Moment-Based Discontinuous Phasor Transform and its Application to the Steady-State Analysis of Inverters and Wireless Power Transfer Systems

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***Abstract*—Power electronic devices are inherently discontinuous systems. Square waves, produced by interconnected transistors, are commonly used to control inverters. This paper proposes a novel phasor transform, based on the theory of moments, which allows to analyze the steady-state behavior of discontinuous power electronic devices in closed-form, *i.e.*, without approximations. In the first part of this paper, it is shown that the phasors of an electric circuit are the moments on the imaginary axis of the linear system describing the circuit. Exploiting this observation, in the second part of this paper, we focus on the analysis of circuits powered by discontinuous sources. The new "discontinuous phasor transform" is defined and the v–i characteristics for inductors, capacitors, and resistors are described in terms of this new phasor transform. Since the new quantities maintain their physical meaning, the instantaneous power and average power can be computed in the phasor domain. The analytic potential of the new tool is illustrated studying the steady-state response of power inverters and of wireless power transfer systems with non-ideal switches.**

***Index Terms*—Discontinuous phasor transform, frequency-domain analysis, moment matching, switched systems.**

I. Introduction

POWER converters are inherently discontinuous[1] systems: power electronics is described as *a branch of electrical and electronic engineering concerned with the analysis, simulation, design, manufacture, and application of switching-mode power converters* [1]–[3]. Since the transient response of electric circuits decays rapidly to zero, the steady-state analysis of power electronic devices is of paramount importance for their design [4]–[7]. However, a trade-off between the level of the approximation and the difficulty of the analysis has usually to be made. Among the techniques that have been developed for this analytic task, the phasor transform represents a powerful and flexible mathematical tool, which has been used for the study of the steady-state behavior of circuits powered by sinusoidal sources [8], [9]. The phasor transform greatly simplifies the dynamic analysis because it changes integro-differential equations in algebraic equations, which are computationally and analytically more easily solvable. In recent years, several generalizations of the phasor transform have been proposed specifically for the analysis of power electronic devices [10]–[13]. These methods are based on the assumption that the phasor is a function of time and that the input signal can be described by a complex exponential. Thus, it is assumed that the input of the circuit is a smooth signal. Periodic discontinuous signals can be analyzed using the Fourier series, obtaining a steady-state description of the response as the sum of the phasors at infinitely many frequencies. However, practically, only an approximation can be obtained, and moreover, high harmonics are usually ignored because they are numerically difficult to compute [10]. In addition, we note that these papers do not explicitly define the time-varying phasor itself. Only the inverse phasor transform is given and the phasor is found comparing two transformed quantities. This forces the designer to recognize and transform known subcircuits. For instance, in [12], nine categories of subcircuits have been identified. One of those, which is not transformed, *entails thousands of different configurations*.

This paper originates from the observation that a phasor is what in the model reduction theory is known as the *moment* of a system. The model reduction problem consists in finding a simplified description of a dynamical system maintaining at the same time specific properties. It has been extensively studied exploiting a plethora of methods, among which there are, precisely, the moment matching methods (for an extensive literature review on model reduction, see [14]). In this paper, it is shown that the phasors of an electric circuit are the components of the unique solution of a Sylvester equation (which are the moments of the system representing the circuit, see, *e.g.*, [15]). The Sylvester equation itself is proved to be the phasor transformed system describing the electric circuit. Exploiting this equivalence, the instantaneous power and the average power are defined utilizing the moments. The interest in revisiting the notion of phasor with the moment theory is twofold:

1) a direct consequence is that the Sylvester equation offers very efficient algorithms to compute the phasors of large-scale systems [14, Section 14.1], [16];
2) the most relevant reason that justifies the interest in this equivalence is that the description of moment has been recently generalized beyond linear systems [15],

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[1]With some abuse of terminology, the word "discontinuous" is used for, e.g., sources, generators, components, to mean that the generated signal is discontinuous.

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[17]–[19]. In particular, in [20] and [21], the moments have been generalized to systems driven by discontinuous signals.

Thus, exploiting the equivalence discussed, and exploiting the results given in [20], a new phasor transform is given. The new phasor transform has the following advantages:

1) differently from other analysis methods, like the state-space averaging, the new phasor transform describes the steady-state response of an electric circuit powered by any periodic discontinuous signal in closed-form, *i.e.*, without approximations;
2) the capability of describing the steady-state response of any electrical circuit without approximations extends also to the study of the behavior of non-ideal switches;
3) differently from [10]–[12], the definition of the phasor itself is given; this has the consequence that it is not necessary to recognize subcircuits (although it is still possible to proceed in this way): the phasor can be computed directly for any linear circuits of whatever complexity;
4) the formula that we provide defines the phasors of all the currents and of the integrals of all the currents in the circuit; as a consequence the steady-state v–i characteristics of all the electric quantities of the circuit can be reconstructed.

To illustrate the utilization of the results of this paper, the steady-state behavior of a Class D *CLL* resonant inverter and of a wireless power transfer system with non-ideal switches (subject to the reverse recovery effect) is analyzed. We show that the new phasor transform is particularly useful when there is little damping in the circuit (scenario which "preserves" the discontinuous behavior that approximating series solutions cannot fully capture).

The rest of this paper is organized as follows. This section continues with a precise formulation of our problem and aim. Then, the circuits that are analyzed in this paper are described and the definition of moments as given in [15] is recalled. In Section II, the equivalence between moments and phasors is proved. The use of the results is illustrated by means of an example and the definition of power is given in terms of moments. In Section III, the definition of phasor and of the inverse phasor transform are generalized to linear circuits powered by discontinuous periodic signals. The phasors of resistors, inductors, and capacitors are given and the characterization of the power is described in the new phasor domain. Extensive simulations show the capability of the new phasor definition in describing the steady-state response of power electronic devices. Section V contains some concluding remarks.

Notation. We use standard notation. $\mathbb{R}_{\geq 0}$ denotes the set of non-negative real numbers; $\mathbb{C}_{<0}$ denotes the set of complex numbers with negative real part. The symbol I denotes the identity matrix and $\sigma(A)$ denotes the spectrum of the matrix $A \in \mathbb{R}^{n\times n}$. The symbol $\Re[z]$ indicates the real part of the complex number z, $\Im[z]$ denotes its imaginary part, and j denotes the imaginary unit. The symbol ϵ_k indicates a vector with the kth element equal to 1 and with all the other elements equal to 0. Given a function $f(t)$, $\overline{F}(\omega)$ represents its phasor at ω, whereas $< f(t) >$ indicates its time average.

A. Problem and Aim

All currents and voltages in linear circuits are described by linear differential equations of the form

$$a_n \frac{d^n}{dt^n} f + a_{n-1} \frac{d^{n-1}}{dt^{n-1}} f + \ldots + a_0 f = u \tag{1}$$

where $f : \mathbb{R} \to \mathbb{R}$ represents a current or voltage, $u(t) \in \mathbb{R}$ is a current or voltage input, and $a_i \in \mathbb{R}$, with $i = 0, \ldots, n$. Without loss of generality, we assume that $a_n \neq 0$. In the analysis of circuits is of interest to study the steady-state response of the system[2], which intuitively can be described as the response of the system when the transient response has vanished. If the input u is a sinusoidal signal of amplitude a_u, angular frequency ω and phase ϕ, then a classical tool for the steady-state analysis of (1) is the phasor of $f(t)$. This is usually introduced by means of the inverse phasor transform, which is defined in his simplest form as

$$f(t) = \Re\left[\overline{F} e^{j\omega t}\right] \tag{2}$$

where[3] $\overline{F} : \mathbb{C} \to \mathbb{C}$ is called the phasor of $f(t)$. Several generalizations of this transform have been proposed. The most general form presented in the literature is the so-called power-invariant (inverse) phasor transform [11] [12], described by

$$f(t) = \Re\left[\overline{F}(t) e^{j\omega(t)}\right] \tag{3}$$

where $\omega(t)$ is nonconstant and the phasor $\overline{F}(t)$ is a function[4] of time and ω. From (3), we note that the analysis of a circuit with the phasor transform is based on the assumption that the input is a complex exponential with frequency $\omega(t)$, *i.e.*, $u(t) = a_u e^{j(\omega(t)+\phi)}$.

In this paper, we revisit and extend the notion of phasor. In the first part of this paper, we show that the phasors of an electric circuit are the moments at $j\omega$ of the linear system describing the circuit. In the second part of this paper, we eliminate the assumption that the source has to be described by a complex exponential extending the notion of phasor, as defined in (3), to a general class of input signals including discontinuous signals, such as square waves and triangular waves.

For the sake of clarity, we now define what we mean with phasor transform of a linear system.

Definition 1: Consider a linear, single-input, single-output, continuous-time, system described by the equations

$$\dot{x} = Ax + Bu, \qquad y = Hx \tag{4}$$

with[5] $x(t) \in \mathbb{C}^n$, $u(t) \in \mathbb{C}$, $y(t) \in \mathbb{C}$, $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times 1}$, and $H \in \mathbb{C}^{1\times n}$. The *phasor transform of the linear system (4)* for the source $u(t) = a_u e^{j(\omega t+\phi)}$, with $a_u \in \mathbb{R}$, $\omega \in \mathbb{R}$ and $\phi \in \mathbb{R}$, is

$$\begin{aligned} \overline{X} j\omega e^{j\omega t} &= A\overline{X} e^{j\omega t} + B a_u e^{j\phi} e^{j\omega t} \\ \overline{Y} e^{j\omega t} &= H\overline{X} e^{j\omega t}. \end{aligned} \tag{5}$$

[2]Note that this means to find the particular solution of (1).

[3]Phasors are frequency dependent. We omit the argument ω from the phasor $\overline{F}(\omega)$ for ease of notation.

[4]We again omit the argument ω from the phasor $\overline{F}(t, \omega)$ for ease of notation.

[5]Usually the state $x(t)$ of a dynamical system represents real quantities. However, herein, we use the complex domain because the quantities involved in the phasor analysis are complex valued.

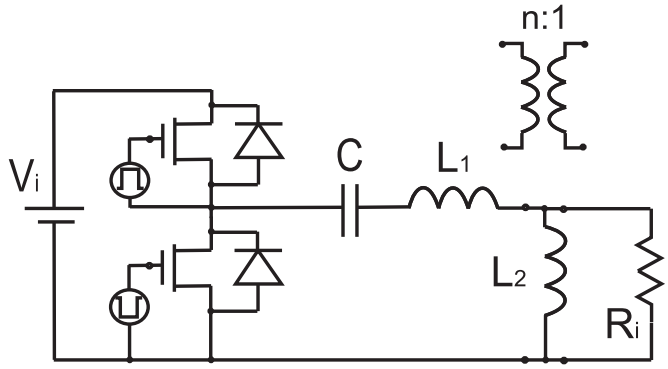


Fig. 1. Class D *CLL* resonant inverter.

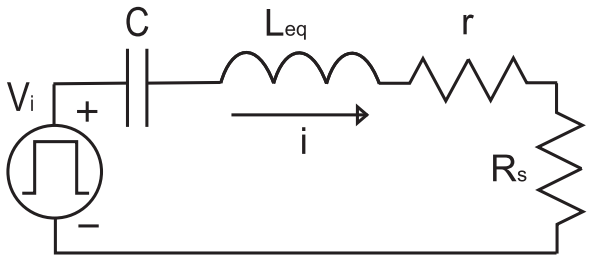


Fig. 2. Equivalent circuit of a Class D *CLL* resonant inverter as derived in [1].

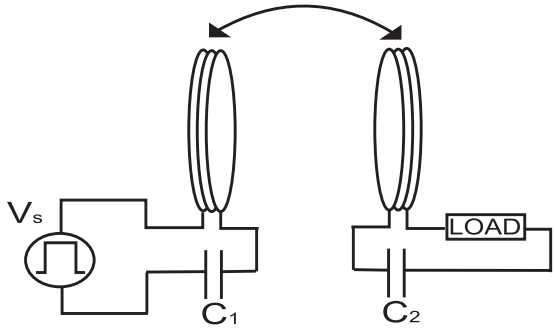


Fig. 3. Wireless power transfer system.

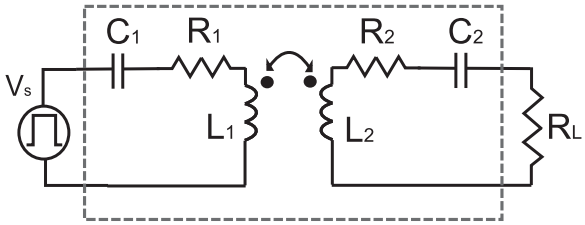


Fig. 4. Equivalent circuit of the wireless power transfer system in Fig. 3.

In what follows, we only assume that the components in the circuit are ideal. We do not make any other simplifying assumption since we aim to describe exactly the behavior of the circuit. In Section IV-B, we relax the hypothesis that the components are ideal considering non-ideal switches.

Remark 1: The theory is developed applying the Kirchhoff's Voltage Law. We assume that the sources are voltage sources and the state variables and the output are currents. As a consequence we show that the moments are the phasors of the currents. An equivalent analysis based on the Kirchhoff's Current Law can be derived.

Remark 2: For the sake of clarity, we consider the single phase case. The results can extend to the multiphase framework either following the approach in [12], or the multi-input approach presented in [22].

B. Case Studies

To illustrate the results of this paper, we study the steady-state response of resonant circuits. The interest in using this typology of circuits lies upon the fact that these are at the basis of many power converters, which are normally driven by square waves. For instance, Fig. 1 shows a Class D *CLL* resonant inverter. The circuit is powered by a dc voltage V_i. The voltage across the capacitor C, which is the output of the subcircuit constituted by the two transistors, is a square wave. Thus, an equivalent circuit can be derived as shown in Fig. 2 (see [1] for the derivation and the definition of the equivalent quantities). We note that the analysis of the Class D *CLL* resonant inverter (and of many other Class D inverters) boils down to the analysis of an *RLC* circuit driven by a square wave. The techniques presented in the second part of this paper are developed for the study of this type of circuits. Moreover, *RLC* circuits arise also in other power electronic applications. For instance, Fig. 3 shows a wireless power transfer system [23], [24] and Fig. 4 shows its equivalent circuit which consists of two coupled *RLC* circuits.

The two examples are used as follows. In Section II, we use the circuit of Fig. 4 because it offers more mathematical complexity, and thus, it is more helpful for showing some important details. In Section III, we first use the circuit of Fig. 2 illustrating the technique for different Q factors[6]. We shows that as the Q factor decreases, *i.e.*, the damping increases, the difference in quality of the approximation between the Fourier series and our solution grows. Finally, we analyze the circuit of Fig. 4 driven by non-ideal switches. We show that the technique presented in this paper can describe, without approximations, the reverse recovery effect, which is present in the majority of commercial switches [25].

C. Moments of Linear Systems

In this section, we recall the notion of moment for linear systems as presented in [15]. Consider system (4), let $W(s) = H(sI - A)^{-1}B$ be the associated transfer function and assume that (4) is minimal, *i.e.*, controllable and observable.

Definition 2: Let $s_i \in \mathbb{C}$, with $s_i \notin \sigma(A)$. The *0-moment of system (4) at* s_i is the complex number $\eta_0(s_i) = H(s_i I - A)^{-1}B$. The *$k$-moment of system (4) at* s_i is the complex number

$$\eta_k(s_i) = \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}(H(sI - A)^{-1}B)\right]_{s=s_i} \tag{6}$$

with $k \geq 1$ integer.

[6]The Q factor, or quality factor, is a parameter that describes the damping of an oscillator or resonator. For an ideal series RLC circuit, we use the definition $\mathrm{Q} = \frac{1}{R}\sqrt{\frac{L}{C}}$.

In [15] (see also [26] and [27]), a characterization of the moments of system (4) has been given in terms of the solution of a Sylvester equation as follows.

Lemma 1: [15]. Consider system (4), $s_i \in \mathbb{C}\backslash\sigma(A)$ for all $i = 1, \ldots, \eta$. There exists a one-to-one relation between the moments $\eta_0(s_1), \ldots, \eta_{k_1-1}(s_1), \ldots, \eta_0(s_\eta), \ldots, \eta_{k_\eta-1}(s_\eta)$ and the matrix $H\Pi$, where Π is the unique solution of the Sylvester equation

$$A\Pi + B\Gamma = \Pi\Sigma \tag{7}$$

with $\Sigma \in \mathbb{C}^{\nu\times\nu}$ any non-derogatory[7] matrix with characteristic polynomial

$$p(s) = \prod_{i=1}^{\eta}(s - s_i)^{k_i} \tag{8}$$

where $\nu = \sum_{i=1}^{\eta} k_i$, and Γ is such that the pair (Γ, Σ) is observable.

In [15], it has also been noted that the moments of system (4) are in one-to-one relation with the well-defined steady-state response of the output of the interconnection between a signal generator with dynamic matrix Σ and output matrix Γ (with the properties described in Lemma 1) and system (4). This interpretation of the notion of moment relies upon the center manifold theory [28], it has the advantage that it can be extended to nonlinear systems and it is of particular interest for the aims of this paper.

Theorem 1: [15]. Consider system (4) and suppose $s_i \in \mathbb{C}\backslash\sigma(A)$ for all $i = 1, \ldots, \eta$, with $\sigma(A) \subset \mathbb{C}_{<0}$. Let $\Sigma \in \mathbb{C}^{\nu\times\nu}$ be any non-derogatory matrix with characteristic polynomial (8). Consider the interconnection of system (4) with the system

$$\dot{\zeta} = \Sigma\zeta, \qquad u = \Gamma\zeta \tag{9}$$

with Γ and $\zeta(0)$ such that the triple $(\Gamma, \Sigma, \zeta(0))$ is minimal. Then, there exists a one-to-one[8] relation between the moments $\eta_0(s_1)$, $\ldots$, $\eta_{k_1-1}(s_1)$, $\ldots$, $\eta_0(s_\eta)$, $\ldots$, $\eta_{k_\eta-1}(s_\eta)$ and the steady-state response of the output y of such interconnected system.

II. Electrical Equivalent of the Moment Theory

In this section, we show that the theory of moments developed to solve the model reduction problem has an electric equivalent. In particular, we prove that the phasors of an electric circuit, as defined in (2), are the moments of the system describing the circuit when a single complex interpolation point is selected. Then, exploiting the equivalence between moments and phasors, we define the power using the moments. Revisiting these results is instrumental in the rest of this paper when the notion of phasor is extended to non-conventional sources: knowing the relation between moments and power is essential to give a physical meaning to the newly defined quantities. To streamline the presentation, we introduce the following definition.

[7]A matrix is non-derogatory if its characteristic and minimal polynomials coincide.

[8]By one-to-one relation, we mean that the moments are uniquely determined by the steady-state response of $y(t)$ and *vice versa*.

Definition 3: The system (4) and the generator (9) are said to be in the *real convention* if the matrices A, B, H, Γ, and Σ have real entries. They are said to be in the *mixed convention* if the matrices A, B, and H have real entries and the matrices Γ and Σ have complex entries. They are said to be in the *complex convention* if the matrices A, B, H, Γ, and Σ have complex entries.

Note that in the real convention and in the mixed convention, for all integers k with $2 \leq 2k \leq n$, the component x_{2k} of x is a current i_k, whereas the component x_{2k-1} of x is the integral $\int_{t_0}^{t} i_k(\tau)d\tau$.

A. Equivalence Between Moments and Phasors

Herein, we show that writing the phasor transform of a linear electric circuit is equivalent to writing the associated Sylvester equation. Moreover, the components of the solution of this Sylvester equation are the phasors of all the currents (and of the integrals of the currents) in the circuit.

Proposition 1: Consider the source $u(t) = a_u e^{j(\omega t+\phi)}$, with $a_u \in \mathbb{R}$, $\omega \in \mathbb{R}$ and $\phi \in \mathbb{R}$, and assume $j\omega \notin \sigma(A)$. The phasor transform of system (4) (see Definition 1) written in the mixed convention coincides with the Sylvester equation (7) with $\Sigma = j\omega$ and $\Gamma = a_u e^{j\phi}$. The components of Π, which is the unique solution of (7), are the phasors of the currents and of the integrals of the currents in the circuit.

Proof: We first compute the phasor transform of system (4) for the source $u = a_u e^{j(\omega t+\phi)}$, namely

$$\begin{aligned} \overline{X} j\omega e^{j\omega t} &= A\overline{X}e^{j\omega t} + Ba_u e^{j\phi}e^{j\omega t} \\ \overline{Y} e^{j\omega t} &= H\overline{X}e^{j\omega t} \end{aligned}$$

by Definition 1. Since $e^{j\omega t} \neq 0$ for all $t \in \mathbb{R}$, it can be canceled out yielding

$$\begin{aligned} \overline{X} j\omega &= A\overline{X} + Ba_u e^{j\phi} \\ \overline{Y} &= H\overline{X}. \end{aligned}$$

Thus, the phasors of all the currents (and their integrals) in the circuit and the phasor of the output current are given by

$$\begin{aligned} \overline{X} &= (j\omega I - A)^{-1} Ba_u e^{j\phi} \\ \overline{Y} &= H(j\omega I - A)^{-1} Ba_u e^{j\phi} \end{aligned}$$

respectively. Consider now the signal generator (9) with

$$\Sigma = j\omega, \ \Gamma = a_u e^{j\phi}.$$

The associated Sylvester equation (7) is

$$A\Pi + Ba_u e^{j\phi} = \Pi j\omega$$

which, if $j\omega \notin \sigma(A)$, has the unique solution

$$\Pi = (j\omega I - A)^{-1} Ba_u e^{j\phi}$$

which concludes the proof. ■

Corollary 1: The phasor of the output response y of system (4) is the moment of the system at $j\omega$, namely $\overline{Y} = H\Pi$. The *inverse phasor transform* of the output current y of system (4) is

$$y(t) = \Re\left[H\Pi e^{\Sigma t}\right].$$

Proof: The first claim follows noting that the phasor of the output response of system (4) is given by

$$\overline{Y} = H\Pi = H(j\omega I - A)^{-1} B a_u e^{\phi}.$$

To prove the second claim, we note that

$$\Re\left[H\Pi e^{\Sigma t}\right] = \Re\left[\overline{Y} e^{j\omega t}\right]$$

which is the inverse phasor transform of the output response of system (4). ■

Remark 3: Phasors are a very special case of moments of linear systems. In fact, they are the moments at the single complex interpolation point $\Sigma = j\omega$.

Remark 4: The higher order derivatives $\frac{d^n x}{dt^n}$ and the integral $\int_{t_0}^{t} x(\tau)d\tau$ are transformed in the phasor domain into $(j\omega)^n \overline{X}$ and $\frac{1}{j\omega}\overline{X}$, respectively. Similarly they are transformed in the "moment domain" into $\Sigma^n \Pi$ and $\Sigma^{-1}\Pi$, respectively.

Remark 5: The relation between moments and phasors established with the Sylvester equation (7) gives the advantage of computing the phasors using Arnoldi and Lanczos algorithms [14, Section 14.1] or the techniques presented in [16]. These techniques are very efficient methods to determine the moments, and thus, the phasors, of very large-scale systems, in which the matrix inversion may be computationally expensive.

Remark 6: The main advantage of the equivalence that we have established is that this lends itself to the extension of the phasor transform beyond the linear framework (and the discontinuous framework of this paper) to circuits with nonlinear elements and circuits with delays. In fact, the moment theory has been extended to nonlinear systems [15], systems with delays [19], and differential-algebraic systems [29].

We present now a worked out example that shows how to apply these results and highlights a few important aspects regarding the "convention" used in the approach.

Example 1: Fig. 4 illustrates a wireless power transfer system [24] consisting of two coils. In this example, we assume that an ac sinusoidal voltage source with an amplitude of V_s and an angular frequency of ω is applied to the transmitter coil on the input side. A load resistor R_L is connected to the receiving coil on the output side. By applying the Kirchhoff's Voltage Law to the two coils, we obtain the system of equations

$$\begin{aligned} R_1 i_1 + L_1 \frac{di_1}{dt} + \frac{1}{C_1}\int i_1 dt + M_{12}\frac{di_2}{dt} &= u(t) \\ M_{21}\frac{di_1}{dt} + R_{2L} i_2 + L_2 \frac{di_2}{dt} + \frac{1}{C_2}\int i_2 dt &= 0 \end{aligned} \qquad (10)$$

where i_1 and i_2 are the currents flowing in the coils 1 and 2, R_1 and R_2 are the resistances, $R_{2L} = R_2 + R_L$, L_1 and L_2 are the self-inductances, C_1 and C_2 are the capacitances, and $M_{12} = M_{21}$ are the mutual inductances between the two coils. We are interested in determining the amplitude and phase of the steady-state current in the receiving coil, *i.e.,* the phasor $\overline{I}_2$.

We start by solving the problem with the phasor transform approach. Transforming the differential equations (10), we obtain the complex algebraic system

$$\begin{aligned} Z_1 \overline{I}_1 + j\omega M_{12}\overline{I}_2 &= V_s \\ j\omega M_{21}\overline{I}_1 + Z_2 \overline{I}_2 &= 0 \end{aligned}$$

where $Z_1 = R_1 + j\omega L_1 - j\frac{1}{\omega C_1}$ and $Z_2 = R_{2L} + j\omega L_2 - j\frac{1}{\omega C_2}$. Solving with respect to $\overline{I}_2$ yields

$$\overline{I}_2 = \frac{-j\omega M_{21}}{Z_1 Z_2 + \omega^2 M_{21} M_{12}} V_s.$$

Now, we compute the moment of the differential system (10) at

$$\dot{\zeta} = \Sigma\zeta,\ u = \Gamma\zeta. \qquad (11)$$

We begin using the "mixed convention" that, as highlighted later, is the most useful among the three representations. Thus, consider $\Gamma = V_s$ and $\Sigma = j\omega$ and the state

$$\begin{aligned} x_1(t) &= \int_{t_0}^{t} i_1(\tau)\, d\tau,\ x_2(t) = i_1(t) \\ x_3(t) &= \int_{t_0}^{t} i_2(\tau)\, d\tau,\ x_4(t) = i_2(t). \end{aligned}$$

System (10) can be represented by the first-order system of differential equations (4) with

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{L_2}{C_1\hat{L}} & -\frac{R_1 L_2}{\hat{L}} & \frac{M_{12}}{C_2\hat{L}} & \frac{M_{12}R_{2L}}{\hat{L}} \\ 0 & 0 & 0 & 1 \\ \frac{M_{21}}{C_1\hat{L}} & \frac{R_1 M_{21}}{\hat{L}} & -\frac{L_1}{C_2\hat{L}} & -\frac{L_1 R_{2L}}{\hat{L}} \end{bmatrix} \\ B &= \begin{bmatrix} 0 & \frac{L_2}{\hat{L}} & 0 & -\frac{M_{21}}{\hat{L}} \end{bmatrix}^{\top} \\ H &= \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned} \qquad (12)$$

where $\hat{L} = L_1 L_2 - M_{12} M_{21} \neq 0$. The solution of the Sylvester equation (7) is given by

$$\Pi = (j\omega - A)^{-1} B V_s$$

and the moment of the system at (Γ, Σ) is given by

$$H\Pi = \epsilon_4 \Pi = \frac{j\omega^3 M_{21}}{D\hat{L}} V_s$$

with

$$D = -\frac{\omega^2}{\hat{L}}\left(Z_1 Z_2 + \omega^2 M_{12} M_{21}\right)$$

the determinant of the matrix $(j\omega I - A)$. In a similar way, we can prove that

$$\begin{bmatrix} \Pi_1 & \Pi_2 & \Pi_3 & \Pi_4 \end{bmatrix}^{\top} = \begin{bmatrix} \frac{1}{j\omega}\overline{I}_1 & \overline{I}_1 & \frac{1}{j\omega}\overline{I}_2 & \overline{I}_2 \end{bmatrix}^{\top}.$$

It is interesting to explore which relation between phasors and moments holds when the "real convention" is used. Consider

system (4) with the matrices given in (12) and the matrices of the signal generator (11) given by

$$\Gamma = \begin{bmatrix} V_s & 0 \end{bmatrix}, \ \Sigma = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}.$$

The input of the system is $u = V_s \cos(\omega t)$ instead of $u = V_s \cos(\omega t) + jV_s \sin(\omega t)$. It comes with no surprises that the phasors are related to the solution of the Sylvester equation by the relations

$$\begin{bmatrix} \Pi_{11} + j\Pi_{12} \\ \Pi_{21} + j\Pi_{22} \\ \Pi_{31} + j\Pi_{32} \\ \Pi_{41} + j\Pi_{42} \end{bmatrix} = \begin{bmatrix} \frac{1}{j\omega}\overline{I}_1 \\ \overline{I}_1 \\ \frac{1}{j\omega}\overline{I}_2 \\ \overline{I}_2 \end{bmatrix}.$$

Finally, we investigate the use of the "complex convention". Consider the coordinates $x_1(t) = i_1(t)$, $x_2(t) = i_2(t)$, and the signal generator (11) with $\Gamma = V_s$ and $\Sigma = j\omega$. System (10) can be represented by a system of integro-differential equations given by

$$\begin{aligned} \dot{x}_1 = \ & \frac{1}{\hat{L}}\left(-R_1 L_2 x_1 - \frac{L_2}{C_1}\int x_1 dt + M_{12}R_{2L}x_2 \right. \\ & \left. + \frac{M_{12}}{C_2}\int x_2 dt + L_2 u\right) \\ \dot{x}_2 = \ & \frac{1}{\hat{L}}\left(-R_{2L} L_1 x_2 - \frac{L_1}{C_2}\int x_2 dt + M_{21}R_1 x_1 \right. \\ & \left. + \frac{M_{21}}{C_1}\int x_1 dt - M_{21} u\right). \end{aligned}$$

Exploiting Remark 4, we can write the Sylvester equation and find its solution, namely

$$\begin{bmatrix} \Pi_1 \\ \Pi_2 \end{bmatrix} = \begin{bmatrix} j\omega\hat{L} + R_1 L_2 + \frac{L_2}{j\omega C_1} & -M_{12}R_{2L} - \frac{M_{12}}{j\omega C_2} \\ -M_{21}R_1 - \frac{M_{21}}{j\omega C_1} & j\omega\hat{L} + L_1 R_{2L} + \frac{L_1}{j\omega C_2} \end{bmatrix}^{-1} \times \begin{bmatrix} L_2 V_s \\ -M_{21} V_s \end{bmatrix}$$

showing easily that

$$\begin{bmatrix} \Pi_1 & \Pi_2 \end{bmatrix}^\top = \begin{bmatrix} \overline{I}_1 & \overline{I}_2 \end{bmatrix}^\top.$$

Remark 7: When the system is linear and the source generates a sinusoidal signal, the real convention and the mixed convention are redundant since each current appears twice in the matrix Π (as $\overline{I}_k$ and $\frac{1}{j\omega}\overline{I}_k$). However, when the input is not a complex exponential, the steady-state of the current i_k and of its time integral are not anymore linked by a simple scaling factor.

B. Definition of Power From the Moments

The phasor analysis is useful to determine the instantaneous and average power absorbed by a load Z at steady-state. Exploiting the relation between phasors and moments, we can define these two quantities with respects to the moments. The instantaneous power is defined as

$$p(t) = v(t)i(t) = \Re\left[\overline{V}e^{j\omega t}\right]\Re\left[\overline{I}e^{j\omega t}\right]. \tag{13}$$

Exploiting the properties of the real-part operator, we write the instantaneous power as

$$p(t) = \frac{1}{2}\Re\left[\overline{V}^*\overline{I}e^{j\omega t}e^{-j\omega t}\right] + \frac{1}{2}\Re\left[\overline{V}\overline{I}e^{j2\omega t}\right]. \tag{14}$$

Using Euler's formula, it can be proved that the average power

$$P = <p(t)> = \frac{1}{2}\Re\left[\overline{V}^*\overline{I}\right] \tag{15}$$

is equal to the first term of (14), and that the second term of (14) has zero average.

Since we have proved that $\overline{I} = H\Pi$, the instantaneous power is described by

$$p(t) = \frac{1}{2}\Re\left[\frac{(H\Pi)^*}{Z}H\Pi\right] + \frac{1}{2}\Re\left[\frac{H\Pi}{Z}H\Pi e^{j2\omega t}\right]. \tag{16}$$

with Z the complex impedance.

Remark 8: The equivalence between the average power P and $\frac{1}{2}\Re[\overline{V}^*\overline{I}]$ is not a definition. The relation is a consequence of the properties of the complex exponential, as highlighted in (14), and we may expect that this relation does not hold if the input is not a complex exponential.

III. Generalizing the Phasor to Sources in Explicit Form

Now that we have linked the theory of phasors with the moment theory, we are able to extend the phasor analysis to more general classes of sources. In this section, instead of considering sinusoidal sources, we study any periodic source which has the following explicit[9] representation

$$\zeta(t) = \Lambda(t)\zeta(0), \ u = \Gamma\zeta \tag{17}$$

with $\Lambda(t)$ such that $\Lambda(t) = \Lambda(t - T)$, for $t \geq T$. This is a general representation of any periodic signal that is linear with respect the initial condition. This class includes possibly discontinuous signals, such as square waves and triangular waves, which are of great interest in circuit analysis. The extension of the theory to this class of signals is possible thanks to the recent developments presented in [20] and [21], which extend the notion of moment to this general class of signal generators. In this paper, we develop an electric equivalent of the theory therein, yielding an extension of the phasor transform to this class of signals. We begin by giving the definition of the discontinuous phasor. We show that multiplying the phasor with the source signal and taking the real part yields the steady-state current. Then,

[9]See [20], [21], [30], and [31], for the definition of explicit and implicit forms. Briefly, a system in implicit form is described by an ordinary differential equation; a system in explicit form may not have a differential representation.

we show the relation between voltage and current for resistors, inductors, and capacitors. Finally, we show that the phasor can be used to compute the steady-state instantaneous and average power. In summary, we extend all the features and properties of the phasor analysis to discontinuous sources.

Remark 9: Since (17) is a periodic signal, a classical phasor analysis can be carried out exploiting the Fourier series of the signal. However, the approach that we propose achieves a closed-form expression of the phasors and of the steady-state quantities. No approximations are introduced and non-ideal components can be considered. A comparison between the new results and the approximation given by the Fourier series is illustrated in the examples.

Remark 10: The state-space averaging method is commonly used as a modeling technique for power electronic converters [3]. However, note that while the averaging method is an approximation, the method we propose is exact. More in detail, the state-space averaging technique provides a reliable approximation of the steady-state behavior of the circuit when the natural frequencies of the converter (the poles of its transfer function) are much smaller than the switching frequency. The averaging method fails to give a good approximation when the natural frequencies of the system are close or above one half of the switching frequency [32]. On the contrary, the discontinuous phasor transform does not have this restriction since it is not an approximation but an exact description of the steady-state behavior of any linear circuit powered by any discontinuous source.

Remark 11: We focus on periodic signals only because these are the most common sources in power electronics. Note that the analysis of this section can be extended to any, possibly non-periodic, signal generators described by an equation of the form $\zeta(t) = \Lambda(t)\zeta(0)$, as discussed in [20] and [21].

Remark 12: In the remaining of the section, we use the mixed convention. A similar analysis for the real convention can be derived. The complex convention presents more difficulties because Corollary 2 does not have an equivalent in this convention.

A. Definition of the Discontinuous Phasor

Since the definition of moment given in this section is based on the existence of the steady-state response of system (4) driven by (17), we need to introduce further hypotheses on the class of input signals (17).

Assumption 1: The vector $\zeta(t)$ defined in (17) has a strictly proper Laplace transform with non-negative poles.

Assumption 1 is a standard condition for the existence of a well-defined steady-state response of the state of system (4) driven by (17) [33], [34].

Assumption 2: The matrix valued function $\Lambda(t)$ is non-singular for all $t \geq 0$.

Assumption 2 is essential to have uniqueness of the solution $\zeta(t)$ of (17). Note, in fact, that it is always satisfied by "properly written" generators of the form (9).

Assume now that there exists a set $\mathcal{T} \subset \mathbb{R}_{\geq 0}$ in which $\Lambda(t)$ is differentiable with respect to t and consider the time-varying system described by the equation

$$\dot{\varpi}(t) = G(t)^{\top}\varpi(t) \tag{18}$$

with $G(t) = -\dot{\Lambda}(t)\Lambda(t)^{-1}$. Let $\Phi(t)$ be the transition matrix (see [35], for its definition) of system (18).

Assumption 3: The function G is piecewise continuous with respect to t. Moreover, there exist $T \geq 0$ and a polynomial $q(t)$ such that $||\Phi(t)|| \leq q(t)$ for all $t \geq T$.

This last technical assumption guarantees that the norm of $\varpi(t)$ in system (18) does not diverge to infinity exponentially [35, Section 29] and it is needed to guarantee that the steady-state response x_s of system (4) driven by (17) can be written as $x_s(t) = \Pi(t)\omega(t)$ for all $t \geq 0$, for some matrix valued function $\Pi(t)$. Moreover, the piecewise continuity of G guarantees that the steady-state response is unique.

Remark 13: It is always possible to describe square waves, triangular waves, and any periodic signal that is almost everywhere differentiable in the mixed convention to satisfy Assumptions 1–3.

We define now the phasor in the case in which the signal generator is described by (17).

Definition 4: Consider system (4) and the signal generator (17). Assume Assumptions 1–3 hold, $\sigma(A) \subset \mathbb{C}_{<0}$ and $\Lambda(t)$ is almost everywhere differentiable. The components of the function

$$\Pi_{\infty}(t) = (I - e^{AT})^{-1}\left[\int_{t-T}^{t} e^{A(t-\tau)}B\Gamma\Lambda(\tau)d\tau\right]\Lambda(t)^{-1} \tag{19}$$

are the *discontinuous phasors* of all the currents and of all the integrals of the currents in system (4) for the source $\Lambda(t)$. The *discontinuous inverse phasor transform* of the steady-state output current $i(t)$ of system (4) is

$$i(t) = \Re\left[\overline{I}(t)\Lambda(t)\right] \tag{20}$$

with $\overline{I}(t) = H\Pi_{\infty}(t)$.

Remark 14: Like in the sinusoidal case, the instantaneous currents are recovered multiplying the phasor with the source and taking the real part.

Remark 15: Differently from the sinusoidal case, the phasor $\overline{I}(t)$ is a time-dependent periodic function. Note that if $\Lambda(t)$ is sinusoidal, (19) defines the usual constant phasor and Π_{∞} solves the Sylvester equation (7).

Remark 16: The matrix $\Phi(t)$ used in Assumption 3 does not appear explicitly in the following. However, it plays a role in the proof of the existence and properties of $\Pi_{\infty}(t)$ introduced in Definition 4 (see [21, Theorem 2] for the details).

Remark 17: The subscript "∞" in Π_{∞} is inherited from [20] and [21] in which it is shown that there exists a family of matrix valued functions $\Pi(t)$ parameterized in $\Pi(0) \in \mathbb{R}^{n\times\nu}$ for which there exist, if Assumptions 1–3 hold, a unique $\Pi_{\infty}(0)$ such that, for any $\Pi(0)$, $\lim_{t\to+\infty}\Pi(t) - \Pi_{\infty}(t) = 0$, where $\Pi_{\infty}(t)$ is the solution of (19). The dependence from $\Pi_{\infty}(0)$ disappears in the periodic case we are considering.

Remark 18: The definition of the function $\Pi_{\infty}(t)$ can be given as in (19) or, alternatively, as the unique solution of

$$\dot{\Pi}(t) = A\Pi(t) + B\Gamma - \Pi(t)\dot{\Lambda}(t)\Lambda(t)^{-1} \tag{21}$$

with the initial condition $\Pi(0) = \Pi_\infty(0)$ computed from (19) in $t = 0$.

Remark 19: The inverse phasor transform (3) introduced in [10]–[12] is a particular case of the more general phasor transform we have introduced. In fact, (3) is recovered when $\Lambda(t) = e^{j\omega(t)}$. Note however, that the new result have a remarkable advantage. In [10]–[12], the inverse phasor transform is introduced but the phasor itself (*i.e.,* the direct phasor transform) is not defined. Thus, to apply the results of those papers one must recognize simple subcircuits and find the v–i characteristics in the phasor domain of the subcircuit. Although this is possible also with the new transform (20) (and it is done in the following), it is not necessary. In fact, since we have the definition of the phasor (19) from the system matrices A and B, we can obtain the phasors of all the currents and voltages in the circuit in closed-form without the need of decomposing the circuit in pre-classified subcircuits.

Now that we have defined the discontinuous phasor and the discontinuous inverse phasor transform, we extend the properties of the phasor circuit analysis.

B. Inductance, Capacitance, and Resistance

Following [10]–[12], we describe the v–i characteristics of some common subcircuits, which constitute power electronic devices. As already remarked, this is not strictly necessary if we want to compute simply the phasors of the overall circuit. However, describing simple circuits in the phasor domain improves the understanding of this new tool. Moreover, to be useful for applications, we need to be able to compute the voltage across an inductor, capacitor, and resistor given the phasor of the current that flows through these components. This is of paramount importance to be able to define the power, and more in general, to make this mathematical extension an accurate description of the physical quantities in the circuit. The expressions that relate voltage and current in an inductor, capacitor, and resistor are, respectively

$$v = L\frac{di}{dt}, \qquad v = \frac{1}{C}\int_0^t i\,d\tau, \qquad v = Ri. \tag{22}$$

Utilizing the phasor transform (2), it can be proved that the relations

$$\overline{V} = j\omega L\overline{I}, \qquad \overline{V} = \frac{1}{j\omega C}\overline{I}, \qquad \overline{V} = R\overline{I} \tag{23}$$

hold. With the phasor transform (3), these become

$$\begin{aligned} \overline{V}(t) &= L\dot{\overline{I}}(t) + j\omega(t)L\overline{I}(t) \\ \dot{\overline{V}}(t) + j\omega(t)\overline{V}(t) &= \frac{1}{C}\overline{I}(t) \\ \overline{V}(t) &= R\overline{I}(t). \end{aligned} \tag{24}$$

When the source is described by the generator (17), these relations may not hold anymore. Consider, for instance, a square wave. Exploiting the Fourier series, we note that this signal is described by infinitely many frequencies ω_k. As noted in [20], this observation suggests that we could describe a square wave by means of an infinite dimensional system. As a consequence, the matrix Π would have infinitely many rows and columns and the phasor would be the sum of infinitely many frequencies. It is exactly for this inability to deal with this type of signals without approximations that Definition 4 has been introduced. In fact, exploiting the discontinuous phasor transform, we obtain the following exact relations.

Theorem 2: Consider the first equation in (22). The relation

$$\overline{V}(t) = L\dot{\overline{I}}(t) + L\frac{\dot{\Lambda}(t)}{\Lambda(t)}\overline{I}(t) \tag{25}$$

holds.

Proof: Consider the first equation in (22). This is a scalar system with $\overline{I}(t) = \Pi_\infty(t) \in \mathbb{C}, A = 0, B = \frac{1}{L}, \Gamma = \overline{V}(t)$. The derivative of the current is

$$\begin{aligned} \frac{di}{dt} &= \frac{d}{dt}\Re\left[\overline{I}(t)\Lambda(t)\right] = \Re\left[\frac{d}{dt}\left[\overline{I}(t)\Lambda(t)\right]\right] \\ &= \Re\left[\dot{\overline{I}}(t)\Lambda(t) + \overline{I}(t)\dot{\Lambda(t)}\right]. \end{aligned}$$

Substituting (21) into the last expression, yields

$$\begin{aligned} &\Re\left[\left(A\overline{I}(t) + B\Gamma\right)\Lambda(t)\right] \\ &\quad = \Re\left[\left(\dot{\overline{I}}(t) + \overline{I}(t)\dot{\Lambda}(t)\Lambda(t)^{-1}\right)\Lambda(t)\right] \end{aligned}$$

from which we recognize, by comparison with (20), that $\dot{\overline{I}}(t) + \overline{I}(t)\dot{\Lambda}(t)\Lambda(t)^{-1}$ is the phasor. ■

Remark 20: If $\Lambda(t) = e^{j\omega t}$, then $\dot{\overline{I}}(t) = 0$, $\dot{\Lambda}(t)\Lambda(t)^{-1} = j\omega$ and (25) becomes the first relation in (23). If we use the phasor transform (3) in which the phasor $I(t)$ is not constant, (25) reduces to the first relation in (24).

Theorem 3: Consider the second equation in (22). The relation

$$\dot{\overline{V}}(t) + \frac{\dot{\Lambda}(t)}{\Lambda(t)}\overline{V}(t) = \frac{1}{C}\overline{I}(t) \tag{26}$$

holds.

Proof: It is similar to the proof of Theorem 2. ■

Remark 21: In the mixed convention, the components with odd indices of Π_∞, computed from (19), are those functions that multiplied by Λ give the steady state of the integrals of the currents. The following result holds.

Corollary 2: In the mixed convention the components with odd indices of Π_∞, computed with (19), are the phasors of the integrals of the currents in the circuit. Thus, for the current i_k, which flows in the capacitance C_k, the relation

$$\frac{\epsilon_{2k-1}^\top \Pi_\infty(t)}{C_k} = \overline{V}_k(t) \tag{27}$$

holds, where $\overline{V}_k$ is the phasor of the voltage across the capacitor C_k.

This last result provides a way to compute the phasor of the voltage across a capacitor. In fact, this value comes directly from solving (19).

Remark 22: In [10]–[12], a discussion regarding the selection of the initial conditions of (24) is missing. Being the phasor defined only as the solution of the differential equations (24), it

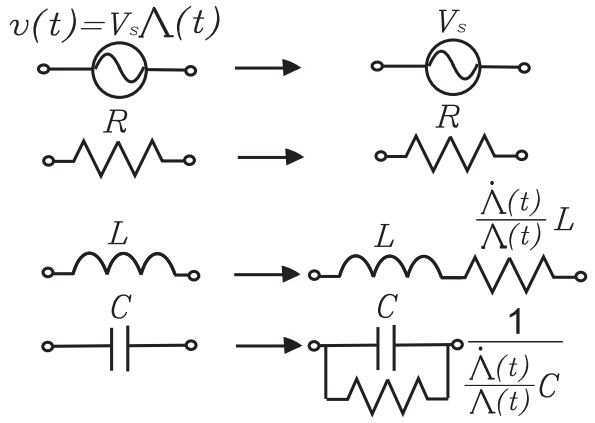


Fig. 5. Phasor models for basic circuit elements.

is not clear how the correct solution is selected among the many identified by the differential equations. With the result we have presented in Remark 18, the initial conditions can be computed as $\Pi(0) = \Pi_\infty(0)$, using (19) for $t = 0$. In fact, this is the most efficient way to compute the phasors, namely solving the differential equations in the phasor domain, *e.g.,* (25) and (26), and using the computational expensive (19) only for $t = 0$.

Theorem 4: Consider the third equation in (22). The relation

$$\overline{V} = R\overline{I}(t) \tag{28}$$

holds.

Proof: The statement holds trivially noting that the multiplication by a real constant and the real-part operator commutes. ■

Fig. 5 shows the phasor equivalent of these circuit elements. Having defined the differential operator, the integral operator and the multiplicative operator, the v–i characteristics of other common circuits in the phasor domain can be easily obtained. For instance, transformers and gyrators, in which voltages and currents are related by a multiplication factor, show phasor equations similar to (28).

Remark 23: In [10] and [12], transformers with variable transformation ratio $s(t)$ are considered for their ability of replacing switches [36]. They are not needed in the present framework since the new phasors we have introduced naturally embed switches. Thus, these time-varying transformers are not analyzed further in this paper. Note anyway that since their v–i characteristics are given by the multiplication of two time-varying functions, *i.e.,* $v_t(t) = s(t)i_t(t)$, they present equations in the phasor domain similar to the ones developed in the following section for the power $p(t) = v(t)i(t)$, replacing $p(t)$ with $v_t(t)$ and $v(t)$ with $s(t)$.

C. Instantaneous, Average, and Reactive Power

Using the phasor transform (20), the instantaneous power is defined as

$$p(t) = v(t)i(t) = \Re\left[\overline{V}(t)\Lambda(t)\right]\Re\left[\overline{I}(t)\Lambda(t)\right] \tag{29}$$

which, exploiting the properties of the real-part operator, yields

$$p(t) = \frac{1}{2}\Re\left[\overline{V}(t)^*\overline{I}(t)\Lambda(t)\Lambda(t)^*\right] + \frac{1}{2}\Re\left[\overline{V}(t)\overline{I}(t)\Lambda(t)^2\right]. \tag{30}$$

As in the sinusoidal case, the instantaneous power is separated in two terms: the average of the first term is equal to the average power, whereas the average of the second term is zero. However, differently from the sinusoidal case, the first term is not constant, in general, and thus, it is not equal to the average power. Hence, the average power and the reactive power are defined as follows.

Definition 5: In the phasor domain identified by the phasor transform (20), the average power P and the reactive power Q are defined as

$$\begin{aligned} P &= \ <p(t)>= \frac{1}{2}\left\langle\Re\left[\overline{V}(t)^*\overline{I}(t)\Lambda(t)\Lambda(t)^*\right]\right\rangle \\ Q &= \ \frac{1}{2}\left\langle\Im\left[\overline{V}(t)^*\overline{I}(t)\Lambda(t)\Lambda(t)^*\right]\right\rangle. \end{aligned} \tag{31}$$

The aforementioned equations (31) are consistent with the usual definition of average power and reactive power in the complex exponential case. For the non-exponential case, (31) generalize the respective relations achievable with the phasor transforms (2) and (3). In fact, both of these always show a time-invariant transform of the average power [12]. Moreover, the sum of the average powers (or reactive powers) generated by the Fourier series expansion of the periodic signal Λ converges to the value given in (31). One can say more when specific signals are considered. For instance, if the input signal is a square wave, the following result holds.

Proposition 2: Assume that the signal generator (17) produces a square wave that takes values in the discrete set $\{-1, 1\}$. The average power and reactive power are described by

$$P = \tfrac{1}{4}\Re\left[\overline{V}(t)^*\overline{I}(t)\right], \qquad Q = \tfrac{1}{4}\Im\left[\overline{V}(t)^*\overline{I}(t)\right]. \tag{32}$$

In fact, note that the term $\Lambda(t)\Lambda(t)^*$ in (31) is constant and equal to 1 if the driving signal is a complex exponential, it is constant and equal to $\frac{1}{2}$ if the driving signal is a square wave and it is a parabola if the driving signal is a triangular wave.

IV. Application of the Discontinuous Phasor Transform to the Analysis of Inverters and Wireless Power Transfer Systems

We now apply the newly defined discontinuous phasor transform to the steady-state description of the currents flowing in a Class D resonant inverter controlled by square waves and a wireless power transfer system with non-ideal switches (affected by the reverse recovery effect).

A. Analysis of a Resonant Inverter

The equation

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int i_1\,dt = u(t) \tag{33}$$

describes the circuit shown in Fig. 2. Consider now that we are interested in studying the steady-state behavior of this circuit

for a particular switching function. The switching function is determined by the designer on the base of the specific application under analysis. Thus, we assume that the switching function is a square wave with given angular frequency ω, *i.e.,* $\sqcap(\omega t + \pi/2)$. This signal does not satisfy Assumption 2. However, this issue can be easily solved considering the extended signal $\sqcap(\omega t + \pi/2) + j \sqcap(\omega t)$. This complex signal is never equal to zero and Assumption 2 is satisfied. Note that this is in line with the smooth case when the source is described by $e^{j\omega t} = \sin(\omega t + \pi/2) + j \sin(\omega t)$.

Thus, we write (33) in the form (4) using the mixed convention and we consider the signal generator described by the equations

$$\begin{aligned} u(t) &= V_s \zeta(t) \\ \zeta(t) &= \sqcap\left(\omega t + \frac{\pi}{2}\right) + j \sqcap(\omega t) \end{aligned} \tag{34}$$

with $V_s \in \mathbb{R}$ and $\sqcap(t)$ implemented as the function *square* of MATLAB. Note that in the real convention, this signal is realized by the system

$$\begin{aligned} u(t) &= V_s \begin{bmatrix} 1 & 0 \end{bmatrix} \zeta(t) \\ \zeta(t) &= \begin{bmatrix} \sqcap\left(\omega t + \frac{\pi}{2}\right) & -\sqcap(\omega t) \\ \sqcap(\omega t) & \sqcap\left(\omega t + \frac{\pi}{2}\right) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \end{aligned} \tag{35}$$

In fact, in the smooth case the matrix Λ would reduce to the usual rotation matrix and $u(t) = \cos(\omega t)$. However, in the following, we use the mixed convention that, as already pointed out, is more compact and it results in more efficient computation.

1) Analysis of a Resonant Inverter With $\mathrm{Q} = 0.1313$*:* The parameters for the simulation have been selected as $L = 229.3\,\mu\mathrm{H}$, $C = 10\,\mu\mathrm{F}$, $R = 36.47\,\Omega$, $V_s = 100\,\mathrm{V}$, and $\omega = \frac{1}{\sqrt{LC}}$. Note that the inverter has a low quality factor of $\mathrm{Q} = 0.1313$. The phasors are computed using (19). The formula has been implemented in MATLAB with the function *integral* with the option *ArrayValued*. Fig. 6 shows the real component (top graph) and the imaginary component (bottom graph) of the phasor $\overline{I}(t)$ computed as $\epsilon_1^\top \Pi_\infty$. Note that the phasor is time-dependent, periodic (with period one fourth of the period of the input source) and discontinuous. The other electrical quantities of the circuit in the phasor domain can be computed as well. For instance, Fig. 7 shows the phasor $\overline{V}_C(t) = \epsilon_2^\top \Pi_\infty / C$, computed as (27), of the voltage across the capacitor C. Fig. 8 (top graph) shows the time histories of the steady-state current $i(t)$ (solid/blue line), the steady-state current i^∞ computed from the phasor $\overline{I}(t)$ as $i^\infty = \Re\left[\overline{I}(t)\Lambda(t)\right]$ (dotted/red line) and the approximated steady-state current computed with the Fourier series with one harmonic $\tilde{i}^1(t)$ (dashed/black line), three harmonics $\tilde{i}^3(t)$ (bold dashed/black line), and seven harmonics $\tilde{i}^7(t)$ (dash-dotted/black line). Note that the transient of $i(t)$ has been eliminated simulating 50 periods of the switching input and considering only the last period. Fig. 8 (bottom graph) shows the corresponding relative errors (with same color coding), namely, $\varepsilon_r^\infty = |i - i^\infty|/|i|$ and $\varepsilon_r^k = |i - \tilde{i}^k|/|i|$, with $k = 1, 3,$ and 7. The steady-state of the current $i(t)$ is exactly described by i^∞ (the error is identically zero). The steady-state is

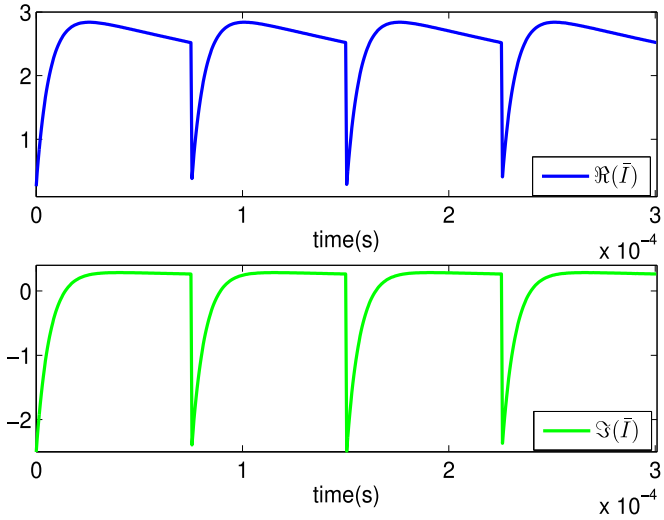


Fig. 6. Real part (top graph) and imaginary part (bottom graph) of the phasor $\overline{I}(t)$.

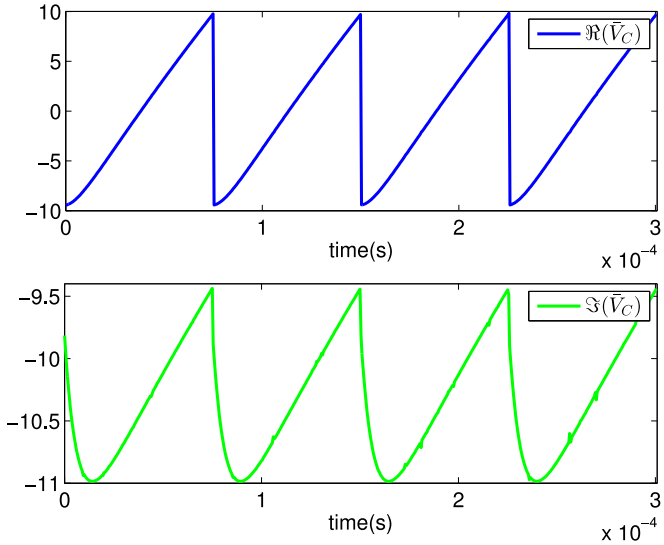


Fig. 7. Real part (top graph) and imaginary part (bottom graph) of the phasor $\overline{V}_C(t)$.

approximated by the Fourier series increasingly better as the number of terms grows. However, note that at the points of discontinuity a finite number of Fourier terms cannot approximate perfectly the steady-state, as it can be noted by the peaks of ε_r^7 (dash-dotted/black line). Fig. 9 shows the absorbed power by the load R. The solid line shows the actual steady-state instantaneous power computed from the current $i(t)$. The dotted line and dash-dotted line show the steady-state instantaneous power $p(t)$ computed as (30) and the steady-state average power P computed as (32), respectively. The figure confirms that the quantities that we have defined maintain their physical meaning since the power relations keep holding in the phasor domain.

2) Analysis of a Resonant Inverter With $\mathrm{Q} = 0.4789$ *and* $\mathrm{Q} = 0.0042$*:* We expect that as the "severity" of the discontinuity increases in the system, the discrepancy between the

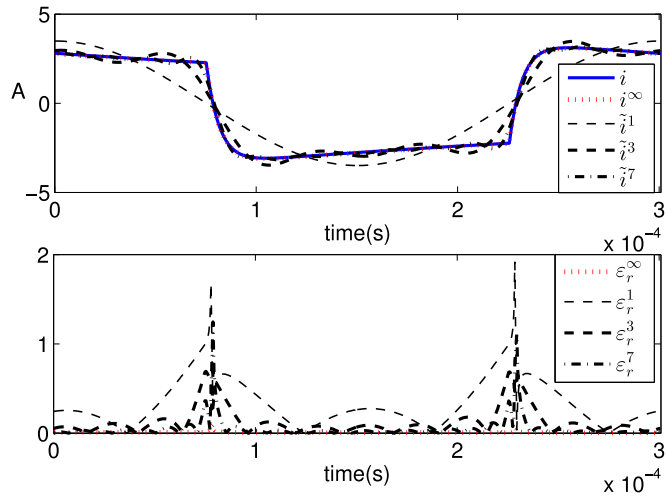


Fig. 8. Top graph: time histories of the steady-state current $i(t)$ (solid/blue line), of the steady-state current $i^\infty(t)$ (dotted/red line) and of the approximated steady-state current computed with the Fourier series with one harmonic $\tilde{i}^1(t)$ (dashed/black line), three harmonics $\tilde{i}^3(t)$ (bold dashed/black line), and seven harmonics $\tilde{i}^7(t)$ (dash-dotted/black line). Bottom graph: resulting relative errors (with same color coding).

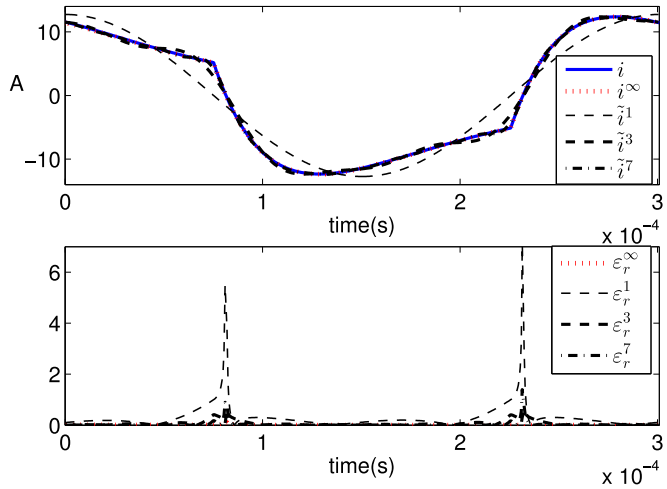


Fig. 10. Top graph: time histories of the steady-state current $i(t)$ (solid/blue line), of the steady-state current $i^\infty(t)$ (dotted/red line) and of the approximated steady-state current computed with the Fourier series with one harmonic $\tilde{i}^1(t)$ (dashed/black line), three harmonics $\tilde{i}^3(t)$ (bold dashed/black line), and seven harmonics $\tilde{i}^7(t)$ (dash-dotted/black line). Bottom graph: resulting relative errors (with same color coding).

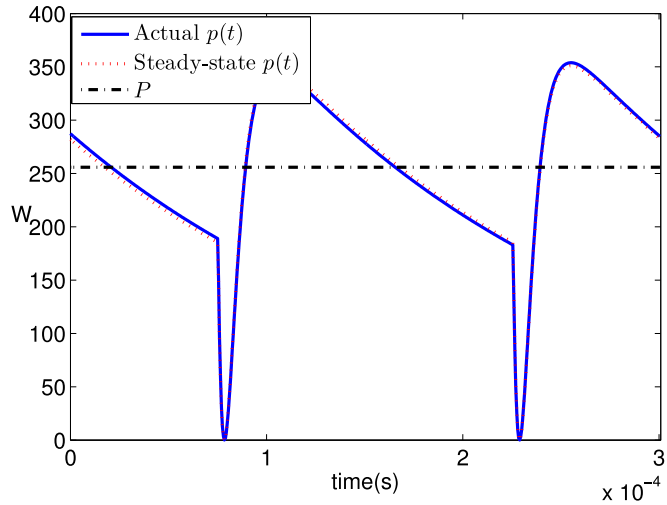


Fig. 9. Instantaneous power absorbed by R (solid/blue line) computed from the current $i(t)$. Instantaneous power (dotted/red line) and average power (dash-dotted/black line) computed from the phasor $\overline{I}(t)$.

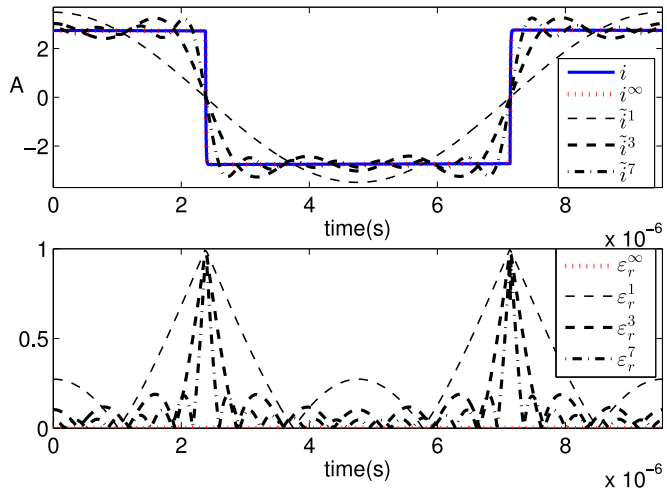


Fig. 11. Top graph: time histories of the steady-state current $i(t)$ (solid/blue line), of the steady-state current $i^\infty(t)$ (dotted/red line) and of the approximated steady-state current computed with the Fourier series with one harmonic $\tilde{i}^1(t)$ (dashed/green line), three harmonics $\tilde{i}^3(t)$ (bold dashed/green line), and seven harmonics $\tilde{i}^7(t)$ (dash-dotted/black line). Bottom graph: resulting relative errors (with same color coding).

Fourier series and the discontinuous phasor response grows. We can check this changing a few parameters: for example, setting $R = 10\,\Omega$ ($\mathrm{Q} = 0.4789$) in the first case and $L = 229.3$ nH and $R = 36.47\,\Omega$ ($\mathrm{Q} = 0.0042$) in the second case. Fig. 10 shows the quantities in Fig. 8 for the intermediate quality factor $\mathrm{Q} = 0.4789$. Note that since the curves are smoother, the current computed with the Fourier series approaches the current computed with the new phasor with fewer harmonics. On the other hand, Fig. 11 shows the quantities in Fig. 8 for the lower quality factor $\mathrm{Q} = 0.0042$. In this case, the error of the Fourier series increases. These figures show that the new discontinuous phasor is particularly useful to describe underdamped circuits that present "severe" discontinuities.

B. Analysis of a Wireless Power Transfer System With Non-ideal Switches

Consider now system (10) represented in the mixed convention by system (4) with the matrices as in (12) and assume that the circuit contains non-ideal switches. In particular, we study the effect of the reverse recovery time of the switching diode BAS19 [37]. The reverse recovery effect consists in an overshoot of the switching signal at the switching time. Most commercially available switching diodes have a reverse recovery time ranging from few nanoseconds to one

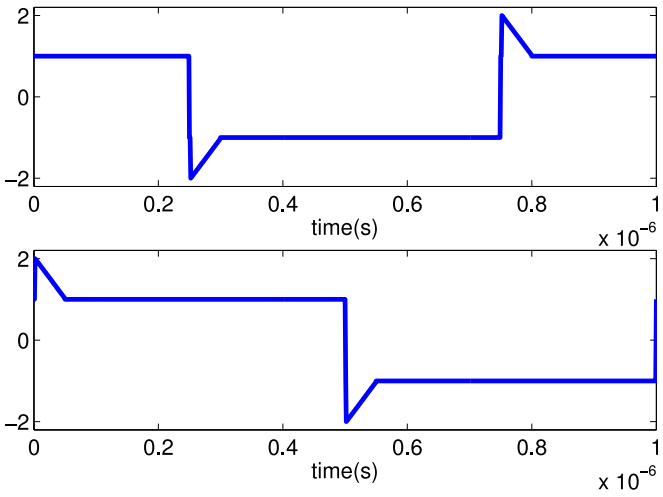


Fig. 12. Real part (top graph) and imaginary part (bottom graph) of the signal representing a non-ideal switch with reverse recovery effect.

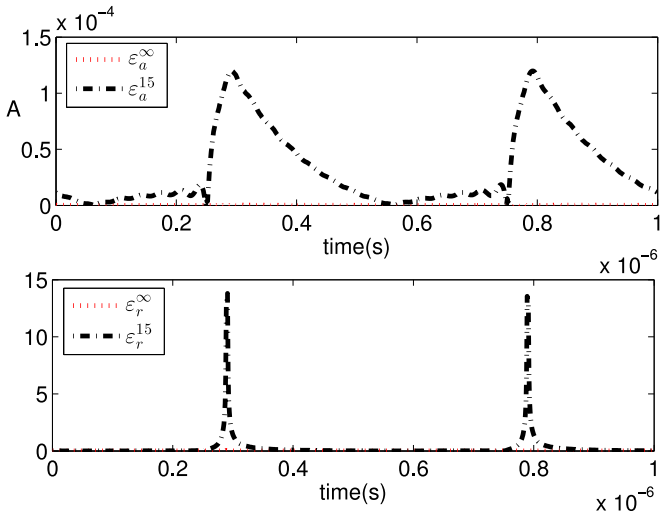


Fig. 14. Time histories of the absolute errors between the steady-state current $i_2(t)$ and i_2^{∞} (dotted/red line) and $\tilde{i}_2^{15}(t)$ (dash-dotted/black line), respectively.

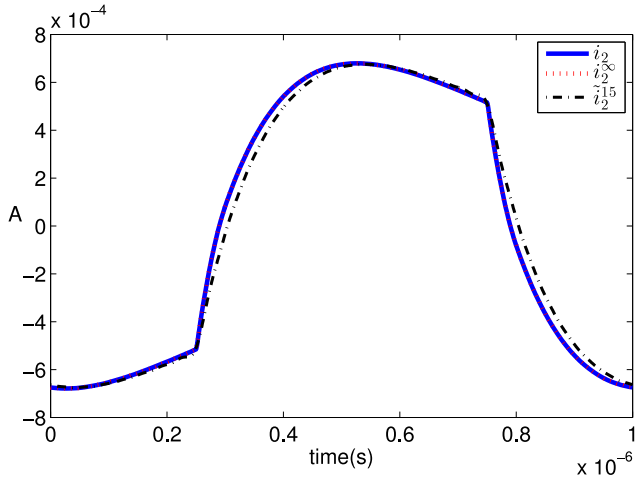


Fig. 13. Time histories of the steady-state current $i_2(t)$ (solid/blue line), of the steady-state current $i_2^{\infty}(t)$ (dotted/red line) and of the approximated steady-state current $\tilde{i}_2^{15}(t)$ computed with the Fourier series with fifteen harmonics (dash-dotted/black line).

microsecond [25]. The diode BAS19, which is considered a fast diode, has a reverse recovery time of 50 ns that becomes relevant at megahertz switching frequencies at which some wireless power transfer systems operate [38]. Fig. 12 shows the behavior of the diode BAS19 at the frequency $f = 1\,\text{MHz}$. This has been modeled with (34) in which we have added a sawtooth wave at the switching times. The parameters for the simulation have been selected as $L_1 = L_2 = 8.203\,\text{mH}$, $M_{12} = M_{21} = 1.545\,\text{mH}$, $C_1 = 1.029\,\text{nF}$, $C_2 = 1.024\,\text{nF}$, $R1 = R_{2L} = 33.576\,\text{k}\Omega$, $V_s = 230\,\text{V}$, and $\omega = 2\pi f$. Fig. 13 shows the time histories of 50th period of the current i_2, the steady-state $i_2^{\infty} = \Re\left[\overline{I}_2(t)\Lambda(t)\right]$, and the approximation $\tilde{i}_2^{15}(t)$ computed with the Fourier series of an ideal switch with fifteen harmonics. Fig. 14 shows the corresponding absolute errors (top) and relative error (bottom). Note that, the new discontinuous phasor can deal with the discontinuous non-ideal source without approximations. On the other hand, the Fourier series cannot fully describe this source.

V. Conclusion

We have proposed a novel phasor transform that allows to analyze in closed-form the steady-state behavior of discontinuous power electronic devices. We have shown that the phasors of an electric circuit are the moments on the imaginary axis of the linear system describing the circuit. Exploiting this relation, we have analyzed circuits powered by discontinuous sources. A new "discontinuous phasor transform" has been defined and the v–i characteristics for inductors, capacitors, and resistors have been described in terms of this new phasor transform. The new quantities maintain their physical meaning: the instantaneous power, the average power, and the reactive power in the phasor domain have been defined. We have illustrated the use of this mathematical tool studying the steady-state response of power inverters and of wireless power transfer systems with non-ideal switches. The equivalence that we have established between moments and phasors allows to extend the phasor transform beyond the linear framework and the discontinuous framework (considered in this paper) to circuits with nonlinear elements and circuits with delays, see [15], [19], and [29].

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