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Propositional Fuzzy Logics:
Tableaux and Strong Completeness

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Abstract

In his famous book *Mathematical Fuzzy Logic*, Petr Hájek defined a new fuzzy logic, which he called BL. It is weaker than the three fundamental fuzzy logics Product, Łukasiewicz and Gödel, which are in turn weaker than classical logic, but axiomatic systems for each of them can be obtained by adding axioms to BL. Thus, Hájek placed all these logics in a unifying axiomatic framework.

In this dissertation, two problems concerning BL and other fuzzy logics have been considered and solved. One was to construct tableaux for BL and for BL with additional connectives. Tableaux are automatic systems to verify whether a given formula must have given truth values, or to build a model in which it does not have these specific truth values. The other problem that was solved is to construct strongly standard complete axiomatic systems for BL, Łukasiewicz and Product logics, which was done by extending Hájek's axiomatic systems for them by an infinitary rule.

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1. INTRODUCTION

Most expressions of natural language involve vagueness to some degree. Vagueness has been investigated by philosophy since antiquity. There are classical problems involving vague terms that have been causing long-lasting debates, such as the sorites paradox, also known as the paradox of the heap. The story is that from a heap grains are removed individually, that is one by one; the question is at which stage of removal, the heap ceases to be a heap. This paradox is attributed to Eubulides of Miletus, an ancient philosopher and a student of Euclid, contemporary to Aristotle (see [Kee00], [Luk11]).

A vague term F such as 'tall', 'red' and 'tadpole' has three characteristics: (1) it admits borderline cases, for which we cannot definitely say whether or not a given object can have the attribute F (something may be neither red nor not red), (2) it lacks sharp boundaries between F and non- F objects (there is no well defined extension of the concept of red in the real world), (3) it is susceptible to sorites paradoxes. Because of property (1), these terms pose a challenge to classical logic and semantics as propositions (formulas) containing vague terms cannot assume one of the two truth values, i.e. truth or falsity, and thus violate the classical principle of bivalence. See [Kee00], [KS96].

Russell in [Rus23], similarly to Frege (1923, cited in [Kee00] p. 9) argues that vagueness is a property of a language, not of objects. Both treat the phenomenon as a defect of natural languages and Russell adds that “whatever vagueness is to be found in my words must be attributed to our ancestors for not having been predominantly interested in logic” (p. 61 in [KS96] reprinted from [Rus23]). Moreover, Russell claims that even scientific terms such as metre or second are vague as they are the result of empirical observations and these cannot be fully precise (that is requiring one fact to verify them). However, his definition of vagueness is more of a definition of polysemy or ambiguity.

In contrast, linguistics and psycholinguists consider vagueness in lexicons as efficiency and economy of natural languages for everyday communication rather than a flaw of not having all precise terms at their disposal. They reject the idea of a vague concept having infinitely many meanings but adopt the view of unfixed boundaries between F and non- F objects as in the aforementioned characteristic (2) of a vague term. Vagueness, according to them, is inherent flexibility of word meanings and is a source of fuzziness alongside meaning variation, i.e. variation of word meanings within a language community, ignorance, i.e. partial knowledge of word meanings, and pragmatism, i.e. sloppy use. See [Löb13].

The philosophical theories aim to identify the logic and semantics for a vague language and to address the sorites paradoxes, while the goal of the linguistic and psycholinguistic theories is to use vagueness to solve issues such as categorisation by introducing Prototype Theory. Both approaches require the tools which mathematicians can provide (for

the argument to use fuzzy logic in linguistics see [Sau11]). We will have a closer look at all three disciplines below.

1.1. Search for explanation of Vagueness. In this section we present philosophical, linguistic and mathematical approaches to vagueness. This summary is by no mean exhaustive, but it outlines some important background motivation for researchers solving problems in fuzzy logics, a field equipping us with tools to reason with vague statements.

1.1.1 Philosophical Theories of Vagueness.

There was a long break between the sorites paradoxes and the first philosophers making an attempt to construct theories to explain them (see [Hyd07], [KS96], [Wil94]). In 1923 Russell tackles the definition of vague terms in terms of isomorphism: a vague term is assigned to many different representations in the world; he uses an example of a smudged photograph, and a vague term such as tall, which can represent two people of different heights ([Rus23]). However, before him Peirce in 1902 and Wells in 1908 discovered properties of vague terms such as borderline cases and lack of sharp boundaries (see [Bla37] reprinted in [KS96]). In the same paper Black ([Bla37]) defines consistency of application of a vague term L to an object x in terms of odds for users' choosing that L applies to x and it seems to be one of the first attempts to assign numerical values to statements containing vague terms, though without any calculus to reason with vague propositions.

From the epistemic view, vagueness is defined as ignorance about the sharply bounded extensions of a predicate. The advantage of this approach is that it preserves classical logic and semantics and explain higher-order vagueness (borderline cases have also precise but unstable boundaries). However, it is considered not credible to admit that there is a precise point at which tall changes to not-tall. The supporters of the view argue pragmatically that the lack of this precise point is due to the difference between the meaning of a vague predicate and its use: words owe their meaning to the way speakers use them, which may demonstrate their ignorance about the state of affairs in the world or the truth-conditions for the statement to be true. See [Car69], [Wil92] reprinted in [KS96], [Wil94], respectively.

Another approach to dealing with borderline cases is presented by supervaluationism. The statements involving them are neither true nor false: they are truth-values gaps. The supporters of this theory claim that there are supertrue and superfalse truth values. If one is to sharpen the boundaries of a predicate's application and non-application to an object, a supertrue (superfalse, resp.) value is assigned to the statement containing it if in all such 'sharpenings', the value of the statement involving the said predicate and the object is true (false, respectively), and neither false nor true otherwise. What

is interesting is that a disjunctive statement may be supertrue even though neither of the disjunct propositions may be, e.g. ‘Jones is tall or he is not’ is true under all ways of sharpening ‘tall’. Similar phenomena can be observed for universal and existential quantifiers. See [Meh58], [Fin75] reprinted in [KS96].

In his paper [Fin75], Fine outlines a detailed theory. He introduces the specification space with specific properties containing a non-empty set of specification points which are partially ordered by the number of ‘sharpenings’ making it true. The specification point is partial if its truth value is a gap, otherwise it is complete. The base-point is a specification point with a propositional atom before any ‘sharpening’ occurred. The base-point is extended by other specification points with no fewer ‘sharpenings’. Fine includes a ‘definitely’ operator D by saying that DA is true at a specification point iff A is true at its base-point, otherwise it is false (compare it with the Baaz operator Δ to be seen in Chapter 3). Without the operator, supervaluationism retains classical logic — that is for any set of statements Γ , we can deduce a consequence α using supervaluation or classical logic (see [KS96] p. 30) — but clearly its semantics is not classical as shown by the disjunctive statements. The theory however does not address higher-order vagueness (see [KS96] pp. 34-36).

There is an alternative approach to supervaluationism called subvaluationism which makes an attempt to explain vagueness using paraconsistent logic, a logic that admits contradictions. Instead of a truth-value gap, we have a truth-value glut assigned to statements that are both true and false. Therefore, we see that the (sub)true truth-value is assigned to statements which are true in some of the aforementioned ‘sharpenings’ and could be false in others. See [Cob11], [Kee00], [Pri07].

Now we turn to degree theories which gave rise to many-valued logics. As summarised in [KS96] pp. 35-49, the common characteristics of these logics are: (1) they admit at least three truth values, (2) most of them are truth-functional: i.e. the truth-value of the complex statement is calculated from its components (for arguments against applicability of truth-functionality of degree theories to vagueness see [Edg96]), (3) they should meet a normality constraint and extend classical semantics, in the sense that if all propositional atoms in a given statement are true or false, the statement should have the same value as it has when employing classical semantics, (4) they are truth-preserving: that is, for a valid argument, a conclusion has a truth value either equal to the least truth value of one of its premises or to one of a number of designated truth values that all the premises take, (5) they have fuzzy interpretations of universal and existential quantifiers as well as predicates: supposing that F is a predicate and x an object, Fx is assigned a value from $[0,1]$ and as classically, $\exists xFx$, $\forall xFx$ are the greatest lower bound of Fx and the least upper bound of Fx , respectively, (6) they do not deal well with higher-order vagueness if the set of truth-values is $[0,1]$ as they impose sharp boundaries between false (truth value 0) and true (truth value 1) statements as well as statements with other

precise truth values (see [Sai90], [Wri76] reprinted in [KS96]). For some degree theories employed to explain vagueness see [Tye94] (three-valued logic), [Mac76] (infinite-valued logic) reprinted in [KS96].

1.1.2 Linguistic Theories about Vagueness.

As has been mentioned above, vagueness is considered by linguistics as a trait of language and the view that all sentences have precise and determined meanings or that such a state is desirable is erroneous (see [Lyo81] p. 203 cited in [Cha94]). Moreover, vague expressions occur in a wide range of contexts, albeit favoured in spoken rather than written language, and cannot be considered as exceptions, and they are even present in the language of analysis as well ([Cha94]).

There are four factors attributed to vagueness: (1) generic character of words, which enables social communication in terms of coping with the infinite variety of our experiences, (2) meaning is never homogeneous, (3) there is a lack of sharp boundaries in the non-linguistic world: this is debated upon by philosophers as an ontic view of vagueness (see [Eva78], [Lew88], [PW95] reprinted in [KS96], and [KS96] pp. 49-57), (4) lack of familiarity with the word's meaning (see [Ull62] p. 118 cited in [Cha94]). The last factor is explained by a lexical gap or being due to pragmatic purposes. The lexical gap arises when the language in question may not have a suitable word, or it could be due to memory loss or ignorance of the speaker.

To explain pragmatic purposes we will discuss briefly some theory of conversation. Grice in [Gri75] introduced the Cooperative Principle, an assumption that speakers cooperate in achieving successful communication, and divided it into four maxims: (1) the Maxim of Quality, according to which one needs to be truthful and avoid lies or giving information for which one lacks evidence, (2) the Maxim of Quantity, according to which one's conversational contribution needs to be appropriately informative, no more and no less than required, (3) the Maxim of Relevance, according to which one needs to stay on the topic, (4) the Maxim of Manner, according to which one needs to avoid obscurity of expression and ambiguity (not to be confused with vagueness), and be brief and orderly. Violation of any of these maxims leads to conversational implicatures, when the meaning of the speaker is implied rather than overtly stated.

In the context of Gricean theory of conversation, the speaker may want to be cautious in making generalisations, which is typical of scientific discourse: thus they conform to the Maxim of Quality. It could be that the subject of conversation may not require precision and the speaker gives the right amount of information following the Maxim of Quantity. The speaker may want to maintain the atmosphere — informality or politeness — and may violate the Maxim of Manner ([Cha94], [Leh75], [MS97]). Therefore we see clearly that vague terms are mutually understood and are purposefully used. ‘They

are part of the linguistic repertoire of the competent language user, who uses them to accomplish particular communicative goals' ([Cha94] p. 197). Moreover, L2 speakers (speakers of a given language as their second language) need to learn to use appropriate vague expressions in order not to be considered to violate the Maxim of Quality ([Cha94]).

According to general cognitive economy, there is a basic level of abstraction at which one gets the most information with the least effort. Therefore, there may or may not be vagueness *de re* (ontic view), but vagueness *de dicto* reflects one's perception of the world. On the basis of this principle, Rosch and her followers constructed a prototype theory of categorisation, where basic-level objects called prototypes carry the most information about a given category. They have the highest category cue validity, i.e. the frequency of use of a given object (cue) as a predictor of a given category. This represents the vertical dimension of categories in the perceived world structure, on top of which we have basic-level objects with most common attributes and similarity in shapes. Due to the horizontal dimension of categories, the basic-level objects are the most differentiated from each other. Now, since similarity is a matter of degree, so is category membership. Thus, the members of a given category are of a graded status connected by family resemblance, a notion due to Wittgenstein (see [Wit97] notes 66-71), and categories have fuzzy boundaries and lack necessary conditions for memberships since less typical examples may lack some of attributes that prototypes have. Vagueness is defined as inherent flexibility of word meaning (categories) and the less prototypical an object is the more vague it is. See [Löb13], [RMD⁺76], [Ros75]), [Ros78].

1.1.3 Mathematical Approaches to Vagueness.

So far we have seen some philosophical theories which try to define what vagueness is and how to explain it, and some linguistic theories whose objective is to explain why vagueness exists in human languages and what its purpose is. We will now have a closer look at some tools produced by mathematicians to model vagueness or at least some of its properties (e.g. [HN03]).

One of the properties of vagueness is gradualness, where the membership of an object or the truth of a sentence is a matter of degree. Although in the concept of vagueness the membership is ill-known, it gave rise to interest in the gradualness and fuzziness issue, that is soft classification of objects or the study of sentences whose truth value may not be truth or falsity. This issue has been a subject of study in many disciplines such as many-valued and fuzzy logics, where there are more than two truth values of formulas. It is also our primary subject of study in this thesis; however our research is focused on two theoretical issues of mathematical fuzzy logics, the name that we will adopt here as an umbrella term for fuzzy logic BL, Gödel logic, Product logic and Łukasiewicz logic, which will be defined and discussed later in this thesis.

The starting point of constructing tools for reasoning with vague statements is classical logic. In classical logic, the set D of truth values consists of two: truth, which is interpreted as 1, and falsity, which is interpreted as 0. We extend the set D to cover all degrees of truth. The structure of this set is influenced by intuitions about truth values. If we believe that we can always say that the statement α is truer than a statement β , then we will need a linearly ordered set, otherwise just a partially ordered set: in the extreme case all non-fully true truth values are less than 1 and non-fully false truth values are greater than 0, and there is no more comparison between the truth values. Note that the symbols 1 and 0 denote the greatest and the least element in D , respectively. The predicates are functions from the set of tuples of objects, to which they assign elements from D . Then we need to construct a structure for reasoning (D, C) , where C consists of the interpretation of connectives in a given fuzzy propositional logic. Usually in fuzzy logics the nullary connectives $\top$ and $\perp$ are interpreted as 1 and 0 (the greatest and the least values of D , resp.). For example in Zadeh logic ([Smi17]), there are three other main connectives: negation $\neg$, disjunction $\vee$ and conjunction $\wedge$, and implication is defined as disjunction of the negated antecedent and the consequent. Their interpretation is as follows. Let V assign truth values from $[0, 1]$ to propositional atoms. Then we can inductively define $V^{([0,1],C)}(\alpha)$ for a formula α of the logic: $V^{([0,1],C)}(\neg\alpha) = 1 - V^{([0,1],C)}(\alpha)$, $V^{([0,1],C)}(\alpha \vee \beta) = \max\{V^{([0,1],C)}(\alpha), V^{([0,1],C)}(\beta)\}$ and $V^{([0,1],C)}(\alpha \wedge \beta) = \min\{V^{([0,1],C)}(\alpha), V^{([0,1],C)}(\beta)\}$, where α, β are formulas of the logic. We may extend the structure to cover the interpretation of quantifiers. Now we can reason deductively. For example, suppose ‘Adam is bald’ has the truth value 0.5 and ‘Adam is tall’ has the truth value 0.7. Then in Zadeh logic, the statements ‘Adam is tall and bald’ and ‘Adam is not tall’ have the truth values 0.5 and 0.3, respectively.

We can consider at least two types of fuzzy logic that arose firstly motivated by providing the tools to solve problems involving vagueness. One arises from a tradition started by Lofti Zadeh’s famous 1965 paper *Fuzzy sets*, in which he introduces the notion of fuzzy sets (for which the membership of an object is gradual), defines set and algebraic operations on them and discusses some of their properties. The other tradition has its roots in Jan Łukasiewicz’s three-valued logic introduced in his 1920 paper *O logice trójwartościowej* and in an independently developed many-valued logic by Emil Post in his 1921 paper *Introduction to a general theory of elementary propositions*. At the heart of these traditions lies rejection of the principle of bivalence (a statement is either true or false). In the case of fuzzy sets, it manifests itself in considering that an element belongs to a set to a certain degree, and in the case of many-valued logic, we have more than two classical truth values, which are truth (usually associated with number 1) and falsity (usually associated with number 0). For the historical exposition and comparisons of these types of fuzzy sets and fuzzy logics, see [BBP⁺16], [CHN11], [DEGP07], [DOP00], [DP15], [HK95], [Kle97], [Mal07], [Zad96].

There are other approaches to generalisation of the classical notion of sets which constitute the basis of vague reasoning. Among them, the rough set philosophy emerged due to Zdzisław Pawlak to classify objects which are not noticeably different into imprecise concepts. Those concepts are defined by lower and upper approximation (sets of objects), that is those objects that surely represent the concept and those that possibly represent the concept, respectively. See [Orl87], [Paw82], [Paw91], [Paw93], [Pol02], [Sko96]. The rough set theory approach seems to be important to a variety of applications such as artificial intelligence (AI) and cognitive sciences, machine learning, knowledge acquisition, decision analysis, knowledge discovery from databases, expert systems, decision support systems, inductive reasoning, and pattern recognition (see [PGBSZ95]). For comparison between fuzzy sets and rough sets, see [DP90].

One of the most significant contributions to the second aforementioned field of fuzzy logic investigation is Petr Hájek's *Metamathematics of Fuzzy Logic* published in 1998 (see [CHN11], [Got06], [MOG09]), in which the author introduces the mathematical fuzzy logic BL (Basic Fuzzy Logic). This logic is central to this thesis. It has binary connectives $\&$ (strong conjunction) and $\rightarrow$ (implication), and nullary connective $\bar{0}$ (falsity); other connectives are defined in terms of these three connectives. The additional conjunction $\&$ has semantics which may cover the cases where we want the conjunction of two formulas to be possibly strictly less 'true' than any of the conjuncts. Following the rules of the syntax of this logic, formulas of BL can be constructed. The semantics of BL is constructed using an evaluation of propositional atoms and a function defined on the interval of reals $[0,1]$ called a t-norm, a notion introduced in [Men42]; different t-norms are discussed in [KRP00]. This function provides semantics for strong conjunction, and when it is left-continuous, its derivative, called a residuum, provides semantics to the implication. The truth value of falsity is 0. The syntax and semantics of BL are discussed in Chapter 3 of this thesis.

In the aforementioned book, Hájek constructs an axiomatic system from whose axioms, using its inference rule, new theorems (formulas) can be proved (derived). He also demonstrates that by adding some axioms, we can have axiomatic systems for other mathematical fuzzy logics introduced earlier to deal with the problem of vagueness, namely, Łukasiewicz, Product and Gödel logics (see [LT30] quoted in [RR58] and [Háj98b], [HGE96], [Dum59] and Chapter 5 of this thesis). The semantics of these logics are given by specific t-norms: the Łukasiewicz, product and Gödel t-norms. He also proves the decomposition theorem (see Chapter 2), which states that each continuous t-norm can be decomposed into countably many Łukasiewicz and product t-norms operating on intervals of $[0,1]$. If two elements of $[0,1]$ are not from the same Łukasiewicz or Product interval then we apply the Gödel t-norm, and otherwise the one operating on this interval. It is a significant achievement to provide a common axiomatic system for these logics and also show that semantically they are linked together by the

decomposition theorem.

The current interests of mathematical fuzzy logicians diverge from solving issues of vagueness, but instead focus on solving technical issues. The following two sections of this chapter look at the place of the mathematical fuzzy logics that are discussed in this thesis and some achievements of mathematical fuzzy logicians, which will be followed by presenting two technical issues that are solved in this thesis.

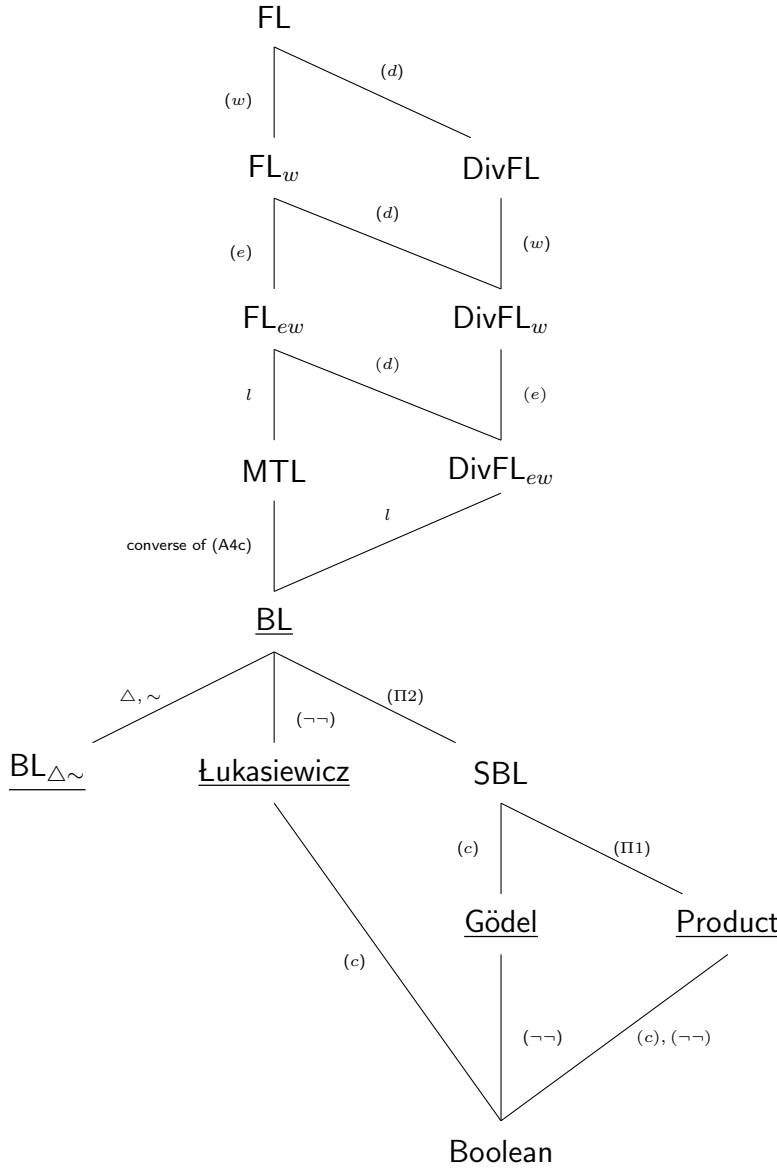
1.2. Place of mathematical fuzzy logics among substructural logics. Substructural logics are non-classical logics that lack structural rules (axioms), which we discuss below, present in classical logic (Boolean in Figure 1). Figure 1 places the logics discussed in the thesis (underlined in Figure 1) among other substructural logics and shows their relation to the classical logics. It is by no means exhaustive. It was constructed on the basis of [BCH11] and [GJ09]. We will discuss the terms used in Figure 1 below.

Although in this thesis the semantics of the fuzzy logics considered is built from t-norms, one can use a more general semantics based on algebras. An FL-algebra, which is used as semantics for a fundamental substructural logic FL (see Figure 1), is a residuated lattice expanded by a constant 0 (the truth-value of falsity). A residuated lattice is a lattice with a monotone monoidal operation $\star$ (semantics of strong conjunction) with unit e , and left and right residuals of $\star$ (semantics for left and right implication). The existence of two residuals is due to the operation $\star$ being non-commutative in general. An FL-algebra is bounded and integral when its lattice reduct is a bounded lattice (with 0 as its least element) and the unit of $\star$ is its greatest element, respectively. These algebras provide the semantics of the logic FL_w , which is the logic FL with the added axiom (w), representing boundedness and integrality of the logic. An FL-algebra is commutative if its operation $\star$ is commutative. In Figure 1 we extend logic FL_w by adding the axiom (e) representing commutativity of strong conjunction and obtain FL_{ew} . When strong conjunction is commutative, the residuals collapse and may be viewed as one implication operation. For the full expositions of the axioms see [BCH11] page 57.

The linear FL_{ew} -algebras are the semantics of MTL, which is the logic of all left-continuous t-norms and is called monoidal t-norm logic. However, more generally a semantics is given by MTL-algebras, which are FL_{ew} -algebras satisfying the prelinearity axiom and need not be linear. The restriction of algebras to linear algebras is denoted by l along the relevant paths in Figure 1. In MTL and the weaker logics just discussed, weak conjunction $\wedge$ is an operation, whose semantics is lattice meet. When we further add to MTL the converse of axiom (A4c) (see page 30 of [BCH11]), weak conjunction becomes definable in terms of strong conjunction and implication, and MTL logic with the converse of (A4c) is called BL.

Another path from the logic FL to BL leads via the logic of divisible residuated lattices, which we called DivFL, by adding relevant axioms as shown in Figure 1. The

Figure 1: Relative position of mathematical fuzzy logics among substructural logics



axiom denoted as (d) in Figure 1 is called the axiom of divisibility (see the axiom in [GJ09]). Once we have this axiom in DivFL_{ew} logic, then the logic BL arises. Its algebraic semantics, defined in Chapter 6, is a BL-algebra.

In this thesis in Chapter 4 we construct a tableau for $\text{BL}_{\Delta\sim}$ with a specific semantics of LP-norms, defined in Chapter 2. This logic is the extension of BL by having two additional connectives: the Baaz connective Δ and the involutive negation $\sim$ (see Chapter 3).

The logic BL can be extended by adding an axiom $(\neg\neg)$ which is the ‘classicality’ of residual negation, that is its involutuality (see Chapter 5). Once this axiom is added, we have Łukasiewicz logic and its algebraic semantics is the MV-algebra (defined in Chapter 6). The axiom $(\Pi2)$ expresses the strictness of residual negation, and adding this axiom to BL yields the logic SBL of continuous t-norms with strict residual negation.

Extending SBL by the axiom (c), which expresses idempotence of strong conjunction, gives Gödel logic (with the Heyting algebra as its semantics). The axiom (II1) expresses cancellation of non-zero elements and added to BL yields Product logic (with the product algebra as its semantics defined in Chapter 6). Finally adding appropriate axioms to Łukasiewicz, Gödel and Product logics as shown in Figure 1 yields the classical logic, also known as Boolean.

1.3. Achievements of the thesis in the context of earlier works. There are four areas of research interest for mathematical fuzzy logicians ([BCH11] p. 68): (1) proving completeness of axiomatic systems with respect to some distinguished semantics, (2) functional representation of functions $f : [0, 1]^n \rightarrow [0, 1]$, where $n \geq 0$ is an integer, by formulas of prominent fuzzy logics, (3) the proof theory, (4) computational complexity. In this thesis we contributed to areas described in (1) and to some extent to areas described in (3) and (4), and thus in this section we will concentrate on discussing some relevant achievements in these areas. For more information on the achievements in all these areas see [CHN11]. We will discuss earlier work and also outline our contributions.

For the logics BL, Łukasiewicz, Product and Gödel, Hájek in [Háj98b] proves soundness. Namely, if a formula ψ is provable from an arbitrary set Γ of formulas, then for every evaluation of propositional atoms and for every continuous t-norm, all formulas in Γ having the truth value 1 implies ψ having the truth value 1. This includes the Łukasiewicz t-norm, the product t-norm, the Gödel t-norm and an arbitrary continuous t-norm with respective axioms in each case added to Hájek's axiomatic system of BL. See the axiomatic systems and the aforementioned so-called soundness proofs in Chapter 5 of this thesis.

Hájek also shows the converse of soundness of his axiomatic systems for Gödel logic. This is called strong standard completeness and will be fully defined in Chapter 5 of this thesis. He demonstrates by counter-examples that this property of the axiomatic systems for Łukasiewicz and Product logics can fail when Γ is an infinite set, so they are not strong standard complete (see Remark 3.2.14 and Corollary 4.1.18 of [Háj98b]). Strong standard completeness also fails for BL (see [BM11], p. 365). Infinitary rules need to be added to axiomatic systems for BL, Łukasiewicz and Product logics to achieve strong standard completeness (see [Mon07], p. 249).

In our thesis we take this further step to prove that when Hajek's axiomatic system is extended by a particular infinitary rule, the aforementioned converse holds for any countable set of formulas, including infinite sets (see Chapter 6); that is, we obtain strong standard completeness. In [Mon07] and [VBEG17] some attempts were made to achieve strong standard completeness for BL and MTL by extending the language of the logics by adding some connectives or formulas representing truth constants and also including some infinitary rules for them (see more in the introduction to Chapter

5 of this thesis). In this thesis we do not extend the language of the logics. See the introduction to Chapter 5 for more background and technical motivation.

To complete a picture of the landscape of methods of establishing standard completeness for fuzzy logics, we should also acknowledge the achievements of the proof theory. Fuzzy logics such as Łukasiewicz logic, Gödel logic and Product logic have elegant hypersequent calculus formations (see [MOG09], [Met11]). Among the rules of these calculi, there are cut and density rules, which express the transitivity of deduction and the density of the algebras used as semantics of the logics, respectively. Lemma 6.3 of the thesis resembles the cut rule, and the density rule is similar to the rule (D^∞) , quoted from [VBEG17] and presented in the introduction to Chapter 5 of this thesis. Some substructural logics are proved to be standard complete via use of their hypersequent calculi with cut and density elimination (see [MM07], [MOG09], [Met11]). However, the authors of [MOG09] say that ‘it is unclear whether density elimination can be obtained or is useful in these cases [i.e. Łukasiewicz logic and Product logic]’ ([MOG09] p. 134) and in the case of BL we have no guarantee to obtain an algebra of the same class (see [MOG09], p. 134). Therefore, it is reasonable to take a different approach to establish strong standard completeness for BL, Łukasiewicz and Product logics, which is one of the achievements of our research presented in this thesis (the details are in Chapters 5 and 6).

To discuss computational complexity of the logics, we need to briefly recall definitions of a satisfiable formula and tautologies (see more detailed definitions in Chapter 3 of this thesis). We will limit the definition to standard semantics for the mathematical fuzzy logics of this thesis. A formula is positively satisfiable (resp. satisfiable) if its truth value is positive (resp. 1) with respect to some evaluation of propositional atoms and some continuous t-norm. A formula is a positive tautology (resp. tautology) if its truth value is positive (resp. 1) with respect to every evaluation of propositional atoms and every continuous t-norm. The continuous t-norms in these definitions are those relevant to a given logic, e.g. for Łukasiewicz logic, the continuous t-norm is the Łukasiewicz t-norm. It is proven that (1) a formula is positively satisfiable if and only if its negation is not a tautology, and (2) a formula is a positive tautology if and only if its negation is not satisfiable. The sets of positively satisfiable and satisfiable formulas for BL, Łukasiewicz, Product and Gödel logics are **NP**-complete and the sets of positive tautologies and tautologies for these logics are **coNP**-complete (see references in [CHN11] pp. 77-78).

Our tableau for BL in Chapter 4 of this thesis confirms the computational complexity results, and the second part of this chapter offers a first approach to assessing computational complexity for $BL_{\Delta\sim}$ although no complexity results were formulated in this thesis. However, our tableau calculi are more natural and intuitive reasoning systems than the ones existing in the current state of the art as they are based very directly on the semantics of BL and $BL_{\Delta\sim}$ and can be used as a tool in solving vagueness is-

sues in a language. Our tableau calculi produce inequalities expressing ranges of values that propositional atoms (representing statements in a natural language) need to satisfy. They only order the endpoints of intervals (ranges) containing arithmetic expressions involving the propositional atoms of the formula for which the tableau is constructed. In this way vagueness existing in the language is preserved and the tableaux open a way to a quantitative measure of the vagueness inherent in a given formula. A full discussion of earlier works on tableaux is presented in the introduction to Chapter 4, in which we also claim that our system is less complex than other tableau calculi.

1.4. Publication status. As can be clearly seen, the problem of the equivalence between semantic and syntactic systems of logics is considered to be one of the first issues that logicians try to tackle and solve. We extend the axiomatic system for BL introduced by Hájek by an infinitary inference rule, a rule by which we can derive a new formula from infinitely many formulas, and we show that given any finite or countably infinite set of formulas, and given a formula whose truth value is 1 whenever every formula in the set has truth value 1, then the formula is provable from the set. The results are presented in the paper [Ku18]. The results were also presented in a conference paper [Ku16].

Another interesting aspect in the research into logics in general is to construct a tableau by which one can verify that a given formula of a logic is true or build a model, in which its truth value is less than 1 (true). This is done here by defining a tableau for BL, which is one of the achievements of this research. For $BL_{\Delta\sim}$ (with a specific semantics) we defined a tableau for finitely many formulas to show that their truth values may come from given subsets of reals in $[0,1]$ by building a model or to prove that they cannot come from this given subset of reals. The research was inspired by the previous research of the author of this thesis, published in [KPS13], and the current results are partially presented in [Ku14a] and [Ku14b].

1.5. Contribution. The main body of Chapters 2 and 3 provides the details of the background necessary for Chapters 4, 5 and 6. The success of the presentation of background theories is due to scrupulous and multiple readings of Professor Ian Hodkinson and his invaluable suggestions. Any remaining errors are entirely the author's.

Chapter 4 presents entirely novel material, in which we define two tableaux, one for verifying that a formula has truth value 1 or less than 1 and the other for verifying that finitely many formulas have truth values belonging to a certain subset of $[0,1]$. I must acknowledge that construction of the tableaux, examples and proofs that the tableaux are sound and complete were positively influenced by supervision and ideas of Professor Ian Hodkinson.

The same impact is present in the construction of an extended axiomatic system for

BL, proofs of formulas in the system modelled on natural deduction, and the proof that the axiomatic system is standard strongly complete with respect to every continuous t-norm. We also showed strong standard completeness for Łukasiewicz and Product logics with respect to Łukasiewicz and Product t-norms. This material is also entirely novel and presented in Chapters 5 and 6. Our original idea for the extended system included Hájek’s seven axioms and three inference rules: modus ponens, an infinitary rule and a prelinearity rule. However, thanks to ideas suggested by Referees of my paper [Kul18] on this topic submitted to a journal, the prelinearity rule can be deleted since it is derivable from Hájek’s axiomatic system enriched only by the infinitary rule.

Other than the aforementioned contributors to the outcomes of the research, the work is the author’s own and otherwise appropriately referenced.

1.6. Structure of the dissertation. The thesis consists of seven chapters. Chapter 2 presents background theory for t-norms, the derivatives of left continuous t-norms and the Decomposition Theorem, and Chapter 3 provides a quick review of syntax and semantics of fuzzy logics: BL, Łukasiewicz, Product and Gödel logics, and defines the notions of 1-tautology, validity and $\mathcal{K}$ -satisfiability.

The research has two main achievements: (1) constructing tableaux for fuzzy logics, and (2) proving strong standard completeness for t-norms. The outcomes of the first part are presented in Chapter 4, which is divided into further two parts preceded by a brief literature review in the area. In Section 4.2 we present a tableau for a formula of BL to show that its true value is 1 by closing the tableau, or to show that its value is less than 1 by constructing a model. In the following section, Section 4.3, we define a tableau for K -satisfiability for $BL_{\Delta, \sim}$, fuzzy logic BL extended by two connectives $\Delta, \sim$. In both sections we also familiarise the reader with the method by producing several examples. Also, we prove that the tableaux are sound and complete with respect to a continuous t-norm.

Chapter 5 focuses on presentation of axioms of fuzzy logics and showing their soundness; we define axiomatic systems for fuzzy logics and what it means for them to be generally sound and complete as well as showing by counterexamples that some fuzzy logics are not strongly standard complete. Then we use one of those examples and extend the axiomatic system proposed by Hájek by an infinitary rule, enabling us to prove strong standard completeness for these extended axiomatic systems for logics: BL, Łukasiewicz and Product, which we achieved in Chapter 6.

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2. BACKGROUND THEORY FOR T-NORMS

In this chapter we recall and review knowledge about continuous t-norms and their derivatives (Sections 2.1 and 2.2), which are the functions providing semantics to (among others) the conjunction and implication connectives of the fuzzy logic BL (details of which will be presented in Chapter 3). Section 2.3 will introduce the notion of a proto-t-norm and recall the basic notion of isomorphism between proto-t-norms. Both are needed eventually in our contributions presented in Chapters 4 and 6.

The main theme of Section 2.4 is to recall the Decomposition Theorem, which is expressed as Theorem 2.33. This is a fundamental, striking and influential result about t-norms, but since we will not use it explicitly later, we will not prove it. Nevertheless, the ideas that went into it and into Lemma 2.28 led to Theorem 3 of [BHMV02], which we reproduce in Theorem 4.10. Theorem 4.10 will be a vital part of our tableau work.

In the layman's terms, the Decomposition Theorem says that any continuous t-norm can be decomposed into a combination of the three 'fundamental' t-norms: the Gödel t-norm, the Łukasiewicz t-norm and the product t-norm, which have explicit formulas. The first phrasing of the Decomposition Theorem occurred in [MS57] as Theorem B. Another formulation of the theorem is given as Remark 2.1.15 and Theorem 2.1.16 in [Háj98b]. Another version of the Decomposition Theorem appears in [MOG09] as Theorem 2.20, and we expressed it as Theorem 2.35. In [CHN11], it is claimed in Theorem 1.1.7 that the original Decomposition Theorem given in [MS57] is expressed as Corollary 2.37. Since the theorem has got different formulations, in this section we provide our version.

In the same section we show that Theorem 2.35 can be derived from the Decomposition Theorem and show its other direction, expressed as Theorem 2.36. We will use the latter theorem in Chapters 4 and 6, together with an overt formula for the residuum (Theorem 2.38). In this section we also define a class of continuous t-norms and call them LP-norms. In the aforementioned theorems and Remarks 2.41 and 2.42, we provide an overt formulas for an LP-norm and its residuum. They are used in defining the tableau calculi.

Further informal explanation will be given later on each item.

2.1. t-norm. The notion of a triangular norm, a t-norm for short, was introduced in [Men42] as a way of generalising the triangular inequality from classical metric spaces to probabilistic metric spaces. However, the set of properties of t-norms was different from the definition adopted in logic: a t-norm was not necessarily associative and the boundary condition was different. A continuous t-norm will give semantics to the connective $\&$ (strong conjunction) of the fuzzy logic BL. We need to introduce the notation that will be used throughout the dissertation: for $a, b \in \mathbb{R}$ with $a < b$, we write $[a, b] = \{x \in \mathbb{R} :$

$a \leq x \leq b\}$, $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$, analogously for $[a, b)$, (a, b) .

Definition 2.1. A *t-norm* is a binary operation $\star : [0, 1]^2 \rightarrow [0, 1]$ such that for all $x, y, z \in [0, 1]$ the following axioms are satisfied:

- T1** $x \star y = y \star x$ (commutativity)
- T2** $x \star (y \star z) = (x \star y) \star z$ (associativity)
- T3** $x \star y \leq x \star z$ if $y \leq z$ (monotonicity)
- T4** $x \star 1 = x$ (boundary condition)

There exist uncountably many t-norms. The whole families of t-norms are considered in [KRP00]; we will have a look at the fundamental ones, which will be used in the Decomposition Theorem (Theorem 2.33). This theorem will constitute the basis of defining LP-norms, a class of continuous t-norms, and those will be used in constructing tableau calculi (Chapter 4) and proving strong standard completeness (Chapter 6). We will also look at some properties of these fundamental t-norms.

Example 2.2. The following are fundamental t-norms: the Łukasiewicz t-norm, the product t-norm (also called Goguen) and the Gödel t-norm (also called the minimum t-norm for obvious reasons).

Łukasiewicz $x \star_L y = \max(0, x + y - 1)$

Product $x \star_P y = x \cdot y$

Gödel $x \star_G y = \min(x, y)$

One easily checks that axioms T1-T4 are satisfied for these t-norms. We prove the satisfiability of axiom T2 for the Łukasiewicz t-norm.

$x \star_L (y \star_L z) = \max\{0, x + \max\{0, y + z - 1\} - 1\} = \max\{0, \max\{x - 1, x + y + z - 2\}\} = \max\{0, x - 1, x + y + z - 2\} = \max\{0, x + y + z - 2\}$ since $x - 1 \leq 0$. Similarly we show that $(x \star_L y) \star_L z = \max\{0, x + y + z - 2\}$. $\square$

Remark 2.3. Let $\star$ be a t-norm. From Definition 2.1 we deduce: (1) the following additional boundary conditions: $0 \star x = x \star 0 = 0$ and $1 \star x = x$, (2) monotonicity in both arguments: $x_1 \star y_1 \leq x_2 \star y_2$ if $x_1 \leq x_2$ and $y_1 \leq y_2$.

Proposition 2.4. 1. For all $x, y \in [0, 1]$ and for any t-norm $\star$, $x \star y \leq x \star_G y$.

2. The only t-norm satisfying $x \star x = x$ for all $x \in [0, 1]$ is the Gödel t-norm.

Proof. Easy to show. $\square$

It can easily be shown that the three fundamental t-norms are continuous; the proof is presented in Appendix A. This is a crucial element in the proof of a consequence to the Decomposition Theorem, Theorem 2.36, whose proof is in Appendix B. Theorem 2.36 is used in later chapters.

Theorem 2.5. The Gödel t-norm, the product t-norm and the Łukasiewicz t-norm are continuous.

We will now discuss some algebraic properties of t-norms, some of which will be used in the formulation of the Decomposition Theorem. The remarks and examples that follow the definitions of those properties are used in some proofs of the aforementioned theorem.

Notation 2.6. Let $a \in [0, 1]$ and let $\star$ be a t-norm. We define $a_\star^{(n)}$ for every $n < \omega$ inductively: $a_\star^{(0)} = 1$ and $a_\star^{(n+1)} = a_\star^{(n)} \star a$.

Definition 2.7. Let $\star$ be a t-norm.

1. An element $a \in [0, 1]$ is called an *idempotent element* of $\star$ iff $a \star a = a$. The numbers 0 and 1 are called *trivial* idempotent elements of $\star$, and each idempotent element in $(0, 1)$ is called a *non-trivial* idempotent element of $\star$.
2. An element $a \in (0, 1)$ is called a *nilpotent element* of $\star$ iff there exists some $n \in \mathbb{N}$ such that $a_\star^{(n)} = 0$.

Remark 2.8. 1. If $a \in [0, 1]$ is an idempotent element of a t-norm $\star$, then, by induction on n , we have $a_\star^{(n)} = a$ for all $n \in \mathbb{N}, n > 1$. Thus, no element of $(0, 1)$ is both idempotent and nilpotent.

2. If a t-norm $\star$ has a nilpotent element a , then there is always an element $b \in (0, 1)$ such that $b_\star^{(2)} = 0$. Indeed, take the least $n > 1$ such that $a_\star^{(n)} = 0$. Then take $b = a_\star^{(n-1)}$, which is positive and such that $b_\star^{(2)} = 0$.
3. If $a \in (0, 1)$ is a nilpotent element of a t-norm $\star$, then each number $b \in (0, a)$ is also a nilpotent element of $\star$. Indeed, by monotonicity, $b_\star^{(n)} \leq a_\star^{(n)} = 0$.

Example 2.9. 1. Each element $a \in [0, 1]$ is an idempotent element of the Gödel t-norm by Proposition 2.4. There are no nilpotent elements of the Gödel t-norm by Remark 2.8 point 1.

2. All elements $a \in (0, 1)$ are nilpotent elements of the Łukasiewicz t-norm. It is easy to show that $a_L^{(n)} = \max\{0, na - (n - 1)\}$ for $a \in (0, 1)$ and $n \geq 0$. We need to find n such that $na - (n - 1) < 0$. Thus, take any $n \in \mathbb{N}$ such that $n > \frac{1}{1-a}$. The only idempotent elements of the Łukasiewicz t-norm are trivial by Remark 2.8 point 1.

3. The only idempotent elements of the product t-norm are trivial; $a \star_P a = a^2$ and $a \star_P a = a$ imply $a = 1$ or $a = 0$. There are no nilpotent elements of the product t-norm since $a_P^{(n)} = a^n > 0$ as $a \in (0, 1)$ (see Remark 2.8 points 2 and 3).

Definition 2.10. For an arbitrary t-norm $\star$ we consider the following properties:

1. The t-norm $\star$ is said to be *strictly monotone* iff

$$(SM) \quad x \star y < x \star z \text{ whenever } x > 0 \text{ and } y < z.$$

2. The t-norm $\star$ is called *Archimedean* iff

$$(AP) \quad \text{for each } (x, y) \in (0, 1)^2 \text{ there is an } n \in \mathbb{N} \text{ with } x_\star^{(n)} < y.$$

3. The t-norm $\star$ has the *limit property* iff

$$(LP) \quad \text{for all } x \in (0, 1), \lim_{n \rightarrow \infty} x_\star^{(n)} = 0.$$

Example 2.11. We will illustrate the properties defined in Definition 2.10.

1. The Gödel t-norm has none of these properties. To show that it does not have (SM) property, take $x = y = \frac{1}{2}$ and $z = \frac{3}{4}$. To show that it does not have (AP), take $x = \frac{1}{2}$ and $y = \frac{1}{3}$. By the fact that all elements of the Gödel t-norm are idempotent, the t-norm does not have (LP).
2. The Łukasiewicz t-norm does not have (SM) property: take $x = z = \frac{1}{2}$ and $y = \frac{1}{3}$. By the fact that all elements of the Łukasiewicz t-norm are nilpotent, the t-norm has (AP) and (LP) properties.
3. The product t-norm has all of these properties. It is easy to check that it has (SM) by the virtue of properties of multiplication on real numbers. We check that the t-norm has (AP). It is obvious for any pair $(x, y) \in (0, 1)^2$ such that $x \leq y$. Consider any pair $(x, y) \in (0, 1)^2$ such that $y < x$. Take any integer $N > \frac{\ln y}{\ln x}$. We check that the t-norm has (LP). Recall that $\lim_{n \rightarrow \infty} x^n = 0$ iff $\forall \epsilon > 0 \exists N \forall n > N |x^n - 0| < \epsilon$. It is obvious for $\epsilon \geq 1$. For $0 < \epsilon < 1$, apply (AP).

Proposition 2.12. Every Archimedean t-norm has no idempotent elements except for 0 and 1.

Proof. Let $\star$ be an Archimedean t-norm. Suppose $0 < a < 1$ is an idempotent element of $\star$. Then $a_\star^{(n)} = a$. Take $b = \frac{a}{2}$. Then there is a pair $(a, \frac{a}{2}) \in (0, 1)^2$ such that (AP) does not hold. Contradiction. $\square$

Proposition 2.13. Let $\star$ be a t-norm. Then $\star$ is Archimedean iff it has the limit property.

Proof. Suppose that $\star$ is an Archimedean t-norm. Define $f_x(n) = x_\star^{(n)}$. The function is non-increasing and bounded by 0. Thus $\lim_{n \rightarrow \infty} f_x(n)$ exists. Let us call it $\bar{x}$. By (AP), $\bar{x} < y$ for every $y > 0$. Thus $\bar{x} = 0$.

Conversely, suppose that $\star$ has (LP). Take arbitrary $x, y \in (0, 1)$. Since as $n \rightarrow \infty$, $x_\star^{(n)} \rightarrow 0$, there exists m such that $x_\star^{(m)} < y$. $\square$

Definition 2.14. 1. A t-norm $\star$ is called *strict* iff it is continuous and strictly monotone.

2. A t-norm $\star$ is called *nilpotent* iff it is continuous and each $a \in (0, 1)$ is a nilpotent element of $\star$.

Remark 2.15. By Theorem 2.5, and Examples 2.9 and 2.11, we conclude that the Łukasiewicz t-norm is nilpotent and the product t-norm is strict.

2.2. Derivatives of a left continuous t-norm. In this section we will define some other operations in terms of a t-norm, which will provide the semantics of some connectives of the fuzzy logic BL. The first of them is the residuum, which gives the semantics to the connective $\rightarrow$. Its definition captured in Proposition 2.16 (rephrased from Lemma 2.1.4 of [Háj98b]) is followed by Lemma 2.17 (rephrased from Lemmas 2.1.6 and 2.1.7 of [Háj98b]) used extensively among other places in the proof of strong standard completeness (Chapter 6).

Proposition 2.16. Let $\star$ be a left continuous t-norm. Then there is a unique operation $\Rightarrow: [0, 1]^2 \rightarrow [0, 1]$ satisfying for all $x, y, z \in [0, 1]$,

$$(x \star z) \leq y \Leftrightarrow z \leq (x \Rightarrow y). \quad (1)$$

This unique operation satisfying (1) is given by $x \Rightarrow y = \sup\{z | x \star z \leq y\}$ and is called the *residuum* of the left continuous t-norm $\star$.

Proof. Take arbitrary $x, y, z \in [0, 1]$.

($\Rightarrow$) Suppose that $x \star z \leq y$. Then, obviously $z \leq \sup\{z | x \star z \leq y\} = x \Rightarrow y$.

($\Leftarrow$) Suppose that $z \leq x \Rightarrow y$. Since $\star$ is non-decreasing, commutative and left continuous,

$$x \star z \leq x \star (x \Rightarrow y) = x \star \sup\{z | x \star z \leq y\} = \sup\{x \star z | x \star z \leq y\} \leq y.$$

We prove the uniqueness. If $\Rightarrow_1, \Rightarrow_2$ satisfy (1), then for all z we have $z \leq x \Rightarrow_1 y$ iff $x \star z \leq y$ iff $z \leq x \Rightarrow_2 y$. Take $z = x \Rightarrow_1 y$ to get $x \Rightarrow_1 y \leq x \Rightarrow_2 y$, and similarly for $\geq$. $\square$

Lemma 2.17. For each continuous t-norm $\star$ and its residuum $\Rightarrow$, for all $x, y, u \in [0, 1]$,

1. $x \leq y$ iff $(x \Rightarrow y) = 1$,
2. $1 \Rightarrow x = x$.
3. If $x \leq y$, then $x = y \star (y \Rightarrow x)$.
4. If $x \leq u \leq y$ and u is idempotent, then $x \star y = x$.
5. If $x \leq y_\star^{(n)}$ for every $n < \omega$, then there is an idempotent $e \in [0, 1]$ such that $x \leq e \leq y$.

Proof. 1. By axiom T4 and Proposition 2.16, $x \leq y$ is equivalent to $x \star 1 \leq y$ and thus it is equivalent to $1 \leq (x \Rightarrow y)$.

2. By axiom T4 and Proposition 2.16, $x = \sup\{z \mid z \star 1 \leq x\} = 1 \Rightarrow x$.
3. Take arbitrary $x, y \in [0, 1]$ such that $x \leq y$. Define a function $f : [0, 1] \rightarrow [0, y]$ by $f(z) = z \star y$. By definition of a t-norm, $f(1) = 1 \star y = y$ and $f(0) = 0 \star y = 0$. Then by continuity of f and the intermediate value theorem, there exists at least one z such that $0 \leq f(z) = x \leq y$. We can take the supremum of these arguments z . That is by the fact that $\star$ is non-decreasing, $\sup\{z \mid z \star y \leq x\}$, which is by Proposition 2.16, $y \Rightarrow x$. Thus, $x = y \star (y \Rightarrow x)$.
4. Take arbitrary $x, y, u \in [0, 1]$ with $x \leq u \leq y$ and with u idempotent. First, we show that $(\#) x = x \star u$. By the previous statement, commutativity of t-norm and $u = u \star u$, $x \star u = u \star (u \Rightarrow x) \star u = u \star (u \Rightarrow x) = x$. By definition of a t-norm, $x \star y \leq x \star 1 = x$. Also, by $(\#) x = x \star u \leq x \star y$. Thus, $x = x \star y$.
5. Take $e = \inf\{y_\star^{(n)} : n < \omega\}$. Obviously, $x \leq e \leq y_\star^{(n)} \leq y$ for every $n < \omega$. We show that e is an idempotent. By continuity of $\star$,

$$e \star e = \inf\{y_\star^{(n)} : n < \omega\} \star \inf\{y_\star^{(m)} : m < \omega\} = \inf\{y_\star^{(n+m)} : n + m < \omega\} = e.$$

□

Since each of the fundamental t-norms is continuous, we can find formulas for their residua expressed in Proposition 2.18 (rephrased from Theorem 2.1.8 of [Háj98b]), which will be used in proofs in Chapters 4 and 6.

Proposition 2.18. The following operations $\Rightarrow : [0, 1]^2 \rightarrow [0, 1]$ are the residua of the Łukasiewicz t-norm, the Gödel t-norm and the product t-norm, respectively.

1. Łukasiewicz implication: $x \Rightarrow y = 1 - x + y$ if $y < x$ and $x \Rightarrow y = 1$ otherwise. Succinctly, it is $x \Rightarrow y = \min\{1, 1 - x + y\}$.

2. Gödel implication: $x \Rightarrow y = y$ if $y < x$ and $x \Rightarrow y = 1$ otherwise.
3. Goguen implication: $x \Rightarrow y = \frac{y}{x}$ if $y < x$ and $x \Rightarrow y = 1$ otherwise.

Proof. For $x \leq y$, we proved in Lemma 2.17 that $x \Rightarrow y = 1$. Take $y < x$. By Lemma 2.17, $y = x \star (x \Rightarrow y)$.

1. By definition of the Łukasiewicz t-norm, $y = \max\{0, x + (x \Rightarrow y) - 1\}$. There are two cases $y = 0$ and $y > 0$. (1) Suppose $y = 0$. Then $x \star (x \Rightarrow 0) = 0$. Since $x > y \geq 0$, then by Example 2.11 and Remark 2.3 applied to $x \star (x \Rightarrow 0) = x \star 0$, $x \Rightarrow 0 = 0$ and by definition of residuum, $x \Rightarrow 0 = \sup\{z \mid x \star_L z \leq 0\}$. Thus, $x + z - 1 \leq 0$, and then $z \leq 1 - x$. Since the maximal z is $x \Rightarrow 0$, $x \Rightarrow 0 = 1 - x + 0$. (2) Suppose $y > 0$. Then $y = x + (x \Rightarrow y) - 1$. Thus, $x \Rightarrow y = 1 - x + y$.
2. By definition of the Gödel t-norm, $y = \min\{x, x \Rightarrow y\}$. Since $x > y$, $y = x \Rightarrow y$.
3. By definition of the product t-norm, $y = x \cdot (x \Rightarrow y)$. Thus, $\frac{y}{x} = x \Rightarrow y$ by $x > y \geq 0$.

□

Now, we will introduce the precomplement, which gives the semantics to the connective $\neg$, and find the formulas for the precomplements of the fundamental t-norms (see Proposition 2.20 rephrased from Lemma 2.1.13 of [Háj98b]). Then we look at definitions of operations $\max, \min$ as defined in terms of the t-norms and their residua (see Proposition 2.21 rephrased from Lemma 2.1.10 of [Háj98b]); they will give the semantics of the connectives $\wedge, \vee$, respectively.

Definition 2.19. The residuum $\Rightarrow$ defines its corresponding unary operation of *precomplement* $(-)x = (x \Rightarrow 0)$.

Proposition 2.20. The following operations $(-) : [0, 1] \rightarrow [0, 1]$ are precomplements of the respective t-norms.

1. Łukasiewicz negation: $(-)x = 1 - x$
2. Gödel and Goguen negations: $(-)0 = 1, (-)x = 0$ for $x > 0$.

Proof. 1. $x \Rightarrow 0 = \min\{1, 1 - x\} = 1 - x$ since $1 - x \leq 1$.

2. Directly from the definition of the respective implications.

□

Proposition 2.21. Let $L = ([0, 1], \star, \Rightarrow, \min, \max, 0, 1)$ be an algebra with the domain $[0, 1]$, a continuous t-norm $\star$, its residuum $\Rightarrow$, binary operations $\min, \max$ defined as usual, and constants $0, 1$. Then

1. $\min\{x, y\} = x \star (x \Rightarrow y)$
2. $\max\{x, y\} = \min\{((x \Rightarrow y) \Rightarrow y), ((y \Rightarrow x) \Rightarrow x)\}$

Proof. 1. Suppose that $x \leq y$. Then $\min\{x, y\} = x$ and by Lemma 2.17, $x \Rightarrow y = 1$. Thus, $\min\{x, y\} = x = x \star 1 = x \star (x \Rightarrow y)$. Now, suppose that $x > y$. Then $\min\{x, y\} = y$ and by Lemma 2.17, $y = x \star (x \Rightarrow y)$.

2. Without loss of generality, suppose that $x \leq y$; the case of $x > y$ is analogous. Then by Lemma 2.17, $x \Rightarrow y = 1$ and $1 \Rightarrow y = y$. So $y = ((x \Rightarrow y) \Rightarrow y)$. Now, obviously $x \leq x$. By Lemma 2.17, $x = y \star (y \Rightarrow x)$. Thus $y \star (y \Rightarrow x) \leq x$. By Proposition 2.16, this is equivalent to $y \leq (y \Rightarrow x) \Rightarrow x$. Finally, $\max\{x, y\} = y = \min\{y, ((y \Rightarrow x) \Rightarrow x)\} = \min\{((x \Rightarrow y) \Rightarrow y), ((y \Rightarrow x) \Rightarrow x)\}$.

□

2.3. Proto-t-norms and isomorphisms. We will generalise the notion of t-norms to intervals other than just $[0,1]$ since it will be used in the proof of strong standard completeness. We will also need the basic notion of isomorphism between proto-t-norms, which will be used in the proof of the lemma supporting the proofs of completeness of tableau calculi (Lemma 3.13) and the proof of strong standard completeness.

Definition 2.22. Let $a, b \in \mathbb{R}$ with $a < b$, and $\star : [a, b]^2 \rightarrow [a, b]$ be a function. We say that $\star$ is a *proto-t-norm* iff it satisfies T1-T3 of Definition 2.1 (recalled below) and T4', for all $x, y, z \in [a, b]$.

- | | | |
|-----|---|----------------------|
| T1 | $x \star y = y \star x$ | (commutativity) |
| T2 | $x \star (y \star z) = (x \star y) \star z$ | (associativity) |
| T3 | $x \star y \leq x \star z$ if $y \leq z$ | (monotonicity) |
| T4' | $x \star b = x$ | (boundary condition) |

We can observe that a t-norm is a proto-t-norm defined on $[0,1]$.

We will need to extend the definition of powers and the Archimedean property from t-norms to proto-t-norms (Notation 2.6 and Definition 2.10 point 2).

Definition 2.23. Let $a, b \in \mathbb{R}$ with $a < b$, and let $\star : [a, b]^2 \rightarrow [a, b]$ be a proto-t-norm. For $x \in [a, b]$ and $n < \omega$, we define $x_\star^{(n)}$ by induction: $x_\star^{(0)} = b$ and $x_\star^{(n+1)} = x_\star^{(n)} \star x$. We also say that $\star$ is Archimedean iff for each $x, y \in (a, b)$, there is $n < \omega$ such that $x^{(n)} < y$.

For a t-norm Definition 2.23 agrees with Notation 2.6 and Definition 2.10 point 2.

Definition 2.24. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $\star_1 : [a, b]^2 \rightarrow [a, b]$ and $\star_2 : [c, d]^2 \rightarrow [c, d]$ be functions.

1. An *isomorphism* from $\star_1$ to $\star_2$ is a strictly increasing bijection $f : [a, b] \rightarrow [c, d]$ such that for every $x, y \in [a, b]$ we have $f(x \star_1 y) = f(x) \star_2 f(y)$.
2. We say that $\star_1$ and $\star_2$ are *isomorphic* iff there is some isomorphism from $\star_1$ to $\star_2$.

Note that an isomorphism from $\star_1$ to $\star_2$ is just an isomorphism in the usual sense from the structure $([a, b], \star_1, \leq)$ to the structure $([c, d], \star_2, \leq)$. In Lemma 2.25, we observe that the isomorphisms preserve the important properties of functions.

Lemma 2.25. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, and let $\star_1 : [a, b]^2 \rightarrow [a, b]$ and $\star_2 : [c, d]^2 \rightarrow [c, d]$ be functions. Let f be an isomorphism from $\star_1$ to $\star_2$. Then:

1. if $\star_1$ is continuous then so is $\star_2$,
2. if $\star_1$ is a proto-t-norm then so is $\star_2$,
3. if $\star_1$ is an Archimedean proto-t-norm then so is $\star_2$.

Proof. Easy. □

In Lemma 2.26 we can observe that the reverse isomorphisms and compositions of isomorphisms are also isomorphisms.

Lemma 2.26. Let $a, b, c, d, k, l \in \mathbb{R}$ with $a < b$, $c < d$ and $k < l$, and let $\star_1 : [a, b]^2 \rightarrow [a, b]$, $\star_2 : [c, d]^2 \rightarrow [c, d]$ and $\star_3 : [k, l]^2 \rightarrow [k, l]$ be functions. Let f be an isomorphism from $\star_1$ to $\star_2$ and g be an isomorphism from $\star_2$ to $\star_3$. Then:

1. f^{-1} is an isomorphism from $\star_2$ to $\star_1$,
2. the composition $g \circ f$ is an isomorphism from $\star_1$ to $\star_3$.

Proof. Easy. □

Remark 2.27. Since $\star_1$ in Lemma 2.26 is clearly isomorphic to itself, it follows from Lemma 2.26 that the relation ‘isomorphic’ is an equivalence relation on the set of binary functions on non-trivial closed bounded intervals of $\mathbb{R}$.

In the following lemma, we give an important non-trivial example of two isomorphic proto-t-norms (cf. Lemma 2.1.23 of [Háj98b]).

Lemma 2.28. Let $\ast : [0.5, 1]^2 \rightarrow [0.5, 1]$ be given by $x \ast y = \max(xy, 0.5)$. Then $\ast$ and the Łukasiewicz t-norm are isomorphic.

Proof. It is easy to check that $\ast$ is a proto-t-norm. Take $f : [0, 1] \rightarrow [0.5, 1]$ to be given by $f(x) = 2^{x-1}$. The function f is clearly a strictly increasing bijection. Then,

$$f^{-1}(f(x) \ast f(y)) = \log_2(f(x) \ast f(y)) + 1 = \log_2(\max\{0.5, 2^{x-1}2^{y-1}\}) + 1 =$$

$$\max\{-1, (x-1) + (y-1)\} + 1 = \max\{0, x+y-1\},$$

which is obviously the Łukasiewicz t-norm. By Lemma 2.26, $*$ and the Łukasiewicz t-norm are isomorphic. $\square$

The lemma above also is an example of defining a new proto-t-norm from a given one via a strictly increasing bijection, which we will formally capture in the lemma below.

Lemma 2.29. Let $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$, let $\star_2 : [c, d]^2 \rightarrow [c, d]$ be a function and let $f : [a, b] \rightarrow [c, d]$ be a strictly increasing bijection. Define $\star_1 : [a, b]^2 \rightarrow [a, b]$ by

$$x \star_1 y = f^{-1}(f(x) \star_2 f(y)),$$

for $x, y \in [a, b]$. Then f is an isomorphism from $\star_1$ to $\star_2$.

Proof. Easy. $\square$

Remark 2.30. By Lemmas 2.25 and 2.26 point 1, if $\star_2$ in Lemma 2.29 is continuous, a proto-t-norm, or Archimedean, then so is $\star_1$.

We now illustrate an application of Lemmas 2.29 and 2.30 in constructing t-norms with desired properties.

Example 2.31. We can derive a strict t-norm isomorphic to the product t-norm. Suppose that $f(x) = 2^x - 1$ and $\star_P$ is the product t-norm. Then,

$$x * y = \log_2((2^x - 1)(2^y - 1) + 1)$$

is a t-norm, which is isomorphic to $\star_P$.

2.4. Decomposition Theorem. In this section we present an important theorem saying that any continuous t-norm $\star$ is isomorphic to a combination of the Gödel t-norm $\star_G$, the Łukasiewicz t-norm $\star_L$ and the product t-norm $\star_P$ (the Decomposition Theorem, Theorem 2.33). The nature of this combination is explained in the aforementioned theorem. The consequence of this theorem expressed in Theorems 2.35 together with 2.36 are summarised in Corollary 2.37, saying that a continuous t-norm is equivalent to the ordinal sum of a family of t-norms isomorphic to either the Łukasiewicz t-norm or the product t-norm, and will be used in constructing tableau calculi in Chapter 4 and the proof of strong standard completeness in Chapter 6. The definition of the ordinal sum is given in Definition 2.34. In the Decomposition Theorem and later we use the notion of a contact interval (also known as a component) which we define below.

Definition 2.32. Let $0 \leq u < v \leq 1$. We call $[u, v]$ a contact interval of a t-norm $\star$ if u, v are idempotents and no element of (u, v) is an idempotent.

Theorem 2.33. (Decomposition Theorem) Let $\star$ be a continuous t-norm. Then:

1. For all elements $x, y \in [0, 1]$, if there is no contact interval I such that $x, y \in I$, then $x \star y = x \star_G y$, where $\star_G$ is the Gödel t-norm.
2. For each contact interval I , the restriction of $\star$ to I is isomorphic¹ either to $\star_P$ (the product t-norm) or to $\star_L$ (the Łukasiewicz t-norm).
3. There are countably many contact intervals of $\star$.

Theorem 2.33 shows that any continuous t-norm $\star$ is isomorphic to a combination of the Gödel t-norm $\star_G$, Łukasiewicz t-norm $\star_L$ and Product t-norm $\star_P$. Now, we will define an ordinal sum of a family of t-norms and show that a continuous t-norm is the ordinal sum of t-norms that are isomorphic to either the product t-norm or the Łukasiewicz t-norm (see [MOG09]), which will help us in constructing tableau calculi and proving the strong standard completeness theorem in Chapters 4 and 6.

Definition 2.34. Let $([a_n, b_n])_{n \in \mathfrak{C}}$ with $0 \leq a_n < b_n \leq 1$ for every $n \in \mathfrak{C}$ be a family of pairwise distinct intervals such that every two distinct interval have disjoint interiors. Then the ordinal sum $\sum_{n \in \mathfrak{C}}([a_n, b_n], \star_n)$ of a family of t-norms $(\star_n)_{n \in \mathfrak{C}}$ is the function $\star : [0, 1]^2 \rightarrow [0, 1]$ defined by

$$x \star y = \begin{cases} a_n + (b_n - a_n) \cdot \left(\frac{x - a_n}{b_n - a_n} \star_n \frac{y - a_n}{b_n - a_n} \right) & \text{if } x, y \in [a_n, b_n] \\ \min(x, y) & \text{otherwise.} \end{cases} \quad (2)$$

We will now turn to Theorems 2.35 and 2.36, which together say, as expressed in Corollary 2.37, that every continuous t-norm is equal to the ordinal sum of a family of t-norms isomorphic to either the Łukasiewicz t-norm or the product t-norm (c.f. Section 2.2.2 of [MOG09]). Since the proof of Theorem 2.35 is relatively short, we will present it here, while the proof of Theorem 2.36 is Appendix B.

Theorem 2.35. Every continuous t-norm is the ordinal sum of a family of t-norms each of which is isomorphic to either the Łukasiewicz t-norm or the product t-norm.

Proof. Take an arbitrary continuous t-norm $\star$. Let $([a_n, b_n])_{n \in \mathfrak{C}}$ be the family of all contact intervals of $\star$. Then by Theorem 2.33, $\star|_{[a_n, b_n]}$, which for simplicity we will denote $\star_n$, is isomorphic to either the Łukasiewicz t-norm or the product t-norm via an isomorphism that we will denote by f_n . Note that $f_n : [a_n, b_n] \rightarrow [0, 1]$. Thus, $x \star_n y = f_n^{-1}(f_n(x) \star_I f_n(y))$, where $\star_I$ is either the Łukasiewicz or the product t-norm as given by $\star_n$.

Take an arbitrary $[a_n, b_n]$. We will show that $x \star_n y$ is $a_n + (b_n - a_n) \cdot \left(\frac{x - a_n}{b_n - a_n} \star_n \frac{y - a_n}{b_n - a_n} \right)$, where $\star_n$ is a t-norm isomorphic to either the Łukasiewicz t-norm or the product t-norm.

¹To be defined in Definition 2.24.

Let us define $h_n : [a_n, b_n] \rightarrow [0, 1]$ as $h_n(x) = \frac{x-a_n}{b_n-a_n}$ for $x \in [a_n, b_n]$. Obviously, it is a strictly increasing bijection. Now, $g_n = f_n \circ h_n^{-1} : [0, 1] \rightarrow [0, 1]$ is a strictly increasing bijection. We will define a t-norm $*_n$ from the family $\sum_{n \in \mathfrak{C}}([a_n, b_n], *_n)$, in the following way:

$$x *_n y = g_n^{-1}(g_n(x) \star_I g_n(y)),$$

where $\star_I$ is the Łukasiewicz t-norm if $*_n$ is isomorphic to the Łukasiewicz t-norm or the product t-norm if $*_n$ is isomorphic to the product t-norm and $x, y \in [0, 1]$.

By Lemma 2.29, $*_n$ is a well-defined t-norm isomorphic to either the Łukasiewicz t-norm or the product t-norm.

Of course, $f_n^{-1} = h_n^{-1} \circ g_n^{-1}$ and $f_n = g_n \circ h_n$. Thus, $x \star_n y = f_n^{-1}(f_n(x) \star_I f_n(y)) = (h_n^{-1} \circ g_n^{-1})((g_n \circ h_n)(x) \star_I (g_n \circ h_n)(y)) = a_n + (b_n - a_n) \cdot (\frac{x-a_n}{b_n-a_n} \star_n \frac{y-a_n}{b_n-a_n})$. $\square$

Now we express the converse of Theorem 2.35 in the theorem below, whose proof is in Appendix B.

Theorem 2.36. Every ordinal sum of a family of t-norms, each of which is isomorphic to either the Łukasiewicz t-norm or the product t-norm, is a continuous t-norm.

Corollary 2.37. The following two are equivalent:

1. The function $\star : [0, 1]^2 \rightarrow [0, 1]$ is a continuous t-norm.
2. The function $\star$ is the ordinal sum of a family of t-norms each of which is isomorphic to either the Łukasiewicz t-norm or the product t-norm.

Proof. By Theorems 2.35 and 2.36. $\square$

Theorem 2.38. Let $\star$ be the ordinal sum $\sum_{n \in \mathfrak{C}}([a_n, b_n], *_n)$ of t-norms $(*_n)_{n \in \mathfrak{C}}$ such that $*_n$ is either isomorphic to the Łukasiewicz t-norm or isomorphic to the product t-norm. Let for each n , g_n denote the isomorphisms for $*_n$. Then the residuum for $\star$ is given by

$$x \Rightarrow_{\star} y = \begin{cases} 1 & \text{if } x \leq y \\ a_n + (b_n - a_n)g_n^{-1}\left(g_n\left(\frac{x-a_n}{b_n-a_n}\right) \Rightarrow_I g_n\left(\frac{y-a_n}{b_n-a_n}\right)\right) & \text{if } x > y \text{ and } x, y \in [a_n, b_n] \\ y & \text{otherwise,} \end{cases} \quad (3)$$

where $\Rightarrow_I$ is either the Łukasiewicz implication if $*_n$ is isomorphic to the Łukasiewicz t-norm, or Goguen implication if $*_n$ is isomorphic to the product t-norm.

Proof. Recall that $x \Rightarrow_{\star} y = \sup\{z \in [0, 1] | x \star z \leq y\}$ since $\star$ is a continuous t-norm by Theorem 2.36. The case for $x \leq y$ is obvious. Let $x > y$.

Suppose that $x, y \in [a_n, b_n]$. We may define $x \Rightarrow_{\star} y = \max\{\sup\{z \in [0, a_n] | x \star z \leq y\}, \sup\{z \in [a_n, b_n] | x \star z \leq y\}, \sup\{z \in (b_n, 1] | x \star z \leq y\}\}$. It is easy to check that

$\sup\{z \in [0, a_n] | x \star z \leq y\} = a_n$, and $\{z \in (b_n, 1] | x \star z \leq y\} = \emptyset$. We will consider $\sup\{z \in [a_n, b_n] | x \star z \leq y\}$.

Suppose that on $[a_n, b_n]$, $\star_n$ is isomorphic to the product t-norm. Then

$$z \star x = a_n + (b_n - a_n)g_n^{-1}\left(g_n\left(\frac{z - a_n}{b_n - a_n}\right) \cdot g_n\left(\frac{x - a_n}{b_n - a_n}\right)\right),$$

and hence

$$z \leq a_n + (b_n - a_n)g_n^{-1}\left(\frac{g_n\left(\frac{y - a_n}{b_n - a_n}\right)}{g_n\left(\frac{x - a_n}{b_n - a_n}\right)}\right)$$

since g_n is strictly increasing bijection, $z \star x \leq y$ and $x > a_n$. Therefore, $x \Rightarrow_\star y = a_n + (b_n - a_n)g_n^{-1}\left(\frac{g_n\left(\frac{y - a_n}{b_n - a_n}\right)}{g_n\left(\frac{x - a_n}{b_n - a_n}\right)}\right)$ as $a_n \leq a_n + (b_n - a_n)g_n^{-1}\left(\frac{g_n\left(\frac{y - a_n}{b_n - a_n}\right)}{g_n\left(\frac{x - a_n}{b_n - a_n}\right)}\right)$.

Suppose that on $[a_n, b_n]$, $\star_n$ is isomorphic to the Łukasiewicz t-norm. Then,

$$z \star x = a_n + (b_n - a_n)g_n^{-1}\left(\max\{0, g_n\left(\frac{z - a_n}{b_n - a_n}\right) + g_n\left(\frac{x - a_n}{b_n - a_n}\right) - 1\}\right),$$

and hence

$$z \leq a_n + (b_n - a_n)g_n^{-1}\left(1 - g_n\left(\frac{x - a_n}{b_n - a_n}\right) + g_n\left(\frac{y - a_n}{b_n - a_n}\right)\right)$$

since g_n is strictly increasing bijection, $z \star x \leq y$ and $x > a_n$. Therefore $x \Rightarrow_\star y = a_n + (b_n - a_n)g_n^{-1}\left(1 - g_n\left(\frac{x - a_n}{b_n - a_n}\right) + g_n\left(\frac{y - a_n}{b_n - a_n}\right)\right)$ as $a_n \leq a_n + (b_n - a_n)g_n^{-1}\left(1 - g_n\left(\frac{x - a_n}{b_n - a_n}\right) + g_n\left(\frac{y - a_n}{b_n - a_n}\right)\right)$.

Thus,

$$x \Rightarrow_\star y = a_n + (b_n - a_n)g_n^{-1}\left(g_n\left(\frac{x - a_n}{b_n - a_n}\right) \Rightarrow_K g_n\left(\frac{y - a_n}{b_n - a_n}\right)\right).$$

Suppose now that there is no $[a_n, b_n]$ such that $x, y \in [a_n, b_n]$ and $z \in [0, 1]$ is such that $x \star z \leq y$. Then, $x \Rightarrow_\star y = y$ since by Proposition 2.4 $x \star z \leq \min\{x, z\}$ and thus we have $x \star z \leq \min\{y, z\}$ by assumption of $x > y$.

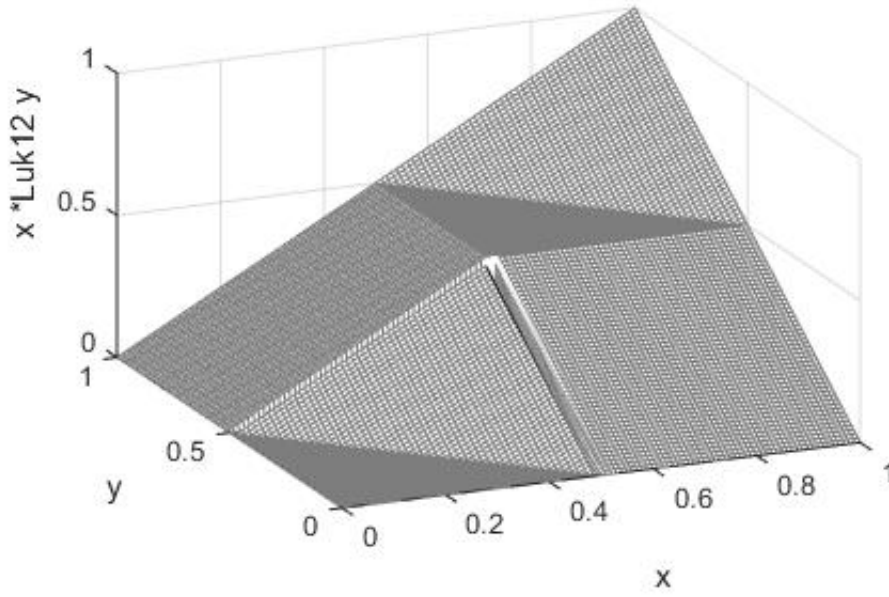
This completes the proof of the theorem. $\square$

We will now illustrate the consequence of the Decomposition Theorem and its converse in Example 2.39.

Example 2.39. In Figure 2, we represented the following t-norm:

$$x \star_{Luk12} y = \begin{cases} \max\{0, x + y - \frac{1}{2}\} & \text{if } x, y \in [0, \frac{1}{2}], \\ \max\{\frac{1}{2}, x + y - 1\} & \text{if } x, y \in [\frac{1}{2}, 1], \\ \min\{x, y\} & \text{otherwise.} \end{cases} \quad (4)$$

Figure 2: Graph of t-norm $\star_{Luk12}$



We calculate a few values:

$$\frac{1}{4} \star_{Luk12} \frac{1}{4} = \max\left\{0, \frac{1}{4} + \frac{1}{4} - \frac{1}{2}\right\} = 0$$

$$\frac{1}{3} \star_{Luk12} \frac{1}{4} = \max\left\{0, \frac{1}{3} + \frac{1}{4} - \frac{1}{2}\right\} = \frac{1}{12}$$

$$\frac{3}{4} \star_{Luk12} \frac{4}{5} = \max\left\{\frac{1}{2}, \frac{3}{4} + \frac{4}{5} - 1\right\} = \frac{11}{20}$$

$$\frac{3}{4} \star_{Luk12} \frac{3}{5} = \max\left\{\frac{1}{2}, \frac{3}{4} + \frac{3}{5} - 1\right\} = \frac{1}{2}$$

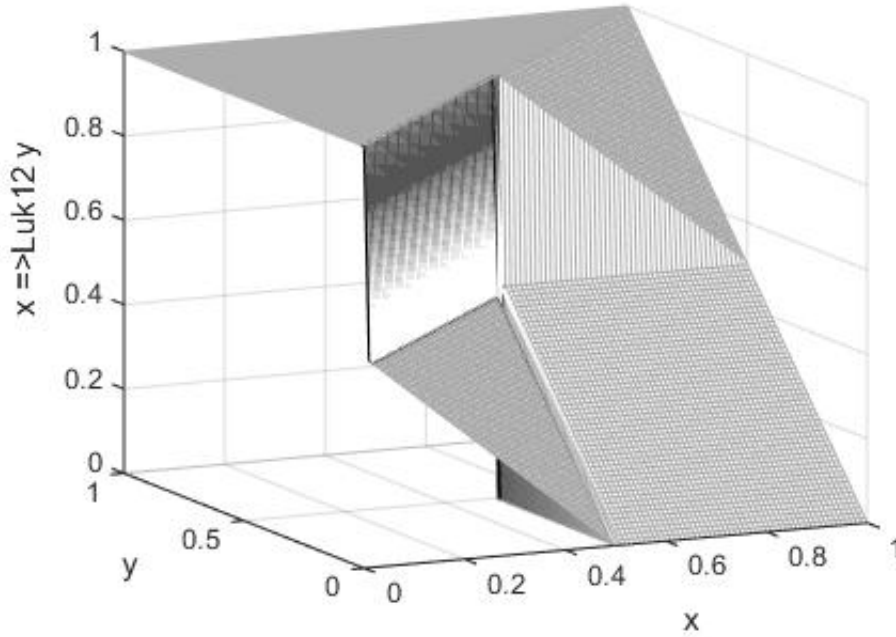
$$\frac{1}{4} \star_{Luk12} \frac{3}{4} = \min\left\{\frac{1}{4}, \frac{3}{4}\right\} = \frac{1}{4}$$

In Figure 3, we graph the residuum $\Rightarrow_{Luk12}$ of the t-norm $\star_{Luk12}$, which is given by

the following formula:

$$x \Rightarrow_{Luk12} y = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{1}{2} - x + y & \text{if } x > y \text{ and } x, y \in [0, \frac{1}{2}], \\ 1 - x + y & \text{if } x > y \text{ and } x, y \in [\frac{1}{2}, 1], \\ y & \text{otherwise.} \end{cases}$$

Figure 3: Graph of residuum $\Rightarrow_{Luk12}$



We calculate a few values:

$$\frac{1}{5} \Rightarrow_{Luk12} \frac{1}{4} = 1$$

$$\frac{1}{3} \Rightarrow_{Luk12} \frac{1}{4} = \frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{5}{12}$$

$$\frac{4}{5} \Rightarrow_{Luk12} \frac{3}{4} = 1 - \frac{4}{5} + \frac{3}{4} = \frac{19}{20}$$

$$\frac{3}{4} \Rightarrow_{Luk12} \frac{1}{4} = \frac{1}{4}$$

Note that on both intervals $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$ we defined a scaled Łukasiewicz t-norm and a scaled Łukasiewicz implication in the $\star_{Luk12}, \Rightarrow_{Luk12}$, respectively.

Let us define a special class of continuous t-norms, which we call LP-norms. They will be employed in constructing tableau calculi.

Definition 2.40. A t-norm $\star : [0, 1]^2 \rightarrow [0, 1]$ is said to be an *LP-norm* iff it is the ordinal sum of a family of t-norms each of which is either the Łukasiewicz or the product t-norm.

Note that the t-norm defined in Example 2.39 is an LP-norm. Below Remarks 2.41 and 2.42 provide an overt formulas for an LP-norm, which we will use in both tableau calculi.

Remark 2.41. If $*_n$ in the equation (2) in Definition 2.34 is the Łukasiewicz t-norm, then

$$a_n + (b_n - a_n) \cdot \left(\frac{x - a_n}{b_n - a_n} *_n \frac{y - a_n}{b_n - a_n} \right) = \max\{a_n, x + y - b_n\}.$$

If $*_n$ is the product t-norm, then

$$a_n + (b_n - a_n) \cdot \left(\frac{x - a_n}{b_n - a_n} *_n \frac{y - a_n}{b_n - a_n} \right) = a_n + \frac{(x - a_n) \cdot (y - a_n)}{b_n - a_n}.$$

Remark 2.42. If in the equation (3) in Definition 2.38, g_n is the identity and $*_n$ is the Łukasiewicz t-norm, then

$$a_n + (b_n - a_n)g_n^{-1} \left(g_n \left(\frac{x - a_n}{b_n - a_n} \right) \Rightarrow_I g_n \left(\frac{y - a_n}{b_n - a_n} \right) \right) = b_n - x + y.$$

If in the equation (3), g_n is the identity and $*_n$ is the product t-norm, then

$$a_n + (b_n - a_n)g_n^{-1} \left(g_n \left(\frac{x - a_n}{b_n - a_n} \right) \Rightarrow_I g_n \left(\frac{y - a_n}{b_n - a_n} \right) \right) = a_n + \frac{(y - a_n) \cdot (b_n - a_n)}{x - a_n}.$$

3. BACKGROUND THEORY FOR FUZZY LOGICS

This chapter recalls some fundamental notions for fuzzy logics relating to their syntax and semantics. Section 3.1 is our presentation of the already known syntax and semantics of the fuzzy logics BL, Łukasiewicz, Product, Gödel and $BL_{\Delta\sim}$, enriched by our examples presented in Section 3.2. The semantics for $BL_{\Delta\sim}$ used in Chapter 4 will eventually be different from the standard one (see e.g. [CHN11]). The reason for the choice of the semantics will be explained in Section 3.4. In this section we also prove lemmas used in the proofs of completeness of tableau calculi in Chapter 4. Section 3.3 will provide the known definitions of 1-tautology, validity and $\mathcal{K}$ -satisfiability.

3.1. Fuzzy logics. The basic fuzzy logic BL is a propositional logic in which the truth values of formulas are from the interval $[0, 1]$, where 0 represents false and 1 represents true, and the other values are intermediate. The logic was introduced by Petr Hájek in [Háj98b] as a generalisation of Łukasiewicz logic, Product logic and Gödel logic. We will also present an extended language of BL, $BL_{\Delta\sim}$ (see more about these logics in [CHN11]).

Let $PROP$ be a set of countably infinitely many propositional atoms. Note that $\bar{0}, \bar{1}$ are symbols for *falsum* and *verum*, respectively.

Definition 3.1. The sets $\mathcal{F}, \mathcal{F}^e$ of formulas of BL and $BL_{\Delta\sim}$, respectively, are defined inductively as follows.

1. $\mathcal{F}$ is the least set such that:

- $PROP \cup \{\bar{0}\} \subseteq \mathcal{F}$,
- if $\psi, \varphi \in \mathcal{F}$, then $\psi \& \varphi, \psi \rightarrow \varphi \in \mathcal{F}$.

2. $\mathcal{F}^e$ is the least set such that:

- $\mathcal{F} \subseteq \mathcal{F}^e$,
- if $\psi, \varphi \in \mathcal{F}^e$, then $\psi \& \varphi, \psi \rightarrow \varphi, \Delta\psi, \sim\psi \in \mathcal{F}^e$.

We will use the following abbreviations: the nullary connective $\bar{1}$, the unary connective $\neg$, and the binary connectives $\vee, \wedge, \underline{\vee}, \equiv$. They will denote the following formulas:

- $\bar{1}$ is $\bar{0} \rightarrow \bar{0}$,
- $\neg\varphi$ is $\varphi \rightarrow \bar{0}$,
- $\varphi \wedge \psi$ is $\varphi \& (\varphi \rightarrow \psi)$,

- $\varphi \vee \psi$ is $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$,
- $\psi \underline{\vee} \varphi$ is $\neg(\neg\psi \& \neg\varphi)$, and
- $\varphi \equiv \psi$ is $(\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$.

However, in Chapter 4 we will treat $\vee, \wedge$ and $\bar{1}$ as primitive not as abbreviations.

The connectives $\&, \wedge$ are called strong and weak conjunctions, respectively, and $\underline{\vee}, \vee$ are called strong and weak disjunctions, respectively. The unary connective, negation $\sim$, is called the involutive negation. We will call a set of formulas a *theory* (as in Definition 2.2.17 of [Háj98b]).

The semantics of the logics is based on a continuous t-norm and its derivatives as defined below. We only need to introduce a new function $\Delta : [0, 1] \rightarrow [0, 1]$, which is defined as $\Delta 1 = 1$ and $\Delta x = 0$ if $x \in [0, 1)$. This function will give the semantics of the Baaz connective Δ . This connective can be interpreted as ‘definitely true’ (see the philosophical theory called Supervaluation discussed in Chapter 1.1.1).

Definition 3.2. Let $V : PROP \rightarrow [0, 1]$ be an assignment from propositional atoms to $[0, 1]$, $\star$ be a continuous t-norm and $\Rightarrow_\star$ its residuum. Then $V_\star, V_\star^e$ defined on $\mathcal{F}, \mathcal{F}^e$, respectively, with values in $[0, 1]$ are given by:

- $V_\star^e(p) = V(p)$ if $p \in PROP$,
- $V_\star^e(\bar{0}) = 0$,
- For any $\psi, \varphi \in \mathcal{F}^e$,
 - $V_\star^e(\psi \& \varphi) = V_\star^e(\psi) \star V_\star^e(\varphi)$,
 - $V_\star^e(\psi \rightarrow \varphi) = V_\star^e(\psi) \Rightarrow_\star V_\star^e(\varphi)$,
 - $V_\star^e(\Delta \psi) = \Delta V_\star^e(\psi)$,
 - $V_\star^e(\sim \psi) = \ominus V_\star^e(\psi)$,
- $V_\star = V_\star^e \upharpoonright_{\mathcal{F}}$.

In Chapter 4, $\ominus x = 1 - x$ for all $x \in [0, 1]$.² By Proposition 2.21, we have $V_\star^e(\psi \vee \varphi) = \max\{V_\star^e(\psi), V_\star^e(\varphi)\}$ and $V_\star^e(\psi \wedge \varphi) = \min\{V_\star^e(\psi), V_\star^e(\varphi)\}$. Also, if $\star$ is the Łukasiewicz t-norm, $V_\star^e(\varphi \underline{\vee} \psi) = \min\{1, V_\star^e(\varphi) + V_\star^e(\psi)\}$.

The syntax of Łukasiewicz, Product and Gödel logics is that of BL and the semantics is given by $V_\star$, where $\star$ is the Łukasiewicz, product and Gödel t-norm, respectively. Since strong disjunction has nice properties only in Łukasiewicz logic, we will not use it in other fuzzy logics.

²Note that this is only one possible interpretation of the involutive negation (see [CHN11]).

3.2. Examples of evaluation. We will now present some examples, in which we will calculate the value of the expression $V_\star((p \rightarrow q) \rightarrow q)$ for some evaluations $V : PROP \rightarrow [0, 1]$ and some continuous t-norms. In the first example the value of the expression is $V(q)$ for every continuous t-norm $\star$. In the second we show that its value depends on the choice of a fundamental continuous t-norm and in the case of the product t-norm also on some further constraints on the assignment V . In the third example, we will present a non-fundamental continuous t-norm and calculate the values of the expression for different assignments $V : PROP \rightarrow [0, 1]$.

Example 3.3. Let $p, q \in PROP$ and $V : PROP \rightarrow [0, 1]$ be such that $V(p) \leq V(q)$. We show that

$$V_\star((p \rightarrow q) \rightarrow q) = V_\star(q)$$

for any continuous t-norm. By Lemma 2.17 points 1 and 2,

$$V_\star((p \rightarrow q) \rightarrow q) = \left(V_\star(p) \Rightarrow_\star V_\star(q) \right) \Rightarrow_\star V_\star(q) = 1 \Rightarrow_\star V_\star(q) = V_\star(q).$$

Example 3.4. Let $p, q \in PROP$ and $V : PROP \rightarrow [0, 1]$ be such that $V(p) > V(q)$. The following equation

$$V_\star((p \rightarrow q) \rightarrow q) = V_\star(p)$$

is true for the Łukasiewicz t-norm, for the product t-norm iff $V(q) > 0$ or $V(p) = 1$, and for the Gödel t-norm iff $V(p) = 1$.

Let $\star$ be the Łukasiewicz t-norm. Then by Proposition 2.18 point 1,

$$V_\star((p \rightarrow q) \rightarrow q) = 1 - (1 - V_\star(p) + V_\star(q)) + V_\star(q) = V_\star(p).$$

Take $\star$ to be the product t-norm. Then by Proposition 2.18 point 3 if $V(q) > 0$,

$$V_\star((p \rightarrow q) \rightarrow q) = V_\star(q) / (V_\star(q) / V_\star(p)) = V_\star(p),$$

and if $V(p) = 1$,

$$V_\star((p \rightarrow q) \rightarrow q) = \left(1 \Rightarrow_\star V_\star(q) \right) \Rightarrow_\star V_\star(q) = V_\star(q) \Rightarrow_\star V_\star(q) = 1 = V_\star(p).$$

Otherwise,

$$V_\star((p \rightarrow q) \rightarrow q) = (V_\star(p) \Rightarrow_\star 0) \Rightarrow_\star 0 = 0 \Rightarrow_\star 0 = 1 > V_\star(p).$$

If $\star$ is the Gödel t-norm, then by Proposition 2.18 point 2,

$$V_\star((p \rightarrow q) \rightarrow q) = (V_\star(p) \Rightarrow_\star V_\star(q)) \Rightarrow_\star V_\star(q) = V_\star(q) \Rightarrow_\star V_\star(q) = 1.$$

If $V(p) = 1$, then it is true, otherwise it is false.

Example 3.5. For different assignments $V : PROP \rightarrow [0, 1]$, we will find the values of $V_\star((p \rightarrow q) \rightarrow q)$ with the t-norm $\star$ defined as follows:

$$x \star y = \begin{cases} 2xy & \text{if } x, y \in [0, \frac{1}{2}], \\ \max\{\frac{1}{2}, x + y - 1\} & \text{if } x, y \in [\frac{1}{2}, 1], \\ \min\{x, y\}, & \text{otherwise.} \end{cases}$$

It is easy to check that $\star$ is an LP-norm (see Definition 2.40, Remarks 2.41 and 2.42) and by Theorem 2.36, it is a continuous t-norm. By Theorem 2.38, we can find its residuum

$$x \Rightarrow_\star y = \begin{cases} 1 & \text{if } x \leq y, \\ \frac{1}{2}(y/x) & \text{if } x > y \text{ and } x, y \in [0, \frac{1}{2}], \\ 1 - x + y & \text{if } x > y \text{ and } x, y \in [\frac{1}{2}, 1], \\ y, & \text{otherwise.} \end{cases}$$

Let $V : PROP \rightarrow [0, 1]$. First, we consider $V(p) \leq V(q)$. Then, the results of Example 3.3 apply. Now, suppose $V(p) > V(q)$. Then $V(p) > 0$.

Case 1: $V(p), V(q) \in [0, \frac{1}{2}]$. Then

$$V_\star(p \rightarrow q) = \frac{1}{2}V_\star(q)/V_\star(p) \geq V_\star(q)$$

and if $V_\star(q) > 0$, then

$$V_\star((p \rightarrow q) \rightarrow q) = V_\star(p).$$

If $V_\star(q) = 0$, then

$$V_\star((p \rightarrow q) \rightarrow q) = 1.$$

Case 2: $V(p), V(q) \in [\frac{1}{2}, 1]$. Then

$$V_\star((p \rightarrow q) \rightarrow q) = 1 - (1 - V_\star(p) + V_\star(q)) + V_\star(q) = V_\star(p).$$

Case 3: $V(p) \geq \frac{1}{2} > V(q)$. Then

$$V_\star(p \rightarrow q) = V_\star(q).$$

Thus

$$V_\star((p \rightarrow q) \rightarrow q) = 1.$$

3.3. 1-tautology, validity and $\mathcal{K}$ -satisfiability. In this section we will define these notions and provide some examples to illustrate them. We will also explain the choice of non-standard semantics of $\text{BL}_{\Delta\sim}$ that will be used in Section 4.3.

Chapters 5 and 6 are devoted to BL, Łukasiewicz and Product logics and will use the notions of 1-tautology and validity only. Note that Łukasiewicz and Product logics have the same syntax as BL, but their semantics is based on the Łukasiewicz and product t-norms, respectively, therefore only the notion of 1-tautology applies to them.

Definition 3.6. Let $\star$ be a continuous t-norm. A formula ψ of $\text{BL}_{\Delta\sim}$ is a 1-tautology of $\star$ iff for all assignments $V : \text{PROP} \rightarrow [0, 1]$, $V_{\star}^e(\psi) = 1$.

Note that obviously 1-tautologies of the Łukasiewicz and product t-norms are also 1-tautologies of a continuous t-norm.

Definition 3.7. A formula ψ of $\text{BL}_{\Delta\sim}$ is *valid* iff it is a 1-tautology of every continuous t-norm.

In proving soundness of the tableau calculus introduced in Section 4.2, we will try to construct a model $\mathcal{A} = ([0, 1], \star, \Rightarrow, \min, \max, 0, 1, V)$,³ where $\star$ is a continuous t-norm, $\Rightarrow$ is its residuum, and $V : \text{PROP} \rightarrow [0, 1]$ is an assignment, and witnessing that a formula $\psi \in \mathcal{F}$ is not valid. This means that we try to find a continuous t-norm $\star$ and an assignment V such that $V_{\star}(\psi) < 1$. In proving completeness of the tableau calculus, we are given a model $\mathcal{A}$ in which ψ has value less than 1.

Example 3.8 shows a valid formula.

Example 3.8. Let φ, θ be formulas of $\text{BL}_{\Delta\sim}$. Then $((\theta \rightarrow \varphi) \& \theta) \rightarrow \varphi$ is valid.

Take an arbitrary continuous t-norm $\star$ and an assignment $V : \text{PROP} \rightarrow [0, 1]$. By properties of continuous t-norms, the semantics of $\text{BL}_{\Delta\sim}$ and Proposition 2.16, $V_{\star}((\theta \rightarrow \varphi) \& \theta) \leq V_{\star}(\varphi)$. By the semantics of $\text{BL}_{\Delta\sim}$ and Proposition 2.17 point 1, this is equivalent to $V_{\star}(((\theta \rightarrow \varphi) \& \theta) \rightarrow \varphi) = 1$, which completes the proof.

The following example will demonstrate the difference between validity in BL and 1-tautology of a continuous t-norm.

Example 3.9. Let $p \in \text{PROP}$. Let $\star$ be the Łukasiewicz t-norm. We show that $\neg\neg p \rightarrow p$ is a 1-tautology of $\star$. Take any $V : \text{PROP} \rightarrow [0, 1]$. Then by Proposition 2.20 point 1 and Lemma 2.17 point 1,

$$V_{\star}(\neg\neg p \rightarrow p) = (1 - (1 - V_{\star}(p))) \Rightarrow_{\star} V_{\star}(p) = V_{\star}(p) \Rightarrow_{\star} V_{\star}(p) = 1.$$

³In Section 4.2, the primitive connectives in the fuzzy logic will be $\&, \rightarrow, \wedge, \vee, \bar{0}$ and $\bar{1}$, which explains the use of additional function symbols $\min, \max$ and the constant 1 in the model.

Let $\star$ be the Gödel t-norm and let $V : PROP \rightarrow [0, 1]$ be such that $0 < V_\star(p) < 1$. Then by Proposition 2.20 point 2 and Lemma 2.17 point 2, $V_\star(\neg p) = 0$, $V_\star(\neg\neg p) = 1$ and $V_\star(\neg\neg p \rightarrow p) = V_\star(p) < 1$. Therefore, $\neg\neg p \rightarrow p$ is not valid.

Let us focus our attention on $\mathcal{K}$ -satisfiability, a notion used in Section 4.3.

Definition 3.10. Let $\mathcal{K} \subseteq [0, 1]$. Let Ψ be a set of formulas of BL ($BL_{\Delta\sim}$) and let $\star$ be a continuous t-norm. The set Ψ is said to be $\mathcal{K}$ -satisfiable with respect to $\star$ iff there exists an assignment $V : PROP \rightarrow [0, 1]$ such that for every $\psi \in \Psi$, $V_\star(\psi) \in \mathcal{K}$. We say that a formula ψ is $\mathcal{K}$ -satisfiable with respect to $\star$ if $\{\psi\}$ is $\mathcal{K}$ -satisfiable with respect to $\star$.

We can clearly see that the notion of validity of a formula ψ is equivalent to there being no continuous t-norm with respect to which ψ is $[0, 1]$ -satisfiable.

The following example illustrates the notion of $\mathcal{K}$ -satisfiability.

Example 3.11. Take $\mathcal{K} = [0, \frac{1}{2}] \cup \{\frac{3}{4}\}$, $\Psi = \{\Delta p, \sim(p \& p), q\}$ and the t-norm $\star$ defined as follows:

$$x \star y = \begin{cases} \max\{0, x + y - \frac{1}{2}\} & \text{if } x, y \in [0, \frac{1}{2}], \\ \frac{1}{2} + 2(x - \frac{1}{2})(y - \frac{1}{2}) & \text{if } x, y \in [\frac{1}{2}, 1], \\ \min\{x, y\}, & \text{otherwise.} \end{cases}$$

It is easy to check that $\star$ is an LP-norm (see Definition 2.40, Remarks (2.41) and (2.42)) and by Theorem 2.36, it is a continuous t-norm.

We show that Ψ is $\mathcal{K}$ -satisfiable with respect to $\star$. Take $V : PROP \rightarrow [0, 1]$ such that $V(p) = \frac{1}{2}$, $V(q) = \frac{3}{4}$. Then

$$\begin{aligned} V_\star(\Delta p) &= \Delta V_\star(p) = \Delta \frac{1}{2} = 0 \in \mathcal{K}, \\ V_\star(\sim(p \& p)) &= 1 - V_\star(p \& p) = 1 - V_\star(p) \star V_\star(p) = 1 - \max\left\{0, \frac{1}{2} + \frac{1}{2} - \frac{1}{2}\right\} = \frac{1}{2} \in \mathcal{K}, \\ V_\star(q) &= \frac{3}{4} \in \mathcal{K}. \end{aligned}$$

In proving soundness of the tableau calculus introduced in Section 4.3, we will try to construct a model $\mathcal{A} = ([0, 1], \star, \Rightarrow, \min, \max, \Delta, \ominus, 0, 1, V)$,⁴ where $\star$ is an LP-norm, $\Rightarrow$ is its residuum, Δ gives the semantics to the Baaz connective, $\ominus$ gives the semantics to the involutive negation, and $V : PROP \rightarrow [0, 1]$ is an assignment, witnessing that Ψ is $\mathcal{K}$ -satisfiable with respect to $\star$. This means that we try to find an LP-norm $\star$ and an assignment V such that $V_\star(\psi) \in \mathcal{K}$ for all $\psi \in \Psi$. In proving completeness of the tableau calculus, we are given such a model $\mathcal{A}$.

⁴In Section 4.3, the primitive connectives in the fuzzy logic will be $\&, \rightarrow, \wedge, \vee, \Delta, \sim, \bar{0}$ and $\bar{1}$, which explains the use of additional function symbols $\min, \max$ and the constant 1 in the model.

3.4. Background lemmas for completeness of tableau calculi. Both tableau calculi presented in Sections 4.2 and 4.3 are constructed on the basis of LP-norms and their residua. In the case of the tableau calculus for BL (Section 4.2) we use Theorem 3 of [BHMV02], cited as Theorem 4.10, together with Lemma 3.13, to prove its completeness for all continuous t-norms (not only for its subclass of LP-norms). However, as shown in Remark 3.14 arising from the claim in the proof of Lemma 3.13, we cannot extend this result to $\text{BL}_{\Delta\sim}$ in Section 4.3 using Lemma 3.13 below. Therefore, the semantics of the strong conjunction in $\text{BL}_{\Delta\sim}$ will be given by an LP-norm.

To prove Lemma 3.13, we will need Lemma 3.12. The idea behind this lemma is that an isomorphism of continuous t-norms also preserves residuum and the constant 0, so is an isomorphism from the algebra $\mathcal{A}_1 = ([0, 1], \star_1, \Rightarrow_1, 0)$ to the algebra $\mathcal{A}_2 = ([0, 1], \star_2, \Rightarrow_2, 0)$. Since evaluating formulas of $\mathcal{F}$ via $(V_1)_{\star_1}$ is essentially evaluating them in $(\mathcal{A}_1, V_1)$, and analogously for $(V_2)_{\star_2}$ (cf. Lemma 2.3.8 of [Háj98b]), it follows that the isomorphism also preserves values of formulas of $\mathcal{F}$. However, to avoid unnecessary definition at this stage, we prove the lemma directly. We will make explicit use of algebras in Chapter 6.

Lemma 3.12. Let $\star_1, \star_2$ be continuous t-norms and let f be an isomorphism from $\star_1$ to $\star_2$. Let $V_1, V_2 : [0, 1] \rightarrow [0, 1]$ be evaluations such that $f(V_1(p)) = V_2(p)$ for every $p \in \text{PROP}$. Then $f((V_1)_{\star_1}(\psi)) = (V_2)_{\star_2}(\psi)$ for every $\psi \in \mathcal{F}$.

Proof. The cases for atomic formulas are obvious. Assume the induction hypothesis for θ, φ . Then by properties of a t-norm, the induction hypothesis, the semantics of BL and the fact that f is an isomorphism from $\star_1$ to $\star_2$, $(V_2)_{\star_2}(\theta \& \varphi) = f((V_1)_{\star_1}(\theta \& \varphi))$. Let $\psi = \theta \rightarrow \varphi$. Take an arbitrary $z \in [0, 1]$. Then by the semantics of BL and Proposition 2.16, $z \leq (V_2)_{\star_2}(\theta \rightarrow \varphi)$ iff $z \star_2 (V_2)_{\star_2}(\theta) \leq (V_2)_{\star_2}(\varphi)$. By the induction hypothesis, this is equivalent to $z \star_2 f((V_1)_{\star_1}(\theta)) \leq f((V_1)_{\star_1}(\varphi))$. Since f is a bijection, there exists $t \in [0, 1]$ such that $t = f^{-1}(z)$. Therefore, $z \star_2 f((V_1)_{\star_1}(\theta)) \leq f((V_1)_{\star_1}(\varphi))$ iff $f(t) \star_2 f((V_1)_{\star_1}(\theta)) \leq f((V_1)_{\star_1}(\varphi))$. By properties of f , this is equivalent to $t \star_1 (V_1)_{\star_1}(\theta) \leq (V_1)_{\star_1}(\varphi)$. By Proposition 2.16, the semantics of BL, we have $t \leq (V_1)_{\star_1}(\theta \rightarrow \varphi)$, and by properties of f , we have further equivalence $z \leq f((V_1)_{\star_1}(\theta \rightarrow \varphi))$. Since z is arbitrary, $(V_2)_{\star_2}(\theta \rightarrow \varphi) = f((V_1)_{\star_1}(\theta \rightarrow \varphi))$. $\square$

The following lemma is used in the proof of completeness of the tableau calculus for BL in Section 4.2. The idea is that any two ordinal sums of ‘similar’ t-norms are isomorphic, so Lemma 3.12 applies.

Lemma 3.13. Let $\star_1$ and $\star_2$ be the ordinal sums $\sum_{n \in \mathfrak{C}}([a_n, b_n], \star_{1,n})$ and $\sum_{n \in \mathfrak{C}}([a_n, b_n], \star_{2,n})$, respectively, of t-norms $(\star_{1,n})_{n \in \mathfrak{C}}$ and $(\star_{2,n})_{n \in \mathfrak{C}}$, respectively, such that for each $n \in \mathfrak{C}$, $\star_{1,n}$ and $\star_{2,n}$ are either both isomorphic to the Łukasiewicz t-norm or both isomorphic

to the product t-norm. Let $V_1 : PROP \rightarrow [0, 1]$. Then there exists $V_2 : PROP \rightarrow [0, 1]$ and a strictly increasing bijection $f : [0, 1] \rightarrow [0, 1]$ such that for every $\psi \in \mathcal{F}$,

$$(V_2)_{\star_2}(\psi) = f((V_1)_{\star_1}(\psi)).$$

Proof. Let $\star_0$ be the LP-norm $\sum_{n \in \mathfrak{C}}([a_n, b_n], \star_{0,n})$ such that $\star_{0,n}$ is the Łukasiewicz t-norm iff $\star_{1,n}$ is isomorphic to the Łukasiewicz t-norm, and $\star_{0,n}$ is the product t-norm iff $\star_{1,n}$ is isomorphic to the product t-norm. First, we will find an isomorphism f_1 from $\star_1$ to $\star_0$, and $V_0 : PROP \rightarrow [0, 1]$.

Let $h_n : [a_n, b_n] \rightarrow [0, 1]$ be the function such that $h_n(x) = \frac{x-a_n}{b_n-a_n}$ defined for each $n \in \mathfrak{C}$. Let $g_n : [0, 1] \rightarrow [0, 1]$ be a strictly increasing bijection such that $x \star_{1,n} y = g_n^{-1}(g_n(x) \star_I g_n(y))$, where $\star_I$ is either the Łukasiewicz t-norm if $\star_{1,n}$ is isomorphic to the Łukasiewicz t-norm or the product t-norm if $\star_{1,n}$ is isomorphic to the product t-norm. We define function $f_1 : [0, 1] \rightarrow [0, 1]$ in the following way: $f_1(x) = (h_n^{-1} \circ g_n \circ h_n)(x)$ if $x \in (a_n, b_n)$ for some (necessarily unique) $n \in \mathfrak{C}$ and $f_1(x) = x$, otherwise. By Lemma 2.26, the two t-norms $\star_1$ and $\star_0$ are isomorphic via f_1 .

We define $V_0 : PROP \rightarrow [0, 1]$ as $V_0(p) = f_1(V_1(p))$ for all $p \in PROP$. By Lemma 3.12 $(V_0)_{\star_0}(\psi) = f_1((V_1)_{\star_1}(\psi))$ for all $\psi \in \mathcal{F}$.

Similarly, we can find a strictly increasing bijection $f_2 : [0, 1] \rightarrow [0, 1]$ and $V_2 : PROP \rightarrow [0, 1]$ such that $V_2(p) = f_2^{-1}(V_0(p))$ for all $p \in PROP$. Then $f_2^{-1}((V_0)_{\star_0}(\psi)) = (V_2)_{\star_2}(\psi)$ for all $\psi \in \mathcal{F}$. Therefore f is $f_2^{-1} \circ f_1$ and V_2 is $(f_2^{-1} \circ f_1)(V_1)$.

This completes the proof of the lemma. $\square$

Remark 3.14. The statement of Lemma 3.13 can be proven for BL enriched by the connective Δ . It cannot be proved for any extension of BL with the connective $\sim$ as we would need that $f_1((V_1)_{\star_1}(\sim \varphi)) = (V_0)_{\star_0}(\sim \varphi)$ to hold for any $\varphi \in \mathcal{F}$.

To show that this can fail, take $\star_1$ to be the ordinal sum $\sum_{n=1}^2([a_n, b_n], \star_{1,n})$, where $a_1 = 0, b_1 = a_2 = \frac{1}{2}, b_2 = 1$ and $\star_{1,1}$ is isomorphic to the product t-norm via function $g_1 : [0, 1] \rightarrow [0, 1]$ and $\star_{1,2}$ is isomorphic to the Łukasiewicz t-norm via function $g_2 : [0, 1] \rightarrow [0, 1]$, where $g_1(x) = g_2(x) = \sqrt{x}$. Thus:

$$h_1(x) = \frac{x-0}{1/2-0} = 2x,$$

$$h_1^{-1}(x) = \frac{1}{2}x,$$

$$h_2(x) = \frac{x-1/2}{1-1/2} = 2x - 1,$$

$$h_2^{-1}(x) = \frac{1}{2}(x + 1).$$

Thus $f_1(x) = \frac{1}{2}\sqrt{2x}$ if $x \in [0, \frac{1}{2}]$ and $f_1(x) = \frac{1}{2}(\sqrt{2x-1} + 1)$ if $x \in [\frac{1}{2}, 1]$. Note that $f_1(\frac{1}{2}) = \frac{1}{2}$ as required.

Take $V_1(p) = \frac{1}{3}$, where $p \in PROP$.

$$\begin{aligned} f((V_1)_{\star_1}(\sim p)) &= f_1(\ominus V_1(p)) = f_1(1 - V_1(p)) = f_1(1 - \frac{1}{3}) = f_1(\frac{2}{3}) = \frac{1}{2}(\sqrt{2 \cdot \frac{2}{3}} - 1 + 1) \\ &= \frac{3 + \sqrt{3}}{6} \approx 0.789 \end{aligned}$$

$$\begin{aligned} (V_0)_{\star_0}(\sim p) &= \ominus V_0(p) = 1 - V_0(p) = 1 - f_1(V_1(p)) = 1 - f_1(\frac{1}{3}) = 1 - \frac{1}{2}\sqrt{2 \cdot \frac{1}{3}} \\ &= \frac{6 - \sqrt{6}}{6} \approx 0.592 \end{aligned}$$

Therefore $f_1((V_1)_{\star_1}(\sim p)) \neq (V_0)_{\star_0}(\sim p)$.

4. TABLEAUX FOR FUZZY LOGICS

In this chapter we will present our approach to constructing tableaux calculi for BL and $BL_{\Delta\sim}$ logics. The idea behind the construction is to use the converse of the consequence of the Decomposition Theorem (see Theorem 2.36) in order to substitute selected subformulas of the formulas at the root of the tableau involving the strong conjunction and implication connectives (and other relevant connectives if necessary) with arithmetic expressions. For example, let p, q be atomic and suppose that $p \& q < \frac{1}{2}$ is at the root. Then $p \& q < \frac{1}{2}$ can be substituted by $\max\{0, p + q - 1\} < \frac{1}{2}$ if a branch with the Łukasiewicz t-norm is selected. Once all subformulas are substituted and there are no more logical connectives, we have inequalities. If the inequalities can be solved, the formula at the root has a model, and if they cannot, then we show that a model does not exist. In our example, we can have $p = \frac{1}{3}, q = 1$, which is not a unique model. If the example was $p \& q \rightarrow p < 1$, obtaining no model for $p \& q \rightarrow p < 1$ would mean that $p \& q \rightarrow p$ is valid. The examples presented here are oversimplified and a lot of steps and details are omitted. The procedure will be further explained in Informal Descriptions in Sections 4.2.1 and 4.3.1, which precede a formal definition for each tableau.

4.1. Related Works. Research in tableau systems for many-valued logics started soon after Gentzen introduced a proof-theoretic system in the 1930s, though these two systems exhibit some differences. The aim for both tableau systems and Gentzen systems is to construct a proof of a formula or a counter-model for it through automatic deduction; in their modern form Gentzen systems use hypersequents in their rules while tableau systems use inequalities, equations and formulas. There are several approaches to constructing tableau systems for finitely-valued systems (see [Häh93], [HEI97], [Häh99]) and infinitely-valued logical systems ([Häh99], [Oli03]), the latter developed for Łukasiewicz logic with Kripke semantics developed in [Urq86] and [Sco74]. A simplified version of Gentzen systems was introduced by [RS59] and was adapted as dual tableaux, top-down validity checkers (at the root there is a formula whose truth value is 1) which are dual to unsatisfiability checkers (at the root we have a formula whose truth value is less than 1). Both systems were proved equivalent for first-order logic with identity and without function symbols (see [GPO07]). An overview of current research in dual tableaux, which include tableaux for modal, temporal, many-valued and fuzzy logics, is in [OGP11], [GPO11]. An overview of the state of the art of research in Gentzen systems for fuzzy logics is in [MOG09].

In 2003 Montagna, Pinna and Tiezzi developed a tableau calculus for the fuzzy logics BL and BL_{Δ} based on sequents of metaformulas ([MPT03]), which they call TBL. TBL is based on semantics of ordinal sums of Łukasiewicz t-norms each scaled

to $[0, 1], [1, 2], \dots, [n-1, n]$, where n is a positive integer, and the validity of a formula $\varphi \circ \psi$ ($\circ \in \{\&, \rightarrow\}$ and $\varphi, \psi \in \mathcal{F}$) is reduced by the rules to the validity of φ and ψ . Their system is much more complex both in terms of number of rules and their complexity than our tableau calculus for the fuzzy logic BL presented in Section 4.2; it takes them at least twice as much space to explain the system. Also, the link between the semantics and the rules is less transparent in TBL than in our system. However, similarly to our tableau for BL, it requires solving a set of linear inequalities to confirm the validity of the formula at the root or construct a counter-model.

In [BM08], first published in 2006 in *the arXiv* (a repository of electronic preprints), Bova and Montagna introduced a proof system for Hájek's logic BL based on a relational hypersequents framework called RHBL, similar to TBL and based on the same semantics, but with the possible number of nodes in a branch reduced to at most n , where n is the number of connectives in the formula at the root, which demonstrates lowering the upper bounds with respect to the provability and unprovability problems of BL. This is achieved by our tableau as well. The complexity of the rules and their transparency has not improved in comparison with TBL.

In [Vet07], Vetterlein constructed analytic proof calculi based on relational hypersequents introduced in [BM08] for the logic ML, called rHML, whose language contains $\&, \rightarrow$ and an additional connective ∇ , and whose semantics is based on finite ordinal sums of Łukasiewicz t-norms in addition to the interpretation of ∇ as a function mapping a truth value t to the greatest idempotent below t . This operator ∇ is Montagna's storage operator (see [Mon07]) applied to ordinal multiples of Łukasiewicz algebras, since the new operator excludes those ordinal sums with the product algebra (see [Vet07] p. 5). He claims that the validity of a formula with respect to BL in RHBL and rHML can be effectively decided, with a decision procedure that is exponential.

Then, Vidal, Bou and Godo engaged in ongoing work on an SMT-based solver for a wide family of continuous t-norm based fuzzy logics ([VBG12]). Their solver tests satisfiability, tautologicity and logical consequence of formulas, though in this particular paper it is not clear how this was achieved and whether their solver is sound and complete as there is no proof of that in their paper.

In 2013 Kułacka, Pattinson and Schröder developed a labelled tableau algorithm for Łukasiewicz Fuzzy $\mathcal{ALC}$ that calls only on (pure) linear programming, and this only to decide atomic clashes ([KPS13]). Our approach to tableau calculi presented in Sections 4.2 and 4.3 is similar, though applied to the propositional fuzzy logic BL and not a modal Łukasiewicz logic.

The aforementioned authors also have not tackled the problem of extending their calculus to finitely many formulas and $\mathcal{K}$ -satisfiability, which we will present in Section 4.3. In this section we also make a first attempt to create a tableau calculus for the logic $BL_{\Delta, \sim}$ with semantics of LP-norms.

4.2. Tableau calculus for fuzzy logic BL. In this part of the thesis we present a tableau calculus for BL. We show that it is sound and complete with respect to continuous t-norms, and demonstrate the refutational procedure and the search for models procedure on selected examples. The idea of the calculus is based on the converse of the consequence of the Decomposition Theorem for a continuous t-norm, by which this operation is shown to be equivalent to the ordinal sum of a family of t-norms defined on countably many intervals. The content of this section originated in [Kuł14a], presenting ideas which are entirely novel. In the paper we only show that closing a tableau for a formula is equivalent to showing validity of the formula with respect to LP-norms. Now, having proved Lemma 3.13, we can also show that a formula is valid with respect to every continuous t-norm. In the approach presented here, however, we use Theorem 3 of [BHMV02] (cited as Theorem 4.10), which says that if a formula is a tautology of the ordinal sum of all finite families of Łukasiewicz t-norms, then it is valid. Therefore, we will only need to use Branch Expansion Rules based on the Łukasiewicz t-norm and its residuum (see Definition 4.3).

4.2.1 Defining the calculus

In this section we will introduce a tableau calculus for showing that a formula ψ of BL is valid or that there is a counter-model, in which the truth value of the formula ψ is less than 1. First we define a tableau formula, then we define a translation function, by which we will get a term for a tableau formula at the root of a tree (also defined below) from ψ .

Definition 4.1. (Tableau formula)

Let $L_0 = Par \cup \{+, -, \min, \max, \leq, <, 0, 1\}$ and $L_1 = L_0 \cup \{\star, \Rightarrow\}$ be signatures, where Par is a set of constants (parameters), $0, 1$ are constants, $+, -, \min, \max, \star, \Rightarrow$ are binary function symbols and $\leq, <$ are binary relation symbols. Let Var be an infinite set of variables.

1. If x, y are L_1 -terms, then $x \leq y, x < y, x = y$ are tableau formulas.
2. Let E be a set of tableau formulas $s \leq t, s < t, s = t$, where s, t are L_0 -terms. Let $\sigma : Var \rightarrow [0, 1]$ be a mapping, and $\mathcal{M} = (\mathbb{R}, +, -, \min, \max, \leq, <, 0, 1, \rho)$ be an L_0 -structure, where $+, -, \min, \max, 0, 1, \leq, <$ are interpreted as usual and $\rho : Par \rightarrow [0, 1]$ is a function. The pair $(\mathcal{M}, \sigma)$ is a *solution* of E iff $\mathcal{M}, \sigma \models \bigwedge E$. By convention $\bigwedge \emptyset = \top$ (verum).⁵ We denote by $\llbracket z \rrbracket^{\mathcal{M}, \sigma}$ the value of L_0 -term z in $\mathcal{M}$ under the assignment σ .

Definition 4.2. (Translation function)

Let $\mathcal{F}$ be the set of formulas of BL and $\mathbb{T}$ be the set of L_1 -terms. Let $\mu : PROP \rightarrow Var$ (we will write $\mu(p)$ as μ_p) be a one-to-one mapping assigning variables to propositional

⁵Note that $\bigwedge$ is a symbol interpreted as conjunction in two-valued logic.

atoms. Then, we define a translation function $\tau : \mathcal{F} \rightarrow \mathbb{T}$, inductively:

- | | |
|--|---|
| 1. $\tau(\bar{0}) = 0, \tau(\bar{1}) = 1,$ | 4. $\tau(\psi \rightarrow \varphi) = \tau(\psi) \Rightarrow \tau(\varphi),$ |
| 2. $\tau(p) = \mu_p$ for every $p \in PROP,$ | 5. $\tau(\psi \vee \varphi) = \max\{\tau(\psi), \tau(\varphi)\},$ |
| 3. $\tau(\psi \& \varphi) = \tau(\psi) \star \tau(\varphi),$ | 6. $\tau(\psi \wedge \varphi) = \min\{\tau(\psi), \tau(\varphi)\}.$ |

Note that $\neg, \underline{\vee}, \equiv$ are treated as abbreviations. We will call the terms defined in points 1 and 2 *atomic terms*.

Let us recall some definitions from graph theory, which we will modify for the purposes of defining the calculus. A *graph* is a structure (N, E) , where N is a set of nodes, and E is a set of edges such that $E \subseteq N \times N$ and $\neg E(n, n)$ for all $n \in N$. A node $n' \in N$ is a *successor* of $n \in N$ iff there is an edge e such that $(n, n') = e$. A node $n' \in N$ is a *predecessor* of $n \in N$ iff there is an edge e' such that $(n', n) = e'$. A *path* from $n \in N$ is a sequence of nodes $n_0 = n, n_1, \dots, n_k, \dots$ such that $k \geq 0$ and n_i is a predecessor of n_{i+1} for $i = 0, 1, \dots, k, \dots$. A *leaf* is a node with no successors, and a *root* is a node with no predecessors. A *branch* is either a path from a root to a leaf if the latter exists, or otherwise an infinite path from a root. We will call a *tree* an acyclic connected graph (N, E) , in which there is exactly one root and if a node is not a root, then it has exactly one predecessor. In a tree (N, E) the *depth* of a node $n \in N$, denoted by $d(n)$, is the number of nodes on the unique path from the root to n . The *height* of a node $n \in N$ within a branch $\mathcal{B}$, denoted by $h(n, \mathcal{B})$, is the number of nodes on the path from n to a leaf of branch $\mathcal{B}$ if $\mathcal{B}$ is finite, and undefined if $\mathcal{B}$ is infinite.

Informal description of the tableau calculus. The idea behind our tableau calculus is to try to construct a model in which the value of the initial formula is less than 1. The t-norm in the model can be taken to be an ordinal sum of finitely many Łukasiewicz t-norms represented in the tableau by selecting parameters which, after applying the map $\rho : Par \rightarrow [0, 1]$, will become endpoints of the intervals in the ordinal sum. We will do it in the following way:

1. We translate the initial formula into an L_1 -term t using the translation τ and stipulate (in the root) that $t < 1$.
2. We choose an L_1 -term $x \star y$ or $x \Rightarrow y$, occurring in the current node of the tableau, where x, y are L_0 -terms.
3. We consider the following three scenarios, which correspond to branching in the tableau:

- (a) x, y belong to the same Łukasiewicz contact interval (see Definition 2.32). We select a new pair of parameters a, b with $a < b$ to define the endpoints of this contact interval and place them together between other pairs introduced earlier, or in front of all them, or at the end of all of them. Alternatively, they may be the same as another pair. We place x, y in an order (if necessary) between the new parameters and substitute an appropriate term for all terms that are the same as the one selected. We can do this, because $\star, \Rightarrow$ are defined on a single contact interval by simple L_0 -terms. Thus, we have eliminated one occurrence of $\star$ or $\Rightarrow$.
 - (b) There is a parameter between x and y . We consider all the cases that satisfy this condition. We also mark x, y so later they will not be accidentally placed in the same interval, which we illustrate in Example 4.7. We substitute an appropriate term for all terms that are the same as the one selected. Again, the values of $\star, \Rightarrow$ on elements of $[0, 1]$ in different Łukasiewicz contact intervals are defined by simple L_0 -terms, and we have eliminated one occurrence of $\star$ or $\Rightarrow$.
 - (c) In case of $x \Rightarrow y$ when $x \leq y$, we substitute 1 for all such terms.
4. We proceed until all terms are L_0 -terms.
 5. Then we solve the linear inequalities to find a model if one exists. If the inequalities have no solution, then no model exists and the initial formula is valid.

Point 2 above is described formally in Definition 4.3. This completes the informal description.

Definition 4.3. A *tableau* $\mathcal{T}$ for a formula ψ of BL is a tree whose nodes are sets of tableau formulas and whose root is $\{\tau(\psi) < 1\}$, and on which the branch expansion rules have been fully applied. The rules of branch expansion are defined below, where $x, y, 0, 1$ are L_0 -terms, and Γ is a set of tableau formulas. Multiple inequalities should be understood in the usual way, e.g. instead of writing $a \leq c, c = d, d < f$, we write $a \leq c = d < f$.

A term t is said to *occur* in a node Γ if t is a subterm of some term occurring in some tableau formula in Γ .

Now, we choose a leaf Γ of the current tree and an ‘active’ term $x \star y$ or $x \Rightarrow y$ occurring in Γ , where x, y are L_0 -terms. We apply each of Cases 1-11 below to Γ wherever they are applicable. We continue the procedure until all formulas in leaves are L_0 -formulas.

The following Cases 1-5 are for selecting parameters. Suppose that parameters $a_0 < b_0 \leq a_1 < \dots \leq a_{n-1} < b_{n-1}$ ($n \geq 0$) have been selected in the previous steps leading to Γ . Note that we can rename the parameters consistently wherever they occur in the

node Γ , and if some of them are the same (as a result of applying Case 2 below), we give them the same name.

Let $a, b \in \text{Par} - \{a_i, b_i | 0 \leq i \leq n-1\}$ be distinct. When $n \geq 1$ we define an *inclusion set* $\mathcal{I}$ to be any of the following sets:

Case 1. $\{0 \leq a < b \leq a_0\}$,

Case 2. $\{a = a_i < b = b_i\}$ for each $0 \leq i \leq n-1$,

Case 3. $\{b_i \leq a < b \leq a_{i+1}\}$ for each $0 \leq i \leq n-2$,

Case 4. $\{b_{n-1} \leq a < b \leq 1\}$.

If no parameters have been selected in the previous steps (i.e. if $n = 0$), then the unique inclusion set $\mathcal{I}$ is

Case 5. $\{0 \leq a < b \leq 1\}$.

The motivation for the following conditions is to take care of the cases when the two L_0 -terms x, y in the active term have values in different Łukasiewicz contact intervals. Consequently they are not in the same interval (a_i, b_i) (unlike in the cases above), which is equivalent to either x is not in any such interval (Cases 6, 9, 10) or x is in such an interval and y is not in the same one (the other cases). We define an *exclusion set* $\mathcal{J}$ to be any of the following sets:

Case 6. $\{0 \leq x \leq a_0\}$,

Case 7. $\{a_i \leq x \leq b_i, y \leq a_i\}$ for each $0 \leq i \leq n-1$,

Case 8. $\{a_i \leq x \leq b_i, b_i \leq y\}$ for each $0 \leq i \leq n-1$,

Case 9. $\{b_i \leq x \leq a_{i+1}\}$ for each $0 \leq i \leq n-2$,

Case 10. $\{b_{n-1} \leq x \leq 1\}$.

If no parameters have been selected in the previous steps, then the unique exclusion set $\mathcal{J}$ is the empty set (Case 11.). Note that the exclusion sets include only already chosen parameters $a_0, b_0, \dots, a_{n-1}, b_{n-1}$.

Marking Procedure. If Cases 6, 9, 10 or 11 are applied in branch $\mathcal{B}$ at node Γ to the active term with L_0 -terms x, y , then we mark x, y as $x^{\Gamma'}, y^{\Gamma'}$ at the nodes Γ' expanding by these cases from Γ . We have decided that x, y are not in the same contact interval and call them *marked*.

Branch expansion rules. Rules $(\star)$ and $(\Rightarrow)$ are used in the case when the active terms are $x \star y$ and $x \Rightarrow y$, respectively. For rule $(\star)$ we have subrules L., min., and for rule $(\Rightarrow)$ additionally a subrule All. For the subrule L. of all rules, we have Cases 1-5, and for the subrule min., we have Cases 6-11. We obtained the branch expansion rules using Remarks 2.41 and 2.42.

We will use the notation $\gamma[v/t]$ to denote the result of substituting the term v for

each occurrence of the term t (if any) in the formula γ .

Rule ($\star$): A branch with the leaf Γ expands following the subrules:

L. $\mathcal{I} \cup \{a \leq x \leq b, a \leq y \leq b\} \cup \{\gamma[\max\{a, x + y - b\}/x \star y] : \gamma \in \Gamma\}$ for each inclusion set $\mathcal{I}$,

min. $\mathcal{J} \cup \{\gamma[\min\{x, y\}/x \star y] : \gamma \in \Gamma\}$ for each exclusion set $\mathcal{J}$.

Rule ($\Rightarrow$): A branch with the leaf Γ expands following the subrules:

All. $\{x \leq y, \gamma[1/x \Rightarrow y] : \gamma \in \Gamma\}$,

L. $\mathcal{I} \cup \{a \leq y < x \leq b\} \cup \{\gamma[b - x + y/x \Rightarrow y] : \gamma \in \Gamma\}$ for each inclusion set $\mathcal{I}$,

min. $\mathcal{J} \cup \{y < x\} \cup \{\gamma[y/x \Rightarrow y] : \gamma \in \Gamma\}$ for each exclusion set $\mathcal{J}$.

Note that each of the branch expansion rules applied to Γ adds several new nodes to the current tree, all of which are the successors of Γ and whose exact number depends on the number of parameters already at Γ as more of Cases 1-11 will be used.

This completes the definition of a tableau.

In the following definition we will define a ‘nice’ solution of an L_0 -term of a node. The reason for this definition is that, as demonstrated in Example 4.7, at some node we can have terms x, y for which we decide that they are not in the same interval, thus they are marked, yet because of further branch expansion, Cases 1-5 could have been applied to active terms involving x, y and thus they could be in the same interval. Since this is clearly a contradiction, we will close such a branch (see Definition 4.4).

Definition 4.4. For each branch $\mathcal{B}$ of a tableau $\mathcal{T}$ and each node $n \in \mathcal{B}$, we consider the set $n \upharpoonright_{L_0}$ of L_0 -formulas in n .

1. A solution $(\mathcal{M}, \sigma)$ of $n \upharpoonright_{L_0}$ is *nice* iff for every marked x^Γ, y^Γ of L_0 -terms, where $\Gamma \in \mathcal{B}$, there are no parameters a, b with $a < b$ occurring at any node of $\mathcal{B}$ such that we have $\llbracket a \rrbracket^{\mathcal{M}, \sigma} \leq \llbracket x^\Gamma \rrbracket^{\mathcal{M}, \sigma}, \llbracket y^\Gamma \rrbracket^{\mathcal{M}, \sigma} \leq \llbracket b \rrbracket^{\mathcal{M}, \sigma}$.
2. We say that $\mathcal{B}$ is *closed* if for some node $n \in \mathcal{B}$, $n \upharpoonright_{L_0}$ has no nice solution, otherwise it is *open*. Tableau $\mathcal{T}$ is *closed* if it only contains closed branches.⁶ A tableau $\mathcal{T}$ is *open* if it has an open branch.

4.2.2 Examples

Now we will show a tableau constructed for the prelinearity axiom with propositional atoms to illustrate the applicability of the rules. The example was generated by a Python

⁶By Tarski’s theorem [Tar51] on decidability of the first-order theory of $(\mathbb{R}, +, \cdot)$, it is decidable whether a tableau is closed.

script, which is included in Appendix A. The names for the nodes consist of 1 (root) followed by pairs of letters and numbers. These pairs indicate the depth of the node: e.g. 1L0 has depth 1, 1L0L2M0 has depth 3. The letters show the branch extension rules that have been used so far: e.g. 1L0A0M0 means that Rule L has been used for substitution of the first connective, then Rule All for the next connective, and Rule min for the third connective. Numbers that follow each letter refer to Cases 1-11; the actual number of them depends on the previous and current rules being used and on the number of parameters at the preceding node:

1. The only number that follows A is 0.
2. After L, 0 indicates Case 5 if no parameters were introduced in the previous steps, or Case 1 if there were. Then $1, \dots, n$ indicate all n Cases 2, numbers $n+1, \dots, n+m$ all m Cases 3, and $n+m+1$ Case 4.
3. After M, 0 indicates Case 11 if no parameters were introduced in the previous steps, or Case 6 if there were. Then $1, \dots, n$ indicate all n Cases 7, numbers $n+1, \dots, n+m$ all m Cases 8, $n+m+1, \dots, n+m+p$ all p Cases 9, and $n+m+p+1$ Case 10.

In Examples 4.5 and 4.6 we omitted marking the subterms of tableau formulas (see Marking Procedure in Definition 4.3), which we will demonstrate in Example 4.7.

Example 4.5. Let us build a tableau for the formula $(p \rightarrow q) \vee (q \rightarrow p)$. Due to the length of the example, we included only the parts that seem to illustrate rules and cases well. In Table 1, we show the partial tableau extending from node 1 (root) of depth 1 and Table 2 shows the partial tableau extending from the node 1L0.

Table 1: Partial tableau extending from 1 of depth 1.

Node n	Formulas in n	Rule applied
Start of branches 1A0A0, 1A0L0, 1A0M0		
1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$	
1A0	$\mu_p \leq \mu_q, \max\{1, \mu_q \Rightarrow \mu_p\} < 1$	Rule $(\Rightarrow)$ All.

Continued on next page

Table 1 – continued from previous page

Node n	Formulas in n	Rule applied
Start of branches 1L0A0, 1L0L0, 1L0L1, 1L0L2, 1L0M0, 1L0M1, 1L0M2, 1L0M3		
1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$	Rule ($\Rightarrow$) L. Case 5
1L0	$0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_q \Rightarrow \mu_p\} < 1$	

Start of branches 1M0A0, 1M0L0, 1M0M0

1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$	Rule ($\Rightarrow$) min. Case 11
1M0	$\mu_q < \mu_p, \max\{\mu_q, \mu_q \Rightarrow \mu_p\} < 1$	

Table 2: Partial tableau extending from 1L0 of depth 2.

Node n	Formulas in n	Rule applied	Branch Closure
Branch 1L0A0			
1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$	Rule ($\Rightarrow$) L. Case 5	
1L0	$0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_q \Rightarrow \mu_p\} < 1$		
1L0A0	$\mu_q \leq \mu_p,$ $0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, 1\} < 1$	Rule ($\Rightarrow$) All.	closed

Continued on next page

Table 2 – continued from previous page

Node n	Formulas in n	Rule applied	Branch closure

Branch 1L0L0

1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$		
1L0	$0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_q \Rightarrow \mu_p\} < 1$	Rule ($\Rightarrow$) L. Case 5	
1L0L0	$0 \leq a_1 < b_1 \leq a_0, a_1 \leq \mu_p < \mu_q \leq b_1,$ $0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, b_1 - \mu_p + \mu_q\} < 1$	Rule ($\Rightarrow$) L. Case 1	closed

Branch 1L0L1

1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$		
1L0	$0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_q \Rightarrow \mu_p\} < 1$	Rule ($\Rightarrow$) L. Case 5	
1L0L1	$a_1 = a_0 < b_0 = b_1, a_1 \leq \mu_p < \mu_q \leq b_1,$ $0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, b_1 - \mu_p + \mu_q\} < 1$	Rule ($\Rightarrow$) L. Case 2	closed

Branch 1L0L2

1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$		
1L0	$0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$	Rule ($\Rightarrow$) L. Case 5	

Continued on next page

Table 2 – continued from previous page

Node n	Formulas in n	Rule applied	Branch closure
1L0L2	$\max\{b_0 - \mu_q + \mu_p, \mu_q \Rightarrow \mu_p\} < 1$ $b_0 \leq a_1 < b_1 \leq 1, a_1 \leq \mu_p < \mu_q \leq b_1,$ $0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, b_1 - \mu_p + \mu_q\} < 1$	Rule $(\Rightarrow)$ L. Case 4	closed

Branch 1L0M0

1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$		
1L0	$0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_q \Rightarrow \mu_p\} < 1$	Rule $(\Rightarrow)$ L. Case 5	
1L0M0	$0 \leq \mu_q \leq a_0, \mu_p < \mu_q,$ $0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_p\} < 1$	Rule $(\Rightarrow)$ min. Case 6	closed

Branch 1L0M1

1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$		
1L0	$0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_q \Rightarrow \mu_p\} < 1$	Rule $(\Rightarrow)$ L. Case 5	
1L0M1	$a_0 \leq \mu_q \leq b_0, \mu_p \leq a_0, \mu_p < \mu_q,$ $0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_p\} < 1$	Rule $(\Rightarrow)$ min. Case 7	closed

Branch 1L0M2

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Table 2 – continued from previous page

Node n	Formulas in n	Rule applied	Branch closure
1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$		
1L0	$0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_q \Rightarrow \mu_p\} < 1$	Rule $(\Rightarrow)$ L. Case 5	
1L0M2	$a_0 \leq \mu_q \leq b_0, b_0 \leq \mu_p, \mu_p < \mu_q,$ $0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_p\} < 1$	Rule $(\Rightarrow)$ min. Case 8	closed

Branch 1L0M3

1	$\max\{\mu_p \Rightarrow \mu_q, \mu_q \Rightarrow \mu_p\} < 1$		
1L0	$0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_q \Rightarrow \mu_p\} < 1$	Rule $(\Rightarrow)$ L. Case 5	
1L0M3	$b_0 \leq \mu_q \leq 1, \mu_p < \mu_q,$ $0 \leq a_0 < b_0 \leq 1, a_0 \leq \mu_q < \mu_p \leq b_0,$ $\max\{b_0 - \mu_q + \mu_p, \mu_p\} < 1$	Rule $(\Rightarrow)$ min. Case 10	closed

This tableau is closed as all its branches are closed (once all branches are extended). All branches are closed since all leaves contain $\mu_p < \mu_q$ and either $\mu_q < \mu_p$ or $\mu_q \leq \mu_p$.

Example 4.6. In this example we show a branch for the tableau for formula $r \rightarrow (p \& q)$, which remains open and we will construct $\sigma : VAR \rightarrow [0, 1]$ and $\rho : PAR \rightarrow [0, 1]$ that satisfy the leaf of the branch.

Table 3: Branch 1L0M0.

Node n	Formulas in n	Rule applied	Branch Closure
Branch 1L0M0			
1	$\mu_r \Rightarrow (\mu_p \star \mu_q)$		
1L0	$0 \leq a_0 < b_0 \leq 1,$ $a_0 \leq \mu_p \leq b_0, a_0 \leq \mu_q \leq b_0,$ $\mu_r \Rightarrow \max\{a_0, \mu_p + \mu_q - b_0\} < 1$	Rule $(\star)$ L. Case 5	
1L0M0	$b_0 \leq \mu_r \leq 1,$ $\max\{a_0, \mu_p + \mu_q - b_0\} < \mu_r,$ $0 \leq a_0 < b_0 \leq 1,$ $a_0 \leq \mu_p \leq b_0, a_0 \leq \mu_q \leq b_0,$ $\max\{a_0, \mu_p + \mu_q - b_0\} < 1$	Rule $(\Rightarrow)$ min. Case 10	open

A solution is $\sigma : VAR \rightarrow [0, 1]$ such that $\sigma(\mu_p) = \frac{1}{4}, \sigma(\mu_q) = \frac{1}{2}, \sigma(\mu_r) = \frac{3}{4}$ and a map $\rho : Par \rightarrow [0, 1]$ is such that $\rho(a_0) = 0, \rho(b_0) = \frac{1}{2}$. It is worth noticing that there are other σ, ρ that satisfy inequalities in node 1L0M0.

Example 4.7. Suppose that $t, u, v \in Var$ are pairwise distinct and suppose that $t \star u, t \star v, u \star v$ are subterms of tableau formulas in tableau $\mathcal{T}$ at node Γ .

1. Suppose at node Γ the active term is $t \star u$ and suppose that parameters $a_0 < b_0 \leq a_1 < \dots \leq a_{n-1} < b_{n-1}$ ($n \geq 1$) have been selected in the previous steps. We apply Rule $(\star)$ min. Case 10. Thus,

$$\Gamma_1 = \{b_{n-1} \leq t^\Gamma \leq 1\} \cup \{\gamma[\min\{t^\Gamma, u^\Gamma\}/t^\Gamma \star u^\Gamma : \gamma \in \Gamma\}$$

2. Now, at Γ_1 , we select $t^\Gamma \star v$ as the active term. We apply Rule $(\star)$ Case 4. Thus,

$$\begin{aligned} \Gamma_2 = & \{b_{n-1} \leq a < b \leq 1\} \cup \{a \leq t^\Gamma \leq b, a \leq v \leq b\} \cup \\ & \{\gamma[\max\{a, t^\Gamma + v - b\}/t^\Gamma \star v] : \gamma \in \Gamma_1\} \end{aligned}$$

3. Next at Γ_2 , we select $u^\Gamma \star v$ as the active term. We apply Rule $(\star)$ Case 2 and the new parameters c, d are such that $a = c, b = d$. Thus,

$$\Gamma_3 = \{c = a < d = b\} \cup \{c \leq u^\Gamma \leq d, c \leq v \leq d\} \cup \{\gamma[\max\{c, u^\Gamma + v - d\}/u^\Gamma \star v] : \gamma \in \Gamma_2\}$$

4. At branch Γ_3 we have $\{c = a < d = b, c \leq u^\Gamma \leq d, a \leq t^\Gamma \leq b\} \subseteq \Gamma_3$. Thus there is no nice solution σ of $\Gamma_3 \upharpoonright_{L_0}$ and the branch is closed.

4.2.3 Soundness and completeness of the calculus

In the proof of soundness and completeness theorem of the tableau calculus, we will use a lemma showing the equivalence between existence of an open branch of a tableau with root $\{\tau(\psi) < 1\}$ and existence of a model in which the formula ψ is not valid. This, however, needs to be preceded by a definition of satisfiability of a branch.

Definition 4.8. Let $\mathcal{B}$ be a branch of a tableau $\mathcal{T}$. Let $0 \leq a_0 < b_0 \leq \dots \leq a_{n-1} < b_{n-1} \leq 1$, where $n \geq 0$, be the inequalities between the parameters introduced by branch extension rules in $\mathcal{B}$. Let $l_{\mathcal{B}}$ be the leaf of the branch. Suppose that $(\mathcal{M}, \sigma)$ is a nice solution of $l_{\mathcal{B}}$ (see Definition 4.4).

1. We expand the model $\mathcal{M}$ to an L_1 -structure $\mathcal{M}_{\mathcal{B}} = (\mathbb{R}, +, -, \min, \max, \leq, <, 0, 1, \rho, \star_{\mathcal{B}}, \Rightarrow_{\mathcal{B}})$, where the functions $\star_{\mathcal{B}}, \Rightarrow_{\mathcal{B}}$ are defined on $\mathbb{R} \times \mathbb{R}$ as follows:

$$v \star_{\mathcal{B}} w = \begin{cases} \max\{\rho(a_k), v + w - \rho(b_k)\} & \text{if } v, w \in [\rho(a_k), \rho(b_k)], 0 \leq k \leq n-1, \\ \min(v, w) & \text{otherwise,} \end{cases}$$

$$v \Rightarrow_{\mathcal{B}} w = \begin{cases} 1 & \text{if } v \leq w \\ \rho(b_k) - v + w & \text{if } \rho(a_k) \leq w < v \leq \rho(b_k), 0 \leq k \leq n-1, \\ w & \text{otherwise.} \end{cases}$$

Remark: This is easily checked to be well defined, although k in $\star_{\mathcal{B}}$ is not always unique.

2. A node S of a branch $\mathcal{B}$ is $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$ iff $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge S$, where the L_1 -structure $\mathcal{M}_{\mathcal{B}}$ is constructed from $\mathcal{M}$ as in point 1.

Now, we say that branch $\mathcal{B}$ is *satisfiable* iff there exist $\mathcal{M}, \sigma$ such that all nodes of $\mathcal{B}$ are $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$. We denote by $\llbracket z \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$ the value of L_1 -term z in $\mathcal{M}_{\mathcal{B}}$ under the assignment σ .

Remark 4.9. By Theorems 2.36 and 2.38, $\star_{\mathcal{B}}$ is a continuous t-norm and $\Rightarrow_{\mathcal{B}}$ is its residuum when both functions are restricted to $[0,1]$.

Below we paraphrase Theorem 3 of [BHMV02], which we will be using in the proof of Lemma 4.12.

Theorem 4.10. Let $\star_{kL}$ be a t-norm whose idempotents are $0, 1/k, 2/k, \dots, (k-1)/k, 1$ for some $0 < k < \omega$ and is isomorphic to the Łukasiewicz t-norm on each of its contact intervals $[i/k, (i+1)/k]$ for $0 \leq i \leq k-1$ (see Definition 2.32). If $\varphi \in \mathcal{F}$ is not valid, then for some $0 < k < \omega$, φ is not a 1-tautology of $\star_{kL}$.

Remark 4.11. By Corollary 2.37, each $\star_{kL}$ is a continuous t-norm.

Lemma 4.12. Let $\mathcal{T}$ be a tableau with a root $\{\tau(\psi) < 1\}$. Then the following are equivalent:

1. $\mathcal{T}$ has an open branch.
2. There is a model $\mathcal{A} = ([0, 1], \star, \Rightarrow, \min, \max, 0, 1, V)$, where $\star$ is a continuous t-norm and $\Rightarrow$ is its residuum, $V : PROP \rightarrow [0, 1]$ such that $V_{\star}(\psi) < 1$.

Proof. Suppose that branch $\mathcal{B}$ of tableau $\mathcal{T}$ is open. So there is a nice solution $(\mathcal{M}, \sigma)$ of leaf $l_{\mathcal{B}}$. We will construct a model $\mathcal{A}$ such that $V_{\star}(\psi) < 1$. First, we put $V(p) = \sigma(\mu_p)$ for all $p \in PROP$. We define the operation $\star : [0, 1]^2 \rightarrow [0, 1]$ as the restriction of $\star_{\mathcal{B}}$ to $[0,1]$ and the operation $\Rightarrow : [0, 1]^2 \rightarrow [0, 1]$ as the restriction of $\Rightarrow_{\mathcal{B}}$ to $[0,1]$ (see Definition 4.8). By Remark 4.9 this is a well-defined model.

Claim 1. Let θ be a subformula of ψ . Then $V_{\star}(\theta) = \llbracket \tau(\theta) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$.

Proof of the claim. We prove by induction on θ . The base case for propositional atoms is given, and $V_{\star}(\bar{0}) = 0 = \llbracket 0 \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \tau(\bar{0}) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$ by Definitions 3.2, 4.8 and 4.2. Similarly for $\theta = \bar{1}$. Assume the induction hypothesis for θ, φ . We show the claim for $\theta \rightarrow \varphi$. The other cases for $\theta \& \varphi, \theta \wedge \varphi, \theta \vee \varphi$ are similar. By Definition 3.2, the above definition of $\Rightarrow$, the induction hypothesis, and Remark 4.9,

$$\begin{aligned} V_{\star}(\theta \rightarrow \varphi) &= V_{\star}(\theta) \Rightarrow V_{\star}(\varphi) = V_{\star}(\theta) \Rightarrow_{\mathcal{B}} V_{\star}(\varphi) \\ &= \llbracket \tau(\theta) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} \Rightarrow_{\mathcal{B}} \llbracket \tau(\varphi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}. \end{aligned}$$

Now, by Definition 4.8,

$$\llbracket \tau(\theta) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} \Rightarrow_{\mathcal{B}} \llbracket \tau(\varphi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \tau(\theta) \Rightarrow \tau(\varphi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma},$$

and by Definition 4.2,

$$\llbracket \tau(\theta) \Rightarrow \tau(\varphi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \tau(\theta \rightarrow \varphi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma},$$

which concludes the proof of the claim.

Claim 2. Every node $m_{\mathcal{B}}$ of $\mathcal{B}$ is $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$.

Proof of claim. By induction on $h(m_{\mathcal{B}}, \mathcal{B})$. By choice of $\mathcal{M}$ and σ , $l_{\mathcal{B}}$ is $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$. Let $m'_{\mathcal{B}}$ be the node of $\mathcal{B}$ such that $h(m'_{\mathcal{B}}, \mathcal{B}) = h(m_{\mathcal{B}}, \mathcal{B}) + 1$, where $m_{\mathcal{B}}$ is assumed inductively to be $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$. Since $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge m_{\mathcal{B}}$, $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge (m_{\mathcal{B}} \cap m'_{\mathcal{B}})$ and $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge (m_{\mathcal{B}} - m'_{\mathcal{B}})$. By inspecting $m_{\mathcal{B}}$ and $m'_{\mathcal{B}}$, we know which tableau formulas γ are in $m'_{\mathcal{B}} - m_{\mathcal{B}}$ and thus by Definition 4.3, which branch expansion rule has been applied to generate $m_{\mathcal{B}}$ from its predecessor $m'_{\mathcal{B}}$. We know which parameters of $\{a_i, b_i : i \in \mathfrak{C}\}$ occur in $m'_{\mathcal{B}}$. We show the claim for one rule and one case. The other rules and cases can be shown analogously. Suppose that $0 \leq a_0 < b_0 \leq a_1 < b_1 \leq 1 \in m'_{\mathcal{B}}$. Suppose that the active term in $m'_{\mathcal{B}}$ is $x \Rightarrow y$, where x, y are L_0 -terms, and the rule and case used to generate $m_{\mathcal{B}}$ is $(\Rightarrow)$ min. Case 9, where $i = 0$. We have x, y marked and let us denote $x^{m'_{\mathcal{B}}}$ and $y^{m'_{\mathcal{B}}}$ as u, v , respectively. Thus, in $m_{\mathcal{B}} - m'_{\mathcal{B}}$ we have (1) $b_0 \leq u \leq a_1$, (2) $v < u$, (3) $\gamma[v/u \Rightarrow v]$ for all $\gamma \in m'_{\mathcal{B}} - m_{\mathcal{B}}$. Since $(\mathcal{M}, \sigma)$ is nice, there are no parameters a, b with $a < b$ occurring in $l_{\mathcal{B}}$ such that $\llbracket u \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}, \llbracket v \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} \in [\rho(a), \rho(b)]$. By Definitions 4.1, 4.4 and 4.8, $\llbracket u \Rightarrow v \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket u \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} \Rightarrow_{\mathcal{B}} \llbracket v \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket v \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$. So for every formula $\gamma \in m'_{\mathcal{B}} - m_{\mathcal{B}}$, we have $\mathcal{M}_{\mathcal{B}}, \sigma \models \gamma$ if and only if $\mathcal{M}_{\mathcal{B}}, \sigma \models \gamma[v/u \Rightarrow v]$. Since $\gamma[v/u \Rightarrow v] \in m_{\mathcal{B}} - m'_{\mathcal{B}}$, the induction hypothesis gives $\mathcal{M}_{\mathcal{B}}, \sigma \models \gamma[v/u \Rightarrow v]$. So $\mathcal{M}_{\mathcal{B}}, \sigma \models \gamma$. This holds for all $\gamma \in m'_{\mathcal{B}} - m_{\mathcal{B}}$. Therefore $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge (m'_{\mathcal{B}} - m_{\mathcal{B}})$. This completes the proof of the claim.

Therefore, in particular by Claim 2, the root of $\mathcal{T}$ is $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$, and thus by Definition 4.8 and Claim 1, $\llbracket \tau(\psi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star}(\psi) < 1$.

Conversely, suppose that there is a model $\mathcal{A} = ([0, 1], \star, \Rightarrow, \min, \max, 0, 1, V)$, where $\star$ is a continuous t-norm and $\Rightarrow$ is its residuum, and $V : PROP \rightarrow [0, 1]$, such that $V_{\star}(\psi) < 1$. Therefore by Theorem 4.10, there exists $k < \omega$ and $V' : PROP \rightarrow [0, 1]$ such that $V'_{\star_{kL}}(\psi) < 1$ and by Lemma 3.13, there is an LP-norm $\star_0$ and $V_0 : PROP \rightarrow [0, 1]$ such that $V_{\star_0}(\psi) < 1$, where $\star_0$ is the ordinal sum $\sum_{i=0}^{k-1} ([i/k, (i+1)/k], \star_i)$ of Łukasiewicz t-norms. We know that for every node $n_{\mathcal{B}}$ of a branch $\mathcal{B}$, we have $n_{\mathcal{B}} \upharpoonright_{L_0} \subseteq l_{\mathcal{B}}$, where $l_{\mathcal{B}}$ is the leaf of the branch. Therefore, a branch is open if its leaf $l_{\mathcal{B}}$ has a nice solution. That is, we need to find a branch $\mathcal{B}$ and a nice solution $(\mathcal{M}, \sigma)$ of $l_{\mathcal{B}}$. First, we take $\sigma(\mu_p) = V(p)$ for all $p \in PROP$ and $\sigma(x) = 0$ for all $x \in Var - \mu(PROP)$. Now in (1) below, we construct the structure $\mathcal{M}$; the process reduces to constructing the mapping $\rho : Par \rightarrow [0, 1]$. At the same time we will be selecting nodes on a branch, we will call it $\mathcal{B}$. Then, in (2) we show that branch $\mathcal{B}$ is open.

We write Λ for the root of the tableau. We let $\circ$ range over $\{\star, \Rightarrow\}$, and $(\#, \#)$

range over $\{(\star, \&), (\Rightarrow, \rightarrow), (\min, \wedge), (\max, \vee)\}$ – that is, $\#$ is the L_1 -function symbol corresponding to the fuzzy connective $\sharp$, and vice versa. Note that when $\# \in \{\min, \max\}$, it is an L_0 -function symbol.

Let φ be a subformula of ψ . We define an L_1 -term φ_Γ for each node Γ of the tableau by induction on $d(\Gamma)$. We let $\varphi_\Lambda = \tau(\varphi)$. If Γ is a successor of Γ' , the active term at Γ' was $x \circ y$, and the branch expansion rule that constructed Γ from Γ' replaced $x \circ y$ in formulas in Γ' by the L_0 -term z , then we define $\varphi_\Gamma = \varphi_{\Gamma'}[z/x \circ y]$.

Remark. Note that for an L_1 -term t ,

$$t[z/x \circ y] = \begin{cases} t, & \text{if } t \in Var \cup \{0, 1\}, \\ z, & \text{if } t = x \circ y, \\ f(t_1[z/x \circ y], \dots, t_n[z/x \circ y]) & \text{if } t = f(t_1, \dots, t_n) \neq x \circ y. \end{cases}$$

Claim 3. For each node Γ of the tableau, each term $t\#u$ occurring in Γ , where $\# \notin L_0$, is equal to $(\theta\#\varphi)_\Gamma$ for some (not necessarily unique) subformula $\theta\#\varphi$ of ψ .

Proof of the claim. By induction on $d(\Gamma)$. The only non-atomic terms occurring in the root $\{\tau(\psi) < 1\}$ are of the form $\tau(\theta\#\varphi)$ for some subformula $\theta\#\varphi$ of ψ , and $\tau(\theta\#\varphi) = (\theta\#\varphi)_\Lambda$ by definition above. If Γ is a successor of Γ' , suppose that the active term at Γ' was $x \circ y$, and that the branch expansion rule that constructed Γ from Γ' replaced $x \circ y$ in formulas in Γ' by L_0 -term z . By inspection of the tableau rules, we see that $t\#u = (t'\#u')[z/x \circ y]$ for some $t'\#u'$ occurring in Γ' . By induction hypothesis, $t'\#u' = (\theta\#\varphi)_{\Gamma'}$ for some subformula $\theta\#\varphi$ of ψ . So $t\#u = (\theta\#\varphi)_{\Gamma'}[z/x \circ y] = (\theta\#\varphi)_\Gamma$. This completes the proof of the claim.

(1) We will assign values of parameters occurring on $\mathcal{B}$ under ρ to elements of $\{0, 1/k, \dots, (k-1)/k, 1\}$. Suppose we have already selected the sequence of nodes $n_0, \dots, n_l$, where $l \geq 0$, n_0 is the root of $\mathcal{T}$ and n_{i+1} is a successor of n_i for all $0 \leq i < l$. That is, branch $\mathcal{B}$ is partially defined and ρ is defined for all parameters in these nodes. By comparing n_l and its successors, we deduce what the active term is and thus which of the branch expansion rules has been applied. Suppose it is $(\star)$; the other case is similar. Let Γ_1 be the set of L_1 -terms in n_l that are not in any of the successors of n_l . Let the active term at n_l be $x \star y$, say. This term occurs in n_l . So by the claim above, we can choose a subformula $\varphi \& \theta$ of ψ with $x \star y = (\varphi \& \theta)_{n_l}$. We know whether or not there is $0 \leq i \leq k-1$ such that $V_{\star_0}(\varphi), V_{\star_0}(\theta) \in [i/k, (i+1)/k]$. If there is such i , we choose the least one and select the subrule L. of $(\star)$, and depending on the relation of the values of parameters occurring on n_l to $i/k, (i+1)/k$, we select the node, say n'_l , resulting from Cases 1-5. We then assign to the parameters at n'_l that do not occur in n_l , say a, b with $a < b$, values $\rho(a) = i/k, \rho(b) = (i+1)/k$. Suppose now that there is no such i . Then there are

$0 \leq i, j \leq k - 1$ such that $i \neq j$ and $V_{\star_0}(\varphi) \in [i/k, (i+1)/k]$, $V_{\star_0}(\theta) \in [j/k, (j+1)/k]$. We know the relations among the values of parameters occurring in n_l , thus we know which of Cases 6-11 match these values under $V_{\star_0}$. Therefore, we can select the subsequent node. In this case there are no new parameters to assign. We have now selected node n_{l+1} in the path from the root. The procedure terminates at a leaf, where there are no L_1 -terms occurring, at which point we selected all nodes in branch $\mathcal{B}$. We also partially defined the function ρ . To the parameters that have not received values under ρ in this procedure, we assign 0. So $\rho : Par \rightarrow [0, 1]$ is now fully defined. We have now constructed an L_0 -structure $\mathcal{M} = (\mathbb{R}, +, -, \min, \max, \leq, <, 0, 1, \rho)$.

(2) To show that $\mathcal{M}, \sigma \models \bigwedge l_{\mathcal{B}}$, where $l_{\mathcal{B}}$ is the leaf of branch $\mathcal{B}$, it is sufficient to prove that for all nodes $n_{\mathcal{B}}$ in $\mathcal{B}$, we have $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge n_{\mathcal{B}}$ and $(\mathcal{M}, \sigma)$ is a nice solution. First, we show Claim 4, which we will need to prove Claim 5, completing this direction of the lemma.

Claim 4. Let the node $\Gamma \in \mathcal{B}$ be arbitrary. Then

1. For each subformula $\theta \# \varphi$ of ψ , if $(\theta \# \varphi)_{\Gamma}$ is not an L_0 -term, then $(\theta \# \varphi)_{\Gamma} = \theta_{\Gamma} \# \varphi_{\Gamma}$.
2. For each subformula φ of ψ , if φ_{Γ} is an L_0 -term, then $\llbracket \varphi_{\Gamma} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star_0}(\varphi)$.
3. For each subformula φ of ψ , we have $\llbracket \varphi_{\Gamma} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \varphi_{\Lambda} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$.
4. For each subformula φ of ψ , φ_{Γ} occurs in Γ .
5. For each subformula φ of ψ , we have $\llbracket \tau(\varphi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star_0}(\varphi)$.

Proof of the claim. We prove 1-4 by induction on $d(\Gamma)$. First, let $\Gamma = \Lambda$.

1. By definition of τ .
2. By induction on φ .

The base case. For $\varphi = p \in PROP$, by definitions of p_{Λ} , the translation function τ (Definition 4.2) and σ , we have

$$\llbracket p_{\Lambda} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \tau(p) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \mu_p \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \sigma(\mu_p) = V_{\star_0}(p).$$

The proof for $\varphi = \bar{0}, \bar{1}$ is easy.

The inductive case. Suppose $\theta \# \varphi$ is a subformula of ψ such that $(\theta \# \varphi)_{\Lambda} = \theta_{\Lambda} \# \varphi_{\Lambda}$ is an L_0 -term, and assume the result for θ, φ . Then $\#$ is an L_0 function symbol, that is, $\# \in \{\wedge, \vee\}$. If $\# = \wedge$, we have $\llbracket (\theta \wedge \varphi)_{\Lambda} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \min\{\theta_{\Lambda}, \varphi_{\Lambda}\} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \min\{\llbracket \theta_{\Lambda} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}, \llbracket \varphi_{\Lambda} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}\}$. By the induction hypothesis,

$$\min\{\llbracket \theta_{\Lambda} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}, \llbracket \varphi_{\Lambda} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}\} = \min\{V_{\star_0}(\theta), V_{\star_0}(\varphi)\} = V_{\star_0}(\theta \wedge \varphi).$$

The other case can be proven analogously.

3. Obvious.

4. By definition $\varphi_\Lambda = \tau(\varphi)$, which is a subterm of $\tau(\psi)$, and $\Lambda = \{\tau(\psi) < 1\}$.

Now suppose that Γ is the successor of $\Gamma' \in \mathcal{B}$. Suppose that the active term at Γ' was $x \circ y$ for some L_0 -terms x, y , and the branch expansion rule that constructed Γ from Γ' replaced $x \circ y$ in formulas in Γ' by some L_0 -term z . Assume inductively that 1-4 hold for Γ' . We prove them for Γ .

1. Suppose that $(\theta\sharp\varphi)_\Gamma$ is not an L_0 -term. Then by the induction hypothesis,

$$(\theta\sharp\varphi)_\Gamma = (\theta\sharp\varphi)_{\Gamma'}[z/x \circ y] = (\theta_{\Gamma'}\sharp\varphi_{\Gamma'})[z/x \circ y] = \theta_{\Gamma'}[z/x \circ y]\sharp\varphi_{\Gamma'}[z/x \circ y] = \theta_\Gamma\sharp\varphi_\Gamma.$$

Otherwise $(\theta\sharp\varphi)_\Gamma$ would be an L_0 -term.

2. In general $\star_{\mathcal{B}}$ interpreted in $\mathcal{M}_{\mathcal{B}}$ may be different from $\star_0$.

Let ζ be a subformula of ψ , and suppose that $\zeta_\Gamma = \zeta_{\Gamma'}[z/x \circ y]$ is an L_0 -term. We need to prove that $\llbracket \zeta_\Gamma \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star_0}(\zeta)$. We will do this by induction on ζ .

There are three cases following Remark preceding Claim 3.

Case 1: ζ is atomic. Then $\zeta_{\Gamma'}$ is obviously an L_0 -term. Then, $\zeta_\Gamma = \zeta_{\Gamma'}[z/x \circ y] = \zeta_{\Gamma'}$, so inductively, $\llbracket \zeta_\Gamma \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \zeta_{\Gamma'} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star_0}(\zeta)$.

Case 2: $\zeta = \theta\sharp\varphi$ and $\zeta_{\Gamma'} = x \circ y$. Since $(\theta\sharp\varphi)_{\Gamma'} = \zeta_{\Gamma'} = x \circ y$, which is not an L_0 -term, by the induction hypothesis for Part 1, we have $(\theta\sharp\varphi)_{\Gamma'} = \theta_{\Gamma'}\sharp\varphi_{\Gamma'}$. As this is $x \circ y$, we have $\theta_{\Gamma'} = x$ and $\varphi_{\Gamma'} = y$, and $\sharp = \circ$. As x, y are L_0 -terms, so by the induction hypothesis we have $V_{\star_0}(\theta) = \llbracket x \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$ and $V_{\star_0}(\varphi) = \llbracket y \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$. In **(1)** we chose some subformula $\theta'\sharp\varphi'$ of ψ with $(\theta'\sharp\varphi')_{\Gamma'} = x \circ y$, and arranged that if $V_{\star_0}(\theta') = \llbracket x \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$ and $V_{\star_0}(\varphi') = \llbracket y \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$, then

$$\llbracket z \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star_0}(\theta')\#_0 V_{\star_0}(\varphi'), \quad (\dagger)$$

where $\#_0$ is $\star_0$ if $\sharp = \star$, and the residuum of $\star_0$ if $\sharp$ is $\Rightarrow$. Again, by the induction hypothesis for Part 1, $\theta'_{\Gamma'} = x$ and $\varphi'_{\Gamma'} = y$ are L_0 -terms. So by the induction hypothesis $V_{\star_0}(\theta') = \llbracket x \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$ and $V_{\star_0}(\varphi') = \llbracket y \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$. So $(\dagger)$ holds. Therefore,

$$\begin{aligned} \llbracket \zeta_\Gamma \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} &= \llbracket z \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star_0}(\theta')\#_0 V_{\star_0}(\varphi') = \llbracket x \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}\#_0 \llbracket y \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} \\ &= V_{\star_0}(\theta)\#_0 V_{\star_0}(\varphi) = V_{\star_0}(\theta\sharp\varphi) = V_{\star_0}(\zeta). \end{aligned}$$

Case 3: $\zeta = \theta \# \varphi$ and $\zeta_{\Gamma'} \neq x \circ y$. If $\zeta_{\Gamma'}$ is an L_0 -term, we use the proof of Case 1. So we assume that it is not an L_0 -term. By the induction hypothesis to Part 1, $\zeta_{\Gamma'} = \theta_{\Gamma'} \# \varphi_{\Gamma'}$. This is not $x \circ y$, so

$$\zeta_{\Gamma} = \zeta_{\Gamma'}[z/x \circ y] = (\theta_{\Gamma'} \# \varphi_{\Gamma'})[z/x \circ y] = \theta_{\Gamma'}[z/x \circ y] \# \varphi_{\Gamma'}[z/x \circ y] = \theta_{\Gamma} \# \varphi_{\Gamma}.$$

This is an L_0 -term, so $\# \in \{\min, \max\}$ and hence $\# \in \{\wedge, \vee\}$. Assume $\# = \wedge$; the other case can be proven analogously. Then $\# = \min$ and $\theta_{\Gamma}, \varphi_{\Gamma}$ are L_0 -terms, so by induction hypothesis,

$$\begin{aligned} \llbracket \zeta_{\Gamma} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} &= \llbracket \min\{\theta_{\Gamma}, \varphi_{\Gamma}\} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \min\{\llbracket \theta_{\Gamma} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}, \llbracket \varphi_{\Gamma} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}\} \\ &= \min\{V_{\star_0}(\theta), V_{\star_0}(\varphi)\} = V_{\star_0}(\theta \wedge \varphi) = V_{\star_0}(\zeta). \end{aligned}$$

3. In (1) we chose some subformula $\theta' \# \varphi'$ of ψ with $(\theta' \# \varphi')_{\Gamma'} = x \circ y$, and arranged that if $V_{\star_0}(\theta') = \llbracket x \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$ and $V_{\star_0}(\varphi') = \llbracket y \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$, then

$$\llbracket z \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket x \circ y \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} \quad (\dagger)$$

By the induction hypothesis for Part 1, $\theta'_{\Gamma'} = x$ and $\varphi'_{\Gamma'} = y$. As x, y are L_0 -terms, by the inductive hypothesis for Part 2 we have $V_{\star_0}(\theta') = \llbracket x \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$ and $V_{\star_0}(\varphi') = \llbracket y \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$. So $(\dagger)$ holds. Thus by $(\dagger)$ and the induction hypothesis,

$$\llbracket \varphi_{\Gamma} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \varphi_{\Gamma'}[z/x \circ y] \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \varphi_{\Gamma'} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \varphi_{\Lambda} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}.$$

4. By the induction hypothesis and inspection of the tableau rules.

This completes the induction. Now we prove Part 5. Let $l_{\mathcal{B}}$ be the leaf of $\mathcal{B}$. By Part 4, $\varphi_{l_{\mathcal{B}}}$ occurs in $l_{\mathcal{B}}$. Since only L_0 -terms occur in $l_{\mathcal{B}}$, $\varphi_{l_{\mathcal{B}}}$ is an L_0 -term. By definition of φ_{Λ} and Parts 3 and 2, $\llbracket \tau(\varphi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \varphi_{\Lambda} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \varphi_{l_{\mathcal{B}}} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star_0}(\varphi)$. This proves the claim.

Claim 5. For every node $n_{\mathcal{B}}$ of $\mathcal{B}$, $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge n_{\mathcal{B}}$ and $(\mathcal{M}, \sigma)$ is a nice solution.

Proof of the claim. By induction on $d(n_{\mathcal{B}})$. For the base case: $d(n_{\mathcal{B}}) = 0$, i.e. $n_{\mathcal{B}} = \Lambda$. By Claim 4 Part 5 and the initial assumption, $\llbracket \tau(\psi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star_0}(\psi) < 1$. Therefore $\mathcal{M}_{\mathcal{B}}, \sigma \models \tau(\psi) < 1$, i.e. $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge n_{\mathcal{B}}$.

Assume the induction hypothesis for a non-leaf node $n_{\mathcal{B}}$ of branch $\mathcal{B}$. We need to show that if $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge n_{\mathcal{B}}$, then $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge n'_{\mathcal{B}}$, where $n'_{\mathcal{B}} \in \mathcal{B}$ and $d(n'_{\mathcal{B}}) = d(n_{\mathcal{B}}) + 1$.

By inspecting $n_{\mathcal{B}}, n'_{\mathcal{B}}$, we know which tableau formulas γ are in $n_{\mathcal{B}} - n'_{\mathcal{B}}$. Suppose that $(\varphi \& \theta)_{n_{\mathcal{B}}} = x \star y$ is the active term of $\gamma \in n_{\mathcal{B}} - n'_{\mathcal{B}}$; the other cases are similar. By (1) above, we know which case and which subrule of which branch expansion rule are used to generate $n'_{\mathcal{B}}$. Thus, suppose that it was Rule $(\star)$ L. Case 1. We will show

one more case below and other cases can be proven analogously. Let a, b be the new parameters occurring at n'_B . Thus, $\{0 \leq a < b \leq a_0\} \subseteq n'_B - n_B$, where a_0 is a parameter occurring at n_B . By (1) above, we know that $0 \leq \rho(a) < \rho(b) \leq \rho(a_0)$. The other elements of $n'_B - n_B$ are (a) $a \leq x \leq b$, (b) $a \leq y \leq b$, (c) $\gamma[\max\{a, x + y - b\}/x \star y]$ for all $\gamma \in n_B - n'_B$. By (1) above and Claim 4 Part 2, the L_0 -formulas in (a) and (b) are $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$ and x, y are not marked since $V_{\star_0}(\theta), V_{\star_0}(\varphi) \in [\rho(a), \rho(b)]$. Now we show that L_1 -formulas in (c) are $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$. Let $z = \max\{a, x + y - b\}$. By (1) above, z was substituted for the active term $x \star y$ in the step from n_B to n'_B . So $(\varphi \& \theta)_{n'_B} = (\varphi \& \theta)_{n_B}[z/x \star y] = (x \star y)[z/x \star y] = z$. By Claim 4 Part 3, $\llbracket z \rrbracket^{\mathcal{M}_B, \sigma} = \llbracket (\varphi \& \theta)_{n'_B} \rrbracket^{\mathcal{M}_B, \sigma} = \llbracket (\varphi \& \theta)_{n_B} \rrbracket^{\mathcal{M}_B, \sigma} = \llbracket x \star y \rrbracket^{\mathcal{M}_B, \sigma}$. Now by the induction hypothesis, all $\gamma \in n_B - n'_B$ are $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$. The formulas in (c) are of the form $\gamma[z/x \star y]$ for $\gamma \in n_B - n'_B$. Since $\llbracket z \rrbracket^{\mathcal{M}_B, \sigma} = \llbracket x \star y \rrbracket^{\mathcal{M}_B, \sigma}$, all formulas in (c) are $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$.

Now, suppose that the rule that was selected is Rule $(\star)$ min. Case 6. The marked terms are $x^{n'_B}$ and $y^{n'_B}$. We denote $x^{n'_B}$ and $y^{n'_B}$ as u, v , respectively. Thus $n'_B - n_B = \{0 \leq u \leq a_0\} \cup \{\gamma[\min\{u, v\}/u \star v] : \gamma \in n_B\}$. By (1) above, we know that there are $0 \leq i, j \leq k-1$ such that $i \neq j$, $(i+1)/k \leq \rho(a_0)$, and $V_{\star_0}(\theta) \in [i/k, (i+1)/k]$, $V_{\star_0}(\varphi) \in [j/k, (j+1)/k]$. By (1) above and Claim 4 Part 2, the L_0 -formulas in $n'_B - n_B$ are $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$. By (1) above and Claim 4 Part 3, the L_1 -formulas in $n'_B - n_B$ are $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$ using the same argument as above. Thus, $\mathcal{M}_B, \sigma \models \bigwedge n'_B$.

We show that $(\mathcal{M}, \sigma)$ is a nice solution with respect to all parameters in n'_B . By (1) and Claim 4 Part 2, if terms x, y are marked by $\Gamma \in \mathcal{B}$, then there is no $i < k$ such that $\llbracket x \rrbracket^{\mathcal{M}_B, \sigma}, \llbracket y \rrbracket^{\mathcal{M}_B, \sigma} \in [i/k, (i+1)/k]$. Since for all parameters $a_j < b_j \in \mathcal{B}$, we have $\rho(a_j) = i/k$ and $\rho(b_j) = (i+1)/k$ for some $i < k$, it follows that there is no pair of parameters $a_j < b_j$ such that $\mathcal{M}_B, \sigma \models a_j \leq x, y \leq b_j$. So $(\mathcal{M}_B, \sigma)$ is nice.

This concludes the proof of the claim.

By Claim 5, in particular $\mathcal{M}, \sigma \models \bigwedge n_B \upharpoonright_{L_0}$ for all $n_B \in \mathcal{B}$. We have now proved that in particular $\mathcal{M}, \sigma \models \bigwedge l_B$. Thus, there is a nice solution $(\mathcal{M}, \sigma)$. Therefore $\mathcal{B}$ is open. $\square$

Theorem 4.13. (Soundness and Completeness)

Let ψ be a formula of BL. Then every tableau with the root $\{\tau(\psi) < 1\}$ is closed iff ψ is valid with respect to every continuous t-norm.

Proof. Immediate from Lemma 4.12. $\square$

Remark 4.14. The findings of Section 4.2 can be easily extended to BL_Δ due to Remark 3.14.

4.3. Tableau calculus for $\mathcal{K}$ -satisfiability for $\text{BL}_{\Delta\sim}$. In the previous section we defined a tableau calculus and proved its soundness and completeness with respect to

continuous t-norms. This enabled us to demonstrate that a given formula of BL is either valid or we could find a model, in which its truth value is less than 1. The generalisation presented below works in two ways: (1) we will have a tableau calculus for a finite set Ψ of formulas of $\text{BL}_{\Delta\sim}$, (2) we will have a calculus to show that they are $\mathcal{K}$ -satisfiable for $\mathcal{K} \subseteq [0, 1]$ with respect to some class of continuous t-norms $\star$, namely LP-norms. That is, there exists an assignment $V : \text{PROP} \rightarrow [0, 1]$ such that for every $\psi \in \Psi$, $V_\star(\psi) \in \mathcal{K}$. We are aware that the logic $\text{BL}_{\Delta\sim}$ presented here has different semantics from the classical one (see [Háj98b]). The semantics of the strong conjunction for our $\text{BL}_{\Delta\sim}$ is LP-norms and for the classical one is the semantics of all continuous t-norms. To the best of our knowledge there is no tableau calculus for the classical $\text{BL}_{\Delta\sim}$. Our approach is an attempt to tackle this problem. We hope it may lead in future to a tableau calculus for the classical semantics. Also, we adapted a specific involutive negation (see [CN11]). This material is presented in [Ku14b], which contains ideas that are entirely novel.

4.3.1 Defining the calculus

To be able to accommodate set-satisfiability within a tableau calculus we will express a subset of $[0, 1]$ as a union of subintervals of $[0, 1]$.

Lemma 4.15. Any subset $\mathcal{K}$ of $[0, 1]$ can be expressed as a union of pairwise disjoint maximal subintervals of $\mathcal{K}$.

Proof. We will inductively build the union. *General case.* Suppose that we have built a set of intervals whose union we call $\hat{\mathcal{K}} \subseteq \mathcal{K}$. Take any $k_0 \in \mathcal{K} - \hat{\mathcal{K}}$. Let $k^- = \inf\{k : k \leq k_0 \wedge \forall t : k < t \leq k_0 \rightarrow t \in \mathcal{K}\}$ and $k^+ = \sup\{k : k_0 \leq k \wedge \forall t : k_0 \leq t < k \rightarrow t \in \mathcal{K}\}$. If $k^-, k^+ \in \mathcal{K}$, then we have a closed interval $[k^-, k^+]$, if $k^- \in \mathcal{K}, k^+ \notin \mathcal{K}$, then a right-open interval $[k^-, k^+)$, if $k^- \notin \mathcal{K}, k^+ \in \mathcal{K}$, then a left-open interval $(k^-, k^+]$, and if $k^-, k^+ \notin \mathcal{K}$, then we have an open interval (k^-, k^+) . We add the interval with the endpoints k^-, k^+ to the union $\hat{\mathcal{K}}$. This concludes the general case. When $\mathcal{K} - \hat{\mathcal{K}} = \emptyset$, the process terminates and we have selected a union of pairwise disjoint maximal subintervals, which is obviously equal to $\mathcal{K}$. $\square$

We will now fix $\mathcal{K}$ and a set of formulas Ψ of $\text{BL}_{\Delta\sim}$, and express the complement $\mathcal{K}' = [0, 1] - \mathcal{K}$ as a union of pairwise disjoint maximal subintervals $\{J_i : i \in I\}$ of $\mathcal{K}'$. The left endpoint of each J_i is denoted by j_i^- and the right endpoint of J_i is denoted by j_i^+ .

Informal description of the tableau calculus. The purpose of this calculus is to construct an LP-norm $\star$ and an assignment $V : \text{PROP} \rightarrow [0, 1]$ such that $V_\star(\psi) \in \mathcal{K}$ for each $\psi \in \Psi$. We will achieve this in the following way:

1. We translate each $\psi \in \Psi$ as in Definition 4.17.
2. For each translated ψ , say x , we define a ‘disjunct formula’ $\eta_{J_i}(x)$, a formula saying $x \notin J_i$, with $J_i \in \mathcal{K}'$ as in Definition 4.16 point 1. To illustrate this step, suppose that $\mathcal{K} = [\frac{1}{2}, \frac{3}{4})$. Then the complement of $\mathcal{K}$ is $\mathcal{K}' = [0, \frac{1}{2}) \cup [\frac{3}{4}, 1]$ and $J_1 = [0, \frac{1}{2})$, $J_2 = [\frac{3}{4}, 1]$. We note that we are proving $\mathcal{K}$ -satisfiability of ψ so $x \in \mathcal{K}$ means $x \notin [0, \frac{1}{2})$ and $x \notin [\frac{3}{4}, 1]$. The disjunct formula $\eta_{J_1}(x)$ of $x \notin [0, \frac{1}{2})$ is $(0 < x) \vee (x \geq \frac{1}{2})$. Using the Split Rule (see Definition 4.18), we split this disjunct formula into either $x < 0$ or $x \geq \frac{1}{2}$, so exactly one of them occurs in the successors of the node containing $\eta_{J_1}(x)$. The goal of this step is to generate finite branches, whose cardinality depends on the total number of connectives in the formulas of Ψ . However, there may possibly be an infinite number of branches due to $\mathcal{K}'$.
3. Once there are no more disjunct formulas, we select an L_1 -term $x \star y$, $x \Rightarrow y$ or Δx , where x, y are L_0 -terms (L_0 - and L_1 -terms are defined in Definition 4.16).
4. We consider the following four scenarios as per Definition 4.18, the first three of which are applicable to L_1 -terms $x \star y$, $x \Rightarrow y$, and the fourth one to Δx .
 - (a) The terms x, y belong to the same Łukasiewicz (product, resp.) contact interval. We select a new pair of parameters a, b with $a < b$ to define the endpoints of this contact interval and place them together between other pairs (Case 3), or in front of all them (Case 1), or at the end of all of them (Case 4). Alternatively, they may be the same as another Łukasiewicz (product, resp.) pair (Case 2). If no parameters were selected in previous branches, then we apply Case 10. We place x, y in an order (if necessary) between the new parameters and substitute for all terms that are the same as the one selected an appropriate term, which is the application of Rules $(\star) \text{ L (P, resp.)}$ or $(\Rightarrow) \text{ L (P, resp.)}$. As in Section 4.2.1, we can do this, because $\star, \Rightarrow$ are defined on a single contact interval by simple L_0 -terms. Thus we eliminate one occurrence of $\star$ or $\Rightarrow$.
 - (b) There is a parameter between x and y . We consider all the cases that satisfy this condition (Cases 5-9 and 11). We also mark x, y so later they will not be accidentally placed in the same interval (Marking Procedure). We substitute for all terms that are the same as the one selected an appropriate term, which is the application of Rules $(\star) \text{ min}$ or $(\Rightarrow) \text{ min}$. Again, the values of $\star, \Rightarrow$ on elements of $[0, 1]$ in different Łukasiewicz (product, resp.) contact intervals are defined by simple L_0 -terms, and we have eliminated one occurrence of $\star, \Rightarrow$.
 - (c) In case of $x \Rightarrow y$ when $x \leq y$, we substitute 1 for all such terms, which is Rule $(\Rightarrow) \text{ All}$.

- (d) To term Δx , we apply Rule (Δ) , which says that if $x \geq 1$, then we substitute 1 for all such terms, and otherwise we substitute 0.
5. We proceed until all terms are L_0 -terms.
6. Then we solve the linear inequalities to find a model if one exists. If it does not, then the initial formula is valid.

This completes the informal description.

We will recall the definition of a tableau formula and a translation function that are necessary for defining $\mathcal{K}$ -tableau of a set of formulas. We will adapt the same notions of graph theory as in Section 4.2.1. We modified these definitions to suit our purpose.

Definition 4.16. (Tableau formula)

Let I be the above index set, and $Const = \{c_i^-, c_i^+ : i \in I\}$, where c_i^-, c_i^+ are pairwise distinct constants. Let $L_0 = Par \cup \{+, -, \cdot, \div, \min, \max, \leq, <\} \cup Const$ and $L_1 = L_0 \cup \{\star, \Rightarrow, \Delta\}$ be signatures, where Par is a set of constants (parameters), $+, -, \cdot, \div, \min, \max, \star, \Rightarrow$ are binary function symbols, Δ is a unary function symbol and $\leq, <$ are binary relation symbols. Let Var be a set of variables.

1. Let x be an L_1 -term. A *disjunct formula* $\eta_{J_i}(x)$ is a formula saying $x \notin J_i$ and defined as $(x \diamond_{J_i}^- c_i^-) \vee (c_i^+ \diamond_{J_i}^+ x)$, where
 - (a) $\diamond_{J_i}^-$ is $<$ and $\diamond_{J_i}^+$ is $<$ if $J_i = [j_i^-, j_i^+]$,
 - (b) $\diamond_{J_i}^-$ is $\leq$ and $\diamond_{J_i}^+$ is $<$ if $J_i = (j_i^-, j_i^+]$,
 - (c) $\diamond_{J_i}^-$ is $<$ and $\diamond_{J_i}^+$ is $\leq$ if $J_i = [j_i^-, j_i^+)$,
 - (d) $\diamond_{J_i}^-$ is $\leq$ and $\diamond_{J_i}^+$ is $\leq$ if $J_i = (j_i^-, j_i^+)$.
2. If x, y are L_1 -terms, then $x \leq y, x < y, x = y, \eta_{J_i}(x)$, ($i \in I$), are *tableau formulas*. If x, y are L_0 -terms, then $x \leq y, x < y, x = y$ are *L_0 -formulas*.
3. An L_0 -structure $\mathcal{M}$ is called *standard* iff it is of the form

$$(\mathbb{R}, +, -, \cdot, \div, \min, \max, \leq, <, 0, 1, \rho, (j_i^-, j_i^+) : i \in I),$$

where $+, -, \cdot, \div, \min, \max, 0, 1, \leq, <$ are the usual functions with $x \div 0$ assigned to 0⁷ for any $x \in \mathbb{R}$, $\rho : Par \rightarrow [0, 1]$ is a function, and, c_i^-, c_i^+ are interpreted in $\mathcal{M}$ as j_i^-, j_i^+ , respectively.

4. Let E be a set of tableau formulas e of the form $s \leq t, s < t, s = t$, where s, t are L_0 -terms. Let $\sigma : Var \rightarrow [0, 1]$ be a mapping and $\mathcal{M}$ a standard L_0 -structure. The pair $(\mathcal{M}, \sigma)$ is a *solution* of E iff $\mathcal{M}, \sigma \models \bigwedge E$. By convention $\bigwedge \emptyset = \top$ (verum). We denote by $\llbracket z \rrbracket^{\mathcal{M}, \sigma}$ the value of L_0 -term z in $\mathcal{M}$ under the assignment σ .

⁷This is only to ensure that $\div$ is a total function. Below, all divisors are non-zero.

Definition 4.17. (Translation function)

Let $\mathcal{F}$ be the set of formulas of $\text{BL}_{\Delta\sim}$ and $\mathbb{T}$ be the set of L_1 -terms. Let $\mu : \text{PROP} \rightarrow \text{Var}$ (we will write $\mu(p)$ as μ_p) be a one-to-one mapping assigning variables to propositional atoms. Let $\psi, \varphi \in \mathcal{F}$. Then, we define a translation function $\tau : \mathcal{F} \rightarrow \mathbb{T}$, inductively:

- | | |
|---|---|
| 1. $\tau(\bar{0}) = 0, \tau(\bar{1}) = 1,$ | 5. $\tau(\psi \vee \varphi) = \max\{\tau(\psi), \tau(\varphi)\},$ |
| 2. $\tau(p) = \mu_p$ for every $p \in \text{PROP},$ | 6. $\tau(\psi \wedge \varphi) = \min\{\tau(\psi), \tau(\varphi)\},$ |
| 3. $\tau(\psi \& \varphi) = \tau(\psi) \star \tau(\varphi),$ | 7. $\tau(\Delta\psi) = \Delta\tau(\psi),$ |
| 4. $\tau(\psi \rightarrow \varphi) = \tau(\psi) \Rightarrow \tau(\varphi),$ | 8. $\tau(\sim\psi) = 1 - \tau(\psi).$ |

Note that $\neg, \underline{\vee}, \equiv$ are treated as abbreviations. We will call the terms defined in points 1 and 2 *atomic terms*.

The definition below is a much extended version of the tableau calculus in [Kul14a] presented in the previous section; it incorporates additional connectives and the extended notion of satisfiability. The exposition of branch expansion rules is more compact than in [Kul14a] with new rules for splitting and for the additional connectives. This compactness of the rules enables us for a finite set of formulas to have infinitely many finite branches as the number of branches depends on the connectives in the L_1 -terms in the root as well as on the set $\mathcal{K}$, and this was not necessary for the tableau calculus in [Kul14a]. Finiteness of the branches of a $\mathcal{K}$ -tableau is essential for the proof of Theorem 4.24.

Definition 4.18. A $\mathcal{K}$ -tableau $\mathcal{T}$ for Ψ is a tree whose nodes are sets of tableau formulas and whose root is

$$\{\eta_{J_i}(\tau(\psi)) : i \in I, \psi \in \Psi\},$$

and on which the branch expansion rules⁸ have been fully applied. Let Γ be a set of tableau formulas.

First, we will well-order Ψ , and then we will be selecting $\psi \in \Psi$ one by one and applying Split Rule to all $\eta_{J_i}(\tau(\psi)), i \in I$ simultaneously in all current nodes.

Split Rule. For each $S \subseteq I$, there is a successor of $\Gamma \cup \{\eta_{J_i}(\tau(\psi)) : i \in I\}$ given by:

$$S_\psi. \Gamma \cup \{\tau(\psi) \diamond_{J_i}^- c_i^- : i \in S\} \cup \{c_i^+ \diamond_{J_i}^+ \tau(\psi) : i \in I - S\}$$

where $\diamond_{J_i}^-, \diamond_{J_i}^+$ are as defined in Definition 4.16.

We know that $\eta_{J_i}(\tau(\psi))$ is $(\tau(\psi) \diamond_{J_i}^- c_i^-) \vee (c_i^+ \diamond_{J_i}^+ \tau(\psi))$ for every $i \in I, \psi \in \Psi$. Split Rule S_ψ means that for every $i \in I$, we add only one element from $\{\tau(\psi) \diamond_{J_i}^- c_i^-, c_i^+ \diamond_{J_i}^+ \tau(\psi)\}$ to the successor node. Since $S \in \wp(I)$, there are $2^{|I|}$ successor nodes for each

⁸We use branch expansion rules to generate nodes in a branch.

$\psi \in \Psi$. To illustrate this rule further, suppose that $\mathcal{K}'$ consists of two intervals $[j_1^-, j_1^+]$ and (j_2^-, j_2^+) . Then the set S can be one of the following $\emptyset, \{1\}, \{2\}$ and $\{1, 2\}$. We have the following cases:

- $S = \emptyset$. Then the new node contains $\Gamma \cup \{c_1^+ < \tau(\psi), c_2^+ \leq \tau(\psi)\}$.
- $S = \{1\}$. Then the new node contains $\Gamma \cup \{\tau(\psi) < c_1^-, c_2^+ \leq \tau(\psi)\}$.
- $S = \{2\}$. Then the new node contains $\Gamma \cup \{c_1^+ < \tau(\psi), \tau(\psi) \leq c_2^-\}$.
- $S = \{1, 2\}$. Then the new node contains $\Gamma \cup \{\tau(\psi) < c_1^-, \tau(\psi) \leq c_2^-\}$.

Next, when there is no more $\eta_{J_i}(\tau(\psi)), i \in I, \psi \in \Psi$ in the current nodes, we will apply the other branch expansion rules.

The multiple inequality should be understood in the usual way, e.g. instead of writing $a \leq c, c = d, d < f$, we write $a \leq c = d < f$. Let x, y be L_0 -terms. Let $K, K_0, \dots, K_{n-1} \in \{L, P\}$ be the labels as shown in the branch expansion rules, where L and P stand for the Łukasiewicz and product contact intervals, respectively. Suppose that parameters $a_0^{K_0} < b_0^{K_0} \leq a_1^{K_1} < \dots \leq a_{n-1}^{K_{n-1}} < b_{n-1}^{K_{n-1}}$ ($n \geq 1$) have been selected in the previous steps. We will use the following sets $\mathcal{I}^K$ ($\mathcal{J}$ respectively) in the subrules **L**, **P** (**min**, respectively) of the branch expansion rules $\star, \Rightarrow$, for which we have chosen the active term (one undergoing substitution) of the form $x \star y, x \Rightarrow y$.

In the following case, $a^K, b^K \in Par$ are new distinct parameters. Then we define inclusion sets $\mathcal{I}^K$ and exclusion sets $\mathcal{J}$.

- Case 1. $\mathcal{I}^K = \{0 \leq a^K < b^K \leq a_0^{K_0}\}$,
- Case 2. $\mathcal{I}^K = \{a^K = a_i^{K_i} < b^K = b_i^{K_i}\}$ for each $0 \leq i \leq n-1$ such that $K = K_i$,
- Case 3. $\mathcal{I}^K = \{b_i^{K_i} \leq a^K < b^K \leq a_{i+1}^{K_{i+1}}\}$ for each $0 \leq i \leq n-2$,
- Case 4. $\mathcal{I}^K = \{b_{n-1}^{K_{n-1}} \leq a^K < b^K \leq 1\}$,
- Case 5. $\mathcal{J} = \{0 \leq x \leq a_0^{K_0}\}$,
- Case 6. $\mathcal{J} = \{a_i^{K_i} \leq x \leq b_i^{K_i}, y \leq a_i^{K_i}\}$ for each $0 \leq i \leq n-1$,
- Case 7. $\mathcal{J} = \{a_i^{K_i} \leq x \leq b_i^{K_i}, b_i^{K_i} \leq y\}$ for each $0 \leq i \leq n-1$,
- Case 8. $\mathcal{J} = \{b_i^{K_i} \leq x \leq a_{i+1}^{K_{i+1}}\}$ for each $0 \leq i \leq n-2$,
- Case 9. $\mathcal{J} = \{b_{n-1}^{K_{n-1}} \leq x \leq 1\}$.

If no parameters have been selected in the previous steps, then the unique inclusion set $\mathcal{I}^K = \{0 \leq a^K < b^K \leq 1\}$ (Case 10.) and the unique exclusion set $\mathcal{J} = \emptyset$ (Case 11.). The motivation for the cases are the same as in Definition 4.3.

Marking Procedure. If Cases 5, 8, 9 and 11 are applied in branch $\mathcal{B}$ at node Γ to the active term with L_0 -terms x, y , then we mark x, y as $x^{\Gamma'}, y^{\Gamma'}$ at the nodes Γ' expanding by these cases from Γ . As in Definition 4.3, these terms are not in the same interval and we call x, y *marked*.

We will use the notation $\gamma[v/t]$ to denote the result of substituting the term v for each occurrence of the term t (if any) in the formula γ .

If a node consists wholly of L_0 -formulas, it is a leaf and no rules are applied to it. Otherwise, we choose an active term t of the form $x \star y, x \Rightarrow y$, or Δx (where x, y are L_0 -terms) that occurs in at least one formula in the node, and apply the rule below according to the form of the active term.

Rule $(\star)$. A branch with a node Γ expands following the subrules:

L. $\mathcal{I}^L \cup \{a^L \leq x \leq b^L, a^L \leq y \leq b^L\} \cup \{\gamma[\max\{a^L, x + y - b^L\}/x \star y] : \gamma \in \Gamma\}$ for each inclusion set $\mathcal{I}^L$,

P. $\mathcal{I}^P \cup \{a^P \leq x \leq b^P, a^P \leq y \leq b^P\} \cup \{\gamma[a^P + \frac{(x-a^P)(y-a^P)}{b^P-a^P}/x \star y] : \gamma \in \Gamma\}$ for each inclusion set $\mathcal{I}^P$,⁹

min. $\mathcal{J} \cup \{\gamma[\min\{x, y\}/x \star y] : \gamma \in \Gamma\}$ for each exclusion set $\mathcal{J}$.

Rule $(\Rightarrow)$. A branch with a node Γ expands following the subrules:

All. $\{x \leq y\} \cup \{\gamma[1/x \Rightarrow y] : \gamma \in \Gamma\}$,

L. $\mathcal{I}^L \cup \{a^L \leq y < x \leq b^L\} \cup \{\gamma[b^L - x + y/x \Rightarrow y] : \gamma \in \Gamma\}$ for each inclusion set $\mathcal{I}^L$,

P. $\mathcal{I}^P \cup \{a^P \leq y < x \leq b^P\} \cup \{\gamma[a^P + \frac{(y-a^P)(b^P-a^P)}{x-a^P}/x \Rightarrow y] : \gamma \in \Gamma\}$ for each inclusion set $\mathcal{I}^P$,¹⁰

min. $\mathcal{J} \cup \{y < x\} \cup \{[y/x \Rightarrow y] : \gamma \in \Gamma\}$ for each exclusion set $\mathcal{J}$.

Rule (Δ) . A branch with a node Γ expands following the subrules:

$\Delta 1$. $\{1 \leq x\} \cup \{\gamma[1/\Delta x] : \gamma \in \Gamma\}$,

$\Delta 2$. $\{x < 1\} \cup \{\gamma[0/\Delta x] : \gamma \in \Gamma\}$.

Note that the actual number of new nodes generated by rules $\star$ and $\Rightarrow$ will depend on how many parameters are in the current node as these influence the number of different

⁹Note that $a^P < b^P \in \mathcal{I}^P$ so the denominator is plainly non-zero.

¹⁰Again, the denominator $x - a^P$ here is plainly constrained by the preceding relation to be non-zero.

$\mathcal{I}^L, \mathcal{I}^P, \mathcal{J}$. That is it depends on how many cases of the subrules we can apply. For example, if there are four parameters in the current node, $0 \leq a_0^L < b_0^L \leq a_1^L < b_1^L \leq 1$, subrules L. and P. will have five (Cases 1, 2, 2, 3, 4) and three (Cases 1, 3, 4) cases, respectively, and the subrule min. will have seven cases (Cases 5, 6, 6, 7, 7, 8, 9). So we get a total of 15 successors for the current node generated by Rule $(\star)$ and 16 successors generated by Rule $(\Rightarrow)$.

This concludes Definition 4.18.

Definition 4.19. For each branch $\mathcal{B}$ of a $\mathcal{K}$ -tableau $\mathcal{T}$ and each node $n \in \mathcal{B}$, we consider the set of L_0 -formulas in n , $n|_{L_0}$.

1. A solution $(\mathcal{M}, \sigma)$ (see Definition 4.16 point 4) of $n|_{L_0}$ is *nice* iff for every marked pair x^Γ, y^Γ of L_0 -terms, where $\Gamma \in \mathcal{B}$, there are no parameters a, b with $a < b$ occurring at any node of $\mathcal{B}$ such that $\llbracket a \rrbracket^{\mathcal{M}, \sigma} \leq \llbracket x^\Gamma \rrbracket^{\mathcal{M}, \sigma}, \llbracket y^\Gamma \rrbracket^{\mathcal{M}, \sigma} \leq \llbracket b \rrbracket^{\mathcal{M}, \sigma}$.
2. We say that $\mathcal{B}$ is *closed* if for some node $n \in \mathcal{B}$, $n|_{L_0}$ has no nice solution, otherwise it is *open*. A $\mathcal{K}$ -tableau $\mathcal{T}$ is *closed* if it only contains closed branches. A $\mathcal{K}$ -tableau $\mathcal{T}$ is *open* if it has an open branch.

4.3.2 Example

Now we will show an example to demonstrate the applicability of the rules. We will not show a whole $\mathcal{K}$ -tableau for Ψ , just an open branch. Note that the root is $\{\eta_{J_i}(\tau(\psi)) : i \in I, \psi \in \Psi\}$ and we assumed that $|\Psi|$ is finite. Let L be a list of formulas in Ψ . The names of the nodes in a branch are constructed as follows $1S1S_1S2S_2 \dots S|\Psi|S_{|\Psi|}R$, where tuples S_kS_k are for k th formula of L and $S_k \subseteq I$ and R is constructed of pairs of letters and numbers as described in Section 4.2.2.

Example 4.20. Let $\mathcal{K} = [\frac{1}{2}, \frac{3}{4}] \cup \{1\}$ and $\Psi = \{\bar{1} \rightarrow p \& r, \Delta r \rightarrow (p \vee q)\}$. Let us build a $\mathcal{K}$ -tableau for Ψ . $J_1 = [0, \frac{1}{2}]$, $J_2 = (\frac{3}{4}, 1)$. Let $\psi_1 = \bar{1} \rightarrow p \& r$ and $\psi_2 = \Delta r \rightarrow (p \vee q)$. In Table 4, we show the partial tableau extending from node 1 (root) of depth 1, Table 5 shows a closed branch of the tableau extending from the node $1S1\{2\}S2\emptyset$ and Table 6 shows an open branch of the tableau extending from the node $1S1\{2\}S2\emptyset$.

Table 4: Partial tableau extending from 1 of depth 1.

Node n	Formulas in n	Rule applied
1	$1 \Rightarrow (\mu_p \star \mu_r) < 0 \vee \frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$	

Continued on next page

Table 4 – continued from previous page

Node n	Formulas in n	Rule applied
1S1{1, 2}	$1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4} \vee 1 \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Split Rule. $S_{\psi_1} = \{1, 2\}$
1	$1 \Rightarrow (\mu_p \star \mu_r) < 0 \vee \frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4} \vee 1 \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Split Rule. $S_{\psi_1} = \{2\}$
1S1{2}	$\frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Split Rule. $S_{\psi_1} = \{1\}$
1	$1 \Rightarrow (\mu_p \star \mu_r) < 0 \vee \frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4} \vee 1 \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Split Rule. $S_{\psi_1} = \{1\}$
1	$1 \Rightarrow (\mu_p \star \mu_r) < 0,$ $1 \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Split Rule. $S_{\psi_1} = \{1\}$
1	$1 \Rightarrow (\mu_p \star \mu_r) < 0 \vee \frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4} \vee 1 \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	

Continued on next page

Table 4 – continued from previous page

Node n	Formulas in n	Rule applied
1S1 $\emptyset$	$\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$ $\frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Split Rule. $S_{\psi_1} = \emptyset$

It is worth noticing that branches extending from 1S1{1,2} and 1S1{1} will eventually close.

Table 5: A closed branch extending from 1S1{2}S2 $\emptyset$.

Node n	Formulas in n	Rule applied
1	$1 \Rightarrow (\mu_p \star \mu_r) < 0 \vee \frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4} \vee 1 \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	
1S1{2}	$\frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Split Rule. $S_{\psi_1} = \{2\}$
1S1{2}S2 $\emptyset$	$\frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4},$ $\frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Split Rule. $S_{\psi_2} = \emptyset$
1S1{2}S2 $\emptyset$ -M0	$\frac{1}{2} \leq 1 \Rightarrow \min\{\mu_p^{1S1\{2\}S2\emptyset M0}, \mu_r^{1S1\{2\}S2\emptyset M0}\},$ $1 \Rightarrow \min\{\mu_p^{1S1\{2\}S2\emptyset M0}, \mu_r^{1S1\{2\}S2\emptyset M0}\} \leq \frac{3}{4},$ $\frac{1}{2} \leq \Delta\mu_r^{1S1\{2\}S2\emptyset M0} \Rightarrow \max\{\mu_p^{1S1\{2\}S2\emptyset M0}, \mu_q\},$ $1 \leq \Delta\mu_r^{1S1\{2\}S2\emptyset M0} \Rightarrow \max\{\mu_p^{1S1\{2\}S2\emptyset M0}, \mu_q\}$	Rule ($\star$) min. Case 11 Remark: μ_p, μ_r are now marked.

Continued on next page

Table 5 – continued from previous page

Node n	Formulas in n	Rule applied
1S1{2}S2 $\emptyset$ -M0A0	$1 \leq \min\{\mu_p^{1S1\{2\}S2\emptyset M0}, \mu_r^{1S1\{2\}S2\emptyset M0}\},$ $\frac{1}{2} \leq 1,$ $1 \leq \frac{3}{4},$ $\frac{1}{2} \leq \Delta\mu_r^{1S1\{2\}S2\emptyset M0} \Rightarrow \max\{\mu_p^{1S1\{2\}S2\emptyset M0}, \mu_q\},$ $1 \leq \Delta\mu_r^{1S1\{2\}S2\emptyset M0} \Rightarrow \max\{\mu_p^{1S1\{2\}S2\emptyset M0}, \mu_q\}$	Rule ($\Rightarrow$) All

This branch is closed since $1S1\{2\}S2\emptyset M0A0|_{L_0}$ has no (nice) solution as $1 \leq \frac{3}{4}$ is false.

Table 6: An open branch extending from 1S1{2}S2 $\emptyset$.

Node n	Formulas in n	Rule applied
1	$1 \Rightarrow (\mu_p \star \mu_r) < 0 \vee \frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4} \vee 1 \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	
1S1{2}	$\frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} < 0 \vee \frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $\Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\} \leq \frac{3}{4} \vee 1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Split Rule. $S_{\psi_1} = \{2\}$
1S1{2}S2 $\emptyset$	$\frac{1}{2} \leq 1 \Rightarrow (\mu_p \star \mu_r),$ $1 \Rightarrow (\mu_p \star \mu_r) \leq \frac{3}{4},$ $\frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Split Rule. $S_{\psi_2} = \emptyset$
1S1{2}S2 $\emptyset$ -L0	$0 \leq a_0^L < b_0^L \leq 1,$ $a_0^L \leq \mu_p \leq b_0^L,$ $a_0^L \leq \mu_r \leq b_0^L,$ $\frac{1}{2} \leq 1 \Rightarrow \max\{a_0^L, \mu_p + \mu_r - b_0^L\},$ $1 \Rightarrow \max\{a_0^L, \mu_p + \mu_r - b_0^L\} \leq \frac{3}{4},$	Rule ($\star$) L. Case 10

Continued on next page

Table 6 – continued from previous page

Node n	Formulas in n	Rule applied
	$\frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	
1S1{2}S2 $\emptyset$ - L0L1	$a_0^L = a_1^L < b_1^L = b_0^L,$ $a_1^L \leq \max\{a_0^L, \mu_p + \mu_r - b_0^L\} < 1 \leq b_1^L,$ $0 \leq a_0^L < b_0^L \leq 1,$ $a_0^L \leq \mu_p \leq b_0^L,$ $a_0^L \leq \mu_r \leq b_0^L,$ $\frac{1}{2} \leq b_1^L - 1 + \max\{a_0^L, \mu_p + \mu_r - b_0^L\},$ $b_1^L - 1 + \max\{a_0^L, \mu_p + \mu_r - b_0^L\} \leq \frac{3}{4},$ $\frac{1}{2} \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\},$ $1 \leq \Delta\mu_r \Rightarrow \max\{\mu_p, \mu_q\}$	Rule ($\Rightarrow$) L. Case 2
1S1{2}S2 $\emptyset$ - L0L1D1	$1 \leq \mu_r$ $\frac{1}{2} \leq 1 \Rightarrow \max\{\mu_p, \mu_q\},$ $1 \leq 1 \Rightarrow \max\{\mu_p, \mu_q\},$ $a_0^L = a_1^L < b_1^L = b_0^L,$ $a_1^L \leq \max\{a_0^L, \mu_p + \mu_r - b_0^L\} < 1 \leq b_1^L,$ $0 \leq a_0^L < b_0^L \leq 1,$ $a_0^L \leq \mu_p \leq b_0^L,$ $a_0^L \leq \mu_r \leq b_0^L,$ $\frac{1}{2} \leq b_1^L - 1 + \max\{a_0^L, \mu_p + \mu_r - b_0^L\},$ $b_1^L - 1 + \max\{a_0^L, \mu_p + \mu_r - b_0^L\} \leq \frac{3}{4}$	Rule (Δ) $\Delta 1$
1S1{2}S2 $\emptyset$ - L0L1D1A0	$1 \leq \max\{\mu_p, \mu_q\},$ $\frac{1}{2} \leq 1,$ $1 \leq 1$ $1 \leq \mu_r$ $a_0^L = a_1^L < b_1^L = b_0^L,$ $a_1^L \leq \max\{a_0^L, \mu_p + \mu_r - b_0^L\} < 1 \leq b_1^L,$ $0 \leq a_0^L < b_0^L \leq 1,$ $a_0^L \leq \mu_p \leq b_0^L,$ $a_0^L \leq \mu_r \leq b_0^L,$ $\frac{1}{2} \leq b_1^L - 1 + \max\{a_0^L, \mu_p + \mu_r - b_0^L\},$ $b_1^L - 1 + \max\{a_0^L, \mu_p + \mu_r - b_0^L\} \leq \frac{3}{4}$	Rule ($\Rightarrow$) All

Continued on next page

Table 6 – continued from previous page

Node n	Formulas in n	Rule applied

No more branch expansion rules can be applied to node $1S1\{2\}S2\emptyset L0L1D1A0$. Let $\{(\mu_p, \frac{1}{2}), (\mu_q, 1), (\mu_r, 1)\} \subseteq \sigma$ and $\mathcal{M} = (\mathbb{R}, +, -, \cdot, \div, \leq, <, 0, 1, \rho, (0, \frac{1}{2}), (\frac{3}{4}, 1))$, where $\{(a_0^L, 0), (b_0^L, 1), (a_1^L, 0), (b_1^L, 1)\} \subseteq \rho$. Then

$$\mathcal{M}, \sigma \models e,$$

for all $e \in \{0 \leq a_0^L < b_0^L \leq 1, a_0^L = a_1^L < b_1^L = b_0^L, a_0^L \leq \mu_p \leq b_0^L, a_0^L \leq \mu_r \leq b_0^L, a_1^L \leq \max\{a_0^L, \mu_p + \mu_r - b_0^L\} < 1 \leq b_1^L, \frac{1}{2} \leq b_1^L - 1 + \max\{a_0^L, \mu_p + \mu_r - b_0^L\}, b_1^L - 1 + \max\{a_0^L, \mu_p + \mu_r - b_0^L\} \leq \frac{3}{4}, 1 \leq \mu_r, 1 \leq \max\{\mu_p, \mu_q\}, \frac{1}{2} \leq 1, 1 \leq 1\}$. Therefore the branch with node $1S1\{2\}S2\emptyset L0L1D1A0$ is open, so the $\mathcal{K}$ -tableau for Ψ is open. Note that this is not the only possible solution and it is not the only open branch.

4.3.3 Soundness and completeness of the calculus

In this section we will show that every finite set Ψ of formulas of $BL_{\Delta\sim}$ is $\mathcal{K}$ -satisfiable for an arbitrary $\mathcal{K} \subseteq [0, 1]$ with respect to LP-norms iff we can construct an open $\mathcal{K}$ -tableau for Ψ . Since Ψ is finite, the $\mathcal{K}$ -tableaux we will work with in this section all have finite branches.

Example 4.21. Let $\mathcal{K} = [\frac{1}{2}, \frac{3}{4}] \cup \{1\}$. The set $\Psi = \{\bar{1} \rightarrow p \& r, \Delta r \rightarrow (p \vee q)\}$ is $\mathcal{K}$ -satisfiable with respect to an LP-norm. Take $V(p) = \frac{1}{2}, V(q) = 1, V(r) = 1$ and Lukasiewicz t-norm as $\star$. Then

$$\begin{aligned} V_\star(\bar{1} \rightarrow p \& r) = 1 &\Rightarrow (V(p) \star V(r)) = 1 \Rightarrow \max\{0, V(p) + V(r) - 1\} \\ &= 1 \Rightarrow \max\{0, \frac{1}{2} + 1 - 1\} = 1 \Rightarrow \frac{1}{2} = 1 - 1 + \frac{1}{2} = \frac{1}{2}, \\ V_\star(\Delta r \rightarrow (p \vee q)) &= \Delta V(r) \Rightarrow \max\{V(p), V(q)\} \\ &= 1 \Rightarrow \max\left\{\frac{1}{2}, 1\right\} = 1 \Rightarrow 1 = 1 \end{aligned}$$

and thus $V_\star(\psi) \in \mathcal{K}$ for all $\psi \in \Psi$. We use the model constructed in Example 4.20 to find values $V(p), V(q), V(r)$ and an LP-norm.

Definition 4.22. Let $\mathcal{B}$ a branch of a tableau $\mathcal{T}$. Let $\{a_i^{K_i}, b_i^{K_i} : 0 \leq i < n\}$, where $a_i^{K_i} < b_i^{K_i}$ and $K_i \in \{L, P\}$, be the parameters introduced by branch extension rules in $\mathcal{B}$. Let $l_{\mathcal{B}}$ be the leaf of the branch. Suppose that $(\mathcal{M}, \sigma)$ is a nice solution of $l_{\mathcal{B}}$ (see Definition 4.19).

1. We expand the model $\mathcal{M}$ to an L_1 -structure $\mathcal{M}_{\mathcal{B}} = (\mathbb{R}, +, -, \cdot, \div, \min, \max, \leq, <, 0, 1, \rho, (j_i^-, j_i^+) : i \in I, \star_{\mathcal{B}}, \Rightarrow_{\mathcal{B}}, \Delta)$, where j_i^-, j_i^+ are endpoints of interval J_i , such that for every $v, w \in \mathbb{R}$,

$$v \star_{\mathcal{B}} w = \begin{cases} \max\{\rho(a_k^L), v + w - \rho(b_k^L)\} & \text{if } v, w \in [\rho(a_k^L), \rho(b_k^L)], 0 \leq k \leq n-1, \\ \rho(a_k^P) + \frac{(v - \rho(a_k^P)) \cdot (w - \rho(a_k^P))}{\rho(b_k^P) - \rho(a_k^P)} & \text{if } v, w \in [\rho(a_k^P), \rho(b_k^P)], 0 \leq k \leq n-1, \\ \min(v, w) & \text{otherwise,} \end{cases}$$

$$v \Rightarrow_{\mathcal{B}} w = \begin{cases} 1 & \text{if } v \leq w, \\ \rho(b_k^L) - v + w & \text{if } \rho(a_k^L) \leq w < v \leq \rho(b_k^L), 0 \leq k \leq n-1, \\ \rho(a_k^P) + \frac{(w - \rho(a_k^P)) \cdot (\rho(b_k^P) - \rho(a_k^P))}{v - \rho(a_k^P)} & \text{if } \rho(a_k^P) \leq w < v \leq \rho(b_k^P), 0 \leq k \leq n-1, \\ w & \text{otherwise,} \end{cases}$$

$$\Delta v = \begin{cases} 1 & \text{if } 1 \leq v, \\ 0 & \text{if } v < 1. \end{cases}$$

Remark: This is easily checked to be well defined, although k in $\star_{\mathcal{B}}$ is not always unique.

2. $\mathcal{M}_{\mathcal{B}}, \sigma \models z_1 \vee z_2$ iff $\mathcal{M}_{\mathcal{B}}, \sigma \models z_1$ or $\mathcal{M}_{\mathcal{B}}, \sigma \models z_2$.
3. A node S of branch $\mathcal{B}$ is $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$ iff for all $s \in S$

$$\mathcal{M}_{\mathcal{B}}, \sigma \models s,$$

where the L_1 -structure $\mathcal{M}_{\mathcal{B}}$ is constructed from $\mathcal{M}$ as in point 1.

We say that branch $\mathcal{B}$ is *satisfiable* iff there exist $\mathcal{M}, \sigma$ such that all nodes of $\mathcal{B}$ are $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$. We denote by $\llbracket z \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$ the value of L_1 -term z in $\mathcal{M}_{\mathcal{B}}$ under the assignment σ .

Remark 4.23. By Theorems 2.36 and 2.38, and Remarks 2.41 and 2.42, $\star_{\mathcal{B}}$ is an LP-norm and $\Rightarrow_{\mathcal{B}}$ is its residuum when both functions are restricted to $[0, 1]$.

Theorem 4.24. (Soundness and Completeness)

Let $\mathcal{K} \subseteq [0, 1]$. Let Ψ be a finite set of $\text{BL}_{\Delta \sim}$ formulas and $\mathcal{T}$ be a $\mathcal{K}$ -tableau, whose root is $\{\eta_i(\tau(\psi)) : i \in I, \psi \in \Psi\}$ constructed as in Definition 4.18. Then the following are equivalent:

1. $\mathcal{T}$ has an open branch.
2. Ψ is $\mathcal{K}$ -satisfiable with respect to an LP-norm.

Proof. There are finitely many formulas in Ψ , therefore each branch of $\mathcal{T}$ is finite.¹¹ Suppose that branch $\mathcal{B}$ of tableau $\mathcal{T}$ is open. Since it is finite, a leaf $l_{\mathcal{B}}$ of branch $\mathcal{B}$ exists. Also, for every node n of branch $\mathcal{B}$, $n \upharpoonright_{L_0} \subseteq l_{\mathcal{B}}$. Since $\mathcal{B}$ is open, there is a nice solution $(\mathcal{M}, \sigma)$ of $l_{\mathcal{B}}$. We will construct a model $\mathcal{A} = ([0, 1], \star, \Rightarrow, \min, \max, \Delta, \ominus, 0, 1, V)$ such that $V_{\star}(\psi) \in \mathcal{K}$ for every $\psi \in \Psi$. First, we put $V(p) = \sigma(\mu_p)$ for all $p \in PROP$. We define operation $\star : [0, 1]^2 \rightarrow [0, 1]$ as $\star_{\mathcal{B}}$ and operation $\Rightarrow : [0, 1]^2 \rightarrow [0, 1]$ as $\Rightarrow_{\mathcal{B}}$ (see Definition 4.22). By Remark 4.23 this is a well-defined model.

Claim 1. Let θ be a subformula of ψ . Then $V_{\star}(\theta) = \llbracket \tau(\theta) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$.

Proof of the claim. We prove by induction on θ . The base case for propositional atoms is given and $V_{\star}(\bar{0}) = 0 = \llbracket 0 \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \tau(\bar{0}) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$ by Definitions 3.2, 4.22 and 4.17. Similarly for $\theta = \bar{1}$. Assume induction hypothesis for θ, φ . We show the claim for $\Delta\theta$. The other cases for $\theta \& \varphi, \theta \rightarrow \varphi, \theta \wedge \varphi, \theta \vee \varphi, \sim\theta$ are similar (cf. Lemma 4.12 Claim 1). By Definition 3.2 and the inductive hypothesis,

$$V_{\star}(\Delta\theta) = \Delta V_{\star}(\theta) = \Delta \llbracket \tau(\theta) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}.$$

Now, by Definition 4.22,

$$\Delta \llbracket \tau(\theta) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \Delta \tau(\theta) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma},$$

and by Definition 4.17,

$$\llbracket \Delta \tau(\theta) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \tau(\Delta\theta) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma},$$

which concludes the proof of the claim.

Claim 2. Every node $m_{\mathcal{B}}$ of $\mathcal{B}$ is $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$.

Proof of claim. By induction on $h(m_{\mathcal{B}}, \mathcal{B})$. By choice of $\mathcal{M}$ and σ , $l_{\mathcal{B}}$ is $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$. Let $m'_{\mathcal{B}}$ be the node of $\mathcal{B}$ such that $h(m'_{\mathcal{B}}, \mathcal{B}) = h(m_{\mathcal{B}}, \mathcal{B}) + 1$, where $m_{\mathcal{B}}$ is assumed to be $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$. Since $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge m_{\mathcal{B}}$, $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge (m_{\mathcal{B}} \cap m'_{\mathcal{B}})$ and $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge (m_{\mathcal{B}} - m'_{\mathcal{B}})$. By inspecting $m_{\mathcal{B}}$ and $m'_{\mathcal{B}}$, we know which tableau formulas γ are in $m'_{\mathcal{B}} - m_{\mathcal{B}}$ and thus by Definition 4.18, which rule has been applied to generate $m_{\mathcal{B}}$ from its predecessor $m'_{\mathcal{B}}$. We show the claim for one rule. The other rules can be shown analogously (see also Claim 2 in the proof of Lemma 4.12). Suppose that $\{\eta_{J_i}(\tau(\psi)) : i \in I\} = m'_{\mathcal{B}} - m_{\mathcal{B}}$. Then for some $S \subseteq I$, $\{\tau(\psi) \Diamond_{J_i}^- c_i^- : i \in S\} \cup \{c_i^+ \Diamond_{J_i}^+ \tau(\psi) : i \in I - S\} = m_{\mathcal{B}} - m'_{\mathcal{B}}$. Thus by Definition 4.22 point 2, $\mathcal{M}_{\mathcal{B}}, \sigma \models \eta_{J_i}(\tau(\psi))$ for all $i \in I$ if $\mathcal{M}_{\mathcal{B}}, \sigma \models \tau(\psi) \Diamond_{J_i}^- c_i^-$ for all $i \in S$ and $\mathcal{M}_{\mathcal{B}}, \sigma \models c_i^+ \Diamond_{J_i}^+ \tau(\psi)$ for all $i \in I - S$.

¹¹Though there may be infinitely many branches with infinite nodes if Ψ is infinite.

Therefore $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge(m'_{\mathcal{B}} - m_{\mathcal{B}})$. This completes the proof of the claim.

Therefore, in particular by Claim 2, the root of $\mathcal{T}$ is $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$, and thus by Definition 4.22 and Claim 1, $\llbracket \tau(\psi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star}(\psi) \in \mathcal{K}$ for every $\psi \in \Psi$. Therefore, Ψ is $\mathcal{K}$ -satisfiable.

Conversely, suppose that there is a model $\mathcal{A} = ([0, 1], \star, \Rightarrow, \min, \max, \Delta, \ominus, 0, 1, V)$, where $\star$ is an LP-norm and $\Rightarrow$ is its residuum, $V : PROP \rightarrow [0, 1]$ such that $V_{\star}(\psi) \in \mathcal{K}$ for all $\psi \in \Psi$. We know that for every node $n_{\mathcal{B}}$ of a branch $\mathcal{B}$, we have $n_{\mathcal{B}} \upharpoonright_{L_0} \subseteq l_{\mathcal{B}}$, where $l_{\mathcal{B}}$ is the leaf of the branch, which exists since all branches are finite. Therefore, a branch is open if its leaf $l_{\mathcal{B}}$ has a nice solution $(\mathcal{M}, \sigma)$. Therefore we need to construct such an L_0 -structure $\mathcal{M}$ and a mapping σ . First, we take $\sigma(\mu_p) = V(p)$ for all $p \in PROP$ and $\sigma(x) = 0$ for all $x \in Var - \mu(PROP)$. In (1) below, we construct a standard structure $\mathcal{M}$; that is we find the mapping $\rho : PAR \rightarrow [0, 1]$. At the same time we choose nodes on a branch, say $\mathcal{B}$. Then, in (2) we show that branch $\mathcal{B}$ is open.

We write Λ for the root of the tableau. We let $\circ$ range over $\{\star, \Rightarrow\}$, and $(\#, \#)$ range over $\{(\star, \&), (\Rightarrow, \rightarrow), (\min, \wedge), (\max, \vee)\}$ (cf. Lemma 4.12).

Let φ be a subformula of ψ , and let t be a tableau formula. We define an L_1 -term φ_{Γ} for each node Γ of the tableau by induction on $d(\Gamma)$. We let $\varphi_{\Lambda} = \tau(\varphi)$. If Γ is a successor of Γ' , the active term at Γ' was $x \circ y$, and the branch expansion rule that constructed Γ from Γ' replaced $x \circ y$ in formulas in Γ' by the L_0 -term z , then we define $\varphi_{\Gamma} = \varphi_{\Gamma'}[z/x \circ y]$. If Γ is a successor of Γ' , the active term at Γ' was Δx , and the branch expansion rule that constructed Γ from Γ' replaced Δx in formulas in Γ' by the L_0 -term z , then we define $\varphi_{\Gamma} = \varphi_{\Gamma'}[z/\Delta x]$.

Remark. Note that for an L_1 -term t ,

$$t[z/x \circ y] = \begin{cases} t, & \text{if } t \in Var \cup \{0, 1\}, \\ z, & \text{if } t = x \circ y, \\ f(t_1[z/x \circ y], \dots, t_n[z/x \circ y]) & \text{if } t = f(t_1, \dots, t_n) \neq x \circ y. \end{cases}$$

and

$$t[z/\Delta x] = \begin{cases} t, & \text{if } t \in Var \cup \{0, 1\}, \\ z, & \text{if } t = \Delta x, \\ f(t_1[z/\Delta x], \dots, t_n[z/\Delta x]) & \text{if } t = f(t_1, \dots, t_n) \neq \Delta x. \end{cases}$$

A term t is said to occur in a node Γ if t is a subterm of some term occurring in some tableau formula in Γ .

Claim 3. For each node Γ of the tableau,

1. each term $t\#u$ occurring in Γ , where $\# \notin L_0$, is equal to $(\theta\#\varphi)_\Gamma$ for some (not necessarily unique) subformula $\theta\#\varphi$ of ψ .
2. each term Δt occurring in Γ is equal to $(\Delta\theta)_\Gamma$ for some (not necessarily unique) subformula $\Delta\theta$ of ψ .

Proof of the claim. The proof is easily adapted from the proof of Claim 3 of Lemma 4.12.

(1) Since we know the values $V_\star(\psi)$ for all $\psi \in \Psi$, we can select the appropriate node from the nodes which were generated by Split Rules applied to each $\eta_i(\tau(\psi)), i \in I$ for each $\psi \in \Psi$. To illustrate the latter, suppose that $j_1^+ \leq V_\star(\psi) \leq j_2^-$ for some $j_1^+, j_2^- \in [0, 1]$ and some $\psi \in \Psi$. Suppose $J_1 = [0, j_1^+)$, $J_2 = (j_2^-, 1]$ and we have $\eta_{J_1}(\tau(\psi))$ as $(\tau(\psi) < 0) \vee (c_1^+ \leq \tau(\psi))$ and $\eta_{J_2}(\tau(\psi))$ as $(\tau(\psi) \leq c_2^-) \vee (1 < \tau(\psi))$. Let ψ have number l given by well-ordering of Ψ . Then we choose node $1...S\{2\}...$, which will have $c_1^+ \leq \tau(\psi), \tau(\psi) \leq c_2^-$. We will interpret c_1^+, c_2^- as j_1, j_2 in $\mathcal{M}$.

Then we proceed in the following way. Let $\star$ be $\sum_{i \in \mathfrak{C}} ([\alpha_i^{K_i}, \beta_i^{K_i}], \ast_i)$. We will assign values of parameters occurring on $\mathcal{B}$ under ρ to elements of $\{\alpha_i^{K_i}, \beta_i^{K_i} : i \in \mathfrak{C}\}$. Suppose we selected the sequence of nodes $n_1, \dots, n_l$, where $l \geq 1$, n_1 is the root of $\mathcal{T}$ and n_{i+1} is a successor of n_i for all $1 \leq i < l$. That is branch $\mathcal{B}$ is partially defined and ρ is defined for all parameters in these nodes. By comparing n_l and its successors, we deduce what the active term is and thus which of the branch expansion rules has been applied. Suppose it is $(\Rightarrow)$, the other cases are similar (see also the proof of Lemma 4.12). Let Γ_1 be the set of L_1 -terms in n_l that are not in any of the successors of n_l . Let the active term at n_l be $x \Rightarrow y$, say. This term occurs in n_l . So by the claim above, we can choose a subformula $\psi_1 \rightarrow \psi_2$ of ψ with $x \Rightarrow y = (\psi_1 \rightarrow \psi_2)_{n_l}$.

We know whether or not there is $i \in \mathfrak{C}$ such that $V_\star(\psi_1), V_\star(\psi_2) \in [\alpha_i^{K_i}, \beta_i^{K_i}]$, and whether or not $V_\star(\psi_1) \leq V_\star(\psi_2)$. If $V_\star(\psi_1) > V_\star(\psi_2)$ and if there is such i ¹² and $K_i = L$ (or $K_i = P$), we select the subrule L. of $(\Rightarrow)$ (the subrule P. of $(\Rightarrow)$, respectively), and depending on the relation of $\alpha_i^{K_i}, \beta_i^{K_i}$ to those $\alpha_j^{K_j}, \beta_j^{K_j}$ that are the values of parameters occurring on n_l , we select the node, say n'_l , resulting from Cases 1-4, and 10. Let ρ assign to the parameters at n'_l that do not occur in n_l , say a, b with $a < b$, values $\rho(a) = \alpha_i^{K_i}, \rho(b) = \beta_i^{K_i}$. Suppose now there is no $i \in \mathfrak{C}$ such that $V_\star(\psi_1), V_\star(\psi_2) \in [\alpha_i^{K_i}, \beta_i^{K_i}]$, and $V_\star(\psi_1) > V_\star(\psi_2)$. We know the relations among $V_\star(\psi_1), V_\star(\psi_2)$, and the values of parameters occurring in n_l , thus we know which of Cases 5-9, and 11 match these values in the model $\mathcal{A}$. The remaining case is $V_\star(\psi_1) \leq V_\star(\psi_2)$, for which there is only one successor. Therefore, we can select the subsequent node. We

¹²If there are two such i we choose the least one.

have now selected the next node in the path from the root. The procedure terminates at a leaf, where there are only L_0 -terms occurring, at which point we selected all nodes in branch $\mathcal{B}$. We also partially defined the function ρ . To the parameters that have not received values under ρ in this procedure, we assign 0. We have now constructed a standard L_0 -structure $\mathcal{M}$.

(2) To be able to show that $\mathcal{M}, \sigma \models e$ for all $e \in l_{\mathcal{B}}$, where $l_{\mathcal{B}}$ is the leaf of branch $\mathcal{B}$, it is sufficient to prove that for all nodes $n_{\mathcal{B}}$ in $\mathcal{B}$, we have $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge n_{\mathcal{B}}$. We also show that $(\mathcal{M}, \sigma)$ is a nice solution. First, we show Claim 4.

Claim 4. Let $\Gamma \in \mathcal{B}$ be arbitrary. Then

1. For each subformula $\theta \# \varphi$ ($\Delta \theta$) of ψ , if $(\theta \# \varphi)_{\Gamma}$ ($(\Delta \theta)_{\Gamma}$, resp.) is not an L_0 -term, then $(\theta \# \varphi)_{\Gamma} = \theta_{\Gamma} \# \varphi_{\Gamma}$ ($(\Delta \theta)_{\Gamma} = \Delta \theta_{\Gamma}$, resp.).
2. For each subformula φ of ψ , if φ_{Γ} is an L_0 -term, then $\llbracket \varphi_{\Gamma} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star}(\varphi)$.
3. For each subformula φ of ψ , we have $\llbracket \varphi_{\Gamma} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = \llbracket \varphi_{\Lambda} \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma}$.
4. For each subformula φ of ψ , φ_{Γ} occurs in Γ .
5. For each subformula φ of ψ , we have $\llbracket \tau(\varphi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} = V_{\star}(\varphi)$.

Proof of the claim. The proof is easily adapted from the proof of Claim 4 of Lemma 4.12.

Claim 5. For every node $n_{\mathcal{B}}$ of $\mathcal{B}$, $\mathcal{M}_{\mathcal{B}}, \sigma \models \bigwedge n_{\mathcal{B}}$ and $(\mathcal{M}, \sigma)$ is a nice solution.

Proof of the claim. By induction on $d(n_{\mathcal{B}})$. For the base case: $d(n_{\mathcal{B}}) = 0$. By Claim 4 part 5 and the assumption, $\llbracket \tau(\psi) \rrbracket^{\mathcal{M}_{\mathcal{B}}, \sigma} \in \mathcal{K}$ for every $\psi \in \Psi$. Therefore $\mathcal{M}_{\mathcal{B}}, \sigma \models \{\eta_{J_i}(\tau(\psi)) : i \in I\}$.

Assume the induction hypothesis for a non-leaf node $n_{\mathcal{B}}$ of branch $\mathcal{B}$. We need to show that if $\mathcal{M}_{\mathcal{B}}, \sigma \models f$ for all $f \in n_{\mathcal{B}}$, then $\mathcal{M}_{\mathcal{B}}, \sigma \models f'$ for all $f' \in n'_{\mathcal{B}}$, where $n'_{\mathcal{B}} \in \mathcal{B}$ and $d(n'_{\mathcal{B}}) = d(n_{\mathcal{B}}) + 1$.

By inspecting $n_{\mathcal{B}}, n'_{\mathcal{B}}$, we know which formulas are in $n_{\mathcal{B}} - n'_{\mathcal{B}}$. Let $x = (\psi_1)_{n_{\mathcal{B}}}, y = (\psi_2)_{n_{\mathcal{B}}}$ as chosen in (1) above. Suppose that $(\psi_1 \rightarrow \psi_2)_{n_{\mathcal{B}}} = x \Rightarrow y$ is the active term of $\gamma \in n_{\mathcal{B}} - n'_{\mathcal{B}}$; the other cases are similar (see the proof of Lemma 4.12). By (1) above, we know which subrule of the branch expansion rule ($\Rightarrow$) and which of its cases are used to generate $n'_{\mathcal{B}}$. Thus, suppose that it was subrule L. and Case 1 (again, the other cases are similar). Let a, b be the new parameters occurring at $n'_{\mathcal{B}}$. Thus, $\{0 \leq a < b \leq a_0^L\} \subseteq n'_{\mathcal{B}} - n_{\mathcal{B}}$, where a_0^L is a parameter occurring at $n_{\mathcal{B}}$. By (1) above, we know that $0 \leq \rho(a) < \rho(b) \leq \rho(a_0^L)$. The other elements of $n'_{\mathcal{B}} - n_{\mathcal{B}}$ are (a) $a \leq y < x \leq b$, and (b) the elements that belong to $\{\gamma[b - x + y/x \Rightarrow y] : \gamma \in n_{\mathcal{B}} - n'_{\mathcal{B}}\}$. By (1) above Claim 4 part 2, the L_0 -formulas in (a) are $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$ and x, y

are not marked since $V_\star(\psi_1), V_\star(\psi_2) \in [\rho(a), \rho(b)]$. By **(1)** above and Claim 4 part 3, the L_1 -formulas in (b) are $\mathcal{B}$ -satisfiable via $\mathcal{M}, \sigma$ (the argument is similar to the one used in Claim 5 of Lemma 4.12). We can show that $(\mathcal{M}, \sigma)$ is nice as in Claim 5 of Lemma 4.12. This completes the proof of the claim.

By Claim 5, we showed also that $\mathcal{M}, \sigma \models f'$ for all $f' \in n'_\mathcal{B} \upharpoonright_{L_0}$ and thus in particular $\mathcal{M}, \sigma \models e$ for all $e \in l_\mathcal{B}$. Thus, there is an L_0 -structure modelling $l_\mathcal{B}$, $\mathcal{M}$, and a nice mapping σ such that $\mathcal{M}, \sigma \models e$ for all $e \in l_\mathcal{B}$. Therefore $\mathcal{B}$ is open. $\square$

Corollary 4.25. Let $\mathcal{K} = [0, 1)$. Let Ψ be a finite set of $\text{BL}_{\Delta\sim}$ formulas and $\mathcal{T}$ be a $\mathcal{K}$ -tableau, whose root is $\{\eta_i(\tau(\psi)) : i \in I, \psi \in \Psi\}$ constructed as in Definition 4.18. Then every such tableau is closed iff every $\psi \in \Psi$ is 1-tautology with respect to every LP-norm.

Proof. By Theorem 4.24 and Definition 3.10. $\square$

5. PROOF SYSTEMS FOR FUZZY LOGICS

Investigating strong standard completeness (defined later in this chapter) in fuzzy logics captures the interest of many logicians. We will briefly look at some of the known positive and negative results for propositional fuzzy logics.

An axiomatic system consists of a set of formulas called axioms, and inference rules by which from axioms we can derive new formulas. Such a system is called strongly standard complete if from an arbitrary set T of formulas we can derive a formula θ using axioms and inference rules, whenever the truth value of θ is 1 given that the truth values of all formulas from T are 1 (more precise definitions will follow later in this chapter). The axiomatic system for Gödel logic as defined by Hájek is strongly standard complete (see [Háj98b]), but the axiomatic systems for Łukasiewicz, Product and BL logics defined by Hájek do not have this property (see [Háj98b] and Examples 5.15, 5.16 and 5.17). However, the picture changes if infinitary rules (rules by which from infinitely many formulas we derive a new formula) are allowed.

To achieve a strong standard completeness result for continuous t-norms, Montagna in [Mon07] extended the language of BL by introducing a new unary connective $*$, called a storage operator, defined as follows: for a formula φ , φ^* is the greatest idempotent below φ . He introduced an infinitary rule (R) to the axiomatic system of his extended BL (which he denotes by BL^*):

$$\frac{\varphi \vee (\alpha \rightarrow \beta^n) : n < \omega}{\varphi \vee (\alpha \rightarrow \beta^*)} \quad (R)$$

He shows that the axiomatic system of BL^* with (R) is strongly standard complete. Montagna similarly extended the language of Łukasiewicz and Product logics, and showed that their axiomatic systems, when extended by the rule (R) and some additional axioms about the connective $*$, are also strongly standard complete.

Another result presented in [VBEG17] demonstrates that an expansion of MTL (Monoidal t-norm based logic) is strongly complete with respect to left continuous t-norms. The authors achieved this by expanding the language by the Baaz connective Δ (see Section 3.1) and countably many (Pavelka-style) truth-constants with one infinitary rule based on the Takeuti-Titani density rule (see [TT84]); the infinitary rule is shown below:

$$\frac{(\varphi \rightarrow \bar{c}) \vee (\bar{c} \rightarrow \psi) : c \in Q_*}{\varphi \rightarrow \psi}, \quad (D^\infty)$$

where Q_* is the subalgebra of $([0, 1], \cap, \cup, *, \Rightarrow, \Delta)$ generated by the rational numbers in $[0, 1]$ (see pp. 133-134 of [VBEG17]).

In both cases the authors extended the language by introducing either a unary con-

nective or nullary connectives and then adding infinitary rules to their axiomatic systems to achieve the desired result. We will not extend the language of fuzzy logics: we will only add one infinitary rule to Hajek's axiomatic system.

In this chapter we will look at the axioms and the axiomatic systems that have been devised for these logics as presented by Hájek in [Háj98b]. We will then extend these axiomatic systems by an infinitary rule suggested by a counterexample for strong standard completeness of BL. We will prove that our extended axiomatic systems for BL, Łukasiewicz, and Product logics are sound, and in Chapter 6 we prove that they are strongly standard complete. The extended axiomatic systems for fuzzy logics presented in this chapter constitute a part of a conference talk (see [Ku16]) and is submitted to Fuzzy Sets and Systems¹³.

5.1. Axioms for fuzzy logics. In this section, we define a set of formulas (Definition 5.1), which we call BL-axioms, and we prove that they are valid (Lemma 5.2 proven in Lemma 2.2.6 of [Háj98b]). We also recall some other formulas and prove that they are 1-tautologies of some continuous t-norms (Lemma 5.3 contained in Lemma 4.1.2, Theorems 3.2.13 and 4.2.17 of [Háj98b]). In Section 5.2 these formulas will become axioms of the axiomatic systems defined there, and Lemmas 5.2 and 5.3 will be used in a proof of general soundness of these axiomatic systems (see Lemma 5.13).

For the rest of the thesis, we fix $PROP$ to be a countably infinite set of propositional atoms and we let $\mathcal{F}$ denote the set of all formulas constructed as in Definition 3.1. However, henceforth we will treat $\bar{0}, \&, \rightarrow$ as primitive connectives and we will regard the other connectives in Definition 3.1 as abbreviations in the following way: $\bar{1}$ as $\bar{0} \rightarrow \bar{0}$, $\neg\varphi$ as $\varphi \rightarrow \bar{0}$, $\varphi \wedge \psi$ as $\varphi \& (\varphi \rightarrow \psi)$, $\varphi \vee \psi$ as $((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi)$, $\psi \underline{\vee} \varphi$ as $\neg(\neg\psi \& \neg\varphi)$, and $\varphi \equiv \psi$ as $(\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi)$. We call an element of $\mathcal{F}$ a *formula* and denote it by $\alpha, \beta, \gamma, \varphi, \psi$, etc. We also define for any formula φ , φ^n for $n < \omega$ inductively by $\varphi^0 = \bar{1}$, $\varphi^{n+1} = \varphi^n \& \varphi$.

Definition 5.1. We will call the following formulas *BL-axioms*. Let $\alpha, \beta, \gamma \in \mathcal{F}$ be arbitrary.

$$(A1) \quad (\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))$$

$$(A2) \quad \alpha \& \beta \rightarrow \alpha$$

$$(A3) \quad \alpha \& \beta \rightarrow \beta \& \alpha$$

$$(A4) \quad \alpha \& (\alpha \rightarrow \beta) \rightarrow \beta \& (\beta \rightarrow \alpha)$$

$$(A5a) \quad (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\alpha \& \beta \rightarrow \gamma)$$

¹³The referees consider the material worthy publishing.

$$(A5b) \quad (\alpha \& \beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow (\beta \rightarrow \gamma))$$

$$(A6) \quad ((\alpha \rightarrow \beta) \rightarrow \gamma) \rightarrow (((\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \gamma)$$

$$(A7) \quad \bar{0} \rightarrow \alpha$$

BL-axiom (A1) expresses transitivity of the implication ($\rightarrow$), (A2) – the entailment of the first factor for the strong conjunction ($\&$), (A3) – commutativity of the strong conjunction ($\&$), (A4) – commutativity of the weak conjunction ($\wedge$), (A5a) and (A5b) – the residuation condition, (A6) – semilinearity, and (A7) expresses the *ex falso quodlibet* law. We use the numbering of axioms introduced in [Háj98b].

We also consider the following additional axioms, in which $\alpha, \beta, \gamma \in \mathcal{F}$ are arbitrary:

$$(\neg\neg) \quad \neg\neg\alpha \rightarrow \alpha$$

$$(II1) \quad \neg\neg\gamma \rightarrow ((\alpha \& \gamma \rightarrow \beta \& \gamma) \rightarrow (\alpha \rightarrow \beta))$$

$$(II2) \quad \alpha \wedge \neg\alpha \rightarrow \bar{0}$$

$$(G) \quad \alpha \rightarrow \alpha \& \alpha$$

Axiom $(\neg\neg)$ expresses involution of the negation ($\neg$) for Łukasiewicz logic. Axioms (II1) and (II2) express cancellation of non-zero elements and strictness of residual negation, respectively, for Product logic, and finally axiom (G) expresses that every element is an idempotent in Gödel logic.

Lemma 5.2. (see Lemma 2.2.6 of [Háj98b])

The BL-axioms are valid.

Proof. Take an arbitrary evaluation $V : PROP \rightarrow [0, 1]$ and continuous t-norm $\star$. Let $\chi \in \mathcal{F}$ be an axiom; we need to show that $V_\star(\chi) = 1$. Let $\alpha, \beta, \gamma \in \mathcal{F}$ be arbitrary.

(A1) By Definition 3.2, $V_\star((\alpha \rightarrow \beta) \rightarrow ((\beta \rightarrow \gamma) \rightarrow (\alpha \rightarrow \gamma))) = (V_\star(\alpha) \Rightarrow_\star V_\star(\beta)) \Rightarrow_\star ((V_\star(\beta) \Rightarrow_\star V_\star(\gamma)) \Rightarrow_\star (V_\star(\alpha) \Rightarrow_\star V_\star(\gamma)))$. Since, obviously, $V_\star(\alpha) \Rightarrow_\star V_\star(\beta) \leq V_\star(\alpha) \Rightarrow_\star V_\star(\beta)$, by Proposition 2.16, $V_\star(\alpha) \star (V_\star(\alpha) \Rightarrow_\star V_\star(\beta)) \leq V_\star(\beta)$. Therefore by monotonicity, associativity and boundary condition of $\star$, $V_\star(\alpha) \star (V_\star(\alpha) \Rightarrow_\star V_\star(\beta)) \star (V_\star(\beta) \Rightarrow_\star V_\star(\gamma)) \star 1 \leq V_\star(\gamma)$. Using Proposition 2.16 three times, we have $1 \leq (V_\star(\alpha) \Rightarrow_\star V_\star(\beta)) \Rightarrow_\star ((V_\star(\beta) \Rightarrow_\star V_\star(\gamma)) \Rightarrow_\star (V_\star(\alpha) \Rightarrow_\star V_\star(\gamma)))$, which proves that the axiom is valid.

(A2) By Definition 3.2, $V_\star(\alpha \& \beta \rightarrow \alpha) = V_\star(\alpha) \star V_\star(\beta) \Rightarrow_\star V_\star(\alpha)$. By monotonicity, associativity and boundary condition of $\star$, $V_\star(\alpha) \star V_\star(\beta) \star 1 \leq V_\star(\alpha)$. Using Proposition 2.16, $1 \leq V_\star(\alpha) \star V_\star(\beta) \Rightarrow_\star V_\star(\alpha)$.

(A3) By Definition 3.2, $V_*(\alpha \& \beta \rightarrow \beta \& \alpha) = V_*(\alpha) \star V_*(\beta) \Rightarrow_* V_*(\beta) \star V_*(\alpha)$. By commutativity, associativity and boundary condition of $\star$, $V_*(\alpha) \star V_*(\beta) \star 1 \leq V_*(\beta) \star V_*(\alpha)$. Using Proposition 2.16, $1 \leq V_*(\alpha) \star V_*(\beta) \Rightarrow_* V_*(\beta) \star V_*(\alpha)$.

(A4) By Definition 3.2, $V_*(\alpha \& (\alpha \rightarrow \beta) \rightarrow \beta \& (\beta \rightarrow \alpha)) = V_*(\alpha) \star (V_*(\alpha) \Rightarrow_* V_*(\beta)) \Rightarrow_* V_*(\beta) \star (V_*(\beta) \Rightarrow_* V_*(\alpha))$. If $V_*(\alpha) \leq V_*(\beta)$, then by Lemma 2.17, $1 = V_*(\alpha) \Rightarrow_* V_*(\beta)$ and $V_*(\alpha) = V_*(\beta) \star (V_*(\beta) \Rightarrow_* V_*(\alpha))$, and thus by boundary condition of $\star$, $V_*(\alpha) \star (V_*(\alpha) \Rightarrow_* V_*(\beta)) = V_*(\beta) \star (V_*(\beta) \Rightarrow_* V_*(\alpha))$, and $1 = V_*(\alpha) \star (V_*(\alpha) \Rightarrow_* V_*(\beta)) \Rightarrow_* V_*(\beta) \star (V_*(\beta) \Rightarrow_* V_*(\alpha))$. Similarly if $V_*(\beta) \leq V_*(\alpha)$.

(A5a) and (A5b) Immediate by Definition 3.2 and Proposition 2.16.

(A6) By Definition 3.2, $V_*(((\alpha \rightarrow \beta) \rightarrow \gamma) \rightarrow (((\beta \rightarrow \alpha) \rightarrow \gamma) \rightarrow \gamma)) = ((V_*(\alpha) \Rightarrow_* V_*(\beta)) \Rightarrow_* V_*(\gamma)) \Rightarrow_* (((V_*(\beta) \Rightarrow_* V_*(\alpha)) \Rightarrow_* V_*(\gamma)) \Rightarrow_* V_*(\gamma))$. If $V_*(\alpha) \leq V_*(\beta)$, then by Lemma 2.17, (1) $V_*(\alpha) \Rightarrow_* V_*(\beta) = 1$, and (2) $1 \Rightarrow_* V_*(\gamma) = V_*(\gamma)$, so by soundness of BL-axiom (A2), associativity and boundary condition of $\star$, $V_*(\gamma) \star ((V_*(\beta) \Rightarrow_* V_*(\alpha)) \Rightarrow_* V_*(\gamma)) \star 1 \leq V_*(\gamma)$. Therefore by commutativity of $\star$ and Proposition 2.16 applied twice, $1 \leq V_*(\gamma) \Rightarrow_* (((V_*(\beta) \Rightarrow_* V_*(\alpha)) \Rightarrow_* V_*(\gamma)) \Rightarrow_* V_*(\gamma))$. By soundness of BL-axioms (A5a) and (A5b), commutativity of $\star$, and by (1) and (2), $((V_*(\alpha) \Rightarrow_* V_*(\beta)) \Rightarrow_* V_*(\gamma)) \Rightarrow_* (((V_*(\beta) \Rightarrow_* V_*(\alpha)) \Rightarrow_* V_*(\gamma)) \Rightarrow_* V_*(\gamma)) = ((V_*(\beta) \Rightarrow_* V_*(\alpha)) \Rightarrow_* V_*(\gamma)) \Rightarrow_* (((V_*(\alpha) \Rightarrow_* V_*(\beta)) \Rightarrow_* V_*(\gamma)) \Rightarrow_* V_*(\gamma))$, and we can prove analogously the case of $V_*(\beta) \leq V_*(\alpha)$.

(A7) Immediate by Definition 3.2, Proposition 2.16, Remark 2.3 and since values of V_* are non-negative.

□

Lemma 5.3. 1. Axiom $(\neg\neg)$ is a 1-tautology of the Łukasiewicz t-norm (see Lemma 3.2.13 of [Háj98b]).

2. Axiom (Π1) is a 1-tautology of the product t-norm (see Lemma 4.1.2 of [Háj98b]).

3. Axiom (Π2) is a 1-tautology of the product t-norm (see Lemma 4.1.2 of [Háj98b]).

4. Axiom (G) is a 1-tautology of the Gödel t-norm (see Theorem 4.2.17(1) of [Háj98b]).

Proof. Let $\star$ be a continuous t-norm. First, $0 \Rightarrow_* 0 = 1$ by soundness of BL-axiom (A7). By Proposition 2.16, $1 \Rightarrow_* 0 = \sup\{z \in [0, 1] : 1 \star z \leq 0\}$ and $1 \Rightarrow_* 1 = \sup\{z \in [0, 1] : 1 \star z \leq 1\}$. Thus $1 \Rightarrow_* 0 = 0$ and $1 \Rightarrow_* 1 = 1$ by boundary condition of $\star$. These facts will be used below. Now let $V : PROP \rightarrow [0, 1]$ be an arbitrary evaluation.

1. Let $\star$ be the Łukasiewicz t-norm. By Definition 3.2 and definition of $\neg$, $V_*(\neg\neg\alpha \rightarrow \alpha) = ((V_*(\alpha) \Rightarrow_* 0) \Rightarrow_* 0) \Rightarrow_* V_*(\alpha)$. We show that $(V_*(\alpha) \Rightarrow_* 0) \Rightarrow_* 0 = V_*(\alpha)$

and thus by Lemma 2.17, we have $((V_*(\alpha) \Rightarrow_* 0) \Rightarrow_* 0) \Rightarrow_* V_*(\alpha) = 1$. By Proposition 2.20, $(V_*(\alpha) \Rightarrow_* 0) \Rightarrow_* 0 = 1 - (1 - V_*(\alpha)) = V_*(\alpha)$.

2. Let $\star$ be the product t-norm. By Definition 3.2, $V_*(\neg\neg\gamma \rightarrow ((\alpha \& \gamma \rightarrow \beta \& \gamma) \rightarrow (\alpha \rightarrow \beta))) = V_*(\neg\neg\gamma) \Rightarrow_* ((V_*(\alpha) \star V_*(\gamma) \Rightarrow_* V_*(\beta) \star V_*(\gamma)) \Rightarrow_* (V_*(\alpha) \Rightarrow_* V_*(\beta)))$. Assume $V_*(\gamma) = 0$. Then $V_*(\neg\neg\bar{0}) = (0 \Rightarrow_* 0) \Rightarrow_* 0 = 1 \Rightarrow_* 0 = 0$. Thus, by soundness of BL-axiom (A7) and Lemma 2.17, $V_*(\neg\neg\gamma) \Rightarrow_* ((V_*(\alpha) \star V_*(\gamma) \Rightarrow_* V_*(\beta) \star V_*(\gamma)) \Rightarrow_* (V_*(\alpha) \Rightarrow_* V_*(\beta))) = 1$. Assume now that $V_*(\gamma) > 0$. Thus $V_*(\neg\neg\gamma) = (V_*(\gamma) \Rightarrow_* 0) \Rightarrow_* 0 = 0 \Rightarrow_* 0 = 1$. Now if $1 \leq V_*(\alpha) \star V_*(\gamma) \Rightarrow_* V_*(\beta) \star V_*(\gamma)$, then by Lemma 2.17, $V_*(\alpha) \star V_*(\gamma) = V_*(\alpha) \cdot V_*(\gamma) \leq V_*(\beta) \star V_*(\gamma) = V_*(\beta) \cdot V_*(\gamma)$. Thus we can divide the last inequality by $V_*(\gamma)$ and obtain $V_*(\alpha) \leq V_*(\beta)$. Therefore by Lemma 2.17, $1 = V_*(\alpha) \Rightarrow_* V_*(\beta)$. Consequently, $V_*(\neg\neg\gamma) \Rightarrow_* ((V_*(\alpha) \star V_*(\gamma) \Rightarrow_* V_*(\beta) \star V_*(\gamma)) \Rightarrow_* (V_*(\alpha) \Rightarrow_* V_*(\beta))) = 1 \Rightarrow_* (1 \Rightarrow_* 1) = 1$. If $V_*(\alpha) \star V_*(\gamma) \Rightarrow_* V_*(\beta) \star V_*(\gamma) < 1$, then by Proposition 2.16, $V_*(\alpha) \star V_*(\gamma) = V_*(\alpha) \cdot V_*(\gamma) > V_*(\beta) \star V_*(\gamma) = V_*(\beta) \cdot V_*(\gamma)$. Then $V_*(\alpha) \cdot V_*(\gamma) \Rightarrow_* V_*(\beta) \cdot V_*(\gamma) = \frac{V_*(\beta) \cdot V_*(\gamma)}{V_*(\alpha) \cdot V_*(\gamma)} = \frac{V_*(\beta)}{V_*(\alpha)} = V_*(\alpha) \Rightarrow_* V_*(\beta)$. Therefore, by Lemma 2.17, $(V_*(\alpha) \cdot V_*(\gamma) \Rightarrow_* V_*(\beta) \cdot V_*(\gamma)) \Rightarrow_* (V_*(\alpha) \Rightarrow_* V_*(\beta)) = 1$. Consequently, $V_*(\neg\neg\gamma) \Rightarrow_* ((V_*(\alpha) \star V_*(\gamma) \Rightarrow_* V_*(\beta) \star V_*(\gamma)) \Rightarrow_* (V_*(\alpha) \Rightarrow_* V_*(\beta))) = 1 \Rightarrow_* 1 = 1$.
3. Let $\star$ be the product t-norm. By Definitions 3.1 and 3.2, $V_*(\alpha \wedge \neg\alpha \rightarrow \bar{0}) = V_*(\alpha) \star (V_*(\alpha) \Rightarrow_* (V_*(\alpha) \Rightarrow_* 0)) \Rightarrow_* 0$. Assume $V_*(\alpha) = 0$. Then by Remark 2.3, commutativity, associativity and boundary condition of $\star$, $1 \star V_*(\alpha) \star (V_*(\alpha) \Rightarrow_* (V_*(\alpha) \Rightarrow_* 0)) = 0$ and thus by Proposition 2.16, $1 \leq V_*(\alpha) \star (V_*(\alpha) \Rightarrow_* (V_*(\alpha) \Rightarrow_* 0)) \Rightarrow_* 0$. Suppose $V_*(\alpha) > 0$. Then $V_*(\alpha) \Rightarrow_* 0 = 0$ and by Remark 2.3, associativity, commutativity and boundary condition of $\star$, $1 \star V_*(\alpha) \star (V_*(\alpha) \Rightarrow_* (V_*(\alpha) \Rightarrow_* 0)) = 0$. Therefore by Proposition 2.16, $1 \leq V_*(\alpha) \star (V_*(\alpha) \Rightarrow_* (V_*(\alpha) \Rightarrow_* 0)) \Rightarrow_* 0$.
4. Let $\star$ be the Gödel t-norm. By Definition 3.2, $V_*(\alpha \rightarrow \alpha \& \alpha) = V_*(\alpha) \Rightarrow_* V_*(\alpha) \star V_*(\alpha)$. Since $\star$ is Gödel t-norm, $V_*(\alpha) = \min\{V_*(\alpha), V_*(\alpha)\} = V_*(\alpha) \star V_*(\alpha)$. By Lemma 2.17, $1 = V_*(\alpha) \Rightarrow_* V_*(\alpha) \star V_*(\alpha)$.

□

In Definition 3.2, $\wedge$ and $\vee$ were primitive operations and so their semantics were defined explicitly. Since we are now taking $\wedge, \vee$ to be abbreviations, their semantics are now determined by the semantics of the primitive operations $\&, \rightarrow$. However, as shown in Lemma 5.4, in fact their semantics are unchanged. We will use the semantics of these operations $(\wedge, \vee)$ in a proof of general soundness in the next section (Lemma 5.13). We will also prove that if $\star$ is the Łukasiewicz t-norm, $V_*(\alpha \underline{\vee} \beta) = \min\{1, V_*(\alpha) + V_*(\beta)\}$, which is used in Example 5.15.

Lemma 5.4. Let $V : PROP \rightarrow [0, 1]$ be an arbitrary evaluation and $\star$ a continuous t-norm. Then,

1. $V_\star(\alpha \wedge \beta) = \min\{V_\star(\alpha), V_\star(\beta)\}$,
2. $V_\star(\alpha \vee \beta) = \max\{V_\star(\alpha), V_\star(\beta)\}$,
3. $V_\star(\alpha \underline{\vee} \beta) = \min\{1, V_\star(\alpha) + V_\star(\beta)\}$ when $\star$ is the Łukasiewicz t-norm.

Proof. 1. By definition of $\wedge$, $V_\star(\alpha \wedge \beta) = V_\star(\alpha) \star (V_\star(\alpha) \Rightarrow_\star V_\star(\beta))$. Assume $V_\star(\alpha) \leq V_\star(\beta)$. Then by boundary condition of $\star$ and Lemma 2.17,

$$\min\{V_\star(\alpha), V_\star(\beta)\} = V_\star(\alpha) = V_\star(\alpha) \star 1 = V_\star(\alpha) \star (V_\star(\alpha) \Rightarrow_\star V_\star(\beta)).$$

The other case we prove analogously using Lemma 5.2.

2. By definition of $\vee$, $V_\star(\alpha \vee \beta) = V_\star(((\alpha \rightarrow \beta) \rightarrow \beta) \wedge ((\beta \rightarrow \alpha) \rightarrow \alpha))$. By point 1,

$$V_\star(((\alpha \rightarrow \beta) \rightarrow \beta) \wedge ((\beta \rightarrow \alpha) \rightarrow \alpha)) = \min\{V_\star((\alpha \rightarrow \beta) \rightarrow \beta), V_\star((\beta \rightarrow \alpha) \rightarrow \alpha)\}.$$

Suppose that $V_\star(\alpha) \leq V_\star(\beta)$. Then by Definition 3.2 and Lemma 2.17, $V_\star((\alpha \rightarrow \beta) \rightarrow \beta) = V_\star(\beta)$. By Lemma 2.17 and assumption, $V_\star(\beta) \star (V_\star(\beta) \Rightarrow_\star V_\star(\alpha)) = V_\star(\alpha)$. Thus, by Proposition 2.16, $V_\star(\beta) \leq (V_\star(\beta) \Rightarrow_\star V_\star(\alpha)) \Rightarrow_\star V_\star(\alpha) = V_\star((\beta \rightarrow \alpha) \rightarrow \alpha)$. Therefore, $\min\{V_\star((\alpha \rightarrow \beta) \rightarrow \beta), V_\star((\beta \rightarrow \alpha) \rightarrow \alpha)\} = V_\star(\beta) = \max\{V_\star(\alpha), V_\star(\beta)\}$. Similarly, we prove for the other case.

3. Let $\star$ be the Łukasiewicz t-norm. By definition of $\underline{\vee}$, $V_\star(\alpha \underline{\vee} \beta) = V_\star(\neg\alpha) \star V_\star(\neg\beta) \Rightarrow_\star 0$. Then, $V_\star(\neg\alpha) = 1 - V_\star(\alpha)$, $V_\star(\neg\beta) = 1 - V_\star(\beta)$, and $(V_\star(\neg\alpha) \star V_\star(\neg\beta)) \Rightarrow_\star 0 = 1 - \max\{0, 1 - V_\star(\alpha) + 1 - V_\star(\beta) - 1\} = \min\{1, V_\star(\alpha) + V_\star(\beta)\}$.

□

5.2. Axiomatic systems for fuzzy logics. In this section we recall the definition of Hilbert-style axiomatic systems for logics (Definition 5.5). We will define semantic consequence (from a set T of formulas whose truth-value is 1 we conclude that a given formula has a truth-value 1; the precise definition will follow) and syntactic consequence (a proof of a formula from T). We show that the axiomatic systems for the aforementioned logics are generally sound (Lemma 5.13): that is, syntactic consequence implies semantic consequence.

Then we will recall the definition of standard completeness (Definition 5.14). We may consider three types of completeness for logics: (1) weak standard completeness: that is, semantic consequence implies syntactic consequence from the empty set, (2) finitely strong standard completeness: that is, semantic consequence implies syntactic

consequence from any finite set of formulas, and (3) strong standard completeness: that is, semantic consequence implies syntactic consequence from an arbitrary set of formulas. These names for types of standard completeness come from [Mon07]; the term ‘standard’ indicates that the domain of algebras (models for semantic consequence) is with respect to $[0,1]$.

We remark that there exists a notion of strong non-standard completeness introduced by Flaminio in 2007, which is a property of an axiomatic system of a fuzzy logic defined as follows: an axiomatic system of a fuzzy logic is strongly non-standard complete if it is complete in a strong sense with respect to algebras whose universe is a non-Archimedean extension $[0,1]^*$ of the real unit interval $[0,1]$ (see [Fla07]). Strong non-standard completeness is beyond the scope of this thesis.

We can recall that Hájek’s axiomatic systems (see BL, L, P in Definition 5.6) are finitely strongly complete (see Theorem 1.2.4 of [BCH11]), but they are not strongly standard complete (see Examples 5.15, 5.16 and 5.17) with the exception of Gödel logic (see G in Definition 5.6) which is strongly standard complete (see Theorems 1.1.18 and 1.2.4 of [BCH11]). However, when a certain infinitary rule (from infinitely many premises we derive a new formula) is added to the system, then the extended axiomatic system is strongly standard complete, as we will prove in Chapter 6. The motivation for this infinitary rule will be provided by Example 5.17.

Below we quote Definition 1.2.1 of [BCH11] of Hilbert-style axiomatic systems.

Definition 5.5. A *Hilbert-style calculus* (or *axiomatic system*) is given by a set of axioms and a set of derivation rules. *Axioms* are selected formulae in a given language. *Derivation rules* are pairs consisting of a finite set of formulae (called the *premises* of the rule) and a single formula (called the *conclusion* of the rule).

To fully define Hilbert-style axiomatic systems for the fuzzy logics we will consider, we recall the inference rule (the derivation rule in Definition 5.5) *Modus Ponens* (MP) and provide the aforementioned infinitary rule (Inf), which has an infinitary set of premises (contrary to Definition 5.5):

$$\frac{\alpha, \alpha \rightarrow \beta}{\beta} \quad (\text{MP})$$

$$\frac{\varphi \vee (\alpha \rightarrow \beta^n) : n < \omega}{\varphi \vee (\alpha \rightarrow \alpha \& \beta)} \quad (\text{Inf})$$

In the definition below we use the notation $\mathcal{H}$ for history (a set of formulas derived up to a point). We say that a *theory* is a subset of $\mathcal{F}$ and denote theories by T, S, U , etc.

Definition 5.6. (Syntactic consequence)

Fix a theory $\mathcal{A}$. Let T be a theory.

- A *finite proof from T* is a finite sequence of formulas $\varphi_i \in \mathcal{F}(i \leq n)$ for some $n < \omega$ such that for each $i \leq n$ and letting $\mathcal{H} = \{\varphi_j : j < i\}$, at least one of the following holds:

1. $\varphi_i \in \mathcal{A} \cup T$,
2. there is a formula χ such that $\chi, \chi \rightarrow \varphi_i \in \mathcal{H}$ (that is φ_i is derived from φ_j, φ_k by (MP), where $j, k < i$ and $\varphi_j = \chi, \varphi_k = \chi \rightarrow \varphi_i$).

The *length* of the proof is n .

- An *infinitary proof from T* is a possibly infinite sequence of formulas $\varphi_i \in \mathcal{F}(i \leq \xi)$ for some ordinal ξ such that for each $i \leq \xi$ and letting $\mathcal{H} = \{\varphi_j : j < i\}$, at least one of the following holds:

1. $\varphi_i \in \mathcal{A} \cup T$,
2. there is a formula χ such that $\chi, \chi \rightarrow \varphi_i \in \mathcal{H}$ (that is φ_i is derived from φ_j, φ_k by (MP), where $j, k < i$ and $\varphi_j = \chi, \varphi_k = \chi \rightarrow \varphi_i$),
3. there are formulas φ, α, β such that $\varphi_i = \varphi \vee (\alpha \rightarrow \alpha \& \beta)$ and $\varphi \vee (\alpha \rightarrow \beta^n) \in \mathcal{H}$ for each $n < \omega$ (that is φ_i is derived by (Inf) from formulas φ_{j_n} ($n < \omega$) by (Inf), where $j_n < i$ and $\varphi_{j_n} = \varphi \vee (\alpha \rightarrow \beta^n)$ for each $n < \omega$).

The *length* of the infinitary proof is ξ .

We call φ_i the *i*th *proof element*. For a formula φ , a *finite (infinitary, resp.) proof of φ from T* is a finite (infinitary, resp.) proof from T ending in φ . If there is a finite (infinitary, resp.) proof of φ from T , we write $T \vdash_{ASL} \varphi$ in the following way:

Axiomatic system ASL	$\vdash_{ASL}$	the theory $\mathcal{A}$	Inference rules used in derivation of φ	type of proof
BL^+	$\vdash_{BL^+}$	BL-axioms	(MP), (Inf)	infinitary
L^+	$\vdash_{L^+}$	BL-axioms, $(\neg\neg)$	(MP), (Inf)	infinitary
P^+	$\vdash_{P^+}$	BL-axioms, $(\Pi 1), (\Pi 2)$	(MP), (Inf)	infinitary
BL	$\vdash_{BL}$	BL-axioms	(MP)	finite
L	$\vdash_L$	BL-axioms, $(\neg\neg)$	(MP)	finite
P	$\vdash_P$	BL-axioms, $(\Pi 1), (\Pi 2)$	(MP)	finite
G	$\vdash_G$	BL-axioms, (G)	(MP)	finite

We write $\vdash_{ASL} \varphi$ if T is $\emptyset$. This concludes the definition.

BL-axioms (A2) and (A3) that are part of our axiomatic systems have been proven to be redundant in Hájek's system (see [CHN11] p. 15), as they are provable from the

other BL-axioms. The other axioms constitute the minimal independent set of axioms for BL. Note that we do not extend the axiomatic system G by the infinitary rule since it is already strongly standard complete.

Remark 5.7. Let T be a theory. $T \vdash_{BL+} \varphi$ ($T \vdash_{L+} \varphi, T \vdash_{P+} \varphi$, resp.) if $T \vdash_{BL} \varphi$ ($T \vdash_L \varphi, T \vdash_P \varphi$, resp.). This remark will be used implicitly in Chapter 6.

Example 5.8. Let $p, q, r, s \in PROP$ and $T = \{p \rightarrow q, p, q \rightarrow r \& s\}$. We will show that $T \vdash_{BL} s$. We start with $\mathcal{H} = \emptyset$. We build a proof by constructing a sequence of formulas.

1. p since $p \in T$, and now $\mathcal{H} = \{p\}$,
2. $p \rightarrow q$ since $p \rightarrow q \in T$, and now $\mathcal{H} = \{p, p \rightarrow q\}$,
3. q since there is a formula $\chi = p$ with $\chi, \chi \rightarrow q \in \mathcal{H}$, and now $\mathcal{H} = \{p, p \rightarrow q, q\}$,
4. $q \rightarrow r \& s$ since $q \rightarrow r \& s \in T$, and now $\mathcal{H} = \{p, p \rightarrow q, q, q \rightarrow r \& s\}$,
5. $r \& s$ since there is a formula $\chi = q$ with $\chi, \chi \rightarrow r \& s \in \mathcal{H}$, and now $\mathcal{H} = \{p, p \rightarrow q, q, q \rightarrow r \& s, r \& s\}$,
6. $r \& s \rightarrow s$ since $r \& s \rightarrow s$ is a BL-axiom, and now $\mathcal{H} = \{p, p \rightarrow q, q, q \rightarrow r \& s, r \& s, r \& s \rightarrow s\}$,
7. s since there is a formula $\chi = r \& s$ with $\chi, \chi \rightarrow s \in \mathcal{H}$, and now $\mathcal{H} = \{p, p \rightarrow q, q, q \rightarrow r \& s, r \& s, r \& s \rightarrow s, s\}$.

Now, we rewrite the proof in a more compact form, where (T) will denote assumptions coming from the set T .

1.	$p \rightarrow q$	(T)
2.	p	(T)
3.	q	(MP) 1,2
4.	$q \rightarrow r \& s$	(T)
5.	$r \& s$	(MP) 3,4
6.	$r \& s \rightarrow s$	(A2)
7.	s	(MP) 5,6

This concludes the example.

The following lemma will be used in Examples 5.15, 5.16 and 5.17.

Lemma 5.9. Let $ASL \in \{BL, L, P, G\}$, and $T \subseteq \mathcal{F}, \varphi \in \mathcal{F}$. If $T \vdash_{ASL} \varphi$, then there is finite $T_0 \subseteq T$ with $T_0 \vdash_{ASL} \varphi$.

Proof. Let $T \vdash_{ASL} \varphi$. We prove by induction on proof length that $T_0 \vdash_{ASL} \varphi$ for some finite $T_0 \subseteq T$.

1. If $\varphi \in \mathcal{A}$, with $\mathcal{A}$ defined as in Definition 5.6, then we can take $T_0 = \emptyset$.
2. If $\varphi \in T$, then we can take $T_0 = \{\varphi\}$.
3. Assume $T \vdash_{ASL} \chi$ and $T \vdash_{ASL} \chi \rightarrow \varphi$ for some $\chi \in \mathcal{F}$ by shorter proofs. Then by the induction hypothesis, there exist finite $T_1, T_2 \subseteq T$ such that $T_1 \vdash_{ASL} \chi$ and $T_2 \vdash_{ASL} \chi \rightarrow \varphi$. Let $\chi_0, \dots, \chi_n, \chi$ for some $n < \omega$ be a proof of χ from T_1 , and $\theta_0, \dots, \theta_m, \chi \rightarrow \varphi$ for some $m < \omega$ be a proof of $\chi \rightarrow \varphi$ from T_2 in ASL . Then $\chi_0, \dots, \chi_n, \chi, \theta_0, \dots, \theta_m, \chi \rightarrow \varphi, \varphi$ is a proof of φ from $T_1 \cup T_2$ in ASL and $T_1 \cup T_2$ is a finite subset of T . Note that φ is derived from $\chi, \chi \rightarrow \varphi$ by (MP).

□

Now, we turn our attention to the definitions of semantic consequence and general soundness, and we prove that the extended axiomatic systems ASL , where $ASL \in \{BL^+, L^+, P^+\}$, are generally sound.

Definition 5.10. (Semantic consequence)

For a class K of continuous t-norms, a theory T and a formula φ , we define $T \models_K \varphi$ iff for every t-norm $\star \in K$ and all evaluations $V : PROP \rightarrow [0, 1]$, if $V_\star(\theta) = 1$ for every $\theta \in T$, then $V_\star(\varphi) = 1$. If $T = \emptyset$, we write $\models_K \varphi$. We define the following classes of continuous t-norms:

- $K(BL) = K(BL^+) = \{\text{all continuous t-norms}\}$,
- $K(L) = K(L^+) = \{\text{Łukasiewicz t-norm}\}$,
- $K(P) = K(P^+) = \{\text{product t-norm}\}$,
- $K(G) = \{\text{Gödel t-norm}\}$.

Remark 5.11. Any class of t-norms is also a set.

Definition 5.12. (General Soundness)

We say that an axiomatic system $ASL \in \{BL^+, L^+, P^+, BL, L, P, G\}$ is *generally sound* if for every theory T and every $\psi \in \mathcal{F}$, $T \models_{K(ASL)} \psi$ whenever $T \vdash_{ASL} \psi$.

Lemma 5.13. (General Soundness)

Each axiomatic system $ASL \in \{BL^+, L^+, P^+, BL, L, P, G\}$ is generally sound.

Proof. Let $T \subseteq \mathcal{F}$, $\varphi \in \mathcal{F}$ be given and fix ASL as in the lemma. Suppose that $T \vdash_{ASL} \varphi$, and assume inductively that if $T \vdash_{ASL} \psi$ then $T \models_{K(ASL)} \psi$ for every $\psi \in \mathcal{F}$ with a shorter proof from T than φ . We show that $T \models_{K(ASL)} \varphi$. There are four cases.

1. If $\varphi \in \mathcal{A}$, where the set $\mathcal{A}$ is as in Definition 5.6, then by Lemmas 5.2 and 5.3, φ is a 1-tautology for every t-norm in $K(ASL)$, so $T \models_{K(ASL)} \varphi$.
2. If $\varphi \in T$, then obviously $T \models_{K(ASL)} \varphi$.
3. Suppose that $T \vdash_{ASL} \psi$ and $T \vdash_{ASL} \psi \rightarrow \varphi$ by shorter proofs. Then by the inductive hypothesis, $T \models_{K(ASL)} \psi$ and $T \models_{K(ASL)} \psi \rightarrow \varphi$. Take any t-norm $\star \in K(ASL)$ and any evaluation $V : PROP \rightarrow [0, 1]$. Assume $V_\star(\theta) = 1$ for every $\theta \in T$. Then (1) $V_\star(\psi) = 1$ and (2) $V_\star(\psi \rightarrow \varphi) = 1$. By Definition 3.2 and Lemma 2.17, (2) becomes $V_\star(\psi) \leq V_\star(\varphi)$. So by (1), $1 \leq V_\star(\varphi)$. Thus $T \models_{K(ASL)} \varphi$.
4. Suppose that $\varphi = \psi \vee (\alpha \rightarrow \alpha \& \beta)$ and $T \vdash_{ASL} \psi \vee (\alpha \rightarrow \beta^n)$ for each $n < \omega$ by shorter proofs. Then by the inductive hypothesis $T \models_{K(ASL)} \psi \vee (\alpha \rightarrow \beta^n)$ for each $n < \omega$. We need to show that $T \models_{K(ASL)} \psi \vee (\alpha \rightarrow \alpha \& \beta)$. We take any t-norm $\star \in K(ASL)$ and an evaluation $V : PROP \rightarrow [0, 1]$. Assume that $V_\star(\theta) = 1$ for all $\theta \in T$. We show that $V_\star(\psi \vee (\alpha \rightarrow \alpha \& \beta)) = 1$. There are two cases: (1) $V_\star(\psi) = 1$ or (2) $V_\star(\psi) < 1$. In case (1), by Lemma 5.4, $V_\star(\psi \vee (\alpha \rightarrow \alpha \& \beta)) = 1$. In case (2), we use $T \models_{K(ASL)} \psi \vee (\alpha \rightarrow \beta^n)$ for each $n < \omega$, from which we get $V_\star(\psi \vee (\alpha \rightarrow \beta^n)) = 1$. Since $V_\star(\psi) < 1$, we have $V_\star(\alpha \rightarrow \beta^n) = 1$ by Lemma 5.4. By Definition 3.2 and Lemma 2.17, $V_\star(\alpha) \leq V_\star(\beta)_\star^{(n)}$ for each $n < \omega$ (see Notation 2.6). We define $e = \inf\{V_\star(\beta)_\star^{(n)}\}$. By Lemma 2.17, e is an idempotent. Since $V_\star(\alpha) \leq e \leq V_\star(\beta)$ and by Lemma 2.17, we conclude that $V_\star(\alpha) = V_\star(\alpha) \star V_\star(\beta)$. Therefore by Definition 3.2 and Lemma 2.17, $V_\star(\alpha \rightarrow \alpha \& \beta) = 1$ and by Lemma 5.4, $V_\star(\psi \vee (\alpha \rightarrow \alpha \& \beta)) = 1$. In both cases (1) and (2), we conclude that $T \models_{K(ASL)} \psi \vee (\alpha \rightarrow \alpha \& \beta)$. Thus $T \models_{K(ASL)} \varphi$.

□

We will recall now a definition of different types of standard completeness of axiomatic systems for the discussed fuzzy logics (see e.g. [Mon07]).

Definition 5.14. (Completeness)

Let $ASL \in \{BL, BL^+, L, L^+, P, P^+, G\}$.

1. We say that ASL is *weakly standard complete* if for every $\psi \in \mathcal{F}$, $\vdash_{ASL} \psi$ whenever $\models_{K(ASL)} \psi$.
2. We say that ASL is *finitely strongly standard complete* if for every $\psi \in \mathcal{F}$ and every finite $T \in \wp(\mathcal{F})$, $T \vdash_{ASL} \psi$ whenever $T \models_{K(ASL)} \psi$.
3. We say that ASL is *strongly standard complete* if $\models_{K(ASL)} \subseteq \vdash_{ASL}$.

Hájek gives a counterexample (see [Háj98b] p. 75) to show that the axiomatic system L for Łukasiewicz logic is not strongly standard complete, which we reproduce below.

Example 5.15. Let $\star$ be the Łukasiewicz t-norm. Let $p \in PROP$, and $p^{(n)\vee}$ be inductively defined as follows: $p^{(1)\vee} = p$ and $p^{(n+1)\vee} = p^{(n)\vee} \vee p$. Let $T = \{p^{(n)\vee} \rightarrow q : n \geq 1\} \cup \{\neg p \rightarrow q\}$ with $p, q \in PROP$. We show that $T \models_{K(L)} q$. Take an arbitrary $V : PROP \rightarrow [0, 1]$ such that $V_\star(\varphi) = 1$ for every $\varphi \in T$. We show that $V(q) = 1$. If $V(p) = 0$, then $V_\star(\neg p) = 1$ and $1 = V_\star(\neg p \rightarrow q) = V_\star(\neg p) \Rightarrow_\star V(q) = 1 \Rightarrow_\star V(q) = V(q)$. Since by Lemma 5.4, $V_\star(p^{(2)\vee}) = \min\{1, 2 \cdot V(p)\}$, it is easy to show by induction on n , that $V_\star(p^{(n)\vee}) = \min\{1, n \cdot V(p)\}$. If $V(p) > \frac{1}{n}$, then $V_\star(p^{(n)\vee}) \geq \min\{1, n \cdot \frac{1}{n}\} = 1$. Thus $1 = V_\star(p^{(n)\vee} \rightarrow q) = 1 - V_\star(p^{(n)\vee}) + V(q) = 1 - 1 + V(q) = V(q)$.

But for any finite subset T_0 of T , we can find sufficiently small $V(p)$ so that $V(q) < 1$ with $V_\star(\theta) = 1$ for all $\theta \in T_0$. Thus $T_0 \not\models_{K(L)} q$, and hence $T_0 \not\models_L q$ by soundness. Therefore $T \not\models_L q$ by Lemma 5.9.

We constructed an example to show that the axiomatic system P for Product logic is also not strongly standard complete.

Example 5.16. Let $\star$ be the product t-norm. Let $T = \{(p^n \rightarrow r) \rightarrow q : n \geq 0\} \cup \{p \rightarrow q, \neg r \rightarrow q\}$ with $p, q, r \in PROP$. We show that $T \models_{K(P)} q$. Take an arbitrary $V : PROP \rightarrow [0, 1]$ such that $V_\star(\varphi) = 1$ for every $\varphi \in T$. We show that $V(q) = 1$. If $V(p) = 1$, then $1 = V(p) \Rightarrow V(q) = 1 \Rightarrow V(q) = V(q)$. If $V(r) = 0$, then $1 = V_\star(\neg r) \Rightarrow V(q) = 1 \Rightarrow V(q) = V(q)$. If $V(p) < 1$ and $V(r) > 0$, then we can find n such that $V(p)_\star^{(n)} \leq V(r)$, and thus $1 = V_\star(p^n \rightarrow r) \Rightarrow V(q) = 1 \Rightarrow V(q) = V(q)$.

But for any finite subset T_0 of T , we can choose V so that $0 < V(p) \leq V(q) < 1$ and $0 < V(r) < V(p)_\star^{(n)} \cdot V(q)$ for all n such that $(p^n \rightarrow r) \rightarrow q \in T_0$. Then $V_\star(\theta) = 1$ for all $\theta \in T_0$ and $V_\star(q) < 1$. Thus $T_0 \not\models_{K(P)} q$ and hence $T_0 \not\models_P q$ by soundness. Therefore $T \not\models_P q$ by Lemma 5.9.

We show that the axiomatic system for BL is not strongly standard complete by our counterexample below, which could also be used for the axiomatic systems L and P .

Example 5.17. Let $ASL \in \{BL, L, P\}$. Let $T = \{p \rightarrow q^n : n < \omega\}$ with $p, q \in PROP$. We show that $T \models_{K(ASL)} p \rightarrow p \& q$. Take an arbitrary $V : PROP \rightarrow [0, 1]$ and a t-norm $\star \in K(ASL)$ such that for all $\varphi \in T$, $V_\star(\varphi) = 1$, which means that for all $n < \omega$, $V(p) \leq V(q)_\star^{(n)}$. By Lemma 2.17 (points 4 and 5), $V_\star(p \rightarrow p \& q) = 1$. Thus $T \models_{K(ASL)} p \rightarrow p \& q$.

Now, take a finite $T_0 \subseteq T$. Then there is a maximal n such that $p \rightarrow q^n \in T_0$. Take $\star$ to be the Łukasiewicz t-norm or the product t-norm. Let $V : PROP \rightarrow [0, 1]$ be such that $V(q)_\star^{(n+1)} < V(p) = V(q)_\star^{(n)}$. Thus $V_\star(\theta) = 1$ for all $\theta \in T_0$. Also,

$$V(q)_\star^{(n+1)} = V(q)_\star^{(n)} \star V(q) = V(p) \star V(q) = V_\star(p \& q) < V(p).$$

So $V_\star(p \rightarrow p \& q) < 1$ and hence $T_0 \not\models_{K(L)} p \rightarrow p \& q$ and $T_0 \not\models_{K(P)} p \rightarrow p \& q$, and

consequently $T_0 \not\models_{K(BL)} p \rightarrow p \& q$. Thus $T_0 \not\models_{ASL} p \rightarrow p \& q$ by soundness. Therefore $T \not\models_{ASL} p \rightarrow p \& q$ by Lemma 5.9.

This example suggested the infinitary rule (Inf) for the extended axiomatic systems BL^+, L^+, P^+ .

6. STRONG STANDARD COMPLETENESS RESULTS

This chapter focuses on the proof of strong standard completeness of the extended axiomatic systems BL^+, L^+, P^+ of the fuzzy logics: BL, Łukasiewicz and Product, respectively. Section 6.1 covers some preliminary results used in the following sections. The results presented in Section 6.2 prove strong standard completeness of the extended axiomatic systems BL^+, L^+, P^+ . The contents of this chapter constitute part of a conference talk [Kuł16], and a paper [Kuł18].

6.1. Preliminaries. This section will provide some necessary tools to prove strong standard completeness for the extended axiomatic systems BL^+, L^+, P^+ , which will be achieved in the following section. First in Lemma 6.1, we recall some useful provable formulas of BL from [Háj98b], which we will use in the subsequent lemmas leading to the proof of strong standard completeness. Then in Lemmas 6.2, 6.3, 6.5, 6.6 and 6.7, we will prove some properties of BL^+ .

Lemma 6.1. Let $\varphi, \varphi_1, \varphi_2, \psi, \psi_1, \psi_2, \chi$ be formulas. Then,

- (T1) 2.2.7(1) of [Háj98b] $\vdash_{BL} \psi \rightarrow (\varphi \rightarrow \psi)$,
- (T2) 2.2.7(2) of [Háj98b] $\vdash_{BL} (\psi \rightarrow (\varphi \rightarrow \chi)) \rightarrow (\varphi \rightarrow (\psi \rightarrow \chi))$,
- (T3) 2.2.7(3) of [Háj98b] $\vdash_{BL} \psi \rightarrow \psi$,
- (T4) 2.2.8(4) of [Háj98b] $\vdash_{BL} \psi \& (\psi \rightarrow \varphi) \rightarrow \varphi$,
- (T5) 2.2.8(5) of [Háj98b] $\vdash_{BL} \psi \rightarrow (\varphi \rightarrow \psi \& \varphi)$,
- (T6) 2.2.8(6) of [Háj98b] $\vdash_{BL} (\psi \rightarrow \varphi) \rightarrow (\psi \& \chi \rightarrow \varphi \& \chi)$,
- (T7) 2.2.8(7) of [Háj98b] $\vdash_{BL} ((\psi_1 \rightarrow \psi_2) \& (\varphi_1 \rightarrow \varphi_2)) \rightarrow (\psi_1 \& \varphi_1 \rightarrow \psi_2 \& \varphi_2)$,
- (T8) 2.2.9(9) of [Háj98b] (i) $\vdash_{BL} \varphi \wedge \psi \rightarrow \psi$, (ii) $\vdash_{BL} \varphi \wedge \psi \rightarrow \varphi$,
- (T9) 2.2.10(13) of [Háj98b] (i) $\vdash_{BL} \varphi \rightarrow \varphi \vee \psi$, (ii) $\vdash_{BL} \psi \rightarrow \varphi \vee \psi$,
(iii) $\vdash_{BL} \varphi \vee \psi \rightarrow \psi \vee \varphi$,
- (T10) 2.2.10(14) of [Háj98b] $\vdash_{BL} (\varphi \rightarrow \psi) \rightarrow (\varphi \vee \psi \rightarrow \psi)$,
- (T11) 2.2.10(15) of [Háj98b] $\vdash_{BL} (\varphi \rightarrow \psi) \vee (\psi \rightarrow \varphi)$,
- (T12) 2.2.11(16') of [Háj98b] $\vdash_{BL} (\psi \rightarrow \chi) \& (\varphi \rightarrow \chi) \rightarrow ((\psi \vee \varphi) \rightarrow \chi)$,
- (T13) 2.2.12(17) of [Háj98b] $\vdash_{BL} \varphi \rightarrow \neg \neg \varphi$,

(T14) 2.2.14 of [Háj98b] (i) $\vdash_{BL} \bar{1}$, (ii) $\vdash_{BL} \psi \rightarrow \bar{1} \& \psi$, (iii) $\vdash_{BL} (\bar{1} \rightarrow \psi) \rightarrow \psi$,
 (iv) $\vdash_{BL} \psi \rightarrow \bar{1}$,¹⁴

(T15) 2.2.15(22) of [Háj98b] (i) $\vdash_{BL} \varphi \vee (\psi \vee \chi) \rightarrow (\varphi \vee \psi) \vee \chi$,
 (ii) $\vdash_{BL} (\varphi \vee \psi) \vee \chi \rightarrow \varphi \vee (\psi \vee \chi)$,

(T16) 2.2.16(24) of [Háj98b] (i) $\vdash_{BL} \varphi \equiv \varphi$, (ii) $\vdash_{BL} (\varphi \equiv \chi) \rightarrow (\chi \equiv \varphi)$,
 (iii) $\vdash_{BL} ((\varphi \equiv \psi) \& (\psi \equiv \chi)) \rightarrow (\varphi \equiv \chi)$,

(T17) 2.2.16(26) of [Háj98b] $\vdash_{BL} (\varphi \equiv \psi) \rightarrow (\varphi \& \chi \equiv \psi \& \chi)$,

(T18) 2.2.16(27) of [Háj98b] $\vdash_{BL} (\varphi \equiv \psi) \rightarrow (\varphi \rightarrow \chi \equiv \psi \rightarrow \chi)$,

(T19) 2.2.16(28) of [Háj98b] $\vdash_{BL} (\varphi \equiv \psi) \rightarrow (\chi \rightarrow \varphi \equiv \chi \rightarrow \psi)$,

(T20) $\vdash_{BL} \bar{0} \vee \varphi \rightarrow \varphi$,

(T21) $\vdash_{BL} (\alpha \rightarrow \beta) \vee \gamma \rightarrow (\alpha \vee \gamma \rightarrow \beta \vee \gamma)$.

Proof. Points (T1)-(T19) hold by Lemmas 2.2.8-2.2.16 of Hájek's book [Háj98b] as shown above. Point (T20) can be easily obtained from (T10) and BL-axiom (A7). We will prove point (T21).

First we will prove (A) $\vdash_{BL} (\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\beta \vee \gamma))$.

1. $\beta \rightarrow \beta \vee \gamma$ (T9)
2. $(\alpha \rightarrow \beta) \rightarrow [(\beta \rightarrow \beta \vee \gamma) \rightarrow (\alpha \rightarrow (\beta \vee \gamma))]$ (A1)
3. $[(\alpha \rightarrow \beta) \rightarrow [(\beta \rightarrow \beta \vee \gamma) \rightarrow (\alpha \rightarrow (\beta \vee \gamma))]] \rightarrow$ (T2)
 $[(\beta \rightarrow \beta \vee \gamma) \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\beta \vee \gamma))]]$
4. $(\beta \rightarrow \beta \vee \gamma) \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\beta \vee \gamma))]$ (MP) 2,3
5. $(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow (\beta \vee \gamma))$ (MP) 1,4

Now, we show that (B) $\vdash_{BL} (\alpha \rightarrow \beta) \rightarrow (\alpha \vee \gamma \rightarrow \beta \vee \gamma)$. By point (T12), we have $\vdash_{BL} (\alpha \rightarrow \beta \vee \gamma) \& (\gamma \rightarrow \beta \vee \gamma) \rightarrow (\alpha \vee \gamma \rightarrow \beta \vee \gamma)$. By BL-axiom (A5b) and (MP), we have $\vdash_{BL} (\alpha \rightarrow \beta \vee \gamma) \rightarrow [(\gamma \rightarrow \beta \vee \gamma) \rightarrow (\alpha \vee \gamma \rightarrow \beta \vee \gamma)]$. By BL-axiom (A1), (A) above and (MP), we have $\vdash_{BL} (\alpha \rightarrow \beta) \rightarrow [(\gamma \rightarrow \beta \vee \gamma) \rightarrow (\alpha \vee \gamma \rightarrow \beta \vee \gamma)]$. By point (T2), we get $\vdash_{BL} (\gamma \rightarrow \beta \vee \gamma) \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \vee \gamma \rightarrow \beta \vee \gamma)]$. Since by point (T9), $\vdash_{BL} \gamma \rightarrow \beta \vee \gamma$, using (MP) we get (B) as required.

As above we get $\vdash_{BL} \gamma \rightarrow \beta \vee \gamma$ and so by point (T1) and (MP), we have $\vdash_{BL} \alpha \vee \gamma \rightarrow (\gamma \rightarrow \beta \vee \gamma)$. By point (T2), we get $\vdash_{BL} \gamma \rightarrow (\alpha \vee \gamma \rightarrow \beta \vee \gamma)$. Now using point (T12) and (B), we get $\vdash_{BL} (\alpha \rightarrow \beta) \vee \gamma \rightarrow (\alpha \vee \gamma \rightarrow \beta \vee \gamma)$ as required. $\square$

¹⁴The last provability is absent in [Háj98b], but it can be easily proved by point (T14) (i), point (T1) and (MP).

We will now establish some properties of the axiomatic system BL^+ : namely, monotonicity (Lemma 6.2), transitivity (Lemma 6.3), Proof by Cases Property (Lemma 6.5), weak disjunction (Lemma 6.6) and Prelinearity (Lemma 6.7). For a theory $\Gamma \subseteq \mathcal{F}$ and a formula $\varphi \in \mathcal{F}$, we write Γ, φ to abbreviate $\Gamma \cup \{\varphi\}$. In this and the next section, we fix any $\vdash \in \{\vdash_{BL^+}, \vdash_{L^+}, \vdash_{P^+}\}$ in the sense of Definition 5.6.

Lemma 6.2. (monotonicity)

For any theories $T, U \subseteq \mathcal{F}$, and formulas $\varphi \in \mathcal{F}$, if $T \vdash \varphi$ and $T \subseteq U$, then $U \vdash \varphi$.

Proof. A proof from T is also a proof from its superset. $\square$

Lemma 6.3. (transitivity)

For any theories $T, U \subseteq \mathcal{F}$, and formulas $\varphi \in \mathcal{F}$, if $U \vdash \varphi$ and $T \vdash \theta$ for each $\theta \in U$, then $T \vdash \varphi$.

Proof. Assume that (1) $U \vdash \varphi$, and (2) $T \vdash \theta$ for every $\theta \in U$. By (1), we know that a proof π of φ from U exists. Take a proof element θ of π such that $\theta \in U$. We apply the following procedure (#) to this proof element. By (2), there is a proof π_θ of θ from T . Add π_θ immediately before θ . This ends procedure (#). Do the same for all $\theta \in U$ that are proof elements of π . As a result we get a proof of φ from T . $\square$

Let us now recall Proof by Cases Property (henceforth PCP; see more in [CN11] e.g. Theorem 1.2.11 or [CN13]): for any theory $\Gamma \subseteq \mathcal{F}$, and formulas $\varphi, \chi, \psi \in \mathcal{F}$,

$$\text{PCP} \quad \text{If } \Gamma, \varphi \vdash \chi \text{ and } \Gamma, \psi \vdash \chi, \text{ then } \Gamma, \varphi \vee \psi \vdash \chi$$

In our proof that PCP is a property of each $ASL \in \{BL^+, L^+, P^+\}$, we will need Lemma 6.4. Note that our proof below does not use the Local Deduction Theorem, Theorem 1.2.10 of [CN11] or Theorem 2.2.18 of [Háj98b], which is used in the proofs of PCP in the aforementioned manuscripts.

Lemma 6.4. For any theory $\Gamma \subseteq \mathcal{F}$, and formulas $\alpha, \beta, \varphi, \psi \in \mathcal{F}$, (P) if $\Gamma, \alpha \vdash \varphi$ and $\Gamma, \beta \vdash \psi$, then $\Gamma, \alpha \vee \beta \vdash \varphi \vee \psi$.

Proof. We show that (P) holds by induction on the proof length, which is an ordinal. Assume that $\Gamma, \alpha \vdash \varphi$ and $\Gamma, \beta \vdash \psi$, and assume inductively that (P) is true (i) for ψ and all formulas φ' with shorter proofs from $\Gamma \cup \{\alpha\}$ than φ , and (ii) for φ and for all formulas ψ' with shorter proofs from $\Gamma \cup \{\beta\}$ than ψ . We show that $\Gamma, \alpha \vee \beta \vdash \varphi \vee \psi$. One or more of the following cases holds:

Case 1: $\varphi = \alpha$ and $\psi = \beta$. Then, we plainly have $\Gamma, \alpha \vee \beta \vdash \varphi \vee \psi$.

Case 2: φ is an axiom or $\varphi \in \Gamma$. Then $\Gamma \vdash \varphi$. By Lemma 6.1 point (T9), $\Gamma \vdash \varphi \vee \psi$. So by monotonicity (Lemma 6.2), $\Gamma, \alpha \vee \beta \vdash \varphi \vee \psi$.

Case 3: φ follows by rule (MP) from formulas $\theta, \theta \rightarrow \varphi$ earlier in the proof. So, $\Gamma, \alpha \vdash \theta$ and $\Gamma, \alpha \vdash \theta \rightarrow \varphi$ by shorter proofs. By induction hypothesis, $\Gamma, \alpha \vee \beta \vdash \theta \vee \psi$ and $\Gamma, \alpha \vee \beta \vdash (\theta \rightarrow \varphi) \vee \psi$. By Lemma 6.1 point (T21), monotonicity (Lemma 6.2), and rule (MP), $\Gamma, \alpha \vee \beta \vdash \varphi \vee \psi$.

Case 4: φ follows by (Inf) from formulas earlier in the proof. So $\varphi = \theta \vee (\rho \rightarrow \rho \& \sigma)$ for some ρ, σ, θ , where $\Gamma, \alpha \vdash \theta \vee (\rho \rightarrow \sigma^n)$ for each $n < \omega$, by a shorter proof. By induction hypothesis, $\Gamma, \alpha \vee \beta \vdash (\theta \vee (\rho \rightarrow \sigma^n)) \vee \psi$ for each $n < \omega$. By Lemma 6.1 points (T9) and (T15), BL-axiom (A1), and (MP) and (Inf) rules, we show that $\Gamma, \alpha \vee \beta \vdash (\theta \vee (\rho \rightarrow \rho \& \sigma)) \vee \psi$, that is $\Gamma, \alpha \vee \beta \vdash \varphi \vee \psi$. Below, lines 1-5 are provable from $\Gamma, \alpha \vee \beta$ for every $n < \omega$.

1.	$(\theta \vee (\rho \rightarrow \sigma^n)) \vee \psi$	assumption (C)
2.	$[(\theta \vee (\rho \rightarrow \sigma^n)) \vee \psi] \rightarrow [\psi \vee (\theta \vee (\rho \rightarrow \sigma^n))]$	(T9)
3.	$\psi \vee (\theta \vee (\rho \rightarrow \sigma^n))$	(MP) 1,2
4.	$[\psi \vee (\theta \vee (\rho \rightarrow \sigma^n))] \rightarrow [(\psi \vee \theta) \vee (\rho \rightarrow \sigma^n)]$	(T15)
5.	$(\psi \vee \theta) \vee (\rho \rightarrow \sigma^k)$	(MP) 1,2
6.	$(\psi \vee \theta) \vee (\rho \rightarrow \rho \& \sigma)$	(Inf) 3
7.	$[(\psi \vee \theta) \vee (\rho \rightarrow \rho \& \sigma)] \rightarrow [\psi \vee (\theta \vee (\rho \rightarrow \rho \& \sigma))]$	(T15)
8.	$\psi \vee (\theta \vee (\rho \rightarrow \rho \& \sigma))$	(MP) 4,5
9.	$[\psi \vee (\theta \vee (\rho \rightarrow \rho \& \sigma))] \rightarrow [(\theta \vee (\rho \rightarrow \rho \& \sigma)) \vee \psi]$	(T9)
10.	$(\theta \vee (\rho \rightarrow \rho \& \sigma)) \vee \psi$	(MP) 8,9

Case 5: ψ is an axiom or $\psi \in \Gamma$. Similar to Case 2.

Case 6: ψ follows by rule (MP) from formulas $\theta, \theta \rightarrow \psi$ earlier in the proof. Similar to Case 3.

Case 7: ψ follows by (Inf) from formulas earlier in the proof. Similar to Case 4.

Therefore $\Gamma, \alpha \vee \beta \vdash \varphi \vee \psi$ as claimed. This completes the induction and the proof. $\square$

Lemma 6.5. PCP holds in each $ASL \in \{BL^+, L^+, P^+\}$.

Proof. Let $\Gamma \subseteq \mathcal{F}$ and $\varphi, \chi, \psi \in \mathcal{F}$. Assume $\Gamma, \varphi \vdash \chi$ and $\Gamma, \psi \vdash \chi$. By definition of $\vee$ and Lemma 6.1 points (T3) and (T10) and (MP), we get $(I) \vdash \chi \vee \chi \rightarrow \chi$. Now, by Lemma 6.4, from $\Gamma, \varphi \vdash \chi$ and $\Gamma, \psi \vdash \chi$, we get $\Gamma, \varphi \vee \psi \vdash \chi \vee \chi$. By (MP) from (I), we get $\Gamma, \varphi \vee \psi \vdash \chi$. Since $\vdash \in \{\vdash_{BL^+}, \vdash_{L^+}, \vdash_{P^+}\}$ is arbitrary, this proves the lemma. $\square$

Lemma 6.6. (weak disjunction)

Let $\Gamma \subseteq \mathcal{F}$ and let $\varphi, \psi \in \mathcal{F}$. If (A) $\Gamma \vdash \varphi \vee \psi$ and (B) $\Gamma, \psi \vdash \varphi$, then $\Gamma \vdash \varphi$.

Proof. Obviously, $\Gamma, \varphi \vdash \varphi$, and by (B), $\Gamma, \psi \vdash \varphi$, thus by PCP (Lemma 6.5), $\Gamma, \varphi \vee \psi \vdash \varphi$. So by (A) and transitivity (Lemma 6.3), $\Gamma \vdash \varphi$. $\square$

The property Prelinearity below (henceforth (Prelin)) is a property of Hájek's axiomatic system BL (see for example [BCH11] p. 20). In our proof we do not use the Local Deduction Theorem as he did, and we have an additional inference rule, namely (Inf), in the axiomatic system BL^+, L^+, P^+ . For $\Gamma \subseteq \mathcal{F}$ and $\alpha, \beta, \varphi \in \mathcal{F}$,

$$\frac{\Gamma, \alpha \rightarrow \beta \vdash \varphi \text{ and } \Gamma, \beta \rightarrow \alpha \vdash \varphi}{\Gamma \vdash \varphi} \quad (\text{Prelin})$$

Lemma 6.7. Property (Prelin) holds in BL^+, L^+, P^+ .

Proof. Let $\Gamma \subseteq \mathcal{F}$ and let $\alpha, \beta, \varphi \in \mathcal{F}$. Assume that $\Gamma, \alpha \rightarrow \beta \vdash \varphi$ and $\Gamma, \beta \rightarrow \alpha \vdash \varphi$, which are the premises of (Prelin). We show that the conclusion holds. By PCP (Lemma 6.5), $\Gamma, (\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha) \vdash \varphi$. By Lemma 6.1 point (T11), $\vdash (\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$, and thus by monotonicity (Lemma 6.2), $\Gamma \vdash (\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$. Therefore, by transitivity (Lemma 6.3), $\Gamma \vdash \varphi$, which is the conclusion in (Prelin). Since $\vdash \in \{\vdash_{BL^+}, \vdash_{L^+}, \vdash_{P^+}\}$ is arbitrary, this proves the lemma. $\square$

6.2. Strong standard completeness. In this section we will prove the strong standard completeness theorem for BL^+, L^+, P^+ .

Theorem 6.8. BL^+, L^+, P^+ are strongly standard complete.

We prove it by contrapositive in the following way. We take any theory T_0 and formula ψ_0 and assume that $T_0 \not\vdash \psi_0$, where $\vdash \in \{\vdash_{BL^+}, \vdash_{L^+}, \vdash_{P^+}\}$. We will define and construct a prelinearly and deductively closed theory¹⁵ $T^* \supseteq T_0$ such that $T^* \not\vdash \psi_0$ (Subsection 6.2.1), with respect to which we define equivalence classes (Subsection 6.2.2), and group them into clusters following a particular relation (Subsection 6.2.3). We discuss the properties of clusters (Subsection 6.2.4) necessary to construct from them linear Archimedean product and MV-algebras (Subsection 6.2.5). We will map each of these onto either standard Łukasiewicz or product algebras using Proposition 4 of [Mon07]. From these we construct a continuous t-norm using the converse of the consequence of Decomposition Theorem (Theorem 2.36), and an evaluation $V : PROP \rightarrow [0, 1]$ such that $V_*(\theta) = 1$ for all $\theta \in T^* \supseteq T_0$, but $V_*(\psi_0) < 1$, which shows that $T_0 \not\vdash \psi_0$, where $\models = \models_{K(ASL)}$ if $\vdash = \vdash_{ASL}$ with $ASL \in \{BL^+, L^+, P^+\}$ (Subsection 6.2.6). For L^+, P^+ we will show that they have a restricted number of clusters, which will be mapped onto appropriate algebras (Subsections 6.2.7 and 6.2.8).

¹⁵It would be an alternative approach to prove that the axiomatic systems for logics BL, Łukasiewicz and Product, BL^+, L^+, P^+ , respectively, are semilinear and then use Theorem 3.1.7 of [CN11], but it seems that the proof of semilinearity would be the same as that of Theorem 6.10.

So take any $T_0 \in \wp(\mathcal{F})$, $\psi_0 \in \mathcal{F}$. We assume that $T_0 \not\vdash \psi_0$, where $\vdash \in \{\vdash_{BL+}, \vdash_{L+}, \vdash_{P+}\}$ continues to be fixed as in Section 6.1.

6.2.1 A prelinearly and deductively closed theory T^*

We will construct a prelinearly and deductively closed theory, which we will call T^* .

Definition 6.9. We say that a theory $T \subseteq \mathcal{F}$ is

1. *prelinearly closed* if for each $\varphi, \psi \in \mathcal{F}$, we have $\varphi \rightarrow \psi \in T$ or $\psi \rightarrow \varphi \in T$,
2. *deductively closed* if $\varphi \in T$ whenever $T \vdash \varphi$.

We remark that because of the presence of (Inf), it is not true in general that if $T_0 \subseteq T_1 \subseteq \dots$ and $T_n \not\vdash \psi_0$ for all $n < \omega$, then $\bigcup_{n < \omega} T_n \not\vdash \psi_0$, so the proof of Theorem 6.10 proceeds differently (see p. 261 of [Mon07]).

Theorem 6.10. There is a prelinearly closed theory $T^* \supseteq T_0$ with $T^* \not\vdash \psi_0$.

Proof. We will inductively construct a chain of theories $T_0 \subseteq T_1 \subseteq \dots$ and formulas $\psi_0, \psi_1, \dots$ with the following properties:

- (P1) $T_n \not\vdash \psi_n$ for each $n < \omega$,
- (P2) for each $n > 0$, $\psi_n = \psi_{n-1} \vee \chi$ and some formula χ . Note that the choice of χ depends on n .

Obviously, properties (P1) and (P2) hold for T_0, ψ_0 . Now, we will enumerate the formulas in $\mathcal{F}$ as $(\alpha_i : i < \omega)$. Suppose that for some n , we defined T_m, ψ_m for every $m \leq n$ with properties (P1) and (P2). We will now define T_{n+1}, ψ_{n+1} .

1. If $T_n \cup \{\alpha_n\} \not\vdash \psi_n$, then $T_{n+1} = T_n \cup \{\alpha_n\}$ and $\psi_{n+1} = \psi_n \vee \bar{0}$.
By Lemma 6.1 points (T20), (T9) and BL-axiom (A1), $\vdash \psi_{n+1} \rightarrow \psi_n$. Since $T_{n+1} \not\vdash \psi_n$ by definition of T_{n+1} , it follows that $T_{n+1} \not\vdash \psi_{n+1}$. Therefore property (P1) is satisfied. Property (P2) is satisfied by the definition of ψ_{n+1} .
2. Assume $(\#) T_n \cup \{\alpha_n\} \vdash \psi_n$. Then $T_{n+1} = T_n$ and for ψ_{n+1} we consider two cases:
 - (a) Suppose that α_n is $\varphi \vee (\alpha \rightarrow \alpha \& \beta)$ for some formulas φ, α, β .

Claim 1. There is $k < \omega$ such that $T_{n+1} \not\vdash \psi_n \vee (\varphi \vee (\alpha \rightarrow \beta^k))$.

Proof of claim. Suppose for contradiction that (C) $T_{n+1} \vdash \psi_n \vee (\varphi \vee (\alpha \rightarrow \beta^k))$ for every $k < \omega$. We can show that $T_n \vdash \psi_n \vee \alpha_n$ similarly to the proof of Case 4 of Lemma 6.4. By $(\#) T_n \cup \{\alpha_n\} \vdash \psi_n$. Thus by Lemma 6.6, $T_n \vdash \psi_n$, contradicting (P1). This proves the claim.

Using Claim 1, choose the least k such that $T_{n+1} \not\vdash \psi_n \vee (\varphi \vee (\alpha \rightarrow \beta^k))$, and define $\psi_{n+1} = \psi_n \vee (\varphi \vee (\alpha \rightarrow \beta^k))$. Then (P1) and (P2) clearly hold.

- (b) If α_n is not $\varphi \vee (\alpha \rightarrow \alpha \& \beta)$ for any formulas φ, α, β , define $\psi_{n+1} = \psi_n \vee \bar{0}$, which satisfies (P2). By Lemma 6.1 points (T20), (T9) and BL-axiom (A1), $\vdash \psi_{n+1} \rightarrow \psi_n$. Since $T_{n+1} \not\vdash \psi_n$ by (P1) and the definition of T_{n+1} , it follows that $T_{n+1} \not\vdash \psi_{n+1}$. Therefore property (P1) is also satisfied.

This completes the inductive definition of T_n, ψ_n for all $n < \omega$.

Claim 2. Let $U \in \wp(\mathcal{F})$ and $n < \omega$. If $U \vdash \psi_n$, then $U \vdash \psi_m$ for every $m \geq n$.

Proof of claim. We will show by induction on k that $U \vdash \psi_{n+k}$ for every $k < \omega$. The base case for $k = 0$ is obvious. Assume the inductive hypothesis: (C) $U \vdash \psi_{n+k}$. We show that $U \vdash \psi_{n+k+1}$, where $\psi_{n+k+1} = \psi_{n+k} \vee \chi$ for some formula χ . We use Lemma 6.1 point (T9).

1. ψ_{n+k} assumption (C)
2. $\psi_{n+k} \rightarrow \psi_{n+k} \vee \chi$ (T9)
3. $\psi_{n+k} \vee \chi$ (MP) 1,2

Define $T^* = \bigcup_{n < \omega} T_n$.

Claim 3. T^* is prelinearly closed.

Proof of claim. Suppose for contradiction that there are $\alpha, \beta \in \mathcal{F}$ such that $\alpha \rightarrow \beta, \beta \rightarrow \alpha \notin T^*$. By construction, this means that there are $n, m < \omega$ such that $T_n \cup \{\alpha \rightarrow \beta\} \vdash \psi_n$ and (a) $T_m \cup \{\beta \rightarrow \alpha\} \vdash \psi_m$. Without loss of generality suppose that $m \geq n$. Then by monotonicity (Lemma 6.2), (b) $T_m \cup \{\alpha \rightarrow \beta\} \vdash \psi_n$. From (b) and Claim 2, we get $T_m \cup \{\alpha \rightarrow \beta\} \vdash \psi_m$. From this and (a), we get $T_m \vdash \psi_m$ by (Prelin) (Lemma 6.7), contradicting property (P1).

In the next claim we show that T^* is deductively closed. We will use this to prove that $T^* \not\vdash \psi_0$.

Claim 4. T^* is deductively closed.

Proof of claim. Let $(\varphi_i : i \leq \xi)$ be an infinitary proof from T^* . We show by induction on $i \leq \xi$ that $\varphi_i \in T^*$ for each i . We assume the inductive hypothesis for all $j < i$. Let $\mathcal{H} = \{\varphi_j : j < i\}$. We show that $\varphi_i \in T^*$. We consider the 4 cases of Definition 5.6, where we assume $\mathcal{A}$ to be the set of all axioms.

1. $\varphi_i \in \mathcal{A}$. Suppose for contradiction that $\varphi_i \notin T^*$. Then there is $n < \omega$ such that (A) $T_n \cup \{\varphi_i\} \vdash \psi_n$. By monotonicity (Lemma 6.2), $T_n \vdash \varphi_i$. So (B) $T_n \vdash \theta$ for every $\theta \in T_n \cup \{\varphi_i\}$. Thus by transitivity (Lemma 6.3) from (A) and (B), $T_n \vdash \psi_n$ contradicting (P1).
2. $\varphi_i \in T^*$. Obvious.

3. There is a formula χ such that $\chi, \chi \rightarrow \varphi_i \in \mathcal{H}$. By inductive hypothesis, $\chi, \chi \rightarrow \varphi_i \in T^*$. Suppose for contradiction $\varphi_i \notin T^*$. This means that for some $n < \omega$, $T_n \cup \{\varphi_i\} \vdash \psi_n$. Now choose $m \geq n$ such that $\chi, \chi \rightarrow \varphi_i \in T_m$. By monotonicity (Lemma 6.2), (A) $T_m \cup \{\varphi_i\} \vdash \psi_n$. Since $T_m \vdash \chi$ and $T_m \vdash \chi \rightarrow \varphi_i$, (B) $T_m \vdash \varphi_i$ by rule (MP). As in point 1 above, we can show that $T_m \vdash \psi_n$ from (A) and (B). Now by Claim 2, $T_m \vdash \psi_m$ contradicting (P1).
4. There are formulas φ, α, β such that $\varphi_i = \varphi \vee (\alpha \rightarrow \alpha \& \beta)$ and $\varphi \vee (\alpha \rightarrow \beta^l) \in \mathcal{H}$ for all $l < \omega$. Suppose for contradiction $\varphi_i \notin T^*$. Pick $n < \omega$ such that $\varphi_i = \alpha_n$. By construction, $\psi_{n+1} = \psi_n \vee (\varphi \vee (\alpha \rightarrow \beta^k))$ for some $k < \omega$. Since inductively $\varphi \vee (\alpha \rightarrow \beta^l) \in T^*$ for all $l < \omega$, we choose $m > n$ such that (C) $\varphi \vee (\alpha \rightarrow \beta^k) \in T_m$. We show that $T_m \vdash \psi_m$ contradicting (P1). We use Lemma 6.1 point (T9).

1.	$\varphi \vee (\alpha \rightarrow \beta^k)$	assumption (C)
2.	$[\varphi \vee (\alpha \rightarrow \beta^k)] \rightarrow [\psi_n \vee (\varphi \vee (\alpha \rightarrow \beta^k))]$	(T9)
3.	$\psi_n \vee (\varphi \vee (\alpha \rightarrow \beta^k))$	(MP) 1,2
4.	ψ_m	Claim 2

Now if $T^* \vdash \varphi$, then there is a proof ending in φ . By induction, $\varphi \in T^*$, which concludes the proof of the claim.

We showed that T^* is prelinearly and deductively closed. We need to show that $T^* \not\vdash \psi_0$. Suppose that $T^* \vdash \psi_0$; then by Claim 4, $\psi_0 \in T^*$ and consequently there exists $n < \omega$ such that $\psi_0 \in T_n$. Thus $T_n \vdash \psi_0$ and by Claim 2, $T_n \vdash \psi_n$, which contradicts (P1). This completes the proof of the theorem. $\square$

6.2.2 Equivalence classes with respect to T^*

We will recall the definition of a particular Lindenbaum-Tarski algebra below in Definition 6.11 (see Definition 2.3.11 of [Háj98b]) and a BL-algebra (see Definitions 2.3.2 and 2.3.3 of [Háj98b]). We show that an algebra with the domain being the set of equivalence classes of formulas with respect to T^* is a linearly-ordered BL-algebra.

Definition 6.11. Let T be a fixed theory over BL. For each formula φ , let $[\varphi]_T$ be the set of all formulas χ such that $T \vdash \varphi \equiv \chi$, and let $L_T = \{[\varphi]_T : \varphi \in \mathcal{F}\}$. We define:

$$\begin{aligned}
\hat{0} &= [\bar{0}]_T, \\
\hat{1} &= [\bar{1}]_T, \\
[\theta]_T \star [\chi]_T &= [\theta \& \chi]_T, \\
[\theta]_T \Rightarrow [\chi]_T &= [\theta \rightarrow \chi]_T, \\
[\theta]_T \cap [\chi]_T &= [\theta \wedge \chi]_T, \\
[\theta]_T \cup [\chi]_T &= [\theta \vee \chi]_T.
\end{aligned}$$

We denote the algebra $(L_T, \cap, \cup, \star, \Rightarrow, \hat{0}, \hat{1})$ by $\mathbb{L}_T$.

It can be checked using Lemma 6.1 point (T16) that for all formulas φ, χ , we have $T \vdash \varphi \equiv \chi$ iff $[\varphi]_T = [\chi]_T$. It now follows by Lemma 6.1 points (T17), (T18) and (T19) that the above definitions of $\star$ and $\Rightarrow$ are sound. Soundness of the definitions of $\cap, \cup$ now follows since $\wedge, \vee$ are abbreviations. We remark that $T \vdash \varphi \Leftrightarrow [\varphi]_T = \hat{1}$.

Definition 6.12. An algebra $\mathbb{L} = (L, \cap, \cup, \star, \Rightarrow, 0, 1)$ with binary operations $\cap, \cup, \star, \Rightarrow$ on L and constants $0, 1 \in L$ is said to be a *BL-algebra* iff the following hold:

1. $(L, \cap, \cup, 0, 1)$ is a lattice. Its ordering is defined as follows. For $x, y \in L_T$, we define

- (a) $x \leq y$ iff $x \cap y = x$,
- (b) $x < y$ iff $x \leq y$ and $x \neq y$.

$(L, \cap, \cup, 0, 1)$ has the greatest element 1 and least element 0.

2. $(L, \star, 1)$ is a commutative semigroup with unit element 1.
3. (residuation) For all x, y, z , $z \leq (x \Rightarrow y)$ iff $x \star z \leq y$.
4. For all $x, y \in L$, $x \cap y = x \star (x \Rightarrow y)$.
5. (pre-linearity) For all $x, y \in L$, $(x \Rightarrow y) \cup (y \Rightarrow x) = 1$.

If $L = [0, 1]$, then $\mathbb{L}$ is called a *standard* BL-algebra.

Lemma 6.13. $\mathbb{L}_{T^*}$ is a linearly-ordered BL-algebra (BL-chain).

Proof. By Lemma 2.3.12 of [Háj98b], $\mathbb{L}_{T^*}$ is a BL-algebra. So $(L_{T^*}, \cap, \cup, \hat{0}, \hat{1})$ is a lattice. We need to show that it is linearly ordered. It is enough to show that $[\varphi]_{T^*} \cap [\chi]_{T^*} \in \{[\varphi]_{T^*}, [\chi]_{T^*}\}$ for any $\varphi, \chi \in \mathcal{F}$. Suppose that $[\varphi \rightarrow \chi]_{T^*} = \hat{1}$. Then

$$[\varphi]_{T^*} \cap [\chi]_{T^*} = [\varphi \wedge \chi]_{T^*} = [\varphi \& (\varphi \rightarrow \chi)]_{T^*} = [\varphi \& \bar{1}]_{T^*} = [\varphi]_{T^*}.$$

The case $[\chi \rightarrow \varphi]_{T^*} = \hat{1}$ is similar, and since T^* is prelinearly closed, at least one of the cases holds. $\square$

We define a linear pre-order $\leq$ in $\mathcal{F}$ (justified by Lemmas 6.13 and 6.15), which will be used in the lemmas about the properties of the relation $\sim$, a relation to be defined below.

Definition 6.14. For $\alpha, \beta \in \mathcal{F}$, we define:

1. $\alpha \leq \beta$ iff $T^* \vdash \alpha \rightarrow \beta$,
2. $\alpha < \beta$ iff $\alpha \leq \beta$ and $\beta \not\leq \alpha$,

Lemma 6.15. Let $\alpha, \beta \in \mathcal{F}$. Then $\alpha \leq \beta$ iff $[\alpha]_{T^*} \leq [\beta]_{T^*}$.

Proof. Assume $\alpha \leq \beta$. By Lemma 6.1 points (T18) and (T19), $[\alpha]_{T^*} \leq [\beta]_{T^*}$. Assume $[\alpha]_{T^*} \leq [\beta]_{T^*}$. By definition of $\equiv$, Lemma 6.1 points (T8) and (T14), $\alpha \leq \beta$. (Cf. Lemma 2.3.12 of [Háj98b].) $\square$

6.2.3 Clusters of equivalent formulas

We define an equivalence relation and group elements of L_{T^*} into its equivalence classes called clusters. Then we study how $\star, \Rightarrow$ behave when applied to elements in the same cluster, or in different clusters (see Lemmas 6.19, 6.22 and 6.23).

Definition 6.16. Let $a, b \in L_{T^*}$.

1. We define $a^{*(n)}$ for $n < \omega$ inductively as follows: $a^{*(0)} = \hat{1}$ and $a^{*(n+1)} = a^{*(n)} \star a$.
2. We define a binary relation $\sim$ on L_{T^*} as $a \sim b$ iff $\exists n < \omega (a^{*(n)} \leq b \leq a \text{ or } b^{*(n)} \leq a \leq b)$.

Note that $([\varphi]_{T^*})^{*(n)} = [\varphi^n]_{T^*}$ for each $\varphi \in \mathcal{F}$ and $n < \omega$.

Lemma 6.17. The relation $\sim$ is an equivalence relation.

Proof. 1. Reflexivity and symmetry. Easy.

2. Transitivity. Let $a, b, c \in L_{T^*}$. Suppose that $a \leq b \leq c$ (other cases are analogous). Assume $a \sim b, b \sim c$. Thus there exist $m, n < \omega$ such that (a) $b^{*(n)} \leq a$, (b) $c^{*(m)} \leq b$. By Lemma 6.1 point (T7) and (b), $c^{*(m \cdot n)} \leq b^{*(n)}$, and so $a \sim c$ since (a) holds and by transitivity of $\leq$.

$\square$

Definition 6.18. A *cluster* is an equivalence class of $\sim$.

We will describe informally what the clusters look like. $\{\hat{1}\}$ is a cluster by itself – the top cluster. Since $[\psi_0]_{T^*} \neq \hat{1}$, there is at least one other cluster. The cluster containing $\hat{0}$ is the bottom cluster: it is $\{a \in L_{T^*} : a^{*(n)} = \hat{0} \text{ for some } n < \omega\}$, which may be just $\{\hat{0}\}$, or may be bigger. Depending on T^* , these two might be the only clusters, or there might be finitely or infinitely many other clusters in between them. But there are only countably many clusters altogether, because $\mathcal{F}$ and hence L_{T^*} are countable.

Lemma 6.19. Let $\alpha, \beta \in \mathcal{F}$ and C be a cluster. If $[\alpha]_{T^*}, [\beta]_{T^*} \in C$, then $[\alpha]_{T^*} \star [\beta]_{T^*} \in C$.

Proof. Assume $[\alpha]_{T^*} \sim [\beta]_{T^*}$. Then by Lemma 6.15, there is $n < \omega$ such that (1) $\beta^n \leq \alpha \leq \beta$ or (2) $\alpha^n \leq \beta \leq \alpha$. Suppose without loss of generality that (1) holds. We use Lemma 6.1 point (T6).

- | | | |
|----|---|----------------|
| 1. | $\beta^n \rightarrow \alpha$ | assumption (1) |
| 2. | $\alpha \rightarrow \beta$ | assumption (1) |
| 3. | $(\beta^n \rightarrow \alpha) \rightarrow (\beta^n \& \beta \rightarrow \alpha \& \beta)$ | (T6) |
| 4. | $\beta^{n+1} \rightarrow \alpha \& \beta$ | (MP) 1,3 |
| 5. | $\alpha \& \beta \rightarrow \beta \& \alpha$ | (A3) |
| 6. | $\beta \& \alpha \rightarrow \beta$ | (A2) |
| 7. | $(\alpha \& \beta \rightarrow \beta \& \alpha) \rightarrow ((\beta \& \alpha \rightarrow \beta) \rightarrow (\alpha \& \beta \rightarrow \beta))$ | (A1) |
| 8. | $(\beta \& \alpha \rightarrow \beta) \rightarrow (\alpha \& \beta \rightarrow \beta)$ | (MP) 5,7 |
| 9. | $\alpha \& \beta \rightarrow \beta$ | (MP) 6,8 |

Thus by points 4 and 9 in the proof above and by Lemma 6.15, we have $[\alpha]_{T^*} \star [\beta]_{T^*} \sim [\beta]_{T^*}$. $\square$

The following two lemmas will be used in Lemmas 6.22 and 6.23 discussing further properties of the clusters.

Lemma 6.20. Let $\alpha, \beta \in \mathcal{F}$. If $\alpha \leq \beta$, then $T^* \vdash \beta \& (\beta \rightarrow \alpha) \equiv \alpha$.

Proof. We use Lemma 6.1 points (T2), (T5), (T4).

- | | | |
|-----|---|------------|
| 1. | $\beta \& (\beta \rightarrow \alpha) \rightarrow \alpha$ | (T4) |
| 2. | $\alpha \& (\alpha \rightarrow \beta) \rightarrow \beta \& (\beta \rightarrow \alpha)$ | (A4) |
| 3. | $[\alpha \& (\alpha \rightarrow \beta) \rightarrow \beta \& (\beta \rightarrow \alpha)] \rightarrow [\alpha \rightarrow ((\alpha \rightarrow \beta) \rightarrow \beta \& (\beta \rightarrow \alpha))]$ | (A5b) |
| 4. | $\alpha \rightarrow ((\alpha \rightarrow \beta) \rightarrow \beta \& (\beta \rightarrow \alpha))$ | (MP) 2,3 |
| 5. | $[\alpha \rightarrow ((\alpha \rightarrow \beta) \rightarrow \beta \& (\beta \rightarrow \alpha))] \rightarrow [(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta \& (\beta \rightarrow \alpha))]$ | (T2) |
| 6. | $(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta \& (\beta \rightarrow \alpha))$ | (MP) 4,5 |
| 7. | $\alpha \rightarrow \beta$ | assumption |
| 8. | $\alpha \rightarrow \beta \& (\beta \rightarrow \alpha)$ | (MP) 7,6 |
| 9. | $(\beta \& (\beta \rightarrow \alpha) \rightarrow \alpha) \rightarrow [(\alpha \rightarrow \beta \& (\beta \rightarrow \alpha)) \rightarrow (\beta \& (\beta \rightarrow \alpha) \equiv \alpha)]$ | (T5) |
| 10. | $(\alpha \rightarrow \beta \& (\beta \rightarrow \alpha)) \rightarrow (\beta \& (\beta \rightarrow \alpha) \equiv \alpha)$ | (MP) 1,9 |
| 11. | $\beta \& (\beta \rightarrow \alpha) \equiv \alpha$ | (MP) 8,10 |

$\square$

Lemma 6.21. Let $\alpha, \beta \in \mathcal{F}$. If for each $n < \omega$, $\alpha \leq \beta^n$, then $T^* \vdash \alpha \& \beta \equiv \alpha$

Proof. We use Lemma 6.1 points (T9), (T20). Below, lines 1-3 are provable from T^* for every $n < \omega$.

1.	$\alpha \rightarrow \beta^n$	assumption
2.	$(\alpha \rightarrow \beta^n) \rightarrow (\bar{0} \vee (\alpha \rightarrow \beta^n))$	(T9)
3.	$\bar{0} \vee (\alpha \rightarrow \beta^n)$	(MP) 1,2
4.	$\bar{0} \vee (\alpha \rightarrow \alpha \& \beta)$	(Inf) 3
5.	$(\bar{0} \vee (\alpha \rightarrow \alpha \& \beta)) \rightarrow (\alpha \rightarrow \alpha \& \beta)$	(T20)
6.	$\alpha \rightarrow \alpha \& \beta$	(MP) 4,5
7.	$\alpha \& \beta \rightarrow \alpha$	(A2)
8.	$(\alpha \& \beta \rightarrow \alpha) \rightarrow ((\alpha \rightarrow \alpha \& \beta) \rightarrow (\alpha \& \beta \equiv \alpha))$	(T5)
9.	$(\alpha \rightarrow \alpha \& \beta) \rightarrow (\alpha \& \beta \equiv \alpha)$	(MP) 7,8
10.	$\alpha \& \beta \equiv \alpha$	(MP) 6,9

□

Lemma 6.22. Let $\alpha, \beta \in \mathcal{F}$ and C be a cluster. If $[\alpha]_{T^*}, [\beta]_{T^*} \in C$ and $\alpha < \beta$, then $[\beta]_{T^*} \Rightarrow [\alpha]_{T^*} \in C$.

Proof. Suppose for contradiction that $[\beta]_{T^*} \Rightarrow [\alpha]_{T^*} \not\sim [\alpha]_{T^*}$. By Lemma 6.1 point (T1), $\alpha \leq \beta \rightarrow \alpha$. Let $n < \omega$. If $(\beta \rightarrow \alpha)^n \leq \beta$, then since $[\beta]_{T^*} \sim [\alpha]_{T^*}$, there is $m < \omega$ with $(\beta \rightarrow \alpha)^m \leq \alpha$, so $[\beta]_{T^*} \Rightarrow [\alpha]_{T^*} \sim [\alpha]_{T^*}$ contradicting our assumption. So by linearity of the pre-order $\leq$, we must have $\beta \leq (\beta \rightarrow \alpha)^n$ for every $n < \omega$. By Lemma 6.21, we have that $T^* \vdash \beta \& (\beta \rightarrow \alpha) \equiv \beta$. By Lemma 6.20, $T^* \vdash \beta \& (\beta \rightarrow \alpha) \equiv \alpha$ since $\alpha \leq \beta$. Therefore by Lemma 6.1 point (T16) and (MP), $T^* \vdash \alpha \equiv \beta$ contradicting $\alpha < \beta$. □

Lemma 6.23. Let $\alpha, \beta \in \mathcal{F}$. If $\alpha < \beta$ and there is no cluster C such that $[\alpha]_{T^*}, [\beta]_{T^*} \in C$, then (A) $[\alpha]_{T^*} \star [\beta]_{T^*} = [\alpha]_{T^*}$ and (B) $[\beta]_{T^*} \Rightarrow [\alpha]_{T^*} = [\alpha]_{T^*}$.

Proof. Take an arbitrary $n < \omega$. Since T^* is prelinearly closed, either (1) $\beta^n \leq \alpha$ or (2) $\alpha \leq \beta^n$. If (1) holds, then $[\alpha]_{T^*} \sim [\beta]_{T^*}$, but we assumed $[\alpha]_{T^*} \not\sim [\beta]_{T^*}$. So it must be (C) $\alpha \leq \beta^n$ for all $n < \omega$. Thus (A) holds by Lemma 6.21.

Now, we will prove (B). By Lemma 6.1 point (T1), $\alpha \leq \beta \rightarrow \alpha$. We need to prove $\beta \rightarrow \alpha \leq \alpha$, and then by Lemma 6.1 point (T5) and (MP), we have (B). First, by Lemma 6.20, $[\beta]_{T^*} \star ([\beta]_{T^*} \Rightarrow [\alpha]_{T^*}) = [\alpha]_{T^*}$.

Now, suppose for contradiction that $\alpha < \beta \rightarrow \alpha$. There are two cases: (1) $[\beta]_{T^*} \not\sim [\beta]_{T^*} \Rightarrow [\alpha]_{T^*}$ and (2) $[\beta]_{T^*} \sim [\beta]_{T^*} \Rightarrow [\alpha]_{T^*}$. In case (1), by (A) $T^* \vdash \beta \& (\beta \rightarrow \alpha) \equiv \beta$, where $\eta = \beta$ if $\beta \leq \beta \rightarrow \alpha$, or $\eta = \beta \rightarrow \alpha$ otherwise. Now, $\alpha < \eta$ since we assumed

that $\alpha < \beta$ and $\alpha < \beta \rightarrow \alpha$, which contradicts $T^* \vdash \beta \& (\beta \rightarrow \alpha) \equiv \alpha$ (Lemma 6.20). In case (2), by Lemma 6.19, $[\beta]_{T^*} \star ([\beta]_{T^*} \Rightarrow [\alpha]_{T^*}) \sim [\beta]_{T^*}$ contradicting $[\alpha]_{T^*} \not\sim [\beta]_{T^*}$ as we have $[\beta]_{T^*} \star ([\beta]_{T^*} \Rightarrow [\alpha]_{T^*}) = [\alpha]_{T^*}$. Therefore $\beta \rightarrow \alpha \leq \alpha$ and consequently $T^* \vdash (\beta \rightarrow \alpha) \equiv \alpha$. $\square$

6.2.4 Properties of clusters

We will now look at two different types of clusters and we will prove some facts about them in Lemmas 6.25 – 6.30, which will be used in constructing the functions that map clusters into appropriate standard algebras.

Definition 6.24. We say that a cluster is *Łukasiewicz* iff there is $x \in C$ such that $x^{*(2)} = x$ and we call x an idempotent, otherwise the cluster is *product*.

We remark that $\hat{0}, \hat{1}$ are idempotents so their clusters are Łukasiewicz.

There is a unique idempotent in any Łukasiewicz cluster C . Suppose for contradiction there are two idempotents $x_1 < x_2$. Then since they are in the same cluster, there exists k such that $x_2^{*(k)} \leq x_1 < x_2$. But $x_2^{*(k)} = x_2$. Contradiction. We denote the unique idempotent of Łukasiewicz cluster C by 0_C . In this subsection we will use $\neg x$ to denote $x \Rightarrow 0_C$ for $x \in C$.

Lemma 6.25. Let C be a Łukasiewicz cluster. Then

1. $0_C \leq x$ for $x \in C$,
2. $0_C = x \star 0_C$ for every $x \in C$,
3. for every $x \in C$, there is $n < \omega$ with $x^{*(n)} = 0_C$.

Proof. 1. Suppose for contradiction there is $x \in C$ such that $x < 0_C$. Then since $x, 0_C \in C$, so that $x \sim 0_C$, there exists n such that $0_C^{*(n)} \leq x$. But $0_C^{*(n)} = 0_C$. Contradiction.

2. By definition of 0_C , and BL-axioms (A1), (A2) and (A3), $0_C = 0_C \star 0_C \leq x \star 0_C \leq 0_C$.

3. For all $k < \omega$, $0_C = 0_C^{*(k)} \leq x^{*(k)}$ by point 1 and Lemma 6.1 point (T7). Since $x \in C$, by definition of cluster, there is n such that $x^{*(n)} \leq 0_C \leq x$. $\square$

Lemma 6.26. Let C be a Łukasiewicz cluster such that $C \neq \{\hat{1}\}$. For any $x \in C$, $\neg x > 0_C$.

Proof. If $x = 0_C$, then $\neg 0_C = \hat{1} > 0_C$. Assume that $x > 0_C$. By Lemma 6.25, there exists the least $n > 1$ such that $x^{*(n)} = 0_C$. Therefore by residuation, $x \Rightarrow 0_C \geq x^{*(n-1)} > 0_C$. $\square$

Lemma 6.27. Let C be any cluster. Suppose that $x, y \in C$ and $(\#) x \star y = y$. Then y is an idempotent. Hence C is Łukasiewicz and $y = 0_C$.

Proof. Since by assumption $(\#)$, $y \leq x \star y$ and by properties of $\star$, $x \star y \leq x$, we have $y \leq x$. Hence since $x \sim y$, (1) there exists n such that $x^{\star(n)} \leq y \leq x$. We can also show by induction on $k < \omega$ that $x^{\star(k)} \star y = y$ from assumption $(\#)$. By properties of $\star$, $x^{\star(k)} \star y \leq x^{\star(k)}$, and thus (2) $y \leq x^{\star(k)}$ for all $k \geq 0$. Therefore by (1) and (2) for sufficiently large m , we have $x^{\star(m)} = y$ and thus $y = x^{\star(m)} \star y = y \star y$. Thus y is an idempotent. By definition, C is Łukasiewicz and $y = 0_C$. $\square$

Lemma 6.28. Let C be a Łukasiewicz cluster. For any $x, y \in C$, if $x < y$, then $\neg y < \neg x$.

Proof. Plainly, $x \not\sim \hat{1}$, so $\hat{1} \notin C$. Assume $x = 0_C$ and $0_C < y$. Then $\neg x = \hat{1}$, and since $y \Rightarrow 0_C \in C$, $y \Rightarrow 0_C < \hat{1}$. Therefore, $\neg y < \neg x$. Assume $0_C < x < y$. By Lemma 6.20, $x = y \star (y \Rightarrow x)$. Thus $\neg x = \neg(y \star (y \Rightarrow x))$. Thus by BL-axiom (A5b), Lemma 6.1 point (T2), $\neg x = y \star (y \Rightarrow x) \Rightarrow 0_C = (y \Rightarrow x) \Rightarrow (y \Rightarrow 0_C) = (y \Rightarrow x) \Rightarrow \neg y$. By commutativity of $\star$, $\neg x \star (y \Rightarrow x) = (y \Rightarrow x) \star ((y \Rightarrow x) \Rightarrow \neg y)$. Again by BL-axiom (A4), $(y \Rightarrow x) \star ((y \Rightarrow x) \Rightarrow \neg y) = \min\{\neg y, y \Rightarrow x\}$. If $\neg x \star (y \Rightarrow x) = y \Rightarrow x$, then either (1) $\neg x = \hat{1}$ or (2) by Lemma 6.27, $y \Rightarrow x = 0_C$. Both cases lead to contradiction since we assumed $x > 0_C$. Therefore $\neg x \star (y \Rightarrow x) = \neg y$ and since $y \Rightarrow x < \hat{1}$ and $0_C < \neg x < \hat{1}$ by Lemma 6.26, $\neg y < \neg x$. $\square$

Lemma 6.29. Let C be a Łukasiewicz cluster and $x \in C$. Then $x = \neg\neg x$.

Proof. First if $x = 0_C$, then $0_C \Rightarrow 0_C = \hat{1}$ and $\hat{1} \Rightarrow 0_C = 0_C$ by Lemma 6.23 if $C \neq \{\hat{1}\}$, and trivially if $C = \{\hat{1}\}$. Assume $x > 0_C$. Then $C \neq \{\hat{1}\}$. By Lemma 6.1 point (T4), $x \star \neg x = x \star (x \Rightarrow 0_C) \leq 0_C$ and by residuation, $x \leq \neg\neg x$. We show $\neg\neg x \leq x$. By Lemma 6.20 and Lemma 6.26, $\neg x \star (\neg x \Rightarrow 0_C) = 0_C$. Then by residuation, $\neg x \leq \neg\neg\neg x$, which is equivalent to the negation of the statement $\neg\neg\neg x < \neg x$. By Lemma 6.28, it is not true that $x < \neg\neg x$, equivalent to $\neg\neg x \leq x$. $\square$

Lemma 6.30. Let C be a cluster and $x, y, z \in C$. Suppose that $x \star z$ is not idempotent and $x \star z \leq y \star z$. Then $x \leq y$.

Proof. Suppose for contradiction $x > y$. Then by monotonicity of $\star$, $x \star z \geq y \star z$ and consequently by the assumption, $x \star z = y \star z$. Since $y < x$, by Lemma 6.20, $x \star (x \Rightarrow y) = y$. Now by associativity and commutativity of $\star$, $x \star z \star (x \Rightarrow y) = x \star (x \Rightarrow y) \star z = y \star z = x \star z$. Then by Lemmas 6.19, 6.22 and Lemma 6.27, $x \star z$ is idempotent, which contradicts the assumption. $\square$

6.2.5 Linear Archimedean product and MV-algebras

We will construct linear Archimedean product and MV-algebras (recalled in Definition 6.31 from [Mon07] page 250).

- Definition 6.31.** 1. A BL-algebra is called an *MV-algebra* iff it satisfies $\neg\neg x = x$ and it is called a *standard MV-algebra* iff its domain is $[0,1]$.
2. A BL-algebra is called a *product algebra* iff it satisfies $\neg x \cup ((x \Rightarrow x \star y) \Rightarrow y) = 1$ ¹⁶ and it is called a *standard product algebra* iff its domain is $[0,1]$.

We recall the definition of Archimedean BL-algebra (see page 255 of [Mon07], cf. Definition 2.23 of this thesis).

Definition 6.32. A BL-algebra $\mathbb{L} = (L, \cap, \cup, \star, \Rightarrow, 0, 1)$ is said to be *Archimedean* iff for every $x, y \in L - \{0, 1\}$, there is a positive integer n such that $x^{\star(n)} < y$.

First let $C \subseteq L_{T^*}$ be a product cluster. Let $C^+ = C \cup \{\hat{0}, \hat{1}\}$. We check that C^+ is closed under each operation $\sharp \in \{\star, \Rightarrow, \cap, \cup\}$. Since we interpret $\cap, \cup$ as $\min, \max$, respectively, $x \cap y = \min\{x, y\} \in C^+$ and $x \cup y = \max\{x, y\} \in C^+$ for any $x, y \in C^+$. Take any $x, y \in C$. Then $x \star y \in C \subseteq C^+$ by Lemma 6.19 and if $x < y$, then $y \Rightarrow x \in C \subseteq C^+$ by Lemma 6.22. If $y \leq x$, then $y \Rightarrow x = \hat{1} \in C^+$. Also, for every $x \in C^+$, $\hat{0} \star x = x \star \hat{0} = \hat{0} \in C^+$ and $\hat{1} \star x = x \star \hat{1} = \hat{1} \Rightarrow x = x \in C^+$ and $\hat{0} \Rightarrow x = x \Rightarrow \hat{1} = \hat{1} \in C^+$. For $x \in C$, $x \Rightarrow \hat{0} = \hat{0} \in C^+$ by Lemma 6.23 since $\hat{0} \notin C$.

For each $\sharp \in \{\star, \Rightarrow, \cap, \cup\}$, we let $\sharp'$ denote its restriction to C^+ . Then $\mathcal{C} = (C^+, \cap', \cup', \star', \Rightarrow', \hat{0}, \hat{1})$ is a subalgebra of $\mathbb{L}_{T^*}$.

Lemma 6.33. $\mathcal{C}$ is a linear Archimedean product algebra.

Proof. First we check that $\mathcal{C}$ is a linear product algebra. Since $\mathcal{C}$ is a subalgebra of $\mathbb{L}_{T^*}$, $\mathcal{C}$ is a linear BL-algebra by Lemma 6.13.

We show that $\mathcal{C}$ satisfies $\forall x, y \in C^+ ((\neg x) \cup ((x \Rightarrow x \star y) \Rightarrow y)) = \hat{1}$. It is easy to check when $x \in \{\hat{0}, \hat{1}\}$ or $y = \hat{1}$. If $y = \hat{0}$ and $x \in C$, then $x \star \hat{0} = \hat{0}$, and since $x, \hat{0}$ are in different clusters, by Lemma 6.23 we have $x \Rightarrow \hat{0} = \hat{0}$. Thus $\max\{x \Rightarrow \hat{0}, (x \Rightarrow x \star \hat{0}) \Rightarrow \hat{0}\} = \max\{\hat{0}, \hat{1}\} = \hat{1}$.

Take $x, y \in C$. Obviously, $x \star y \leq x$. By Lemmas 6.19 and 6.22, $x \star y, x \Rightarrow x \star y \in C$. Then $x \star (x \Rightarrow x \star y) = x \star y$ by Lemma 6.20, and since $x \star y$ is not idempotent, by Lemma 6.30, $x \Rightarrow x \star y = y$. We have $x \Rightarrow \hat{0} = \hat{0}$ by Lemma 6.23. Therefore $\max\{x \Rightarrow \hat{0}, (x \Rightarrow x \star y) \Rightarrow y\} = \max\{\hat{0}, \hat{1}\} = \hat{1}$. Thus $\mathcal{C}$ is a linear product algebra.

We show that $\mathcal{C}$ is Archimedean. That is, for every $x, y \in C$, there is a positive integer n such that $x^{\star(n)} < y$. If $x \leq y$, then $(\#)$ since there are no idempotents in C and by properties of $\star$, $x^{\star(2)} < x$. Thus, $n = 2$. If $y < x$, then by definition of cluster, there exists $m < \omega$ such that $x^{\star(m)} \leq y < x$. Take $n = 2m$, and by $(\#)$, $x^{\star(n)} < y$. This completes the proof of the lemma. $\square$

¹⁶Note that in [Háj98b] a product algebra is defined as a BL-algebra satisfying $\neg\neg z \leq ((x \star z \Rightarrow y \star z) \Rightarrow (x \Rightarrow y))$ and $x \cap \neg x = 0$. However, we use the definition of [Mon07] since we will use Montagna's result from this paper.

Now let $C \subseteq L_{T^*}$ be a Łukasiewicz cluster with idempotent 0_C (the minimal element). In this case we let $C^+ = C \cup \{\hat{1}\}$. It can be checked that C^+ is closed under each operation $\sharp \in \{\star, \Rightarrow, \cap, \cup\}$ similarly to product clusters (note that $0_C \in C$). We define a subalgebra $\mathcal{C}^- = (C^+, \cap', \cup', \star', \Rightarrow', \hat{1})$ of the $\hat{0}$ -free reduct $\mathbb{L}_{T^*}^- = (L_{T^*}, \cap, \cup, \star, \Rightarrow, \hat{1})$ of $\mathbb{L}_{T^*}$, where we let $\sharp'$ denote the restriction of $\sharp$ to C^+ . Let $\mathcal{C} = (C^+, \cap', \cup', \star', \Rightarrow', 0_C, \hat{1})$.

Lemma 6.34. $\mathcal{C}$ is a linear Archimedean MV-algebra.

Proof. We show that $\mathcal{C}$ is a linear MV-algebra. 0_C is the minimal element of C^+ , thus $\mathcal{C}$ is a BL-algebra since $(C^+, \cap', \cup', 0_C, \hat{1})$ is a lattice with the largest element $\hat{1}$ and the least element 0_C and other properties of a BL-algebra are valid since they are valid in $\mathcal{C}^-$ as it is a subalgebra of $\mathbb{L}_{T^*}^-$ (cf. Definition 6.12). It is also linear by Lemma 6.13. By Lemma 6.29, and because $(\hat{1} \Rightarrow 0_C) \Rightarrow 0_C = \hat{1}$, $\mathcal{C}$ is an MV-algebra. Therefore it is a linear MV-algebra.

We show that $\mathcal{C}$ is Archimedean. That is, for every $x, y \in C - \{0_C, \hat{1}\}$, there is a positive integer n such that $x^{\star(n)} < y$. By Lemma 6.25, there exists $n < \omega$ such that $x^{\star(n)} = 0_C < y$. $\square$

Definition 6.35. An algebra is called *trivial* if its domain consists of one element. Otherwise it is called *non-trivial*.

Remark 6.36. If $C = \{\hat{1}\}$, then the algebra $\mathcal{C}$ is a trivial MV-algebra, since $0_{\{\hat{1}\}} = \hat{1}$. For each cluster $C \neq \{\hat{1}\}$, $\mathcal{C}$ is non-trivial since $\hat{1} \notin C$.

6.2.6 Constructing a continuous t-norm and an evaluation

We will embed each $\mathcal{C}$ into $[0, 1]_P$ or $[0, 1]_L$ (see Definition 6.37) if $\mathcal{C}$ is a product algebra or an MV-algebra, respectively. Then we will construct a continuous t-norm and an evaluation, and thus we will complete the proof of Theorem 6.8 for BL^+ .

There are two cases:

1. The set of clusters is $\{\{\hat{0}\}, \{\hat{1}\}\}$. The evaluation $V : PROP \rightarrow [0, 1]$ is such that $V(p) = 0$ for all $p \in \hat{0}$ and $V(p) = 1$ for $p \in \hat{1}$, and any continuous t-norm $*$ can be used to build the standard model. It is easy to check that $V_*(\theta) = 1$ for all $\theta \in T_0$, and $V_*(\psi_0) = 0$. This proves Theorem 6.8 in this case.
2. There is a cluster $C \notin \{\{\hat{0}\}, \{\hat{1}\}\}$.

For the rest of Section 6.2, we assume that there is a cluster $C \notin \{\{\hat{0}\}, \{\hat{1}\}\}$.

Definition 6.37. For each product cluster C , we define $S_C = [0, 1]_P$, where $[0, 1]_P$ denotes the standard product algebra whose t-norm is the product t-norm. For each Łukasiewicz cluster $C \neq \{\hat{1}\}$, let $S_C = [0, 1]_L$, where $[0, 1]_L$ denotes the standard MV-algebra whose t-norm is the Łukasiewicz t-norm. We leave $S_{\{\hat{1}\}}$ undefined.

Definition 6.38. We define the order of clusters C, D : $C < D$ if for all $x \in C$ and all $y \in D$, $x < y$.

We will use Montagna's Proposition 4 of [Mon07], which we quote below as Proposition 6.39. Note that every standard MV-algebra is isomorphic to $[0, 1]_L$, and every standard product algebra is isomorphic to $[0, 1]_P$ (see e.g p. 250 of [Mon07]).

Proposition 6.39. An MV-chain (a product chain respectively) is Archimedean iff it can be embedded in the MV-algebra (product algebra respectively) on $[0, 1]$ by a complete embedding.

So for each cluster $C \neq \{\hat{1}\}$, there is an algebra embedding $\zeta_C : \mathcal{C} \rightarrow S_C$. We leave $\zeta_{\{\hat{1}\}}$ undefined.

For each cluster $C \neq \{\hat{1}\}$:

- We choose $a_C, b_C \in [0, 1]$ as follows:
 1. Suppose that $\hat{0} \in C$. Then $a_C = 0$. If $C = \{\hat{0}\}$, then $b_C = 0$. Otherwise $b_C \in (0, 1]$.
 2. For every cluster C such that $\hat{0} \notin C$: $a_C, b_C \in [0, 1]$ such that $a_C < b_C$, and assume that $a_C > 0$ if C is Łukasiewicz. If there exist clusters D, E such that $D < C < E$, then $b_D \leq a_C$ and $b_C \leq a_E$. This is possible since the set of clusters is countable.
- Let $*_C$ be the t-norm (product or Łukasiewicz) on S_C .

Using Definition 2.34, let $*$ be the ordinal sum

$$\sum_{C \neq \{\hat{0}\}, \{\hat{1}\}} ([a_C, b_C], *_C)$$

of the $*_C$ for all clusters $C \neq \{\hat{1}\}$ with $a_C < b_C$. By assumption, there is at least one cluster $C \neq \{\hat{0}\}, \{\hat{1}\}$, so the sum is well defined. By Theorem 2.36, $*$ is a continuous t-norm on $[0, 1]$. Therefore, it has a residuum $\Rightarrow_*$.

Let $\lambda_C : [0, 1] \rightarrow [a_C, b_C]$ be defined as $\lambda_C(x) = a_C + (b_C - a_C) \cdot x$ for all $x \in [0, 1]$. We define a map $\xi : L_{T^*} \rightarrow [0, 1]$ by

$$\xi(x) = \begin{cases} 1 & \text{if } x = \hat{1}, \\ (\lambda_C \circ \zeta_C)(x) & \text{otherwise, where } C \text{ is the unique cluster with } x \in C. \end{cases}$$

We show that $\xi : L_{T^*} \rightarrow ([0, 1], \cap, \cup, *, \Rightarrow_*, 0, 1)$ is an embedding by Lemmas 6.40 and 6.41.

Lemma 6.40. The map ξ is an injection and order-preserving.

Proof. Let $x < y$ in $\mathbb{L}_{T^*}$. We show that $\xi(x) < \xi(y)$. There are two cases: (1) $x \sim y$ and (2) $x \not\sim y$. In case (1), since $x < y$, then the cluster $C \supseteq \{x, y\}$ is different from $\{\hat{0}\}, \{\hat{1}\}$. Since ζ_C is an embedding and λ_C is an increasing linear map, $\lambda_C(\zeta_C(x)) = \xi(x) < \xi(y) = \lambda_C(\zeta_C(y))$.

Now, we prove case (2). If $x \not\sim y$, then there are clusters $C < D$ such that $x \in C$ and $y \in D$. Thus $\xi(x) \in [a_C, b_C]$ and $\xi(y) \in [a_D, b_D]$. Therefore $\xi(x) \leq b_C \leq a_D \leq \xi(y)$. We need to show that $\xi(x) \neq \xi(y)$. We know that $x \neq \hat{1}$. Thus $\zeta_C(x) < \zeta_C(\hat{1}) = b_C$ if $a_C < b_C$, and therefore $\xi(x) < \xi(y)$. If $a_C = b_C$, then $C = \{\hat{0}\}$ and $\xi(x) = 0$. We need to show that $\xi(y) > 0$. If $a_D > 0$, then obviously $0 < a_D \leq \xi(y)$. If $a_D = 0$, then D is product by definition and thus $\hat{0} \notin D$, but $\hat{0} \in D^+$. Thus since $\hat{0} < y$, $\zeta_D(\hat{0}) < \zeta_D(y)$ since ζ_D is an embedding. Therefore $0 = \xi(\hat{0}) < \xi(y)$ since $\xi(y) = \lambda_D(\zeta_D(y))$ and λ_D is an increasing linear map. $\square$

Lemma 6.41. The map ξ is a homomorphism from $\mathbb{L}_{T^*}$ into $([0, 1], \cap, \cup, *, \Rightarrow_*, 0, 1)$.

Proof. By assumption, $\zeta_C : \mathcal{C} \rightarrow S_C$ is an algebra embedding, so it is a homomorphism.

The map λ_C is clearly a homomorphism from the $\{\cap, \cup, *_C\}$ -reduct of S_C into $([a_C, b_C], \cap, \cup, *_\lambda)$, where $*_C$ is the product t-norm (Łukasiewicz t-norm, resp.), S_C is $[0, 1]_P$ ($[0, 1]_L$, resp.) and $*_{\lambda_C} : [a_C, b_C] \rightarrow [a_C, b_C]$ is the proto-t-norm given by formula $x *_\lambda y = \lambda_C(\lambda_C^{-1}(x) *_C \lambda_C^{-1}(y))$ for $x, y \in [a_C, b_C]$. If $y < x$ in $[a_C, b_C]$, then $x \Rightarrow_{*\lambda_C} y = \lambda_C(\lambda_C^{-1}(x) \Rightarrow_{*_C} \lambda_C^{-1}(y))$ by Theorem 2.38, where g_C in that theorem is the identity function.

By definition of ξ we have

$$\xi \upharpoonright C = \lambda_C \circ \zeta_C.$$

The compositions of two homomorphisms is a homomorphism. So $\xi \upharpoonright C$ preserves $\cap, \cup, *$ and if $y < x$ in $\mathcal{C}$, then $\xi(x \Rightarrow y) = \xi(x) \Rightarrow_* \xi(y)$.

Now let C, D be clusters such that $x \in C, y \in D$ and $D < C$. Then by Lemma 6.23, definition of $*$ and Theorem 2.38, $\xi(x) * \xi(y) = \xi(x \star y)$ and $\xi(x) \Rightarrow_* \xi(y) = \xi(x \Rightarrow y)$.

Now suppose that $x \leq y$. Then since ξ preserves the order,

$$\xi(x \Rightarrow y) = \xi(\hat{1}) = 1 = \xi(x) \Rightarrow_* \xi(y)$$

By definition $\xi(\hat{1}) = 1$ and we show that $\xi(\hat{0}) = 0$. If $\hat{0} \in C$, then

$$\xi(\hat{0}) = (\lambda_C(\zeta_C(\hat{0}))) = \lambda_C(0) = a_C.$$

But $a_C = 0$ by the choice of a_C . $\square$

We define $V(p) = \xi([p]_{T^*})$ for each $p \in PROP$. As ξ is a homomorphism, a simple

induction on the complexity of formulas shows $V_*(\varphi) = \xi([\varphi]_{T^*})$ for every formula φ . We only check $\bar{0}$, $\&$, $\rightarrow$ since these are the primitive connectives of formulas.

1. Let C be the cluster such that $\hat{0} \in C$.

$$V_*(\bar{0}) = 0 = \zeta_C(\hat{0}) = 0 + (b_C - 0) \cdot \zeta_C(\hat{0}) = \xi(\hat{0}) = \xi([\bar{0}]_{T^*}).$$

2. Inductively assume the result for φ, χ . By semantics of $\&$, the induction hypothesis, the fact that ξ is a homomorphism and the definition of $\star$ in $\mathbb{L}_{T^*}$,

$$V_*(\varphi \& \chi) = V_*(\varphi) * V_*(\chi) = \xi([\varphi]_{T^*}) * \xi([\chi]_{T^*}) = \xi([\varphi]_{T^*} \star [\chi]_{T^*}) = \xi([\varphi \& \chi]_{T^*})$$

3. We can show $V_*(\varphi \rightarrow \chi) = \xi([\varphi \rightarrow \chi]_{T^*})$ similarly to point 2.

So $V_*(\varphi) = \xi([\varphi]_{T^*}) = \xi(\hat{1}) = 1$ for all $\varphi \in T^* \supseteq T_0$. Since $T^* \not\models \psi_0$, we have $[\psi_0]_{T^*} < \hat{1}$ in $\mathbb{L}_{T^*}$ so as ξ is injective, $V_*(\psi_0) = \xi([\psi_0]_{T^*}) < 1$. This completes the proof of Theorem 6.8 for BL^+ .

We will focus now on completing the proof for L^+, P^+ . We are assuming that there is at least one cluster $C \neq \{\hat{0}\}, \{\hat{1}\}$.

6.2.7 Completing the proof for L^+

We prove Theorem 6.8 for L^+ . We will show that axiom $(\neg\neg)$ restricts the number of clusters to two: C with $\hat{0} \in C$ and $\{\hat{1}\}$.

Lemma 6.42. For L^+ , if C is a cluster, then either $\hat{0} \in C$ or $\hat{1} \in C$.

Proof. Suppose that $\hat{0} \notin C$. We show that $\hat{1} \in C$. Since $\hat{0} \notin C$, there is a cluster D such that $D < C$ and $\hat{0} \in D$. By Lemma 6.23, $x \Rightarrow \hat{0} = \hat{0}$ for any $x \in C$. Thus $(x \Rightarrow \hat{0}) \Rightarrow \hat{0} = \hat{1}$. But for all $\varphi \in \mathcal{F}$, by axiom $(\neg\neg)$ and by Lemma 6.1 point (T13), $T^* \vdash_{L^+} ((\varphi \rightarrow \bar{0}) \rightarrow \bar{0}) \equiv \varphi$. Thus, $(x \Rightarrow \hat{0}) \Rightarrow \hat{0} = x$. Thus $x = \hat{1}$. Hence $\hat{1} \in C$. $\square$

Thus for the cluster $C \ni \hat{0}$, we choose $b_C = 1$, and once it is set, it follows that $*$ is the Łukasiewicz t-norm. This completes the proof of Theorem 6.8 for L^+ .

6.2.8 Completing the proof for P^+

We prove Theorem 6.8 for P^+ . We will show that axioms (II1), (II2) restrict the number of clusters to three: $\{\hat{0}\}, \{\hat{1}\}$ and a product cluster C . Thus for cluster C , we choose $a_C = 0, b_C = 1$, and thus $*$ is the product t-norm.

Lemma 6.43. There are exactly two Łukasiewicz clusters: $\{\hat{0}\}, \{\hat{1}\}$.

Proof. Let C be a cluster such that $\hat{0} \in C$. First we show that $C = \{\hat{0}\}$. Suppose for contradiction that there is $[\varphi]_{T^*} \in C$ such that $\varphi > \bar{0}$. Since the only idempotent of the cluster is $\hat{0}$, $\varphi^2 < \varphi$. Then by Lemma 6.28, $(\#) \neg\varphi < \neg\varphi^2$. By definition of $\wedge$, $\varphi \wedge \neg\varphi \rightarrow \bar{0} = (\varphi \& (\varphi \rightarrow (\varphi \rightarrow \bar{0}))) \rightarrow \bar{0}$. Now by BL-axioms (A5a) and (A5b), Lemma 6.1 points (T17), (T18) and (T2),

$$\begin{aligned} T^* \vdash_{P^+} (\varphi \& (\varphi \rightarrow (\varphi \rightarrow \bar{0}))) \rightarrow \bar{0} &\equiv (\varphi \& (\varphi^2 \rightarrow \bar{0})) \rightarrow \bar{0} \\ T^* \vdash_{P^+} (\varphi \& (\varphi^2 \rightarrow \bar{0})) \rightarrow \bar{0} &\equiv \varphi \rightarrow ((\varphi^2 \rightarrow \bar{0}) \rightarrow \bar{0}) \\ T^* \vdash_{P^+} \varphi \rightarrow ((\varphi^2 \rightarrow \bar{0}) \rightarrow \bar{0}) &\equiv (\varphi^2 \rightarrow \bar{0}) \rightarrow (\varphi \rightarrow \bar{0}) \end{aligned}$$

which together with $(\#)$ leads to $\varphi \wedge \neg\varphi \rightarrow \bar{0} < \bar{1}$ contradicting axiom (II2).

Now we show that there are no other idempotents except for $\hat{0}, \hat{1}$. Take an idempotent $[e]_{T^*} \in \mathbb{L}_{T^*}$ such that $[e]_{T^*} > \hat{0}$. Let $\chi = \theta = e, \varphi = \bar{1}$. Since there is at most one idempotent in a cluster and $[e]_{T^*} > \hat{0}$, $[e]_{T^*}, \hat{0}$ are in different clusters. Take an arbitrary $x \in L_{T^*}$ such that $x > \hat{0}$. Then there are two cases:

1. $x \star [\chi]_{T^*} = \min\{x, [\chi]_{T^*}\}$ if $x, [\chi]_{T^*}$ in different clusters.
2. $x \star [\chi]_{T^*} > \hat{0}$ if $x, [\chi]_{T^*}$ in the same cluster.

In both cases $x \star [\chi]_{T^*} > \hat{0}$. Then by residuation $\neg[\chi]_{T^*} < x$ for all $x > \hat{0}$ and thus $T^* \vdash_{P^+} \neg\chi \equiv \bar{0}$. Therefore $T^* \vdash_{P^+} \neg\neg\chi \equiv \bar{1}$. Also $T^* \vdash_{P^+} \varphi \& \chi \equiv e$ and $T^* \vdash_{P^+} \theta \& \chi \equiv e$ and thus $T^* \vdash_{P^+} \varphi \& \chi \rightarrow \theta \& \chi \equiv \bar{1}$. Therefore

$$T^* \vdash_{P^+} \neg\neg\chi \rightarrow ((\varphi \& \chi \rightarrow \theta \& \chi) \rightarrow (\varphi \rightarrow \theta)) \equiv \bar{1} \rightarrow (\bar{1} \rightarrow (\bar{1} \rightarrow e))$$

and thus $T^* \vdash_{P^+} \neg\neg\chi \rightarrow ((\varphi \& \chi \rightarrow \theta \& \chi) \rightarrow (\varphi \rightarrow \theta)) \equiv e$. All by Lemma 6.1. Therefore $T^* \vdash_{P^+} e \equiv \bar{1}$ by axiom (II1). $\square$

Lemma 6.44. There is at most one product cluster.

Proof. Suppose for contradiction that there are two product clusters C, D such that $\{\hat{0}\} < C < D < \{\hat{1}\}$. Take $[\chi]_{T^*} \in C, [\theta]_{T^*} \in D$ and let $\varphi = \bar{1}$. Thus by Lemma 6.23, $[\varphi]_{T^*} \star [\chi]_{T^*} = [\chi]_{T^*}, [\theta]_{T^*} \star [\chi]_{T^*} = [\chi]_{T^*}, [\varphi]_{T^*} \Rightarrow [\theta]_{T^*} = [\theta]_{T^*}$ and $([\chi]_{T^*} \Rightarrow [\chi]_{T^*}) \Rightarrow [\theta]_{T^*} = [\theta]_{T^*}$. Then

$$T^* \vdash_{P^+} \neg\neg\chi \rightarrow ((\varphi \& \chi \rightarrow \theta \& \chi) \rightarrow (\varphi \rightarrow \theta)) \equiv \neg\neg\chi \rightarrow \theta.$$

Since $[\chi]_{T^*} \neq \hat{0}$, $T^* \vdash_{P^+} \neg\neg\chi \equiv \bar{1}$ and axiom (II1) does not hold. Contradiction. Therefore there is at most one product cluster. $\square$

This completes the proof of Theorem 6.8.

7. CONCLUSION AND FURTHER RESEARCH

This research project focused on propositional fuzzy logics: BL, $\text{BL}_{\Delta\sim}$, Łukasiewicz, Product and Gödel logics. We proved strong standard completeness for axiomatic systems of BL, Łukasiewicz and Product logics, and constructed tableau calculi for BL and $\text{BL}_{\Delta\sim}$. In this chapter we summarise the results addressing the original research questions and what remains to be done.

To prove strong standard completeness, we introduced an infinitary inference rule using one counterexample, an example which shows that Hájek's axiomatic system for BL is not strongly standard complete. We also achieved this result extending axiomatic systems with the aforementioned rule for Łukasiewicz and Product logics. This is a very significant result as previous attempts at achieving strong standard completeness were extending the language of the fuzzy logics, which we did not need to do. This work may possibly be extended to cover cases of uncountable sets of propositional atoms, or other fuzzy logics stronger than BL. One of those could be $\text{BL}_{\Delta\sim}$.

We defined two types of tableaux: (1) a tableau for a formula ψ of BL to demonstrate that it is valid with respect to continuous t-norms or to construct a model in which its truth value is less than 1, (2) a tableau to demonstrate or refute that a finite set Ψ of formulas of $\text{BL}_{\Delta\sim}$ is $\mathcal{K}$ -satisfiable. If $\mathcal{K} = [0, 1)$ and we can close every tableau with a suitable root (see Section 4.3), then the formula ψ is valid with respect to every continuous t-norm. Even though other tableaux exist, the significance of the ones presented in the thesis is that they are based very directly on the semantics of these fuzzy logics, by which they may become a tool in the research into vagueness. We claimed that our tableaux are less complex than similar ones in the current state of the art.

There are several routes by which the aforementioned area can be further researched. Computational Complexity is one such area. [Han11] presents the state-of-art of complexity of propositional fuzzy logics. It has been previously proven that the axiomatic systems for BL, Product and Łukasiewicz logics are finitely strongly standard complete (see [Háj98b], [Háj98a], [CEGT00]), and Gödel logic is strongly standard complete (see [Háj98b]). We know that the complexity of deciding validity in $[0, 1]_{\mathbb{L}}$ is **coNP**-complete, and thus due to finite strong standard completeness of Łukasiewicz logic, the complexity issue of deciding validity in Łukasiewicz logic is **coNP**-complete. Similarly for the other logics that we mentioned (see [Han11], [BHMV02]). It can be seen that our first tableau also gives a **coNP** procedure to decide validity. The open question remains what computational complexity our second tableau presented in Section 4.3 enjoys.

As we mentioned in Section 4.3, extending the second tableau to continuous t-norms (not just LP-norms) may result in finding a tableau for $\text{BL}_{\Delta\sim}$ with the standard semantics. Another aspect that is worth pursuing is construction of similar tableaux for

(fragments of) predicate logic, modal logic or temporal logic, as well as other propositional fuzzy logics, which hierarchy we can appreciate in [CHN11]. Although we have written code to generate examples for tableaux for continuous t-norms in Python, we have not expanded this using non-linear programming to solve the constructed inequalities with parameters (endpoints of intervals) and variables representing propositional atoms. This could be another problem to solve: to apply an appropriate algorithm to find values of parameters and variables. That can potentially also be extended to the aforementioned logics.

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APPENDIX A

A proof of Theorem 2.5 is provided below.

Theorem 2.5. The Gödel t-norm, the product t-norm and the Łukasiewicz t-norm are continuous.

Proof. First, we recall definitions of continuity of a function defined for two variables, which will be used below.

Function $f(x, y)$ is continuous on the interval $[0, 1]^2$

iff

$$\forall (x_0, y_0) \in [0, 1]^2 \quad \lim_{(x, y) \rightarrow (x_0, y_0)} f(x, y) = f(x_0, y_0).$$

iff

$$(\forall \epsilon > 0)(\exists \delta > 0) \left(\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta \rightarrow |f(x, y) - f(x_0, y_0)| < \epsilon \right).$$

We show that the Gödel t-norm is continuous. Take arbitrary $\epsilon > 0$. Take $\delta = \frac{1}{2} \min\{|y_0 - x_0|, \epsilon\}$ if $|y_0 - x_0| > 0$ and $\delta = \epsilon$, otherwise. Assume $(x - x_0)^2 + (y - y_0)^2 < \delta^2$. Then $|x - x_0| < \delta$ and $|y - y_0| < \delta$. We show $|x \star_G y - x_0 \star_G y_0| < \epsilon$. First, we prove it for $x_0 = y_0$.

$$|\min(x, y) - x_0| \leq \max\{|x - x_0|, |y - y_0|\} < \epsilon.$$

Without loss of generality, assume that $x_0 < y_0$. The case of $x_0 > y_0$ is analogous. By definition of the Gödel t-norm, $|\min(x, y) - \min(x_0, y_0)| = |\min(x, y) - x_0|$. By the definition of δ , we know that $x < y$. Thus,

$$|\min(x, y) - x_0| = |x - x_0| < \delta < \epsilon.$$

We show that the product t-norm is continuous. Take arbitrary $\epsilon > 0$. Take $\delta = \frac{1}{3} \min\{\epsilon, 1\}$. Assume $(x - x_0)^2 + (y - y_0)^2 < \delta^2$. Then $|x - x_0| < \delta$ and $|y - y_0| < \delta$. We show $|x \star_P y - x_0 \star_P y_0| < \epsilon$. By definition of the product t-norm, $|x \star_P y - x_0 \star_P y_0| = |xy - x_0 y_0|$. Thus,

$$|xy - x_0 y_0| = |(x - x_0)(y - y_0) + x y_0 + x_0 y - 2 x_0 y_0| \leq |x - x_0| |y - y_0| + |x - x_0| y_0 + |y - y_0| x_0,$$

and

$$|x - x_0| |y - y_0| + |x - x_0| y_0 + |y - y_0| x_0 \leq \delta^2 + \delta \cdot y_0 + \delta \cdot x_0 \leq \delta^2 + 2\delta < \epsilon.$$

We show that the Łukasiewicz t-norm is continuous. Take arbitrary $\epsilon > 0$. Take $\delta = \frac{1}{2} \min\{\epsilon, |x_0 + y_0 - 1|\}$ if $1 \neq x_0 + y_0$ and $\delta = \frac{1}{2}\epsilon$, otherwise. Assume $(x - x_0)^2 + (y - y_0)^2 < \delta^2$. Then $|x - x_0| < \delta$ and $|y - y_0| < \delta$. We show $|x \star_L y - x_0 \star_L y_0| < \epsilon$. First, we prove for $x_0 + y_0 = 1$.

$$|\max\{0, x + y - 1\} - 0| \leq |\max\{0, (x - x_0) + (y - y_0)\}| \leq 2 \max\{|x - x_0|, |y - y_0|\} < \epsilon.$$

Now, consider $1 \neq x_0 + y_0$. By definition of the Łukasiewicz t-norm, $|x \star_L y - x_0 \star_L y_0| = |\max\{0, x + y - 1\} - \max\{0, x_0 + y_0 - 1\}|$. We show that, by definition of δ , the signs of $x + y - 1$ and $x_0 + y_0 - 1$ are the same. Suppose that $x_0 + y_0 - 1 > 0$, then $-\frac{1}{2}(x_0 + y_0 - 1) < x - x_0, -\frac{1}{2}(x_0 + y_0 - 1) < y - y_0$. Therefore $-(x_0 + y_0 - 1) < x + y - (x_0 + y_0)$, and consequently $0 < x + y - 1$. Suppose that $x_0 + y_0 - 1 < 0$, then $x - x_0 < -\frac{1}{2}(x_0 + y_0 - 1), y - y_0 < -\frac{1}{2}(x_0 + y_0 - 1)$. Therefore $x + y - (x_0 + y_0) < -(x_0 + y_0 - 1)$, and consequently $x + y - 1 < 0$. Thus, either $x + y - 1$ and $x_0 + y_0 - 1$ are both less than 0 or both positive. In the first case

$$|\max\{0, x + y - 1\} - \max\{0, x_0 + y_0 - 1\}| \leq |0 - 0| < \epsilon.$$

In the latter case,

$$|\max\{0, x + y - 1\} - \max\{0, x_0 + y_0 - 1\}| \leq |(x - x_0) + (y - y_0)| \leq |x - x_0| + |y - y_0| < \epsilon.$$

This concludes the proof. $\square$

APPENDIX B

A proof Theorem 2.36 is provided below.

Theorem 2.36. Every ordinal sum of a family of t-norms isomorphic to either the Łukasiewicz t-norm or the product t-norm is a continuous t-norm.

Proof. Let $\star$ be the ordinal sum $\sum_{n \in \mathfrak{C}} ([a_n, b_n], *_n)$ of t-norms $(*_n)_{n \in \mathfrak{C}}$ such that $*_n$ is either isomorphic to the Łukasiewicz t-norm or isomorphic to the product t-norm. It is obvious that $\star$ is a well-defined function on $[0, 1]^2$ with values in $[0, 1]$. We will show that $\star$ is a continuous t-norm in two steps: (1) we prove that it is a continuous function and (2) we prove that it is a t-norm. We will denote the isomorphisms for each $*_n$, by g_n , and we will denote by $h_n : [a_n, b_n] \rightarrow [0, 1]$ a function $h_n(x) = \frac{x - a_n}{b_n - a_n}$ for $x \in [a_n, b_n]$. Note that $g_n, h_n, g_n^{-1}, h_n^{-1}$ are all strictly increasing bijections.

We will prove that $\star$ is a continuous function. Let $x, y \in [0, 1]$. We will show that for every sequence $(x_k)_{k \in \mathbb{N}}$ and every sequence $(y_k)_{k \in \mathbb{N}}$, if $\lim_{k \rightarrow \infty} x_k = x$ and $\lim_{k \rightarrow \infty} y_k = y$, then $\lim_{k \rightarrow \infty} x_k \star y_k = x \star y$. Suppose that there is at least one interval $[a_n, b_n]$ for some $n \in \mathfrak{C}$, otherwise we deal with the Gödel t-norm that is continuous by Theorem 2.5.

- (1) Suppose that $x, y \in [a_n, b_n]$ for some $n \in \mathfrak{C}$. We show the case $x = y = a_n$ as the other cases are analogous.

Take arbitrary sequences $(x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} x_k = x$ and $\lim_{k \rightarrow \infty} y_k = y$. We take subsequences (A) $(x'_k)_{k \in \mathbb{N}} \uparrow x$ and (B) $(x''_k)_{k \in \mathbb{N}} \downarrow x$ and for all $k \in \mathbb{N}$, $x''_k < b_n$. At least one of them exists.

In case of (A) we take $(y'_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$, $x'_k \leq y'_k$. Then by Proposition 2.4, $\lim_{k \rightarrow \infty} x'_k \star y'_k \leq \lim_{k \rightarrow \infty} x'_k \star_G y'_k \lim_{k \rightarrow \infty} x'_k = x = x \star y$.

In case of (B) we will select $(y''_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$, either (B1) $y''_k < a_n < x''_k$, or (B2) $a_n < y''_k < x''_k$. At least (B1) or (B2) is true. Then for (B1) $\lim_{k \rightarrow \infty} x''_k \star y''_k = \lim_{k \rightarrow \infty} y''_k = y = x \star y$. In case of (B2) $\lim_{k \rightarrow \infty} x''_k \star y''_k = x \star y$ by Lemma 2.25 since $g_n, h_n, g_n^{-1}, h_n^{-1}$ are all strictly increasing bijections, and by Theorem 2.5.

- (2) Suppose that $x_0, y_0 \in (a_n, b_n)$ for some $n \in \mathfrak{C}$.

Take arbitrary sequences $(x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} x_k = x_0$ and $\lim_{k \rightarrow \infty} y_k = y_0$. We take subsequences $(x'_k)_{k \in \mathbb{N}}$ and $(y'_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$, $a_n < x'_k, y'_k < b_n$. Then $\lim_{k \rightarrow \infty} x'_k \star y'_k = x_0 \star y_0$ by Lemma 2.25 and Theorem 2.5.

- (3) Suppose there is $[a_n, b_n]$ such that one of its endpoint, say c , is such that $\min\{x_0, y_0\} < c < \max\{x_0, y_0\}$. Suppose that $\min\{x_0, y_0\} = x_0$ and $\max\{x_0, y_0\} = y_0$. The other case is analogous.

Take arbitrary sequences $(x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} x_k = x_0$ and $\lim_{k \rightarrow \infty} y_k = y_0$. We take subsequences $(x'_k)_{k \in \mathbb{N}}$ and $(y'_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$, $x'_k < c < y'_k$. Then $\lim_{k \rightarrow \infty} x'_k \star y'_k = \lim_{k \rightarrow \infty} x'_k = x_0 = x_0 \star y_0$.

- (4) Suppose that $b_n < x_0, y_0 < a_{n+1}$, where b_n is either 0 or the right endpoint of an interval and a_{n+1} is either 1 or the left endpoint of an interval. Suppose that $x_0 \leq y_0$. The other case $y_0 \leq x_0$ is similar. Take arbitrary sequences $(x_k)_{k \in \mathbb{N}}, (y_k)_{k \in \mathbb{N}}$ such that $\lim_{k \rightarrow \infty} x_k = x_0$ and $\lim_{k \rightarrow \infty} y_k = y_0$. We take subsequences $(x'_k)_{k \in \mathbb{N}}$ and $(y'_k)_{k \in \mathbb{N}}$ such that for every $k \in \mathbb{N}$, $x'_k < y'_k$. Then $\lim_{k \rightarrow \infty} x'_k \star y'_k = \lim_{k \rightarrow \infty} x'_k = x_0 = x_0 \star y_0$. This completes the proof that $\star$ is continuous.

Now we need to check the properties of a t-norm. We show the claim that for every $x, y \in [0, 1]$,

$$x \star y \leq x. \quad (\#)$$

Suppose that $x, y \in [a_n, b_n]$. Then $x \star y = a_n + (b_n - a_n)g_n^{-1} \left(g_n \left(\frac{x - a_n}{b_n - a_n} \right) \star_I g_n \left(\frac{y - a_n}{b_n - a_n} \right) \right)$, where $\star_I$ is either the Łukasiewicz t-norm or the product t-norm depending on $\ast_n$. By properties of t-norms $a_n + (b_n - a_n)g_n^{-1} \left(g_n \left(\frac{x - a_n}{b_n - a_n} \right) \star_I g_n \left(\frac{y - a_n}{b_n - a_n} \right) \right) \leq a_n + (b_n - a_n)g_n^{-1} \left(g_n \left(\frac{x - a_n}{b_n - a_n} \right) \right) = x$. The case that there is no $[a_n, b_n]$ that x, y belong to, $x \star y = \min\{x, y\} \leq x$. This completes the proof of the claim.

- (1) We show that $\star$ is commutative. It is clear from the definition of the ordinal sum.
- (2) We show that $\star$ is associative. We need to show that $(x \star y) \star z = x \star (y \star z)$. We will consider five cases. Assume that $n \in \mathfrak{C}$.

- (a) Suppose that $x, y, z \in [a_n, b_n]$. Then

$$\frac{x \star y - a_n}{b_n - a_n} = g_n^{-1} \left(g_n \left(\frac{x - a_n}{b_n - a_n} \right) \star_I g_n \left(\frac{y - a_n}{b_n - a_n} \right) \right),$$

where $\star_I$ is either the Łukasiewicz t-norm or the product t-norm depending on $\ast_n$, and thus

$$(x \star y) \star z = a_n + (b_n - a_n)g_n^{-1} \left(g_n \left(\frac{x - a_n}{b_n - a_n} \right) \star_I g_n \left(\frac{y - a_n}{b_n - a_n} \right) \star_I g_n \left(\frac{z - a_n}{b_n - a_n} \right) \right).$$

Similarly, we show

$$x \star (y \star z) = a_n + (b_n - a_n)g_n^{-1} \left(g_n \left(\frac{x - a_n}{b_n - a_n} \right) \star_I g_n \left(\frac{y - a_n}{b_n - a_n} \right) \star_I g_n \left(\frac{z - a_n}{b_n - a_n} \right) \right).$$

- (b) Suppose that $x, y \in [a_n, b_n]$ and $z \notin [a_n, b_n]$. There are two subcases: either $z < a_n$ or $z > b_n$. The former yields $(x \star y) \star z = z$ and $x \star (y \star z) = z$. The latter yields $(x \star y) \star z = x \star y$ and $x \star (y \star z) = x \star y$.
- (c) Suppose that $y, z \in [a_n, b_n]$ and $x \notin [a_n, b_n]$. The proof is similar to (b).
- (d) Suppose that $x, z \in [a_n, b_n]$ and $y \notin [a_n, b_n]$. The proof is similar to (b).
- (e) There is no $[a_n, b_n]$ such that at least two of x, y, z belong to it. Then $(x \star y) \star z = \min\{x, y, z\} = x \star (y \star z)$.
- (3) We show that $\star$ is monotonic. Assume that $y \leq z$. We need to show that $x \star y \leq x \star z$. We will consider four cases.
- (a) Suppose that $x, y, z \in [a_n, b_n]$. Then $x \star y \leq x \star z$ by the fact that $g_n, h_n, g_n^{-1}, h_n^{-1}$ are all strictly increasing bijections.
- (b) Suppose that $x, y \in [a_n, b_n]$ and $z \notin [a_n, b_n]$. Since by (#), $x \star y \leq x$ and $x \star y \leq y \leq z$, by definition of the ordinal sum, $x \star y \leq \min\{x, z\} = x \star z$.
- (c) Suppose that $x, z \in [a_n, b_n]$ and $y \notin [a_n, b_n]$. Then $y < a_n$ by assumption and $x \star z \geq a_n$. Thus $x \star z \geq y = \min\{x, y\} = x \star y$.
- (d) There is no interval $[a_n, b_n]$ such that x and at least one of y, z belong to it. Then $x \star y = \min\{x, y\} \leq \min\{x, z\} = x \star z$.
- (4) We show that $\star$ has a boundary condition $x \star 1 = x$. If there exists a contact interval $[a_n, 1]$ such that x is in it, then $x \star 1 = x$ by properties of $g_n, h_n, g_n^{-1}, h_n^{-1}$ and t-norms. Otherwise, $x \star 1 = \min\{x, 1\} = x$.

This completes the proof of the theorem. □

APPENDIX C

We include a Python script consisting of two parts: the main script (the first four pages) and the auxiliary script containing functions used by the main script (the remaining five pages). The code can be run under Idle 2.7. The program was used to generate examples for Chapter 4.

```
"""
From a given Root create Leaves using branch expansion rules.
"""

# FUNCTIONS

import functions

def branch_rule(symbol, node):
    """
    Apply a branch expansion rule.
    """

    #identify the position of symbol
    marker = formula.find(symbol)
    test = marker > 0

    if test:
        lnFormula = len(formula)
        Int = Tree[Branch]['intervals']
        lnInt = len(Int)
        test_star = symbol == '*'

        #update the history item in the new branch of Tree
        History_new = Tree[Branch]['history'].copy()
        History_node = node[:]
        History_intervals = Int[:]
        History_label = Branch
        History_new.update({History_label: {'node':
History_node, 'intervals': history_intervals}})

        #find the number of Lukasiewicz and Product
intervals introduced to date
        Luk_int, Prod_int = functions.intervals(Int, lnInt)

        #identify the position of the end of the term
identified for substitution
        k = marker
        marker_end = functions.end_term(formula, lnFormula,
k)

        #identify the position of the beginning of the term
identified for substitution
        k = marker
        marker_begin = functions.start_term(formula,
lnFormula, k)

        #define the term for substitution
        x = formula[marker_begin + 1: marker]
        y = formula[marker + 1: marker_end]
        term = x + symbol + y

        #define marker for the new interval
        new_Luk = 'L' + str(Luk_int + 1) + 'L'
        new_Prod = 'P' + str(Prod_int + 1) + 'P'
```

```

        #create new nodes with substituted terms
        node_Luk = functions.substitute(x, y, term, node,
lnNode, symbol, 'Luk', new_Luk, new_Prod)
        node_Prod = functions.substitute(x, y, term, node,
lnNode, symbol, 'Prod', new_Luk, new_Prod)
        node_min = functions.substitute(x, y, term, node,
lnNode, symbol, 'min', new_Luk, new_Prod)
        if not test_star:
            node_All = functions.substitute(x, y, term,
node, lnNode, symbol, 'All', new_Luk, new_Prod)

        #create new branches of Tree with updated History
and inequalities for intervals
        Tree[Branch + 'A0'] = {'node': node_All,
'intervals': Int, 'history': History_new}
        #create a new interval if no intervals introduced to
date
        if lnInt == 0:
            node_Luk.extend(['0<=a' + new_Luk, 'a' +
new_Luk + '<b' + new_Luk,
                                'b' + new_Luk + '<=1', 'a' +
new_Luk + '<=' + y, x + '<=b' + new_Luk])
            node_Prod.extend(['0<=a' + new_Prod, 'a' +
new_Prod + '<b' + new_Prod,
                                'b' + new_Prod + '<=1', 'a'
+ new_Prod + '<=' + y, x + '<=b' + new_Prod])
            if test_star:
                node_Luk.extend(['a' + new_Luk +
'<=' + x, y + '<=b' + new_Luk])
                node_Prod.extend(['a' + new_Prod +
'<=' + x, y + '<=b' + new_Prod])

            #creating nodes
            Tree[Branch + 'L0'] = {'node': node_Luk,
'intervals': ['L1L'], 'history': History_new}
            Tree[Branch + 'P0'] = {'node': node_Prod,
'intervals': ['P1P'], 'history': History_new}
            Tree[Branch + 'M0'] = {'node': node_min,
'intervals': [], 'history': History_new}
        else:
            #Lukasiewicz and Product cases
            #insert a new interval if some intervals
exist

Tree.update(functions.new_intervals(test_star, x, y, Int, lnInt,
'L', new_Luk, node_Luk, Branch, History_new))

Tree.update(functions.new_intervals(test_star, x, y, Int, lnInt,
'P', new_Prod, node_Prod, Branch, History_new))
        #overwrite an existing interval

Tree.update(functions.old_intervals(test_star, x, y, Int, lnInt,
'L', new_Luk, node_Luk, Branch, History_new))

Tree.update(functions.old_intervals(test_star, x, y, Int, lnInt,
'P', new_Prod, node_Prod, Branch, History_new))

```

```

        #Godel cases
        #x not in any of the existing intervals
        Tree.update(functions.not_in_interval(x, y,
Int, lnInt, node_min, Branch, History_new))
        #x in one of the existing intervals and y
less than the left endpoint of the interval
        Tree.update(functions.to_left(x, y, Int,
lnInt, node_min, Branch, History_new))
        #x in one of the existing intervals and y
greater than the right endpoint of the interval
        Tree.update(functions.to_right(x, y, Int,
lnInt, node_min, Branch, History_new))

        #delete used nodes
        del Tree[Branch]

    return test

# ROOT

Root = ['(p1p>(p1p*p2p))<1']
Root_intervals = []
History = {}

Leaves = {}
Tree = {}
Branch = '1'

Tree[Branch] = {'node': Root, 'intervals': Root_intervals,
'history': History}

while Tree:
    Branch = Tree.keys()[0]
    #apply a branch expansion rule to the first symbol
    j = 0
    node = Tree[Branch]['node']
    lnNode = len(node)

    while j < lnNode:
        formula = node[j]
        if branch_rule('*', node): break
        if branch_rule('>', node): break
        j += 1

    else:
        Leaves.update({Branch: Tree[Branch]})
        del Tree[Branch]

#save the output to a text file

import json
output = open('BL.txt', 'w')
```

```
output.write('\n')
output.write('Root: ' + str(Root) + '\n\n')
output.write('Intervals: ' + str(Root_intervals) + '\n\n')
json.dump(Leaves, fp = output, indent = 4)
output.close()

print(open('BL.txt').read())
```

```
"""
The set of all functions used by BL module.
"""

# FUNCTIONS

def end_term(formula, lnFormula, k):
    """
    Identify the position of the end of the term identified for
    substitution.
    """
    counter = 1
    while not counter == 0 and k < lnFormula:
        k += 1
        if formula[k] == ')' and counter >= 1:
            counter -= 1
        if formula[k] == '(':
            counter += 1
        if counter == 0: break
    return k

def intervals(Int, lnInt):
    """
    Find the number of Lukasiewicz and Product intervals
    introduced in the preceding node.
    """
    Luk_int = 0
    Prod_int = 0
    for i in range(lnInt):
        name = Int[i]
        current = int(name[1:-1])
        if name[0] == 'L' and current > Luk_int:
            Luk_int = current
        if name[0] == 'P' and current > Prod_int:
            Prod_int = current
    return Luk_int, Prod_int

def new_intervals(test_star, x, y, Int, lnInt, name, new_int, node,
Branch, history):
    """
    Insert a new interval before or after or in-between the
    existing intervals.
    """
    tree = {}
    for counter in range(0, lnInt + 1):
        New_Branch = Branch + name + str(counter)
        intervals = Int[:]
        intervals.insert(counter, new_int)
        node_new = node[:]
        node_new.extend(['a' + new_int + '<=' + y, x + '<=b'
+ new_int])
        if test_star: node_new.extend(['a' + new_int + '<='
+ x, y + '<=b' + new_int])
        if counter == 0:
```

```

        node_new.extend(['0<=a' + new_int, 'a' +
new_int + '<b' + new_int, 'b' + new_int + '<=a' +
                                Int[0]])
        if counter == lnInt:
            node_new.extend(['b' + Int[-1] + '<=a' +
new_int,
                                'a' + new_int + '<b' +
new_int, 'b' + new_int + '<=1'])
            if lnInt > 1 and not counter == 0 and counter <
lnInt:
                node_new.extend(['b' + Int[counter-1] +
'<=a' + new_int,
                                'a' + new_int + '<b' +
new_int, 'b' + new_int + '<=a' +
                                Int[counter]])
            tree[New_Branch] = {'node': node_new, 'intervals':
intervals, 'history': history}
            return tree

def not_in_interval(x, y, Int, lnInt, node, Branch, history):
    """
    Create branches for Godel case with x not in any of the
    existing intervals.
    """
    tree = {}
    for counter in range(0, lnInt + 1):
        New_Branch = Branch + 'M' + str(counter)
        node_new = node[:]
        if counter == 0:
            node_new.extend(['0<=' + x, x + '<=a' +
Int[0]])
            if counter == lnInt:
                node_new.extend(['b' + Int[-1] + '<=' + x, x
+ '<=1'])
            if not counter == 0 and counter < lnInt and lnInt >
1:
                node_new.extend(['b' + Int[counter-1] + '<='
+ x, x + '<=a' + Int[counter]])
            tree[New_Branch] = {'node': node_new, 'intervals':
Int, 'history': history}
            return tree

def old_intervals(test_star, x, y, Int, lnInt, name, new_int, node,
Branch, history):
    """
    Overwrite the existing intervals
    """
    tree = {}
    new_branch = lnInt
    for counter in range(0, lnInt):
        if Int[counter][0] == name:
            new_branch += 1
            New_Branch = Branch + name + str(new_branch)
            intervals = Int[:]
            intervals.pop(counter)
            intervals.insert(counter, new_int)

```

```

        node_new = node[:]
        node_new.extend(['a' + new_int + '<=' + y, x
+ '<=b' + new_int])
        if test_star: node_new.extend(['a' + new_int
+ '<=' + x, y + '<=b' + new_int])
        node_new.extend(['a' + new_int + '=a' +
Int[counter],
                                'a' + new_int + '<b' +
new_int,
                                'b' + new_int + '=b' +
Int[counter]])
        tree[New_Branch] = {'node': node_new,
'intervals': intervals, 'history': history}
        return tree

def substitute(x, y, term, node, lnNode, symbol, name, new_Luk,
new_Prod):
    """
    Substitute all occurrences of the term in the node as the
    branch expansion rule dictates for a given symbol and a given name.
    """
    counter = 0
    node_sub = node[:]
    while counter < lnNode:
        if node_sub[counter].find(term) > -1:
            if symbol == '*' and name == 'Luk':
                term_sub =
node_sub[counter].replace('(' + term + ')', 'max{a' + new_Luk + ','
+ x + '+' + y + '-b' + new_Luk + '}')
            if symbol == '*' and name == 'Prod':
                term_sub =
node_sub[counter].replace(term, 'a' + new_Prod + '+div{(' + x + '-a'
+ new_Prod +
                                ').( '
+ y + '-a' + new_Prod + '),b' + new_Prod +
                                '-a'
+ new_Prod + '})')
            if symbol == '*' and name == 'min':
                term_sub =
node_sub[counter].replace('(' + term + ')', 'min{' + x + ',' + y +
'}')
            if symbol == '>' and name == 'Luk':
                term_sub =
node_sub[counter].replace(term, 'b' + new_Luk + '-' + x + '+' + y)
            if symbol == '>' and name == 'Prod':
                term_sub =
node_sub[counter].replace(term, 'a' + new_Prod + '+div{(' + y + '-a'
+ new_Prod +
                                ').(b' + new_Prod + '-a' + new_Prod + '),(' + x +
                                '-a'
+ new_Prod + '})')
            if symbol == '>' and name == 'min':
                term_sub =
node_sub[counter].replace('(' + term + ')', y)

```

```

        if symbol == '>' and name == 'All':
            term_sub =
node_sub[counter].replace('(' + term + ')', '1')
            node_sub.pop(counter)
            node_sub.insert(counter, term_sub)
            counter += 1
    if symbol == '>':
        if name == 'All':
            node_sub.append(x + '<=' + y)
        else: node_sub.append(y + '<' + x)
    return node_sub

def start_term(formula, lnFormula, k):
    """
    Identify the position of the start of the term identified for
    substitution.
    """
    counter = 1
    while not counter == 0 and k > 0:
        k -= 1
        if formula[k] == ')' and counter >= 1:
            counter += 1
        if formula[k] == '(':
            counter -= 1
        if counter == 0: break
    return k

def to_left(x, y, Int, lnInt, node, Branch, history):
    """
    Create branches for Godel case with x in one of the existing
    intervals and y less than the left endpoint of the interval.
    """
    tree = {}
    new_branch = lnInt
    for counter in range(0, lnInt):
        new_branch += 1
        New_Branch = Branch + 'M' + str(new_branch)
        node_new = node[:]
        node_new.extend(['a' + Int[counter] + '<=' + x, x +
'<=b' + Int[counter],
                                y + '<=a' + Int[counter]])
        tree[New_Branch] = {'node': node_new, 'intervals':
Int, 'history': history}
    return tree

def to_right(x, y, Int, lnInt, node, Branch, history):
    """
    Create branches for Godel case with x in one of the existing
    intervals and y greater than the right endpoint of the interval.
    """
    tree = {}
    new_branch = 2 * lnInt
    for counter in range(0, lnInt):
        new_branch += 1
        New_Branch = Branch + 'M' + str(new_branch)

```

```
        node_new = node[:]
        node_new.extend(['a' + Int[counter] + '<=' + x, x +
'<=b' + Int[counter],
                                'b' + Int[counter] + '<=' + y])
        tree[New_Branch] = {'node': node_new, 'intervals':
Int, 'history': history}
    return tree
```