

Collisional phenomena and quasi-steady separation in high Reynolds number flows

A thesis submitted for the degree of
Doctor of Philosophy in Mathematics of
Imperial College London

by

Alexandre. F. J. DJEHIZIAN

Department of Mathematics
Imperial College London
London SW7 2AZ
United Kingdom

April 2021

I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

Alexandre DJEHIZIAN

Copyright declaration

The copyright of this thesis rests with the author. Unless otherwise indicated, its contents are licensed under a Creative Commons Attribution-Non Commercial 4.0 International Licence (CC BY-NC). Under this licence, you may copy and redistribute the material in any medium or format. You may also create and distribute modified versions of the work. This is on the condition that: you credit the author and do not use it, or any derivative works, for a commercial purpose. When reusing or sharing this work, ensure you make the licence terms clear to others by naming the licence and linking to the licence text. Where a work has been adapted, you should indicate that the work has been changed and describe those changes. Please seek permission from the copyright holder for uses of this work that are not included in this licence or permitted under UK Copyright Law.

To Jeanne, Eugène, Frédéric and my mum.

Abstract

Researchers have understood the significance of boundary layers in the process of separation since Prandtl (1904) seminal paper. Nowadays, society brings up new challenges for both researchers and engineers, and understanding the underlying issues of flow separation is essential, as this area covers a large spectrum of applications. This thesis focuses on understanding unsteady boundary-layer separation with large Reynolds number Re , using theoretical and numerical tools. In particular, employing perturbation methods allows us to deduce many mathematical and physical information about the problems studied that experiments would fail to highlight.

The first part of this thesis deals with the unsteady flow generated by the rotation of a plane disk of finite radius. A free jet is generated from the rim of the disk, and it propagates away as time increases with a finite speed. Solving numerically the boundary-layer equations reveals that at the front of the jet, termed as a “pseudo-shock”, the axial velocity is singular, while the radial and circumferential velocities experience a jump. On the pseudo-shock’s scale, we show that as the jet penetrates the stagnant fluid, it ejects the fluid from the boundary layer into the region at the outer edge of the boundary layer, which is governed by the Euler equations. Their solution show that the ejected fluid is redirected upstream and reinjected back into the boundary layer through the “entrainment effect” process. An unsteady wall jet is discussed in Part II where similar conclusions are drawn overall.

The third part concerns the quasi-steady, supersonic flow past a flat plate where a shock wave/expansion fan impinges upon the boundary layer and moves in the downstream direction $\hat{x}$ parallel to the plate, with the dimensionless velocity $\bar{V}_{sh}$. This situation may occur when a shock wave, generated by an aircraft, hits another one during formation flying. Assuming $Re^{-1/8} \ll \bar{V}_{sh} \ll 1$, the interaction region may be described in the framework of the *triple-deck theory*, and the lower deck of the three-tiered structure is divided into two regions: an inviscid one, and a near-wall viscous sublayer *1b*. Setting the strength of the expansion fan such that the velocity at the outer edge of *1b* is equal to zero at $\hat{x} = 0$, the streamline slope in *1b* develops a singularity. A new shorter region is added to smooth it out, and where the pressure grows in the upstream direction, such that it finishes at a finite position where it becomes singular.

Acknowledgements

First and foremost, I would like to express my sincere appreciation to my supervisor, Pr. Anatoly Ruban. Throughout my entire thesis, he was truly involved in sharing his knowledge and experience with me. I cannot count the number of hours we spent discussing fluid dynamics and mathematics in his office, and I feel very privileged to have received his teaching and shared his passion. In addition to his flawless dedication, he always knew how to provide clear directives so that my research can progress in the right direction. At the same time, he offered me enough freedom to be creative, make my own mistakes, and in the end become a better doctoral student. I could have not wished for a better supervisor, and Pr. Ruban is, and will remain someone that I appreciate as a leading expert in the field, but also as a person. Thank you professor.

I would like to extend my deepest thanks to my family:

To Jeanne. To you who always looked after me in both good and hard times. We have gone through a lot together, and no matter what happened in life, I was always utterly convinced that everything was going to be alright, because I knew that you would always be there for me. I cannot thank you enough for everything you have done, and I am very proud to have shared the past fourteen years by your side.

To Eugène. To you who showed me that age is just a number. To these amazing moments with a glass (more likely the whole bottle) of whisky, gossiping about the whole world. To all these memorable breakfasts with the Aubagne dream team, getting hammered by 11 a.m. And to you, who made me understand that life, after all, is a wonderful adventure, and that being happy only requires the sun, cicadas, and a good glass of Ricard.

To Frédéric. My Godfather and Cousin. Thanks for sharing your knowledge with me and for our interesting chats. I always enjoy receiving your photographs of your work, it is a great reminder of home.

To Lili. For being the coolest cat in the world.

A huge thanks goes to my friends:

To Connor. Nineteen years of friendship and many more to come, filled with fantastic

and warm memories. I consider you to be part of the family. Boris approves, beurek and Irn-Bru for everybody!

To Jonas. For being so nice and kind. Every time we meet, it seems like no time has passed since we last saw each other. Thank you for all the laughs, and for being like a brother to me.

To Fabrice. For your kindness and support without asking for anything in return.

To Quentin. Thank you for all the years we spent at each others' house while growing up and for being such a great and funny friend.

To Charlotte. One of my oldest friends. I would have been so happy to be told as a kid that after all these years we would still be such close friends.

To Clémence. I still see in my head when we were little, going to school together. Such great memories, thank you for being a long time friend.

To Thibaut. Thank you for your support and your good music suggestions.

To Oumaima. Thank you for our endless calls, chatting about all the topics in the universe. You know that you can always count on me.

To Jean-Baptiste. Thank you for being such a great friend. I always enjoy very much chatting about all our shared passions. I have great memories of all the moments in Douai, London and Paris. Looking forward to going to Andertons together!

To Fatma. Thank you for your infectious happiness and laughs and your good advice. Thanks for being so kind and also for your amazing pastries (that I cannot wait to try again!)

To Ronan. For being constantly happy, and being someone making the most of the simple things in life. Thank you for these crazy and wild weekends in London.

To Loïs. For being so chill and funny. I have a great memory of all the laughs and parties we had together.

To Jonathan. For being funny and so trustworthy. Thank you for these amazing holidays in Nice, doing longboarding, playing ukulele and having a blast at Wayne's bar.

To Grace and Jure. Thank you for all the good moments in Gradpad, I had a lot of fun with you.

To Julie. Thank you for all our pleasant chats, and for always taking an interest in what is going on in my life.

To Claude and Denis. Thank you for all the times you looked after me when I was a child. You are always so welcoming and you make me feel like part of the family.

To Gilles and Catherine. For your rock solid great sense of humour and for all the times you made sure I was feeling at home.

To Mike and Margaret. For being so kind, lively, and always smiling.

To Jonas and Elena. For being so happy and for all your good vibes.

To Ruth, Tim, and Tom. Thank you for your kindness and your hospitality.

Last but not least, a special thanks to Joely. The past years would not have been the same without you. You always make me smile and laugh, and you have become a vital part of my life. Now the past is filled with lovely memories with you, I cannot wait to find out what the future holds for us.

Contents

Abstract	5
Acknowledgements	6
List of Figures	12
List of Tables	16
 I Discontinuous solutions of the unsteady boundary-layer equations for a rotating disk of finite radius	 17
1 Introduction	18
1.1 Overview of the problem	20
1.2 Problem outline	22
2 Problem formulation	25
2.1 Flow regime a	31
2.2 Flow regime b	34
2.3 Entrainment effect	42
3 Computational analysis	48
4 Numerical results	57
5 Internal structure of the pseudo-shock	68
5.1 New mathematical model in the vicinity of the pseudo-shock	68
5.2 Prediction of the flow behaviour on the pseudo-shock's scale	72
5.3 Solving numerically the governing equations using a stream-function vorticity formulation	76
6 Wake immediately downstream of the disk edge	86
7 Discussion	93

II	The unsteady analogue of the Glauert laminar jet	95
8	Introduction	96
9	Problem formulation	98
9.1	Formulation of the governing equations of the flow	98
9.2	Boundary-layer scale	102
9.3	Entrainment effect	104
10	Computational methods to solve numerically the boundary-layer equations for the wall jet	106
11	Numerical results	110
12	Internal structure of the pseudo-shock for the plane wall jet	115
12.1	In the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock	115
12.2	Near-wall viscous sublayer	120
13	Discussion	123
III	Quasi-steady boundary-layer separation in supersonic flow with a downstream moving separation point	124
14	Introduction	125
14.1	Overview of the problem	125
14.2	Problem outline	131
15	Problem formulation	133
16	Flow analysis in the inviscid region 1a	146
17	Numerical solution of the triple-deck problem	154
18	Flow analysis in the viscous region 1b	159
18.1	Governing equations in the viscous region 1b	159
18.2	Region 1b'	163
18.3	Displacement effect of region 1b	166
18.4	Displacement effect of region 1a	169

19 Flow near the singularity. Inviscid region C	172
19.1 Computation of the pressure distribution in region C	181
19.2 Analysis of the singularity	186
20 Discussion	188
21 Conclusion of the thesis	190
 Appendix	 195
A Thomas algorithm	195
B Von Mises variables transformation	197
C The trapezoidal rule	199

List of Figures

1.1	Illustration of the changing velocity profile when the boundary layer is subject to a strong adverse pressure gradient leading to separation. Region 3 corresponds to the external inviscid flow. Region 2 is the boundary-layer, whose dimensionless thickness is $O\left(Re^{-1/2}\right)$, where Re is the Reynolds number. Region 1 is the separation region, where recirculation regions develop. S is where the flow detaches from the body surface, and it is defined as the point where the skin friction is equal to zero for the first time.	19
1.2	Cylindrical coordinates. On the drawing on the right, the disk is being observed from the top.	23
2.1	F_r plotted against η	34
2.2	f plotted against η	35
2.3	F_z plotted against η	35
2.4	Problem layout during regime b . A free jet is generated by the rotation of the disk. The fluid is stagnant in region 3. Region 2 corresponds to the outer inviscid region where the entrainment effect takes place. Region 1 is the viscous boundary layer. The part for negative Z has been omitted.	38
3.1	Stencil in Z -direction. A second-order accuracy scheme is used between the nodes $k+1/2$ and $k-1/2$ for first-order derivatives with respect to Z . The values of U and W at the nodes $k+1/2$ and $k-1/2$ are calculated through an arithmetic mean between k and $k+1$, and between k and $k-1$, respectively.	50
4.1	Radial velocity along the jet axis at time $t = 4, 6, 8, 10, 12$. Discontinuity develops at the position of the pseudo-shock. Immediately before the shock, the fluid is stagnant.	58
4.2	Circumferential velocity along the jet axis at time $t = 4, 6, 8, 10, 12$	58

4.3	Axial velocity at $Z = Z_{\max}$ at time $t = 4, 6, 8, 10, 12$. Behind the pseudo-shock, it is negative due to the entrainment effect taking place at the outer edge of the boundary layer. There is a burst at the pseudo-shock position due to the discontinuity of the radial velocity component.	59
4.4	Streamline pattern on the boundary-layer scale at $t = 12$ in the case of the flow generated by the rotation of the disk	60
4.5	Radial velocity on the jet axis at time $t = 5$ for different meshes. Δr is the mesh parameter which has the most impact on the sharpness of the discontinuity at the position of the pseudo-shock. There is almost no difference between the plots for $\Delta r = 0.001$ and $\Delta r = 0.002$, where we can observe a sharp gradient at the position of the pseudo-shock. . .	61
4.6	Axial velocity at $Z = Z_{\max}$ at time $t = 4$ for different values of Δr	62
4.7	Radial velocity for small values of time.	63
4.8	Comparison of the Kármán's solution F with U/r at $t = 4$ and $t = 6$ for $r = 0.5$	64
4.9	Streamline pattern of the flow in region 2 at $t = 6$. At the pseudo-shock's position $r_s(t = 6) = 1.564$, fluid particles are ejected from the boundary layer. This happens when the jet penetrates the stagnant fluid, which acts as a point source. These fluid particles are redirected upstream of the shock, and they are injected again into the boundary layer further upstream.	65
4.10	Zoom closer to the point source.	66
5.1	Evolution of the shock's position with time	70
5.2	Inner structure of the pseudo-shock. A uniform velocity profile enters the domain when $X \rightarrow \infty$. The initial velocity profile when $X \rightarrow -\infty$ is first positive, and turns negative at point E . We expect a stagnation point at point S . A large recirculation eddy is formed within the DSF area.	73
5.3	Streamline pattern of the inner structure of the pseudo-shock generated by the rotation of the disk at $t = 8$. We observe a clear dividing streamline and a recirculation region below it.	83
5.4	Influence of ν_a on the dividing streamline (along which $\tilde{\psi} = 0$).	83
5.5	Velocity and pressure along $Z = 0$ plotted with respect to X in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock.	84

6.1	Axial velocity on the boundary-layer scale at $Z = 0.02$ and $t = 10$. A singularity, represented by a “spike” which becomes bigger as Δr becomes smaller, appears immediately downstream of the disk edge at $r = 1^+$	88
6.2	Solutions u_1 and u_2 of the set of equations (6.14a)-(6.14e)	91
9.1	Problem layout. Fluid is injected from a narrow slit in the impermeable barrier on the $(O, \hat{y})$ axis. Contrary to the free jet generated by the rotation of the disk presented in Part I, this wall jet is not symmetric; the matching condition with the stagnant fluid still holds at the upper edge of the jet, but on the other side of it, the no-slip condition must be satisfied on the motionless wall surface.	99
11.1	Longitudinal velocity at $Y = 1$ for time $t = 4, 6, 8, 10, 12$	111
11.2	Lateral velocity at $Y = Y_{\max}$ for time $t = 4, 6, 8, 10, 12$	112
11.3	Dimensionless skin friction for time $t = 4, 6, 8, 10, 12$	112
11.4	Streamline pattern of the wall jet on the boundary-layer scale at $t = 12$	113
11.5	Longitudinal velocity u for small values of time t at $Y = 1$	113
11.6	The entrainment effect occurring at the outer edge of the boundary layer. Like for the case of the rotating disk, the pseudo-shock acts as a point source.	114
12.1	Inner structure of the pseudo-shock	117
12.2	Streamline pattern on the pseudo-shock's scale at $t = 6$. Along the red streamline, $\tilde{\psi} = -0.05$, and along the blue streamline, $\tilde{\psi} = -0.01$. The three streamlines below the blue streamline have been plotted for $\tilde{\psi} = 0$, $\tilde{\psi} = 0.1$ and $\tilde{\psi} = 0.2$, starting from the top one to the bottom one. The streamlines above the red streamline have been plotted from $\tilde{\psi} = -0.1$ to $\tilde{\psi} = -2.1$, with the interval $\Delta\tilde{\psi} = -0.1$ between each streamline. . .	119
12.3	Streamline pattern in the near-wall viscous sublayer.	122
14.1	Boundary-layer separation on a downstream moving wall.	129
15.1	Problem layout. In the Figure, an expansion fan is generated from an expansion ramp, and it moves with the velocity $\hat{V}_{\text{sh}}$ downstream of the flow.	135
15.2	Three-tiered structure of the interaction region	138
15.3	Splitting of region 1	145

16.1	The longitudinal velocity profile U_* for different values of positive X_* .	153
17.1	The streamline pattern in the boundary layer exposed to an impinging expansion fan	156
17.2	Pressure variation in the interaction region. Dashed line shows the theoretical solution (16.24)	157
18.1	Formation of region $1b'$. The variable η is of order one in region $1b'$ in the limits as $\Psi \rightarrow -\infty$ and $X_* \rightarrow 0^+$. It will be introduced in Section 18.2.	161
18.2	Formation of regions $1a'$ and $1b'$	164
19.1	Formation of region C . It is an extension of regions $1a'$ and $1b'$, and it will smooth out the solution when approaching $X_* = 0^+$	173
19.2	Streamline pattern in region C	177
19.3	Pressure variation in region C for $\Psi_M = -0.4$ and $\chi = 3.0, 4.0$ and 5.0 .	185
19.4	Logarithm of the pressure $\check{P}$ plotted with respect to the logarithm of $\check{X} - \check{X}_0$ for $\chi = 3, 5$. $\check{X}_0$ is the position of the singularity, found numerically.	187

List of Tables

4.1	Mesh parameters for the rotating disk on the boundary-layer scale . . .	57
4.2	Mesh comparison. The bold quantities correspond to the mesh parameter changed as compared to the previous mesh.	61
5.1	Comparison between $V_{s,\text{inv}}(t)$ and $V_{s,\text{num}}(t)$	76
5.2	Mesh parameters used to solve the discretised governing equations in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock	81
11.1	Mesh parameters for the wall jet on the boundary-layer scale	110
12.1	Mesh parameters used to solve numerically the governing equations (12.6a)- (12.6h) in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock.	119
19.1	Mesh parameters for calculation of the pressure $\check{P}$ in region C . We recall that $\chi = 2 \ln(\sigma) $	185

Part I

Discontinuous solutions of the
unsteady boundary-layer equations
for a rotating disk of finite radius

Chapter 1

Introduction

In the course of this thesis, we are interested in understanding the complex behaviour of unsteady boundary-layer separation in high Reynolds number flows. Flow separation is a common phenomenon in fluid dynamics; in fact, many limitations must be imposed on the geometry of the body placed in the flow in order to avoid separation. When the boundary layer breaks away from the body surface, it has a significant impact on the flow. For instance, if we consider the flow separation from the wing of an aircraft with an angle of attack which is too large, we observe a significant distortion of the streamline pattern. Eddies and recirculation regions erupt from the boundary layer, leading to a decrease of the lift, and a large growth of the drag. Another example is a pump. Fluid sucked into a pipe thanks to a pump may separate if the edges at the entrance of the pipe are too sharp, or if the local curvature is too large. This would have a direct impact on the yield of the pump, being less efficient. An aircraft undergoing an increased drag and a smaller lift will have to consume more fuel, implying an increase cost for the company. In addition to yield and economy issues, the third problem, and probably the most important one, is about security. Flow separation may lead to a loss of control of a plane, impacting the security of the passengers. As the aerofoils are changing to adjust to the new concept of laminar wing, understanding of boundary-layer separation is essential with nowadays new challenges.

In his seminal paper presented at the third International Congress of Mathematicians in Heidelberg, Prandtl (1904) introduced the concept of boundary layers in order to explain the process of flow separation in high Reynolds number flows. A boundary layer is a thin region in the direct neighbourhood of a body placed in a flow, where the viscous terms become in balance with the convective terms. It allows the smoothing of the velocity from the external inviscid region, up to the body surface where the no-slip condition must be satisfied. Although Prandtl did not fully appreciate the subtleties of the physical and mathematical aspects, it became clear that boundary layers play a key role in flow separation. Since then, many physicists and mathematicians have studied

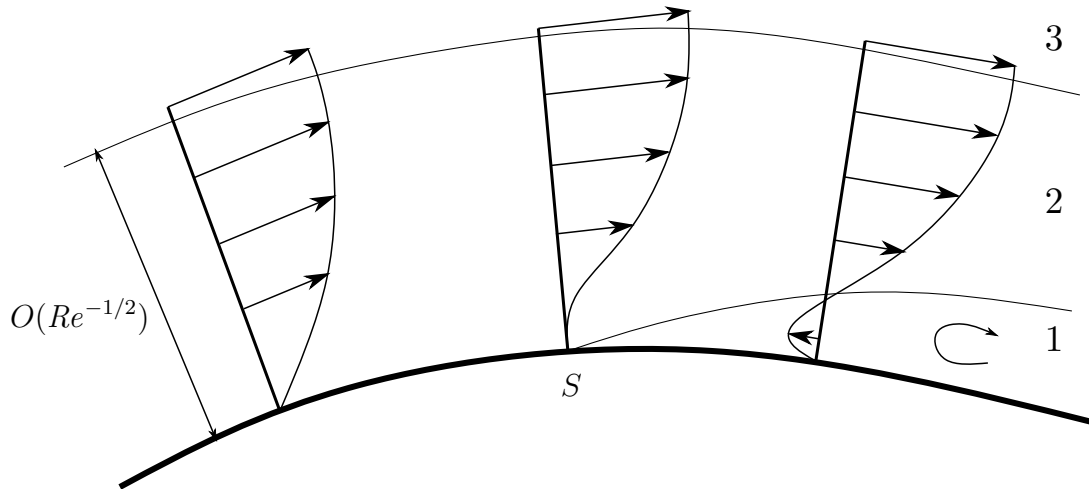


Figure 1.1: Illustration of the changing velocity profile when the boundary layer is subject to a strong adverse pressure gradient leading to separation. Region 3 corresponds to the external inviscid flow. Region 2 is the boundary-layer, whose dimensionless thickness is $O(Re^{-1/2})$, where Re is the Reynolds number. Region 1 is the separation region, where recirculation regions develop. S is where the flow detaches from the body surface, and it is defined as the point where the skin friction is equal to zero for the first time.

the subject both theoretically and experimentally, and nowadays, the steady case can be described by the triple-deck theory (this theory will be described more in details in Chapter 15. It shows that, if the Reynolds number is assumed to be large, when the flow is subject to a strong enough adverse pressure gradient, boundary-layer separation may be described by an interaction between the boundary layer and the external inviscid region, termed as the *viscous-inviscid interaction*. Even though the triple-deck model works when the separation point is motionless or when it moves “slowly”, it is not valid in the case of unsteady flows. The unsteady case is more complicated, and the current literature, despite continuous development of the subject, is still in its infancy (see Sychev (1979, 1984, 1987), van Dommelen and Shen (1983b), Sychev et al. (1998)). This thesis focuses on two aspects of unsteady boundary-layer separation. The first one concerns collisional singularities, presented in two distinct problems in Part I and Part II, respectively. The second aspect, presented in Part III, deals with understanding if and when the boundary layer can develop a singularity characteristic of the flow separations from the surface of a body moving upstream of the oncoming flow.

1.1 Overview of the problem

Flow separation has been a particularly active field of research in fluid dynamics for years. Although the first separated model was suggested by Kirchhoff (1869), Prandtl (1904)'s renowned boundary-layer theory was the cornerstone of the development of the mathematical and physical analysis of the separation process. In his seminal paper, Prandtl presented his results about the incompressible flow past a rigid flat plate at high Reynolds number. Even though he managed to deduce the boundary-layer equations governing the flow in the direct neighbourhood of the body surface, he did not solve them; that was Blasius (1908), Prandtl's PhD student, who solved them a few years later using a similarity solution. Instead, Prandtl focused on the separation process in detail. He stated that boundary-layer separation was due to a gradual increase of the pressure: under this condition, the fluid particles adjacent to the body surface undergo a deceleration and the velocity profile starts to change. The ones closer to the surface of the body are more affected by the adverse pressure gradient, as they have less kinetic energy: therefore, they are more sensitive to pressure variations. Further downstream, they stop at the point of separation S ; see Figure 1.1. This causes the boundary layer to break away from the body surface at the separation point, leading to a reverse flow region downstream and containing recirculation regions and eddies. According to Prandtl, the separation point S is defined as the point where the skin friction is equal to zero for the first time, and its position could be deduced by solving the boundary-layer equations.

Although Prandtl's analysis described rather well the physical processes leading to flow separation, a number of aspects remained unclear, especially regarding the fact that the recirculation regions do not stay confined within the boundary layer; instead, they erupt from it, and their characteristic size is comparable with the size of the body. Later, it became obvious that the Prandtl (1904)'s theory was contradictory because of the Goldstein (1948)'s singularity, located at the point of zero skin friction. Now, we know that separation is not caused by a gradual pressure rise but by a sharp pressure increase in a small region in the vicinity of the separation point where the interaction region forms. We also know that separation is the result of an interaction between the boundary layer and the external inviscid flow, termed as the *viscous-inviscid interaction*. The mathematical theory of the viscous-inviscid interaction, based on asymptotic analysis, is called the *triple-deck theory*. It was formulated simultaneously by Neiland

(1969b) and Stewartson and Williams (1969) for the self-induced separation in a supersonic flow. To these two publications were added the contributions of Stewartson (1969) and Messiter (1970) who both worked on the incompressible flow near the trailing edge of a flat plate. A detailed description of the triple-deck theory will be presented in Chapter 15.

Numerous researchers have contributed to the triple-deck model ever since, and it became clear that the viscous-inviscid interaction plays a significant role in many fluid flows. However, the theory of separated flows remains incomplete, as the triple-deck model is mainly reliable for two-dimensional and steady flows, or when the separation point moves slowly. There is currently much less theoretical work on the subject for three-dimensional and/or unsteady flows, which appear to be way more complicated than their steady two-dimensional counterparts. The addition of one more independent spatial and/or temporal variable may have major impact of the flow behaviour. This is partly due to the fact that it may change the type of the governing equations of the flow. Thus, it may lead to alternative forms of separation which do not even involve flow reversal. A prime example of such an alternative is the so-called *collisional singularity*, often developed in three-dimensional boundary-layers, and highlighted by Stewartson et al. (1980), Kluwick and Wohlfahrt (1986) and Allen and Riley (1994). This phenomenon occurs when fluid particles in the boundary layer are deflected in the “spanwise” direction causing the fluid to move away, in the normal direction from the body surface, leading to displacement of the streamlines from it, and eruption of a jet. The same phenomenon may be observed in two-dimensional unsteady boundary-layers. Several authors focused on these particular type of flows. Among them, Degani et al. (1996) studied the unsteady boundary-layer separation from the leading edge of a pitching up aerofoil. Their numerical results show that if the angle of attack exceeds a critical angle α^* which depends on the aerofoil shape, then a singularity develops at a finite time t_0 in the boundary-layer; this triggers a viscous-inviscid interaction process.

Another case of collisional singularity has been studied in detail by Ruban and Vonatsos (2008). They were the first researchers to prove that Prandtl’s boundary-layer equations admitted a class of discontinuous solutions. In addition to the proof of existence, they also showed that under certain conditions, these solutions are the only possible solutions of the boundary-layer equations. Their paper focuses on the unsteady analogue of the Schlichting jet (1933). Schlichting considered a semi-infinite region filled with fluid, bounded by a flat and impermeable barrier. He investigated the two-dimensional, steady free jet emerging from a narrow slit in the barrier. He

showed that the solution of the boundary-layer equations could be found analytically, and that it exists in the form of a similarity solution. Ruban and Vonatsos (2008) considered that initially, the narrow slit is closed and that the fluid in the region is stagnant. At time $\hat{t} = 0$, the slit is opened and a free jet starts to propagate as time increases. The solution of the unsteady boundary-layer equations shows a well-established front coinciding with a discontinuity in the longitudinal velocity. They termed these discontinuities as a “pseudo-shock”. Part I and Part II of this thesis deal with an unsteady free jet, and an unsteady wall jet, respectively. The introduction of an additional independent time variable allows for discontinuities to develop in the solution of the boundary-layer equations. It is crucial to track such discontinuities, as one necessary condition for separation is to observe singularities developing due to discontinuities. We will show that pseudo-shocks are related to collisional singularity topics in Chapters 5 and 12, related to the internal structure of the pseudo-shock for the free jet and the wall jet, respectively. When these jets propagate as time increases, they penetrate the initially stagnant fluid in the surrounding, leading to a jump in the streamwise velocity components, while the “spanwise” velocity component develop a singularity. The collision between the propagating pseudo-shock and the stagnant fluid is the reason why pseudo-shocks are linked to collisional phenomena. Furthermore, by introducing an additional region in the vicinity of the pseudo-shock structure in order to smooth out the singularity, we will show that fluid erupts from the boundary layer at the position of the pseudo-shock into the surrounding area. Thus, collisional phenomena can be categorised as a new type of separation. Although the pseudo-shock is formed at the front of the jet in the case of Ruban and Vonatsos, it may also develop in the middle of the jet if the momentum flux increases with time (see Hewitt and Duck (2011)). Even though the pseudo-shock in Hewitt and Duck is collisional, it is a different type of singularity as Hewitt and Duck’s shock develops at a finite time, while Ruban and Vonatsos’ shock develops immediately after the narrow slit is opened at $\hat{t} = 0$. Similarly to Ruban and Vonatsos (2008), this part of the thesis shows the existence of discontinuous solutions.

1.2 Problem outline

Here, we shall study the free jet generated by the rotation of a plane disk of finite radius. We shall show that when the jet penetrates the stagnant fluid, it ejects fluid in

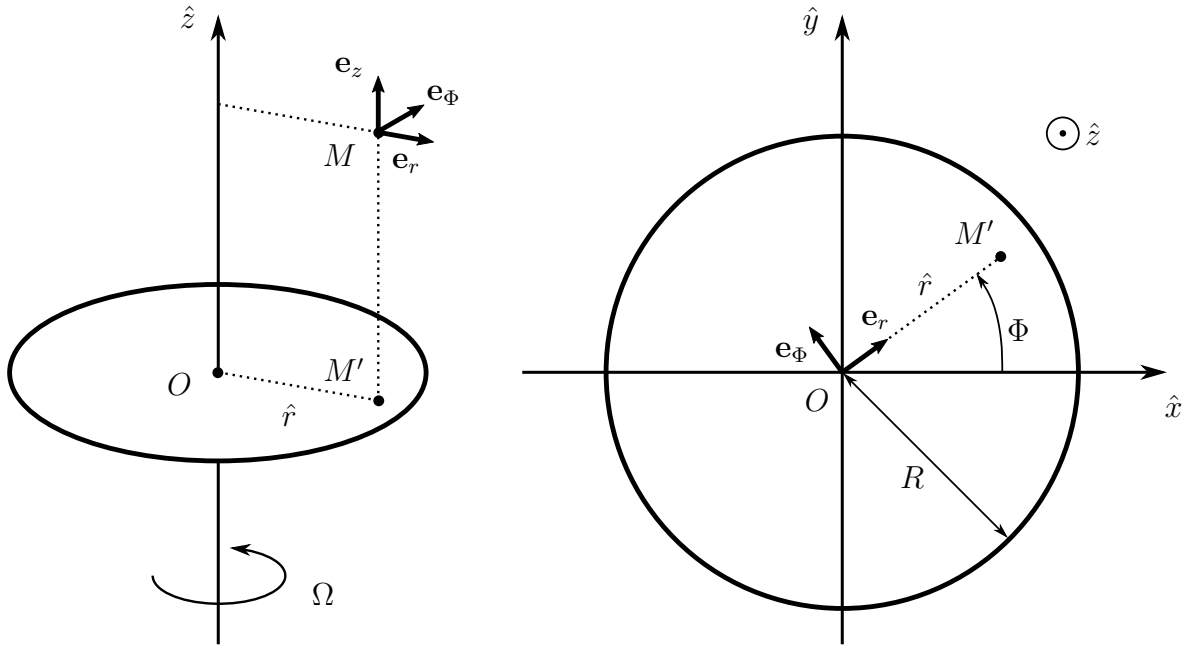


Figure 1.2: Cylindrical coordinates. On the drawing on the right, the disk is being observed from the top.

the surrounding area. This fluid is injected back into the boundary layer through the region outside of it, where the entrainment effect takes place. The jet propagates along the positive $\hat{r}$ - and Φ -directions as time $\hat{t}$ increases, where $\hat{r}$ and Φ correspond to the radial and circumferential directions respectively; see Figure 1.2.

It is observed on the boundary-layer's scale, as the governing equations are hyperbolic in $(\hat{r}, \hat{t})$ in this region, allowing for discontinuities to develop. The analysis is performed by assuming that the Reynolds number Re is large. This assumption allows us to reduce the Navier-Stokes equations to the boundary-layer equations. After establishing the set of boundary-layer equations, we express them in conservative form (since discontinuities are expected) in Chapter 2, and then we solve these equations numerically in Chapter 3. The plots of the solution are presented and discussed in Chapter 4. We shall see that a jet with a well-established front is generated from the rim of the disk, and it propagates away in radial and circumferential directions with increasing time. The radial and circumferential components of the velocity are discontinuous, similarly to shock waves in gas dynamics. Immediately before the discontinuity, the fluid is at rest. Following Ruban and Vonatsos (2008), we shall term these discontinuities as “pseudo”-shock for the following reason: the discontinuity of classical shock waves in gas dynamics may be smoothed out, and their characteristic thickness are compa-

rable with the molecular mean free path, which is an $O(Re^{-1})$ quantity. Hence, the continuum hypothesis is not applicable, and the Boltzmann equation of the kinetic gas theory must be used in order to describe the internal structure of the shocks. In our case, we shall deduce in Chapter 5 that the characteristic length of the pseudo-shock is an $O(Re^{-1/2})$ quantity. Therefore, the shock is smoothed out over a much larger length, and the Navier-Stokes equations may be used to describe the internal structure of the shock.

It will be shown in Chapter 5 that the flow in the region of the pseudo-shock may be treated as inviscid, and quasi-steady if studied in the reference frame moving with the the leading edge of the jet. These simplifications allowed us to determine the speed of the shock, which is exactly equal to half the radial velocity on the jet axis, immediately behind the pseudo-shock. Turner (1962) considered a roof tank, filled with a salted solution, which is placed on top of a large tank filled with fresh water. An orifice is placed at the bottom of the roof tank, and a tap is placed nearby that orifice to preclude the salted solution to flow down. At time $\hat{t} = 0$, the tap is opened and a jet starts to flow down the large tank. We will see in Section 5.2 that this result is in close agreement with the experimental observations of Turner (1962)

The next step is to solve numerically, by applying pseudo-transient and multigrid splitting methods, the governing equations of the flow in the region in the vicinity of the pseudo-shock, written with a stream function-vorticity formulation. The streamline pattern of the flow reveals a dividing streamline, delimiting the jet front, and a recirculation region in the pseudo-shock. The plot of the pressure on the jet axis in the region in the vicinity of the pseudo-shock reveals that the pressure reaches its maximum when the streamwise velocity becomes zero on the jet axis; see at the end of Chapter 5. Finally, in Chapter 6, the wake immediately after the rim of the disk will be analysed. The flow appears to be quasi-steady and quasi-two-dimensional, and the solution is found similar to that of Goldstein (1930)'s wake.

Let us notice that this rotating disk problem is similar to the problem of the impulsively rotating sphere, which has been investigated by various authors. In particular, recently, Calabretto et al. (2019) investigated the flow experimentally and numerically using Navier-stokes simulations. The approach of this thesis is different as our goal is to obtain analytically the asymptotic expansions of the flow functions in the limit as $Re \rightarrow \infty$.

Chapter 2

Problem formulation

We place a thin plane disk of finite radius R into an infinite region filled with an initially stagnant fluid. At time $\hat{t} = 0$, the disk is brought into rotation impulsively around the axis perpendicular to its surface and passing through its center, at the constant angular velocity Ω . Owing to the action of the viscous forces, the fluid particles adjacent to the disk start to rotate, but their trajectories are not circular: they also tend to move away from the axis of rotation due to inertia forces. In addition to the layer of rotating fluid developing above and below the disk, a jet is generated from the rim of the rotating disk, and it propagates radially (and circumferentially) away as time increases.

Because of the topology of the flow, it is convenient to use cylindrical polar coordinates $(\hat{r}, \Phi, \hat{z})$ such that the origin O coincides with the center of the disk, and the axis $(O, \hat{z})$ coincides with the axis of rotation of the disk. This coordinate system is presented in Figure 1.2.

Thus, a point M of space may be located by the three coordinates $\hat{r}$, Φ and $\hat{z}$ such that $\hat{z}$ is the distance between M and M' , where M' is the projection of M to the $(\hat{r}, \Phi)$ plane, $\hat{r}$ measures the distance from M to the axis of rotation $(O\hat{z})$ while Φ is the angle between the $\hat{x}$ -axis and the line (OM') . The unit vectors along the $\hat{r}$ -, Φ - and $\hat{z}$ - axes are denoted respectively $\mathbf{e}_r$, $\mathbf{e}_\Phi$ and $\mathbf{e}_z$.

The problem is assumed to be axisymmetric as the solution is invariant with respect to an arbitrary rotation through an angle $\Delta\Phi$, so all the flow quantities are independent of the circumferential coordinate Φ :

$$\frac{\partial}{\partial \Phi} = 0. \quad (2.1)$$

Moreover, the flow is symmetric with respect to the plane containing the disk, so we shall study the problem for positive $\hat{z}$ only, the solution for negative $\hat{z}$ may be deduced by symmetry.

Our first goal is to solve the unsteady boundary-layer equations dictating the behaviour of the flow. Contrary to their steady counterpart, they are not simply parabolic; they have elements of hyperbolicity due to the sub-hyperbolic operators

$$\frac{\partial \widehat{V}_r}{\partial \widehat{t}} + \widehat{V}_r \frac{\partial \widehat{V}_r}{\partial \widehat{r}}, \quad \text{and} \quad \frac{\partial \widehat{V}_\Phi}{\partial \widehat{t}} + \widehat{V}_r \frac{\partial \widehat{V}_\Phi}{\partial \widehat{r}}$$

in the radial and circumferential momentum equations (with $\widehat{V}_r$ and $\widehat{V}_\Phi$ being the radial and circumferential velocity components, respectively). Because of the parabolic and hyperbolic natures of the boundary-layer equations, they are said to be of mixed-type. It is because of their hyperbolic feature that we expect discontinuities to develop in the flow fields. Indeed, the projected characteristics curves coincide with the trajectories of the fluid particles in the $(\widehat{r}, \widehat{t})$ plane. The solution evolves along each projected characteristic curve (but the value of the solution is different on different characteristics). When they intersect, the solution develops a collisional singularity, followed by a discontinuity in the velocity field. This discontinuity has been termed as a “pseudo”-shock by Ruban and Vonatsos (2008), as mentioned earlier. Naturally, the third dimension $\widehat{z}$ with the axial velocity $\widehat{V}_z$ must be also considered so that mass conservation can be satisfied.

Tracking these discontinuities is crucial within the scope of this thesis. Indeed, flow separation from the axis of the jet is due to a sharp pressure increase, distributed over a short distance. This large pressure gradient implies we will observe discontinuities. Obviously, the boundary-layer equations are not valid in the vicinity of these discontinuities. When Prandtl (1904) formulated his boundary-layer theory, he assumed that boundary-layer separation was triggered by a gradual increase of the pressure and that all the flow longitudinal gradients were small as compared to the ones in transverse direction. However, this assumption does not hold if discontinuities form in the flow fields. As pointed out by Ruban and Vonatsos (2008) in their study of the unsteady analogue of the laminar free jet formulated by Schlichting (1933), discontinuities may develop under certain circumstances owing to the nature of the governing equations. We will show in Chapter 5 that in the vicinity of the pseudo-shock’s position, the flow is inviscid in the leading-order approximation. The pressure gradient is restored into these equations, and it is “large” where the discontinuities are located.

We consider that the fluid is viscous, incompressible, Newtonian, and we neglect the

body forces acting on the fluid such as gravity. Let us denote by ρ , μ , ν , the density, dynamic viscosity and kinematic viscosity, respectively. The quantity p_∞ corresponds to the pressure in the stagnant part of the fluid. We also introduce $\widehat{V}_r$, $\widehat{V}_\Phi$, $\widehat{V}_z$, $\widehat{p}$ the radial, circumferential, axial components of the velocity, and the pressure, respectively. The hat stands for dimensional quantities. The flow is described by the unsteady Navier-Stokes equations written in cylindrical polar coordinates as

$$\frac{\partial \widehat{V}_r}{\partial \widehat{t}} + \widehat{V}_r \frac{\partial \widehat{V}_r}{\partial \widehat{r}} + \widehat{V}_z \frac{\partial \widehat{V}_r}{\partial \widehat{z}} - \frac{\widehat{V}_\Phi^2}{\widehat{r}} = -\frac{1}{\rho} \frac{\partial \widehat{p}}{\partial \widehat{r}} + \nu \left(\frac{\partial^2 \widehat{V}_r}{\partial \widehat{z}^2} + \frac{\partial^2 \widehat{V}_r}{\partial \widehat{r}^2} + \frac{1}{\widehat{r}} \frac{\partial \widehat{V}_r}{\partial \widehat{r}} - \frac{\widehat{V}_r}{\widehat{r}^2} \right), \quad (2.2a)$$

$$\frac{\partial \widehat{V}_\Phi}{\partial \widehat{t}} + \widehat{V}_r \frac{\partial \widehat{V}_\Phi}{\partial \widehat{r}} + \widehat{V}_z \frac{\partial \widehat{V}_\Phi}{\partial \widehat{z}} + \frac{\widehat{V}_r \widehat{V}_\Phi}{\widehat{r}} = \nu \left(\frac{\partial^2 \widehat{V}_\Phi}{\partial \widehat{z}^2} + \frac{\partial^2 \widehat{V}_\Phi}{\partial \widehat{r}^2} + \frac{1}{\widehat{r}} \frac{\partial \widehat{V}_\Phi}{\partial \widehat{r}} - \frac{\widehat{V}_\Phi}{\widehat{r}^2} \right), \quad (2.2b)$$

$$\frac{\partial \widehat{V}_z}{\partial \widehat{t}} + \widehat{V}_r \frac{\partial \widehat{V}_z}{\partial \widehat{r}} + \widehat{V}_z \frac{\partial \widehat{V}_z}{\partial \widehat{z}} = -\frac{1}{\rho} \frac{\partial \widehat{p}}{\partial \widehat{z}} + \nu \left(\frac{\partial^2 \widehat{V}_z}{\partial \widehat{z}^2} + \frac{\partial^2 \widehat{V}_z}{\partial \widehat{r}^2} + \frac{1}{\widehat{r}} \frac{\partial \widehat{V}_z}{\partial \widehat{r}} \right), \quad (2.2c)$$

$$\frac{\partial \widehat{V}_r}{\partial \widehat{r}} + \frac{\partial \widehat{V}_z}{\partial \widehat{z}} + \frac{\widehat{V}_r}{\widehat{r}} = 0. \quad (2.2d)$$

One must formulate boundary conditions in order to get a full description of the problem.

As we discussed earlier, a layer of rotating fluid is formed above and below the rotating disk due to the action of viscous forces, and the fluid particles adjacent to the disk surface follow its rotation, that is why the no-slip condition must be satisfied on the disk surface. Besides, the impermeability condition also holds as no fluid particle will flow through the disk. Hence, for all $\widehat{t} \geq 0$,

$$\widehat{V}_\Phi = \Omega \widehat{r}, \quad \widehat{V}_r = \widehat{V}_z = 0, \quad \text{at } \widehat{z} = 0, \quad \widehat{r} \in [0, R]. \quad (2.2e)$$

On the axis of the jet propagating in the radial direction away from the disk rim, the symmetry conditions hold

$$\widehat{V}_z = \frac{\partial \widehat{V}_r}{\partial \widehat{z}} = \frac{\partial \widehat{V}_\Phi}{\partial \widehat{z}} = 0, \quad \text{at } \widehat{z} = 0, \quad \widehat{r} > R. \quad (2.2f)$$

Far from the disk, the fluid is stagnant

$$\widehat{V}_\Phi = 0, \quad \text{at } \widehat{r}^2 + \widehat{z}^2 = \infty. \quad (2.2g)$$

The motion of the fluid in the radial direction is caused by the centrifugal forces $\widehat{V}_\Phi^2/\widehat{r}^2$,

which is the fourth term on the left-hand side of the radial momentum equation (2.2a). According to that and to (2.2g), far from the disk the centrifugal term vanishes, so we deduce that the radial component of the velocity is equal to 0 outside the layer of rotating fluid

$$\widehat{V}_r = 0, \quad \text{at } \widehat{r}^2 + \widehat{z}^2 = \infty. \quad (2.2h)$$

Regarding the axial velocity component $\widehat{V}_z$, it is determined with the help of the continuity equation (2.2d). Far from the disk, $\widehat{V}_z$ satisfies

$$\widehat{V}_z = 0, \quad \text{at } \widehat{r}^2 + \widehat{z}^2 = \infty. \quad (2.2i)$$

The axial momentum equation (2.2c) is kept to compute the pressure field $\widehat{p}$. Thus, we do not need a boundary condition for $\widehat{V}_z$ for large values of $\widehat{z}$. Since the flow is at rest far from the disk, the pressure there is kept constant

$$\widehat{p} = p_\infty, \quad \text{at } \widehat{r}^2 + \widehat{z}^2 = \infty. \quad (2.2j)$$

On the axis of rotation of the disk, the fluid particles cannot move across, and a symmetry condition holds

$$\widehat{V}_r = \widehat{V}_\Phi = \frac{\partial \widehat{V}_z}{\partial \widehat{r}} = 0, \quad \text{at } \widehat{r} = 0. \quad (2.2k)$$

The boundary-value problem (2.2a)-(2.2k) is now fully established. It describes the flow motion everywhere in the domain for all values of $\widehat{t} \geq 0$. When the disk starts rotating at $\widehat{t} = 0$ with the constant angular velocity Ω , a layer of rotating fluid develops adjacent to the disk. Its thickness, which was initially equal to 0, grows as time increases. Based on the numerical results presented in Chapter 4, if Ω is kept constant for long enough, a steady state is reached: then the sought flow functions do not vary in time anymore, including the thickness of the viscous rotating fluid layer. Regarding the unsteady solution of the boundary-value problem (2.2a)-(2.2k), one must distinguish two flow regimes depending on the order of magnitude of time $\widehat{t}$. The first one is when $\widehat{t}$ is small, such that the convective terms of the momentum equations (2.2a), (2.2b) and (2.2c) are negligible as compared to the local accelerations $\partial \widehat{V}_r / \partial \widehat{t}$, $\partial \widehat{V}_\Phi / \partial \widehat{t}$ and $\partial \widehat{V}_z / \partial \widehat{t}$, respectively. A second regime must be considered when $\widehat{t}$ becomes larger and that the convective terms cannot be neglected anymore. Instead, they are of same order as the

local accelerations. For the sake of understanding, the former one may be called regime *a*, while the second one will be called regime *b*. During both regimes, the layer of moving fluid may be divided into two regions: a thin viscous layer adjacent to the disk surface and the jet axis $\hat{z} = 0$, and an outer inviscid layer whose thickness is of order R and where the entrainment effect takes place. A detailed discussion of these two regions will be presented later on. For now, the order of magnitudes of $\hat{t}$ corresponding to both regimes *a* and *b* may be determined by performing an inspection analysis of the sought flow functions in the viscous layer of rotating fluid.

For $\hat{t} \geq 0$, the circumferential velocity component must satisfy the no-slip condition (2.2e) on the disk surface, and it tends to 0 outside the viscous layer according to the boundary condition (2.2g). Thus,

$$\hat{V}_\Phi \sim \Omega \hat{r}. \quad (2.3)$$

Since the fluid is brought into motion due to the action of viscous forces, the thickness $\hat{z}$ of the viscous layer may be estimated by balancing the local acceleration with the viscous term in the circumferential momentum equation (2.2b)

$$\frac{\partial \hat{V}_\Phi}{\partial \hat{t}} \sim \nu \frac{\partial^2 \hat{V}_\Phi}{\partial \hat{z}^2}.$$

It results that

$$\hat{z} \sim \sqrt{\nu \hat{t}}. \quad (2.4)$$

As expected, $\hat{z}$ tends to 0 during regime *a* (that is when $\hat{t}$ is small), and $\hat{z} \ll \hat{r}$. During regime *b* when $\hat{t}$ becomes larger, $\hat{z}$ increases and satisfies $\hat{z} \sim \sqrt{\nu/\Omega}$.

The movement of fluid in the radial direction is dictated by the centrifugal term $\hat{V}_\Phi^2/\hat{r}$, so this term must be comparable with the local acceleration in the radial momentum equation (2.2a)

$$\frac{\partial \hat{V}_r}{\partial \hat{t}} \sim \frac{\hat{V}_\Phi^2}{\hat{r}}.$$

Using (2.3), we deduce that

$$\hat{V}_r \sim \Omega^2 \hat{r} \hat{t}. \quad (2.5)$$

The axial velocity component may be evaluated using the continuity equation (2.2d) by applying the principle of least degeneration:

$$\frac{\partial \hat{V}_r}{\partial \hat{r}} \sim \frac{\partial \hat{V}_z}{\partial \hat{z}}.$$

Using (2.4) and (2.5) for $\widehat{z}$ and $\widehat{V}_r$ in the above equation, leads to the conclusion that

$$\widehat{V}_z \sim \Omega^2 \nu^{1/2} \widehat{t}^{3/2}. \quad (2.6)$$

Finally, the pressure variation must be estimated. One can do this by studying the flow in the outer inviscid region where the entrainment effect takes place. Relation (2.6) is also valid in the outer region because the entrainment effect causes an injection of fluid into the viscous layer, so the axial velocity component is of same order in both regions. We balance the pressure gradient and the local acceleration in the axial momentum equation (2.2c)

$$\frac{\partial \widehat{V}_z}{\partial \widehat{t}} \sim \frac{1}{\rho} \frac{\partial \widehat{p}}{\partial \widehat{z}}.$$

Substitution of (2.6) and taking into account that in the inviscid region, $\widehat{z} \sim \widehat{r}$, it leads to

$$\widehat{p} - p_\infty \sim \rho \Omega^2 \widehat{r} \sqrt{\nu \widehat{t}}. \quad (2.7)$$

Thus, estimates of all flow quantities in the viscous layer have been determined, and they are all valid for regime *a*. The order of magnitude of $\widehat{t}$ during regime *b* can be found by balancing the convective terms with the local acceleration in the radial momentum equation (2.2a)

$$\frac{\partial \widehat{V}_r}{\partial \widehat{t}} \sim \widehat{V}_r \frac{\partial \widehat{V}_r}{\partial \widehat{r}}.$$

Using (2.5), it results that during regime *b*

$$\widehat{t} \sim \frac{1}{\Omega}. \quad (2.8)$$

Therefore, immediately after the disk is brought into rotation and as long as $\widehat{t} \ll 1/\Omega$, the flow is under regime *a*, implying that the convective terms are disregarded. Then, $\widehat{t}$ keeps increasing, and once $\widehat{t} \sim 1/\Omega$, regime *b* is reached and the convective terms become comparable with the local acceleration. If $\widehat{t}$ keeps increasing further, then the steady state is reached, and we shall see later that the solution satisfies the solution of Kármán (1921) for $|\widehat{r}| \leq R$. For now, we shall study both regimes *a* and *b*.

2.1 Flow regime a

Let us start with regime a when $\hat{t}$ is small, that is when $\hat{t} \ll 1/\Omega$. We shall find an analytical solution for regime a in the viscous layer for $|\hat{r}| \leq 1$ as the solution admits a self-similar form. However, for $\hat{r} > 1$, our arguments will not hold, and we will not be able to express the solution in an analytic form. Since it is easier to handle dimensionless quantities, let us introduce the dimensionless quantities t , r , z , V_r , V_Φ , V_z and p based on (2.3), (2.4), (2.5), (2.6) and (2.7):

$$\begin{aligned}\hat{t} &= \alpha t, \quad \hat{r} = Rr, \quad \hat{z} = \sqrt{\nu\alpha}z, \\ \hat{V}_r &= \Omega^2 R \alpha V_r, \quad \hat{V}_\Phi = \Omega R V_\Phi, \quad \hat{V}_z = \Omega^2 \nu^{1/2} \alpha^{3/2} V_z, \\ \hat{p} - p_\infty &= \rho \Omega^2 R \sqrt{\nu\alpha} p,\end{aligned}\tag{2.9}$$

where α is a constant whose dimension is time. Its order of magnitude determines whether regime a or b is considered. Substituting (2.9) into equations (2.2a)-(2.2d), and assuming that $\alpha \ll 1/\Omega$ (that is regime a is considered), we arrive at the conclusion that

$$\frac{\partial V_r}{\partial t} - \frac{V_\Phi^2}{r} = \frac{\partial^2 V_r}{\partial z^2}, \tag{2.10a}$$

$$\frac{\partial V_\Phi}{\partial t} = \frac{\partial^2 V_\Phi}{\partial z^2}, \tag{2.10b}$$

$$\frac{\partial p}{\partial z} = 0, \tag{2.10c}$$

$$\frac{\partial V_r}{\partial r} + \frac{\partial V_z}{\partial z} + \frac{V_r}{r} = 0. \tag{2.10d}$$

We notice that the centrifugal term in the circumferential momentum equation (2.10b) has been disregarded due to its small order of magnitude in the limit as $\hat{t} \ll 1/\Omega$. However the centrifugal term in the radial momentum equation (2.10a) is of same order as the local acceleration, and so it cannot be neglected as the solution of this equation without the term V_Φ^2/r would be $V_r = 0$, which is a contradiction. The system of equations (2.10a)-(2.10d) with the corresponding boundary conditions (2.2e)-(2.2k) may be solved analytically. Let us start with the circumferential equation (2.10b). Guided by (2.3), we seek the solution for V_Φ in the form

$$V_\Phi = r F_\Phi(t, z). \tag{2.11}$$

Substitution of (2.11) into the circumferential momentum equation (2.10b) allows us to express it and its corresponding boundary conditions as a boundary-value problem in the unknown F_Φ :

$$\frac{\partial F_\Phi}{\partial t} = \frac{\partial^2 F_\Phi}{\partial z^2}, \quad (2.12a)$$

$$F_\Phi = 1, \quad \text{at } z = 0, \quad (2.12b)$$

$$F_\Phi = 0, \quad \text{at } z = \infty, \quad (2.12c)$$

$$F_\Phi = 0, \quad \text{at } t = 0 \text{ except at } z = 0. \quad (2.12d)$$

The boundary-value problem (2.12) can be solved by considering the following argument: since the thickness $\hat{z}$ of the viscous layer of rotating fluid is small as compared to the disk radius R (we recall that during regime a , $\hat{z} \sim \sqrt{\nu \hat{t}}$ and that $\hat{r} \sim R$), one can consider that there is no characteristic length scale involved in the problem under the assumption $\hat{t} \ll 1/\Omega$. Furthermore, there is no characteristic time scale which can be assigned to the problem. Therefore, the solution must admit a self-similar form. This means that if the solution is known at one specific time $\hat{t}$, then the solution at another time may be found by making use of affine transformations. Let

$$F_\Phi = \mathcal{F}(t, z). \quad (2.13)$$

be the solution of (2.12). We shall write the affine transformations by introducing the constants A , B and C such that

$$t = A\bar{t}, \quad z = B\bar{z}, \quad F_\Phi = C\bar{F}_\Phi. \quad (2.14)$$

We substitute (2.14) into the system (2.12), and we choose the constants B and C such that they satisfy

$$B = \sqrt{A}, \quad C = 1,$$

where A is arbitrary. In that case, the transformed problem (involving the transformed variables $\bar{t}$, $\bar{z}$ and $\bar{F}_\Phi$) is identical to the original one. Hence, since it is identical, it admits (2.13) as a solution too, which can now be rewritten as

$$\bar{F}_\Phi = \mathcal{F}(\bar{t}, \bar{z}). \quad (2.15)$$

Going back to the original variables, and setting A equal to $\hat{t}$, the solution F_Φ has the following form

$$F_\Phi = \mathcal{F}\left(1, \frac{z}{\sqrt{t}}\right) = f(\eta), \quad (2.16)$$

where η is called the similarity variable given by

$$\eta = \frac{z}{\sqrt{t}}. \quad (2.17)$$

We substitute (2.16) and (2.17) into the original problem (2.12), which leads to the following ordinary linear differential equation for f :

$$f''(\eta) + \frac{1}{2}\eta f'(\eta) = 0, \quad (2.18a)$$

$$f(0) = 1, \quad (2.18b)$$

$$f(\infty) = 0. \quad (2.18c)$$

The solution of this boundary-value problem can be written as follows

$$f(\eta) = 1 - \frac{1}{\sqrt{\pi}} \int_0^\eta \exp\left(-\frac{1}{4}\xi^2\right) d\xi. \quad (2.19)$$

Hence, we found an analytical solution for V_Φ such that $V_\Phi = rf(\eta)$ with f being given by (2.19).

Guided by (2.9), we seek the solution for the radial velocity component in the form

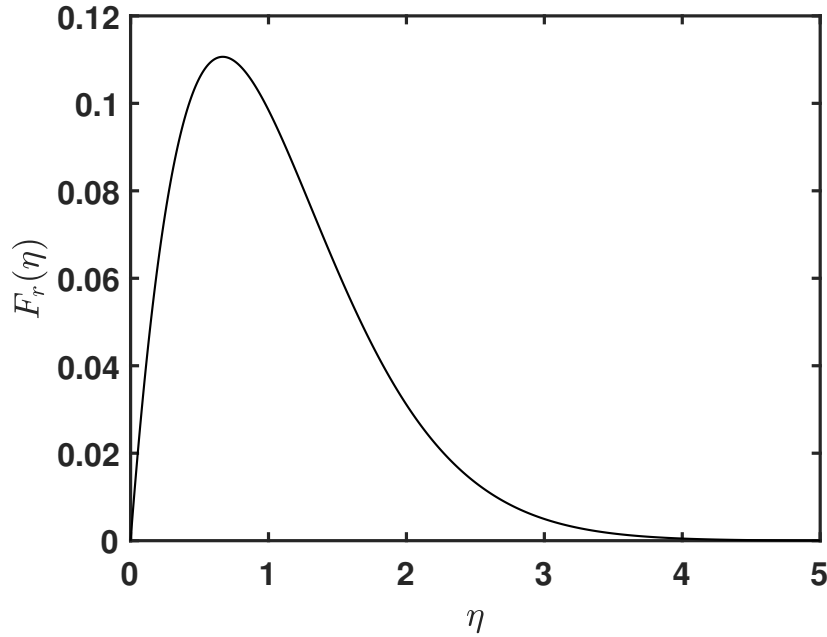
$$V_r = rtF_r(\eta). \quad (2.20)$$

Substitution of (2.20), (2.16) and (2.17) into the radial momentum equation (2.10a) with its related boundary conditions leads to

$$F_r''(\eta) + \frac{1}{2}\eta F_r'(\eta) - F_r(\eta) = -f^2(\eta), \quad (2.21a)$$

$$F_r(0) = 0, \quad (2.21b)$$

$$F_r(\infty) = 0. \quad (2.21c)$$

Figure 2.1: F_r plotted against η

The boundary-value problem (2.21) can be solved numerically. Finally, we seek the axial velocity component in the form

$$V_z = t^{3/2} F_z(\eta). \quad (2.22)$$

Then the continuity equation yields

$$F'_z(\eta) = -2F_r(\eta), \quad (2.23a)$$

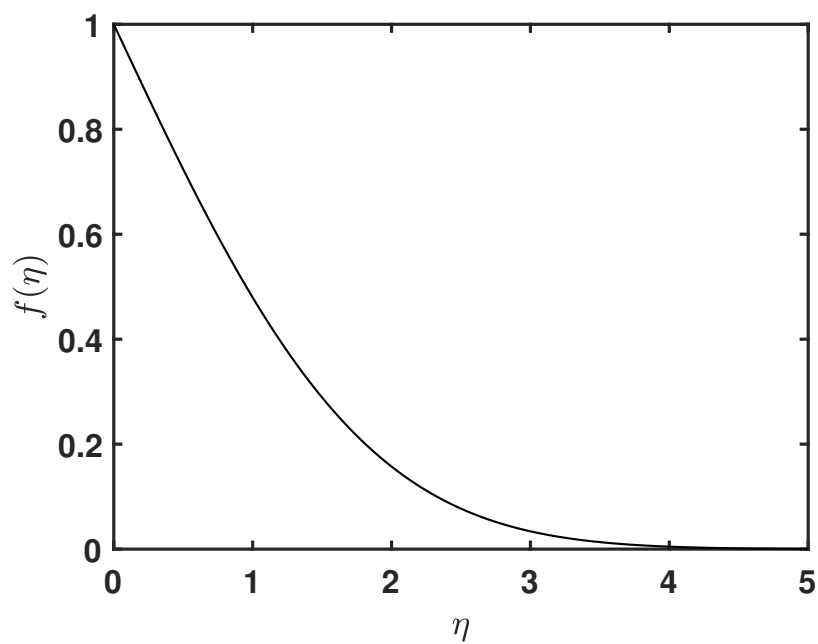
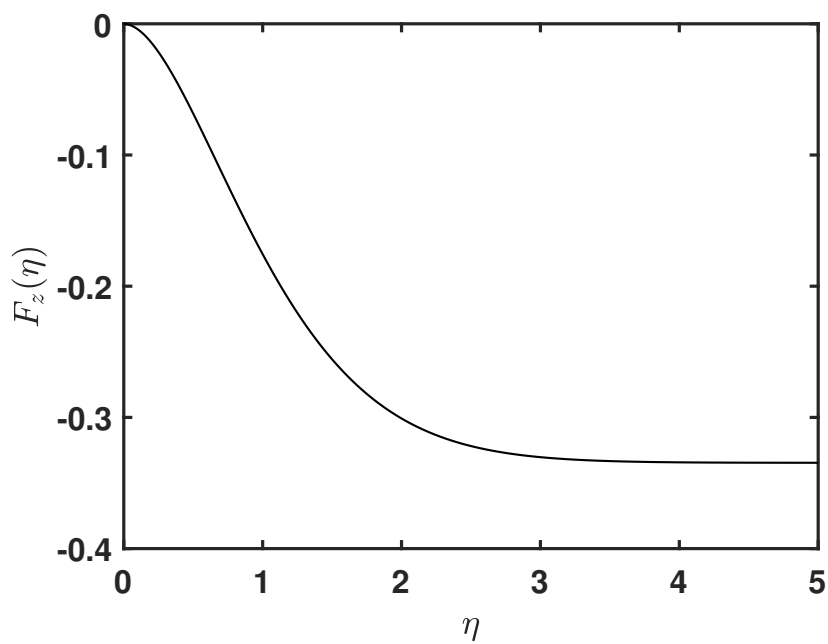
$$F_z(0) = 0. \quad (2.23b)$$

The solutions F_r , f and F_z have been plotted in Figures 2.1, 2.2 and 2.3.

Thus, we found an analytical solution for regime a which is valid in the limit as $\hat{t} \ll 1/\Omega$.

2.2 Flow regime b

Now, let us focus on regime b . We need to write the Navier-Stokes equations (2.2a)-(2.2k) in dimensionless form. With that purpose in mind, let us introduce the dimen-

Figure 2.2: f plotted against η Figure 2.3: F_z plotted against η

dimensionless quantities r , z , V_r , V_Φ , V_z and p such that

$$\begin{aligned}\hat{r} &\longrightarrow R r, & \hat{z} &\longrightarrow R z, \\ \hat{V}_r &\longrightarrow \Omega R V_r, & \hat{V}_\Phi &\longrightarrow \Omega R V_\Phi, & \hat{V}_z &\longrightarrow \Omega R V_z, \\ \hat{p} &\longrightarrow p_\infty + \rho \Omega^2 R^2 p.\end{aligned}\tag{2.24}$$

The circumferential angle Φ is already dimensionless. It remains to introduce a dimensionless time t . As we explained earlier, during regime b , the convective terms are in balance with the local acceleration, and $\hat{t} \sim 1/\Omega$. Hence, we can introduce the dimensionless time t such that

$$\hat{t} \longrightarrow \frac{1}{\Omega} t.\tag{2.25}$$

Substitution of (2.24) and (2.25) into (2.2) leads to the dimensionless unsteady Navier-Stokes equations

$$\frac{\partial V_r}{\partial t} + V_r \frac{\partial V_r}{\partial r} + V_z \frac{\partial V_r}{\partial z} - \frac{V_\Phi^2}{r} = -\frac{\partial p}{\partial r} + \frac{1}{Re} \left(\frac{\partial^2 V_r}{\partial z^2} + \frac{\partial^2 V_r}{\partial r^2} + \frac{1}{r} \frac{\partial V_r}{\partial r} - \frac{V_r}{r^2} \right),\tag{2.26a}$$

$$\frac{\partial V_\Phi}{\partial t} + V_r \frac{\partial V_\Phi}{\partial r} + V_z \frac{\partial V_\Phi}{\partial z} + \frac{V_r V_\Phi}{r} = \frac{1}{Re} \left(\frac{\partial^2 V_\Phi}{\partial z^2} + \frac{\partial^2 V_\Phi}{\partial r^2} + \frac{1}{r} \frac{\partial V_\Phi}{\partial r} - \frac{V_\Phi}{r^2} \right),\tag{2.26b}$$

$$\frac{\partial V_z}{\partial t} + V_r \frac{\partial V_z}{\partial r} + V_z \frac{\partial V_z}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re} \left(\frac{\partial^2 V_z}{\partial z^2} + \frac{\partial^2 V_z}{\partial r^2} + \frac{1}{r} \frac{\partial V_z}{\partial r} \right),\tag{2.26c}$$

$$\frac{\partial V_r}{\partial r} + \frac{\partial V_z}{\partial z} + \frac{V_r}{r} = 0.\tag{2.26d}$$

where Re is the Reynolds number defined as

$$Re = \frac{\Omega R^2}{\nu}.$$

In what follows, we shall assume that Re is large. If Re is not assumed to be large, then the analysis cannot be conducted analytically as perturbation methods require a small parameter (namely, $1/Re$ when $Re \gg 1$).

The dimensionless equations (2.26a)-(2.26d) are subject to the following boundary

conditions:

$$V_\Phi = r, \quad V_r = V_z = 0, \quad \text{at } z = 0, \quad r \in [0, 1], \quad (2.26e)$$

$$\frac{\partial V_r}{\partial z} = \frac{\partial V_\Phi}{\partial z} = V_z = 0, \quad \text{at } z = 0, \quad r > 1, \quad (2.26f)$$

$$V_r = V_\Phi = 0, \quad \text{at } r^2 + z^2 = \infty, \quad (2.26g)$$

$$p = 0, \quad \text{at } r^2 + z^2 = \infty, \quad (2.26h)$$

$$V_r = V_\Phi = \frac{\partial V_z}{\partial r} = 0, \quad \text{at } r = 0. \quad (2.26i)$$

Writing problem (2.26a)-(2.26i) in dimensionless form made the coefficient $1/Re$ appear. It multiplies the viscous terms on the right-hand side of the radial, circumferential and axial momentum equations, respectively (2.26a), (2.26b) and (2.26c). Since the Reynolds number is assumed to be large, then we are dealing with a singular perturbation problem. In order to solve it, we shall follow the Prandtl's hierarchical approach, which is based on the method of matched asymptotic expansions. Even though the factor $1/Re$ tends to 0 as $Re \rightarrow \infty$, we cannot simply neglect the viscous terms in the momentum equations (2.26a), (2.26b) and (2.26c) for the following reason: there exists a small region in the vicinity of the disk and the jet axis termed as the boundary layer where the viscous terms are large, and they become comparable with the convective terms due to the second order derivatives. Hence, we shall introduce an inner viscous region in the direct neighbourhood of the disk surface and the jet axis, and an outer region where the flow is inviscid. The solution may be written in terms of asymptotic expansions for the outer and inner regions, as functions of spatial and time variables, and also as functions of the large parameter Re . They will be called respectively the outer and inner expansions. The matching conditions for the leading order terms of the outer and inner expansions are given by the Prandtl's matching rule, which was introduced by Prandtl (1904) in his study of fluid flows with small viscosity. It states that the inner limit of the leading order term of the outer expansion is equal to the outer limit of the leading order term of the inner expansion.

The structure of the flow is sketched in Figure 2.4. The region 3 is where the fluid is stagnant, so the velocity vector $\mathbf{V}$ is equal to the zero vector everywhere in this region. Region 1_{ps} corresponds to the inner structure of the pseudo-shock, and it will be studied in Chapter 5. Region 1_{gw} , investigated in Chapter 6, lies in the wake, immediately downstream of the disk edge.

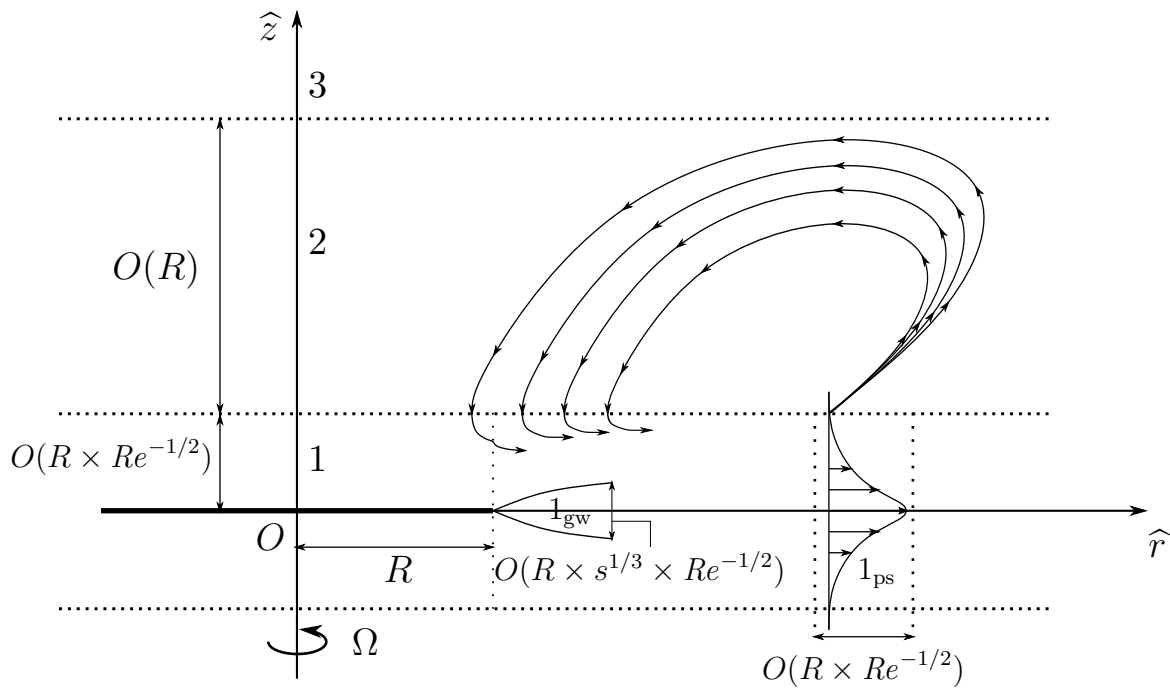


Figure 2.4: Problem layout during regime *b*. A free jet is generated by the rotation of the disk. The fluid is stagnant in region 3. Region 2 corresponds to the outer inviscid region where the entrainment effect takes place. Region 1 is the viscous boundary layer. The part for negative Z has been omitted.

We shall introduce the outer region 2 in Figure 2.4, where

$$r, z = O(1). \quad (2.27)$$

We will prove in Section 2.3 that the flow is indeed inviscid in region 2 in the limit (2.27), where the viscous terms in (2.26a), (2.26b) and (2.26c) are negligible, and where the Navier-Stokes equations become the Euler equations. Consequently, when employing the Prandtl's hierarchical approach and introducing region 2 (at this stage, the boundary layer has not been added yet), the impermeability condition $V_z = 0$ on the disk surface holds, but the no-slip condition (2.26e) cannot be satisfied as the flow is inviscid. From that, it follows that one needs to consider an additional region, called region 1, where the flow would be viscous so that the no-slip condition can be satisfied. Naturally, when region 1 has been added, the “bottom” of region 2 is not located anymore on the disk surface, but it is instead located at the outer edge of region 1; this implies that the impermeability condition for V_z is no longer valid at the bottom of region 2 when region 1 is also considered. Instead, a matching condition with region 1 must be applied for V_z ; see Section 2.3.

Let us introduce the inner region in the direct neighbourhood of the disk (that is region 1), and corresponding to the boundary layer. We define a new scaled variable Z such that in this inner region

$$r, Z = \frac{z}{\delta(Re)} = O(1), \quad (2.28)$$

where δ is a quantity depending on the Reynolds number and corresponding to the thickness of the boundary layer, such that

$$\delta(Re) \ll 1, \quad \text{as } Re \rightarrow \infty. \quad (2.29)$$

The inner expansions which are valid inside the boundary layer have the following form in the leading order

$$\begin{aligned} V_r &= \sigma(Re)\mathcal{V}_{r,0}(r, Z, t) + \cdots, \\ V_\Phi &= \eta(Re)\mathcal{V}_{\Phi,0}(r, Z, t) + \cdots, \\ V_z &= \xi(Re)\mathcal{V}_{z,0}(r, Z, t) + \cdots, \\ p &= \chi(Re)\mathcal{P}_0(r, Z, t) + \cdots, \end{aligned} \quad \text{as } Re \rightarrow \infty, \quad (2.30)$$

where σ , η , ξ and χ are quantities to determine by mathematical and physical arguments. Since the no-slip condition holds in this region, as r is of order one, substitution of V_Φ into (2.26e) gives

$$\eta(Re) = 1. \quad (2.31)$$

Then, the centrifugal terms need to be in balance with the convective terms into the radial momentum equation (2.26a), leading to

$$\sigma(Re) = 1. \quad (2.32)$$

Now, by taking (2.32) into account, let us replace (2.28) and (2.30) into the continuity equation (2.26d)

$$\frac{\partial \mathcal{V}_{r,0}}{\partial r} + \frac{\xi(Re)}{\delta(Re)} \frac{\partial \mathcal{V}_{z,0}}{\partial Z} + \frac{\mathcal{V}_{r,0}}{r} = 0. \quad (2.33)$$

By applying the principle of least degeneration, we must set

$$\xi(Re) = \delta(Re). \quad (2.34)$$

Indeed, if we had assumed that $\xi(Re) \ll \delta(Re)$, then equation (2.33) would have been written

$$\frac{\partial \mathcal{V}_{r,0}}{\partial r} + \frac{\mathcal{V}_{r,0}}{r} = 0.$$

This is an ordinary differential equation of order one, and its solution is $h_0(Z, t)/r$, where h_0 is a function to determine. So under the assumption $\xi(Re) \ll \delta(Re)$, the continuity equation gives the radial component of the velocity $\mathcal{V}_{r,0} = h_0(Z, t)/r$. Nevertheless, the radial momentum equation allows us to get $\mathcal{V}_{r,0}$ as well. Therefore we have two different equations for $\mathcal{V}_{r,0}$: there is a contradiction.

On the other hand, if we had assumed that $\xi(Re) \gg \delta(Re)$, then equation (2.33) would have degenerated to

$$\frac{\partial \mathcal{V}_{z,0}}{\partial Z} = 0.$$

By using the boundary conditions, this yields to $\mathcal{V}_{z,0} = 0$ everywhere in the boundary layer, which is a contradiction too. Consequently, necessarily, $\xi(Re) = \delta(Re)$.

Substitution of (2.28) and (2.30) into the radial momentum equation (2.26a), and taking

into account (2.31), (2.32) and (2.34) yields

$$\begin{aligned} \frac{\partial \mathcal{V}_{r,0}}{\partial t} + \mathcal{V}_{r,0} \frac{\partial \mathcal{V}_{r,0}}{\partial r} + \mathcal{V}_{z,0} \frac{\partial \mathcal{V}_{r,0}}{\partial Z} - \frac{\mathcal{V}_{\Phi,0}^2}{r} = -\chi(Re) \frac{\partial \mathcal{P}_0}{\partial r} + \\ \frac{1}{Re \delta^2(Re)} \frac{\partial^2 \mathcal{V}_{r,0}}{\partial Z^2} + \frac{1}{Re} \left(\frac{\partial^2 \mathcal{V}_{r,0}}{\partial r^2} + \frac{1}{r} \frac{\partial \mathcal{V}_{r,0}}{\partial r} - \frac{\mathcal{V}_{r,0}}{r^2} \right). \end{aligned} \quad (2.35)$$

Since we are working in the limit as $Re \rightarrow \infty$, the three last terms on the right-hand side of equation (2.35) may be disregarded. Furthermore, we introduced the boundary layer so that the no-slip condition can be satisfied on the disk surface. So the remaining viscous term in (2.35) must be kept in balance with the convective terms. Hence we set

$$\delta(Re) = Re^{-1/2}. \quad (2.36)$$

This means that the thickness of the boundary layer is an $O(Re^{-1/2})$ quantity. Then, combining (2.34) with (2.36) gives

$$\xi(Re) = Re^{-1/2}. \quad (2.37)$$

It remains to determine $\chi(Re)$. As usual when dealing with boundary-layer theory, the pressure field is found by analysing the flow in the inviscid outer region. It will be shown later that the pressure in the outer region is an $O(Re^{-1/2})$ quantity. Hence,

$$\chi(Re) = Re^{-1/2}. \quad (2.38)$$

Therefore, in the inner region the dimensionless Navier-Stokes equations reduce to the

Prandtl's boundary-layer equations as

$$\frac{\partial \mathcal{V}_{r,0}}{\partial t} + \frac{\partial(\mathcal{V}_{r,0}^2)}{\partial r} + \frac{\partial(\mathcal{V}_{r,0}\mathcal{V}_{z,0})}{\partial Z} + \frac{\mathcal{V}_{r,0}^2 - \mathcal{V}_{\Phi,0}^2}{r} = \frac{\partial^2 \mathcal{V}_{r,0}}{\partial Z^2}, \quad (2.39a)$$

$$\frac{\partial \mathcal{V}_{\Phi,0}}{\partial t} + \frac{\partial(\mathcal{V}_{r,0}\mathcal{V}_{\Phi,0})}{\partial r} + \frac{\partial(\mathcal{V}_{\Phi,0}\mathcal{V}_{z,0})}{\partial Z} + \frac{2\mathcal{V}_{r,0}\mathcal{V}_{\Phi,0}}{r} = \frac{\partial^2 \mathcal{V}_{\Phi,0}}{\partial Z^2}, \quad (2.39b)$$

$$\frac{\partial \mathcal{P}_0}{\partial Z} = 0, \quad (2.39c)$$

$$\frac{\partial \mathcal{V}_{r,0}}{\partial r} + \frac{\partial \mathcal{V}_{z,0}}{\partial Z} + \frac{\mathcal{V}_{r,0}}{r} = 0, \quad (2.39d)$$

$$\mathcal{V}_{\Phi,0} = r, \quad \mathcal{V}_{r,0} = \mathcal{V}_{z,0} = 0, \quad \text{at } Z = 0, \quad r \in [0, 1], \quad (2.39e)$$

$$\frac{\partial \mathcal{V}_{r,0}}{\partial Z} = \frac{\partial \mathcal{V}_{\Phi,0}}{\partial Z} = \mathcal{V}_{z,0} = 0, \quad \text{at } Z = 0, \quad r > 1, \quad (2.39f)$$

$$\mathcal{V}_{r,0} = \mathcal{V}_{\Phi,0} = 0, \quad \text{at } Z = \infty, \quad (2.39g)$$

$$\mathcal{P}_0 = \lim_{z \rightarrow 0} p'_0, \quad \text{at } Z = \infty, \quad (2.39h)$$

$$\mathcal{V}_{r,0} = \mathcal{V}_{\Phi,0} = 0, \quad \text{at } r = 0, \quad (2.39i)$$

where the term $\lim_{z \rightarrow 0} p'_0$ in the matching condition for the pressure field corresponds to the inner limit of the leading order term of the outer expansion for the pressure. The radial (2.39a) and circumferential (2.39b) momentum equations have been written in conservative form by making use of the continuity equation (2.39d).

2.3 Entrainment effect

Let us get back to the outer region 2; see Figure 2.4. Outside the boundary layer 1, there is a region where there is a slow movement of fluid which is observed due to the entrainment effect. The fluid particles are sucked down, and once they reach the boundary layer, they undergo a significant acceleration owing to the action of viscous forces, and they are redirected away from the axis of rotation of the disk. Let us introduce a new scaled independent variable z' , such that in region 2

$$z' = \frac{z}{\delta'(Re)} = O(1), \quad (2.40)$$

where it is assumed that $\delta'(Re) \gg Re^{-1/2}$. Let us notice that if we had imposed that $\delta'(Re) = Re^{-1/2}$ or $\delta'(Re) \ll Re^{-1/2}$, it would mean that we would observe the flow on the boundary-layer scale, which is not relevant when trying to analyse the flow structure

in region 2. The axial velocity component is comparable in order of magnitude with $Re^{-1/2}$ since it is what comes into the boundary layer, and there is a matching for V_z between regions 1 and 2. The asymptotic expansions of the sought flow functions in region 2 have the following form

$$\begin{aligned} V_r &= \sigma'(Re)\mathcal{V}'_{r,0}(r, z', t) + \cdots, \\ V_\Phi &= \eta'(Re)\mathcal{V}'_{\Phi,0}(r, z', t) + \cdots, \\ V_z &= Re^{-1/2}\mathcal{V}'_{z,0}(r, z', t) + \cdots, \\ p &= \chi'(Re)\mathcal{P}'_0(r, z', t) + \cdots, \end{aligned} \quad \text{as } Re \rightarrow \infty, \quad (2.41)$$

where σ' , η' and χ' are to be found. Substitution of (2.40) and (2.41) into the continuity equation (2.26d) yields

$$\sigma'(Re)\frac{\partial \mathcal{V}'_{r,0}}{\partial r} + \frac{Re^{-1/2}}{\delta'(Re)}\frac{\partial \mathcal{V}'_{z,0}}{\partial z'} + \sigma'(Re)\frac{\mathcal{V}'_{r,0}}{r} = 0.$$

Using the principle of least degeneration, we get

$$\sigma'(Re)\delta'(Re) = Re^{-1/2}. \quad (2.42)$$

If $\sigma'(Re) \gg 1$ or $\sigma'(Re) = 1$, then $\delta'(Re) \ll Re^{-1/2}$ or $\delta'(Re) = Re^{-1/2}$, respectively. In both cases, it is contradictory with our assumption $\delta'(Re) \gg Re^{-1/2}$ discussed earlier. Hence, necessarily

$$\sigma'(Re) \ll 1. \quad (2.43)$$

Let us substitute (2.41) and (2.40) into the radial momentum equation (2.26a). In the limit as $Re \rightarrow \infty$ and (2.43), it reduces to

$$\sigma'(Re)\frac{\partial \mathcal{V}'_{r,0}}{\partial t} - \eta'^2(Re)\frac{\mathcal{V}'_{\Phi,0}}{r} = -\chi'(Re)\frac{\partial \mathcal{P}'_0}{\partial r}.$$

We notice that the viscous terms are negligible. The same applies to the circumferential and axial momentum equations. Thus, the Navier-Stokes equations become the Euler equations, and the flow inside region 2 may be treated as inviscid.

Let us now prove that the leading-order circumferential velocity component is identically zero in region 2. We recall that the circulation at time t along a closed fluid

contour $\mathcal{C}$ is given by

$$\Gamma(t) = \oint_{(\mathcal{C})} \mathbf{V} \cdot d\mathbf{r},$$

where $d\mathbf{r}$ is an infinitesimal increment of the position vector $\mathbf{r}$ along $\mathcal{C}$, and $\mathbf{V}$ is the velocity vector. The flow in region 2 is inviscid, incompressible, and without any body force acting on the fluid, so according to the Kelvin's circulation theorem, the circulation Γ along any closed fluid contour $\mathcal{C}$ is invariant with time:

$$\frac{d\Gamma(t)}{dt} = 0.$$

Initially, before the disk starts rotating, the fluid is at rest, so the velocity vector is equal to the zero vector everywhere in the domain. Hence, along any closed fluid contour $\mathcal{C}_0$, the circulation is equal to zero. According to the Kelvin's circulation theorem, the circulation along any closed contour $\mathcal{C}$, in the inviscid region, will remain equal to zero after the disk starts rotating:

$$\oint_{(\mathcal{C})} \mathbf{V} \cdot d\mathbf{r} = 0.$$

Using Stokes theorem, and recalling that the vorticity field $\boldsymbol{\omega}$ is defined as the curl of the velocity vector, it results that

$$\iint_{(\sigma)} \boldsymbol{\omega} \cdot d\mathbf{S} = 0,$$

where σ is a surface based on $\mathcal{C}$, and $d\mathbf{S} = \mathbf{n}dS$ is a small element of σ , with $\mathbf{n}$ being a normal unit vector to σ . Since $\mathcal{C}$ is arbitrary, then we deduce that the flow is irrotational in region 2, that is

$$\boldsymbol{\omega} = 0. \tag{2.44}$$

Since the flow is irrotational, there exists a scalar function φ called the velocity potential, such that

$$\mathbf{V} = \nabla\varphi. \tag{2.45}$$

In cylindrical polar coordinates, (2.45) is written as

$$\mathbf{V} = \frac{\partial\varphi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\varphi}{\partial\Phi}\mathbf{e}_\Phi + \frac{\partial\varphi}{\partial z}\mathbf{e}_z. \tag{2.46}$$

As the flow is axisymmetric, the second term on the right-hand side of (2.46) is nought. Consequently in region 2, $V'_{\Phi,0} = 0$ everywhere. This means that in the leading order

approximation, there is no contribution from the circumferential velocity. However, the higher-order terms of the asymptotic expansion of V_Φ in (2.41) are not necessarily equal to zero.

Let us now try to obtain the solution in region 2. Since the flow is irrotational, the second component of the vorticity field ω gives

$$\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r} = 0. \quad (2.47)$$

Besides, since the flow is incompressible and axisymmetric, from the continuity equation (2.26d) one can define the dimensionless stream-function ψ such that

$$rV_r = \frac{\partial \psi}{\partial z}, \quad rV_z = -\frac{\partial \psi}{\partial r}. \quad (2.48)$$

By substituting (2.41) into the second equation in (2.48), we deduce the form of the asymptotic expansion of ψ in region 2

$$\psi = Re^{-1/2}\psi' + \dots, \quad \text{as } Re \rightarrow \infty. \quad (2.49)$$

The substitution of (2.48) into (2.47) yields

$$\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} = 0. \quad (2.50)$$

The equation (2.50) can be solved numerically with the right boundary conditions, and the solution ψ will allow us to plot the streamline pattern in region 2. Let us notice that the left-hand side of (2.50) is different from $\nabla^2 \psi$ written in cylindrical polar coordinates because of the minus sign. Substitution of (2.40) and (2.49) into equation (2.50) leads to

$$\delta'(Re) = 1, \quad (2.51)$$

which means that the thickness of region 2 is comparable with its length. Consequently, it follows from equation (2.42) that

$$\sigma'(Re) = Re^{-1/2}. \quad (2.52)$$

So the radial velocity component has the same order of magnitude as the axial velocity component.

Finally, we balance the local acceleration in the radial momentum equation with the pressure gradient, yielding

$$\chi'(Re) = Re^{-1/2}. \quad (2.53)$$

All the asymptotic expansions of the sought flow functions are now known. Substitution of (2.40) and (2.41) into the dimensionless Euler equations leads to

$$\frac{\partial \mathcal{V}'_{r,0}}{\partial t} = -\frac{\partial \mathcal{P}'_0}{\partial r}, \quad (2.54a)$$

$$\frac{\partial \mathcal{V}'_{z,0}}{\partial t} = -\frac{\partial \mathcal{P}'_0}{\partial z'}, \quad (2.54b)$$

$$\frac{\partial \mathcal{V}'_{r,0}}{\partial r} + \frac{\partial \mathcal{V}'_{z,0}}{\partial z'} + \frac{\mathcal{V}'_{r,0}}{r} = 0. \quad (2.54c)$$

The system of equations (2.54) involves primitive variables. A more straightforward way to get the solution in region 2 is to solve equation (2.50), which is equivalent to (2.54) but written in terms of the stream-function. Replacing z' , and ψ by their asymptotic representations (2.40) and (2.49) respectively, and working with the $O(Re^{-1/2})$ terms, the governing equation in region 2 may be written as follows

$$\frac{\partial^2 \psi'}{\partial r^2} - \frac{1}{r} \frac{\partial \psi'}{\partial r} + \frac{\partial^2 \psi'}{\partial z'^2} = 0. \quad (2.55a)$$

The differential equation (2.55a) is elliptic, and it requires two boundary conditions in r , and two boundary conditions in z' to be solved.

At $z' = 0$, the matching condition with the solution from the boundary layer must be satisfied. We recall that $V_z = O(Re^{-1/2})$ inside the boundary layer and in region 2 (but not on the pseudo-shock's scale where we will see in Chapter 5 that $V_z = O(1)$). Hence, the matching condition for the axial velocity component can be written as

$$\mathcal{V}'_{z,0} = \lim_{Z \rightarrow \infty} \mathcal{V}_{z,0} \quad \text{at } z' = 0.$$

Consequently, the matching condition related to ψ' at $z' = 0$ has the form

$$\lim_{Z \rightarrow \infty} \mathcal{V}_{z,0} = -\frac{1}{r} \frac{\partial \psi'}{\partial r} \quad \text{at } z' = 0. \quad (2.55b)$$

On the axis of rotation of the disk, we recall that the radial velocity component is equal

to 0 as the flow is axisymmetric, consequently

$$\psi' = 0 \quad \text{at } r = 0. \quad (2.55c)$$

When z' is large, the solution must match with the stagnant fluid of region 3. Furthermore, when r is large positive, the fluid is also at rest since it is located before the front of the pseudo-shock. Even though the velocity components tend to zero when r or z' are large, it would not be correct to set $\psi' = 0$ when r or z' are large, since the distances over which we would integrate the velocity in order to obtain ψ' also become large. Instead, we write

$$\frac{\partial \psi'}{\partial r} = \frac{\partial \psi'}{\partial z'} = 0, \quad \text{at } r^2 + z'^2 = \infty. \quad (2.55d)$$

In the next chapter, we shall present the numerical techniques employed to solve the sets of equations (2.39a)-(2.39i) governing the flow in the boundary layer, and (2.55a)-(2.55d) governing the inviscid flow in region 2. The sought flow functions found numerically will be presented in Chapter 4.

Chapter 3

Computational analysis

We shall solve the boundary-value problem (2.39a)-(2.39i) numerically with the help of the following finite-difference method. For the sake of simplicity, we shall denote the velocity components related to $\mathcal{V}_{r,0}$, $\mathcal{V}_{\Phi,0}$ and $\mathcal{V}_{z,0}$ as U , V and W , respectively, and use P for the pressure $\mathcal{P}_0$. We shall employ a uniform grid

$$\{r_i, Z_k, t_n\} = \{\Delta r(i-1), \Delta Z(k-1), \Delta t(n-1)\}, \quad (3.1)$$

such that

$$i \in \llbracket 1; N_r + 1 \rrbracket, \quad k \in \llbracket 1; N_Z + 1 \rrbracket, \quad n \in \llbracket 1; N_t + 1 \rrbracket, \quad (3.2)$$

and where Δr , ΔZ and Δt are small grid spacings in r -direction, Z -direction and for time, respectively. N_r , N_Z and N_t are integers which determine the number of nodes in the r -, Z - and t -directions, respectively.

For the time derivatives, a backward Euler scheme was employed.

$$\frac{\partial U}{\partial t} = \frac{U_{i,k}^n - U_{i,k}^{n-1}}{\Delta t}, \quad (3.3a)$$

where $U_{i,k}^n$ is the approximated value of U at the (i, k, n) node:

$$U_{i,k}^n \approx U(r_i, Z_k, t_n).$$

In this study we adopted an implicit scheme, leading to a more stable numerical process. We chose a first-order scheme for $\partial d / \partial t$ following Ruban and Vonatsos (2008) fast monotonic convergence. The numerical error is smaller near the discontinuity with such a scheme and the solution is well resolved.

Regarding space derivatives, we shall use second order schemes for $\partial / \partial Z$ and $\partial^2 / \partial Z^2$, while a first order scheme will be employed for $\partial / \partial r$. The reason why we chose to use

a first order scheme for $\partial/\partial r$ instead of a second order one is because of the hyperbolic nature of the operators $\partial U/\partial t + U\partial U/\partial r$ and $\partial V/\partial t + U\partial V/\partial r$. For each time t_0 , at the position of the pseudo-shock $r_s(t_0)$, we expect to observe discontinuities of the flow quantities U and V for the pair $(t_0, r_s(t_0))$. So higher order schemes would develop oscillations nearby the pseudo-shock and the error brought by a second order scheme would be very large in that case. There is a lot of evidence in gas dynamics that using higher order schemes produce oscillations in the vicinity of discontinuities.

Discretisation of the boundary-layer equations (2.39a)-(2.39i) leads to a set of non-linear algebraic equations for $U_{i,k}^n$, $V_{i,k}^n$ and $W_{i,k}^n$. When solving these equations, we shall use iterative techniques which rely on linearisation of our equations. In particular, when dealing with U^2 in (2.39a), we apply the Newton's method which consists in using the Taylor expansion around an approximate value of U , denoted by U^- , as follows

$$\begin{aligned} U^2 &= (U^-)^2 + 2U^-(U - U^-) + O((U - U^-)^2), \\ &\approx -(U^-)^2 + 2U^-U. \end{aligned}$$

Consequently, the r -gradient of U^2 can be written

$$\begin{aligned} \frac{\partial(U^2)}{\partial r} &= \frac{(U_{i,k}^n)^2 - (U_{i-1,k}^n)^2}{\Delta r}, \\ &= \frac{2(U^-)_{i,k}^n U_{i,k}^n}{\Delta r} - \frac{((U^-)_{i,k}^n)^2 + (U_{i-1,k}^n)^2}{\Delta r}. \end{aligned} \quad (3.3b)$$

The term UW in (2.39a) may be linearised by decoupling the equations of the system (2.39a)-(2.39i). After decoupling, UW becomes W^-U , where W^- is a computed approximation of W . Hence, the Z -derivative of UW may be written as

$$\begin{aligned} \frac{\partial(UW)}{\partial Z} &= \frac{U_{i,k+\frac{1}{2}}^n W_{i,k+\frac{1}{2}}^n - U_{i,k-\frac{1}{2}}^n W_{i,k-\frac{1}{2}}^n}{\Delta Z}, \\ &= \frac{(W^-)_{i,k+1}^n + (W^-)_{i,k}^n}{4\Delta Z} U_{i,k+1}^n + \frac{(W^-)_{i,k+1}^n - (W^-)_{i,k-1}^n}{4\Delta Z} U_{i,k}^n, \\ &\quad - \frac{(W^-)_{i,k}^n + (W^-)_{i,k-1}^n}{4\Delta Z} U_{i,k-1}^n. \end{aligned} \quad (3.3c)$$

We used the stencil shown in Figure 3.1 in Z -direction. The terms at $k + 1/2$ and $k - 1/2$ have been calculated by an arithmetic mean.

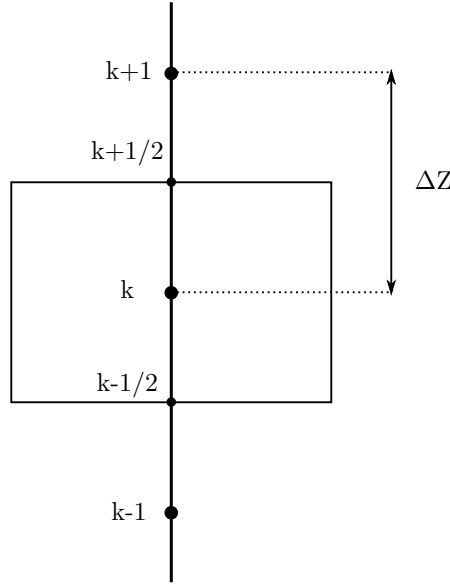


Figure 3.1: Stencil in Z -direction. A second-order accuracy scheme is used between the nodes $k + 1/2$ and $k - 1/2$ for first-order derivatives with respect to Z . The values of U and W at the nodes $k + 1/2$ and $k - 1/2$ are calculated through an arithmetic mean between k and $k + 1$, and between k and $k - 1$, respectively.

The last two terms in (2.39a) are discretised as follows

$$\begin{aligned} \frac{U^2 - V^2}{r} &= \frac{(U_{i,k}^n)^2 - ((V^-)_{i,k}^n)^2}{r_i}, \\ &= \frac{2(U^-)_{i,k}^n U_{i,k}^n}{r_i} - \frac{((U^-)_{i,k}^n)^2 + ((V^-)_{i,k}^n)^2}{r_i}, \end{aligned} \quad (3.3d)$$

$$\frac{\partial^2 U}{\partial Z^2} = \frac{1}{(\Delta Z)^2} U_{i,k-1}^n - \frac{2}{(\Delta Z)^2} U_{i,k}^n + \frac{1}{(\Delta Z)^2} U_{i,k+1}^n, \quad (3.3e)$$

where V^- is the computed approximation of V for the current iteration. The circumferential momentum equation (2.39b) is dealt with in a similar way.

As we discussed earlier, owing to the nature of the momentum equations, discontinuities can develop, and we expect a pseudo-shock to propagate radially away from the edge of the disk as time t increases. Unlike the steady Navier-Stokes equations which are elliptic, the unsteady boundary-layer equations (2.39a) and (2.39b) are of mixed-type: parabolic, and hyperbolic in r and t . One of the features of such partial differential equations is that if the radial velocity component remains positive, then the solution at one point depends on the boundary conditions upstream of that point only, and it will not be influenced by changes of boundary conditions downstream of it, contrary

to elliptic equations where the solution at one point depends on all the other points of the domain. This feature allows us to use marching methods to solve numerically the boundary-value problem (2.39a)-(2.39i): at each time step n , the solution is calculated along the computational line $r = r_i$ for all $k \in \llbracket 1; N_Z + 1 \rrbracket$. Once the solution on r_i is found, we move to the next computational line $r = r_{i+1}$ and so on. Let us describe in more detail the process of obtaining the solution along a fixed computational line r_i . Iterative techniques are the tools to solve systems of nonlinear equations. As we saw earlier, decoupling the equations and also using the Newton's method allowed us to linearise them. For fixed n and i , we solve sequentially (2.39a), (2.39b) and (2.39d), giving us $U_{i,k}^n$, $V_{i,k}^n$ and $W_{i,k}^n$ respectively for the current iteration, and for all $k \in \llbracket 1; N_Z + 1 \rrbracket$. At each new iteration, the computed approximations U^- , V^- and W^- get systematically closer to the exact solutions U , V and W respectively, and we continue the iterations until the absolute residual $\|\mathbf{F}(U^-, V^-, W^-)\|$ becomes smaller than a small parameter ε , where $\mathbf{F}$ is a column vector resulting of the concatenation of $(F_U)_{i,k}^n$ and $(F_V)_{i,k}^n$ for all k and for fixed i and n

$$\mathbf{F} = \begin{pmatrix} (F_U)_{i,1}^n \\ (F_U)_{i,2}^n \\ \vdots \\ (F_U)_{i,N_Z+1}^n \\ (F_V)_{i,1}^n \\ \vdots \end{pmatrix},$$

and where $(F_U)_{i,k}^n$ and $(F_V)_{i,k}^n$ are defined such that

$$\begin{aligned} (F_U)_{i,k}^n(U, V, W) = & \frac{U_{i,k}^n - U_{i,k}^{n-1}}{\Delta t} + \frac{(U_{i,k}^n)^2 - (U_{i-1,k}^n)^2}{\Delta r} + \frac{W_{i,k+1}^n + W_{i,k}^n}{4\Delta Z} U_{i,k+1}^n \\ & - \frac{W_{i,k+1}^n - W_{i,k-1}^n}{4\Delta Z} U_{i,k}^n - \frac{W_{i,k}^n + W_{i,k-1}^n}{4\Delta Z} U_{i,k-1}^n + \frac{(U_{i,k}^n)^2 - (V_{i,k}^n)^2}{r_i} \\ & - \frac{1}{(\Delta Z)^2} U_{i,k-1}^n + \frac{2}{(\Delta Z)^2} U_{i,k}^n - \frac{1}{(\Delta Z)^2} U_{i,k+1}^n, \quad (3.4a) \end{aligned}$$

$$\begin{aligned}
(F_V)_{i,k}^n(U, V, W) = & \frac{V_{i,k}^n - V_{i,k}^{n-1}}{\Delta t} + \frac{U_{i,k}^n V_{i,k}^n - U_{i-1,k}^n V_{i-1,k}^n}{\Delta r} + \frac{W_{i,k+1}^n + W_{i,k}^n}{4\Delta Z} V_{i,k+1}^n \\
& - \frac{W_{i,k+1}^n - W_{i,k-1}^n}{4\Delta Z} V_{i,k}^n - \frac{W_{i,k}^n + W_{i,k-1}^n}{4\Delta Z} V_{i,k-1}^n + \frac{2U_{i,k}^n V_{i,k}^n}{r_i} \\
& - \frac{1}{(\Delta Z)^2} V_{i,k-1}^n + \frac{2}{(\Delta Z)^2} V_{i,k}^n - \frac{1}{(\Delta Z)^2} V_{i,k+1}^n. \quad (3.4b)
\end{aligned}$$

The right-hand sides of (3.4a) and (3.4b) correspond to the discretised radial and circumferential momentum equations respectively, employing the finite difference schemes presented earlier but prior to linearisation. If the exact solutions U , V and W are used in (3.4a) and (3.4b) for the corresponding (i, k, n) node, then F_U and F_V are both equal to zero. However, using instead the computed approximations U^- , V^- and W^- , then F_U and F_V are not equal to zero. This is why our goal with iterative techniques is to find the computed approximations U^- , V^- , and W^- for each node (i, k, n) such that F_U and F_V are almost zero. The accuracy of the iterative methods depends on the small parameter ε .

At each iteration, U is obtained in the following way. For a fixed time step n and a fixed computational line r_i , discretisation and linearisation of the radial momentum equation (2.39a) leads to a system of linear algebraic equations which can be expressed in compact form as

$$A_U(U^-, V^-, W^-)X_U = B_U(U^-, V^-, W^-), \quad (3.5)$$

where X_U is a vector containing the unknowns $U_{i,k}^n$ for fixed i and n , and for all $k \in \llbracket 1; N_Z + 1 \rrbracket$ such that

$$X_U = \begin{pmatrix} U_{i,1}^n \\ U_{i,2}^n \\ \vdots \\ U_{i,N_Z+1}^n \end{pmatrix}.$$

A_U is a tridiagonal matrix and B_U is a column vector. Both of them are known as they depend on the known computed approximations U^- , V^- and W^- . The system (3.5) can be solved by making use of the Thomas algorithm (see Appendix A), which is a simplified form of the Gaussian elimination.

Afterwards, we obtain V for the current iteration in the same way: discretisation and linearisation of the circumferential momentum equation (2.39b) yields to a system

of linear algebraic equations written as

$$A_V(U^-, V^-, W^-)X_V = B_V(U^-, V^-, W^-), \quad (3.6)$$

where X_V is a vector containing the unknowns $V_{i,k}^n$ such that

$$X_V = \begin{pmatrix} V_{i,1}^n \\ V_{i,2}^n \\ \vdots \\ V_{i,N_Z+1}^n \end{pmatrix}.$$

As in the radial momentum equation case, A_V is a known tridiagonal matrix and B_V is a known column vector. Therefore, the Thomas algorithm may be also used to solve system (3.6).

The axial velocity component is calculated for the current iteration using the continuity equation (2.39d):

$$W_{i,k}^n = W_{i,k-1}^n - \Delta Z \left(\frac{(U^-)_{i,k}^n - U_{i-1,k}^n}{\Delta r} + \frac{(U^-)_{i,k}^n}{r_i} \right), \quad \forall k \in \llbracket 2; N_Z + 1 \rrbracket. \quad (3.7)$$

We could have discretised $\partial(UW)/\partial Z$ with the following scheme instead

$$\frac{\partial(UW)}{\partial Z} = \frac{U_{i,k+1}^n W_{i,k+1}^n - U_{i,k-1}^n W_{i,k-1}^n}{\Delta Z}, \quad (3.8)$$

but the scheme (3.3c) is more convenient. Indeed, even though both schemes are of second order accuracy, the difference is that by using the one presented in Figure 3.1, we made $U_{i,k}^n$ appear explicitly. The involvement of $U_{i,k}^n$ is important for the following reason. At each iteration, the system of linear algebraic equations (3.5) is solved thanks to the Thomas algorithm. We recall that A_U is a known tridiagonal matrix of the form

$$A_U = \begin{pmatrix} b_1 & c_1 & 0 & 0 & \dots & 0 \\ a_2 & b_2 & c_2 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & a_{N_Z} & b_{N_Z} & c_{N_Z} \\ 0 & \dots & 0 & 0 & a_{N_Z+1} & b_{N_Z+1} \end{pmatrix}. \quad (3.9)$$

The Thomas algorithm is very powerful as it restricts the memory allocation required since only three column vectors need to be created in order to define the matrix A_U instead of allocating the memory of a whole matrix. Also, it is particularly computationally efficient as it requires an order $O(N_Z + 1)$ calculations instead of $O((N_Z + 1)^3)$ for the Gauss method with partial pivoting. Therefore, the Thomas algorithm is efficient, but in order to prevent ill-conditioning (and so round-off contamination), the matrix A_U must be diagonally dominant, that is for all $i \in \llbracket 2; N_Z \rrbracket$

$$|b_i| > |a_i| + |c_i|. \quad (3.10)$$

The values of a_1 and c_{N_Z+1} are set equal to 0. Thus, approximating $\partial(UW)/\partial Z$ with the scheme (3.3c) instead of scheme (3.8) makes the matrix A_U more diagonally dominant, implying a better conditioning of the Thomas algorithm. The same arguments apply to the term $\partial(VW)/\partial Z$ into (2.39b).

Let us now discuss the numerical methods for the system of equations (2.55a)-(2.55d) governing the flow in region 2, where the entrainment effect takes place. Again, we shall use the finite difference method to solve this system. At each time step n , we shall use a uniform grid

$$\{r_i, z'_k\} = \{\Delta r(i - 1), \Delta z'(k - 1)\}, \quad (3.11)$$

with

$$i \in \llbracket 1; N_r + 1 \rrbracket, \quad k \in \llbracket 1; N_{z'} + 1 \rrbracket, \quad (3.12)$$

and where Δr , $\Delta z'$ are small grid spacings in r -direction and z' -direction, respectively. Thus, at the node (i, k, n) , we have $\psi'(r, z', t) \approx \psi'_{i,k}{}^n$.

Let us discretise equation (2.55a) by using second order accuracy schemes such that for all $(i, k) \in \llbracket 2; N_r \rrbracket \times \llbracket 2; N_{z'} \rrbracket$ (the superscript for the time step has been omitted here),

$$\frac{\psi'_{i-1,k} - 2\psi'_{i,k} + \psi'_{i+1,k}}{(\Delta r)^2} - \frac{1}{r_i} \frac{\psi'_{i+1,k} - \psi'_{i-1,k}}{2\Delta r} + \frac{\psi'_{i,k-1} - 2\psi'_{i,k} + \psi'_{i,k+1}}{(\Delta z')^2} = 0, \quad (3.13)$$

leading to a system of linear algebraic equations of the form

$$A\Psi' = B, \quad (3.14)$$

where A is a known square matrix, B is a known column vector and Ψ' is a vector of unknowns:

$$\Psi' = \begin{pmatrix} \psi'_{1,1} \\ \psi'_{1,2} \\ \vdots \\ \psi'_{N_r+1, N_{z'}+1} \end{pmatrix}. \quad (3.15)$$

Equation (2.55a) is of elliptic type, therefore the solution at one point depends on all the other points of the domain. Hence, we shall solve it with the help of an iterative technique in order to avoid memory allocation issues and to remain computationally efficient. Iterative methods are the tools to linearise sets of nonlinear algebraic equations, but they can also be applied to solve sets of linear algebraic equations as they can be much more effective than the classic Gauss elimination. Thus, employing iterative techniques for the set of equations (3.14), at a fixed time step, starting from an initial guess, the solution is systematically approached at each new iteration, and the iterations are performed until the maximum of the absolute value of the solution change $\max_{i,k} |\psi'_{i,k}{}^{(l+1)} - \psi'_{i,k}{}^{(l)}|$ becomes smaller than a prescribed small parameter ε . Here, l denotes the iteration number. We used the successive overrelaxation method (SOR) which evaluates Ψ' at the new iteration $(l+1)$ as a weighted average of Ψ' at the previous iteration (l) and Ψ' given by the Gauss-Seidel method. The solution is calculated by the Gauss-Seidel method at the new iteration $(l+1)$ for the node (i, k) as

$$\begin{aligned} \psi'_{i,k}{}^{(l+1)} = s_{rz} \left[\left(\frac{1}{(\Delta r)^2} + \frac{1}{2r_i \Delta r} \right) \psi'_{i-1,k}{}^{(l+1)} + \frac{1}{(\Delta z')^2} \psi'_{i,k-1}{}^{(l+1)} \right. \\ \left. + \left(\frac{1}{(\Delta r)^2} - \frac{1}{2r_i \Delta r} \right) \psi'_{i+1,k}{}^{(l)} + \frac{1}{(\Delta z')^2} \psi'_{i,k+1}{}^{(l)} \right], \quad (3.16) \end{aligned}$$

where we set

$$s_{rz} = \frac{0.5}{\frac{1}{(\Delta r)^2} + \frac{1}{(\Delta z')^2}}.$$

Let us denote $\psi'^{(*)}$ the Gauss-Seidel solution given by (3.16), and let us introduce the overrelaxation factor ω^* . The solution on the $(l+1)$ iteration is taken to be

$$\psi'_{i,k}{}^{(l+1)} = \omega^* \psi'_{i,k}{}^{(*)} + (1 - \omega^*) \psi'_{i,k}{}^{(l)}. \quad (3.17)$$

The relaxation parameter must satisfy $0 < \omega^* < 2$ to guarantee the convergence of the

SOR method. Obviously the goal of the SOR method is to accelerate the convergence of the Gauss-Seidel method, therefore our relaxation parameter will satisfy $1 < \omega^* < 2$.

In Chapter 4, based on the numerical techniques presented in this chapter, we shall see the numerical results of the boundary-layer equations; in particular, we are interested in plotting the radial and circumferential velocity components on the jet axis in order to observe that the front of the jet is propagating along both radial and circumferential directions at a finite speed, but also we want to show that there is a discontinuity at the position of the pseudo-shock. The axial velocity component will be plotted at the outer edge of boundary layer, and it will reveal a singularity at the position of the pseudo-shock. The numerical results of the set of equations (2.55a)-(2.55d) governing the flow in region 2 where the entrainment effect takes place will be shown. Finally, we shall compare the numerical solution of the boundary-layer equations above the disk surface with the Kármán (1921) solution.

Chapter 4

Numerical results

The discretisation procedure of the set of boundary-layer equations (2.39a)-(2.39i) introduces truncation errors. The difference between the exact solution and the approximate numerical solution, termed the solution error, depends mainly on the grid spacings and the value of the higher order derivatives. Obviously, the solution error can be minimised by using a more refined grid; say, (3.3a) is accurate to within $O\left(\frac{\partial^2 U}{\partial t^2} \Delta t\right)$. Consequently, a balance between accuracy and computational efficiency must be found. We used the mesh parameters of Table 4.1. The matching conditions (2.39g) for U and V at $Z = \infty$ turn to Dirichlet boundary conditions computationally speaking. Therefore, the upper limit of the computational domain, denoted as $Z_{\max}$, must be large enough to satisfy (2.39g). To satisfy this requirement, we shall pick a $Z_{\max}$ when U and V decreased at least to 1% of their maximum values. The results of the calculations with the mesh presented in Table 4.1 are displayed in Figures 4.1, 4.2 and 4.3. The radial velocity U in Figure 4.1 is plotted against r on the jet axis for different values of time t . We observe as expected a pseudo-shock propagating at finite speed as time increases in the positive r -direction. It consists of a discontinuity in U , such that everywhere before the shock, the flow is at rest. Immediately behind the shock, the velocity is finite and positive. The same features have been observed in the circumferential velocity component V , plotted against r on the jet axis and for different values of time in Figure 4.2. Regarding the axial velocity component, we plotted it at the outer edge of the boundary layer against r for the corresponding times t . The results are shown

Δt	Δr	ΔZ	$Z_{\max}$	ε
0.05	0.001	0.02	35	10^{-5}

Table 4.1: Mesh parameters for the rotating disk on the boundary-layer scale

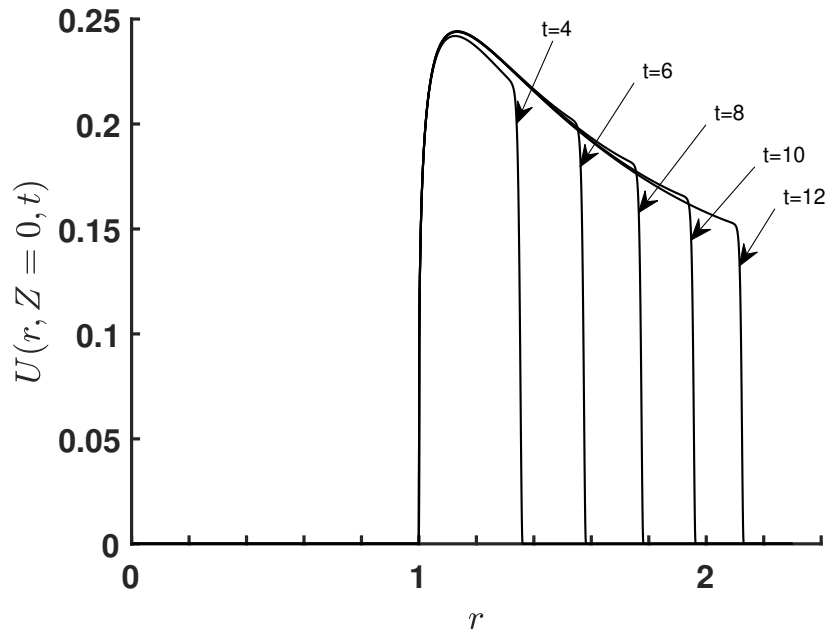


Figure 4.1: Radial velocity along the jet axis at time $t = 4, 6, 8, 10, 12$. Discontinuity develops at the position of the pseudo-shock. Immediately before the shock, the fluid is stagnant.

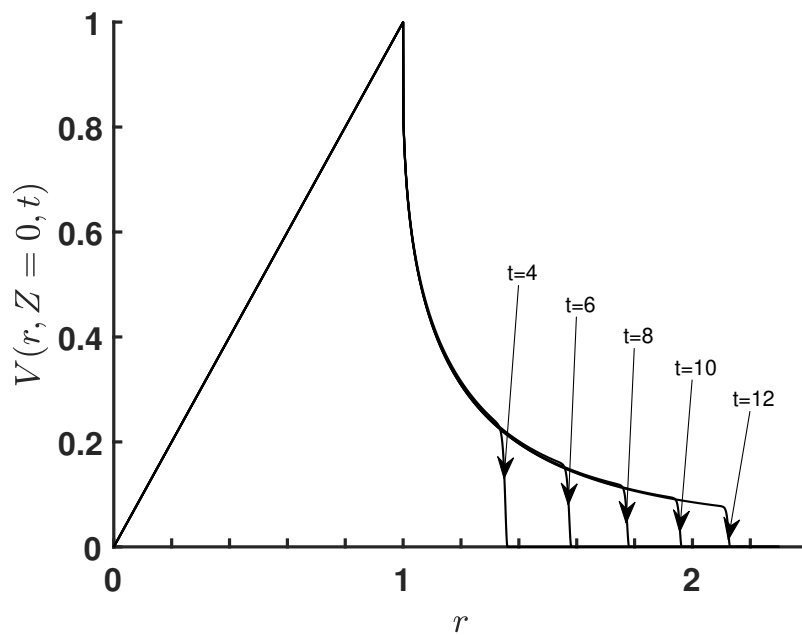


Figure 4.2: Circumferential velocity along the jet axis at time $t = 4, 6, 8, 10, 12$

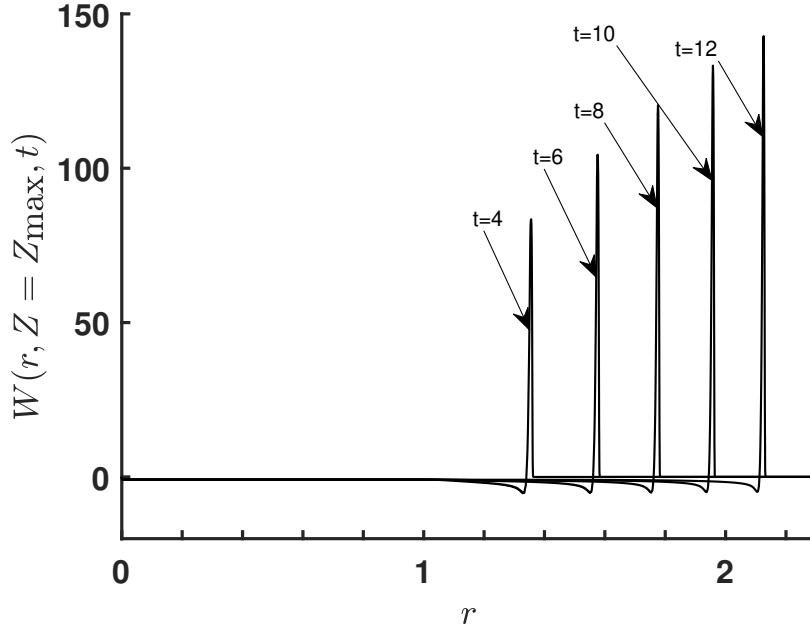


Figure 4.3: Axial velocity at $Z = Z_{\max}$ at time $t = 4, 6, 8, 10, 12$. Behind the pseudo-shock, it is negative due to the entrainment effect taking place at the outer edge of the boundary layer. There is a burst at the pseudo-shock position due to the discontinuity of the radial velocity component.

in Figure 4.3. Starting from $r = 0$, W is negative and remains globally constant up to the vicinity of the shock, where it decays slightly before bursting at the position of the shock. Immediately downstream of the shock, it is equal to 0. The negative sign of W in the main part of the boundary layer confirms the analysis performed in Chapter 2 about the entrainment effect. The spike for W is actually a singularity. Indeed, the numerical results are obtained on a finite grid, leading to round-off errors. Therefore, the slope of the tangent at the position of the pseudo-shock in Figures 4.1 and 4.2 is very large negative, but it is not infinite as we expect from the exact solution due to the discontinuity. If the exact solution for the radial velocity were plotted, then gradient of U would be infinite negative at the position of the pseudo-shock, and W would not show a spike, but a singularity instead at the position of the shock (see Figure 4.6) as W is obtained from the continuity equation (2.39d).

It follows from the continuity equation (2.39d) that there exists a scalar function called the stream-function and denoted ψ such that

$$rU = \frac{\partial \psi}{\partial Z}, \quad rW = -\frac{\partial \psi}{\partial r}. \quad (4.1)$$

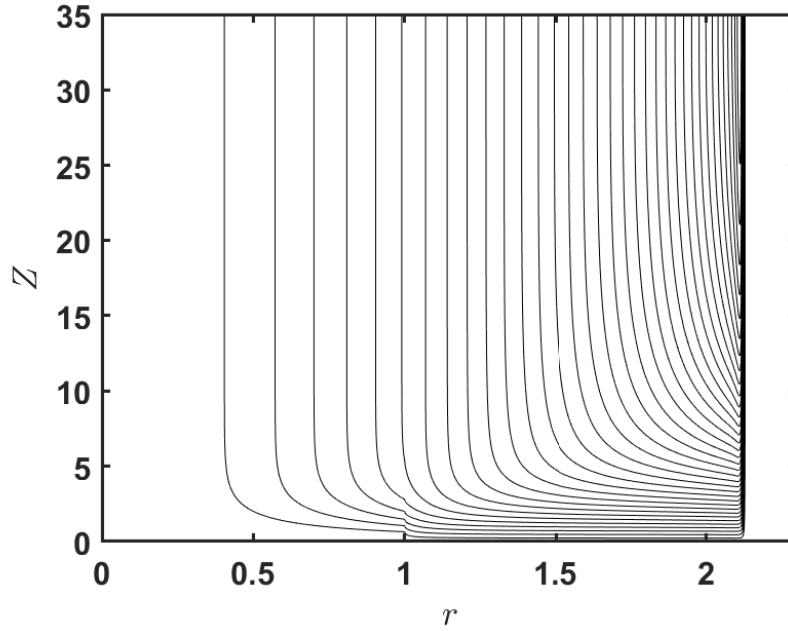


Figure 4.4: Streamline pattern on the boundary-layer scale at $t = 12$ in the case of the flow generated by the rotation of the disk

The streamlines are defined as lines $\mathcal{L}$ where the velocity vector is tangent to $\mathcal{L}$ at each point. The stream-function ψ being constant along each streamline, one can plot the iso- ψ lines in order to see the flow behaviour at a fixed time t . We plotted the streamline pattern in Figure 4.4 at $t = 12$. As we expected, we see the pseudo-shock at $r \approx 1.48$, and the flow is stagnant for all $r > 1.48$.

We repeated our calculations with different meshes so that we can see the impact of grid spacings change. We performed our calculations with the meshes presented in Table 4.2. We plotted the radial velocity component, plotted at $Z = 0$ against r at time $t = 5$, for the meshes of Table 4.2. The results are displayed in Figure 4.5. We can see that at the position of the pseudo-shock, Δr has an impact on the slope of radial velocity (the same phenomenon is also observed for the circumferential velocity): their derivatives with respect to r become larger in absolute value for a smaller value of Δr at the pseudo-shock's position. This is consistent with the fact that discretising our problem with the finite difference method introduced a truncation error $O(\Delta t, \Delta r, (\Delta Z)^2)$, so the quotient $\Delta U / \Delta r$ must tend to $-\infty$ at the pseudo-shock's position as the grid spacing Δr tends to 0. A change of ΔZ did not produce any significant difference.

	Δt	Δr	ΔZ	$Z_{\max}$	ε
Mesh 1	0.05	0.001	0.02	35	10^{-5}
Mesh 2	0.05	0.002	0.02	35	10^{-5}
Mesh 3	0.05	0.01	0.02	35	10^{-5}
Mesh 4	0.1	0.01	0.02	35	10^{-5}
Mesh 5	0.1	0.01	0.005	35	10^{-5}

Table 4.2: Mesh comparison. The bold quantities correspond to the mesh parameter changed as compared to the previous mesh.

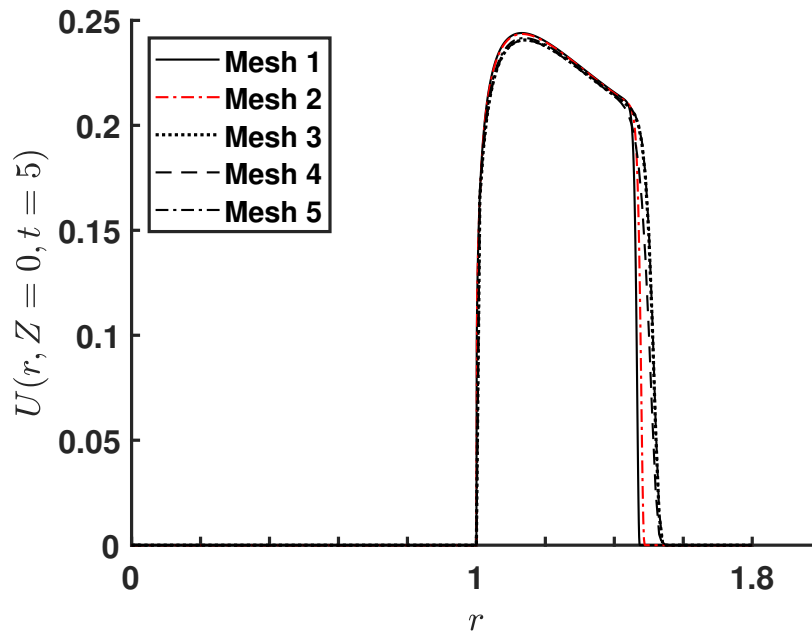


Figure 4.5: Radial velocity on the jet axis at time $t = 5$ for different meshes. Δr is the mesh parameter which has the most impact on the sharpness of the discontinuity at the position of the pseudo-shock. There is almost no difference between the plots for $\Delta r = 0.001$ and $\Delta r = 0.002$, where we can observe a sharp gradient at the position of the pseudo-shock.

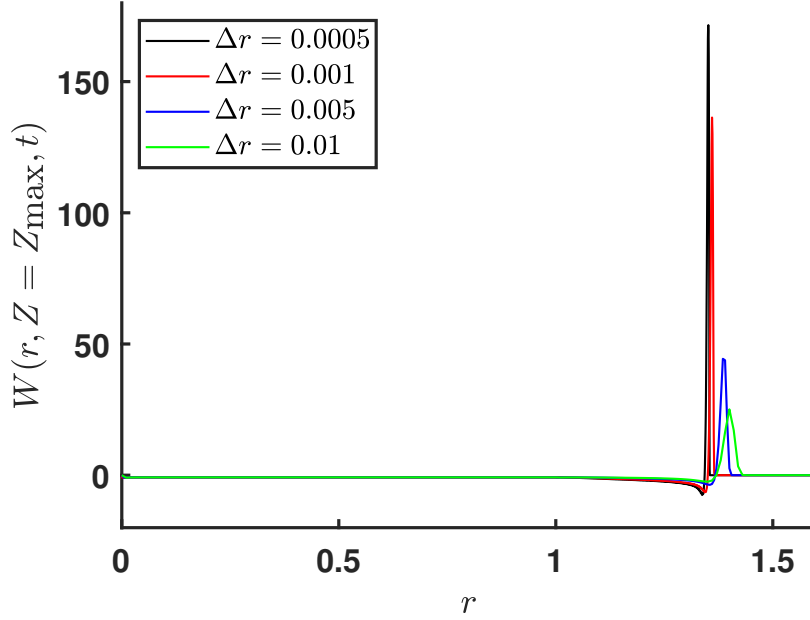


Figure 4.6: Axial velocity at $Z = Z_{\max}$ at time $t = 4$ for different values of Δr .

The parameter $Z_{\max}$ has been chosen such that it is large enough so that the velocity components U and V are almost 0 at Z close to $Z_{\max}$, as discussed earlier.

Figure 4.6 shows the impact of changing the grid spacing Δr on the spike of W . We can see that as Δr becomes smaller, the spike for W at the position of the pseudo-shock becomes bigger. Therefore, as $\Delta r \rightarrow 0$, W would become infinite at the shock's position.

Let us notice that our problem cannot be linked to Figure 11 of Hewitt and Duck (2011) as, in their case, they have a finite time singularity contrary to our problem. Indeed, we plot in Figure 4.7 the radial velocity component on the jet axis for smaller values of time. We observe that the pseudo-shock forms immediately downstream of the edge of the disk, as soon as the disk is impulsively brought into rotation, even though it is not well pronounced for small values of time such as $t = 0.5$. Unlike Hewitt and Duck (2011) where the pseudo-shock forms at a finite time (and consequently, everything is smooth before that critical time), in our case it forms as soon as the jet leaves the edge of the disk.

One last interesting plot on the boundary-layer scale is the comparison of our results with the Kármán flow. Kármán (1921) solved the Navier-Stokes equations for the case of a disk of infinite radius placed in an infinite region filled with an initially stagnant

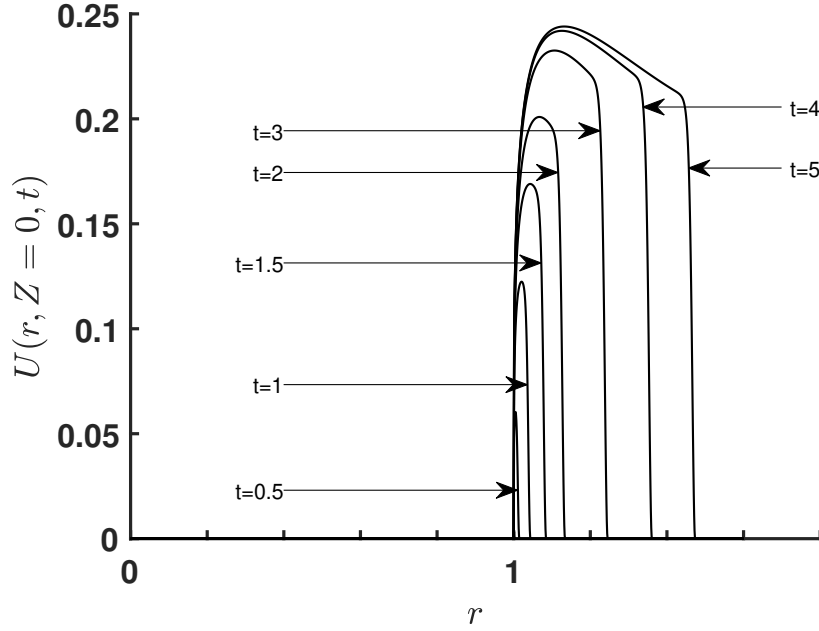


Figure 4.7: Radial velocity for small values of time.

fluid, and he studied the steady solution. The notation, the coordinate system and the reference frame is the same as the ones we introduced in the case of the disk with finite radius. Kármán sought the solution in the following form

$$\hat{V}_r = \Omega \hat{r} F(\zeta), \quad \hat{V}_\Phi = \Omega \hat{r} G(\zeta), \quad \hat{V}_z = \sqrt{\Omega \nu} H(\zeta), \quad (4.2a)$$

with the functions F , G , and H being determined numerically, and where the variable ζ is defined such that

$$\zeta = \sqrt{\frac{\Omega}{\nu}} \hat{z}. \quad (4.2b)$$

We wish to compare our unsteady solution for different values of time with the Kármán steady solution. This will be done for $|r| \leq 1$ because of the following reason: we recall that the unsteady boundary-layer equations are hyperbolic in $\hat{r}$ and $\hat{t}$, therefore if the radial velocity component remains positive, then the solution at one point depends on the boundary conditions upstream of that point. Any perturbation downstream will not have any impact on the solution at that point. Thus, the flow “does not know” that the disk is of finite radius, as long as $|r| \leq 1$. Hence, we expect to recover the Kármán’s solution when t becomes sufficiently large. Let us scale the quantities in (4.2a) and (4.2b) so that they can match the variables we are dealing with in our numerical

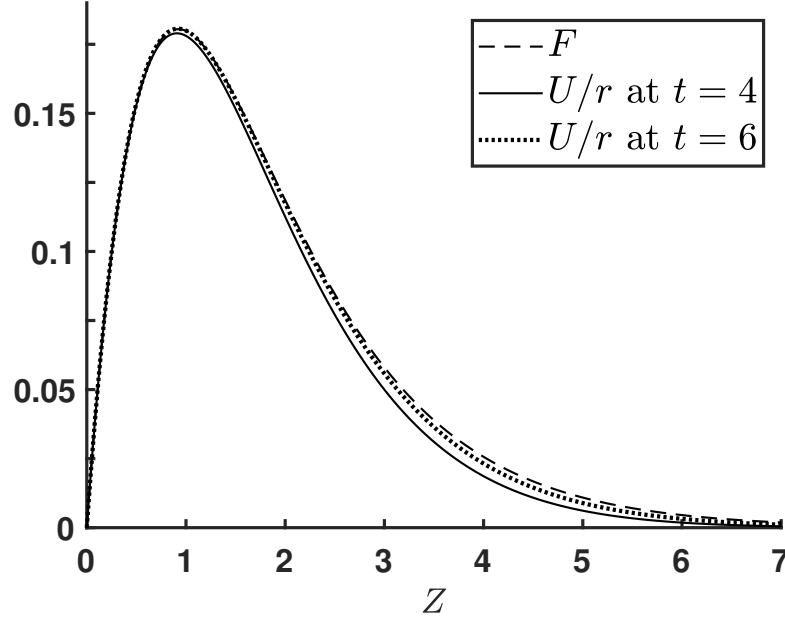


Figure 4.8: Comparison of the Kármán's solution F with U/r at $t = 4$ and $t = 6$ for $r = 0.5$

analysis of a disk of finite radius. We recall that we introduced the dimensionless variable z such that $\hat{z} = Rz$, and that the Reynolds number is defined such that $Re = \Omega R^2/\nu$. Consequently, the variable (4.2b) becomes

$$\zeta = Z, \quad (4.3)$$

where $Z = z/Re^{-1/2}$. Thus, on the boundary-layer scale, the flow quantities in (4.2a) have the following form

$$U = rF(Z), \quad V = rG(Z), \quad W = H(Z). \quad (4.4)$$

Now, we can compare our radial and circumferential velocity components with the Kármán solution by dividing them with r . The comparison has been performed for U in Figure 4.8 for two values of time, where we chose $r = 0.5$. We can see that our numerical solution at times $t = 4$ and $t = 6$ is very close to the Kármán's steady solution. The maximum of the difference between the Kármán solution and U/r at $t = 6$ is about 30%. Our calculations showed that from larger times, our numerical solution approaches more and more the Kármán's solution, with a maximum difference of less

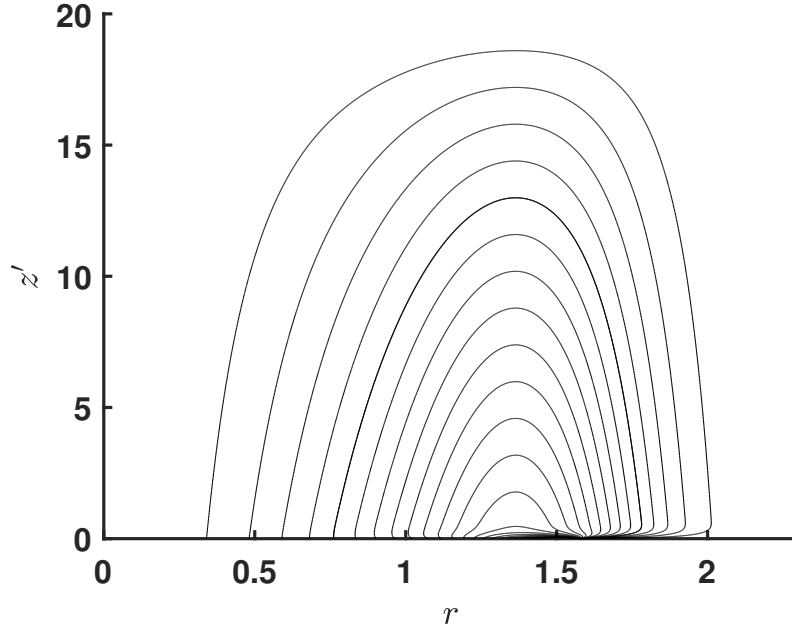


Figure 4.9: Streamline pattern of the flow in region 2 at $t = 6$. At the pseudo-shock's position $r_s(t = 6) = 1.564$, fluid particles are ejected from the boundary layer. This happens when the jet penetrates the stagnant fluid, which acts as a point source. These fluid particles are redirected upstream of the shock, and they are injected again into the boundary layer further upstream.

than 4% at $t = 12$.

In Figure 4.9, we show the solution of equations (2.55a)-(2.55d) governing the flow in region 2 at $t = 6$. A uniform grid was employed, where the grid spacings in r and z' direction are $\Delta r = \Delta z' = 0.001$. The upper limit of the domain has been chosen as $z'_{\max} = 20$. The streamlines have been plotted from $\psi' = 0.05$ to $\psi' = 1.15'$, with the interval $\Delta\psi' = 0.05$ between each streamline. The results of Figure 4.10 show that for the flow in region 2, the pseudo-shock serves as a point source. It injects the fluid into region 2 and then this fluid is entrained back into the boundary layer upstream of the pseudo-shock. The point source is located at a different position when the streamline pattern is plotted at another time as the pseudo-shock moves in the positive r -direction as time increases. A zoom on Figure 4.10 is shown in Figure 4.10. There is a divergence of the streamlines close to the point source. It resembles closely to the two-dimensional potential source; see Section 3.4 of Ruban and Gajjar (2014). When solving equation (2.55a) governing the flow in the inviscid region 2, one does

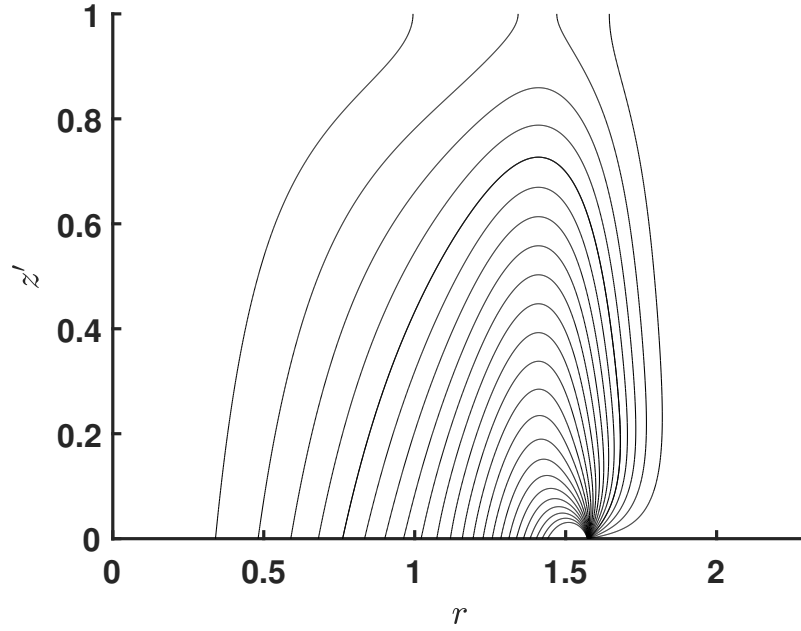


Figure 4.10: Zoom closer to the point source.

not require an explicit matching condition with the solution inside the pseudo-shock. Indeed, near a singularity, the second term in equation (2.55a) is small as compared to the first and third terms. This implies that the singularity here is the same as those of the Laplace equation. The inner structure of the pseudo-shock will be analysed in Chapter 5, and we will show that on the pseudo-shock's scale, $r - r_s(t) = O(Re^{-1/2})$ and $z = O(Re^{-1/2})$, where $r_s(t)$ is the position of the pseudo-shock at time t . One only needs to know the fluid mass flux from the pseudo-shock, given by

$$\int_{r_s - \varepsilon}^{r_s + \varepsilon} V_z dr, \quad (4.5)$$

where $Re^{-1/2} \ll \varepsilon \ll 1$. The integration in equation (4.5) is performed along the outer edge of the boundary layer. The above statement relies on the following well-known properties of the Laplace equation. This equation allows only for singularities of certain types. They are: a potential source, a potential vortex, a dipole, a quadrupole, and more; see p161 – 164 from Ruban and Gajjar (2014). No other type of singularity is possible. Moreover, if the integral in (4.5) is finite, then automatically we are dealing with a source. The strength of the source is uniquely defined by equation (4.5). Thus, in order to find the solution in the outer inviscid region 2, we only need to know the

distribution of V_z at the outer edge of the boundary layer.

In a nutshell, the numerical results presented in this chapter revealed that the rotation of the disk generates a jet propagating at a finite speed in both radial and circumferential directions as time increases. This is expressed by a clear discontinuity at the position of the pseudo-shock $r_s(t)$ when observing the radial and circumferential velocity components plotted with respect to r on the jet axis. Immediately downstream of this discontinuity, the fluid is at rest. Regarding the axial velocity component, a singularity develops at $r = r_s(t)$ when plotted at the outer edge of the boundary layer: the Prandtl's boundary-layer theory is not applicable, and this is why a new mathematical model is required in the vicinity of this singularity. This new model is presented in the next chapter, allowing us to analyse the inner structure of the pseudo-shock.

The analysis of region 2 and solving the Euler equations written in terms of the stream-function showed that at $r = r_s(t)$, fluid erupts at the outer edge of the boundary layer and it is redirected away forming large recirculation regions on both sides of the disk. The fluid is reinjected back further upstream into the boundary layer; this is the expression of the entrainment effect.

Chapter 5

Internal structure of the pseudo-shock

5.1 New mathematical model in the vicinity of the pseudo-shock

In this chapter, we shall turn our attention to region 1_{ps} (see Figure 2.4). The boundary-layer theory formulated by Prandtl (1904) is based on the assumption that the longitudinal derivatives of the fluid-dynamic functions are small in comparison with the ones in transverse direction. Indeed, in our case,

$$\frac{\partial \widehat{V}_r}{\partial \widehat{r}} \ll \frac{\partial \widehat{V}_r}{\partial \widehat{z}}, \quad (5.1)$$

because $\widehat{z} = RRe^{-1/2}Z$, with the Reynolds number being large. However, the numerical results showed us that the rotation of the disk generates a pseudo-shock propagating away from the rim of the disk as time increases. This shock consists of a discontinuity in the velocity components: therefore, the assumption of the smallness of the longitudinal gradients is valid everywhere in the boundary layer, except in a small region in the vicinity of the pseudo-shock where they become equal in order of magnitude to the transverse gradients. The boundary-layer theory is not valid in this region, so we need to predict the form of the asymptotic solution there. In this new region, we need to allow the pressure to change across the jet. For this to happen, the length $\Delta \widetilde{r}$ of the pseudo-shock should be comparable with the thickness of the jet. Hence,

$$\Delta \widetilde{r} \sim Re^{-1/2}. \quad (5.2)$$

For a detailed discussion of this issue, the reader is referred to Ruban and Vonatsos (2008).

Let us notice that the longitudinal extent $\Delta \widetilde{r}$ of this pseudo-shock is much larger than the characteristic thickness of shock waves in classical gas dynamics. Indeed, a

normal shock consists of a jump in the sought flow functions, and this jump may be smoothed out by observing it at a different scale, allowing the determination of the characteristic thickness of the region in the vicinity of the shock. The thickness of the classical shock in gas dynamics is an $O(Re^{-1})$ quantity and it is comparable with the molecular mean free path λ_m . As a consequence, the continuum hypothesis is not valid for describing the internal structure of this shock. In the case of our pseudo-shock generated by the rotation of the disk, its thickness is much larger than an $O(Re^{-1})$ quantity. This allows us to use the Navier-Stokes equations when describing its internal structure, and this is why we are referring to it as a “pseudo”-shock.

We need to introduce a new scaled variable X accordingly to (5.2). Since we want to study the flow behaviour in the vicinity of the moving pseudo-shock, it is convenient to perform our analysis in the reference frame moving with the shock. Thus, we introduce the characteristic coordinates as

$$Z = \frac{z}{Re^{-1/2}} = O(1), \quad X = \frac{r - r_s(t)}{Re^{-1/2}}, = O(1), \quad (5.3)$$

where $r_s(t)$ is the pseudo-shock's position at time t . Thanks to the numerical results we obtained in Chapter 4, we were able to plot the shock's position against time, as presented in Figure 5.1. The position of the pseudo-shock was determined by tracking the singularity in the axial velocity component W for each time step. We may write the asymptotic expansions of the fluid-dynamic functions on the pseudo-shock's scale, in the reference frame moving with it

$$\begin{aligned} V_r &= V_s(t) + \tilde{\mathcal{V}}_{r,0}(X, Z, t) + \cdots, \\ V_\Phi &= \tilde{\mathcal{V}}_{\Phi,0}(X, Z, t) + \cdots, \\ V_z &= \tilde{\xi}(Re)\tilde{\mathcal{V}}_{z,0}(X, Z, t) + \cdots, \\ p &= \tilde{\chi}(Re)\tilde{\mathcal{P}}_0(X, Z, t) + \cdots, \end{aligned} \quad \text{as } Re \rightarrow \infty, \quad (5.4)$$

where $\tilde{\xi}$ and $\tilde{\chi}$ must be found using mathematical and physical arguments. $V_s(t)$ is the shock speed at time t . The radial component of the velocity V_r is of order one because it varies from an order one quantity upstream of the shock, to 0 in front it (in a fixed reference frame). The circumferential velocity component V_Φ also remains an order one quantity.

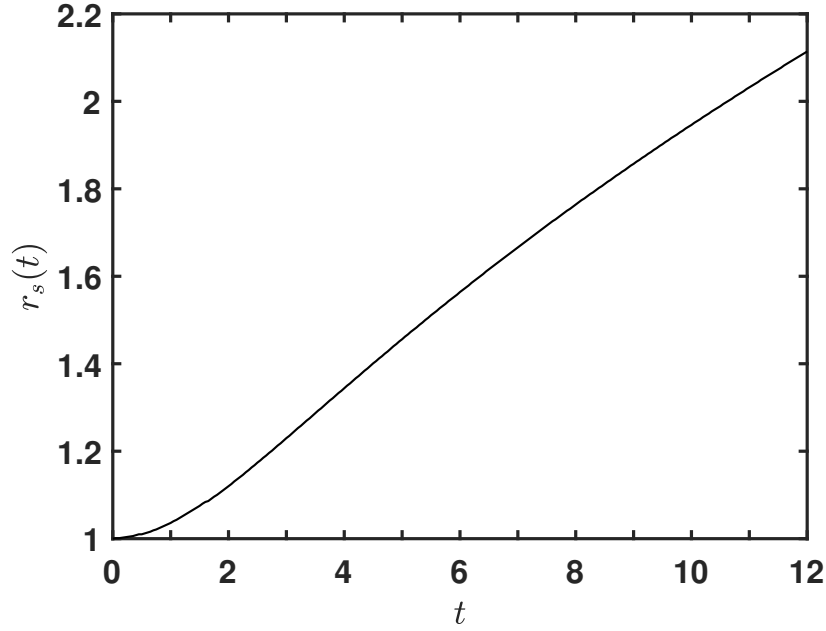


Figure 5.1: Evolution of the shock's position with time

Substitution of (5.3) and (5.4) into the continuity equation (2.26d) gives

$$Re^{1/2} \frac{\partial \tilde{\mathcal{V}}_{r,0}}{\partial X} + \tilde{\xi}(Re) Re^{1/2} \frac{\partial \tilde{\mathcal{V}}_{z,0}}{\partial Z} + \frac{\tilde{\mathcal{V}}_{r,0}}{r_s(t)} = 0.$$

Using the principle of least degeneration, it follows that

$$\tilde{\xi}(Re) = 1. \quad (5.5)$$

We balanced the second term with the first one, and not with the third one because, otherwise, it would have given a degenerated equation

$$\frac{\partial \tilde{\mathcal{V}}_{r,0}}{\partial X} = 0.$$

This equation states that the quantity $\tilde{\mathcal{V}}_{r,0}$ does not depend on X , which is a contradiction since $\tilde{\mathcal{V}}_{r,0}$ is of order one behind the pseudo-shock, and it must decay to 0 in front of it.

Let us substitute (5.3) and (5.4) into the radial momentum equation (2.26a). Keeping

in mind that $\tilde{\xi}$ is given by (5.5), we find that

$$\begin{aligned} V'_s(t) + \frac{\partial \tilde{\mathcal{V}}_{r,0}}{\partial t} + \frac{1}{Re^{-1/2}} \left(\tilde{\mathcal{V}}_{r,0} \frac{\partial \tilde{\mathcal{V}}_{r,0}}{\partial X} + \tilde{\mathcal{V}}_{z,0} \frac{\partial \tilde{\mathcal{V}}_{r,0}}{\partial Z} \right) - \frac{\tilde{\mathcal{V}}_{\Phi,0}^2}{r_s^2(t)} = \\ - \frac{\tilde{\chi}(Re)}{Re^{-1/2}} \frac{\partial \tilde{\mathcal{P}}_0}{\partial X} + \frac{\partial^2 \tilde{\mathcal{V}}_{r,0}}{\partial Z^2} + \frac{\partial^2 \tilde{\mathcal{V}}_{r,0}}{\partial X^2} + \frac{1}{Re} \left(\frac{Re^{1/2}}{r_s(t)} \frac{\partial \tilde{\mathcal{V}}_{r,0}}{\partial X} - \frac{\tilde{\mathcal{V}}_{r,0}}{r_s^2(t)} \right). \end{aligned}$$

The four last terms on the right-hand side, as well as the first two terms on the left-hand side may be all disregarded in the limit as $Re \rightarrow \infty$. The pseudo-shock we observed in our numerical results can only be explained by a sharp increase of the pressure. Consequently, in the inner structure of the shock, the pressure gradients must be in balance with the convective terms. To restore the pressure gradients, we have to set

$$\tilde{\chi}(Re) = 1. \quad (5.6)$$

With the information gathered above, the flow in the region in the vicinity of the pseudo-shock is governed by:

$$\tilde{\mathcal{V}}_{r,0} \frac{\partial \tilde{\mathcal{V}}_{r,0}}{\partial X} + \tilde{\mathcal{V}}_{z,0} \frac{\partial \tilde{\mathcal{V}}_{r,0}}{\partial Z} = - \frac{\partial \tilde{\mathcal{P}}_0}{\partial X}, \quad (5.7a)$$

$$\tilde{\mathcal{V}}_{r,0} \frac{\partial \tilde{\mathcal{V}}_{\Phi,0}}{\partial X} + \tilde{\mathcal{V}}_{z,0} \frac{\partial \tilde{\mathcal{V}}_{\Phi,0}}{\partial Z} = 0, \quad (5.7b)$$

$$\tilde{\mathcal{V}}_{r,0} \frac{\partial \tilde{\mathcal{V}}_{z,0}}{\partial X} + \tilde{\mathcal{V}}_{z,0} \frac{\partial \tilde{\mathcal{V}}_{z,0}}{\partial Z} = - \frac{\partial \tilde{\mathcal{P}}_0}{\partial Z}, \quad (5.7c)$$

$$\frac{\partial \tilde{\mathcal{V}}_{r,0}}{\partial X} + \frac{\partial \tilde{\mathcal{V}}_{z,0}}{\partial Z} = 0. \quad (5.7d)$$

The system of equations (5.7) does not depend on time, so the problem governing the flow in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock is steady in the leading order approximation: the problem is then said to be quasi-steady. If we had not studied the flow in the reference frame moving with the shock, the problem would have been unsteady in the leading order approximation. Let us also notice that $\tilde{\mathcal{V}}_{\Phi,0}$ appears only in the circumferential momentum equation (5.7b). As a result, the solution for the radial and axial velocity components, and for the pressure may be calculated independently of $\tilde{\mathcal{V}}_{\Phi,0}$ using (5.7a), (5.7c) and (5.7d). The problem is said to be quasi-two-dimensional. The solution $\tilde{\mathcal{V}}_{\Phi,0}$ may be determined thereafter by solving equation (5.7b). Thus, the flow in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock

is inviscid, quasi-steady and quasi-two-dimensional in the leading order approximation when observed in the reference frame moving with the shock.

5.2 Prediction of the flow behaviour on the pseudo-shock's scale

We shall now focus on the set of equations (5.7a), (5.7c) and (5.7d) as it can be solved independently as a two-dimensional system. For the sake of simplicity, the flow functions $\tilde{\mathcal{V}}_{r,0}$, $\tilde{\mathcal{V}}_{\Phi,0}$, $\tilde{\mathcal{V}}_{z,0}$ and $\tilde{\mathcal{P}}_0$ will be denoted $\tilde{U}$, $\tilde{V}$, $\tilde{W}$ and $\tilde{P}$, respectively. Since the flow is two-dimensional and incompressible, one can define the stream-function $\tilde{\psi}$ such that

$$\tilde{U} = \frac{\partial \tilde{\psi}}{\partial Z}, \quad \tilde{W} = -\frac{\partial \tilde{\psi}}{\partial X}. \quad (5.8)$$

Since the flow is quasi-steady and dictated by the inviscid flow theory, the Bernoulli equation is applicable:

$$H(\tilde{\psi}) = \frac{1}{2} (\tilde{U}^2 + \tilde{W}^2) + \tilde{P}, \quad (5.9)$$

where the quantity $H(\tilde{\psi})$ is constant along each streamline.

Our task now is to predict theoretically the flow behaviour in the $Re^{-1/2} \times Re^{-1/2}$ region. We shall also determine the speed of the shock $V_s(t)$ by theoretical means, and compare it with the experimental results of Turner (1962). Naturally, the speed of the pseudo-shock $V_s(t_n)$ may be also determined numerically at time t_n from its position $r_s(t_n)$ (see Figure 5.1) determined numerically when we solved the boundary-layer equations (2.39a)-(2.39i) in Chapter 4 using finite difference methods:

$$V_s(t_n) \approx \frac{r_s^n - r_s^{n-l}}{l\Delta t}, \quad (5.10)$$

where l is an integer such that $l \geq 1$. The reason why we did not set $l = 1$ necessarily is to obtain a more accurate result when determining $V_s(t_n)$. Indeed, we solved the boundary-layer equations numerically on a finite grid. Therefore, we may obtain the same position of the pseudo-shock for two consecutive time steps.

Immediately behind the shock, the radial velocity component inside the main part

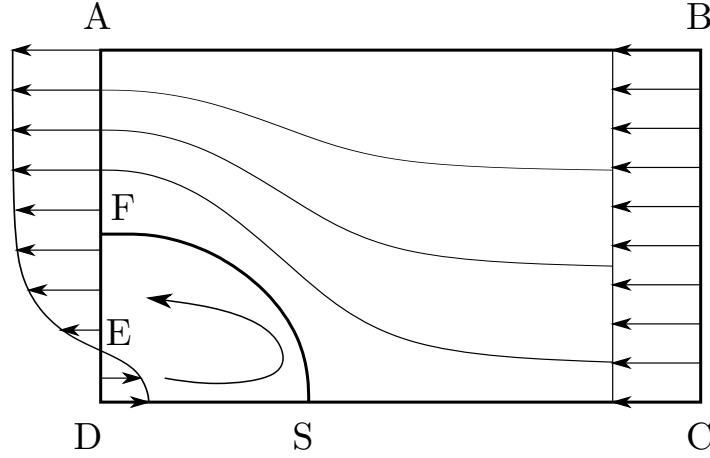


Figure 5.2: Inner structure of the pseudo-shock. A uniform velocity profile enters the domain when $X \rightarrow \infty$. The initial velocity profile when $X \rightarrow -\infty$ is first positive, and turns negative at point E . We expect a stagnation point at point S . A large recirculation eddy is formed within the DSF area.

of the boundary layer is $U(r_s^-(t), Z, t)$, so the Prandtl's matching rule yields to

$$U(r_s^-(t), Z, t) = V_s(t) + \tilde{U}(-\infty, Z, t), \quad (5.11a)$$

$$0 = \tilde{W}(-\infty, Z, t). \quad (5.11b)$$

In front of the shock, the matching is performed with the stagnant fluid, where the radial velocity component is $U(r_s^+(t), Z, t)$. So the matching with the flow downstream of the shock gives

$$0 = V_s(t) + \tilde{U}(\infty, Z, t), \quad (5.11c)$$

$$0 = \tilde{W}(\infty, Z, t). \quad (5.11d)$$

Let us denote by $\tilde{U}_+$ and $\tilde{U}_-$ the radial components of the velocity on the jet axis at $X = -\infty$ and $X = \infty$, respectively. According to matching conditions (5.11a) and (5.11c), we have

$$\tilde{U}_+ = U(r_s^-(t), 0, t) - V_s(t), \quad (5.12a)$$

$$\tilde{U}_- = -V_s(t). \quad (5.12b)$$

The inner structure of the pseudo-shock is sketched in Figure 5.2. The matching conditions (5.11c) and (5.11d) show that downstream of the shock, in the reference frame

moving with it, there is a uniform stream between B and C with the speed $-V_s(t)$. On the jet axis DC , the radial velocity component $\tilde{U}$ varies from a positive value equal to $\tilde{U}_+$ at point D , to a negative value at point C equal to $\tilde{U}_-$. Therefore, there must exist a stagnation point S on the jet axis, somewhere between D and C . This stagnation point S is due to the collision of the fluid elements coming from ED , and the ones coming from BC . We expect to observe a dividing streamline FS and recirculation regions developing in the region delimited by FSD . The location of point F in Figure 5.2 corresponds to the point where the stream-function $\tilde{\psi}$ is equal to 0 at $X = -\infty$ and $Z \neq 0$. Indeed, along the jet axis DC , the symmetry condition holds

$$\frac{\partial \tilde{U}}{\partial Z} = \tilde{W} = 0, \quad \text{at } Z = 0. \quad (5.13)$$

Consequently, the stream-function is equal to 0 everywhere on the jet axis. At $X = -\infty$, $\tilde{U}$ is positive at $Z = 0$ and decreases up to point E where it turns 0, and keep decreasing and tends to $-V_s(t)$ as $Z \rightarrow \infty$. Hence, according to this observation, and using the definition of $\tilde{\psi}$ formulated in (5.8), we expect that $\tilde{\psi}$ at $X = -\infty$ which is equal to 0 at $Z = 0$, increases up to point E , and starts decreasing from point E as Z increases. Besides, $\tilde{\psi}$ is a continuous function of Z , it is monotonically decaying from E to A such that $\tilde{\psi} > 0$ at point E , and $\tilde{\psi} < 0$ at point A . So according to the intermediate value theorem, there exists a unique point, denoted F , where $\tilde{\psi} = 0$ at $X = -\infty$ between E and A . Since the stream-function is constant along each streamline, then we deduce that the dividing streamline is FS .

Let us apply the Bernoulli equation (5.9) to the streamline DS . At point D , the radial velocity is given by (5.12a), while the axial velocity is given by (5.11b). Moreover, we proved earlier that the pressure P is constant and equal to 0 everywhere in the main part of the boundary layer, so $\tilde{P}_D = 0$. Point S is a stagnation point, so the velocity is 0 by definition. However, the pressure at point S , denoted as $\tilde{P}_S$, cannot be 0. The Bernoulli equation applied to the streamline DS reduces to

$$\frac{1}{2} \tilde{U}_+^2 = \tilde{P}_S. \quad (5.14)$$

At point C , the velocity components are given by (5.11c) and (5.11d). The pressure at point C is equal to 0, as the fluid is stagnant there. Using the Bernoulli equation on

streamline SC leads to a second equation involving $\tilde{P}_S$

$$\tilde{P}_S = \frac{1}{2}\tilde{U}_-^2. \quad (5.15)$$

Combining equation (5.14) with (5.15), it follows that

$$\tilde{U}_+ = \pm \tilde{U}_-. \quad (5.16)$$

Keeping in mind the geometry of the flow in Figure 5.2, we have to choose

$$\tilde{U}_+ = -\tilde{U}_-. \quad (5.17)$$

It is easily deduced from (5.12a) and (5.12b) that

$$V_s(t) = \frac{1}{2}U(r_s(t), 0, t). \quad (5.18)$$

This is in close agreement with the experiments of Turner (1962). His apparatus consisted of two tanks: a small tank is filled with salted water and it is placed on top of a large tank filled with fresh water. At time $\hat{t} = 0$, an orifice in the small tank is opened and a jet starts to flow into the large tank. His measurements show that the speed of the jet front $V_s(t)$ is

$$V_s(t) = 0.49 U(r_s(t), 0, t).$$

Moreover, since we were able to track the position $r_s(t)$ of the pseudo-shock numerically (see Figure 5.1), we can compare the speed of the pseudo-shock $V_{s,\text{inv}}(t)$ given by equation (5.18) and obtained by our analysis of the $Re^{-1/2} \times Re^{-1/2}$ inner region in the leading-order approximation, and the one found by differentiation with respect to time of $r_s(t)$, denoted as $V_{s,\text{num}}(t)$, and given by equation (5.10). The comparison is presented in Table 5.1. The value of $U(r_s(t), 0, t)$ used in equation (5.18) is the value of the radial velocity on the jet axis on the boundary-layer scale, immediately upstream of the discontinuity.

In the next section, we will solve the modified Euler equations (5.7) governing the flow in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock. In particular, we are interested in plotting the streamline pattern of the flow in this region so that we can verify that it is consistent with our theoretical predictions discussed earlier

t	$V_{s,\text{inv}}(t)$	$V_{s,\text{num}}(t)$	Relative error
4	0.1094	0.112	2.38%
6	0.1002	0.108	7.78%
8	0.0904	0.096	6.19%
10	0.0824	0.088	6.8%
12	0.0758	0.08	5.54%

Table 5.1: Comparison between $V_{s,\text{inv}}(t)$ and $V_{s,\text{num}}(t)$.

and summarised in Figure 5.2. We will make sure that the speed of the pseudo-shock determined numerically by differentiating the shock's position with respect to time coincides well with formula (5.18). The speed value will be used in the matching conditions at $X = \pm\infty$ for the radial velocity.

5.3 Solving numerically the governing equations using a stream-function vorticity formulation

Our task now is to solve the system of equations (5.7a), (5.7c) and (5.7d). It is currently written in what we call the primitive-variable formulation, that is the unknowns are the velocity and the pressure fields. We can turn the system into a more convenient form where the pressure will not be explicit anymore thanks to the so called stream-function vorticity formulation. The stream-function has already been defined in (5.8), and we now define the scalar vorticity field ω such that

$$\omega = \frac{\partial \widetilde{W}}{\partial X} - \frac{\partial \widetilde{U}}{\partial Z}. \quad (5.19)$$

Let us perform cross-differentiation on equations (5.7a) and (5.7c): firstly we apply the differential operator $\partial/\partial Z$ to equation (5.7a), then we apply the differential operator $\partial/\partial X$ to equation (5.7c), and finally we perform the subtraction of the resulting axial momentum equation with the resulting radial momentum equation. Taking into account that $\partial \widetilde{U}/\partial X + \partial \widetilde{W}/\partial Z$ is equal to 0 according to the continuity equation (5.7d), the cross-differentiation leads to

$$\widetilde{U} \frac{\partial \omega}{\partial X} + \widetilde{W} \frac{\partial \omega}{\partial Z} = 0. \quad (5.20)$$

Equation (5.20) is the steady vorticity transport equation for inviscid flows. The velocity components $\tilde{U}$ and $\tilde{W}$ in (5.20) are given by (5.8) where they are written in terms of the stream-function.

By substituting (5.8) into the definition of the scalar vorticity field (5.20), we find the stream-function vorticity relation, which is a Poisson equation

$$\frac{\partial^2 \tilde{\psi}}{\partial X^2} + \frac{\partial^2 \tilde{\psi}}{\partial Z^2} = -\omega. \quad (5.21)$$

Combining (5.20), (5.21) and (5.8) together, we arrive at the following set of equations

$$\tilde{U} \frac{\partial \omega}{\partial X} + \tilde{W} \frac{\partial \omega}{\partial Z} = 0, \quad (5.22a)$$

$$\frac{\partial^2 \tilde{\psi}}{\partial X^2} + \frac{\partial^2 \tilde{\psi}}{\partial Z^2} = -\omega, \quad (5.22b)$$

$$\tilde{U} = \frac{\partial \tilde{\psi}}{\partial Z}, \quad \tilde{W} = -\frac{\partial \tilde{\psi}}{\partial X}. \quad (5.22c)$$

Therefore, the system of equations (5.7a), (5.7c) and (5.7d) written with the primitive-variable formulation has been rearranged by using the stream-function vorticity formulation. Equation (5.22a) shows us that ω is constant along each streamline.

Before turning to the boundary conditions required to solve the system of equations (5.22a), (5.22b) and (5.22c), we shall first describe the numerical methods that will be used to solve this problem.

First of all, a modification on equation (5.22a) is required. Calculations (not presented in this thesis) were performed to try to solve the inviscid problem (5.22a)-(5.22c) with the appropriate boundary conditions. They showed that the solution of (5.22a) develops oscillations and the numerical problem cannot be solved. We do not know why oscillations appear. It could be that the steady problem is ill-posed. To suppress this instability, we introduced an artificial viscosity ν_a , which turns the hyperbolic equation (5.22a) into an elliptic equation:

$$\tilde{U} \frac{\partial \omega}{\partial X} + \tilde{W} \frac{\partial \omega}{\partial Z} = \nu_a \left(\frac{\partial^2 \omega}{\partial X^2} + \frac{\partial^2 \omega}{\partial Z^2} \right). \quad (5.23)$$

Obviously, ν_a must be kept small as the flow is supposed to be inviscid in the leading-order approximation. There is very small dependence on ν_a when it is small enough;

see Figure 5.4 later in this section.

To solve (5.23), (5.22b) and (5.22c), we applied the pseudo-transient method. It was introduced by Burggraf (1966), and it consists of writing the unsteady analogues of both equations (5.23) and (5.22b). In this approach, one marches in time until the steady state is reached; then time plays the role of an iteration parameter. Hence, we shall add the false time derivatives $\partial\omega/\partial t$ to the left-hand side of (5.23), and $\partial\tilde{\psi}/\partial t$ to the right-hand side of (5.22b). It should be noted that adding $\partial\tilde{\psi}/\partial t$ does not turn (5.22b) into an equation modeling a real unsteady flow. Thus, the variable t here should be termed as an artificial time. With the addition of false time derivatives $\partial\omega/\partial t$ and $\partial\tilde{\psi}/\partial t$, it turned both equations (5.23) and (5.22b) into parabolic type, allowing us to use marching methods, which is equivalent to using a single iterative procedure. By doing that, we are not modeling the unsteady solution, but it is not an issue as only the steady solution is of interest in our case. The pseudo-transient method allows us to obtain a fast convergence, contrary to the double iteration scheme used to solve the classic unsteady problem. Mallinson and de Vahl Davis (1973) applied the pseudo-transient method to the problem of the buoyancy-driven motion of a viscous incompressible fluid in an enclosure. According to them, one needs to multiply the time derivative $\partial\omega/\partial t$ by the parameter ν_a as this will affect the adjustment of the development of the transient vorticity ω .

By taking account of the modifications stated above, the problem written with the help of the pseudo-transient method has the following form

$$\nu_a \frac{\partial\omega}{\partial t} + \tilde{U} \frac{\partial\omega}{\partial X} + \tilde{W} \frac{\partial\omega}{\partial Z} = \nu_a \left(\frac{\partial^2\omega}{\partial X^2} + \frac{\partial^2\omega}{\partial Z^2} \right), \quad (5.24a)$$

$$\frac{\partial\tilde{\psi}}{\partial t} - \frac{\partial^2\tilde{\psi}}{\partial X^2} - \frac{\partial^2\tilde{\psi}}{\partial Z^2} = \omega, \quad (5.24b)$$

$$\tilde{U} = \frac{\partial\tilde{\psi}}{\partial Z}, \quad \tilde{W} = -\frac{\partial\tilde{\psi}}{\partial X}. \quad (5.24c)$$

There is another main advantage in applying the pseudo-transient method: since our problem is two-dimensional, we shall use multidimensional splitting techniques. A prime example is the alternating direction implicit (ADI) method, introduced by Peaceman and Rachford (1955) in its original form, and later formulated in its generalised form by Douglas and Gunn (1964). The ADI method may be described as follows: after having discretised the system of equations (5.24a)-(5.24c) with finite differences, each time step Δt is divided into two half-steps $\Delta t/2$. During the first half-step, equations

(5.24a) and (5.24b) are solved implicitly in X (and explicitly in Z), while during the second half-step, the equations are solved implicitly in Z (and explicitly in X). The equations (5.24a) may be linearised using for $\tilde{U}$ and $\tilde{W}$ their values at the previous time step, as time plays the role of an iteration parameter. There are two benefits of solving the system with the ADI method at each time iteration. The first one is that a full step is unconditionally stable, even though each half-step is conditionally stable, but only a full step is of interest here. The second advantage of the ADI method is that it allows us to use the Thomas algorithm (that was already introduced in Chapter 3) at each half-step. The association of the pseudo-transient and ADI methods is interesting regarding the conditioning of the Thomas algorithm. We already discussed this aspect in Chapter 3, and thanks to the introduction of time derivatives, the condition $|b_i| > |a_i| + |c_i|$ is more likely to be satisfied, implying prevention of round-off contamination.

Let us now formulate the boundary conditions for equations (5.24a)-(5.24c).

On the jet axis, a symmetry condition holds, so the axial velocity component is equal to 0: so according to equation (5.24c), $\tilde{\psi}$ is equal to 0 as well on the jet axis. Regarding ω , since $\partial\tilde{U}/\partial Z = 0$ at $Z = 0$ due to the symmetry condition applied on the jet axis, we have $\omega = 0$ along $Z = 0$:

$$\tilde{\psi} = \omega = 0, \quad \text{at } Z = 0. \quad (5.24d)$$

Far from the jet axis, the solution matches with the external inviscid flow, which appears to be uniform in the reference frame moving with the shock, such that the radial velocity component satisfies $\tilde{U}|_{Z=\infty} = -V_s(t)$. Hence, by substituting $\tilde{U}|_{Z=\infty}$ into the first equation in (5.24c) and also into (5.19), we get

$$\frac{\partial\tilde{\psi}}{\partial Z} = -V_s(t), \quad \omega = 0, \quad \text{at } Z = \infty. \quad (5.24e)$$

At $X = -\infty$, the solution matches with the one immediately behind the pseudo-shock. Since the matching is performed in the reference frame moving with the shock, it is convenient to introduce the function U_I such that

$$U_I(Z, t) = U(r_s(t), Z, t) - V_s(t). \quad (5.24f)$$

Thus, the boundary conditions at $X = -\infty$ may be written as

$$\tilde{\psi} = \int_0^Z U_I(Z, t) dZ, \quad \omega = -\frac{\partial U_I}{\partial Z}, \quad \text{at } X = -\infty. \quad (5.24g)$$

In the first equation of (5.24g), the constant of integration has been set equal to 0 because $\tilde{\psi} = 0$ at $Z = 0$ according to the boundary condition (5.24d). In the second equation of (5.24g), the term $\partial \tilde{W} / \partial X$ at $X = -\infty$ has been omitted for the following reason: in the main part of the boundary layer, outside of the region of the pseudo-shock, the axial component of the velocity is of the order of $Re^{-1/2}$, contrary to the radial component which is of order one. Consequently, since we are working in the limit as $Re \rightarrow \infty$, the term $\partial \tilde{W} / \partial X$ is negligible as compared to $\partial \tilde{U} / \partial Z$ at $X = -\infty$.

At $X = \infty$, the flow is uniform, so we have

$$\tilde{\psi} = -V_s(t)Z, \quad \omega = 0, \quad \text{at } X = \infty. \quad (5.24h)$$

Regarding the initial condition for ω and $\tilde{\psi}$ at $t = 0$, it will not influence the steady solution, however it is important to choose it so that it is not too far from the steady solution if we want a quicker convergence of the pseudo-transient iteration process. We suggest that for all Z , a linear interpolation is performed between ω and $\tilde{\psi}$ given at $X = -\infty$ by (5.24g), and ω and $\tilde{\psi}$ given at $X = \infty$ by (5.24h), respectively.

The set of equations (5.24a)-(5.24c) is discretised by approximating the derivatives with finite differences. We employed a uniform grid:

$$\{X_i, Z_k, t_n\} = \{-X_{\max} + \Delta X(i-1), \Delta Z(k-1), \Delta t(n-1)\}, \quad (5.25)$$

such that

$$i \in \llbracket 1; N_X + 1 \rrbracket, \quad k \in \llbracket 1; N_Z + 1 \rrbracket, \quad n \in \llbracket 1; \widetilde{N}_t \rrbracket. \quad (5.26)$$

ΔX , ΔZ and Δt are small grid spacings in X -direction, Z -direction and for time, respectively. $N_X + 1$ is the number of nodes in the X -direction.

Regarding the left and right limits of the domain, we performed our calculations between $-X_{\max}$ and $X_{\max}$ where we set $X_{\max}$ positive. The number of time steps $\widetilde{N}_t$ is chosen large enough for the steady state to be achieved. This is carried out by monitoring the maximum of the absolute values of both solution changes: the steady

Δt	ΔX	ΔZ	ν_a	$X_{\max}$	$Z_{\max}$	ε
0.005	0.1	0.02	0.0001	20	35	10^{-5}

Table 5.2: Mesh parameters used to solve the discretised governing equations in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock

state is reached when $\max_{i,k} |\omega_{i,k}^{n+1} - \omega_{i,k}^n|$ and $\max_{i,k} |\tilde{\psi}_{i,k}^{n+1} - \tilde{\psi}_{i,k}^n|$ both become smaller than a small parameter ε . The discretised system may be written as follows

$$\begin{aligned} & \nu_a \frac{\omega_{i,k}^{n+1} - \omega_{i,k}^n}{\Delta t} + \tilde{U}_{i,k}^n \frac{\omega_{i+1,k}^{n+1} - \omega_{i-1,k}^{n+1}}{2\Delta X} + \tilde{W}_{i,k}^n \frac{\omega_{i,k+1}^{n+1} - \omega_{i,k-1}^{n+1}}{2\Delta Z} = \\ & \nu_a \left(\frac{\omega_{i-1,k}^{n+1} - 2\omega_{i,k}^{n+1} + \omega_{i+1,k}^{n+1}}{(\Delta X)^2} + \frac{\omega_{i,k-1}^{n+1} - 2\omega_{i,k}^{n+1} + \omega_{i,k+1}^{n+1}}{(\Delta Z)^2} \right), \end{aligned} \quad (5.27a)$$

$$\frac{\tilde{\psi}_{i,k}^{n+1} - \tilde{\psi}_{i,k}^n}{\Delta t} - \frac{\tilde{\psi}_{i-1,k}^{n+1} - 2\tilde{\psi}_{i,k}^{n+1} + \tilde{\psi}_{i+1,k}^{n+1}}{(\Delta X)^2} - \frac{\tilde{\psi}_{i,k-1}^{n+1} - 2\tilde{\psi}_{i,k}^{n+1} + \tilde{\psi}_{i,k+1}^{n+1}}{(\Delta Z)^2} = \omega_{i,k}^{n+1}, \quad (5.27b)$$

$$\tilde{U}_{i,k}^n = \frac{\tilde{\psi}_{i,k+1}^n - \tilde{\psi}_{i,k-1}^n}{2\Delta Z}, \quad \tilde{W}_{i,k}^n = -\frac{\tilde{\psi}_{i+1,k}^n - \tilde{\psi}_{i-1,k}^n}{2\Delta X}. \quad (5.27c)$$

The streamline pattern on the pseudo-shock's scale at $t = 8$ is displayed in Figure 5.3. The pseudo-shock's position $r_s(t = 8)$ is equal to 1.746, and its speed has been determined numerically using equation (5.10), and we found that it is equal to 0.096 at $t = 8$. The speed given by equation (5.18) using the value for the radial velocity found by solving the governing equations on the boundary-layer scale gives instead a speed equal to 0.0904, which corresponds to a difference of 6.19%. Thus, both values coincide rather well with one another. Furthermore, even though the governing equations (5.22a)-(5.22c) are quasi-steady in the leading order approximation, we had to plot the streamline pattern at a specific time ($t = 8$ here) because the initial velocity profile U_I imposed by the matching condition at $X = -\infty$ as well as the shock speed $V_s(t)$ both depend on time. We recall that U_I is given by (5.24f), so here it is written as

$$U_I(Z, t = 8) = U(r = 1.746, Z, t = 8) - 0.096, \quad \text{for all } Z \geq 0.$$

The mesh used to plot Figure 5.3 is presented in Table 5.2. The upper limit of the domain $Z_{\max}$ as well as the grid spacing in Z -direction remain the same as in the solution of the boundary-layer equations in Chapter 4. In Figure 5.3, we can see as expected

that there is a dividing streamline along which the stream-function is equal to zero. A recirculation region is observed below this dividing streamline. In the jet region, the streamlines have been plotted for $\tilde{\psi} = 0.05$ and $\tilde{\psi} = 0.1$. The rest of the streamlines (that is, apart from the area of the recirculation region and the dividing streamline) corresponds to $\tilde{\psi} = -0.05$, and $\tilde{\psi}$ between -0.1 and -1.5 , with the interval $\Delta\tilde{\psi} = -0.1$ between each streamline. The choice of the time step Δt is important for the convergence of the pseudo-transient method: a smaller Δt implies a slower convergence, but also a bigger radius of convergence. In particular, regarding the artificial viscosity ν_a , we were able to set it equal to 0.001 without any problem for convergence, even with a coarse grid $\Delta t = 0.1$ and $\Delta X = 0.1$. However when we kept decreasing it further, the solution change started to oscillate and our computer code did not converge. One way to remedy this issue is to decrease our parameter Δt . Thus, we managed to get a stable and convergent solution with a small artificial viscosity $\nu_a = 0.0001$ when $\Delta t = 0.005$. A smaller artificial viscosity implies that the tangent to the dividing streamline at the stagnation point is more vertical, which is what we expect to see at the stagnation point; see Figure 5.4. A change in the grid spacing ΔX did not produce any noticeable variation in the streamline pattern. We would have expected that the streamlines when $X \rightarrow -\infty$ become parallel to the X -axis, as $W = O(1)$ in region 1_{ps} , while $W = O(Re^{-1/2})$ in the main part of region 1. But instead, the tangents to the streamlines do not go to zero in Figure 5.3. More precisely, the behaviour of the streamlines is parabolic when $X \rightarrow -\infty$. This is a version of the *free streamline theory* (see equation (3.8.54) in Ruban and Gajjar (2014)). The free streamline theory is applicable to inviscid flows that have a region of constant pressure. One of the well-known results of this theory is that the streamlines adjacent to the region of constant pressure are parabolic.

Now that the velocity is known everywhere in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock, let us determine the pressure field $\tilde{P}$ in this region. Let us denote by $\tilde{U}_{\text{jet}}$ and $\tilde{P}_{\text{jet}}$ the radial velocity and the pressure at $Z = 0$. Then using definition (5.8), we have

$$\tilde{U}_{\text{jet}} = \left. \frac{\partial \tilde{\psi}}{\partial Z} \right|_{Z=0}. \quad (5.28)$$

We recall that the Bernoulli function $\tilde{H}(\tilde{\psi})$ given by (5.9) is constant along each streamline, in particular, at $Z = 0$ where $\tilde{\psi} = 0$. Along this streamline, the axial velocity component $\tilde{W}$ is equal to zero to satisfy the symmetry condition. Hence, from (5.9), it

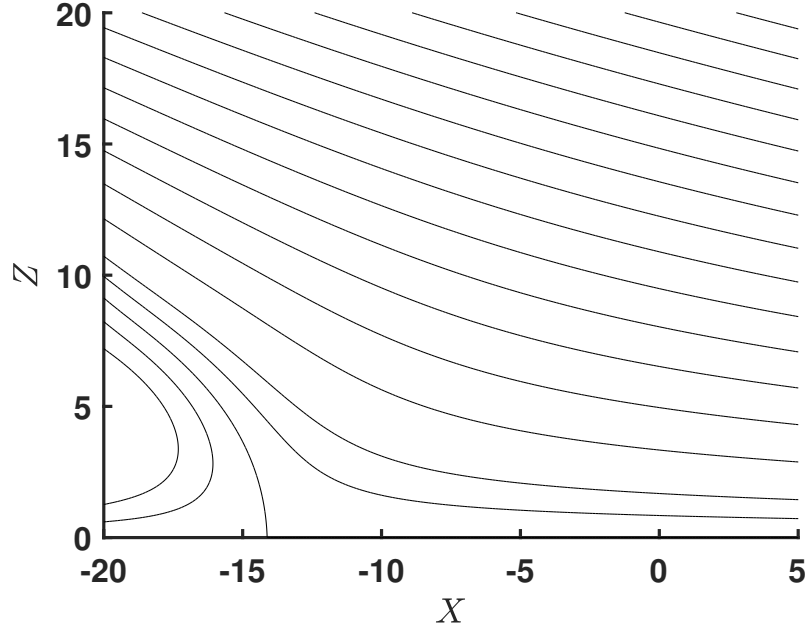
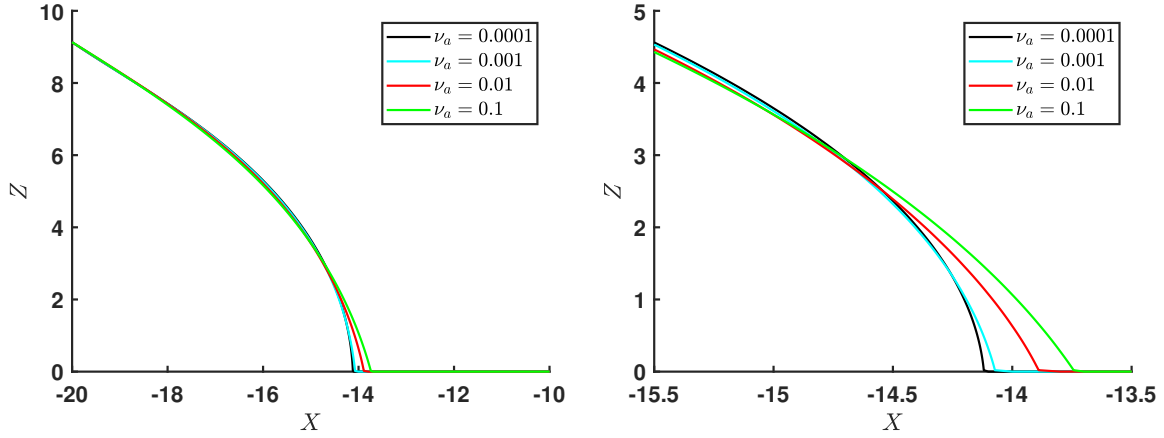


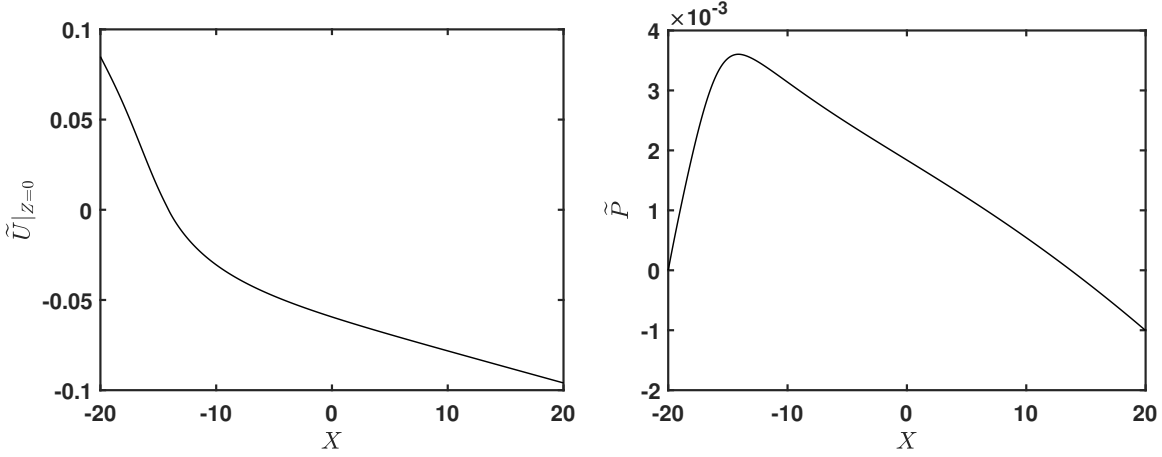
Figure 5.3: Streamline pattern of the inner structure of the pseudo-shock generated by the rotation of the disk at $t = 8$. We observe a clear dividing streamline and a recirculation region below it.



(a) Dividing streamlines plotted for different values of ν_a

(b) Zoom in the vicinity of the point where the dividing streamline meets the other streamline along which $\tilde{\psi} = 0$ along the jet axis. We can observe that a smaller value of ν_a results in a more vertical tangent at that point.

Figure 5.4: Influence of ν_a on the dividing streamline (along which $\tilde{\psi} = 0$).



(a) Longitudinal velocity $\tilde{U}_{\text{jet}}$ plotted with respect to X . It decreases from $U_I|_{Z=0} = +V_s(t)$ at $X_{\min}$ to $-V_s(t)$ at $X_{\max}$

(b) Pressure field $\tilde{P}_{\text{jet}}$ plotted with respect to X . It is equal to zero at $X_{\min}$, then it grows until a maximum value $\tilde{P}_{\max}$, and then it decreases and it reaches almost zero at $X_{\max}$.

Figure 5.5: Velocity and pressure along $Z = 0$ plotted with respect to X in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock.

results that

$$\tilde{P}_{\text{jet}} = \frac{1}{2} \left(U_I^2(Z=0, t) - \tilde{U}_{\text{jet}}^2 \right) = \frac{1}{2} \left(V_s^2(t) - \tilde{U}_{\text{jet}}^2 \right). \quad (5.29)$$

The second equality in (5.29) comes from the definition of the speed of the shock $V_s(t)$ shown in equation (5.18). Equation (5.29) shows that the pressure at $Z = 0$ becomes zero when $\tilde{U}_{\text{jet}}^2 = V_s^2 = U_I^2$. This happens at exactly two points; at $X = \pm\infty$.

The plots of $\tilde{U}_{\text{jet}}$ and $\tilde{P}_{\text{jet}}$ against X are presented in Figure 5.5. In order to compute the pressure $\tilde{P}_{\text{jet}}$, we chose the first equality in (5.29). Hence, the pressure is equal to zero at $X_{\min}$. It grows until reaching a maximum value $\tilde{P}_{\max} \approx 0.0036$ at $X_S = -14.1$. The point X_S corresponds to the stagnation point, where the velocity $\tilde{U}_{\text{jet}}$ becomes zero. Downstream of this point, the pressure decreases until reaching approximately -0.001 at $X_{\max}$. $\tilde{P}_{\text{jet}}$ is not equal to zero at $X_{\max}$ for the following reason: we used the first equality in (5.29) when calculating the pressure, and in our numerical results, $|V_s(t)|$ is slightly different from $|U_I|_{Z=0}$. The variation of $\tilde{P}_{\text{jet}}$ shows that the flow is subject to an adverse pressure gradient until $X = X_S$ where the stagnation point is reached. Downstream of it, the pressure gradient is favourable and $\tilde{U}_{\text{jet}}$ is negative.

In this chapter, the inner structure of the pseudo-shock was analysed. Due to

the singularity of the axial velocity component at the position of the pseudo-shock, and highlighted in Chapter 4, the Prandtl's model was not applicable anymore in the vicinity of this singularity; a new region must be considered in order to smooth it out. With that purpose in mind, a new scaled variable X was then introduced in (5.3) such that it is of order one in the vicinity of the singularity. Furthermore, in order to explain the sharp variation of the radial and circumferential velocity components at the position of the pseudo-shock, the pressure gradient must be restored into the governing equations. Our analysis revealed that this new region's dimensions are $Re^{-1/2} \times Re^{-1/2}$ and that the flow is inviscid and quasi-two-dimensional in the leading-order approximation, and quasi-steady in the reference frame moving with the shock. Solving numerically the governing equations and plotting the streamline pattern in Figure 5.3 showed a dividing streamline delimiting the jet front, and two recirculation regions are formed on both sides of the disk (only one is visible in Figure 5.3 as the domain for negative Z is not represented here). Finally, we were able to plot the radial velocity and the pressure with respect to X along the jet axis using the Bernoulli equation. We conclude that the pressure grows in the positive X -direction until the stagnation point S is reached. From that point, it decays and it reaches zero when X is large positive. The next chapter will focus on region 1_{gw} in the wake immediately downstream of the disk rim.

Chapter 6

Wake immediately downstream of the disk edge

Here our goal is to study the flow in region 1_{gw} (see Figure 2.4), that is the vicinity of $r = 1^+$, and to determine the asymptotic expansions of the sought flow functions. We expect the flow to act in a similar way in comparison with the classical Goldstein (1930)'s wake who studied the wake immediately downstream of a flat plate. Nevertheless, our case presented in this chapter is slightly different as we are dealing with a three-dimensional and unsteady problem. Shidlovskii (1977) studied the double deck theory for a jet emerging from a rotating disk. He considered a small region near the edge of the disk. The work presented in this chapter is different as the region we shall consider is bigger than the one of Shidlovskii (1977).

Firstly, let us set the variable s such that

$$s = r - 1. \quad (6.1)$$

The flow in region 1 is governed by the boundary-layer equations (2.39a)-(2.39i). For $|r| \leq 1$, the no-slip condition (2.39e) must be satisfied on the disk surface: in particular, the radial velocity must be equal to zero. The solution of the boundary-layer equations we are dealing with is sought in the class of continuous functions with respect to r :

$$\mathcal{V}_{r,0}(r, Z, t) = \mathcal{V}_{r,0}(Z, t) + o(1), \quad \text{as } s \rightarrow 0^+. \quad (6.2)$$

where $\mathcal{V}_{r,0}(Z, t)$ behaves near the disk surface as follows

$$\mathcal{V}_{r,0}(Z, t) = \lambda_1 Z + O(Z^2), \quad \text{as } Z \rightarrow 0, \quad (6.3)$$

with $\lambda_1 = \left. \frac{\partial \mathcal{V}_{r,0}}{\partial Z} \right|_{Z=0}$ being the dimensionless radial skin friction on the disk surface.

Immediately downstream of the edge of the disk at $r = 1^+$, the deficit of the velocity

near $Z = 0$ cannot cease to exist once the fluid leaves the edge of the disk. Hence, as we saw with our numerical results presented in Chapter 4, the boundary layer still exists for $r > 1$ and it is still governed by the boundary-layer equations, meaning that (6.2) and (6.3) are still valid. However (6.3) does not satisfy the symmetry condition (2.39f) which must hold in the wake at $Z = 0$. Thus, we are dealing with a singular perturbation problem, so one must introduce a viscous sub-layer whose thickness becomes smaller as s tends to 0. Such a sublayer was first introduced by Goldstein (1930) in his analysis of the wake downstream the trailing edge of a flat plate. Let us determine the thickness of this new region that we shall call from now on region 1b. Since the flow is viscous, we must balance the convective and viscous terms of the momentum equations: from the radial momentum equation (2.39a) we have

$$\mathcal{V}_{r,0} \frac{\partial \mathcal{V}_{r,0}}{\partial r} \sim \frac{\partial^2 \mathcal{V}_{r,0}}{\partial Z^2}.$$

The term $\mathcal{V}_{r,0}$ is comparable with Z according to (6.3), therefore the thickness of the viscous-sublayer in the wake is estimated as

$$Z \sim s^{1/3}. \quad (6.4)$$

Correspondingly, we introduce a new scaled variable ζ such that in region 1b

$$\zeta = \frac{Z}{s^{1/3}} = O(1), \quad \text{as } s \rightarrow 0^+. \quad (6.5)$$

We recall that inside the boundary layer, the stream-function is defined by (4.1). Using the first equation in (4.1), and knowing that $\mathcal{V}_{r,0} \sim Z$, it results that in region 1b

$$\psi \sim s^{2/3}. \quad (6.6)$$

This suggests that the asymptotic expansion of the stream-function in region 1b has the following form

$$\psi = s^{2/3} f_0(\zeta, t) + \dots, \quad \text{as } s \rightarrow 0^+, \quad (6.7)$$

where f_0 is a function to determine later. Substitution of (6.7) into (4.1) leads to the

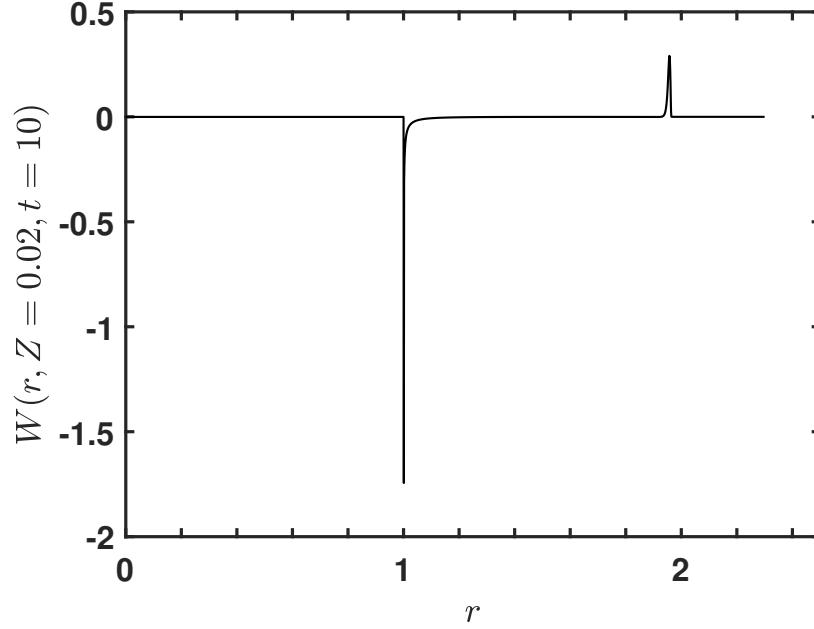


Figure 6.1: Axial velocity on the boundary-layer scale at $Z = 0.02$ and $t = 10$. A singularity, represented by a “spike” which becomes bigger as Δr becomes smaller, appears immediately downstream of the disk edge at $r = 1^+$.

asymptotic expansions of the radial and axial velocity components in region 1b:

$$\left. \begin{aligned} \mathcal{V}_{r,0} &= s^{1/3} \frac{\partial f_0}{\partial \zeta} + \cdots, \\ \mathcal{V}_{z,0} &= \frac{1}{3} s^{-1/3} \left(\zeta \frac{\partial f_0}{\partial \zeta} - 2f_0 \right) + \cdots, \end{aligned} \right\} \text{ as } s \rightarrow 0^+. \quad (6.8)$$

Let us notice that the singularity in the asymptotic expansion for $\mathcal{V}_{z,0}$ presented in (6.8) is in good agreement with our numerical results obtained in Chapter 4. Indeed, Figure 6.1 shows the axial velocity on the boundary-layer scale for $Z = 0.02$. We can observe that immediately downstream of the disk edge, the axial velocity becomes singular.

It remains to find the asymptotic expansion of the circumferential velocity component. The equation governing $\mathcal{V}_{\Phi,0}$ is parabolic, so it also does not allow for discontinuities in r . Since $\mathcal{V}_{\Phi,0}$ remains smooth in Z , we can use a Taylor series expansion about $Z = 0$ such that

$$\mathcal{V}_{\Phi,0} = 1 + \lambda_2 Z + O(Z^2) \quad \text{as } Z \rightarrow 0, \quad (6.9)$$

where $\lambda_2 = \left. \frac{\partial \mathcal{V}_{\Phi,0}}{\partial Z} \right|_{Z=0}$. Thus, $\mathcal{V}_{\Phi,0} \sim 1$, and we may seek its asymptotic expansion in region 1b in the form

$$\mathcal{V}_{\Phi,0} = f_1(\zeta, t) + s^{1/3} f_2(\zeta, t) + \cdots, \quad \text{as } s \rightarrow 0^+, \quad (6.10)$$

where f_1 and f_2 are order one functions.

Substitution of (6.8) and (6.10) into the boundary-layer equations leads to the following set of equations for the unknowns f_0 and f_1

$$\frac{\partial^3 f_0}{\partial \zeta^3} + \frac{2}{3} f_0 \frac{\partial^2 f_0}{\partial \zeta^2} - \frac{1}{3} \left(\frac{\partial f_0}{\partial \zeta} \right)^2 = 0, \quad (6.11a)$$

$$\frac{\partial^2 f_1}{\partial \zeta^2} + \frac{2}{3} f_0 \frac{\partial f_1}{\partial \zeta} = 0, \quad (6.11b)$$

The boundary conditions for (6.11a) and (6.11b) are

$$\frac{\partial^2 f_0}{\partial \zeta^2} = \frac{\partial f_1}{\partial \zeta} = f_0 = 0, \quad \text{at } \zeta = 0, \quad (6.11c)$$

$$\frac{\partial f_0}{\partial \zeta} = \lambda_1 \zeta + \cdots, \quad \text{as } \zeta \rightarrow \infty, \quad (6.11d)$$

$$f_1 = 1 + \cdots, \quad \text{as } \zeta \rightarrow \infty. \quad (6.11e)$$

Since there are no derivatives with respect to time, the flow near the disk rim is quasi-steady. It is also quasi-two-dimensional according to (6.8) in the limit as $s \rightarrow 0^+$. Equation (6.11a) involves only one unknown function f_0 , so it may be solved independently of equation (6.11b).

Equation (6.11a) is a nonlinear differential equation of third-order in the unknown f_0 . We can use shooting methods to solve it. This technique consists in rewriting equation (6.11a) as a set of three first-order differential equations. Let us define vector $\mathbf{u}$ as

$$\mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} f_0 \\ \frac{\partial f_0}{\partial \zeta} \\ \frac{\partial^2 f_0}{\partial \zeta^2} \end{pmatrix}. \quad (6.12)$$

Then we will have

$$\frac{\partial u_1}{\partial \zeta} = u_2, \quad (6.13a)$$

$$\frac{\partial u_2}{\partial \zeta} = u_3, \quad (6.13b)$$

$$\frac{\partial u_3}{\partial \zeta} + \frac{2}{3}u_1u_3 - \frac{1}{3}u_2^2 = 0. \quad (6.13c)$$

The boundary conditions for these equations are written as:

$$u_1 = u_3 = 0, \quad \text{at } \zeta = 0, \quad (6.13d)$$

$$u_2 = \lambda_1 \zeta + \cdots, \quad \text{as } \zeta \rightarrow \infty. \quad (6.13e)$$

The set of equations (6.13a)-(6.13e) can be solved numerically by firstly treating it as an initial value problem with boundary conditions

$$u_1 = u_3 = 0, \quad u_2 = 1, \quad \text{at } \zeta = 0.$$

The boundary condition (6.13e) has been omitted and replaced by $u_2|_{\zeta=0} = 1$, where 1 has been chosen arbitrarily. It will be shown that once the functions u_1 , u_2 and u_3 are determined in all the domain, no matter the value assigned to $u_2|_{\zeta=0}$, the boundary condition (6.13e) can be satisfied by making use of affine transformations. We introduce a uniform grid $\zeta_k = \Delta\zeta(k-1)$ where $\Delta\zeta$ is the grid spacing in the ζ -direction, and $k \in \llbracket 1; N_\zeta + 1 \rrbracket$ where N_ζ is an integer. We represent (6.13a)-(6.13c) as

$$\frac{(u_1)_k - (u_1)_{k-1}}{\Delta\zeta} = (u_2)_{k-1}, \quad (6.14a)$$

$$\frac{(u_2)_k - (u_2)_{k-1}}{\Delta\zeta} = (u_3)_{k-1}, \quad (6.14b)$$

$$\frac{(u_3)_k - (u_3)_{k-1}}{\Delta\zeta} + \frac{2}{3}(u_1)_{k-1}(u_3)_{k-1} - \frac{1}{3}((u_2)_{k-1})^2 = 0. \quad (6.14c)$$

We used forward finite difference schemes, turning (6.14a)-(6.14c) to a set of linear algebraic equations. This set is associated with the boundary conditions:

$$(u_1)_1 = (u_3)_1 = 0, \quad (6.14d)$$

$$(u_2)_1 = 1. \quad (6.14e)$$

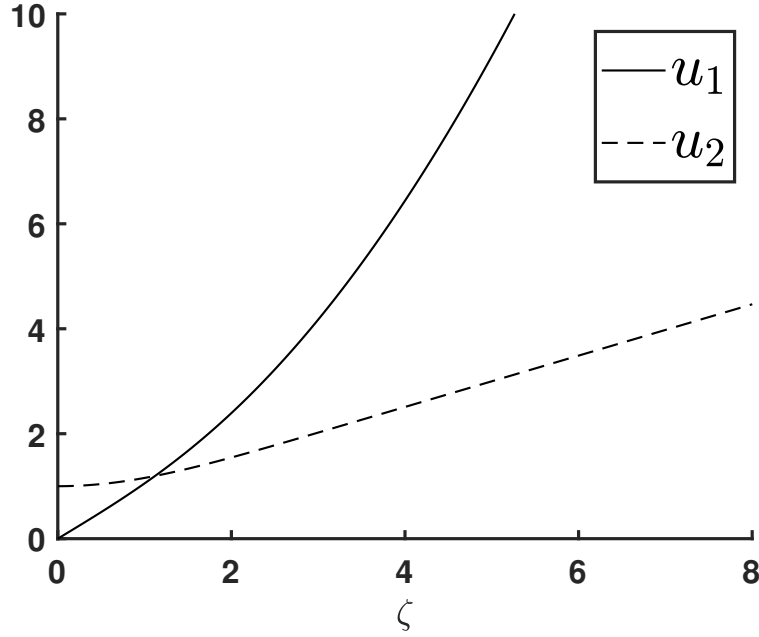


Figure 6.2: Solutions u_1 and u_2 of the set of equations (6.14a)-(6.14e)

One can use marching procedure to solve the system (6.14a)-(6.14e). The solution for u_1 , u_2 and u_3 is constructed by progressing from one computational point ζ_k to the following one ζ_{k+1} . The numerical results are displayed in Figure 6.2. We used a grid spacing $\Delta\zeta = 0.001$, and we plotted the solution for $\zeta \in [0; \zeta_{\max}]$. Thus, we found the solution for f_0 (let us denote it $\bar{f}_0$), with the modified boundary condition $u_2 = 1$ at $\zeta = 0$. Let us perform affine transformations

$$f_0 = A\bar{f}_0, \quad \zeta = B\bar{\zeta}. \quad (6.15)$$

It is easily seen that equation (6.11a) remains unchanged if

$$B = \frac{1}{A}, \quad (6.16)$$

where A is arbitrary.

Now, we need to satisfy the boundary condition (6.13e). By making use of (6.16), we notice that

$$\frac{\partial^2 f_0}{\partial \zeta^2} = A^3 \frac{\partial^2 \bar{f}_0}{\partial \bar{\zeta}^2}.$$

Setting $\zeta_{\max}$ in the above equation, we see that the constant A can be found from

$$A^3 \left. \frac{\partial^2 \bar{f}_0}{\partial \bar{\zeta}^2} \right|_{\zeta=\zeta_{\max}} = \lambda_1. \quad (6.17)$$

Setting constant A this way, the boundary condition (6.13e) is satisfied.

Let us now focus on equation (6.11b) which can be solved analytically. Let us set $F_1 = \partial f_1 / \partial \zeta$, then (6.11b) can be rewritten as

$$\frac{\partial F_1}{\partial \zeta} + \frac{2}{3} f_0 F_1 = 0. \quad (6.18a)$$

We assign to it the boundary condition (6.11c)

$$F_1 = 0 \quad \text{at } \zeta = 0. \quad (6.18b)$$

The general solution of (6.18a) has the form

$$F_1 = C(t) e^{-\frac{2}{3} \int_0^\zeta f_0(\eta) d\eta}, \quad (6.19)$$

where $C(t)$ is a function of time t . With the boundary condition (6.18b) taken into account, we deduce that $C(t) = 0$ for all t , hence $F_1 = 0$ for all ζ and t . Consequently, by making use of the matching condition (6.11e), we find that

$$f_1 = 1 \quad \forall \zeta. \quad (6.20)$$

The solution we found resembles the one from the Goldstein (1930)'s wake past a flat plate. Furthermore, surely, one could compare the radial and axial velocity we found by analysing the wake immediately downstream of the disk edge with the radial and axial velocity found numerically by solving the boundary-layer equations. However, this would require a particularly refined mesh in the vicinity of $r = 1^+$, which may require substantial computation time.

Chapter 7

Discussion

Part I focuses on a class of discontinuous solutions of the classical boundary-layer equations. Such solutions were proved to exist for the first time by Ruban and Vonatsos (2008), who dealt with the unsteady analogue of the Schlichting (1933) jet. It is due to the hyperbolic nature of the unsteady boundary-layer equations that they could observe discontinuous solutions. Similarly, in this study we are concerned with the unsteady flow set into motion by a plane disk of finite radius rotating with a constant angular velocity. The fluid, initially stagnant, is brought into motion owing to the viscous forces. Due to the hyperbolic nature of the operator $\partial \widehat{V}_r / \partial \widehat{t} + \widehat{V}_r \partial \widehat{V}_r / \partial \widehat{r}$, a jet is generated and it disturbs the stagnant fluid as it propagates away from the disk edge as time increases. We show in Chapter 2 that the structure of the flow can be divided into three regions: (i) a viscous layer 1 corresponding to the boundary layer and adjacent to the disk surface and the jet axis, (ii) an outer inviscid region 2 where the entrainment effect takes place, and (iii) the rest of the domain 3 where the fluid is stagnant. We studied the viscous layer 1 by solving numerically the unsteady boundary-layer equations. The numerical methods are presented in Chapter 3 and the results are shown in Chapter 4. The solution showed that, similarly to Ruban and Vonatsos (2008), a free jet develops from the rim of the disk, and it has a well-established front represented by a discontinuity in the radial and circumferential velocity components. This jet is similar to the shock waves that form in supersonic flows, however it cannot be termed strictly speaking as a “shock”, as its thickness is much larger than the characteristic thickness of usual shock waves. On top of the rotating disk, a layer of rotating fluid develops as time $\widehat{t}$ increases, and the flow in this region reaches its steady state when t is large enough. Then, the solution reproduces the one given by Kármán (1921) for the case of the steady flow due to the rotation of a disk of infinite radius. Thereafter, solving the governing equations of the inviscid region 2 where the entrainment effect takes place, revealed that the pseudo-shock acted as a point source, where fluid is emitted from there, and which is redirected and reinjected into the boundary layer, upstream of the pseudo-shock.

Afterwards, we studied the “internal structure” of the shock in Chapter 5, corresponding to the region in the vicinity of the pseudo-shock where the discontinuity is located. It appeared to be a $Re^{-1/2} \times Re^{-1/2}$ region, and we found that the flow on that scale could be treated as inviscid, and also quasi-steady if observed in the reference frame moving with the jet front. Using these simplifications, we found the theoretical speed of the shock, which matched with the experimental observation of Turner (1962). Furthermore, this allowed us to simplify the governing equations in this region to the steady modified Euler equations (the right-hand side of the circumferential momentum equation (5.7b) is equal to zero as the problem is axisymmetric) that we solved numerically using the stream function-vorticity formulation. Finally, we studied the region immediately downstream of the disk rim. The flow is quasi-steady and quasi-two-dimensional, and the governing equations are similar to the Goldstein (1930)’s wake. We show in Figure 6.1 that the axial velocity obtained numerically on the boundary-layer scale is in good agreement with the asymptotic behaviour in the Goldstein’s wake.

This problem belongs to the class of flows developing the so-called *collisional singularity*, studied for a three-dimensional boundary layer by Stewartson et al. (1980), Kluwick and Wohlfahrt (1986) and Allen and Riley (1994). This phenomenon has been observed in the case of a two-dimensional boundary layer by Degani et al. (1996), and later by Ruban and Vonatsos (2008). The current literature about collisional singularity is not dense. Stewartson was enthusiastic about that phenomenon, but he died unfortunately three years later after his publication Stewartson et al. (1980), and few researchers have focused on this subject since. This is why we are optimistic that this field may be developed by other researchers in the future. For example, the analysis of the rotating disk presented in Part I of this thesis may be applied to the problem of the rotating sphere. Similarly to the rotating disk case, a sphere is placed in a large reservoir of an initially stagnant fluid, and at time $\hat{t} = 0$, the sphere is impulsively brought into rotation around its axis with constant angular velocity. Two boundary layers are formed, one at each pole, and they develop along the sphere surface, until colliding at the equator. It would be interesting, for future publications, to determine whether the solution can be constructed after the time of collision.

Part II

The unsteady analogue of the Glauert laminar jet

Chapter 8

Introduction

In this chapter, we are interested in studying the unsteady near-wall jet. This type of jet is different from the free jet generated by the rotating disk presented in Part I. While for a free jet there are no boundaries restricting its propagation, a near-wall jet propagates parallel to a solid boundary on one side, while the other side remains unrestricted. This situation may occur in many situations. For example when a jet of air, created by the vertical take-off of a fighter plane, strikes the ground at right angle: it generates a wall jet propagating radially over the ground. In everyday life, this also happens when a jet of water (created by opening a tap for instance) falls into a sink which is partially filled with water. There also exists a sub-category of wall jets, namely, plane wall jets. Such jets are found when, for example, a large tank filled with liquid is separated into two parts by an impermeable barrier. If the water levels are different in each part, and that the barrier is slightly lifted up, a plane wall jet will propagate along the surface of the bottom of the tank. This part of this thesis focuses on such jets. For that, we shall consider the unsteady analogue of the laminar Glauert jet (1956). Glauert showed that in the case of a steady, laminar wall jet (and plane wall jet), the boundary-layer equations may be solved analytically in the form of a self-similar solution. Our unsteady case appears to be more complex: the introduction of a third independent variable, namely time $\hat{t}$, precludes an analytic solution. Moreover, the boundary-layer equations become of mixed-type: parabolic, and hyperbolic in $(\hat{x}, \hat{t})$, where $\hat{x}$ is the coordinate measured along the wall. This allows for discontinuities to develop, and, similarly to the case of the rotating disk of Part I, we expect a jet to develop, propagating from a narrow slit over the wall as time increases. The main difference with the free jet of Part I is that on one side of the jet, the no-slip condition must be satisfied on the motionless wall surface. Recently, Naqavi et al. (2018) performed direct numerical simulations to study the turbulent steady and unsteady wall jet when the Reynolds number is equal to 7500. Our approach is different as our goal is to deduce the asymptotic expansions of the solution in the limit as $Re \rightarrow \infty$. Moreover, the wall jet considered in this part

remains laminar and it never becomes turbulent.

In Chapter 9, we shall present the problem and write the governing equations of the flow. Then, after introducing the dimensionless Navier-Stokes equations and assuming that the Reynolds number Re is large, we will conduct our analysis based on Prandtl's hierarchical approach; the boundary layer is introduced in the direct neighbourhood of the wall surface, and the boundary-value problem of the flow will be formulated. We shall also analyse the region lying between the boundary layer and the stagnant surrounding flow. In this region of slow moving fluid, the entrainment effect process takes place, injecting fluid into the boundary layer. In Chapter 10, the numerical methods to solve the boundary-layer equations are presented, while in Chapter 11, the results of the computation are presented. It is shown that a jet emerges from the slit and it propagates over the wall surface as time increases. The longitudinal velocity component and the dimensionless skin friction both experience discontinuities at the position of the "pseudo-shock". Chapter 12 focuses on the investigation of the internal structure of the pseudo-shock. The analysis is performed in the reference frame moving with the latter. Also, the viscous sublayer in the vicinity of the shock is analysed numerically.

Chapter 9

Problem formulation

9.1 Formulation of the governing equations of the flow

Let us consider a semi-infinite region delimited by a wall, and divided into two by a flat and impermeable barrier which is perpendicular to the wall, as depicted in Figure 9.1. The regions on both sides of the barrier are filled with the same, initially stagnant fluid. The fluid on the left-hand side of the barrier is kept at higher pressure. At time $\hat{t} = 0$, a narrow slit is opened (see Figure 9.1) and a jet starts to propagate along the wall as time increases. Similarly to Part I, region 1 corresponds to the boundary layer, where the flow is governed by the classical Prandtl's boundary-layer equations. This will be studied in Section 9.2. In region 2, the flow is inviscid and fluid erupts from the boundary layer and is redirected away, and reinjected back into the boundary layer through the entrainment effect process. We will deal with that region in Section 9.3. In region 3, the fluid is at rest.

We shall use Cartesian coordinates $(\hat{x}, \hat{y}, \hat{z})$, where the origin O of the reference frame is located at the intersection between the wall and the barrier. The $\hat{x}$ -axis is aligned with the wall surface, while the $\hat{y}$ -axis coincides with the barrier. The problem is assumed to be two-dimensional. Then, all the flow quantities are independent of $\hat{z}$ and the “spanwise” component of the velocity $\hat{w}$ is equal to zero

$$\frac{\partial}{\partial \hat{z}} = 0, \quad \hat{w} = 0. \quad (9.1)$$

Similarly to Part I, our goal is to solve the unsteady boundary-layer equations governing the flow, and to study the region in the vicinity of the jet head. The difference with the rotating disk, which is a free jet, is that the problem studied here deals with a wall jet. We would like to know if the same results can be observed in the case when

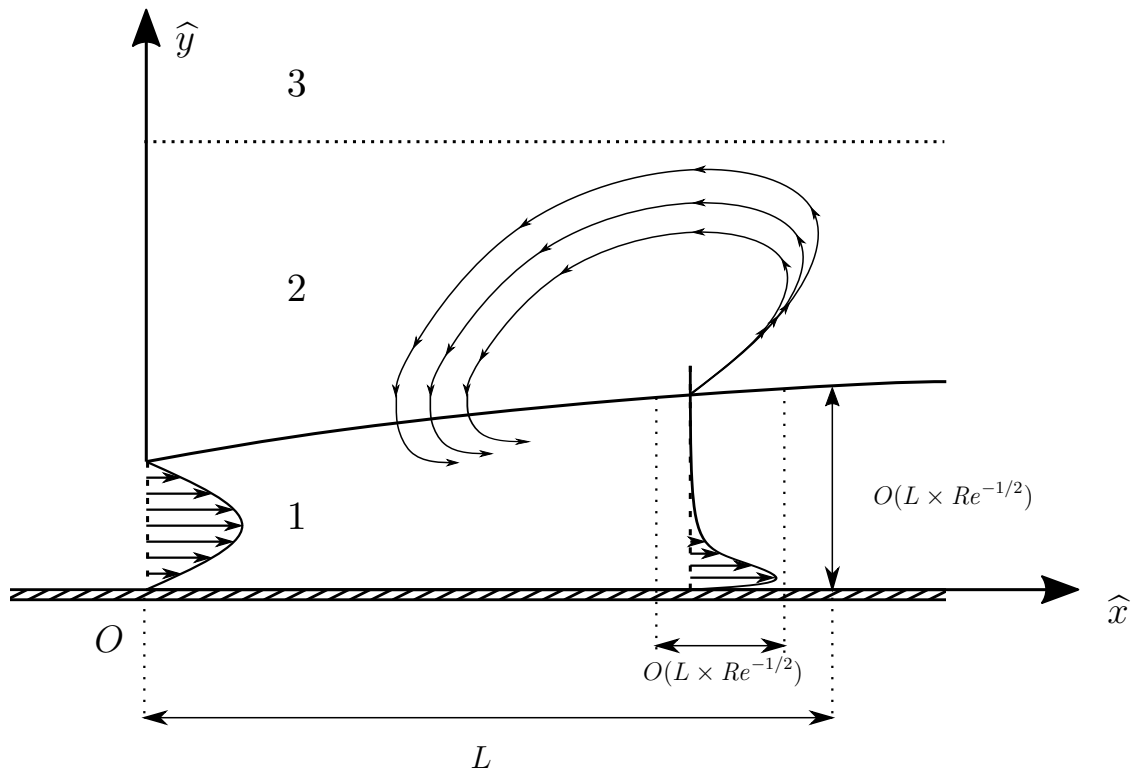


Figure 9.1: Problem layout. Fluid is injected from a narrow slit in the impermeable barrier on the $(O, \hat{y})$ axis. Contrary to the free jet generated by the rotation of the disk presented in Part I, this wall jet is not symmetric; the matching condition with the stagnant fluid still holds at the upper edge of the jet, but on the other side of it, the no-slip condition must be satisfied on the motionless wall surface.

a solid wall is present. We shall see that due to the hyperbolic nature of the unsteady boundary-layer equations, a pseudo-shock propagates along the wall as time increases. This shock may be tracked by observing the discontinuity of the velocity at its position, in the same way as for the rotating disk. Thereafter, the region in the vicinity of the pseudo-shock will be studied, and the governing equations, written using the vorticity stream-function formulation, will be solved applying iterative techniques. Finally, we shall introduce an additional viscous near-wall sublayer on the pseudo-shock's scale, as the flow in this region is inviscid in the leading-order approximation, and that the viscous terms must be recovered nearby the wall.

The fluid is assumed to be viscous, incompressible, Newtonian, and the body forces acting on it are neglected. As usual, the density, the dynamic viscosity and the kinematic viscosity are denoted ρ , μ and ν , respectively. The pressure in the stagnant fluid above the jet is denoted by p_∞ . The longitudinal and lateral components of the velocity respectively are $\hat{u}$ and $\hat{v}$, and the pressure in the fluid is $\hat{p}$. The hat symbol on top of the flow functions means that they are dimensional quantities. The flow is governed by the unsteady two-dimensional Navier-Stokes equations. For an incompressible flow they are written as follows

$$\frac{\partial \hat{u}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} = -\frac{1}{\rho} \frac{\partial \hat{p}}{\partial \hat{x}} + \nu \left(\frac{\partial^2 \hat{u}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{y}^2} \right), \quad (9.2a)$$

$$\frac{\partial \hat{v}}{\partial \hat{t}} + \hat{u} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} = -\frac{1}{\rho} \frac{\partial \hat{p}}{\partial \hat{y}} + \nu \left(\frac{\partial^2 \hat{v}}{\partial \hat{x}^2} + \frac{\partial^2 \hat{v}}{\partial \hat{y}^2} \right), \quad (9.2b)$$

$$\frac{\partial \hat{u}}{\partial \hat{x}} + \frac{\partial \hat{v}}{\partial \hat{y}} = 0. \quad (9.2c)$$

The flow is subject to the following boundary conditions:

On the wall, the no-slip and impermeability conditions must be satisfied

$$\hat{u} = \hat{v} = 0, \quad \text{at } \hat{y} = 0. \quad (9.2d)$$

Far from the wall, the fluid is stagnant. Hence,

$$\hat{u} = \hat{v} = 0, \quad \text{at } \hat{x}^2 + \hat{y}^2 = \infty. \quad (9.2e)$$

Far from the jet, the fluid is stagnant, so the boundary condition for the pressure may

be written as

$$\hat{p} = p_\infty, \quad \text{at } \hat{r}^2 + \hat{z}^2 = \infty. \quad (9.2f)$$

We also need to prescribe an initial profile at $\hat{x} = 0$, however, it should be done on the boundary-layer scale, as the dimension of the slit is supposed to be small.

Our first goal is to express the boundary-value problem (9.2a)-(9.2f) in a dimensionless form. Let us introduce the dimensionless quantities u, v, p, t, x, y such that

$$\begin{aligned} \hat{x} &\longrightarrow Lx, & \hat{y} &\longrightarrow Ly, & \hat{t} &\longrightarrow \frac{L}{V_\infty}t \\ \hat{u} &\longrightarrow V_\infty u, & \hat{v} &\longrightarrow V_\infty v, \\ \hat{p} &\longrightarrow p_\infty + \rho V_\infty^2 p, \end{aligned} \quad (9.3)$$

where L is characteristic distance from the slit to an observation point, and V_∞ a typical velocity in the jet. Substitution of (9.3) into the governing equations (9.2a)-(9.2c) leads to

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right), \quad (9.4a)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re} \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right), \quad (9.4b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (9.4c)$$

where Re is the Reynolds number such that

$$Re = \frac{V_\infty L}{\nu}. \quad (9.5)$$

In what follows, Re is assumed to be large.

The next two sections will consist of describing the asymptotic behaviour of the flow on the boundary-layer scale and also in the inviscid region 2 where the entrainment effect takes place.

9.2 Boundary-layer scale

Similarly to the rotating disk case of Part I, by using the arguments of Prandtl's hierarchical approach when $\Re$ is assumed to be large, the problem can be described by splitting the flow domain into three distinct regions. Outside of the jet, the flow is at rest, and the velocity vector $\mathbf{V}$ is equal to the zero vector everywhere. Similarly to Section 2.2, we introduce two other regions. An outer region where the viscous terms in the momentum equations (9.4a) and (9.4b) are disregarded, and an inner region in the direct neighbourhood of the wall where these viscous terms cannot be neglected anymore, and where they become of same order than the convective ones. In the outer region, we have

$$x, y = O(1). \quad (9.6)$$

The Navier-Stokes equations become the Euler equations and the flow is described by the inviscid flow theory. In this region, the entrainment effect takes place, meaning that fluid particles are sucked down into the jet. Since the flow is inviscid, the no-slip condition (9.2d) cannot be satisfied on the wall surface. This is why we need to introduce an inner region close to the wall, corresponding to the boundary layer, where

$$x, Y = \frac{y}{Re^{-1/2}} = O(1). \quad (9.7)$$

Using the principle of least degeneration, we are able to deduce, in a similar way to that in Section 2.2 of Part I, the form of the asymptotic expansions of the sought flow functions in the boundary layer. They have the following form

$$\begin{aligned} u &= u(x, Y, t) + \cdots, \\ v &= Re^{-1/2}V(x, Y, t) + \cdots, \quad \text{as } Re \rightarrow \infty. \\ p &= Re^{-1/2}P(x, Y, t) + \cdots, \end{aligned} \quad (9.8)$$

Substitution of (9.8) into the dimensionless Navier-Stokes equations (9.4a)-(9.4c) leads to the boundary-layer equations, written in conservative form as follows

$$\frac{\partial u}{\partial t} + \frac{\partial(u^2)}{\partial x} + \frac{\partial(uV)}{\partial Y} = \frac{\partial^2 u}{\partial Y^2}, \quad (9.9a)$$

$$\frac{\partial P}{\partial Y} = 0, \quad (9.9b)$$

$$\frac{\partial u}{\partial x} + \frac{\partial V}{\partial Y} = 0 \quad (9.9c)$$

The conservative form of the longitudinal momentum equation (9.9a) has been deduced by making use of the continuity equation (9.9c). Due to the leading-order term of the pressure asymptotic expansion in (9.8) being $O(Re^{-1/2})$, there is no pressure gradient in the stream-wise direction in (9.9a).

The set of boundary-layer equations (9.9a)-(9.9c) are solved with the following boundary conditions

$$u = V = 0, \quad \text{at } Y = 0, \quad (9.9d)$$

$$u = 0, \quad \text{as } Y \rightarrow \infty. \quad (9.9e)$$

We assume that a parabolic profile is imposed in the slit

$$V = 0, \quad u = \begin{cases} Y(1 - Y), & \text{for } Y \in [0, 1], \\ 0, & \text{for } Y > 1 \end{cases}, \quad \text{at } x = 0. \quad (9.9f)$$

Finally, the matching condition for the pressure is written as

$$P = \lim_{y \rightarrow 0} p'_0, \quad \text{as } Y \rightarrow \infty. \quad (9.9g)$$

The right-hand side in (9.9g) represents the inner limit of the leading order term of the outer expansion of the pressure. We imposed a parabolic profile in the boundary condition (9.9f) for u across the slit, located at $x = 0$ for $Y \in [0, 1]$. Other initial profiles are possible (such as a uniform velocity profile for $Y \in [0, 1]$). In this flow, the jet propagates along a wall, contrary to a free-jet. Since the wall is motionless, it slows down significantly the fluid due to the action of viscous forces and the no-slip condition (9.9d) which must be satisfied. Thus, unless the speed of the pseudo-shock is larger (by adding for instance a multiplying constant in (9.9f)), a particularly refined mesh is necessary in the x -direction when determining numerically the solution of the

boundary-layer equations (9.9a)-(9.9g). A direct consequence is that the computation of the solution takes a larger time to compute than if we had considered a “stronger” initial velocity profile through the slit, and also that it requires more memory allocation. The reason why we did not add a multiplying factor in the initial boundary condition (9.9f) is to keep all the flow quantities of order one.

9.3 Entrainment effect

Let us now focus on the flow region lying between the boundary layer and the region where the fluid is stagnant. We shall call it region 2. The analysis is the same as the one presented in Section 2.3 of Part I. We introduce a new independent variable y' such that in region 2,

$$y' = y = O(1), \quad (9.10)$$

The sought flow functions may be written in terms of asymptotic expansions valid in region 2:

$$\begin{aligned} u &= Re^{-1/2}U'(x, y', t) + \cdots, \\ v &= Re^{-1/2}V'(x, y', t) + \cdots, \quad \text{as } Re \rightarrow \infty, \\ p &= Re^{-1/2}P'(x, y', t) + \cdots, \end{aligned} \quad (9.11)$$

The order of magnitude of the pressure has been determined by balancing the local acceleration in the longitudinal momentum equation with the pressure derivative.

The flow in region 2 may be treated as inviscid in the leading-order approximation, and from Kelvin’s circulation theorem the flow is irrotational. We define the scalar vorticity field as

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0. \quad (9.12)$$

From the continuity equation (9.4c), we recall that there exists a scalar function ψ called the stream-function, such that

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \quad (9.13)$$

Substitution into the the scalar vorticity field leads to the Laplace equation

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \quad (9.14)$$

From the definition in (9.13), and recalling that $v = O(Re^{-1/2})$ in region 2, it results that $\psi \sim Re^{-1/2}$ and that the asymptotic expansion of the stream-function in region 2 has the form

$$\psi = Re^{-1/2}\psi' + \dots, \text{ as } Re \rightarrow \infty. \quad (9.15)$$

Substitution into the Laplace equation gives

$$\frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y'^2} = 0. \quad (9.16a)$$

Equation (9.16a) is solved with the following boundary conditions.

- At $y' = 0$, one must match the lateral velocity v to the solution of the boundary layer, so $V'(x, y' = 0, t) = \lim_{Y \rightarrow \infty} V$. Hence

$$\frac{\partial \psi'}{\partial x} = - \lim_{Y \rightarrow \infty} V, \text{ at } y' = 0. \quad (9.16b)$$

- At $x = 0$, the flat barrier is impermeable, so $u' = 0$ and

$$\psi' = 0, \text{ at } x = 0. \quad (9.16c)$$

- Far from the jet, we match with the stagnant fluid:

$$\frac{\partial \psi'}{\partial x} = \frac{\partial \psi'}{\partial y'} = 0. \quad (9.16d)$$

In Sections 9.2 and 9.3, we determined the set of boundary-layer equations (9.9a)-(9.9g) and the set of equations (9.16a)-(9.16d) governing the inviscid flow in region 2, respectively. We shall introduce the numerical methods in order to solve them in Chapter 10, and the numerical results will be presented in Chapter 11.

Chapter 10

Computational methods to solve numerically the boundary-layer equations for the wall jet

The set of equations (9.9a)-(9.9g) may be solved numerically by applying the finite difference method. Let us define a uniform grid

$$\{x_i, Y_j, t_n\} = \{\Delta x(i-1), \Delta Y(j-1), \Delta t(n-1)\}, \quad (10.1)$$

such that

$$i \in [1, N_x + 1], \quad j \in [1, N_Y + 1], \quad n \in [1, N_t + 1], \quad (10.2)$$

and where Δx , ΔY and Δt are small grid spacings in x -, Y - and t -directions, respectively. N_x , N_Y and N_t are integers such that there are $N_x + 1$, $N_Y + 1$ and $N_t + 1$ mesh points in x -, Y - and t -directions, respectively. The approximate solution at the node (i, j, n) is written

$$u_{i,j}^n \approx u(x_i, Y_j, t_n).$$

The numerical methods to solve (9.9a)-(9.9g) are the same as the ones presented in Chapter 3. A backward Euler scheme is applied to the time derivative $\partial u / \partial t$. The nonlinear terms u^2 and uV are linearised; this is performed for the former one by using the Newton's method, where we write the Taylor expansion of u around a computed approximation, denoted as u^- , and such that we keep the linear terms. The quantity uV in the momentum equation (9.9a) is linearised by replacing it with uV^- , where V^- is a computed approximation of V . By doing this, we decouple the system, and for a fixed mesh point i , u and V are computed sequentially, using the momentum equation (9.9a) to determine u , and afterwards the continuity equation (9.9c) to find V . This process is iterated until convergence is reached. A first-order scheme is used for $\partial u / \partial t$

and $\partial u / \partial x$ while second-order schemes are used for $\partial u / \partial Y$ and $\partial^2 u / \partial Y^2$:

$$\frac{\partial u}{\partial t} = \frac{u_{i,j}^n - u_{i,j}^{n-1}}{\Delta t}, \quad (10.3a)$$

$$\begin{aligned} \frac{\partial(u)^2}{\partial x} &= \frac{(u_{i,j}^n)^2 - (u_{i-1,j}^n)^2}{\Delta x}, \\ &= \frac{2(u^-)_{i,j}^n u_{i,j}^n - ((u^-)_{i,j}^n)^2 + (u_{i-1,j}^n)^2}{\Delta x}, \end{aligned} \quad (10.3b)$$

$$\begin{aligned} \frac{\partial(uV)}{\partial Y} &= \frac{u_{i,j+\frac{1}{2}}^n (V^-)_{i,j+\frac{1}{2}}^n - u_{i,j-\frac{1}{2}}^n (V^-)_{i,j-\frac{1}{2}}^n}{\Delta Y}, \\ &= \frac{(V^-)_{i,j+1}^n + (V^-)_{i,j}^n}{4\Delta Y} u_{i,j+1}^n + \frac{(V^-)_{i,j+1}^n - (V^-)_{i,j-1}^n}{4\Delta Y} u_{i,j}^n - \\ &\quad \frac{(V^-)_{i,j}^n + (V^-)_{i,j-1}^n}{4\Delta Y} u_{i,j-1}^n, \end{aligned} \quad (10.3c)$$

$$\frac{\partial^2 u}{\partial Y^2} = \frac{1}{(\Delta Y)^2} u_{i,j+1}^n - \frac{2}{(\Delta Y)^2} u_{i,j}^n + \frac{1}{(\Delta Y)^2} u_{i,j-1}^n. \quad (10.3d)$$

The solution is constructed by progressing in time. At each time step n , the solution is computed sequentially along each computational line $x = x_i$. Discretisation and linearisation of the longitudinal momentum equation (9.9a) leads to a system of linear algebraic equations in the unknown $u_{i,j}^n$ (where i and n are fixed) which can be solved by applying the Thomas algorithm (see Appendix A). We iterate until the absolute residual $\|F_u\| (u^-, V^-)$ becomes smaller than a small parameter ε , where $F_u = (F_u)_{i,j}^n$ is a column vector (with i and n fixed) such that for all $j \in \llbracket 2, N_Y + 1 \rrbracket$,

$$\begin{aligned} (F_u)_{i,j}^n &= \frac{u_{i,j}^n - u_{i,j}^{n-1}}{\Delta t} + \frac{(u_{i,j}^n)^2 - (u_{i-1,j}^n)^2}{\Delta x} + \frac{V_{i,j+1}^n + V_{i,j}^n}{4\Delta Y} u_{i,j+1}^n + \\ &\quad \frac{V_{i,j+1}^n - V_{i,j-1}^n}{4\Delta Y} u_{i,j}^n - \frac{V_{i,j}^n + V_{i,j-1}^n}{4\Delta Y} u_{i,j-1}^n - \\ &\quad \frac{1}{(\Delta Y)^2} u_{i,j+1}^n + \frac{2}{(\Delta Y)^2} u_{i,j}^n - \frac{1}{(\Delta Y)^2} u_{i,j-1}^n. \end{aligned} \quad (10.4)$$

The right-hand side of (10.4) corresponds to the discretised longitudinal momentum equation (9.9a) using the finite-difference schemes presented above, and prior to linearisation. Similarly than the explanations given in Chapter 3, if the exact solutions U and V are used in (10.4), then F_u is equal to zero. But using instead the computed approximations U^- and V^- calculated at each iteration of the iterative process, F_u is not equal to zero. Our goal is to find the appropriate U^- and V^- for each mesh node

(i, j, n) such that F_u is almost equal to zero, in accordance with a small parameter ε .

Going back to the description of the computational method to solve the boundary-layer equations, once u is found for the current computational line, $V_{i,j}^n$ may be deduced by using the continuity equation for all $j \in \llbracket 2, N_Y + 1 \rrbracket$

$$V_{i,j}^n = V_{i,j-1}^n - \frac{\Delta Y}{\Delta x} (u_{i,j}^n - u_{i-1,j}^n). \quad (10.5)$$

The finite difference method is also used to compute numerically the solution of equation (9.16a) of region 2, written in terms of the stream-function. A uniform mesh is employed:

$$\{x_i, y_j'\} = \{\Delta x(i-1), \Delta y'(j-1)\}, \quad (10.6)$$

where

$$i \in \llbracket 1; N_x + 1 \rrbracket, \quad j \in \llbracket 1; N_{y'} + 1 \rrbracket, \quad (10.7)$$

such that Δx and $\Delta y'$ are small grid spacings in x - and y' -directions, respectively. At a fixed time step, the exact solution $\psi(x, y', t)$ is approximated by $(\psi')_{i,j}^n$.

The second order derivatives are discretised using second-order accurate central difference schemes. As a result, equation (9.16a) gives for all $(i, j) \in \llbracket 2; N_x \rrbracket \times \llbracket 2; N_{y'} \rrbracket$

$$\frac{\psi'_{i-1,j} - 2\psi'_{i,j} + \psi'_{i+1,j}}{(\Delta x)^2} + \frac{\psi'_{i,j-1} - 2\psi'_{i,j} + \psi'_{i,j+1}}{(\Delta y')^2} = 0. \quad (10.8)$$

The SOR method allows us to solve easily this set of linear algebraic equations. We set

$$s_{xy} = \frac{1}{\frac{2}{(\Delta x)^2} + \frac{2}{(\Delta y')^2}},$$

and proceed from iteration (k) to a new iteration $(k+1)$ as follows

$$\psi'_{i,j}{}^{(k+1)} = s_{xy} \left[\frac{\psi'_{i-1,j}{}^{(k+1)} + \psi'_{i+1,j}{}^{(k)}}{(\Delta x)^2} + \frac{\psi'_{i,j-1}{}^{(k+1)} + \psi'_{i,j+1}{}^{(k)}}{(\Delta y')^2} \right]. \quad (10.9)$$

If we denote this solution by $\psi'^{(*)}$ corresponding to the solution given by the Gauss-

Seidel method, then the SOR method may be written as

$$\psi'_{i,j}{}^{(k+1)} = \omega^* \psi'_{i,j}{}^{(*)} + (1 - \omega^*) \psi'_{i,j}{}^{(k)}, \quad (10.10)$$

where ω^* is a suitably chosen over-relaxation parameter.

The next chapter focuses on presenting the numerical solutions of the governing equations in regions 1 and 2.

Chapter 11

Numerical results

Contrary to the free-jet generated by the rotation of the disk that we studied in Part I, the jet we are dealing with here propagates along a motionless wall. Due to the action of the viscous forces and the no-slip condition (9.9d) which must be satisfied on the wall surface, the fluid is slowed down, and the pseudo-shock's speed $V_s(t)$ is smaller than the one of the free-jet. Therefore, a particularly refined mesh is necessary in the x -direction. For our calculations, we used the mesh parameters presented in Table 11.1. We notice that the upper limit of the domain $Y_{\max} = 1400$ is much larger than in the rotating disk case (where we recall that we had chosen $Z_{\max} = 35$). Even though the matching condition $u = 0$ at $Y = \infty$ is imposed in the code and will be satisfied in any case, picking a large value for $Y_{\max}$ here is necessary. When plotting u with respect to Y immediately upstream of the pseudo-shock's position, it appears that the damping of u as Y increases is slower than for the rotating disk case. We found that if $Y_{\max}$ is not large enough, then u will develop oscillations when Y is close to $Y_{\max}$, and it would subsequently lead to errors for V according to (10.5). These errors have a direct impact on the streamlines pattern of the flow. That is why it is important to choose a $Y_{\max}$ large enough.

We plotted in Figure 11.1 the longitudinal velocity component u against x at $Y = 1$ for different values of time t . Similarly to the jet generated by the rotating disk, we observe a pseudo-shock propagating along the x -axis as time increases, and clear discontinuities are shown at the position of the pseudo-shock. The lateral velocity component V is plotted against x at $Y = Y_{\max}$ in Figure 11.2 for the corresponding

Δt	Δx	ΔY	$Y_{\max}$	ε
0.02	0.0005	0.025	1400	10^{-5}

Table 11.1: Mesh parameters for the wall jet on the boundary-layer scale

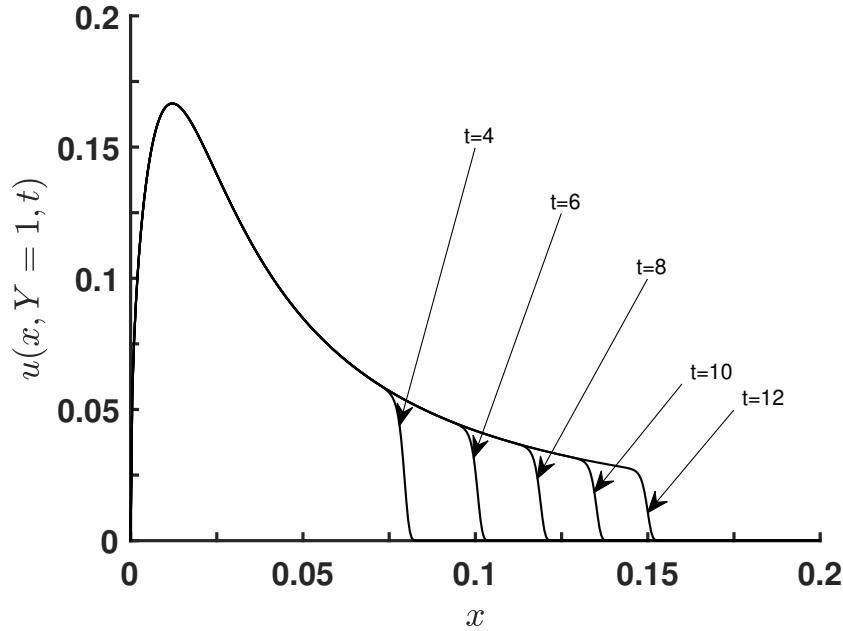


Figure 11.1: Longitudinal velocity at $Y = 1$ for time $t = 4, 6, 8, 10, 12$.

times t . Due to the entrainment effect, V remains negative and almost constant from $x = 0$ up to the position of the pseudo-shock. Then it undergoes a small decay, followed by a burst at the position of the shock.

In Figure 11.3, we show the dimensionless skin friction $\tau = \frac{\partial u}{\partial Y}|_{Y=0}$. It also experiences discontinuities at the position of the pseudo-shock.

The streamline pattern in the boundary layer is displayed in Figure 11.4 for $t = 12$.

Similarly to the pseudo-shock of Part I, one wishes to determine whether it forms immediately after the narrow slit is opened, or if it develops at a finite time. We plot in Figure 11.5 the longitudinal velocity for small values of time at $Y = 1$. We can observe that, similarly to Figure 4.7 in Chapter 4, the pseudo-shock forms as soon as the slit opens. contrary to Hewitt and Duck (2011) where their pseudo-shock develops for a finite time.

In Figure 11.6, we plotted the streamline pattern at $t = 6$ in region 2, that is where the entrainment effect takes place. Similarly to the free jet of Part I, fluid is ejected from the pseudo-shock, and it penetrates the stagnant region. The ejected fluid particles are then redirected and reinjected into the jet upstream of the pseudo-shock.

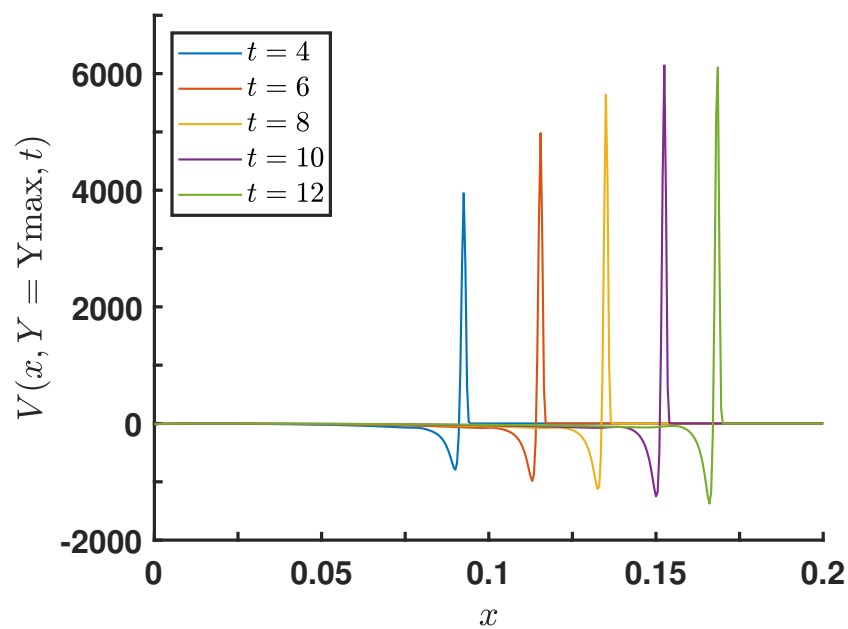


Figure 11.2: Lateral velocity at $Y = Y_{\max}$ for time $t = 4, 6, 8, 10, 12$.

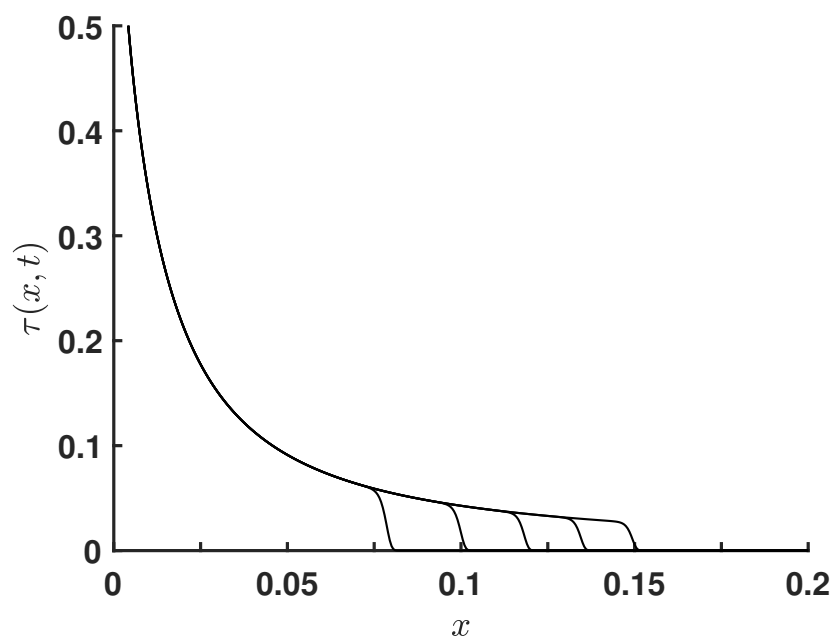


Figure 11.3: Dimensionless skin friction for time $t = 4, 6, 8, 10, 12$.

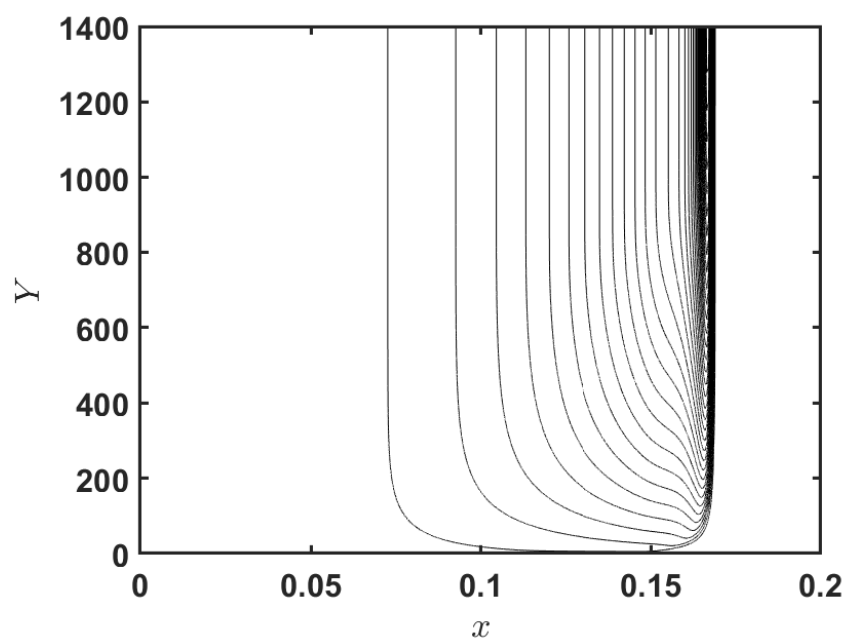


Figure 11.4: Streamline pattern of the wall jet on the boundary-layer scale at $t = 12$

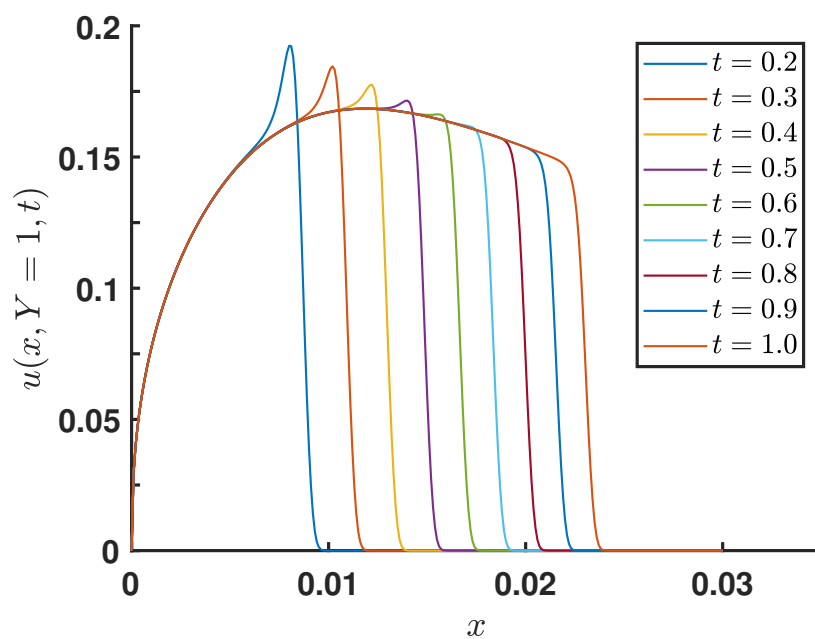
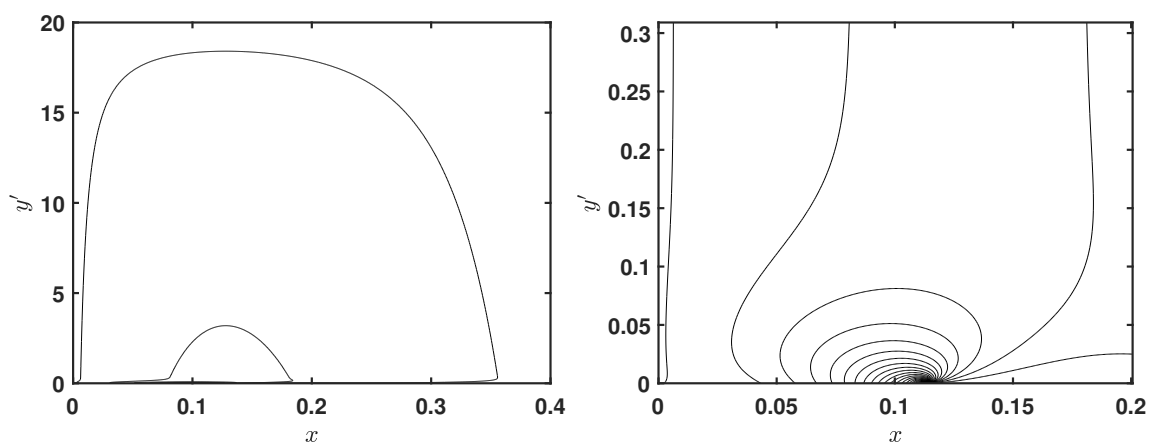


Figure 11.5: Longitudinal velocity u for small values of time t at $Y = 1$.



(a) Streamline pattern in region 2.

(b) Zoom in the vicinity of the $y' = 0$.

Figure 11.6: The entrainment effect occurring at the outer edge of the boundary layer. Like for the case of the rotating disk, the pseudo-shock acts as a point source.

Chapter 12

Internal structure of the pseudo-shock for the plane wall jet

12.1 In the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock

It was shown in the previous chapter that the opening of the narrow slit produced a jet with a well-established front propagating along the plane wall as time increases. At the jet front, the velocity develops a discontinuity termed a pseudo-shock, as shown in Figure 11.1. We are interested in the flow behaviour in the vicinity of the pseudo-shock. Prandtl's boundary-layer theory is not applicable in this region, as the longitudinal derivatives become of the same order as the lateral ones. Thus, a new mathematical model is required. For that purpose, we shall use a similar process to the one we introduced in Chapter 5. In the vicinity of the shock, we need to reintroduce the pressure gradient into the momentum equations. Hence, it is necessary that the characteristic length of this region is

$$\Delta X = O(Re^{-1/2}). \quad (12.1)$$

We shall perform our analysis in the reference frame moving with the pseudo-shock. Correspondingly, we introduce a new scaled variable X , such that

$$X = \frac{x - x_s(t)}{Re^{-1/2}}, \quad (12.2)$$

where $x_s(t)$ is the position of the shock and X is an order one quantity in the region nearby the pseudo-shock.

The asymptotic expansions of the flow functions in the region considered are

$$\left. \begin{aligned} u &= V_s(t) + \tilde{U}(X, Y, t) + \cdots, \\ v &= \tilde{V}(X, Y, t) + \cdots, \\ p &= \tilde{P}(X, Y, t) + \cdots, \end{aligned} \right\} \quad \text{as } Re \rightarrow \infty, \quad (12.3)$$

where $V_s(t)$ is the speed of the pseudo-shock. Substitution of (12.3) into the Navier-Stokes equations, and using (12.2) and the definition of Y in (9.7), shows that the flow obeys inviscid flow theory, and that the Navier-Stokes equations become the steady Euler equations in the leading-order approximation:

$$\tilde{U} \frac{\partial \tilde{U}}{\partial X} + \tilde{V} \frac{\partial \tilde{U}}{\partial Y} = -\frac{\partial \tilde{P}}{\partial X}, \quad (12.4a)$$

$$\tilde{U} \frac{\partial \tilde{V}}{\partial X} + \tilde{V} \frac{\partial \tilde{V}}{\partial Y} = -\frac{\partial \tilde{P}}{\partial Y}, \quad (12.4b)$$

$$\frac{\partial \tilde{U}}{\partial X} + \frac{\partial \tilde{V}}{\partial Y} = 0. \quad (12.4c)$$

There is no derivative with respect to time involved in both momentum equations (12.4a) and (12.4b) as according to the asymptotic expansions (12.3) we have

$$\frac{\partial u}{\partial t} = V'_s(t) + \frac{\partial \tilde{U}}{\partial t} - Re^{1/2} V_s(t) \frac{\partial \tilde{U}}{\partial X} + \cdots, \quad (12.5a)$$

$$\frac{\partial v}{\partial t} = \frac{\partial \tilde{V}}{\partial t} - Re^{1/2} V_s(t) \frac{\partial \tilde{V}}{\partial X} + \cdots. \quad (12.5b)$$

As we can notice, the terms $\partial \tilde{U}/\partial t$ and $\partial \tilde{V}/\partial t$ are negligible as compared to the convective terms and the pressure derivatives.

The inner structure of the pseudo-shock is sketched in Figure 12.1. The initial profile of $\tilde{U}$ when X tends to $-\infty$, denoted as $\tilde{U}_I$, varies as follows with respect to Y : $\tilde{U}_I$, which is negative at $Y = 0$, grows until $Y = Y_G$ where it is positive. Then it decays, and it tends to $-V_s(t)$ when Y tends to infinity. Remember that $V_s(t)$ is obtained as described in Chapter 11. We recall that, according to the continuity equation (12.4c), we can define the stream-function $\tilde{\psi}$ such that $\tilde{U} = \partial \tilde{\psi} / \partial Y$ and $\tilde{V} = -\partial \tilde{\psi} / \partial X$. Since the impermeability condition $\tilde{V} = 0$ must be satisfied on the rigid wall, $\tilde{\psi} = 0$ at $Y = 0$. Then it decays, grows again monotonically up to a positive value, and finally it decays towards $-\infty$ when Y tends to infinity. This means that, apart from the point $Y = 0$,

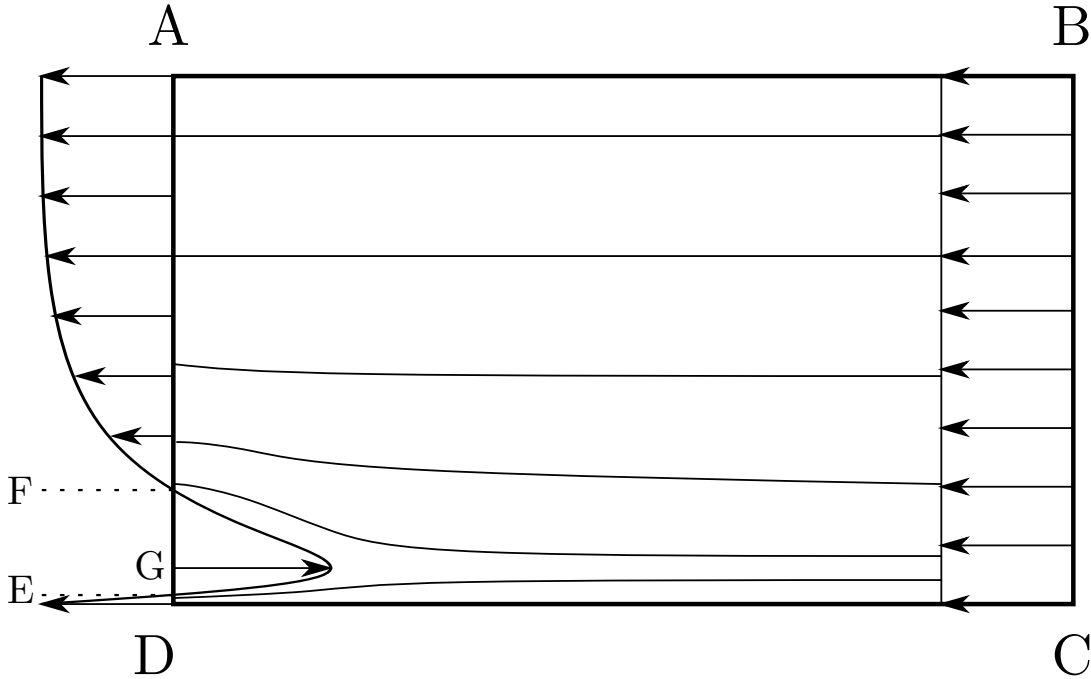


Figure 12.1: Inner structure of the pseudo-shock

there exists two other distinct values Y such that $\tilde{\psi} = 0$ when $X \rightarrow -\infty$, say Y_E and Y_F . The numerical results will show us that, instead of observing a dividing streamline from a point between (AD) and a point between (DC) as for the rotating disk, we shall observe a streamline emerging from point Y_E , and leaving the domain at point Y_F .

Let us solve numerically the problem in the $Re^{-1/2} \times Re^{-1/2}$ region. We shall use the same numerical methods presented in Chapter 5 of Part I. The Euler equations are rewritten in terms of the stream-function vorticity formulation, and we add an artificial viscosity:

$$\nu_a \frac{\partial \omega}{\partial t} + \tilde{U} \frac{\partial \omega}{\partial X} + \tilde{V} \frac{\partial \omega}{\partial Y} = \nu_a \left(\frac{\partial^2 \omega}{\partial X^2} + \frac{\partial^2 \omega}{\partial Y^2} \right), \quad (12.6a)$$

$$\frac{\partial \tilde{\psi}}{\partial t} - \frac{\partial^2 \tilde{\psi}}{\partial X^2} - \frac{\partial^2 \tilde{\psi}}{\partial Y^2} = \omega, \quad (12.6b)$$

$$\tilde{U} = \frac{\partial \tilde{\psi}}{\partial Y}, \quad \tilde{V} = -\frac{\partial \tilde{\psi}}{\partial X}, \quad (12.6c)$$

with

$$\omega = \frac{\partial \tilde{V}}{\partial X} - \frac{\partial \tilde{U}}{\partial Y}. \quad (12.6d)$$

They are subject to the following boundary conditions:

On the plate surface, the impermeability condition holds, and we set a soft boundary condition for the scalar vorticity field

$$\tilde{\psi} = \frac{\partial \omega}{\partial Y} = 0, \quad \text{at } Y = 0. \quad (12.6e)$$

It should be noted that the condition on ω is needed because of the artificial viscosity in the vorticity transport equation (12.6a). It only influences a thin layer $Y \sim \nu_a^{1/2}$ in the flow. For more details, see Kevorkian and Cole (1981).

Far from the wall surface, the solution matches with the external inviscid flow. In the reference frame moving with the shock, the longitudinal velocity component satisfies $\tilde{U}|_{Y=\infty} = -V_s(t)$. Thus,

$$\frac{\partial \tilde{\psi}}{\partial Y} = -V_s(t), \quad \omega = 0, \quad \text{at } Y = \infty. \quad (12.6f)$$

At $X = -\infty$, the initial profile for U is given by $\tilde{U}_I(Y, t) = \lim_{x \rightarrow x_s^-} u(x, Y, t) - V_s(t)$. Moreover, the lateral velocity component is of order $O(Re^{-1/2})$ on the boundary-layer scale, so

$$\tilde{\psi} = \int_0^Y \tilde{U}_I(Y, t) dY, \quad \omega = -\frac{\partial \tilde{U}_I}{\partial Y}, \quad \text{at } X = -\infty. \quad (12.6g)$$

At $X = \infty$, the flow is uniform, and we have

$$\tilde{\psi} = -V_s(t)Y, \quad \omega = 0, \quad \text{at } X = \infty. \quad (12.6h)$$

The set of equations (12.6a)-(12.6h) is solved using the pseudo-transient method associated to the ADI method, in the exact same way as in Section 5.3 in Part I. The grid of Table 12.1 was used. We plotted the streamline pattern in the inner structure of the shock at $t = 6$. Its position is $x_s(t = 6) = 0.0965$, and its speed is $V_s(t = 6) = 0.0107$. Contrary to the inner structure of the free jet plotted in Figure 5.3 in Part I, the streamline along which $\tilde{\psi} = 0$ does not form a dividing streamline. Interestingly, instead of a dividing streamline observed in the case of the rotating disk (see Figure 5.3 on page 83), here, we see that the blue streamline for $\tilde{\psi} = -0.01$ takes

Δt	ΔX	ΔY	ν_a	$X_{\max}$	$Y_{\max}$	ε
0.1	0.1	0.05	0.01	20	1400	10^{-5}

Table 12.1: Mesh parameters used to solve numerically the governing equations (12.6a)-(12.6h) in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock.

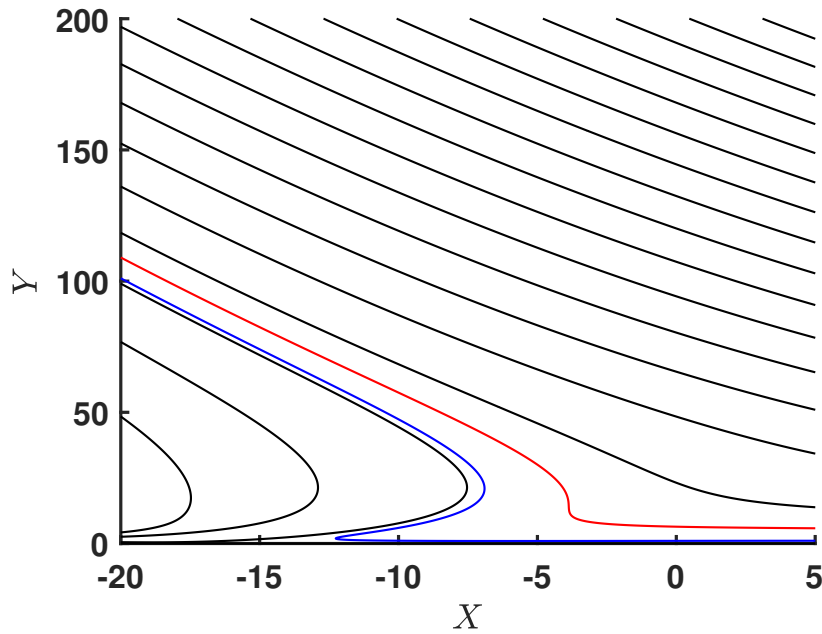


Figure 12.2: Streamline pattern on the pseudo-shock's scale at $t = 6$. Along the red streamline, $\tilde{\psi} = -0.05$, and along the blue streamline, $\tilde{\psi} = -0.01$. The three streamlines below the blue streamline have been plotted for $\tilde{\psi} = 0$, $\tilde{\psi} = 0.1$ and $\tilde{\psi} = 0.2$, starting from the top one to the bottom one. The streamlines above the red streamline have been plotted from $\tilde{\psi} = -0.1$ to $\tilde{\psi} = -2.1$, with the interval $\Delta\tilde{\psi} = -0.1$ between each streamline.

an unusual zigzag form. It enters the domain from $X_{\max}$, and it initially aims to exit the domain at $X_{\min}$ below the shock. However, it is too close to the shock, and it is entrained back at the top of the shock; the fluid particles are redirected further upwards.

12.2 Near-wall viscous sublayer

In the previous section, we proved that in the leading order approximation, the Navier-Stokes equations become the Euler equations in the vicinity of the pseudo-shock. The flow may then be treated as inviscid. Nevertheless, the no-slip condition on the wall surface should be satisfied. This is why we must introduce a new layer in the direct neighbourhood of the wall where the viscous terms become comparable with the convective terms in the Navier-Stokes equations. Let us call this new layer region 1_{vis} . Let N be a new scaled variable such that in region 1_{vis}

$$N = \frac{y}{\check{\delta}(Re)} = O(1), \quad (12.7)$$

where $\check{\delta}(Re) \ll Re^{-1/2}$ as $Re \rightarrow \infty$.

On the wall surface, the longitudinal velocity must be equal to $-V_s(t) = O(1)$. And it varies from this value to $\tilde{U}|_{Y=0} = O(1)$ at the outer edge of region 1_{vis} . Therefore, u is of order one quantity in this region. Since the viscous terms must be restored, we balance them with the convective terms in the dimensionless longitudinal Navier-Stokes equation (9.4a). Recalling that $x = O(Re^{-1/2})$, we deduce that

$$\check{\delta}(Re) = Re^{-3/4}. \quad (12.8)$$

Applying the principle of least degeneration into the continuity equation, we find that the lateral velocity satisfies $v \sim Re^{-1/4}$. Finally, the pressure field is still of order one.

Let us write the asymptotic expansions of the sought flow functions in region 1_{vis} as

$$\begin{aligned} u &= \check{u}(X, N, t) + \cdots, \\ v &= Re^{-1/4} \check{v}(X, N, t) + \cdots, \quad \text{as } Re \rightarrow \infty, \\ p &= \check{p}(X, N, t) + \cdots, \end{aligned} \quad (12.9)$$

We substitute (12.9) into the Navier-Stokes equations, and working with the leading

order terms, they reduce to the boundary-layer equations:

$$\check{u} \frac{\partial \check{u}}{\partial X} + \check{v} \frac{\partial \check{u}}{\partial N} = -\frac{d\check{p}}{dX} + \frac{\partial^2 \check{u}}{\partial N^2}, \quad (12.10a)$$

$$\frac{\partial \check{p}}{\partial N} = 0, \quad (12.10b)$$

$$\frac{\partial \check{u}}{\partial X} + \frac{\partial \check{v}}{\partial N} = 0. \quad (12.10c)$$

They are solved with the following boundary conditions:

On the wall surface, the no-slip and the impermeability conditions hold

$$\check{u} = -V_s(t), \quad \check{v} = 0, \quad \text{at } N = 0. \quad (12.10d)$$

At the outer edge of region 1_{vis}, the solution must match with the solution in the main part of the pseudo-shock

$$\check{u} = \tilde{U}|_{Y=0}, \quad \text{at } N = \infty. \quad (12.10e)$$

When X is large, the solution must match with the oncoming uniform flow

$$\check{u} = -V_s(t), \quad \text{at } X = \infty. \quad (12.10f)$$

The set of equations (12.10a)-(12.10f) may be solved numerically using the same techniques presented in Section 3 of Part I. The difference is that, this time the pressure derivative $d\check{p}/dX$ appears on the right-hand side of the longitudinal momentum equation (12.10a). However, this is not an issue as it is known for all X : at the outer edge of region 1_{vis}, the Bernoulli equation holds along the streamline where $\tilde{\psi} = 0$, so we have

$$\frac{d\check{p}}{dX} = \frac{\partial \tilde{P}}{\partial X} \bigg|_{Y=0} = -\tilde{U} \frac{\partial \tilde{U}}{\partial X} \bigg|_{Y=0}. \quad (12.11)$$

Marching methods are used to calculate the solution by progressing along each computational line $X = X_i$, and we start from the right-hand side of the domain. The streamline pattern of the flow in the near-wall viscous sublayer is shown in Figure 12.3. The streamlines at the bottom of the viscous region, that is, close to the moving plate surface (we recall that in the reference frame moving with the pseudo-shock, the plate is moving in the upstream direction with the velocity V_s), are almost parallel. Thus, the fluid enters the domain from the right-hand side where a uniform profile $-V_s(t)$ is imposed, and it leaves the domain on the left-hand side, where the velocity tends to

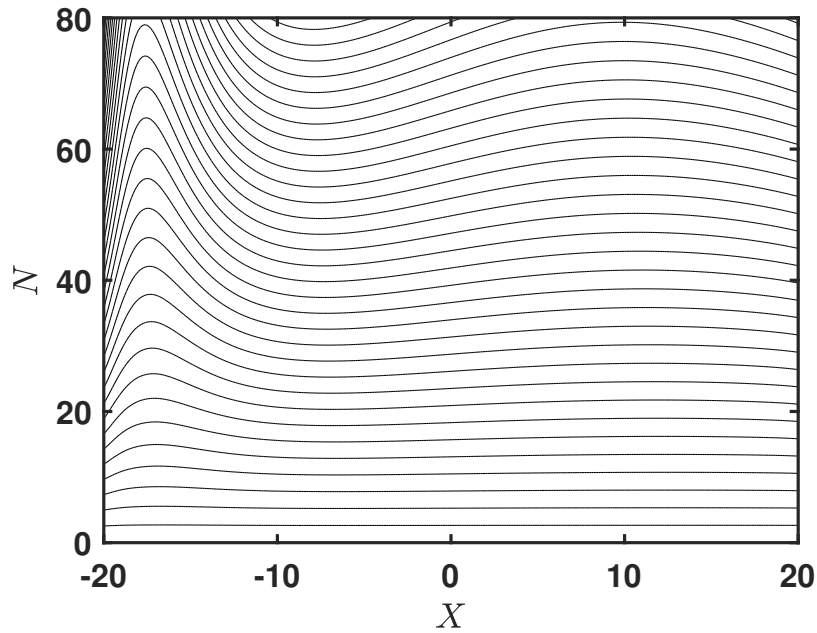


Figure 12.3: Streamline pattern in the near-wall viscous sublayer.

$-V_s(t)$ when N is close to zero. When approaching the outer edge of region 1_{vis} , there are small perturbations due to the matching condition (12.10e), but everything remains smooth.

Chapter 13

Discussion

In this part, our attention is with the unsteady analogue of the laminar Glauert jet. Dealing with unsteady two-dimensional boundary-layer equations implies that discontinuities are allowed to develop in the flow fields. This feature is of great interest as our goal is to understand better unsteady separation phenomena, and discontinuities in the velocity field mean that a sharp variation of the pressure are observed. The work presented in this part follows directly from the analysis in Part I where we studied the free jet generated by the rotation of a plane disk of finite radius. The wall jet problem presents many similarities with the rotating disk problem: the opening of the narrow slit generates a jet propagating along the x -axis as time increases. The longitudinal velocity is discontinuous at the pseudo-shock's position, and the lateral velocity component experiences a singularity at this position (alongside Part I, the radial and circumferential velocities were discontinuous, and the axial velocity was singular at the shock's position). The discontinuity was termed as a "pseudo-shock", following Ruban and Vonatsos (2008) study of the unsteady analogue of the Schlichting (1933) laminar jet. This denomination was employed because this jet is similar to shock waves in gas dynamics. However, the length of our shock is too large, and the continuum hypothesis can be applied, contrary to shock waves. Similarly to Part I, we studied the inner structure of the pseudo-shock where the flow may be treated as inviscid. The difference with the free jet is that the symmetry condition no longer holds; instead, the impermeability condition must be applied to the wall surface. The underlying consequence is that immediately upstream of the pseudo-shock, the stream-function is equal to zero at three distinct points (instead of two if we compare with the free jet). Thus, we did not observe a dividing streamline, as with the free jet. As the jet propagates, it ejects fluid particles when it penetrates the stagnant region. This fluid is redirected and reinjected back into the jet, upstream of the pseudo-shock. Finally, we finish our analysis by introducing a near-wall viscous sublayer in the direct neighbourhood of the wall surface.

Part III

Quasi-steady boundary-layer separation in supersonic flow with a downstream moving separation point

Chapter 14

Introduction

14.1 Overview of the problem

Boundary-layer separation is a complex process which occurs in most fluid flows; in fact, significant restrictions must be imposed on the shape of the body placed in the flow in order to avoid separation. It is important for the engineers and the researchers to fully comprehend the whole separation process, as it may have a significant impact on the flow. Indeed, when the boundary layer breaks away from the body surface, this leads to two distinct phenomena: distortion of the streamline pattern due to the displacement effect of the boundary layer, and modification of the fluid-dynamic forces acting on the body. This leads to various consequences, such as the eruption of eddies comparable in size with the body surface. Therefore, having a deep understanding of this subject is essential, as the results may be used in a wide range of applications, such as in the aeronautic and military sectors or in energy industry.

The first separated solution was described by Kirchhoff (1869), dealing with the flow past a flat plate which is perpendicular to the oncoming flow, and where separation took place at the plate's edges. Thereafter, his model was applied to various body shapes, in particular, Levi-Civita (1907) studied the separation in the case of a flow past a circular cylinder. He proved that the Euler equations admit a class of separated solutions with the position of the separation point, located on the cylinder surface, playing the role of a free parameter. However, Kirchhoff's model could not explain the physical processes of separation, and it did not allow for the position of the separation point to be identified.

The cornerstone of the theory of separation is the paper of Prandtl (1904). He introduced the concept of the boundary layer to explain the separation process, and this may be described by considering the flow past a solid body at high Reynolds number Re . In the boundary layer, the velocity varies from zero on the body surface where the no-slip condition holds, to the value dictated by the inviscid flow theory at the outer edge of the boundary layer. At the front of the body, the fluid accelerates,

so the boundary layer is subject to a negative (*favourable*) pressure gradient. Further downstream, the pressure starts to rise. The fluid starts to decelerate and it is subject to a positive (*adverse*) pressure gradient. The fluid particles with less kinetic energy (that is, the ones in the direct neighbourhood of the body surface) are more sensitive to pressure variations. Consequently, they are strongly decelerated as compared to the fluid particles close to the outer edge of the boundary layer. Then, the velocity profile gradually changes until the point of separation S is reached. This point is defined such that it corresponds to the location on the body surface where the skin friction $\tau_w = \mu \frac{\partial u}{\partial n}|_{n=0}$ becomes equal to zero for the first time. Here, the coefficient μ stands for the dynamic viscosity, u is the velocity component tangent to the surface of the body, and n is the coordinate orthogonal to it. Upstream of S , τ_w is positive, and immediately downstream of that point, τ_w is negative, signifying that a reverse flow region develops. It contains recirculation regions and eddies, which is characteristic of separated flows.

Prandtl's boundary-layer theory describes the physical processes involved in flow separation rather well. However, it appears to be inconsistent regarding several points. Indeed, Prandtl's boundary-layer theory is applicable when the longitudinal gradients are small in comparison with the ones in transverse direction. Consequently, his theory cannot be applied to fluid flows with local singularities in the fluid-dynamic functions in the vicinity of corners, or more generally in the vicinity of bodies with large local curvatures; but also, the Prandtl's theory cannot be used nearby regions of interaction between the boundary layer and the external inviscid flow, and flows near separation and reattachment points. Another inconsistency of Prandtl's model concerns recirculation regions; in practical cases, when the flow detaches from body surface, the eddies formed in the separation region downstream of the separation point are comparable in size with the characteristic length of the body. But according to Prandtl's theory, the development of such recirculation regions implies that necessarily, the boundary layer should break away from the body surface, and it does not explain why the eddies would not stay confined inside the boundary layer, whose thickness is $O(Re^{-1/2})$. Nevertheless, Blasius (1908) proved that the development of eddies in the boundary layer does not imply that there is separation. He considered a circular cylinder, initially placed in an infinite region filled with stagnant fluid. At time $\hat{t} = 0$, the cylinder is impulsively brought into motion with the uniform velocity $\mathbf{V}_0$, and a boundary layer is formed around the surface of the cylinder. At $t = 0.644$, the skin friction turns zero at the rear stagnation point, and two symmetric recirculation regions develop inside the boundary layer as they expand in the upstream direction. Nonetheless, they stay confined in the

boundary layer, while the latter remains thin. The boundary layer breaks away for a larger dimensionless time, namely, at $t \approx 3$, when eddies erupt from the boundary layer at the angle $\theta = 111^\circ$ from the front stagnation point. This result has been confirmed in various experiments, such as in Nakayama (1988). He observed the eddies eruption to coincide with the moment when the solution of the boundary-layer equations become singular. This sort of behaviour has been observed in other flows. In particular, Walker (1978) focused on the finite time singularity in the boundary layer exposed to a rectilinear vortex. A description of the singularity was given by van Dommelen and Shen (1980), and the conclusions have been restudied using analytical as well as numerical methods by several authors, namely, van Dommelen and Shen (1982, 1983a,b), Cowley (1983), Elliott et al. (1983), Ingham (1984), Peridier et al. (1991), Christov and Tsankov (1993) and Cassel et al. (1996).

Regarding steady flows, the failing of Prandtl's theory to explain the eruption of eddies was mentioned by Howarth (1938) and Hartree (1939). They assumed that the velocity $U_e(s)$ at the outer edge of the boundary layer was linearly decreasing with s , where s is the coordinate measured along the body surface. They found that the solution develops a singularity at the point of zero skin friction $s = s_0$. The form of this singularity was found analytically by Landau and Lifshitz (1944), who deduced that, immediately upstream of the singularity, the dimensionless skin friction τ_w decreases as the square root of $s_0 - s$, that is $\tau_w \approx \sqrt{s_0 - s}$ as s tends to s_0^- . Furthermore, they showed that the velocity component V_0 perpendicular to the plate surface undergoes an unbounded growth when approaching the singularity, such that $V_0 \sim 1/\sqrt{s_0 - s}$ as s tends to s_0^- . The form of V_0 upstream of the singularity explains the eruption of eddies when τ_w is equal to zero for the first time. Goldstein (1948) confirmed this result a few years later, but more importantly, he found out that the singularity precludes the solution to be constructed downstream of the point of zero skin friction, implying that new mathematical models are required to describe the separation process.

It appears that Prandtl's boundary-layer theory cannot be applied in the vicinity of the separation point due to the singularity: the longitudinal gradients of the flow functions are no longer small as compared to the ones in transverse direction. An important point that Prandtl did not take into account, is that there exists an interaction between the boundary layer and the outer inviscid flow, termed as the *viscous-inviscid interaction*. The mathematical description of this interaction is based on asymptotic methods, and it is named the *triple-deck theory*. It was formulated simultaneously by Neiland (1969b) and Stewartson and Williams (1969) for the self-induced separation in

a supersonic flow. To these two publications were added the contributions of Stewartson (1969) and Messiter (1970) who both worked on the incompressible flow near the trailing edge of a flat plate. From there, researchers understood that the viscous-inviscid interaction was an essential key to describe flow separation.

Although the triple-deck theory describes well the separation process, it is only valid when the flow is steady, or when the separation point moves slowly. The unsteady flow cases appear to be more complicated. This is partly due to the introduction of an additional independent variable (namely time), which modifies the type of the governing equations. In the current literature, researchers have focused on two distinct types of unsteady boundary-layer separation. The first one may be called “incipient separation”. It is when separation occurs at a given time, and at a specific point on the body surface. For instance, it is what happens at time $t \approx 3$ for the case of the impulsively started cylinder discussed earlier. The second category of separation concerns flows with “developed separation”. A prime example is the flow past a circular cylinder at sufficiently large Reynolds number; a Kármán vortex street develops in the wake of the cylinder. This is due to vortices forming in the boundary layer through an accumulation of vorticity. When the circulation around each individual vortex reaches a critical value, the vortex erupts from the boundary layer and another one develops again into the boundary layer. The difference with the incipient separation is that the separation point moves up and down the cylinder surface. The eruption of vortices from the boundary layer allows us to infer that the solution develops a singularity when approaching the separation point. Otherwise, the eddies would stay confined within the boundary layer.

Under certain conditions, an analysis may be established between unsteady flows on a motionless wall and steady flows on a moving wall. The analogy was highlighted by Sears (1956) and Moore (1958) for the first time. This may be justified through the following argument. The eruption of eddies implies that the lateral velocity component becomes singular at the separation point. Hence, the longitudinal gradient of the longitudinal velocity component is infinite at this same position. Thus, the local acceleration becomes negligible in comparison with the convective terms in the reference frame moving with the separation point, and the flow may be treated as quasi-steady in this moving frame. However, the triple-deck model is not applicable because in the reference frame moving with the separation point, the plate is no longer motionless. Figure 14.1 illustrates the velocity profile change in the case of the boundary-layer separation from a downstream moving wall. Due to viscosity, the no-slip condition holds

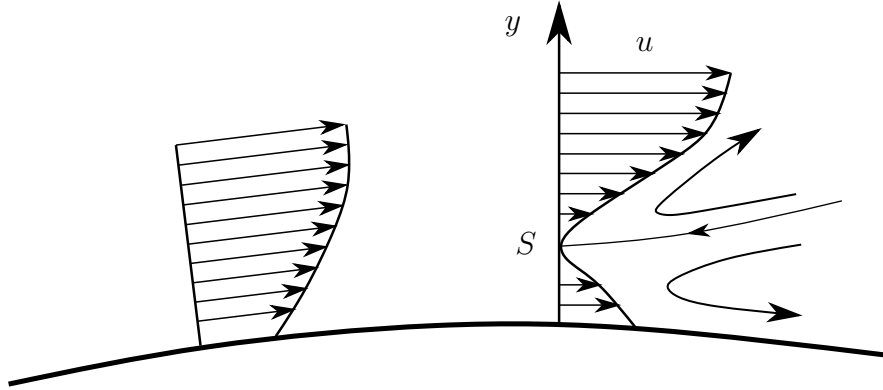


Figure 14.1: Boundary-layer separation on a downstream moving wall.

on the surface of the body, and the fluid particles adjacent to it move in the downstream direction. Hence, the separation point cannot be located on the body surface. Instead, it is located in the middle of the boundary layer, as suggested for the first time by Rott (1956), Sears (1956) and Moore (1958). When the flow is subject to an adverse (positive) pressure gradient, the fluid particles start to slow down. Obviously, the ones around the middle of the boundary layer will be more affected, and the velocity profile will show a minimum at a distance $n_{\min}$ from the wall. If the pressure gradient is strong enough, the minimum velocity keeps decreasing with s , the coordinate along the wall surface, until the separation point $(s_S, n_{\min}(s_S))$ is reached, where the longitudinal velocity u is equal to zero. The so-called Moore-Rott-Sears condition

$$u = \frac{\partial u}{\partial n} = 0, \quad (14.1)$$

holds at this point.

Later, Sychev, Vic. V. (1980) confirmed that the solution of the boundary-layer equations develops a singularity at the Moore-Rott-Sears point, for the case of the downstream moving wall. He showed that the minimum velocity $u_{\min}$ decreases as $\sqrt{s_S - s}$. Moreover, he proved that the singularity precludes the solution to be constructed downstream of the separation point, implying that a new mathematical model is necessary to describe the separation process. Thus, a new theory was developed with the contributions of Sychev (1979, 1984, 1987) and van Dommelen and Shen (1983b). They showed that, similarly to the case of the motionless wall, there exists an interaction between the boundary layer and the external inviscid flow. Nevertheless, the boundary layer is subject to an extreme adverse pressure gradient, further upstream of

the interaction region. Therefore, the fluid particles are strongly decelerated, and the viscous-inviscid interaction is altered, making the boundary layer predominantly inviscid. The interested reader about this theory is referred to Chapter 5 of the monograph by Sychev et al. (1998).

Ruban et al. (2011) presented an analogue of this theory for supersonic flows. They studied the supersonic flow past a flat plate, where a moving shock wave impinges upon the boundary layer. In the reference frame of the shock wave, the plate moves in the downstream direction. The authors considered initially that the dimensionless pressure jump across the shock is of order $Re^{-1/4}$, and the dimensionless shock speed V_{sh} is $O(Re^{-1/8})$, where we recall that Re is the Reynolds number which is assumed to be large. These assumptions allowed the authors to describe the equations thanks to the triple-deck theory. The problem was solved numerically using windward finite difference for derivatives with X , where X is the coordinate along the plate surface, and Chebyshev collocation for derivatives with respect to Y , the coordinate normal to the plate. For more information about this particular numerical method, the reader is referred to method *B* described in Korolev et al. (2002). The results showed that when the plate is motionless, the dimensionless skin friction $\tau_w = \frac{\partial U}{\partial Y}|_{Y=0}$ (where U is the longitudinal velocity) plotted with respect to X intersects the abscissa four times. The first and last points of intersection give the separation and reattachment positions of the primary separation bubble. The two other points correspond to a second smaller separation bubble embedded inside the primary one. Since the flow in the bubble region is rather slow, the pressure plot develops a characteristic plateau. When the plate starts moving in the downstream direction, Ruban et al. showed that the skin friction no longer shows where the recirculation regions (or bubbles) are. Instead, it shows if the fluid layer adjacent to the wall moves faster ($\tau_w > 0$) or slower ($\tau_w < 0$), corresponding to the left and right-hand sides in Figure 14.1, respectively. Some criteria allowed them to judge if the flow is separated or not, but a more rigorous analysis is necessary to be sure about that. The transition between the attached and separated forms of the flow takes place when U is positive everywhere, except at one point where the Moore-Rott-Sears condition (14.1) holds. Of course, they found that the shock strength must be stronger to trigger separation for a larger speed of the plate. After solving numerically the triple-deck problem for different values of the plate speed, Ruban et al. assumed that the speed of the shock was large such that $V_{sh} \gg Re^{-1/8}$, which allowed them to find an analytic solution. They showed that under this assumption, the lower deck of the triple-deck structure was divided into two regions: one where the flow is inviscid,

and a near-wall viscous layer. More importantly, they found that the streamline slope of the viscous sub-layer becomes singular at $X = 0$ when the strength of the shock wave is chosen such that the velocity at the outer edge of this viscous layer is equal to zero at $X = 0$. Overall, their work showed that physical processes involved are similar to the ones of the subsonic case.

The case of the upstream moving wall appears to be more complicated. Zhuk (1982) was the first researcher to present numerical results of the triple-deck equations for the upstream moving wall case. More recently, Yapalparvi and van Dommelen (2012) repeated the calculations. The results show that when increasing the speed of the plate (and by extension, the speed of the shock wave), it suppressed the separation. They also concluded that the Moore-Rott-Sears condition (14.1) could not be used to know if the flow was separated, contrary to the case of the downstream moving wall. This part of this thesis shows that if instead of an impinging shock wave, an expansion fan impinges the boundary layer, then the flow topology is totally different, and when the strength of the expansion fan is chosen specifically, a singularity develops in the boundary layer, as described by Neiland (1969a).

14.2 Problem outline

In this part, we shall study the case of the supersonic flow past a flat plate, where a shock/expansion fan impinges upon the boundary layer. The shock/expansion fan moves with the velocity $\widehat{V}_{sh}$ in the downstream direction. In the reference frame moving with the shock, the plate is no longer motionless, and it moves in the upstream direction. In Chapter 15, we shall apply the triple-deck theory to describe the problem by assuming that the dimensionless speed of the shock/expansion fan V_{sh} is an $O(Re^{-1/8})$ quantity. Then, we shall assume that $1 \gg V_{sh} \gg Re^{-1/8}$, which will allow us to solve the problem analytically. Under this condition, the near-wall viscous sublayer of the triple deck structure splits into two regions: an inviscid layer *1a*, as well as a near-wall viscous layer called *1b*. Chapter 16 deals with the inviscid region *1a*. The velocity components and the pressure distribution in this region are determined analytically. It will be shown that no separation can be expected if a shock wave impinges upon the boundary layer. However, an expansion fan can be generated instead of a shock wave. We can set its strength to a specific value such that the velocity at the outer edge of the boundary layer becomes zero at the point where the expansion fan hits the boundary layer. If

the strength of the expansion fan is equal to another value instead, then we will show that no singularity of the streamline slope can be expected in the near-wall viscous sublayer, implying that separation would not be possible. In Chapter 17, we go back to the problem when the speed of the expansion fan is still an order $Re^{-1/8}$ quantity, and the problem is solved numerically. When this speed increases, the jump of the pressure distribution is more strongly emphasised. Chapter 17 does not appear before because our goal is to compare the pressure determined numerically for increasing values of the speed of the plate surface, with the analytical pressure obtained by investigating region 1a; we expect the numerical pressure to coincide more and more with the pressure found by assuming $1 \gg V_{sh} \gg Re^{-1/8}$ as the speed of the plate becomes larger. In Chapter 18, the viscous near-wall sublayer is analysed. It is shown that the displacement effect of the boundary layer is due to a small region lying between regions 1a and 1b, close to $\hat{x} = 0$, and called region 1b'. Furthermore, the streamline slope in this region becomes singular at $\hat{x} = 0$ when the strength of the expansion fan is set such that the longitudinal velocity at the outer edge of the viscous region 1b is equal to zero at $\hat{x} = 0$. In Chapter 19, we introduce a final region called C in the vicinity of $\hat{x} = 0^+$ in order to smooth out the singularity of the streamline slope, which gives more information about the topology of the flow. In that region, the pressure is increasing in the negative $\hat{x}$ direction, meaning that the fluid particles close to the plate surface are subject to an adverse pressure gradient. We will show that the pressure becomes singular for a finite value of $\hat{x}$.

Chapter 15

Problem formulation

Let us consider the two-dimensional supersonic flow past a flat plate aligned with the oncoming uniform flow. A compression shock/expansion wave, generated by a wedge/expansion ramp placed above the plate, as depicted in Figure 15.1, perturbs the flow and impinges upon the boundary layer formed along the surface of the body at point I . We assume that the ramp, and so the shock, is moving downstream with the velocity $\widehat{V}_{\text{sh}}$. Thus, when the problem is studied in the reference frame moving with the shock/expansion fan, the plate moves in the upstream direction. Our interest is to study the incipience of the separation triggered by the shock/expansion wave. With that purpose in mind, we shall assume firstly that $\widehat{V}_{\text{sh}}/\widehat{V}_{\infty}$ is an $O(Re^{-1/8})$ quantity, where $\widehat{V}_{\infty}$ is the free-stream velocity, and Re is the Reynolds number and it is assumed to be large. It will be defined properly in terms of the problem quantities later. The condition on $\widehat{V}_{\text{sh}}/\widehat{V}_{\infty}$ can be written mathematically in the following form:

$$\frac{\widehat{V}_{\text{sh}}}{\widehat{V}_{\infty}} = Re^{-1/8} C_s U_w, \quad (15.1)$$

where C_s is a scaling constant of order one and which will be written explicitly later in this chapter. The quantity U_w corresponds to the scaled speed of the plate, which is of order one under assumption (15.1). This assumption allows us to describe the interaction between the shock/expansion fan and the boundary layer in the framework of the *triple-deck theory* (this theory will be discussed in this chapter). The resulting interaction problem will be deduced thereafter. This interaction problem cannot be solved analytically. However, an analytical analysis may be performed by starting from the triple-deck equations, and by assuming that

$$Re^{-1/8} \ll \frac{\widehat{V}_{\text{sh}}}{\widehat{V}_{\infty}} \ll 1. \quad (15.2)$$

This condition on the speed of the shock/expansion fan allows us to extend the domain of validity of the triple-deck theory; this will be discussed in more detail at the end of this chapter. Moreover, under assumption (15.2), the quantity U_w is no longer of order one, but instead we have

$$U_w \rightarrow \infty.$$

Therefore, U_w is used as a large parameter to apply perturbation methods so that the problem can be solved analytically.

The assumptions (15.1) or (15.2) about having a “slow” moving shock/expansion fan hitting the boundary layer are of interest in many supersonic and quasi-steady practical flows. For example, during an airshow; supersonic planes generate shock waves which can hit the boundary layer formed in the vicinity of nearby planes’ surface. This may trigger separation. Furthermore, even though aircraft seem to be motionless as compared to each other during a formation flying, they never fly at the exact same speed. Therefore, in the event of a shock wave, generated by an aircraft from a formation, hits another plane of the same formation, this shock wave moves slowly when observed in the reference frame moving with the plane generating this shock.

As introduced earlier in Section 14.2, under assumption (15.2), we will prove in Chapter 16 that the lower deck of the triple-deck structure is divided into an inviscid region 1a and a near-wall viscous sublayer 1b. We will perform the asymptotic analysis of region 1a in Chapter 16, where the asymptotic expansions of the solutions will be found. In Chapter 17, the interaction problem will be solved numerically, that is under assumption $\widehat{V}_{sh}/\widehat{V}_\infty = O(Re^{-1/8})$. The reason why Chapter 17 does not appear before, is because we wish to compare the pressure distribution obtained for different values of U_w by solving the interaction problem numerically, with the analytical pressure found by investigating the inviscid region 1a after assuming (15.2). We expect that, as U_w becomes larger, the numerical pressure coincides more and more with the analytical pressure (which is only valid when assuming (15.2)), as when $U_w \rightarrow \infty$, condition (15.2) is satisfied.

We introduce a Cartesian coordinate system $(\widehat{x}, \widehat{y})$ moving with the shock/expansion fan, such that its origin I' is the projection of I on the plate surface. We chose the coordinate $\widehat{x}$ to be measured along the plate surface from point I' , while $\widehat{y}$ is orthogonal to the $\widehat{x}$ -axis. The velocity components in the $\widehat{x}$ - and $\widehat{y}$ -directions are denoted as $\widehat{u}$ and $\widehat{v}$, respectively. The stream function is denoted as $\widehat{\psi}$. We also introduce the fluid pressure, density, dynamic viscosity and enthalpy, denoted as $\widehat{p}$, $\widehat{\rho}$, $\widehat{\mu}$ and $\widehat{h}$, respectively. The

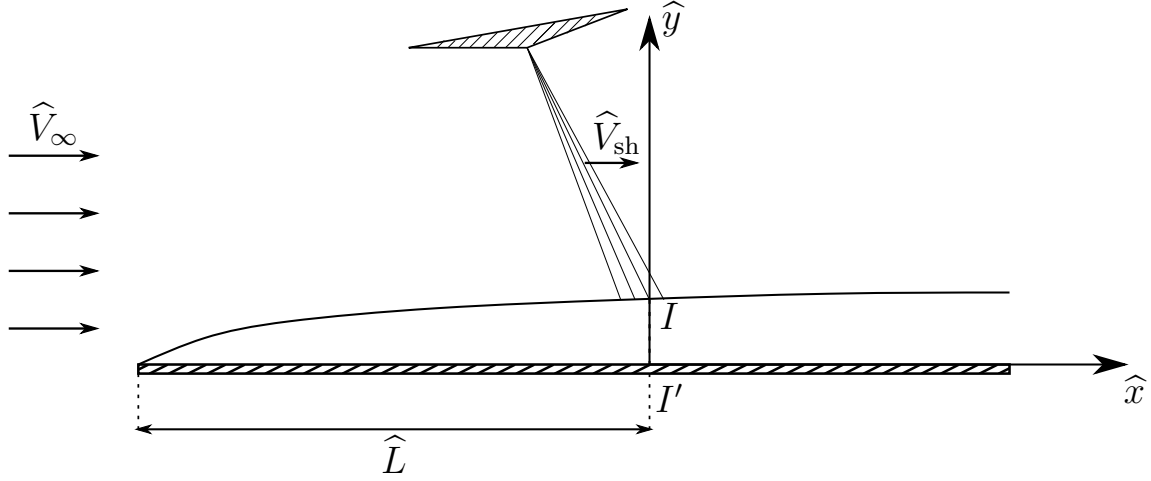


Figure 15.1: Problem layout. In the Figure, an expansion fan is generated from an expansion ramp, and it moves with the velocity $\hat{V}_{sh}$ downstream of the flow.

“hat” stands for dimensional quantities. The flow is assumed to be compressible, so $\hat{\rho}$ and $\hat{\mu}$ are not constants, and they depend on the variables $\hat{x}$, $\hat{y}$ and $\hat{t}$. We also assume that the fluid is Newtonian, and that the body forces $\mathbf{f}$ can be neglected. We shall study the flow in the reference frame moving with the shock/expansion fan. Hence, in that moving frame, the plate is no longer considered as motionless and its velocity vector $\hat{\mathbf{U}}_w$ is equal to the opposite of $\hat{\mathbf{V}}_{sh}$. The speed and strength of the shock/expansion fan can vary in time, but the dimensionless characteristic time scale t must satisfy

$$t \gg \frac{Re^{-1/2}}{\bar{U}_w^2}, \quad (15.3)$$

where $\hat{U}_w = \hat{V}_\infty \bar{U}_w$. Let us notice that $\bar{U}_w$ is different from U_w : while U_w is linked to $\hat{V}_{sh}/\hat{V}_\infty$ by an order of magnitude $Re^{-1/8}$ according to (15.1), $\bar{U}_w$ is the dimensionless form of $\hat{U}_w$, that is, essentially $\bar{U}_w$ is $\hat{V}_{sh}/\hat{V}_\infty$. Under assumption (15.3), the flow may be treated as quasi-steady in the reference frame moving with the shock/expansion fan. The compressible Navier-Stokes equations governing the flow may be written in the following form

$$\hat{\rho} \left(\hat{u} \frac{\partial \hat{u}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{u}}{\partial \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{x}} + \frac{\partial}{\partial \hat{x}} \left[\hat{\mu} \left(\frac{4}{3} \frac{\partial \hat{u}}{\partial \hat{x}} - \frac{2}{3} \frac{\partial \hat{v}}{\partial \hat{y}} \right) \right] + \frac{\partial}{\partial \hat{y}} \left[\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right) \right], \quad (15.4a)$$

$$\hat{\rho} \left(\hat{u} \frac{\partial \hat{v}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{v}}{\partial \hat{y}} \right) = -\frac{\partial \hat{p}}{\partial \hat{y}} + \frac{\partial}{\partial \hat{y}} \left[\hat{\mu} \left(\frac{4}{3} \frac{\partial \hat{v}}{\partial \hat{y}} - \frac{2}{3} \frac{\partial \hat{u}}{\partial \hat{x}} \right) \right] + \frac{\partial}{\partial \hat{x}} \left[\hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right) \right], \quad (15.4b)$$

$$\begin{aligned} \hat{\rho} \left(\hat{u} \frac{\partial \hat{h}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{h}}{\partial \hat{y}} \right) &= \hat{u} \frac{\partial \hat{p}}{\partial \hat{x}} + \hat{v} \frac{\partial \hat{p}}{\partial \hat{y}} + \frac{1}{Pr} \left[\frac{\partial}{\partial \hat{x}} \left(\hat{\mu} \frac{\partial \hat{h}}{\partial \hat{x}} \right) + \frac{\partial}{\partial \hat{y}} \left(\hat{\mu} \frac{\partial \hat{h}}{\partial \hat{y}} \right) \right] \\ + \hat{\mu} \left(\frac{4}{3} \frac{\partial \hat{u}}{\partial \hat{x}} - \frac{2}{3} \frac{\partial \hat{v}}{\partial \hat{y}} \right) \frac{\partial \hat{u}}{\partial \hat{x}} &+ \hat{\mu} \left(\frac{4}{3} \frac{\partial \hat{v}}{\partial \hat{y}} - \frac{2}{3} \frac{\partial \hat{u}}{\partial \hat{x}} \right) \frac{\partial \hat{v}}{\partial \hat{y}} + \hat{\mu} \left(\frac{\partial \hat{u}}{\partial \hat{y}} + \frac{\partial \hat{v}}{\partial \hat{x}} \right)^2, \end{aligned} \quad (15.4c)$$

$$\frac{\partial (\hat{\rho} \hat{u})}{\partial \hat{x}} + \frac{\partial (\hat{\rho} \hat{v})}{\partial \hat{y}} = 0, \quad (15.4d)$$

$$\hat{h} = \frac{\gamma}{\gamma - 1} \frac{\hat{p}}{\hat{\rho}}. \quad (15.4e)$$

They are the $\hat{x}$ -, $\hat{y}$ -momentum equations, the energy equation, the continuity equation and the state equation, respectively. The quantity γ is the specific heat ratio, and Pr is the Prandtl number.

We introduce a characteristic length scale denoted by $\hat{L}$ as the distance between the leading edge of the plate and point I' . In the unperturbed flow in front of the plate, the pressure, the density and the dynamic viscosity are $\hat{p}_\infty$, $\hat{\rho}_\infty$ and $\hat{\mu}_\infty$, respectively. We introduce the following dimensionless quantities x , y , u , v , ψ , p , ρ , μ and h such that

$$\begin{aligned} \hat{x} &\longrightarrow \hat{L}x, \quad \hat{y} \longrightarrow \hat{L}y, \quad \hat{u} \longrightarrow \hat{V}_\infty u, \quad \hat{v} \longrightarrow \hat{V}_\infty v, \\ \hat{p} &\longrightarrow \hat{p}_\infty + \hat{\rho}_\infty \hat{V}_\infty^2 p, \\ \hat{\psi} &\longrightarrow \hat{\rho}_\infty \hat{V}_\infty \hat{L} \Psi, \quad \hat{\rho} \longrightarrow \hat{\rho}_\infty \rho, \quad \hat{\mu} \longrightarrow \hat{\mu}_\infty \mu, \quad \hat{h} \longrightarrow \hat{V}_\infty^2 h. \end{aligned} \quad (15.5)$$

The Reynolds number is defined in the following way

$$Re = \frac{\hat{\rho}_\infty \hat{L} \hat{V}_\infty}{\hat{\mu}_\infty}, \quad (15.6)$$

and it is assumed to be large.

Since we are dealing with a supersonic flow, the free-stream Mach number M_∞ is greater than one and has a finite value

$$M_\infty = \frac{\hat{V}_\infty}{\sqrt{\frac{\gamma \hat{p}_\infty}{\hat{\rho}_\infty}}} > 1. \quad (15.7)$$

Here and in the following, we assume that the fluid is a perfect gas.

Substitution of (15.5) into the Navier-Stokes equations (15.4a)-(15.4e) leads to the dimensionless Navier-Stokes equations:

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \frac{1}{Re} \left(\frac{\partial}{\partial x} \left[\mu \left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \frac{\partial v}{\partial y} \right) \right] + \frac{\partial}{\partial y} \left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \right), \quad (15.8a)$$

$$\rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \frac{1}{Re} \left(\frac{\partial}{\partial y} \left[\mu \left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \frac{\partial u}{\partial x} \right) \right] + \frac{\partial}{\partial x} \left[\mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \right] \right), \quad (15.8b)$$

$$\begin{aligned} \rho \left(u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} \right) &= u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + \frac{1}{Re} \left(\frac{1}{Pr} \left[\frac{\partial}{\partial x} \left(\mu \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial y} \left(\mu \frac{\partial h}{\partial y} \right) \right] \right. \\ &\left. + \mu \left(\frac{4}{3} \frac{\partial u}{\partial x} - \frac{2}{3} \frac{\partial v}{\partial y} \right) \frac{\partial u}{\partial x} + \mu \left(\frac{4}{3} \frac{\partial v}{\partial y} - \frac{2}{3} \frac{\partial u}{\partial x} \right) \frac{\partial v}{\partial y} + \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 \right), \end{aligned} \quad (15.8c)$$

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0, \quad (15.8d)$$

$$h = \frac{1}{(\gamma - 1)M_\infty^2} \frac{1}{\rho} + \frac{\gamma}{\gamma - 1} \frac{p}{\rho}. \quad (15.8e)$$

Now the dimensionless Navier-Stokes equations have been stated, the mathematical analysis of the flow may be performed by either applying the limit $Re \rightarrow \infty$ directly to the Navier-Stokes equations (15.8a)-(15.8e), or starting with the triple-deck theory. We chose to deal with the latter. First of all, let us assume that

$$\bar{V}_{sh} = O(Re^{-1/8}), \quad (15.9)$$

where $\bar{V}_{sh}$ is the dimensionless speed of the shock/expansion fan, such that $\hat{V}_{sh} = \hat{V}_\infty \bar{V}_{sh}$. This assumption allows us to describe the interaction between the shock/expansion fan by the classical *triple-deck theory*. It was formulated simultaneously by Neiland (1969b) and Stewartson and Williams (1969) for the self-induced separation in a supersonic flow. In addition to these publications, Stewartson (1969) and Messiter (1970) both contributed to the triple-deck theory by working on the incompressible flow near the trailing edge of a flat plate. Since we are studying the flow in the reference frame moving with the shock/expansion fan, the main difference with the classical triple-deck model is that the wall is no longer motionless; it moves with the velocity $\hat{\mathbf{U}}_w$. The triple-deck theory is a mathematical model based on asymptotic analysis which allows

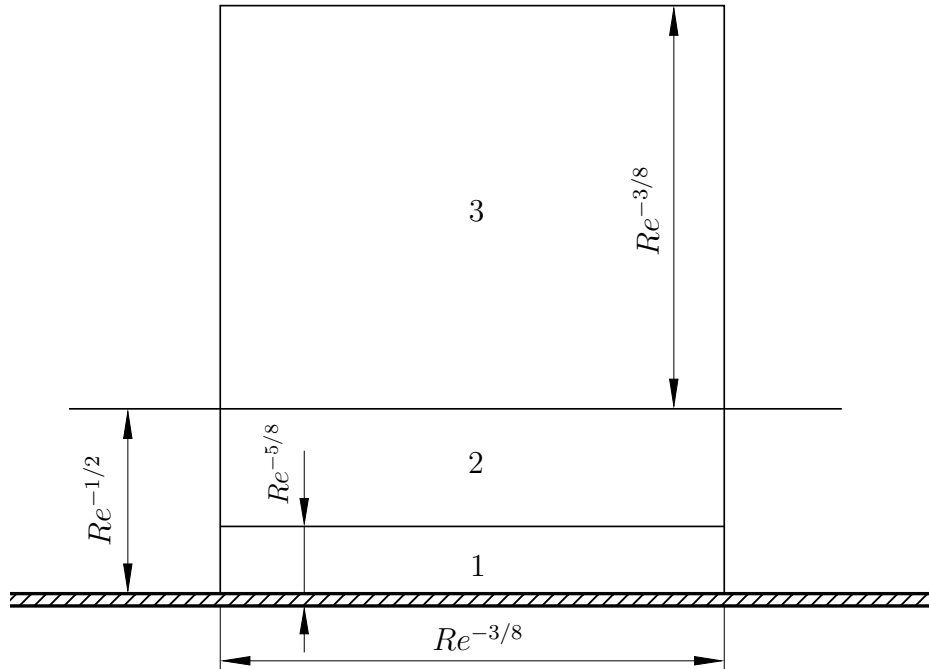


Figure 15.2: Three-tiered structure of the interaction region

us to describe the phenomenon of interaction between the boundary layer and the external inviscid flow, termed as the *viscous-inviscid interaction*. The region where the viscous-inviscid interaction takes place is called the *interaction region*. We may summarise the theory as follows: we consider the flow past a flat plate and we assume that the pressure increases for some reason over a short distance $\Delta x \ll 1$ at the outer edge of the boundary layer. The pressure variation Δp is small, and its increment causes the fluid particles to slow down and the boundary layer to thicken locally due to the displacement of streamlines from the plate. An *inspection analysis* may be applied to the problem in order to measure the order of magnitude of the variations of the fluid dynamic functions in terms of Δp . Balancing the convective terms with the pressure gradient in (15.8a), and recalling that the longitudinal velocity is an order one quantity everywhere in the boundary layer except near the plate surface, it emerges that the velocity variation Δu , which is due to the pressure increase, satisfies $\Delta u \sim \Delta p$. However, this relation loses its validity in the direct neighbourhood of the plate surface, where the nonlinear effects become important, as the longitudinal velocity u becomes comparable with its variation Δu . Hence, a near-wall sublayer must be introduced adjacent to the wall, and the inspection analysis reveals that in this region, we have $\Delta u \sim \sqrt{\Delta p}$. Therefore, the interaction region is divided into three layers as shown in Figure 15.2. The

longitudinal extent Δx of the interaction region is comparable with $Re^{-3/8}$. Region 3 is the external region where the solution is dictated by the inviscid flow theory. Region 2 is the main part of the boundary layer in the interaction region, and the flow may be considered as inviscid in the leading-order approximation. Region 1 corresponds to the near-wall viscous sublayer whose thickness is estimated as $O(Re^{-5/8})$ and where the nonlinear effects cannot be neglected anymore. In this region, the fluid particles are especially sensitive to pressure variations. Thus, if the pressure rises, it will cause a strong deceleration of the fluid and a noticeable displacement of the streamlines in this sublayer. Region 2 also produces a displacement effect of the boundary layer, but its impact is negligible as compared to the contribution of region 1. Region 2 only carries the deformation produced by the thickening of all the fluid filaments of region 1. Region 3 converts the streamline deformations into pressure perturbation which will influence regions 2 and 1. This whole process is universal; no matter how separation is triggered in a flow (boundary layer subject to an impinging shock wave, flow past a forward facing step, or more generally, when the flow is subject to a strong adverse pressure gradient), the description of the viscous-inviscid interaction is the same. That is why it is termed as the *self-induced separation*. In region 1 the asymptotic expansions of the solution of the Navier-Stokes equations (15.8a)-(15.8e) has the form (for detailed explanations about the theory and the form of the asymptotic expansions, the reader is referred to Section 2 of Ruban (2018))

$$\begin{aligned}
 x &= 1 + Re^{-3/8} \frac{\mu_w^{-1/4} \rho_w^{-1/2}}{\lambda^{5/4} \beta^{3/4}} X, & y &= Re^{-5/8} \frac{\mu_w^{1/4} \rho_w^{-1/2}}{\lambda^{3/4} \beta^{1/4}} Y, \\
 u &= Re^{-1/8} \frac{\mu_w^{1/4} \rho_w^{-1/2}}{\lambda^{-1/4} \beta^{1/4}} U + \dots, & v &= Re^{-3/8} \frac{\mu_w^{3/4} \rho_w^{-1/2}}{\lambda^{-3/4} \beta^{-1/4}} V + \dots, \\
 p &= Re^{-1/4} \frac{\mu_w^{1/2}}{\lambda^{-1/2} \beta^{1/2}} P + \dots, & \rho &= \rho_w + \dots, \quad \mu = \mu_w + \dots,
 \end{aligned} \tag{15.10}$$

where λ , ρ_w and μ_w are respectively the dimensionless skin friction, density and dynamic viscosity on the plate surface, immediately before the interaction region. The free-stream Mach number has an influence through the constant $\beta = \sqrt{M_\infty^2 - 1}$. The difference between our problem and the triple-deck model when the plate is motionless is that addition of the plate speed in the no-slip condition and some matching conditions; see below.

The flow in region 1 is given by the boundary-layer equations

$$U \frac{\partial U}{\partial X} + V \frac{\partial U}{\partial Y} = -\frac{dP}{dX} + \frac{\partial^2 U}{\partial Y^2}, \quad (15.11a)$$

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0. \quad (15.11b)$$

We notice that the continuity equation (15.11b) is written in its incompressible form. Although we are dealing with a compressible flow, the thickness of the near-wall viscous sublayer 1 is comparable to $Re^{-5/8}$, and the velocity in this region is small, namely $u \sim Re^{-1/8}$. Consequently, the flow in region 1 may be treated as incompressible.

Since the flow is viscous in region 1, the no-slip and the impermeability conditions must be satisfied on the plate surface

$$U = -U_w, \quad V = 0, \quad \text{at } Y = 0. \quad (15.11c)$$

As we said earlier, the flow is observed in the reference frame moving with the shock/expansion fan, so the plate is no longer motionless, and it moves in the upstream direction. The quantity U_w may be defined as follows

$$U_w = Re^{1/8} \frac{\mu_w^{-1/4} \rho_w^{1/2}}{\lambda^{1/4} \beta^{-1/4}} \bar{V}_{sh}. \quad (15.11d)$$

Far from region 1, the far-field attenuation condition must hold. The flow immediately upstream or downstream of the interaction region follows the classical boundary-layer theory of a flow past a flat plate. The thickness of region 1 is $O(Re^{-5/8})$, which implies that it is small as compared to $Re^{-1/2}$, the thickness of the unperturbed boundary layer. Hence, the asymptotic behaviour of U when $|X|$ is large is given by the representation of the Blasius solution at the bottom of the boundary layer. When observed in the reference frame moving with the shock/expansion fan, it is written as

$$U = Y - U_w, \quad \text{at } X = \pm\infty. \quad (15.11e)$$

The matching condition at the outer edge of region 1 with the main part of the boundary layer has the following form

$$U = Y + A(X) + \dots, \quad \text{as } Y \rightarrow \infty. \quad (15.11f)$$

The function A is the so-called *displacement function*. We omitted to subtract U_w in (15.11f) and instead incorporate it into the function $A(X)$. The displacement function appears in the interaction law, which is deduced by analysing the flow in region 3, and it is referred to as Ackeret's formula

$$P = P_S \mathcal{H}(X) - A'(X). \quad (15.11g)$$

where the function $\mathcal{H}$ is the Heaviside function such that

$$\mathcal{H}(X) = \begin{cases} 0, & \text{if } X < 0, \\ 1, & \text{if } X \geq 0. \end{cases} \quad (15.11h)$$

The Ackeret formula comes from the upper inviscid and supersonic deck of the triple-deck structure. When considering a supersonic and steady flow, the boundary-layer equations are still parabolic. However, in supersonic flow, there are other equations to take into account; namely, the velocity potential equation which is hyperbolic in the case of a supersonic flow. The solution of the velocity potential equation is used to find the Ackeret formula. For the sake of simplicity, we directly consider it without focusing on the outer inviscid region 3. The interested reader is referred to subsection 3.2.3 of Ruban (2015). The first term on the right-hand side of (15.11g) represents the pressure jump due to the impinging shock/expansion fan, where P_S is the shock's strength. It is positive for a shock, and negative for the expansion wave. It is related to the pressure jump $\Delta \hat{p}_{\text{sh}}$ across the shock/expansion fan, and it may be defined as follows

$$P_S = \frac{\Delta \hat{p}_{\text{sh}} Re^{1/4}}{\hat{\rho}_{\infty} \hat{V}_{\infty}^2} \frac{\mu_w^{-1/2}}{\lambda^{1/2} \beta^{-1/2}} \quad (15.11i)$$

The second term on the right-hand side of (15.11g) represents the displacement effect of the boundary layer. Ackeret's formula is found by solving the velocity potential in the outer inviscid and supersonic flow. The interested reader is referred to sub-Section 3.2 of Ruban (2015).

Finally we must write a boundary-condition for P describing the state of the flow downstream of the interaction region. We will assume that

$$P = P_S, \quad \text{at } X = \infty. \quad (15.11j)$$

The interaction problem (15.11a)-(15.11j) is now fully described, and it is valid under

condition (15.9). It is applicable for a perfect gas flow, and it remains valid whether the wall is adiabatic and/or if the wall temperature is kept constant, at least locally in the triple-deck region. In Chapter 17, we shall compute numerically the solution of this interaction problem for different values of U_w . These numerical results will allow us to confirm the validity of our analytical solution that we shall write in the limit (15.2). The set of equations (15.11a)-(15.11j) is written in terms of Cartesian coordinates, and this will be used to compute the numerical solutions of Chapter 17. However, for theoretical analysis in the limit (15.2), it is more convenient to write the set of equations (15.11a)-(15.11j) using von Mises variables (s, Ψ) (see Appendix B), where Ψ is the stream function, defined from the continuity equation (15.11b) as

$$U = \frac{\partial \Psi}{\partial Y}, \quad V = -\frac{\partial \Psi}{\partial X}. \quad (15.12)$$

There are several benefits when using these variables instead of Cartesian coordinates. Mainly, it will simplify the governing equations; in particular, the lateral component of the velocity V will vanish from the momentum equation (15.11a). Thus, if the pressure is known, the asymptotic expansion of the longitudinal velocity component U may be found directly from the momentum equation.

Using von Mises variables, the momentum and continuity equations, (15.11a) and (15.11b) respectively, can be written as

$$U \frac{\partial U}{\partial X} = -\frac{dP}{dX} + U \frac{\partial}{\partial \Psi} \left(U \frac{\partial U}{\partial \Psi} \right), \quad (15.13a)$$

$$\frac{\partial}{\partial \Psi} \left(\frac{V}{U} \right) = \frac{\partial}{\partial X} \left(\frac{1}{U} \right). \quad (15.13b)$$

They are solved in association with Ackeret's formula, which may be written as follows

$$P = P_S \mathcal{H}(X) + \lim_{\Psi \rightarrow \infty} \frac{V}{U}. \quad (15.13c)$$

They are subject to the no-slip and impermeability conditions (15.11c):

$$U = -U_w, \quad V = 0, \quad \text{at } \Psi = 0. \quad (15.13d)$$

The matching condition (15.11e) with the unperturbed flow, downstream or upstream

of the interaction region, in terms on von Mises variables is

$$U|_{X=\pm\infty} = \begin{cases} -\sqrt{2\Psi + U_w^2}, & \text{if } \Psi \in [0, -\frac{1}{2}U_w^2], \\ \sqrt{2\Psi + U_w^2}, & \text{if } \Psi > -\frac{1}{2}U_w^2. \end{cases} \quad (15.13e)$$

The matching condition at the outer edge of the viscous sublayer 1 (15.11f) is now written as

$$U = \sqrt{2\Psi} + \dots, \quad \text{as } \Psi \rightarrow \infty. \quad (15.13f)$$

The boundary condition (15.11j) remains unchanged.

The interaction problem (15.13a)-(15.13f) written in terms of von Mises variables is similar to the one formulated by Ruban et al. (2011) who also dealt with the case of the supersonic flow past a flat plate, where an oblique shock impinges upon the boundary layer. The difference is that they assumed that the shock moved in the upstream direction, which implies that when considering the flow in the reference frame moving with the shock, the plate would move in the downstream direction. Furthermore, they showed that separation was possible only when a shock wave whose scaled strength $P_s/U_w^2 = 1/2$ was hitting the boundary layer. The mathematical description of the separation process is somewhat different from that presented by Sychev, Vic. V. (1979, 1983, 1984, 1987) for the incompressible flow. However, the physical processes leading to the separation are rather similar. In our case, we considered the case of an upstream moving wall. We shall show that a shock wave does not trigger any separation: instead, and surprisingly enough, the boundary layer breaks away from the plate when subject to an expansion fan. As we discussed earlier, an analytical solution may be found in the limit as $Re^{-1/8} \ll \frac{\hat{V}_{sh}}{\hat{V}_\infty} \ll 1$. Our approach consists of assuming that $\bar{V}_{sh} = O(Re^{-1/8})$ first, which allowed us to write the interaction problem (15.13a)-(15.13f) applying the triple-deck theory. We recall that U_w and $\bar{V}_{sh}$ are related by

$$\bar{V}_{sh} = Re^{-1/8} \frac{\mu_w^{1/4} \rho_w^{-1/2}}{\lambda^{-1/4} \beta^{1/4}} U_w.$$

Thereafter, we will assume that U_w is large

$$U_w \rightarrow \infty. \quad (15.14)$$

This assumption will allow the limit (15.2) to be satisfied while the triple-deck model remains valid, according to the *Kaplan's Extension theorem*. This theorem may be sum-

marised as follows: perturbation methods often lead to asymptotic expansions which fail to remain uniformly valid in the region of interest. When trying to solve such problems, one applies singular perturbation methods. When using for example the method of matched asymptotic expansions, one defines an outer and an inner region, with associated outer and inner expansions, respectively. Let us suppose that the outer region is defined for a variable x such that $x = O(1)$, while the inner region is defined when the scaled variable $X = x/\varepsilon$ is of order one, where ε is a small parameter which stands for the thickness of the inner region. Even when the outer and inner expansions are known, there exists a region between the outer and inner region, called an overlap region, where we do not know the form of the asymptotic expansions. However, we may set a new scaled variable X_η which satisfies $X_\eta = x/\eta(\varepsilon)$, and in order to ensure that we are lying in the overlap region, we impose the following condition on η : $\varepsilon \ll \eta(\varepsilon) \ll 1$. Substitution of $x = \eta(\varepsilon)X_\eta$ into the outer and inner expansion of a function leads to the exact same asymptotic expansion. The important result here is that the inner and outer expansions are valid in wider regions than the ones that were initially intended.

In the limit (15.14), we shall show that the lower deck of the triple-deck model (region 1 on Figure 15.2) splits into two regions: an inviscid upper layer and a thinner viscous near-wall sublayer. Henceforth, they will be referred to as region 1a and region 1b, respectively (see Figure 15.3). We shall introduce them sequentially, starting with region 1a in Chapter 16, and continuing with region 1b in Chapter 18. We also recall that in Chapter 17, we shall get back to the interaction problem when $\bar{V}_{sh} = O(Re^{-1/8})$. In that case, $U_w = O(1)$. The reason why we get back to this assumption is to be able to compare the pressure found numerically by solving the interaction problem for different values of U_w with the analytic pressure found by investigating the inviscid region 1a which is valid as $U_w \rightarrow \infty$. We expect that as U_w becomes larger, the numerical pressure coincides more and more with the analytic pressure.

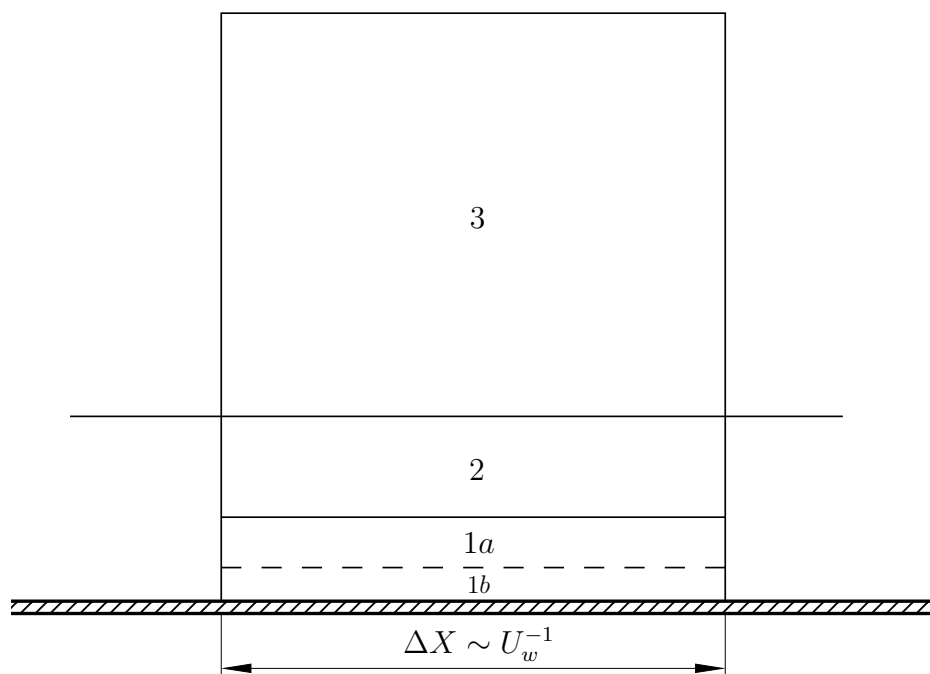


Figure 15.3: Splitting of region 1

Chapter 16

Flow analysis in the inviscid region

1a

In this chapter, we may investigate the lower deck of the triple-deck structure in the limit as

$$U_w \rightarrow \infty. \quad (16.1)$$

The form of the asymptotic solution of the interaction problem (15.13a)-(15.13f) in region 1a may be found by applying the principle of least degeneration. We start with the initial condition (15.13e). In order to avoid degeneration, we have to set

$$\Psi \sim U_w^2, \quad \text{and } U \sim U_w. \quad (16.2)$$

Since separation is a nonlinear process, the longitudinal velocity and its variation are in balance with one another, and the convective term must be of same order as the pressure gradient in the momentum equation (15.13a), leading to

$$P \sim U_w^2. \quad (16.3)$$

As usual, the scaling for the lateral velocity component V is found using the continuity equation. By balancing the terms on each side of (15.13b), we find that

$$V \sim \frac{U_w^2}{\Delta X}, \quad (16.4)$$

where ΔX is the longitudinal extent of the interaction region.

Substitution of (16.4) into the Ackeret's formula (15.13c), and balancing the term on the left-hand side with the second term of the right-hand side leads to

$$\Delta X \sim \frac{1}{U_w}. \quad (16.5)$$

Substituting (16.5) back into (16.4) gives

$$V \sim U_w^3. \quad (16.6)$$

This suggests that, in region 1a, when U_w is large, the asymptotic solution of the interaction problem (15.13a)-(15.13f) may be sought in the following form

$$\left. \begin{aligned} X &= \frac{1}{U_w} X_*, & \Psi &= U_w^2 \Psi_*, \\ U &= U_w U_* + \dots, & V &= U_w^3 V_* + \dots, & P &= U_w^2 P_* + \dots, \end{aligned} \right\} \text{ as } U_w \rightarrow \infty. \quad (16.7)$$

Substitution of the scaled variables and the asymptotic expansions (16.7) into the interaction problem (15.13a)-(15.13f) leads in the leading order approximation to the following set of equations

$$U_* \frac{\partial U_*}{\partial X_*} = -\frac{dP_*}{dX_*}, \quad (16.8a)$$

$$\frac{\partial}{\partial \Psi_*} \left(\frac{V_*}{U_*} \right) = \frac{\partial}{\partial X_*} \left(\frac{1}{U_*} \right), \quad (16.8b)$$

$$P_* = \bar{P}_S \mathcal{H}(X_*) + \lim_{\Psi_* \rightarrow \infty} \frac{V_*}{U_*}. \quad (16.8c)$$

These equations are solved with the following boundary conditions

$$V_* = 0, \quad \text{at } \Psi_* = 0, \quad (16.8d)$$

$$U_* = \begin{cases} -\sqrt{2\Psi_* + 1}, & \text{if } \Psi_* \in [0, -\frac{1}{2}], \\ \sqrt{2\Psi_* + 1}, & \text{if } \Psi_* > -\frac{1}{2}, \end{cases} \quad \text{at } X_* = \infty, \quad (16.8e)$$

$$U_* = \sqrt{2\Psi_*} + \dots, \quad \text{as } \Psi_* \rightarrow \infty, \quad (16.8f)$$

$$P_* = \bar{P}_S, \quad \text{at } X_* = \infty. \quad (16.8g)$$

Here, we have introduced the quantity $\bar{P}_S$, referred to the scaled strength of the shock/expansion fan, such that

$$P_S = U_w^2 \bar{P}_S, \quad (16.9)$$

with $\bar{P}_S$ being an order one parameter.

We notice that the viscous terms were neglected in the momentum equation (16.8a), implying that the flow can be treated as inviscid in the leading order approximation in

region 1a. Hence, the no-slip condition cannot be satisfied on the plate surface; only the impermeability condition $V_* = 0$ holds at $\Psi_* = 0$. This means that one needs to introduce thereafter a viscous sublayer, called region 1b, in the direct neighbourhood of the moving plate, so that the no-slip condition can be satisfied. This region will smooth out the solution when approaching the plate surface. Furthermore, the far-field attenuation (15.13e) cannot be satisfied both upstream and downstream of the interaction region. We shall assume henceforward that the longitudinal velocity is undisturbed in the flow downstream of the interaction region, as stated in the boundary condition (16.8e). This means that no additional boundary layer is developing downstream and that the flow follows the Blasius solution. However, upstream of the interaction region, we cannot draw conclusions about the flow behaviour. For additional details of this issue, the reader is referred to Kirsten (2018).

We shall now find analytically the solution of the problem (16.8a)-(16.8g). Firstly, integration of the momentum equation (16.8a) with respect to X_* yields the Bernoulli equation, such that the quantity

$$H_*(\Psi_*) = \frac{1}{2}U_*^2 + P_* \quad (16.10)$$

is constant along each streamline. The far-field attenuation (16.8e) and the boundary condition (16.8g) allow us to obtain the form of H_* by setting $X_* = \infty$ in (16.10)

$$H_*(\Psi_*) = \Psi_* + \frac{1}{2} + \bar{P}_S. \quad (16.11)$$

Substitution of (16.11) into (16.10) and solving the resulting equation for U_* gives

$$U_* = \begin{cases} -\sqrt{2\Psi_* - 2P_* + 2\left(\frac{1}{2} + \bar{P}_S\right)}, & \text{for } \Psi_* \in [0, P_* - \left(\frac{1}{2} + \bar{P}_S\right)], \\ \sqrt{2\Psi_* - 2P_* + 2\left(\frac{1}{2} + \bar{P}_S\right)}, & \text{for } \Psi_* > P_* - \left(\frac{1}{2} + \bar{P}_S\right). \end{cases} \quad (16.12)$$

It remains to find the pressure distribution P_* . Integration of the continuity equation (16.8b) with the initial condition (16.8d) gives

$$\frac{V_*}{U_*} = \int_0^{\Psi_*} \frac{\partial}{\partial X_*} \left(\frac{1}{U_*} \right) d\Psi_*. \quad (16.13)$$

We must distinguish between two cases, depending on the sign of U_* in (16.12). We start with the first interval $\Psi_* \in [0, P_* - \left(\frac{1}{2} + \bar{P}_S\right)]$. Then U_* is negative, and it follows

that

$$\frac{V_*}{U_*} = \frac{dP_*}{dX_*} \left[\frac{1}{\sqrt{2\Psi_* - 2P_* + 2\left(\frac{1}{2} + \bar{P}_S\right)}} - \frac{1}{\sqrt{-2P_* + 2\left(\frac{1}{2} + \bar{P}_S\right)}} \right]. \quad (16.14)$$

At the position of the flow reversal where $U_* = 0$, the first term on the right-hand side of (16.14) vanishes, leading to

$$\frac{V_*}{U_*} = -\frac{dP_*}{dX_*} \frac{1}{\sqrt{-2P_* + 2\left(\frac{1}{2} + \bar{P}_S\right)}}. \quad (16.15)$$

Now, we assume that $\Psi_* > P_* - \left(\frac{1}{2} + \bar{P}_S\right)$. Then U_* is positive, and we have

$$\frac{V_*}{U_*} = -\frac{dP_*}{dX_*} \left(\frac{1}{\sqrt{2\Psi_* - 2P_* + 2\left(\frac{1}{2} + \bar{P}_S\right)}} + \frac{1}{\sqrt{-2P_* + 2\left(\frac{1}{2} + \bar{P}_S\right)}} \right). \quad (16.16)$$

We set $\Psi_* = \infty$ in (16.16) and we substitute the result into Ackeret's formula (16.8c), leading to the following equation for P_* :

$$P_* = \bar{P}_S \mathcal{H}(X_*) - \frac{1}{\sqrt{-2P_* + 2\left(\frac{1}{2} + \bar{P}_S\right)}} \frac{dP_*}{dX_*}. \quad (16.17)$$

The general solution of (16.17) can be written as

$$P_* = \begin{cases} 0, & \text{if } X_* < 0, \\ \frac{1}{2} + \bar{P}_S - \frac{1}{2} \left(\frac{1 - Ce^{-X_*}}{1 + Ce^{-X_*}} \right)^2, & \text{if } X_* > 0, \end{cases} \quad (16.18)$$

where C is a constant. Since P_* is continuous everywhere, in particular at $X_* = 0$, we have to set

$$C = \frac{1 - \sqrt{1 + 2\bar{P}_S}}{1 + \sqrt{1 + 2\bar{P}_S}}. \quad (16.19)$$

The direction of variation of the pressure distribution P_* on the interval $X_* \in [0, \infty)$ depends on the value of the constant C . Two cases must be considered according to the sign of $\bar{P}_S$.

In the case of an impinging shock, $\bar{P}_S$ is positive, and it follows from (16.19) that

$C \in [-1, 0]$. Therefore, it follows from (16.18) that P_* undergoes a monotonic growth, and that the flow in the viscous sublayer 1b (see Figure 15.3) moving in the negative X_* -direction, is subject to a favourable (negative) pressure gradient. The separation of the boundary layer cannot be expected under this condition. On the other hand, if we consider the case of an expansion fan, then $\bar{P}_S$ is negative (but it cannot be less than $-1/2$ due to the square root in (16.19)) and $C \in]0, 1]$. Hence, (16.18) shows that P_* monotonically decreases on $X_* \in [0, \infty)$. The flow in region 1b appears to be under the action of an adverse (positive) pressure gradient as it moves in the direction opposite to the X_* -axis. Similarly to the case of the downstream moving wall Ruban et al. (2011), a singularity is expected to develop in region 1b when the velocity $U_e(X_*)$ at its outer edge becomes equal to zero.

The expression for $U_e(X_*)$ may be found by setting $\Psi_* = 0$ in the solution (16.12) for U_* in region 1a. We substitute P_* from (16.18), which gives for positive X_*

$$U_e(X_*) = -\frac{1 - Ce^{-X_*}}{1 + Ce^{-X_*}}. \quad (16.20)$$

Consequently, when dealing with an expansion fan impinging upon the boundary layer, $U_e(X_*)$ first turns zero at $X_* = 0$ if we assume that the strength of the fan satisfies

$$\bar{P}_S = -\frac{1}{2}. \quad (16.21)$$

In that case, $C = 1$ and U_e may be written in the following way

$$U_e(X_*) = \begin{cases} 0, & \text{for } X_* < 0 \\ -\frac{1 - e^{-X_*}}{1 + e^{-X_*}}, & \text{for } X_* \geq 0. \end{cases} \quad (16.22)$$

We notice that $U_e(X_*)$ is negative for all $X_* \geq 0$.

Finally, setting $\bar{P}_S = -1/2$ in (16.11), we see that $H_*(0) = 0$, and then it follows from (16.10) that

$$P_*(X_*) = -\frac{1}{2}U_e^2(X_*), \quad \forall X_*. \quad (16.23)$$

This allows us to obtain an analytical expression for the pressure distribution in the

inviscid region 1a:

$$P_* = \begin{cases} 0, & \text{if } X_* < 0, \\ -\frac{1}{2} \left(\frac{1 - e^{-X_*}}{1 + e^{-X_*}} \right)^2, & \text{if } X_* > 0. \end{cases} \quad (16.24)$$

The pressure P_* is continuous at $X_* = 0$, and it is equal to 0. For $X_* \geq 0$, it decreases with X_* .

This completes the flow analysis in region 1a in terms of von Mises variables. Since the flow in this region is inviscid, we shall add in Chapter 18 a viscous sublayer, called region 1b, where the viscous and convective terms are in balance with one another. In the case where the scaled strength of the expansion fan is $\bar{P}_S = -1/2$, it will be shown that the streamline slope V/U at the outer edge of region 1b develops a singularity at $X_* = 0$.

Before proceeding to the next chapter, it is interesting to observe the behaviour of the solution U_* in Cartesian coordinates X_* and Y_* , where Y_* corresponds to the direction orthogonal to the plate surface in region 1a. According to the initial boundary condition (15.11e), Y_* is defined such that $Y = U_w Y_*$. Let us convert the longitudinal velocity U_* in region 1a into Cartesian coordinates. By considering that $\bar{P}_S = -1/2$ and that P_* and $U_e(X_*)$ are linked by equation (16.23), the solution (16.12) for U_* may be rewritten as follows

$$U_* = \begin{cases} -\sqrt{2\Psi_* + U_e^2(X_*)}, & \text{for } \Psi_* \in \left[0, -\frac{1}{2}U_e^2(X_*)\right], \\ \sqrt{2\Psi_* + U_e^2(X_*)}, & \text{for } \Psi_* > -\frac{1}{2}U_e^2(X_*). \end{cases} \quad (16.25)$$

We recall that the stream function Ψ is defined by (15.12), and using the equations (16.7) and the relation $Y = U_w Y_*$, we deduce that U_* and Ψ_* are related by

$$U_* = \frac{\partial \Psi_*}{\partial Y_*}. \quad (16.26)$$

Two cases must be considered when replacing U_* in (16.26), depending on the value of Ψ_* . Nevertheless, in both cases, substitution of (16.25) into (16.26) and integrating

yields to

$$\Psi_* = \frac{1}{2} \left(\left(Y_* + U_e(X_*) \right)^2 - U_e^2(X_*) \right). \quad (16.27)$$

Furthermore, when $\Psi_* \in [0, -\frac{1}{2}U_e^2(X_*)]$, the corresponding interval for Y_* is $Y_* \in [0, -U_e(X_*)]$, while when Ψ_* belongs to the second interval $\Psi_* > -\frac{1}{2}U_e^2(X_*)$, we have $Y_* > -U_e(X_*)$. Taking this into account, and substituting (16.27) into (16.25), we deduce that the expression of the longitudinal velocity U_* , which is valid everywhere in region 1a, has the following simple form in Cartesian coordinates

$$U_* = Y_* + U_e(X_*). \quad (16.28)$$

The lateral velocity component $V_*(X_*, Y_*)$ can be deduced by substituting (16.28) into the continuity equation written in Cartesian coordinates and valid in region 1a:

$$\frac{\partial U_*}{\partial X_*} + \frac{\partial V_*}{\partial Y_*} = 0.$$

Taking into account that the impermeability condition must hold on the plate surface, we can deduce that

$$V_* = -U'_e(X_*)Y_*. \quad (16.29)$$

Let us notice that combining (15.10) and (16.7), we obtain the following order of magnitudes for the dimensionless longitudinal and lateral velocity components in region 1a:

$$\begin{aligned} u &\sim Re^{-1/8}U_w, \\ v &\sim Re^{-3/8}U_w^3. \end{aligned}$$

Thus, under assumption (16.1) (coming from the condition $Re^{-1/8} \ll \bar{V}_{sh} \ll 1$ as discussed earlier), it results that

$$v \ll u, \quad \text{in region 1a.} \quad (16.30)$$

We sketched in Figure 16.1 the velocity profile U_* for different values of positive X_* . For a fixed X_* , the velocity U_* varies linearly in Y_* ; it is negative at $Y_* = 0$ (except for $X_* \leq 0$ where it is always equal to 0), until Y_* reaches the value $-U_e(X_*)$ where U_* becomes nought. For all $Y_* > -U_e(X_*)$, U_* is strictly positive. In Figure 16.1, at

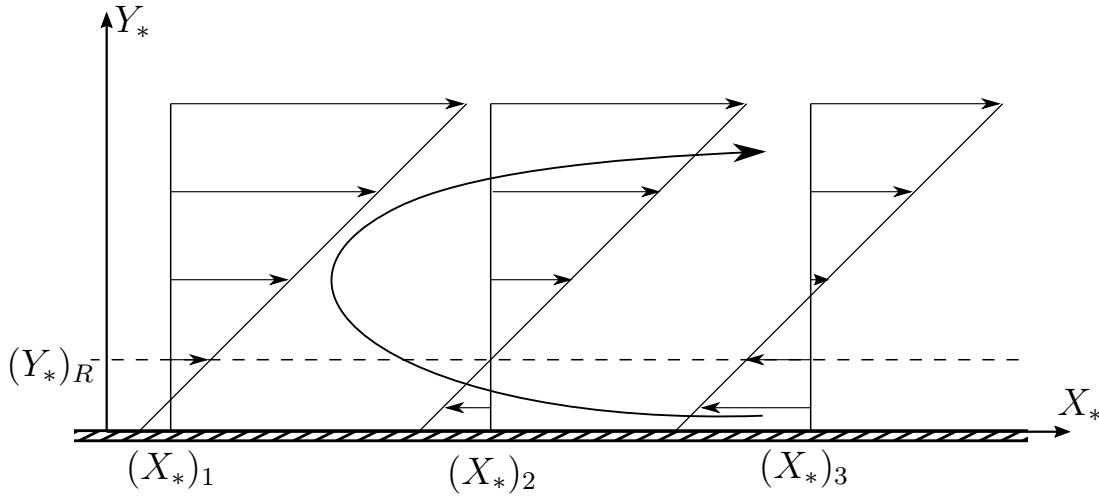


Figure 16.1: The longitudinal velocity profile U_* for different values of positive X_* .

$Y_* = (Y_*)_R$, the velocity U_* is strictly negative at $(X_*)_3$, then it turns zero at $(X_*)_2$, and it is strictly positive at $(X_*)_1$. Therefore, the fluid particles near the plate surface which are moving upstream of the flow, stop at some point and are redirected downstream of the flow further away from the body surface, as shown by the redirection arrow in Figure 16.1. Obviously, the no-slip condition is not satisfied at $Y_* = 0$ as the flow in region 1a is inviscid in the leading order approximation. The introduction of a viscous sublayer 1b will remedy this issue.

In the next chapter, we shall go back to the interaction problem (15.13a)-(15.13f), valid when $U_w = O(1)$, and we will solve it numerically for different values of U_w . When $U_w = O(1)$, neither the splitting of the lower deck of the triple-deck structure applies, nor the scaling (16.7). The goal of Chapter 17 is twofold: firstly, we wish to compare the pressure (16.24) we found analytically by studying region 1a with the pressure obtained by solving the interaction problem (15.13a)-(15.13f). We expect the latter to coincide more and more with (16.24) as U_w increases. The other goal of Chapter 17 is to confirm the prediction of Figure 19.2; this figure will show us the streamline pattern in the region added to smooth out the singularity of the streamline slope in region 1b. This singularity will be highlighted in Chapter 18.

Chapter 17

Numerical solution of the triple-deck problem

In this chapter, we go back to the interaction problem (15.11a)-(15.11j) where we assume that $U_w = O(1)$, that is, $\bar{V}_{sh} = O(Re^{-1/8})$. We are interested in solving it numerically for different values of U_w by making use of the technique developed by Kravtsova et al. (2005). This is based on Newton-Raphson method applied to the governing set of equations (15.11a)-(15.11j) which have been initially discretised using finite difference methods and a uniform finite grid $\{X_i\}$ where $i \in \llbracket 1; N \rrbracket$. The pressure equation is given by the Hilbert integral from the linear theory:

$$\frac{dP}{dX} = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{A''(s)}{s - X} ds. \quad (17.1)$$

Thus, for a A_i given, where A_i is the displacement function at mesh point i , one can compute the inviscid pressure gradient from (17.1) and denoted as $\left. \frac{dP}{dX} \right|_{\text{inv}}(X_i)$. On the other hand, we can also find a viscous pressure gradient by solving the boundary-layer equations which are valid in the lower deck of the triple-deck model. This pressure gradient is denoted as $\left. \frac{dP}{dX} \right|_{\text{v}}(X_i)$.

The solution is constructed by using marching methods. At each mesh point i , our task is to find a function A_i which satisfies $|F_k(A_i)| < \varepsilon$, where ε is a small parameter, and where $F_k(A_i)$ satisfies

$$F_k(A_i) = \left. \frac{dP}{dX} \right|_{\text{inv}}(X_i) - \left. \frac{dP}{dX} \right|_{\text{v}}(X_i), \quad k \in \llbracket 1; N \rrbracket. \quad (17.2)$$

With the exact solution, $F_k = 0$ as we recall that the pressure does not change across the triple-deck structure. The approximated solution A_i is found by using the Newton-Raphson method: function F_k is Taylor expanded around a computed approximation of A_i , denoted as A_i^- , and the linear terms are kept and we set the result equal to zero.

Then the resulting set of equations is solved using a two steps procedure (where we introduced $\delta A_i = A_i - A_i^-$):

$$\text{We solve } F_k(A_i^-) + \frac{\partial F_k}{\partial A_i}(A_i^-)\delta A_i = 0, \text{ with respect to } \delta A_i,$$

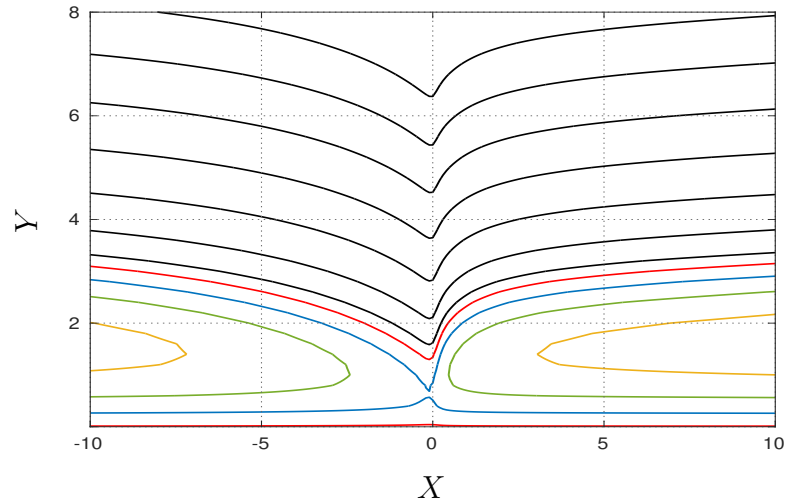
$$\text{We update } A_i \leftarrow A_i^- + \delta A_i.$$

The set of equations is solved at each new iteration until the condition $|F(A_i)| < \varepsilon$ is satisfied. Usually, we have between 12 and 15 iterations before the code converges with the accuracy $\varepsilon = 10^{-6}$. An initial guess is necessary for the displacement function A when solving the problem. Since this method is sensitive to the initial guess, as usual when employing the iterative Newton's method, we could not compute directly the case for $U_w = 1.0$. Instead, the calculations were performed with a small value of the speed of the plate U_w , and then we increased it gradually. We started the calculations for the case $U_w = 0.1$ with an initial guess for the displacement function $A = 0$. The results are presented in Figures 17.1 and 17.2. The strength of the expansion fan P_S was chosen such that

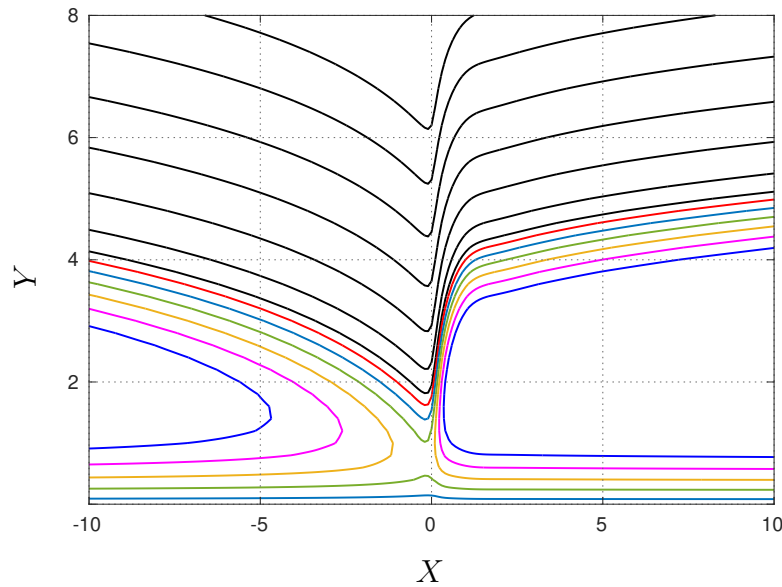
$$P_S = -\frac{1}{2}U_w^2. \quad (17.3)$$

Figures 17.1 show the streamline pattern of the flow for $X \in [-10, 10]$ for $U_w = 2.0$ and $U_w = 3.0$. We can see in both figures that two recirculation regions are developing upstream and downstream of the impinging expansion fan; we shall term them as the secondary and primary eddy, respectively. We can notice that the primary eddy becomes larger when U_w increases, while the secondary eddy becomes slightly smaller. The development of the primary eddy may be explained as follows. Close to the plate surface, the fluid particles moves in the negative X -direction due to the viscous forces and the movement of the plate. When they encounter a growing pressure, they start to decelerate; the fluid particles in the direct neighbourhood of the plate surface have enough kinetic energy to overcome the growing pressure, and keep moving in the downstream direction. However, according to the matching condition (15.11e), the longitudinal velocity increases when moving away from the surface of the plate when X is large. Therefore, there are always fluid particles with smaller initial velocity, and the smaller the initial velocity is, the earlier they will stop and turn back to form the primary eddy.

Regarding the secondary eddy, we can notice in Figures 17.1 that the fluid particles far from the plate keep traveling in the positive X -direction. Nevertheless, the ones



(a) $U_w = 2.0$. The streamlines shown in black correspond to the following values of the stream function: 0.001; 1; 3; 6; 10; 15; 20, going from the inner part to the outer part. For the coloured lines these are: -0.4 ; -0.8 ; -1.2 ; -1.6 , going from the red one to the orange one.



(b) $U_w = 3.0$. The streamlines shown in black correspond to the following values of the stream function: 0.001; 1; 3; 6; 10; 15; 20, from the inner part to the outer part. For the coloured lines these are: -0.4 ; -0.8 ; -1.2 ; -1.6 ; -2 ; -2.4 , starting from the red one to the dark blue one.

Figure 17.1: The streamline pattern in the boundary layer exposed to an impinging expansion fan

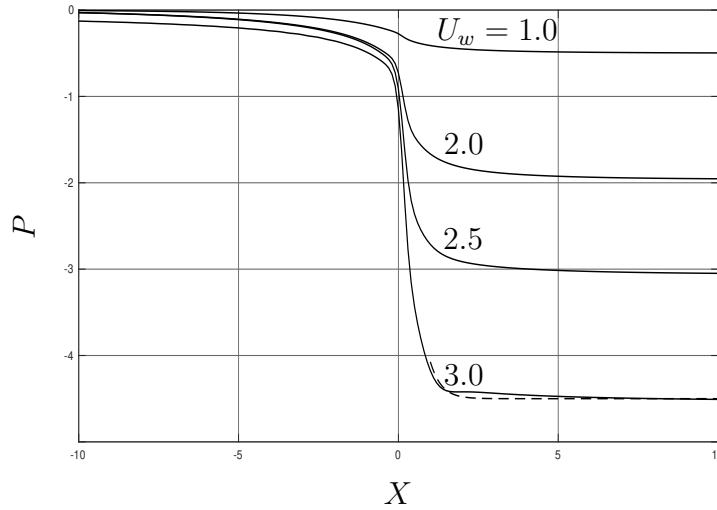


Figure 17.2: Pressure variation in the interaction region. Dashed line shows the theoretical solution (16.24)

closer to the plate surface are slowed down when approaching the impinging expansion fan, and they are redirected back in the negative X -direction, despite being subject to a favourable pressure gradient. This phenomenon can be explained by noticing that the numerical results showed that the viscous forces were particularly important in the secondary eddy before its “collision” with the primary eddy. When $U_w \rightarrow \infty$, as we showed in the previous chapter that the flow is inviscid in the leading-order approximation, we expect the secondary eddy to vanish in the limit as $U_w \rightarrow \infty$.

Figure 17.2 shows the pressure distribution in the interaction region. As expected from equation (17.3), the drop in the pressure at $X = 0$ is more important when U_w becomes larger; a discontinuity starts to appear in the pressure gradient as U_w becomes larger, but not in the pressure. The dashed line corresponds to the analytic pressure (16.24) that we found by assuming that $U_w \rightarrow \infty$. Obviously, the analytic pressure plotted through the dashed line has been rescaled according to (16.7), such that $X = \frac{1}{U_w} X_*$ and $P = U_w^2 P_*$, so that the comparison with the pressure obtained numerically can be performed on the same scale. As we can see in Figure 17.2, the plot of the numerical pressure for $U_w = 3.0$ coincides well with the dashed line. The pressure tends to -4.5 when X becomes “large” positive. This is consistent with equation (16.24) where the rescaled P tends to $-U_w^2/2 = -4.5$. We have not kept plotting the dashed line when X approaches 0^+ as this is not relevant; indeed, the dashed line does not coincide well with the plot of the numerical pressure anymore. However, this is to be expected as

the comparison is performed for $U_w = 3.0$. We would need much larger values of U_w in order to have a good matching between the two curves, as we recall that (16.24) is valid when $U_w \rightarrow \infty$.

Solving numerically the interaction problem (15.11a)-(15.11j) in this chapter allowed us to confirm that the analytic pressure found when assuming $U_w \rightarrow \infty$ was consistent with the numerical results as U_w increases. The streamline patterns in Figure 17.1 will be useful to deduce the topology of the flow in an additional region on a smaller scale in Chapter 19. More details are given about this region at the beginning of the next chapter, where we will return back to the case $U_w \rightarrow \infty$, and we shall investigate the viscous sublayer 1b, introduced by applying the Prandtl's hierarchical approach so that the no-slip condition can be satisfied on the plate surface.

Chapter 18

Flow analysis in the viscous region 1b

In this chapter, we return back to the study of the interaction problem (15.13a)-(15.13f) in the limit as $U_w \rightarrow \infty$. As stated in Chapter 16, we must introduce a viscous sublayer adjacent to the plate surface so that the no-slip condition (15.13d) can be satisfied; see Figure 15.3. Our goal is to find the asymptotic expansions of the sought flow functions as $X_* \rightarrow 0^+$. This point is of interest as we recall that we set the scaled strength of the expansion fan equal to $-1/2$ in Chapter 16. Under this condition, we showed that the velocity U_e at the outer edge of the viscous sublayer 1b becomes zero at $X_* = 0$. We shall prove in Section 18.3 that the fact that $U_e(X_* = 0) = 0$ leads to a singularity in the streamline slope V/U of region 1b at $X = 0$. We will add a new region in Chapter 19, called region C , so that the singularity can be smoothed out. The streamline pattern of the flow in this region will be deduced from the asymptotic expansions of the flow functions in regions 1a and 1b. Moreover, the numerical results obtained in Chapter 17 will confirm our prediction of the topology of the flow in region C .

18.1 Governing equations in the viscous region 1b

The longitudinal extent of region 1b is the same as that of region 1a. Since we want to perform the matching with region 1a, the scaling for U in region 1b is the same as for the inviscid region 1a. The aim of the introduction of region 1b is to satisfy the no-slip condition. Consequently, the convective term must be in balance with the viscous one in the momentum equation (15.13a). Since $\Delta X \sim 1/U_w$ and $U \sim U_w$, it is easily found that in region 1b,

$$\Psi = O(1). \quad (18.1)$$

The order of magnitude of the lateral velocity V is found from the continuity equation (15.13b):

$$V \sim U_w. \quad (18.2)$$

Finally, the pressure does not change across region 1b, and it remains the same as in region 1a.

Hence, the asymptotic expansions of the sought flow functions in region 1b have the form

$$\left. \begin{aligned} X &= \frac{1}{U_w} X_*, \\ U &= U_w \tilde{U} + \dots, \quad V = U_w \tilde{V} + \dots, \quad P = U_w^2 P_* + \dots, \end{aligned} \right\} \text{ as } U_w \rightarrow \infty. \quad (18.3)$$

Substitution of (18.3) into the momentum and continuity equations (15.13a)-(15.13b) gives

$$\tilde{U} \frac{\partial \tilde{U}}{\partial X_*} - U_e \frac{dU_e}{dX_*} = \tilde{U} \frac{\partial}{\partial \Psi} \left(\tilde{U} \frac{\partial \tilde{U}}{\partial \Psi} \right), \quad (18.4a)$$

$$\frac{\partial}{\partial \Psi} \left(\frac{\tilde{V}}{\tilde{U}} \right) = \frac{\partial}{\partial X_*} \left(\frac{1}{\tilde{U}} \right). \quad (18.4b)$$

The pressure gradient has been written in terms of U_e by using the Bernoulli relation (16.23). The governing equations (18.4a) and (18.4b) may be solved with the following boundary conditions

$$\tilde{U} = -1, \quad \tilde{V} = 0, \quad \text{at } \Psi = 0, \quad (18.4c)$$

$$\tilde{U} = -1, \quad \text{at } X = \infty, \quad (18.4d)$$

$$\tilde{U} = U_e(X_*), \quad \text{at } \Psi = -\infty. \quad (18.4e)$$

Let us notice that the outer edge of region 1b corresponds to when Ψ is large and negative (and not positive) because the matching condition for the longitudinal velocity gives

$$U_e(X_*) = \left. \frac{\partial \Psi}{\partial \tilde{Y}} \right|_{\tilde{Y}=\infty},$$

where $\tilde{Y} = U_w Y$.

Our goal is to study the asymptotic solution of the interaction problem (18.4a)-(18.4e) at the outer edge of region 1b, in the vicinity of $X_* = 0$, that is in the neighbourhood of where $U_e(X_*)$ becomes nought. We will see in Section 18.3 that the streamline slope of region 1b is singular at $X_* = 0$ in that case. In order to find the asymptotic

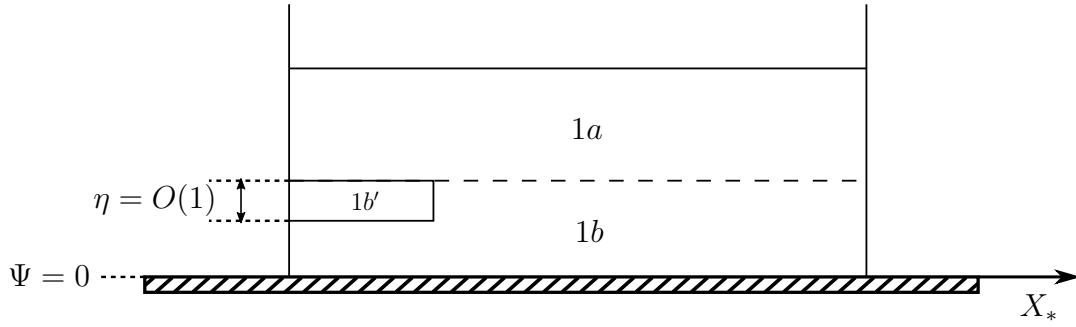


Figure 18.1: Formation of region $1b'$. The variable η is of order one in region $1b'$ in the limits as $\Psi \rightarrow -\infty$ and $X_* \rightarrow 0^+$. It will be introduced in Section 18.2.

solution in this area, we shall firstly predict the form of the asymptotic expansions of the solution in region $1b$ when X_* is large and positive, while Ψ remains of order one. This will allow us to predict the form of the solution in the limit as $\Psi \rightarrow -\infty$ and $X_* = O(1)$. Finally, we will make X_* small by using a double limit procedure; the introduction of an additional region lying at the outer edge of region $1b$ when $X_* \rightarrow 0^+$ will be required. We shall call it region $1b'$; see Figure 18.1. The reason why we calculate the solution for large values of X_* beforehand, is because this is where the boundary layer is formed. The asymptotic expansion as X_* approaches 0^+ is deduced from the one when X_* is large. If we had determined straight away the expansion as $X_* \rightarrow 0^+$, the resulting asymptotic expansion would have been wrong.

Let us start with the analysis of the flow in region $1b$ in the limit as

$$X_* \rightarrow \infty, \quad \Psi = O(1). \quad (18.5)$$

The velocity $U_e(X_*)$ at the outer edge of region $1b$ is given by (16.22), and it is easily found from it that

$$U_e(X_*) = -1 + 2e^{-X_*} + \dots, \quad \text{as } X_* \rightarrow \infty. \quad (18.6)$$

This allows us to predict the form of the asymptotic expansion of the longitudinal velocity $\tilde{U}$ in the limit (18.5):

$$\tilde{U} = -1 + f(\Psi)e^{-X_*} + \dots, \quad \text{as } X_* \rightarrow \infty, \quad \Psi = O(1), \quad (18.7)$$

where f is a function to be determined such that $f(\Psi) \rightarrow 2$ as $\Psi \rightarrow -\infty$.

In order to find function f , we substitute (18.7) into the momentum equation (18.4a), and in the leading order approximation it reduces to an ordinary linear differential equation of second-order

$$f''(\Psi) - f(\Psi) = -2. \quad (18.8)$$

The boundary conditions for this equation are found from (18.4c) and (18.4e) to be

$$f(0) = 0, \quad f(-\infty) = 2. \quad (18.9)$$

Solving the boundary-value problem (18.8)-(18.9) show that

$$f(\Psi) = 2(1 - e^\Psi). \quad (18.10)$$

Hence, the asymptotic expansion of $\tilde{U}$ in the limit (18.5) may be written as

$$\tilde{U} = -1 + 2(1 - e^\Psi)e^{-X_*} + \dots, \quad \text{as } X_* \rightarrow \infty, \quad \Psi = O(1) \quad (18.11)$$

Let us now consider the limit

$$\Psi \rightarrow -\infty, \quad X_* = O(1). \quad (18.12)$$

This places us at the outer edge of region 1b. At this stage, it is convenient to rewrite the Bernoulli equation by replacing P_* using equation (16.23):

$$\tilde{H} = \frac{1}{2}\tilde{U}^2 - \frac{1}{2}U_e^2(X_*). \quad (18.13)$$

We substitute (18.13) into the momentum equation (18.4a), leading to

$$\frac{\partial \tilde{H}}{\partial X_*} = \tilde{U} \frac{\partial^2 \tilde{H}}{\partial \Psi^2}. \quad (18.14)$$

Substituting (18.11) and (18.6) into the modified Bernoulli equation (18.13), we find that

$$\tilde{H} = 2e^{\Psi-X_*} + \dots, \quad \text{as } X_* \rightarrow \infty. \quad (18.15)$$

Therefore, we look for an asymptotic expansion of $\tilde{H}$ in the limit (18.12) in the following form

$$\tilde{H} = h(X_*)e^\Psi + \dots, \quad \text{as } \Psi \rightarrow -\infty, \quad (18.16)$$

where h is a function to be found such that

$$h(X_*) = 2e^{-X_*} + \dots, \quad \text{as } X_* \rightarrow \infty. \quad (18.17)$$

Substitution of (18.16) into (18.14) yields, in the leading order approximation, the following ordinary differential equation of first-order in the unknown h which is valid for $X_* \in [0, \infty)$:

$$h'(X_*) + \frac{1 - e^{-X_*}}{1 + e^{-X_*}} h(X_*) = 0. \quad (18.18)$$

The solution of the boundary-value problem (18.18)-(18.17) may be written as

$$h(X_*) = \frac{2}{(e^{X_*/2} + e^{-X_*/2})^2}, \quad \text{for } X_* \in [0, \infty). \quad (18.19)$$

Thus, the modified Bernoulli function may be written in the limit (18.12) as

$$\tilde{H} = \frac{2e^\Psi}{(e^{X_*/2} + e^{-X_*/2})^2} + \dots, \quad \text{as } \Psi \rightarrow -\infty, \quad X_* \in [0, \infty). \quad (18.20)$$

We may substitute it back into (18.13), and solving it with respect to $\tilde{U}$, we find that

$$\tilde{U} = -\sqrt{U_e^2(X_*) + \frac{4e^\Psi}{(e^{X_*/2} + e^{-X_*/2})^2}} + \dots, \quad \text{as } \Psi \rightarrow -\infty, \quad X_* \in [0, \infty). \quad (18.21)$$

18.2 Region 1b'

We are now interested in the behaviour of (18.21) when X_* becomes small. The Taylor expansion of $U_e(X_*)$ around $X_* = 0^+$ may be written as

$$U_e(X_*) = -\frac{1}{2}X_* + O(X_*^3), \quad \text{as } X_* \rightarrow 0^+. \quad (18.22)$$

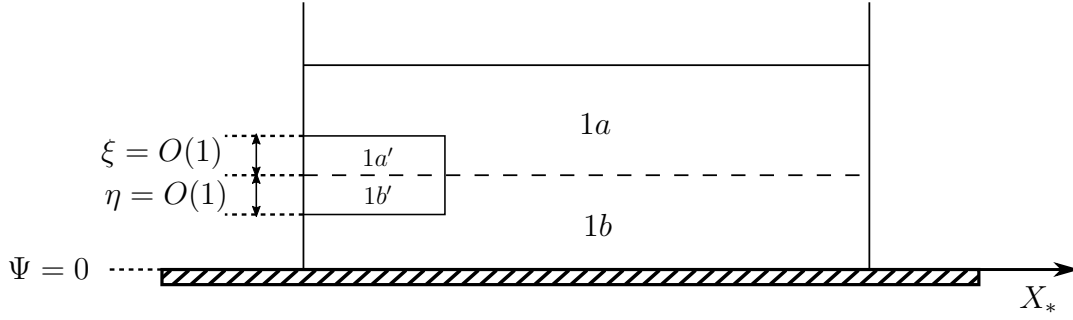


Figure 18.2: Formation of regions 1a' and 1b'.

In this limit, (18.21) is reduced to

$$\tilde{U} = -\sqrt{\frac{1}{4}X_*^2 + e^\Psi + \dots}, \quad \text{as } \Psi \rightarrow -\infty, X_* \rightarrow 0^+. \quad (18.23)$$

We have used a double limit procedure to deduce the asymptotic expansion (18.23): firstly, we wrote $\tilde{U}$ by assuming that Ψ tends to minus infinity while X_* remains of order one. Thereafter, we made X_* small. The asymptotic expansion (18.23) remains valid as long as the second term in the square root is small as compared to the first term. However, there exists a small region lying at the outer edge of region 1b and when X_* tends to 0^+ , where both terms become of same order; we shall call it region 1b' (see Figure 18.2). Thus, we introduce a function Φ , such that in region 1b'

$$\Phi(\eta) = \frac{e^\Psi}{X_*^2} = O(1), \quad \text{as } \Psi \rightarrow -\infty, X_* \rightarrow 0^+, \quad (18.24)$$

where Φ is an arbitrary function of the independent variable η . For convenience, we choose $\Phi(\eta) = e^\eta$, which leads to

$$\eta = \Psi - 2 \ln(X_*). \quad (18.25)$$

Substitution of (18.25) into (18.23) shows that, approaching region 1b', $\tilde{U}$ has the form

$$\tilde{U} = -X_* \sqrt{\frac{1}{4} + e^\eta + \dots}. \quad (18.26)$$

Hence, we look for $\tilde{U}$ in region 1b' in the following form

$$\tilde{U} = -X_* F(\eta) + \dots, \quad \text{as } X_* \rightarrow 0^+, \quad (18.27)$$

where F is a function to determine, such that

$$F(\eta) = \sqrt{\frac{1}{4} + e^\eta} + \dots, \quad \text{as } \eta \rightarrow -\infty. \quad (18.28)$$

The function F may be found by substituting (18.27) and (18.22) into the momentum equation (18.4a):

$$\tilde{U} \frac{\partial \tilde{U}}{\partial X_*} = -X_* F(\eta) \left(-F(\eta) + 2F'(\eta) \right) + \dots, \quad (18.29a)$$

$$U_e \frac{dU_e(X_*)}{dX_*} = \frac{1}{4} X_* + \dots, \quad (18.29b)$$

$$\tilde{U} \frac{\partial}{\partial \Psi} \left(\tilde{U} \frac{\partial \tilde{U}}{\partial \Psi} \right) = -X_*^3 F(\eta) \left(F(\eta) F'(\eta) \right)' + \dots. \quad (18.29c)$$

We may notice that the viscous term can be disregarded in the leading order approximation, which implies that the flow in region 1b' can be considered as inviscid. Retaining the $O(X_*)$ terms provides a nonlinear first-order ordinary differential equation in $F(\eta)$:

$$2F(\eta)F'(\eta) - F^2(\eta) = -\frac{1}{4}. \quad (18.30)$$

The solution of (18.30) satisfying the matching condition (18.28) is

$$F(\eta) \equiv \sqrt{\frac{1}{4} + e^\eta}. \quad (18.31)$$

Thus, the asymptotic expansion of $\tilde{U}$ in region 1b' has the following form

$$\tilde{U} = -X_* \sqrt{\frac{1}{4} + e^\eta} + \dots, \quad \text{as } X_* \rightarrow 0^+. \quad (18.32)$$

In order to complete the analysis in region 1b, let us consider the limit

$$\Psi = O(1), \quad X_* \rightarrow 0^+. \quad (18.33)$$

The velocity $U_e(X_*)$ at the outer edge of region 1b is regular for all positive X_* . Hence, the solution in region 1b may be written in terms of a Taylor expansion, with X_* being

small, such that

$$\tilde{U}(X_*, \Psi) = \tilde{U}_0(\Psi) + X_* \tilde{U}_1(\Psi) + \cdots, \quad X_* \rightarrow 0^+. \quad (18.34)$$

We substitute (18.34) and (18.22) into the momentum equation (18.4a), and in the leading order approximation it leads to

$$\tilde{U}_1 = \frac{1}{2} \frac{d^2}{d\Psi^2} (\tilde{U}_0^2). \quad (18.35)$$

It remains to find the asymptotic behaviours of both $\tilde{U}_0$ and $\tilde{U}_1$ at the outer edge of region 1b. Firstly, let us write the asymptotic expansion of $\tilde{U}$ at the lower edge of region 1b', that is, as η tends to plus infinity: according to (18.32), we have

$$\tilde{U} = -X_* e^{\eta/2} + \cdots = -e^{\Psi/2} + \cdots. \quad (18.36)$$

The matching between (18.36) and (18.34) shows that

$$\tilde{U}_0(\Psi) = -e^{\Psi/2} + \cdots, \quad \text{as } \Psi \rightarrow -\infty. \quad (18.37)$$

Substitution of (18.37) into (18.35) gives

$$\tilde{U}_1 = \frac{1}{2} e^{\Psi} + \cdots, \quad \text{as } \Psi \rightarrow -\infty. \quad (18.38)$$

The asymptotic expansions of the flow functions have been determined in the limits of interest in regions 1a, 1b', and in the main part of region 1b. The next two sections will focus on calculating the displacement effects of regions 1b and 1a as $X_* \rightarrow 0^+$.

18.3 Displacement effect of region 1b

The streamline slope V/U quantifies the displacement effect of the boundary layer due to the impinging expansion fan, which causes the streamlines to be displaced from the plate surface. The quantity V/U corresponds to the angle between the plate surface and the tangent to the streamlines. Our goal is to compute and to compare V/U produced in regions 1a and 1b when X_* is small and positive, in order to determine the

dominant contribution of these two regions on the displacement effect of the boundary layer. This section focuses on the calculation of V/U at the outer edge of region 1b as $X_* \rightarrow 0^+$. However, as we showed earlier, the region 1b may be subdivided into two parts: the main part of region 1b, and an inviscid small region called region 1b', lying at the outer edge of region 1b. For that reason, we shall compute V/U independently for these two regions. The next section will focus on the calculation of the streamline slopes of regions 1a and 1b.

Let us start by integrating the continuity equation (18.4b) with respect to Ψ . Taking into account the boundary conditions (18.4c) on the plate surface, the streamline slope in region 1b can be written as follows

$$\frac{\tilde{V}}{\tilde{U}} = - \int_0^\Psi \frac{1}{\tilde{U}^2} \frac{\partial \tilde{U}}{\partial X_*} d\psi. \quad (18.39)$$

We shall divide the integration interval into two distinct intervals depending on the region considered

$$\Psi \in [0, \Psi_\Delta], \text{ and } \Psi \in [\Psi_\Delta, \Psi], \quad (18.40)$$

where we set

$$\Psi_\Delta = -\Delta + 2 \ln(X_*). \quad (18.41)$$

The parameter Δ is negative, and it satisfies the condition

$$1 \ll -\Delta \ll -2 \ln(X_*). \quad (18.42)$$

The first interval in (18.40) corresponds to the main part of region 1b, while the second one refers to region 1b'.

Let us start with the first interval $\Psi \in [0, \Psi_\Delta]$. We substitute the asymptotic expansion (18.34) for $\tilde{U}$ into (18.39). We have

$$- \int_0^{\Psi_\Delta} \frac{1}{\tilde{U}^2} \frac{\partial \tilde{U}}{\partial X_*} d\psi = - \int_0^{\Psi_\Delta} \frac{\tilde{U}_1}{\tilde{U}_0^2} d\psi + \dots, \text{ as } X_* \rightarrow 0^+. \quad (18.43)$$

Furthermore, according to the asymptotic behaviours of $\tilde{U}_0$ and $\tilde{U}_1$, we have

$$\frac{\tilde{U}_1}{\tilde{U}_0^2} \rightarrow \frac{1}{2} + \dots, \text{ as } \psi \rightarrow \infty. \quad (18.44)$$

We rearrange the integral on the right-hand side of (18.43) such that

$$-\int_0^{\Psi_\Delta} \frac{\tilde{U}_1}{\tilde{U}_0^2} d\psi = -\int_0^{\Psi_\Delta} \left(\frac{\tilde{U}_1}{\tilde{U}_0^2} - \frac{1}{2} \right) d\psi - \int_0^{\Psi_\Delta} \frac{1}{2} d\psi, \quad (18.45a)$$

$$= -\int_0^{\Psi_\Delta} \left(\frac{\tilde{U}_1}{\tilde{U}_0^2} - \frac{1}{2} \right) d\psi - \ln(X_*) + \frac{\Delta}{2}, \quad (18.45b)$$

$$= -\ln(X_*) + \frac{\Delta}{2} + D + \cdots, \text{ as } X_* \rightarrow 0^+, \quad (18.45c)$$

with constant D defined such that

$$D = -\int_0^{-\infty} \left(\frac{\tilde{U}_1}{\tilde{U}_0^2} - \frac{1}{2} \right) d\psi. \quad (18.46)$$

Let us now turn to the second interval $\Psi \in [\Psi_\Delta, \Psi]$. We recall that in region 1b', the solution has the form (18.27), such that the function F is given by (18.31). We perform a change of variable $\eta = \Psi - 2 \ln(X_*)$ with $\eta_\Delta = \Psi_\Delta - 2 \ln(X_*) = -\Delta$:

$$\frac{\partial}{\partial X_*} \int_{\Psi_\Delta}^{\Psi} \frac{d\psi}{\tilde{U}(X_*, \psi)} = -\frac{1}{X_*^2} \int_{\eta_\Delta}^{\eta} \left(-\frac{1}{F(\xi)} + \frac{2F'(\xi)}{F^2(\xi)} \right) d\xi, \quad (18.47a)$$

$$= -\frac{2}{X_*^2} \left[\ln \left(\frac{\sqrt{\frac{1}{4} + e^\xi} + \frac{1}{2}}{\sqrt{\frac{1}{4} + e^\xi} - \frac{1}{2}} \right) - \frac{1}{F(\xi)} \right]_{\eta_\Delta}^{\eta}, \quad (18.47b)$$

$$= -\frac{2}{X_*^2} \left(\ln \left(\frac{\sqrt{\frac{1}{4} + e^\eta} + \frac{1}{2}}{\sqrt{\frac{1}{4} + e^\eta} - \frac{1}{2}} \right) - \ln \left(\frac{\sqrt{\frac{1}{4} + e^{-\Delta}} + \frac{1}{2}}{\sqrt{\frac{1}{4} + e^{-\Delta}} - \frac{1}{2}} \right) - \frac{2}{\sqrt{1 + 4e^\eta}} + \frac{2}{\sqrt{1 + 4e^{-\Delta}}} \right). \quad (18.47c)$$

Furthermore, we recall that Δ satisfies the conditions (18.42), and also that at the outer edge of region 1b', $\eta \rightarrow -\infty$. Therefore, the streamline slope at the outer edge of region 1b' may be written as

$$\frac{\partial}{\partial X_*} \int_{\Psi_\Delta}^{\Psi} \frac{d\psi}{\tilde{U}(X_*, \psi)} = \frac{2\Psi}{X_*^2} - \frac{4}{X_*^2} (\ln(X_*) - 1) + \cdots, \text{ as } \Psi \rightarrow -\infty, X_* \rightarrow 0^+. \quad (18.48)$$

Equations (18.48) and (18.45c) show that there is a singularity at $X_* = 0$, and that the streamline slope produced by the main part of region 1b is negligible in comparison with the one of region 1b'. This implies that the displacement effect of region 1b is predominantly due to the contribution of region 1b'. Substitution of (18.48) into (18.39), and rescaling $\tilde{U}$ and $\tilde{V}$ using (18.3), we may conclude that at the outer edge of region 1b'

$$\frac{V}{U} = \frac{2\Psi}{X_*^2} - \frac{4}{X_*^2} \left(\ln(X_*) - 1 \right) + \dots, \text{ as } \Psi \rightarrow -\infty, X_* \rightarrow 0^+. \quad (18.49)$$

18.4 Displacement effect of region 1a

In this section, we shall compare the displacement effects of regions 1a and 1b. To deal with the streamline slope of region 1a, we start by integrating the continuity equation (16.8b) with respect to Ψ_*

$$\frac{V_*}{U_*} = \frac{\partial}{\partial X_*} \int_0^{\Psi_*} \frac{d\psi_*}{U_*(X_*, \psi_*)}. \quad (18.50)$$

The first step is to match the streamline slope at the lower edge of region 1a with (18.49). This implies that both Ψ_* and X_* must be small. However, as we showed in Chapter 16, the longitudinal velocity (16.25) is written in two different ways depending on the value of Ψ_* : U_* is negative if $\Psi_* \in [0, -\frac{1}{2}U_e^2(X_*)]$, and it is positive for $\Psi_* \in [-\frac{1}{2}U_e^2(X_*), +\infty]$. Therefore, the streamline slope (18.50) in region 1a may also be expressed by one of two formulas, depending on the interval considered, namely

$$\frac{V_*}{U_*} = -\frac{\partial}{\partial X_*} \int_0^{\Psi_*} \frac{d\psi_*}{\sqrt{2\psi_* + U_e^2(X_*)}}, \text{ for } \Psi_* \in \left[0, -\frac{1}{2}U_e^2(X_*)\right], \quad (18.51a)$$

$$\begin{aligned} \frac{V_*}{U_*} &= -\frac{\partial}{\partial X_*} \int_0^{-\frac{1}{2}U_e^2(X_*)} \frac{d\psi_*}{\sqrt{2\psi_* + U_e^2(X_*)}} + \\ &\frac{\partial}{\partial X_*} \int_{-\frac{1}{2}U_e^2(X_*)}^{\Psi_*} \frac{d\psi_*}{\sqrt{2\psi_* + U_e^2(X_*)}}, \text{ for } \Psi_* \in \left[-\frac{1}{2}U_e^2(X_*), +\infty\right]. \end{aligned} \quad (18.51b)$$

To perform the matching with region 1b, we need to choose what equation to use out of (18.51a) and (18.51b). Equation (18.51a) is valid everywhere between region 1b and the point of reversal of the flow, while equation (18.51b) is valid above the point of reversal. Thus, the latter cannot be used for matching. We recall that, $U_e(X_*)$ is given

by (16.22), and it follows that $U_e(X_*) \rightarrow 0$ as $X_* \rightarrow 0^+$. Consequently, we shall apply a double limit procedure for matching, leading to both $\Psi_* \rightarrow 0$ and $X_* \rightarrow 0^+$. We also recall that the asymptotic expansion of the velocity at the outer edge of the viscous sublayer 1b when $X_* \rightarrow 0^+$ is $O(X_*)$ according to (18.22). Thus, we need to know how small Ψ_* is as compared to X_* , in the limit as $\Psi_* \rightarrow 0$ and $X_* \rightarrow 0^+$.

Let us write the asymptotic expansion of the quotient $1/U_*$ when Ψ_* and X_* are both small, and ignoring the sign of U_* for the moment:

$$\frac{1}{U_*} = \pm \left(\frac{2}{X_*} - \frac{8\Psi_*}{X_*^3} + \cdots \right), \quad \text{as } \Psi_* \rightarrow 0, X_* \rightarrow 0^+. \quad (18.52)$$

The asymptotic formula (18.52) has been found by using a double limit procedure: it was assumed firstly that Ψ_* tends to zero while X_* remains of order one. Afterwards, we assumed that X_* tends to 0^+ . Hence, strictly speaking, the expression (18.52) follows the definition of an asymptotic expansion assuming that the leading-order term is much larger than the second one, that is if $\Psi_*/X_*^2 \ll 1$, as $\Psi_* \rightarrow 0$ and $X_* \rightarrow 0^+$. Then, there exists a region, that we shall call from now on region 1a', lying between regions 1b' and 1a and where the terms Ψ_* and X_*^2 become of same order (see Figure 18.2). We introduce a new independent variable, denoted ξ , such that in region 1a', we have

$$\xi = \frac{\Psi_*}{X_*^2} = O(1), \quad \text{as } X_* \rightarrow 0^+, \Psi_* \rightarrow 0. \quad (18.53)$$

Thus, the matching of the streamline slopes between regions 1a and 1b must be performed when ξ tends to 0, that is when $\Psi_* \ll X_*^2$. Under this assumption, the streamline slope is determined at the lower edge of region 1a when X_* is small (which corresponds actually to the bottom of region 1a'). Furthermore, $U_e(X_*) = O(X_*)$ as $X_* \rightarrow 0^+$, so this implies that $\Psi_* \ll U_e^2(X_*)$ when $\xi \rightarrow 0$. Therefore, it confirms that the right equation to use to perform the matching of the streamline slope of region 1a with the one of region 1b is equation (18.51a), and rescaling the variables by applying (16.7), we deduce the streamline slope at the bottom of region 1a when X_* is small

$$\frac{V}{U} = \frac{2\Psi}{X_*^2} + \cdots, \quad \text{as } X_* \rightarrow 0^+, \Psi \rightarrow 0. \quad (18.54)$$

By comparing the streamline slope (18.54) of the lower edge of region 1a with (18.49), corresponding to the one at the outer edge of region 1b, we notice that the leading-order term of both expansions coincide with one another. It comes from the integration of

the continuity equation and it stands for the inertia term. Thus, the remainder of the asymptotic expansion (18.49) is the displacement effect of region 1b

$$\left. \frac{V}{U} \right|_{1b} = -\frac{4}{X_*^2} \left(\ln(X_*) - 1 \right) + \dots, \quad \text{as } X_* \rightarrow 0^+. \quad (18.55)$$

Finally, we shall evaluate the displacement effect of region 1a. Although equation (18.54) shows the streamline slope at the bottom of region 1a when X_* is small, it does not correspond to the displacement effect of region 1a. In order to find it, we use equation (18.51b). We find

$$\frac{V_*}{U_*} = -U'_e(X_*) + \frac{U_e(X_*)U'_e(X_*)}{\sqrt{2\Psi_* + U_e^2(X_*)}}, \quad (18.56)$$

and this equation is valid “almost” everywhere in region 1a. We make X_* tends to 0^+ in (18.56), and recalling that the asymptotic expansion of $U_e(X_*)$ when X_* is small positive is given by (18.22). Hence

$$\frac{V_*}{U_*} = \frac{1}{2} + \frac{1}{2\sqrt{1+8\xi}}. \quad (18.57)$$

Thus, making ξ large in (18.57) and going back to unscaled variables by using (16.7), we can conclude that

$$\left. \frac{V}{U} \right|_{1a} = \frac{1}{2}U_w^2 + \dots, \quad \text{as } X_* \rightarrow 0^+. \quad (18.58)$$

We also notice that, according to (18.57), the main contribution of the displacement effect of region 1a is produced by region 1a'.

Comparison of (18.58) and (18.55) which are the displacement effects of regions 1a and 1b, respectively, shows that in the limit as $U_w \gg 1$, the contribution of region 1a is prevailing in comparison to region 1b as long as X_* is not too small. Nevertheless, no matter how large U_w is, there exists a region that we will call region C , where we cannot neglect the contribution of region 1b. In the next Chapter 19, we shall analyse the flow behaviour in this new region C .

Chapter 19

Flow near the singularity. Inviscid region C

In this chapter, we shall introduce a new region C , whose goal is to smooth out the singularity of the streamline slope discussed in Chapter 18. We shall determine the form of the asymptotic expansions of the flow functions in this region; in particular, we will compute numerically the pressure variation. The topology of the flow will be deduced from the asymptotic expansions of regions $1a$ and $1b$ determined in Chapters 18 and 16.

Let us introduce a new longitudinal variable denoted $\check{X}$, such that in region C we have

$$\check{X} = \frac{X_*}{\sigma(U_w)} = O(1), \quad (19.1)$$

where $\sigma(U_w) \ll 1$ as $U_w \rightarrow \infty$, and such that the scaling parameter σ satisfies the equation

$$\frac{|\ln(\sigma)|}{\sigma^2} = \frac{1}{2}U_w^2. \quad (19.2)$$

This ensures that the streamline slope angles produced by the displacement effects of regions $1a$ and $1b$ through (18.58) and (18.55) respectively, are of same order. The formation of region C is shown in Figure 19.1.

New asymptotic expansions for the fluid-dynamic functions are required so that we can comprehend the flow behaviour nearby the singularity. As usual, the form of the asymptotic solution may be predicted from the behaviour of the solution on approach to region C . The stream filaments that emerge from regions $1a'$ and $1b'$ pass through region C . We shall rewrite the solution of region $1a'$ when approaching region C , that is when X_* becomes comparable with σ . Since region $1a'$ lies inside region $1a$, equation (16.25) is also valid in region $1a'$ as long as ξ remains of order one. Hence, in the limit

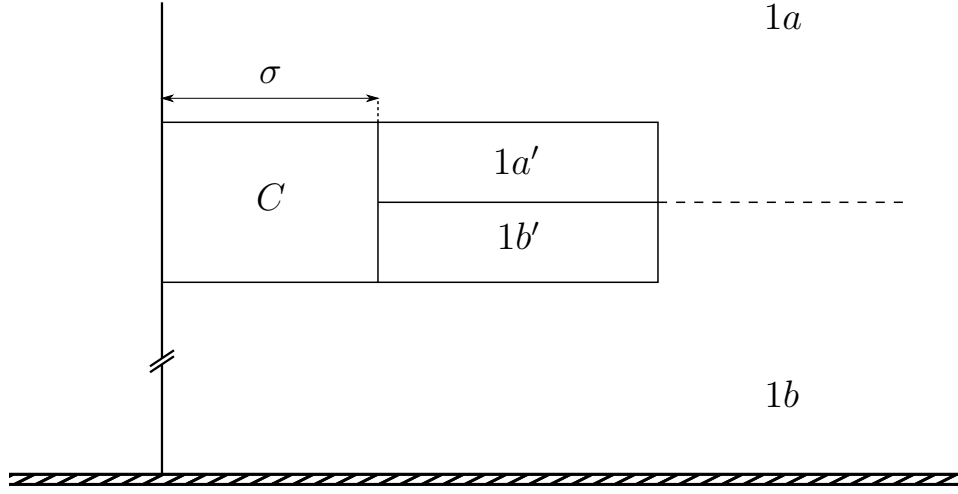


Figure 19.1: Formation of region C . It is an extension of regions $1a'$ and $1b'$, and it will smooth out the solution when approaching $X_* = 0^+$

as $X_* \rightarrow 0^+$ using (18.22) and the definition of ξ (18.53),

$$U_*(X_*, \xi) = \begin{cases} -\frac{1}{2}X_*\sqrt{1+8\xi} + \dots, & \text{for } \xi \in [0, -\frac{1}{8}], \\ \frac{1}{2}X_*\sqrt{1+8\xi} + \dots, & \text{for } \xi > -\frac{1}{8}. \end{cases} \quad (19.3)$$

Thus, in terms of the original variables (16.7), when approaching region C , the solution has the following form

$$U = \begin{cases} -U_w\sigma\frac{1}{2}\check{X}\sqrt{1+8\xi} + \dots, & \text{for } \xi \in [0, -\frac{1}{8}], \\ U_w\sigma\frac{1}{2}\check{X}\sqrt{1+8\xi} + \dots, & \text{for } \xi > -\frac{1}{8}, \end{cases} \quad (19.4a)$$

$$P = -U_w^2\sigma^2\frac{1}{8}\check{X}^2 + \dots, \quad (19.4b)$$

where

$$\xi = \frac{\Psi}{U_w^2\sigma^2\check{X}^2}. \quad (19.4c)$$

The asymptotic expansion (19.4b) has been found by recalling that P_* and $U_e(X_*)$ are related to one another according to relation (16.23). The asymptotic behaviour of $U_e(X_*)$ when X_* is small is given by (18.22). Consequently, the asymptotic solution in

region C may be sought in the form

$$\left. \begin{aligned} U &= U_w \sigma \check{U} + \dots, & P &= U_w^2 \sigma^2 \check{P} + \dots, \\ X &= \frac{\sigma}{U_w} \check{X}, & \Psi &= U_w^2 \sigma^2 \check{\Psi}. \end{aligned} \right\} \quad (19.5)$$

Since region C is an extension of regions $1a'$ and $1b'$, we must consider the matching with both regions.

We start with region $1a'$. We rewrite (19.4a)-(19.4c) in terms of the variables of region C , leading to the conclusion that

$$\left. \begin{aligned} \check{U} &= \begin{cases} -\frac{1}{2} \check{X} \sqrt{1 + 8\xi} + \dots, & \text{for } \xi \in [0, -\frac{1}{8}], \\ \frac{1}{2} \check{X} \sqrt{1 + 8\xi} + \dots, & \text{for } \xi > -\frac{1}{8}, \end{cases} \\ \check{P} &= -\frac{1}{8} \check{X}^2 + \dots, \end{aligned} \right\} \text{ as } \check{X} \rightarrow \infty. \quad (19.6a)$$

Here, ξ is written in terms of $\check{\Psi}$ and $\check{X}$ as

$$\xi = \frac{\check{\Psi}}{\check{X}^2}. \quad (19.6b)$$

Let us turn now to the matching conditions with the solution in region $1b'$. We rewrite the solution for this region, given by (18.32), using the rescaling (19.1) as well as the original variable U using (18.3). Hence, it gives us the solution approaching region C :

$$U = -U_w \sigma \frac{1}{2} \check{X} \sqrt{1 + 4e^\eta} + \dots. \quad (19.7)$$

It results that the matching condition, when $\check{X}$ is large, of the solution in region C with the one of region $1b'$ is given by

$$\check{U} = -\frac{1}{2} \check{X} \sqrt{1 + 4e^\eta} + \dots, \quad \text{as } \check{X} \rightarrow \infty. \quad (19.8)$$

The similarity variable η has been defined in relation (18.25). Let us rewrite it in terms

of the scaled variables $\check{X}$ and $\check{\Psi}$, by substituting (19.1) and (19.5) into it:

$$\eta = U_w^2 \sigma^2 \check{\Psi} - 2 \ln(\sigma) - 2 \ln(\check{X}). \quad (19.9)$$

Furthermore, $2 \ln(\sigma) = -U_w^2 \sigma^2$ according to (19.1). Consequently,

$$\eta = -2 \ln(\check{X}) + \chi \left(1 + \check{\Psi}\right), \quad (19.10)$$

with $\chi = 2|\ln(\sigma)|$.

We also need to predict the form of the asymptotic expansion for the lateral velocity component in region C . We recall that Ψ is defined by (15.12): using (19.5), (16.7) and (19.1), one can see that V is an order $U_w^3 \sigma$ quantity. Consequently, in region C , the lateral velocity component should be sought in the form

$$V = U_w^3 \sigma \check{V}. \quad (19.11)$$

Let us now focus on determining the analytical form of $\check{U}$. We shall use composite asymptotic expansions as C lies in the overlap regions of both regions $1a'$ and $1b'$. Since the viscous terms have been disregarded in the leading order approximation in $1a'$ and $1b'$, we expect the flow to be inviscid in region C as well. This can be proved by substituting (19.5) into the momentum equation (15.13a), and working with the leading-order terms, the momentum equation becomes

$$\check{U} \frac{\partial \check{U}}{\partial \check{X}} = -\frac{d\check{P}}{d\check{X}}. \quad (19.12)$$

Integrating (19.12) with respect to $\check{X}$, we arrive at the Bernoulli equation

$$\check{H}(\check{\Psi}) = \frac{1}{2} \check{U}^2 + \check{P}, \quad (19.13)$$

where $\check{H}(\check{\Psi})$ is constant along each streamline. The function $\check{H}$ may be calculated by adding the value it takes by matching it with region $1a'$ with the one it takes by matching it in region $1b'$. Following this step, the common value it takes in the overlap region between $1a'$ and $1b'$ must be subtracted so that it is not counted twice.

Let us start by matching function $\check{H}$ with region $1a'$: we substitute the matching conditions (19.6a) for $\check{U}$ and $\check{P}$ into (19.13), and we replace the similarity variable ξ by its definition (19.6b), leading to

$$\check{H}_{1a'} = \check{\Psi}. \quad (19.14)$$

Similarly, when matching with region $1b'$ using (19.8) for $\check{U}$, and replacing the similarity variable η with its expression (19.10), it yields to

$$\check{H}_{1b'} = \frac{1}{2}e^{\chi(1+\check{\Psi})}. \quad (19.15)$$

We notice that the matching between (19.14) and (19.15) in the overlap region between $1a'$ and $1b'$ may be performed by setting $\check{\Psi} \rightarrow 0$ and $\check{\Psi} \rightarrow -\infty$, respectively:

$$\lim_{\check{\Psi} \rightarrow 0} \check{H}_{1a'} = \lim_{\check{\Psi} \rightarrow -\infty} \check{H}_{1b'} = 0.$$

As a result, the common value in the overlap region is equal to zero, and adding (19.14) and (19.15) together gives the composite Bernoulli function

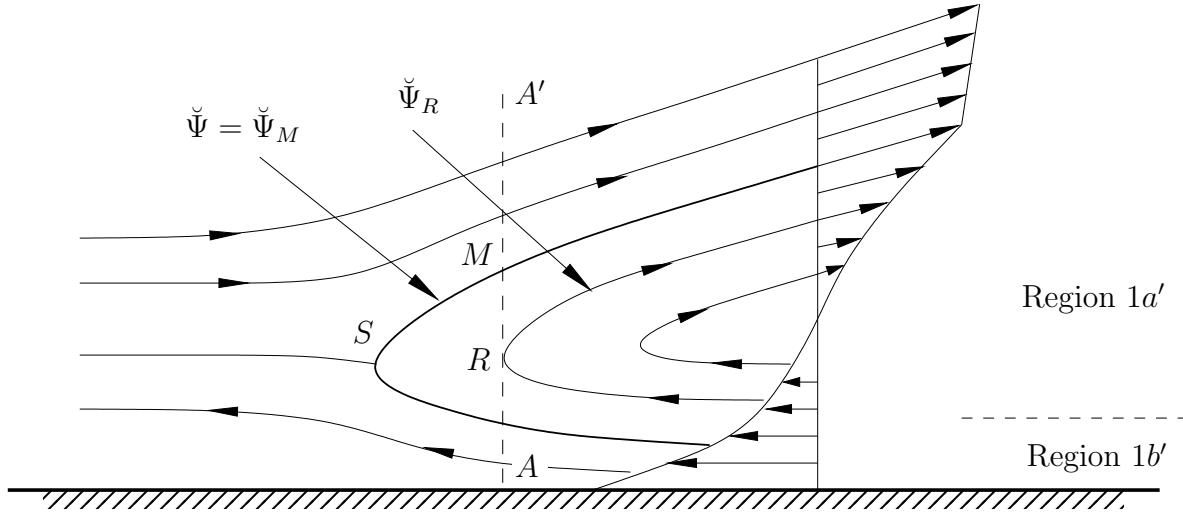
$$\check{H} = \check{\Psi} + \frac{1}{2}e^{\chi(1+\check{\Psi})}. \quad (19.16)$$

Substitution of (19.16) into (19.13), and solving the resulting equation for $\check{U}$, leads to the expression for $\check{U}$

$$\check{U} = \pm \sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}. \quad (19.17)$$

The sign of the velocity $\check{U}$ in (19.17) depends on the location where the flow is observed in region C . In order to know where $\check{U}$ is positive and where it is negative, let us focus on the topology of the streamlines in region C . With that purpose in mind, we shall make use of the sketch of Figure 19.2, which depicts the expected streamline pattern of region C . The numerical results of the interaction problem in Chapter 17 confirmed the prediction of Figure 19.2.

In equation (18.32), we have shown that the velocity $\tilde{U}$ was negative in region $1b'$. This means that the fluid particles in this region move in the direction opposite to the flow outside of the boundary layer. According to the pressure distribution (16.18) when the strength $\bar{P}_S$ of the expansion fan is set equal to $-1/2$, these fluid particles are subject to an adverse (positive) pressure gradient as they move in the negative X_* -direction and then, they undergo a deceleration until they stop and are redirected in


 Figure 19.2: Streamline pattern in region C .

the positive X_* -direction. Thus, for any cross-section AA' in region C (see Figure 19.2), there exists a point R such that

$$\check{U}|_R = 0.$$

Therefore, for any $\check{X}$ in region C , the value of the stream-function $\check{\Psi}_R$ passing through point R may be found by setting (19.17) equal to zero. This leads to the following equation linking the pressure $\check{P}$ with $\check{\Psi}_R$ at cross-section AA'

$$2\check{\Psi}_R + e^{\chi(1+\check{\Psi}_R)} - 2\check{P} = 0. \quad (19.18)$$

We notice that $\check{\Psi}_R$ and $\check{P}$ are both functions of $\check{X}$.

It remains to find the pressure distribution $\check{P}(\check{X})$. Since region C is rather short, then the flow in this region is subject to the conditions of the “condensed flow regime”, formulated by Bogolepov and Neiland (1971). They proved that if a subsonic or supersonic flow is considered along a plate with a surface roughness, then the description of the flow around the roughness depends on the roughness length Δx . If $\Delta x \sim Re^{-3/8}$, then the interaction region is described by the classical triple-deck theory. The pressure perturbations are caused by the wall surface, and also by the displacement effect of the lower tier of the triple-deck structure. The flow regime with $\Delta x \ll Re^{-3/8}$ may be obtained from the standard triple-deck ($\Delta x \sim Re^{-3/8}$) by reducing δx . Bogolepov and Neiland showed that the effect of the wall roughness must be compensated by the

displacement effect of the lower tier, otherwise the pressure would be very large. This would not be acceptable as a large pressure cannot be sustained by a “slow” flow in the lower tier. For them, the pressure equation is different as our problem is different. Instead of a wall roughness, the plate is flat, but an expansion fan hits the boundary layer. In the short region C , $X \ll Re^{-3/8}$, and in order to avoid large pressure, the term of the displacement effect of the boundary layer in the Ackeret formula must be constant. Thus, under the assumption of condensed flow regime, the streamline slope at the outer edge of region C is constant, that is

$$\left. \frac{V}{U} \right|_{\check{Y}=\infty} = \text{constant}. \quad (19.19)$$

Matching with the streamline slope (18.58) in region $1a$ gives us the value of the constant on the right-hand side of (19.19):

$$\left. \frac{V}{U} \right|_{\check{Y}=\infty} = \frac{1}{2} U_w^2. \quad (19.20)$$

Scaling U and V on the left-hand side of (19.19) with the help of (19.5) and (19.11), respectively, we deduce that in region C , the following equation holds

$$\left. \frac{\check{V}}{\check{U}} \right|_{\check{Y}=\infty} = \frac{1}{2}. \quad (19.21)$$

The term on the left-hand side of (19.21) may be found using the continuity equation (15.13b). Substitution of (19.5) and (19.11) into (15.13b) shows that the continuity equation assumes in region C the following form

$$\frac{\partial}{\partial \check{\Psi}} \left(\frac{\check{V}}{\check{U}} \right) = \frac{\partial}{\partial \check{X}} \left(\frac{1}{\check{U}} \right). \quad (19.22)$$

The quotient $\check{V}/\check{U}$ can be retrieved by integrating equation (19.22) with respect to $\check{\Psi}$. However, for each fixed $\check{X}$, three integration intervals must be distinguished. The first one corresponds to AR on Figure 19.2, and it lies below the reversal point R . The fluid particles move from the right to the left across the section AR , so for this integration interval, we shall use the velocity $\check{U}$, given by equation (19.17), with a negative sign. Above point R , the velocity $\check{U}$ is positive. The second interval is RM , and the velocity

is given by (19.17) with a positive sign. M corresponds to the point on the cross-section AA' where the last streamline originating from region $1b'$ passes through it. The stream-function is constant along this streamline, and it takes the value $\check{\Psi}_M$, such that

$$2\check{\Psi}_M + e^{\chi(1+\check{\Psi}_M)} - 2\check{P}_M = 0, \quad (19.23)$$

where $\check{P}_M$ is the maximum value of the pressure $\check{P}$ in region C . In Figure 19.2, it corresponds to the pressure at $\check{X} = \check{X}_S$, where $\check{X}_S$ is the coordinate of the stagnation point S on the $\check{X}$ -axis. The pressure $\check{P}_M$ can be defined by the strength of the expansion fan

$$P_S = -\frac{1}{2}U_w^2 - U_w^2\sigma^2\check{P}_M. \quad (19.24)$$

Finally, the last interval on AA' lies above point M , and it is crossed by the streamlines coming from the unperturbed flow in the region before the expansion fan, and where the Bernoulli function $\check{H}$ is equal to $\check{\Psi}$, as it is in region $1a'$. However, the numerical results of the triple-deck problem that we obtained in Figure 17.1 show that there is a shift between the streamlines upstream and downstream of the impinging expansion fan. Therefore, we need to introduce a shift $\check{\Psi}_0$ in the Bernoulli function such that

$$\check{H} = \check{\Psi} + \check{\Psi}_0. \quad (19.25)$$

The constant $\check{\Psi}_0$ can be found in the following way. We recall that $\check{H}$ is constant along each streamline. In particular, let us consider the streamline passing through the stagnation point S (and turning back, going through M). Then,

$$\check{H} = \check{\Psi}_M + \check{\Psi}_0. \quad (19.26)$$

Recalling that $\check{H}$ is given by (19.13), and that at point S , $\check{U} = 0$ and $\check{P} = \check{P}_M$, it results that

$$\check{P}_M = \check{\Psi}_M + \check{\Psi}_0. \quad (19.27)$$

We substitute (19.27) into (19.23), leading to

$$\check{\Psi}_0 = \frac{1}{2}e^{\chi(1+\check{\Psi}_M)}. \quad (19.28)$$

The last step is to substitute (19.26) into (19.13) by using (19.28) as a value for $\check{\Psi}_0$.

This gives:

$$\check{\Psi}_M + \frac{1}{2}e^{\chi(1+\check{\Psi}_M)} = \frac{1}{2}\check{U}^2 + \check{P}.$$

We solve this equation with respect to $\check{U}$ and we keep the positive solution, as the longitudinal velocity $\check{U}$ is positive at point M and above it. This yields to the expression of the velocity on the interval above point M

$$\check{U} = \sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi}_M)} - 2\check{P}}. \quad (19.29)$$

Thus, we have determined the expressions of $\check{U}$ for the three distinct integration intervals, and the streamline slope (19.21) in region C may be written as

$$\frac{\check{V}}{\check{U}} = \frac{\partial}{\partial \check{X}} \left[- \int_{+\infty}^{\check{\Psi}_R} \frac{d\check{\Psi}}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}} + \int_{\check{\Psi}_R}^{\check{\Psi}_M} \frac{d\check{\Psi}}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}} + \int_{\check{\Psi}_M}^{\check{\Psi}} \frac{d\check{\Psi}}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi}_M)} - 2\check{P}}} \right],$$

or, equivalently,

$$\frac{\check{V}}{\check{U}} = \frac{\partial}{\partial \check{X}} \left[\int_{\check{\Psi}_M}^{+\infty} \frac{d\check{\Psi}}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}} + 2 \int_{\check{\Psi}_R}^{\check{\Psi}_M} \frac{d\check{\Psi}}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}} + \int_{\check{\Psi}_M}^{\check{\Psi}} \frac{d\check{\Psi}}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi}_M)} - 2\check{P}}} \right]. \quad (19.30)$$

Furthermore, we showed in relation (19.21) that in region C , the streamline slope was equal to $1/2$ when $\check{Y} = \infty$. Consequently, setting $\check{\Psi} = \infty$ in (19.30) yields to a nonlinear

ordinary differential equation in $\check{P}$

$$\begin{aligned} \frac{1}{2} = \frac{d}{d\check{X}} & \left[2 \int_{\check{\Psi}_R}^{\check{\Psi}_M} \frac{d\check{\Psi}}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}} \right] \\ & + \frac{d\check{P}}{d\check{X}} \left[\int_{\check{\Psi}_M}^{+\infty} \frac{d\check{\Psi}}{\left[2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P} \right]^{3/2}} + \frac{1}{\sqrt{2\check{\Psi}_M + e^{\chi(1+\check{\Psi}_M)} - 2\check{P}}} \right]. \end{aligned} \quad (19.31a)$$

Our goal is to solve equation (19.31a). We recall that $\check{\Psi}_R$ and $\check{P}$ are related to one another by

$$2\check{\Psi}_R + e^{\chi(1+\check{\Psi}_R)} - 2\check{P} = 0. \quad (19.31b)$$

Equations (19.31a) and (19.31b) should be solved with the matching condition (19.6a):

$$\check{P} = -\frac{1}{8}\check{X}^2 + \dots, \quad \text{as } \check{X} \rightarrow \infty. \quad (19.31c)$$

We recall that the matching condition (19.31c) comes from the matching with the analytic pressure distribution deduced in Chapter 16 when X_* becomes small.

19.1 Computation of the pressure distribution in region C

The first integral on the right-hand side of (19.31a) needs a special treatment, as the function $1/\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}$ becomes singular at $\check{\Psi} = \check{\Psi}_R$. Using the linearity rule of the integral, we split the integration interval into two intervals such that the first one is in the vicinity of the singularity:

$$\check{I}_1 = \int_{\check{\Psi}_R}^{\check{\Psi}_M} \frac{d\check{\Psi}}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}}, \quad (19.32a)$$

$$= \int_{\check{\Psi}_R}^{\check{\Psi}_R+\delta} \frac{d\check{\Psi}}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}} + \int_{\check{\Psi}_R+\delta}^{\check{\Psi}_M} \frac{d\check{\Psi}}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}}, \quad (19.32b)$$

$$= \check{I}_{1,1} + \check{I}_{1,2}, \quad (19.32c)$$

where δ is a small parameter. We set $\check{f}_{\check{I}_1}(\check{X}, \check{\Psi}) = 2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}$. Let us write the Taylor expansion of $\check{f}_{\check{I}_1}$ around $\check{\Psi}_R$

$$\begin{aligned} \check{f}_{\check{I}_1,1}(\check{X}, \check{\Psi}) &= \check{f}_{\check{I}_1,1}|_{\check{\Psi}=\check{\Psi}_R} + \left. \frac{\partial \check{f}_{\check{I}_1,1}}{\partial \check{\Psi}} \right|_{\check{\Psi}=\check{\Psi}_R} (\check{\Psi} - \check{\Psi}_R) + \cdots, \quad \text{as } \check{\Psi} \rightarrow \check{\Psi}_R, \\ &= \left(2 + \chi e^{\chi(1+\check{\Psi}_R)} \right) (\check{\Psi} - \check{\Psi}_R) + \cdots, \quad \text{as } \check{\Psi} \rightarrow \check{\Psi}_R, \end{aligned}$$

where we recall that $\check{f}_{\check{I}_1,1}|_{\check{\Psi}=\check{\Psi}_R} = 0$ according to (19.31b). We substitute $\check{f}_{\check{I}_1,1}$ into $\check{I}_{1,1}$, and it leads to

$$\check{I}_{1,1} = 2\sqrt{\frac{\delta}{2 + \chi e^{\chi(1+\check{\Psi}_R)}}}. \quad (19.33)$$

We apply the differential operator $d/d\check{X}$ to $\check{I}_{1,1}$

$$\frac{d\check{I}_{1,1}}{d\check{X}} = -\frac{\sqrt{\delta}\chi^2 e^{\chi(1+\check{\Psi}_R)}}{\left(2 + \chi e^{\chi(1+\check{\Psi}_R)}\right)^{3/2}} \frac{d\check{\Psi}_R}{d\check{X}}. \quad (19.34)$$

Moreover, differentiating (19.31b) with respect to $\check{X}$ links $d\check{\Psi}_R/d\check{X}$ to $d\check{P}/d\check{X}$

$$\frac{d\check{\Psi}_R}{d\check{X}} = \frac{2}{2 + \chi e^{\chi(1+\check{\Psi}_R)}} \frac{d\check{P}}{d\check{X}}. \quad (19.35)$$

Thus, substitution of (19.35) into (19.34) yields to

$$\frac{d\check{I}_{1,1}}{d\check{X}} = -\frac{2\sqrt{\delta}\chi^2 e^{\chi(1+\check{\Psi}_R)}}{\left(2 + \chi e^{\chi(1+\check{\Psi}_R)}\right)^{5/2}} \frac{d\check{P}}{d\check{X}}. \quad (19.36)$$

Let us now focus on the calculation of the term $d\check{I}_{1,2}/d\check{X}$. The function

$$\check{g}_{\check{I}_{1,2}}(\check{X}, \check{\Psi}) = \frac{1}{\sqrt{2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}}},$$

is defined and continuous on the interval $\check{\Psi} \in [\check{\Psi}_R + \delta, \check{\Psi}_M]$. Hence, we can apply the *Leibniz integral rule* when calculating $d\check{I}_{1,2}/d\check{X}$. We recall that this rule is about differentiation under the integral sign, where the differentiation is applied to a proper

integral. The rule can be stated as follows:

$$\begin{aligned} \frac{d\check{I}_{1,2}}{d\check{X}} = \check{g}_{\check{I}_{1,2}}(\check{X}, \check{\Psi}_M) \frac{d}{d\check{X}} \check{\Psi}_M - \check{g}_{\check{I}_{1,2}}(\check{X}, \check{\Psi}_R + \delta) \frac{d}{d\check{X}} (\check{\Psi}_R(\check{X}) + \delta) + \\ \int_{\check{\Psi}_R + \delta}^{\check{\Psi}_M} \frac{\partial}{\partial \check{X}} (\check{g}_{\check{I}_{1,2}}(\check{X}, \check{\Psi})) d\check{\Psi}. \end{aligned} \quad (19.37)$$

The first term of the right-hand side of (19.37) is equal to zero as $\check{\Psi}_M$ is constant. Afterwards, when replacing $d\check{\Psi}_R/d\check{X}$ by $d\check{P}/d\check{X}$ by applying (19.35), it follows that

$$\begin{aligned} \frac{d\check{I}_{1,2}}{d\check{X}} = \frac{d\check{P}}{d\check{X}} \left(-\frac{2}{\left(2 + \chi e^{\chi(1+\check{\Psi}_R)}\right) \sqrt{2(\check{\Psi}_R + \delta) + e^{\chi(1+\check{\Psi}_R+\delta)}} - 2\check{P}} + \right. \\ \left. \int_{\check{\Psi}_R + \delta}^{\check{\Psi}_M} \frac{d\check{\Psi}}{\left(2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}\right)^{3/2}} \right). \end{aligned} \quad (19.38)$$

It is convenient to introduce the following variables

$$A = \frac{4\sqrt{\delta}\chi^2 e^{\chi(1+\check{\Psi}_R)}}{\left(2 + \chi e^{\chi(1+\check{\Psi}_R)}\right)^{5/2}}, \quad (19.39a)$$

$$B = \frac{4}{\left(2 + \chi e^{\chi(1+\check{\Psi}_R)}\right) \sqrt{2(\check{\Psi}_R + \delta) + e^{\chi(1+\check{\Psi}_R+\delta)}} - 2\check{P}}, \quad (19.39b)$$

$$C = \frac{1}{\sqrt{2\check{\Psi}_M + e^{\chi(1+\check{\Psi}_M)}} - 2\check{P}}, \quad (19.39c)$$

$$\check{I}_2 = \int_{\check{\Psi}_R + \delta}^{\check{\Psi}_M} \frac{d\check{\Psi}}{\left(2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}\right)^{3/2}}, \quad (19.39d)$$

$$\check{I}_3 = \int_{\check{\Psi}_M}^{+\infty} \frac{d\check{\Psi}}{\left(2\check{\Psi} + e^{\chi(1+\check{\Psi})} - 2\check{P}\right)^{3/2}}. \quad (19.39e)$$

This allows us to write equation (19.31a) as

$$\frac{1}{2} = \frac{d\check{P}}{d\check{X}} \left[-A - B + 2\check{I}_2 + \check{I}_3 + C \right]. \quad (19.40)$$

We shall solve the set of equations (19.40)-(19.31b)-(19.31c) by using the finite difference method. Let us introduce a uniform grid $\check{X}_i = \check{X}_{\max} - \Delta\check{X}(\check{N}_X + 1 - i)$ such that $i \in \llbracket 1; \check{N}_X + 1 \rrbracket$, where $\Delta\check{X}$ is a small grid spacing in $\check{X}$ -direction and $\check{X}_{\max}$ is the right-hand side limit of the domain. The term $d\check{P}/d\check{X}$ in (19.40) may be discretised by using a first-order finite difference scheme

$$\frac{d\check{P}}{d\check{X}} = \frac{\check{P}_i - \check{P}_{i-1}}{\Delta\check{X}}, \quad (19.41)$$

where $\check{P}_i \approx \check{P}(\check{X}_i)$ is the approximate solution of $\check{P}$ at the mesh point i . We shall use marching methods to compute $\check{P}$ throughout the domain: we start our calculations from a large $\check{X} = \check{X}_{\max}$ where $\check{P}|_{\check{X}_{\max}}$ satisfies the matching condition (19.31c). Then, the solution is calculated sequentially, going from one computational line $\check{X}_i$ to the next one $\check{X}_{i-1}$ by applying an explicit scheme. Thus, we discretise equation (19.40) and we use (19.41), and at each computational line $\check{X}_{i-1}$, the pressure $\check{P}_{i-1}$ is given by

$$\check{P}_{i-1} = \check{P}_i - \frac{\Delta\check{X}}{2(-A - B + 2\check{I}_2 + \check{I}_3 + C)}, \quad \text{for all } i \in \llbracket 2; \check{N}_X + 1 \rrbracket, \quad (19.42a)$$

$$\check{P}_{\check{N}_X+1} = -\frac{1}{8}X_{\max}^2. \quad (19.42b)$$

Since $\check{I}_2$ and $\check{I}_3$ are not improper integrals, they can both be computed numerically by using the trapezoidal rule (see Appendix C). Once $\check{P}_{i-1}$ has been determined, before moving to the next computational line, $(\check{\Psi}_R)_{i-1}$ must be calculated. With that purpose in mind, we use the Newton's method to linearise equation (19.31b), that is, we write a Taylor expansion around a computed approximation of $\check{\Psi}_R$, denoted $(\check{\Psi}_R)^-$, and we keep the linear terms. We solve the resulting equation with respect to $\check{\Psi}_R$

$$\check{\Psi}_R = (\check{\Psi}_R)^- - \frac{2(\check{\Psi}_R)^- + e^{\chi(1+(\check{\Psi}_R)^-)} - 2\check{P}}{2 + \chi e^{\chi(1+(\check{\Psi}_R)^-)}}. \quad (19.43)$$

In order to find $\check{\Psi}_R$ at the current mesh point $i - 1$, we iterate until

$$\left| 2 \left((\check{\Psi}_R)^- \right)_{i-1} + e^{\chi(1+(\check{\Psi}_R)^-)_{i-1}} - 2\check{P}_{i-1} \right| < \varepsilon,$$

$\check{X}_{\max}$	$\Delta\check{X}$	δ	$\check{\Psi}_M$	ε
30	0.02	0.05	-0.4	10^{-6}

Table 19.1: Mesh parameters for calculation of the pressure $\check{P}$ in region C . We recall that $\chi = 2|\ln(\sigma)|$

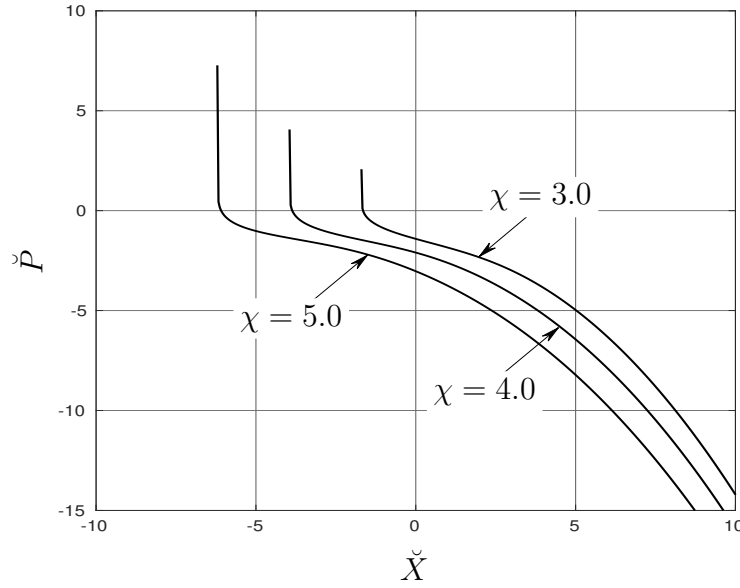


Figure 19.3: Pressure variation in region C for $\Psi_M = -0.4$ and $\chi = 3.0, 4.0$ and 5.0 .

where ε is a small constant.

For our numerical computation, the parameters presented in Table 19.1 have been used. The results of the calculations are displayed in Figure 19.3, where the pressure distribution $\check{P}$ was plotted against $\check{X}$ for $\Psi_M = -0.4$ and for different values of χ . A particularly refined partition must be used when using the trapezoidal rule if we want the problem to be mesh independent. For large values of $\check{X}$, it was found that the results match the asymptote $\check{P} = -(1/8)\check{X}^2$ when $\check{X}$ is close to $\check{X}_{\max}^-$. When χ becomes larger, the graphs start to deviate from one another. $\check{P}$ increases monotonically in the negative $\check{X}$ -direction. Furthermore, the solution finishes at a finite position $\check{X}_0$. According to equation (19.31a), the problem is invariant with respect to a shift in $\check{X}$ -direction. Therefore, the graphs of the pressure for different values of χ may be shifted so that the finite position $\check{X}_0$ coincides with $\check{X} = 0$. It results that, after shifting the graph of $\check{P}$ for a fixed χ , the pressure grows in the negative $\check{X}$ -direction, and it reaches $\check{P}_M$ at $\check{X} = 0$. Then for negative $\check{X}$, there is no interaction between the boundary layer

and the external inviscid flow, and the pressure is constant and equal to $\check{P}_M$. Even though there is a singularity when computed numerically, we can stop our pressure at the value we wish, since we recall that the strength of the expansion fan is given by (19.24); so we can control $\check{P}_M$ and set it to the value we wish. We do not want $\check{P}$ to go to infinity, but instead we want it to stop at $\check{P}_M$, as it is the maximum of the pressure in region C . Consequently, when $\check{P}_M = O(1)$, the problem is fully described. However, if $\check{P}_M$ becomes large, there is a singularity in $\check{P}$. Then, an additional region would need to be introduced in the vicinity of this singularity, where dP/dY should be restored in the governing equations.

19.2 Analysis of the singularity

Here, we shall deduce an analytical form of the pressure $\check{P}$ in the vicinity of the singularity. When P_M tends to infinity, $\check{P}$ develops a singularity and the contribution of region $1a$ is mainly due to the region lying inside the loop formed by the streamline where the stream-function is constant and equal to Ψ_M (see Figure 19.2). This allows us to neglect the second and third integrals on the right-hand side of (19.31a). Furthermore, the streamlines originating from region $1a$ cannot penetrate the region where $\check{P}$ is large. Hence, the contribution of region $1a$ in the composite Bernoulli function, given by (19.14), can be neglected. It results that (19.31a) reduces to

$$\frac{1}{4} = \frac{d}{d\check{X}} \int_{\check{\Psi}_R}^{\infty} \frac{d\check{\Psi}}{\sqrt{e^{\chi(1+\Psi)} - 2\check{P}}}, \quad (19.44)$$

and under the previous assumptions, Ψ_R is now defined as

$$e^{\chi(1+\Psi_R)} - 2\check{P} = 0. \quad (19.45)$$

The integral on the right-hand side of (19.44) may be integrated as follows

$$\int_{\check{\Psi}_R}^{\infty} \frac{d\check{\Psi}}{\sqrt{e^{\chi(1+\Psi)} - 2\check{P}}} = \frac{\pi}{\chi\sqrt{2\check{P}}}. \quad (19.46)$$

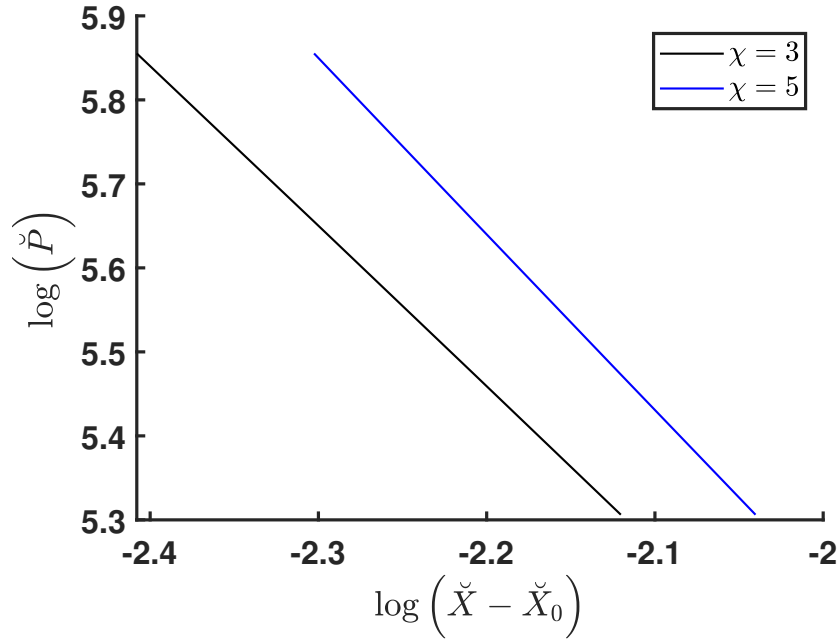


Figure 19.4: Logarithm of the pressure $\check{P}$ plotted with respect to the logarithm of $\check{X} - \check{X}_0$ for $\chi = 3, 5$. $\check{X}_0$ is the position of the singularity, found numerically.

Substituting (19.46) into (19.44) leads to

$$\check{P} = \frac{8\pi^2}{\chi^2 (\check{X} - \check{X}_0)^2}, \quad \text{as } \check{X} \rightarrow \check{X}_0^+, \quad (19.47)$$

where we recall that $\check{X}_0$ corresponds to the position of the singularity. The analytical asymptotic behaviour of $\check{P}$ (19.47) when $\check{X}$ approaches $\check{X}_0^+$ coincides very well with our numerical results. Indeed, applying the logarithm function to equation (19.47) yields to

$$\log(\check{P}) = \text{constant} - 2 \log(\check{X} - \check{X}_0), \quad \text{as } \check{X} \rightarrow \check{X}_0^+. \quad (19.48)$$

We plot in Figure 19.4 $\log(\check{P})$ with respect to $\log(\check{X} - \check{X}_0)$ for $\chi = 3$ and $\chi = 5$. The values for $\check{P}$ we chose are in the vicinity of the singularity. The slopes we obtained numerically are equal to approximately -1.97 and -2.09 , for $\chi = 3$ and $\chi = 5$, respectively. This is consistent with the value -2 of the slope in equation (19.48). Let us notice that the results of Figure 19.4 are valid when solving equation (19.44).

Chapter 20

Discussion

This problem describes the interaction between the boundary layer and the external supersonic flow, where an impinging expansion fan impinging upon the boundary layer. Our objective in this part is to understand if and when the boundary layer can separate from a rigid body surface that moves in the direction opposite to the main flow external to the boundary layer. The reason why the flow is chosen to be supersonic is twofold. First, in a supersonic flow the region of interest is relatively small which makes it easier to formulate the governing equations and boundary conditions. Second, we wanted to study both the flow with adverse pressure gradient and the flow with favourable pressure gradient. As we have seen, the adverse and favourable pressure gradients can be easily achieved in supersonic flows by imposing either a shock wave or an expansion fan impinging on the boundary layer.

The interaction of a shock wave with the boundary layer is a classical problem of the triple-deck theory. Strictly speaking this theory is valid when the non-dimensional wall velocity $\bar{U}_w = \hat{V}_{sh}/\hat{V}_\infty$ is an order $O(Re^{-1/8})$ quantity. In Chapter 17, we presented the results of the numerical solution of the triple-deck equations for the boundary layer on an upstream moving wall (when studied in the reference frame moving with the shock/expansion fan, the plate is no more motionless and it moves in the upstream direction). The case of an impinging shock, when the boundary layer is exposed to the adverse pressure gradient, was analysed earlier by Zhuk (1982) and Yapalparvi and van Dommelen (2012). Their calculations (and our theoretical arguments) suggest that the increasing wall speed suppresses the separation. Keeping this in mind, the main attention in the present work is focused on the boundary layer exposed to an expansion fan. We found that two recirculation regions form in the boundary layer; one lies downstream of the impinging expansion fan and the other upstream of the fan. The former grows as the wall speed $|U_w|$ increases, while the latter becomes relatively small. The observed flow behaviour resembles that in the so-called ‘pseudo-shock’ Ruban and Vonatsos (2008), which establishes a link to the “collisional separation” as described by

Stewartson et al. (1980) and Kluwick and Wohlfahrt (1986). Indeed, there is a collision between the flow entrained by the plate, with the flow moving in the downstream direction.

These conclusions are confirmed by theoretical analysis of the flow that has been performed under the assumption that the wall speed is larger than that in the triple-deck theory, namely, $1 \gg \widehat{V}_{\text{sh}}/\widehat{V}_{\infty} \gg Re^{-1/8}$. In this case the viscous lower tier of the triple-deck structure splits into the inviscid and viscous layers. The pressure distribution along the interaction region can be found by analysing the interaction of the inviscid layer with the external supersonic flow; this analysis was performed in Chapter 16. Still, to satisfy the no-slip condition on the body surface, one has to study the flow behaviour in the near-wall viscous layer. We found in Chapter 18 that the solution for the viscous layer remains smooth in the case of impinging shock, but develops a singularity when the boundary layer interacts with the expansion fan. A detailed analysis of the flow behaviour near the singularity shows that a new shorter region, called region C , should be introduced in the vicinity of the separation point. This region is studied in details in Chapter 19. In this vicinity the flow is governed by the integro-differential equation (19.31a). We found numerically and by means of theoretical analysis of (19.31a) that solution encounters yet another singularity (19.47). The physical processes that lead to its formation are essentially the same as those leading to a singularity in the expansion solution of the classical triple-deck theory; see Neiland (1969a) and Neiland and Sychev (1966). Nevertheless, when $\check{P}_M$, the maximum pressure in region C , is of order one, then the pressure grows in the negative $\check{X}$ -direction, and it stops at $\check{P}_M$. Then, for further upstream $\check{X}$, the pressure is constant, and it is equal to $\check{P}_M$: this means that the problem is fully described when $\check{P}_M = O(1)$ and that there is no singularity in that case. A singularity arises when $\check{P}_M$ is large. Under this condition, the equation for the pressure reduces to (19.44). In order to finish the analysis of the problem, a new region should be introduced in the vicinity of this singularity to smooth it out, and where the derivative of P with respect to Y should be reintroduced into the governing equations. This task is left for future publications.

Chapter 21

Conclusion of the thesis

This thesis focuses on the complex topic of unsteady boundary-layer separation with large Reynolds number Re , using asymptotic and numerical methods. This is performed through three distinct problems presented in Parts I, II and III. Studying these three problems analytically shows that asymptotic methods allow us to understand the behaviour of unsteady flows with large Reynolds number, and in particular, unsteady boundary-layer separation, without making practical costly experiments or involving heavy computations by solving the Navier-Stokes equations. The simplification of these equations employing the method of matched asymptotic expansions, leading to the boundary-layer equations, turn a set of elliptic equations into a set of mixed-type equations which are no longer elliptic. Thus, discontinuities are to be expected due to the sub-hyperbolic operator of the unsteady three- or two-dimensional incompressible boundary-layer equations (as treated in Parts I and II, respectively) or supersonic and quasi-steady boundary-layer equations (as dealt with in Part III). It is particularly important to track discontinuities in the flow functions, as a sharp variation of the velocity is related to a large pressure gradient; and separation occurs when the flow is subject to a “large” adverse pressure gradient. Understanding flow separation is important as nowadays society brings new challenges, and this topic is concerned with many practical situations. For example, the dimples of a golf ball create a turbulent boundary layer of air that clings to the ball’s surface. This allows the boundary layer to detach further downstream, leading to a decrease of the size of the wake. Another example, at a larger scale, concerns the shock wave generated by a supersonic aircraft that may hit another aircraft flying nearby with almost the same speed. The perturbation of the boundary layer may result in flow separation and cause a crash. This topic is addressed in Part III of this thesis.

Part I deals with the unsteady and incompressible free jet generated by the rotation of a plane disk of finite radius. The disk, placed in a large reservoir filled with an initially stagnant fluid, is impulsively brought into rotation around its axis at time

$\hat{t} = 0$. This problem is a case of *collisional singularity*, discussed by Stewartson et al. (1980), Kluwick and Wohlfahrt (1986), Allen and Riley (1994), and more recently by Hewitt and Duck (2011). Using the Prandtl's hierarchical approach, the domain of the flow may be divided into three regions (see Figure 2.4). Region 1 is the boundary layer and its thickness is $O(Re^{-1/2})$. It is located in the vicinity of the disk surface and the jet axis, and it is governed by the three-dimensional boundary-layer equations written in cylindrical coordinates. At the outer edge of the boundary layer lies region 2, where the flow is inviscid and governed by the modified Euler equations, and where the fluid particles are subject to the entrainment effect process. Its thickness is $O(1)$. Finally, region 3 is where the flow is stagnant. Solving numerically the unsteady boundary-layer equations using finite difference methods revealed that a pseudo-shock propagates in both radial and circumferential directions as time increases with a finite speed. The term “pseudo-shock” has been introduced by Ruban and Vonatsos (2008) in order to distinguish it from classical shock waves in gas dynamics, as their thickness is much smaller than the one of the pseudo-shock. The pseudo-shock forms as soon as the jet leaves the disk edge, contrary to Hewitt and Duck (2011) where their pseudo-shock develops at a finite time. Plotting the radial and circumferential velocity components with respect to r , the dimensionless radial direction, and along the jet axis, shows that they experience a discontinuity at the position of the pseudo-shock, such that immediately downstream of it, the flow is at rest. This discontinuity, developed due to elements of hyperbolicity of the unsteady boundary-layer equations, represents the front of the jet, and it propagates as time increases. The plot of the velocity component parallel to the disk axis at the outer edge of the boundary layer shows that it develops a singularity at the position of the pseudo-shock. Therefore, a new region, called 1_{ps} , needs to be introduced in order to smooth out this singularity, and to analyse the inner structure of the pseudo-shock. With that purpose in mind, we reintroduce the pressure gradient into the momentum equations as the discontinuity (and singularity) in the velocity components can only be explained by a strong pressure variation. The asymptotic analysis of this new region, whose dimensions are $Re^{-1/2} \times Re^{-1/2}$, reveals that the flow is inviscid in the leading-order approximation, and it may be treated as quasi-steady and quasi-two-dimensional when observed in the reference frame moving with the shock. After being rewritten in terms of the stream-function vorticity relation and adding an artificial small viscosity to preclude the problem to be ill-posed, the governing equations in the $Re^{-1/2} \times Re^{-1/2}$ region are solved employing multigrid splitting techniques associated with the pseudo-transient method. In the plot of the streamline

pattern, we can observe that as the jet penetrates the stagnant fluid, it ejects the fluid from the boundary layer into the surrounding area. In order to understand what happens to the ejected fluid, we investigate the inviscid region 2 at the outer edge of the boundary layer by solving numerically the governing equations applying the successive over-relaxation method. We find that the pseudo-shock acts as a point source, where the streamlines diverge from that point. Then the fluid is redirected away due to the entrainment effect, and it is reinjected back into the boundary layer further upstream. As a result, in the $Re^{-1/2} \times Re^{-1/2}$ region on the pseudo-shock's scale, we observe a dividing streamline delimiting the front of the jet, and two recirculation regions are formed within the pseudo-shock, one of each side of the jet axis. Finally, we plot the pressure distribution on the pseudo-shock's scale. We find that the pressure grows from zero in the positive radial direction up to its maximum located along the line $X = X_S$. Point S , whose coordinates in the (X, Z) -plane are $(X_S, 0)$, corresponds to the stagnation point where the dividing streamline crosses the jet axis. Downstream of that point, the pressure decays up to zero. The work presented in Part I could be expanded to the case of a rotating sphere, generating two boundary layers developing along its surface and colliding at the equator, generating a jet.

Part II deals with a rather similar problem even though there are clear differences. This problem concerns an unsteady and incompressible wall jet. We considered a semi-infinite region filled with an initially stagnant fluid, and separated by a vertical and impermeable barrier. A narrow slit is opened at $\hat{t} = 0$, and a wall jet propagates as time increases at a finite speed with a well-established front. Similarly to Part I, using the Prandtl's hierarchical approach, we find that the domain is divided into three regions: region 1 being the boundary layer, region 2 where the flow is inviscid and where the entrainment process takes place, and region 3 where the flow is stagnant. The problem is two-dimensional, and solving numerically the boundary-layer equations shows a discontinuity at the position of the pseudo-shock in the longitudinal velocity, while the lateral velocity component is singular at this position. The main difference with Part I is that the problem is no longer symmetric; while the matching conditions with region 2 are of same nature, on the other side of the jet, the no-slip condition must be satisfied on the motionless plate surface. A new region in the vicinity of the pseudo-shock, called 1_{ps} is introduced in order to smooth out the singularity, and as for Part I, the flow is inviscid and quasi-steady in the leading-order approximation when observed in the reference frame moving with the pseudo-shock. However, the speed of the shock cannot be determined analytically. We also show that no dividing streamline

is expected neither. Solving numerically the governing equations allows us to plot the streamline pattern in the $Re^{-1/2} \times Re^{-1/2}$ region in the vicinity of the pseudo-shock, and we can see that the shock penetrates the stagnant fluid, but no recirculation regions appear contrary to the rotating disk case. We finish the analysis by introducing a near-wall viscous layer in order to restore the viscous terms into the momentum equation, so that the no-slip condition can be satisfied on the plate surface. It appears that the solution is smooth in this region.

In Part III, we consider the supersonic, compressible and unsteady flow past a motionless flat plate, where a shock wave/expansion fan impinges upon the boundary layer at $x = 0$ (where x is the dimensionless coordinate along the plate surface), and it moves in the downstream direction with the dimensionless velocity $\bar{V}_{sh}$. When the dimensionless time t satisfies $t \gg Re^{-1/4}$, the problem may be treated as quasi-steady in the reference frame moving with the shock/expansion fan. In that reference frame, the plate is no longer motionless, but it moves in the upstream direction. The case of a downstream moving plate has been studied by Ruban et al. (2011). Yapalparvi and van Dommelen (2012) dealt with the case of an upstream moving wall numerically only, and when considering a shock wave. The problem is solved analytically in Part III, and we show that no separation can be expected when considering a shock wave. However, when an expansion fan hits the boundary layer, if its strength is set properly, then separation may happen. Firstly, we assume that $\bar{V}_{sh} = O(Re^{-1/8})$, where we recall that Re is the Reynolds number, such that $\bar{V}_{sh} = C \times Re^{-1/8} U_w$ where C is a constant. Under this assumption, the interaction region may be described by the *triple-deck theory*. After writing the interaction problem, we assume that $Re^{-1/8} \ll \bar{V}_{sh} \ll 1$, which amounts to assume that $U_w \rightarrow \infty$. The previous assumption allows us to obtain a large parameter U_w , and to conduct the study analytically, starting from the triple-deck equations. We show that when $U_w \rightarrow \infty$, the lower deck of the triple-deck structure may be divided into two regions: an inviscid region *1a*, and a near-wall viscous sublayer *1b*. We start our analysis with region *1a*, and we find analytically the asymptotic solution using the von Mises variables. We deduce that if the strength of the expansion fan is set such that the velocity at the outer edge of region *1b* is equal to zero at $X_* = 0$ (X_* being scaled with respect to x), then the streamline slope in region *1b* becomes singular at $X_* = 0$. Comparing the streamline slopes of regions *1a* and *1b* when X_* approaches zero, we determine that, no matter the value of U_w , the displacement effect of region *1a* is predominant in comparison with the one of region *1b*. However, there exists a small region, called region *C*, where the contributions of regions *1a* and *1b* become

of the same order. This new region C allows us to smooth out the singularity of the streamline slope of region $1b$. In this region, we add a small perturbation $\varepsilon \check{P}_M$ (where $\varepsilon \ll 1$) to the strength of the expansion fan. We compute the pressure numerically in this region, and we find that it grows in the negative $\check{X}$ -direction, and it finishes at a finite value of $\check{X} = \check{X}_0$ where the pressure becomes equal to $\check{P}_M$. Therefore, when $\check{P}_M = O(1)$, the pressure distribution is smooth and the problem is solved. However when $\check{P}_M$ becomes large, the pressure in region C becomes singular at $\check{X}_0$. For future publications, a new region needs to be introduced to smooth out the singularity, and where the derivative of the pressure with respect to the coordinate normal to the plate must be reintroduced into the momentum equation.

Appendix A

Thomas algorithm

The Thomas algorithm allows to solve efficiently systems of linear algebraic equations of the form

$$AX = B, \quad (\text{A.1})$$

where X is a vector of unknowns such that $X \in \mathcal{M}_{N,1}(\mathbb{R})$ with $N \in \mathbb{N}^*$. $A \in \mathcal{M}_N(\mathbb{R})$ is a known tridiagonal matrix, and $B \in \mathcal{M}_{N,1}(\mathbb{R})$ is a known column vector such that

$$A = \begin{pmatrix} b_1 & c_1 & 0 & 0 & \dots & 0 \\ a_2 & b_2 & c_2 & 0 & \dots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & \dots & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & \dots & 0 & 0 & a_N & b_N \end{pmatrix}, \quad B = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ \vdots \\ r_{N-1} \\ r_N \end{pmatrix}, \quad (\text{A.2})$$

respectively.

The Thomas algorithm is a variant of the Gauss elimination which takes advantage of the banded nature of matrix A . We recall that a matrix is said to be sparse and banded if there are few nonzero terms, and if these terms are clustered around the diagonal of the matrix. While the Gauss method with partial pivoting involves $O(N^3)$ calculations, the Thomas algorithm only requires $O(N)$ calculations, which makes it way more efficient than the Gauss method. Furthermore, the Thomas algorithm requires less memory allocation as only three column vectors must be defined instead of a whole matrix.

This algorithm involves of two stages.

Stage 1

The first stage, often termed as “sweeping forward”, consists in rewriting the matrix equation (A.1) in the form $UX = \rho$, where U is an upper triangular matrix such that the terms on the diagonal are all equal to 1.

• Row 1. The first line L_1 is divided by b_1 : $L_1 \leftarrow L_1/b_1$, and the first equation is rewritten as

$$x_1 + \gamma_1 x_2 = \rho_1, \quad \text{with } \gamma_1 = \frac{c_1}{b_1}, \quad \rho_1 = \frac{r_1}{b_1}.$$

• Row 2. We perform the transformation $L_2 \leftarrow L_2 - a_2 L_1$ on the second line L_2 . Hence, the second equation has the form

$$(b_2 - a_2 \gamma_1) x_2 + c_2 x_3 = r_2 - a_2 \rho_1.$$

Then, the second line is divided by $b_2 - a_2 \gamma_1$: $L_2 \leftarrow L_2 / (b_2 - a_2 \gamma_1)$. Now, the second equation is written

$$x_2 + \gamma_2 x_3 = \rho_2, \quad \text{with } \gamma_2 = \frac{c_2}{b_2 - a_2 \gamma_1}, \quad \rho_2 = \frac{r_2 - a_2 \rho_1}{b_2 - a_2 \gamma_1}.$$

• The same process is reiterated from row 3 to row $N - 1$. The transformations on line i are performed by involving line $i - 1$.

• Row N . We eliminate a_N using the transformation $L_N \leftarrow L_N - a_N L_{N-1}$. Thus, the last equation is now written

$$x_N = \rho_N, \quad \text{with } \rho_N = \frac{r_N - a_N \rho_{N-1}}{b_N - a_N \gamma_{N-1}}.$$

Stage 2

The second stage is a “backward sweep”, starting from line N up to line 1.

- Row N . The last equation has been already solved during stage 1: $x_N = \rho_N$.
- Row $N - 1$. We rearrange line $N - 1$ to get the unknown x_{N-1}

$$x_{N-1} = \rho_{N-1} - \gamma_{N-1} x_N.$$

- We reiterate this process until line 1.

Appendix B

Von Mises variables transformation

This part of the appendix explains how to turn a partial differential equation from Cartesian coordinates into Von Mises variables. We will pick the example of the momentum equation (15.11a). The two components of the velocity vectors are involved into it, namely

$$U = U(X, Y), \quad V = V(X, Y), \quad (\text{B.1})$$

which are both functions of X and Y .

Von Mises variables are a pair of independent variables (s, ψ) defined such that

$$s = X, \quad (\text{B.2a})$$

$$U = \frac{\partial \psi}{\partial Y}, \quad V = -\frac{\partial \psi}{\partial X}, \quad (\text{B.2b})$$

where we recall that ψ is the stream function.

The chain rule shall be used in order to convert the differential operators $\partial/\partial X$, $\partial/\partial Y$ and $\partial^2/\partial Y^2$:

$$\begin{aligned} \frac{\partial U}{\partial X} &= \frac{\partial U}{\partial s} \frac{\partial s}{\partial X} + \frac{\partial U}{\partial \psi} \frac{\partial \psi}{\partial X} \\ &= \frac{\partial U}{\partial X} - V \frac{\partial U}{\partial \psi}, \end{aligned} \quad (\text{B.3a})$$

$$\begin{aligned} \frac{\partial U}{\partial Y} &= \frac{\partial U}{\partial s} \frac{\partial s}{\partial Y} + \frac{\partial U}{\partial \psi} \frac{\partial \psi}{\partial Y} \\ &= U \frac{\partial U}{\partial \psi}, \end{aligned} \quad (\text{B.3b})$$

$$\begin{aligned} \frac{\partial^2 U}{\partial Y^2} &= \frac{\partial}{\partial s} \left(U \frac{\partial U}{\partial \psi} \right) \frac{\partial s}{\partial Y} + \frac{\partial}{\partial \psi} \left(U \frac{\partial U}{\partial \psi} \right) \frac{\partial \psi}{\partial Y} \\ &= U \frac{\partial}{\partial \psi} \left(U \frac{\partial U}{\partial \psi} \right). \end{aligned} \quad (\text{B.3c})$$

Hence, the momentum equation written in terms of von Mises variables has the following form

$$U \frac{\partial U}{\partial X} = -\frac{dP}{dX} + U \frac{\partial}{\partial \psi} \left(U \frac{\partial U}{\partial \psi} \right). \quad (\text{B.4})$$

Appendix C

The trapezoidal rule

The trapezoidal rule allows to compute numerically an approximation of the integral of a function f which is continuous, and defined everywhere on a closed integration interval $[a, b]$. This method consists of splitting the integration interval into a set of small intervals where the function f is approximated on each sub-interval by a linear function. Let $[c, d]$, a sub-interval of the closed interval $[a, b]$, such that $a \leq c < d \leq b$. The trapezoidal rule on the closed interval $[c, d]$ approximates the integral of function f by

$$\int_c^d f(x)dx \approx \int_c^d (mx + n)dx, \quad (\text{C.1})$$

where $(m, n) \in \mathbb{R}^2$, and such that $f(c) = mc + n$ and $f(d) = md + n$. Then, it follows that

$$\int_c^d f(x)dx \approx \frac{1}{2}(d - c)\left(f(c) + f(d)\right). \quad (\text{C.2})$$

The smaller the quantity $|d - c|$ is, the more accurate the approximation of the integral of f will be.

Thus, if the closed interval $[a, b]$ is split into n sub-intervals $[c_1, c_2]$, $[c_2, c_3]$, $\dots$ $[c_n, c_{n+1}]$ such that $c_1 = a$ and $c_{n+1} = b$, then an approximation of the integral of function f on segment $[a, b]$ can be written as

$$\int_a^b f(x)dx = \frac{1}{2} \sum_{i=1}^n (c_{i+1} - c_i) \left(f(c_{i+1}) + f(c_i) \right). \quad (\text{C.3})$$

Bibliography

- Allen, T. and Riley, N. (1994). The three-dimensional boundary layer on a yawed body of revolution. *Journal of Engineering Mathematics*, 28:345–364.
- Blasius, H. (1908). Grenzsichten in flüssigkeiten mit kleiner reibung. *Zeitschrift für angewandte Mathematik und Physik*, 56:1–37.
- Bogolepov, V. V. and Neiland, V. Y. (1971). Supersonic viscous flow over a small roughnesses on the surface of a body. *Tr. TsAGI*, No. 1363.
- Burggraf, O. D. (1966). Analytical and numerical studies of the structure of steady separated flows. *Journal of Fluid Mechanics*, 24:113–151.
- Calabretto, S. A. W., Denier, J. P., and Levy, B. (2019). An experimental and computational study of the post-collisional flow induced by an impulsively rotated sphere. *Journal of Fluid Mechanics*, 881:772–793.
- Cassel, K., Smith, F., and Walker, J. (1996). The onset of instability in unsteady boundary-layer separation. *J. Fluid Mech.*, 315:223–256.
- Christov, C. and Tsankov, I. (1993). Numerical investigation of the laminar boundary layer flow around an impulsively moved circular cylinder. *Computer Meth. Appl. Mech. Eng.*, 109(1-2):1–15.
- Cowley, S. J. (1983). Computer extension and analytic continuation of Blasius’ expansion for impulsive flow past a circular cylinder. *J. Fluid Mech.*, 135:389–405.
- Degani, A. T., Li, Q., and Walker, J. D. A. (1996). Unsteady separation from the leading edge of a thin airfoil. *Physics of Fluids*, 8:704.
- Douglas, J. and Gunn, J. (1964). A general formulation of alternating direction methods. *Numerische Mathematik*, 6:428–453.
- Elliott, J. W., Smith, F. T., and Cowley, S. J. (1983). Breakdown of boundary layers: (i) on moving surfaces; (ii) in semi-similar unsteady flow; (iii) in fully unsteady flow. *Geophys. Astrophys. Fluid Dyn.*, 25(1-2):77–138.

- Glauert, M. B. (1956). The wall jet. *Journal of Fluid Mechanics*, 1:625–643.
- Goldstein, S. (1930). Concerning some solutions of the boundary layer equations in hydrodynamics. *Mathematical Proceedings of the Cambridge Philosophical Society*, 26:1–30.
- Goldstein, S. (1948). On laminar boundary-layer flow near a position of separation. *Q. J. Mech. Appl. Math.*, 1(1):43–69.
- Hartree, D. R. (1939). A solution of the laminar boundary-layer equations for retarded flow. *British R and M. Aero. Res. Coun. Rep. Memo.* 2426 (issued 1949).
- Hewitt, R. E. and Duck, P. W. (2011). Pulsatile jets. *Journal of Fluid Mechanics*, 670:240–259.
- Howarth, L. (1938). On the solution of the laminar boundary layer equations. *Proc. Roy. Soc. London, A* 164:547–579.
- Ingham, D. (1984). Unsteady separation. *J. Comput. Phys.*, 53(1):90–99.
- Kevorkian, J. and Cole, J. D. (1981). *Perturbation methods in applied mathematics*. Springer-Verlag.
- Kirchhoff, G. (1869). Zur theorie freier flüssigkeitsstrahlen. *Journal für die reine und angewandte Mathematik*, 70:289–298.
- Kirsten, J. (2018). *Viscous-Inviscid Interaction on Moving Walls*. PhD thesis, Imperial College London.
- Kluwick, A. and Wohlfahrt, H. (1986). Hot-wire-anemometer study of the entry flow in a curved duct. *Journal of Fluid Mechanics*, 165:335–353.
- Korolev, G. L., Gajjar, J. S. B., and Ruban, A. I. (2002). Once again on the supersonic flow separation near a corner. *Journal of Fluid Mechanics*, 463:173–199.
- Kravtsova, M. A., Zametaev, V. B., and Ruban, A. I. (2005). An effective numerical method for solving viscous-inviscid interaction problems. *Phil. Trans. R. Soc. A*, 363(1830):1157–1167.
- Kármán, T. V. (1921). Über laminare und turbulente reibung. *Zeitschrift für Angewandte Mathematik und Mechanik*, 1:233–252.

- Landau, L. D. and Lifshitz, E. M. (1944). *Mechanics of Continuous Media*. Gostekhizdat, Moscow.
- Levi-Civita, T. (1907). Scie e leggi di resistenza. *Rendconte de Circolo Matematico di Palermo*, XXIII:1–37.
- Mallinson, G. D. and de Vahl Davis, G. (1973). The method of the false transient for the solution of coupled elliptic equations. *Journal of Computational Physics*, 12:435–461.
- Messiter, A. F. (1970). Boundary-layer flow near the trailing edge of a flat plate. *SIAM Journal on Applied Mathematics*, 18:241–257.
- Moore, F. K. (1958). On the separation of the unsteady laminar boundary layer. In Görtler, H., editor, *Boundary Layer Research*, pages 296–311. Springer-Verlag.
- Nakayama, Y. (1988). *Visualized Flow. Fluid Motion in Basic and Engineering Situations Revealed by Flow Visualization*. Compiled by the Japan Society of Mechanical Engineers. Pergamon Press.
- Naqavi, I. Z., Tyacke, J. C., and Tucker, P. G. (2018). Direct numerical simulation of a wall jet: flow physics. *Journal of Fluid Mechanics*, 852:507–542.
- Neiland, V. (1969a). Asymptotic theory for calculating heat fluxes near the corner of a body contour. *Izv. Akad. Nauk SSSR, Mech. Zhidk. Gaza*, 4:53–60.
- Neiland, V. (1969b). Theory of laminar boundary layer separation in supersonic flow. *Izv. Akad. Nauk SSSR, Mech. Zhidk. Gaza*, 4:33–35.
- Neiland, V. and Sychev, V. V. (1966). Asymptotic solutions of the Navier–Stokes equations in regions with large local perturbations. *Izv. Akad. Nauk SSSR, Mech. Zhidk. Gaza*, 1(4):43–49.
- Peaceman, D. W. and Rachford, Jr, H. H. (1955). The numerical solution of parabolic and elliptic differential equations. *Journal of the Society for Industrial and Applied Mathematics*, 3:28–41.
- Peridier, V., Smith, F., and Walker, J. (1991). Vortex-induced boundary-layer separation. Part 1. The unsteady limit problem $Re \rightarrow \infty$. *J. Fluid Mech.*, 232:99–131.
- Prandtl, L. (1904). Über flüssigkeitsbewegung bei sehr kleiner Reibung. In *Verh. III. Intern. Math. Kongr., Heidelberg*, pages 484–491. Teubner, Leipzig, 1905.

- Rott, N. (1956). Unsteady viscous flow in the vicinity of a stagnation point. *Q. Appl. Math.*, 13(4):444–451.
- Ruban, A. I. (2015). *Fluid Dynamics - Part 2: Asymptotic Problems of Fluid Dynamics*. Oxford University Press.
- Ruban, A. I. (2018). *Fluid Dynamics - Part 3: Boundary Layers*. Oxford University Press.
- Ruban, A. I., Araki, D., Yapalparvi, R., and Gajjar, J. S. B. (2011). On unsteady boundary-layer separation in supersonic flow. part 1. upstream moving separation point. *Journal of Fluid Mechanics*, 678:124–155.
- Ruban, A. I. and Gajjar, S. B. (2014). *Fluid Dynamics - Part 1: Classical Fluid Dynamics*. Oxford University Press.
- Ruban, A. I. and Vonatsos, K. N. (2008). Discontinuous solutions of the boundary-layer equations. *Journal of Fluid Mechanics*, 614:407–424.
- Schlichting, H. (1933). Laminare strahlausbreitung. *Zeitschrift für Angewandte Mathematik und Mechanik*, 13:260–263.
- Sears, W. R. (1956). Some recent developments in airfoil theory. *Journal of the Aeronautical Sciences*, 23:490–499.
- Shidlovskii, V. P. (1977). The structure of the viscous fluid flow near the edge of a rotating disk. *Journal of Applied Mathematics and Mechanics*, 41:464–472.
- Stewartson, K. (1969). On the flow near the trailing edge of a flat plate ii. *Mathematika*, 16:106–121.
- Stewartson, K., Cebeci, T., and Chang, K. C. (1980). A boundary-layer collision in a curved duct. *The Quarterly Journal of Mechanics and Applied Mathematics*, 33:59–75.
- Stewartson, K. and Williams, P. G. (1969). Self-induced separation. *Proc. Roy. Soc. London, A* 312:181–206.
- Sychev, V. V., Ruban, A. I., Sychev, Vic. V., and Korolev, G. L. (1998). *Asymptotic Theory of Separated Flows*. Cambridge University Press.

- Sychev, Vic. V. (1979). Asymptotic theory of nonstationary separation. *Mekh. Zhid. Gaza*, pages 21–32. (English translation *Fluid Dyn.* **14** (6), 829–838).
- Sychev, Vic. V. (1980). On certain singularities in solutions of the boundary-layer equations on a moving surface. *Prikl. Mat. Mekh.*, 44(5):831–838.
- Sychev, Vic. V. (1983). Theory of unsteady boundary-layer separation and wake breakdown. *Usp. Mekh.*, 6:13–51.
- Sychev, Vic. V. (1984). On the asymptotic theory of laminar separation from a moving surface. *Prikl. Mat. Mekh.*, 48(2):247–253. (English translation *J. Appl. Math. Mech.* **48** (2), 171–176).
- Sychev, Vic. V. (1987). Analytical solution of the problem of flow near the boundary-layer separation point on a moving wall. *Prikl. Mat. Mekh.*, 51(3):519–521. (English translation *J. Appl. Math. Mech.* **51** (3), 405–407).
- Turner, J. S. (1962). The ‘starting plume’ in neutral surroundings. *Journal of Fluid Mechanics*, 13:356–368.
- van Dommelen, L. and Shen, S. (1980). The spontaneous generation of the singularity in a separating laminar boundary layer. *J. Comput. Phys.*, 38(2):125–140.
- van Dommelen, L. and Shen, S. (1982). The genesis of separation. In Cebeci, T., editor, *Numerical and Physical Aspects of Aerodynamic Flows*, pages 293–311. Springer.
- van Dommelen, L. and Shen, S. (1983a). Boundary layer separation singularities for an upstream moving wall. *Acta Mechanica*, 49(3-4):241–254.
- van Dommelen, L. and Shen, S. (1983b). An unsteady interactive separation process. *AIAA J.*, 21(3):358–362.
- Walker, J. (1978). The boundary layer due to rectilinear vortex. *Proc. R. Soc. Lond. A*, 359(1697):167–188.
- Yapalparvi, R. and van Dommelen, L. L. (2012). Numerical solution of unsteady boundary-layer separation in supersonic flow: upstream moving wall. *J. Fluid Mech.*, 706:413–430.
- Zhuk, V. (1982). On local recirculation zones in the supersonic boundary layer on a moving wall. *Zh. Vychisl. Mat. & Mat. Fiz.*, 22:1255–1260.