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THE CONSTRUCTION OF CHEBYSHEV APPROXIMATIONS IN THE COMPLEX PLANE

by

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ABSTRACT

This thesis is concerned with practical methods for the construction of best and near-best uniform approximations on regions of the complex plane.

The first chapter sets out the necessary theory concerning the possibility of approximation and the existence, uniqueness and characterisation of best approximations. An algorithm is derived for polynomial approximation similar to the exchange algorithm of Remes for real approximation. This involves less computation than existing methods, and numerical examples show it to be more effective, providing best rather than near-best approximations in some cases. The method is generalised in Chapter 3 to rational approximation with equal success.

A different approach is then used to reduce the polynomial approximation problem to a sequence of quadratically constrained mathematical programs. An efficient algorithm for their solution is described and analysed. The method is extended in Chapter 5 to the rational problem using a form of the differential correction algorithm. Several best approximations are computed by these algorithms on various regions of the complex plane.

The last two chapters investigate methods for finding near-best approximations. The use of truncated Faber series in approximation is described together with its relation to best approximation. Finally it is shown that approximations may be obtained for the disc and ellipse by economising Chebyshev or rational Chebyshev approximations. These simply obtained approximations are shown to be very nearly optimal.

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CONTENTS

Title Page			1
Abstract			2
Acknowledgements			3
Contents			4
Preface			7
Chapter 1	The Theoretical Basis for Complex Approximation		
	1.1	The problem	10
	1.2	Possibility of approximation	13
	1.3	Existence and uniqueness of best approximation	14
	1.4	Characterisation of best polynomial approximation	17
	1.5	Characterisation of best rational approximation	22
Chapter 2	Constructive Methods for Best Polynomial Approximation Based on Characterisation		
	2.1	Introduction	26
	2.2	The computation of extremal signatures	29
	2.3	Polynomial approximation on an interval of the imaginary axis	35
	2.4	Polynomial approximation on more general regions of the complex plane	41
	2.5	Polynomial approximation on the unit disc	42
	2.6	Polynomial approximation on regions other than a disc	51

Chapter 3 Constructive Methods for Best Rational Approximation Based on Characterisation

3.1 Introduction 53

3.2 Computation of the extremal signature 55

3.3 An exchange algorithm for the unit disc 57

3.4 The exchange algorithm for other regions 62

3.5 Conclusion 69

Chapter 4 The Construction of Best Polynomial Approximations by Programming Methods

4.1 Introduction and history 70

4.2 Existing algorithms and their limitations 73

4.3 The new algorithm - description and convergence 76

4.4 The new algorithm - implementation 81

4.5 Approximation on a continuum 86

Chapter 5 The Construction of Best Rational Approximations by Programming Methods

5.1 Introduction 99

5.2 Choice of algorithm 100

5.3 Implementation and convergence 103

5.4 Numerical examples 114

5.5 The continuous problem 118

5.6 Concluding remarks 127

Chapter 6 Near-best Polynomial Approximation Based on Truncated Series Expansions or Interpolation

6.1 Introduction 128

6.2 Near-best approximation 131

6.3 The role of Faber series in near-best approximation 134

6.4 Norms of Faber projections 144

6.5 Interpolatory projections 151

Chapter 7 Economisation and Related Techniques

7.1 Introduction 156

7.2 Possibility of economisation in the complex plane 159

7.3 Asymptotic economisation on the unit disc 163

7.4 Asymptotic economisation on an interval of the real axis 172

7.5 Asymptotic economisation on the ellipse 176

7.6 Economisation of a rational function on the unit disc 180

7.7 Economisation of a rational function on an interval of the real axis 188

7.8 Conclusion 191

References 192

PREFACE

It is not the purpose of this thesis to enlarge on or even expound the vast theory of uniform approximation of functions of a complex variable that has been developed over the last century. Rather, the aim is to present practical algorithms for the construction of best or almost best approximations. Since it is readily available in the literature, the theoretical basis for the problem is treated as briefly as possible.

Philip Davis has said* "with a flip of the wrist it is possible to turn out yards of polynomial and rational approximations". While this is true in the real case of which he was speaking, the situation regarding complex approximation is very different. For instance, we are not aware of any published algorithms for the construction of the best Chebyshev approximation to a continuous function in even such a simple region as the unit disc!

The objective of this thesis is partly to collect together some of the scattered results concerning the construction of approximations in the complex plane and to discuss their merits. Because of this it has been felt necessary to indicate exactly which parts of the thesis are original, although we believe a collection and review of existing methods has not been hitherto presented.

*DAVIS, P. J. [1965] Approximation theory and electronic computers in "Approximation of Functions", ed. Garabedian, H.L., p. 155, Elsevier Pub. Co.

In Chapter 2, the formulae for extremal signatures have been simplified and explicit formulae given for polynomials and polynomial-rational sums. An alternative algorithm to that proposed by Dierieck and Kamp [1971] for the determination of the argument θ^* of the error in polynomial approximation on the imaginary axis is described and analysed. The exchange algorithm for the disc is a new method for constructing best polynomial approximations, as is that for the best rational approximation in Chapter 3.

The algorithm in Chapter 4 for computing best approximations on a finite point set is similar to although developed independently of the algorithm of Barrodale, Delves and Mason [1977] for the same problem. We have not seen the first algorithm of Remes used before to construct best approximations on a general compact subset of the complex plane.

The use of the differential correction algorithm in Chapter 5 for rational approximation on a finite point set and the verification of its convergence is new, as is the extension to more general sets.

Most of the ideas in Chapter 6 are well known, although they have not all been explicitly applied to the problem of near-best approximation. Some Faber polynomials are exhibited for some simple regions other than those for which they are usually given in standard texts. The explicit calculation of projection norms for various interpolation schemes in the case of the semi disc are believed to be original.

Practically no attempt has been made before to use economisation in the complex plane, although it is a well established technique in the real case. Most of the economisation methods are therefore new, as are the asymptotic economisation methods for both polynomial and rational approximation, although the idea has again been borrowed from the real case. In some cases, the complex methods provide alternative proofs (but not new methods) for the corresponding real problems.

Finally, unless otherwise stated, all the numerical results are original as are the Algol and Fortran programs used to compute them.

CHAPTER 1

THE THEORETICAL BASIS FOR COMPLEX APPROXIMATION

1.1 The Problem

The general setting for the work contained in this thesis is the space C(B) of complex valued functions f continuous on a compact subset B of the complex z-plane. Two classes of approximating function are considered. The first is P_n, the n dimensional linear space of generalised polynomials

$$p_n(z) = \sum_{k=1}^{n} a_k \phi_k(z) \quad \text{where} \quad \phi_k(z) \in C(B).$$

We shall often use the special case where $\phi_k(z) = z^{k-1}$. It is then convenient to let p_n denote the $n+1$ dimensional space of algebraic polynomials

$$p_n(z) = \sum_{k=0}^{n} a_k z^k .$$

The second is V, the class of bounded rational functions $r_{m,n}(z)$ on B where

$$r_{m,n}(z) = \frac{\sum_{k=0}^{m} a_k z^k}{\sum_{k=0}^{n} b_k z^k}, \qquad b_0 = 1 .$$

When the distinction is unimportant, we shall let A denote either class of approximating function.

The problem considered is that of finding functions $A(z) \in A$ which are close to $f \in A$ in some sense. The measure of approximation is the Chebyshev (uniform or ℓ_∞) norm.

$$\| f - A \| = \max_{z \in B} | f(z) - A(z) | .$$

1.1 The Problem (cont'd)

Before an attempt is made to construct such approximations, there are four questions which need to be answered.

(i) *The possibility of approximation.* In order to obtain useful approximations, it is necessary to know whether there are functions belonging to A that are sufficiently close to f. That is, given an arbitrary small quantity $\varepsilon > 0$, does there exist a function $A(z) \in A$ such that

$$\| f - A \| < \varepsilon \ ?$$

(ii) *Existence of best approximations.* Suppose now that the values of m and n are fixed in the definition of the class A of approximating functions. Among such approximations those nearest to f will be of most practical use. In particular we would like to know whether there is an $A^*(z) \in A$ with the property that

$$\| f - A^* \| \leq \| f - A \| \quad \text{for all } A \in A. \tag{1.1.1}$$

If such an A^* exists, then it is called the best (Chebyshev or uniform) approximation to f.

(iii) *Uniqueness.* A best approximation may exist but fail to be unique in the sense that there is more than one $A^*(z)$ satisfying (1.1.1). This would not normally affect the usefulness of any one of these best approximations, but the lack of uniqueness will usually complicate any algorithm used to find it.

1.1 The Problem (cont'd)

(iv) *Characterisation.* Suppose we are given an approximation $A(z) \in A$. Can we determine whether or not it is a best approximation to f? In some cases there are simple tests which can be applied, based on certain characteristics which the best approximation is known to possess. Sometimes, but not always, this provides a method for constructing best approximations.

In the case where B is an interval I of the real axis and f is a continuous real-valued function on I, then approximation is certainly possible, and best approximations are known to exist uniquely and to have simple characterisations. Moreover this characterisation leads to efficient algorithms for their construction.

In the complex plane, the situation is less straightforward. A brief outline of the known results concerning the above four questions will now be given to form a basis for the computational methods described in the rest of the thesis.

1.2 Possibility of Approximation

The possibility of approximation by rational functions has been investigated by Walsh from whom we give a typical result.

Theorem 1.1 (Walsh [1929])

Let the function $f(z)$ be analytic on the closed subset B of the extended plane. If $\epsilon > 0$ is arbitrary, then there exists a rational function $r(z)$ such that

$$| f(z) - r(z) | < \epsilon \qquad \text{for all} \quad z \in B .$$

The corresponding result for polynomial approximation has been developed in stages starting with the classical paper of Runge in 1885 for Jordan regions but not finally exhibited in its most general form until the paper of Mergelyan in 1952.

Theorem 1.2 (Mergelyan [1952])

If B is a compact set not separating the plane, and if $f(z)$ is continuous on B and analytic in the interior points of B, then $f(z)$ can be uniformly approximated on B as closely as desired by a polynomial in z.

1.3 Existence and Uniqueness of Best Approximations

For approximation by generalised polynomials, the existence of a best approximation is assured. The following result is taken from Smirnov and Lebedev [1966, p.430].

Theorem 1.3

Let B be a compact set and let $f(z)$ and $\phi_k(z)$, $(k = 1, 2, \ldots, n)$ be continuous functions on B. If P_n is the class of polynomials

$$p_n(z) = \sum_{k=1}^{n} a_k \phi_k(z)$$

then there exists a polynomial $p_n^*(z) \in P_n$ of best approximation to $f(z)$ on B.

To ensure uniqueness, we need the concept of a Chebyshev system.

Definition 1.1

The system of functions $\phi_k(z)$, $(k = 1, 2, \ldots, n)$ on a compact set B is said to be a Chebyshev system on B if and only if each polynomial

$$p_n(z) = \sum_{k=1}^{n} a_k \phi_k(z),$$

not identically zero, has at most $n-1$ distinct zeros on B.

There is an intimate relation between Chebyshev systems and the uniqueness of best approximations.

Theorem 1.4

For every function $f \in c(B) - P_n$ to have a unique best approximation it is necessary and sufficient for the system $\phi_k(z)$ $(k = 1, 2, \ldots, n)$ to form a Chebyshev system.

The sufficiency was proved by Tonelli [1908] and the necessity by Haar [1918].

1.3 Existence and Uniqueness of Best Approximations (cont'd)

Note 1

The condition$ for a system to be a Chebyshev system is often called the Haar condition, and the space P_n is then called a Haar subspace.

Note 2

The condition is certainly satisfied for the case when $\phi_k(z) = z^{k-1}$.

It is interesting that the conditions for a unique *best* approximation to exist are weaker than those for the existence of an *arbitrarily close* approximation, for which analyticity at interior points of B is necessary.

For rational approximation the situation is less favourable as the following theorem shows.

Theorem 1.5 Walsh [1960]

Let the function f be continuous on a compact set B which is dense in itself. Then there exists a rational function $r_{m,n}(z) \in V$ of best approximation to f on B.

One consequence of this theorem is that best approximations are not guaranteed if B contains isolated points, and Walsh [1960, p.353] gives an example for which the best approximation fails to exist in such a case. This result is disappointing since many numerical methods are based on finding best approximations on finite point sets. In addition, best approximations by rational functions may not be unique, and an example is given by Walsh [1960, p.356]. With these limitations

1.3 Existence and Uniqueness of Best Approximations (cont'd)

on the theory of rational approximation, it is perhaps surprising that the constructive methods in Chapter III and Chapter V have been so successful.

We end this section with three definitions.

Definition 1.2

The error $\| f - p_n^* \|$ in the best polynomial approximation is called the *degree of approximation of* f *on* B by polynomials of degree n, and denoted by $E_n(f ; B)$. The corresponding value for $\| f - r_{m,n}^* \|$ is denoted by $E_{m,n}(f ; B)$.

Definition 1.3

The set $D = \{ z : | f(z) - p^*(z) | = \| f - p^* \|, \ z \in B \}$ is called the *extremal set* for the best approximation of f on B.

Definition 1.4

A *characteristic set* $\overline{D} \subset D$ is defined as the set of points $\{z_k\}_1^r$ such that $E_n(f,\overline{D}) = E_n(f,D)$ and for an arbitrary subset $D^* \subset D$, $E_n(f,D^*) < E_n(f,D)$. In other words, the best approximation of f on D is also the best approximation of f on $\overline{D}$.

Every characteristic set is known to contain r points where r satisfies the inequality

$$1 \leq r \leq 2n + 1 .$$

1.3 Existence and Uniqueness of Best Approximations (cont'd)

If the system $\{\phi_k(z)\}$ form a Chebyshev system, then the characteristic set contains at least $n+1$ points, so that r satisfies the inequality

$$n+1 \leqslant r \leqslant 2n+1 .$$

Proofs of these inequalities may be found in Smirnov and Lebedev [1966].

1.4 Characterisation of Best Polynomial Approximations

The fundamental result is that of Kolmogorov [1948].

Theorem 1.6

The generalised polynomial $p_n^* \in P_n$ is a best approximation to $f \in C(B) - P_n$ if and only if

$$\inf_{z \in D} \operatorname{Re}\{\overline{(f(z) - p_n^*(z))}\, p_n(z)\} \leqslant 0$$

for all $p_n \in P_n$ where D is the extremal set.

The following lemma enables us to restate this theorem in a more convenient form.

Lemma (Smirnov and Lebedev [1966])

Suppose that $u_{i,j} (i = 1, 2, \ldots, r;\ j = 1, 2, \ldots n)$, are given numbers. For real numbers $\delta_i \geqslant 0$ $(i = 1, 2, \ldots, r)$, not all zero, to exist such that

$$\sum_{i=1}^{r} \delta_i u_{ij} = 0, \qquad (j = 1, 2, \ldots, n),$$

it is necessary and sufficient that for an arbitrary system of numbers c_j $(j = 1, 2, \ldots, n)$, the inequalities

$$\operatorname{Re} \sum_{j=1}^{n} c_j u_{ij} > 0, \qquad (i = 1, 2, \ldots r)$$

should not be satisfied for all i.

1.4 Characterisation of Best Polynomial Approximations (cont'd)

Theorem 1.7

The generalised polynomial $p_n^* \in P_n$ is a best to $f \in C(B) - P_n$ if and only if there exists a set $\{z_k\}_{k=1}^{r} \in D$, $1 \leq r \leq 2n+1$, and numbers $\delta_i > 0$, $(i = 1, 2, \ldots, r)$ such that

$$\sum_{i=1}^{r} \delta_i \{ (f(z_i) - p_n^*(z_i)) \}^{-} \phi_j(z_i) = 0, \quad (j = 1, 2, \ldots, n)$$

or equivalently

$$\sum_{i=1}^{r} \delta_i \{ f(z_i) - p_n^*(z_i) \}^{-} p_n(z_i) = 0,$$

for all $p_n \in P_n$.

The conditions of Theorem 1.7 can be restated using the concept of an extremal signature.

Definition 1.5

A *signature* $\sum$ is a finite collection of ordered pairs $\{(z_j, \varepsilon_j) ; j = 1, \ldots, r\}$ where z_j are distinct points of the z-plane and ε_j are complex signs, that is numbers such that $|\varepsilon_j| = 1$.

Definition 1.6

A signature $\sum$ is called *extremal* (for P_n) if there exist non-zero complex numbers $\zeta_1, \zeta_2, \ldots, \zeta_r$ with

$$\operatorname{sgn}(\zeta_j) = \bar{\varepsilon}_j \quad (j = 1, 2, \ldots r)$$

such that

$$\sum_{j=1}^{r} \zeta_j p_n(z_j) = 0 \tag{1.4.1}$$

for every $p_n \in P_n$.

1.4 Characterisation of Best Polynomial Approximations

Definition 1.6 (cont'd)

Relating the quantities $\delta_k\{f(z_k) - p_n^*(z_k)\}$ and τ_k, it will be seen that Theorem 1.7 can be stated in the following form.

Theorem 1.8 (see for example Rivlin and Weiss [1968])
If $f \in C(B)$ and a polynomial $p_n^* \in P_n$ can be found such that $f - p_n^*$ attains its maximum absolute value on the points of an extremal signature, with the indicated signa, then p_n^* is a best approximation to f. Conversely, if p_n^* is a best approximation to f, some subset of the points for which $|f(z) - p_n^*(z)|$ attains its maximum, endowed with the signa $\operatorname{sgn}\{f(z) - p_n^*(z)\}$ is an extremal signature.

Some examples will now be given to illustrate the great diversity in the behaviour of r and ε_j.

Example 1

Let B be a closed interval of the real line, f a real-valued function and p_n a real-valued generalised polynomial where the $\{\phi_k(x)\}$ form a Chebyshev system. Then the value of r in Theorems 1.7 and 1.8 is exactly $n+1$, and every set of $n+2$ points equipped with alternating signs $\sum = \{(x_r, (-1)^r); r = 1, 2, \ldots, n+1\}$ is an extremal signature.

1.4 Characterisation of Best Polynomial Approximations (cont'd)

Example 2

For $\rho > 0$, the set

$$\{(\rho e^{\frac{2\pi i k}{n+2}}, e^{-\frac{2\pi i k}{n+2}}); \quad k = 0, 1, \ldots, n+1\}$$

is an extremal signature $\sum$ for P_n, the space of algebraic polynomials of degree at most n. It follows from Theorem 1.8 that the best approximation to z^{n+1} on $B: |z| \leq \rho$ is $p_n^*(z) = 0$.

The next two examples are due to Videnskii [1956].

Example 3

In contrast to the alternation of real approximation, consider the approximation of z^{n+1} by algebraic polynomials of degree n on the set

$$z_0 = 0, \quad z_\nu = e^{\frac{2\pi\nu i}{n+1}} \qquad (\nu = 1, 2, \ldots, n+1).$$

For this problem, the set $\{(0, -1), (e^{\frac{2\pi\nu i}{n+1}}, 1), \nu = 1, \ldots, n+1\}$, that is with $\varepsilon_0 = -1$, $\varepsilon_1 = \ldots = \varepsilon_{n+1} = 1$, is an extremal signature since if $0 \leq k \leq n$ and with $\zeta_j = \overline{\varepsilon}_j$ $(j = 0, 1, \ldots, n+1)$,

$$\sum_{\nu=0}^{n+1} \zeta_j z_j^k = \sum_{\nu=1}^{n+1} 1.e^{\frac{2\pi\nu i}{n+1}} = 0.$$

Moreover $z_0^{n+1} - \frac{1}{2} = -\frac{1}{2}$, $z_\nu^{n+1} - \frac{1}{2} = \frac{1}{2}$ $(\nu = 1, \ldots, n+1)$ and therefore by Theorem 1.8, $p_n^*(z) = \frac{1}{2}$ is the required best approximation.

1.4 Characterisation of Best Polynomial Approximation (cont'd)

The following example shows that the characteristic set may contain more than $n+1$ points.

Example 4

Let $f(z) = z^5$, and let $B = \{e^{2\pi iv/6}\}_{v=0}^{5}$. The best approximation to f on B by polynomials of degree 3 is $p_3^*(z) = 0$ with a degree of approximation equal to 1; on any subset of B containing 5 points, the degree of approximation is $3(2-\sqrt{3})$ which is less than 1.

Finally we illustrate a class of problem where r is near to its upper limit.

Example 5

Let B be the ellipse C_ρ, the image of $|w| = \rho$, $0 < \rho \leq 1$ under the mapping

$$z = \frac{1}{2}(w + 1/w).$$

The points $z_k = \frac{1}{2}(\rho + 1/\rho)\cos k\pi/(n+1) - i\frac{1}{2}(1/\rho - \rho)\sin k\pi/(n+1)$ lie on C_ρ. It is easily verified that the set $\{z_j, (-1)^j\}_{j=0}^{2n+1}$ is an extremal signature for P_n, the space of algebraic polynomials of degree at most n, so that $r = 2n+2$ (remember for P_n in this case, the bounds on r are $n+2 \leq r \leq 2n+3$).

By invoking Theorem 1.8, we see that in C_ρ (and therefore inside C_ρ plus C_ρ) the best approximation to z^{n+1} from P_n is

$$p_n^*(z) = z^{n+1} - T_{n+1}^*(z)$$

where $T_{n+1}^*(z) = 2^{-n}T_{n+1}(z)$ and $T_n(z)$ is the classical Chebyshev polynomial defined by $T_k(\cos\theta) = \cos k\theta$.

The degree of approximation is

$$\max_{z \in C_\rho} |T_{n+1}(z)| = \frac{1}{2}(\rho^{n+1} + \rho^{-(n+1)}).$$

1.5 Characterisation of Best Rational Approximation

In view of the problems associated with existence and uniqueness of best rational approximations, it is hardly surprising that we do not have a satisfactory characterisation either. In the case of real approximation, certain results are known (see for example Meinardus [1967, chapter 9]). In the complex case, the lack of uniqueness naturally suggests that a global theory will have to be abandoned in favour of a study of locally best approximations.

The first attempt to characterise best rational approximations was that by Williams [1968] under somewhat stringent conditions.

Definition 1.7

Let $Z = z_1, z_2, \ldots, z_N$, $N = n + m + 2$ consist of distinct points of B and satisfy the following conditions:

(a) there exists a unique best rational approximation $R_Z \in V$ of f on Z;

(b) R_Z has no common factors, and $a_m \neq 0$, $b_n \neq 0$.

Then Z is called an admissible point set of B, abbreviated to a.p.s.

Theorem 1.9 (Williams [1968])

Given $f \in C(B) - V$, let Z be an a.p.s. of B and let the functions

$$\{\phi_j(z)\} = \{zf(z), z^2f(z), \ldots, z^nf(z), 1, z, \ldots, z^m\}$$

form a Chebyshev set on Z. Let $R_Z = p^*/q^* \in V$ be a best approximation of f on Z. Then

(i) $|f(z_j) - p^*(z_j)/q^*(z_j)| = h(Z)$, $j = 1, \ldots, N$;

(ii) there exists an extremal signature $\{(z_j, \varepsilon_j), j=1, \ldots, N\}$ with respect to $\{\phi_j(z)\}$ such that

$$\operatorname{sgn}\{f(z_j) - p^*(z_j)/q^*(z_j)\} = \varepsilon_j \operatorname{sgn}(\overline{q^*(z_j)}), \quad j = 1, \ldots, N.$$

1.5 Characterisation of Best Rational Approximation (cont'd)

More recently, some results on the characterisation of locally best approximations have appeared, the first by Dunham [1965] who gives the following theorem as a special case of a more general theory of non-linear approximation. We first define a locally best approximation.

Definition 1.8

Denote by $R(E)$ the rational function

$$\frac{p(z)}{q(z)} = \frac{\sum_{k=0}^{m} a_k z^k}{\sum_{k=0}^{n} b_k z^k} \quad , \quad b_0 = 1 ,$$

where $E = (a_0, a_1, \ldots, a_m, b_1, \ldots, b_n)$ is taken from $\mathcal{E}$, a subset of complex $N(= n + m + 2)$ space.

Then $R(E^*)$ is a locally best approximation to f if

$$\| f - R(E^*) \| \leq \| f - R(E) \|$$

for all $E(\neq E^*)$ in a neighbourhood of E^*.

After defining the local Haar property at $E \in \mathcal{E}$ on B, and showing that this condition is satisfied for $R(E)$ provided $R(E)$ is non-degenerate, $a_n b_m \neq 0$ and the parameter space $\mathcal{E}$ is equal to $\{E : q(z) \neq 0 \text{ for } z \in B\}$, Dunham proves a result which may be written in the following form.

Theorem 1.10

A necessary and sufficient condition for $R(E^*) = p^*/q^*$ to be locally best to $f \notin R(E)$ on B is that for all $E \in \mathcal{E}$,

$$\min_{z \in D} \operatorname{Re} \{ \overline{(f(z) - p^*(z)/q^*(z))} \cdot (q^*(z)p(z) - p^*(z)q(z))/(q^*(z))^2 \} \leq 0 \qquad (1.5.1)$$

where $D = \{ z : |f(z) - p^*(z)/q^*(z)| = \| f - p^*/q^* \| \}$.

NOTE: This theorem has also been given by Ellacott and Williams [1976].

1.5 Characterisation of Best Rational Approximation (cont'd)

Suppose now that in Theorem 1.10, the set B is a finite point set and the extremal set D is $\{z_k\}_{k=1}^{M}$. (We make the usual assumption about the existence of a best approximation on such a set.)

Then (1.5.1) becomes

$$\min_{1 \le \ell \le M} \operatorname{Re}\left\{ \left(f_\ell - \frac{p^*_\ell}{q^*_\ell}\right)\left(\frac{q^*_\ell p_\ell - p^*_\ell q_\ell}{q^{*2}_\ell}\right)\right\} \le 0 \text{ for every } E \in E$$

or

$$\min_{1 \le \ell \le M} \operatorname{Re}\left\{\left(f_\ell - \frac{p^*_\ell}{q^*_\ell}\right)\left(\frac{p_\ell}{q^*_\ell} - \frac{p^*_\ell q_\ell}{q^{*2}_\ell}\right)\right\} \le 0 \text{ for every } E \in E \qquad (1.5.2)$$

where $p_\ell = p(z_\ell)$ etcetera.

Expanding p_ℓ and q_ℓ in (1.5.2), we see that

$$\min_{1 \le \ell \le M} \operatorname{Re}\left\{\left(f_\ell - \frac{p^*_\ell}{q^*_\ell}\right)\left(\frac{\sum_{k=0}^{m} a_k z^k}{q^*_\ell} - \frac{p^*_\ell}{q^{*2}_\ell}\sum_{k=0}^{n} b_k z^k\right)\right\} \le 0. \qquad (1.5.3)$$

Writing

$$u_{\ell k} = z_\ell^k (f_\ell - p^*_\ell/q^*_\ell)/q^*_\ell \quad , \quad (k = 0, 1, \ldots, m),$$

$$u_{\ell k+m} = z^k_\ell p^*_\ell (f_\ell - p^*_\ell/q^*_\ell)/q^{*2}_\ell \quad , \quad (k = 1, 2, \ldots, n),$$

and $c_k = a_k$, $(k = 0, 1, \ldots, m)$, $c_{k+m} = -b_k$, $(k = 1, 2, \ldots, n)$,

(1.5.3) becomes

$$\min_{1 \le \ell \le M} \operatorname{Re} \sum_{k=0}^{m+n} u_{\ell k} c_k \le 0$$

which by the lemma of section 1.4 implies the existence of real $\delta_\ell \ge 0$ $(\ell = 1, 2, \ldots, M)$, not all zero such that

$$\sum_{\ell=1}^{M} \delta_\ell u_{\ell k} = 0, \quad (k = 0, 1, \ldots, n+m).$$

1.5 Characterisation of Best Rational Approximation (cont'd)

Reverting to the original notation, this is equivalent to

$$\sum_{\ell=1}^{M} \delta_\ell z_\ell^{\,k} (f_\ell - p^*_\ell/q^*_\ell)/q^*_\ell = 0$$

and (1.5.4)

$$\sum_{\ell=1}^{M} \delta_\ell z_\ell^{\,k} p^*_\ell (f_\ell - p^*_\ell/q^*_\ell)/q^{*2}_\ell = 0 .$$

These equations may be restated in terms of an extremal signature. Define the set of functions $\{\psi_j(z)\}$ by

$$\{\psi_j(z)\} = \{\frac{1}{q^*(z)}, \frac{z}{q^*(z)}, \ldots, \frac{z^m}{q^*(z)}, z\frac{p^*(z)}{q^*(z)^2}, z^2\frac{p^*(z)}{q^*(z)^2} \ldots$$

$$\ldots z^n \frac{p^*(z)}{q^*(z)^2} \} . \qquad (1.5.5)$$

Theorem 1.11

If a rational function $p^*(z)/q^*(z)$ can be found such that $f(z) - p^*(z)/q^*(z)$ attains its maximum absolute value on the points of an extremal signature (with respect to $\{\psi_j\}$), with the indicated signa, then $p^*(z)/q^*(z)$ is a locally best rational approximation to $f(z)$.

Further discussion of the two forms of characterisation given by Theorems 1.9 and 1.11 will be deferred until Chapter 3, where it will be shown that there is a considerable computational advantage in Theorem 1.11.

We are now in a position to investigate how best approximations might actually be constructed, and this is the concern of the rest of this thesis.

CHAPTER 2

CONSTRUCTIVE METHODS FOR BEST POLYNOMIAL APPROXIMATION BASED ON CHARACTERISATION

2.1 Introduction

Having outlined the theoretical basis for approximation in the complex plane, we turn our attention to a discussion of existing algorithms based on characterisation, and the development of a new algorithm. We shall assume unless otherwise stated that the approximating polynomial is

$$p_n(z) = \sum_{k=1}^{n} a_k \phi_k(z) \qquad (2.1.1)$$

where the functions $\{\phi_k(z)\}_1^n$ form a Chebyshev set.

In the problem of approximating a real-valued function on an interval [a,b] of the real axis, it is well known that the best approximation has an error which exhibits alternate signs on a characteristic set of $n+1$ points. This forms the basis for the exchange algorithm of Remes for constructing best approximations. This is among the most powerful methods available for approximation; indeed in their comprehensive survey of methods for constructing minimax approximations, Lee and Roberts [1973] found that when the second algorithm of Remes was successful, no other method could compete with it in terms of computation time.

In the complex case, there are however two serious difficulties. The first is that the number of points r in the characteristic set is unknown, except that it satisfies the inequality

$$n+1 \leq r \leq 2n+1 \qquad (2.1.2)$$

2.1 Introduction (cont'd)

The second difficulty is that even if the value of r was known, the computation of an extremal signature when r exceeds $n+1$ does not seem to be possible. In view of this, we are generally forced to construct approximations on signatures of only $n+1$ points, whether or not this is justified by the theory. It is therefore necessary to investigate the implications of this assumption before proceeding further.

There are two ways of studying the effect of using an extremal signature based on a restricted number of points. It should be noted that the existence of the best approximation in such a case is guaranteed from Theorem 1.3 of Chapter 1.

Lemma 2.1

Let B be a compact set containing more than $n+1$ points, A any subset of B containing $n+1$ distinct points $\{z_k\}_1^{n+1}$ and p_A the polynomial such $f-p_A$ attains its maximum absolute value $h(A)$ on an extremal signature $\sum$ based on $\{z_k\}_1^{n+1}$. Define the set D as

$$D = \{z : |f(z) - p_A(z)| \leq h(A), \quad z \in B\} \tag{2.1.3}$$

Then p_A is the best approximation to f on D.

Proof

By the definition of D, $f-p_A$ attains its maximum absolute value on the points of an extremal signature $\sum$ and so the lemma follows by Theorem 1.8 of Chapter 1.

2.1 Introduction

Lemma 2.1

Proof (cont'd)

In practical examples, we hope that the set D might be similar to the set B, although D is not necessarily connected.

The second interpretation is very important for much of the work based on characterisation for it provides immediate *a posteriori* upper and lower bounds for the degree of approximation $E_n(f ; B)$. This is called an inclusion interval by Williams [1968].

Lemma 2.2

With the same notation as lemma 2.1,

$$h(A) \leq E_n(f ; B) \leq \max_{z \in B} |f(z) - p_A(z)| \qquad (2.1.4)$$

Proof

The l.h. inequality is proved by noting that $h(A) = E_n(f ; B)$ and that the degree of approximation on a subset cannot exceed that on the whole set. The r.h. inequality follows immediately from the definition of $E_n(f ; B)$.

Corollary

If

$$h(A) = \max_{z \in B} |f(z) - p_A(z)|, \quad = \| f - p_A \|$$

then p_A is the best approximation to f on B.

On the other hand, if the inclusion interval (2.1.4) is relatively small, say for example

$$\left| \frac{\| f - p_A \| - h(A)}{\| f - p_A \|} \right| < 0.1 \qquad (2.1.5)$$

then p_A may be regarded as an 'almost-optimal' approximation.

2.2 The Computation of Extremal Signatures

The equations (1.4.1) defining an extremal signature may be conveniently written as

$$\sum_{k=1}^{r} \delta_k e^{-i\theta_k} p(z_k) = 0 \quad \text{for every} \quad p \in P \tag{2.2.1}$$

where

$$\zeta_k = \delta_k e^{-i\theta_k} \tag{2.2.2}$$

and δ_k is a real positive number $(k = 1, 2, \ldots, r)$. The best approximation p^* is then determined from the equations

$$f(z_k) - p^*(z_k) = h(D) e^{i\theta_k} \quad (k = 1, 2, \ldots, r). \tag{2.2.3}$$

In view of the discussion earlier in the chapter, we shall assume unless otherwise stated that $r = n + 1$. Expressing (2.2.1) in the equivalent form

$$\sum_{k=1}^{n+1} \delta_k e^{-i\theta_k} \phi_s(z_k) = 0, \quad (s = 1, 2, \ldots, n+1), \tag{2.2.4}$$

$p(z_k)$

the Chebyshev condition for the system $\{\phi_s(z)\}_1^n$ ensures that there is a unique solution except for a non-zero factor, and that no δ_k is zero. Normalising the equations so that $\zeta_{n+1} = \delta_{n+1} e^{-i\theta_{n+1}} = 1$, the ζ_k, $(k = 1, 2, \ldots, n)$ are found from the equations

$$\begin{bmatrix} \phi_1(z_1) & \phi_1(z_2) & \cdots & \phi_1(z_n) \\ \phi_2(z_1) & \phi_2(z_2) & \cdots & \phi_2(z_n) \\ \vdots & \vdots & & \vdots \\ \phi_n(z_1) & \phi_n(z_2) & \cdots & \phi_n(z_n) \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \vdots \\ \zeta_n \end{bmatrix} = \ominus \begin{bmatrix} \phi_1(z_{n+1}) \\ \phi_2(z_{n+1}) \\ \vdots \\ \phi_n(z_{n+1}) \end{bmatrix} \tag{2.2.5}$$

$\Leftarrow$ 2.2.4

$$\phi_i(z_j) = e^{j(\omega_j \times T_i)}$$

2.2 The Computation of Extremal Signatures (cont'd)

NOTE 1

If z_{n+1} is allowed to vary while $z_1, z_2, \ldots, z_n$ remain fixed, the matrix factorisation needs only to be performed once, the ζ_k's being found merely by forward and backward substitutions.

NOTE 2

The choice of the $n+1$'th element of ζ for normalisation is of course quite arbitrary. If z_j is required to vary, then it is best to normalise by making $\zeta_j = 1$.

Another way of expressing the solution (Videnski [1956]), although not very suitable for computation, is to use Cramer's rule. The numbers $\delta_k e^{-i\theta_k}$ are proportional to the determinants

$$\Delta_k = (-1)^k \left| \phi_s(z_j) \right| , \quad (s = 1, 2, \ldots, n; \; j = 1, 2, \ldots, k-1, k+1, n+1) \qquad (2.2.6)$$

In certain special, and fortunately important cases, the above methods of solution may be considerably simplified. We shall consider two of the most useful types of approximation as illustrations.

(a) Approximation by Algebraic Polynomials

Here $\phi_k(z) = z^k , \quad (k = 0, 1, \ldots, n)$ (2.2.7)

NOTE: We have adopted the more usual convention of letting the suffix start at 0, so that the space P now has dimension $n+1$. Accordingly r now satisfies the inequality

$$n+2 \leq r \leq 2n+3 \qquad (2.2.8)$$

and the restricted signature contains $n+2$ points.

Using Lagrange's interpolation formula for an arbitrary polynomial $p_n(z)$ of degree n at the points $z_1, z_2, \ldots, z_{n+2}$ we have

2.2 The Computation of Extremal Signatures

(a) Approximation by Algebraic Polynomials (cont'd)

$$p_n(z) = \sum_{k=1}^{n+2} \frac{p_n(z_k)Q(z)}{Q'(z_k)(z-z_k)} \qquad (2.2.9)$$

where $Q(z) = \prod_{\nu=1}^{n+2} (z - z_\nu)$.

Equating coefficients of z^{n+1} on each side of equation (2.2.9) gives

$$0 = \sum_{k=1}^{n+2} \frac{p_n(z_k)}{Q'(z_k)} \quad . \qquad (2.2.10)$$

Suppose $\frac{1}{Q'(z_k)} = \delta_k e^{-i\theta_k}$, $(r = 1, \ldots, n+2)$. $\qquad (2.2.11)$

Then we can write (2.2.10) in the form

$$\sum_{k=1}^{n+2} \delta_k e^{-i\theta_k} p_n(z_k) = 0$$

for all polynomials of degree n, which is exactly the property (2.2.1) of the extremal signature in this case. Since $Q'(z_k)$ is just

$$\prod_{\substack{\nu=1 \\ \nu \neq k}}^{n+2} (z_k - z_\nu) \qquad (2.2.12)$$

this provides a rapid computation of the extremal signature with $O(n^2)$ operations. By expanding the determinants Δ_k in (2.2.6) we can also show that

$$\frac{1}{Q'(z_k)} = c\,\Delta_k \quad (k = 1, 2, \ldots, n+2) \qquad (2.2.13)$$

where

$$c = \prod_{\nu=1}^{n+1} \prod_{s=\nu+1}^{n+2} (z_\nu - z_s).$$

2.2 The Computation of Extremal Signatures

(a) Approximation by Algebraic Polynomials (cont'd)

This is the one case where we can make at least some progress when $r > n+2$. If $U(z)$ denotes an arbitrary polynomial of degree $r-n-2$, then again using Lagrange's formula,

$$U(z)p_n(z) = \sum_{k=1}^{r} \frac{U(z_k)p_n(z_k)Q(z)}{Q'(z_k)(z-z_k)} \tag{2.2.14}$$

where

$$Q(z) = \prod_{\nu=1}^{r} (z-z_\nu).$$

Equating coefficients of z^{n+r-1} in (2.2.14), we have

$$0 = \sum_{k=1}^{r} \frac{U(z_k)p_n(z_k)}{Q'(z_k)} \tag{2.2.15}$$

which can again be written as

$$\sum_{k=1}^{r} \delta_k e^{-i\theta_k} p_n(z_k) = 0. \tag{2.2.16}$$

This time however,

$$\delta_k e^{-i\theta_k} = \frac{U(z_k)}{Q'(z_k)}$$

and $U(z)$ would have to be determined in conjunction with the (overdetermined) equations

$$f(z_k) - p(z_k) = h(D)e^{i\theta_k} \quad (k = 1, 2, \ldots, r). \tag{2.2.17}$$

The resulting equations for p and U are non-linear, and no progress has yet been made on their solution, even when the value of r is known.

2.2 The Computation of Extremal Signatures (cont'd)

(b) Approximation by a Polynomial-Rational Sum

Here

$$\{\phi_j(x)\}_1^n = \left\{ 1, z, z^2, \ldots, z^m, \frac{1}{z-\alpha_1}, \frac{1}{z-\alpha_2}, \frac{1}{z-\alpha_t} \right\} \qquad (2.2.17)$$

where $n = m + t + 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_t$ are fixed numbers Using the determinantal equations (2.2.6), $\delta_k e^{-i\theta_k}$ is proportional to Δ_k where

$$\Delta_k = \begin{vmatrix} 1 & z_1 & \cdots & z_1^m & \frac{1}{z_1-\alpha_1} & \cdots & \frac{1}{z_1-\alpha_t} \\ 1 & z_2 & \cdots & z_2^m & \frac{1}{z_2-\alpha_1} & \cdots & \frac{1}{z_2-\alpha_t} \\ \vdots & & & & & & \\ 1 & z_{k-1} & \cdots & z_{k-1}^m & \frac{1}{z_{k-1}-\alpha_1} & \cdots & \frac{1}{z_{k-1}-\alpha_t} \\ 1 & z_{k+1} & \cdots & z_{k+1}^m & \frac{1}{z_{k+1}-\alpha_1} & \cdots & \frac{1}{z_{k+1}-\alpha_t} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 1 & z_{n+1} & \cdots & z_{n+1}^m & \frac{1}{z_{n+1}-\alpha_1} & \cdots & \frac{1}{z_{n+1}-\alpha_t} \end{vmatrix} \qquad (2.2.18)$$

By successive subtraction of the columns containing rational terms, it is not difficult to show that

$$\Delta_k = \frac{\prod_{s=2}^{t} \prod_{r=1}^{s-1} (\alpha_r - \alpha_s) \times \det(z_\nu^j)}{\prod_{\substack{r=1 \\ r\neq k}}^{n} \prod_{s=1}^{t} (z_r - \alpha_s)}$$

$$(j = 0, 1, \ldots, n-1, \nu = 1, 2, \ldots, k-1, k+1, \ldots, n+1)$$

2.2 The Computation of Extremal Signatures

(b) Approximation by a Polynomial-Rational Sum (cont'd)

or

$$\Delta_k = \frac{C}{Q'(z_k)} \prod_{s=1}^{t} (z_k - \alpha_s) \qquad (2.2.19)$$

where C is a number independant of k. Hence we can in this case also compute the extremal signature in a simple manner without resorting to the solution of equations (2.2.5).

NOTE

The expression for Δ_k is unchanged if instead of $\phi_j(z) = z^j$, $\phi_j(z)$ is any polynomial of degree j.

This follows simply from the elementary properties of determinants.

2.3 Polynomial Approximation on an Interval of the Imaginary Axis

Dierieck and Kamp [1971] have proposed an ingenious algorithm for approximation by algebraic polynomials along the imaginary axis based on characterisation. This is of practical importance for the approximation of a complex transfer function. The set B for simplicity has been taken as

$$B = \{ i\omega : 0 \leq \omega \leq 1 \}.$$

A polynomial approximation for $f(i\omega)$ has to be found of the form

$$p(i\omega) = \sum_{k=1}^{n} a_k (i\omega)^k .$$

Let the characteristic set D be $\{i\omega_k\}^{n+2}$, $\omega_1 < \omega_2 < \ldots < \omega_{n+2}$, where in common with Dierieck and Kamp we have made the usual restriction on the number of points in D. Then

$$\prod_{\substack{\nu=1 \\ \nu \neq k}}^{n+2} (i\omega_k - i\omega_\nu) = (i)^{n+1}(-1)^{n+2-k} = C(-1)^k$$

where C is independent of k, and thus the numbers ζ_k are proportional to $(-1)^k$. Therefore

$$\varepsilon_k = (-1)^k e^{i\theta^*}$$

where θ^* is an unknown.

Equations (2.2.3) defining the best approximation can now be written as

$$\sum_{\nu=0}^{n} a_\nu (i\omega_k)^\nu - f(i\omega_k) = (-1)^k h(D) e^{i\theta^*}, \quad (k=1, \ldots, n+2) \qquad (2.3.1)$$

Define the *projection operators* P_1^θ and P_2^θ by

$$\begin{bmatrix} P_1^\theta \phi \\ P_2^\theta \phi \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} R_e \phi \\ I_m \phi \end{bmatrix} \qquad (2.3.2)$$

2.3 Polynomial Approximation on an Interval of the Imaginary Axis (cont'd)

The projection of equations (2.3.1) with $\theta = \theta^*$ yields

$$P_1^{\theta^*} p(i\omega_k) - P_1^{\theta^*} f(i\omega_k) = (-1)^k h(D)\, P_1^{\theta^*} e^{i\theta^*} = (-1)^k h(D) \qquad (2.3.3)$$

$$P_2^{\theta^*} p(i_{\ k}) - P_2^{\theta^*} f(i_{\ k}) = (-1)^k h(D)\, P_2^{\theta^*} e^{i\theta^*} = 0. \qquad (2.3.4)$$

Now $P_1^{\theta} p(i\omega)$ is a polynomial $\sum_{\nu=0}^{n} \alpha_\nu \omega^\nu$ with real coefficients,

and $P_1^{\theta} f(i\omega)$ is a real function of the (real) variable ω.

Moreover from (2.3.3)

$$\left| P_1^{\theta^*} p(i\omega_k) - P_1^{\theta^*} f(i\omega_k) \right| = h(D), \quad \omega_k \in D$$

and

$$\left| P_1^{\theta^*} p(i\omega) - P_1^{\theta^*} f(i\omega) \right| = \left| R_e\, e^{-i\theta^*} (p(i\omega) - f(i\omega)) \right|$$

$$\leq \left| e^{i\theta} (p(i\omega) - f(i\omega)) \right| = \left| p(i\omega) - f(i\omega) \right|$$

$$\leq h(D) \quad \text{for all } i\omega \in B. \qquad (2.3.5)$$

It follows that $P_1^{\theta^*} p$ is the best Chebyshev approximation to $P_1^{\theta^*} f$; since this is just a real approximation problem we can apply the second Remes' algorithm (for example) and obtain the coefficients α_ν and, which is more important, the characteristic set D. Thus, if we can find the value of θ^*, we have essentially solved the approximation problem on the imaginary axis.

Dierieck and Kamp determine θ^* by using equation (2.3.4) for which the right hand side, by the uniqueness of the best approximation only vanishes when $\theta = \theta^*$. The equation is solved by the Regula Falsi. Numerical experience shows this to be the least satisfactory part of their algorithm, the slow convergence necessitating rather a large number of real approximation problems. We conclude our study of their method by showing that θ^* can be found quite simply by successive substitution.

2.3 Polynomial Approximation on an Interval of the Imaginary Axis (cont'd)

Theorem 2.1

Let D_v be a set of $n+2$ distinct points of B, an approximation to the characteristic set D for the best approximation p^* of f on B. Let the degree of approximation to f on D_v be $h(D_v)$ with a corresponding value of $\theta = \theta_v$ in (2.3.1). If the best approximation of $P_1^{\theta_v} f(i\omega)$ by a real polynomial has a degree of approximation equal to λ_{v+1} with characteristic set D_{v+1}, then

(i) $\{h(D_v)\}$ is an increasing sequence;

(ii) $\{h(D_v)\}$ is bounded above;

(iii) $h(D_v) \to h(D)$ and $D_v \to D$ as $v \to \infty$.

Proof:

On the set D_v we have

$$p(i\omega_k) - f(i\omega_k) = (-1)^k h(D_v) e^{i\theta_v}, \quad i\omega_k \in D_v \tag{2.3.6}$$

and by projecting the equations

$$P_1^{\theta_v} p(i\omega_k) - P_1^{\theta_v} f(i\omega_k) = (-1)^k h(D_v).$$

By the theorem of de la Vallee Poussin [1919] for real approximation

$$h(D_v) < \lambda_{v+1}. \tag{2.3.7}$$

On the set D_{v+1}, we then have

$$P_1^{\theta_v} p(i\omega_k) - P_1^{\theta_v} f(i\omega_k) = (-1)^k \lambda_{v+1}, \; i\omega_k \in D_{v+1} \tag{2.3.8}$$

whereas

$$p(i\omega_k) - f(i\omega_k) = (-1)^k h(D_{v+1}) e^{i\theta_{v+1}}, \; i\omega_k \in D_{v+1}. \tag{2.3.9}$$

Projecting (2.3.9) gives

$$P_1^{\theta_v} p(i\omega_k) - P_1^{\theta_v} f(i\omega_k) = (-1)^k h(D_{v+1}) P_1^{\theta_v} e^{i\theta_{v+1}}. \tag{2.3.10}$$

2.3 Polynomial Approximation on an Interval of the Imaginary Axis (cont'd)

Theorem 2.1

Proof: (cont'd)

Comparing (2.3.8) with (2.3.10) we see that

$$\lambda_{v+1} = h(D_{v+1})\, P_1^{\theta_v} e^{i\theta_{v+1}}$$

or

$$\lambda_{v+1} = h(D_{v+1}) \cos(\theta_v - \theta_{v+1}) \qquad (2.3.11)$$

Therefore

$$\lambda_{v+1} \leq h(D_{v+1}). \qquad (2.3.12)$$

Combining (2.3.7) and (2.3.12) we have

$$h(D_{v+1}) \geq h(D_v) \qquad (2.3.13)$$

which proves (i).

To prove (ii), use (2.1.4) to give

$$h(D_v) \leq E(f; B).$$

Thus the sequence $\{h(D_v)\}$ is convergent.

Finally, project (2.3.9) using $P_2^{\theta_v}$, giving

$$P_2^{\theta_v} p(i\omega_k) - P_2^{\theta_v} f(i\omega_k) = (-1)^k h(D_{v+1}) P_2^{\theta_v} e^{i\theta_{v+1}}$$

$$= (-1)^k h(D_{v+1}) \sin(\theta_v - \theta_{v+1}) \qquad (2.3.14)$$

Since from (2.3.7) and (2.3.12),

$$h(D_v) \leq \lambda_{v+1} \leq h(D_{v+1}),$$

$h(D_v)$ and λ_v tend to the same limit as $v \to \infty$, and so (2.3.11) implies that θ_v tends to a limit θ^*, the right hand side of (2.3.14) therefore $\to 0$ as $v \to \infty$, which as we have seen in the discussion of Dˇerieck and Kamp's algorithm only vanishes when p is the (unique) best approximation to f on B. This concludes the proof.

Theorem 2.1 (cont'd)

To illustrate this theorem we have computed some of the approximations used by Dierieck and Kamp.

Example 1

The approximation of $e^{i\omega\tau}$ for $\omega \in [0,1]$ with $\tau = 2.7732$. Table 1 gives the values of $h(D)$ and $\| f - p^* \|$, and the final values of θ. As a comparison, the corresponding results of Dierieck and Kamp are given alongside.

Table 1

Degree	h(D)	$\| f - p^* \|$	θ^* in Degrees	h(D) D and K values	θ^*
3	0.0174 7956 7	0.0174 7956 8	79.4463 28°	0.0174 793	79.4463 3°
4	0.0024 6331 8	0.0024 6331 8	-10.5536 72°	0.0024 6133	-10.5536 8°
5	0.0002 8794 3	0.0002 8794 3	-100.5536 7 °	0.0002 8778	-100.5536 8°

The closeness of $h(D)$ and $\| f - p^* \|$ suggests that we have obtained the best approximation in each case, and therefore the assumption that D contains $n+2$ points is justified in these examples. There is no apparent reason for the discrepancies in the values of $h(D)$.

To illustrate the monotone converge of $\{h(D_v)\}$, the values of $h(D_v)$ and θ_v are given in Table 2 for $n = 4$. The iteration is initiated with equally spaced values for D_0.

Table 2

v	$h(D_v)$	θ_v
1	0.0015 3815 79	-10.5536 72
2	0.0024 6331 80	-10.5536 71
3	0.0024 6331 80	-10.5536 71

2.3 Polynomial Approximation on an Interval of the Imaginary Axis

Theorem 2.1

Example 1 (cont'd)

The convergence is very fast and is probably second order, although no attempt has been made to prove this.

Example 2

The approximation of $(0.5 + \tan i\omega)/(2 + \tan i\omega)$ for $\omega \in [0,1]$. Table 3 gives the values of $h(D)$ and $\| f - p^* \|$, and the final values of θ.

Table 3

Degree	h(D)	$\| f - p^* \|$	θ^* Degrees	h(D) D and K values	θ^*
2	0.794259 E-2	0.794259 E-2	-31.9064	0.794248 E-2	-31.9541
3	0.190817 E-2	0.190817 E-2	121.0050	0.190702 E-2	121.2691
4	0.393588 E-3	0.393588 E-3	-88.0597	0.393390 E-3	-91.1165

Once again there is a small discrepancy between our values and those of Dierieck and Kamp; this may be due to their using insufficient iterations of the regula falsi iteration which has only first order convergence.

2.4 Polynomial Approximations on more general regions of the Complex Plane

Consider the approximation of a function f analytic on a domain S bounded by a piecewise-smooth Jordan curve C. In this form, the only method based on characterisation is that of Williams [1968].

We work entirely with a characteristic set E containing $n+2$ points $z_1, z_2, \ldots, z_{n+2}$ and we look for the set E* which maximises $|h(E)|$ over all $E \in C$. The function $|h(E)|$ is maximised by the simple method of performing a sequence of univariate maximisations with respect to each of $z_1, \ldots, z_{n+2}$ in turn, a method which was proposed by Curtis and Frank [1959] in the real case. To be of practical use, we need efficient methods for maximising $|h(z)|$ where $E = \{z_1, \ldots, z_{j-1}, z, z_{j+1}, \ldots z_{n+2}\}$. The function $h(z)$ is defined by equations (2.2.3). Fortunately, for algebraic polynomials $h(z)$ can be computed independently of equations (2.2.3) which only need to be solved once to determine the final approximation $p^*(z)$.

Solving (2.2.3) by Cramer's rule gives

$$h(E) = \frac{f_1\Delta_1 + f_2\Delta_2 + \ldots + (-1)^{n+2} f_{n+2}\Delta_{n+2}}{e^{i\theta_1}\Delta_1 + e^{i\theta_2}\Delta_2 + \ldots + e^{i\theta_{n+2}}\Delta_{n+2}} . \qquad (2.4.1)$$

But $\arg\Delta_k = -\theta_k$ and so $\Delta_k = |\Delta_k| e^{-i\theta_k}$. Thus

$$h(E) = \frac{f_1\Delta_1 + f_2\Delta_2 + \ldots + (-1)^{n+2} f_{n+2}\Delta_{n+2}}{|\Delta_1| + |\Delta_2| + \ldots + |\Delta_{n+2}|} . \qquad (2.4.2)$$

For algebraic polynomials, we have from (2.2.13) that $\Delta_k = \Delta/(Q'(z_k))$ and so

2.4 Polynomial Approximations on more general regions of the Complex Plane (cont'd)

$$|h(E)| = \frac{|f_1/Q'(z_1) + f_2/Q'(z_2) + \ldots + f_{n+2}/Q'(z_{n+2})|}{|Q'(z_1)| + \ldots\ldots\ldots + |Q'(z_{n+2})|}$$

For other sets $\{\phi_k(z)\}$ of approximating functions, we may have to resort to the solution of equations (2.2.5) for the extremal signature, although rearranging the equations so that z_j is the last equation when maximising with respect to z_j gives some economisation in operations.

We now describe a multiple-exchange algorithm which avoids the rather inefficient univariate maximisations.

2.5 Polynomial approximation on the unit disc

The algorithm will first be presented in its simplest form. Let $f(z)$ be a function such that $f(\bar{z}) = \overline{f(z)}$. Since C is symmetric about the real axis, the characteristic set D is also symmetric about the real axis, and we therefore only consider sets E of $n+2$ points which have the same property. This implies that all the approximations have real coefficients and that $\theta^* = 0$. (Williams [1968]). The important cases when $p(z)$ is either an algebraic polynomial or a polynomial-rational sum have the interesting property that θ_k depends only on z_k and is independent of the other points of E. This leads to an ascent algorithm which does not involve the explicit maximisation of $h(E)$. Let us first verify our assertion about θ_k.

(not proved) Let $z_1, z_2, \ldots, z_{n+2}$ (where $z_k = e^{i\alpha_k}$) be arranged on the unit disc so that $\alpha_1 < \alpha_2 < \ldots\ldots \alpha_{n+2}$.

2.5 Polynomial approximation on the unit disc (cont'd)

Then

$$Q'(z_k) = \prod_{\substack{\ell = 1 \\ \ell \neq k}}^{n+2} (e^{i\alpha_k} - e^{i\alpha_\ell})$$

$$= \prod_{\substack{\ell = 1 \\ \ell \neq k}}^{n+2} 2i \sin \frac{(\alpha_k - \alpha_\ell)}{2} \{\cos \frac{\alpha_k + \alpha_\ell}{2} + i \sin \frac{\alpha_k + \alpha_\ell}{2}\}$$

and so

$$\arg Q'(z_k) = (n+1)\frac{\pi}{2} + k\pi + \sum_{\substack{\ell = 1 \\ \ell \neq k}}^{n+2} \frac{\alpha_k + \alpha_\ell}{2}$$

$$= \frac{n\alpha_k}{2} + (n+1)\frac{\pi}{2} + k\pi, \qquad (2.5.1)$$

which depends only on α_k.

Theorem 2.2

Let $E = \{ z_k = e^{i\alpha_k} \}_{k=1}^{n+2}$ be a set of $n+2$ points on the unit circle symmetric about the real axis, and let f be an analytic function such that $f(\bar{z}) = \overline{f(z)}$. Let $p(z)$ be the best approximation to f on E with $r(z) = f(z) - p(z)$ and with degree of approximation $h(E)$. Define z_k' as follows for $k = 1, 2, \ldots, n+2$:

(i) $s = \operatorname{sgn}\{ e^{-i\theta_k} r(z_k) \}$;

(ii) α_k' is the value of α which maximises

$F(\alpha) = \operatorname{Re}\{ s e^{-i\theta} r(z) \}$

in the neighbourhood $(\alpha_{k-1}, \alpha_{k+1})$ of α_k where θ is based on $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \ldots \alpha_{n+2}\}$ and $z = e^{i\alpha}$;

(iii) $z_k' = e^{i\alpha_k'}$.

2.5 Polynomial approximation on the unit disc

Theorem 2.2 (cont'd)

If $E' = \{ z_1', z_2', \ldots, z_{n+2}' \}$, then

(a) E' consists of $n+2$ distinct points;

(b) $h(E') \geq h(E)$.

Proof

The equations defining $h(E)$ are from (2.2.3)

$$f(z_k) - p(z_k) = th(E)e^{i\theta_k}, \quad (k = 1, 2, \ldots, n+2)$$

where $t = \pm 1$ is included to ensure $h(E) > 0$ ($h(E)$ is real in this case). Therefore

$$e^{-i\theta_k} r(z_k) = th(E)$$

and so $s = t$. Also

$$F(\alpha) = h(E) \quad \text{at} \quad \alpha = \alpha_k .$$

Moreover, because of the term $k\pi$ in (2.5.1), $F(\alpha)$ at any point α in the interval $\alpha_k \leq \alpha \leq \alpha_{k+1}$ considered as a neighbourhood of α_k has the same magnitude but opposite sign to $F(\alpha)$ considered as a neighbourhood of α_{k+1}. This precludes the possibility of α_k' and α_{k+1}' coinciding, and a similar argument shows that α_k' cannot coincide with α_{k-1}'. This proves assertion (a).

The equations defining $h(E')$ are

$$f(z_k) - p'(z_k') = h(E')s'e^{i\theta_k}, \quad (k = 1, 2, \ldots, n+2)$$

where θ_k' is based on $\{\alpha_1', \alpha_2', \ldots, \alpha_{n+2}'\}$. Defining Δ_k as the determinant $(-1)^k \left| z_j^s \right|$ $(s = 1, \ldots, n; j = 1, \ldots k-1, k+1, \ldots, n+2)$, we have from (2.4.2)

2.5 Polynomial approximation on the unit disc

Theorem 2.2

Proof (cont'd)

$$h(E') = s' \sum_{k=1}^{n+2} f(z_k')\Delta_k' \Big/ \sum_{k=1}^{n+1} |\Delta_k'| \quad \checkmark$$

$$= s' \sum_{k=1}^{n+2} r(z_k')\Delta_k' \Big/ \sum_{k=1}^{n+2} |\Delta_k'| \quad *$$

since $h(E')$ is zero for any polynomial of degree n by definition. Using the facts that $\arg \Delta_k' = \theta_k'$ and $h(E')$ is real,

$$h(E') = s' \sum_{k=1}^{n+2} r(z_k')|\Delta_k'|e^{i\theta_k'} \Big/ \sum_{k=1}^{n+2} |\Delta_k'|$$

$$= s' \operatorname{Re} \sum_{k=1}^{n+2} r(z_k')\, e^{i\theta_k'}|\Delta_k'| \Big/ \sum_{k=1}^{n+2} |\Delta_k'| .$$

But from condition (ii) of the theorem,

$$\operatorname{Re}\{r(z_k')\, e^{i\theta_k'}\} \geq h(E),$$

and so $s' = 1$ and

$$h(E') \geq h(E).$$

This completes the proof.

Note

There is obviously no need to locate the precise maxima of $\operatorname{Re}\{s\, e^{-i\phi} r(e^{i\alpha})\}$ at every step of the algorithm.

2.5 Polynomial approximation on the unit disc

Theorem 2.2 (cont'd)

To illustrate theorem 2.2, several approximations have been computed. This algorithm, when applicable, gives much smaller inclusion intervals than Williams' method, and the successive values of $h(E)$ have the characteristic rapid convergence of the second Remes' algorithm which is known to be quadratic for sufficiently smooth functions (see for example Veidinger [1960]).

When the precision of the computation (37 binary digits) and also the approximate location of the maxima of the error term (obtained by subtabulation on a set of 1800 points on the unit circle and a subsequent parabolic fit), the closeness of $h(D)$ and $\|f - p^*\|$ leads us to believe that in all the examples considered, the characteristic set contains $n+2$ points. Moreover we conjecture that this is true for all approximation on the unit disc. This has not been proved, although no counter-example has been found.

The results of Williams [1968] are given as a comparison. The examples indicate that the size of his final inclusion intervals are not due to the characteristic set containing insufficient points.

2.5 Polynomial approximation on the unit disc

Example 1

The approximation of e^z on the unit disc

Degree n	Initial Set		Final Set		Number of iterations	Williams' result	
	h(E)	$\|\| f - p_n \|\|$	h(E)	$\|\| f - p_n \|\|$		h(E)	$\|\| f - p_n \|\|$
4	0.833 E-2	1.115 E-2	0.8568 659 E-2	0.8568 665 E-2	4	0.857 E-2	0.898 E-2
6	0.198 E-3	0.251 E-3	0.2015 46 E-3	0.2015 48 E-3	3	0.201 E-3	0.212 E-3
8	0.276 E-5	0.334 E-5	0.2783 5 E-5	0.2783 8 E-5	3	0.278 E-3	0.303 E-5
10	0.253 E-7	0.292 E-7	0.25 E-7	0.26 E-7	3	0.252 E-7	0.268 E-7

2.5 Polynomial approximation on the unit disc (cont'd)

One example will serve to illustrate the convergence of the algorithm.

Example 2

The approximation of e^z on the unit disc by a polynomial of degree 4

Iteration	h(E)
1	0.8333 3584 E-2
2	0.8568 4784 E-2
3	0.8568 6590 E-2
4	0.8568 6590 E-2

The final approximation is

$$p_4^*(z) = 1.0000\,0000\,6 + 0.9999\,9986\,40\,z + 0.4999\,9717\,z^2 + 0.1666\,3267\,07\,z^3 + 0.0430\,8345\,09\,z^4 .$$

Example 3

The approximation of $\ln(1+z/2)$ on the unit disc

n	Final Characteristic Set		Number of Iterations
	h(E)	$\|\| f - p_n \|\|$	
4	0.7592 0765 E-2	0.7592 0765 E-2	4
6	0.1382 8623 E-2	0.1383 8636 E-2	4

For the function $1/(z-a)$, there are analytic expressions known for the best approximation and its error (see Example 2 of Section 6.1) which can be used as a comparison with the numerical results.

2.5 Polynomial approximation on the unit disc (cont'd)

Example 4

Approximation of $1/(z-2)$ on the unit disc

n	Final Characteristic Set		Williams' results		Exact $E_n(f)$
	h(E)	$\|f - p_n\|$	h(E)	$\|f - p_n\|$	
2	0.8333 33 E-1	0.8333 46 E-1	0.833 E-1	0.859 E-1	0.8333 33 E-1
4	0.2083 33 E-1	0.2083 35 E-1	-	-	0.2083 33 E-1
6	0.5208 33 E-2	0.5208 37 E-2	0.521 E-2	0.527 E-2	0.4208 33 E-2
8	0.1302 08 E-2	0.1302 10 E-2	0.1302 E-2	0.1314 E-2	0.1302 08 E-2

Example 5

Approximation of $\ln(1 + z/2)$ on the unit disc by a polynomial rational sum of the form

$$p_n(z) = \sum_{k=0}^{n-1} a_k z^k + a_n/(z+2)$$

Degree n	Initial Set		Final Set	
	h(E)	$\|f - p_n\|$	h(E)	$\|f - p_n\|$
4	0.147 E-2	0.491 E-2	0.2130 12 E-2	0.2130 33 E-2
6	0.175 E-3	0.698 E-3	0.2684 40 E-3	0.2684 42 E-3

The coefficients of $p_6(z)$ are listed below

2.5 Polynomial approximation on the unit disc

Example 5 (cont'd)

k	a_k
0	0.1543 5399 63
1	0.4228 2300 88
2	-0.0864 1152 67
3	0.0223 7250 24
4	-0.0059 7817 36
5	0.0014 7640 67
6	-0.3087 0799 83

A comparison with approximations by algebraic polynomials of the same degree given in example three shows the advantage of including a rational term when the function being approximated has a singularity near the unit disc.

One interesting feature of approximation on the disc is the remarkable uniformity of the error function on the boundary of the disc. We conclude with two examples of this phenomenon.

Example 6

For all $z = e^{i\theta}$, $0 \leq \theta \leq 2\pi$ we have

(a) $0.856865\,E\text{-}2 \leq |\, e^z - p_4^*(z) \,| \leq 0.856866\,E\text{-}2$

(b) $0.138381\,E\text{-}2 \leq |\, \ln(1 + z/2) - p_6^*(z) \,| \leq 0.138386\,E\text{-}2$

2.6 Polynomial Approximation on regions other than a disc

The proof of the convergence of the algorithm in section 2.5 is only valid for the disc, since otherwise θ_k is not solely dependent on z_k. Nevertheless the algorithm is immediately applicable to any of the regions in section 2.4 (provided that z_k is redefined as a boundary point of the relevant region), and all the numerical results show convergence properties similar to those for the disc. For such regions, the inclusion intervals are a little smaller than those of Williams for corresponding problems, but the main factor governing the sizes of the inclusion intervals is the number of points in the true characteristic set exceeding n + 2.

Example 1

Approximation of e^z on the unit semidisc
$\{ |z| \leq 1, \quad \mathrm{Re}\, z \geq 0 \}$

n	Initial Set		Final Set		Williams' Results	
	h(E)	$\|f - p_n\|$	h(E)	$\|f - p_n\|$	h(E)	$\|f - p_n\|$
4	0.333E-2	0.587E-2	0.346E-2	0.451E-2		
6	0.393E-4	0.997E-4	0.504E-4	0.528E-4	0.475E-4	0.739E-4

Example 2

Approximation of $1/\sqrt{1+2z}$ on the unit semidisc
by a polynomial of degree 6

Initial Set		Final Set		Williams' Results	
h(E)	$\|f - p_6\|$	h(E)	$\|f - p_6\|$	h(E)	$\|f - p_6\|$
0.154E-1	0.345E-1	0.168E-1	0.177E-1	0.163E-1	0.232E-1

2.6 Polynomial Approximation on regions other than a disc (cont'd)

Example 3

Approximation of e^z on a 'standard' ellipse with semimajor axis = $\sqrt{2}$ and semiminor axis = 1.

The approximations are compared with those of Geddes [1973] using a Chebyshev projection $C_n f$ (see chapters 6 and 7).

n	Initial Set		Final Set		Geddes' Results
	h(E)	$\|f - p_n\|$	h(E)	$\|f - p_n\|$	$\|f - C_n f\|$
2	0.3086 E 0	0.4806 E 0	0.3402 E 0	0.3441 E 0	0.43 E 0
4	0.2198 E-1	0.3365 E-1	0.2317 E-1	0.2320 E-1	0.28 E-1
6	0.7563 E-3	0.1118 E-2	0.7820 E-3	0.7822 E-2	0.90 E-3
8	-	-	0.1560 E-4	0.1561 E-4	0.17 E-4

In Chapter four of the thesis, some of the approximations in this section will be compared with the best approximation, and the restricted characteristic sets compared with the true characteristic sets.

CHAPTER 3

CONSTRUCTIVE METHODS FOR BEST RATIONAL APPROXIMATION BASED ON CHARACTERISATION

3.1 Introduction

The construction of best rational approximations is the most difficult yet often the most useful of the problems within the scope of this thesis. Throughout this chapter we shall restrict ourselves to the case when the rational approximation r_{mn} is the quotient

$$r_{mn}(z) = \sum_{k=0}^{m} a_k z^k / (1 + \sum_{k=1}^{n} b_k z^k), \quad N = n + m + 2 \tag{3.1.1}$$

where $a_m b_n = 0$ and r_{mn} is bounded on a compact subset B of the z-plane on which we require the best approximation to some analytic function f. The only significant work concerned with methods based on characterisation seems to be that of Williams [1972] using the characterisation of Theorem 1.9. The principle of his method is similar to that of his method for polynomial approximation in that we work on finite subsets E of N points and try to maximise the degree of approximation h(E) over all possible subsets of B. This is achieved, or rather attempted, by making univariate maximisations with respect to each point of E. The extremal signature is constructed from equations (2.2.5) resulting in

$$f(z_k) - p(z_k)/q(z_k) = h(E)e^{i\phi k}\, \mathrm{sgn}(\overline{q(z_k)}) \quad (k = 1, 2, .., N). \tag{3.1.2}$$

These equations are solved iteratively by taking an initial approximation to the denominator q(z) in the right hand side, and solving the resulting equations, which can be posed as an eigenvalue problem, by inverse iteration. This "inner iteration" is the same as that performed in real approximation using the second

3.1 Introduction (cont'd)

Remes algorithm. The resulting function $q(z)$ is then used in the right hand side of the next iteration, and the method continues until it converges.

The inclusion interval

$$h(E) \leq E_{mn}(f\,;B) \leq \max_{z\in B} \left| f(z) - r_{mn}(z) \right|$$

still holds since the proof of the corresponding result (2.1.4) for polynomials is still valid for rational approximation. This enables us to judge at least a posteriori the success or otherwise of the method in any particular application.

There are three principal difficulties with the algorithm.

(a) It makes the assumption that the characteristic set has N points.

(b) There appears to be no simple method for computing $h(E)$ without solving equations (3.1.2) explicitly as there is in the polynomial case using (2.4.2). Indeed, since we need $q(z)$ for the next step of the iteration, it is unlikely that the complete solution of (3.1.2) can be avoided.

(c) Even when $r = N$, the maximising process is not always very effective in minimising $\| f - r_{mn} \|$.

There seems to be no answer to the first objection. We have already discussed in the polynomial case the difficulties of dealing with extremal signatures based on more than the minimum number of points. Although Williams does compute "best" approximations in cases where the characteristic set has more than N points, particularly on the semidisc, such problems are much more effectively dealt with by

3.1 Introduction (cont'd)

the methods of Chapter 5. We have therefore restricted our discussion mainly to the disc, where we conjecture that N-points are always sufficient for the characteristic set. Fortunately, it is precisely in this case that an alternative algorithm is possible which overcomes problems (c) and (d) above.

Before presenting this algorithm, we return to the alternative form of the characterisation given in Theorem 1.11 which leads to a simpler method for calculating the extremal signature than Theorem 1.9, for which the solution of equations (2.2.5) is the only possibility.

3.2 The Computation of the Extremal Signature

Using Theorem 1.11, the best rational approximation on a set of N-points may be determined from the equations

$$f(z_k) - p^*(z_k)/q^*(z_k) = h(D)e^{i\theta_k} \quad (k = 1, 2, \ldots, N) \tag{3.2.1}$$

where the θ_k are defined by

$$\sum_{k=1}^{N} \delta_k e^{-i\theta_k} \psi_s(z_k) = 0, \quad (s = 1, 2, \ldots, m+n+1) \tag{3.2.2}$$

As in the case of approximation by algebraic polynomials, the equations (3.2.2) can be solved explicitly because of the special form of the set $\{\psi_j(z)\}_{j=1}^{N-1}$. The solution of (3.2.2) is known to exist because the functions $\{\psi_j(z)\}$ form a Chebyshev set (Ellacott and Williams [1967]).

From (2.2.6), the numbers $\delta_k e^{-i\theta_k}$ are proportional to the determinants

$$\Delta_k = (-1)^k \left| \psi_s(z_j) \right|, \quad (s = 1, 2, \ldots, m+n+1; \quad j = 1, 2, \ldots, k-1, k+1, \ldots, N). \tag{3.2.3}$$

3.2 The Computation of the Extremal Signature (cont'd)

Using (1.5.5), and writing p_k for $p(z_k)$ etc,

$$\Delta_k = (-1)^k \begin{vmatrix} 1/q_1^* & z_1/q_1^* & \cdots & z_1^m/q_1^* & z_1 p_1^*/q_1^{*2} & \cdots & z_1^n p_1^*/q_1^{*2} \\ 1/q_2^* & z_2/q_2^* & \cdots & z_2^m/q_2^* & z_2 p_2^*/q_2^{*2} & \cdots & z_2^n p_2^*/q_2^{*2} \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 1/q_{k-1}^* & z_{k-1}/q_{k-1}^* & \cdots & z_{k-1}^m/q_{k-1}^* & z_{k-1}p_{k-1}^*/q_{k-1}^{*2} & \cdots & z_{k+1}^n p_{k+1}^*/q_{k+1}^{*2} \\ 1/q_{k+1}^* & z_{k+1}/q_{k+1}^* & \cdots & z_{k+1}^m/q_{k+1}^* & z_{k+1}p_{k+1}^*/q_{k+1}^{*2} & \cdots & z_{k+1}^n p_{k+1}^*/q_{k+1}^* \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ 1/q_N^* & z_N/q_N^* & \cdots & z_N^m/q_N^* & z_N p_N^*/q_N^{\,2} & \cdots & z_N^n p_N^*/q_N^{*2} \end{vmatrix}$$

Thus

$$\Delta_k\, q_1^{*2} q_2^{*2} \cdots q_{k-1}^{*2} q_{k+1}^{*2} \cdots q_N^{*2}$$

$$= \begin{vmatrix} q_1^* & z_1 q_1^* & \cdots & z_1^m q_1^* & z_1 p_1^* & \cdots & z_1^n p_1^* \\ q_2^* & z_2 q_2^* & \cdots & z_2^m q_2^* & z_2 p_2^* & \cdots & z_2^n p_2^* \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ q_{k-1}^* & z_{k-1}q_{k-1}^* & \cdots & z_{k-1}q_{k-1}^* & z_{k-1}p_{k-1}^* & \cdots & z_{k-1}^n p_{k-1}^* \\ q_{k+1}^* & z_{k+1}q_{k+1}^* & \cdots & z_{k+1}q_{k+1}^* & z_{k+1}p_{k+1}^* & \cdots & z_{k+1}^n p_{k+1}^* \\ \cdot & & & & & & \\ \cdot & & & & & & \\ \cdot & & & & & & \\ q_N^* & z_N q_N^* & \cdots & z_N q_N^* & z_N p_N^* & \cdots & z_N^n p_N^* \end{vmatrix}$$

= (using elementary properties of determinants)

3.2 The Computation of the Extremal Signature (cont'd)

$$A \begin{vmatrix} 1 & z_1 & \cdots & \cdots & z_1^{n+m} \\ 1 & z_2 & \cdots & \cdots & z_2^{n+m} \\ \vdots & & & & \\ 1 & z_{k-1} & \cdots & \cdots & z_{k-1}^{n+m} \\ 1 & z_{k+1} & \cdots & \cdots & z_{k+1}^{n+m} \\ \vdots & & & & \\ 1 & z_N & \cdots & \cdots & z_N^{n+m} \end{vmatrix} .$$

where A depends on the coefficients of p and q but is independent of k. Using (2.2.13), we find that

$$\Delta_k = c\, q_k^{*2} / Q'(z_k) \tag{3.2.4}$$

where

$$Q(z) = \prod_{\nu=1}^{N} (z - z_\nu) \quad \text{and} \quad c \text{ is a constant.}$$

3.3 An Exchange Algorithm for the Unit Disc

As in the polynomial case, the success of the algorithm depends on the fact that θ_k depends only on α_k (where $z_k = e^{i\alpha_k}$) and is independent of the other points of E. To be more explicit, using the same assumptions and techniques of section 2.5, we find that

$$e^{i\theta_k} = e^{i\phi_k} \operatorname{sgn} \{ \overline{q}(z_k) \}^2 \tag{3.3.1}$$

3.3 An Exchange Algorithm for the Unit Disc (cont'd)

where $\phi_k = N\alpha_k/2 + (N+2)\pi/2 + k\pi$. The best approximation on a fixed set E of N points is then computed in a manner analagous to that of Williams. The equations defining $h(E)$ may be written by theorem 1.11 as

$$f(z_k) - p(z_k)/q(z_k) = h(E)e^{i\theta_k}\,\mathrm{sgn}\{\bar{q}(z_k)\}^2 \qquad (3.3.2)$$
$$(k = 1, 2, \ldots, N).$$

These equations are solved iteratively as

$$f(z_k) - p^{(\nu+1)}(z_k)/q^{(\nu+1)}(z_k) =$$
$$= h^{\nu+1}(E)e^{i\theta_k}\,\mathrm{sgn}\{\bar{q}^{(\nu)}(z_k)\}^2, \qquad (3.3.3)$$
$$(k = 1, 2, \ldots, N)$$

the resulting equations for $p^{(\nu+1)}$, $q^{(\nu+1)}$ and $h^{\nu+1}(E)$ being posed as an eigenvalue problem and solved by inverse iteration. In common with Williams, we are unable to establish the convergence of the iteration (3.3.3) although we find, as he did, that convergence is fast and that $\{h^{\nu}(E)\}$ is monotone decreasing to $h(E)$. The fact that any $h^{\nu}(E)$ must exceed $h(E)$ is due to $h(E)$ being $\max_E |f - p^{(\nu)}/q^{(\nu)}|$ and this by definition cannot be less than $h(E)$.

An illustration of the convergence is provided by the following example, where $f(z) = e^z$, $m = n = 2$ and E is the set of points $\{z_k = e^{ik\pi/3}\}_{k=0}^{5}$, equally spaced on the unit circle. The initial value for $q(z)$ was taken from the $(2-2)$ Pade approximant.

3.3 An Exchange Algorithm for the Unit Disc (cont'd)

ν	$h(\nu)$
1	0.13891 E-2
2	0.11086 E-2
3	0.11085 E-2
4	0.11085 E-2

If E is now changed by a small amount, the previous denominator can be used to start the new iteration, and the convergence is even more rapid. The main concern of this section however is the development of an ascent algorithm which does not involve the univariate maximisation technique.

Theorem 3.1

Let $E = \{ z_k = e^{i\alpha_k} \}_{k=1}^{N}$ be a set of N points on the unit circle symmetric about the real axis, and let f be an analytic function such that $f(\bar{z}) = \overline{f(z)}$. Let $r_{mn}(z)$ be the best approximation to f on E with $g(z) = f(z) - r_{mn}(z)$. Define z_k' as follows for $k = 1, 2, \ldots, N$:

(i) $s = \operatorname{sgn} \{ e^{-i\theta_k} g(z_k) \}$;

(ii) α_k' is the value of α which maximises

$$F(\alpha) = \operatorname{Re} \{ s e^{-i\theta} g(z) \}$$

in the neighbourhood $(\alpha_{k-1}, \alpha_{k+1})$ of α_k where θ is based on $\{\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \ldots, \alpha_N\}$ and $z = e^{i\alpha}$;

(iii) $z_k' = e^{i\alpha_k'}$.

3.3 An Exchange Algorithm for the Unit Disc

Theorem 3.1 (cont'd)

If $E' = \{ z_1', z_2', \ldots, z_N' \}$, then

(a) E' consists of N distinct points;

(b) $h(E') \geq h(E)$,

provided that the approximations are such that $\arg\{ q(z_k) \}$ and $\arg\{ q(z_k') \}$ do not differ by more than $\pi/2$.

Proof

Statement (a) follows by the same argument as the corresponding part of theorem 2.2.

The equations defining $h(E)$ and $h(E')$ are from (3.3.2)

$$f(z_k) - p(z_k)/q(z_k) = h(E) e^{i\phi_k} \operatorname{sgn}\{ \overline{q}(z_k) \}^2 \qquad (3.3.4)$$

and

$$f(z_k') - p'(z_k')/q'(z_k') = h(E') e^{i\phi'_k} \operatorname{sgn}\{ \overline{q}'(z_k') \}^2 \qquad (3.3.5)$$

Let us assume

$$h(E') < h(E) \qquad (3.3.6)$$

and show that this leads to a contradiction. Let the maximum value of $F(\alpha)$ in the neighbourhood of α_k, found in step (ii) of the theorem, be d_k where $d_k \geq h(E)$ by definition. Thus

$$f(z_k') - p(z_k')/q(z_k') = d_k e^{i\phi_k'} \operatorname{sgn}\{ \overline{q}(z_k') \}^2 . \qquad (3.3.7)$$

Subtracting (3.3.7) from (3.3.5), we have

$$p(z_k')/q(z_k') - p'(z_k')/q'(z_k') = h(E') e^{i\phi_k'} \operatorname{sgn}\{ \overline{q}(z_k') \}^2 - d_k e^{-i\phi_k'} \operatorname{sgn}\{ \overline{q}'(z_k') \}^2$$

3.3 An Exchange Algorithm for the Unit Disc

Theorem 3.1

Proof (cont'd)

Multiplying through by $q(z_k')\,q'(z_k')$ gives

$$p(z_k')q'(z_k') - p'(z_k')q(z_k') = h(E')e^{i\phi_k'} q'(z_k')q(z_k')\,\mathrm{sgn}\{\overline{q}(z_k')\}^2 - d_k e^{i\phi_k'} q'(z_k')q(z_k')\,\mathrm{sgn}\{\overline{q}'(z_k')\}^2 \qquad (3.3.8)$$

Let $q(z_k') = |q(z_k')|e^{i\beta_k}$ and $q'(z_k) = |q'(z_k')|e^{i\beta_k'}$.

Then

$$e^{-i\phi_k'}\{p(z_k')q'(z_k') - p'(z_k')\,q(z_k')\} = |q(z_k')q'(z_k')|\{h(E)e^{i(\beta_k - \beta_k')} - d_k e^{-i(\beta_k' - \beta_k)}\}. \qquad (3.3.9)$$

Since $p(z)q'(z) - p'(z)q(z)$ is a polynomial of degree $m + n$ $(= N-2)$, using the definition of ϕ_k and the property of extremal signatures, there exist positive numbers δ_k, $(k = 1, 2, \ldots, N)$, not all zero, such that

$$\sum_{k=1}^{N} \delta_k e^{-i\phi_k}\{p(z_k')q'(z_k') - p'(z_k')q(z_k')\} = 0 \qquad (3.3.10)$$

The right hand side of (3.3.9) under the same summation becomes

$$\sum_{k=1}^{N} \delta_k |q(z_k')\,q'(z_k')|\{h(E)e^{i(\beta_k' - \beta_k)} - d_k e^{-i(\beta_k' - \beta_k)}\}$$

and the real part of this expression is

$$\sum_{k=1}^{N} \delta_k |q(z_k')q'(z_k')|\{h(E) - d_k\}\cos(\beta_k' - \beta_k) \qquad (3.3.11)$$

where $|\beta_k' - \beta_k| < \pi/2$.

3.3 An Exchange Algorithm for the Unit Disc

Theorem 3.1

Proof (cont'd)

Since $\delta_k \geq 0$ for all values of k with at least one strict inequality, and $h(E') < h(E) \leq d_k$ by our assumption (3.3.6), we deduce that expression (3.3.11) is negative in contradiction to (3.3.10). Thus

$$h(E') \geq h(E).$$

The algorithm performs in practice in a manner similar to that of the Remes exchange algorithm for real rational approximation. Many approximations have been computed using the algorithm of this section, and some results are presented giving the initial inclusion interval, the final inclusion, the number of iterations used in the exchange algorithm and the final inclusion given by Williams [1972] as a comparison. The final inclusion is quoted up to the first significant figure in which $h(E)$ and $\| f - r_{mn} \|$ differ. In each case the algorithm is started with a set of points equally spaced on the unit circle.

As in the polynomial case, we conjecture on the basis of the proximity of $h(E)$ and $\| f - r_{mn} \|$ in the numerical results that best rational approximations can be constructed on a disc using sets of only N points. We attribute the sizes of Williams' inclusion intervals to the inefficient maximising procedure for $h(E)$ rather than to an insufficient number of points.

The initial denominator $q(z)$ is taken in all cases as the denominator of the relevant Padé approximation.

Example 1 $f(z) = e^z$

m=n	Initial Set		Final Set		Number of Iterations	Williams' Results	
	h(E)	$\|\| f - r_{mn} \|\|$	h(E)	$\|\| f - r_{mn} \|\|$		h(E)	$\|\| f - r_{mn} \|\|$
1	0.6812 E-1	2.6066 E-1	0.8479 7754 E-1	0.8479 7754 E-1	6	0.84 E-1	1.04 E-1
2	0.1109 E-2	0.3540 E-2	0.1393 2840 E-2	0.1393 2841 E-2	4	0.139 E-2	0.163 E-2
3	0.788 E-5	2.620 E-5	0.9932 E-5	0.9934 E-5	4	0.99 E-5	1.14 E-5
4	0.32 E-7	1.05 E-7	0.394 E-7	0.399 E-7	4	0.393 E-7	0.455 E-7

$$r_{22}(z) = \frac{1.0000\,01457 + 0.5161\,9398\,90z + 0.0920\,8496\,38z^2}{1.0 - 0.4837\,8755\,83z + 0.0757\,4425\,00z^2}$$

Rational approximations on a disc have errors which exhibit remarkable uniformity in norm. For instance,

$$\min_{|z|=1} | e^z - r_{22}(z) | = 0.1393\,19\,E\text{-}1$$

$$\max_{|z|=1} | e^z - r_{22}(z) | = 0.1393\,28\,E\text{-}1 .$$

3.3 An Exchange Algorithm for the Unit Disc (cont'd)

Example 2 $f(z) = \ln(1 + z/2)$

m=n	Initial Set		Final Set		Number of Iterations	Williams' Results	
	h(E)	$\| f - r_{mn} \|$	h(E)	$\| f - r_{mn} \|$		h(E)	$\| f - r_{mn} \|$
1	0.100 E-1	1.071 E-1	0.1285 3684 E-1	0.1285 3685 E-1	4	0.128 E-1	0.138 E-1
2	0.139 E-3	1.610 E-3	0.2463 44 E-3	0.2463 45 E-3	5	0.244 E-3	0.300 E-3
3	0.175 E-5	2.523 E-5	0.455 39 E-5	0.455 43 E-5	5	0.450 E-5	0.579 E-5

$$r_{33}(z) = \frac{0.5000\,0000z + 0.2661\,6585z^2 + 0.0271\,7121z^3}{1.0 + 0.7823\,3177z + 0.1665\,9276z^2 + 0.0077\,0738z^3}$$

The final set E for r_{33} is

$\{e^{i\theta} : \theta = 0,\ 68.2191°,\ 119.9987°,\ 154.5565°,\ 180°,\ 205.4435°,\ 240.0013°,\ 291.7809°\}$.

3.3 An Exchange Algorithm for the Unit Disc (cont'd)

Example 3 $f(z) = \tan z$

m=n	Initial Set		Final Set		Number of Iterations	Williams' Results	
	h(E)	$\| f - r_{mn} \|$	h(E)	$f - r_{mn}$		h(E)	$\| f - r_{mn} \|$
2	0.202 E-1	0.379 E-1	0.2395 5279 E-1	0.2395 5280 E-1	4	0.235 E-1	0.320 E-1
3	0.547 E-3	1.319 E-3	0.6592 89 E-3	0.6592 93 E-3	3	0.650 E-3	0.848 E-3
4	0.851 E-5	2.38 E-5	0.1031 3 E-4	0.1031 4 E-4	4	0.102 E-4	0.134 E-4
5	0.85 E-7	2.61 E-7	0.103 E-6	0.107 E-6	4	0.103 E-6	0.134 E-6

$$r_{44}(z) = \frac{-0.0000\,0000\,05 + 0.9999\,9999\,83z + 0.0000\,0071\,60z^2 - 0.0956\,3553\,37z^3 + 0.0000\,0004\,91z^4}{1.0 - 0.0000\,0071\,62z - 0.4289\,6900\,05z^2 + 0.0000\,0028\,78z^3 + 0.0096\,5859\,05z^4}$$

3.4 The exchange algorithm for other regions

In view of the success and relative simplicity of the algorithm based on theorem 3.1, we have used it on a number of examples where the convergence has not been proved. These are approximations on an ellipse and semi-disc. Numerical evidence suggests that for such problems, subsets of N points are not sufficient to compute best approximations on the whole region. Nevertheless, some of the results are very good, especially on the ellipse and this is demonstrated by the following examples.

Example 1

$$f(z) = e^z$$

Approximation on an ellipse, semi major axis length 2 along real axis and semi minor axis length 1 along imaginary axis.

$$m = n = 2$$

Initial Set		Final Set		Number of Iterations
h(E)	$\| f - r_{mn} \|$	h(E)	$\| f - r_{mn} \|$	
0.0537 E-1	0.6447 E-1	0.1099 E-1	0.1136 E-1	6

$$r_{22}(z) = \frac{1.0002\,0554 + 0.5553\,6804\, z + 0.1059\,5259\, z^2}{1.0 \quad - 0.4493\,5230\, z + 0.0605\,6426\, z^2}$$

We note that the norm of the error in the (2-2) Padé approximant is about 4.3!

3.4 The exchange algorithm for other regions (cont'd)

Example 2

$$f(z) = \ln(1 + z/2)$$

Approximation on an ellipse, semi minor axis length 1 along the real axis and semi major axis length 2 along the imaginary axis.

m = n	Initial Set		Final Set		Number of Iterations
	h(E)	$\|\| f - r_{mn} \|\|$	h(E)	$\|\| f - r_{mn} \|\|$	
2	0.0628 E-2	0.3865 E-2	0.1233 E-2	0.1308 E-2	6
3	0.3917 E-4	0.6394 E-4	0.4075 E-4	0.4229 E-4	4

$$r_{33}(z) = \frac{-0.0000\,0589 + 0.4999\,8450z + 0.2550\,2643z^2 + 0.0246\,0656z^3}{1.0 + 0.7600\,9524z + 0.1559\,4149z^2 + 0.0069\,1533z^3}$$

Final set E.

m = n = 2. $\{\cos\theta + 2i\sin\theta : \theta = 74.81°, 132.04°, 167.77°, 192.23°, 227.96°, 285.19°\}$.

m = n = 3 $\{\cos\theta + 2i\sin\theta : \theta = 0°, 84.83°, 128.22°, 159.61°, 180°, 200.39°, 231.78°, 275.17°\}$.

3.4 The exchange algorithm for other regions (cont'd)

Example 3

$$f(z) = e^z$$

Approximation on the semi disc $\{ |z| \leq 1, \mathrm{Re}(z) \geq 0\}$. This example and the next were also considered by Williams [1968].

m=n	Initial Set		Final Set		Number of Iterations	Williams' Results	
	h(E)	$\|f - r_{mn}\|$	h(E)	$\|f - r_{mn}\|$		h(E)	$\|f - r_{mn}\|$
2	0.570 E-3	0.986 E-3	0.574 E-3	0.953 E-3	5	0.588 E-3	0.967 E-3
3	0.230 E-5	0.406 E-5	0.259 E-5	0.306 E-5	4	0.231 E-5	0.449 E-5

Example 4

$$f(z) = \ln(1 + z)$$

Approximation on the semi disc $\{ |z| \leq 1, \mathrm{Re}(z) \geq 0\}$.

m=n	Initial Set		Final Set		Number of Iterations	Williams' Results	
	h(E)	$\|f - r_{mn}\|$	h(E)	$\|f - r_{mn}\|$		h(E)	$\|f - r_{mn}\|$
2	0.270 E-3	0.313 E-3	0.272 E-3	0.283 E-3	5	0.272 E-3	0.326 E-3

$$r_{22}(z) = \frac{0.0002\,8260 + 0.9983\,0556\, z + 0.3870\,9593\, z^2}{1.0 + 0.8820\,8010\, z + 0.1178\,4848\, z^2}$$

Final set $E = \{ e^{i\theta}, \theta = \pm 46.34 \; ; \pm i, \; \pm 0.4284\, i \}$

3.5 Conclusion

The methods based on characterisation are probably the most efficient available for constructing both polynomial and rational approximations on circular regions and provide very good approximations on nearly circular regions. For other regions, the methods are of only limited value, quite apart from the theoretical difficulties involved in the convergence of the algorithms in these cases. Further remarks about the method will be deferred until the end of Chapter 5.

CHAPTER 4

THE CONSTRUCTION OF BEST POLYNOMIAL APPROXIMATIONS BY PROGRAMMING METHODS

4.1 Introduction and History

That Chebyshev polynomial approximation can be achieved in the complex plane by programming methods is not a new idea. If D is a finite point set containing at least $n+1$ points, then the approximation of a function f by $p \in P_n$, where P_n is the space of generalised polynomials of degree n, can be simply posed as a non-linear program with linear objective function and quadratic constraints.

Let D be the set $\{ z_k \}_{k=1}^{m}$ $(m \geq n+1)$. Then the best approximation problem is to find

$$\min_{p \in P_n} \max_{j} | p(z_j) - f(z_j) | \quad (j = 1, 2, \ldots, m) \tag{4.1.1}$$

which may be written as

$$\min \quad d$$

subject to

$$| p(z_j) - f(z_j) | \leq d \quad (j = 1, 2, \ldots, m) \tag{4.1.2}$$

or

$$\min \quad d$$

subject to

$$\left| \sum_{k=1}^{n} a_k \phi_k (z_j) - f(z_j) \right| \leq d \quad (j = 1, 2, \ldots, m) \tag{4.1.3}$$

where $\{ \phi_k (z) \}_{k=1}^{n}$ is a set of linearly independent functions

4.1 Introduction and History (cont'd)

spanning P_n. Introduce the notation

$$\phi_k(z_j) = c_{jk}\,,\ f(z_j) = b_j (j = 1, 2, \ldots, m;\ k = 1, 2, \ldots, n). \tag{4.1.4}$$

Then the approximation problem consists of determining the point $a^* = (a_1^*, a_2^*, \ldots, a_n^*)$ such that

$$\max_j \left| \sum_{k=1}^{n} c_{jk} a_k^* - b_j \right| = \min_a \max_j \left| \sum_{k=1}^{n} c_{jk} a_k - b_j \right| \tag{4.1.5}$$

which can be considered as the minimax solution of the set of inconsistent complex linear equations

$$Ca = b \tag{4.1.6}$$

where

$$C = (c_{jk})\,,\ a = (a_k) \text{ and } b = (b_k).$$

Finally defining

$$a_k = \alpha_k + i\beta_k\,,\quad c_{jk} = u_{jk} + iv_{jk}\,,\ b_j = \gamma_j + i\delta_j \tag{4.1.7}$$

and noting that the problem is not essentially changed if we replace the modulus in (4.1.1), (4.1.2), (4.1.3) and (4.1.5) by its square, we arrive at the problem

$$\min d$$

subject to

$$\left| \sum (\alpha_k + i\beta_k)(u_{jk} + iv_{jk}) - (\gamma_j + i\delta_j) \right|^2 \leq d \tag{4.1.8}$$

or

$$\min d$$

subject to

$$F_j(\alpha, \beta) \leq d\,,\quad (j = 1, 2, \ldots, m). \tag{4.1.9}$$

The inequalities can now be seen to be quadratic in the unknowns α_k, β_k $(k = 1, 2, \ldots, n)$.

4.1 Introduction and History (cont'd)

Most of the work in this field was done originally by Russian mathematicians, notably Zuhovickii. The first work was published by Zuhovickii [1961] followed by Zuhovickii, Poljak and Primak [1963] and Zuhovickii and Avdeeva [1964]. In common with us, they were interested also in the 'continuous' problem of approximation on a general compact set B. Their approach was merely to take a finite subset of B and to argue that if $f(z)$ and $\{\phi_k(z)\}_{k=1}^{n}$ are continuous of B, then the solution of the problem on the subset will differ little from that on the whole of B, provided that the subset is sufficiently dense in B. We shall see in section (4.5) that there is a more precise relation between the continuous problem and its discrete analogue.

The solution of (4.1.8) is not trivial, and several algorithms have been proposed. We shall examine them in the next section. There are two principal redeeming features of the non-linearity. The first is that the constraints are convex functions, and the second is that they are quadratic.

A simple example will clarify the above ideas. To find the best linear approximation to $1/(2-z)$ on the semidisc $\{|z| \leq 1, \mathrm{Re}(z) \geq 0\}$ we might, using symmetry, take a finite subset $D = \{1, e^{j\pi/4}, i, 0.5i, 0\}$. The problem is then to find a_1 and a_2 to minimise

$$\max_{z \in D} \; | a_1 + a_2 z - 1/(2-z) |$$

(where a_1 and a_2 are real in the case). This is equivalent to minimising

4.1 Introduction and History (cont'd)

$$d$$

subject to

$$|a_1 - \tfrac{1}{2}|^2 \leq d$$

$$|a_1 + a_2 i/2 - 1/(2 - i/2)|^2 \leq d$$

$$|a_1 + a_2 i - 1/(2 - i)|^2 \leq d$$

$$|a_1 + a_2(1 + i)/\sqrt{2} - 1/(2 - (1+i)/\sqrt{2})|^2 \leq d$$

$$|a_1 + a_2 - 1|^2 \leq d$$

or

$$\min d$$

subject to

$$(a_1 - \tfrac{1}{2})^2 \leq d$$

$$(a_1 - 8/9)^2 + (a_2/2 - 2/9)^2 \leq d$$

$$(a_1 - 2/5)^2 + (a_2 - 1/5)^2 \leq d$$

$$(a_1 - (16 + 3\sqrt{2})/34)^2 + (a_2 - (5\sqrt{2} + 4)/34)^2 \leq d$$

$$(a_1 + a_2 - 1)^2 \leq d.$$

4.2 Existing Algorithms and their Limitations

The first attempt at finding a Chebyshev solution of equation (4.1.6) was the interesting algorithm of Zuhovickii [1961]. Using the notation of (4.1.7), equation (4.1.6) can be expanded in the form of a set of m pairs of real linear equations

$$\left.\begin{aligned} u_{j1}\alpha_1 - v_{j1}\beta_1 + u_{j2}\alpha_2 - v_{j2}\beta_2 + \ldots + u_{jn}\alpha_n - v_{jn}\beta_n &= \gamma_j \\ v_{j1}\alpha_1 + u_{j1}\beta_1 + v_{j2}\alpha_2 + u_{j2}\beta_2 + \ldots + v_{jn}\alpha_n - u_{jn}\beta_n &= \delta_j \end{aligned}\right\} \qquad (4.2.1)$$

where $j = 1, 2, \ldots, m$.

4.2 Existing Algorithms and their Limitations (cont'd)

Each pair of equations represents a pair of mutually orthogonal $(2n - 1)$-dimensional planes of a real $2n$-dimensional Euclidean space, so that the intersection of each such pair is a $(2n - 2)$-dimensional plane of this space. The magnitude

$$\left| \sum_{k=1}^{n} c_{jk} a_k^0 - b_j \right| =$$

$$\sqrt{\left\{ \sum_{k=1}^{n} (u_{jk}\alpha_k^0 - v_{jk}\beta_k^0) - \gamma_j \right\}^2 + \left\{ \sum_{k=1}^{n} (v_{jk}\alpha_k^0 + u_{jk}\beta_k^0) - \delta_j \right\}^2} \quad (4.2.2)$$

represents the distance from the point

$$\alpha^0 = (\alpha_1^0, \beta_1^0, \ldots, \alpha_n^0, \beta_n^0)$$

to the $(2n - 2)$-dimensional plane (4.2.1) multiplied by

$$\sqrt{\sum_{k=1}^{n} |c_{jk}|^2} \ .$$

We shall therefore call (4.2.2) the *weighted* distance from a^0 to the plane (or just the distance where there is no confusion).

The algorithm then consists of moving such that the distances from a subset of the planes are equal, and greater than the distances from the remaining planes until the maximum distance from any plane is minimised. It could therefore be regarded as a non-linear simplex type algorithm. Zuhovickii uses geometrical arguments to show that the minimum point is the same as that for a subset of r of the planes such that $r \leq 2n + 1$ (or $n + 1 \leq r \leq 2n + 1$ if the equations satisfy the Haar condition). However we have not been able to implement this algorithm directly because of the difficulty of moving along non-linear manifolds. Indeed Zuhovickii himself has not been able to do so (private communication).

4.2 Existing Algorithms and their Limitations (cont'd)

An alternative method was then published by Zuhovickii, Poljak and Primak [1963] which relaxes the condition of remaining in the manifold when searching for a minimum. By using linear approximations, they produce a perfectly feasible algorithm which can easily be implemented by linear programming. Their work was published at about the same time as, although developed independently of, Zoutenjik's method of feasible directions (Zoutenjik [1960]). Their method, and that of Zuhovickii and Avdeeva [1964], are essentially variants of Zoutenjik's method, as is that of Barrodale, Delves and Mason [1977] which differs only in that it includes linearisations of all constraints at each step, instead of just the active ones. Unfortunately, our experience with all of these methods is that the convergence can be slow, and this has been confirmed by Pironneau and Polak [1973] who show that the class of feasible direction methods converge at best linearly, and that some variants may have sub-linear convergence. Since it is vital for the development of a practical method for best approximation to have an efficient algorithm for the solution of the non-linear program (4.1.8) we have rejected the feasible direction methods.

There are now many algorithms available for non-linear programming such as the gradient projection method and the reduced gradient method. We have achieved some success with the exact penalty method, developed for equality constraints by Fletcher [1970,1971] and Fletcher and Lill [1970], and extended to inequality constraints by Fletcher [1972]. Their methods seek to unify the Lagrangian and Penalty function approaches to non-linear programming.

4.2 Existing Algorithms and their Limitations (cont'd)

For the problem

$$\min F(x)$$

subject to

$$c_i(x) \geq 0 \qquad (i = 1, 2, \ldots, m)$$

a function of the form

$$\phi(x) = F - cN^{+}g + q\,c^{T}N^{+}N^{+T}c$$

has to be minimised (an unconstrained problem) where N^{+} is the generalised inverse of a matrix N whose columns are the normals to the constraints. Although the convergence of this method is good, it has the serious disadvantage that the 'nice' quadratic nature of the constraints is lost in the construction of ϕ.

More recently, an algorithm has been published by Robinson [1972] which appears to be ideal for our problem, and is by far the most successful algorithm we have tried for the solution of (4.1.8).

4.3 The New Algorithm-description and convergence

Robinson's method solves the non-linear programming problem

$$\min \{\, \theta(x) \mid g(x) \leq 0 \,\}$$

where θ and g are differentiable functions from $\mathbb{R}^n$ to $\mathbb{R}$ and $\mathbb{R}^m$ to R respectively (the original algorithm also included equality constraints which do not concern us here). His method, which is iterative, requires initial estimates x_0 of the Kuhn-Tucker point and λ^0 of the vector of multipliers associated with g, and under certain conditions converges to a Kuhn-Tucker point x^* with associated λ^*.

4.3 The New Algorithm-description and convergence (cont'd)

The algorithm consists of solving a sequence of linearly constrained problems

$$\min\{\theta(x) + \lambda^{(k)T}\{g(x) - Lg(x_k, x)\} \mid Lg(x_k, x) \leq 0\}$$

where $Lg(x_k, x) = g(x_k) + g'(x_k)(x - x_k)$ is a linearlisation of $g(x)$ about x_k. The crucial feature of the method is the way it transfers the non-linearily of the constraints into the objective function. Its relevance to our problem should now be clear. The problem (4.1.8) now becomes simply a sequence of quadratic programs, the quantity $g(x) - Lg(x_k, x)$ being merely the second derivative term of the Taylor expansion of g about x_k and can be found algebraically.

For the problem (4.1.9), Robinson's algorithm becomes

$$\min\{d + \lambda^{(k)T}\{F(\alpha, \beta) - LF(\alpha^{(k)}, \beta^{(k)}, \alpha, \beta)\} \mid LF(\alpha^{(k)}, \beta^{(k)}, \alpha, \beta) \leq d\} \quad (k = 0, 1, 2, \ldots) \qquad (4.3.1)$$

Since the solution of (4.1.9) represents the best approximation by a polynomial on a finite point set, the solution is unique (assuming the Haar condition is satisfied). For sufficiently good starting approximations, second order convergence to this solution is guaranteed (Robinson [1972]). However, global convergence of Robinson's method is not assured (Fletcher [1964, p.231]), and even for a convex problem, certain initial choices for λ may lead to subprograms in (4.3.1) with no solution (Robinson - private communication). To ensure convergence from an arbitrary starting point, the algorithm needs certain modifications, and we have adopted the following changes. To make certain that each sub-program has a solution, we introduce norms on the variation in α and β of the form $\|\alpha - \alpha^{(k)}\| \leq h$ and $\|\beta - \beta^{(k)}\| \leq h$

4.3 The New Algorithm-description and convergence (cont'd)

where h is a positive constant. To ensure that progress is made towards the solution of the original problem (4.1.9) at each iteration, the resulting values of $\alpha - \alpha^{(k)}$ and $\beta - \beta^{(k)}$ are regarded as search directions in which we look for a reduction in the quantity

$$\max_j \; F_j(\alpha , \beta) \quad (j = 1, 2, \ldots, m) .$$

The algorithm will now be restated in full, and its convergence proved.

Algorithm 4.1

Step 1 : Set $k = 0$ and choose initial values of $\alpha^{(0)}$, $\beta^{(0)}$ and $\lambda^{(0)}$. Choose $h > 0$.

Step 2 : Set $\delta^{(k)} = \max_j F_j(\alpha^{(k)}, \beta^{(k)})$.

Step 3 : Find a Kuhn-Tucker point $(\alpha^*, \beta^*, \lambda^*)$ of the subprogram

$$\min\{d + \lambda^{(k)T}.\{F(\alpha,\beta) - LF(\alpha^{(k)},\beta^{(k)},\alpha,\beta) \mid LF(\alpha^{(k)},\beta^{(k)},\alpha,\beta) \leq d,$$
$$\| \alpha^{(k)} - \alpha \| \leq h, \| \beta^{(k)} - \beta \| \leq h\}.$$

Step 4 : Choose $\mu \; (0 < \mu \leq 1)$ so that with

$$\alpha^{(k+1)} = \alpha^{(k)} + \mu(\alpha^* - \alpha^{(k)}), \quad \beta^{(k+1)} = \beta^{(k)} + \mu(\beta^* - \beta^{(k)})$$

we have

$$\max_j F_j(\alpha^{(k+1)}, \beta^{(k+1)}) < \delta^{(k)}.$$

Step 5 : Stop if algorithm has converged; otherwise set $k := k + 1$, $\lambda^{(k)} = \lambda^*$ and return to step 2.

4.3 The New Algorithm-description and convergence (cont'd)

Theorem 4.1

Using the notation of the algorithm, the sequence $\{\delta^{(k)}\}$ is monotone decreasing and converges to $\delta^* = \min_{\alpha,\beta} \max_j F_j(\alpha,\beta)$.

Proof

If the term involving $\lambda^{(k)}$ is removed from the objective function in step 3, the method becomes essentially that of Barrodale, Delves and Mason [1977], and the proof follows a somewhat similar argument. Our method could therefore be regarded as a variant of theirs in which the curvature of the constraints is taken into account.

Denote the objective function in step 3 of the algorithm by

$$\phi_k(\alpha, \beta, d) = d + Q_k(\alpha,\beta)$$

and let its minimum value subject to the given constraints be $\phi_k(\alpha^*, \beta^*, d^*)$. Since the constraints are satisfied by the values $\alpha = \alpha^{(k)}$, $\beta = \beta^{(k)}$, $d = \delta^{(k)}$ and $\phi_k(\alpha^{(k)},\beta^{(k)},\delta^{(k)}) = \delta^{(k)}$, we must have $\phi_k(\alpha^*,\beta^*, d^*) \leq \delta^{(k)}$.

There are two cases to consider.

(a) $\phi_k(\alpha^*, \beta^*, d^*) < \delta^{(k)}$. Since $Q_k(\alpha,\beta)$ is by its construction non-negative, this condition implies $d^* < \delta^{(k)}$.

(b) $\phi_k(\alpha^*, \beta^*, d^*) = \delta^{(k)}$. Since $Q_k(\alpha^*, \beta^*, d^*) = 0$ and $d^* = \delta^{(k)}$ or $Q_k(\alpha^*, \beta^*, d^*) > 0$ and $d^* < \delta^{(k)}$.

The outcome of the minimisation is then either that $d^* = \delta^{(k)}$ or $d^* < \delta^{(k)}$, and we consider these two possibilities separately.

4.3 The New Algorithm-description and convergence

Theorem 4.1

Proof (cont'd)

Define the set J as $\{j : F_j(\alpha^{(k)}, \beta^{(k)}) = \delta^{(k)}\}$.

If $d^* = \delta^{(k)}$, then there are no values of α, β such that $LF_j(\alpha^{(k)}, \beta^{(k)}, \alpha, \beta) < \delta^{(k)}$ for all $j \in J$, and so for any α, β, there is a suffix $t \in J$ such that

$$LF_t(\alpha^{(k)}, \beta^{(k)}, \alpha, \beta) \geq \delta^{(k)}. \tag{4.3.1}$$

Hence for sufficiently small $\lambda > 0$ and still for all α, β,

$$F_t(\alpha^{(k)} + \lambda\alpha, \beta^{(k)} + \lambda\beta) \geq \delta^{(k)}$$

which is sufficient to ensure that $\alpha^{(k)}, \beta^{(k)}$ is a local minimum of $F_t(\alpha,\beta)$ and therefore of

$$\max_j F_j(\alpha,\beta) \tag{4.3.2}$$

But (4.3.2) is a convex function, so that $\alpha^{(k)}, \beta^{(k)}$ is the unique global minimum of (4.3.2) with

$$\delta^{(k)} = \delta^*. \tag{4.3.3}$$

If $d^* < \delta^{(k)}$ then for all $j \in J$, and all α, β,

$$LF_j(\alpha^{(k)}, \beta^{(k)}, \alpha, \beta) < \delta^k \tag{4.3.4}$$

and so for sufficiently small λ, we have

$$F_j(\alpha^{(k)} + \lambda\alpha, \beta^{(k)} + \lambda\beta) < \delta^{(k)}$$

for both $j \in J$ from (4.3.4) and for $j \notin J$ by definition of J. Therefore

$$\delta^{(k+1)} < \delta^{(k)}. \tag{4.3.5}$$

The proof now follows from (4.3.3), (4.3.5) and the non-negativity of $F(\alpha,\beta)$.

4.4 The New Algorithm-Implementation

As we have already noted, each subprogram of the algorithm is a quadratic program, and a careful choice of quadratic programming algorithm thus has to be made. The particular form of the quadratic programming problem may be written as

$$\begin{aligned} &\min \quad x^T Ax + b^T x \\ &\text{subject to} \quad C^T x \leq d \end{aligned} \tag{4.4.1}$$

where A is positive semi-definite since the objective function is a linear combination of convex functions. The method adopted is that of Fletcher [1971a]. It has the following advantages:

(a) it is numerically stable for the general quadratic programming problem, and therefore in particular for the solution of (4.4.1);

(b) it is not dependent on a linear programming code;

(c) it gives, as a by-product of the method, the Lagrange multipliers at the solution which are needed to set up the next subprogram.

Fletcher's algorithm is based on the classical formulation of the minimisation of a quadratic function subject to linear euqality constraints. For inequality constraints, we keep a basis of active constraints which are treated as equalities. The solution with such a basis is called a *feasible solution to an equality problem* (FSEP). The signs of the Lagrange multipliers associated with the equality problem determine whether the FSEP is the solution of the quadratic program. The algorithm moves from one FSEP to another by systematically adding and subtracting constraints from the basis, and Fletcher shows that this can be

4.4 The New Algorithm-Implementation (cont'd)

done in such a way that the result is a descent algorithm. The solution in each new basis can be found economically without recomputing ab initio the solution of the equality problem.

Our algorithm is initiated by estimating the r constraints in the problem (4.1.9) which are going to be active, and then setting the corresponding λ s to the value $1/r$, the rest being zero. If no estimate is available, we merely set all the λs to $1/m$.

Extensive computation using an ALGOL program based on this algorithm has shown that:

(a) except possibly in the first few iterations, a full step ($\mu = 1$) in the descent direction always resulted in a reduction in $\max_j F_j(\alpha,\beta)$ in which case no univariate search was necessary;

(b) using $\mu = 1$ throughout the algorithm never failed to give a convergent iteration;

(c) with the indicated choice of λs in the initial step, all the subprograms had bounded solutions without involving the artificial normalising constraints $\| \alpha^{(k)} - \alpha \| < d$ $\| \beta^{(k)} - \beta \| \leq d$ in step 2 of the algorithm.

Some numerical results are given to illustrate the progress of the algorithm in practise. The subscripts of the linearised constraints which are active at each iteration are also indicated to show how quickly the algorithm picks out the final active set (which, at convergence, are also the active constraints of the original problem).

4.4 The New Algorithm-Implementation (cont'd)

Example 1

Approximation of $1/\sqrt{1+2z}$ by a polynomial of degree 4 on the subset $D = \{1, e^{i\pi/6}, e^{i\pi/3}, i, i/2, 0\}$ of the semidisc $B = \{|z| \leq 1, \mathrm{Re}(z) \geq 0\}$. The initial approximation was found by interpolation at the zeros of $T_5(z; B)$. (see 7.2)

Iteration Number	Suffices of active set	$\max_j F_j(\alpha,\beta)$
0		0.0038 5841
1	3456	0.0019 7995
2	13456	0.0018 9401
3	13456	0.0017 9111
4	13456	0.0017 8987
5	13456	0.0017 8987

Values of λ_i		
i	Initial	Final
1	1/6	0.0536 7816
2	1/6	0
3	1/6	0.1295 4318
4	1/6	0.1630 9147
5	1/6	0.3633 4690
6	1/6	0.2903 4029

$$p^*(z) = 0.9576\,9319 - 0.7255\,5967z + 0.5972\,4825z^2 - 0.3758\,8002z^3 + 0.1661\,5532z^4 .$$

$$\max_{z\in D} |f(z) - p_4^*(z)| = 0.0423\,0681 .$$

4.4 The New Algorithm-Implementation

Example 1 (cont'd)

It is interesting that for the whole semidisc,

$$\| f(z) - p_4(z) \| = 0.0425\,8619$$

so that even on such a coarse subset, we already have a better approximation than that of Williams using characterisation, who obtained the inclusion interval [0.0418, 0.0440] for this problem.

Example 2

Approximation of z^8 by a polynomial of degree 7 on the set $D = \{1, e^{i\pi/12}, e^{2i\pi/12}, e^{3i\pi/12}, e^{4i\pi/12}, e^{5i\pi/12}, i, i/2, 0\}$.

The initial approximation was taken from the Faber polynomial* of degree 8 for the semidisc.

*see 6.3

Iteration Number	Suffices of active set	$\max_j F_j(\alpha,\beta)$
0		0.0343 5500
1	2345 6789	0.0197 0757
2	2345 6789	0.0178 9074
3	2345 6789	0.0177 9557
4	2345 6789	0.0177 9506
5	2345 6789	0.0177 9506

In the following two examples the original algorithm of Robinson has been used without insisting on a descent at each iteration.

4.4 The New Algorithm-Implementation (cont'd)

Example 3

Approximation of e^z by a polynomial of degree 4 on the set $\{1, e^{i\pi/6}, e^{i\pi/3}, i, i/2, 0\}$. The initial approximation was $e^z \simeq 1 + z + z^2/2 + z^3/6 + z^4/24$.

Iteration Number	Suffices of active set	$\max_j F_j(\alpha,\beta)$
0		0.9765 E-4
1	12345	0.3670 E-4
2	12346	0.7571 E-4
3	12345	0.7884 E-4
4	12346	0.3234 E-4
5	12346	0.1579 E-4
6	12346	0.1355 E-4
7	12346	0.1349 E-4
8	12346	0.1349 E-4

Example 4

Approximation of $1/\sqrt{1 + 2z}$ by a polynomial of degree 8 on $D = \{1, e^{i\pi/}, e^{2i\pi/12}, e^{3i\pi/12}, e^{4i\pi/12}, e^{5i\pi/12}, i, 4i/5, 3i/5, 2i/5, i/5, 0\}$.

Iteration Number	Suffices of active set	$\max_j F_j(\alpha,\beta)$
0		2.5730 E-4
1	2 3 4 5 6 7 8 9 10 12	1.1161 E-4
2	6 7 8 9 10 11 12	0.6724 E-4
3	1 3 4 6 7 8 9 10	0.7372 E-4
4	2 4 5 6 7 8 9 10 11 12	0.5695 E-4
5	1 3 4 5 6 7 8 9 10 12	0.5144 E-4
6	1 3 4 5 6 7 8 9 10 12	0.5061 E-4
7	1 3 4 5 6 7 8 9 10 12	0.5061 E-4

4.5 Approximation on a Continuum

We now return to the original problem of finding the best approximation to a function on an arbitrary compact subset B of the complex plane. Any approach based on a finite but "large" subset of B will give a correspondingly "large" number of constraints in (4.1.9) which is obviously undesirable. Strictly speaking, we only need the number of points in the characteristic set, which does not exceed $2n + 1$, but it is unlikely that their values will be known

Although the majority of real approximations are computed by the second (exchange) algorithm of Remes, there is also a lesser known first algorithm of Remes which is of more general applicability than the second and which does not depend on the characterisation of the best approximation. This algorithm has been successfully used by Watson [1975] to compute multivariate best approximations. To approximate a function f by a generalised polynomial

$p(z) = \sum_{i=1}^{n} a_i \phi_i(z)$, (where f is continuous on B and $\{\phi_i(z)\}_{i=1}^{n}$ is a set of linearly independent continuous functions on B), Using the first algorithm of Remes, we take an initial discrete subset of B containing at least $n + 1$ points which are supplemented at each iteration by the global error of the current approximation. Defining

$$r(a, z) = \sum_{i=1}^{n} a_i \phi_i(z) - f(z)$$

and

$$\Delta(a) = \max_{z \in B} |r(a, z)|,$$

the algorithm may formally be stated as follows.

4.5 Approximation on a Continuum (cont'd)

First Algorithm of Remes

At the k'th step we are given a finite subset $B^{(k)}$ of B. Find the vector $a^{(k)}$ to minimise

$$\Delta^k(a) = \max_{z \in B^{(k)}} \{|r(a, z)|\}$$

Select $z^{(k)} \in B$ to maximise $|r(a^{(k)}, z)|$, so that $|r(a^{(k)}, z^{(k)})| = \Delta(a^{(k)})$. Now restart with the finite set $B^{(k+1)} = B^{(k)} \cup \{z^{(k)}\}$.

Theorem 4.2

The sequence $\Delta^k(a^{(k)})$ is monotone increasing and converges to $\min_a \Delta(a)$. Also the sequence $\{a^{(k)}\}$ is bounded and its cluster points minimise $\Delta(a)$. Moreover, if the base functions $\phi_i(z)$ satisfy the Haar condition, the vectors $\{a^{(k)}\}$ converge to the coefficients of the unique best approximation.

The proof of theorem 4.2 given by Cheney [1960, p.96] is perfectly valid in the complex plane.

When implementing this algorithm, certain modifications can be made in the spirit of those suggested by Watson in his multivariate method. When searching for $z^{(k)} \in B$ to maximise $|r(a^{(k)}, z)|$ we note also the position of other local maxima and include these as well as the global maximum. These are usually found as a matter of course in the search for the global maximum; this clearly cannot upset the convergence and the only effect the modification can have is to accelerate the convergence.

4.5 Approximation on a Continuum (cont'd)

In practice, we start the algorithm with a subset of m points where $n + 1 \leq m \leq 2n + 1$. Computational experience indicates that the Remes algorithm (with the suggested modifications) is quite quickly convergent, and very acceptable approximations may be obtained in only a few iterations. It is tempting to reduce the number of constraints by dropping points from $B^{(k)}$ where the error is small, and which are therefore likely to correspond to inactive constraints. Although this has been tried, it needs to be done with care, and of course invalidates the convergence proof.

Since there is relatively little variation in $a^{(k)}$ from one iteration to the next, the algorithm (4.1) for constructing best approximations on a finite set converges very rapidly, usually in only two or three iterations.

Some examples are presented using the above ideas, the first being given in rather more detail than the others to show the exact working of the algorithm.

Example 1

Approximation of e^z by a polynomial of degree 4 on the semi-disc $B = \{|z| \leq 1, \mathrm{Re}(z) \geq 0\}$. Underlining indicates points corresponding to active constraints in subset approximations.

4.5 Approximation on a Continuum (cont'd)

Example 1 (cont'd)

Iteration 1

$B^{(1)} = \{e^{i\theta}, \theta = \underline{0}°, \underline{30}°, \underline{60}°, \underline{90}°; 0.5i, \underline{0}\}$

$\max_{z\epsilon B^{(1)}} | f(z) - p_4^{(1)}(z) | = 0.367273\,E\text{-}2$

$\max_{z\epsilon B} | f(z) - p_4^{(1)}(z) | = 0.394044\,E\text{-}2$

Local maxima of $| f(z) - p_4^{(1)}(z) |$ on B are $0.394044\,E\text{-}2$ at $\theta = 19.159$ and $0.383139\,E\text{-}2$ at $\theta = 65.757$ (found by subtabulation of the error on a subset of 200 points, parabolic fit in the neighbourhood of the discrete maxima and subsequent checking by further subtabulation).

Iteration 2

$B^{(2)} = \{e^{i\theta}, \quad = 0°, \underline{19.159}°, \underline{30}°, \underline{60}°, \underline{65.757}°, \underline{90}°;$

$0.5i, \underline{0}\}$

$\max_{z\epsilon B^{(2)}} | f(z) - p_4^{(2)}(z) | = 0.377441\,E\text{-}2$

$\max_{z\epsilon B} | f(z) - p_4^{(2)}(z) | = 0.384312\,E\text{-}2$

Local maxima of $| f(z) - p_4^{(2)}(z) |$ on B are $0.384312\,E\text{-}2$ at $24.517°$ and 0.381199 at $63.020°$.

4.5 Approximation on a Continuum

Example 1 (cont'd)

Iteration 3

$B^{(3)} = \{e^{i\theta}, \theta = 0°, \underline{19.159°}, \underline{24.517°}, 30°, 60°, \underline{63.02°}, 65.757°, \underline{90°}; 0.5i, \underline{0}\}$

$$\max_{z\varepsilon B^{(3)}} | f(z) - p_4^{(3)}(z) | = 0.379242\,E\text{-}2$$

$$\max_{z\varepsilon B} | f(z) - p_4^{(3)}(z) | = 0.380886\,E\text{-}2$$

Local maxima are 0.380386 E-2 at 21.865° and 0.379695 at 63.946°.

Iteration 4

$B^{(4)} = \{e^{i\theta}; \underline{19.159°}, \underline{21.865°}, \underline{24.517°}, 60°, 63.02°, \underline{63.946°}, 65.757°, \underline{90°}; 0\}$ (at this stage, points which are obviously not going to appear in the final characteristic set have been dropped).

$$\max_{z\varepsilon B^{(4)}} | f(z) - p_4^{(4)}(z) | = 0.379824\,E\text{-}2$$

$$\max_{z\varepsilon B} | f(z) - p_4^{(4)}(z) | = 0.380213\,E\text{-}2$$

Local maxima are 0.380213 E-2 at 20.525° and 0.379864 E-2 at 63.793°.

4.5 Approximation on a Continuum

Example 1 (cont'd)

Iteration 5

$B^{(5)}$ = { 19.159°, 20.525°, 21.865°, 24.517°, 60°, 63.02°, 63.793°, 63.946°, 65.757°, 90°; 0 }

$$\max_{z\in B^{(5)}} \left| f(z) - p_4^{(5)}(z) \right| = 0.379883\,E\text{-}2$$

$$\max_{z\in B} \left| f(z) - p_4^{(5)}(z) \right| = 0.379982\,E\text{-}2$$

Local maxima are 0.380213 E-2 at 21.20° and 0.379884 at 63.77°.

Iteration 6

$B^{(6)}$ = {20.525°, 21.20°, 21.865°, 63.793°, 63.77°, 63.946°, 90°; 0 }

$$\max_{z\in B^{(6)}} \left| f(z) - p_4^{(6)}(z) \right| = 0.379920\,E\text{-}2$$

$$\max_{z\in B} \left| f(z) - p_4^{(6)}(z) \right| = 0.379921\,E\text{-}2$$

The table below shows the number N of iterations used in algorithm 4.1 for the solution of the discrete approximation problem at each iteration of the Remes algorithm.

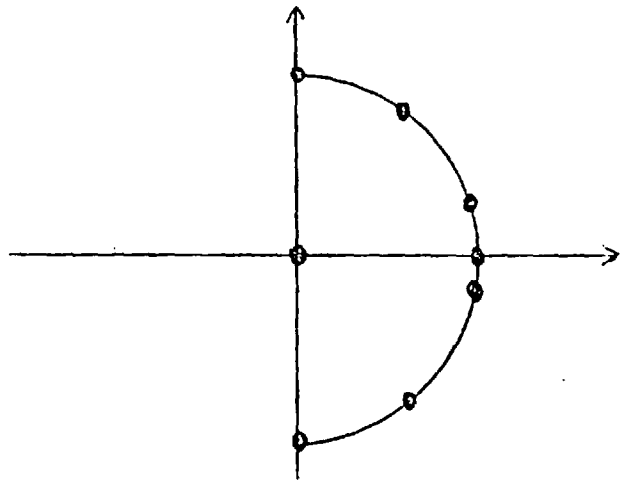
k	N
1	8
2	3
3	3
4	3
5	3
6	2

4.5 Approximation on a Continuum

Example 1 (cont'd)

$$p_4^{(6)}(z) = 1.0037\,9920 + 0.9871\,3539z + 0.5223\,3205z^2 + 0.1419\,5339z^3 + 0.0596\,4899z^4 .$$

This example shows clearly why the characterisation based method of Chapter 2 cannot give an optimal approximation, for which the characteristic set has the 8 points indicated on the diagram.



In Chapter 2 we obtained the inclusion (0.346, 0.451)E-2 on a set of $n+2 = 6$ points.

4.5 Approximation on a Continuum (cont'd)

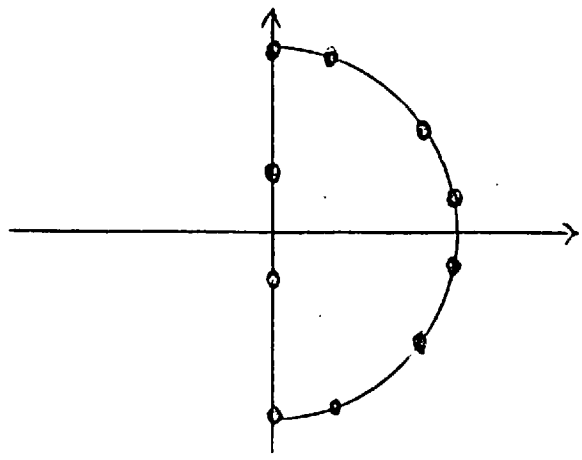
Example 2

Approximation of e^z by a polynomial of degree 6 on the semi-disc.

Remes' Iteration number - k	$\max_{z\in B(k)} \lvert f(z)-p^{(k)}(z)\rvert$	$\max_{z\in B} \lvert f(z)-p^{(k)}(z)\rvert$	Number of iterations of algorithm 4.1 to find $p_6^{(k)}(z)$
1	0.4743 E-4	0.6990 E-4	10
2	0.4924 E-4	0.5678 E-4	3
3	0.5033 E-4	0.5159 E-4	3
4	0.5064 E-4	0.5091 E-4	2
5	0.5068 E-4	0.5076 E-4	2
6	0.5069 E-4	0.5071 E-4	2
7	0.5070 E-4	0.5070 E-4	2

$$p_6^{(7)}(z) = 1.0000\,5063 + 0.9997\,4101z + 0.5006\,6107z^2 + 0.1655\,7197z^3 + 0.0429\,4488z^4 + 0.0072\,6677z^5 + 0.0019\,9563z^6.$$

The characteristic set is indicated on the diagram, and contains $r = 10 =$ points $= n + 4$ points.



4.5 Approximation on a Continuum (cont'd)

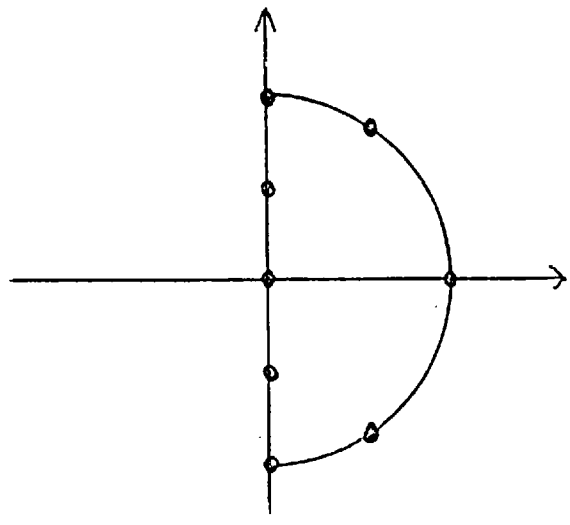
Example 3

Approximation of $1/\sqrt{1+2z}$ by a polynomial of degree 4 on the semi disc.

Remes' Iteration number - k	$\max_{z\in B(k)} \lvert f(z)-p_4^{(k)}(z)\rvert$	$\max_{z\in B} \lvert f(z)-p_4^{(k)}(z)\rvert$	Number of iterations of algorithm 4.1 to find $p_6^{(k)}(z)$
1	0.423068 E-1	0.425862 E-1	5
2	0.423309 E-1	0.423600 E-1	4
3	0.423388 E-1	0.423448 E-1	3
4	0.423407 E-1	0.423421 E-1	3
5	0.423410 E-1	0.423414 E-1	2
6	0.423412 E-1	0.423412 E-1	2

$$p_4^{(6)}(z) = 0.957659 - 0.724654z + 0.597509z^2 - 0.377420z^3 + 0.166597z^4 .$$

The characteristic set contains $8 = n + 4$ points.



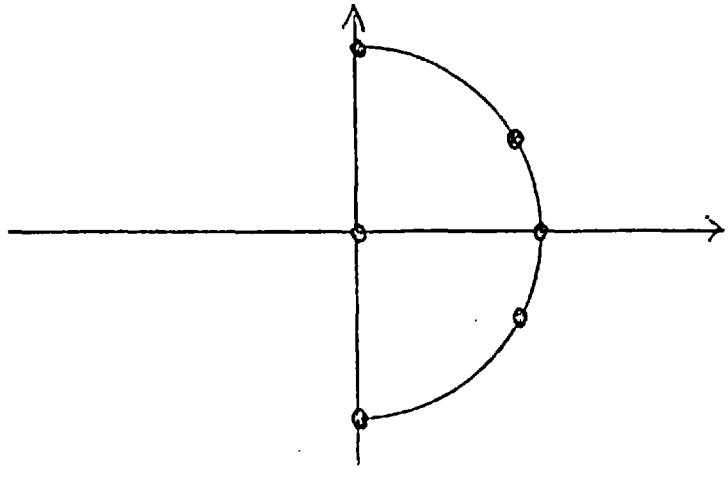
4.5 Approximation on a Continuum (cont'd)

Example 4

To find the Chebyshev polynomial of degree 4 for the semi-disc by constructing the best approximation of degree 3 to $f(z) = z^4$.

Remes' Iteration number - k	$\max_{z\in B(k)} \lvert f(z)-p_3^{(k)}(z)\rvert$	$\max_{z\in B} \lvert f(z)-p_4^{(k)}(z)\rvert$	Number of iterations of algorithm 4.1 to find $p_4^{(k)}(z)$
1	0.4237 4132	0.4373 2665	5
2	0.4281 1873	0.4295 1161	3
3	0.4285 3944	0.4286 9102	3
4	0.4285 8618	0.4286 0250	3
5	0.4285 9118	0.4285 9295	3
6	0.4285 9172	0.4285 9190	3
7	0.4285 9178	0.4285 9180	3
8	0.4285 9179	0.4285 9179	3

The characteristic set contains 6 = n + 3 points.



The Chebyshev polynomial is $T_4(z;B) = z^4$

$- 1.5142\,8994z^3 + 1.6855\,8549z^2 - 1.1712\,9554z + 0.4285\,9179$.

4.5 Approximation on a Continuum (cont'd)

Example 5

As a final illustration of the use of the Remes algorithm, the Chebyshev polynomials $T_k(z;B)$ for the semi disc $B = \{z : |z| \leq 1, \mathrm{Re}(z) \geq 0\}$ have been computed up to degree 8. The results are given in table 4.1. As an indication of their accuracy, the final inclusion interval for the Remes iteration is also given for each polynomial in table 4.2.

4.5 Approximation on a Continuum (cont'd)

Table 4.1

Coefficients of Chebyshev polynomials

$$T_k(z\,;B) = \sum_{i=0}^{k} a_i^{(k)} z^k$$

for $1 \leq k \leq 8$.

k	i	$a_i^{(k)}$
1	0	0.0
	1	1.0
2	0	0.4142 1356
	1	-0.5857 8644
	2	1.0
3	0	-0.4858 6512
	1	1.0
	2	-1.0864 3133
	3	1.0
4	0	0.4285 9179
	1	-1.1712 9554
	2	1.6855 8549
	3	-1.5142 8994
	4	1.0
5	0	-0.3145 6354
	1	1.1334 7188
	2	-2.0695 5879
	3	2.4223 7830
	4	-1.8794 2716
	5	1.0
6	0	0.2345 7300
	1	-1.0454 5226
	2	2.3253 4778
	3	-3.3400 5290
	4	3.3233 4349
	5	-2.2630 6161
	6	1.0
7	0	-0.1765 9236
	1	0.9288 5234
	2	-2.4383 6116
	3	4.1570 7478
	4	-5.0174 8590
	5	4.3682 8413
	6	-2.6476 8982
	7	1.0
8	0	0.1340 3425
	1	-0.8100 0630
	2	2.4463 4299
	3	-4.8236 9049
	4	6.8259 8391
	5	-7.1710 0377
	6	5.5664 9644
	7	-3.0340 1391
	8	1.0

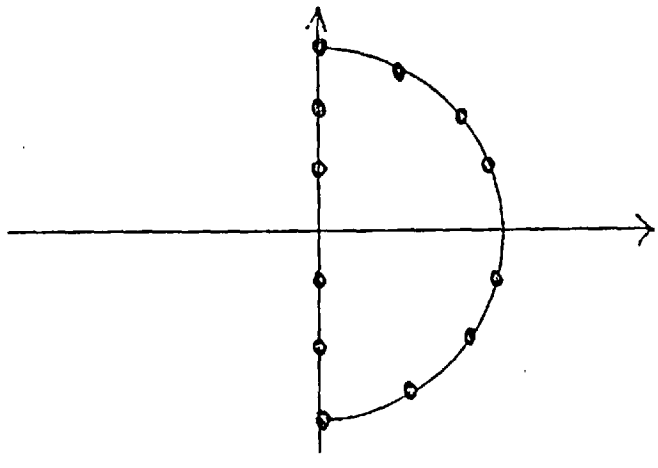
4.5 Approximation on a Continuum (cont'd)

Table 4.2

Inclusion intervals for $T_k(z\,;B)$. (D represents the Characteristic set as computed by the Remes algorithm.)

k	$\max_{z\in D} \lvert T_k(z\,;B)\rvert$	$\max_{z\in B} \lvert T_k(z\,;B)\rvert$
1	1.0	1.0
2	0.8284 2712	0.8284 2712
3	0.6005 6621	0.6005 6621
4	0.4285 9179	0.4285 9179
5	0.3145 6354	0.3145 6355
6	0.2346 9750	0.2346 9750
7	0.1768 8179	0.1768 8181
8	0.1341 4311	0.1341 4315

Note: For $T_7(z\,;B)$, which is constructed from a best approximation of degree 6, the characteristic set



contains 12 points, so that this is another example where the characteristic set has almost the maximum possible size (for this example, we have $8 \leq r \leq 13$).

CHAPTER 5

THE CONSTRUCTION OF BEST RATIONAL APPROXIMATIONS BY PROGRAMMING METHODS

5.1 Introduction

In contrast to the case of polynomial approximation, the use of programming methods for complex rational approximation does not seem to have been previously considered. Since any method based on programming can only construct best approximations on a finite point set, there are the usual problems with existence, and a locally best approximation may not be globally best.

The initial problem of this chapter is to find an approximation of the form

$$r_{mn}(z) = p(z)/q(z) = \sum_{r=0}^{m} a_r z^r \Big/ \sum_{r=0}^{n} b_r z^r$$

to a function f continuous on a finite set D of M points where $M \geq N$ $(= m + n + 2)$ such that

$$\max_{1 \leq j \leq m} | f(z_j) - r_{mn}(z_j) | = \| f - r_{mn} \| \qquad (5.1.1)$$

is minimised subject to $q(z)$ being bounded on D. To make any progress, we shall have to assume that f and D are such that a solution exists. The motivation for the consideration of programming based methods is (like the polynomial case) the avoidance of the theoretically unjustifiable limitations of working with approximations based on a restricted number of points as is necessary with methods connected with characterisation.

5.2 Choice of Algorithm

In the corresponding problem where D is a real point set, and f and r_{mn} are real functions, there are three principal methods available for the minimisation of (5.1.1). These have been listed and tested by Lee and Roberts [1973].

The general idea of each algorithm is to produce an iteration, each step of which is a linear problem and which, in the real case, is amenable to solution by linear programming. Apart from details such as the normalisation of the approximations and starting conditions, the methods are based on the following iterations:

(i) Loeb's Algorithm (see for example Barrodale & Mason [1970])

Find p_{k+1} and q_{k+1} so that

$$\max_j \frac{|f(x_j)q(x_j) - p(x_j)|}{|q_k(x)|}$$

is minimised over all polynomials p(x), q(x).

(ii) The Osborne-Watson Algorithm

Find δp_k and δq_k so that

$$\max_j \left| f(x_j) - \frac{p_k(x_j)}{q_k(x_j)} - \frac{q_k(x_j)\delta p(x_j) - p_k(x_j)\delta q\ (x_j)}{q_k^{\,2}\ (x_j)} \right| \qquad (5.2.1)$$

is minimised over all polynomials δp, δq. Then set

$p_{k+1} = p_k + \lambda_k \delta p_k$, $q_{k+1} = q_k + \lambda_k \delta q_k$ where

$$\left\| f - \frac{p_k + \lambda_k \delta p_k}{q_k + \lambda_k \delta q_k} \right\| \leq \left\| f - \frac{p_k + \lambda \delta p_k}{q_k + \lambda \delta q_k} \right\| \qquad (5.2.2)$$

for all $\lambda \in [0, 1]$.

5.2 Choice of Algorithm (cont'd)

(iii) The ~~Deferred~~ Differential Correction Method

(a) The original method (abbreviated to ODC)
(Cheney and Loeb [1961])

Find p_{k+1}, q_{k+1} to minimise

$$\max_j \frac{|f(x_j)\, q(x_j) - p(x_j)| - \Delta_k\, q(x_j)}{q_k(x_j)} \qquad (5.2.3)$$

where $\Delta_k = \| f - p_k/q_k \|$ (5.2.4)

(b) The later method (abbreviated to DC)
(Cheney and Loeb [1962])

This method is the same as the ODC method except that the term $q_k(x_j)$ in the denominator of (5.2.3) is omitted.

Each of these methods can be adapted for use in the complex plane, but any algorithm which is unsatisfactory in the real case is unlikely to provide a good algorithm for complex approximation. We have therefore rejected Loeb's algorithm because it is not guaranteed to converge (Lee and Roberts [1973]).

The convergence of the Osborne-Watson algorithm is guaranteed in the real case under certain conditions, but Osborne and Watson say "the values of λ may become very small and progress can be intolerably slow; consequently the initial approximation is often critical, especially when a large number of parameters are involved." Nevertheless this was the method adopted by Ellacott and Williams [1976] who have proved the algorithm convergent in the complex plane

5.2 Choice of Algorithm (cont'd)

and successfully used it to construct best approximations. They do not, however, report on the rate of convergence of their method except to say that a maximum of 20 iterations were used. They have also used an entirely different approach to the linear sub-problems which are solved by a form of Lawson's algorithm .

In the light of the comments made by Lee and Roberts, and the results of Ellacott and Williams not being available at the time this work was carried out, we have chosen the ODC method. This method has an interesting history in that the original method was superseded because the DC method was considered to have more certain convergence properties. However the ODC Method was reinvestigated by Barrodale, Powell and Roberts [1972] who showed that not only did the ODC method have sure convergence properties but that under reasonable conditions (the existence of a non-degenerate best approximation) the method was quadratically convergent, a property not shared by the later version. The numerical tests of Lee and Roberts [1973] also confirmed its vast superiority over the later method, yet without requiring any significant increase in computation. This led to their recommending the use of the ODC method for problems where the Remes exchange algorithm failed.

5.3 Implementation and Convergence

In order to adapt the ODC algorithm for use in the complex plane, we square the moduli in (5.2.3) and (5.2.4), and the complete algorithm becomes:

Step 1 : choose initial approximations $p_0(z)$ and $q_0(z)$;

Step 2 : set $k = 0$;

Step 3 : set $\Delta_0 = \| f - p_0/q_0 \|^2$;

Step 4 : find $p_{k+1}(z)$ and $q_{k+1}(z)$ to minimise

$$\max_j \left\{ \frac{|f(z_j)q(z_j) - p(z_j)|^2 - \Delta_k |q(z_j)|^2}{|q_k(z_j)|^2} \right\}; \qquad (5.3.1)$$

Step 5 : set $\Delta_{k+1} = \| f - p_{k+1}/q_{k+1} \|^2$

Step 6 : if Δ_k and Δ_{k+1} are sufficiently close then stop, else increase k by 1 and return to Step 4.

Before establishing the convergence of the algorithm, we need to discuss the choice of p_0 and q_0. In the absence of any known initial approximation, we have followed the suggestion of Lee and Roberts [1973] and obtained p_0 and q_0 by minimising

$$\| fq - p \| \quad \text{subject to} \quad b_0 = 1. \qquad (5.3.2)$$

This frequently provides a good initial approximation to f, although it is possible for q to be zero on D. In this case we borrow an idea from Ellacott and Williams [1976] and choose

$$\frac{p_0}{q_0} = \frac{\lambda \overline{p}_0}{(1-\lambda) + \lambda \overline{q}}$$

where $\overline{p}_0, \overline{q}_0$ is the solution of (5.3.2), $0 \leq \lambda \leq 1$ and λ is chosen as the largest value so that p_0/q_0 is pole free on D.

5.3 Implementation and Convergence (cont'd)

This is certainly possible because $\lambda = 0$ at least gives an admissible starting approximation, albeit a poor one. The actual minimisation of (5.3.2) corresponds to a linear approximation problem with respect to the space spanned by the functions

$$\{\phi\} = \{1, z, \ldots, z^m, zf, z^2f, \ldots, z^nf\}.$$

A unique best approximation will therefore exist if the set $\{\phi\}$ is a Chebyshev set (we note that this is the same condition imposed by Williams [1972] in his characterisation based method) and is readily computed using the algorithm in the previous chapter. This starting algorithm has been successfully used for all the numerical examples.

Returning to the main algorithm, the convergence properties are similar to those for the real case, and are given by the following result.

Theorem 5.1

Let $f \in C[D]$ have a unique non-degenerate best approximation r^*_{mn}. Assume that $q_k(z_j) \neq 0$, $1 \leq j \leq M$. If $\Delta_k \neq \Delta^*$ then

(i) $q_{k+1}(z_j) \neq 0$, $1 \leq j \leq M$;

(ii) $\Delta_{k+1} < \Delta_k$;

(iii) the sequence $\{\Delta_k\}$ converges to Δ^*.

5.3 Implementation and Convergence

Theorem 5.1 (cont'd)

Proof

With suitable modifications, the proof corresponds to that of Barrodale, Powell and Roberts [1972] for the real case.

Let $p^{\dagger}(z)/q^{\dagger}(z)$ be any approximation such that

$$q^{\dagger}(z_j) \neq 0, \quad (1 \leq j \leq M) \tag{5.3.3}$$

and

$$\max_j \; |q^{\dagger}(z_j)| = 1 \tag{5.3.4}$$

whose maximum error $\Delta^{\dagger}$ on D is less than Δ_k.

Define

$$\ell^{\dagger} = \min_j |q^{\dagger}(z_j)|^2 > 0$$

and u as the maximum attainable value of any polynomial $q(z)$ satisfying (5.3.4). Because of the minimum property of p_{k+1} and q_{k+1}, we have the inequality

$$\max_j \left\{ \frac{|f(z_j)q_{k+1}(z_j) - p_{k+1}(z_j)|^2 - \Delta_k |q_{k+1}(z_j)|^2}{|q_k(z_j)|^2} \right\}$$

$$\leq \max_j \left\{ \frac{|f(z_j)q^{\dagger}(z_j) - p^{\dagger}(z_j)|^2 - \Delta_k \; q^{\dagger}(z_j)|^2}{|q_k(z_j)|^2} \right\} \tag{5.3.5}$$

$$\leq \max_j \left\{ \left[\left| f(z_j) - \frac{p^{\dagger}(z_j)}{q^{\dagger}(z_j)} \right|^2 - \Delta_k \right] \frac{|q^{\dagger}(z_j)|^2}{|q_k(z_j)|^2} \right\}$$

$$\leq \max_j \; (\Delta^{\dagger} - \Delta_k) \frac{|q^{\dagger}(z_j)|^2}{|q_k(z_j)|^2}$$

$$= -(\Delta_k - \Delta^{\dagger}) \min_j \frac{|q^{\dagger}(z_j)|^2}{|q_k(z_j)|^2}$$

$$< -\ell^{\dagger}(\Delta_k - \Delta^{\dagger})/u^2. \tag{5.3.6}$$

5.3 Implementation and Convergence

Theorem 5.1

Proof (cont'd)

From (5.3.5) and (5.3.6) it follows that

$$\Delta_k \frac{|q_{k+1}(z_j)|^2}{|q_k(z_j)|^2} \geq \ell^\dagger(\Delta_k - \Delta^\dagger)/u^2, \quad 1 \leq j \leq M \tag{5.3.7}$$

and so $q_{k+1}(z_j) \neq 0$ for $1 \leq j \leq M$ which proves (i).

Also from (5.3.5) and (5.3.7) we have that

$$\{|f(z_j)q_{k+1}(z_j) - p_{k+1}(z_j)|^2 - \Delta_k |q_{k+1}(z_j)|^2\} / |q_k(z_j)|^2 < 0$$

for all values of j, and so

$$\left| f(z_j) - \frac{p_{k+1}(z)}{q_{k+1}(z)} \right|^2 < \Delta_k, \quad (1 \leq j \leq M).$$

Since Δ_{k+1} maximises the left hand side of this inequality, it follows that

$$\Delta_{k+1} < \Delta_k$$

which proves (ii).

Since the sequence $\{\Delta_k\}$ decreases monotonically and is bounded below (by Δ^*), it has a limit $\overline{\Delta}$ say. To prove (iii) we make the assumption that $\overline{\Delta} > \Delta^*$ and show that this leads to a contradiction. Let $p^\dagger(z)/q^\dagger(z)$ be any approximation satisfying (5.3.3) and (5.3.4) with maximum error $\Delta^\dagger$ on D is less than $\overline{\Delta}$. Combining (5.3.5) and (5.3.6) we have, for all j,

$$\left| f(z_j) - \frac{p_{k+1}(z_j)}{q_{k+1}(z_j)} \right|^2 - \Delta_k \leq - \frac{\ell^\dagger(\Delta_k - \Delta^\dagger)}{u^2} \frac{|q_k(z_j)|^2}{|q_{k+1}(z_j)|^2}.$$

5.3 Implementation and Convergence

Theorem 5.1

Proof (cont'd)

Let ξ_k be the member of D for which $|q_k(z_j)/q_{k+1}(z_j)|$ is least. Then, in particular,

$$\Delta_{k+1} - \Delta_k \leq -\frac{\ell^\dagger(\Delta_k - \Delta^\dagger)}{u^2} \frac{|q_k(\xi_k)|^2}{|q_{k+1}(\xi_k)|^2}$$

or

$$\Delta_{k+1} - \Delta_k \leq -\frac{\ell^\dagger(\bar{\Delta} - \Delta^\dagger)}{u^2} \frac{|q_k(\xi_k)|^2}{|q_{k+1}(\xi_k)|^2}.$$

Since $\{\Delta_k\}$ converges and $(\bar{\Delta} - \Delta^\dagger) > 0$, it follows that

$$\lim_{k\to\infty} \left| \frac{q_k(\xi_k)}{q_{k+1}(\xi_k)} \right| = 0. \tag{5.3.8}$$

From (5.3.7) we have

$$|q_{k+1}(z_j)|^2 \geq \frac{\ell^\dagger(\Delta_k - \Delta^\dagger)}{u^2\Delta_k} |q_k(z_j)|^2 \tag{5.3.9}$$

$$\geq c\,|q_k(z_j)|^2, \quad (1 \leq j \leq M).$$

where c is the positive constant

$$c = \min\left\{\frac{1}{2},\ \ell^\dagger(\bar{\Delta} - \Delta^\dagger)/(u^2\bar{\Delta})\right\}.$$

Because of (5.3.8), there is a constant K such that

$$\left| \frac{q_k(\xi_k)}{q_{k+1}(\xi_k)} \right| \leq c^{M/2} \quad \text{for all} \quad k \geq K. \tag{5.3.10}$$

5.3 Implementation and Convergence

Theorem 5.1

Proof (cont'd)

Combining (5.3.9) and (5.3.10), we find that

$$\prod_{j=1}^{M} |q_{k+1}(z_j)|^2 \geq \frac{c^{M-1}}{c^M} \prod_{j=1}^{M} |q_k(z_j)|^2$$

$$= \frac{1}{c} \prod_{j=1}^{M} |q_k(z_j)|^2$$

$$\geq 2 \prod_{j=1}^{M} |q_k(z_j)|^2 .$$

It follows that $\|q_k\|$ increases indefinitely as $k \to \infty$ in contradiction to the normalisation. Thus the assumption that $\overline{\Delta} > \Delta^*$ is invalid, and so $\overline{\Delta} = \Delta^*$ which proves (iii).

Note 1

The proof of quadratic convergence does not appear to carry over to the complex plane because of the lack of a theorem corresponding to the strong uniqueness of real rational approximation on which the proof depends. However, the convergence appears to be quadratic in all the examples we have tried. This was also reported by Kaufman and Taylor [1974] for the use of the ODC algorithm in rational approximation of functions of more than one variable, where the proof of quadratic convergence is not valid either.

5.3 Implementation and Convergence (cont'd)

Note 2

Property 1 of theorem 5.1 shows that provided the initial approximation is such that $q_0(z_j) \neq 0$, $(j = 1, \ldots, M)$, then $q_k(z_j) \neq 0$ for any k. If $0 \in D$ (and a linear transformation will always ensure that this condition is satisfied), then $b_0 \neq 0$ for every approximation $p_k(z)/q_k(z)$ and we may therefore use the alternative normalisation $b_0 = 1$. We have used this condition for all the numerical examples, since it simplifies the minimisation of (5.3.1) without any problems arising. It would be possible for some $b_r (1 \leq r \leq n)$ to become very large in a near-degenerate problem in which case it might be advisable to return to the original normalisation.

Note 3

The function to be minimised in (5.3.1) is a quadratic in $a_r (r = 0, 1, \ldots, m)$ and $b_r (r = 1, 2, \ldots, n)$ but unlike the polynomial case (or even Loeb's algorithm) it is not necessarily convex. The convergence of the algorithm which accomplishes the minimisation (Robinson's method, as in Chapter 4) is only assured for a sufficiently good initial approximation. It is a tribute to the robustness of Robinson's method that in no example considered has there been any difficulty with the minimisation.

A simple example will show the occurrence of non-convexity, although the function (5.3.1) still has a unique global minimisation in this example. Consider the

5.3 Implementation and Convergence

Note 3 (cont'd)

approximation of $1+z$ by a function of the form $a/(1+bz)$ on the set $D = \{0, 1, i, -i\}$ where we may take a and b to be real. The initial approximation is found by minimising

$$\max_j \; |(1 + z_j)(1 + bz_j) - a|^2$$

$$= \max \{ (1 - a)^2, (2 + 2b - a)^2 , (1 - b - a)^2 + (1+b)^2 \}$$

The minimisation of this convex function is easily found to be $a = 1\frac{1}{2}$, $b = -\frac{1}{2}$ and

$$\Delta_0 = \max_{z \in D} \left| (1 + z) - \frac{1\frac{1}{2}}{1 - \frac{1}{2}z} \right|^2 = 1.$$

The first step of the ODC algorithm becomes the minimisation of

$$\max_{z \in D} \left\{ \frac{|(1 + z)(1 + bz) - a|^2 - |1 + bz|^2}{|1 - \frac{1}{2}z|^2} \right\}$$

$$= \max\{(1 - a)^2 - 1, 4[(2 + 2b - a)^2 - (1 + b)^2], \tfrac{4}{5}[(1 - b - a)^2 + 2b]\} \qquad (5.3.11)$$

This has a unique minimum at $a = 1.2866$, $b = -0.4905$. To demonstrate the form of the function (5.3.11), it has been sketched in graphs 1 - 3 in the neighbourhood of the minimum for

(i) variations in β with $a = 1.2866$, $b = -0.4905 + \beta$;

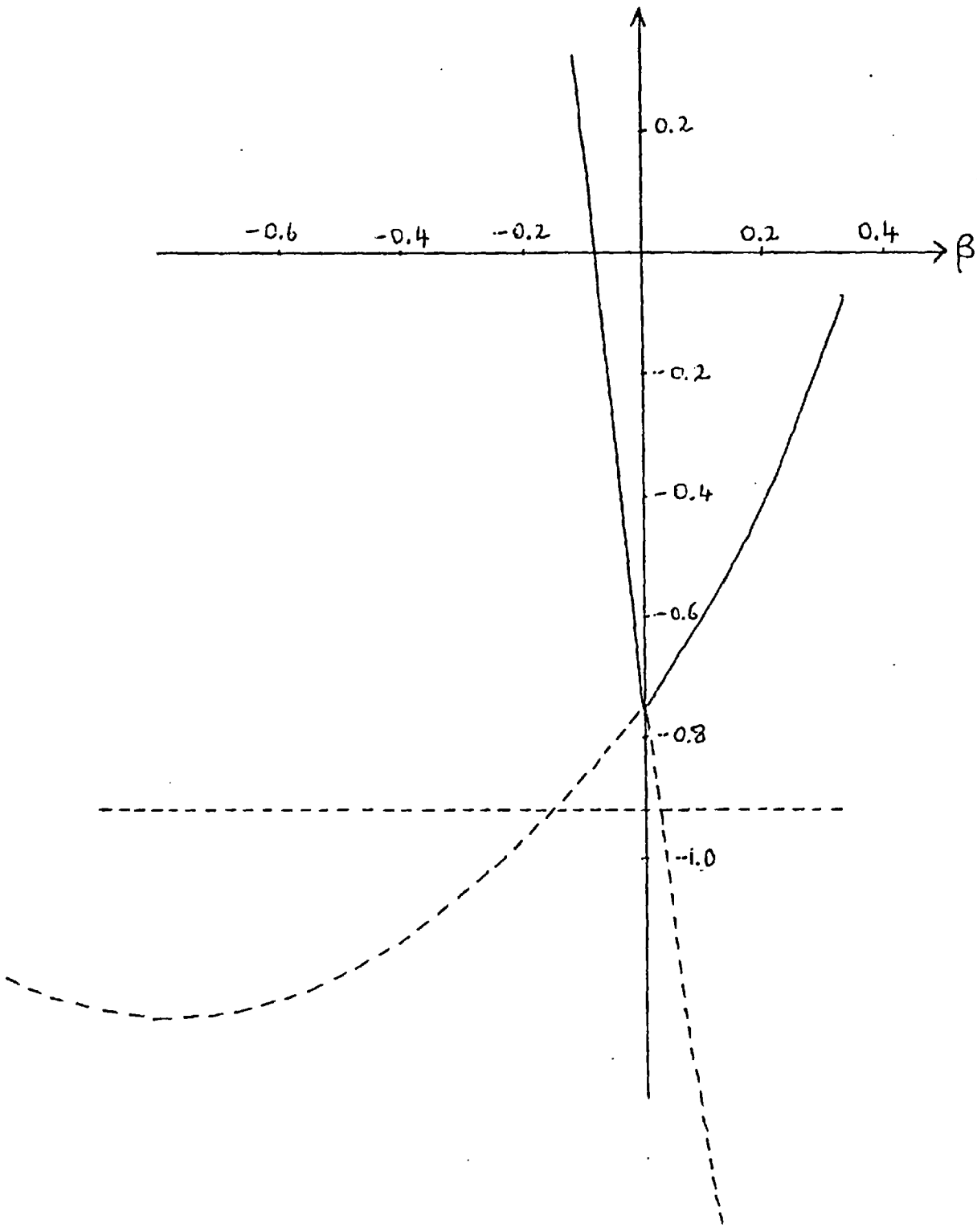
(ii) variations in α with $a = 1.2866 + \alpha$, $b = -0.4905$;

(iii) variations in β with $\alpha = 2\beta$, $a = 1.2866 + \alpha$, $b = -0.4905 + \beta$.

The last graph clearly shows the lack of convexity.

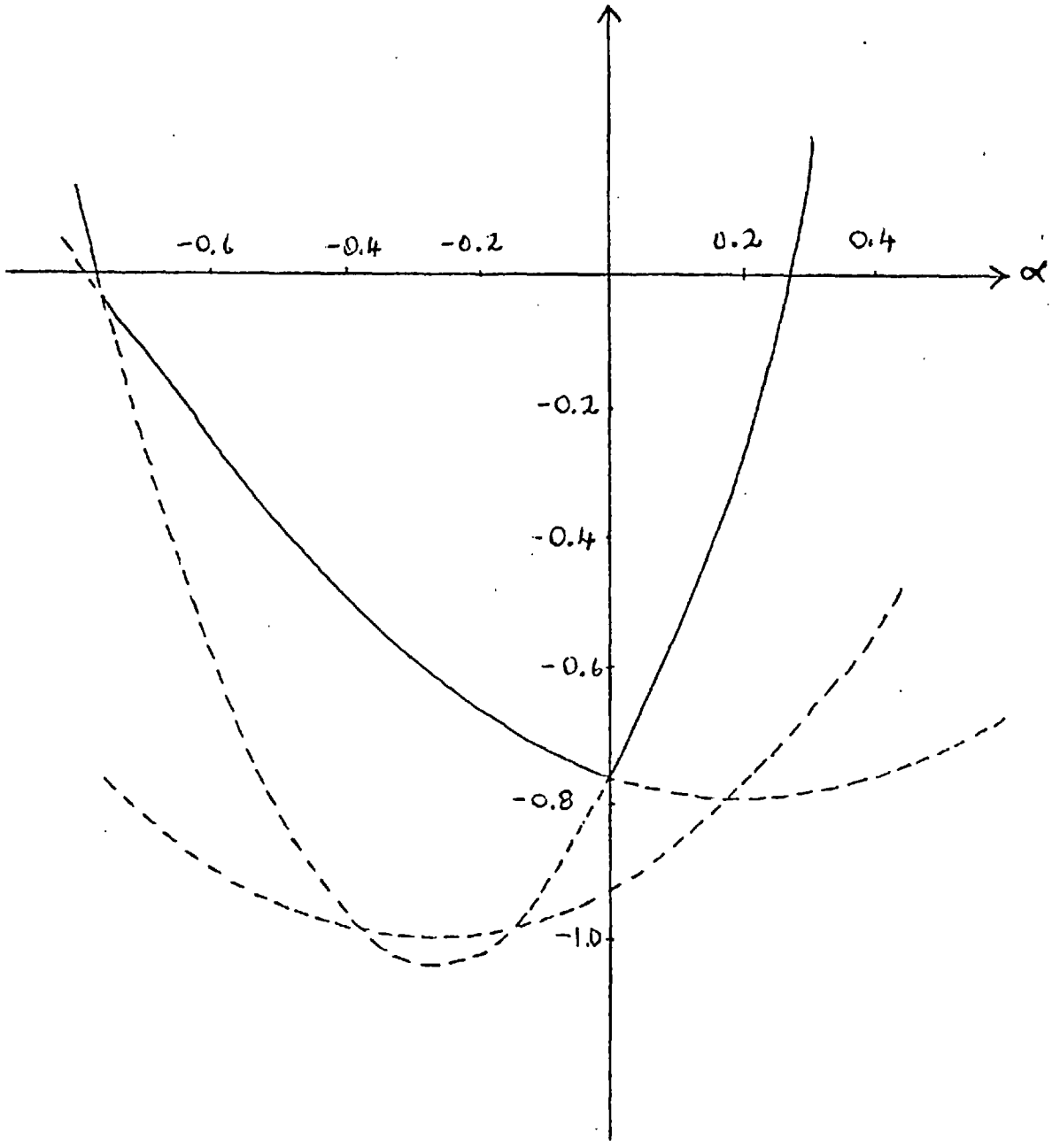
GRAPH 1

Sketch of (5.3.11) with
$a = 1.2866, \quad b = -0.4905 + \beta.$



GRAPH 2

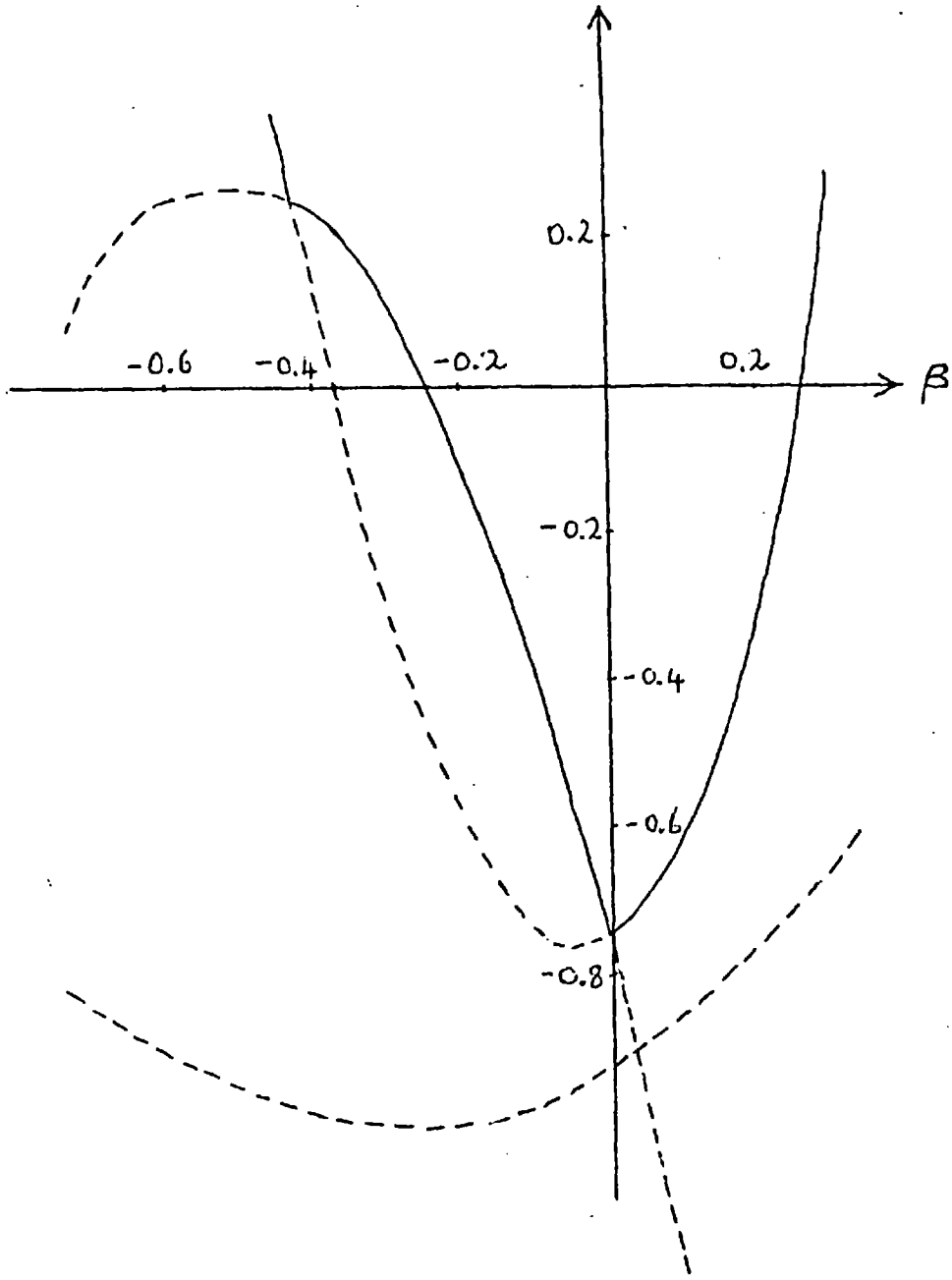
Sketch of (5.3.11) with
$a = 1.2866 + \alpha$, $b = -0.4905$



GRAPH 3

Sketch of (5.3.11) with

$a = 1.2866 + \alpha$, $b = -0.4905 + \beta$, where $\alpha = 2\beta$.



5.3 Implementation and Convergence

For this approximation, we find

$$\| f - p_1/q_1 \| = 0.5254.$$

The progress of the method may be seen by constructing the best approximation which, for this example, can be directly found by minimising

$$\max_{z \in D} \left| (1 + z) - \frac{a}{1 + bz} \right|$$

$$= \max\left\{ (a-1)^2, \left(\frac{a}{1+b} - 2 \right)^2, \frac{(a-1+b)^2 + (1+b)^2}{1+b^2} \right\} \qquad (5.3.12)$$

The direct problem is non-linear, but in this particular case may be solved to give $a = 1.432$, $b = -0.432$ with

$$\| f - p^*/q^* \| = 0.5214,$$

so that p_1/q_1 is nearly optimal.

5.4 Numerical Examples

To illustrate the working of the ODC algorithm, several approximations have been computed, mainly on subsets of the boundary of the semidisc $\{ z : |z| \leq 1, \ \mathrm{Re}(z) \geq 0 \}$. The reason for using subsets containing relatively few points will be apparent in the next section. The results demonstrate the fast (and apparently quadratic) convergence of the sequence $\{\Delta_k\}$, the algorithm being initiated in each case by minimising (5.3.2). In no example did this fail to give a pole-free initial approximation. The number of iterations K_{min} of the algorithm from Chapter 4 used to solve the quadratic subproblems (5.3.1) and (5.3.2) are also noted to show that the lack of convexity (except in the first step) does not seem to

5.4 Numerical Examples (cont'd)

inhibit the convergence (or even the rate of convergence). The asterisks on the points of D denote those corresponding to active constraints at the minimum of (5.3.1) when $\Delta = \Delta^*$. In other words, the best approximation on D is also the best approximation on the subset D^* of asterisked points.

Example 1

Approximation of e^z on the sets: (θ in degrees throughout)

$\{e^{i\theta} : \theta = 0, \pm 22.5^*, \pm 45^*, \pm 67.5 ; iy : y = \pm 1^*, \pm 2/3, \pm 1/3, 0\}$ $m = n = 1$

$\{e^{i\theta} : \theta = 0^*, \pm 30^*, \pm 60^* ; iy : y = \pm 1^*, \pm 0.5, 0^*\}$ $m = n = 2$

$\{e^{i\theta} : \theta = 0, \pm 15^*, \pm 30, \pm 45^*, \pm 60, \pm 75^* ; iy : y = \pm 1^*, \pm 0.5^*, 0^*\}$ $m = n = 3$

m = n	Iteration k	Δ_k	K_{min}	$\| f - r_{mn}^* \|$
1	0	0.6384 1961 E-2	7	
	1	0.5593 0410 E-2	4	
	2	0.5590 9259 E-2	3	
	3	0.5590 9257 E-2	2	
	4	0.5590 9257 E-2	1	0.7477 E-1
2	0	0.419 E-6	8	
	1	0.419 E-6	8	0.6477 E-2
3	0	0.69 E-11	9	
	1	0.69 E-11	4	0.2607 E-5

5.4 Numerical Examples (cont'd)

Example 2

Approximation of $\ln(1+z)$ on the set

$\{e^{i\theta} : \theta = 0, \pm 22.5^*, \pm 45^*, \pm 67.5^* ; iy : y = \pm 1^*, \pm 0.75, \pm 0.5^*, \pm 0.25, 0^*\}$

m=n	Iteration k	Δ_k	K_{min}	$\| f - r_{mn}^* \|$
2	0	0.7522 E-7	3	
	1	0.7521 E-7	1	0.2743 E-3

Example 3

Approximation of $1/\sqrt{1+2z}$ on the sets

$\{e^{i\theta} : \theta = 0^*, \pm 30, \pm 60 ; iy : y = \pm 1^*, \pm 0.5^*, 0^*\}$ $m = n = 1$

$\{e^{i\theta} : \theta = 0^*, \pm 30^*, \pm 60 ; iy : y = \pm 1^*, \pm 0.5^*, 0^*\}$ $m = n = 2$

$\{e^{i\theta} : \theta = 0^*, \pm 30, \pm 60^* ; iy : y = \pm 1^*, \pm 0.5^*, 0^*\}$ $m = n = 3$

m=n	Iteration k	Δ_k	K_{min}	$\| f - r_{mn}^* \|$
1	0	0.1984 111 E-3	6	
	1	0.1846 656 E-3	3	
	2	0.1846 653 E-3	2	
	3	0.1846 653 E-3	1	0.1359 E-1
2	0	0.3157 E-6	7	
	1	0.2886 E-6	4	
	2	0.2885 E-6	3	0.5371 E-3
3	0	0.319 E-9	7	
		0.284 E-9	4	
		0.284 E-9	3	0.1685 E-4

No problems were encountered in this example, which failed to converge in the method based on characterisation when $m = n = 2$ or 3.

5.4 Numerical Examples (cont'd)

Example 4

The approximation of $\tan 1.5z$ on the following subsets of the unit disc by the rational function $zr_{mn}(z^2)$

$\{e^{i\theta} : \theta = 0^*, \pm 15^*, \pm 30, \pm 60, \pm 90^*, \pm 120, \pm 150, \pm 165^*, \pm 180^*\}$ $m = n = 1$

$\{e^{i\theta} : \theta = 0^*, \pm 15^*, \pm 30^*, \pm 60^*, \pm 90^*, \pm 120^*, \pm 150^*, \pm 165^*, \pm 180^*\}$ $m = n = 2$

m=n	Iteration k	Δ_k	K_{min}	$\| f - r^*_{mn} \|$
1	0	0.1144 5584 8 E-1	13	
	1	0.2292 902 E-3	11	
	2	0.1662 054 E-3	8	
	3	0.1661 927 E-3	2	
	4	0.1661 927 E-3	1	0.1289 E-1
2	0	2.273 E-10	10	
	1	0.668 E-10	4	
	2	0.668 E-10	2	0.817 E-5

This example was chosen because there is a pole of $f(z)$ near the unit disc (at $z = \pm 1.0472$), but this did not cause any serious difficulties.

5.5 The Continuous Problem

The ultimate aim of this chapter is to exhibit an algorithm for constructing a best rational approximation on a continuum, and the general comments at the beginning of section 4.5 are also applicable here. Under certain conditions, the first algorithm of Remes may be used for rational approximation, and provides a practical method for constructing very nearly optimal approximations on only relatively small subsets of B.

First, define

$$r(c, z) = \sum_{r=0}^{m} a_r z^r \Big/ \sum_{r=0}^{n} b_r z^r - f(z)$$

where $c = (a_0, a_1, \ldots, a_m, b_0, b_1, \ldots, b_m)$

and

$$\delta(c) = \max_{z \in B} |r(c, z)|$$

(we use δ rather than Δ to avoid confusion with the Δ_k used in the ODC algorithm).

First Algorithm of Remes

At the k'th step we are given a finite subset $B^{(k)}$ of B.

Find the vector $c^{(k)}$ to minimise

$$\delta^k(c) : \max_{z \in B^{(k)}} |r(c, z)| .$$

Select $z^{(k)} \in B$ to maximise $|r(c^{(k)}, z)|$, so that

$$|r(c^{(k)}, z^{(k)})| = \delta(c^{(k)}) .$$

Now start with the finite set $B^{(k+1)} = B^{(k)} \cup \{z^{(k)}\}$.

5.5 The Continuous Problem (cont'd)

Theorem 5.2

If

(i) f has a unique non-degenerate best approximation r^* on B,

and

(ii) there exists a constant $\alpha > 0$ such that for every k, $|q_k(z)| > \alpha$ for all $z \in B^{(k)}$,

then the sequence $\{\delta^k(c^{(k)})\}$ is monotone increasing and converges to $p = \min_c \delta(c)$. Also the vectors $\{c^{(k)}\}$ converge to the coefficients of the best approximation.

Proof

Whereas the proof of this theorem for polynomial approximation extends directly to the complex domain, the case of rational approximation needs certain modifications and we therefore give a complete proof.

Since by construction, each approximation has the property that $|c_i^{(k)}| \leq 1$, the vectors $c^{(k)}$ are bounded. The monotone nature of the sequence $\{\delta^k(c^{(k)})\}$ follows from the inclusions $B^{(k)} \subset B^{(k+1)} \subset B$, so that

$$\delta^k(c) \leq \delta^{k+1}(c) \leq \delta(c).$$

Hence

$$\delta^k(c^{(k)}) \leq \delta^{k+1}(c^{(k+1)}) \leq p$$

and thus $\{\delta^k(c^{(k)})\}$ is monotone increasing with a limit of say $p - \varepsilon (\varepsilon \geq 0)$.

5.5 The Continuous Problem (cont'd)

Theorem 5.2

Proof (cont'd)

Since z^{ℓ} $(0 \leq \ell \leq m+n)$ is bounded on B and $|c_i| \leq 1$, $|d_i| \leq 1$, there exists a positive constant F such that

$$|r(d,z) - r(c,z)| \leq \frac{F}{\alpha^2} |d - c|, \quad z \in B,$$

for all approximations with denominator bounded below in modulus by α, or

$$|r(d,z) - r(c,z)| \leq G|d - c| \text{ say}.$$

Therefore

$$|r(d,z)| \leq |r(c,z)| + G|d - c|$$

and

$$\delta(d) \leq \delta(c) + G|b - c|.$$

We make the assumption that $c > 0$ and show that this leads to a contradiction.

Let d denote any cluster point of the sequence $\{c^{(k)}\}$. For any $\zeta > 0$, there is an index k such that $|d - c^{(k)}| < \zeta$ and an index $i > k$ such that $|d - c^{(i)}| < \zeta$. Thus $|c^{(k)} - c^{(i)}| < 2\zeta$, and

$$\begin{aligned} p \leq \delta(d) &\leq \delta(c^{(k)}) + G\zeta \\ &= r(c^{(k)}, z^{(k)}) + G\zeta \\ &= r(c^{(i)}, z^{(k)}) + 3G\zeta \\ &\leq \delta^i(c^{(i)}) + 3G\zeta \end{aligned}$$

or

$$p \leq \delta(d) \leq p - \varepsilon + 3G\zeta.$$

If ζ is chosen so that $3G\zeta < \varepsilon$, then this is a contradiction, and we conclude that $\varepsilon = 0$ and $\delta(d) = p$. Moreover, since we

5.5 The Continuous Problem

Theorem 5.2

Proof (cont'd)

have assumed in the statement of the theorem that there is a unique point c^* for which $\delta(c^*) = \min_c \delta(c)$, the bounded sequence $\{c^{(k)}\}$ can only have one cluster point to which $\{c^{(k)}\}$ converges. This completes the proof.

A few examples will demonstrate the effectiveness of this algorithm in constructing best approximations. For each step of the algorithm, we indicate the number of iterations of the ODC algorithm used to find $c^{(k)}$, the number of points $M^{(k)}$ in the subset $B^{(k)}$, and also the inclusion

$$\delta^k(c^{(k)}) \leq p \leq \delta(c^{(k)})$$

which shows how the Remes algorithm is progressing. Whenever possible, the final results are compared with those from Chapter 3 obtained by characterisation.

5.5 The Continuous Problem (cont'd)

Example 1

Approximation of e^z on the semi disc $\{|z| \leq 1, \mathrm{Re}(z) \geq 0\}$

m=n	Iteration k	$M^{(k)}$	W	$\delta^k(c^{(k)})$	$\delta(c^{(k)})$	Final inclusion by characterisation
1	1	14	4	0.748 E-1	0.817 E-1	
	2	16	4	0.762 E-1	0.774 E-1	
	3	18	4	0.767 E-1	0.769 E-1	
2	1	10	3	0.6477 20 E-3	0.7791 46 E-3	
	2	14	3	0.6830 79 E-3	0.6911 75 E-3	
	3	18	2	0.6843 69 E-3	0.6862 68 E-3	
	4	14	2	0.6849 28 E-3	0.6850 35 E-3	
	5	10	2	0.6849 61 E-3	0.6849 83 E-3	
	6	14	2	0.6849 68 E-3	0.6849 68 E-3	0.574 E-3, 0.953 E-3
3	1	16	3	0.2607 E-5	0.2760 E-5	
	2	18	3	0.2640 E-5	0.2663 E-5	
	3	26	3	0.2644 E-5	0.2650 E-5	
	4	24	3	0.2646 E-5	0.2647 E-5	0.259 E-5, 0.306 E-5

Coefficients for $r_{22}(z)$ and $r_{33}(z)$

m = n	k	$a^{(k)}$	$b^{(k)}$
2	0	1.0006 8497	1.0
	1	0.5350 3372	-0.4629 8543
	2	0.1076 4972	0.0676 5754
3	0	0.9999 9735	1.0
	1	0.5215 8283	-0.4784 2931
	2	0.1113 5511	0.0898 1217
	3	0.0105 1217	-0.0067 9289

5.5 The Continuous Problem

Example 1 (cont'd)

Note 1

For r_{22} and r_{33}, even the initial approximation is better than that obtained by characterisation.

Note 2

In each case, two or three iterations of the Remes' algorithm provide a very acceptable approximation.

Note 3

When constructing $B^{(k+1)}$ from $B^{(k)}$, points which are obviously going to remain inactive are omitted, which accounts for $M^{(k+1)}$ sometimes being less than $M^{(k)}$.

5.5 The Continuous Problem (cont'd)

Example 2

Approximation of $1/\sqrt{1+2z}$ on the semi disc .

m=n	Iteration k	M(k)	W	$\delta^k(c^{(k)})$	$\delta(c^{(k)})$	Final inclusion by characterisation
1	1	10	5	0.1359 E-1	0.1686 E-1	
	2	12	5	0.1524 E-1	0.1530 E-1	
	3	14	3	0.1526 E-1	0.1527 E-1	
	4	10	3	0.1527 E-1	0.1527 E-1	
2	1	10	4	0.5371 E-3	0.5529 E-3	
	2	16	4	0.5417 E-3	0.5451 E-3	
	3	20	3	0.5420 E-3	0.5426 E-3	
	4	22	3	0.5424 E-3	0.5424 E-3	0.506 E-3, 0.933 E-3
3	1	10	3	0.1685 E-4	0.4696 E-4	
	2	14	3	0.2061 E-4	0.2127 E-4	
	3	22	3	0.2082 E-4	0.2091 E-4	
	4	28	3	0.2084 E-4	0.2085 E-4	0.170 E-4, 0.554 E-4

Coefficients for r_{11}, r_{22} and r_{33}

\ m=n	k	$a^{(k)}$	$b^{(k)}$
1	0	0.9899 3294	1.0
	1	0.3435 1446	1.2500 9733
2	0	0.9994 6459	1.0
	1	1.1352 6751	2.1298 6717
	2	0.1249 4725	0.7803 3902
3	0	0.9999 7916	1.0
	1	1.9763 9623	2.9761 1590
	2	0.8543 3618	2.3322 9310
	3	0.0460 3551	0.4060 7149

5.5 The Continuous Problem

Example 2 (cont'd)

Note 1

The notes on the previous example also apply here.

Note 2

The presence of the singularity at $z = -0.5$ does not materially affect the convergence of either the ODC or the Remes' iteration. Williams [1972] reported difficulty with this example in part of his method. The inclusions based on characterisation are taken from his results.

Example 3

Approximation of $\ln(1+z)$ on the semi disc

m=n	Iteration k	$M^{(k)}$	W	$\delta^k(c^{(k)})$	$\delta(c^{(k)})$	Final inclusion by characterisation
2	1	16	2	0.2743 E-3	0.2813 E-3	
	2	22	2	0.2771 E-3	0.2781 E-3	
	3	26	2	0.2772 E-3	0.2774 E-3	0.272 E-3, 0.283 E-3

$$r_{22}(z) = \frac{0.0002\,7718 + 0.9983\,2189\,z + 0.3871\,2144\,z^2}{1.0 + 0.8821\,2763\,z + 0.1178\,4426\,z^2}$$

5.5 The Continuous Problem (cont'd)

Example 4

Approximation of $\tan 1.5z$ on the unit disc by the rational function $zr_{mn}(z^2)$.

m=n	Iteration k	$M^{(k)}$	W	$\delta^k(c^{(k)})$	$\delta(c^{(k)})$
1	1	16	6	0.1289 E-1	0.2401 E-1
	2	20	4	0.1315 E-1	0.1315 E-1
2	1	16	4	0.817 E-5	4.961 E-5
	2	24	3	0.953 E-5	0.960 E-5

Coefficients for r_{11} and r_{22}

m=n	k	$a^{(k)}$	$b^{(k)}$
1	0	1.5009 7723	1.0
	1	-0.2543 6320	-0.9115 1404
2	0	1.5000 0000	1.0
	1	-0.3774 6921	-1.0016 4606
	2	0.0084 2112	0.0818 4740

5.6 Concluding Remarks

The results of this chapter lead us to believe that the algorithms used probably provide the best method for constructing best rational approximations in the complex plane. The method is free from the restrictions imposed by those based on characterisation, and there seems to be no theoretical or practical justification for using methods based on characterisation for any region which is not circular or at least nearly circular. For the semidisc, for instance, even a course subset of about 2N (where $N = n+m+2$) points usually gives a better approximation than that using characterisation.

The only possible modification of the method would be to use an alternative algorithm for the construction of the best approximation on the subsets of B. The main, indeed only, contender must then be the Osborne-Watson algorithm used by Ellacott and Williams. The ODC algorithm used in the complex plane appears in practise to maintain the quadratic convergence of the corresponding real algorithm, and is thus faster than the algorithm of Osborne and Watson. However, the quadratic subprograms in their algorithm would be certainly convex, whereas in the ODC algorithm they are not, although in no case has this affected the convergence or even the rate of convergence of the algorithm used to solve these subprograms.

CHAPTER 6

NEAR-BEST POLYNOMIAL APPROXIMATION BASED ON TRUNCATED SERIES EXPANSIONS OR INTERPOLATION

6.1 Introduction

It is well known that a function f which is analytic on a domain G may be expanded in a Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

which converges uniformly to f on the disc $|z - z_0| < \Delta$ where z_0 is an arbitrary point of G and Δ is the distance between z and the boundary of G. One of the simplest polynomial approximations is therefore obtained by truncating such a series after a suitable number of terms.

Much work has been done in the real variable case to develop other series expansions of a function with a view to representing the function approximately by the truncation of such a series. The most commonly used series is the Chebyshev expansion

$$f(x) = \sum_{n=0}^{\infty} a_n T_n(x), \qquad (-1 \leq x \leq 1),$$

where $T_n(x)$ is the Chebyshev polynomial

$$T_n(x) = \cos(n \cos^{-1} x), \qquad (-1 \leq x \leq 1).$$

For many functions a truncated Chebyshev series provides a significantly better approximation than the Taylor series truncated after the same number of terms. Because Chebyshev series are derived much more simply than the best approximation, they have been widely used in the approximation of elementary functions (see for example Clenshaw [1962] and Luke [1969]).

6.1 Introduction (cont'd)

In the complex plane the situation is somewhat different. The Chebyshev polynomials are not generally known for an arbitrary domain, except possibly by numerical construction, and there are no straightforward ways of finding the corresponding expansions. There are however certain exceptions. In the case of a disc, $|z-\alpha| \leq R$, the Chebyshev polynomials are simply

$$T_n(z) = (z - \alpha)^n$$

and the Chebyshev expansion coincides with the Taylor expansion. It is not surprising therefore that for a disc, the truncated Taylor series provides a relatively good approximation as we shall see later. Moreover these approximations are actually best in the sense of least squares. The only other domain for which the Chebyshev polynomials have a simple form is the ellipse (and the degenerate cases of intervals of the real or imaginary axes) for which they are the suitable transformed classical Chebyshev polynomials (Rivlin [1974]). Before we discuss the exact nature of such approximations and their relation to best approximations, it is worthwhile noting some interesting examples.

Example 1 (Walsh [1935a])

Let $f(z)$ be analytic at the origin, let $P_n^*(z)$ be the polynomial of degree n of best approximation to $f(z)$ on the set $|z| = r$, and let $P_n(z)$ be the sum of the first $n+1$ terms of the Maclaurin development of $f(z)$. Then as r approaches zero we have

$$\lim_{r \to 0} P_n^*(z) = P_n(z) .$$

6.1 Introduction (cont'd)

Example 2 (Rivlin [1967])

This is a function for which a slight perturbation of the partial sums of the Taylor series gives a best approximation (the degree of approximation was given by Alper [1959]).

If $0 < A < 1$, the best uniform approximation by polynomials of degree at most n on D to the function

$$f(z) = 1/(1 - Az)$$

is

$$P_n(z) = 1 + Az + \ldots + A^{n-1}z^{n-1} + (A^n/(1-A^2))z^n ,$$

and the degree of approximation is $A^{n+1}/(1 - A^2)$.

6.2 Near-Best Approximation

The theory which enables us to measure more precisely the relation between best approximations and those obtained from truncated series or interpolation is due to Cheney and Price [1970].

Let X be a linear space normed by $||\,.\,||$; define a subspace Y and map $M: X \to Y$. For $x \in X$, Mx is said to be an approximation to x from Y with error $E = ||\, x - Mx\,||$. Let $y^B \in Y$ be the best approximation to x from Y. Then

$$||\, x - y^B\,|| \leq ||\, x - y\,|| \quad \text{for all } y \in Y.$$

An approximation Mx is said to be near-best within a relative distance ρ if (Geddes and Mason [1975])

$$||\, x - Mx\,|| \leq (1 + \rho)\, ||\, x - y^B\,||\,.$$

For practical purposes we are often content with approximations where $\rho < 9$ so that not more than one decimal place is lost in passing from the best to the near-best approximation.

The class of approximation maps which we shall consider are projections, that is a map $P: X \to Y$ which is bounded, linear and idempotent. A projection satisfies the inequality

$$||\, x - Px\,|| \leq (||\, 1 - P\,||)\,(x - y^B)$$

or since

$$||\, I - P\,|| \leq 1 + ||P\,||,$$

we have

$$||\, x - Px\,|| \leq (1 + ||\, P\,||)(x - y^B)\,.$$

We therefore seek projections with a small norm. The possibility of obtaining near-best approximations in the real variable may be seen from the results of Powell [1967]. In this case X is the space of continuous functions on the interval $[-1, 1]$ with the Chebyshev norm,

6.2 Near-Best Approximation (cont'd)

and Y is the subspace of polynomials of degree n. If $T_n f$ denotes the partial sum of the Chebyshev expansion of $f \in X$ and $I_n f$ denotes the polynomial interpolating f at the zeros of the Chebyshev polynomial $T_{n+1}(x)$, then Powell's results are equivalent to

$$\| f - T_n f \| \leq 3.5 \| f - B_n f \|$$

and

$$\| f - I_n f \| \leq 4 \| f - B_n f \| \quad \text{when } n \leq 20.$$

It should be noted that $\| T_n \|$ and $\| I_n \|$ tend to infinity with n, but sufficiently slowly to give an acceptably small norm for values of n likely to be encountered in practice.

The corresponding theory for the complex plane has been developed and illustrated by Geddes [1973, 1975, 1976]. We shall concentrate initially on projections associated with series. Geddes has analysed the family $\sum$ of projections $P_n : A(B) \to P_n$ where B is a bounded closed domain of the complex plane, A is the linear space of functions continuous in A and analytic in the interior of B and P_n is the subspace of algebraic polynomials of degree not exceeding n. The two cases considered by Geddes are the disc and the ellipse. In the first case the projection $T_n : A(B) \to P_n$ which projects f onto its Taylor polynomial is shown to be minimal in in the sense that $\| T_n \| \leq \| P_n \|$ for all $P_n \in \sum$. The bound $\| T_n \| \leq \tau_n$ where

$$\tau_n = \frac{4}{\pi^2} + \ln n + \eta + o(1)$$

where $\eta = 0.989$ and $o(1)$ tends to zero monotonically to zero as n increases indefinitely. An explicit computation of $\| T_n \|$ shows it to be less than 2.223 for $n \leq 20$ and 3.789 for $n \leq 1000$

6.2 Near-Best Approximation (cont'd)

so that the Taylor projection can certainly be regarded as giving a near-best approximation.

In the case of the ellipse, the partial sums of the Chebyshev series $C_n f$ are shown to be near-best approximations with

$$\| C_n \| \leq \gamma_n$$

where

$$\gamma_n = \frac{4}{\pi^2} \ln n + \eta + o(1), \quad \eta = 1.27033$$

and $0(1)$ is defined as above. In particular $\gamma_n = 2.5$ when $n = 20$ and 4.070 when $n = 1000$. In this case however, C_n is not believed to be minimal.

6.3 The role of the Faber series in Near-Best Approximation

The main limitation of Geddes work is that it is restricted to the cases when B is a disc or an ellipse. The most natural generalisation of these projections is the Faber projection

$$F_n : A(B) \to P_n$$

which projects f onto the partial sum of its Faber expansion. We shall deduce Geddes results as special cases. The Faber polynomials were originally defined by Faber [1903 and 1907]; convenient expositions may be found in Markushevich [1967] or Smirnov and Lebedev [1966]. We shall give some basic definitions and a few pertinent results.

Let B be a bounded closed set of points in the z-plane with a simply connected domain B' as its complement. Let

$$z = \psi(w) = w + b_0 + \frac{b_1}{w} + \ldots$$

map $1-1$ conformally the domain $|w| > \rho$ onto the domain B' so that $\psi(\infty) = \infty$. Let $w = \Phi(z)$ be the inverse function with expansion in the neighbourhood of $z = \infty$ of the form

$$\Phi(z) = z + a_0 + \frac{a_1}{z} + \ldots .$$

We shall denote by Γ_r the image of the circle $|w| = r$, $r > \rho$ when mapped by $z = \psi(w)$, and the bounded domain with boundary Γ_r by B_r. The boundary of B itself is denoted by Γ or Γ_ρ. The quantity ρ is variously known as the capacity or transfinite diameter of B.

The Faber polynomial $P_k(z)$ may be defined as the principal part of the expansion of $\{\Phi(z)\}^k$ about $z = \infty$. A summary of the properties of $P_k(z)$ relevant to our investigations is as follows.

6.3 The role of the Faber series in Near-Best Approximation (cont'd)

Property 1

$$P_k(z) = \frac{1}{2\pi i} \int_{|t|=r'} \frac{t\psi'(t)}{\psi(t)-z} t^{k-1}\, dt, \quad z \in B_r, \rho \leq r \leq r'$$

or equivalently

$$P_k(z) = \frac{1}{2\pi i} \int_{\Gamma_{r'}} \frac{\Phi(\zeta)^k}{\zeta - z}\, d\zeta$$

Property 2

$$P_k(z) = P_k(\psi(\tau)) = \tau^k + c_k + \frac{c_{k+1}}{\tau} + \ldots$$

Property 3

The Faber polynomials for B are also those for the domain

$$B_r(\rho < r < \infty).$$

Property 4

If the function $f(z)$ is analytic in the domain

$$I(B_R), \quad \rho < R < \infty,$$

and there is a singular point on the contour line L_R, then $f(z)$ can be expanded into a series of Faber polynomials

$$f(z) = \sum_{k=0}^{\infty} a_k P_k(z)$$

where

$$a_k = \frac{1}{2\pi i} \int_{|t|=r_1} f(\psi(t)) \cdot \frac{dt}{t^{k+1}}, \quad \rho < r_1 < R.$$

6.3 The role of the Faber series in Near-Best Approximation (cont'd)

The series converges uniformly on $I(B_R)$ and diverges outside B_R.

In Section 6.4 we shall investigate the relation between a truncated Faber expansion and the best polynomial approximation. Before doing so, a few examples of Faber polynomials will illustrate the techniques involved for different domains.

Example 1

B is the disc $| z - z_0 | \leq \rho$.

$w = \Phi(z) = z - z_0$ and $\{\Phi(z)\}^k = (z - z_0)^k$.

Thus

$$P_k(z) = (z - z_0)^k .$$

Example 2

B is an ellipse with foci at ± 1.

$z = \psi(w) = w + \frac{1}{4w}$ and $w = \Phi(z) = \frac{1}{2}(z + \sqrt{z^2 - 1})$

where the branch of the root is such that

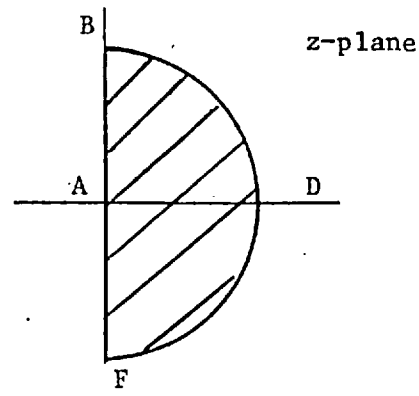
$$\lim_{z \to \infty} \frac{\sqrt{z^2 - 1}}{z} = 1 .$$

Then

$$P_k(z) = \frac{1}{2^k} \{ (z + \sqrt{z^2 - 1})^k + (z - \sqrt{z^2 - 1})^k \}$$

since $P_k(\psi(\tau)) = \tau^k + 2^{-2k} \tau^{-k}$ which satisfies Property 2. These Faber polynomials may readily be shown to be the (suitably normalised) classical Chebyshev polynomials for the interval $[-1, 1]$; this follows by writing $z = \cos\theta$ when $P_k(\cos\theta) = \frac{1}{2^k} (2 \cos k\theta) = 2^{-(k-1)} \cos k\theta$.

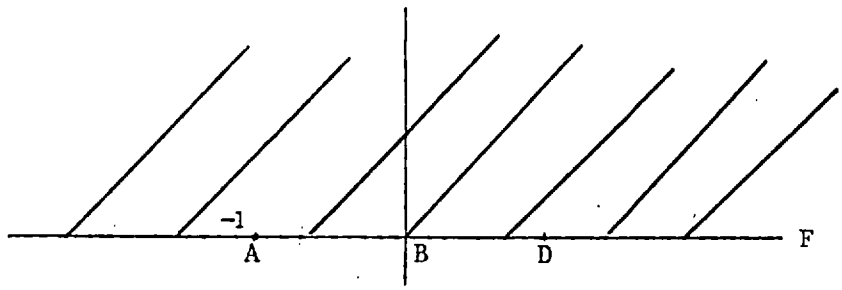
6.3 The role of the Faber series in Near-Best Approximation (cont'd)



Example 3

B is the unit semidisc $|z| \leq 1$, $\mathrm{Re}(z) \geq 0$.

Here the mapping function is a little more complicated. We first map B' onto the upper half plane (Kober [1957]) such that the arc BDF is mapped on the positive part of the real axis in the ζ-plane.



The required mapping is

$$\zeta = \left\{ i \left(\frac{z - i}{z + i} \right) \right\}^{2/3}$$

A bilinear transform will then map the upper half plane onto the exterior of the unit circle where for convenience we let $\zeta = \pm 1$ map onto $w = \pm 1$. This is accomplished by the transformation

$$w = \frac{1 - \alpha\zeta}{\zeta - \alpha} \quad \text{where} \quad \alpha = e^{i\pi/3}.$$

The whole mapping is then

$$w = \frac{\alpha(z - i)^{2/3} - \bar{\alpha}(z + i)^{2/3}}{(z + 1)^{2/3} - (z - i)^{2/3}}$$

6.3 The role of the Faber series in Near-Best Approximation

Example 3 (cont'd)

To expand w in powers of $1/z$ we write w as

$$w = \frac{\alpha(1 - i/z)^{2/3} - \bar{\alpha}(1 + i/z)^{2/3}}{(1 + i/z)^{2/3} - (1 - i/z)^{2/3}}$$

and then multiply numerator and denominator by

$$(i + i/z)^{4/3} + (1 + i/z)^{2/3}\,(1 - i/z)^{2/3} + (1 - i/z)^{2/3}$$

which makes the denominator $4i/z$. Some manipulation yields

$$w = \frac{3\sqrt{3}z}{4} - \frac{1}{2} - \frac{\sqrt{3}}{4z} + \frac{\sqrt{3}z}{4}\left(1 + \frac{1}{z^2}\right)^{2/3}\left\{\left(1 + \frac{i}{z}\right)^{2/3} + \left(1 - \frac{i}{z}\right)^{2/3}\right\}$$

or

$$w = \frac{3\sqrt{3}}{4}\left\{z - \frac{2}{3\sqrt{3}} + \frac{5}{27z} - \frac{32}{729z^3} + \frac{416}{19683z^5} - \frac{2272}{177147z^7} + \frac{62864}{4782969z^9} - \ldots\right\}.$$

Thus the expansion inside tbe bracket maps the exterior of the unit semidisc onto $|w| > 4/3\sqrt{3}$, the transfinite diameter of the semidisc. A computer program has been written to generate the Faber polynomials from an expansion, and the first nine are listed below.

NOTE: A similar technique may be used to generate the Faber polynomials for a sector.

Table 1 Faber polynomials $P_n(z) = \sum_{k=0}^{n} a_k^{(n)} z^k$

for the unit semidisc $(0 \leq n \leq 9)$.

n	k	$a_k^{(n)}$
1	0	
	1	1.0
2	0	0.5185 1852
	1	-0.7698 0036
	2	1.0
3	0	-0.4846 8912
	1	1.0
	2	-1.1547 0054
	3	1.0
4	0	0.3813 4431
	1	-1.0834 2273
	2	1.6296 2963
	3	-1.5396 0072
	4	1.0
5	0	-0.2777 1947
	1	1.0562 4143
	2	-1.9957 7871
	3	2.4074 0741
	4	-1.9245 0090
	5	1.0
6	0	0.2022 0495
	1	-0.9693 7823
	2	2.2263 3745
	3	-3.2787 7931
	4	3.3333 3333
	5	-2.3094 0108
	6	1.0

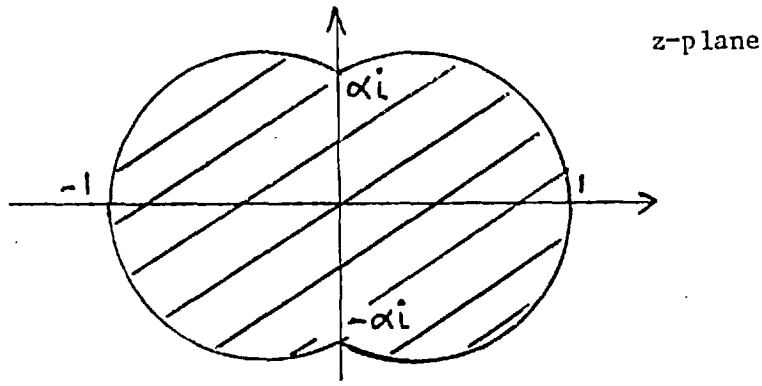
n	k	$a_k^{(n)}$
7	0	-0.1540 1483
	1	0.8624 1935
	2	-2.3321 0438
	3	4.0617 2840
	4	-4.9894 4677
	5	4.4074 0741
	6	-2.6943 0126
	7	1.0
8	0	0.1218 2350
	1	-0.7557 5990
	2	2.3398 8721
	3	-4.7011 6763
	4	6.7544 5816
	5	-7.1848 0335
	6	5.6296 2963
	7	-3.0792 0144
	8	1.0
9	0	-0.9652 1811
	1	0.6556 9273
	2	-2.2766 6608
	3	5.1742 1125
	4	-8.4963 1508
	5	10.5185 1852
	6	-9.9218 7130
	7	7.0000 0000
	8	-3.4641 0162
	9	1.0

6.3 The role of the Faber series in Near-Best Approximation (cont'd)

Example 4

B is the region enclosed by the circles $|z \pm \alpha| = \alpha\sqrt{2}$

where $\alpha = \sqrt{2} - 1$.



This domain has a simple mapping

$$w = z - \frac{\alpha^2}{z}$$

which maps B' onto $|w| \geq 2\alpha$.

The first few Faber polynomials are

$$P_0(z) = 1\ , \quad P_1(z) = z\ , \quad P_2(z) = z^2 - 2\alpha^2\ , \quad P_3(z) = z^3 - 3\alpha^2 z,$$

$$P_4(z) = z^4 - 4z^2\alpha^2 + 6\alpha^4\ , \quad P_5(z) = z^5 - 5\alpha^2 z^3 + 10\alpha^4 z\ .$$

Example 5

B is a region bounded by an arbitrary polygon.

The method of constructing the Faber polynomials is generally rather difficult, and has been outlined by Nepritvorennaja [1973] without giving any computational details. Let the polygon have vertices A_i and exterior angles $\pi\alpha_i$ $(i = 1, \ldots n)$. Then we map B onto the exterior of the unit circle using a variant of the Schwarz-Christoffel transform

$$z = \psi(w) = D_1 \int_{a_1}^{w} (1 - \frac{a_1}{w})^{\alpha_1 - 1} \ldots (1 - \frac{a_n}{w})^{\alpha_n - 1}\, dw + D_2 \qquad (6.3.1)$$

6.3 The Role of the Faber series in Near-Best Approximation

Example 5 (cont'd)

where a_i is the point on the unit circle corresponding to A_i. The transformation itself is not at all trivial, and the details of the technique for constructing the a_i is given in Nepritvorennaja [1970]. We then expand the integrand and integrate term-by-term. Since

$$\sum_{i=1}^{n} a_i(\alpha_i - 1) = 0,$$

there is no logarithmic term, and so

$$z = d\psi(w) = dw + d_0 + \frac{d_1}{w} + \frac{d_2}{w} + \ldots . \qquad (6.3.2)$$

The final mapping

$$w = c\Phi(z) = cz + c_0 + \frac{c_1}{z} + \frac{c_2}{z^2} + \ldots . \qquad (6.3.3)$$

is obtained by reversal of the series (6.3.2.). The very complex equations which perform this reversal are given by Nepritvorennaja [1973].

To illustrate this method, we have computed the first few Faber polynomials for the square $|\mathrm{Re}(z)| \leq 1$, $|\mathrm{Im}(z)| \leq 1$. The mapping function (6.3.1) is

$$z = D \int_1^w (1 + \frac{1}{w^4})^{\frac{1}{2}} \, dz$$

$$= D(w - \frac{1}{6w^3} + \frac{1}{8w^7} - \frac{1}{176w^{11}} + \frac{1}{384w^{15}} - \frac{7}{4864w^{19}} \ldots) \qquad (6.3.4)$$

which maps the exterior of the square onto the exterior of the circle $w = 1$. Using $w = 1$ in (6.3.3) and summing the series by the Euler transform, we find that $D = 1.1803\,4060$.

6.3 The Role of the Faber series in Near-Best Approximation

Example 5 (cont'd)

Writing $z' = z/D$,

$$z' = w - \frac{1}{6w^3} + \ldots .$$

Substituting into (6.3.3) and equating coefficients of z', z'^{-3}, z'^{-7}, ... gives the equations

$$d_3 + c_3 = 0$$

$$d_7 - 3c_3d_3 + c_7 = 0$$

$$d_{11} - 3c_3d_7 + 6c_3d_3^2 - 7c_7d_3 + c_{11} = 0$$

$$d_{15} + c_3(-3d_{11} + 12d_3d_7 - 10d_3^3) + c_7(-7d_7 + 28d_3^2) - 11c_{11}d_3 + c_{15} = 0$$

$$d_{19} + c_3(-3d_{15} + 6d_7^2 + 12d_3d_{11} - 30d_3^2d_7 + 15d_3^4)$$
$$+ c_7(-7d_{11} + 56d_3d_7 - 84d_3^3) + c_{11}(-11d_7 + 66d_3^2) - 15c_{15}d_3 + c_{19} = 0,$$

with solution

$$c_3 = 1/6, \quad c_7 = -17/168, \quad c_{11} = \frac{1163}{11088}, \quad c_{15} = -\frac{106537}{798336} .$$

The final mapping is then

$$w = z' + \frac{c_3}{z^{13}} + \frac{c_7}{z^{17}} + \frac{c_{11}}{z'^{11}} + \frac{c_{15}}{z'^{15}} \quad \ldots .$$

$$= \frac{z}{D} + \frac{c_3D^3}{z^3} + \frac{c_7D^7}{z^7} + \frac{c_{11}D^{11}}{z^{11}} + \frac{c_{15}D^{15}}{z^{15}} \quad \ldots$$

so that

$$\Phi(z) = z + \frac{D^4}{6z^3} - \frac{17D^8}{168z^7} + \frac{1163D^{12}}{11088z^{11}} - \frac{106537D^{16}}{798336z^{15}} + \ldots .$$

6.3 The Role of the Faber series in Near-Best Approximation (cont'd)

Table 2 Faber Polynomials

$$P_n(z) = \sum_{k=0}^{[n/4]} a_k^{(n)} z^{n-4k}, \quad (0 \leq n \leq 16)$$

for the unit square.

n	k	$a_k^{(n)}$
1	0	1.0
2	0	1.0
3	0	1.0
4	0	1.0
	1	1.2940 1146
5	0	1.0
	1	1.6175 1433
6	0	1.0
	1	1.9410 1720
7	0	1.0
	1	2.2645 2006
8	0	1.0
	1	2.5880 2293
	2	-0.1196 0469
9	0	1.0
	1	2.9115 2679
	2	0.3363 8819
10	0	1.0
	1	3.2350 2866
	2	0.8070 3518
11	0	1.0
	1	3.5585 3153
	2	1.5623 3627

n	k	$a_k^{(n)}$
12	0	1.0
	1	3.8820 3439
	2	2.3322 9147
	3	0.3728 5461
13	0	1.0
	1	4.2055 3726
	2	3.2069 0077
	3	0.4144 0504
14	0	1.0
	1	4.5290 4012
	2	4.1861 6417
	3	0.6155 6186
15	0	1.0
	1	4.8525 4299
	2	5.2700 8168
	3	1.0101 8099
16	0	1.0
	1	5.1760 4586
	2	6.4586 5329
	3	1.6321 1832
	4	-0.4090 0103

6.3 The Role of the Faber series in Near-Best Approximation (cont'd)

If B is a real interval, a near-best rational approximation may often be obtained from the Chebyshev series, known as a rational Chebyshev approximation. Its derivation depends on the possibility of expressing the product $T_m(x)T_n(x)$ as the sum of Chebyshev polynomials. In the complex plane, the corresponding property for Faber polynomials seems to be limited to the disc and ellipse. The absence of a relation of this kind for other Faber polynomials precludes the development of a rational Faber expansion for general regions of the complex plane.

6.4 Norms of Faber Projections

There are at least three known bounds for the norm of the Faber projection, one due to Alper and Ivanov [1953] and the other two to Kovari and Pommerenke [1967]. Although not couched in terms of projections, the projection norms can be deduced from their results.

Theorem 6.1 (Alper and Ivanov)

If the domain D is bounded by a finite number of differentiable arcs, making angles between themselves, the exterior of which do not exceed $\lambda\pi (0 < \lambda \leq 1)$, and $f(z)$ is analytic in the region D and continuous in the closed domain $\overline{D}$, then

$$\left| \sum_{k=0}^{n} a_k P_k(z) \right| \leq C \, \|f\| \, (\ln n)^2 \quad (n > 1)$$

for all $z \in \overline{D}$.

6.4 Norms of Faber Projections (cont'd)

Theorem 6.2 (Kovari and Pommerenke)

Let K be any continuum whose complement is connected. For any function analytic in the interior of K and continuous on K,

$$\left| f(z) - \sum_{k=0}^{n} a_k P_k(z) \right| \leq An^{\alpha} \rho_n(f, K), \qquad z \in K$$

where $\alpha < \frac{1}{2}$ and $\rho_n(f, K)$ is the degree of approximation of f on K by polynomials of degree not exceeding n.

Theorem 6.3 (Kovari and Pommerenke)

Suppose K is a closed Jordan domain whose boundary Γ is of bounded rotation; for any function analytic in the interior of K and continuous on K, we have

$$\left| f(z) - \sum_{k=0}^{n} a_k P_k(z) \right| \leq (A \ln n + B) \rho_n (f, K)$$

where A and B depend on the domain K.

Asymptotically, the best estimate is provided by Theorem 6.3 followed by Theorems 6.1 and 6.2. The conditions under which the bound in Theorem 6.2 is valid are of course less stringent than the other two, but for the simple domains considered in Section 6.2, all the theorems are valid.

In order to obtain more information about the bound of Theorem 6.3, we shall investigate exactly how the constraints A and B depend on the domain K.

6.4 Norms of Faber Projections (cont'd)

Theorem 6.3 is based on the following lemma (Kovari and Pommerenke [1967].

"With the notation of Theorem 6.3, and denoting the n^{th} partial sum of the Faber expansion by $S_n(z)$, for $z \in K$,

$$|f(z) - S_n(z)| \leq \rho_n (1 + L_n)$$

where

$$L_n = \frac{1}{2\pi} \int_{|t|=1} \left| \sum_{k=0}^{\infty} \frac{F_K(z)}{t^{K+1}} \right| |dt| \qquad (6.4.1)$$

is the 'Le$begue-Constant' of the system of Faber polynomials."

At this point we recall the concept of a curve of bounded variation (Radon [1919]).

Definition

Let $\Gamma^* : z(\tau)$ be a smooth Jordan curve, and let $\theta(\tau)$ denote the angle between the positive real axis and the tangent of Γ^* at $z(\tau)$. Then

$$\int_{\Gamma^*} |d\theta| = \int_{\Gamma^*} \left| \frac{d\theta}{d\tau} \right| d\tau$$

is the total rotation of Γ^*. Now let Γ be an arbitrary Jordan curve, and let $z = \psi(\zeta)$ map $|\zeta| > \rho$ onto the exterior of Γ, and let Γ_r $(r > \rho)$ denote the level curves of the mapping. Let $V(r)$ be the total rotation of Γ_r. If $V(r)$

6.4 Norms of Faber Projections

Definition (cont'd)

is bounded, Γ is said to be of bounded rotation, and

$$V = \sup_{r>\rho} V(r) = \lim_{r\to\rho} V(r)$$

is called the total rotation of Γ.

From the definition, it follows (Paatero [1933]) that the rotation V is at least 2π, is exactly 2π if and only if the region is convex and never exceeds 4π for a simply connected region. Also (Radon [1919])

$$\int_\Gamma |\, d_w \arg (w - w)| \leq V \tag{6.4.2}$$

and (Pommerenke [1965])

$$P_k(\psi(e^{i\theta})) = \frac{1}{\pi} \int_0^{2\pi} e^{iks} d_s v(s,\theta) \tag{6.4.3}$$

where

$$v(s,\theta) = \arg\{\psi(e^{is}) - \psi(e^{i\theta})\}.$$

We then have from (6.4.1)

$$L_n = \frac{1}{2\pi} \int_0^{2\pi} \Big| \sum_{k=0}^{n} P_K(\psi(e^{i\phi}))e^{-ik\theta} \Big| \, d\theta,$$

whence by (6.4.3),

$$L_n = \frac{1}{2\pi^2}\int_0^{2\pi} \Big| \int_0^{2\pi} \sum_{k=0}^{n} e^{ik(s-\theta)} d_s v(s,\phi) \Big| \, d\theta$$

$$= \frac{1}{2\pi^2} \int_0^{2\pi} \left\{ \int_0^{2\pi} \left| \frac{\sin\frac{n+1}{2}(s-\theta)}{\sin\frac{s-\theta}{2}} \right| d\theta \right\} |d_s v(s,\phi)|$$

$$= \frac{1}{\pi^2} \int_0^{2\pi} \left\{ \int_0^{\pi} \left| \frac{\sin(n+1)x}{\sin x} \right| dx \right\} |d_s v(s,\phi)|$$

6.4 Norms of Faber Projections (cont'd)

Using (6.4.2) and a standard inequality for the integral (see for example Geddes and Mason [1975]),

$$L_n \leq \frac{V}{\pi} \left\{ \frac{4}{\pi^2} \ln n + \eta + o(1) \right\} \qquad (6.4.4)$$

where $\eta = 0.98941 \ldots$ and $o(1)$ decreases monotonically from 0.166 at $\eta = 2$ to zero as n tends to infinity. This bound is twice that obtained by Geddes [1973] for the unit disc and not quite twice that obtained by him for the ellipse. However this bound is valid for any domain with a boundary of bounded rotation. We note in passing that there is a factor of 2 missing in the results of Kovari and Pommerenke [1967] which we originally believed to reduce exactly to those of Geddes for a disc.

For a convex domain ($V = 2\pi$) a few values of the bound (6.4.4) are given in Table .

TABLE 3

Bounds on L_n for a convex domain

n	bound
2	2.872
4	3.284
10	3.922
20	4.446
50	5.166
100	5.720
1000	7.578

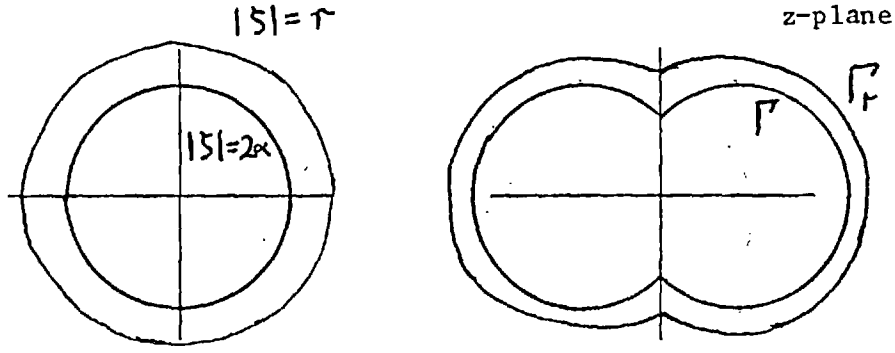
6.4 Norms of Faber Projections (cont'd)

Thus the error in the truncated Faber expansion of degree n does not exceed the error in the best polynomial approximation of the same degree by a factor of more than ten for any value of n likely to be used in practise.

For the domain considered in example 4 of Section 6.3, $V \doteq 2\pi$. The function

$$z = \psi(\zeta) = \frac{1}{2}\{\zeta^2 + \sqrt{\zeta^2 + 4\alpha^2}\} =$$

maps $|\zeta| = 2\alpha$ onto the curve Γ, and $|\zeta| = r$ $(r > 2\alpha)$ onto Γ_r.



From Markushevich [1967], the tangent to Γ_r has inclination

$$\phi = \theta + \pi/2 + \arg \psi'(re^{i\theta})$$

and so

$$\frac{\partial \phi}{\partial \theta} = 1 + \frac{\partial}{\partial \theta}\{\arg \psi'(re^{i\theta})\}$$

$$= 1 + \mathrm{Re}\{\zeta \frac{d}{d\zeta}[\ln \psi'(\zeta)]\}$$

$$= 1 + \mathrm{Re}\{\zeta \frac{\psi''(\zeta)}{\psi'(\zeta)}\}$$

$$= \mathrm{Re}\{\frac{4\alpha^2}{\zeta^2 + 4\alpha^2} + \frac{\zeta}{\sqrt{\zeta^2 + 4\alpha^2}}\}$$

6.4 Norms of Faber Projections (cont'd)

Thus V(r), the total rotation of Γ_r is

$$\int_0^{2\pi} \left\{ \mathrm{Re}\left[\frac{4\alpha^2}{r^2e^{2i\theta} + 4\alpha^2} + \frac{re^{i\theta}}{\sqrt{r^2e^{2i\theta} + 4\alpha^2}} \right] \right\} d\theta \qquad (6.4.5)$$

$$= 4 \left\{ \int_0^{\pi/2-\varepsilon} .d\theta \qquad \int_{\pi/2-\varepsilon}^{\pi/2} .d\theta \right\}$$

where . denotes the integrand of (6.4.5). Therefore

$$V = 4 \int_0^{\pi/2-\varepsilon} \left\{ \frac{1}{2} + \frac{\cos(\theta/2)}{\sqrt{2\cos\theta}} \right\} d\theta + 4 \lim_{p\to 1} \int_0^{\varepsilon} \left\{ \frac{1}{p^2 - e^{2i\theta}} + \frac{p}{p^2 - e^{2i\theta}} \right\} d\phi$$

where $\phi = \pi/2 - \theta$ and $p = r/2\alpha$,

so that $V = 3\pi$.

For this domain, therefore, the bounds in Table 3 need to be multiplied by 1.5, but $1 + L_n$ still remains less than 10 if n is less than 100. Indeed, if n is less than 20, $1 + L_n < 10$ for any simply connected domain (where $V \leq 4\pi$).

6.5 Interpolating Projections

It has already been seen in Section 6.1 that a near-best approximation on the real interval $[-1,1]$ can be obtained by interpolation in the zeros of $T_{n+1}(x)$. The most significant corresponding work in the complex plane is again that of Geddes [1973]. Before looking at this, we shall briefly outline the scattered results previously known about interpolation in the complex plane. These results suggest choices of interpolation points which might fruitfully be investigated. There are three particular results pertinent to our work, which can all be found in the classic essay of Walsh [1935a]. The first is a beautiful result due to Runge [1904] and Walsh [1932].

Theorem 6.4

Let the function $f(z)$ be analytic for $|z| < R > 1$ but have a singularity on the circle $|z| = R$. Let $p_n(z)$ be the polynomial of degree n which coincides with $f(z)$ in the $(n+1)$st roots of unity. Then the sequence $p_n(z)$ converges to $f(z)$ everywhere interior to $|z| = R$, uniformly on any closed point set interior to the circle $|z| = R$.

The extension to a more general region bounded by a Jordan curve C is due to Fejer [1918].

Theorem 6.5

Let C be an arbitrary Jordan curve in the z-plane and $f(z)$ on arbitrary function analytic on and within C. Then if $p_n(z)$ is a polynomial of degree n found by interpolation to $f(z)$ in $n+1$ equally spaced points on C, the sequence $p_n(z)$ converges to $f(z)$ uniformly in the closed interior of C.

NOTE: By 'equally spaced points on C' we mean the $n+1$ points which correspond to the points $w^{n+1} = 1$ when the exterior of C is mapped onto the exterior of

6.5 Interpolating Projections

Theorem 6.5 (cont'd)

$|w| = 1$ so that the points at infinity correspond to each other.

As an alternative to using points on C, there is a result of Faber [1920].

Theorem 6.6

If C is an analytic Jordan curve and if f(z) is analytic on and within C, then the sequence of polynomials $p_n(z)$ of degree n found by interpolation in the roots of $T_{n+1}(z)$, (the Chebyshev polynomial of degree $n+1$ for C) converges to f(z) in C.

If an attempt is to be made to construct near-best approximations by interpolation, it is also instructive to know something about the zeros of the errors in best approximations. There are several indications that the situation is not nearly so simple as that for the real case. One positive result was given by Motzkin and Walsh [1961] who show that for a function analytic in the neighbourhood of the origin, the best approximation on the disc $|z| \leq \varepsilon$, $(\varepsilon > 0)$, for sufficiently small ε has an error with at least $n+1$ zeros interior to the circle $|z| = \rho\varepsilon$ where $\rho(\varepsilon)$ approaches zero with ε. A potentially more useful result which overcomes the restriction of the small disc was given much later by Saff [1973]. Although overcoming the need for the region to be small, his results place restrictions on the approximated function and are only valid for sufficiently large n. More precisely, if $a_{n+1}/a_n \to 0$ as $n \to \infty$, then the best approximation has an error with exactly $n+1$ zeros in $|z| < 1+\varepsilon$ for each $n > n_\varepsilon$, and if $n^{\frac{1}{2}}a_{n+1}/a_n \to 0$ as $n \to \infty$, then the error has exactly $n+1$ zeros in $|z| < 1$ and no zeros on $|z| = 1$. To quote a particular example, for $n \geq 6$, the best approximation of degree n to e^z interpolates e^z in exactly $n+1$ points in the unit disc.

6.5 Interpolating Projections (cont'd)

In case we expect these special results to be indicative of the general case, we have the contrasting result of Al'per [1969], that for each n, there is a function $f(z)$ depending on n which is analytic in $|z| < 1$, continuous on $|z| \leq 1$ and such that the error in the best approximation of degree k ($k < n$) has no zeros in $|z| \leq 1$! In practise, we are of course unlikely to know the zeros of the best approximation, even if there are the correct number to enable an approximation to be constructed. In the complex plane, even the idea of using the zeros of the Chebyshev polynomials is made difficult by the fact that the Chebyshev polynomials are not generally known in an analytic form.

The earlier work of this chapter does suggest that we might try the zeros of the Faber polynomial for a given region, but once again there are difficulties. Kovari and Pommerenke [1967] show that the zeros of the Faber polynomial $P_n(z)$ for a convex region K always lie in K, but this is not necessarily true for non-convex regions. Ullman [1972] has conjectured that the zeros always lie in the convex hull $H(K)$ of K, but this has been disproved by a counter-example given by Goodman [1974]. Even for convex regions, our numerical results show that this approach is inferior to that of Theorem 6.5.

The relation between best approximations and those constructed by interpolation has been investigated by Geddes [1973] for the disc and ellipse. Define the Lagrange interpolating projection

$$L_n : A(C) \to \Pi_n \quad \text{by}$$

$$(L_n f)(z) = \sum_{k=0}^{n} f(z_k)\,\ell_k(z)$$

where $$\ell_k(z) = \frac{\prod_{\substack{i=1 \\ i \neq k}}^{n} (z - z_i)}{\prod_{\substack{i=1 \\ i \neq k}}^{n} (z_k - z_i)}$$

6.5 Interpolating Projections (cont'd)

It follows that $\| L_n \| \leq \| \Lambda_n \|$ where

$$\Lambda_n(z) = \sum_{k=0}^{n} | \ell_k(z) | . \qquad (6.5.1)$$

For the disc Geddes shows, using the interpolation scheme of Theorem 6.4, that $\| L_n \| < \phi_n$ where

$$\phi_n = \frac{2}{\pi} \ell n \; n + \eta + o(1),$$

$\eta = 0.07998 \ldots$ and $o(1) \to 0$ as $n \to \infty$.

For the ellipse, the "equally spaced" nodes corresponding to Theorem 6.5 are shown to be the roots of $T_{n+1}(z) = \cos h ((n+1) \ell n \rho)$ where ρ is the eccentricity of the ellipse. Numerical computation of $\| \Lambda_n \|$ shows it to be large when ρ approaches 1, and an alternative scheme is proposed based on the images of $w^{n+1} = i$ for which $\| \Lambda_n \|$ does not exceed the acceptably small value of 3.2 for any ρ and $1 \leq n \leq 10$.

In the spirit of Geddes, we have taken the semidisc and computed bounds for the projection norms directly from (6.5.1), which are displayed in Table 4.

TABLE 4

Values of $\| \Lambda_n \|$ for selected values of n and with interpolation nodes as:

(A) images of $w^{n+1} = 1$;

(B) images of $w^{n+1} = -1$;

(C) roots of $T_{n+1}(z)$, the Chebyshev polynomial of degree $n+1$ for the semidisc.

6.5 Interpolating Projections

TABLE 4
(cont'd)

degree	Bounds on projection norm		
	A	B	C
4	3.06	2.16	4.75
6	3.16	2.25	10.78
7	3.50	2.56	16.86
9	3.57	2.52	-
10	3.81	2.84	-
12	3.87	2.77	-
16		2.95	-

NOTE 1

As with the ellipse, the choice of $w^{n+1} = 1$ is not necessarily the best.

NOTE 2

Interpolation in the roots of Chebyshev polynomials is unsatisfactory for the semidisc.

NOTE 3

The non-monotone behaviour of the bounds as n increases is probably due to the different ways in which the images of $w^{n+1} = \pm 1$ distribute themselves around the 'corners' $\pm i$ of the semidisc for different values of n.

CHAPTER 7

ECONOMISATION AND RELATED TECHNIQUES

7.1 Introduction

One of the most popular methods for constructing polynomials has been that of economisation. The method was known as early as 1938 (see for example Lanczos [1938] and Lanczos [1956]) and has since been expounded many times.

Let $f : [-1, 1] \to \mathbb{R}$ be a function with power series representation

$$f(x) = \sum_{k=0}^{\infty} a_k x^k ,$$

uniformly convergent on $[-1, 1]$. Polynomial approximations to any specified accuracy can be simply obtained by truncation of the series at an appropriate point, giving a polynomial of degree N say. The basic problem of economisation is to replace $P_N(x)$ by a polynomial of degree $N-1$ with a minimum error, namely so that

$$\max_{-1 \leq x \leq 1} | P_N(x) - P_{N-1}(x) | \qquad (7.1.1)$$

is minimised. This problem is completely solved by taking

$$P_{N-1}(x) = P_N(x) - \frac{a_N}{2^{N-1}} T_N(x) \qquad (7.1.2)$$

where $T_N(x)$ is the classical Chebyshev polynomial of degree N for the interval $[-1, 1]$ and a_N is the coefficient of x^N in $P_N(x)$. The process can be continued to give a polynomial $P_{N-2}(x)$ so that

$$\max_{-1 \leq x \leq 1} | P_{N-1}(x) - P_{N-2}(x) |$$

is minimised, but unfortunately this is not equivalent to obtaining a best approximation of $P_N(x)$ by polynomials of degree $N-2$.

7.1 Introduction (cont'd)

For functions such as the exponential with a fairly quickly convergent Taylor expansion, the method provides a simple construction of near-best approximations. However, for many functions, including standard functions such as $\ln(1 + x)$ on $[0, 1]$, the convergence of the Taylor series is so slow as to make the method impractical in its usual form. The economisation of a slowly convergent series, or a series which converges only over part of the range for which the approximation is required, is the subject of a paper by Thatcher [1964]. To find the coefficients c_q in the expansion:

$$f(x) = \sum_{q=0}^{\infty}{}' c_q T_q(x) \tag{7.1.3}$$

from the series

$$f(x) = \sum_{r=0}^{\infty} a_r x^r \tag{7.1.4}$$

he substitutes directly for x^n in terms of $T_q(x)$ for $0 \leq q \leq n$, namely

$$x^n = \sum_{q=0}^{n} \theta_{n,q} T_q(x), \tag{7.1.5}$$

whence we find by substituting (7.1.5) into (7.1.4) and comparing with (7.1.3) that

$$c_q = \sum_{r=q}^{\infty} a_r \theta_{r,q} \ . \tag{7.1.6}$$

As $\theta_{j+1,q}/\theta_{j,q} \to 1$ as $j \to \infty$, the convergence of the series (7.1.6) is governed by that of (7.1.2) for $x = 1$. For rapidly convergent series, this may be summed directly, but for the cases of interest where the series converges slowly, the coefficients may be evaluated by using some transformation of the partial sums. Suggested methods are the Euler transform for alternating series or the e_k - transformation of Wynn [1956] and Shanks [1955].

7.1 Introduction (cont'd)

The first attempt to use a related technique for rational functions was that of Maehly [1960] who developed a method for the economisation of continued fractions. A somewhat simpler analysis for Padé approximations has been given by Ralston [1963]. The approximations considered have the form

$$R_{m,n}(x) = \frac{P_m(x)}{Q_n(x)}$$

where $P_m(x)$ and $Q_n(x)$ are polynomials of degree m and n respectively. In each case the problem solved was not that of obtaining an approximation $R_{m-1,n-1}(x)$ so that

$$\max_{-1 \leq x \leq 1} |R_{m,n}(x) - R_{m-1,n-1}(x)| \qquad (7.1.7)$$

is a minimum, which would be the exact analog of the polynomial problem. Instead the problem is posed in the following way. A Padé approximation of index N to $f(x)$ has the property that

$$\lim_{x \to 0} \frac{f(x) - R_{m,n}(x)}{x^{N+1}} d_{N+1}$$

or

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon\mu) - R_{m,n}(\varepsilon\mu)}{\varepsilon^{N+1}} = d_{N+1}\mu^{N+1}.$$

The 'economised' rational function $C_{m,n}(x)$ is derived so that

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon\mu) - C_{m,n}(\varepsilon\mu)}{\varepsilon^{N+1}} = \frac{d_{N+1} T_{N+1}(\mu)}{2^N}.$$

Since, for all polynomials of degree $N+1$ with coefficient of μ^{N+1} equal to unity, $T_{N+1}(\mu)/2^N$ has the smallest maximum value in $[-1,1]$, for sufficiently small ε, the maximum error of $C_{m,n}(x)$ on $[-\varepsilon, \varepsilon]$ will be less than that of $R_{m,n}(x)$. Because of its dependence on a small ε, this is clearly a less satisfactory result than the corresponding polynomial problem. For this reason, we shall return later to the direct problem.

7.2 Possibility of Economisation in the Complex Plane

The difficulties encountered in finding best approximations by both polynomial and rational functions in the complex plane makes it worthwhile reinvestigating the economisation technique to see if any useful constructive methods can be obtained. Our general conclusion will be that this is indeed so.

Let us first confine our discussion to the unit disc. The success of the economisation method for polynomial approximation depends on the fact that the Taylor expansion does not 'belong' to an interval I of the real line but rather to a disc, whereas the validity of the economised polynomial is restricted to I itself. It is clear that in the case of the real interval $[-1, 1]$, it would not be possible to economise the truncated Chebyshev approximation

$$f(x) = \sum_{k=0}^{N+1} a_k T_k(x) , \tag{7.2.1}$$

to a polynomial of degree N except by deleting the last term, since the resulting error is then directly $a_{N+1}T_{N+1}(x)$ as required by the economisation method. Yet if we start with a truncated expansion of a function $f \in A(C)$,

$$f(z) = \sum_{k=0}^{N+1} a_k z^k , \tag{7.2.2}$$

this is precisely the situation which pertains, because the polynomial z^k is the Chebyshev polynomial of degree k for the unit disc and (7.2.2) is already the truncated Chebyshev expansion of f on D. The only method of economisation as is usually defined would therefore be to truncate the Taylor expansion further. We therefore proceed as follows.

7.2 Possibility of Economisation in the Complex Plane (cont'd)

Let $T_k(z ; B)$ be the Chebyshev polynomial of degree k for the region B, and let f be analytic on B, then f has a representation

$$f(z) = \sum_{k=0}^{\infty} a_k T_k(z ; B) , \tag{7.2.3}$$

Let $P_{N+r}(z)$ be a polynomial approximation to $f(z)$ obtained by truncating the series, that is

$$P_{N+r}(z) = \sum_{k=0}^{N+r} a_k T_k(z ; B) \tag{7.2.4}$$

We may extend the concept of economisation to that of replacing $P_{N+r}(z)$ by $Q_N(z)$, a polynomial of degree N such that

$$\max_{z \in B} | P_{N+r}(z) - Q_N(z) | \tag{7.2.5}$$

is minimised, or equivalently, to finding $Q_N(z)$ such that

$$\max_{z \in B} \left| \sum_{k=N+1}^{N+r} a_k T_k(z ; B) - Q_N(z) \right| \tag{7.2.6}$$

is minimised.

When $r = 1$, we have the immediate solution obtained by choosing $Q(z) = 0$ since by definition $T_{N+1}(z ; B)$ is a minimising polynomial for B.

When $r = 2$, we have to choose $Q_N(z)$ so that

$$\max_{B} \left| a_{N+2} T_{N+2}(z ; B) + a_{N+1} T_{N+1}(z ; B) - Q_N(z) \right|$$

is a minimum, or equivalently, assuming $a_{N+2} \neq 0$, to minimise

$$\max_{B} \left| T_{N+2}(z ; B) + \alpha T_{N+1}(z ; B) - Q_N(z) \right| \tag{7.2.7}$$

7.2 Possibility of Economisation in the Complex Plane (cont'd)

If B is the real interval $[-1, 1]$, then this is the classical Zolotareff problem, although we shall refer to it as such for any region B. The real problem was first solved by Zolotareff [1877] and an account can be found in Achieser [1956]. For sufficiently small α, the solution can be expressed quite simply in terms of Chebyshev polynomials. Disappointingly, in cases of interest where the Chebyshev expansion is quickly convergent, we expect $\alpha = a_{N+2}/a_{N+1}$ to be large, and then no elementary solution exists, although it can be derived in terms of elliptic functions.

Ideally, we would now like to present a complete solution of the above problem (7.2.5) for all values of $r > 1$ and any region B. However, since we do not know an explicit expression for $T_k(z ; B)$ in general, we are constrained to concentrate on the disc and ellipse.

A solution to the Zolotareff problem for a disc has recently been published by Ryzakov [1974]. The problem is posed as that of finding

$$\mu_n(\lambda,1) = \inf_{P_n \in \mathcal{P}_n(\lambda,1)} \sup_{|z| \leq 1} |P_n(z)|$$

with

$$\Pi_n(\lambda, z) \in \mathcal{P}_n(\lambda,1) \text{ such that}$$

$$\mu_n(\lambda, 1) = \sup_{|z| \leq 1} |\Pi_n(\lambda, z)|$$

where $\mathcal{P}_n(\lambda, 1)$ is the set of polynomials

$$P_n(z) = \sum_{k=0}^{n} p_k z^k$$

7.2 Possibility of Economisation in the Complex Plane (cont'd)

with $p_n = \lambda$, $p_{n-1} = 1$. His solution depends on showing that the set $\{ z_k(t) \}_{k=1}^{n}$ with $z_k(t) = e^{i(2k-n-1)t}$, $k = 1, 2, \ldots, n$, $t \in [\pi/(n+2), \pi/n]$ is characteristic. Unfortunately, the proof does not hold if $n > 2$ and the supremum does not occur on the correct point set. For example, if $n = 4$ and $\lambda = \frac{1}{2}$, the characteristic set in Ryzakov's proof should be $\{e^{-3ti}, e^{-ti}, e^{ti}, e^{3ti}\}$ where $t = 36.17258247°$ and $\mu_4(\frac{1}{2}, 1) = 1.21912153$; in fact his $\pi_4(\frac{1}{2}, z)$ has a maximum of 1.2862.

The correct solution computed by the exchange algorithm of Chapter 2 has a characteristic set $\{ z_k = e^{itk} \}_{k=1}^{4}$ where $t_1 = 35.0701°$, $t_2 = 111.2802°$, $t_3 = -t_1$ and $t_4 = -t_2$, with $\mu_4(\frac{1}{2}, 1) = 1.2223014$. A careful investigation of this problem has led us to believe that no analytic solution of the problem exists for a disc.

It is, however, possible to construct an approximate solution to the minimisation problem (7.2.6) for the disc and ellipse which is quite adequate for practical purposes.

7.3 Asymptotic Economisation on the Unit Disc

For the unit disc, the problem (7.2.6) may be restated as that of finding $Q_n(z)$ such that

$$\max_{z \in D} \left| \sum_{k=n+1}^{n+r} a_k z^k + Q_n(z) \right| \qquad (7.3.1)$$

is a minimum. Since an analytic solution is not possible for $r > 1$, and the numerical solution for particular values of $a_K (k = n+1, \ldots, n+r)$ involves considerable computation, we shall find a rational function which is a good approximation to $\sum_{k=n+1}^{n+r} a_k z^k$ and has a fractional part which is small in D for large N. Its integral part is then the polynomial approximation required. This approximation is not optimal, but asymptotically optimal.

We begin with two theorems due to Carathéodory and Fejér [1911].

Theorem 7.1

For any polynomial $c_0 + c_1 z + \ldots + c_{n-1} z^{n-1} \not\equiv 0$, there exists a unique fraction of the form

$$R(z) = \lambda \frac{\bar{\alpha}_{n-1} + \ldots + \bar{\alpha}_0 z^{n-1}}{\alpha_0 + \ldots + \alpha_{n-1} z^{n-1}} \qquad \lambda > 0,$$

that is regular in $|z| < 1$ and that has as its first n coefficients in its expansion about $z = 0$ the numbers $c_0, c_1, \ldots, c_{n-1}$. Out of all functions $f(z) = c_0 + c_1 z + \ldots$ that are regular in $|z| < 1$ and have the same first n coefficients $c_0, c_1, \ldots, c_{n-1}$, this function and only it minimises the quantity

$$M = \max_{|z| < 1} |f(z)| .$$

7.3 Asymptotic Economisation on the Unit Disc (cont'd)

Theorem 7.2

The number $\lambda(c_0, \ldots, c_{n-1})$ is equal to the largest positive root of the equation of degree 2n

$$\left|\begin{array}{cccc|cccc} -\lambda & 0 & \ldots & 0 & c_0 & c_1 & \ldots & c_{n-1} \\ 0 & -\lambda & \ldots & 0 & 0 & c_0 & \ldots & c_{n-2} \\ \vdots & & & \vdots & & & & \\ 0 & 0 & \ldots & -\lambda & 0 & 0 & \ldots & c_0 \\ \hline \bar{c}_0 & 0 & \ldots & 0 & -\lambda & 0 & \ldots & 0 \\ \bar{c}_1 & \bar{c}_0 & \ldots & 0 & 0 & -\lambda & \ldots & 0 \\ \vdots & & & \vdots & \vdots & & & \vdots \\ \bar{c}_{n-1} & \bar{c}_{n-2} & \ldots & \bar{c}_0 & 0 & 0 & \ldots & -\lambda \end{array}\right| = 0$$

If all the $c_0\, c_1, \ldots, c_{n-1}$ are real, then $\lambda(c_0, \ldots c_{n-1})$ is also the greatest of the absolute values of the roots of the equation of degree n

$$\left|\begin{array}{ccccc} -\lambda & 0 & \ldots & \ldots & c_0 \\ \vdots & -\lambda & \ldots & c_0 & c_1 \\ \vdots & \vdots & \iddots & \iddots & \vdots \\ \vdots & \vdots & \iddots & c_0-\lambda & \vdots \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ c_0 & c_1 & \ldots & \ldots & c_{n-1}-\lambda \end{array}\right| = 0 .$$

A simple proof of theorems 7.1 and 7.2 is given by Goluzin [1946].

Now

$$\max_{z \in D} \left| \sum_{k=n+1}^{n+r} a_k z^k + Q_n(z) \right|$$

$$= \max_{z \in c} \left| \sum_{k=n+1}^{n+r} a_k z^k + Q_n(z) \right|$$

7.3 Asymptotic Economisation on the Unit Disc (cont'd)

$$= \max_{|z|=1} \left| z^{n+r} \left\{ \sum_{k=n+1}^{n+r} \frac{a_k}{z^{n+r-k}} + \sum_{k=0}^{n} \frac{a_k'}{z^{n+r-k}} \right\} \right|$$

$$= \max_{|z|=1} \left| \sum_{k=0}^{r-1} \frac{b_k}{z^k} + \sum_{k=r}^{n+r} \frac{c_k}{z^k} \right|$$

where

$$b_k = a_{n+r-k} \qquad (7.3.2)$$

and

$$c_k = a'_{n+r-k}$$

$$= \max_{|z|=1} \left| \sum_{k=0}^{r-1} b_k w^k + \sum_{k=r}^{n+r} c_k w^k \right| \qquad (7.3.3)$$

where

$$w = 1/z .$$

The corresponding problem of minimising

$$\max_{|w|=1} \left| \sum_{k=0}^{r-1} b_k w^k + \sum_{k=r}^{\infty} c_k w^k \right| \qquad (7.3.4)$$

is solved by theorems 7.1 and 7.2 which give both the coefficients c_k and the maximum λ. The coefficients of $Q_n(z)$ are then found from (7.3.2).

The difference $R_n(w)$ between the expressions inside the moduli signs in (7.3.3) and (7.3.4) is given by

$$R_n(w) = \sum_{k=n+r+1}^{\infty} c_k w^k . \qquad (7.3.5)$$

The behaviour of $R_n(w)$ for large n can be deduced with the help of the following lemma.

7.3 Asymptotic Economisation on the Unit Disc (cont'd)

Lemma 7.3

Let the rational function

$$\frac{z^q \prod_{k=1}^{m} (1 - z\bar{t}_k)}{\sum_{k=1}^{m} (t_k - z)}$$

where q and m are positive integers and $|t_k| < 1$, $(k = 1,2, \ldots m)$, be written as

$$P_q(z) + R(z)$$

where $P_q(z)$ is a polynomial of degree q (the integral part of the function) and $R(z)$ is the remainder (the fractional part).

Then

$$\max_{|z|=1} \left| R(z) \right| \to 0$$

uniformly as $q \to \infty$.

PROOF

The remainder $R(z)$ can be written as

$$R(z) = \frac{P_{m-1}(z)}{\prod_{k=1}^{m} (t_k - z)}$$

where $P_{m-1}(z)$ is a polynomial of degree $m-1$ at most. Let $\{s_r\}_{r=1}^{n}$ be the set of distinct numbers of the set $\{t_k\}_{k=1}^{m}$ and let t_k occur m_k times. Then $R(z)$ has the partial fraction representation

$$R(z) = \sum_{r=1}^{n} \sum_{j=1}^{m_r} \frac{a_{r_j}}{(s_r - z)^j} \qquad (7.3.5)$$

where

$$\sum_{r=1}^{n} m_r = m.$$

7.3 Asymptotic Economisation on the Unit Disc

Lemma 7.3 (cont'd)

Since we require the limit of $R(z)$ as $q \to \infty$, there is no restriction on assuming

$$q > \max_{1 \leq r \leq n} m_r \ . \tag{7.3.6}$$

Multiplying through and equating numerators,

$$z^q \prod_{r=1}^{n} (1 - z\bar{s}_r)^{m_r} + \prod_{r=1}^{n} (s_r - z)^{m_r} P_q(z)$$

$$\sum_{r=1}^{n} \sum_{j=1}^{m_r} a_{rj} \prod_{\substack{i=1 \\ i \neq r}}^{n} (s_i - z)^{m_i} (s_r - z)^{m_r - j} \tag{7.3.7}$$

To find $a_{p,j}$, $(j = 1, \ldots, m_p)$, $1 \leq p \leq n$, let $z = s_p$.

Then

$$s_p^q \prod_{r=1}^{n} (1 - s_p\bar{s}_r) = \sum_{j=1}^{m_p} a_{pj} (s_p - z)^{m_p - j} = a_{p,m_p}.$$

Therefore

$$a_{p,m_p} = O(s_p)^q \ , \tag{7.3.9}$$

the precise formula being unimportant.

If $m_p > 1$, we find a_{p,m_p-j}, $(j = 1, \ldots, m_p - 1)$ by differentiating j times and letting $z = s_p$, giving

$$a_{p,m_p-j} = O(s_p)^{q-j} \tag{7.3.9}$$

where $q - j$ is positive by (7.3.7). Since $|s_p| < 1$, the terms (finite in number) $a_{r,j} (r = 1, \ldots, n;\ j = 1, \ldots, m_r)$ tend to zero as q tends to infinity, and $|s_r - z|$ is bounded away from zero on $|z| = 1$. Thus from (7.3.5) we have

$|R(z)| \to 0$ uniformly as $q \to \infty$ •

7.3 Asymptotic Economisation on the Unit Disc (cont'd)

Theorem 7.4

$$\max_{|w|=1} \left| R_n(w) \right| \to 0 \quad \text{uniformly as} \quad n \to \infty .$$

PROOF

Using the notations of theorems 7.1 and 7.2,

$$R_n(w) = \lambda \left(\frac{\bar{\alpha}_{r-1} + \ldots + \bar{\alpha}_0 w^{r-1}}{\alpha_0 + \ldots + \alpha_{r-1} w^{r-1}} \right) - P_{n+r}(w) \qquad (7.3.10)$$

where $P_{n+r}(w)$ is a polynomial such that the expansion of $R_n(w)$ in ascending powers of w has its first $n+r+1$ coefficients equal to zero. The function

$$\lambda \left(\frac{\bar{\alpha}_{r-1} + \ldots + \bar{\alpha}_0 w^{r-1}}{\alpha_0 + \ldots + \alpha_{r-1} w^{r-1}} \right)$$

is analytic for $|w| < 1$ and its modulus is equal to λ on $|w| = 1$. Therefore if the denominator is factorised as

$$\mu \prod_{k=1}^{r-1} (t_k w - 1) ,$$

then $|t_k| < 1$ for $k = 1, 2, \ldots, r-1$, and the numerator is

$$\lambda \bar{\mu} \prod_{k=1}^{r-1} (w - t_k) .$$

Writing $w = 1/z$ and multiplying through by z^{n+r} gives

$$z^{n+r} R_n\left(\frac{1}{z}\right) = \lambda z^{n+r} \frac{\bar{\mu} \prod_{k=1}^{r-1} (1 - z\bar{t}_k)}{\mu \prod_{k=1}^{r-1} (t_k - z)} - Q_{n+r}(z)$$

where $Q_{n+r}(z) = z^{n+r} P_{n+r}\left(\frac{1}{z}\right)$ is a polynomial of degree $n+r$ and is the integral part of the rational function.

7.3 Asymptotic Economisation on the Unit Disc (cont'd)

Theorem 7.4 (cont'd)

Thus

$$\max_{|w|=1} \left| R_n(w) \right| = \max_{|z|=1} \left| z^{n+r} R_n\left(\frac{1}{z}\right) \right| = \max_{|z|=1} R_n\left(\frac{1}{z}\right)$$

tends to zero uniformly by Lemma 7.3.

The power of the asymptotic economisation method can be illustrated by a few numerical examples.

EXAMPLE 1

To approximate e^z on D by a polynomial of degree 4.

We choose $n = 4$ and $r = 4$, in which case we are effectively approximating

$$\frac{z^5}{5!} + \frac{z^6}{6!} + \frac{z^7}{7!} + \frac{z^8}{8!}$$

by a polynomial of degree 4 .

The value of λ is the largest eigenvalue of a 4×4 matrix which is most conveniently found by the power method. This also gives the eigenvector from which the coefficients of the approximation are calculated. In no numerical example have we found slow convergence of the power method due to a subdominant eigenvalue being near the dominant. For this example the progress of the power method is given in Table 1.

TABLE 1

Successive Estimates for λ_{max}

Iteration Number	λ_{max} estimate
1	0.2951 3887 E - 2
2	0.8602 8162 E - 2
3	0.8568 5155 E - 2
4	0.8568 6534 E - 2
5	0.8568 6529 E - 2
6	0.8568 6529 E - 2

3 Asymptotic Economisation on the Unit Disc

Theorem 7.4

EXAMPLE 2 (cont'd)

The approximating polynomial is

$$Q_4(z) = -0.0000\,0063\,11 + 0.0000\,0268\,00z + 0.0000\,0368\,96z^2$$

$$+0.0000\,3419\,55z^3 - 0.0141\,6750\,7z^4$$

l the corresponding approximation to e^z, obtained by subtracting

(z) from the Taylor expansion of degree 4, is

$$P_4(z) = 1.0000\,0063\,11 + 0.9999\,9732\,00z + 0.49999\,9631\,04z^2$$

$$0.1666\,3247\,12z^3 + 0.0430\,8341\,74z^4,$$

.th

$$\max_{z \in \Sigma} |e^z - P_4(z)| = |e^z - P_4(z)| = 0.8571\,6784\,E-2.$$

Table 2 gives the errors in polynomial approximations of degree 4 to e^z obtained by taking r = 2, 3, 4, 5 and 6.

TABLE 2

The effect of increasing the value of r in the asymptotic economisation

r	λ	$\|\| e^z - P_4(z) \|\|$
2	0.8558 7189 E - 2	0.8824 6518 E - 2
3	0.8568 3957 E - 2	0.8596 1971 E - 2
4	0.8568 6529 E - 2	0.8571 6784 E - 2
5	0.8568 6578 E - 2	0.8569 0236 E - 2
6	0.8568 6579 E - 2	0.8568 6692 E - 2

The last value compares favourably with the error in the best approximation computed by the exchange algorithm of Chapter 2 as 0.8568 6590. Thus the solution of a simple matrix eigenvalue problem of degree 6 for only its dominant eigenvalue and corresponding eigenvector gives an approximation which is within 0.0012% of the best, obtained at considerable computational expense.

7.3 Asymptotic Economisation on the Unit Disc

Theorem 7.4

EXAMPLE 1 (cont'd)

A comparison of λ with $\| e^z - P_4(z) \|$ in Table 2 also shows that the remainder term in Theorem 7.4 is very small, despite the moderate value of n in this example.

EXAMPLE 2

To approximate $\ln(1 + z/2)$ by polynomials of degree 6.

The Taylor expansion converges more slowly than in the first example, and so slightly more terms of the series have been used.

TABLE 3

Errors in economised polynomials of degree 6 for $\ln(1 + z/2)$ on D

r	$\| \ln(1 + z/2) - P_4(z) \|$
2	0.1696 2080 E - 2
6	0.1400 9229 E - 2
10	0.1384 1394 E - 2

Since the best approximation has an error norm of 0.1383 864 E - 2, the final approximation with $r = 10$ is within 0.02% of the best.

In view of these results, we feel justified in calling approximations obtained by asymptotic economisation "almost-best" approximations. They are considerably closer than the "near-best" approximations of Geddes [1973] where errors of up to 1000% were considered reasonable.

7.4 Asymptotic Economisation on an interval of the real axis

The solution of problem (7.2.5) when B is the interval [-1,1] has been considered by Elliott [1973] and Elliott and Lam [1973]. By relating this problem to that for the unit disc, a considerable simplification can be made in their analysis.

Following an idea of Bernstein [1926], Elliott and Lam consider the function

$$F_{k,\ell}(x) = \cos[k\theta - \sum_{s=1}^{\ell} \delta_s(\theta)], \quad 0 \leq \theta \leq \pi \tag{7.4.1}$$

where

$$x = \cos\theta, \quad \delta_s(\theta) = i\,\ell n\left(\frac{t_s e^{i\theta} - 1}{t_s - e^{i\theta}}\right), \tag{7.4.2}$$

and the set $T_\ell = \{t_1, \ldots, t_\ell\}$ satisfies the conditions

$$\begin{aligned} &\text{(i)} \quad 0 < |t_s| < 1, \quad s = 1, \ldots, \ell; \\ &\text{(ii)} \quad \text{if } t_s \in T_\ell \text{ and } \mathrm{Im}(t_s) \neq 0, \text{ then } \bar{t}_s \in T_\ell. \end{aligned} \tag{7.4.3}$$

The branch of the logarithm is chosen so that $\delta_s(0) = 0$ and $\delta_s(\pi) = \pi$. Since $\delta_s(\theta)$ is a (real) continuous function of θ, the function $k\theta - \sum_{s=1}^{\ell} \delta_s(\theta)$ takes on all values in $[0, (k-\ell)\pi]$ as θ varies from 0 to π. At the $k - \ell + 1$ points where $k\theta - \sum_{s=1}^{\ell} \delta_s(\theta)$ takes the values $0, \pi, 2\pi, \ldots, (k-\ell)\pi$, the function $F_{k,\ell}(x)$ takes the values $(-1)^j$.

Elliott and Lam [1973] show that

(a) $F_{k,\ell}(x)$ is a rational function of x, being the quotient of a polynomial of degree $k + \ell$ and one of degree ℓ, and may therefore be written as

7.4 Asymptotic Economisation on an interval of the real axis (cont'd)

$$F_{k,\ell}(x) = G_{k,\ell}(x) + R_{k,\ell}(x) \quad . \tag{7.4.4}$$

where $G_{k,\ell}$ is a polynomial of degree k and $R_{k,\ell}(x)$ is the remainder;

(b) $R_{k,\ell}(x)$ tends to zero uniformly on $[-1,1]$ as $k \to \infty$, and so the behaviour of $G_{k,\ell}(x)$ will, for large n, resemble that of $F_{k,\ell}(x)$ which oscillates between the extreme values of ± 1, taking these values alternately in at least $k - \ell + 1$ points of $[-1,1]$;

(c) if $\sum_{k=0}^{N+r} a_k T_k(x)$ is written as $Q_N(x) + E_{N,r}\, G_{N+r,r}(x)$ then $E_{N,r}$ is the eigenvalue of a certain matrix, and because of the oscillation properties of $G_{N+r,r}(x)$, $Q_N(x)$ may be regarded as an approximate solution of the minimisation problem (in other words, $Q_N(x)$ has the approximate characterisation of the polynomial of best approximation to $\sum_{k=0}^{N+r} a_k T_k(x)$).

A simple proof of these results is obtained by noting that:

$$\begin{aligned} F_{k,\ell}(x) &= \cos\Big[k\theta - \sum_{s=1}^{\ell} \delta_s(\theta)\Big] \\ &= R_e \Big\{ e^{ik\theta} \cdot \prod_{s=1}^{\ell} e^{-i\delta_s(\theta)} \Big\} \\ &= R_e \Big\{ e^{ik\theta} \prod_{s=1}^{\ell} \left(\frac{t_s e^{i\theta} - 1}{t_s - e^{i\theta}} \right) \Big\} \\ &= R_e \Big\{ z^k \prod_{s=1}^{\ell} \left(\frac{t_s z - 1}{t_s - z} \right) \Big\} \text{ where } z = e^{i\theta}. \end{aligned} \tag{7.4.5}$$

7.4 Asymptotic Economisation on an interval of the real axis (cont'd)

$$\text{Since } \sum_{k=0}^{N+r} a_k T_k(x) = \sum_{k=0}^{N+r} a_k T_k(\cos\theta)$$

$$= \sum_{k=0}^{N+r} a_k \cos k\theta$$

$$= \mathrm{Re}\left\{ \sum_{k=0}^{N+r} a_k z^k \right\}, \qquad (7.4.6)$$

$$R_{k,\ell}(x) = \mathrm{Re}\{R(z)\},$$

where $R(z)$ is defined as in Lemma 7.3. The conditions on t_s are satisfied by (7.4.3), and so $R(z)$ tends to zero uniformly as k tends to infinity by the same lemma. Therefore $R_{k,\ell}(x)$ has the same property on $[-1,1]$.

Although we have not used the minimum property of

$$z^k \prod_{s=1}^{\ell} \left(\frac{t_s z - 1}{t_s - z} \right),$$

the fraction of this form which agrees with $\sum_{k=0}^{N+r} a_k z^k$ and is regular in $|z| = 1$ is unique by theorem 7.1, and we deduce that $E_{n,r}$ is the eigenvalue of largest modulus of the matrix defined in theorem 7.2.

The computation of the economised polynomials on $[-1,1]$ is therefore identical to that for the disc, and provides a quick construction of "almost-best" approximation from the truncated Chebyshev series.

.4 Asymptotic Economisation on an interval of the real axis (cont'd)

EXAMPLE

Asymptotic economisation of e^x on $[-1,1]$.

TABLE 4

Errors in economised polynomials

r	λ	$\| e^x - P_4(x) \|$
0		0.591 E-3
2	0.547 E-3	0.550 E-3
3	0.547 E-3	0.547 E-3

The error in the best polynomial approximation of degree 4, computed by the Remes exchange algorithm, has a norm of 0.547 E-3.

7.5 Asymptotic Economisation on the Ellipse

The Chebyshev polynomials for the ellipse are the same as the classical Chebyshev polynomials for a real interval (see example 5 of section 1.4). A different normalisation than usual is convenient here. The Chebyshev polynomial for the ellipse with foci $-\xi$ and ξ is the same as that for the interval $[-\xi, \xi]$, namely $\beta_k T_k(z/\xi)$. In particular, if ξ is chosen as $2\rho/(1+\rho^2)$ where ρ is the eccentricity, then the ellipse (which will be denoted by C_ρ) always has its semi-major axis equal to 1 and its semi minor axis equal to $(1-\rho^2)/(1+\rho^2)$. If $z \in C_\rho$, then

$$z = \frac{\rho^2 e^{i\theta} + e^{-i\theta}}{\rho^2 + 1}, \quad 0 \leq \theta \leq 2\pi \tag{7.5.1}$$

and

$$T_k(z\,;\,C_\rho) = \frac{e^{2k}e^{ik\theta} + e^{-ik\theta}}{\rho^{2k} + 1}$$

$$= \left(\frac{1}{\rho^k} + \rho^k\right) T_k\left(\frac{(1+\rho^2)z}{2\rho}\right) \tag{7.5.2}$$

where $T_k(x)$ is the classical Chebyshev polynomial for the interval $[-1, 1]$. With this normalisation,

$$\| T_k(z\,;\,C_\rho) \| = \max_{z \in C_\rho} | T_k(z\,;\,C_\rho) | = 1.$$

As ρ increases from 0 to 1, C_ρ varies continuously from the disc $(\rho = 0)$ to the interval $[-1, 1]$ $(\rho = 1)$.

At the end of the preceding section, it was found that the method for economisation of an approximation on a real interval was identical to that on a disc. It might therefore be expected

7.5 Asymptotic Economisation on the Ellipse (cont'd)

that the same calculation would yield economised approximations on an ellipse. We need to choose $Q_n(z)$ in order to minimise

$$\max_{z \in C_\rho} \left| \sum_{k=n+1}^{n+r} a_k T_k(z ; C_\rho) + Q_n(z) \right|$$

or equivalently, to choose a_k' $(k = 0, 1, \ldots, n)$ to minimise

$$\max_{z \in C_\rho} \left| \sum_{k=n+1}^{n+r} a_k \left(\frac{\rho^{2k} e^{ik\theta} + e^{-ik\theta}}{\rho^{2k} + 1} \right) + \sum_{k=0}^{n} a_k' \left(\frac{\rho^{2k} e^{ik\theta} + e^{-ik\theta}}{\rho^{2k} + 1} \right) \right|,$$

where

$$z = \frac{\rho^2 e^{i\theta} + e^{-i\theta}}{\rho^2 + 1}$$

$$= \max_{z \in C_\rho} \left| \sum_{k=n+1}^{n+r} a_k \{ \cos k\theta - i \left(\frac{1 - \rho^{2k}}{1 + \rho^{2k}} \right) \sin k\theta \} \right.$$

$$\left. + \sum_{k=0}^{n} a_k' \{ \cos k\theta - i \left(\frac{1 - \rho^{2k}}{1 + \rho^{2k}} \right) \sin k\theta \} \right| . \qquad (7.5.3)$$

It is not possible to minimise this expression, even in an asymptotic sense as for the cases when $\rho = 0$ or 1 since the bracket $(1 - \rho^{2k})/(1 + \rho^{2k})$ now depends on k. However, it is still possible to show that a similar reduction in the value of (7.5.3) is possible over the uneconomised polynomial ($a_k' = 0$) to that for a disc, and that the expression (7.5.3) is still bounded asymptotically by λ, where λ is computed using the same formula as for the disc.

The expression inside the modulus sign of (7.5.3) is

$$\left\{ \sum_{k=n+1}^{n+r} a_k \cos k + \sum_{k=0}^{n} a_k' \cos k\theta \right\} \quad \ldots\ \textit{(expression continued overleaf)}$$

7.5 Asymptotic Economisation on the Ellipse (cont'd)

$$-i\left\{\sum_{k=n+1}^{n+r} a_k\left(\frac{1-\rho^{2k}}{1+\rho^{2k}}\right)\sin k\theta + \sum_{k=0}^{n} a_k'\left(\frac{1-\rho^{2k}}{1+\rho^{2k}}\right)\sin k\theta\right\}$$

which is bounded asymptotically by λ, where λ is found using theorem 7.2.

A few examples will be given to demonstrate the effectiveness of the economisation.

Example 1

To approximate e^z on the ellipse semi major axis = [0, 1], semi minor axis = [0, 0.5i], foci $\pm\sqrt{3}/2$ ($\rho = 1/\sqrt{3}$) by a polynomial of degree 4.

r	$\| e^z - p_4(z) \|$
0	0.2333 E-2
2	0.21096 E-2
4	0.20807 E-2
6	0.20805 E-2

The final approximation

$$e^z = 1.1904\,7433 + 1.0967\,2589\,T_1(z; C_\rho) +$$
$$0.3324\,9450\,T_2(z; C_\rho) + 0.0763\,9490\,T_3(z; C_\rho) +$$
$$0.0141\,1181\,T_4(z; C_\rho)$$

where $T_k(z; C_\rho) = (3^{k/2} + 3^{-k/2})T_k(2z/\sqrt{3})$

{or $T_1 = z$, $T_2 = (8z^2 - 3)/5$, $T_3 = (32z^3 - 18z)/14$,

$T_4 = (128z^4 - 96z^2 + 9)/41$}

has an error with 10 extrema on C_ρ, each of which are between 0.20804 E-2 and 0.20806 E-2 in modulus.

7.5 Asymptotic Economisation on the Ellipse (cont'd)

Example 2

The approximation of $1/(4-z)$ on the "standard" ellipse with foci ± 1, semi major axis $= \sqrt{2}$, semi minor axis $= 1$, $\rho = \sqrt{2} - 1$.

This example was also considered by Geddes [1973], where approximations correspond to the uneconomised polynomial ($r = 0$). In each case, the approximations are compared with those computed by the algorithm of Chapter 2, which the inclusion intervals show to be almost optimal for this problem.

n	r	$\|f - p_n(z)\|$	Inclusion interval
4	0	1.009 E-3	
	2	0.8554 E-3	[0.7726 E-3,
	3	0.8000 E-3	0.7736 E-3]
	4	0.7815 E-3	
5	0	0.309 E-3	
	2	0.2625 E-3	[0.2369 E-3,
	3	0.2454 E-3	0.2371 E-3]
	4	0.2396 E-3	
6	0	0.941 E-4	
	2	0.805 E-4	[0.726 E-4,
	3	0.752 E-4	0.727 E-4]
	4	0.734 E-4	

In this example, the economised polynomials provide approximations which (when $r = 4$) are each within about 1 per cent of the best approximation. Considering the simplicity of their construction compared with the approximations computed using characterisation or programming, this is a very reasonable result.

7.6 Economisation of a rational Function on the unit disc

The usual methods for economisation of rational functions were derived only for small intervals, strictly only as the interval length tends to zero. However, it is possible to economise a rational function on the unit disc directly. Given a rational function $R_{n,n}(z)$, $(n > 1)$, regular on the unit disc, we shall construct an economical function $R_{n-1,n-1}(z)$ such that $R_{n,n}(z) - R_{n-1,n-1}(z)$ is a minimising function.

We begin with a result to be found in Achieser [1953, p. 243] which uses the same function as Theorem 7.1.

Theorem 7.5

Let the points $z_1, z_2, \ldots, z_n$ lie outside the unit disc. Then for $N \geq n$,

$$\min_{\substack{A_i \\ 1 \leq i \leq N}} \max_{|z| = 1} \frac{|z^N + A_1 z^{N-1} + \ldots + A_N|}{|(z - z_1) \ldots (z - z_n)|} = \frac{1}{|z_1 \ldots z_n|},$$

where the numerator is

$$z^{N-n} \frac{(z\bar{z}_1 - 1) \ldots (z\bar{z}_n - 1)}{\bar{z}_1 \ldots \bar{z}_n}$$

For our purpose, this result may be rewritten as follows.

Corollary 7.5

Let $1 + \alpha_1 z + \ldots + \alpha_n z^n$ have all its zeros outside the unit disc. Then for $N \geq n$

$$\min_{\substack{A_i \\ 1 \leq i \leq N}} \max_{|z| = 1} \frac{|z^N + A_1 z^{N-1} + \ldots + A_N|}{|1 + \alpha_1 z + \ldots + \alpha_n z^N|} = 1$$

where the numerator is

$$z^{N-n}(z^n + \bar{\alpha}_1 z^{n-1} + \ldots + \bar{\alpha}_n).$$

7.6 Economisation of a rational Function on the unit disc

Corollary 7.5 (cont'd)

We now demonstrate the economisation of a rational function $R_{n,n}(z)$, $n > 1$, with real coefficients by writing

$$R_{n,n}(z) - R_{n-1,n-1}(z) = \lambda \left(\frac{z^k + \alpha_1 z^{k-1} + \ldots + \alpha_k}{1 + \alpha_1 z + \ldots + \alpha_k z^k} \right) \qquad (7.6.1)$$

where $k = 2n-1$, since this gives an error of the form required by the Corollory 7.5. Let

$$R_{n,n}(z) = \frac{\sum_{r=0}^{n} a_r z^r}{\sum_{r=0}^{n} b_r z^r} \quad \text{and} \quad R_{n-1,n-1}(z) = \frac{\sum_{r=0}^{n-1} c_r z^r}{\sum_{r=0}^{n-1} d_r z^r}$$

where $b_0 = d_0 = 1$. Expanding the left hand side of (7.6.1) and comparing with the right hand side gives the equations

$$\left[\begin{array}{c|c} B_1 & A_1 \\ \hline B_2 & A_2 \end{array}\right] \begin{bmatrix} c \\ d \end{bmatrix} = -\lambda \left[\begin{array}{c|c} 0 & B_3 \\ \hline 0 & B_4 \end{array}\right] \begin{bmatrix} c \\ d \end{bmatrix} \qquad (7.6.2)$$

where

$$B_1 = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ -b_1 & -1 & & 0 \\ \vdots & \vdots & & \vdots \\ -b_{n-1} & -b_{n-2} & \cdots & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ a_1 & a_0 & & 0 \\ \vdots & \vdots & & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} -b_n & -b_{n-1} & \cdots & -b_1 \\ 0 & -b_n & \cdots & -b_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & -b_n \end{bmatrix}, \quad A_2 = \begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 \\ 0 & a_n & \cdots & a_2 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & a_n \end{bmatrix},$$

7.6 Economisation of a rational Function on the unit disc (cont'd)

$$B_3 = \begin{vmatrix} 0 & 0 & & b_n \\ 0 & 0 & b_n & b_{n-1} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ b_n & b_{n-1} & b_2 & b_1 \end{vmatrix} \qquad B_4 = \begin{vmatrix} b_{n-1} & b_{n-2} & & 1 \\ b_{n-2} & b_{n-3} & 1 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & 0 & 0 \end{vmatrix},$$

$$c = [c_0, c_1, \ldots, c_n]^T \quad \text{and} \quad d = [1, d_1, \ldots, d_{n-1}]^T. \tag{7.6.3}$$

Since $b_n \neq 0$, B_1 and B_2 are non-singular, and so (7.6.2) can be written as

$$(B_1^{-1} A_1 - B_2^{-1} A_2)d = \lambda(B_1^{-1} B_3 - B_2^{-1} B_4)d, \tag{7.6.4}$$

an eigenvalue problem of order $n-1$.

The question now arises as to which is the required eigenvalue of (7.6.4). It is essential that the error function is pole-free in D, and since the norm of the error in the economisation is λ, we would like to choose the smallest in modulus of the real eigenvalues. This question is at least partly answered by the following theorem.

Theorem 7.6

Let $\lambda B(z)$ and $\lambda^* B^*(z)$ be two right hand sides of (7.6.1) corresponding to two distinct eigenvalues of (7.6.2). If $B(z)$ is pole-free on D, whilst $B^*(z)$ has at least one pole in D, then $|\lambda^*| > |\lambda|$.

Proof

Define

$$\phi(z) = \lambda B(z) - \lambda^* B^*(z). \tag{7.6.5}$$

On $|z| = 1$, we have

$$|\phi(z)| \geq \big||\lambda B(z)| - |\lambda^* B^*(z)|\big|$$
$$= |\lambda - \lambda^*| \neq 0.$$

7.6 Economisation of a rational Function on the unit disc (cont'd)

Theorem 7.6 (cont'd)

Therefore $\phi(z)$ does not vanish on $|z| = 1$. As z moves round $|z| = 1$, $\arg \phi(z)$ increases by $2\pi(\tau - \sigma)$ where τ is the number of zeros and σ the number of poles in $|z| < 1$, each root being counted according to its multiplicity.

Now $B(z)$ has the form

$$\prod_{r=\emptyset|}^{2n-1} \frac{(zz_r - 1)}{(z - z_r)} ,$$

and since $B(z)$ has no poles in $|z| < 1$, it must have $(2n-1)$ zeros in $|z| < 1$.

Furthermore, from (7.6.1) we see that

$$\phi(z) = \lambda B(z) - \lambda^* B^*(z) = R^*_{n-1,\,n-1}(z) - R_{n-1,\,n-1}(z)$$

where $R_{n-1,n-1}(z)$ and $R^*_{n-1,\,n-1}(z)$ are the economised functions corresponding to λ and λ^* respectively, and so $\phi(z)$ has at least one pole in $|z| < 1$ and at most $2n-2$ zeros in $|z| < 1$.

Therefore as z moves round $|z| = 1$, $\arg B(z)$ increases by $(2n-1)2\pi$ and $\arg\phi(z)$ increases by at most $(2n-3)2\pi$, which together imply that

$$\arg \left\{ \frac{B(z)}{\phi(z)} \right\}$$

increases by at least 4π. Consequently, the circle $|z| = 1$ includes a point $z = z_0$ such that $\lambda B(z)/\phi(z)$ is negative.

Thus from (7.6.5)

$$\begin{aligned} |\lambda^*| &= |\lambda^* B^*(z_0)| = |\lambda B(z_0) - \phi(z_0)| \\ &= |\lambda B(z_0)| \cdot \left| 1 - \frac{\phi(z_0)}{\lambda B(z_0)} \right| \\ &> |\lambda B(z_0)| = |\lambda| , \end{aligned}$$

and so

$$|\lambda^*| > |\lambda| .$$

7.6 Economisation of a rational Function on the unit disc (cont'd)

Theorem 7.6 (cont'd)

We have not been able to prove the existence of a pole-free solution. The significance of this theorem is that if the solution corresponding to the smallest eigenvalue has poles in D, then the problem cannot be solved, since the theorem shows that no larger eigenvalue could correspond to a pole-free solution.

Before proceeding to numerical illustrations, we make two observations about this result. Firstly, in no numerical example has the real eigenvalues of smallest modulus failed to produce a pole-free solution. Secondly, if this result is thought to be unsatisfactory, then an even worse situation exists with respect to best rational approximation on a real point set. In this case it is known (Maehly [1963] and Werner [1963]) that at most one eigenvalue leads to a pole-free approximation. Moreover there may not be a pole-free solution, and that when there is, it may not correspond to the eigenvalue of smallest modulus.

The method does not extend to the determination of $R_{n-2,n-2}(z)$ such that

$$\max_{z\in D} \left| R_{n,n}(z) - R_{n-2,n-2}(z) \right|$$

is minimised, but the economisation may of course be applied successively to $R_{n,n}(z)$, $R_{n-1,n-1}(z)$ and so on.

Example 1 $f(z) = e^z$.

We begin with the [3,3] Padé approximant, obtained by truncating the continued fraction. Equations (7.6.2) are solved, as in all the numerical examples, by Newton's method using as initial approximation $\lambda = 0$ and $R_{22}(z)$ the [2,2] Padé approximant.

7.6 Economisation of a rational Function on the unit disc (cont'd)

Example 1 (cont'd)

The iteration converges in 3 steps to $\lambda = 0.1398\,5942\,E-2$ and

$$R_{2,2}(z) = \frac{1.0 + 0.5163\,4215\,z + 0.0921\,5674\,z^2}{1.0 + 0.4836\,4207\,z + 0.0756\,7457\,z^2} .$$

For this approximation

$$\max_{z\in D} \left| e^z - R_{2,2}(z) \right| = 0.1405\,316\,E-2$$

against $0.3996\,1141E-2$ for the [2,2] Padé approximant.

Example 2 $f(z) = e^z$

Comparison of economised rational approximations with the best rational approximation $R^*_{r,r}(z)$, calculated from the algorithm of Chapter 3.

Table 1

Successive economisation of an n,n Pade approximant $R_{n,n}(z) \to R_{n-1,n-1}(z) \to \ldots\ldots \to R_{r,r}(z)$.

r	n	$\max_{z\in D} \left\| e^z - R_{r,r}(z) \right\|$	$\max_{z\in D} \left\| e^z - R^*_{r,r}(z) \right\|$
2	2	0.3996 E-2	0.1393 E-2
	3	0.1405 E-2	
	4	0.1403 E-2	
3	3	0.2803 E-4	0.9934 E-5
	4	0.9980 E-45	
	5	0.9971 E-45	
	6	0.9971 E-5	
4	4	1.105 E-7	0.396 E-7
	5	0.397 E-7	
	6	0.397 E-7	

7.6 Economisation of a rational Function on the unit disc (cont'd)

Example 4 $f(z) = \ln(1 + z/2)$

A comparison of economised rational functions with the best approximation.

We make a slight diversion before completing this example. A technique for accelerating the convergence of a Chebyshev series on the interval $[a, b]$ has been given by Scraton [1960], based on a bilinear transformation of the interval onto itself. It is particularly effective when the function being approximated has a singularity near the interval $[a, b]$. The free parameter is chosen to move the singularity of the transformed function away from the interval. A similar transformation can be applied to the unit disc. The rate of convergence of the Padé approximation to a function is governed by the distance of the nearest singularity to D. In addition, the bilinear transformation of an [n,n] Pade approximant leaves the degree unchanged.

This example will serve to illustrate the details of the method. For the function $\ln(1 + z/2)$, the [1,1] Padé approximant is

$$R_{1,1}(z) = \frac{2z}{4 + z} .$$

The bilinear transformation

$$w = \frac{z + R}{1 + Rz}$$

maps $|z| \leq 1$ onto $|w| \leq 1$. Now f has singularities at $z = -2$ and at infinity. These map into $w_1 = (-2 + R)/(1 - 2R)$ and $w_2 = 1/R$. The value of R is chosen to maximise $\min\{|w_1|, |w_2|\}$ namely $R = 2 - \sqrt{3}$ which makes $w_1 = -(2 + \sqrt{3})$, $w_2 = (2 + \sqrt{3})$. We can now find the transformed [1,1] Padé approximant, denoted by $R^T_{1,1}(z)$. Since $z = (w - R)/(1 - Rw)$,

7.6 Economisation of a rational Function on the unit disc (cont'd)

Example 4 (cont'd)

$$\ln(1 + z/2) = \ln \{1 + \tfrac{1}{2}(w - R) / (1 - Rw)\}$$

$$= \ln \frac{\sqrt{3}}{2} \left(\frac{1 + Rw}{1 - Rw} \right).$$

Thus

$$R^{T}_{1,1}(w) = \ln(\sqrt{3}/2) + 2Rw$$

$$= \ln(\sqrt{3}/2) + 2R \left(\frac{z + R}{1 + Rz} \right)$$

$$R^{T}_{1,1}(z) = \frac{-0.002\,474\,967 + 0.497\,356\,295\,4\, z}{1 + 0.267\,949\,192\,4\, z} .$$

The other Padé approximants can be similarly constructed and economised in the usual way.

Table 2

Comparison of the errors in Padé, transformed Padé and economised transformed Padé approximations to $\ln(1 + z/2)$

r	n	$\max_{z \in D} \lvert \ln(1 + z/2) - R_{r,r}(z) \rvert$	$\max_{z \in D} \lvert \ln(1 + z/2) - R^{*}_{r,r}(z) \rvert$
1	1	0.265 E-1	0.128537 E-1
	1 (transformed)	0.134077 E-1	
	3 (transformed)	0.128542 E-1	
	5 (transformed)	0.128537 E-1	
3	3	2.55 E-5	0.45543 E-5
	3 (transformed)	0.51431 E-5	
	5 (transformed)	0.45560 E-5	
	7 (transformed)	0.45557 E-5	

These examples show how a simple technique can give almost-best rational approximations on the disc.

7.7 Economisation of a rational function on an interval of the real axis

In the light of the results of the preceeding section concerning the disc, we return to the direct problem of economisation on a finite real interval. By "direct" we mean that it does not depend on a small interval or on n being large. For convenience the interval will be taken as $[-1, 1]$. Given a rational function $S_{n,n}(x)$ with no poles on the interval $[-1, 1]$, we require $S_{n-1,n-1}(x)$ so that

$$\max_{x\in[-1,1]} \left| S_{n,n}(x) - S_{n-1,n-1}(x) \right| \tag{7.7.1}$$

is a minimum.

Now the function $F_{k,\ell}(x)$ as defined in (7.4.1) attains the values ± 1 alternately at $k-\ell+1$ points of $[-1, 1]$. By examining the definition of $F_{k,\ell}(x)$ in terms of $\delta_s(\theta)$ in (7.4.2), it can be seen that $F_{0,\ell}(x)$ attains the values ± 1 alternately at $\ell-1$ points of $[-1, 1]$. But from (7.4.5), $F_{0,\ell}(x)$ is the real part of

$$\left\{ \frac{z^{\ell} + \alpha_1 z^{\ell-1} + \ldots + \alpha_{\ell}}{1 + \alpha_1 z + \ldots + \alpha_{\ell} z^{\ell}} \right\} \quad \text{where } z = e^{i\theta} \tag{7.7.2}$$

Suppose it were possible to express $S_{n,n}(x)$ as the real part of $R_{n,n}(z)$ where $x = \cos\theta$, $z = e^{i\theta}$ and $R_{n,n}(z)$ is analytic on D. Then if $R_{n-1,n-1}(z)$ is constructed from (7.6.1), $S_{n-1,n-1}(x) = \mathrm{Re}\{ R_{n-1,n-1}(z) \}$ is the solution to (7.7.1) since $S_{n-1,n-1}(x)$ has the required characterisation of a rational function of best approximation to $S_{n,n}(x)$.

The problem is therefore reduced to the expression of $S_{n,n}(x)$ as $\mathrm{Re}\{ R_{n,n}(x) \}$. Let

$$S_{n,n}(x) = \frac{{\sum_{k=0}^{n}}' u_k T_k(x)}{{\sum_{k=0}^{n}}' v_k T_k(x)} \quad \text{and} \quad R_{n,n}(z) = \frac{\sum_{k=0}^{n} a_k z^k}{\sum_{k=0}^{n} b_k z^k} \; ; \tag{7.7.3}$$

7.7 Economisation of a rational function on an interval of the real axis (cont'd)

then with $z = e^{i\theta}$ it is easy to show that the coefficients $b_k (k = 0, \ldots, n)$, satisfy the non-linear equations

$$\sum_{k=0}^{n-r} b_k b_{k+r} = v_r \quad (r = 0, \ldots, n), \tag{7.7.4}$$

while the coefficients $a_k (k = 0, \ldots, n)$ satisfy the linear equations

$$\sum_{k=0}^{n-r} (a_k b_{k+r} + b_k a_{k+r}) = u_r \quad (r = 0, \ldots, n) \tag{7.7.5}$$

assuming the b's have been found.

Although the system (7.7.4) is only mildly non-linear and may be readily solved by Newton's method, we have been unable to prove anything about the existence of a solution in general. It is possible that a slightly different formulation of the problem might lead to a simpler analysis.

We conclude with a simple illustrative example.

Example. $f(x) = e^x$.

The [2,2] rational Chebyshev approximation

$$\frac{1.001005 + 0.482571\,T_1(x) + 0.039719\,T_2(x)}{1.0 - 0.478306\,T_1(x) + 0.038734\,T_2(x)}$$

is found by solving (7.7.4) and (7.7.5) to be the real part of

$$\frac{1.266066 + 0.815126\,z + 0.016147\,z^2}{1.0 - 0.248954\,z + 0.020575\,z^2}$$

where $x = \cos\theta$ and $z = e^{i\theta}$. The algorithm of section (7.6) then gives an economised function

7.7 Economisation of a rational function on an interval of the real axis (cont'd)

Example (cont'd)

$$\frac{1.265966 + 0.838684\,z}{1.0 - 0.231624\,z} \quad \text{with} \quad \lambda = 0.020996$$

whose real part

$$\frac{1.017137 + 0.517682\,x}{1.0 - 0.439661\,x}$$

is the desired approximation to e^x on $[-1,1]$. This approximation has a maximum error of 0.02107 on $[-1,1]$ which should be compared with 0.02097 for the best approximation.

It is probable that the method could be extended to the economisation of a rational function on an ellipse in the same way as the extension was possible for polynomial approximations.

7.8 Conclusion

This chapter has necessarily had to be compressed and should be regarded as an indication of what can be done in the way of constructing simple yet almost-best approximations. There is scope for more analysis in the case of the ellipse and also rational functions $R_{m,n}(z)$ for $m \neq n$, and there is sufficient numerical evidence to justify further investigation. If a method for economising a Faber series could be derived, then almost all approximations could be found by the methods of this chapter, and the algorithms of Chapters 4 and 5 only used when a very precise best approximation is required.

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