THE BACKGROUND RADIATION IN HOMOGENEOUS ISOTROPIC WORLD-MODELS

Thesis

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ABSTRACT

The problem of background radiation in homogeneous isotropic world-models due to radiation emitted by extragalactic nebulae is considered in more detail and greater generality than seems to have been attempted previously.

In Chapter I a brief historical survey is made of former theoretical and empirical attempts to determine the cosmical background radiation in both the optical and radio regions of the spectrum.

In Chapter II the general formula for the background spectrum in homogeneous isotropic world-models is calculated on the assumptions that the galaxies radiate as point-sources and that intergalactic space is totally transparent. These calculations are made for various classes of world-model, including models for which the number-density of light sources is constant in co-moving co-ordinates, the steady-state world-model with continual creation, and static world-models in which the frequency of a travelling photon decays with time. With the same assumptions, general formulae for the total background radiation in these models is derived in Chapter III; the general formula for the rate of reception of information in these models is also derived.

The total background radiation in specific world-models is calculated in Chapter IV, assuming that on the average the sources do not evolve with time and that the average source is a black-body radiator at $6000^{\circ}K$ with total luminosity 10^{10} Suns. The models for which these calculations have been made include the Milne, Page, de Sitter, Dirac,

Lemaître and steady-state world-models. In Chapter V numerical results and graphs of the background spectra of these models are given.

In Chapter VI the effect on the background radiation of absorption due to dust in the arms of spiral galaxies and in intergalactic space is considered in a general preliminary way. Also the effect of evolution is considered in a particular model (the Milne model) and the possible combined effects of absorption and evolution are briefly considered.

ACKNOWLEDGEMENTS

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PARTS OF THE THESIS WHICH ARE ORIGINAL

Chapter I is mainly historical. There is a reference to Halley which does not seem to have been mentioned before.

Chapter II is mainly original.

Chapter III (from Section 2 onwards) is original, and contains some new general theorems.

Chapter IV consists mainly of new calculations. Theorem VIII is original. Theoretical results for the Lemaître model in Section 4 are original. The criticism of Metzner's and Morrison's work at the end of this Chapter is also original.

Chapter V contains original numerical results and graphs.

Chapter VI is entirely original.

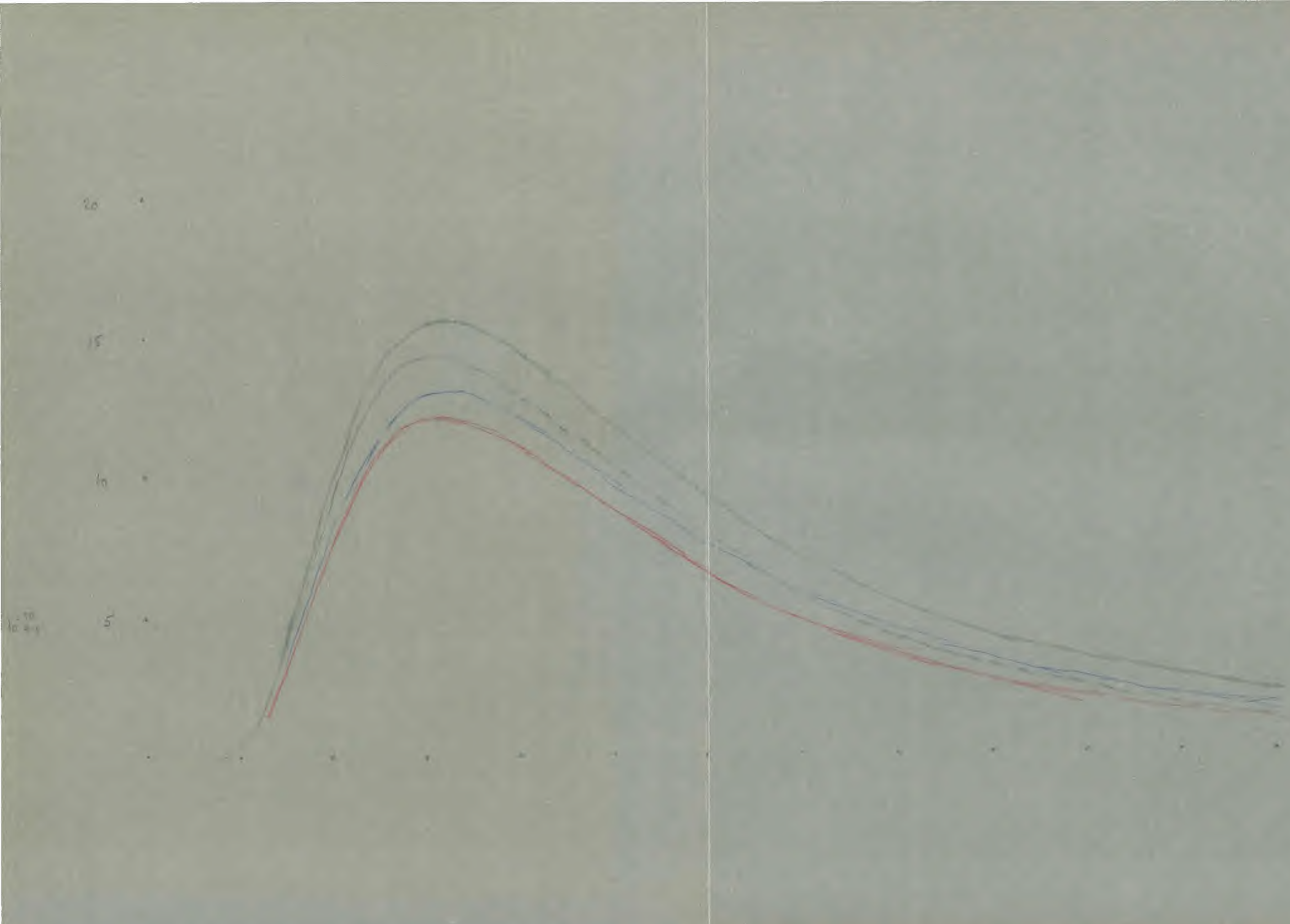

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THE BACKGROUND RADIATION IN HOMOGENEOUS ISOTROPIC WORLD-MODELS

Chapter I

The Problem of Background Radiation

1. Introduction. The background intensity of radiation due to extra-galactic sources is a phenomenon of considerable cosmological interest. Even from the simple observation that the night sky is dark on a moonless night, far-reaching conclusions about the nature of the universe emerge. It may not therefore be too much to hope that when this subject is considered in more detail, both theoretically and observationally, it will throw valuable light on many cosmological problems. Different theoretical world-models predict different results for the background brightness and spectrum, and observations of the background radiation may therefore enable us to discriminate between these models. Moreover, the problem of background radiation is one which must eventually be faced. For, other attempts to find which world-models are in closest accord with the empirical data concern only those galaxies which can be distinguished individually with modern instruments. It is conceivable that a particular model may give a good fit for such data and yet diverge considerably from the actual universe at greater distances. On the other hand, the background brightness and spectrum essentially depend on conditions in the more remote regions of the universe.

The theoretical problem of determining the total intensity of background radiation was examined in a simple form using Newtonian concepts

of space and time by W. Olbers (1) in a classic paper published in 1826. He considered a static world-model filled with stars whose average brightness and density are constant in both space and time, and calculated the absolute intensity of the background radiation assuming that space is Euclidean and that light from a star will be absorbed by any other star it chances to meet on the way. Olbers was greatly mystified by the incongruous result he obtained from such natural assumptions, namely, that the brightness of the night sky would be the same as the average brightness of the stars. He explained away this paradoxical result by postulating the existence of a tenuous gas, too thin to be detected, which absorbs the bulk of the radiation when it travels very long distances. The postulated gas, however, will not produce the desired effect, because it will heat up until a temperature is reached at which it radiates as much energy as it receives.

Many 18th century writers had discussed this same problem now associated with the name of Olbers. As early as 1720 Halley (2) came across Olbers's problem when attempting to vindicate the hypothesis that the sphere of fixed stars is infinite, for he writes, "if we suppose all the Fixt Stars to be as far from one another, as the nearest of them is from the Sun; that is, if we may suppose the Sun to be one of them, at a greater distance their Disks and Light will be diminish'd in the proportion of Squares, and the Space to contain them will be increased in the same proportion." Unfortunately Halley then erroneously argued

that the light from the distant stars would not be observable, for he says, "their Beams, aided by any help yet known, are not sufficient to move our Sense; after the same manner as a small Telescopical fixt Star is by no means perceivable to the naked Eye."

Assuming that the stars are equidistant from one another, Halley deduced in a second paper that there could not be more than thirteen stars on a sphere with radius equal to the distance of separation between the stars, and with the Sun at its centre. Assuming further that the stars have the same brightness as the sun, Halley identified these thirteen stars with the 1st magnitude stars on the celestial sphere. Halley accounted for the fact that there are more than thirteen 1st magnitude stars by relaxing the conditions that the stars are equidistant and have the same brightness. Halley then made the following interesting observations on Olbers's problem regarding successive shells, again drawing the erroneous conclusion that the light from the distant stars in a static infinite universe will not be visible:- "If therefore the Number of them be supposed Thirteen, omitting Nicities in a Matter of such Irregularity, at twice the distance from the Sun there may be placed four times as many, or 52; which with the same allowance, would nearly represent the number of the Stars we find to be of the 2d magnitude: so 9 x 13 or 117, for those at three times the distance: and at ten times the distance 100 x 13 or 1300 Stars; which distance may perhaps diminish the light of any of the Stars of the first magnitude to that of the sixth, it being but the hundredth part of what, at their present distance, they appear with.

But if, since we have room enough for it, we should suppose the Sphere continued to 10 times the last, or 100 times the first distance, the number of Stars would be 130 000, and they would appear with the 10 000th part of the Light of a first magnitude Star, as we now see it. This is so small a pulle of Light, that it may well be questioned, whether the Eye, assisted with any artificial help, can be made sensible thereof."

C. V. L. Charlier (3) in 1922 postulated a static hierarchical world-model in which Olbers paradox may be removed, when classical concepts of physics are used. The world-model is largely of historical interest, however, since it merely "saves" the phenomenon and does not predict or explain many of the remaining large scale features of the universe, e.g. the universal red-shift of the extragalactic nebulae. In this world-model, stars cluster together to form galaxies of the first order, in a similar manner first order galaxies cluster together to form second order galaxies and so on ad infinitum. Charlier showed that, if R_i is the average radius of the ith order galaxies and N_i the average number of the $(i-1)$th galaxies forming an ith order galaxy, then provided

$$\frac{R_i}{R_{i-1}} > \sqrt{N_i} \qquad (1.1)$$

for $i = 1, 2, 3, \ldots$ the criterion of Olbers (that the night sky must be dark) is satisfied. Further, Charlier showed that if the inequality (1.1) holds, then a further paradox due to Seeliger (4) - that the law

of Newtonian gravitation in a static infinite universe in which the total mass is infinite leads to an infinite attraction at every point - is also overcome.

Recently, the problem of the darkness of the night sky has been considered anew by H. Bondi (5) in the light of modern knowledge. Since no satisfactory explanation resolving Olbers's paradox of the type suggested by Olbers himself has been found, Bondi concludes that one or more of the original assumptions must be altered. These assumptions may be tabulated as follows:-

(i) The average brightness and density of the stars is constant throughout space.

(ii) The average brightness and density of the stars is constant for all time.

(iii) The stars have no systematic motions.

(iv) The geometry of space is Euclidean.

(v) The known laws of physics apply.

Bondi decided to confine attention to homogeneous and isotropic world-models and further, he decided to retain assumption (v). I shall discuss the possibility of abandoning this assumption later in this thesis, but normally it will be assumed to hold implicitly.

It should be noted that although the paradox can still be derived if (i) and (ii) are modified so that the averages mentioned are the same merely in those regions from which light reaches us at the present moment,

such a state of affairs is excluded if the Copernican principle, that the Earth cannot be in an exceptional position in the universe, is accepted.

The paradox remains even if (iv) is no longer retained and one of the other two possible geometries of a homogeneous isotropic universe is accepted instead of Euclidean geometry. Bondi gives a proof of this only for the simple case of a static model in which the stars are considered as points (and therefore do not obscure the background radiation). It will be shown in Chapter II that, not only does the geometry of the universe make no difference to the background radiation in a static universe, but that the result holds for any homogeneous and isotropic universe irrespective of whether the sources of radiation have an evolutionary history, or have a systematic motion apart from smaller individual proper motions.

Almost all modern cosmological theories are based on the Cosmological Principle which states that the general aspect of the universe is the same throughout space,* apart from minor local irregularities. The

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* The standard treatment implies the existence of a cosmic or public time. In terms of this the first of Olbers's assumptions can be appropriately reformulated: the average brightness and density of the "stars" at a given cosmic epoch is the same from the standpoint of each. By "stars" we mean, in fact, clusters of galaxies.

Cosmological Principle together with the assumptions of homogeneity and isotropy imply the validity of (i) and in order to resolve the paradox of Olbers, either (ii) or (iii) must be abandoned, or else both, since they are the only uninvestigated assumptions remaining.

The steady-state theory of Bondi and Gold goes a stage further than other cosmological theories in being based on a more restricted principle known as the Perfect Cosmological Principle which states that the general aspect of the universe is the same throughout space and also throughout time apart from local irregularities. This implies that (ii) must be retained and therefore (iii) must be dropped if the background radiation is to be of a reasonable magnitude, well below the estimate of Olbers. The Perfect Cosmological Principle together with the Principle of Continual Creation determines the form of the large scale motion which is found to be a particular type of expansion.

The total background radiation in the steady-state world-model is derived theoretically in Chapter III when the effect of intergalactic absorption and the finite size of the galaxies is neglected, and is given explicitly by formula (3.15). Since the Perfect Cosmological Principle is adopted in this world-model, the physical quantities involved are time-independent and are therefore the same as those that are observed at the present epoch in our neighbourhood. It is found upon substituting empirical values into formula (3.15) that the density of background radiation is indeed well below Olbers's estimate and is of the order of 10^{-13} erg. cm^{-3}. The recession of the galaxies has caused their spectra

to be reddened and hence they have lost some of their energy. This loss is sufficient to keep the radiation density at a value low enough to resolve the paradox. The total background radiation in the steady-state world-model when the obscuring effects of intergalactic matter and the finite size of the galaxies are included has also been discussed tentatively in Chapter VI. Unfortunately, very little is known at present about intergalactic absorption and it is therefore not yet possible to obtain even the correct order of magnitude of the total background radiation by means of this formula.

Suppose next that, while retaining postulate (iii), we abandon postulate (ii) so that the average brightness of the galaxies is now assumed to be a function of the time. Equation (3.7) in Chapter III shows that the radiation density in a static universe can be of the empirically correct order of magnitude provided the average brightness of the galaxies decreases rapidly enough as we extrapolate back into the distant past. Furthermore, if in such a universe the galaxies were all created 10^{10} years ago (which is approximately the current estimate for the age of our Galaxy (6)), but since then have been radiating uniformly, the density of radiation could still be of the right order of magnitude.

Finally, suppose that postulates (ii) and (iii) are discarded. We note that postulate (ii) makes two assertions. In rejecting it, in conjunction with our rejection of (iii), I shall consider models in which the density varies with time but in which the average brightness of a galaxy may or may not change with time. In justifying relaxation of

postulates (iii) in general, I shall appeal to the usual interpretation of the red-shifts observed in the spectra of the extragalactic nebulae (when compared with some reference spectrum) as due to the Doppler effect associated with recessional motion. From time to time alternative explanations of the extragalactic red-shifts have, however, been suggested. These have usually depended on some modification of postulate (v). For example, in Milne's theory, when his τ-scale of time is used, the galaxies remain at rest, and the frequency of a photon is conserved. The frequency of an atomic transition, however, increases with the passage of time. The red-shift is caused by the increase in the frequency of the spectral lines emitted from the atoms used to form a comparison spectrum. The particular case of static homogeneous isotropic world-models in which the frequency of a photon diminishes with time whilst the frequency of atomic transitions remains constant will be considered later. Confining attention to the Doppler interpretation of the red-shift, we note that in analysing his original uniformly expanding world-model Milne made a pioneer contribution to our subject (7). Since, the radiation sources in this model are confined to the interior of an expanding Euclidean sphere of radius ct, at epoch t, and their number density at distance r from the observer is proportional to

$$\frac{t}{(t^2 - r^2/c^2)^2}, \qquad (1.2)$$

it follows that they form an open set and so their total number is

infinite. Since Milne assumed implicitly that the average absolute luminosity of a galaxy does not change significantly with time, he was obliged to show that the total intensity of radiation at any point in the model is finite. He found that the apparent luminosity of a source receding with velocity V is reduced as compared with that of a similar stationary source at the same distance by a factor

$$\left(\frac{1 - V/c}{1 + V/c}\right)^2, \tag{1.3}$$

and by a further factor

$$\left(\frac{1 + V/c}{Vt_0}\right)^2 \tag{1.4}$$

due to distance, where t_0 is the epoch at which the observer receives the radiation. Since the total number of sources with velocities in the range V to V + dV is proportional to

$$\frac{V^2 dV}{(1 - V^2/c^2)^2}, \tag{1.5}$$

corresponding to the spatial density given by (1.2), it followed on combining (1.3), (1.4) and (1.5) and integrating over the range (0, c) in V that the total intensity at the origin (and hence, by the Cosmological Principle at any point of the model) was finite.

Milne's original model consisted of fundamental particles (idealisations of galaxies) receding uniformly in Euclidean space. In the usual technique of relativistic cosmology, the fundamental particles are represented by points embedded in an expanding space. I shall show that, using this technique, a strikingly simple formula (given by equation (3.7))

may be derived for the total intensity of background radiation in homogeneous isotropic world-models in which the density of radiation sources is constant in co-moving co-ordinates. Provided the world-model is of this type, the formula (3.7) can be applied to any expanding and evolutionary world-model, where the terms expanding and evolutionary have been used in the following distinct senses:-

(a) by an expanding world-model is meant a world-model in which the average number-density of galaxies (or clusters of galaxies) tends to diminish with time.

(b) by an evolutionary world-model is meant a world-model in which there is on the average a statistical change with time in the intrinsic characteristics of galaxies and clusters of galaxies.

Thus there are four types that can arise, namely:-

(1) expanding and evolutionary world-models, e.g. the Lemaître world-model,

(2) non-expanding and non-evolutionary world-models, e.g. the Einstein static universe, the steady-state model with continual creation,

(3) non-expanding and evolutionary world-models, and

(4) expanding and non-evolutionary world-models.

With the aid of formula (3.7) we are able, in particular, to establish finite sky brightness, in general, for any homogeneous isotropic model of types (1) and (4), expanding from an initial state

(e.g. point singularity) at a finite past epoch. This result may be regarded as a generalisation of Milne's, which was of course confined to a uniformly expanding model of type (4).

It is remarkable that, when temporal uniformity of absolute luminosity of individual sources is assumed, the calculated numerical values of the radiation density in many of these models are very nearly equal, whereas the corresponding value for the steady-state model is somewhat smaller. (The comparisons are, of course, all made for the same empirically given value of Hubble's constant in the law relating red-shift and distance.) This result has been confirmed in a paper by Hetzner and Morrison (8) entitled "The Flow of Information in Cosmological Models". In this paper they calculated the total intensity of background radiation, due to radiation sources with a Doppler-shift less than any given value, for the world-models* of Milne, Page, Einstein-de Sitter, Dirac, de Sitter and also the steady-state world-model of Bondi and Gold. The effect of one radiation source obscuring another was neglected in the calculations, and it was assumed that the expanding models considered were of type (4). Unfortunately, Hetzner and Morrison made an error when deriving the formula for the number of

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* A more complete description of these models is given in Bondi's book. We shall show that, given the number density of the radiation sources, it is enough to know the analytical form of the expansion factor when calculating the background radiation.

radiation sources in a given range of Doppler-shift in the steady-state model, with the result that their value is lower than the actual one. When the correction is applied it is found that the steady-state model and Dirac's model of type (4) yield the same value for the background radiation. Further, if it is assumed in the Dirac model that the emission spectra of the radiation sources have always remained the same since the time of creation, the Dirac model and the steady-state model yield the same background spectrum and this spectrum is displaced less to the red than are the background spectra obtained from the other world-models of type (4) considered by Metzner and Morrison. This indicates that in both the steady-state model and the Dirac model the background radiation is supplied mainly by the foreground galaxies. The slightly larger value for the radiation density in the other world-models may be primarily explained by the fact that the light received at the Earth from more distant galaxies was emitted at a correspondingly earlier epoch when the density of light sources and therefore the average emission per unit volume of space was greater, whilst these same quantities remain constant in the steady-state world-model. The radiation from these distant regions will therefore produce more radiation in the red end of the spectrum in the evolutionary models than in the steady-state world-model. The complete explanation, however, is more difficult to find. The rate of recession increases rapidly with distance in the Dirac model, with the result that the

weakening effect of recession on the radiation due to the very distant galaxies more than counteracts the strengthening effect due to the increase in the number of sources per unit volume. It is shown in Chapter IV that with increasing epoch of observation in expanding world-models whose expansion factor R takes the special form $R = at^p$ (where t is a cosmic time, a and p are constants) the total background intensity increases when $p > \frac{1}{3}$, decreases when $p < \frac{1}{3}$, and is constant when $p = \frac{1}{3}$. The last case, $p = \frac{1}{3}$, represents the Dirac universe. We conclude that for world-models of the type $R = at^p$ the weakening effect of recession and the strengthening effect of increasing number density must balance when $p = \frac{1}{3}$, the latter effect predominating when $p > \frac{1}{3}$. All the world-models of this particular type* which were considered by Metzner and Morrison had $p \geqslant \frac{1}{3}$.

If the effect of the finite size of the galaxies is taken into account, more absorption by them of radiation from remoter sources will occur in the expanding models than in the steady-state model. Consequently, so long as we assume that all sources in the former radiate uniformly, i.e. models of type (4), the total background intensity of radiation will be almost the same for all models considered by Metzner and Morrison. This result is important because any significant difference in practice between the empirical total background intensity and this theoretical value

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* When $p < \frac{1}{3}$ the time scale is rather short.

would be evidence for the evolution of individual sources as well as evidence against the steady-state theory.

2. The Background Spectrum. The first attempt to calculate theoretically the spectral distribution of the background radiation seems to have been made by Leontovaki (9) in 1936. Leontovaki confined his investigation of this problem to Milne's uniformly expanding world-model of type (4) containing point sources, so that the effect of one galaxy obscuring another was neglected. He assumed that on the average the sources emit light of the same continuous spectral distribution as the Sun's. Although the theoretical formula which he derived for the spectral intensity of background radiation appears to have been misprinted, his numerical results, which were obtained by a rough graphical method of integration, are correct.

Leontovaki presented his results in graphical form. He expressed the spectral intensity of the extragalactic sources as a function of the logarithm of the wavelength, because the resulting curve is bodily displaced without distortion as a consequence of the Doppler shift, and this simplifies numerical calculations. Leontovaki thus obtained a graph of background spectral intensity against logarithmic wavelength which he converted to the more familiar graph of spectral intensity and wavelength. He plotted the continuous solar spectrum and assumed that it also roughly represented the radiation from a stationary galaxy. He then plotted the spectrum of radiation from all galaxies in Milne's model (type 4) and also that of the background radiation in this model

due to all galaxies receding with Doppler shifts exceeding some given cut-off value. In order to make a comparison between these three spectral distributions, the three curves were plotted in such a way that the areas under each were equal.

Leontovski found that the mean shift between the spectrum of a stationary galaxy and the background spectrum due to those galaxies which are so distant that they cannot be resolved with a telescope (which, in 1936, corresponded to a "cut-off" Doppler ratio due to recession of rather more than a quarter the speed of light) is about half the length of the visible spectrum. Leontovski wrote in his letter, "to a most sensitive eye the background of the galaxies would appear as dark red." Although Leontovski neglected the effect of the nearer galaxies obscuring more distant ones in his calculations, he mentioned that this effect should be included and observed that it would tend to diminish the redness of the background. This result will be discussed theoretically in Chapter VI.

More recently, McVittie and Wyatt (10) have considered the same problem for expanding homogeneous isotropic world-models of general relativity in which the extragalactic radiation sources are regarded as point sources. They showed first that it is possible to calculate the radiation for specific models when the average luminosity and spectrum of a galaxy in a localised region are given as well as the distribution of galaxies and their large scale motion. They pointed out that it was not possible to use a general approximation method because no restriction

can be placed on the red-shifts in the way demanded by such a method. McVittie and Eyatt then confined their attention to the Milne model and therefore did not show that, in general, the background radiation is independent of the curvature index, nor did they discuss how to obtain, in general, the limits of the relevant integrals. These questions are considered in the present thesis. They obtained the background spectrum in Milne's uniformly expanding model due to sources radiating in the visual region and also that due to sources radiating in the radio region of the spectrum. Although the possible time dependence of the luminosity and spectrum of the galaxies was taken into account when the general result was derived, in their numerical calculations they assumed that there was no systematic change in these quantities with time. Since they maintained that in order to calculate the background spectral intensity a priori knowledge of the luminosity and spectral functions must be given, in the visual region they took the profile of the spectrum of an average galaxy to be the same as that of a blackbody at effective temperature $5000^{\circ}K$ and the total luminosity to be 10^{10} times the luminosity of the Sun. They concluded from their numerical calculations on a Milne universe that it would be possible at high galactic latitudes to separate the background radiation from both the zodiacal light and diffuse light from the Galaxy. They suggested that measurements could be made photo-electrically from outside the atmosphere with a high-resolution instrument, and they maintained that light from galactic stars would produce a negligible effect at galactic latitudes as low as 30°.

3. The Background Radio Spectrum. In studying background radiation in the radio region of the spectrum, McVittie and Wyatt took the average spectrum of the galaxies to be proportional to an inverse power of the frequency over a given frequency range and zero outside that range. (The power to which the frequency is raised is nearly -1). The average monochromatic power output at 100 megacycles per second appears to be of the order of 10^{22} watts per cycle per second. On this basis and assuming that on the average the sources do not change significantly with lapse of time, it was shown that the background spectrum in the Milne model would be detectable at radio frequencies. (In particular, they found the contribution to the background radiation at radio wavelengths from extremely remote galaxies due to the large red-shift of their visual radiation is negligible.)

An attempt has been made by Costain (11) to measure the background radio spectrum. The isotropic component from extragalactic sources may be distinguished from the galactic halo component only if their spectra are different, otherwise it is only possible to place an upper limit on the magnitude of the extragalactic component (12). Furthermore, it is found that, because of the difficulty in accounting for the background noise from aerials and receivers, absolute measurements of background radiation made by various radio astronomers at different frequencies yields a wide range of values for the spectral index. In order to overcome this difficulty Costain used identical aerial systems when measuring the background radiation at different frequencies. He

found that the spectral index of the background extragalactic radiation is about -1, and pointed out that this value provides important information about the nature of discrete radio sources (13), and is also useful as a means for distinguishing between different types of cosmological model.

The total background radio emission from extragalactic radio sources has been considered theoretically by Shakeshaft (14) and also by Priester (15). Shakeshaft in 1954 calculated the total background radio radiation for a world-model associated with a de Sitter metric in which the radio sources are non-evolutionary and conserved, and also for the steady-state model, in order to account for the amount of radio noise observed towards the galactic poles, which at that time could not be accounted for by galactic sources. This additional radio radiation corresponded to a brightness temperature T of about 600°K at 100 megacycles, where T is defined by the Rayleigh-Jeans Law:

$$T = \frac{B(\nu)c^2}{2k\nu^2}, \qquad (1.6)$$

where ν is the frequency, $B(\nu)$ the flux-density per steradian, k is Boltzmann's constant and c the velocity of light. Shakeshaft estimated the radiation per unit volume due to collisions between galaxies and found that the relativistic model could account for the observed polar intensity but that the steady-state model could not. This result is not conclusive, however, because the radio emission from the Galaxy extends over a much larger distance than was once thought, which would account for most of the observed polar intensity.

4. Observations of the Background Radiation. As already mentioned, McVittie and Wyatt have suggested that measurements of the background radiation at optical wavelengths should be made photoelectrically outside the Earth's atmosphere. As early as 1901, however, S. Newcomb (16) had attempted to measure this radiation, and pointed out that knowledge of the total amount of light received from all the stars would have important consequencies on the theories of the structure of the universe. Using optical methods, Newcomb began by comparing various regions of the sky and found surprisingly little difference between the brightness of the galaxy and the rest of the sky. In fact he found the absolute brightness at the galactic poles to be equal to 90 stars of tenth magnitude per square degree, and about twice this amount in the galactic plane. In the following years many attempts were made to measure the absolute brightness of the sky and it was soon discovered that there were strong contributions to this radiation that had been overlooked. It was not until 1937 that Elvey and Roach (17) managed to disentangle those contributions using a recording photoelectric photometer. By making recordings at different hours of the night throughout the year they constructed, by analytical methods, isophotes of the zodiacal light and the light from the galactic system. After the effect of the stars in the Galaxy had been removed, they found an excess of light which includes the light from extragalactic nebulae and light produced by the scattering of starlight by interstellar matter. Isophotes of this excess light which includes deviations in the mean number of stars, are given in Fig. 11 of their paper "A Photoelectric Study of the Light of the Night Sky".

The problem of measuring the background radiation due to extra-galactic nebulae was taken up again in 1956 by Baum (18, 19) who also pointed out the cosmological importance of such an observation. Baum suggested that the experiment should be carried out with an artificial Earth satellite because at the Earth's surface the air glow, even on a moonless night, dwarfs the integrated background radiation. Furthermore, the radiation can be observed over a much wider wavelength range since there would be no atmospheric absorption. The radiation received by the satellite must then be distinguished from the zodiacal light and the light from our own Galaxy. The instrument required for the observation would be a simple and compact light meter consisting of a small lens, some tiny colour filters, a red-sensitive photomultiplier, a radium paint button for standardisation and a small circuit, the total power output requirements for the system being less than a watt. The signal output of the photomultiplier is fed into a telemetering channel which would consist of streams of pulses whose frequency is an accurate measure of the light received. A diaphragm placed at the focus of the lens would restrict the field of view to a few degrees and the incidental spin of the satellite sweeps the field of view across the Earth's atmosphere whilst the fluctuation in brightness with variation in galactic latitude would provide a means of eliminating the contribution to the background radiation from the stars in the local galaxy and the zodiacal light.

Baum gave strong reasons why we should like to know the brightness and colour of the extragalactic background radiation. First, the

observation may enable us to determine which world-model gives the best fit to the universe. Secondly, the empirical results may be used to distinguish between models which fit other observations which extend as far as we can see with modern telescopes but not all the way to the horizon. At present the situation is that many different theoretical world-models fit the observations made in the range of telescopes whilst at greater distances they predict totally different results. Therefore the final identification of a particular world-model with the actual universe cannot be made until either the present observations are greatly improved or an observation is made that depends on the state of the universe beyond the range of ordinary optical and radio telescopes. For this reason an observation on the brightness and colour of the night sky is very important. Thirdly, we should like to know the absolute surface brightness of the regions between the galaxies, which at present is unknown. Recent measurements of surface brightness in the tenuous outskirts of galaxies suggest that they have no real boundaries, in fact they seem to spread out indefinitely and blend into one another. There may even be more stars in intergalactic space than within the conventional boundaries assigned to galaxies. Baum suggested that in order to distinguish between the effect of an intergalactic population from that due to the universe beyond the range of modern telescopes, the brightness of the extragalactic sky must be observed in more than two colours.

Baum estimated the feasibility of the experiment by making a rough calculation of the minimum signal expected from extragalactic sources and

comparing it with the amount of foreground light expected at high galactic latitudes in our own galaxy. Star counts indicate that the integrated foreground light of the Galaxy in both the red and the infra-red regions of the spectrum is about 6 magnitudes per square degree. Using Shane's and Wirtanen's (20) counts of galaxies down to the 18th photographic magnitude, Baum calculated that the brightness in the red region of the spectrum is 10th magnitude per square degree whilst in the infrared region it is 9th magnitude per square degree. In order to account for galaxies with apparent magnitudes between the 18th and 23rd magnitude (the threshold of the 200-inch Mount Palomar telescope), these values will be increased by about 2 magnitudes. Then according to Baum, the additional extragalactic contribution due to galaxies fainter than the 23rd magnitude will be about 15 per cent of the total background radiation. It should therefore be possible to measure the extragalactic contribution provided the dependence of the galactic contribution on galactic latitude is known.

5. The Residual Background Radiation. There is a peculiar difficulty associated with the observation of the background radiation due to thermal emission by the galaxies which does not appear to have been discussed by previous investigators. Some cosmological theories postulate the existence of radiation left over from the initial singular state (epoch of "creation"), but they predict that the present state of this remnant radiation closely approximates to blackbody radiation at a few degrees Kelvin. For example, Gamow (21) postulates the existence of blackbody

radiation which at the earliest epoch is at a sufficiently high temperature to produce the nuclear synthesis of the various elements from protons and neutrons. For adiabatically expanding relativistic world-models in which matter and radiation are homogeneously and isotropically distributed, Herman and Alpher (22) found that if matter and radiation were not converted into each other the remnant radiation is of the same total intensity as the calculated background intensity of the light sources but is effectively blackbody radiation at a temperature of about two degrees Kelvin and therefore will have a negligible effect on the total intensity observed in the visual and radio regions.

The object of the present thesis is to develop with greater generality and in more detail than seems to have been attempted previously the theory of background radiation in homogeneous isotropic world-models. In particular attention has been paid to the possible effects of absorption of the background radiation due to the presence of dust in the arms of spiral galaxies and in intergalactic space. Owing to the complexity of the problem the calculations have been made in two stages. In the first, I have considered the total intensity and spectrum of background radiation in various world-models, neglecting evolutionary effects and possible obscuring effects due to dust in the arms of spiral galaxies and intergalactic matter. Under these assumptions the problem has been considered in both the visual and radio region of the spectrum.

The radio region, however, has not been considered in such detail since, (compared with most light sources) many radio sources may be short-lived and it is therefore probably not safe to assume that radio sources are non-evolutionary. In the second stage a preliminary attempt is made to take into account the possible effects of obscuration of the background radiation by dust in spiral galaxies and by intergalactic matter, and also the secular evolution of the galaxies. It is found that, in expanding world-models in which the galaxies were more luminous in the past than now, the background radiation is greater at longer wavelengths than it would be if the evolutionary effects were absent. On the other hand the effect of absorption by dust is to decrease the background radiation at longer wavelengths. Consequently, in such cases, the effects of evolution and absorption will tend to counterbalance one another. Therefore, the results of our calculations for idealized models in which the sources are regarded as points and their evolution is neglected are perhaps somewhat more realistic than might otherwise be thought.

References

1. Olbers, W., (1826), Astronomische Jahrbuch, 110.
2. Halley, E., (1720), Philosophical Transactions, 31, pp 22-26.
3. Charlier, C. V. L., (1922), Arkiv för Matematik, Astronomi och Fusik, 16, No. 22 (Stockholm).
4. Seeliger, H., (1895), Astronomische Nachrichten, 137, No. 3273, 130.
5. Bondi, H., (1960), "Cosmology", (Cambridge University Press).
6. Hoyle, F., Proceedings of the Royal Society, Series A, 260, 201.
7. Milne, E. A., (1933) Zeitschrift für Astrophysik, 6, 1.
8. Metzner, A. W. K., Morrison, P., (1959), Monthly Notices of the Royal Astronomical Society, 119, 657.
9. Leontovski, M., (1936), Nature, 137, 904.
10. McVittie, G. C., Wyatt, S. P., (1959), Astrophysical Journal, 130, 1.
11. Costain, C. H., (1960), Monthly Notices of the Royal Astronomical Society, 120, 248.
12. Baldwin, J. E., (1955), Monthly Notices of the Royal Astronomical Society, 115, 691.
13. Ryle, M., (1958), Proceedings of the Royal Society, Series A, 248, 289.
14. Shakeshaft, J. R., (1954), Philosophical Magazine, 45, 1136.
15. Priester, W., (1954), Zeitschrift für Astrophysik, 34, 295.
16. Newcomb, S., (1901), Astrophysical Journal, 14, 297.
17. Elvey, C. T., Roach, F. E., (1937), Astrophysical Journal, 85, 213.
18. Baum, W. A. (1956) Publications of the Astronomical Society of the Pacific, 68, 118.

19. Baum, W. A., (1956), Astronomical Journal, 61, 171

20. Shane, C. D., Wirtanen, C. A., (1954), Astronomical Journal, 59, 285.

21. Gamow, G., (1948), Nature, 162, 680

22. Alpher, R. A., Herman, R. C., (1949), Physical Review, 75, 1089.

Chapter II

General Formula for the Spectral Distribution of Background Radiation in Various Classes of World-Model

Throughout this thesis attention will be restricted to homogeneous isotropic world-models which obey the Cosmological Principle. Robertson and Walker (1, 2) have shown that such models are associated with a line element that is expressible in the form

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2}\left\{\frac{dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)}{(1 + kr^2/4)^2}\right\}, \qquad (2.1)$$

where t is a cosmic time, R(t) is a function of t that increases monotonically during expansion, c the local speed of light, (r, θ, φ) are dimensionless spatial co-ordinates and k can take the values 1, 0, -1 and denotes the constant curvature of $d\sigma^2$, where

$$d\sigma^2 = \frac{dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)}{(1 + kr^2/4)^2}. \qquad (2.2)$$

The radiation sources in these world-models have constant spatial co-ordinates (r, θ, φ) throughout the history of the model. An observer O is introduced at the origin of the co-ordinates (r, θ, φ). Since we are only considering models in which the Cosmological Principle holds, any point in the model can be chosen as origin. Let the number of extra-galactic radiation sources lying within the range (r, r + dr) according to O be denoted by dN. The average intensity of the radiation (energy per second) emitted by the sources in the shell (r, r + dr) at time t

in the waveband $(\lambda, \lambda + d\lambda)$ will be $F(\lambda, t)\, d\lambda$, say. $F(\lambda, t)$ is called the radiation-density. When the systematic effect of evolution of the sources is either negligible or statistically zero throughout the past history of the model (as in the steady-state model of continual creation), then the average intensity of the sources will be constant in time, and t will not appear explicitly in F.

The total flux (energy per unit area per second) received at 0 at time t_0 from a source in the shell $(r, r + dr)$ emitting at time t is obtained by dividing $F(\lambda, t)\, d\lambda$ by the factor $4\pi D^2$, where

$$D = \frac{R_0^2\, r}{R(1 + kr^2/4)}, \qquad (2\cdot3)$$

and $R = R(t)$ and $R_0 = R(t_0)$. The quantity D is often referred to as the luminosity-distance, because it is defined in such a manner that in the models we are considering the intensity of radiation from distant sources varies as the inverse square of their luminosity-distance D. The factor $4\pi D^2$ therefore automatically accounts for the effects of both recession and distance on the radiation from the sources.

The waveband of the radiation received at 0 at time t_0 due to the sources in the shell emitting in the waveband $(\lambda, \lambda + d\lambda)$ is denoted by $(\lambda_0, \lambda_0 + d\lambda_0)$, where

$$\lambda_0 = \frac{R_0 \lambda}{R}, \qquad (2\cdot4)$$

and

$$d\lambda_0 = \frac{R_0\, d\lambda}{R}. \qquad (2\cdot5)$$

Hence, the total flux received at 0 from all the sources in the shell $(r, r + dr)$ will be *

$$\frac{F(\lambda, t)\, d\lambda\, dN}{4\pi D^2} = F\left(\frac{R\lambda_0}{R_0}, t\right) \frac{R d\lambda_0\, dN}{4\pi R_0 D^2}. \qquad (2\cdot6)$$

The flux received in isotropic models in solid angle Ω at 0 is a factor $\frac{\Omega}{4\pi}$ of (2·6).

1. Expanding Models with Conservation of Matter. For the present, it will be assumed that the number of radiation-sources per unit volume of the 3-space given by equation (2·2) in the homogeneous, isotropic world-models of metric given by equation (2·1) is the same throughout the model and does not change with time. (This assumption is normally made in the analysis of homogeneous isotropic world-models, but in fact may not be equally valid for all types of radiation sources. In particular, in an evolutionary world-model the number of intense radio-sources, the so-called "radio stars", whether due to collisions of galaxies or to peculiar properties of elliptical nebulae at a certain stage in their history, may have been greater per unit volume of the co-ordinate space (r, θ, φ) in the past than now). Therefore, we may write for expanding models

$$dN = \frac{4\pi n_0 R_0^3 r^2\, dr}{(1 + kr^2/4)^3}, \qquad (2\cdot7)$$

where n_0 is the number of sources per unit proper volume at time t_0.

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* The obscuring effects of the intermediate galaxies and intergalactic matter are neglected. These effects will be considered in Chapter VI.

Since in this type of cosmological model the paths of photons are given by the null geodesics of metric (2·1), it follows that for the path of radiation arriving at 0

$$\frac{dr}{(1 + kr^2/4)} = - \frac{cdt}{R(t)}, \tag{2·8}$$

and since (2·8) holds at every point along the null geodesic,

$$\int_0^r \frac{dr}{(1 + kr^2/4)} = \int_t^{t_0} \frac{cdt}{R(t)}, \tag{2·9}$$

which gives the time of emission of a radiation source at distance r when the signal is received by 0 at time t_0.

Hence, in expanding world-models, we deduce from (2·6) and (2·7), with the aid of (2·3) and (2·8), that the flux of radiation received at a given epoch t_0 at 0 in the waveband $(\lambda_0, \lambda_0 + d\lambda_0)$ in solid angle Ω from sources emitting at times in the range (t - dt, t) can be expressed as

$$F\left(\frac{R\lambda_0}{R_0}, t\right)\left(\frac{R}{R_0}\right)^2 n_0 \, cdt \, d\lambda_0 \frac{\Omega}{4\pi}. \tag{2·10}$$

We observe that the total radiation received at a given epoch from sources emitting in a given range of time (t - dt, t) is independent of the curvature index k. This result, which does not seem to have been pointed out by previous writers, is a generalization of the rule characterising successive shells in discussions of Olbers's paradox in a universe without expansion, but unlike the situation analysed in these discussions the total radiation received is not the same from each shell.

Integrating equation (2·10) over the range of radiation sources from which radiation reaches 0 at the present epoch t_0, we obtain $I(\lambda_0, t_0)\, d\lambda_0$

the total radiation flux* in the waveband $(\lambda_o, \lambda_o + d\lambda_o)$ which is given by

$$I(\lambda_o, t_o)\, d\lambda_o = \frac{n_o c\, d\lambda_o}{4\pi} \Omega \int_{t'}^{t_o} F\left(\frac{R\lambda_o}{R_o}, t\right)\left(\frac{R}{R_o}\right)^2 dt, \qquad (2\cdot11)$$

where t' is the proper time of emission† of radiation from the most distant radiation sources visible at time t_o. Clearly the particular value of t' will depend on the model considered.

The justification of formula (2·11) in the case of models with positive curvature requires further consideration. For, if radiation can circumnavigate the universe, then at a given instant t_o the observer may receive radiation, from sources of given r emitted at different epochs‡ $t_r^{(m)}$, which has made m circuits of the universe, where m = 1, 2, 3, ... We observe first that, if $t_o^{(1)} < t \leq t_o$, formula (2·10) applies to radiation received by 0 at time t_o which has made not more than one circuit. Similarly, for $t_o^{(m+1)} < t \leq t_o^{(m)}$, a formula similar to (2·10) applies to radiation received by 0 at $t_o^{(m)}$ which has made not more than one circuit.

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* The flux per unit waveband-width $I(\lambda_o, t_o)$ is referred to as the flux-density.

† In the case of a closed world-model where the curvature index k = +1, t' is the earliest proper time of emission of radiation from the most distant radiation sources.

‡ The epoch of emission will depend on whether the topology of the model is elliptical or spherical.

If the sources at r emit radiation at time $t_r^{(m)}$ of wavelength $\lambda_r^{(m)}$ and this radiation is received by 0 at $t_o^{(m)}$ at wavelength $\lambda_o^{(m)}$, then

$$\frac{\lambda_r^{(m)}}{R(t_r^{(m)})} = \frac{\lambda_o^{(m)}}{R(t_o^{(m)})} . \tag{2·12}$$

Hence, the energy received per unit area per second at 0 at $t_o^{(m)}$ in the waveband $(\lambda_o^{(m)}, \lambda_o^{(m)} + d\lambda_o^{(m)})$ from sources emitting in the time range $(t_r^{(m)} - dt_r^{(m)}, t_r^{(m)})$ is given by (2·10) with t replaced by $t_r^{(m)}$, t_o replaced by $t_o^{(m)}$, λ_o replaced by $\lambda_o^{(m)}$, and $\Omega = 4\pi$, namely

$$F\left\{\frac{\lambda_o^{(m)} R(t_r^{(m)})}{R(t_o^{(m)})}, t_r^{(m)}\right\} n(t_o^{(m)}) \frac{R^2(t_r^{(m)})}{R^2(t_o^{(m)})} d\lambda_o^{(m)} c dt_r^{(m)}, \tag{2·13}$$

where $n(t_o^{(m)})$ is the number of sources in unit proper volume at time $t_o^{(m)}$.

Since we are considering models in which matter is conserved, we have the condition that

$$n(t_o^{(m)}) R^3(t_o^{(m)}) = n_o R_o . \tag{2·14}$$

Using formula (2·14), (2·13) can be expressed in the following form,

$$\frac{n_o R_o^3}{R^3(t_o^{(m)})} F\left\{\lambda_o^{(m)} \frac{R(t_r^{(m)})}{R(t_o^{(m)})}, t_r^{(m)}\right\} \frac{R^2(t_r^{(m)})}{R^2(t_o^{(m)})} c dt_r^{(m)} d\lambda_o^{(m)} . \tag{2·15}$$

This radiation makes m complete circuits of the universe before arriving at 0 at time t_o in the waveband $(\lambda_o, \lambda_o + d\lambda_o)$, where

$$\frac{\lambda_o}{R_o} = \frac{\lambda_o^{(m)}}{R(t_o^{(m)})} . \tag{2·16}$$

Its intensity at 0 after this number of circuits will be given by multiplying (2·14) by $\left\{\frac{R(t_o^{(m)})}{R_o}\right\}^4$, two of these factors $\frac{R(t_o^{(m)})}{R_o}$ being due to the usual number effect and energy-frequency effect, respectively,

and the other two due to the diminution of apparent solid angle subtended at 0 by the source at time t_0 as compared with that at time $t_0^{(m)}$.

Consequently, the energy flux in the waveband $(\lambda_0, \lambda_0 + d\lambda_0)$ received by 0 at time t_0 due to radiation emitted by sources in the proper time interval $(t - dt, t)$, where $t_0^{(m+1)} < t \leq t_0^{(m)}$, is

$$n_0 F\left(\frac{\lambda_0 R}{R_0}, t\right) \left(\frac{R}{R_0}\right)^2 c\,dt\,d\lambda_0, \qquad (2\cdot17)$$

for $m = 1, 2, 3, \ldots$. Since, as previously shown, this result holds for $t_0^{(1)} < t \leq t_0$, equation (2·11) is the appropriate formula for the spectrum of the background radiation in expanding models when the curvature index is +1 as well as 0 and -1.

To complete the derivation of the general formula for the background spectrum in expanding models it remains to describe the method of obtaining t'. The relation between the distance of a galaxy and the time of emission t at which light must be emitted by that galaxy in order to reach 0 at time t_0 is given by

$$\sigma(r) = \int_t^{t_0} \frac{c\,dt}{R}, \qquad (2\cdot18)$$

where $\sigma(r)$ is defined in terms of co-ordinate distance r by the relation

$$\sigma(r) = \int_0^r \frac{dr}{1 + kr^2/4}, \qquad (2\cdot19)$$

and lies in the range $0 \leq \sigma \leq \infty$ whatever the curvature index may be.

The determination of the most distant radiation source that can be seen by 0 at time t_0 depends on the largest value of $\sigma(r)$, and this

corresponds to the earliest value t' of t in equation (2·18). This will depend on the existence and significance of a zero of R(t). If an initial zero of this function indicates an epoch of world-creation (in other words, if the model can be assumed to exist only for subsequent values of t) then $t' = t_0 - T$, where T is the age of the world-model, and hence if $t_0 = T$ it follows that $t' = 0$. On the other hand, if the model exists both before and after a zero-singularity at $t = t^*$ of R(t), we must consider whether account should be taken of radiation from sources which were emitting before the singular event at $t = t^*$. On physical grounds, it might be argued that this radiation would be absorbed by the highly condensed matter and that instead we have to consider the remnant radiation from the singular state. Unless otherwise stated, it will be assumed that no radiation reaches the observer at $t = t_0$ which has passed through a singular state.

Formula (2·11) takes a somewhat simpler form when we change over from wavelength (λ) to frequency (ν). Let the average intensity of the radiation emitted by the sources in the shell (r, r + dr) at time t in the range of frequencies ($\nu - d\nu, \nu$) be $G(\nu, t)\, d\nu$, let the flux of radiation received at 0 at t_0 in the range of frequencies ($\nu_0 - d\nu_0, \nu_0$) in solid angle Ω be $J(\nu_0, t)\, d\nu_0$, and let $\lambda = \frac{c}{\nu}$, and $d\lambda = -\frac{c\,d\nu}{\nu^2}$. Then, since the frequency range ($\nu - d\nu, \nu$) corresponds to the range of wavelength ($\lambda, \lambda + d\lambda$), the following two identities hold:

$$J(\nu_0, t_0)\, d\nu_0 = I(\lambda_0, t_0)\, d\lambda_0, \tag{2·20}$$

$$G(\nu, t)\, d\nu = F(\lambda, t)\, d\lambda. \tag{2·21}$$

Substituting for I and F in (2·11) using the relations (2·20) and (2·21), it follows that

$$J(\nu_o,t_o)d\nu_o = \frac{n_o e\, d\nu_o \Omega}{4\pi}\int_{t'}^{t_o} G\left(\frac{\nu_o R_o}{R}, t\right)dt \qquad (2\cdot22)$$

Radio-astronomers, who work in terms of frequencies, would measure the flux-density $J(\nu_o, t_o)$ (given by (2·22)) in solid angle Ω in watts (metres)$^{-1}$ (cycle per sec)$^{-1}$. Optical astronomers, on the other hand, normally work in wavelengths and would measure $I(\lambda_o, t_o)$ the flux-density in solid angle Ω (given by equation (2·11)) in ergs (centimetres)$^{-2}$ (seconds)$^{-1}$ (Angstroms)$^{-1}$.

So far we have considered only the background spectrum from the whole observable universe, but it is also useful to introduce the residual background spectrum, corresponding to a cut-off at some particular lower limit of spectral shift, e.g. $\frac{\delta\lambda}{\lambda} = 1/5$. Suppose in general the cut-off is at $\frac{\delta\lambda}{\lambda} = z$, then if t" is the epoch of emission of light which arrives at 0 at t_o from those sources receding with a red-shift of amount z, the residual background spectrum is obtained by integrating expression (2·10) from t' to t", where t' is the time of emission of radiation from the most distant source which arrives at 0 at time t_o, and t" is the time of emission of radiation from the sources with spectral shift z which arrives at 0 at time t_o. Clearly the contribution to the background radiation from those sources with spectral shift less than z has been omitted. If $I(\lambda_o, t_o)\, d\lambda_o$ denotes the residual background spectral radiation received by 0 at time t_o from all those radiation sources with spectral shift greater than z, then

$$I(\lambda_o, t_o)d\lambda_o = \frac{n_o c\, d\lambda_o}{4\pi}\Omega \int_{t'}^{t''} F\left(\frac{\lambda_o R}{R_o}, t\right)\left(\frac{R}{R_o}\right)^2 dt, \qquad (2\cdot23)$$

where t" satisfies the equation

$$\frac{R_o}{R(t'')} = 1 + z, \qquad (2\cdot24)$$

and therefore

$$t'' = R^{-1}\left\{\frac{R_o}{1+z}\right\}. \qquad (2\cdot25)$$

2. Steady-state Model with Continual Creation. In the case of the steady-state model, formulae (2·11), (2·22) and (2·23) for the background radiation in expanding models with continual creation must be modified. In the steady-state model the number of sources per unit proper volume is a constant, n_o, and hence

$$dN = 4\pi n_o R^3 r^2 dr. \qquad (2\cdot26)$$

The luminosity distance is given by equation (2·3) with k = 0 and the equation of the path of a photon satisfies (2·8) with k = 0. Hence, for the steady-state model,

$$D = \frac{R_o^2 r}{R}, \qquad (2\cdot27)$$

and

$$dr = -\frac{c\,dt}{R}. \qquad (2\cdot28)$$

Since the Perfect Cosmological Principle holds in this model, the average intensity ($F(\lambda, t)\, d\lambda$) emitted by the sources is independent of the epoch of emission (t). Further, the background radiation at the observer 0 will be independent of the time of reception t_o. Substituting for dN, D and dr given by equations (2·26), (2·27) and (2·28), respectively, expression (2·6) becomes

$$F\left(\frac{\lambda_o R}{R_o}\right)\ n_o\left(\frac{R}{R_o}\right)^5\ c\,dt\,d\lambda_o, \qquad (2\cdot29)$$

which is the total flux received at 0 in the waveband $(\lambda_o, \lambda_o + d\lambda_o)$ from all sources in the shell (r, r + dr) in the steady-state model. The expansion factor R takes the specific form

$$R = e^{Ht}, \qquad (2\cdot30)$$

where H is Hubble's constant [*] in the steady-state model.

Upon integrating (2·29) over all values of t earlier than the epoch t_o and substituting for R using equation (2·30) we obtain the total background flux at 0 at time t_o. Denoting the total background flux-density in solid angle Ω in the steady-state model by $I(\lambda_o)$, then

$$I(\lambda_o)d\lambda_o = \frac{n_o c\, d\lambda_o}{4\pi}\Omega\int_{-\infty}^{t_o} F(\lambda_o e^{H(t - t_o)})e^{5H(t - t_o)}dt. \qquad (2\cdot31)$$

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[*] From his observations of extragalactic nebulae, Hubble found that to a first approximation the spectral red-shift of the galaxies is directly proportional to their distance. More precisely, if t is the time of emission of radiation from a source, t_o the time of reception at 0, $\ell = r R_o$ the proper distance of the source from 0, and z the spectral red-shift, then

$$cz = H\ell,$$

where H is a constant for all clusters. This relation between the red-shift and distance of the extragalactic nebulae is known as Hubble's Law and the constant H is called Hubble's constant. In the steady-state model H must be independent of time, but in other models it may vary with time.

The integration is performed over all epochs earlier than t_0, because it follows from (2·28) that the path of a photon arriving at 0 at $t = t_0$ is given by

$$r = \frac{c}{H}\left(e^{-Ht} - e^{-Ht_0}\right), \tag{2·32}$$

and this equation is satisfied by all positive values of r, and these correspond to all values of t in the range $t_0 \geq t > -\infty$.

As a check on formula (2.31) we observe that the integral is independent of the epoch of observation t_0, for $t - t_0$ in the integrand can be replaced by t giving

$$I(\lambda_0)d\lambda_0 = \frac{n_0 c\, d\lambda_0 \Omega}{4\pi}\int_{-\infty}^{0} F(\lambda_0 e^{Ht})e^{5Ht}dt. \tag{2·33}$$

The residual background radiation in the steady-state world-model when there is a cut-off at $\delta\lambda/\lambda = z$ is obtained by integrating (2·29) from $-\infty$ to t" where t" is the time of emission of radiation from those sources with a spectral shift of amount z which arrives at 0 at time t_0. Hence

$$e^{H(t_0 - t")} = 1 + z,$$

or
$$t" = t_0 - \frac{1}{H}\log_e(1 + z). \tag{2·34}$$

Hence on replacing $t - t_0$ by t we obtain the formula

$$I(\lambda_0)d\lambda_0 = \frac{n_0 c\, d\lambda_0 \Omega}{4\pi}\int_{-\infty}^{-\frac{1}{H}\log_e(1+z)} F(\lambda_0 e^{Ht})e^{5Ht}dt. \tag{2·35}$$

3. Static World-Models with Loss of Energy by Travelling Photons.

It is of particular interest to derive the general formula for the background spectrum for homogeneous isotropic world-models in which the

red shift of the spectra of the extragalactic nebulae is due to some physical cause other than recession.*

Non-recessional theories may be subdivided into two types, (a) and (b), say. In type (a) the red-shifts are attributed to some secular variation in one of the "constants" of nature, whereas in type (b) the red-shift depends on some factor influencing photons as they travel through intergalactic space. By a suitable choice of natural measurements in type (a) it is often possible to arrive at a recessional theory. I shall consider the simpler hypothesis of type (b): that over intergalactic distances the frequency of a photon decreases with the passage of time, whilst all the constants of nature are secular invariants.

Since we are now considering static homogeneous isotropic world-models, we shall restrict attention to the Robertson-Walker line element (1·1) in the special form

$$ds^2 = dt^2 - \frac{R_0^2}{c^2}\left\{\frac{dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)}{(1 + kr^2/4)^2}\right\}, \qquad (2\cdot36)$$

where R_0 is a constant scale factor.

Let the number of sources of radiation in the shell $(r, r + dr)$ be dN. Let us consider radiation emitted by these sources at time t in the waveband $(\lambda, \lambda + d\lambda)$ and suppose that the observer O receives this radiation at time t_0 in the waveband $(\lambda_0, \lambda_0 + d\lambda_0)$. If it is assumed

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* Although no physical explanation of the extragalactic red-shift, other than the effect of recession, has been generally accepted, an alternative process cannot yet be ruled out.

that the red-shift is independent of wavelength, then λ_o and λ must be connected by a functional relationship of the form

$$\frac{\lambda_o}{S(t_o)} = \frac{\lambda}{S(t)} . \qquad (2\cdot37)$$

Since S is independent of λ, it follows, on differentiating (2·37), that

$$\frac{d\lambda_o}{S(t_o)} = \frac{d\lambda}{S(t)} . \qquad (2\cdot38)$$

If we assume further that the red-shift is independent of the epoch of emission t, then G. J. Whitrow (3) has shown that S(t) takes the specific form

$$S(t) = e^{at}, \qquad (2\cdot39)$$

where a is a constant, but we shall take the general form of S until we come to make numerical calculations.

The flux of radiation received at O at time t_o from sources in the shell (r, r + dr) in the waveband $(\lambda_o, \lambda_o + d\lambda_o)$ in solid angle Ω is obtained from the left hand side of equation (2·6) with the aid of equations (2·37) and (2·38), and is given by

$$\frac{\Omega}{4\pi} F\left(\frac{\lambda_o S}{S_o}, t\right) \frac{S}{S_o} d\lambda_o \frac{dN}{4\pi D^2}, \qquad (2\cdot40)$$

where $S_o = S(t_o)$ and $S = S(t)$.

The luminosity-distance D for static models with decaying photons is given by

$$D = \frac{R_o r}{1 + kr^2/4} \left(\frac{S_o}{S}\right)^{\frac{1}{2}}, \qquad (2\cdot41)$$

where the term $\frac{R_o r}{1 + kr^2/4}$ takes into account the diminution in the intensity of radiation due to distance and the term $\left(\frac{S_o}{S}\right)^{\frac{1}{2}}$ the diminution

in intensity due to the decrease in frequency of photons with the passage of time. Since the model is static there is no number effect.

The number of sources per unit spatial volume of world-models associated with metric (2·36) is

$$dN = \frac{4\pi n_0 R_0^3 r^2 dr}{(1 + kr^2/4)^3}, \qquad (2·42)$$

where n_0 is the constant number of sources per unit proper volume. Hence (2·40) becomes

$$\frac{\Omega}{4\pi} F(\frac{\lambda_0 S}{S_0}, t)(\frac{S}{S_0})^2 d\lambda_0 \frac{n_0 R_0 dr}{(1 + kr^2/4)}, \qquad (2·43)$$

where (2·41) and (2·42) have been used to eliminate D and dN.

The equation of a light path is a null geodesic of the metric (2·36), therefore for an incoming ray

$$\frac{dr}{1 + kr^2/4} = -\frac{cdt}{R_0}. \qquad (2·44)$$

Eliminating dr from (2·43) using (2.44) we find that the radiation received at t_0 in the waveband $(\lambda_0, \lambda_0 + d\lambda_0)$ from sources emitting in the time interval $(t - dt, t)$ in solid angle Ω is

$$\frac{\Omega}{4\pi} F(\frac{\lambda_0 S}{S_0}, t)(\frac{S}{S_0})^2 d\lambda_0 n_0 c dt. \qquad (2·45)$$

The background flux-density $I(\lambda_0, t_0)$ at 0 at time t_0 is obtained by integrating (2·45) from t_1 to t_0, where t_1 is the epoch of emission from the most distant source that can be observed by 0 at t_0. For static models, $t_1 = -\infty$, hence

$$I(\lambda_0, t_0) d\lambda_0 = \frac{n_0 \Omega c d\lambda_0}{4\pi} \int_{-\infty}^{t_0} F(\frac{\lambda_0 S}{S_0}, t)(\frac{S}{S_0})^2 dt, \qquad (2·46)$$

which is independent of the curvature index k as in the case of the expanding models with conservation of matter.

The background spectrum $I(\lambda_0, t_0)\, d\lambda_0$ when there is a cut-off at $\delta\lambda/\lambda = z$ is obtained similarly by integrating (2·45) from $-\infty$ to t'' where t'' satisfies the relation

$$\frac{S_0}{S(t'')} = 1 + z \qquad (2\cdot47)$$

and

$$I(\lambda_0, t_0)d\lambda_0 = \frac{n_0 \Omega c d\lambda_0}{4\pi} \int_{-\infty}^{t''} F\left(\frac{\lambda_0 S}{S_0}, t\right)\left(\frac{S}{S_0}\right)^2 dt_0 \,. \qquad (2\cdot48)$$

We note the similarity which exists between formula (2·46) and the corresponding result given by equation (2·11) for the background spectrum in expanding models when S in equation (2·46) is replaced by the expansion factor R. An exact equality does not exist, however, because the lower limit of integration in formula (2·11) is not the same for all expanding world-models. If, however, we take $S = e^{at}$ then the formula is identical with that which we would obtain for an expanding model* with expansion factor $R = e^{Ht}$ provided the constant a takes the same numerical value as Hubble's constant H. In general it is impossible to compare the theoretical formulae for the spectral background radiation in homogeneous isotropic world-models because the lower limits of integration are different for

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* We shall refer to expanding world-models with conservation of matter and expansion factor $R = e^{Ht}$ as de Sitter metric models. If in addition the curvature index is zero and the model is subject to the laws of general relativity the model is the de Sitter model and there is no matter in it.

different models, and the integral cannot be evaluated explicitly. Consequently, recourse must be made to numerical evaluation of the formulae before a comparison can be made between various world-models, and this task is deferred until Chapter IV.

References

1. Robertson, H. P., Astrophysical Journal (1935), 82, 284.
2. Walker, A. G., Proclamation of the London Mathematical Society (2), (1936), 42, 90.
3. Whitrow, G. J., Monthly Notices of the Royal Astronomical Society, (1954), 114, 180.

Chapter III

General Formulae for the Total Intensity of Background Radiation in Various Classes of World-Model

In this chapter the general formula for the total intensity of background radiation in homogeneous and isotropic world-models will be derived, and the closely related problem of the rate of reception of information from extragalactic sources, recently investigated by Metzner and Morrison, will be discussed.

1. Milne's Uniformly Expanding World-Model. Following the classical discussion by Olbers, the first significant modern theoretical contribution to the problem of the total background radiation in the universe, apart from a paper by C. V. L. Charlier in 1922 (1) on hierarchical world-models, was due to Milne (2). In Milne's uniformly expanding model the number of sources receding from the observer O with velocities in the range (V, V + dV) is given by

$$dN = \frac{AV^2\, dV}{(1 - V^2/c^2)^2}\,, \qquad (3\cdot1)$$

where A is a constant and c the velocity of light. As Milne remarked, when V = c, dN given by equation (3·1) becomes infinite, and therefore it might happen that the radiation from the very distant extragalactic radiation sources would be infinite in this uniformly expanding world-model. If this model were to be regarded as a useful approximation to the empirical universe, it was necessary for Milne to show that the total

background brightness in it is finite. Milne assumed implicitly that the average total radiation from the extragalactic sources is constant for all epochs of emission. The intensity of radiation from a source which is receding from O with velocity V in Milne's model is reduced by a factor

$$\left(\frac{1 - V/c}{1 + V/c}\right)^2 , \tag{3·2}$$

due to the effect of recession and by a further factor

$$\left(\frac{1 + V/c}{Vt_o}\right)^2, \tag{3·3}$$

due to distance, where t_o is the epoch of reception of radiation at O.

Combining expressions (3·1), (3·2) and (3·3) together with the average luminosity L of the sources in the world-model, we find the flux of radiation from all the sources lying in the range (V, V + dV) to be

$$L \frac{AV^2\, dV}{(1 - V^2/c^2)^2} \left(\frac{1 - V/c}{1 + V/c}\right)^2 \left(\frac{1 + V/c}{Vt_o}\right)^2 . \tag{3·4}$$

Integrating (3·4) over the range of velocities (0, c), we obtain the total intensity of background radiation received by O at epoch t_o. Denoting the total intensity received by O in solid angle Ω at epoch t_o by Q, we find that

$$\begin{aligned} Q &= \frac{\Omega}{4\pi} \frac{AL}{t_o^2} \int_o^c \frac{dV}{(1 + V/c)^2} \\ &= \frac{\Omega}{4\pi} \frac{AL}{t_o^2} \frac{c}{2} , \end{aligned} \tag{3·5}$$

which is finite. This result was obtained by Milne.

2. Homogeneous Isotropic Expanding Models (with conservation of matter). In the general case of an expanding evolutionary world-model associated with metric (2·1), the formula for the total background radiation received at epoch t_0 in solid angle Ω by an observer O can be obtained by integrating $I(\lambda_0, t_0)$ given by equation (2·11) over all λ_0. Denoting this quantity by Q,

$$Q = \int_0^\infty I(\lambda_0, t_0)\, d\lambda_0. \tag{3·6}$$

Substituting for $I(\lambda_0, t_0)$ from equation (2·11), it follows that for expanding evolutionary world-models,

$$Q = n_0 c \frac{\Omega}{4\pi} \int_{t'}^{t_0} L(t) \frac{R}{R_0}\, dt, \tag{3·7}$$

where

$$L(t) = \int_0^\infty F\left(\frac{\lambda_0 R}{R_0}, t\right) \frac{R}{R_0}\, d\lambda_0, \tag{3·8}$$

and t' is the epoch of emission of radiation from the most distant source observed at the present time t_0.

On changing the variable in (3·8) by replacing $\frac{R\lambda_0}{R_0}$ by λ_0, we deduce that

$$L(t) = \int_0^\infty F(\lambda_0, t)\, d\lambda_0, \tag{3·9}$$

so that L(t) is the average luminosity of the light sources emitting at proper time t.

Olbers's criterion that Q given by (3·7) is to be finite will be satisfied if

$$\int_{t'}^{t_0} L R\, dt \tag{3·10}$$

is finite.

If there is no systematic evolutionary effect throughout the history of the model, so that F is independent of t, then L given by equation (3·9) will be a constant. In this case the condition that Q is finite is that

$$\int_{t'}^{t} R\,dt \qquad (3.11)$$

is finite.

With the aid of conditions (3·10) and (3·11) a number of useful theorems on background radiation in homogeneous isotropic world-models may be derived when absorption effects due to the finite size of the extragalactic nebulae and to intergalactic matter are neglected.

For example, if the model expands from a singularity at a particular epoch - which can be taken to be zero - and if there is no systematic evolutionary effect throughout the history of the model, then since from the analysis of Chapter II $t' = 0$, it follows that, if $R(t)$ is everywhere finite in the range $t' \leq t \leq t_o$, (3·11) will be finite and hence Q must be finite. Thus we have the following theorem:

Theorem I. In any expanding homogeneous isotropic world-model in which the number of sources (galaxies) and their average luminosity are conserved, and $R(t) = 0$ for $t = 0$ and finite for $0 \leq t \leq t_o$, where t_o is the present epoch, the total intensity of background radiation at time t_o is finite.

This theorem is a generalization of Milne's result [*] for a uniformly

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[*] It is interesting to note that for many of these models, including Milne's, the number-density of sources diverges for an infinite Doppler shift, e.g. Page's Model.

expanding world-model with L independent of t and R = ct. Moreover even if L varies with t, provided the quantity LR is always finite we have the following theorem:

Theorem II. In any expanding homogeneous isotropic world-model in which the number of sources is conserved, if $R(t) = 0$ for $t = 0$ and $L(t)R(t)$ is finite for $0 \leq t \leq t_0$ where $L(t)$ is the average luminosity of those sources which emit radiation at time t that arrives at the observer at the present epoch t_0, the total intensity of background radiation at time t_0 is finite.

Thus there is finite sky brightness in any world-model expanding from a singularity that occurred at a finite epoch in the past provided the quantity LR is always finite. On the other hand, in the case of an oscillating universe there may be infinite sky brightness. For those world-models which have been oscillating without singularity ($R > 0$) for an infinite time in the past, t' is negatively infinite. Hence using condition (3·11) we have the following theorem:

Theorem III. In any oscillating homogeneous isotropic world-model with an infinite past for which

(i) $R_{max} \geqslant R(t) \geqslant R_{min} > 0$ for all values of $t \leq t_0$ where R_{max} is finite, and t_0 is the present epoch,

(ii) the average luminosity of the sources does not change significantly with time,

the total intensity of background radiation is infinite.

Clearly if L varies with t condition (3·10) yields the two following theorems:

Theorem IV. In any oscillating homogeneous isotropic world-model with an infinite past for which

(i) $R_{max} \geqslant R(t) \geqslant R_{min} > 0$, for all $t \leqslant t_o$, where t_o is the present epoch and R_{max} is finite,

(ii) $L_{max} \geqslant L(t) \geqslant L_{min} > 0$, for all $t \leqslant t_o$, where L_{max} is finite,

the total intensity of background radiation is infinite.

Theorem V. In any oscillating homogeneous isotropic world-model with an infinite past for which

(i) $R_{max} \geqslant R(t) \geqslant R_{min} > 0$ for all $t \leqslant t_o$, where t_o is the present epoch and R_{max} is finite,

(ii) $L(t) > 0$ for an interval of time δt not vanishingly small in each oscillation ($\delta t > 0$),

the total intensity of background radiation is infinite.

It is of interest to compute the formula for the total background radiation from all sources with red-shift greater than some cut-off value $\delta\lambda/\lambda = z$ in homogeneous isotropic world-models with conservation of matter. Integrating over all wavelengths the background spectrum in solid angle Ω from such sources, which is given by equation (2·23) we find that

$$Q = \int_0^\infty I(\lambda_o, t_o)\, d\lambda_o = n_o c \frac{\Omega}{4\pi} \int_{t'}^{t''} L(t) \frac{R}{R_o}\, dt, \qquad (3·12)$$

where Q is the total background radiation in solid angle Ω from all sources with red-shift greater than z, $L(t)$ is given by (3·8) and is therefore the average luminosity of the sources emitting at time t, and t''

given by equation (2·25) is the epoch of emission from those sources with red-shift z whose light reaches 0 at t_0.

3. The Steady-State Model (with continual creation). In the case of the steady-state theory we must use equation (2·33) in place of (2·11) when evaluating the total background intensity. Consequently, writing Q for the total background intensity in solid angle Ω in this model we have

$$Q = \int_0^\infty I(\lambda_0)d\lambda_0 = n_0 c \frac{\Omega}{4\pi} \int_{-\infty}^{0} L e^{4Ht}\, dt, \tag{3·13}$$

where

$$L = \int_0^\infty F(e^{Ht}\lambda_0)\, d\,(e^{Ht}\lambda_0) = \int_0^\infty F(\lambda)d\lambda \tag{3·14}$$

is the average constant luminosity of the galaxies. Integrating (3·13) we obtain the following result:

Theorem VI. In the steady-state world-model with continual creation the total background intensity of radiation in solid angle Ω is

$$Q = \frac{n_0 cL}{4H}\left(\frac{\Omega}{4\pi}\right), \tag{3·15}$$

where n_0 is the number of galaxies in unit proper volume, c the velocity of light, L the average luminosity of the galaxies and H is Hubble's constant.

The total intensity of background radiation in solid angle Ω from all sources beyond the cut-off distance defined by $\delta\lambda/\lambda = z$ is obtained similarly by integrating (2·35) over λ_0 we find that

$$Q = \frac{n_0 cL\Omega}{4\pi} \int_{-\infty}^{-\frac{1}{H}\log_e(1+z)} e^{4Ht} dt = \frac{n_0 cL}{4H} \frac{1}{(1+z)^4}\left(\frac{\Omega}{4\pi}\right). \tag{3·16}$$

4. Static Models with Loss of Energy by Travelling Photons. The total background intensity of radiation Q in solid angle Ω in non-expanding world-models in which the frequency of photons diminishes in transit is obtained by integrating (2.46) over all λ_o. Hence

$$Q = \int_0^\infty I(\lambda_o, t_o)\, d\lambda_o$$
$$= n_o c \frac{\Omega}{4\pi} \int_{-\infty}^{t_o} L(t) \frac{S}{S_o}\, dt, \qquad (3\cdot17)$$

where $L(t) = \int_0^\infty F\left(\frac{\lambda_o S}{S_o}, t\right) \frac{S}{S_o} d\lambda_o = \int_0^\infty F(\lambda, t)\, d\lambda,$ (3·18)

is the average luminosity of the sources at proper time t, and $S(t)$ is a function of time and plays a similar role to the expansion factor in expanding models.

We note that equation (3·17) is similar to equation (3·7) for the total intensity of background radiation in expanding world-models as in the corresponding formulae for the spectral distribution in the two types of model. Clearly from equation (3·17) we can derive the following theorem:-

Theorem VII. A necessary condition for finite background radiation in static non-evolutionary world-models in which the frequency of travelling photons diminishes with the passage of time is that

$$S(t) \longrightarrow 0 \text{ as } t \longrightarrow -\infty .$$

The background intensity in solid angle Ω when there is a cut-off at $\delta\lambda/\lambda = z$ is obtained by integrating (2·48), the result is

$$Q = n_o c \frac{\Omega}{4\pi} \int_{-\infty}^{t''} L(t) \frac{S}{S_o}\, dt, \qquad (3\cdot19)$$

where t" satisfies (2·47) and is the epoch of emission of radiation from sources which arrives at O at epoch t_o with red-shift $\delta\lambda/\lambda = z$.

5. Flow of Information in World-Models. It is interesting to see how our work compares with that of Metzner and Morrison (3) who have approached the problem from the point of view of information theory. They have considered the rate of reception of information in a given frequency band from extragalactic radiation sources of given Doppler shift. The rate of reception of information by an observer with a telescope (or radio telescope) from an object of given type diminishes the farther away the object is. Metzner and Morrison have investigated the way in which the information rate approaches the limiting zero rate at the horizon of various world-models. They considered in this investigation homogeneous isotropic world-models associated with the line element given by equation (2·1), and they assumed that the radiation sources in these world-models are non-evolutionary. Further they neglected the obscuring effect of nearby radiation sources from more distant ones.

We shall now derive the general formulae for the information rate of a single source with given [x] Doppler ratio, and the information rate from all sources with Doppler ratio greater than a given value. Since Metzner and Morrison considered specific world-models they did not obtain these formulae.

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[x] It is more convenient in the following analysis to use Doppler ratio Z defined by $Z = \frac{\lambda + \delta\lambda}{\lambda}$ than the Doppler shift $\frac{\delta\lambda}{\lambda} = Z - 1$.

Denoting the apparent luminosity in a well-defined frequency band $(\nu_o, \nu_o - d\nu_o)$ of a single source with Doppler ratio Z by $b(Z)$, then the rate of reception of information in this band (4) is given by

$$C(Z) = d\nu_o \log \left[1 + \frac{b(Z)}{N}\right], \qquad (3.20)$$

where N is the noise power (associated with the measurement of $b(Z)$, which we will treat as a constant. If the base of the logarithm is 2 or e then the units of $C(Z)$ are in bits per second or natural units, respectively.

The information rate $C(Z)$ from a single source with large Z was calculated by Metzner and Morrison for particular world-models, including the steady-state model with continual creation.

For sufficiently large Z equation (3.20) may be written

$$C(Z) = \frac{d\nu_o}{N} \; b(Z), \qquad (3.21)$$

and we note that the information rate from a remote single source is proportional to its apparent luminosity.

Turning now to the consideration of total background radiation, we note that the rate of reception of information from all sources with Doppler ratio not less than Z is given by

$$J(Z) = \int_Z^\infty C(Z)\, n(Z)\, dZ, \qquad (3.22)$$

where $n(Z)\,dZ$ is the number of sources with Doppler ratios in the range $(Z, Z + dZ)$. For sufficiently large Z, however, we have seen that $C(Z)$ is proportional to $b(Z)$. Therefore, the total information rate from all sources with Doppler ratio greater than Z is proportional to the total

background radiation from those sources Q, (say), and from (3.22) we see that

$$J(Z) = \frac{d\nu_o}{N} Q \qquad (3\cdot23)$$

In the next chapter we shall derive the values of the total information rate from all sources with Doppler ratio greater than a given large value Z for special classes of world-model and compare our results with Metzner's and Morrison's.

References

1. Charlier, C. V. L., Arkiv für Matematic, Astronomi och Fysik, (1922), 16, No. 22.
2. Milne, E. A., Zeitscrift für Astrophysik, (1933), 6, 1.
3. Metzner, A. W. K., and Morrison P., Monthly Notices Royal Astronomical Society, (1959), 119, 657.
4. Bell, D. A., "Information Theory and its Engineering Applications", London (1956).

Chapter IV

Calculations of Total Intensity of Radiation in Particular World-Models

In this chapter the total background intensity will be calculated for a number of particular world-models. Neglecting the possible absorption effects due to the finite size of the galaxies and to intergalactic matter, and assuming, except where otherwise stated, that L, the average luminosity of the sources, does not change significantly with time, we shall consider:-

(i) Models for which the expansion-factor R(t) is proportional to t^p, where p is a constant.

(ii) Models for which the red-shift factor R(t), or S(t), is an exponential function of t.

(iii) The steady-state model with continual creation.

(iv) Models for which the expansion factor R(t) is determined by the field equations of General Relativity.

The required formula in cases (i), (ii) and (iii) are easily derived from our previous general results, but those in case (iv) require somewhat lengthier calculations.

1. Calculations for models in which $R(t) = at^p$, where a and p are constants.

From equation (3·7) the total background intensity of radiation per unit solid angle in expanding world-models with conservation of matter in which the radiation sources are not evolving with time is given by

$$Q = \frac{n_o c L}{4\pi} \int_{t'}^{t_o} \frac{R}{R_o}\, dt, \qquad (4\cdot1)$$

where L is the average luminosity of the sources and t' is the epoch of emission of the most distant sources that can be observed at the present epoch t_o.

In this particular type of world-model, t' = 0 and (4·1) becomes

$$Q = \frac{n_o c L}{4\pi} \int_0^{t_o} \left(\frac{t}{t_o}\right)^p dt$$

$$= \frac{n_o c L}{4\pi} \frac{t_o}{p+1}. \qquad (4\cdot2)$$

It is interesting to note from this formula that since n_o is proportional to t^{-3p}, it follows that Q is proportional to t_o^{1-3p}. Thus when p = 1, we find that Q is proportional to $\frac{1}{4}t_o^2$ which agrees with the result obtained by Milne for his model, see equation (3·5). More generally, we obtain the following theorem:-

Theorem VIII. In world-models for which $R(t) = t^p$ and L is independent of t, the background brightness diminishes with t if $p > \frac{1}{3}$, does not change if $p = \frac{1}{3}$ (Dirac's Model) and increases indefinitely with t if $p < \frac{1}{3}$.

If the present value of Hubble's constant in the red-shift distance law is denoted by H_o, then

$$H_o = \frac{\dot{R}_o}{R_o} = \frac{p}{t_o}. \qquad (4\cdot3)$$

Eliminating t_o from (4·2) using (4·3), it follows that

$$Q = \frac{p}{p+1} \left(\frac{n_o c L}{4\pi H_o}\right). \qquad (4\cdot4)$$

The term $\frac{n_0 c L}{4\pi H_0}$ in (4·4) may be evaluated directly from observational data. We will take the average luminosity of the light sources to be (1)

$$L = \alpha \times 10^{10} L_{\odot}, \qquad (4\cdot5)$$

(where $L_{\odot} = 3\cdot86 \times 10^{33}$ erg. sec.$^{-1}$ is the luminosity of the sun), the number-density of galaxies at the present epoch to be (2)

$$\begin{aligned} n_0 &= \beta \text{ galaxies per cubic megaparsec} \\ &= \beta \times 3\cdot42 \times 10^{-74} \text{ galaxies cm}^{-3}, \end{aligned} \qquad (4\cdot6)$$

and the present value of Hubble's constant to be (3, 4)

$$\begin{aligned} H_0 &= \gamma \times 100 \text{ km sec}^{-1} \text{ Mpc}^{-1} \\ &= \gamma \times 3\cdot24 \times 10^{-18} \text{ sec}^{-1}, \end{aligned} \qquad (4\cdot7)$$

where α, β, and γ are numerical constants, which allow for uncertainties in the observed values of L, n_0 and H_0, respectively, and are of the order of unity. Hence we find that

$$\frac{n_0 c L}{4\pi H_0} = \frac{\alpha\beta}{\gamma} \times 9\cdot73 \times 10^{-4} \text{ erg cm}^{-2} \text{ sec}^{-1} \text{ ster}^{-1} \qquad (4\cdot8)$$

The total flux of background radiation is obtained by multiplying (4·8) by the factor $p/(p+1)$. We note that this factor is $\frac{1}{2}$ when $n = \frac{1}{3}$ (Dirac Model), $\frac{1}{2}$ when $n = 1$ (Milne uniformly expanding model) and approximately unity when n is large. It is also interesting to note from (4·2) that when $p = 0$ - and hence R is a constant - the total flux of background radiation per unit solid angle in a static universe in which the sources began emitting at epoch $t = 0$ and have continued emitting at a constant rate up to the present epoch t_0 is given by

$$Q = \frac{n_0 c L t_0}{4\pi}. \qquad (4\cdot9)$$

This well-known result serves as a check on formula (4·2).

Numerical values for the total background radiation for particular models of the type $R = at^p$ are given in Table I assuming that $\propto \beta / \gamma = 1$.

Our theoretical result for the total flux of background radiation in the Milne uniformly expanding model, namely, $\frac{1}{2}\frac{n_0 c L}{4\pi H_0}$ may be derived from McVittie's and Wyatt's (5) formula for flux-density in this model, and this serves as an independent check on our results. The flux-density obtained by McVittie and Wyatt is given by equation (19) in their paper, namely,

$$\frac{dw}{4\pi} n_0 \, df_0 \int_0^{t_0} c L_i \psi\left(\frac{t_0 f_0}{t_i}\right) dt_i, \qquad (4\cdot10)$$

where the symbols used are related to ours by the relations,

$$dw = \Omega, \; n_0 = n_0, \; df_0 = d\nu_0, \; L_i = L, \; t_i = t, \qquad (4\cdot11)$$

and ψ is defined by the equation

$$\psi(\nu) = \frac{G(\nu)}{L}, \qquad (4\cdot12)$$

$G(\nu)d\nu$ being the average luminosity of the sources in the frequency range $(\nu - d\nu, \nu)$.

Integrating (4·12) over all frequencies, we must have

$$\int_0^\infty \psi(\nu)\, d\nu = 1. \qquad (4\cdot13)$$

Hence integrating (4·10) over all frequencies, we find that (in our notation) the flux of background radiation in unit solid angle is given by

$$\frac{n_0 c L}{4\pi} \int_0^{t_0} \frac{t}{t_0} dt = \frac{1}{2}\left(\frac{n_0 c L t_0}{4\pi}\right), \qquad (4\cdot14)$$

in agreement with our previous result (4·2) with $p = 1$.

If a cut-off at $\delta\lambda/\lambda = z$ is introduced, i.e. only background radiation from remote galaxies with red-shifts in excess of z is considered, the radiation from these galaxies in unit solid angle is given by equation (3·12) with $\Omega = 1$, namely

$$Q = \frac{n_0 c L}{4\pi} \int_0^{t''} \left(\frac{t}{t_0}\right)^p dt, \qquad (4·15)$$

where t" (which satisfies equation (2·24)), is the epoch of emission of radiation from galaxies with red-shift z which reaches the observer at t_0, hence

$$(t'')^p = \frac{t_0^p}{1+z}, \qquad (4·16)$$

and

$$Q = \frac{p}{p+1} \left(\frac{n_0 c L}{4\pi H_0}\right) \frac{1}{(1+z)^{\left(1+\frac{1}{p}\right)}}, \qquad (4·17)$$

we find. Numerical values for the background radiation when there is a cut-off at z = 0·2 are given in Table II.

2. Calculations for models in which R(t) is an exponential function of t.

As stated at the end of Chapter III, we shall refer to expanding world-models with conservation of matter and expansion factor $R = e^{Ht}$ as de Sitter metric models. The special case with zero curvature index which is subject to the laws of general relativity, is the de Sitter model and there is no matter in it.

It was shown in Chapter III that, apart from the lower limit of integration, the formula for the background radiation in static world-models in which the wavelength of a photon increases with time according to the law, namely

$$\frac{\lambda_0}{S_0} = \frac{\lambda}{S(t)} \qquad (4·18)$$

is the same as that for an expanding model with expansion factor R in which matter is conserved provided S = R. In particular, if the static model is non-evolutionary then S takes the specific form

$$S = e^{at}, \tag{4.19}$$

where a is a constant. Expanding S to the first order we may derive Hubble's red-shift distance law with a equal to Hubble's constant. Hence, since the lower limit of integration in the de Sitter metric model and the static non-evolutionary model is the same, the background flux per unit solid angle Q in these models is given by the same formula. Using (4.1) with $R = e^{Ht}$ and $t' = -\infty$ then

$$Q = \frac{n_0 c L}{4\pi} \int_{-\infty}^{t_0} e^{H(t - t_0)} dt$$

$$= \frac{n_0 c L}{4\pi H}. \tag{4.20}$$

We note that this result is also the same as that for the static non-evolutionary model described above in which there is no red-shift and the sources have been radiating from time t = 0 up to the present epoch t_0, where $t_0 = \frac{1}{H}$.

When there is a cut-off at $\delta\lambda/\lambda = z$, we may use equation (3.12) with $R = e^{Ht}$ and $t' = -\infty$. Then

$$Q = \frac{n_0 c L}{4\pi} \int_{-\infty}^{t''} e^{H(t - t_0)} dt, \tag{4.21}$$

where

$$e^{Ht''} = \frac{e^{Ht}}{1 + z}, \tag{4.22}$$

and therefore

$$Q = \frac{n_0 c L}{4\pi H} \cdot \frac{1}{1 + z}. \tag{4.23}$$

3. Calculations for the steady-state model with continual creation.

The formula for the background radiation in the steady-state model is given by equation (3·15), namely

$$Q = \tfrac{1}{4}\frac{n_0 c L}{4\pi H} . \qquad (4\cdot24)$$

When there is a cut-off at $\delta\lambda/\lambda = z$, the result is given by (3·16), namely

$$Q = \tfrac{1}{4}\left(\frac{n_0 c L}{4\pi H}\right)\frac{1}{(1+z)^4} \qquad (4\cdot25)$$

We note that expression (4·24) and (4·35) agree with the corresponding results for the Dirac model given by formulae (4·4) and (4·17), respectively when $p = \frac{1}{3}$ and $H = H_0$.

Numerical values for the total background radiation in the models considered above with $\alpha\beta/\gamma = 1$ are given in Table I. The background radiation in these models when there is a cut-off at red-shift $\delta\lambda/\lambda = 0\cdot2$ is given in Table II.

TABLE I

The Total Background Radiation in Particular World-Models

MODEL	R(or S)	AGE	Q	ρ	m	T
de Sitter Metric	e^{Ht}	Infinite	$9\cdot7 \times 10^{-4}$	$4\cdot5 \times 10^{-34}$	94·6	2·71
Decaying Photon	e^{at}	Infinite	$9\cdot7 \times 10^{-4}$	$4\cdot5 \times 10^{-34}$	94·6	2·71
Page	at^2	$1\cdot9 \times 10^{10}$	$6\cdot5 \times 10^{-4}$	$3\cdot0 \times 10^{-34}$	63·4	2·45
Milne	ct	$9\cdot7 \times 10^{9}$	$4\cdot9 \times 10^{-4}$	$2\cdot3 \times 10^{-34}$	47·8	2·28
Einstein-de Sitter	$at^{\frac{2}{3}}$	$6\cdot5 \times 10^{9}$	$3\cdot9 \times 10^{-4}$	$1\cdot8 \times 10^{-34}$	38·1	2·14
Dirac	$at^{\frac{1}{3}}$	$3\cdot2 \times 10^{9}$	$2\cdot4 \times 10^{-4}$	$1\cdot1 \times 10^{-34}$	23·4	1·91
Steady-state	e^{Ht}	Infinite	$2\cdot4 \times 10^{-4}$	$1\cdot1 \times 10^{-34}$	23·4	1·91

TABLE II

The Background Radiation in Particular World-Models

when there is a cut-off at $\delta\lambda/\lambda = 0{\cdot}2$

MODEL	R(or S)	AGE	Q	ρ	m	T
de Sitter Metric	e^{Ht}	Infinite	$8{\cdot}1 \times 10^{-4}$	$3{\cdot}8 \times 10^{-34}$	79·1	2·58
Decaying Photon	e^{at}	Infinite	$8{\cdot}1 \times 10^{-4}$	$3{\cdot}8 \times 10^{-34}$	79·1	2·58
Page	at^2	$1{\cdot}9 \times 10^{10}$	$4{\cdot}9 \times 10^{-4}$	$2{\cdot}3 \times 10^{-34}$	47·8	2·29
Milne	ct	$9{\cdot}7 \times 10^{9}$	$3{\cdot}4 \times 10^{-4}$	$1{\cdot}6 \times 10^{-34}$	33·2	2·09
Einstein-de Sitter	$at^{\frac{2}{3}}$	$6{\cdot}5 \times 10^{9}$	$2{\cdot}5 \times 10^{-4}$	$1{\cdot}2 \times 10^{-34}$	24·4	1·90
Dirac	$at^{\frac{1}{2}}$	$3{\cdot}2 \times 10^{9}$	$1{\cdot}2 \times 10^{-4}$	$0{\cdot}56 \times 10^{-34}$	11·7	1·58
Steady-state	e^{Ht}	Infinite	$1{\cdot}2 \times 10^{-4}$	$0{\cdot}56 \times 10^{-34}$	11·7	1·58

Notes on Tables I and II

Column Q is the flux of radiation per unit solid angle and is given in erg sec^{-1} cm^{-2} $ster^{-1}$. Column ρ is the mass density of radiation in gm cm^{-3}, and is obtained by multiplying column Q by $4\pi/c^3$. Column m denotes the number of stars of the tenth photovisual magnitude per square degree which would produce a total flux of radiation equal to the background radiation (although the spectral distribution would be different). The method of computing this measure of brightness is as follows:-

The sun has an absolute photovisual magnitude of -26·86 and an area of 0·223 square degrees. If L_* denotes the luminosity of a star of photovisual magnitude 10, then

$$L_\odot = L_* \times 10^{-0{\cdot}4(-26{\cdot}86-10)}$$

where $L_\odot$ is the luminosity of the sun.

The luminosity of the sun per square degree is

$$\frac{L_\odot}{0{\cdot}223} = \frac{L_* \times 10^{15{\cdot}16}}{0{\cdot}223}$$

$$= 10^{15} \times 6{\cdot}19 \text{ stars of magnitude 10 per square degree.}$$

The flux of radiation at the surface of the sun is

$$\frac{L_\odot}{4\pi R_\odot^2} = 10^{10} \times 6{\cdot}34 \text{ erg sec}^{-1} \text{ cm}^{-1} \text{ ster}^{-1},$$

where $R_\odot = 6{\cdot}96 \times 10^{10}$ cm is the radius of the sun.

If the background flux is given by Q, then the background radiation is a fraction $Q/6{\cdot}34 \times 10^{10}$ of the sun's. Therefore the background brightness is equivalent to $\frac{Q}{10^{10} \times 6{\cdot}34} 6{\cdot}19 \times 10^{15} = Q \times 9{\cdot}76 \times 10^{4}$ stars of tenth magnitude per square degree.

Column T in Tables I and II gives the effective blackbody temperature of the background radiation in degrees Kelvin. By definition, the flux of background radiation per unit solid angle $Q = \frac{\sigma}{\pi} T^4$, where $\sigma = 5{\cdot}67 \times 10^{-5}$ erg cm^{-2} sec^{-1} deg^{-4} is the Stefan-Boltzmann constant.

Finally, it should be noted that the age of the model is given in years, corresponding to $\alpha = 1$, whereas the columns for Q, ρ, m and T depend only on the condition $\alpha\beta/\gamma = 1$. If $\alpha\beta/\gamma = k$, Q, ρ and m must be multiplied by k, and T by $k^{\frac{1}{4}}$.

We notice from Table I that the greatest background radiation is found in the de Sitter metric model and the Decaying Photon model with $S = e^{at}$, and is least in the steady-state model with continual creation and in the

Dirac model. Further, on comparing Table I with Table II, we notice that when there is a cut-off at $\delta\lambda / \lambda = 0{\cdot}2$ the residual background radiation in the steady-state model and Dirac model is about half the total background radiation, whilst in the de Sitter metric model and Decaying Photon model the residual background radiation is more than 80 per cent of the total background radiation, the other cases mentioned being intermediate. This last result indicates that the background radiation in the steady-state and Dirac models comes mainly from the foreground galaxies, whilst in the de Sitter metric and Decaying Photon models it comes mainly from the more distant galaxies.

It is interesting to compare the effective temperature T for the total background radiation given in the last column of Table I with that derived by Eddington (6) in 1926 for the total radiation received from the Galaxy. Assuming the light from the Galaxy to be equal to 2000 stars of apparent bolometric magnitude 1·0, Eddington found that the effective temperature of this radiation is $3^{\circ}{\cdot}18$K. Using the Stefan-Boltzmann law, we find that this radiation is about five times as intense as the total background radiation in the world models considered in Table I.

In 1937 Elvey and Roach (7) found that at galactic latitudes greater than 30° the residual* background radiation (due to deviations in the mean number of stars, to extragalactic nebulae, and to starlight scattered by interstellar matter) is less than 25 stars of tenth magnitude per square

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* When the effects due to airglow and zodiacal light have been subtracted.

degree. Assuming that $\alpha\beta/\gamma$ is unity, we see from the m-column in Table I that the background radiation in the steady-state model and the Dirac model is of this order of magnitude, whilst in the de Sitter metric and Decaying Photon models it is nearly four times as large. The background radiation in these models, however, comes mainly from the infra-red region of the spectrum and this was not observed by Elvey and Roach.

Nevertheless, if we conclude that comparison of our results in Table I with their observations implies that $\alpha\beta/\gamma < 1$, we can deduce that $\alpha\beta < 1$, since γ is now believed to be approximately unity. Consequently, $\alpha < \frac{1}{\beta}$ Since, in principle β can be determined directly by counts of galaxies in our neighbourhood, this means that we have an upper limit for α. We therefore conclude that, if the universe approximates in the large to an isotropic homogeneous world-model (of expanding, steady-state or decaying photon type) with negligible intergalactic absorption of light, our interpretation of the results obtained by Elvey and Roach indicates an upper limit to the average absolute magnitude of a galaxy throughout the entire range of the universe observable with a telescope of unlimited power. For example, if the number of galaxies in our neighbourhood is 0·5 per cubic megaparsec, then $\beta = \frac{1}{2}$. Hence $\alpha < 2$, and so the average absolute luminosity of a galaxy would be less than 2×10^{10} times that of the sun. This would be the average absolute magnitude of a galaxy throughout the entire past history of the universe.

4. Calculations for models in which R(t) is determined by the field equations of General Relativity.

The expansion factor R in homogeneous isotropic models of general relativity satisfies (11) the following equation:-

$$\left(\frac{dR}{dt}\right)^2 = \tfrac{1}{3}K\rho c^2 R^2 - kc^2 + \tfrac{1}{3}\Lambda R^2, \qquad (4\cdot26)$$

where $K = 8\pi G/c^2$, G is the gravitational constant, c the velocity of light, ρ is the sum of the mass density of matter and radiation, Λ is the cosmological constant, and k = 0, + 1 or - 1 the curvature index.

The energy of matter and radiation E in volume R^3 is given by

$$E = c^2\rho R^3, \qquad (4\cdot27)$$

and therefore (4·26) may be written in the alternative form

$$\frac{dR}{dt} = \pm\left\{\tfrac{1}{3}\frac{KE}{R} - kc^2 + \tfrac{1}{3}\Lambda R^2\right\}^{\frac{1}{2}}, \qquad (4\cdot28)$$

where the plus sign is taken when the model is expanding and the minus sign when the model is contracting.

Using (4·28), we may write equation (3·7) for the total background radiation in expanding world models in the alternative form

$$Q = \frac{n_o e}{4\pi}\int_{R(t')}^{R_o} L\,\frac{R}{R_o}\,\frac{|dR|}{\left\{\frac{KE}{3R} - kc^2 + \Lambda R^2\right\}^{\frac{1}{2}}}, \qquad (4\cdot29)$$

where t' is the epoch of emission of the most distant sources that can be seen by the observer O at time t_o.

In general the integral in (4·29) may be evaluated numerically for particular values of k and Λ when the functional dependences of E and L - the average luminosity of the sources at time t - are known in terms of R.

In particular, we will consider those models for which the energy per unit proper volume of space is conserved, and therefore from equation (4·27)

$$\rho R^3 = \rho_0 R_0^3. \tag{4·30}$$

In Table III is a brief summary of the types of behaviour that arise for such models with particular values of the curvature index when the cosmological constant is positive, zero and negative (12). Models for which the expansion factor starts from 0 and increases to infinity are said to be Expanding I, whilst models for which R starts from some non-zero value and increases indefinitely are called Expanding II.

When there is a cut-off at $\frac{\delta\lambda}{\lambda} = z$, the background radiation is given by equation (3.12), and using equation (4·28) we find that

$$Q = \frac{n_0 c}{4\pi} \int_{R(t')}^{R(t'')} L \frac{R}{R_0} \frac{|dR|}{\left\{\frac{KE}{3R} - kc^2 + \Lambda R^2\right\}^{\frac{1}{2}}}, \tag{4·31}$$

where

$$E = c^2 \rho_0 R_0^3, \tag{4·32}$$

and t" is the epoch of emission of radiation from those sources with red-shift z that arrives at the observer at time t_0, and satisfies

$$\frac{R(t'')}{R_0} = \frac{\lambda}{\lambda + \delta\lambda} = \frac{1}{1+z}. \tag{4·33}$$

TABLE III

Behaviour of Relativistic World-Models

k	$\Lambda > 0$	$\Lambda = 0$	$\Lambda < 0$
1	Expanding I Expanding II Oscillating	Oscillating	Oscillating
0	Expanding I	Expanding I	Oscillating
-1	Expanding I	Expanding I	Oscillating

The two classes of relativity world-model for which (a) $\Lambda = 0$, and (b) $\Lambda \neq 0$, in which energy is conserved will now be considered in turn.

(a) Models for which $\Lambda = 0$

(i) Curvature index $k = 0$. This model is the Einstein-de Sitter model and has already been considered among the models of type $R = at^p$ where a and p are constants. It is instructive, however, to derive the total flux of background radiation in unit solid angle using equation (4·29). Putting k and Λ equal to zero we find that for non-evolutionary sources,

$$Q = \frac{n_o cL}{4\pi} \int_{R(t')}^{R_o} \frac{R}{R_o} \left\{ \frac{3R}{K\rho_o R_o^3} \right\}^{\frac{1}{2}} dR . \qquad (4·34)$$

Since sources can be seen that began emitting immediately after the singularity, $R(t') = 0$. Using (4·26), the present value of Hubble's constant H_o is given by

$$H_o = \frac{\dot{R}_o}{R_o} = (\tfrac{1}{3} K \rho_o)^{\frac{1}{2}}, \qquad (4·35)$$

and therefore

$$Q = \frac{2}{5} \left(\frac{n_o cL}{4\pi H_o} \right), \qquad (4·36)$$

in agreement with (4·4) when $p = \frac{2}{3}$.

If there is a cut-off at $\delta\lambda/\lambda = z$, the upper limit in (4·34) is replaced by $R(t'')$ where

$$\frac{R(t'')}{R_o} = \frac{1}{1+z} . \qquad (4·37)$$

Hence we find that

$$Q = \frac{2}{5} \left(\frac{n_o cL}{4\pi H_o} \right) \frac{1}{(1+z)^{5/2}} , \qquad (4·38)$$

in agreement with (4·17) when $p = \frac{2}{3}$

(ii) Curvature index $k = +1$. This model, known as the Einstein oscillating model, has a parametric solution which may be expressed as follows:

$$R = A(1 - \cos\theta), \tag{4·39}$$

$$ct = A(\theta - \sin\theta), \tag{4·40}$$

where

$$A = \frac{1}{6} K \rho_o R_o^3, \tag{4·41}$$

and θ is a parameter. Clearly this model has a period of oscillation T where

$$T = 2\pi A/c. \tag{4·42}$$

Since both R and t have been expressed parametrically, it is simplest to use equation (3·7) when deriving the background radiation. Assuming that the sources are non-evolutionary, (3·7) becomes

$$Q = \frac{n_o L A^2}{4\pi R_o} \int_{\theta'}^{\theta_o} (1 - \cos\theta)^2 \, d\theta, \tag{4·43}$$

where θ_o and θ' are given by the equations

$$ct_o = A(\theta_o - \sin\theta_o), \tag{4·44}$$

and

$$ct' = A(\theta' - \sin\theta'), \tag{4·45}$$

respectively, and t_o and t' have their usual meaning. The equation of a light path is given by

$$\int_0^r \frac{dr}{1 + r^2/4} = \int_t^{t_o} \frac{c\,dt}{R}. \tag{4·46}$$

Using (4·39) and (4·40) we find that

$$2 \tan^{-1} r/2 = \int_t^{t_o} \frac{c\,dt}{R} = \int_\theta^{\theta_o} d\theta = (\theta_o - \theta), \tag{4·47}$$

where θ_o satisfies (4·44). Without loss of generality we may take $\theta = 0$ at the beginning of the present oscillation; hence $t = 0, \theta_o \leq 2\pi$, and (4·47)

is satisfied by all θ in the range $(0,\theta_0)$. Consequently, the earliest epoch of emission of light from a source which reaches the observer at the present epoch is $t = 0$, and therefore from (4·43), the background radiation in the Einstein oscillating universe, assuming that no radiation reaches us from epochs earlier than $t = 0$, is given by

$$Q = \frac{n_0 L A^2}{4\pi R_0} \left(\frac{3\theta_0}{2} - 2\sin\theta_0 + \tfrac{1}{4}\sin 2\theta_0\right). \qquad (4\cdot48)$$

If H_0 is the present value of Hubble's constant, then from (4·26) with $k = +1$, $\Lambda = 0$

$$H_0^2 = \left(\frac{\dot{R}^2}{R^2}\right)_0 = \tfrac{1}{3}K c^2 \rho_0 - \frac{c^2}{R_0^2}, \qquad (4\cdot49)$$

and for a real solution

$$H_0^2 < \tfrac{1}{3}K c^2 \rho_0 . \qquad (4\cdot50)$$

Suppose that H_0 = 100 kilometres per second per megaparsec = $3\cdot24 \times 10^{-18}$ sec^{-1}, then since $K = 8\pi G/c^2 = 1\cdot86 \times 10^{-27}$,

$$\rho_0 > 1\cdot88 \times 10^{-29} \text{ gm cm}^{-3}. \qquad (4\cdot51)$$

We shall now calculate the background flux of radiation per unit solid angle when $H_0 = 3\cdot24 \times 10^{-18}$ sec^{-1} and $\rho_0 = 2 \times 10^{-29}$ gm cm^{-3} using equation (4·48). Substituting these values in (4·49) we find that $R_0 = 3\cdot70 \times 10^{28}$ cm, and from (4·41) $A = 3\cdot14 \times 10^{29}$. From (4·39) at $t = t_0$,

$$R_0 = A(1 - \cos\theta_0). \qquad (4\cdot52)$$

Hence, substituting the values derived for A and R_0 we find that

$$\theta_0 = 0.4765.$$

Therefore, using (4·44), the present age of the model is given by

$$t_o = \frac{A}{c}(\theta_o - \sin\theta_o) = 2{\cdot}63 \times 10^{17}\text{sec.}$$

$$= 8{\cdot}3 \times 10^9 \text{ years.} \qquad (4{\cdot}53)$$

The model expands for a period $\frac{1}{2}T = 1{\cdot}05 \times 10^{12}$yrs. using (4·42). Assuming the values for n_o and L given by (4·6) and (4·5), respectively, we find that, with $\alpha = \beta = 1$

$$Q = 3{\cdot}37 \times 10^{-4} \text{ erg cm}^{-2} \text{ sec}^{-1} \text{ ster}^{-1}, \qquad (4{\cdot}54)$$

using (4·43). We find from (4·4) that this is comparable with the total background flux in a model of type $R = at^p$ with $p \sim \frac{1}{2}$.

When there is a cut-off at $\delta\lambda/\lambda = z$, the residual background radiation is given by (4·43) with the upper limit of the integral replaced by θ'' where

$$\frac{1}{1+z} = \frac{R(t'')}{R_o} = \frac{1-\cos\theta''}{1-\cos\theta_o}, \qquad (4{\cdot}55)$$

and t'' is the epoch of emission of radiation from those sources having a red-shift z, which arrives at 0 at t_o. With z = 0·2 we find from (4·55) that $\theta'' = 0.4355$, and on replacing θ_o by θ'' in (4·43) we find that

$$Q = 2{\cdot}14 \times 10^{-4} \text{ erg sec}^{-1}\text{cm}^{-2}\text{ster}^{-1}.$$

(iii) <u>Curvature index k = -1</u>. By analogy with the last example we may express R and t in parametric form as follows:

$$R = B(\cosh\theta - 1), \qquad (4{\cdot}56)$$

$$ct = B(\sinh\theta - \theta), \qquad (4{\cdot}57)$$

where $\theta \geqslant 0$ is a parameter and

$$B = \frac{1}{6}\kappa\rho_o R_o^3 \qquad (4{\cdot}58)$$

This model is of Expanding I type and from (4·56) and (4·57) we see that near the singularity when θ is small,

$$R \propto t^{\frac{2}{3}}.$$

(since $\cos h\theta \sim 1 + \frac{1}{2}\theta^2$ and $\sin h\theta \sim \theta$) whilst when θ is large

$$R \propto t$$

(since $\cos h\theta \sim \frac{1}{2}e^{\theta}$ and $\sin h\theta \sim \frac{1}{2}e^{\theta}$).

If H_o is the present value of Hubble's constant, then from (4·26)

$$H_o^2 = \left(\frac{\dot{R}_o}{R_o}\right)^2 = \tfrac{1}{3} K \rho_o c^2 + \frac{c^2}{R_o^2}, \qquad (4·59)$$

and for a real solution $H_o^2 > \frac{1}{3} c^2 K \rho_o$ (4·60)

Taking $H_o = 100$ km sec^{-1}mpc^{-1}, then from (4·59)

$$\rho_o < 1·88 \times 10^{-29} \text{gm cm}^{-3}$$

We shall now calculate the total background flux per unit solid angle when $H_o = 100$ km sec^{-1}mpc^{-1} and $\rho_o = 10^{-31}$gm cm^{-3}. Substituting in (4·59) we find that $R_o = 9·29 \times 10^{27}$cm, and hence using (4·58) we find that $B = 2·48 \times 10^{25}$. At the present epoch t_o (4·56) becomes

$$R_o = B(\cosh \theta_o - 1) \qquad (4·61)$$

and therefore from this equation we find that $\theta_o = 6·62$. Using (4·57) we obtain

$$t_o = 9·7 \times 10^9 \text{ yrs.} \qquad (4·62)$$

The background flux of radiation for non-evolutionary sources is given by (3·7) which becomes, using (4·56) and (4·57)

$$Q = \frac{n_o L B^2}{4\pi R_o} \int_{\theta'}^{\theta_o} (\cos h\theta - 1)^2 \, d\theta, \qquad (4·63)$$

where $$ct' = B(\sinh \theta' - \theta') \qquad (4·64)$$

and t' is the epoch of emission of radiation from the most distant sources that reaches 0 at t_o.

The equation of a light path is given by

$$\int_0^r \frac{dr}{1-r^2/4} = \int_t^{t_o} \frac{cdt}{R} = \int_\theta^{\theta_o} d\theta = (\theta_o - \theta). \qquad (4\cdot65)$$

From (4·64) we see that the most distant sources that can be seen at the present epoch is given by $\theta = 0$. Therefore $\theta' = 0$ and (4·63) becomes

$$Q = \frac{n_o L B^2}{4\pi R_o} \left(\frac{3}{2}\theta_o + \frac{1}{4} \sinh 2\theta_o - 2 \sinh \theta_o \right). \qquad (4\cdot66)$$

Substituting the values found above for B, R_o and θ_o, and taking the values for L and n_o given by (4·5) and (4·6) with $\alpha = \beta = 1$ we find that

$$Q = 4\cdot86 \times 10^{-4} \text{ erg cm}^{-2}\text{sec}^{-1}\text{ster}^{-1}. \qquad (4\cdot67)$$

We note from Table I that this is numerically equal to the total flux of background radiation per unit solid angle in the Milne uniformly expanding model when the same assumptions are made about n_o, L and H_o. (This is a confirmation of the numerical calculations since when θ is large R = ct as in the Milne model).

In the case of a cut-off at $\delta\lambda/\lambda = 0\cdot2$, θ_o in (4·66) is replaced by θ'' where

$$\frac{1}{1\cdot2} = \frac{\cosh \theta'' - 1}{\cosh \theta_o - 1}. \qquad (4\cdot68)$$

The background radiation is found to be

$$Q = 3\cdot38 \times 10^{-4} \text{ erg cm}^{-2}\text{sec}^{-1}\text{ster}^{-1} \qquad (4\cdot69)$$

on taking the same values for the empirical constants.

Again we note that this is equal to the background radiation in the Milne model when there is a cut-off $\delta\lambda/\lambda = 0\cdot2$.

The background radiation has also been calculated when the density of matter takes the values 10^{-29} and 10^{-30} gm cm^{-3} respectively and the results

which are given in Table IV show that the background radiation is insensitive to the density of matter and radiation. This Table gives the background radiation in the relativistic models with zero cosmological constant considered above. The first column gives the name of the model, the second column gives the curvature index whilst the remaining columns Q, ρ, m and T have the same meaning as those in Tables I and II. Table V gives the background radiation in these models when there is a cut-off at $\delta\lambda/\lambda = 0{\cdot}2$. In these tables $\alpha\beta = 1$, $\gamma = 1$. If $\alpha\beta = k$, $\gamma = 1$ the values of Q, ρ and m are multiplied by k and T by $k^{\frac{1}{4}}$.

We note from Tables IV and V that the background radiation in these models with the assumptions made about the light sources is restricted to a narrow range of values.

TABLE IV

Background Radiation in Relativistic Models for which $\Lambda = 0$

Model	k	Q	ρ	m	T	Age 10^9 yrs.
Einstein-de Sitter	0	$3{\cdot}9 \times 10^{-4}$	$1{\cdot}8 \times 10^{-34}$	38.1	2·16	6.5
Einstein Oscillating	+1	$3{\cdot}4 \times 10^{-4}$	$1{\cdot}6 \times 10^{-34}$	33·2	2·08	8.3
Expanding I (i) $\rho_0 = 10^{-29}$	-1	$4{\cdot}2 \times 10^{-4}$	$2{\cdot}0 \times 10^{-34}$	41.0	2·20	7.3
(ii) $\rho_0 = 10^{-30}$	-1	$4{\cdot}8 \times 10^{-4}$	$2{\cdot}2 \times 10^{-34}$	46.8	2·27	9.1
(iii) $\rho_0 = 10^{-31}$	-1	$4{\cdot}9 \times 10^{-4}$	$2{\cdot}3 \times 10^{-34}$	47.8	2·28	9.7

TABLE V

Background Radiation in Relativistic models for which $\Lambda = 0$ with cut-off $\delta\lambda/\lambda = 0{\cdot}2$

Model	k	Q	ρ	m	T	Age 10^9 yrs.
Einstein-de Sitter	0	$2{\cdot}5 \times 10^{-4}$	$1{\cdot}2 \times 10^{-34}$	24·4	1·93	6.5
Einstein Oscillating	+1	$2{\cdot}1 \times 10^{-4}$	$0{\cdot}98 \times 10^{-34}$	20·5	1·85	8.3
Expanding I (i) $\rho_0 = 10^{-29}$	-1	$2{\cdot}8 \times 10^{-4}$	$1{\cdot}3 \times 10^{-34}$	27·3	1·92	7.3
(ii) $\rho_0 = 10^{-30}$	-1	$3{\cdot}35 \times 10^{-4}$	$1{\cdot}5 \times 10^{-34}$	32·2	2·07	9.1
(iii) $\rho_0 = 10^{-31}$	-1	$3{\cdot}41 \times 10^{-4}$	$1{\cdot}6 \times 10^{-34}$	33·2	2·09	9.7

(b) Relativistic Models for which $\Lambda \neq 0$

Many, but not all, theorists now believe that the cosmological constant must be zero, but even if we relax this condition we find that only one model with $\Lambda \neq 0$ has been extensively studied. This is the Lemaître model, of curvature index $k = +1$, which is of type Expanding I: it explodes from a singular state, passes through a phase in which it behaves like a static Einstein universe and then begins expanding again.* The values of the parameters Λ and R in the static phase will be denoted by Λ_e and R_e respectively. In this model the derivative of R, given by (4·26), has a double root at $R = R_e$ (9), and therefore we find that

$$\left(\frac{dR}{dt}\right)^2 = \frac{c^2}{3RR_e^2}(R - R_e)^2(R + 2R_e), \tag{4·70}$$

$$R_e = \tfrac{1}{2}\kappa\rho_o R_o^3, \tag{4·71}$$

and

$$\Lambda_e = \frac{c^2}{R_e^2}. \tag{4·72}$$

Let H_o be the present value of Hubble's constant, then using (4·70)

$$H_o^2 = \frac{\dot{R}_o^2}{R_o^2} = \frac{c^2}{3R_o^3R_e^2}(R_o - R_e)^2(R_o + 2R_e)$$

$$= \frac{c^2(q-1)^2(q+2)}{3R_o^3q}, \tag{4·73}$$

where

$$q = \frac{R_o}{R_e}, \tag{4·74}$$

and using (4·71) and (4·74), (4·67) becomes

$$H_o^2 = \tfrac{1}{6}\kappa\rho_o c^2(q-1)^2(q+2). \tag{4·75}$$

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* Eddington considered this model too, but restricted attention to the phase following the Einstein state.

If H_o and ρ_o are known empirically, then with the aid of equations (4·74), (4·71) and (4·72) the constants R_o, R_e and Λ_e may be determined as follows:- From equation (4·73) q may be found, and since (4·71) may be written in the form

$$\frac{1}{q} = \tfrac{1}{2}\kappa\rho_o R_o^2,$$

we may determine R_o. Hence R_e and Λ_e may be found from (4·74) and (4·72) respectively.

The background brightness per unit solid angle in relativistic world-models is given by (4·29) which for the Lemaître model reduces to

$$Q = \frac{n_o}{4\pi}\int_{R'}^{R_o} L\frac{R}{R_o}\left(\frac{3R}{R+2R_e}\right)^{\frac{1}{2}}\frac{R_e dR}{R-R_e}, \qquad (4\cdot76)$$

where $R' = R(t')$ and t' is the epoch of emission of radiation from the most distant sources that reaches the observer at t_o. Since the extragalactic nebulae were formed after the static phase, $R' = R_e + \Delta$ where Δ is a small positive constant which must be determined. If we assume that when the extragalactic nebulae are formed they have an average luminosity L, and if we assume further that they do not evolve, then

$$Q = \frac{n_o L}{4\pi}\int_{R_e+\Delta}^{R_o}\frac{R}{R_o}\left(\frac{3R}{R+2R_e}\right)^{\frac{1}{2}}\frac{R_e dR}{(R-R_e)} \qquad (4\cdot77)$$

which may be integrated analytically as follows:-

$$\int_{R_e+\Delta}^{R_o}\frac{R^2 dR}{(R-R_e)(R^2+2RR_e)^{\frac{1}{2}}} = \int_{R_e+\Delta}^{R_o}\frac{(R+R_e)dR}{(R^2+2RR_e)^{\frac{1}{2}}} + \int_{R_e+\Delta}^{R_o}\frac{R_e^2 dR}{(R-R_e)(R^2+2RR_e)^{\frac{1}{2}}}$$

$$= \left[(R^2+2RR_e)^{\frac{1}{2}}\right]_{R_e+\Delta}^{R_o} - \int_{\mu_e}^{\mu_o}\frac{R_e^2 d\mu}{(3\mu^2R_e^2+4\mu R_e+1)^{\frac{1}{2}}} \qquad (4\cdot78)$$

where $\mu^{-1} = R - R_e$, $\mu_o^{-1} = R_o - R_e$ and $\mu_e^{-1} = \Delta$. In order to evaluate the

integral on the right hand side of (4·72) we put $\sqrt{3}\,R_e\mu + 2 = \sqrt{3}\cosh\theta$, then

$$\int_{\mu_e}^{\mu_o} \frac{R_e^2 d\mu}{(3\mu^2 R_e^2 + 4\mu R_e + 1)^{\frac{1}{2}}} = \int_{\theta_e}^{\theta_o} \frac{R_e^2 d\theta}{\sqrt{3} R_e} = \frac{R_e}{\sqrt{3}} (\theta_o - \theta_e), \tag{4·79}$$

where $\sqrt{3}\,R_e\mu_o + 2 = \sqrt{3}\cosh\theta_o$ and $\sqrt{3}\,R_e\mu_e + 2 = \sqrt{3}\cosh\theta_e$. Hence

$$Q = \frac{n_o L \sqrt{3} R_e}{4\pi R_o} \left\{ R_o \left(1 + \frac{2R_e}{R_o}\right)^{\frac{1}{2}} - (R_e + \Delta)\left(1 + \frac{2R_e}{R_e + \Delta}\right)^{\frac{1}{2}} \right\}$$

$$+ \frac{n_o L R_e^2}{4\pi R_o} \left\{ \cosh^{-1}\left(\frac{R_e}{\Delta} + \frac{2}{\sqrt{3}}\right) - \cosh^{-1}\left(\frac{R_e}{R_o - R_e} + \frac{2}{\sqrt{3}}\right) \right\}. \tag{4·80}$$

We note from this expression that Q depends critically on the particular value chosen for Δ.

From equation (4·70) we find that

$$c\,dt = \frac{R_e dR}{R - R_e} \left(\frac{3R}{R + 2R_e}\right)^{\frac{1}{2}}, \tag{4·81}$$

and integrating (4·81) over the interval of time (t', t_o) we get

$$c(t_o - t') = \int_{R_e+\Delta}^{R_o} \frac{\sqrt{3}\,R_e R dR}{(R - R_e)(R^2 + 2RR_e)^{\frac{1}{2}}} \tag{4·82}$$

Now $(t_o - t_1) = \tau$ (say) is the age of the most remote galaxies observable in the model; Δ depends on this age. We find that

$$\int_{R_e+\Delta}^{R_o} \frac{\sqrt{3}R_e R dR}{(R-R_e)(R^2+2RR_e)^{\frac{1}{2}}} = \int_{R_e+\Delta}^{R_o} \frac{\sqrt{3}R_e dR}{(R^2+2RR_e)^{\frac{1}{2}}} + \int_{R_e+\Delta}^{R_o} \frac{\sqrt{3}R_e^2 dR}{(R-R_e)(R^2+2RR_e)^{\frac{1}{2}}} \tag{4·83}$$

From (4·79) it was shown that

$$\int_{R_e+\Delta}^{R_o} \frac{\sqrt{3}R_e^2 dR}{(R-R_e)(R^2+2RR_e)^{\frac{1}{2}}} = R_e \left\{ \cosh^{-1}\left(\frac{R_e}{\Delta} + \frac{2}{\sqrt{3}}\right) - \cosh^{-1}\left(\frac{R_e}{R_o - R_e} + \frac{2}{\sqrt{3}}\right) \right\}. \tag{4·84}$$

Further, $$\int_{R_e+\Delta}^{R_o} \frac{\sqrt{3}R_e dR}{(R^2+2RR_e)^{\frac{1}{2}}} = \int_{\phi_e}^{\phi_o} \sqrt{3}R_e d\phi = \sqrt{3}R_e(\phi_o - \phi_e), \tag{4·85}$$

where $R + R_e = R_e \cosh\phi$, $R_o + R_e = R_e \cosh\phi_o$ and $2R_e + \Delta = R_e \cosh\phi_e$.

Hence, using (4.85) and (4.84), (4.82) becomes

$$c\tau = \sqrt{3}R_e\left\{\cosh^{-1}\left(\frac{R_o+R_e}{R_e}\right) - \cosh^{-1}\left(\frac{2R_e+\Delta}{R_e}\right)\right\}$$

$$+ R_e\left\{\cosh^{-1}\left(\frac{R_e}{\Delta}+\frac{2}{\sqrt{3}}\right) - \cosh^{-1}\left(\frac{R_e}{R_o-R_e}+\frac{2}{\sqrt{3}}\right)\right\}. \qquad (4.86)$$

In the case of a cut-off at $\delta\lambda/\lambda = z$, the upper limit in (4.77) is replaced by R", where $R'' = R(t'')$ and t" is the epoch of emission of the radiation from those galaxies with red-shift z which arrives at 0 at t_o. Therefore,

$$\frac{R_o}{R''} = 1 + z, \qquad (4.87)$$

and (4.80) is replaced by

$$Q = \frac{n_o L\sqrt{3}R_e}{4\pi R_o}\left\{R''\left(1+\frac{2R_e}{R''}\right)^{\frac{1}{2}} - (R_e+\Delta)\left(1+\frac{2R_e}{R_e+\Delta}\right)^{\frac{1}{2}}\right\}$$

$$+ \frac{n_o L R_e^2}{4\pi R_o}\left\{\cosh^{-1}\left(\frac{R_e}{\Delta}+\frac{2}{\sqrt{3}}\right) - \cosh^{-1}\left(\frac{R_e}{R''-R_e}+\frac{2}{\sqrt{3}}\right)\right\} \qquad (4.88)$$

Numerical values for the background radiation in the Lemaître model have been calculated when the density ρ_o takes the values 10^{-29}, 10^{-30} and 10^{-31} gm cm^{-3} respectively, and the background radiation is given in Table VI. The background radiation when there is a cut-off at $\delta\lambda/\lambda = 0.2$ is given in Table VII. It has been assumed in these calculations that, as before, the average luminosity of the sources is $\alpha \times 10^{10} \times L_\odot$ (where $L_\odot$ is the luminosity of the sun) and the number density of galaxies is β galaxies per cubic megaparsec, where α and β are constants. In Tables VI and VII the product $\alpha\beta$ has been taken to be unity. For Hubble's constant we have assumed the value $H_o = 100$ km sec^{-1} mpc^{-1}. The constant $\Delta = R_e - R'$ has been chosen so that in all cases the age of the oldest galaxies is 10^{10} years. The red-shift of these galaxies is

given by $\delta\lambda/\lambda = \frac{R_0}{R'} - 1 = \frac{R_0}{R_e + \Delta} - 1$ and the numerical value of this red-shift is included in Table VI. It is interesting to note that when the density is 10^{-29} gm cm^{-3} the red-shift of the oldest galaxy is about 0·8. Red-shifts as large as 0.461 have been measured by Minkowski (10) from spectrograms of abnormal galaxies with strong emission lines.

TABLE VI

Total Background Radiation in the Lemaître Model

ρ_0	10^{-29}	10^{-30}	10^{-31}
R_0	0.74×10^{28}	1.73×10^{28}	3.83×10^{28}
R_e	3.75×10^{27}	4.82×10^{27}	5.22×10^{27}
Δ	0.083	0.481	1.698
$\delta\lambda/\lambda$	0.822	1.425	1.722
Q	7.09×10^{-4}	6.61×10^{-4}	6.38×10^{-4}
ρ	3.30×10^{-34}	3.07×10^{-34}	2.97×10^{-34}
m	69.2	64.5	62.3
T	2.50	2.46	2.44

TABLE VII

Background Radiation in the Lemaître Model when there is a cut-off at $\delta\lambda/\lambda = 0.2$

ρ_0	10^{-29}	10^{-30}	10^{-31}
Q	5.40×10^{-4}	4.97×10^{-4}	4.76×10^{-4}
ρ	2.51×10^{-34}	2.31×10^{-34}	2.22×10^{-34}
m	52.7	48.5	46.5
T	2.34	2.29	2.27

Notes on Tables VI and VII

The density of matter and radiation ρ_o is given in gm cm^{-3}. R_o and R_e are given in cm. Q is the flux per unit solid angle in c.g.s. units. ρ is the mass density of background radiation in gm cm^{-3}. The number of tenth magnitude stars per square degree equal to the background radiation is m. T is the effective temperature of the background radiation. In these tables $\alpha\beta = 1$, $\gamma = 1$. If $\alpha\beta = k$, $\gamma = 1$ the values of Q, ρ and m are multiplied by k and T by $k^{\frac{1}{4}}$

We note from Table VI and VII that the background flux is comparatively insensitive to the density ρ_o over the density range considered, but we find that it is very sensitive to the age of the oldest galaxies. For example, if we double our previous value for the age of the oldest galaxies taking $\tau = 2 \times 10^{10}$ yrs; then we find that for the case $\rho_o = 10^{-30}$ gm cm^{-3} the background flux has increased by 50 per cent to the value

$$Q = 10.0 \times 10^{-4} \text{erg cm}^{-2}\text{sec}^{-1}\text{ster}^{-1}.$$

($\rho = 4.7 \times 10^{-34}$ gm cm^{-3}, m = 97.6 stars of tenth magnitude per square degree T = 2.73°K). On the other hand, if $\tau = 10 \times 10^{10}$ yrs., we find that when $\rho_o = 10^{-30}$ gm cm^{-3}

$$Q = 60.1 \times 10^{-4} \text{erg cm}^{-2}\text{sec}^{-1}\text{ster}^{-1}.$$

($\rho = 2.8 \times 10^{-33}$ gm cm^{-3}, m = 587 stars of tenth magnitude per square degree T = 4.27°K) which is three times as great as Eddington's value for the light from the Galaxy.

Comparing the background radiation in the Lemaître model - when the oldest galaxies are 10^{10} yrs. old and the density lies in the range 10^{-29} to 10^{-31} gm cm^{-3} - with the models considered in Table I we find that the total background radiation in the Lemaître model lies very close to that of the Page world-model.

In general, when $\Lambda \neq 0$, the background radiation given by (4·29) does not yield a simple analytical solution; instead elliptical functions must be used. In such cases it would be simplest to integrate (4·29) numerically when the constants R_o and Λ have been determined. In order to determine R_o and Λ it is not enough (in general) to know H_o and ρ_o. If the acceleration parameter defined by $q = \frac{-\ddot{R}_o}{R_o H_o^2}$ is known, however, we may use in addition to (4·26) a second fundamental equation given by (8)

$$\frac{\dot{R}^2}{R^2} + \frac{2\ddot{R}}{R} + \kappa p = \frac{-kc^2}{R^2} + \Lambda , \qquad (4·89)$$

where p is the pressure of matter and radiation and may be taken to be zero (11). The constants Λ and R_o may then be determined by inserting the present values of H_o, ρ_o and q_o in (4·26) and (4·89) and solving these equations simultaneously.

5. The Flow of Information in Particular World-Models.

It was shown in Chapter III (see equation 3·23) that $J(Z)$ the information rate from all sources with Doppler ratio greater than Z, (when Z is sufficiently large), is given by

$$J(Z) = \frac{d\nu_o Q}{N} , \qquad (4·90)$$

where $d\nu_o$ is the width of the receiving frequency band, Q the total background radiation from all sources with a Doppler ratio larger than the cut-off value Z and N is the noise power.

We shall now derive $J(Z)$ for the particular models considered in the first section of this Chapter. In these models the total background radiation Q when there is a cut-off at Doppler shift $\delta\lambda/\lambda = z$ has been derived. If galaxies with Doppler shift z have Doppler ratio Z, so that

$$Z = \frac{\lambda + \delta\lambda}{\lambda} = 1 + z, \tag{4·91}$$

we find that:-

(i) Using equation (4·17) and substituting in (4·90), the total information rate in world-models for which the expansion factor $R = at^p$ (a and p constants) is given by

$$J(Z) = \frac{d\nu_o}{N} \frac{p}{p+1} \left(\frac{n_o c L}{4\pi H_o}\right) \frac{1}{Z^{1+\frac{1}{p}}}. \tag{4·92}$$

(ii) The total information rate in world-models for which the red-shift factor R, or S, is equal to e^{Ht} where H is a constant is given by (using (4·23) in (4·90)),

$$J(Z) = \frac{d\nu_o}{N} \left(\frac{n_o c L}{4\pi H}\right) \frac{1}{Z}. \tag{4·93}$$

(iii) The total information rate in the steady state model with continual creation (using (4·25) in (4·90)) is given by

$$J(Z) = \frac{d\nu_o}{N} \frac{1}{4} \left(\frac{n_o c L}{4\pi H}\right) \frac{1}{Z^4}. \tag{4·94}$$

Metzner and Morrison have calculated the Z-dependence for large Z of the total information rate in the Dirac, Einstein-de Sitter, Milne, Page, de Sitter and steady-state world-models. On comparing our results (i), (ii), and (iii) with theirs, we notice that Metzner and Morrison obtained the value Z^{-6} instead of Z^{-4} in the steady-state model, and their

results for the remaining models is a factor Z^{-1} smaller than ours. The difference of a factor Z between Metzner's and Morrison's results and ours is due to the fact that they have considered the frequency band of emission to be constant, whilst we have taken the receiving frequency band to be constant. The reason for the additional discrepancy of a factor Z^{-1} between their result and ours for the steady-state model is that an error appears to have been made by Metzner and Morrison when deriving n(Z). The number of sources in the steady-state model with continual creation is constant in unit proper volume. Metzner and Morrison calculated proper volume in terms of proper distance $l_s = rR$, where R is evaluated at the epoch of emission, but the formula for proper volume $4\pi l_s^2\, dl_s$ - which they used - should be replaced by $4\pi R^3 r^2 dr$.

In fact $4\pi l_s^2\, dl_s = 4\pi R^2 r^2 d(Rr)$, and

$$d(Rr) = Rdr + rdR.$$

Along the incoming radiation path

$$\frac{cdt}{R} = -dr.$$

Also in the steady-state model $\dot{R} = HR$, and hence $dR = HRdt$. Further

$$r = \int_t^{t_o} \frac{cdt}{R} = \int_R^{R_o} \frac{cdR}{HR^2} = \frac{c}{H}\left(\frac{1}{R} - \frac{1}{R_o}\right).$$

Consequently $d(Rr) = \frac{R^2}{R_o^2} dr$, and

$$4\pi l_s^2 dl_s = 4\pi \frac{R^4}{R_o^4} r^2 dr$$

which differs by a factor $Z = \frac{R}{R_o}$ from the correct formula $4\pi R^3 r^2 dr$.

Having cleared up this discrepancy, there is now complete agreement between our results and those of Metzner and Morrison.

References

1. de Vaucouleurs, G., (1959), Handbuch der Physik, Volume 53, p.360 (Berlin: Julius Springer).
2. Allen, C. W., (1955), "Astrophysical Quantities", (London: Athlone Press).
3. Sandage, A., (1958), Astrophysical Journal, 127, 513.
4. de Vaucouleurs, G., (1958), Nature, 182, 1478.
5. McVittie, G. C., Wyatt, S. P., (1959), Astrophysical Journal, 130, 1.
6. Eddington, A. S., (1926), "The Internal Constitution of the Stars", Cambridge University Press.
7. Elvey, C. T., Roach, F. E., (1937), Astrophysical Journal, 85, 213.
8. McVittie, G. C., (1956), "General Relativity and Cosmology", (London: Chapman and Hall).
9. Robertson, H. P., (1933), Reviews of Modern Physics, 5, 62.
10. Minkowski, R., (1960), Astrophysical Journal, 132, 908.
11. Sandage, A., (1961), Astrophysical Journal, 133, 355.

Chapter V

Calculations of the Spectral Distribution of Background Radiation in Particular World-Models

1. Introduction. In this Chapter we shall consider first the visual background spectrum due to extragalactic nebulae in particular homogeneous isotropic world-models, when the effects of intergalactic absorption are neglected. Secondly, we shall consider briefly the background spectrum in the radio region. In the visual region of the spectrum we shall consider in our numerical calculations the same world-models for which the total background radiation was derived in the last Chapter.

2. Models for which matter is conserved. The general formula for the spectrum of background radiation in expanding world-models for which matter is conserved, neglecting intergalactic absorption, was derived in Chapter II, and is given by

$$I(\lambda_0, t_0)d\lambda_0 = \frac{n_0 c}{4\pi} d\lambda_0 \Omega \int_{t_1}^{t_0} F\left(\frac{R\lambda_0}{R_0}, t\right) \frac{R^2}{R_0^2} dt, \qquad (5\cdot1)$$

where $F(\lambda,t)$ is the intensity of radiation at wavelength λ in unit wave-band width of an average galaxy at time t, t_1 is the earliest epoch of emission of light from a galaxy which can be observed at the present epoch, and $I(\lambda_0,t_0)$ is the intensity of background radiation at wavelength λ_0 in unit waveband-width at the present epoch t_0 in solid angle Ω. Clearly, from equation (5·1), in order to determine $I(\lambda_0,t_0)$ we must know $F(\lambda,t)$. In principle, F may be determined at the present epoch from observations on nearby galaxies. At previous epochs it must be determined theoretically

using current theories of stellar evolution. For the moment we shall assume, unless otherwise stated, that on the average the galaxies are not evolving, so that F is independent of t. We shall also assume that an average galaxy in our neighbourhood has the same spectral distribution of radiation as a black-body at effective temperature 6000°K (1) and a total luminosity of 10^{10} Suns. Hence, we shall take

$$F(\lambda,t) = F(\lambda) = 10^{10} L_{\odot} \left(\frac{c_1}{\pi T}\right)^4 \frac{15}{\lambda^5 (e^{c_1/\lambda T} - 1)}, \qquad (5\cdot2)$$

where $L_{\odot} = 3\cdot86 \times 10^{33}$ erg sec^{-1} is the luminosity of the Sun, $T = 6000^{\circ}$K and $c_1 = 1\cdot439$ is a radiation constant.

The numerical calculations are simplified if we write $F(\lambda) = f(\log_e \lambda)$, so that the flux of radiation received per unit wavelength per unit solid angle may be written, using (5·1), in the form

$$I(\lambda_o, t_o) = \frac{n_o c}{4\pi} \int_{t_1}^{t_o} f\left(\log_e \lambda_o + \log_e \frac{R}{R_o}\right) \frac{R^2}{R_o^2} dt. \qquad (5\cdot3)$$

If we are considering a cut-off at $\delta\lambda/\lambda = z$, then the upper limit t_o in (5·3) is replaced by t', where

$$\frac{R_o}{R(t')} = 1 + z. \qquad (5\cdot4)$$

3. Models for which $R = at^p$ where a and p are constants. For this particular class of world-model, $t_1 = 0$. If we put $e^{-x} = \frac{R}{R_o} = \frac{t^p}{t_o^p}$, we find that (5·3) becomes

$$I(\lambda_o, t_o) = \frac{n_o c}{4\pi p} \int_0^{\infty} f(\log_e \lambda_o - x) e^{-(2 + \frac{1}{p})x} dx. \qquad (5\cdot5)$$

In the case of a cut-off at $\delta\lambda/\lambda = z$, we have

$$I(\lambda_o, t_o) = \frac{n_o c}{4\pi p} \int_{x'}^{\infty} f(\log_e \lambda_o - x) e^{-(2 + \frac{1}{p})x} dx, \qquad (5\cdot6)$$

where

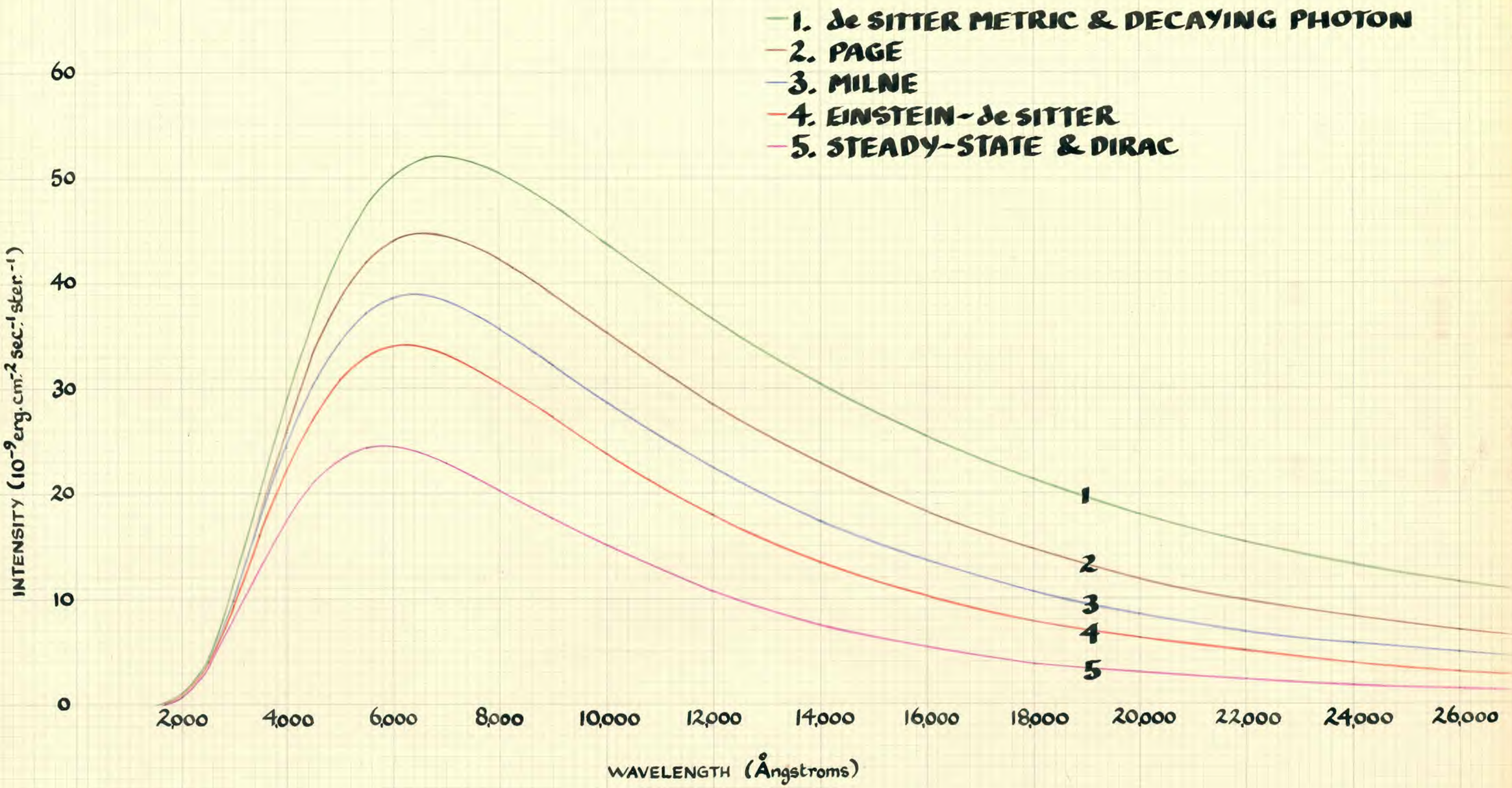
$$x' = \log(1 + z). \qquad (5\cdot7)$$

The background spectra have been calculated, using equations (5·5) and (5·6) for the particular world-models of Dirac ($p = \frac{1}{3}$), Einstein-de Sitter ($p = \frac{2}{3}$), Milne ($p = 1$) and Page ($p = 2$), with the above assumption that the radiation sources behave like black-body radiators with effective temperature 6000°K and total luminosity 10^{10} Suns. For the purpose of these calculations it was assumed further that the number-density of galaxies is 1 mpc^{-3} and Hubble's constant is 100 km sec^{-1} mpc^{-1}. The results are shown in graph A over the range of wavelengths 2,000 Å to 26,000 Å, extending from the ultraviolet to the infrared region of the spectrum. In graph B is shown the spectra when there is a cut-off at $\delta\lambda/\lambda = 0{\cdot}2$. If, instead of the above values, we took the total luminosity of the average source to be $\alpha \times 10^{10} L_{\odot}$, the number-density of galaxies to be β mpc^{-3} and Hubble's constant to be $\gamma \times 100$ km sec^{-1} mpc^{-1} then the flux-density $I(\lambda_0, t_0)$ is merely multiplied by the constant factor $\alpha\beta/\gamma$.

Table VIII gives the wavelength λ_{max} at which the maximum intensity I_{max} occurs for the models considered in graph A, and I_{max} the intensity at λ_{max}, and also $\lambda_{\frac{1}{2}}$, the wavelength dividing the spectrum into two parts, in each of which the total radiation is the same. Table IX gives the same information as Table VIII when there is a cut-off at $\delta\lambda/\lambda = 0{\cdot}2$.

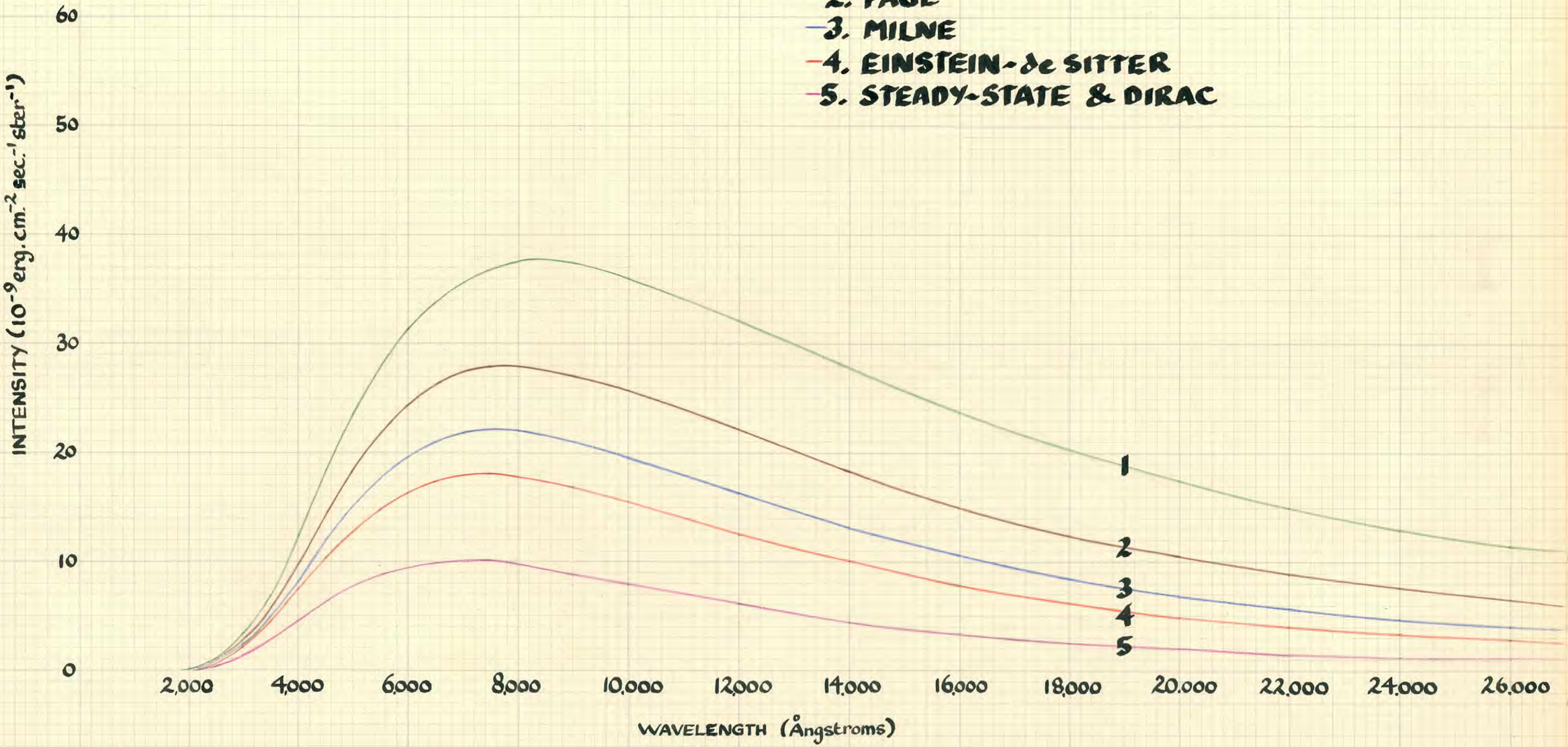
Finally, the complete spectrum in the Milne uniformly expanding model has been given in graph C, together with the spectrum of a black-body at 6000°K with the same total radiation for comparison. We note that the background spectrum in the Milne model derived by us agrees quantitatively with that derived by Leontovski (1) and also with that derived by McVittie and Wyatt (2) who assumed that the spectrum of the average source is a black-body at 5000°K.

GRAPH 'A' BACKGROUND SPECTRUM IN PARTICULAR WORLD-MODELS



GRAPH 'B' BACKGROUND SPECTRUM IN PARTICULAR WORLD-MODELS WITH CUT-OFF $\delta\lambda/\lambda = 0{\cdot}2$

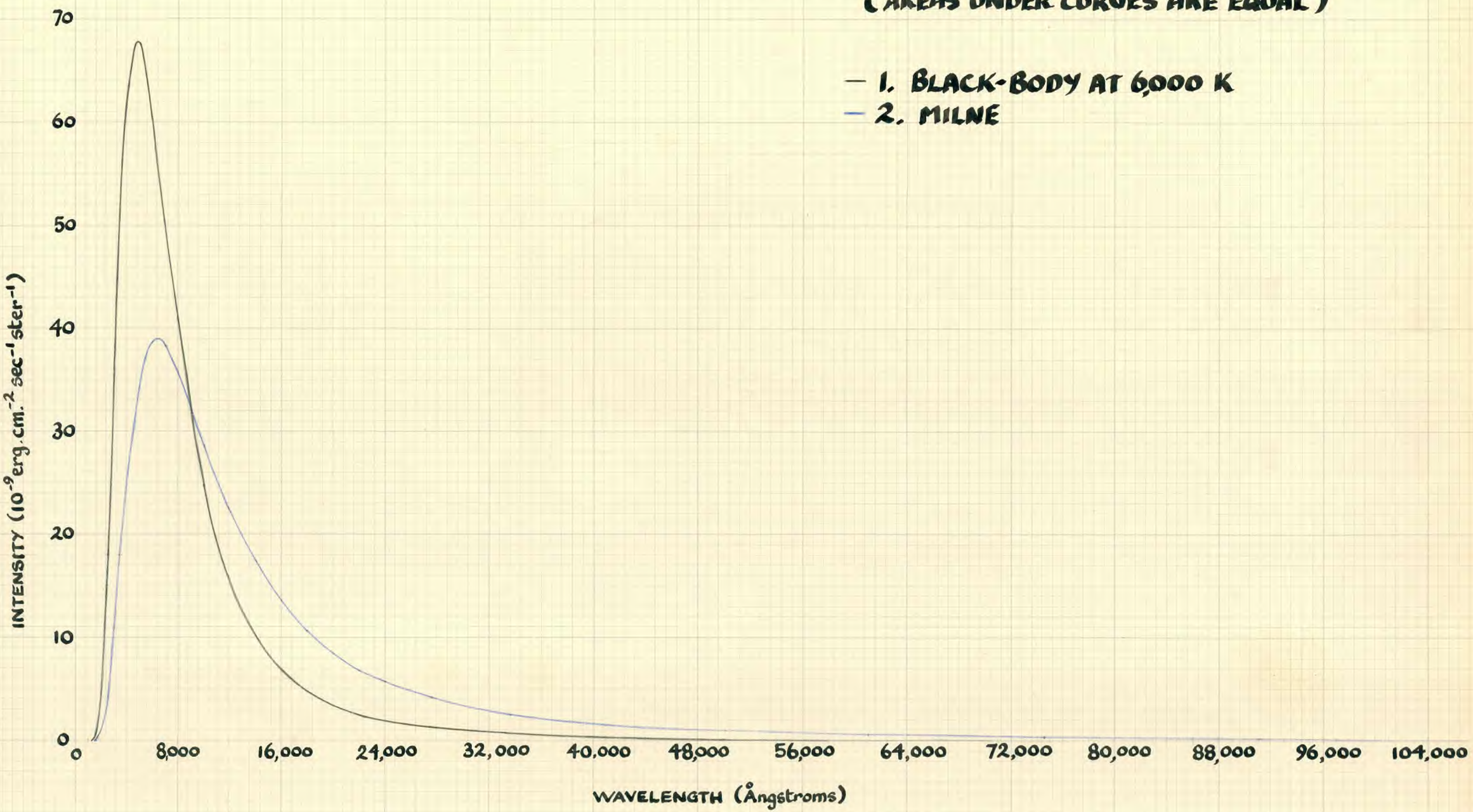
1. de SITTER METRIC & DECAYING PHOTON
2. PAGE
3. MILNE
4. EINSTEIN-de SITTER
5. STEADY-STATE & DIRAC



GRAPH 'C'

THE TOTAL BACKGROUND SPECTRUM IN THE MILNE UNIFORMLY EXPANDING WORLD-MODEL COMPARED WITH THE SPECTRUM OF A BLACK-BODY AT 6,000°K

(AREAS UNDER CURVES ARE EQUAL)



4. Models for which the red-shift factor is an exponential function of t. We shall consider first the de Sitter metric models, where $R = e^{Ht}$ and H is Hubble's constant. In this case $t_1 = -\infty$, and if we make the substitution $x = H(t_0 - t)$, then we find that (5.3) becomes

$$I(\lambda_0, t_0) = \frac{n_0 c}{4\pi} \int_0^\infty f(\log_e \lambda_0 - x) e^{-2x} dx. \qquad (5.8)$$

When there is a cut-off at $\delta\lambda/\lambda = z$ we replace the lower limit in (5.8) by $x' = \log(1 + z)$.

In the static decaying-photon models for which $S = e^{at}$ where S satisfies (2.37) we find from (2.46) that the intensity of background radiation in unit waveband width per unit solid angle is given by

$$I(\lambda_0, t_0) = \frac{n_0 c}{4\pi} \int_{t_1}^{t_0} f(\log_e \frac{S\lambda_0}{S_0}) \frac{S^2}{S_0^2} dt. \qquad (5.9)$$

Hence, if we put $x = a(t_0 - t)$, then since $t_1 = -\infty$, (5.9) becomes

$$I(\lambda_0, t_0) = \frac{n_0 c}{4\pi} \int_0^\infty f(\log_e \lambda_0 - x) e^{-2x} dx, \qquad (5.10)$$

which is the same result as that obtained for the de Sitter metric models. When there is a cut-off at $\delta\lambda/\lambda = z$ the lower limit in (5.10) is replaced by x' where

$$x' = \log(1 + z). \qquad (5.11)$$

The numerical results for the background spectra in the de Sitter metric models and the decaying-photon models have been included in graphs A and B and in Tables VIII and IX with the same assumptions about the average light sources made above.

5. The steady-state model with continual creation. The background spectrum in the steady-state model is given by (2.33) and the average spectral intensity of the sources is independent of the epoch of emission.

If we put $x = H(t_0 - t)$, then in unit solid angle the background flux-density is given by

$$I(\lambda_0) = \frac{n_0 c}{4\pi} \int_0^\infty f(\log_e \lambda_0 - x) e^{-5x} dx \qquad (5.12)$$

When there is a cut-off at $\delta\lambda/\lambda = z$, the lower limit in (5.12) is replaced by $\log(1 + z)$.

We note that this formula is identical with that obtained for a model of the type $R = at^p$ when $p = \frac{1}{3}$, i.e. the Dirac model. The background spectrum in the steady-state model, with the same assumptions as before concerning the radiation sources, is included in graphs A and B and in Tables VIII and IX.

TABLE VIII

Background Spectrum in Particular World-Models

Model	λ_{max}	I_{max}		$\lambda_{\frac{1}{2}}$
		F	=	
de Sitter metric	6,800	52×10^{-9}	49	15,000
Decaying Photon	6,800	52×10^{-9}	49	15,000
Page	6,500	43×10^{-9}	38	11,800
Milne	6,400	38×10^{-9}	33	10,500
Einstein-de Sitter	6,300	33×10^{-9}	28	9,800
Dirac	5,800	24×10^{-9}	19	8,800
Steady-state	5,800	24×10^{-9}	19	8,800

TABLE IX

Background Spectrum in Particular World-Models when there is a cut-off at $\delta\lambda/\lambda = 0.2$

Model	λ_{max}	I_{max}		$\lambda_{\frac{1}{2}}$
		F	m	
de Sitter metric	8,300	38×10^{-9}	49	17,200
Decaying Photon	8,300	38×10^{-9}	49	17,200
Page	7,800	28×10^{-9}	32	14,300
Milne	7,600	21×10^{-9}	22	12,700
Einstein-de Sitter	7,500	18×10^{-9}	19	11,700
Dirac	7,400	10×10^{-9}	11	10,400
Steady-state	7,400	10×10^{-9}	11	10,400

Notes on Tables VIII & IX

The wavelength λ_{max} at which maximum intensity occurs is given in Angstroms. The maximum intensity I_{max} expressed in erg cm^{-2} sec^{-1} $ster^{-1}$ Å^{-1} is given by column F. Column m gives the number of stars of tenth magnitude per square degree which are equivalent to I_{max} at λ_{max}, assuming that the stars are black-body radiators at 6000°K. The wavelength $\lambda_{\frac{1}{2}}$ (given in Angstroms) divides the spectrum in two equal parts.

From Tables VIII and IX and from Graphs A and B, we see that for the models considered the wavelength at which maximum intensity* occurs is confined to the narrow range 5,800 to 6,800 Å when there is no cut-off.

* For a black-body at 6000°K, λ_{max} = 4,800 Å, whilst $\lambda_{\frac{1}{2}}$ = 6,800 Å.

When there is a cut-off of amount $\delta\lambda/\lambda = 0.2$, λ_{max} lies in the narrow range 7,400 to 8,300 Å. On the other hand, $\lambda_{\frac{1}{2}}$ (which divides the spectrum into two equal parts) has a wide range of values, so that for example (if we consider the case without cut-off) in the steady-state model $\lambda_{\frac{1}{2}}$ is in the infrared at 8,800 Å, whereas at the other extreme, in the de Sitter metric model, $\lambda_{\frac{1}{2}}$ is in the far infrared at 15,000 Å. This indicates that a greater proportion of background radiation comes from extremely distant sources in the de Sitter metric model than in the steady-state model. We note also from Tables VIII and IX that the maximum intensity I_{max} is least in the steady-state and Dirac models and greatest in the de Sitter metric and decaying-photon models. When there is no cut-off, I_{max} in the de Sitter metric model is twice as large as I_{max} in the steady-state model, and three times as large when there is a cut-off at $\delta\lambda/\lambda = 0.2$.

The reason why I_{max} is greater in the de Sitter metric model than in the steady-state model is that in the past the de Sitter metric model has had a higher number-density, whilst in the steady-state model the number-density has remained constant. Therefore the luminosity per unit volume in the de Sitter model was greater in the past than at present, but has always remained the same in the steady-state model. Consequently the contribution to the background spectrum from sources with a given red-shift $\delta\lambda/\lambda = z$ in the de Sitter model is greater than from the corresponding sources in the steady-state model. When there is a cut-off this effect is more pronounced because the contribution from the nearby galaxies is removed, and their number-density is approximately the same in both types of model. More generally, we see from this that among the

expanding models the more highly accelerated will have the higher number-densities in the past than now and hence will give higher values of I_{max}. This result is demonstrated in Tables VIII and IX.

When actual measurements are made, some cut-off must be imposed because the light from nearby galaxies will be markedly anisotropic. Consequently, we see from Graph B and the above discussion that measurements will have to be made at several wavelengths, particularly at some very long wavelengths, in excess of 10,000 Å (i.e. 1μ), in order to obtain significant criteria for distinguishing between different world-models.

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6. Models for which the expansion factor R is determined by the field equations of General Relativity. We shall confine attention to the relativistic models considered in Chapter IV for which the total background radiation was derived. It will, however, not be necessary to consider the Einstein-de Sitter model ($\Lambda = 0, k = 0$) since it has already been included among the models of type $R = at^p$ when $p = \frac{2}{3}$.

(i) Model for which $\Lambda = 0$, $k = +1$

The background spectrum in this model is given by equation (5.3) with $t_1 = 0$. Using the parametric equations (4.39) and (4.40) and putting $e^{-x} = \frac{R}{R_0}$, we find that

$$e^{-x} = \frac{A}{R_0}(1 - \cos\theta), \tag{5.13}$$

and therefore

$$\theta = \cos^{-1}\left(1 - \frac{R_0 e^{-x}}{A}\right). \tag{5.14}$$

Hence, we find that (5.3) may be written

$$I(\lambda_o, t_o) = \frac{n_o R_o}{4\pi} \int_0^\infty f(\log_e \lambda_o - x) \frac{e^{-3x} dx}{(\frac{2A}{R_o} e^x - 1)^{\frac{1}{2}}}, \qquad (5.15)$$

since, using the parametric relations,

$$\frac{R^2}{R_o^2} dt = \frac{A^3}{cR_o^2}(1 - \cos\theta)^3 d\theta$$

$$= \frac{R_o}{c} \frac{e^{-3x} dx}{(\frac{2A}{R_o} e^x - 1)^{\frac{1}{2}}},$$

on using (5.13) and (5.14).

The background spectrum when there is a cut-off at $\delta\lambda/\lambda = z$ is obtained by replacing the lower limit in (5.15) by log (1 + z). The background spectrum has been calculated for this model, assuming as above that n_o = 1 galaxy mpc^{-3}, H = 100 km sec^{-1} mpc^{-1}, and L = $10^{10} L_\odot$. Further, we have assumed that the density $\rho_o = 2 \times 10^{-29}$ gm cm^{-3} as in Chapter IV, and we find that $2A/R_o = 17.0$. The total background spectrum is given in Graph D, and when there is a cut-off of amount $\delta\lambda/\lambda = 0.2$ the spectrum is given in Graph E. Clearly, from (5.15) if A/R_o is large enough, the exponential term behaves like $e^{-3\frac{1}{2}x}$, as in the spectrum of the Einstein-de Sitter model given by (5.5) when $p = \frac{2}{3}$, the same assumptions being made as before concerning the light sources.

(ii) Model for which $\Lambda = 0$, $k = -1$

In this case, using (5.3) and the parametric equations (4.56) and (4.57) we find, on putting $e^{-x} = R/R_o$, that

$$I(\lambda_o, t_o) = \frac{n_o R_o}{4\pi} \int_0^\infty f(\log_e \lambda_o - x) \frac{e^{-3x} dx}{(\frac{2A}{R_o} e^{-x} + 1)^{\frac{1}{2}}}. \qquad (5.16)$$

If we consider a cut-off at $\delta\lambda/\lambda$, the lower limit in (5.16) is replaced by log (1 + z).

The background spectrum has been calculated numerically using (5.16), making the same assumptions about n_0, H and L as in case (i). For the density ρ_0 we have taken the two values 10^{-31}, and 10^{-29} gm cm^{-3} respectively. When $\rho_0 = 10^{-31}$ gm cm^{-3} we find that $2B/R_0 = 5.3 \times 10^{-3}$. Since B/R_0 is small, we find that the integral (5.16) behaves like

$$\int_0^\infty f(\log\lambda_0 - x)e^{-3x}dx,$$

and therefore the spectrum of background radiation in this model is proportional to that in the Milne model, which is given by (5.5) with $p = 1$. In Chapter IV we found that, if we take the density ρ_0 to be 10^{-31} gm cm^{-3}, the total radiation is equal to that in the Milne model (with the same values for n_0, H and L) and therefore the spectra in the two models are identical.

In graphs D and E are given the spectra of the relativistic models for which $\Lambda = 0$, $k = 0, +1, -1$ together with the de Sitter model and steady-state model for comparison (the same values for n_0, H and L taken in all cases). We note that the variation between the models with $\Lambda = 0$ is not so pronounced as that between models of type $R = at^p$.

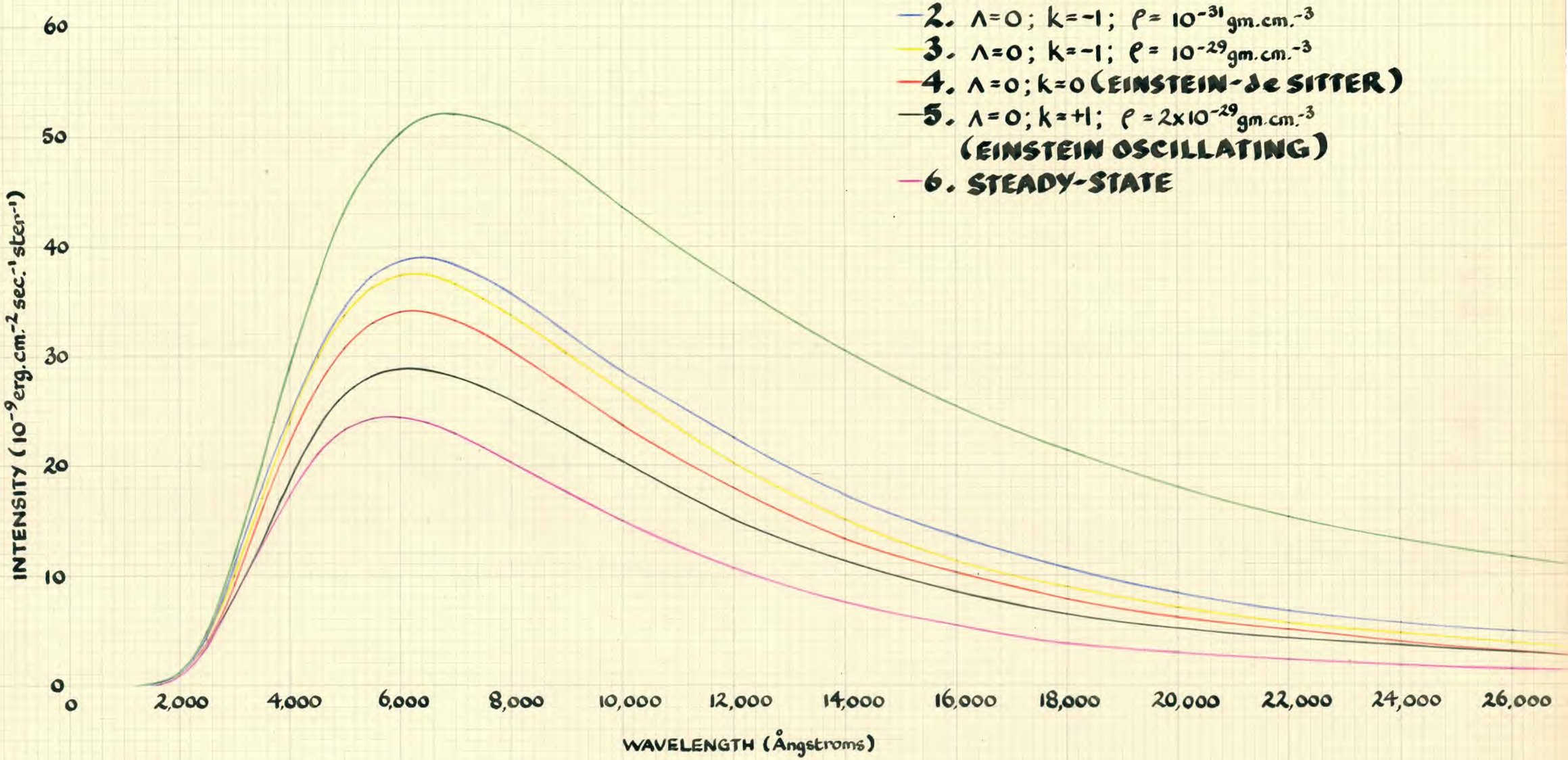
(iii) The Lemaître Model.

Since matter is conserved in this model, the background spectrum is given by (5.3). In order to evaluate (5.3) we may use the fundamental equation derived from the field equations of general relativity and given in this case by equation (4.28), namely

$$\left(\frac{dR}{dt}\right)^2 = \frac{c^2(R - Re)^2}{3RRe^2}(R + 2Re), \qquad (5.17)$$

where R_e is the value of R at the static Einstein stage. Eliminating t from (5.3) using (5.17) we obtain

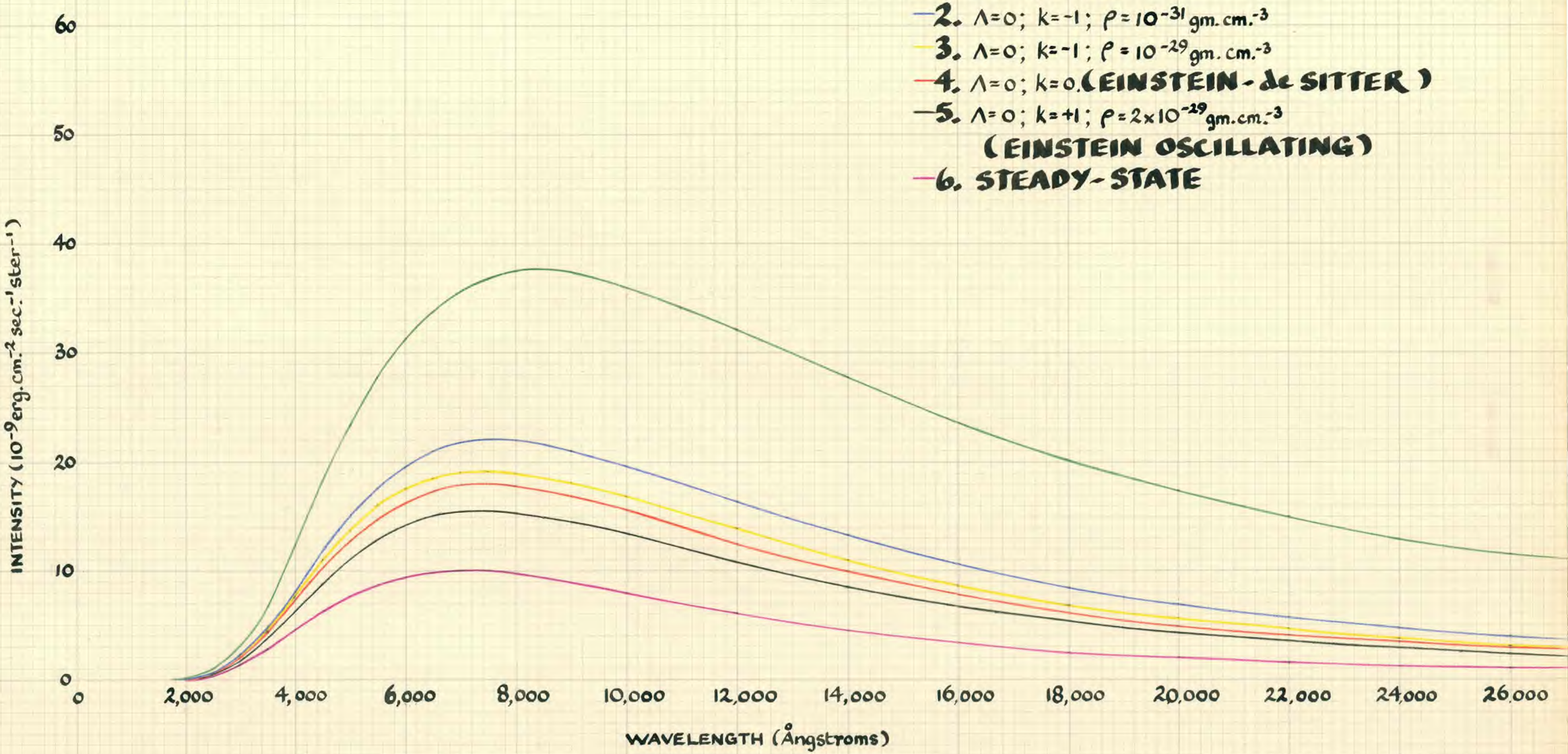
GRAPH 'D' BACKGROUND RADIATION IN RELATIVISTIC WORLD-MODELS WITH $\Lambda=0$



GRAPH 'E'

BACKGROUND RADIATION IN RELATIVISTIC WORLD-MODELS WITH $\Lambda=0$ & CUT-OFF $\delta\lambda/\lambda=0.2$

1. de SITTER
2. $\Lambda=0$; $k=-1$; $\rho=10^{-31}$ gm. cm.$^{-3}$
3. $\Lambda=0$; $k=-1$; $\rho=10^{-29}$ gm. cm.$^{-3}$
4. $\Lambda=0$; $k=0$. (EINSTEIN - de SITTER)
5. $\Lambda=0$; $k=+1$; $\rho=2\times10^{-29}$ gm. cm.$^{-3}$ (EINSTEIN OSCILLATING)
6. STEADY-STATE



$$I(\lambda_o, t_o) = \frac{n_o}{4\pi} \int_{R'}^{R_o} f(\log_e \lambda_o + \log \frac{R}{R_o}) \frac{R^2}{R_o^2} \left(\frac{3R}{R + 2R_e}\right)^{\frac{1}{2}} \frac{R_e dR}{(R - R_e)}, \quad (5.18)$$

where $R' = R(t')$ and t' is the earliest epoch of emission of radiation arriving at 0 at t_o. Putting $e^{-x} = R/R_o$ then

$$I(\lambda_o, t_o) = \frac{n_o R_e \sqrt{3}}{4\pi} \int_0^{x'} f(\log_e \lambda_o - x) \frac{e^{-3\frac{1}{2}x} dx}{(e^{-x} + 2R_e/R_o)^{\frac{1}{2}}(e^{-x} - R_e/R_o)}. \quad (5.19)$$

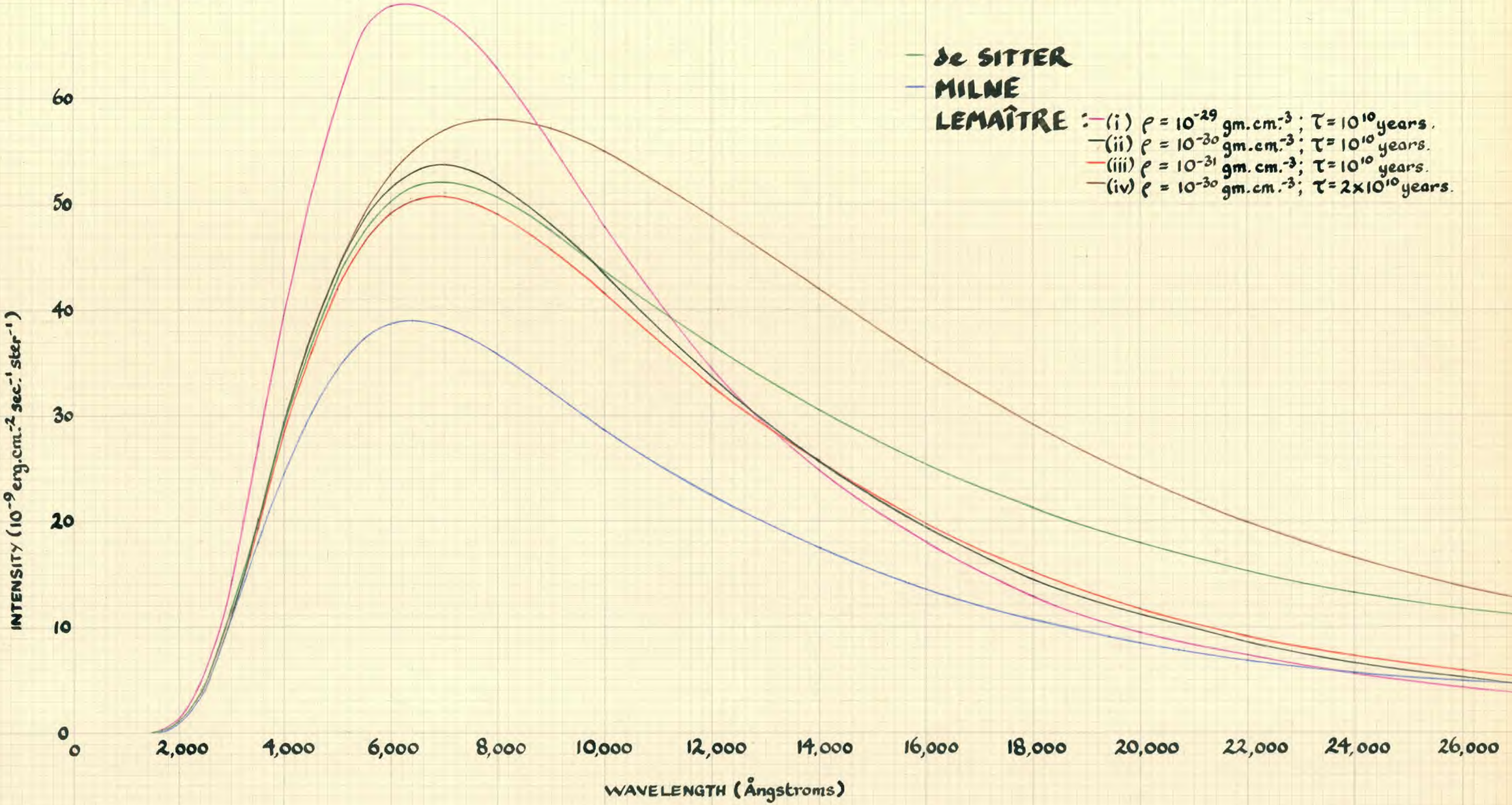
The background spectrum in this model has been calculated when the density $\rho_o = 10^{-29}$, 10^{-30} and 10^{-31} gm cm^{-3}, respectively, assuming that the average total luminosity of the sources is $10^{10} L_\odot$, and that they are black bodies at 6000°K, and that n_o = 1 galaxy mpc^{-3}, H = 100 km sec^{-1} mpc^{-1}. In order to evaluate (5.19) we must know x', and we determine x' by assuming (as in Chapter IV) that the galaxies began emitting radiation 10^{10} years ago. The results for the Lemaître model are given in Graphs F and G together with the Milne uniformly expanding model and the de Sitter metric model for comparison.

We see from the graphs that, assuming that the galaxies were created 10^{10} years ago, the background spectra in the cases where in c.g.s. units, $\rho_o = 10^{-30}$ and 10^{-31} show little difference, whilst there is a marked difference if $\rho_o = 10^{-29}$, despite the fact that, as shown in Chapter IV Tables VI and VII, the total intensity is nearly the same in all cases. If, $\rho_o = 10^{-30}$ or 10^{-31}, the background spectra follow closely the spectrum in the de Sitter metric model for wavelengths less than 10,000 Å (1μ). At longer wavelengths the spectra fall off more rapidly and eventually approach the background spectrum of the Milne model.

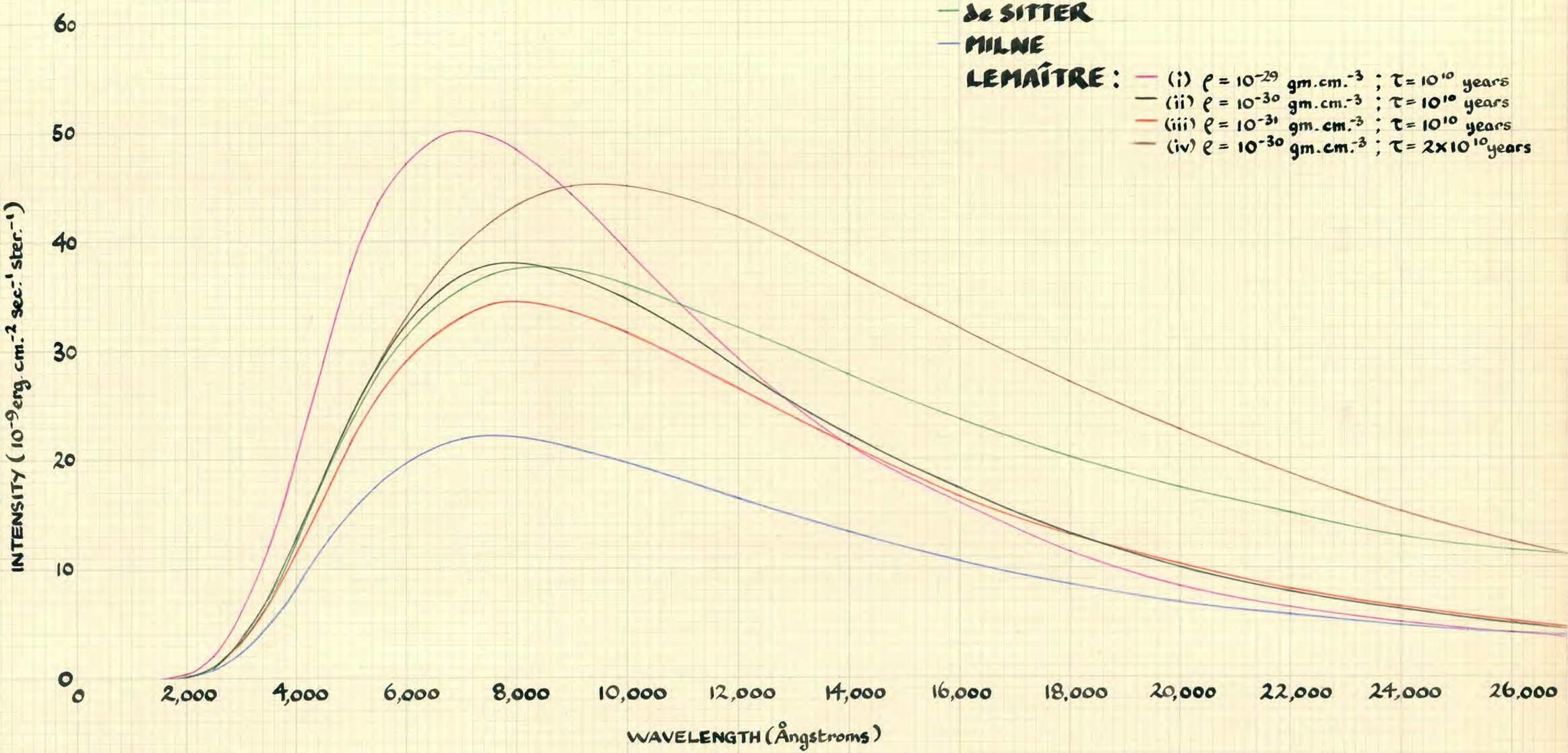
On the other hand, if $\rho_o = 10^{-29}$, the maximum is higher and sharper than if we take $\rho_o = 10^{-30}$ or 10^{-31}, and the spectrum falls off more

GRAPH 'F'

BACKGROUND SPECTRUM IN THE LEMAÎTRE MODEL



GRAPH 'G' BACKGROUND SPECTRUM IN THE LEMAÎTRE MODEL WITH CUT-OFF $\delta\lambda/\lambda = 0{\cdot}2$



rapidly at longer wavelengths. The reason for this is that, assuming that the galaxies were created 10^{10} years ago, the value of R at this epoch in the past in the model with $\rho_o = 10^{-29}$ is much closer to R_o than in the corresponding models with $\rho_o = 10^{-30}$ or $\rho_o = 10^{-31}$. If $\rho_o = 10^{-29}$, the wavelength at which maximum radiation is received from the most distant galaxies, assuming that they are black-body radiators at 6000 Å, is about 9000 Å, and no galaxy has a red-shift greater than $\delta\lambda/\lambda = 0.82$, approximately.

If, however, we assume that the galaxies were created 2×10^{10} years ago, we get a somewhat similar effect in the model with $\rho_o = 10^{-30}$. We find that the spectrum behaves like the de Sitter metric model up to 6000 Å, then as we go to longer wavelengths the intensity at first increases to greater values than the de Sitter metric model and finally at very long wavelengths falls back to the de Sitter value (not shown on graph). The maximum of the black-body spectrum for the most distant sources in this case occurs at about 16,000 Å, and the effect of this contribution can be seen in Graph E. The effect is more noticeable in Graph F where the influence of nearby galaxies has been eliminated by introducing a spectral cut-off.

7. The Background Spectrum in the Radio Region. An increase in radio noise towards the galactic poles has been observed for some years. The most probable explanation is that this increase is due to an isotropic component of extragalactic radio sources. It has been shown that the radiation from normal galaxies is inadequate to account for the observations, and therefore the background radiation must be due to discrete

radio sources which the Australian workers have called Class II (3). The number of Class II objects discovered is now roughly equal to the number of naked eye stars and out of these only fifty or so objects have been identified optically. By studying a number of nearby Class II (4) objects it is found that the emission of energy at radio frequency approximates to the law

$$G(\nu) = C\nu^{-x}, \tag{5.20}$$

where C is a constant, and the spectral index x is about unity. If we assume that the number of radio sources per unit proper volume is conserved, then the background radio spectrum in expanding world-models is given by (2.22) which may be written

$$J(\nu_o, t_o) = \frac{n_o c C \Omega}{4\pi} \int_{t_1}^{t_o} \nu_o^{-x} \left(\frac{R}{R_o}\right)^{x} dt$$

$$= AC\nu_o^{-x}, \tag{5.21}$$

where

$$A = \frac{n_o c \Omega}{4\pi} \int_{t_1}^{t_o} \left(\frac{R}{R_o}\right)^{x} dt. \tag{5.22}$$

We see from (5.21) that, in expanding models, the background radio spectrum is of the same form as the sources, with the same spectral index x. Since x is approximately unity, the factor A given by (5.22) becomes

$$A = \frac{n_o c \Omega}{4\pi} \int_{t_1}^{t_o} \frac{R}{R_o}\, dt. \tag{5.23}$$

From equation (4.1) we see that A is proportional to the total background radiation from non-evolutionary light sources. For the models considered in Table I in Chapter IV we see that A is greatest in the de Sitter metric and decaying photon models and least in the steady-state and Dirac models being four times as great in the former as in the latter. The problem of

measuring the absolute intensity of radio sources, however, is extremely difficult, and it is not yet possible to decide which world-model approximates closest to the universe.

If most Class II radio sources were colliding galaxies then it would follow that in expanding models the number-density of Class II objects will increase as we look into the past because the more congested the galaxies are the greater the number of collisions. Formula (5.21) would no longer apply because n_0 would be a function of t. In fact the appropriate formula in this and other cases where n_0 is a function of t is

$$J(\nu_0, t_0) = \frac{cC\nu_0^{-x}\Omega}{4\pi}\int_{t_1}^{t_0} n_0(t)\left(\frac{R}{R_0}\right)^x dx \qquad (5.24)$$

Again we note that the background radio spectrum is proportional to ν_0^{-x}. Costain (5) has observed that, to within the observational errors, the spectral index of the background radio spectrum is equal to that of the sources.

In the case of the steady-state model we may use equation (2.33) to obtain the background spectrum in the radio region. On changing from wavelengths to frequencies we find that

$$J(\nu_0) = \frac{n_0 cC\Omega\nu_0^{-x}}{4\pi}\int_{-\infty}^{t_0}\left(\frac{R}{R_0}\right)^x\left(\frac{R}{R_0}\right)^3 dt$$

$$= \frac{n_0 cC\Omega\nu_0^{-x}}{16\pi H}, \qquad (5.25)$$

approximately, since x is nearly unity and $R = e^{Ht}$, where we are assuming that the radio sources are non-evolutionary and that the number-density per unit proper volume is constant. We cannot, however, necessarily apply the usual assumption, when considering the steady-state model, that the

average luminosity of the radio sources per unit proper volume is a constant. Hoyle and Narlikar (6) have found, when using Hoyle's theory of the formation of galaxies in the steady-state model, that account has to be taken of the probability of a galaxy becoming a radio source at some time in its history, and a number of possible solutions can arise.

It would seem that when considering background radiation at radio wavelengths we are on even more uncertain ground than when considering radiation at optical wavelengths. Many radio-sources may be short lived, whereas it is probably safe to assume that most light-sources have far longer lives.

References.

1. Leontovski, M., (1936), Nature, 137, 904.
2. McVittie, G. C., Wyatt, S. P., (1959), Astrophysical Journal, 130, 1.
3. Mills, B. Y., Slee, O. B., Hill, E. R., (1958), Australian Journal of Physics, 11, 360.
4. Whitfield, G. R., (1957), Monthly Notices of the Royal Astronomical Society, 117, 680.
5. Costain, C. H., (1960), Monthly Notices of the Royal Astronomical Society, 120, 248.
6. Hoyle, F., Narlikar, J. V., (1961) Monthly Notices of the Royal Astronomical Society, 123, 133.

Chapter VI

The Effects of Absorption and Evolution on Background Radiation

1. Introduction. In previous chapters the calculations for the background radiation have been based on a simplified approach in which the galaxies are idealized as points. It has also been assumed that space is totally transparent, and the possibility of significant effects due to evolution of galaxies has been neglected. In the present state of knowledge, it is hardly possible to do more than consider in a general preliminary way the kind of modifications of our previous work which might result if account were taken of the presence of absorbing material in the arms of spiral galaxies and in intergalactic space and of systematic evolution of galaxies.

The presence of absorbing clouds of gas and dust in spiral galaxies becomes apparent when they are seen edge-on. Number counts of galaxies observed through the arms of nearby spiral galaxies which are seen face-on show a decrease compared with number counts made outside, indicating that between 10 and 20 per cent of the light passing through the spiral arms has been absorbed. In our own galaxy, Hubble (1) has calculated that the intensity of light, passing right through our region of the Galaxy at right angles to the galactic plane, will be reduced by 40 per cent, and that this absorption is non-selective in wavelength. On the other hand, Holmberg (2) has found that elliptical galaxies are effectively transparent.

Obscuration by diffuse intergalactic material would vary enormously with the form of the material. For a given mass of intergalactic material, dust particles with diameters of 10^{-5}cm (comparable with the wavelength of light) would be the most efficient absorbers. Larger lumps of solid matter or gas in the form of molecules, atoms or electrons would be far less efficient absorbers. Opinion is divided among observational astronomers as to whether significant quantities of intergalactic material exist. The only direct evidence of intergalactic matter is (a) the presence of intergalactic filaments connecting one galaxy to another, which has recently been investigated by Zwicky (3), and (b) the presence of neutral hydrogen in a number of clusters of galaxies revealed by Heeschen's observations (4) of the 21 cm emission line of neutral hydrogen.

Zwicky (5) has also tried to detect the presence of intergalactic material from a statistical analysis of number counts of galaxies and clusters of galaxies. Essentially the method depends on the hypothesis that deviations from a random distribution of galaxies can be interpreted as evidence that intergalactic material exists. He concludes that there is evidence of dust spread through large clusters, which in the case of the Coma cluster absorbs 0.6 magnitudes (over 40 per cent). He writes (p.108), "Dust seems to be not only spread throughout the large clusters of galaxies but often appears to be strongly concentrated in the space between and around two or more massive neighbouring galaxies, as may

be surmised from preliminary counts of distant galaxies in the localities mentioned."

With the 48-inch Schmidt telescope at Mount Palomar, Zwicky (3) has found about one cluster of galaxies per square degree, whilst very few other clusters of galaxies could be identified with the 200-inch Mount Palomar telescope. Zwicky concludes that there is enough intergalactic dust, probably within the large clusters and in smaller concentration between them, to interfere with the apparent structure and therefore identifiability of very distant clusters of galaxies, although brighter members of these clusters may still be visible in the 200-inch telescope.

2. The Effect of Absorption in Expanding World-Models. As in Chapter II we shall consider expanding world-models associated with the line element (2.1). Without loss of generality we may consider an observer O at the origin of the spatial co-ordinates (r, θ, φ). Suppose that at the present epoch t_o, O receives a light signal which was emitted at epoch t' by a galaxy G whose radial co-ordinate distance is r'. At a later epoch t the signal will have reached the point P on the light path GO (see Figure 1) at distance r from O.

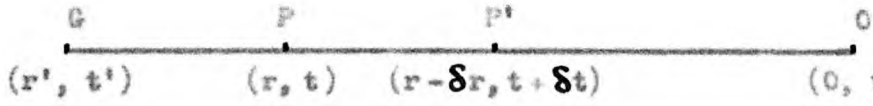


Figure 1

Along the incoming light path GO, the current co-ordinates (r, t) satisfy the relation

$$\frac{dr}{1 + kr^2/4} = -\frac{cdt}{R(t)} . \qquad (6.1)$$

Let the function $p(t - t')$ of the variable $(t - t')$ be the probability $Pr\left\{\text{Non-absorption in } t - t'\right\}$ that the light signal has not been absorbed by a galaxy in the time interval $(t - t')$. Let P' be a neighbouring point to P on the light path GO with co-ordinates $(r - \delta r, t + \delta t)$ then

$$Pr\left\{\text{Non-absorption in } t + \delta t - t'\right\} = Pr\left\{\text{Non-absorption in } t - t'\right\} \times Pr\left\{\text{Non-absorption in } \delta t\right\},$$

since the events are statistically independent. Hence,

$$p(t + \delta t - t') = p(t - t')\, p(\delta t). \qquad (6.2)$$

The probability that the light signal will be absorbed by a galaxy when it passes from P to P' is given by* the effective absorbing cross-section presented by all the galaxies in the shell $(r, r - \delta r)$ divided by the total surface area of the shell at time t, i.e.

$$Pr\left\{\text{Absorption in } \delta t\right\} = \frac{A\delta n\,(1 + kr^2/4)^2}{4\pi R^2 r^2},$$

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* We are assuming here that the thickness of the shell is sufficiently thin to neglect the effect of one galaxy standing in front of another.

where $^{\ddagger}A$ is the average total effective absorbing area of the galaxies (and may be a function of time and wavelength), $\delta n = \frac{4\pi n_0 R_0^3 r^2 \delta r}{(1 + kr^2/4)^3}$ is the number of galaxies in the shell $(r, r - \delta r)$, and $\frac{4\pi R^2 r^2}{(1 + kr^2/4)^2}$ is the surface area. Therefore,

$$\begin{aligned} p(\delta t) &= \Pr\left\{\text{Non-absorption in } \delta t\right\} \\ &= 1 - \Pr\left\{\text{Absorption in } \delta t\right\} \\ &= 1 - \frac{An_0R_0^3\delta r}{R^2(1+kr^2/4)} \\ &= 1 - \frac{An_0R_0^3c\delta t}{R^3}, \end{aligned} \tag{6.4}$$

using (6.1).

Therefore, using (6.2) and (6.4) we find that

$$p(t + \delta t - t') = p(t - t')\left(1 - \frac{An_0R_0^3c\delta t}{R^3}\right),$$

whence

$$\frac{p(t + \delta t - t') - p(t - t')}{\delta t} = - p(t - t') \frac{An_0R_0^3c}{R^3} \tag{6.5}$$

In the limit at $\delta t \to 0$, equation (6.5) becomes

$$\frac{d}{dt}\left\{p(t - t')\right\} = -p(t - t') \frac{An_0R_0^3c}{R^3}. \tag{6.6}$$

Integrating (6.6) we find that

$$p(t - t') = \exp\left\{ - n_0R_0^3c \int_{t'}^{t} \frac{A dt}{R^3}\right\}, \tag{6.7}$$

where the constant of integration has been put equal to unity since $p(0) = 1$.

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‡ We note that A does not undergo the general expansion of the universe.

The probability that the light signal reaches 0 at t_0 is therefore given by

$$p(t_0 - t') = \exp\left\{ - n_0 R_0^3 \int_{t'}^{t_0} \frac{A\,dt}{R^3} \right\}, \qquad (6.8)$$

or equivalently the fraction of the total light emitted by a galaxy at time t which arrives at 0 at t_0 without being absorbed by intermediate galaxies is given by

$$p(t_0 - t) = \exp\left\{ - n_0 R_0^3 \int_{t}^{t_0} \frac{A\,dt}{R^3} \right\} . \qquad (6.9)$$

If we now consider radiation received by 0 from galaxies in the shell $(r, r + \delta r)$ then the flux of radiation at 0 without absorption is given by (2.10), and a fraction $p(t_0 - t)$ of this radiation actually reaches 0, the remainder being absorbed by intermediate galaxies. Hence equation (2.12) for the total background radiation in expanding world-models is replaced by

$$I(\lambda_0, t_0)\, d\lambda_0 = \frac{n_0 \epsilon}{4\pi} \frac{d\lambda_0}{\lambda_0} \Omega \int_{t_1}^{t_0} p(t_0 - t)\, F\left(\frac{R_0}{R_0}\lambda, t\right) \frac{R^2}{R_0^2}\, dt. \qquad (6.10)$$

Any absorption due to intergalactic dust grains will produce an additional effect to that already considered in equation (6.3). Let the mass of a dust grain be m and its effective cross-section a. Then, if ρ is the density of dust grains at P, the number of dust grains is ρ/m cm^{-3}. As the light signal travels from P to P', it passes per unit area $(\rho/m)c\delta t$ dust grains, and therefore the probability of absorption by a dust grain is $\frac{a}{m}\rho c\delta t$, where we have neglected the effect of one dust grain lying in front of another which would reduce the effective absorbing area of the grains. This expression, however, becomes exact as $\delta t \to 0$. Since we are considering the model to be expanding and matter conserved,

$$\rho R^3 = \rho_0 R_0^3, \qquad (6.11)$$

where ρ_0 is the present density of intergalactic dust. Hence we must add the term $\frac{a\rho_0}{m}\ \frac{\sigma\delta t}{R^3}$ to the right hand side of (6.3).

If we assume that the dimensions of intergalactic dust grains are similar to interstellar dust grains, and take the effective absorbing radius of a dust grain to be 10^{-5} cm, the mean mass of a dust grain to be 10^{-13}gm (6), and the mean density of dust grains in intergalactic space to be 10^{-31} gm cm^{-3}, then the number of grains per cm^3 is 10^{-18}, and the absorption of light in our neighbourhood due to intergalactic dust per unit area per cm of travel is

$$10^{-18}\pi(10^{-5})^2 \sim 3 \times 10^{-28}.$$

Hence using (6.9), and taking $R = R_0$, we find that in our own locality of the universe, light can travel a distance 7.5×10^8 pc through intergalactic dust before its intensity is reduced by 50 per cent.

On the other hand, neglecting the effect of intergalactic absorption we can calculate the effective absorbing area of an average spiral galaxy seen face-on. Taking the average radius of a spiral galaxy to be 10 kpc (7), and assuming that 40 per cent of light passing through the disc of a typical spiral is absorbed, then an upper limit to the effective absorbing area would be $300 \times 0.4 \sim 100$ kpc^2. In our own neighbourhood of the universe, the effective absorption per unit area per cm of travel due to dust in the arms of spiral galaxies is given by $n_0 A$ where n_0 is the number of spiral galaxies per cm^3 and

$A\text{cm}^2$ is the average effective cross-sectional absorbing area. Taking $n_o = \frac{1}{2}\ \text{mpc}^{-3}$ (7) and $A = 100\ \text{kpc}^2$ then $n_o A = 1.5 \times 10^{-29}\text{cm}^{-1}$. Hence using (6.9), and considering local conditions only, by taking $R = R_o$, we find that light would travel a distance $\frac{\log_e 2}{n_o A} \sim 1.5 \times 10^9$ cm. If we take a somewhat lower value, say 20 per cent, for the absorption of light passing through the disc of a galaxy, then $A \sim 60\ \text{kpc}^2$ and if we reduce A by a further 50 per cent to account for the random orientation of spiral galaxies then the distance light would travel before its intensity is reduced by half becomes $\sim 5.9 \times 10^9$ pc which is nearly 8 times farther than in the case of a dust filled universe considered above.

The total background radiation Q_a (say) when the effects of absorption are included is obtained by integrating (6.10) over all λ_o, and if we assume that A is independent of λ_o we find that

$$Q_a = \frac{n_o c \Omega}{4\pi} \int_{t_1}^{t_o} p(t - t_o)\, L \frac{R}{R_o}\, dt, \qquad (6.12)$$

where

$$L = \int_0^\infty F(\lambda_o, t)\, d\lambda_o \qquad (6.13)$$

is the average luminosity of the sources at time t_o.

Confining attention for the moment to the Milne model, if Q is the total background radiation in the Milne model when the sources are taken to be points and the effects of evolution are assumed to be negligible, and Q_1 is the modified Q when the effects of intergalactic dust are taken into account, then we find that if we take the effective

absorption per unit area per cm of travel a $\rho_{o/m}$ to be 3×10^{-28} cm^{-1},

$$\frac{Q_1}{Q} = 0.34 \qquad (6.14)$$

If Q_2 denotes the modified Q when the effects of dust in spiral galaxies is taken into account, neglecting the effects of intergalactic dust then we find that

$$\frac{Q_2}{Q} = 0.83 , \qquad (6.15)$$

where we have assumed that the effective absorption per unit area per cm of travel due to galaxies $n_o A = 1.5 \times 10^{-29}$ cm^{-1}.

On comparing (6.14) with (6.15), we note that the effect of intergalactic dust on the background radiation could be very much greater than the effect of dust in the arms of spiral galaxies.

3. The Effect of Absorption in the Steady-State World-Model. We must first derive the probability of absorption in this model. From Figure 1, the probability that a light signal travelling from P to a neighbouring point P' will be absorbed is given by the total effective absorbing cross-section of the galaxies in the shell $(r, r - \delta r)$ divided by the surface area of the shell at time t_o, i.e.

$$\Pr\left\{ \text{Absorption in } \delta t \right\} = \frac{A \delta n}{4\pi R^2 r^2}, \qquad (6.16)$$

where A is the average effective absorbing cross-section of the galaxies, $\delta n = 4\pi n_o R^3 r^2 dr$ is the number of galaxies in the shell $(r, r - \delta r)$ and $4\pi R^2 r^2$ is the area of the shell at time t. Hence

$$\Pr\left\{ \text{Absorption in } \delta t \right\} = n_o A R \delta r = n_o A c \delta t, \qquad (6.17)$$

since along an incoming light path

$$\frac{c\,\delta t}{R} = \delta r. \tag{6.18}$$

Therefore, if $p(t_o - t) = Pr\left\{\text{Non-absorption in } t_o - t\right\}$, we find using (6.2) and (6.17) that

$$p(t + \delta t - t') = p(t - t')\,(1 - n_o A c\,\delta t), \tag{6.19}$$

and in the limit as $\delta t \longrightarrow 0$,

$$\frac{dp}{dt} = - n_o A c p \tag{6.20}$$

Integrating (6.20) we find that since $p(0) = 1$,

$$p(t_o - t_{//}) = \exp\left\{-An_o c(t_o - t)\right\}. \tag{6.21}$$

Hence, when the effect of absorption due to dust in the arms of spiral galaxies is taken into account, equation (2.31) for the total background spectrum is replaced by

$$I(\lambda_o)d\lambda_o = \frac{n_o c \mathcal{R} d\lambda_o}{4\pi}\int_{-\infty}^{t_o} F(\lambda_o e^{H(t - t_o)}) \exp\left\{(5H + An_o c)(t - t_o)\right\}\,dt$$

$$= \frac{n_o c \mathcal{R} d\lambda_o}{4\pi}\int_{0}^{\infty} F(\lambda_o e^{-Ht}) \exp\left\{-(5H + An_o c)t\right\}\,dt. \tag{6.22}$$

The additional effect of absorption by intergalactic dust grains may be calculated in the same manner as in the previous Section. We find that the probability of absorption by a dust grain as light travels from P to P' is given by a $\rho_o c\delta t/m$, where ρ_o is the constant density of intergalactic dust, and this expression is added to the right hand side of (6.17).

Integrating (6.22) over all wavelengths, we obtain the total background radiation Q_a which is given by

$$Q_a = \frac{n_0 e \Omega L}{4\pi} \int_0^\infty \exp\left\{-(4H + An_0)t\right\} dt$$

$$= \frac{n_0 e \Omega L}{4\pi} \left(\frac{1}{4H + An_0 e}\right) . \qquad (6.23)$$

As before let Q denote the total background radiation when the sources are taken to be points, and Q_1 the modified Q when the effect of intergalactic dust is taken into account. Taking $H = 100$ km sec^{-1} mpc^{-1} and $a\rho_0/m = 3 \times 10^{-28}$ cm^{-1}, we find that

$$\frac{Q_1}{Q} = 0.59 \qquad (6.24)$$

On the other hand, if Q_2 is the modified Q when the effect of dust in spiral galaxies is taken into account, then taking $n_0 A = 1.5 \times 10^{-29}$ cm^{-1} we find that

$$\frac{Q_2}{Q} = 0.966 \qquad (6.25)$$

From (6.25) we see that in the steady-state model the effect of dust in the arms of spiral galaxies is negligible whilst from (6.24) we see that the effect of intergalactic dust could be significant. Furthermore, compared with the Milne model the effect of dust is very much less in the steady-state model, since in this model most of the background radiation comes from foreground galaxies where the effects of absorption are small, whereas in the Milne model a larger proportion of the background radiation comes from remote galaxies where the effects of absorption are large.

4. The Possible Effect of Evolution of Galaxies in the Milne Uniformly Expanding World-Model. We shall finally consider in a preliminary way the possible effects of the evolution of galaxies in Milne's world-model. In previous chapters we have assumed that the spectrum of an average galaxy is the same as that of a black-body at $T = 6{,}000^{\circ}K$. In a more exact treatment account must be taken of the deviation from a black-body spectrum, e.g. the Balmer discontinuity at $3647\,\mathring{A}$, and also the effect of the rate of star formation on the luminosity function of galaxies (which has been considered recently by M. Schmidt (8)), and the effect of evolution of stars. Sandage has remarked (9) that an understanding of the evolution of galaxies is the most important of the unsolved problems blocking the solution of the cosmological problem in its first approximation. We shall therefore confine attention to a special case in which the average galaxy is a black-body radiating at temperature $T(t)^{\circ}K$ at cosmological time t, where T decreases linearly at a rate of $100^{\circ}K$ in 10^9 years to the present value of $6{,}000^{\circ}K$. We shall assume further that the total luminosity of the galaxies obeys Stefan's law, and that at the present epoch it is equivalent to about 10^{10} Suns. (On this assumption, if the age of the model is 10^{10} years, this would imply that the temperature of an average galaxy immediately after the epoch of world-creation, if it were already in existence, would have been $7{,}000^{\circ}K$ and its luminosity 1.85×10^{10} Suns). We have assumed that the average effective temperature of the galaxies increases as we go into the past because current theories of nucleogenesis in stars indicate that the earlier generations of stars have a higher

hydrogen to metal abundance than present generation stars, and will be intrinsically brighter than present stars.

As before, let Q denote the total background radiation in the Milne model when the sources are taken to be point-sources radiating like black-bodies at T = 6000°K. If Q* denotes the modified Q when the above (hypothetical) effects of evolution are taken into account, then we find that, taking Hubble's constant H to be 100 km sec^{-1} mpc^{-1},

$$\frac{Q^*}{Q} = 1.25, \tag{6.26}$$

and we note that this amount of evolution although slight has a significant effect on the total background radiation.

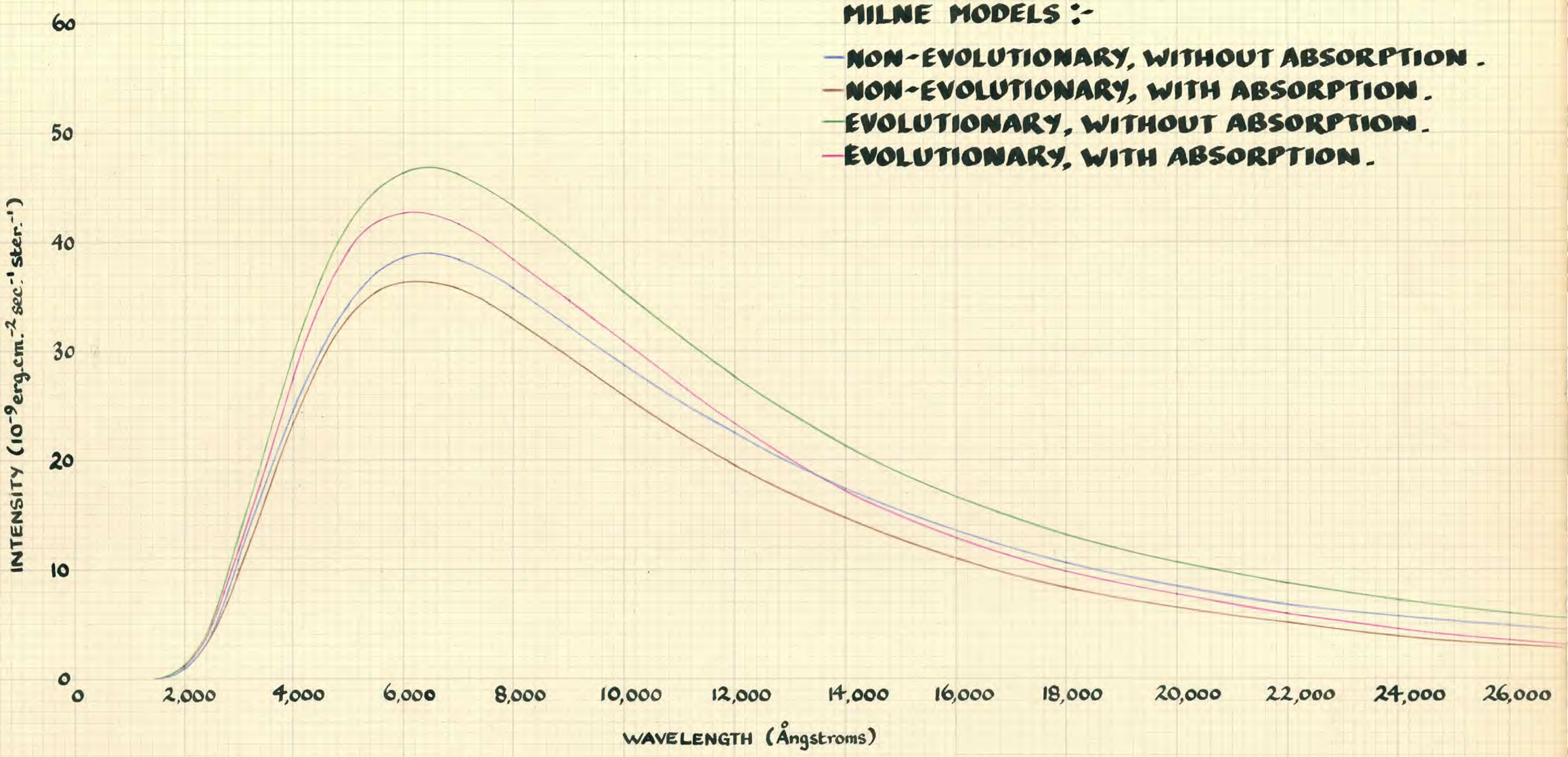
If Q_2^* denotes the modified Q when the effects of both absorption and evolution are included we find that, taking $n_o A = 1.5 \times 10^{-29} cm^{-1}$,

$$\frac{Q_2^*}{Q} = 0.998, \tag{6.27}$$

and we note that in this particular case the effects of absorption and evolution almost counterbalance one another.

The background spectra in the Milne model in which the above effects of absorption and evolution are taken into account have also been calculated and the results are given in Graph H. In all the cases considered it has been assumed that Hubble's constant H = 100 km sec^{-1} mpc^{-1}, and the number density of galaxies n_o = 1 mpc^{-3}. We see from Graph H that, compared with the model in which the effects of evolution and absorption are neglected, the background spectrum in the non-evolutionary model in which the effect of absorption is taken into

GRAPH 'H' BACKGROUND SPECTRUM IN MILNE'S UNIFORMLY EXPANDING WORLD-MODEL



account is less at longer wavelengths and the wavelength λ_{max} of maximum intensity is shifted to shorter wavelengths. On the other hand, the background spectrum in the model in which the hypothetical evolutionary effects are considered but absorption is neglected is greater at longer wavelengths and λ_{max} is shifted to longer wavelengths. When the effects of both absorption and evolution are considered, however, we note that although the total background radiation in this model is very nearly equal to that in the model in which these effects are neglected, the background spectrum is somewhat different, being greater at shorter wavelengths and less at longer wavelengths, with λ_{max} approximately the same in both cases.

References

1. Hubble, E. P., (1936), "The Realm of the Nebulae", Yale University Press.
2. Holmberg, E., (1947), Meddelanden Lund, Serie II, Band 2, No. 120.
3. Zwicky, F., (1959), Handbuch der Physik, 53, p.390, (Berlin: Julius Springer).
4. Heeschen, D. S., (1956), Astrophysical Journal, 124, 660.
5. Zwicky, F., (1957), "Morphological Astronomy", (Berlin: Julius Springer).
6. Allen, C. W., (1955), "Astrophysical Quantities", p.226, (London: Athlone Press).
7. de Vaucouleurs, G., (1958), "L'Exploration des Galaxies Voisines", (Paris: Masson et Cie).
8. Schmidt, M., (1959), Astrophysical Journal, 129, 243.
9. Sandage, A., (1961), Astrophysical Journal, 134, 916.