

1 **Stabilized semi-implicit material point method for hydro-mechanical**
2 **coupled large deformation soil-structure interaction analyses**

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12 **Abstract**

13 Hydro-mechanical coupled analysis of geotechnical problems generally poses stability
14 challenges, particularly for soil-structure interaction problems involving large deformations.
15 This study presents a stabilized semi-implicit MPM (Material Point Method) treatment based
16 on the incremental fractional step approach to suppress the arbitrary and unphysically pressure
17 oscillations commonly encountered in explicit MPM formulations. The proposed treatment
18 alleviates the restrictive constraints on time increment durations imposed by geomaterial
19 permeability. A frictional contact algorithm incorporating a penalty function is also shown to
20 effectively reduce the formation of spurious gaps and numerical fluctuations over soil-
21 structure contact areas. The stability of the semi-implicit MPM treatment is enhanced by
22 introducing artificial compressibility into the pressure Poisson equation, integrating the APIC
23 (Affine Particle-in-Cell) scheme for particle-grid information transfer, as well as adopting the
24 B-bar approach to mitigate volumetric locking. Additionally, an axisymmetric formulation is
25 developed to enable more efficient modeling of selected classical geotechnical problems. The
26 effectiveness and robustness of the proposed framework are demonstrated against a series of
27 benchmark test cases including coupled, axisymmetric, dynamic, large deformation and soil-
28 structure interaction simulations considering the Modified Cam-clay constitutive model.

29
30 **Keywords:** Hydro-mechanical coupling; Large deformations; Soil-structure interaction;
31 Material point method; Fractional step method

1. Introduction

Analyses of large deformation, geotechnical problems are commonly required to enable efficient geotechnical engineering solutions to be developed for high value projects involving liquid-filled porous media such as offshore pile driving (Ceccato et al., 2016; Fu et al., 2024), drag anchor installation and pullout (Liang et al., 2021) or pipeline installation (Li et al., 2021). However, such analyses are difficult to undertake because of the involved multi-physics coupling phenomenon and intricate soil-structure interactions. Among potential numerical methods of analysis, the material point method (MPM) is well-suited for simulating large deformation problems in history-dependent granular materials (Sulsky et al., 1994). By utilizing an Eulerian background mesh to solve the governing equations and Lagrangian particles to update the material kinematics respectively, MPM can alleviate the mesh distortion issues that are commonly encountered in conventional mesh-based methods, such as the finite element method (FEM) (Zienkiewicz and Shiomi, 1984; Markert et al., 2010; Smith et al., 2014), without introducing additional costs of remeshing implicit in, for example, Arbitrary Lagrangian-Eulerian (ALE) (Benson, 1989; Nazem et al., 2006; Yang et al., 2020) or particle finite element method (PFEM) analyses (Zhang et al., 2013; Monforte et al., 2018; Yuan et al., 2022). MPM has therefore become increasingly popular in modeling progressive failures of multiphase granular soils with soil-structure interactions (Yerro et al., 2015; Bandara and Soga, 2015; Ceccato et al., 2016; Martinelli and Galavi, 2022).

Existing MPM treatments of hydro-mechanical coupled large deformation problems employ a diverse range of basic governing equations, time integration, spatial discretization, and stabilization algorithms (Zheng et al., 2024). Most reported two-phase MPM approaches adopt explicit time integration schemes and consider the fluid phase as being weakly compressible (Zhang et al., 2009; Bisht et al., 2021). While relatively simple to implement, the critical time step size is restricted in explicit MPM treatments by not only the wave propagation speed in the porous medium but also the material permeabilities, which can render the scheme inefficient for low-permeability porous material. Additionally, the coupled MPM formulations must satisfy the Ladyzhenskaya–Babuška–Brezzi (LBB), or ‘Inf-Sup’ stability condition in order to ensure the convergence of the solution for nearly incompressible fluids (Gresho and Sani, 1998). Various stabilization algorithms have been proposed to suppress unphysical pore pressure oscillations and numerical instabilities that may arise, including the reduced integration approach (Bandara and Soga, 2015; Bandara et al., 2016), the multi-field variational principle (Chen et al., 2018), polynomial pressure projection (Zhao and Choo, 2020), and cell variation averaging (Zheng et al., 2021; Wang et al., 2023).

Alternatively, semi-implicit MPM treatments employing a fractional step approach offer a potential way to effectively address the numerical stability and efficiency issues in coupled

large deformation modeling (Yamaguchi et al., 2020; Kularathna et al., 2021; Yuan et al., 2023). This fractional step approach, originally proposed by Chorin (1968), decouples the pressure and kinematic fields by splitting the governing equations into two computational steps: a prediction step and a correction step involving intermediate variables. In the prediction step, the intermediate velocities are solved while ignoring the pore pressure gradient term, and in the correction step they are corrected explicitly by considering the impact of pore water pressure using values obtained by implicitly solving the pressure Poisson equation. This mixed explicit and implicit time integration allows larger time steps to be employed than in the fully explicit schemes and often requires fewer iterative solutions than fully implicit schemes. Depending on how the pore pressure terms are treated in the governing equations, the fractional step approach can be further distinguished as either incremental or non-incremental (Kularathna and Soga, 2017). As the non-incremental fractional step method often suffers from the “splitting errors” (Zienkiewicz et al., 2014) and excessive pore pressure dissipation (Yamaguchi et al., 2020; Wang and Wang, 2023) with first-order accuracy, the second-order incremental scheme is adopted for the studies reported herein which can achieve a certain pressure stability provided the time increments are set appropriately with respect to the critical time step (Kularathna et al., 2021; Yuan et al., 2023; Liang et al., 2023; Yu et al., 2024). To further suppress unphysical pressure oscillations, especially for low-permeability materials, Hidano et al. (2024) employed a sub-grid approach in which two grids with different spatial resolutions are constructed to separately compute the pore pressure and velocity fields. This approach necessitates significant adjustments to the computational procedures that may be avoided by the simple adoption of an artificial raised fluid compressibility in the pressure Poisson equation (Markert et al., 2010; Yuan et al., 2023).

Although the semi-implicit MPM analyses based on the fractional step approaches have been applied successfully to various geotechnical problems, a rigorous contact algorithm tailored to the semi-implicit MPM framework is required for the accurate simulation of soil-structure interactions. Additionally, most existing semi-implicit MPM formulations are formulated for two-dimensional (2D) plane strain/stress or three-dimensional (3D) scenario, thereby limiting their applicability to axisymmetric problems such as pile installation and cone penetration. While axisymmetric MPM formulations have been proposed for explicit coupled analyses (Sulsky and Schreyer, 1996; Nairn and Guilkey, 2015; Martinelli and Galavi, 2022), their extension to semi-implicit schemes is not straightforward owing to the additional splitting procedures and the solution of the pressure Poisson equation.

Another commonly encountered issue in MPM analyses is the occurrence of checkerboard stress patterns induced by kinematic locking. These unphysical stress oscillations can be effectively mitigated by adopting stabilization strategies such as the B-bar method (Bisht et al., 2021), the F-bar method (Telikicherla and Moutsanidis, 2022; Zhao et al.,

2023), or the Hu-Washizu multi-field variational formulation (Mast et al., 2012). Additionally, the accuracy and stability of MPM are also closely related to the information transfer scheme between material points and the background grid. Two schemes are commonly used for transferring kinematic variables from grid nodes to particles: the PIC (particle-in-cell) scheme (Brackbill and Ruppel, 1986) and the FLIP (fluid implicit particle) transfer (Brackbill, 1987), along with various extensions proposed to improve their performance.

Considering all these issues, a stabilized semi-implicit MPM is developed to enable the hydro-mechanical coupled modelling of large deformation problems involving soil-structure interactions and advanced soil models. The incremental fractional step approach is employed to address issues associated with numerical accuracy and stability in conjunction with: (i) a pressure stabilization approach, (ii) a locking-mitigation technique, and (iii) a stable particle kinematics transfer scheme. To properly detect the contacts between soil and structure, the grid-based frictional contact algorithm is enhanced by incorporating a penalty function to effectively reduce the appearance of spurious gaps and numerical fluctuations (Ma et al., 2014). Additionally, an axisymmetric formulation is developed to enable the model capacity to deal economically with axisymmetric problems.

The paper is organized as follows. Section 2 outlines the governing equations, derived from mixture theory, that are implemented in the code along with discretized formulations and contact algorithms. Section 3 introduces numerical stabilization strategies incorporated into the semi-implicit MPM framework. Section 4 provides the detailed implementation of axisymmetric formulations in the coupled MPM. Several validation tests are then explored in Section 5 to test the model performance against a range of geomaterial loading cases involving gradually increasing loading complexity. In Section 6, the coupled MPM treatment is applied to simulate large displacement strip footing penetration and displacement pile installation processes to demonstrate the robustness and capability of this method in dealing with soil-structure interactions and material nonlinearity. Finally, Section 7 summarizes the conclusions drawn from this study.

2. Coupled MPM formulations

This section summarises the established formulations required for semi-implicit MPM analyses of fully saturated porous media. A single layer of material points is adopted to represent both solid and liquid phases (i.e., single-point two-phase) as schematically shown in Fig. 1. Subsections 2.1-2.4 briefly review the key mathematics for a two-phase semi-implicit MPM (i.e., governing equations and spatio-temporal discretizations) as employed in earlier studies (Kularathna et al., 2021; Zheng et al., 2024). Subsection 2.5 incorporates the grid-based contact algorithm into the semi-implicit MPM framework to address the soil-structure interaction.

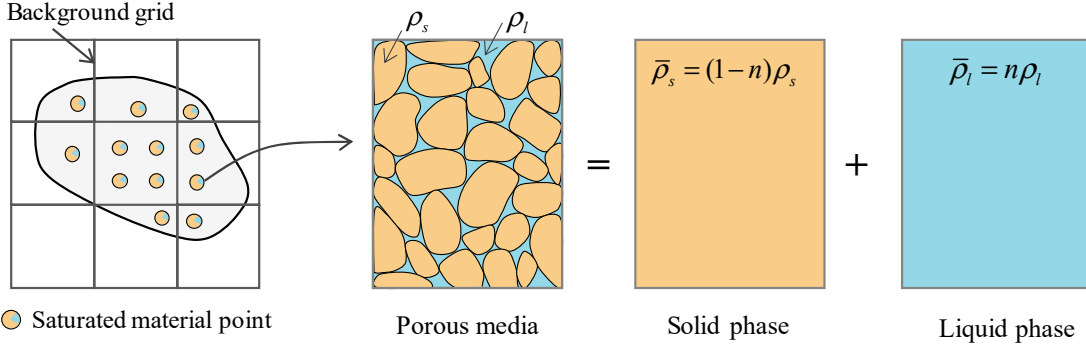


Fig. 1. Single-point MPM description for modeling two-phase porous media.

2.1. Governing equations

According to mixture theory, saturated soil is homogenized as a continuum medium consisting of solid skeleton and liquid phase. The partial density of each component $\bar{\rho}_\alpha$ ($\alpha \in \{s, l\}$, indicating solid and liquid phases, respectively) is represented by its volume fraction and intrinsic density ρ_α :

$$\bar{\rho}_s = (1-n)\rho_s, \quad \bar{\rho}_l = n\rho_l, \quad \rho_m = (1-n)\rho_s + n\rho_l \quad (1)$$

where n is the porosity, and ρ_m is the overall density of the mixture.

Terzaghi's effective stress is utilized for saturated porous media, and the partial stresses of soil and liquid are defined as,

$$\boldsymbol{\sigma}_s = \boldsymbol{\sigma}' - (1-n)p\mathbf{I}, \quad \boldsymbol{\sigma}_l = -np\mathbf{I} \quad (2)$$

where $\boldsymbol{\sigma}$ is the total stress tensor (positive for tension), $\boldsymbol{\sigma}'$ is the effective stress tensor, p is the pore pressure (positive for compression), and $\mathbf{I}$ is the identity tensor.

Assuming incompressible soil grains and liquid while neglecting the porosity gradient, the mass balance equation of the mixture can be formulated as:

$$\nabla \cdot [(1-n)\mathbf{v}_s + n\mathbf{v}_l] = 0 \quad (3)$$

The motions of the solid and liquid phase are governed by the momentum conservation:

$$(1-n)\rho_s \frac{D_s \mathbf{v}_s}{Dt} = \nabla \cdot \boldsymbol{\sigma}_s + (1-n)\rho_s \mathbf{b} + \mathbf{f}_d \quad (4a)$$

$$n\rho_l \frac{D_l \mathbf{v}_l}{Dt} = \nabla \cdot \boldsymbol{\sigma}_l + n\rho_l \mathbf{b} - \mathbf{f}_d \quad (4b)$$

where $\mathbf{b}$ is the body force vector, and $\mathbf{f}_d$ is the drag force due to solid-liquid interactions, and

$\frac{D_\alpha(*)}{Dt} = \frac{\partial(*)}{\partial t} + \mathbf{v}_\alpha \cdot \nabla(*)$ indicates the material time derivative of $(*)$ with respect to α phase.

As the laminar flow is considered in this study, the drag force satisfies the Darcy's law:

$$\mathbf{f}_d = \frac{n^2 \rho_l \mathbf{g}}{k} (\mathbf{v}_l - \mathbf{v}_s) \quad (5)$$

165 where g is the gravitational acceleration ($g = 9.81\text{m/s}^2$) and k is the hydraulic conductivity.
 166 The sum of Eqs. (4a) and (4b) gives the momentum balance equation of the mixture:

$$167 \quad (1-n)\rho_s \frac{D_s \mathbf{v}_s}{Dt} + n\rho_l \frac{D_l \mathbf{v}_l}{Dt} = \nabla \cdot (\boldsymbol{\sigma}' - p\mathbf{I}) + \rho_m \mathbf{b} \quad (6)$$

168 2.2. Time discretization

169 By utilizing the forward Euler scheme for the velocity variables in Eqs. (4b) and (6), the
 170 time-discretized forms of the momentum equations can be written as:

$$171 \quad (1-n)\rho_s \frac{\mathbf{v}_s^{k+1} - \mathbf{v}_s^k}{\Delta t} + n\rho_l \frac{\mathbf{v}_l^{k+1} - \mathbf{v}_l^k}{\Delta t} = \nabla \cdot (\boldsymbol{\sigma}'^k - p^{k+1}\mathbf{I}) + \rho_m \mathbf{b} \quad (7a)$$

$$172 \quad n\rho_l \frac{\mathbf{v}_l^{k+1} - \mathbf{v}_l^k}{\Delta t} = -n\nabla p^{k+1} + n\rho_l \mathbf{b} - \frac{n^2 \rho_l g}{k} (\mathbf{v}_l^{k+1} - \mathbf{v}_s^{k+1}) \quad (7b)$$

173 where superscript k indicates the discrete approximation at time t_k and Δt is the time increment.

174 The fractional step algorithm (Chorin, 1968) is employed to decouple the pore pressure
 175 and kinematical variables in the solid-fluid coupled system, facilitating the equal-order
 176 interpolation of displacement and pressure fields. Following the incremental fractional step
 177 procedure, the momentum equations are split into a prediction step and a correction step by
 178 introducing the intermediate velocity $\mathbf{v}_\alpha^*$. The drag force term in Eq. (5) is formulated based
 179 on the intermediate velocity, which enables the time step size being independent of
 180 permeability while ensuring accuracy (Markert et al., 2010; Kularathna et al., 2021). The split
 181 momentum equations at the prediction step can be expressed as,

$$182 \quad (1-n)\rho_s \frac{\mathbf{v}_s^* - \mathbf{v}_s^k}{\Delta t} + n\rho_l \frac{\mathbf{v}_l^* - \mathbf{v}_l^k}{\Delta t} = \nabla \cdot (\boldsymbol{\sigma}'^k - p^k \mathbf{I}) + \rho_m \mathbf{b} \quad (8a)$$

$$183 \quad n\rho_l \frac{\mathbf{v}_l^* - \mathbf{v}_l^k}{\Delta t} = -n\nabla p^k + n\rho_l \mathbf{b} - \frac{n^2 \rho_l g}{k} (\mathbf{v}_l^* - \mathbf{v}_s^*) \quad (8b)$$

184 and the correction parts for the mixture and liquid phase are expressed as:

$$185 \quad (1-n)\rho_s \frac{\mathbf{v}_s^{k+1} - \mathbf{v}_s^*}{\Delta t} + n\rho_l \frac{\mathbf{v}_l^{k+1} - \mathbf{v}_l^*}{\Delta t} = -\nabla(p^{k+1} - p^k) \quad (9a)$$

$$186 \quad n\rho_l \frac{\mathbf{v}_l^{k+1} - \mathbf{v}_l^*}{\Delta t} = -n\nabla(p^{k+1} - p^k) \quad (9b)$$

187 Substituting Eq. (9) into Eq. (3), the pressure Poisson equation can be derived from the
 188 mass balance equation of the mixture in accordance with the implicit scheme:

$$189 \quad \Delta t \left(\frac{1-n}{\rho_s} + \frac{n}{\rho_l} \right) \nabla^2 (p^{k+1} - p^k) = (1-n) \nabla \cdot \mathbf{v}_s^* + n \nabla \cdot \mathbf{v}_l^* \quad (10)$$

2.3. Spatial discretization

The spatial discretization employed in MPM is similar to that in FEM, except that the numerical integrations are performed at the moving material points rather than fixed Gauss quadrature points. The standard Galerkin procedure and MPM discretization techniques are adopted to spatially discretize the governing equations (further details are provided in Kularathna et al. (2021)). To mitigate the cell-crossing noise associated with the C^0 basis functions, this study employs the Generalized Interpolation Material Point (GIMP) method (Bardenhagen and Kober, 2004) and B-Spline MPM (Gan et al., 2018; Steffen et al., 2008). The momentum balance equations at the prediction step in Eq. (8) are spatially discretized as:

$$\mathbf{M}_{s,ij}(\mathbf{v}_{s,j}^* - \mathbf{v}_{s,j}^k) + \mathbf{M}_{l,ij}(\mathbf{v}_{l,j}^* - \mathbf{v}_{l,j}^k) = \Delta t(\mathbf{f}_i^{ext} + \mathbf{f}_i^{int}) \quad (11a)$$

$$\mathbf{M}_{l,ij}(\mathbf{v}_{l,j}^* - \mathbf{v}_{l,j}^k) + \mathbf{Q}_{ij}\Delta t(\mathbf{v}_{l,j}^* - \mathbf{v}_{s,j}^*) = \Delta t(\mathbf{f}_{l,i}^{ext} + \mathbf{f}_{l,i}^{int}) \quad (11b)$$

where $\mathbf{v}_{\alpha,i}$, $\mathbf{f}_{\alpha,i}^{int}$ and $\mathbf{f}_{\alpha,i}^{ext}$ represent the nodal velocity, internal and external force for phase α at node i , respectively, and the nodal matrices associated with mass and drag force are denoted as $\mathbf{M}_\alpha$ and $\mathbf{Q}$ being:

$$\mathbf{M}_{\alpha,ij} = \sum_{p=1}^{N_p} S_{ip} S_{jp} \bar{\rho}_\alpha V_p \quad (12a)$$

$$\mathbf{Q}_{ij} = \sum_{p=1}^{N_p} n_p^2 \frac{\rho_l g}{k} S_{ip} S_{jp} V_p \quad (12b)$$

$$\mathbf{f}_i^{int} = -\sum_{p=1}^{N_p} \nabla S_{ip} : (\boldsymbol{\sigma}_p^k - p_p^k \mathbf{I}) V_p \quad (12c)$$

$$\mathbf{f}_i^{ext} = \sum_{p=1}^{N_p} S_{ip} \rho_m \mathbf{b}_p V_p + \sum_{p=1}^{N_p} S_{ip} \bar{\mathbf{t}}_p V_p h^{-1} \quad (12d)$$

$$\mathbf{f}_{l,i}^{int} = \sum_{p=1}^{N_p} \nabla S_{ip} : (n_p p_p^k \mathbf{I}) V_p \quad (12e)$$

$$\mathbf{f}_{l,i}^{ext} = \sum_{p=1}^{N_p} S_{ip} n_p \rho_l \mathbf{b}_p V_p + \sum_{p=1}^{N_p} S_{ip} V_p h^{-1} \bar{p}_p \hat{\mathbf{n}} \quad (12f)$$

where S_{ip} is the shape function evaluated at the position of a material point p ; subscripts i and p denote node and material point, respectively; N_p is the total number of material points associated with node i , V_p is the particle volume, $\bar{\mathbf{t}}_p$ and $\bar{p}_p$ denote the surface traction forces for the mixture and liquid phase, respectively, and $\hat{\mathbf{n}}$ is the surface outward unit normal vector. It should be noted that, although the drag force term is implicitly evaluated based on

the intermediate velocities, the above momentum equations can still be transformed into nodal-wise explicit expressions by replacing $\mathbf{M}_\alpha$ and $\mathbf{Q}$ with lumped matrices. Accordingly, this nodal-wise scheme is adopted here, as it simplifies the computational procedure associated with the implicit solution of nonsymmetric matrices and improves computational efficiency without compromising prediction accuracy.

Similarly, the pressure Poisson equation in Eq. (10) can be spatially discretized as:

$$\mathbf{L}_{ij} \Delta t (p_j^{k+1} - p_j^k) = \mathbf{N}_{s,ij} \mathbf{v}_{s,j}^* + \mathbf{N}_{l,ij} \mathbf{v}_{l,j}^* \quad (13)$$

with

$$\mathbf{L}_{ij} = \sum_{p=1}^{N_p} \left(\frac{1-n_p}{\rho_{s,p}} + \frac{n_p}{\rho_{l,p}} \right) \nabla S_{ip} \nabla S_{jp} V_p \quad (14a)$$

$$\mathbf{N}_{s,ij} = \sum_{p=1}^{N_p} \nabla S_{ip} S_{jp} (1-n_p) V_p \quad (14b)$$

$$\mathbf{N}_{l,ij} = \sum_{p=1}^{N_p} \nabla S_{ip} S_{jp} n_p V_p \quad (14c)$$

Finally, velocities are updated with an explicit correction procedure as shown in Eq. (9), which is spatially discretized as:

$$\mathbf{M}_\alpha (\mathbf{v}_{\alpha,i}^{k+1} - \mathbf{v}_{\alpha,i}^*) = \mathbf{G}_{\alpha,ij} \Delta t (p_j^{k+1} - p_j^k) \quad (15)$$

with

$$\mathbf{G}_{\alpha,ij} = \sum_{p=1}^{N_p} S_{ip} \phi_{\alpha,p} V_p \nabla S_{jp} \quad (16)$$

where $\phi_{\alpha,p}$ denotes the volumetric fraction of α phase and expresses as $\phi_{l,p} = 1 - \phi_{s,p} = n_p$, and the lumped mass matrix is used in the correction step.

2.4. Contact algorithm

The widely employed grid-based contact algorithm (Bardenhagen et al., 2000) enhanced by the penalty function modification is incorporated into the semi-implicit MPM framework for modeling potential separation and frictional sliding between the impermeable rigid structure and the deformable saturated medium. First, the enforcement of contact constraints on interface nodes is determined through the impenetrability detection condition,

$$(\mathbf{v}_{r,i} - \mathbf{v}_{\alpha,i}) \cdot \mathbf{n}_{r,i} > 0 \quad (17)$$

where $\mathbf{n}_{r,i}$ is the outward unit vector normal to the rigid body at node i and determined from its mass gradient, and $\mathbf{v}_{r,i}$ denotes the nodal velocity vector of rigid body.

Once the interface node is detected as in contact, the nodal velocity and acceleration of

the deformable body in the prediction and correction steps are adjusted to avoid interpenetration. To reduce premature contact detection and spurious contact gaps, issues that are commonly encountered with node-based contact algorithms, the penalty function proposed by Ma et al. (2014) is employed in the velocity correction, which incorporates a weighing factor f_p , to account for the distance between the contact node and the structure surface.

The fractional step approach decouples the pressure and kinematic fields by splitting the governing equations into two computational steps: a prediction step and a correction step. Soil-structure contact detection needs to be performed for both of these steps. Therefore, the correction of the nodal intermediate velocity $\Delta \mathbf{v}_{\alpha,i}^{predict}$ and the corresponding contact force $\mathbf{f}_{\alpha,i}^{predict}$ applied to α phase are calculated as:

$$\Delta \mathbf{v}_{\alpha,i}^{predict} = f_p (\Delta \mathbf{v}_{\alpha,i} \cdot \mathbf{n}_{r,i} \cdot \mathbf{n}_{r,i} + \Delta \mathbf{v}_{\alpha,i} \cdot \mathbf{n}_{r,i} \cdot \mu \boldsymbol{\omega}) \quad (18a)$$

$$\mathbf{f}_{\alpha,i}^{predict} = \frac{m_{\alpha,i}}{\Delta t} \Delta \mathbf{v}_{\alpha,i}^{predict} \quad (18b)$$

with

$$\begin{cases} f_p = 1 - \left(\frac{\min(d, L)}{L} \right)^k \\ \Delta \mathbf{v}_{\alpha,i} = \mathbf{v}_{r,i} - \mathbf{v}_{\alpha,i}^* \\ \boldsymbol{\omega} = \frac{\Delta \mathbf{v}_{\alpha,i} \times \mathbf{n}_{r,i}}{\|\Delta \mathbf{v}_{\alpha,i} \times \mathbf{n}_{r,i}\|} \\ \mu = \min \left(\mu_c, \frac{\|\Delta \mathbf{v}_{\alpha,i} \times \mathbf{n}_{r,i}\|}{\Delta \mathbf{v}_{\alpha,i} \cdot \mathbf{n}_{r,i}} \right) \end{cases} \quad (19)$$

in which f_p is the penalty factor, $m_{\alpha,i}$ is the nodal mass for the α phase, L and d denote the element size and the distance from the contact node to the structure surface, respectively, k is the penalty power coefficient and is set to 7 in this study, $\boldsymbol{\omega}$ is the unit tangent vector of the contact surface, μ represents the mobilized interfacial friction from sticking state or frictional sliding state, and μ_c is the Coulomb friction coefficient. The first term on the right-hand side of Eq. (18a) enforces the impenetrability condition through normal velocity correction, while the second term adjusts the tangential velocity based on the Coulomb friction criterion. Notice that for simplification, the lumped mass matrix is employed to allow explicit and nodal-wise correction of nodal velocities, otherwise, the global nodal momentum equations need to be solved iteratively to compute contact forces with the consistent matrix.

Following the same procedure, the corrected nodal velocity $\Delta \mathbf{v}_{\alpha,i}^{correct}$ and contact force $\mathbf{f}_{\alpha,i}^{correct}$ at the correction step are calculated as:

$$\Delta \mathbf{v}_{\alpha,i}^{correct} = f_p (\Delta \mathbf{v}_{\alpha,i} \cdot \mathbf{n}_{r,i} \cdot \mathbf{n}_{r,i} + \Delta \mathbf{v}_{\alpha,i} \cdot \mathbf{n}_{r,i} \cdot \mu \boldsymbol{\omega}) \quad (20a)$$

$$\mathbf{f}_{\alpha,i}^{correct} = \frac{m_{\alpha,i}}{\Delta t} \Delta \mathbf{v}_{\alpha,i}^{correct} \quad (20b)$$

with $\Delta \mathbf{v}_{\alpha,i} = \mathbf{v}_{r,i} - \mathbf{v}_{\alpha,i}^{k+1}$. The nodal contact force $\mathbf{f}_i^{contact}$ acting on the saturated media is determined by assembling the contributions from all components,

$$\mathbf{f}_i^{contact} = \mathbf{f}_{s,i}^{predict} + \mathbf{f}_{l,i}^{predict} + \mathbf{f}_{s,i}^{correct} + \mathbf{f}_{l,i}^{correct} \quad (21)$$

3. Numerical stabilization schemes

This section presents the specific stabilization treatments incorporated into the proposed semi-implicit framework to enhance numerical robustness and computational stability. The adopted stabilization strategies are intended to suppress pore pressure oscillations, mitigate volumetric locking of the solid phase, and improve the particle-grid kinematic transfer scheme. Although individual stabilization techniques have been applied to address specific numerical issues in previous studies (Bandara and Soga, 2015; Markert et al. 2010; Jiang et al., 2015), this section reports their comprehensive application to achieve improved numerical stability and accuracy in hydro-mechanically coupled large-deformation geotechnical analyses.

3.1. Pore pressure stabilization strategy

Although the fractional step method satisfies the Inf-Sup compatibility condition, most hydromechanically coupled large-deformation MPM analyses involving elastoplastic materials require the time step to be smaller than the critical time step to ensure accurate explicit stress integration. However, the adoption of time steps smaller than the critical values may trigger unphysical pore pressure oscillations and so require additional stabilization treatment (Codina, 2001). In this study, the pressure oscillation is tackled by introducing the artificial compressibility into the pressure Poisson equation for its simplicity and efficiency (Markert et al. 2010; Yuan et al. 2023). Accordingly, Eq. (10) is rewritten as,

$$\Delta t \left(\frac{1-n}{\rho_s} + \frac{n}{\rho_l} \right) \nabla^2 (p^{n+1} - p^n) + \varepsilon \nabla^2 (p^{n+1} - p^n) = (1-n) \nabla \cdot \mathbf{v}_s^* + n \nabla \cdot \mathbf{v}_l^* \quad (22)$$

where ε is a sufficiently small parameter controlling the level of artificial compressibility, this term prevents the stiffness term from approaching zero under infinitely small time steps to enhance the numerical stability, and the value of ε is determined as,

$$\varepsilon \geq \beta_0 \Delta t_{cr} \left(\frac{1-n}{\rho_s} + \frac{n}{\rho_l} \right) - \Delta t \left(\frac{1-n}{\rho_s} + \frac{n}{\rho_l} \right), \quad \Delta t_{cr} = \frac{L_{min}}{\sqrt{(K + 4G/3) / \bar{\rho}_s}} \quad (23)$$

in which Δt_{cr} is the critical time step restricted by the Courant-Friedrichs-Levy (CFL) condition, L_{min} is the smallest element size, K and G are the solid bulk modulus and shear modulus, respectively, and the degree of artificial compressibility is adjusted via the parameter β_0 . The quantity β_0 is not a true material property. Instead it is introduced as a pragmatic device to ensure numerical stability. Its magnitude is tuned adaptively to find a value which is

sufficiently large to stabilize unphysical pressure oscillations, yet small enough to avoid degrading the accuracy of the volume-preserving solution. Referring to [Markert et al. \(2010\)](#) and [Yuan et al. \(2023\)](#), the value of β_0 is typically chosen within the range $[1, 10]$, which is generally effective to balance numerical accuracy and computational stability in hydromechanically coupled large-deformation MPM analyses.

3.2. Volumetric locking mitigation technique

MPM solutions for nearly incompressible materials are susceptible to volume locking, where excessive stiffness easily causes kinematic locking and strain field errors. The B-bar approach, originally proposed by [Hughes \(1980\)](#), is one of the commonly used techniques to alleviate volumetric locking and checkerboard stress distributions in MPM analysis of solid phase problems ([Bandara and Soga, 2015](#); [Bisht et al., 2021](#)). The approach modifies the conventional B-matrix in a B-bar matrix to calculate the strain increments of particles and nodal internal forces of the solid phase. Employing reduced integration for the volumetric component and full integration for the deviatoric component, the specific form of the B-bar matrix for the plane-strain configuration is:

$$\bar{\mathbf{B}} = \begin{bmatrix} \frac{1}{2}(\nabla S_{ic,x} + \nabla S_{ip,x}) & \frac{1}{2}(\nabla S_{ic,y} - \nabla S_{ip,y}) \\ \frac{1}{2}(\nabla S_{ic,x} - \nabla S_{ip,x}) & \frac{1}{2}(\nabla S_{ic,y} + \nabla S_{ip,y}) \\ \nabla S_{ip,y} & \nabla S_{ip,x} \end{bmatrix} \quad (24)$$

where $\nabla S_{ip,x}$ and $\nabla S_{ip,y}$ denote the gradient of shape functions evaluated at the material point p in the x and y direction, respectively, and $\nabla S_{ic,x}$ and $\nabla S_{ic,y}$ are the gradients of shape function evaluated at the element center. The evaluation of $\nabla S_{ic,x}$ and $\nabla S_{ic,y}$ is not straightforward, as the material point domain in GIMP scheme may stretch across multiple elements. The contribution from the portions of the particle domain lying within individual element are computed separately and assembled to determine $\nabla S_{ic,x}$ and $\nabla S_{ic,y}$ (see [Coombs et al. \(2018\)](#) and [Bisht et al. \(2021\)](#) for further details).

3.3. APIC transfer

The accuracy and stability of MPM analyses are also related to the information transfer scheme adopted between material points and background grid. To balance energy conservation and numerical stability, the APIC scheme proposed by [Jiang et al. \(2015\)](#) is adopted in this study. In APIC, the transfer operator of velocity is enriched by augmenting each particle with a locally affine description to enhance momentum conservation properties. The nodal velocity is obtained by mapping the locally affine velocity $\mathbf{v}_p^{affine}$ to the grid nodes,

$$\mathbf{v}_{\alpha,i} = \sum_{p=1}^{N_p} S_{ip} m_{\alpha,p} \mathbf{v}_{\alpha,p}^{affine} / m_{\alpha,i} \quad (25)$$

with

$$m_{\alpha,i} = \sum_{p=1}^{N_p} S_{ip} \bar{\rho}_\alpha V_p, \quad \mathbf{v}_{\alpha,p}^{affine} = \mathbf{v}_{\alpha,p} + \mathbf{C}_{\alpha,p} (\mathbf{x}_i - \mathbf{x}_p) \quad (26)$$

where $\mathbf{C}_{\alpha,p}$ denotes the linear transformation of α phase at material point p , $\mathbf{v}_{\alpha,p}$ represents the translational velocity, and $\mathbf{x}$ is the spatial position vector. The locally affine velocity field is composed of a translational component and a linear component. If affine matrix $\mathbf{C}_{\alpha,p} = 0$, the particle-grid transfer scheme would degrade to the PIC mode. Specifically, the affine matrix $\mathbf{C}_{\alpha,p}$ is calculated as:

$$\mathbf{C}_{\alpha,p} = \mathbf{B}_{\alpha,p} (\mathbf{D}_p)^{-1} \quad (27)$$

with

$$\mathbf{B}_{\alpha,p} = \sum_{i=1}^{N_n} S_{ip} \mathbf{v}_{\alpha,i} (\mathbf{x}_i - \mathbf{x}_p)^T, \quad \mathbf{D}_p = \sum_{i=1}^{N_n} S_{ip} (\mathbf{x}_i - \mathbf{x}_p) (\mathbf{x}_i - \mathbf{x}_p)^T \quad (28)$$

in which $\mathbf{B}_{\alpha,p}$ represents the affine state containing the angular momentum, $\mathbf{D}_p$ is the inertia-like matrix derived by preserving affine motion.

3.4. Computational procedure

The proposed stabilized semi-implicit MPM approach is summarized below as Algorithm 1. Following the finite-strain theory and accounting for substantial configuration rotations, the effective stress is calculated with the Cauchy stress and updated via the Jaumann rate:

$$\boldsymbol{\sigma}'_p^{k+1} = \boldsymbol{\sigma}'_p^k + \Delta t \boldsymbol{\sigma}'_p + \Delta t (\boldsymbol{\sigma}'_p^k \cdot \mathbf{W}_p^k - \mathbf{W}_p^k \cdot \boldsymbol{\sigma}'_p^k) \quad (29)$$

where the vorticity tensor $\mathbf{W}_p^k$ is calculated as:

$$\mathbf{W}_p^k = \frac{1}{2} \sum_{i=1}^{N_n} (\nabla S_{ip} \mathbf{v}_{s,i}^k - (\nabla S_{ip} \mathbf{v}_{s,i}^k)^T) \quad (30)$$

The velocity, position and pore pressure of the material points are updated using:

$$\mathbf{v}_{\alpha,p}^{k+1} = \sum_{i=1}^{N_n} S_{ip} \mathbf{v}_{\alpha,i}^{k+1} \quad (31a)$$

$$\mathbf{x}_p^{k+1} = \mathbf{x}_p^k + \Delta t \sum_{i=1}^{N_n} S_{ip} \mathbf{v}_{s,i}^{k+1} \quad (31b)$$

$$p_p^{k+1} = p_p^k + \sum_{i=1}^{N_n} S_{ip} (p_i^{k+1} - p_i^k) \quad (31c)$$

The particle volume, density and porosity are updated using:

$$V_p^{k+1} = V_p^k (1 + \Delta \epsilon_{vol,p}^k) \quad (32a)$$

$$\rho_p^{k+1} = \frac{\rho_p^k}{(1 + \Delta \epsilon_{vol,p}^k)} \quad (32b)$$

$$n_p^{k+1} = 1 - \frac{1 - n_p^k}{(1 + \Delta \epsilon_{vol,p}^k)} \quad (32c)$$

where N_n denotes the total number of computational nodes associated with particle p ,

$\Delta \epsilon_{vol,p} = \det(\Delta \mathbf{F}_p) - 1$ is the incremental volumetric strain, and the incremental deformation

gradient is obtained from $\Delta \mathbf{F}_p = \mathbf{I} + \Delta t \sum_{i=1}^{N_n} \nabla S_{ip} \mathbf{v}_{s,i}^k$.

Algorithm 1 Implementation procedure of the coupled semi-implicit MPM.

-
1. **Initialization** at $t = 0$:
 2. Initialize variables at material points and grid nodes;
 3. Determine the time increment Δt – Eq. (23);
 4. **While** $0 < t < t_{end}$ **do**:
 5. Map particle mass and momentum to grid nodes – Eqs. (25)-(26);
 6. Compute strain increments and effective stress based on constitutive model;
 7. Solve prediction equations for intermediate velocity– Eq. (11);
 8. **If** contact-detection condition is satisfied at the prediction step– Eq. (17):
 9. Correct intermediate velocity and compute contact force – Eq. (18);
 10. Solve pressure Poisson equation for incremental pore pressure– Eq. (13);
 11. Solve corrector equations to obtain nodal velocities – Eq. (15);
 12. **If** contact-detection condition is satisfied at the correction step– Eq. (17):
 13. Correct final velocity and compute contact force – Eq. (20);
 14. Update particle velocity and affine matrix – Eqs. (31a) and (27);
 15. Update particle position, pore pressure and other variables – Eqs. (31b)-(32);
 16. Reset background grid and update time $t = t + \Delta t$.
 17. **End**
-

365

366 4. Axisymmetric formulation

367 To date, the proposed computational framework has been developed for mainly 2D MPM
 368 analyses. While extensions to enable full 3D simulations are under-consideration, they fall
 369 outside the scope of the present study. The proposed approach helps to avoid the mesh
 370 distortion issues widely encountered in mesh-based methods for large deformation problems,

but it can pose other challenges in 3D analyses regarding numerical stability and high computational costs. Specifically, the semi-implicit MPM scheme involves frequent particle-grid mappings, nodal-wise momentum updates, and an iterative solution for the pore pressure Poisson equation. These steps are computationally expensive and time-consuming. Nevertheless, the proposed approach is well-suited for future GPU parallelization solutions, through a matrix-free solution strategy for the pore pressure equation, which can significantly reduce the computational time.

However, many important practical geotechnical prototype problems can already be modelled effectively using an axisymmetric approach. An axisymmetric MPM formulation was originally proposed by [Sulsky and Schreyer \(1996\)](#) and subsequently reformulated within the GIMP framework by [Nairn and Guilkey \(2015\)](#). While these axisymmetric formulations are straightforwardly applicable in explicit coupled MPM ([Bisht et al., 2021](#); [Martinelli and Galavi, 2022](#)), their implementation within the semi-implicit framework requires additional considerations, particularly in the treatment of particle volumes (mass) and strains, the pressure Poisson equation and nodal internal forces, which need to account for cylindrical volume integration and axisymmetric gradient operators.

Axisymmetric formulations are characterized in the r - z plane (where r and z represent the radial and axial directions respectively), with the circumferential direction θ in the cylindrical coordinates being taken as 1 radian. The particle volume in a 1-radian wedge is calculated as:

$$V_p = \int_{\Omega} \chi_p(r, z) r dr dz = A_p \bar{r}_p \quad (33)$$

where χ_p is the particle characteristic function in GIMP, and chosen as the Dirac delta function for standard MPM or B-spline MPM, and A_p and $\bar{r}_p$ denote the r - z plane area and radial position of particle p , respectively.

In axisymmetric MPM, an extra shape function T_{ip} is required to indicate the circumferential variable, which is generally expressed as $T_{ip} = S_{ip}/r_p$, and the shape function S_{ip} and gradient ∇S_{ip} are computed according to the planar formulations. In the GIMP method, the axisymmetric shape functions and gradient (S_{ip} , ∇S_{ip} and T_{ip}) need to be reformulated following the procedure developed by [Nairn and Guilkey \(2015\)](#).

The components of strain increment $\Delta \boldsymbol{\varepsilon}$ in the axisymmetric coordinate are evaluated by:

$$(\Delta \boldsymbol{\varepsilon}_{s,p})_{rr} = \Delta t \sum_{i=1}^{N_n} \nabla S_{ip,r} (\mathbf{v}_{s,i})_r \quad (34a)$$

$$(\Delta \boldsymbol{\varepsilon}_{s,p})_{zz} = \Delta t \sum_{i=1}^{N_n} \nabla S_{ip,z} (\mathbf{v}_{s,i})_z \quad (34b)$$

$$(\Delta \boldsymbol{\varepsilon}_{s,p})_{rz} = \Delta t \sum_{i=1}^{N_n} (\nabla S_{ip,r} (\mathbf{v}_{s,i})_z + \nabla S_{ip,z} (\mathbf{v}_{s,i})_r) \quad (34c)$$

$$(\Delta \boldsymbol{\varepsilon}_{s,p})_{\theta\theta} = \Delta t \sum_{i=1}^{N_n} T_{ip} (\mathbf{v}_{s,i})_r \quad (34d)$$

where $(\mathbf{v}_{s,i})_r$ and $(\mathbf{v}_{s,i})_z$ are the radial and axial components of nodal velocity, respectively, $\nabla S_{ip,r}$ and $\nabla S_{ip,z}$ are the gradients of shape functions evaluated in r and z directions, respectively, and the extra circumferential strain increment $(\Delta \boldsymbol{\varepsilon}_{s,p})_{\theta\theta}$ requires to be taken into account compared to the plane-strain condition.

For the axisymmetric pressure Poisson equation, the discretized form of Laplacian operator $\mathbf{L}$ on the left-hand side of Eq. (13) remains identical to that of the plane-strain formulation (Smith et al., 2014), except for the integration weighting V_p . In contrast, the divergence terms on the right-hand side, which characterize the volume variation, are modified to account for the circumferential deformation, given as follows,

$$\mathbf{L} \Delta t (p_j^{n+1} - p_j^n) = N_s \mathbf{v}_{s,j}^* + N_l \mathbf{v}_{l,j}^* + N_{s,\theta} (\mathbf{v}_{s,j}^*)_r + N_{l,\theta} (\mathbf{v}_{l,j}^*)_r \quad (35a)$$

$$N_{s,\theta} = \sum_{p=1}^{N_p} T_{ip} S_{jp} (1 - n_p) V_p \quad (35b)$$

$$N_{l,\theta} = \sum_{p=1}^{N_p} T_{ip} S_{jp} n_p V_p \quad (35c)$$

The internal forces of solid and liquid phases (Eq. 12) are reformulated after incorporating the circumferential stress $(\boldsymbol{\sigma}_{\alpha,p})_{\theta\theta}$,

$$(\mathbf{f}_{\alpha,i}^{int})_r = - \sum_{p=1}^{N_p} V_p (\nabla S_{ip,r} (\boldsymbol{\sigma}_{\alpha,p})_{rr} + \nabla S_{ip,z} (\boldsymbol{\sigma}_{\alpha,p})_{rz} + T_{ip} (\boldsymbol{\sigma}_{\alpha,p})_{\theta\theta}) \quad (36a)$$

$$(\mathbf{f}_{\alpha,i}^{int})_z = - \sum_{p=1}^{N_p} V_p (\nabla S_{ip,r} (\boldsymbol{\sigma}_{\alpha,p})_{rz} + \nabla S_{ip,z} (\boldsymbol{\sigma}_{\alpha,p})_{zz}) \quad (36b)$$

Similarly, the corresponding terms in the correction step of Eq. (15) are also modified by evaluating the contribution of the circumferential pore pressure increment,

$$(\mathbf{G}_{\alpha,ij})_r = \sum_{p=1}^{N_p} S_{ip} \phi_{\alpha,p} V_p (\nabla S_{jp,r} + \nabla S_{jp,z} + T_{jp}) \quad (37a)$$

$$(\mathbf{G}_{\alpha,ij})_z = \sum_{p=1}^{N_p} S_{ip} \phi_{\alpha,p} V_p (\nabla S_{jp,r} + \nabla S_{jp,z}) \quad (37b)$$

Besides above adjustments, the traction boundary also requires proper prescription based on the specific particle positions (Nairn and Guilkey, 2015). If the B-bar technique is adopted,

the expression for the axisymmetric condition is adjusted to:

$$\bar{\mathbf{B}} = \frac{1}{3} \begin{bmatrix} \nabla S_{ic,r} + 2\nabla S_{ip,r} + T_{ic} - T_{ip} & \nabla S_{ic,z} - \nabla S_{ip,z} \\ \nabla S_{ic,r} - \nabla S_{ip,r} + T_{ic} - T_{ip} & \nabla S_{ic,z} + 2\nabla S_{ip,z} \\ \nabla S_{ic,r} - \nabla S_{ip,r} + T_{ic} + T_{ip} & \nabla S_{ic,z} - \nabla S_{ip,z} \\ 3\nabla S_{ip,z} & 3\nabla S_{ip,r} \end{bmatrix} \quad (38)$$

in which variables with the subscript p indicate values evaluated at the particle position, while those with the subscript c are evaluated at the element center.

5. Analysis of coupled loading benchmark

The performance of the MPM approach set out above is tested by considering a range of benchmark problems involving linear elastic porous media, starting with classical, small displacement, linearly elastic one-dimensional consolidation analysis before moving to Cryer's consolidation problem, dynamic elastic wave propagation and the large displacement, elastic, block slumping problem. Comparisons with results obtained from other analytical and numerical methods have been carried out to validate the simulation accuracy. The Authors' in-house MPM code is developed in Fortran by adopting Algorithm 1 in combination with established algorithms as cited in the corresponding references.

5.1. One-dimensional consolidation

In the first example, the classical one-dimensional consolidation problem is simulated through 2D plane strain MPM analysis. As shown in Fig. 2, a constant vertical stress increment of 10 kPa is applied to the top surface of a saturated 1.0 m high and 0.02 m wide soil column which is not allowed to displace laterally. Only top surface drainage is allowed and roller boundaries are applied to the two lateral sides with the base being fully constrained. The computational domain is discretized by structured meshes with linear quadrilateral elements of $0.02 \text{ m} \times 0.02 \text{ m}$ size, each containing 2×2 material points. The solid phase is treated as linearly elastic, and the model parameters are set as follows: Young's modulus $E = 1 \times 10^7 \text{ Pa}$, Poisson's ratio $\nu = 0.3$, soil grain density $\rho_s = 2650 \text{ kg/m}^3$, pore fluid density $\rho_l = 1000 \text{ kg/m}^3$, initial porosity $n_0 = 0.3$ and hydraulic conductivity $k = 1.0 \times 10^{-3} \text{ m/s}$. The initial pore pressure and effective stress are both set to zero, and the gravity is ignored.

Fig. 3 shows the spatial distribution of pore pressure along the column depth at different dimensionless time T , (i.e. $T = c_v t / H^2$, where c_v is the coefficient of consolidation and H is the column height). As can be seen, the numerical results from the coupled MPM match well with Terzaghi's analytical solutions. One main advantage of the semi-implicit approach over explicit MPM treatments is that the soil permeability is far less sensitive to the time step size. A series of consolidation tests with different hydraulic conductivities are simulated with fixed time steps of $\Delta t = 1.0 \times 10^{-4} \text{ s}$. The simulated pore pressure of a selected material point (i.e.

Point A at $h = 0.5$ m) in the middle of the specimen and the corresponding analytical solution are plotted in Fig. 4. The numerical results show close agreement with the theoretical solution under various permeabilities.

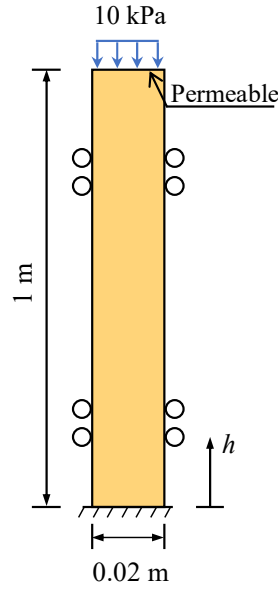


Fig. 2. Geometry and boundary conditions for one-dimensional consolidation test.

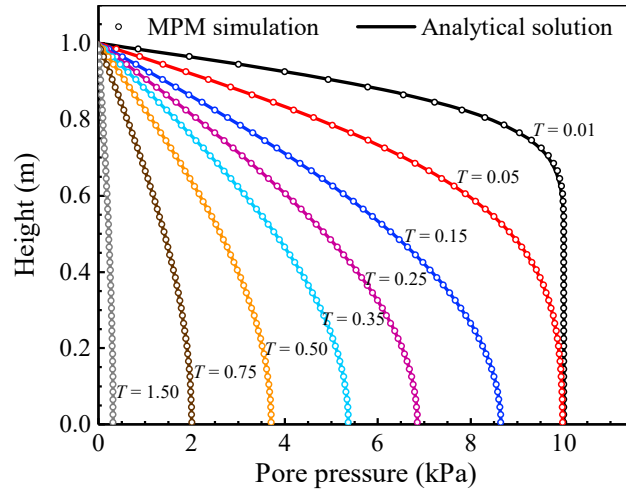


Fig. 3. Comparison of pore pressure profiles obtained from MPM simulations and Terzaghi's solution.

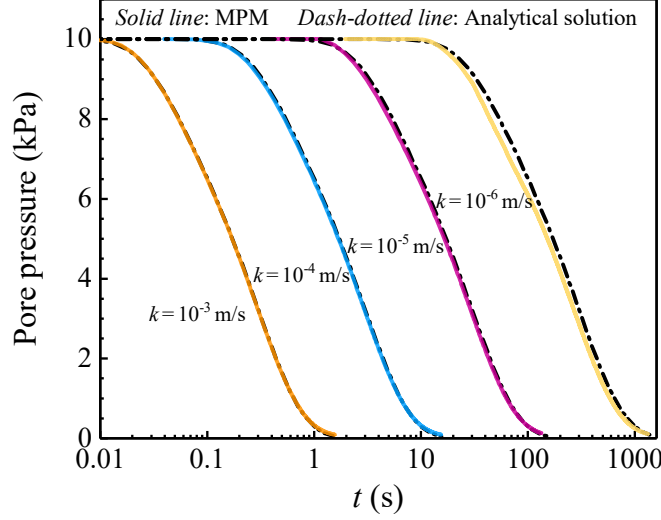


Fig. 4. Pore pressure dissipation at Point A under various hydraulic conductivities.

5.2. Cryer's problem

The classical coupled axisymmetric problem considered by Cryer (1963) is addressed next. The semi-implicit MPM and simulation results are compared with the theoretical solution to verify the coupled axisymmetric formulation. Fig. 5 shows how the axisymmetric two-dimensional model is established in the cylindrical coordinate (r, z, θ) with $\theta = 1$ radian and radius $R = 0.5$ m. Free drainage is allowed at the outer surface, and an impermeable boundary assigned along the vertical axis of the sphere. An isotropic pressure increment $P_0 = 10$ kPa is applied around the sphere surface. The background grid is constructed with an element size of $0.05 \text{ m} \times 0.05 \text{ m}$ and 4 material points per element. To improve the precision of external pressure application, material points near the exterior surface are uniformly repositioned, allowing for the computation of particle traction based on a constant curvature. The initial pore pressure and stress are set to zero. The material properties for the isotropic elastic sphere are set as: $E = 1 \times 10^7 \text{ Pa}$, $\rho_s = 2143 \text{ kg/m}^3$, $\rho_l = 1000 \text{ kg/m}^3$, $n_0 = 0.3$ and $k = 1.0 \times 10^{-3} \text{ m/s}$. Various Poisson's ratios are considered for comparison.

Fig. 6 plots the variation of pore pressure at the sphere center with dimensionless time T (i.e. $T = c_v t / R^2$) for various Poisson's ratios. The MPM results show a good agreement with Cryer's analytical results. Fig. 7 illustrates the distribution of pore pressure inside the sphere at $T = 0.05, 0.15$ and 0.20 (i.e. $t = 0.0091 \text{ s}, 0.0182 \text{ s}, 0.0273 \text{ s}$) for $\nu = 0.3$. It is also observed that the induced pore pressure near the sphere center exceeds the applied surface traction and then gradually dissipates to zero.

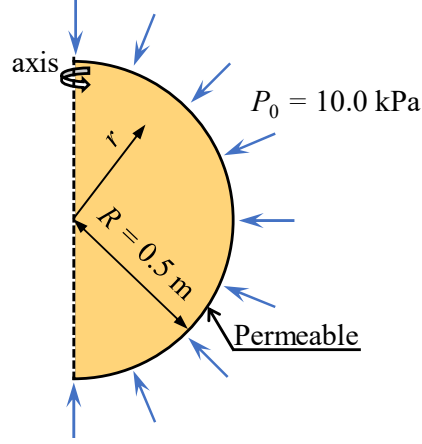


Fig. 5. Geometry and boundary conditions for Cryer's spherical consolidation problem.

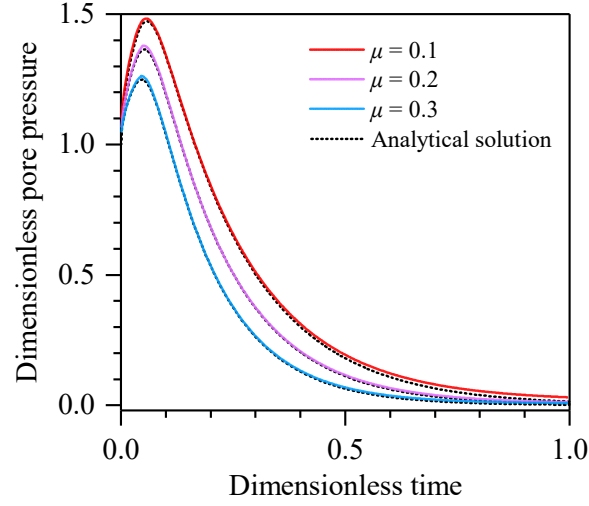


Fig. 6. Pore pressure evolution at the center of sphere for different Poisson's ratios.

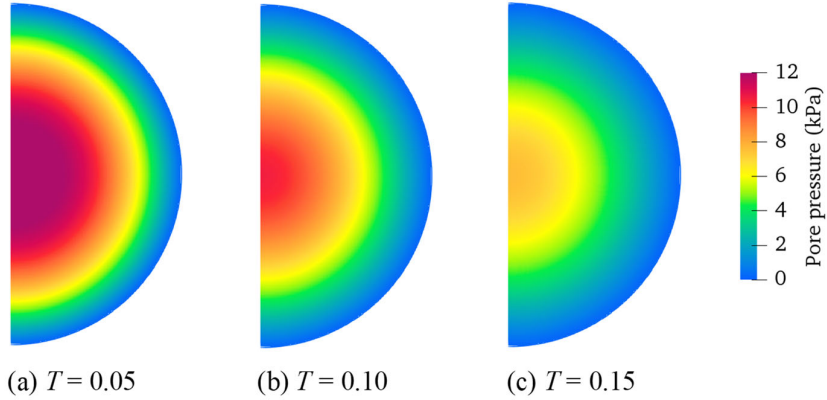


Fig. 7. Distribution of pore pressure inside the sphere at different times for $\nu = 0.3$.

5.3. Wave propagation

Wave propagation, which can provide critical tests of how numerical schemes perform (Markert et al., 2010; Yuan et al., 2022), is considered here to investigate the performance of the developed MPM code in 2D dynamic hydro-mechanical analysis. The geometry, boundary

and loading conditions are depicted in Fig. 8. The saturated rectangular domain has width and height of 21 m and 10 m, respectively, and is discretized using structured quadrilateral 0.125 m× 0.125 m (4 material points) elements. The lateral boundaries are constrained for horizontal motions, while the bottom boundary is fully fixed. Pore fluid can only drain from the top surface. A sinusoidal load pulse (in kPa) is imposed at the middle of the top surface with a width of 1 m and a magnitude prescribed as:

$$q(t) = 100 \times \sin(25\pi t) \times H(0.04 - t) \quad (39)$$

where $H(*)$ denotes the Heaviside step function. The soil is modeled as linear elastic with model parameters consistent with previous studies (Markert et al., 2010; Yuan et al., 2022): $E = 14.5$ MPa, $\nu = 0.3$, $\rho_s = 2000$ kg/m³, $\rho_l = 1000$ kg/m³, $n_0 = 0.33$ and $k = 1.0 \times 10^{-2}$ m/s.

Point A at (5, 10) and Point B at (8, 8), as identified in Figure 8 are selected to illustrate the local soil displacement and pore pressure responses during wave propagation. The captured horizontal and vertical displacements ($u_x - u_y$) at point A and pore pressure evolution at point B are plotted in Fig. 9, where the MPM results match both the FEM results reported by Markert et al. (2010) and SPFEM results by Yuan et al. (2022) well, showing the elliptic motions induced by traveling surface Rayleigh waves at point A. Fig. 10 illustrates the pore pressure distributions at $t = 0.05, 0.10, 0.15$ and 0.20 s from both the MPM and previous SPFEM analysis, which shows satisfactory agreements and no spurious pore pressure oscillations are observed in MPM results.

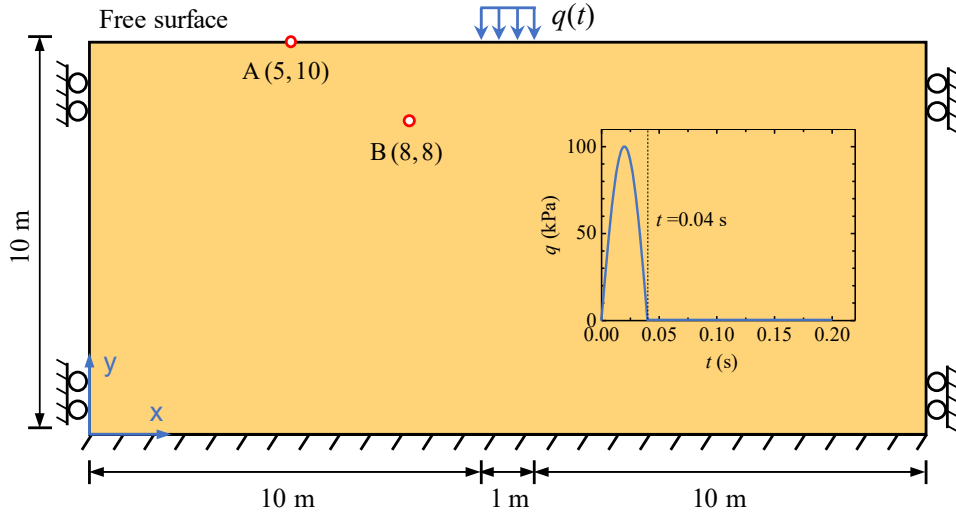


Fig. 8. Model setups for the wave propagation problem.

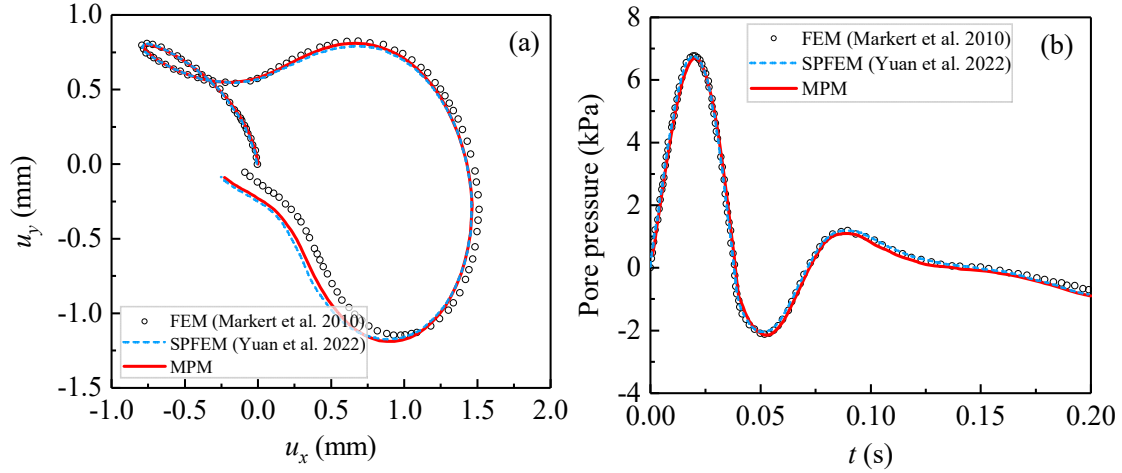


Fig. 9. Dynamic response of the selected points: (a) motion at Point A; (b) pore pressure at Point B.

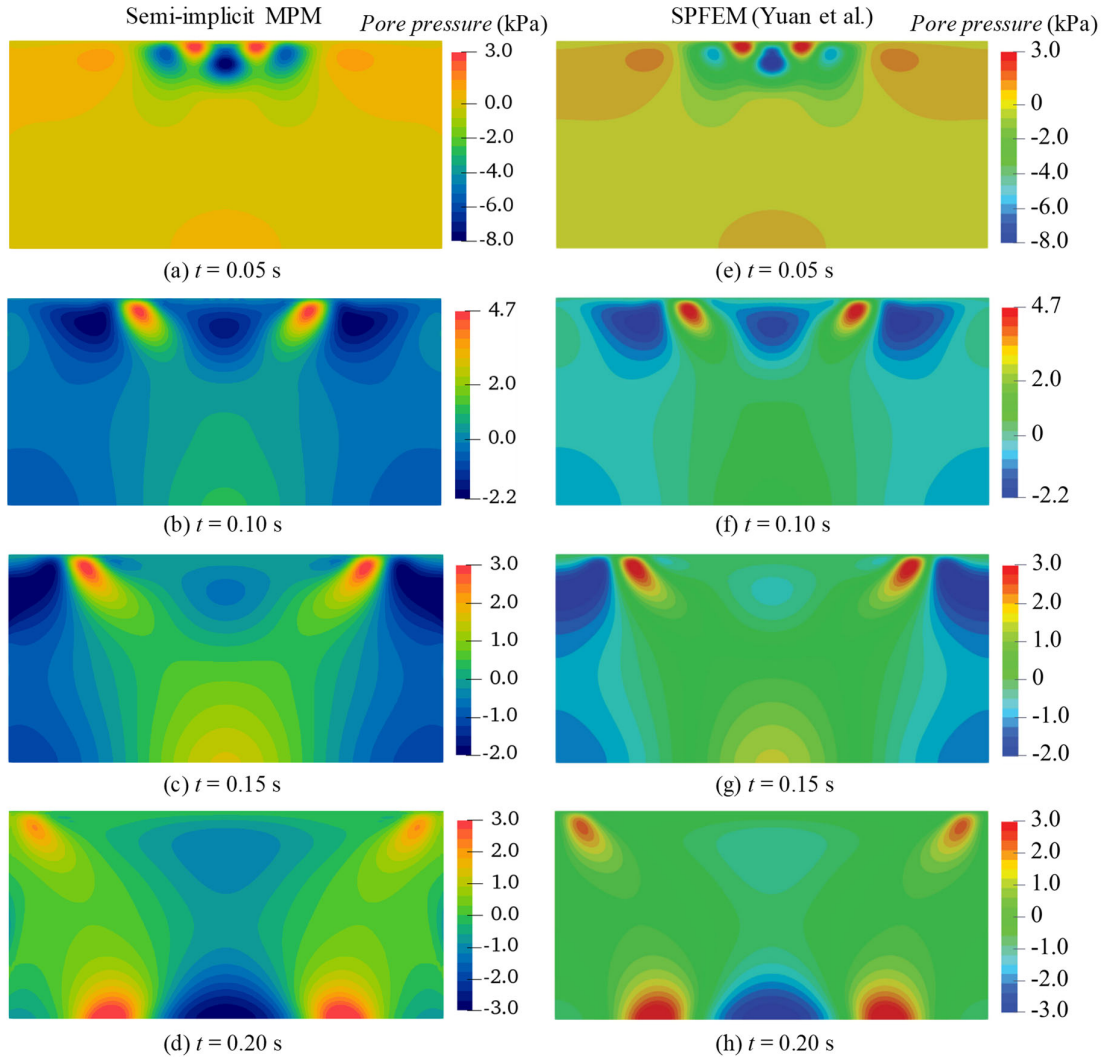


Fig. 10. Pore pressure from: (a)-(d) MPM and (e)-(h) SPFEM by Yuan et al. (2022).

5.4. Slumping block

Moving to consider larger deformation cases, the 2D self-weight block slumping problem is investigated to test the efficacy of the adopted coupled MPM stabilization techniques, considering a saturated $4 \text{ m} \times 2 \text{ m}$ block of very soft, but linear elastic and highly permeable, soil consolidating under gravity (Zhao and Choo, 2020; Zheng et al., 2021; Yuan et al., 2022). Taking advantage of the symmetry of this problem, only the right half of the block is modelled (Fig. 11(a)). The computational domain is discretized by structured quadrilateral $0.05 \text{ m} \times 0.05 \text{ m}$ elements containing 4×4 material points. The left and bottom boundaries are impermeable and supported by rollers, while the top and right boundaries allow for free movement and drainage. According to Zheng et al. (2021), material properties were assigned as follows: $E = 100 \text{ kPa}$, $\nu = 0.3$, $\rho_s = 2650 \text{ kg/m}^3$, $\rho_l = 1000 \text{ kg/m}^3$, $n_0 = 0.4$ and $k = 1.0 \times 10^{-4} \text{ m/s}$. Gravity is applied with a ramp-like loading function to avoid dynamic effects (Fig. 11(b)) and the simulation is terminated at $t = 0.5 \text{ s}$.

Fig. 12 exhibits the pore pressure distribution at different time intervals obtained from MPM analyses with and with considering artificial compressibility. Since the adopted time step $\Delta t = 8 \times 10^{-4} \text{ s}$ is smaller than the critical step size $\Delta t_{cr} = 0.0054 \text{ s}$, pronounced pressure oscillations that are inherent to the incremental formulation are observed when the artificial compressibility is not incorporated, which are evidently alleviated by introducing the artificial compressibility. Moreover, comparable results are obtained from the parametric analyses with $\beta_0 = 4$ and $\beta_0 = 8$, which both show smooth and consistent pore pressure predictions. This demonstrates that introducing an appropriately chosen artificial compressibility can simultaneously suppress the pore pressure oscillations and maintain the computational accuracy for hydromechanically coupled large-deformation MPM analyses of elastoplastic materials.

Fig. 13 compares the effective stress distribution at $t = 0.5 \text{ s}$ computed with and without the B-bar formulation in the GIMP scheme to assess its effectiveness in mitigating volumetric locking effects. As can be observed, the standard GIMP solution exhibits non-physical stress oscillations with a checkerboard pattern, while the B-bar GIMP result shows less oscillatory stress fields, validating the B-bar method for reducing the volumetric locking issues.

Results from other numerical approaches, such as FEM and PFEM, are also presented in this section to validate the effectiveness of the proposed MPM for hydro-mechanically coupled problems, without necessarily exploring the relative computational efficiency of the methods. Fig. 14 compares the pressured-stabilized results with PFEM analysis (Yuan et al., 2022), demonstrating excellent consistency in the predicted pore pressure evolution at three selected locations (Points A, B and C, as marked in Fig. 11).

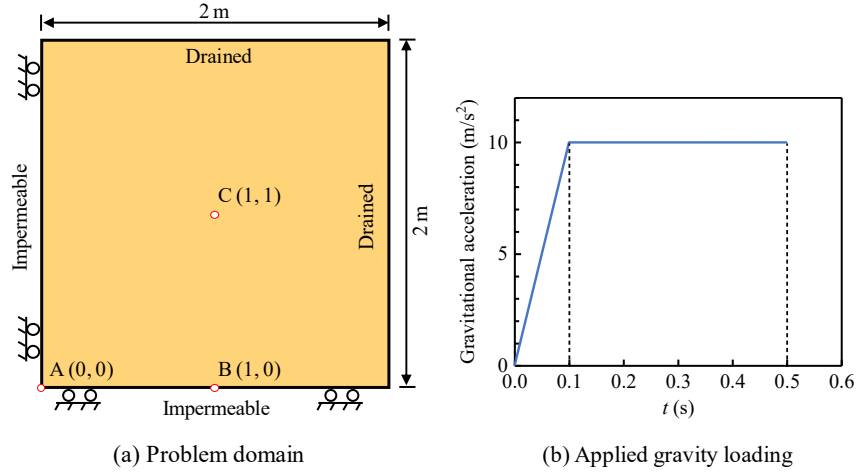


Fig. 11. Model setups for the slumping block problem: (a) geometry and boundary conditions (b) history of the applied gravity loading.

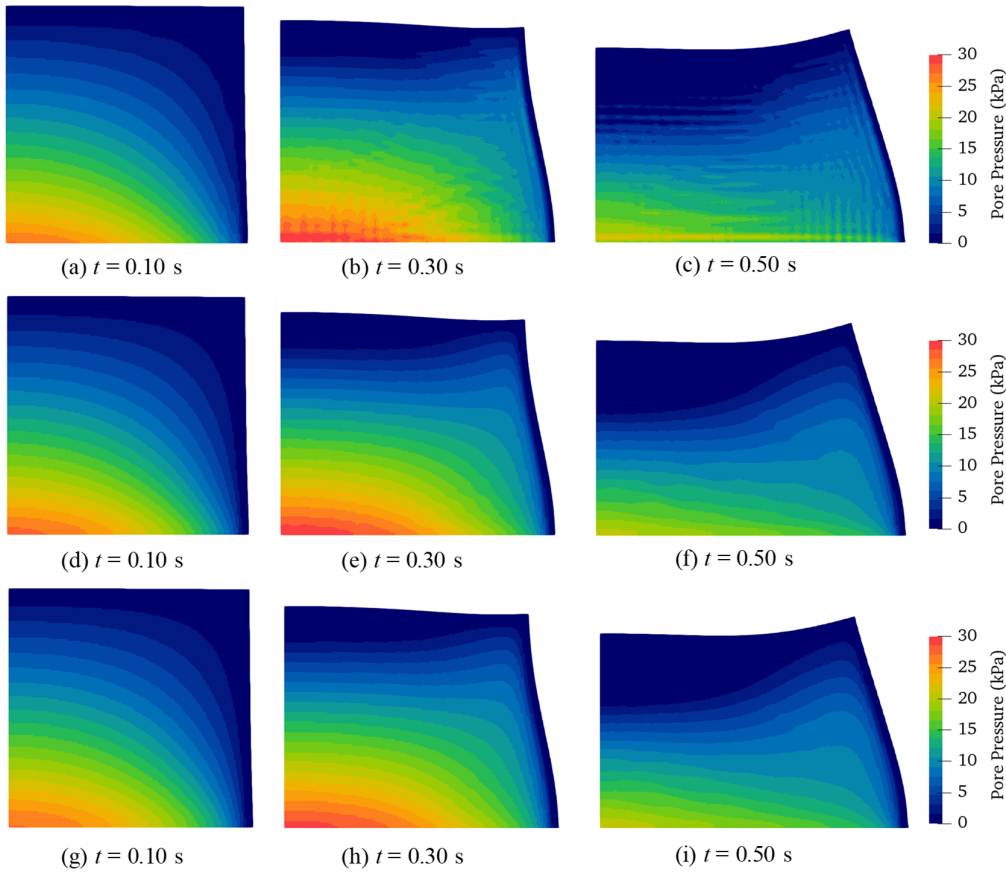


Fig. 12. Time sequence of pore pressure contours obtained from the coupled MPM: (a-c) without artificial compressibility; (d-f) with $\beta_0 = 4$; and (g-i) $\beta_0 = 8$.

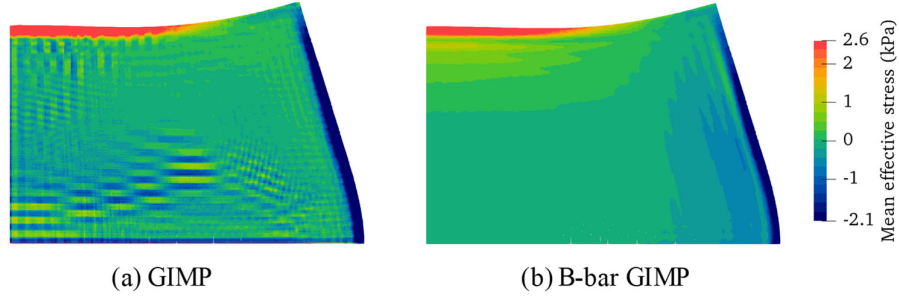


Fig. 13. Mean effective stress at $t = 0.5$ s (a) without B-bar and (b) with B-bar.

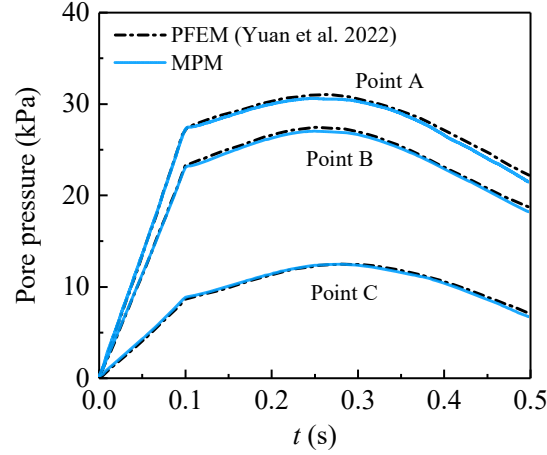


Fig. 14. Pore pressure evolution at the selected points (Points A, B and C in Fig. 11).

6. Soil-structure interaction problems

Having demonstrated in Section 5 the ability of the coupled semi-implicit MPM treatment to cope with a range of linear elastic soil problems, this Section tests its efficiency to cope with large deformations that involve more representative non-linear geomaterial behaviour as well as soil-structure interaction. The classical form of the Modified Cam-clay (MCC) model proposed by Roscoe and Burland (1968) was implemented in MPM code to better capture the behavior of relatively soft fine-grained soil. An explicit Euler scheme with sub-stepping and error control was used to integrate the stress-strain relationship at the element level (Sloan, 1987). A rigid strip footing problem is considered first, covering both small and large deformation regimes. Following this, the installation process of a stiff closed ended displacement pile is explored as this practically important case allowed an evaluation of the MPM as a potential tool for investigating large deformation problems under axisymmetric conditions.

6.1. Rigid strip footing

Plane strain strip footing analyses were performed with the proposed coupled MPM, covering the geometry and boundary conditions illustrated in Fig. 15. The footing width $B = 1$ m and the width and height of the computational domain both set to $W = H = 4B$ for the

small deformation analysis and extended out to $W = 12B$ and $H = 10B$ for the large deformation. The lateral boundaries are supported by rollers and the bottom boundary is fixed. Only the top surface outside of the footing permits for drainage and ground to either side is subject to a uniform vertical surcharge $q = 10$ kPa. The rigid footing is pushed vertically into a saturated MCC soil with a specified constant velocity. To be compatible with Monforte et al. (2023), gravity is neglected and initial pore pressures set to zero, while the initial effective vertical and horizontal stresses are both set to 10 kPa. The geomaterial properties are taken as follows: the slope of unloading-reloading line $\kappa = 0.01$, the slope of normal compression line $\lambda = 0.1$, the slope of critical state line $M = 1$, Poisson's ratio $\nu = 0.3$, initial preconsolidation pressure $p_{c0} = 15$ kPa, soil grain density $\rho_s = 2650$ kg/m³, fluid density $\rho_l = 1000$ kg/m³, initial porosity $n_0 = 0.45$, and permeability $k = 1.0 \times 10^{-10}$ m/s.

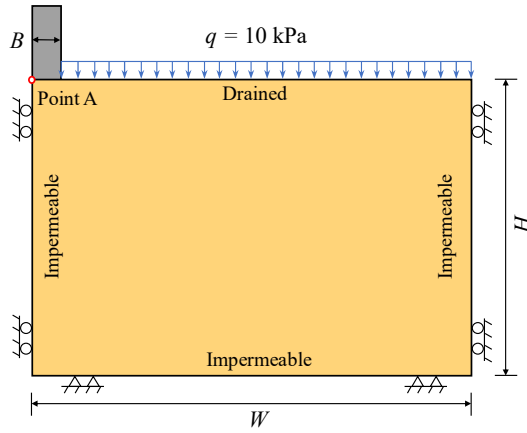


Fig. 15. Sketch of the geometry and boundary conditions for the strip footing model.

6.1.1. Small deformation analysis

The small deformation analysis of footing penetration involved displacing the rigid strip vertically to a maximum penetration depth = 0.05 m at a velocity of 0.05 m/s. To be consistent with the simulation by Monforte et al. (2023), an interface friction coefficient of $\mu = 1.0$ was adopted to match approximately the full adherence condition between the footing and soil.

Fig. 16 illustrates the evolution of contact force and pore pressure at the base of the footing (Point A in Fig. 15) during penetration, which approximately matches the results obtained by Monforte et al. (2023) using node-based smoothed finite element with variable element types and sizes. The pore pressure contour at the end of penetration from MPM results is depicted in Fig. 17 along with the previous FEM results, showing good consistency between the two numerical results. The displacement field, deviatoric strain and effective stress fields at the end of the simulation are plotted in Fig. 18, which nicely match the reported FEM results.

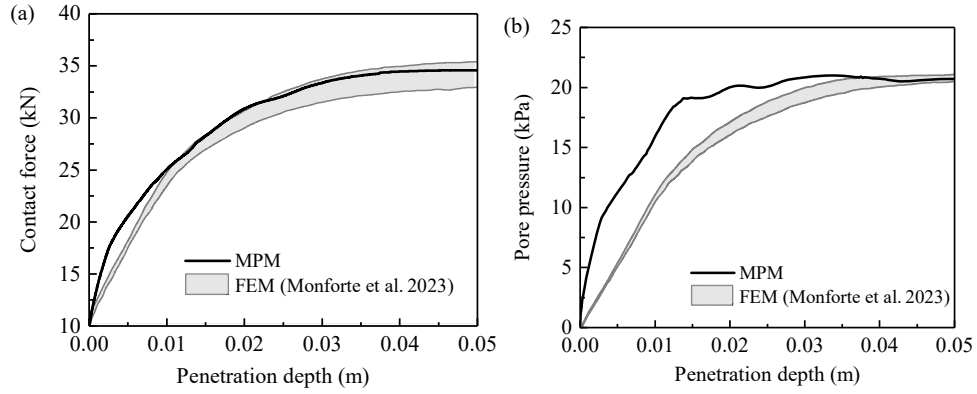


Fig. 16. Evolution of (a) contact force and (b) pore pressure at point A during penetration.

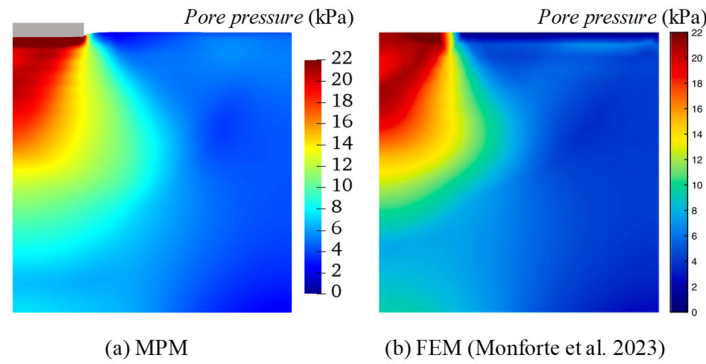


Fig. 17. Pore pressure at the end of penetration obtained from (a) the proposed MPM and (b) FEM results by Monforte et al. (2023).

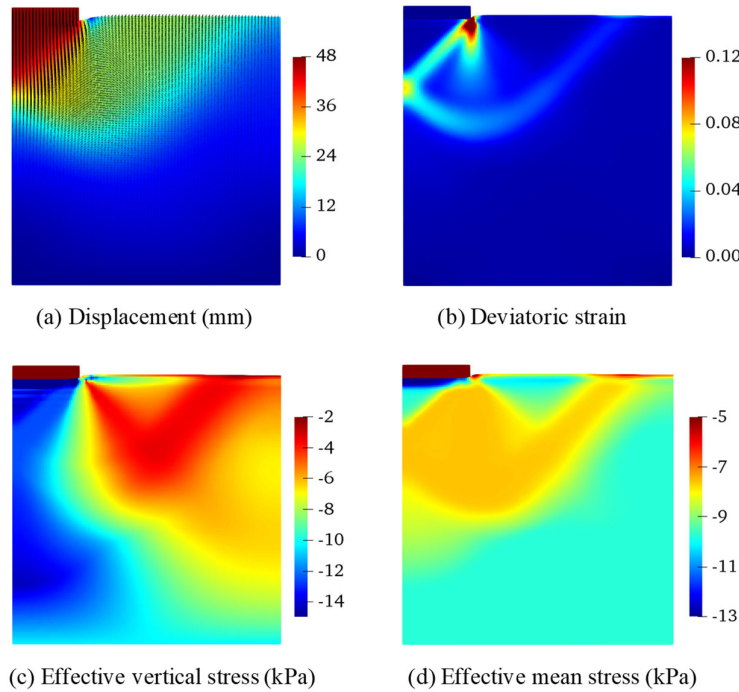


Fig. 18. Results at the end of small deformation analysis: (a) displacement, (b) deviatoric strain, (c-d) effective vertical and mean stress.

6.1.2. Large deformation analysis

The footing problem was further explored by advancing to greater penetration depths to assess the numerical stabilization and accuracy in simulating soil-structure interactions subjected to extreme deformations. The model settings were as in the small deformation analysis, except for the extended computational domain. Additionally, for computational efficiency, the penetration velocity was doubled to 0.10 m/s as the final penetration depth extended to 2 m. Parametric analyses were performed to explore the impact of the interfacial friction coefficient μ on the contact force. Fig. 19 depicts the load-displacement curves obtained from the coupled B-bar GIMP simulations with different interfacial friction coefficients. While the small deformation analysis stabilized to a finite limit, the large deformation resistances increased steadily at an almost constant rate with penetration depth and appear insensitive to μ . Furthermore, a comparative analysis of the contact algorithm with and without penalty function modification for $\mu = 0.1$ case demonstrates its effectiveness in suppressing contact oscillations caused by premature contact during large deformation process.

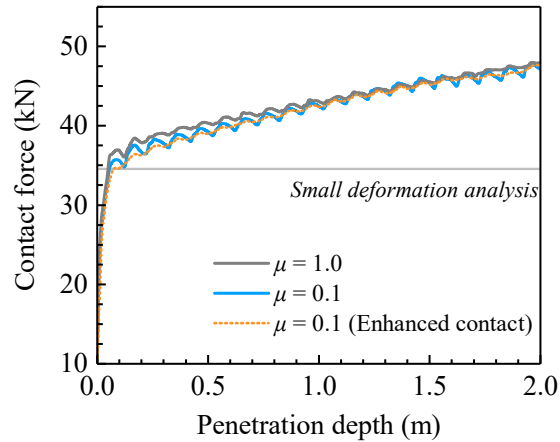


Fig. 19. Load-displacement response for the large penetration analysis.

Fig. 20 illustrates the numerical results of the rigid footing at the end of penetration, displaying smooth and continuous distributions of the displacement and deviatoric strain contours. The pore pressure and effective vertical stress field show comparable patterns to those found in the small deformation analysis, which further verifies the stability and accuracy of the proposed method for hydro-mechanical coupled soil-structure interaction problems involving large deformations.

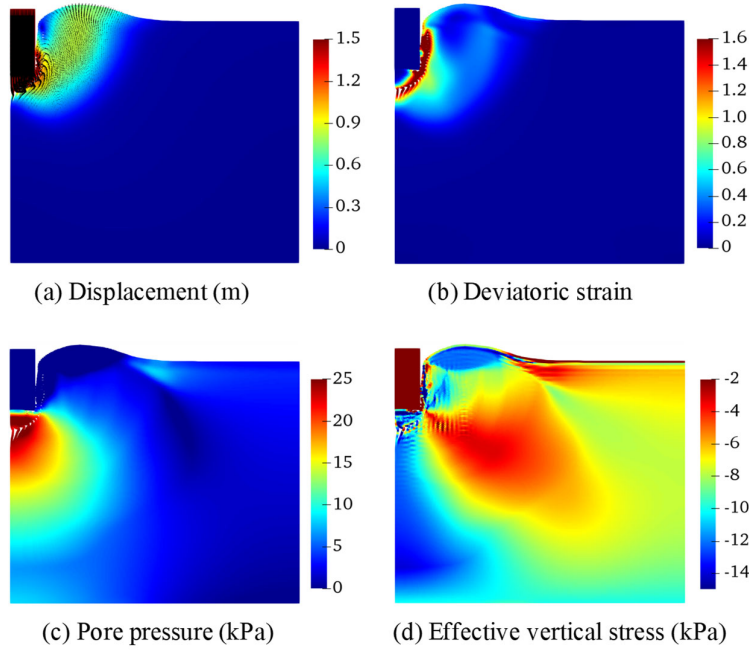


Fig. 20. Simulations results of strip footing under large penetration at the final time.

6.2. Rapid installation of pile

The proposed MPM in conjunction with axisymmetric formulation is adopted to investigate the rapid installation process of a rigid displacement pile into saturated soil. The model configurations align with those of the ALE-based numerical model developed by Sabetamal et al. (2014) for comparison purposes. The geometry and boundary conditions are depicted in Fig. 21. The pile with a diameter $D = 0.4$ m and 60° apex angle was advanced monotonically over 1 s into the saturated soil to a prescribed depth of $8.75D$. To avoid additional numerical fluctuations, the pile tip was initially embedded into the soil, and the penetration velocity increased linearly from 0 to $8.75D/s$ over the initial 0.1 s. Only the top ground surface was allowed to drain, while other boundaries were kept impermeable. A constant surcharge of $q = 70$ kPa was applied at the ground surface and gravity is considered for both phases. Initially, the geostatic stress and pore pressure are allocated to the material points to ensure stress equilibrium. The undisturbed initial earth pressure coefficient K_0 was assumed to be 1.0. The computational domain is discretized using structured rectangular grids, with 4 material points per element and a grid size of $L = D/10$. The mechanical behavior of soils is modelled by the classical MCC model with parameters: $M = 0.984$, $\kappa = 0.05$, $\lambda = 0.29$, $\nu = 0.3$, $\rho_s = 3800$ kg/m³, $\rho_l = 1000$ kg/m³, $n_0 = 0.643$, $k = 1.0 \times 10^{-8}$ m/s, and an overconsolidation ratio $OCR = 2.0$, which leads to p_{c0} increasing linearly with depth. The contact friction between the pile and soil is neglected by setting the interfacial friction coefficient to zero (i.e., $\mu = 0.0$).

Given the lack of shaft resistance, the pile tip resistance q_t (defined as tip load divided by

base plan area) is equivalent to the resistance expected from a ‘super-large’ 0.4 m diameter CPT probe and makes up the total load that applied at the pile head to advance the pile to depth. Fig. 22 shows the soil resistance q_t versus the dimensionless penetration depth L_p/D , where L_p refers to the distance from the pile tip to the ground surface, which closely resemble those reported by Sabetamal et al. (2014). The linear increase of initial stress q_t reflects the gains in soil effective stresses, p_{c0} and undrained shear strength with depth.

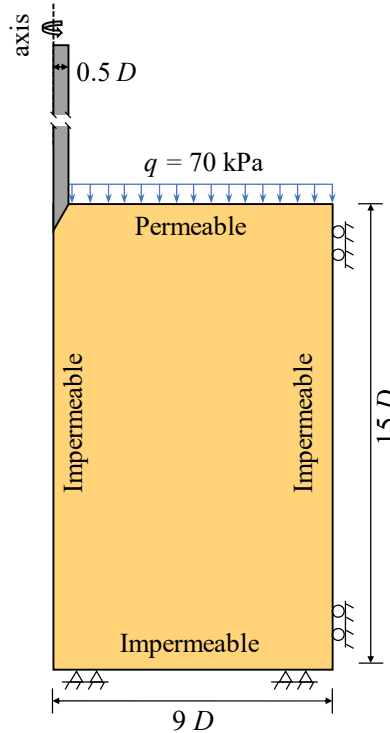


Fig. 21. Sketch of the geometry and boundary conditions for the pile installation analysis.

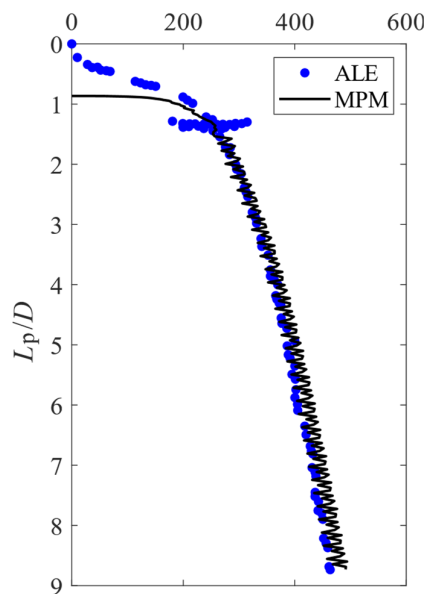


Fig. 22. Soil penetration resistance predicted by ALE (Sabetamal et al. 2014) and MPM.

Fig. 23 illustrates the evolutions of excess pore pressure, radial displacement, total radial stress and effective radial stress at penetration depths of $3.50D$ and $8.75D$. As can be seen, the maximum excess pore pressures are generated in the vicinity of the pile tip, and excess pore pressures accumulate along the shaft as the pile penetrates. The mobilized radial displacement decreases with increasing distance from the pile shaft, which exhibits a spindle-shaped pattern similar to that observed in sands (Gao et al., 2022). Additionally, the total radial stress shows a distribution pattern similar to the excess pore pressure, which accounts for a major proportion of the total stress due to the approximately undrained installation conditions. The effective radial stresses show marked concentrations near the edge of the pile tip.

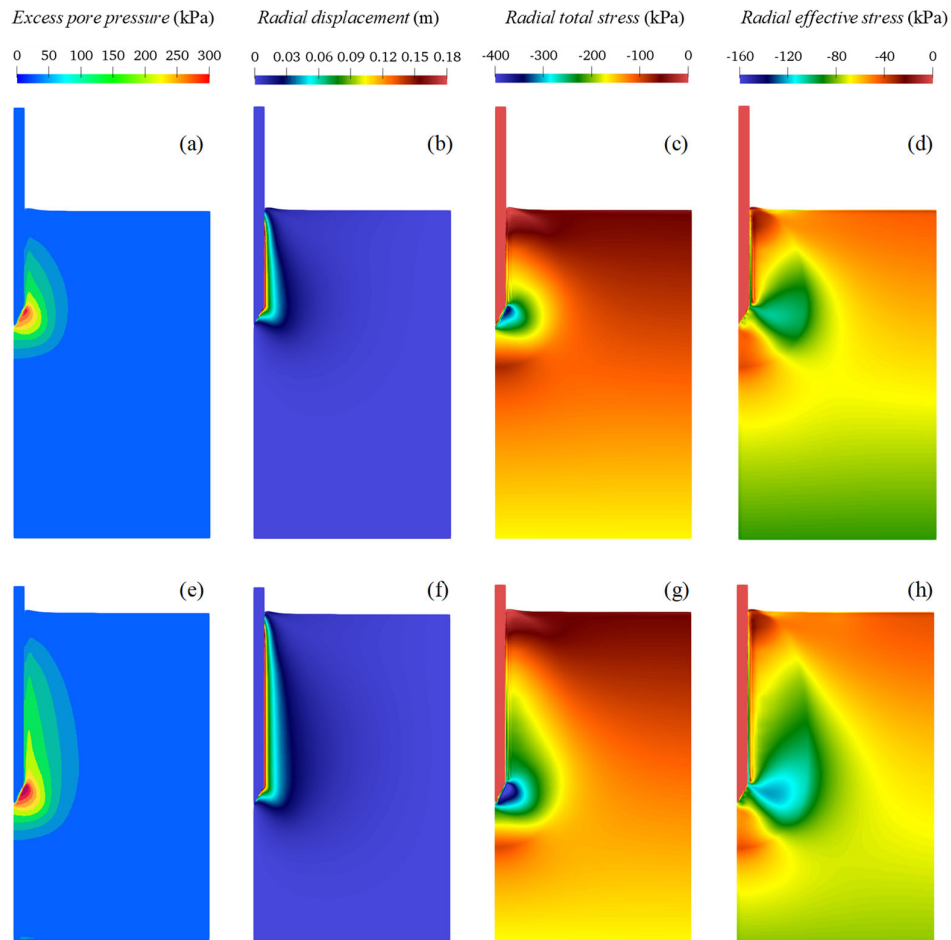


Fig. 23. Distribution of excess pore pressure, radial displacement, total radial stress and effective radial stress for pile driving depth of (a-d) $3.50D$ and (e-h) $8.75D$.

Fig. 24 depicts the excess pore pressure evolution during pile installation at various radial locations, considering soil depths of $2.5D$ and $5.0D$. The zone of greatest pore pressure change extends to a radial range of $3D$ from the pile shaft. Local excess pore pressures undergo rapid increases as the pile tip penetrates into each soil level, but decline sharply as the pile tip extends to greater depth and tend towards steady-state values by the end of the very rapid installation process. The abrupt decreases in excess pore pressure are most pronounced within a radial

distance of $1D$ from the pile shaft, while the more radially distant excess pore pressures tend towards steadier distributions.

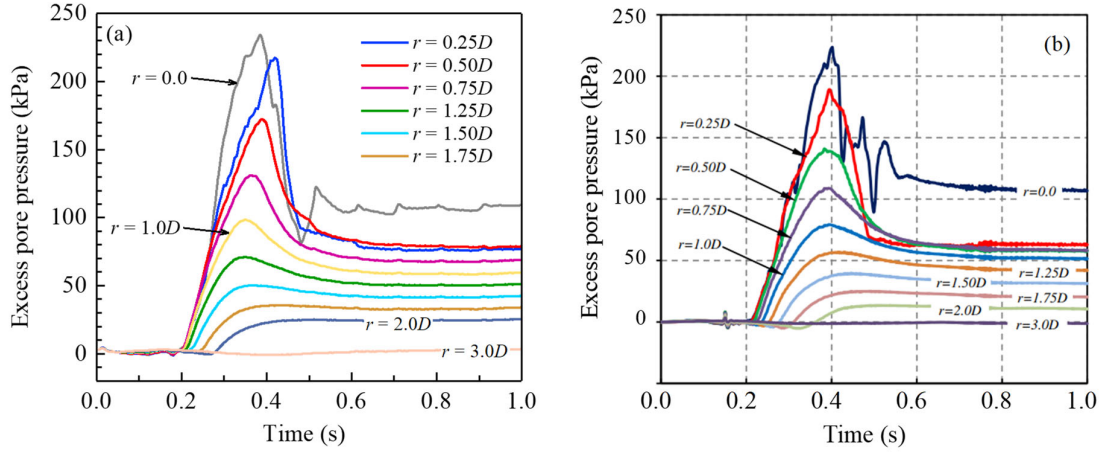


Fig. 24. Evolution of excess pore pressure at different radial distances for depths of $2.5D$ obtained from (a) the proposed MPM and (b) ALE results by et al. (2014).

7. Summary and conclusions

A stabilized semi-implicit MPM modelling framework has been developed for 2D and axisymmetric conditions to simulate the hydro-mechanical coupled large-deformation problems involving soil–structure interaction. The main conclusions of this study are summarized as follows:

1. A stabilized semi-implicit MPM framework based on incremental fractional step method is proposed by integrating a penalty-based frictional contact algorithm, artificial fluid compressibility, the APIC scheme, and B-bar stabilization techniques. These combined strategies effectively suppress nonphysical pore pressure oscillations, alleviate volumetric locking, and reduce checkerboard stress patterns, thereby enhancing numerical stability and accuracy.

2. The proposed framework is systematically validated through a series of benchmark problems of increasing complexity, ranging from one-dimensional consolidation to pile installation simulations involving advanced soil constitutive models. The numerical results show good agreement with analytical solutions and previously published high-fidelity numerical results, demonstrating the robustness and accuracy of the method.

3. The semi-implicit MPM framework provides a reliable and stable numerical tool that is suitable for a broad class of hydro-mechanically coupled large-deformation problems. The 2D axisymmetric formulations offers an efficient alternative for simulating certain practical geotechnical prototype problems, such as displacement pile installation and cone penetration testing.

4. Although the proposed framework is computationally more efficient than its fully explicit counterpart, full 3D simulations remain challenging due to the high computation cost.

This limitation could potentially be mitigated by the incorporation of GPU parallelization techniques, which constitutes a future extension of the present work.

Data availability statement

Some or all data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

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