

PERTURBATIVE TECHNIQUES IN QUANTUM CHROMODYNAMICS

by

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ABSTRACT

This thesis is concerned with several problems relating to the applications of perturbation theory to Quantum Chromodynamics (QCD).

The first chapter is related to a well-known prediction of QCD: the prediction of scaling violations in the moments of structure functions of hadrons, measurable in inclusive deep inelastic scattering (DIS). Despite good quantitative agreement between theory and experiment, there has been much debate as to how well these results test QCD. In chapter one we consider the claim that the data can be reproduced by making only a few general assumptions, which can be justified independently of QCD. It is shown here that this claim is false, as additional, unjustifiable assumptions are necessary if definite predictions are to be made.

Chapter two contains an example of the use of the 'leading log approximation' (LLA) in QCD perturbation theory. The scaling violations of inclusive DIS have been derived in this approximation. Here a semi-inclusive deep inelastic process is considered, in which a large invariant mass lepton pair is observed in the final state. It is shown that the QCD corrections factorize, giving rise to scaling violations, as in inclusive DIS. The apparent contradiction of this result with the phenomena of colour oscillations is also discussed.

The final chapter considers the problem of infrared divergences in perturbative QCD. For many processes these divergences cancel when an indefinite number of low-energy gluons are allowed for in the final state. This is the Bloch-Nordsieck theorem. Here it is shown that this theorem fails for processes involving at least two coloured particles in the initial state. At the level of two soft gluons the non-leading divergences fail to cancel. The possible implications of this result for perturbative QCD are discussed.

P R E F A C E

The work in this thesis was carried out in the Department of Physics, Imperial College, between October 1978 and May 1981, under the supervision of Dr. I.G. Halliday. The material presented is the original work of the author, except where otherwise stated. The work of Chapter 3 was done in collaboration with Drs. S. Gendron, I.G. Halliday and C.T. Sachrajda.

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To my parents

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INTRODUCTION

There is currently great hope that Quantum Chromodynamics (QCD), [1], a non-abelian gauge theory based on the gauge group $SU(3)$, is the correct theory of strong interactions. The fundamental quanta of QCD, the quarks antiquarks and gluons, carry a 'colour' charge which generates their interactions. Hadrons are believed to be colour singlet bound states of these particles. The apparent absence of free coloured particles in the real world has led to the belief that colour is 'confined' within the hadron. However, it has not yet been established whether QCD predicts confinement.

Our current inability to deal with the confinement problem greatly restricts the predictive power of QCD. Most of the predictions made to date have used perturbative techniques which in some sense 'by-pass' this problem. In this thesis we will consider several problems related to this approach.

QCD perturbation theory builds on old parton model [2] concepts. This earlier model was motivated by the observed 'scaling' of hadron structure functions in deep inelastic scattering experiments at SLAC [3]. Here a hadron is probed with a highly virtual boson, of invariant mass-squared q^2 . 'Scaling', first predicted by Bjorken [4], is the statement that the hadron structure functions become independent of q^2 as $-q^2$ becomes increasingly large. This is naturally explained in the parton model by assuming the hadron is made up of point-like particles, the partons, and interacts by emitting a single collinear parton with a probability dependent only on the parton and hadron types, and their momentum ratio. In deep inelastic scattering (DIS) this 'active' parton subsequently interacts with the virtual boson, the remaining partons playing a passive role. In this model the structure functions are a measure of the probability distribution functions for the hadron to emit a parton, and hence independent of q^2 .

This parton model picture is assumed to apply to all hard scattering processes, these being characterized by some large invariant mass scale. In DIS this is $Q^2 = -q^2 \gg (\text{hadron mass})^2$. The parton model assumes that the confinement mechanism is independent of the particular hard scattering process. All effects of confining physics are contained in the parton distribution functions. Hence hard scattering cross-sections are given by convolutions of process-independent distribution functions with parton interaction cross-sections. These depend on the particular hard scattering process, but are calculable perturbatively. The parton distribution functions are not calculable in this model.

Clearly the parton model alone is not a complete theory, having nothing to say about the mechanism by which the hadron constituents, which have negligible interactions at high energies, can yet be confined within the hadron. This picture can be understood within the framework of QCD, in principle a complete theory of strong interactions. There, renormalization group [5] arguments show [6] that the effective QCD coupling is energy dependent, and decreases towards zero for increasingly high energies. This is the famous property of 'asymptotic freedom', possessed only by non-abelian gauge theories. At low energy scales it is plausible, but unproven, that the effective coupling becomes strong, which could explain confinement.

Furthermore, the property of asymptotic freedom, combined with the method of the α operator product expansion [7], led to quantitative predictions for DIS [8]. These showed that the parton model picture is essentially preserved in QCD, the only new feature being a slow Q^2 dependence now appearing in the hadron structure functions. Subsequent analyses of structure function measurements [9 - 12] show good quantitative agreement with the QCD predictions. This was initially hailed as a major success for QCD, but there has since been much debate as to how significant these results are [13].

Thus, the factorization of the long distance confining physics from the short distance physics, the essence of the parton model assumption, was seen to be preserved in QCD, in this process. It

was then natural to ask whether this result was valid for a wider class of processes. The original renormalization group (RG) and operator product expansion (OPE) methods of [8] are not very useful for investigating this problem, being only directly applicable to inclusive DIS [8] and e^+e^- annihilation [14]. More recently alternative approaches have been developed which have much wider applicability.

The first of these was the 'leading log approximation' (LLA) to perturbative QCD. This diagrammatic approach was originally introduced by Gribov and Lipatov [15], who studied DIS in pseudoscalar and neutral vector meson field theories. Their results were subsequently reproduced by Christ, Hasslacher and Mueller [16], using OPE and RG techniques. The development of the LLA in QCD was stimulated by the works of Politzer [17] and Sachrajda [18], who showed that factorization was valid in low orders of perturbation theory in the Drell-Yan [19] process. The complete equivalence of the LLA to the RG and OPE approach to inclusive DIS in QCD has now been established in a number of works [20 - 22]; more general hard scattering processes have been treated [20, 21, 23] and the method extended to include non-leading corrections [20, 23]. The results of these works show that QCD preserves the parton model picture in many, and possibly all processes which admit a parton model interpretation [20, 23]. Furthermore, the Q^2 -dependence of the parton distribution functions is predicted to be the same in any such process. Thus, the leading QCD corrections to the parton model, the mass singularities, factorize and are 'universal'. A simple probabilistic interpretation of this picture has emerged [24].

The belief that factorization holds for many processes has led to the wide applicability of QCD perturbation theory [25], since, because of factorization, one need only apply it to the short distance, high energy region where the effective coupling is small. The long distance effects, which are not yet fully understood, are process independent, so it suffices to measure them experimentally in one process.

The most general proof of factorization in QCD [23] relies on the assumption that all infrared divergences of perturbation theory are cancelled via the Bloch-Nordsieck [26] mechanism. This assumption has since been proven false [27 - 30]. Infrared divergences arise in any field theory with massless quanta, such as Quantum Electrodynamics (QED). The Bloch-Nordsieck theorem [26] states that, if one considers only inclusive cross-sections in which an indefinite number of low energy, massless quanta are allowed for in the final state, the divergences cancel. This has long been known to be true in QED [31]. The philosophy behind this approach is that any detector has finite energy resolution, so one can never detect all the 'soft' quanta, and only such inclusive cross-sections are experimentally meaningful. Of course, as far as QCD is concerned one never detects quarks and gluons, only hadrons, but perturbation theory takes no account of this, so it is still reasonable to consider only such inclusive processes.

In chapter one we will consider a claim [32] that the results of some of the famous scaling violation experiments [9, 10] described earlier, do not in fact test QCD at all, but only some weaker, general assumptions which can be justified without reference to QCD. We show [33] that these assumptions alone do not explain the data.

In chapter two we use the LLA to show how the factorization we have described emerges in QCD for a specific semi-inclusive deep inelastic process, one for which the renormalization group and operation product expansion approach of [8] is not applicable. We consider the case where the active quark in DIS radiates a timelike photon, which subsequently decays to a final state muon pair. An apparent contradiction between the result of factorization for this process, and some work by Lipkin on colour oscillations [34], will be discussed. The process considered here is in fact also covered by the more general works of refs. [20, 23], but this work was done independently.

Finally, in chapter three it is shown [28] that the Bloch-Nordsieck theorem fails in perturbative QCD. For a process involving two quarks in the initial state, and at the level of two soft gluons, non-leading divergences fail to cancel. This result was originally established by Doria et al. [27]. We agree with their final results, but not some of their intermediate expressions.

CHAPTER ONE

MOMENTS OF STRUCTURE FUNCTIONS AND

EXPERIMENTAL TESTS OF QCD

1.1 INTRODUCTION

As described in the introduction to this thesis, an important result of QCD has been the prediction [8] of the high energy behaviour of hadron structure functions. Recent results of the BEBC [9] and CDHS [10] collaborations, who measure the non-singlet structure function $x F_3(x, Q^2)$ for neutrino scattering on nucleons, show good quantitative agreement with the QCD predictions. This was initially believed to be a major success for the theory. However, in order to establish how strong the evidence is for QCD here, we must consider whether we really need QCD to explain these results.

Work has been done in this area by Glück and Reya [35], who consider whether various non-asymptotically free field theories are compatible with the data. They find that measurements on $x F_3(x, Q^2)$ are insensitive to the underlying field theory. However, this result lies on the assumption that, in the non-asymptotically free theories, in which the effective coupling does not tend to zero at high energies, it instead tends to a small, fixed value. This smallness of the effective coupling is necessary to produce approximate scaling in such theories.

We shall here be considering a slightly different approach due to Harari [32], who claims that only a few very general assumptions are necessary to understand the $x F_3$ data. These assumptions all have some theoretical justification having little or no connection with QCD. They lead to bounds within which the data should lie. In fact the bounds encompass all the data, and the QCD predictions, but the data points are spread rather randomly within these bounds, showing no clear preference for the QCD predictions. Harari therefore claims that these results provide evidence only for the general assumptions, and merely a consistency check for QCD.

We have shown [33], however, that Harari's general assumptions do not, in fact, restrict the data to lie within his quoted bounds. The point is that only by making an additional, unjustifiable assumption can bounds be derived. This assumption can be changed at will, provided it remains consistent with the justifiable assumptions. Such a change may lead to different bounds. We will

here construct an example where this is the case. Furthermore, our new bounds do not overlap with Harari's bounds, and miss the data and QCD predictions entirely. Therefore, Harari's general assumptions can lead to 'predictions' inconsistent with the xF_3 data, so it is not true that the data test only these general assumptions.

It should be noted, however, that there are many other factors to be considered before one can establish how well these results test QCD. The possible importance of corrections to the original predictions, due to higher order terms in the QCD perturbation series, and non-leading twist effects, has to be allowed for [36]. These problems have led to much debate as to the importance of these tests [13].

In section two we outline the background to this famous test of QCD. Deep inelastic scattering is described and the QCD predictions quoted. In section three we give Harari's assumptions and outline the derivation of his bounds. Section four contains our counter example. In section five we discuss the importance of Harari's claim. For completeness we outline the sources of Harari's general assumptions in a separate appendix.

1.2 DEEP INELASTIC SCATTERING AND QCD

In deep inelastic scattering we study the inclusive, lepton-hadron scattering process shown in fig. 1.1, in the Bjorken limit [4], defined by

$$Q^2 = -q^2 \rightarrow \infty, \quad \text{with } x = \frac{Q^2}{2q \cdot p} \text{ fixed.} \quad (1.1)$$

Here p and q are respectively the hadron and virtual boson

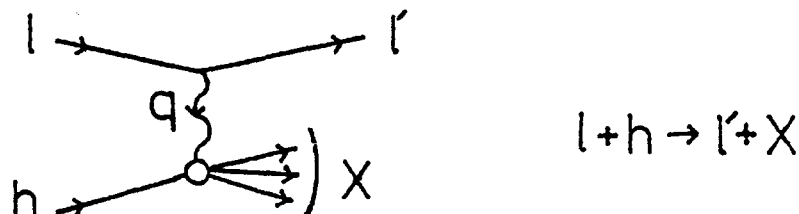


Fig. 1.1 The deep inelastic scattering process. Here l, l' denote leptons, h a hadron, the wavy line a virtual boson, and X the unobserved final state particles.

four momenta. The cross-section for this process is proportional to a hadron tensor $W_{\mu\nu}$, given by

$$W_{\mu\nu} = \frac{1}{\pi} \text{Im } T_{\mu\nu} \quad (1.2)$$

$$T_{\mu\nu} = i \int d^4y e^{iq \cdot y} \langle p | T (J_\mu(y) J_\nu(0)) | p \rangle$$

where J_μ is the hadronic electromagnetic or weak current, depending on the nature of the incident lepton. By Lorentz invariance $W_{\mu\nu}$ has the decomposition

$$W_{\mu\nu} = - \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) W_1(x, Q^2) + \left(p_\mu - \frac{p \cdot q}{q^2} q_\mu \right) \left(p_\nu - \frac{p \cdot q}{q^2} q_\nu \right) W_2(x, Q^2) + i \epsilon_{\mu\nu\alpha\beta} p^\alpha q^\beta \frac{W_3(x, Q^2)}{2M} \quad (1.3)$$

Here M is the hadron mass, and the parity-violating W_3 term occurs only for neutrino and antineutrino scattering, with a minus and plus sign respectively. The structure functions are defined by

$$M W_1(x, Q^2) = F_1(x, Q^2),$$

$$\nu W_{2,3}(x, Q^2) = F_{2,3}(x, Q^2), \quad \text{with} \quad \nu = 2p \cdot q \quad (1.4)$$

'Scaling' is the statement that [4]

$$F_i(x, Q^2) \rightarrow F_i(x) \quad \text{as} \quad Q^2 \rightarrow \infty \quad (1.5)$$

Experimentally scaling is violated, the structure functions showing a weak Q^2 dependence. This is understood quantitatively in QCD [8]. There the operator product expansion [7] applied to

$T_{\mu\nu}$ decomposed as in (1.3), gives

$$T_i(x, Q^2) = \sum_{n=0}^{\infty} \frac{1}{x^n} A_n f_n^i(Q^2) + \text{terms non-leading in Bjorken limit} \quad (1.6)$$

Here the A_n 's are matrix elements of composite operators, not calculable perturbatively. The $f_n^i(Q^2)$ functions are Fourier transforms of the coefficient functions in the operator product expansion. They satisfy a renormalization group equation [5] which gives their Q^2 dependence. For a flavour non-singlet structure function, such as the neutrino function T_3 , this takes a particularly simple form [8],

$$f_n^i(Q^2) = f_n^i(Q_0^2) \ln(Q^2/Q_0^2)^{-a_n} \quad (1.7)$$

where the a_n 's, known as anomalous dimensions, are calculable in QCD perturbation theory [8]. The sum on n in (1.6) is unravelled by taking moments [16, 37]:

$$\begin{aligned} A_n f_n^i(Q^2) &= \frac{1}{2\pi i} \oint dx x^{n-1} T_i(x, Q^2) \\ &= 2 \int_0^1 dx x^{n-1} F_i(x, Q^2) \\ &= 2M_n(Q^2) \end{aligned} \quad (1.8)$$

Thus $M_n(Q^2)$, the n th moment, has a calculable Q^2 dependence [8]:

$$M_n(Q^2) = M_n(Q_0^2) \ln(Q^2/Q_0^2)^{-a_n} \quad (1.9)$$

This prediction is conveniently tested by using the experimental results to plot a graph of $\ln M_n$ versus $\ln M_{n,1}$, as a function of Q^2 . The slope is predicted to be equal to the calculable ratio of anomalous dimensions. These slopes have recently been measured $^{\nu N}$ for the non-singlet, neutrino-nucleon structure function $x F_3^{\nu N}(x, Q^2)$, which averages over proton and neutron targets.

$$x F_3^{\nu N}(x, Q^2) = \frac{1}{2} (x F_3^{\nu p}(x, Q^2) + x F_3^{\nu n}(x, Q^2)) \quad (1.10)$$

The measured slopes are in good agreement with the QCD predictions. For the remainder of this chapter, we shall be attempting to predict the values of these slopes from very general assumptions only.

Note an alternative definition of the moment has been introduced by Nachtmann [38], to take into account finite hadron mass effects. Eq. (1.8) is known as the Cornwall-Norton moment.

1.3 GENERAL ASSUMPTIONS : HARARI'S METHOD

Harari's approach [32] is to make four very general assumptions concerning the form of the non-singlet structure function $x F_3(x, Q^2)$. He then 'guesses' a form for the structure function consistent with these assumptions. It is then straightforward to evaluate the moments and derive bounds for the slope of the graph of $\ln M_n$ versus $\ln M_{n,1}$.

The assumptions are:

$$\text{i) } x F_3(x, Q_0^2) \sim x^{\frac{1}{2}} \quad \text{as } x \rightarrow 0 ; \quad (1.11)$$

$$\text{ii) } x F_3(x, Q_0^2) \sim (1-x)^3 \quad \text{as } x \rightarrow 1 ; \quad (1.12)$$

$$\text{iii) the Gross-Llewellyn-Smith (GLS) sum rule [39] holds for all } Q^2, \text{ that is } \int_0^1 dx F_3(x, Q^2) = 3 \quad (1.13)$$

and

(iv) the structure function shifts towards small x as Q^2 increases.

In (i) and (ii), Q_0^2 is a fixed value of Q^2 which is arbitrary apart from the requirement that it lie within the presently accessible experimental region.

These assumptions have little or nothing to do with QCD. Their justifications are outlined briefly in an appendix to this chapter.

With these assumptions in mind Harari proposes the following form for the structure function:

$$xF_3(x, Q^2) = Cx^{\frac{1}{2}f}(1-x)^{3g} \quad (1.14)$$

where f and g are slowly varying functions of Q^2 satisfying

$$\begin{aligned} f(Q_0^2) &= g(Q_0^2) = 1 \\ f'(Q_0^2) \quad g'(Q_0^2) &\geq 0 \end{aligned} \quad (1.15)$$

but are otherwise arbitrary. The prime here denotes the Q^2 derivative. The conditions (1.15) are imposed to make the structure function move to the left with Q^2 . The normalization constant $C(Q^2)$ is fixed by (1.13), the GLS sum rule.

The above expressions easily lead to an expression for the moments:

$$\begin{aligned} M_n(Q^2) &= \int_0^1 dx \, x^{n-1} (xF_3(x, Q^2)) \\ &= \frac{3B(\frac{1}{2f} + n - 1, 3g + 1)}{B(\frac{1}{2f}, 3g + 1)} \end{aligned} \quad (1.16)$$

$$\text{where} \quad B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \quad (1.17)$$

is the Euler beta function, and $\Gamma(x)$ the gamma function.

The slope of the graph of $\ln M_n$, versus $\ln M_n$ is given by

$$\frac{P_{n'}}{P_n} = \frac{\frac{d}{dQ^2} \ln M_{n'}(Q^2)}{\frac{d}{dQ^2} \ln M_n(Q^2)}$$

$$= \frac{\left[\psi(n' - \frac{1}{2}) - \psi(\frac{1}{2}) \right] + (6g'_{f'} - 1) \left[\psi(n' + 3\frac{1}{2}) - \psi(4\frac{1}{2}) \right]}{\left[\psi(n - \frac{1}{2}) - \psi(\frac{1}{2}) \right] + (6g'_{f'} - 1) \left[\psi(n + 3\frac{1}{2}) - \psi(4\frac{1}{2}) \right]} \quad (1.18)$$

using (1.16), where

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x). \quad (1.19)$$

Bounds for the slopes at $Q^2 = Q_0^2$ can now be derived using (1.15) which implies $0 \leq g'_{f'} \leq \infty$. The resulting bounds are listed in table 1, and displayed graphically in Fig. 1.2, in comparison with the published results of the CDHS [10] and BEBC [9] experimental groups, the QCD predictions, and new bounds derived in the following section. We see that Harari's bounds take in all the data and the QCD predictions. Furthermore, though the data is consistent with QCD it is spread rather uniformly within the bounds, showing no clear preference for the QCD values. It therefore appears that the results test only Harari's general assumptions (i) to (iv), and no more. We will show in the next section that this is not the case.

TABLE 1

Two sets of lower and upper bounds for p_n/p_n , one resulting from Harari's work [32], and a new set from our work, are here listed in comparison with the QCD predictions [8], and the published results of the BEBC [9] and CDHS [10] groups. The CDHS analysis evaluates both Nachtmann (N) and Cornwall-Norton (C-N) moments. These numbers are displayed graphically in figure 1.2.

	<u>Harari's bounds</u>		<u>New bounds</u>		<u>QCD</u>
	<u>Lower</u>	<u>Upper</u>	<u>Lower</u>	<u>Upper</u>	<u>Prediction</u>
P5/P3	1.18	1.71	1.76	2.01	1.46
P6/P4	1.10	1.45	1.50	1.63	1.29
P6/P3	1.22	2.00	2.10	2.47	1.62
P7/P3	1.26	2.26	2.41	2.88	1.76

Experimental Data

	<u>CDHS (N)</u>	<u>CDHS (C-N)</u>	<u>BEBC (N)</u>
P5/P3	1.34 ± 0.12	1.58 ± 0.12	1.50 ± 0.08
P6/P4	1.18 ± 0.09	1.34 ± 0.07	1.29 ± 0.06
P6/P3	1.38 ± 0.15	1.76 ± 0.15	
P7/P3			1.84 ± 0.20

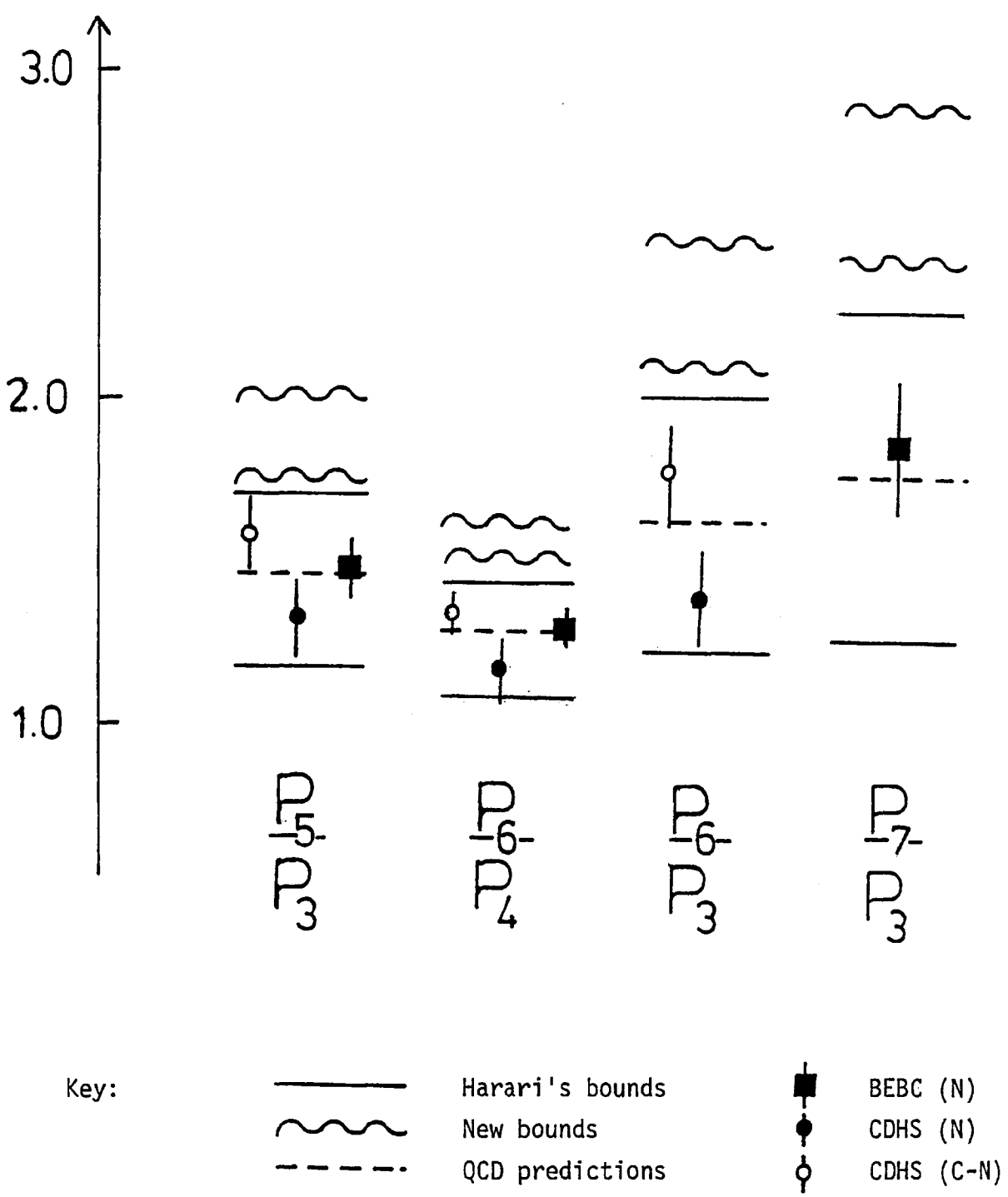


Fig. 1.2 A graphical display of the information listed in Table 1.

1.4 A COUNTER EXAMPLE.

We now show that it is possible to retain Harari's assumptions, (i) to (iv), yet derive bounds for the slopes of the log - log plot which do not take in the data. We do this by choosing a form for the structure function which is different from Harari's (1.14) but still satisfies assumptions (i) to (iv).

We choose

$$xF_3(x, Q^2) = C(Q^2) x^{-3/2} (x^2 + \sin^2 \alpha) (1-x)^\beta \quad (1.20)$$

where now α and β are arbitrary, slowly varying functions of Q^2 which obey

$$\alpha(Q_0^2) = \pi, \quad \beta(Q_0^2) = 3,$$

$$\alpha'(Q_0^2) \geq 0, \quad \beta'(Q_0^2) \geq 0. \quad (1.21)$$

This and Harari's structure function (1.14) are compared with the BEBC data for $Q^2 = Q_0^2 = 20 \text{ GeV}^2$ in figure (1.3). Harari's is an excellent fit to the data. Ours is a slightly poorer fit, but the important point is that it is consistent with Harari's general assumptions. From here on it is straightforward to derive

bounds for the slopes in the manner described in the previous section.

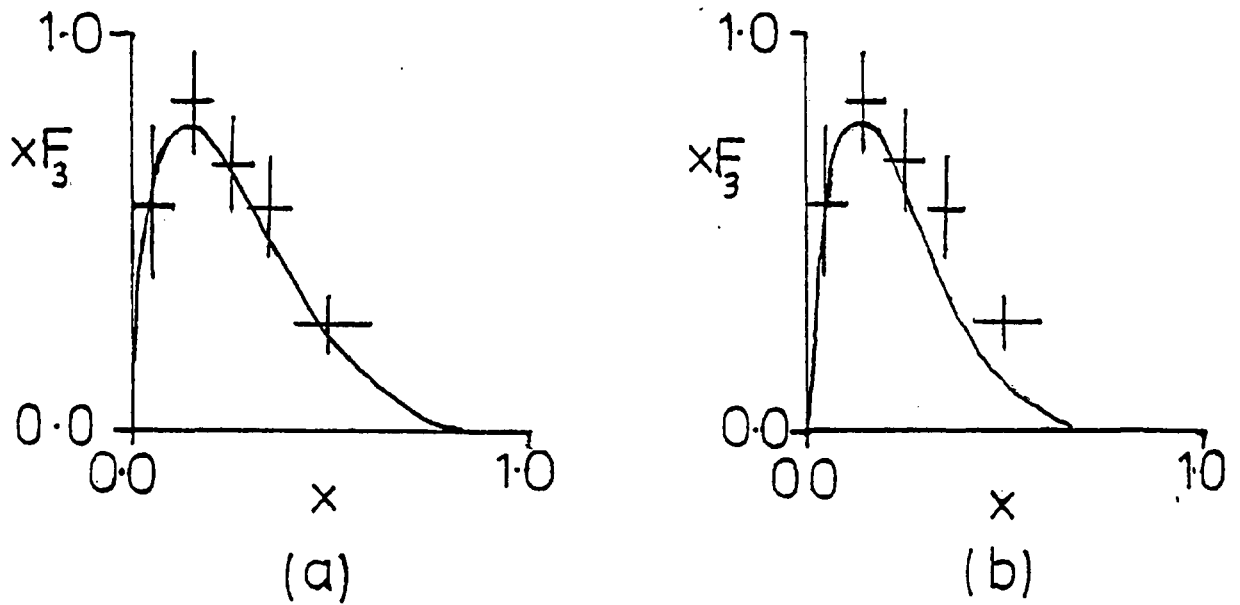


Fig. 1.3 A comparison of Harari's structure function (a), and ours (b), with the BEBC [9] measurements for $Q^2 = Q_0^2 = 20 \text{ GeV}^2$

The only minor difference is that the integrals cannot be done directly, so we write $\sin^2(\alpha x) = \frac{1}{2} (1 - \cos(2\alpha x))$ and expand this as a power series in $(2\alpha x)^2$. The resulting n th moment is given by

$$M_n(Q^2) = \frac{3B(n-\frac{1}{2}, \nu+1) \times (1 + \frac{1}{2} \sum_{m=1}^{\infty} C_m(n, Q^2))}{B(\frac{1}{2}, \nu+1) \times (1 + \frac{1}{2} \sum_{m=1}^{\infty} C_m(1, Q^2))} \quad (1.22)$$

where

$$C_m(n, Q^2) = \frac{(-)^{m+1} (2\alpha)^{2m} B(n+2m-5/2, \nu+1)}{(2m)! B(n-\frac{1}{2}, \nu+1)} \quad (1.23)$$

and $B(x,y)$ is again the Euler beta function (see (1.17)). The resulting expression for the bounds of the slope of the log - log plot, p_n'/p_n contains infinite series, but the sums converge quite rapidly, and are not difficult to compute.

We thus arrive at the new bounds given in Table 1 and Fig. 1.2. There we see that these new bounds are completely incompatible with Harari's. The region they allow misses both the data and the QCD predictions. We therefore conclude that our chosen form for the structure function (1.20, 21) is wrong. It is, however, consistent with the general assumptions (i) to (iv). It is therefore clear that these assumptions alone do not restrict the slopes to lie within the bounds derived by Harari. Thus, the conclusion that the experimental results of [9, 10] provide evidence only for these general assumptions is false.

1.5 COMMENTS

We have concluded that we need more than just the general assumptions of Harari [32] to explain the data of [9, 10]. We cannot here make any stronger conclusions concerning the evidence these results provide for QCD, because of the ambiguities in the QCD predictions [13] mentioned in the introduction.

In addition, we note that what Harari's work does show is that there are possible forms of the structure functions which have apparently no connection with QCD, yet reproduce the QCD predictions for the Q^2 dependence of the moments. Indeed Harari's proposed structure function (1.14, 15) is similar in form to the Buras-Gaemers parametrization [40], already known to reproduce the QCD predictions. So perhaps Harari's results are not too surprising.

However, the importance of this ambiguity depends on the questions one asks. If one believes that the interactions of the nucleon are governed by an underlying field theory, then the only important question is - what is that field theory? Then

Harari's example, which does not attempt to answer this question, is not relevant, though perhaps the work of Glück and Reya [35] is.

Alternatively, if one is not convinced that strong interactions should be described by a field theory, then examples such as Harari's must be taken seriously. Indeed it is not difficult to write down arbitrary forms for the structure functions which reproduce the measured Q^2 dependence of the moments. In this sense one might take the view that the experimental results can provide no more than a consistency check on QCD, regardless of higher order corrections etc. However, one would not in general have any reason to believe such arbitrary structure functions to be physically realistic. The importance of Harari's example is that he claims his predictions to be theoretically justified. We have shown that this is not the case.

APPENDIX I

Here we outline the theoretical basis for Harari's general assumptions (i) to (iv).

$$(i) \ xF_3(x, Q_0^2) \sim x^{\frac{1}{2}} \text{ near } x = 0.$$

This results from Regge theory arguments [41] . $x \rightarrow 0$ for fixed Q^2 is equivalent to $s = (p + q)^2 \rightarrow \infty$ for fixed $t = Q^2$ (see Fig. 1.1), so this is the Regge limit of deep inelastic scattering. Now the optical theorem relates the structure functions to the total cross-section for virtual boson scattering on hadrons. This in turn is given by Regge theory in the Regge limit. One obtains [41]

$$F_3(x) \sim x^{-\alpha(0)}$$

where $\alpha(0)$ is given by the leading Regge trajectory. The leading trajectory is that of the pomeron, with $\alpha(0) = 1$. This is assigned to the 'sea' partons since they are known to dominate at small x . These partons make no contribution to flavour non-singlet structure functions however, so for these one takes the next to leading trajectory, with $\alpha(0) = \frac{1}{2}$, given by experiment. This leads to assumption (i), which is also supported by empirical evidence.

$$(ii) \ xF_3(x, Q_0^2) \sim (1 - x)^3 \text{ near } x = 1.$$

This is given by several arguments: the Drell-Yan [42] and West [43] connection, Bloom-Gilman duality [44] and dimensional counting [45] . The essential point is that $x \rightarrow 1$ for fixed $s = (p + q)^2 = Q^2 (\frac{1}{x} - 1)$ is the exclusive limit of deep inelastic scattering, since for $x = 1$ the hadron is scattered elastically. Demanding a smooth inclusive - exclusive connection then relates the structure functions to the square of the hadron form factor, $F^2(Q^2)$, measured in the exclusive process. This is then given,

for example, by the dimensional counting rules [45] , which assume scale invariance of the confining mechanism. Using $Q^2 \sim (1 - x)^{-1}$ in the exclusive limit then gives assumption (ii).

$$(iii) \quad \int_0^1 dx F_3^{\nu N}(x) Q^2 = 3 \quad .$$

This is the Gross-Llewellyn-Smith sum rule [39] and is a parton model result. There, $F_3^{\nu N}(x)$ is given by [2]

$$F_3^{\nu N}(x) = (q_i(x) - \bar{q}_i(x)) \quad (1.24)$$

where $q_i(x)$ ($\bar{q}_i(x)$) denotes the probability distribution functions for a quark (antiquark) of flavour 'i' and fractional momentum x within the proton. Thus the integral over all x 'counts' baryon number, and so gives (iii) .

(iv) The structure function shifts towards $x = 0$ with increasing Q^2 . This is expected from general parton model/field theoretic ideas. As Q^2 increases, it is reasonable to expect an increase in the brehmsstrahlung of low energy partons by the hadron constituents, in any field theory. The parton model interpretation of the structure functions, as in (1.24) for example, then leads one to expect them to increase for small x , and decrease for large x , with increasing Q^2 .

CHAPTER TWO

TIMELIKE PHOTON BREHMSSTRAHLUNG IN DEEP

INELASTIC SCATTERING, AND LEADING LOG QCD

2.1 INTRODUCTION

In the LLA to QCD [20 - 23] one considers the QCD corrections to a parton model cross-section in perturbation theory. The convergence of the perturbation expansion is in general spoilt by the appearance of large logarithmic factors, the 'mass singularities', coming from the low momentum region $|k^2| \ll Q^2$. In processes characterized by just one large invariant mass scale, Q^2 , the dominant terms at n th order in the perturbation series typically have an order of magnitude $(\alpha_s(Q^2) \ln Q^2)^n$, where $\alpha_s(Q^2)$ is the effective QCD coupling, evaluated at the point Q^2 . Since $\alpha_s(Q^2) \sim \frac{1}{\ln Q^2}$ as $Q^2 \rightarrow \infty$ [6],

the smallness of the running coupling is balanced by the large logarithm, and such terms are all of order unity, for all n . In the LLA [15], we sum these terms to all orders of perturbation theory, neglecting non-leading logarithms of order $(\alpha_s(Q^2))^n (\ln Q^2)^{n-1}$ or smaller. This summation is greatly simplified in a suitable non-covariant 'axial' gauge, in which all the leading diagrams have the same simple 'ladder' structure. These gauges were originally used by Lipatov [46].

If, on summing the leading mass singularities, we obtain the old parton model cross-section multiplied by a new Q^2 dependent factor, we have factorization. This correction factor can then be 'absorbed' into the non-calculable parton distribution functions, giving them a scale-breaking Q^2 -dependence. If this QCD correction factor is process independent, for all processes admitting a parton model interpretation, then one expects to observe the same Q^2 dependence in the parton distribution functions in all such processes, and we have a consistent parton model picture. In this manner, the parton model picture is preserved in leading log QCD. Furthermore, once these mass singularities have been removed in this way, one can reasonably hope that the remaining perturbation series will be convergent.

We will here be proving the factorization of mass singularities in leading log QCD for a specific semi-inclusive deep inelastic

process, one in which a muon pair resulting from timelike photon decay is observed in the final state, as shown in Figure 2.1. We shall only consider the case where the time-like photon is produced by bremsstrahlung from the 'active' quark, in the parton model picture. This process has also been covered by some of the more general works on the LLA [20, 23], but this work was done independently of these.

In order to prove factorization for this process, we

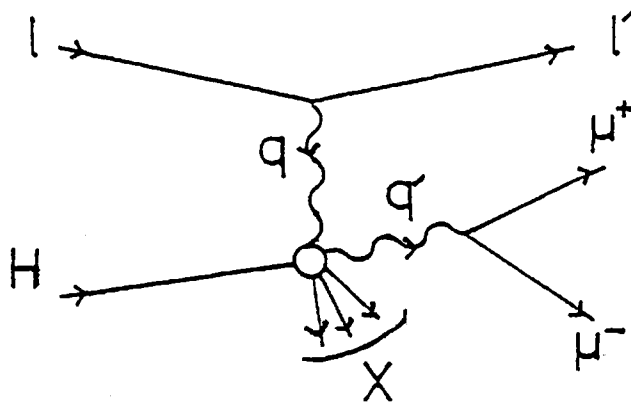


Fig. 2.1 Muon pair production via timelike photon production and decay, in deep inelastic scattering. The wavy lines denote photons, H is a hadron, l, l' are leptons, and X the unobserved final state particles. The 'blob' denotes the hadron-photon interaction.

will restrict our attention to the kinematic region in which there is just one large invariant mass scale. Specifically (see Fig. 2.1)

$$q'^2 \sim 0(Q^2) \quad (2.1)$$

Then only one type of large logarithm can occur, $\ln Q^2$.

Note the general case of deep inelastic multi-muon production in the parton model has been treated by Barger, Keung and Phillips [47].

There is an apparent contradiction between our factorization result for the process of Fig. 2.1, and the work of Lipkin's [34] on colour oscillations. Briefly, Lipkin shows that the colour of a quark measured in a colour-singlet state oscillates rapidly, making any colour dependence of the quark electric charge essentially unobservable. Such a colour dependence is predicted in the old Han-Nambu [48] model of hadron structure, but not in the standard Gell-Mann model [49]. Thus Lipkin's results suggest that one cannot easily distinguish between these models experimentally. However, the factorization of QCD corrections in our second order electromagnetic process leads us to expect a dependence on the fourth power of the quark charge, averaged over colour. This is a model dependent quantity, so we have a contradiction. The problem will be seen to arise because colour oscillations are essentially non-perturbative phenomena, of which we do not yet have a complete understanding.

In section two we will do the parton model calculation for the process of figure 2.1, essentially to set up the kinematics. In section three we will describe the methods of the LLA [15] in some detail, showing how the logarithms arise, and describing how they factorize for the standard, inclusive deep inelastic case in QCD [20 - 23]. In section four we apply these methods to our semi-inclusive process, and there prove factorization. In section five we consider whether this process will be straightforward to observe experimentally, since many other mechanisms lead to the same final state. Unfortunately, this does not appear to be the case. Section six contains a discussion on the contradiction of these results with Lipkin's work on colour oscillations.

2.2 PARTON MODEL CALCULATION

We first consider the parton model calculation for the subprocess

$$\gamma^*(q) + p \rightarrow \gamma^*(q') + X \quad (2.2)$$

$\downarrow$
 $\mu^+ \mu^-$

illustrated in figure 2.2. Here p denotes an incoming quark. The timelike (q') photon is assumed to be produced by brehmsstrahlung. The cross-section will be calculated in the usual deep inelastic limit of

$$Q^2 = -q^2 \rightarrow \infty \quad \text{for fixed} \quad x = \frac{Q^2}{2p \cdot q}, \quad (2.3)$$

$$p^2 = m^2 \ll Q^2,$$

where m is the incoming quark mass. We also impose the

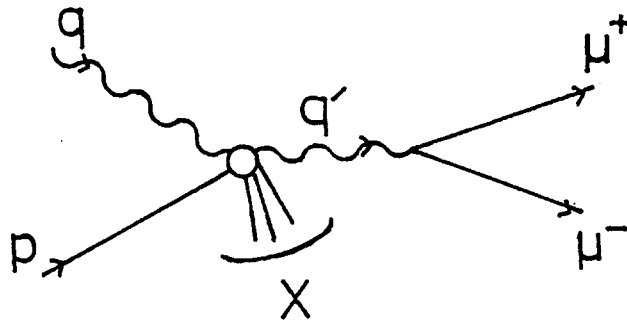


Fig. 2.2 The Feynman diagram for the quark, virtual photon subprocess contributing to the process shown in Fig. 2.1. Here 'p' labels an incoming quark line carrying momentum p . Otherwise the symbols are as in Fig. 2.1

condition (see (2.1))

$$q'^2 = \tau Q^2 \quad (2.5)$$

Where τ is fixed and finite, so that there is only one large mass scale in the process.

On summing over all unobserved final states X , the cross-section for (2.1) is related to a contribution to the forward $\gamma^* p$ amplitude by the optical theorem, as shown in figure 2.3. In the parton model the contribution due to photon brehmsstrahlung is given by the sum of the diagrams of figure 2.4. Note the diagrams given by interchanging the spacelike photons do not contribute, by kinematics.

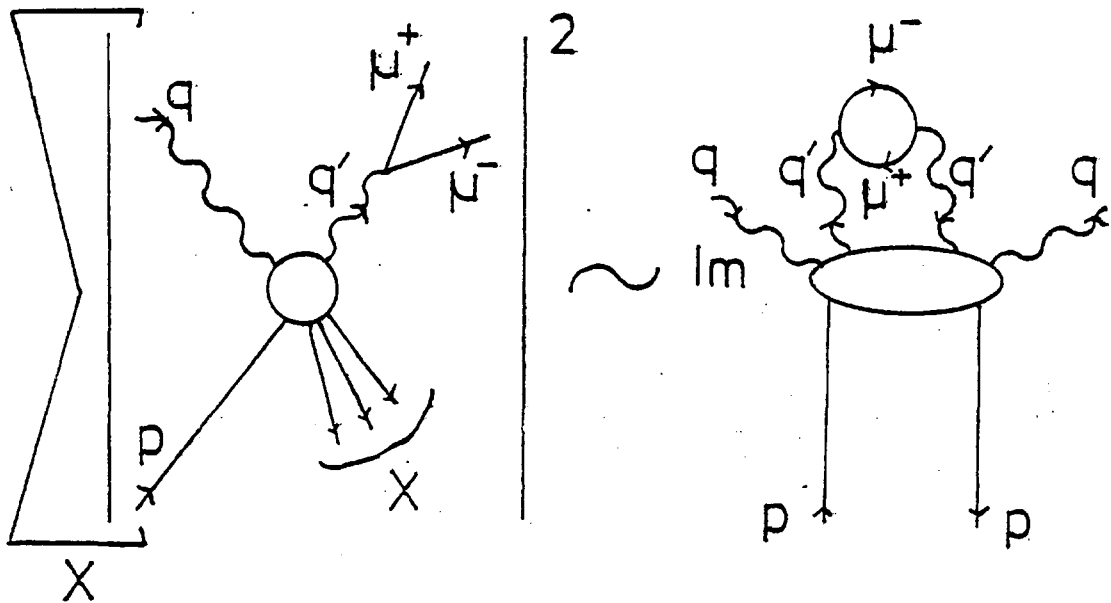
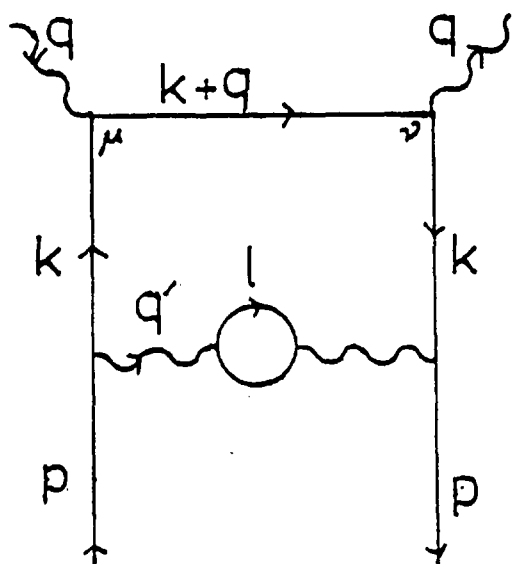
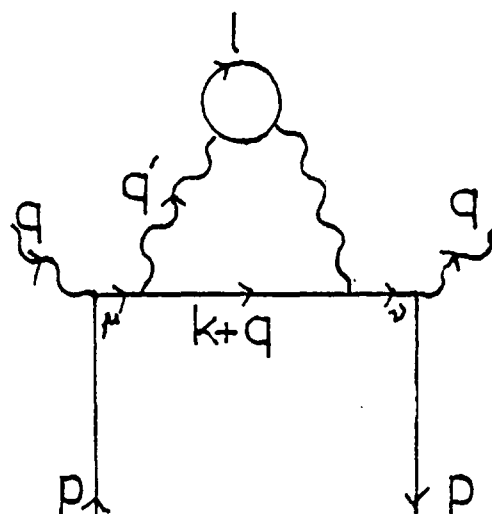


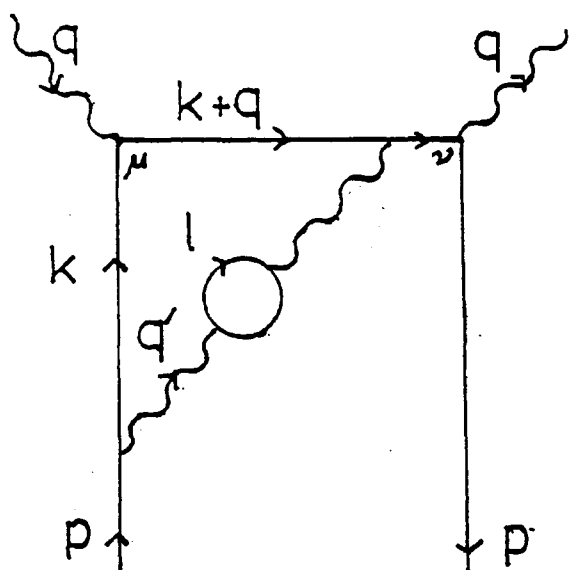
Fig. 2.3 An example of the optical theorem. On the left hand side the cross-section is to be evaluated, summed over all X , and on the right, the imaginary part of the amplitude is to be taken.



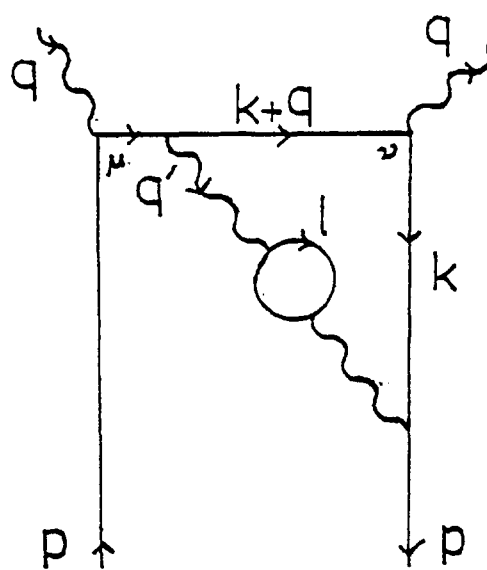
(a)



(b)



(c)



(d)

Fig. 2.4 The diagrams contributing to the right hand diagram of Fig. 2.3 in the parton model. All lines are labelled by their momenta. μ, ν are the Lorentz indices of the spacelike photon vertices.

Denoting the amplitude for the sum of these diagrams by $f_{\mu\nu}(x, Q^2)$, then the quantity we aim to calculate, which we label $A_{\mu\nu}(x, \tau, Q^2)$, is given by

$$2me^2 A_{\mu\nu}(x, \tau, Q^2) = \frac{1}{\pi} \frac{d}{d\tau'^2} \text{Im} f_{\mu\nu}(x, Q^2) \Big|_{\tau'^2 = \tau Q^2} \quad (2.6)$$

where e is the electron charge. $A_{\mu\nu}$ has the Lorentz decomposition

$$\begin{aligned} A_{\mu\nu}(x, \tau, Q^2) = & -A_1(x, \tau, Q^2) \frac{(g_{\mu\nu} - q_\mu q_\nu)}{Q^2} \\ & + \frac{A_2(x, \tau, Q^2)}{m^2} (p_\mu - \frac{p \cdot q}{Q^2} q_\mu) (p_\nu - \frac{p \cdot q}{Q^2} q_\nu) \end{aligned} \quad (2.7)$$

by gauge invariance. The constants in (2.6) and (2.7) have been chosen so that A_1 , A_2 are defined in analogy with the quark structure functions measured in inclusive deep inelastic scattering (see (1.3)).

For the evaluation of A_1 of A_2 it is convenient to use Sudakov variables [50]. We thus decompose our loop momenta, k and l (defined in figure 2.4), as

$$k = \alpha \bar{q} + \beta p + \underline{\kappa}, \quad l = \gamma \bar{q} + \delta p + \underline{\Lambda} \quad (2.8)$$

$$\text{where } \bar{q} = q + x p \quad (2.9)$$

and $\underline{\kappa}$, $\underline{\Lambda}$ are 3-vectors satisfying

$$\underline{\kappa} \cdot p = \underline{\kappa} \cdot q = \underline{\Lambda} \cdot p = \underline{\Lambda} \cdot q = 0. \quad (2.10)$$

$$\text{Note } \bar{q}^2 \sim O(p^2) \ll Q^2, \quad p \cdot q \sim O(Q^2) \quad (2.11)$$

We shall everywhere discard terms of order m^2 relative to those of order Q^2 .

Now from (2.8, 11) the phase space becomes

$$d^4k d^4l = (p \cdot q)^2 d\alpha d\beta d^2\kappa d\gamma d\delta d^2\Lambda. \quad (2.12)$$

The α and γ integrations are done by closing contours. The condition that the initial integrand have poles in both the upper and lower halves of the complex α, γ planes leads to the restriction

$$0 < \delta < 1 - \beta. \quad (2.13)$$

We now close the α, γ contours in their respective upper and lower half planes, picking up the poles at

$$l^2 = \gamma \delta 2 p \cdot q - \Lambda^2 = m_\mu^2 - i\varepsilon, \quad (2.14)$$

$$(l - q')^2 = (\alpha + \gamma)(\delta - 1 + \beta) 2 p \cdot q - (\Lambda + \kappa)^2 = m_\mu^2 - i\varepsilon,$$

where m_μ is the muon mass.

Using these relations we now find

$$\begin{aligned} q'^2 &= (p - k)^2 = -\alpha(1 - \beta) 2 p \cdot q - \kappa^2 \\ &= \frac{(1 - \beta)^2}{\delta(1 - \beta - \delta)} \left(\Lambda + \frac{\delta \kappa}{1 - \beta} \right)^2 \end{aligned} \quad (2.15)$$

discarding mass terms. Fixing q'^2 as in (2.5), we can now take

the imaginary part by the substitution

$$\frac{1}{((k+q)^2 - m^2 + i\epsilon)} \longrightarrow -i\pi \delta((k+q)^2 - m^2) \quad (2.16)$$

The constraints (2.5), (2.14 to 16) can be used to eliminate all but the β and δ integrations. These conditions restrict the Sudakov variables α , β to the regions

$$-1 < \alpha < -\frac{\tau x}{1-x}, \quad x < \beta < 1 - \tau x. \quad (2.17)$$

These inequalities imply that, in all four parton diagrams, the internal quark lines are far off-shell, since they require:

(i) $\frac{\tau x}{1-x} < 1$, which implies $(p+q)^2 > q'^2 \sim 0(Q^2)$, and

(ii) neither α nor β are ever vanishingly small, so

$$k^2 = 2\alpha\beta p \cdot q - k'^2 \sim 0(Q^2) \text{ always.}$$

Note we do not include the 'cut' quark line, on which the imaginary part is taken, as an internal line. Thus internal quark lines in the diagrams of figures 2.4 all carry momenta k or $(p+q)$, and are far off-shell. This result will make the leading log QCD calculation of section four very simple.

Note (2.17) also implies $x < \frac{1}{1+\tau}$, so in practice only $\tau < 1$

will be useful, otherwise the allowed x range becomes very small.

We now evaluate the remaining δ and β integrals. The δ integral takes the form

$$\int_0^{1-\beta} d\delta \operatorname{Tr} \psi \gamma_\rho (\psi - q') \gamma_\sigma \propto \frac{(g_{\rho\sigma} - q'_\rho q'_\sigma)}{q'^2} \quad (2.18)$$

by current conservation at the electromagnetic vertex. Multiplying through by $g^{\rho\sigma}$ now gives the constant of proportionality to be $\frac{4q'^2(1-\beta)}{3}$. The $q'_\rho q'_\sigma$ terms can be

discarded, also by current conservation. The resulting integral is now straightforward but tedious. We find

$$mA_1 = f \frac{\alpha^2}{12\pi^2 q'^2} \times P_1(x, \gamma) \quad (2.19)$$

$$\frac{p \cdot q}{m} A_2 = f \frac{\alpha^2}{12\pi^2 q'^2} \times 2x P_2(x, \gamma) \quad (2.20)$$

where f is the fractional quark charge, $\alpha = \frac{e^2}{4\pi}$, the QED coupling, and

$$\begin{aligned} P_1(x, \gamma) = & 1 + x + 2\gamma x(2-3x-3\gamma x) - \frac{x}{(1-\gamma x)} \\ & + \frac{1}{2(1-x)} (3 + 2x(2-3\gamma x)) - \frac{\gamma x}{(1-x)^2} (1-\gamma x) - \frac{(\gamma x)^2}{2(1-x)^3} \\ & - \left(1 + x - 2\gamma x(1+2x-3\gamma x^2) - 2\frac{(1-\gamma x)^2}{(1-x)}\right) \ln \left(\frac{(1-x)(1-\gamma x)}{\gamma x^2}\right) \end{aligned} \quad (2.21)$$

$$\begin{aligned}
P_2(x, \tau) = & 1 + 3x + 6\tau x(2 - 3x - 3\tau x) + \frac{x}{(1 - \tau x)} \\
& - \frac{3}{2(1-x)} (1 + 2\tau x(2 - 3\tau x)) + \frac{\tau x}{(1-x)^2} (1 + \tau x) + \frac{(\tau x)^2}{2(1-x)^3} \\
& - \left[(1 + x - 2\tau x(1 + 6x - 9\tau x^2)) - 2 \frac{(1 - \tau x)^2}{(1-x)} \right] \ln \left(\frac{(1-x)(1-\tau x)}{\tau x^2} \right).
\end{aligned}
\tag{2.22}$$

Note the logarithms here are not large, because τ , x are finite.

Finally, we give for completeness the parton model prediction for the full process (see fig. 2.1). This is easily derived from the parton model results for inclusive deep inelastic scattering [2], since A_1 , A_2 have been defined in analogy with the quark structure functions. We thus find

$$\begin{aligned}
\frac{d\sigma^H}{dQ^2 dx d\tau} = & \frac{\alpha^4}{3\pi(Q^2)^2 \tau} \times \sum_{i=1}^{N_f} f_i^4 \int \frac{dz}{z} \left(q_i^H(z) + \bar{q}_i^H(z) \right) \\
& \times \left((1-y)^2 P_2\left(\frac{x}{z}, \tau\right) + y^2 P_1\left(\frac{x}{z}, \tau\right) \right).
\end{aligned}
\tag{2.23}$$

Here $y = \frac{p \cdot q}{p \cdot l}$, where p and l are now the four momenta of the target hadron, $p \cdot l$, and incident lepton respectively. 'i' labels quark flavour, and N_f is the number of flavours. The $q_i^H(z)$ ($\bar{q}_i^H(z)$) are the usual probability distribution functions for finding a quark (antiquark) of flavour 'i' and fractional momentum z in a hadron H [2].

Note also that if the incident lepton is a muon, the amplitude should be antisymmetrized with respect to interchange of the two final state muons. We have not taken this fact into account here. Some justification for neglecting this effect is given in [47].

2.3 THE LEADING LOG APPROXIMATION AND QCD

We will here describe the methods of the LLA [15], using the standard case of inclusive DIS as an example. More details will be found in the original QCD papers [20 - 23]. This discussion will be similar to those of [15, 21].

The large logarithms of QCD perturbation theory arise from two rather different sources. The so-called 'collinear' logarithms arise when a quark near its mass-shell radiates a gluon with four-momentum almost proportional to its own. Then there are 'ultraviolet' logarithms, which are due to propagator and vertex corrections, and come from the high momentum end of the integration range.

First consider the diagrams of figure 2.5. Figure 2.5 (a) is the parton model contribution to the hadron structure functions measured in DIS. Figure 2.5 (b) is a lowest order QCD correction term. Note that the discontinuity in $s=(p+q)^2$ is to be taken. Working in the usual deep inelastic limit (2.3, 4) we introduce the Sudakov decomposition of momenta as in the previous section. Thus we write (see (2.8 to 10))

$$p' = \alpha \bar{q} + \beta p + \underline{k}, \quad (2.24)$$

$$d^4 p' = p \cdot q \, d\alpha d\beta d^2 \underline{k}. \quad (2.25)$$

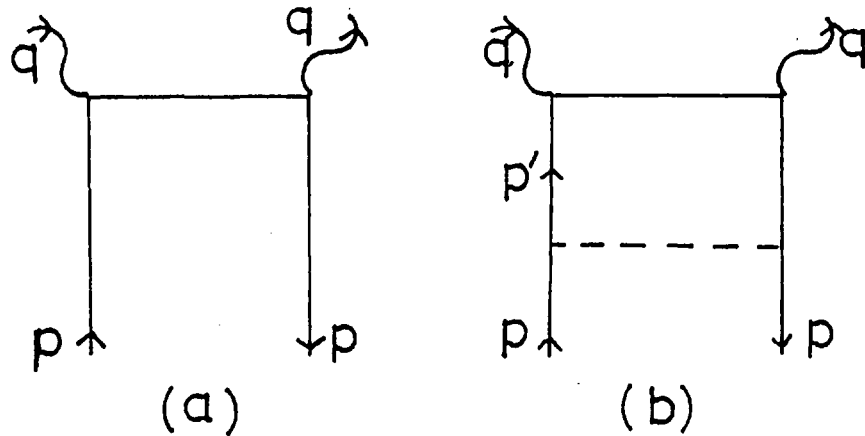


Fig 2.5 Contributions to the structure functions of deep inelastic scattering in (a) the parton model, and (b) lowest order QCD. The dashed line is a gluon, and the solid lines quarks. In this and subsequent diagrams of similar form the horizontal lines are on shell.

Doing the integral by closing the contour in the upper half plane puts the gluon on shell, resulting in

$$p'^2 = \frac{-\underline{k}^2}{1-\beta} \quad \text{for} \quad \underline{k}^2 \gg m^2 \quad (2.26)$$

Now the $\underline{k}^2$ s in the trace contribute a term of order $\underline{k}^2$ to the numerator, so the $\underline{k}^2$ integral contains a term

$$\int_{\gg m^2}^{\ll Q^2} \frac{d\underline{k}^2}{\underline{k}^2} \sim \ln Q^2/m^2, \quad (2.27)$$

where the upper limit comes from putting the top quark line on shell, to obtain the s-discontinuity. This is called a collinear logarithm, as it comes from the region

$$m^2 \ll \underline{k}^2 \ll Q^2,$$

$$|\alpha| \approx \frac{\underline{k}^2}{(1-\beta)2p \cdot q} \ll 1, \quad (2.28)$$

which implies, by (2.24)

$$p' \approx \beta p, \quad \text{where now} \quad x < \beta < 1 \quad (2.29)$$

Because of this collinear structure, the amplitude of figure 2.5 (b) is essentially proportional to the parton amplitude, figure 2.5(a), so successive gluon emissions produce further logarithms. For example, the n gluon 'ladder' diagram of figure 2.6 (a) has an amplitude of order $(\ln Q^2)^n$, coming from the region

$$m^2 \ll \underline{k}_1^2 \ll \dots \ll \underline{k}_n^2 \ll Q^2$$

$$1 > \beta_1 > \dots > \beta_n > x \quad (2.30)$$

$$|\alpha_1| \ll |\alpha_2| \ll \dots \ll |\alpha_n| \ll 1$$

where 'i' labels the momentum on the 'ith' rung of the ladder (see figure 2.6 (a)).

In a general covariant gauge diagrams of more complicated structure also give leading logarithms. These are the 'crossed' ladders, such as figure 2.6 (b), and also 'collinear' virtual gluon correction terms. These logarithms are associated with the unphysical longitudinal gluon polarization, and can be summed using Ward identities [15]. More simply however, there exists a special class of non-covariant gauges [46], the axial gauges,

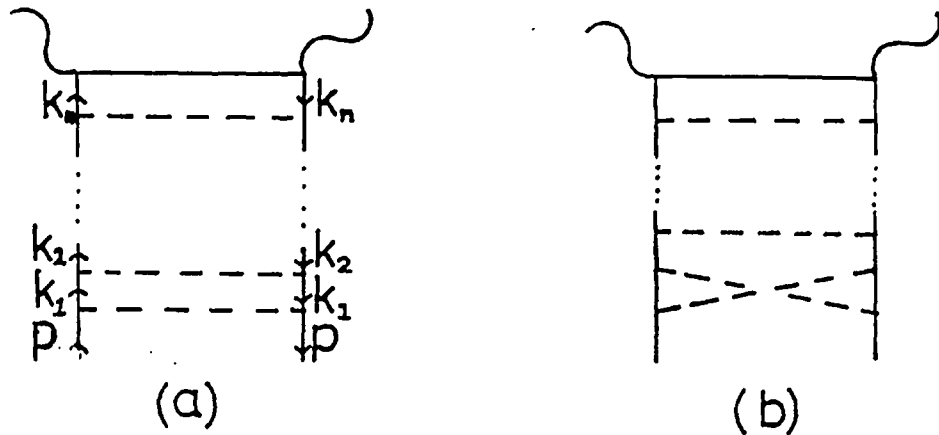


Fig. 2.6 Nth order QCD corrections to the structure functions. (a) is an uncrossed, and (b) a crossed 'ladder' diagram.

in which these terms are non-leading. These gauges are specified by a four-vector n_μ satisfying $\epsilon \cdot n = 0$, where ϵ_μ is the gluon polarization four-vector. A suitable choice of n_μ thus eliminates the unphysical polarizations.

In general ladder diagrams can have both quark and gluon rungs. One must also consider gluon initiated ladders, since gluons will be present in the target hadron 'sea'.

Now since the outgoing particles, that is to say those on the rungs of the ladder, are on-shell, they can also radiate 'collinear' gluons, as in figure 2.7 for example. However, we need not deal with such diagrams explicitly, since they can be included by replacing each 'bare' rung propagator with the corresponding full propagator. We then require the discontinuity of the full propagator in the LLA.

Thus the total contribution of collinear logarithms to the LLA is given by the sum of general, uncrossed ladder diagrams, with full propagators on the rungs of the ladder.

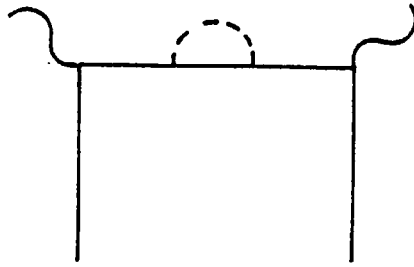


Figure 2.7 'Collinear' gluon radiation by an outgoing particle. The gluon and central quark line are on shell.

Finally, one must introduce virtual propagator and vertex corrections, since these give logarithms of the form $\ln Q^2/\mu^2$, $-\mu^2$ being the renormalization point. Thus we introduce full propagators and vertices everywhere along the sides of the ladder.

Now for large momentum scales the forms of these full propagators and vertices, in the LLA, are known from renormalization group results [6]. It turns out [20 - 22] that the only effect of all such insertions in the ladder is to replace the fixed QCD coupling by the effective, running coupling, evaluated at the relevant momentum scale, and also to cancel the wave-function renormalization factors associated with the external quark lines.

So finally one must sum all uncrossed ladder diagrams with the QCD running coupling at the vertices. The series is found to exponentiate [20 - 22], so can be summed to all orders. For our purpose it is not necessary to pursue these calculations here. The important point to note is that all the dominant diagrams in a non-covariant gauge have a straight ladder structure.

Because of the collinearity of the momenta on the sides of these ladders (see (2.29), (2.30)), these diagrams have amplitudes which are convolutions of 'bare' parton amplitudes with QCD ladder amplitudes. The convolution is unravelled by taking moments as in (1.8), and thus the QCD corrections to the moments just modify the parton model result by a power of $\ln Q^2$, leading to expressions like (1.9). In this manner the QCD corrections factorize from the parton model prediction, and can be absorbed in the parton distribution functions, so the parton model picture is preserved in this process.

2.4 THE LEADING LOG APPROXIMATION AND TIMELIKE PHOTON BREHMSSTRAHLUNG

We now return to the parton model diagrams for our lepton pair production process (see Fig. 2.4) and investigate how they can be 'dressed' in QCD to produce leading logarithms.

First we try to add one gluon to the parton diagrams in a way that generates a 'collinear' logarithm. In the previous section we saw that only quarks whose momentum l satisfies $l^2 \ll Q^2$ (see (2.26, 28)) can radiate gluons in the LLA. We have also seen that, in the parton diagrams of Fig. 2.4, quarks on internal lines are far off-shell, since (see (i), (ii))

$$lk^2, (p+q)^2 \sim O(Q^2) \quad \text{always}$$

Therefore, gluons cannot be radiated from these lines in the LLA. This is true in any gauge.

The remaining quark lines to be considered are the external quark legs, and the outgoing, 'cut' quark line. These lines are on-shell, so gluon radiation from these lines can give rise to collinear divergences. To deal with these it is simplest to work in the axial gauge specified by the vector q . Then the gluon polarization tensor is given by [46]

$$D_{\mu\nu}(k, q) = -g_{\mu\nu} + \frac{k_\mu q_\nu + q_\mu k_\nu}{k \cdot q} - q^2 \frac{k_\mu k_\nu}{(k \cdot q)^2} \quad (2.31)$$

where k is the gluon four-momentum.

First consider gluon radiation from the external legs, as in Figs. 2.8 (a) and (b). The discussion of the previous section should make it clear that the 'ladder' gluon of Fig. 2.8 (a) gives a logarithm from the region (see (2.24 to 29))

$$p^2 \ll p'^2 \ll Q^2,$$

$$p' \approx y p, \quad 0 < y < 1. \quad (2.32)$$

However, in an axial gauge, where 'crossed' diagrams are non-leading we do not expect Fig. 2.8 (b) to give a logarithm.

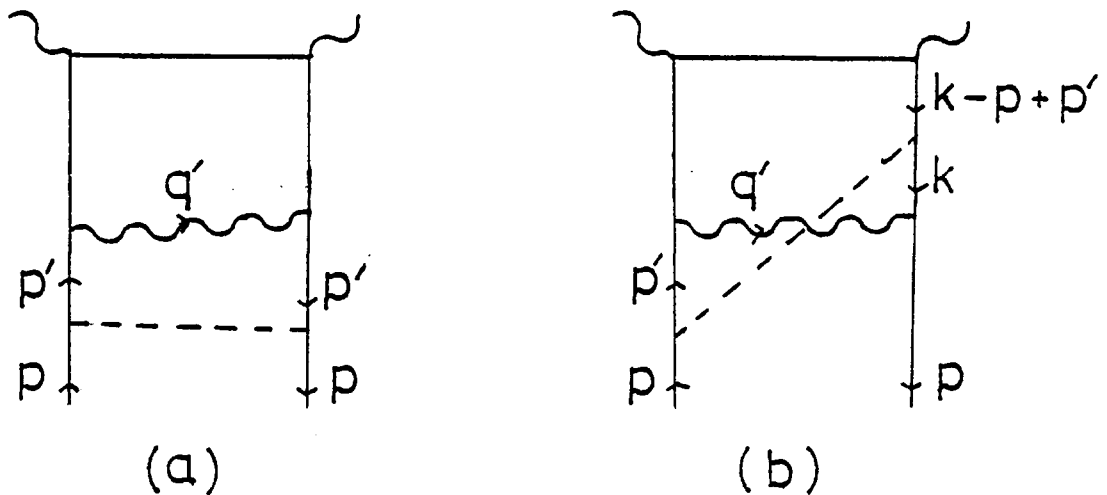


Fig. 2.8 Examples of single gluon corrections to the parton model diagrams of Fig. 2.4, due to gluon radiation by the incoming quark. The muon loop is not displayed in the diagrams of this section.

To confirm this, recall that the collinear logarithm came from a term in the trace of order $\underline{k}^2$, (see (2.24)). In a general gauge there is also a contribution to the trace which is independent of $\underline{k}$, but this is absent in an axial gauge. Thus to achieve an integrand of order $\frac{1}{\underline{k}^2}$, which we need to produce a logarithm

(see (2.27)), we need two denominator factors proportional to $\underline{k}^2$. In the region (2.32) we have $p'^2 \propto \underline{k}^2$ (see (2.26)) so the symmetric diagram, Fig. 2.8 (a), gives a logarithm. But in figure 2.8 (b), $(k - p + p')^2 \sim O(Q^2) \gg \underline{k}^2$, since $|k^2| \sim O(Q^2)$, so the extra $\frac{1}{\underline{k}^2}$ factor cannot be achieved, and this diagram does

not give a logarithm. Similar arguments dismiss all other diagrams in which a gluon radiated from one external quark leg is not absorbed on the other.

The only other possible source of collinear logarithms is from the outgoing quark, i.e. on the top quark line. Similar arguments to the above show that only a collinear gluon radiated and reabsorbed on this line will have a logarithm. For example, Fig. 2.9 is important in the region

$$l^2 = m^2 \ll (k+q)^2 \ll Q^2,$$

$$k+q \approx y l, \quad 0 < y < 1. \quad (2.33)$$

(Note that there is actually no logarithm in the imaginary part of the amplitude here [21, 22] since the transverse momentum in $(k+q)^2$ is essentially fixed by the various kinematic constraints.

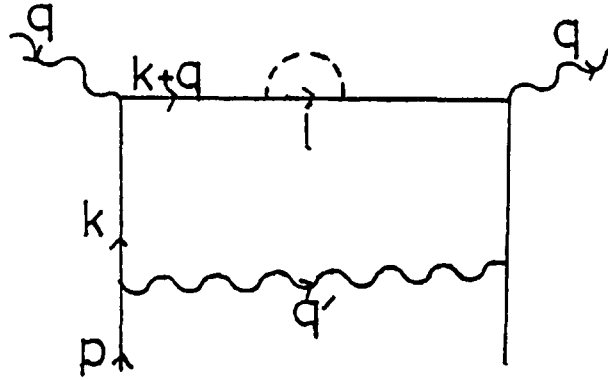


Fig. 2.9 Collinear gluon radiation by the outgoing quark.

The logarithm arises when we take moments, as in (1.8), since integrating over x for fixed Q^2 is equivalent to integrating over $s = (p + q)^2$, allowing $(k + q)^2$ to vary).

Thus the only important single gluon corrections are associated with a 'ladder' gluon attached to a parton diagram, and a propagator correction to the top quark line. It is therefore apparent that the situation is entirely analogous to the inclusive deep inelastic case described in the previous section. Thus higher order corrections are generated by adding more rungs to the QCD 'ladder', and introducing full propagators and vertices everywhere. These insertions can be removed as before, to leading log, leaving diagrams which are just QCD ladders attached to parton model diagrams. Therefore the QCD corrections to this process factorize from the parton model prediction, in the manner described in section 2.3.

Furthermore, it is not necessary for us to evaluate these ladders. Because of the factorization property, the ladder amplitude here will give rise to the same $\ln Q^2$ correction factor as is predicted in inclusive DIS. Therefore, on absorbing this factor into the parton distribution functions, we see that the same scaling violations (i.e. Q^2 dependence) are expected in both processes. Thus one expects to measure the same Q^2 dependent distribution functions in both processes, so we have

a consistent parton model picture. The precise quantitative form of these scaling violations (see 1.8) is derived in many places in the literature (8, 20 - 22).

One minor point to note is that, since $x < \frac{1}{1+\tau}$ (see (2.17)), the analogue of the n th moment is

$$M_n(\tau, Q^2) = \int_0^{1/(1+\tau)} dx x^{n-1} \frac{d\sigma(x, \tau, Q^2)}{dQ^2 dx d\tau} \quad (2.34)$$

so this quantity will only be useful for small n .

2.5 EXPERIMENTAL CONSIDERATIONS

We now consider whether the factorization demonstrated in the previous section will have experimentally observable consequences. This will be so only if the mechanism we have considered can be distinguished experimentally from any other processes which lead to the same final state in DIS, or if the other processes have very much smaller cross-sections. The general case of muon pair production in DIS has been fully analysed in [47]. There are three main sources of muon pairs:

- (i) hadronic final state interactions;
- (ii) heavy quark production and decay; and
- (iii) electromagnetic processes.

The results of [47] indicate that mechanism (i) will not give rise to muon pairs of large invariant mass, so we will discuss only the remaining two sources.

First note that the cross-section for the electromagnetic process we have been discussing is very severely damped at high energies by the far off-shell photon propagators, both contributing factors of order $1/Q^2$. Any process not subject to such a severe

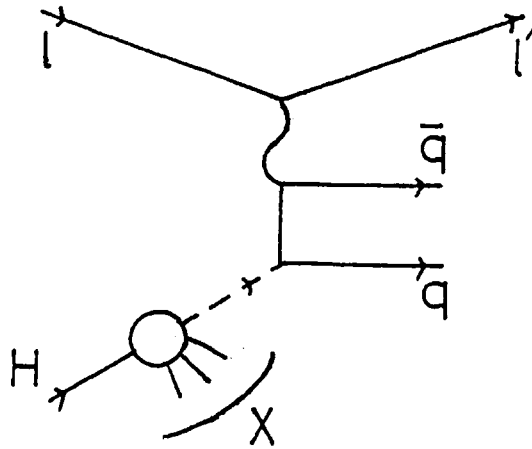


Fig. 2.10 The photon-gluon fusion mechanism. $q, \bar{q}$ denote a heavy quark and antiquark respectively.

damping will thus become relatively more important at sufficiently high energies. One expects this to be the case for mechanism (ii), heavy quark production and decay. This is expected to occur via the photon-gluon fusion mechanism illustrated in Fig. 2.10, where muons will result from the weak decay of the heavy quarks. The principal contribution will of course come from charm production.

We have no reason to expect such contributions to be unimportant at high energies. However, it may be possible to distinguish between processes involving charm and the electromagnetic process in which we are interested. For example, one expects charm decay to lead to final state strange particles, which may be experimentally detectable. The angular distribution of muon pairs should also be different in the two processes. In the centre of mass of the photon-hadron system one would expect the muons from charm decay to be essentially in the forward direction, i.e. with low transverse momentum relative to the hadron direction of motion. In contrast, the kinematics of the parton model calculation of section 2.2 lead to (see (2.15))

$$q'^2 = \frac{(1-\beta)^2}{\delta(1-\beta-\delta)} \left(\Lambda + \frac{\delta K}{(1-\beta)} \right)^2$$

(2.35)

$$\underline{k}^2 = \frac{(\beta - x)}{(1 - x)} + (1 - \beta - \gamma x) - 2 p \cdot q \quad (2.36)$$

Where $\underline{\Lambda}$, $\underline{k} + \underline{\Lambda}$ are the transverse momenta of the muons relative to the $\underline{p}$ direction. Thus we have

$$\underline{\Lambda}^2 \sim O(\underline{k} + \underline{\Lambda})^2 \sim O(Q^2)$$

except in the limit

$$\frac{\delta}{(1 - \beta)} \ll 1, \quad 1 - \frac{\delta}{(1 - \beta)} \ll 1.$$

Thus the muons produced electromagnetically will in general have very high transverse momentum, which should distinguish them from those resulting from charm decay.

We now consider the alternative electromagnetic processes which contribute. There are two types of mechanism; timelike photon brehmsstrahlung from the incident or emergent lepton, and the Bethe-Heitler photon fusion process, examples of which are shown in Figs. 2.11(a) and (b) respectively. In these processes the spacelike photon now carries momentum $k = q - q'$, where q' is, as usual, the difference between incident and emergent lepton momenta. Now in terms of the Sudakov variable β of Section 2.2 (see (2.8)), we find

$$(q - q')^2 = \frac{Q^2}{x(1 - x)} + (\beta - 1 + \gamma x) + O(m^2) \quad (2.37)$$

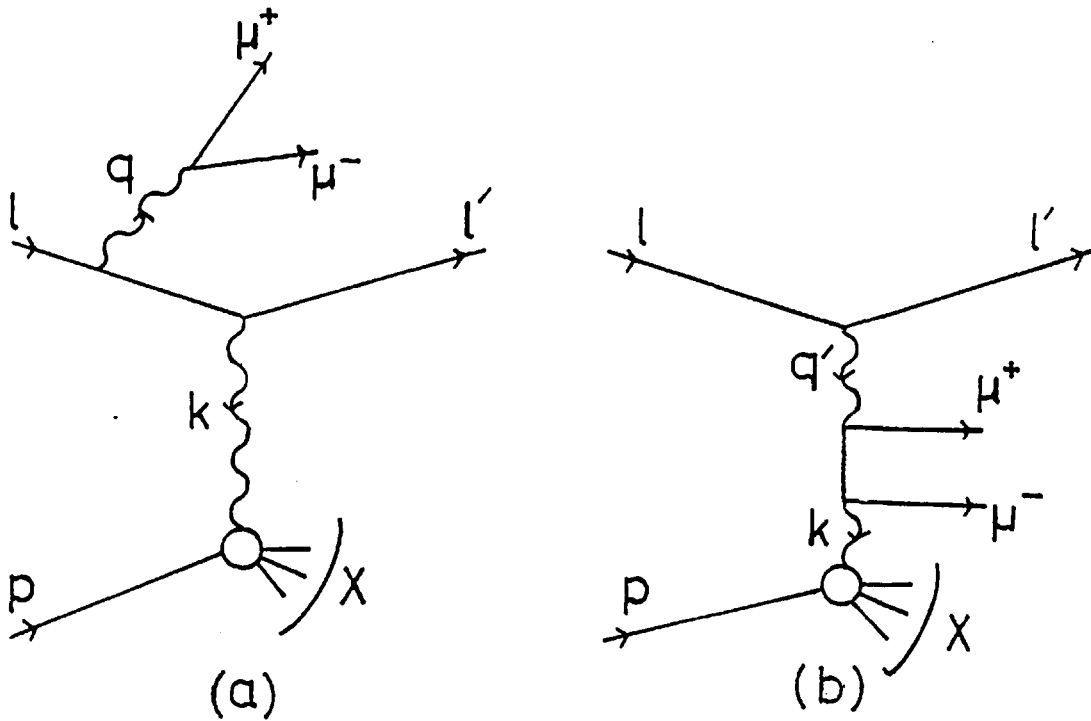


Fig. 2.11 The production of muon pairs by (a) lepton bremsstrahlung, and (b) the Bethe-Heitler process.

where, recalling (2.17) ,

$$\kappa < \beta < 1 - \tau\kappa . \quad (2.38)$$

Thus $|(q-q')^2| \sim O(Q^2)$ except in the region $\beta \rightarrow 1 - \tau\kappa$. This is the forward region, since (2.36) implies $k^2 \ll Q^2$ here. So in the high transverse momentum region, these processes show a similar propagator suppression to our original electromagnetic process, and one might reasonably expect their cross-sections to be of a similar order of magnitude. Moreover there is no obvious way of distinguishing these processes experimentally, for example by the angular distribution of the muon pairs.

Thus it seems unlikely that the quark bremsstrahlung process we have considered will be straightforward to observe experimentally. A full numerical calculation would clarify this point.

2.6 COLOUR OSCILLATIONS

It has been demonstrated by Lipkin [34] that the colour of a quark confined to a singlet system oscillates rapidly, leading to coherence effects between quarks of different colours. In consequence, only the colour average of any colour-dependent quantity is experimentally observable, at least at present day energies.

The importance of this result is that distinguishing between different quark models now becomes difficult. In the standard Gell-Mann [49] model of hadron structure the quarks have fractional colour-independent electric charges. However, in the Han-Nambu [48] model, the quarks have integral but colour-dependent charges. The average over colour of the quark charge is the same in both models, so if colour oscillations are present the old Han-Nambu model cannot be easily dismissed. For example, it has been stressed [34] that first and second order electromagnetic processes are insensitive to any colour dependence of the quark charge, if they take place at an energy below the threshold for production of states which are not colour singlets. Such a threshold, which may exist if colour confinement is not complete, is clearly above the energies reached to date.

This result is in clear contradiction with our expectations for the second order, photon-hadron electromagnetic process considered in sections 2.2 and 2.4. There, the parton model predicted a cross-section proportional to the fourth power of the quark charge averaged over colour, $\langle Q_c^4 \rangle$. Furthermore, leading log QCD corrections do not change this factor, since, as we have seen, the only effect of these corrections is to multiply the parton model prediction by a power of $\ln Q^2$.

Since $\langle Q_c^4 \rangle$ is a model-dependent quantity, we thus can, in principle, distinguish between the two quark models in this process. This should be true regardless of the energy scale of the interaction, Q^2 , although of course this should be high enough for QCD perturbation theory to be valid. In contrast, colour oscillations tell us that such a process is not sensitive to the colour dependence of the quark charge at currently accessible energies.

The source of this contradiction is not hard to find. The important points in Lipkin's derivation [34] of these results are that quarks can only be observed in a colour singlet state at present experimental energies, with any colour non-singlet states having a very much higher mass. Colour oscillations are therefore a result of colour confinement, and essentially non-perturbative phenomena. On the other hand, our prediction was based on the parton model assumption, which treats partons as 'quasi-free' objects. It is therefore not surprising that the effects of colour oscillations were nowhere to be seen in our calculation.

Lipkin himself pointed out this contradiction [34] between the naive parton model and the colour oscillation phenomena, explaining it with reasoning similar to the above. The point is that, in the parton model, when we calculate hadronic interaction cross-sections by convoluting parton distribution functions with parton interaction cross-sections, we take the incoherent average over parton colours. Thus interference effects between incoming partons of different colours, which will be present if there are colour oscillations, are neglected in this approach. For example, the parton model prediction for inclusive deep inelastic scattering is proportional to $\langle Q_c^2 \rangle$, the colour average of the squared quark charge, a model dependent quantity. But colour oscillations would replace this factor by $\langle Q_c \rangle^2$, which is model independent.

So in part we can understand the absence of colour oscillations in our calculations as being due to the neglect of coherence effects in the naive parton model. However, there is a further contradiction,

since even allowing for the fact that we may have 'missed' interference effects between incoming quarks of different colour, we still do not expect the quark colour to change between the two electromagnetic interactions of our second order process, as a consequence of the leading log QCD result. Thus we still expect a sensitivity to the colour dependence of the quark charge. In contrast, Lipkin claims that colour oscillations should destroy any correlation between the quark colour at the two electromagnetic vertices.

The point to be made here is that our results are based on a diagrammatic calculations, which is essentially perturbative in character. All non-perturbative effects have been lumped into the parton distribution functions, which we do not know how to calculate at present. Hence we do not expect non-perturbative effects, like colour oscillations to show up 'within' the diagrammatic calculation, so the quark colour does not oscillate between photon vertices in our Feynman diagram. To put it another way, it is not meaningful to consider colour oscillations between the vertices of a diagrammatic calculation, since if non-perturbative effects are important here then the perturbative diagrammatic calculation is surely invalid.

In conclusion, we recall that that all parton model and perturbative QCD calculations rest on the belief that the long distance 'confining' physics factorizes out from the short distance physics in a process independent manner, for all hard scattering processes. We then apply perturbation theory to the short distance region. If non-perturbative phenomena like colour oscillations are important in this region, then this procedure must be incorrect, so contradictory results are obtained. If the factorization procedure is correct, then the colour dependence of the quark charge is in principle observable in the process we have considered. Clearly only a better understanding of the confining mechanism will resolve this problem.

CHAPTER THREE

A COUNTER EXAMPLE TO THE BLOCH-NORDSIECK

THEOREM IN NON-ABELIAN GAUGE THEORIES

3.1 INTRODUCTION

The belief that the mass singularities of QCD perturbation theory factorize and are universal in all hard scattering processes [20, 23] has been the basis of much of QCD phenomenology. This important result rests on the assumption that all infrared divergences are cancelled by the Bloch-Nordsieck [26] mechanism, as they are known to in Quantum Electrodynamics (QED) [31]. In QCD this cancellation of divergences between real production and virtual processes has been proven for processes involving no hadronic particles in the initial state, such as e^+e^- annihilation [51], and also where there is just one incoming coloured particle, as in the inclusive deep inelastic process $\gamma^* + \text{quark} \rightarrow \text{anything}$ [52]. More recently Doria and collaborators [27] have shown that the Bloch-Nordsieck theorem fails for processes involving two coloured particles in the initial state. In the presence of two 'soft' gluons, non-leading infrared divergences fail to cancel. Here 'soft' means low momentum particles.

The importance of this surprising result demanded that it be checked, and we here repeat this calculation. We retain the methods of Doria et al. almost entirely, but find several serious errors in their work. Surprisingly, we still agree with their final result.

The process we consider is $q^\alpha + q^a \rightarrow \gamma^* + X$, where α, a label the colour of the quarks q . γ^* denotes a virtual photon and X all other final state particles. On summing over all X the cross-section is given, by the generalised optical theorem [53], as a discontinuity in the forward $q^\alpha + q^a + \gamma^*$ amplitude, shown in Fig. 3.1. We assume that, in the absence of soft gluons, this amplitude is described by a 'hard' amplitude, H , which is infrared finite, and does not have a singular behaviour when the momenta entering it vary slightly off mass shell. This hard amplitude is then 'dressed'

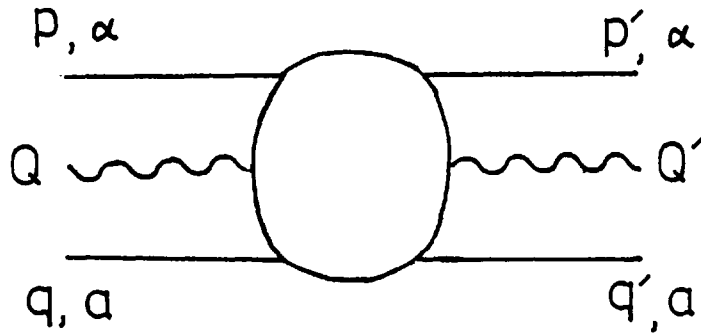


Fig. 3.1 The forward $q^\alpha + q^a + \gamma^*$ process.
The solid lines denote quarks, and the wavy lines photons. The blob contains all possible interactions.

with soft gluons in all possible ways, at fourth order in g , the QCD coupling. Thus on taking the discontinuity, both real and virtual gluon corrections to our original process are included, as demanded by the Bloch-Nordsieck theorem. It will be seen that we need only explicitly consider soft gluons which interact with the external lines of H .

The hard amplitude is a four index matrix, $H_{ab}^{\alpha\beta}$, the labels referring to the colours of the quarks external to H .

We calculate the cross-section for the incoherent sum over quark colours, which corresponds to taking the trace over the colour index of each incoming quark. This procedure, at order g^4 , leads to colour matrix structures which can all be reduced to linear combinations of the following group theoretical weights;

$$C_2(G) H_{ab}^{\alpha\beta} (T_i)^{\beta\alpha} (T_i)_{ba} ,$$

$$H_{ab}^{\alpha\beta} \{T_i, T_j\}^{\beta\alpha} \{T_i, T_j\}_{ba} ,$$

(3.1)

$$C_2(F) H_{aa}^{\alpha\beta} (T_i)^{\beta\alpha} (T_i)_{ba} ,$$

$$C_2(F)^2 H_{aa}^{\alpha\alpha} , \quad C_2(G) C_2(F) H_{aa}^{\alpha\alpha} ,$$

$$\text{where} \quad f_{ijk} f_{ijl} = C_2(G) \delta_{kl} , \quad (3.2)$$

$$(T_i T_i)_{ab} = C_2(F) \delta_{ab} ,$$

$$\text{and} \quad [T_i, T_j] = i f_{ijk} T_k .$$

The matrices T_i are the generators, and the f_{ijk} the structure constants, of a non-abelian group. We do not restrict this to be $SU(3)$, the gauge group of QCD.

Only the first two terms will be considered here. The first term occurs only in non-abelian theories, and its infrared divergent coefficient is found to be non-vanishing. The second term is abelian in character, because of the symmetry under interchange of i and j . Its coefficient is easily shown to be infrared finite by standard QED methods. This is used as a check on our method.

Since the numerical value of each group theoretical weight depends on the particular non-abelian group chosen, our uncancelled infrared divergence cannot in general cancel with the coefficient of one of the other group theory factors. The possibility of cancellation for a particular non-abelian group is excluded in [54], where the infrared divergences associated with each of the remaining group theory factors is shown to be zero.

Our final result for the infrared divergent contribution to the cross-section is

$$\text{flux} \times \frac{d\sigma}{d^3Q} = \frac{1}{(n-4)} \times \frac{g_s^4}{32\pi^2} \times \bar{H} \left(\frac{1-\beta}{\beta} \right) \left(2 - \frac{1}{\beta} \ln \left(\frac{1+\beta}{1-\beta} \right) \right), \quad (3.3)$$

where $\beta = |q_1|/q_0 < 1$, q_μ being the four-momentum of one quark in the rest frame of the other. The quarks are assumed to be on mass-shell. We have here used dimensional regularization, as we do throughout, in $n > 4$ dimensions. $\bar{H}$ is the basic hard process dressed as in the first line of (3.1).

Note our methods suggest this result is a property only of the initial state, so should apply to any process involving two incoming quarks. Doria et al. have also considered quark-antiquark annihilation into a virtual photon [29], and found (3.3) changes only by a sign.

In section two we set up the calculation. Various tricks are used to reorganise the graphs in a way that leads to a great deal of calculation. These methods are essentially due to Doria et al. [27], building on work by Libby and Sterman [52]. We take special care to ensure these methods work at the level of non-leading divergences. In section three the remaining graphs are evaluated for the non-abelian group theory factor. Here our techniques differ from those of Doria et al. [27], and we disagree with some of their intermediate

results. In section four we check the methods of sections two and three, by proving cancellation of infrared divergences for the abelian factor using both these methods and standard QED methods. Section five contains some comments on our results and its implications.

3.2 GRAPHICAL REORGANISATION

We now set out to reduce the amount of explicit calculation that need be done, by cancelling integrands as far as possible. It should be pointed out that we believe there must exist a more efficient cancellation technique than the one used here, since our final result is very much simpler than our cancellations would lead one to expect. Note some simplifications have been shown to occur in the Coulomb gauge by Doria et al. [29].

The cross-section for the process $q+q \rightarrow \gamma^* + X$ is given by Mueller's generalised optical theorem [53]:

$$\text{flux} \frac{d\sigma}{d^3Q} = \frac{1}{2i} \Delta_{\mathcal{S}} A(s_R^+, \mathcal{S}, s_L^-) \quad (3.4)$$

Here the kinematics are as in figure (3.1). $A(s_R^+, \mathcal{S}, s_L^-)$ is almost the forward amplitude for $q + q + \gamma^*$ scattering, the only difference being that all its left and right hand invariants, which include $s_L = (p+q)^2$ and $s_R = (p'+q')^2$ respectively, are to be evaluated above and below their respective cuts. $\mathcal{S} = (p+q-Q)^2$ is the missing mass squared, and $\Delta_{\mathcal{S}}$ denotes the $\mathcal{S}$ -discontinuity of the amplitude A . Note left and right hand momentum pairs like p, p' are distinguished only for pedagogical reasons. Numerically they will be equal.

Soft gluon corrections to our basic 'hard' process are now generated by dressing the corresponding hard forward amplitude H with soft gluons. Infrared divergences arise only when these couple to quarks on mass-shell, since then one obtains singular fermion propagators as the soft gluon momenta go to zero. Now the amplitude

is given by some Feynman integral, and so is analytic everywhere away from normal physical threshold energies. Thus if we consider only energies away from such thresholds in our hard amplitude, we may analytically continue the integration contour away from the physical region. Then when we take the $\mathcal{S}$ -discontinuity of H , particles on internal lines cannot go on shell. Thus soft gluons only lead to divergences when they couple to the external lines of H . Note that, had we taken the $\mathcal{S}$ -discontinuity without distorting the contour, the quarks on internal lines of H corresponding to the outgoing quarks of our original process would go on shell. Thus soft gluons can cause divergences when they couple to H , but these divergences must cancel among themselves, and we need not consider them explicitly. This simple cancellation of divergences associated with final state particles is common to many processes.

Now the hard amplitude H depends on the momentum fed into it, and when dressed with two soft gluons this varies by soft gluon momenta. Thus in general we need $H(p+k, q+k')$, where k, k' are soft momenta, and p, q are the external mass-shell quark momenta. The divergences arise for $k, k' \rightarrow 0$ so we may always take the soft momenta to be small relative to p and q . By assumption H does not have a singular behaviour when the momenta entering it vary slightly off mass-shell, so we may expand H as:

$$H(p+k, q+k') \approx H(p, q) + \text{terms proportional to soft momenta.}$$

The latter terms may be discarded in the spirit of the eikonal approximation, discussed below. Thus H is essentially constant, and we may treat it in our graphs as an effective four-quark vertex.

We now make the standard eikonal approximation. This involves discarding soft gluon momenta relative to the 'hard' quark momenta p, q wherever possible. Thus we write fermion propagator denominators as $(p+k)^2 - m^2 + i\epsilon \approx 2p \cdot k$. In numerators we write $\not{p} + \not{k} \approx \not{p}$, which greatly simplifies the gamma matrix structure, since then $(\not{p} + \not{k} + m)\gamma_\mu \approx 2p_\mu$. Since the divergences arise for $k \rightarrow 0$, these

approximations should not change the infrared singularity structure of our integrals. However, some care is required because, for two gluons, we can have integrals with up to three logarithmic divergences, and we do not want to make approximations which are not valid for non-leading divergences. In fact, numerator terms proportional to soft momenta can lead to single divergences like $\int \frac{dk}{k} \int dk'$, where $k=|k|$. This diverges as $k \rightarrow 0$ for fixed k' . It is therefore analogous to the case of one soft gluon and one hard. These divergences must therefore cancel among themselves, since in the presence of only one soft gluon, the non-abelian nature of the theory has no effect, and we have cancellation by the QED result.

We now re-write the cross-section (3.4) as

$$2i \text{flux} \frac{d\sigma}{d^3Q} = \Delta_{\mathcal{S}} \left(A(s_L^+, \mathcal{S}, s_R^+) - \Delta_{s_R} A(s_L^+, \mathcal{S}, s_R) \right). \quad (3.5)$$

We now demonstrate that, as a result of the eikonal approximation, we need only explicitly need take the $\mathcal{S}$ -discontinuity of the hard amplitude; the rest is automatic. Consider for example the contribution to $A(s_L^+, \mathcal{S}, s_R^+)$ shown in Fig. 3.2. Taking the $\mathcal{S}$ -discontinuity explicitly puts the k gluon on shell with energy $k_0=|k|$. Alternatively, using ordinary Feynman rules everywhere gives a k_0 integral of the form

$$I = \int \frac{dk_0}{(k^2+i\epsilon)(-2p \cdot k+i\epsilon)(-2p \cdot (k+k')+i\epsilon)} \quad (3.6)$$

Now even though we have made the eikonal approximation, we can still let the integration range extend to infinity, and thus evaluate the integral by closing contours. This is a standard procedure in infrared work, and clearly will not affect the infrared singularity structure of I . Again, there may be non-leading errors induced in the full graphical amplitude, but they come from the

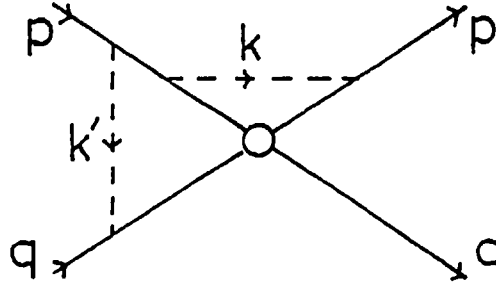


Fig. 3.2 A graph which contributes to the amplitude $A(s_L^+, \mathcal{J}, s_R^+)$. The dashed lines denote gluons, and the circle represents the hard amplitude H .

region $|k_0| \gg$ soft momenta. As we have seen such terms may be discarded. Thus, in (3.6), closing the k_0 contour in the lower half of the complex plane picks up the pole at $k_0 = |k| - i\epsilon$, exactly as was given by taking the $\mathcal{J}$ -discontinuity explicitly. Note this only works because of the eikonal approximation. In the full amplitude there would have been more than one pole in the lower half plane.

Similar reasoning applies to all the diagrams contributing to $A(s_L^+, \mathcal{J}, s_R^+)$. It also applies to the Δ_{s_R} term after taking the s_R discontinuity. Thus we may drop the $\Delta_{\mathcal{J}}$ in (3.5), as long as we replace the hard amplitude by its $\mathcal{J}$ -discontinuity.

Therefore

$$\text{flux} \frac{d\sigma}{d^3Q} = \left(A(s_L^+, \mathcal{J}, s_R^+) - \Delta_{s_R} A(s_L^+, \mathcal{J}, s_R) \right) \frac{\Delta_{\mathcal{J}} H}{2i}, \quad (3.7)$$

where we have extracted the hard amplitude factor from the A 's $\frac{\Delta_{\mathcal{J}} H}{2i}$ is now a real constant, the $H_{ab}^{\alpha\beta}$ of the introduction.

Finally, the number of graphs contributing to the first term in (3.7) may be greatly reduced using an identity due to Libby and Sterman [52]. This is

$$\begin{aligned}
& \frac{1}{(a_1 + i\epsilon)(a_1 + a_2 + 2i\epsilon) \dots \dots (a_1 + a_2 + \dots + a_n + ni\epsilon)} \\
& + \frac{1}{(-a_1 - i\epsilon)(a_2 + i\epsilon)(a_2 + a_3 + 2i\epsilon) \dots \dots (a_2 + a_3 + \dots + a_n + (n-1)i\epsilon)} \\
& + \frac{1}{(-a_1 - a_2 - 2i\epsilon)(-a_2 - i\epsilon)(a_3 + i\epsilon) \dots \dots (a_3 + \dots + a_n + (n-2)i\epsilon)} \\
& + \text{etc.} = 0
\end{aligned} \tag{3.8}$$

This result is true for all n , and is proven by evaluating the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(x + i\epsilon)(x + a_1 + 2i\epsilon) \dots \dots (x + a_1 + a_2 + \dots + a_n + (n+1)i\epsilon)}$$

in two ways, by closing the contour in both the upper and lower half planes in turn.

To see how this identity is useful consider diagrams of the type shown in Fig. 3.3. Here n gluons are attached to a single quark line, r to the right of the hard process and $n-r$ to the left. Because of the trace over external quark colour, the colour matrix structure of this diagram is independent of r . Now assume we have ordinary Feynman propagators on the left, and anti-Feynman propagators, with $-i\epsilon$ prescriptions, on the right. Then letting $a_j = 2p \cdot k_j$ in (3.8), we see that each term in (3.8) corresponds to the quark propagator factors for a given value of r in Fig. 3.3. Thus the Libby-Sterman identity (3.8) tells us that the sum over all r of Fig. 3.3 is zero.

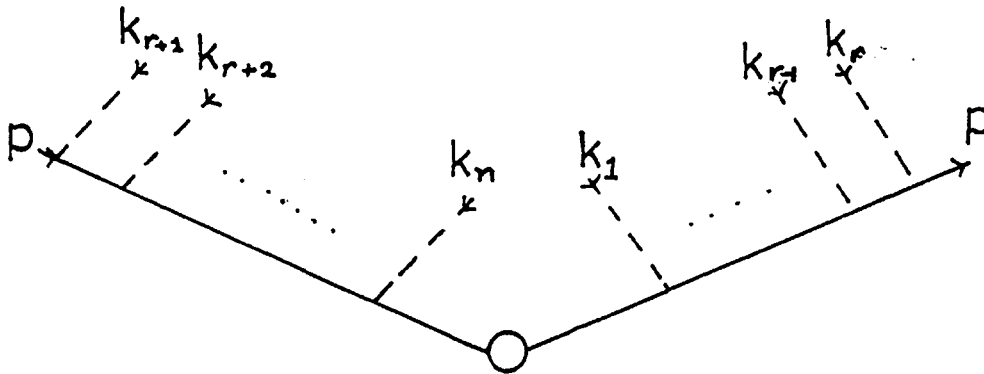


Fig. 3.3 n soft gluons attached to a quark line. k_i labels soft gluon momenta.

We wish to apply this identity to the graphs of $A(s_L^+, \mathcal{J}, s_R^+)$. Here we have ordinary Feynman propagators to the right of the hard amplitude. We therefore re-write propagators on the right using the identity

$$\frac{i}{(x+i\epsilon)} - \frac{i}{(x-i\epsilon)} = 2\pi \delta(x) \quad (3.9)$$

wherever necessary to allow the cancellation to proceed. We thus combine graphs of the same colour matrix structure, and differing only in the way gluons are attached to one fermion line, obtaining cancellations like the one illustrated in Fig. 3.4. Since we have two fermion lines, the Libby-Sterman cancellation can be applied twice to the whole set of graphs. Note that the identity is only useful because we do not have to take the $\mathcal{J}$ -discontinuity explicitly, otherwise the gluon propagators would differ as we moved them from left to right.

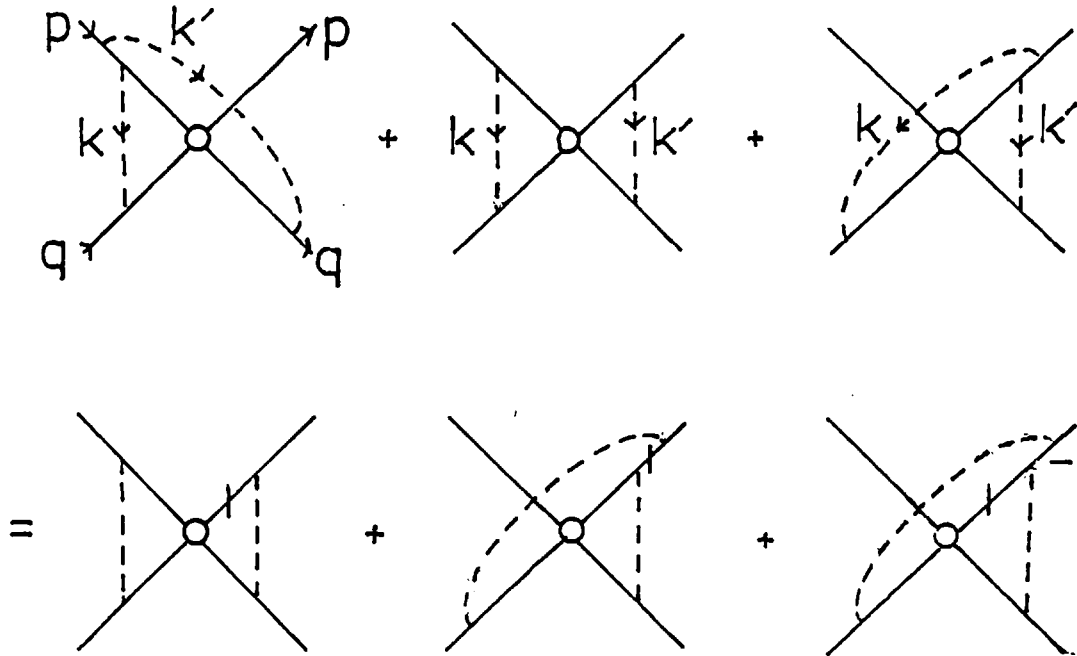


Fig. 3.4 An example of the Libby-Sterman cancellation. Here $|$ denotes a mass-shell quark line, and $-$ an anti-Feynman propagator. Henceforth the circle represents the 'cut' hard amplitude.

After cancelling as far as possible, the remaining graphs contributing to the non-abelian group theoretical weight in $A(s_L^+, \mathcal{J}, s_R^+)$ are shown in Fig. 3.5.

Graphs having s_R discontinuities which contribute to the non-abelian factor are shown in Fig. 3.6. Since H is at this stage a real constant, we can only take the s_R discontinuity to the right of H , and not through it.

Note we have not displayed graphs obtainable by interchange of p and q , since each amplitude is clearly invariant under this interchange.

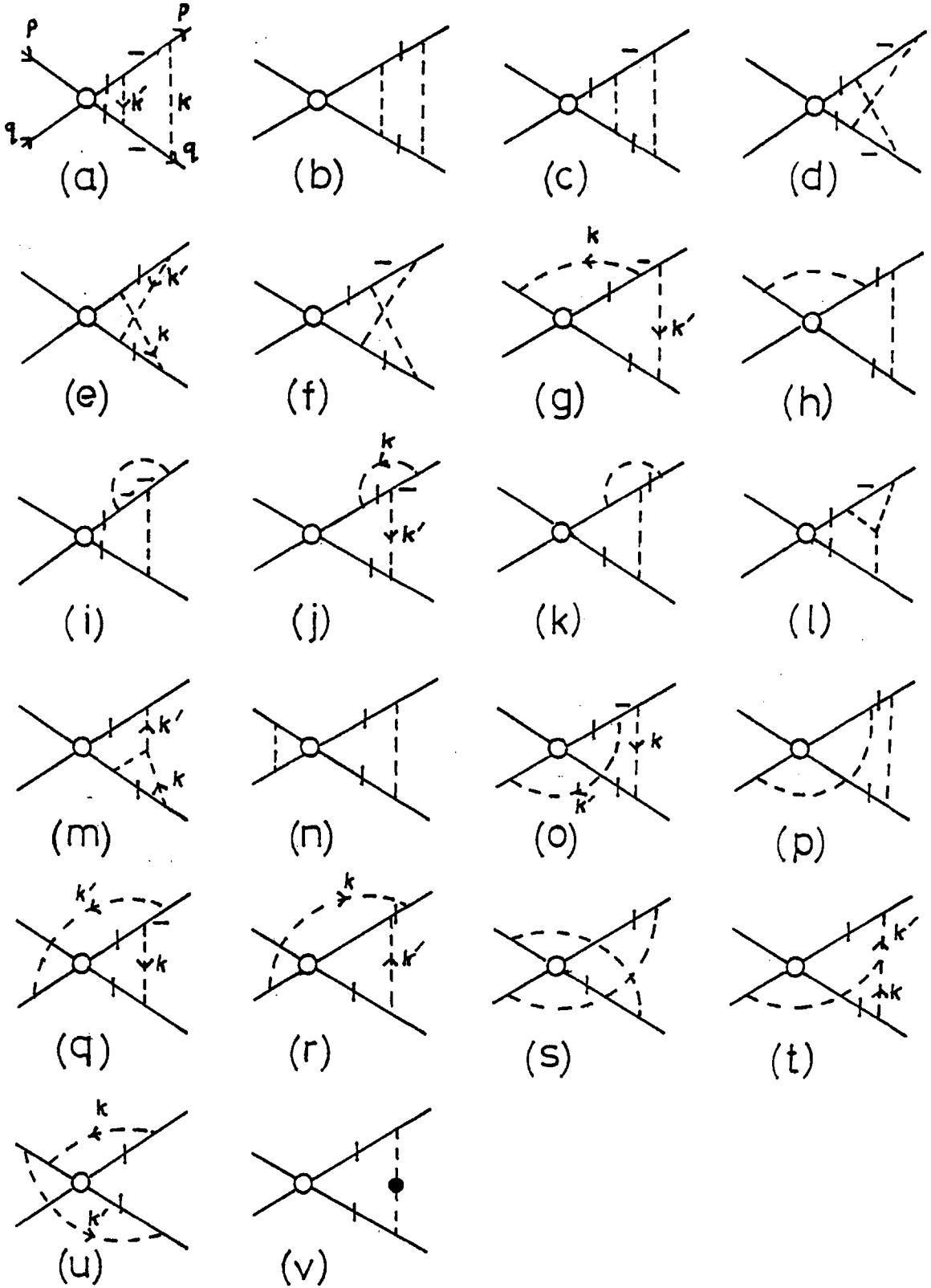


Fig. 3.5 Graphs contributing to the amplitude $A(s_L^+, \delta, s_R^+)$ after the Libby-Sterman cancellation. $- \bullet -$ denotes the $O(g^2)$ gluon self energy due to ghost and gluon loops.

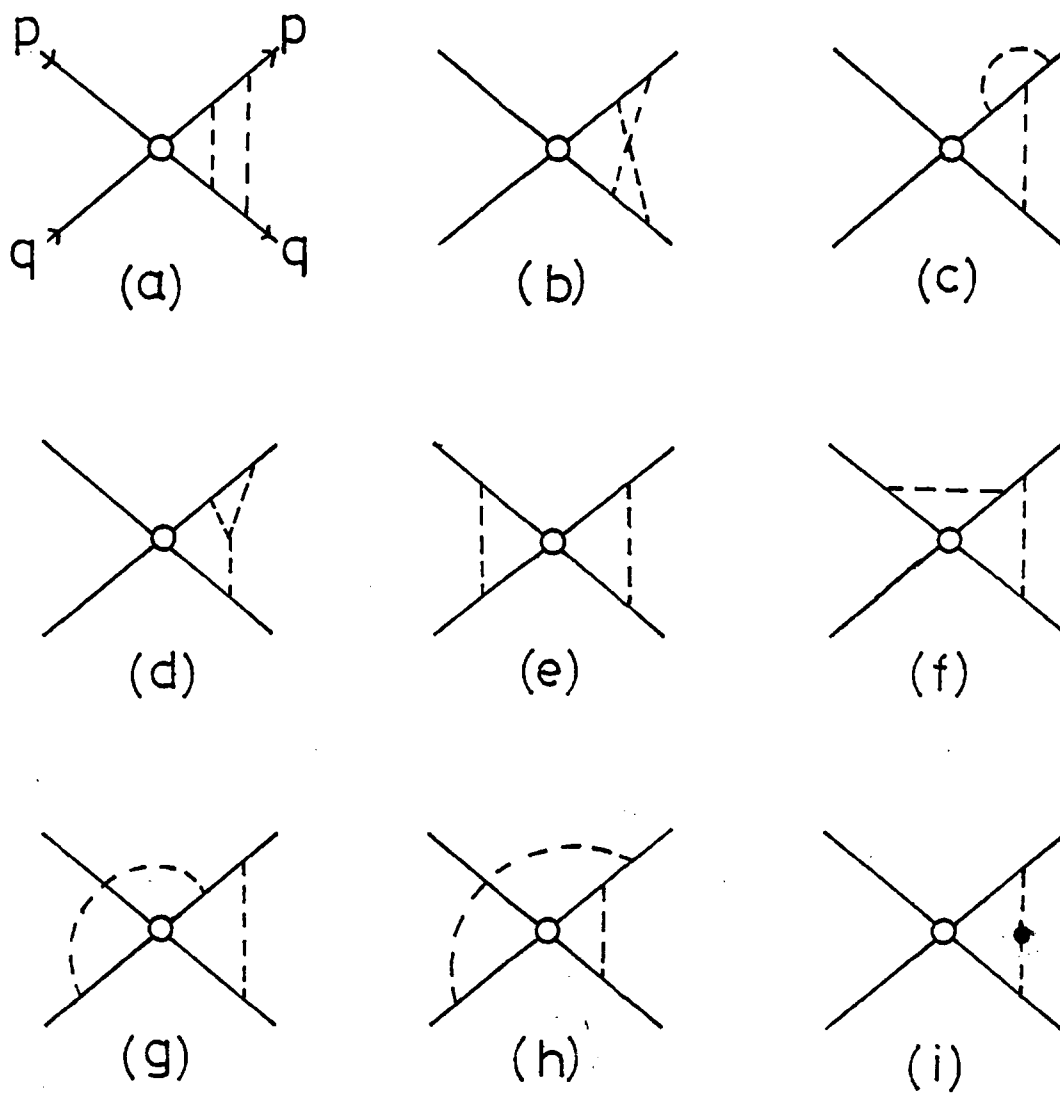


Fig. 3.6 Graphs having s_R discontinuities.

3.3. EVALUATION OF NON-ABELIAN TERM

First consider the graphs of Fig. 3.6, whose s_R discontinuities we wish to evaluate. Our hard amplitude H forms an effective four point vertex which is real. The usual Feynman rules would put a factor of 'i' at such a vertex. Elsewhere we have the standard Feynman rules, corresponding to S-matrix elements. It follows that the graphs of (a), (b), (c), (d) and (j) correspond to effective T matrix elements, since $S = iT$. These T-matrices are real analytic, so their s_R discontinuities are pure imaginary. Since the cross-section is real they may be discarded. Similar arguments eliminate the graphs of (f), (g) and (h). This time taking the s_R discontinuity gives graphs of the form $iT_1T_2^*$, and in all cases T_1 and T_2 are real T-matrices, corresponding either to the tree graphs or to graphs having all invariants below threshold. These imaginary properties can easily be checked explicitly.

Only one graph, (e), survives, since in this case T_1 has an imaginary part. The non-vanishing real part comes from the 'cuts' shown in Fig. 3.7. This graph is easily seen to cancel one from the other set of graphs, Fig. 3.5 (n).

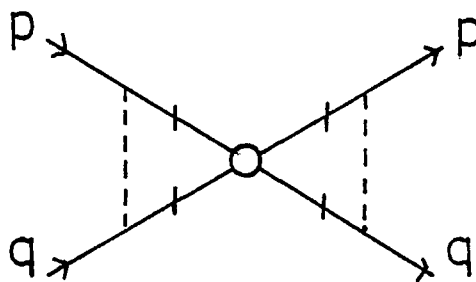


Fig. 3.7 The only non-vanishing real contribution to the s_R discontinuity of Fig. 3.6.

Now consider Fig. 3.5. Again, since we only want the real part we may discard the graphs (d), (h), (i), (l), (p) and (v), which all take the form iT_1T_2 , with T_1, T_2 being real T-matrices.

We now calculate the contribution of the remaining diagrams to the non-abelian group theoretical factor given in the first line of (3.1).

On comparing the integrands for different graphs we find immediately the following simplifications:

$$(c)=(q) \quad (o)=(f) \quad (g)=(j) \quad (k)=(u) \quad (m)=(t) \quad (e)=(s)=-(r) \quad (3.10)$$

Since (r) has an inverse under interchange of p and q, and (e) and (s) do not, as they are self-inverse, we see that these graphs cancel; $(e)+(s)+2(r)=0$. Thus we need only evaluate explicitly the graphs (a), (b), (c), (f), (g), (j), (k) and (m). We write the contribution of a general graph (i) to the non-abelian group theory factor in the form

$$C_2(G)(T_i)^{ab}(T_i)_{ab} H^{ab} \times \frac{g^4}{(2\pi)^{2(n-1)}} A_i, \quad (3.11)$$

where n is the space-time dimension, and g the QCD coupling. We take $n > 4$ to regulate the infrared divergences, which show up as poles at $n = 4$.

First consider (a). With the momentum labelling shown we find

$$A_a = -\frac{1}{4} \int \frac{d^n k d^n k' \delta(2p \cdot (k+k')) \delta(2q \cdot (k+k')) (4p \cdot q)^2}{(k^2 + i\epsilon)(k'^2 + i\epsilon)(2p \cdot k - i\epsilon)(-2q \cdot k - i\epsilon)} \quad (3.12)$$

where the factor of $-1/4$ comes from the colour matrix structure, and the Feynman gauge has been used, as it will be throughout.

It is convenient to evaluate our integrals in the Lorentz frame defined by

$$p^\mu = (m, \mathbf{0}), \quad q^\mu = (q_0, 0, 0, q_z), \quad q_z > 0. \quad (3.13)$$

Thus

$$A_a = -\frac{1}{4} \int \frac{d^n k d^n k' \delta(2q \cdot (k+k')) 4q_0^2}{(k_0 - i\epsilon)(k_0^2 - k^2 + i\epsilon)(k_0^2 - k'^2 + i\epsilon)(-2q_0 k_0 + 2q_z \cdot k - i\epsilon)} \quad (3.14)$$

We now do the k_0 integral by closing the contour in the upper half plane. To avoid ambiguity the ϵ 's in the gluon propagators must be kept unequal here. Then there are three poles in the upper half plane, but only the residue at $k_0 = i\epsilon$ contributes, the other two cancelling each other. Thus

$$A_a = -\frac{1}{4} - 2\pi i \int \frac{d^{n-1} k d^{n-1} k' \delta(2q \cdot (k+k')) 4q_0^2}{k^2 k'^2 (2q_z \cdot k - i\epsilon)}. \quad (3.15)$$

Taking the real part by letting we find $\frac{1}{(2q_z \cdot k - i\epsilon)} \rightarrow i\pi \delta(2q_z \cdot k)$

$$\text{Re } A_a = \frac{\pi^2}{2\beta^2} \left(\int \frac{d^{n-1} \underline{k}}{k^2} \delta(k_z) \right)^2 \quad (3.16)$$

where $\beta = \frac{q_z}{q_0}$, which satisfies $0 < \beta < 1$.

Up till this point we have implicitly assumed the existence of a small momentum cut-off $|\underline{k}|, |\underline{k}'| < \Delta \ll q_z$, since the eikonal approximation is only valid with some such restriction. We now find it convenient to remove this restriction and integrate over all momenta. To suppress the ultraviolet divergences which will now arise as a result of these approximations we introduce a cut-off factor, $\exp - (\underline{k}^2 + \underline{k}'^2)/\Lambda^2$, into the integrands, which has essentially no effect on their infrared structure. Since this regularization is forced on us by the eikonal approximation, which is valid in the same region for all graphs, we use the same value of Λ^2 for all graphs. Our final answer is then independent of this non-covariant cut-off.

The rationale behind the above manipulation is that it enables us to do our n-dimensional integrals very carefully. We found throughout these calculations that this was extremely important. Wrong answers were obtained unless we let $n \rightarrow 4$ in a controlled manner. Extending the range of momentum integrals allowed us to use Feynman parametrization, in which n-dimensional integrals are straightforward. This avoided the difficulties of angular integrals in more than three space dimensions.

We now return to the calculation of A_a . We now have, effectively

$$A_a = \frac{\pi^2}{2\beta^2} \left(\int \frac{d^{n-1} \underline{k}}{k^2} \delta(k_z) e^{-\underline{k}^2/\Lambda^2} \right)^2 \quad (3.17)$$

Writing

$$\frac{1}{k^2} = \int_0^\infty d\gamma e^{-\gamma k^2}, \quad (3.18)$$

the momentum integral becomes a straightforward Gaussian, and we obtain

$$\begin{aligned} A_a &= \frac{\pi^n}{2\beta^2} \left(\int_0^\infty d\gamma \left(\gamma + \frac{1}{\Lambda^2} \right)^{\frac{1-n}{2}} \right)^2 \\ &= \frac{2\pi^n \Lambda^{2(n-4)}}{\beta^2 (n-4)^2}. \end{aligned} \quad (3.19)$$

Now turn to (b).

$$\begin{aligned} A_b &= -\frac{1}{4} \int \frac{d^n k \, d^n k' \, \delta(2p \cdot k) \, \delta(2q \cdot k) \, (4p \cdot q)^2}{(2p \cdot k' + i\epsilon)(-2q \cdot k' + i\epsilon)(k^2 + i\epsilon)(k'^2 + i\epsilon)} \\ &= q_0^2 \int \frac{d^{n-1} \underline{k} \, \delta(2q \cdot k)}{k^2} \int \frac{d^n k'}{(k_0' + i\epsilon)(-2q_0 k_0' + 2q \cdot k' + i\epsilon)(k_0'^2 - \underline{k}^2 + i\epsilon)} \end{aligned} \quad (3.20)$$

Closing the k_0' contour in the lower half plane, we pick up three poles. Only the residue at $k_0' = -i\epsilon$ contributes, the other two residues being pure imaginary. We therefore find

$$A_b = \frac{\pi i}{2\beta^2} \int \frac{d^{n-1}k}{k^2} \delta(k_z) \int \frac{d^{n-1}k'}{k'^2 \times (k_z' + i\epsilon)} \quad (3.21)$$

and thus by (3.15)

$$\text{Re } A_a = \text{Re } A_b. \quad (3.22)$$

Now consider (c) and (f).

$$A_c + A_f = \frac{1}{4} \int \frac{d^n k d^n k' \delta(2p \cdot (k+k')) \delta(2q \cdot k) (4p \cdot q)^2}{(k^2 + i\epsilon)(k'^2 + i\epsilon)(-2q \cdot (k+k') + i\epsilon)} \quad (3.23)$$

$$\times \left(\frac{1}{(2p \cdot k' + i\epsilon)} + \frac{1}{(2p \cdot k' - i\epsilon)} \right)$$

where the first and second terms in the large brackets come from (c) and (f) respectively. So defining $A_{c+f} = A_c + A_f$, with the momentum cut-off introduced as in (3.17), we find

$$A_{c+f} = -\frac{1}{2\beta^2} \times \int \frac{d^{n-1}k d^{n-1}k' \times \exp -((k^2 + k'^2)/\Lambda^2)}{(\beta^2 k_z^2 - k^2)(\beta^2 k_z^2 - k'^2 + i\epsilon)} \quad (3.24)$$

$$\times \frac{k_z}{(k_z + k_z' + i\epsilon) k_z^2}$$

Note we discard $i\epsilon$'s as they become irrelevant, when the delta function constraints make it impossible for the associated discontinuity to be taken.

The poles at $n=4$ of (3.24) are evaluated in appendix 3.1, with the result

$$A_{c+f} = -\frac{\pi^n}{\beta^2} \Lambda^{2(n-4)} \cdot \left(\frac{1}{(n-4)^2} + \frac{1}{2(n-4)} \ln \left(\frac{1+\beta}{1-\beta} \right) \right). \quad (3.25)$$

Now consider (g) and (k).

$$A_g = -\frac{1}{2} \int \frac{d^n k d^n k' \delta(2q \cdot k') \delta(2p \cdot (k+k')) 4m^2 \times 4p \cdot q}{(k^2 + i\epsilon)(k'^2 + i\epsilon)(2p \cdot k' - i\epsilon)(2p \cdot k + i\epsilon)} \quad (3.26)$$

$$A_k = -\frac{1}{2} \int \frac{d^n k d^n k' \delta(2p \cdot k) \delta(2q \cdot k') 4m^2 \times 4p \cdot q}{(k^2 + i\epsilon)(k'^2 + i\epsilon)(2p \cdot k' + i\epsilon)(2p \cdot (k+k') + i\epsilon)} \quad (3.27)$$

Proceeding as before we arrive at

$$A_g = \frac{1}{2\beta^2} \int \frac{d^{n-1} k d^{n-1} k' \exp - \left[(k^2 + k'^2)/\Lambda^2 \right]}{(\beta^2 k_z'^2 - k^2 + i\epsilon)(\beta^2 k_z'^2 - k'^2)(k_z' + i\epsilon)^2} \quad (3.28)$$

$$A_k = \frac{1}{2\beta^2} \int \frac{d^{n-1} k d^{n-1} k' \exp - \left[(k^2 + k'^2)/\Lambda^2 \right]}{k_z^2 (\beta^2 k_z'^2 - k'^2)(k_z' + i\epsilon)^2} \quad (3.29)$$

These are evaluated in appendix 3.2. Both have an unphysical pole at $n=3$, but these cancel in the sum $A_g + A_k$. Only A_g has an $n=4$ pole. Thus

$$A_g = \frac{\pi^4}{\beta} (n-4). \quad (3.30)$$

Finally we come to (m).

$$A_m = -\frac{1}{2} \int \frac{d^n k d^n k' \delta(2p \cdot k) \delta(2q \cdot k') \cdot 8(2m^2 q \cdot k + (p \cdot q)(p \cdot k'))}{(k^2 + i\varepsilon)(k'^2 + i\varepsilon)((k' - k)^2 + i\varepsilon)(2p \cdot k' + i\varepsilon)} \quad (3.31)$$

which becomes

$$\begin{aligned} A_m &= \frac{1}{2} \int \frac{d^{n-1} \underline{k} d^{n-1} \underline{k}' (k'_z - 2k_z) \exp - [(k^2 + k'^2)/\Lambda^2]}{k^2(\beta^2 k_z'^2 - k'^2)(\beta^2 k_z'^2 - (k' - k)^2 + i\varepsilon)(k'_z + i\varepsilon)} \\ &= \frac{\pi^4}{2} \frac{1}{(n-4)} \times \left(\frac{1}{\beta} \ln \left(\frac{1+\beta}{1-\beta} \right) - 2 \right) \end{aligned} \quad (3.32)$$

as shown in appendix 3.3.

We now evaluate the final sum of the amplitudes A_i . Taking into account the equalities of (3.10, 22), and also factors of two from the inverse graphs, we arrive at

$$\begin{aligned}
 A_{\text{tot}} &= A_a + A_b + 4(A_{c+f} + A_g + A_m) \\
 &= -\frac{4\pi^4}{(n-4)} \cdot \frac{(1-\beta)}{\beta} \cdot \left(\frac{1}{2\beta} \ln \left(\frac{1+\beta}{1-\beta} \right) - 1 \right). \quad (3.33)
 \end{aligned}$$

Multiplying by the factor in (3.11), now with $n=4$, we obtain the full infrared divergent contribution to the cross-section as quoted in the introduction, (3.3).

3.4 ABELIAN EXAMPLE

We now calculate the infrared divergent contribution to the 'abelian' group theoretical weight factor given in the second line of (1.1). This is very easily shown to be zero by more direct QED type arguments, so we have a check on our methods.

The graphs which contribute are shown in Fig. 3.8, where graphs obtained by up-down inversion, and also interchange of left and right, are also to be included. To evaluate the contribution to the cross-section directly, we now take their $\mathcal{J}$ -discontinuity explicitly. Thus the ordinary Feynman rules are now complex-conjugated to the right of the hard vertex, and gluons crossing it are on shell. Therefore left - right interchange is equivalent to complex conjugation.

The group theoretical weight factor considered here is abelian in the sense that the colour matrices associated with all the graphs of Fig. 3.8 make the same contribution to this weight factor, $1/4$. This allows us to combine pairs of graphs as shown, using the identity

$$\begin{aligned}
 & \frac{1}{(2p \cdot k + i\epsilon)} \frac{1}{(2p \cdot (k+k') + i\epsilon)} + \frac{1}{(2p \cdot k' + i\epsilon)} \frac{1}{(2p \cdot (k+k') + i\epsilon)} \\
 &= \frac{1}{(2p \cdot k + i\epsilon)} \frac{1}{(2p \cdot k' + i\epsilon)}
 \end{aligned}
 \tag{3.34}$$

This is a standard trick in QED. Thus two successive gluon emissions are independent when we sum over all possible gluon

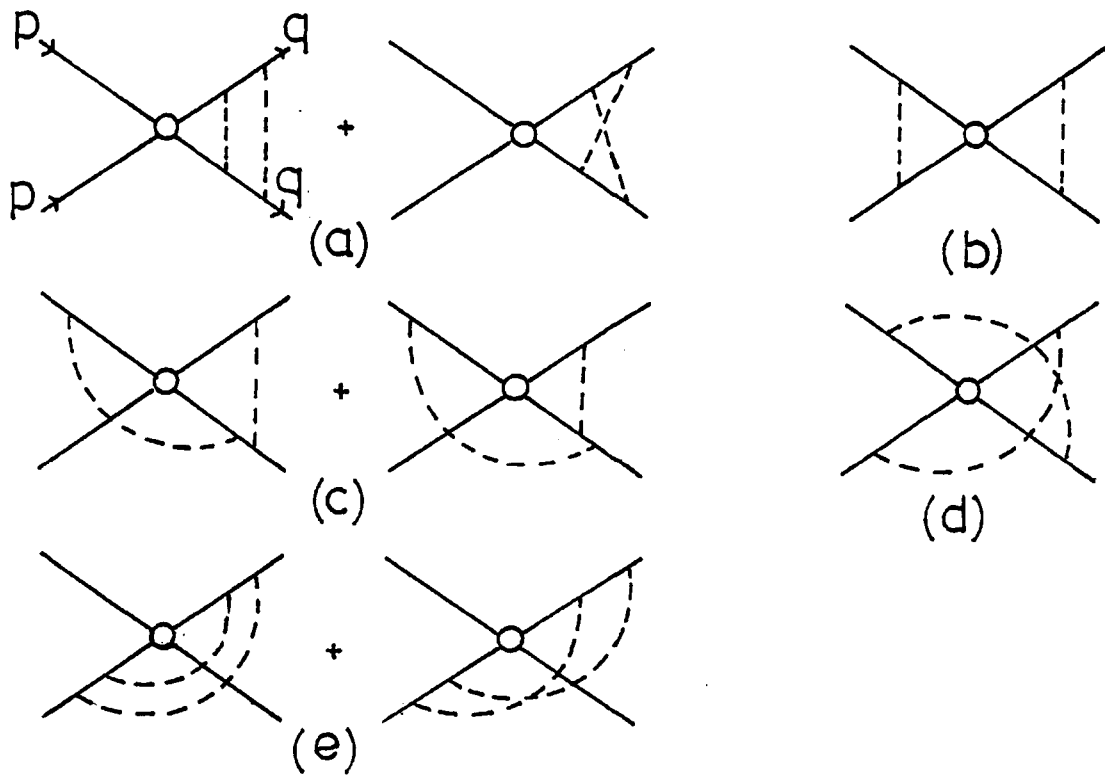


Fig. 3.8 Diagrams contributing to the abelian group theory factor, on taking their $\mathcal{S}$ -discontinuity. Graphs obtained by left - right inversion, and also up-down inversion, are also to be included.

orderings, so the integrals decouple, and everything can be expressed in terms of the following:

$$I_{\pm} = \int \frac{d^4 k}{(k^2 \pm i\epsilon)(2p \cdot k - i\epsilon)(-2q \cdot k - i\epsilon)}$$

$$I_{\delta}^{\pm} = 2\pi \int \frac{d^4 k \delta(k^2) \Theta(\pm k_0)}{(2p \cdot k)(-2q \cdot k)} \quad (3.35)$$

which satisfy

$$I_{-}^{*} = I_{+}, \quad I_{\delta}^{+} = I_{\delta}^{-} = I_{\delta}, \quad I_{+} - I_{-} = -2i I_{\delta}. \quad (3.36)$$

Note the delta function in the definition of $I_{\delta}^{\pm}$ makes $i\epsilon$'s irrelevant there. In terms of these integrals we find for the $\mathcal{B}$ -discontinuities of Fig. 3.8

$$(a) = -\frac{1}{2} I_{-}^2 \quad (b) = I_{+} I_{-} \quad (c) = i I_{-} I_{\delta}$$

$$(d) = I_{\delta}^2 \quad (e) = \frac{1}{2} I_{\delta}^2. \quad (3.37)$$

To evaluate the total abelian term we must add in the complex conjugate of (a) and (c), and the up-down inverses of (c) and (e).

Then the sum is easily seen to be zero, using (3.36).

We will now derive this result using our original method. All graphs contributing to the abelian factor contribute also to the non-abelian factor, so there are no new graphs to calculate. However, some of the graphical amplitudes now change sign, since the colour matrix factors make different contributions to the abelian and non-abelian terms. From Fig. 3.6 we require the s_R discontinuities of (a), (b), (e), (g) and (h), of which only (e) has a non-vanishing real part, as we saw in the previous section. From Fig. 3.5 we need (a) to (f) and (n) to (s) inclusive, of which (d) and (p) are pure imaginary, and (e) cancels (n) as before. Of the remaining graphs, the contribution of (a), (b), (c), (q) and (r) change sign relative to the calculation of the $C_2(G)$ term. Denoting the amplitudes corresponding to the A_i of the previous section by B_i , we now have

$$B_a = B_b = -A_a = -\frac{\pi^2}{2\beta^2} \left(\int \frac{d^{n-1}k}{k^2} S(k_z) \right)^2 \quad (3.38)$$

by (3.16, 22). For (c), (f), (o) and (q) we have

$$B_c = B_q = -A_c, \quad B_f = B_o = A_f, \quad (3.39)$$

so by (3.23) their total is

$$\begin{aligned} B_{c+f+o+q} &= \frac{1}{2} \int \frac{d^n k d^n k' S(2p \cdot (k+k')) S(2q \cdot k) (4p \cdot q)^2}{(k^2 + i\epsilon)(k'^2 + i\epsilon)(-2q \cdot (k+k') + i\epsilon)} \\ &\quad \times \left(\frac{-1}{2p \cdot k' + i\epsilon} + \frac{1}{2p \cdot k' - i\epsilon} \right) \end{aligned} \quad (3.40)$$

Using
$$\frac{-1}{2p \cdot k' + i\epsilon} + \frac{1}{2p \cdot k' - i\epsilon} = 2\pi i \delta(2p \cdot k')$$

and taking the real part by the replacement

$$\frac{1}{-2q \cdot (k+k') + i\epsilon} \rightarrow -i\pi \delta(2q \cdot (k+k'))$$

the other $i\epsilon$'s being irrelevant because of the δ -function constraints, the integrals decouple and we obtain

$$\beta_{c+f+o+q} = \frac{\pi^2}{\beta^2} \left(\int \frac{d^{n-1} \underline{k} \delta(k_z)}{k^2} \right)^2 = -2\beta_a. \quad (3.41)$$

Finally we need (e), (r) and (s).

$$\begin{aligned} \beta_e = \beta_r = \beta_s &= \frac{1}{4} \int \frac{d^n k d^n k' \delta(2p \cdot k') \delta(2q \cdot k) (4p \cdot q)^2}{(k^2 + i\epsilon)(k'^2 + i\epsilon)(2p \cdot (k+k') + i\epsilon)(-2q \cdot (k+k') + i\epsilon)} \\ &= \frac{(4p \cdot q)^2}{4} \int \frac{d^n k \delta(2q \cdot k)}{(k^2 + i\epsilon)(2p \cdot k + i\epsilon)} \times \int \frac{d^n k' \delta(2p \cdot k')}{(k'^2 + i\epsilon)(-2q \cdot k' + i\epsilon)} \end{aligned} \quad (3.42)$$

Symmetrizing under $k \rightarrow -k$ gives

$$\frac{1}{2p \cdot k + i\epsilon} \rightarrow -i\pi \delta(2p \cdot k).$$

Proceeding similarly with the k' integral, we find

$$\begin{aligned} \beta_e &= -\frac{\pi^2}{4} \left(4p \cdot q \int \frac{d^n k \delta(2p \cdot k) \delta(2q \cdot k)}{(k^2 + i\epsilon)} \right)^2 \\ &= -\frac{\pi^2}{4\beta^2} \left(\int \frac{d^{n-1} \underline{k} \delta(k_z)}{k^2} \right)^2 = \frac{\beta_a}{2} \end{aligned} \quad (3.43)$$

Thus everything is expressible in terms of β_a , and we can now evaluate the sum. Introducing factors of two for the inverses of (c), (f), (o), (q) and (r), and taking into account the equalities (3.38, 42), gives

$$\beta_{tot} = 2\beta_a + 2\beta_{c+f+o+q} + 4\beta_e = 0 \quad (3.44)$$

as required by our simpler QED-like argument.

3.5 DISCUSSION

Infrared divergences arise when we do perturbative calculations in all field theories with massless gauge bosons, because in such theories the interaction is a long range one which modifies the asymptotic states of the theory. Standard perturbation theory neglects this effect, and asymptotic states are assumed to be free particle states. Infrared divergences are a manifestation of this false assumption.

In QED this infrared problem is well understood. The long range force causes soft photon radiation to persist asymptotically, giving rise to asymptotic states containing an infinite number of soft photons. These are the so-called 'coherent' states [55, 56]. In practice however, it is usually simpler to do perturbation theory in the usual fashion, with free particle asymptotic states, and cancel the resulting infrared divergences via the Bloch-Nordsieck mechanism, which is known always to work in QED [31]. We can do this because the soft photons have no physical effect; they are undetectable.

The infrared structure of non-abelian gauge theories has long been known to be very much more complicated than in QED. Indeed, because of the conjectured confinement property, asymptotic states with quark and gluon quantum numbers are not even expected to exist in QCD. The additional complications arise because the gauge bosons of non-abelian theories are 'charged'; the gluon carries colour, and is therefore self-coupled. Thus in QCD if soft gluon radiation persists asymptotically, so also do the soft gluon self-interactions, leading to very severe infrared problems. This situation arises when we do perturbation theory on free quarks and gluons, because

long range confining effects are then not taken into account.

In QED the long range force has no physical effect other than to modify the asymptotic states, and we can in some sense ignore the soft photons, as described above. However, in QCD the persistence of soft gluon radiation means the long range force is a colour changing one, so the colour of a quark is not constant asymptotically. Perturbation theory with states of fixed colour takes no account of this. Furthermore, if it were possible to do scattering experiments with free quarks, we could not take quarks of fixed colour, since soft gluon radiation would be undetectable. Because of this it is usual to consider only colour averaged cross-sections in which we average over initial quark colours and sum over final colours, as has been done in this work. Indeed it is not difficult to find examples of non-cancelling infrared divergences if the colours are in some way restricted [28]. A more rigorous mathematical justification for this colour-averaging procedure has been given by Strocchi [57].

Note this colour changing effect is closely related to the colour oscillations derived by Lipkin, described in section 3.6. Lipkin's colour oscillations are a consequence of colour confinement. The infrared colour changing effects described here are a consequence of the non-abelian nature of the theory, which we hope leads to confinement.

Of course the real problem is that we do not know how to calculate properly in the infrared region. Strong, confining forces are expected to dominate here, and perturbation theory is not justified. Indeed, we have worked throughout with the fixed QCD coupling, and there is no reason to believe this should be small. However, we can usually get round these problems, since, because of the factorization property of hard scattering processes in QCD [20, 23], the long distance confining physics is believed to be independent of the short distance physics. One may then usefully apply perturbation theory to the short distance 'hard' process, where we have asymptotic freedom. However for this factorization to hold in general, it is necessary that all infrared divergences are cancelled. Our results therefore show that there may be problems associated with processes involving two hadrons in the initial state, such as the Drell-Yan [19] process. In a complete calculation, with quark confinement properly dealt with, it is plausible that the $n=4$ pole in our result will be replaced by a term like $\log R$, R being the spatial extent of the hadron. This

reflects the fact that confinement will suppress radiation on wavelengths greater than R . Such a term could be an additional source of large logarithms, and consequently spoil factorization. However, the coefficient of our uncanceled infrared divergence (3.3) is small at high energies, since

$$\frac{1-\beta}{\beta} \sim O\left(\frac{m^2}{s^2}\right) \quad \text{for} \quad s=(p+q)^2 \gg m^2,$$

so such a term may still be suppressed. Note terms suppressed at high energies have not yet been calculated in QCD; they are the "non-leading twist" terms.

We now consider the KLN theorem [58], which states that, in any field theory, all infrared divergences cancel when the appropriate sum over almost degenerate initial and final states is taken. We have allowed for soft gluons only in the final state, this being the situation most relevant physically. It has been demonstrated in [29] that cancellation does occur when soft gluons are included also in the initial state. Thus our result is consistent with the KLN theorem. Of course we do expect some incoming gluons in QCD, but there is no reason to expect their "flux" to be exactly as required for a KLN type cancellation.

We now mention some other possible ways 'round' the problem. It has been conjectured, and partially demonstrated [30], that the divergences may exponentiate when summed to all orders of perturbation theory. Then since

$$\lim_{n \rightarrow 4^+} \exp\left(-\frac{1}{n-4}\right) = 0$$

the divergences cancel. In this case the divergences should not spoil factorization.

Also, some attempt has recently been made to construct the QCD analogues of the 'coherent' states of QED [55, 56]. The Fadeev and Kulish [56] construction of the asymptotic states in QED has been

generalized to QCD [59], enabling 'asymptotic' states containing an infinite number of soft gluons to be defined in QCD perturbation theory. It has been shown [60] that, if the incoming quarks in the process we have considered are replaced by these coherent quark states, the divergent factor (3.3) is cancelled. However, it is not at all clear what is the physical significance of these states in QCD, since the only asymptotic states observed experimentally are the hadron states. The construction of [59] is perturbative, so non-perturbative long range effects, which one hopes lead to hadron formation, do not appear. Also, some points of the Fadeev and Kulish construction do not obviously carry over to the QCD case.

APPENDIX 3.1

We will here evaluate the $n=4$ poles of the integral (see(3.24)).

$$A_{c+f} = \frac{-1}{2\beta^2} \int \frac{d^{n-1} \underline{k} d^{n-1} \underline{k}' \exp -[(\underline{k}^2 + \underline{k}'^2)/\Lambda^2]}{(\beta^2 k_z^2 - \underline{k}^2)(\beta^2 k_z^2 - \underline{k}'^2 + i\epsilon)(k_z + k_z' + i\epsilon)} k_z \quad (3.45)$$

We write

$$\begin{aligned} \frac{1}{\beta^2 k_z^2 - \underline{k}^2} &= \int_0^\infty d\alpha e^{-\alpha(\underline{k}^2 - \beta^2 k_z^2)} \\ \frac{1}{k_z} &= \int_0^\infty d\gamma e^{-\gamma k_z^2} \\ \frac{1}{\beta^2 k_z^2 - \underline{k}'^2 + i\epsilon} &= -i \int_0^\infty d\mu e^{-i\mu(\underline{k}'^2 - \beta^2 k_z^2 + i\epsilon)} \\ \frac{1}{k_z + k_z' + i\epsilon} &= -i \int_0^\infty dv e^{iv(k_z + k_z' + i\epsilon)} \end{aligned} \quad (3.46)$$

This last equation allows us to replace k_z by $\frac{-i\partial}{\partial v}$ in the numerator of (3.45).

With the above substitutions the $\underline{k}$, $\underline{k}'$ integrals of (3.45) become pure Gaussians, so are straightforward to evaluate. We find

$$\begin{aligned}
A_{c+f} = & \frac{i\pi^{n-1}}{2\beta^2} \int_0^\infty d\alpha d\gamma d\mu dv \left(\alpha + \frac{1}{\lambda^2}\right)^{-\frac{(n-2)}{2}} \left(i\mu + \frac{1}{\lambda^2}\right)^{-\frac{(n-1)}{2}} \\
& \times \left(\frac{1}{\lambda^2} + \alpha(1-\beta^2) + \gamma - i\mu\beta^2\right)^{-1/2} \frac{\partial}{\partial v} \exp -\frac{v^2}{4}\left(\frac{1}{C} + \frac{1}{C'}\right)
\end{aligned}
\tag{3.47}$$

with $C = \frac{1}{\lambda^2} + i\mu$

and $C' = \frac{1}{\lambda^2} + \gamma + \alpha(1-\beta^2) - i\mu\beta^2$.

Here the $i\epsilon$'s have been discarded, since for $\mu, v \rightarrow \infty$ the integrals are convergent.

Evaluating the partial derivative, and then scaling α, μ, γ by $\frac{1}{\lambda^2}$, and v by $\frac{1}{\lambda}$ now gives

$$A_{c+f} = \frac{-i\pi^{n-1} \lambda^{2(n-4)}}{4\beta^2} \int_0^\infty \frac{d\alpha d\gamma d\mu dv}{(1+\alpha)^{\frac{(n-2)}{2}} C'^{3/2} C^{\frac{n-1}{2}}} v \exp -\left[\frac{v^2}{4}\left(\frac{1}{C} + \frac{1}{C'}\right)\right] \tag{3.48}$$

where now C, C' are as in (3.47) but with $\frac{1}{\lambda^2}$ replaced by 1.

The v integral can now be easily evaluated, giving

$$\begin{aligned}
A_{c+f} = & \frac{-i\pi^{n-1} \lambda^{2(n-4)}}{2\beta^2} \int_0^\infty d\alpha d\mu d\gamma (1+\alpha)^{-\frac{n-2}{2}} (1+i\mu)^{-\frac{n-1}{2}} \\
& \times C'^{-1/2} (C+C')^{-1}
\end{aligned}
\tag{3.49}$$

The μ contour may now be rotated to run just to the right of the negative imaginary axis, giving $\mu \rightarrow -i\mu$ in the above. We now have

$$A_{c+f} = \frac{-\pi^{n-1}}{2\beta^2} \Lambda^{2(n-4)} \int_0^\infty d\alpha d\mu d\gamma (1+\alpha)^{-\left(\frac{n-2}{2}\right)} (1+\mu)^{-\left(\frac{n-3}{2}\right)} \\ \times (1 + \alpha(1-\beta^2) + \gamma - \mu\beta^2)^{-\frac{1}{2}} (2 + \gamma + (\alpha+\mu)(1-\beta^2))^{-1} \quad (3.50)$$

We now do the γ integral, which is of the form

$$\int \frac{d\gamma}{(a+b\gamma)(A+B\gamma)^{1/2}} = 2 \tan^{-1} \left[\frac{b(A+B\gamma)}{(aB - Ab)} \right]^{1/2} \\ \times [b(aB - Ab)]^{-1/2} \quad (3.51)$$

for $\frac{aB - Ab}{b} > 0$, $A + B\gamma > 0$.

Since we only require the real part of this integral, the integration range is now restricted by the condition that the argument of the square root in (3.50) be positive. There are two allowed regions:

- a) $1 + \alpha(1-\beta^2) - \mu\beta^2 < 0$, $\gamma\beta^2 \geq \mu\beta^2 - 1 - \alpha(1-\beta^2)$;
 b) $1 + \alpha(1-\beta^2) - \mu\beta^2 > 0$, $\gamma \geq 0$.

Thus we find

$$A_{c+f} = \frac{-\pi^{n-1}}{\beta^2} \Lambda^{2(n-4)} (I + J) \quad (3.52)$$

where

$$I = \int_0^\infty d\alpha \int_0^\infty d\mu \frac{1 + \alpha(1-\beta^2)}{\beta^2} (1+\alpha)^{-\left(\frac{n-2}{2}\right)} (1+\mu)^{-\left(\frac{n-1}{2}\right)} \\ \times \left(\frac{\pi}{2} - \tan^{-1} \left[\frac{(1 + \alpha(1-\beta^2) - \mu\beta^2)}{(1+\mu)} \right]^{1/2} \right) \quad (3.53)$$

and

$$J = \frac{\pi}{2} \int_0^\infty d\alpha \int_{\frac{1+\alpha(1-\beta^2)}{\beta}}^\infty d\mu (1+\alpha)^{-\left(\frac{n-2}{2}\right)} (1+\mu)^{-\left(\frac{n-2}{2}\right)} \quad (3.54)$$

We first consider J. Doing the trivial μ integration gives

$$J = \frac{\pi}{(n-4)} \left(\frac{\beta^2}{1-\beta^2} \right)^{\frac{n-4}{2}} K, \quad (3.55)$$

where

$$K = \int_0^\infty d\alpha (1+\alpha)^{-\left(\frac{n-2}{2}\right)} (z+\alpha)^{-\left(\frac{n-4}{2}\right)} \quad (3.56)$$

with $z = \frac{1+\beta^2}{1-\beta^2} > 1.$

Now substituting $v = \alpha + \frac{(1+z)}{2}$ into K, we find

$$K = \frac{1}{2} \int_{\frac{1+z}{2}}^\infty dv^2 \left(1 + \frac{z-1}{2v} \right) \left(v^2 - \frac{(z-1)^2}{4} \right)^{-\left(\frac{n-2}{4}\right)} \quad (3.57)$$

The contribution from the second term in the numerator has no pole at $n=4$, so we may let $n=4$ here. Thus

$$\begin{aligned} K &\rightarrow \frac{1}{2} \int_{\frac{1+z}{2}}^\infty dv^2 \left(v^2 - \frac{(z-1)^2}{4} \right)^{-\left(\frac{n-2}{2}\right)} \\ &\quad + \frac{1}{2} (z-1) \int_{\frac{1+z}{2}}^\infty dv \left(v^2 - \frac{(z-1)^2}{4} \right)^{-1} \\ &= \frac{1}{n-4} \end{aligned} \quad (3.58)$$

Thus, using (3.55),

$$J = \frac{2\pi}{(n-4)^2} \left(1 + \frac{(n-4)}{2} \ln \frac{\beta^2}{1-\beta^2} \right) \quad (3.59)$$

where we have kept only $n=4$ pole terms, as usual.

We now consider I. In (3.53) we use

$$\frac{\pi}{2} - \tan^{-1} x = \cot^{-1} x,$$

and substitute for μ with

$$t^2 = (1 + \alpha(1-\beta^2) - \mu\beta^2)(1+\mu)^{-1}.$$

Then

$$I = 2 \int_0^\infty d\alpha (1+\alpha)^{-\frac{n-2}{2}} (1+\beta^2 + \alpha(1-\beta^2))^{-\frac{n-4}{2}} \\ \times \int_0^{[1+\alpha(1-\beta^2)]^{1/2}} dt \, t \, \cot^{-1} t \, (\beta^2 + t^2)^{\frac{n-6}{2}}. \quad (3.60)$$

The pole term clearly comes from the region $\alpha \rightarrow \infty$, so we need only consider $\alpha \geq \Delta$, where Δ is large but otherwise arbitrary. Now we interchange the order of integration, with

$$\int_\Delta^\infty d\alpha \int_0^{[1+\alpha(1-\beta^2)]^{1/2}} dt = \int_0^{[1+\Delta(1-\beta^2)]^{1/2}} dt \int_\Delta^\infty d\alpha + \int_{[1+\Delta(1-\beta^2)]^{1/2}}^\infty dt \int_{\frac{t^2-1}{1-\beta^2}}^\infty d\alpha$$

Since α is large the integrand of (3.60) simplifies and the α integrals are trivial. We obtain

$$\begin{aligned}
 I &= \frac{2}{n-4} \int_0^\infty dt \, t \, (\beta^2 + t^2)^{-1} \cot^{-1} t \\
 &= \frac{\pi}{n-4} \ln \left(\frac{1+\beta}{\beta} \right)
 \end{aligned}
 \tag{3.62}$$

by contour integration.

Now combining (3.52,59,62), we obtain the result

$$A_{c+f} = -\frac{\pi^n \Lambda^{2(n-4)}}{\beta^2 (n-4)^2} \times \left(1 + \frac{n-4}{2} \ln \frac{1+\beta}{1-\beta} \right)
 \tag{3.63}.$$

APPENDIX 3.2

Here we evaluate the $n=4$ poles in the sum of the integrals A_g, A_k (see (3.28, 29)). First consider A_g :

$$A_g = \frac{1}{2\beta^2} \int \frac{d^{n-1}k \, d^{n-1}k' \exp - [(k^2 + k'^2)/\Lambda^2]}{(\beta^2 k_z'^2 - k^2 + i\varepsilon)(\beta^2 k_z'^2 - k'^2)(k_z' + i\varepsilon)^2} \quad (3.64)$$

Rewriting denominators as before, we find

$$\begin{aligned} A_g &= -\frac{i}{2\beta^2} \int_0^\infty d\alpha d\mu dv \, v \, \int d^{n-1}k \, d^{n-1}k' \exp - \left\{ \right. \\ &\quad \left. \alpha(k'^2 - \beta^2 k_z'^2) + i\mu(k^2 - \beta^2 k_z'^2 - i\varepsilon) - i\nu(k_z' + i\varepsilon) + \frac{(k^2 + k'^2)}{\Lambda^2} \right\} \\ &= -\frac{i}{2\beta^2} \pi^{n-1} \int_0^\infty d\alpha d\mu dv \, v \, \left(\frac{1}{\Lambda^2} + i\mu \right)^{-\frac{(n-1)}{2}} \left(\frac{1}{\Lambda^2} + \alpha \right)^{-\frac{(n-2)}{2}} \\ &\quad \times C^{-1/2} \exp - \frac{v^2}{4C} \end{aligned} \quad (3.65)$$

where $C = \alpha(1 - \beta^2) + \frac{1}{\Lambda^2} - i\mu\beta^2$,

and the ε 's have been discarded because the above integrals are convergent.

We now do the trivial v integration, and then scale α, μ by $\frac{1}{\Lambda^2}$. This results in

$$A_g = -\frac{i\pi^{n-1}}{\beta^2} \Lambda^{2(n-4)} \int_0^\infty \frac{d\alpha d\mu \left(1 + \alpha(1 - \beta^2) - i\mu\beta^2 \right)^{1/2}}{(1 + i\mu)^{(n-1)/2} (1 + \alpha)^{(n-2)/2}} \quad (3.66)$$

We can now rotate the μ contour, with $\mu \rightarrow -i\mu$. The argument of the square root in (3.66) is now real, but of indefinite sign. Since we only want the real part of A_g we only consider the region where the square root has positive argument, i.e.

$$1 + \alpha(1 - \beta^2) - \mu\beta^2 \geq 0.$$

$$\text{So} \quad \mu \leq \frac{1 + \alpha(1 - \beta^2)}{\beta^2}. \quad (3.67)$$

We now have

$$\begin{aligned} A_g = & - \frac{\pi^{(n-1)/2}}{\beta^2} \Lambda^{2(n-1)} \int_0^\infty d\alpha (1 + \alpha)^{-(\frac{n-2}{2})} \\ & \times \int_0^{\frac{1 + \alpha(1 - \beta^2)}{\beta^2}} d\mu (1 + \alpha(1 - \beta^2) - \mu\beta^2)^{1/2} (1 + \mu)^{-(\frac{n-1}{2})}. \end{aligned} \quad (3.68)$$

At this point let us consider A_k (see (3.29)).

$$A_k = \frac{1}{2\beta^2} \int \frac{d^{n-1} \underline{k} d^{n-1} \underline{k}' \exp -[(\underline{k}^2 + \underline{k}'^2)/\Lambda^2]}{\underline{k}^2 (\beta^2 \underline{k}_2'^2 - \underline{k}'^2) (\underline{k}_2' + i\varepsilon)^2} \quad (3.69)$$

The momentum integrals here have decoupled. We have

$$\begin{aligned} \int \frac{d^{n-1} \underline{k}}{\underline{k}^2} e^{-\underline{k}^2/\Lambda^2} &= \int_0^\infty d\mu \int d^{n-1} \underline{k} \exp -[\underline{k}^2 (\frac{1}{\Lambda^2} + \mu)] \\ &= \pi^{\frac{n-1}{2}} \int_0^\infty d\mu \left(\frac{1}{\Lambda^2} + \mu \right)^{-(\frac{n-1}{2})} \end{aligned} \quad (3.70)$$

and

$$\begin{aligned}
 & \int \frac{d^{n-1} \underline{k}' e^{-\underline{k}'^2/\Lambda^2}}{(\beta^2 k_z'^2 - \underline{k}'^2)(k_z' + i\epsilon)}^2 = \int_0^\infty d\alpha d\nu \cdot \nu \cdot \int d^{n-1} \underline{k}' \\
 & \times \exp - \left\{ \frac{\underline{k}'^2}{\Lambda^2} - i\nu(k_z' + i\epsilon) + \alpha(\underline{k}'^2 - \beta^2 k_z'^2) \right\} \\
 & = \pi^{\frac{n-1}{2}} \int_0^\infty d\alpha d\nu \cdot \nu \left(\alpha + \frac{1}{\Lambda^2} \right)^{-(\frac{n-2}{2})} \left(\alpha(1-\beta^2) + \frac{1}{\Lambda^2} \right)^{-1/2} \\
 & \times \exp - \left[\frac{\nu^2}{4 \left[\alpha(1-\beta^2) + \frac{1}{\Lambda^2} \right]} \right] \\
 & = 2\pi^{\frac{n-1}{2}} \int_0^\infty d\alpha \left(\frac{1}{\Lambda^2} + \alpha \right)^{-(\frac{n-2}{2})} \left(\frac{1}{\Lambda^2} + \alpha(1-\beta^2) \right)^{1/2}.
 \end{aligned} \tag{3.71}$$

Combining the above three equations, and scaling α, μ by $\frac{1}{\Lambda^2}$ we now find

$$A_K = \frac{\pi^{n-1}}{\beta^2} \Lambda^{2(n-4)} \int_0^\infty \frac{d\alpha d\mu (1 + \alpha(1-\beta^2))^{1/2}}{(1+\alpha)^{(n-2)/2} (1+\mu)^{(n-1)/2}} \tag{3.72}$$

Comparing this with (3.68) for A_g we see two very similar expressions. Combining the two we obtain

$$\begin{aligned}
 A_g + A_K &= -\frac{\pi^{n-1}}{\beta^2} \Lambda^{2(n-4)} \int_0^\infty \frac{d\alpha}{(1+\alpha)^{(n-2)/2}} \times \int_0^\infty \frac{d\mu}{(1+\mu)^{(n-1)/2}} \times \left\{ \right. \\
 & \left. (1 + \alpha(1-\beta^2) - \mu\beta^2)^{1/2} \Theta(1 + \alpha(1-\beta^2) - \mu\beta^2) - (1 + \alpha(1-\beta^2))^{1/2} \right\}.
 \end{aligned} \tag{3.73}$$

Now make the substitution $\mu \rightarrow \frac{\mu \beta^2}{1 + \alpha(1 - \beta^2)}$. We find

$$A_g + A_k = -\frac{\pi}{\beta} \Lambda^{2(n-4)} \int_0^\infty \frac{d\alpha}{(1+\alpha)^{(n-2)/2}} \left(\frac{1 + \alpha(1 - \beta^2)}{\beta^2} \right)^{-\left(\frac{n-4}{2}\right)} \\ \times \int_0^\infty \frac{d\mu}{(\mu + \varepsilon)^{(n-1)/2}} \times \left((1-\mu)^{1/2} \theta(1-\mu) - 1 \right), \quad (3.74)$$

where $\varepsilon = \beta^2 (1 + \alpha(1 - \beta^2))^{-1}$.

We now see that each separate term in the μ integral has an unphysical pole at $n=3$, coming from the region $\mu \rightarrow 0$, but these poles cancel in the sum. There is also an $n=4$ pole coming from the region $\alpha \rightarrow \infty$, so we need only consider large α . Consequently $\varepsilon \ll 1$, and the μ integral is essentially independent of α in the vicinity of $n=4$. It is also convergent here, so we set $n=4$ and evaluate

$$\int_0^\infty \frac{d\mu}{(\mu + \varepsilon)^{3/2}} \left((1-\mu)^{1/2} \theta(1-\mu) - 1 \right) = -\pi + O(\varepsilon). \quad (3.75)$$

The $O(\varepsilon)$ terms do not contribute to the $n=4$ pole, as anticipated. Note it would be wrong to set $n=4$ here if the μ integral were strongly α dependent for large α , since this could affect the n dependence of the resulting α integral.

Finally, for large α we have

$$A_g + A_k = \frac{\pi}{\beta} \Lambda^{2(n-4)} \left(\frac{1 - \beta^2}{\beta^2} \right)^{-\frac{n-4}{2}} \times \int_0^\infty d\alpha \alpha^{-(n-3)} \\ = \frac{\pi}{\beta} \frac{1}{n-4} \quad (3.76)$$

APPENDIX 3.3

Here we evaluate the poles at $n=4$ of

$$A_m = \frac{1}{2} \int \frac{d^{n-1} \underline{k} d^{n-1} \underline{k}' (k'_z - 2k_z) \exp - [(k^2 + k'^2)/\Lambda^2]}{k^2 (\beta^2 k_z'^2 - k'^2) (\beta^2 k_z'^2 - (k' - k)^2 + i\varepsilon) (k'_z + i\varepsilon)} \quad (3.77)$$

Re-writing denominators as before, we find

$$A_m = \frac{1}{2} \int_0^\infty d\alpha d\gamma d\delta d\mu \int d^{n-1} \underline{k} d^{n-1} \underline{k}' (k'_z - 2k_z) \\ \times \exp - \left\{ \frac{(k^2 + k'^2)}{\Lambda^2} + \alpha k^2 + \gamma (k'^2 - \beta^2 k_z'^2) - i\delta (k'_z + i\varepsilon) \right. \\ \left. - i\mu (\beta^2 k_z'^2 - (k - k')^2 + i\varepsilon) - i\sigma k_z \right\} \Big|_{\sigma=0} \quad (3.78)$$

Where we have introduced σ so that we may let

$$(k'_z - 2k_z) \rightarrow -i \left(\frac{\partial}{\partial \delta} - 2 \frac{\partial}{\partial \sigma} \right) \quad (3.79)$$

in the numerator of the above. Then the momentum integrals become pure Gaussians, and we obtain

$$A_m = -\frac{i\pi^{n-1}}{2} \int_0^\infty \frac{d\alpha d\gamma d\delta d\mu}{C^{(n-2)/2} C'^{1/2}} \left(\frac{\partial}{\partial \delta} - 2 \frac{\partial}{\partial \sigma} \right) \exp - \left(\frac{D}{C'} \right) \Big|_{\sigma=0} \quad (3.80)$$

Here C and C' are the determinants of the Gaussians in k_i, k_i' for $i \neq z$, and k_z, k_z' respectively:

$$C = (\alpha + \frac{1}{\Lambda^2})(\gamma + \frac{1}{\Lambda^2}) + i\mu(\alpha + \gamma + \frac{2}{\Lambda^2})$$

$$C' = (\frac{1}{\Lambda^2} + \alpha + i\mu)(\frac{1}{\Lambda^2} + (\gamma + i\mu)(1 - \beta^2)) + \mu^2 \quad (3.81)$$

and $\mathcal{D} = \frac{\delta^2}{4} \left(\frac{1}{\Lambda^2} + \alpha + i\mu \right) + \frac{i\mu\sigma\delta}{2} + \text{term quadratic in } \sigma$

We have here discarded the ϵ 's in the exponent, since the integrals in (3.80) are clearly convergent. We now evaluate the partial derivatives and do the integration. We have

$$\int_0^\infty d\delta \frac{\partial}{\partial \delta} e^{-\mathcal{D}/C'} \Big|_{\sigma=0} = - e^{-\mathcal{D}/C'} \Big|_{\sigma=\delta=0} = -1$$

and

$$\int_0^\infty d\delta \frac{\partial}{\partial \sigma} e^{-\mathcal{D}/C'} \Big|_{\sigma=0} = -\frac{i\mu}{2C'} \int_0^\infty d\delta \delta \exp -\frac{\delta^2}{4} \left[\frac{1}{\Lambda^2} + \alpha + i\mu \right]$$

$$= -i\mu \left(\frac{1}{\Lambda^2} + \alpha + i\mu \right)^{-1} \quad (3.82)$$

Now scaling α, μ, γ by $\frac{1}{\Lambda^2}$, (3.80) becomes

$$A_m = \frac{i\pi^{n-1}}{2} \Lambda^{2(n-4)} \int_0^\infty \frac{d\alpha d\gamma d\mu}{C^{(n-2)/2} C'^{1/2}} \frac{(1 + \alpha + i\mu)^{-1} (1 + \alpha - i\mu)}{C^{(n-2)/2} C'^{1/2}} \quad (3.83)$$

with C, C' as in (3.81) but with $\frac{1}{\Lambda^2}$ replaced by 1.

We can now rotate the contour to run just to the right of the negative imaginary axis, giving $\mu \rightarrow -i\mu$ in the above. C' now becomes

$$C' = (1 + \alpha + \mu)(1 + (\gamma + \mu)(1 - \beta^2)) - \mu^2 \quad (3.84)$$

Since we only want the real part of the integral we restrict the integration range to $C' \geq 0$. Now since the n dependent factor $C^{(n-1)/2}$ has no zero in this integration region, the only $n=4$ singularity can come from the $\alpha, \gamma, \mu \rightarrow \infty$ region. We therefore introduce into A_m the factor

$$1 = \int_0^\infty d\rho \delta(\rho - \alpha - \gamma - \mu) \quad (3.85)$$

and scale $\alpha, \gamma, \mu \rightarrow \rho\alpha, \rho\gamma, \rho\mu$. Now the singularities come only from the large ρ region, so we need only consider $\rho \gg 1$. This gives

$$\begin{aligned} A_m \approx & \frac{\pi^{n-1}}{2} \Lambda^{2(n-4)} \int_\Delta^\infty d\rho \rho^{3-n} \int_0^1 d\alpha d\gamma d\mu \delta(1 - \alpha - \gamma - \mu) \\ & \times (\alpha\gamma + \mu(\alpha + \gamma))^{-\frac{(n-2)}{2}} ((\alpha + \mu)(\gamma + \mu)(1 - \beta^2) - \mu^2)^{-1/2} \\ & \times (\alpha - \mu)(\alpha + \mu)^{-1}, \end{aligned} \quad (3.86)$$

where Δ is large but otherwise arbitrary. Now the ρ integral gives the $n=4$ pole, so for the residue we may set $n=4$ elsewhere. Thus

$$\begin{aligned} A_m \rightarrow & \frac{\pi^3}{2(n-4)} \int_0^1 d\alpha d\gamma d\mu \delta(1 - \alpha - \gamma - \mu) (\alpha\gamma + \mu(\alpha + \gamma))^{-1} \\ & \times ((\alpha + \mu)(\gamma + \mu)(1 - \beta^2) - \mu^2)^{-1/2} (\alpha - \mu)(\alpha + \mu)^{-1}. \end{aligned} \quad (3.87)$$

We now write

$$[(\alpha + \gamma)\mu + \alpha\gamma]^{-1} = 2 \int_0^\infty d\rho \rho \exp - \rho^2 [(\alpha + \gamma)\mu + \alpha\gamma] \quad (3.88)$$

and scale α, γ, μ by ρ^{-1} . (3.87) becomes

$$A_m = \frac{\pi^3}{n-4} \int_0^\infty \frac{d\alpha d\gamma d\mu \exp - [(\alpha+\gamma)\mu + \alpha\gamma]}{[(\alpha+\gamma)(\mu+\gamma)(1-\beta^2) - \mu^2]^{1/2}} \times \frac{(\alpha-\mu)}{(\alpha+\mu)} \quad (3.89)$$

We now scale α, μ by μ and do the μ integration, finding

$$A_m = \frac{\pi^3}{2(n-4)} \int_0^\infty \frac{d\alpha d\gamma (\alpha-1)}{[(\alpha+1)(\gamma+1)(1-\beta^2) - 1]^{1/2} (\alpha+1) (\alpha+\gamma+\alpha\gamma)} \quad (3.90)$$

Now substitute $x = \alpha + 1$, $y = (\alpha + 1)(\gamma + 1)$. Then

$$\begin{aligned} A_m &= \frac{\pi^3}{2(n-4)} \int_1^\infty \frac{dx}{x^2} (x-2) \int_x^\infty \frac{dy}{(y(1-\beta^2)-1)^{1/2} (y-1)} \\ &= \frac{\pi^3}{2(n-4)} \int_1^\infty \frac{dy}{[y(1-\beta^2)-1]^{1/2} [y-1]} \int_1^y \frac{dx}{x^2} (x-2) \\ &= \frac{\pi^3}{2(n-4)} \int_1^\infty \frac{dy}{[y(1-\beta^2)-1]^{1/2} [y-1]} \left[\ln y + \frac{2}{y} - 2 \right] \\ &= \frac{\pi^3}{2(n-4)} (I_1 + I_2), \end{aligned} \quad (3.91)$$

where I_1 corresponds to the contribution of the $\ln y$ term, and I_2 to the remainder. Substituting $t^2 = y(1-\beta^2)-1$, we find

$$\text{Re } I_2 = -4 \int_0^\infty \frac{dt}{(t^2+1)} = -2\pi \quad (3.92)$$

$$\begin{aligned}
\text{and } \operatorname{Re} I_1 &= 2 \int_0^{\infty} dt \ln \left(\frac{t^2+1}{1-\beta^2} \right) \frac{1}{(t^2+\beta^2)} \\
&= \int_{-\infty}^{\infty} dt \ln \left[\frac{(1+it)(1-it)}{(1-\beta^2)} \right] \times \frac{1}{\beta^2+t^2} \\
&= 2 \operatorname{Re} \int_{-\infty}^{\infty} dt \ln \left[\frac{1+it}{(1-\beta^2)^{1/2}} \right] \times \frac{1}{\beta^2+t^2} \\
&= 2 \operatorname{Re} \left\{ 2\pi i \ln \left[\frac{1+\beta}{(1-\beta^2)^{1/2}} \right] \frac{1}{2i\beta} \right\} \\
&= \frac{\pi}{\beta} \ln \frac{1+\beta}{1-\beta} .
\end{aligned} \tag{3.93}$$

where we have closed the contour in the upper half plane, picking up the pole at $t = i\beta$.

Thus finally, combining the above three equations gives

$$A_m = \frac{\pi^4}{2(n-4)} \times \left(-2 + \frac{1}{\beta} \ln \left| \frac{1+\beta}{1-\beta} \right| \right) \tag{3.94}$$

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