

NONLINEAR STOCHASTIC TRANSPORT PARTIAL DIFFERENTIAL EQUATIONS: WELL-POSEDNESS AND APPLICATIONS TO DATA ASSIMILATION

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BY
OANA-ANDREA LANG

DEPARTMENT OF MATHEMATICS
IMPERIAL COLLEGE
180 QUEEN'S GATE, LONDON SW7 2AZ

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or results from the work of other people, published or otherwise, are fully acknowledged and cited appropriately.

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Nonlinear Stochastic Transport Partial Differential Equations: Well-posedness and Applications to Data Assimilation

ABSTRACT

In this thesis I study analytical properties and applications to data assimilation for nonlinear stochastic transport partial differential equations which originate in fluid dynamics. The thesis has two parts.

In the first part (Chapters 3-5) I focus on theoretical results for three two-dimensional stochastic transport models. I prove analytical properties (existence, uniqueness, continuity with respect to initial conditions) for the corresponding classes of stochastic partial differential equations (SPDEs) driven by transport noise: the stochastic Euler equation (SE), the stochastic great lake equation (SGLE), and the stochastic rotating shallow water (SRSW) model. The well-posedness strategy is based on constructing an approximating sequence of solutions which is proven to be relatively compact and to converge to the solution of the original equation in a suitable sense. The solution is global for the 2D SE equation and local for the SGLE and for the SRSW model.

In the second part (Chapter 6) I prove the applicability of one of these models (the SRSW model) in a data assimilation setting. I implement a data assimilation methodology based on a bootstrap particle filter combined with two additional procedures (tempering and jittering). The methodology is tested first on the Lorenz '63 model and then applied to the SRSW model.

To my parents.

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NOTATION

We summarise the notation used throughout the manuscript. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space and X a generic Banach space. Throughout the thesis C (or $\mathcal{C}$), is a generic notation for constants whose values can change from line to line.

- We denote by $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ the d -dimensional torus. In our case $d = 2$.
- Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$, $d > 0$ be a multi-index of length $|\alpha| = \sum_{j=1}^d \alpha_j$. Then $\partial^\alpha = \partial^{(\alpha_1, \alpha_2, \dots, \alpha_d)} = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_d^{\alpha_d}$ denotes the differential operator of order $|\alpha|$, with $\partial^{(0,0,\dots,0)} f = f$ for any function f defined on $\mathbb{T}^d$, and $\partial_i^{\alpha_i} = \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}}$, $x \in \mathbb{T}^d$. In this thesis, $d = 2$ and $|\alpha| \leq k$.
- $L^p(\mathbb{T}^2; \mathcal{X})$ is the class of all measurable p -integrable functions f defined on $\mathbb{T}^2$, with values in $\mathcal{X}^1$ (p is a positive real number). The space is endowed with its canonical norm $\|f\|_p = \left(\int_{\mathbb{T}^2} \|f\|_{\mathcal{X}}^p dx \right)^{1/p}$. Conventionally, for $p = \infty$ we denote by L^∞ the space of essentially bounded measurable functions.
- $L_b^p(\mathbb{T}^2; \mathcal{X})$ is the class of all measurable p -integrable functions f defined on $\mathbb{T}^2$, with values in $\mathcal{X}$ (p is a positive real number). The space is endowed with its canonical weighted norm $\|f\|_{b,p} = \left(\int_{\mathbb{T}^2} \|f\|_{\mathcal{X}}^p b(x) dx \right)^{1/p}$. The weight b is such that $0 < b_{min} \leq b(x) \leq b_{max}$ for any $x \in \mathbb{T}^2$, and b_{min} and b_{max} are two constants.
- For $a_1, a_2 \in L^2(\mathbb{T}^2)$, we denote by $\langle \cdot, \cdot \rangle$ the scalar product

$$\langle a_1, a_2 \rangle := \int_{\mathbb{T}^2} a_1(x) \cdot a_2(x) dx.$$

¹Here and later whenever the space $\mathcal{X}$ coincides with the Euclidean space $\mathbb{R}$ or $\mathbb{R}^2$, it is omitted from the notation: For example $L^p(\mathbb{T}^2; \mathcal{X})$ becomes $L^p(\mathbb{T}^2)$, etc.

- For $a_1, a_2 \in L_b^2(\mathbb{T}^2)$, we denote by $\langle \cdot, \cdot \rangle_b$ the scalar product

$$\langle a_1, a_2 \rangle_b := \int_{\mathbb{T}^2} a_1(x) \cdot a_2(x) b(x) dx.$$

- $\mathcal{W}^{m,p}(\mathbb{T}^2)$ is the Sobolev space of functions $f \in L^p(\mathbb{T}^2)$ such that $D^\alpha f \in L^p(\mathbb{T}^2)$ for $0 \leq |\alpha| \leq m$, where $D^\alpha f$ is the distributional derivative of f . The canonical norm for this space is $\|f\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_p^p \right)^{1/p}$, with m a positive integer and $1 \leq p < \infty$. A detailed presentation of Sobolev spaces can be found in [1].
- $\mathcal{W}_b^{m,p}(\mathbb{T}^2)$ is the Sobolev space of functions $f \in L_b^p(\mathbb{T}^2)$ such that $D^\alpha f \in L_b^p(\mathbb{T}^2)$ for $0 \leq |\alpha| \leq m$, where $D^\alpha f$ is the distributional derivative of f and $|\alpha|$ is the length of the multi-index α . The canonical norm for this space is $\|f\|_{b,m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_{b,p}^p \right)^{1/p}$, with m a positive integer and $1 \leq p < \infty$. We denote by $\mathcal{W}_b^{-m,p}$ the dual of $\mathcal{W}_b^{m,p}$. A detailed presentation of weighted Sobolev spaces can be found in [71], respectively.
- $C^m(\mathbb{T}^2; \mathcal{X})$ is the (vector) space of all $\mathcal{X}$ -valued functions f which are continuous on $\mathbb{T}^2$ with continuous partial derivatives $D^\alpha f$ of orders $|\alpha| \leq m$, for $m \geq 0$. $C^\infty(\mathbb{T}^2; \mathcal{X})$ is regarded as the intersection of all spaces $C^m(\mathbb{T}^2; \mathcal{X})$. Note that on $\mathbb{T}^2$ all continuous functions are bounded.
- $C([0, \infty); \mathcal{X})$ is the space of continuous functions from $[0, \infty)$ to $\mathcal{X}$ equipped with the uniform convergence norm over compact subintervals of $[0, \infty)$.
- $L^p(0, T; \mathcal{X})$ is the space of measurable functions from $[0, \infty)$ to $\mathcal{X}$ such that the norm

$$\|f\|_{L^p(0,T;\mathcal{X})} = \left(\int_0^t \|f(t)\|_{\mathcal{X}}^p dt \right)^{1/p}$$

is finite.

- $D([0, \infty); \mathcal{X})$ is the space of càdlàg functions that is functions $f : [0, \infty) \rightarrow \mathcal{X}$ which are right-continuous and have limits to the left, endowed with the Skorokhod topology. This topology is a natural choice in this case because its underlying metric transforms $D([0, \infty); \mathcal{X})$ into a complete separable metric space. For further details see [45] Chapter 3, Section 5, pp. 117-118.

-
- $\mathbb{D}_1 := \{u \in C^\infty(\mathbb{T}^2) : \nabla \cdot (bu) = 0\}$.
 - $\mathbb{D}_2 :=$ the completion of $\mathbb{D}_1$ in the L_b^2 - norm.
 - $\mathbb{D}_3 :=$ the completion of $\mathbb{D}_1$ in the $\mathcal{W}_b^{1,2}$ - norm.
 - $\mathcal{P} :=$ the L_b^2 - orthogonal projection onto $\mathbb{D}_2$.
 - $\mathcal{D}(-\Delta) := \{f \in (L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2)) \times L^2(\mathbb{T}^2) : \sum_{j=1}^{\infty} \alpha_j^2 \|u^j\|_2^2 < \infty\}$. Here $(\alpha_j)_j$ is a sequence of eigenvalues such that $-\Delta \varphi_j = \alpha_j \varphi_j$ where $(\varphi_j)_j$ is a standard orthonormal basis of the space $L^2(\mathbb{T}^2)$.
 - $\hat{z} \times u := (-u^2, u^1)$ for a two-dimensional vector $u = (u^1, u^2)$ and a unit vector $\hat{z}$.

"We have to remember that what we observe is not nature itself, but nature exposed to our method of questioning."

W. Heisenberg in *Physics and Philosophy: The Revolution in Modern Science*

1

INTRODUCTION

Our understanding of the surrounding world is increasingly accurate, yet imperfect. Pure and applied mathematicians around the globe devote their time to designing, implementing, and testing sophisticated mathematical models, in their quest for capturing the full complexity of nature. Exploring the turbulent behaviour and the apparently (i.e. as exposed to our *method of questioning*) chaotic variability of the multi-scale physical processes which govern the atmospheric and oceanic dynamics is one of the most adventurous scientific journeys one can undertake. This complexity can be translated into mathematical language using nonlinear stochastic partial differential equations. Nonlinear transport equations, in particular, play a central role in modelling a broad range of phenomena such as flood waves, shock waves, chemical reactions, gas dynamics, etc. In a deterministic setting such an equation can be expressed as

$$u_t + uu_x = 0 \tag{1.1}$$

where $u(t, x)$ is the velocity of the fluid flow which depends on time and space, u_t is the partial time derivative and u_x is the partial spatial derivative (see e.g. [83]). The applicability of a continuous model hinges upon its time and space discretization. In spite of considerable

progress made in this direction over the last decade, there are still numerous small-scale physical processes which cannot be resolved using present models: turbulent multi-scale motions, friction, radiation, evaporation, tropical cumulus convection, condensation, etc. ([65]). Such subgrid-scale processes can have a substantial influence on the larger-scale resolved components, and therefore will affect the overall proficiency of the model. Their accurate representation is the subject of intensive research, but exploring the chaotic intrinsic variability of oceans and atmosphere is too difficult when using models grounded in exactitude and determinism. In response to this, modern stochastic approaches are currently being developed with the aim of providing better strategies for encoding randomness into a priori deterministic models: see [62], [80]. Such approaches are calibrated by the incorporation of high-resolution real data into classical deterministic models and will replace the purely theoretical insights. The reason is that real observations contain the full physical and chemical structure of the atmosphere/ocean, while the model solved on a computer is just a low-resolution version of this - a *method of questioning* that could be improved by the knowledge learnt from real data. In these new approaches, stochasticity is introduced at a fundamental level of the model derivation in such a way that the resulting equations preserve the original physical properties of the system (e.g. Kelvin circulation theorem [62], energy conservation [80]) and they can incorporate a priori unresolved fluid degrees of freedom (see [30]).

A wealth of new stochastic partial differential equations arise in this context (see [62], [80], [63], [86]). The analytical and numerical properties of these new models are yet to be explored. In this thesis we address this problem from two perspectives: a purely theoretical one in which we prove well-posedness properties of solutions of these stochastic partial differential equations, and an applied one in which we investigate the applicability of one of these models in a data assimilation setting.

The models we have studied have been derived from physical principles using the Stochastic Advection by Lie Transport (SALT) approach (see [62], [86]). In this setting stochasticity is introduced at the level of the Lie derivative along the velocity vector field:

$$dy_t := u_t(y_t)dt + \sum_i \xi_i(y_t) \circ dW_t^i, \quad (1.2)$$

where y is the newly perturbed stochastic trajectory, u is the deterministic fluid velocity, W_t^i are independent Brownian motions, and $(\xi_i)_i$ are time-independent divergence-free vector fields which depend on the underlying physics and can be learnt from real data. The noise $\sum_i \xi_i(y_t) \circ dW_t^i$ is introduced using Stratonovich integration and we use the notation ' $\circ$ ' to specify this. It is known in the literature as *transport noise*. This terminology highlights the fact that the noise preserves the transport character of the original partial differential equation.

1.1 MAIN RESULTS

The thesis has two parts: the first part covers theoretical results and the second part covers methodology and numerics. In the first part I focus on three classes of two-dimensional stochastic partial differential equations (SPDEs) driven by transport noise: the stochastic Euler equation (SE), the stochastic great lake equation (SGLE), and the stochastic rotating shallow water (SRSW) model. In the second part I implement a data assimilation methodology based on particle filters combined with tempering and jittering. This methodology does not require any other approximation methods. It is proven to work well first on the Lorenz '63 model and then applied to the SRSW model.

1.1.1 ANALYTICAL PROPERTIES

In the following I give details of the three different theoretical frameworks and the corresponding main results.

1. The stochastic 2D Euler vorticity equation. The first equation treated here is a stochastic version of the classical 2D Euler equation. We treat the Euler equation in vorticity form, see equation (1.3) below, on a 2D torus $\mathbb{T}^2$. I prove that there exists a unique global strong solution for the stochastic two-dimensional Euler equation driven by transport noise, when the fluid is incompressible. In particular, I show that the initial smoothness of the solution is preserved. This is the best one could achieve in this framework, as the equation is truly of transport type, so we cannot expect the initial condition to be smoothed out. The result we obtain is important from a practical perspective. For example, when a finite element numerical approximation is used, the smoothness of the solution and the choice of the basis, see [18], governs

the rate of convergence of the numerical approximation.

The proof of the main result is based on approximating the solution of the original Euler equation with a family of viscous solutions which is proved to be relatively compact. Relative compactness is obtained using the mild form of the approximating equations.

Although the model is not data-driven in itself, the methodology I developed does not impose unrealistic conditions on the driving stochastic vectors. As a consequence, the same analytical properties hold true when the stochastic parameters are estimated from real data, provided that they are sufficiently smooth and their norms decay sufficiently fast. In particular we lift a certain condition (see Remark 3.2 for details) that was assumed in an earlier paper by Brzeźniak, Flandoli, Maurelli [21] to prove existence of a distributional solution for (1.3). In this sense the work presented here extends the result in [21]. However we note that in [21] the existence of a solution of equation (1.3) is shown for an initial condition (albeit distributional solution only) in $L^\infty(\mathbb{T}^2)$.

Let us state in the following the main result of this part of the thesis. We denote by ω the vorticity of the incompressible fluid, that is $\omega_t = \text{curl } u_t$. In the following k is an arbitrary positive integer $k \geq 2$. For $u \in L^2(\mathbb{T}^2)$, the operator $\mathcal{L}_u : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$ given by $\mathcal{L}_u \omega := u \cdot \nabla \omega$ can be interpreted as the Lie derivative of ω along u . In a similar way, for $\xi_i \in L^2(\mathbb{T}^2)$ we can define $\mathcal{L}_i : \mathcal{W}^{k,2}(\mathbb{T}^2) \subset L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$ such that $\mathcal{L}_i \omega := \xi_i \cdot \nabla \omega$. The specific properties of $\mathcal{L}_i$ can be found in Chapter 2 Section 2.6. Then the main result reads as follows:

Theorem 1.1. *Under certain conditions on the vector fields $(\xi_i)_i$ the two-dimensional stochastic Euler vorticity equation*

$$d\omega_t + \mathcal{L}_{u_t} \omega_t dt + \sum_{i=1}^{\infty} \mathcal{L}_i \omega_t \circ dW_t^i = 0, \quad \omega_0 \in \mathcal{W}^{k,2}(\mathbb{T}^2), \quad (1.3)$$

admits a unique global (in time) solution in the space $\mathcal{W}^{k,2}(\mathbb{T}^2)$. Moreover, ω_t is a continuous function of the initial condition.

The contents of this part of the thesis are subject of the paper [34] which is under revision for publication in the journal *Stochastics and Partial Differential Equations: Analysis and Computations*.

2. The stochastic great lake equation. This model is more complicated than the 2D Euler

equation as the domain now allows for a depth-varying bottom b (SGLE reduces to SE when b is constant). In addition, the relationship between the fluid velocity and the fluid vorticity is no longer of the form $\omega = \text{curl } u$ (see Remark 1.5 below). As a result, the equation becomes more challenging from a technical perspective. The novelty here is that the analysis is no longer carried out on flat spaces, but on weighted Sobolev spaces. The choice of the weight is given by the function b . By introducing the additional bottom topography component we pay a price: the global well-posedness is lost. Instead we show that the stochastic great lake equation admits a pathwise unique and probabilistically strong solution only locally in time.

One of the key points here is to prove that the smoothness of the velocity vector field is still controlled by the smoothness of its corresponding vorticity. We do this by exploiting the more complicated (but still linear) relationship between the velocity and the vorticity vector fields, by using Lax-Milgram-type arguments. Another important ingredient for the proof of well-posedness is the fact that the bottom topography function affects just the geometry of the ambient space, but not its topology.

Remark 1.2. *The stochastic part does not interfere with this generalisation. This is due to the fact that the stochasticity is introduced as a constraint within a variational principle such that the physical meaning of the quantities of interest has not been altered. As a result, the vorticity field can be recovered from the deterministic, unperturbed velocity without any stochasticity involved.*

The main result of this part of the thesis reads as follows:

Theorem 1.3. *Under certain conditions on the vector fields $(\xi_i)_i$, the vorticity equation of the two-dimensional stochastic great lake system*

$$d\omega_t + \mathcal{L}_{u_t}\omega_t dt + \sum_i \mathcal{L}_i\omega_t \circ dW_t^i = 0 \quad (1.4a)$$

$$\nabla \cdot (bu) = 0 \quad (1.4b)$$

$$\omega = b^{-1} \text{curl } v \quad (1.4c)$$

$$v = u + \frac{1}{6} \delta^2 b^2 \nabla (\nabla \cdot u). \quad (1.4d)$$

admits a pathwise unique local strong solution which belongs to the weighted Sobolev space $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$.

This work is subject of the paper [35] which has been submitted for publication in the *Romanian*

3. The stochastic rotating shallow water (SRSW) model. This model incorporates the effects of rotation (ϵ, f) , gravity (g) , pressure (p) and varying surface height (not just depth) (h) on the dynamics of the fluid. These three properties make the model more complex, but they are essential in developing a realistic model for geophysical fluid dynamics. Also, the stochastic term is more complicated in this case. The added components make the analysis much more delicate. The solution is shown to exist in weak form and not in strong form as in the previous two cases, see Theorem 1.4 below.

The strategy resembles the one presented in [34] and [35], though it involves a nontrivial generalisation to the case of compressible fluids. This is the most challenging model of the three covered in this thesis. Nevertheless, I succeeded in proving existence and uniqueness of a viscous local strong solution in the Sobolev space $\mathcal{W}^{1,2}(\mathbb{T}^2)$. The operators $\mathcal{L}_u$ and $\mathcal{L}_i$ are as above. We also define $\mathcal{A}_i : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$, $\mathcal{A}_i v := v_1 \cdot \nabla \xi_i^1 + v_2 \cdot \nabla \xi_i^2$ where $v = (v_1, v_2)$ and $\xi_i = (\xi_i^1, \xi_i^2)$. The following is the main result of this part of the thesis:

Theorem 1.4. *Under certain conditions on the vector fields $(\xi_i)_i$, the stochastic rotating shallow water system*

$$dv_t + [\mathcal{L}_{u_t} v_t + f \hat{z} \times u_t + \nabla p_t] dt + \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i) v_t] \circ dW_t^i = \nu \Delta v_t dt \quad (1.5a)$$

$$dh_t + \nabla \cdot (h_t u_t) dt + \sum_{i=1}^{\infty} [\nabla \cdot (\xi_i h_t)] \circ dW_t^i = \eta \Delta h_t dt \quad (1.5b)$$

admits a pathwise unique local weak solution which belongs to the Sobolev space $\mathcal{W}^{1,2}(\mathbb{T}^2)^3$.

Remark 1.5. *The links between the models covered by our analysis can be summarised as follows:*

For the first three models the velocity vector field u can be recovered from the fluid vorticity ω using an operator denoted by K . What differs in each case is the form of the operator K : the more physical details we capture within the model (e.g. bottom topography b , aspect ratio δ , replace incompressibility condition $\nabla \cdot u = 0$ by $\nabla \cdot (bu) = 0$), the more complicated form K has. Nonetheless, the existence and uniqueness analysis can be carried out as long as the smoothness of the velocity can be controlled by the smoothness of the vorticity:

$$\|K\omega\|_{(b),k,p} \leq C \|\omega\|_{(b),k-1,p}. \quad (1.6)$$

Euler	Lake	Great Lake	SRSW
$\omega = \operatorname{curl} u$	$\omega = b^{-1} \operatorname{curl} u$	$\omega = b^{-1} \operatorname{curl} v$ $v = u + \frac{1}{6} \delta^2 b^2 \nabla(\nabla \cdot u)$ $=: \mathcal{M}u$	$\omega = z \cdot \operatorname{curl} v$ $v = \epsilon u + \mathcal{R}$ time-evolution for $(h + b)$
$u = \nabla^\perp \psi$	$u = b^{-1} \nabla^\perp \psi$	$u = b^{-1} \nabla^\perp \psi$	
$u = K\omega$	$u = K\omega$	$u = K\omega$	...
$K = \operatorname{curl}^{-1}$	$K = \operatorname{curl}^{-1} b$	$K = (\operatorname{curl} \mathcal{M})^{-1} b$	
$\nabla \cdot u = 0$	$\nabla \cdot (bu) = 0$	$\nabla \cdot (bu) = 0$	time-evolution for $\nabla \cdot u$

The case of the lake equation is not explicitly calculated, as on the two-dimensional torus this is a trivial generalisation of the 2D Euler case. A more detailed explanation of all terms can be found in the corresponding chapters below.

For the rotating shallow water model the reasoning is more delicate. This is mainly due to the fact that the fluid is now fully compressible, which makes the analysis of the nonlinear term much harder. Moreover, the velocity equation contains a pressure term whose evolution is given by another SPDE (see Chapter 5).

This work is the subject of the paper *Stochastic advection by Lie transport (SALT) for the one-layer viscous rotating shallow water equations* ([36]) to be submitted shortly.

WELL-POSEDNESS STRATEGY

All three theoretical results use a common methodology for proving existence and uniqueness of solutions. Our strategy is briefly described below.

We start by constructing an approximating sequence of solutions. This is done by linearising, truncating, and adding viscosity terms to the original equation. We show that each of these approximating SPDEs admits a global unique smooth solution. Then we prove a couple of a priori estimates. These are used together with specific stochastic criteria for relative compactness (see Section 2.5 below) to show that the approximating sequence of SPDEs is relatively compact/tight in a relevant space of functions. The relative compactness property ensures that we can extract a convergent subsequence of approximating solutions. Therefore we can pass to the limit to obtain a (proved to be also unique) solution of a truncated version of the original equation. In our

methodology this solutions is weak in probabilistic sense. Nonetheless, in parallel we prove that the solutions is pathwise unique. Due to this and using a Yamada-Watanabe argument (see [99]) we can conclude that the solution is actually strong in probabilistic sense. Finally, we decide whether the truncation can be fully removed, to obtain a global solution, or it can be lifted only up to a stopping time, to obtain a local solution.

1.1.2 APPLICATIONS TO DATA ASSIMILATION

In the second part of this thesis we develop a data assimilation methodology for the stochastic rotating shallow water model. The methodology is based on a bootstrap particle filter combined with tempering and jittering. The suitability of these two methods used in tandem for high-dimensional problems has been proven in [9] and tested in [30], [31], [92]. In [31] the method is used for the stochastic incompressible 2D Euler model with damping and forcing, while in [92] it is tested for the 2D quasi-geostrophic model. We have no knowledge of any previous application of this methodology for the stochastic rotating shallow water system. Our aim is to show the applicability of the particle filter described in Chapter 6 and the stochastic rotating shallow water model running alongside. We start by applying the particle filter to the classical Lorenz '63 model and then use this as a stepping stone for the SRSW model. The results are encouraging in both cases.

The data assimilation methodology has been implemented in Fortran and the model has been solved on a staggered grid using a stochastic Runge-Kutta scheme of order 4. The domain corresponds to a strip between 30 and 60 degrees north latitude with free-slip boundaries at the north and south sides. The SRSW model makes the problem particularly harder as it can allow for gravity waves. However, we assume that the stochastic vector fields are in geostrophic balance. The observation process consists of noisy pointwise measurements of the pressure field. In this context, the role of the stochastic parameters $(\xi_i)_i$ is twofold: improved representation of the missing physics and increased variability in the cloud of particles. In order to assess the uncertainty of the model, we plot the ensemble of particles one standard deviation region about the ensemble mean and compare it with the truth. The output shows that the ensemble of stochastic paths successfully tracks the truth, without any extra approximations required (see Chapter 5).

For the purpose of this chapter the *truth* is a pathwise realisation of the SRSW model. Due to the well-posedness result proven in Chapter 5, the equations are satisfied pointwise and the stochastic integrals are well-defined. The periodicity of the domain also inspires a rigorous numerical setup for generating the stochastic forcing. Moreover, the solution constructed in Chapter 5 is strong in probabilistic sense. This ensures that fixing an a priori working probability space and a driving Brownian motion is mathematically justified. Nonetheless, a classical difficulty which arises in both the deterministic and the stochastic setting concerns the global/character of the solution. The solution of the SRSW is defined only up to a positive stopping time which depends on the size of the initial data. Based on previous efforts made in this sense - some of them presented in the literature (see e.g. [26], [29]) and some of them being carried out in our own investigations - we believe that the existence of a solution could be established for a longer time interval through a judicious choice of the stochastic parameters. A rigorous further investigation is required for this and it is subject to future research.

1.2 THESIS STRUCTURE

In Chapter 2 we introduce some background material and preparatory results which will be used in the rest of the thesis. In Chapter 3 we prove well-posedness for the 2D stochastic Euler equation (Theorem 1.1). In Chapter 4 we prove existence of a unique local solution for the stochastic great lake equation (Theorem 1.3). In Chapter 5 we prove existence of unique maximal solution for the stochastic rotating shallow water system (Theorem 1.4). In Chapter 6 we implement the data assimilation methodology for the stochastic rotating shallow water model, using the Lorenz '63 model as a stepping stone. Chapter 8 is an appendix where we collect a couple of results and derivations which are useful for the overall understanding of the thesis.

2

PRELIMINARIES

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In this chapter we introduce some background material and a couple of preparatory results which will be used throughout the rest of the thesis.

2.1 SETUP

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space and $(W^i)_{i \in \mathbb{N}}$ a sequence of independent one-dimensional Brownian motions adapted to the complete and right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$. In other words, $\mathcal{S} := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, (W^i)_{i \in \mathbb{N}})$ is a fixed *stochastic basis*. The elementary events corresponding to the underlying probability space are denoted by $w \in \Omega$. For any Hilbert space H , we say that a random variable X is H -valued if it is defined on the probability space Ω and it takes values in the space H .

2.2 CYLINDRICAL BROWNIAN MOTION

The driving processes involved in the equations studied in this thesis are infinite-dimensional. We briefly present below the classical generalisation of a Brownian motion to an infinite-dimensional setting. More detailed versions can be found in [40], [75], [61]. Let $\mathfrak{U}$ be a separable Hilbert space endowed with an orthonormal basis $(e_r)_{r \in \mathbb{N}}$. Then the process $W_t(w, \cdot)$ formally defined as

$$W_t(w, \cdot) = \sum_{r=1}^{\infty} e_r(\cdot) W_t^r(w) \quad (2.1)$$

where $(W^r)_{r \in \mathbb{N}}$ is a sequence of independent real-valued Brownian motions, is known in the literature as a *cylindrical Brownian motion* (or *cylindrical Wiener process*). As pointed out in [61], the series $\sum_{r=1}^{\infty} W_t^r(w) e_r(\cdot)$ does not converge in any sense in the space $\mathfrak{U}$. However, a new Hilbert space $\mathfrak{U}_0 \supset \mathfrak{U}$ endowed with a suitable norm can be constructed such that the convergence holds in $\mathfrak{U}_0$. This is defined as

$$\mathfrak{U}_0 := \left\{ f = \sum_{r=1}^{\infty} a_r e_r : \sum_{r=1}^{\infty} \frac{a_r^2}{r^2} < \infty \right\} \quad (2.2)$$

with the corresponding norm

$$\|f\|_{\mathfrak{U}_0}^2 := \sum_{r=1}^{\infty} \frac{a_r^2}{r^2} \quad (2.3)$$

In this context, we denote the set of *Hilbert-Schmidt operators* from $\mathfrak{U}$ to $\mathfrak{U}_0^1$ by

$$L_2(\mathfrak{U}, \mathfrak{U}_0) := \left\{ T \in L(\mathfrak{U}, \mathfrak{U}_0) : \sum_{r=1}^{\infty} \|Te_r\|_{\mathfrak{U}_0}^2 < \infty \right\}, \quad (2.4)$$

endowed with the norm

$$\|T\|_{L_2(\mathfrak{U}, \mathfrak{U}_0)}^2 := \sum_{r=1}^{\infty} \|Te_r\|_{\mathfrak{U}_0}^2. \quad (2.5)$$

Following [40], the trajectories of the process W belong to the space $C_{loc}([0, \infty); \mathfrak{U}_0)$. We use this property in Chapter 5. The cylindrical Brownian motion defined in (2.1) is usually identified with the sequence of real-valued Brownian motions $(W^r)_r$.

2.3 STOCHASTIC TRANSPORT USING ITÔ AND STRATONOVICH INTEGRALS

The stochastic partial differential equations studied in this thesis have been originally written (see [62]) in Stratonovich form:

$$X_t = X_0 + \int_0^t A_s(X_s) ds + \sum_{i=1}^{\infty} \int_0^t B_s^i(X_s) \circ dW_s^i \quad (2.6)$$

where A and B^i are nonlinear operators, $(W^i)_i$ are 1-dimensional independent Brownian motions and $\circ$ denotes Stratonovich integration. In order for the stochastic process X to be well-defined the following conditions need to be imposed on A , B^i and X_0 :

- a. X_0 is an $\mathcal{F}_0$ -measurable H -valued random variable.
- b. $(A_s(X_s))_s$ is an H -valued $(\mathcal{F}_t)_t$ -progressively measurable process such that

$$\mathbb{P} \left(\int_0^t \|A_s(X_s)\|_H ds < \infty, \forall t \geq 0 \right) = 1. \quad (2.7)$$

- c. $(B_s^i(X_s))_s$ is an $L_2(\mathfrak{U}, H)$ -valued $(\mathcal{F}_t)_t$ -progressively measurable process such that

$$\mathbb{P} \left(\sum_{i=1}^{\infty} \int_0^t \|B_s^i(X_s)\|_{L_2(\mathfrak{U}, H)}^2 ds < \infty, \forall t \geq 0 \right) = 1. \quad (2.8)$$

¹Similarly $L_2(\mathfrak{U}, H)$ can be defined for any Hilbert space H . We presented in Section 2.2 the abstract, classical setting, in the spirit of [40]. For the purpose of this thesis H can be associated with the space $L^2(\mathbb{T}^2)$.

d. $(B_s^i(X_s))_s$ is a semimartingale and the quadratic variation process $[B^i(X), W]_t$ is such that

$$\mathbb{P} \left(\sum_{i=1}^{\infty} \|[B^i(X), W^i]_t\|_H^2 ds < \infty, \forall t \geq 0 \right) = 1. \quad (2.9)$$

The last two conditions ensure that the stochastic Stratonovich integral involved in (2.6) is a semimartingale. The Itô equivalent of this Stratonovich integral (see e.g. [6]) is given by

$$\int_0^t B_s^i(X_s) \circ dW_s^i = \int_0^t B_s^i(X_s) dW_s^i + \frac{1}{2} [B^i(X), W^i]_t$$

where $\int_0^t B_s^i(X_s) dW_s^i$ is the classical Itô integral and $[B^i(X), W^i]_t$ is the quadratic variation of the two processes $B^i(X)$ and W^i (see e.g. [40]). In our case the operator B^i is intrinsically linked (see [62]) with the Lie derivative along a Stratonovich stochastic vector field dy_t given by

$$dy_t := u_t(y_t)dt + \sum_i \xi_i(y_t) \circ dW_t^i$$

where u is the deterministic velocity of the fluid flow and $(\xi_i)_i$ are time-independent and divergence-free vector fields. The drift velocity u is embedded in A while the diffusive terms ξ_i are embedded in B^i . We distinguish two particular cases, one which corresponds to the stochastic vorticity Euler equation (Chapter 3) and the stochastic great lake equation (Chapter 4), and the other one which corresponds to the stochastic rotating shallow water system (Chapters 5 and 6). In what follows $\xi_i : \mathbb{T}^2 \rightarrow \mathbb{R}^2$, $\mathcal{L}_i : \mathcal{W}^{k,2}(\mathbb{T}^2) \subset L^2(\mathbb{T}^2) \rightarrow \mathbb{T}^2$ is a first-order operator, and $\mathcal{A}_i : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$ is a zero-order operator. Their properties are explained in detail in Section 2.6. We distinguish two cases:

- *Example 1.* $B_s^i(X_s) := -\mathcal{L}_i X_s$ where $\mathcal{L}_i X_s = \mathcal{L}_{\xi_i} X_s := \xi_i \cdot \nabla X_s$ is the Lie derivative of ξ_i along X_s . Then

$$[B^i(X), W^i]_t = -[\mathcal{L}_i X, W^i]_t = -\mathcal{L}_i[X, W^i]_t = -\int_0^t \mathcal{L}_i^2 X_s ds$$

since $[X, W^i]_t = \int_0^t \mathcal{L}_i X_s ds$. In differential form this reads as

$$B_t^i(X_t) \circ dW_t^i = \mathcal{L}_i X_t \circ dW_t^i = \mathcal{L}_i X_t dW_t^i - \frac{1}{2} \mathcal{L}_i^2 X_t dt.$$

- *Example 2.* $B_s^i(X_s) := -(\mathcal{L}_i + \mathcal{A}_i)X_s$ where $\mathcal{L}_i$ is as before and $\mathcal{A}_i X_s := X_s^1 \nabla \xi_i^1 + X_s^2 \nabla \xi_i^2$ for $X_s = (X_s^1, X_s^2)$ and $\xi_i = (\xi_i^1, \xi_i^2)$. In this case we obtain

$$B_t^i(X_t) \circ dW_t^i = (\mathcal{L}_i + \mathcal{A}_i)X_t \circ dW_t^i = (\mathcal{L}_i + \mathcal{A}_i)X_t dW_t^i - \frac{1}{2}(\mathcal{L}_i + \mathcal{A}_i)^2 X_t dt.$$

The explicit calculations can be found in Chapter 5.

The Stratonovich integral is sometimes preferred for modelling in fluid dynamics due to its formal analogies with the Riemann-Stieltjes integral: the transformation under composition with a C^3 map is similar for the two integrals and therefore the chain rule of Stratonovich calculus holds in its standard form. However, in contrast to its Itô counterpart, the Stratonovich integral is not a martingale. Even though the equations look more complicated, the martingale property is essential for the theoretical analysis. For this reason, we work with the Itô version of equation (2.6).

The original approach for deriving these type of equations using stochastic variational principles can be found in [62]. A more general framework which extends the analysis to semi-martingale driven variational principles is presented in [86]. A detailed derivation of the stochastic vorticity Euler equation is presented in [37], while the derivation of the great lake equation can be found in [63]. However, we have not seen a detailed derivation of the stochastic rotating shallow water model in any published work. We include in Appendix the calculations corresponding to the SRSW system, for completeness.

2.4 WELL-POSEDNESS FOR NONLINEAR (TRANSPORT) SPDES

2.4.1 SOLUTION TYPES

From the PDE standpoint we work with *strong* and *weak* solutions, while from a probabilistic standpoint we work with *pathwise* (*strong in probabilistic sense*) and *martingale* (*weak in probabilistic sense*) solutions. Consider the stochastic partial differential equation (2.6).

Definition 2.1. (*Pathwise solutions*) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a fixed stochastic basis and W a fixed Brownian motion with respect to this basis. Let ζ be an $\mathcal{F}_0$ -measurable random variable. A stochastic process X is a pathwise solution/probabilistic strong solution of equation (2.6) with initial condition ζ if

- i. X is an $(\mathcal{F}_t)_t$ -adapted process with $\mathbb{P}$ -a.s. continuous trajectories.
- ii. $\mathbb{P}(X_0 = \zeta) = 1$.
- iii. $(A_s(X_s))_s$ and $(B_s^i(X_s))_s$ are two processes which satisfy conditions (2.7) and (2.8), (2.9), respectively.
- iv. The identity (2.6)

$$X_t = X_0 + \int_0^t A_s(X_s) ds + \sum_{i=1}^{\infty} \int_0^t B_s^i(X_s) \circ dW_s^i$$

holds $\mathbb{P}$ -almost surely in H for all $t \in [0, \infty)$.

Definition 2.2. (*Martingale solutions*) A triple $\left((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}), \tilde{X}, \tilde{W}\right)$ is a martingale/probabilistic weak solution of equation (2.6) if

- i. $\left((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}), \tilde{X}, \tilde{W}\right)$ is a stochastic basis.
- ii. $\tilde{W}$ is an $(\tilde{\mathcal{F}}_t)_t$ -adapted Brownian motion on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.
- iii. $\tilde{X}$ is an $(\tilde{\mathcal{F}}_t)_t$ -adapted stochastic process with $\tilde{\mathbb{P}}$ -almost surely continuous trajectories.
- iv. $(A_s(\tilde{X}_s))_s$ and $(B_s^i(\tilde{X}_s))_s$ are two processes such that

a. $(A_s(\tilde{X}_s))_s$ is H -valued and $(\tilde{\mathcal{F}}_t)_t$ -progressively measurable process with

$$\mathbb{P} \left(\int_0^t \|A_s(\tilde{X}_s)\|_H ds < \infty, \forall t \geq 0 \right) = 1. \quad (2.10)$$

b. $(B_s^i(\tilde{X}_s))_s$ is $L_2(\mathfrak{U}, H)$ -valued and $(\tilde{\mathcal{F}}_t)_t$ -progressively measurable process such that

$$\mathbb{P} \left(\sum_{i=1}^{\infty} \int_0^t \|B_s^i(\tilde{X}_s)\|_{L_2(\mathfrak{U}, H)}^2 ds < \infty, \forall t \geq 0 \right) = 1. \quad (2.11)$$

c. $(B_s^i(\tilde{X}_s))_s$ is a semimartingale and the quadratic variation process $[B^i(\tilde{X}), \tilde{W}]_t$ is such that

$$\mathbb{P} \left(\sum_{i=1}^{\infty} \|[B^i(\tilde{X}), \tilde{W}]_t\|_H^2 ds < \infty, \forall t \geq 0 \right) = 1. \quad (2.12)$$

v. The identity

$$\tilde{X}_t = \tilde{X}_0 + \int_0^t A_s(\tilde{X}_s) ds + \sum_{i=1}^{\infty} \int_0^t B_s(\tilde{X}_s) \circ d\tilde{W}_s^i \quad (2.13)$$

holds $\tilde{\mathbb{P}}$ -almost surely in H for all $t \in [0, \infty)$.

Definition 2.3. A random variable $\tau : \Omega \rightarrow [0, \infty]$ is called an $(\mathcal{F}_t)_t$ -stopping time provided that for any $t \in [0, \infty)$ the event $\{\tau \leq t\}$ belongs to the sigma-algebra $\mathcal{F}_t$.

Definition 2.4. A local solution of equation (2.6) is given by a couple (X, τ) where $\tau : \Omega \rightarrow [0, \infty]$ is a stopping time and $X : \Omega \times [0, \tau) \times \mathbb{T}^2 \rightarrow H$ is an $(\mathcal{F}_t)_t$ -adapted stochastic process such that for any bounded stopping time $\tau' \leq \tau$ the identity

$$X_{t \wedge \tau'} = X_0 + \int_0^{t \wedge \tau'} A_s(X_s) ds + \sum_{i=1}^{\infty} \int_0^{t \wedge \tau'} B_s^i(X_s) \circ d\tilde{W}_s^i \quad (2.14)$$

holds $\mathbb{P}$ -almost surely in H for all $t \in [0, \infty)$.

Definition 2.5. When $\tau = \infty$ in the above definition, the solution is called global.

We provide exact and more detailed definitions of solutions in each chapter, depending on the particular form of the equation which is studied in the corresponding part of the thesis.

2.4.2 WELL-POSEDNESS STRATEGY - REMARKS

We construct the solution of an SPDE as the limit of an approximating sequence of solutions with good compactness properties. This sequence converges to a truncated solution. Then we prove that we can lift the truncation either completely, to obtain a global solution, or just up to a stopping time, to obtain a local solution. The form of the truncation is usually dictated by the form of the nonlinear term. Proving that the approximating solutions are contained in a compact set allows us to infer the existence of a convergent subsequence. In order to show that any convergent subsequence converges to the same limit and this limit satisfies the original SPDE, we use the pathwise uniqueness property of the targeted limit and the *continuity* of the approximating solutions.

Definition 2.6. (*Pathwise uniqueness*) Equation (2.6) has a pathwise unique solution if any two solutions X^1 and X^2 with the property that $X_0^1 = X_0^2$ $\mathbb{P}$ -almost surely, are indistinguishable i.e.

$$\mathbb{P}(X_t^1 = X_t^2, \text{ for all } t \geq 0) = 1.$$

Definition 2.7. (*Uniqueness in law*) Equation (2.6) has a unique in law solution if any two (martingale) solutions $((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}), X^1, W)$ and $((\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t \geq 0}, \tilde{\mathbb{P}}), X^2, \tilde{W})$, with the same initial distribution, have the same law.

Theorem 2.8. (*Yamada-Watanabe ([67], [99])*) Pathwise uniqueness implies uniqueness in law.

As an immediate corollary, weak existence (in probabilistic sense) and pathwise uniqueness imply strong existence. Detailed proofs can be found in [67] Section 5.3 and [99] for the infinite-dimensional setting.

A couple of technical classical results are needed in order to carry out the arguments.

Theorem 2.9. (*Burkholder-Davis-Gundy inequality ([40])*) Let H be a separable Hilbert space and $0 < p < \infty$. Then there exists a positive constant $C(p)$ such that for every stochastic process B which is $(\mathcal{F}_t)_t$ -progressively measurable, with

$$\mathbb{E} \left[\int_0^T \|B_s\|_{L_2(\mathfrak{U}, H)}^2 ds \right] < \infty \tag{2.15}$$

the following inequality holds

$$\mathbb{E} \left[\sup_{t \in [0, T]} \left\| \int_0^t B_s dW_s \right\|_H^p \right] \leq C(p) \mathbb{E} \left[\int_0^T \|B_s\|_{L_2(\mathfrak{U}, H)}^2 ds \right]^{p/2}. \quad (2.16)$$

Relation (2.15) gives actually the necessary condition for the Itô stochastic integral $\int_0^T B_s dW_s$ to be a well-defined martingale (see [40] Section 4.2).

Theorem 2.10. *(Convergence of the stochastic integrals ([72], Theorem 4.2)) Let $(\mathcal{F}_t^n)_n$ be a sequence of filtrations and $(X^n)_n$ a sequence of $\mathcal{F}_t^n$ -adapted processes with càdlàg trajectories. Let W be a standard Brownian motion. If (X^n, W) converges in distribution to (X, W) , in the Skorokhod topology, then $(X^n, W, \int X^n dW)$ converges in distribution to $(X, W, \int X dW)$ in the Skorokhod topology. If the first convergence holds in probability, then the convergence of the stochastic integrals holds also in probability.*

In a stochastic framework we need to take care not only of the space and time variables, but also of the random one. Two typical difficulties arise here: a lack of classical topology on the probability space and a lack of classical time differentiability for the driving Brownian motion. The first one is overcome by using Prokhorov's and Skorokhod's theorems in tandem. The second one is addressed in Section 2.5.

Theorem 2.11. *(Prokhorov's theorem ([13])) Let (H, d) be a complete and separable metric space and $\mathcal{G}$ a collection of probability measures on H . Then $\mathcal{G}$ is relatively compact if and only if $\mathcal{G}$ is tight.*

Note that a probability measure m is said to be *tight* if for any $\epsilon > 0$ there exists a compact set $K \subset H$ such that $m(K) \geq 1 - \epsilon$. A family of probability measures $\mathcal{G}$ is *tight* if for any $\epsilon > 0$ there exists a compact set $K \subset H$ such that

$$\inf_{m \in \mathcal{G}} m(K) \geq 1 - \epsilon.$$

Theorem 2.12. *(Skorokhod's representation's theorem ([42])) Let (H, d) be a complete and separable metric space and $(m^n)_{n \in \mathbb{N}}$ probability measures on H such that m^n converges in distribution to m . Then there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and some H -valued random variables*

$(\tilde{X}^n)_{n \in \mathbb{N}}$ and $\tilde{X}$ defined on $\tilde{\Omega}$, such that the law of $\tilde{X}^n$ is m^n , for any $n \geq 0$, the law of X is m , and $\lim_{n \rightarrow \infty} \tilde{X}^n(w) = \tilde{X}(w)$ in H $\tilde{\mathbb{P}}$ -almost surely.

2.5 CRITERIA FOR RELATIVE COMPACTNESS IN A STOCHASTIC SETTING

Establishing criteria for relative compactness is more delicate in a stochastic setting, compared to a deterministic setting. One of the main problems is that the Brownian motion does not admit a classical time derivative. It is therefore impossible to control the stochastic integral in the Sobolev space $\mathcal{W}^{k,2}(0, T; H)$ with $k \geq 1$, where H is a general Hilbert space. In this context, all stochastic relative compactness arguments try to account for the lack of classical pathwise differentiability and to provide tools for overcoming it. This is done indeed by using pathwise Hölder continuity (rather than differentiability of the Brownian path): the trajectories of the Brownian motion are γ -Hölder continuous with exponent $\gamma \in (0, 1/2)$ (see e.g [67] or [90]).

The result which establishes conditions for the existence of a regular (Hölder continuous) modification of a stochastic process is known as the *Kolmogorov-Čentsov theorem* or the *Kolmogorov continuity theorem* (Theorem 2.8 in [67]):

Theorem 2.13. *Let $X = (X_t)_{t \geq 0}$ be a H -valued stochastic process such that for all $s, t \in [0, T]$*

$$\mathbb{E} [\|X_t - X_s\|_H^\alpha] \leq C_1 |t - s|^{1+\beta}$$

for some positive constants α, β, C_1 . Then there exists a continuous modification $Y = (Y_t)_{t \geq 0}$ of X which has locally $\mathbb{P}$ -almost surely δ -Hölder continuous trajectories with $\delta \in (0, \frac{\beta}{\alpha})$. That is, there exists an almost surely positive random variable $\tau(w)$ and a positive constant C_2 such that

$$\mathbb{P} \left(w : \sup_{\substack{s, t \in [0, T] \\ t-s \in (0, \tau(w))}} \frac{\|Y_t(w) - Y_s(w)\|_H}{|t - s|^\delta} \leq C_2 \right) = 1.$$

We have used two different relative compactness criteria in our endeavour to prove existence of a solution of a stochastic partial differential equation. The first one explores in more depth the probabilistic nature of the solution. It relies on a time-uniform control of the conditional expectation corresponding to the time increments of the solution, by means of constructing

a random variable, say γ , which is independent of these increments and has good properties in expectation. The construction of this random variable is realised step by step by controlling each term of the equation in a suitable norm. The beauty of this approach resides in its generality: one can control a very broad class of stochastic processes, driven by however wild types of noise, as long as this auxiliary random variable γ can be constructed.

Theorem 2.14 ([45] Theorem 8.6). *Let (H, d) be a complete and separable metric space, $(X^\alpha)_\alpha$ a family of processes with càdlàg sample paths, and suppose that for every $\eta > 0$ and any rational $t \geq 0$ there exists a compact set $K_{\eta,t} \subset H$ such that $\sup_\alpha \mathbb{P}(X_t^\alpha \notin K_{\eta,t}) \leq \eta$. Then the following two statements are equivalent:*

- a. $(X^\alpha)_\alpha$ is relatively compact.
- b. For each $T > 0$ there exists $\beta > 0$ and a family $(\gamma_\delta^\alpha)_{0 < \delta < 1}$ of nonnegative random variables such that $\mathbb{E}[\tilde{d}^\beta(X_t^\alpha, X_{t+u}^\alpha) | \mathcal{F}_t^\alpha] \leq \mathbb{E}[\gamma_\delta^\alpha | \mathcal{F}_t^\alpha]$ and $\lim_{\delta \rightarrow 0} \limsup_\alpha \mathbb{E}[\gamma_\delta^\alpha] = 0$ for $t \in [0, T]$, $u \in [0, \delta]$, where $\tilde{d} = d \wedge 1$.

The restriction imposed by this criterion is the fact that a priori the solution is required to be a process with càdlàg trajectories. The second approach is based on embedding properties for fractional Sobolev spaces $\mathcal{W}^{\beta,p}$ with $\beta \in (0, 1)$ and $p \geq 2$, endowed with the norm

$$\|f\|_{\mathcal{W}^{\beta,p}(0,T;H)} := \int_0^T \|f_t\|_H^p dt + \int_0^T \int_0^T \frac{\|f_t - f_s\|_H^p}{|t - s|^{1+\beta p}} dt ds$$

It is easy to see that for an $(\mathcal{F}_t)_t$ -progressively measurable process $B = (B_s)_s$ such that (2.15) holds and for $\beta \in (0, 1/2)$, $p \geq 2$, one has

$$\int_0^t B_s dW_s \in L^p(\Omega; \mathcal{W}^{\beta,p}(0, T; H)).$$

This is intuitively true due to the $1/2$ -Hölder continuity of trajectories of the Brownian motion.

It can be proven using the Burkholder-Davis-Gundy inequality, see e.g. [50]. Let $s \leq t$. Then

$$\begin{aligned}\mathcal{N} &:= \mathbb{E} \left[\int_0^T \int_0^T \frac{\left\| \int_0^t B_r dW_r - \int_0^s B_r dW_r \right\|_H^p}{|t-s|^{1+\beta p}} dt ds \right] = \mathbb{E} \left[\int_0^T \int_0^T \frac{\left\| \int_s^t B_r dW_r \right\|_H^p}{|t-s|^{1+\beta p}} dt ds \right] \\ &= \int_0^T \int_0^T \frac{\mathbb{E} \left[\left\| \int_s^t B_r dW_r \right\|_H^p \right]}{|t-s|^{1+\beta p}} dt ds\end{aligned}$$

By the Burkholder-Davis-Gundy inequality we know that

$$\mathbb{E} \left[\left\| \int_s^t B_r dW_r \right\|_H^p \right] \leq C(p) \mathbb{E} \left[\left(\int_s^t \|B_r\|_{L_2(\mathfrak{U}, H)}^2 dr \right)^{p/2} \right]$$

therefore

$$\begin{aligned}\mathcal{N} &\leq C(p) \int_0^T \int_0^T \frac{\mathbb{E} \left[\left(\int_s^t \|B_r\|_{L_2(\mathfrak{U}, H)}^2 dr \right)^{p/2} \right]}{|t-s|^{1+\beta p}} dt ds \\ &= C(p) \mathbb{E} \left[\int_0^T \int_0^T \frac{\left(\int_0^t \|B_r\|_{L_2(\mathfrak{U}, H)}^2 dr - \int_0^s \|B_r\|_{L_2(\mathfrak{U}, H)}^2 dr \right)^{p/2}}{|t-s|^{1+\beta p}} dt ds \right].\end{aligned}$$

For $p = 2$ we have

$$\begin{aligned}\mathcal{N} &= C \mathbb{E} \left[\int_0^T \int_0^T \frac{\int_s^t \|B_r\|_{L_2(\mathfrak{U}, H)}^2 dr}{|t-s|^{1+2\beta}} dt ds \right] \\ &= C \int_0^T \int_0^T \frac{\mathbb{E} \left[\int_s^t \|B_r\|_{L_2(\mathfrak{U}, H)}^2 dr \right]}{|t-s|^{1+2\beta}} dt ds \\ &= C \int_0^T \int_0^T \int_s^t \frac{\mathbb{E} \left[\|B_r\|_{L_2(\mathfrak{U}, H)}^2 \right]}{|t-s|^{1+2\beta}} dr dt ds \\ &\leq C \mathbb{E} \left[\int_0^T \|B_r\|_{L_2(\mathfrak{U}, H)}^2 dr \right]\end{aligned}$$

For $p > 2$

$$\mathcal{N} \leq C(p) \mathbb{E} \left[\left\| \int_0^t \|B_r\|_{L_2(\mathfrak{U}, H)}^2 dr \right\|_{\mathcal{W}^{1, p/2}} \right]$$

and this is covered by the Burkholder-Davis-Gundy inequality.

Remark 2.15. *B has to be a progressively measurable process in $L^p(\Omega \times [0, T]; L_2(\mathfrak{U}, H))$ in order for the above stochastic integral to be well-defined in $L^p(\Omega \times [0, T]; H)$ (see [40]).*

The second approach is inspired by the classical Arzela-Ascoli theorem and a deterministic embedding theorem:

Theorem 2.16. ([94] Theorem 13.2)) *Let H_1, H_2 be two reflexive Banach spaces such that the embedding $H_1 \hookrightarrow H_2$ is compact. Let $\mathcal{H}$ be a bounded set of functions in $L^1(\mathbb{R}, H_1) \cap L^p(\mathbb{R}, H_2), p > 1$ such that*

$$i. \int_{-\infty}^{\infty} \|h(t+s) - h(s)\|_{H_2}^p ds \rightarrow 0 \text{ as } t \rightarrow 0 \text{ uniformly for } h \in \mathcal{H}$$

ii. the support of the functions $h \in \mathcal{H}$ is included in a fixed compact $[-T, T] \subset \mathbb{R}$.

Then the set $\mathcal{H}$ is relatively compact in $L^p(\mathbb{R}, H_2)$. That is, we have the compact embedding

$$L^1(\mathbb{R}, H_1) \cap L^p(\mathbb{R}, H_2) \hookrightarrow L^p(\mathbb{R}, H_2).$$

Given the fact that the stochastic integral belongs to the Sobolev space $\mathcal{W}^{\beta, p}$, we would like to be able to replace one of the L^p spaces above by $\mathcal{W}^{\beta, p}$. This would help us in proving that a sequence of solutions is relatively compact. The proof of the above theorem is based on first showing that an associated set $\mathcal{H}_a = \{T_a h, h \in \mathcal{H}\}, a > 0$, with $T_a h : \mathbb{R} \rightarrow H_2, (T_a h)(s) = \frac{1}{2a} \int_{s-a}^{s+a} h(t) dt = \frac{1}{2a} \int_{-a}^a h(s+t) dt$, is relatively compact in $L^p(\mathbb{R}, H_2)$. The complete proof can be found in [94] pp.97. By modifying the argument for uniform convergence and equicontinuity in a convenient way, one can replace the more general condition *i.* and use instead the fact that

$h \in \mathcal{W}^{\beta,p}(0, T; H_2)$:

$$\begin{aligned}
\int_0^T \|T_a h(s) - h(s)\|_{H_2}^p ds &\leq \int_0^T \frac{1}{2a} \int_{-a}^a \|h(t+s) - h(s)\|_{H_2}^p dt ds \\
&\leq \frac{1}{2a} \int_{-a}^a \int_0^T \|h(t+s) - h(s)\|_{H_2}^p ds dt \\
&\leq \frac{a^{\beta p}}{2} \int_0^T \int_{-a}^a \frac{\|h(t+s) - h(s)\|_{H_2}^p}{a^{1+\beta p}} dt ds \\
&\leq \frac{a^{\beta p}}{2} \int_0^T \int_{-a}^a \frac{\|h(t+s) - h(s)\|_{H_2}^p}{|t|^{1+\beta p}} dt ds \\
&\leq \frac{a^{\beta p}}{2} \int_0^T \int_{-a}^a \frac{\|h(r) - h(s)\|_{H_2}^p}{|r-s|^{1+\beta p}} dr ds \\
&\leq \frac{a^{\beta p}}{2} \|h\|_{\mathcal{W}^{\beta,p}(0,T;H_2)}^p.
\end{aligned}$$

Then one obtains the following embedding:

$$L^p(\mathbb{R}, H_1) \cap \mathcal{W}^{\beta,p}(\mathbb{R}, H_2) \hookrightarrow L^p(\mathbb{R}, H_2).$$

This modification has been presented in [50] for three Banach spaces:

Theorem 2.17. *Let $H_1 \subset H_2 \subset H_3$ be three Banach spaces and $p \in (1, \infty), \beta \in (0, 1)$. Assume that H_1 and H_3 are reflexive and the embedding $H_1 \hookrightarrow H_2$ is compact. Then the embedding*

$$L^p(0, T; H_1) \cap \mathcal{W}^{\beta,p}(0, T; H_3) \hookrightarrow L^p(0, T; H_2)$$

is compact.

The result is first proved for $H_2 = H_3$ as explained above and then it is extended to the third space using an interpolation. Using the fact that $\mathcal{W}^{\beta,p}(0, T; H_3)$ is continuously embedded into $C^\delta([0, T]; H_3)$ for all $\delta \in (0, \beta p - 1)$ the following result is also true ([50]):

Theorem 2.18. *If $H_1 \subset H_2$ are two Banach spaces such that the embedding $H_1 \hookrightarrow H_2$ is compact, and $\beta \in (0, 1), p > 1$ are such that $\beta p > 1$, then the embedding*

$$\mathcal{W}^{\beta,p}(0, T; H_1) \hookrightarrow C([0, T]; H_2) \tag{2.17}$$

is compact.

We will also use the following theorem:

Theorem 2.19. (*Gagliardo-Nirenberg [55]*) Let $u \in L^q$ and $D^m u \in L^r$ with $1 \leq q, r \leq \infty$. Then there exists a constant C such that the following inequalities hold for $D^j u$ with $0 \leq j < m$:

$$\|D^j u\|_p \leq C \|D^m u\|_r^a \|u\|_q^{1-a}$$

where $a \in [j/m, 1]$ is defined such that $\frac{1}{p} = \frac{j}{2} + a\left(\frac{1}{r} - \frac{m}{2}\right) + \frac{1-a}{q}$.

2.6 PREPARATORY RESULTS

The stochastic part of the three models studied in this thesis is introduced using a sequence of divergence-free vector fields $(\xi_i)_i$ ([62]). These are time-independent and in practice they can be derived from the underlying physics. We present in this section their mathematical properties.

Definition 2.20. (*The operators $\mathcal{L}_i$ and $\mathcal{A}_i$*) We define the following differential operators:

$$\mathcal{L}_i : \mathcal{W}^{k,2}(\mathbb{T}^2) \subset L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2), \quad k \geq 1 \quad (2.18)$$

$$\mathcal{L}_i f := \xi_i \cdot \nabla f$$

$$\mathcal{A}_i : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2) \quad (2.19)$$

$$\mathcal{A}_i f := f_1 \cdot \nabla \xi_i^1 + f_2 \cdot \nabla \xi_i^2$$

More precisely

$$(\mathcal{A}_i f)_r = \sum_{\beta=1}^2 f_\beta (\partial_r \xi_i^\beta). \quad (2.20)$$

Assumptions

The vector fields $\xi_i : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ are chosen to be time-independent and divergence-free and for any $f \in \mathcal{W}^{2,2}(\mathbb{T}^2)$ we have

$$\sum_{i=1}^{\infty} \|\mathcal{L}_i f\|_2^2 \leq C \|f\|_{1,2}^2 \quad (2.21a)$$

$$\sum_{i=1}^{\infty} \|\mathcal{L}_i^2 f\|_2^2 \leq C \|f\|_{2,2}^2 \quad (2.21b)$$

$$\sum_{i=1}^{\infty} \|\xi_i\|_{k+1,\infty}^2 < \infty. \quad (2.21c)$$

Remark 2.21. If $(W^i)_{i \in \mathbb{N}}$ is a sequence of independent Brownian motions on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, then provided $f \in L^2(0, T; W^{2,2}(\mathbb{T}^2, \mathbb{R}))$, the first two conditions in (2.21) ensure that the infinite sums of stochastic integrals

$$\sum_{i=1}^{\infty} \int_0^t \mathcal{L}_i f_s dW_s^i \quad (2.22)$$

are well defined and belong to $L^2(0, T; L^2(\mathbb{T}^2, \mathbb{R}))$. Similarly, the processes $s \mapsto \mathcal{L}_i^2 f_s$ are well-defined and belong to $L^2(0, T; L^2(\mathbb{T}^2, \mathbb{R}))$ provided the solution of the stochastic partial differential equation belongs to a suitably chosen space. In particular, the Itô correction is well defined. The third condition is needed for proving a number of required a priori estimates (see below).

Remark 2.22. The condition that the family $(\xi_i)_i$ is divergence-free is important for obtaining good a priori estimates, see Proposition 2.23. This analysis could be extended to the case where $(\xi_i)_i$ are not divergence-free, as long as we make sure that $\nabla \cdot \xi_i$ is controlled correspondingly. Nevertheless, for Chapters 3 and 4 we require the Biot-Savart law (see Proposition 3.17) or an equivalent extension (see Proposition 4.9) to hold true. These two propositions are not valid in the case where the velocity u is not divergence-free and zero-average. Removing the divergence-free condition on $(\xi_i)_i$ would break these two conditions. We believe that a further extension could be achieved by splitting the velocity field into a divergence-free, zero-average part and a non-divergence-free, non-zero-average one. However, this requires a careful control of all terms. Since the study of the case where the family $(\xi_i)_i$ is beyond the scope of this thesis, we leave it for further investigation.

Proposition 2.23. Let $f \in \mathcal{W}^{2,2}(\mathbb{T}^2)$.

- i. The dual $\mathcal{L}_i^* : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$ of the operator $\mathcal{L}_i$ is given by $-\mathcal{L}_i$. As a result, the following relation is true: $\langle f, \mathcal{L}_i^2 f \rangle + \langle \mathcal{L}_i f, \mathcal{L}_i f \rangle = 0$.
- ii. There exists a constant C such that:

$$\left| \langle \partial^k f, \partial^k \mathcal{L}_i^2 f \rangle + \langle \partial^k \mathcal{L}_i f, \partial^k \mathcal{L}_i f \rangle \right| \leq C \|f\|_{k,2}^2.$$

- iii. The dual $\mathcal{A}_i^* : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$ of the operator $\mathcal{A}_i$ is given by the matrix $\nabla \xi_i$ with compo-

nents given by

$$(\mathcal{A}_i^* f)_r = \sum_{\beta=1}^2 f_\beta (\partial_\beta \xi_i^r).$$

Note that $\mathcal{A}_i f$ can be written as $\mathcal{A}_i f = \mathbf{A} f$ where the components of the matrix $\mathbf{A}$ are given by (2.20). Then $\mathcal{A}_i^* f = \mathbf{A}^T f$.

Proof *i.* We can write

$$\begin{aligned} \langle \mathcal{L}_i a, b \rangle &= \langle \xi_i \cdot \nabla a, b \rangle = \int_{\mathbb{T}^2} (\xi_i \cdot \nabla a) \cdot b dx = \sum_{\alpha} \int_{\mathbb{T}^2} (\xi_i \cdot \nabla a)_{\alpha} b_{\alpha} dx \\ &= \sum_{\alpha} \sum_j \int_{\mathbb{T}^2} \xi_i^j ((\partial_j a_{\alpha}) b_{\alpha}) dx = \sum_{\alpha} \sum_j \int_{\mathbb{T}^2} \left(\xi_i^j (-a_{\alpha} (\partial_j b_{\alpha})) - a_{\alpha} b_{\alpha} (\partial_j \xi_i^j) \right) dx \\ &= \sum_{\alpha} \sum_j \int_{\mathbb{T}^2} \left(a_{\alpha} (-\xi_i^j (\partial_j b_{\alpha})) - a_{\alpha} b_{\alpha} (\partial_j \xi_i^j) \right) dx = \sum_{\alpha} \int_{\mathbb{T}^2} \left(a_{\alpha} (-\xi_i \cdot \nabla b)_{\alpha} - a_{\alpha} b_{\alpha} (\nabla \cdot \xi_i) \right) dx \\ &= \int_{\mathbb{T}^2} a \cdot ((-\xi_i \cdot \nabla b)) dx = \langle a, -(\xi_i \cdot \nabla b) \rangle \\ &= \langle a, -\mathcal{L}_i b \rangle. \end{aligned}$$

We have used the fact that $\operatorname{div}(\xi_i) = \nabla \cdot \xi_i = 0$. Therefore $\mathcal{L}_i^* = -\mathcal{L}_i$. Then

$$\begin{aligned} \langle \mathcal{L}_i^2 f, f \rangle + \langle \mathcal{L}_i f, \mathcal{L}_i f \rangle &= \langle \mathcal{L}_i f, \mathcal{L}_i^* f \rangle + \langle \mathcal{L}_i f, \mathcal{L}_i f \rangle \\ &= \langle \mathcal{L}_i f, -\mathcal{L}_i f \rangle + \langle \mathcal{L}_i f, \mathcal{L}_i f \rangle = 0. \end{aligned}$$

ii. We have

$$\partial^k \mathcal{L}_i^2 f = \left((\partial^k \mathcal{L}_i) \mathcal{L}_i \right) f = \left((\partial^k \mathcal{L}_i + \mathcal{L}_i \partial^k - \mathcal{L}_i \partial^k) \mathcal{L}_i \right) f = ((\mathcal{L}_i \partial^k + L_i) \mathcal{L}_i) f$$

where $L_i := \partial^k \mathcal{L}_i - \mathcal{L}_i \partial^k$. One can write

$$\begin{aligned}
\langle \partial^k \mathcal{L}_i^2 f, \partial^k f \rangle + \langle \partial^k \mathcal{L}_i f, \partial^k \mathcal{L}_i f \rangle &= \langle (\mathcal{L}_i \partial^k \mathcal{L}_i) f, \partial^k f \rangle + \langle (L_i \mathcal{L}_i) f, \partial^k f \rangle + \langle \partial^k \mathcal{L}_i f, \partial^k \mathcal{L}_i f \rangle \\
&= \langle \partial^k \mathcal{L}_i f, \mathcal{L}_i^* \partial^k f \rangle + \langle (L_i \mathcal{L}_i) f, \partial^k f \rangle + \langle \partial^k \mathcal{L}_i \omega_t, \partial^k \mathcal{L}_i \omega_t \rangle \\
&= \langle \partial^k \mathcal{L}_i f, -\mathcal{L}_i \partial^k f \rangle + \langle (L_i \mathcal{L}_i) f, \partial^k f \rangle + \langle \partial^k \mathcal{L}_i f, \partial^k \mathcal{L}_i f \rangle \\
&= \langle \partial^k \mathcal{L}_i f, -(\partial^k \mathcal{L}_i - \partial^k \mathcal{L}_i + \mathcal{L}_i \partial^k) f \rangle + \langle (L_i \mathcal{L}_i) f, \partial^k \omega_t \rangle \\
&\quad + \langle \partial^k \mathcal{L}_i f, \partial^k \mathcal{L}_i f \rangle \\
&= -\langle \partial^k \mathcal{L}_i f, \partial^k \mathcal{L}_i f \rangle + \langle \partial^k \mathcal{L}_i f, L_i f \rangle + \langle (L_i \mathcal{L}_i) f, \partial^k f \rangle \\
&\quad + \langle \partial^k \mathcal{L}_i f, \partial^k \mathcal{L}_i f \rangle \\
&= \langle \partial^k \mathcal{L}_i f, L_i f \rangle + \langle (L_i \mathcal{L}_i) f, \partial^k f \rangle.
\end{aligned}$$

Using the same strategy we have

$$\begin{aligned}
\langle \partial^k \mathcal{L}_i f, L_i f \rangle + \langle L_i \mathcal{L}_i f, \partial^k f \rangle &= \langle L_i f, \partial^k \mathcal{L}_i f \rangle - \langle L_i f, \mathcal{L}_i \partial^k f \rangle + \langle L_i' f, \partial^k f \rangle \\
&= \langle L_i f, (\partial^k \mathcal{L}_i - \mathcal{L}_i \partial^k) f \rangle + \langle L_i' f, \partial^k f \rangle \\
&= \langle L_i f, L_i f \rangle + \langle L_i' f, \partial^k f \rangle.
\end{aligned}$$

where $L_i' := L_i \mathcal{L}_i - \mathcal{L}_i L_i$ and we have rewritten the two terms as follows:

$$\begin{aligned}
\langle \partial^k \mathcal{L}_i f, L_i f \rangle &= \langle \mathcal{L}_i f, \partial^k L_i f \rangle, \\
\langle L_i \mathcal{L}_i f, \partial^k f \rangle &= \langle (\mathcal{L}_i L_i - \mathcal{L}_i L_i + L_i \mathcal{L}_i) f, \partial^k f \rangle \\
&= \langle \mathcal{L}_i L_i f, \partial^k f \rangle + \langle L_i' f, \partial^k f \rangle \\
&= \langle L_i f, \mathcal{L}_i^* \partial^k f \rangle + \langle L_i' f, \partial^k f \rangle \\
&= \langle L_i f, -\mathcal{L}_i \partial^k f \rangle + \langle L_i' f, \partial^k f \rangle.
\end{aligned}$$

Due to our assumptions on ξ_i each term can be bounded by $\|f\|_{k,2}^2$, so the required inequality is proven.

iii. By definition, $\langle \mathcal{A}_i f, g \rangle = \langle f_1 \nabla \xi_i^1, g \rangle + \langle f_2 \nabla \xi_i^2, g \rangle$. We analyse each term separately.

$$\begin{aligned}
\langle f_1 \nabla \xi_i^1, g \rangle &= \int_{\mathbb{T}^2} f_1 \nabla \xi_i^1 \cdot g dx = \sum_{\alpha=1}^2 \int_{\mathbb{T}^2} (f_1 \nabla \xi_i^1)_\alpha g_\alpha dx \\
&= \int_{\mathbb{T}^2} (f_1 \nabla \xi_i^1)_1 g_1 + (f_1 \nabla \xi_i^1)_2 g_2 dx \\
&= \int_{\mathbb{T}^2} (f_1 \partial_1 \xi_i^1) g_1 + (f_1 \partial_2 \xi_i^1) g_2 dx
\end{aligned}$$

With similar calculations for the second term we obtain

$$\langle f_2 \nabla \xi_i^2, g \rangle = \int_{\mathbb{T}^2} (f_2 \partial_1 \xi_i^2) g_1 + (f_2 \partial_2 \xi_i^2) g_2 dx.$$

Now remark that

$$\begin{aligned}
\int_{\mathbb{T}^2} \sum_{\alpha} f_{\alpha} \left(\sum_j g_j \partial_j \xi_i^{\alpha} \right) &= \int_{\mathbb{T}^2} \sum_{\alpha} f_{\alpha} \left(\sum_j g_j \partial_j \xi_i \right)_{\alpha} \\
&= \langle f, \sum_j g_j \partial_j \xi_i \rangle \\
&= \langle f, g_1 \partial_1 \xi_i + g_2 \partial_2 \xi_i \rangle \\
&= \langle f, g \cdot (\partial_1 \xi_i \ \partial_2 \xi_i) \rangle \\
&= \langle f, g \cdot \nabla \xi_i \rangle
\end{aligned}$$

That is

$$\langle \mathcal{A}_i f, g \rangle = \langle f, \nabla \xi_i \cdot g \rangle.$$

Remark 2.24. We highlight the fact that $\mathcal{A}_i$ is a zero-order operator and therefore the order of the operator $\mathcal{L}_i + \mathcal{A}_i$ is dictated by the order of $\mathcal{L}_i$. This remains true also for $(\mathcal{L}_i + \mathcal{A}_i)^2$, as $\mathcal{A}_i^2 f = (f_1 \partial_1 \xi_i^1 + f_2 \partial_1 \xi_i^2) \nabla \xi_i^1 + (f_1 \partial_2 \xi_i^2 + f_2 \partial_2 \xi_i^2) \nabla \xi_i^2$. Taking into account ii. we have

$$\langle \nabla(\mathcal{L}_i + \mathcal{A}_i) f, \nabla(\mathcal{L}_i + \mathcal{A}_i) f \rangle \leq C \|f\|_{1,2}^2 \quad (2.23)$$

A higher order control of the form

$$\langle \partial^k f, \partial^k \mathcal{A}_i^2 f \rangle + \langle \partial^k \mathcal{A}_i f, \partial^k \mathcal{A}_i f \rangle \leq C \|f\|_{k,2}^2 \quad (2.24)$$

can be proven, although we do not need this stronger result for the purpose of this thesis.

3

THE STOCHASTIC EULER EQUATION WITH TRANSPORT NOISE

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In this chapter we prove that the stochastic 2D Euler equation driven by transport noise is well-posed in the Sobolev space $\mathcal{W}^{k,2}(\mathbb{T}^2)$, $k \geq 2$. The content of this chapter is the subject of the paper [34].

3.1 INTRODUCTION

Consider the two dimensional Euler equation modelling an incompressible flow perturbed by transport type stochasticity

$$d\omega_t + u_t \cdot \nabla \omega_t dt + \sum_{i=1}^{\infty} (\xi_i \cdot \nabla \omega_t) \circ dW_t^i = 0$$

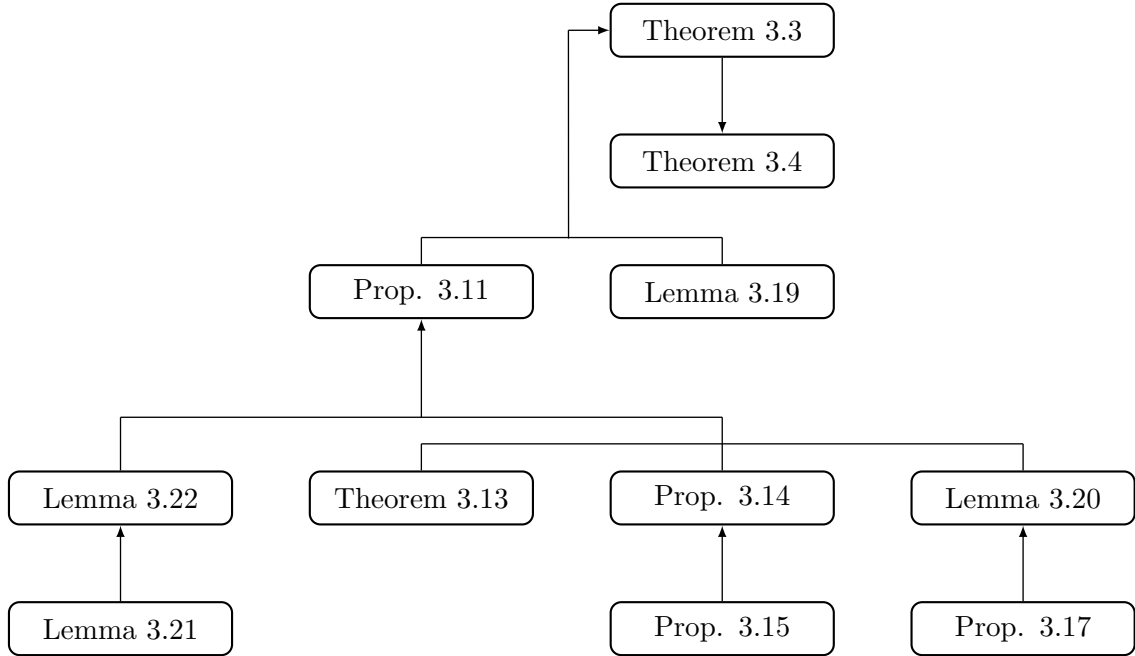
with initial condition ω_0 , where ξ_i are time-independent divergence-free vector fields and the processes $W^i, i \geq 1$ are independent Brownian motions. Classically, u_t stands for the velocity of an incompressible fluid and $\omega_t = \text{curl } u_t$ is the corresponding fluid vorticity. The stochastic part considered here follows the Stochastic Advection by Lie Transport (SALT) theory (see [62], [37], [51], [30]) and corresponds to a stochastic integral of Stratonovich type.

The Euler equation is used to model the motion of an incompressible inviscid fluid. A representative aspect in this context is the study of the fluid vortex dynamics modelled by vorticity equation. There is a vast literature on well-posedness in the deterministic setting, see e.g. [78], [64], [44], [94], [68], [69], [5], [14], and references therein.

The introduction of stochasticity into ideal fluid dynamics has received special attention over the past two decades. On one hand, comprehensive physical models can be obtained when the stochastic term accounts for physical uncertainties ([62], [30], [31], [37]), whilst, in some cases, the regularity properties of the deterministic solution can be improved when the right type of stochasticity is added ([51], [49], [103], [41]). Global existence of smooth solutions for the stochastic Euler equation with multiplicative noise in both 2D and 3D has been obtained in [103]. In [12], a weak solution of the Euler equation with additive noise is constructed as an inviscid limit of stochastic damped 2D Navier-Stokes equations. A martingale solution constructed also as a limit of Navier-Stokes equations but with cylindrical noise can be found in [22]. Existence

and uniqueness results with different variations in terms of stochastic forcing and approximations can be found in [81], [27], [82], [10] and references therein. An overview of results on this topic is provided in [11].

The analysis of nonlinear stochastic partial differential equations with noise of transport type has recently expanded substantially, see e.g., [62], [37], [30], [31], [8], [2], [3], [54]. Existence of a solution for the two-dimensional stochastic Euler equation with noise of transport type has been considered in [21]. While in [21] the authors prove the existence and pathwise uniqueness of a distributional solution in $L^\infty(\mathbb{T}^2)$, in this chapter we are concerned with the existence of a strong solution and give conditions under which the solution enjoys smoothness properties¹. In [48], a random point vortices system is used to construct a so-called ρ - white noise solution. Local well-posedness and a Beale-Kato-Majda blow-up criterion for the three-dimensional case in the space $\mathcal{W}^{2,2}(\mathbb{T}^2)$ has been obtained in [37]. Full well-posedness for a point vortex dynamics system based on this equation has been proven in [52]. The linear case has been considered in [49] and in [53]. We provide below a diagram which explains the dependency chain for the results presented in this chapter:



In the sequel, $\mathbb{T}^2$ is the two-dimensional torus, $k \geq 0$ is a fixed positive integer and $\mathcal{W}^{k,2}$ is the

¹In other words, we identify conditions under which the (strong) solution of the two-dimensional stochastic Euler equation with noise of transport type belongs to the Sobolev space $\mathcal{W}^{k,2}$ with k arbitrarily high.

usual Sobolev space. The of this chapter is the following:

Theorem: Under the conditions stated in Section 2.6 on the vector fields $(\xi_i)_i$, the two-dimensional stochastic Euler vorticity equation

$$d\omega_t + u_t \cdot \nabla \omega_t dt + \sum_{i=1}^{\infty} (\xi_i \cdot \nabla \omega_t) \circ dW_t^i = 0, \quad \omega_0 \in \mathcal{W}^{k,2}(\mathbb{T}^2) \quad (3.1)$$

admits a unique global (in time) solution in the space $\mathcal{W}^{k,2}(\mathbb{T}^2)$. Moreover, ω_t is a continuous function of the initial condition.

Remark 3.1. *As stated above, the stochastic terms in (3.1) are stochastic integrals of Stratonovich type. The corresponding Itô form of equation (3.1) is*

$$d\omega_t + u_t \cdot \nabla \omega_t dt + \sum_{i=1}^{\infty} \xi_i \cdot \nabla \omega_t dW_t^i = \frac{1}{2} \sum_{i=1}^{\infty} \xi_i \cdot \nabla (\xi_i \cdot \nabla \omega_t) dt. \quad (3.2)$$

The assumptions we impose on the vector fields $(\xi_i)_i$ are described in Chapter 2, Section 2.6. In short, the $(\xi_i)_i$ parameters are assumed to be sufficiently smooth and to have norms which decay sufficiently fast as i increases, see condition (2.21). This ensures that the infinite sums in (3.1) and (3.2) make sense in $\mathcal{W}^{k,2}(\mathbb{T}^2)$.

Remark 3.2. *Importantly, we do not require the additional assumption²*

$$\sum_{i=1}^{\infty} \xi_i(x) \xi_i^T(x) = cI_2 \quad (3.3)$$

used in [21]. As a result, in the Itô version (3.2) of the SPDE, the term $\frac{1}{2} \sum_{i=1}^{\infty} \xi_i \cdot \nabla (\xi_i \cdot \nabla \omega_t)$ does not reduce to $c\Delta \omega_t$. This would simplify the analysis as, in this case, the Laplacian commutes with higher order derivatives. Moreover, it commutes with the operation of convolution with the Biot-Savart kernel, an essential ingredient used in [21]. The general term $\frac{1}{2} \sum_{i=1}^{\infty} \xi_i \cdot \nabla (\xi_i \cdot \nabla \omega_t)$ makes the analysis harder. We succeed in controlling it by considering it in tandem with the term $\sum_{i=1}^{\infty} \int_{\mathbb{T}^2} (\xi_i \cdot \nabla \omega_t)^2 dx$ coming from the quadratic variation of the stochastic integrals (see Lemma 3.20 and Proposition 2.23 i.) appearing in the evolution equation for the process $t \mapsto \|\omega_t\|^2$.

²Here c is a non-negative constant, I_2 is the identity matrix, and $\xi(x)^T$ is the transpose of $\xi(x)$.

Another technical difficulty appears when trying to control the high-order derivatives of the vorticity. Nonetheless, this is achieved through a set of inequalities (see Lemma 3.20 and Proposition 2.23) that have first been introduced in the literature by Gyöngy and Krylov ([56], [57], [58]) and recently used by Crisan, Flandoli and Holm ([37]). Again, we emphasize that these rather surprising inequalities hold true without imposing assumptions on the driving vectors $(\xi_i)_i$ other than on their smoothness and summability. This finding is particularly important when using this model for the purpose of uncertainty quantification and data assimilation: for example, in [30], [31], [32], the driving vectors $(\xi_i)_i$ are *estimated* from the data and not a priori chosen. The methodology used in these papers does not naturally lead to driving vector fields that satisfy assumption (3.3) so removing it is essential for our research programme.

We emphasize that the appearance of the second order differential operator $\omega \mapsto \frac{1}{2} \sum_{i=1}^{\infty} \xi_i \cdot \nabla (\xi_i \cdot \nabla \omega)$ in the Itô version of the Euler equation does not give the equation a parabolic character, even if one assumes the restriction (3.3) with c chosen strictly positive. Equation (3.1) is truly a transport type equation and one cannot expect the initial condition to be smoothed out. The best scenario is to prove that the initial level of smoothness of the solution is preserved. This is indeed the main finding of our research.

Let $(W^i)_{i \in \mathbb{N}}$ be a sequence of independent Brownian motions on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Provided $\omega \in L^2(0, T; W^{2,2}(\mathbb{T}^2, \mathbb{R}))$, the first two conditions in (2.21) ensure that the infinite sums of stochastic integrals

$$\sum_{i=1}^{\infty} \int_0^t \mathcal{L}_i \omega_s dW_s^i \quad (3.4)$$

are well defined and belong to $L^2(0, T; L^2(\mathbb{T}^2, \mathbb{R}))$. Similarly, the processes $s \mapsto \mathcal{L}_i^2 \omega_s$ are well-defined and belong to $L^2(0, T; L^2(\mathbb{T}^2, \mathbb{R}))$ provided the solution of the stochastic partial differential equation (3.1) belongs to a suitably chosen space (see Definition 3.5 below). In particular, the Itô correction in (3.2) is well defined. The third condition is needed for proving a number of required a priori estimates (see Lemma 3.20 in Section 3.6). In the following definition and throughout this chapter, $k \geq 2$ is a fixed integer.

3.2 MAIN RESULTS

Let us start by re-stating the existence and uniqueness result, this time with complete details:

Theorem 3.3. *If $\omega_0 \in \mathcal{W}^{k,2}(\mathbb{T}^2)$ is a divergence-free function then the two-dimensional stochastic Euler vorticity equation (3.1)*

$$d\omega_t + \mathcal{L}_{u_t}\omega_t dt + \sum_{i=1}^{\infty} \mathcal{L}_i \omega_t \circ dW_t^i = 0$$

admits a unique global $\mathcal{F}_t$ -adapted divergence-free solution $\omega = \{\omega_t, t \in [0, \infty)\}$ with values in the space $C([0, \infty); \mathcal{W}^{k,2}(\mathbb{T}^2))$. In particular, if $k \geq 4$ the solution is classical.

The proof of Theorem 3.3 is contained in Sections 3.2.1 and 3.2.2. We state next a result that shows the continuity with respect to initial conditions:

Theorem 3.4. *Let $\omega, \tilde{\omega}$ be two $C([0, \infty); \mathcal{W}^{k,2}(\mathbb{T}^2))$ -solutions of equation (3.1). Define A^3 as the process $A_t := \int_0^t \|\omega_s\|_{k,2} ds$, for any $t \geq 0$. Then there exists a positive constant C independent of the two solutions such that*

$$\mathbb{E}[e^{-CA_t} \|\omega_t - \tilde{\omega}_t\|_{k,2}^2] \leq \|\omega_0 - \tilde{\omega}_0\|_{k,2}^2. \quad (3.5)$$

The proof of this theorem is incorporated in Section 3.3.3.

Definition 3.5. *a. A strong solution of the stochastic partial differential equation (3.1) is an $(\mathcal{F}_t)_t$ -adapted process $\omega : \Omega \times \mathbb{T}^2 \rightarrow \mathbb{R}$ with trajectories in the space $C([0, \infty); \mathcal{W}^{k,2}(\mathbb{T}^2))$, such that the identity⁴*

$$\omega_t = \omega_0 - \int_0^t \mathcal{L}_{u_s} \omega_s ds - \sum_{i=1}^{\infty} \int_0^t \mathcal{L}_i \omega_s dW_s^i + \frac{1}{2} \sum_{i=1}^{\infty} \int_0^t \mathcal{L}_i^2 \omega_s ds$$

with $\omega|_{t=0} = \omega_0$, holds $\mathbb{P}$ -almost surely in $L^2(\mathbb{T}^2; \mathbb{R})$.

³Note that there is no link between the process $(A_t)_t$ defined here and the operator A used in Chapter 2 to denote a generic coefficient corresponding to a general SPDE.

⁴Here and everywhere else u is implicitly defined as the velocity field whose vorticity is ω , in other words $\omega_t = \text{curl } u_t = \nabla^\perp u = \partial_2 u_1 - \partial_1 u_2$. See further details in Remarks 3.6 and 3.16.

- b. A weak/distributional solution of equation (3.1) is an $(\mathcal{F}_t)_t$ - adapted process $\omega : \Omega \times \mathbb{T}^2 \rightarrow \mathbb{R}$ with trajectories in the set $C([0, \infty); \mathcal{W}^{k,2}(\mathbb{T}^2))$, which satisfies the equation in the weak topology of $L^2(\mathbb{T}^2, \mathbb{R})$ i.e.

$$\langle \omega_t, \varphi \rangle = \langle \omega_0, \varphi \rangle - \int_0^t \langle \omega_s, \mathcal{L}_{u_s}^* \varphi \rangle ds - \sum_{i=1}^{\infty} \int_0^t \langle \omega_s, \mathcal{L}_i^* \varphi \rangle dW_s^i + \frac{1}{2} \sum_{i=1}^{\infty} \int_0^t \langle \omega_s, \mathcal{L}_i^* \mathcal{L}_i^* \varphi \rangle ds$$

holds $\mathbb{P}$ - almost surely for all $\varphi \in C^\infty(\mathbb{T}^2, \mathbb{R})$.

- c. A weak probabilistic/martingale solution of equation (3.1) is a triple $(\check{\omega}, \{\check{W}^i\}_i), (\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}}), (\check{\mathcal{F}}_t)_t$ such that $(\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}})$ is a probability space, $(\check{\mathcal{F}}_t)_t$ is a filtration defined on this space, $\check{\omega}$ is a continuous $(\check{\mathcal{F}}_t)_t$ -adapted real valued process $\check{\omega} : \Omega \times \mathbb{T}^2 \rightarrow \mathbb{R}$ with trajectories in the set $C([0, \infty); \mathcal{W}^{k,2}(\mathbb{T}^2))$, $\{\check{W}^i\}_i$ are independent $(\check{\mathcal{F}}_t)_t$ -adapted Brownian motions and the identity

$$\check{\omega}_t = \check{\omega}_0 - \int_0^t \mathcal{L}_{\check{u}_s} \check{\omega}_s ds - \sum_{i=1}^{\infty} \int_0^t \mathcal{L}_i \check{\omega}_s d\check{W}_s^i + \frac{1}{2} \sum_{i=1}^{\infty} \int_0^t \mathcal{L}_i^2 \check{\omega}_s ds$$

with $\check{\omega}|_{t=0} = \check{\omega}_0$, holds $\check{\mathbb{P}}$ -almost surely in $L^2(\mathbb{T}^2; \mathbb{R})$.⁵

- d. A classical solution of equation (3.1) is an $(\mathcal{F}_t)_t$ - adapted process $\omega : \Omega \times \mathbb{T}^2 \rightarrow \mathbb{R}$ with trajectories of class $C([0, \infty); C^2(\mathbb{T}^2; \mathbb{R}))$.

Remark 3.6. The velocity field u is not uniquely identified through the equation $\omega = \text{curl } u$. Indeed any two velocity fields that differ by a constant will lead to the same vorticity map ω . Instead we identify u through the "explicit" formula $u = \nabla^\perp \Delta^{-1} \omega$, see details in Remark 3.16 in Section 3.6. in particular, since u and ω are defined in terms of partial derivatives of other functions, on the torus, they must have zero average:

$$\int_{\mathbb{T}^2} u^1(x) dx = \int_{\mathbb{T}^2} u^2(x) dx = \int_{\mathbb{T}^2} \omega(x) dx = 0.$$

Note that if ω_0 has zero average, then ω_t will have zero average, as it is immediate that all the terms appearing in the Euler equation (either (3.1) or (3.2)) have zero average.

⁵We use the "check" notation in the description of the various components of a weak probabilistic solution, to emphasize that the existence of a weak solution does not guarantee that, for a given set of Brownian motions $(W^i)_i$ defined on a (possibly different) probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a solution of (3.1) will exist. Clearly strong solvability implies weak solvability.

Remark 3.7. Note that $\omega_t \in \mathcal{W}^{k,2}(\mathbb{T}^2)$ implies $u_t \in \mathcal{W}^{k+1,2}(\mathbb{T}^2)$ (see Section 3.6). By standard Sobolev embedding theorems $\mathcal{W}^{k+1,2}(\mathbb{T}^2) \hookrightarrow \mathcal{W}^{k,2}(\mathbb{T}^2) \hookrightarrow C_B(\mathbb{T}^2)$ for $k \geq 2$, hence the terms $\mathcal{L}_{u_t}\omega_t = u_t \cdot \nabla \omega_t \in L^2(\mathbb{T}^2, \mathbb{R})$ and $\langle \omega_s, \mathcal{L}_{u_s}^* \varphi \rangle$ are well defined.

Remark 3.8. Naturally, if ω_t is a strong solution in the sense of Definition 3.5, then it is also a weak/distributional solution. In this sense, our result enhances the solution properties presented in [21] at the expense of stronger assumptions on the initial condition of the stochastic partial differential equation, but without the need to impose the additional constraint (3.3). Note also that if ω_t is a weak/distributional solution with paths in $C([0, T]; \mathcal{W}^{k,2}(\mathbb{T}^2))$ then, by integration by parts, the equation has a strong solution.

3.2.1 PATHWISE UNIQUENESS OF SOLUTION FOR THE EULER EQUATION

The uniqueness of the solution of equation (3.1) is an immediate corollary of inequality (3.5). However, the proof of (3.5) requires the existence of an approximating sequence which is constructed as part of the existence results. We present below a direct proof which avoids this, given the fact that pathwise uniqueness is required for the proof of existence of a strong (probabilistic) solution.

Suppose that equation (3.1) admits two global $\mathcal{F}_t$ -adapted solutions ω_1 and ω_2 with values in the space $C([0, \infty); \mathcal{W}^{k,2}(\mathbb{T}^2))$ and let $\bar{\omega} := \omega^1 - \omega^2$. Consider the corresponding velocities u^1 and u^2 such that $\text{curl } u^1 = \omega^1$, $\text{curl } u^2 = \omega^2$ and $\bar{u} := u^1 - u^2$. Since both ω^1 and ω^2 satisfy (3.2), their difference satisfies

$$d\bar{\omega}_t + (\mathcal{L}_{\bar{u}_t}\omega_t^1 + \mathcal{L}_{u_t^2}\bar{\omega}_t)dt + \sum_{i=1}^{\infty} \mathcal{L}_i \bar{\omega}_t dW_t^i - \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_i^2 \bar{\omega}_t dt = 0.$$

By an application of the Itô formula one obtains

$$\begin{aligned} d\|\bar{\omega}_t\|_2^2 &= -2 \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle dW_t^i - 2 \langle \bar{\omega}_t, \mathcal{L}_{\bar{u}_t}\omega_t^1 + \mathcal{L}_{u_t^2}\bar{\omega}_t \rangle dt \\ &\quad + \sum_{i=1}^{\infty} (\langle \bar{\omega}_t, \mathcal{L}_i^2 \bar{\omega}_t \rangle + \langle \mathcal{L}_i \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle) dt. \end{aligned}$$

Note that the last term in the above identity is null (see Proposition 2.23 and Lemma 3.20)⁶ and that

$$|\langle \bar{\omega}_t, \mathcal{L}_{\bar{u}_t} \omega_t^1 \rangle| \leq \|\bar{\omega}_t\|_2 \|\bar{u}_t\|_4 \|\nabla \omega_t^1\|_4 \leq C \|\bar{\omega}_t\|_2^2 \|\omega_t^1\|_{k,2}.$$

This is true since by the Sobolev embedding theorem (see [1] Theorem 4.12 case A) one has $\|\nabla \omega_t^1\|_4 \leq C \|\omega_t^1\|_{k,2}$ and using also the Biot-Savart law one has $\|\bar{u}_t\|_4 \leq C \|\bar{u}_t\|_{1,2} \leq C \|\bar{\omega}_t\|_2$. Finally, observe that $\langle \bar{\omega}_t, \mathcal{L}_{u_t^2} \bar{\omega}_t \rangle = -\frac{1}{2} \int_{\mathbb{T}^2} (\nabla \cdot u_t^2) (\bar{\omega}_t)^2 dx = 0$ since $\operatorname{div} u_t^2 = 0$. It follows that

$$d\|\bar{\omega}_t\|_2^2 + 2 \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle dW_t^i = -2 \langle \bar{\omega}_t, \mathcal{L}_{\bar{u}_t} \omega_t^1 \rangle dt \leq C \|\omega_t^1\|_{k,2} \|\bar{\omega}_t\|_2^2 dt. \quad (3.6)$$

Since we only have a priori bounds for the expected value of the process $t \rightarrow \|\omega_t^1\|_{k,2}$ and not for its pathwise values, the uniqueness cannot be deduced through a Gronwall type argument. Instead, we proceed as follows: let A be the process defined as $A_t := \int_0^t C \|\omega_s^1\|_{k,2} ds$, for any $t \geq 0$. This is an increasing process that stays finite $\mathbb{P}$ -almost surely for all $t \geq 0$ as the paths of ω^1 are in $C([0, \infty); \mathcal{W}^{k,2}(\mathbb{T}^2))$. By integration by parts,

$$d(e^{-A_t} \|\bar{\omega}_t\|_2^2) = e^{-A_t} (d\|\bar{\omega}_t\|_2^2 - C \|\bar{\omega}_t\|_2^2 \|\omega_t^1\|_{k,2} dt) \leq -2e^{-A_t} \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle dW_t^i.$$

This leads to

$$\mathbb{E}[e^{-A_t} \|\bar{\omega}_t\|_2^2] \leq \mathbb{E} \left[\int_0^t -2e^{-A_s} \sum_{i=1}^{\infty} \langle \bar{\omega}_s, \mathcal{L}_i \bar{\omega}_s \rangle dW_s^i \right] = 0.$$

We conclude that $e^{-A_t} \|\bar{\omega}_t\|_2^2 = 0$, and since e^{-A_t} cannot be null due to the finiteness of A_t we deduce that $\|\bar{\omega}_t\|_2^2 = 0$ almost surely, which gives the claim.

The above argument uses the fact that the terms $\langle \bar{\omega}_t, \mathcal{L}_i^2 \bar{\omega}_t \rangle$ and $\langle \mathcal{L}_i \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle$ are well defined. In other words, even though we only wish to control the L^2 -norm of the vorticity, we have to resort to higher order derivatives. This is permitted as we assumed that $\omega \in \mathcal{W}^{k,2}(\mathbb{T}^2)$ for $k \geq 2$. By applying a similar argument, to control the $\mathcal{W}^{k,2}(\mathbb{T}^2)$ -norm of the vorticity we would need to control terms of the form $\langle \partial^k \bar{\omega}_t, \partial^k \mathcal{L}_i^2 \bar{\omega}_t \rangle$ and $\langle \partial^k \mathcal{L}_i \bar{\omega}_t, \partial^k \mathcal{L}_i \bar{\omega}_t \rangle$. This is no longer allowed because we do not have enough smoothness in the system. To overcome this difficulty we will

⁶The application of the Lemma requires that the two solutions ω^1 and ω^2 belong to $\mathcal{W}^{k,2}(\mathbb{T}^2)$ with $k \geq 2$. To deduce (3.5) we need a similar control (albeit not an identity) for higher order derivatives. This is done by using the approximating sequence constructed in Section 3.3 and then taking the limit. This is the reason why we cannot prove directly (3.5).

make use of a smooth approximating sequence for the vorticity equation, see Section 3.3.3.

Remark 3.9. *The above uniqueness result is somewhat stronger than the uniqueness deduced from inequality (3.5). It shows that a solution of (3.1) will be unique in the larger space $L^2(\mathbb{T}^2)$ rather than in the space $\mathcal{W}^{k,2}(\mathbb{T}^2)$. Nevertheless, inequality (3.5) shows the continuity of the solution with respect to initial conditions.*

Remark 3.10. *We note that, in contrast to the deterministic version of the Euler equation, the minimal k that ensures the existence of a strong solution is $k = 2$. This is because of the occurrence of the term $\mathcal{L}_i^2 \omega$ in the Itô version of the equation. Moreover, in order for the Stratonovich integral in equation (3.1) to make sense, we require $k \geq 3$ as the term $\mathcal{L}_i^3 \omega$ appears in the evolution equation for the covariation between $\mathcal{L}_i \omega$ and W^i required for the rigorous definition of the Stratonovich integral. Nonetheless, the methodology from this chapter can be used to cover initial conditions $\omega_0 \in \mathcal{W}^{k,2}(\mathbb{T}^2)$ with $k < 2$. In this case we have to content ourselves with weak/distributional solutions. Whilst this is not the subject of this thesis, such a solution can be shown to exist as long as the product $\omega \mathcal{L}_u \varphi$ makes sense in a suitably chosen sense. We need to interpret the nonlinear term in a weak form as a generalised function and replace it with $\int_0^t \langle u_s \omega_s, \nabla \varphi \rangle ds$. Then the same methodology can be applied as long as we can control the $\|\omega\|_{W^{-1,2}}$ norm. This is possible, for example, if we assume $\omega_0 \in L^4(\mathbb{T}^2)$.*

3.2.2 EXISTENCE OF SOLUTION FOR THE EULER EQUATION

The existence of the solution of equation (3.1) is proved by first showing that a truncated version of it has a solution, and then removing the truncation. In particular we will truncate the nonlinear term in (3.1) by using a smooth function f_R equal to 1 on $[0, R]$, equal to 0 on $[R + 1, \infty)$, and decreasing on $[R, R + 1]$, for arbitrary $R > 0$. We then have the following:

Proposition 3.11. *If $\omega_0 \in \mathcal{W}^{k,2}(\mathbb{T}^2)$ is a divergence-free function then the following equation*

$$d\omega_t^R + f_R(\|\omega_t^R\|_{k-1,2}) \mathcal{L}_{u_t^R} \omega_t^R dt + \sum_{i=1}^{\infty} \mathcal{L}_i \omega_t^R \circ dW_t^i = 0 \quad (3.7)$$

admits a unique global $\mathcal{F}_t$ -adapted divergence-free solution $\omega^R = \{\omega_t^R, t \in [0, \infty)\}$ with values in the space $C([0, \infty); \mathcal{W}^{k,2}(\mathbb{T}^2))$. In particular, if $k \geq 4$, the solution is classical.

Remark 3.12. *The truncation in terms of the norm $\|\omega_t^R\|_{k-1,2}$ and not $\|\omega_t^R\|_{k,2}$ is not incidental as it suffices to control the norm $\|u_t^R\|_{k,2}$ (see Proposition 3.17).*

We prove Proposition 3.11 in Section 3.3. For now let us proceed with the proof of global existence for the solution of the Euler equation (3.1). Define $\tau_R(\omega) := \inf_{t \geq 0} \{\|\omega_t^R\|_{k-1,2} \geq R\}$. Observe that on $[0, \tau_R]$, $f_R(\|\omega_t^R\|_{k-1,2}) = 1$, and therefore, on $[0, \tau_R]$ the solution of the truncated equation (3.7) is, in fact a solution of (3.1) with all required properties. It therefore makes sense to define the process $\omega = \{\omega_t, t \in [0, \infty)\}$ $\omega_t = \omega_t^R$ for $t \in [0, \tau_R]$. This definition is consistent as, following the uniqueness property of the solution of the truncated equation (see Section 3.3.1), $\omega_t^R = \omega_t^{R'}$ for $t \in [0, \tau_{\min(R, R')}]$. The process ω defined this way is a solution of the Euler equation (3.1) on the interval $[0, \sup_{R>0} \tau_R)$. To obtain a global solution we need to prove that $\sup_{R>0} \tau_R = \infty$ $\mathbb{P}$ -almost surely. Let $\mathcal{A} := \{\omega \in \Omega \mid \sup_{R>0} \tau_R(\omega) < \infty\}$. Then

$$\mathcal{A} = \bigcup_{N>0} \left\{ \sup_R \tau_R(\omega) \leq N \right\} = \bigcup_N \bigcap_R \{|\tau_R(\omega)| \leq N\}$$

and

$$\mathbb{P}(|\tau_R(\omega)| \leq N) = \mathbb{P}\left(\sup_{t \in [0, N]} \|\omega_t^R\|_{k-1,2} > R\right).$$

In order to finish the proof of global existence we use Lemma 3.19 and the fact that

$$\mathbb{P}(\|\omega_t^R\|_{k-1,2} > R) \leq \frac{\mathbb{E}\left[\ln\left(\|\omega_t^R\|_{k-1,2}^2 + e\right)\right]}{R^2 + e} \leq \frac{\mathcal{C}(\omega_0, T)}{R^2 + e} \xrightarrow{R \rightarrow \infty} 0.$$

It follows that

$$\mathbb{P}\left(\bigcap_R \{|\tau_R(\omega)| \leq N\}\right) = \lim_{R \rightarrow \infty} \mathbb{P}(|\tau_R(\omega)| \leq N) = 0.$$

and therefore $\mathbb{P}(\mathcal{A}) = 0$. This concludes the global existence for the solution of the equation (3.1).

3.3 ANALYSIS OF THE TRUNCATED EQUATION

3.3.1 UNIQUENESS OF SOLUTION FOR THE TRUNCATED EQUATION

In this section we prove uniqueness of the solution of equation (3.7). We use a similar strategy as the one used to prove the uniqueness of the solution of the (un-truncated) Euler equation (3.1). Suppose that equation (3.7) admits two global $\mathcal{F}_t$ -adapted solutions $\omega^{1,R}$ and $\omega^{2,R}$ with values in the space $C([0, \infty); \mathcal{W}^{k,2}(\mathbb{T}^2))$. We prove that $\omega^{1,R}$ and $\omega^{2,R}$ must coincide. In the following, we will formally drop the dependence on R of the two solutions. As above, let $\bar{\omega} := \omega^1 - \omega^2$ and consider the corresponding velocities u^1 and u^2 such that $\text{curl } u^1 = \omega^1$, $\text{curl } u^2 = \omega^2$ and $\bar{u} := u^1 - u^2$. Since both ω^1 and ω^2 satisfy (3.7), their difference satisfies

$$d\bar{\omega}_t + ((f_R(\omega_t^1)\mathcal{L}_{u_t^1} - f_R(\omega_t^2)\mathcal{L}_{u_t^2})\bar{\omega}_t^1 + f_R(\omega_t^2)\mathcal{L}_{u_t^2}\bar{\omega}_t)dt + \sum_{i=1}^{\infty} \mathcal{L}_i \bar{\omega}_t dW_t^i - \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_i^2 \bar{\omega}_t dt = 0.$$

where $f_R(\omega) = f_R(\|\omega\|_{k-1,2})$. By an application of the Itô formula and after eliminating the null terms (see Lemma 3.20, Remark 3.23, and use the fact that u_t^2 is divergence-free), one obtains

$$d\|\bar{\omega}_t\|_2^2 + 2 \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle dW_t^i = -2 \langle \bar{\omega}_t, (f_R(\omega_t^1)\mathcal{L}_{u_t^1} - f_R(\omega_t^2)\mathcal{L}_{u_t^2})\bar{\omega}_t^1 \rangle dt$$

One can show that (see [37] for a proof) there exists a constant $C = C(R)$ such that

$$\|f_R(\omega_t^1)u_t^1 - f_R(\omega_t^2)u_t^2\|_4 \leq C\|\bar{\omega}_t\|_{k-1,2}$$

and to finally deduce that

$$|\langle \bar{\omega}, (f_R(\omega_t^1)\mathcal{L}_{u_t^1} - f_R(\omega_t^2)\mathcal{L}_{u_t^2})\bar{\omega}_t^1 \rangle| \leq \|\bar{\omega}_t\|_2 \|\bar{u}_t\|_4 \|\nabla \omega_t^1\|_4 \leq C\|\bar{\omega}_t\|_2 \|\bar{\omega}_t\|_{k-1,2} \|\omega_t^1\|_{k,2}.$$

It follows that

$$d\|\bar{\omega}_t\|_2^2 + 2 \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle dW_t^i \leq C\|\omega_t^1\|_{k,2} \|\bar{\omega}_t\|_2^2 dt.$$

Similar arguments are used to control $\|\partial^\alpha \bar{\omega}_t\|_2^2$ where α is a multi-index with $|\alpha| \leq k-1$ and to

deduce that there exists a constant C such that

$$d\|\partial^\alpha \bar{\omega}_t\|_2^2 + 2 \sum_{i=1}^{\infty} \langle \partial^\alpha \bar{\omega}_t, \partial^\alpha \mathcal{L}_i \bar{\omega}_t \rangle dW_t^i \leq C \|\omega_t^1\|_{k,2} \|\bar{\omega}_t\|_2^2 dt,$$

where we use the control (see Lemma 3.20)

$$\langle \partial^\alpha \bar{\omega}_t, \partial^\alpha \mathcal{L}_i^2 \bar{\omega}_t \rangle + \langle \partial^\alpha \mathcal{L}_i \bar{\omega}_t, \partial^\alpha \mathcal{L}_i \bar{\omega}_t \rangle \leq C \|\bar{\omega}\|_{k,2}^2.$$

Some care is required for the case when $|\alpha| = k - 1$ as $\partial^\alpha \mathcal{L}_i^2 \bar{\omega}_t$ is no longer well-defined. In this case, by using the weak form of the equation (3.7) to rewrite $\langle \partial^\alpha \bar{\omega}_t, \partial^\alpha \mathcal{L}_i^2 \bar{\omega}_t \rangle$ as $-\langle \partial^{\alpha_1} \partial^\alpha \bar{\omega}_t, \partial^{\alpha_2} \mathcal{L}_i^2 \bar{\omega}_t \rangle$ we can proceed as above by using that

$$-\langle \partial^{\alpha_1} \partial^\alpha \bar{\omega}_t, \partial^{\alpha_2} \mathcal{L}_i^2 \bar{\omega}_t \rangle + \langle \partial^\alpha \mathcal{L}_i \bar{\omega}_t, \partial^\alpha \mathcal{L}_i \bar{\omega}_t \rangle \leq C \|\bar{\omega}_t\|_{k,2}^2.$$

The above is true for functions in $\mathcal{W}^{k+1,2}(\mathbb{T}^2)$ and, by the continuity of both sides in the above inequality, it is also true for functions in the larger space $\mathcal{W}^{k,2}(\mathbb{T}^2)$, since $\mathcal{W}^{k+1,2}(\mathbb{T}^2)$ is dense in $\mathcal{W}^{k,2}(\mathbb{T}^2)$. The proof is concluded in an identical manner as that for the uniqueness of the Euler equation (see Section (3.2.1))

3.3.2 EXISTENCE OF SOLUTION FOR THE TRUNCATED EQUATION

The strategy of proving that the truncated equation (3.7) has a solution is to construct an approximating sequence of processes that will converge in distribution to a solution of (3.7). This justifies the existence of a weak solution. Together with the pathwise uniqueness of the solution of this equation, we then deduce that strong uniqueness holds.

Recall that $\omega_0 \in \mathcal{W}^{k,2}(\mathbb{T}^2)$. Let $(\omega_0^n)_n \in C^\infty(\mathbb{T}^2)$ be a sequence such that $\omega_0^n \xrightarrow{n \rightarrow \infty} \omega_0$ in $\mathcal{W}^{k,2}(\mathbb{T}^2)$. For any $t \geq 0$ we construct the sequence $(\omega_t^{\nu_n, R, n})_{n \geq 0}$ with $\omega_t^{\nu_0, R, 0} := \omega_0^0$, and for $n \geq 1$ and $\omega_0^{\nu_n, R, n} := \omega_0^n$,

$$d\omega_t^{\nu_n, R, n} = (\nu_n \Delta \omega_t^{\nu_n, R, n} - f_R(\omega_t^{\nu_{n-1}, R, n-1}) \mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n}) dt - \sum_{i=1}^{\infty} \mathcal{L}_i \omega_t^{\nu_n, R, n} \circ dW_t^i, \quad (3.8)$$

where $\nu_n = \frac{1}{n}$ is the viscous parameter ($n > 0$) and $u_t^{\nu_{n-1}, R, n-1} = \text{curl}^{-1}(\omega_t^{\nu_{n-1}, R, n-1})$ ⁷. Also $f_R(\omega_t^{\nu_n, R, n}) := f_R(\|\omega_t^{\nu_n, R, n}\|_{k-1, 2})$. The corresponding Itô form of equation (3.8) is ⁸

$$d\omega_t^{\nu_n, R, n} = (\nu_n \Delta \omega_t^{\nu_n, R, n} + P_t^{n-1, n}(\omega_t^{\nu_n, R, n}))dt - \sum_{i=1}^{\infty} \mathcal{L}_{\xi_i} \omega_t^{\nu_n, R, n} dW_t^i, \quad (3.9)$$

where $P_t^{n-1, n}(\omega_t^{\nu_n, R, n})$ is defined as

$$P_t^{n-1, n}(\omega_t^{\nu_n, R, n}) := -f_R(\omega_t^{\nu_{n-1}, R, n-1}) \mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n} + \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_{\xi_i}^2 \omega_t^{\nu_n, R, n}, \quad t \geq 0. \quad (3.10)$$

Theorem 3.13. *If $\omega_0^{\nu_n, R, n} \in C^\infty(\mathbb{T}^2)$ is a divergence-free function with null spatial mean, then the two-dimensional stochastic vorticity equation (3.9) admits a unique global $\mathcal{F}_t$ -adapted solution $\omega^{\nu_n, R, n} = \{\omega_t^{\nu_n, R, n}, t \in [0, \infty)\}$ in the space $C([0, \infty); C^\infty(\mathbb{T}^2))$.*

The proof of this theorem is provided in Section 3.4.

Proposition 3.14. *The laws of the family of solutions $(\omega^{\nu_n, R, n})_{\nu_n \in [0, 1]}$ is relatively compact in the space of probability measures over $D([0, T], L^2(\mathbb{T}^2))$ for any $T \geq 0$.*

The proof of Proposition 3.14 is left for Section 3.5.

Proof of existence of the solution of equation (3.7)

Using a diagonal subsequence argument we can deduce from Proposition 3.14 and the fact that $\lim_{n \rightarrow \infty} \omega_0^{\nu_n, R, n} = \omega_0$ the existence of a subsequence $(\omega^{\nu_{n_j}})_j$ with $\lim_{j \rightarrow \infty} \nu_{n_j} = 0$, which is convergent in distribution over $D([0, \infty), L^2(\mathbb{T}^2))$. We show that the limit of the corresponding distributions is the distribution of a stochastic process that solves (3.7). This justifies the existence of a weak (probabilistic) solution. By using the Skorokhod representation theorem (see [13] Section 6, pp. 70), there exists a space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence of processes $(\tilde{\omega}^{\nu_n, R, n}, \tilde{u}^{\nu_n, R, n}, (\tilde{W}^i)_i, n = 1, \infty)$ which has the same distribution as that of the original converging subsequence and which converges (when $n \rightarrow \infty$) almost surely to a triplet $(\tilde{\omega}^R, \tilde{u}^R, (\tilde{W}^i)_i)$ in $D([0, T], L^2(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathbb{R}^{\mathbb{N}})$. Note $\omega^{\nu_n, R, n}$ and $\tilde{\omega}^{\nu_n, R, n}$ have the same distribution, so

⁷The operator curl^{-1} is the convolution with the Biot-Savart kernel, see Remark 3.16 for details.

⁸The stochastic Itô integral is understood here in the usual sense, see [40].

that for any test function $\varphi \in C^\infty(\mathbb{T}^2)$ we have

$$\begin{aligned} \langle \tilde{\omega}_t^{\nu_n, R, n}, \varphi \rangle &= \langle \tilde{\omega}_0^{\nu_n, R, n}, \varphi \rangle + \nu_n \int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n}, \Delta \varphi \rangle ds \\ &\quad + \int_0^t f_R(\tilde{\omega}_s^{\nu_{n-1}, R, n-1}) \langle \tilde{\omega}_s^{\nu_n, R, n}, \mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \varphi \rangle ds \\ &\quad + \frac{1}{2} \sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n}, \mathcal{L}_i^2 \varphi \rangle ds + \sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n}, \mathcal{L}_i \varphi \rangle d\tilde{W}_s^i. \end{aligned} \quad (3.11)$$

Note that there exists a constant $C = C(T)$ such that

$$\sup_{n \geq 1} \tilde{\mathbb{E}} \left[\sup_{s \in [0, T]} \|\tilde{\omega}_s^{\nu_n, R, n}\|_{k, 2}^4 \right] \leq C, \quad (3.12)$$

where $\tilde{\mathbb{E}}$ is the expectation with respect to $\tilde{\mathbb{P}}$. We prove this in Lemma 3.20 for the original sequence, but since $\tilde{\omega}^{\nu_n, R, n}$ satisfies the same SPDE, the same a priori estimates hold for $\tilde{\omega}^{\nu_n, R, n}$. Since the space of continuous functions is a subspace of the space of càdlàg functions and the Skorokhod topology relativised to the space of continuous functions coincides with the uniform topology, it follows that the sequence $(\tilde{\omega}^{\nu_n, R, n}, \tilde{u}^{\nu_n, R, n}, (\tilde{W}^i)_i, n = 1, \infty)$ converges (when $n \rightarrow \infty$) $\tilde{\mathbb{P}}$ -almost surely to $(\tilde{\omega}^R, \tilde{u}^R, (\tilde{W}^i)_i)$ also in the uniform norm. It also holds that

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[\int_0^t \|\tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R\|_2^2 ds \right] = 0$$

and since

$$\begin{aligned} \sum_{i=1}^{\infty} \tilde{\mathbb{E}} \left[\int_0^t (\langle \tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R, \mathcal{L}_i \varphi \rangle)^2 ds \right] &\leq \sum_{i=1}^{\infty} \|\mathcal{L}_i \varphi\|_2^2 \tilde{\mathbb{E}} \left[\int_0^t \|\tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R\|_2^2 ds \right] \\ &\leq C \|\varphi\|_{1, 2}^2 \tilde{\mathbb{E}} \left[\int_0^t \|\tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R\|_2^2 ds \right] \end{aligned} \quad (3.13)$$

also the limit of the right hand side of (3.13) converges to 0 (we use here the control $\sum_{i=1}^{\infty} \|\mathcal{L}_i \varphi\|_2^2 \leq C \|\varphi\|_{1, 2}^2$ assumed in (2.21)). Now Theorem 4.2 in [72] allows us to conclude that the term

$$\sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n}, \mathcal{L}_i \varphi \rangle d\tilde{W}_s^i$$

converges to $\sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^R, \mathcal{L}_i \varphi \rangle d\tilde{W}_s^i$ in distribution. By a similar application of the Skorokhod representation theorem, we can also assume that on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, the term $\sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n}, \mathcal{L}_i \varphi \rangle d\tilde{W}_s^i$ converges to $\sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^R, \mathcal{L}_i \varphi \rangle d\tilde{W}_s^i$ $\tilde{\mathbb{P}}$ -almost surely (as well as in $L^2(\tilde{\mathbb{P}})$). Let us prove the convergence of the remaining terms in (3.11) :

- $\tilde{\omega}^{\nu_n, R, n}$ converges $\tilde{\mathbb{P}}$ -almost surely to $\tilde{\omega}^R$ in $D([0, \infty), L^2(\mathbb{T}^2))$. Since φ is bounded, it follows that $\langle \tilde{\omega}_t^{\nu_n, R, n}, \varphi \rangle \xrightarrow{n \rightarrow \infty} \langle \tilde{\omega}_t^R, \varphi \rangle$ and $\langle \tilde{\omega}_0^{\nu_n, R, n}, \varphi \rangle \xrightarrow{n \rightarrow \infty} \langle \tilde{\omega}_0^R, \varphi \rangle$, $\tilde{\mathbb{P}}$ -almost surely (as well as in $L^2(\tilde{\mathbb{P}})$), for any $\varphi \in C^\infty(\mathbb{T}^2)$.
- The second term on the right hand side of (3.11) converges to 0 when $n \rightarrow \infty$ because the integral is uniformly bounded in $L^2(\tilde{\mathbb{P}})$ (again, because of (3.12)) and $\nu_n \rightarrow 0$ when $n \rightarrow \infty$.
- Using the fact that $\tilde{u}^{\nu_{n-1}, R, n-1}$ is the convolution between $\tilde{\omega}^{\nu_{n-1}, R, n-1}$ and the Biot-Savart kernel we obtain that $\tilde{u}^{\nu_n, R, n}$ converges to $\tilde{u}^R$, $\tilde{\mathbb{P}}$ -almost surely (as well as in $L^2(\tilde{\mathbb{P}})$). Moreover, one can write

$$\begin{aligned} \int_0^t \langle \tilde{u}_s^{\nu_{n-1}, R, n-1} \cdot \nabla \tilde{\omega}_s^{\nu_n, R, n} - \tilde{u}_s^R \cdot \nabla \tilde{\omega}_s^R, \varphi \rangle ds &= \int_0^t \langle (\tilde{u}_s^{\nu_{n-1}, R, n-1} - \tilde{u}_s^R) \cdot \nabla \tilde{\omega}_s^{\nu_n, R, n}, \varphi \rangle ds \\ &\quad - \int_0^t \langle \tilde{u}_s^R \cdot (\nabla \tilde{\omega}_s^{\nu_n, R, n} - \nabla \tilde{\omega}_s^R), \varphi \rangle ds. \end{aligned}$$

We have

$$\begin{aligned} |\langle (\tilde{u}_s^{\nu_{n-1}, R, n-1} - \tilde{u}_s^R) \cdot \nabla \tilde{\omega}_s^{\nu_n, R, n}, \varphi \rangle| &= |\langle (\tilde{u}_s^{\nu_{n-1}, R, n-1} - \tilde{u}_s^R) \cdot \tilde{\omega}_s^{\nu_n, R, n}, \nabla \varphi \rangle| \\ &\leq \|\nabla \varphi \cdot \tilde{\omega}_s^{\nu_n, R, n}\|_2 \|\tilde{u}_s^{\nu_{n-1}, R, n-1} - \tilde{u}_s^R\|_2 \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

and

$$\begin{aligned} |\langle \tilde{u}_s^R \cdot (\nabla \tilde{\omega}_s^{\nu_n, R, n} - \nabla \tilde{\omega}_s^R), \varphi \rangle| &= |\langle \tilde{u}_s^R \cdot (\tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R), \nabla \varphi \rangle| \\ &\leq \|\nabla \varphi \cdot \tilde{u}_s^R\|_2 \|\tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R\|_2 \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

for $0 \leq t \leq T$.

- Lastly, the integrals coming from the Itô correction term are treated in a similar fashion:

$$\begin{aligned}
|\langle \xi_i \cdot \nabla(\xi_i \cdot \nabla \tilde{\omega}_s^{\nu_n, R, n}) - \xi_i \cdot \nabla(\xi_i \cdot \nabla \tilde{\omega}_s^R), \varphi \rangle| &= |\langle \xi_i \cdot \nabla \tilde{\omega}_s^{\nu_n, R, n} - \xi_i \cdot \nabla \tilde{\omega}_s^R, \xi_i \cdot \nabla \varphi \rangle| \\
&= |\langle \tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R, \xi_i \cdot \nabla(\xi_i \cdot \nabla \varphi) \rangle| \\
&\leq \|\xi_i \cdot \nabla(\xi_i \cdot \nabla \varphi)\|_2 \|\tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R\|_2 \xrightarrow{n \rightarrow \infty} 0
\end{aligned}$$

since $\|\xi_i \cdot \nabla(\xi_i \cdot \nabla \varphi)\|_2$ is finite by the initial assumptions made on $(\xi_i)_i$.

We have shown so far that there exists a weak/distributional solution in the sense of Definition 3.5. part b. on the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. However, since $\tilde{\omega}^R$ belongs to the space $\mathcal{W}^{k,2}(\mathbb{T}^2) \hookrightarrow C_B^{k-m}(\mathbb{T}^2)$ the solution is also strong, again, as a solution on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ (and not on the original space). It follows that $(\tilde{\omega}, \tilde{u}, (\tilde{W}^i)_i)$ is a weak/probabilistic solution of the truncated Euler equation (3.7) in the sense of Definition 3.5 part c. Together with the pathwise uniqueness proved in Section 3.3.2 and using the Yamada-Watanabe theorem for the infinite-dimensional setting (see, for instance, [99]) we conclude the existence of a strong solution of the truncated Euler equation, again in the sense of Definition 3.5, part a. Now using the embedding $\mathcal{W}^{k,2}(\mathbb{T}^2) \hookrightarrow C_B^{k-m}(\mathbb{T}^2)$ with $2 \leq m \leq k$ and $k \geq 4$ we conclude that the solution is classical when $k \geq 4$.

3.3.3 CONTINUITY WITH RESPECT TO INITIAL CONDITIONS

In this section we prove Theorem 3.4. As stated in the theorem, let $\omega, \tilde{\omega}$ be two $C([0, \infty); \mathcal{W}^{k,2}(\mathbb{T}^2))$ -solutions of equation (3.1) and define A as the process $A_t := \int_0^t \|\omega_s\|_{k,2} ds$, for any $t \geq 0$. Let $\omega^R, \tilde{\omega}^R$ be their corresponding truncated versions and also let $(\omega_t^{\nu_n, R, n})_{n \geq 0}$ and $(\tilde{\omega}_t^{\nu_n, R, n})_{n \geq 0}$ be, respectively, the corresponding sequences constructed as in Section 3.3.2 on the same space after the application of the Skorohod representation theorem. By Fatou's lemma, applied twice, we deduce that

$$\begin{aligned}
\mathbb{E}[e^{-CA_t} \|\omega_t - \tilde{\omega}_t\|_{k,2}^2] &\leq \mathbb{E}[\liminf_n e^{-CA_t^n} \|\omega_t^{\nu_n, R, n} - \tilde{\omega}_t^{\nu_n, R, n}\|_{k,2}^2] \\
&\leq \liminf_n \mathbb{E}[e^{-CA_t^n} \|\omega_t^{\nu_n, R, n} - \tilde{\omega}_t^{\nu_n, R, n}\|_{k,2}^2]
\end{aligned}$$

where A^n is the process defined by $A_t^n := \int_0^t \|\omega_s^{\nu_n, R, n}\|_{k,2} ds$, for any $t \geq 0$. Following a similar proof with that of the uniqueness of the Euler equation, one then deduces that there exists a

positive constant C independent of the two solutions and independent of R and n such that

$$\mathbb{E}[e^{-CA_t^n} \|\omega_t^{\nu_n, R, n} - \tilde{\omega}_t^{\nu_n, R, n}\|_{k,2}^2] \leq \|\omega_0 - \tilde{\omega}_0\|_{k,2}^2.$$

which gives the result. We emphasize that we use here the fact that the processes $(\omega_t^{\nu_n, R, n})_{n \geq 0}$ and $(\tilde{\omega}_t^{\nu_n, R, n})_{n \geq 0}$ take values in $\mathcal{W}^{k+2,2}(\mathbb{T}^2)$ as an essential ingredient, a property that was not true for either the solution of the Euler equation or its truncated version.

3.4 ANALYTICAL PROPERTIES OF THE APPROXIMATING SEQUENCE OF SOLUTIONS

3.4.1 EXISTENCE AND UNIQUENESS OF THE APPROXIMATING SEQUENCE

We show that the sequence $(\omega_t^{\nu_n, R, n})_n$ given by formula (3.9), that is,

$$d\omega_t^{\nu_n, R, n} = (\nu_n \Delta \omega_t^{\nu_n, R, n} + P_t^{n-1, n}(\omega_t^{\nu_n, R, n}))dt - \sum_{i=1}^{\infty} \mathcal{L}_{\xi_i} \omega_t^{\nu_n, R, n} dW_t^i$$

with $\omega_0^{\nu_n, R, n} = \omega_0^n$, is smooth. The equation above is a particular case of equation (1.1) – (1.2) in Chapter 4, Section 4.1, pp.129 in [100]. All assumptions required by Theorem 1 and Theorem 2 in [100], Chapter 4, are fulfilled. Therefore there exists a unique solution $\omega_t^{\nu_n, R, n}$ which belongs to the class $L^2([0, T], \mathcal{W}^{k,2}(\mathbb{T}^2)) \cap C([0, T], \mathcal{W}^{k-1,2}(\mathbb{T}^2))$ and satisfies equation (3.9) for all $t \in [0, T]$ and for all ω in $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$.

Furthermore, since the conditions are fulfilled for all $k \in \mathbb{N}$, using Corollary 3 at pp. 141 in [100], we obtain that $\omega_t^{\nu_n, R, n}$ is $\mathbb{P}$ - a.s. in $C([0, T], C^\infty(\mathbb{T}^2))$. Note that $u_t^{\nu_{n-1}, R, n-1} \in C^\infty(\mathbb{T}^2)$ for any $n \geq 1$, using the Biot-Savart law and an inductive argument. One has $u_t^{\nu_{n-1}, R, n-1} = K \star \omega_t^{\nu_{n-1}, R, n-1}$ with K being the Biot-Savart kernel defined in Section 3.6. The convolution between K and $\omega_t^{\nu_{n-1}, R, n-1}$ is commutative, so we have

$$u_t^{\nu_{n-1}, R, n-1}(x) = \int_{\mathbb{T}^2} K(y) \omega_t^{\nu_{n-1}, R, n-1}(x - y) dy.$$

Since $\omega_t^{\nu_{n-1}}$ is in $C^\infty(\mathbb{T}^2)$ by Corollary 3 (at step $n - 1$), and using the fact that $K \in L^1(\mathbb{T}^2)$ we conclude that $u_t^{\nu_{n-1}, R, n-1} \in C^\infty(\mathbb{T}^2)$. This, together with the initial assumptions (2.21), ensures that all the coefficients of equation 3.9 are infinitely differentiable. The uniform boundedness is

ensured by the truncation $f_R(\omega_t^{\nu_{n-1},R,n-1})$, as proven in Lemma 3.20 from Section 3.6.

3.4.2 CONTINUITY OF THE APPROXIMATING SEQUENCE

Proposition 3.15. *There exists a constant $C = C(T)$ independent of n and R such that*

$$\mathbb{E}[\|\omega_t^{\nu_n,R,n} - \omega_s^{\nu_n,R,n}\|_{L^2}^4] \leq C(t-s)^2, \quad t, s \in [0, T].$$

In particular, by the Kolmogorov-Čentsov criterion (see [67]), the processes $\omega^{\nu_n,R,n}$ have continuous trajectories in $L^2(\mathbb{T}^2)$.

Proof Consider $s \leq t$. Then

$$\omega_t^{\nu_n,R,n} - \omega_s^{\nu_n,R,n} = \nu_n \int_s^t \Delta \omega_p^{\nu_n,R,n} dp - \int_s^t P_p^{n-1,n}(\omega_p^{\nu_n,R,n}) dp - \sum_{i=1}^{\infty} \int_s^t \mathcal{L}_i \omega_p^{\nu_n,R,n} dW_p^i \quad (3.14)$$

We will estimate the expected value of each of these terms. For the first term we have

$$\begin{aligned} \mathbb{E} \left[\left\| \int_s^t \nu_n \Delta \omega_p^{\nu_n,R,n} dp \right\|_{L^2}^4 \right] &\leq (t-s)^4 \mathbb{E} \left[\sup_{p \in [s,t]} \|\Delta \omega_p^{\nu_n,R,n}\|_2^4 \right] \\ &\leq T^2(t-s)^2 \mathbb{E} \left[\sup_{p \in [0,t]} \|\omega_p^{\nu_n,R,n}\|_{k,2}^4 \right] \\ &\leq C(t-s)^2. \end{aligned}$$

Next we have

$$\begin{aligned} &\mathbb{E} \left[\left\| \int_s^t f_R(\omega_p^{\nu_{n-1},R,n-1}) \mathcal{L}_{u_t^{\nu_{n-1},R,n-1}} \omega_p^{\nu_n,R,n} dp \right\|_{L^2}^4 \right] \\ &\leq (t-s)^4 \mathbb{E} \left[\sup_{p \in [s,t]} \|f_R(\omega_p^{\nu_{n-1},R,n-1}) u_p^{\nu_{n-1},R,n-1} \cdot \nabla \omega_p^{\nu_n,R,n}\|_{L^2}^4 \right] \\ &\leq T^2(t-s)^2 \mathbb{E} \left[\sup_{p \in [0,T]} \|\omega_p^{\nu_n,R,n}\|_{k,2}^4 \right] \\ &\leq C(t-s)^2. \end{aligned} \quad (3.15)$$

The penultimate inequality is true given that

$$\|f_R(\omega_p^{\nu_{n-1},R,n-1}) u_p^{\nu_{n-1},R,n-1} \cdot \nabla \omega_p^{\nu_n,R,n}\|_{L^2}^2 \leq \|u_p^{\nu_{n-1},R,n-1}\|_{\infty}^2 \|\nabla \omega_p^{\nu_n,R,n}\|_2^2 \leq \|\omega_p^{\nu_n,R,n}\|_{k,2}^2$$

since

$$\|\nabla u_p^{\nu_{n-1}, R, n-1}\|_\infty \leq C \|\nabla u_p^{\nu_{n-1}, R, n-1}\|_{k,2} \leq C \|u_p^{\nu_{n-1}, R, n-1}\|_{k+1,2} \leq C \|\omega_p^{\nu_{n-1}, R, n-1}\|_{k,2} < \infty,$$

and the last inequality is true due to the a priori estimates proved in Lemma 3.20 v. Similarly, we can prove that

$$\mathbb{E} \left[\left\| \int_s^t \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_i^2 \omega_p^{\nu_n, R, n} dp \right\|_{L^2}^4 \right] \leq C(t-s)^2.$$

which, together with (3.15) gives a control on the second term of (3.14). For the last term we use the Burkholder-Davis-Gundy inequality and obtain

$$\begin{aligned} \mathbb{E} \left[\left\| \int_s^t \sum_{i=1}^{\infty} \xi_i \cdot \nabla \omega_p^{\nu_n, R, n} dW_p^i \right\|_{L^2}^4 \right] &\leq C \mathbb{E} \left[\left(\int_s^t \sum_{i=1}^{\infty} \|\xi_i \cdot \nabla \omega_p^{\nu_n, R, n}\|_2^2 dp \right)^2 \right] \\ &\leq C(t-s)^2 \mathbb{E} \left[\sup_p \sum_{i=1}^{\infty} \|\xi_i \cdot \nabla \omega_p^{\nu_n, R, n}\|_2^4 \right] \\ &\leq C(t-s)^2 \mathbb{E} \left[\sup_p \|\omega_p^{\nu_n, R, n}\|_{k,2}^4 \right] \\ &\leq C(t-s)^2 \end{aligned}$$

due to the initial assumption (2.21) and the a priori estimates (3.20). The conclusion now follows by a direct application of the Kolmogorov-Čentsov criterion.

3.5 RELATIVE COMPACTNESS OF THE APPROXIMATING SEQUENCE OF SOLUTIONS

In this section we prove that the approximating sequence of solutions constructed in Section 3.3.2 is relatively compact in the space $D([0, T], L^2(\mathbb{T}^2))$.

Proof of Proposition 3.14

In order to prove relative compactness we use Kurtz' criterion for relative compactness (see Theorem 2.13 in Chapter 2). To do so we need to show that, for every $\eta > 0$ there exists a compact set $K_{\eta, t} \subset L^2(\mathbb{T}^2)$ such that $\sup_n \mathbb{P}(\omega_t^{\nu_n, R, n} \notin K_{\eta, t}) \leq \eta$. The compact we use is

$$K_{\eta, t} := \left\{ \omega \in \mathcal{W}^{k,2}(\mathbb{T}^2) \mid \|\omega\|_{k,2} < \left(\frac{C}{\eta} \right)^{\frac{1}{4}} \right\}$$

where C is the constant appearing in the a priori estimates (3.20). By a Sobolev compact embedding theorem, $K_{\eta,t}$ is a compact set in $L^2(\mathbb{T}^2)$ and

$$\sup_n \mathbb{P}(\omega_t^{\nu_n, R, n} \notin K_{\eta, t}) = \sup_n \mathbb{P}(\|\omega_t^{\nu_n, R, n}\|_{k, 2} \geq \eta) \leq \sup_n \frac{\eta}{C} E \left[\sup_{t \in [0, T]} \|\omega_t^{\nu_n, R, n}\|_{k, 2}^4 \right] \leq \eta.$$

To prove relative compactness, we need to justify part b) of Kurtz' criterion, as per Theorem 2.13. For this we will show that there exists a family $(\gamma_\delta^n)_{0 < \delta < 1}$ of nonnegative random variables such that

$$\mathbb{E}[\|\omega_{t+l}^{\nu_n, R, n} - \omega_t^{\nu_n, R, n}\|_2^2 | \mathcal{F}_t] \leq \mathbb{E}[\gamma_\delta^n | \mathcal{F}_t]$$

and $\lim_{\delta \rightarrow 0} \limsup_n \mathbb{E}[\gamma_\delta^n] = 0$ for $t \in [0, T]$. We will use the mild form of equation (3.9), that is

$$\omega_t^{\nu_n, R, n} = S^n(t) \omega_0^{\nu_n, R, n} - \int_0^t S^n(t-s) P_s^{n-1, n}(\omega_s^{\nu_n, R, n}) ds - \sum_{i=1}^{\infty} \int_0^t S^n(t-s) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i,$$

with $P_s^{n-1, n}$ as defined in (3.10) and $S^n(t) := e^{\nu_n \Delta t}$. One has

$$\begin{aligned} \|\omega_{t+l}^{\nu_n, R, n} - \omega_t^{\nu_n, R, n}\|_2^2 &\leq C \left(\|(S^n(t+l) - S^n(t)) \omega_0^{\nu_n, R, n}\|_2^2 \right. \\ &\quad + \left\| \int_0^t (S^n(t+l-s) - S^n(t-s)) P_s^{n-1, n}(\omega_s^{\nu_n, R, n}) ds \right\|_2^2 \\ &\quad + \left\| \int_t^{t+l} S^n(t+l-s) P_s^{n-1, n}(\omega_s^{\nu_n, R, n}) ds \right\|_2^2 \\ &\quad + \left\| \sum_{i=1}^{\infty} \int_0^t (S^n(t+l-s) - S^n(t-s)) \mathcal{L}_{\xi_i} \omega_s^{\nu_n, R, n} dW_s^i \right\|_2^2 \\ &\quad \left. + \left\| \sum_{i=1}^{\infty} \int_t^{t+l} S^n(t+l-s) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i \right\|_2^2 \right) \end{aligned}$$

We will estimate each term separately. For the first term we have

$$\mathbb{E}[\|(S^n(t+l) - S^n(t)) \omega_0^{\nu_n, R, n}\|_2^2 | \mathcal{F}_t] \leq \|(S^n(l) - 1) \omega_0^{\nu_n, R, n}\|_2^2$$

For the second term,

$$\begin{aligned} &\mathbb{E} \left[\left\| \int_0^t (S^n(t+l-s) - S^n(t-s)) P_s^{n-1, n}(\omega_s^{\nu_n, R, n}) ds \right\|_2^2 \middle| \mathcal{F}_t \right] \\ &\leq T \mathbb{E} \left[\int_0^T \|(S^n(l) - 1) P_s^{n-1, n}(\omega_s^{\nu_n, R, n})\|_2^2 ds \middle| \mathcal{F}_t \right] \end{aligned}$$

For the third term we have

$$\begin{aligned}
& \mathbb{E} \left[\left\| \int_t^{t+l} S^n(t+l-s) P_s^{n-1,n}(\omega_s^{\nu_n, R, n}) ds \right\|_2^2 \middle| \mathcal{F}_t \right] \\
& \leq \mathbb{E} \left[l^2 \sup_{s \in [t, t+l]} \|S^n(t+l-s) P_s^{n-1,n}(\omega_s^{\nu_n, R, n})\|_2^2 \middle| \mathcal{F}_t \right] \\
& \leq \mathbb{E} \left[Cl^2 \sup_{s \in [t, t+l]} \|P_s^{n-1,n}(\omega_s^{\nu_n, R, n})\|_2^2 \middle| \mathcal{F}_t \right] \\
& \leq \mathbb{E} \left[Cl^2 \sup_{s \in [0, T+1]} \|P_s^{n-1,n}(\omega_s^{\nu_n, R, n})\|_2^2 \middle| \mathcal{F}_t \right]
\end{aligned}$$

Note that (see [40] and [59]) using the fact that

$$\int_{a_1}^{a_2} (a_2 - r)^{\alpha-1} (r - a_1)^{-\alpha} dr = C(\alpha)$$

where $C(\alpha)$ is a constant which depends on $\alpha > 0$ only, and also using the semigroup property $S^n(t-s) = S^n(t-r)S^n(r-s)$ for $s < r < t$, one can write

$$\begin{aligned}
& \int_0^t S^n(t-s) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i = C(\alpha)^{-1} \int_0^t S^n(t-s) \left(\int_s^t (t-r)^{\alpha-1} (r-s)^{-\alpha} dr \right) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i \\
& = C(\alpha)^{-1} \int_0^t \left(\int_s^t (t-r)^{\alpha-1} (r-s)^{-\alpha} S^n(t-r) S^n(r-s) dr \right) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i \\
& = C(\alpha)^{-1} \int_0^t \left(\int_0^r (t-r)^{\alpha-1} (r-s)^{-\alpha} S^n(t-r) S^n(r-s) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i \right) dr \\
& = C(\alpha)^{-1} \int_0^t \left(S^n(t-r) \left(\int_0^r (r-s)^{-\alpha} S^n(r-s) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i \right) (t-r)^{\alpha-1} \right) dr \\
& = C(\alpha)^{-1} \int_0^t S^n(t-r) z(r) (t-r)^{\alpha-1} dr
\end{aligned}$$

where

$$z(r) := \int_0^r (r-s)^{-\alpha} S^n(r-s) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i.$$

We choose $\alpha \in (0, 1/2)$ such that all integrals are well-defined. For more details on the formula above - also called *the factorisation formula* - see [40] or [59]. Then the fourth term can be

estimated as follows:

$$\begin{aligned}
& \left\| \int_0^t (S^n(t+l-s) - S^n(t-s)) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i \right\|_2^2 \\
& \leq \int_0^t \|(t+l-r)^{\alpha-1} S^n(t+l-r)z(r) - (t-r)^{\alpha-1} S^n(t-r)z(r)\|_2^2 dr \\
& = \int_0^t \|((t+l-r)^{\alpha-1} S^n(l) - (t-r)^{\alpha-1}) S^n(t-r)z(r)\|_2^2 dr \\
& \leq \int_0^T \|((t+l-r)^{\alpha-1} S^n(l) - (t-r)^{\alpha-1}) S^n(t-r)z(r)\|_2^2 dr
\end{aligned}$$

and therefore

$$\begin{aligned}
& \mathbb{E} \left[\left\| \int_0^t (S^n(t+l-s) - S^n(t-s)) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i \right\|_2^2 \middle| \mathcal{F}_t \right] \\
& \leq \mathbb{E} \left[\int_0^T \|((t+l-r)^{\alpha-1} S^n(l) - (t-r)^{\alpha-1}) S^n(t-r)z(r)\|_2^2 dr \middle| \mathcal{F}_t \right]
\end{aligned}$$

with $z(r) := \int_0^r (r-s)^{-\alpha} S^n(r-s) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i$. For the fifth term we have

$$\begin{aligned}
\int_t^{t+l} \|S^n(t+l-s) \mathcal{L}_i \omega_s^{\nu_n, R, n}\|_2^2 ds & \leq \int_t^{t+l} \|\mathcal{L}_i \omega_s^{\nu_n, R, n}\|_2^2 ds \leq l^2 \sup_{s \in [t, t+l]} \|\mathcal{L}_i \omega_s^{\nu_n, R, n}\|_2^2 \\
& \leq l^2 \sup_{s \in [0, T+1]} \|\mathcal{L}_i \omega_s^{\nu_n, R, n}\|_2^2.
\end{aligned}$$

so

$$\mathbb{E} \left[\left\| \int_t^{t+l} S^n(t+l-s) \mathcal{L}_i \omega_s^{\nu_n, R, n} dW_s^i \right\|_2^2 \middle| \mathcal{F}_t \right] \leq \mathbb{E} \left[l^2 \sup_{s \in [0, T+1]} \|\mathcal{L}_i \omega_s^{\nu_n, R, n}\|_2^2 \middle| \mathcal{F}_t \right]$$

Define

$$\begin{aligned}
\gamma_l^{\nu_n} &:= \|(S^n(l) - 1) \omega_0^{\nu_n, R, n}\|_2 + \int_0^T \|(S^n(l) - 1) P_s^{n-1, n}(\omega_s^{\nu_n, R, n})\|_2^2 ds \\
&+ Cl^2 \sup_{s \in [0, T+1]} \|P_s^{n-1, n}(\omega_s^{\nu_n, R, n})\|_2^2 \\
&+ l^2 \sup_{s \in [0, T+1]} \|\mathcal{L}_i \omega_s^{\nu_n, R, n}\|_2^2 + \int_0^T \|((t+l-r)^{\alpha-1} S^n(l) - (t-r)^{\alpha-1}) S^n(t-r)z(r)\|_2^2 dr
\end{aligned}$$

and $\gamma_\delta^n := \sup_{l \in [0, \delta]} \gamma_l^{\nu_n}$. From Lemma 3.20 we deduce that there exist two constants c_1 and c_2 such that $\mathbb{E}[\sup_s \|P_s^{n-1, n}(\omega_s^{\nu_n, R, n})\|_2^2] \leq c_1$ and $\mathbb{E}[\sup_s \|\mathcal{L}_i \omega_s^{\nu_n, R, n}\|_2^2] \leq c_2$. The integrands in the integrals above converge pointwise to 0 when $l \rightarrow 0$ due to the strong continuity of the semigroup S^n . At

the same time, they are bounded by integrable functions, therefore the convergence is uniform in space by the dominated convergence theorem. Then the requirement

$$\lim_{\delta \rightarrow 0} \limsup_{\nu_n} \mathbb{E}[\gamma_\delta^n] = 0$$

is met. In conclusion all the conditions required by Kurtz' criterion are fulfilled and therefore $(\omega_t^{\nu_n, R, n})_{\nu_n}$ is relatively compact.

3.6 A PRIORI ESTIMATES AND THE BIOT-SAVART LAW

In this section we prove the a priori estimates used in the proof of existence of a solution for the Euler equation and we also review some fundamental results mentioned before. We start by introducing the Biot-Savart operator which establishes the connection between the velocity vector field u and the vorticity vector field ω .

Remark 3.16 (The Biot-Savart kernel). *The vorticity field corresponding to a 2D incompressible fluid is conventionally regarded as a scalar quantity $\omega = \text{curl } u = \partial_2 u_1 - \partial_1 u_2$ (formally it is a vector $(0, 0, \partial_2 u_1 - \partial_1 u_2)$ orthogonal to $u = (u_1, u_2, 0)$ [78]). It is known (see [78], [52]) that if $\psi : \mathbb{T}^2 \times [0, \infty) \rightarrow \mathbb{R}$ is a solution for $\Delta\psi = -\omega$ then $u = \nabla^\perp \psi$ solves $\omega = \text{curl } u$, so $u = -\nabla^\perp \Delta^{-1} \omega$. It is worth mentioning that the existence of a (unique, up to an additive constant) stream function ψ - and therefore the reconstruction of u from ω is ensured by the incompressibility condition $\text{div } u = 0$ [78]. A periodic, distributional solution of $\Delta\psi = -\omega$ is given by ([52])*

$$\psi(x) = (G \star \omega)(x)$$

where G is the Green function of the operator $-\Delta$ on $\mathbb{T}^2$, $G(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{e^{ik \cdot x}}{\|k\|^2}$. Then the vector field $u = \nabla^\perp \psi$ is uniquely derived from ω as follows:

$$u(x) = (K \star \omega)(x) = \int_{\mathbb{T}^2} K(x - y) \omega(y) dy \tag{3.16}$$

where K is the so-called Biot-Savart kernel

$$K(x) = \nabla^\perp G(x) = \sum_{k \in \mathbb{Z}^2 \setminus \{0\}} \frac{ik^\perp}{\|k\|^2} e^{ik \cdot x}$$

with $k = (k_1, k_2)$, $k^\perp = (k_2, -k_1)$. It is known (see, for example [79]) that G is smooth everywhere except at $x = 0$, and that $K \in L^1(\mathbb{T}^2)$.

For the following result we recall a few elementary results of Fourier analysis. We embed $L^2(\mathbb{T}^2)$ into $L^2(\mathbb{T}^2; \mathbb{C})$ and consider the basis of functions $\{e^{2\pi i \xi \cdot x}; \xi \in \mathbb{Z}^2\}$. Then every $f \in L^2(\mathbb{T}^2; \mathbb{C})$ can be expressed as

$$f(x) = \sum_{\xi \in \mathbb{Z}^2} \widehat{f}(\xi) e^{2\pi i \xi \cdot x}$$

where $\widehat{f}(\xi) = \int_{\mathbb{T}^2} e^{-2\pi i \xi \cdot x} f(x) dx$, $\xi \in \mathbb{Z}^2$ are the corresponding Fourier coefficients. We have the classical Parseval identity (see e.g. [78])

$$\int_{\mathbb{T}^2} |f(x)|^2 dx = \sum_{\xi \in \mathbb{Z}^2} |\widehat{f}(\xi)|^2.$$

If $v \in L^2(\mathbb{T}^2; \mathbb{R}^2)$ is a vector field with components v_i , $i = 1, 2$, we write $\widehat{v}(\xi) = \int_{\mathbb{T}^2} e^{2\pi i \xi \cdot x} v(x) dx$ and we have, in a similar way, that $\int_{\mathbb{T}^2} |v(x)|^2 dx = \sum_{\xi \in \mathbb{Z}^2} |\widehat{v}(\xi)|^2$. Since u and ω are partial derivatives of other functions on the torus, they must have zero average:

$$\int_{\mathbb{T}^2} u^1(x) dx = \int_{\mathbb{T}^2} u^2(x) dx = \int_{\mathbb{T}^2} \omega(x) dx = 0.$$

Hence $\widehat{u^1}(0,0) = \widehat{u^2}(0,0) = \widehat{\omega}(0,0) = 0$ and the term corresponding to $\xi = (0,0)$ does not appear in the Fourier expansion for u_1 , u_2 and, respectively ω .

We introduce, for every $s \geq 0$, the fractional Sobolev space $W^{s,2}(\mathbb{T}^2; \mathbb{C})$ to be the space of all functions $f \in L^2(\mathbb{T}^2; \mathbb{C})$ such that

$$\sum_{\xi \in \mathbb{Z}^2} |\xi|^{2s} |\widehat{f}(\xi)|^2 < \infty.$$

There exist a constant $C > 1$ (see e.g. [47]) such that if $s \in \mathbb{N}$ then

$$C^{-1} \|f\|_{s,2}^2 \leq \sum_{\xi \in \mathbb{Z}^2} (1 + |\xi|^{2s}) \left| \widehat{f}(\xi) \right|^2 < C \|f\|_{s,2}^2$$

Therefore (see e.g. [47]) we can extend the norm $\|f\|_{s,2}$ defined for $s \in \mathbb{N}$ to arbitrary $s > 0$ to be given by

$$\|f\|_{s,2}^2 = \sum_{\xi \in \mathbb{Z}^2} (1 + |\xi|^{2s}) \left| \widehat{f}(\xi) \right|^2$$

We denote by $W_{\sigma}^{s,2}(\mathbb{T}^2, \mathbb{R}^2)$ the space of all zero mean divergence free (divergence in the sense of distribution) vector fields $v \in L^2(\mathbb{T}^2; \mathbb{R}^2)$ such that all components v_i , $i = 1, 2$ belong to $W^{s,2}(\mathbb{T}^2; \mathbb{R})$. For a vector field $v \in W_{\sigma}^{s,2}(\mathbb{T}^2, \mathbb{R}^2)$ the norm $\|v\|_{s,2}$ is defined by the identity $\|v\|_{s,2}^2 = \sum_{i=1}^2 \|v_i\|_{s,2}^2$, where $\|v_i\|_{s,2}^2$ is defined above. We thus have again $\|v\|_{s,2}^2 := \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} |\xi|^{2s} (|\widehat{v}_1(\xi)| + |\widehat{v}_2(\xi)|)^2$. For $f \in W^{s,2}(\mathbb{T}^3; \mathbb{C})$, we denote by $(-\Delta)^{s/2} f$ the function of $L^2(\mathbb{T}^3; \mathbb{C})$ with Fourier coefficients $|\xi|^s \widehat{f}(\xi)$. For even integers $s \in \mathbb{N}$, this definition coincides with the classical definition. Similarly, we write $-\Delta^{-1} f$ for the function having Fourier coefficients $|\xi|^{-2} \widehat{f}(\xi)$. We use the same notations for vector fields, meaning that the operations are made componentwise.

The Biot-Savart operator is the reconstruction of a zero mean divergence free vector field u from a divergence free vector field ω such that $\text{curl } u = \omega$. As stated in Remark 3.16 on the 2D torus it is given by $u = -\text{curl } \Delta^{-1} \omega$. It follows that the Fourier coefficients of u are given by $\widehat{u}(\xi) = |\xi|^{-2} \xi^{\perp} \widehat{\omega}(\xi)$, where $\xi^{\perp} = (\xi_1, \xi_2)^{\perp} = (\xi_2, -\xi_1)$.

In the next proposition we highlight the smoothing properties of the Biot-Savart kernel K .

Proposition 3.17 (The Biot-Savart law, [37], [78]). *Let u be the divergence-free, zero average, vector field defined as $u = -\text{curl } \Delta^{-1} \omega$. Then, for any $s \geq 0$, there exists a constant $C_{s,2}$, independent of u such that*

$$\|u\|_{s+1,2} \leq C_{s,2} \|\omega\|_{s,2}. \quad (3.17)$$

Proof Using the definition given above of $\|u\|_{s+1,2}^2$, the formula which relates $\widehat{u}(\xi)$ and $\widehat{\omega}(\xi)$

and the rule $|a \times b| \leq |a| |b|$, we get

$$\begin{aligned}
\|u\|_{s+1,2}^2 &= \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} (1 + |\xi|^{2s+2}) |\widehat{u}(\xi)|^2 \\
&= \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} (1 + |\xi|^{2s+2}) |\xi|^{-4} |\xi^\perp \widehat{\omega}(\xi)|^2 \\
&\leq \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} (|\xi|^{-2} + |\xi|^{2s}) |\widehat{\omega}(\xi)|^2 \\
&\leq \sum_{\xi \in \mathbb{Z}^2 \setminus \{0\}} (1 + |\xi|^{2s}) |\widehat{\omega}(\xi)|^2
\end{aligned}$$

and the latter is precisely equal to $\|\omega\|_{s,2}$.

Remark 3.18. *The norm $\|\cdot\|_{m,2}$ is equivalent to the norm defined as $|||f||| := \|f\|_2 + \|D^m f\|_2$, therefore it is enough to show that all properties hold for the L^2 norm of f and for the L^2 norm of the maximal derivative $D^m f$ (see [20] pp. 217).*

Let ω_t be the solution of the Euler equation (3.1) and $\omega_t^{\nu_n, R, n}$ the solution of the linear approximating equation (3.8). In the following lemmas we collect a number of identities and a priori estimates. Lemma 3.19 in particular is essential in proving that the solution of the Euler equation is global.

Lemma 3.19. *There exists a constant $\mathcal{C} = \mathcal{C}(\omega_0, T)$ which is independent of the truncation radius R such that*

$$\mathbb{E} \left[\sup_{s \in [0, t]} (\ln(e + \|\omega_s\|_{k,2}^2)) \right] \leq \mathcal{C}(\omega_0, T).$$

Proof By the Biot-Savart law

$$u(x) = \int_{\mathbb{T}^2} K(x-y) \omega(y) dy = -\frac{1}{4\pi} \int_{\mathbb{T}^2} \frac{(x-y)^\perp}{|x-y|^2} \omega(y) dy$$

We use the truncation

$$z_\epsilon(x) = \begin{cases} 1, & \text{if } |x| \leq \epsilon \\ 0, & \text{if } |x| > 2\epsilon \end{cases}$$

with $|\nabla z_\epsilon(x)| \leq \frac{C}{\epsilon}$, $\epsilon \leq 1$. Let

$$u^1(x) := -\frac{1}{4\pi} \int_{\mathbb{T}^2} z_\rho(x-y) K(x-y) \omega(y) dy$$

$$u^2(x) := -\frac{1}{4\pi} \int_{\mathbb{T}^2} (1 - z_\epsilon(x-y)) K(x-y) \omega(y) dy.$$

Then

$$\begin{aligned} \nabla u^1(x) &:= -\frac{1}{4\pi} \int_{\mathbb{T}^2} z_\epsilon(x-y) K(x-y) \nabla \omega(y) dy \\ &\leq \left(\int_{\mathbb{T}^2} (z_\epsilon(x-y) K(x-y))^{4/3} dy \right)^{3/4} \left(\int_{\mathbb{T}^2} (\nabla \omega(y))^4 dy \right)^{1/4} \\ &\leq \|K\|_{L^{4/3}(\{y: |x-y| < 2\epsilon\})} \|\nabla \omega\|_{L^4(\{y: |x-y| < 2\epsilon\})} \\ &\leq C\epsilon^{2/3} \|\nabla \omega\|_4 \\ &\leq C\epsilon^{2/3} \|u\|_{3,2} \end{aligned}$$

given that $\|\nabla \omega\|_4 \leq C\|\nabla \omega\|_{1,2} \leq C\|\omega\|_{2,2} \leq C\|u\|_{3,2}$ by the Sobolev embedding $W^{1,2} \hookrightarrow L^4$ and the Biot-Savart law. So

$$\|\nabla u^1(x)\|_\infty \leq C\epsilon^{2/3} \|u\|_{3,2}.$$

For the second integral

$$\nabla u^2(x) := -\frac{1}{4\pi} \int_{\mathbb{T}^2} \nabla ((1 - z_\epsilon(x-y)) K(x-y)) \omega(y) dy$$

we use the fact that

$$|\nabla K(x-y)| \leq \frac{C}{|x-y|^2}$$

so

$$\|\nabla u^2(x)\|_\infty \leq C(1 - \ln \epsilon) \|\omega\|_\infty.$$

That is

$$\|\nabla u\|_\infty \leq C \left(\epsilon^{2/3} \|u\|_{3,2} + (1 - \ln \epsilon) \|\omega\|_\infty \right)$$

If $\|u\|_{3,2} \leq 1$ we choose $\epsilon = 1$, otherwise we choose $\epsilon = \frac{1}{\|u\|_{3,2}^{3/2}}$. We have

$$\|\nabla u\|_\infty \leq C \left(1 + \left(1 + \frac{3}{2} \ln \|u\|_{3,2} \right) \|\omega\|_\infty \right)$$

Using the inequality $1 + \ln^+ x \leq C \ln(e + x)$ for sufficiently large C , we have

$$\begin{aligned}\|\nabla u_t\|_\infty &\leq C (1 + (1 + \ln^+ \|u_t\|_{3,2}) \|\omega_t\|_\infty) \\ &\leq C (1 + \ln(e + \|u_t\|_{3,2}) \|\omega_t\|_\infty)\end{aligned}$$

and using the transport property and Biot-Savart $\|u_t\|_{3,2} \leq \|\omega_t\|_{2,2} \leq \|\omega_t\|_{2,2}$ we have

$$\|\nabla u_t\|_\infty \leq C (1 + \ln(e + \|\omega_t\|_{k,2}) \|\omega_0\|_\infty) \quad (3.18)$$

By the Itô formula

$$\begin{aligned}d \ln(e + \|\omega_t\|_{k,2}^2) &\leq \frac{1}{e + \|\omega_t\|_{k,2}^2} d\|\omega_t\|_{k,2}^2 - \frac{2}{(e + \|\omega_t\|_{k,2}^2)^2} \sum_{k=1}^{\infty} \left(|\langle \partial^k \mathcal{L}_i \omega_t, \partial^k \omega_t \rangle| + |\langle \mathcal{L}_i \omega_t, \omega_t \rangle| \right)^2 dt \\ &\leq \frac{C}{e + \|\omega_t\|_{k,2}^2} (\|\nabla u_t\|_\infty + \|\omega_t\|_\infty) \|\omega_t\|_{k,2}^2 dt + dY_t \\ &\leq C (\|\omega_t\|_\infty + C_1 (1 + \|\omega_t\|_\infty \ln(e + \|\omega_t\|_{k,2}^2))) dt + dY_t \\ &\leq C (C_1 + \|\omega_t\|_\infty + \|\omega_t\|_\infty \ln(e + \|\omega_t\|_{k,2}^2)) dt + dY_t \\ &\leq (C_1 \ln(e + \|\omega_t\|_{k,2}^2) + \|\omega_t\|_\infty \ln(e + \|\omega_t\|_{k,2}^2) + \|\omega_t\|_\infty \ln(e + \|\omega_t\|_{k,2}^2)) dt + dY_t \\ &\leq C_2 (1 + \|\omega_t\|_\infty) \ln(e + \|\omega_t\|_{k,2}^2) dt + dY_t\end{aligned}$$

where

$$Y_t := \sum_{i=1}^{\infty} \int_0^t \frac{2 (\langle \partial^k \mathcal{L}_i \omega_s, \partial^k \omega_s \rangle + \langle \mathcal{L}_i \omega_s, \omega_s \rangle)}{e + \|\omega_s\|_{k,2}^2} dW_s^k.$$

Define

$$Z_t := \int_0^t (1 + \|\omega_s\|_\infty) ds.$$

We have

$$e^{-C_2 Z_t} \ln(e + \|\omega_t\|_{k,2}^2) \leq \ln(e + \|\omega_0\|_{k,2}^2) + \int_0^t e^{-C_2 Z_s} dY_s \quad (3.19)$$

By the Burkholder-Davis-Gundy inequality

$$\mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s dY_r \right| \right] \leq \alpha \mathbb{E} \left[\left\langle \int_0^\cdot dY_r \right\rangle_t^{1/2} \right]$$

We control the quadratic variation of the stochastic integral using the fact that

$$\sum_{i=1}^{\infty} \|\xi_i\|_{3,2} < \infty$$

and

$$\left| \langle \mathcal{L}_i \omega_t, \omega_t \rangle + \langle \partial^k \mathcal{L}_i \omega_t, \partial^k \omega_t \rangle \right| \leq C \|\xi_i\|_{3,2}^2 \|\omega_t\|_{k,2}^2 < \infty$$

since

$$|\langle \mathcal{L}_i \omega_t, \omega_t \rangle| \leq \|\nabla \xi_i\|_{\infty} \|\omega_t\|_2^2$$

and

$$\left| \langle \partial^k \mathcal{L}_i \omega_t, \partial^k \omega_t \rangle \right| \leq \|\xi_i\|_{3,2} \|\omega_t\|_{k,2}^2.$$

We write

$$\begin{aligned} \left[\int_0^\cdot dY_s \right]_t &\leq \sum_{i=1}^{\infty} \int_0^t \frac{4 \left(\langle \partial^k \mathcal{L}_i \omega_s, \partial^k \omega_s \rangle + \langle \mathcal{L}_i \omega_s, \omega_s \rangle \right)^2}{\left(e + \|\omega_s\|_{k,2}^2 \right)^2} dW_s^k \\ &\leq 4C \sum_{i=1}^{\infty} \|\xi_i\|_{3,2}^2 \int_0^t \frac{\|\omega_s\|_{k,2}^4}{\left(e + \|\omega_s\|_{k,2}^2 \right)^2} ds \\ &\leq Ct. \end{aligned}$$

In consequence, the Burkholder-Davis-Gundy inequality gives

$$\mathbb{E} \left[\sup_{s \in [0,t]} \left| \int_0^s dY_r \right| \right] \leq C\sqrt{t}.$$

Using this in 3.19 we obtain

$$\mathbb{E} \left[\sup_{s \in [0,t]} \left(e^{-C \int_0^s (1 + \|\omega_r^R\|_{\infty}) dr} \ln(e + \|\omega_s^R\|_{k,2}^2) \right) \right] \leq \ln(e + \|\omega_0\|_{k,2}^2) + C\sqrt{t}.$$

By the Fatou lemma this holds also for the original solution ω

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t]} \left(e^{-C \int_0^s (1 + \|\omega_r\|_\infty) dr} \ln(e + \|\omega_s\|_{k,2}^2) \right) \right] \\ & \leq \mathbb{E} \left[\sup_{s \in [0, t]} \liminf_{R \rightarrow \infty} \left(e^{-C \int_0^s (1 + \|\omega_r^R\|_\infty) dr} \ln(e + \|\omega_s^R\|_{k,2}^2) \right) \right] \leq \ln(e + \|\omega_0\|_{k,2}^2) + C\sqrt{t}. \end{aligned}$$

Using the transport property $\|\omega_t\|_\infty = \|\omega_0\|_\infty$ we conclude that there exists a constant $\mathcal{C}(\omega_0, T)$ which is independent of the truncation radius R such that

$$\mathbb{E} \left[\sup_{s \in [0, t]} (\ln(e + \|\omega_s\|_{k,2}^2)) \right] \leq \mathcal{C}(\omega_0, T).$$

Lemma 3.20. *Let ω be the solution of the Euler equation (3.1) and $\omega^{\nu_n, R, n}$ the solution of the linear approximating equation 3.8. Then the following properties hold:*

i. *If $\omega_t, \omega_t^{\nu_n, R, n} \in \mathcal{W}^{k,2}(\mathbb{T}^2)$ then the following L^2 -transport formulae⁹ hold $\mathbb{P}$ - almost surely:*

$$\|\omega_t\|_{L^2} = \|\omega_0\|_{L^2} \quad \text{and} \quad \|\omega_t^{\nu_n, R, n}\|_{L^2} \leq \|\omega_0\|_{L^2}.$$

ii. *If $\omega \in \mathcal{W}^{k,2}(\mathbb{T}^2)$, then $(P_t^{n-1, n})_t$ defined in 3.10 and $(\mathcal{L}_i \omega_t^{\nu_n, R, n})_t$ are processes with paths taking values in $L^2(\mathbb{T}^2)$.*

iii. *There exists a constant C_1 such that:*

$$|\langle \partial^k \omega_t^{\nu_n, R, n}, \partial^k (\mathcal{L}_i^2 \omega_t^{\nu_n, R, n}) \rangle + \langle \partial^k (\mathcal{L}_i \omega_t^{\nu_n, R, n}), \partial^k (\mathcal{L}_i \omega_t^{\nu_n, R, n}) \rangle| \leq C_1 \|\omega_t^{\nu_n, R, n}\|_{k,2}^2.$$

iv. *There exist some constants C_2 and C'_2 such that:*

$$\begin{aligned} & |\langle \partial^k \omega_t^{\nu_n, R, n}, f_R(\omega_t^{\nu_{n-1}, R, n-1}) \partial^k (\mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}}} \omega_t^{\nu_n, R, n}) \rangle| \\ & \leq C_2 \|\partial^k u_t^{\nu_n, R, n-1}\|_2^a \|u_t^{\nu_n, R, n-1}\|_2^{1-a} \|\omega_t^{\nu_n, R, n}\|_{k,2}^2 \end{aligned}$$

⁹Actually analogous L^p -transport formulae hold. We prove such formulae in Chapter 4 Section 4.10.

with $0 < a \leq 1$, and

$$|\langle \partial^k \omega_t^{\nu_n, R, n}, f_R(\omega_t^{\nu_{n-1}, R, n-1}) \partial^k (\mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n}) \rangle| \leq C'_2 \|\omega_t^{\nu_n, R, n}\|_{k,2}^2.$$

v. There exists a constant $C(T)$ independent of n such that

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\omega_t^{\nu_n, R, n}\|_{k,2}^4 \right] \leq C(T).$$

Proof of Lemma 3.20

i. By the Itô formula

$$\begin{aligned} d\|\omega_t^{\nu_n, R, n}\|_2^2 &= -2 \sum_i \langle \omega_t^{\nu_n, R, n}, \mathcal{L}_i \omega_t^{\nu_n, R, n} \rangle dW_t^i - 2 \langle \omega_t^{\nu_n, R, n}, f_R(\omega_t^{\nu_{n-1}, R, n-1}) \mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n} \rangle dt \\ &\quad + 2 \langle \omega_t^{\nu_n, R, n}, \nu_n \Delta \omega_t^{\nu_n, R, n} \rangle dt + \sum_i \left(\langle \omega_t^{\nu_n, R, n}, \mathcal{L}_i^2 \omega_t^{\nu_n, R, n} \rangle dt + \langle \mathcal{L}_i \omega_t^{\nu_n, R, n}, \mathcal{L}_i \omega_t^{\nu_n, R, n} \rangle dt \right). \end{aligned}$$

The last sum has been proved to be 0 at i. and $\langle \omega_t^{\nu_n, R, n}, \mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n} \rangle dt = 0$ by integration by parts since $u_t^{\nu_{n-1}, R, n-1}$ is divergence-free. Similarly $\langle \omega_t^{\nu_n, R, n}, \mathcal{L}_i \omega_t^{\nu_n, R, n} \rangle = 0$ since the vector fields ξ_i are assumed to be divergence-free. Therefore

$$\|\omega_t^{\nu_n, R, n}\|_2 = \|\omega_0\|_2^2 - 2 \int_0^t \nu_n (\nabla \omega_s^{\nu_n, R, n})^2 ds \leq \|\omega_0\|_2^2 \quad \mathbb{P} - \text{almost surely.}$$

The calculations are similar for the Euler equation, but there are no viscous terms, hence

$$\|\omega_t\|_2 = \|\omega_0\|_2^2 \quad \mathbb{P} - \text{almost surely.}$$

ii. One has

$$\|u_t^{\nu_{n-1}, R, n-1} \cdot \nabla \omega_t^{\nu_n, R, n}\|_2^2 \leq C \|u_t^{\nu_{n-1}, R, n-1}\|_\infty^2 \|\nabla \omega_t^{\nu_n, R, n}\|_2^2 \leq C \|\omega_t^{\nu_n, R, n}\|_{k,2}^4$$

due to Hölder's inequality and the following inequalities:

a) $\|u_t^{\nu_{n-1}, R, n-1}\|_\infty \leq C \|\nabla u_t^{\nu_{n-1}, R, n-1}\|_\infty$ by Poincaré's inequality

(or using directly $\|u_t^{\nu_{n-1}, R, n-1}\|_\infty \leq \|u_t^{\nu_{n-1}, R, n-1}\|_{k,2} \leq \|\omega_t^{\nu_{n-1}, R, n-1}\|_{k-1,2}$)

b) $\|\nabla u_t^{\nu_{n-1}, R, n-1}\|_\infty \leq C \|\nabla u_t^{\nu_{n-1}, R, n-1}\|_{k,2}$ by the Sobolev embedding theorem $\mathcal{W}^{k,2}(\mathbb{T}^2) \hookrightarrow L^\infty(\mathbb{T}^2)$ ([1], Theorem 4.12, case A, $m = k, p = n = 2, q = \infty$).

c) $\|\nabla u_t^{\nu_{n-1},R,n-1}\|_{k,2} \leq \|u_t^{\nu_{n-1},R,n-1}\|_{k+1,2}$ by the definition of the Sobolev norm

d) $\|u_t^{\nu_{n-1},R,n-1}\|_{k+1,2} \leq C\|\omega_t^{\nu_{n-1},R,n-1}\|_{k,2}$ by the Biot-Savart law.

All the other terms which involve ξ_i stay in L^2 according to the initial assumptions 2.21, therefore the conclusion holds also for the process $(\mathcal{L}_u \omega_t^{\nu_n,R,n})_t$.

iii. This is proven in Chapter 2, Section 2.6, Proposition 2.23 ii. We take $f := \omega_t^{\nu_n,R,n}$.

iv. We have

$$\begin{aligned}
& |\langle \partial^k \omega_t^{\nu_n,R,n}, f_R(\omega_t^{\nu_{n-1},R,n-1}) \partial^k (\mathcal{L}_{u_t^{\nu_{n-1},R,n-1}} \omega_t^{\nu_n,R,n}) \rangle| \\
&= |\langle \partial^k \omega_t^{\nu_n,R,n}, f_R(\omega_t^{\nu_{n-1},R,n-1}) \sum_{|\beta| \leq k} C_k^\beta (\partial^\beta u_t^{\nu_{n-1},R,n-1}) \cdot (\partial^{k-\beta} (\nabla \omega_t^{\nu_n,R,n})) \rangle| \\
&= |f_R(\omega_t^{\nu_{n-1},R,n-1}) \sum_{|\beta| \leq k} C_k^\beta \langle \partial^k \omega_t^{\nu_n,R,n}, \partial^\beta u_t^{\nu_{n-1},R,n-1} \cdot \partial^{k-\beta} (\nabla \omega_t^{\nu_n,R,n}) \rangle| \\
&= \left| f_R(\omega_t^{\nu_{n-1},R,n-1}) \sum_{|\beta| \leq k} C_k^\beta \int_{\mathbb{T}^2} \partial^k \omega_t^{\nu_n,R,n} \cdot \partial^\beta u_t^{\nu_{n-1},R,n-1} \cdot \partial^{k-\beta} (\nabla \omega_t^{\nu_n,R,n}) dx \right| \\
&\leq f_R(\omega_t^{\nu_{n-1},R,n-1}) \sum_{|\beta| \leq k} C_k^\beta \int_{\mathbb{T}^2} |\partial^k \omega_t^{\nu_n,R,n} \cdot \partial^\beta u_t^{\nu_{n-1},R,n-1} \cdot \partial^{k-\beta} (\nabla \omega_t^{\nu_n,R,n})| dx
\end{aligned}$$

Using Hölder and Cauchy-Schwartz inequalities one has

$$\begin{aligned}
& \int_{\mathbb{T}^2} |\partial^k \omega_t^{\nu_n,R,n} \cdot \partial^\beta u_t^{\nu_{n-1},R,n-1} \cdot \partial^{k-\beta} (\nabla \omega_t^{\nu_n,R,n})| dx \\
&\leq \|\partial^k \omega_t^{\nu_n,R,n}\|_2 \|\partial^\beta u_t^{\nu_{n-1},R,n-1} \cdot \partial^{k-\beta} (\nabla \omega_t^{\nu_n,R,n})\|_2 \\
&\leq \|\partial^k \omega_t^{\nu_n,R,n}\|_2 \|\partial^\beta u_t^{\nu_{n-1},R,n-1}\|_4 \|\partial^{k-\beta} (\nabla \omega_t^{\nu_n,R,n})\|_4 \\
&\leq \|\omega_t^{\nu_n,R,n}\|_{k,2} \|\partial^\beta u_t^{\nu_{n-1},R,n-1}\|_4 \|\partial^{k-\beta} (\nabla \omega_t^{\nu_n,R,n})\|_4
\end{aligned}$$

By the Gagliardo-Nirenberg inequality

$$\begin{aligned}
& \|\partial^\beta u_t^{\nu_{n-1},R,n-1}\|_4 \leq C \|\partial^{\beta+1} u_t^{\nu_{n-1},R,n-1}\|_2^a \|u_t^{\nu_{n-1},R,n-1}\|_2^{1-a} \\
&\leq C \|u_t^{\nu_{n-1},R,n-1}\|_{\beta+1,2}^a \|u_t^{\nu_{n-1},R,n-1}\|_2^{1-a} \\
&\leq C \|u_t^{\nu_{n-1},R,n-1}\|_{k+1,2}^a \|u_t^{\nu_{n-1},R,n-1}\|_2^{1-a} \\
&\leq C \|\omega_t^{\nu_{n-1},R,n-1}\|_{k,2}^a \|u_t^{\nu_{n-1},R,n-1}\|_2^{1-a} \\
&\leq C \|\omega_t^{\nu_{n-1},R,n-1}\|_{k,2}
\end{aligned}$$

since $\|u_t^{\nu_{n-1},R,n-1}\|_2^{1-a} \leq \|u_t^{\nu_{n-1},R,n-1}\|_{2,2}^{1-a} \leq \|u_t^{\nu_{n-1},R,n-1}\|_{k+1,2}^{1-a} \leq C \|\omega_t^{\nu_{n-1},R,n-1}\|_{k,2}^{1-a}$ by the

Biot-Savart law. Similarly

$$\begin{aligned}
\|\partial^{k-\beta}(\nabla \omega_t^{\nu_n, R, n})\|_4 &\leq C\|\partial^{k-\beta+1}(\nabla \omega_t^{\nu_n, R, n})\|_2^a \|\nabla \omega_t^{\nu_n, R, n}\|_2^{1-a} \\
&\leq C\|\nabla \omega_t^{\nu_n, R, n}\|_{k-\beta+1, 2}^a \|\omega_t^{\nu_n, R, n}\|_{2, 2}^{1-a} \\
&\leq C\|\omega_t^{\nu_n, R, n}\|_{k, 2}^a \|\omega_t^{\nu_n, R, n}\|_{k, 2}^{1-a} \\
&= C\|\omega_t^{\nu_n, R, n}\|_{k, 2}
\end{aligned}$$

Therefore

$$\begin{aligned}
|\langle \partial^k \omega_t^{\nu_n, R, n}, f_R(\omega_t^{\nu_{n-1}, R, n-1}) \partial^k (\mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n}) \rangle| &\leq f_R(\omega_t^{\nu_{n-1}, R, n-1}) \|\omega_t^{\nu_{n-1}, R, n-1}\|_{k, 2} \|\omega_t^{\nu_n, R, n}\|_{k, 2}^2 \\
&\leq \tilde{C} \|\omega_t^{\nu_n, R, n}\|_{k, 2}^2.
\end{aligned}$$

The above inequalities do not hold properly when $\beta \in \{0, 1\}$ and $\omega_t^{\nu_n, R, n}, \omega_t^{\nu_{n-1}, R, n-1} \in \mathcal{W}^{k, 2}(\mathbb{T}^2)$.

Therefore we consider these two cases separately. For $\beta = 0$:

$$\begin{aligned}
\int_{\mathbb{T}^2} \partial^k \omega_t^{\nu_n, R, n} \cdot u_t^{\nu_{n-1}, R, n-1} \cdot \partial^k (\nabla \omega_t^{\nu_n, R, n}) dx &= \int_{\mathbb{T}^2} u_t^{\nu_{n-1}, R, n-1} \cdot \nabla (\partial^k \omega_t^{\nu_n, R, n}) \cdot \partial^k \omega_t^{\nu_n, R, n} dx \\
&= \frac{1}{2} \int_{\mathbb{T}^2} u_t^{\nu_{n-1}, R, n-1} \cdot \nabla |\partial^k \omega_t^{\nu_n, R, n}|^2 dx \\
&= -\frac{1}{2} \int_{\mathbb{T}^2} (\partial^k \omega_t^{\nu_n, R, n})^2 (\nabla \cdot u_t^{\nu_{n-1}, R, n-1}) dx \\
&= 0.
\end{aligned}$$

For $\beta = 1$:

$$\begin{aligned}
\|\nabla u_t^{\nu_{n-1}, R, n-1} \cdot \partial^k \omega_t^{\nu_n, R, n}\|_2 &\leq \|\nabla u_t^{\nu_{n-1}, R, n-1}\|_\infty^2 \|\partial^k \omega_t^{\nu_n, R, n}\|_2 \\
&\leq C \|\nabla u_t^{\nu_{n-1}, R, n-1}\|_{k, 2} \|\omega_t^{\nu_n, R, n}\|_{k, 2} \\
&\leq C \|u_t^{\nu_{n-1}, R, n-1}\|_{k+1, 2} \|\omega_t^{\nu_n, R, n}\|_{k, 2} \\
&\leq C \|\omega_t^{\nu_{n-1}, R, n-1}\|_{k, 2} \|\omega_t^{\nu_n, R, n}\|_{k, 2}.
\end{aligned}$$

We used the embedding $\mathcal{W}^{k, 2} \hookrightarrow L^\infty$ and the Biot-Savart law. We need this property for relative compactness (see Proposition 3.14).

v. After applying the Itô formula we obtain

$$\begin{aligned}
\|\partial^k \omega_t^{\nu_n, R, n}\|_2^2 &= \|\partial^k \omega_0^{\nu_n, R, n}\|_2^2 + 2\nu_n \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, \partial^{k+2} \omega_s^{\nu_n, R, n} \rangle ds \\
&\quad - 2 \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, f_R(\omega_s^{\nu_{n-1}, R, n-1}) \partial^k \mathcal{L}_{u_s^{\nu_{n-1}, R, n-1}} \omega_s^{\nu_n, R, n} \rangle ds \\
&\quad + \sum_{i=1}^{\infty} \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i^2 \omega_s^{\nu_n, R, n} \rangle ds \\
&\quad + \sum_{i=1}^{\infty} \int_0^t \langle \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n} \rangle ds \\
&\quad - 2 \sum_{i=1}^{\infty} \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n} \rangle dW_s^i.
\end{aligned}$$

We analyse each term. One can write

$$\langle \partial^k \omega_s^{\nu_n, R, n}, \partial^{k+2} \omega_s^{\nu_n, R, n} \rangle = -\|\partial^{k+1} \omega_s^{\nu_n, R, n}\|_2^2 \leq 0.$$

We want to estimate the other terms independently of ν_n . All terms are estimated above. After summing up we have

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \in [0, t]} \|\omega_s^{\nu_n, R, n}\|_{k, 2}^2 \right] &\leq \mathbb{E} [\|\partial^k \omega_0^{\nu_n, R, n}\|_2^2] + C_2'(T) \int_0^t \mathbb{E} \left[\sup_{s \in [0, t]} \|\omega_s^{\nu_n, R, n}\|_{k, 2}^2 \right] ds \\
&\quad + C_1 \int_0^t \mathbb{E} \left[\sup_{s \in [0, t]} \|\omega_s^{\nu_n, R, n}\|_{k, 2}^2 \right] ds \\
&\quad + 2\mathbb{E} \left[\sup_{s \in [0, t]} \sum_{i=1}^{\infty} \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n} \rangle dW_s^i \right].
\end{aligned}$$

Let

$$\mathcal{B}_t := \sum_{i=1}^{\infty} \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n} \rangle dW_s^i \quad \text{and} \quad \beta_t := \|\omega_t^{\nu_n, R, n}\|_{k, 2}^2.$$

$\mathcal{B}_t$ is a local martingale We have

$$\beta_t \leq \beta_0 + 2\mathcal{B}_t + (C_2' + C_1) \int_0^t \beta_s ds$$

and by Gronwall lemma

$$\mathbb{E}\left[\sup_{s \in [0, t]} \beta_s^2\right] \leq 2e^{2(C'_2 + C_1)t} \left(\beta_0^2 + 4\mathbb{E}\left[\sup_{s \in [0, t]} \mathcal{B}_s^2\right]\right).$$

Using the Burkholder-Davis-Gundy inequality there exists a constant $\tilde{\alpha}$ such that

$$\mathbb{E}\left[\sup_{s \in [0, t]} \mathcal{B}_s^2\right] \leq \tilde{\alpha} \mathbb{E}\left[\sqrt{\langle \mathcal{B} \rangle_s^2}\right].$$

It is easy to show that $\mathbb{E}[\langle \mathcal{B} \rangle_s] \leq C_2^2 \mathbb{E}[\sup_{s \in [0, t]} \beta_s^2]$: following the same calculations as those from step v. we can see that

$$\langle \partial^k \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n} \rangle \leq C_2 \|\omega_s^{\nu_n, R, n}\|_{k, 2}^2$$

if instead of making use of the truncation corresponding to $\|u_s^{\nu_{n-1}, R, n-1}\|_{k, 2}$ we take into account assumption 2.21. Hence

$$\langle \mathcal{B} \rangle_s \leq C_2^2 \sup_{s \in [0, t]} \|\omega_s^{\nu_n, R, n}\|_{k, 2}^4 = C_2^2 \sup_{s \in [0, t]} \beta_s^2$$

and the above inequality follows. In conclusion

$$\mathbb{E}\left[\sup_{s \in [0, t]} \beta_s^2\right] \leq \tilde{C}_1(T) \beta_0^2 + \tilde{C}_2(T) \int_0^t \mathbb{E}\left[\sup_{s \in [0, t]} \beta_s^2\right] ds$$

with $\tilde{C}_1(T) := 2 \exp(2(C'_2 + C_1)T)$ and $\tilde{C}_2(T) := 8\tilde{\alpha}C_2^2 \exp(2(C'_2 + C_1)T)$. Using again Gronwall's inequality we have

$$\mathbb{E}\left[\sup_{s \in [0, t]} \|\omega_s^{\nu_n, R, n}\|_{k, 2}^4\right] \leq C(T)$$

with $C(T) := \tilde{C}_1(T) \beta_0^2 \exp(\tilde{C}_2(T)T)$.

The following lemma is instrumental in showing that the limit of the approximating sequence satisfies the Euler equation in $\mathcal{W}^{k, 2}(\mathbb{T}^2)$ although the relative compactness property holds in $D([0, T], L^2(\mathbb{T}^2))$. It is also essential when proving a priori estimates for the truncated solution ω^R (see Lemma 3.22).

Lemma 3.21.

i. Assume that $(a_n)_n$ is a sequence of functions such that $\lim_{n \rightarrow \infty} a_n = a$ in $L^2(\mathbb{T}^2)$ and $\sup_{n \geq 1} \|a_n\|_{s,2} < \infty$ for $s \geq 0$. Then $a \in \mathcal{W}^{s,2}(\mathbb{T}^2)$ and $\|a\|_{s,2} \leq \sup_{n \geq 1} \|a_n\|_{s,2}$. Moreover, $\lim_{n \rightarrow \infty} a_n = a$ in $\mathcal{W}^{s',2}(\mathbb{T}^2)$ for any $s' < s$.

ii. Assume that $a_n : \Omega \mapsto \mathcal{W}^{s,2}(\mathbb{T}^2)$ is a sequence of measurable maps such that, $\lim_{n \rightarrow \infty} a_n = a$ in $L^2(\mathbb{T}^2)$, $\mathbb{P}$ -almost surely or $\lim_{n \rightarrow \infty} a_n = a$ in distribution. Further assume that $\sup_{n \geq 1} \mathbb{E}[\|a_n\|_{s,2}^2] < \infty$. Then, $\mathbb{P}$ -almost surely, $a \in \mathcal{W}^{s,2}(\mathbb{T}^2)$ and $\mathbb{E}[\|a\|_{s,2}^2] \leq \sup_{n \geq 1} \mathbb{E}[\|a_n\|_{s,2}^2]$

Proof of Lemma 3.21

i. Since $\lim_{n \rightarrow \infty} a_n = a$ in $L^2(\mathbb{T}^2)$ it follows that for arbitrary $\lambda \in \mathbb{Z}^2$ we have

$$\lim_{n \rightarrow \infty} \widehat{a_n}(\lambda) = \lim_{n \rightarrow \infty} \int_{\mathbb{T}^2} e^{2\pi i \lambda \cdot x} a_n(x) dx = \int_{\mathbb{T}^2} e^{2\pi i \lambda \cdot x} a(x) dx = \widehat{a}(\lambda)$$

Therefore by Fatou's lemma

$$\begin{aligned} \|a\|_{s,2}^2 &= \sum_{\lambda \in \mathbb{Z}^2} (1 + |\lambda|^{2s}) |\widehat{a}(\lambda)|^2 = \sum_{\lambda \in \mathbb{Z}^2} (1 + |\lambda|^{2s}) \liminf_{n \rightarrow \infty} |\widehat{a_n}(\lambda)|^2 \\ &\leq \liminf_{n \rightarrow \infty} \sum_{\lambda \in \mathbb{Z}^2} (1 + |\lambda|^{2s}) |\widehat{a_n}(\lambda)|^2 = \liminf_{n \rightarrow \infty} \|a_n\|_{s,2}^2 \leq \sup_{n \geq 1} \|a_n\|_{s,2}^2. \end{aligned}$$

For the second part we can write

$$\|a_n - a\|_{s',2} = \sum_{\lambda \in \mathbb{Z}^2, |\lambda| \leq M} (1 + |\lambda|^{2s'}) |(\widehat{a_n} - \widehat{a})\lambda| + \sum_{\lambda \in \mathbb{Z}^2, |\lambda| \geq M} (1 + |\lambda|^{2s'}) |(\widehat{a_n} - \widehat{a})\lambda|.$$

Note that

$$\begin{aligned} \sum_{\lambda \in \mathbb{Z}^2, |\lambda| \geq M} (1 + |\lambda|^{2s'}) |(\widehat{a_n} - \widehat{a})\lambda| &\leq \sum_{\lambda \in \mathbb{Z}^2, |\lambda| \geq M} \frac{1 + |\lambda|^{2s}}{1 + M^{2(s-s')}} |(\widehat{a_n} - \widehat{a})\lambda| \\ &\leq \frac{1}{1 + M^{2(s-s')}} \left(\sup_{n \geq 1} \|a_n\|_{s,2}^2 + \|a\|_{s,2}^2 \right) \end{aligned}$$

and we can choose M such that the last term is strictly smaller than $\frac{\epsilon}{2}$. Likewise, n can be chosen such that

$$\sum_{\lambda \in \mathbb{Z}^2, |\lambda| \leq M} (1 + |\lambda|^{2s'}) |(\widehat{a_n} - \widehat{a})\lambda| < \frac{\epsilon}{2}$$

hence a_n converges to a in $\mathcal{W}^{s',2}$.

ii. As above it follows that

$$\mathbb{E}[|\widehat{a}(\lambda)|^2] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[|\widehat{a}_n(\lambda)|^2]$$

and therefore

$$\mathbb{E}[\|a\|_{s,2}^2] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[\|a_n\|_{s,2}^2] \leq \sup_{n \geq 1} \mathbb{E}[\|a_n\|_{s,2}^2].$$

Lemma 3.22. *Let ω_t^R be the solution of the truncated Euler equation (3.7). There exists a constant $\tilde{C}(T)$ independent of R such that*

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|\omega_t^R\|_{k,2}^2\right] \leq \tilde{C}(T).$$

Proof. This is a direct consequence of Lemma 3.21 ii. with

$$a_n := a_{n,t} := \sup_t \omega_t^{\nu_n, R, n} \quad \text{and} \quad a := a_t := \sup_t \omega_t^R.$$

since $\lim_{n \rightarrow \infty} \omega_t^{\nu_n, R, n} = \omega_t^R$ $\mathbb{P}$ -almost surely in $L^2(\mathbb{T}^2)$ (see Section 3.3), and $\sup_{n \geq 1} \mathbb{E}[\|a_n\|_{s,2}^2] < \infty$ by Lemma 3.20 v. and Proposition 3.14.

Remark 3.23. *i. In Lemma 3.22 we cannot follow an approach identical to the one used in Lemma 3.20 v., due to a lack of smoothness in the truncated equation (3.7). This difficulty could be overcome if we embed $L^2(\mathbb{T}^2)$ into $L^2(\mathbb{T}^2; \mathbb{C})$, consider the basis of functions $\{e^{2\pi i \xi \cdot x}; \xi \in \mathbb{Z}^2\}$, and express ω_t^R as $\sum_{\xi \in \mathbb{Z}^2} |\xi|^{2k} \langle \omega_t^R, \varphi_\xi \rangle \varphi_\xi$, where $\varphi_\xi : \mathbb{T}^2 \mapsto \mathbb{C}$, $\varphi_\xi(x) = e^{2\pi i \xi \cdot x}$, $x \in \mathbb{T}$ and $\xi \in \mathbb{Z}^2$. Then $\partial^k \omega_t^R$ will be square integrable if and only if $\sum_{\xi \in \mathbb{Z}^2} |\xi|^{2k} \langle \omega_t^R, \varphi_\xi \rangle^2 < \infty$ and we can finish the proof using the weak form of the truncated equation. However, for the time being we prefer a direct argument using Lemma 3.21 as presented above because it doesn't require the introduction of any additional technical elements.*

ii. Except the estimates which involve the second order operator $\mathcal{L}_i^2$, all the other estimates derived in Lemma 3.20 for $\omega^{\nu_n, R, n}$ hold also for ω^R .

Remark 3.24. *We denote by $(S^n(t))_t$ the semigroup of the generator $A := \nu_n \Delta$. This semigroup is strongly continuous (see [77]) and for any $f \in L^2(\mathbb{T}^2)$ it is true that*

$$\|S^n(t)f\|_{k,2} \leq \|f\|_{k,2}.$$

To show the last property consider $\hat{f}$ the Fourier transform of f and use Parseval's formula to deduce that

$$\begin{aligned}\|\partial^k S^n(t)f\|_2 &= \|S^n(t)(\partial^k f)\|_2 = (2\pi)^2 \|\widehat{S^n(t)(\partial^k f)}(y)\|_2 \\ &= (2\pi)^2 \|e^{-ty^2} \widehat{(\partial^k f)}(y)\|_2 \leq (2\pi)^2 \|\widehat{\partial^k f}\|_2 = \|\partial^k f\|_2.\end{aligned}$$

4

THE GREAT LAKE EQUATION WITH TRANSPORT NOISE

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In this chapter we prove that the stochastic great lake equation driven by transport noise admits a unique strong solution local in the Sobolev space $\mathcal{W}^{k,2}(\mathbb{T}^2)$, $k \geq 2$. The content of this chapter is the subject of the paper [35].

4.1 INTRODUCTION

Consider the following two-dimensional stochastic system which models the evolution of the vorticity corresponding to an inviscid fluid on the two-dimensional torus $\mathbb{T}^2$:

$$d\omega_t + \mathcal{L}_{u_t}\omega_t dt + \sum_i \mathcal{L}_i\omega_t \circ dW_t^i = 0 \quad (4.1)$$

$$\nabla \cdot (bu) = 0, \quad \nabla \cdot (b\xi_i) = 0$$

$$\omega = b^{-1} \operatorname{curl} v$$

$$v = u + \frac{1}{6}\delta^2 b^2 \nabla(\nabla \cdot u).$$

The initial condition ω_0 is assumed to be an element of the weighted Sobolev space $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$. The velocity of the fluid is denoted by u_t and ω_t stands for the corresponding vorticity. As before, the term $\mathcal{L}_{u_t}\omega_t$ can be interpreted as the Lie derivative which expresses the change of vorticity along the flow generated by the velocity vector field u_t , while $\mathcal{L}_i\omega_t$ is a circulation-preserving perturbation of this field ([37], [62]). The model presented here has been derived using the SALT approach ([63],[62]) and it is inspired by the so-called *great lake equations* (see [24], [25], [74] for the deterministic case). The great lake equations model the circulation of an inviscid fluid in a shallow water basin with varying bottom topography b . The stochastic counterpart has been described in [63] for a three-dimensional box. The domain we consider in this chapter is the two-dimensional torus. The vector fields ξ_i are time-independent and divergence-free and can be associated with uncertainty induced by missing physics or incomplete data (see [30], [31]). The processes $W^i, i \geq 1$, are independent Brownian motions and δ is the aspect ratio of the domain (that is the ratio between vertical and horizontal length scales). The function b is fixed in time and there exist two constants b_{min} and b_{max} such that $0 < b_{min} \leq b(x) \leq b_{max}$ for any $x \in \mathbb{T}^2$.

If b is constant then the great lake equations formally reduce to the classical 2D Euler equations ([74]).

The stochastic part follows the stochastic advection by Lie transport theory presented in [62] and [86] and is represented by a stochastic integral of Stratonovich type. Well-posedness for the deterministic version of this system has been proven in [74]. In Chapter 3 we have considered a similar stochastic vorticity equation but without taking into account the bottom topography b .

In this chapter we prove that the system (4.1) admits a pathwise unique and probabilistically strong solution in the Sobolev space $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$, the initial smoothness of the system being preserved. For a detailed analysis we use the Itô form of the vorticity equation. As in the case of the Euler equation presented in Chapter 3, we must control the second order term which comes from the Itô correction and involves the operator $\mathcal{L}_i^2$. We do this by combining it with the quadratic variation of the stochastic integral. In order to control the higher-order derivatives of the vorticity we prove a set of inequalities for the operator $\mathcal{L}_i$ (see Lemma 4.17), based on specific smoothness and summability assumptions (see Assumptions 4.3). These inequalities and assumptions are similar to those proven in Chapter 2 Section 2.6 but they are adapted to the case where the Sobolev space $\mathcal{W}^{k,2}(\mathbb{T}^2)$ is endowed with the weight function b .

Note that the classical incompressibility condition $\nabla \cdot u = 0$ is not true here, a fact which usually generates technical difficulties when trying to control the nonlinear term and to obtain uniform a priori bounds. However, we have the weighted incompressibility condition $\nabla \cdot (bu) = 0$. Due to this, working in the weighted Sobolev space $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$ comes as natural and we are able to extend the techniques from Chapter 3 to this more general case. One of the key points here is to prove that the vorticity to stream function operator can be generalised (see Section 4.5) and therefore the smoothness of the velocity vector field is still controlled by the smoothness of its corresponding vorticity. We will show that even in a stochastic framework, b affects just the geometry of the ambient space, not the topology, a consequence of the non-degeneracy of b . In [74] it has been shown that this property holds in a deterministic framework. In contrast to the result from Chapter 3 and following the idea from [74] we control the smoothness of the velocity vector field using Lax-Milgram-type arguments instead of Fourier analysis techniques. We cannot present Chapter 3 as a particular case of Chapter 4, as the solution is global in the first case and only local in the second case. In particular, the a priori estimates for the great lake equation

depend on the truncation radius R (see Lemma 4.18), and therefore a direct correspondent of Lemma 3.19 from Chapter 3 does not exist in this chapter. This is the main reason why the arguments used in Section 3.2.2 hold only up to a stopping time and we obtain a local solution.

In the sequel, $\mathbb{T}^2$ is the two-dimensional torus, $k \geq 2$ is a fixed positive integer and $\mathcal{W}_b^{k,2}$ is a weighted Sobolev space (see Section 4.2). The main result of this chapter reads as follows:

Theorem: Under certain conditions on the vector fields $(\xi_i)_i$, the vorticity equation of the two-dimensional stochastic great lake system 4.1 admits a pathwise unique local strong solution which belongs to the space $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$.

Remark 4.1. *The corresponding Itô form of the above evolution equation for ω is*

$$d\omega_t + \mathcal{L}_{u_t}\omega_t dt + \sum_{i=1}^{\infty} \mathcal{L}_i \omega_t dW_t^i = \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_i^2 \omega_t dt. \quad (4.2)$$

The assumptions on the vector fields $(\xi_i)_i$ are described in Section 4.2. In brief, they are assumed to be sufficiently smooth and their corresponding norms to decay sufficiently fast as i increases, so that the infinite sums in (4.1), respectively, in (4.2), make sense in the right spaces, (see condition (4.3) below).

Since u is such that $\nabla \cdot (bu) = 0$, there exists ([78]) a stream function ψ such that $u = b^{-1}\nabla^\perp \psi$. According to [24], when using the weighted incompressibility condition $\nabla \cdot (bu) = 0$, the relationship between v and u can be defined through a linear operator $\mathcal{M}$ as follows:

$$v = \mathcal{M}u = u + \delta^2 b^{-1} \left[-\frac{1}{3} \nabla(b^3 \nabla \cdot u) - \frac{1}{2} \nabla(b^2 u \cdot \nabla b) + \frac{1}{2} b^2 (\nabla \cdot u) \nabla b + b(u \cdot \nabla b) \nabla b \right].$$

The introduction of a stream function here allows one to recover u from ω . Qualitatively, if $u = K\omega$ and ω is the solution of the vorticity equation, we want to show that K has good regularity properties and that it imposes the right incompressibility conditions on u . Note that ψ is the solution of the elliptic problem

$$\omega = b^{-1} \operatorname{curl}(\mathcal{M}(b^{-1} \nabla^\perp \psi)) \quad \text{on } \mathbb{T}^2.$$

The great lake equation is more general than the Euler equation in the sense that the link between

the vorticity field and the velocity field is not given by the curl operator only, but by a more general (linear) operator which is proved to have good regularity properties.

The chapter is organised as follows: In Section 4.2 we introduce the main assumptions and key notations. In Section 4.3 we introduce the main results. In Section 4.4 we show that the solution of the great lake equation is almost surely pathwise unique. Section 4.5 contains the regularity properties of the vorticity to stream function operator. In Section 4.7 one can find the proof for existence of a strong solution (in the sense of Definition 4.2) and for uniqueness, of a truncated version of the great lake equation. In Section 4.6 we explain how the local solution of the great lake equation is obtained from the global solution of its truncated version. In Section 4.8 we show continuity with respect to initial conditions for the original equation. In Section 4.9 we show that the family of approximating solutions is relatively compact. In Section 4.10 one can find a couple of technical properties which are essential in proving a priori estimates.

4.2 PRELIMINARIES

Assumptions on the vector fields $(\xi_i)_i$

The vector fields $\xi_i : \mathbb{T}^2 \rightarrow \mathbb{R}^2$ are chosen to be time-independent quantities such that $\nabla \cdot (b\xi_i) = 0$, which need to be specified from the underlying physics. We assume that for any $f \in \mathcal{W}_b^{2,2}(\mathbb{T}^2)$:

$$\sum_{i=1}^{\infty} \|\mathcal{L}_i f\|_{b,2}^2 \leq C \|f\|_{b,1,2}^2, \quad \sum_{i=1}^{\infty} \|\mathcal{L}_i^2 f\|_{b,2}^2 \leq C \|f\|_{b,2,2}^2, \quad \sum_{i=1}^{\infty} \|\xi_i\|_{k+1,\infty}^2 < \infty. \quad (4.3)$$

These assumptions would be of particular interest when using this model as a signal process in stochastic filtering applications: see for instance [30], [31], [32] in the case of the stochastic 2D Euler equation.

Let $\{W^i\}_{i \in \mathbb{N}}$ be a sequence of independent Brownian motions on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Provided that $\omega \in L_b^2(0, T; \mathcal{W}_b^{2,2}(\mathbb{T}^2, \mathbb{R}))$, the first two conditions in (4.3) ensure that the infinite sums of stochastic integrals

$$\sum_{i=1}^{\infty} \int_0^t \xi_i \cdot \nabla \omega_s dW_s^i \quad (4.4)$$

are well defined and belong to $L_b^2(0, T; L_b^2(\mathbb{T}^2, \mathbb{R}))$. Similarly, the processes $s \mapsto \xi_i \cdot \nabla(\xi_i \cdot \nabla \omega_s)$ are well-defined and belong to $L_b^2(0, T; L_b^2(\mathbb{T}^2, \mathbb{R}))$. In particular, the Itô correction in (4.2) is well defined. The third condition is needed for proving a number of required a priori estimates (see Lemma 4.18). In the following definitions $k \geq 2$ is a fixed integer.

Definition 4.2.

- a. A strong solution of the stochastic partial differential equation (4.2) is an $(\mathcal{F}_t)_t$ -adapted process $\omega : \Omega \times \mathbb{T}^2 \rightarrow \mathbb{R}$ with trajectories in the space $C([0, \infty); \mathcal{W}_b^{k,2}(\mathbb{T}^2))$, such that the identity

$$\omega_t = \omega_0 - \int_0^t u_s \cdot \nabla \omega_s ds + \sum_{i=1}^{\infty} \int_0^t \xi_i \cdot \nabla \omega_s dW_s^i + \frac{1}{2} \sum_{i=1}^{\infty} \int_0^t \xi_i \cdot \nabla(\xi_i \cdot \nabla \omega_s) ds$$

with $\omega|_{t=0} = \omega_0$, holds $\mathbb{P}$ -almost surely in $L_b^2(\mathbb{T}^2; \mathbb{R})$.

- b. A weak/distributional solution of equation (4.2) is an $(\mathcal{F}_t)_t$ -adapted process $\omega : \Omega \times \mathbb{T}^2 \rightarrow \mathbb{R}$ with trajectories in the set $C([0, \infty); \mathcal{W}_b^{k,2}(\mathbb{T}^2))$, $k \geq 2$ fixed, which satisfies the equation in the weak topology of $L_b^2(\mathbb{T}^2, \mathbb{R})$ i.e.

$$\langle \omega_t, \varphi \rangle_b = \langle \omega_0, \varphi \rangle_b - \int_0^t \langle \omega_s, \mathcal{L}_{u_s}^* \varphi \rangle_b ds + \sum_{i=1}^{\infty} \int_0^t \langle \omega_s, \mathcal{L}_i^* \varphi \rangle_b dW_s^i + \frac{1}{2} \sum_{i=1}^{\infty} \int_0^t \langle \omega_s, \mathcal{L}_i^* \mathcal{L}_i^* \varphi \rangle_b ds$$

holds $\mathbb{P}$ -almost surely for all $\varphi \in C^\infty(\mathbb{T}^2, \mathbb{R})$.

- c. A weak probabilistic/martingale solution of equation (4.2) is a triple $(\tilde{\omega}, \{\tilde{W}^i\}_i), (\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}), (\tilde{\mathcal{F}}_t)_t$ such that $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ is a probability space, $(\tilde{\mathcal{F}}_t)_t$ is a filtration defined on this space, $\tilde{\omega}$ is a continuous $(\tilde{\mathcal{F}}_t)_t$ -adapted real-valued process $\tilde{\omega} : \Omega \times \mathbb{T}^2 \rightarrow \mathbb{R}$ with trajectories in the set $C([0, \infty); \mathcal{W}_b^{k,2}(\mathbb{T}^2))$, $k \geq 2$ fixed, $\{\tilde{W}^i\}_i$ are independent $(\tilde{\mathcal{F}}_t)_t$ -adapted Brownian motions and the identity

$$\omega_t = \omega_0 - \int_0^t u_s \cdot \nabla \omega_s ds + \sum_{i=1}^{\infty} \int_0^t \xi_i \cdot \nabla \omega_s dW_s^i + \frac{1}{2} \sum_{i=1}^{\infty} \int_0^t \xi_i \cdot \nabla(\xi_i \cdot \nabla \omega_s) ds$$

with $\omega|_{t=0} = \omega_0$, holds $\tilde{\mathbb{P}}$ -almost surely in $L_b^2(\mathbb{T}^2; \mathbb{R})$.

- d. A classical solution of equation (4.2) is an $(\mathcal{F}_t)_t$ -adapted process $\omega : \Omega \times \mathbb{T}^2 \rightarrow \mathbb{R}$ with

trajectories of class $C([0, \infty); C^2(\mathbb{T}^2; \mathbb{R}))$.

Remark 4.3. The velocity field v is not uniquely identified through the equation $\omega = b^{-1} \text{curl } v$. Indeed any two velocity fields that differ by a constant will lead to the same vorticity map ω . Instead we identify v through the "explicit" formula $v = \mathcal{M}(b^{-1} \nabla^\perp \psi)$, where ψ is a stream function which exists due to the fact that $\nabla \cdot (bu) = 0$.

Remark 4.4. Note that $\omega_t \in \mathcal{W}_b^{k,2}(\mathbb{T}^2)$ implies $u_t \in \mathcal{W}_b^{k+1,2}(\mathbb{T}^2)$ (see Section 4.5). By standard Sobolev embedding theorems $\mathcal{W}_b^{k+1,2}(\mathbb{T}^2) \hookrightarrow \mathcal{W}_b^{k,2}(\mathbb{T}^2) \hookrightarrow C(\mathbb{T}^2)$ for $k \geq 2$, hence the terms $\mathcal{L}_{u_t} \omega_t = u_t \cdot \nabla \omega_t \in L_b^2(\mathbb{T}^2, \mathbb{R})$ and $\langle \omega_s, \mathcal{L}_{u_s}^* \varphi \rangle_b$ are well defined.

Remark 4.5. Naturally, if ω_t is a strong solution in the sense of Definition 4.2, then it is also a weak/distributional solution. Note also that if ω_t is a weak/distributional solution with paths in $C([0, T]; \mathcal{W}_b^{k,2}(\mathbb{T}^2))$ then, by integration by parts, it is also a strong solution.

4.3 MAIN RESULTS

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P})$ be a filtered probability space and $k \geq 2$ a fixed integer as above. We restate the main result of this chapter:

Theorem 4.6. If $\omega_0 \in \mathcal{W}_b^{k,2}(\mathbb{T}^2)$ such that $\nabla \cdot (b\omega_0) = 0$, then the two-dimensional stochastic great lake equation (4.1)

$$d\omega_t + \mathcal{L}_{u_t} \omega_t dt + \sum_{i=1}^{\infty} \mathcal{L}_i \omega_t \circ dW_t^i = 0$$

admits a pathwise unique and probabilistically strong local $\mathcal{F}_t$ -adapted solution $\omega = \{\omega_t, t \in [0, \infty)\}$ with values in the space $C([0, \infty); \mathcal{W}_b^{k,2}(\mathbb{T}^2))$. In particular, if $k \geq 4$ the solution is classical.

Moreover, we have continuity with respect to initial conditions:

Theorem 4.7. Let $\omega, \tilde{\omega}$ be two $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$ -solutions of the great lake equation (4.1). Define the process $B = (B_t)_t$ ¹ as $B_t := \int_0^t \|\omega_s\|_{b,k,2} ds$, for any $t \geq 0$. Then there exists a positive constant $\mathcal{C}$ independent of the two solutions ω and $\tilde{\omega}$ such that

¹Note that there is no link between the process $(B_t)_t$ defined here and the operator B used in Chapter 2 to denote a generic coefficient corresponding to a more general SPDE.

$$\mathbb{E}[e^{-cB_t} \|\omega_t - \tilde{\omega}_t\|_{b,k,2}^2] \leq \|\omega_0 - \tilde{\omega}_0\|_{b,k,2}^2. \quad (4.5)$$

The proof of these two theorems are covered in Sections 4.4, 4.6, and 4.7.

4.4 PATHWISE UNIQUENESS

In this section we prove that any two solutions ω^1 and ω^2 defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$, driven by the same Brownian motion $(W^i)^i$, and with $\mathbb{P}$ -almost surely the same initial conditions ω_0^1 and ω_0^2 , are indistinguishable, that is

$$\mathbb{P}(\omega_t^1 = \omega_t^2 \text{ for all } t \geq 0) = 1.$$

From a probabilistic perspective this means that the solution is pathwise unique. This result is essential for the proof of existence of a strong solution in the sense of Definition 4.2.

Suppose that equation (4.1) admits two $\mathcal{F}_t$ -adapted solutions ω^1 and ω^2 with values in the space $C([0, \infty); \mathcal{W}_b^{k,2}(\mathbb{T}^2))$ and let $\bar{\omega} := \omega^1 - \omega^2$. Consider the corresponding velocities u^1 and u^2 such that $b^{-1} \text{curl}(\mathcal{M}u^1) = \omega^1$, $b^{-1} \text{curl}(\mathcal{M}u^2) = \omega^2$ and $\bar{u} := u^1 - u^2$. Since both ω^1 and ω^2 satisfy (4.2), their difference satisfies

$$d\bar{\omega}_t + (\mathcal{L}_{\bar{u}_t} \omega_t^1 + \mathcal{L}_{u_t^2} \bar{\omega}_t) dt + \sum_{i=1}^{\infty} \mathcal{L}_i \bar{\omega}_t dW_t^i - \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_i^2 \bar{\omega}_t dt = 0.$$

By the Itô formula one obtains

$$\begin{aligned} d\|\bar{\omega}_t\|_{b,2}^2 &= -2 \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle_b dW_t^i - 2 \langle \bar{\omega}_t, \mathcal{L}_{\bar{u}_t} \omega_t^1 + \mathcal{L}_{u_t^2} \bar{\omega}_t \rangle_b dt \\ &\quad + \sum_{i=1}^{\infty} (\langle \bar{\omega}_t, \mathcal{L}_i^2 \bar{\omega}_t \rangle_b + \langle \mathcal{L}_i \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle_b) dt. \end{aligned}$$

Note that the last term in the above identity is null (see Lemma 4.17) and that

$$|\langle \bar{\omega}_t, \mathcal{L}_{\bar{u}_t} \omega_t^1 \rangle_b| \leq \|\bar{\omega}_t\|_{b,2} \|\bar{u}_t\|_{b,4} \|\nabla \omega_t^1\|_{b,4} \leq C \|\bar{\omega}_t\|_{b,2}^2 \|\omega_t^1\|_{b,k,2}.$$

This is true since by the Sobolev embedding theorem (see [1] Theorem 4.12 case A) one has

$\|\nabla \omega_t^1\|_{b,4} \leq C\|\omega_t^1\|_{b,k,2}$ and due to the smoothness properties proved in Section 4.5 one has $\|\bar{u}_t\|_{b,4} \leq C\|\bar{u}_t\|_{b,1,2} \leq C\|\bar{\omega}_t\|_{b,2}$.

Finally, observe that $\langle \bar{\omega}_t, \mathcal{L}_{u_t^2} \bar{\omega}_t \rangle_b = -\frac{1}{2} \int_{\mathbb{T}^2} (\nabla \cdot (b u_t^2)) (\bar{\omega}_t)^2 dx = 0$ since $\nabla \cdot (b u_t^2) = 0$. It follows that

$$d\|\bar{\omega}_t\|_{b,2}^2 + 2 \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle_b dW_t^i = -2 \langle \bar{\omega}_t, \mathcal{L}_{\bar{u}_t} \omega_t^1 \rangle_b dt \leq C\|\omega_t^1\|_{b,k,2} \|\bar{\omega}_t\|_{b,2}^2 dt. \quad (4.6)$$

Since we only have a priori bounds for the expected value of the process $t \rightarrow \|\omega_t^1\|_{b,k,2}$ and not for its pathwise values, the uniqueness cannot be deduced through a classical Gronwall-type argument. Instead, we proceed as follows: let $B = (B_t)_t$ be the process defined as $B_t := \int_0^t C\|\omega_s^1\|_{b,k,2} ds$, for any $t \geq 0$. This is an increasing process that stays finite $\mathbb{P}$ -almost surely for all $t \geq 0$ as the paths of ω^1 are in $C([0, \infty); \mathcal{W}_b^{k,2}(\mathbb{T}^2))$. By integration by parts,

$$d(e^{-B_t} \|\bar{\omega}_t\|_{b,2}^2) = e^{-B_t} (d\|\bar{\omega}_t\|_{b,2}^2 - C\|\bar{\omega}_t\|_{b,2}^2 \|\omega_t^1\|_{k,2} dt) \leq -2e^{-A_t} \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle_b dW_t^i.$$

This leads to

$$\mathbb{E}[e^{-B_t} \|\bar{\omega}_t\|_{b,2}^2] \leq \mathbb{E}\left[-2e^{-B_t} \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle_b dW_t^i\right] = 0.$$

We conclude that $e^{-B_t} \|\bar{\omega}_t\|_{b,2}^2 = 0$, and since e^{-B_t} cannot be null due to the finiteness of B_t we deduce that $\|\bar{\omega}_t\|_{b,2}^2 = 0$ almost surely, which gives the claim.

4.5 THE VORTICITY TO STREAM FUNCTION MAP

In this section we provide a generalisation of the Biot-Savart law from Chapter 3 Section 3.6, which is the main ingredient used to extend the well-posedness properties to a more general class of stochastic partial differential equations. Note that the stochastic part does not interfere with this generalisation. This is due to the fact that stochasticity has been introduced as a constraint within a variational principle such that the physical meaning of the quantities of interest has not been altered: see [62], [86]. More precisely, $\omega = b^{-1} \text{curl } v$ where v depends on the deterministic, unperturbed velocity vector field u only, without any stochasticity involved. The central result is Proposition 4.9 ².

Lemma 4.8. *The restriction of the operator $\mathcal{M}$ to the space $\mathbb{D}_2$ is a continuous, positive, in-*

²The proofs from this section follow closely the arguments from [74] Section 2.

vertible and self-adjoint operator such that for every $u \in L_b^2(\mathbb{T}^2)$ one has

$$\mathcal{P}\mathcal{M}\mathcal{P}u = \mathcal{P} \left(u + \frac{\delta^2}{3} \nabla b (\nabla b \cdot \mathcal{P}u) \right).$$

Proof: Let $u, v \in \mathbb{D}_1$. We have

$$\mathcal{M}u = u + \delta^2 \left(-\frac{1}{3} b^{-1} \nabla (b^3 \nabla \cdot u) - \frac{1}{2} b^{-1} \nabla (b^2 u \cdot \nabla u) + \frac{1}{2} b (\nabla \cdot u) \nabla b + (u \cdot \nabla b) \nabla b \right)$$

and the bilinear form ℓ considered below can be written as

$$\begin{aligned} \ell(u, v) &:= \langle \mathcal{M}u, v \rangle_b = \int_{\mathbb{T}^2} \mathcal{M}u \cdot v b dx \\ &= \langle u, v \rangle_b - \frac{\delta^2}{3} \int_{\mathbb{T}^2} b^{-1} \nabla (b^3 \nabla \cdot u) \cdot v b dx - \frac{\delta^2}{2} \int_{\mathbb{T}^2} b^{-1} \nabla (b^2 u \cdot \nabla u) \cdot v b dx \\ &\quad + \frac{\delta^2}{2} \int_{\mathbb{T}^2} b^2 (\nabla \cdot u) (\nabla b \cdot v) dx + \delta^2 \int_{\mathbb{T}^2} b (u \cdot \nabla b) (\nabla b \cdot v) dx \\ &= \langle u, v \rangle_b + \frac{\delta^2}{3} \int_{\mathbb{T}^2} b^3 (\nabla \cdot u) (\nabla \cdot v) dx + \frac{\delta^2}{2} \int_{\mathbb{T}^2} b^2 (u \cdot \nabla b) (\nabla \cdot v) dx \\ &\quad + \frac{\delta^2}{2} \int_{\mathbb{T}^2} b^2 (\nabla \cdot u) (v \cdot \nabla v) dx + \delta^2 \int_{\mathbb{T}^2} b (u \cdot \nabla b) (v \cdot \nabla b) dx \\ &= \langle u, v \rangle_b - \frac{\delta^2}{3} \int_{\mathbb{T}^2} b (u \cdot \nabla b) (v \cdot \nabla b) dx - \frac{\delta^2}{2} \int_{\mathbb{T}^2} b (u \cdot \nabla b) (v \cdot \nabla b) dx \\ &\quad - \frac{\delta^2}{2} \int_{\mathbb{T}^2} b (u \cdot \nabla b) (v \cdot \nabla b) dx + \delta^2 \int_{\mathbb{T}^2} b (u \cdot \nabla b) (v \cdot \nabla b) dx \\ &= \langle u, v \rangle_b + \frac{\delta^2}{3} \langle u \cdot \nabla b, v \cdot \nabla b \rangle_b \\ &\leq c \|u\|_{b,2} \|v\|_{b,2}. \end{aligned}$$

The calculations have been simplified by the fact that $u, v \in \mathbb{D}_1$ implies $\nabla \cdot u = -b^{-1} u \cdot \nabla b$ and $\nabla \cdot v = -b^{-1} v \cdot \nabla b$. Therefore ℓ is continuous and symmetric and due to the Poincaré inequality it is also coercive. By the Lax-Milgram theorem (see for instance [20], pp. 140, Corollary 5.8) we can conclude that the operator $\mathcal{M}$ is invertible.

Proposition 4.9. *For every $\omega \in \mathcal{W}_b^{-1,2}(\mathbb{T}^2)$ there exists a unique function $u \in \mathbb{D}_2$ such that $u = K\omega$. Moreover, K is continuous and*

$$\|K\omega\|_{b,k,p} \leq C \|\omega\|_{b,k-1,p}.$$

Proof: Let $u, v \in \mathbb{D}_1$. Then (see [79]) there exist $\varphi_1, \varphi_2 \in C^\infty(\mathbb{T}^2)$ such that $bu = \nabla^\perp \varphi_1$ and $bv = \nabla^\perp \varphi_2$. We integrate by parts and use the previous lemma to write

$$\begin{aligned}\tilde{\ell}(\varphi_1, \varphi_2) &:= \langle \varphi_1, \omega \rangle_b = \left\langle \varphi_1, b^{-1} \operatorname{curl} \left(\mathcal{M} \left(b^{-1} \nabla^\perp \varphi_2 \right) \right) \right\rangle_b \\ &= \int_{\mathbb{T}^2} \left(b^{-1} \nabla^\perp \varphi_1 \right) \cdot \mathcal{M} \left(b^{-1} \nabla^\perp \varphi_2 \right) b dx \\ &\leq C \|u\|_{b,2} \|v\|_{b,2}.\end{aligned}$$

On the other hand, we can also write

$$\left\langle \varphi_1, b^{-1} \operatorname{curl} \left(\mathcal{M} \left(b^{-1} \nabla^\perp \varphi_2 \right) \right) \right\rangle_b \leq C \|\varphi_1\|_{b,1,2} \|\varphi_2\|_{b,1,2}$$

and by the Poincaré inequality

$$\|\varphi_1\|_{b,1,2}^2 \leq C \left\langle \varphi_1, b^{-1} \operatorname{curl} \left(\mathcal{M} \left(b^{-1} \nabla^\perp \varphi_1 \right) \right) \right\rangle_b$$

which explains also the elliptic character of the operator. The same estimates hold for $\varphi_1, \varphi_2 \in \mathcal{W}^{1,2}(\mathbb{T}^2)$ by a density argument. Therefore the bilinear form $\tilde{\ell}$ is continuous and coercive. By the Lax-Milgram theorem, for any $\omega \in \mathcal{W}^{-1,2}(\mathbb{T}^2)$ there exists a unique $u \in \mathbb{D}_2$ such that $u = K\omega$ and $K : \mathcal{W}^{-1,2}(\mathbb{T}^2) \rightarrow \mathbb{D}_2$ is continuous. Note that the problem

$$\omega = b^{-1} \operatorname{curl} \left(\mathcal{M} (b^{-1} \nabla^\perp \psi) \right) \quad \text{on } \mathbb{T}^2$$

is elliptic and therefore by regularity theorems for elliptic problems (see [47] Section 6.3 or [104] Section 6.2.8) we actually have

$$\|\psi\|_{b,k,p} \leq C \|\omega\|_{b,k-2,p}.$$

4.6 LOCAL EXISTENCE FOR THE GREAT LAKE EQUATION

The existence of the solution of equation (4.2) is proved by first showing that a truncated version of it has a solution, and then removing the truncation up to a positive stopping time. In particular we truncate the nonlinear term in (4.2) by using a smooth function f_R equal to 1 on $[0, R]$, equal to 0 on $[R+1, \infty)$, and decreasing on $[R, R+1]$, for arbitrary $R > 0$, with $f_R(u_t^R) := f_R(\|u_t^R\|_{b,k,2})$.

Then we have the following:

Proposition 4.10. *If $\omega_0 \in \mathcal{W}_b^{k,2}(\mathbb{T}^2)$ such that $\nabla \cdot (b\omega_0) = 0$, then the following equation*

$$d\omega_t^R + f_R(u_t^R)\mathcal{L}_{u_t^R}\omega_t^R dt + \sum_{i=1}^{\infty} \mathcal{L}_i\omega_t^R \circ dW_t^i = 0 \quad (4.7)$$

admits a unique global $\mathcal{F}_t$ -adapted solution $\omega^R = \{\omega_t^R, t \in [0, \infty)\}$ with values in the space $C([0, \infty); \mathcal{W}_b^{k,2}(\mathbb{T}^2))$. In particular, if $k \geq 4$, the solution is classical.

Remark 4.11. *Observe that, by definition, the truncation function f_R depends on the norm $\|\omega_t^R\|_{b,k-1,2}$ and not on the norm $\|\omega_t^R\|_{b,k,2}$. This is not incidental as it suffices to control the norm $\|u_t^R\|_{b,k,2}$ (see Section 4.5).*

We prove Proposition 4.10 in Section 4.7. For now let us proceed with the proof of local existence for the solution of the stochastic great lake equation (4.2). Define the stopping time

$$\tau_R(\omega) := \inf_{t \geq 0} \left\{ \|\omega_t^R\|_{b,k-1,2} \geq \frac{R}{C} \right\}$$

where C is such that $\|\nabla u\|_{\infty} \leq C\|u\|_{b,k,2}$. Observe that such a constant exists due to the Sobolev embedding $\mathcal{W}_b^{k,2} \hookrightarrow L^{\infty}$ and to the regularity properties proven in Section 4.5, since $\|\nabla u\|_{\infty} \leq C\|\nabla u\|_{b,k,2} \leq C\|u\|_{b,k+1,2} \leq C\|u\|_{b,k,2}$.

Lemma 4.12. *Let $\omega_0 \in \mathcal{W}_b^{k,2}(\mathbb{T}^2)$ and $\omega^R : \Omega \times [0, \infty) \times \mathbb{T}^2 \rightarrow \mathbb{R}$ be a global $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$ -solution of the truncated equation (4.7) and $\omega : \Omega \times [0, \tau_R] \times \mathbb{T}^2 \rightarrow \mathbb{T}^2 \rightarrow \mathbb{R}$ be the restriction of ω^R to the time interval $[0, \tau_R]$. Then ω is a local $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$ -solution of the original great lake equation (4.2).*

Proof Observe that for $t \in [0, \tau_R]$ we have $\|\nabla u_t\|_{\infty} \leq C\|\omega_t\|_{b,k,2} \leq R$. Thus, $f_R(u_t^R) = 1$ and taking into account the pathwise uniqueness property we conclude that the truncated equation coincides with the original equation.

Remark 4.13. *If we dispense with the requirement of showing the existence of a strong solution, then similar uniqueness and relative compactness arguments can be used to show the existence of a unique global weak solution, provided $\omega_0 \in L^{\infty}(\mathbb{T}^2)$. The key result here is the fact that*

the L^∞ -norm of the solution of (4.2) as well as that of any of its truncated versions remains constant in time (see Lemma 4.17):

$$\begin{aligned}\|\omega_t\|_\infty &= \|\omega_0\|_\infty \\ \|\omega_t^R\|_\infty &\leq \|\omega_0^R\|_\infty.\end{aligned}$$

This is an extension of the result in [21]. In particular, one can show the existence of a global solution $\omega_t \in \mathcal{W}_b^{-2,2}(\mathbb{T}^2)$.

4.7 UNIQUENESS AND EXISTENCE FOR THE TRUNCATED EQUATION

In this section we prove that the truncated equation admits a global unique solution. Any global solution for the truncated equation, restricted to the corresponding stopping time, is a local solution for the original great lake equation.

4.7.1 PATHWISE UNIQUENESS FOR THE TRUNCATED EQUATION

The strategy for showing that any solution of the truncated equation is pathwise unique is similar to the one presented in Section 4.4. The only difference arises due to the truncated terms. We have

$$d\|\bar{\omega}_t\|_{b,2}^2 + 2 \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle_b dW_t^i = -2 \langle \bar{\omega}_t, (f_R(\omega_t^1) \mathcal{L}_{u_t^1} - f_R(\omega_t^2) \mathcal{L}_{u_t^2}) \omega_t^1 \rangle_b dt$$

One can show that (see [37] for a proof) there exists a constant $C = C(R)$ such that

$$\|f_R(\omega_t^1) u_t^1 - f_R(\omega_t^2) u_t^2\|_{b,4} \leq C \|\bar{\omega}_t\|_{b,k-1,2}$$

and therefore

$$|\langle \bar{\omega}, (f_R(\omega_t^1) \mathcal{L}_{u_t^1} - f_R(\omega_t^2) \mathcal{L}_{u_t^2}) \omega_t^1 \rangle_b| \leq \|\bar{\omega}_t\|_{b,2} \|\bar{u}_t\|_{b,4} \|\nabla \omega_t^1\|_{b,4} \leq C \|\bar{\omega}_t\|_{b,2} \|\bar{\omega}_t\|_{b,k-1,2} \|\omega_t^1\|_{b,k,2}.$$

We deduce that

$$d\|\bar{\omega}_t\|_{b,2}^2 + 2 \sum_{i=1}^{\infty} \langle \bar{\omega}_t, \mathcal{L}_i \bar{\omega}_t \rangle_b dW_t^i \leq C \|\omega_t^1\|_{b,k,2} \|\bar{\omega}_t\|_{b,2}^2 dt.$$

Similar arguments are used to control $\|\partial^\alpha \bar{\omega}_t\|_{b,2}^2$ where α is a multi-index with $|\alpha| \leq k-1$ and to deduce that there exists a constant $C = C(R)$ such that

$$d\|\partial^\alpha \bar{\omega}_t\|_{b,2}^2 + 2 \sum_{i=1}^{\infty} \langle \partial^\alpha \bar{\omega}_t, \partial^\alpha \mathcal{L}_i \bar{\omega}_t \rangle_b dW_t^i \leq C \|\omega_t^1\|_{b,k,2} \|\bar{\omega}_t\|_{b,2}^2 dt,$$

where we use the control (see Lemma 4.17)

$$\langle \partial^\alpha \bar{\omega}_t, \partial^\alpha \mathcal{L}_i^2 \bar{\omega}_t \rangle_b + \langle \partial^\alpha \mathcal{L}_i \bar{\omega}_t, \partial^\alpha \mathcal{L}_i \bar{\omega}_t \rangle_b \leq C \|\bar{\omega}_t\|_{b,k,2}^2.$$

We need to pay special attention to the case when $|\alpha| = k-1$, as $\partial^\alpha \mathcal{L}_i^2 \bar{\omega}_t$ is no longer well defined. In this case we use the weak form of equation (4.7) to rewrite $\langle \partial^\alpha \bar{\omega}_t, \partial^\alpha \mathcal{L}_i^2 \bar{\omega}_t \rangle_b$ as $-\langle \partial^{\alpha_1} \partial^\alpha \bar{\omega}_t, \partial^{\alpha_2} \mathcal{L}_i^2 \bar{\omega}_t \rangle_b$ and then we can proceed as above by using that

$$-\langle \partial^{\alpha_1} \partial^\alpha \bar{\omega}_t, \partial^{\alpha_2} \mathcal{L}_i^2 \bar{\omega}_t \rangle_b + \langle \partial^\alpha \mathcal{L}_i \bar{\omega}_t, \partial^\alpha \mathcal{L}_i \bar{\omega}_t \rangle_b \leq C \|\bar{\omega}_t\|_{b,k,2}^2.$$

The above control is true for functions in $\mathcal{W}_b^{k+1,2}(\mathbb{T}^2)$ and, by the continuity of both sides in the above inequality, it is also true for functions which belong to the larger space $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$, since $\mathcal{W}_b^{k+1,2}(\mathbb{T}^2)$ is dense in $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$. The proof is now concluded in an identical manner as that for the uniqueness of the original equation.

4.7.2 EXISTENCE OF SOLUTION FOR THE TRUNCATED EQUATION

Given the fact that the topology of $\mathcal{W}_b^{k,2}(\mathbb{T}^2)$ is equivalent to the topology of $\mathcal{W}^{k,2}(\mathbb{T}^2)$ for a nondegenerate weight b (see [71]), the proof is similar to the proof from [34]. The strategy is to construct an approximating sequence of processes that will converge in distribution to a solution of the equation (4.7). This justifies the existence of a weak solution. Together with the pathwise uniqueness obtained in Section 4.7.1, we then deduce that a strong unique solution exists. For any $t \geq 0$ we construct the sequence $(\omega_t^{\nu_n, R, n})_{n \geq 0}$ with $\omega_t^{\nu_0, R, 0} := \omega_0^0$ and for $n \geq 1$:

$$\begin{aligned} \omega_0^{\nu_n, R, n} &:= \omega_0^n \\ d\omega_t^{\nu_n, R, n} &= \left(\nu_n \Delta \omega_t^{\nu_n, R, n} - f_R(u_t^{\nu_{n-1}, R, n-1}) \mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n} \right) dt - \sum_{i=1}^{\infty} \mathcal{L}_i \omega_t^{\nu_n, R, n} \circ dW_t^i, \end{aligned} \quad (4.8)$$

where $\nu_n = \frac{1}{n}$ is the viscous parameter and $u_t^{\nu_{n-1}, R, n-1} = K\omega_t^{\nu_{n-1}, R, n-1}$. The corresponding Itô form of equation (4.8) is ³

$$d\omega_t^{\nu_n, R, n} = \left(\nu_n \Delta \omega_t^{\nu_n, R, n} + P_t^{n-1, n}(\omega_t^{\nu_n, R, n}) \right) dt - \sum_{i=1}^{\infty} \mathcal{L}_i \omega_t^{\nu_n, R, n} dW_t^i, \quad (4.9)$$

where $P_t^{n-1, n}(\omega_t^{\nu_n, R, n})$ is defined as

$$P_t^{n-1, n}(\omega_t^{\nu_n, R, n}) := -f_R(u_t^{\nu_{n-1}, R, n-1}) \mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n} + \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_i^2 \omega_t^{\nu_n, R, n}, \quad t \geq 0. \quad (4.10)$$

Theorem 4.14. *If $\omega_0^{\nu_n, R, n} \in C^\infty(\mathbb{T}^2)$ is such that $\nabla \cdot (b\omega_0^{\nu_n, R, n}) = 0$, then the two-dimensional stochastic vorticity equation (4.9) admits a unique global $\mathcal{F}_t$ -adapted solution $\omega_t^{\nu_n, R, n} = \{\omega_t^{\nu_n, R, n}, t \in [0, \infty)\}$ which belongs to the space $C([0, \infty); C^\infty(\mathbb{T}^2))$.*

The stochastic equation (4.9) is a particular case of the more general equation (1.1) – (1.2) analysed in Chapter 4, Section 4.1, pp.129 from [100]. All the assumptions required by Theorem 1 and Theorem 2 in [100], Chapter 4, are fulfilled. Therefore there exists a unique solution $\omega_t^{\nu_n, R, n}$ which belongs to the class $L^2([0, T], \mathcal{W}_b^{k, 2}(\mathbb{T}^2)) \cap C([0, T], \mathcal{W}_b^{k-1, 2}(\mathbb{T}^2))$ and satisfies equation (4.9) for all $t \in [0, T]$ and for all ω in $\Omega' \subset \Omega$ with $\mathbb{P}(\Omega') = 1$. Furthermore, since the conditions are fulfilled for all $k \in \mathbb{N}$, using Corollary 3 from pp. 141 in [100], we obtain that $\omega_t^{\nu_n, R, n}$ is $\mathbb{P}$ -a.s. in $C([0, T], C^\infty(\mathbb{T}^2))$. Note that $u_t^{\nu_{n-1}, R, n-1} \in C^\infty(\mathbb{T}^2)$ for any $n \geq 1$, using the regularity properties of the vorticity to stream function operator and an inductive argument. One has $u_t^{\nu_{n-1}, R, n-1} = K\omega_t^{\nu_{n-1}, R, n-1}$ where K is the vorticity to stream function operator as before. We have

$$\|K\omega_t^{\nu_{n-1}, R, n-1}\|_{b, k+1, p} \leq C \|\omega_t^{\nu_{n-1}, R, n-1}\|_{b, k, p}.$$

Since $\omega_t^{\nu_{n-1}}$ belongs to $C^\infty(\mathbb{T}^2)$ we deduce that $u_t^{\nu_{n-1}, R, n-1}$ also belongs to $C^\infty(\mathbb{T}^2)$. This, together with the initial assumptions (4.3) ensures that all the coefficients of equation 4.9 are infinitely differentiable. The uniform boundedness is ensured by the truncation $f_R(u_t^{\nu_{n-1}, R, n-1})$, as proved in Lemma 4.17.

Remark 4.15. *(Continuity of the approximating sequence). There exists a constant $C = C(T)$*

³The stochastic Itô integral is understood here in the usual sense, see [40].

independent of n and R such that

$$\mathbb{E}[\|\omega_t^{\nu_n, R, n} - \omega_s^{\nu_n, R, n}\|_{L_b^2}^4] \leq C(t-s)^2, \quad t, s \in [0, T].$$

In particular, by the Kolmogorov-Čentsov criterion (see [67]), the processes $\omega^{\nu_n, R, n}$ have continuous trajectories in $L_b^2(\mathbb{T}^2)$. An explicit proof of this fact is based on L^2 -estimates for each term and can be found in [34] Section 5.2.

Proposition 4.16. *The laws of the family of solutions $(\omega^{\nu_n, R, n})_{\nu_n \in [0, 1]}$ is relatively compact in the space of probability measures over $D([0, T], L_b^2(\mathbb{T}^2))$ for any $T \geq 0$.*

Proof of existence for the solution of equation (4.7)

From Proposition 4.16 and the fact that $\lim_{n \rightarrow \infty} \omega_0^{\nu_n, R, n} = \omega_0$ we can deduce, using a diagonal subsequence argument, the existence of a subsequence $(\omega^{\nu_{n_j}})_j$ with $\lim_{j \rightarrow \infty} \nu_{n_j} = 0$ which is convergent in distribution over $D([0, \infty), L_b^2(\mathbb{T}^2))$. We show that the limit of the corresponding distributions is the distribution of a stochastic process which solves the truncated equation (4.7). This justifies the existence of a weak probabilistic solution. By using the Skorokhod representation theorem (see [13]), there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and a sequence of processes $(\tilde{\omega}^{\nu_n, R, n}, \tilde{u}^{\nu_n, R, n}, (\tilde{W}^i)_i, n = 1, \infty)$ which has the same distribution as that of the original converging subsequence and which converges, for $n \rightarrow \infty$, almost surely to a triplet $(\tilde{\omega}^R, \tilde{u}^R, (\tilde{W}^i)_i)$ in $D([0, T], L_b^2(\mathbb{T}^2) \times \mathcal{W}_b^{1,2}(\mathbb{T}^2) \times \mathbb{R}^{\mathbb{N}})$. Observe that $\omega^{\nu_n, R, n}$ and $\tilde{\omega}^{\nu_n, R, n}$ have the same distribution, therefore for any test function $\varphi \in C^\infty(\mathbb{T}^2)$ we have

$$\begin{aligned} \langle \tilde{\omega}_t^{\nu_n, R, n}, \varphi \rangle_b &= \langle \tilde{\omega}_0^{\nu_n, R, n}, \varphi \rangle_b + \nu_n \int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n}, \Delta \varphi \rangle_b ds \\ &\quad + \int_0^t f_R(\tilde{u}_s^{\nu_{n-1}, R, n-1}) \langle \tilde{\omega}_s^{\nu_n, R, n}, \mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \varphi \rangle_b ds \\ &\quad + \frac{1}{2} \sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n}, \mathcal{L}_i^2 \varphi \rangle_b ds + \sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n}, \mathcal{L}_i \varphi \rangle_b d\tilde{W}_s^i. \end{aligned} \tag{4.11}$$

Observe also that there exists a constant $C = C(R, T)$ such that

$$\sup_{n \geq 1} \tilde{\mathbb{E}} \left[\sup_{s \in [0, T]} \|\tilde{\omega}_s^{\nu_n, R, n}\|_{b, k, 2}^4 \right] \leq C, \tag{4.12}$$

where $\tilde{\mathbb{E}}$ is the expectation with respect to $\tilde{\mathbb{P}}$. We prove this in Lemma 4.17 for the original sequence, but since $\tilde{\omega}^{\nu_n, R, n}$ satisfies the same SPDE, the same a priori estimates hold for $\tilde{\omega}^{\nu_n, R, n}$. We know that the space of continuous functions is a subspace of the space of càdlàg functions, therefore the Skorokhod topology relativised to the space of continuous functions coincides with the uniform topology. It follows that the sequence $(\tilde{\omega}^{\nu_n, R, n}, \tilde{u}^{\nu_n, R, n}, (\tilde{W}^i)_i, n = 1, \infty)$ converges $\tilde{\mathbb{P}}$ -almost surely to $(\tilde{\omega}^R, \tilde{u}^R, (\tilde{W}^i)_i)$ when $n \rightarrow \infty$, in the uniform norm. It also holds that

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[\int_0^t \|\tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R\|^2 ds \right] = 0$$

and since

$$\begin{aligned} \sum_{i=1}^{\infty} \tilde{\mathbb{E}} \left[\int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R, \mathcal{L}_i \varphi \rangle_b^2 ds \right] &\leq \sum_{i=1}^{\infty} \|\mathcal{L}_i \varphi\|_{b,2}^2 \tilde{\mathbb{E}} \left[\int_0^t \|\tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R\|^2 ds \right] \\ &\leq C \|\varphi\|_{b,1,2}^2 \tilde{\mathbb{E}} \left[\int_0^t \|\tilde{\omega}_s^{\nu_n, R, n} - \tilde{\omega}_s^R\|^2 ds \right] \end{aligned} \quad (4.13)$$

also the limit of the right hand side of (4.13) converges to 0 (we use here the control $\sum_{i=1}^{\infty} \|\mathcal{L}_i \varphi\|_{b,2}^2 \leq C \|\varphi\|_{b,1,2}^2$ assumed in (4.3)). Now Theorem 4.2 in [72] allows us to conclude that the stochastic term

$$\sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n}, \mathcal{L}_i \varphi \rangle_b d\tilde{W}_s^i$$

converges in distribution to $\sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^R, \mathcal{L}_i \varphi \rangle_b d\tilde{W}_s^i$. Using a similar application of the Skorokhod

representation theorem we can also assume that on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, the term $\sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^{\nu_n, R, n}, \mathcal{L}_i \varphi \rangle_b d\tilde{W}_s^i$ converges to $\sum_{i=1}^{\infty} \int_0^t \langle \tilde{\omega}_s^R, \mathcal{L}_i \varphi \rangle_b d\tilde{W}_s^i$ $\tilde{\mathbb{P}}$ -almost surely (and also in $L_b^2(\tilde{\mathbb{P}})$).

The convergence of the remaining terms is shown in a similar fashion and similar to [34]. The only difference is that we have weighted scalar products and norms, but given the fact that b is a bounded nondegenerate function, all the convergences hold as requested.

We have proven so far that there exists a weak/distributional solution in the sense of Definition 4.2. part b. on the probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. Moreover, since $\tilde{\omega}^R$ belongs to the space $\mathcal{W}_b^{k,2}(\mathbb{T}^2) \hookrightarrow C^{k-m}(\mathbb{T}^2)$ the solution is also strong, as a solution on the space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. It follows that $(\tilde{\omega}, \tilde{u}, (\tilde{W}^i)_i)$ is a weak/probabilistic solution of the truncated equation (4.7) in the sense of Definition 4.2 part c. Together with the pathwise uniqueness proved in Section 4.7.2 and using

the Yamada-Watanabe theorem for the infinite-dimensional setting (see, for instance, [99]) we conclude the existence of a strong solution of the truncated vorticity equation, again in the sense of Definition 4.2, part a. Now using the embedding $\mathcal{W}_b^{k,2}(\mathbb{T}^2) \hookrightarrow C^{k-m}(\mathbb{T}^2)$ with $2 \leq m \leq k$ and $k \geq 4$ we conclude that the solution is classical when $k \geq 4$.

4.8 CONTINUITY WITH RESPECT TO INITIAL CONDITIONS

In this section we prove Theorem 4.7. Let $\omega, \tilde{\omega}$ be two $C([0, \infty); \mathcal{W}_b^{k,2}(\mathbb{T}^2))$ -solutions of equation (4.2) and define the process $A = (A_t)_t$ such that $A_t := \int_0^t \|\omega_s\|_{b,k,2}^2 ds$, for any $t \geq 0$. Let $\omega^R, \tilde{\omega}^R$ be their corresponding truncated versions and also let $(\omega_t^{\nu_n, R, n})_{n \geq 0}$ and $(\tilde{\omega}_t^{\nu_n, R, n})_{n \geq 0}$ be, respectively, the corresponding sequences constructed as in Section 4.7.2 on the same space after the application of the Skorokhod representation theorem. By Fatou's lemma, applied twice, we deduce that

$$\begin{aligned} \mathbb{E}[e^{-CA_t} \|\omega_t - \tilde{\omega}_t\|_{b,k,2}^2] &\leq \mathbb{E}[\liminf_n e^{-CA_t^n} \|\omega_t^{\nu_n, R, n} - \tilde{\omega}_t^{\nu_n, R, n}\|_{b,k,2}^2] \\ &\leq \liminf_n \mathbb{E}[e^{-CA_t^n} \|\omega_t^{\nu_n, R, n} - \tilde{\omega}_t^{\nu_n, R, n}\|_{b,k,2}^2] \end{aligned}$$

where $A^n = (A_t^n)_t$ is the process defined by $A_t^n := \int_0^t \|\omega_s^{\nu_n, R, n}\|_{b,k,2}^2 ds$, for any $t \geq 0$. Following a similar proof with that of the uniqueness of the vorticity equation presented in Section 4.4, we deduce that there exists a positive constant C independent of the two solutions and independent of R and n such that

$$E[e^{-CA_t^n} \|\omega_t^{\nu_n, R, n} - \tilde{\omega}_t^{\nu_n, R, n}\|_{b,k,2}^2] \leq \|\omega_0 - \tilde{\omega}_0\|_{b,k,2}^2$$

which gives the result. We emphasize that we use here the fact that the processes $(\omega_t^{\nu_n, R, n})_{n \geq 0}$ and $(\tilde{\omega}_t^{\nu_n, R, n})_{n \geq 0}$ take values in $\mathcal{W}_b^{k+2,2}(\mathbb{T}^2)$ as an essential ingredient, a property that was not true for either the solution of the original great lake equation (4.2) or its truncated version.

4.9 RELATIVE COMPACTNESS

In this section we prove that the approximating sequence of solutions constructed in Section 4.7.2 is relatively compact in the space $D([0, T], L_b^2(\mathbb{T}^2))$. To do so we use Kurtz' criterion for

relative compactness (see Chapter 2 Theorem 2.14).

Proof of Proposition 4.16

We need to show that, for every $\eta > 0$, there exists a compact set $K_{\eta,t} \subset L_b^2(\mathbb{T}^2)$ such that $\sup_n \mathbb{P}(\omega_t^{\nu_n,R,n} \notin K_{\eta,t}) \leq \eta$. We define the compact

$$K_{\eta,t} := \left\{ \omega \in \mathcal{W}_b^{k,2}(\mathbb{T}^2) \mid \|\omega\|_{b,k,2} < \left(\frac{C}{\eta} \right)^{\frac{1}{4}} \right\}$$

where C is the constant which appears in the a priori estimates (Lemma 4.17). By a Sobolev compact embedding theorem, $K_{\eta,t}$ is a compact set in $L_b^2(\mathbb{T}^2)$ and

$$\sup_n \mathbb{P}(\omega_t^{\nu_n,R,n} \notin K_{\eta,t}) = \sup_n \mathbb{P}(\|\omega_t^{\nu_n,R,n}\|_{b,k,2} \geq \eta) \leq \sup_n \frac{\eta}{C} E \left[\sup_{t \in [0,T]} \|\omega_t^{\nu_n,R,n}\|_{b,k,2}^4 \right] \leq \eta.$$

To prove relative compactness, we need to justify part b) of Kurtz' criterion, as per Theorem 8.6 in [45]. For this we will show that there exists a family $(\gamma_\delta^n)_{0 < \delta < 1}$ of nonnegative random variables such that

$$\mathbb{E}[\|\omega_{t+l}^{\nu_n,R,n} - \omega_t^{\nu_n,R,n}\|_{b,2}^2 | \mathcal{F}_t] \leq \mathbb{E}[\gamma_\delta^n | \mathcal{F}_t]$$

and $\lim_{\delta \rightarrow 0} \limsup_n \mathbb{E}[\gamma_\delta^n] = 0$ for $t \in [0, T]$. We will use the mild form of equation (4.9), that is

$$\omega_t^{\nu_n,R,n} = S^n(t) \omega_0^{\nu_n,R,n} - \int_0^t S^n(t-s) P_s^{n-1,n}(\omega_s^{\nu_n,R,n}) ds - \sum_{i=1}^{\infty} \int_0^t S^n(t-s) \mathcal{L}_i \omega_s^{\nu_n,R,n} dW_s^i,$$

with $P_s^{n-1,n}$ as defined in (4.10) and $S^n(t) := e^{\nu_n \Delta t}$. One has

$$\begin{aligned} \|\omega_{t+l}^{\nu_n,R,n} - \omega_t^{\nu_n,R,n}\|_{b,2}^2 &\leq C \left(\|(S^n(t+l) - S^n(t)) \omega_0^{\nu_n,R,n}\|_{b,2}^2 \right. \\ &\quad + \left\| \int_0^t (S^n(t+l-s) - S^n(t-s)) P_s^{n-1,n}(\omega_s^{\nu_n,R,n}) ds \right\|_{b,2}^2 \\ &\quad + \left\| \int_t^{t+l} S^n(t+l-s) P_s^{n-1,n}(\omega_s^{\nu_n,R,n}) ds \right\|_{b,2}^2 \\ &\quad + \left\| \sum_{i=1}^{\infty} \int_0^t (S^n(t+l-s) - S^n(t-s)) \mathcal{L}_i \omega_s^{\nu_n,R,n} dW_s^i \right\|_{b,2}^2 \\ &\quad + \left\| \sum_{i=1}^{\infty} \int_t^{t+l} S^n(t+l-s) \mathcal{L}_i \omega_s^{\nu_n,R,n} dW_s^i \right\|_{b,2}^2 \Big) \end{aligned}$$

The calculations follow as per Chapter 3 Section 3.5. The process $\gamma_l^{\nu_n}$ is defined as

$$\begin{aligned} \gamma_l^{\nu_n} &:= \|(S^n(l) - 1)\omega_0^{\nu_n, R, n}\|_{b,2} + \int_0^T \|(S^n(l) - 1)P_s^{n-1, n}(\omega_s^{\nu_n, R, n})\|_{b,2}^2 ds \\ &\quad + Cl^2 \sup_{s \in [0, T+1]} \|P_s^{n-1, n}(\omega_s^{\nu_n, R, n})\|_{b,2}^2 \\ &\quad + l^2 \sup_{s \in [0, T+1]} \|\mathcal{L}_i \omega_s^{\nu_n, R, n}\|_{b,2}^2 + \int_0^T \|((t+l-r)^{\alpha-1} S^n(l) - (t-r)^{\alpha-1}) S^n(t-r) z(r)\|_{b,2}^2 dr \end{aligned}$$

and $\gamma_\delta^n := \sup_{l \in [0, \delta]} \gamma_l^{\nu_n}$. From Lemma 4.17 we deduce that there exist two constants c_1 and c_2 such that $\mathbb{E}[\sup_s \|P_s^{n-1, n}(\omega_s^{\nu_n, R, n})\|_{b,2}^2] \leq c_1$ and $\mathbb{E}[\sup_s \|\mathcal{L}_i \omega_s^{\nu_n, R, n}\|_{b,2}^2] \leq c_2$. The integrands in the integrals above converge pointwise to 0 when $l \rightarrow 0$ due to the strong continuity of the semigroup S^n . At the same time, they are bounded by integrable functions, therefore the convergence is uniform in space by the dominated convergence theorem. Then the requirement

$$\lim_{\delta \rightarrow 0} \limsup_{\nu_n} \mathbb{E}[\gamma_\delta^n] = 0$$

is met. In conclusion all the conditions required by Kurtz' criterion are fulfilled and therefore $(\omega_t^{\nu_n, R, n})_{\nu_n}$ is relatively compact.

4.10 A PRIORI ESTIMATES

Lemma 4.17.

Let ω_t be the solution of the vorticity equation (4.2) and $\omega_t^{\nu_n, R, n}$ the solution of the linear approximating equation (4.8). Then the following properties hold:

i. For any $f \in \mathcal{W}_b^{2,2}(\mathbb{T}^2)$ we have

$$\langle f, \mathcal{L}_i^2 f \rangle_b + \langle \mathcal{L}_i f, \mathcal{L}_i f \rangle_b = 0.$$

ii. If $\omega_t, \omega_t^{\nu_n, R, n} \in \mathcal{W}_b^{k, 2}(\mathbb{T}^2)$ then the following transport formulae hold $\mathbb{P}$ -almost surely:

$$\begin{aligned}\|\omega_t\|_{L_b^p} &= \|\omega_0\|_{L_b^p} \\ \|\omega_t^{\nu_n, R, n}\|_{L_b^p} &\leq \|\omega_0\|_{L_b^p} \\ \|\omega_t\|_\infty &= \|\omega_0\|_\infty \\ \|\omega_t^{\nu_n, R, n}\|_\infty &\leq \|\omega_0\|_\infty.\end{aligned}$$

provided that the initial condition is in $L^\infty(\mathbb{T}^2)$.

iii. If $\omega \in \mathcal{W}_b^{k, 2}(\mathbb{T}^2)$, then $(P_t^{n-1, n})_t$ defined in 4.10 and $(\mathcal{L}_i \omega_t^{\nu_n, R, n})_t$ are processes with paths taking values in $L_b^2(\mathbb{T}^2)$.

iv. There exists a constant C_1 such that:

$$|\langle \partial^k \omega_t^{\nu_n, R, n}, \partial^k (\mathcal{L}_i^2 \omega_t^{\nu_n, R, n}) \rangle_b + \langle \partial^k (\mathcal{L}_i \omega_t^{\nu_n, R, n}), \partial^k (\mathcal{L}_i \omega_t^{\nu_n, R, n}) \rangle_b| \leq C_1 \|\omega_t^{\nu_n, R, n}\|_{b, k, 2}^2.$$

v. There exist some constants C_2 and C'_2 such that:

$$\begin{aligned}& |\langle \partial^k \omega_t^{\nu_n, R, n}, f_R(u_t^{\nu_{n-1}, R, n-1}) \partial^k (\mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n}) \rangle_b| \\ & \leq C_2 \|\partial^k u_t^{\nu_n, R, n-1}\|_{b, 2}^a \|u_t^{\nu_n, R, n-1}\|_{b, 2}^{1-a} \|\omega_t^{\nu_n, R, n}\|_{b, k, 2}^2\end{aligned}$$

with $0 < a \leq 1$, and

$$|\langle \partial^k \omega_t^{\nu_n, R, n}, f_R(u_t^{\nu_{n-1}, R, n-1}) \partial^k (\mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n}) \rangle_b| \leq C'_2 \|\omega_t^{\nu_n, R, n}\|_{b, k, 2}^2.$$

The technical details of the proof of the above inequalities are similar to the ones proven in Lemma 3.20 from Chapter 3. The only difference is given by the fact the on the right hand side we always have also the norm of the bounded nondegenerate function b , which does not alter qualitatively the result. However, a slightly more significant difference appears in the control of the higher derivatives of the Lie derivative $\mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n}$ where instead of the divergence-free assumption from Chapter 3 we use the regularity properties proven in Section 4.5. However, we provide below the proof for part ii.

Proof for ii: By the Itô formula

$$\begin{aligned}
d\|\omega_t^{\nu_n, R, n}\|_{b,2}^2 &= -2 \sum_i \langle \omega_t^{\nu_n, R, n}, \mathcal{L}_i \omega_t^{\nu_n, R, n} \rangle_b dW_t^i - 2 \langle \omega_t^{\nu_n, R, n}, f_R(u_t^{\nu_{n-1}, R, n-1}) \mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n} \rangle_b dt \\
&\quad + 2 \langle \omega_t^{\nu_n, R, n}, \nu_n \Delta \omega_t^{\nu_n, R, n} \rangle_b dt \\
&\quad + \sum_i \left(\langle \omega_t^{\nu_n, R, n}, \mathcal{L}_i^2 \omega_t^{\nu_n, R, n} \rangle_b dt + \langle \mathcal{L}_i \omega_t^{\nu_n, R, n}, \mathcal{L}_i \omega_t^{\nu_n, R, n} \rangle_b dt \right).
\end{aligned}$$

The last sum has been proved to be 0 at i. and $\langle \omega_t^{\nu_n, R, n}, \mathcal{L}_{u_t^{\nu_{n-1}, R, n-1}} \omega_t^{\nu_n, R, n} \rangle_b dt = 0$ by integration by parts since $bu_t^{\nu_{n-1}, R, n-1}$ is divergence-free. Similarly $\langle \omega_t^{\nu_n, R, n}, \mathcal{L}_i \omega_t^{\nu_n, R, n} \rangle_b = 0$ since the vector fields ξ_i are assumed to be such that $\nabla \cdot (b\xi_i) = 0$. Therefore

$$\|\omega_t^{\nu_n, R, n}\|_{b,2}^2 = \|\omega_0\|_{b,2}^2 - 2 \int_0^t \nu_n (\nabla \omega_s^{\nu_n, R, n})^2 ds \leq \|\omega_0\|_{b,2}^2 \quad \mathbb{P}\text{-almost surely.}$$

The calculations are similar for the non-viscous equation, but there are no viscous terms, hence

$$\|\omega_t\|_{b,2}^2 = \|\omega_0\|_{b,2}^2 \quad \mathbb{P}\text{-almost surely.}$$

In 4.10 we raise both sides to the power $p/2$ with $p > 2$ and we use the Minkovski inequality to obtain

$$\|\omega_t^{\nu_n, R, n}\|_{b,2}^p = \|\omega_0\|_{b,2}^p - 2 \left(\int_0^t \nu_n (\nabla \omega_s^{\nu_n, R, n})^2 ds \right)^{p/2} \leq \|\omega_0\|_{b,2}^p \quad \mathbb{P}\text{-almost surely}$$

and

$$\|\omega_t\|_{b,2}^p = \|\omega_0\|_{b,2}^p$$

$\mathbb{P}$ -almost surely. Given the fact that $\omega_t^{\nu_n, R, n}, \omega_t \in L_b^p(\mathbb{T}^2)$ for any $p \geq 1$ and $\mathcal{W}_b^{1,p} \hookrightarrow L^\infty$ we conclude that $\omega_t^{\nu_n, R, n}, \omega_t \in L^\infty(\mathbb{T}^2)$ and we can pass to the limit $p \rightarrow \infty$ to deduce that

$$\begin{aligned}
\|\omega_t\|_\infty &= \|\omega_0\|_\infty \\
\|\omega_t^{\nu_n, R, n}\|_\infty &\leq \|\omega_0\|_\infty.
\end{aligned}$$

Proposition 4.18. *There exists a constant $C(R, T)$ independent of n such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|\omega_t^{\nu_n, R, n}\|_{b, k, 2}^4 \right] \leq C(R, T).$$

Proof: After applying the Itô formula to the approximating sequence (4.8) we obtain

$$\begin{aligned} \|\partial^k \omega_t^{\nu_n, R, n}\|_{b, 2}^2 &= \|\partial^k \omega_0^{\nu_n, R, n}\|_{b, 2}^2 + 2\nu_n \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, \partial^{k+2} \omega_s^{\nu_n, R, n} \rangle_b ds \\ &\quad - 2 \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, f_R(u_s^{\nu_n-1, R, n-1}) \partial^k \mathcal{L}_{u_s^{\nu_n-1, R, n-1}} \omega_s^{\nu_n, R, n} \rangle_b ds \\ &\quad + \sum_{i=1}^{\infty} \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i^2 \omega_s^{\nu_n, R, n} \rangle_b ds \\ &\quad + \sum_{i=1}^{\infty} \int_0^t \langle \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n} \rangle_b ds \\ &\quad - 2 \sum_{i=1}^{\infty} \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n} \rangle_b dW_s^i. \end{aligned}$$

We analyse each term. One can write

$$\langle \partial^k \omega_s^{\nu_n, R, n}, \partial^{k+2} \omega_s^{\nu_n, R, n} \rangle_b = -\|\partial^{k+1} \omega_s^{\nu_n, R, n}\|_{b, 2}^2 \leq 0.$$

We want to estimate the other terms independently of ν_n . All terms are estimated above. After summing up we have

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t]} \|\omega_s^{\nu_n, R, n}\|_{b, k, 2}^2 \right] &\leq \mathbb{E} [\|\partial^k \omega_0^{\nu_n, R, n}\|_{b, 2}^2] + C'_2(T) \int_0^t \mathbb{E} \left[\sup_{s \in [0, t]} \|\omega_s^{\nu_n, R, n}\|_{b, k, 2}^2 \right] ds \\ &\quad + C_1 \int_0^t \mathbb{E} \left[\sup_{s \in [0, t]} \|\omega_s^{\nu_n, R, n}\|_{b, k, 2}^2 \right] ds + 2\mathbb{E} \left[\sup_{s \in [0, t]} \sum_{i=1}^{\infty} \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n} \rangle_b dW_s^i \right]. \end{aligned}$$

Let

$$\mathcal{B}_t := \sum_{i=1}^{\infty} \int_0^t \langle \partial^k \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n} \rangle_b dW_s^i \quad \text{and} \quad \beta_t := \|\omega_t^{\nu_n, R, n}\|_{b, k, 2}^2.$$

$\mathcal{B}_t$ is a local martingale. We have

$$\beta_t \leq \beta_0 + 2\mathcal{B}_t + (C'_2 + C_1) \int_0^t \beta_s ds$$

and by Gronwall lemma

$$\mathbb{E}\left[\sup_{s \in [0, t]} \beta_s^2\right] \leq 2e^{2(C'_2 + C_1)t} \left(\beta_0^2 + 4\mathbb{E}\left[\sup_{s \in [0, t]} \mathcal{B}_s^2\right]\right).$$

Using the Burkholder-Davis-Gundy inequality there exists a constant $\tilde{\alpha}$ such that

$$\mathbb{E}\left[\sup_{s \in [0, t]} \mathcal{B}_s^2\right] \leq \tilde{\alpha} \mathbb{E}\left[\sqrt{\langle \mathcal{B} \rangle_s^2}\right].$$

We can show that $\mathbb{E}[\langle \mathcal{B} \rangle_s] \leq C_2^2 \mathbb{E}[\sup_{s \in [0, t]} \beta_s^2]$: with the same calculations as those from Lemma 3.20 v. we can see that

$$\langle \partial^k \omega_s^{\nu_n, R, n}, \partial^k \mathcal{L}_i \omega_s^{\nu_n, R, n} \rangle_b \leq C_2 \|\omega_s^{\nu_n, R, n}\|_{b, k, 2}^2$$

if instead of making use of the truncation corresponding to $\|u_s^{\nu_{n-1}, R, n-1}\|_{b, k, 2}$ we take into account assumption 4.3. Hence

$$\langle \mathcal{B} \rangle_s \leq C_2^2 \sup_{s \in [0, t]} \|\omega_s^{\nu_n, R, n}\|_{b, k, 2}^4 = C_2^2 \sup_{s \in [0, t]} \beta_s^2$$

and the above inequality follows. In conclusion

$$\mathbb{E}\left[\sup_{s \in [0, t]} \beta_s^2\right] \leq \tilde{C}_1(T) \beta_0^2 + \tilde{C}_2(T) \int_0^t \mathbb{E}\left[\sup_{s \in [0, t]} \beta_s^2\right] ds$$

with $\tilde{C}_1(T) := 2 \exp(2(C'_2 + C_1)T)$ and $\tilde{C}_2(T) := 8\tilde{\alpha}C_2^2 \exp(2(C'_2 + C_1)T)$. Using again Gronwall's inequality we have

$$\mathbb{E}\left[\sup_{s \in [0, t]} \|\omega_s^{\nu_n, R, n}\|_{b, k, 2}^4\right] \leq \mathcal{C}(R, T)$$

with $\mathcal{C}(R, T) := \tilde{C}_1(T) \beta_0^2 \exp(\tilde{C}_2(T)T)$.

The following lemma is essential when showing that the limit of the approximating sequence satisfies the great lake equation in $\mathcal{W}_b^{k, 2}(\mathbb{T}^2)$ although the relative compactness property holds in $D([0, T], L_b^2(\mathbb{T}^2))$.

Given the fact that the proof is similar to the proof of Lemma 3.20 from Chapter 3, we do not redo it here.

Lemma 4.19.

i. Assume that $(a_n)_n$ is a sequence of functions such that $\lim_{n \rightarrow \infty} a_n = a$ in $L_b^2(\mathbb{T}^2)$ and $\sup_{n \geq 1} \|a_n\|_{b,s,2} < \infty$ for $s \geq 0$. Then $a \in \mathcal{W}_b^{s,2}(\mathbb{T}^2)$ and $\|a\|_{b,s,2} \leq \sup_{n \geq 1} \|a_n\|_{b,s,2}$. Moreover, $\lim_{n \rightarrow \infty} a_n = a$ in $\mathcal{W}_b^{s',2}(\mathbb{T}^2)$ for any $s' < s$.

ii. Assume that $a_n : \Omega \mapsto \mathcal{W}_b^{s,2}(\mathbb{T}^2)$ is a sequence of measurable maps such that, $\lim_{n \rightarrow \infty} a_n = a$ in $L_b^2(\mathbb{T}^2)$, $\mathbb{P}$ -almost surely or $\lim_{n \rightarrow \infty} a_n = a$ in distribution. Further assume that $\sup_{n \geq 1} \mathbb{E}[\|a_n\|_{b,s,2}^2] < \infty$. Then, $\mathbb{P}$ -almost surely, $a \in \mathcal{W}_b^{s,2}(\mathbb{T}^2)$ and $\mathbb{E}[\|a\|_{b,s,2}^2] \leq \sup_{n \geq 1} \mathbb{E}[\|a_n\|_{b,s,2}^2]$.

Remark 4.20. The norm $\|\cdot\|_{b,m,2}$ is equivalent to the norm defined as $|||f||| := \|f\|_{b,2} + \|D^m f\|_{b,2}$, therefore it is enough to show that all properties hold for the L_b^2 norm of f and for the L_b^2 norm of the maximal derivative $D^m f$ (see [20] pp. 217).

5

THE STOCHASTIC ROTATING SHALLOW WATER EQUATIONS WITH TRANSPORT NOISE

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In this chapter we prove existence and uniqueness of a maximal pathwise solution for the viscous stochastic rotating shallow water system driven by transport noise.

5.1 INTRODUCTION

The stochastic rotating shallow water equations model the evolution of a compressible rotating fluid below a free surface. The typical vertical length scale is assumed to be much smaller than the horizontal one, hence the *shallow* aspect. This model is a simplification of the primitive equations which are known for their complicated and computationally expensive structure ([65]). Albeit its simplified form, the SRSW system retains key aspects of the atmospheric and oceanic dynamics. In its original form it allows for gravity and Rossby waves which play a highly important role in climate and weather modelling ([102]). In this chapter we consider a viscous version of the stochastic rotating shallow water model on the two-dimensional torus $\mathbb{T}^2$:

$$dv_t + [u_t \cdot \nabla v_t + f \hat{z} \times u_t + \nabla p_t] dt + \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i)v_t] \circ dW_t^i = \nu \Delta v_t dt \quad (5.1a)$$

$$dh_t + \nabla \cdot (h_t u_t) dt + \sum_{i=1}^{\infty} [\nabla \cdot (\xi_i h_t)] \circ dW_t^i = \eta \Delta h_t dt \quad (5.1b)$$

where

- $u = (u^1, u^2)$ is the horizontal fluid velocity vector field, $v := \epsilon u + \mathcal{R}$, with $\text{curl } \mathcal{R} = f \hat{z}$
- ϵ is the Rossby number, a dimensionless number which describes the effects of rotation on the fluid flow: a small Rossby number ($\epsilon \ll 1$) suggests that the rotation dominates over the advective terms; it can be expressed as $\epsilon = \frac{U}{fL}$ where U is a typical scale for horizontal speed and L is a typical length scale.
- f is the Coriolis parameter, $f = 2\Theta \sin \varphi$ where Θ is the rotation rate of the Earth and φ is the latitude; $f \hat{z} \times u = (-fu^2, fu^1)$, where $\hat{z}$ is a unit vector pointing away from the centre of the Earth. For the analytical analysis we assume f to be constant.

-
- $\mathcal{R}$ corresponds to the vector potential for the (divergence-free) rotation rate about the vertical direction, and it is chosen such that $\nabla \mathcal{R} = 0$
 - h is the total depth
 - $p := \frac{h-b}{\epsilon \mathcal{F}}$, ∇p is the pressure term, b is the bottom topography as in the previous chapter.
 - $\mathcal{F}$ is the Froude, a dimensionless number which relates to the stratification of the flow. It can be expressed as $\mathcal{F} = \frac{U}{NH}$ where H is the typical vertical scale and N is the buoyancy frequency.
 - $\mathcal{L}_i : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$, $\mathcal{L}_i v = \xi_i \cdot \nabla v$ is a second order operator, as described in Chapter 2.
 - $\mathcal{A}_i : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$, $\mathcal{A}_i v = v^1 \nabla \xi_i^1 + v^2 \nabla \xi_i^2$ is a zero-order operator, as described in Chapter 2.
 - ν, η are positive constants which correspond to fluid viscosities.

This single-layered model has been derived using the SALT approach, as described in [62], [86], [63]. A detailed derivation can be found in the Appendix, Section 8.2.

To our best knowledge, this specific form of the SRSW model, driven by this particular type of stochasticity, has not been considered before. A stochastic version of the viscous rotating shallow water system with external forcing and multiplicative noise has been studied in [75]. A rotating shallow water model driven by Lévy noise has been considered in [39]. As pointed out in [75], the number of results available in the literature on stochastic shallow water equations is limited. In the deterministic case, existence of solutions under certain conditions and without rotation was proven in [84]. Smooth approximate solutions for the 2D deterministic rotating shallow water system have been constructed in [19]. Long time existence for rapidly rotating deterministic shallow water and Euler equations has been shown in [29].

As compared to the two models studied in Chapters 3 and 4, this analysis is more difficult from both a deterministic and a stochastic perspective. First of all, the fluid is now compressible, which makes the control of the nonlinear terms harder. The smoothness induced here by the viscous term helps in balancing the nonlinear effects possibly generated by $\nabla \cdot u$. Secondly, the stochastic part contains an extra term ($\mathcal{A}_i$, see Section 5.3) whose effects need to be attentively

taken into account when deriving a priori estimates. Let aside the lack of incompressibility, there is also no Biot-Savart-type relationship involving the velocity vector field. This implies on the one hand that an L^p -transport property is out of reach and on the other hand that a control of the higher order derivatives is much more difficult. We prove that this system admits a maximal solution in the Sobolev space $\mathcal{W}^{1,2}(\mathbb{T}^2)^3$. Our approach resembles the approach followed in [75] for obtaining a priori estimates (as the deterministic terms are very similar) and for proving relative compactness for the approximating sequence of solutions. However, we use a different approximation method, extending the methodology developed in Chapters 3 and 4 to a system of SPDEs which model a compressible fluid under the effects of rotation. At the same time, the main proof of existence of a truncated solution (see Theorem 5.4) builds upon the arguments developed in Chapters 3 and 4. Moreover, the structure of the noise is different in our case and we do not impose Lipschitz conditions. The noise can be written as a sum of two operators: the second order operator $\mathcal{L}_i$ which has been seen before in the SE and the SGLE models, and a new zero-order operator $\mathcal{A}_i$ whose properties are described in Chapter 2 Section 2.6

5.2 STRATEGY & STRUCTURE

In a broad sense the strategy is similar to the one developed in Chapters 3 and 4. We construct sequences of approximating solutions which converge in a suitable sense to a truncated form of the original SRSW system. Then we show that the truncation can be lifted. As opposed to the case of the Euler equation, here we can lift the truncation only up to a positive stopping time. This is due to a lack of transport properties for both variables, which derive from the compressibility condition and the form of the nonlinear terms. Therefore we obtain a local solution for the original system (5.1). This is proven in Section 5.4. Nonetheless, in Section 5.4.1 we show that this solution is maximal and due to the pathwise uniqueness property from Section 5.4.2, it is also strong in probabilistic sense. In Section 5.4.3 we show global existence for the truncated model, while in Section 5.4.4 we prove the main a priori estimates. In Section 5.5 we study the analytical properties of the approximating sequence: relative compactness (Section 5.5.1) and a priori estimates (Section 5.5.2).

5.3 PRELIMINARIES

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P}, (W^i)_i)$ be a fixed stochastic basis as before and $(\xi_i)_i$ a sequence of vector fields which satisfy the usual assumptions from Chapter 2 and from Section 5.3.2, are predictable and $\mathcal{F}_t$ -adapted. Consider the SRSW system on the time interval $[0, \infty)$ and the initial conditions $v_0 = (v_0^1, v_0^2) \in \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$, $h_0 \in \mathcal{W}^{1,2}(\mathbb{T}^2)$ with law m_0 i.e. $m_0(A) = \mathbb{P}((v_0, h_0) \in A)$ for any Borel set $A \subset (L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2)) \times L^2(\mathbb{T}^2)$.

5.3.1 DEFINITIONS

We define the following notions of solutions:

Definition 5.1.

- a. A strong/pathwise local solution of the SRSW system is given by a triple $((v, h), \tau)$ where $\tau : \Omega \rightarrow [0, \infty]$ is a strictly positive stopping time and $v : \Omega \times [0, \tau] \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$, $h : \Omega \times [0, \tau] \times \mathbb{T}^2 \rightarrow \mathbb{R}$ are two processes with initial conditions v_0, h_0 such that $v_{t \wedge \tau}(\cdot)$ and $h_{t \wedge \tau}(\cdot)$ are $\mathcal{F}_t$ -adapted for any $t \geq 0$, with

$$v_{t \wedge \tau}(\cdot) \in L^2(\Omega; C([0, T]; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2))) \quad (5.2a)$$

$$h_{t \wedge \tau}(\cdot) \in L^2(\Omega; C([0, T]; \mathcal{W}^{1,2}(\mathbb{T}^2))) \quad (5.2b)$$

$$v_{t \wedge \tau}(\cdot) \in L^2(\Omega; L^2(0, T; \mathcal{D}(-\Delta))) \quad (5.2c)$$

$$h_{t \wedge \tau}(\cdot) \in L^2(\Omega; L^2(0, T; \mathcal{D}(-\Delta))) \quad (5.2d)$$

and the SRSW system is satisfied locally i.e. for any bounded stopping time $\tau' \leq \tau$

$$\begin{aligned} \langle v_{t \wedge \tau'}, \varphi \rangle &= \langle v_0, \varphi \rangle + \int_0^{t \wedge \tau'} \langle v_s, \mathcal{L}_{u_s}^* \varphi \rangle ds + \int_0^{t \wedge \tau'} \langle u_s, f \hat{z} \times \varphi \rangle ds + \int_0^{t \wedge \tau'} \langle p_s, \nabla \varphi \rangle ds \\ &\quad - \int_0^{t \wedge \tau'} \langle v_s, \nu \Delta \varphi \rangle ds + \sum_{i=1}^{\infty} \int_0^{t \wedge \tau'} \langle v_s, (\mathcal{L}_i^* + \mathcal{A}_i^*) \varphi \rangle dW_s^i \\ &\quad - \frac{1}{2} \sum_{i=1}^{\infty} \int_0^{t \wedge \tau'} \langle v_s, (\mathcal{L}_i^* + \mathcal{A}_i^*)^2 \varphi \rangle ds \\ \langle h_{t \wedge \tau'}, \varphi \rangle &= \langle h_0, \varphi \rangle + \int_0^{t \wedge \tau'} \langle h_s u_s, \nabla \varphi \rangle ds - \int_0^{t \wedge \tau'} \langle h_s, \eta \Delta \varphi \rangle ds \\ &\quad + \sum_{i=1}^{\infty} \int_0^{t \wedge \tau'} \langle h_s, \mathcal{L}_i^* \varphi \rangle dW_s^i - \frac{1}{2} \sum_{i=1}^{\infty} \int_0^{t \wedge \tau'} \langle h_s, (\mathcal{L}_i^*)^2 \varphi \rangle ds \end{aligned}$$

hold $\mathbb{P}$ -almost surely, as identities in $L^2(\mathbb{T}^2; \mathbb{R}^2), L^2(\mathbb{T}^2; \mathbb{R})$, for all test functions $\varphi \in \mathcal{W}^{2,2}(\mathbb{T}^2)$.

b. If $\tau = \infty$ then the solution (v, h) is called a global solution.

c. A strong/pathwise maximal solution of the SRSW system is given by a triple $((v, h), \mathcal{T})$ where $\mathcal{T} : \Omega \rightarrow [0, \infty]$ is a non-negative stopping time and $v : \Omega \times [0, \mathcal{T}) \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$, $h : \Omega \times [0, \mathcal{T}) \times \mathbb{T}^2 \rightarrow \mathbb{R}$ are two processes such that there exists an increasing sequence of stopping times $(\tau^n)_n$ with the following properties:

- i. $\mathcal{T} = \lim_{n \rightarrow \infty} \tau^n$ and $\mathbb{P}(\mathcal{T} > 0) = 1$
- ii. $((v, h), \tau^n)$ is a pathwise local solution of the SRSW system for every $n \in \mathbb{N}$
- iii. if $\mathcal{T} < \infty$ then

$$\sup_{t \in [0, \mathcal{T}]} \|v_t\|_{1,2}^2 + \sup_{t \in [0, \mathcal{T}]} \|h_t\|_{1,2}^2 + \nu \int_0^{\mathcal{T}} \|\Delta v_t\|_2^2 dt + \eta \int_0^{\mathcal{T}} \|\Delta h_t\|_2^2 dt = \infty.$$

d. A weak/martingale local solution of the SRSW system is given by a triple

$((\check{v}, \check{h}), (\check{W}^i)_i), (\check{\Omega}, \check{\mathcal{F}}, \check{\mathbb{P}}), (\check{\mathcal{F}}_t)_t$ such that $(\check{\Omega}, \check{\mathcal{F}}, (\check{\mathcal{F}}_t)_t, \check{\mathbb{P}}, (\check{W}^i)_i)$ is a stochastic basis, $\check{v}, \check{h}$ are two continuous $(\check{\mathcal{F}}_t)_t$ -adapted real valued processes $\check{v} : \check{\Omega} \times \mathbb{T}^2 \rightarrow \mathbb{R}^2$, $\check{h} : \check{\Omega} \times \mathbb{T}^2 \rightarrow \mathbb{R}$ which satisfy (5.2) for a stopping time $\tau : \check{\Omega} \rightarrow [0, \infty]$, $(\check{W}^i)_i$ are independent $(\check{\mathcal{F}}_t)_t$ -adapted Brownian motions, and the identities

$$\begin{aligned} \langle \check{v}_{t \wedge \tau}, \varphi \rangle &= \langle \check{v}_0, \varphi \rangle + \int_0^{t \wedge \tau} \langle \check{v}_s, \mathcal{L}_{\check{u}_s}^* \varphi \rangle ds + \int_0^{t \wedge \tau} \langle \check{u}_s, f \hat{z} \times \varphi \rangle ds + \int_0^{t \wedge \tau} \langle \check{p}_s, \nabla \varphi \rangle ds \\ &\quad - \int_0^{t \wedge \tau} \langle \check{v}_s, \nu \Delta \varphi \rangle ds + \sum_{i=1}^{\infty} \int_0^{t \wedge \tau} \langle \check{v}_s, (\mathcal{L}_i^* + \mathcal{A}_i^*) \varphi \rangle d\check{W}_s^i \\ &\quad - \frac{1}{2} \sum_{i=1}^{\infty} \int_0^{t \wedge \tau} \langle \check{v}_s, (\mathcal{L}_i^* + \mathcal{A}_i^*)^2 \varphi \rangle ds \\ \langle \check{h}_{t \wedge \tau}, \varphi \rangle &= \langle \check{h}_0, \varphi \rangle + \int_0^{t \wedge \tau} \langle \check{h}_s \check{u}_s, \nabla \varphi \rangle ds - \int_0^{t \wedge \tau} \langle \check{h}_s, \eta \Delta \varphi \rangle ds \\ &\quad + \sum_{i=1}^{\infty} \int_0^{t \wedge \tau} \langle \check{h}_s, \mathcal{L}_i^* \varphi \rangle d\check{W}_s^i - \frac{1}{2} \sum_{i=1}^{\infty} \int_0^{t \wedge \tau} \langle \check{h}_s, (\mathcal{L}_i^*)^2 \varphi \rangle ds \end{aligned}$$

hold $\check{\mathbb{P}}$ -almost surely in $L^2(\mathbb{T}^2; \mathbb{R}^2), L^2(\mathbb{T}^2; \mathbb{R})$, for all test functions $\varphi \in \mathcal{W}^{2,2}(\mathbb{T}^2)$.

Remark 5.2. • *Part c. of the definition above states that $\mathcal{T}$ is the largest stopping time with properties i. and ii., that is for any other triple $((v', h'), \mathcal{T}')$ which satisfies i. and ii. we necessarily have $\mathcal{T}' \leq \mathcal{T}$ $\mathbb{P}$ -a.s., and $v = v', h = h'$ on $[0, \mathcal{T}']$.*

- *The first two definitions of solutions are established with respect to a fixed stochastic basis, the solutions being strong in probabilistic sense. The solution defined at d. is weak in probabilistic sense, meaning that $(\check{v}, \check{h})$ are not necessarily adapted to the original filtration $(\mathcal{F}_t)_t$ generated by the driving Brownian motion which corresponds to the SRSW system (5.1).*
- *Note that throughout this chapter we work only with weak solutions in the PDE sense. This is a consequence of the way in which the compressibility property of the fluid flow interferes with the degree of regularity of a corresponding solution.*

5.3.2 ASSUMPTIONS AND REMARKS

The $(\xi_i)_i$ vector fields are divergence-free and they satisfy the assumptions explained in Chapter 2, Section 2.6. However, note that for $f \in \mathcal{W}^{1,2}(\mathbb{T}^2)$ the term $\mathcal{L}_i^2 f$ makes sense only in weak form, that is $\langle \mathcal{L}_i^2 f, \varphi \rangle = \langle \mathcal{L}_i f, \mathcal{L}_i^* \varphi \rangle$ where φ is a test function from $\mathcal{W}^{1,2}(\mathbb{T}^2)$. For this reason condition (2.21b)

$$\sum_{i=1}^{\infty} \|\mathcal{L}_i^2 f\|_2^2 \leq C \|f\|_{2,2}^2$$

needs to be replaced with the weaker form

$$\sum_{i=1}^{\infty} \|\langle \mathcal{L}_i^2 f, \varphi \rangle\|_2^2 \leq C \|f\|_{1,2}^2 \|\varphi\|_{1,2}^2. \quad (5.5)$$

If we choose the test function $\varphi \in \mathcal{W}^{2,2}(\mathbb{T}^2)$ then we can also write $\langle \mathcal{L}_i^2 f, \varphi \rangle = \langle f, (\mathcal{L}_i^2)^* \varphi \rangle$. Note that condition (5.5) meets our needs as in this case we require the solution to be only weak in PDE sense. Thus also the Itô correction is well-defined in this context.

Similarly, the nonlinear term as well as all the other terms which involve derivatives of v and h make sense in weak form as expressed in Definition 5.1. Nevertheless, v and h are functions which belong also to the space $\mathcal{D}(-\Delta) := \{f \in (L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2)) \times L^2(\mathbb{T}^2) : \sum_{j=1}^{\infty} \alpha_j^2 \|w^j\|_2^2 < \infty\}$. Here

$(\alpha_j)_j$ is a sequence of eigenvalues such that $-\Delta\varphi_j = \alpha_j\varphi_j$ where $(\varphi_j)_j$ is a standard orthonormal basis of the space $L^2(\mathbb{T}^2)$.

5.3.3 ITÔ FORMULATION

In this section we provide the Itô version of system (5.1) which will be used for the well-posedness analysis. We highlight one more time that the terms involved must be understood in weak sense from the PDE perspective, as explained above. For brevity, we will omit the test functions in this section. The velocity equation can be written more compactly in the form

$$dv_t + F_t(v_t)dt + \sum_i G_t(v_t) \circ dW_t^i = 0$$

with $v := \epsilon u + \mathcal{R}$, $F_t(v_t) := u_t \cdot \nabla v_t + f \hat{z} \times u_t + \nabla p_t + \nu \Delta v_t$ and $G_t(v_t) := (\xi_i \cdot \nabla) v_t + v_t^1 \nabla \xi_i^1 + v_t^2 \nabla \xi_i^2$, where Then

$$\begin{aligned} \int_0^t G_s(v_s) \circ dW_s^i &= \int_0^t G_s(v_s) dW_s^i + \frac{1}{2} [G(v), W^i]_t \\ &= \int_0^t G_s(v_s) dW_s^i + \frac{1}{2} \int_0^t (\mathcal{L}_i^2 v_s + (\mathcal{L}_i + \mathcal{A}_i)^2 v_s) ds. \end{aligned}$$

The quadratic variation process $[G(v), W^i]_t$ is determined as follows. We know that for an Itô process of the form $v_t = v_0 + \int_0^t F_s ds + \int_0^t G_s \circ dW_s$ the cross-variation process is given by

$$[v, W]_t = \int_0^t G_s ds.$$

In our case

$$\begin{aligned}
(\xi_i \cdot \nabla v + v_j \nabla \xi_i^j) \circ dW_t^i &= (\xi_i \cdot \nabla v) \circ dW_t^i + (v_j \nabla \xi_i^j) \circ dW_t^i \\
&= (\xi_i \cdot \nabla v) dW_t^i - \frac{1}{2} [\xi_i \cdot \nabla v, W^i]_t + (v_j \nabla \xi_i^j) dW_t^i - \frac{1}{2} [v_j \nabla \xi_i^j, W^i]_t \\
&= (\xi_i \cdot \nabla v) dW_t^i - \frac{1}{2} \xi_i \cdot \nabla [v, W^i]_t + (v_j \nabla \xi_i^j) dW_t^i - \frac{1}{2} \nabla \xi_i^j [v_j, W^i]_t \\
&= (\xi_i \cdot \nabla v) dW_t^i - \frac{1}{2} \int_0^t \xi_i \cdot \nabla (\xi_i \cdot \nabla v_s + (v_s)_j \nabla \xi_i^j) ds \\
&\quad + (v_j \nabla \xi_i^j) dW_t^i - \frac{1}{2} \nabla \xi_i^j \int_0^t (G_s)_j ds \\
&= (\xi_i \cdot \nabla v) dW_t^i - \frac{1}{2} \int_0^t \xi_i \cdot \nabla (\xi_i \cdot \nabla v_s + (v_s)_j \nabla \xi_i^j) ds \\
&\quad + (v_j \nabla \xi_i^j) dW_t^i - \frac{1}{2} \sum_j \nabla \xi_i^j \int_0^t \left(\sum_k (\xi_i^k \partial_k (v_s)_j + (v_s)_k \partial_j \xi_i^k) \right) ds \\
&= (\xi_i \cdot \nabla v) dW_t^i - \frac{1}{2} \int_0^t \xi_i \cdot \nabla (\xi_i \cdot \nabla v_s + (v_s)_j \nabla \xi_i^j) ds \\
&\quad + (v_j \nabla \xi_i^j) dW_t^i - \frac{1}{2} \int_0^t (\nabla \xi_i^j) ((\xi_i \cdot \nabla (v_s)_j + (v_s)_k \partial_j \xi_i^k) ds \\
&= (\xi_i \cdot \nabla v + v_j \nabla \xi_i^j) dW_t^i \\
&\quad - \frac{1}{2} \int_0^t \left[\xi_i \cdot \nabla \left(\xi_i \cdot \nabla v_s + (v_s)_j \nabla \xi_i^j \right) + \nabla \xi_i^j \left(\xi_i \cdot \nabla (v_s)_j + (v_s)_k \partial_j \xi_i^k \right) \right] ds \\
&= (\xi_i \cdot \nabla v + v_j \nabla \xi_i^j) dW_t^i \\
&\quad - \frac{1}{2} \int_0^t \left(\mathcal{L}_i^2 v_s + \mathcal{L}_i((v_s)_j \nabla \xi_i^j) + (\nabla \xi_i^j) \mathcal{L}_i(v_s)_j + (\nabla \xi_i^j)((v_s)_k \partial_j \xi_i^k) \right) ds \\
&= (\xi_i \cdot \nabla v + v_j \nabla \xi_i^j) dW_t^i \\
&\quad - \frac{1}{2} \int_0^t \left(\mathcal{L}_i^2 v_s + \mathcal{L}_i(\mathcal{A}_i v_s) + \mathcal{A}_i(\mathcal{L}_i v_s) + \mathcal{A}_i^2 v_s \right) ds \\
&= (\mathcal{L}_i + \mathcal{A}_i) v_t dW_t^i - \frac{1}{2} \int_0^t (\mathcal{L}_i + \mathcal{A}_i)^2 v_s ds.
\end{aligned}$$

The second equation is treated in a similar fashion and since we assume that ξ_i are divergence-free, we obtain $\nabla \cdot (\xi_i h) = h \nabla \cdot \xi_i + \xi_i \cdot \nabla h = \xi_i \cdot \nabla h$ and therefore

$$[\nabla \cdot (\xi_i h), W^i]_t = \xi_i \cdot \nabla [h, W^i]_t = \xi_i \cdot \nabla \int_0^t \xi_i \cdot \nabla h_s ds,$$

therefore

$$\begin{aligned}\int_0^t [\nabla \cdot (\xi_i h_s)] \circ dW_s^i &= \int_0^t \xi_i \cdot \nabla h_s dW_s^i - \frac{1}{2} \int_0^t \xi_i \cdot \nabla (\xi_i \cdot \nabla h_s) ds \\ &= \int_0^t \mathcal{L}_i h_s dW_s^i - \frac{1}{2} \int_0^t \mathcal{L}_i^2 h_s ds.\end{aligned}$$

We deduce that the corresponding Itô form of the system (5.1) is

$$dv_t + [\mathcal{L}_{u_t} v_t + f \hat{z} \times u_t + \nabla p_t - \nu \Delta v_t] dt + \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i) v_t] dW_t^i = \frac{1}{2} \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i)^2 v_t] dt \quad (5.6a)$$

$$dh_t + [\nabla \cdot (h_t u_t) - \eta \Delta h_t] dt + \sum_{i=1}^{\infty} \mathcal{L}_i h_t dW_t^i = \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_i^2 h_t dt. \quad (5.6b)$$

We highlight that we interpret equations (5.6a) and (5.6b) in weak sense, that is

$$\begin{aligned}d\langle v_t, \varphi \rangle &= -\langle v_t, \mathcal{L}_{u_t}^* \varphi \rangle dt - \langle u_t, f \hat{z} \times \varphi \rangle dt - \langle p_t, \nabla \varphi \rangle dt + \langle v_t, \nu \Delta \varphi \rangle dt \\ &\quad - \sum_{i=1}^{\infty} \langle v_t, (\mathcal{L}_i^* + \mathcal{A}_i^*) \varphi \rangle dW_t^i + \frac{1}{2} \sum_{i=1}^{\infty} \langle v_t, (\mathcal{L}_i^* + \mathcal{A}_i^*)^2 \varphi \rangle dt\end{aligned} \quad (5.7a)$$

$$d\langle h_t, \varphi \rangle = -\langle h_t u_t, \nabla \varphi \rangle dt + \langle h_t, \eta \Delta \varphi \rangle dt - \sum_{i=1}^{\infty} \langle h_t, \mathcal{L}_i^* \varphi \rangle dW_t^i + \frac{1}{2} \sum_{i=1}^{\infty} \langle h_t, (\mathcal{L}_i^*)^2 \varphi \rangle dt \quad (5.7b)$$

The reason for having to work with the weak form is that we only assume that $v_t \in \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$, $h_t \in \mathcal{W}^{1,2}(\mathbb{T}^2)$ and $v_t, h_t \in L^2([0, \tau]; \mathcal{D}(-\Delta))$. More precisely we do not assume that $\mathcal{L}_i^2$ and $(\mathcal{L}_i + \mathcal{A}_i)^2$ are well-defined in (5.6a) and (5.6b) respectively. Nevertheless, we will still need to control the evolution of $t \rightarrow \|v_t\|_2^2$, $t \rightarrow \|h_t\|_2^2$, $t \rightarrow \|\nabla v_t\|_2^2$, $t \rightarrow \|\nabla h_t\|_2^2$. In order to do this, we need to use an intermediate step. We take $(\varphi_k)_k$ to be an orthonormal basis in $L^2(\mathbb{T}^2)$ ¹. Then we can express the L^2 norm of v_t, h_t using this basis:

$$\|v_t\|_2^2 = \sum_{k=1}^{\infty} c_k \langle v_t, \varphi_k \rangle^2 =: \sum_{k=1}^{\infty} c_k v_t(\varphi_k)^2 \quad (5.8a)$$

$$\|h_t\|_2^2 = \sum_{k=1}^{\infty} c_k \langle h_t, \varphi_k \rangle^2 =: \sum_{k=1}^{\infty} c_k h_t(\varphi_k)^2 \quad (5.8b)$$

¹For example $\varphi_k(x) := c_k e^{ik \cdot x}$ where c_k is a normalising constant.

Applying the Itô formula we have

$$\begin{aligned}
& d\|v_t(\varphi_k)\|_2^2 + 2\nu\|\nabla v_t(\varphi_k)\|_2^2 dt \\
&= 2 \left\langle v_t(\varphi_k), \left(\mathcal{L}_{u_t} v_t + f\hat{z} \times u_t + \nabla p_t + \frac{1}{2} \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i)^2 v_t] v_t \right) (\varphi_k) \right\rangle dt \\
&+ 2 \sum_{i=1}^{\infty} \langle v_t(\varphi_k), (\mathcal{L}_i + \mathcal{A}_i) v_t(\varphi_k) \rangle dW_t^i + \sum_{i=1}^{\infty} \langle (\mathcal{L}_i + \mathcal{A}_i) v_t(\varphi_k), (\mathcal{L}_i + \mathcal{A}_i) v_t(\varphi_k) \rangle dt,
\end{aligned}$$

and

$$\begin{aligned}
& d\|h_t(\varphi_k)\|_2^2 + 2\eta\|\nabla h_t(\varphi_k)\|_2^2 dt = 2 \left\langle h_t(\varphi_k), \nabla \cdot (h_t u_t)(\varphi_k) + \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_i^2 h_t(\varphi_k) \right\rangle dt \\
&+ 2 \sum_{i=1}^{\infty} \langle h_t(\varphi_k), \mathcal{L}_i h_t(\varphi_k) \rangle dW_t^i + \sum_{i=1}^{\infty} \langle \mathcal{L}_i h_t(\varphi_k), \mathcal{L}_i h_t(\varphi_k) \rangle dt.
\end{aligned}$$

Then using Fatou's lemma we can write

$$\begin{aligned}
d\|v_t\|_2^2 &\leq \sum_k d\|v_t(\varphi_k)\|_2^2 \\
d\|h_t\|_2^2 &\leq \sum_k d\|h_t(\varphi_k)\|_2^2.
\end{aligned}$$

This intermediate step is immediate but lengthy, so in the following we will only work with the corresponding strong versions of the inequalities required.

5.4 EXISTENCE AND UNIQUENESS OF STRONG PATHWISE SOLUTIONS FOR THE SRSW SYSTEM

Theorem 5.3. *Consider the initial conditions $u_0 \in \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$, $h_0 \in \mathcal{W}^{1,2}(\mathbb{T}^2)$ and a fixed stochastic basis $\mathcal{S} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathbb{P}, (W^i)_i)$. Then there exists a unique pathwise maximal solution $(v, h) \in (\mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$ of the stochastic rotating shallow water system 5.1.*

The existence of solution for system (5.1) is proved by first showing that a truncated version of it has a solution and then removing the truncation up to a positive stopping time. In particular we will truncate the non-linear and the pressure terms in (5.1) by using a smooth function $f_R : \mathbb{R}_+ \rightarrow [0, 1]$ equal to 1 on $[0, R]$, equal to 0 on $[R + 1, \infty)$, and decreasing on $[R, R + 1]$,

$f_R(u) := f_R(\|u\|_\infty)$, $f_R(h) := f_R(\|h\|_\infty)$, $f_R(u, h) := f_R(\|u\|_\infty, \|h\|_\infty)$, for arbitrary $R > 0$. We then have the following:

Theorem 5.4. *If the initial conditions (v_0, h_0) belong to $(\mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$ then the following truncated system*

$$dv_t + [f_R(u_t)\mathcal{L}_{u_t}v_t + f\hat{z} \times u_t + f_R(h_t)\nabla p_t + \nu\Delta v_t]dt + \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i)v_t] \circ dW_t^i = 0 \quad (5.9a)$$

$$dh_t + [f_R(u_t, h_t)\nabla \cdot (h_t u_t) + \eta\Delta h_t]dt + \sum_{i=1}^{\infty} [\nabla \cdot (\xi_i h_t)] \circ dW_t^i = 0 \quad (5.9b)$$

admits a unique global pathwise solution $(v^R, h^R) = \{(v_t^R, h_t^R), t \in [0, \infty)\}$.

Proposition 5.5. *Given $(v_0, h_0) \in (\mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$ and $R > 0$, the restriction $(v, h) : \Omega \times [0, \tau_R) \times \mathbb{T}^2 \rightarrow \mathbb{R}^2, \mathbb{R}$ of the global solution $(v, h) : \Omega \times [0, \infty) \times \mathbb{T}^2 \rightarrow \mathbb{R}^2, \mathbb{R}$ corresponding to the truncated equation (5.9) in the space $(\mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$, is a local solution of the original SRSW system (5.1). We denote by τ_R .*

Proof For $t \in [0, \tau_R]$ $f_R(v^R, h^R) = f_R(v^R) = f_R(h^R) = 1$ therefore the two equations are the same.

5.4.1 MAXIMAL SOLUTION FOR THE SRSW SYSTEM

Proposition 5.6. *Given $(v_0, h_0) \in (\mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$ and $R > 0$, there exists a maximal solution $((v, h), \mathcal{T})$ of the original SRSW system (5.1) such that*

$$\limsup_{t \rightarrow \mathcal{T}} (\|v_t\|_{1,2}^2 + \|h_t\|_{1,2}^2) + \limsup_{t \rightarrow \mathcal{T}} \left(\int_0^t \|\Delta v_s\|_2^2 ds + \int_0^t \|\Delta h_s\|_2^2 ds \right) = \infty$$

whenever $\mathcal{T} < \infty$.

Proof If we choose $R = M$ in Proposition 5.5 then (v_M, h_M, τ_M) is a local solution of the SRSW system (5.1), in the space $(\mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$. Consider the set of all stopping times τ such that (v, h, τ) is a local solution of the system (5.1). Denote by $\mathcal{T}$ the supremum of this set. Choose $(\tau_M)_M$ such that

$$\mathcal{T} = \lim_{M \rightarrow \infty} \tau_M.$$

Denote by (v_M, h_M) the local solution corresponding to the stopping time τ_M . Any such a solution is pathwise unique (see next section), therefore for any $N \geq M$ we have

$$v_N|_{[0, \tau_M)} = v_M|_{[0, \tau_M)}$$

That is, (v_N, h_M) is well-defined and the measure of the set

$$\bigcap_{M, N} \{v_N(t \wedge \tau_M \wedge \tau_N) = v_M(t \wedge \tau_M \wedge \tau_N), h_N(t \wedge \tau_M \wedge \tau_N) = h_M(t \wedge \tau_M \wedge \tau_N)\}$$

is equal to 1. Assume that $\mathcal{T} < \infty$. We have shown that the process (v, h) is continuous in time $(v, h) \in C([0, T]; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \times C([0, T]; \mathcal{W}^{1,2}(\mathbb{T}^2))$ therefore for any event from the set described above and for any $0 < t < \mathcal{T}$ we can define

$$\bar{\bar{v}}_t := \lim_{M \rightarrow \infty} v_{t \wedge \tau_M} = \lim_{t \nearrow \mathcal{T}} v_t$$

$$\bar{\bar{h}}_t := \lim_{M \rightarrow \infty} h_{t \wedge \tau_M} = \lim_{t \nearrow \mathcal{T}} h_t.$$

Then there exists $\tilde{\tau}_M$ such that $|\tilde{\tau}_M - \mathcal{T}| \leq \frac{1}{M}$ and

$$L := \sup_{s \in [0, \tilde{\tau}_M]} \left(\|\bar{\bar{v}}_s\|_{1,2}^2 + \|\bar{\bar{h}}_s\|_{1,2}^2 \right) + \int_0^{\tilde{\tau}_M} \|\Delta \bar{\bar{v}}_s\|_2^2 ds + \int_0^{\tilde{\tau}_M} \|\Delta \bar{\bar{h}}_s\|_2^2 ds \geq M. \quad (5.10)$$

Then

$$\begin{aligned} & \limsup_{t \nearrow \mathcal{T}} \left(\|\bar{\bar{v}}_t\|_{1,2}^2 + \|\bar{\bar{h}}_t\|_{1,2}^2 \right) + \limsup_{t \nearrow \mathcal{T}} \left(\int_0^t \|\Delta \bar{\bar{v}}_s\|_2^2 ds + \int_0^t \|\Delta \bar{\bar{h}}_s\|_2^2 ds \right) \\ & \geq \limsup_{M \rightarrow \infty} \left(\|\bar{\bar{v}}_{\tilde{\tau}_M}\|_{1,2}^2 + \|\bar{\bar{h}}_{\tilde{\tau}_M}\|_{1,2}^2 + \int_0^{\tilde{\tau}_M} \|\Delta \bar{\bar{v}}_s\|_2^2 ds + \int_0^{\tilde{\tau}_M} \|\Delta \bar{\bar{h}}_s\|_2^2 ds \right) = \infty. \end{aligned}$$

Therefore $\mathcal{T} < \infty$ implies

$$\limsup_{t \rightarrow \mathcal{T}} \left(\|v_t\|_{1,2}^2 + \|h_t\|_{1,2}^2 \right) + \limsup_{t \rightarrow \mathcal{T}} \left(\int_0^t \|\Delta v_s\|_2^2 ds + \int_0^t \|\Delta h_s\|_2^2 ds \right) = \infty.$$

For the second part of the proof, suppose that there exists $((v', h'), \mathcal{T}')$ such that $\mathcal{T}' > \mathcal{T}$ and $v'|_{[0, \tau)} = v|_{[0, \tau)}, h'|_{[0, \tau)} = h|_{[0, \tau)}$. Therefore $\mathcal{T} < \infty$ and by the continuity of (v', h') on $\{\mathcal{T}' > cT\}$

and the properties described above for $\mathcal{T}$ we deduce that

$$\limsup_{M \rightarrow \infty} L(\bar{v}, \tau_M) = \limsup_{M \rightarrow \infty} L(\bar{v}', \tau_M) = L(\bar{v}', \mathcal{T}) < \infty.$$

But on the other hand $\limsup_{M \rightarrow \infty} L(\bar{v}, \tau_M) = \infty$. We conclude that $((v, h), \mathcal{T})$ is a maximal solution.

5.4.2 PATHWISE UNIQUENESS FOR THE TRUNCATED SRSW SYSTEM

Proposition 5.7. *If a solution of the truncated SRSW system 5.9 exists, then it is pathwise unique.*

Proof Let $v^{R,1}$ and $v^{R,2}$ be two different local solutions of the truncated velocity equation, and $h^{R,1}$ and $h^{R,2}$ two different local solutions of the truncated height equation. We denote the corresponding differences by $\bar{v}^R := v^{R,1} - v^{R,2}$, $\bar{h}^R := h^{R,1} - h^{R,2}$. Then $\bar{v}^R$ and $\bar{h}^R$ satisfy the system:

$$\begin{aligned} d\bar{v}_t^R - \nu \Delta \bar{v}_t^R dt + f k \times \bar{v}_t^R dt + g \nabla \bar{v}_t^R dt + f_R(v^{R,1}, h^{R,1}) v_t^{R,1} \cdot \nabla v_t^{R,1} dt - f_R(v^{R,2}, h^{R,2}) v_t^{R,2} \cdot \nabla v_t^{R,2} dt \\ + \sum_{i=1}^{\infty} (\mathcal{L}_i + \mathcal{A}_i) \bar{v}_t^R dW_t^i - \frac{1}{2} \sum_{i=1}^{\infty} (\mathcal{L}_i + \mathcal{A}_i)^2 \bar{v}_t^R dt = 0. \end{aligned}$$

$$\begin{aligned} d\bar{h}_t^R - \eta \Delta \bar{h}_t^R dt + f_R(v^{R,1}, h^{R,1}) \nabla \cdot (h_t^{R,1} v_t^{R,1}) dt - f_R(v^{R,2}, h^{R,2}) \nabla \cdot (h_t^{R,2} v_t^{R,2}) dt \\ + \sum_{i=1}^{\infty} \mathcal{L}_i \bar{h}_t^R dW_t^i - \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_i^2 \bar{h}_t^R dt = 0 \end{aligned}$$

We apply the Itô formula and obtain

$$\begin{aligned} d\|\bar{v}_t^R\|_{1,2}^2 + d\|\bar{h}_t^R\|_{1,2}^2 + 2\nu|\Delta \bar{v}_t^R|^2 dt + 2\eta\|\Delta \bar{h}_t^R\|^2 dt = -2\langle \Delta \bar{v}_t^R, f k \times \bar{v}_t^R + \nabla \bar{h}_t^R \rangle dt \\ - 2\langle \Delta \bar{v}_t^R, f_R(v^{R,2}) v_t^{R,2} \cdot \nabla v_t^{R,2} - f_R(v^{R,1}) v_t^{R,1} \cdot \nabla v_t^{R,1} \rangle dt \\ - 2\langle \Delta \bar{h}_t^R, f_R(v^{R,2}, h^{R,2}) \nabla \cdot (v_t^{R,2} h_t^{R,2}) - f_R(v^{R,1}, h^{R,1}) \nabla \cdot (v_t^{R,1} h_t^{R,1}) \rangle dt \\ + \sum_{i=1}^{\infty} \langle (\mathcal{L}_i + \mathcal{A}_i) \bar{v}_t^R, (\mathcal{L}_i + \mathcal{A}_i) \bar{v}_t^R \rangle dt + \sum_{i=1}^{\infty} \langle \mathcal{L}_i \bar{h}_t^R, \mathcal{L}_i \bar{h}_t^R \rangle dt \\ + \sum_{i=1}^{\infty} \langle \Delta \bar{v}_t^R, (\mathcal{L}_i + \mathcal{A}_i)^2 \bar{v}_t^R \rangle dt + \sum_{i=1}^{\infty} \langle \Delta \bar{h}_t^R, \mathcal{L}_i^2 \bar{h}_t^R \rangle dt \\ + 2 \sum_{i=1}^{\infty} \langle \Delta \bar{v}_t^R, (\mathcal{L}_i + \mathcal{A}_i) \bar{v}_t^R \rangle dW_t^i + 2 \sum_{i=1}^{\infty} \langle \Delta \bar{h}_t^R, \mathcal{L}_i \bar{h}_t^R \rangle dW_t^i. \end{aligned}$$

We control each term as follows:

- $|\langle \Delta \bar{v}_t^R, f_R(v^{R,2})v_t^{R,2} \cdot \nabla v_t^{R,2} - f_R(v^{R,1})v_t^{R,1} \cdot \nabla v_t^{R,1} \rangle| \leq \frac{\nu}{5} \|\Delta \bar{v}_t^R\|_2^2 + C \|\bar{v}_t^R\|_{1,2}^2$
- $|\langle \Delta \bar{v}_t^R, \nabla \bar{h}_t^R \rangle| \leq \frac{\nu}{5} \|\Delta \bar{v}_t^R\|_2^2 + C \|\bar{h}_t^R\|_{1,2}^2$
- the first nonlinear term

$$\begin{aligned} & |\langle \Delta \bar{v}_t^R, f_R(v^{R,2})v_t^{R,2} \cdot \nabla v_t^{R,2} - f_R(v^{R,1})v_t^{R,1} \cdot \nabla v_t^{R,1} \rangle| \\ & \leq |\langle \Delta \bar{v}_t^R, (f_R(v^{R,1}) - f_R(v^{R,2}))v_t^{R,1} \cdot \nabla \bar{v}_t^R \rangle| + |\langle \Delta \bar{v}_t^R, v_t^{R,1} \cdot \nabla v_t^{R,1} - v_t^{R,2} \cdot \nabla v_t^{R,2} \rangle| \end{aligned}$$

Now for the first term in this sum we have

$$\begin{aligned} & |\langle \Delta \bar{v}_t^R, (f_R(v^{R,1}) - f_R(v^{R,2}))v_t^{R,1} \cdot \nabla \bar{v}_t^R \rangle| \leq \|\bar{v}^R\|_\infty \|\Delta \bar{v}_t^R\|_2 \|v_t^{R,1}\|_\infty \|\nabla v_t^{R,1}\|_2 \\ & \leq \|\bar{v}^R\|_\infty \|\Delta \bar{v}_t^R\|_2 \|\Delta v_t^{R,1}\|_2^{1/2} \|\nabla v_t^{R,1}\|_2^{3/2} \\ & \leq \frac{\nu}{5} \|\Delta \bar{v}_t^R\|_2^2 + C \|\bar{v}^R\|_\infty \|\Delta v_t^{R,1}\|_2^2 \|v_t^{R,1}\|_{1,2}^2 \end{aligned}$$

For the second term of the above sum we have

$$\begin{aligned} & |\langle \Delta \bar{v}_t^R, v_t^{R,1} \cdot \nabla v_t^{R,1} - v_t^{R,2} \cdot \nabla v_t^{R,2} \rangle| \leq |\langle \Delta \bar{v}_t^R, \bar{v}_t^R \cdot \nabla v_t^{R,1} - v_t^{R,2} \cdot \nabla \bar{v}_t^R \rangle| \\ & \leq \|\Delta \bar{v}_t^R\|_2 (\|\bar{v}_t^R\|_4 \|\nabla v_t^{R,1}\|_4 + \|v_t^{R,2}\|_4 \|\nabla \bar{v}_t^R\|_4) \\ & \leq \|\Delta \bar{v}_t^R\|_2 (\|\bar{v}_t^R\|_2^{1/2} \|\nabla \bar{v}_t^R\|_2^{1/2} \|\nabla v_t^{R,1}\|_2^{1/2} \|\Delta v_t^{R,1}\|_2^{1/2} + \|v_t^{R,2}\|_2^{1/2} \|\Delta v_t^{R,2}\|_2^{1/2} \|\nabla \bar{v}_t^R\|_2) \\ & \leq \frac{\nu}{5} \|\Delta \bar{v}_t^R\|_2^2 + C \|\bar{v}_t^R\|_{1,2}^2 (\|v_t^{R,1}\|_{1,2}^2 \|\Delta v_t^{R,1}\|_2^2 + \|v_t^{R,2}\|_{1,2}^2 + \|\Delta v_t^{R,2}\|_2^2) \end{aligned}$$

- the second nonlinear term

$$\begin{aligned} & |\langle \Delta \bar{h}_t^R, f_R(v^{R,2}, h^{R,2}) \nabla \cdot (v_t^{R,2} h_t^{R,2}) - f_R(v^{R,1}, h^{R,1}) \nabla \cdot (v_t^{R,1} h_t^{R,1}) \rangle| \\ & \leq |\langle \Delta \bar{h}_t^R, (f_R(v^{R,1}, h^{R,1}) - f_R(v^{R,2}, h^{R,2})) \nabla \cdot (v_t^{R,2} h_t^{R,2}) \rangle| \\ & \quad + |\langle \Delta \bar{h}_t^R, f_R(v^{R,2}, h^{R,2}) (\nabla \cdot (v_t^{R,2} h_t^{R,2}) - \nabla \cdot (v_t^{R,1} h_t^{R,1})) \rangle| \\ & \leq \eta \|\Delta \bar{h}_t^R\|_2^2 + \frac{\nu}{5} \|\Delta \bar{v}_t^R\|_2^2 \\ & \quad + (\|\bar{v}_t^R\|_\infty + \|\bar{h}_t^R\|_\infty) (\|v_t^{R,1}\|_{1,2}^2 \|\Delta v_t^{R,1}\|_2^2 + \|v_t^{R,2}\|_{1,2}^2 \|\Delta v_t^{R,2}\|_2^2) \\ & \quad + (\|\bar{v}_t^R\|_\infty + \|\bar{h}_t^R\|_\infty) (\|h_t^{R,1}\|_{1,2}^2 \|\Delta h_t^{R,1}\|_2^2 + \|h_t^{R,2}\|_{1,2}^2 \|\Delta h_t^{R,2}\|_2^2) \end{aligned}$$

as before

- the rest of the terms are similar to the ones from the untruncated equation, therefore their control is similar to the control obtained for the a priori estimates in Proposition 5.17.

Define the stopping time

$$\tau_q := \inf_{s \geq 0} \left(\int_0^s \|v_t^{R,1}\|_{1,2}^2 \|\Delta v_t^{R,1}\|_2^2 + \|v_t^{R,2}\|_{1,2}^2 \|\Delta v_t^{R,2}\|_2^2 + \|h_t^{R,1}\|_{1,2}^2 \|\Delta h_t^{R,1}\|_2^2 + \|h_t^{R,2}\|_{1,2}^2 \|\Delta h_t^{R,2}\|_2^2 dt \geq q \right)$$

For $0 \leq t \leq T$ we obtain

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_q]} \|\bar{v}_s^R\|_{1,2}^2 + \sup_{s \in [0, t \wedge \tau_q]} \|\bar{h}_s^R\|_{1,2}^2 + 2\nu \int_0^{t \wedge \tau_q} \|\Delta \bar{v}_s^R\|_2^2 ds + 2\eta \int_0^{t \wedge \tau_q} \|\Delta \bar{h}_s^R\|_2^2 ds \right] \\ & \leq \mathbb{E} [\|\bar{v}_0^R\|_{1,2}^2 + \|\bar{h}_0^R\|_{1,2}^2] \\ & + C\mathbb{E} \left[\int_0^{t \wedge \tau_q} (\|\bar{v}_s^R\|_\infty + \|\bar{h}_s^R\|_\infty) (2 + \|v_s^{R,1}\|_{1,2}^2 \|\Delta v_s^{R,1}\|_2^2 + \|v_s^{R,2}\|_{1,2}^2 \|\Delta v_s^{R,2}\|_2^2) ds \right] \\ & + C\mathbb{E} \left[\int_0^{t \wedge \tau_q} (\|\bar{v}_s^R\|_\infty + \|\bar{h}_s^R\|_\infty) (\|h_s^{R,1}\|_{1,2}^2 \|\Delta h_s^{R,1}\|_2^2 + \|h_s^{R,2}\|_{1,2}^2 \|\Delta h_s^{R,2}\|_2^2) ds \right] \end{aligned}$$

By the stochastic Gronwall lemma we deduce that

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t \wedge \tau_q]} \|\bar{v}_s^R\|_{1,2}^2 + \sup_{s \in [0, t \wedge \tau_q]} \|\bar{h}_s^R\|_{1,2}^2 + 2\nu \int_0^{t \wedge \tau_q} \|\Delta \bar{v}_s^R\|_2^2 ds + 2\eta \int_0^{t \wedge \tau_q} \|\Delta \bar{h}_s^R\|_2^2 ds \right] \\ & \leq C\mathbb{E} [\|\bar{v}_0^R\|_{1,2}^2 + \|\bar{h}_0^R\|_{1,2}^2] = 0 \end{aligned}$$

and therefore the solution is unique.

5.4.3 GLOBAL EXISTENCE FOR THE TRUNCATED SRSW SYSTEM

Proposition 5.8. *The truncated SRSW system 5.9 admits a global pathwise solution (v^R, h^R) with paths in the space $L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \times L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2))$.*

The strategy for proving that the truncated system (5.4) has a solution is to construct an approximating system of processes that will converge in distribution to a solution of (5.4). This justifies the existence of a weak solution. Together with the pathwise uniqueness of the solution of this equation, we then deduce that strong/pathwise existence holds.

Recall that $(v_0, h_0) \in \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$. Let $(v_0^n, h_0^n)_n \in C^\infty(\mathbb{T}^2) \times C^\infty(\mathbb{T}^2)$ be a sequence such

that $(v_0^n, h_0^n) \xrightarrow{n \rightarrow \infty} (v_0, h_0)$ in $\mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)$. For any $t \geq 0$ we construct the sequence $(v_t^n, h_t^n)_{n \geq 0}$ with $v_t^0 := u_0^0, h_t^0 := h_0^0$, and for $n \geq 1$

$$\begin{aligned} dv_t^n + [f_R(u_t^{n-1})\mathcal{L}_{u_t^{n-1}}v_t^n + f\hat{z} \times u_t^n + f_R(h_t^{n-1})\nabla p^{n-1}]dt + \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i)v_t^n] \circ dW_t^i &= \nu \Delta v_t^n dt \\ dh_t^n + f_R(u_t^{n-1}, h_t^{n-1}) (h_t^{n-1}(\nabla \cdot u_t^n) + u_t^{n-1} \cdot \nabla h_t^n) dt + \sum_{i=1}^{\infty} [\nabla \cdot (\xi_i h_t^n)] \circ dW_t^i &= \delta \Delta h_t^n dt. \end{aligned} \quad (5.11)$$

The corresponding Itô form of the system (5.11) is ²

$$\begin{aligned} dv_t^n &= \nu \Delta v_t^n dt + P_t^{n-1,n}(v_t^n)dt - \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i)v_t^n] dW_t^i \\ dh_t^n &= \delta \Delta h_t^n dt + Q_t^{n-1,n}(h_t^n)dt - \sum_{i=1}^{\infty} [\nabla \cdot (\xi_i h_t^n)] dW_t^i \end{aligned} \quad (5.12)$$

where $P_t^{n-1,n}(v_t^n)$ and $Q_t^{n-1,n}(h_t^n)$ are defined, respectively, as follows (for $t \geq 0$):

$$\begin{aligned} P_t^{n-1,n}(v_t^n) &:= -f_R(u_t^{n-1})\mathcal{L}_{u_t^{n-1}}v_t^n - f\hat{z} \times u_t^n - f_R(h_t^{n-1})\nabla p_t^{n-1} + \frac{1}{2} \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i)^2 v_t^n] \\ Q_t^{n-1,n}(h_t^n) &:= -f_R(u_t^{n-1}, h_t^{n-1}) (h_t^{n-1}(\nabla \cdot u_t^n) + u_t^{n-1} \cdot \nabla h_t^n) + \frac{1}{2} \sum_{i=1}^{\infty} \mathcal{L}_i^2 h_t^n \end{aligned}$$

Theorem 5.9. *If $(v_0^n, h_0^n) \in C^\infty(\mathbb{T}^2) \times C^\infty(\mathbb{T}^2)$ then the two-dimensional stochastic rotating shallow water system (5.12) admits a unique global $\mathcal{F}_t$ -adapted solution $(v^n, h^n) = \{(v_t^n, h_t^n), t \in [0, \infty)\}$ in the space $C([0, \infty); C^\infty(\mathbb{T}^2))$.*

Proposition 5.10. *The laws of the family of solutions $(v^n, h^n)_n$ is relatively compact in the space of probability measures over $D([0, T], L^2(\mathbb{T}^2))$ for any $T \geq 0$.*

Proposition 5.10 is proven in the next section.

Proof of Proposition 5.8 We know that $\lim_{n \rightarrow \infty} v_0^n = v_0$ $\mathbb{P}$ -a.s. and $\lim_{n \rightarrow \infty} h_0^n = h_0$ $\mathbb{P}$ -a.s. in $\mathcal{W}^{1,2}(\mathbb{T}^2)$. Also, we show in Section 5.5 that the family of approximate solutions $(v^n, h^n)_n$ is relatively compact in the phase space $\mathcal{E}$. Therefore we can extract a subsequence (v^{n_j}, h^{n_j}) which is convergent in distribution in the phase space $\mathcal{E}$. Let $(\tilde{v}, \tilde{h})$ be the corresponding limit

²The stochastic Itô integral is understood here in the usual sense, see [40].

and μ its distribution. Note that $(\tilde{v}, \tilde{h})$ is a probabilistically weak solution of the truncated SRSW system. By the Skorokhod representation theorem there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a sequence of processes $((\tilde{v}^n, \tilde{h}^n), (\tilde{W}^i)_i)_{n=1}^\infty$ with the same distribution as $((v^n, h^n), (W^i)_i)_{n=1}^\infty$, and a limiting process $(\tilde{v}^R, \tilde{h}^R, (\tilde{W}^i)_i)$ such that $((\tilde{v}^n, \tilde{h}^n), (\tilde{W}^i)_i)_{n=1}^\infty$ converges (when $n \rightarrow \infty$) $\tilde{\mathbb{P}}$ -a.s. to $(\tilde{v}^R, \tilde{h}^R, (\tilde{W}^i)_i)$ in the phase space $\mathcal{E}$. Then for any test function $\varphi \in C^\infty(\mathbb{T}^2)$ we can write

$$\begin{aligned} \langle \tilde{v}_t^n, \varphi \rangle &= \langle \tilde{v}_0^n, \varphi \rangle + \nu \int_0^t \langle \tilde{v}_s^n, \Delta \varphi \rangle ds - \int_0^t f_R(\tilde{u}_s^{n-1}) \langle \tilde{v}_s^n, \mathcal{L}_{\tilde{u}_s^{n-1}}^* \varphi \rangle ds - \int_0^t \langle \tilde{u}_s^n, f \hat{z} \times \varphi \rangle ds \\ &\quad - \int_0^t f_R(h_s^{n-1}) \langle \tilde{p}_s^{n-1}, \nabla \varphi \rangle ds + \frac{1}{2} \sum_{i=1}^\infty \int_0^t \langle \tilde{v}_s^n, (\mathcal{L}_i^* + \mathcal{A}_i^*)^2 \varphi \rangle ds \\ &\quad - \sum_{i=1}^\infty \int_0^t \langle \tilde{v}_s^n, (\mathcal{L}_i^* + \mathcal{A}_i^*) \varphi \rangle dW_s^i \end{aligned} \quad (5.13)$$

$$\begin{aligned} \langle \tilde{h}_t^n, \varphi \rangle &= \langle \tilde{h}_0^n, \varphi \rangle + \eta \int_0^t \langle \tilde{h}_s^n, \Delta \varphi \rangle ds - \int_0^t f_R(\tilde{h}_s^{n-1}) \langle \varphi, \tilde{h}_s^{n-1} \nabla \cdot \tilde{u}_s^n \rangle ds \\ &\quad + \int_0^t f_R(\tilde{u}_s^{n-1}) \langle \tilde{h}_s^n, \mathcal{L}_{\tilde{u}_s^{n-1}}^* \varphi \rangle ds + \frac{1}{2} \sum_{i=1}^\infty \int_0^t \langle \tilde{h}_s^n, (\mathcal{L}_i^*)^2 \varphi \rangle ds \\ &\quad - \sum_{i=1}^\infty \int_0^t \langle \varphi, \nabla \cdot (\xi_i \tilde{h}_s^n) \rangle dW_s^i \end{aligned} \quad (5.14)$$

Given the fact that $(\tilde{v}^n, \tilde{h}^n)$ and (v^n, h^n) satisfy the same equation, all the estimates established for (v^n, h^n) are also true for $\tilde{v}^n, \tilde{h}^n$. Thus, there exist two constants $\mathcal{C}_1(T, R)$ and $\mathcal{C}_2(T, R)$ such that

$$\sup_{n \geq 1} \tilde{\mathbb{E}} \left[\sup_{t \in [0, T]} \|\tilde{v}_t^n\|_{1,2}^2 + \sup_{t \in [0, T]} \|\tilde{h}_t^n\|_{1,2}^2 \right] \leq \mathcal{C}_1(T, R). \quad (5.15)$$

$$\sup_{n \geq 1} \tilde{\mathbb{E}} \left[\int_0^T \|\Delta \tilde{v}_t^n\|_2^2 dt + \int_0^T \|\Delta \tilde{h}_t^n\|_2^2 dt \right] \leq \mathcal{C}_2(T, R), \quad (5.16)$$

which ensures that the corresponding time integrals of the terms involved are uniformly bounded in $L^2(\tilde{\mathbb{P}})$. This will allow us to establish convergence $\tilde{\mathbb{P}}$ -a.s. and in $L^2(\tilde{\mathbb{P}})$. Since we prove (see Section 5.5) relative compactness in the space $\mathcal{E} = L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \cap C(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \times L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2)) \cap C(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2))$ it is also true that

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[\int_0^t \|\tilde{v}_s^n - \tilde{v}^R\|_2^2 ds \right] = 0$$

$$\lim_{n \rightarrow \infty} \tilde{\mathbb{E}} \left[\int_0^t \|\tilde{h}_s^n - \tilde{h}^R\|_2^2 ds \right] = 0.$$

Moreover, by Theorem 2.13 in Chapter 2 and using the initial assumptions from Section 2.6 we deduce that also the stochastic terms $\sum_{i=1}^{\infty} \int_0^t \langle \tilde{v}_s^n, (\mathcal{L}_i^* + \mathcal{A}_i^*) \varphi \rangle dW_s^i$ and $\sum_{i=1}^{\infty} \int_0^t \langle \varphi, \nabla \cdot (\xi_i \tilde{h}_s^n) \rangle dW_s^i$ converge, respectively, to $\sum_{i=1}^{\infty} \int_0^t \langle \tilde{v}_s^R, (\mathcal{L}_i^* + \mathcal{A}_i^*) \varphi \rangle dW_s^i$ and $\sum_{i=1}^{\infty} \int_0^t \langle \varphi, \nabla \cdot (\xi_i \tilde{h}_s^R) \rangle dW_s^i$ in distribution. Here we use the fact that $\nabla \cdot (\xi_i \tilde{h}_s^n) = \xi_i \cdot \nabla \tilde{h}_s^n + \tilde{h}_s^n \nabla \cdot \xi_i$ and consequently $|\langle \nabla \cdot (\xi_i \tilde{h}_s^n), \varphi \rangle| \leq |\langle \xi_i \cdot \nabla \tilde{h}_s^n, \varphi \rangle| + |\langle \tilde{h}_s^n \nabla \cdot \xi_i, \varphi \rangle| = 2|\langle \tilde{h}_s^n, \xi_i \cdot \nabla \varphi \rangle|$. By applying the Skorokhod representation theorem as before we deduce that the convergence holds almost surely on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ and also in $L^2(\tilde{\mathbb{P}})$. Now we can pass to the limit in (5.13) and (5.14). We first analyse the terms from the velocity equation:

- $\tilde{v}^n \xrightarrow[n \rightarrow \infty]{} \tilde{v}^R$ $\tilde{\mathbb{P}}$ -a.s. in $\mathcal{E}_v$ and φ is bounded, therefore $\langle \tilde{v}^n, \varphi \rangle \xrightarrow[n \rightarrow \infty]{} \langle \tilde{v}^R, \varphi \rangle$ and $\langle \tilde{v}_0^n, \varphi \rangle \xrightarrow[n \rightarrow \infty]{} \langle \tilde{v}_0^R, \varphi \rangle$ $\tilde{\mathbb{P}}$ -a.s. and in $L^2(\tilde{\mathbb{P}})$ for any $\varphi \in C^\infty(\mathbb{T})^2$.
- $\nu \int_0^t \langle \tilde{v}_s^n - \tilde{v}_s^R, \Delta \varphi \rangle ds \xrightarrow[n \rightarrow \infty]{} 0$ $\tilde{\mathbb{P}}$ -a.s. and in $L^2(\tilde{\mathbb{P}})$ since $|\langle \tilde{v}_s^n - \tilde{v}_s^R, \Delta \varphi \rangle| \leq \|\tilde{v}_s^n - \tilde{v}_s^R\|_2 \|\Delta \varphi\|_2$ and (5.15) holds.
- $\int_0^t \langle \mathcal{L}_{\tilde{u}_s^{n-1}} \tilde{v}_s^n - \mathcal{L}_{\tilde{u}_s^R} \tilde{v}_s^R, \varphi \rangle ds \xrightarrow[n \rightarrow \infty]{} 0$ $\tilde{\mathbb{P}}$ -a.s. and in $L^2(\tilde{\mathbb{P}})$ since

$$\begin{aligned} |\langle \mathcal{L}_{\tilde{u}_s^{n-1}} \tilde{v}_s^n - \mathcal{L}_{\tilde{u}_s^R} \tilde{v}_s^R, \varphi \rangle| &= |\langle (\tilde{u}_s^{n-1} - \tilde{u}_s^R) \cdot \nabla \tilde{v}_s^n, \varphi \rangle - \langle \tilde{u}_s^R (\nabla \tilde{v}_s^n - \nabla \tilde{v}_s^R), \varphi \rangle| \\ &\leq |\langle (\tilde{u}_s^{n-1} - \tilde{u}_s^R) \cdot \tilde{v}_s^n, \nabla \varphi \rangle| + |\langle \tilde{u}_s^R (\tilde{v}_s^n - \tilde{v}_s^R), \nabla \varphi \rangle| \\ &\leq \|\nabla \varphi \cdot \tilde{v}_s^n\|_2 \|\tilde{u}_s^{n-1} - \tilde{u}_s^R\|_2 + \|\nabla \varphi \cdot \tilde{u}_s^R\|_2 \|\tilde{v}_s^n - \tilde{v}_s^R\|_2 \xrightarrow[n \rightarrow \infty]{} 0 \end{aligned}$$

- $\int_0^t \langle f \hat{z} \times (\tilde{u}_s^n - \tilde{u}_s^R), \varphi \rangle ds \xrightarrow[n \rightarrow \infty]{} 0$ $\tilde{\mathbb{P}}$ -a.s. and in $L^2(\tilde{\mathbb{P}})$ since

$$|\langle f \hat{z} \times (\tilde{u}_s^n - \tilde{u}_s^R), \varphi \rangle| = |\langle \tilde{u}_s^n - \tilde{u}_s^R, f \hat{z} \times \varphi \rangle| \leq \|\varphi\|_{1,2} \|\tilde{u}_s^n - \tilde{u}_s^R\|_2 \xrightarrow[n \rightarrow \infty]{} 0$$

-
- $\int_0^t f_R(h_s^{n-1}) \langle \nabla \tilde{p}_s^{n-1} - \nabla \tilde{p}_s^R, \varphi \rangle ds \xrightarrow[n \rightarrow \infty]{} 0$ $\tilde{\mathbb{P}}$ -a.s. and in $L^2(\tilde{\mathbb{P}})$ since

$$\begin{aligned}
|\langle \nabla \tilde{p}_s^{n-1} - \nabla \tilde{p}_s^R, \varphi \rangle| &= |\langle \tilde{p}_s^{n-1} - \tilde{p}_s^R, \nabla \varphi \rangle| \\
&\leq \|\varphi\|_{1,2} \|\tilde{p}_s^{n-1} - \tilde{p}_s^R\|_2 \xrightarrow[n \rightarrow \infty]{} 0 \\
&\text{due to the fact that } \|\tilde{h}_s^{n-1} - \tilde{h}_s^R\|_2 \xrightarrow[n \rightarrow \infty]{} 0 \quad (\text{see below})
\end{aligned}$$

- $\sum_{i=1}^{\infty} \int_0^t \langle (\mathcal{L}_i + \mathcal{A}_i)^2 (\tilde{v}_s^n - \tilde{v}_s^R), \varphi \rangle ds \xrightarrow[n \rightarrow \infty]{} 0$ $\tilde{\mathbb{P}}$ -a.s. and in $L^2(\tilde{\mathbb{P}})$ since

$$\begin{aligned}
|\langle \mathcal{L}_i^2 \tilde{v}_s^n - \mathcal{L}_i^2 \tilde{v}_s^R, \varphi \rangle| &= |\langle \xi_i \cdot \nabla (\xi_i \cdot \nabla \tilde{v}_s^n) - \xi_i \cdot \nabla (\xi_i \cdot \nabla \tilde{v}_s^R), \varphi \rangle| \\
&= |\langle \xi_i \cdot \nabla \tilde{v}_s^n - \xi_i \cdot \nabla \tilde{v}_s^R, \xi_i \cdot \nabla \varphi \rangle| \\
&= |\langle \tilde{v}_s^n - \tilde{v}_s^R, \xi_i \cdot \nabla (\xi_i \cdot \nabla \varphi) \rangle| \\
&\leq \|\xi_i \cdot \nabla (\xi_i \cdot \nabla \varphi)\|_2 \|\tilde{v}_s^n - \tilde{v}_s^R\|_2 \xrightarrow[n \rightarrow \infty]{} 0
\end{aligned}$$

and

$$\begin{aligned}
|\langle \mathcal{A}_i^2 \tilde{v}_s^n - \mathcal{A}_i^2 \tilde{v}_s^R, \varphi \rangle| &= |\mathcal{A}_i^2 (\tilde{v}_s^n - \tilde{v}_s^R), \varphi| \\
&= |\langle ((\tilde{v}_s^{n,1} - \tilde{v}_s^{R,1}) \partial_1 \xi_i^1 + (\tilde{v}_s^{n,2} - \tilde{v}_s^{R,2}) \partial_1 \xi_i^2) \nabla \xi_i^1 + ((\tilde{v}_s^{n,1} - \tilde{v}_s^{R,1}) \partial_2 \xi_i^1 + (\tilde{v}_s^{n,2} - \tilde{v}_s^{R,2}) \partial_2 \xi_i^2) \nabla \xi_i^2, \varphi \rangle| \\
&\leq \|\tilde{v}_s^n - \tilde{v}_s^R\|_2 \|\xi_i \cdot \nabla \varphi\|_2 \xrightarrow[n \rightarrow \infty]{} 0
\end{aligned}$$

since $\|\xi_i \cdot \nabla \varphi\|_2$ is finite by the initial assumptions made on $(\xi_i)_i$.

In the height equation (5.14) the terms are similar to the ones analysed in Chapter 3, Proposition 3.11 (see Section 3.3.2) and they converge with exactly the same arguments, except for the following:

- $\int_0^t \langle \tilde{h}_s^{n-1} \nabla \cdot \tilde{u}_s^n - \tilde{h}_s^R \nabla \cdot \tilde{u}_s^R, \varphi \rangle ds \xrightarrow[n \rightarrow \infty]{} 0$ $\tilde{\mathbb{P}}$ -a.s. and in $L^2(\tilde{\mathbb{P}})$ since

$$\begin{aligned}
|\langle \tilde{h}_s^{n-1} \nabla \cdot \tilde{u}_s^n, \varphi \rangle| &= |\langle (\tilde{h}_s^{n-1} - \tilde{h}_s^R) \nabla \cdot \tilde{u}_s^n, \varphi \rangle - \langle \tilde{h}_s^R (\nabla \cdot \tilde{u}_s^n - \nabla \cdot \tilde{u}_s^R), \varphi \rangle| \\
&\leq |\langle (\tilde{h}_s^{n-1} - \tilde{h}_s^R) \cdot \tilde{u}_s^n, \nabla \varphi \rangle| + |\langle \tilde{h}_s^R (\tilde{u}_s^n - \tilde{u}_s^R), \nabla \varphi \rangle| \\
&\leq \|\nabla \varphi \cdot \tilde{u}_s^n\|_2 \|\tilde{h}_s^{n-1} - \tilde{h}_s^R\|_2 + \|\nabla \varphi \cdot \tilde{h}_s^R\|_2 \|\tilde{u}_s^n - \tilde{u}_s^R\|_2 \xrightarrow[n \rightarrow \infty]{} 0
\end{aligned}$$

We have constructed a weak solution of the SRSW system. In addition to this, in Section 5.4.2 we have shown that any such a solution is pathwise unique. We now apply the Yamada-Watanabe theorem ([99]) to conclude that the solution is actually strong in probabilistic sense.

5.4.4 A PRIORI ESTIMATES FOR THE SRSW SYSTEM

Proposition 5.11. *There exist two constants $\mathcal{C}^1$ and $\mathcal{C}^2$ such that*

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t \wedge \tau]} \|v_s\|_{1,2}^2 + \sup_{s \in [0, t \wedge \tau]} \|h_s\|_{1,2}^2 \right] &\leq \mathcal{C}^1 \\ \mathbb{E} \left[\nu \int_0^{t \wedge \tau} \|\Delta v_s\|_2^2 ds + \eta \int_0^{t \wedge \tau} \|\Delta h_s\|_2^2 ds \right] &\leq \mathcal{C}^2. \end{aligned} \quad (5.17)$$

Proof of proposition 5.11. We first consider the velocity equation. By the Itô formula one has

$$\begin{aligned} d\|\nabla v_t\|_2^2 + 2\nu\|\Delta v_t\|_2^2 dt &= 2 \left\langle \nabla v_t, \nabla (\mathcal{L}_{u_t} v_t + f \hat{z} \times u_t + \nabla p_t) + \frac{1}{2} \sum_{i=1}^{\infty} \nabla [(\mathcal{L}_i + \mathcal{A}_i)^2 v_t] \right\rangle dt \\ &+ 2 \sum_{i=1}^{\infty} \left\langle \nabla v_t, \nabla ((\mathcal{L}_i + \mathcal{A}_i) v_t) \right\rangle dW_t^i + \sum_{i=1}^{\infty} \left\langle \nabla ((\mathcal{L}_i + \mathcal{A}_i) v_t), \nabla ((\mathcal{L}_i + \mathcal{A}_i) v_t) \right\rangle dt. \end{aligned} \quad (5.18)$$

We first integrate equation 5.18 in time over $[0, s]$ for $0 \leq s \leq t \leq T$, and then we take the supremum in s over $[0, t]$ and the expected value with respect to the probability $\mathbb{P}$:

$$\begin{aligned} \mathbb{E} \left[\sup_{s \in [0, t]} \|\nabla v_s\|_2^2 + 2\nu \int_0^t \|\Delta v_r\|_2^2 dr \right] &\leq \mathbb{E} [\|v_0\|_{1,2}^2] + \mathbb{E} \left[\int_0^t \left(c_1(\nu) \|\Delta v_r\|_2^2 + \sup_{s \in [0, r]} c_2 \|v_s\|_{1,2}^6 \right) dr \right] \\ &+ \mathbb{E} \left[\int_0^t \left(\frac{\nu}{c_6} \|\Delta v_r\|_2^2 + \sup_{s \in [0, r]} \frac{1}{c_6 \nu} \|v_s\|_{1,2}^2 \right) dr \right] \\ &+ \mathbb{E} \left[\int_0^t \left(\frac{\nu}{c_5} \|\Delta v_r\|_2^2 + \sup_{s \in [0, r]} \frac{1}{c_5 \nu} \|p_s\|_{1,2}^2 \right) dr \right] \\ &+ \mathbb{E} \left[C \int_0^t \sup_{s \in [0, r]} \|v_s\|_{1,2}^2 dr \right] \\ &+ \mathbb{E} \left[2 \sup_{s \in [0, t]} \int_0^s \sum_{i=1}^{\infty} \langle \Delta v_r, (\mathcal{L}_i + \mathcal{A}_i) v_r \rangle dW_r^i \right] \end{aligned}$$

with

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s \sum_{i=1}^{\infty} \langle \Delta v_r, (\mathcal{L}_i + \mathcal{A}_i) v_r \rangle dW_r^i \right| \right] &\leq C \mathbb{E} \left[\int_0^t \sum_{i=1}^{\infty} \langle \Delta v_r, (\mathcal{L}_i + \mathcal{A}_i) v_r \rangle^2 dr \right]^{1/2} \\
&\leq C \mathbb{E} \left[\int_0^t \sum_{i=1}^{\infty} \|\Delta v_r\|_2^2 \|(\mathcal{L}_i + \mathcal{A}_i) v_r\|_2^2 dr \right]^{1/2} \\
&\leq C \mathbb{E} \left[\int_0^t \|\Delta v_r\|_2^2 \sum_{i=1}^{\infty} \|(\mathcal{L}_i + \mathcal{A}_i) v_r\|_2^2 dr \right] \\
&\leq C \mathbb{E} \left[\sup_{s \in [0, t]} \|v_s\|_{1,2}^2 \int_0^t \|\Delta v_r\|_2^2 dr \right] \\
&\leq \frac{1}{2} \mathbb{E} \left[\sup_{s \in [0, t]} \|v_s\|_{1,2}^2 \right] + \frac{1}{2} C \mathbb{E} \left[\int_0^t \|\Delta v_r\|_2^2 dr \right] \\
&\leq \frac{C^2}{\epsilon} \mathbb{E} \left[\sup_{s \in [0, t]} \|v_s\|_{1,2}^2 \right] + \epsilon \mathbb{E} \left[\int_0^t \|\Delta v_r\|_2^2 dr \right]
\end{aligned}$$

Remark 5.12. We need to have $\mathbb{E} [\|v_0\|_{1,2}^2 + \|h_0\|_{1,2}^2] < \infty$ in order to be able to apply Gronwall.

Now we consider the height equation. By the Itô formula one has

$$\begin{aligned}
d\|\nabla h_t\|_2^2 + 2\eta\|\Delta h_t\|_2^2 dt &= 2 \left\langle \nabla h_t, \nabla \left(\nabla \cdot (h_t u_t) \right) \right\rangle + \frac{1}{2} \sum_{i=1}^{\infty} \nabla (\mathcal{L}_i^2 h_t) \rangle dt \\
&\quad + 2 \sum_{i=1}^{\infty} \langle \nabla h_t, \nabla (\mathcal{L}_i h_t) \rangle dW_t^i + \sum_{i=1}^{\infty} \langle \nabla (\mathcal{L}_i h_t), \nabla (\mathcal{L}_i h_t) \rangle dt.
\end{aligned} \tag{5.19}$$

We integrate equation (5.19) in time over $[0, s]$ for $0 \leq s \leq t \leq T$, and then we take the supremum in s over $[0, t]$ and the expected value as above:

$$\begin{aligned}
&\mathbb{E} \left[\sup_{s \in [0, t]} \|\nabla h_s\|_2^2 + 2\eta \int_0^t \|\Delta h_s\|_2^2 ds \right] \leq \mathbb{E} [\|h_0\|_{1,2}^2] \\
&\quad + \mathbb{E} \left[\int_0^t \left(c_3(\eta) \|\Delta h_r\|_2^2 + c_4(\nu) \|\Delta v_r\|_2^2 + \sup_{s \in [0, r]} c_5(\|h_s\|_{1,2}^6 + \|v_s\|_{1,2}^6) \right) dr \right] \\
&\quad + \mathbb{E} \left[C \int_0^t \sup_{s \in [0, r]} \|h_s\|_{1,2}^2 dr \right] \\
&\quad + \mathbb{E} \left[2 \sup_{s \in [0, t]} \int_0^s \sum_{i=1}^{\infty} \langle \Delta h_r, \mathcal{L}_i h_r \rangle dW_r^i \right]
\end{aligned}$$

with

$$\mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s \sum_{i=1}^{\infty} \langle \Delta h_r, \mathcal{L}_i h_r \rangle dW_r^i \right| \right] \leq \frac{C^2}{\epsilon} \mathbb{E} \left[\sup_{s \in [0, t]} \|h_s\|_{1,2}^2 \right] + \epsilon \mathbb{E} \left[\int_0^t \|\Delta h_r\|_2^2 dr \right]$$

as above. Summing up

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t]} \|\nabla v_s\|_2^2 + \sup_{s \in [0, t]} \|\nabla h_s\|_2^2 + \nu \int_0^t \|\Delta v_r\|_2^2 dr + \eta \int_0^t \|\Delta h_s\|_2^2 ds \right] \leq \mathbb{E} [\|v_0\|_2^2 + \|h_0\|_2^2] \\ & + \mathbb{E} \left[\int_0^t \sup_{s \in [0, r]} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2)^3 + (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) dr \right] + \mathbb{E} \left[\int_0^t (\|v_r\|_{1,2}^2 + \|h_r\|_{1,2}^2) dr \right] \\ & \leq \mathbb{E} [\|v_0\|_2^2 + \|h_0\|_2^2] + C \mathbb{E} \left[\int_0^t \sup_{s \in [0, r]} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2 + 2) dr \right] \end{aligned}$$

Define

$$\tau := \inf_{r \geq 0} \left\{ \sup_{s \in [0, r]} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) > q \right\}.$$

$$C_1 := \mathbb{E} [\|v_0\|_2^2 + \|h_0\|_2^2]$$

Note that τ is indeed a stopping time: the stochastic process (v, h) is càdlàg in L^2 and the filtration is right-continuous. Then

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t \wedge \tau]} \|\nabla v_s\|_2^2 + \sup_{s \in [0, t \wedge \tau]} \|\nabla h_s\|_2^2 + \nu \int_0^{t \wedge \tau} \|\Delta v_r\|_2^2 dr + \eta \int_0^{t \wedge \tau} \|\Delta h_s\|_2^2 ds \right] \leq C_1 \\ & + C(q^2 + 2) \mathbb{E} \left[\int_0^{t \wedge \tau} \sup_{s \in [0, r]} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) dr \right] \end{aligned}$$

that is

$$\mathbb{E} \left[\sup_{s \in [0, t \wedge \tau]} \|v_s\|_{1,2}^2 + \sup_{s \in [0, t \wedge \tau]} \|h_s\|_{1,2}^2 \right] \leq C_1 + C(q^2 + 2) \int_0^{t \wedge \tau} \mathbb{E} \left[\sup_{s \in [0, r]} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) \right] dr$$

therefore by the deterministic Gronwall lemma for $z_t := \mathbb{E} \left[\sup_{s \in [0, t \wedge \tau]} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) \right]$ we have

$$\mathbb{E} \left[\sup_{s \in [0, t \wedge \tau]} \|v_s\|_{1,2}^2 + \sup_{s \in [0, t \wedge \tau]} \|h_s\|_{1,2}^2 \right] \leq C_1 + C(q^2 + 2) \int_0^t \mathbb{E} \left[\sup_{s \in [0, r \wedge \tau]} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) \right] dr$$

and

$$\mathbb{E} \left[\sup_{s \in [0, t \wedge \tau]} \|v_s\|_{1,2}^2 + \sup_{s \in [0, t \wedge \tau]} \|h_s\|_{1,2}^2 \right] \leq C_1 e^{C(q^2+2)t} \quad (5.20)$$

So we have so far

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t \wedge \tau]} \|v_s\|_{1,2}^2 + \sup_{s \in [0, t \wedge \tau]} \|h_s\|_{1,2}^2 + \nu \int_0^{t \wedge \tau} \|\Delta v_r\|_2^2 dr + \eta \int_0^{t \wedge \tau} \|\Delta h_s\|_2^2 ds \right] \\ & \leq C_1 + C(q^2 + 2) \int_0^t \mathbb{E} \left[\sup_{s \in [0, r \wedge \tau]} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) \right] dr \end{aligned}$$

and we can conclude that

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t \wedge \tau]} \|v_s\|_{1,2}^2 + \sup_{s \in [0, t \wedge \tau]} \|h_s\|_{1,2}^2 + \nu \int_0^{t \wedge \tau} \|\Delta v_r\|_2^2 dr + \eta \int_0^{t \wedge \tau} \|\Delta h_s\|_2^2 ds \right] \\ & \leq C_1 + C_1 e^{C(q^2+2)t} \\ & \leq q \quad \text{when } t \leq \frac{C_1}{C(2+q^2)} \ln \frac{q-C_1}{C_1} \end{aligned}$$

Therefore

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|\nabla v_s\|_2^2 + \sup_{s \in [0, t]} \|\nabla h_s\|_2^2 + \nu \int_0^{t \wedge \tau} \|\Delta v_r\|_2^2 dr + \eta \int_0^{t \wedge \tau} \|\Delta h_s\|_2^2 ds \right] \leq q$$

when t is as above.

Remark 5.13. Note that we can also treat the above equation as a separable differential equation

$$\frac{dy_t}{dt} \leq C_1 + C(q^2 + 2)y_t$$

with

$$y_t := \mathbb{E} \left[\int_0^{t \wedge \tau} \sup_{s \in [0, r]} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) dr \right]$$

Then

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t]} \|\nabla v_s\|_2^2 + \sup_{s \in [0, t]} \|\nabla h_s\|_2^2 + \nu \int_0^{t \wedge \tau} \|\Delta v_r\|_2^2 dr + \eta \int_0^{t \wedge \tau} \|\Delta h_s\|_2^2 ds \right] \\ & \leq C_1 + \frac{C_1}{C(q^2 + 2)} (e^{C(q^2+2)t} - 1) \\ & \leq C_1 + C_1 e^{C(q^2+2)t} \\ & \leq q \quad \text{when } t \leq \frac{C_1}{C(2+q^2)} \ln \frac{q-C_1}{C_1}. \end{aligned}$$

Lemma 5.14. *The following statements are true:*

a. *There exist some constants c_1, c_2 such that*

$$|\langle \nabla v_t, \nabla(\mathcal{L}_{u_t} v_t) \rangle| \leq c_1(\nu) \|\Delta v_t\|_2^2 + c_2 \|v_t\|_{1,2}^6$$

b. *There exist some constants c_3, c_4, c_5 such that*

$$|\langle \Delta h_t, \nabla \cdot (h v_t) \rangle| \leq c_3(\eta) \|\Delta h_t\|_2^2 + c_4(\nu) \|\Delta v_t\|_2^2 + c_5(\|h_t\|_{1,2}^6 + \|v_t\|_{1,2}^6)$$

Proof a. Using Young's inequality and the fact that $v = \epsilon u + \mathcal{R}$:

$$\begin{aligned} |\langle \nabla v_t, \nabla(\mathcal{L}_{u_t} v_t) \rangle| &= |\langle \Delta v_t, (\mathcal{L}_{u_t} v_t) \rangle| \leq \|\Delta v_t\|_2 \|u_t\|_\infty \|\nabla v_t\|_2 \\ &\leq \|\Delta u_t\|_2^{1/2} \|u_t\|_{1,2}^{1/2} \|\Delta v_t\|_2 \|\nabla v_t\|_2 \\ &\leq \|\Delta v_t\|_2^{1/2} \|v_t\|_{1,2}^{1/2} \|\Delta v_t\|_2 \|\nabla v_t\|_2 \\ &\leq \|\Delta v_t\|_2^{3/2} \|v_t\|_{1,2}^{3/2} \\ &\leq \frac{\nu}{c} \|\Delta v_t\|_2^2 + C \|v_t\|_{1,2}^6 \end{aligned}$$

b.

$$\begin{aligned} |\langle \Delta h_t, \nabla \cdot (h v_t) \rangle| &= |\langle \Delta h_t, h_t \nabla \cdot v_t + v_t \nabla h_t \rangle| \leq \|\Delta h_t\|_2 \|h_t \nabla v_t\|_2 + \|\Delta h_t\|_2 \|v_t \nabla \cdot h_t\|_2 \\ &\leq \|\Delta h_t\|_2 \|h_t\|_4 \|\nabla v_t\|_4 + \|\Delta h_t\|_2 \|v_t\|_4 \|\nabla h_t\|_4 \\ &\leq \|\Delta h_t\|_2 \|h_t\|_{1,2} \|\nabla v_t\|_2^{1/2} \|\Delta v_t\|_2^{1/2} + \|\Delta h_t\|_2 \|v_t\|_{1,2} \|\nabla h_t\|_2^{1/2} \|\Delta h_t\|_2^{1/2} \\ &\leq \|\Delta h_t\|_2 \|h_t\|_{1,2} \|\nabla v_t\|_2^{1/2} \|\Delta v_t\|_2^{1/2} + \|\Delta h_t\|_2^{3/2} \|v_t\|_{1,2} \|\nabla h_t\|_2^{1/2} \end{aligned}$$

We have used Ladyzhenskaya's inequality ($n = 2$): $\|f\|_4 \leq C \|f\|_2^{1/2} \|\nabla f\|_2^{1/2}$

Proposition 5.15. *The stopping time τ is almost surely strictly positive.*

Proof We have seen that

$$\begin{aligned} &\sup_{t \in [0, \tau \wedge \ell]} \|v_t\|_{1,2}^2 + \sup_{t \in [0, \tau \wedge \ell]} \|h_t\|_{1,2}^2 + \nu \int_0^{\tau \wedge \ell} \|\Delta v_s\|_2^2 ds + \eta \int_0^{\tau \wedge \ell} \|\Delta h_s\|_2^2 ds \\ &\leq \|v_0\|_2^2 + \|h_0\|_2^2 + C \sup_{t \in [0, \tau \wedge \ell]} \left(\int_0^t (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2 + 2) ds \right) \end{aligned}$$

so

$$\begin{aligned}
& \sup_{t \in [0, \tau \wedge \ell]} \|v_t\|_{1,2}^2 + \sup_{s \in [0, \tau \wedge \ell]} \|h_t\|_{1,2}^2 - \|v_0\|_2^2 + \|h_0\|_2^2 \\
& \leq C \sup_{t \in [0, \tau \wedge \ell]} \left(\int_0^t (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2 + 2) ds \right) \\
& \leq C \int_0^{\tau \wedge \ell} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2 + 2) ds
\end{aligned}$$

and then

$$\begin{aligned}
& \mathbb{P} \left(\sup_{t \in [0, \ell \wedge \tau]} \|v_t\|_{1,2}^2 + \sup_{t \in [0, \ell \wedge \tau]} \|h_t\|_{1,2}^2 - \|v_0\|_2^2 + \|h_0\|_2^2 > 1 \right) \\
& \leq C \mathbb{P} \left(\int_0^{\tau \wedge \ell} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2 + 2) ds > 1 \right) \\
& \leq C \mathbb{E} \left[\int_0^{\tau \wedge \ell} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2 + 2) ds \right] \\
& \leq C \mathbb{E} \left[\ell \sup_{s \in [0, \tau \wedge \ell]} (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2) (\|v_s\|_{1,2}^2 + \|h_s\|_{1,2}^2 + 2) \right] \\
& \leq C \ell \mathcal{C}^1 (\mathcal{C}^1 + 2) \\
& \leq \tilde{C} \ell
\end{aligned}$$

Then we can write

$$\begin{aligned}
\mathbb{P}(\tau = 0) &= \mathbb{P} \left(\bigcap_{\ell} \{\tau < \ell\} \right) = \lim_{\ell \rightarrow 0} \mathbb{P}(\tau < \ell) \\
&\leq \lim_{\ell \rightarrow 0} \mathbb{P} \left(\sup_{t \in [0, \ell \wedge \tau]} \|v_t\|_{1,2}^2 + \sup_{t \in [0, \ell \wedge \tau]} \|h_t\|_{1,2}^2 - \|v_0\|_2^2 + \|h_0\|_2^2 > 1 \right) \\
&\leq \lim_{\ell \rightarrow 0} C \ell = 0
\end{aligned}$$

to conclude that

$$\mathbb{P}(\tau > 0) = 1.$$

5.5 ANALYTICAL PROPERTIES OF THE APPROXIMATING SYSTEM

5.5.1 RELATIVE COMPACTNESS

We define the following processes, to shorten the notation:

$$\begin{aligned} X_t^{v^n} &:= v_0^n + \int_0^t (\nu \Delta v_s^n + P_s^{n-1,n}(v_s^n)) ds \\ Y_t^{v^n} &:= \int_0^t \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i)v_s^n] dW_s^i \\ X_t^{h^n} &:= h_0^n + \int_0^t (\eta \Delta h_s^n + Q_s^{n-1,n}(h_s^n)) ds \\ Y_t^{h^n} &:= \int_0^t \sum_{i=1}^{\infty} [\nabla \cdot (\xi_i h_s^n)] dW_s^i \end{aligned}$$

Lemma 5.16. *Let $\beta \in [0, \frac{1}{2})$ and $p \in [2, \infty)$. There exists a positive constant $\mathcal{C}$ such that*

$$\begin{aligned} \mathbb{E} \left[\|X_t^{v^n}\|_{\mathcal{W}^{1,2}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^2 \right] &\leq \mathcal{C} \\ \mathbb{E} \left[\|X_t^{h^n}\|_{\mathcal{W}^{1,2}(0,T;L^2(\mathbb{T}^2))}^2 \right] &\leq \mathcal{C} \\ \mathbb{E} \left[\|Y_t^{v^n}\|_{\mathcal{W}^{\beta,p}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^p \right] &\leq \mathcal{C} \\ \mathbb{E} \left[\|Y_t^{h^n}\|_{\mathcal{W}^{\beta,p}(0,T;L^2(\mathbb{T}^2))}^p \right] &\leq \mathcal{C}. \end{aligned}$$

The following probability measures can be defined on the path space

$$m_v^n(A) := \mathbb{P}(v^n \in A)$$

$$m_h^n(A) := \mathbb{P}(h^n \in A)$$

$$m_W(A) := \mathbb{P}(W \in A)$$

for any measurable set A .

$$m^n := m_v^n \times m_h^n \times m_W,$$

the path space being given by

$$\mathcal{E} := \mathcal{E}_v \times \mathcal{E}_h \times \mathcal{E}_W$$

with

$$\begin{aligned}\mathcal{E}_v &= L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \cap C(0, T; \bar{\mathcal{W}}^{1,2}(\mathbb{T}^2) \times \bar{\mathcal{W}}^{1,2}(\mathbb{T}^2)) \\ \mathcal{E}_h &= L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2)) \cap C(0, T; \bar{\mathcal{W}}^{1,2}(\mathbb{T}^2))\end{aligned}$$

Remark 5.17. Note that $\mathcal{E}_W = C(0, T; \mathfrak{U}_0)$ ($\mathfrak{U}_0$ is defined in Chapter 2).

Proposition 5.18. The family of measures $(m^n)_n$ is relatively compact in the original phase space $\mathcal{E}$.

Remark 5.19. By Prokhorov's theorem the family of measures $(m^n)_n$ is relatively compact, therefore there exists a convergent subsequence $(m^{n_i})_{n_i}$.

Proof of Proposition 5.18 ³ We will first show that the sequence $(m^n)_n$ is tight. Then by Prokhorov's theorem (see Chapter 2, Theorem 2.11), $(m^n)_n$ is also relatively compact. In order to prove tightness, we need to show the existence of a compact set B_ϵ in $\mathcal{E}$ such that

$$m^n(B_\epsilon^C) \leq \epsilon.$$

Given the structure of the path space, we will show that B_ϵ is of the following form:

$$B_\epsilon := B_\epsilon^v \times B_\epsilon^h \times B_\epsilon^W \times B_\epsilon^W$$

with

$$m_v^n((B_\epsilon^v)^C) \leq \frac{\epsilon}{4}, \quad m_h^n((B_\epsilon^h)^C) \leq \frac{\epsilon}{4}, \quad m_W^n((B_\epsilon^W)^C) \leq \frac{\epsilon}{4}.$$

The sequence (m_W^n) is constant, hence also relatively compact. Then we can apply Prokhorov's theorem directly to conclude that the set B_ϵ^W exists as requested. We define the following sets

$$D_1 := \{f \in L^2(0, T; \mathcal{D}(-\Delta)) \cap \mathcal{W}^{\beta,2}(0, T; L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2)) \text{ such that}$$

$$\|f\|_{L^2(0, T; \mathcal{D}(-\Delta))}^2 + \|f\|_{\mathcal{W}^{\beta,2}(0, T; L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^2 \leq R^2\}$$

$$D_2 := \{f \in L^2(0, T; \mathcal{D}(-\Delta)) \cap \mathcal{W}^{\beta,2}(0, T; L^2(\mathbb{T}^2)) : \|f\|_{L^2(0, T; \mathcal{D}(-\Delta))}^2 + \|f\|_{\mathcal{W}^{\beta,2}(0, T; L^2(\mathbb{T}^2))}^2 \leq R^2\}$$

Using Theorem 2.17 from Chapter 2 we deduce that D_1 is compact in $L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2))$ and D_2 is compact in $L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2))$, as a consequence of the following compact

³The arguments are close to the arguments used in Lemma 4.4. from [75], adapted to our case.

embeddings:

$$L^2(0, T; \mathcal{D}(-\Delta)) \cap \mathcal{W}^{\beta,2}(0, T; L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2)) \hookrightarrow L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2))$$

$$L^2(0, T; \mathcal{D}(-\Delta)) \cap \mathcal{W}^{\beta,2}(0, T; L^2(\mathbb{T}^2)) \hookrightarrow L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2))$$

In order to show compactness in the spaces $L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2))$ and $L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2))$ respectively, it suffices to consider the dynamics generated by v^n in D_1 and the dynamics generated by h^n in D_2 . However, for proving compactness within the spaces $C(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2))$ and $C(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2))$ we will need to analyse separately the dynamics generated by the drift and the one driven by the Brownian motion. This is due to the natural lack of regularity of the Brownian motion, which needs to be addressed carefully in this setting. Therefore

$$\begin{aligned} m_v^n(D_1^C) &= \mathbb{P} \left(\|v_t^n\|_{L^2(0,T;\mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2))} + \|v_t^n\|_{\mathcal{W}^{\beta,2}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))} \geq R^2 \right) \\ &\leq \frac{2}{R^2} \mathbb{E} \left(\|v_t^n\|_{L^2(0,T;\mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2))} + \|v_t^n\|_{\mathcal{W}^{\beta,2}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))} \right) \\ &\leq \frac{C}{R^2} \end{aligned}$$

due to Lemma 5.16. Similarly

$$\begin{aligned} m_h^n(D_2^C) &= \mathbb{P} \left(\|v_t^n\|_{L^2(0,T;\mathcal{W}^{1,2}(\mathbb{T}^2))} + \|v_t^n\|_{\mathcal{W}^{\beta,2}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))} \geq R^2 \right) \\ &\leq \frac{2}{R^2} \mathbb{E} \left[\|v_t^n\|_{L^2(0,T;\mathcal{W}^{1,2}(\mathbb{T}^2))} + \|v_t^n\|_{\mathcal{W}^{\beta,2}(0,T;L^2(\mathbb{T}^2))} \right] \\ &\leq \frac{C}{R^2}. \end{aligned}$$

Now we intend to show that the compact embeddings are true also within the space of time continuous functions. We know from Theorem 2.17 in Chapter 2 that the embeddings

$$\mathcal{W}^{1,2}(0, T; L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2)) \hookrightarrow C(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2))$$

$$\mathcal{W}^{1,2}(0, T; L^2(\mathbb{T}^2)) \hookrightarrow C(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2))$$

are compact. Thus, the sum D_3 of a ball of radius R in $\mathcal{W}^{1,2}(0, T; L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))$ and a ball of radius R in $\mathcal{W}^{\beta,p}(0, T; L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))$ is compact in $C(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2))$. With exactly the same arguments, the sum D_4 of a ball of radius R in $\mathcal{W}^{1,2}(0, T; L^2(\mathbb{T}^2))$ and a ball of radius

R in $\mathcal{W}^{\beta,p}(0, T; L^2(\mathbb{T}^2))$ is compact in $C(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2))$. Then

$$\begin{aligned} m_v^n(D_3^C) &\leq \mathbb{P} \left(\|X_t^{v^n}\|_{\mathcal{W}^{1,2}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^2 \geq R^2 \right) + \mathbb{P} \left(\|Y_t^{v^n}\|_{\mathcal{W}^{\beta,p}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^p \geq R^p \right) \\ &\leq \frac{\mathcal{C}}{R^2} \end{aligned}$$

and

$$\begin{aligned} m_h^n(D_4^C) &\leq \mathbb{P} \left(\|X_t^{h^n}\|_{\mathcal{W}^{1,2}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^2 \geq R^2 \right) + \mathbb{P} \left(\|Y_t^{h^n}\|_{\mathcal{W}^{\beta,p}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^p \geq R^p \right) \\ &\leq \frac{\mathcal{C}}{R^2} \end{aligned}$$

We know so far that $D_1 \cap D_3$ is compact in $L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \cap C(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2))$ and $D_2 \cap D_4$ is compact in $L^2(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2) \times \mathcal{W}^{1,2}(\mathbb{T}^2)) \cap C(0, T; \mathcal{W}^{1,2}(\mathbb{T}^2))$, so

$$m_v^n((D_1 \cap D_3)^C) \leq m_v^n(D_1^C) + m_v^n(D_3^C) \leq \frac{\mathcal{C}}{R^2}$$

and

$$m_h^n((D_2 \cap D_4)^C) \leq m_h^n(D_2^C) + m_h^n(D_4^C) \leq \frac{\mathcal{C}}{R^2}.$$

If we choose the radius $R = \sqrt{\frac{4\mathcal{C}}{\epsilon}}$, $B_\epsilon^v = D_1 \cap D_3$ and $B_\epsilon^h = D_2 \cap D_4$ then we obtain

$$m^n(B_\epsilon^C) \leq \epsilon$$

as requested.

Proof of Lemma 5.16 For the deterministic components one has

$$\begin{aligned} \mathbb{E} \left[\|X_t^{v^n}\|_{\mathcal{W}^{1,2}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^2 \right] &= \mathbb{E} \left[\int_0^T \|X_t^{v^n}\|_{L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2)}^2 dt \right] + \mathbb{E} \left[\int_0^T \left\| \frac{d}{dt} X_t^{v^n} \right\|_{L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2)}^2 dt \right] \\ &\leq \mathcal{C} \end{aligned}$$

and

$$\mathbb{E} \left[\|X_t^{h^n}\|_{\mathcal{W}^{1,2}(0,T;L^2(\mathbb{T}^2))}^2 \right] = \mathbb{E} \left[\int_0^T \|X_t^{h^n}\|_{L^2(\mathbb{T}^2)}^2 dt \right] + \mathbb{E} \left[\int_0^T \left\| \frac{d}{dt} X_t^{h^n} \right\|_{L^2(\mathbb{T}^2)}^2 dt \right] \leq \mathcal{C}$$

due to Proposition 5.20. The stochastic components are controlled using a version of the

Burkholder-Davis-Gundy inequality (see Chapter 2):

$$\begin{aligned}
\mathbb{E} \left[\left\| Y_t^{v^n} \right\|_{\mathcal{W}^{\beta,p}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^p \right] &= \mathbb{E} \left[\left\| \int_0^t \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i)v_s^n] dW_s^i \right\|_{\mathcal{W}^{\beta,p}(0,T;L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^p \right] \\
&\leq a(p, \beta, T) \mathbb{E} \left[\int_0^T \left\| \sum_{i=1}^{\infty} (\mathcal{L}_i + \mathcal{A}_i)v_s^n \right\|_{L_2(\mathfrak{U}, L^2(\mathbb{T}^2) \times L^2(\mathbb{T}^2))}^p dW_s^i \right] \\
&\leq \mathcal{C}
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E} \left[\left\| Y_t^{h^n} \right\|_{\mathcal{W}^{\beta,p}(0,T;L^2(\mathbb{T}^2))}^p \right] &= \mathbb{E} \left[\left\| \int_0^t \sum_{i=1}^{\infty} [\nabla \cdot (\xi_i h_s^n)] dW_s^i \right\|_{\mathcal{W}^{\beta,p}(0,T;L^2(\mathbb{T}^2))}^p \right] \\
&\leq a(p, \beta, T) \mathbb{E} \left[\int_0^T \left\| \sum_{i=1}^{\infty} \nabla \cdot (\xi_i h_s^n) \right\|_{L_2(\mathfrak{U}, L^2(\mathbb{T}^2))}^p dW_s^i \right] \\
&\leq \mathcal{C}
\end{aligned}$$

by the assumptions 2.21 on the vector fields $(\xi_i)_i$ and Proposition 5.22.

5.5.2 A PRIORI ESTIMATES

Proposition 5.20. *There exist two constants $\mathcal{C}_1(R)$ and $\mathcal{C}_2(R)$ such that*

$$\mathbb{E} \left[\sup_{t \in [0,T]} \|v_t^n\|_{1,2}^2 + \sup_{t \in [0,T]} \|h_t^n\|_{1,2}^2 \right] \leq \mathcal{C}_1(R). \quad (5.21)$$

$$\mathbb{E} \left[\int_0^T \|\Delta v_t^n\|_2^2 dt + \int_0^T \|\Delta h_t^n\|_2^2 dt \right] \leq \mathcal{C}_2(R). \quad (5.22)$$

Proof of proposition 5.20. All linear terms are controlled as in Proposition 5.11 (and the bounds are independent of n). For the nonlinear terms we make use of the cutoff functions and obtain global (R -dependent) estimates. We first consider the velocity equation. By the Itô

formula one has

$$\begin{aligned}
& d\|\nabla v_t^n\|_2^2 + 2\nu\|\Delta v_t^n\|_2^2 dt \\
&= 2\left\langle \nabla v_t^n, \nabla(f_R(u_t^{n-1})\mathcal{L}_{u_t^{n-1}}v_t^n + f\hat{z} \times u_t^n + f_R(h_t^{n-1})\nabla p_t^{n-1}) + \frac{1}{2}\sum_{i=1}^{\infty}\nabla[(\mathcal{L}_i + \mathcal{A}_i)^2 v_t^n] \right\rangle dt \\
&+ 2\sum_{i=1}^{\infty}\left\langle \nabla v_t^n, \nabla\left((\mathcal{L}_i + \mathcal{A}_i)v_t^n\right) \right\rangle dW_t^i + \sum_{i=1}^{\infty}\left\langle \nabla\left((\mathcal{L}_i + \mathcal{A}_i)v_t^n\right), \nabla\left((\mathcal{L}_i + \mathcal{A}_i)v_t^n\right) \right\rangle dt.
\end{aligned} \tag{5.23}$$

We first integrate equation (5.23) in time over $[0, s]$ for $0 \leq s \leq t \leq T$, and then we take the supremum in s over $[0, t]$ and the expected value with respect to the probability $\mathbb{P}$:

$$\begin{aligned}
& \mathbb{E}\left[\sup_{s \in [0, t]} \|\nabla v_s^n\|_2^2 + 2\nu \int_0^s \|\Delta v_r^n\|_2^2 dr\right] \\
& \leq \mathbb{E}\left[\|\nabla v_0^n\|_2^2\right] + \mathbb{E}\left[R \int_0^t \left(\frac{\nu}{c_1}\|\Delta v_r^n\|_2^2 + \sup_{s \in [0, r]} \frac{1}{c_1\nu}\|v_s^n\|_{1,2}^2\right) dr\right] \\
& + \mathbb{E}\left[\int_0^t \left(\frac{\nu}{c_6}\|\Delta v_r^n\|_2^2 + \sup_{s \in [0, r]} \frac{1}{c_6\nu}\|v_s^n\|_{1,2}^2\right) dr\right] \\
& + \mathbb{E}\left[\int_0^t \left(\frac{\nu}{c_5}\|\Delta v_r^n\|_2^2 + \sup_{s \in [0, r]} \frac{1}{c_5\nu}\|p_s^{n-1}\|_{1,2}^2\right) dr\right] \\
& + \mathbb{E}\left[C \int_0^t \sup_{s \in [0, r]} \|v_s^n\|_{1,2}^2 dr\right] \\
& + \mathbb{E}\left[2 \sup_{s \in [0, t]} \int_0^s \sum_{i=1}^{\infty} \langle \Delta v_r^n, (\mathcal{L}_i + \mathcal{A}_i)v_r^n \rangle dW_r^i\right]
\end{aligned}$$

with

$$\begin{aligned}
\mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s \sum_{i=1}^{\infty} \langle \Delta v_r^n, (\mathcal{L}_i + \mathcal{A}_i) v_r^n \rangle dW_r^i \right| \right] &\leq C \mathbb{E} \left[\int_0^t \sum_{i=1}^{\infty} \langle \Delta v_r^n, (\mathcal{L}_i + \mathcal{A}_i) v_r^n \rangle^2 dr \right]^{1/2} \\
&\leq C \mathbb{E} \left[\int_0^t \sum_{i=1}^{\infty} \|\Delta v_r^n\|_2^2 \|(\mathcal{L}_i + \mathcal{A}_i) v_r^n\|_2^2 dr \right]^{1/2} \\
&\leq C \mathbb{E} \left[\int_0^t \|\Delta v_r^n\|_2^2 \sum_{i=1}^{\infty} \|(\mathcal{L}_i + \mathcal{A}_i) v_r^n\|_2^2 dr \right] \\
&\leq C \mathbb{E} \left[\sup_{s \in [0, t]} \|v_s^n\|_{1,2}^2 \int_0^t \|\Delta v_r^n\|_2^2 dr \right] \\
&\leq \frac{1}{2} \mathbb{E} \left[\sup_{s \in [0, t]} \|v_s^n\|_{1,2}^2 \right] + \frac{1}{2} C \mathbb{E} \left[\int_0^t \|\Delta v_r^n\|_2^2 dr \right] \\
&\leq \frac{C^2}{\epsilon} \mathbb{E} \left[\sup_{s \in [0, t]} \|v_s^n\|_{1,2}^2 \right] + \epsilon \mathbb{E} \left[\int_0^t \|\Delta v_r^n\|_2^2 dr \right]
\end{aligned}$$

Now we consider the height equation. By the Itô formula one has

$$\begin{aligned}
&d\|\nabla h_t^n\|_2^2 + 2\delta \|\Delta h_t^n\|_2^2 dt \\
&= 2 \left\langle \nabla h_t^n, f_R(u_t^{n-1}, h_t^{n-1}) \nabla (h_t^{n-1} (\nabla \cdot u_t^n) + u_t^{n-1} \cdot \nabla h_t^n) + \frac{1}{2} \sum_{i=1}^{\infty} \nabla (\mathcal{L}_i^2 h_t^n) \right\rangle dt \\
&+ 2 \sum_{i=1}^{\infty} \langle \nabla h_t^n, \nabla (\mathcal{L}_i h_t^n) \rangle dW_t^i + \sum_{i=1}^{\infty} \langle \nabla (\mathcal{L}_i h_t^n), \nabla (\mathcal{L}_i h_t^n) \rangle dt.
\end{aligned} \tag{5.24}$$

We integrate equation (5.24) in time over $[0, s]$ for $0 \leq s \leq t \leq T$, and then we take the supremum in s over $[0, t]$ and the expected value as above:

$$\begin{aligned}
&\mathbb{E} \left[\sup_{s \in [0, t]} \|\nabla h_s^n\|_2^2 + 2\delta \int_0^t \|\Delta h_s^n\|_2^2 ds \right] \\
&\leq \mathbb{E} [\|h_0^n\|_{1,2}^2] + \mathbb{E} \left[\int_0^t \sup_{s \in [0, r]} 2Rp \|\Delta h_s^n\|_2^2 + Rq \|h_s^n\|_{1,2}^2 + Rq \|v_s^n\|_{1,2}^2 dr \right] \\
&+ \mathbb{E} \left[C \int_0^t \sup_{s \in [0, r]} \|h_s^n\|_{1,2}^2 dr \right] \\
&+ \mathbb{E} \left[2 \sup_{s \in [0, t]} \int_0^s \sum_{i=1}^{\infty} \langle \Delta h_r^n, \mathcal{L}_i h_r^n \rangle dW_r^i \right]
\end{aligned}$$

with

$$\mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s \sum_{i=1}^{\infty} \langle \Delta h_r^n, \mathcal{L}_i h_r^n \rangle dW_r^i \right| \right] \leq \frac{C^2}{\epsilon} \mathbb{E} \left[\sup_{s \in [0, t]} \|h_s^n\|_{1,2}^2 \right] + \epsilon \mathbb{E} \left[\int_0^t \|\Delta h_r^n\|_2^2 dr \right]$$

as above. We end up with

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t]} \|v_s^n\|_{1,2}^2 + \sup_{s \in [0, t]} \|h_s^n\|_{1,2}^2 \right] + \mathbb{E} \left[\int_0^t \|\Delta v_s^n\|_2^2 ds + \int_0^t \|\Delta h_s^n\|_2^2 ds \right] \\ & \leq C_1(R) + C(R) \int_0^t \mathbb{E} \left[\sup_{s \in [0, r]} \|v_s^n\|_{1,2}^2 + \sup_{s \in [0, r]} \|h_s^n\|_{1,2}^2 \right] dr \end{aligned}$$

which by Gronwall's inequality gives

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|v_s^n\|_{1,2}^2 + \sup_{s \in [0, t]} \|h_s^n\|_{1,2}^2 \right] \leq C_1(R) e^{C(R)t}$$

and therefore also

$$\mathbb{E} \left[\int_0^t \|\Delta v_s^n\|_2^2 ds + \int_0^t \|\Delta h_s^n\|_2^2 ds \right] \leq C_2(R).$$

Lemma 5.21. *The following statements are true:*

a. *There exists a constant c_1 such that*

$$|\langle \nabla v_t^n, f_R(u_t^{n-1}) \nabla (\mathcal{L}_{u_t^{n-1}} v_t^n) \rangle| \leq R \left(\frac{\nu}{c_1} \|\Delta v_t^n\|_2^2 + \frac{1}{c_1 \nu} \|v_t^n\|_{1,2}^2 \right)$$

b. *There exists some constants c_5, c_6 such that*

$$b1. |\langle \nabla v_t^n, \nabla (\nabla p_t^{n-1}) \rangle| \leq \frac{\nu}{c_5} \|\Delta v_t^n\|_2^2 + \frac{1}{c_5 \nu} \|p_t^{n-1}\|_{1,2}^2$$

$$b2. |\langle \nabla v_t^n, \nabla (f \hat{z} \times u_t^n) \rangle| \leq \frac{\nu}{c_6} \|\Delta v_t^n\|_2^2 + \frac{1}{c_6 \nu} \|v_t^n\|_{1,2}^2$$

c. *There exists some constants p, q such that*

$$|\langle \nabla h_t^n, f_R(u_t^{n-1}, h_t^{n-1}) \nabla (h_t^{n-1} (\nabla \cdot u_t^n) + u_t^{n-1} \cdot \nabla h_t^n) \rangle| \leq 2Rp \|\Delta h_t^n\|_2^2 + Rq \|h_t^n\|_{1,2}^2 + Rq \|v_t^n\|_{1,2}^2$$

Proof of Lemma 5.21.

a. We have

$$\begin{aligned}
|\langle \nabla v_t^n, f_R(u_t^{n-1}) \nabla (\mathcal{L}_{u_t^{n-1}} v_t^n) \rangle| &= \langle \Delta v_t^n, f_R(u_t^{n-1}) \mathcal{L}_{u_t^{n-1}} v_t^n \rangle \leq f_R(u_t^{n-1}) \|u_t^{n-1}\|_\infty \|\nabla v_t^n\|_2 \|\Delta v_t^n\|_2 \\
&\leq R \|\nabla v_t^n\|_2 \|\Delta v_t^n\|_2 \\
&\leq R(1/c_1 \nu \|\nabla v_t^n\|_2^2 + \nu/c_1 \|\Delta v_t^n\|_2^2) \\
&\leq R \left(\frac{1}{c_1 \nu} \|v_t^n\|_{1,2}^2 + \frac{\nu}{c_1} \|\Delta v_t^n\|_2^2 \right).
\end{aligned}$$

b. Using Young's inequality one can write

b1.

$$|\langle \nabla v_t^n, \nabla(\nabla p_t^{n-1}) \rangle| = |\langle \Delta v_t^n, \nabla p_t^{n-1} \rangle| \leq \frac{\nu}{c_5} \|\Delta v_t^n\|_2^2 + \frac{1}{c_5 \nu} \|\nabla p_t^{n-1}\|_2^2 \leq \frac{\nu}{c_5} \|\Delta v_t^n\|_2^2 + \frac{1}{c_5 \nu} \|p_t^{n-1}\|_{1,2}^2$$

b2.

$$|\langle \nabla v_t^n, \nabla(f \hat{z} \times u_t^n) \rangle| = |\langle \Delta v_t^n, f \hat{z} \times u_t^n \rangle| \leq \frac{\nu}{c_6} \|\Delta v_t^n\|_2^2 + \frac{1}{c_6 \nu} \|\nabla u_t^n\|_2^2 \leq \frac{\nu}{c_6} \|\Delta v_t^n\|_2^2 + \frac{1}{c_6 \nu} \|u_t^n\|_{1,2}^2.$$

c. We have

$$\begin{aligned}
&|\langle \nabla h_t^n, f_R(u_t^{n-1}, h_t^{n-1}) \nabla (h_t^{n-1}(\nabla \cdot u_t^n) + u_t^{n-1} \cdot \nabla h_t^n) \rangle| \\
&= |\langle \Delta h_t^n, f_R(h_t^{n-1}, u_t^{n-1})(h_t^{n-1}(\nabla \cdot u_t^n) + u_t^{n-1} \cdot \nabla h_t^n) \rangle| \\
&=: d_1 + d_2
\end{aligned}$$

$$\begin{aligned}
d_1 &\leq f_R(h_t^{n-1}, u_t^{n-1}) \|\Delta h_t^n\|_2 \|h_t^{n-1}\|_\infty \|\nabla \cdot u_t^n\|_2 \\
&\leq R \|\Delta h_t^n\|_2 \|u_t^n\|_{1,2} \\
&\leq R(p \|\Delta h_t^n\|_2^2 + q \|v_t^n\|_{1,2}^2) \\
d_2 &\leq f_R(h_t^{n-1}, u_t^{n-1}) \|\Delta h_t^n\|_2 \|u_t^{n-1}\|_\infty \|\nabla h_t^n\|_2 \\
&\leq R \|\Delta h_t^n\|_2 \|\nabla h_t^n\|_2 \\
&\leq R(p \|\Delta h_t^n\|_2^2 + q \|h_t^n\|_{1,2}^2)
\end{aligned}$$

So

$$|\langle \nabla h_t^n, f_R(u_t^{n-1}, h_t^{n-1}) \nabla (h_t^{n-1}(\nabla \cdot u_t^n) + u_t^{n-1} \cdot \nabla h_t^n) \rangle| \leq 2Rp \|\Delta h_t^n\|_2^2 + Rq \|h_t^n\|_{1,2}^2 + Rq \|v_t^n\|_{1,2}^2.$$

Proposition 5.22. *There exist a constant $\mathcal{C}_3(R)$ such that*

$$\mathbb{E} \left[\sup_{t \in [0, T]} \|v_t^n\|_{1,2}^p + \sup_{t \in [0, T]} \|h_t^n\|_{1,2}^p \right] \leq \mathcal{C}_3(R). \quad (5.25)$$

Proof The proof follows the same strategy as the proof of Proposition 5.20. In order to control the p th moment of (v^n, h^n) , we raise the estimates

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t]} \|v_s^n\|_{1,2}^2 + \sup_{s \in [0, t]} \|h_s^n\|_{1,2}^2 \right] + \mathbb{E} \left[\int_0^t \|\Delta v_s^n\|_2^2 ds + \int_0^t \|\Delta h_s^n\|_2^2 ds \right] \\ & \leq C_1 + \mathbb{E} \left[\int_0^t (\|v_s^n\|_{1,2}^2 + \|h_s^n\|_{1,2}^2) ds \right] \\ & + \mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s \sum_{i=1}^{\infty} \langle \Delta v_r^n, (\mathcal{L}_i + \mathcal{A}_i) v_r^n \rangle dW_r^i \right| \right] + \mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s \sum_{i=1}^{\infty} \langle \Delta h_r^n, \mathcal{L}_i h_r^n \rangle dW_r^i \right| \right] \end{aligned}$$

to the power $p/2$. Then we apply Minkowski's inequality for $p > 2$ and the Burkholder-Davis-Gundy inequality for $p/2$. So

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t]} \|v_s^n\|_{1,2}^p + \sup_{s \in [0, t]} \|h_s^n\|_{1,2}^p \right] \leq C_1 + \mathbb{E} \left[\int_0^t (\|v_s^n\|_{1,2}^p + \|h_s^n\|_{1,2}^p) ds \right] \\ & + \mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s \sum_{i=1}^{\infty} \langle \Delta v_r^n, (\mathcal{L}_i + \mathcal{A}_i) v_r^n \rangle dW_r^i \right|^{p/2} \right] + \mathbb{E} \left[\sup_{s \in [0, t]} \left| \int_0^s \sum_{i=1}^{\infty} \langle \Delta h_r^n, \mathcal{L}_i h_r^n \rangle dW_r^i \right|^{p/2} \right] \end{aligned}$$

and using the Burkholder-Davis-Gundy inequality we obtain

$$\begin{aligned} & \mathbb{E} \left[\sup_{s \in [0, t]} \|v_s^n\|_{1,2}^p + \sup_{s \in [0, t]} \|h_s^n\|_{1,2}^p \right] + \mathbb{E} \left[\int_0^t \|\Delta v_s^n\|_2^p ds + \int_0^t \|\Delta h_s^n\|_2^p ds \right] \\ & \leq C_1(R) + C(R) \int_0^t \mathbb{E} \left[\sup_{s \in [0, r]} \|v_s^n\|_{1,2}^p + \sup_{s \in [0, r]} \|h_s^n\|_{1,2}^p \right] dr \end{aligned}$$

which by Gronwall's inequality gives

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|v_s^n\|_{1,2}^p + \sup_{s \in [0, t]} \|h_s^n\|_{1,2}^p \right] \leq \mathcal{C}_3(R).$$

Remark 5.23. *This result holds also for the original equation but up to a stopping time.*

6

DATA ASSIMILATION FOR THE STOCHASTIC ROTATING SHALLOW WATER MODEL

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6.3	Applications for the Stochastic Rotating Shallow Water Model	152
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In this chapter we construct a robust particle filter for the stochastic rotating shallow water model. This is based on adaptive tempering and jittering and it is first tested for the Lorenz 1963 model.

6.1 INTRODUCTION

Let X and Z be two processes defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. X is usually called the *signal process* or the *truth* and Z is the *observation process*. In our case X is given by a pathwise solution of the stochastic rotating shallow water system computed on a staggered grid. We have proven in Chapter 5 that such a solution exists.¹ The nonlinear filtering problem consists in finding the best approximation of the posterior distribution of the signal X_t given the observations $Z_1, Z_2, \dots, Z_t$.² The posterior distribution is usually denoted by π_t . We denote the dimension of the state space by d_X and the dimension of the observation space by d_Z .

A *particle filter* is a sequential Monte Carlo method in which the posterior is approximated using a set of *particles*, that is random measures of the form

$$\pi_t \approx \sum_{\ell} w_t^{\ell} \delta(x_t^{\ell})$$

where δ is the Dirac delta function, $w_t^1, w_t^2, \dots$ are the *weights* of the particles and $x_t^1, x_t^2, \dots$ are their corresponding positions [4]. One can make inferences about the signal process using Bayes' theorem, the time-evolution induced by the model, and observations [88], [4]. Observations are modelled as noisy measurements of the truth, using the *observation operator*:

$$\mathcal{H} : \mathbb{R}^{d_X} \rightarrow \mathbb{R}^{d_Z} \tag{6.1}$$

$$Z_t = \mathcal{H}(X_t) + V_t \tag{6.2}$$

where $(V_t)_{t \geq 0}$ are independent identically distributed random variables with standard normal distribution and $\mathcal{H}$ is a Borel-measurable function. Observations are incorporated into the system at *assimilation times*. The ensemble of particles is evolved between assimilation times according to the law of the signal. At each assimilation time the observation is incorporated into the system

¹Note that in Chapter 5 we work with infinite dimensional function spaces, while here the state space will be $\mathbb{R}^{d_X}$.

²For a mathematical introduction on the subject see [4]. For an introduction from the data assimilation perspective see [96] and [88].

through the *likelihood function*:

$$g_t^{z_t} : \mathbb{R}^{d_x} \rightarrow [0, 1], \quad g_t^{z_t}(x) = g_t(z_t - \mathcal{H}(x_t)) = \mathbb{P}(Z_t \in dz_t | X_t = x_t) \quad (6.3)$$

that is

$$\int_A g(z_t - \mathcal{H}(x_t)) dz_t = \mathbb{P}(Z_t \in A | X_t = x_t) \quad (6.4)$$

where $A \in \mathcal{B}(\mathbb{R}^{d_z})$. The following recursion formula holds (see [4])

$$\pi_t = g_t \star \pi_{t-1} \mathcal{K}_t \quad (6.5)$$

where

$$\mathcal{K}_t : \mathbb{R}^{d_x} \times \mathcal{B}(\mathbb{R}^{d_x}) \rightarrow [0, 1], \quad \mathcal{K}_t(X_{t-1}, B) = \mathbb{P}(X_t \in B | X_{t-1}) \quad (6.6)$$

for any measurable set $B \in \mathcal{B}(\mathbb{R}^{d_x})$ and by ' $\star$ ' we denoted the projective product as defined in the Appendix (Section 8.3). Schematically, this can be arranged as

$$\pi_{t-1}^{a, z_{0:t-1}} \xrightarrow[\substack{\text{model} \\ \text{forecast} \\ \text{prediction}}]{\mathcal{K}_t} \pi_{t-1}^{a, z_{0:t-1}} \mathcal{K}_t =: \pi_t^b =: p_t \xrightarrow[\substack{\text{assimilation} \\ \text{analysis} \\ \text{update}}]{\text{tempering, } g_t^{z_t} \star} g_t^{z_t} \star \pi_t^b = \pi_t^{a, z_{0:t}}. \quad (6.7)$$

The indices a and b stand for *assimilation* and *background* respectively. Here $p_t = p_t^{Z_{0:t-1}}$ is the prior distribution of the signal, that is a probability measure which belongs to the space $\mathcal{P}(\underbrace{\mathbb{R}^{d_x} \times \mathbb{R}^{d_x} \times \dots \times \mathbb{R}^{d_x}}_{t+1})$ with $t = 0, 1, \dots$, and corresponds to the path space generated by $X_0, X_1, \dots, X_t$. By $Z_{0:t}$ we mean the random vector $(Z_0, Z_1, \dots, Z_t)$ with the corresponding observed values $z_{0:t}$. Taking into account the definition of the projective product, the recursion formula (6.5) can also be written as

$$\pi_t(B) = \frac{\int_B g_t^{z_t}(x_t) p_t(dx_t)}{\int_{\mathbb{R}^{d_x}} g_t^{z_t}(x_t) p_t(dx_t)} = \beta_t^{-1} \int_B g_t^{z_t}(x_t) p_t(dx_t) \quad (6.8)$$

where $B \in \mathcal{B}(\mathbb{R}^{d_x})$ and $\beta_t := \int_{\mathbb{R}^{d_x}} g_t^{z_t}(x_t) p_t(dx_t)$ is a normalising constant. We are looking for

an approximation of the form

$$\pi_t \approx \pi_t^N = \sum_{\ell=1}^N w_t^\ell \delta(x_t^\ell). \quad (6.9)$$

The typical particle filter runs as follows. Each particle is propagated forward in time using the model (SRSW or Lorenz '63 in our case). We model the evolution of the signal discretely in time through a map $\mathcal{M}_t$. In the following, $\mathcal{M}_t$ is a discrete realisation of the Lorenz '63 or the SRSW model. We have

$$\mathcal{M}_t : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_x} \quad (6.10)$$

and if $x_{t_1}^\ell, x_{t_2}^\ell$ is the position of the particle ℓ at time t_1 respectively t_2 then

$$x_{t_2}^\ell = \mathcal{M}_{t_2}(x_{t_1}^\ell). \quad (6.11)$$

The particle trajectory is independent of the trajectory of the signal. At each assimilation time, every particle is weighted depending on the likelihood of its position, given the observation. The weight measures how close the particle trajectory is to the signal trajectory.

$$\pi_{t_2}^a = \sum_{\ell=1}^N w_{t_2}^\ell \delta(\mathcal{M}_{t_2}(x_{t_1}^\ell)) = \sum_{\ell=1}^N w_{t_2}^\ell \delta(x_{t_2}^\ell) \quad (6.12)$$

Particles which are close to the *truth* and therefore have higher weights will be multiplied, while those which are far away will be eliminated. There are several studies ([95],[97]) which have shown that in high-dimensional spaces the tendency is to have one particle gaining a weight close to one while all the others are discarded, having weight close to zero. On the other and, in [4] it has been proven that a particle filter will provide a good approximation of the posterior distribution only when enough particles are used, which explains the difficulty of the problem at hand. The standard rapid divergence of the particles' trajectory from the signal trajectory is known in the literature as *the curse of dimensionality* [95], [96]. Nonetheless, considerable progress has been made in this direction over the last years. A state-of-the-art analysis of the most recent efforts on tackling the filter degeneracy problem can be found in [97] and [95]. Innovative remedies arise from different directions: optimal transportation [88], tempering [9],[66],[30],[31], localisation [28],[87], model reduction [30], data assimilation as a boundary value problem [89], jittering [9], nudging [92], proposal densities [98]. Some of them have been tested operationally: [87].

Our work is part of these attempts of developing a particle filter methodology for high dimensional models originating in (stochastic) fluid dynamics. We have implemented a particle filter with adaptive tempering and jittering for the stochastic rotating shallow water model³. The suitability of these two methods used in tandem for high-dimensional problems has been proven in [9] and tested in [30], [31], [92]. In [31] the method is used for the stochastic incompressible 2D Euler model with damping and forcing, while in [92] it is tested for the 2D quasi-geostrophic model. We have no knowledge of any previous application of this methodology for the stochastic rotating shallow water system. The complexity of this model makes the problem harder. The aim of this chapter is to show the applicability of the particle filter described below and the stochastic rotating shallow water model running alongside. We start with first coupling the particle filter with the classical Lorenz '63 model and then we move to the targeted SRSW model. The results are encouraging in both cases.

In tempering, an artificial dynamics is introduced via a sequence of artificial target distributions between prior and posterior. Each intermediate distribution has a characteristic *temperature* chosen such that a reasonable number of good particles survive. We first follow a resampling procedure⁴ in which the particles with low weights are replaced with particles with higher weights such that at the end an ensemble of equal-weighted particles is obtained. However, the particles can be concentrated in the wrong place. In order to quantify the spread of the weights with respect to the posterior, we use the *effective sample size* statistic:

$$ess(\mathbf{w}) = \frac{1}{N \sum_{\ell=1}^N (w^\ell)^2} \quad (6.13)$$

The *ess* will be chosen to be above a certain threshold in order for the ensemble of particles to be oriented in the right direction. In tempering one increases gradually the variance of the distribution using a sequence of *temperatures* $0 = \phi_0 < \phi_1 < \dots \phi_R = 1$ to ensure that the *ess* remains above the chosen threshold. Once the temperature is (dynamically) chosen, a resam-

³The code used throughout this chapter was initially based on a code provided by Prof. Peter Jan van Leeuwen for the deterministic rotating shallow water model. This code has been repeatedly modified, extended to the stochastic case, and adapted to our purposes. The particle filter methodology with tempering and jittering has been implemented specifically for the SRSW model.

⁴This is a static procedure.

pling procedure is applied. The output is a sequence of *tempered posterior distributions* with corresponding normalised tempered weights. In this case the intermediate tempered posterior distribution at the r^{th} tempering step is given by

$$\pi_t^r(B) = \frac{\int_B (g_t^{z_t}(x_t))^{\phi_r} p_t(dx_t)}{\int_{\mathbb{R}^{d_X}} (g_t^{z_t}(x_t))^{\phi_r} p_t(dx_t)} \quad (6.14)$$

for any $B \in \mathcal{B}(\mathbb{R}^{d_X})$. We denote by $\mathbf{x} = (x^\ell)_{\ell=1}^N$ the ensemble of particles. Then

$$\pi_{t_i}^{r,N} = \sum_{\ell=1}^N w_{t_i}^{r,\ell}(\phi_r, \mathbf{x}) \delta(x_{t_i}^\ell) \quad (6.15)$$

where

$$w_{t_i}^{r,\ell}(\phi_r, \mathbf{x}) = \left(g_{t_i}^{z_{t_i}}(x_{t_i}^\ell) \right)^{\phi_r - \phi_{r-1}}, \quad \text{with} \quad \sum_l w_{t_i}^\ell = 1. \quad (6.16)$$

The corresponding *ess* is given by

$$ess_i(\phi^r, \mathbf{x}) := \|w_{t_i}(\phi_r, \mathbf{x})\|_{\ell^2}^{-1}. \quad (6.17)$$

The following is the tempering and jittering algorithm (see e.g. [66],[30]):

1. At initial time $t = 0$: sample N particles from the prior distribution.
2. On the time interval $(t_{i-1}, t_i]$: we have an ensemble $\mathbf{x}$ of particles with positions $(x_{t_{i-1}}^\ell)_\ell$ and we want to assimilate observational data z_{t_i} in order to obtain a new ensemble $(x_{t_i}^\ell)_\ell$ that defines $\pi_{t_i}^N$:
 - 2.1. Evolve $x_{t_{i-1}}^\ell \xrightarrow[SRSW, Lorenz63]{SPDE} x_{t_i}^\ell$.
 - 2.2. Set temperature $\phi = 1$.
 - 2.3. While $ess_i(\phi, \mathbf{x}) < N_{threshold}$ do
 - * Find $\phi' \in (1-\phi, 1)$ such that $ess_i(\phi' - (1-\phi), \mathbf{x}) \approx N_{threshold}$. Resample according to $w_{t_i}^\ell(\phi' - (1-\phi), \mathbf{x})$ and apply MCMC with jittering if required (i.e. if there are duplicates) $\Rightarrow$ a new ensemble $\mathbf{x}(\phi')$.

* Set $\phi = 1 - \phi'$ and $\mathbf{x} = \mathbf{x}(\phi)$.

2.4. If $ess_i \geq N_{threshold}$ then Stop and go to the $(i + 1)^{th}$ filtering step with $(x_{t_i}^\ell, w_i^\ell)_\ell$.

The jittering procedure is standard (see e.g. [30]) but we will briefly explain the idea. Without jittering we have

$$x_{t_2}^\ell = \mathcal{M}(x_{t_1}^\ell) \quad (6.18)$$

which can also be written as

$$x_{t_2}^\ell = \mathcal{M}(x_{t_1}^\ell, W(t_1 : t_2)) \quad (6.19)$$

where W is the driving Brownian motion of the model and with $W(t_1 : t_2)$ we denoted the Brownian path between t_1 and t_2 . In this case the prior distribution is given by

$$p_{t_2} = \frac{1}{N} \sum_{\ell=1}^N \delta(x_{t_2}^\ell). \quad (6.20)$$

The problem is that sometimes the particles end up in the same place and we can have a large number of duplicates. In order to overcome this issue we modify the last part of the particle trajectory using a *jittering parameter* ρ and a Gaussian random variable Z which is orthogonal to W . Then the new dynamics is given by

$$\tilde{x}_{t_2}^\ell = \mathcal{M}_{t_2} \left(x_{t_1}^\ell, \rho W(t_1 : t_2) + \sqrt{1 - \rho^2} Z(t_1 : t_2) \right) \quad (6.21)$$

where

$$x_{t_i}^\ell = \mathcal{M}_{t_i} \left(x_{t_{i-1}}^\ell, W(t_{i-1} : t_i) \right). \quad (6.22)$$

6.2 APPLICATIONS FOR THE LORENZ 1963 MODEL

6.2.1 MODEL DESCRIPTION

The *Lorenz '63* model is a classical nonlinear three-dimensional model that is a precursor of turbulence theory which has been introduced in [76]. It reads

$$\frac{dx}{dt} = \alpha(y - x) \tag{6.23a}$$

$$\frac{dy}{dt} = (\beta - z)x - y \tag{6.23b}$$

$$\frac{dz}{dt} = xy - \gamma z \tag{6.23c}$$

where α, β, γ are real positive parameters. The model is well-known for the broad spectrum of patterns displayed for different values of α, β, γ and its well-known *butterfly attractor*. The original values chosen by Lorenz in [76] were $\alpha = 10, \beta = 28$ and $\gamma = \frac{8}{3}$. For a discussion on the behaviour of the solutions for different parameter values see e.g. [93]. This model is implemented using a Runge-Kutta scheme of order 4, with initial conditions $x_0 = 1.508870, y_0 = -1.531271, z_0 = 25.46091$. The details of the Runge-Kutta scheme can be found in Appendix. The Lorenz '63 model contained random perturbations at every time step. After using the Runge-Kutta scheme to implement the three variables from system (6.23), we generate a random field and perturb the system in a manner which is similar to the one explained in (6.21), with $\rho = 0.99$.

6.2.2 DATA ASSIMILATION RESULTS

We perform the data assimilation analysis using an ensemble of 50 particles which evolve for 500 time steps and each time step has size 0.01. In the standard scenario (Figures 6.4-6.7) all three variables of the system are observed every 20 time steps. The initial uncertainty is equal to 1, while the observational uncertainty and the model error are equal to 0.1. We present below a couple of scenarios obtained for different values of these parameters. The output is displayed for the first and the third variable. The ensemble of particles is plotted one standard deviation region around the ensemble mean. We first show in Figures 6.1-6.3 how the model evolves without any assimilation of data . As it can be seen in Figure 6.1 and 6.2, in the absence of the data

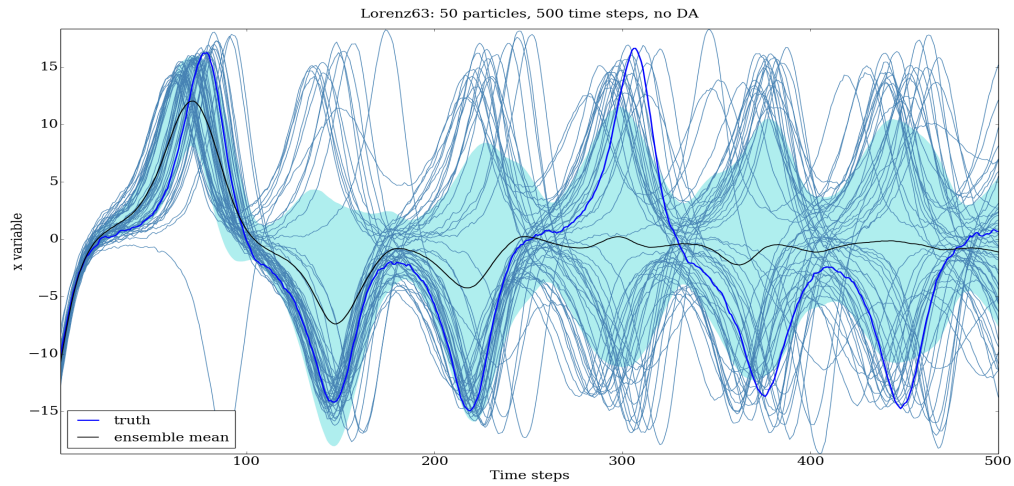


Figure 6.1: Evolution of the Lorenz '63 model without any data assimilation. Displayed: x variable. The particles fill out the attractor.

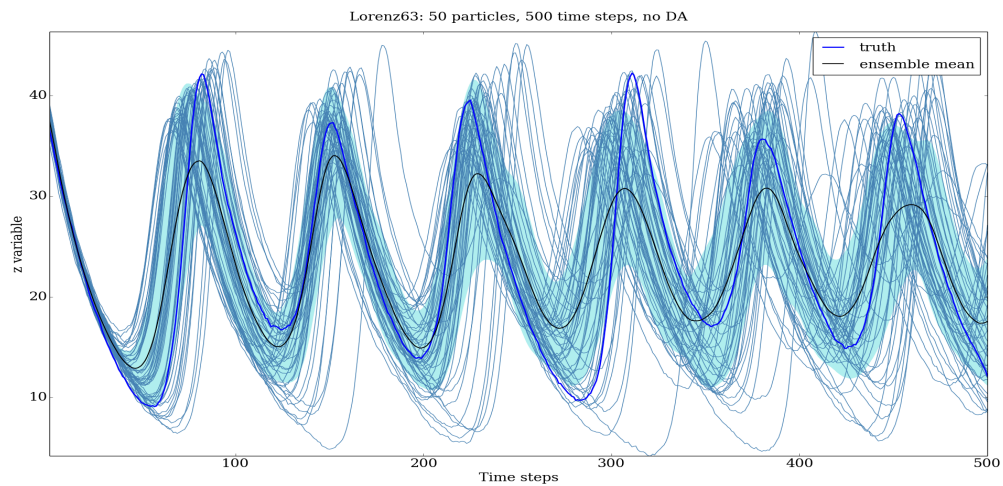


Figure 6.2: Evolution of the Lorenz '63 model without any data assimilation. Displayed: z variable.

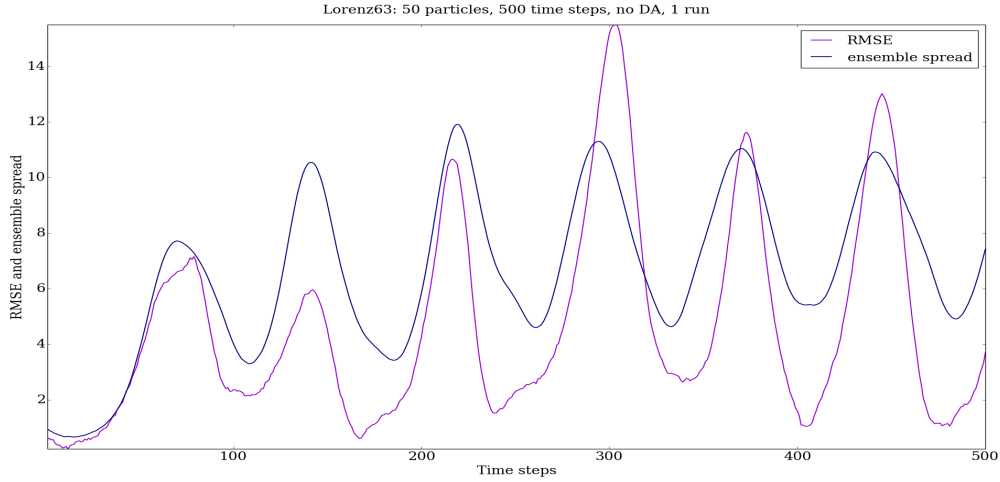


Figure 6.3: Evolution of the Lorenz '63 model for 500 time steps without any data assimilation. Displayed: RMSE and ensemble spread.

assimilation step, the particles spread gradually around the entire attractor. This behaviour is exhibited when plotting both the first variable x and the third variable z . In other words, the uncertainty increases (at times quite dramatically). The root mean square error (RMSE) and the ensemble spread (ES) is plotted in Figure 6.3. In this case the RMSE and the ES oscillate around a value which is comparable with the size of the attractor (the particles *fill out* the attractor). Compared to the previous scenario, we show in Figures 6.4-6.7 that by assimilating observations every 20 time steps, we manage to substantially reduce the uncertainty. Through a repeated application of the forecast/data assimilation step, the cloud of particles successfully tracks the truth, evolving around the attractor set. In Figure 6.6 we exhibit the first three instances to emphasize the dramatical reduction of uncertainty resulting from the application of the particle filter with tempering and jittering, as described in Section 6.1. We plot in Figure 6.7 the RMSE and the ES. We can see that the RMSE and the ES are comparable. This is a feature highly appreciated by data assimilation practitioners because it shows that our estimate of the uncertainty as measured by the width of the ensemble is a good estimate of the actual error in the ensemble mean. Based on this measure of success we can conclude that the particle filter performs satisfactory in the case where all the three variables of the system are observed. For the following scenario (Figures 6.8-6.11) we increase the observational uncertainty to 5, while observing all three variables. Decreasing the accuracy of the observations reduces the efficacy of

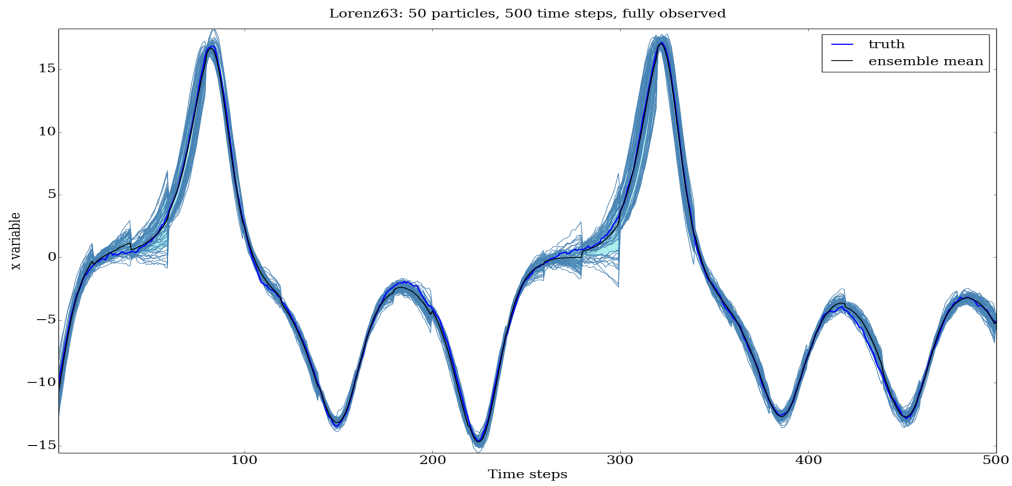


Figure 6.4: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Displayed: x variable. The initial uncertainty and the observational uncertainty are both equal to 1, the model error is equal to 0.1. Uncertainty is substantially reduced at assimilation times.

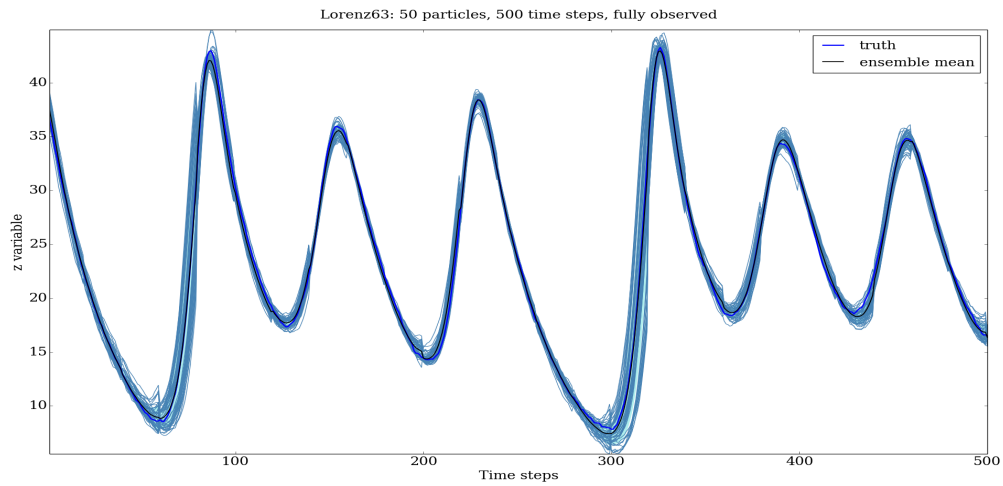


Figure 6.5: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Displayed: z variable. The initial uncertainty and the observational uncertainty are both equal to 1, the model error is equal to 0.1. Uncertainty is substantially reduced at assimilation times.

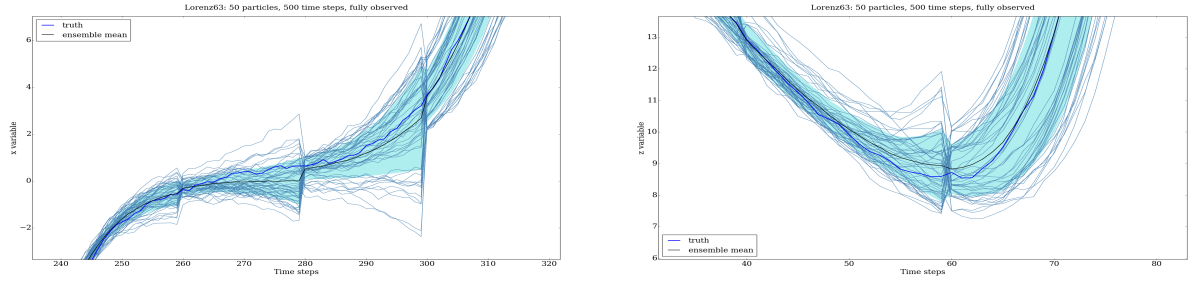


Figure 6.6: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Displayed: x enlarged (left) and z enlarged (right). Instances of the tempering procedure.

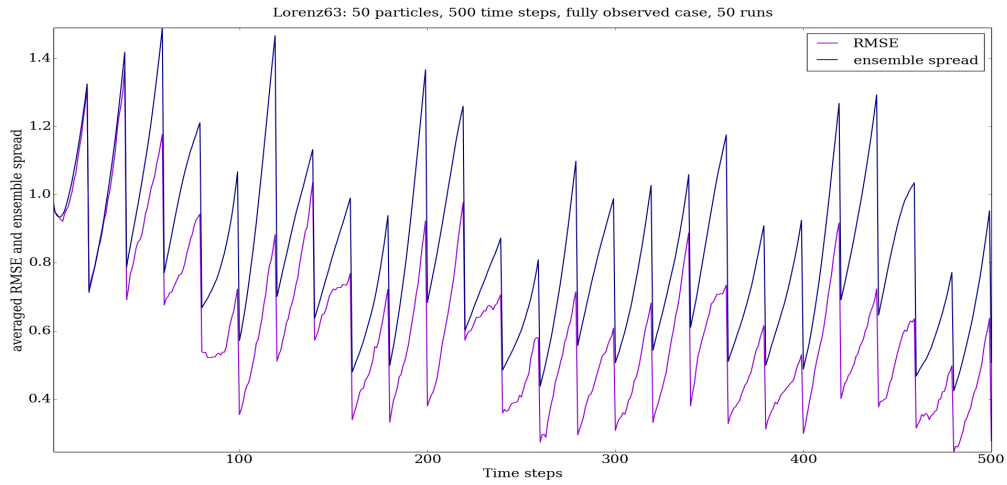


Figure 6.7: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Displayed: averaged (50 runs) RMSE and ensemble spread. The initial uncertainty and the observational uncertainty are both equal to 1, the model error is equal to 0.1. Uncertainty is substantially reduced at assimilation times.

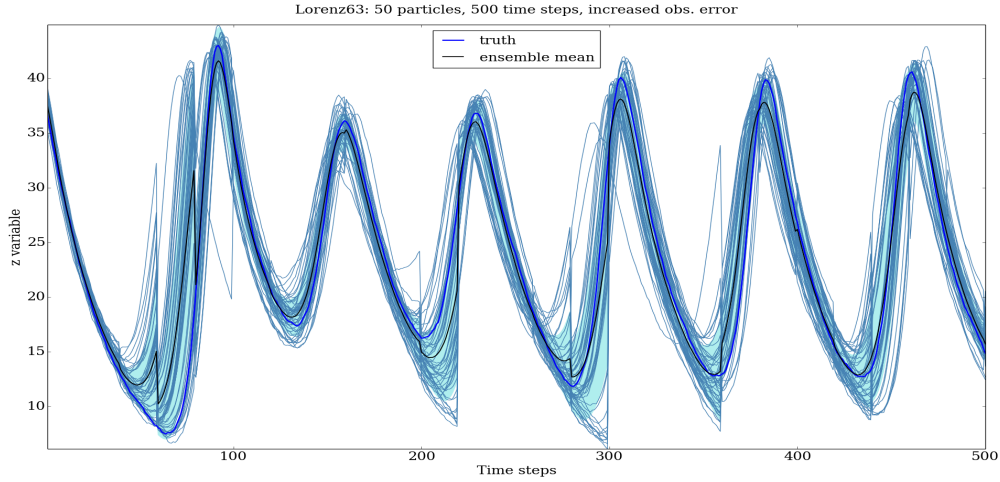


Figure 6.8: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observational uncertainty is increased to 5. Displayed: z variable. The accuracy of the observations decreases.

the data assimilation procedure. Figures 6.8-6.9 show that the reduction in the uncertainty is not as large as in the previous case. This can be seen better in Figure 6.8 for the z variable. In figure 6.10 we plot the RMSE and the ES. As one would expect, both the RMSE and the ES have higher-amplitude oscillations around the data assimilation times as the observational uncertainty is now much larger than before. For the following scenario (Figures 6.12-6.15) we increase both the observational uncertainty and the initial uncertainty and we observe all three variables. As a result, the uncertainty reduction is now even smaller than in the previous case. Nonetheless, the particle filter is efficient enough to eventually constrain the particles to follow the truth. This can be observed attentively in Figure 6.18. In the following scenario (Figures 6.16-6.19) observational uncertainty is equal to 0.5, while all other parameters are as in the standard setting. The observational and initial error being now much smaller than in the previous case, the ensemble of particles follows the truth uniformly. It tends to diverge from the truth just before the assimilation time, but the particle filter is very efficient in pushing the particles back, closer to the true trajectory. This can be easily observed in Figures 6.16-6.18. In Figure 6.19 we can notice that also RMSE and the ES is greatly decreased, varying now between 0 and 3.

In all previous settings the observation operator was linear. We now perform a couple of tests (Figures 6.20-6.27) with nonlinear observation operators. We first make the observations fully

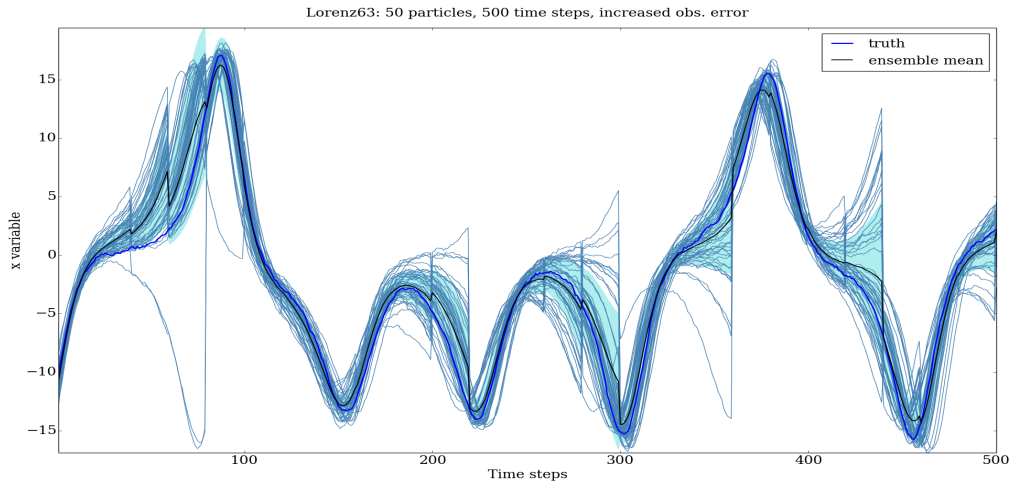


Figure 6.9: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observational uncertainty is increased to 5. Displayed: x variable. The accuracy of the observations decreases.

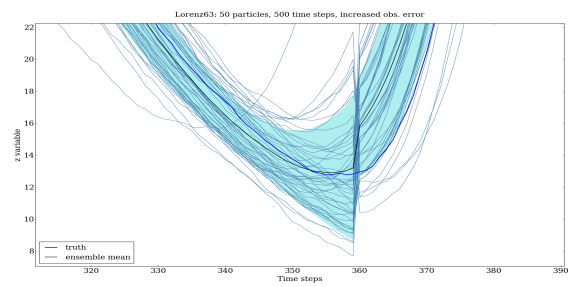
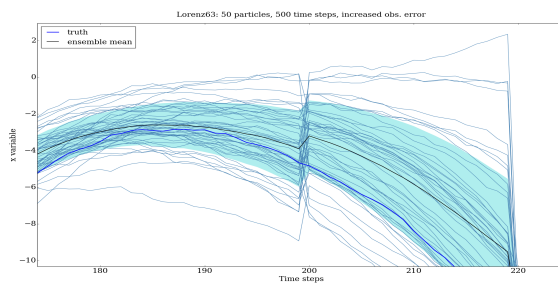


Figure 6.10: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observational uncertainty is increased to 5. Displayed: x enlarged (left) and z enlarged (right). Instances of the tempering procedure.

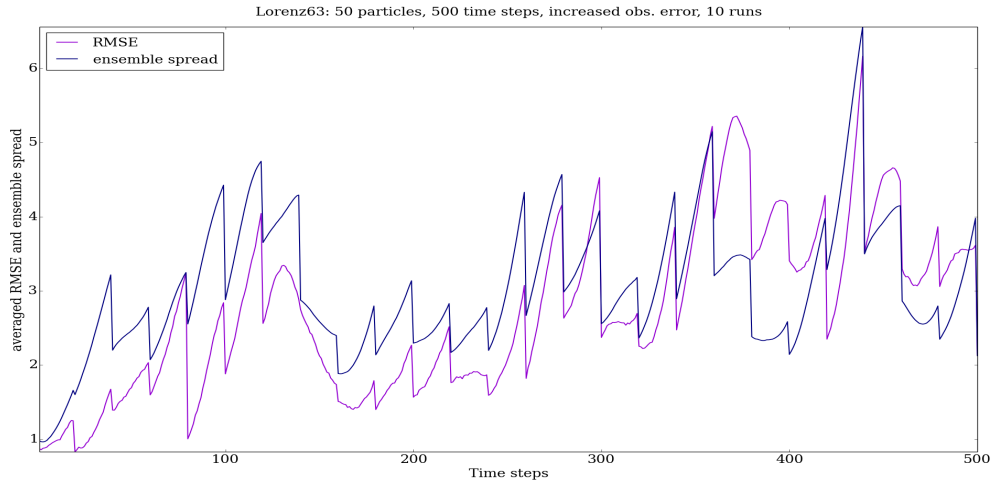


Figure 6.11: Evolution of the Lorenz '63 model for 500 time steps. Observational uncertainty is increased to 5. Displayed: averaged (10 runs) RMSE and ensemble spread. RMSE and ES have higher-amplitude oscillations around the data assimilation times.

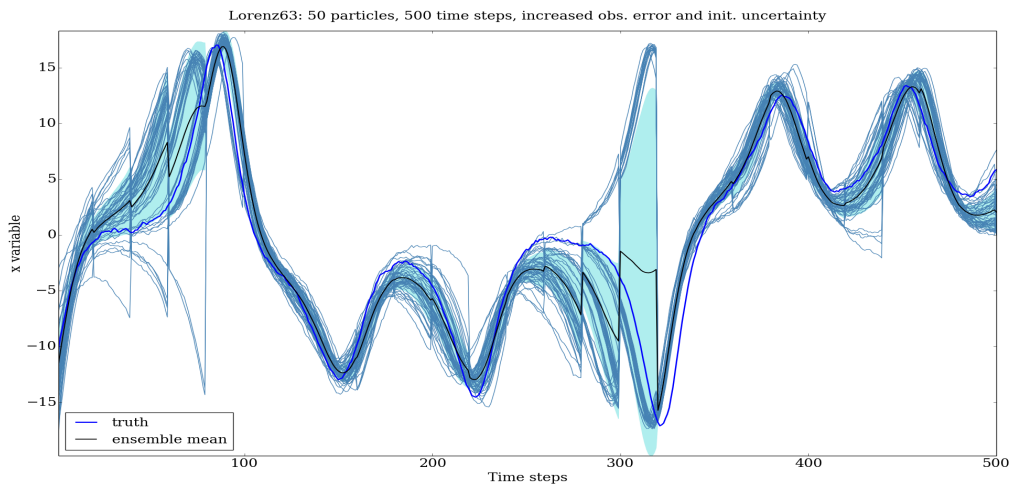


Figure 6.12: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observational uncertainty is increased to 5, initial uncertainty is increased to 3. Displayed: x variable. Small uncertainty reduction, but the particle filter constrains the particles to follow the truth.

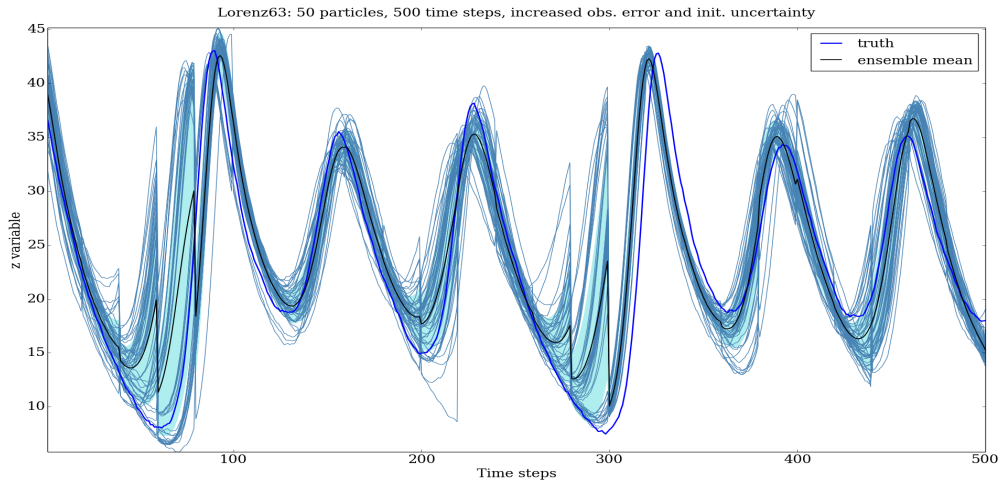


Figure 6.13: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observational uncertainty is increased to 5, initial uncertainty is increased to 3. Displayed: z variable. Small uncertainty reduction, but the particle filter constrains the particles to follow the truth.

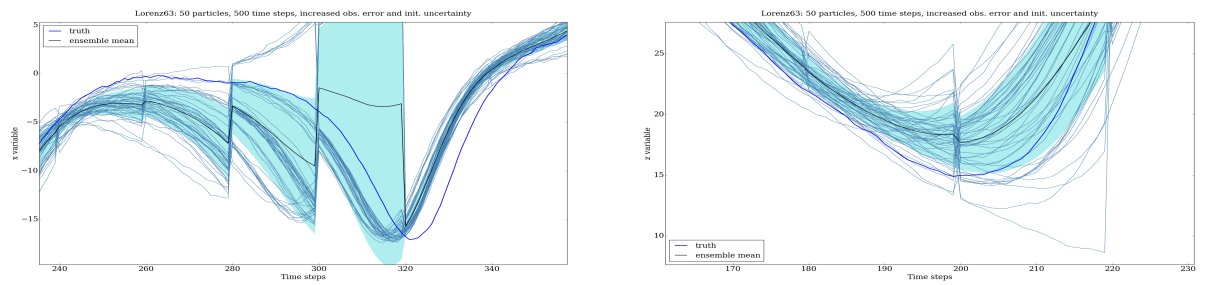


Figure 6.14: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observational uncertainty is increased to 5, initial uncertainty is increased to 3. Displayed: x enlarged (left) and z enlarged (right).

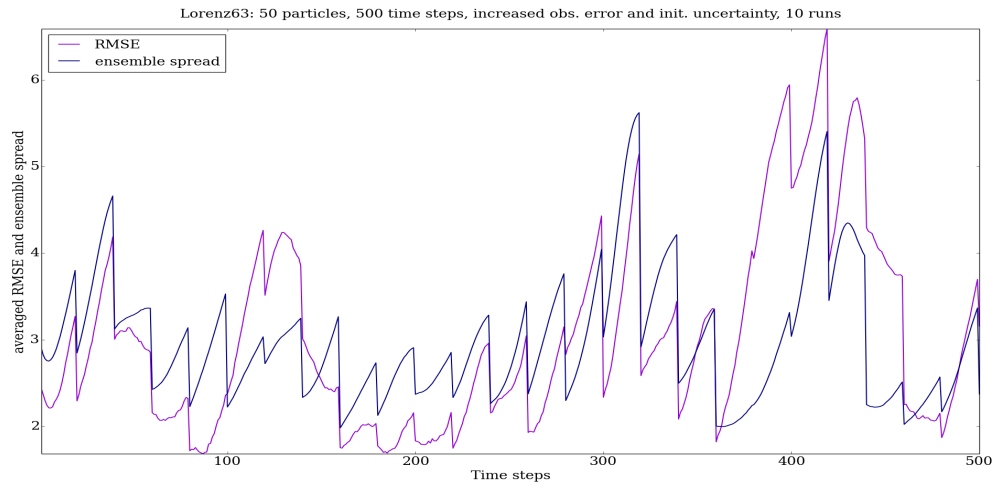


Figure 6.15: Evolution of the Lorenz '63 model for 500 time steps. Observational uncertainty is increased to 5, initial uncertainty is increased to 3. Displayed: averaged (10 runs) RMSE and ensemble spread. Small uncertainty reduction, but the particle filter constrains the particles to follow the truth.

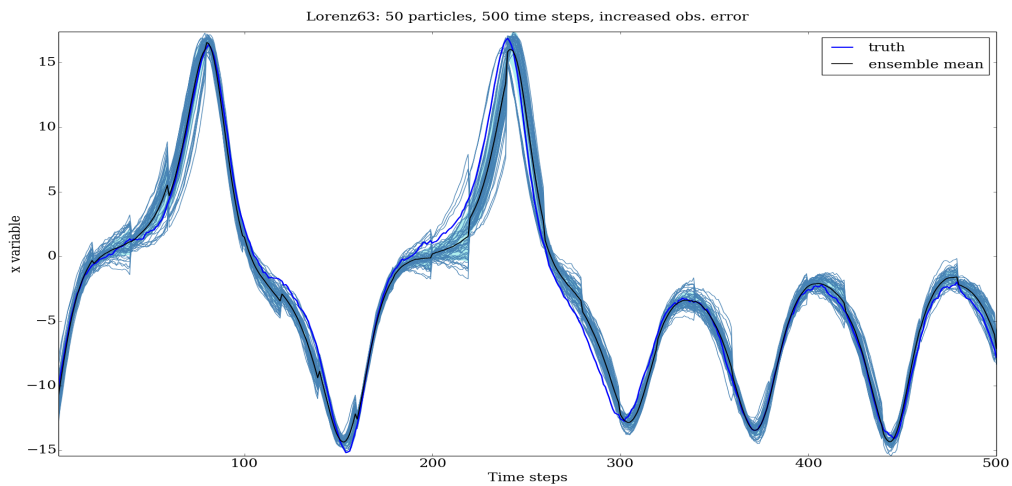


Figure 6.16: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observational uncertainty is increased to 0.5. Displayed: x variable.

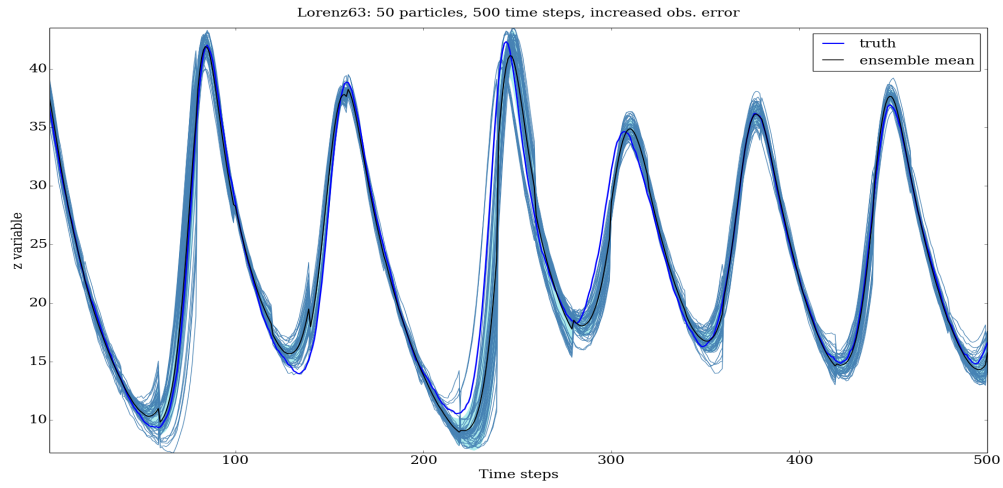


Figure 6.17: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observational uncertainty is increased to 0.5. Displayed: z variable.

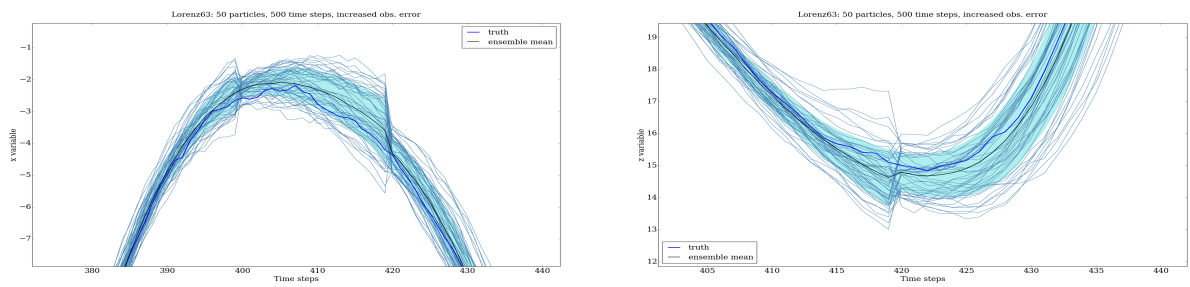


Figure 6.18: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observational uncertainty is increased to 0.5. Displayed: x enlarged (left) and z enlarged (right).

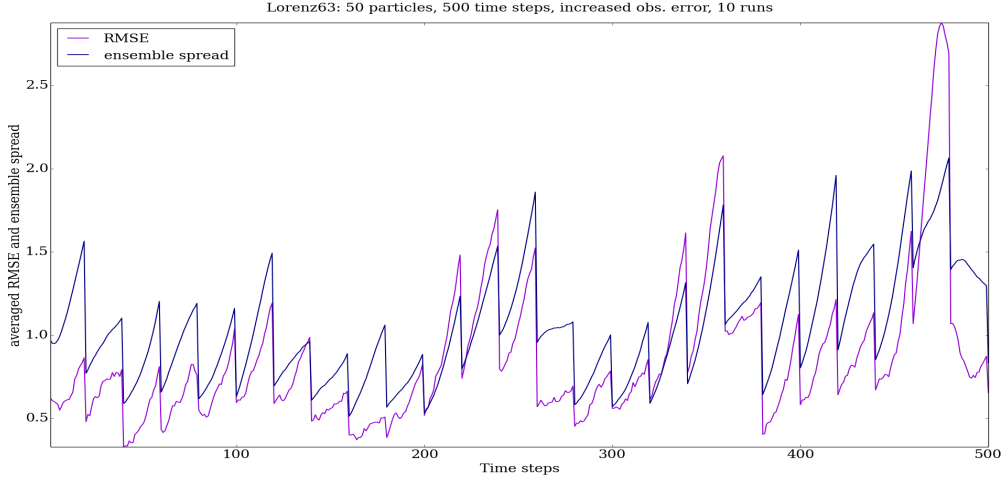


Figure 6.19: Evolution of the Lorenz '63 model for 500 time steps. Observational uncertainty is increased to 0.5. Displayed: RMSE and ensemble spread.

nonlinear (Figures 6.20-6.23). More precisely, the observation operator becomes

$$\mathcal{H}(x, y, z) = (x^2, y^2, z^2)$$

with observations given by

$$Z = \mathcal{H}(X_t) + V_t$$

as explained in Section 6.1. As we can see from Figure 6.20 the information available is not enough to keep the posterior concentrated around the truth at all times, as the system misses information in which wing of the butterfly it is. This is the case particularly with the x direction. The behaviour of the posterior in the z direction is much better, as one can see in Figure 6.21. The values of the RMSE and the ES are plotted in Figure 6.23. We can see that both can become very large, essentially of the order of the support of the diffusion (the attractor), as the particles can spread around the entire attractor. We now make the observation nonlinear just for the first variable (Figures 6.24-6.27). That is,

$$\mathcal{H}(x, y, z) = (x^2, y, z).$$

In contrast to the previous case, the particle filter performs much better. The (partial) linearity of the observation operator imposes a massive uncertainty reduction, the posterior distribution

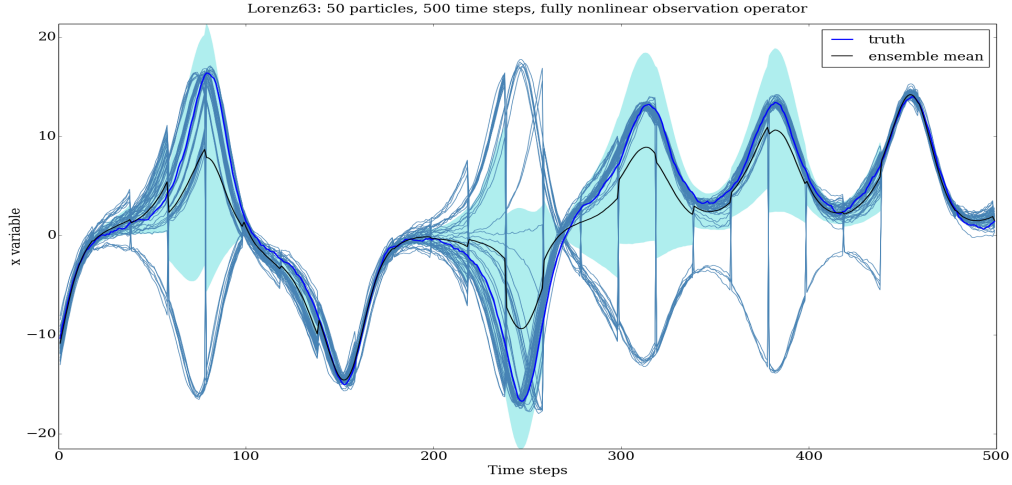


Figure 6.20: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observations are denoted by z_1, z_2, z_3 . Fully nonlinear observation operator i.e. $z_1 = x^2, z_2 = y^2, z_3 = z^2$ plus noise. Displayed: x variable. The particle filter struggles to keep the particles concentrated around the truth.

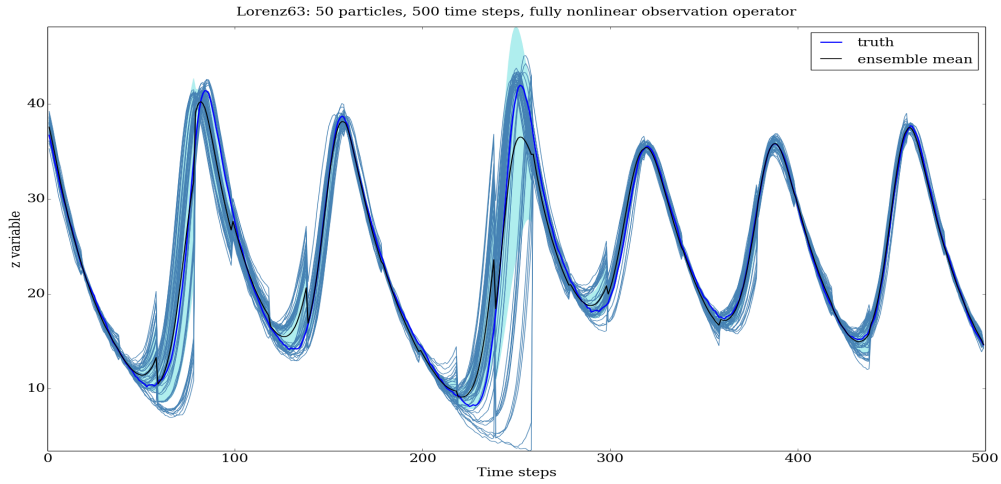


Figure 6.21: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observations are denoted by z_1, z_2, z_3 . Fully nonlinear observation operator i.e. $z_1 = x^2, z_2 = y^2, z_3 = z^2$ plus noise. Displayed: z variable.

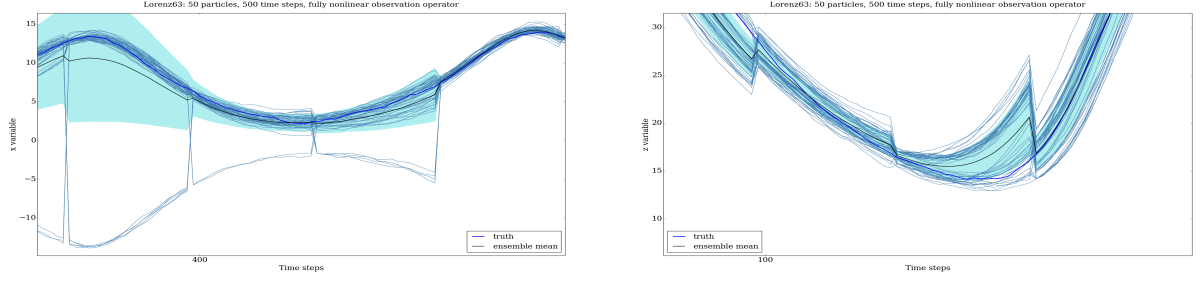


Figure 6.22: Evolution of the Lorenz '63 model for 500 time steps when all 3 variables are observed every 20 time steps. Observations are denoted by z_1, z_2, z_3 . Fully nonlinear observation operator i.e. $z_1 = x^2, z_2 = y^2, z_3 = z^2$ plus noise. Displayed: x enlarged (left) and z enlarged (right).

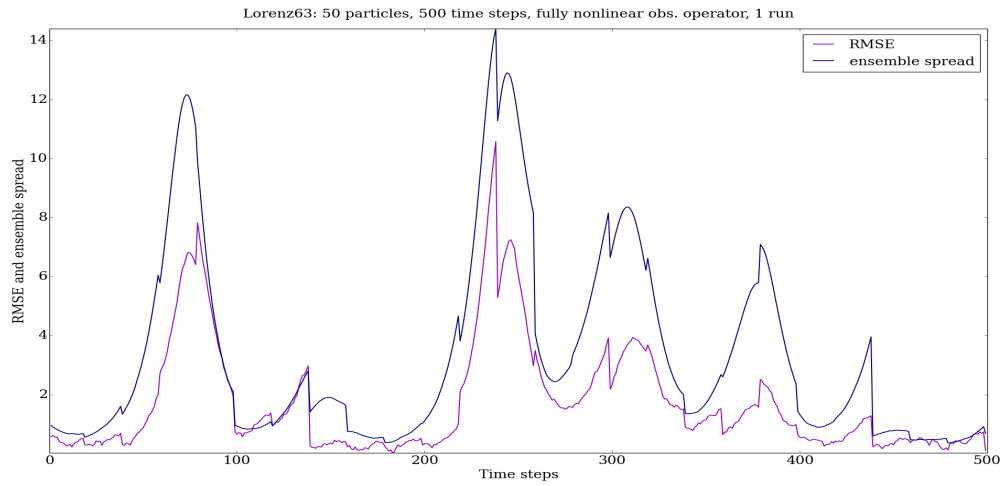


Figure 6.23: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observations are denoted by z_1, z_2, z_3 . Fully nonlinear observation operator i.e. $z_1 = x^2, z_2 = y^2, z_3 = z^2$ plus noise. Displayed: RMSE and ensemble spread.

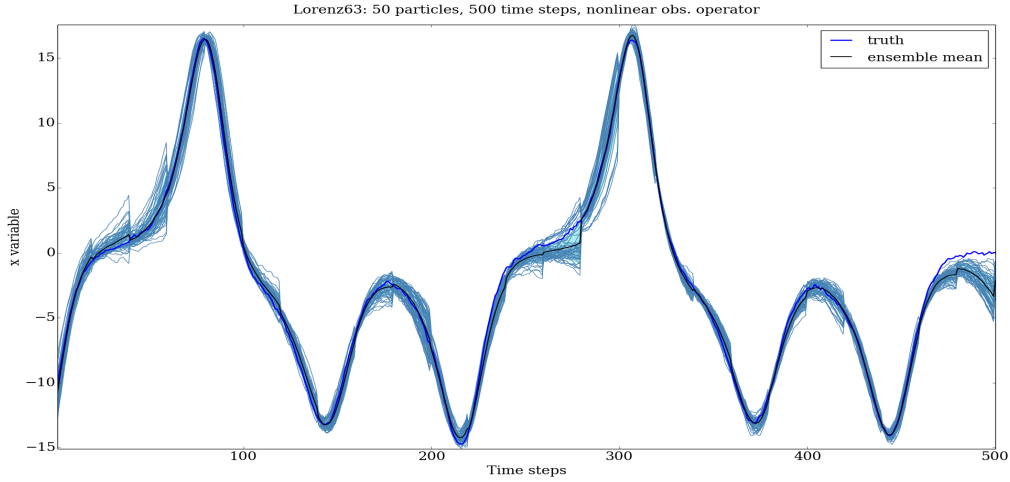


Figure 6.24: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observations are denoted by z_1, z_2, z_3 . Nonlinear observation operator for the first variable i.e. $z_1 = x^2, z_2 = y, z_3 = z$ plus noise. Displayed: x variable. The particles are concentrated around the truth.

being now much closer to the truth. The cloud of particles remains concentrated around the signal trajectory and the data assimilation steps keep the uncertainty in check. Both the x and the z variable (Figures 6.24-6.25) are successfully tracked, as highlighted in Figure 6.26. The RMSE and the ES plotted in Figure 6.27 confirm the findings: most of the time, the RMSE remains in the interval $[0, 1]$ with rare excursions away from this interval. The ensemble spread remains also small, with some oscillations around the assimilation time. The underlying reason for this behaviour is that in this case the system knows in which wing it is via the linear y observation.

We conclude that our methodology based on a bootstrap particle filter with tempering and jittering can be successfully used to assimilate data for the Lorenz '63 model. The performance of the data assimilation methodology is shown to be influenced by the following factors:

- Observational uncertainty: we test this for three values of the observation error (0.1, 0.5, and 5); the results improve as the observation error is reduced.
- Initial uncertainty: we test this in two cases (initially this parameter is set to 1, and then increased to 3). The results improve as the initial uncertainty is reduced, although that advantage decreases over time as the initial uncertainty is forgotten.
- Linearity of the observation operator - three observation operators are used: a fully linear

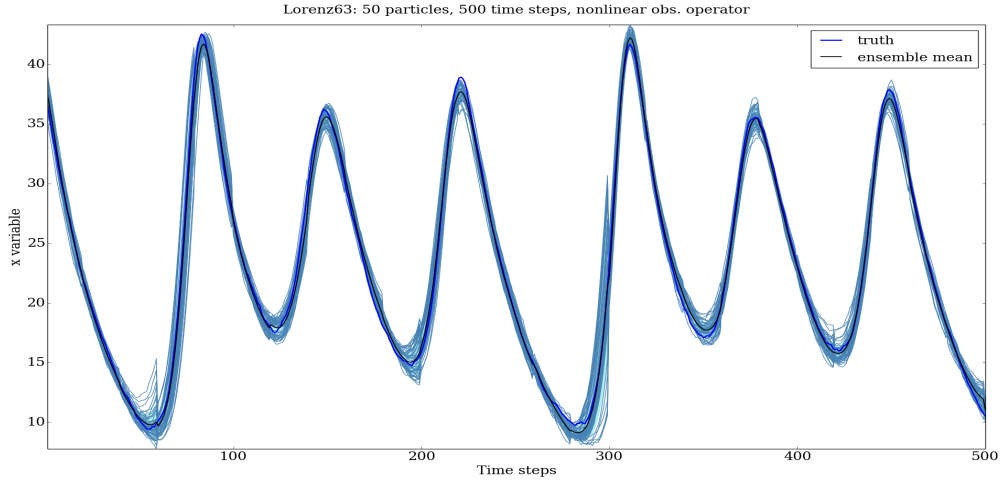


Figure 6.25: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observations are denoted by z_1, z_2, z_3 . Nonlinear observation operator for the first variable i.e. $z_1 = x^2, z_2 = y, z_3 = z$ plus noise. Displayed: z variable. The particles are concentrated around the truth.

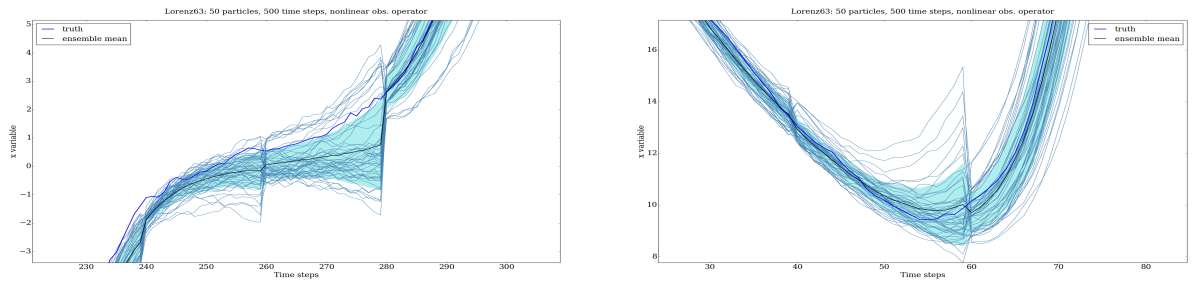


Figure 6.26: Evolution of the Lorenz '63 model for 500 time steps when all 3 variables are observed every 20 time steps. Observations are denoted by z_1, z_2, z_3 . Nonlinear observation operator for the first variable i.e. $z_1 = x^2, z_2 = y, z_3 = z$ plus noise. Displayed: x enlarged (left) and z enlarged (right).

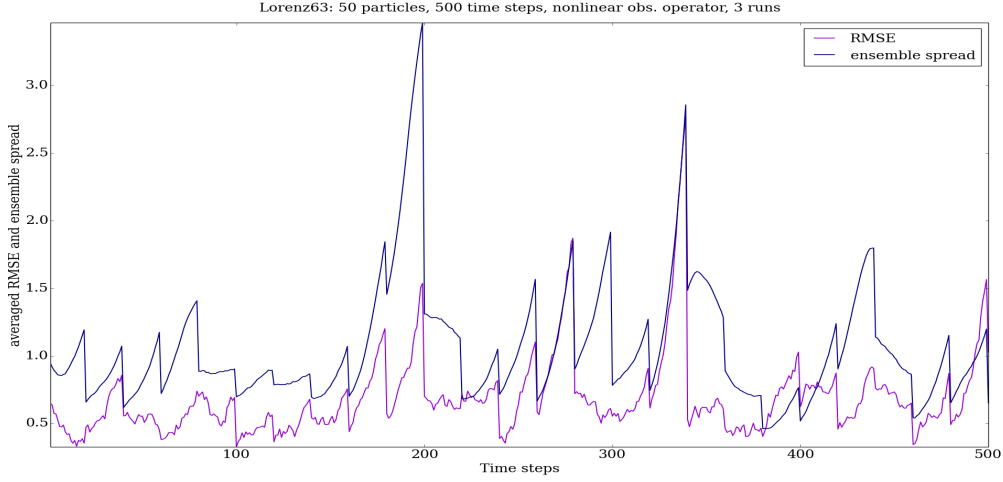


Figure 6.27: Evolution of the Lorenz '63 model for 500 time steps, all 3 variables are observed every 20 time steps. Observations are denoted by z_1, z_2, z_3 . Fully nonlinear observation operator i.e. $z_1 = x^2, z_2 = y, z_3 = z$ plus noise. Displayed: averaged (3 runs) RMSE and ensemble spread.

one ($\mathcal{H}(x, y, z) = (x, y, z)$), a partially linear one ($\mathcal{H}(x, y, z) = (x^2, y, z)$), and a fully nonlinear one ($\mathcal{H}(x, y, z) = (x^2, y^2, z^2)$). We obtain the best results in the linear case. However, it is worth highlighting that we obtain good results also in the nonlinear case, where we see a bimodal distribution. In contrast to this, a standard Kalman filter would probably fail, due to the difficulties generated by multiple modes.

In future research we intend to explore also the influence of other parameters such as: the number of particles, the size of data assimilation window, the *ess* threshold, the model error.

6.3 APPLICATIONS FOR THE STOCHASTIC ROTATING SHALLOW WATER MODEL

6.3.1 MODEL DESCRIPTION

The rotating shallow water model (RSW) is a classical nonlinear fluid dynamics model which contains key aspects of the oceanic and atmospheric dynamics. A detailed analytical description of this model has been provided in Chapter 5. From a numerical perspective, challenges are generated especially by the nonlinear advective terms. Nonlinear advection is a dispersive process. While in a linear case the wave-like solutions have constant amplitude and propagate at constant

speed, in the nonlinear setting waves of different wavenumbers can propagate at varying speeds. The short waves can be amplified by the nonlinear structure, with direct impact on the accuracy of the solution ([43]). These intricacies can be overcome by making use of the natural dominant balances which appear in atmosphere and oceans (geostrophic and hydrostatic balance) and by implementing the models using suitable numerical schemes.

The stochastic rotating shallow water model used in our work is given by the following set of equations:

$$dv_t + [u_t \cdot \nabla v_t + f \hat{z} \times u_t + \nabla p_t] dt + \sum_{i=1}^{\infty} [(\mathcal{L}_i + \mathcal{A}_i)v_t] \circ dW_t^i = \nu \Delta v_t dt \quad (6.24a)$$

$$dh_t + \nabla \cdot (h_t u_t) dt + \sum_{i=1}^{\infty} [\nabla \cdot (\xi_i h_t)] \circ dW_t^i = 0 \quad (6.24b)$$

where $\mathcal{L}_i v := \xi_i \cdot \nabla v$, $\mathcal{A}_i v := v_j \nabla \xi_i^j = \sum_{j=1}^2 v_j \nabla \xi_i^j$, $v = (v^1, v^2)$, $v := \epsilon u + \mathcal{R}$, $u = (u^1, u^2)$ is the horizontal fluid velocity vector, $p := \frac{h-b}{\epsilon \mathcal{F}}$ is the pressure term, h is the total depth, $b = b(x)$ is the bottom topography, $x = (x^1, x^2)$, $f = f(x)$ is the Coriolis parameter, $\hat{z}$ is a unit vector pointing away from the centre of the Earth, $\epsilon \ll 1$ is the Rossby number, $\mathcal{F}$ is the Froude number, $\mathcal{R} = \mathcal{R}(x)$ is the vector potential of the divergence-free rotation rate about the vertical direction, with $\text{curl } \mathcal{R}(x) = f(x)\hat{z}$, and ν is a positive constant which corresponds to the fluid viscosity.

We start with a velocity vector field which is in geostrophic balance at the initial time $t = 0$. Nonetheless, the geostrophic balance condition does not hold for the velocity field from $t = 1$ onwards, the fields evolving according to the full nonlinear equations (6.24). On the other hand, the random fields are assumed to stay in geostrophic balance at each time step and we give the explicit formula for this in the next subsection, equations (6.34). Intuitively, the geostrophic balance can be explained as follows ([101] pp. 58): initially there exists a pressure gradient within the fluid, which generates a fluid dynamics from high-pressure regions towards low-pressure regions; while the fluid flow evolves in time, it is deflected by the Coriolis force; in the Northern hemisphere the Coriolis force deflects the fluid to the right, while the pressure gradient does not change direction much, so that the two forces can come into equilibrium. The resulting direction

of the fluid motion is perpendicular on both the Coriolis force and the pressure force. In the atmosphere and oceans this balance tends to be stable, especially away from the boundaries. Small disturbances to it lead to the appearance of gravity waves. The geostrophic balance is dominant as the wind and the currents are usually weak in comparison to the speed of the Earth rotation ([101], pp.59). The Rossby number in this case is small ($\epsilon \sim 10^{-1}$), meaning that the rotation dominates the advective part and it is balanced by the pressure gradient force ([38], pp. 86). If the rotation is fast, then the horizontal flow is in 'near' geostrophic balance, that is 'nearly' divergence-free ([102], pp.95). We also assume geostrophic balance for the stochastic forcing described in Section 6.3.1, to strongly reduce the generation of artificial gravity waves.

We initialise the model by first calculating the Coriolis parameter and the pressure field for each grid point. Then we generate $v_0 = (v_0^1, v_0^2)$ using the geostrophic balance condition. The periodicity of the domain inspires a rigorous numerical setup for generating the stochastic forcing. This is described in Section 6.3.1. The system (6.24) is implemented on a staggered grid, using a Runge-Kutta scheme of order 4. The domain corresponds to a strip situated between 30 and 60 degrees north latitude. The numerical scheme is described in Appendix. A staggered grid (also known as Arakawa C-grid) is usually preferred when implementing weather prediction models due to the low dispersion errors and the lack of computational modes associated with it ([38], pp. 313).

STOCHASTIC FORCING

For a state vector X of dimension d the nonlinear stochastic forcing is given by

$$\sum_{i=1}^{\infty} \mathcal{B}_i(\xi_i) X_t dW_t^i \sim \sqrt{\mathcal{Q}} dW(X) \sim \left(\sum_{i=1}^{\infty} \mathcal{B}_i(\xi_i) dW^i \right) X =: R_i X \quad (6.25)$$

where $\mathcal{B}_i : L^2(\mathbb{T}^2) \rightarrow L^2(\mathbb{T}^2)$ is a general operator which depends on the vector fields ξ_i . In our case $\mathcal{B}_i$ can be given by $\mathcal{L}_i$ or by $\mathcal{L}_i + \mathcal{A}_i$. With $\sim$ we denoted the equivalence in notation of the terms above, it has no mathematical meaning other than expressing different ways of rewriting the same stochastic term. Then we interpret $\sum_{i=1}^{\infty} \mathcal{B}_i(\xi_i) dW_t^i$ as a random operator applied to the state vector X and we denote it by R_i . Here dW is a d -dimensional vector of independent Brownian motions and $\mathcal{Q}$ is a space-covariance operator. We discretize the SPDE in time and

space, and therefore $\mathcal{Q}$ becomes a covariance matrix of dimension d^2 .

Given the fact that generating $\sqrt{\mathcal{Q}}dW$ is computationally expensive, we first do this in the spectral space, and then return to the physical space. The covariance matrix is symmetric and circulant. We determine it in the spectral space for a system which is periodic in both the x and y directions, using the fast Fourier transform. Since our original domain is not periodic, we then truncate the resulting field to the physical domain, in order to avoid a periodic random field. In particular, we compute the Fourier transform of a column corresponding to the circulant matrix. This column has a Gaussian correlation structure. At every time step, we generate a Gaussian random field (again in the spectral space) and then perform the multiplication between the column of the covariance matrix and this newly generated random field. Finally, we use the inverse fast Fourier transform to return to the physical domain.

It is known (see [91]) that any continuous random field R_i can be expressed as a Fourier-Stieltjes integral over a complex-valued Fourier increment dY_{R_i} :

$$R_i(\mathbf{x}) = R_i(x, y) = \int_{-\infty}^{\infty} e^{i\boldsymbol{\mu} \cdot \mathbf{x}} dY_{R_i}(\boldsymbol{\mu}) \quad i = 1, 2, \dots \quad (6.26)$$

where the random component Y_{R_i} must satisfy the following properties:

$$\begin{aligned} \mathbb{E}[Y_{R_i}(\boldsymbol{\mu})] &= 0 \\ \mathbb{E}[Y_{R_i}(\boldsymbol{\mu})Y_{R_j}^*(\boldsymbol{\mu}')] &= 0 \quad \text{for } \boldsymbol{\mu} \neq \boldsymbol{\mu}' \\ \mathbb{E}[Y_{R_i}(\boldsymbol{\mu})Y_{R_j}^*(\boldsymbol{\mu}')] &= Q_{ij}(\boldsymbol{\mu})d\boldsymbol{\mu} \end{aligned} \quad (6.27)$$

where $Y_{R_j}^*$ is the complex conjugate of Y_{R_j} , Q_{ij} is the covariance between the two processes R_i and R_j , $n_{\mathbf{x}}$ is the dimension of the spatial vector $\mathbf{x}$, and $\boldsymbol{\mu} = (\mu_1, \mu_2, \dots, \mu_{n_{\mathbf{x}}})$ corresponds to the wave number. Therefore, the key ingredient in generating the random fields $(R_i)_i$ is the accurate retrieval of the processes $Y_{R_i}(\boldsymbol{\mu})$. In practice this should be effectuated in discrete time, so that one can implement the integral (6.26) numerically. Inspired by the periodic structure of the domain, one way of efficiently performing this is by using a complex-valued spectral decomposition ([46]). We know from classical Fourier analysis that the vector fields R_i can also be expressed as

$$R_i(\mathbf{x}) = \int_{-\infty}^{\infty} e^{i\boldsymbol{\mu} \cdot \mathbf{x}} \hat{R}_i(\boldsymbol{\mu}) d\boldsymbol{\mu} \quad (6.28)$$

where $\hat{R}_i$ is the Fourier transform of R_i . Therefore we can write

$$R_i(x_{k_1}, y_{k_2}) = \sum_{j,k} \hat{R}_i(\mu_j, \mu_k) e^{i(\mu_j x_{k_1} + \mu_k y_{k_2})} \Delta \boldsymbol{\mu} \quad (6.29)$$

with $x_{k_1} = k_1 \Delta x$, $y_{k_2} = k_2 \Delta y$, $\mu_j = \frac{2\pi j}{n \Delta x}$, $\mu_k = \frac{2\pi k}{m \Delta y}$, $\Delta \boldsymbol{\mu} = \Delta \mu_j \Delta \mu_k = \frac{(2\pi)^2}{nm \Delta x \Delta y}$ and

$$\hat{R}_i(\mu_j, \mu_k) = \frac{C}{\sqrt{\Delta \boldsymbol{\mu}}} e^{-\frac{\mu_j^2 + \mu_k^2}{\sigma^2} + 2\pi i \varphi_{jk}} \quad (6.30)$$

where (n, m) is the dimension of the grid and $\varphi_{jk} \in [0, 1]$ is a random number which introduces a random phase shift for any given wave number. Therefore

$$\begin{aligned} R_i(x_{k_1}, y_{k_2}) &= C \sqrt{\Delta \boldsymbol{\mu}} \sum_{j,k} e^{-\frac{\mu_j^2 + \mu_k^2}{\sigma^2} + 2\pi i \varphi_{jk}} e^{i(\mu_j x_{k_1} + \mu_k y_{k_2})} \\ &= |\Delta \boldsymbol{\mu}|^{1/2} \sum_{j,k} e^{\varphi_{jk}(\boldsymbol{\mu})} C e^{i\boldsymbol{\mu} \cdot \mathbf{x}} \\ &\approx \Delta Y_{R_i}(\boldsymbol{\mu}) \approx dY_{R_i}(\boldsymbol{\mu}) \end{aligned} \quad (6.31)$$

where φ_{jk} is a random process which corresponds to the (discretized) wave number domain with volume element given by $|\Delta \boldsymbol{\mu}| = \prod_{j=1}^{n_x} \Delta \mu_j$ for any fixed wave number $\boldsymbol{\mu}$. Therefore

$$\begin{aligned} R_i(\mathbf{x}) &= \int_{-\infty}^{\infty} e^{i\boldsymbol{\mu} \cdot \mathbf{x}} dY_{R_i}(\boldsymbol{\mu}) \\ &\approx \sum_{j,k} e^{\varphi_{jk}(\boldsymbol{\mu})} C e^{i\boldsymbol{\mu} \cdot \mathbf{x}} |\Delta \boldsymbol{\mu}|^{1/2} \end{aligned} \quad (6.32)$$

that is, a summation over all the discrete wave numbers of the grid. By the central limit theorem, R_i are normally distributed. In order for the expression in 6.32 to be well-defined, ΔY_{R_i} and ΔY_{R_j} must be orthogonal in the limit as the discretization becomes finer ([91]). An efficient way of computing 6.32 is by using a fast Fourier transform algorithm. One has

$$\begin{aligned} \mathbb{E}[R_i(x) R_j^*(x')] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\boldsymbol{\mu} \cdot \mathbf{x} - i\boldsymbol{\mu}' \cdot \mathbf{x}'} \mathbb{E}[dY_{R_i}(\boldsymbol{\mu}) dY_{R_j}^*(\boldsymbol{\mu}')] \\ &= \int_{-\infty}^{\infty} e^{i\boldsymbol{\mu} \cdot (\mathbf{x} - \mathbf{x}')} Q_{ij}(\boldsymbol{\mu}) (d\boldsymbol{\mu}) \end{aligned} \quad (6.33)$$

that is the covariance of R_i and R_j^* can be written as the inverse Fourier transform of Q_{ij} .

More details on random field generators can be found in [91] and [46]. As mentioned in the previous section, we assume that the dominant balance for the random vectors in this system is the *geostrophic balance*, that is the pressure gradient force and the Coriolis force balance each other. We have three random vectors: R^{v^1}, R^{v^2} corresponding to each component of the velocity $v = (v^1, v^2)$, and R^p which corresponds to the pressure field p . We obtain them as follows: first, we compute R^p using equation (6.29); then we use R^p to compute R^{v^1} and R^{v^2} under the geostrophic balance assumption:

$$R^{v^1} = -\frac{g}{f} \frac{\partial R^p}{\partial y} \quad (6.34a)$$

$$R^{v^2} = \frac{g}{f} \frac{\partial R^p}{\partial x}. \quad (6.34b)$$

CONNECTION WITH SALT

The SRSW model considered here has been derived using the SALT approach described in [62] (see Appendix, Section 8.2). Equivalently, it can be derived using a more general semi-martingale driven variational principle as described in [86]. Stochasticity is introduced in the advection part of the dynamics to model the uncertain transport behaviour within the fluid flow. To the best of our knowledge this is the first implementation of the SALT SRSW model in a data assimilation setting. Numerical implementations and particle filter algorithms for other SALT models (2D Euler, SQG) have been extensively developed in [30]-[33]. Our numerical implementation of the stochastic forcing is equivalent to the one described in [30] up to an isomorphism. More precisely, if one extends the approach described in [30] to a periodic domain, then the stochastic vector fields correspond to the Fourier modes of the transport covariance matrix and they form a basis of the underlying space. This basis may be different from the basis chosen by us above. However, an isomorphism can be established between any two orthogonal bases corresponding to the same underlying space.

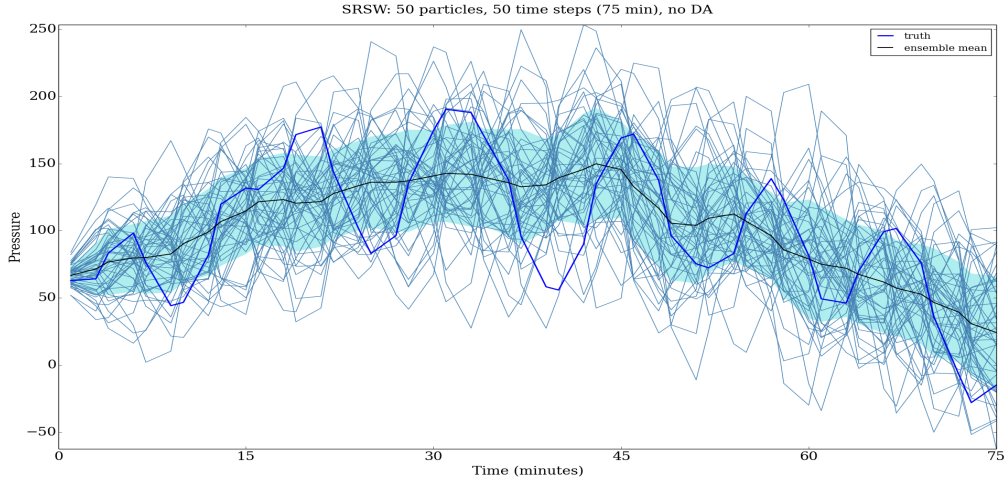


Figure 6.28: Evolution of the SRSW model for 50 time steps. No assimilation of data. Displayed: pressure field. The ensemble mean is far from the truth.

6.3.2 DATA ASSIMILATION RESULTS

We perform the data assimilation analysis using an ensemble of 50 particles. We plot the ensemble of particles one standard deviation region about the ensemble mean and compare it with the truth. The *truth* is a pathwise realisation of the SRSW model. In the *standard* setting (Figures 6.30-6.32) we observe the system for 50 time steps, with a time step size of 90 seconds (hence we observe every 1.25 hour). We use one observation at each analysis time, and the observational uncertainty and the initial uncertainty are equal to 1. The stochastic model error is set to 200 metres. As mentioned, the random forcing is implemented using a stochastic advection velocity in SALT. We generate these stochastic advection fields by first generating a stochastic pressure field of amplitude 200 m, and then determining the velocity fields in the zonal and meridional directions. These then form the stochastic advection fields that are applied in all three evolution equations.

We first show the output obtained when no data is assimilated. Figure 6.28 contains the evolution of the ensemble of particles corresponding to the pressure field at a grid point in the middle of the domain. We can see that the prior distribution is spread out and the truth is not successfully tracked (the ensemble mean remains far from the truth).

Now we start to observe the system every 10 time steps (Figures 6.30-6.32), so every 15 minutes.

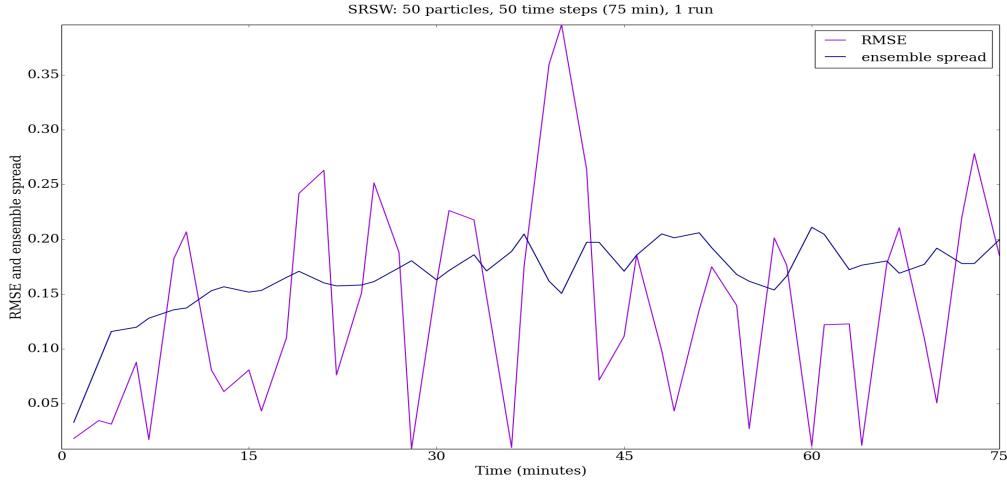


Figure 6.29: Evolution of the SRSW model for 50 time steps. No assimilation of data.
Displayed: RMSE and ensemble spread.

In Figure 6.30 one can see that the assimilation of data improves the performance of the ensemble of particles: the standard deviation is reduced and the particle trajectories are *corrected* at assimilation times (see Figure 6.31).

We then decrease the stochastic model error to 50 and observe the system every 5 time steps (Figures 6.33-6.35). In figure 6.33 one can notice that the particle filter efficiency is significantly improved by the data assimilation time window: by observing the system more often we obtain a cloud of particles which are better concentrated around the truth. This success is due also to the fact that we decreased the model error to 50. The RMSE and SE illustrated in Figure 6.35 are much smaller than before.

Remark 6.1. *Note that all plots are rescaled automatically.*

From Figure 6.36 we can easily learn that the particle filter is robust with respect to the SRSW model error: although we increased the model error 5 times (compared to the *standard scenario* described in the beginning), the truth is still tracked by the ensemble of particles. This is remarkable, as we increased again the data assimilation window and the system is observed only every 10 time steps. Pointwise, the particle filter does not always succeed in capturing the truth, but overall the cloud of particles performs well: the true trajectory is most of the time within the one standard deviation region around the ensemble mean. This is however done with

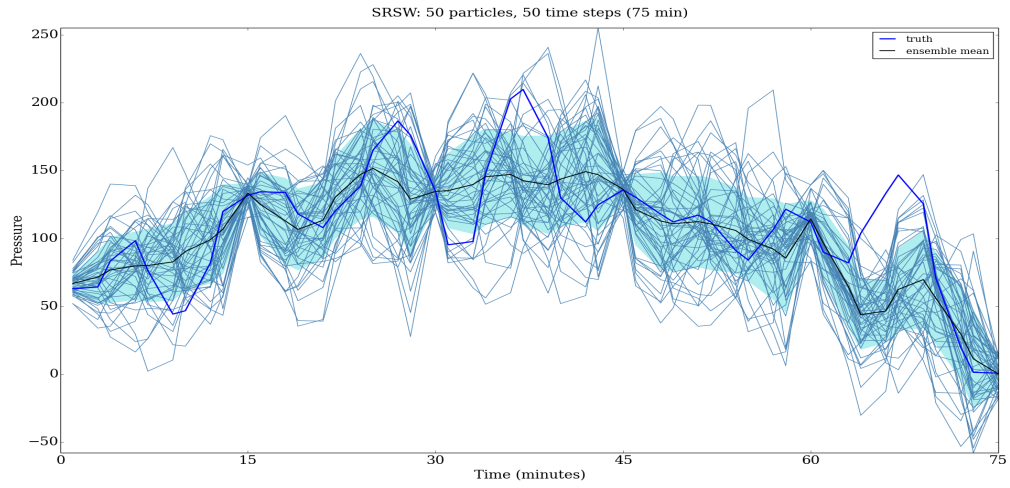


Figure 6.30: Evolution of the SRSW model for 50 time steps. The system is observed every 10 time steps. Displayed: pressure field. The ensemble of particles is drawn towards the truth at observation times.

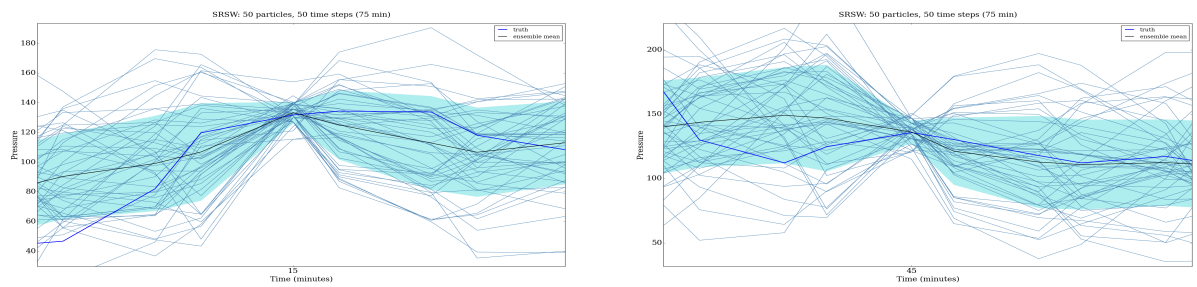


Figure 6.31: Evolution of the SRSW model for 50 time steps. The system is observed every 10 time steps. Displayed: pressure field - enlarged. Instances of the tempering procedure.

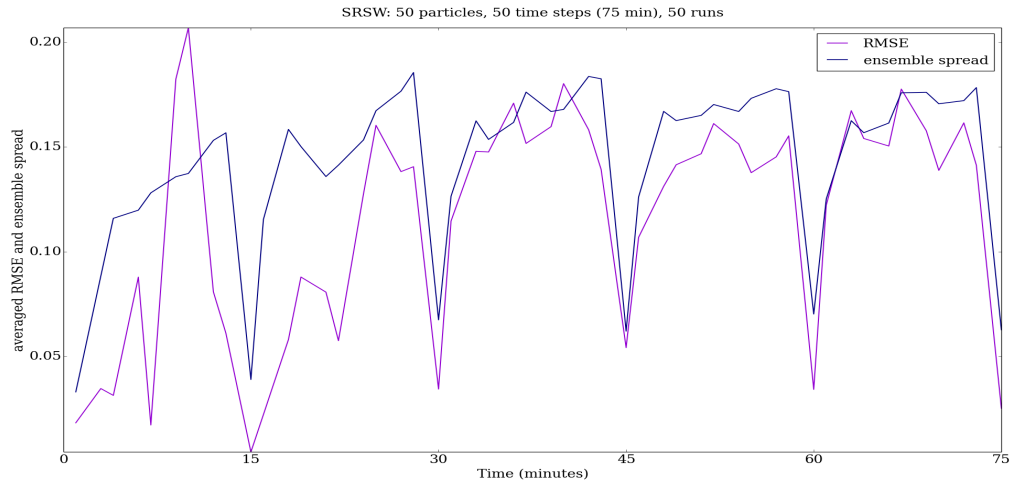


Figure 6.32: Evolution of the SRSW model for 50 time steps. The system is observed every 10 time steps. Displayed: averaged (50 runs) RMSE and ensemble spread. The RMSE and the ensemble spread are comparable and both decrease at assimilation times.

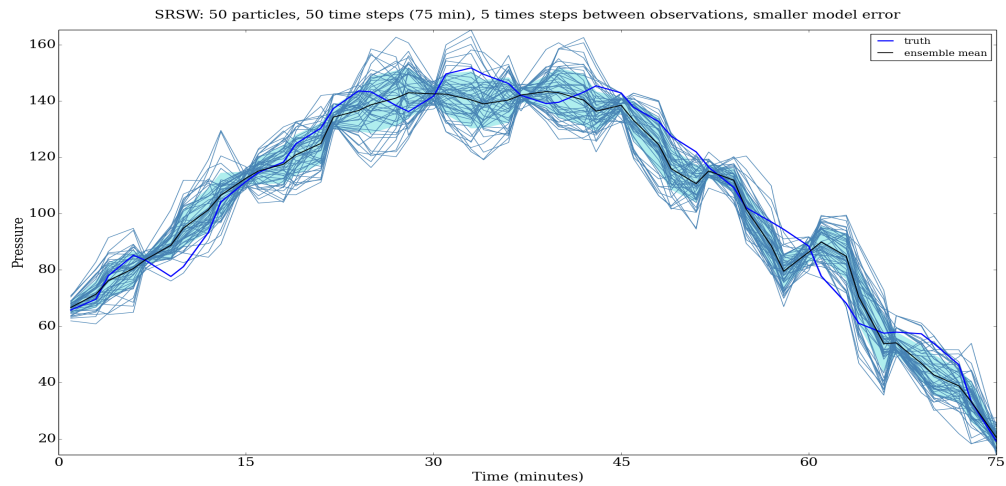


Figure 6.33: Evolution of the SRSW model for 50 time steps. The system is observed every 5 time steps. The model error is decreased to 50. Displayed: pressure field. The particle filter efficacy is improved by the time window and the decreased model error.

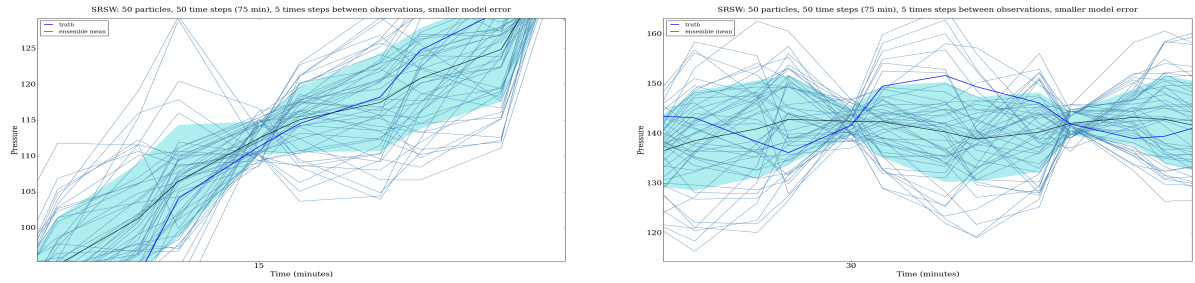


Figure 6.34: Evolution of the SRSW model for 50 time steps. The system is observed every 5 time steps. The model error is decreased to 50. Displayed: pressure field - enlarged.

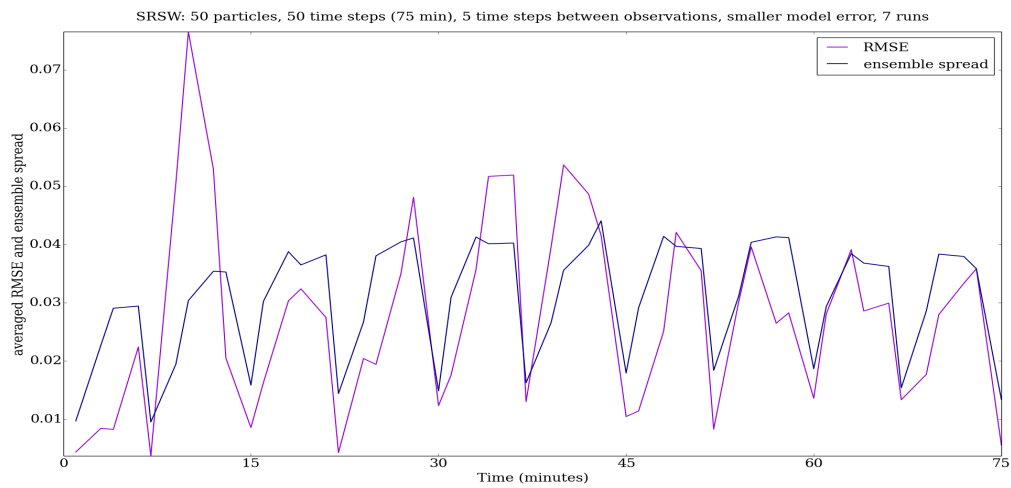


Figure 6.35: Evolution of the SRSW model for 50 time steps. The system is observed every 5 time steps. The model error is decreased to 50. Displayed: averaged (7 runs) RMSE and ensemble spread.

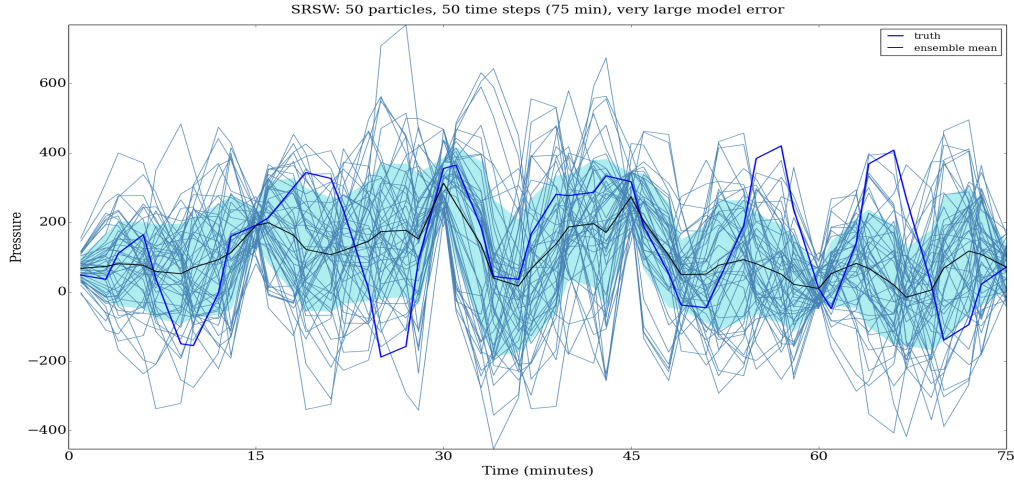


Figure 6.36: Evolution of the SRSW model for 50 time steps. The system is observed every 10 time steps. The model error is increased to 1000. Displayed: pressure field. The particle filter is robust with respect to the model error.

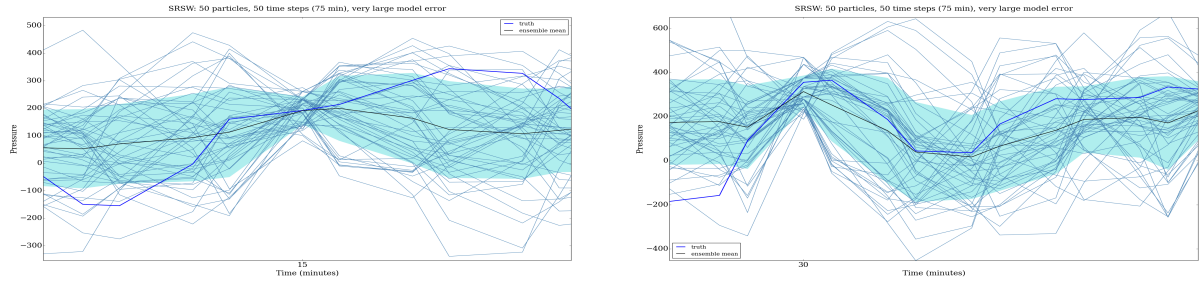


Figure 6.37: Evolution of the SRSW model for 50 time steps. The system is observed every 10 time steps. The model error is increased to 1000. Displayed: pressure field - enlarged.

less accuracy than before but both the RMSE and ES tend to be stable (Figure 6.38). For the next scenario (Figures 6.39-6.41) the system is observed every 5 time steps and we use 100 observations at each analysis time. Figure 6.39 shows that a drastic increase in the number of observations (100 now as opposed to 1 as we had before) greatly improves the efficiency of the particle filter. In Figure 6.41 one can see that the ensemble spread is substantially reduced. The accuracy is now also much better, as emphasized in Figure 6.40. The true innovation is that although we do not perform any localisation, we can assimilate 100 observations without filter degeneracy. This suggest that this particle filter is an important step towards beating the *curse of dimensionality* that plagued particle filtering for so long. We show below (Figures 6.42-6.44) the evolution of the SRSW model for 100 time steps, when observed every 5 time steps, with 5

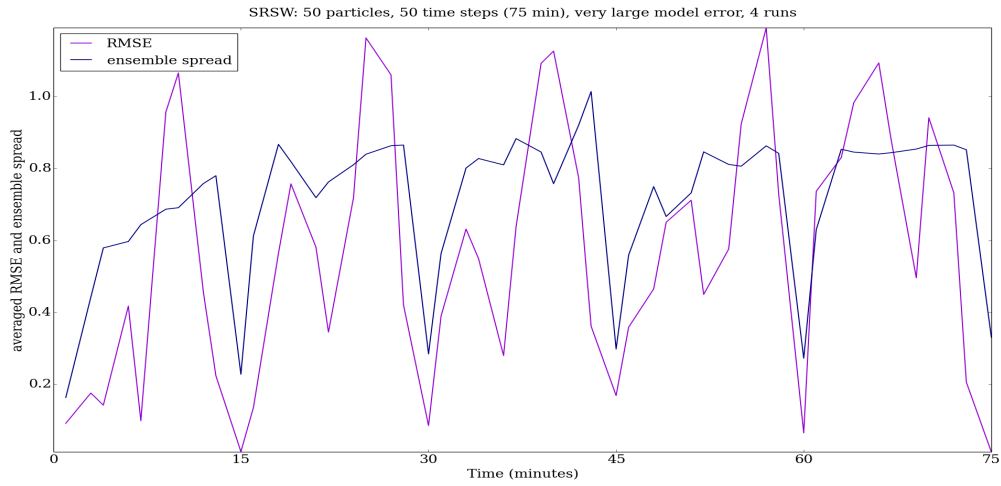


Figure 6.38: Evolution of the SRSW model for 50 time steps. The system is observed every 10 time steps. The model error is increased to 1000. Displayed: averaged (4 runs) RMSE and ensemble spread. The particle filter is robust with respect to the model error.

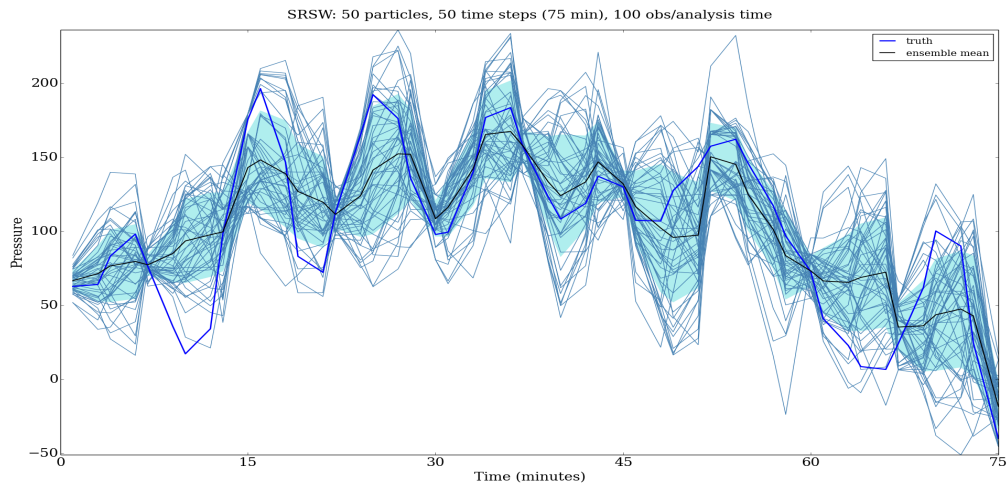


Figure 6.39: Evolution of the SRSW model for 50 time steps. The system is observed every 5 time steps, with 100 observations/analysis time. Displayed: pressure field. A large number of observations improves the performance of the particle filter.

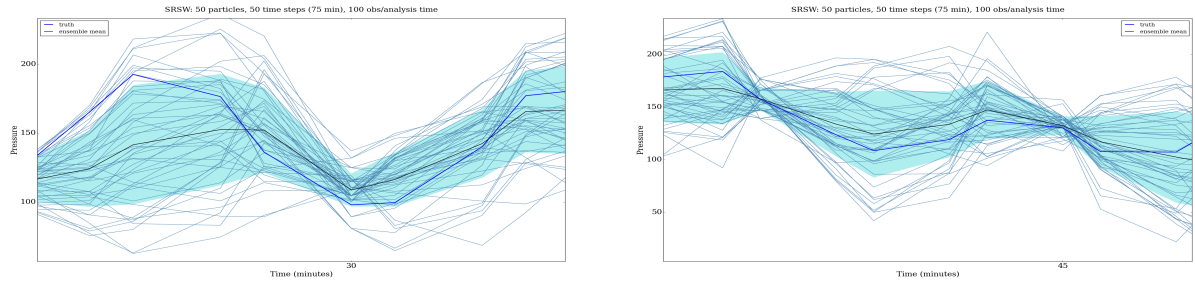


Figure 6.40: Evolution of the SRSW model for 50 time steps. The system is observed every 5 time steps, with 100 observations/analysis time. Displayed: pressure field - enlarged.

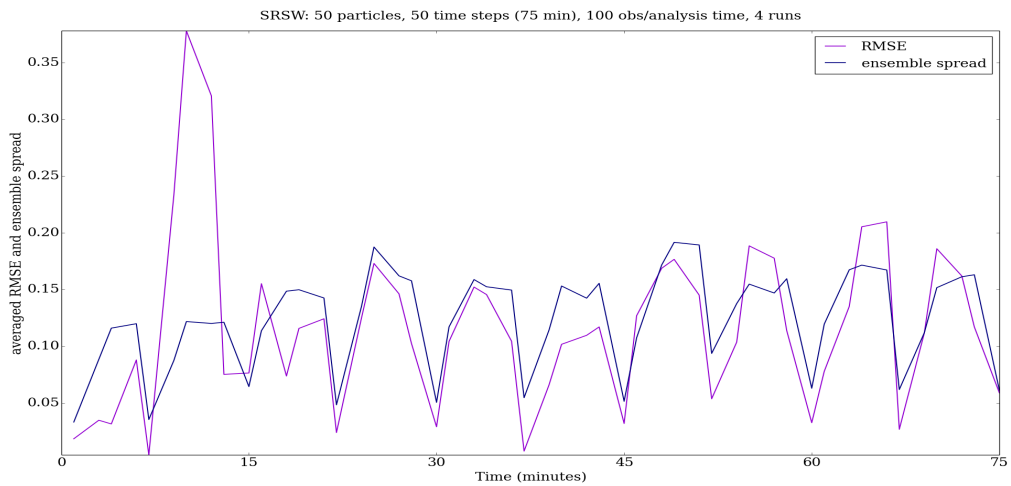


Figure 6.41: Evolution of the SRSW model for 50 time steps. The system is observed every 5 time steps, with 100 observations/analysis time. Displayed: averaged (4 runs) RMSE and ensemble spread. A large number of observations improves the performance of the particle filter.

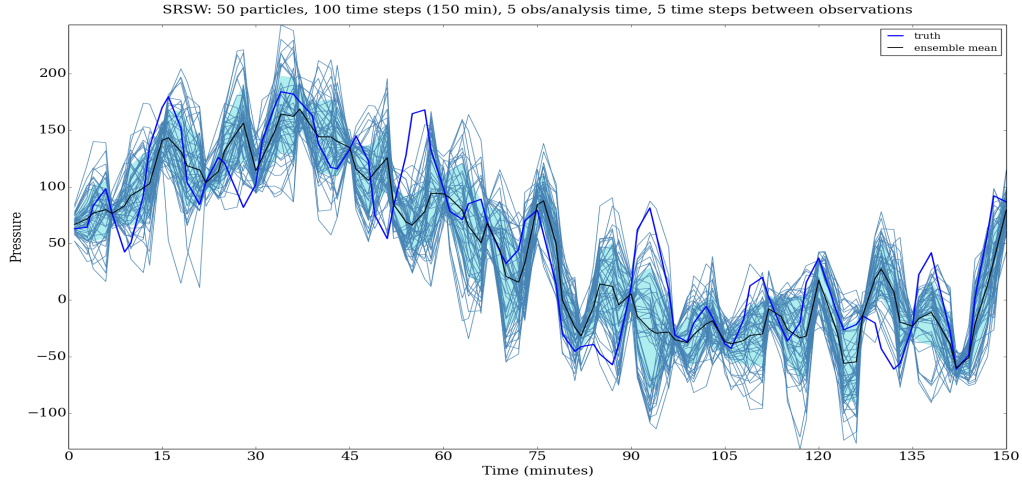


Figure 6.42: Evolution of the SRSW model for 100 time steps. The system is observed every 5 time steps, with 5 observations/analysis time. Displayed: pressure field. The cloud of particles follows the truth most of the time.

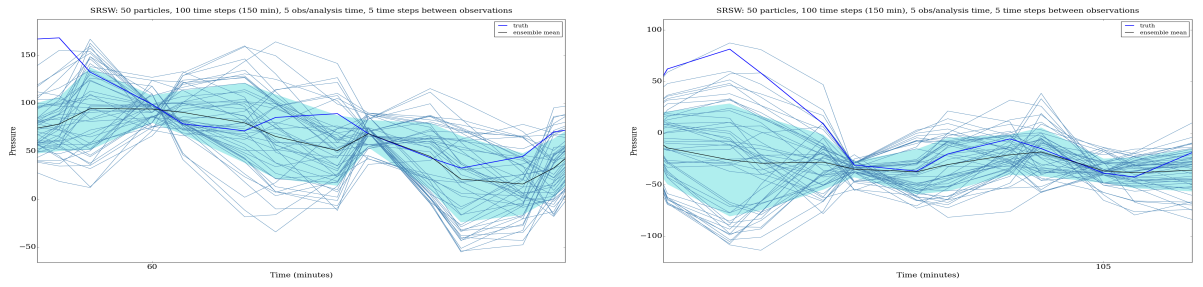


Figure 6.43: Evolution of the SRSW model for 100 time steps. The system is observed every 5 time steps, with 5 observations/analysis time. Displayed: pressure field - enlarged. Instances of the tempering procedure.

observations at each analysis time. Although the particles have a constant tendency to diverge from the truth, especially after the first assimilation step, by observing the system quite often and making the observations informative enough, we capture the truth most of the time. We conclude that our particle filter based on tempering and jittering can be successfully used also for the SRSW model. We highlight that this particle filter has been designed without using any other approximation methods (such as localisation). According to our results, there are a couple of parameters which have a relevant influence on the proficiency of the particle filter:

- DA time window: the results are better when we observe the system every 5 time steps, compared to when we observe it every 10 time steps.

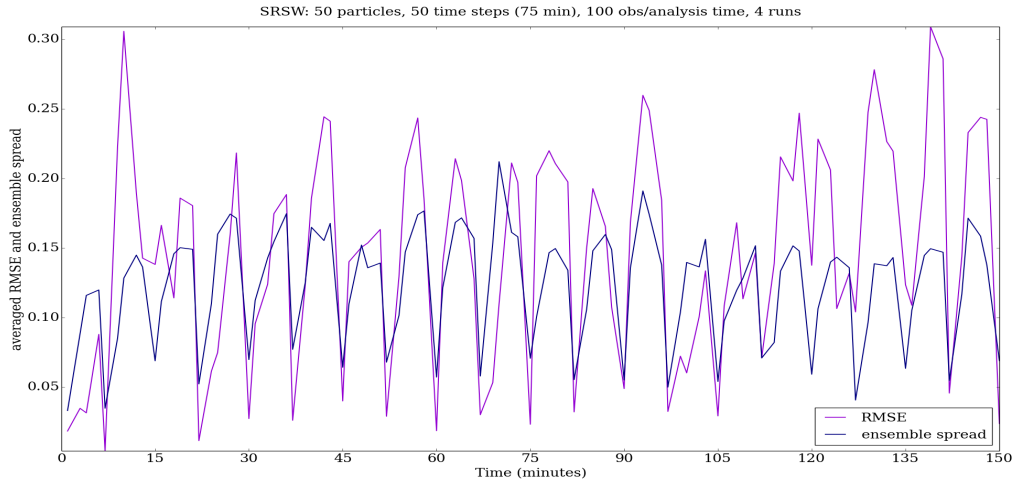


Figure 6.44: Evolution of the SRSW model for 50 time steps. The system is observed every 5 time steps, with 5 observations/analysis time. Displayed: averaged (4 runs) RMSE and ensemble spread. The RMSE and the ensemble spread are comparable, the RMSE tends to increase towards the end of the data assimilation time window, but then it decreases again.

- Number of observations per assimilation time: we test this for 1 observation, 5 observations, and 100 observations, respectively, per analysis time. The best output is obtained in the last case. However, by reducing the number of observations from 100 to 5, but observing the system every 5 time steps, the truth can be well tracked for a long period of time (Figure 6.42).
- Model error: the particle filter is shown to be robust with respect to increases in model error (compare Figures 6.36 and 6.38 with the rest of the experiments, for instance).

The experiments show that the system is underobserved. This is to be expected, as the number of degrees of freedom is very high ($\sim 133,440$). Nonetheless, the findings in this chapter are sufficiently promising to encourage further in-depth investigations based on this particle filter. We intend to explore also the influence of other parameters such as: the number of particles, observational and initial uncertainty, the *ess* threshold, variations of the stochastic forcing. Moreover, we intend to use this work as a stepping stone for modelling a slice of the real atmosphere using pressure observations collected by DWD using commercial aircraft.

7

SUMMARY AND OUTLOOK

In this thesis I have studied analytical properties and applications to data assimilation for three nonlinear stochastic transport partial differential models which originate in fluid dynamics and are driven by transport noise. Broadly speaking, the aim of these new stochastic equations is to respond to the challenge of successfully encoding randomness into a priori deterministic models. However, there are not too many mathematical results on this topic so far. The results I have presented in this thesis constitute one of the very first contributions towards a rigorous understanding and numerically validated methodology for this type of SPDEs.

In the first part (Chapters 3-5) I have developed an analytical methodology to construct a pathwise unique strong solution for three stochastic fluid dynamics models: the stochastic 2D Euler equation, the stochastic great lake equation, and the stochastic rotating shallow water model. The well-posedness strategy is based on constructing an approximating sequence of solutions which is proven to be relatively compact and to converge to the solution of the original equation in a suitable sense. The solution is global for the 2D Euler equation and local for the stochastic great lake equation and for the stochastic rotating shallow water model. The framework is applicable also to other types of SPDEs: I am currently working on extending

these techniques to SPDEs derived using the Location Uncertainty (see [80]) approach.

In the second part (Chapter 6) I have proven the applicability of one of these transport models (the stochastic rotating shallow water model) in a data assimilation setting. In this sense, I have implemented a data assimilation methodology based on a bootstrap particle filter combined with tempering and jittering, without any other approximations. The methodology has been shown to work first on the Lorenz '63 model and then applied to the stochastic rotating shallow water model. I have run more than 200 experiments overall. Based on these experiments, my conclusion is that the particle filter implemented in Chapter 6 is robust and can be used as a springboard for modelling a part of the real atmosphere. This is part of a collaboration project with Roland Potthast from DWD.

8

APPENDIX

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8.1 THE RUNGE-KUTTA APPROXIMATION SCHEME

For a dynamical system

$$\frac{dX}{dt} = F(X, t) \quad (8.1)$$

the Runge-Kutta scheme of order 4 (see e.g. [43]) is given by

$$X^{n+1} = X^n + \frac{\Delta t}{6}(c_1 + 2c_2 + 2c_3 + c_4) \quad (8.2)$$

where $c_i, i \in \{1, 2, 3, 4\}$ are the intermediate steps

$$c_1 = F(X^n, t) \quad (8.3a)$$

$$c_2 = F\left(X^n + \frac{\Delta t}{2}c_1, t + \frac{\Delta t}{2}\right) \quad (8.3b)$$

$$c_3 = F\left(X^n + \frac{\Delta t}{2}c_2, t + \frac{\Delta t}{2}\right) \quad (8.3c)$$

$$c_4 = F(X^n + \Delta t c_3, t + \Delta t) \quad (8.3d)$$

with $X^n = X(t_n) = X_{t_n}$ and $t_n = n\Delta t$ for $n \in \{0, 1, 2, \dots, N-1\}$, $t_N = T$, $\Delta t = t_{n+1} - t_n$. Each of these intermediate steps is solved using a standard Euler approximation scheme (see e.g. [70]).

We study SPDEs of the form

$$F(X, t) = A(X, t)dt + \sum_i B^i(X, t) \circ dW_t^i. \quad (8.4)$$

We show below that from a numerical perspective it is enough to first consider the SPDE in a simplified Itô form

$$F(X, t) = A(X, t)dt + \sum_i B^i(X, t)dW_t^i \quad (8.5)$$

as due to the specific structure of the Runge-Kutta of order 4 scheme, the Itô correction associated with the Stratonovich integral from equation (8.4) appears automatically in the second iteration.

The discretized version of the equation (8.5) is given by

$$F(X^n, t) = A(X^n)\Delta t + \sum_i B(X^n)\Delta W^i \quad (8.6)$$

where $\Delta W^i = W_{t_{n+1}}^i - W_{t_n}^i$. Then ¹

$$c_1 = F(X^n) = A(X^n)\Delta t + \sum_i B^i(X^n)\Delta W^i \quad (8.7a)$$

$$c_2 = F\left(X^n + \frac{c_1}{2}\right) = F\left(X^n + \frac{1}{2}\left(A(X^n)\Delta t + \sum_i B^i(X^n)\Delta W^i\right)\right) \quad (8.7b)$$

$$c_3 = F\left(X^n + \frac{c_2}{2}\right) \quad (8.7c)$$

Let

$$\begin{aligned} \ell_1 &:= X^n \\ \ell_2 &:= X^n + \frac{F(X^n)}{2} = X^n + \frac{1}{2}\left(A(X^n)\Delta t + \sum_i B^i(X^n)\Delta W^i\right) \\ \ell_3 &:= X^n + \frac{F(\ell_2)}{2} = X^n + \frac{1}{2}F\left(X^n + \frac{1}{2}\left(A(X^n)\Delta t + \sum_i B^i(X^n)\Delta W^i\right)\right) \\ &= X^n + \frac{1}{2}F(X^n) + \frac{1}{4}F\left(A(X^n)\Delta t + \sum_i B^i(X^n)\Delta W^i\right) \\ &= X^n + \frac{1}{2}A(X^n)\Delta t + \frac{1}{2}\sum_i B^i(X^n)\Delta W^i \\ &\quad + \frac{1}{4}A^2(X^n)(\Delta t)^2 + \frac{1}{4}\sum_i AB^i(X^n)\Delta W^i\Delta t + \frac{1}{4}\sum_i B^i(A(X^n))\Delta t\Delta W^i \\ &\quad + \frac{1}{4}\sum_i (B^i)^2(X^n)(\Delta W^i)^2 \\ &= X^n + \frac{1}{2}\left(A(X^n)\Delta t + \sum_i B^i(X^n)\Delta W^i + \frac{1}{2}\sum_i (B^i)^2(X^n)(\Delta W^i)^2\right) \\ &\quad + \frac{1}{2}\left(A^2(X^n)(\Delta t)^2 + \frac{1}{2}\sum_i A(B^i(X^n))\Delta W^i\Delta t + \frac{1}{2}\sum_i B^i(A(X^n))\Delta t\Delta W^i\right). \end{aligned}$$

Then

$$\ell_3 = X^n + \frac{1}{2}\left(A(X^n)\Delta t + \sum_i B^i(X^n)\Delta W^i + \frac{1}{2}\sum_i (B^i)^2(X^n)(\Delta W^i)^2\right) + \text{higher order terms}$$

and therefore we have recovered the Itô correction $\frac{1}{2}\sum_i (B^i)^2(X^n)(\Delta W^i)^2$. This is known (see [70]) as *Heun's method* or the *improved Euler method*. Formally it is based on introducing an

¹We drop the time dependence for the moment.

auxiliary variable (we neglect the dependence on i here as it is not essential)

$$X^* := X^n + A(X^n)\Delta t + B(X^n)\Delta W.$$

Then in the Euler scheme used for calculating the intermediate steps $c_i, i \in \{1, 2, 3, 4\}$ we have

$$\begin{aligned} X^{n+1} &= X^n + \frac{\Delta t}{2}(A(X^n) + A(X^*)) + \frac{\Delta W}{2}(B(X^n) + B(X^*)) \\ &= X^n + A(X^n)\Delta t + B(X^n)\Delta W + \frac{1}{2}B^2(X^n)(\Delta W)^2 \\ &\quad + \frac{1}{2}A^2(X^n)(\Delta t)^2 + \frac{1}{2}A(B(X^n))\Delta t\Delta W + \frac{1}{2}(B(A)(X^n))\Delta t\Delta W \end{aligned}$$

which is similar to what we had before.

8.2 DERIVATION OF THE STOCHASTIC ROTATING SHALLOW WATER MODEL

2

8.2.1 DETERMINISTIC SETTING

The dynamics of a classical two-dimensional rotating shallow water deterministic system is given (see [38], Chapter 2) by

$$\epsilon \frac{Du}{Dt} + f\hat{z} \times u + \nabla p = 0 \tag{8.8a}$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (hu) = 0 \tag{8.8b}$$

where $\frac{D}{Dt} := \frac{\partial}{\partial t} + u \cdot \nabla$ is the material derivative, $u = (u^1, u^2)$ is the horizontal fluid velocity vector, $p := \frac{h-b}{\epsilon\mathcal{F}}$ is the pressure term, h is the total depth, $b = b(x)$ is the bottom topography, $x = (x^1, x^2)$, $f = f(x)$ is the Coriolis parameter, $\hat{z}$ is a unit vector pointing away from the centre of the Earth, ϵ is the Rossby number, $\mathcal{F}$ is the Froude number. The velocity equation can be

²The first part of this section is based on the lectures given by Prof. Darryl Holm within the Mathematics of Planet Earth Centre for Doctoral Training, see e.g. [38] Chapter 2. The second part is based on [62] and on a couple of discussions I had with Erwin Luesink on the derivation of SALT models, with particular focus on the rotating shallow water system. The calculations are my own.

rewritten in *curl* form as

$$\partial_t(\epsilon u + \mathcal{R}) - u \times \text{curl}(\epsilon u + \mathcal{R}) + \nabla \left(p + \frac{\epsilon}{2}|u|^2 \right) = 0 \quad (8.9)$$

where $\mathcal{R} = \mathcal{R}(x)$ is the vector potential of the divergence-free rotation rate about the vertical direction, with $\text{curl } \mathcal{R}(x) = f(x)\hat{z}$. We show that (8.8a) and (8.9) are equivalent. Equation (8.8a) can be written as

$$\epsilon \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) + f\hat{z} \times u + \nabla p = 0.$$

The nonlinear term can be expressed as

$$u \cdot \nabla u = \omega \times u + \nabla \left(\frac{|u|^2}{2} \right)$$

where $\omega = \text{curl } u$, since

$$\frac{1}{2} \nabla(u \cdot u) = (u \cdot \nabla)u + u \times (\nabla \times u).$$

Therefore

$$\begin{aligned} \epsilon \partial_t u + \epsilon \left(\omega \times u + \nabla \left(\frac{|u|^2}{2} \right) \right) + f\hat{z} \times u + \nabla p &= 0 \\ \partial_t(\epsilon u) + \epsilon \text{curl } u \times u + \nabla \left(\frac{\epsilon}{2}|u|^2 \right) + f\hat{z} \times u + \nabla p &= 0 \\ \partial_t(\epsilon u) - u \times \text{curl}(\epsilon u) + \nabla \left(\frac{\epsilon}{2}|u|^2 \right) + f\hat{z} \times u + \nabla p &= 0 \\ \partial_t(\epsilon u) - u \times \text{curl}(\epsilon u) + f\hat{z} \times u + \nabla \left(\frac{\epsilon}{2}|u|^2 + p \right) &= 0 \\ \partial_t(\epsilon u) - u \times \text{curl}(\epsilon u) - u \times f\hat{z} + \nabla \left(\frac{\epsilon}{2}|u|^2 + p \right) &= 0 \\ \partial_t(\epsilon u) - u \times (\text{curl}(\epsilon u) + f\hat{z}) + \nabla \left(\frac{\epsilon}{2}|u|^2 + p \right) &= 0 \\ \partial_t(\epsilon u) - u \times (\text{curl}(\epsilon u) + \text{curl } \mathcal{R}) + \nabla \left(\frac{\epsilon}{2}|u|^2 + p \right) &= 0 \end{aligned}$$

given that that $\text{curl } \mathcal{R} = f\hat{z}$. That is

$$\partial_t(\epsilon u) - u \times \text{curl}(\epsilon u + \mathcal{R}) + \nabla \left(\frac{\epsilon}{2}|u|^2 + p \right) = 0.$$

Since $\mathcal{R}$ is time-independent this can be rewritten as

$$\partial_t(\epsilon u + \mathcal{R}) - u \times \text{curl}(\epsilon u + \mathcal{R}) + \nabla \left(\frac{\epsilon}{2} |u|^2 + p \right) = 0 \quad (8.10)$$

which is what we call *the curl form of equation (8.8a)*.

The rotating shallow water equations are typically derived from momentum and mass continuity equations. Alternatively, they can be derived using variational principles as in [38] Chapter 2. We follow this second approach.

Let $\mathcal{M}$ be a manifold and $\mathcal{V}(\mathcal{M})$ a vector space over $\mathcal{M}$ with its dual denoted by $\mathcal{V}^*(\mathcal{M})$. $\mathcal{V}$ is the space of all advected quantities including the depth h . A vector field U on a manifold $\mathcal{M}$ is a map $U : \mathcal{M} \rightarrow T\mathcal{M}$ which assigns a vector $U(r)$ to any point $r \in \mathcal{M}$. Here $T\mathcal{M}$ is the tangent bundle $T\mathcal{M} = \bigcup_{r \in \mathcal{M}} T_r\mathcal{M}$ where $T_r\mathcal{M}$ is the tangent space to $\mathcal{M}$ in r . The real vector space of vector fields on a manifold $\mathcal{M}$ is denoted by $\mathcal{X}(\mathcal{M})$. For any L^2 -integrable real functions $a \in \mathcal{V}, b \in \mathcal{V}^*$ one can define the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{V}^* \times \mathcal{V}} : \mathcal{V}^* \times \mathcal{V} \rightarrow \mathbb{R}$

$$\langle a, b \rangle = \int \langle a(x), b(x) \rangle_{\mathcal{V}^* \times \mathcal{V}} dx$$

The *variational derivative* $\frac{\delta F}{\delta \varphi} \in \mathcal{V}^*(\mathcal{M})$ of a $\mathcal{V}(\mathcal{M})$ -valued real functional $F[\varphi]$ of smooth functions φ is given by

$$\delta F[\varphi] = \lim_{\zeta \rightarrow 0} (F[\varphi + \zeta \delta \varphi] - F[\varphi]) = \left\langle \frac{\delta F}{\delta \varphi}, \delta \varphi \right\rangle.$$

One can define the Lagrangian $l : \mathcal{X}(\mathbb{R}^2) \times \mathcal{V}(\mathbb{R}^2) \rightarrow \mathbb{R}$ which in this context will depend on the fluid velocity $u \in \mathcal{X}(\mathbb{R}^2)$ and the depth $h \in \mathcal{V}(\mathbb{R}^2)$. The depth h can be regarded as an advected quantity. Then an action function can be defined as the time integral of the Lagrangian (see e.g. [60])

$$\mathcal{I} = \int_0^T l(u, h) dt. \quad (8.11)$$

By applying Hamilton's principle we have

$$\delta \mathcal{I} = 0 \quad (8.12)$$

We assume that the total depth h satisfies the continuity equation

$$\partial_t h + \nabla \cdot (hu) = 0.$$

Then equation (8.12) is equivalent to $\int_0^T \delta l(u, h) dt = 0$ that is

$$\int_0^T \left\langle \frac{\delta l}{\delta u}, \delta u \right\rangle + \left\langle \frac{\delta l}{\delta h}, \delta h \right\rangle dt = 0$$

One has (see [38] pp. 125)

$$\delta u = \frac{d}{d\epsilon} \Big|_{\epsilon=0} u = \partial_t v + u \cdot \nabla v - v \cdot \nabla u$$

$$\delta h = \frac{d}{d\epsilon} \Big|_{\epsilon=0} h = -\nabla \cdot (hv)$$

where $v \in \mathcal{X}(\mathbb{R}^2)$ is a variational vector field which vanishes at $t = 0$ and $t = T$. Using Einstein's standard summation notation we can write

$$\int_0^T \left(\left\langle \frac{\delta l}{\delta u^i}, \partial_t v^i + u^j \partial_j v^i - v^j \partial_j u^i \right\rangle + \left\langle \frac{\delta l}{\delta h}, -\partial_i (h v^i) \right\rangle \right) dt = 0.$$

Integrating by parts we obtain

$$\int_0^T \left\langle -\partial_t \frac{\delta l}{\delta u^i} - \partial_j \left(\frac{\delta l}{\delta u^i} u^j \right) - \frac{\delta l}{\delta u^k} \partial_i u^k + h \partial_i \frac{\delta l}{\delta h}, v^i \right\rangle dt + \left\langle \frac{\delta l}{\delta u^i}, v^i \right\rangle \Big|_0^t = 0.$$

This implies that

$$\partial_t \frac{\delta l}{\delta u^i} + \partial_j \left(\frac{\delta l}{\delta u^i} u^j \right) + \frac{\delta l}{\delta u^k} \partial_i u^k = h \partial_i \frac{\delta l}{\delta h}$$

which is the *Euler-Poincaré equation for ideal fluid dynamics* ([38]). But we also know that

$$\partial_t h + \partial_j (h u^j) = 0$$

so the Euler-Poincaré equation can be written as

$$\frac{D}{Dt} \frac{1}{h} \frac{\delta l}{\delta u} + \frac{1}{h} \frac{\delta l}{\delta u^j} \nabla u^j - \nabla \frac{\delta l}{\delta h} = 0$$

where $\frac{D}{Dt}$ is, as usual, the material derivative. Using the fundamental vector identity of fluid dynamics

$$(x \cdot \nabla)y + y_j \nabla x^j = -x \times (\nabla \times y) + \nabla(x \cdot y)$$

with $x = u$ and $y = \frac{1}{h} \frac{\delta l}{\delta u}$ we obtain

$$\frac{\partial}{\partial t} \left(\frac{1}{h} \frac{\delta l}{\delta u} \right) - u \times \text{curl} \left(\frac{1}{h} \frac{\delta l}{\delta u} \right) + \nabla \left(u \cdot \frac{1}{h} \frac{\delta l}{\delta u} - \frac{\delta l}{\delta h} \right) = 0. \quad (8.13)$$

For the rotating shallow water system the Lagrangian is given by

$$l(u, h) = \int \frac{\epsilon}{2} h |u|^2 + hu \cdot \mathcal{R}(x) - \frac{(h - b(x))^2}{2\epsilon \mathcal{F}} d^2 x dt$$

and then

$$\begin{aligned} \frac{\delta l}{\delta u} &= \epsilon h u + h \mathcal{R} = h(\epsilon u + \mathcal{R}) \\ \frac{\delta l}{\delta h} &= \frac{\epsilon}{2} |u|^2 + u \cdot \mathcal{R} - \frac{h - b}{\epsilon \mathcal{F}}. \end{aligned}$$

Let $p := \frac{h-b}{\epsilon \mathcal{F}}$ as before. Following the steps described above we have

$$\delta \mathcal{I} = \int_0^T \int \left\langle h(\epsilon u + \mathcal{R}), \delta u \right\rangle + \left\langle \frac{\epsilon}{2} |u|^2 + u \cdot \mathcal{R} - p, \delta h \right\rangle d^2 x dt = 0$$

that is

$$\int_0^T \int \left(h(\epsilon u + \mathcal{R}) \cdot \delta u + \left(\frac{\epsilon}{2} |u|^2 + u \cdot \mathcal{R} - p \right) \cdot \delta h \right) d^2 x dt = 0.$$

In a similar way we obtain

$$\frac{\partial}{\partial t} \left(\frac{1}{h} h(\epsilon u + \mathcal{R}) - u \times \text{curl} \left(\frac{1}{h} h(\epsilon u + \mathcal{R}) \right) \right) + \nabla \left(u \cdot \frac{1}{h} h(\epsilon u + \mathcal{R}) - \frac{\epsilon}{2} |u|^2 - u \cdot \mathcal{R} + p \right) = 0$$

$$\partial_t(\epsilon u + \mathcal{R}) - u \times \text{curl}(\epsilon u + \mathcal{R}) + \nabla \left(\frac{\epsilon}{2} |u|^2 + u \cdot \mathcal{R} - u \cdot \mathcal{R} + p \right) = 0$$

$$\partial_t(\epsilon u + \mathcal{R}) - u \times \text{curl}(\epsilon u + \mathcal{R}) + \nabla \left(\frac{\epsilon}{2} |u|^2 + p \right) = 0$$

which is exactly the curl form (8.10) of the velocity equation.

8.2.2 STOCHASTIC SETTING

For a vector space $\mathcal{V}$, $(x, y) \in T^*\mathcal{V}$, and $z \in \mathcal{X}$ the *diamond operation* is defined as $\diamond : T^*\mathcal{V} \rightarrow \mathcal{X}^*$

$$\langle x \diamond y, z \rangle_{\mathcal{V}} := \langle x, -\mathcal{L}_z y \rangle$$

where $\langle \cdot, \cdot \rangle_{\mathcal{V}} : T^*\mathcal{V} \times T\mathcal{V} \rightarrow \mathbb{R}$, $\langle \cdot, \cdot \rangle_{\mathcal{X}} : \mathcal{X}^* \times \mathcal{X} \rightarrow \mathbb{R}$, $x \diamond y \in \mathcal{X}^*$ and $\mathcal{L}_x y$ is the Lie derivative of an element $y \in \mathcal{V}^*$ with respect to a vector field $x \in \mathcal{X}$. $\mathcal{L}_x^T$ is the L^2 -adjoint of $\mathcal{L}_x$. We now consider the action $\mathcal{I}$ given by

$$\mathcal{I} = \int (l(u, h) + \langle q, dh + \mathcal{L}_{dy_t} h \rangle_{\mathcal{V}}) dt$$

where

$$dy_t = u dt + \sum_i \xi_i \circ dW_t^i \in \mathcal{X}$$

and q is the dual of h . In what follows the ad operation is the adjoint representation corresponding to a Lie algebra $\mathfrak{g}$ associated with a Lie group G , $ad : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$. It can be identified with the canonical Lie bracket (see [104] Proposition 3.47): for any $x, y \in \mathfrak{g}$ we have

$$ad_x y = [x, y].$$

The ad^* operation is given by $ad^* : \mathcal{X} \times \mathcal{X}^* \rightarrow \mathcal{X}^*$

$$\langle ad_x^* m, y \rangle = \langle m, ad_x y \rangle.$$

In particular $ad_x^* y = \mathcal{L}_x y$ for $y \in \mathcal{X}^*$. Then can write

$$\begin{aligned}
\langle q, dh + \mathcal{L}_{dy_t} h \rangle &= \langle q, dh \rangle + \langle q, \mathcal{L}_{dy_t} h \rangle \\
&= \langle q, dh \rangle - \langle q, -\mathcal{L}_{dy_t} \eta \rangle \\
&= \langle q, dh \rangle - \langle q \diamond h, dy_t \rangle \\
&= \langle q, dh \rangle - \langle q \diamond h, udt + \xi_i \circ dW_t^i \rangle \\
&= \langle q, dh \rangle - \langle q \diamond h, udt \rangle - \langle p \diamond h, \xi_i \circ dW_t^i \rangle \\
&= \langle q, dh \rangle + \langle q, \mathcal{L}_{udt} h \rangle - \langle q \diamond h, \xi_i \circ dW_t^i \rangle \\
&= \langle q, dh + \mathcal{L}_{udt} h \rangle - \langle q \diamond h, \xi_i \circ dW_t^i \rangle \\
&= \left\langle q, \frac{dh}{dt} + \mathcal{L}_u h \right\rangle dt - \langle q \diamond h, \xi_i \rangle \circ dW_t^i.
\end{aligned}$$

Then

$$\mathcal{I} = \int \left(l(u, h) + \left\langle q, \frac{dh}{dt} + \mathcal{L}_u h \right\rangle \right) dt - \int \langle q \diamond h, \xi_i \rangle \circ dW_t^i.$$

After applying Hamilton's variational principle $\delta \mathcal{I} = 0$ we obtain

$$\begin{aligned}
&\delta \int \left(l(u, h) + \left\langle q, \frac{dh}{dt} + \mathcal{L}_u h \right\rangle \right) dt - \delta \int \langle q \diamond h, \xi_i \rangle \circ dW_t^i = 0 \\
&\int \left\langle \frac{\delta l}{\delta u}, \delta u \right\rangle + \left\langle \frac{\delta l}{\delta h}, \delta h \right\rangle + \left\langle \delta q, \frac{dh}{dt} + \mathcal{L}_u h \right\rangle + \left\langle q, \delta \left(\frac{dh}{dt} + \mathcal{L}_u h \right) \right\rangle - \delta \int \langle q \diamond h, \xi_i \rangle \circ dW_t^i = 0 \\
&\int \left(\left\langle \frac{\delta l}{\delta u}, \delta u \right\rangle + \left\langle \frac{\delta l}{\delta h}, \delta h \right\rangle + \left\langle \delta q, \frac{dh}{dt} + \mathcal{L}_u h \right\rangle + \left\langle -\frac{dq}{dt} + \mathcal{L}_u^T p, \delta h \right\rangle + \langle -q \diamond h, \delta u \rangle \right) dt -
\end{aligned}$$

that is

$$\int \langle -\mathcal{L}_{\xi_i} h, \delta q \rangle - \langle \mathcal{L}_{\xi_i}^T p, \delta h \rangle \circ dW_t^i = 0,$$

due to the fact that

$$\begin{aligned}
\left\langle q, \delta \left(\frac{dh}{dt} + \mathcal{L}_u h \right) \right\rangle &= \left\langle q, \delta \left(\frac{dh}{dt} \right) \right\rangle + \langle q, \delta(\mathcal{L}_u h) \rangle \\
&= \left\langle -\frac{dq}{dt}, \delta h \right\rangle + \langle q, \mathcal{L}_u(\delta h) \rangle + \langle q, \mathcal{L}_{\delta u} h \rangle \\
&= \left\langle -\frac{dq}{dt} + \mathcal{L}_u^T p, \delta h \right\rangle + \langle -q \diamond h, \delta u \rangle.
\end{aligned}$$

That is, $\delta\mathcal{I} = 0$ is equivalent to

$$\int \left(\left\langle \frac{\delta l}{\delta u} dt - q \diamond h, \delta u \right\rangle + \left\langle \frac{\delta l}{\delta h} dt - dq + \mathcal{L}_{dy_t}^T p, \delta h \right\rangle + \langle dh + \mathcal{L}_{dy_t} h, \delta q \rangle \right) dt = 0$$

which gives

$$\frac{\delta l}{\delta u} dt = q \diamond h = h(\epsilon u + \mathcal{R}) \quad (8.14)$$

$$\frac{\delta l}{\delta h} dt = dq + \mathcal{L}_{dy_t}^T p = \frac{\epsilon}{2} |u|^2 + u \cdot \mathcal{R} - p \quad (8.15)$$

and

$$dh + \mathcal{L}_{dy_t} h = 0. \quad (8.16)$$

The momentum m is defined as

$$m := \frac{\delta l}{\delta u} = q \diamond h = h(\epsilon u + \mathcal{R}).$$

We show that

$$dm + \mathcal{L}_{dy_t} m = \frac{\delta l}{\delta h} \diamond h dt.$$

We want

$$dm - \frac{\delta l}{\delta h} \diamond h dt = -\mathcal{L}_{dy_t} m.$$

Since both $\frac{\delta l}{\delta h} \diamond h$ and $\mathcal{L}_{dy_t}$ appear in the definition of the $\diamond$ operator (into a scalar product), it

makes sense to investigate $\langle dm - \frac{\delta l}{\delta h} \diamond h, z \rangle$. We have

$$\begin{aligned}
\left\langle dm - \frac{\delta l}{\delta h} \diamond h, z \right\rangle &= \left\langle d(q \diamond h) - \frac{\delta l}{\delta h} \diamond h, z \right\rangle \\
&= \left\langle (\mathcal{L}_{dy_t}^T p) \diamond h + \frac{\delta l}{\delta h} \diamond h + q \diamond (dh) - \frac{\delta l}{\delta h} \diamond h, z \right\rangle \\
&= \langle (\mathcal{L}_{dy_t}^T p) \diamond h + q \diamond (dh), z \rangle \\
&= \langle (\mathcal{L}_{dy_t}^T q) \diamond h - q \diamond \mathcal{L}_{dy_t} h, z \rangle \\
&= \langle (\mathcal{L}_{dy_t}^T q) \diamond h, z \rangle + \langle -q \diamond \mathcal{L}_{dy_t} h, z \rangle \\
&= \langle \mathcal{L}_{dy_t}^T p, -\mathcal{L}_z h \rangle + \langle q, \mathcal{L}_z(\mathcal{L}_{dy_t} h) \rangle \\
&= \langle q, -\mathcal{L}_{dy_t}(\mathcal{L}_z h) \rangle + \langle q, \mathcal{L}_z(\mathcal{L}_{dy_t} h) \rangle \\
&= \langle q, (-\mathcal{L}_{dy_t} \mathcal{L}_z + \mathcal{L}_z \mathcal{L}_{dy_t}) h \rangle \\
&= \langle q, ad_{dy_t} z \rangle \quad \text{where } ad_{dy_t} z := -\mathcal{L}_{dy_t} \mathcal{L}_z + \mathcal{L}_z \mathcal{L}_{dy_t} \\
&= -\langle q \diamond h, ad_{dy_t} z \rangle \\
&= -\langle ad_{dy_t}^*(q \diamond h), z \rangle \quad \text{use } ad_{dy_t}^* = \mathcal{L}_{dy_t} \\
&= \langle \mathcal{L}_{dy_t} m, z \rangle. \quad (\text{q.e.d.})
\end{aligned}$$

Consider now the evolution of $\frac{m}{h}$:

$$\begin{aligned}
d\left(\frac{m}{h}\right) + \mathcal{L}_{dy_t}\left(\frac{m}{h}\right) &= \frac{1}{h} dm - m d\left(\frac{1}{h}\right) + \frac{1}{h} \mathcal{L}_{dy_t} m + m \mathcal{L}_{dy_t}\left(\frac{1}{h}\right) \\
&= \frac{1}{h} \left(dm - \frac{m}{h} (dh + \mathcal{L}_{dy_t} h) + \mathcal{L}_{dy_t} m \right) \\
&= \frac{1}{h} (dm + \mathcal{L}_{dy_t} m)
\end{aligned}$$

that is

$$d\left(\frac{m}{h}\right) + \mathcal{L}_{dy_t}\left(\frac{m}{h}\right) = \frac{1}{h} \frac{\delta l}{\delta h} \diamond h = h \nabla \frac{\delta l}{\delta h} \quad (8.17)$$

Seeing that $\frac{m}{h} = \epsilon u + \mathcal{R}$ and

$$\frac{\delta l}{\delta h} \diamond h = h \nabla \frac{\delta l}{\delta h} = h \nabla \left(\frac{\epsilon}{2} |u|^2 + u \cdot \mathcal{R} - \frac{h-b}{\epsilon \mathcal{F}} \right) dt,$$

relation (8.17) becomes

$$d(\epsilon u + \mathcal{R}) + \mathcal{L}_{dy_t}(\epsilon u + \mathcal{R}) = \nabla \left(\frac{\epsilon}{2} |u|^2 + u \cdot \mathcal{R} - \frac{h-b}{\epsilon \mathcal{F}} \right) dt.$$

Note that

$$\mathcal{L}_{dy_t}(\epsilon u + \mathcal{R}) = dy_t \cdot \nabla(\epsilon u + \mathcal{R}) + (\epsilon u + \mathcal{R})_j \nabla dy_t^j, \quad (8.18)$$

using

$$\begin{aligned} \mathcal{L}_X(v \cdot dx) &= (X \cdot \nabla v + v_j \nabla X^j) \cdot dx \\ &= (-X \times \text{curl} v + \nabla(X \cdot v)) \cdot dx. \end{aligned}$$

For these reasons (8.17) becomes

$$\begin{aligned} d(\epsilon u + \mathcal{R}) + dy_t \cdot \nabla(\epsilon u + \mathcal{R}) + (\epsilon u + \mathcal{R})_j \nabla dy_t^j &= \nabla \left(\frac{\epsilon}{2} |u|^2 + u \cdot \mathcal{R} - p \right) dt \\ d(\epsilon u + \mathcal{R}) - dy_t \times \text{curl}(\epsilon u + \mathcal{R}) + \nabla(dy_t \cdot (\epsilon u + \mathcal{R})) &= \nabla \left(\frac{\epsilon}{2} |u|^2 + u \cdot \mathcal{R} - p \right) dt \\ \epsilon du - dy_t \times \text{curl}(\epsilon u + \mathcal{R}) + \nabla(udt \cdot (\epsilon u + \mathcal{R})) + \sum_i \nabla(\xi_i \circ dW_t^i \cdot (\epsilon u + \mathcal{R})) \\ &= \nabla \left(\frac{\epsilon}{2} |u|^2 + u \cdot \mathcal{R} - p \right) dt. \end{aligned}$$

But

$$\begin{aligned} \nabla(udt \cdot (\epsilon u + \mathcal{R})) &= \epsilon \nabla(u \cdot u) dt + \nabla(u \cdot \mathcal{R}) dt \\ &= \epsilon \nabla |u|^2 dt + \nabla(u \cdot \mathcal{R}) dt. \end{aligned}$$

Therefore we have

$$\epsilon du - dy_t \times \text{curl}(\epsilon u + \mathcal{R}) + \sum_i \nabla(\xi_i \circ dW_t^i \cdot (\epsilon u + \mathcal{R})) = \nabla \left(\frac{\epsilon}{2} |u|^2 - p \right) dt,$$

which is the stochastic velocity equivalent of RSW.

8.3 THE PROJECTIVE PRODUCT

Let $(\Omega, \mathcal{F})$ be a measurable space. Then $\mathcal{B}(\mathcal{F})$ is the space of measurable and bounded functions $\psi : \Omega \rightarrow \mathbb{R}$. This space is endowed with the supremum norm

$$\|\psi\|_\infty := \sup_{w \in \Omega} |\psi(w)|.$$

For all functions $\psi \in \mathcal{B}(\mathcal{F})$ and all measures μ on $\mathcal{F}$, we define

$$\mu(\psi) := \int \psi d\mu.$$

A *Markov kernel* is a mapping $\mathcal{K} : \Omega \times \mathcal{F} \rightarrow [0, 1]$ such that

- i) $\mathcal{K}(w, \cdot) \in \mathcal{P}(\mathcal{F})$ for all $w \in \Omega$
- ii) $\mathcal{K}(\cdot, B)$ is measurable for all $B \in \mathcal{F}$.

For any Markov kernel we define the following two mappings:

- 1) $\mathcal{B}(\mathcal{F}) \ni \psi \rightarrow \mathcal{K}(\psi) \in \mathcal{B}(\mathcal{F})$,

$$\mathcal{K}(\psi)(\omega) := \int \psi(x) \mathcal{K}(\omega, dx)$$

- 2) $\mathcal{P}(\mathcal{F}) \ni \mu \rightarrow \mu\mathcal{K} \in \mathcal{P}(\mathcal{F})$ such that for all $\psi \in \mathcal{B}(\mathcal{F})$

$$(\mu\mathcal{K})(\psi) = \int \left[\int \psi(x) \mathcal{K}(w, dx) \right] \mu(dw).$$

If $\psi \geq 0$ is a μ -integrable function such that $\mu(\psi) > 0$, then one can define the *projective product* between ψ and μ as follows: $\mathcal{P}(\mathcal{F}) \ni \mu \rightarrow \psi \star \mu \in \mathcal{P}(\mathcal{F})$ such that

$$(\psi \star \mu)(\phi) = \frac{\mu(\psi\phi)}{\mu(\psi)}$$

for all $\phi \in \mathcal{B}(\mathcal{F})$. In other words, $\psi \star \mu : \mathcal{B}(\mathcal{F}) \rightarrow \mathbb{R}$ such that

$$(\psi \star \mu)(B) = \frac{\int_B \psi(x) \mu(dx)}{\int_\Omega \psi(x) \mu(dx)}$$

for any $B \in \mathcal{B}(\mathcal{F})$.

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