Continuous spectrum or measurable reducibility for quasiperiodic cocycles in $\mathbb{T}^d \times SU(2)$

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Abstract

We continue our study of the local theory for quasiperiodic cocycles in $\mathbb{T}^d \times G$, where $G = SU(2)$, over a rotation satisfying a Diophantine condition and satisfying a closeness-to-constants condition, by proving a dichotomy between measurable reducibility (and therefore pure point spectrum), and purely continuous spectrum in the space orthogonal to $L^2(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d \times G)$. Subsequently, we describe the equivalence classes of cocycles under smooth conjugacy, as a function of the parameters defining their K.A.M. normal form. Finally, we derive a complete classification of the dynamics of one-frequency ($d = 1$) cocycles over a Recurrent Diophantine rotation.

All theorems will be stated sharply in terms of the number of frequencies d, but in the proofs we will always assume $d = 1$, for simplicity in expression and notation.

Contents

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1 Introduction and statement of the results

This article continues the work taken up by the author in [Kar15], [Kar14a] and [Kar14b]. This work used and developed the techniques of [Kri99a], [Kri01], [Eli02] and [Fra04] among others, and this part was motivated by [dA13], where conditions for the existence of absolutely continuous spectrum are given.

We think that the K.A.M.-theoretical, or local, part of the theory of such dynamical systems from the topological point of view is concluded with the following theorem and with the classification and path connectedness theorems that we state later on. On the other hand, the metric abundance of reducible cocycles in the C^∞ category is an open question (see [Kri99a] for the proof of this theorem in the analytic category).

Theorem 1.1. *Let $\alpha \in \mathbb{T}^d$ satisfy a Diophantine condition $DC(\gamma, \tau)$. Then, there exist $\varepsilon > 0$ and $s_0 \in \mathbb{N}^*$, depending on d, γ, τ, such that every cocycle $(\alpha, A.e^{F(\cdot)})$ with $A \in G = SU(2)$, $\|F\|_0 < \varepsilon$ and $\|F\|_{s_0} < 1$ satisfies the following dichotomy:*

1. *either the cocycle is measurably reducible and therefore has pure-point spectrum,*
2. *or the spectrum of the associated Koopman operator is purely continuous in the space orthogonal to $L^2(\mathbb{T}^d) \hookrightarrow L^2(\mathbb{T}^d \times G, \mathbb{C})$. Such cocycles are weak mixing in the fibers, and they are not strong mixing.*

The neighborhood of constants described in the statement of the theorem will be referred to as the *K.A.M.* or the *local regime* and denoted by $\mathcal{N}$. Conjugations of the size of those produced by the K.A.M. scheme when the product thus constructed converges, or of those who reduce a given cocycle to its K.A.M. normal form, satisfy a condition of the type $|Y|_0 < C$ and $|Y|_{s_0 - \xi\gamma} < 1$, for some constants $\epsilon \leq C < 1$ and $1 \leq \xi < s_0/\gamma$ depending on γ, τ and d. Conjugations satisfying such estimates will be referred to as *close-to-the-identity conjugations*. They form a contractible set in $C^\infty(\mathbb{T}^d, G)$ denoted by $\mathcal{V}$.

We make the following remarks. Firstly, measurable reducibility, as in item 1, is equivalent to a condition which is explicit in terms of the output of the K.A.M. scheme with the parameters of theorem 1.1, (cf. [Kar14b] or the discussion below). The question of differentiable rigidity of measurable reducibility was investigated in [Kar14a], where we proved that measurable reducibility to a full measure set of constants $DC_\alpha \subset G$ implies in fact smooth reducibility (see thm 1.6 below). On the other hand, in [Kar14b] it was shown that for constants in the generic set $\mathcal{L}_a = G \setminus DC_\alpha$, reducibility is not rigid. Moreover, the construction of the measurable transfer function under the relevant condition shows that its form is quite special. The assumption that a measurable conjugation reduces a smooth cocycle imposes some constraints on the former, which consequently will not be very wild.

Secondly, the non-existence of a measurable conjugation reducing the coycle to a constant, as in item 2, is equivalent to the complementary condition to the one in item 1, and thus equally explicit. In the complementary space of the one bearing the purely continuous spectrum (i.e. in $L^2(\mathbb{T}^d)$, the Kronecker factor of the cocycle), the spectrum is pure-point, because of the quasiperiodic dynamics within the torus.

The proof is based on the use of the K.A.M. normal form, introduced in [Kar14b], in order to prove that the existence of an eigenfunction of the Koopman operator associated to a given cocycle implies the condition for measurable reducibility, provided that the eigenfunction depends non-trivially on the variable in the fibers. The K.A.M. normal form, though not essential to the proof, greatly simplifies the calculations and elucidates the geometry of the problem. The existence of the K.A.M. normal form is a corollary of the almost reducibility theorem, first obtained in [Eli02], and its function in our study, put informally, is to separate the close-to-the-identity part of the conjugation (the one in $\mathcal{V}$) from the far-from-the-identity part, both constructed almost exactly as in [Eli02], and keep the second which contains all the interesting information. The purpose of the subsequent analysis is to establish the (necessary and sufficient, as it turns out) conditions under which some rearrangement of these far-from-the-identity conjugations converges in some function space. The (necessary and sufficient) condition for measurable reducibility that we referred to above, in this context, is that the angles θ_i between the successive far-from-the-identity conjugations at the steps i and $i+1$ of the scheme be square summable. Still informally, θ_i estimates the commutator appearing in the following sequence of operations: solve an equation in the coordinates of the i-th step of the scheme, make one step of scheme, solve the same equation with greater precision, undo the change of coordinates and check the compatibility of the expressions. We remark that cohomology in $C^\infty(\mathbb{T}, G)$ over the rotation $x \mapsto x + \alpha$ seems to demand only one step of the scheme (see the proof of thms 1.1 and 1.2 in [Kar14b] where the rearrangement of the far-from-the-identity conjugations is described) for the estimate to be effective, while cohomology in C^∞ or in $L^2(\mathbb{T}^d \times G, \mathbb{C})$ over any given cocycle seems to demand two steps of the scheme (see e.g. the proof of thm 2 herein).

The proof that non-reducible cocycles are not strong mixing uses the fact that a dynamical system is stong mixing iff the iterates of the Koopman operator associated to it converge to 0 in the weak operator topology. This property is incompatible with rigidity, in the sense that for every cocycle $(\alpha, A(\cdot))$ in the local regime, there exists a sequence of iterates $m_j \to \infty$ such that

$$(\alpha, A(\cdot))^{m_j} \overset{C^\infty}{\to} (0, Id)$$

Combining the results obtained in the literature mentioned in the beginning of this section with thm 1.1, we obtain the following picture for the dynamics of cocycles in $\mathbb{T}^d \times G$, close to a constant, supposing that we have run the K.A.M. scheme ([Kri99a], [Eli02], [Kar14a], [Kar14b], [Kar15]).

Theorem 1.2 ([Kri99a],[Eli02],[Kar14b]). *Every cocycle in the K.A.M. regime, $\mathcal{N}$, is almost reducible, and conjugate to a cocycle in K.A.M. normal form.*

The next theorems concern the reducibility of cocycles, via transfer functions of regularity L^2, C^∞, or intermediate Sobolev regularity $H^s, 0 < s < \infty$.

Theorem 1.3 ([Kar14b]). *A given cocycle is measurably reducible iff the angles of successive far-from-the-identity conjugations are summable in ℓ^2. We can then construct a sequence of C^∞ smooth conjugations converging in L^2 toward a reducing conjugation.*

The following is a corollary of the proof of the previous theorem.

Corollary 1.4 ([Kar14b]). *If the angles are in fact summable in higher regularity $h^\sigma, 0 < \sigma \leq \infty$ (i.e. if $\{N_{n_i}^\sigma \theta_i\} \in \ell^2$), then the cocycle is H^σ-reducible and the conjugation constructed as above converges in H^σ.*

A particular case of the corollary is that of the occurrence of a finite number of far-from-the-identity conjugations, where summability in every H^σ is trivial. This always occurs when the cocycle is reducible to a constant which is "more Diophantine than the frequency" (cf. [Kar14a]).

Theorem 1.5 ([Kar14b]). *For any $\sigma, 0 \leq \sigma \leq \infty$, cocycles that are reducible in any given regularity H^σ, and not any higher, are dense in $\mathcal{N}$. Reducibility in any given regularity is an F_σ condition.*

The following theorem states that for "a generic" reducible cocycle measurable conjugation is not rigid, while "for almost every" one, it is.

Theorem 1.6 ([Kar14a], [Kar14b]). *Reducibility in regularity $0 \leq \sigma < \infty$ can only occur when the constant cocycle (α, A) to which the cocycle is reduced is Liouville with respect to α, i.e. when $A \in \mathcal{L}_\alpha$. If the constant cocycle is Diophantine with respect to α, i.e. if $A \in DC_\alpha = G \setminus \mathcal{L}_\alpha$, then measurable reducibility implies smooth reducibility.*

The following theorems, starting with thm 1.1 concern non-reducible cocycles. Herein, we prove that all cocycles that are not measurably reducible are weak mixing in the fibers, and therefore uniquely ergodic in the whole space for the product of the Haar measures. Non-reducibility is a G_δ condition, and is also dense in $\mathcal{N}$. This theorem strengthens H. Eliasson's theorem in two ways. The condition we obtain is striclty looser than the one given by H. Eliasson, as well as being optimal. Moreover, we prove that the cocycles satisfying this condition are not only uniquely ergodic, but in fact weak mixing. Generic cocycles have a stronger property of ergodicity.

Theorem 1.7 ([Kar14b]). *Distributional unique ergodicity is also equivalent to a G_δ-dense condition, which is stricter than the one for weak mixing.*

We refer the reader unfamiliar with the notions to [Kar14b] and to the bibliography therein for the definition of DUE and the related concepts.

Distributional unique ergodicity is equivalent to an explicit condition on the asymptotic repartition of the angles between successive far-from-the-identity conjugations. Relaxation of this condition in a controlled way gives the following theorem.

Theorem 1.8 ([Kar14b]). *In the border between unigue ergodicity in the space of distributions and in the classical sense, countably infinite invariant distributions of arbirtarily high orders are created.*

We have also proved that DUE cocycles are not Cohomologically Stable, since the following theorem holds. The Diophantine condition of the theorem is stricter than DC_α, see paragraph 2.3.

Theorem 1.9 ([Kar14b]). *A cocycle in $\mathcal{N}$ is cohomologically stable iff it is C^∞ reducible to a constant cocycle (α, A) inducing a Diophantine rotation on its invariant tori $\mathcal{T}_{(\alpha,A)} \approx \mathbb{T}^d \times \mathbb{S}^1 \hookrightarrow \mathbb{T}^d \times G$.*

The following theorems concern the topology of the different conjugacy classes, and the way they lie in $\mathcal{N}$.

Theorem 1.10. *Given any cocycle $(\alpha, A'(\cdot)) \in \mathcal{N}$, one can construct a continuous path $[0,1] \to \mathcal{N}$ such that $(\alpha, A_0(\cdot)) = (\alpha, Id)$, for all $t \in [0,1)$ the cocycle $(\alpha, A_t(\cdot))$ is C^∞ reducible, and $(\alpha, A_1(\cdot))$ is the K.A.M. normal form of $(\alpha, A'(\cdot))$.*

The path is in fact piecewise C^∞. If we allow the path to exit the K.A.M. regime, we can obtain more.

Theorem 1.11. *Given any cocycle $(\alpha, A'(\cdot)) \in \mathcal{N}$, one can construct a continuous path $[0,1] \to SW^\infty_\alpha(\mathbb{T}^d, G)$ such that $(\alpha, A_0(\cdot)) = (\alpha, Id)$, for all $t \in [0,1)$ the cocycle $(\alpha, A_t(\cdot))$ is C^∞ conjugate to (α, Id), and $(\alpha, A_1(\cdot))$ is the K.A.M. normal form of $(\alpha, A'(\cdot))$.*

The path $\gamma\colon [0,1) \to C^\infty(\mathbb{T}^d, G)$ acting on (α, Id) and producing

$$(\alpha, A_t(\cdot)) = Conj_{\gamma(t)}(\alpha, Id)$$

is continuous (in fact piecewise C^∞), and degenerates in a prescribed way when the target cocycle is not C^∞ reducible. At time $t = 1^-$ it may exit C^∞ into a function space of lower regularity, or a space of distributions[1]. The path in the conjugacy space may exit $\mathcal{V}$, and in general will do so. The path in SW^∞_α will consequently exit $\mathcal{N}$ and reenter, in general an infinite number of times. Since the space of conjugations taking a cocycles to their respective normal forms is contractible (it is the space $\mathcal{V}$), we immediately obtain the following corollary.

Corollary 1.12. *The target of the path can be the cocycle $(\alpha, A'(\cdot))$ itself, while the same as in theorem 1.10 (resp. thm 1.11) holds for all times $t \in [0,1)$.*

[1] In fact the limit is always well defined in $H^{-d/2}(\mathbb{T}^d, G)$.

We can in fact obtain the following, stronger theorem, which establishes a way in which the topology of any two classes share some properties.

Theorem 1.13. *Given any two cocycles $(\alpha, A_0(\cdot))$ and $(\alpha, A'(\cdot))$, in $\mathcal{N}$, one can construct a continuous path $[0,1] \to \mathcal{N}$, and such that:*

1. *for every $t \in [0,1)$ the normal form of the cocycle $(\alpha, A_t(\cdot))$ has the same as tail as that of $(\alpha, A_0(\cdot))$, and thus the same dynamical properties.*
2. *$(\alpha, A_1(\cdot))$ is the K.A.M. normal form of $(\alpha, A'(\cdot))$, and thus conjugate to it via a conjugation in $\mathcal{V}$.*

As before, the target cocycle can be the cocycle $(\alpha, A'(\cdot))$.

Informally, we can connect any two types of dynamical behavior without exiting $\mathcal{N}$. The following theorem says that if we allow the path to exit $\mathcal{N}$, we can do the same thing with conjugacy classes.

Theorem 1.14. *Given any two cocycles $(\alpha, A_0(\cdot))$ and $(\alpha, A'(\cdot))$, in $\mathcal{N}$, one can construct a continuous path $[0,1] \to SW^{\infty}_{\alpha}$ such that:*

1. *for every $t \in [0,1)$, the cocycle $(\alpha, A_t(\cdot))$ is conjugate to $(\alpha, A_0(\cdot))$.*
2. *$(\alpha, A_1(\cdot))$ is the K.A.M. normal form of $(\alpha, A'(\cdot))$, and thus conjugate to it via a conjugation in $\mathcal{V}$.*

As before, the target cocycle can be the cocycle $(\alpha, A'(\cdot))$.

Again, a path in the space of conjugations acting on $(\alpha, A_0(\cdot))$ is constructed with the same properties as in theorem 1.11.

These two last theorems illustrate the necessity of a transversality condition for obtaining a full-measure reducibility theorem for one-parameter families of cocycles as in [Kri99b] or [Kri99a]. On the other hand, the K.A.M. normal form should be expected to depend badly on parameters along a generic family, so our approach is not expected to be well adapted to the metric point of view.

Finally, we give a satisfactory classification of conjugation classes.

Theorem 1.15. *Two cocycles in K.A.M. normal form are C^{∞} conjugate iff the parameters defining the normal forms satisfy the following properties.*

1. *The resonant steps n_i^j, $j = 1, 2$, are the same, except for a finite number.*
2. *The angles between successive conjugations*

$$\theta_i^j = \arctan \frac{|\hat{F}_i^j(k_i^j)|}{|\epsilon_i^j|}, j = 1, 2$$

are equal up to $O(N_{i-1}^{-\infty})$.

3. *The "rotation numbers"*

$$a_i^j = k_i^j \alpha \mod \mathbb{Z} \simeq \sqrt{|\hat{F}_{i-1}^j(k_{i-1}^j)|^2 + (\epsilon_{i-1}^j)^2}, j = 1,2$$

satisfy

$$a_i^1 - a_i^2 = \tilde{k}_i \alpha \mod \mathbb{Z}$$

and

$$k_i^1 - k_i^2 = \tilde{k}_i$$

Here, $\tilde{k}_i \in \mathbb{Z}$ has to be such that $N_{i-1} < k_i^j \leq N_i, j = 1,2$, and $\tilde{k}_i \in \mathbb{Z}^*$ are such that

$$\{|\tilde{k}_i|^s \theta_{i-1}\} \in \ell^2, \forall s > 0$$

4. *The arguments of $\hat{F}_i^j(k_i^j) \in \mathbb{C} \setminus \{0\}$, φ_i, are equal up to $O(N_{i-1}^{-\infty})$.*

Thus, the cocycles in the K.A.M. regime are parametrized by the action of conjugations in $\mathcal{V}$ composed with constant ones on the right, and the parameters θ_i, a_i and φ_i, that have nonetheless to respect the limitations of a K.A.M. scheme with given parameters.

Classification up to H^σ conjugation is obtained by replacing the $O(N_i^{-\infty})$ by the respective $O(N_i^{-\sigma})$ ones.

Of course, the K.A.M. scheme has some tolerance with respect to the size of some of the parameters. For example, the inequality $N_{i-1} < k_{i+1}^j \leq N_i, j = 1,2$ can be violated to a certain extent without any significant consequences, so this classification should be taken with a grain of salt.

We observe that every representative of a constant cocycle (α, A_d) with $A_d \in DC_\alpha$ has a finite normal form, modulo the action of conjugations in $\mathcal{V}$, and are therefore defined by a finite number of parameters in the parameter space. Therefore, in a certain sense, the orbits of Liouvillean cocycles in the local regime, even under C^∞ conjugations, are bigger than the orbits of Diophantine ones, since they have representatives in the parameter space that are not finitely determined for any choice of parameters for the K.A.M. scheme. Let us call $\mathcal{P}$ the space of K.A.M. normal forms modulo the action of close-to-the-identity C^∞ conjugations. It is formed by the data $\{k_i, \epsilon_i, \phi_i, \theta_i\}$, where k_i is the resonant frequency, ϵ_i is the distance from the exact resonance, ϕ_i is the argument of the resonant mode, and θ_i is the angle defined in fig. 1. Let us also call $\tilde{\mathcal{P}}$ the reduced parameter space, where we omit the parameter ϕ_i which irrelevant for the dynamical properties of the cocycle.

Corollary 1.16. *The orbit of each constant cocycle (α, A_d) with $A_d \in DC_\alpha$ has countably many representatives in $\tilde{\mathcal{P}}$. The orbit of each constant cocycle (α, A_l) with $A_l \in \mathcal{L}_\alpha$ has uncountably many representatives in the same space.*

This corollary shows in fact that the orbit of a Liouville constant exits the K.A.M. regime and re-enters many more times than the orbit of a Diophantine one, which is constrained by the differentiable rigidity theorem, 1.6, to leave to infinity as the norms of conjugations grow. We also obtain the following corollary.

Corollary 1.17. *In $\mathcal{P}$, every conjugation class is dense. Every class is totally disconnected in $\tilde{\mathcal{P}}$*

The density could be viewed as an infinite-dimensional analogue of the density of $x + \beta\mathbb{Z} \mod 1$ in $[0, 1]$, for each $x \in [0, 1]$ and for $\beta \in \mathbb{R} \setminus \mathbb{Q}$ fixed, though the analogy is quite loose.

We topologize the parameter space in a way that is compatible with the mapping of the parameters into a space of C^∞ functions, i.e. closeness means $O(N_{n_i}^{-\infty})$-closeness, and the h^s norms use $N_{n_i}^s$ as weights. The formal definition would be tedious and we omit it.

The infinite dimensionality comes from the infinite number of significant "rotation numbers" a_i at each step of the K.A.M. scheme. This theorem and its corollary show that there should be no reasonable way of defining a fibered rotation number for non-reducible cocycles, as we can for $SL(2, \mathbb{R})$ cocycles, see [Her83] and [JM82].

Theorem 1.18. *Every conjugacy class is dense in $\mathcal{N}$. Total disconnectedness is lost because of the action of conjugations in $\mathcal{V}$. Conjugacy classes are not locally connected around any point.*

The proof of these last results implies the next theorem, which illustrates at what point all classes and all dynamical behaviours are indistinguishable, at least before having iterated the dynamical system an infinite number of times.

Theorem 1.19. *Given any cocycle $(\alpha, A(\cdot)) \in \mathcal{N}$ and every conjugacy class $\mathcal{C}$ represented in $\mathcal{N}$, the cocycle $(\alpha, A(\cdot))$ is almost conjugate to $\mathcal{C}$.*

The precise definition of *almost conjugation*, a generalization of almost reducibility, is given in def. 2.7. The theorem is in fact slightly stronger than a corollary of the almost reducibility theorem, and we stress it since the analysis of the K.A.M. normal form shows that, in fact, any class of cocycles can serve as the linear model, admittedly using as a basis the class of constant cocycles.

All of the above theorems hold for any fixed number of frequencies $d \in \mathbb{N}^*$ and a bigger d only results in a smaller neighborhood of constants where they hold true, due to Sobolev injection theorems. If, now, we restrict ourselves to the one-frequency case ($d = 1$), we can use the powerful tool of renormalization ([Kri01], [AK06], [FK09], see also [Kar15]) which we can combine with the work of [Fra00], [Fra04] and [dA13] and obtain the following picture, which fills the total space $SW_\alpha^\infty(\mathbb{T}, G)$, provided that $\alpha \in RDC$.

Theorem 1.20 ([Kri01], [Fra04], [Kar15])**.** *The picture described in theorems 1.1 up to 1.19 holds true in an open dense subset of the total space $SW_\alpha^\infty(\mathbb{T}, G)$, provided that $\alpha \in RDC$ (we stress that $d = 1$ in the global theorems).*

We have also identified the cocycles in the complementary set to that which is renormalized into the K.A.M. regime (under the standing arithmetic assumption).

Theorem 1.21 ([Kri01], [Kar15])**.** *The total space $SW^{\infty}_{\alpha}(\mathbb{T},G)$, $\alpha \in RDC$, is filled up by the countable union of immersed Fréchet manifolds corresponding to the conjugacy classes of periodic geodesics of G of degree $r \in \mathbb{N}^*$. These manifolds are of codimension $2r$.*

The spectral properties of these cocycles where studied by K. Fraczek and R. T. de Aldecoa.

Theorem 1.22 ([Fra00], [Fra04], [dA13])**.** *The cocycles described in the previous theorem have purely absolutely continuous spectrum when restricted in $L^2(\mathbb{T}) \times \mathcal{E}_{2m+1}$, for every $m \in \mathbb{N}$.*

Finally, we think that a picture similar to the one above should hold when $G = SU(2)$ is replaced by any semisimple compact Lie group (cf. [Kri99a], [Kar15]), at least in the K.A.M. regime. The phenomena observed in the more general case should consist of combinations and interactions between different behaviors obsverved in $SU(2)$, respecting the conditions of linear dependence and (non-)commutativity between the different root spaces. However, in the neighborhood of singular geodesics interesting phenomena may appear, caused by the interaction of the strong mixing with the weak mixing part of the dynamics. The analysis of such systems seems to be difficult.

Further, and certainly non-exhaustive, literature in the subject includes the works of Cl. Chavaudret ([Cha11],[Cha12],[Cha13]), also in collaboration with St. Marmi ([CM12]) and with L. Stolovich ([CS]), H. Eliasson ([Eli88], [Eli92a], [Eli01], [Eli92b], see also [Eli09]), X. Hou and J. You ([HY09], [HY12]) and of G. Popov ([HP13]), Q. Zhou ([YZ13]), and the paper of Avila-Fayad-Kocsard [AFK12], which triggered this finer study that we took up in our recent papers.

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2 Notation and definitions

2.1 The group $SU(2)$

The matrix group $G = SU(2) \approx \mathbb{S}^3 \subset \mathbb{C}^2$ is the multiplicative group of unitary 2×2 matrices of determinant 1. We will denote the matrix $S \in G$, $S = \begin{pmatrix} z & w \\ -\bar{z} & \bar{w} \end{pmatrix}$, where $(z,w) \in \mathbb{C}^2$ and $|z|^2 + |w|^2 = 1$, by $\{z,w\}_G$. The subscript will be suppressed from the notation, unless necessary. When coordinates in $\mathbb{C}^2$ are fixed, the circle $\mathbb{S}^1$ is naturally embedded in G as the group of diagonal matrices, which is a maximal torus (i.e. a maximal abelian subgroup) of G.

The Lie algebra $g = su(2)$ is naturally isomorphic to $\mathbb{R}^3 \approx \mathbb{R} \times \mathbb{C}$ equipped with its vector and scalar product. The element $s = \begin{bmatrix} it & u \\ -\bar{u} & -it \end{bmatrix}$ will be denoted by $\{t, u\}_g \in \mathbb{R} \times \mathbb{C}$. The scalar product is defined by

$$\langle \{t_1, u_1\}, \{t_2, u_2\} \rangle = t_1 t_2 + \mathcal{R}(u_1 \bar{u}_2) = t_1 t_2 + \mathcal{R}u_1.\mathcal{R}u_2 + \mathcal{I}u_1.\mathcal{I}u_2$$

Mappings with values in g will be denoted by

$$U(\cdot) = \{U_t(\cdot), U_z(\cdot)\}_g$$

in these coordinates, where $U_t(\cdot)$ is a real-valued and $U_z(\cdot)$ is a complex-valued function.

The adjoint action of the group on its algebra is pushed-forward to the action of $SO(3)$ on $\mathbb{R} \times \mathbb{C}$. In particular, the diagonal matrices, of the form $S = \exp(\{2i\pi s, 0\}_g)$, we have $Ad(S).\{t, u\} = \{t, e^{4i\pi s} u\}$.

2.2 Functional Spaces

We will consider the space $C^\infty(\mathbb{T}, g)$ equipped with the standard maximum norms

$$\|U\|_s = \max_{0 \leq \sigma \leq s} \max_{\mathbb{T}} |\partial^\sigma U(\cdot)|$$

for $s \geq 0$, and the Sobolev norms

$$\|U\|_{H^s}^2 = \sum_{k \in \mathbb{Z}^d} (1 + |k|^2)^s |\hat{U}(k)|^2$$

where $\hat{U}(k) = \int U(\cdot) e^{-2i\pi kx}$ are the Fourier coefficients of $U(\cdot)$. The fact that the injections $H^{s+d/2}(\mathbb{T}^d, g) \hookrightarrow C^s(\mathbb{T}^d, g)$ and $C^s(\mathbb{T}^d, g) \hookrightarrow H^s(\mathbb{T}^d, g)$ for all $s \geq 0$ are continuous is classical. By abusing the notation, we will note $H^0 = L^2$. We will denote the corresponding spaces of complex sequences by lowercase letters,

$$h^s = \{f \in \ell^2, \sum (1+n)^{2s} |f_n|^2 < \infty\}$$

For this part, see [Fol95] and [SW71]. In view of the identification $G \approx \mathbb{S}^3$, with normalized measure, the space $C^\infty((G))$ of smooth $\mathbb{C}$-valued functions defined on G, can be identified with $C^\infty(\mathbb{S}^3)$, and the identification is an isometry between the L^2 spaces.

Let us give a convenient basis for $C^\infty(\mathbb{S}^3)$. Given a system of coordinates (ζ, ω) in $\mathbb{C}^2$, we can define an orthonormal basis for $\mathcal{P}_m$, the space of homogeneous polynomials of degree m, by $\{\psi_{l,m}\}_{0 \leq l \leq m}$ where $\psi_{l,m}(\zeta, \omega) = \sqrt{\frac{(m+1)!}{l!(m-l)!}} \zeta^l \omega^{m-l}$. The group G acts on $\mathcal{P}_m$ by

$$\{z, w\}.\phi(\zeta, \omega) = \phi(z\zeta + w\omega, -\bar{w}\zeta + \bar{z}\omega)$$

and the resulting representation is noted by π_m. For m fixed, we can define the matrix coefficients relative to the basis by $\pi_m^{j,p}\{z, \bar{z}, w, \bar{w}\} \mapsto \langle \{z, w\}.\psi_{j,m}, \psi_{p,m} \rangle$. The matrix coefficients are harmonic functions of $z, \bar{z}, w, \bar{w}$, and are of bidegree $(m-p, p)$, i.e. they are homogeneous of degree $m-p$ in (z, w), and homogeneous of degree p in $(\bar{z}, \bar{w})$, and they generate the space $\mathcal{E}_m$. We thus obtain the decomposition $L^2 = \oplus_{m\in\mathbb{N}}\mathcal{E}_m$

Therefore, given a system of coordinates in $\mathbb{C}^2$, a function $f \in L^2(\mathbb{S}^3)$ can be written in the form

$$f(z, \bar{z}, w, \bar{w}) = \sum_{m\in\mathbb{N}} \sum_{0\leq p\leq m} \sum_{0\leq j\leq m} f^m_{j,p} \pi^{j,p}_m(z, \bar{z}, w, \bar{w})$$

where $f^m_{j,p} \in \mathbb{C}$ are the Fourier coefficients. The functions $\pi^{j,p}_m(z, \bar{z}, w, \bar{w})$ are the eigenfunctions of the Laplacian on $\mathbb{S}^3$ and consequently smooth (in fact real analytic), and they form an orthonormal basis for $L^2(\mathbb{S}^3)$. In higher regularity, they generate a dense subspace of C^∞.

The group G acts on $C^\infty(G) \equiv C^\infty(\mathbb{S}^3)$ by pullback: if $A \in G$ and $\begin{pmatrix}\zeta\\ \omega\end{pmatrix} \in \mathbb{S}^3$, then, for $\phi : \mathbb{S}^3 \to \mathbb{C}$,

$$(A.\phi)\left(\begin{pmatrix}\zeta\\ \omega\end{pmatrix}\right) = \phi\left(A^* \begin{pmatrix}\zeta\\ \omega\end{pmatrix}\right)$$

If coordinates are chosen so that $A = \{e^{2i\pi a}, 0\}$ is diagonal, then

$$A.\phi(\zeta, \omega) = \phi(e^{-2i\pi a}\zeta, e^{2i\pi a}\omega)$$

and A then acts on harmonic functions by

$$A.\pi^{j,p}_m(z, \bar{z}, w, \bar{w}) = e^{-2i\pi(m-2p)a}\pi^{j,p}_m(z, \bar{z}, w, \bar{w})$$

where $m-2p = m-p-p$ is the difference of the degree of homogeneity in $(\bar{z}, \bar{w})$ and (z, w). Therefore, the harmonics in these coordinates are eigenvectors for the associated operator. In particular, if a is irrational, the eigenvectors for the eigenvalue 1 are exactly the elements $\pi^{j,m/2}_m$, $0 \leq j \leq m$.

The group of symmetries of $\mathbb{C}.\psi_{m/2,m}$ is exactly the normalizer of $\mathcal{T}$, the torus of matrices commuting with A. We revisit the following lemma from [Kar14b]. It examines the effect of changes of coordinates on the eigenvectors for the eigenvalue 1, $\psi_{m/2,m}$.

Lemma 2.1. *For a given $m > 0$ and even, $\pi_{m/2,m}(D.\psi_{m/2,m}) = 1$ iff D is in the normalizer $\mathcal{N}_\mathcal{T}$ of $\mathcal{T}$. The derivative of the norm of the projection at $D \equiv Id \in G \mod \mathcal{N}_\mathcal{T}$ is negative and $\pi_{m/2,m}(D.\psi_{m/2,m}) < 1$ when $D \notin \mathcal{N}_\mathcal{T}$.*

Proof. Call $l = m/2$ and calculate the projection:

$$\pi_{l,m}(\{z, w\}_G.\psi_{l,m}) = \sum_0^l (-1)^i \binom{l}{i}^2 |z|^{2(l-i)}|w|^{2i}\psi_{l,m} = p_l(|z|, |w|)\psi_{l,m}$$

The factor of the projection, p_l, is a Legendre polynomial in the variable $|z|^2$ and $|w|^2 = 1 - |z|^2$. The conclusion follows from the properties of Legendre polynomials. ∎

Returning to more general facts from calculus, the C^s norms for functions in $C^\infty(G)$ are defined in a classical way, and the Sobolev norms are defined by imposing a rate of decay on $f_m^{j,p}$, the coefficients of the harmonics in the expansion of f, $\|f\|_{H^s}^2 = \sum_{m,j,p}(1+m^2)^s |f_m^{j,p}|^2$.

Finally, we will use the truncation operators for mappings $\mathbb{T} \to g$:

$$\begin{aligned} T_N f(\cdot) &= \sum_{|k|\leq N} \hat{f}(k) e^{2i\pi k \cdot} \\ \dot{T}_N f(\cdot) &= T_N f(\cdot) - \hat{f}(0) \\ R_N f(\cdot) &= \sum_{|k|>N} \hat{f}(k) e^{2i\pi k \cdot} \end{aligned}$$

These operators satisfy the estimates

$$\|T_N f(\cdot)\|_{C^s} \leq C_s N \|f(\cdot)\|_{C^s} \quad (1)$$

$$\|R_N f(\cdot)\|_{C^s} \leq C_{s,s'} N^{s-s'+2} \|f(\cdot)\|_{C^{s'}} \quad (2)$$

2.3 Arithmetics, continued fraction expansion

For this section we refer the reader to [Khi63]. Let us introduce some notation. For $\alpha \in \mathbb{R}^*$, define $|||\alpha|||_{\mathbb{Z}} = dist(\alpha, \mathbb{Z}) = \min_{\mathbb{Z}} |\alpha - l|$, $[\alpha]$ the integer part of α, $\{\alpha\}$ its fractional part and $G(\alpha) = \{\alpha^{-1}\}$, the Gauss map.

Consider $\alpha \in \mathbb{T} \setminus \mathbb{Q}$ fixed, and let $\alpha_n = G^n(\alpha) = G(\alpha_{n-1})$, $a_n = [\alpha_{n-1}^{-1}]$. The following definition is classical.

Definition 2.1. *We will denote by $DC(\gamma, \tau)$ the set of numbers α in $\mathbb{T} \setminus \mathbb{Q}$ such that for any $k \neq 0$, $|\alpha k|_{\mathbb{Z}} \geq \frac{\gamma^{-1}}{|k|^\tau}$. Such numbers are called Diophantine.*

The set $DC(\gamma, \tau)$, for $\tau > 2$ fixed and $\gamma \in \mathbb{R}_+^*$ small is of positive Haar measure in $\mathbb{T}$, and $\cup_{\gamma>0} DC(\gamma, \tau)$ is of full Haar measure. The numbers that do not satisfy any Diophantine condition are called *Liouvillean*. They form a residual set of 0 Lebesgue measure.

The following definition concerns the preservation of Diophantine properties when the algorithm of continued fractions is applied to the number.

Definition 2.2. *We will denote by $RDC(\gamma, \tau)$ the full measure set of recurrent Diophantine numbers, i.e. the α in $\mathbb{T} \setminus \mathbb{Q}$ such that $G^n(\alpha) \in DC(\gamma, \tau)$ for infinitely many n.*

In contexts where the parameters γ and τ are not significant, they will be omitted in the notation of both sets.

Finally, we will need to approximate the eigenvalues of matrices in G with iterates of α, and thus need the following notion, which is looser than $(\alpha, a) \in \mathbb{T}^{d+1}$ being Diophantine.

Definition 2.3. *We will denote by $DC_\alpha(\gamma, \tau)$ the set of elements A of G satisfying the following property. If $A = D\{e^{2i\pi a}, 0\}D^*$ for some $D \in G$, then for $k \neq 0$,*

$$|||a - k\alpha||| \geq \frac{\gamma^{-1}}{|k|^\tau}$$

Such numbers are called Diophantine with respect to α.

2.4 Cocycles in $\mathbb{T}^d \times SU(2)$

2.4.1 Definition of the dynamics

Let $\alpha \in \mathbb{T}^d \equiv \mathbb{R}^d/\mathbb{Z}^d$, $d \in \mathbb{N}^*$, be an irrational rotation. If we also let $A(\cdot) \in C^\infty(\mathbb{T}^d, G)$, the couple $(\alpha, A(\cdot))$ acts on the fibered space $\mathbb{T}^d \times G \to \mathbb{T}^d$ by

$$(\alpha, A(\cdot)).(x, S) = (x + \alpha, A(x).S), (x, S) \in \mathbb{T}^d \times G$$

We will call such an action a *quasiperiodic cocycle over* α (or simply a cocycle). The space of such actions is denoted by $SW^\infty_\alpha(\mathbb{T}^d, G)$, most times abbreviated to SW^∞_α. The number $d \in \mathbb{N}^*$ is the number of frequencies of the cocycle.

The space $\bigcup_{\alpha \in \mathbb{T}^d} SW^\infty_\alpha$ will be denoted by SW^∞. The space SW^∞_α inherits the topology of $C^\infty(\mathbb{T}^d, G)$, and SW^∞ has the standard product topology of $\mathbb{T}^d \times C^\infty(\mathbb{T}^d, G)$. We note that cocycles are defined over more general maps and in more general contexts of regularity and structure of the basis and fibers.

The cocycle acts on any product space $\mathbb{T}^d \times E$, provided that $G \curvearrowright E$, in an obvious way. The particular case which will be important in this article is the representation of G on $L^2(G)$, and the resulting action of the cocycle on $L^2(\mathbb{T}^d \times G)$.

The n-th iterate of the action is given by

$$\begin{aligned}(\alpha, A(\cdot))^n.(x, S) &= (n\alpha, A_n(\cdot)).(x, S) = (x + n\alpha, A_n(x).S) \\ &= (x + n\alpha, A(\cdot + (n-1)\alpha)...A(\cdot).S)\end{aligned}$$

if $n > 0$. Negative iterates are the inverses of positive ones:

$$(\alpha, A(\cdot))^{-n} = ((\alpha, A(\cdot))^n)^{-1} = (-n\alpha, A^*(\cdot - n\alpha)...A^*(\cdot - \alpha))$$

2.4.2 Conjugation and reducibility

The cocycle $(\alpha, A(\cdot))$ is called a constant cocycle if $A(\cdot) = A \in G$ is a constant mapping. In that case, the quasiperiodic product reduces to a simple product of matrices, $(\alpha, A)^n = (n\alpha, A^n)$.

The group $C^\infty(\mathbb{T}^d, G) \hookrightarrow SW^\infty(\mathbb{T}^d, G)$ acts by *fibered conjugation*: Let $B(\cdot) \in C^\infty(\mathbb{T}^d, G)$ and $(\alpha, A(\cdot)) \in SW^\infty(\mathbb{T}^d, G)$. Then we define

$$\begin{aligned}Conj_{B(\cdot)}.(\alpha, A(\cdot)) &= (0, B(\cdot)) \circ (\alpha, A(\cdot)) \circ (0, B(\cdot))^{-1} \\ &= (\alpha, B(\cdot + \alpha).A(\cdot).B^{-1}(\cdot))\end{aligned}$$

The dynamics of $Conj_{B(\cdot)}.(\alpha, A(\cdot))$ and $(\alpha, A(\cdot))$ are essentially the same, since

$$(Conj_{B(\cdot)}.(\alpha, A(\cdot)))^n = (n\alpha, B(\cdot + n\alpha).A_n(\cdot).B^{-1}(\cdot))$$

Definition 2.4. *Two cocycles $(\alpha, A(\cdot))$ and $(\alpha, \tilde{A}(\cdot))$ in SW^{∞}_{α} are conjugate iff there exists $B(\cdot) \in C^s(\mathbb{T}^d, G)$ such that $(\alpha, \tilde{A}(\cdot)) = Conj_{B(\cdot)}.(\alpha, A(\cdot))$. We will use the notation*

$$(\alpha, A(\cdot)) \sim (\alpha, \tilde{A}(\cdot))$$

to state that the two cocycles are conjugate to each other.

Since constant cocycles are a class whose dynamics can be analysed, we give the following definition.

Definition 2.5. *A cocycle will be called reducible iff it is conjugate to a constant.*

In contrast with the greater part of the literature, in this article reducible means that the transfer function is at least measurable, whenever its regularity is not mentioned. In this article, cocycles are always C^{∞} smooth, but the smoothness of conjugations may vary from $H^0 \equiv L^2$ to C^{∞}.

Due to the fact that not all cocycles are reducible (e.g. generic cocycles in $\mathbb{T} \times \mathbb{S}^1$ over Liouvillean rotations, but also cocycles over Diophantine rotations, even though this result is hard to obtain, see [Eli02], [Kri01]) we also need the following concept, which has proved to be central in the study of such dynamical systems.

Definition 2.6. *A cocycle $(\alpha, A(\cdot))$ is said to be almost reducible if there exists a sequence of conjugations $B_n(\cdot) \in C^{\infty}$, such that $Conj_{B_n(\cdot)}.(\alpha, A(\cdot))$ becomes arbitrarily close to constants in the C^{∞} topology, i.e. iff there exists (A_n), a sequence in G, such that*

$$A_n^* \left(B_n(\cdot + \alpha) A(\cdot) B_n^*(\cdot)\right) \overset{C^{\infty}}{\to} Id$$

When this property is established in a K.A.M. constructive way, we can compare the size of $F_n(\cdot) \in C^{\infty}(\mathbb{T}^d, g)$, the error term which makes this last limit into an equality, with the rate of growth of the conjugation B_n, and obtain that

$$Ad(B_n(\cdot)).F_n(\cdot) = B_n(\cdot).F_n(\cdot).B_n^*(\cdot) \overset{C^{\infty}}{\to} 0$$

In this case, almost reducibility in the sense of the definition above and almost reducibility in the sense that "the cocycle can be conjugated arbitrarily close to reducible cocycles" are equivalent.

Herein, we will prove a more general statement, concerning conjugation close to any conjugacy class, where the same considerations on the error term apply.

Definition 2.7. *Let $(\alpha, A(\cdot))$ be a given cocycle, and $\mathcal{C}$ a given class cocycles up to conjuation. The cocycle $(\alpha, A(\cdot))$ is said to be almost conjugate to $\mathcal{C}$ if there exists a sequence of conjugations $B_n(\cdot) \in C^{\infty}$ and a sequence of cocycles $(\alpha, C_n(\cdot)) \in \mathcal{C}$, such that $Conj_{B_n(\cdot)}.(\alpha, A(\cdot))$ becomes arbitrarily close to $(\alpha, C_n(\cdot))$ in the C^{∞} topology.*

2.4.3 Review of the K.A.M. scheme and of the normal form

Local conjugation. Let $(\alpha, Ae^{F(\cdot)}) = (\alpha, A_1 e^{F_1(\cdot)}) \in SW^\infty(\mathbb{T}, G)$ be a cocycle over a Diophantine rotation satisfying some smallness conditions to be made more precise later on, and suppose, moreover, that $A = \{e^{2i\pi a}, 0\}$ is diagonal. The goal is to conjugate the cocycle ever closer to constant cocyles by means of an iterative scheme. This is obtained by iterating the following lemma, for the detailed proof of which we refer to [Kri99a], [Eli02] or [Kar15]. For the sake of completeness, we sketch the proof, following the notation of [Kar14b].

Lemma 2.2. *Let $\alpha \in DC(\gamma, \tau)$ and $K \geq C\gamma N^\tau$. Let, also, $(\alpha, Ae^{F_1(\cdot)}) \in SW^\infty(\mathbb{T}^d, G)$ with*

$$c_0 K N^{s_0} \varepsilon_{1,0} < 1$$

where c_0, s_0 depend on γ, τ and d, and $\varepsilon_{1,s} = \|F_1\|_s$. Then, there exists a conjugation $G(\cdot) = G_1(\cdot) \in C^\infty(\mathbb{T}^d, G)$ such that

$$G_1(\cdot + \alpha).A_1.e^{F_1(\cdot)}.G_1^*(\cdot) = A_2 e^{F_2(\cdot)}$$

and such that the mappings $G_1(\cdot)$ and $F_2(\cdot)$ satisfy the following estimates

$$\begin{aligned} \|G_1(\cdot)\|_s &\leq c_{1,s}(N^s + K N^{s+d/2}\varepsilon_{1,0}) \\ \varepsilon_{2,s} &\leq c_{2,s} K^2 N^{2\tau+d}(N^s \varepsilon_{1,0} + \varepsilon_{1,s})\varepsilon_{1,0} + C_{s,s'} K^2 N^{s-s'+2\tau+d}\varepsilon_{1,s'} \end{aligned}$$

where $s' \geq s$, and $\varepsilon_{2,s} = \|F_2(\cdot)\|_s$

If we suppose that $Y(\cdot) : \mathbb{T} \to g$ can conjugate $(\alpha, A_1 e^{F_1(\cdot)})$ to $(\alpha, A_2 e^{F_2(\cdot)})$, with $\|F_2(\cdot)\| \ll \|F_1(\cdot)\|$, then it must satisfy the functional equation

$$A_1^* e^{Y(\cdot+\alpha)} A_1 e^{F_1(\cdot)} e^{-Y(\cdot)} = A_1^* A_2 e^{F_2(\cdot)}$$

Linearization of this equation under the assumption that all C^0 norms are smaller than 1 gives

$$Ad(A_1^*)Y(\cdot + \alpha) + F_1(\cdot) - Y(\cdot) = \exp^{-1}(A_1^* A_2)$$

The equation for the diagonal coordinate is a linear cohomological one, and after truncation in the frequency domain it reads

$$Y_t(\cdot + \alpha) - Y_t(\cdot) + \dot{T}_N F_{1,t}(\cdot) = 0$$

and the solution as well as the estimates it satisfies are classical. The rest satisfies the estimate of eq. 1, and the mean value $\hat{F}_{1,t}(0)$ is an obstruction and will be integrated in $\exp^{-1}(A_1^* A_2)$.

The equation concerning the non-diagonal part is a twisted cohomological equation, whose twist depends on the linear model (α, A_1), and it reads

$$e^{-4i\pi a_1} Y_z(\cdot + \alpha) - Y_z(\cdot) + F_{1,z}(\cdot) = 0 \qquad (3)$$

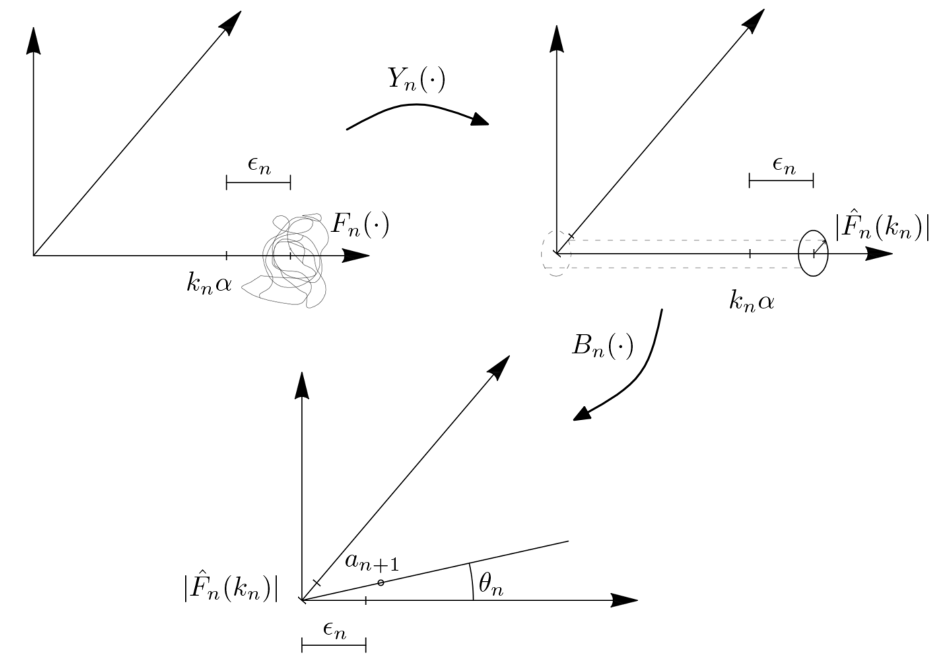

Figure 1: The n-th step of the K.A.M. scheme

or, in the frequency domain,

$$(e^{2i\pi(k\alpha-2a_1)}-1)\hat{Y}_z(k)=-\hat{F}_{1,z}(k),\ k\in\mathbb{Z} \tag{4}$$

If for some k_1 we have

$$|k_1\alpha-2a_1|_{\mathbb{Z}}< K^{-1}=N^{-\nu}$$

with $\nu>\tau$ to be fixed, we declare the corresponding Fourier coefficient $\hat{F}_{1,z}(k_1)$ a resonance, and integrate it to the obstructions. We know by [Eli02] that such a k_1 (called a *resonant mode*), if it exists and satisfies $0<k_1\leq N$, is unique in $\{k\in\mathbb{Z},|k-k_1|\leq 2N\}$. We can thus write

$$a_1=k_1\alpha \mod \mathbb{Z}+\epsilon_1$$

and call ϵ_1 the distance to the exact resonance. If we now call $T_{2N}^{k_1}$ the truncation operator projecting on the frequencies $0<|k-k_1|\leq 2N$ if k_1 exists, the equation

$$e^{-4i\pi a_1}Y_z(\cdot+\alpha)-Y_z(\cdot)=-T_{2N}^{k_1}F_{1,z}(\cdot)$$

can be solved and the solution satisfies the announced estimates.

In total, the equation that can be solved with good estimates is

$$Ad(A_1^*)Y(\cdot+\alpha)-Y(\cdot)+F_1(\cdot)=\{\hat{F}_{1,t}(0),\hat{F}_{1,z}(k_1)e^{2i\pi k_1\cdot}\}+\{R_NF_{1,t}(\cdot),R_{2N}^{k_1}F_{1,z}(\cdot)\}$$

with $\|Y(\cdot)\|_s \leq C_s N^{s+\nu+1/2}\varepsilon_{1,0}$, and thus there exists $F_2'(\cdot)$, a "quadratic" term, such that

$$e^{Y(\cdot+\alpha)}A_1 e^{F_1(\cdot)}e^{-Y(\cdot)} = \{e^{2i\pi(a+\hat{F}_{1,t}(0))},0\}_G.e^{\{0,\hat{F}_{1,z}(k_1)e^{2i\pi k_1\cdot}\}_g}e^{F_2'(\cdot)}$$

If k_1 exists and is non-zero, iteration of local conjugation is impossible. On the other hand, the conjugation $B(\cdot) = \{e^{-2i\pi k_1\cdot/2},0\}$ is such that, if we call $F_1'(\cdot) = Ad(B(\cdot)).F_1(\cdot) = \{F_{1,t}(\cdot), e^{-2i\pi k_1\cdot}F_{1,z}(\cdot)\}$, similarly for $Y(\cdot)$, and $A_1' = B(\alpha)A_1 = \{e^{2i\pi(a-k_1\alpha/2)}\}$, they satisfy the equation

$$\begin{aligned} Ad((A')^*)Y'(\cdot+\alpha) - Y'(\cdot) + F_1'(\cdot) &= \{\hat{F}_t(0), \hat{F}_z(k_1)\} + \{R_N F_t(\cdot), e^{-2i\pi k_1\cdot}R_{2N}^{k_1}F_z(\cdot)\} \\ &= \{\hat{F}_{1,t}'(0), \hat{F}_{1,z}'(0)\} + \{R_N F_{1,t}'(\cdot), \tilde{R}_{2N}^{k_1}F_{1,z}(\cdot)\} \end{aligned}$$

where $\tilde{R}_{2N}^{k_1}$ is a dis-centered rest operator. The equation for primed variables can be obtained from eq. 3 by applying $Ad(B(\cdot))$ and using that $B(\cdot)$ is a morphism and commutes with A_1. This implies that

$$\begin{aligned} Conj_{B(\cdot)}(\alpha, A_1.\exp(\{\hat{F}_t(0), \hat{F}_z(k_1)e^{2i\pi k_1\cdot}\})) &= (\alpha, A_1'.\exp(\{\hat{F}_t(0), \hat{F}_z(k_1)\}) \\ &= (\alpha, A_2) \end{aligned}$$

that is, $B(\cdot)$ reduces the initial constant perturbed by the obstructions to a cocycle close to (α, Id). If $B(\cdot)$ happens to be 2-periodic, we post-conjugate with $C(\cdot)\colon 2\mathbb{T} \to G$, a torus morphism commuting with A_2 and algebraically conjugate to

$$\begin{pmatrix} e^{i\pi\cdot} & 0 \\ 0 & e^{-i\pi\cdot} \end{pmatrix}$$

This conjugation adds $\pm i\pi\alpha$ to the arguments of the eigenvalues of A_2 and restores 1-periodicity without deteriorating the estimates (see also [Kar15]), as it only shifts the frequencies of the perturbation of A_2 by 1. We let the reader convince themselves that this conjugations does not influence the estimates or the proofs of the theorems of the article and we will omit it in the rest of the arguments.

The K.A.M. scheme and normal form. If we define the following set of parameters, we can iterate lemma 2.2. Let $N_{n+1} = N_n^{1+\sigma} = N^{(1+\sigma)^{n-1}}$, where $N = N_1$ is big enough and $0 < \sigma < 1$, and $K_n = N_n^{\nu}$, for some $\nu > \tau$. If we suppose that $(\alpha, A_n e^{F_n(\cdot)})$ satisfies the hypotheses of lemma 2.2 for the corresponding parameters, then we obtain a mapping $G_n(\cdot) = B_n(\cdot)e^{Y_n(\cdot)}$ that conjugates it to $(\alpha, A_{n+1}e^{F_{n+1}(\cdot)})$, and we use the notation $\varepsilon_{n,s} = \|F_n\|_s$.

If we suppose that the initial perturbation small in small norm: $\varepsilon_{1,0} < \epsilon < 1$, and not big in some bigger norm: $\varepsilon_{1,s_0} < 1$, where ϵ and s_0 depend on the choice of parameters, then we can prove (see [Kar15] and, through it, [FK09]), that the lemma can be iterated into a scheme, and moreover

$$\begin{aligned} \varepsilon_{n,s} &= O(N_n^{-\infty}) \text{ for every fixed } s \text{ and} \\ \|G_n\|_s &= O(N_n^{s+\lambda}) \text{ for every } s \text{ and some fixed } \lambda > 0 \end{aligned}$$

We sum these inequalities up by saying that the norms of perturbations decay exponentially, while conjugations grow polynomially.

This fact allows us to obtain the normal form as follows. The product of conjugations produced by the scheme at the n-th step is written in the form $H_n(\cdot) = B_n(\cdot)e^{Y_n(\cdot)}...B_1(\cdot)e^{Y_1(\cdot)}$, where the $B_j(\cdot)$ reduce the resonant modes. We can rewrite the product in the form

$$B_n(\cdot)\dots B_1(\cdot).e^{\tilde{Y}_n(\cdot)}\dots e^{\tilde{Y}_2(\cdot)}e^{Y_1(\cdot)}$$

where $\tilde{Y}_j(\cdot) = \prod_{j-1}^{1} Ad(B_i^*(\cdot)).Y_j(\cdot)$. Since the $Y_j(\cdot)$ converge exponentially fast to 0 (they are conjugations comparable with F_j with a fixed loss of derivatives) in C^∞, and since the algebraic conjugation deteriorates the C^s norms by a factor of the order of N_{n-1}^{s+d}, $\prod_\infty^1 \exp(\tilde{Y}_j(\cdot))$ always converges, say to $D(\cdot) \in C^\infty(\mathbb{T}^d, G)$, even if the $H_n(\cdot)$ do not. The cocycle $Conj_{D(\cdot)}(\alpha, A^{F(\cdot)})$ is the *K.A.M. normal form* of the cocycle $(\alpha, A^{F(\cdot)})$, and it has the property that the K.A.M. scheme applied to it consists only in the reduction of resonant modes.

Notation 2.1. *For a cocycle in normal form, we relabel the indexes as* $(\alpha, A_{n_i}e^{F_{n_i}}) = (\alpha, A_i e^{F_i})$*, where* n_i *is a step where a reduction of a resonant mode takes place.*

In the language of fig. 1 a cocycle in normal form, after the successive conjugations up to the step i and in the first order of magnitude looks like a circle around the origin in the plane tangent to a resonant sphere $\{S.\{2\pi(k_i\alpha + \epsilon_i, 0\}_g.S^*\}_{S\in G}$. Its radius is $|\hat{F}_i(k_i)|$. The reduction of the resonant mode drives $k_i\alpha$ to 0, and reduces the rest of the perturbation to the point of coordinates $\{2\pi\epsilon_i, \hat{F}_i(k_i)\}_g$. The picture repeats itself if we zoom in in order to see the finer scales of the dynamics, and the first part of the picture (the reduction by the close-to-the-identity transformation $Y_n(\cdot)$) never occurs.

At the step i, we will assume that the constant $A_i = \{e^{2i\pi k_i\alpha}, 0\}$ is the exact resonance, and the first order perturbation

$$e^{F_i(\cdot)} = e^{\{2i\pi\epsilon_i, 0\}}.e^{\{0, \hat{F}_i(k_i)e^{2i\pi k_i \cdot}\}}$$

contains the distance from the exact resonance, $\{2i\pi\epsilon_i, 0\}$.[2]

3 Proof of weak mixing

Let $f \in L^2(\mathbb{T}\times G, \mathbb{C})$ be an eigenfunction of $U = U_{(\alpha, A(\cdot))}$, with explicit dependence on S. We remark that any eigenfunction depending non-trivially on S also depends non-trivially on x, unless $A(\cdot) \equiv A \in G$ is constant. Since each subspace $L^2(\mathbb{T}) \times \mathcal{E}_m$ is invariant under U, for each m fixed, we can suppose that there exists $m \in \mathbb{N}^*$ such that $f \in L^2(\mathbb{T}) \times \mathcal{E}_m$. The function f then admits a development

$$f(x,S) = \sum_{\substack{k\in\mathbb{Z}\\ 0\leq j,p\leq m}} f_{j,p}^{m,k} e^{2i\pi kx}\pi_m^{j,p}(z,\bar{z},w,\bar{w}) = \sum_{\substack{k\in\mathbb{Z}\\ 0\leq j,p\leq m}} f_{j,p}^{k} e^{2i\pi kx}\pi^{j,p}(z,\bar{z},w,\bar{w})$$

[2]This choice interferes with the estimates only when the cocycle is C^∞ reducible.

where we have dropped m from the notation since it is considered to be fixed. It will be replaced by i, the index of the step of the K.A.M. scheme.

The equation satisfied by an eigenfunction of the Koopman operator U is

$$f(x-\alpha, A^{-1}(x).S) = \lambda f(x,S)$$

for some fixed $\lambda \in \mathbb{S}^1$.

For a constant cocycle $(\alpha, A(\cdot)) \equiv (\alpha, A)$, the following lemma is immediate, under the assumption that

$$A = \begin{pmatrix} e^{2i\pi a} & 0 \\ 0 & e^{-2i\pi a} \end{pmatrix}$$

is a diagonal matrix in the coordinates that we introduce.

Lemma 3.1. *Let (α, A) be a constant cocycle, and consider the canonical basis of $\mathcal{E}_m$, formed by the functions $\pi^{j,p} = \pi^{j,p}_m$. Then, for every $k \in \mathbb{Z}$ and $0 \leq j, p \leq m$, the function*

$$e^{2i\pi kx}\pi^{j,p}(z, \bar{z}, w, \bar{w}) \in L^2(\mathbb{T}) \times \mathcal{E}_m$$

is an eigenfunction of the Koopman operator $U_{(\alpha,A)}$, with eigenvalue

$$e^{-2i\pi(k\alpha+(m-2p)a)}$$

The proof of the lemma is by immediate calculation (or see [Kar14b]) and points to the proof of the main theorem of the paper, where the assumption that the cocycle $(\alpha, A(\cdot))$ be in K.A.M. normal form becomes relevant. The argument is the following, and it is to be compared with the proof of thm 1.9. If f is an eigenfunction of the operator associated to the cocycle $(\alpha, A(\cdot))$, and if $(\alpha, A_i e^{F_i(\cdot)}) \overset{H_i^*}{\sim} (\alpha, A(\cdot))$ at the n_i-th step of the K.A.M. scheme, then, $f_i = f \circ (Id, H_i^*)$ is an eigenfunction of $U_i = U_{(\alpha, A_i e^{F_i(\cdot)})}$. Since $F_i(\cdot)$ is very small, f_i should be close to an eigenfunction of the operator $U_i' = U_{(\alpha, A_i)}$. The corresponding eigenvalues of the exact eigenfunctions will be distinct, since α is supposed Diophantine. Since this approximation converges exponentially fast for $n \to \infty$, and since the support in the frequencies in $L^2(\mathbb{T})$ is related with the summability of the angles, we obtain the announced theorem.

We now make the argument precise. Clearly, $f_0 = f$ is an eigenfunction of the operator $U_0 = U_{(\alpha, A(\cdot))}$ and for the eigenvalue λ iff $f_i = f \circ (Id, H_i^*)$ is an eigenfunction of the operator

$$U_i = U_{(\alpha, A_i e^{F_i(\cdot)})}, i \in \mathbb{N}$$

for the same eigenvalue. Then, linearization with respect to the dynamics gives

$$\tilde{U}_i f_i = \lambda f_i + O_{L^2}(N_i^{-\infty}) \tag{5}$$

where $\tilde{U}_i = U_{(\alpha, A_i)}$ and the constants on the O_{L^2} depend on the norm of the function f. Linearization is possible because f depends in a C^∞ (in fact real analytic) way on the variable in G, and the L^2 character of the function may only be due to the slow decay of the Fourier coefficients in the variable in $\mathbb{T}$.

We will also use the fact that, if the cocycle is not measurably reducible, resonances appear rarely.

Lemma 3.2. *Let $(\alpha, A(\cdot))$ be in normal form and not measurably reducible. Then,*

$$n_{i+1} - n_i \to \infty$$

Proof. The condition that the cocycle is not measurably reducible is equivalent to

$$\{\theta_i\} = \{\arctan \frac{|\hat{F}(k_i)|}{|\epsilon_i|}\} \notin \ell^2$$

Since $|\hat{F}(k_i)| = O(|k_i|^{-\infty})$ and $|\hat{F}(k_i)| > 0$ for all i, we obtain that, also, $|\epsilon_i| = O(|k_i|^{-\infty})$. This implies that $|||k_{i+1}\alpha||| = O(|k_i|^{-\infty})$. Since, however

$$\frac{\log|k_{i+1}|}{\log|k_i|} \approx (1+\sigma)^{n_{i+1}-n_i}$$

and $\alpha \in DC$, we can conclude. □

In fact, this argument can be applied as soon as the cocycle is not C^∞ reducible.

Let us now apply the operator T_{N_i} on eq. 5 and use the fact that it commutes with $\tilde{U}_i$ to obtain

$$\tilde{U}_i T_{N_i} f_i = \lambda T_{N_i} f_i + O_{L^2}(N_i^{-\infty})$$

Consequently, since the inverse of $\tilde{U}_i \circ T_{N_i}$ does not magnify the error term outside of $O_{L^2}(N_i^{-\infty})$, $T_{N_i} f_i$ is $O_{L^2}(N_i^{-\infty})$-close to an eigenfunction of $\tilde{U}_i \circ T_{N_i}$. Since the eigenvalues of $\tilde{U}_i \circ T_{N_i}$ are separated by $\gamma^{-1}.N_i^{-m\tau}$, we find that the eigenvalues of $\tilde{U}_i \circ T_{N_i}$ and their corresponding eigenfunctions are $O(N_i^{-\infty})$ and $O_{L^2}(N_i^{-\infty})$ good approximations of the corresponding objects for U_i. Therefore, since L^2-norms in the original coordinates and those of the i-th step of the K.A.M. scheme are the same, the same approximation holds also for the operator U and the eigenfunctions transformed accordingly.

We now compare the equations 5 at the steps n_i and $n_{i+1} \gg n_i$, and examine how the divergence of the product of conjugations sends L^2 mass to infinity, thus contradicting the initial assumption that $f \in L^2$.

Let us express the eigenfunctions of $\tilde{U}_i$ in a coordinate system where (α, A_i) is diagonal. There exists $l_i \in \mathbb{N}$, such that, up to $O(N_i^{-\infty})$,[3]

$$f_i(x, S_i) = \sum_{\substack{k+(m-2p)k_i = l_i \\ 0 \leq j,p \leq m \\ |k| < N_i}} f_{j,p}^{i,k} e^{2i\pi kx} \pi_i^{j,p}(S_i)$$

[3]The upper index k in the Fourier coefficients $f_{j,p}^{i,k}$ is in fact redundant.

Let us, now, apply the transformation $B_i(\cdot)$ which conjugates $(\alpha A_i e^{F_i(\cdot)})$ to $(\alpha A_{i+1} e^{F_{i+1}(\cdot)})$. In the new coordinates,

$$\begin{aligned} f_i(x,\tilde{S}_i) &= \sum_{\substack{k+(m-2p)k_i=l_i \\ 0\leq j,p\leq m \\ |k|<N_i}} f^{i,k}_{j,p} e^{2i\pi(k+(m-2p)k_i)x} \pi_i^{j,p}(\tilde{S}_i) \\ &= \sum_{0\leq j,p\leq m} \tilde{f}^{i}_{j,p} e^{2i\pi l_i x} \pi_i^{j,p}(\tilde{S}_i) \end{aligned}$$

If we let D_i be such that $D_i A_i D_i^*$ is diagonal in the coordinates where A_{i+1} is diagonal, then D_i is θ_i-away from a diagonal matrix in the same coordinates. If we also apply D_i, we obtain the new coordinates (x, S_{i+1}) where the cocycle $(\alpha, A(\cdot))$ is represented by $(\alpha, A_{i+1}e^{F_{i+1}(\cdot)})$ and A_{i+1} is diagonal, and the expression

$$f_i(x,S_{i+1}) = \sum_{\substack{0\leq j,p\leq m \\ |k|<N_i}} \tilde{f}^{i+1,k}_{j,p} e^{2i\pi l_i x} \pi_{i+1}^{j,p}(S_{i+1})$$

By our observation, the formula above should coincide up to $O_{L^2}(N_{i+1}^{-\infty})$ with $T_{2mN_i} f_{i+1}(x, S_{i+1})$, where

$$\tilde{U}_{i+1} T_{N_{i+1}} f_{i+1} = \lambda T_{N_{i+1}} f_{i+1} + O_{L^2}(N_{i+1}^{-\infty})$$

for the same eigenvalue λ, eventually up to $O(N_{i+1}^{-\infty})$.

The incompatibility between the two representations arises from the transformation rule of the $\pi^{j,p}$ under a change of basis. More precisely, the only functions that are eigenfunctions for the operator $\tilde{U}_{i+1}$ for the eigenvalue $e^{2i\pi l_i\alpha}$ are the functions $e^{2i\pi l_i \cdot}\pi_{i+1}^{j,m/2}, 0 \leq j \leq m$, and m and even number. Therefore, the compatibility of the two expressions for the eigenfunction would impose that

$$f_i(x,S_{i+1}) = \sum_{\substack{0\leq j\leq m \\ |k|<N_i}} \tilde{f}^{i+1,k}_{j,m/2} e^{2i\pi l_i x} \pi_{i+1}^{j,m/2}(S_{i+1})$$

When we insert D_i^* in this expression in order to undo the change of coordinates $\tilde{S}_i \mapsto S_{i+1}$, we constrain the coefficients $f^{i,k}_{j,p}$ in the image of $\oplus_{0\leq j\leq m}\mathbb{C}\pi_{i+1}^{j,m/2}$ under the change of coordinates, always up to $O(N_i^{-\infty})$.

Now, the same must hold when we compare the expressions obtained at the steps $i+1$ and $i+2$. Comparison between the constraint on the coefficients at the step $i+2$, i.e.

$$f_i(x,S_{i+2}) = \sum_{\substack{0\leq j\leq m \\ |k|<N_i}} \tilde{f}^{i+2,k}_{j,m/2} e^{2i\pi l_i x} \pi_{i+2}^{j,m/2}(S_{i+2})$$

shows that the space admissible at the step $i+1$ is restricted. When we project the preimage of the vector $\zeta_{i+2}^{m/2}\omega_{i+2}^{m/2}$ under the change of coordinates $\tilde{S}_{i+1} \mapsto$

S_{i+2}, the norm of the vector shrinks by a factor $\geq O(\theta_{i+1}^2)$, i.e.

$$|\langle (D_{i+1})_* (\zeta_{i+2}^{m/2} \omega_{i+2}^{m/2}), \zeta_{i+1}^{m/2} \omega_{i+1}^{m/2} \rangle| \leq \frac{(m/2)!}{(m+1)!} (1 - O(\theta_{i+1}^2))$$

(see lem. 2.2 of [Kar14b] and lem. 2.1).

Since the different constraints on the coefficients are imposed in different scales of the dynamics for every different i, or equivalently since they correspond to frequencies in $\mathbb{Z}^d$ belonging to distant shells, these constraints are independent from one another. Therefore, if the angles are not summable in ℓ^2, the intersection of the constraints is empty and there exists no eigenfunction in L^2. On the other hand, if the angles are summable in ℓ^2, the procedure converges and produces an eigenfunction as should be expected.

Finally, for any given cocycle in $\mathcal{N}$, we prove the existence of a subsequence of iterates accumulating to $(0, Id)$ in the C^∞ topology.

Proposition 3.3. *All cocycles in the K.A.M. regime are rigid.*

Proof. Every cocycle is almost reducible to a resonant one

$$(\alpha, A(\cdot)) \overset{H_i^*}{\sim} (\alpha, A_i) + O(N_{i+1}^{-\infty})$$

where $A_i = \{e^{2i\pi k_{i+1}\alpha}, 0\}$ up to $O(N_i^{-\infty})$. Since, in the case where $(\alpha, A(\cdot))$ is not C^∞ reducible, $n_{i+1} \gg n_i$, there exists an iterate $n_i < m_i \ll n_{i+1}$ such that $(\alpha, A(\cdot))^{m_i} = (m_i\alpha, O(N_i^{-\infty}))$, and $m_i \alpha \to 0$ when $i \to \infty$.[4] □

4 The topology of congucacy classes

In this section we sketch a proof of theorems 1.10, 1.11, 1.13, 1.14 and 1.15, corollary 1.17 and theorem 1.18.

The conjugations that act on a K.A.M. normal form at step i of the scheme are:

1. Far-from-the-identity conjugations commuting with the constant A_i

$$B'(\cdot) = \begin{pmatrix} e^{2i\pi k_i' \cdot /2} & 0 \\ 0 & e^{-2i\pi k_i' \cdot /2} \end{pmatrix}$$

   where $k_i' \in \mathbb{Z}$ is such that that $N_{i-1} < k_i + k_i' \leq N_i$.

2. Constant conjugations commuting with A_i.

3. Conjugations such as those constructed in the proof of thm 1.3.

4. Conjugations regaining periodicity, if necessary.

[4]In fact, $\nu > \tau$ is sufficient for obtaining rigidity for any cocycle, but it seems cumbersome, unless the cocycle is not C^∞ reducible.

These conjugations act as follows.

1. Translation of the resonance by $k_i'\alpha/2$ and of the corresponding resonant frequency by k_i'. They satisfy the estimate $N_{i-1}^{s+1/2} \lesssim \|B'\|_s \lesssim N_i^{s+1/2}$.

2. Multiplication of $\hat{F}_i(k_i)$ by a complex number in $\mathbb{S}^1$. This only changes the argument of $\hat{F}_i(k_i)$, and such a conjugation can be introduced as a phase in $B_i(\cdot)$. They satisfy the estimate $\|B''\|_s \lesssim N_{i-1}^{s+1/2}|\theta_{i-1}|$.

3. The conjugations of the third kind are constructed as follows. Consider a one-parameter subgroup $\{D_i^t\}_{t\in[0,1]}$, of minimal length such that $D_i A_{i+1} D_i^*$ is diagonal in the coordinates where A_i is diagonal and $D_i = D_i^1$. Then, the path
$$t \mapsto B_i^*(\cdot)e^{tD_{i+1}}B_i(\cdot)$$
when it acts by conjugation on $(\alpha, A_i e^{F_i(\cdot)})$, transforms the parameters of the normal form as follows
$$\begin{aligned} \theta_i^t &= (1-t)\theta_i \\ \frac{|\hat{F}_i^t(k_i)|}{|\epsilon_i^t|} &= \arctan \theta_i^t \\ \sqrt{|\hat{F}_i^t(k_i)|^2 + (\epsilon_i^t)^2} &\equiv \sqrt{|\hat{F}_i^0(k_i)|^2 + (\epsilon_i^0)^2} \simeq a_{i+1} \end{aligned}$$
i.e. the angle between A_i and A_{i+1} is driven to zero, the rotation number at the step $i+1$ is added to the one at the step i and the following resonances are translated by k_i. These conjugations affect the norms by a factor
$$O_{H^s}(|\theta_i^t - \theta_i^0|N_{i-1}^s) = O_{H^s}(|t\theta_i^0|N_{i-1}^s)$$
when θ_i is close to 0, and therefore will be close to the identity whenever t is small enough.

The fact that these three conjugations are the only ones who act on the parameter space of normal forms follows from the following. In view of item 1, we can assume that the resonant mode is the same for both forms. Then, we can apply item 3 to each one, in order to obtain a resonant constant, but with the resonant mode disactivated. If these two cocycles are conjugate to each other, then their arguments $a_i^j + a_{i+1}^j$, for $j = 1, 2$ must be equal up to $k_i'\alpha$, with k_i' not too big (see again item 1). Since resonances are unique for this size of k_i', no other conjugation can act on the space of normal forms.

These facts prove thm 1.15, by applying the same procedure as for the construction of the K.A.M. normal form, its corollary 1.17, and thm 1.18. Corollary 1.16 is proved by combining the estimates above with the proof of thm 1.6, where it is proved that the K.A.M. scheme produces only a finite number of resonances for cocycles reducible to a constant in CD_α, and, "generically", an infinite one if the constant is in $\mathcal{L}_\alpha$.

The construction of the paths is carried out by partitioning the interval $[0,1]$ into dyadic intervals, and then continuously deforming the parameters of the i-th step of the K.A.M. scheme for $t \in [2^{i-1}, 2^i]$ in a continuous way from those of the original cocycle to those corresponding to the normal form of the target.

Let us sketch the proof of thm 1.10. First, connect the Id with A_1 with a continuous path, say the shortest one parameter connecting the two elements. This part cannot be obtained by acting by conjugation. Then, activate corresponding mode of the normal form by a remarametrization of, say

$$t \to \{e^{2i\pi t\epsilon_1}, 0\}.\{0, e^{\{0, t\hat{F}_1 e^{2i\pi k_1 \cdot}\}}\}$$

Proceed by induction.

The proof of theorem 1.11 replaces the first step by the following one. Let $B_{12}^t(\cdot)\colon [0,1/4] \to C^\infty$ be a path such that $B_{12}^0(\cdot) \equiv Id$, and $B_{12}^{1/4}(\cdot) \equiv B_1.B_2(\cdot)$. This is possible, since $B_1.B_2(\cdot)$ is homotopic to constant mappings. This conjugation transforms the constant cocycle (α, Id) into $(\alpha, \{e^{2i\pi(a_1+a_2)}, 0\})$, where we recall that the constants in the normal form are exactly resonant, but the path exists $\mathcal{N}$. Then, we can activate the mode in the normal form with a path $[1/4, 1/2] \to C^\infty$ acting by a conjugation as in item 3. This will drive the constant part to $\{e^{2i\pi k_1 \alpha}, 0\} = A_1$, and add the pertrurbation

$$\{e^{2i\pi\epsilon_1}, 0\}.\{0, e^{\{0, \hat{F}_1 e^{2i\pi k_1 \cdot}\}}\}$$

Proceed by induction.

The proofs of the two other similar theorems 1.13 and 1.14, are only slightly more complicated.

References

[dA13] R. T. de Aldecoa, *The absolute continuous spectrum of skew products of compact Lie groups*, 2013. arXiv 1307.7348.

[AFK12] A. Avila, B. Fayad, and A. Kocsard, On manifolds supporting distributionally uniquely ergodic diffeomorphisms, *arXiv:1211.1519* (2012).

[AK06] A. Avila and R. Krikorian, Reducibility or nonuniform hyperbolicity for quasiperiodic Schrödinger cocycles, *Ann. of Math. (2)* **164** (2006), 911–940. MR 2259248 (2008h:81044).

[Cha11] C. Chavaudret, Reducibility of quasiperiodic cocycles in linear Lie groups, *Ergodic Theory Dynam. Systems* **31** (2011), 741–769. MR 2794946.

[Cha12] C. Chavaudret, Almost reducibility for finitely differentiable $SL(2,\mathbb{R})$-valued quasi-periodic cocycles, *Nonlinearity* **25** (2012), 481–494. MR 2876877.

[Cha13] C. CHAVAUDRET, Strong almost reducibility for analytic and Gevrey quasi-periodic cocycles, *Bull. Soc. Math. France* **141** (2013), 47–106. MR 3031673.

[CM12] C. CHAVAUDRET and S. MARMI, Reducibility of cocycles under a brjuno-russmann arithmetical condition, *JOURNAL OF MODERN DYNAMICS* **6** (2012), 59–78.

[CS] C. CHAVAUDRET and L. STOLOVITCH, Analytic reducibility of resonant cocycles to a normal form.

[Eli88] L. H. ELIASSON, Perturbations of stable invariant tori for Hamiltonian systems, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **15** (1988), 115–147 (1989). MR 1001032 (91b:58060).

[Eli92a] L. H. ELIASSON, Floquet solutions for the 1-dimensional quasi-periodic Schrödinger equation, *Comm. Math. Phys.* **146** (1992), 447–482. MR 1167299 (93d:34141).

[Eli92b] L. H. ELIASSON, Floquet solutions for the one-dimensional quasiperiodic Schödinger equation, *Communications in Mathematical Physics* **146** (1992).

[Eli01] L. H. ELIASSON, Almost reducibility of linear quasiperiodic systems, *Proceedings of Symposia in Pure Mathematics* **69** (2001).

[Eli02] L. H. ELIASSON, Ergodic skew-systems on $T^d \times SO(3, R)$, *Ergodic Theory and Dynamical Systems* **22** (2002), 1429–1449.

[Eli09] L. H. ELIASSON, Résultats non-perturbatifs pour l'équation de Schrödinger et d'autres cocycles quasi-périodiques [d'après Avila, Bourgain, Jitomirskaya, Krikorian, Puig], *Astérisque* (2009), Exp. No. 988, viii, 197–217 (2010), Séminaire Bourbaki. Vol. 2007/2008. MR 2605323 (2011g:37057).

[FK04] B. FAYAD and A. KATOK, Constructions in elliptic dynamics, *Ergodic Theory and Dynamical Systems* **24** (2004), 1477–1520.

[FK09] B. FAYAD and R. KRIKORIAN, Rigidity results for quasiperiodic $\mathrm{SL}(2, \mathbb{R})$-cocycles, *J. Mod. Dyn.* **3** (2009), 497–510. MR 2587083 (2011f:37046).

[Fol95] G. B. FOLLAND, *A course in abstract harmonic analysis*, *Studies in Advanced Mathematics*, CRC Press, Boca Raton, FL, 1995. MR 1397028 (98c:43001).

[Fra00] K. FRACZEK, On cocycles with values in the group $SU(2)$, *Monaschefte für Mathematik* (2000), 279–307.

[Fra04] K. FRACZEK, On the degree of cocycles with values in the group $SU(2)$, *Israel Journal of Mathematics* **139** (2004), 293–317.

[Her83] M. R. Herman, Une mthode pour minorer les exposants de lyapounov et quelques exemples montrant le caractre local d'un thorme d'arnold et de moser sur le tore de dimension 2., *Commentarii mathematici Helvetici* **58** (1983), 453–502 (fre). Available at http://eudml.org/doc/139950.

[HP13] X. Hou and G. Popov, Rigidity of the reducibility of gevrey quasi-periodic cocycles on u (n), *arXiv preprint arXiv:1307.2954* (2013).

[HY09] X. Hou and J. You, Local rigidity of reducibility of analytic quasi-periodic cocycles on $\mathrm{U}(n)$, *Discrete Contin. Dyn. Syst.* **24** (2009), 441–454. MR 2486584 (2010e:37024).

[HY12] X. Hou and J. You, Almost reducibility and non-perturbative reducibility of quasi-periodic linear systems, *Invent math* **190** (2012), 209–260.

[JM82] R. Johnson and J. Moser, The rotation number for almost periodic potentials, *Comm. Math. Phys.* **84** (1982), 403–438. Available at http://projecteuclid.org/euclid.cmp/1103921211.

[Kar14a] N. Karaliolios, *Differentiable rigidity for quasi-periodic cocycles in semisimple compact Lie groups*, 2014. arXiv 1407.4799.

[Kar14b] N. Karaliolios, *Invariant distributions for quasiperiodic cocycles in* $\mathbb{T}^d \times SU(2)$, 2014. arXiv 1407.4763.

[Kar15] N. Karaliolios, *Global aspects of the reducibility of quasiperiodic cocycles in semisimple compact Lie groups*, 2015. arXiv 1505.04562.

[KT06] A. Katok and J.-P. Thouvenot, Spectral properties and combinatorial constructions in ergodic theory, *Handbook of dynamical systems* **1** (2006), 649–743.

[Khi63] A. Y. Khintchine, *Continued Fractions*, P. Noordhoff Ltd, 1963.

[Kri99a] R. Krikorian, Réductibilité des systèmes produits-croisés à valeurs dans des groupes compacts, *Astérisque* (1999), vi+216. MR 1732061 (2001f:37030).

[Kri99b] R. Krikorian, Réductibilité presque partout des flots fibrés quasi-périodiques à valeurs dans des groupes compacts, *Ann. Sci. École Norm. Sup. (4)* **32** (1999), 187–240. MR 1681809 (2000h:37033).

[Kri01] R. Krikorian, Global density of reducible quasi-periodic cocycles on $T^1 \times SU(2)$, *Annals of Mathematics* **154** (2001), 269–326.

[Par04] W. Parry, *Topics in ergodic theory*, **75**, Cambridge university press, 2004.

[SW71] E. M. Stein and G. Weiss, *Introduction to Fourier analysis on Euclidean spaces*, Princeton University Press, Princeton, N.J., 1971, Princeton Mathematical Series, No. 32. MR 0304972 (46 #4102).

[YZ13] J. You and Q. Zhou, Embedding of analytic quasi-periodic cocycles into analytic quasi-periodic linear systems and its applications, *Communications in Mathematical Physics* **323** (2013), 975–1005.