CONTRIBUTIONS TO RISK-SENSITIVE ASSET MANAGEMENT

by

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Abstract

Modern Portfolio Theory, pioneered by Markowitz, Sharpe and Lintner, has had a considerable impact on financial theory and practice. Its tenets permeate the field of financial economics and form the foundation of the investment decision-making process. However, one of its key limitations is that it restricts the decision making process to a single allocation decision and a single time period.

Whilst the traditional remedy, a continuous time model using the classical theory of stochastic control, is too restrictive for use in the investment-making process, a solution using risk-sensitive control to formulate the investment problem appears promising. Bielecki and Pliska pioneered this approach. They developed risk-sensitive asset management (RSAM) to address continuous-time asset allocation problems in an incomplete market, where the dynamics of securities prices is a function of some underlying macroeconomic, microeconomic or statistical variables which are referred to collectively as 'valuation factors'.

Our research extends the risk-sensitive investment management theory and its applications. First, we look at a new class of control problems designed to address the questions of allocating assets to outperform a given liability. We show that when assets, liabilities and economic factors are modelled with diffusion processes an analytical solution exists, which enables us to gain economic insights into the nature of the investment process and to propose a mutual fund theorem.

We also present a general treatment of the jump diffusion optimal investment problem. This extends the underlying market model by addressing the possibility of random jumps in both asset prices and valuation factors levels. Because an analytical

solution does not generally exist in this setting, we prove that the value function of the control problem is the unique continuous viscosity solution of the associated risk-sensitive Hamilton-Jacobi-Bellman partial integro-differential equation (RS HJB PIDE).

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Index of Notations

Denote by

- $\mathbb{R}^n$, the n-dimensional Euclidian space, with elements $x = (x_1, \ldots, x_n)$;
- $\|x\|$, the Euclidian norm of x, i.e. for $x \in \mathbb{R}^n$, $\|x\| := \langle x, x \rangle^{\frac{1}{2}} = \sqrt{\sum_{i=1}^n x_i^2}$;
- $\|x\|_\infty$, the L^∞ norm;
- $C^K(Q)$, the set of k times continuously differentiable functions on a set Q;
- $USC(Q)$, the set of upper semicontinuous functions on a given set Q and by $LSC(Q)$ the set of lower semicontinuous functions on a given set Q;
- $\mathcal{S}_n$, the set of all $n \times n$ symmetric matrices;
- I_n, the $n \times n$ identity matrix;
- M', the transpose of a matrix M;
- $\mathbf{1}$, the unit vector of appropriate dimension;
- $\mathscr{B}_R(y) := \{x : \|x - y\| < R\}$, the ball of radius $R > 0$ centered at y, by $\mathscr{B}_R := \{x : \|x\| < R\}$, the ball of radius R centered at the origin and by $\mathscr{B}$ the unit ball in the appropriate number of dimensions;
- $1_Q(x)$ is the indicator function: $1_Q(x) = 1$ if $x \in Q$ and 0 otherwise;
- $\mathcal{B}_{\mathbf{Z}}$, the σ-algebra of Borel sets of a topological set $\mathbf{Z}$;

For a given function Φ, we denote by

- $D\Phi$, the gradient vector defined as $D\Phi := \left(\frac{\partial \Phi}{\partial x_1}, \ldots, \frac{\partial \Phi}{\partial x_i}, \ldots, \frac{\partial \Phi}{\partial x_n}\right)'$;
- $D^2\Phi$, the Hessian matrix defined as $D^2\Phi := \left[\frac{\partial^2 \Phi}{\partial x_i x_j}\right]$, $i, j = 1, \ldots, n$.

In ordinary differential equations, we will denote the time derivative $\frac{dy}{dt}$ by $\dot{y}$ for .

Chapter 1

Introduction

Modern Portfolio Theory, pioneered by Markowitz, Sharpe and Lintner, has had a considerable impact on financial theory and practice. Its tenets permeate the field of financial economics and form the foundation of the investment decision-making process. However, one of its limitations is that it restricts the decision making process to a single allocation decision and a single time period.

The traditional remedy, a continuous time model using the classical theory of stochastic control, as proposed by Merton [39], suffers from several restrictions which make it ill-suited for practical use in the investment-making process. First, the theory established by Merton requires that the investor knows both his/her utility function and degree of risk aversion. But does anyone truly know what his/her utility function and degree of risk aversion are? In addition, even in its simplest setting, the Merton approach results in a nonlinear control problem. Strict assumptions and a substantial degree of mathematical stylization are therefore required if one is to derive an analytical solution to the investment problem. The alternative would be to solve more realistic problems via numerical and computational methods. But this approach may not be practical either. Because the dimensionality of the problem is equal to the number of assets in the investment universe, one is rapidly confronted with massive computational problems that Bellman eloquently named the 'curse of dimensionality'.

In this thesis, we advocate an alternative approach to the optimal investment problem based on the theory of risk-sensitive control. Risk-sensitive control can be defined as a generalization of classical stochastic control in which the degree of risk aversion or risk tolerance of the optimizing agent is explicitly parameterized in the objective criterion and influences directly the outcome of the optimization. Bielecki and

Pliska [15] pioneered the application of risk–sensitive control to asset management by addressing continuous-time asset allocation problems in an incomplete market, where the dynamics of securities prices is a function of some underlying macroeconomic, microeconomic or statistical variables, referred to collectively as 'valuation factors'. With an appropriately chosen reward function, risk-sensitive control is a direct extension of mean-variance analysis and is consistent with utility maximization *à la* Merton, while retaining an intuitive interpretation as a maximization of long-term (risk-sensitive) return on investment. In addition, once an appropriate change of measure due to Kuroda and Nagai (see [37]) has been performed, the risk-sensitive investment problem can be expressed as a linear control problem. This linear problem is both much simpler and more robust than the nonlinear problem of the Merton approach. From a numerical and computational standpoint, risk-sensitive control is also appealing as the effective dimensionality of the problem is the relatively small number of underlying valuation factors rather than the relatively large number of assets in the investment universe. This significant reduction in the dimensionality of the problem, compared to the Merton approach, might just be enough to help us set up realistic problems while keeping the curse of dimensionality at bay. For all these reasons, the risk-sensitive approach is particularly interesting and promising from an investment management perspective.

Our research extends risk-sensitive investment management theory and its applications. First, we introduce and solve a new class of control problems designed to address the case of a funded investor attempting to allocate his/her assets to outperform a given liability. In order to maximize the risk-sensitive return on equity of his/her portfolio, the investor can not control only the asset allocation but also the

degree of leverage. We show that, when assets and factors are modelled using pure diffusion processes, an analytical solution exists. We subsequently analyze the solution in order to draw economic insights about the nature of the investment problem and of the optimal strategy, and summarize our findings into a mutual fund theorem.

Next, we present a treatment of the jump diffusion optimal investment problem. This approach transforms the underlying market model by allowing for the possibility of random jumps in both asset prices and the level of the valuation factors. In our analysis, we characterize jumps using random Poisson measures, a wide class of processes which includes the Poisson, compounded Poisson and Lévy processes. This choice enables us to model a wide range of specifications. Since an analytical solution does not generally exist in this setting, we prove that the value function of the control problem is the unique continuous viscosity solution of the associated risk-sensitive Hamilton-Jacobi-Bellman Partial Integro Differential Equation (RS HJB PIDE). This result is important for two reasons. First, proving that the value function is the unique solution of the associated HJB PIDE conclude the resolution of our control problem. Second, the fundamental 'stability' result of Barles and Souganidis [9] closely connects the notion of viscosity solution with the convergence of numerical schemes, and opens the way to a numerical resolution of the jump-diffusion risk-sensitive optimal investment problem. Finally, we extend our results to solve jump-diffusion versions of the benchmarked asset management model considered by Davis and Lleo [21] and of the asset and liability problem previously solved in Chapter 3 for the diffusion case.

The thesis is organized as follows. Chapter 2 introduces risk-sensitive control, its application to asset management and some recent development in the field. In

Chapter 3, we address a new problem concerned with asset and liability management. In Chapter 4, we present an extension of the framework first introduced by Bielecki and Pliska by bringing into the analysis the possibility of jumps in both asset prices and factor levels. In this chapter, we derive the risk-sensitive Hamilton-Jacobi-Bellman integro partial differential equation and solve the resulting control problem. In Chapter 5, we show that the value function of the jump diffusion risk sensitive asset management control problem is the unique continuous viscosity solution of the associated Hamilton-Jacobi-Bellman integro partial differential equation. In Chapter 6, we adapt the results derived in Chapters 4 and 5 to solve a jump-diffusion version of the benchmarked risk sensitive asset management and risk-sensitive asset and liability management problems.

Chapter 2

An Overview of Risk-Sensitive Asset Management

In this chapter, we outline of some of the main concepts in investment theory, introduce risk sensitive control, give an overview of risk-sensitive asset management and finally review some recent developments in the field of risk-sensitive asset management.

2.1 Risk-Sensitive Control

Risk-sensitive control is defined as a generalization of classical stochastic control in which the degree of risk aversion or risk tolerance of the optimizing agent is explicitly parameterized in the objective criterion and influences directly the outcome of the optimization.

In risk-sensitive control, the decision maker's objective is to select a control policy $h(t)$ to maximize the criterion

$$J(x,t,h;\theta) := -\frac{1}{\theta}\ln \mathrm{E}\left[e^{-\theta F(t,x,h)}\right] \tag{2.1}$$

where

- t is the time and x is the state variable;
- F is a given reward function, and;
- the risk sensitivity $\theta \in]-1,0[\cup]0,\infty)$ is an exogenous parameter representing the decision maker's degree of risk aversion.

A Taylor expansion of the previous expression around $\theta = 0$ evidences the vital

role played by the risk sensitivity parameter:

$$J(x,t,h;\theta) = \mathbf{E}\left[F(x,t,h)\right] - \frac{\theta}{2}\mathbf{Var}\left[F(x,t,h)\right] + O(\theta^2) \tag{2.2}$$

when

- $\theta \to 0$, we are in the "risk-null" case which corresponds to classical stochastic control;

- $\theta < 0$, we obtain the "risk-seeking" case corresponding to a maximization of the expectation of a convex decreasing function of $F(t,x,h)$;

- $\theta > 0$, we get the"risk-averse" case corresponding to a minimization of the expectation of a convex increasing function of $F(t,x,h)$.

Therefore, risk-sensitive control differs from traditional stochastic control in that it explicitly models the risk-aversion of the decision maker as an integral part of the control framework, rather than importing it in the problem via an externally defined utility function.

Jacobson [33] took the firts steps toward the development of a consistent risk-sensitive control theory by solving the Linear-Exponential-of-Quadratic Regulator (LEQR) problem, which is the risk-sensitive equivalent of the traditional Linear Regulator problem found in classical control literature. Bensoussan and Van Schuppen [12] and Whittle [45] greatly contributed to the development of the theory in the complete observation case, while a succinct treatment of the partial observation LEQR case can be found in Bensoussan [10].

Recent developments in the field of risk-sensitive control have been mostly concerned with exploring two avenues. The first is the formulation of a general theory for the solution of the risk sensitive Hamilton-Jacobi-Bellman Partial Differential Equation. While the LEQR case admits an analytical solution and a number of other cases admit a classical $C^{1,2}$ solution, the formalization of a consistent theory has proved elusive. Indeed, proving the existence and uniqueness of a classical solution often carry restriction which exclude the LEQR problem, such as assumptions on the behaviour of the reward function or bounds on the state space. Bensoussan, Frehse and Nagai [11] achieved an important step by developing a generalized theory of classical solutions which includes the LEQR problem as a subcase. However, we can still expect developments in this area as the number of applications of risk-sensitive control, and therefore the range of reward functions, grows.

The second avenue of research is concerned with the application of risk-sensitive control theory to solve specific scientific, engineering, economics and financial problems. Risk-sensitive control was first applied to solve a financial problem by Lefebvre and Montulet [38] in a corporate finance context and by Fleming [28] in a portfolio selection context. However, Bielecki and Pliska [15] were the first to regard the continuous time risk-sensitive control as a practical tool that could be used to solve the 'real world' portfolio selection problems. Kuroda and Nagai [37], Nagai and Peng [40] as well as Fleming and Sheu [26] and [27] have provided major contributors to the field. Recently, risk sensitive control has found new applications in economics, through the works of Hansen and Sargent [30].

2.2 The Risk-Sensitive Asset Management Model

Since we are mainly concerned with the behaviour of risk-averse investors, from this point onward, we will assume that

Assumption 2.1 (Standing Assumption).

$$\theta > 0$$

Bielecki and Pliska [15] chose the reward function to be the logarithm of the investor's wealth V, i.e.

$$F(t, x, h) = \ln V(t, x, h)$$

The investor's objective is therefore to maximize the risk-sensitive (log) return of his/her portfolio.

With this choice of reward function, we can express the control criterion defined in equation (2.1) as

$$J(x, t, h; \theta) := -\frac{1}{\theta} \ln \mathbf{E}\left[e^{-\theta \ln V(t,x,h)}\right] \tag{2.3}$$

The expectation

$$\mathbf{E}\left[e^{-\theta \ln V(t,x,h)}\right] = \mathbf{E}\left[V(t, x, h)^{-\theta}\right] =: U_\theta(V(t, x, h)) \tag{2.4}$$

can then be interpreted as the expected utility of time t wealth under the power utility (HARA) function.

In addition, a Taylor expansion of the criterion around $\theta = 0$ to the second order yields

$$J(x,t;\theta) = \mathbf{E}\left[\ln V(t,x,h)\right] - \frac{\theta}{2}\mathbf{Var}\left[\ln V(t,x,h)\right] + O(\theta^2) \tag{2.5}$$

From this expansion, we recover the Kelly criterion portfolio in the limit as $\theta \to 0$. [1] Moreover, when $\ln V$ is normally distributed, Bielecki and Pliska [16] showed that the risk-sensitive approach is equivalent to the mean-variance optimization.

2.2.1 Asset and Factor Dynamics

Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ be the underlying probability space. Bielecki and Pliska [15] proposed a factor model with $m > 0$ assets and $n \geq 0$ factors in which the prices of the risky assets follow a SDE of the form

$$\frac{dS_i(t)}{S_i(t)} = (a + AX(t))_i dt + \sum_{k=1}^{M} \sigma_{ik} dW_k(t), \quad S_i(0) = s_i, \quad i = 1, \ldots, m \tag{2.6}$$

where

- $M := n + m$;
- $X(t)$ is the $\mathbb{R}^n$-valued factor process with components $X_j(t),\ j = 1, \ldots, n$ and with dynamics defined below in (2.8);
- $a \in \mathbb{R}^m$;
- $A \in \mathbb{R}^{m \times n}$;
- $\Sigma := [\sigma_{ij}],\ i = 1, \ldots, m,\ j = 1, \ldots, M$; and

[1] The initial treatment of the Kelly criterion can be found in Kelly [35]. Ziemba [46] gives a detailed discussion of the Kelly criterion and of related strategies in a finance context.

- $W(t)$ is a $\mathbb{R}^M$-valued $(\mathcal{F}_t)$-Brownian motion with components $W_k(t)$, $k = 1, \ldots, M$.

To this investment universe, one could either add a risk-free asset, as in Kuroda and Nagai [37] or a short-term 'money market' asset satisfying the equation

$$\frac{dS_0(t)}{S_0(t)} = (a_0 + A_0' X(t))\, dt, \qquad S_0(0) = s_0 \tag{2.7}$$

where $a_0 \in \mathbb{R}$ is a scalar constant and $A_0 \in \mathbb{R}^n$ is a n-element column vector, as in Bielecki, Chancelier, Pliska and Sulem [13] or Davis and Lleo [21]. Note that if we set $A_0 = 0$ and $a_0 = r$, then equation (2.7) can be interpreted as the dynamics of a risk-free rate.

The drift of the asset prices depends on n valuation factors modelled as affine processes with constant diffusion

$$dX(t) = (b + BX(t))dt + \Lambda dW(t), \qquad X(0) = x \tag{2.8}$$

where

- $b \in \mathbb{R}^n$;
- $B \in \mathbb{R}^{n \times n}$;
- $\Lambda := [\Lambda_{ij}]$, $i = 1, \ldots, n$, $j = 1, \ldots, M$.

These factors must be specified, but they could include macroeconomic, microeconomic or statistical variables. Although no econometric study has been undertaken to identify the specific factors, one could in principle use economic variables such as GDP, long and short-term interest rates or inflation. We will assume throughout that

the factors are observable. [2].

We assume throughout the thesis that

Assumption 2.2.

$$\Sigma\Sigma' > 0$$

The effect of this assumption is to prevent redundant assets. For example, we will not able to model in our investment market a share and an option or futures on that share. However, this assumption leaves us free to model a wide range of assets such as shares, bonds and commodities products as well as related indexes. This in turn provides enough freedom to solve the type of asset allocation question most long-term investors, such as endowment funds and mutual funds, would face in reality. Indeed, in the asset management industry, derivatives are not considered as a separate asset class. Rather, the use of the derivative markets is left as a tactical decision to the asset manager who will often use them to fine tune the risk profile of his/her portfolio or get some exposure to assets when the underlying cash market is illiquid.

Two types of asset classes have been considered so far: equities (see for example Bielecki and Pliska [15] and the empirical studies by Bielecki, Pliska and Sherris [18]) and fixed income (see Bielecki and Pliska [17] and the empirical study by Bielecki, Harris, Li and Pliska [14]). In particular, to model fixed income securities, Bielecki and Pliska [17] applied the elegant concept of rolling horizon bonds proposed by Rutkowski [42], treating rolling horizon bonds as securities and their yields as factors.

[2] Nagai and Peng [40] have used linear filtering to address the partial information case in which assets prices are directly observable but factor levels are not.

2.2.2 Portfolio Dynamics

Let $\mathcal{G}_t := \sigma((S(s), X(s)), 0 \leq s \leq t)$ be the sigma-field generated by the security and factor processes up to time t.

The allocation of wealth among the assets is defined by an $\mathbb{R}^m$-valued stochastic process h, where the ith component $h_i(t)$ denotes the proportion of current wealth invested in the ith risky security at time t, $i = 1, \ldots, m$. The proportion invested in the money market account is $h_0(t) = 1 - \sum_{i=1}^{m} h_i(t)$.

Definition 2.3. An investment process $h(t)$ is in class $\mathcal{H}$ if the following conditions are satisfied:

1. $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t \geq 0}$;

2. $\mathbb{P}\left(\int_0^T |h(s)|^2 \, ds < +\infty\right) = 1, \quad \forall T > 0.$

Taking the budget equation into consideration, the wealth, $V(t)$, of the asset only portfolio in response to an investment strategy $h \in \mathcal{H}$ follows the dynamics

$$\frac{dV_t}{V_t} = (a_0 + A_0' X(t))\, dt + h'(t)\, (a - a_0 \mathbf{1} + (A - \mathbf{1} A_0')\, X(t))\, dt + h'(t) \Sigma dW_t$$

where $\mathbf{1} \in \mathbf{R}^m$ denotes the m-element unit column vector. Defining $\hat{a} := a - a_0 \mathbf{1}$ and $\hat{A} := A - \mathbf{1} A_0'$, we can express the portfolio dynamics as

$$\frac{dV_t}{V_t} = (a_0 + A_0' X(t))\, dt + h'(t) \left(\hat{a} + \hat{A} X(t)\right) dt + h'(t) \Sigma dW_t$$

Applying Itô's lemma, the logarithm of the investor's wealth is given by the SDE

$$\begin{aligned}\ln V(t) &= \ln v + \int_0^t (a_0 + A_0' X(s)) + h(s)' \left(\hat{a} + \hat{A} X(s)\right) ds \\ &\quad -\frac{1}{2}\int_0^t h(s)'\Sigma\Sigma' h(s) ds + \int_0^t h(s)'\Sigma dW(s), \end{aligned} \tag{2.9}$$

where $V(0) = v$, h is the m-dimensional vector of portfolio weights and we used the notation $\hat{a} := a - a_0 1$ and $\hat{A} := A - 1A_0'$.

We immediately notice that the equation for V is independent of the asset prices and depends solely on the valuation factors (the state process). As a result, the effective dimension of the risk-sensitive asset management model is the number of factors rather than the number of assets. The limited impact of the number of assets is particularly important since for practical applications we would typically use only a few factors (possibly 3 to 5) to parametrize a large cohort of assets and asset classes (possibly several dozens). The risk-sensitive asset management model is therefore particularly efficient from a computational perspective.

To obtain an equation for the term $e^{-\theta \ln V(t)}$ inside the expectation of the risk-sensitive criterion, we multiply $\ln V(t, x, h)$ by $-\theta$ and then take the exponential. This yields the relation

$$\begin{aligned}e^{-\theta \ln V(t;h)} &= v_0^{-\theta} \exp\left\{\theta \int_0^t g(X_s, h(s); \theta) ds - \theta \int_0^t h(s)'\Sigma dW_s \right. \\ &\quad \left. -\frac{1}{2}\theta^2 \int_0^t h(s)'\Sigma\Sigma' h(s) ds \right\} \end{aligned} \tag{2.10}$$

where

$$g(x,h;\theta) = \frac{1}{2}(\theta+1)\,h'\Sigma\Sigma'h - a_0 - A_0'x - h'(\hat{a}+\hat{A}x) \tag{2.11}$$

2.2.3 Fomulation of the Control Problems

Two related classes of problems have been investigated. Bielecki and Pliska [15] proposed the infinite time horizon class of problems $\mathscr{P}_\theta$:

> For $\theta \in]-1,0[\cup]0,+\infty[$, maximize the risk sensitive expected log-growth rate of the portfolio per unit of time

$$J_\theta(v,x;h) := \liminf_{t\to\infty}\left(-\frac{1}{\theta}\right)t^{-1}\ln \mathrm{E}e^{-\theta \ln V(t,x,h)} \tag{2.12}$$

Kuroda and Nagai [37] considered the finite time horizon class of problems $\mathscr{P}_{\theta,T}$:

> For $\theta \in]-1,0[\cup]0,+\infty[$, maximize the risk sensitive expected log-growth rate of the portfolio over a time horizon T

$$J_{\theta,T}(v,x;h) := \left(-\frac{1}{\theta}\right)\ln \mathrm{E}e^{-\theta \ln V(t,x,h)} \tag{2.13}$$

In fact, for Kuroda and Nagai, solving the finite-horizon problem is a first step. By studying its asymptotic properties, they deduce a solution to the infinite-horizon problem.

In the remainder of this section, we briefly sketch the main steps involved in solving the finite time horizon problem. The full argument will be developed in Chapter 3 as

we set out to solve an Asset and Liability Management problem using the same line of reasoning.

2.2.4 Solving The Finite Horizon Control Problem

Kuroda and Nagai [37] observed that the term

$$\exp\left\{-\theta\int_0^t h(s)'\Sigma dW_s - \frac{1}{2}\theta^2\int_0^t h(s)'\Sigma\Sigma' h(s)ds\right\}$$

in equation (2.10) is an exponential martingale iff

$$\mathbf{E}\left[\exp\left\{-\theta\int_0^t h(s)'\Sigma dW_s - \frac{1}{2}\theta^2\int_0^t h(s)'\Sigma\Sigma' h(s)ds\right\}\right] = 1 \qquad (2.14)$$

When this condition is satisfied, a change of measure is possible. Under the measure $\mathbb{P}_h^\theta$ defined, via the Radon-Nikodým derivative

$$\begin{aligned}\chi_t &:= \left.\frac{d\mathbb{P}_h^\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} \\ &= \exp\left\{-\theta\int_0^t h(s)'\Sigma dW_s - \frac{1}{2}\theta^2\int_0^t h(s)'\Sigma\Sigma' h(s)ds\right\}, \quad \forall t \geq 0\end{aligned}$$

the risk-sensitive criterion can be expressed as

$$I(v,x;h;t,T) = -\frac{1}{\theta}\ln\mathbf{E}_{t,x}^{h,\theta}\left[\exp\left\{\theta\int_t^T g(X_s,h(s);\theta)ds - \theta\ln v\right\}\right] \qquad (2.15)$$

where the factor dynamics under the new measure $\mathbb{P}_h^\theta$ is given by:

$$dX_s = (b + BX_s - \theta\Lambda\Sigma' h(s))\,ds + \Lambda dW_s^\theta \qquad (2.16)$$

and where

$$W_s^\theta = W_t + \theta \int_0^t \Sigma' h(s) ds$$

is a $\mathbb{P}_h^\theta$-Brownian motion.

In this formulation, the problem is a standard Linear Exponential of Quadratic Regulator (LEQR) control problem which can be solved exactly, up to the resolution of a system of Riccati equations.

2.2.5 Uncorrelated Asset and Factor Noise

In their groundbreaking work, Bielecki and Pliska [15] made the assumption that asset and factor noise are uncorrelated (i.e. $\Sigma\Lambda' = 0$) in order to derive their main results. [3]

Adopting the change of measure method proposed by Kuroda and Nagai [37], Davis and Lleo [21] noted that in the special case when $\Sigma\Lambda' = 0$, the evolution of X_t under the measure $\mathbb{P}_h^\theta$ given in equation (2.16) can be expressed as:

$$dX_s = (b + BX_s)\, ds + \Lambda dW_s^\theta$$

The evolution of the state is therefore independent of the control variable h and, as a result, the control problem can be solved through a pointwise maximization of the auxiliary criterion function $I(v, x; h; t, T)$.

[3] This assumption is no longer required if one adopts the more general and powerful change of measure-based method proposed by Kuroda and Nagai [37].

The optimal control h^*, in this case, is the maximizer of the function $g(x;h;t,T)$ defined in equation (2.11)and is given by

$$h^* = \frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left(\hat{a}+\hat{A}x\right)$$

which is a position of $\frac{1}{\theta+1}$ in the log utility (or Kelly) portfolio.

Let $\Phi(t,x)$ be the value function corresponding to the exponential of integral criterion $I(v,x;h;t,T)$. Substituting the value of h^* in the equation for g, we note that

$$\begin{aligned}\Phi(t,x) &= \sup_{h\in\mathcal{A}(T)} I(f_0,x;h;t,T)\\ &= -\frac{1}{\theta}\ln \mathbf{E}_{t,x}^{h,\theta}\left[\exp\left\{\theta\int_t^T g(x,h^*(s);t,T;\theta)ds\right\}v_0^{-\theta}\right]\end{aligned}$$

The PDE for Φ can now be obtained directly via an exponential transformation and an application of Feynman-Kac.

2.2.6 The General Case

In the general case, the value function Φ for the auxiliary criterion function $I(v,x;h;t,T)$, defined as

$$\Phi(t,x) = \sup_{\mathcal{A}(T)} I(v,x;h;t,T)$$

where $\mathcal{A}(T)$ is the set of admissible strategies, satisfies the Hamilton-Jacobi-Bellman Partial Differential Equation (HJB PDE)

$$\frac{\partial\Phi}{\partial t}+\sup_{h\in\mathbb{R}^m} L_t^h\Phi(X(t))=0 \tag{2.17}$$

where

$$\begin{aligned} L_t^h \Phi(t,x) &= \left(b + Bx - \theta \Lambda \Sigma' h(s)\right)' D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) \\ &\quad -\frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi - g(x,h;\theta) \end{aligned}$$

and subject to terminal condition $\Phi(T,x) = \ln v$.

Solving the optimization problem, we find that the optimal investment policy $h^*(t)$ is given by

$$h^*(t) = \frac{1}{\theta+1}\left(\Sigma\Sigma'\right)^{-1}\left[\hat{a} + \hat{A}X(t) - \theta\Sigma\Lambda' D\Phi(t, X(t))\right] \tag{2.18}$$

The solution of the HJB PDE is of the form

$$\Phi(t,x) = x'Q(t)x + x'q(t) + k(t)$$

where $Q(t)$ solves an n-dimensional matrix Riccati equation and $q(t)$ solves a n-dimensional linear ordinary differential equation dependent on Q (see Kuroda and Nagai [37] for the original derivation. Chapter 3 will also address this question from the more general perspective of the ALM problem).

2.3 Mutual Fund Theorem

Obtaining an exact expression for the optimal control, h^* in (2.18), enables us to develop an economic interpretation of the investment policy involved. This interpretation is most commonly expressed as a "mutual fund" or "fund separation" theorem. We now present without proof a Mutual Fund Theorem due to Davis and Lleo (Corol-

lary 2 in [21]):

Theorem 2.4 (Mutual Fund Theorem). *Any portfolio can be expressed as a linear combination of investments into two "mutual funds" with respective risky asset allocations:*

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h^C(t) &= -(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right) \end{aligned} \tag{2.19}$$

and respective allocation to the money market account given by:

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h_0^C(t) &= 1 + \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right) \end{aligned}$$

Moreover, if an investor has a risk sensitivity θ, then the respective weights of each mutual fund in the investor's portfolio are equal to $\frac{1}{\theta+1}$ and $\frac{\theta}{\theta+1}$.

In the risk-sensitive asset management model, the allocation between the two funds is a sole function of the investor's risk sensitivity θ. As $\theta \to 0$, the investor's wealth gets invested in the log-utility portfolio, i.e. portfolio K. As $\theta \to \infty$, the investor's wealth is increasingly invested in portfolio C. Portfolio C's investment policy can be viewed as a strategy which consists in holding a large position in the short-term rate and a set of positions trading on the comovement of assets and valuation factors.

This mutual fund theorem is a generalization case of the result established by Merton. Indeed, if we did not have any valuation factor (i.e. $n = 0$), the risky securities would follow geometric Brownian motions with drift vector $\mu = a$, the

money market account would be the risk-free asset (i.e. $a_0 = r$ and $A_0 = 0$) and there would be no comovement between assets and liabilities (i.e. $\Sigma\Lambda' = 0$). Thus, the fund C would be fully invested in the risk-free asset. In this case, Theorem 2.4 reduces to Merton's mutual fund theorem.

Chapter 3

Risk-Sensitive Asset and Liability Management

In this chapter, we consider a new risk sensitive control problem in which an investor manages a portfolio of assets partly funded by the investor's personal wealth, with the balance funded by a stochastic liability. The objective of the investor is to maximize the risk-sensitive return on equity of the portfolio. To achieve this objective, the investor can not only chose an asset allocation but can also control the degree of leverage of the portfolio by increasing or decreasing the level of liability. To solve this problem we use the change-of-measure method proposed by Kuroda and Nagai [37] and which was originally derived within an asset only framework. With minor modifications to this change-of-measure method, we derive an analytical solution for the ALM problem.

The novelty of our approach is twofold. First, we venture outside of the usual asset only framework to explore the decision making process and deduce optimal strategies for a large class of of funded investors, including insurance companies, hedge funds and other sophisticated investors. Of course, even in the asset only case an investor can take up some degree of leverage by shorting some assets in order to increase their holdings of other assets and by borrowing at the money market rate. However, this type of leverage is endogenous and bound by the budget constraint. The situation our ALM model is different as a stochastic liability, which differs from the money market account, is exogenously imposed on the investor at the onset. The interaction between the asset portfolio and the liability gives rise to an exogenous leverage, which we will model as a control variable.

Second, the introduction of a second control variable, the degree of leverage, enables us to investigate the relation between asset allocation, leverage and risk aversion.

In particular, although we would intuitively expect the risk aversion to have an obvious impact on the *degree* of asset-liability leverage taken by the investor, we find that the risk aversion has a more pronounced effect on the *nature* of the leverage. Indeed, very risk averse investor will seek leverage only to perfect their asset and liability hedging strategy, while less risk averse investors will see leverage as an opportunity to increase their excess returns above the money market rate. The resulting Asset and Liability Management (ALM) model is therefore more sophisticated than its asset only counterpart.

This chapter is organized as follows. Section 1 introduces the general setting of the model. The control problem is formulated in Section 2 and its associated HJB PDE is derived in Section 3. Sections 4 and 5 are respectively concerned with solving the finite time horizon and the infinite-time horizon model. Finally, in Section 6 we explore the economic implications of the model and propose a mutual fund theorem.

3.1 Analytical Setting

3.1.1 Asset Only Setting

To the exception of the addition of a liability, the setting in this chapter is largely unchanged compared to Chapter 3. As a result, we will only point out the key differences. In this ALM model, $M = n + m + 1$, reflecting one Brownian motion for each factor and asset as well as for the liability. Moreover, wet let $\mathcal{G}_t := \sigma((S(s), L(s), X(s)), 0 \leq s \leq t)$ be the sigma-field generated by the security, liability and factor processes up to time t. The dynamics of the liability process $L(s)$ is given in equation (3.1) below.

3.1.2 Liability Modelling

We assume that the liability evolves according to the SDE

$$\frac{dL(t)}{L(t)} = (c + C'X(t))dt + \varsigma' dW(t), \qquad L(0) = l \tag{3.1}$$

where c and A_0 are scalar constants, C is a n-element column vector, and ς is a M-element column vector.

This formulation, which was previously used to model benchmarks (see Davis and Lleo [21]), is wide enough to adequately characterize a wide range of liability structures.

We will also assume that

Assumption 3.1. The Euclidian norm of the projection of ς onto the orthogonal complement to the space spanned by Σ' is strictly positive and hence $\varsigma'(I - \Sigma'(\Sigma\Sigma')^{-1}\Sigma)\varsigma > 0$.

This assumption ensures internal consistency of the financial system by eliminating the possibility of fully replicating the liability through a portfolio of assets. Indeed, if liabilities could be exactly replicated by a portfolio of assets, then the question of an optimal amount of liability and degree of leverage would be irrelevant. Indeed, an investor would be able to accept any arbitrary level of liability, safe in the knowledge that the extra funds generated by issuing the liability can then be used to fully hedge the liability by transacting on the asset market.

3.1.3 Relation with Equity and Leverage

The time t equity, $E(t)$, is the wealth belonging directly to the investor. Equity is defined as the difference between the assets and the liabilities, i.e.

$$E(t) = V(t) - L(t)$$

with

$$E(0) = e_0 = v - l > 0$$

and hence the dynamics of the equity is given in differential form by

$$dE(t) = dV(t) - dL(t)$$

The time t degree of leverage, or leverage ratio, $\rho(t)$, is defined as the ratio of asset value to equity value, i.e.

$$\rho(t) = \frac{V(t)}{E(t)}$$

As a result, we have

$$V(t) = \rho(t)E(t)$$

and

$$L(t) = (\rho(t) - 1)E(t)$$

Note: because in the present model we are dealing with leverage ratio rather than leverage multiplier, a few clarifications might be required. In our analysis, a leverage ratio of 1 represents a portfolio with assets equal to the investor's equity and no liabilities. A leverage ratio greater than 1 indicates a leveraged portfolio and a leverage ratio less than 1 indicates un-levered portfolio. Hence, if an asset manager leverages

a portfolio with liabilities equal to 5 times the portfolio's equity, the corresponding leverage ratio is equal to 6 (i.e. 5 times for the liabilities and 1 time for the equity).

It is important to note that in our model leverage is a control variable, rather than a mere resultant of the dynamics of asset and liabilities. By establishing leverage as a control variable, we make it possible for the investor to choose an optimal level of leverage in addition to the optimal investment strategy. In doing so, we bring the leveraging dimension, a dimension central to sophisticated investment strategies, to life. Formally,

Definition 3.2. A leverage process $\rho(t)$ is in class $\mathcal{R}$ if the following conditions are satisfied:

1. $\rho(t) \in \mathbb{R}^+$;

2. $\rho(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t \geq 0}$;

3. $P\left(\int_0^T |\rho(s)|^2 \, ds < +\infty\right) = 1$;

3.1.4 Equity Dynamics

The dynamics of the equity can be expressed as

$$\begin{aligned} dE(t) &= \left(a_0 + A_0' X(t)\right) V_t dt + h'(t)\left(\hat{a} + \hat{A} X(t)\right) V_t dt + V_t h'(t) \Sigma dW_t \\ &\quad -(c + C' X(t)) L(t) dt - L(t) \varsigma' dW(t) \end{aligned}$$

Rewriting in terms of equity and leverage only, we get

$$\begin{aligned}\frac{dE(t)}{E(t)} &= (c + C'X(t))dt \\ &\quad + \rho(t)\left[(a_0 + A_0'X(t)) + h'(t)\left(\hat{a} + \hat{A}X(t)\right) - (c + C'X(t))\right]dt \\ &\quad + \left[\rho(t)(h'(t)\Sigma - \varsigma') + \varsigma'\right]dW(t), \\ E(0) &= e_0 \end{aligned} \tag{3.2}$$

3.2 Problem Setup

3.2.1 Optimization Criterion

In our analysis, we consider two related classes of risk sensitive control problems. First, we consider the class of problems $\mathcal{P}_{\theta,T}$: for $\theta \in]-1, 0[\cup]0, +\infty[$, maximize the risk sensitive expected log return on equity over a time horizon T

$$J_{\theta,T}(v, x; h, \rho) := \left(-\frac{1}{\theta}\right) \ln \mathbf{E} e^{-\theta \ln E(T;h,\rho)} \tag{3.3}$$

over the set of control $(h, \rho) \in \mathcal{A}_{h,\rho}(T)$ (defined below).

Second, we consider the class of problems $\mathcal{P}_{\theta}$: for $\theta \in]-1, 0[\cup]0, +\infty[$, maximize the risk sensitive expected log return on equity per unit of time

$$J_{\theta}(v, x; h, \rho) := \liminf_{t\to\infty} \left(-\frac{1}{\theta}\right) t^{-1} \ln \mathbf{E} e^{-\theta \ln E(t;h,\rho)} \tag{3.4}$$

over the set of control $(h, \rho) \in \mathcal{A}_{h,\rho}(T)$ (defined below).

3.2.2 Interpretation of the Criterion

From a financial perspective, the quantity $\frac{\ln E(t)}{t}$ represents the average growth rate of equity up to time t. Thus, maximizing $J_{\theta,T}(v,x;h,\rho)$ is equivalent to maximizing the risk-sensitive return on equity (RSROE) over a time horizon T.

The control criterion also has an economic interpretation. Let $E(t)$ be the monetary amount of wealth (i.e. equity) held at time t. Note that

$$\begin{aligned} U_\theta(E) &= \mathrm{E}e^{-\theta \ln E(t)} \\ &= \mathrm{E}\left[E(t)^{-\theta}\right] \end{aligned} \tag{3.5}$$

represents the expected utility of time t wealth under the power utility (HARA) function.

Thus, $J_\theta(E)$ defined as

$$J_\theta(E) := \liminf_{t\to\infty} \left(-\frac{1}{\theta}\right) t^{-1} \ln \mathrm{E}\left[E(t)^{-\theta}\right] \tag{3.6}$$

can also be interpreted as the long-term growth rate of the expected utility under the power utility function of the consumption E.

The risk-sensitive approach equips us with two interpretations for one optimization criterion. While the second interpretation satisfies the tenets of economic theory, the first interpretation supplies sophisticated investors with an intuitive measure of risk-sensitive performance: RSROE.

3.2.3 Risk Sensitive Asset Management as a Special Case of Risk Sensitive Asset and Liability Management

The relationship between the risk sensitive ALM problem we propose and Bielecki and Pliska's risk sensitive asset management becomes clear once we set the liability to 0. In this case, the assets represent the investor's equity, or personal wealth and leverage loses its status of control variable by becoming identically equal to 1. In turn, Bielecki and Pliska's risk sensitive asset management becomes a particular case of the proposed Risk Sensitive Asset and Liability Management (RSALM) model.

3.3 Derivation of the Bellman Equation

3.3.1 Criterion Under the Expectation

By Itô,

$$\begin{aligned}\ln E(t) &= e_0 + \int_0^t (c + C'X(s))\,ds \\ &\quad + \int_0^t \rho(s)\left[(a_0 + A_0'X(t)) + h'(s)(\hat{a} + \hat{A}X(s)) - c - C'X(s)\right]ds \\ &\quad -\frac{1}{2}\int_0^t \left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right]\left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right]'ds \\ &\quad + \int_0^t \left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right]dW_s \qquad (3.7)\end{aligned}$$

Multiplying by $-\theta$ and taking the exponential on both sides of (3.7), we get

$$\begin{aligned}e^{-\theta \ln E(t)} &= e_0^{-\theta}\exp\left\{\theta\int_0^t g(X_s, h(s), \rho(s); \theta)ds - \theta\int_0^t \left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right]dW_s\right. \\ &\quad \left. -\frac{\theta^2}{2}\int_0^t \left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right]\left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right]'ds\right\} \qquad (3.8)\end{aligned}$$

where

$$\begin{aligned} g(x,h,\rho;\theta) &= \frac{1}{2}(\theta+1)\left[\rho(h'\Sigma-\varsigma')+\varsigma'\right]\left[\rho(h'\Sigma-\varsigma')+\varsigma'\right]' \\ &\quad -\rho\left[h'(\hat{a}+\hat{A}x)+a_0-c+(A_0'-C')x\right]-c-C'x \end{aligned} \tag{3.9}$$

3.3.2 Change of Measure

Let $\mathbb{P}^\theta$ be a probability measure on $(\Omega,\mathcal{F})$ such that

$$\begin{aligned} \chi_t &:= \left.\frac{d\mathbb{P}^\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} \\ &= \exp\left\{-\theta\int_0^t\left[\rho(s)(h'(s)\Sigma-\varsigma')+\varsigma'\right]dW_s\right. \\ &\quad \left.-\frac{\theta^2}{2}\int_0^t\left[\rho(s)(h'(s)\Sigma-\varsigma')+\varsigma'\right]\left[\rho(s)(h'(s)\Sigma-\varsigma')+\varsigma'\right]'ds\right\} \qquad \forall t\geq 0 \end{aligned} \tag{3.10}$$

We denote by $\mathcal{A}_{h,\rho}(T)$ the set of investment strategies $(h,\rho)\in\mathcal{H}\times\mathcal{H}$ on $[0,T]$ such that $\mathbb{P}^\theta$ is a probability measure, i.e.

$$\mathbf{E}[\chi_t]=1$$

Under $\mathbb{P}^\theta$,

$$W_t^\theta = W_t+\theta\int_0^t\left[\rho(s)(\Sigma'h(s)-\varsigma)+\varsigma\right]ds$$

is a standard Brownian motion and X_t satisfies the SDE:

$$dX_s=(b+BX_s-\theta\Lambda\left[\rho(s)(\Sigma'h(s)-\varsigma)+\varsigma\right])ds+\Lambda dW_s^\theta \tag{3.11}$$

We can now introduce the auxiliary criterion function under the measure $\mathbb{P}^\theta$:

$$I(e_0, x; h, \rho; t, T) = -\frac{1}{\theta} \ln \mathbf{E}_{t,x}^{h,\theta} \left[\exp \left\{ \theta \int_t^T g(X_s, h(s), \rho(s); \theta) ds - \theta \ln e_0 \right\} \right] \quad (3.12)$$

where $\mathbf{E}_{t,x}^{h,\theta}[\cdot]$ denotes the expectation taken with respect to measure $\mathbb{P}^\theta$.

3.3.3 The HJB Equation

Let Φ be the value function for the auxiliary criterion function $I(e_0, x; h, \rho; t, T)$. Then Φ is defined as

$$\Phi(t, x) = \sup_{\mathcal{A}(T)} I(e_0, x; h, \rho; t, T)$$

and it satisfies the HJB PDE

$$\frac{\partial \Phi}{\partial t} + \sup_{\substack{h \in \mathbb{R}^m \\ \rho \in \mathbb{R}^+}} L_t^h \Phi = 0 \quad (3.13)$$

where

$$\begin{aligned} L_t^{h,\rho} \Phi &= (b + Bx - \theta \Lambda [\rho(t)(\Sigma' h(t) - \varsigma) + \varsigma])' D\Phi + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda' D^2 \Phi \right) \\ &\quad - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi - g(x, h, \rho; \theta) \end{aligned} \quad (3.14)$$

and subject to terminal condition $\Phi(T, x) = \ln e_0 = \ln(v - l)$.

3.4 Solving the Finite Time Horizon Problem

3.4.1 Deriving the Optimal Control

Start by considering the terms inside the supremum:

$$
\begin{aligned}
&\sup_{\substack{h\in\mathbb{R}^m\\ \rho\in\mathbb{R}^+}} L_t^{h,\rho}\Phi \\
= &\sup_{\substack{h\in\mathbb{R}^m\\ \rho\in\mathbb{R}^+}} \Big\{ \left(b+Bx-\theta\Lambda\left[\rho(\Sigma' h-\varsigma)+\varsigma\right]\right)' D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi \\
&-\frac{1}{2}(\theta+1)\left[\rho(h'\Sigma-\varsigma')+\varsigma'\right]\left[\rho(h'\Sigma-\varsigma')+\varsigma'\right]' \\
&+\rho\left[h'(\hat{a}+\hat{A}x)+a_0-c+(A_0'-C')x\right]+c+C'x\Big\} \\
= &\sup_{\substack{h\in\mathbb{R}^m\\ \rho\in\mathbb{R}^+}} \Big\{ (b+Bx)' D\Phi - \theta\rho(h'\Sigma-\varsigma')\Lambda' D\Phi - \theta\varsigma'\Lambda' D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) \\
&-\frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi - \frac{1}{2}(\theta+1)\rho^2(h'\Sigma-\varsigma')(\Sigma' h-\varsigma) \\
&-\frac{1}{2}(\theta+1)\rho\left[(h'\Sigma-\varsigma')\varsigma+\varsigma'(\Sigma' h-\varsigma)\right]-\frac{1}{2}(\theta+1)\varsigma'\varsigma \\
&+\rho\left[h'(\hat{a}+\hat{A}x)+a_0-c+(A_0'-C')x\right]+c+C'x\Big\} \\
= &\ (b+Bx)' D\Phi - \theta\varsigma'\Lambda' D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi - \frac{1}{2}(\theta+1)\varsigma'\varsigma + c + C'x \\
&+ \sup_{\substack{h\in\mathbb{R}^m\\ \rho\in\mathbb{R}^+}} \Big\{ -\theta\rho(h'\Sigma-\varsigma')\Lambda' D\Phi - \frac{1}{2}(\theta+1)\rho^2(h'\Sigma-\varsigma')(\Sigma' h-\varsigma) \\
&-\frac{1}{2}(\theta+1)\rho\left[(h'\Sigma-\varsigma')\varsigma+\varsigma'(\Sigma' h-\varsigma)\right]+\rho\left[h'(\hat{a}+\hat{A}x)+a_0-c+(A_0'-C')x\right]\Big\} \qquad (3.15)
\end{aligned}
$$

A necessary condition for the maximization of $L_t^{h,\rho}\Phi$ is that the derivatives of $L_t^{h,\rho}\Phi$ with respect to h and ρ be equal to 0. We will begin with the derivative with respect to h:

$$\begin{aligned}
&\frac{\partial L_t^{h,\rho}\Phi}{\partial h} \\
= \;& -\theta\rho\Sigma\Lambda' D\Phi - (\theta+1)\,\rho^2\Sigma\Sigma' h + (\theta+1)\,\rho^2\Sigma\varsigma - (\theta+1)\,\rho\Sigma\varsigma + \rho(\hat{a}+\hat{A}x) \\
= \;& 0
\end{aligned}$$

In order to solve this equation for h, we need to distinguish two cases: $\rho = 0$ and $\rho \neq 0$. Indeed, if $\rho = 0$ our problem degenerates to a case where the investor would refuse to invest his/her wealth and simply keep it "under the mattress." If $\rho = 0$, then we note that

$$\frac{\partial L_t^{h,\rho}\Phi}{\partial h} = 0 \qquad \forall h \in \mathbb{R}^m$$

Now what of the derivative of $L_t^{h,\rho}\Phi$ with respect to ρ? This derivative is equal to 0 for all $h \in \mathbb{R}^m$ iff

$$\begin{aligned}
&\frac{\partial L_t^{h,\rho}\Phi}{\partial \rho} \\
= \;& -\theta(h'\Sigma - \varsigma')\Lambda' D\Phi - (\theta+1)\,\rho(h'\Sigma - \varsigma')(\Sigma' h - \varsigma) \\
& -\frac{1}{2}(\theta+1)\left[(h'\Sigma - \varsigma')\varsigma + \varsigma'(\Sigma' h - \varsigma)\right] + h'(\hat{a}+\hat{A}x) + a_0 - c + (A_0' - C')x \\
= \;& 0
\end{aligned}$$

Hence, $\rho = 0$ iff

$$\begin{aligned}
&-\theta(h'\Sigma - \varsigma')\Lambda' D\Phi \\
&-\frac{1}{2}(\theta+1)\left[(h'\Sigma - \varsigma')\varsigma + \varsigma'(\Sigma' h - \varsigma)\right] + h'(\hat{a} + \hat{A}x) + a_0 - c + (A_0' - C')x \\
&= \quad 0
\end{aligned}$$

Thus, $\rho = 0\ \forall h \in \mathbb{R}^m, x \in \mathbb{R}$ if and only if $\varsigma = 0$, $a = a_0(1) = c(1) = 0$, $\hat{A} = (1)A_0' = (1)C' = 0$. This makes intuitive sense. Indeed, the only situation when an investor would refuse to invest and rather keep his/her wealth "under the mattress" is when no long or short investment opportunity exists.

We will now revert to the general case and assume $\rho \neq 0$, then the optimal asset allocation h^* is given by

$$h^* = \frac{1}{\rho}\frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}x - \theta\Sigma\Lambda' D\Phi + (\theta+1)(\rho-1)\Sigma\varsigma\right) \tag{3.16}$$

Indeed, since $\rho > 0$ and by Assumption 2.2, we can see that h^* is a maximizer of relationship (3.15) for a given ρ.

We now substitute the optimal value for h, given in (3.16) into the derivation (3.15) in order to express the derivation as a function of ρ,

$$
\begin{aligned}
&\sup_{\rho\in\mathbb{R}^+} L_t^{h^*,\rho}\Phi \\
= \;& (b+Bx)'\,D\Phi - \theta\varsigma'\Lambda' D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi - \frac{1}{2}\left(\theta+1\right)\varsigma'\varsigma \\
&+c + C'x + \sup_{\rho\in\mathcal{R}}\left\{-\frac{1}{\theta+1}(\hat{a}+\hat{A}x)'(\Sigma\Sigma')^{-1}\Sigma\Lambda' D\Phi\right. \\
&+\frac{1}{\theta+1}\theta D\Phi'\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' D\Phi - \theta(\rho-1)\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' D\Phi \\
&+\theta\rho\varsigma'\Lambda' D\Phi - \frac{1}{2}\frac{1}{\theta+1}(\hat{a}+\hat{A}x)'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}x) \\
&-\frac{1}{2}\frac{1}{\theta+1}\theta^2 D\Phi'\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' D\Phi - \frac{1}{2}\left(\theta+1\right)(\rho-1)^2\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma \\
&+\theta\frac{1}{\theta+1}(\hat{a}+\hat{A}x)'(\Sigma\Sigma')^{-1}\Sigma\Lambda' D\Phi - (\rho-1)(\hat{a}+\hat{A}x)'(\Sigma\Sigma')^{-1}\Sigma\varsigma \\
&+\theta(\rho-1)D\Phi'\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma + \rho\varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}x) \\
&-\rho\theta\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' D\Phi + \rho(\rho-1)\left(\theta+1\right)\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma - \frac{1}{2}\left(\theta+1\right)\rho^2\varsigma'\varsigma \\
&-\varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}x) + \theta\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' D\Phi \\
&-\left(\theta+1\right)(\rho-1)\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma + \left(\theta+1\right)\rho\varsigma'\varsigma \\
&+\frac{1}{\theta+1}(\hat{a}+\hat{A}x)'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}x) - \frac{1}{\theta+1}D\Phi'\Lambda\Sigma'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}x) \\
&\left.+(\rho-1)\varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}x) - \rho(c-a_0+(C'-A_0')x)\right\} \qquad (3.17)
\end{aligned}
$$

Since we are already optimal in the h direction, then all we need to do is to find the optimal value for ρ. This is done, as usual, by taking the derivative of equation (3.17) with respect to ρ, finding ρ^* such that the derivative is equal to 0, and identifying the sufficient conditions for ρ^* to be the optimum. Taking the derivative and setting it to 0, we find that

$$\begin{aligned}\rho^* &= 1+\frac{1}{\theta+1}\frac{1}{\varsigma'\left(I-\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right)\varsigma}\times\left[-\theta\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' D\Phi+\theta\varsigma'\Lambda' D\Phi\right.\\ &\left.+\varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}x)-(c-a_0+(C'-A_0')x)\right]\end{aligned}$$

Note that Assumption 3.1 ensures that we can effectively divide by $\varsigma'\left(I-\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right)\varsigma$ in the previous expression.

Now, we must outline the type of conditions under which ρ^* is the optimum for the problem under consideration. Taking the second derivative of equation (3.17) with respect to ρ we find that

$$\frac{\partial^2 L_t^{h,\rho}\Phi}{\partial\rho^2}<0$$

if and only if

$$(\theta+1)\,\varsigma'\left(I-\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right)\varsigma>0$$

which is true by Assumption 3.1.

To conclude, under Assumptions 2.2 and 3.1, the supremum is reached for

$$h^*=\frac{1}{\rho^*}\frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left(\hat{a}+\hat{A}x-\theta\Sigma\Lambda' D\Phi+(\theta+1)\,(\rho-1)\Sigma\varsigma\right) \tag{3.18}$$

and

$$\begin{aligned}\rho^* &= 1+\frac{1}{\theta+1}\frac{1}{\varsigma'\left(I-\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right)\varsigma}\times\Big[-\theta\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' D\Phi+\theta\varsigma'\Lambda' D\Phi\\ &\quad+\varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}x)-(c-a_0+(C'-A_0')x)\Big]\\ &= 1+\frac{1}{\theta+1}\kappa\left[\theta\varsigma'\Pi\Lambda' D\Phi+\varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}x)-(c-a_0+(C'-A_0')x)\right]\end{aligned} \tag{3.19}$$

where

$$\Pi = I-\Sigma'(\Sigma\Sigma')^{-1}\Sigma \tag{3.20}$$

and

$$\kappa=\frac{1}{\varsigma'\Pi\varsigma} \tag{3.21}$$

Remark 3.3. Note that $\kappa=\frac{1}{\varsigma'\Pi\Pi'\varsigma}$ since Π is a projection and $\Pi\Pi'=\Pi$.

The supremum is given by

$$\begin{aligned}&\sup_{h\in\mathbb{R}^m,\ \rho\in\mathbb{R}^+} L_t^{h,\rho}\Phi\\ &= -\frac{1}{2}(D\Phi)'K_0(\theta)D\Phi+x'K_1(\theta)D\Phi+\frac{1}{2}x'K_2(\theta)x+K_3(\theta)D\Phi+K_4(\theta)x+K_5(\theta)\end{aligned}$$

where

$$K_0(\theta) = \theta\Lambda\left[I-\frac{1}{\theta+1}\left(\Sigma'(\Sigma\Sigma')^{-1}\Sigma+\kappa\Pi'\varsigma\varsigma'\Pi\right)\right]\Lambda' \tag{3.22}$$

$$K_1(\theta) = B' - \frac{1}{\theta+1}\hat{A}'(\Sigma\Sigma')^{-1}\Sigma\Lambda' + \frac{1}{\theta+1}\kappa\left(\hat{A}'(\Sigma\Sigma')^{-1}\Sigma\varsigma - (C - A_0)\right)\varsigma'\Pi\Lambda' \tag{3.23}$$

$$K_2(\theta) = \frac{1}{\theta+1}\hat{A}'(\Sigma\Sigma')^{-1}\hat{A} + \frac{1}{\theta+1}\kappa\left(\hat{A}'(\Sigma\Sigma')^{-1}\Sigma\varsigma - (C - A_0)\right)\left(\hat{A}'(\Sigma\Sigma')^{-1}\Sigma\varsigma - (C - A_0)\right)' \tag{3.24}$$

$$K_3(\theta) = b' - \frac{1}{\theta+1}\hat{a}'(\Sigma\Sigma')^{-1}\Sigma\Lambda' + \frac{1}{\theta+1}\kappa\left[\hat{a}'(\Sigma\Sigma')^{-1}\Sigma\varsigma - (c - a_0)'\right]\varsigma'\Pi\Lambda' \tag{3.25}$$

$$\begin{aligned} K_4(\theta) &= A_0' + \frac{1}{\theta+1}\hat{a}'(\Sigma\Sigma')^{-1}\hat{A} \\ &+\frac{1}{\theta+1}\kappa\left(\hat{a}'(\Sigma\Sigma')^{-1}\Sigma\varsigma - (c - a_0)'\right)\left(\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A} - (C' - A_0')\right) \end{aligned} \tag{3.26}$$

$$\begin{aligned} K_5(\theta) &= \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) + \frac{1}{2}\frac{1}{\theta+1}\hat{a}'(\Sigma\Sigma')^{-1}\hat{a} + a_0 \\ &+\frac{1}{2}\frac{1}{\theta+1}\kappa\left[\hat{a}'(\Sigma\Sigma')^{-1}\Sigma\varsigma - (c - a_0)'\right]'\left[\hat{a}'(\Sigma\Sigma')^{-1}\Sigma\varsigma - (c - a_0)'\right] \end{aligned}$$

3.4.2 Solving the HJB PDE

One could verify that the HJB PDE is solved by taking Φ to be

$$\Phi(x,t) = \frac{1}{2}x'Q(t)x + q'(t)x + k(t) \tag{3.27}$$

where Q is a $n \times n$ symmetric matrix, q is a n-element column vector and k is a scalar, provided that the following three differential equations can be solved:

- a **Riccati equation** related to the coefficient of the quadratic term and used to determine the symmetric non-negative matrix $Q(t)$,

$$\dot{Q}(t) - Q'(t)K_0(\theta)Q(t) + K_1(\theta)Q(t) + Q'(t)K_1'(\theta) + K_2(\theta) = 0 \qquad (3.28)$$

  for $t \in [0, T]$ and with *terminal* condition $Q(T) = 0$;

- a **linear ordinary differential equation** related to the coefficient of the linear term and used to determine the vector $q(t)$,

$$\dot{q}(t)\left(K_1(\theta) - Q(t)K_0(\theta)\right)q(t) + Q(t)K_3'(\theta) + K_4'(\theta) = 0 \qquad (3.29)$$

  with *terminal* condition $q(T) = 0$, and;

- an **ordinary differential equation** related to the constant term and used to determine the constant $k(t)$.

$$\dot{k}(t) - \frac{1}{2}q'(t)K_0(\theta)q(t) + K_3(\theta)q(t) + K_6(\theta) + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'Q(t)\right) = 0 \qquad (3.30)$$

  with *terminal* condition $k(T) = \ln e_0$.

where

$$\begin{aligned} K_6(\theta) &= K_5(\theta) - \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) \\ &= \frac{1}{2}\frac{1}{\theta+1}\hat{a}'(\Sigma\Sigma')^{-1}\hat{a} + a_0 \\ &\quad + \frac{1}{2}\frac{1}{\theta+1}\kappa\left[\hat{a}'(\Sigma\Sigma')^{-1}\Sigma\varsigma - (c-a_0)'\right]'\left[\hat{a}'(\Sigma\Sigma')^{-1}\Sigma\varsigma - (c-a_0)'\right] \end{aligned} \tag{3.31}$$

The matrix $I - \frac{1}{\theta+1}\left(\Sigma'(\Sigma\Sigma')^{-1}\Sigma + \kappa\Pi'\varsigma\varsigma'\Pi\right)$ appearing in the definition of K_0 in equation (3.22) can be rewritten as:

$$\begin{aligned} & I - \frac{1}{\theta+1}\Sigma'(\Sigma\Sigma')^{-1}\Sigma - \frac{1}{\theta+1}\kappa\Pi'\varsigma\varsigma'\Pi \\ = \; & I - \frac{1}{\theta+1}\Sigma'(\Sigma\Sigma')^{-1}\Sigma - \frac{1}{\theta+1}\Pi'\varsigma(\varsigma'\Pi\varsigma)^{-1}\varsigma'\Pi \\ = \; & I - \frac{1}{\theta+1}\Sigma'(\Sigma\Sigma')^{-1}\Sigma - \frac{1}{\theta+1}\Pi'\varsigma\left[(\Pi'\varsigma)'\Pi\varsigma)\right]^{-1}(\Pi'\varsigma') \end{aligned}$$

where the last line follows from Remark 3.3

Now, $\Sigma'(\Sigma\Sigma')^{-1}\Sigma$ can be interpreted as the projection on a space spanned by the matrix Σ' and $\Pi'\varsigma\left[(\Pi'\varsigma)'\Pi\varsigma)\right]^{-1}(\Pi'\varsigma')$ can be interpreted as the projection on a space spanned by the matrix $\Pi'\varsigma$. Therefore $I - \Sigma'(\Sigma\Sigma')^{-1}\Sigma - \Pi'\varsigma\left[(\Pi'\varsigma)'\Pi\varsigma)\right]^{-1}(\Pi'\varsigma')$ is also a projection. Thus, $\Sigma'(\Sigma\Sigma')^{-1}\Sigma$, $\Pi'\varsigma\left[(\Pi'\varsigma)'\Pi\varsigma)\right]^{-1}(\Pi'\varsigma')$ and $I - \Sigma'(\Sigma\Sigma')^{-1}\Sigma - \Pi'\varsigma\left[(\Pi'\varsigma)'\Pi\varsigma)\right]^{-1}(\Pi'\varsigma')$ are positive definite matrices. Then, because $\theta > 0$, $I - \frac{1}{\theta+1}\Sigma'(\Sigma\Sigma')^{-1}\Sigma - \frac{1}{\theta+1}\theta\kappa\Pi'\varsigma\varsigma'\Pi$ is positive definite and therefore the Riccati equation (3.28) has a unique solution $Q(t) \geq 0$ defined for all $t \leq T$. (See Davis [20], Proposition 4.4.2.)

3.4.3 Formalizing the Solution

The following Theorem extends Theorem 2.1 in Kuroda and Nagai [37] to the assets and liability case and can be proved in the same fashion.

Theorem 3.4. *If equation* (3.28) *has a solution* Q*, then the optimal investment strategy* h^* *and optimal leverage* ρ^**, with* $(h^*, \rho^*) \in \mathcal{A}_{h,\rho}(T)$*, respectively defined by*

$$h_t^* = \frac{1}{\rho^*}\frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left[\hat{a} - \theta\Sigma\Lambda' q(t) + (\theta+1)(\rho^*-1)\Sigma\varsigma + \left(\hat{A} - \theta\Sigma\Lambda' Q(t)\right)X_t\right] \tag{3.32}$$

and

$$\begin{aligned}\rho_t^* &= 1 + \frac{1}{\theta+1}\kappa\left[\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a} - (c-a_0) + \theta\varsigma'\Pi\Lambda' q(t)\right.\\ &\quad\left. + \left(\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A} - (C'-A_0') + \theta\varsigma'\Pi\Lambda' Q(t)\right)X_t\right] \end{aligned}\tag{3.33}$$

where κ *is given by* (3.21)*,* Π *is given by* (3.20) *and* q *is a solution of equation* (3.29)*, are optimal for the finite time horizon problem*

$$J_{\theta,T}(v,x;h,\rho) := -\frac{1}{\theta}\ln \boldsymbol{E}e^{-\theta E(T;h,\rho)} = \frac{1}{2}x'Q(0)x + q'(0)x + k(0)$$

where k *is a solution of equation* (3.30)*.*

Proof. This proof is adapted from Kuroda and Nagai.

Consider a portfolio strategy $\{(h,\rho) : h \in \mathcal{H}(T), \rho \in \mathcal{R}(T)\}$ and define

$$\begin{aligned} Z_t &:= Z_t(h) \\ &= \theta \left\{ \int_0^t a_0(X_\tau, h(\tau), \rho(\tau); \theta) d\tau - \int_0^t \left[\rho(h'\Sigma - \varsigma') + \varsigma'\right] dW_\tau \right. \\ &\quad \left. -\frac{\theta}{2} \int_0^t \left[\rho(h'\Sigma - \varsigma') + \varsigma'\right] \left[\rho(h'\Sigma - \varsigma') + \varsigma'\right]' d\tau \right\} \end{aligned}$$

which corresponds to the terms in the $\exp(\cdot)$ of the original criterion under the physical measure $\mathbb{P}$ as given in equation (3.8).

For a function Φ defined by equation (3.27), that is

$$\Phi(t,x) = \frac{1}{2} x' Q(t) x + q'(t) x + k(t)$$

we define the variable χ as

$$\chi(t,x) = -\theta \left(\Phi(t,x) - \ln e_0\right)$$

and the generator L as

$$L\Phi = \frac{1}{2} \mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) + (b + Bx)' D\Phi$$

Applying Itô's Lemma to $\chi(s, X_s)$, we get

$$d\chi(t+s, X_s) = -\theta \left(\frac{\partial \Phi}{\partial t} + L\Phi\right)(t+s, X_s) ds - \theta (D\Phi)'(t+s, X_s) \Lambda dW_s$$

Now, applying Itô to $e^{\chi(s,X_s)}$, we get

$$\begin{aligned}\frac{de^{\chi(t+s,X_s)}}{e^{\chi(t+s,X_s)}} &= -\theta\left(\frac{\partial\Phi}{\partial t}+L\Phi\right)(t+s,X_s)-\theta(D\Phi)'(t+s,X_s)\Lambda dW_s \\ &\quad +\frac{\theta^2}{2}(D\Phi)'\Lambda\Lambda'(D\Phi)(t+s,X_s)ds\end{aligned}$$

Therefore, applying the product rule,

$$\begin{aligned}&\frac{de^{Z_s}e^{\chi(t+s,X_s)}}{e^{Z_s}e^{\chi(t+s,X_s)}} \\ =\ & -\theta\left(\frac{\partial\Phi}{\partial t}+L\Phi\right)(t+s,X_s)-\theta(D\Phi)'(t+s,X_s)\Lambda dW_s+\frac{\theta^2}{2}(D\Phi)'\Lambda\Lambda'(D\Phi)(t+s,X_s)ds \\ & +\theta\left[a_0(X_s,h(s),\rho(s);\theta)ds-\left[\rho(s)(h'(s)\Sigma-\varsigma')+\varsigma'\right]dW_s\right] \\ & +\frac{\theta^2}{4}\left[\rho(s)(h'(s)\Sigma-\varsigma')+\varsigma'\right]\Lambda'(D\Phi)(t+s,X_s)ds \\ =\ & -\theta\left(\frac{\partial\Phi}{\partial t}+L^h_{t+s}\Phi\right)(t+s,X_s)ds-\theta\left(D\Phi'(t+s,X_s)\Lambda+\left[\rho(s)(h'(s)\Sigma-\varsigma')+\varsigma'\right]\right)dW_s\end{aligned}$$

where $L^{h,\rho}_s$ is the HJB operator defined earlier as

$$\begin{aligned}L^{h,\rho}_s\Phi &= \left(b+Bx-\theta\Lambda\left[\rho(s)(\Sigma'h(s)-\varsigma)+\varsigma\right]\right)'D\Phi+\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'D^2\Phi\right) \\ &\quad -\frac{\theta}{2}(D\Phi)'\Lambda\Lambda'D\Phi-g(x,h,\rho;\theta)\end{aligned}$$

Thus, applying Itô to $\ln\left(e^{Z_s}e^{\chi(s,X_s)}\right)$ and integrating over $[0,T]$, we get

$$\begin{aligned}&(Z_T+\chi(T,X_T))-(Z_0+\chi(0,X_0)) \\ &=\int_0^T d\ln\left(e^{Z_T+\chi(T,X_T)}\right) \\ &=-\theta\int_0^T\left(\frac{\partial\Phi}{\partial t}+L^{h,\rho}_s\Phi\right)(s,X_s)ds-\theta\int_0^T\left(D\Phi'(s,X_s)\Lambda+\left[\rho(s)(\Sigma'h(s)-\varsigma)+\varsigma\right]\right)dW_s \\ &-\frac{\theta^2}{2}\int_0^T\left(D\Phi'(s,X_s)\Lambda+\left[\rho(s)(\Sigma'h(s)-\varsigma)+\varsigma\right]\right)\left(D\Phi'(s,X_s)\Lambda+\left[\rho(s)(\Sigma'h(s)-\varsigma)+\varsigma\right]\right)'ds\end{aligned}$$

Hence, taking the exponential of both sides, noting that $Z_0 = 0$ and rearranging, we obtain:

$$\begin{aligned} e^{Z_T+\chi(T,X_T)} &= e^{\chi(0,X_0)} \times \exp\left\{-\theta \int_0^T \left(\frac{\partial \Phi}{\partial t} + L_s^{h,\rho}\Phi\right)(s, X_s)ds\right. \\ &\quad -\theta \int_0^T \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) dW_s \\ &\quad -\frac{\theta^2}{2} \int_0^T \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) \\ &\quad \left.\times \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right)' ds\right\} \end{aligned} \tag{3.34}$$

From the three differential equations (3.28)-(3.29), $\Phi(T, x) = \ln e_0$, so $\chi(T, x) = 0$, and thus

$$\begin{aligned} E_t^{-\theta} &= e_0^{-\theta} e^{Z_T} \\ &= e_0^{-\theta} e^{Z_T+\chi(T,X_T)} \\ &= e_0^{-\theta} e^{-\theta(\Phi(0,x)-\ln e_0)} \times \exp\left\{-\theta \int_0^T \left(\frac{\partial \Phi}{\partial t} + L_s^{h,\rho}\Phi\right)(s, X_s)ds\right. \\ &\quad -\theta \int_0^T \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) dW_s \\ &\quad -\frac{\theta^2}{2} \int_0^T \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) \\ &\quad \left.\times \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right)' ds\right\} \\ &= e^{-\theta\Phi(0,x)} \times \exp\left\{-\theta \int_0^T \left(\frac{\partial \Phi}{\partial t} + L_s^{h}\Phi\right)(s, X_s)ds\right. \\ &\quad -\theta \int_0^T \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) dW_s \\ &\quad -\frac{\theta^2}{2} \int_0^T \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) \\ &\quad \left.\times \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right)' ds\right\} \end{aligned}$$

$$
\begin{aligned}
E_t^{-\theta} &= e^{-\theta\Phi(0,x)} \times \exp\left\{-\theta \int_0^T \left(\frac{\partial \Phi}{\partial t} + L_s^h \Phi\right)(s, X_s) ds\right\} \\
&\quad \times \exp\left\{-\theta \int_0^T \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) dW_s\right. \\
&\quad -\frac{\theta^2}{2} \int_0^T \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) \\
&\quad \left.\times \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right)' ds\right\}
\end{aligned}
$$

Taking the HJB equation (3.13) into account, we note that since $\left(\frac{\partial \Phi}{\partial t} + L_s^{h,\rho}\Phi\right) \leq 0$ then $\exp\left\{-\theta \int_0^T \left(\frac{\partial \Phi}{\partial t} + L_s^{h,\rho}\Phi\right)(s, X_s) ds\right\} \geq 1$ and hence

$$
\begin{aligned}
E_t^{-\theta} &\geq e^{-\theta\Phi(0,x)} \times \exp\left\{-\theta \int_0^T \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) dW_s\right. \\
&\quad -\frac{\theta^2}{2} \int_0^T \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) \\
&\quad \left.\times \left(D\Phi'(s, X_s)\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right)' ds\right\}
\end{aligned}
$$

Substituting the derivatives of the quadratic form given in (3.27) in the equation, we get

$$
\begin{aligned}
E_t^{-\theta} &\geq e^{-\theta\Phi(0,x)} \times \exp\left\{-\theta \int_0^T \left((Q(s)X_s + q)'\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) dW_s\right. \\
&\quad -\frac{\theta^2}{2} \int_0^T \left((Q(s)X_s + q)'\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) \\
&\quad \left.\times \left((Q(s)X_s + q)'\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right)' ds\right\} \qquad (3.35)
\end{aligned}
$$

If $(h^*, \rho^*) \in \mathcal{A}_{h,\rho}(T)$, then

$$\exp\left\{-\theta \int_0^T \left((Q(s)X_s + q)'\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right) dW_s \right.$$
$$-\frac{\theta^2}{2} \int_0^T \left((Q(s)X_s + q)'\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]\right)$$
$$\left. \times \left((Q(s)X_s + q)'\Lambda + [\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma]'\right)' ds \right\}$$

is an exponential martingale. Taking the expectation on (3.35) we finally show that

$$J(v, x; h, \rho; T) \leq \frac{1}{2} x' Q(0) x + q'(0) x + k(0) \tag{3.36}$$

Moreover, we have shown that the pair $(h^*(s,x), \rho^*(s,x))$ with

$$\begin{aligned} h^*(s,x) & \\ = \quad & = \frac{1}{\rho^*(s,x)} \frac{1}{\theta+1} (\Sigma\Sigma')^{-1} \left(\hat{a} + \hat{A}x - \theta\Sigma\Lambda' D\Phi + (\theta+1)\left(\rho^*(s,x) - 1\right)\Sigma\varsigma\right) \\ = \quad & \frac{1}{\rho^*(s,x)} \frac{1}{\theta+1} (\Sigma\Sigma')^{-1} \left[\hat{a} - \theta\Sigma\Lambda' q(s) + (\theta+1)\left(\rho^*(s,x) - 1\right)\Sigma\varsigma + \left(\hat{A} - \theta\Sigma\Lambda' Q(s)\right) x\right] \end{aligned}$$

and

$$\begin{aligned} \rho^*(s,x) \quad = \quad & 1 + \frac{1}{\theta+1}\kappa \left[-\theta\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' D\Phi + \theta\varsigma'\Lambda' D\Phi \right. \\ & \left. + \varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a} + \hat{A}x) - (c - a_0 + (C' - A_0')x)\right] \\ = \quad & 1 + \frac{1}{\theta+1}\kappa \left[+\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a} - \right. \\ & -\theta\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' q(s) + \theta\varsigma'\Lambda' q(s) + \left(\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A} - (C' - A_0')\right. \\ & \left. \left. -\theta\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' Q(s) + \theta\varsigma'\Lambda' Q(s)\right) x\right] \\ = \quad & 1 + \frac{1}{\theta+1}\kappa \left[\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a} - (c - a_0) + \theta\varsigma'\Pi\Lambda' q(t) \right. \\ & \left. + \left(\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A} - (C' - A_0') + \theta\varsigma'\Pi\Lambda' Q(t)\right) x\right] \end{aligned}$$

attains the supremum in the HJB equation (3.13) and therefore equality holds in equation (3.36) for the optimal portfolio strategy (h^*, ρ^*) with h^* and ρ^* respectively defined in equations (3.32) and (3.33).

Central to the admissibility of the portfolio strategy (h^*, ρ^*) is the existence of a probability measure $\mathbb{P}^\theta$. For this probability measure to be well-defined, it is necessary for χ_t defined in (3.10) to be an exponential martingale. We substituting the values for h^* and ρ^* into the equations (3.10) for χ_t and (3.11) for the dynamics of $X(t)$ under $\mathbb{P}^\theta$ and note that X is Gaussian. We then apply Lemma 3.1.1 in [10] to conclude that χ_t is an exponential martingale and that $\mathbb{P}^\theta$ is well defined. Therefore,

$$J(v, x; h^*, \rho^*; T) = \frac{1}{2}x'Q(0)x + q'(0)x + k(0)$$

□

3.5 Uncorrelated Asset, Liability and Factor Noise

In this section, we show that in the special case when asset and factor noise, and liability and factor noise are uncorrelated, i.e.

$$\Lambda\Sigma' = 0$$

$$\Lambda\varsigma' = 0$$

the problem can be solved via a simple pointwise maximization and an application of the Feynman-Kac formula.

From (3.11) it is clear that the evolution of X_t under the measure $\mathbb{P}_h^\theta$ given in

equation (3.10) can be expressed as:

$$dX_s = (b + BX_s)\,ds + \Lambda dW_s^\theta$$

The evolution of the state is therefore independent of the control variables h and ρ and, as a result, the control problem can be solved through a pointwise maximization of the auxiliary criterion function $I(v, x; h; t, T)$.

The optimal controls h^* and ρ^* are the maximizers of the function $g(x; h, \rho; \theta)$ defined in (3.9). They are given by

$$h^* = \frac{1}{\rho}\frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}x + (\theta+1)(\rho^* - 1)\Sigma\varsigma\right)$$

and

$$\rho^* = 1 + \frac{1}{\theta+1}\kappa\left[\varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a} + \hat{A}x) - (c - a_0 + (C' - A_0')x)\right]$$

where

$$\kappa = \frac{1}{\varsigma'\Pi\varsigma}$$

and

$$\Pi = I - \Sigma'(\Sigma\Sigma')^{-1}\Sigma$$

We note that the optimal asset allocation is a un-levered position of $\frac{1}{\theta+1}$ in the log utility (or Kelly) portfolio.

Let $\Phi(t, x)$ be the value function corresponding to the exponential of integral criterion $I(e_0, x; h, \rho; t, T)$. Substituting the value of h^* and ρ^* in the equation for g,

we note that

$$\begin{aligned}\Phi(t,x) &= \sup_{\mathcal{A}(T-t)} I(e_0,x;h,\rho;t,T) \\ &= -\frac{1}{\theta}\ln \mathrm{E}_{t,x}^{h,\theta}\left[\exp\left\{\theta\int_t^T g(X_s,h^*(s),\rho^*(s);\theta)ds\right\}e_0^{-\theta}\right]\end{aligned}$$

Let $\tilde{\Phi}(t,x) = e^{-\theta\Phi(t,x)}$, then

$$\tilde{\Phi}(t,x) = \mathrm{E}_{t,x}^{h,\theta}\left[\exp\left\{\theta\int_t^T g(X_s,h^*(s),\rho^*(s);\theta)ds\right\}e_0^{-\theta}\right]$$

and applying the Feynman-Kac formula, we obtain

$$\frac{\partial\tilde{\Phi}}{\partial t} + (b+Bx)'\,D\tilde{\Phi} + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\tilde{\Phi}\right) + \theta g(x,h^*,\rho^*;\theta)\tilde{\Phi} = 0$$

subject to the terminal condition $\tilde{\Phi}(T,x) = e_0^{-\theta}$.

Reversing the exponential transformation, dividing by $-\theta\tilde{\Phi}$ and rearranging we obtain the PDE

$$\frac{\partial\Phi}{\partial t} + L_t\Phi = 0 \tag{3.37}$$

where

$$L_t\Phi = (b+Bx)'\,D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi - g(x,h^*,\rho^*;\theta) \tag{3.38}$$

and subject to the terminal condition $\Phi(T,x) = \ln e_0$.

Since $L_t\Phi = \sup_{h\in\mathbb{R}^m} L_t^h\Phi$ then the PDE (3.38) has a solution Φ of the form (3.27) where $Q(t)$, $q(t)$ and $k(t)$ respectively satisfy equations (3.28), (3.29) and (3.30).

3.6 Asymptotic Behaviour of the Solution as $t \to +\infty$

3.6.1 The Riccati Equation

We start by defining the matrix L by

$$L = \theta\left[I - \frac{1}{\theta+1}\left(\Sigma'(\Sigma\Sigma')^{-1}\Sigma + \kappa\Pi'\varsigma\varsigma'\Pi\right)\right]^{-1} \tag{3.39}$$

so that the matrix $K_0(\theta)$ defined in equation (3.22) can be expressed as

$$K_0(\theta) = \Lambda L^{-1}\Lambda'$$

Furthermore, we observe that since $K_0(\theta) > 0$, it follows that $L > 0$.

In addition, we note that the matrix $K_2(\theta)$ defined in (3.24) is a symmetric non-negative definite matrix. Hence, $K_2(\theta)$ can be written as

$$K_2(\theta) = C'C$$

where C can be found, for example, through a Cholesky decomposition.

The matrix Riccati equation (3.28) can now be rewritten as

$$\dot{Q}(t) - Q'(t)\Lambda L^{-1}\Lambda' Q(t) + K_1(\theta)Q(t) + Q'(t)K_1'(\theta) + CC' = 0, \quad Q(T) = 0 \tag{3.40}$$

To derive an upper bound for $Q(t)$, consider now the linear ODE for a given $M \times n$ matrix N:

$$\dot{P} + (K_1 - \Lambda N)'P + P(K_1 - \Lambda N) + C'C + N'LN = 0, \quad P(T) = 0 \tag{3.41}$$

with solution

$$P(t) = \int_t^T e^{(s-t)(K_1-\Lambda N)'}(C'C + N'LN)e^{(s-t)(K_1-\Lambda N)}ds$$

then by Lemma 4.1 in Kuroda and Nagai [37], we conclude that

$$Q(t) \leq P(t),\ t \in [0,T] \tag{3.42}$$

.

Let N be a $M \times n$ matrix and N_1 be a $n \times n$ matrix. In order to apply the same line of reasoning as Kuroda and Nagai in their Proposition 2.2 (see [37]), we need to find a minimal condition for the matrices

$$K_1(\theta) - \Lambda N$$

and

$$K_1'(\theta) - C'N_1$$

to be stable (see Definition A.1).

Such a set of conditions could be to find 2 matrices N and N_1 such that

$$N_1 C = \Lambda N \tag{3.43}$$

and

$$K_1(\theta) - \Lambda N_1 \qquad \text{is stable} \tag{3.44}$$

A few observations before proceeding further. First, this minimal condition is not the only way to find N and N_1 as it is not necessary that $N_1 C = \Lambda N$ to satisfy both stability conditions. Nevertheless, this condition is relatively systematic, seems to ease rather than impair the search for N and N_1 and is in keeping with the spirit of Kuroda and Nagai. Second, Condition (3.43) represents an undetermined system of equations, having, potentially, an infinite number of solutions. The question becomes then to find a solution satisfying Condition (3.44). There is, however no guarantee that such a solution exists. In the following we will therefore have to assume:

Assumption 3.5. Assume there exists a $M \times n$ matrix N and a $n \times n$ matrix N_1 such that Conditions (3.43) and (3.44) are satisfied.

3.6.2 Main Result

The main result is analogous to Proposition 2.2 in Kuroda and Nagai [37].

(i). By assumption 3.5, there exists a $n \times n$ matrix N_1 such that $K_1'(\theta) - C'N_1$ is stable. Hence (C', K_1') is stabilizable (see Definition A.4), which implies that (C, K_1) is detectable (see Definition A.5). Consequently, Theorem 4.2 in Kuroda and Nagai [37] applies: $\exists \lim_{T \to +\infty} Q(0,T) := \tilde{Q}$, which is a solution of the Riccati equation

$$K_1'\tilde{Q} + \tilde{Q}K_1 - \tilde{Q}K_0\tilde{Q} + K_2(\theta) = 0$$

Now, by assumption 3.5, there exists a $M \times n$ matrix N such that $K_1 - \Lambda N$ is

stable. The asymptotic term

$$\tilde{P} = \int_t^{+\infty} e^{(s-t)(K_1-\Lambda N)'}(C'C + N'LN)e^{(s-t)(K_1-\Lambda N)}ds$$

is therefore well defined and estimate (3.42) holds asymptotically, i.e.

$$\tilde{Q} \leq \int_t^{+\infty} e^{(s-t)(K_1-\Lambda N)'}(C'C + N'LN)e^{(s-t)(K_1-\Lambda N)}ds$$

In addition, by Remark 1 (ii). following Lemma 4.1 in Kuroda and Nagai [37], we observe that since $L > 0$, then $Q(t) \geq 0,\ t > 0$. In particular, $\tilde{Q} \geq 0$ and we can conclude that

$$0 \leq \tilde{Q} \leq \int_t^{+\infty} e^{(s-t)(K_1-\Lambda N)'}(C'C + N'LN)e^{(s-t)(K_1-\Lambda N)}ds$$

(ii). We now look at the asymptotic behaviour of the linear ODE for $q(t)$. Set $\bar{q} = q(T-t)$ then $\bar{q}$ satisfies the linear ODE

$$\begin{aligned} \dot{\bar{q}} &= (K_1(\theta) - Q(T-t)K_0(\theta))\,\bar{q}(t) + Q(T-t)K_3'(\theta) + K_4'(\theta) = 0 \\ \bar{q}(0) &= 0 \end{aligned}$$

Setting $N_2(t) = Q(T-t;T)$, it follows from Theorem 4.2 in Kuroda and Nagai [37] that $N_2(t)$ converges to $\tilde{Q}$ as $T \to +\infty$ and $K_1^* - N_2(t)K_0 = K_1^* - P(T-t;T)\Lambda L^{-1}\Lambda'$ to a stable matrix $K_1^* - \tilde{Q}K_0$. Therefore, by Lemma 4.4, $\exists \lim_{T\to+\infty} q(t;T) = \lim_{T\to+\infty} q(T-t) = \tilde{q}$ which satisfies

$$\left(K_1(\theta) - \tilde{Q}(T-t)K_0(\theta)\right)\tilde{q} + \tilde{Q}(T-t)K_3'(\theta) + K_4'(\theta) = 0$$

Moreover, $\lim_{T\to+\infty} \frac{k(0;T)}{T} = \lim_{T\to+\infty} \frac{k(0)-k(T)}{T} = \lim_{T\to+\infty} \frac{1}{T}\int_0^T -\dot{k}(s)ds$ and $\dot{k}(t) = \dot{k}(t;T)$ converges to a constant $-\bar{r}(\theta)$ defined by

$$\bar{r}(\theta) = -\frac{1}{2}\tilde{q}'(t)K_0(\theta)\tilde{q} + K_3(\theta)\tilde{q} + K_6(\theta) + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\tilde{Q}\right)$$

as $T \to +\infty$ and we see that $\lim_{T\to+\infty} \frac{k(0;T)}{T}$ as $T \to +\infty$.

(iii). Assume that (K_1', C') is controllable (see Definition A.6), then by Remarks 1 of Lemma 1 and Remark 2 of Lemma 3 in Kuroda and Nagai [37], we see that $\tilde{Q} > 0$.

We can now formalize this derivation into the following proposition:

Proposition 3.6. *(i). If there exists a $M \times n$ matrix N and a $n \times n$ matrix N_1 such that Conditions (3.43) and (3.44) are satisfied, then $Q(0) = Q(0;T)$ converges as $T \to +\infty$ to a nonegative definite matrix $\tilde{Q}$, which is a solution of the algebraic Riccati equation:*

$$K_1'\tilde{Q} + \tilde{Q}K_1 - \tilde{Q}K_0\tilde{Q} + K_2(\theta) = 0 \tag{3.45}$$

Moreover, $\tilde{Q}$ satisfies the estimate

$$0 \le \tilde{Q} \le \int_t^{+\infty} e^{(s-t)(K_1-\Lambda N)'}(C'C + N'LN)e^{(s-t)(K_1-\Lambda N)}ds \tag{3.46}$$

(ii). In addition, as $T \to +\infty$, $q(0) = q(0;T)$ converges to a constant vector $\tilde{q}$, which satisfies

$$\left(K_1(\theta) - \tilde{Q}(T-t)K_0(\theta)\right)\tilde{q} + \tilde{Q}(T-t)K_3'(\theta) + K_4'(\theta) = 0 \tag{3.47}$$

Moreover, $\frac{k(0;T)}{T}$ *converges to a constant* $\bar{r}(\theta)$ *defined by*

$$\bar{r}(\theta) = -\frac{1}{2}\tilde{q}'(t)K_0(\theta)\tilde{q} + K_3(\theta)\tilde{q} + K_6(\theta) + \frac{1}{2}tr\left(\Lambda\Lambda'\tilde{Q}\right)$$

(iii). If, in addition to Conditions (3.43) *and* (3.44) *we assume that*

$$(K_1', C') \quad \textit{is controllable} \tag{3.48}$$

then the solution $\tilde{Q}$ *of equation* (3.45) *is strictly positive definite.*

3.6.3 Ergodic Bellman Equation

From the Proposition, we deduce the ergodic Bellman PDE:

$$\begin{aligned} \bar{r} = \quad & \sup \quad L_t^{h,\rho}\Phi, \quad x \in \mathbb{R}^n \\ & h \in \mathbb{R}^m \\ & \rho \in \mathbb{R}^+ \end{aligned} \tag{3.49}$$

where

$$\begin{aligned} L_t^{h,\rho}\Phi = \quad & \left(b + Bx - \theta\Lambda\left[\rho(s)(\Sigma' h(s) - \varsigma) + \varsigma\right]\right)' D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) \\ & -\frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi - g(x, h, \rho; \theta) \end{aligned} \tag{3.50}$$

has a solution given by the pair of functions $w = \frac{1}{2}x'\tilde{Q}x + \tilde{q}x$ and $\bar{r}(\theta)$ defined above.

We now define $\mathcal{A}$ as the set of investment strategies h and leverage strategies ρ such that $(h, \rho) \in \mathcal{A}_{h,\rho}(T)\ \forall T$. Such strategy is said to be admissible. We can now

extend Theorem 2.3 from Kuroda and Nagai [37] to:

Theorem 3.7. *(i). Supposing that Assumption 3.5 holds, we have*

$$\sup_{(h,\rho)\in\mathcal{A}} J(v,x;h,n u) \leq \bar{r}(\theta)$$

(ii). If, in addition to Assumption (3.5)*, we assume Condition* (3.48)*, that*

$$(K_1', C') \quad \textit{is controllable} \tag{3.51}$$

and also assume that

$$\hat{Q}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\hat{Q} + \kappa\hat{Q}\Lambda\Pi'\varsigma\varsigma'\Pi\Lambda'\hat{Q} < K_2(\theta) \tag{3.52}$$

where $\hat{Q} = \theta\tilde{Q}$ and $\tilde{Q}$ is the nonegative definite solution of the algebraic Riccati equation (3.45)*, then the investment strategy $\tilde{h}_t$ defined by*

$$h_t^* = \frac{1}{\rho^*}\frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left[\hat{a} - \theta\Sigma\Lambda'\tilde{q} + (\theta+1)(\rho^*-1)\Sigma\varsigma + \left(\hat{A} - \theta\Sigma\Lambda'\tilde{Q}\right)X_t\right] \tag{3.53}$$

and the optimal leverage $\tilde{\rho}^*(t)$ defined by

$$\begin{aligned}\rho_t^* \;=\; & 1 + \frac{1}{\theta+1}\kappa\left[\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a} - (c-a_0) + \theta\varsigma'\Pi\Lambda'\tilde{q}\right. \\ & \left. + \left(\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A} - (C'-A_0') + \theta\varsigma'\Pi\Lambda'\tilde{Q}\right)X_t\right]\end{aligned} \tag{3.54}$$

are optimal for the control problem:

$$\sup_{(h,\rho)\in\mathcal{A}} J(v,x;h,\rho) = J(v,x;\tilde{h},\tilde{\rho}) = \bar{r}(\theta)$$

Proof. The proof follows from Kuroda and Nagai [37] and from the proof of Theorem 1 presented earlier in this paper.

First, note that

$$\begin{aligned}\tilde{\rho}(t)\tilde{h}(t) &= \frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left(\hat{a}-\theta\Sigma\Lambda'\tilde{q}\right)+\frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left(\hat{A}-\theta\Sigma\Lambda'\tilde{Q}\right)x \\ &+\frac{1}{\theta+1}\kappa(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a} \\ &-\frac{1}{\theta+1}\kappa(\Sigma\Sigma')^{-1}\Sigma\varsigma(c-a_0)\tilde{r}(t))+\frac{1}{\theta+1}\kappa(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Pi\Lambda'\tilde{q} \\ &+\frac{1}{\theta+1}\kappa(\Sigma\Sigma')^{-1}\Sigma\varsigma\left(\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A}-(C'-A_0')-\theta\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{Q}\right. \\ &\left.+\theta\varsigma'\Lambda'\tilde{Q}\right)x\end{aligned}$$

First Conclusion:

By Theorem 3.4,

$$J(v,x;h,\rho)\leq\frac{1}{2}x'Q(0)x+q(0)'x+k(0),\quad\forall(h,\rho)\in\mathcal{A}(T)$$

Then by Proposition 3.6, inequality (3.51) holds.

Second Conclusion:

Let $\tilde{h}$ be the investment strategy defined by equation (3.53) and $\tilde{\rho}$ be the leverage defined by equation (3.54).

Set

$$\tilde{Z}_t=Z_t(\tilde{h},\tilde{\rho})$$

and

$$\tilde{\chi}(x)=-\theta w(x)$$

where

$$w(x) = \frac{1}{2}x'\tilde{Q}x + \tilde{q}x$$

Then, the analogue to equation (3.34) is given by

$$\begin{aligned}
&e^{\tilde{Z}_T + \tilde{\chi}(T, X_T)} \\
&= e^{\tilde{\chi}(x)} \times \exp\left\{-\theta \int_0^T \left((Dw)' X_s \Lambda + \left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma - \varsigma') + \varsigma'\right]\right) dW_s \right. \\
&-\frac{\theta^2}{2} \int_0^T \left((Dw)' X_s \Lambda + \left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma - \varsigma') + \varsigma'\right]\right) \\
&\left.\left((Dw)' X_s \Lambda + \left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma - \varsigma') + \varsigma'\right]\right)' ds\right\} e^{-\theta \bar{r}(\theta) T}
\end{aligned}$$

Therefore,

$$\begin{aligned}
E_t^\theta &= e_0^{-\theta} e^{\tilde{Z}_T} \\
&= e_0^{-\theta} e^{\tilde{\chi}(x) - \theta \bar{r}(\theta) T - \theta \int_0^T \left((Dw)' X_s \Lambda + \left[\tilde{\rho}(s)(\Sigma' \tilde{h}(s) - \varsigma) + \varsigma\right]\right) dW_s} \\
&\quad \times \exp\left\{-\frac{\theta^2}{2} \int_0^T \left((Dw)' X_s \Lambda + \left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma - \varsigma') + \varsigma'\right]\right)\right. \\
&\quad \left.\times \left((Dw)' X_s \Lambda + \left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma - \varsigma') + \varsigma'\right]\right)' ds\right\}
\end{aligned}$$

Note that

$$\begin{aligned}
&(Dw)' X_s \Lambda + \left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma - \varsigma') + \varsigma'\right] \\
= &\left(X_s' \tilde{Q} + \tilde{q}'\right) \Lambda + \frac{1}{\theta + 1} \left(\hat{a} + \hat{A} X(s)\right)' (\Sigma\Sigma')^{-1} \Sigma \varsigma - \frac{1}{\theta + 1} \theta \left(\tilde{q} + \tilde{Q} X(s)\right)' \Lambda \Sigma' (\Sigma\Sigma')^{-1} \Sigma \varsigma \\
&+ \frac{1}{\theta + 1} \kappa \left[\varsigma' \Sigma' (\Sigma\Sigma')^{-1} \hat{a} - (c - a_0) + \theta \varsigma' \Pi \Lambda' \tilde{q}\right. \\
&\left. + \left(\varsigma' \Sigma' (\Sigma\Sigma')^{-1} \hat{A} - (C' - A_0') + \theta \varsigma' \Pi \Lambda' \tilde{Q}\right) X_s\right] (\Sigma\Sigma')^{-1} \Sigma \\
&- \frac{1}{\theta + 1} \kappa \left[\varsigma' \Sigma' (\Sigma\Sigma')^{-1} \hat{a} - (c - a_0) + \theta \varsigma' \Pi \Lambda' \tilde{q}\right. \\
&\left. + \left(\varsigma' \Sigma' (\Sigma\Sigma')^{-1} \hat{A} - (C' - A_0') + \theta \varsigma' \Pi \Lambda' \tilde{Q}\right) X_t\right] \varsigma
\end{aligned}$$

By standard argument, $\forall T$,

$$\begin{aligned}
&\mathbf{E}\left[\exp\left\{-\theta\int_0^T\left((Dw)'X_s\Lambda+\left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma-\varsigma')+\varsigma'\right]\right)dW_s\right.\right.\\
&-\frac{\theta^2}{2}\int_0^T\left((Dw)'X_s\Lambda+\left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma-\varsigma')+\varsigma'\right]\right)\\
&\left.\left.\times\left((Dw)'X_s\Lambda+\left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma-\varsigma')+\varsigma'\right]\right)'ds\right\}\right]=1 \qquad (3.55)
\end{aligned}$$

since $Dw(X_t)$, $\tilde{h}_s$, $\tilde{\rho}_s$ and $\tilde{h}_s\tilde{\rho}_s$ are linear with respect to the Gaussian process X_t.

Introduce the probability measure $\mathbb{P}^{\tilde{h},\tilde{\rho}}$:

$$\begin{aligned}
\mathbb{P}^{\tilde{h},\tilde{\rho}}(A) &= \mathbf{E}\left[\exp\left\{-\theta\int_0^T\left((Dw)'X_s\Lambda+\left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma-\varsigma')+\varsigma'\right]\right)dW_s\right.\right.\\
&\quad -\frac{\theta^2}{2}\int_0^T\left((Dw)'X_s\Lambda+\left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma-\varsigma')+\varsigma'\right]\right)\\
&\quad \left.\left.\left((Dw)'X_s\Lambda+\left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma-\varsigma')+\varsigma'\right]\right)'ds\right\};A\right]\\
&\quad A\in\mathcal{F}_T,\forall T
\end{aligned}$$

Kuroda and Nagai noted that "the probability measure can be defined universally with respect to T by taking a canonical space $\Omega=C\left([0;+\infty];\mathbb{R}^M\right),\mathbb{F}_t=\sigma(Y_s;s)$, where $Y_t(\omega)$ is the coordinate map."

Under the probability measure $\mathbb{P}^{h,\tilde{\rho}}$,

$$\begin{aligned}
\hat{W}_t^j &= W_t^i-\langle W_\cdot^i,-\theta\int_0^\cdot\left((Dw)'X_s\Lambda+\left[\tilde{\rho}(s)(\tilde{h}'(s)\Sigma-\varsigma')+\varsigma'\right]\right)dW_s\rangle_t\\
&= W_t^i+\theta\int_0^t\left\{\left[\tilde{\rho}(s)(\Sigma'\tilde{h}(s)-\varsigma)+\varsigma\right]+\Lambda'\left(\tilde{Q}X_s+\tilde{q}\right)\right\}^i ds,\\
&\quad i=1,2,\ldots,M
\end{aligned}$$

is an M dimensional standard Brownian motion and we have the following Lemma

Lemma 3.8. *X_t satisfies*

$$dX_t = \left[\left(K_1(\theta) - K_0(\theta)\tilde{Q}\right) X_t + f\right] dt + \Lambda d\hat{W}_t^\theta$$

for some f depending on $\tilde{q}$ and θ.

Proof. This lemma follows from a routine but tedious calculation.

Under the probability measure $\mathbb{P}^{\tilde{h},\tilde{\rho}}$, X_t satisfies the SDE:

$$\begin{aligned}
dX_t &= \left(b + BX_t - \theta\Lambda\left[\tilde{\rho}(t)(\Sigma'\tilde{h}(t) - \varsigma) + \varsigma\right]\right) dt - \theta\Lambda\Lambda'\left(\tilde{Q}X_t + \tilde{q}\right) + \Lambda d\hat{W}_t^\theta \\
&= \left(b + BX_t - \frac{\theta}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X_t\right) + \frac{\theta^2}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(\tilde{q} + \tilde{Q}X_t\right)\right. \\
&\quad -\frac{\theta}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\kappa\left[\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a}\right. \\
&\quad \left.-(c - a_0) + \theta\varsigma'\Pi\Lambda'\tilde{q} + \left(\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A} - (C' - A_0') + \theta\varsigma'\Pi\Lambda'\tilde{Q}\right) X_t\right]\Sigma\varsigma \\
&\quad +\frac{\theta}{\theta+1}\kappa\Lambda\left[\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a} - (c - a_0) + \theta\varsigma'\Pi\Lambda'\tilde{q}\right. \\
&\quad \left.\left. + \left(\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A} - (C' - A_0') + \theta\varsigma'\Pi\Lambda'\tilde{Q}\right) X_t\right]\varsigma\right) dt \\
&\quad -\theta\Lambda\Lambda'\left(\tilde{Q}X_t + \tilde{q}\right) + \Lambda d\hat{W}_t^\theta
\end{aligned}$$

$$
\begin{aligned}
dX_t &= bdt + BX_t dt - \frac{\theta}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\hat{a}dt - \frac{\theta}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\hat{A}X_t dt \\
&\quad + \frac{\theta^2}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{q}dt + \frac{\theta^2}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{Q}X_t dt \\
&\quad - \frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a}dt + \frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma(c-a_0)dt \\
&\quad - \frac{\theta^2}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Pi\Lambda'\tilde{q}dt - \theta\frac{1}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A}X_t dt \\
&\quad + \frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma(C'-A_0')X_t dt - \frac{\theta^2}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Pi\Lambda'\tilde{Q}X_t dt \\
&\quad + \frac{\theta}{\theta+1}\kappa\Lambda\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a}dt - \frac{\theta}{\theta+1}\kappa\Lambda(c-a_0)dt + \frac{\theta^2}{\theta+1}\kappa\Lambda\theta\varsigma'\Pi\Lambda'\tilde{q}dt \\
&\quad + \frac{\theta}{\theta+1}\kappa\Lambda\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A}X_t dt - \frac{\theta}{\theta+1}\kappa\Lambda(C'-A_0')X_t dt + \frac{\theta^2}{\theta+1}\kappa\Lambda\varsigma'\Pi\Lambda'\tilde{Q}X_t dt \\
&\quad - \theta\Lambda\Lambda'\left(\tilde{Q}X_t + \tilde{q}\right) + \Lambda d\hat{W}_t^\theta
\end{aligned}
$$

$$
\begin{aligned}
dX_t &= BX_t dt - \frac{\theta}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\hat{A}X_t dt + \frac{\theta^2}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{Q}X_t dt \\
&\quad - \frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A}X_t dt + \frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma(C'-A_0')X_t dt \\
&\quad - \frac{\theta^2}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Pi\Lambda'\tilde{Q}X_t dt + \frac{\theta}{\theta+1}\kappa\Lambda\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A}X_t dt \\
&\quad - \frac{\theta}{\theta+1}\kappa\Lambda(C'-A_0')X_t dt + \frac{\theta^2}{\theta+1}\kappa\Lambda\varsigma'\Pi\Lambda'\tilde{Q}X_t dt - \theta\Lambda\Lambda'\tilde{Q}X_t dt + bdt \\
&\quad - \frac{\theta}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\hat{a}dt + \frac{\theta^2}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{q}dt \\
&\quad - \frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a}dt + \frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma(c-a_0)dt \\
&\quad - \frac{\theta^2}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Pi\Lambda'\tilde{q}dt + \frac{\theta}{\theta+1}\kappa\Lambda\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a}dt - \frac{\theta}{\theta+1}\kappa\Lambda(c-a_0)dt \\
&\quad + \frac{\theta^2}{\theta+1}\kappa\Lambda\theta\varsigma'\Pi\Lambda'\tilde{q}dt - \theta\Lambda\Lambda'\tilde{q}dt + \Lambda d\hat{W}_t^\theta
\end{aligned}
$$

$$
\begin{aligned}
dX_t &= BX_t dt - \frac{\theta}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\hat{A}X_t dt + \frac{\theta}{\theta+1}\kappa\Lambda\left(I-\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right)\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{A}X_t dt \\
&\quad -\frac{\theta}{\theta+1}\kappa\left(I-\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right)\Lambda(C'-A_0')X_t dt - \Lambda\left[\theta I-\theta^2\frac{1}{\theta+1}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right]\Lambda'\tilde{Q}X_t dt \\
&\quad +\frac{\theta^2}{\theta+1}\kappa\Lambda\left(I-\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right)\varsigma\varsigma'\Pi\Lambda'\tilde{Q}X_t dt + b dt - \frac{\theta}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\hat{a}dt \\
&\quad +\frac{\theta^2}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{q}dt - \frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a}dt \\
&\quad +\frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma(c-a_0)dt - \frac{\theta^2}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Pi\Lambda'\tilde{q}dt \\
&\quad +\frac{\theta}{\theta+1}\kappa\Lambda\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a}dt - \frac{\theta}{\theta+1}\kappa\Lambda(c-a_0)dt \\
&\quad +\frac{\theta^2}{\theta+1}\kappa\Lambda\theta\varsigma'\Pi\Lambda'\tilde{q}dt - \theta\Lambda\Lambda'\tilde{q}dt + \Lambda d\hat{W}_t^\theta \\
&= \left[\left(K_1(\theta)-K_0(\theta)\tilde{Q}\right)X_t + f\right]dt + \Lambda d\hat{W}_t^\theta
\end{aligned}
$$

Where

$$
\begin{aligned}
f &= b - \frac{\theta}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\hat{a} + \frac{\theta^2}{\theta+1}\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{q} \\
&\quad -\frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a} + \frac{\theta}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma(c-a_0) \\
&\quad -\frac{\theta^2}{\theta+1}\kappa\Lambda\Sigma'(\Sigma\Sigma')^{-1}\Sigma\varsigma\varsigma'\Pi\Lambda'\tilde{q} + \frac{\theta}{\theta+1}\kappa\Lambda\varsigma'\Sigma'(\Sigma\Sigma')^{-1}\hat{a} \\
&\quad -\frac{\theta}{\theta+1}\kappa\Lambda(c-a_0) + \frac{\theta^2}{\theta+1}\kappa\Lambda\theta\varsigma'\Pi\Lambda'\tilde{q} - \theta\Lambda\Lambda'\tilde{q}
\end{aligned}
$$

□

Thus,

$$
-\frac{2}{\theta T}\ln \mathbf{E}\left[E_T^{-\theta}\right] = \frac{c}{T} + \bar{r}(\theta) - \frac{2}{\theta T}\ln \mathbf{E}^{(\tilde{h},\tilde{\rho})}\left[e^{\frac{\theta}{2}X_T'\tilde{Q}X_T+\theta\tilde{q}'X_T}\right]
$$

where $c = c(x, e_0)$.

Now, all we need is to prove that

$$\frac{2}{\theta T}\ln \mathrm{E}^{(\tilde{h},\tilde{\rho})}\left[e^{\frac{\theta}{2}X_T'\tilde{Q}X_T+\theta\tilde{q}'X_T}\right]\to 0 \text{ as } T\to+\infty$$

To achieve this objective, we first consider the mean, $m(t)$, and variance, $U(t)$, of $X(t)$ under the probability measure $\mathbb{P}^{(\tilde{h},\tilde{\rho})}$. $m(t)$ satisfies the ODE

$$\dot{m}(t)=\left(K_1(\theta)-K_0(\theta)\tilde{Q}\right)m(t)+f$$

and therefore

$$m(t)=e^{t\left(K_1(\theta)-K_0(\theta)\tilde{Q}\right)}\left(x+\int_0^t e^{-s\left(K_1(\theta)-K_0(\theta)\tilde{Q}\right)}f\,ds\right)$$

Similarly, $U(t)$ satisfies the ODE

$$\dot{U}(t)=\left(K_1(\theta)-K_0(\theta)\tilde{Q}\right)U(t)+U(t)\left(K_1(\theta)-K_0(\theta)\tilde{Q}\right)'+\Lambda\Lambda'$$

and therefore

$$U(t)=\int_0^t e^{(t-s)\left(K_1(\theta)-K_0(\theta)\tilde{Q}\right)}\Lambda\Lambda' e^{(t-s)\left(K_1(\theta)-K_0(\theta)\tilde{Q}\right)'}ds \tag{3.56}$$

Since $K_1(\theta)-K_0(\theta)\tilde{Q}$ is stable, then $\exists\lim_{t\to+\infty}U(t)=:U$.

To finish proving the second conclusion, we prove a Lemma which extends Lemma 5.1 from Kuroda and Nagai [37].

Lemma 3.9. *Set*

$$\hat{Q} = \theta\tilde{Q}$$

and suppose that

$$\hat{Q}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\hat{Q} + \kappa\hat{Q}\Lambda\Pi'\varsigma\varsigma'\Pi\Lambda'\hat{Q} < K_2(\theta) \tag{3.57}$$

then we have

$$\frac{1}{\theta}\tilde{Q}^{-1} > U \geq U(T)$$

Proof. $\tilde{Q}$ is a solution of (3.45) and

$$\begin{aligned} K_0(\theta) &= \theta\Lambda\left[I - \frac{1}{\theta+1}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\right]\Lambda' - \frac{1}{\theta+1}\theta\kappa\Lambda\Pi'\varsigma\varsigma'\Pi\Lambda' \\ &= \theta\Lambda\Lambda' - \frac{\theta^2}{2(\theta+2)}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda' - \frac{1}{\theta+1}\theta\kappa\Lambda\Pi'\varsigma\varsigma'\Pi\Lambda' \end{aligned}$$

So,

$$K_1'\tilde{Q} + \tilde{Q}K_1 - \tilde{Q}K_0\tilde{Q} + K_2(\theta) = 0$$

Hence,

$$(K_1 - K_0\tilde{Q})'\tilde{Q} + \tilde{Q}(K_1 - K_0\tilde{Q}) + \tilde{Q}K_0\tilde{Q} + K_2(\theta) = 0$$

Expanding,

$$\begin{aligned} &(K_1 - K_0\tilde{Q})'\tilde{Q} + \tilde{Q}K_1 - K_0\tilde{Q}) + \theta\tilde{Q}\Lambda\Lambda'\tilde{Q} - \frac{\theta^2}{2(\theta+2)}\tilde{Q}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{Q} \\ &-\frac{1}{\theta+1}\theta\kappa\tilde{Q}\Lambda\Pi'\varsigma\varsigma'\Pi\Lambda'\tilde{Q} + K_2(\theta) = 0 \end{aligned}$$

Multiplying by θ

$$(K_1 - K_0\tilde{Q})'\hat{Q} + \hat{Q}(K_1 - K_0\tilde{Q}) + \hat{Q}\Lambda\Lambda'\hat{Q} - \frac{\theta^3}{4(\theta+2)}\tilde{Q}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{Q}$$
$$-\theta^2\frac{1}{\theta+1}\kappa\tilde{Q}\Lambda\Pi'\varsigma\varsigma'\Pi\Lambda'\tilde{Q} + \theta K_2(\theta) = 0$$

By Proposition 3.6, we know that $\tilde{Q} > 0$ and is invertible. Hence, because $\theta > 0$, $\hat{Q} > 0$ and is invertible. Therefore, pre and post multiplying by $\hat{Q}^{-1}$ we have,

$$\hat{Q}^{-1}(K_1 - K_0\tilde{Q})' + (K_1 - K_0\tilde{Q})\hat{Q}^{-1} + \Lambda\Lambda' + \frac{1}{\theta+1}\hat{Q}^{-1}\left[-\theta^2\tilde{Q}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{Q}\right.$$
$$\left.-\theta^2\kappa\tilde{Q}\Lambda\Pi'\varsigma\varsigma'\Pi\Lambda'\tilde{Q} + K_2(\theta)\right]\hat{Q}^{-1} = 0$$

Thus,

$$\hat{Q}^{-1} = \int_0^\infty e^{s(K_1-K_0\tilde{Q})}\left[\Lambda\Lambda' + \frac{1}{\theta+1}\hat{Q}^{-1}\left[-\theta^2\tilde{Q}\Sigma'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\tilde{Q}\right.\right.$$
$$\left.\left.-\theta^2\kappa\tilde{Q}\Lambda\Pi'\varsigma\varsigma'\Pi\Lambda'\tilde{Q} + K_2(\theta)\right]\hat{Q}^{-1}\right]e^{s(K_1-K_0\tilde{Q})'}ds$$

Then, by equation (3.56) and Condition (3.57), we can conclude that:

$$\frac{1}{\theta}\tilde{Q}^{-1} > U \geq U(T)$$

□

We will now conclude the proof of the second part of our Theorem. Since (K_1', C') is controlable, so is $(K_1 - K_0\tilde{Q}, \Lambda)$, by Lemma 4.3 in Kuroda and Nagai [37]. This implies that the variance U_t of X_t is nondegenerate. Applying Lemma 3.9, we see

that $\theta\tilde{Q} < U(T)^{-1}$. Therefore,

$$\begin{aligned}
&\mathrm{E}^{(\tilde{h},\tilde{\rho})}\left[\exp\left\{\frac{\theta}{2}X_T'\tilde{Q}X_T + \theta\tilde{q}'X_T\right\}\right] \\
&= \int e^{\frac{\theta}{2}x'\tilde{Q}x+\theta\tilde{q}'x}x\frac{1}{(2\pi)^{n/2}\sqrt{\det U(T)}}e^{=}-\frac{1}{2}(x-m(T))'U(T)^{-1}(x-m(T))dx \\
&= \frac{\sqrt{\det\tilde{U}(T)^{-1}}}{\sqrt{\det U(T)}}e^{\frac{1}{2}\left(\theta\tilde{q}'+m(T)'U(T)^{-1}\right)'\tilde{U}(T)^{-1}\left(\theta\tilde{q}'+m(T)'U(T)^{-1}\right)}e^{-\frac{1}{2}m(T)'U(T)^{-1}m(T)}
\end{aligned}$$

where $\tilde{U}(T) = U(T)^{-1} - \theta\tilde{P}$. Since $U(T) \to U > 0$ and $m(t) \to \int_0^\infty e^{s(K_1-K_0\tilde{Q})}fds$,

$$\frac{1}{\theta T}\ln\mathrm{E}^{(\tilde{h},\tilde{\rho})}\left[\exp\left\{\frac{\theta}{2}X_T'\tilde{Q}X_T + \theta\tilde{q}'X_T\right\}\right] \to 0$$

as $T \to \infty$. This concludes the proof of the Theorem.

□

3.7 Economic Interpretation and Mutual Fund Theorems

The objective of this section is to explore the interplay between risk aversion, optimal investment strategy and leverage as well as to provide some economic intuition for the results derived in previous sections.

Our findings are formalized in a mutual fund theorem and a redefinition of fractional Kelly strategies.

3.7.1 Understanding Optimal Leverage

The optimal leverage is expressed as

$$\rho_t^* = 1 + \frac{1}{\theta+1}\kappa\left[\theta\varsigma'\Pi\Lambda'(Q(t)X_t + q(t)) + \varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a} + \hat{A}X_t) - (c - a_0 + (C' - A_0')X_t)\right]$$

where

$$\Pi = I - \Sigma'(\Sigma\Sigma')^{-1}\Sigma \tag{3.58}$$

and

$$\kappa = \frac{1}{\varsigma'\Pi\varsigma} \tag{3.59}$$

Again, since $\Pi\Pi' = \Pi$,

$$\kappa = \frac{1}{\varsigma'\Pi\Pi'\varsigma} = \frac{1}{\|\Pi'\varsigma\|^2} \tag{3.60}$$

where $\|\cdot\|$ denotes the Euclidian norm.

The coefficient κ is the inverse of the norm of the projection of the vector ς onto the orthogonal complement to the space spanned by the matrix Σ'. Interpreting from a least square perspective, which is a plausible point of view since the primary objective of the portfolio manager is to use assets in order to at the very least match liabilities, this norm can be understood as a measure of the aggregate error incurred in using the asset market to approximate the liability. It is therefore unsurprising that leverage is inversely proportional to this error term: the higher the error, the less adequate the asset market is to replicate the liability and as a consequence, the lower the leverage.

The first term inside the parentheses, $\theta\varsigma'\Pi\Lambda'(Q(t)X_t + q(t))$, relates to the co-

movement of the valuation factors and the aggregate error incurred in using the asset market to estimate the liability. This error is computed as the projection of the vector ς onto the orthogonal complement to the space spanned by the matrix Σ'.

The second term inside the parentheses, $\varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}X_t)$ can be expressed as $\hat{u}'(\hat{a}+\hat{A}X_t)$ where $\hat{u}=(\Sigma\Sigma')^{-1}\Sigma\varsigma$. This represents an unbiased estimator of a linear relationship between asset risks and liability risk $\varsigma=\Sigma' u$. Note however that here, the estimator $\hat{u}$ is not applied to asset risks but to asset returns. This terms therefore represents an estimation of the return of the liability "managed" or "approximated", in an ALM sense, on the assets market.

The second term is then adjusted by the third term, $(c-a_0+(C'-A_0')X_t$, representing the excess return of the liability with respect to the short term rate. Therefore, $\varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}X_t)-(c-a_0+(C'-A_0')X_t)$ can be interpreted as a measure of excess return of the liability "managed" on the assets market with respect to the actual liability incurred.

3.7.2 The Role of Risk Aversion in the Leverage

The role played by risk aversion can be best evidenced by studying the asymptotic behaviour of leverage as $\theta\longrightarrow 0$ and as $\theta\longrightarrow\infty$.

$$\lim_{\theta\longrightarrow+\infty}\rho^* \;=\; 1+\kappa\varsigma'\Pi\Lambda'(Q(t)X_t+q(t))$$

$$\lim_{\theta\longrightarrow 0}\rho^* \;=\; 1+\kappa\left[\varsigma'\Sigma'(\Sigma\Sigma')^{-1}(\hat{a}+\hat{A}X_t)-(c-a_0+(C'-A_0')X_t)\right] \qquad (3.61)$$

These two limits show clearly how risk aversion affects the nature of leverage. When risk aversion tends to 0, leverage is to a large extend a function of the difference between the asset and liability drift, and hence, loosely speaking, of expected return. To the contrary, when risk aversion tend to infinity, leverage is to a large extend a function of asset and liability diffusion, and hence, loosely speaking, of "risk". This observation evidences that both low risk aversion and high risk aversion investors may decide to take on some leverage.

3.7.3 The Role of Leverage in the Investment Strategy

We start by considering the case of an investor who has already determined his/her optimal degree of leverage and analyze the impact that the leverage has on the determination of an optimal investment policy.

The optimal investment strategy, h^*, can be interpreted as the sum of three components. The first component is related to the log utility, or Kelly criterion, investment strategy $(\Sigma\Sigma')^{-1}(\hat{a} + \hat{A}X_t)$. The important element to note is that, in our model, the level of allocation to the log utility investment strategy is not only related to risk-aversion, through the $\frac{1}{\theta+1}$ coefficient, but is also inversely related to leverage, through the $\frac{1}{\rho^*}$ coefficient. Indeed, the effect of the $\frac{1}{\rho^*}$ coefficient is to "un-lever" the investment made in the log utility portfolio. An alternative way of expressing this idea is by observing that whatever the degree of leverage, the investment in the log utility portfolio only scales with the investor's equity.

The second component, $-\theta(\Sigma\Sigma')^{-1}\Lambda'(Q(t)X_t + q(t))$ can be viewed as an adjust-

ment to the log utility portfolio related directly to risk aversion. As expected given its corrective role, its contribution to the asset allocation is inversely related to leverage, and hence can be interpreted as a "un-levered" adjustment. This component corresponds to the "intertemporal hedging term" identified in the financial economics literature.

The third component, $(\theta+1)\frac{\rho^*-1}{\rho^*}(\Sigma\Sigma')^{-1}\Sigma\varsigma$ is the levered portion of the investment strategy. This component is closely linked to the "co-movement" or "co-diffusion" of the asset and liabilities. Therefore,. this component represents the liability-driven part of the investment policy.

To conclude, the role of leverage is not to enable the investor to generate larger returns by taking more risk. The leveraged portion of the asset portfolio is fully invested in a portfolio of liability matching assets. This strategy is represented in the levered liability-driven component of the investment policy. On the contrary, the asset-driven part of the investment strategy, which stems from the log utility portfolio plus a risk-aversion driven intertemporal hedging term, is not levered, and as a result only scales with equity. This observation is essentially consistent with the investment strategies adopted by insurance companies. Insurance companies often manage their asset portfolio through a two tiers approach: the larger portion of the assets is used to immunize the liabilities (the insurance policies issued) while the surplus (equivalent to our notion of equity) is invested in higher return and higher risk assets in order to generate capital appreciation.

3.7.4 Mutual Fund Theorems

Theorem 3.10. *(ALM Mutual Fund Theorem). Given a time t and a state vector $X(t)$, any portfolio can be expressed as a linear combination of investments into three "mutual funds" with respective risky asset allocations:*

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h^C(t) &= -(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right) \\ h^L(t) &= (\Sigma\Sigma')^{-1}\Sigma\varsigma \end{aligned} \tag{3.62}$$

and respective allocation to the money market account given by:

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h_0^C(t) &= 1 + \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right) \\ h_0^L(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\varsigma \end{aligned} \tag{3.63}$$

Moreover, if an investor has a risk sensitivity θ, then the respective weights of each mutual fund in the investor's portfolio are equal to $\frac{1}{(\theta+1)\rho}$, $\frac{\theta}{(\theta+1)\rho}$ and $\frac{\rho-1}{\rho}$ where ρ is the leverage associated with the portfolio.

Proof. The optimal risk-sensitive asset allocation is given by:

$$\begin{aligned} h_t^* &= \frac{1}{\rho^*}\frac{1}{\theta+1}(\Sigma\Sigma')^{-1}\left[\hat{a} - \theta\Sigma\Lambda' q(t) + (\theta+1)\left(\rho^*-1\right)\Sigma\varsigma + \left(\hat{A} - \theta\Sigma\Lambda' Q(t)\right)X_t\right] \\ &= \frac{1}{\rho^*}(\Sigma\Sigma')^{-1}\left(\frac{1}{\theta+1}\left[\hat{a} + \hat{A}X_t - \theta\Sigma\Lambda'\left(q(t) + Q(t)X_t\right)\right] + (\rho^*-1)\Sigma\varsigma\right) \end{aligned}$$

Note that, as should be,

$$\frac{1}{(\theta+1)\rho}+\frac{\theta}{(\theta+1)\rho}+\frac{\rho-1}{\rho}=1$$

Now, we denote by

$$\begin{aligned} h^K(t) &= (\Sigma\Sigma')^{-1}\left(\hat{a}+\hat{A}X(t)\right) \\ h^C(t) &= -(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t)+Q(t)X(t)\right) \\ h^L(t) &= (\Sigma\Sigma')^{-1}\Sigma\varsigma \end{aligned}$$

the risky asset allocation of funds K, C and L. We see the risky allocation of any optimal portfolio is a linear combination of investments in mutual funds K, C and L, with respective asset allocation $\frac{1}{(\theta+1)\rho}$, $\frac{\theta}{(\theta+1)\rho}$ and $\frac{\rho-1}{\rho}$.

Now, since

$$h(t)=\frac{1}{(\theta+1)\rho}h^K(t)+\frac{\theta}{(\theta+1)\rho}h^C(t)+\frac{\rho-1}{\rho}h^L(t)$$

then, by the budget equation

$$\begin{aligned} h_0(t) &= 1-1'h(t) \\ &= \frac{1}{(\theta+1)\rho}\left(1-1'h^K(t)\right)+\frac{\theta}{(\theta+1)\rho}\left(1-1'h^C(t)\right)+\frac{\rho-1}{\rho}\left(1-1'h^L(t)\right) \\ &= \frac{1}{(\theta+1)\rho}h_0^K(t)+\frac{\theta}{(\theta+1)\rho}h_0^C(t)+\frac{\rho-1}{\rho}h_0^L(t) \end{aligned}$$

where

$$\begin{aligned} h_0^K(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\left(\hat{a} + \hat{A}X(t)\right) \\ h_0^C(t) &= 1 + \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\Lambda'\left(q(t) + Q(t)X(t)\right) \\ h_0^L(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\varsigma \end{aligned}$$

□

Remark 3.11. Fund K can be interpreted as the log utility or Kelly criterion fund. Fund C can be interpreted as a "corrective fund" providing an adjustment related to the investor's risk aversion, and fund L represents a liability matching fund whose objective is to track the liabilities closely.

Corollary 3.12. *(Geometric Brownian Motion.) When the risky assets follow a Geometric Brownian Motion with drift vector μ and the money market account is risk-free (i.e. $a_0 = r$ and $A_0 = 0$), then any optimal portfolio can be expressed as a linear combination of investments into two "mutual funds" with respective asset allocations*

$$\begin{aligned} h^K(t) &= \frac{1}{\theta+1}(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1}) \\ h^L(t) &= (\Sigma\Sigma')^{-1}\Sigma\varsigma \end{aligned} \tag{3.64}$$

and respective allocation to the money market account given by:

$$\begin{aligned} h_0^K(t) &= 1 - \frac{1}{\theta+1}\mathbf{1}'(\Sigma\Sigma')^{-1}(\mu - r\mathbf{1}) \\ h_0^C(t) &= 1 - \mathbf{1}'(\Sigma\Sigma')^{-1}\Sigma\varsigma \end{aligned}$$

Moreover, if an investor has a risk sensitivity θ, then the respective weights of each

mutual fund in the investor's portfolio are equal to $\frac{1}{\rho}$ *and* $\frac{\rho-1}{\rho}$ *where* ρ *is the leverage associated with the portfolio.*

3.7.5 Fractional Kelly Strategies Redefined

Kelly criterion investing and fractional Kelly strategy hold a special place in investment theory and practice and a detailed account of Kelly and fractional Kelly strategies is given by Ziemba [46]. A full Kelly strategy, also called a Kelly criterion strategy corresponds to a full investment in the log utility portfolio. A fractional Kelly strategy with fraction k is a strategy investing a proportion k of the wealth in the Kelly or log-utility portfolio and a proportion $1-k$ in the risk-free asset. One of the main issues related to the definition of fractional Kelly strategies is that they are not generally optimal, except in the case in which the dynamics of asset prices follows a geometric Brownian motion. To remedy this problem, Davis and Lleo [21] reinterpreted the mutual fund theorem (Theorem 3) to propose a redefinition of fractional Kelly strategies in the context of benchmarked investing that they name "benchmark adjusted fractional Kelly strategies".

A similar reasoning could be used in the ALM context by reinterpreting Theorem 3.10 to define "asset and liability adjusted fractional Kelly strategies." But, first, we need to extend the definition of fractional Kelly strategies to reflect our factor-driven setting. We will call a fractional Kelly strategy with fraction k a strategy which invests a proportion k of the wealth in the Kelly or log-utility portfolio K (in the notation of Theorem 3.10) and a proportion $1-k$ in intertemporal hedging term C. We note that in the case when asset prices are modelled using geometric Brownian motions, the portfolio C is fully invested in the risk-free asset and our extended

definition of fractional Kelly strategies reverts to the classical definition.

The investor's strategy can then be interpreted as an investment of $\frac{\rho-1}{\rho}$ in the liability-matching fund L and $\frac{1}{\rho}$ in a fractional Kelly portfolio with fraction $k = \frac{1}{\theta+1}$. In the case when asset prices are modelled using geometric Brownian motions, this definition follows directly from the formulation of Corollary 3.12.

3.8 Conclusion

In this chapter, we have shown that the introduction of leverage as an additional control variable complementing the asset allocation vector leads to a formulation of the asset and liability management problem as a maximization of the risk sensitive return on equity. This two-control problem can the be solved exactly through the change of measure technique pioneered by Kuroda and Nagai [37] and yields an elegant analytical solution.

We then drew some insights into the interrelationship between risk aversion, leverage and the asset allocation strategy. In particular, we observed that the degree of risk aversion has a more significant impact on the nature of the leverage decision than on the level of leverage adopted. We also noted that the levered portion of the assets was invested in a liability-driven strategy while the equity was allocated to an asset-driven strategy aiming. This observation is consistent with the traditional behaviour of insurance companies who often manage their asset portfolio through a two tiers approach: the larger portion of the assets is used to immunize their liabilities while their surplus is invested in higher return and higher risk assets in order to generate

capital appreciation.

The findings related to the asset allocation are summarized into a mutual fund theorem which shows that the optimal asset allocation can be decomposed into an allocation to three funds: the Kelly or log utility fund, an intertemporal hedging fund and a liability-driven fund. Finally, we used the mutual fund theorem to derive a new definition of Kelly criterion strategies consistent with the objectives of asset and liability management.

Chapter 4

Jump Diffusion Risk-Sensitive Asset Management: The HJB PDE

We now consider a jump-diffusion version of the risk-sensitive asset management problem. Wan [44] has already briefly sketched a jump-diffusion extension of Bielecki and Pliska's [15] infinite horizon risk-sensitive asset management model. In his approach, jumps, which are modelled using Poisson processes, only affect asset prices, and only one deterministic jump size is allowed. Using the methodology initially proposed by Bielecki and Pliska [15], Wan assumed that the factor and asset noise are uncorrelated to obtain a relation for both the optimal control h^* and the asymptotic growth rate of the investor's wealth, $\rho(\theta)$.

Our treatment, which concerns the finite horizon case, is markedly different. To start with, jumps can occur in both asset prices and factor level. We characterize jumps using random Poisson measures, a wide class of processes which includes the Poisson, compounded Poisson, and Lévy processes. This choice enables us to introduce a distribution of random sizes. Finally, we use Kuroda and Nagai's change of measure method which allows us to derive a general treatment in which asset and factor noise can be correlated.

In this chapter, we derive of the risk-sensitive Hamilton-Jacobi-Bellman Partial Integro Differential Equation (RS HJB PIDE) associated with the control problem. We also show that under given conditions, a unique optimal control policy exists. The chapter is organized as follows. Section 1 introduces the general setting of the model and defines the class of random Poisson measures which will be used to model the jump component of the asset and factor dynamics. The control problem is formulated in Section 2 and its associated RS HJB PIDE is derived. In Section 3, we show the existence of a unique optimal control policy.

4.1 Analytical Setting

4.1.1 Overview

The growth rates of the assets are assumed to depend on n factors $X_1(t), \ldots, X_n(t)$ which follow the dynamics given in equation (4.2) below. As in Kuroda and Nagai's asset-only model, the assets market comprises m risky securities S_i, $i = 1, \ldots m$. In contrast to Kuroda and Nagai, we assume that the money market account process, S_0 is an affine function of the valuation factors, which enables us to easily model a stochastic short rate. Let $N := n + m$.

Let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$ be the underlying probability space. On this space, we define an $\mathbb{R}^M$-valued $(\mathcal{F}_t)$-Brownian motion $W(t)$ with components $W_k(t)$, $k = 1, \ldots, M$. Moreover, let $(\mathbf{Z}, \mathcal{B}_\mathbf{Z})$ be a Borel space[1]. Let $\mathbf{p}$ be an $(\mathcal{F}_t)$-adapted σ-finite Poisson point process on $\mathbf{Z}$ whose underlying point functions are maps from a countable set $\mathbf{D_p} \subset (0, \infty)$ into $\mathbf{Z}$. Define

$$\mathfrak{Z}_\mathbf{p} := \{U \in \mathcal{B}(Z), \mathbb{E}[N_\mathbf{p}(t, U)] < \infty \; \forall t\} \tag{4.1}$$

Consider $N_\mathbf{p}(dt, dz)$, the Poisson random measure on $(0, \infty) \times \mathbf{Z}$ induced by $\mathbf{p}$.

Our analysis will focus on stationary Poisson point processes of class (QL) with associated Poisson random measure $N_\mathbf{p}(dt, dx)$. The class (QL) is defined in [31] (Definition II.3.1 p. 59) as

[1] $\mathbf{Z}$ is a standard measurable (metric or topological) space and $\mathcal{B}_\mathbf{Z}$ is the Borel σ-field endowed to $\mathbf{Z}$.

Definition 4.1. An $(\mathcal{F}_t)$-adapted point process $\mathbf{p}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be *of class (QL)* with respect to $(\mathcal{F}_t)$ if it is σ-finite and there exists $\hat{N}_{\mathbf{p}} = \left(\hat{N}_{\mathbf{p}}(t, U)\right)$ such that

(i.) for $U \in \mathcal{Z}_p$, $t \mapsto \hat{N}_{\mathbf{p}}(t, U)$ is a continuous $(\mathcal{F}_t)$-adapted increasing process;

(ii.) for each t and a.a. $\omega \in \Omega$, $U \mapsto \hat{N}_{\mathbf{p}}(t, U)$ is a σ-finite measure on $(\mathbf{Z}, \mathcal{B}(\mathbf{Z}))$;

(iii.) for $U \in \mathcal{Z}_p$, $t \mapsto \tilde{N}_{\mathbf{p}}(t, U) = N_{\mathbf{p}}(t, U) - \hat{N}_{\mathbf{p}}(t, U)$ is an $(\mathcal{F}_t)$-martingale;

The random measure $\left\{\hat{N}_{\mathbf{p}}(t, U)\right\}$ is called the *compensator* of the point process p.

Because the Poisson point processes we consider are stationary, then their compensators are of the form $\hat{N}_{\mathbf{p}}(t, U) = \nu(U)t$ where ν is the σ-finite characteristic measure of the Poisson point process $\mathbf{p}$.

Finally, for notational convenience, we define the Poisson random measure $\bar{N}_{\mathbf{p}}(dt, dz)$ as

$$\bar{N}_{\mathbf{p}}(dt, dz) = \begin{cases} N_{\mathbf{p}}(dt, dz) - \hat{N}_{\mathbf{p}}(dt, dz) = N_{\mathbf{p}}(dt, dz) - \nu(dz)dt =: \tilde{N}_{\mathbf{p}}(dt, dz) & \text{if } z \in \mathbf{Z}_0 \\ N_{\mathbf{p}}(dt, dz) & \text{if } z \in \mathbf{Z} \backslash \mathbf{Z}_0 \end{cases}$$

where $\mathbf{Z}_0 \subset \mathcal{B}_{\mathbf{Z}}$ such that $\nu(\mathbf{Z} \backslash \mathbf{Z}_0) < \infty$.

4.1.2 Two Relevant Lévy-Related Subcases

We will now describe two possible approaches to model a Lévy-driven dynamics for the risky assets. Both approaches are subcases of our general stationary Poisson point process model.

General Lévy Driven Processes

The following application of stationary Poisson point processes to model Lévy processes can be found in Ikeda and Watanabe [31] and is consistent with the treatment of Lévy processes as presented in other texts such as Applebaum [5].

Let $Y(t)$ be a l-dimensional Lévy process. Let $\mathbf{D}_p = \{t > 0, Y(t) \neq Y(t^-)\}$ and for $t \in \mathbf{D}_p$ define $p(t) = Y(t) - Y(t^-)$, then p is indeed an $(\mathcal{F}_t)$-adapted stationary Poisson point process defined on $\mathbf{Z} = \mathbb{R}^m \backslash \{0\}$. In this case, the characteristic measure ν is the Lévy measure.

We can then define the Lévy-driven counterpart to equations (4.2) and (4.4) given below. Note that when we assume that there is no underlying valuation factor, i.e. $X \equiv 0$, the security price processes follow geometric Lévy processes.

Alternative Characterization of Lévy Driven Processes

In their characterization, Øksendal and Sulem [36] defined a vector of l independent random Poisson measures. Note that this construction precludes simultaneous jumps in two or more of the random Poisson measures and as a result differs from the previous characterization.

Øksendal and Sulem's characterization is therefore equivalent to the following. Let $Y(t)$ be a one-dimensional Lévy process. As above, let $\mathbf{D}_p = \{t > 0, Y(t) \neq Y(t^-)\}$ and for $t \in \mathbf{D}_p$ define $p(t) = Y(t) - Y(t^-)$. In this case, $\mathbf{Z} = \mathbb{R} \backslash \{0\}$ and denote the associated Poisson random measure by $N_{\mathbf{p}}(dt, dx)$.

For $t \in \mathbf{D}_p$ consider the $(\mathcal{F}_t)$-measurable process R_t taking values in $\{1, 2, \ldots, l\}$ with equal probability. For $t \in \mathbf{D}_p$, define a new l-dimensional point process f as:

$$f(t) := \begin{pmatrix} 1_{\{1\}}(R_t) \\ 1_{\{2\}}(R_t) \\ \vdots \\ 1_{\{l\}}(R_t) \end{pmatrix} \times \left(Y(t) - Y(t^-)\right)$$

where $1_Q(x)$ is the indicator function.

The corresponding Poisson random measure by $N_f(dt, dx)$ is defined as

$$N_f(dt, dx) := \begin{pmatrix} 1_{\{1\}}(R_t) \\ 1_{\{2\}}(R_t) \\ \vdots \\ 1_{\{l\}}(R_t) \end{pmatrix} N_{\mathbf{p}}(dt, dx)$$

and ν is the associated l-dimensional Lévy measure.

4.1.3 Factor Dynamics

We assume that the dynamics of the n factors can be expressed as:

$$dX(t) = (b + BX(t^-))dt + \Lambda dW(t) + \int_{\mathbf{Z}} \xi(z)\bar{N}_{\mathbf{p}}(dt, dz), \qquad X(0) = x \tag{4.2}$$

where

- $X(t)$ is the $\mathbb{R}^n$-valued factor process with components $X_j(t),\ j = 1, \ldots, n$
- $b \in \mathbb{R}^n$;
- $B \in \mathbb{R}^{n \times n}$;
- $\Lambda := [\Lambda_{ij}],\ i = 1, \ldots, n,\ j = 1, \ldots, N$; and
- $\xi(z) \in \mathbb{R}^n$ with $-\infty < \xi_i^{min} \leq \xi_i(z) \leq \xi_i^{max} < \infty$ for $i = 1, \ldots, n$.

Moreover, the vector-valued function $\xi(z)$ satisfies:

$$\int_{\mathbf{Z}_0} |\xi(z)|^2 \nu(dz) < \infty$$

(see for example Definition II.4.1 in Ikeda and Watanabe [31] where $\mathbf{F_P}$ and $\mathbf{F_P^{2,loc}}$ are respectively given in equations II(3.2) and II(3.5))

4.1.4 Asset Market Dynamics

Let S_0 denote the wealth invested in the money market account with dynamics given by the equation:

$$\frac{dS_0(t)}{S_0(t)} = (a_0 + A_0' X(t))\, dt, \qquad S_0(0) = s_0 \tag{4.3}$$

where $a_0 \in \mathbb{R}$ is a scalar constant and $A_0 \in \mathbb{R}^n$ is a n-element column vector.

Let $S_i(t)$ denote the price at time t of the ith security, with $i = 1, \ldots, m$. The dynamics of risky security i can be expressed as:

$$\begin{aligned} \frac{dS_i(t)}{S_i(t^-)} &= (a + AX(t))_i dt + \sum_{k=1}^{N} \sigma_{ik} dW_k(t) + \int_{\mathbf{Z}} \gamma_i(z) \bar{N}_{\mathbf{p}}(dt, dz), \\ & S_i(0) = s_i, \quad i = 1, \ldots, m \end{aligned} \tag{4.4}$$

where

- $a \in \mathbb{R}^m$;
- $A \in \mathbb{R}^{m \times n}$;
- $\Sigma := [\sigma_{ij}]$, $i = 1, \ldots, m$, $j = 1, \ldots, N$
- $\gamma(z) \in \mathbb{R}^m$ with $-1 \leq \gamma_i^{min} \leq \gamma_i(z) \leq \gamma_i^{max} < +\infty$ for $i = 1, \ldots, m$.

In addition, the vector-valued function $\gamma(z)$ satisfies:

$$\int_{\mathbf{Z}_0} |\gamma(z)|^2 \nu(dz) < \infty$$

4.1.5 Portfolio Dynamics

In our analysis, we will assume that:

Assumption 4.2. The systematic (factor-driven) and idiosyncratic (asset-driven) jump risks are uncorrelated, i.e $\forall z \in \mathbf{Z}$, $\gamma(z)\xi'(z) = 0$.

Mathematically, this implies that for a given $z \in \mathbf{Z}$, if $\gamma(z) \neq 0$ then $\xi(z) = 0$, and conversely if $\xi(z) \neq 0$ then $\gamma(z) = 0$. This assumption, which will prove sufficient to

show the existence of a unique optimal investment policy, may appear somewhat restrictive as it does not enable us to model a jump correlation structure across factors and assets, although we can model a jump correlation structure within the factors and within the assets. We present a detailed analysis of the control problem in Section 4.3.

Remark 4.3. Assumption (4.2) is automatically satisfied when jumps are allowed only in the security prices and the state variable $X(t)$ is modelled using a diffusion process.

Let $\mathcal{G}_t := \sigma((S(s), X(s)), 0 \leq s \leq t)$ be the sigma-field generated by the security and factor processes up to time t.

Definition 4.4. A control process $h(t)$ is in class $\mathcal{H}$ if the following conditions are satisfied:

1. $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t \geq 0}$;
2. $P\left(\int_0^T |h(s)|^2 \, ds < +\infty\right) = 1, \quad \forall T > 0;$
3. $h'(t)\gamma(z) \geq -1, \quad \forall t > 0, z \in \mathbf{Z}$, a.s. $d\nu$.

We will consider a m-element control vector $\tilde{h}$ defined on the compact set $\overline{\mathcal{O}}$ where $\mathcal{O} := \{]-1,1[\}^m$ and $\overline{\mathcal{O}}$ represents the closure of $\mathcal{O}$. We will also introduce a rescaled m-element control vector h, where the ith component h_i, $i = 1, \ldots, m$, is defined as

$$h_i = \begin{cases} -\infty, & \tilde{h}_i = -1 \\ \tanh^{-1}(\tilde{h}_i), & \tilde{h}_i \in]-1,1[\\ +\infty, & \tilde{h}_i = 1 \end{cases}$$

for $i = 1, \ldots, m$.

Definition 4.5. A control process $\tilde{h}(t)$ is in class $\tilde{\mathcal{H}}$ if the following conditions are satisfied:

1. $\tilde{h}(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t\geq 0}$;

2. $P\left(\int_0^T |h(s)|^2 \, ds < +\infty\right) = 1, \quad \forall T > 0;$

3. $h'(t)\gamma(z) \geq -1, \quad \forall t > 0, z \in \mathbf{Z}$, a.s. $d\nu$.

In our model, we will use h to represent the investment strategy as the allocation of the investor's wealth among the assets, with $h_i, i = 1, \ldots, m$ representing the proportion of the investor's wealth invested in asset i. Note that Condition 3 in Definition 4.5 is necessary to prevent the investor's wealth, which follows the dynamics given in equation (4.5) below, from turning negative as a result of a large jump.

As a result of this construction, h is defined on $\overline{\mathbb{R}^m}$ rather than $\mathbb{R}^m$. The construction will be justified later when we conclude that the optimal asset allocation is an interior point of a set $\overline{J} \subset \mathbb{R}^m$ defined in equation (4.10), and thus that we do not expect to have $h_i = +/-\infty$ for any $i = 1, \ldots, m$. The main advantage of this construction is that the formal control, $\tilde{h}$, is defined on a compact set. This property will be useful when we solve the HJB PIDE associated with this problem using viscosity solutions. However, for practical purpose, the relevant control is the auxiliary vector h which gives us the asset allocation. As a result, we will present our results in terms of the investment strategy h rather than the formal control $\tilde{h}$.

The proportion invested in the money market account is $h_0(t) = 1 - \sum_{i=1}^m h_i(t)$, or equivalently, $\tilde{h}_0(t) = \tanh\left(1 - \sum_{i=1}^m \tanh^{-1} \tilde{h}_i(t)\right)$. Taking this budget equation into consideration, the wealth, $V(t)$ of the investor in response to an investment strategy

$h \in \mathcal{H}$, follows the dynamics

$$\begin{aligned}\frac{dV(t)}{V(t^-)} &= (a_0 + A_0' X(t))\, dt + h'(t)\left(a - a_0 \mathbf{1} + (A - \mathbf{1}A_0')\, X(t)\right) dt \\ &\quad + h'(t)\Sigma dW_t + \int_{\mathbf{Z}} h'(t)\gamma(z)\bar{N}_{\mathbf{p}}(dt, dz)\end{aligned}$$

where $\mathbf{1} \in \mathbf{R}^m$ denotes the m-element unit column vector and with $V(0) = v$. Defining $\hat{a} := a - a_0\mathbf{1}$ and $\hat{A} := A - \mathbf{1}A_0'$, we can express the portfolio dynamics as

$$\frac{dV(t)}{V(t^-)} = (a_0 + A_0' X(t))\, dt + h'(t)\left(\hat{a} + \hat{A}X(t)\right) dt + h'(t)\Sigma dW_t + \int_{\mathbf{Z}} h'(t)\gamma(z)\bar{N}_{\mathbf{p}}(dt, dz) \tag{4.5}$$

4.2 Problem Setup

4.2.1 Optimization Criterion

Similarly to Bielecki and Pliska [15] and Kuroda and Nagai [37], we assume that the objective of the investor is to maximize the long-term risk adjusted growth of his/her portfolio of assets. By Itô, the log of the portfolio value in response to a strategy h is

$$\begin{aligned}\ln V(t) &= \ln v + \int_0^t (a_0 + A_0' X(s)) + h(s)'\left(\hat{a} + \hat{A}X(s)\right) ds - \frac{1}{2}\int_0^t h(s)'\Sigma\Sigma' h(s) ds \\ &\quad + \int_0^t h(s)'\Sigma dW(s) + \int_0^t \int_{\mathbf{Z}_0} \left\{\ln\left(1 + h(s)'\gamma(z)\right) - h(s)'\gamma(z)\right\} \nu(dz) ds \\ &\quad + \int_0^t \int_{\mathbf{Z}} \ln\left(1 + h(s)'\gamma(z)\right) \bar{N}_{\mathbf{p}}(ds, dz)\end{aligned} \tag{4.6}$$

4.2.2 Criterion Under the Expectation

Multiplying by $-\theta$ and taking the exponential on both sides of (4.6), we get

$$\begin{aligned} e^{-\theta \ln V(t)} &= v^{-\theta} \exp\left\{\theta \int_0^t g(X_s, h(s); \theta) ds - \theta \int_0^t h(s)' \Sigma dW_s \right. \\ &\quad -\frac{1}{2}\theta^2 \int_0^t h(s)' \Sigma\Sigma' h(s) ds + \int_0^t \int_{\mathbf{Z}} \ln\left(1 - H(z, h(s); \theta)\right) \tilde{N}_{\mathbf{p}}(ds, dz) \\ &\quad \left. + \int_0^t \int_{\mathbf{Z}} \left\{\ln\left(1 - H(z, h(s); \theta)\right) + H(z, h(s); \theta)\right\} \nu(dz) ds \right\} \end{aligned}$$

where

$$\begin{aligned} g(x, h; \theta) &= \frac{1}{2}(\theta + 1) h' \Sigma\Sigma' h - a_0 - A_0' x - h'(\hat{a} + \hat{A} x) \\ &\quad + \int_{\mathbf{Z}} \left\{ \frac{1}{\theta} \left[\left(1 + h' \gamma(z)\right)^{-\theta} - 1 \right] + h' \gamma(z) 1_{\mathbf{Z}_0}(z) \right\} \nu(dz) \end{aligned}$$

$$H(z, h; \theta) = 1 - \left(1 + h' \gamma(z)\right)^{-\theta} \tag{4.7}$$

4.2.3 Change of Measure

Let $\mathbb{P}_h^\theta$ be the measure on $(\Omega, \mathcal{F})$ defined as

$$\begin{aligned} \chi_t &:= \left. \frac{d\mathbb{P}_h^\theta}{d\mathbb{P}} \right|_{\mathcal{F}_t} \\ &= \exp\left\{ -\theta \int_0^t h(s)' \Sigma dW_s - \frac{1}{2}\theta^2 \int_0^t h(s)' \Sigma\Sigma' h(s) ds \right. \\ &\quad + \int_0^t \int_{\mathbf{Z}} \ln\left(1 - H(z, h(s); \theta)\right) \tilde{N}_{\mathbf{p}}(ds, dz) \\ &\quad \left. + \int_0^t \int_{\mathbf{Z}} \left\{\ln\left(1 - H(z, h(s); \theta)\right) + H(z, h(s); \theta)\right\} \nu(dz) ds \right\}, \quad \forall t \geq 0 \end{aligned} \tag{4.8}$$

For a change of measure to be possible, we must ensure that the following technical condition holds:

$$H(z, h(s); \theta) \leq 1$$

This condition is satisfied iff

$$\begin{aligned} H(z, h(s); \theta) \leq 1 &\Leftrightarrow 1 - (1 + h'(s)\gamma(z))^{-\theta} \leq 1 \\ &\Leftrightarrow (1 + h'(s)\gamma(z))^{-\theta} \geq 0 \\ &\Leftrightarrow h'(s)\gamma(z) \geq -1 \end{aligned} \tag{4.9}$$

a.s. $d\nu$, which was already one of the conditions required for $\tilde{h}$ to be in class $\tilde{\mathcal{H}}$ (Condition 3 in Definition 4.5).

More precisely, define

$$\mathbf{S} := \text{supp}(\nu) \in \mathcal{B}_{\mathbf{Z}}$$

and

$$\tilde{\mathbf{S}} := \text{supp}(\nu \circ \gamma^{-1}) \in \mathcal{B}(\mathbb{R}^m)$$

where supp$(\cdot)$ denotes the measure's support. Let J be the set defined as

$$J := \left\{ h \in \mathbb{R}^m : -1 - h'\psi < 0 \quad \forall \psi \in \tilde{\mathbf{S}} \right\} \tag{4.10}$$

and $\mathcal{K}$ be the set

$$\mathcal{K} := \{ h(t) \in \mathcal{H} : h(t) \in J \quad \forall t \text{ a.s.} \} \tag{4.11}$$

For a given z, the equation $h'\gamma(z) = -1$ describes an Hyperplane in $\mathbb{R}^m$, and as a result, it can be seen that the set J is a convex set in $\mathbb{R}^m$.

We denote by $\mathcal{A}(T)$ the set of investment strategies $h \in \mathcal{H}$ on $[0,T]$ such that $\mathbb{P}_h^\theta$ is an equivalent martingale measure and thus

$$\begin{aligned}\mathbf{E}\left[\exp\left\{-\theta\int_0^t h(s)'\Sigma dW_s - \frac{1}{2}\theta^2\int_0^t h(s)'\Sigma\Sigma' h(s)ds\right.\right.\\ +\int_0^t\int_{\mathbf{Z}}\ln\left(1-H(z,h(s);\theta)\right)\tilde{N}_{\mathbf{p}}(ds,dz)\\ \left.\left.+\int_0^t\int_{\mathbf{Z}}\left\{\ln\left(1-H(z,h(s);\theta)\right)+H(z,h(s);\theta)\right\}\nu(dz)ds\right\}\right]=1 \qquad (4.12)\end{aligned}$$

For $h \in \mathcal{A}(T)$,

$$W_t^\theta = W_t + \theta\int_0^t \Sigma' h(s)ds$$

is a standard Brownian motion under the measure $\mathbb{P}_h^\theta$ and we have

$$\begin{aligned}\int_0^t\int_{\mathbf{Z}}\tilde{N}_{\mathbf{p}}^\theta(ds,dz) &= \int_0^t\int_{\mathbf{Z}}N_{\mathbf{p}}(ds,dz) - \int_0^t\int_{\mathbf{Z}}\left\{1-H(z,h(s);\theta)\right\}\nu(dz)ds\\ &= \int_0^t\int_{\mathbf{Z}}N_{\mathbf{p}}(ds,dz) - \int_0^t\int_{\mathbf{Z}}\left\{\left(1+h'\gamma(z)\right)^{-\theta}\right\}\nu(dz)ds\end{aligned}$$

As a result, $X(s),\ 0 \le s \le t$ satisfies the SDE:

$$\begin{aligned}dX(s) &= \left(b+BX(s^-)-\theta\Lambda\Sigma' h(s)+\int_{\mathbf{Z}}\xi(z)\left[\left(1+h'\gamma(z)\right)^{-\theta}-1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)ds\\ &\quad +\Lambda dW_s^\theta+\int_{\mathbf{Z}}\xi(z)\tilde{N}_{\mathbf{p}}^\theta(ds,dz) \qquad (4.13)\end{aligned}$$

We can now introduce the auxiliary criterion function under the measure $\mathbb{P}_h^\theta$:

$$I(v,x;h;t,T) = -\frac{1}{\theta}\ln\mathbf{E}_{t,x}^{h,\theta}\left[\exp\left\{\theta\int_t^T g(X_s,h(s);\theta)ds-\theta\ln v\right\}\right] \qquad (4.14)$$

where $\mathbf{E}_{t,x}^{h,\theta}[\cdot]$ denotes the expectation taken with respect to the measure $\mathbb{P}_h^\theta$ and with initial conditions (t,x).

4.2.4 Remarks on the Role of the Change of Measure Condition (4.9)

The condition $h'\gamma(z) \geq -1$ is endogenous to the control problem and can be interpreted as a risk management safeguard preventing the investor from investing in some of the portfolios if the jump component of these portfolios could result in the investor's bankruptcy.

One could compare this change of measure condition with the introduction of stopping time to track bankruptcy time and define the solvency region in the Merton approach, since both are related to the risk that the investor might go bankrupt. However, two main differences exist between these approaches. First, in the jump diffusion version of the Merton model, bankruptcy could arise from either large enough jumps in the asset prices or large enough diffusive change in price. However, in the JDRSAM model the only movement which could potentially result in bankruptcy arises from jumps. This is due to the geometric nature of the model which prevents bankruptcy in the pure diffusion case. Second, in the Merton model, the solvency constraint is exogenous to the control problem: it has to be imposed through the introduction of a stopping time. On the other hand, in the JDRSAM model the constraint is endogenous to the control problem: it arises naturally from the change in measure.

The effect of condition (4.9) on the control problem is to impose a set of holding constraints on the portfolio. Turning this observation to our advantage, we deduce that in a risk sensitive setting, a large class of investment problems with holding constraints can be modelled simply by introducing jumps corresponding with the constraints.

For example, consider an investment problems with holding constraints of the form

$$h_i^{\min} \leq h_i \leq h_i^{\max}$$

with $h_i^{\min} < 0 < h_i^{\max}$ for $i = 1, \ldots, m$. This set of constraints can be approximated by introducing a number of mutually exclusive jump scenarios set such that for given z_{i_1}, z_{i_2}, $i = 1, \ldots, m$

$$\begin{aligned} \gamma_i(z_{i_1}) &= -\frac{1}{h_i^{\min}} \\ \gamma_k(z_{i_1}) &= 0 \quad k \neq i \\ \gamma_i(z_{i_2}) &= -\frac{1}{h_i^{max}} \\ \gamma_k(z_{i_2}) &= 0 \quad k \neq i \end{aligned}$$

if $h_i^{\min}, h_i^{\max} \neq 0$. However, we note that to complete our approximation we also need to set the probability of these fictitious scenarios to be arbitrarily small, so as not to have too much impact on the selection of the optimal investment strategy.

Similarly, a holding constraint of the type

$$h_i^{\min} \geq 0$$

can be approximated up to some arbitrarily small error term ϵ_i by a jump scenario z_i

$$\begin{aligned} \gamma_i(z_i) &= -\frac{1}{h_i^{min} + \epsilon_i} \\ \gamma_k(z_i) &= 0 \quad k \neq i \end{aligned}$$

Here, again, to complete our approximation we need to set the probability of these fictitious scenarios to be arbitrarily small.

However, this approach has a limitation as jumps cannot be used to enforce constraints of the type

$$h_i^{\min} \leq h_i \leq h_i^{\max}$$

with $h_i^{\min} > 0$ or $h_i^{\max} < 0$. Indeed, since the policy $h = 0$, i.e. invest all of the wealth in the money market, it cannot result in bankruptcy and therefore is always an acceptable strategy.

4.2.5 The HJB Equation

In this section we derive the risk-sensitive Hamilton-Jacobi-Bellman partial integro differential equation (RS HJB PIDE) associated with the optimal control problem under consideration. Since we do not anticipate that a classical solution generally exists, we will not attempt to derive a verification theorem. Instead, in the next chapter, we will show that the value function Φ is a solution of the RS HJB PIDE in the viscosity sense. In fact, we will show that the value function is the unique continuous viscosity solution of the RS HJB PIDE. This result will in turn justify the association of the RS HJB PIDE with the control problem and replace the verification theorem we would derive if a classical solution existed.

Proposition 4.6. *Let Φ be the value function for the auxiliary criterion function $I(v,x;h;t,T)$ defined in* (4.14)*. Then Φ is defined as*

$$\Phi(t,x) = \sup_{h\in\mathcal{A}(T)} I(v,x;h;t,T)$$

Moreover, if $\Phi \in C^{1,2}([0,T]\times\mathbb{R}^n)$, then Φ satisfies the RS HJB PIDE

$$\frac{\partial\Phi}{\partial t} + \sup_{h\in J} L_t^h \Phi(t,X(t)) = 0 \tag{4.15}$$

where $\mathcal{A}(T)$ was defined above as the set of investment strategies $h\in\mathcal{H}$ on $[0,T]$ such that relation (4.12) *holds, J is defined in* (4.10)*, and*

$$\begin{aligned} L_t^h\Phi(t,x) \;=\; & \left(b + Bx - \theta\Lambda\Sigma' h(s) + \int_{\mathbf{Z}} \xi(z)\left[(1+h'\gamma(z))^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)' D\Phi \\ & + \frac{1}{2} tr\left(\Lambda\Lambda' D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi \\ & + \int_{\mathbf{Z}} \left\{-\frac{1}{\theta}\left(e^{-\theta(\Phi(t,x+\xi(z))-\Phi(t,x))} - 1\right) - \xi'(z)D\Phi\right\}\nu(dz) - g(x,h;\theta) \end{aligned} \tag{4.16}$$

and subject to terminal condition $\Phi(T,x) = \ln v$.

Proof. **Outline -** This derivation can be decomposed in five steps. First, we define $\tilde{\Phi}$ as an exponential transformation of Φ. In the next three steps, we derive the dynamics of the log transformed problem and prove that it is ≥ 0, ≤ 0 and thus, equal to 0. Finally, reversing our exponential transformation, we conclude on the RS HJB PIDE associated with the original control problem.

Step 1: Exponential Transformation

Define $Q := [0,T] \times \mathbb{R}^n$ and denote by $\tilde{I}(v,x,h;t,T)$ the cost function defined as

$$\tilde{I}(v,x,h;t,T) := \mathrm{E}_{t,x}^{h,\theta}\left[\exp\left\{\theta\int_t^T g(s,X_s,h(s);\theta)ds - \theta\ln v\right\}\right]$$

then the value function $\tilde{\Phi}(t,x)$ defined as

$$\tilde{\Phi}(t,x) := \inf_{\mathcal{A}(T)} \tilde{I}(v,x;h;t,T)$$

can be expressed directly as a function of Φ:

$$\begin{aligned}\tilde{\Phi}(t,x) &= \inf_{\mathcal{A}(T)} \tilde{I}(v,x;h;t,T)\\ &= \inf_{\mathcal{A}(T)} \exp\{-\theta I(v,x;h;t,T)\}\\ &= \exp\left\{-\theta \sup_{h\in\mathcal{A}(T)} I(v,x;h;t,T)\right\}\\ &= \exp\{-\theta\Phi(t,x)\} \qquad (4.17)\end{aligned}$$

Step 2: First Inequality

Let $(t,x) \in Q$. Fix $h \in J$ and consider a constant control $\hat{h} = h$. We note that $\hat{h} \in \mathcal{K}$, where $\mathcal{K}$ is defined in (4.11). Moreover, substituting the control in the definition of the Radon-Nikodym derivative χ_t given in equation (4.8), we see that χ_t is indeed a an exponential martingale. As a result, the condition (4.12) is satisfied: the measure P_h^θ is well-defined and $\hat{h}$ is therefore an admissible control.

For $\delta > 0$, define the stopping time

$$\tau_\delta := \inf \{s > t : (s - t, X_s - x) \notin [0, \delta) \times \alpha \mathscr{B}_n\}$$

for a given constant $\alpha > 0$ and where $\mathscr{B}_n$ is the unit ball in $\mathbb{R}^n$.

$\lim_{\delta \to 0} \tau_\delta = 0$ and, as noted by Touzi [43], $\tau_\delta = \delta$ for $\delta \leq \bar{\delta}(\omega)$ sufficiently small.

By the Dynamic Programming Principle for jump-diffusion processes, (see for example Lemma 1.5 and Section 4 in Ishikawa [32] for a statement of this result and a meticulous proof in the context of Lévy processes)

$$\tilde{\Phi}(t,x) \;\leq\; \mathbf{E}_{t,x}^{h,\theta}\left[\exp\left\{\theta \int_t^{\tau_\delta} g(s, X_s, \hat{h}_s; \theta) ds\right\} \tilde{\Phi}(\tau_\delta, X_{\tau_\delta})\right]$$

where $\mathbf{E}_{t,x}^{h,\theta}[\cdot]$ represents the expectation under the measure $\mathbb{P}_h^\theta$ given initial data (t, x).

Thus,

$$0 \;\leq\; \mathbf{E}_{t,x}^{h,\theta}\left[\exp\left\{\theta \int_t^{\tau_\delta} g(s, X_s, \hat{h}_s; \theta) ds\right\} \tilde{\Phi}(\tau, X_\tau^k)\right] - \tilde{\Phi}(t,x)$$

Define $Z(t) = \theta \int_t^{\tau_\delta} g(s, X_s, \hat{h}_s; \theta) ds$, then

$$d\left(e^{Z_s}\right) = \theta g(s, X_s, \hat{h}_s; \theta) e^{Z_s}$$

Also, by Itô, for $s \in [t, \tau_\delta]$,

$$d\tilde{\Phi}_s = \left\{\frac{\partial \tilde{\Phi}}{\partial s} + \mathcal{L}\tilde{\Phi}\right\} ds + D\tilde{\Phi}' \Lambda dW_s^\theta + \int_{\mathbf{Z}} \left\{\tilde{\Phi}(t, X(s^-) + \xi(z)) - \tilde{\Phi}(t, X(s^-))\right\} \tilde{N}_{\mathbf{p}}(dt, dz)$$

where $\mathcal{L}$ is the generator of the state process $X(t)$ defined as:

$$\begin{aligned}\mathcal{L}\tilde{\Phi}(t,x) &:= \left(b + Bx - \theta\Lambda\Sigma' h(s) + \int_{\mathbf{Z}} \xi(z)\left[(1+h'\gamma(z))^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)' D\tilde{\Phi} \\ &\quad + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\tilde{\Phi}\right) + \int_{\mathbf{Z}}\left\{\tilde{\Phi}(x+\xi(z)) - \tilde{\Phi}(x) - \xi'(z)D\tilde{\Phi}\right\}\nu(dz)ds\end{aligned}$$

By the Itô product rule, and since $dZ_s \cdot \tilde{\Phi}_s = 0$, we get

$$d\left(\tilde{\Phi}_s e^{Z_s}\right) = \tilde{\Phi}_s d\left(e^{Z_s}\right) + e^{Z_s} d\tilde{\Phi}_s$$

and hence for $s \in [t, \tau_\delta]$

$$\begin{aligned}\tilde{\Phi}(s, X_s)e^{Z_s} &= \tilde{\Phi}(t,x)e^{Z_t} + \theta\int_t^s \tilde{\Phi}(u, X_u)g(u, X_u, \hat{h}_u;\theta)e^{Z_u}du \\ &\quad + \int_t^s \left(\frac{\partial\tilde{\Phi}}{\partial u}(u, X_u) + \mathcal{L}\tilde{\Phi}(u, X_u)e^{Z_u}\right)du + \int_t^s D\tilde{\Phi}'\Lambda dW_u^\theta \\ &\quad + \int_t^s\int_{\mathbf{Z}}\left\{\tilde{\Phi}(t, X(u^-)+\xi(z)) - \tilde{\Phi}(t, X(u^-))\right\}d\tilde{N}_{\mathbf{p}}(du,dz)du\end{aligned}$$

Noting that $u(t,x)e^{Z_t} = u(t,x)$ and taking the expectation with respect to the initial data (t,x), we get

$$\begin{aligned}&\mathbf{E}_{t,x}^{h,\theta}\left[\tilde{\Phi}(t, X_s)e^{Z_s}\right] \\ &= \tilde{\Phi}(t,x)e^{Z_t} + \mathbf{E}_{t,x}^{h,\theta}\left[\int_t^s\left(\frac{\partial\tilde{\Phi}}{\partial u}(u, X_u) + \mathcal{L}\tilde{\Phi}(u, X_u) + \theta\tilde{\Phi}(s, X_s)g(u, X_u, \hat{h}_u;\theta)\right)e^{Z_u}du\right]\end{aligned}$$

In particular, for $s = \tau_\delta$,

$$\begin{aligned}0 &\leq \mathbf{E}_{t,x}^{h,\theta}\left[\tilde{\Phi}(\tau_\delta, X_{\tau_\delta})e^{Z_{\tau_\delta}}\right] - \tilde{\Phi}(t,x)e^{Z_t} \\ &= \mathbf{E}_{t,x}^{h,\theta}\left[\int_t^{\tau_\delta}\left(\frac{\partial\tilde{\Phi}}{\partial u}(u, X_u) + \mathcal{L}\tilde{\Phi}(u, X_u) + \theta\tilde{\Phi}(u, X_u)g(u, X_u, \hat{h}_u;\theta)\right)e^{Z_u}du\right]\end{aligned}$$

and thus

$$\begin{aligned} 0 &\leq \frac{1}{\delta}\left(\mathbf{E}_{t,x}^{h,\theta}\left[\tilde{\Phi}(\tau_\delta, X_{\tau_\delta})e^{Z\tau_\delta}\right] - \tilde{\Phi}(t,x)e^{Z_t}\right) \\ &= \frac{1}{\delta}\left(\mathbf{E}_{t,x}^{h,\theta}\left[\int_t^{\tau_\delta}\left(\frac{\partial\tilde{\Phi}}{\partial u}(u,X_u) + \mathcal{L}\tilde{\Phi}(u,X_u) + \theta\tilde{\Phi}(u,X_u)g(u,X_u,\hat{h}_u;\theta)\right)e^{Z_u}du\right]\right) \end{aligned}$$

As $\delta \to 0$, $\tau_\delta \to t$

$$\begin{aligned} &\frac{1}{\delta}\left(\mathbf{E}_{t,x}^{h,\theta}\left[\int_t^{\tau_\delta}\left(\frac{\partial\tilde{\Phi}}{\partial u}(u,X_u) + \mathcal{L}\tilde{\Phi}(u,X_u) + \theta\tilde{\Phi}(u,X_u)g(u,X_u,\hat{h}_u;\theta)\right)e^{Z_u}du\right]\right) \\ \to\ &\frac{\partial\tilde{\Phi}}{\partial u}(u,X_u) + \mathcal{L}\partial\tilde{\Phi}(u,X_u) + \theta\partial\tilde{\Phi}(u,X_u)g(u,X_u,\hat{h}_u;\theta) \end{aligned}$$

a.s. by the Mean Value Theorem, since the random variable

$$\frac{1}{\delta}\int_t^{\tau_\delta}\left(\frac{\partial\tilde{\Phi}}{\partial u}(u,X_u) + \mathcal{L}\tilde{\Phi}(u,X_u) + \theta\tilde{\Phi}(u,X_u)g(u,X_u,\hat{h}_u;\theta)\right)e^{Z_u}du$$

is bounded for sufficiently small δ.

Hence, by the Dominated Convergence Theorem, we have

$$\frac{\partial\tilde{\Phi}}{\partial t}(t,X_t) + \mathcal{L}\tilde{\Phi}(t,X_t) + \theta\tilde{\Phi}(t,X_t)g(t,X_t,\hat{h}_t;\theta) \geq 0$$

Step 3: Second Inequality

This step in the proof is a slight adaptation of the proof for classical control problems in Touzi [43]. Let $(t_0,x_0) \in Q$. We intend to prove that

$$\frac{\partial\tilde{\Phi}}{\partial t}(t_0,x_0) + \inf_{h\in\mathcal{H}}\left\{\mathcal{L}^h\tilde{\Phi}(t_0,x_0) + \theta\tilde{\Phi}g(t_0,x_0,h;\theta)\right\} \leq 0$$

by contradiction. Thus, assume that

$$\frac{\partial \tilde{\Phi}}{\partial t}(t_0, x_0) + \inf_{h \in \mathcal{H}} \left\{ \mathcal{L}^h \tilde{\Phi}(t_0, x_0) + \theta \tilde{\Phi} g(t_0, x_0, h; \theta) \right\} > 0 \tag{4.18}$$

For $\varsigma > 0$ define φ as

$$\varphi(t, x) := \tilde{\Phi}(t, x) - \varsigma \frac{1}{2} |x - x_0|^2$$

Thus,

$$\begin{aligned}
\varphi(t, x) &\leq \tilde{\Phi}(t, x) \qquad \forall (t, x) \in Q \\
\left(\tilde{\Phi} - \varphi \right)(t_0, x_0) &= 0 \\
\left(D\tilde{\Phi} - D\varphi \right)(t_0, x_0) &= 0 \\
\left(\frac{\partial \tilde{\Phi}}{\partial t} - \frac{\partial \varphi}{\partial t} \right)(t_0, x_0) &= 0 \\
\left(D^2 \tilde{\Phi} - D^2 \varphi \right)(t_0, x_0) &= \varsigma \mathbf{I}_n
\end{aligned}$$

where $\mathbf{I}_n$ denote the n-dimensional identity matrix.

Our assumption implies that, for small enough $\varsigma > 0$,

$$\frac{\partial \varphi}{\partial t}(t_0, x_0) + \inf_{h \in \mathcal{H}} \left\{ \mathcal{L}^h \varphi(t_0, x_0) + \theta \varphi g(t_0, x_0, h; \theta) \right\} > 0 \tag{4.19}$$

Let $\mathcal{N}_\delta$ be an open neighbourhood of (t_0, x_0) defined for $\delta > 0$ as

$$\mathcal{N}_\delta := \{(t, x) : (t - t_0, x - x_0) \in (-\delta, \delta) \times \delta \mathscr{B}_n, \text{ and (4.19) holds}\} \tag{4.20}$$

For $\rho > 0$, consider the set J^ρ of ρ-optimal controls h^ρ satisfying

$$\tilde{I}(t_0, x_0, h^\rho) \leq \tilde{\Phi}(t_0, x_0) + \rho \tag{4.21}$$

Also, let $\epsilon > 0$, $\epsilon \leq \rho$ be such that

$$\min_{Q \setminus \mathcal{N}_\delta} \left(\tilde{\Phi} - \varphi \right) = \frac{\varsigma}{2} \min_{Q \setminus \mathcal{N}_\delta} |x - x_0|^2 \geq 2\epsilon e^{-\delta\theta M_\delta} > 0 \tag{4.22}$$

where M_δ is defined as

$$M_\delta := \max_{x \in \mathcal{D}, h \in J^\rho} (-g(x, h; \theta), 0)$$

for

$$\mathcal{D} := \{x : (x - x_0) \in (\delta + \zeta)\mathscr{B}_n\}$$

and

$$\zeta := \max_{z \in \mathbf{Z}} \|\xi(z)\|$$

Note that $\zeta < \infty$ by boundedness of $\xi(z)$ and thus $M_\delta < \infty$.

Consider the ϵ-optimal control h^ϵ and denote by X^ϵ the controlled process defined by the control process h^ϵ and introduce the stopping time

$$\tau := \inf \left\{ s > t_0 : (s, \tilde{X}^\epsilon(s)) \notin \mathcal{N}_\delta \right\}$$

Note that since we assumed that $-\infty \leq \xi_i^{\min} \leq \xi_i \leq \xi_i^{\max} < \infty$ for $i = 1, \ldots, n$ and

since ν is assumed to be finite then $X^\epsilon(\tau)$ is also finite and in particular,

$$(\tilde{\Phi} - \varphi)(\tau, \tilde{X}^\epsilon(\tau)) \geq 2\epsilon e^{-\delta\theta M_\delta} \tag{4.23}$$

Let $Z(t) = \theta \int_t^\tau g(u, \tilde{X}_u^\epsilon, h_u^\epsilon; \theta) du$. By (4.23),

$$\tilde{\Phi}(\tau, \tilde{X}^\epsilon(\tau))e^{Z(\tau)} - \tilde{\Phi}(t_0, x)e^{Z(t_0)} \geq \int_{t_0}^\tau d\left(\varphi(u, \tilde{X}^\epsilon(u))e^{Z_u}\right) + 2\epsilon$$

i.e.

$$\tilde{\Phi}(t_0, x_0) \leq \tilde{\Phi}(\tau, \tilde{X}_k^\epsilon(\tau))e^{Z(\tau)} - \int_{t_0}^\tau d\left(\varphi(u, \tilde{X}^\epsilon(u))e^{Z_u}\right) - 2\epsilon$$

Taking expectation with respect to the initial data (t_0, x_0),

$$\tilde{\Phi}(t_0, x_0) \leq \mathbf{E}_{t_0,x_0}^{h,\theta}\left[\tilde{\Phi}(\tau_0, \tilde{X}^\epsilon(\tau))e^{Z(\tau)} - \int_{t_0}^\tau d\left(\varphi(u, \tilde{X}^\epsilon(u))e^{Z_u}\right)\right] - 2\epsilon$$

Note that by the Itô product rule,

$$\begin{aligned} & d\left(\varphi(u, \tilde{X}^\epsilon(u))e^{Z_u}\right) \\ = \; & \varphi_u d\left(e^{Z_u}\right) + e^{Z_u} d\varphi_u \\ = \; & \frac{\partial\varphi}{\partial t}(t, x) + \mathcal{L}^h\varphi(t, x) + \theta g(t, x, h; \theta) \end{aligned}$$

Also note that by definition,

$$\begin{aligned} & \frac{\partial\varphi}{\partial t}(u, \tilde{X}_u) + \mathcal{L}^{h^\epsilon}\varphi(\tau, X_\tau) + \theta\varphi g(u, \tilde{X}_u, h^\epsilon; \theta) \\ \geq \; & \frac{\partial\varphi}{\partial t}(u, \tilde{X}_u) + \inf_{h\in\mathcal{H}}\left\{\mathcal{L}^h\varphi(u, \tilde{X}_u) + \theta\varphi g(u, \tilde{X}_u, h; \theta)\right\} \\ \geq \; & 0 \end{aligned}$$

on $[t_0, \tau]$.

Therefore,

$$\begin{aligned}
\tilde{\Phi}(t_0, x_0) &\leq \mathbf{E}^{h,\theta}_{t_0,x_0}\left[\tilde{\Phi}(\tau, \tilde{X}^\epsilon(\tau))e^{Z(\tau)} - \int_{t_0}^{\tau} d\left(\varphi(u, \tilde{X}^\epsilon(u))e^{Z_u}\right)\right] - 2\epsilon \\
&\leq -2\epsilon + \mathbf{E}^{h,\theta}\left[\exp\left\{\theta\int_{t_0}^{\tau} g(X_u, h^\epsilon(u); \theta)du\right\}\tilde{\Phi}(\tau, \tilde{X}^\epsilon(\tau))\right] \\
&\leq -2\epsilon + \tilde{I}(t_0, x_0, h^\epsilon) \\
&\leq \tilde{\Phi}(t_0, x_0) - \epsilon
\end{aligned}$$

where the third inequality follows from the Dynamic Programming Principle and the last inequality follows from the definition of ϵ-optimal controls in (5.20).

The contradiction is proved.

Step 4: RS HJB PIDE for the Exponentially Transformed Problem

Applying directly the results of steps 2 and 3 yields the RS HJB PIDE associated with the exponentially transformed problem, namely,

$$\begin{aligned}
&\frac{\partial\tilde{\Phi}}{\partial t}(t,x) + \inf_{h\in J}\left\{\left(b + Bx - \theta\Lambda\Sigma' h(s) + \int_{\mathbf{Z}} \xi(z)\left[(1 + h'\gamma(z))^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)' D\tilde{\Phi}\right. \\
&\quad + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\tilde{\Phi}(t,x)\right) + \int_{\mathbf{Z}}\left\{\tilde{\Phi}(t, x + \xi(z)) - \tilde{\Phi}(t,x) - \xi'(z)D\tilde{\Phi}(t,x)\right\}\nu(dz) \\
&\quad \left.+\theta g(x, h; \theta)\tilde{\Phi}(t,x)\right\} \\
= \;& 0
\end{aligned}$$

subject to terminal condition $\tilde{\Phi}(T, x) = v^{-\theta}$.

Step 5: Conclusion - RS HJB PIDE For the JDRSAM

We can now reverse our exponential transformation by taking the logarithm of both sides of (4.17) and multiplying both sides by $-\frac{1}{\theta}$ in order to get back to our original control problem:

$$\Phi(t,x) = -\frac{1}{\theta}\ln\tilde{\Phi}(t,x) = \sup_{h\in\mathcal{A}(T)} -\frac{1}{\theta}\ln\tilde{I}_{\theta,T}(h) = \sup_{h\in\mathcal{A}(T)} -\frac{1}{\theta}\ln \mathrm{E}^{h,\theta}\left[\exp\left\{\theta\int_t^T g(X_s,h(s);\theta)ds\right\}\right]$$

Define $\Phi(t,x) := -\frac{1}{\theta}\ln\tilde{\Phi}(t,x)$, then $\tilde{\Phi}(t,x) = e^{-\theta\Phi(t,x)}$ and

$$\begin{aligned}
\frac{\partial\tilde{\Phi}}{\partial t} &= -\theta\frac{\partial\Phi}{\partial t}\tilde{\Phi} \\
D\Phi &= -\theta D\Phi\cdot\Phi \\
D^2\tilde{\Phi} &= -\theta D^2\Phi\cdot\tilde{\Phi} + \theta^2(D\Phi)'D\Phi\cdot\tilde{\Phi}
\end{aligned}$$

Substituting into the RS HJB PIDE for $\tilde{I}_{\theta,T}(h)$,

$$\begin{aligned}
& -\theta\frac{\partial\Phi}{\partial t}\tilde{\Phi} \\
+ \ & \inf_{h\in J}\left\{-\theta\left(b + BX_s - \theta\Lambda\Sigma' h(s) + \int_{\mathbf{Z}}\xi(z)\left[(1+h'\gamma(z))^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)' D\Phi\cdot\tilde{\Phi}\right. \\
& -\frac{\theta}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\cdot\tilde{\Phi}\right) + \frac{\theta^2}{2}(D\Phi)'\Lambda\Lambda' D\Phi\cdot\tilde{\Phi} \\
& + \int_{\mathbf{Z}}\left\{e^{-\theta\Phi(t,X(s)+\xi(z))} - e^{-\theta\Phi(t,X(s))} + \theta\xi'(z)D\Phi\cdot\tilde{\Phi}\right\}\nu(dz) \\
& \left. +\theta g(X,h;\theta)\tilde{\Phi}\right\} \\
= \ & 0
\end{aligned}$$

dividing by $-\theta\tilde{\Phi}$ and rearranging, we finally obtain the RS HJB PIDE governing

the dynamics of the value function $\Phi(t,x)$ for the criterion $J_{0,T}(v,x;h,\nu)$:

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in J} L_t^h \Phi(t, X(t)) = 0 \tag{4.24}$$

where

$$\begin{aligned} L_t^h \Phi(t,x) \;=\; & \left(b + Bx - \theta \Lambda \Sigma' h(s) + \int_{\mathbf{Z}} \xi(z) \left[(1 + h'\gamma(z))^{-\theta} - 1_{\mathbf{Z}_0}(z) \right] \nu(dz) \right)' D\Phi \\ & + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda' D^2 \Phi \right) - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi \\ & + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(\Phi(t,x+\xi(z)) - \Phi(t,x))} - 1 \right) - \xi'(z) D\Phi \right\} \nu(dz) - g(x, h(t)) \end{aligned} \tag{4.25}$$

and subject to terminal condition $\Phi(T,x) = \ln v$.

□

In the remainder of this chapter, we will use the RS HJB PIDE to prove the existence of a unique optimal control policy. But until we have proved (in the next chapter) that the value function is the solution in some sense of the RS HJB PIDE, we must keep in mind that the relation between the control problem and the RS HJB PIDE is mathematically tenuous.

4.3 Solving the Control Problem

4.3.1 Developing the Supremum

Start by considering the supremum:

$$
\begin{aligned}
& \sup_{h\in J} L_t^h \Phi \\
= \; & \sup_{h\in J} \left\{ \left(b + Bx + \int_{\mathbf{Z}} \xi(z) \left[(1 + h'\gamma(z))^{-\theta} - 1_{\mathbf{Z}_0}(z) \right] \nu(dz) \right)' D\Phi - \theta h'\Sigma\Lambda' D\Phi \right. \\
& + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi \\
& + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta}\left(e^{-\theta(\Phi(t,x+\xi(z))-\Phi(t,x))} - 1 \right) - \xi'(z) D\Phi \right\} \nu(dz) ds \\
& \left. - \frac{1}{2}(\theta+1) h'\Sigma\Sigma' h + a_0 + A_0' x + h'(\hat{a} + \hat{A}x) \right. \\
& \left. - \int_{\mathbf{Z}} \left\{ \frac{1}{\theta}\left[(1 + h'\gamma(z))^{-\theta} - 1 \right] + h'\gamma(z) 1_{\mathbf{Z}_0}(z) \right\} \nu(dz) \right\} \\
= \; & (b + Bx)' D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi + a_0 + A_0' x \\
& + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta}\left(e^{-\theta(\Phi(t,x+\xi(z))-\Phi(t,x))} - 1 \right) - \xi'(z) D\Phi 1_{\mathbf{Z}_0}(z) \right\} \nu(dz) \\
& + \sup_{h\in J} \left\{ -\frac{1}{2}(\theta+1) h'\Sigma\Sigma' h - \theta h'\Sigma\Lambda' D\Phi + h'(\hat{a} + \hat{A}x) \right. \\
& \left. - \frac{1}{\theta}\int_{\mathbf{Z}} \left\{ (1 - \theta\xi'(z) D\Phi) \left[(1 + h'\gamma(z))^{-\theta} - 1 \right] + \theta h'\gamma(z) 1_{\mathbf{Z}_0}(z) \right\} \nu(dz) \right\} \qquad (4.26)
\end{aligned}
$$

4.3.2 Solving the Control Problem

First observe that the term

$$
-\frac{1}{2}(\theta+1) h'\Sigma\Sigma' h - \theta h'\Sigma\Lambda' D\Phi + h'(\hat{a} + \hat{A}x) - \int_{\mathbf{Z}} h'\gamma(z) 1_{\mathbf{Z}_0}(z) \nu(dz)
$$

is strictly concave in h $\forall z \in \mathbf{Z}$ a.s. $d\nu$ since $\theta > 0$ and by Assumption 2.2. Under Assumption 4.2, the nonlinear jump-related term

$$-\frac{1}{\theta}\int_{\mathbf{Z}}\left\{(1-\theta\xi'(z)D\Phi)\left[(1+h'\gamma(z))^{-\theta}-1\right]\right\}\nu(dz)$$

simplifies to

$$-\frac{1}{\theta}\int_{\mathbf{Z}}\left\{\left[(1+h'\gamma(z))^{-\theta}-1\right]\right\}\nu(dz)$$

which is also concave in h $\forall z \in \mathbf{Z}$ a.s. $d\nu$.

Therefore, the supremum is reached for a unique optimal control h^*, which is an interior point of the set J defined in equation (4.10), and the supremum, evaluated at h^*, is finite.

4.3.3 Discussion of the Jump-Related Term

For the purpose of this short section, we will assume that ν is a finite measure. This assumption will help us simplify the notation and lift some technicalities so as not detract from the broader discussion.

The Concavity Condition

Going back to the supremum in (4.26), we will now discuss the nonlinear jump-related term

$$-\frac{1}{\theta}\int_{\mathbf{Z}}\left\{(1-\theta\xi'(z)D\Phi)\left[(1+h'\gamma(z))^{-\theta}-1\right]\right\}\nu(dz) \tag{4.27}$$

Taking a Taylor expansion of the expression (4.27) around $h = 0$, we obtain

$$
\begin{aligned}
& -\frac{1}{\theta}\int_{\mathbf{Z}}\left\{\left(1-\theta\xi'(z)D\Phi\right)\left[\left(1+h'\gamma(z)\right)^{-\theta}-1\right]\right\}\nu(dz) \\
= & \int_{\mathbf{Z}}\left\{\left(1-\theta\xi'(z)D\Phi\right)h'\gamma(z)-\frac{\theta+1}{2}\left(1-\theta\xi'(z)D\Phi\right)\left(h'\gamma(z)\right)^{2}\right. \\
& \left.+\frac{(\theta+1)(\theta+2)}{3!}\left(1-\theta\xi'(z)D\Phi\right)\left(h'\gamma(z)\right)^{3}+\ldots\right\}\nu(dz)
\end{aligned}
$$

For the control problem to have a unique finite solution, we note that the supremum must satisfy the condition

Condition 4.7.

$$
-\frac{1}{\theta}\int_{\mathbf{Z}}\left\{\left(1-\theta\xi'(z)D\Phi\right)\left[\left(1+h'\gamma(z)\right)^{-\theta}-1\right]\right\}\nu(dz) \tag{4.28}
$$

is concave in h.

In fact, since $\left[\left(1+h'\gamma(z)\right)^{-\theta}-1\right]$ is convex in $h\ \forall z\in\mathbf{Z}$ a.s. $d\nu$, then the concavity of (4.28) depends indirectly on the sign of $(1-\theta\xi'(z)D\Phi)$.

However, this dependence is only indirect and should be viewed in the context of an integration whose effect is to sum up the terms $(1-\theta\xi'(z)D\Phi)$ for all the values of $z\in\mathbf{Z}$ weighted by the factor $\left[\left(1+h'\gamma(z)\right)^{-\theta}-1\right]$. As a result, in instances where Condition 4.7 is satisfied, $(1-\theta\xi'(z)D\Phi)$ can still be negative for some values of $z\in\mathbf{Z}$.

At first sight, there does not seem to exist a clear economic interpretation or a simpler criterion to encapsulate this condition. In fact, one cannot check ex-ante that this condition is satisfied. However, we conjecture that the family of investment

policies satisfying this condition is large. Indeed, we can see heuristically that since the value function Φ and the reward function $\ln V(t)$ share the same scaling, then, for "reasonable" (i.e. $o(10^{-1})$ to $o(1)$) jump intensities, $\xi'(z)D\Phi$ should have a scale comparable to that of $d\ln V(t)$. The general interpretation of the $d\ln V(t)$ term is as a percentage change, i.e. as a quantity of $o(10^{-2})$ or $o(10^{-1})$. Hence, for reasonable jump intensities and θ "not too large" (to give a rough yet conservative order of magnitude, say, $\theta < 5$), we can expect the term $1 - \theta\xi'(z)D\Phi$ to be positive for $z \in \mathbf{Z}$ a.s $d\nu$, and the Condition (4.28) to be satisfied.

For somewhat larger theta (say, $5 < \theta < 10$), satisfaction of the condition is still generally expected, although it is more difficult to see since it involves both the effect of the integral and of the "weight" term $\left[(1 + h'\gamma(z))^{-\theta} - 1\right]$. Finally, as θ gets larger and larger, the set of policies satisfying the concavity condition may decrease markedly but even then, there might still exist sets of policies which satisfy Condition (4.28).

Proposed Interpretation of The Nonlinear Jump-Related Term

The nonlinear jump-related term (4.27) can be expanded into the sum

$$-\frac{1}{\theta}\int_{\mathbf{Z}}\left\{\left[(1 + h'\gamma(z))^{-\theta} - 1\right]\right\}\nu(dz)$$
$$+\int_{\mathbf{Z}}\left\{\xi'(z)D\Phi\left[(1 + h'\gamma(z))^{-\theta} - 1\right]\right\}\nu(dz)$$

The first term,

$$-\frac{1}{\theta}\int_{\mathbf{Z}}\left\{\left[(1 + h'\gamma(z))^{-\theta} - 1\right]\right\}\nu(dz)$$

can be interpreted as the risk-sensitive change in the controlled reward function (the investor's log wealth) due to an expected number of jumps in asset prices. An insight into this interpretation can be gained by performing a Taylor expansion of this term around $h = 0$:

$$\begin{aligned} & -\frac{1}{\theta}\int_{\mathbf{Z}}\left\{(1+h'\gamma(z))^{-\theta}-1\right\}\nu(dz) \\ = & \int_{\mathbf{Z}}\left\{h'\gamma(z)-\frac{(\theta+1)}{2}(h'\gamma(z))^2+\frac{(\theta+1)(\theta+2)}{3!}(h'\gamma(z))^3+\ldots\right\}\nu(dz) \end{aligned}$$

In particular, taking the limit as $\theta \to 0^+$, we see that

$$\begin{aligned} & \lim_{\theta\to 0^+} -\frac{1}{\theta}\int_{\mathbf{Z}}\left\{(1+h'\gamma(z))^{-\theta}-1\right\}\nu(dz) \\ = & \int_{\mathbf{Z}}\left\{h'\gamma(z)-\frac{1}{2}(h'\gamma(z))^2+\frac{1}{3}(h'\gamma(z))^3+\ldots\right\}\nu(dz) \\ = & \int_{\mathbf{Z}}\ln(1+h'\gamma(z))\nu(dz) \end{aligned}$$

which is the instantaneous change in the investor's log wealth due to an expected number of jumps (as evidenced by the definition of V(t)).

The second term,

$$\int_{\mathbf{Z}}\left\{\xi'(z)D\Phi\left[(1+h'\gamma(z))^{-\theta}-1\right]\right\}\nu(dz)$$

can be interpreted, roughly speaking, as the difference between the change in the value function due to jumps in the factors under the $\mathbb{P}_h^\theta$ and under the $\mathbb{P}$ measure. Indeed, since

1. $D\Phi$ is the change in the value function given a small change in the components of

the state vector $X(t)$;

2. $\xi(z)$ is the jump intensity vector for the state variable, given $z \in \mathbf{Z}$, and;

3. as a result of the change of measure and of the definition of the compensated Poisson measure under the measure $\mathbb{P}_h^\theta$ as

$$\int_0^t \int_{\mathbf{Z}} \tilde{N}_{\mathbf{p}}^\theta(ds, dz) = \int_0^t \int_{\mathbf{Z}} N_{\mathbf{p}}(ds, dz) - \int_0^t \int_{\mathbf{Z}} \left\{ (1 + h'\gamma(z))^{-\theta} \right\} \nu(dz)ds$$

the term

$$\int_0^t \int_{\mathbf{Z}} \left\{ (1 + h'\gamma(z))^{-\theta} \right\} \nu(dz)ds$$

represents the expected number of jumps on the time interval $[0, t]$ under the measure $\mathbb{P}_h^\theta$.

then the expression $\int_{\mathbf{Z}} \xi'(z) D\Phi \left(1 + h'\gamma(z)\right)^{-\theta} \nu(dz)$ can be viewed as the approximate change in the value function, under the measure $\mathbb{P}_h^\theta$, due to an expected number of jumps in the state variable $X(t)$. This term is then compared with the approximate change in the value function, under the measure $\mathbb{P}$, due to an expected number of jumps in the state variable $X(t)$, which is given by $\int_{\mathbf{Z}} \xi'(z) D\Phi \nu(dz)$.

To conclude, the concavity condition can be interpreted as a constraint on the change of measure, and thus on the definition of an acceptable strategy. A consequence of the change of measure is the transfer of a part of the jump-related drift from the wealth process to the state process. While this jump risk was controllable via the asset allocation when it was in the wealth process, it is no longer as controllable when incorporated in the state process. From an economics perspective, a natural

constraint would therefore be to curb the quantity of risk which can be transferred from the wealth process to the state process, and to link this curb to the risk-sensitive parameter θ. It appears that it is precisely what the concavity constraint is achieving, although it is still unclear why this constraint is not symmetric and tolerate large negative jumps but no large positive jumps.

Conclusion: the Role of Assumption 4.2

While Assumption 4.2 proves sufficient for the concavity condition to be satisfied, as we will see now, it is in fact not necessary. However, from a mathematical perspective, what Assumption 4.2 provides us with is a clear structural condition that we can impose on our model ex-ante and ensures the existence of a unique solution.

From a numerical analysis perspective, our investigation into the properties of the nonlinear jump-related term suggests that we might be able to relax Assumption 4.2 and simply check at each node on the approximating grid that the concavity condition 4.7 is indeed satisfied.

4.4 Conclusion

In this chapter, we have extended the classical risk-sensitive asset management setting to include the possibility of jumps in asset prices and in the level of the valuation factors. We have used the broad class of random Poisson measures as a model for the jumps since it enables us to consider a wide range of specifications.

Under Assumption 4.2, we have shown that a unique optimal control policy exists and the value function, evaluated at this point, is finite. We noted that Assump-

tion 4.2 is sufficient to satisfy the more general, but also more complex, concavity condition 4.7. Although Assumption 4.2 is not a necessary condition, it provides us with is a clear structural condition that we can impose on our model ex-ante and which enures the existence of a unique solution.

We have also remarked that one of the specificities of jump diffusion risk sensitive control is that it disallows any investment policy which may result in the investor's bankruptcy, contrary to the Merton type of approach in which bankruptcy can and does occur. This unique feature can be turned to our advantage and used to model a large class of problems with holding constraints directly, via the introduction of fictitious, low probability, jumps in security prices.

Chapter 5

Jump Diffusion Risk-Sensitive Asset Management: A Viscosity Approach

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In the jump diffusion setting introduced in the previous chapter, neither an analytical nor a classical $C^{1,2}$ solutions may generally exist. As a result, to give a sense to the relation between the value function and the risk sensitive Hamilton-Jacobi-Bellman Partial Integro Differential Equation (RS HJB PIDE), we will consider a class of weak solutions called viscosity solutions, which have gained a widespread acceptance in control theory in recent years (see for example the classical review article by Crandall, Ishii and Lions [19] for an overview of viscosity solutions and the book by Fleming and Soner [29] for a detailed view of their application to control problems).

In this Chapter, we will develop a viscosity solution approach to address a general class of risk sensitive control problems which includes the jump diffusion risk sensitive asset management model as a subcase. The main results are a comparison theorem and the proof that the value function of the control problem under consideration is the unique continuous viscosity solution of the associated RS HJB PIDE.

The chapter is organized as follows. Section 1 introduces the idea of viscosity solution and contains a brief overview of the literature. In Section 2, we present a more general class of problems which includes the jump diffusion risk sensitive asset management model as a subcase. In Section 3, we investigate the properties of the value function, and we prove that the value function can be characterized as a viscosity solution of the associated RS HJB PIDE in Section 4. In Section 5, we present the main result of this chapter: a comparison theorem. In Section 6 we show that the value function of the control problem is the unique continuous viscosity solution of the associated RS HJB PIDE. Finally, in Section 7, we apply these results to finish solving the jump diffusion risk sensitive asset management problem.

5.1 Introduction

In recent years, viscosity solutions have gained a widespread acceptance as an effective technique to obtain a weak sense solution for HJB PDEs when no classical (i.e $C^{1,2}$) solution can be shown to exist, which is the case for many stochastic control problems. Viscosity solutions also have a very practical interest. Indeed, once a solution has been interpreted in the viscosity sense and the uniqueness of this solution has been proved via a comparison result, the fundamental 'stability' result of Barles and Souganidis [9] opens the way to a numerical resolution of the problem through a wide range of schemes. Readers interested in an overview of viscosity solutions should refer to the classic article by Crandall, Ishii and Lions [19], the books by Fleming and Soner [29] and Oksendal and Sulem [36], as well as the notes by Barles [6] and Touzi [43].

While the use of viscosity solutions to solve classical diffusion-type stochastic control problems has been extensively studied and surveyed (see Fleming and Soner [29] and Touzi [43]), the introduction of a jump-related measure makes the jump-diffusion framework more complex as we must contend with a PIDE as opposed to an elliptical PDE. As a result, so far no general theory has been developed to solve jump-diffusion problems. Instead, the assumptions made to derive a comparison result are closely related to what the specific problem allows. Broadly speaking, the literature can be split along two lines of analysis, depending on whether the measure associated with the jumps is assumed to be finite.

In the case when the jump measure is finite, Alvarez and Tourin [1] considered a fairly general setting in which the jump term does not need to be linear in the function

u which solves the PIDE. In this setting, Alvarez and Tourin developed a comparison theorem that they apply to a stochastic differential utility problem. Amadori [3] extended Alvarez and Tourin's analysis to price European options. Barles, Buckdahn and Pardoux [7] studied the viscosity solution of integro-differential equations associated with backward SDEs (BSDEs).

Traditionally, the study of singular measures has focused on the Lévy measure. Pham [41] derived a comparison result for the variational inequality associated with an optimal stopping problem. Jakobsen and Karlsen [34] analyzed in detail the impact of the Lévy measure's singularity and propose a maximum principle. Amadori, Karlsen and La Chioma [4] focused on geometric Lévy processes and the partial integro differential equations they generate before applying their results to BSDEs and to the pricing of European and American derivatives.

Recently, Barles and Imbert [8] have revisited the theory of viscosity solution focusing on PIDEs and more specifically, on the nonlocal operator associated with a measure. Notably, Barles and Imbert proposed a new definition of viscosity solutions combining test functions and semijets, proved a Jensen-Ishii Lemma for elliptical and parabolic second-order PIDEs and gave a number of applications, including the derivation of a comparison theorem. The nonlocal operators Barles and Imbert considered in their examples and applications are of the form

$$\mathcal{I}[u(t,x)] := \int \{u(t,x+\xi(z)) - u(t,x) - \xi'(z)Du\}\,\nu(dz)$$

which is linear in the argument u.

Linearity is a main obstacle for us. Indeed, the jump term arising in the derivation of the risk-sensitive HJB PIDE and defined below in equation (5.7) by

$$\mathcal{I}^{RS}[u(t,x)] := \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(u(t,x+\xi(z))-u(t,x))} - 1 \right) - \xi'(z) Du \right\} \nu(dz)$$

is nonlinear in its argument u. The results of Barles and Imbert are therefore not directly applicable to the risk-sensitive control problem. We will instead construct a line of argument consistent with Alvarez and Tourin [1].

As far as our jump diffusion risk-sensitive control problem is concerned, we will present a treatment which does not restrict the compensator ν. At some point, we will however need ν to be finite. This assumption will only be made for a purely technical reason arising in the proof of the comparison result (Section 5). Since the rest of the story is still valid if ν is not finite, and in accordance with our goal of keeping the discussion as broad as possible, we will write the rest of the chapter in the spirit of a general compensator ν.

5.2 A Slightly More General Class of Problems

We will now extend the scope of our investigation beyond the confines of the asset only jump diffusion risk sensitive asset management problem formulated in the previous chapter by considering a more general risk-sensitive control problem. Indeed, our concern is to derive a comparison theorem able to accommodate a wider class of problems including benchmarked asset management and asset and liability management (which will both be addressed in the next chapter).

5.2.1 Setting

Introduce a reference probability system $\upsilon = (\Omega, \{\mathcal{F}_t\}, \mathbb{P}, W, N_{\mathbf{p}}, h(t))$, where

(i). $(\Omega, \mathcal{F}_T, \mathbb{P})$ is a probability space;

(ii). $\{\mathcal{F}_t\}$ is an increasing family of σ-algebras;

(iii). W is a $\mathcal{F}_t$-adapted $(n+m)$-dimensional Brownian motion;

(iv). $N_{\mathbf{p}}$ is a $\mathcal{F}_t$-adapted Poisson random measure induced by the $\mathcal{F}_t$-adapted point process $\mathbf{p}$;

(v). $h(t)$ is an *admissible* control in the sense of Definition 5.1.

Definition 5.1. A control process $h(t)$ is *admissible* with respect to the reference probability system υ if the following conditions are satisfied:

1. $h(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{F}_t\}_{t \geq 0}$;

2. $P\left(\int_0^T |h(s)|^2 \, ds < +\infty\right) = 1, \quad \forall T > 0$;

3. $h(t,\omega) \in J \subset \mathbb{R}^m$ for a given compact set J, $\forall t \in [0,T]$, $\forall \omega \in \Omega$ a.s.

4. $h'(t)\gamma(z) \geq -1, \quad \forall t > 0, z \in \mathbf{Z}$, a.s. $d\nu$.

Remark 5.2. if J were not a compact set, an additional requirement would need to be satisfied for $h(t)$ to be admissible, namely:

$$\mathbb{E}\left[\int_0^T |h(t)|^k dt\right] < \infty, \quad k = 1, 2, \ldots$$

State Process Dynamics

Assume the dynamics of the state process $X(t)$, under a measure $\mathbb{P}$, is given by the SDE:

$$\begin{aligned} dX(t) &= f(t, X(t^-), h(t))ds + \Lambda(t)dW_t \\ &\quad + \int_{\mathbf{Z}} \xi(z)\tilde{N}_{\mathbf{p}}(dt, dz), \\ X(0) &= x \end{aligned}$$

with $X(t) \in \mathbb{R}^n$, $t \in [0, T]$ and where f defined on $\mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{K}$ is continuous in all its arguments and satisfies the linear growth condition

$$|f(t, x, h)| \leq \bar{C}_f (1 + |t| + |x|) \tag{5.1}$$

$\forall (t, x) \in [t_0, T) \times \mathbb{R}^n$, for some constant $\bar{C}_f > 0$ and where f and Λ defined on $\mathbb{R}^+$ satisfy the Lipschitz condition

$$|f(t, x, h) - f(s, y, h)| + |\Lambda(t, x, h) - \Lambda(s, y, h)| \leq C\,[|t - s| + |x - y|] \tag{5.2}$$

for some constant $C > 0$.

The sets $\mathbf{Z}$ and $\mathbf{Z_0}$, the Poisson point process $N_{\mathbf{p}}(dt, dz)$, the compensator $\nu(dz)$ and the compensated process process $\tilde{N}_{\mathbf{p}}(dt, dz)$ are as in Chapter 4. We also assume that

(i). $\xi_i(z)$ is bounded. More specifically, $-\infty < \xi_i^{\min} \leq \xi_i(z) \leq \xi_i^{\max} < \infty \quad \forall z \in \mathbf{Z}$ a.s. $d(\nu)$.

(ii). the vector-valued function $\xi(z)$ satisfies:

$$\int_{\mathbf{Z}_0} |\xi(z)|^2 \nu(dz) < \infty \tag{5.3}$$

(see for example Definition II.4.1 in Ikeda and Watanabe [31] where $\mathbf{F_P}$ and $\mathbf{F_P^{2,loc}}$ are respectively given in equations II(3.2) and II(3.5))

Remark 5.3. For the uniqueness result derived in Section 5 to hold, we will assume that the measure ν is finite. In that case assumption (5.3) above would be automatically satisfied.

5.2.2 The Optimal Control Problem

Consider the criterion function

$$I(x;h;t,T) = -\frac{1}{\theta} \ln \mathbf{E}_{t,x} \left[\exp \left\{ \theta \int_t^T g(s, X_s, h(s)) ds + \theta g_T(X_T;T) \right\} \right] \tag{5.4}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is the running cost and $g_T : \mathbb{R}^n \to \mathbb{R}$ is the terminal cost. Moreover, we assume that

(i). g is Lipschitz continuous in x and t and continuous in h;

(ii). g_T is continuous on $\mathbb{R}^n$.

This criterion function is well defined on the reference probability system

$$\upsilon = (\Omega, \{\mathcal{F}_t\}, \mathbb{P}, W, N_{\mathbf{p}}, h(t))$$

Let $\mathcal{V}$ be the set of reference probability systems.

Let Φ be the value function for the auxiliary criterion function $I(v, x; h; t, T)$, defined as

$$\Phi(t,x) = \sup_{v \in \mathcal{V}} I(x; h; t, T), \quad (t,x) \in [0,T] \times \mathbb{R}^n$$

Touzi (see p.8 of [43]) pointed out that, contrary to the criterion function $I(x; h; t, T)$, the value function Φ does not depend on past information and is therefore Markovian.

The RS HJB PIDE associated with the value function Φ of the control problem is:

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in J} L_t^h \Phi = 0 \tag{5.5}$$

subject to terminal condition

$$\Phi(T,x) = -g_T(x;T) \tag{5.6}$$

where

$$\begin{aligned} L_t^h \Phi &= f(t,x,h)' D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t) D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda(t)' D\Phi \\ &+ \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta}\left(e^{-\theta(\Phi(t,x+\xi(z))-\Phi(t,x))} - 1\right) - \xi'(z) D\Phi \right\} \nu(dz) - g(t,x,h) \end{aligned} \tag{5.7}$$

Denote by $F(t,x,p,A)$ the functional

$$F(t,x,p,A) := -\sup_{h \in J} \left\{ f(t,x,h)'p + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t)A\right) - \frac{\theta}{2} p'\Lambda\Lambda(t)'p - g(t,x,h) \right\} \tag{5.8}$$

and by $\mathcal{I}^{RS}[u(t,x)]$ the nonlocal operator

$$\mathcal{I}^{RS}[u(t,x)] := \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(u(t,x+\xi(z))-u(t,x))} - 1 \right) - \xi'(z) Du \right\} \nu(dz) \tag{5.9}$$

Finally, fix $h \in J$ and denote by $\mathcal{L}^h$ the integro-differential generator of the state variable X:

$$\begin{aligned} \mathcal{L}^h \varphi &= f(t,x,h)' D\varphi + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda'(t) D^2 \varphi \right) \\ &\quad + \int_{\mathbf{Z}} \left\{ \varphi(x+\xi(z)) - \varphi(x) - \xi'(z) D\varphi \right\} \nu(dz) \end{aligned} \tag{5.10}$$

In this setting, we remark that $\Phi(t,x)$ is locally bounded on $[0,T] \times \mathbb{R}^n$ since g is itself locally bounded on $[0,T] \times \mathbb{R}^n \times J$.

5.2.3 The Exponentially Transformed Control Problem

To prove our viscosity-related results, we will find it easier to construct an argument based on the exponentially transformed value function $\tilde{\Phi}(t,x) := e^{-\theta\Phi(t,x)}$ which satisfies the RS HJB PIDE:

$$\begin{aligned} & -\frac{\partial \tilde{\Phi}}{\partial t} - \inf_{h \in \mathcal{H}} \left\{ f(t,x,h)' D\tilde{\Phi} + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda'(t) D^2 \tilde{\Phi} \right) \right. \\ & \quad + \int_{\mathbf{Z}} \left\{ \tilde{\Phi}(t,x+\xi(z)) - \tilde{\Phi}(t,x) - \xi'(z) D\tilde{\Phi} \right\} \nu(dz) \\ & \quad \left. + \theta g(t,x,h) \tilde{\Phi} \right\} \\ = \; & 0 \end{aligned} \tag{5.11}$$

subject to terminal condition

$$\tilde{\Phi}(T,x) = e^{\theta g_T(x;T)} \tag{5.12}$$

The objective of this transformation, which is equivalent to the more general method proposed by Duffie and Lions [23], is to suppress the quadratic growth term $(D\Phi)'\Lambda\Lambda' D\Phi$ by trading it off against a new linear term: $\theta g\tilde{\Phi}$, thus transforming the quasi-linear PDE (5.5) into the semi-linear PDE (5.11).

By analogy with our original control problem, we denote by $\tilde{F}(t,x,r,p,M)$ the functional

$$\begin{aligned}\tilde{F}(t,x,r,p,M) &:= -\inf_{h\in J}\left\{f(t,x,h)'p+\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t)M\right)+\theta g(t,x,h)r\right\}\\ &= +\sup_{h\in J}\left\{-f(t,x,h)'p-\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t)M\right)-\theta g(t,x,h)r\right\}\end{aligned} \tag{5.13}$$

Remark 5.4. In the pure diffusion case studied up to Chapter 3, Φ has the quadratic form $\Phi(t,x)=\frac{1}{2}x'Q(t)x+q(t)x+k(t)$ where $Q(t)>0$. Hence, in this case, the value function for the exponentially transformed problem $\tilde{\Phi}$ has functional form $\tilde{\Phi}(t,x)=e^{-\frac{\theta}{2}x'Q(t)x-\theta q(t)x-\theta k(t)}$. $\tilde{\Phi}$ is therefore bounded $\forall(t,x)\in[0,T]\times\mathbb{R}^n$ and $lim_{|x|\to\infty}\tilde{\Phi}(t,x)=0$.

5.2.4 Definitions

Before proceeding further, we will introduce the following definition:

Definition 5.5. The upper semicontinuous envelope $u^*(x)$ of a function u at x is defined as

$$u^*(x)=\limsup_{y\to x}u(y)$$

and the lower semicontinuous envelope $u_*(x)$ of $u(x)$ is defined as

$$u_*(x) = \liminf_{y \to x} u(y)$$

Note in particular the fundamental inequality between a function and its upper and lower semicontinuous envelopes:

$$u_* \leq u \leq u^*$$

Definition 5.6 (Parabolic Semijets)**.** Let $u \in USC([0,T] \times \mathbb{R}^n)$ and $(t,x) \in [0,T] \times \mathbb{R}^n$. We define:

- the Parabolic superjet $\mathcal{P}^{2,+}u(t,x)$ as

$$\begin{aligned} \mathcal{P}^{2,+}u(t,x) \;:=\; & \{(p,q,A) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n : \\ & u(s,y) \leq u(s,x) + p(s-t) + \langle q, y-x \rangle + \frac{1}{2} \langle A(y-x), y-x \rangle \\ & + o(|s-t| + |y-x|^2) \text{ as } (s,y) \to (t,x)\} \end{aligned}$$

- the closure of the Parabolic superjet $\overline{\mathcal{P}}^{2,+}u(t,x)$ as

$$\begin{aligned} \overline{\mathcal{P}}^{2,+}u(t,x) \;:=\; & \Big\{(p,q,A) = \lim_{k \to \infty} (p_k, q_k, A_k) \text{ with } (p_k, q_k, A_k) \in \mathcal{P}^{2,+}u(t,x) \\ & \text{and } \lim_{k \to \infty} (t_k, x_k, u(t_k, x_k)) = (x, t, u(t,x))\Big\} \end{aligned}$$

Let $u \in LSC([0,T] \times \mathbb{R}^n)$ and $(t,x) \in [0,T] \times \mathbb{R}^n$. We define:

- the Parabolic subjet $\mathcal{P}^{2,-}u(t,x)$ as $\mathcal{P}^{2,-}u(t,x) := -\mathcal{P}^{2,+}u(t,x)$, and;

- the closure of the Parabolic subjet $\overline{\mathcal{P}}^{2,-}u(t,x)$ as $\overline{\mathcal{P}}^{2,-}u(t,x) = -\overline{\mathcal{P}}^{2,+}u(t,x)$

The purpose of the next two conditions is to ensure that the jump term is semicontinuous at any given point $(t,x) \in [0,T] \times \mathbb{R}^n$ (see Lemma 1 and Conditions (6) and (7) in [1]). In our setting, we note that since the value function Φ and the function $x \mapsto e^x$ are locally bounded, these two conditions are satisfied.

Condition 5.7 (Condition on an Upper Semicontinuous Function u)**.** Let $(t,x) \in [0,T] \times \mathbb{R}^n$ and $(p,q,A) \in \mathcal{P}^{2,+}u(t,x)$, there are $\varphi \in C(\mathbb{R}^n)$, $\varphi \geq 1$ and $R > 0$ such that for

$$((s,y),z) \in (\mathscr{B}_R(t,x) \cap ([0,T] \times \mathbb{R}^n)) \times \mathbf{Z},$$

$$\int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(u(s,y+\xi(z))-u(s,y))} - 1 \right) - \xi'(z)q \right\} \nu(dz) \leq \varphi(y)$$

Condition 5.8 (Condition on a Lower Semicontinuous Function u)**.** Let $(t,x) \in [0,T] \times \mathbb{R}^n$ and $(p,q,A) \in \mathcal{P}^{2,-}u(t,x)$, there are $\varphi \in C(\mathbb{R}^n)$, $\varphi \geq 1$ and $R > 0$ such that for

$$((s,y),z) \in (\mathscr{B}_R(t,x) \cap ([0,T] \times \mathbb{R}^n)) \times \mathbf{Z},$$

$$\int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(u(s,y+\xi(z))-u(s,y))} - 1 \right) - \xi'(z)q \right\} \nu(dz) \geq -\varphi(y)$$

Remark 5.9. Note that the jump-related integral term

$$\int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(u(s,y+\xi(z))-u(s,y))} - 1 \right) - \xi'(z)q \right\} \nu(dz)$$

is well defined when $(p, q, A) \in \mathcal{P}^{2,\pm}u(t,x)$. First, by Taylor,

$$\begin{aligned} & \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(u(s,y+\xi(z))-u(s,y))} - 1 \right) - \xi'(z)q \right\} \nu(dz) \\ = & \int_{\mathbf{Z}} \left\{ (u(s,y+\xi(z)) - u(s,y)) - \frac{\theta}{2} \left(u(s,y+\xi(z)) - u(s,y) \right)^2 \right. \\ & \left. + \frac{\theta^2}{3!} \left(u(s,y+\xi(z)) - u(s,y) \right)^3 + \ldots - \xi'(z)q \right\} \nu(dz) \end{aligned}$$

By definition of the Parabolic superjet $\mathcal{P}^{2,+}u(t,x)$, for $t = s$, the pair (q, A) satisfies the inequality

$$u(s,y+\xi(z)) - u(s,y) - \xi'(z)q \leq \frac{1}{2}\xi'(z)A\xi(z) + o(|\xi(z)|^2)$$

Similarly, by definition of the Parabolic subjet $\mathcal{P}^{2,-}u(t,x)$, for $t = s$, the pair (q, A) satisfies the inequality

$$u(s,y+\xi(z)) - u(s,y) - \xi'(z)q \geq \frac{1}{2}\xi'(z)A\xi(z) + o(|\xi(z)|^2)$$

Thus, if u is a viscosity solution, we have

$$u(s,y+\xi(z)) - u(s,y) - \xi'(z)q = \frac{1}{2}\xi'(z)A\xi(z) + o(|\xi(z)|^2)$$

and the jump-related integral is equal to

$$\begin{aligned} & \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(u(s,y+\xi(z))-u(s,y))} - 1 \right) - \xi'(z)q \right\} \nu(dz) \\ = & \int_{\mathbf{Z}} \left\{ -\frac{\theta}{2} \left(u(s,y+\xi(z)) - u(s,y) \right)^2 + \frac{1}{2}\xi'(z)A\xi(z) + o(|\xi(z)|^2) + o(\theta^2) \right\} \nu(dz) \end{aligned}$$

which is well-defined.

We now give two equivalent definitions of viscosity solutions adapted from Alvarez and Tourin [1]:

- a definition based on the notion of semijets;
- a definition based on the notion of test function.

Before introducing these two definitions, we need to define parabolic semijet of upper semicontinuous and lower semicontinuous functions and to add two additional conditions.

Definition 5.10 (Viscosity Solution (Semijets))**.** A locally bounded function $u \in USC([0,T] \times \mathbb{R}^n)$ satisfying Condition 5.7 is a viscosity subsolution of (5.5), if for all $x \in \mathbb{R}^n$, $u(T,x) \leq -g_T(x)$, and for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $(p,q,A) \in \mathcal{P}^{2,+}u(t,x)$, we have

$$-p + F(t,x,q,A) - \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta}\left(e^{-\theta(u(t,x+\xi(z))-u(t,x))} - 1\right) - \xi'(z)q \right\} \nu(dz) \leq 0$$

A locally bounded function $u \in LSC([0,T] \times \mathbb{R}^n)$ satisfying Condition 5.8 is a viscosity supersolution of (5.5), if for all $x \in \mathbb{R}^n$, $u(T,x) \geq -g_T(x)$, and for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $(p,q,A) \in \mathcal{P}^{2,-}u(t,x)$, we have

$$-p + F(t,x,q,A) - \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta}\left(e^{-\theta(u(t,x+\xi(z))-u(t,x))} - 1\right) - \xi'(z)q \right\} \nu(dz) \geq 0$$

A locally bounded function Φ whose upper semicontinuous and lowersemicontinuous envelopes are a viscosity subsolution and a viscosity supersolution of (5.5) is a viscosity solution of (5.5).

Definition 5.11 (Viscosity Solution (Test Functions)). A locally bounded function $u \in USC([0,T] \times \mathbb{R}^n)$ is a viscosity subsolution of (5.5), if for all $x \in \mathbb{R}^n$, $u(T,x) \leq -g_T(x)$, and for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $\psi \in C^2([0,T] \times \mathbb{R}^n)$ such that $u(t,x) = \psi(t,x)$, $u < \psi$ on $[0,T] \times \mathbb{R}^n \backslash \{(t,x)\}$, we have

$$-\frac{\partial \psi}{\partial t} + F(t,x,D\psi,D^2\psi) - \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(\psi(t,x+\xi(z))-\psi(t,x))} - 1 \right) - \xi'(z) D\psi \right\} \nu(dz) \leq 0$$

A locally bounded function $v \in LSC([0,T] \times \mathbb{R}^n)$ is a viscosity supersolution of (5.5), if for all $x \in \mathbb{R}^n$, $v(T,x) \geq -g_T(x)$, and for all $(t,x) \in [0,T] \times \mathbb{R}^n$, $\psi \in C^2([0,T] \times \mathbb{R}^n)$ such that $v(t,x) = \psi(t,x)$, $v > \psi$ on $[0,T] \times \mathbb{R}^n \backslash \{(t,x)\}$, we have

$$-\frac{\partial \psi}{\partial t} + F(t,x,D\psi,D^2\psi) - \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(\psi(t,x+\xi(z))-\psi(t,x))} - 1 \right) - \xi'(z) D\psi \right\} \nu(dz) \geq 0$$

A locally bounded function Φ whose upper semicontinuous and lower semicontinuous envelopes are a viscosity subsolution and a viscosity supersolution of (5.5) is a viscosity solution of (5.5).

We would have similar definition for the viscosity supersolution, subsolution and solution of equation (5.11). Once again, the superjet and test function formulations are strictly equivalent (see Alvarez and Tourin [1] and Crandall, Ishii and Lions [19]).

Remark 5.12. Alvarez and Tourin [1] showed that Conditions 5.7 and 5.8 are key in guaranteeing that Definitions 5.10 and 5.11 are indeed equivalent.

Remark 5.13. An alternative, more classical, but also more restrictive definition of viscosity solution is as the continuous function which is both a supersolution and a subsolution of (5.5) (see Definition 5.1 in Barles [6]). The line of reasoning we will follow will make full use of the latitude afforded by our definition and we will have to

wait until the comparison result is established in Section 5.5 to prove the continuity of the viscosity solution.

5.3 Properties of the Value Function of the Exponentially Transformed Problem

5.3.1 Growth Bounds

The following assumption relates to the behaviour of the value function $\tilde{\Phi}$ of the exponentially transformed stochastic control problem.

Assumption 5.14. If

(i). $g(t,x,h)$ is locally bounded on $[0,T]\times\mathbb{R}^n\times J$ and is Lipschitz continuous in x;

(ii). *Lower Bound.*

$$\tilde{\Phi}(t,x)\geq 0\ \forall x\in\overline{\mathbb{R}^n}$$

and

$$\tilde{\Phi}(t,x)>0\ \forall x\in\mathbb{R}^n$$

(iii). *Upper Bounds.* for the $2n$ constant controls $\bar{h}^k, k=1,\ldots,2n$ parameterizing the $4n$ functions

$$\alpha^k : t\in[0,T]\to\mathbb{R},\ k=1,\ldots,2n$$

$$\beta^k : t\in[0,T]\to\mathbb{R}^n,\ k=1,\ldots,2n$$

such that for $i = 1, \ldots, n$,

$$\begin{aligned} \beta_{2i-1}(t) &< 0 \\ \beta_{2i}(t) &> 0 \end{aligned}$$

where $\beta_i^k(t)$ denotes the i-th component of the vector $\beta^k(t)$, we have the following upper bounds:

$$\tilde{\Phi}(t,x) \leq e^{\alpha^k(t)+\beta^{k'}(t)x}$$

for each element $x_i, i = 1, \ldots, n$ of x.

Moreover, the following hold as a result of these assumptions:

(i). *Global Upper Bound.* There exists a constant $C_{\tilde{\Phi}} > 0$ such that

$$\tilde{\Phi}(t,x) \leq C_{\tilde{\Phi}}$$

(ii). *Asymptotic Behaviour.* $\lim_{|x|\to\infty} \tilde{\Phi}(t,x) = 0 \ \forall t \in [0,T]$ and this convergence is monotone in x and occurs at an exponential rate.

Remark 5.15. As we will see later, these conditions are satisfied in the asset management case under some very weak conditions.

5.3.2 Viscosity Properties

The theory of viscosity solutions was initially developed for elliptical PDEs of the form

$$H(t,x,u,Du,D^2u) = 0$$

and parabolic PDEs of the form

$$\frac{\partial u}{\partial t} + H(t, x, u, Du, D^2u) = 0$$

for what Crandall, Ishii and Lions [19] termed a "proper" functional $H(t, x, r, p, A)$.

Definition 5.16. A functional $H(t, x, r, p, A)$ is said to be *proper* if it satisfies the following two properties:

1. (degenerate) ellipticity:

$$H(t, x, r, p, A) \leq H(t, x, r, p, B), \qquad B \leq A$$

and

2. monotonicity

$$H(t, x, r, p, A) \leq H(t, x, s, p, A), \qquad r \leq s$$

In our problem, the functional H is defined as

$$\begin{aligned} H(t, x, p, A) &:= F(t, x, p, A) \\ &\quad + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(\Phi(t, x+\xi(z)) - \Phi(t,x))} - 1 \right) - \xi'(z)p \right\} \nu(dz) \\ &= -\sup_{h \in J} \left\{ f(t, x, h)'p + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'(t)A\right) \right. \\ &\quad -\frac{\theta}{2} p'\Lambda\Lambda(t)'p \\ &\quad + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(\Phi(t, x+\xi(z)) - \Phi(t,x))} - 1 \right) - \xi'(z)p \right\} \nu(dz) \\ &\quad \left. -g(t, x, h) \right\} \end{aligned} \tag{5.14}$$

In particular, the diffusion functional F defined in relationship (5.8) is indeed "proper". As a result, we can develop a viscosity approach to show that the value function Φ is the unique solution of the associated RS HJB PIDE.

5.4 The Value Function as a Viscosity Solution

To show that the value function is a (discontinuous) viscosity solution of the associated RS HJB PIDE (5.5), we follow an argument by Touzi [43] which enables us to make a greater use of control theory in the derivation of the proof.

Theorem 5.17. *Assume that*

(i). g is locally bounded on $[0,T] \times \mathbb{R}^n \times J$;

(ii). Φ is locally bounded on $[0,T] \times \mathbb{R}^n$;

(iii). $f(t,x,h)$ and $g(t,x,h)$ are continuous in $(t,x) \in [0,T] \times \mathbb{R}^n$ for fixed $h \in J$;

(iv). F is continuous in $(t,x) \in [0,T] \times \mathbb{R}^n$.

then Φ is a (discontinuous) viscosity solution of the RS HJB PIDE (5.5) *on* $[0,T] \times \mathbb{R}^n$, *subject to terminal condition* (5.6).

Proof. **Outline -** This proof can be decomposed in five steps. First, we define $\tilde{\Phi}$ as an exponential transformation of Φ. In the next three steps, we prove that $\tilde{\Phi}$ is a viscosity solutions of the exponentially transformed RS HJB PIDE by showing that it is 1). a viscosity subsolution, 2). a viscosity supersolution and hence 3). a viscosity solution. Finally, applying a change of variable result, such as Proposition 2.2 in [43], we conclude that Φ is a viscosity solution of the RS HJB PIDE (5.5)

Step 1: Exponential Transformation

In order to prove that the value function Φ is a (discontinuous) viscosity solution of (5.5), we will start by proving that the exponentially transformed value function $\tilde{\Phi}$ is a (discontinuous) viscosity solution of (5.11).

Step 2: Viscosity Subsolution

Let $(t_0, x_0) \in Q := [0, t] \times \mathbb{R}^n$ and $u \in C^{1,2}(Q)$ satisfy

$$0 = (\tilde{\Phi}^* - u)(t_0, x_0) = \max_{(x,t)\in Q}(\tilde{\Phi}^*(t, x) - u(t, x)) \tag{5.15}$$

and hence

$$\tilde{\Phi} \leq \tilde{\Phi}^* \leq u \tag{5.16}$$

on Q

Let (t_k, x_k) be a sequence in Q such that

$$\lim_{k\to\infty}(t_k, x_k) = (t_0, x_0)$$

$$\lim_{k\to\infty} \tilde{\Phi}(t_k, x_k) = \tilde{\Phi}^*(t_0, x_0)$$

and define the sequence $\{\xi\}_k$ as $\xi_k := \tilde{\Phi}(t_k, x_k) - u(t_k, x_k)$. Since u is of class $C^{1,2}$, $\lim_{k\to\infty} \xi_k = 0$.

Fix $h \in J$ and consider a constant control $\hat{h} = h$. Denote by X^k the state process with initial data $X^k_{t_k} = x_k$ and, for $k > 0$, define the stopping time

$$\tau_k := \inf \left\{ s > t_k : (s - t_k, X^k_s - x_k) \notin [0, \delta_k) \times \alpha \mathscr{B}_n \right\}$$

for a given constant $\alpha > 0$ and where $\mathscr{B}_n$ is the unit ball in $\mathbb{R}^n$ and

$$\delta_k := \sqrt{\xi_k} \left(1 - 1_{\{0\}}(\xi_k)\right) + k^{-1} 1_{\{0\}}(\xi_k)$$

From the definition of τ_k, we see that $\lim_{k \to \infty} \tau_k = t_0$.

By the Dynamic Programming Principle,

$$\tilde{\Phi}(t_k, x_k) \;\leq\; \mathbf{E}_{t_k, x_k} \left[\exp \left\{ \theta \int_{t_k}^{\tau_k} g(s, X_s, \hat{h}_s) ds \right\} \tilde{\Phi}(\tau_k, X^k_{\tau_k}) \right]$$

where $\mathbf{E}_{t_k, x_k}[\cdot]$ represents the expectation under the measure $\mathbb{P}$ given initial data (t_k, x_k).

By inequality (5.16),

$$\tilde{\Phi}(t_k, x_k) \;\leq\; \mathbf{E}_{t_k, x_k} \left[\exp \left\{ \theta \int_{t_k}^{\tau_k} g(s, X_s, \hat{h}_s) ds \right\} u(\tau_k, X^k_{\tau_k}) \right]$$

and hence by definition of ξ_k,

$$u(t_k, x_k) + \xi_k \;\leq\; \mathbf{E}_{t_k, x_k} \left[\exp \left\{ \theta \int_{t_k}^{\tau_k} g(s, X_s, \hat{h}_s) ds \right\} u(\tau_k, X^k_{\tau_k}) \right]$$

i.e.

$$\xi_k \leq \mathbf{E}_{t_k,x_k}\left[\exp\left\{\theta\int_{t_k}^{\tau_k} g(s,X_s,\hat{h}_s)ds\right\}u(\tau_k,X^k_{\tau_k})\right]-u(t_k,x_k)$$

Define $Z(t_k)=\theta\int_{t_k}^{\tau_k} g(s,X_s,\hat{h}_s)ds$, then

$$d\left(e^{Z_s}\right):=\theta g(s,X_s,\hat{h}_s)e^{Z_s}ds$$

Also, by Itô,

$$\begin{aligned} du_s &= \left\{\frac{\partial u}{\partial s}+\mathcal{L}u\right\}ds+Du'\Lambda(s)dW_s \\ &\quad+\int_{\mathbf{Z}}\left\{u\left(s,X(s^-)+\xi(z)\right)-u\left(s,X(s^-)\right)\right\}\tilde{N}_{\mathbf{p}}(ds,dz)\end{aligned}$$

for $s\in[t_k,\tau_k]$.

By the Itô product rule, and since $dZ_s\cdot u_s=0$, we get

$$d\left(u_se^{Z_s}\right)=u_sd\left(e^{Z_s}\right)+e^{Z_s}du_s$$

and hence for $t\in[t_k,\tau_k]$

$$\begin{aligned} u(t,X^k_t)e^{Z_t} &= u(t_k,x_k)e^{Z_{t_k}}+\theta\int_{t_k}^{t}u(s,X^k_s)g(s,X^k_s,\hat{h}_s)e^{Z_s}ds \\ &\quad+\int_{t_k}^{t}\left(\frac{\partial u}{\partial s}(s,X^k_s)+\mathcal{L}u(s,X^k_s)e^{Z_s}\right)ds+\int_{t_k}^{t}Du'\Lambda(s)dW_s \\ &\quad+\int_{t_k}^{t}\int_{\mathbf{Z}}\left\{u\left(t,X^k(s^-)+\xi(z)\right)-u\left(t,X^k(s^-)\right)\right\}\tilde{N}_{\mathbf{p}}(dt,dz)\end{aligned}$$

Noting that $u(t_k,x_k)e^{Z_{t_k}}=u(t_k,x_k)$ and taking the expectation with respect to

the initial data (t_k, x_k), we get

$$\begin{aligned} & \mathbf{E}_{t_k,x_k}\left[u(t,X_t)e^{Z_t}\right] \\ = \; & u(t_k,x_k)e^{Z_{t_k}} + \mathbf{E}_{t_k,x_k}\left[\int_{t_k}^{t}\left(\frac{\partial u}{\partial s}(s,X_s) + \mathcal{L}u(s,X_s) + \theta u(s,X_s)g(s,X_s,\hat{h}_s)\right)e^{Z_s}ds\right] \end{aligned}$$

In particular, for $t = \tau_k$,

$$\begin{aligned} \xi_k &\leq \mathbf{E}_{t_k,x_k}\left[u(\tau_k,X_{\tau_k})e^{Z_{\tau_k}}\right] - u(t_k,x_k)e^{Z_{t_k}} \\ &= +\mathbf{E}_{t_k,x_k}\left[\int_{t_k}^{\tau_k}\left(\frac{\partial u}{\partial s}(s,X_s) + \mathcal{L}u(s,X_s) + \theta u(s,X_s)g(s,X_s,\hat{h}_s)\right)e^{Z_s}ds\right] \end{aligned}$$

and thus

$$\begin{aligned} \frac{\xi_k}{\delta_k} &\leq \frac{1}{\delta_k}\left(\mathbf{E}_{t_k,x_k,}\left[u(\tau_k,X_{\tau_k})e^{Z_{\tau_k}}\right] - u(t_k,x_k)e^{Z_{t_k}}\right) \\ &= \frac{1}{\delta_k}\left(\mathbf{E}_{t_k,x_k}\left[\int_{t_k}^{\tau_k}\left(\frac{\partial u}{\partial s}(s,X_s) + \mathcal{L}u(s,X_s) + \theta u(s,X_s)g(s,X_s,\hat{h}_s)\right)e^{Z_s}ds\right]\right) \end{aligned}$$

As $k \to \infty$, $t_k \to t_0$, $\tau_k \to t_0$, $\frac{\xi_k}{\delta_k} \to 0$ and

$$\begin{aligned} & \frac{1}{\delta_k}\left(\mathbf{E}_{t_k,x_k}\left[\int_{t_k}^{t}\left(\frac{\partial u}{\partial s}(s,X_s) + \mathcal{L}u(s,X_s) + \theta u(s,X_s)g(s,X_s,\hat{h}_s)\right)e^{Z_s}ds\right]\right) \\ \to \; & \frac{\partial u}{\partial s}(s,X_s) + \mathcal{L}u(s,X_s) + \theta u(s,X_s)g(s,X_s,\hat{h}_s) \end{aligned}$$

a.s. by the Bounded Convergence Theorem, since the random variable

$$\frac{1}{\delta_k}\int_{t_k}^{t}\left(\frac{\partial u}{\partial s}(s,X_s) + \mathcal{L}u(s,X_s) + \theta u(s,X_s)g(s,X_s,\hat{h}_s)\right)e^{Z_s}ds$$

is bounded for large enough k.

Hence, we conclude that since $\hat{h}_s$ is arbitrary,

$$\frac{\partial u}{\partial s}(s, X_s) + \mathcal{L}u(s, X_s) + \theta u(s, X_s) g(s, X_s, \hat{h}_s) \geq 0$$

.

i.e.

$$-\frac{\partial u}{\partial s}(s, X_s) - \mathcal{L}u(s, X_s) - \theta u(s, X_s) g(s, X_s, \hat{h}_s) \leq 0$$

This argument proves that V is a (discontinuous) viscosity subsolution of the PDE (5.11) on $[0, t) \times \mathbb{R}^n$ subject to terminal condition $\tilde{\Phi}(T, x) = e^{\theta g_T(x;T)}$.

Step 3: Viscosity Supersolution

This step in the proof is a slight adaptation of the proof for classical control problems in Touzi [43]. Let $(t_0, x_0) \in Q$ and $u \in C^{1,2}(Q)$ satisfy

$$0 = (\tilde{\Phi}_* - u)(t_0, x_0) < (\tilde{\Phi}_* - u)(t, x) \text{ for } Q \backslash (t_0, x_0) \tag{5.17}$$

We intend to prove that at (t_0, x_0)

$$\frac{\partial u}{\partial t}(t, x) + \inf_{h \in \mathcal{H}} \left\{ \mathcal{L}^h u(t, x) - \theta g(t, x, h) \right\} \leq 0$$

by contradiction. Thus, assume that

$$\frac{\partial u}{\partial t}(t, x) + \inf_{h \in \mathcal{H}} \left\{ \mathcal{L}^h u(t, x) - \theta g(t, x, h) \right\} > 0 \tag{5.18}$$

at (t_0, x_0).

Since $\mathcal{L}^h u$ is continuous, there exists an open neighbourhood $\mathcal{N}_\delta$ of (t_0, x_0) defined

for $\delta > 0$ as

$$\mathcal{N}_\delta := \{(t,x) : (t-t_0, x-x_0) \in (-\delta,\delta) \times \delta\mathscr{B}_n, \text{ and (5.18) holds}\} \tag{5.19}$$

Note that by (5.17) and since $\Phi > \Phi_* > u$,

$$\min_{Q\setminus\mathcal{N}_\delta} \left(\tilde{\Phi} - u\right) > 0$$

For $\rho > 0$, consider the set J^ρ of ρ-optimal controls h^ρ satisfying

$$\tilde{I}(t_0, x_0, h^\rho) \leq \tilde{\Phi}(t_0, x_0) + \rho \tag{5.20}$$

Also, let $\epsilon > 0$, $\epsilon \leq \gamma$ be such that

$$\min_{Q\setminus\mathcal{N}_\delta} \left(\tilde{\Phi} - u\right) \geq 3\epsilon e^{-\delta\theta M_\delta} > 0 \tag{5.21}$$

where M_δ is defined as

$$M_\delta := \max_{(t,x)\in\mathcal{N}_\delta^J, h\in J^\rho} (-g(x,h), 0)$$

for

$$\mathcal{N}_\delta^J := \{(t,x) : (t-t_0, x-x_0) \in (-\delta,\delta) \times (\zeta+\delta)\mathscr{B}_n\} \tag{5.22}$$

and

$$\zeta := \max_{z\in\mathbb{Z}} \|\xi(z)\|$$

Note that $\zeta < \infty$ by boundedness of $\xi(z)$ and thus $M_\delta < \infty$.

Now let (t_k, x_k) be a sequence in $\mathcal{N}_\delta$ such that

$$\lim_{k\to\infty}(t_k, x_k) = (t_0, x_0)$$

and

$$\lim_{k\to\infty}\tilde{\Phi}(t_k, x_k) = \tilde{\Phi}_*(t_0, x_0)$$

Since $(\tilde{\Phi} - u)(t_k, x_k) \to 0$, we can assume that the sequence (t_k, x_k) satisfies

$$|(\tilde{\Phi} - u)(t_k, x_k)| \leq \epsilon, \qquad \text{for } k \geq 1 \tag{5.23}$$

for ϵ defined by (5.21)

Consider the ϵ-optimal control h_k^ϵ, denote by $\tilde{X}_k^\epsilon$ the controlled process defined by the control process h_k^ϵ and introduce the stopping time

$$\tau_k := \inf\left\{s > \tau_k : (s, \tilde{X}_k^\epsilon(s)) \notin \mathcal{N}_\delta\right\}$$

Note that since we assumed that $-\infty \leq \xi_i^{\min} \leq \xi_i \leq \xi_i^{\max} < \infty$ for $i = 1, \ldots, n$ and since ν is assumed to be bounded then $X(\tau)$ is also finite and in particular,

$$(\tilde{\Phi} - u)(\tau_k, \tilde{X}_k^\epsilon(\tau_k)) \geq (\tilde{\Phi}_* - u)(\tau_k, \tilde{X}_k^\epsilon(\tau_k)) \geq 3\epsilon e^{-\delta\theta M_\delta} \tag{5.24}$$

Choose $\mathcal{N}_\delta^J$ so that $(\tau, \tilde{X}^\epsilon(\tau)) \in \mathcal{N}_\delta^J$. In particular, since $X^\epsilon(\tau)$ is finite then $\mathcal{N}_\delta^J$ can be defined to be a strict subset of Q and we can effectively use the local boundedness of g to establish M_δ.

Let $Z(t_k) = \theta \int_{t_k}^{\bar{\tau}_k} g(s, \tilde{X}_s^\epsilon, h_s^\epsilon) ds$, since $\tilde{\Phi} \geq \tilde{\Phi}_*$ and by (5.23) and (5.24),

$$\begin{aligned}
& \tilde{\Phi}(\tau_k, \tilde{X}_k^\epsilon(\tau_k)) e^{Z(\tau_k)} - \tilde{\Phi}(t_k, x_k) e^{Z(t_k)} \\
\geq \; & u(\tau_k, \tilde{X}_k^\epsilon(\tau_k)) e^{Z(\tau_k)} - \tilde{\Phi}(t_k, x_k) e^{Z(t_k)} + 3\epsilon e^{-\delta\theta M_\delta} e^{Z(\tau_k)} - \epsilon \\
\geq \; & \int_{t_k}^{\tau_k} d\left(u(s, \tilde{X}_k^\epsilon(s)) e^{Z_s}\right) + 2\epsilon
\end{aligned}$$

i.e.

$$\tilde{\Phi}(t_k, x_k) \leq \tilde{\Phi}(\tau_k, \tilde{X}_k^\epsilon(\tau_k)) e^{Z(\tau_k)} - \int_{t_k}^{\tau_k} d\left(u(s, \tilde{X}_k^\epsilon(s)) e^{Z_s}\right) - 2\epsilon$$

Taking expectation with respect to the initial data (t_k, x_k),

$$\tilde{\Phi}(t_k, x_k) \leq \mathbf{E}_{t_k, x_k}\left[\tilde{\Phi}(\tau_k, \tilde{X}_k^\epsilon(\tau_k)) e^{Z(\tau_k)} - \int_{t_k}^{\tau_k} d\left(u(s, \tilde{X}_k^\epsilon(s)) e^{Z_s}\right)\right] - 2\epsilon$$

Note that by the Itô product rule,

$$\begin{aligned}
& d\left(u(s, \tilde{X}_k^\epsilon(s)) e^{Z_s}\right) \\
= \; & u_s d\left(e^{Z_s}\right) + e^{Z_s} du_s \\
= \; & \frac{\partial u}{\partial t}(t, x) + \mathcal{L}^h u(t, x) + \theta g(t, x, h)
\end{aligned}$$

Since we assumed that

$$-\frac{\partial u}{\partial t}(t, x) - \mathcal{L}^h u(t, x) - \theta g(t, x, h) < 0$$

then

$$-\int_{t_k}^{\tau_k} d\left(u(s, \tilde{X}_k^\epsilon(s)) e^{z_s}\right) < 0$$

and therefore

$$\begin{aligned}
\tilde{\Phi}(t_k, x_k) &\leq \mathbf{E}_{t_k,x_k}\left[\tilde{\Phi}(\tau_k, \tilde{X}_k^\epsilon(\tau_k))e^{Z(\tau_k)} - \int_{t_k}^{\tau_k} d\left(u(s, \tilde{X}_k^\epsilon(s))e^{Z_s}\right)\right] - 2\epsilon \\
&\leq -2\epsilon + \mathbf{E}\left[\exp\left\{\theta\int_{t_k}^{\tau_k} g(X_s, h_k^\epsilon(s))ds\right\}\tilde{\Phi}(\tau_k, \tilde{X}_k^\epsilon(\tau_k))\right] \\
&\leq -2\epsilon + \tilde{I}(t_k, x_k, h_k^\epsilon) \\
&\leq \tilde{\Phi}(t_k, x_k) - \epsilon
\end{aligned}$$

where the third inequality follows from the Dynamic Programming Principle and the last inequality follows from the definition of ϵ-optimal controls (see equation (5.20)).

Hence, equation (5.18),

$$\frac{\partial u}{\partial t}(t,x) + \inf_{h\in\mathcal{H}}\left\{\mathcal{L}^h u(t,x) - \theta g(t,x,h)\right\} > 0$$

is false and we have shown that

$$\frac{\partial u}{\partial t}(t,x) + \inf_{h\in\mathcal{H}}\left\{\mathcal{L}^h u(t,x) - \theta g(t,x,h)\right\} \leq 0$$

This argument therefore proves that $\tilde{\Phi}$ is a (discontinuous) viscosity supersolution of the PDE (5.11) on $[0,t)\times\mathbb{R}^n$ subject to terminal condition $\tilde{\Phi}(T,x) = e^{\theta g_T(x;T)}$.

Step 4: Viscosity Solution

Since $\tilde{\Phi}$ is both a (discontinuous) viscosity subsolution and a supersolution of (5.11), it is a (discontinuous) viscosity.

Step 5: Conclusion

Since by assumption Φ is locally bounded, so is $\tilde{\Phi}$. In addition, $\varphi(x) = e^{-\theta x}$ is of class $C_1(\mathbb{R})$. Also we note that $\frac{d\varphi}{dx} < 0$. By the change of variable property (see for example Proposition 2.2 in Touzi [43]), we see that

1. since $\tilde{\Phi}$ is a (discontinuous) viscosity subsolution of (5.11), $\Phi = \varphi^{-1} \circ \tilde{\Phi}$ is a (discontinuous) viscosity supersolution of (5.5);

2. since $\tilde{\Phi}$ is a (discontinuous) viscosity supersolution of (5.11), $\Phi = \varphi^{-1} \circ \tilde{\Phi}$ is a (discontinuous) viscosity subsolution of (5.5).

and therefore Φ is a (discontinuous) viscosity solution of (5.5) on $[0, t) \times \mathbb{R}^n$ subject to terminal condition $\tilde{\Phi}(T, x) = e^{\theta g_T(x;T)}$.

□

We also note the following corollary:

Corollary 5.18. *(i). Φ^* is a upper semicontinuous viscosity subsolution, and;*

(ii). Φ_* is a lower semicontinuous viscosity supersolution

of the RS HJB PIDE (5.5) *on* $[0, T] \times \mathbb{R}^n$, *subject to terminal condition* (5.6).

As a result of this corollary, we note that Φ^*, Φ_* and Φ are respectively a subsolution, a supersolution, and a solution in the classical sense as expressed in Definitions 5.10 and 5.11.

5.5 Comparison Result

Once we have characterized the class of viscosity solutions associated with a a given problem, the next task is to prove that the problem actually admits a unique viscosity

solution by establishing a comparison theorem.

Comparison theorems are the cornerstone of the application of viscosity theory. Their main use is to prove uniqueness, and in our case continuity, of the viscosity solution. Although a set of, by now fairly standard, techniques can be applied in the proof, the comparison theorem *per se* is generally customized to address both the specificities of the PDE and the requirements of the general problem.

We face three main difficulties in establishing a comparison result for our risk-sensitive control problem. The first difficulty lies in the presence of the jump term and of the compensator ν. Since we will assume that the measure is finite, this can be addressed following the general argument proposed by Alvarez and Tourin [1] and Amadori [2]. The second obstacle is the behaviour of the value function Φ as $\|x\| \to \infty$. In the pure diffusion case or LEQR case solved by Kuroda and Nagai [37], the value function is quadratic in the state and is therefore not bounded for $x \in \mathbb{R}^n$. Consequently, there is no reason to expect the integro-differential RS HJB PIDE (5.5) to be bounded. The third hurdle is the presence of an extra non-linearity: the quadratic growth term $(D\Phi)' \Lambda\Lambda' D\Phi$. This extra non-linearity could, in particular, increases the complexity of the derivation of a comparison result for an unbounded value function. Before dealing with the asymptotic growth condition we will therefore need to address this non-linear term. The traditional solution, an exponential change of variable such as the one proposed by Duffie and Lions [23], is equivalent to the log transformation we used to derive the RS HJB PIDE and again to prove that the value function is a viscosity solution of the RS HJB PIDE. However, the drawback of this method is that, by creating a new zeroth order term equal to

the solution multiplied by the cost function g, it imposes a heavy requirement on g for the PDE to satisfy the monotonicity property of Definition 5.16.

To address these difficulties, we will need to adopt a slightly different strategy from the usual proof template for comparison theorems. In particular, we will exploit the properties of the exponentially transformed value function $\tilde{\Phi}$ stated in Assumption 5.14 and alternate between the log transformed RS HJB PIDE(5.11) and the quadratic growth RS HJB PIDE (5.5) through the proof.

Theorem 5.19 (Comparison Result on an Unbounded State Space). *Assume that:*

(i). $g(t,x,h)$ *is locally bounded on* $[0,T]\times\mathbb{R}^n\times J$ *and is Lipschitz continuous in* x*;*

(ii). for some constants $C,\bar{C}_f,\bar{C}_\Lambda>0$

$$\begin{aligned} |f(t,x,h)| &\leq \bar{C}_f\,(1+|t|+|x|) \\ |\Lambda(t)| &\leq \bar{C}_\Lambda\,(1+|t|) \\ |f(t,x,h)-f(s,y,h)|+|\Lambda(t)-\Lambda(s)| &\leq C\,[|t-s|+|x-y|] \end{aligned}$$

$$\forall (t,x)\in[t_0,T)\times\mathbb{R}^n$$

(iii). $g_0\in C(\mathbb{R}^n)$.

(iv). Assumption 5.14 holds.

Let $\tilde{u}=e^{-\theta v}\in USC([0,T]\times\mathbb{R}^n)$ *be a bounded from above viscosity subsolution of* (5.5) *and* $\tilde{v}=e^{-\theta u}\in LSC([0,T]\times\mathbb{R}^n)$ *be a bounded from below viscosity supersolution of* (5.5). *Then*

$$u\leq v \quad \text{on } [0,T]\times\mathbb{R}^n$$

Proof. **Outline -** This proof can be decomposed in seven steps. In the first step, we perform the usual exponential transformation to rewrite the problem for the value function Φ into a problem for the value function $\tilde{\Phi}$. The rest of the proof is done by contradiction. In step 2, we state the assumption we are planning to disprove. The properties of the value function $\tilde{\Phi}$ related to Assumption 5.14 are used in Step 3 to deduce that it is enough to prove the comparison result for Φ on a bounded state space to reach our conclusion. We then double variables in step 4 before finding modulus of continuity for the diffusion and the jump components respectively in steps 5 and 6. Finally, we reach a contradiction in step 7 and conclude the proof.

Step 1: Exponential Transformation

Let $u \in USC([0,T] \times \mathbb{R}^n)$ be a viscosity subsolution of (5.5) and $v \in LSC([0,T] \times \mathbb{R}^n)$ be a viscosity supersolution of (5.5). Define:

$$\tilde{u} := e^{-\theta v}$$

$$\tilde{v} := e^{-\theta u}$$

By the change of variable property (see for example Proposition 2.2 in Touzi [43]), $\tilde{u}$ and $\tilde{v}$ are respectively a viscosity subsolution and a viscosity supersolution of the RS HJB PIDE (5.11) for the exponentially transformed value function $\tilde{\Phi}$.

Thus, to prove that

$$u \leq v \quad \text{on } [0,T] \times \mathbb{R}^n$$

it is sufficient to prove that

$$\tilde{u} \leq \tilde{v} \quad \text{on } [0,T] \times \mathbb{R}^n$$

Step 2: Setting the Problem

As is usual in the derivation of comparison results, we argue by contradiction and assume that

$$\sup_{[0,T]\times\mathbb{R}^n} [\tilde{u}(t,x) - \tilde{v}(t,x)] > 0 \tag{5.25}$$

Step 3: Taking the Behaviour of the Value Function into Consideration

The assertion of this theorem is that the comparison result holds in the class of functions satisfying Assumption 5.14. As a result, we will concentrate our analysis on subsolutions and supersolutions sharing the same growth properties as the exponentially transformed value function $\tilde{\Phi}$ as stated in Assumption 5.14, namely:

$$0 < \tilde{u}(t,x) \leq e^{\alpha^k(t)+\beta^{k'}(t)x} \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n$$

$$0 < \tilde{v}(t,x) \leq e^{\alpha^k(t)+\beta^{k'}(t)x} \quad \forall (t,x) \in [0,T] \times \mathbb{R}^n$$

and

$$\lim_{|x|\to\infty} \tilde{u}(t,x) = \lim_{|x|\to\infty} \tilde{v}(t,x) = 0 \; \forall t \in [0,T] \tag{5.26}$$

for $k = 1, \ldots, 2n$ where α^k and β^k are the functions given in Assumption 5.14. Since (5.26) holds at an exponential rate, then by Assumption (5.25) there exists

$R > 0$, such that

$$\sup_{[0,T]\times\mathbb{R}^n} [\tilde{u}(t,x) - \tilde{v}(t,x)] = \sup_{[0,T]\times\mathcal{B}_R} [\tilde{u}(t,x) - \tilde{v}(t,x)]$$

Hence, it is enough to show a contradiction with respect to the hypothesis

$$\sup_{Q} [\tilde{u}(t,x) - \tilde{v}(t,x)] > 0 \tag{5.27}$$

established on the set $Q := [0,T] \times \mathcal{B}_R$.

Before proceeding to the next step, we will restate assumption (5.27) in terms of u and v as

$$\sup_{Q} [u(t,x) - v(t,x)] > 0 \tag{5.28}$$

Step 4: Doubling of Variables on the Set Q

Let $\eta > 0$ be such that

$$N := \sup_{Q} [u(t,x) - v(t,x) - \varphi(t)] > 0$$

where $\varphi(t) := \frac{\eta}{t}$.

We will now double variables, a technique commonly used in viscosity solutions literature (see e.g. Crandall, Ishii and Lions [19]). Consider a global maximum point

$(t_\epsilon, x_\epsilon, y_\epsilon) \in (0,T] \times \mathcal{B}_R \times \mathcal{B}_R =: Q_d$ of

$$u(t,x) - v(t,y) - \varphi(t) - \epsilon|x-y|^2$$

and define

$$N_\epsilon := \sup_{Q_d} \left[u(t,x) - v(t,y) - \varphi(t) - \epsilon|x-y|^2\right] > 0$$

Note that $N_\epsilon > 0$ for ϵ large enough. Moreover, $N_\epsilon \geq N$ and $N_\epsilon \downarrow 0$ as $\epsilon \to \infty$.

It is well established (see Lemma 3.1 and Proposition 3.7 in [19]) that along a subsequence

$$\lim_{\epsilon \to \infty} (t_\epsilon, x_\epsilon, y_\epsilon) = (\hat{t}, \hat{x}, \hat{x})$$

for some $(\hat{t}, \hat{x}) \in [0,T] \times \mathbb{R}^n$ which is a maximum point of

$$u(t,x) - v(t,x) - \varphi(t)$$

Via the same argument, we also have

$$\lim_{\epsilon \to \infty} \epsilon|x_\epsilon - y_\epsilon|^2 = 0$$

as well as

$$\lim_{\epsilon \to \infty} u(t_\epsilon, x_\epsilon) = u(\hat{t}, \hat{x})$$

and

$$\lim_{\epsilon \to \infty} v(t_\epsilon, x_\epsilon) = v(\hat{t}, \hat{x})$$

In addition, we note that

$$\lim_{\epsilon \to \infty} N_\epsilon = N$$

Applying Theorem 8.3 in Crandall, Ishii and Lions [19] at $(t_\epsilon, x_\epsilon, y_\epsilon)$, we see that there exists $a_\epsilon, b_\epsilon \in \mathbb{R}$ and $A_\epsilon, B_\epsilon \in \mathcal{S}_n$ such that

$$(a_\epsilon, \epsilon(x_\epsilon - y_\epsilon), A_\epsilon) \in \overline{\mathcal{P}}_u^{2,+}$$

$$(b_\epsilon, \epsilon(x_\epsilon - y_\epsilon), B_\epsilon) \in \overline{\mathcal{P}}_v^{2,-}$$

$$a_\epsilon - b_\epsilon = \varphi'(t_\epsilon)$$

and

$$-3\epsilon \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leq \begin{bmatrix} A_\epsilon & 0 \\ 0 & -B_\epsilon \end{bmatrix} \leq 3\epsilon \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}$$

Thus, we have for the subsolution u

$$-a_\epsilon + F(t_\epsilon, x_\epsilon, \epsilon(x_\epsilon - y_\epsilon), A_\epsilon)$$
$$+ \int_{\mathbf{Z}} \left\{ \frac{1}{\theta} \left(e^{-\theta(u(t_\epsilon, x_\epsilon + \xi(z)) - u(t_\epsilon, x_\epsilon))} - 1 \right) + \epsilon \xi'(z)(x_\epsilon - y_\epsilon) \right\} \nu(dz) \leq 0$$

and for the supersolution v,

$$
\begin{aligned}
&-b_\epsilon + F(t_\epsilon, y_\epsilon, \epsilon(x_\epsilon - y_\epsilon), B_\epsilon) \\
&+ \int_{\mathbf{Z}} \left\{ \frac{1}{\theta} \left(e^{-\theta(v(t_\epsilon, y_\epsilon + \xi(z)) - v(t_\epsilon, y_\epsilon))} - 1 \right) + \epsilon \xi'(z)(x_\epsilon - y_\epsilon) \right\} \nu(dz) \geq 0
\end{aligned}
$$

Subtracting these two inequalities,

$$
\begin{aligned}
-\varphi'(t_\epsilon) &= b_\epsilon - a_\epsilon \\
&\leq F(t_\epsilon, y_\epsilon, \epsilon(x_\epsilon - y_\epsilon), B_\epsilon) - F(t_\epsilon, x_\epsilon, \epsilon(x_\epsilon - y_\epsilon), A_\epsilon) \\
&\quad + \int_{\mathbf{Z}} \left\{ \frac{1}{\theta} \left(e^{-\theta(v(t_\epsilon, y_\epsilon + \xi(z)) - v(t_\epsilon, y_\epsilon))} - 1 \right) + \epsilon \xi'(z)(x_\epsilon - y_\epsilon) \right\} \nu(dz) \\
&\quad - \int_{\mathbf{Z}} \left\{ \frac{1}{\theta} \left(e^{-\theta(u(t_\epsilon, x_\epsilon + \xi(z)) - u(t_\epsilon, x_\epsilon))} - 1 \right) + \epsilon \xi'(z)(x_\epsilon - y_\epsilon) \right\} \nu(dz) \\
&= F(t_\epsilon, y_\epsilon, \epsilon(x_\epsilon - y_\epsilon), B_\epsilon) - F(t_\epsilon, x_\epsilon, \epsilon(x_\epsilon - y_\epsilon), A_\epsilon) \\
&\quad + \frac{1}{\theta} \int_{\mathbf{Z}} \left\{ e^{-\theta(v(t_\epsilon, y_\epsilon + \xi(z)) - v(t_\epsilon, y_\epsilon))} \right\} \nu(dz) \\
&\quad - \frac{1}{\theta} \int_{\mathbf{Z}} \left\{ e^{-\theta(u(t_\epsilon, x_\epsilon + \xi(z)) - u(t_\epsilon, x_\epsilon))} \right\} \nu(dz)
\end{aligned} \tag{5.29}
$$

Step 5: Modulus of Continuity

In this step, we focus on the (diffusion) operator F.

$$
\begin{aligned}
&F(t_\epsilon, y_\epsilon, \epsilon(x_\epsilon - y_\epsilon), B_\epsilon) - F(t_\epsilon, x_\epsilon, \epsilon(x - y), A_\epsilon) \\
= &\sup_{h \in J} \left\{ \epsilon f(t_\epsilon, y_\epsilon, h)' (x_\epsilon - y_\epsilon) + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda'(t_\epsilon) B_\epsilon \right) - \frac{\theta}{2} \epsilon^2 (x_\epsilon - y_\epsilon)' \Lambda \Lambda'(t_\epsilon) (x_\epsilon - y_\epsilon) - g(t_\epsilon, y_\epsilon, h) \right\} \\
&- \sup_{h \in J} \left\{ \epsilon f(t_\epsilon, x_\epsilon, h)' (x_\epsilon - y_\epsilon) + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda'(t_\epsilon) A_\epsilon + \delta I_n \right) \right. \\
&\left. - \frac{\theta}{2} \epsilon^2 (x_\epsilon - y_\epsilon)' \Lambda \Lambda'(t_\epsilon) (x_\epsilon - y_\epsilon) - g(t_\epsilon, x_\epsilon, h) \right\}
\end{aligned}
$$

$$
\begin{aligned}
& F(t_\epsilon, y_\epsilon, \epsilon(x_\epsilon - y_\epsilon), B_\epsilon) - F(t_\epsilon, x_\epsilon, \epsilon(x-y), A_\epsilon) \\
\leq\ & \frac{1}{2}|\mathrm{tr}\,(\Lambda\Lambda'(t_\epsilon)B_\epsilon - \Lambda\Lambda'(t_\epsilon)A_\epsilon)| + \sup_{h\in\bar{J}}\{\epsilon|f(t_\epsilon, y_\epsilon, h) - f(t_\epsilon, x_\epsilon, h)||(x_\epsilon - y_\epsilon)|\} \\
& + \sup_{h\in J}\{|g(t_\epsilon, x_\epsilon, h) - g(t_\epsilon, y_\epsilon, h)|\} \\
\leq\ & \frac{1}{2}|\mathrm{tr}\,(\Lambda\Lambda'(t_\epsilon)A_\epsilon - \Lambda\Lambda'(t_\epsilon)B_\epsilon)| + \sup_{h\in\bar{J}}\{\epsilon|f(t_\epsilon, y_\epsilon, h) - f(t_\epsilon, x_\epsilon, h)||(x_\epsilon - y_\epsilon)|\} \\
& + \sup_{h\in J}\{|g(t_\epsilon, x_\epsilon, h) - g(t_\epsilon, y_\epsilon, h)|\}
\end{aligned}
$$

For f, by Assumption (iii),

$$|f(t_\epsilon, y_\epsilon, h) - f(t_\epsilon, x_\epsilon, h)| \leq C_1\,|y_\epsilon - x_\epsilon|$$

for some constant $C_1 > 0$.

Still by Assumption (iii),

$$
\begin{aligned}
& \mathrm{tr}\,(\Lambda\Lambda'(t_\epsilon)A_\epsilon - \Lambda\Lambda'(t_\epsilon)B_\epsilon) \\
=\ & \mathrm{tr}\left(\begin{bmatrix} \Lambda\Lambda'(t_\epsilon) & \Lambda\Lambda'(t_\epsilon) \\ \Lambda\Lambda'(t_\epsilon) & \Lambda\Lambda'(t_\epsilon) \end{bmatrix}\begin{bmatrix} A_\epsilon & 0 \\ 0 & -B_\epsilon \end{bmatrix}\right) \\
\leq\ & 3\epsilon\ \mathrm{tr}\left(\begin{bmatrix} \Lambda\Lambda'(t_\epsilon) & \Lambda\Lambda'(t_\epsilon) \\ \Lambda\Lambda'(t_\epsilon) & \Lambda\Lambda'(t_\epsilon) \end{bmatrix}\begin{bmatrix} I & -I \\ -I & I \end{bmatrix}\right) \\
=\ & 0
\end{aligned}
$$

Finally, for g, by Assumption (ii),

$$|g(t_\epsilon, y_\epsilon, h) - g(t_\epsilon, x_\epsilon, h)| \leq C_2\,|y_\epsilon - x_\epsilon|$$

for some constant $C_2 > 0$.

Combining these estimates, we get

$$\begin{aligned} & F(t_\epsilon, y_\epsilon, \epsilon(x_\epsilon - y_\epsilon), B_\epsilon) - F(t, x_\epsilon, \epsilon(x_\epsilon - y_\epsilon), A_\epsilon) \\ \leq \;& \omega(\epsilon\,|y_\epsilon - x_\epsilon|^2 + |y_\epsilon - x_\epsilon|) \end{aligned} \tag{5.30}$$

for a function $\omega(\zeta) = C\zeta$, with $C = \max[C_1, C_2]$. The function $\omega : [0, \infty) \to [0, \infty)$, which satisfies the condition $\omega(0^+) = 0$, is called a modulus of continuity.

Step 6: The Jump Term

We now consider the jump term

$$\begin{aligned} & \frac{1}{\theta}\int_{\mathbf{Z}} \left\{e^{-\theta(v(t_\epsilon, y_\epsilon + \xi(z)) - v(t_\epsilon, y_\epsilon))} - e^{-\theta(u(t_\epsilon, x_\epsilon + \xi(z)) - u(t_\epsilon, x_\epsilon))}\right\} \nu(dz) \\ = \;& \frac{1}{\theta}\int_{\mathbf{Z}} \left\{e^{-\theta(v(t_\epsilon, y_\epsilon + \xi(z)) - v(t_\epsilon, y_\epsilon))} - e^{-\theta(u(t_\epsilon, x_\epsilon + \xi(z)) - u(t_\epsilon, x_\epsilon) + v(t_\epsilon, x_\delta) - v(t_\epsilon, x_\delta))}\right\} \nu(dz) \end{aligned} \tag{5.31}$$

Since for $\epsilon > 0$ large enough, $u(t, x) - v(t, y) \geq 0$ then

$$u(t_\epsilon, x_\epsilon + \xi(z)) - u(t_\epsilon, x_\epsilon) + v(t_\epsilon, y_\epsilon) - v(t_\epsilon, y_\epsilon + \xi(z)) \leq -(u(t_\epsilon, x_\epsilon) - v(t_\epsilon, y_\epsilon)) + N$$

by definition of N.

Moreover, since $N_\epsilon = \sup_{Q_d}[u(t, x) - v(t, y) - \varphi(t) - \epsilon|x - y|^2] > 0$, then $N_\epsilon \leq$

$u(t_\epsilon, x_\epsilon) - v(t_\epsilon, y_\epsilon)$ and therefore

$$u(t_\epsilon, x_\epsilon + \xi(z)) - u(t_\epsilon, x_\epsilon) + v(t_\epsilon, y_\epsilon) - v(t_\epsilon, y_\epsilon + \xi(z)) \leq N - N_\epsilon$$

for $z \in \mathbf{Z}$.

Thus,

$$e^{-\theta(u(t_\epsilon, x_\epsilon+\xi(z))-u(t_\epsilon,x_\epsilon)+v(t_\epsilon,y_\epsilon)-v(t_\epsilon,y_\epsilon))} \geq e^{-\theta(v(t_\epsilon,y_\epsilon+\xi(z))-v(t_\epsilon,y_\epsilon)+N-N_\epsilon)}$$

and equation (5.31) can be bounded from above by:

$$\begin{aligned}
-\varphi'(t_\delta) &\leq \frac{1}{\theta}\int_{\mathbf{Z}} \left\{e^{-\theta(v(t_\epsilon,y_\epsilon+\xi(z))-v(t_\epsilon,y_\epsilon))} - e^{-\theta(u(t_\epsilon,x_\epsilon+\xi(z))-u(t_\epsilon,x_\epsilon)+v(t_\epsilon,x_\epsilon)-v(t_\epsilon,x_\epsilon))}\right\}\nu(dz) \\
&\leq \frac{1}{\theta}\int_{\mathbf{Z}} \left\{e^{-\theta(v_\epsilon(t_\epsilon,y_\epsilon+\xi(z))-v(t_\epsilon,y_\epsilon))} - e^{-\theta(v(t_\epsilon,y_\epsilon+\xi(z))-v(t_\epsilon,y_\epsilon)+N-N_\delta)}\right\}\nu(dz) \\
&= \frac{1}{\theta}\int_{\mathbf{Z}} \left\{e^{-\theta(v(t_\epsilon,y_\epsilon+\xi(z))-v(t_\epsilon,y_\epsilon))}\left(1 - e^{-\theta(N-N_\epsilon)}\right)\right\}\nu(dz) \\
&= \frac{1}{\theta}\int_{\mathbf{Z}} \left\{e^{-\theta\left(-\frac{1}{\theta}[\ln\tilde{v}(t_\epsilon,y_\epsilon+\xi(z))-\ln\tilde{v}(t_\epsilon,y_\epsilon)]\right)}\left(1 - e^{-\theta(N-N_\epsilon)}\right)\right\}\nu(dz) \\
&= \frac{1}{\theta}\int_{\mathbf{Z}} \left\{\frac{\tilde{v}(t_\epsilon, y_\epsilon + \xi(z))}{\tilde{v}(t_\epsilon, y_\epsilon)}\left(1 - e^{-\theta(N-N_\epsilon)}\right)\right\}\nu(dz) \qquad (5.32)
\end{aligned}$$

By Assumption 5.14 and since $\tilde{v}$ is LSC, then $\exists\lambda > 0 : 0 < \lambda \leq \tilde{v}(t,x) \leq C_{\tilde{\Phi}}\ \forall(t,x) \in Q_d$. As a result,

$$\frac{\tilde{v}(t_\epsilon, y_\epsilon + \xi(z))}{\tilde{v}(t_\epsilon, y_\epsilon)} \leq K$$

for some constant $K > 0$.

In addition, since the measure ν is assumed to be finite and the function $\zeta \mapsto e^{\zeta}$ is continuous, we can establish the following upper bound for the right-hand side of (5.32):

$$\begin{aligned} & \frac{1}{\theta}\int_{\mathbf{Z}} \left\{ e^{-\theta(v(t_\epsilon, y_\epsilon + \xi(z)) - v(t_\epsilon, y_\epsilon))} \left(1 - e^{-\theta(N - N_\epsilon)}\right) \right\} \nu(dz) \\ \leq \;& \omega_R(N - N_\epsilon) \sup_{[0,T]\times\mathbb{R}^n} \nu(\mathbf{Z}) \end{aligned} \tag{5.33}$$

for some modulus of continuity ω_R parameterized by R.

Step 7: Conclusion

We now substitute the upper bound obtained in inequalities (5.30) and (5.33) in (5.29) to obtain:

$$-\varphi'(t_\epsilon) \;\leq\; \omega(\epsilon\,|y_\epsilon - x_\epsilon|^2 + |y_\epsilon - x_\epsilon|) + \omega_R(N - N_\epsilon) \sup_{[0,T]\times\mathbb{R}^n} \nu(\mathbf{Z}) \tag{5.34}$$

Taking the limit superior in inequality (5.34) as $\epsilon \to \infty$ and recalling that

(1). the measure ν is finite;

(2). $\xi_i(z), i = 1, \ldots, m$ is bounded $\forall z \in \mathbf{Z}$ a.s. $d\nu$

we see that

$$\nu(\mathbf{Z}) < \infty$$

Then

$$\lim_{\epsilon \to 0} \omega_R(N - N_\epsilon)\nu(\mathbf{Z}) = 0$$

which leads to the contradiction

$$-\varphi'(t) = \frac{\eta}{t^2} \leq 0$$

We conclude from this that Assumption 5.28 is false and therefore

$$\sup_Q [v(t,x) - u(t,x)] \geq 0 \tag{5.35}$$

Stated differently, we conclude that

$$u \leq v \quad \text{on } [0,T] \times \mathbb{R}^n$$

□

5.6 Uniqueness

Uniqueness is a direct consequence of Theorem 5.19. Another important corollary is the fact that the viscosity solution Φ is in fact continuous on $[0,T] \times \mathbb{R}^n$.

Corollary 5.20 (Uniqueness and Continuity). *The function* $\Phi(t,x)$ *defined on* $[0,T] \times \mathbb{R}^n$ *is the unique continuous viscosity solution of the RS HJB PIDE* (5.5) *subject to terminal condition* (5.6).

Proof. Uniqueness is a standard by-product of Theorem 5.19. Continuity can be proved as follows. By definition of the upper and lower semicontinuous envelopes, recall that

$$\Phi_* \leq \Phi \leq \Phi^*$$

By Corollary 5.18 Φ_* and Φ^* are respectively semicontinuous superolution and

subsolution of the RS HJB PIDE (5.5) subject to terminal condition (5.6)

We note that as a consequence of Theorem 5.19 is that

$$\Phi_* \geq \Phi^*$$

and hence

$$\Phi_* = \Phi^*$$

is a continuous viscosity solution of the RS HJB PIDE (5.5) subject to terminal condition (5.6).

Hence, $\Phi = \Phi_* = \Phi^*$ and it is the unique continuous viscosity solution of the RS HJB PIDE (5.5) subject to terminal condition (5.6).

□

Now that we have proved uniqueness and continuity of the viscosity solution Φ to the RS HJB PIDE (5.5) subject to terminal condition (5.6), we can deduce that the RS HJB PIDE (5.11) subject to terminal condition (5.12) also has a unique continuous viscosity solution. We formalize the uniqueness and continuity of $\tilde{\Phi}$ in the following corollary:

Corollary 5.21 (Uniqueness and Continuity). *The function* $\tilde{\Phi}(t,x)$ *defined on* $[0,T]\times\mathbb{R}^n$ *is the unique continuous viscosity solution of the RS HJB PIDE* (5.11) *subject to terminal condition* (5.12).

5.7 Solving the Jump Diffusion Risk Sensitive Asset Management Problem

5.7.1 Setting

In the general setting introduced in Section 5.2, take

$$\begin{aligned} f(t,x,h) &:= b + BX(s^-) - \theta\Lambda\Sigma' h(s) + \int_{\mathbf{Z}} \xi(z)\left[(1+h'\gamma(z))^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz) \\ \Lambda(t,x,h) &:= \Lambda \\ g(t,x,h) &\quad \text{is defined in (4.7)} \\ g_T(x,T;) &:= -\ln v \end{aligned}$$

and note that

- f satisfies the growth condition (5.1) and f and Λ satisfy the Lipschitz condition (5.2);
- g_T is continuous in $\mathbb{R}^n$;
- g is affine in x and continuous in t and h,

5.7.2 Properties of the Value Function of the Exponentially Transformed Problem

The following proposition details the behaviour of the value function $\tilde{\Phi}$ of the exponentially transformed stochastic control problem and shows that the value function $\tilde{\Phi}$ satisfies Assumption 5.14.

Assumption 5.22. There exists $2n$ constant controls $\bar{h}^k, k = 1, \ldots, 2n$ such that the $2n$ functions $\beta^k(t) : t \mapsto \mathbb{R}^n$ defined by

$$\beta^k(t) = \theta B^{-1} \left(1 - e^{B(T-t)}\right) \left(A_0 + \bar{h}^k \hat{A}\right) \tag{5.36}$$

and $2n$ functions $\alpha^k(t) : t \mapsto \mathbb{R}$ defined by

$$\alpha^k(t) = -\int_t^T q^k(s)ds \tag{5.37}$$

where

$$\begin{aligned} q^k(t) \quad := \quad & \left(b - \theta\Lambda\Sigma'\bar{h} + \int_{\mathbf{Z}} \xi(z) \left[\left(1 + \bar{h}^{k'}\gamma(z)\right)^{-\theta} - 1_{\mathbf{Z}_0}(z)\right] \nu(dz)\right)' \beta^{k'}(t) \\ & + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\beta^{k'}(t)\beta^k(t)\right) + \int_{\mathbf{Z}} \left\{e^{\beta^k \xi(z)} - 1 - \xi'(z)\beta^{k'}(t)\right\} \nu(dz) \\ & + \frac{1}{2}\theta\left(\theta + 1\right)\bar{h}^{k'}\Sigma\Sigma'\bar{h}^k - \theta a_0 - \theta\hat{a} + \theta \int_{\mathbf{Z}} \left\{\frac{1}{\theta}\left[\left(1 + \bar{h}^{k'}\gamma(z)\right)^{-\theta} - 1\right]\right. \\ & \left. + \bar{h}^{k'}\gamma(z)1_{\mathbf{Z}_0}(z)\right\} \nu(dz) \end{aligned}$$

exist and for $i = 1, \ldots, n$ satisfy:

$$\begin{aligned} \beta_i^i(t) \quad &< \quad 0 \\ \beta_i^{n+i}(t) \quad &> \quad 0 \end{aligned} \tag{5.38}$$

where $\beta_j^i(t)$ denotes the j-th component of the vector $\beta^i(t)$.

Remark 5.23. Key to this assumption is the condition (5.38) which imposes a specific constraint one one element of each of the $2n$ vectors $\beta^k(t)$. To clarify the structure of this constraint, define a square $n \times n$ matrix M_β^- as the matrix whose i-th column (with $i = 1, \ldots n$) is the n-element column vector $\beta^i(t)$. Then all the elements $m_{jj}^-, j =$

$1, \ldots, m$ on the diagonal of M_β^- are such that

$$m_{jj}^- = \beta_j^j(t) < 0$$

Similarly, define a square $n \times n$ matrix M_β^+ as the matrix whose i-th column (with $i = 1, \ldots n$) is the n-element column vector $\beta^{n+i}(t)$. Then all the elements $m_{jj}^+, j = 1, \ldots, m$ on the diagonal of M_β^+ are such that

$$m_{jj}^+ = \beta_j^{n+j}(t) > 0$$

Note that there is no requirement for either M_β^- or M_β^+ to have full rank. It would in fact be perfectly acceptable to have rank 1 as a result of column duplication.

Remark 5.24. For the function β in equation (5.36) to exists, B must be invertible. The existence and admissibility of $2n$ constant controls $\bar{h}^k, k = 1, \ldots, 2n$ such that (5.36) satisfies (5.38) is subject to solving a system of at most n inequalities with m variables and $m > n$. The existence of appropriate controls is therefore likely and may only require some adjustments to the elements of the matrices A_0, A, B or to the set J via an adjustment to the maximum jump size allowed.

Proposition 5.25. *Assume that*

(i). $g(t, x, h)$ is locally bounded on $[0, T] \times \mathbb{R}^n \times J$ and is Lipschitz continuous in x;

(ii). Assumption 5.22 holds.

Then,

(i). Lower Bound.

$$\tilde{\Phi}(t, x) \geq 0 \; \forall x \in \mathbb{R}^n$$

and

$$\tilde{\Phi}(t,x) > 0 \; \forall x \in \mathbb{R}^n$$

(ii). Upper Bounds. For the $2n$ constant controls $\bar{h}^k, k = 1, \ldots, 2n$ parameterizing the $4n$ functions

$$\alpha^k : t \in [0,T] \to \mathbb{R}, \; k = 1, \ldots, 2n$$

$$\beta^k : t \in [0,T] \to \mathbb{R}^n, \; k = 1, \ldots, 2n$$

such that for $i = 1, \ldots, n$,

$$\begin{aligned} \beta_i^i(t) &< 0 \\ \beta_i^{n+i}(t) &> 0 \end{aligned}$$

where $\beta_j^i(t)$ denotes the j-th component of the vector $\beta^i(t)$, then we have the following upper bounds:

$$\tilde{\Phi}(t,x) \leq e^{\alpha^k(t)+\beta^{k'}(t)x}$$

in each element $x_i, i = 1, \ldots, n$ of x.

Proof. (i). *Lower Bound.* The first property,

$$\tilde{\Phi}(t,x) \geq 0 \; \forall x \in \mathbb{R}^n$$

follows directly from the definition of $\tilde{\Phi}$ as an exponent

$$\tilde{\Phi}(t,x) = e^{-\frac{1}{\theta}\Phi(t,x)} = \inf_{h \in J} \mathrm{E}_{t,x}\left[\exp\left\{\theta \int_t^T g(s, X_s, h(s))ds - \theta \ln v\right\}\right] \geq 0$$

The second property,

$$\tilde{\Phi}(t,x) > 0 \, \forall x \in \mathbb{R}^n$$

stems from the fact that the optimal control h^* is an interior point of the set J, that g is continuous in x on $\mathbb{R}^n$ and that the state process $X(t)$ has bounded jumps.

(ii). *Upper Bounds.* Setting $\mathbf{Z} = \mathbb{R}^n \backslash \{0\}$ and recalling that the dynamics of the state variable $X(t)$ under the $\mathbb{P}_h^\theta$-measure is given by

$$\begin{aligned} dX(t) &= \left(b + BX(t^-) - \theta \Lambda \Sigma' h(s) + \int_{\mathbf{Z}} \xi(z) \left[(1 + h'\gamma(z))^{-\theta} - 1_{\mathbf{Z}_0}(z) \right] \nu(dz) \right) ds \\ &\quad + \Lambda dW_t^\theta + \int_{\mathbb{R}^n} \xi(z) \tilde{N}_{\mathbf{p}}^\theta(dt, dz) \end{aligned}$$

we note that the associated Lévy measure $\tilde{\nu}$ can be defined via the map:

$$\tilde{\nu} = \nu \circ \xi^{-1} \tag{5.39}$$

We will now limit ourselves the class $\mathcal{H}^c$ of constant controls. By the optimality principle, for an arbitrary admissible constant control policy $\bar{h}$, we have

$$\tilde{\Phi}(t,x) \leq \tilde{I}(x; \bar{h}; t, T) \leq \mathbf{E}_{t,x} \left[\exp \left\{ \theta \int_t^T g(s, X_s, \bar{h}) ds - \theta \ln v \right\} \right] := W(t,x) \tag{5.40}$$

In this setting, we note that the function g is an affine function of the affine process $X(t)$. Affine process theory (See Appendix A in Duffie and Singleton [25], Duffie, Pan and Singleton [24] or Duffie, Filipovic and Schachermayer [22] for more details on the properties of affine processes) leads us to anticipate that

the expectation on the right-hand side of equation (5.40) will take the form

$$W(t,x) = \exp\{\alpha(t) + \beta(t)x\} \tag{5.41}$$

where

$$\alpha : t \in [0,T] \to \mathbb{R}$$

$$\beta : t \in [0,T] \to \mathbb{R}^n$$

are functions solving two ODEs.

Indeed, applying the Feynman-Kac formula, we find that the function $W(t,x)$ satisfies the integro-differential PDE:

$$\begin{aligned}
&\frac{\partial W}{\partial t} + \left(b + BX_s - \theta\Lambda\Sigma'\bar{h} + \int_{\mathbf{Z}} \xi(z)\left[\left(1+\bar{h}'\gamma(z)\right)^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)' DW(t,x) \\
&+\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2 W(t,x)\right) + \int_{\mathbf{Z}} \left\{W(t,x+\xi(z)) - W(t,x) - \xi'(z)DW(t,x)\right\}\nu(dz) \\
&+\theta g(x,\bar{h};\theta)W(t,x) = 0
\end{aligned}$$

subject to terminal condition $\tilde{\Phi}(T,x) = v^{-\theta}$.

Now, taking the candidate solution to be in the form

$$W(t,x) = \exp\{\alpha(t) + \beta(t)x\}$$

we have

$$\begin{aligned}\frac{\partial W}{\partial t} &= \left(\alpha(\dot{t}) + \dot{\beta}(t)x\right) W(t,x) \\ DW &= \beta'(t)W(t,x) \\ D^2W &= \beta'(t)\beta(t)W(t,x)\end{aligned}$$

Substituting into the PDE, we get

$$\begin{aligned}&\left(\alpha(\dot{t}) + \dot{\beta}(t)x\right) W(t,x) \\ &+\left(b + Bx - \theta\Lambda\Sigma'\bar{h} + \int_{\mathbf{Z}} \xi(z)\left[(1+h'\gamma(z))^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)' \beta'(t)W(t,x) \\ &+\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\beta'(t)\beta(t)\right) W(t,x) \\ &+\int_{\mathbf{Z}} \left\{W(t,x+\xi(z)) - W(t,x) - \xi'(z)\beta'(t)W(t,x)\right\}\nu(dz) \\ &+\theta\left(\frac{1}{2}(\theta+1)\,\bar{h}'\Sigma\Sigma'\bar{h} - a_0 - A_0'x - \bar{h}'(\hat{a}+\hat{A}x)\right. \\ &\left.+\int_{\mathbf{Z}} \left\{\frac{1}{\theta}\left[(1+h'\gamma(z))^{-\theta} - 1\right] + \bar{h}'\gamma(z)1_{\mathbf{Z}_0}(z)\right\}\nu(dz)\right) W(t,x) = 0\end{aligned}$$

Dividing by $W(t,x)$ and rearranging, we get

$$\begin{aligned}&\left(\dot{\beta}(t) + B'\beta'(t) - \theta A_0' - \theta\bar{h}'\hat{A}\right)x \\ =\ &-\left(\alpha(\dot{t}) + \left(b - \theta\Lambda\Sigma'\bar{h} + \int_{\mathbf{Z}} \xi(z)\left[(1+\bar{h}'\gamma(z))^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)' \beta'(t)\right. \\ &+\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\beta'(t)\beta(t)\right) + \int_{\mathbf{Z}} \left\{e^{\beta\xi(z)} - 1 - \xi'(z)\beta'(t)\right\}\nu(dz) \\ &+\frac{1}{2}\theta(\theta+1)\,\bar{h}'\Sigma\Sigma'\bar{h} - \theta a_0 - \theta\hat{a} + \theta\int_{\mathbf{Z}} \left\{\frac{1}{\theta}\left[(1+\bar{h}'\gamma(z))^{-\theta} - 1\right]\right. \\ &\left.\left.+\bar{h}'\gamma(z)1_{\mathbf{Z}_0}(z)\right\}\nu(dz)\right)\end{aligned}$$

Since the left-hand side is independent from the right-hand side, then both sides are orthogonal. As a result we now only need to solve the two ODEs

$$\dot{\beta}(t) + B'\beta'(t) - \theta A_0' - \theta \bar{h}' \hat{A} = 0 \tag{5.42}$$

and

$$\begin{aligned} & \dot{\alpha}(t) + \left(b - \theta \Lambda \Sigma' \bar{h} + \int_{\mathbf{Z}} \xi(z) \left[\left(1 + \bar{h}'\gamma(z)\right)^{-\theta} - 1_{\mathbf{Z}_0}(z) \right] \nu(dz) \right)' \beta'(t) \\ & + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda' \beta'(t) \beta(t) \right) + \int_{\mathbf{Z}} \left\{ e^{\beta \xi(z)} - 1 - \xi'(z) \beta'(t) \right\} \nu(dz) \\ & + \frac{1}{2} \theta \left(\theta + 1 \right) \bar{h}' \Sigma \Sigma' \bar{h} - \theta a_0 - \theta \hat{a} + \theta \int_{\mathbf{Z}} \left\{ \frac{1}{\theta} \left[\left(1 + \bar{h}'\gamma(z)\right)^{-\theta} - 1 \right] + \bar{h}'\gamma(z) 1_{\mathbf{Z}_0}(z) \right\} \nu(dz \\ = \; & 0 \end{aligned} \tag{5.43}$$

to obtain the value of $W(t,x)$. The ODE (5.42) for β is linear and admits the solution

$$\beta(t) = \theta B^{-1} \left(1 - e^{B(T-t)}\right) \left(A_0 + \bar{h}^k \hat{A} \right) \tag{5.44}$$

As for the ODE (5.43) for α, we only need to integrate to get

$$\alpha(t) = - \int_t^T q(s) ds \tag{5.45}$$

where

$$\begin{aligned} q(t) \;:=\; & \left(b-\theta\Lambda\Sigma'\bar{h}+\int_{\mathbf{Z}}\xi(z)\left[\left(1+\bar{h}'\gamma(z)\right)^{-\theta}-1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)'\beta'(t) \\ & +\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\beta'(t)\beta(t)\right)+\int_{\mathbf{Z}}\left\{e^{\beta\xi(z)}-1-\xi'(z)\beta'(t)\right\}\nu(dz) \\ & +\frac{1}{2}\theta\left(\theta+1\right)\bar{h}'\Sigma\Sigma'\bar{h}-\theta a_0-\theta\hat{a}+\theta\int_{\mathbf{Z}}\left\{\frac{1}{\theta}\left[\left(1+\bar{h}'\gamma(z)\right)^{-\theta}-1\right]\right. \\ & \left.+\bar{h}'\gamma(z)1_{\mathbf{Z}_0}(z)\right\}\nu(dz) \end{aligned}$$

Observe that $W(t,x)$ is increasing in x_i, the i-th element of x, if $\beta_i > 0$, and conversely, $W(t,x)$ is decreasing in x_i if $\beta_i < 0$

Equations (5.44) and (5.45) are respectively equations (5.36) and (5.37) from Assumption 5.22. By Assumption 5.22, there exists $2n$ constant controls $\bar{h}^k, k = 1,\ldots,2n$ such that for $i = 1,\ldots,n$,

$$\begin{aligned} \beta_i^i(t) &< 0 \\ \beta_i^{n+i}(t) &> 0 \end{aligned}$$

where $\beta_j^i(t)$ denotes the j-th component of the vector $\beta^i(t)$. We can now conclude that we have the following upper bounds

$$\tilde{\Phi}(t,x) \leq e^{\alpha^k(t)+\beta^{k'}(t)x}$$

for each element $x_i, i = 1,\ldots,n$ of x.

□

Remark 5.26. To obtain the upper bounds and the asymptotic behaviour, we do not

need the $2n$ constant controls to be pairwise different. In fact, we need at least 2 different controls and at most $2n$ different controls. Moreover, we could consider wider classes of controls extending beyond constant controls. This would require some modifications to the proof but would also alleviate the assumptions required for the Theorem to hold.

Remark 5.27. For a given constant control $\bar{h}$, equation (5.42) is a linear n-dimensional ODE. However, if in the dynamics of the state variable $X(t)$, Λ and Ξ depended on X, the ODE would be nonlinear. Once ODE (5.42) is solved, obtaining $\alpha(t)$ from equation (5.43) is a simple matter of integration.

Remark 5.28. For a given constant control h, given $x \in \mathbb{R}^n$ and $t \in [0, T]$, the solution of ODE (5.42) is the same whether the dynamics of $S(t)$ and $X(t)$ is the jump diffusion considered here or the corresponding pure diffusion model. The converse is, however, not true since in the pure diffusion setting $h \in \mathbb{R}^m$, while in the jump diffusion case $h \in J \subset \mathbb{R}^m$.

5.7.3 Viscosity Solution of the RS HJB PIDE

We apply directly Theorem 5.17 to show that Φ is a viscosity solution of (4.15). Then, we apply Theorem 5.19 and Corollary 5.20 to prove that Φ is the unique viscosity solution of (4.15) and that it is continuous.

5.8 Conclusion

In the jump-diffusion setting, analytical or even classical $C^{1,2}$ solutions do not generally exist. As a result, we considered a wider class of weak solutions, namely viscosity solutions. We proved that the value function of a class of risk sensitive control problems, which includes the jump diffusion risk sensitive asset management problem as

a subcase, is the unique, continuous, viscosity solution of the associated risk-sensitive Hamilton-Jacobi-Bellman Partial Integro Differential Equation.

The main benefit of having considered a class of problems extending beyond the jump-diffusion risk sensitive asset management problem will be evident in the next chapter, as we apply the methods and results derived in Chapters 4 and 5 to solve the jump diffusion version of the risk sensitive benchmarked asset management problem and of the risk sensitive asset and liability management problem.

Chapter 6

Jump Diffusion Risk-Sensitive Asset Management: Applications

In this final chapter, we apply the methodology and results derived in Chapters 4 and 5 to derive jump diffusion versions of the risk-sensitive benchmarked asset management model proposed by Davis and Lleo [21] and of the risk-sensitive asset and liability management model introduced in Chapter 3.

The chapter is organized as follows. In Section 1, we consider the jump diffusion risk sensitive benchmarked asset management problem before solving the jump diffusion risk sensitive asset and liability management problem in Section 2.

6.1 Benchmarked Asset Management

In this section, we extend the model developed by Davis and Lleo [21] in the diffusion case by allowing jumps in the valuation factors, asset prices, as well as in the value of the benchmark.

6.1.1 Factor, Asset and Portfolio Dynamics

The factors, risky assets and money market asset follow the dynamic given in Sections 4.1.3 and 4.1.4.

6.1.2 Benchmark Dynamics

In a jump-diffusion setting, the value of the liability, $L(t)$, follows the dynamics

$$\frac{dL(t)}{L(t)} = (c + C'X(t))dt + \varsigma' dW(t) + \int_{\mathbf{Z}} \zeta(z)\bar{N}_{\mathbf{p}}(dt, dz), \qquad L(0) = l \tag{6.1}$$

where

- $c \in \mathbb{R}$;

- $C \in \mathbb{R}^n$;
- $\varsigma \in \mathbb{R}^M$; and
- $\zeta(z) \in \mathbb{R}$, with $-1 < \zeta^{min} \leq \zeta(z) \leq \zeta^{max} < +\infty$.

In addition, $\zeta(z)$ satisfies:

$$\int_{\mathbf{Z}_0} |\zeta(z)|^2 \nu(dz) < \infty \tag{6.2}$$

Remark 5.3 in Chapter 5 also applies to assumption 6.2.

6.1.3 Wealth Dynamics and Control Policy

Let $\mathcal{G}_t := \sigma((S(s), X(s), L(s)), 0 \leq s \leq t)$ be the sigma-field generated by the security, benchmark and factor processes up to time t.

To ensure the existence of a unique investment strategy in Section 6.1.8, we will require the following assumption:

Assumption 6.1. The systematic (factor-driven) and idiosyncratic (asset-driven) jump risks are uncorrelated, i.e $\forall z \in \mathbf{Z}$, $\gamma(z)\xi'(z) = 0$ and $\zeta(z)\xi'(z) = 0$.

In order for our control to be defined on a compact set, we adopt the same strategy as in Chapter 4: we define a control process $\tilde{h}$ on the compact set $\overline{\mathcal{O}}$ where $\mathcal{O} :=]-1, 1[^m$ and $\overline{\mathcal{O}}$ represents the closure of $\mathcal{O}$. The asset allocation policy is then defined as

$$h_i = \begin{cases} -\infty, & \tilde{h}_i = -1 \\ \tanh^{-1}(\tilde{h}_i), & \tilde{h}_i \in]-1, 1[\\ +\infty, & \tilde{h}_i = 1 \end{cases}$$

for $i = 1, \ldots, m$

As in the asset only case we introduce the following definition:

Definition 6.2. A control process $\tilde{h}(t)$ is in class $\tilde{\mathcal{H}}$ if the following conditions are satisfied:

1. $\tilde{h}(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t\geq 0}$;

2. $P\left(\int_0^T |h(s)|^2 \, ds < +\infty\right) = 1, \quad \forall T > 0;$

3. $h'(t)\gamma(z) \geq -1, \quad \forall t > 0, z \in \mathbf{Z}$, a.s. $d\nu$.

Finally, the wealth dynamics remains as in Section 4.1.5.

6.1.4 Optimization Criterion

In this setting, the investor's main objective is to maximize the risk adjusted growth of his/her assets relative to the benchmark. To track the achievement of this goal, we introduce the process $F(t;h)$, which corresponds to the log excess return of the asset portfolio over its benchmark, i.e.

$$F(t;h) := \ln \frac{V(t)}{L(t)} = \ln V(t) - \ln L(t)$$

with $F(0;h) := f_0 = \ln \frac{v}{l}$.

By Itô's lemma, the log excess return in response to a strategy h is

$$\begin{aligned}
F(t;h) &= \ln\frac{v}{l} + \int_0^t d\ln V(s) - \int_0^t d\ln L(s) \\
&= \ln\frac{f_0}{+}\int_0^t \left(a_0 + A_0'X(s) + h(s)'\left(\hat{a} + \hat{A}X(s)\right)\right)ds \\
&\quad -\frac{1}{2}\int_0^t h(s)'\Sigma\Sigma'h(s)ds + \int_0^t h(s)'\Sigma dW(s) \\
&\quad + \int_0^t\int_{\mathbf{Z}_0}\left\{\ln\left(1+h'\gamma(z)\right) - h'\gamma(z)\right\}\nu(dz)ds \\
&\quad + \int_0^t\int_{\mathbf{Z}}\ln\left(1+h'\gamma(z)\right)\bar{N}_{\mathbf{p}}(ds,dz) - \int_0^t (c + C'X(s))ds \\
&\quad + \frac{1}{2}\int_0^t \varsigma'\varsigma ds - \int_0^t \varsigma' dW(s) - \int_0^t\int_{\mathbf{Z}_0}\left\{\ln\left(1+\zeta(z)\right) - \zeta(z)\right\}\nu(dz)ds \\
&\quad - \int_0^t\int_{\mathbf{Z}}\ln\left(1+\zeta(z)\right)\bar{N}_{\mathbf{p}}(ds,dz)
\end{aligned} \tag{6.3}$$

We now consider the class of problems $\mathcal{P}_{\theta,T}$: for $\theta \in]0,+\infty[$, maximize the risk sensitive expected log excess return over a time horizon T, and introduce the associated criterion function

$$J_{\theta,T}(v,x;h) := \left(-\frac{1}{\theta}\right)\ln \mathbf{E}e^{-\theta F(T;h)} \tag{6.4}$$

6.1.5 Criterion Under the Expectation

Multiplying by $-\theta$ and taking the exponential on both sides of (6.3), we get

$$\begin{aligned}
e^{-\theta F(t;h)} &= f_0^{-\theta}\exp\left\{\theta\int_0^t g(X_s,h(s);\theta)ds - \theta\int_0^t \left(h(s)'\Sigma - \varsigma'\right)dW_s\right. \\
&\quad -\frac{1}{2}\theta^2\int_0^t (h(s)'\Sigma-\varsigma')(h(s)'\Sigma-\varsigma')'ds \\
&\quad + \int_0^t\int_{\mathbf{Z}}\ln\left(1-H(z,h(s);\theta)\right)\tilde{N}_{\mathbf{p}}(ds,dz) \\
&\quad \left. + \int_0^t\int_{\mathbf{Z}}\left\{\ln\left(1-H(z,h(s);\theta)\right) + H(z,h(s);\theta)\right\}\nu(dz)ds\right\}
\end{aligned}$$

where

$$\begin{aligned} g(x,h;\theta) &= \frac{1}{2}(\theta+1)\,h'\Sigma\Sigma'h - a_0 - A_0'x - h'(\hat{a}+\hat{A}x) \\ &\quad -\frac{1}{2}\theta(h'\Sigma\varsigma+\varsigma'\Sigma'h) + (c+C'x) + \frac{1}{2}(\theta-1)\,\varsigma'\varsigma \\ &\quad + \int_0^t \int_{\mathbf{Z}} \left\{ \frac{1}{\theta}\left[\left(\frac{1+h'\gamma(z)}{1+\zeta(z)} \right)^{-\theta} - 1 \right] + (h'\gamma(z) - \zeta(z))\; 1_{\mathbf{Z}_0}(z) \right\} \nu(dz)ds \end{aligned}$$

$$H(z,h;\theta) := 1 - \left(\frac{1+h'\gamma(z)}{1+\zeta(z)} \right)^{-\theta}$$

6.1.6 Change of Measure

Let $\mathbb{P}_h^\theta$ be the measure on $(\Omega, \mathcal{F})$ defined as

$$\begin{aligned} \chi_t &:= \left. \frac{d\mathbb{P}_h^\theta}{d\mathbb{P}} \right|_{\mathcal{F}_t} \\ &= \exp\left\{ -\theta \int_0^t (h(s)'\Sigma - \varsigma')\, dW_s - \frac{1}{2}\theta^2 \int_0^t (h(s)'\Sigma - \varsigma')(h(s)'\Sigma - \varsigma')' ds \right. \\ &\quad + \int_0^t \int_{\mathbf{Z}} \ln\left(1 - H(z,h(s);\theta)\right) \tilde{N}_{\mathbf{p}}(ds,dz) \\ &\quad \left. + \int_0^t \int_{\mathbf{Z}} \left\{ \ln\left(1 - H(z,h(s);\theta)\right) + H(z,h(s);\theta) \right\} \nu(dz)ds \right\}, \quad \forall t \geq 0 \end{aligned}$$

For a change of measure to be possible, we must ensure that the following technical condition holds:

$$H(z,h(s);\theta) \leq 1$$

This condition is satisfied iff

$$1 - \left(\frac{1+h'\gamma(z)}{1+\zeta(z)} \right)^{-\theta} \leq 1 \;\Leftrightarrow\; \left(\frac{1+h'\gamma(z)}{1+\zeta(z)} \right)^{-\theta} \geq 0 \Leftrightarrow h'\gamma(z) \geq -1$$

a.s. $d\nu$, (since $\zeta(z) > -1 \ \forall z \in \mathbf{Z}$). We observe that this inequality was already required for $\tilde{h}$ to be in class $\tilde{\mathcal{H}}$ (Condition 3 in Definition 6.2).

More precisely, define

$$\mathbf{S} := \text{supp}(\nu) \in \mathcal{B}_{\mathbf{Z}}$$

and

$$\tilde{\mathbf{S}} := \text{supp}(\nu \circ \gamma^{-1}) \in \mathcal{B}(\mathbb{R}^m)$$

where supp$(\cdot)$ denotes the measure's support. Let J be the set defined as

$$J := \left\{ h \in \mathbb{R}^m : -1 - h'\psi < 0 \quad \forall \psi \in \tilde{\mathbf{S}} \right\} \tag{6.5}$$

and $\mathcal{K}$ be the set

$$\mathcal{K} := \{ h(t) \in \mathcal{H} : h(t) \in J \quad \forall t \text{ a.s.} \} \tag{6.6}$$

We denote by $\mathcal{A}(T)$ the set of investment strategies $h \in \mathcal{H}$ on $[0, T]$ such that $\mathbb{P}_h^\theta$ is an equivalent martingale measure and thus

$$\begin{aligned} &\mathrm{E}\left[\exp\left\{-\theta \int_0^t (h(s)'\Sigma - \varsigma')\, dW_s - \frac{1}{2}\theta^2 \int_0^t (h(s)'\Sigma - \varsigma')(h(s)'\Sigma - \varsigma')' ds \right.\right. \\ &+ \int_0^t \int_{\mathbf{Z}} \ln\left(1 - H(z, h(s); \theta)\right) \tilde{N}_{\mathbf{p}}(ds, dz) \\ &\left.\left. + \int_0^t \int_{\mathbf{Z}} \left\{\ln\left(1 - H(z, h(s); \theta)\right) + H(z, h(s); \theta)\right\} \nu(dz) ds \right\}\right] \\ &= 1 \end{aligned} \tag{6.7}$$

For $h \in \mathcal{A}(T)$,

$$W_t^\theta = W_t + \theta \int_0^t (\Sigma' h(s) - \varsigma)\, ds$$

is a standard Brownian motion under the measure $\mathbb{P}_h^\theta$ and we have

$$\begin{aligned}\int_0^t \int_{\mathbf{Z}} \tilde{N}_{\mathbf{p}}^\theta(ds,dz) &= \int_0^t \int_{\mathbf{Z}} N_{\mathbf{p}}(ds,dz) - \int_0^t \int_{\mathbf{Z}} \{1 - H(z,h(s);\theta)\}\,\nu(dz)ds \\ &= \int_0^t \int_{\mathbf{Z}} N_{\mathbf{p}}(ds,dz) - \int_0^t \int_{\mathbf{Z}} \left\{ \left(\frac{1+h'\gamma(z)}{1+\zeta(z)} \right)^{-\theta} \right\} \nu(dz)ds\end{aligned}$$

As a result, $X(s)$, $0 \le s \le t$ satisfies the SDE:

$$\begin{aligned}&dX(s) \\ =\ & \left(b + BX(s^-) - \theta\Lambda\left(\Sigma' h(s) - \varsigma\right) + \int_{\mathbf{Z}} \xi(z) \left[\left(\frac{1+h'\gamma(z)}{1+\zeta(z)} \right)^{-\theta} - 1_{\mathbf{Z}_0}(z) \right] \nu(dz) \right) ds \\ & + \Lambda dW_s^\theta + \int_{\mathbf{Z}} \xi(z) \tilde{N}_{\mathbf{p}}^\theta(ds,dz)\end{aligned}$$

Under the measure $\mathbb{P}_h^\theta$, the auxiliary criterion function associated with the benchmarked problem is defined as

$$I(f_0,x;h;t,T) = -\frac{1}{\theta} \ln \mathbf{E}_{t,x}^{h,\theta} \left[\exp\left\{ \theta \int_t^T g(X_s,h(s);\theta)ds - \theta \ln f_0 \right\} \right] \tag{6.8}$$

where $\mathbf{E}_{t,x}^{h,\theta}[\cdot]$ denotes the expectation taken with respect to measure $\mathbb{P}_h^\theta$ given initial data (t,x).

6.1.7 The HJB Equation

Let Φ be the value function for the auxiliary criterion function $I(f_0,x;h;t,T)$. Then Φ is defined as

$$\Phi(t,x) = \sup_{h \in \mathcal{A}(T)} I(f_0,x;h;t,T)$$

and it satisfies the RS HJB PIDE

$$\frac{\partial \Phi}{\partial t} + \sup_{h \in J} L_t^h \Phi(t, X(t)) = 0 \tag{6.9}$$

where

$$\begin{aligned} & L_t^h \Phi(t,x) \\ = & \left(b + Bx - \theta \Lambda \left(\Sigma' h(s) - \varsigma \right) + \int_{\mathbf{Z}} \xi(z) \left[\left(\frac{1 + h'\gamma(z)}{1 + \zeta(z)} \right)^{-\theta} - 1_{\mathbf{Z}_0}(z) \right] \nu(dz) \right)' D\Phi \\ & + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda' D^2 \Phi \right) - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi \\ & + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(\Phi(t,x+\xi(z)) - \Phi(t,x))} - 1 \right) - \xi'(z) D\Phi \right\} \nu(dz) - g(x,h;\theta) \end{aligned} \tag{6.10}$$

and subject to terminal condition $\Phi(T,x) = \ln f_0 = \ln \frac{v}{l}$.

6.1.8 Solving the Control Problem

The terms inside the sup can be expressed as:

$$\begin{aligned} & \sup_{h \in J} L_t^h \Phi \\ = & \sup_{h \in J} \left\{ \left(b + Bx - \theta \Lambda \left(\Sigma' h(s) - \varsigma \right) + \int_{\mathbf{Z}} \xi(z) \left[\left(\frac{1 + h'\gamma(z)}{1 + \zeta(z)} \right)^{-\theta} - 1_{\mathbf{Z}_0}(z) \right] \nu(dz) \right)' D\Phi \right. \\ & + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda' D^2 \Phi \right) - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi \\ & + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(\Phi(t,x+\xi(z)) - \Phi(t,x))} - 1 \right) - \xi'(z) D\Phi \right\} \nu(dz) \\ & - \frac{1}{2} (\theta + 1) h' \Sigma \Sigma' h + a_0 + A_0' x + h'(\hat{a} + \hat{A} x) \\ & + \frac{1}{2} \theta (h' \Sigma \varsigma + \varsigma' \Sigma' h) - (c + C'x) - \frac{1}{2} (\theta - 1) \varsigma' \varsigma \\ & \left. - \int_0^t \int_{\mathbf{Z}} \left\{ \frac{1}{\theta} \left[\left(\frac{1 + h'\gamma(z)}{1 + \zeta(z)} \right)^{-\theta} - 1 \right] + \left(h'\gamma(z) - \zeta(z) \right) 1_{\mathbf{Z}_0}(z) \right\} \nu(dz) ds \right\} \end{aligned}$$

$$
\begin{aligned}
& \sup_{h\in J} L_t^h \Phi \\
= \; & (b+Bx)'D\Phi + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2\Phi\right) - \frac{\theta}{2}(D\Phi)'\Lambda\Lambda' D\Phi + a_0 + A_0' x \\
& -(c+C'x) - \frac{1}{2}(\theta-1)\varsigma'\varsigma \\
& + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta}\left(e^{-\theta(\Phi(t,x+\xi(z))-\Phi(t,x))} - 1\right) - \xi'(z)D\Phi\, 1_{\mathbf{Z}_0}(z) \right\} \nu(dz) \\
& + \sup_{h\in J} \left\{ -\frac{1}{2}(\theta+1)\, h'\Sigma\Sigma' h + h'(\hat{a}+\hat{A}x) + \frac{1}{2}\theta(h'\Sigma\varsigma + \varsigma'\Sigma' h) - \theta\left(\Sigma' h(s) - \varsigma\right)'\Lambda' D\Phi \right. \\
& -\frac{1}{\theta}\int_{\mathbf{Z}} (1-\theta\xi'(z)D\Phi)\left[\left(\frac{1+h'\gamma(z)}{1+\zeta(z)}\right)^{-\theta} - 1\right]\nu(dz) \\
& \left. -\int_0^t \int_{\mathbf{Z}} \left\{ \left(h'\gamma(z) - \zeta(z)\right) 1_{\mathbf{Z}_0}(z) \right\} \nu(dz)ds \right\} \qquad (6.11)
\end{aligned}
$$

The term

$$
\begin{aligned}
& -\frac{1}{2}(\theta+1)\, h'\Sigma\Sigma' h + h'(\hat{a}+\hat{A}x) + \frac{1}{2}\theta(h'\Sigma\varsigma + \varsigma'\Sigma' h) - \theta\left(\Sigma' h(s) - \varsigma\right)'\Lambda' D\Phi \\
& -\int_0^t \int_{\mathbf{Z}} \left\{ \left(h'\gamma(z) - \zeta(z)\right) 1_{\mathbf{Z}_0}(z) \right\} \nu(dz)ds
\end{aligned}
$$

is concave in h $\forall z \in \mathbf{Z}$ a.s. $d\nu$ and under Assumption 6.1, the jump-related term

$$
-\frac{1}{\theta}\int_{\mathbf{Z}} \left\{ (1-\theta\xi'(z)D\Phi)\left[\left(\frac{1+h'\gamma(z)}{1+\zeta(z)}\right)^{-\theta} - 1\right] \right\} \nu(dz)
$$

simplifies to

$$
-\frac{1}{\theta}\int_{\mathbf{Z}} \left\{ \left[\left(\frac{1+h'\gamma(z)}{1+\zeta(z)}\right)^{-\theta} - 1\right] \right\} \nu(dz)
$$

Therefore, the supremum is reached for a unique optimal control h^*, which is an interior point of the set J defined in equation (6.5), and the supremum evaluated at h^* is finite.

6.1.9 Properties of the Value Function of the Exponentially Transformed Problem

In the benchmark case, Assumption (5.22) and Proposition (5.25) can be restated as

Assumption 6.3. There exists $2n$ constant controls $\bar{h}^k, k = 1, \ldots, 2n$ such that the functions

$$\beta^k(t) = \theta B^{-1}\left(1 - e^{B(T-t)}\right)\left(A_0 + \bar{h}^k\hat{A} - C\right) \tag{6.12}$$

and

$$\alpha^k(t) = -\int_t^T q^k(s)ds \tag{6.13}$$

where

$$\begin{aligned} q^k(t) \;:=\; & \left(b - \theta\Lambda\left(\Sigma'\bar{h}(s) - \varsigma\right) + \int_{\mathbf{Z}} \xi(z)\left[\left(\frac{1+\bar{h}'\gamma(z)}{1+\zeta(z)}\right)^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)'\beta^{k'}(t) \\ & + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\beta^{k'}(t)\beta^k(t)\right) + \int_{\mathbf{Z}}\left\{e^{\beta^k\xi(z)} - 1 - \xi'(z)\beta^{k'}(t)\right\}\nu(dz) \\ & + \frac{1}{2}\theta\left(\theta+1\right)\bar{h}'\Sigma\Sigma'\bar{h} - \theta a_0 + \theta c - \theta\hat{a} \\ & - \frac{1}{2}\theta^2(\bar{h}'\Sigma\varsigma + \varsigma'\Sigma'\bar{h}) + (c + C'x) + \frac{1}{2}\left(\theta - 1\right)\varsigma'\varsigma \\ & + \theta\int_{\mathbf{Z}}\left\{\frac{1}{\theta}\left[\left(\frac{1+\bar{h}'\gamma(z)}{1+\zeta(z)}\right)^{-\theta} - 1\right] + \left(\bar{h}'\gamma(z) - \zeta(z)\right)1_{\mathbf{Z}_0}(z)\right\}\nu(dz) \end{aligned} \tag{6.14}$$

exist and for $i = 1, \ldots, n$ satisfy:

$$\begin{aligned} \beta_i^i(t) &< 0 \\ \beta_i^{n+i}(t) &> 0 \end{aligned} \tag{6.15}$$

where $\beta_j^i(t)$ denotes the j-th component of the vector $\beta^i(t)$.

Note that Remarks 5.23 and 5.24 in Chapter 5 can readily be extended to the benchmark case.

Proposition 6.4. *Assume that*

(i). $g(t,x,h)$ is locally bounded on $[0,T]\times\mathbb{R}^n\times J$ and is Lipschitz continuous in x;

(ii). Assumption 6.3 holds.

Then,

(i). Lower Bound.

$$\Phi(t,x)\geq 0\ \forall x\in\mathbb{R}^n$$

and

$$\tilde{\Phi}(t,x)>0\ \forall x\in\Re:=\{x\in\mathbb{R}^n:\|x\|_\infty<\infty\}$$

(ii). Upper Bounds. For the $2n$ constant controls $\tilde{h}^k, k=1,\ldots,2n$ parameterizing the $4n$ functions

$$\alpha^k: t\in[0,T]\to\mathbb{R},\ k=1,\ldots,2n$$

$$\beta^k: t\in[0,T]\to\mathbb{R}^n,\ k=1,\ldots,2n$$

such that for $i=1,\ldots,n$,

$$\begin{aligned}\beta_i^i(t) &< 0\\ \beta_i^{n+i}(t) &> 0\end{aligned}$$

where $\beta_j^i(t)$ denotes the j-th component of the vector $\beta^i(t)$, then we have the following upper bounds:

$$\tilde{\Phi}(t,x) \leq e^{\alpha^k(t)+\beta^{k'}(t)x}$$

in each element $x_i, i = 1, \ldots, n$ of x.

Proof. Since the proof proceeds in the same was as the proof of Proposition (5.25), we will only indicate the differences between the two proofs.

Upper Bounds. Set $\mathbf{Z} = \mathbb{R}^n \setminus \{0\}$ and consider the class $\mathcal{H}^c$ of constant controls. By the optimality principle, for an arbitrary admissible constant control policy $\bar{h}$, we have

$$\tilde{\Phi}(t,x) \leq \tilde{I}(x;\bar{h};t,T) \leq \mathrm{E}_{t,x}\left[\exp\left\{\theta\int_t^T g(s,X_s,\bar{h})ds - \theta\ln f_0\right\}\right] := W(t,x) \quad (6.16)$$

Since the function g is an affine function of the affine process $X(t)$, we expect that $W(t,x)$ takes the form

$$W(t,x) = \exp\{\alpha(t) + \beta'(t)x\} \quad (6.17)$$

for two functions

$$\alpha : t \in [0,T] \to \mathbb{R}$$

$$\beta : t \in [0,T] \to \mathbb{R}^n$$

to be defined.

Applying the Feynman-Kac formula, we deduce that $W(t,x)$ satisfies the integro-differential PDE:

$$\begin{aligned}
&\frac{\partial W}{\partial t}+\left(b+Bx-\theta\Lambda\left(\Sigma'\bar{h}-\varsigma\right)+\int_{\mathbf{Z}}\xi(z)\left[\left(\frac{1+\bar{h}'\gamma(z)}{1+\zeta(z)}\right)^{-\theta}-1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)'DW\\
&+\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'D^2W\right)+\int_{\mathbf{Z}}\left\{W(t,x+\xi(z))-W(t,x)-\xi'(z)DW(t,x)\right\}\nu(dz)\\
&+\theta g(x,\bar{h};\theta)W(t,x)=0
\end{aligned}$$

subject to terminal condition $\tilde{\Phi}(T,x)=f_0^{-\theta}$.

Substituting the form for the candidate solution into the PDE, dividing by $W(t,x)$ and rearranging, we deduce that β and α respectively satisfy the ODEs:

$$\dot{\beta}(t)+B'\beta(t)-\theta A_0'-\theta\bar{h}'\hat{A}+\theta C=0 \tag{6.18}$$

and

$$\begin{aligned}
&\dot{\alpha(t)}+\left(b-\theta\Lambda\left(\Sigma'\bar{h}-\varsigma\right)+\int_{\mathbf{Z}}\xi(z)\left[\left(\frac{1+\bar{h}'\gamma(z)}{1+\zeta(z)}\right)^{-\theta}-1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)'\beta'(t)\\
&+\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\beta'(t)\beta(t)\right)+\int_{\mathbf{Z}}\left\{e^{\beta\xi(z)}-1-\xi'(z)\beta'(t)\right\}\nu(dz)\\
&+\frac{1}{2}\theta\left(\theta+1\right)\bar{h}'\Sigma\Sigma'\bar{h}-\theta a_0+\theta c-\theta\hat{a}\\
&-\frac{1}{2}\theta^2(\bar{h}'\Sigma\varsigma+\varsigma'\Sigma'\bar{h})+(c+C'x)+\frac{1}{2}\left(\theta-1\right)\varsigma'\varsigma\\
&+\theta\int_{\mathbf{Z}}\left\{\frac{1}{\theta}\left[\left(\frac{1+\bar{h}'\gamma(z)}{1+\zeta(z)}\right)^{-\theta}-1\right]+\left(\bar{h}'\gamma(z)-\zeta(z)\right)1_{\mathbf{Z}_0}(z)\right\}\nu(dz)=0
\end{aligned} \tag{6.19}$$

The ODE (6.18) for β is linear and admits the solution

$$\beta(t) = \theta B^{-1}\left(1 - e^{B(T-t)}\right)\left(A_0 + \bar{h}^k \hat{A} - C\right) \tag{6.20}$$

As for the ODE (6.19) for α, we only need to integrate to get

$$\alpha(t) = -\int_t^T q(s)ds \tag{6.21}$$

where

$$\begin{aligned} q(t) \;:=\; & \left(b - \theta\Lambda\left(\Sigma'\bar{h}(s) - \varsigma\right) + \int_{\mathbf{Z}} \xi(z)\left[\left(\frac{1+\bar{h}'\gamma(z)}{1+\zeta(z)}\right)^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)'\beta'(t) \\ & + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\beta'(t)\beta(t)\right) + \int_{\mathbf{Z}}\left\{e^{\beta\xi(z)} - 1 - \xi'(z)\beta'(t)\right\}\nu(dz) \\ & + \frac{1}{2}\theta\left(\theta+1\right)\bar{h}'\Sigma\Sigma'\bar{h} - \theta a_0 + \theta c - \theta\hat{a} \\ & - \frac{1}{2}\theta^2(\bar{h}'\Sigma\varsigma + \varsigma'\Sigma'\bar{h}) + (c + C'x) + \frac{1}{2}\left(\theta-1\right)\varsigma'\varsigma \\ & + \theta\int_{\mathbf{Z}}\left\{\frac{1}{\theta}\left[\left(\frac{1+\bar{h}'\gamma(z)}{1+\zeta(z)}\right)^{-\theta} - 1\right] + \left(\bar{h}'\gamma(z) - \zeta(z)\right)1_{\mathbf{Z}_0}(z)\right\}\nu(dz) \end{aligned}$$

As expected, equations (6.20) and (6.21) are respectively equations (6.12) and (6.13) defined in Assumption 6.3

There is no further change to the proof.

□

6.1.10 Viscosity Solution of the HJB PDE

Overall, the results developed in Chapter 5 remain valid if we choose:

$$f(t,x,h) := b + Bx - \theta\Lambda\left(\Sigma' h(s) - \varsigma\right) + \int_{\mathbf{Z}} \xi(z)\left[\left(\frac{1+h'\gamma(z)}{1+\zeta(z)}\right)^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)$$

$$\Lambda(t) := \Lambda$$

$g(t,x,h)$ is defined in (6.5)

$$g_T(x,T) := -\theta \ln f_0$$

We also note that

- f satisfies the growth condition (5.1) and f and Λ satisfy the Lipschitz condition (5.2);
- g_T is continuous in $\mathbb{R}^n$;
- g is affine in x and continuous in t and h,

Finally, by proposition (6.4), the value function $\tilde{\Phi}$ satisfies Assumption 5.14. Therefore, by Theorem 5.17, Theorem 5.19 and Corollary 5.20, Φ is the unique viscosity solution of (6.9) and it is a continuous function.

6.2 Asset and Liability Management

In this section, we extend our earlier analysis of the ALM problem, by allowing jumps in the valuation factors, asset prices, as well as in the value of the liability.

6.2.1 Factor, Asset and Portfolio Dynamics

The factors, risky assets and money market asset follow the dynamic given in Sections 4.1.3 and 4.1.4.

6.2.2 Wealth Dynamics and Control Policy

Let $\mathcal{G}_t := \sigma((S(s), X(s), L(s)), 0 \leq s \leq t)$ be the sigma-field generated by the security, liability and factor processes up to time t.

In order for our control to be defined on a compact set, we still define a control process $\tilde{h}$ on the compact set $\overline{\mathcal{O}}$ where $\mathcal{O} := \{]-1, 1[\}^m$. The asset allocation policy is then defined as

$$h_i = \begin{cases} -\infty, & \tilde{h}_i = -1 \\ \tanh^{-1}(\tilde{h}_i), & \tilde{h}_i \in]-1, 1[\\ +\infty, & \tilde{h}_i = 1 \end{cases}$$

for $i = 1, \ldots, m$

In this case, we need to introduce a slightly modified definition:

Definition 6.5. *A control process $\tilde{h}(t)$ is in class $\tilde{\mathcal{H}}$ if the following conditions are satisfied:*

1. $\tilde{h}(t) \in \overline{\mathcal{O}}$;
2. $\tilde{h}(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t \geq 0}$;
3. $P\left(\int_0^T |h(s)|^2 \, ds < +\infty\right) = 1, \quad \forall T > 0$;

The wealth dynamics remains as in section 4.1.5.

6.2.3 Dynamics of the Liability

In a jump-diffusion setting, the value of the liability, $L(t)$, follows the dynamics

$$\frac{dL(t)}{L(t)} = (c + C'X(t))dt + \varsigma' dW(t) + \int_{\mathbf{Z}} \zeta(z)\bar{N}_{\mathbf{p}}(dt, dz), \qquad L(0) = l \tag{6.22}$$

where

- $c \in \mathbb{R}$;
- $C \in \mathbb{R}^n$;
- $\varsigma \in \mathbb{R}^M$; and
- $\zeta(z) \in \mathbb{R}$, with $-1 < \zeta^{min} \leq \zeta(z) \leq \zeta^{max} < +\infty$.

In addition, the $\zeta(z)$ satisfies:

$$\int_{\mathbf{Z_0}} |\zeta(z)|^2 \nu(dz) < \infty \tag{6.23}$$

Remark 5.3 in Chapter 5 also applies to assumption 6.23.

To ensure the existence of a unique ALM strategy in Section 6.2.10, we will require the following assumption:

Assumption 6.6. The systematic (factor-driven) and idiosyncratic (asset-driven and liability-driven) jump risks are uncorrelated, i.e $\forall z \in \mathbb{Z}$, $\gamma(z)\xi'(z) = 0$ and $\zeta(z)\xi'(z) = 0$.

6.2.4 Equity and Leverage

As in the diffusion case, we define the time t equity $E(t)$ as

$$E(t) = V(t) - L(t), \quad E(0) = e_0 = v - l > 0$$

and introduce a second control variable, the time t leverage ratio $\rho(t)$, defined as

$$\rho(t) = \frac{V(t)}{E(t)}$$

Formally, we will consider a scalar control $\tilde{\rho}$ defined on the compact set $[0, 1]$. The actual leverage ratio ρ will then be defined through a rescaling of the formal scalar control vector $\tilde{\rho}$:

$$\rho := -\ln \tilde{\rho}$$

The scalar control $\tilde{\rho}$ satisfies the definition:

Definition 6.7. A leverage process $\tilde{\rho}(t)$ is in class $\tilde{\mathcal{R}}$ if the following conditions are satisfied:

1. $\tilde{\rho}(t) \in \mathbb{R}^+$;

2. $\rho(t)$ is progressively measurable with respect to $\{\mathcal{B}([0,t]) \otimes \mathcal{G}_t\}_{t \geq 0}$;

3. $P\left(\int_0^T |\rho(s)|^2 \, ds < +\infty\right) = 1$;

Once again, this construction enables us to define the control on a compact set, which will be useful when we solve the RS HJB PIDE associated with this problem using viscosity solutions. For practical purpose, the relevant control is the auxiliary vector ρ which gives the leverage ratio. As for h, the construction will be justified when we conclude that the optimal asset allocation is an interior point of a set $\overline{J} \subset \mathbb{R}^m$

defined in equation (6.28),

We now combine the definitions of the two controls into a formal ALM investment policy $\pi(t)$ defined as a pair $(h(t), \rho(t))$ satisfying the following definition:

Definition 6.8. An ALM investment policy $\pi(t) := (h(t), \rho(t))$ is in class Π if the following conditions are satisfied:

1. $\tilde{h}(t)$ is in class $\tilde{\mathcal{H}}$;

2. $\rho(t)$ is in class $\mathcal{R}$;

3. $h'\gamma(z) \geq \frac{-1+(\rho-1)\zeta(z)}{\rho}, \quad \forall t > 0, z \in \mathbf{Z}$, a.s. $d\nu$.

6.2.5 Equity Dynamics

In differential terms, the dynamics of the equity can be expressed as

$$\begin{aligned} dE(t) &= V(t^-)\left[(a_0 + A_0'X(t))\,dt + h'(t)\left(\hat{a} + \hat{A}X(t)\right)dt + h'(t)\Sigma dW_t\right. \\ &\quad \left. + \int_{\mathbf{Z}} h'(t)\gamma(z)\bar{N}_{\mathbf{p}}(dt, dz)\right] - L(t^-)\left[(c + C'X(t))L(t^-)dt\right. \\ &\quad \left. + \varsigma' dW(t) + \int_{\mathbf{Z}} \zeta(z)\bar{N}_{\mathbf{p}}(dt, dz)\right] \end{aligned}$$

Rewriting in terms of equity and leverage only, we get

$$\begin{aligned} \frac{dE(t)}{E(t^-)} &= (c + C'X(t))dt \\ &\quad + \rho(t)\left[(a_0 + A_0'X(t)) + h'(t)\left(\hat{a} + \hat{A}X(t)\right) - (c + C'X(t))\right]dt \\ &\quad + \left[\rho(t)(h'(t)\Sigma - \varsigma') + \varsigma'\right]dW(t) \\ &\quad + \int \left[\rho(t)(h'(t)\gamma(z) - \zeta(z)) + \zeta(z)\right]\bar{N}_{\mathbf{p}}(dt, dz), \\ &\quad E(0) = e_0 \end{aligned} \tag{6.24}$$

6.2.6 Optimization Criterion

The principal objective of funded investors is to maximize their risk-sensitive return on equity. By Itô's lemma, the log return on equity in response to a strategy (h, ρ) is

$$\begin{aligned}
\ln E(t) \;=\; & e_0 + \int_0^t c + C'X(s)ds \\
& + \int_0^t \rho(s)\left[(a_0 + A_0'X(t)) + h'(s)(\hat{a} + \hat{A}X(s)) - c - C'X(s)\right] ds \\
& - \frac{1}{2}\int_0^t \left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right]\left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right]' ds \\
& + \int_0^t \left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right] dW_s \\
& + \int_0^t \int_{\mathbf{Z}_0} \{\ln\left[1 + \zeta(z) + \rho(s)(h'(s)\gamma(z) - \zeta(z))\right] \\
& - \left[\rho(s)(h'(s)\gamma(z) - \zeta(z)) + \zeta(z)\right]\} \nu(dz)ds \\
& + \int_0^t \int_{\mathbf{Z}} \ln\left[1 + \zeta(z) + \rho(s)(h'(s)\gamma(z) - \zeta(z))\right] \bar{N}_{\mathbf{p}}(ds, dz)
\end{aligned} \tag{6.25}$$

We now consider the class of problems $\mathcal{P}_{\theta,T}$: for $\theta \in]0, +\infty[$, maximize the risk sensitive expected log return on equity over a time horizon T, and introduce the related criterion function $J_{\theta,T}$ defined as

$$J_{\theta,T}(v, l, x; h, \rho) := \left(-\frac{1}{\theta}\right) \ln \mathbf{E} e^{-\theta \ln E(t;h,\rho)} \tag{6.26}$$

6.2.7 Criterion Under the Expectation

Multiplying by $-\theta$ and taking the exponential on both sides of (6.25), we get

$$\begin{aligned} e_0^{-\theta \ln E(t)} &= e^{-\theta} \exp\left\{\theta \int_0^t g(X_s, h(s), \rho(s); \theta) ds - \theta \int_0^t \left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right] dW_s \right. \\ &\quad -\frac{\theta^2}{2} \int_0^t \left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right] \left[\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'\right]' ds \\ &\quad + \int_0^t \int_{\mathbf{Z}} \ln\left(1 - H(z, h(s), \rho(s); \theta)\right) \tilde{N}_{\mathbf{p}}(ds, dz) \\ &\quad \left. + \int_0^t \int_{\mathbf{Z}} \left\{\ln\left(1 - H(z, h(s), \rho(s); \theta)\right) + H(z, h(s), \rho(s); \theta)\right\} \nu(dz) ds \right\} \end{aligned}$$

where

$$\begin{aligned} g(x, h, \rho; \theta) &= \frac{1}{2}(\theta + 1) \left[\rho(h'\Sigma - \varsigma') + \varsigma'\right] \left[\rho(h'\Sigma - \varsigma') + \varsigma'\right]' \\ &\quad -\rho\left[h'(\hat{a} + \hat{A}x) + a_0 - c + (A_0' - C')x\right] - c - C'x \\ &\quad + \int_{\mathbf{Z}} \left\{\frac{1}{\theta}\left(\left[1 + \zeta(z) + \rho(s)(h'(s)\gamma(z) - \zeta(z))\right]^{-\theta} - 1\right)\right. \\ &\quad \left. + \left[\rho(s)(h'(s)\gamma(z) - \zeta(z)) + \zeta(z)\right] 1_{\mathbf{Z}_0}(z)\right\} \nu(dz) \end{aligned} \tag{6.27}$$

$$H(z, h, \rho; \theta) := 1 - \left[1 + \zeta(z) + \rho(s)(h'(s)\gamma(z) - \zeta(z))\right]^{-\theta}$$

6.2.8 Change of Measure

Let $\mathbb{P}_h^\theta$ be the measure on $(\Omega, \mathcal{F})$ defined as

$$\begin{aligned}
\chi_t &:= \left.\frac{d\mathbb{P}_h^\theta}{d\mathbb{P}}\right|_{\mathcal{F}_t} \\
&= \exp\left\{-\theta \int_0^t [\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma']\, dW_s \right. \\
&\quad -\frac{\theta^2}{2} \int_0^t [\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma'] [\rho(s)(h'(s)\Sigma - \varsigma') + \varsigma']'\, ds \\
&\quad + \int_0^t \int_{\mathbf{Z}} \ln\left(1 - H(z, h(s), \rho(s); \theta)\right) \tilde{N}_{\mathbf{p}}(ds, dz) \\
&\quad \left. + \int_0^t \int_{\mathbf{Z}} \left\{\ln\left(1 - H(z, h(s), \rho(s); \theta)\right) + H(z, h(s), \rho(s); \theta)\right\} \nu(dz) ds \right\}, \quad \forall t \geq 0
\end{aligned}$$

For a change of measure to be possible, we must ensure that the following technical condition holds:

$$H(z, h, \rho; \theta) \leq 1$$

This condition is satisfied iff

$$\begin{aligned}
1 - [1 + \zeta(z) + \rho(h'\gamma(z) - \zeta(z))]^{-\theta} \leq 1 \quad &\Leftrightarrow \quad 1 + \zeta(z) + \rho(h'\gamma(z) - \zeta(z)) \geq 0 \\
&\Leftrightarrow \quad \rho(h'\gamma(z) - \zeta(z)) \geq -1 - \zeta(z) \\
&\Leftrightarrow \quad \rho h'\gamma(z) \geq -1 + (\rho - 1)\zeta(z) \\
&\Leftrightarrow \quad h'\gamma(z) \geq \frac{-1 + (\rho - 1)\zeta(z)}{\rho}
\end{aligned}$$

a.s. $d\nu$. Note that this condition is already part of Definition 6.8.

To interpret this condition, multiply both sides of the penultimate line by the

equity level $E(t)$ to get

$$\rho E(t)h(t)'\gamma(z) \geq -E(t) + (\rho - 1)E(t)\zeta(z)$$

Thus, this condition states that under any scenario, a downward jump in the value of the assets (equal ρE) cannot result in a bigger loss than the entire equity plus the corresponding jump in the liability value (equal to. $(\rho - 1)E$). In short, the investment policies cannot put the investor at risk of losing more than the value of his/her equity as a result of a jump.

Now, define

$$\mathbf{S} := \text{supp}(\nu) \in \mathcal{B}_{\mathbf{Z}}$$

$$\mathbf{S}_\gamma := \text{supp}(\nu \circ \gamma^{-1}) \in \mathcal{B}(\mathbb{R}^m)$$

and

$$\mathbf{S}_\zeta := \text{supp}(\nu \circ \zeta^{-1}) \in \mathcal{B}(\mathbb{R})$$

where supp$(\cdot)$ denotes the measure's support. Let J be the set defined as

$$J := \left\{(h, \rho) \in \mathbb{R}^m \times \mathbb{R}^+ : -h'\psi + \frac{-1 + (\rho - 1)\varphi}{\rho} < 0 \quad \forall \psi \in \mathbf{S}_\gamma, \varphi \in \mathbf{S}_\zeta\right\} \tag{6.28}$$

and $\mathcal{K}$ be the set

$$\mathcal{K} := \{(h(t), \rho(t)) \in \mathcal{H} \times \mathcal{R} : h(t) \in J \quad \forall t \text{ a.s.}\} \tag{6.29}$$

We denote by $\mathcal{A}(T)$ the set of investment strategies $(h, \rho) \in \mathcal{H} \times \mathcal{R}$ on $[0, T]$ such

that $\mathbb{P}_h^\theta$ is an equivalent martingale measure and thus

$$\begin{aligned}
&\mathbf{E}\left[\exp\left\{-\theta\int_0^t[\rho(s)(h'(s)\Sigma-\varsigma')+\varsigma']\,dW_s\right.\right.\\
&-\frac{\theta^2}{2}\int_0^t[\rho(s)(h'(s)\Sigma-\varsigma')+\varsigma']\,[\rho(s)(h'(s)\Sigma-\varsigma')+\varsigma']'\,ds\\
&+\int_0^t\int_{\mathbf{Z}}\ln\left(1-H(z,h(s),\rho(s);\theta)\right)\tilde{N}_{\mathbf{p}}(ds,dz)\\
&\left.\left.+\int_0^t\int_{\mathbf{Z}}\left\{\ln\left(1-H(z,h(s),\rho(s);\theta)\right)+H(z,h(s),\rho(s);\theta)\right\}\nu(dz)ds\right\}\right]\\
&=1
\end{aligned}\tag{6.30}$$

For $h\in\mathcal{A}(T)$,

$$W_t^\theta=W_t+\theta\int_0^t(\rho(s)[\Sigma'h(s)-\varsigma]+\varsigma)\,ds$$

is a standard Brownian motion under the measure $\mathbb{P}_h^\theta$ and we have

$$\begin{aligned}
\int_0^t\int_{\mathbf{Z}}\tilde{N}_{\mathbf{p}}^\theta(ds,dz) &= \int_0^t\int_{\mathbf{Z}}N_{\mathbf{p}}(ds,dz)-\int_0^t\int_{\mathbf{Z}}\left\{1-H(z,h(s),\rho(s);\theta)\right\}\nu(dz)ds\\
&= \int_0^t\int_{\mathbf{Z}}N_{\mathbf{p}}(ds,dz)-\int_0^t\int_{\mathbf{Z}}\left\{(1+\zeta(z)+\rho(s)[h'(s)\gamma(z)-\zeta(z)])^{-\theta}\right\}\nu(dz)ds
\end{aligned}$$

As a result, $X(s),\ 0\le s\le t$ satisfies the SDE:

$$\begin{aligned}
dX(s) &= \Big(b+BX(s^-)-\theta\Lambda\left(\rho(s)[\Sigma'h(s)-\varsigma]+\varsigma\right)\\
&\quad+\int_{\mathbf{Z}}\xi(z)\left[(1+\zeta(z)+\rho(s)[h'(s)\gamma(z)-\zeta(z)])^{-\theta}-1_{\mathbf{Z}_0}(z)\right]\nu(dz)\Big)\,ds\\
&\quad+\Lambda dW_s^\theta+\int_{\mathbf{Z}}\xi(z)\tilde{N}_{\mathbf{p}}^\theta(ds,dz)
\end{aligned}$$

Under the measure $\mathbb{P}_h^\theta$, the auxiliary criterion function associated with the ALM

problem can be expressed as

$$I(e_0, x; h, \rho; t, T) = -\frac{1}{\theta} \ln \mathrm{E}_{t,x}^{h,\theta} \left[\exp \left\{ \theta \int_t^T g(X_s, h(s), \rho(s); \theta) ds - \theta \ln e_0 \right\} \right] \quad (6.31)$$

where $\mathrm{E}_{t,x}^{h,\theta}[\cdot]$ denotes the expectation taken with respect to measure $\mathbb{P}_h^\theta$ and given initial data (t, x).

6.2.9 The HJB Equation

Let Φ be the value function for the auxiliary criterion function $I(e_0, x; h, \rho; t, T)$. Then Φ is defined as

$$\Phi(t, x) = \sup_{\mathcal{A}(T)} I(v, x; h, \rho; t, T)$$

and it satisfies the RS HJB PIDE

$$\frac{\partial \Phi}{\partial t} + \sup_{\pi \in J} L_t^h \Phi(t, X(t)) = 0 \quad (6.32)$$

where

$$\begin{aligned} & L_t^h \Phi(t, x) \\ = \; & (b + Bx - \theta \Lambda (\rho(s)[\Sigma' h(s) - \varsigma] + \varsigma) \\ & + \int_{\mathbf{Z}} \xi(z) \left[(1 + \zeta(z) + \rho(s)[h'(s)\gamma(z) - \zeta(z)])^{-\theta} - 1_{\mathbf{Z}_0}(z) \right] \nu(dz) \Big)' D\Phi \\ & + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda' D^2 \Phi \right) - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi - g(x, h, \rho; \theta) \\ & + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(\Phi(t, x+\xi(z)) - \Phi(t,x))} - 1 \right) - \xi'(z) D\Phi \right\} \nu(dz) \end{aligned} \quad (6.33)$$

and subject to terminal condition $\Phi(T, x) = \ln e_0 = \ln(v - l)$.

This result could be proved in exactly the same way as Proposition 4.6, by simply considering an investment policy $\pi = (h, \rho)$ instead of the allocation strategy h.

6.2.10 Solving the Control Problem

The terms inside the sup can be expressed as:

$$
\begin{aligned}
& \sup_{\pi \in J} L_t^h \Phi \\
= \; & \sup_{\pi \in J} \{ (b + Bx - \theta \Lambda \left(\rho(s)[\Sigma' h(s) - \varsigma] + \varsigma \right) \\
& + \int_{\mathbf{Z}} \xi(z) \left[(1 + \zeta(z) + \rho(s)[h'(s)\gamma(z) - \zeta(z)])^{-\theta} - 1_{\mathbf{Z}_0}(z) \right] \nu(dz) \Big)' D\Phi \\
& + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda' D^2 \Phi \right) - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi \\
& + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(\Phi(t, x + \xi(z)) - \Phi(t,x))} - 1 \right) - \xi'(z) D\Phi \right\} \nu(dz) \\
& - \frac{1}{2} (\theta + 1) \left[\rho(h'\Sigma - \varsigma') + \varsigma' \right] \left[\rho(h'\Sigma - \varsigma') + \varsigma' \right]' \\
& + \rho \left[h'(\hat{a} + \hat{A}x) + a_0 - c + (A_0' - C')x \right] + c + C'x \\
& - \int_{\mathbf{Z}} \left\{ \frac{1}{\theta} \left(\left[1 + \zeta(z) + \rho(s)(h'(s)\gamma(z) - \zeta(z)) \right]^{-\theta} - 1 \right) \right. \\
& \left. + \left[\rho(s)(h'(s)\gamma(z) - \zeta(z)) + \zeta(z) \right] 1_{\mathbf{Z}_0}(z) \right\} \nu(dz) \} \\
= \; & (b + Bx)' D\Phi + \frac{1}{2} \mathrm{tr} \left(\Lambda \Lambda' D^2 \Phi \right) - \frac{\theta}{2} (D\Phi)' \Lambda \Lambda' D\Phi + c + C'x \\
& + \int_{\mathbf{Z}} \left\{ -\frac{1}{\theta} \left(e^{-\theta(\Phi(t, x + \xi(z)) - \Phi(t,x))} - 1_{\mathbf{Z}_0}(z) \right) - \xi'(z) D\Phi \right\} \nu(dz) \\
& + \sup_{\pi \in J} \left\{ -\frac{1}{2} (\theta + 1) \left[\rho(h'\Sigma - \varsigma') + \varsigma' \right] \left[\rho(h'\Sigma - \varsigma') + \varsigma' \right]' \right. \\
& - \theta \left(\rho(s)[\Sigma' h(s) - \varsigma] + \varsigma \right)' \Lambda' D\Phi + \rho \left[h'(\hat{a} + \hat{A}x) + a_0 - c + (A_0' - C')x \right] \\
& - \frac{1}{\theta} \int_{\mathbf{Z}} (1 - \theta \xi'(z) D\Phi) \left[(1 + \zeta(z) + \rho(s)[h'(s)\gamma(z) - \zeta(z)])^{-\theta} - 1 \right] \nu(dz) \\
& \left. - \int_{\mathbf{Z}} \left[\rho(s)(h'(s)\gamma(z) - \zeta(z)) + \zeta(z) \right] 1_{\mathbf{Z}_0}(z) \nu(dz) \right\} \qquad (6.34)
\end{aligned}
$$

The term

$$-\frac{1}{2}(\theta+1)\left[\rho(h'\Sigma-\varsigma')+\varsigma'\right]\left[\rho(h'\Sigma-\varsigma')+\varsigma'\right]'$$
$$-\theta\left(\rho(s)[\Sigma' h(s)-\varsigma]+\varsigma\right)'\Lambda' D\Phi+\rho\left[h'(\hat{a}+\hat{A}x)+a_0-c+(A_0'-C')x\right]$$
$$-\int_{\mathbf{Z}}\left[\rho(s)(h'(s)\gamma(z)-\zeta(z))+\zeta(z)\right]1_{\mathbf{Z}_0}(z)\nu(dz)$$

is concave in h and ρ $\forall z \in \mathbf{Z}$ a.s. $d\nu$ and under Assumption 6.6, the nonlinear jump-related term

$$-\frac{1}{\theta}\int_{\mathbf{Z}}(1-\theta\xi'(z)D\Phi)\left[(1+\zeta(z)+\rho(s)[h'(s)\gamma(z)-\zeta(z)])^{-\theta}-1\right]\nu(dz)$$

simplifies to

$$-\frac{1}{\theta}\int_{\mathbf{Z}}\left\{(1+\zeta(z)+\rho(s)[h'(s)\gamma(z)-\zeta(z)])^{-\theta}-1\right\}\nu(dz)$$

Therefore, the supremum is reached for a unique optimal control $\pi^* = (h^*, \rho^*)$, which is an interior point of the set J defined in equation (6.28), and the supremum evaluated at π^* is finite.

6.2.11 Properties of the Value Function of the Exponentially Transformed Problem

In the ALM case, Assumption (5.22) and Proposition (5.25) can be restated as

Assumption 6.9. There exists $2n$ pairs of constant controls $\bar{\pi}^k = (\bar{\rho}^k, \bar{h}^k), k = 1, \ldots, 2n$ such that the functions

$$\beta^k(t) = \theta B^{-1}\left(1-e^{B(T-t)}\right)\left(\bar{\rho}^k A_0' + \bar{\rho}^k \bar{h}^{k\prime}\hat{A} - (\bar{\rho}^k - 1)C\right) \tag{6.35}$$

and

$$\alpha^k(t) = -\int_t^T q(s)ds \tag{6.36}$$

where

$$\begin{aligned} q(t) \quad := \quad & \left(b + Bx - \theta\Lambda\left(\bar{\rho}^k[\Sigma'\bar{h}^k - \varsigma] + \varsigma\right)\right. \\ & \left. + \int_{\mathbf{Z}} \xi(z)\left[\left(1 + \zeta(z) + \bar{\rho}^k[\bar{h}^{k'}\gamma(z) - \zeta(z)]\right)^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)'\beta^{k'}(t) \\ & + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\beta^{k'}(t)\beta^k(t)\right) + \int_{\mathbf{Z}}\left\{e^{\beta^k\xi(z)} - 1 - \xi'(z)\beta^{k'}(t)\right\}\nu(dz) \\ & \frac{1}{2}\theta\left(\theta+1\right)\left[\bar{\rho}^k(\bar{h}^{k'}\Sigma - \varsigma') + \varsigma'\right]\left[\bar{\rho}^k(\bar{h}^{k'}\Sigma - \varsigma') + \varsigma'\right]' \\ & -\theta\bar{\rho}^k a_0 - \theta\bar{\rho}^k\hat{a} + \theta(\bar{\rho}^k - 1)C \\ & + \int_{\mathbf{Z}}\left\{\frac{1}{\theta}\left(\left[1 + \zeta(z) + \bar{\rho}^k(\bar{h}^{k'}\gamma(z) - \zeta(z))\right]^{-\theta} - 1\right)\right. \\ & \left. + \left[\bar{\rho}^k(\bar{h}^{k'}\gamma(z) - \zeta(z)) + \zeta(z)\right]1_{\mathbf{Z}_0}(z)\right\}\nu(dz) \end{aligned}$$

exist and for $i = 1, \ldots, n$ satisfy:

$$\begin{aligned} \beta_i^i(t) \quad &< \quad 0 \\ \beta_i^{n+i}(t) \quad &> \quad 0 \end{aligned} \tag{6.37}$$

where $\beta_j^i(t)$ denotes the j-th component of the vector $\beta^i(t)$.

Note that Remarks 5.23 and 5.24 in Chapter 5 can readily be extended to the ALM case.

Proposition 6.10. *Assume that*

(i). $g(t,x,\pi)$ is locally bounded on $[0,T]\times\mathbb{R}^n\times J^{\Pi}$ and is Lipschitz continuous in x;

(ii). Assumption 6.9 holds.

Then,

(i). Lower Bound.

$$\tilde{\Phi}(t,x)\geq 0\ \forall x\in\mathbb{R}^n$$

and

$$\tilde{\Phi}(t,x)>0\ \forall x\in\mathfrak{R}:=\{x\in\mathbb{R}^n:\|x\|_\infty<\infty\}$$

(ii). Upper Bounds. For the $2n$ constant controls $(\bar{\rho}^k,\bar{h}^k), k=1,\ldots,2n$ parameterizing the $4n$ functions

$$\alpha^k: t\in[0,T]\to\mathbb{R},\ k=1,\ldots,2n$$

$$\beta^k: t\in[0,T]\to\mathbb{R}^n,\ k=1,\ldots,2n$$

such that for $i=1,\ldots,n$,

$$\begin{aligned}\beta_i^i(t) &< 0\\ \beta_i^{n+i}(t) &> 0\end{aligned}$$

where $\beta_j^i(t)$ denotes the j-th component of the vector $\beta^i(t)$, then we have the following upper bounds:

$$\tilde{\Phi}(t,x)\leq e^{\alpha^k(t)+\beta^{k'}(t)x}$$

in each element $x_i, i = 1, \ldots, n$ *of* x.

Proof. Since the proof proceeds in the same way as the proof of Proposition (5.25), we will only indicate the differences.

Upper Bounds. Set $\mathbf{Z} = \mathbb{R}^n \setminus \{0\}$ and consider the class Π^c of constant controls $\bar{\pi} := (\bar{\rho}, \bar{h})$. By the optimality principle, for an arbitrary admissible constant control policy $\bar{h}$, we have

$$\tilde{\Phi}(t,x) \leq \tilde{I}(x;\bar{h};t,T) \leq \mathbf{E}_{t,x}\left[\exp\left\{\theta \int_t^T g(s, X_s, \bar{h})ds - \theta \ln f_0\right\}\right] := W(t,x) \quad (6.38)$$

Since the function g is an affine function of the affine process $X(t)$, we expect that $W(t,x)$ takes the form

$$W(t,x) = \exp\{\alpha(t) + \beta(t)x\} \quad (6.39)$$

for two functions

$$\alpha : t \in [0,T] \to \mathbb{R}$$

$$\beta : t \in [0,T] \to \mathbb{R}^n$$

are functions solving two ODEs.

Applying the Feynman-Kac formula, we deduce that $W(t,x)$ satisfies the integro-

differential PDE:

$$\frac{\partial W}{\partial t} + \left(b + Bx - \theta\Lambda\left(\bar{\rho}[\Sigma'\bar{h} - \varsigma] + \varsigma\right)\right.$$
$$\left. + \int_{\mathbf{Z}} \xi(z)\left[\left(1 + \zeta(z) + \bar{\rho}[\bar{h}'\gamma(z) - \zeta(z)]\right)^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)' DW$$
$$+\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda' D^2 W\right) + \int_{\mathbf{Z}} \{W(t, x + \xi(z)) - W(t, x) - \xi'(z)DW(t, x)\}\,\nu(dz)$$
$$+\theta g(x, \bar{h}, \bar{\rho}; \theta)W(t, x) = 0$$

subject to terminal condition $\tilde{\Phi}(T, x) = f_0^{-\theta}$.

Substituting the form for the candidate solution into the PDE, dividing by $W(t, x)$ and rearranging, we deduce that C and c respectively satisfy the ODEs:

$$\dot{\beta}(t) + B'\beta'(t) - \theta\bar{\rho}A_0' - \theta\bar{\rho}\bar{h}'\hat{A} + \theta(\bar{\rho} - 1)C = 0 \tag{6.40}$$

and

$$\dot{\alpha}(t) + \left(b + Bx - \theta\Lambda\left(\bar{\rho}[\Sigma'\bar{h} - \varsigma] + \varsigma\right)\right.$$
$$\left. + \int_{\mathbf{Z}} \xi(z)\left[\left(1 + \zeta(z) + \bar{\rho}[\bar{h}'\gamma(z) - \zeta(z)]\right)^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)' \beta'(t)$$
$$+\frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\beta'(t)\beta(t)\right) + \int_{\mathbf{Z}} \left\{e^{\beta\xi(z)} - 1 - \xi'(z)\beta'(t)\right\}\nu(dz)$$
$$\frac{1}{2}\theta\left(\theta + 1\right)\left[\bar{\rho}(\bar{h}'\Sigma - \varsigma') + \varsigma'\right]\left[\bar{\rho}(\bar{h}'\Sigma - \varsigma') + \varsigma'\right]'$$
$$-\theta\bar{\rho}a_0 - \theta\bar{\rho}\hat{a} + \theta(\bar{\rho} - 1)C$$
$$+\int_{\mathbf{Z}} \left\{\frac{1}{\theta}\left(\left[1 + \zeta(z) + \bar{\rho}(\bar{h}'\gamma(z) - \zeta(z))\right]^{-\theta} - 1\right)\right.$$
$$\left. + \left[\bar{\rho}(\bar{h}'\gamma(z) - \zeta(z)) + \zeta(z)\right] 1_{\mathbf{Z}_0}(z)\right\}\nu(dz) = 0 \tag{6.41}$$

The ODE (6.40) for β is linear and admits the solution

$$\beta(t) = \theta B^{-1}\left(1 - e^{B(T-t)}\right)\left(\bar{\rho}A_0' + \bar{\rho}\bar{h}'\hat{A} - (\bar{\rho}-1)C\right) \tag{6.42}$$

As for the ODE (6.41) for α, we only need to integrate to get

$$\alpha(t) = -\int_t^T q(s)ds \tag{6.43}$$

where

$$\begin{aligned}
q(t) \;:=\; & \left(b + Bx - \theta\Lambda\left(\bar{\rho}[\Sigma'\bar{h} - \varsigma] + \varsigma\right)\right. \\
& \left. + \int_{\mathbf{Z}} \xi(z)\left[\left(1 + \zeta(z) + \bar{\rho}[\bar{h}'\gamma(z) - \zeta(z)]\right)^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz)\right)' \beta'(t) \\
& + \frac{1}{2}\mathrm{tr}\left(\Lambda\Lambda'\beta'(t)\beta(t)\right) + \int_{\mathbf{Z}}\left\{e^{\beta\xi(z)} - 1 - \xi'(z)\beta'(t)\right\}\nu(dz) \\
& \frac{1}{2}\theta\left(\theta+1\right)\left[\bar{\rho}(\bar{h}'\Sigma - \varsigma') + \varsigma'\right]\left[\bar{\rho}(\bar{h}'\Sigma - \varsigma') + \varsigma'\right]' \\
& -\theta\bar{\rho}a_0 - \theta\bar{\rho}\hat{a} + \theta(\bar{\rho}-1)C \\
& + \int_{\mathbf{Z}}\left\{\frac{1}{\theta}\left(\left[1 + \zeta(z) + \bar{\rho}(\bar{h}'\gamma(z) - \zeta(z))\right]^{-\theta} - 1\right)\right. \\
& \left. + \left[\bar{\rho}(\bar{h}'\gamma(z) - \zeta(z)) + \zeta(z)\right] 1_{\mathbf{Z}_0}(z)\right\}\nu(dz)
\end{aligned}$$

As expected, equations (6.42) and (6.43) are respectively equations (6.35) and (6.36) from Assumption 6.9

There is no change to the rest of the proof.

□

6.2.12 Viscosity Solution of the HJB PDE

The results developed in Chapter 5 remain valid with minor modifications to the proofs if instead of a single control h, we consider a more general investment policy π, and in particular, if we choose $\pi(t) = (h(t), \rho(t)) \in \Pi$ and $(\rho, h) \in \mathbb{R}^+ \times J =: J^{\Pi}$.

Theorem 5.17 can be extended to the ALM setting without much difficulty by substituting the investment policy $\pi = (\rho, h)$ for h in the proofs.

Theorem 6.11. *Assume that*

(i). g is locally bounded on $[0,T] \times \mathbb{R}^n \times J^{\Pi}$;

(ii). Φ is locally bounded on $[0,T] \times \mathbb{R}^n$;

(iii). $f(t,x,\pi)$ and $g(t,x,\pi)$ are continuous in $(t,x) \in [0,T] \times \mathbb{R}^n$ for fixed $\pi \in J^{\Pi}$;

(iv). F is continuous in $(t,x) \in [0,T] \times \mathbb{R}^n$.

then Φ is a (discontinuous) viscosity solution of the RS HJB PIDE (5.5) *on* $[0,T] \times \mathbb{R}^n$, *subject to terminal condition* (5.6).

Since the proof of Theorem 5.19 does not rely on arguments involving the specific control variable, the result can readily be extended to the ALM case by simply changing Assumption *(ii).* as follows:

Theorem 6.12 (Comparison Result on an Unbounded State Space(ALM)). *Assume that:*

(i). $g(t,x,\pi)$ is locally bounded on $[0,T] \times \mathbb{R}^n \times J^{\Pi}$ and is Lipschitz continuous in x;

(ii). for some constants $C, \bar{C}_f, \bar{C}_\Lambda > 0$

$$|f(t,x,\pi)| \leq \bar{C}_f\,(1+|t|+|x|)$$

$$|\Lambda(t)| \leq \bar{C}_\Lambda\,(1+|t|)$$

$$|f(t,x,\pi)-f(s,y,\pi)|+|\Lambda(t)-\Lambda(s)| \leq C\,[|t-s|+|x-y|]$$

$$\forall (t,x) \in [t_0,T)\times\mathbb{R}^n$$

(iii). $g_0 \in C(\mathbb{R}^n)$.

(iv). Assumption 5.14 holds.

Let $u \in USC([0,T]\times\mathbb{R}^n)$ *be a bounded from above viscosity subsolution of* (5.5) *and* $v \in LSC([0,T]\times\mathbb{R}^n)$ *be a bounded from below viscosity supersolution of* (5.5). *Then*

$$u \leq v \quad on\ [0,T]\times\mathbb{R}^n$$

Theorem 6.12 can be proved in the same way as Theorem (5.19), the only minor difference being that we need to consider a policy π rather that the policy h.

We can finally obtain uniqueness and continuity as a corollary to Theorem 6.12:

Corollary 6.13 (Uniqueness and Continuity). *The function* $\Phi(t,x)$ *defined on* $[0,T]\times\mathbb{R}^n$ *is the unique continuous viscosity solution of the RS HJB PIDE* (5.5) *subject to terminal condition* (5.6).

6.2.13 Concluding on the ALM Case

If we choose:

$$\begin{aligned} f(t,x,\pi) &:= b + BX(s^-) - \theta\Lambda\left(\rho(s)[\Sigma' h(s) - \varsigma] + \varsigma\right) \\ &\quad + \int_{\mathbf{Z}} \xi(z)\left[(1+\zeta(z)+\rho(s)[h'(s)\gamma(z)-\zeta(z)])^{-\theta} - 1_{\mathbf{Z}_0}(z)\right]\nu(dz) \\ \Lambda(t) &:= \Lambda \\ g(t,x,\pi) &\quad \text{is defined in (6.27)} \\ g_T(x,T) &:= -\theta \ln e_0 \end{aligned}$$

We also note that in the ALM case, We also note that

- f satisfies the growth condition (5.1) and f and Λ satisfy the Lipschitz condition (5.2);
- g_T is continuous in $\mathbb{R}^n$;
- g is affine in x and continuous in t and h,

Finally, by proposition (6.4), the value function $\tilde{\Phi}$ satisfies Assumption 5.14. Therefore, by Theorem 5.17, Theorem 5.19 and Corollary 5.20, Φ is the unique viscosity solution of (6.9) and it is continuous.

Extending our earlier results into a more general policy setting pays an additional dividend since we are now in a position to use directly Theorems 6.11 and 6.12 as well as Corollary 6.13 to establish existence, uniqueness and continuity of a viscosity solution to a risk-sensitive control problem subject to an arbitrary number of control variables. Assumption 6.10 and Proposition 6.36 will still need to be tailored to the specific control problem and to the structure of the policy.

6.3 Conclusion

In this chapter, we have applied the methodology and results derived in Chapters 4 and 5 to solve jump-diffusion versions of the risk sensitive benchmarked asset management problem formulated by Davis and Lleo [21], and of the risk sensitive asset and liability management problem solved in the diffusion case in Chapter 3.

In the benchmarked asset management problem, proving that the value function is the unique continuous viscosity solution of the associated Hamilton-Jacobi-Bellman Partial Integro Differential Equation could be done via a direct application of the results derived in Chapters 4 and 5.

The asset and liability management problem, on the other hand, required a slight extension of these results in order to accommodate the existence of a second control, the leverage ratio.

Chapter 7

Conclusion

Risk-sensitive control provides a promising theoretical framework to help address complex investment management problems. Its close relationship with mean-variance analysis and utility maximization and the fact that the effective dimensionality of the control problem is the number of factors rather than the number of assets make it particularly appealing.

In the diffusion case, our treatment of the asset and liability case further illustrates the flexibility of Bielecki and Pliska's original setting and of Kuroda and Nagai change-of-measure method. Indeed, the two-control asset and liability problem can be solved in the same way as the original asset only case or the benchmarked case. While the addition of a liability process and of a second control has increased the complexity of the problem, the complexity of solution method has remained unchanged. This versatility benefits the development of economic insights into the nature of the optimal asset allocation and optimal leverage as well as the formulation of mutual fund theorem. In particular, we have noted that the allocation strategies for the asset only, benchmarked and ALM case all share the same fundamental structure, namely an allocation to the log utility portfolio, plus some additional allocations related to the benchmarked and ALM problems.

In the jump-diffusion case, analytical or even classical $C^{1,2}$ solutions do not generally exist. However, we have shown that even in the general random Poisson measure-based jump-diffusion setting we considered, the risk-sensitive asset management problem can be solved. Indeed, under some general conditions, a unique optimal control exists and the value function is the unique continuous viscosity solution of the risk sensitive Hamilton-Bellman-Jacobi integro partial differential equation associated with

the control problem. As in the diffusion case, these properties conveniently extend to the benchmarked and the ALM problems. The control problems could then be obtained numerically.

We have not yet exhausted the potential of risk-sensitive control and a number of important questions, such as the inclusion of investment constraints, or the issue of consumption, remain unanswered. In fact, if anything, the resolution of problems extending beyond the confines of the asset only case and the formulation of a jump-diffusion framework have significantly broadened the scope of potential research questions, by enabling not only a more precise formulation of the control problems and a more accurate modelling of the existing asset classes, but also by prompting investigations into the modelling of additional asset classes, such as credit securities.

Appendix A

Definitions from Dynamic System Theory

Definition A.1 (Stability). A $n \times n$ matrix L is *stable* if all the solutions to the ODE

$$\dot{x}(t) = Lx(t)$$

converge to 0 as $t \to \infty$.

Remark A.2. The matrix L is stable iff all its eigenvalues have negative real parts.

Remark A.3. A symmetric matrix L is stable iff

$$(-1)^k |L_k| > 0, \qquad k = 1, 2, \ldots$$

where $|L_k|$ denotes the k-th order principal leading minor of A.

The following definitions can be found in the Appendix of [40]

Definition A.4 (Stabilizability). The pair (L, M) of $n \times n$ matrix L and $n \times l$ matrix M is said to be *stabilizable* if there exists a $l \times n$ matrix K such that $L - MK$ is stable.

Definition A.5 (Detectability). The pair (L, F) of $n \times n$ matrix L and $l \times n$ matrix F is said to be *detectable* if (L', F') is stabilizable.

Definition A.6 (Controllability). The pair (L, M) of $n \times n$ matrix L and $n \times l$ matrix M is said to be *controllable* if the $n \times nl$ matrix

$$[MLML^2M \ldots L^{n-1}M] \tag{A.1}$$

has rank n.

Definition A.7 (Observability). The pair (L, F) of $n \times n$ matrix L and $l \times n$ matrix F is said to be *observable* if (L', F') is controllable.

Moreover, we have the following properties:

Properties A.8. (i). if the pair (L, M) is controllable, then it is stabilizable.

(ii). if the pair (L, F) is observable, then it is detectable.

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