

A stiffness reduction method for the in-plane design of structural steel elements

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Abstract

Stiffness reduction offers a practical means of considering the detrimental influence of geometrical imperfections, residual stresses and the spread of plasticity in the analysis and design of steel structures. In this paper, a stiffness reduction approach is presented, which utilises Linear Buckling Analysis (LBA) and Geometrically Nonlinear Analysis (GNA) in conjunction with developed stiffness reduction functions for the design of columns and beam-columns in steel frames. This approach eliminates the need for modelling geometrical imperfections and requires no member buckling checks. For columns, inelastic flexural buckling loads can be obtained using LBA with appropriate stiffness reduction, while GNA with stiffness reduction is required to determine an accurate prediction of beam-column failure. The accuracy and practicality of the proposed method is shown in several examples, including regular and irregular members. For the latter case in particular, it is found that the proposed approach provides more accurate capacity predictions than traditional design methods, when compared to results generated by means of nonlinear finite element modelling.

Keywords:

Stiffness reduction; steel columns; steel beam-columns; inelastic buckling

1. Introduction

The development of plasticity within the members of a steel frame may significantly influence its overall response, generally resulting in a member force distribution different from that obtained through elastic analysis. In current design specifications [1, 2], this is implicitly accounted for by employing effective length or notional load approaches in conjunction with member design equations. An alternative design strategy is based upon the use of stiffness reduction concepts [3–9]. Recent studies have focused on the enhancement of plastic hinge analysis through member stiffness reduction to take account of the spread of plasticity along

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the member length [10]. Orbison [11] investigated the use of a stiffness reduction function obtained from the Column Research Council (CRC) column strength curve [12], while Liew [13] proposed a refined plastic hinge approach using a smooth stiffness reduction function for plastic hinges in conjunction with a stiffness reduction function derived from the LRFD inelastic flexural buckling formulation [14] for members. Ziemian and McGuire [15] developed a stiffness reduction function considering the combined influence of minor axis bending and compression. Using the refined plastic hinge approach [13], Landesmann and Batista [16] derived stiffness reduction expressions using the European column buckling curves in lieu of the LRFD column buckling curve. Barszcz and Gizejowski [17] proposed theoretical models for compression members using the European buckling curves to determine different stiffness reduction expressions for axial and flexural stiffness as functions of member non-dimensional slenderness. Finally, Zubychan [18, 19] proposed stiffness reduction functions to capture the development of plasticity at cross-sectional level.

The above studies have generally focused on the development of stiffness reduction schemes for use in plastic hinge analysis, which is not widely used in practice. More recently, Maleck [20] and Surovek-Maleck and White [21, 22] proposed the use of stiffness reduction in conjunction with Geometrically Nonlinear Analysis (GNA) to account for the detrimental influence of the spread of plasticity on the response of steel frames. This method, which is included in the two most recent versions of the AISC-360 specification, including AISC-360-10 [2], can be readily applied using conventional structural analysis software, enabling the design of members without having to consider increased effective lengths associated with the sway buckling mode. Nevertheless, the stiffness reduction scheme suggested by Surovek-Maleck and White [2] does not fully capture the detrimental influence of the spread of plasticity, geometrical imperfections and residual stresses. Thus, this method still requires the use of column strength equations for member design, and may lead to overly conservative strength predictions when compared against accurate results from Geometrically and Materially Nonlinear Analyses with Imperfections (GMNIA).

To extend previous stiffness reduction approaches, a stiffness reduction method is presented herein that utilises more advanced stiffness reduction functions to capture fully the detrimental influence of the spread of plasticity, residual stresses and geometrical imperfections on the capacity of columns and beam-columns. According to the proposed approach, Linear Buckling Analysis (LBA) with stiffness reduction is used for the design of columns, while the use of Geometrically Nonlinear Analysis (GNA) with stiffness reduction and without considering imperfections is proposed for the design of beam-columns. Note that both LBA and GNA are elastic analysis methods. In the latter case, the section forces at the most heavily loaded cross-section are checked against the ultimate cross-section resistance. Unlike the design approach of Surovek-Maleck and White [21], the method proposed in this study does not require the use of column strength equations; instead only cross-section checks are necessary. Furthermore, the proposed approach does not require the explicit modelling of geometrical imperfections, and hence avoids the need to identify a suitable shape and direction.

In the following sections of this paper, an accurate finite element modelling approach used to analyse steel members is first described. The developed stiffness reduction equations for

members under axial load, bending and combined axial load and bending are then presented. A series of design examples for regular and irregular members and a simple frame are shown to illustrate the application of the method. In all cases, the proposed approach is compared against accurate GMNIA predictions and the results obtained using current EN 1993-1-1 [1] formulations.

2. Finite element modelling

In this section, the key characteristics of the finite element modelling approach used in this study to represent the response of steel columns and beam-columns are presented. The models, which allow for geometrical and material nonlinearities and include residual stresses and geometrical imperfections, are initially validated against experimental results from the literature, and then used later in the paper for comparisons with the proposed design equations.

2.1. Development of finite element models

In the numerical simulations, the finite element analysis software Abaqus [23] was used, and the steel members were modelled utilising elastic-plastic beam elements. Such elements are suitable for analysing members which are not susceptible to local buckling, as is the case for all members considered in this study. The adopted element is named B31OS in the Abaqus element library [23] and is a Timoshenko beam element that accounts for transverse shear deformations and warping rotation. Thirty-three integration points were used for each flange and web of a generic I section to represent accurately the variation of strains and stresses within the cross-section and to capture the spread of plasticity. For numerical integration over the element length, the Simpson method [23] with one integration point located in the middle of each element was chosen. The Poisson's ratio was taken as 0.3 in the elastic range and 0.5 in the plastic range by defining the effective Poisson's ratio as 0.5 to allow for the change of cross-sectional area under load. The tri-linear elastic-plastic stress-strain relationship shown in Fig. 1 was employed, where E is the Young's modulus, E_{sh} is the strain hardening modulus, f_y and ϵ_y are the yield stress and strain respectively, ϵ_{sh} is the strain value at the onset of strain hardening. The parameters f_u and ϵ_u correspond to the ultimate stress and strain respectively. E_{sh} was assumed to be 2% of E and ϵ_{sh} was taken as $10\epsilon_y$, conforming to the ECCS recommendations [24] for hot-rolled structural steel. Isotropic hardening and the von Mises yield criterion with associated plastic flow were assumed. The engineering stress-strain model shown in Fig. 1 was then transformed into the true stress-strain model according to the constitutive formulation used in Abaqus [23], which is based upon the Cauchy stress-strain assumption. S235 steel grade was used in all the simulations.

In the numerical models, the ECCS [24] residual stress patterns illustrated in Fig. 2 were employed to define the initial stress values at the section integration points through the SIGINI user subroutine [23]. The initial geometric member imperfections (i.e. out-of-straightness) were assumed to be half-sine waves in shape and $1/1000$ of the corresponding member length in magnitude [25]. These imperfections and residual stresses are used in all GMNIA conducted throughout this paper unless otherwise stated.

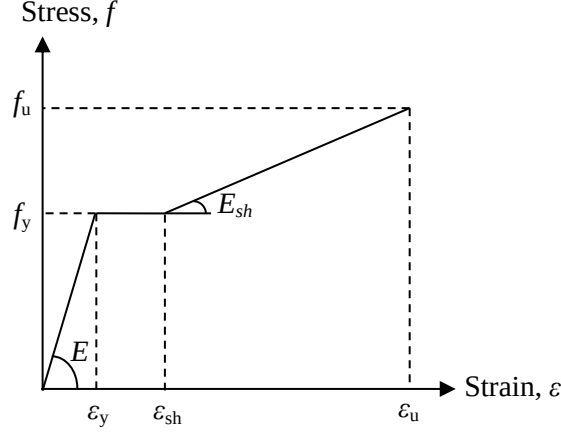


Figure 1: Material stress-strain curves used in finite element models

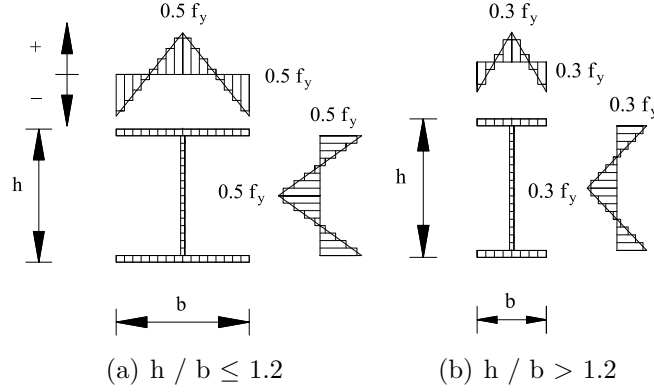


Figure 2: Residual stress patterns applied to finite element models (+ ve = tension; - ve=compression)

2.2. Validation of finite element models

To validate the adopted finite element modelling approach, the experiments of Van Kuren and Galambos [26] on steel beam-columns were considered (Fig. 3). Additional results of the experiments are also provided in Galambos and Lay [27]. In the tests, an axial load was applied to the column first and then the bending moment, which was applied to only one end, was increased up to collapse; the specimens were restrained in the out-of-plane direction. Fig. 3 shows the experimental and numerical normalised moment-deformation curves for test specimens A5 and A7, in which N and M_y are the applied axial load and major axis bending moment, N_{pl} and M_{pl} are the yield load and plastic moment capacity and θ is the end rotation. Both specimens have a non-dimensional slenderness $\bar{\lambda} = 1.23$, where $\bar{\lambda} = \sqrt{A f_y / N_{cr}}$, in which A is the cross-sectional area, f_y is the material yield stress and N_{cr} is the elastic buckling load of the member. The numerical curves were obtained through GMNIA.

The close agreement between the experimental and numerical results shown in the figure indicates that the adopted finite element description can accurately predict the physical

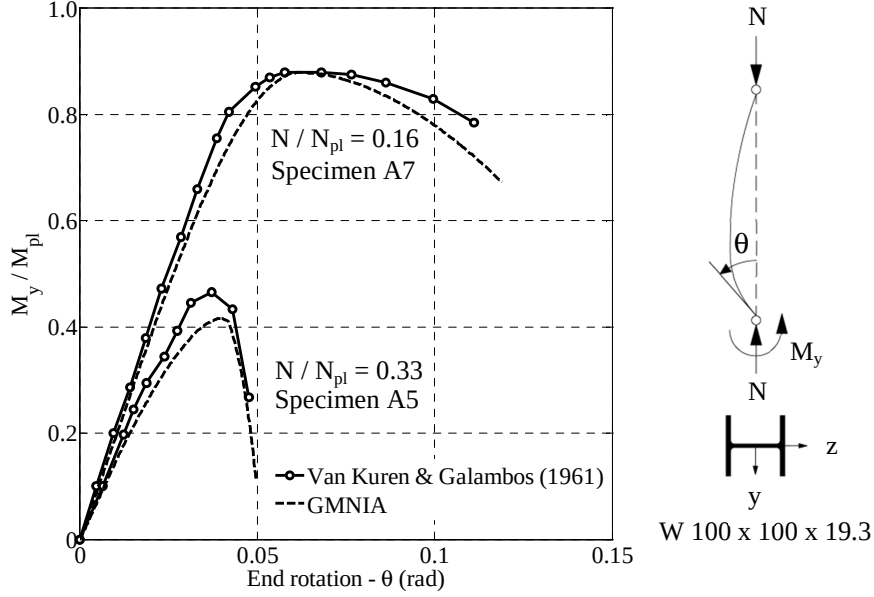


Figure 3: Comparison between the normalised moment-deformation paths of the FE models and those from the experiments of Van Kuren and Galambos [26]

response of steel beam-columns. The discrepancy for the specimen A5, whose axial load level is $N/N_{pl} = 0.33$, may result from the difference between the geometrical imperfections assumed in the numerical model and the actual values, which are not reported in Van Kuren and Galambos [26]. On the other hand, owing to a relatively small axial load, geometrical imperfections are of less significance for the A7 specimen, thus resulting in a very accurate numerical prediction of the capacity of the member.

3. Stiffness reduction under axial loading

This section addresses the derivation of the stiffness reduction function for steel members under axial loading and illustrates the use of the proposed stiffness reduction function for the calculation of the inelastic flexural buckling capacity of steel columns. Having validated the finite element models in Section 2, the accuracy of the stiffness reduction function is assessed using results obtained through GMNIA.

3.1. Derivation of stiffness reduction function for axial load

The stiffness reduction function for a member under axial loading τ_N is derived using the European column buckling curves given in EN 1993-1-1 [1]. As shown in Fig. 4, where N_{Ed} is the applied axial load, the inelastic buckling load of a column can be expressed as the elastic buckling load multiplied by a stiffness reduction factor τ_N .

The stiffness reduction function τ_N , which corresponds to the ratio between the reduced modulus E_r and the Young's modulus E , can be calculated by considering the inelastic and elastic critical buckling loads of the member, $N_{cr,i}$ and N_{cr} respectively, and applying eq. (1), where χ is the buckling reduction factor.

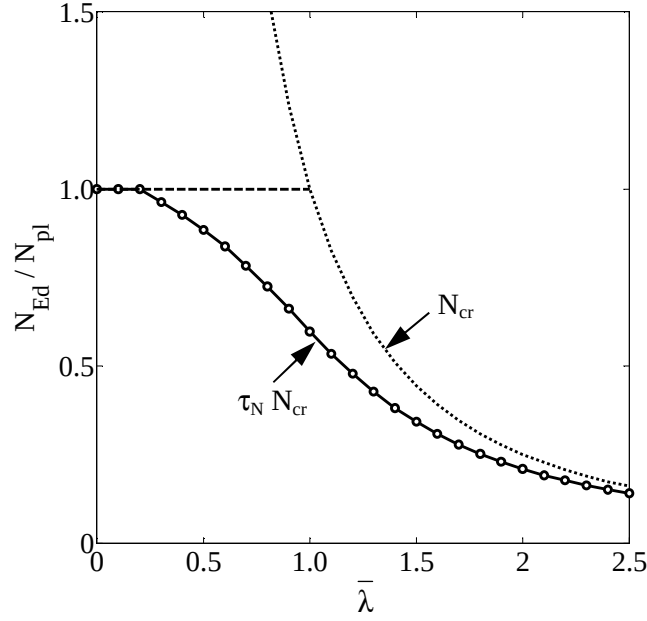


Figure 4: Use of a stiffness reduction factor τ_N to transform elastic flexural buckling loads N_{cr} to inelastic flexural buckling loads $N_{cr,i}$

Table 1: Imperfection factors α for flexural buckling from EN 1993-1-1 [1]

Buckling curve	a_0	a	b	c	d
α	0.13	0.21	0.34	0.49	0.76

$$\tau_N = \frac{E_r}{E} = \frac{N_{cr,i}}{N_{cr}} = \chi \bar{\lambda}^2 \quad (1)$$

The buckling reduction factor χ can be obtained from the Perry-Robertson expression [28] given by eq. (2), which is the basis of the European column curves where α is the imperfection factor. Five values of α are given in EN 1993-1-1 [1] to define 5 different buckling curves - see Table 1.

$$\chi = \frac{1}{\phi + \sqrt{\phi^2 - \bar{\lambda}^2}} \quad \text{where} \quad \phi = 0.5 \left[1 + \alpha(\bar{\lambda} - 0.2) + \bar{\lambda}^2 \right] \quad (2)$$

The Perry-Robertson equation can also be expressed as shown in eq. (3).

$$\chi + \chi \frac{\alpha(\bar{\lambda} - 0.2)}{1 - \chi \bar{\lambda}^2} = 1 \quad (3)$$

6

Rearranging eq. (3) in terms of $\bar{\lambda}$ gives:

$$\bar{\lambda}^2 = \frac{4\psi^2}{\alpha^2 \chi^2 \left[1 + \sqrt{1 - 4\psi \frac{(\chi-1)}{\chi \alpha^2}} \right]^2} \quad \text{where } \psi = 1 + 0.2\alpha\chi - \chi \quad (4)$$

Substituting eq. (4) into eq. (1) and with $\chi = N_{Ed}/N_{pl}$ gives eq. (5), where τ_N is expressed as a function of the ratio between the applied axial load N_{Ed} and the yield load $N_{pl} = Af_y$ as well as the imperfection factor α . Inclusion of the imperfection factor in the proposed expression for τ_N allows an implicit consideration of the effects of residual stresses and geometrical imperfections. For stocky members ($\bar{\lambda} \leq 0.2$), EN 1993-1-1 allows the full cross-sectional resistance to be used, in which case $N_{Ed}/N_{pl} = 1.0$ and $\tau_N = 0.04$.

$$\tau_N = \frac{4\psi^2}{\alpha^2 N_{Ed}/N_{pl} \left[1 + \sqrt{1 - 4\psi \frac{(N_{Ed}/N_{pl}-1)}{\alpha^2 N_{Ed}/N_{pl}}} \right]^2} \quad \text{but } \tau_N \leq 1 \quad \text{where } \psi = 1 + 0.2\alpha \frac{N_{Ed}}{N_{pl}} - \frac{N_{Ed}}{N_{pl}} \quad (5)$$

For members subjected to pure compression, the stiffness reduction method provides the inelastic buckling load rather than the full load-displacement path as the member is assumed to be perfectly straight. The strength of the column may be checked using eq. (6), where the inelastic flexural buckling load amplifier $\alpha_{cr,i}$ of a column must be greater than or equal to 1.0. To determine the ultimate inelastic buckling strength of a column, an iterative procedure should be followed, where the applied axial load is increased until satisfying $\alpha_{cr,i} = N_{cr,i}/N_{Ed} = 1.0$.

$$\alpha_{cr,i} = \frac{N_{cr,i}}{N_{Ed}} = \frac{\tau_N N_{cr}}{N_{Ed}} \geq 1.0 \quad \text{but } \tau_N N_{cr} \leq N_{pl} \quad (6)$$

Similar to above, a stiffness reduction function $\tau_{N,CRC}$ has previously been derived for the Column Research Council (CRC) column strength equation [12] as given by eq. (7) [10, 11, 29]. Kim and Chen [30] proposed to reduce $\tau_{N,CRC}$ by a factor of 0.85 to account for geometrical imperfections. A stiffness reduction function $\tau_{N,LRFD}$ has also been derived based on the LRFD column buckling curve [14], which is shown in eq. (8) and takes into account the effects of both residual stresses and geometrical imperfections [13], though note that the LRFD code [14] applies a reduction factor of 0.877 to N_{cr} . The comparison between the stiffness reduction function developed in this study and those found in the literature is illustrated by Fig. 5.

$$\tau_{N,CRC} = 1.0 \quad \text{if } \frac{N_{Ed}}{N_{pl}} \leq 0.5$$

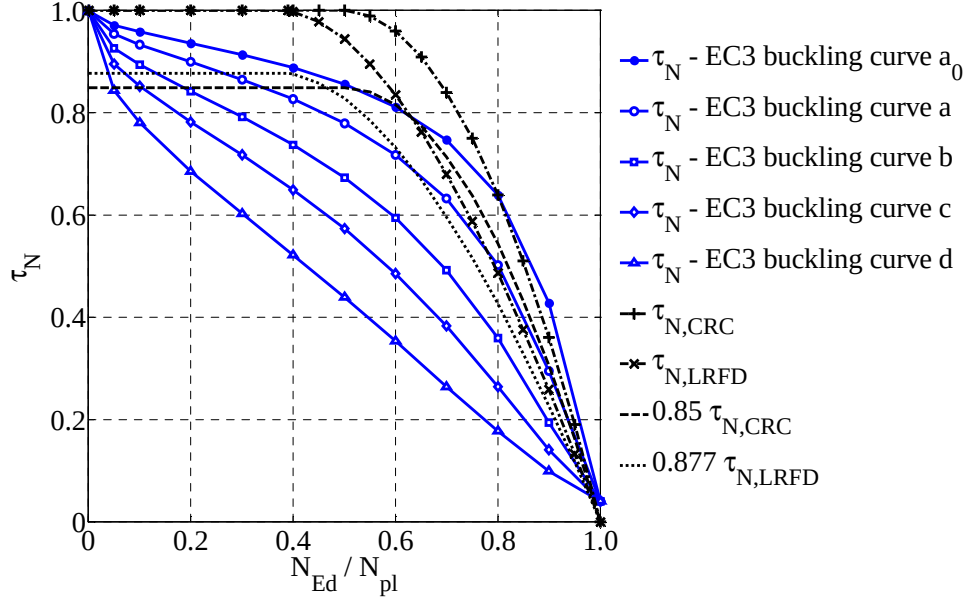


Figure 5: Comparison between the proposed stiffness reduction function due to axial load based on the EN 1993-1-1 (EC3) [1] buckling curves with reduction functions from the literature based on the CRC and AISC LRFD buckling curves

$$\tau_{N,CRC} = 4 \frac{N_{Ed}}{N_{pl}} \left(1 - \frac{N_{Ed}}{N_{pl}} \right) \quad \text{if} \quad \frac{N_{Ed}}{N_{pl}} > 0.5 \quad (7)$$

$$\begin{aligned} \tau_{N,LRFD} &= 1.0 \quad \text{if} \quad \frac{N_{Ed}}{N_{pl}} \leq 0.39 \\ \tau_{N,LRFD} &= -2.724 \frac{N_{Ed}}{N_{pl}} \ln \left(\frac{N_{Ed}}{N_{pl}} \right) \quad \text{if} \quad \frac{N_{Ed}}{N_{pl}} > 0.39 \end{aligned} \quad (8)$$

It should be noted that use of the derived stiffness reduction function τ_N provides an exact match to the European buckling curves, but relies on the suitability of these curves to capture accurately the behaviour of real columns. Choice of buckling curve in EN 1993-1-1 [1] is made through a buckling curve selection table, which has been developed on the basis of extensive experimental, numerical and probabilistic studies [31]. Using multiple buckling curves, τ_N considers different magnitudes and patterns of residual stresses and different buckling directions. Landesmann and Batista [16] also proposed a stiffness reduction function based on the European buckling curves, for use with refined plastic hinge analysis, and suggesting stiffness reduction when N_{Ed}/N_{pl} exceeds a certain value. However, this may lead to the overestimation of the strength for inelastic flexural buckling if geometrical imperfections are not accounted for.

3.2. Application of the derived stiffness reduction function to inelastic flexural buckling

While the proposed stiffness reduction function τ_N yields the same results as those obtained through the Eurocode 3 (EC3) provisions [1] in the case of regular members, it is

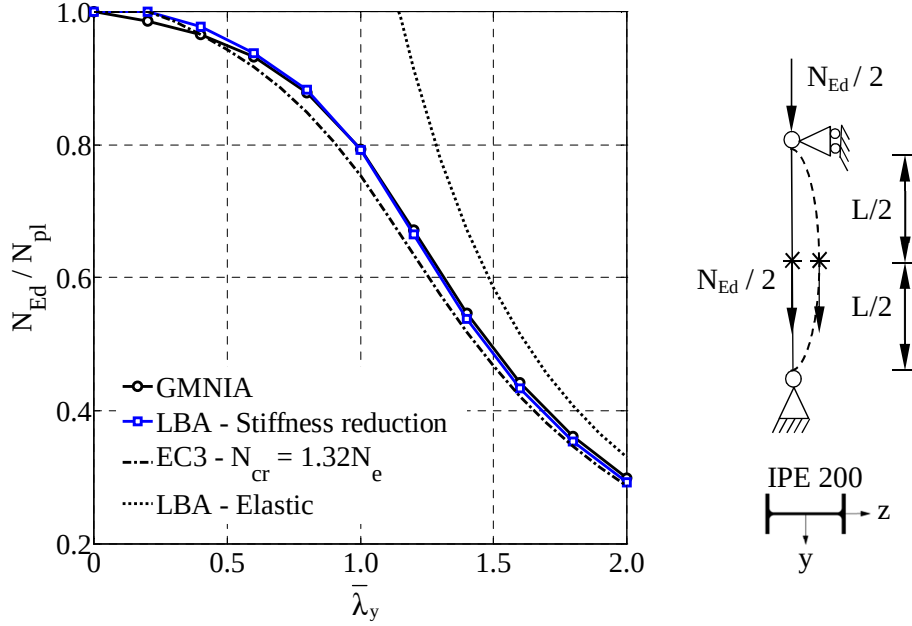


Figure 6: Comparison of the results obtained through LBA with stiffness reduction (LBA-SR) with those determined from GMNIA and EN 1993-1-1 (EC3) [1] for a column subjected to an end and intermediate compressive force

shown in this section that incorporation of the stiffness reduction function τ_N into Linear Buckling Analysis (LBA) leads to more accurate results for the inelastic buckling of columns with irregular geometry, as well as various boundary and loading conditions. When using current design methods, irregular problems can generally only be solved by making conservative assumptions or simplifications to the structural system. Thus, the use of LBA in conjunction with the stiffness reduction function given in eq. (5), which will be henceforth referred to as LBA-SR, may provide both an accurate and practical way to solve this kind of problem. Herein, the proposed stiffness reduction method is applied to three examples of irregular columns including a column with an intermediate compressive load, a column with an intermediate elastic lateral restraint and a column with a non-uniform cross-section. Below, in addition to irregular columns, the influence of the development of plasticity in beams on the strength of a connected column in a steel frame is also assessed.

3.2.1. Column with an intermediate compressive load

The accuracy of the proposed stiffness reduction method is first assessed by analysing a column loaded with two point loads, applied at one end and at the column mid-height - see Fig. 6. The inelastic buckling loads for columns with an IPE 200 profile and different non-dimensional slenderness were obtained through LBA-SR combined with the developed stiffness reduction function, and compared against values determined through GMNIA and by applying the Eurocode 3 [1] flexural buckling equations. When using the proposed approach, since the axial force is not uniform within the column, different stiffness reduction factors given by eq. (5) were applied to the two parts of the column with uniform axial force.

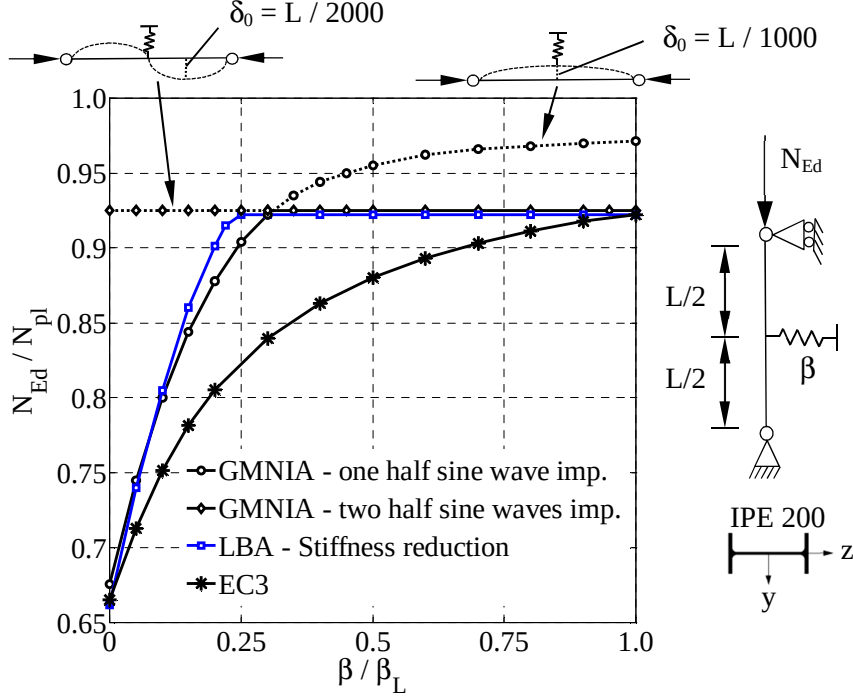


Figure 7: Comparison of the results obtained through LBA with stiffness reduction (LBA-SR) with those from GMNIA and EN 1993-1-1 (EC3) for a column with an intermediate elastic restraint ($\bar{\lambda}_y = 1.0$)

In a similar fashion, the inelastic buckling load calculated according to Eurocode 3 utilised the elastic buckling load N_{cr} for a column with an intermediate compressive load, which is $N_{cr} = 1.32N_e$ [32] where $N_e = \pi^2 EI / L^2$. The results of this comparison are shown in Fig. 6 for different non-dimensional slenderness values, determined by neglecting the intermediate load. Both calculation methods provide good results, but the results obtained through LBA-SR may be seen to be closer to the GMNIA curve than the Eurocode 3 (buckling curve a) predictions. This confirms the accuracy of the proposed approach in predicting inelastic buckling loads for columns loaded by intermediate compressive loads.

3.2.2. Column with an intermediate elastic lateral restraint

When a column is not restrained by a fully rigid support, the stiffness of the support should be included into the analytical or numerical description of the column. Herein, a column with an IPE 200 cross-section and non-dimensional member slenderness $\bar{\lambda}_y = 1.0$, restrained by an elastic lateral support at the mid-height, is investigated. Note that the column is fully restrained in the out-of-plane direction. The non-dimensional slenderness for the column was calculated neglecting the elastic restraint. The inelastic buckling load of the restrained member was determined through GMNIA and the proposed approach considering different values for the stiffness β of the restraint. In the GMNIA, two imperfection shapes were considered - a single half-sine wave and two half-sine waves over the member length. For the Eurocode 3 [1] calculations, the elastic buckling load used to determine the slenderness

of the restrained member was determined through LBA. The results of the analyses are shown in Fig. 7, where it can be observed that the full half-sine wave imperfection is critical up to a normalised restrained stiffness $\beta/\beta_L = 0.3$ where $\beta_L = 16\pi^2 EI/L^3$ is the threshold stiffness that leads to elastic buckling of the column in the second mode. For larger β/β_L values, the two half-sine wave imperfection yields the lower column capacity. Comparison of the results obtained through LBA-SR and GMNIA indicates that the proposed approach provides accurate predictions of inelastic buckling loads, with a maximum discrepancy of 2.6 %. Conversely, use of the Eurocode 3 [1] flexural buckling equations with elastic buckling loads determined through LBA leads to overly conservative predictions, with a maximum underestimation of the ultimate load determined through GMNIA of 9.2 %

It is noteworthy that the magnitude of the elastic restraint stiffness required to force inelastic buckling in the second mode $\beta_{L,inelastic}$ is smaller than that required for elastic buckling β_L , as seen in Fig. 7. This is due to the fact that for a column that experiences stiffness reduction due to the development of plasticity, the effectiveness of the support provided by the intermediate elastic restraint increases, since the ratio of the restraint stiffness to the member stiffness becomes larger. This behaviour is captured accurately by LBA-SR as shown in Fig. 7. Similar behaviour may also be seen in columns in steel frames that undergo stiffness reduction, while connected members remain elastic; in such instances, the end restraint afforded to the column effectively increases. This behaviour was observed by Yura [33] and Disque [34], who proposed methods for the determination of the inelastic effective lengths considering the stiffness reduction in the column. Trahair and Hancock [7] also used a similar approach to that presented in this paper to analyse columns with end restraints, where equivalent observations were made.

3.2.3. Column with a varying cross-section

The use of the stiffness reduction function τ_N to determine the inelastic buckling loads of columns with a non-uniform cross-section (i.e. stepped columns) is addressed in this section. In general, the strength of a column can be increased if additional plates are used in its middle portion where the deflections and section forces usually assume their maximum values. Herein, a column with an IPE 200 cross-section restrained in the out-of-plane direction and strengthened through the addition of plates over a length equal to 50% of the total length around the mid-height, is investigated. The ratio between the second moment of area of the original cross-section I_{z1} and that of the strengthened section I_{z2} is equal to 0.40 and the ratio between the areas A_1/A_2 is equal to 0.53. Note that in the presentation of the results, the non-dimensional slenderness $\bar{\lambda}_z$ is shown with respect to the original area of the cross-section A_1 and the elastic buckling load of the unstrengthened column. Fig. 8 shows the results obtained through LBA-SR, GMNIA, LBA and the Eurocode 3 flexural buckling equations, where $N_{pl,1} = A_1 f_y$. In the first case, two different stiffness reduction factors were used for the portions of the column with the original and increased cross-sections. Good correlation between LBA-SR and GMNIA can be observed, where the proposed approach slightly overestimates column capacity by up to 2% for small slenderness values, but is very accurate over the remainder of the slenderness range. On the other hand, the use of the Eurocode 3 [1] formulae, with the elastic buckling load of the strengthened column taken as

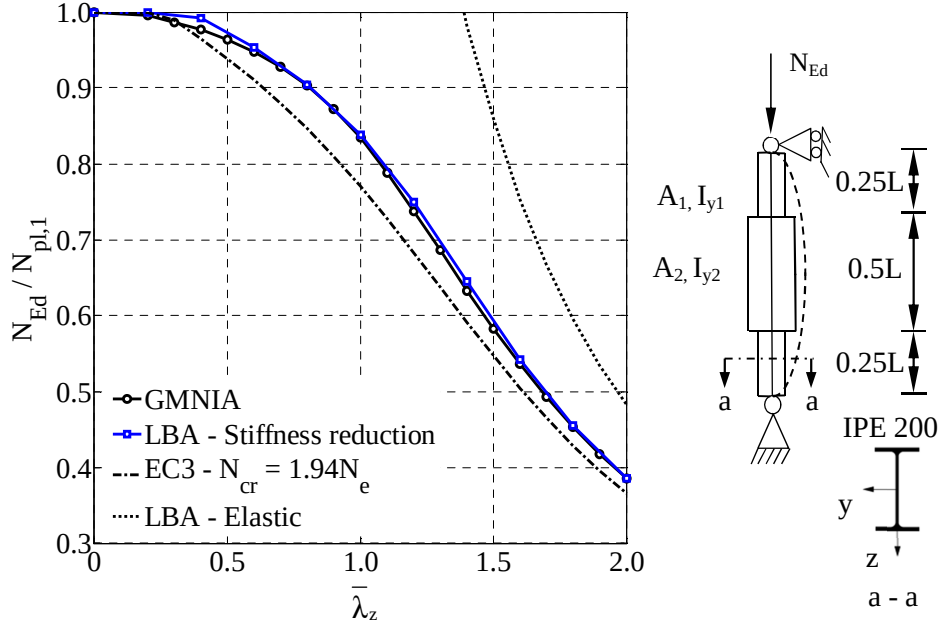


Figure 8: Comparison of the results obtained through LBA with stiffness reduction (LBA-SR) with those of GMNIA and EN 1993-1-1 (EC3) for a stepped column

$N_{cr} = 1.94N_e$ [32], where N_e is the elastic buckling load of the original column, leads to less accurate predictions.

3.2.4. Influence of beam plasticity on the strength of a column

To show the influence of the development of plasticity in the surrounding members of a column on its strength, a simple braced frame, which is restrained in the out-of-plane direction, is analysed in this section. As shown in Fig. 9, the frame is subjected to axial compression only. The magnitude of the axial load in the column and beam is varied to determine the influence of the relative stiffness reduction on the column strength. Non-proportional loading was applied in GMNIA, where the axial load in the beam N_{beam} was applied before the axial load in the column N_{column} . The imperfect shape adopted in the GMNIA is shown in Fig. 9. As can be seen from the figure, the strength of the column considerably decreases when the axial load in the beam becomes significant. While LBA-SR could capture this strength reduction, the Eurocode 3 [1] flexural buckling equations with elastic buckling loads obtained through LBA could not. This is due to the use of the elastic buckling loads without accounting for the development of plasticity in the beam.

It is worth noting that when the column is subjected to a large axial load and the beam subjected to a small axial load, the resistance of the column determined through GMNIA and LBA-SR is larger than that calculated through Eurocode 3, with the use of elastic buckling loads. This relates to similar behaviour to that discussed in the section 3.2.2, where the influence of the elastic restraint, here afforded by the connected beam, effectively increases when the column undergoes stiffness reduction.

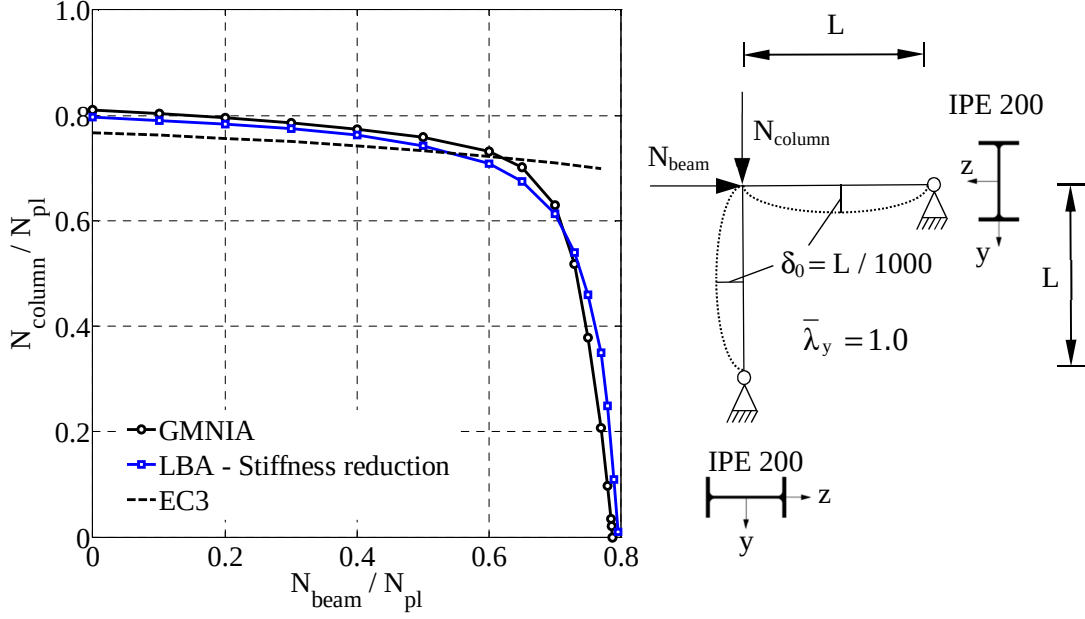


Figure 9: Comparison of the results obtained through LBA with stiffness reduction (LBA-SR) with those of GMNIA and EN 1993-1-1 (EC3) for a braced frame

4. Stiffness reduction under bending

This section addresses the effects of the development of plasticity in steel members under bending. For a member sufficiently restrained against out-of-plane instability effects, stiffness reduction under constant bending can be investigated considering the moment-curvature ($M - \varphi$) relationship. This depends on the corresponding cross-section geometry, material model and distribution and magnitude of residual stresses, but not on the geometrical imperfections. Applying bending moment incrementally, the change in flexural stiffness can be determined, with the tangent stiffness EI_r defined as $EI_r = dM_{Ed}/d\varphi$. The ratio between the flexural stiffness at a particular bending moment value and the initial flexural stiffness then provides the stiffness reduction factor due to bending $\tau_M = EI_r/EI$. Assuming the ECCS residual stress models given in Fig. 2, this leads to the stiffness reduction patterns shown in Fig. 10. The reduction in flexural stiffness of the cross-section can be considered in three stages: elastic, primary plastic and secondary plastic [35]. In the elastic stage, since yielding has yet to develop, the stiffness reduction factor is equal to 1.0. When the applied bending reaches the value corresponding to the limit ϕ (Fig. 10), yielding initiates and the stiffness of the cross-section decreases in the primary plastic stage. Note that due to the presence of residual stresses, yielding develops prior to the attainment of the traditional elastic moment capacity M_{el} . The yielding patterns of the cross-section in the primary plastic stage are shown in Fig. 10 for major and minor axis bending. The black regions represent the areas where plasticity develops, indicating yielding initially occurs in both flanges for the case of major axis bending, while it is limited to the regions around the flange tips which contain compressive residual stresses for the case of minor axis bending.

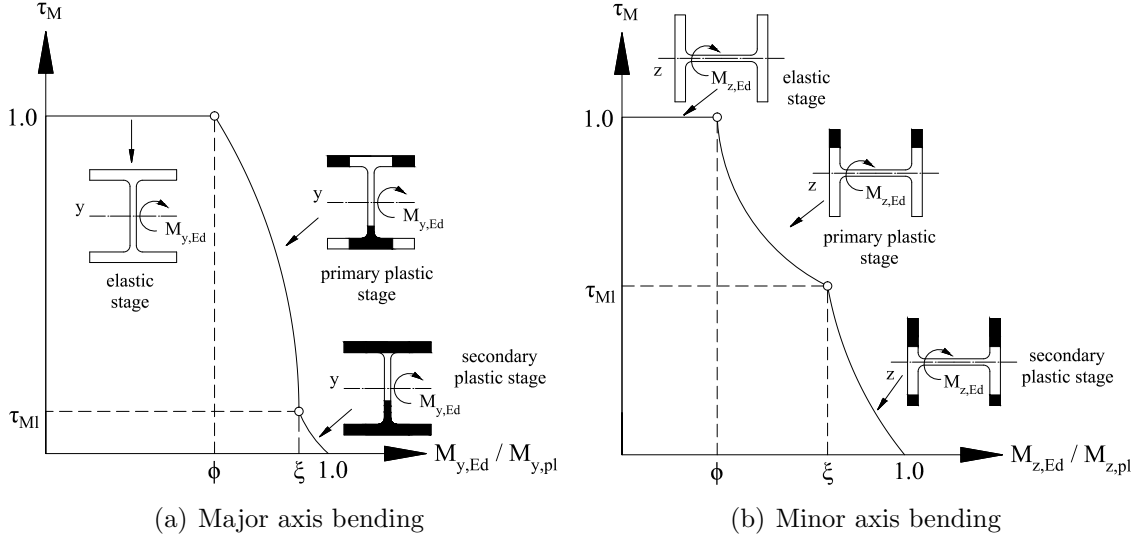


Figure 10: Patterns of stiffness reduction due to pure bending τ_M in I-sections

The secondary plastic stage starts when the applied bending reaches the limit value corresponding to ξ (Fig.10). For the case of major axis bending moment, this limit is defined as when both flanges are fully yielded, while for minor axis bending, the limit is defined as when the flange tips on the tension side of the cross-section reach yield. Since the flanges provide the largest contribution to the flexural stiffness of the cross-section, the secondary plastic stage develops quite late in the case of major axis bending compared to minor axis bending. Thus, the reduced stiffness at the onset of the secondary plastic stage τ_{Ml} is considerably smaller for the case of major axis bending.

Since the same general stiffness reduction patterns shown in Fig. 10 are exhibited by all I-shaped cross-sections, expressions for the stiffness reduction due to bending can be calibrated on the basis of moment-curvature relationships generated through finite element analysis. This approach was followed by Zubyan [18, 19] to develop an expression for the reduction of stiffness in I-sections under pure bending, assuming the ECCS residual stress distributions [24] shown in Fig. 2. This function is utilised in the present study and is given in eq. (9). The model parameters τ_{Ml} , ξ , ϕ , β , δ are given in Table 2 for major and minor axis bending of I-sections. Two sets of model parameters are provided, for $h/b \leq 1.2$ and $h/b > 1.2$, which correspond to the different residual stress patterns shown in Fig. 2.

$$\begin{aligned}
 \tau_M &= 1.0 \quad \text{if} \quad M_{Ed}/M_{pl} \leq \phi \\
 \tau_M &= (1 - \tau_{Ml}) \left[1 - \left(\frac{M_{Ed}/M_{pl} - \phi}{\xi - \phi} \right)^\beta \right]^{1/\beta} + \tau_{Ml} \quad \text{if} \quad \phi \leq M_{Ed}/M_{pl} \leq \xi \\
 \tau_M &= \tau_{Ml} \left[1 - \left(\frac{M_{Ed}/M_{pl} - \xi}{1 - \xi} \right) \right]^{1/\delta} \quad \text{if} \quad \xi \leq M_{Ed}/M_{pl} \quad (9)
 \end{aligned}$$

The ability of the stiffness reduction function of eq. (9) to capture accurately the effects

Table 2: Parameters for the stiffness reduction function for I-sections in bending [18, 19]

	Major axis bending					Minor axis bending				
	τ_{Ml}	ϕ	ξ	β	δ	τ_{Ml}	ϕ	ξ	β	δ
$h/b \leq 1.2$	0.04	$0.5 \frac{W_{el,y}}{W_{pl,y}}$	0.98	1.5	1.0	0.5	$0.5 \frac{W_{el,z}}{W_{pl,z}}$	0.8	0.85	0.6
$h/b > 1.2$	0.08	$0.7 \frac{W_{el,y}}{W_{pl,y}}$	0.95	1.5	1.0	0.6	$0.7 \frac{W_{el,z}}{W_{pl,z}}$	0.75	0.85	0.6

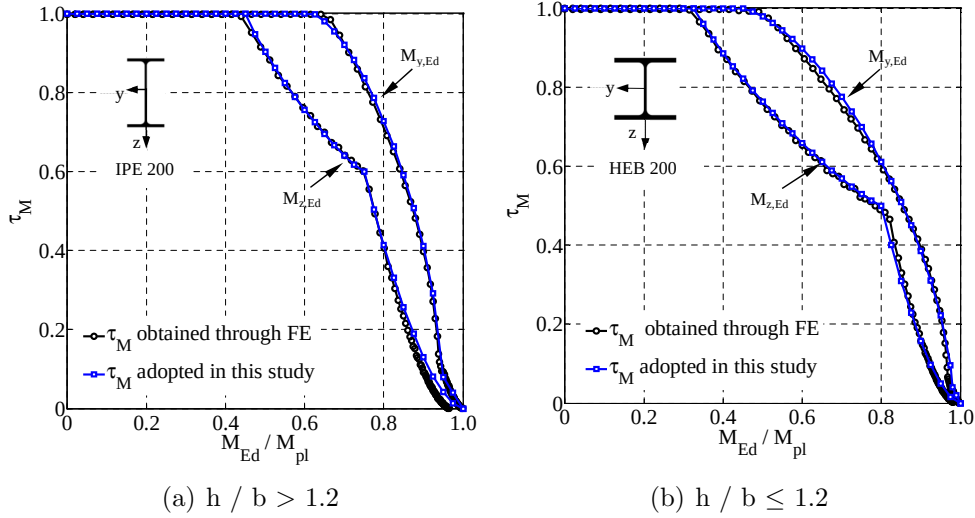


Figure 11: Comparison of the adopted stiffness reduction functions due to bending [18, 19] with those obtained through finite element analysis

of the spread of plasticity through the cross-section depth was verified using the results of finite element analysis. A wide range of European IPE and HE sections were analysed in this study, employing 313 section integration points for each cross-section (105 integration points in both flanges and web). As can be seen from Fig. 11, the adopted stiffness reduction function is accurate, providing results in close agreement to the finite element predictions for both major and minor axis bending and for narrow and wide flange sections.

5. Stiffness reduction under combined axial load and bending

This section addresses the in-plane design of beam-columns through the stiffness reduction method. After introducing ultimate cross-section strength interaction equations, a stiffness reduction function for combined bending and compression is developed. The accuracy of the proposed design approach using the developed stiffness reduction functions in conjunction with elastic, but Geometrically Nonlinear Analysis (GNA-SR) is investigated for different cross-section shapes and axes of bending. Finally, the influence of the moment gradient on the beam-column strength and the modification of the stiffness reduction function to account for this influence is examined.

5.1. Interaction equations

Before focusing on stiffness reduction due to combined bending and axial load, it is necessary to first consider ultimate cross-section strength interaction equations. Numerous interaction equations to define the ultimate capacity of different cross-section shapes under combined loading can be found in the literature. In this study, the continuous interaction equations proposed by Duan and Chen [36] were chosen because of their accuracy and simple form, which is important to achieve practicality in the proposed method. These expressions are given by eq. (10) and eq. (11), for axial load plus major axis bending and axial load plus minor axis bending respectively, A_w and A_f are the areas of the web and flange respectively.

$$1/\alpha_{ult,c} = \left(\frac{N_{Ed}}{N_{pl}} \right)^{1.3} + \frac{M_{y,Ed}}{M_{pl,y}} = 1 \quad (10)$$

$$1/\alpha_{ult,c} = \left(\frac{N_{Ed}}{N_{pl}} \right)^{\gamma} + \frac{M_{z,Ed}}{M_{pl,z}} = 1, \quad \gamma = 2 + 1.2 \left(\frac{A_w}{A_f} \right) \quad (11)$$

5.2. Derivation of stiffness reduction function for combined bending and axial load

To derive the stiffness reduction function considering the influence of combined bending and axial load τ_{MN} , the stiffness reduction functions for pure bending τ_M and pure axial load τ_N described in the previous sections were utilised. The proposed expression for the stiffness reduction function considering both bending and axial load is given by eq. (12).

$$\tau_{MN} = \tau_M \tau_N \left\{ 1 - \left(\frac{N_{Ed}}{N_{pl}} \right)^{\eta} \left(\frac{M_{Ed}}{M_{pl}} \right)^{\rho} \right\} \quad (12)$$

In this form of equation, τ_{MN} has two anchor points: when the bending moment is zero $\tau_{MN} = \tau_M$ and when the axial force is zero $\tau_{MN} = \tau_N$. These two anchor points are used in conjunction with the applied force and moment values to obtain τ_{MN} . The factors η and ρ in eq. (12) are given in Table 3 for major and minor axis bending. These were derived by calibration using the interaction equations provided in eq. (10) and eq. (11) and the results obtained through GMNIA of simply supported beam-columns with different European cross-sections subjected to axial load plus uniform bending.

5.3. Application of stiffness reduction function to beam-columns under axial load and uniform bending

In this section, the accuracy of the proposed stiffness reduction function τ_{MN} is assessed for the in-plane design of beam-columns, restrained in the out-of-plane direction. Geometrically Nonlinear Analysis with stiffness reduction (GNA-SR) was performed on a series of simply-supported beam-columns under axial load and uniform bending moment. This is an elastic but Geometrically Nonlinear Analysis, where geometrical imperfections are not explicitly modelled, but the elastic modulus of the material is multiplied by the stiffness

Table 3: Proposed values for η and ρ parameters for stiffness reduction function due to combined axial force and bending

	Major axis bending		Minor axis bending	
	η	ρ	η	ρ
$h/b \leq 1.2$	0.5	0.9	0.5	0.5
$h/b > 1.2$	0.8	1.0	0.5	0.55

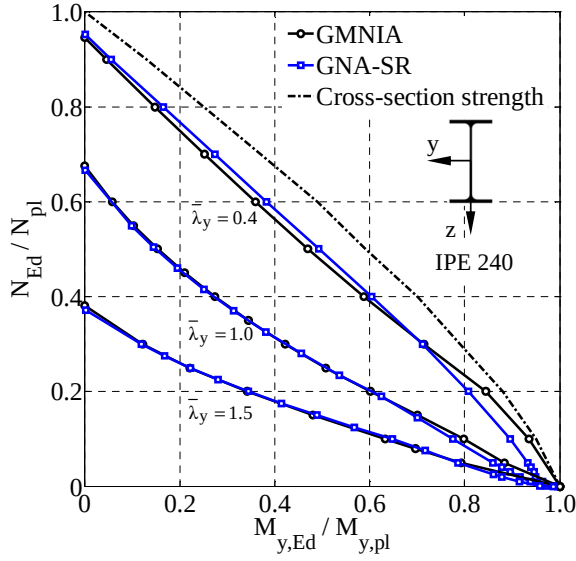
Table 4: Range of European cross-section dimensions considered to assess the accuracy of the proposed stiffness reduction method

	$h/b \leq 1.2$ - 10 sections		$h/b > 1.2$ - 25 sections	
	Max	Min	Max	Min
h/b	1.2	0.91	3.34	1.22
b/t_f	25	5.3	22.22	7.73
h/t_w	37.7	10	60.63	17.95

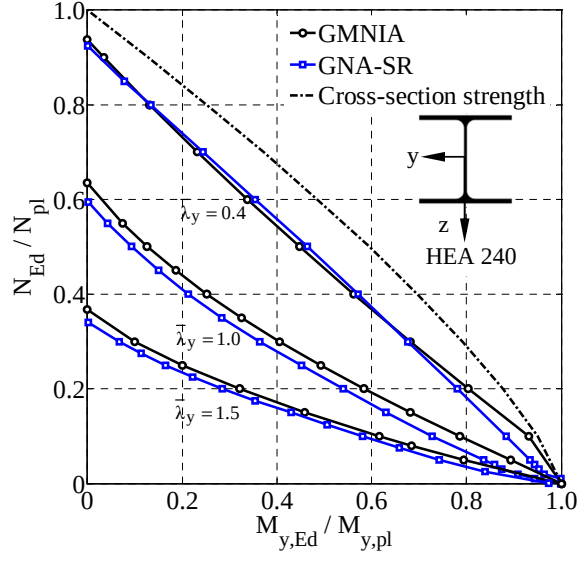
reduction factor τ_{MN} provided in eq. (12) to account for the spread of plasticity, residual stresses and geometrical imperfections. The maximum load carrying capacity of the member is achieved when the sections forces at the most heavily loaded cross-section are equal to the corresponding ultimate cross-section resistance given by eq. (10) and eq. (11).

Comparisons between the results obtained through GNA-SR and GMNIA for major axis bending are shown in Fig. 12. It may be seen that GNA-SR provides accurate strength predictions for beam-columns with IPE 240 (Fig. 12a) and HEA 240 cross-sections (Fig. 12b), when different ratios of bending to axial force, as well as different slenderness values, are considered. Fig. 12a shows just a slight overestimation of the IPE 240 beam-column strength for a non-dimensional slenderness $\bar{\lambda}_y = 0.4$ and low bending moment values, while Fig. 12b displays slightly over-conservative predictions of the HEA 240 beam-column strength for slenderness values $\bar{\lambda}_y = 1.0$ and $\bar{\lambda}_y = 1.5$. This is due principally to the underestimation of the strong axis flexural buckling strength of columns with a wide flange cross-section (e.g. HEA 240) given by the Eurocode 3 buckling curves. The slight underestimation of strength for high bending moment values relates largely to the assumed cross-section strength interaction equation.

The results obtained through GNA-SR and GMNIA for simply supported beam-columns subjected to uniform minor axis bending and compression are shown in Fig. 13. The GNA-SR predictions, which were calculated using the cross-section strength given in eq. (11), may be seen to be in very close agreement to those those obtained through GMNIA. Spread of plasticity, leading to stiffness reduction, and instability effects transform the convex cross-section interaction surface of a beam-column to a concave shape for intermediate-to-high slenderness values. GNA with the proposed stiffness reduction function captures this trans-

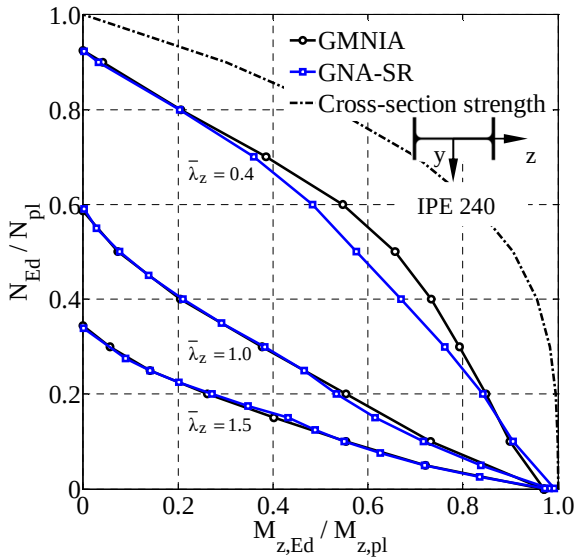


(a) IPE 240

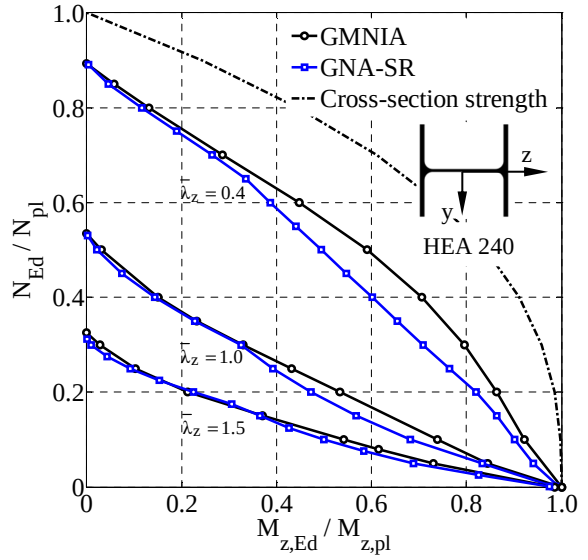


(b) HEA 240

Figure 12: Comparison of the results obtained through Geometrically Nonlinear Analysis with stiffness reduction (GNA-SR) with those obtained through GMNIA for simply-supported beam-columns subjected uniform major axis bending and axial compression



(a) IPE 240



(b) HEA 240

Figure 13: Comparison of the results obtained through Geometrically Nonlinear Analysis with stiffness reduction (GNA-SR) with those obtained through GMNIA for simply-supported beam-columns subjected uniform minor axis bending and axial compression

Table 5: Number of beam-column tests performed to assess the accuracy of the proposed stiffness reduction method

	Major axis bending		Minor axis bending	
	$\bar{\lambda}_y$	N	$\bar{\lambda}_z$	N
$h/b \leq 1.2$	0.4	90	0.4	80
	1.0	50	1.0	50
	1.5	30	1.5	30
$h/b > 1.2$	0.4	216	0.4	216
	1.0	144	1.0	120
	1.5	72	1.5	72

formation accurately, as shown in Fig. 13.

The proposed stiffness reduction method was also evaluated for a series of 35 European IPE and HE cross-sections. The range of cross-section properties of the all analysed members are reported in Table 4 where b and h are the cross-section width and depth, respectively, and t_w and t_f are the web and flange thickness respectively. The results obtained through GNA-SR were compared with those of GMNIA for 1350 simply-supported beam-columns subjected to axial compression plus uniform bending as shown in Table 5, where N is the number of beam-columns analysed for a particular group. Three non-dimensional member slenderness values, $\bar{\lambda} = 0.4, 1.0$ and 1.5 , were considered to cover the response of beam-columns with low, intermediate and high slenderness. A summary of the comparison of the results of GNA-SR (the proposed stiffness reduction method) with those of GMNIA is provided in Table 6. The accuracy of the method was assessed using eq. (13) where ϵ is the ratio between R_{GNA-SR} and R_{GMNIA} where R_{GNA-SR} and R_{GMNIA} are the radial distances measured from the origin to the interaction curves determined through GNA-SR and GMNIA, respectively. Note that these distances were normalised through N_{Ed}/N_{pl} and M_{Ed}/M_{pl} . Values of ϵ larger than 1.0 indicate unconservative strength predictions. ϵ_{av} , ϵ_{cov} represent the average and coefficient of variation (COV) of ϵ respectively, while ϵ_{max} , ϵ_{min} are the maximum and minimum of ϵ values in the range considered.

$$\epsilon = \frac{R_{GNA-SR}}{R_{GMNIA}} \quad (13)$$

As seen in Table 6, the proposed method is very accurate for cross-sections with aspect ratios h/b larger than 1.2, with ϵ_{av} values close to 1.0 and with minimal scatter. Though the method is less accurate for cross-sections with $h/b \leq 1.2$ particularly for minor axis bending, the accuracy remains good. Moreover, in all of the considered wide range cases, the unconservative errors were less than 4% (i.e. $\epsilon \leq 1.04$), which shows the reliability of the proposed stiffness reduction method.

For the purpose of comparing the stiffness reduction method with the methods given in Eurocode 3 [1] for the design of beam-columns, the results obtained through Annexes A

Table 6: Summary of comparison of the results obtained through GNA-SR (the proposed stiffness reduction method) with those of GMNIA and Eurocode 3 (EC3) Annexes A and B for simply supported beam-columns subjected to axial compression and uniform bending moment

		Major axis bending					Minor axis bending				
		$\bar{\lambda}_y$	ϵ_{av}	ϵ_{cov}	ϵ_{max}	ϵ_{min}	$\bar{\lambda}_z$	ϵ_{av}	ϵ_{cov}	ϵ_{max}	ϵ_{min}
Proposed stiffness reduction method	$h/b \leq 1.2$	0.4	0.99	0.021	1.02	0.95	0.4	0.95	0.032	1.00	0.88
		1.0	0.95	0.009	0.97	0.93	1.0	0.97	0.026	1.01	0.93
		1.5	0.96	0.009	0.98	0.94	1.5	0.97	0.031	1.02	0.95
	$h/b > 1.2$	0.4	1.00	0.025	1.04	0.94	0.4	0.97	0.030	1.03	0.92
		1.0	0.99	0.010	1.02	0.97	1.0	0.99	0.012	1.02	0.97
		1.5	1.00	0.008	1.02	0.99	1.5	1.00	0.014	1.02	0.97
EC 3 Annex A	$h/b \leq 1.2$	0.4	1.00	0.025	1.07	0.95	0.4	0.99	0.023	1.03	0.93
		1.0	0.96	0.009	0.98	0.94	1.0	1.01	0.015	1.03	0.98
		1.5	0.96	0.014	0.98	0.91	1.5	0.99	0.005	1.00	0.98
	$h/b > 1.2$	0.4	0.98	0.024	1.03	0.93	0.4	0.97	0.043	1.07	0.87
		1.0	0.97	0.009	0.99	0.94	1.0	1.00	0.021	1.05	0.97
		1.5	0.97	0.012	0.99	0.94	1.5	0.98	0.020	1.02	0.95
EC 3 Annex B	$h/b \leq 1.2$	0.4	0.97	0.016	1.00	0.93	0.4	0.89	0.053	0.97	0.81
		1.0	0.95	0.011	0.99	0.93	1.0	0.93	0.051	1.01	0.87
		1.5	0.96	0.007	0.98	0.95	1.5	0.96	0.024	0.99	0.92
	$h/b > 1.2$	0.4	0.96	0.028	1.01	0.90	0.4	0.88	0.076	1.00	0.76
		1.0	0.98	0.019	1.02	0.94	1.0	0.94	0.043	1.01	0.89
		1.5	0.98	0.013	1.02	0.97	1.5	0.97	0.025	1.02	0.94

and B of EN 1993-1-1 [1] were compared against the results obtained through GMNIA in Table 6 where ϵ is the ratio between the radial distances measured from the origin to the interaction curves determined through Annexes A or B and GMNIA respectively. It is seen from the table that the stiffness reduction method is more accurate in all cases in comparison to the Annex B of Eurocode 3 [1]. Moreover, the method is more accurate than the Annex A in the majority of the considered cases.

Finally, it should be emphasised that in the application of the stiffness reduction method, the stiffness of the beam-column is reduced from the onset of the analysis on the basis of the applied bending and compression prior to any deformation (i.e. without accounting for second-order effects). Then to determine the load-carrying capacity of a beam-column, the applied bending and axial load should be increased until the ultimate capacity at the most

heavily loaded section is reached (i.e. $\alpha_{ult,c} = 1.0$).

5.4. Moment gradient effect

Variation of bending moment along the member length may lead to a change in the required stiffness reduction, since the development of plasticity will be different from that in beam-columns subjected to constant bending. Thus, in general, the influence of moment gradient has to be accounted for to achieve accurate results for beam-columns subjected to a non-uniform bending moment. One approach to consider this is to apply different stiffness reduction values τ_{MN} to different parts of the member, considering the average bending moment for each part. An alternative way is to incorporate moment gradient factors into the calculation of the stiffness reduction factors. The latter option is chosen in this study as it would generally be easier to implement in practice. In this respect, the formula proposed by Austin [37], which has been extensively used in previous studies to account for the influence of bending moment gradient in the design of beam-columns, is considered. Austin's expression is given in eq. (14), where μ is the ratio between the smaller and larger bending moments at the member ends, with a linear moment gradient between those points.

$$C_m = 0.6 + 0.4\mu \quad \text{but} \quad C_m \geq 0.4 \quad (14)$$

Austin's moment gradient factor formula can only be applied to members with linear moment gradients. As a more general moment gradient formula, the equation put forward by Kirby and Nethercot [38], appearing slightly modified in [2], can be used for members subjected to both linear and nonlinear moment gradients. Kirby and Nethercot's formula is given in eq. (15) where M_{Ed} is the absolute value of maximum moment along the member length, and M_A , M_B and M_C are the absolute values of moment at the quarter point, centreline and three-quarter point of the member length respectively.

$$C_m = \frac{2.5M_{Ed} + 3M_A + 4M_B + 3M_C}{12.5M_{Ed}} \quad (15)$$

It is proposed that either eq. (14) or eq. (15) may be adopted in the stiffness reduction method developed herein to account for bending moment gradient effects. Specifically, in the calculation of τ_M and τ_{MN} , the bending moment values need to be factored by C_m as shown in eq. (16) and eq. (17), where M_{Ed} is the largest bending moment value along the member length.

$$\begin{aligned} \tau_M &= (1 - \tau_{Ml}) \left[1 - \left(\frac{C_m M_{Ed}/M_{pl} - \phi}{\xi - \phi} \right)^\beta \right]^{1/\beta} + \tau_{Ml} \quad \text{if} \quad \phi \leq C_m M_{Ed}/M_{pl} \leq \xi \\ \tau_M &= \tau_{Ml} \left[1 - \left(\frac{C_m M_{Ed}/M_{pl} - \xi}{1 - \xi} \right)^\delta \right]^{1/\delta} \quad \text{if} \quad \xi \leq C_m M_{Ed}/M_{pl} \end{aligned} \quad (16)$$

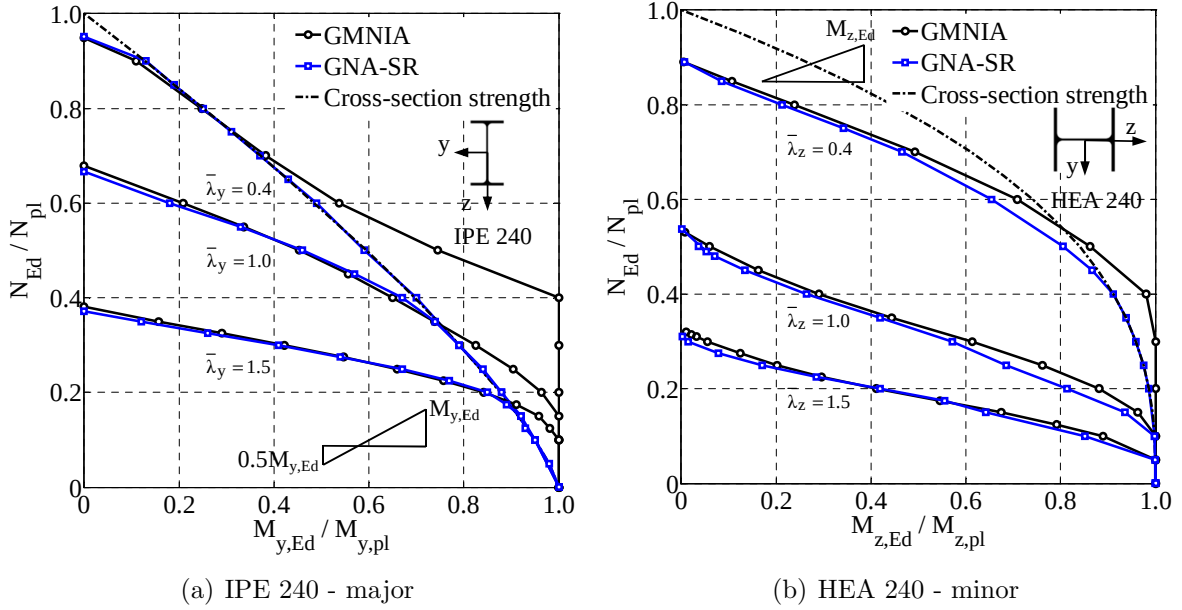


Figure 14: Comparison of the results obtained through Geometrically Nonlinear Analysis with stiffness reduction (GNA-SR) with those obtained through GMNIA for simply-supported beam-columns subjected to axial compression and varying bending along the length

$$\tau_{MN} = \tau_M \tau_N \left\{ 1 - \left(\frac{N_{Ed}}{N_{pl}} \right)^\eta \left(\frac{C_m M_{Ed}}{M_{pl}} \right)^\rho \right\} \quad (17)$$

Following the procedure described, the strength of a series of beam-columns subjected to compression and varying bending moment along the member length were calculated through the GNA with the stiffness reduction function τ_{MN} taking into account the moment gradient through eq. (14). A comparison of the results obtained through GNA-SR and GMNIA is shown in Fig. 14, where it is seen that accurate results can be achieved. Note that the results from GMNIA for stocky members with low axial loads exceed the cross-section strength due to the occurrence of strain hardening and that the maximum moments have been limited to M_{pl} in Fig. 14. Although the Austin formula is used in the present, the Kirby and Nethercot formula was incorporated into the stiffness reduction functions in Kucukler et al. [39] where a more comprehensive verification study of the procedure proposed in this section was performed considering a wide range of bending moment diagrams. Similarly accurate results to those presented herein were achieved, confirming that the proposed procedure is general and can be applied to different shapes of bending moment diagrams, provided suitable C_m factors are obtained.

6. Conclusions

This study focused on the prediction of the steel column and beam-column capacity through a stiffness reduction method. Stiffness reduction functions were developed for pure

compression, pure bending and combined bending and compression. In the case of pure compression, the Eurocode 3 [1] column buckling curves were utilised. Stiffness reduction functions for pure bending were determined considering the moment-curvature response for a wide-range of European cross-section shapes, while stiffness reduction expressions for combined bending and compression were determined by calibration against GMNIA results. For the design of columns, the use of linear buckling analysis in conjunction with the derived stiffness reduction function due to axial loading (LBA-SR) was proposed. Since LBA-SR provides the same results as the flexural buckling equations of Eurocode 3 [1] for regular problems, the accuracy of the method was assessed for some irregular problems, such as columns with elastic lateral restraints and stepped columns, showing more accurate results than traditional methods. For the design of beam-columns, the use of Geometrically Nonlinear Analysis with stiffness reduction (GNA-SR), where the geometrical imperfections are not explicitly modelled, was proposed. In this case, only the section forces at the most heavily loaded section are checked against the ultimate cross-section resistance. The proposed stiffness reduction method was verified against GMNIA results of 1350 beam-columns subjected to uniform bending and compression, covering a wide range of cross-section properties. Stiffness reduction functions accounting for the influence of bending moment gradient were also investigated. Accurate results were achieved utilising simple moment gradient factors.

A key aspect of the proposed method is that it can be applied using conventional structural analysis software packages to perform LBA-SR and GNA-SR, thus offering a very practical means of determining the capacity of columns and beam-columns. In comparison to previous approaches, the stiffness reduction scheme proposed in this study enables the design of members without the use of column strength equations, but only cross-section checks, which may bring about significant improvements in terms of both accuracy and practicality, particularly when designing steel frames. Future research will be directed towards the consideration of out-of-plane effects and application of the method to the design of steel frames.

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