1.0858E+11	6.7078E+05	4.1437E+00	2.5597E-05	1.5813E-10
	0.33	1.6440E+14	1.0156E+09	6.2737E+03	3.8756E-02	2.3941E-07	1.4790E-12	9.1362E-18
	0.67	9.4987E+06	5.8678E+01	3.6248E-04	2.2392E-09	1.3833E-14	8.5450E-20	5.2787E-25
	1.00	5.4881E-01	3.3903E-06	2.0943E-11	1.2938E-16	7.9921E-22	4.9371E-27	3.0499E-32

For $idxX1 = 2$, $idxX2 = 0$ we get

	X2	168.1312						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
75.3638	-1.00	7.4863E+52	5.0644E+40	3.4260E+28	2.3177E+16	1.5679E+04	1.0606E-08	7.1751E-21
	-0.67	2.6257E+47	1.7763E+35	1.2016E+23	8.1289E+10	5.4991E-02	3.7201E-14	2.5166E-26
	-0.33	9.2095E+41	6.2301E+29	4.2146E+17	2.8511E+05	1.9288E-07	1.3048E-19	8.8267E-32
	0.00	3.2301E+36	2.1851E+24	1.4782E+12	1.0000E+00	6.7649E-13	4.5764E-25	3.0959E-37
	0.33	1.1329E+31	7.6641E+18	5.1847E+06	3.5074E-06	2.3727E-18	1.6051E-30	1.0858E-42
	0.67	3.9736E+25	2.6881E+13	1.8185E+01	1.2302E-11	8.3220E-24	5.6297E-36	3.8085E-48
	1.00	1.3937E+20	9.4283E+07	6.3781E-05	4.3147E-17	2.9189E-29	1.9746E-41	1.3358E-53

For $idxX1 = 2$, $idxX2 = 1$ we get

	X2	110.0000						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
75.3638	-1.00	1.7834E+40	1.9462E+32	2.1238E+24	2.3177E+16	2.5292E+08	2.7601E+00	3.0120E-08
	-0.67	6.2550E+34	6.8259E+26	7.4490E+18	8.1289E+10	8.8709E+02	9.6806E-06	1.0564E-13
	-0.33	2.1939E+29	2.3941E+21	2.6127E+13	2.8511E+05	3.1114E-03	3.3954E-11	3.7053E-19
	0.00	7.6948E+23	8.3971E+15	9.1636E+07	1.0000E+00	1.0913E-08	1.1909E-16	1.2996E-24
	0.33	2.6989E+18	2.9452E+10	3.2140E+02	3.5074E-06	3.8275E-14	4.1769E-22	4.5581E-30
	0.67	9.4660E+12	1.0330E+05	1.1273E-03	1.2302E-11	1.3425E-19	1.4650E-27	1.5987E-35
	1.00	3.3201E+07	3.6231E-01	3.9538E-09	4.3147E-17	4.7085E-25	5.1383E-33	5.6073E-41

For $idxX1 = 2$, $idxX2 = 2$ we get

	X2	71.9676						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
75.3638	-1.00	2.9479E+32	1.8211E+27	1.1250E+22	6.9494E+16	4.2930E+11	2.6520E+06	1.6382E+01
	-0.67	1.0340E+27	6.3872E+21	3.9457E+16	2.4374E+11	1.5057E+06	9.3015E+00	5.7460E-05
	-0.33	3.6265E+21	2.2403E+16	1.3839E+11	8.5490E+05	5.2811E+00	3.2624E-05	2.0153E-10
	0.00	1.2720E+16	7.8574E+10	4.8539E+05	2.9985E+00	1.8523E-05	1.1443E-10	7.0686E-16
	0.33	4.4612E+10	2.7559E+05	1.7025E+00	1.0517E-05	6.4967E-11	4.0133E-16	2.4792E-21
	0.67	1.5647E+05	9.6661E-01	5.9712E-06	3.6887E-11	2.2787E-16	1.4076E-21	8.6956E-27
	1.00	5.4881E-01	3.3903E-06	2.0943E-11	1.2938E-16	7.9921E-22	4.9371E-27	3.0499E-32

Fibre step 1

Moving to the previous time-step (fibre 1) we follow the procedure for minimising the value function. The results obtained for this time-step are shown below.

For $idxX1 = 0, idxX2 = 0$ we get

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
115.1910	-1.00	8.8069.E+64	8.2887.E+62	1.5907.E+61	1.2309.E+63	1.8685.E+65	2.8366.E+67	4.3063.E+69
minMOpt	-0.67	8.5537.E+63	8.0504.E+61	1.5450.E+60	1.1955.E+62	1.8148.E+64	2.7550.E+66	4.1825.E+68
1.4094E+12	-0.33	8.3161.E+62	7.8268.E+60	1.5013.E+59	1.1611.E+61	1.7626.E+63	2.6758.E+65	4.0622.E+67
minH1	0.00	9.9645.E+61	9.3780.E+59	1.6253.E+58	1.1278.E+60	1.7120.E+62	2.5990.E+64	3.9456.E+66
0.00	0.33	4.3564.E+62	4.0998.E+60	3.9312.E+58	1.1108.E+59	1.6809.E+61	2.5517.E+63	3.8738.E+65
minH2	0.67	9.6552.E+63	9.0865.E+61	8.5537.E+59	4.5714.E+58	5.7183.E+60	8.6810.E+62	1.3179.E+65
-0.33	1.00	2.1788.E+65	2.0504.E+63	1.9301.E+61	7.9262.E+59	9.2762.E+61	1.4082.E+64	2.1378.E+66

and the corresponding adjusted part of the $mOpt$ matrix is

	X2	135.9942						
X1	vHold	-0.67	-0.33	0.00	0.33	0.67	1.00	1.33
115.1910	-1.00	1.6253.E+58	1.1683.E+48	8.3984.E+37	6.0371.E+27	4.3396.E+17	3.1195.E+07	2.2424.E-03
	-0.67	4.1612.E+49	2.9912.E+39	2.1502.E+29	1.5456.E+19	1.1110.E+09	7.9865.E-02	5.7410.E-12
	-0.33	1.0653.E+41	7.6581.E+30	5.5049.E+20	3.9571.E+10	2.8445.E+00	2.0447.E-10	1.4698.E-20
	0.00	2.7275.E+32	1.9606.E+22	1.4094.E+12	1.0131.E+02	7.2824.E-09	5.2349.E-19	3.7630.E-29
	0.33	6.9830.E+23	5.0196.E+13	3.6082.E+03	2.5937.E-07	1.8645.E-17	1.3402.E-27	9.6340.E-38
	0.67	1.7878.E+15	1.2851.E+05	9.2378.E-06	6.6404.E-16	4.7734.E-26	3.4313.E-36	2.4665.E-46
	1.00	4.5771.E+06	3.2901.E-04	2.3651.E-14	1.7001.E-24	1.2221.E-34	8.7847.E-45	6.3147.E-55

For $idxX1 = 0, idxX2 = 1$ we get

	X2	88.9744						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
115.1910	-1.00	1.3250.E+57	9.4516.E+57	2.5148.E+59	6.7240.E+60	1.7978.E+62	4.8070.E+63	1.2853.E+65
minMOpt	-0.67	1.2869.E+56	9.1798.E+56	2.4425.E+58	6.5307.E+59	1.7461.E+61	4.6688.E+62	1.2483.E+64
1.5157E+09	-0.33	1.2499.E+55	8.9158.E+55	2.3723.E+57	6.3429.E+58	1.6959.E+60	4.5345.E+61	1.2124.E+63
minH1	0.00	1.2140.E+54	8.6595.E+54	2.3041.E+56	6.1605.E+57	1.6472.E+59	4.4041.E+60	1.1776.E+62
0.67	0.33	1.1801.E+53	8.4108.E+53	2.2379.E+55	5.9836.E+56	1.5999.E+58	4.2777.E+59	1.1437.E+61
minH2	0.67	1.3780.E+52	8.2559.E+52	2.1939.E+54	5.8659.E+55	1.5684.E+57	4.1935.E+58	1.1213.E+60
-1.00	1.00	5.3660.E+52	2.7623.E+52	6.7210.E+53	1.7967.E+55	4.8040.E+56	1.2845.E+58	3.4344.E+59

and $mOpt$ is

	X2	88.9744						
X1	vHold	0.00	0.33	0.67	1.00	1.33	1.67	2.00
115.1910	-1.67	1.3780.E+52	3.1837.E+45	7.3557.E+38	1.6995.E+32	3.9265.E+25	9.0717.E+18	2.0959.E+12
	-1.33	3.5280.E+43	8.1510.E+36	1.8832.E+30	4.3510.E+23	1.0053.E+17	2.3225.E+10	5.3660.E+03
	-1.00	9.0323.E+34	2.0868.E+28	4.8214.E+21	1.1139.E+15	2.5737.E+08	5.9462.E+01	1.3738.E-05
	-0.67	2.3125.E+26	5.3427.E+19	1.2344.E+13	2.8519.E+06	6.5891.E-01	1.5223.E-07	3.5172.E-14
	-0.33	5.9203.E+17	1.3678.E+11	3.1603.E+04	7.3015.E-03	1.6869.E-09	3.8975.E-16	9.0048.E-23
	0.00	1.5157.E+09	3.5019.E+02	8.0909.E-05	1.8693.E-11	4.3189.E-18	9.9784.E-25	2.3054.E-31
	0.33	3.8806.E+00	8.9657.E-07	2.0714.E-13	4.7858.E-20	1.1057.E-26	2.5547.E-33	5.9023.E-40

For $idxX1 = 1, idxX2 = 0$ we get

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	7.0204.E+57	6.6068.E+55	6.2177.E+53	7.9663.E+51	3.2113.E+53	4.8744.E+55	7.3999.E+57
minMOpt	-0.67	1.2112.E+57	1.1398.E+55	1.0727.E+53	1.3745.E+51	5.5431.E+52	8.4136.E+54	1.2773.E+57
1.0000	-0.33	2.0973.E+56	1.9738.E+54	1.8575.E+52	2.4001.E+50	9.9003.E+51	1.5027.E+54	2.2813.E+56
minH1	0.00	4.4508.E+55	4.1886.E+53	3.9421.E+51	7.1660.E+49	5.2475.E+51	7.9658.E+53	1.2093.E+56
0.00	0.33	9.4888.E+55	8.9299.E+53	8.4055.E+51	3.2921.E+50	3.7972.E+52	5.7646.E+54	8.7513.E+56
minH2	0.67	9.2965.E+56	8.7489.E+54	8.2352.E+52	3.3749.E+51	3.9472.E+53	5.9923.E+55	9.0970.E+57
0.00	1.00	9.7245.E+57	9.1516.E+55	8.6143.E+53	3.5330.E+52	4.1331.E+54	6.2745.E+56	9.5254.E+58

with $mOpt$ equal to

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	7.1660.E+49	5.1512.E+39	3.7028.E+29	2.6617.E+19	1.9133.E+09	1.3754.E-01	9.8866.E-12
	-0.67	2.4001.E+43	1.7252.E+33	1.2402.E+23	8.9147.E+12	6.4082.E+02	4.6064.E-08	3.3112.E-18
	-0.33	8.0384.E+36	5.7783.E+26	4.1536.E+16	2.9858.E+06	2.1463.E-04	1.5428.E-14	1.1090.E-24
	0.00	2.6923.E+30	1.9353.E+20	1.3911.E+10	1.0000.E+00	7.1883.E-11	5.1672.E-21	3.7144.E-31
	0.33	9.0170.E+23	6.4817.E+13	4.6593.E+03	3.3492.E-07	2.4075.E-17	1.7306.E-27	1.2440.E-37
	0.67	3.0200.E+17	2.1709.E+07	1.5605.E-03	1.1217.E-13	8.0634.E-24	5.7963.E-34	4.1665.E-44
	1.00	1.0115.E+11	7.2708.E+00	5.2265.E-10	3.7570.E-20	2.7006.E-30	1.9413.E-40	1.3955.E-50

For $idxX1 = 1$, $idxX2 = 1$ we get

	X2	88.9744						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	1.6746.E+45	1.3876.E+44	1.6017.E+45	4.2727.E+46	1.1424.E+48	3.0545.E+49	8.1670.E+50
minMOpt	-0.67	2.8891.E+44	2.3938.E+43	2.7631.E+44	7.3708.E+45	1.9708.E+47	5.2693.E+48	1.4089.E+50
0.0002	-0.33	5.0028.E+43	4.1385.E+42	4.7667.E+43	1.2715.E+45	3.3997.E+46	9.0901.E+47	2.4305.E+49
minH1	0.00	1.0614.E+43	8.0763.E+41	8.2275.E+42	2.1935.E+44	5.8649.E+45	1.5681.E+47	4.1928.E+48
0.67	0.33	2.2606.E+43	1.1206.E+42	1.4665.E+42	3.7864.E+43	1.0123.E+45	2.7068.E+46	7.2372.E+47
minH2	0.67	2.2146.E+44	1.0470.E+43	7.4699.E+41	6.7858.E+42	1.8081.E+44	4.8345.E+45	1.2926.E+47
-0.33	1.00	2.3166.E+45	1.0942.E+44	5.3022.E+42	3.8288.E+42	9.5857.E+43	2.5627.E+45	6.8520.E+46

and the corresponding part of $mOpt$ is

	X2	88.9744						
X1	vHold	-0.67	-0.33	0.00	0.33	0.67	1.00	1.33
86.8123	-1.67	7.4699.E+41	1.7259.E+35	3.9874.E+28	9.2126.E+21	2.1285.E+15	4.9176.E+08	1.1362.E+02
	-1.33	2.5019.E+35	5.7803.E+28	1.3355.E+22	3.0855.E+15	7.1288.E+08	1.6470.E+02	3.8053.E-05
	-1.00	8.3793.E+28	1.9360.E+22	4.4729.E+15	1.0334.E+09	2.3876.E+02	5.5163.E-05	1.2745.E-11
	-0.67	2.8064.E+22	6.4840.E+15	1.4981.E+09	3.4611.E+02	7.9966.E-05	1.8475.E-11	4.2686.E-18
	-0.33	9.3994.E+15	2.1716.E+09	5.0174.E+02	1.1592.E-04	2.6783.E-11	6.1879.E-18	1.4296.E-24
	0.00	3.1481.E+09	7.2734.E+02	1.6804.E-04	3.8825.E-11	8.9701.E-18	2.0725.E-24	4.7882.E-31
	0.33	1.0544.E+03	2.4360.E-04	5.6282.E-11	1.3003.E-17	3.0043.E-24	6.9412.E-31	1.6037.E-37

Fibre step 0

At the final fibre step (fibre 0) we get (for $idxX1 = 0$ and $idxX2 = 0$)

Timesteps	Holding Steps	Option Price
2	7	13.3227
25	25	8.0552
50	50	7.2058
Theoretical value		6.3180

Table 3.15: Option price convergence

	X2	110.0000						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	6.8133.E+57	1.3958.E+56	1.1889.E+55	5.9436.E+56	3.9094.E+58	2.5716.E+60	1.6916.E+62
minMOpt	-0.67	8.4618.E+56	1.7335.E+55	1.4765.E+54	7.3817.E+55	4.8553.E+57	3.1938.E+59	2.1009.E+61
6925.7021	-0.33	1.0509.E+56	2.1530.E+54	1.8337.E+53	9.1677.E+54	6.0300.E+56	3.9666.E+58	2.6093.E+60
minH1	0.00	1.3082.E+55	2.6800.E+53	2.2791.E+52	1.1389.E+54	7.4909.E+55	4.9276.E+57	3.2414.E+59
0.33	0.33	2.1127.E+54	4.3273.E+52	3.1089.E+51	1.4631.E+53	9.6233.E+54	6.3303.E+56	4.1641.E+58
minH2	0.67	8.1908.E+54	1.6766.E+53	4.9083.E+51	9.7259.E+52	6.3931.E+54	4.2055.E+56	2.7664.E+58
-0.33	1.00	1.2983.E+56	2.6575.E+54	7.4083.E+52	1.2970.E+54	8.5246.E+55	5.6076.E+57	3.6887.E+59

while the final values of the $mOpt$ matrix are

	X2	110.0000						
X1	vHold	-0.67	-0.33	0.00	0.33	0.67	1.00	1.33
100.0000	-1.33	3.1089.E+51	1.0919.E+43	3.8351.E+34	1.3470.E+26	4.7310.E+17	1.6617.E+09	5.8362.E+00
	-1.00	6.4088.E+43	2.2509.E+35	7.9059.E+26	2.7768.E+18	9.7528.E+09	3.4254.E+01	1.2031.E-07
	-0.67	1.3211.E+36	4.6402.E+27	1.6298.E+19	5.7241.E+10	2.0105.E+02	7.0613.E-07	2.4801.E-15
	-0.33	2.7234.E+28	9.5654.E+19	3.3596.E+11	1.1800.E+03	4.1445.E-06	1.4556.E-14	5.1126.E-23
	0.00	5.6142.E+20	1.9719.E+12	6.9257.E+03	2.4325.E-05	8.5436.E-14	3.0007.E-22	1.0539.E-30
	0.33	1.1573.E+13	4.0649.E+04	1.4277.E-04	5.0144.E-13	1.7612.E-21	6.1858.E-30	2.1726.E-38
	0.67	2.3858.E+05	8.3795.E-04	2.9431.E-12	1.0337.E-20	3.6306.E-29	1.2752.E-37	4.4788.E-46

The price of the option premium is given by $optionPrem = mOpt(3,3) = 1179.9970$.

In order to calculate the option price for the specific example, we need to calculate the following function

$$optValue = \exp(-r * T) * \ln\left(\frac{optionPrem}{marketPrem}\right) / \gamma \Rightarrow$$

$$optValue = \exp(-0.06) * \ln\left(\frac{1179.9970}{1}\right) * 2 = 13.3227$$

This is the approximate price of a spread option. The theoretical price of a spread option with the same input parameters is 6.3180. Although this might seem far from the answer obtained using this algorithm, it must be taken into account that a small number of time-steps and a coarse asset holding discretisation was used. A summary of how the option price changes with increasing time-steps and holding discretisations, can be seen in Table 3.15.

Obviously, the higher the discretisation the closer the price gets to the theoretical value of 6.3180. This is shown later in this chapter, in \he results section. A numerical example demonstrating the pricing of the same option subject to interrupted trading can be found in the Appendix, section Appendix III.

3.6 Results

In this section, the results obtained for the spread option model, pricing options on two assets with interrupted trading, will be presented. Prior to presenting any results, a thorough check on the feasibility of the model will be performed. It is important to check that the model behaves in a proper fashion, in that it converges (in specific special or extreme cases) to values that can be calculated in proven ways. For example, in the case when one of the two assets used in pricing a two asset option has an initial value of zero, then the option price should revert to the price of the single asset model. Furthermore, when no interrupts occur during the life of this option, the value should converge to the theoretical Black & Scholes option price.

3.6.1 Convergence to single asset option prices

We will first test the convergence to single asset options, both with continuous and interrupted trading. For this set of examples the price of the second asset will be set to zero ($S_2 = 0$) throughout the entire investment horizon (i.e. the second asset is absent). This is an important set of results as it will quickly identify any erroneous behaviour or non-convergence of the model. The check against the single asset model (with interrupts) will be used to asses the ability of the model to price this type of options properly.

No trading interrupts

Firstly, the convergence to the Black & Scholes price for a single asset will be presented. Here, it is assumed that no trading interruption period intervenes during the life of the option. It is expected that the model will converge to the 'theoretical' price of a single asset model, since (second asset price is zero) no adjustment is required in the second asset holding. This means that the premium associated with writing the option will be equal to the premium of the single asset model of the previous chapter, while the premium associated with the market is the same as before, since the market contains only one risky asset.

In the first example, the convergence of the model to single asset options is investigated through the increase of time-steps, while keeping the number of asset holding steps constant. In particular, the input set used in this example is the following, $S_1 = 100$, $\sigma_1 = 0.2$, $\mu_1 = r$, $K = 100, T = 1, r = 0.06$. In order to avoid any interrupts, the trading stop time ($tStp$) and the trade resumption time ($tStr$) are set to 1, while $\gamma = 0.5$. 50 holding steps are used in this example.

The prices obtained by the two asset model (referred to as f2IT) are compared to three different models (Figure 3.19). A single asset Black & Scholes (referred to as f1BS, from now on) spot model is used to obtain the 'theoretical' value of the option; a single asset Cox-Ross-Rubinstein (f1CRR) model and a two asset Clewlow & Strickland model (referred to as f2CRR to denote that it is a lattice based method) are used in order to compare the trend of the convergence for the different time-steps.. As expected, as the number of time-steps used increases, the price of the f2IT model converges to the price of the f1BS.

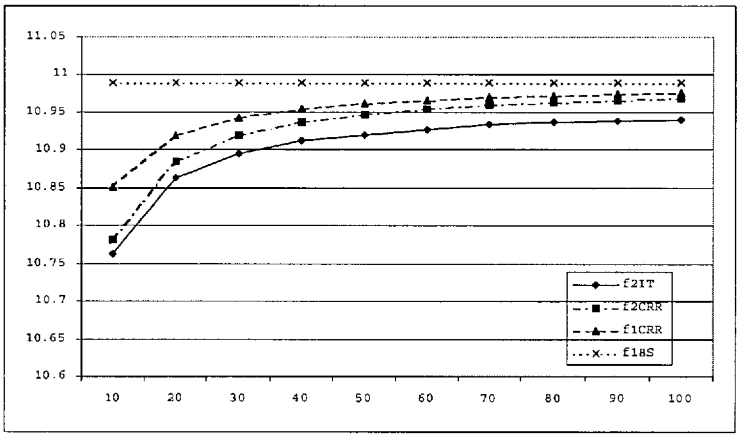


Figure 3.19: Convergence to single asset options - increasing timesteps

Obviously, the f1CRR and f2CRR models converge to the f1BS price as well, but their usefulness lies in that they show the trend of convergence as the number of time-steps becomes bigger. Clearly, the f2IT model follows their trend, even though the number of holding steps is small.

In the next example, the convergence of the model to the f1BS price is investigated when the number of holding steps is increased, while keeping the number of time-steps constant. The same set of inputs is used for this example, while the number of time-steps is set to 50.

In Figure 3.19 we can see, that for a low number of holding steps (10 to 30) the price obtained by the f2IT model is extremely inaccurate. Nevertheless, as the number of holding steps reaches a reasonable amount (50 to 70) convergence starts to occur, while for high values of holding steps (90 and 100) the price is almost exact with the f1CRR and f2CRR models. In this figure the f1CRR and f2CRR model yield constant prices since a constant time-step is used.

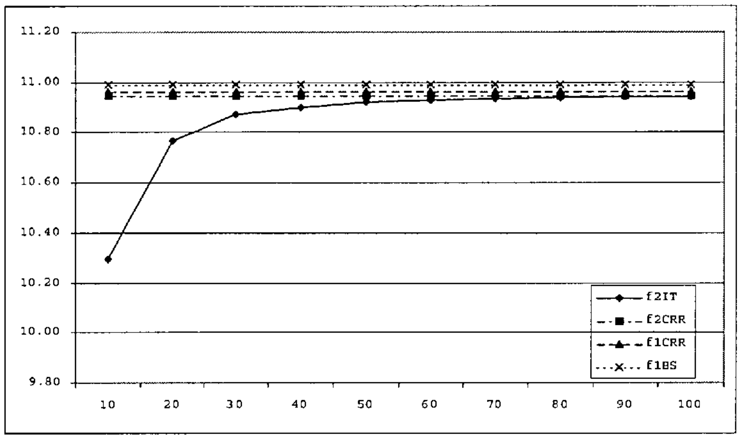


Figure 3.20: Convergence to single asset options - increasing holding steps

Trading interrupts

We now turn our attention to the ability of the model to price trading interrupts in a single asset setting. In order to verify the results obtained from the f2IT model, the single asset model (referred to as f1IT) of the previous chapter will be used. Checking that the two models converge, in the simple case of no trading interrupts, is redundant, as this has already been done through checking for convergence to the f1CRR model, which is equivalent to the f1IT in this base case. The set of inputs used in this example is the same, for all asset prices and option terms specifications, to the previous examples. In order to create a trading interruption period $tStp$ is set to 0.5, while $tStr = 1$. Again the convergence will be investigated in two ways. By increasing the time-steps and by increasing the number of holding steps. In the first example, the number of holding steps is fixed to 50.

As it can be seen from Figure 3.21, both models move in the same direction, while the distance between them seems to become narrower as the number of steps increases,

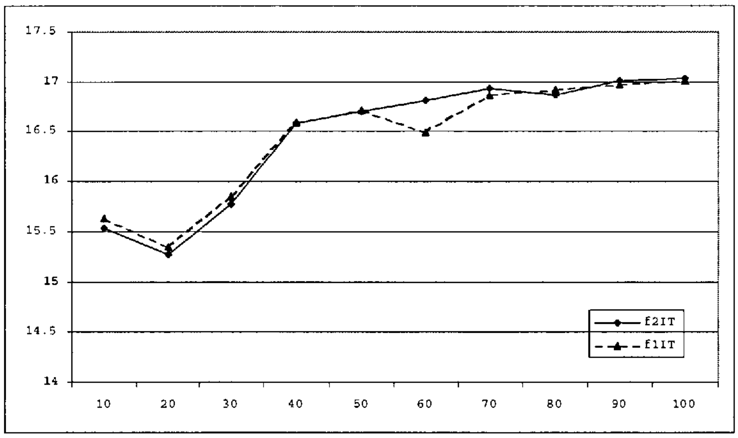


Figure 3.21: Trading interrupts convergence - increasing timesteps

apart from the outlier point for 60 time-steps.. The main point of this example is that both models move in the same direction.

The next example (Figure 3.22) investigates how the f2IT model converges to the f1IT model (for a constant time-step number) as the number of holding steps increases. The time-steps are fixed to 50. In contrast to the previous example, the distance between the two model decreases, an effect expected since the number of time-steps increases. We can again draw the conclusion that the f2IT model is behaving properly since both models move in the same direction.

It has now been demonstrated that the f2IT model converges to single asset models. Furthermore, it has be demonstrated that it handles trading interrupts in the same way that the f1IT model does. All these checks are reassuring as they confirm the feasibility of the model. We will now move to a more complex problem and check how the f2IT model converged to two asset models in a two asset setting. This is a useful check since the base case for the model should yield the theoretical price of a two asset option.

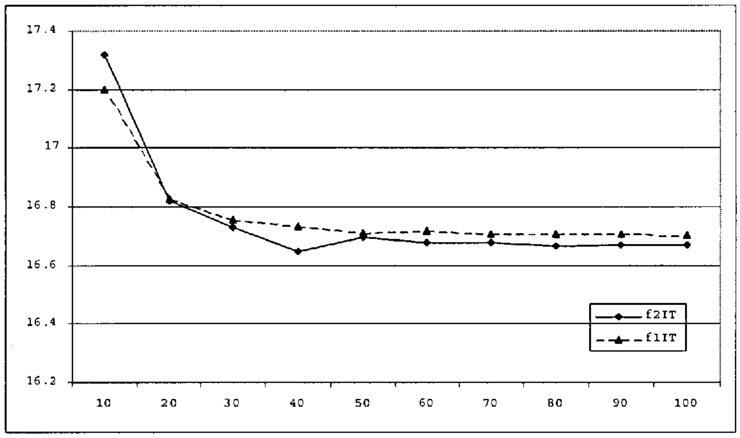


Figure 3.22: Trading interrupts convergence - increasing holding steps

3.6.2 Convergence to two asset models

No trading interrupts

For this set of examples we need to change the input set, as we need to specify the price and statistics of the second asset. The set used here is the following; $S_1 = 100$, $\sigma_1 = 0.2$, $\mu_1 = r$, $S_2 = 100$, $\sigma_2 = 0.2$, $\mu_2 = r$, $K = 0$, $T = 1$, $r = 0.06$ and $\rho = -0.5$. In order to avoid any interrupts, the trading stop time ($tStp$) and the trade resumption time ($tStr$) are reset to 1, while $\gamma = 0.5$. In the next example the convergence of the f2IT model is demonstrated to the two asset Black & Scholes model (f2BS) and the f2CRR. We will investigate this effect in two steps, firstly by checking the time-step convergence and then the holding step convergence.

In this example the number of holding steps is set to 50. It can be seen from Figure 3.23 that the convergence to the 'theoretical' f2BS price is strong, nevertheless a significant number of time-steps is required in order to decrease the error between the f2IT and the f2BS prices. Specifically, the error for 100 time-steps between f2IT and f2BS is 0.87%,

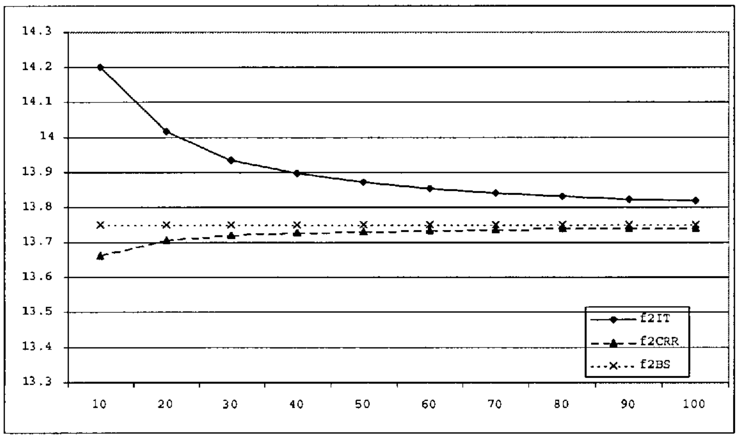


Figure 3.23: Convergence to two asset models - increasing timesteps

while the error for 130 time-steps drops to 0.69% (not shown on the figure). A factor affecting the error between the two values is the number of holding steps, which as it increases it brings the error (at least to the f2CRR model) down.

This effect is demonstrated in Figure 3.24, were the value of the f2IT model converges to both the f2CRR and the f2BS prices. As with the single asset examples presented earlier, the prices are quite far apart for small steps of holding discretisation. Nevertheless, once a reasonable amount of steps is used (60 or 70), the model converges to a steady price, the f2CRR price.

Trading interrupts

We will now use the f2IT model to price two asset options (particularly spread options) in the case where some interruption occurs during the life of the option. In the cases examined here we do not have a model to check the prices obtained, but the results drawn from the checks performed earlier in this section confirm that the model will behave properly. The tests that are crucial for this part are the convergence to the single asset model

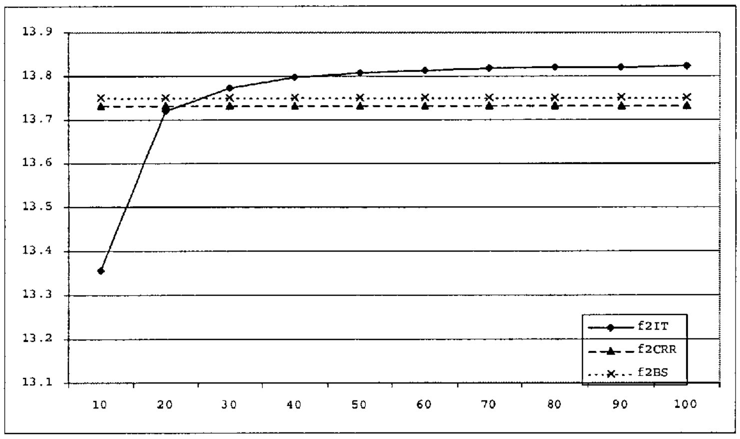


Figure 3.24: Convergence to two asset models - increasing holding steps

when interruptions occur during the life of the option and the convergence to the two asset models in the base case of no interruptions occurring. The latter test is useful because it confirms that as the interruption period gets smaller, the price of the f2IT model must converge to the price of a f2BS or f2CRR model. In this set of examples the following set of inputs is used: $S_1 = 100$, $\sigma_1 = 0.2$, $\mu_1 = r$, $S_2 = 100$, $\sigma_2 = 0.2$, $\mu_2 = r$, $K = 0$, $T = 1$, $r = 0.06$, $\rho = 0.5$ and $\gamma = 0.5$. The number of time-steps is set to 50 and the number of asset holding steps are set to 50.

In Figure 3.25, the time trading resumes is fixed to the expiry of the option ($tStr = 1$), while the time trading is interrupted is increasing towards the expiry of the option. It can be observed that the price indeed drops as the duration of the interruption decreases and specifically, the f2IT price converges to the prices of the f2BS and the f2CRR models.

In the next example (Figure 3.26), the situation is reversed. That is the interruption starting period is fixed to the pricing time of the option ($tStp = 0$) while the trading resumption time is increased towards the expiry of the option.

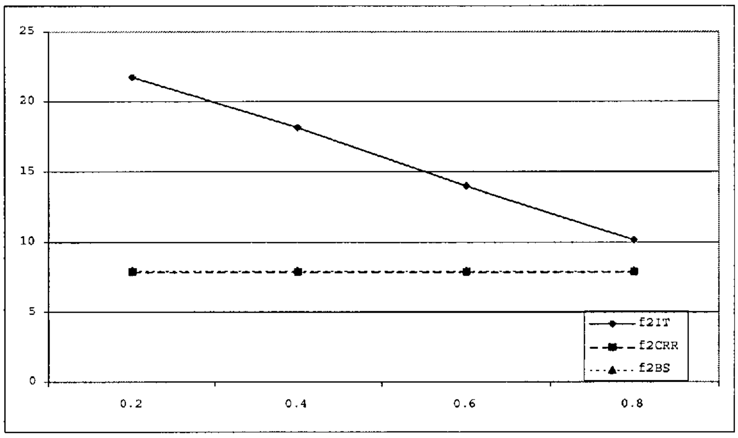


Figure 3.25: Pricing trading interrupts - increasing time interrupt starts

Again it can be immediately observed, that the price of the option reverts to the price of the 'theoretical' f2BS model and the f2CRR model as the duration of the interruption is decreased.

3.6.3 Effect of multiple trading interruptions

In the previous chapter, it was demonstrated that a decrease in value of the option can be achieved by splitting an interruption period into smaller periods, while keeping the overall interruption period equal. This effect is of potentially high value, since it can be applied in many cases, particularly scheduling. In order to investigate whether this effect carries through to this model, four sets of results are presented. The first two sets of results, deal with the change of option price with increasing volatility in the first and the second asset respectively. The third set of results deals with the sensitivity of the option price change with respect to changing correlation between the two assets and the final set of results demonstrates how different levels of risk aversion affect the change in the option value. For all the results presented in this section, the reduction, in percentage terms, is

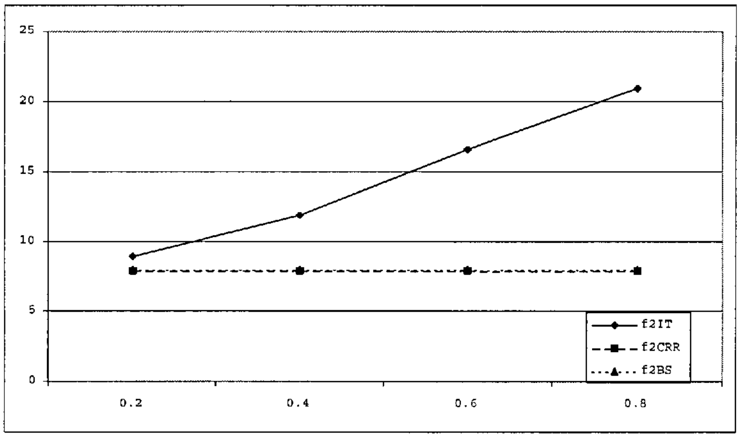


Figure 3.26: Pricing trading interrupts - increasing trading resumption time

represented by the doted line on the figures, where the level of percentage reduction is charted on the right-hand side of the figure.

The parameter set for this section is common, unless otherwise stated. The assets' spot prices are set to 20 and the strike price of the option is set to 0. The two volatilities are set to 20%, while the two drifts are set equal to the risk free interest rate. The correlation of the two assets' returns is set to 50%, the risk free interest rate is set to 3% and the risk aversion factor is set to 0.5. The options expire in one year. The initial period of interruption is a twelve week period, which is split into smaller periods, similarly to the previous chapter. The durations, starting and ending times for these periods are given in Table 3.16.

Sensitivity to the volatility of the first asset

The sensitivity to the volatility of the first asset is tested against an increasing level, starting from 10% and it is increased to 20% and finally to 30%. The three figures, Figure 3.27, Figure 3.28 and Figure 3.29, show the increase in the change of the option price when the initial interruption period is separated into smaller periods.

dur	12	dur	6	dur	4
start	end	start	end	start	end
0.3846	0.6154	0.2500	0.3654	0.1923	0.2692
		0.6346	0.7500	0.4615	0.5385
				0.7308	0.8077
dur	**3**	**dur**	**2**	**dur**	**1**
start	end	start	end	start	end
0.1548	0.2125	0.0971	0.1356	0.0769	0.0962
0.3663	0.4240	0.2510	0.2894	0.1538	0.1731
0.5779	0.6356	0.4048	0.4433	0.2308	0.2500
0.7894	0.8471	0.5587	0.5971	0.3077	0.3269
		0.7125	0.7510	0.3846	0.4038
		0.8663	0.9048	0.4615	0.4808
				0.5385	0.5577
				0.6154	0.6346
				0.6923	0.7115
				0.7692	0.7885
				0.8462	0.8654
				0.9231	0.9423

Table 3.16: Durations, starting and ending times

The option prices, for the three cases (volX1 = 10%, volX1 = 20% and volX1 = 30%) are decreasing as the number of periods of interruption increases. This is exactly the same effect as observed for the single asset model. Furthermore, the change of the value of the options is getting bigger as the volatility increases.

More specifically, for a volatility of 10%,the option value decreases by 1.09% through the process of separating the single twelve week interruption period into twelve single week periods of interruption. As the volatility increases, this reduction increases to 1.29% for a volatility of 20% and finally to 1.78% for a volatility of 30%. It is also worth mentioning that, the reduction in the option price is monotonous for increasing number of interruption periods. These results suggest that there is a bigger gain by following such a procedure when the first asset is more volatile.

Sensitivity to the volatility of the second asset

In Figure 3.30, Figure 3.31 and Figure 3.32 the effect of the second asset's volatility on the change of the option price is shown. The results were obtained for similar levels of volatility as for the first asset.

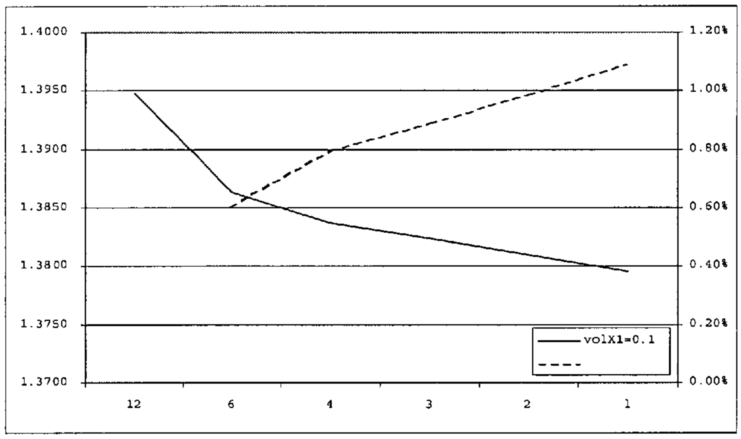


Figure 3.27: Volatility sensitivity for first asset (volX1 = 10%)

These results are very similar to the results obtained for the first asset's volatility level. In fact, the option prices are almost identical, as well as the change in option value. This suggests that it is irrelevant which asset has a high volatility. The main result from these two sets is that it is advantageous to distribute periods of interruption when volatility is high.

Sensitivity to correlation

In this set of results, the effect of the correlation between the two assets' returns is investigated. These results can be useful for applications where extreme events, such as near perfect positive or negative correlation occurs, or completely uncorrelated assets are considered. In order to demonstrate this effect three points are used, a correlation of 50%, 0% and −50%.

The results obtained for the correlation sensitivity are shown in Figure 3.33, Figure 3.34 and Figure 3.35. The reduction of the option prices is linear, since the biggest reduction is observed for the negatively correlated assets. The reduction of the option price between

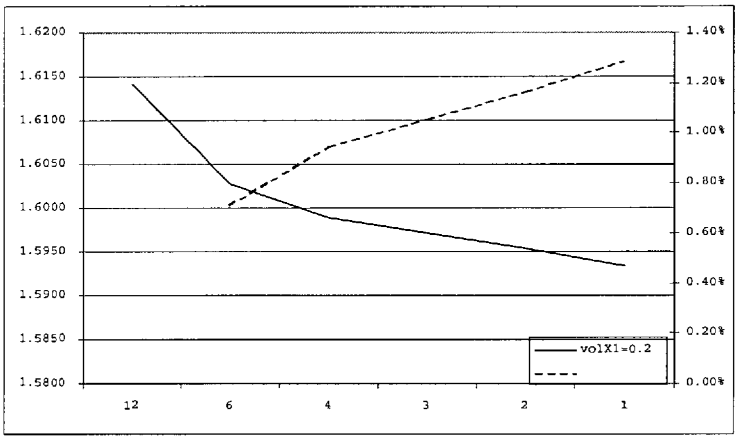


Figure 3.28: Volatility sensitivity for first asset (volX1 = 20%)

the single twelve week period and the twelve single week interruptions is 1.29% for the positively correlated assets. As the assets tend towards being more negatively correlated this reduction becomes higher, since for 0% correlation the reduction is 1.91% and 2.35% for the negatively correlated assets. The lowest reduction is associated with the positively correlated assets, where coincidentally we observe the lowest option prices.

These results are intuitively correct, since one would benefit more from spreading periods of interruption in the cases where the two assets are negatively correlated. If we consider that the options priced here are spread options, two assets moving in opposite directions could introduce more risk over a large period of no trading, by widening the spread significantly.

Sensitivity to the risk aversion factor

The final set of results, demonstrates the effect that the risk aversion factor has on the reduction of the option prices. The three figures (Figure 3.36, Figure 3.37 and Figure 3.38) show the results obtained for increasing γ. The first figure uses a γ of 0.2, the second uses

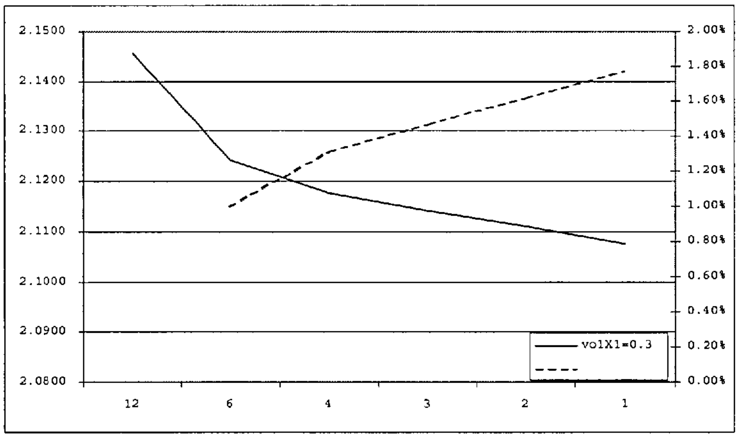


Figure 3.29: Volatility sensitivity for first asset (volX1 = 30%)

0.5 and the third has a γ of 0.7.

The effect that the risk aversion factor has on the actual option prices is quite small. Nevertheless, the reduction for higher values of risk aversion is significantly high. The change between the single twelve week period interruption, for a low 0.2 γ, and the equivalent twelve one-week trading interrupts is 0.47%. The corresponding change, for a higher γ of 0.7, is 1.91%, which is more than double. On the other hand, the option prices for the higher risk aversion factor are bigger than those for lower γ.

In this section, the convergence of the two asset model was investigated. After the preliminary verification that it does converge to the 'theoretical' values for simple cases (one asset options), it was tested against the 'theoretical' values of two asset options (spread options). Results were provided on how the value of the option is increased when periods of trading interruption occur. Finally, it was demonstrated how the value of the option can be decreased by spreading the trading interruption periods across the year and how the main parameters for pricing the option, namely the volatility of the two assets, their

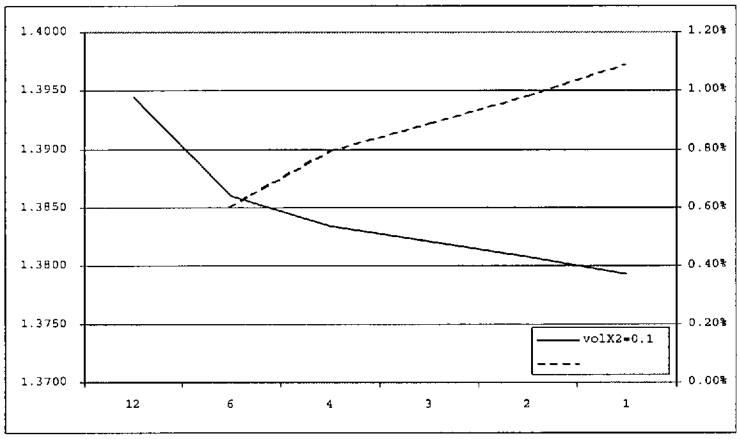


Figure 3.30: Volatility sensitivity for second asset (volX2 = 10%)

correlation and the risk aversion factor, affect this reduction.

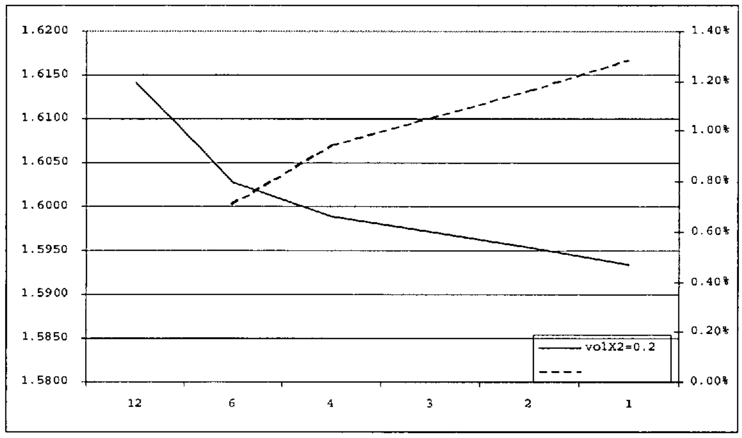


Figure 3.31: Volatility sensitivity for second asset (volX2 = 20%)

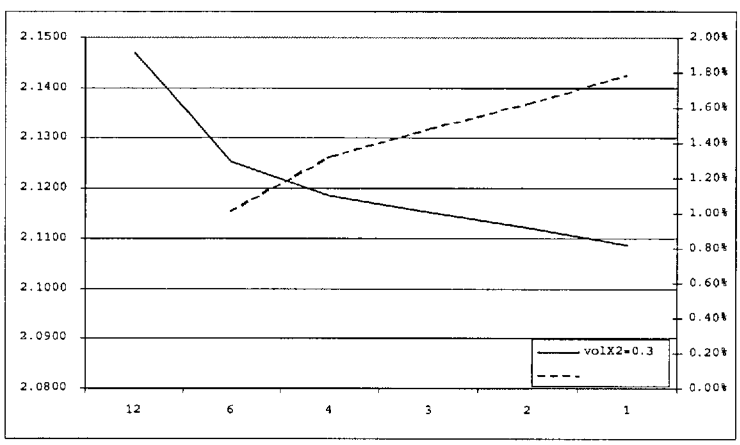


Figure 3.32: Volatility sensitivity for second asset (volX2 = 30%)

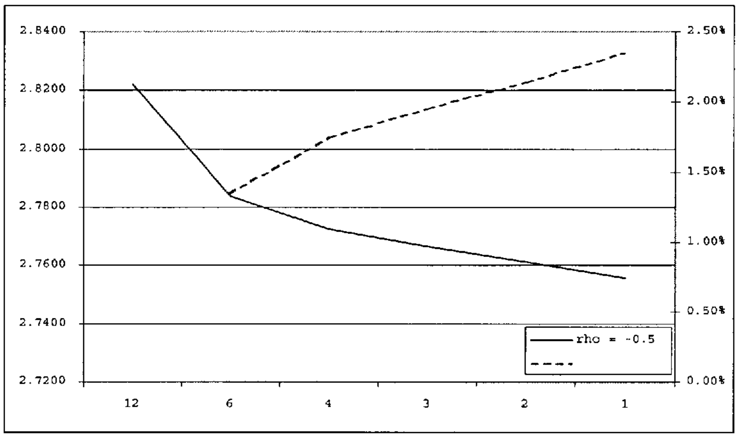


Figure 3.33: Correlation sensitivity (rho = -50%)

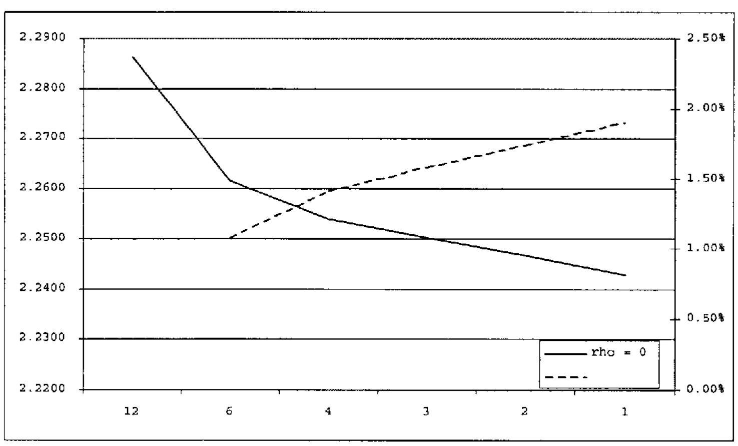


Figure 3.34: Correlation sensitivity (rho = 0%)

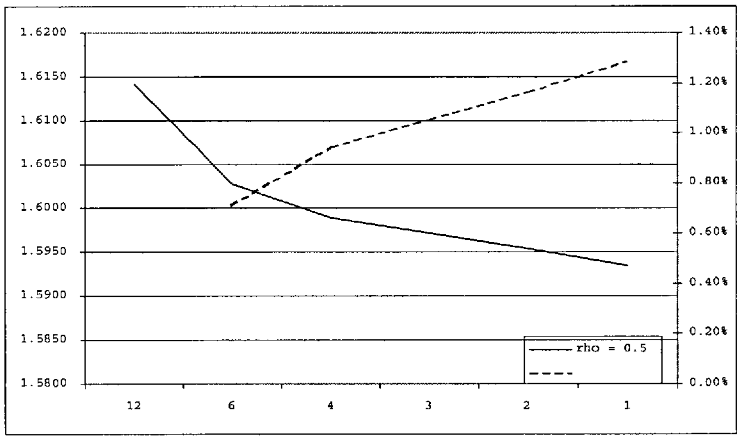


Figure 3.35: Correlation sensitivity (rho = 50%)

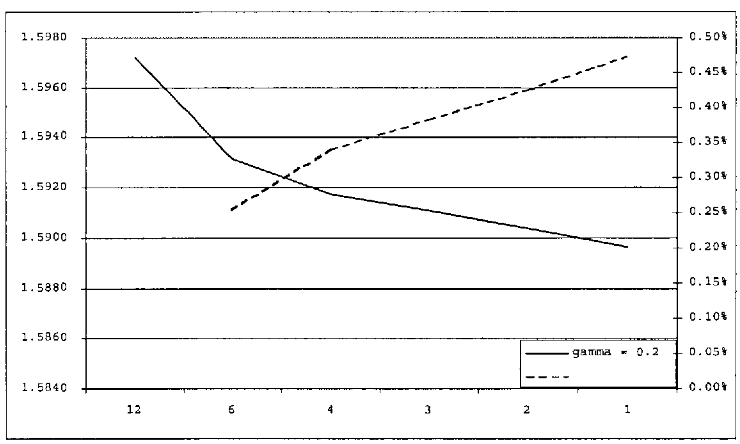


Figure 3.36: Risk aversion sensitivity (gamma = 0.2)

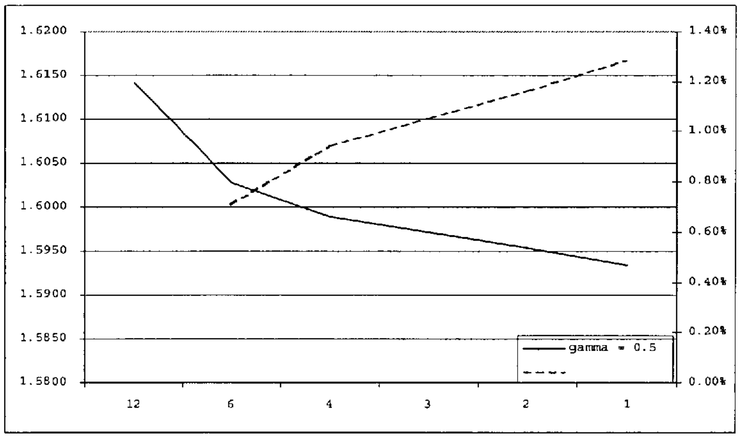


Figure 3.37: Risk aversion sensitivity (gamma = 0.5)

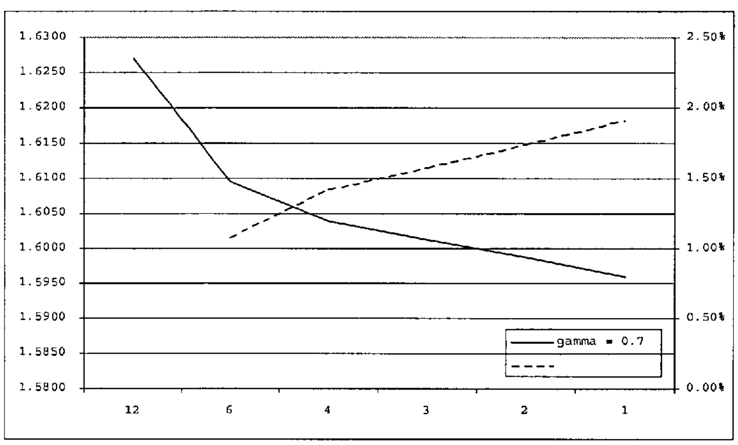


Figure 3.38: Risk aversion sensitivity (gamma = 0.7)

Chapter 4
An application of the two-asset model

We will now apply the model developed in the previous chapter to a real world case. We will consider the example of a petroleum refinery and try to price the optionality associated with the scheduling of its maintenance periods. Firstly, some background will be provided for the operation of the plant and then the context of the problem considered in this chapter will be given.

Some use of financial contracts will be required in order to define the setting of the problem. The contract particularities will be provided prior to the specification of the problem and the outline of the proposed solution. Finally, the results drawn for this example will be presented and discussed in detail at the end of this chapter.

4.1 Profit margin and crack spread

In this example an effort is made to price the optionality associated with shutdown periods of a petroleum refining plant. Petroleum refineries require periods of maintenance throughout the year, sometimes disrupting the production of refined products. The maintenance is essential to the normal operation of the plant and thus may not be able to be rescheduled or ignored. In this example it is assumed that the plant has a scheduled maintenance timetable, which dictates that the plant must stop operating for a small period of time at the end of a month. We will assume that four days are required for the essential work to be carried out.

Petroleum refining plants use crude oil as a feedstock which they process in order to produce heating oil and unleaded gasoline. This makes the plant exposed to two sources of risk, the floating price of crude oil and the floating prices of its products. The crude oil is placed in a different market than its products (such as heating oil and gasoline)and the two markets are operating independently, obeying their own economic and regulatory rules. The difference between the purchasing of the feedstock and the sale of its products generates the revenue (or margin) of the plant. Typically, petroleum refineries use crude oil to produce heating oil and gasoline in a proportion of 3 crude oil barrels to 2 barrels of gasoline and 1 heating oil barrel. The process of converting crude oil into its products is called 'cracking'. Hence the margin generated by refining crude oil into heating oil and gasoline is called the crack spread. Crack spreads are tradeable futures contracts in many markets, such as the NYMEX. One of the most popular crack spreads is the 3:2:1 crack spread which includes three crude oil, two gasoline and one heating oil futures, reflecting the refining process of a typical petroleum refinery. Other combinations of crack spreads include the 5:3:2 crack spread, which can be used by refineries that produce a lower ratio between gasoline and heating oil.

Due to the difference in nature of crude oil and its products, the quotation of their prices is different as well. Crude oil prices are quoted in dollars per barrel ($\$/bbl$), while heating oil and gasoline are quoted in cents per gallon (c/gal). In order to calculate the crack spread, the gasoline and heating oil futures prices need to be converted into $\$/bbl$

Month	Crude Oil	Unl Gasoline	Heating Oil	Crack Spread	
Mixture	3	2	1	3:2:1	\$/bbl
Oct-03	32.07	0.9370	0.8415	17.84	5.95
Nov-03	31.65	0.8675	0.8490	13.58	4.53
Dec-03	31.10	0.8360	0.8555	12.86	4.29
Jan-04	30.52	0.8199	0.8550	13.22	4.41
Feb-04	29.88	0.8179	0.8445	14.53	4.84

Table 4.17: Crack spread futures

prices. The conversion factor between c/gal and $\$/bbl$ is 0.42 ($1c/gal = 0.42\$/bbl$) for the commodities considered here. Then, taking into account the ratio of the commodities involved we can calculate the spread in the following way. Assume that the current price of crude oil is $\$32.07/bbl$. With heating oil at $84.15c/gal$ and gasoline at $93.70c/gal$, the crack spread is $2*93.70c/gal*0.42\frac{\$/bbl}{c/gal} + 84.15c/gal*0.42\frac{\$/bbl}{c/gal} - 3*\$32.07/bbl = \$17.84$ per three barrels of crude oil ($\$5.95/bbl$). In fact, from recent data from the NYMEX the crack spreads for the period from October 2003 to February 2004 are shown in Table 4.17.

In order to eliminate one source of risk the plant manager could buy an oil swap, a contract which converts the floating price of crude oil into a fixed price. Purchasing an oil swap has some positive and negative effects on the management of the plant. One of the positive effects of purchasing an oil swap is that one of the main overheads of the plant (the purchasing of its feedstock) is fixed and therefore the manager can assign cash reserves to other operations rather than the risk management of the oil price. Further, it can possibly generate some income for the plant in the cases when the market (spot) price of crude oil is above the level of the swap. As a downside, the plant may, at certain times, be paying more for its feedstock than the spot market. The major problem generated by purchasing a swap is that it offers no protection against a reduction of the margin between the feedstock and the products. A more effective method for locking in the margin of the plant is by trading in crack spread options; options on the difference of heating oil or gasoline and crude oil. Crack spread options are described in more detail, later in this chapter.

The plant produces V_b barrels of combined product daily at the ratio of two barrels of gasoline to one barrel of heating oil, thus requiring V_b barrels of crude oil per day for its normal production. It is assumed that the plant can inelastically buy crude oil and sell all its products in the market, therefore leaving out any storage or any offer and demand pricing issues. In order to simplify the problem, it is also assumed that the plant products are sold at the end of the month at the spot price of the day. In order to avoid cashflow mismatching, it is assumed that the plant pays for the oil used throughout the month on the same day, again based on the spot price of the day. In order to calculate the profit of the plant, one could take the current futures prices for the required period as an estimate of the spot price for the cashflows day and calculate the margin for the plant. Then by multiplying by the required volumes, the profit of the plant can be calculated.

It is easy to extract what the optionality of the plant is (associated with the margin of the plant). We can price a spread option (such as a crack spread option) based on a crude oil futures contract and one of the products of the plant and combine multiples of these options to reflect the production process of the plant. This gives the optionality of the plant if there were no maintenance periods throughout this period. Nevertheless, a conventional option pricing model cannot be used to price this optionality if there were any trading gaps (such as a maintenance period) during this period. In this case we can use the model developed in the previous chapter, to determine how the price is affected by such events.

We can calculate how much this mismatching costs to the plant by comparing the values of spread options based on continuous production and the price of spread options with restricted production during the periods of maintenance. By definition of the problem, if the plant manager were to sell spread options between the feedstock crude oil and the output products, he would end up with a zero profit strategy since the plant operations would form a natural hedge for the spread options (if the plant were not to stop producing). Therefore, the price of the continuous production spread options acts as a price indicator of the plant's value under normal daily operation. By comparing it to the price of spread options with interrupted production (the a priori know periods of maintenance) can give us an indication of the value lost in such scenarios.

It is appropriate, at this stage, to offer some background to the crack spread options included in this example case. Some detailing will be provided for the crack spread option, which is a derivative based on crack spreads as discussed earlier in this section.

Crack spread option

Most of the background required to understand crack spread options has been covered already in this section by the detailed description of the crack spread futures. Crack spread options resemble normal spread options (typically of the American type), but in contrast to the crack spread futures, only exist between a pairwise matching between crude oil and either heating oil or gasoline. The pricing of financial spread options is outside the scope of this chapter and thus omitted. It is mentioned though, that the payoff of a spread option dependent on two assets X_1 and X_2 is given by the following equation

$$SO_{t_0} = \max\left[X_{1,T} - k * X_{2,T} - K, 0\right] \exp\left(-r * (T - t_0)\right)$$

where k is a positive constant, K is the strike of the spread option, r is the risk-free interest rate and T is the maturity of the contract.

Crack spread options can be used by refineries in order to fix the lower boundary of their production margin, while they can be combined in numerous ways to produce different instruments tailored to the specific needs of the refinery. As an example, a call spread option (bought) can be combined with a put spread option (sold at a higher strike) in order to reduce the cost of the strategy, since some of the potential gain associated with purchasing the call option, is given away by the sale of the put.

Crack spread options can also be combined in order to produce a suitable hedge for or simulate a petroleum refining plant with the production proportions mentioned previously. We can produce a hedge for (or simulate) the plant by selling (purchasing) a call crack spread option between heating oil and crude oil and two call spread options between gasoline and crude oil, a strategy which in total gives one short (long) heating oil, two short (long) gasoline and three long (short) crude oil futures contracts.

There is a variety of intercommodity spread options, similar to the crack spread option, used across the utility world. The most common include the following:

1) Spark spread option: A spread option between electricity prices and natural gas prices to reflect the production of electricity using natural gas.

2) Frac spread option: A spread option between propane and natural gas to reflect the production of propane.

More information on crack spreads and their derivatives can be found in [31] .

4.2 Problem set-up

In order to complete the preliminary analysis of this example we need to provide some context for the parameters that will be used for pricing the contracts required. We will be pricing the optionality of the plant over the month of December 2003, hence the specific month futures price for the NYMEX light sweet crude oil, heating oil and unleaded gasoline are required as an index of the spot price at the beginning of the month. In NYMEX it is specified that the futures contracts expire close to the end of the month (crude oil futures expire three days prior to the 25th calendar day while heating oil and unleaded gasoline expire on the last trading day) preceding the delivery month. This means that the futures price for delivery in December, expire at the end of November and we can use the December delivery futures as an estimate of the spot price close to the beginning of the December. In order to price the spread options, we also require the volatilities of the three commodities and the two correlations between crude oil and heating oil and between crude oil and gasoline. In order to obtain the volatilities, option contracts on these commodities will be used to extract the implied volatilities. For the correlations, the spread options available in the market will used and the implied correlations will be extracted by using the futures prices and the implied volatilities extracted. From data obtained from the NYMEX website, we get the dataset seen in Table 4.18. The volatilities were implied by using an American style option on futures contracts, while the correlations were implied by using an American style option on two assets. The reason behind using these models, is that both single commodity and crack spread options in the market, are of the American style. The two-asset American style option model was obtained from Clewlow and

Commodity	**Price**	**Volatility**	**Correlation**
Crude Oil	30.77	0.2761	
Heating Oil	35.34	0.3060	0.9132
Unl Gasoline	34.59	0.2963	0.7877

Table 4.18: NYMEX data

Strickland [8] (chapter 2.11). More on extracting implied data, from market data can be found in [13] , [12] and [14] .

We will now prices two crack spread options, one between heating oil and crude oil (HCO) and one between unleaded gasoline and crude oil (GCO) for a continuous trading example. The options will be priced for one month (December) with the asset prices and statistics listed in Table 4.18, while $T = 31/365$ and $r = 1.25\%$. Since continuous production is assumed, the starting and stopping times for the interruption period are set equal to the expiry of the options. Additionally, the options will be priced for two risk aversion factors 0.2 and 0.8 to investigate the behaviour of the plant manager towards risk.

The options are priced using 62 time-steps (increments of half day) and 20 steps for the asset discretisation. The option prices obtained for the lower gamma (0.2) are 0.5669 for the HCO and 0.7364 for the GCO. The equivalent prices calculated with a continuous time spread model are 0.5662 for the HCO and 0.7349 for the GCO. From now on we will ignore the continuous time spread option for the reason that this model is inapplicable to the rest of this analysis and in order to avoid convergence errors.

Having calculated the continuous production option prices, we can calculate the optionality of the plant. We need two CGOs and one HCO in order to simulate the production process of the plant. This means that the plant option is equal to $2 * 0.7364 + 0.5669 =$

Option	Price ,γ=0.2 ($)	Price ,γ=0.8 ($)
HCO	0.5669	0.5723
GCO	0.7364	0.7418
Optionality	2.0397/3bbl 0.6799/bbl	2.0559/3bbl 0.6853/bbl

Table 4.19: Continuous production option values

Case I		Case II		Case III	
start	end	start	end	start	end
0.07397	0.08493	0.03699	0.04247	0.01849	0.02123
		0.07945	0.08493	0.03973	0.04247
				0.06096	0.06370
				0.08219	0.08493

Table 4.20: Interruption times

2.0397 per three barrels of oil. This gives $\$0.6799/bbl$ of oil. For ease of reference we will refer to the continuous production case as the base case. The summary of the results for the two risk aversion factors, for the basic case, are shown in Table 4.19.

If a period of interruption is introduced, such as a four days maintenance, we can re-price the options with the modified starting and stopping times and compare the value of the new optionality and the base case optionality. This can give us a measure of the cost of this maintenance period to the plant.

4.3 Results

We will now investigate how different settings of maintenance periods affect the plant optionality. We will start from the first case, which is a continuous period of maintenance of four days (Case I) and then split the maintenance period into two periods of two days (Case II) and four periods of one day (Case III). All cases will be priced under the two risk aversion factors.

The times used for the three cases are listed in Table 4.20. By pricing the two options

Option	Price ,γ=0.2 ($)	Price ,γ=0.8 ($)
HCO Case I	0.5675	0.5749
HCO Case II	0.5672	0.5736
HCO Case III	0.5671	0.5730
GCO Case I	0.7375	0.7466
GCO Case II	0.7369	0.7440
GCO Case III	0.7367	0.7429
Optionality Case I	2.0425	2.0682
	0.6808	0.6894
Optionality Case II	2.0411	2.0616
	0.6804	0.6872
Optionality Case III	2.0404	2.0589
	0.6801	0.6863

Table 4.21: Option values for the three cases

(HCO and GCO) for the two risk aversion factors the results shown in Table 4.21 were obtained. The first thing one can notice, is that the right column yields higher results than the column for a lower risk aversion factor. That is expected, as a more risk averse individual would require a more effective management of risk, hence the higher option premiums.

A result, expected from the previous two chapters, is that the option premiums reduce, as the number of periods of interruption increases. The reduction in the spread option price between Case I and Case II is 0.06% for $\gamma = 0.2$ and 0.24% for $\gamma = 0.8$ for the HCO and 0.08% for $\gamma = 0.2$ and 0.35% for $\gamma = 0.8$ for the GCO. Case III yields even bigger reductions (in comparison to Case I), where, for the HCO the option price is reduced by 0.08% for $\gamma = 0.2$ and 0.34% for $\gamma = 0.8$ and for the GCO the option price is reduced by 0.11% for $\gamma = 0.2$ and 0.49% for $\gamma = 0.8$.

From the results in Table 4.21, it can be seen that the maintenance period of Case I adds $\$0.0009/bbl$ in the $\gamma = 0.2$ case and $\$0.0041/bbl$ for the $\gamma = 0.8$ case. The maintenance schedule of Case II adds $\$0.0005/bbl$ for $\gamma = 0.2$ and $\$0.0019/bbl$ for $\gamma = 0.8$. Case III has the smallest effect of the three cases, since it only adds $\$0.0002/bbl$ for $\gamma = 0.2$ and only $\$0.0010/bbl$ for $\gamma = 0.8$.

By comparing the optionality of the base case and Case I, we can see that the value of such a schedule (in terms of how much this maintenance schedule adds to the optionality of the plant) is bigger than the subsequent Case II and Case III (smaller increase of the three schedules). This leads to the conclusion that spreading maintenance periods throughout the month, can significantly affect the value of the plant. The changes in value obtained might seem low, but by considering that these differences refer to the per barrel change, they can amount to significant savings for a large plant.

Obviously, several different cases can also be considered under this set-up, such as having restrictions placed, so that some periods cannot be shifted in time or split and optimise the rest of the periods to minimise the increase caused by production disruptions.

Chapter 5
Conclusions

In this thesis, one of the main assumptions in the Black & Scholes model [5] was investigated. The argument of market completeness was challenged, in the sense that the underlyings of options contracts are not always available for trading, therefore the hedging strategy suggested by the Black & Scholes model may not always be applicable. In that context, the theory of utility maximisation was used in order to create option pricing models.

The field of trading interruption has not been thoroughly researched, therefore practices from other areas were borrowed in order to solve the problem. The theory of portfolio pricing was initially investigated both in a complete market sense, where methods such as the martingale approach of Karatzas, Lehoczky and Shreve [19] and Pliska [34] and in a more general setting, using stochastic control, developed by Merton [29] . Most of the transaction costs literature is developed based on the stochastic control method. Papers such as [23] , [16] , [11] , and [2] are all based on the concept of portfolio utility maximisation. In this thesis, these concepts were used in a new setting, namely the pricing of options that cannot be continuously hedged by their underlying.

In the second chapter, two models were presented. Both models can handle the pricing of options based on one underlying which experiences some period of interrupted trading. The first model is the general approach to the optimisation problem, using a general utility function. The basis for this model is a simple portfolio consisting of a cash amount (or a bond) and a risky asset (such as a stock or some commodity). The aim of this model is to maximise the expected value of this basic portfolio when it is processed through some generic utility function. Then the premium associated with selling a call vanilla option is added to the portfolio, which has the effect of increasing the portfolio value. The premium is then minimised in order to make the value of the modified portfolio equal to the basic portfolio. The minimised premium is then the price at which the writer of the option is indifferent between holding the basic portfolio and holding the modified portfolio (including the optimised option premium). This model was tested under three different utility functions, all having a hyperbolic ARA measure. The sensitivity of the model against the risk aversion factor and the initial wealth of the portfolio was investigated in the continuous and interrupted settings. The results showed that the logarithmic utility function was insensitive to the risk aversion factor, while the negative exponential utility function was insensitive to the initial endowment in the portfolio. The optimisation problem variables were the holding of the risky asset for each time-step and for each path of a decomposed binomial lattice and the initial endowment of the portfolio. This prohibited the use of large numbers of time-steps as the number of optimisation variables grows at a rate of

$2^N * (N+1) + 1$ where N is the number of time-steps used for the binomial lattice. Nevertheless, it formed the basis for further investigation into the simplifications arising from the use of the negative exponential utility function.

The use of this utility function, not only reduces the optimisation problem by one factor (the initial endowment in the portfolio), but also yields a more precise solution to the problem. This forms the second model, derived in the second chapter, which is much easier to implement and can yield very accurate results, as larger numbers of steps can be used both on a binomial or a trinomial lattice. The second model, still obeys to basic rules of the first model, but is much more flexible in its application. Firstly, it was established that when the restriction of interrupted trading was abolished, the prices obtained from this model reverted to the continuous time, 'theoretical' Black & Scholes prices. Under uninterrupted trading, the effect of the risk aversion factor was found to be minimal. Then, the effect that different periods of interruption have on the model was investigated. This showed that the longer a period of interruption lasts the higher the option price is, while some small effect associated with the positioning of the interruption was found. That showed that an interruption of the same length, yields a higher option price when it occurs closer to the expiry of the option, nevertheless this effect was very small. The final investigation for this model, dealt with the effect of breaking up a period of interruption into smaller periods. The total period of interruption was always kept constant, but the number of interruption periods was increased. This strategy showed that the option price can be lowered with increasing number of periods, which suggests that having a single period of interruption can generate more risk than having more interruptions of smaller sizes.

The results from these two initial models encouraged the development of a third model, capable of tackling more general problems, since the portfolio assumed by the writer of the option, was expanded to include one more risky asset. This portfolio can be used to price options of a much wider class. Options such as spread and exchange options can be priced based on this portfolio. The methodology followed to develop this two-asset model is closely related with the methodology followed to develop the second model of the second chapter. The principle is the same, to find the premium for the option that makes the writer of such an option indifferent between holding either of the two portfolios. In this chapter, only the negative exponential utility function was used, since the model for a general utility function would yield a vast optimisation problem, as the holdings for two assets (hence a two-asset binomial lattice is required) for every node and every time-step of the lattice need to be optimised. The form of the solution is very similar to the one found in the previous chapter, apart from the fact that, the wealth process now includes two risky assets. The relation between the two assets can have a general form, but the only case investigated was the difference between their prices at the expiry of the option (spread option). The model was exhaustively tested in order to establish its convergence properties to single asset models (the case when the initial price of one of the assets is 0) for both cases of continuous and interrupted trading. Then the convergence properties to two asset continuous trading models was investigated. Once it was established that the model does indeed converge to these test cases, it was presented with cases of interrupted trading in two-asset cases. The results obtained, resembled the ones obtained for the one-asset model (with negative exponential utility) in that the price of the option increases as the period of interruption increases. The effect of the positioning of an interruption period showed similar behaviour to the effect observed in the single asset model, but again was found to be very small. Finally, the effect of splitting a fixed interruption period into smaller parts, while keeping the overall interruption time constant, was investigated. As expected from the previous analysis, the option prices were reduced for increasing number of interruption periods. Nevertheless, the reduction achieved by this strategy in the two-asset setting was much bigger than the reduction observed in the single asset setting.

The results obtained through simulating the three models, encourage the opinion that the pricing of trading interruptions using these models is a valid approach. Furthermore, these models provide an additional tool, in an area of option pricing inadequately researched. One of the main, interesting points arising throughout this work, is how the appetite of the holder of the portfolio towards risk affects the option prices. In the cases when there are no interruptions throughout the life of the option, this effect is minimal, therefore only becomes an issue when interruptions occur. This is definitely one of the points requiring further research, since different levels of risk aversion yield different option prices, which generates a mismatch and a deficit against complete market models. The area of market microstructure and games between market makers and investors could potentially lead to an acceptable measure which would provide a risk aversion factor suitable for the prevailing market conditions (market volatility). A good reference on these subjects is [32].

The models developed here have a wide range of applications, from simple financial option pricing to solving scheduling problems. In the fourth chapter, an example application is provided, where the two-asset model was used to show how the maintenance schedule for a petroleum refinery can be altered to reduce its effect on the value of the plant. This example was used, since the setting of a petroleum refinery is an ideal case for the model. The inherent optionality in the plant, which profits from the difference in price between the selling of its products and the purchasing of its feedstock, was exploited to price crack spread options, which are the commodity industry standard for pricing option on the difference between heating oil or gasoline and crude oil. The word exploited is used here in order to reflect the fact that the plant operations form a natural hedge if the plant manager were to sell a crack spread option. For simplicity, the example was considered over a single month of operation, where some assumptions were made to simplify the problem. The comparison between the basic option value of the plant (no maintenance period) and the subsequent cases with maintenance periods, was done using solely the model developed here in order to avoid any numerical differences arising from the discretisation of the problem. The maintenance period was initially assumed to last for four full days and was then split into two periods of two days and four periods of one day. The last case (4×1 week) yielded the lowest cost for the maintenance period, something which was expected from the analysis of the second and third chapters.

There are several parts of this thesis that can be extended in order to yield new research. Firstly, the most obvious extension, is to modify the models in order to price different types of options, such as American and Asian options. This can be of great significance, since these models can have a number of applications in the commodities sector, where most of options are of the American type, while Asian contracts are becoming increasingly popular. Secondly, the models can be extended in the direction of a multiperiod exercise specification (such as a Bermudan option) which can be used to solve scheduling problems in a multiperiod setting. Further to that, the area of other utility functions, such as the power utility or the logarithmic utility functions could be investigated. The insensitivity of the logarithmic utility to the risk aversion factor makes it a prime candidate for pricing options that need to be 'fairly' priced, but has to be approximated in some way if the problem faced in the second chapter (the large scale optimisation problem) is to be overcome. The power utility on the other hand, which is affected by both initial wealth and risk aversion factor, can be used in optimisation problems that require sensitivity to both parameters.

Chapter 6
Appendices

6.1 Appendix I - Convergence of single asset model under continuous trading

In this appendix, the numerical convergence of the single-asset model (IT1) to the Black-Scholes model ([5]), denoted as BS, and the CRR model ([9]), denoted as CRR, will be investigated. In order to provide some consistency, the following inputs were used for all the following examples, apart from where specified otherwise. The price of the options investigated in this section, will be the price of a call option on an underlying asset, whose current price is 50, continuous yield is 0% and volatility of returns is 20%. The strike price is set to 50, as well. The expiry of the option is one year, and the risk-free rate is 6%. The nominal number of time-steps is set to 200. For the single-asset model, the additional parameters required are set to 0.5 for gamma and 100 holding steps are used. Additionally, in order to assess the numerical convergence to the two continuous trading models, the trading start and stop times are set to the maturity. As an indication, the run time for an IT model calculation, with 200 time-steps and 100 holding steps requires around 1 second.

The first step is to investigate the convergence of the IT1 model, with increasing number of time-steps.

From Figure 6.39, it can be seen that the error between IT1 and BS is decreasing for bigger time-steps, while overall is very small. Additionally, the error between the IT1 and the CRR models is almost zero.

The next test is used to demonstrate the convergence properties of the model, for in-the-money and out-of-the-money options. For this example, the price of the underlying is varied from 25 to 75, while the strike is kept at a constant value of 50.

It can be seen, that although there is some relative error for deep out-of-the-money options between IT1 and BS, the error is eliminated for in-the-money options. Additionally, the difference between the IT1 and CRR models are negligible.

In the following figure (Figure 6.41), the effect of volatility on the convergence of the models is investigated.

It can be observed that some slight discrepancies can be observed for large volatility parameters. However, for levels of volatility up to 100%, all three models seem to be subject to very small discrepancies (less than 0.1%).

In the next investigation (Figure 6.42), the effect of the risk-free rate will be demonstrated..

Throughout the investigated range, the error between IT1 and BS models, for increasing interest rate, is small. The error between the IT1 and the CRR model is negligible.

The final investigation involves increasing the value of the underling's yield (Figure 6.43).

Although some valuation differences between the BS and the IT1 model can be observed, the differences between the IT1 and the CRR models are very small.

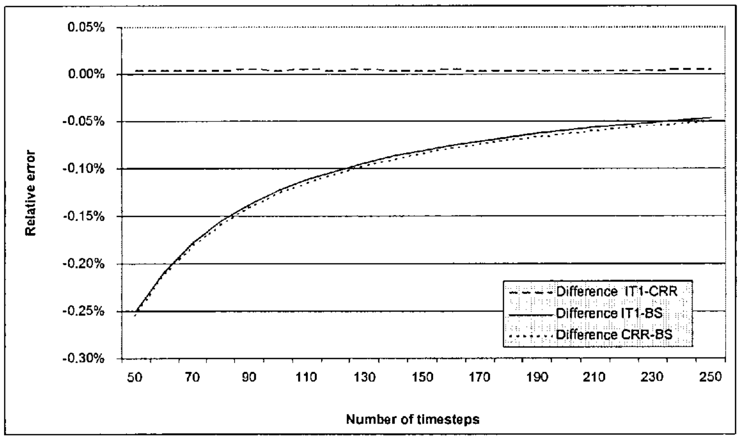


Figure 6.39: Relative error for increasing number of time-steps

6.2 Appendix II - Numerical stability of two asset model

In this appendix, the numerical convergence of the two asset interrupted trading model is demonstrated in relation to a two asset Black-Scholes model (BS2) and a two asset CRR model (CRR2). In order to demonstrate the properties of the convergence, five sets of results will be provided. The behaviour of the model for two at-the-money options will be provided, for changing levels of the two underlying assets and one example where the options move from in-the-money to out-of-the-money to in the money will be shown firstly. Then, the importance of the number of time-steps in relation to the holding steps will be shown. Finally, the effect of the volatility level on the convergence will be shown.

The basis for all the following examples will be the following set of parameters: the price of the two assets is set to 50, and the strike level is set to 0. The two continuous yields will be 0% and the two volatility parameters are set to 20%. The correlation between the two assets is assumed to be -50%, while all options mature in one year and the risk-free rate is set to 6%. The time-steps used for the calculations are 150, unless otherwise stated.

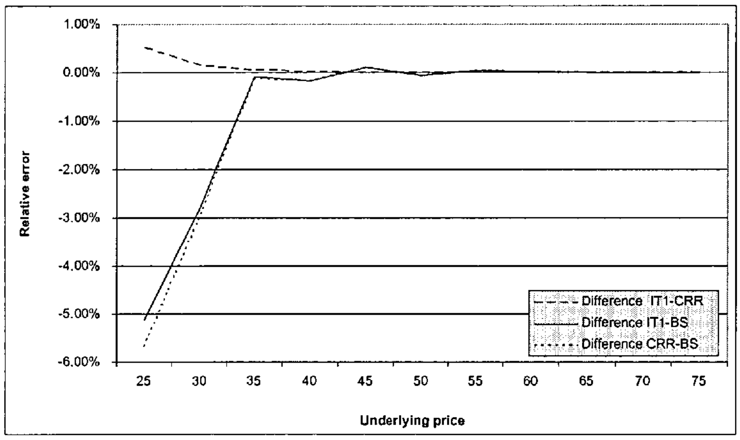


Figure 6.40: Relative error for increasing underlying price

In order to investigate the numerical convergence to the two continuous trading models (BS2 and CRR2) the trading stop and starting times are set equal to the maturity (one year). The holding steps, unless otherwise stated, are set to 50.

The first set of results concerns the impact of the first assets price fluctuation. By adjusting the strike price accordingly (so that it reflects the difference between the two asset prices), the options valued are at-the-money options.

The differences observed between the IT2 and CRR2 models in Figure 6.44 are between 0.03% and 0.1%. The behaviour of the relative differences between the IT2 and the BS2 models resembles that between the BS2 and the CRR2 models.

In Figure 6.45, a similar behaviour can be observed for an increasing price for the second underlying. Again the strike of the option is adjusted to reflect the difference in price between the two assets. All the relative differences fluctuate in the same ranges as with Figure 6.44.

The final comparison, in terms of price differentials, is the option price fluctuation

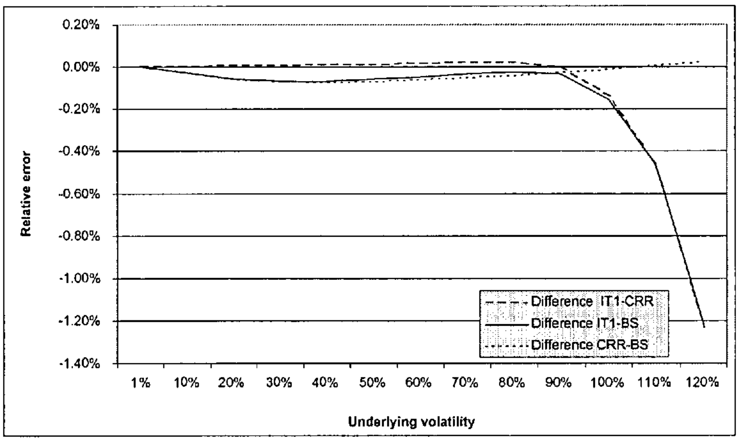


Figure 6.41: Relative error for increasing volatility

between in-the-money and out-of-the-money options. This is shown in Figure 6.46.

Although the relative differences observed for out-of-the-money options between IT2 and BS2 are higher than in the previous two cases, it can be seen that the behaviour of the IT2 and CRR2 are closely matched. It must be noted, that this behaviour is consistent with the case of the single asset options, where the larger differences were observed for the out-of-the-money options.

In the next example (Figure 6.47), the relationship between the number of time-steps and the number of holding steps is investigated. In this example, the number of holding steps is increased to 75 and the number of time-steps is decreased to 100.

From this example, it can easily be seen that the most significant factor in the valuation is the number of time-steps. The relative differences observed in Figure 6.47 are higher than those observed in Figure 6.44.

The final investigation in the convergence properties of the IT2 model is the behaviour of the model with increasing volatility. The effect of the volatility is the same for both

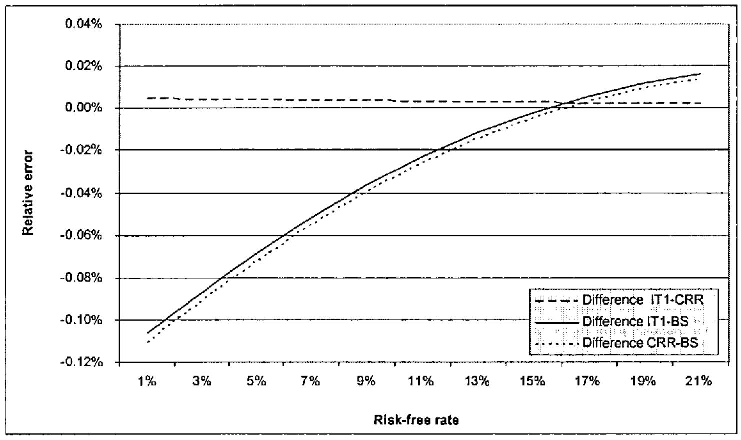


Figure 6.42: Relative error for increasing risk-free rate

assets, hence only the volatility for the first underlying will be demonstrated. In this example (Figure 6.48), the range of volatility that produces small differences is tighter than in the single asset case.

The range that the range is within a reasonable range is from 0% up to 90%, where the difference just exceeds 1%. As with the single asset model, the behaviour of the IT2 model is inconsistent with that of the CRR2 model, whose differences with the BS2 model seem to decrease.

6.3 Appendix III - Pricing an option on two assets with interrupts

We will now use this algorithm to price the same option when trading is interrupted during fibre step 1.

This represents a trading interruption between times 0.33 and 0.67. The prices for the market and the option premium at the initial fibre step (fibre 2) are the same so we start at fibre step 1 for the market premium. When trading is interrupted, the portfolio is left to diffuse while no rebalancing takes place. This is reflected in the expression used

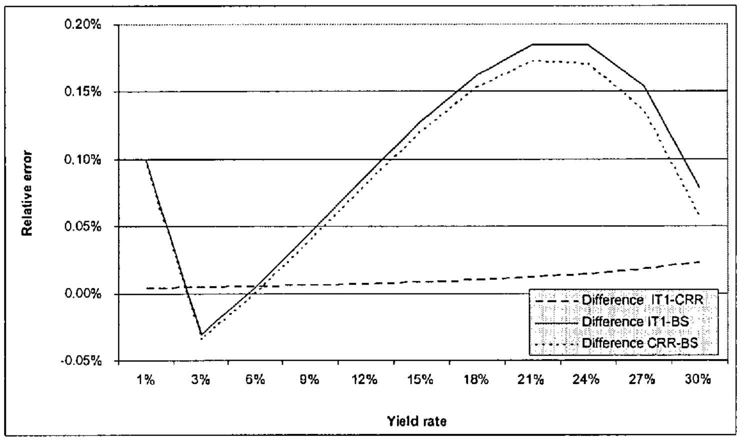


Figure 6.43: Relative error for increasing yield

to recalculate the elements of the $mOpt$ matrix (at point idxX1=0, idxX2=1, idxH1=3, idxH2=0)

$$mOpt(3,7) = pUU * mOpt(3,7) + pUD * mOpt(3,17) + pDU * mOpt(10,7) + pDD * mOpt(10,17) \Rightarrow$$

$$mOpt(3,7) = 0.4192*(7.6948E+23)+0.1515*(4.2420E+15)+0.0985*(7.6948E+23) + 0.3308 * (4.2420E + 15) \Rightarrow$$

$$mOpt(3,7) = 3.9836E + 23$$

Market premium calculation

Fibre step 1

For $idxX1 = 0, idxX2 = 0$ we get

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
115.1910	-1.00	8.8069.E+64	5.9578.E+52	4.0305.E+40	3.7120.E+28	1.0756.E+20	1.1736.E+12	1.2807.E+04
	-0.67	2.1899.E+55	1.4815.E+43	1.0022.E+31	9.2303.E+18	2.6747.E+10	2.9184.E+02	3.1847.E-06
	-0.33	5.4509.E+45	3.6875.E+33	2.4946.E+21	2.3026.E+09	6.7132.E+00	7.3247.E-08	7.9932.E-16
	0.00	1.6722.E+36	1.1312.E+24	7.6529.E+11	1.0000.E+00	5.2638.E-09	5.7439.E-17	6.2682.E-25
	0.33	1.8717.E+28	1.2662.E+16	8.5671.E+03	2.4945.E-08	2.0899.E-16	2.2806.E-24	2.4888.E-32
	0.67	1.0620.E+21	7.1845.E+08	4.8612.E-04	1.4331.E-15	1.2051.E-23	1.3151.E-31	1.4352.E-39
	1.00	6.1356.E+13	4.1507.E+01	2.8085.E-11	8.2799.E-23	6.9629.E-31	7.5983.E-39	8.2919.E-47

For $idxX1 = 0, idxX2 = 1$ we get

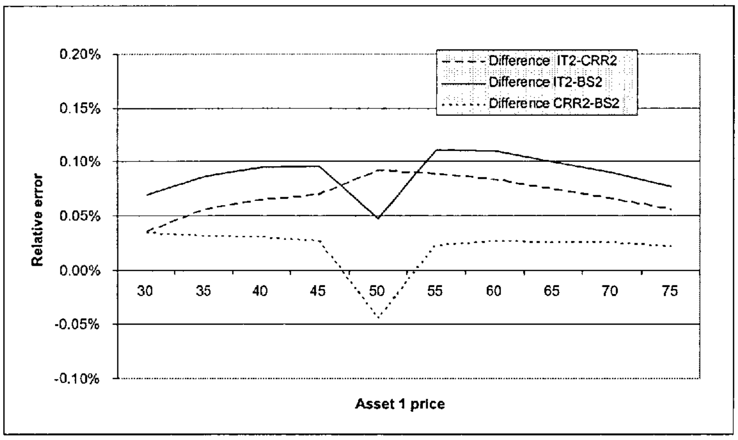


Figure 6.44: Relative error for increasing Asset 1 price

	X2	88.9744						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
115.1910	-1.00	2.0980.E+52	2.2895.E+44	2.5000.E+36	3.7120.E+28	6.1175.E+22	3.7607.E+17	2.3232.E+12
	-0.67	5.2168.E+42	5.6930.E+34	6.2166.E+26	9.2303.E+18	1.5212.E+13	9.3518.E+07	5.7770.E+02
	-0.33	1.2985.E+33	1.4170.E+25	1.5474.E+17	2.3026.E+09	3.8179.E+03	2.3472.E-02	1.4499.E-07
	0.00	3.9834.E+23	4.3470.E+15	4.7516.E+07	1.0000.E+00	2.9852.E-06	1.8406.E-11	1.1370.E-16
	0.33	4.4586.E+15	4.8657.E+07	5.3407.E-01	2.4945.E-08	1.1837.E-13	7.3081.E-19	4.5146.E-24
	0.67	2.5299.E+08	2.7609.E+00	3.0307.E-08	1.4331.E-15	6.8255.E-21	4.2142.E-26	2.6033.E-31
	1.00	1.4616.E+01	1.5951.E-07	1.7510.E-15	8.2799.E-23	3.9436.E-28	2.4348.E-33	1.5041.E-38

For $idxX1 = 1, idxX2 = 0$ we get

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	7.0204.E+57	4.7492.E+45	3.2128.E+33	2.9590.E+21	8.5743.E+12	9.3553.E+04	1.0209.E-03
	-0.67	4.0564.E+50	2.7441.E+38	1.8564.E+26	1.7100.E+14	4.9569.E+05	5.4084.E-03	5.9021.E-11
	-0.33	2.3526.E+43	1.5915.E+31	1.0767.E+19	1.0000.E+07	2.9652.E-02	3.2353.E-10	3.5306.E-18
	0.00	1.6722.E+36	1.1312.E+24	7.6529.E+11	1.0000.E+00	5.2638.E-09	5.7439.E-17	6.2682.E-25
	0.33	1.1940.E+30	8.0772.E+17	5.4652.E+05	1.5387.E-06	1.2757.E-14	1.3922.E-22	1.5192.E-30
	0.67	3.9179.E+24	2.6504.E+12	1.7933.E+00	5.2829.E-12	4.4416.E-20	4.8469.E-28	5.2893.E-36
	1.00	1.3726.E+19	9.2855.E+06	6.2828.E-06	1.8523.E-17	1.5576.E-25	1.6998.E-33	1.8549.E-41

For $idxX1 = 1, idxX2 = 1$ we get

	X2	88.9744						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	1.6724.E+45	1.8250.E+37	1.9929.E+29	2.9590.E+21	4.8766.E+15	2.9979.E+10	1.8519.E+05
	-0.67	9.6632.E+37	1.0545.E+30	1.1515.E+22	1.7100.E+14	2.8192.E+08	1.7331.E+03	1.0706.E-02
	-0.33	5.6044.E+30	6.1160.E+22	6.6786.E+14	1.0000.E+07	1.6862.E+01	1.0367.E-04	6.4044.E-10
	0.00	3.9834.E+23	4.3470.E+15	4.7516.E+07	1.0000.E+00	2.9852.E-06	1.8406.E-11	1.1370.E-16
	0.33	2.8443.E+17	3.1039.E+09	3.4062.E+01	1.5387.E-06	7.2256.E-12	4.4611.E-17	2.7558.E-22
	0.67	9.3332.E+11	1.0185.E+04	1.1181.E-04	5.2829.E-12	2.5156.E-17	1.5532.E-22	9.5946.E-28
	1.00	3.2698.E+06	3.5683.E-02	3.9170.E-10	1.8523.E-17	8.8219.E-23	5.4469.E-28	3.3648.E-33

These are the final values in the $mOpt$ matrix for this time-step, which now are going to be used in the final time-step for pricing the option.

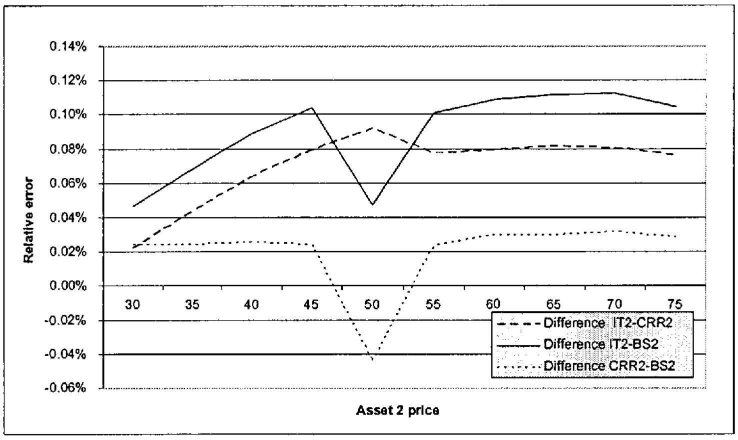


Figure 6.45: Relative error for increasing Asset 2 price

Fibre step 0

In the final step of the algorithm, trading is allowed, therefore the values in the $mOpt$ matrix will be updated by minimising the value through adjusting the portfolio and reading the value for zero holding.

For $idxX1 = 0$, $idxX2 = 0$ we get

	X2	110.0000						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	3.6918.E+64	7.1107.E+60	1.3696.E+57	4.8894.E+53	6.1206.E+55	1.0661.E+59	1.8750.E+62
minMOpt	-0.67	4.4532.E+62	8.5773.E+58	1.6521.E+55	5.8980.E+51	7.3835.E+53	1.2861.E+57	2.2619.E+60
1.0000	-0.33	5.3825.E+60	1.0367.E+57	1.9969.E+53	7.1605.E+49	9.0751.E+51	1.5809.E+55	2.7804.E+58
minH1	0.00	9.8816.E+58	1.9033.E+55	3.6662.E+51	2.6347.E+48	1.0821.E+51	1.8960.E+54	3.3348.E+57
0.00	0.33	6.9459.E+59	1.3378.E+56	2.5778.E+52	8.6240.E+49	8.7680.E+52	1.5404.E+56	2.7093.E+59
minH2	0.67	1.0377.E+62	1.9986.E+58	3.8511.E+54	1.4066.E+52	1.4699.E+55	2.5826.E+58	4.5422.E+61
0.00	1.00	1.7615.E+64	3.3928.E+60	6.5376.E+56	2.3915.E+54	2.5003.E+57	4.3930.E+60	7.7264.E+63

We now adjust the portfolio for the optimal holding $minH1$ and $minH2$, which gives the following values for $mOpt$

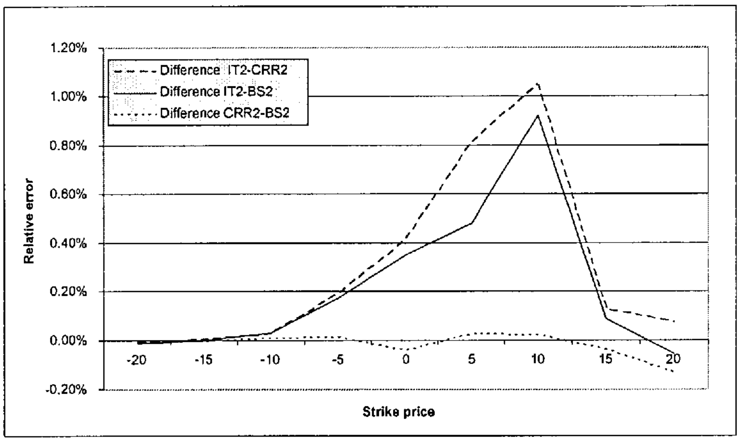


Figure 6.46: Relative error for increasing strike

X1	X2	110.0000						
	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	2.6347.E+48	9.2536.E+39	3.2501.E+31	1.1415.E+23	4.0094.E+14	1.4082.E+06	4.9460.E-03
	-0.67	5.4312.E+40	1.9076.E+32	6.6999.E+23	2.3532.E+15	8.2651.E+06	2.9029.E-02	1.0196.E-10
	-0.33	1.1196.E+33	3.9324.E+24	1.3812.E+16	4.8510.E+07	1.7038.E-01	5.9842.E-10	2.1018.E-18
	0.00	2.3080.E+25	8.1063.E+16	2.8472.E+08	1.0000.E+00	3.5123.E-09	1.2336.E-17	4.3327.E-26
	0.33	4.7578.E+17	1.6711.E+09	5.8693.E+00	2.0614.E-08	7.2403.E-17	2.5430.E-25	8.9317.E-34
	0.67	9.8079.E+09	3.4448.E+01	1.2099.E-07	4.2495.E-16	1.4926.E-24	5.2422.E-33	1.8412.E-41
	1.00	2.0218.E+02	7.1013.E-07	2.4942.E-15	8.7602.E-24	3.0768.E-32	1.0807.E-40	3.7956.E-49

We can read the value for the market premium for zero holding in both assets which gives $marketPrem = 1$.

The procedure must be repeated in order to calculate the option premium.

Option premium calculation

As with the market premium calculation, the values in the final fibre step are not affected by the trading interruption at fibre steps 2. Therefore, we will start by presenting the middle fibre (fibre 1).

Fibre step 1

For $idxX1 = 0$, $idxX2 = 0$ we get

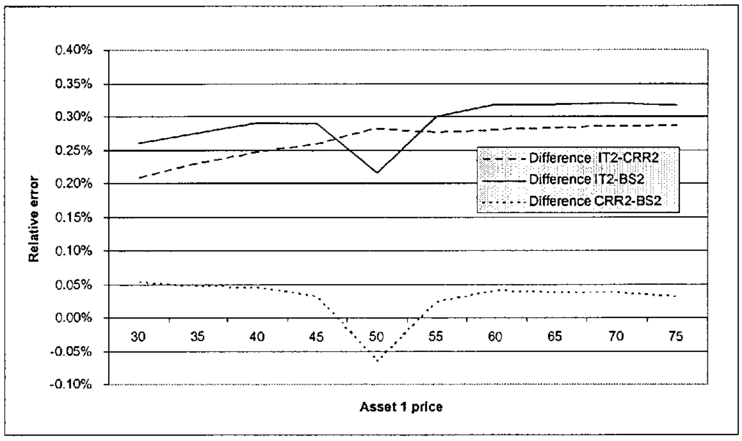


Figure 6.47: Relative error for increasing Asset 1 price

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
115.1910	-1.00	8.8069.E+64	5.9582.E+52	8.2196.E+40	4.5719.E+32	4.9889.E+24	5.4442.E+16	5.9411.E+08
	-0.67	2.1899.E+55	1.4816.E+43	2.0439.E+31	1.1368.E+23	1.2405.E+15	1.3538.E+07	1.4773.E-01
	-0.33	5.4509.E+45	3.6877.E+33	5.0848.E+21	2.8268.E+13	3.0847.E+05	3.3662.E-03	3.6735.E-11
	0.00	1.6722.E+36	1.1313.E+24	1.4094.E+12	7.0296.E+03	7.6706.E-05	8.3708.E-13	9.1348.E-21
	0.33	1.8717.E+28	1.2662.E+16	8.7273.E+03	1.7727.E-06	1.9281.E-14	2.1041.E-22	2.2962.E-30
	0.67	1.0620.E+21	7.1845.E+08	4.8616.E-04	1.8677.E-15	1.6794.E-23	1.8327.E-31	1.9999.E-39
	1.00	6.1356.E+13	4.1507.E+01	2.8085.E-11	8.2907.E-23	6.9747.E-31	7.6112.E-39	8.3059.E-47

For $idxX1 = 0, idxX2 = 1$ we get

	X2	88.9744						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
115.1910	-1.00	1.3250.E+57	2.1837.E+51	1.3424.E+46	8.2927.E+40	5.1228.E+35	3.1646.E+30	1.9549.E+25
	-0.67	3.2948.E+47	5.4299.E+41	3.3380.E+36	2.0620.E+31	1.2738.E+26	7.8689.E+20	4.8610.E+15
	-0.33	8.1927.E+37	1.3502.E+32	8.3003.E+26	5.1274.E+21	3.1675.E+16	1.9567.E+11	1.2087.E+06
	0.00	2.0372.E+28	3.3574.E+22	2.0639.E+17	1.2750.E+12	7.8761.E+06	4.8655.E+01	3.0056.E-04
	0.33	5.0701.E+18	8.3488.E+12	5.1324.E+07	3.1705.E+02	1.9585.E-03	1.2099.E-08	7.4740.E-14
	0.67	1.5157.E+09	2.0981.E+03	1.2881.E-02	7.9574.E-08	4.9157.E-13	3.0366.E-18	1.8759.E-23
	1.00	1.5111.E+01	1.7972.E-06	1.0103.E-11	6.2401.E-17	3.8548.E-22	2.3813.E-27	1.4710.E-32

For $idxX1 = 1, idxX2 = 0$ we get

	X2	135.9942						
X1	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	7.0204.E+57	4.7492.E+45	3.2128.E+33	2.9590.E+21	8.5743.E+12	9.3553.E+04	1.0209.E-03
	-0.67	4.0564.E+50	2.7441.E+38	1.8564.E+26	1.7100.E+14	4.9569.E+05	5.4084.E-03	5.9021.E-11
	-0.33	2.3526.E+43	1.5915.E+31	1.0767.E+19	1.0000.E+07	2.9652.E-02	3.2353.E-10	3.5306.E-18
	0.00	1.6722.E+36	1.1312.E+24	7.6529.E+11	1.0000.E+00	5.2638.E-09	5.7439.E-17	6.2682.E-25
	0.33	1.1940.E+30	8.0772.E+17	5.4652.E+05	1.5387.E-06	1.2757.E-14	1.3922.E-22	1.5192.E-30
	0.67	3.9179.E+24	2.6504.E+12	1.7933.E+00	5.2829.E-12	4.4416.E-20	4.8469.E-28	5.2893.E-36
	1.00	1.3726.E+19	9.2855.E+06	6.2828.E-06	1.8523.E-17	1.5576.E-25	1.6998.E-33	1.8549.E-41

For $idxX1 = 1, idxX2 = 1$ we get

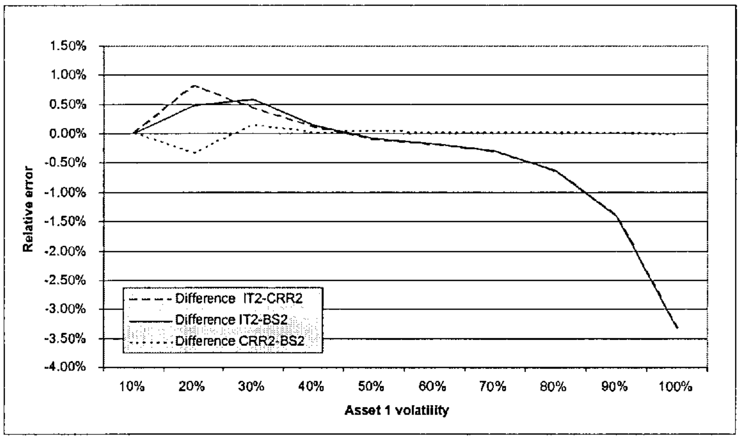


Figure 6.48: Relative error for increasing Asset 1 volatility

X1	X2	88.9744						
	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
86.8123	-1.00	1.6746.E+45	3.2059.E+37	8.5500.E+31	5.2694.E+26	3.2552.E+21	2.0109.E+16	1.2422.E+11
	-0.67	9.6761.E+37	1.8523.E+30	4.9400.E+24	3.0445.E+19	1.8807.E+14	1.1618.E+09	7.1771.E+03
	-0.33	5.6119.E+30	1.0726.E+23	2.8542.E+17	1.7591.E+12	1.0866.E+07	6.7127.E+01	4.1468.E-04
	0.00	3.9877.E+23	7.0103.E+15	1.6500.E+10	1.0164.E+05	6.2784.E-01	3.8785.E-06	2.3959.E-11
	0.33	2.8445.E+17	3.2579.E+09	9.8501.E+02	5.8760.E-03	3.6296.E-08	2.2422.E-13	1.3851.E-18
	0.67	9.3332.E+11	1.0194.E+04	1.6804.E-04	3.5269.E-10	2.1713.E-15	1.3413.E-20	8.2857.E-26
	1.00	3.2698.E+06	3.5684.E-02	3.9950.E-10	6.6650.E-17	3.8553.E-22	2.3813.E-27	1.4710.E-32

Fibre step 0

For $idxX1 = 0, idxX2 = 0$ we get

X1	X2	110.0000						
	vHold	-1.00	-0.67	-0.33	0.00	0.33	0.67	1.00
100.0000	-1.00	3.6918.E+64	7.2054.E+60	1.6488.E+62	2.8999.E+65	5.1005.E+68	8.9709.E+71	1.5778.E+75
minMOpt	-0.67	4.4532.E+62	8.6914.E+58	1.9889.E+60	3.4980.E+63	6.1524.E+66	1.0821.E+70	1.9032.E+73
7.8342E+06	-0.33	5.3825.E+60	1.0505.E+57	2.3991.E+58	4.2194.E+61	7.4212.E+64	1.3053.E+68	2.2957.E+71
minH1	0.00	9.8816.E+58	1.9199.E+55	2.8939.E+56	5.0896.E+59	8.9518.E+62	1.5745.E+66	2.7692.E+69
0.33	0.33	6.9459.E+59	1.3379.E+56	3.5167.E+54	6.1398.E+57	1.0799.E+61	1.8993.E+64	3.3406.E+67
minH2	0.67	1.0377.E+62	1.9986.E+58	3.8940.E+54	7.5477.E+55	1.3274.E+59	2.3347.E+62	4.1063.E+65
-0.33	1.00	1.7615.E+64	3.3928.E+60	6.5376.E+56	1.0023.E+55	1.5923.E+58	2.8002.E+61	4.9250.E+64

After adjusting the portfolio, $mOpt$ becomes

X1	X2	110.0000						
	vHold	-0.67	-0.33	0.00	0.33	0.67	1.00	1.33
100.0000	-1.33	3.5167.E+54	1.2352.E+46	4.3382.E+37	1.5237.E+29	5.3516.E+20	1.8796.E+12	6.6018.E+03
	-1.00	7.2495.E+46	2.5462.E+38	8.9430.E+29	3.1410.E+21	1.1032.E+13	3.8748.E+04	1.3609.E-04
	-0.67	1.4944.E+39	5.2489.E+30	1.8435.E+22	6.4750.E+13	2.2742.E+05	7.9876.E-04	2.8055.E-12
	-0.33	3.0807.E+31	1.0820.E+23	3.8003.E+14	1.3348.E+06	4.6881.E-03	1.6466.E-11	5.7833.E-20
	0.00	6.3507.E+23	2.2305.E+15	7.8342.E+06	2.7516.E-02	9.6643.E-11	3.3944.E-19	1.1922.E-27
	0.33	1.3091.E+16	4.5981.E+07	1.6150.E-01	5.6722.E-10	1.9922.E-18	6.9973.E-27	2.4576.E-35
	0.67	2.6987.E+08	9.4787.E-01	3.3292.E-09	1.1693.E-17	4.1069.E-26	1.4424.E-34	5.0663.E-43

The option premium is given by $mOpt(3,3) = 1.3348.E + 06$. The option value is

Timesteps	Holding Steps	Option Price
2	7	26.5658
25	25	16.2672
50	50	14.4295

Table 6.22: Option price convergence

given by

$$optValue = \exp(-r * T) * \ln\left(\frac{optionPrem}{marketPrem}\right) / \gamma \Rightarrow$$

$$optValue = 2 * \exp(-0.06) * \ln\left(\frac{1.3348.E{+}06}{1}\right) = 26.5658$$

As with the continuous trading example, this value is not exact as a very coarse discretisation was used. Unfortunately, we do not have a theoretical value to check this price against. The number of steps will be increased in both dimensions in order to get a more accurate price. The summary of the results can be seen in Table 6.22. Using 25 steps for the lattice and 25 steps to discretise the holdings in the assets, the result becomes 16.2672, while by using 50 time-steps and 50 holding steps, the value drops to 14.4295.

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