

TRANSVERSE FLUX FOUR-LIMBED LINEAR
INDUCTION MOTORS

By

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ABSTRACT

This thesis is concerned with the numerical analysis of the electromagnetic field of the four-limbed transverse flux plate levitator. Particular attention is paid to the characteristics of the lateral stability. The analysis is limited to the two-dimensional case only, the potential function employed being the magnetic vector potential.

Initially the results of experimental investigations are presented and the similar nature of the lateral stability of a single-phase and a three-phase machine established. By using simplified lumped circuit models, expressions are derived for the levitation force of two classes of axially symmetric plate levitator. Although the lumped circuit method is based on many assumptions, the results obtained give a good indication of the performance characteristics. In particular the method indicates that a circle diagram can be useful.

In the remainder of the thesis, rigorous numerical analysis is undertaken using two different methods. In the first, which is the finite difference method, a transformation technique is used to represent the open boundary. Agreement between the computed and experimental results is given wherever possible. The second method of analysis used is the finite element method. It is shown that this second method is computationally cheaper, yielding more accurate results. Finally it is shown that removing the outermost iron limbs of the machine can improve its performance.

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TABLE OF CONTENTS

	<u>Page</u>
Title	1
Abstract	2
Acknowledgements	3
Table of Contents	4
 <u>Chapter 1:</u>	
Introduction	10
1.1 Historical Background of Linear Induction Motors	10
1.2 Historical Background of Transverse Flux Linear Induction Motors	10
1.3 Available Analysis of T.F.M.	14
1.4 Methods Available Employing Algebraic Analysis	15
1.5 Field Solution by Numerical Methods	15
1.6 About this Thesis	16
1.7 Contribution to Existing Knowledge of T.F.M.	17
 <u>Chapter 2:</u>	
An Experimental Study of Xi-Core	20
2.1 Introduction	20
2.2 Experiments with Levitated Plates	22
2.3 Low Power Experiments	27
2.3.1 Normal Force Variation with Airgap	27
2.3.2 Input Impedance Measurements	30
2.4 Measurement of Goodness Factor	31
2.4.1 Effect of Plate Width on G	32
2.4.2 Effect of Frequency on G	32
2.4.3 Effect of Airgap Value on G	32
2.4.4 Effect of Plate Thickness on G	33
2.4.5 Goodness Factor and G	33
2.5 Lateral Force Measurements	37
2.5.1 The Significance of the Cross-over Point	43
2.5.2 General Form of the Lateral Force Variation	45
2.6 Lateral Stability and Stored Energy	46
2.7 A Method of increasing Lateral Stability	50
2.8 Experiments on Three-Phase Xi-Core	50
2.9 Conclusions	51

	<u>Page</u>
<u>Chapter 3:</u> Simple Models using Coils	54
3.1 Introduction	54
3.2 Review of Published Work	54
3.3 Extension of Theory for Two Coils	56
3.3.1 The Model and the Assumptions	56
3.3.2 The Parameters at the Input Terminals	58
3.3.3 Force under All Conditions	59
3.3.4 Stability of Two-Coil Model	59
3.4 Verification of the Force Expression	61
3.5 A Lumped Circuit Model of the Xi-Core	63
3.5.1 The Parameters at the Input Terminals	65
3.5.2 Levitation Force of the Xi-Core	67
3.6 Verification of the Force Expression	68
3.7 Conclusions	68
 <u>Chapter 4:</u> Theory of Finite Difference Analysis	 71
4.1 Introduction	71
4.2 The Electromagnetic Field Equations	72
4.3 Problem Description by Vector Potential	72
4.3.1 Significance of the Grad ϕ Component	75
4.3.2 Field Equations in Terms of A	75
4.4 Finite Difference Equations from Ampere's Law	76
4.4.1 Calculation of Admittance Conductance Values	79
4.4.2 Evaluation of Datum Node Potential	83
4.5 Method of Representing an Open Boundary	83
4.5.1 The Mathematical Transformation	84
4.5.2 Network Interpretation of the Transformation	84
4.6 Solution of the Assembled Network	88
4.6.1 The S.O.R. Method in Network Terms	89
4.6.2 Influence of Material Properties on Convergence	92
4.7 Fine Mesh Expansion	95

	<u>Page</u>
4.8	Methods of Calculating Forces and Impedance 95
4.8.1	Calculation of Flux Densities 96
4.8.2	Force from Maxwell Stress Method 96
4.8.3	Force from $J \times B$ Method 99
4.8.4	Calculation of Input Impedance 99
4.9	Conclusions 100
<u>Chapter 5:</u>	Results of Finite Difference Analysis 101
5.1	Introduction 101
5.1.1	Verification of Picture Frame Transformation 101
5.1.2	Material Properties and Convergence Time 104
5.2	The Xi-Core on Open Circuit 106
5.2.1	Computed and Experimental Results 106
5.2.2	The Core Flux Distributions 109
5.2.3	The Nature of the Airgap Field 112
5.3	The Xi-Core with Symmetrical Plates 113
5.3.1	Computed and Experimental Results 113
5.3.2	Space Distribution of Flux Densities 117
5.3.3	Space Distribution of Induced Currents 122
5.3.4	Computed Stress Distributions 124
5.4	The Xi-Core with Asymmetrical Plates 128
5.4.1	Computed and Experimental Results 128
5.4.2	Space Distribution of Flux Densities 128
5.4.3	Space Distribution of Induced Current Densities 131
5.4.4	Stress Distributions 135
5.5	Factors Determining Lateral Stability 141
5.5.1	Effect of Frequency on Lateral Stability 141
5.5.2	Effect of Plate Width on Lateral Stability 143
5.5.3	Effect of Airgap on Lateral Stability 149
5.5.4	Effect of Stored Energy on Stability 149
5.5.5	Lateral Force Stiffness 154
5.6	Conclusions 156

	<u>Page</u>
<u>Chapter 6:</u> Theory of Finite Element Analysis	159
6.1 Introduction	159
6.2 The Euler-Lagrange Equation	159
6.2.1 The Functional of a given P.D.E.	161
6.2.1.1 Magnetostatic Problems	161
6.2.1.2 Eddy Current Problems	163
6.3 Approximations Needed for Functional Minimisation	165
6.3.1 Space Discretisation	165
6.3.2 Local Space Variation	166
6.3.2.1 Local Space Variation in Cartesian Co-ordinates	166
6.3.2.2 Local Space Variation in Triangle Area Co-ordinates	168
6.4 Direct Minimisation of Functionals	169
6.4.1 Functional Minimisation for One Element	170
6.4.1.1 Network Analogue of an Element	172
6.4.2 Functional Minimisation of the Full Model	173
6.4.2.1 Elimination of the Datum Node	175
6.4.2.2 The Sparsity Structure of the S-Matrix	176
6.4.2.3 Problems Posed by the Open Boundary	177
6.5 The Theory of the Infinite Boundary	179
6.5.1 Division into Interior and Exterior Regions	181
6.5.2 Subdivision of Exterior Region	181
6.5.3 The S-Matrix for any Exterior Region	183
6.5.4 Succession Combination of Exterior Region	186
6.5.5 Generating the $[S_{11}]$ Matrix by Recursion	191
6.5.6 Attaching the $[S_{11}]$ Matrix to the Interior S-Matrix	192
6.6 Automatic Generation of Infinite Boundary	194
6.6.1 Location of Star-Point	195
6.6.2 Determination of Mapping Ratio	195
6.6.3 Re-numbering the Inner Boundary Nodes	197

	<u>Page</u>
6.6.4	Generation of Exterior Nodes 197
6.6.5	Formation of Exterior Elements 198
6.6.6	Obtaining the $[S_{11}]$ Matrix after Recursion 198
6.7	Sparse Matrix Solution 199
6.7.1	Compact Storage of S-Matrix 199
6.7.1.1	Storage in General Case 200
6.7.1.2	Retrieval in the General Case 201
6.7.2	Solution of Compact Stored Equations 201
6.7.2.1	An Example of Modified Gaussian Elimination 201
6.7.2.2	General Case of Modified Gaussian Elimination 205
6.7.2.3	Back Substitution 207
6.8	Calculation of Flux Densities, Forces and Input Impedance 208
6.8.1	Flux Densities 208
6.8.2	Forces 209
6.8.3	Input Impedance 212
<u>Chapter 7:</u>	Results of Finite Element Analysis 214
7.1	Introduction 214
7.1.1	Method of Setting Up the Model 215
7.1.2	Verification of Infinite Boundary 217
7.2	Comparison of Computed and Experimental Results 217
7.2.1	Xi-Core with Symmetrical Plates 217
7.2.2	Xi-Core with Asymmetrical Plates 223
7.3	Stress Distributions 230
7.3.1	Symmetrical Plates 230
7.3.2	Asymmetrical Plates 232
7.4	Lateral Force Stiffness 234
7.5	Rolling Stability 236
7.6	Effect of Changes in Xi-Core Geometry 238
7.6.1	A Change in Rotor Geometry 238
7.6.2	A Change in Stator Geometry 244
7.7	Conclusions 249

	<u>Page</u>
<u>Chapter 8:</u> Discussion, Further Work and Conclusions	252
8.1 The Performance of the Xi-Core	252
8.1.1 The Effect of the Goodness Factor	252
8.1.2 The Merits of the Finite Difference Method	254
8.1.3 The Merits of the Finite Element Method	254
8.1.4 The Stator Geometry of the Single-Phase Xi-Core	255
8.2 Further Work	256
8.3 Conclusions	257
Appendix A	258
Appendix B	261
Appendix C	270
References	275

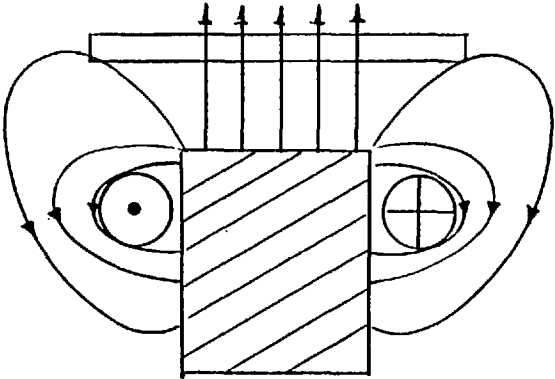
INTRODUCTION1.1 HISTORICAL BACKGROUND OF LINEAR
INDUCTION MOTORS

The idea of obtaining direct linear motion, using the electromagnetic energy conversion principles of the rotary induction machine, originated at the turn of the century. However, the idea remained largely dormant, until about 1946 when Westinghouse^{1.1} made a full scale machine to launch aircraft from ships. Shortly after this, E.R. Laithwaite (now Professor of Heavy Electrical Engineering at Imperial College, London) began fundamental research into linear induction motors. He soon established that the design principles required for the linear induction motor are radically different from those pertaining to the rotary induction motor. It is the ignorance of this fact that prevented his predecessors from constructing efficient devices. Since 1947 up to the present time he has been the leading research worker in this field and is largely responsible for the current worldwide interest in high-speed ground transportation schemes, in most of which linear induction motors provide the thrust. Among the many papers and books published by him on linear induction machines, those relevant to this thesis are quoted at appropriate places.

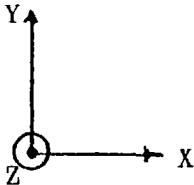
1.2 HISTORICAL BACKGROUND OF TRANSVERSE
FLUX LINEAR INDUCTION MOTORS

In the conventional linear induction motor with long pole pitch the increasing leakage flux of the long magnetic circuits remains a major problem. A long pole pitch, however, is a vital requirement for high speed machines operating on a 50 Hz supply.

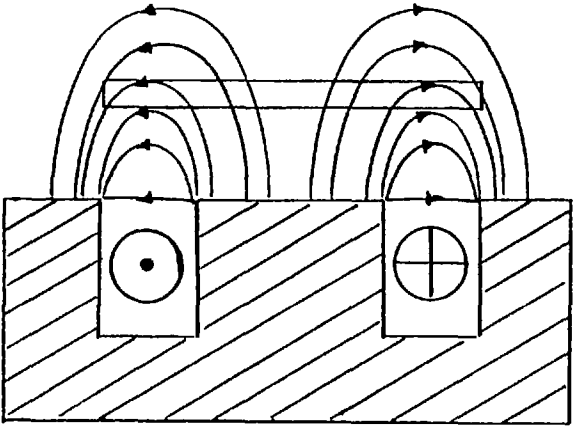
Major proportion of the flux going in the Z-direction to a distance dependent on pole pitch.



A.F.M.



Major proportion of the flux avoiding the Z-direction irrespective of pole pitch.



T.F.M.

Figure 1.1: The Evolution of the first T.F.M.

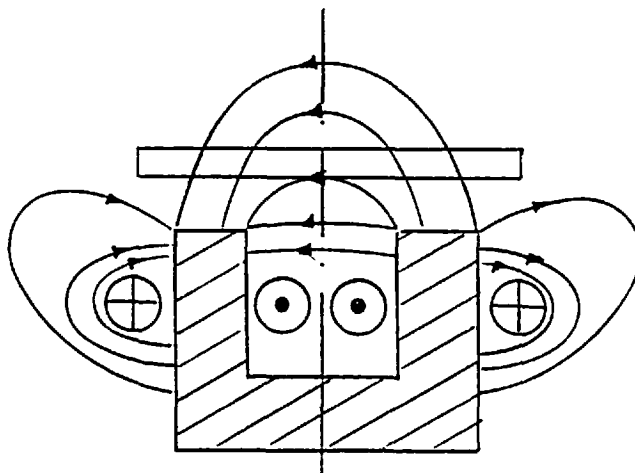


Figure 1.2: The second T.F.M. (the U-core).

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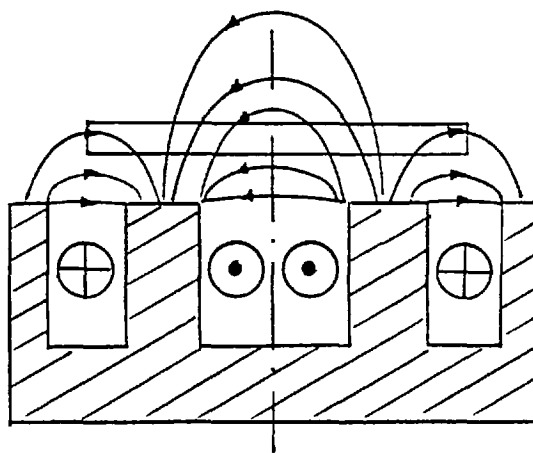


Figure 1.3: The third T.F.M. (the Xi-core).

While investigating this problem, linear induction motors with transverse magnetic circuits were discovered and reported in 1971 in a paper by Laithwaite, Eastham and co-workers^{1.2}. The authors reported the discovery of the first transverse flux machine (T.F.M.) which was the result of adding two extra iron limbs to enclose the overhang of a conventional linear axial flux machine (A.F.M.). This is illustrated in Fig. 1.1.

Thus it is seen that the first T.F.M. geometry was obtained by enhancing the leakage flux of the A.F.M. and consequently changing the magnetic circuit to the transverse section. The major advantage of this new magnetic circuit, as reported by the inventors, was that machines with long pole pitches could be built requiring less iron. The authors did not consider the levitation force or the lateral stability of this first T.F.M., which they named the "E-core" machine.

The first T.F.M. which provides thrust, levitation and lateral stability was introduced in 1974 by two of the inventors of the first T.F.M., Eastham and Laithwaite^{1.5}. This second T.F.M. was named the "C-core" by the inventors, but is also known as the U-core. The latter name is used throughout this thesis. The characteristic feature of the U-core shown in Fig. 1.2 is that the secondary plates could be levitated, guided and propelled along the length of the machine, in much the same way as a piece of wood floating down a river. Therefore this machine was described as an electromagnetic river or "Mag-river" for short.

The evolution of the two T.F.M.s as described by the inventors resulted from a purely experimental approach aided by past experience with A.F.M. designs. Using the same approach a third

T.F.M. geometry was designed (at Imperial College) by adding extra iron limbs to the U-core machine to enclose its overhang. This machine is called the Xi-core (Fig. 1.3) and until recently, relatively little has been published on it.

The Xi-core, too, has been found experimentally to guide, levitate and propel secondary plates. Thus, to-date three T.F.M.s have been developed:

1. The E-core,
2. The U-core,
3. The Xi-core.

Of these, only the last two belong to the Mag-River category.

1.3 AVAILABLE ANALYSIS OF T.F.M.s

The first published analysis of a T.F.M. is that due to Wyatt^{1.4}, who analysed the E-core machine numerically, using a $T-\Omega$ formulation for its field. His analysis showed the inherent instability of a single-phase E-core. Closely following this, Greatorex^{1.5} published the results of research carried out on the E-core and the Xi-core. He adopted an analytical approach using $T-\Omega$ formulation on simplified models of the two machines. The two models are capable of approximately representing the two machines with symmetrically placed rotor plates. The lateral stability is therefore investigated only experimentally. Besides these two, there is little published analysis available, chiefly due to the fact that T.F.M.s have come into being only during the last few years.

1.4 METHODS AVAILABLE EMPLOYING ALGEBRAIC ANALYSIS

There are many analytical methods available for the analysis of A.F.M.s. A complete bibliography is given by Nasar^{1.6}. To adopt one or more of these for the analysis of T.F.M.s further work needs to be carried out. If entry and exit effects can be neglected, then the two-dimensional theory as published by Freeman and Lowther^{1.7}, among others, can be used as a first approximation. As a next step, the Freeman and Papageorgiou work^{1.8,1.9} could perhaps prove useful.

1.5 FIELD SOLUTION BY NUMERICAL METHODS

The finite difference numerical method of solving electromagnetic field problems was originally developed for solution by hand. Solution of eddy current problems in this manner was pioneered by Sen and Adkins^{1.10}. The first successful exploitation of the digital computer for the solution of the eddy currents in a conductor in an iron slot, using finite difference method, was reported by Mamak and Laithwaite^{1.11}. Subsequent developments of the finite difference method have been in the two areas of the formulation of the problem and the methods of solving the resulting simultaneous equations. In the former area, a significant development worthy of mention is the network approach due to Carpenter^{1.12}. In this method the field problem is translated into a problem of solving for the node potentials of an admittance network whose topology corresponds to that of the actual machine. In the area of solution of the simultaneous equations the major development is the successive-over-relaxation method, and this is the method used almost universally now, in finite difference analysis.

A comparatively recent numerical method is the finite element technique, first introduced by Zienkiewicz and Cheung^{1.13} for the analysis of structural problems in Civil Engineering but subsequently extended to many other branches of science. This method differs from the finite difference, in that while the latter describes the field in differential form, the former expresses it in the integral form, thus eliminating the boundary conditions at material interfaces. Silvester and Chari^{1.14} have carried out extensive finite element solution of the fields of rotating machines, including those that are magnetically saturated. Chari^{1.15} has reported the first successful application of the finite element to the solution of eddy current problems. Closely following this, Carpenter^{1.16} presented a network interpretation of the finite element method when applied to eddy current problems. The next significant progress made in the finite element analysis, which is relevant to this thesis, is the introduction of a "ballooning" technique to deal with open boundaries, by Silvester, Lowther, Carpenter and Wyatt^{1.17}.

1.6 ABOUT THIS THESIS

The research undertaken in this thesis is mainly centred on the Xi-core T.F.M., with particular reference to its lateral stability. Experimental investigations carried out on the single-phase machine and a three-phase version of it are described first. Although the thrust obtained in the three-phase machine is absent in the single-phase machine, the lateral stability characteristics of the two machines are shown to be similar. All subsequent analysis is restricted to the single-phase machine only. A simplified analytical method is presented using a lumped circuit model to

represent axially symmetric E-core and the Xi-core machines. The results obtained by this simplified model are shown to be in agreement with the observed behaviour. This suggested the applicability of the circle diagram by Freeman and Lowther^{1.18} to T.F.M.s as well. Therefore the impedance loci with frequency variation was examined for both the single-phase and three-phase machines, and the results are included in an appendix. In the remainder of the thesis, the theory and results of the numerical analysis, using both the finite difference and finite element techniques are given. The transformation technique used to represent the open boundary is originally due to Dr. C.J. Carpenter (at Imperial College), but was translated into a numerical algorithm by Dr. A. Wyatt, who made it available for this research. The conclusions reached on the performance of the Xi-core, in the light of knowledge gained by the analysis, are presented.

1.7 CONTRIBUTION TO EXISTING KNOWLEDGE OF T.F.M.

The salient features of this thesis, which the author considers as original contributions to the existing knowledge of T.F.M.s, are:

1. The results presented in Section 2.5.2 which show that for a given Xi-core, the lateral force/lateral displacement curves can have only one of three forms. Of these, that corresponding to stable plates has an approximately sinusoidal variation given by:

$$F_x = F_{x_{\max}} \sin\left(\frac{2\pi}{0.55} \cdot \frac{x}{L}\right)$$

The transition from one form to another occurs with changes in the Goodness Factor.

2. The results presented in Section 3.3.2, that a necessary, but not sufficient, condition for the E-core rotors to be stable is Goodness Factor > 1 .
3. The force expressions derived in Chapter 3, which enable the calculation of the levitation force of the E-core, and Xi-core, approximately, from a knowledge of the rate of change of input impedance.
4. The normal force distributions of Section 5.3.4, which explains the inherent rolling instability of the Xi-core rotors, with decreasing plate widths.
5. The new rotor geometry analysed in Section 7.6.1, which shows a method of increasing the lateral stability of wide rotor plates of the Xi-core.
6. Section 7.6.2, which shows that the Xi-core is inferior to the U-core in most respects.

Other features to be found in this thesis, which also form a contribution to knowledge, are:

1. The derivation of finite difference equations directly by the application of Ampere's law, without invoking the duality principle.
2. The use of a fine mesh expansion technique to obtain more accurate finite difference solutions in the region of interest, from the complete coarse mesh solution.
3. The description of the theory of the "ballooning" technique, in terms of admittance networks.

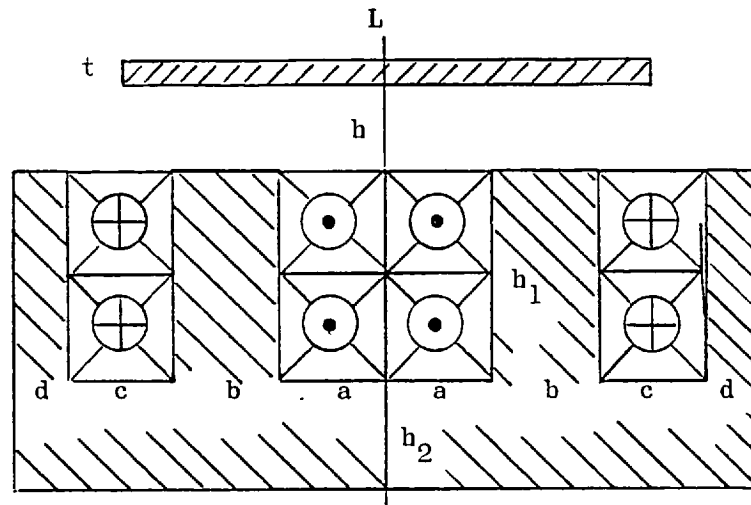
4. The development of the automatic generation of the infinite boundary, and showing its practical applications.

AN EXPERIMENTAL STUDY OF Xi-CORE2.1 INTRODUCTION

The experimental investigation carried out on the Xi-core is presented in this chapter. The constructional details of the machine are given in Fig. 2.1. All the experimental work described refers to the Xi-core with airbacked rotor plates, since the investigation is confined to rotor configurations which provide levitation and lateral stability. In the opening sections of the chapter, the results of experiments carried out while the secondary plates were levitating, are described. These establish the magnitude of the various quantities (kVA/kg, lift, etc.) which are the performance indices of the machine. It is shown that as a first approximation, the levitation force may be assumed proportional to the power dissipated in the rotor.

Subsequent sections describe the experiments conducted at low power input levels, without levitating the plates. A dimensionless ratio derived from the input impedance is shown to possess all the characteristics of the Goodness Factor, and is subsequently treated as the Goodness Factor. The lateral force measurements made are shown to suggest that increasing the Goodness of the machine is to bring about lateral instability. Despite several efforts, a common limiting value of the Goodness Factor, above which the rotor plates become unstable, could not be found.

It is shown empirically that all plates that are stable have a lateral force (F_x)/lateral displacement (x) characteristic which is approximately given by:



$$a = b = c = h_2 = 0.025 \text{ m.}$$

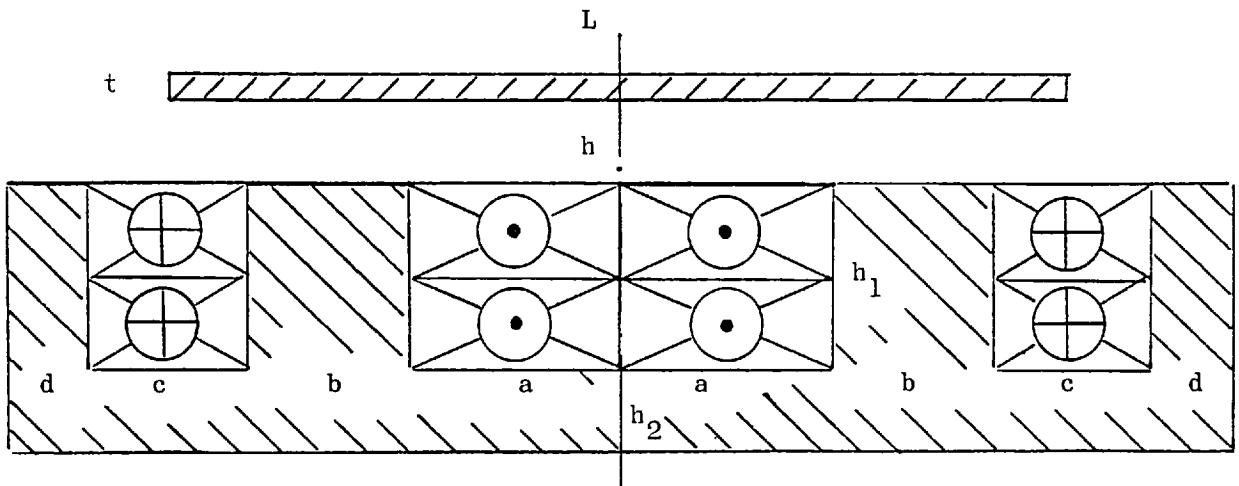
$$h_1 = 0.050 \text{ m.}$$

$$d = 0.0127 \text{ m.}$$

Stator coils 16 turns each.

Length of stator 0.457 m.

Figure 2.1: The single-phase Xi-core.



$$a = b = c = 0.038 \text{ m.} ; d = h_2 = 0.019 \text{ m.} ; h_1 = 0.044 \text{ m.}$$

Stator coils 19 turns each; Pole pitch 0.148 m.

Distribution in 5 slots/pole/phase.

Length of stator 0.595 m.

Figure 2.2: The three-phase Xi-core.

$$F_x = F_{x,\max} \sin\left(\frac{2\pi}{0.35} \cdot \frac{x}{L}\right)$$

A method of stabilising a marginally stable plate is described briefly. This will be discussed in more detail in a later chapter theoretically. By evaluating the stored energy in the field, it is shown that the stable plates are those for which the symmetrical position is one of maximum stored energy, as would be expected.

The Xi-core can be made to provide thrust by using a three-phase stator winding. In such a three-phase machine, the transverse (Fig. 2.1) component of the excitation current, that would result, will provide additional lift force, but not lateral force. Therefore it may be possible to apply the results of the study of lateral stability of a single-phase Xi-core, to a three-phase machine as well.

In order to confirm this view, the lateral force/lateral displacement characteristics of a three-phase Xi-core were obtained experimentally. The constructional details of this machine are given in Fig. 2.2. The results of these experiments are included to provide quantitative confirmation of the foregoing argument.

2.2 EXPERIMENTS WITH LEVITATED PLATES

Several plates of widths ranging from 102 mm to 155 mm (i.e. from $2(a+b)$ to $2(a+b+c)$ in Fig. 2.1) were levitated at various heights by altering the supply voltage to obtain the required currents. The variation of the levitation height with input current is shown in Fig. 2.3 for these plate widths. The power losses in the secondary plates were measured directly by making use of the shadow coils in the machine. Since the experiment was conducted with the

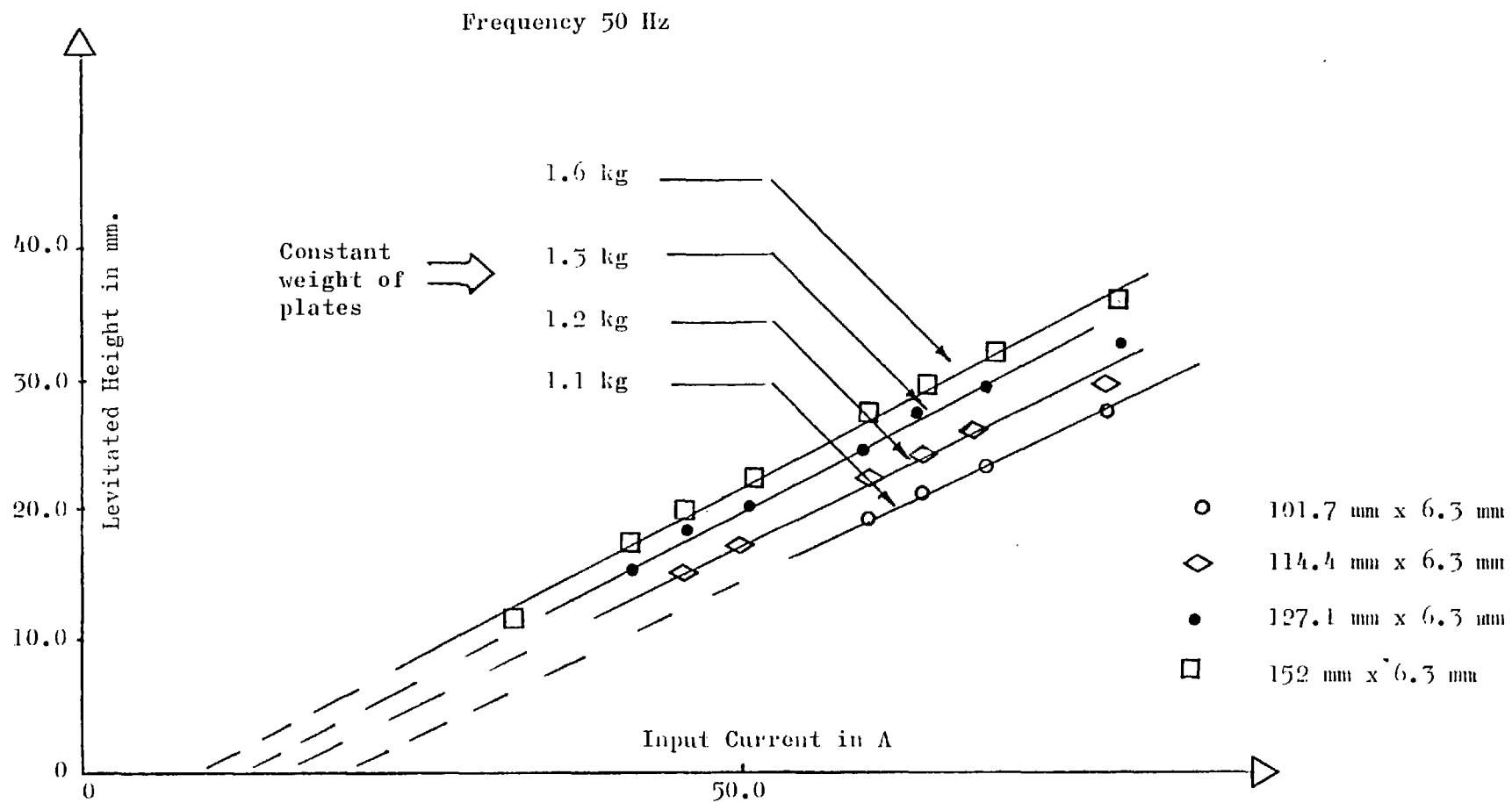


Figure 2.3: Levitation heights of different rotors at 50 Hz.

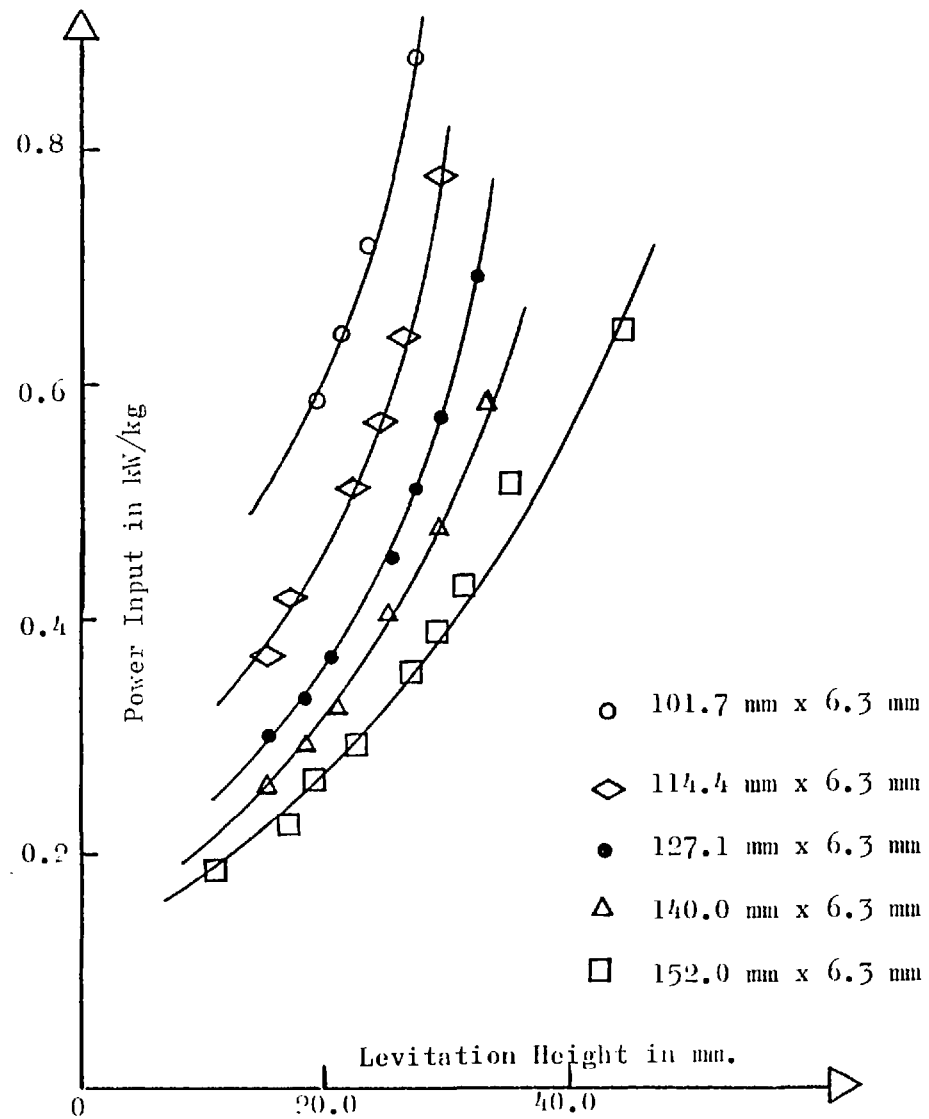


Figure 2.5: Power input required for different plates.

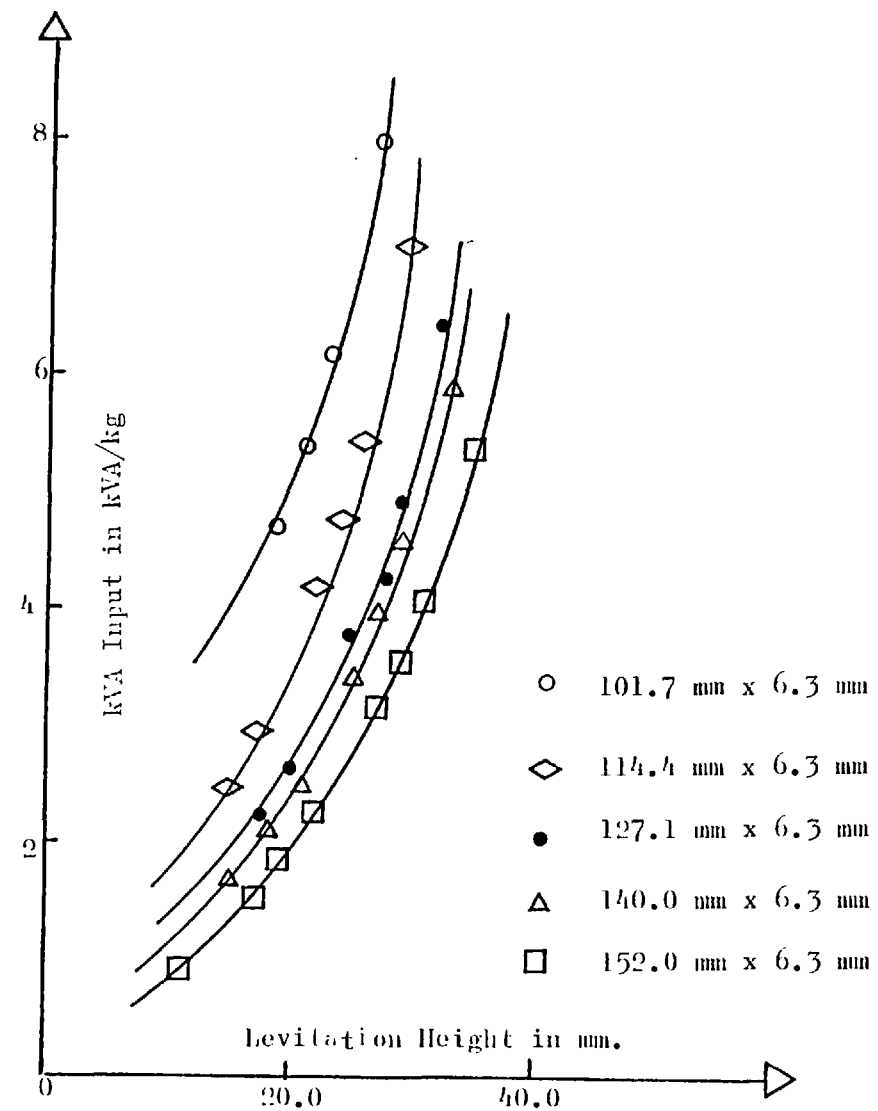


Figure 2.4: kVA required for different plates.

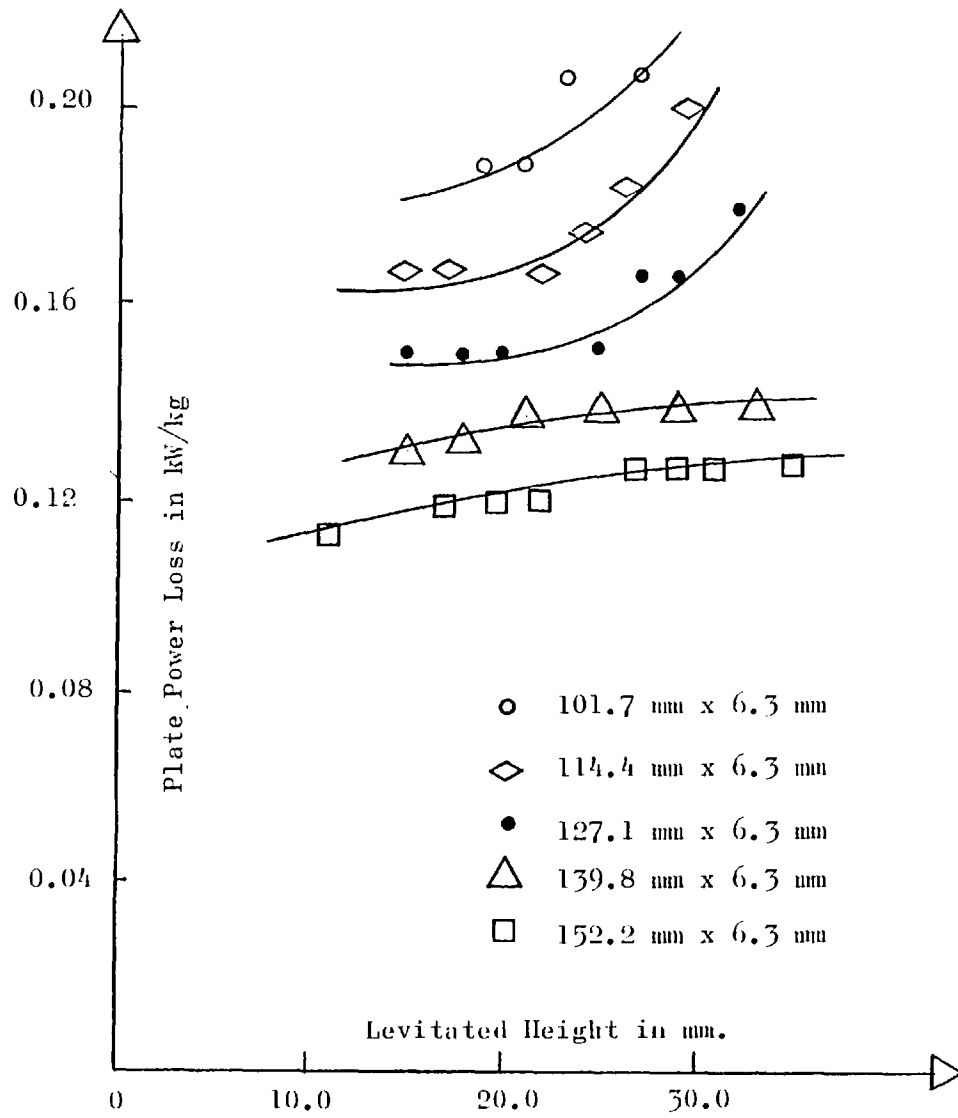


Figure 2.6: The variation of plate power loss with airgap.

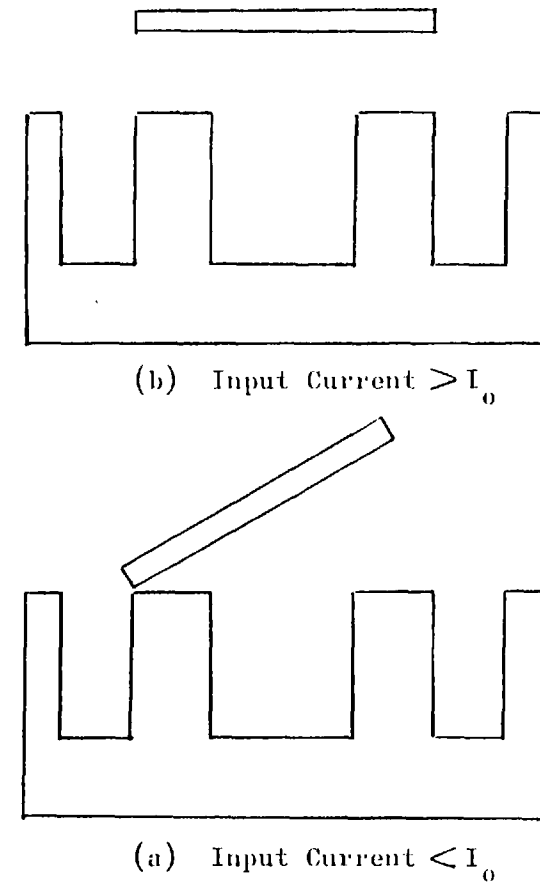


Figure 2.7: The observed behaviour of a narrow plate

plates floating, the levitation force for each of the plates was equal to its weight. Hence by dividing the measured kVA, kW and plate power loss by the weight of the plates, the curves of kVA/kg, kW/kg and plate loss/kg vs. levitation heights were obtained. These are shown in Figs. 2.4, 2.5 and 2.6, respectively.

Considering the curves of Fig. 2.3 as a first approximation, they can be assumed linear. On this assumption the curves are given by the empirical equation:

$$h = K_1 I + K_2 I_0 \quad (2.1)$$

K_1 and K_2 are constants with dimensions. If the magnitude of K_2 is always made equal to unity, then I_0 becomes the minimum current required to levitate the plates. The constants K_1 , K_2 and I_0 evaluated for the four plates of Fig. 2.3 are:

Plate Width mm	K_1 mm/A	K_2 mm/A	I_0 A
101.7	0.50	1.0	20.0
114.4	0.50	1.0	15.0
127.1	0.53	1.0	12.5
152.5	0.52	1.0	10.0

Table 2.1

When the excitation current is increased gradually, the wider plates were found to rise slowly with the plates remaining horizontal. For the narrow plates, however, the plate behaviour illustrated in Fig. 2.7 was observed.

Examination of the kVA/kg, kW/kg and plate power loss/kg characteristics shows that with increasing levitation heights, the kVA required increases rapidly. The same tendency is observed in kW requirements. By contrast, the plate power losses are seen to vary little, especially for the wider plates. To make a quantitative comparison of the relative changes, consider the widest plate (152.5 mm) at two airgap values of 10 mm and 40 mm. To increase the levitation height from 10 mm to 40 mm, the input kVA needs to be increased from 1 to 7, which is a sevenfold increase. The input kW increases from about 0.15 to 0.60, which is a fourfold increase. The plate power loss, however, increases from about 110 W to 125 W only, which is a negligible change by comparison. Therefore, to levitate a constant load at different increasing airgaps, a nearly constant plate power loss, but a rapidly increasing kVA need to be supplied. The total power losses to be supplied will still be large due to the ohmic losses in the stator winding.

2.3 LOW POWER EXPERIMENTS

In the foregoing section, in the experiments described, the rotor plates were found to heat up quickly due to the large input currents. Hence an accurate measurement of forces becomes difficult. In order to overcome this, and at the same time to obtain the variation of normal force with airgap for various plate widths, experiments were carried out at low power levels.

2.3.1 Normal Force Variation with Airgap

The normal force variation with airgap at a constant excitation current was obtained for three plates, falling into the

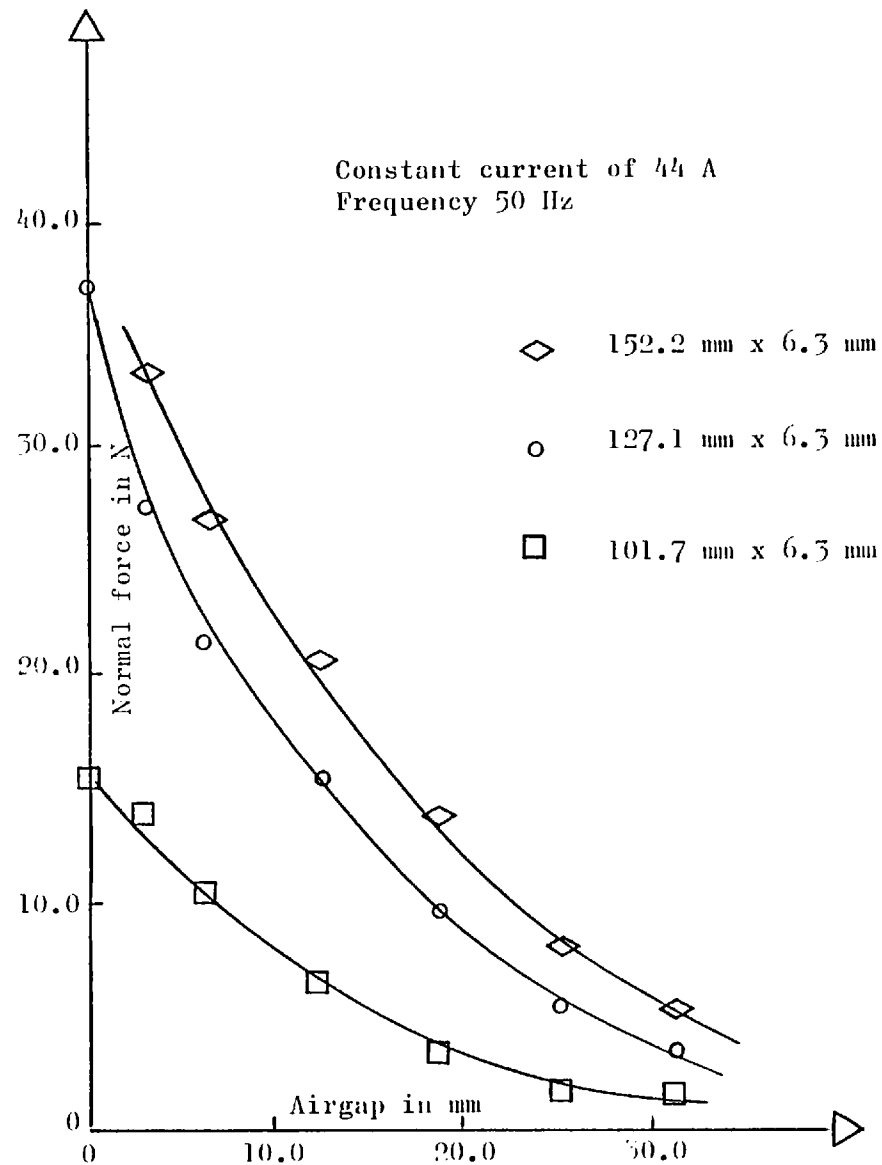


Figure 2.8: Normal force variation with airgap.

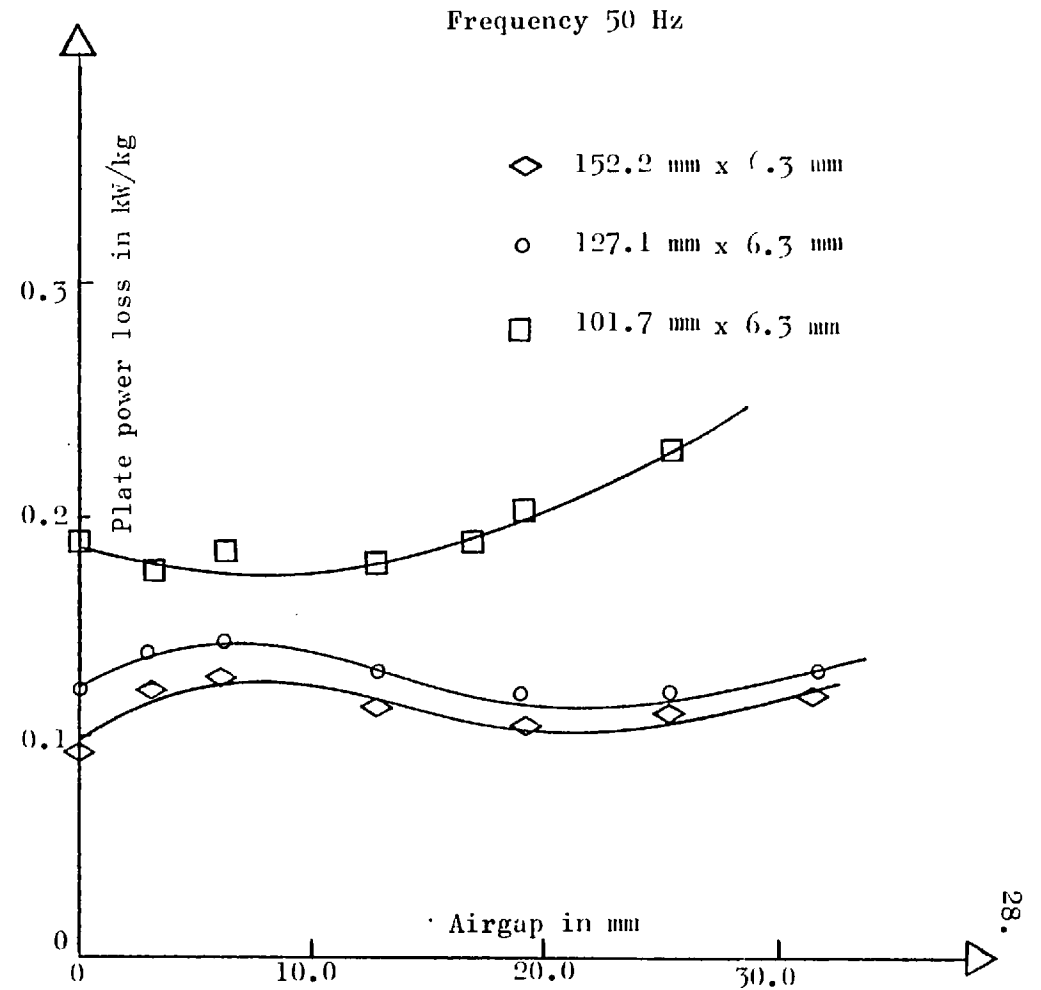


Figure 2.9: Plate power loss variation with airgap.

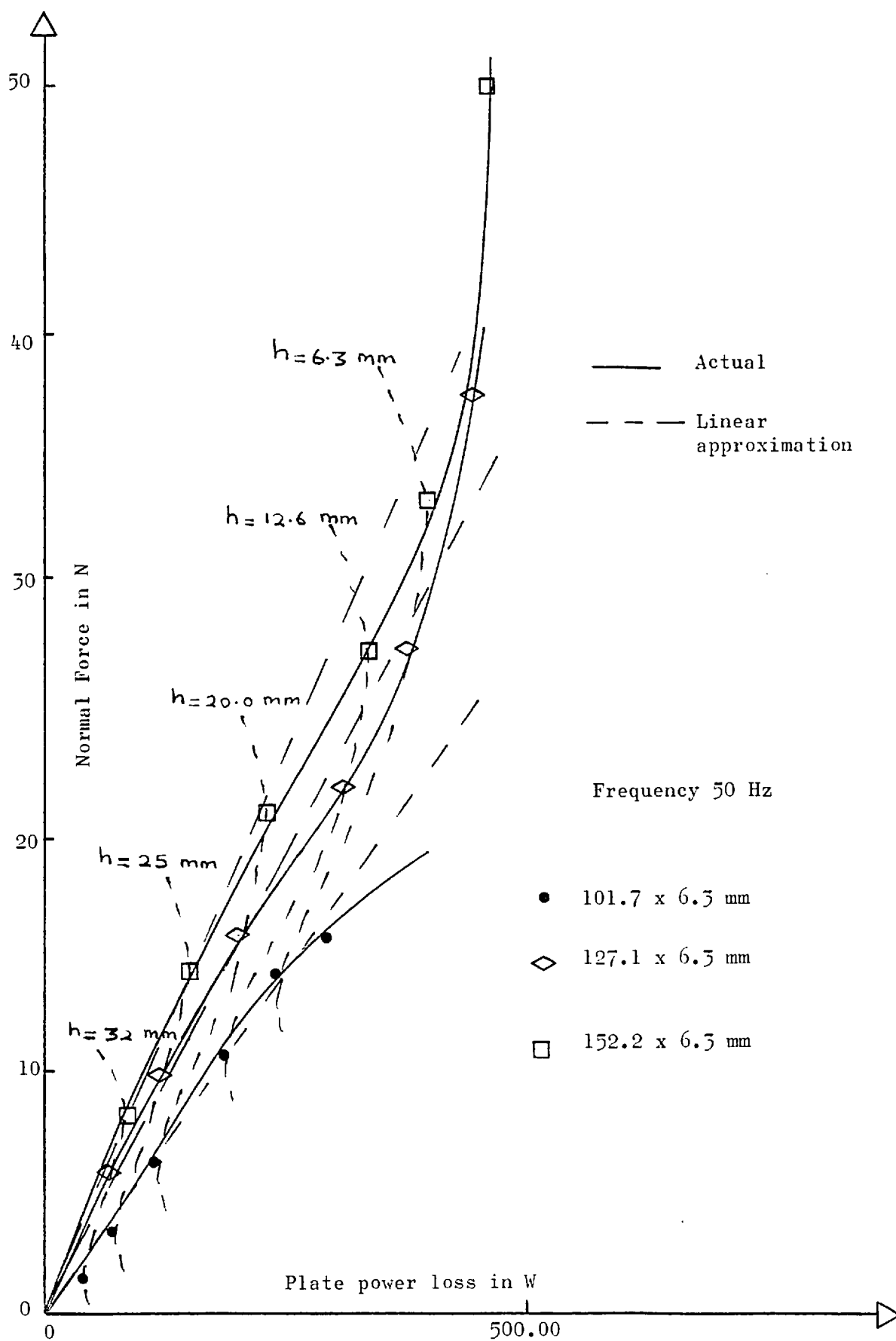


Figure 2.10: Normal Force Variation with Plate Power Loss.

narrow (102 mm), intermediate (127 mm) and wide (152.5 mm) categories. The plates were clamped to a weighing scale, with the plate replacing one of the scale pans. The stator of the Xi-core was placed underneath the plate without touching the plate and with a clearance equal to the required value of airgap. The vertical force was measured before and after energising the stator, the difference being the normal force on the plate at the airgap used. Since forces are proportional to the square of the input current, all the forces measured thus were scaled up to the values corresponding to a current of 44 A. The measured forces are shown in Fig. 2.8. The curves show that at any airgap, if the width of a plate is allowed to increase from 102 mm, then the normal force increases rapidly until the plate width becomes 127 mm. Thereafter further increase in width causes a less rapid increase in the normal force.

2.3.2 Input Impedance Measurements

Low power input impedance measurements were carried out for several plates with the plate resting on non-magnetic supports, at the required airgap value. From these results, in conjunction with the normal force vs. airgap curves of Fig. 2.8, a more precise variation of plate power loss/kg vs. levitation height can be obtained indirectly as follows. Consider the 101.7 mm wide plate in Fig. 2.8 for example. At any airgap value (h_1 , say) let the normal force be F_1 , at the excitation current of 44 A. From the measured input impedance results, let the effective resistance (i.e. resistive part of the input impedance) be R_1 at this airgap of h_1 . Then the plate power loss/kg is $(44)^2 \cdot R_1 / F_1$, at the height of h_1 , F_1 having been converted to kg.

The variation of the plate power loss/kg with levitation height obtained in this manner is shown in Fig. 2.9. These curves confirm the results of Fig. 2.6, but show the variations at small airgaps in more detail. Since these results firmly establish that the power loss in the plate levitated at different heights, is approximately constant, it follows that one can write (also approximately):

$$\text{Levitation Force} \propto \text{Plate Power Loss}$$

The extent to which one can assume proportionality between levitation force and plate power loss is illustrated in Fig. 2.10. This shows the measured force vs. measured plate loss at several airgaps. It is seen that except at very small airgaps, the supposition appears to be valid.

2.4 MEASUREMENT OF GOODNESS FACTOR

It will be shown later in Chapter 3 that a dimensionless ratio G derived from the input impedance is the time constant of decay of rotor currents multiplied by the angular frequency of supply. For the present, however, G is introduced simply as:

$$G = \frac{X_o - X_{in}}{R_{in}}$$

where, X_o = Input reactance without the plate,

X_{in} = Input reactance within the plate,

R_{in} = Input resistance within the plate.

Using the results of low power impedance measurements, let us examine the characteristics of G . For this purpose the following tests were carried out:

- 1) Measurement of X_{in} and R_{in} at 50 Hz, for several plate widths in symmetrical position, all plates being of the same thickness but at varying airgaps.
- 2) Repetition of the above tests, but with the plate thickness halved.
- 3) Measurement of X_{in} and R_{in} for one of the plate widths used in (1) for one particular airgap at several frequencies.

2.4.1 Effect of Plate Width on G

When the plate thickness, airgap value and supply frequency are maintained constant, the variation of G with plate width is shown in Fig. 2.11. The variations show that provided the plate width is between the limits $2(a+b)$ and $2(a+b+c)$, (Fig. 2.1), one can write, approximately, $G \propto L$.

2.4.2 Effect of Frequency on G

When the plate width, thickness and airgap value are maintained constant, the variation of G with frequency is shown in Fig. 2.12. It is seen that $G \propto f$.

2.4.3 Effect of Airgap Value on G

When the plate width, its thickness, and supply frequency are held constant, the variation of G with airgap is shown in Fig. 2.13 and Fig. 2.14. Though it is not possible to obtain the exact mathematical variation, one can see that it is $G \propto h^n$, where $n \leq 1$.

2.4.4 Effect of Plate Thickness on G

To determine the effect of plate thickness, consider two plates of the same width, at the same airgap, but of thicknesses t_1 and t_2 . If G_{t_1} and G_{t_2} are the values of G for the two cases, then:

$$\frac{G_{t_1}}{G_{t_2}} = \left(\frac{t_1}{t_2}\right)^n$$

Choosing $t_1 = 2t_2$, and plotting G_{t_1} vs. G_{t_2} (Fig. 2.15) for several plates, the value of the unknown n was obtained as 0.73.

2.4.5 Goodness Factor and G

The ratio G will be hereinafter treated as the goodness factor of the machine since:

- (a) In Chapter 3, G is shown as the rotor time constant multiplied by the angular frequency, by using simplified models for the Xi-core.
- (b) Freeman and Lowther^{2.1} have shown more rigorously that $(X_o - X_{in})/R_{in}$ for a conventional linear induction motor excited by a travelling wave is the Laithwaite Goodness Factor^{2.2}.

Therefore the Goodness Factor of the particular Xi-core being investigated is, with airbacked rotors:

$$G \propto \omega L t^{0.73} h^n$$

where, ω = angular frequency,
 L = plate width, but between the limits mentioned in 2.4.1,
 t = plate thickness,
 h = airgap,
 n = an index < 1 .

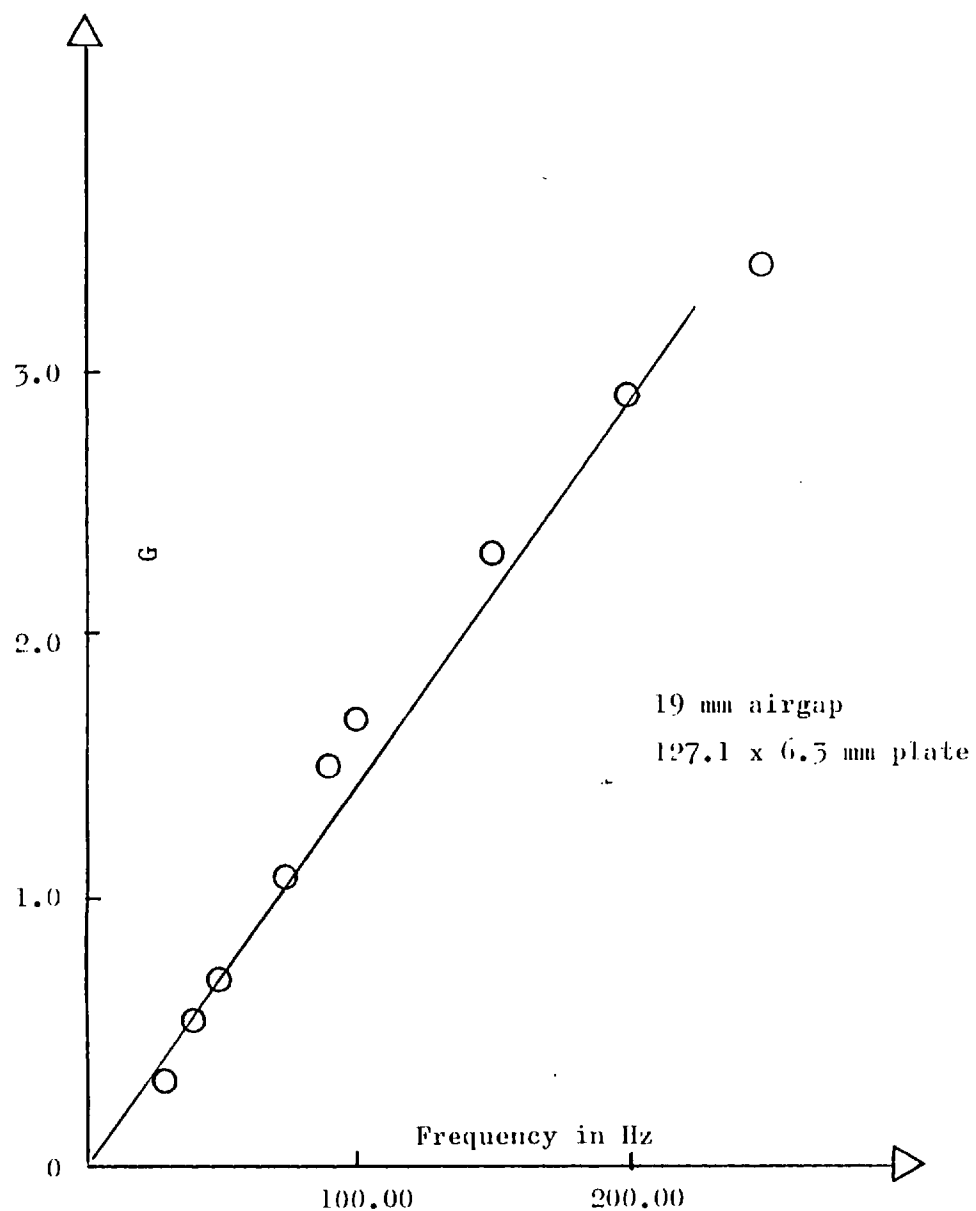


Figure 2.12: Variation of Goodness Factor with Frequency.

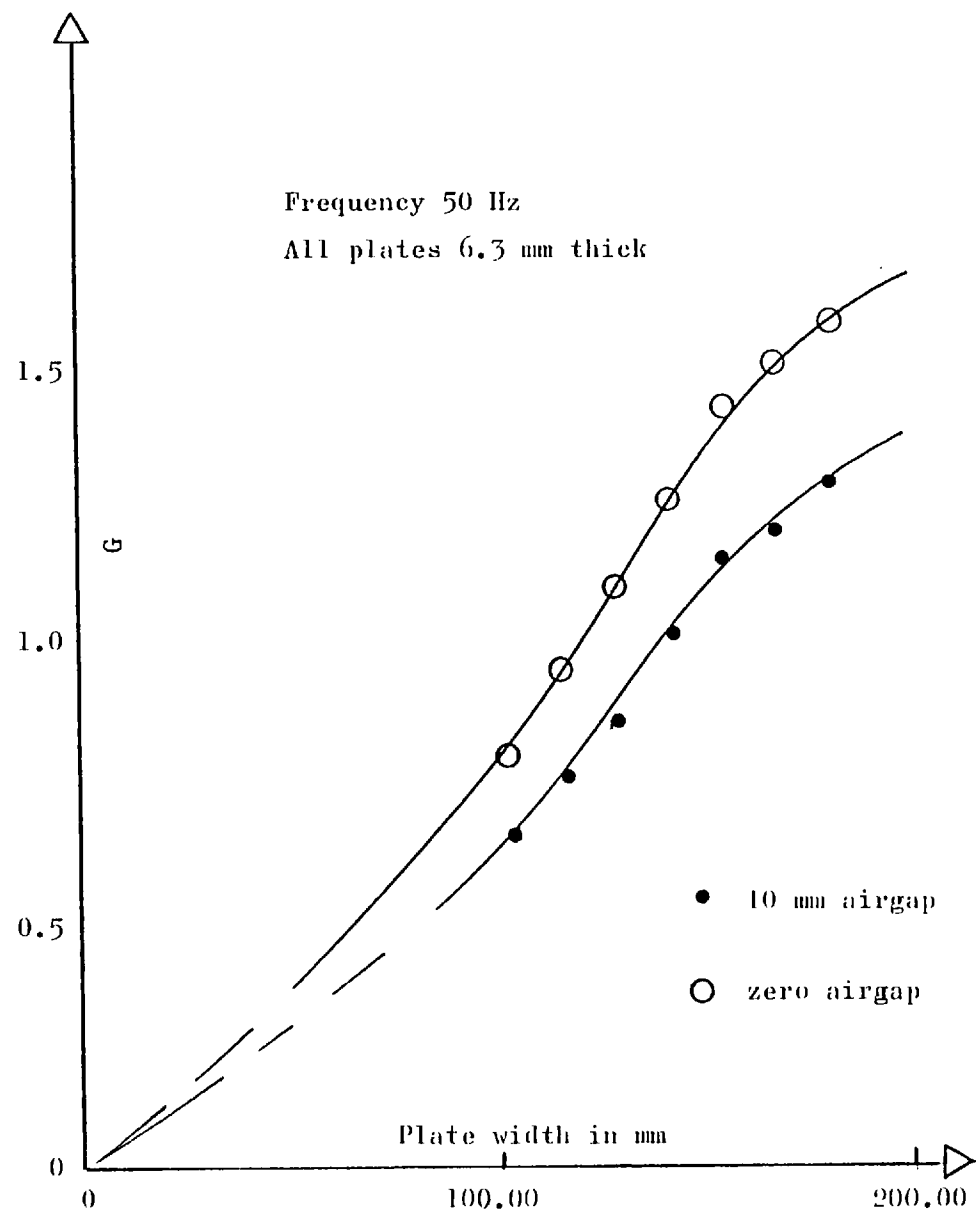


Figure 2.11: Variation of Goodness Factor with plate Width.

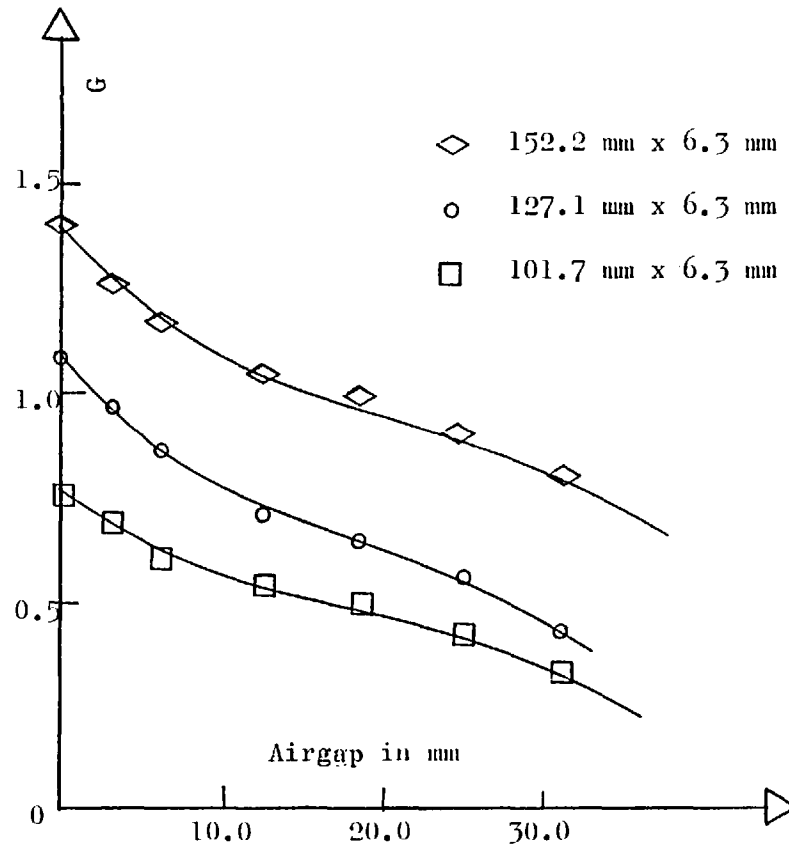


Figure 2.13: Variation of Goodness Factor with Airgap.

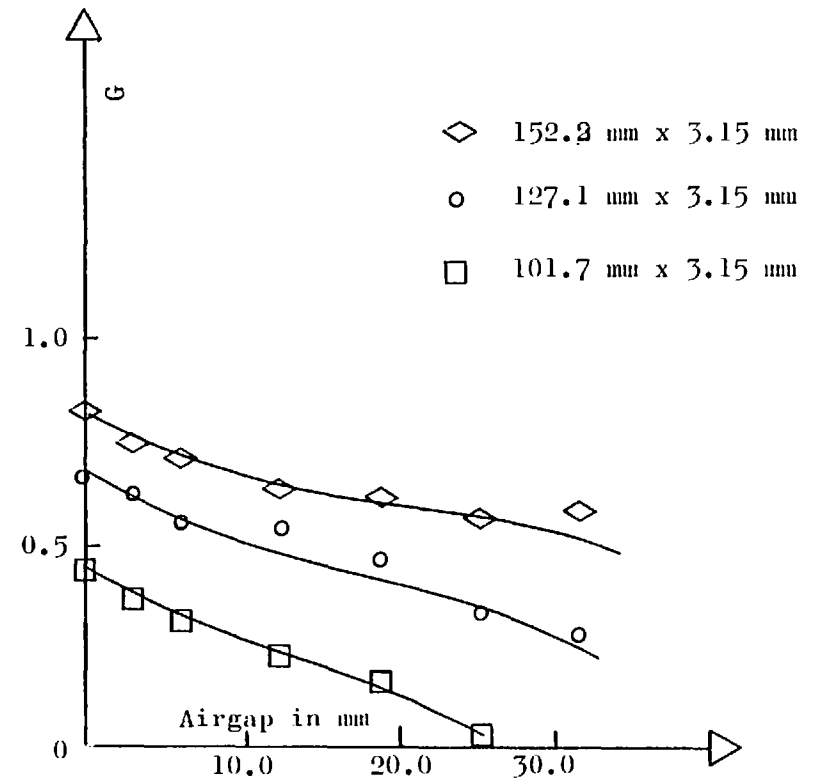


Figure 2.14: Variation of Goodness Factor with Airgap.

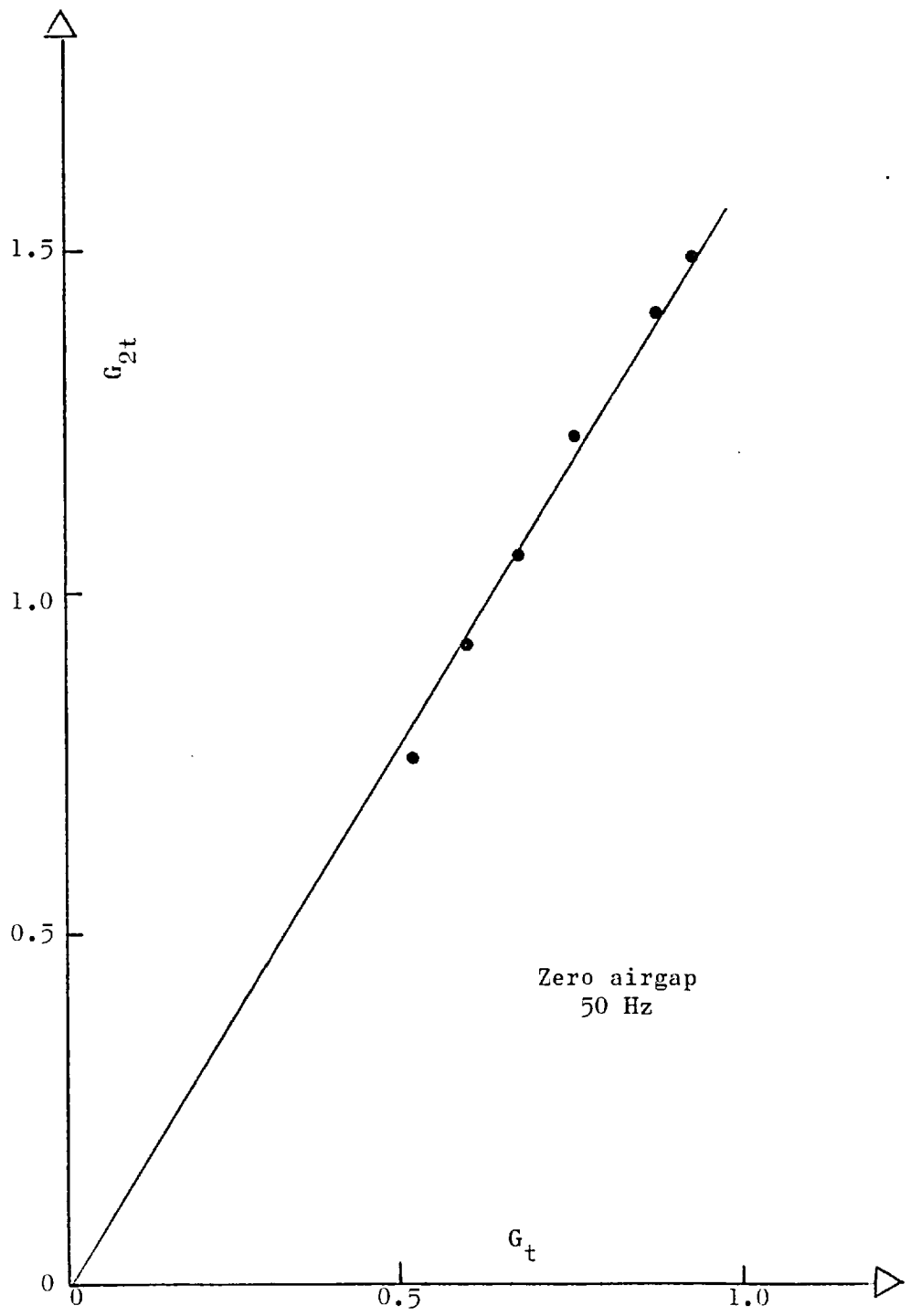


Figure 2.15: Effect of plate thickness on G .

2.5 LATERAL FORCE MEASUREMENTS

In order to measure the lateral force variation, with lateral displacement, an experimental rig (Fig. 2.16) was constructed. Secondary plates could be rigidly clamped to this rig and the whole plate arrangement was free to slide horizontally on linear bearings. A force transducer was rigidly fixed to this bearing-mounted plate structure. The rig thus produces a d.c. voltage of appropriate sign on the force transducer when the stator is energised. This d.c. voltage is proportional to the lateral force, the proportionality constant being obtained by a calibration procedure. The required airgap and rotor displacement were achieved by positioning the stator underneath the plate. Using this rig the following tests were carried out:

- (a) Force/displacement measurements at 50 Hz for several plate widths, all of the same thickness and at a constant airgap.
- (b) Force/displacement measurements at 50 Hz, for the same plates used in (a) but at a larger airgap value.
- (c) Force/displacement measurements at several frequencies for three of the plate widths used in (a) and (b), the airgap being as in (b).
- (d) Force/displacement measurements at 50 Hz for two plates at the same airgap value, as in (b), but halving the plate thickness.
- (e) Force/displacement measurements at 50 Hz, and 200 Hz for one of the plate widths used in (d) at the same airgap.

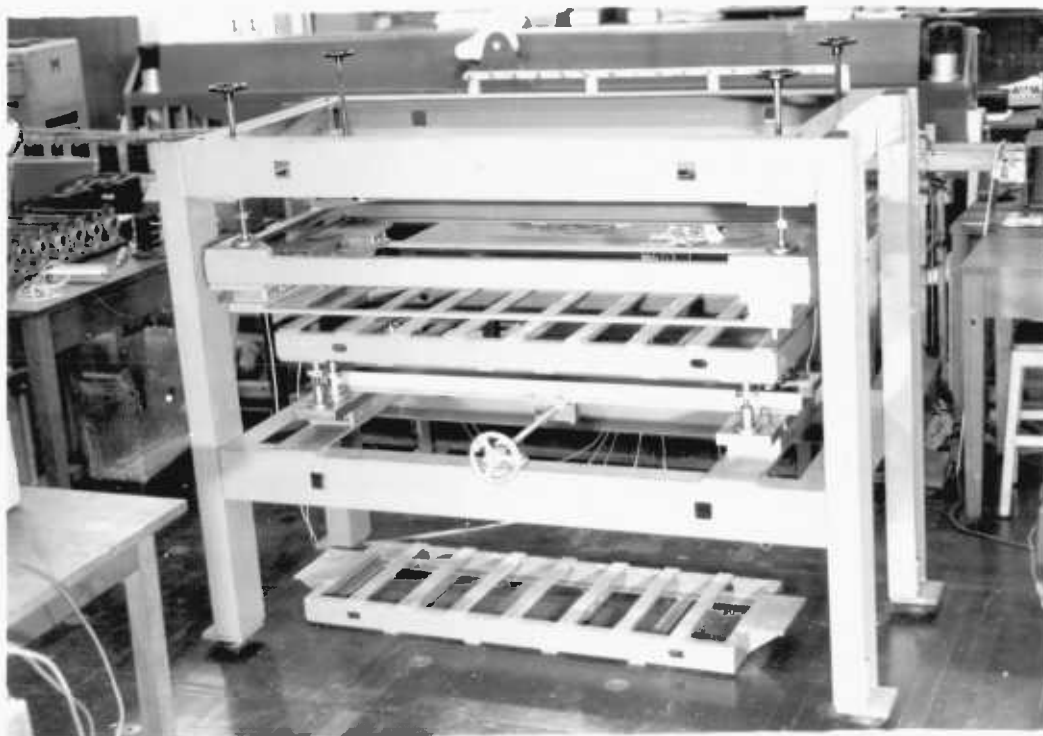


Figure 2.16: The static force measuring rig.

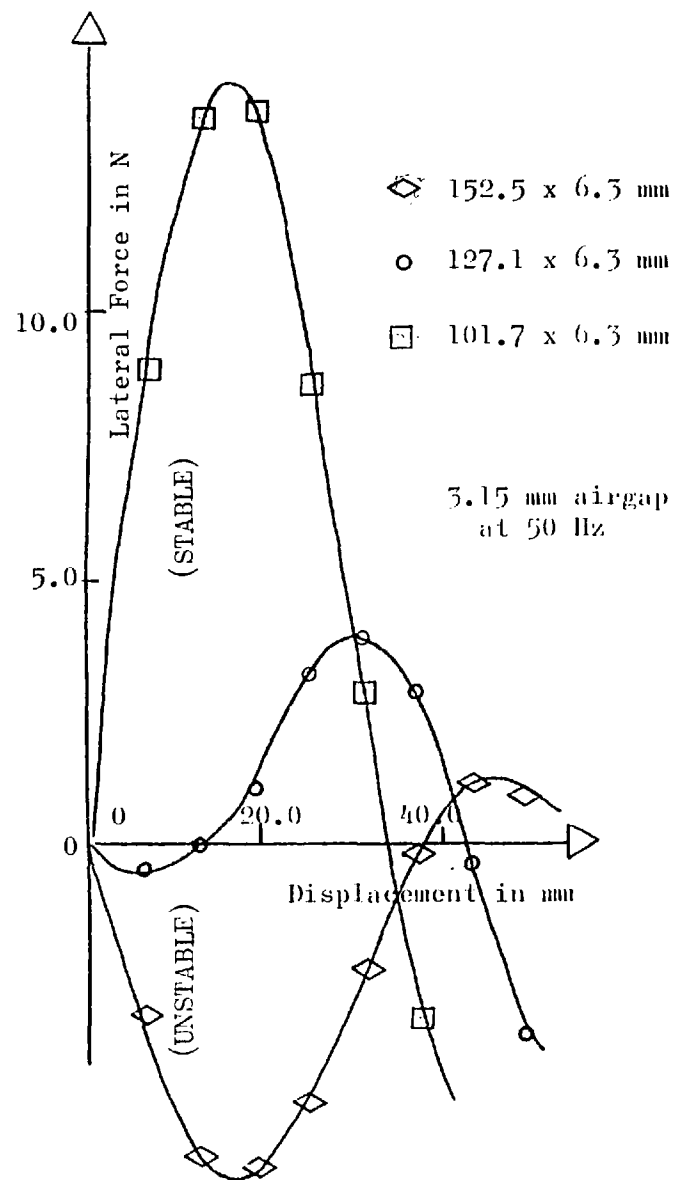


Figure 2.17: Lateral Stability characteristics.

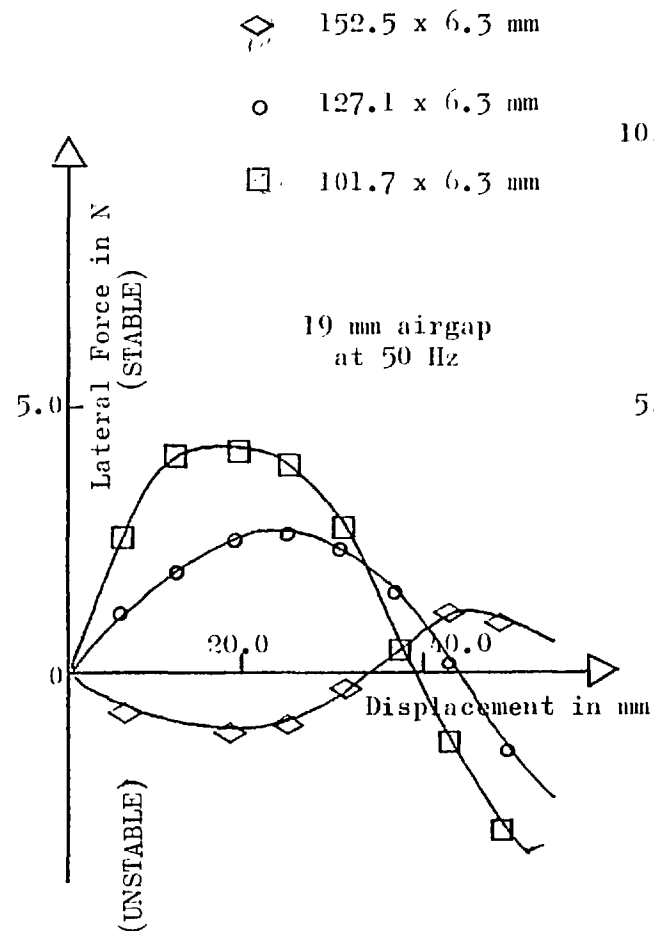


Figure 2.18: Lateral Stability characteristics.

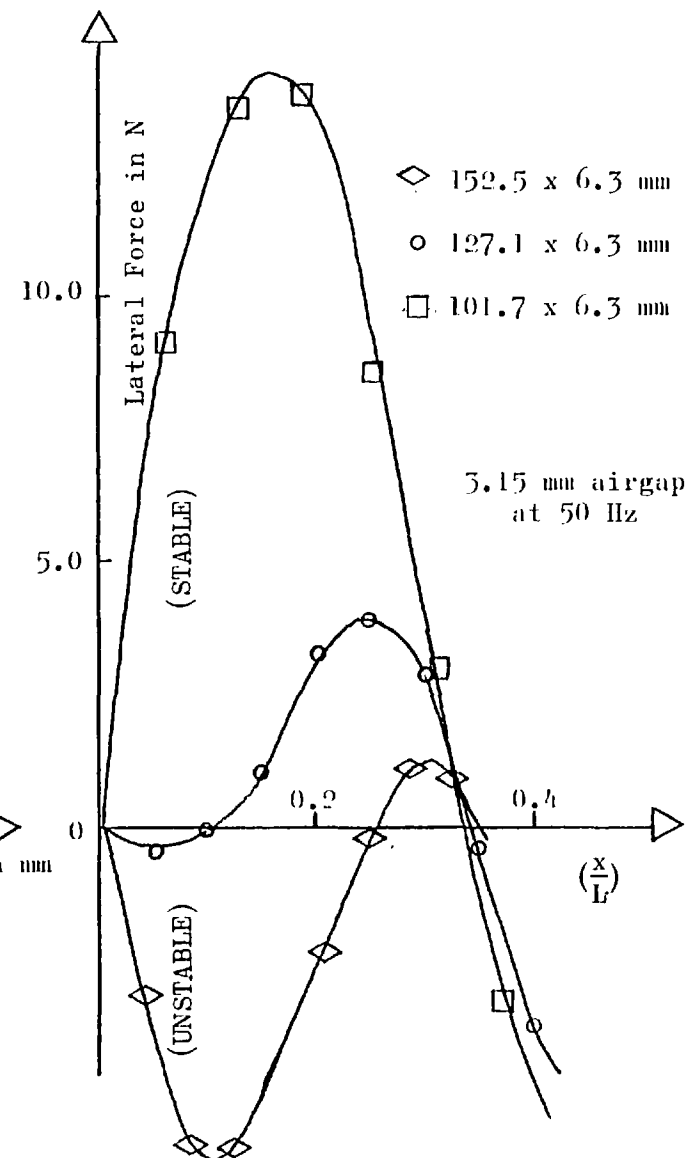


Figure 2.19: Lateral Stability characteristics.

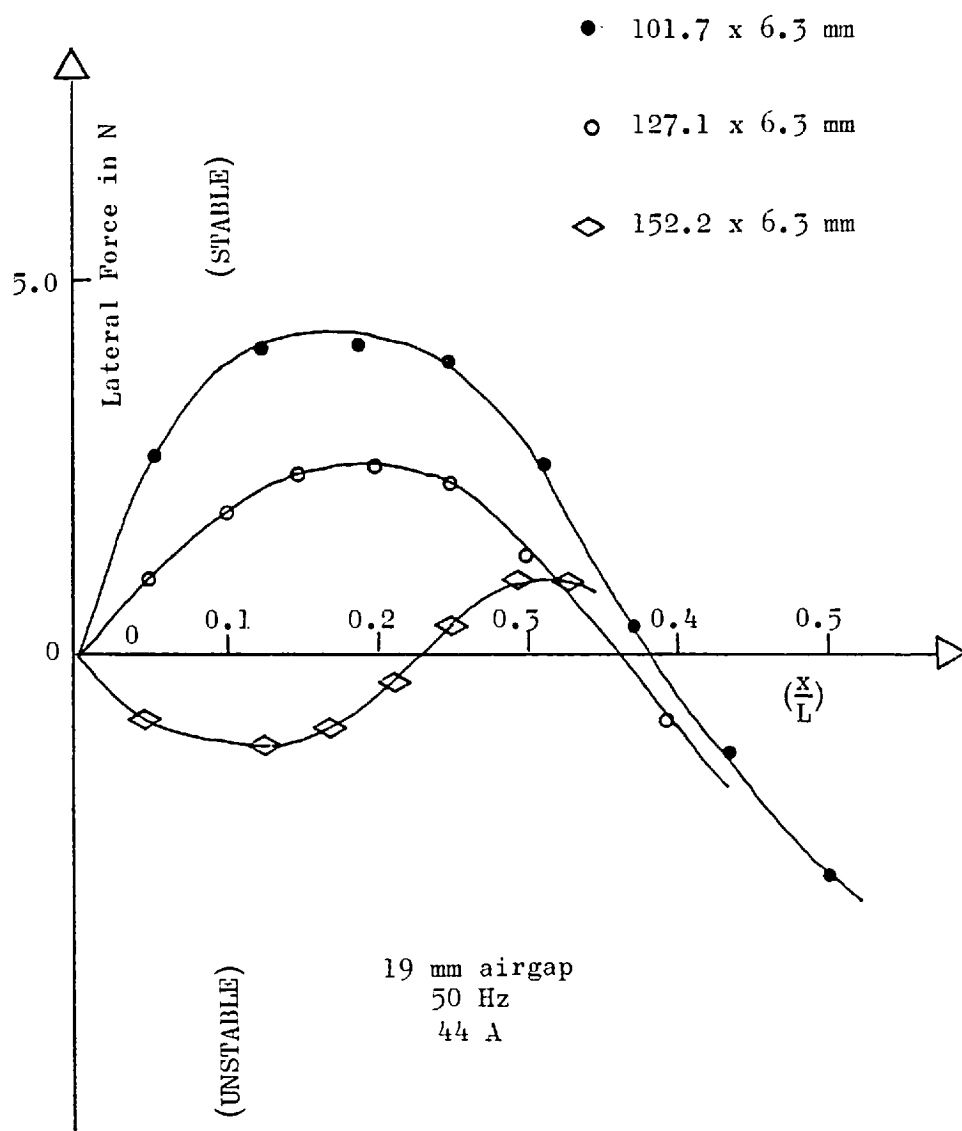


Figure 2.20: Lateral Stability Characteristics.

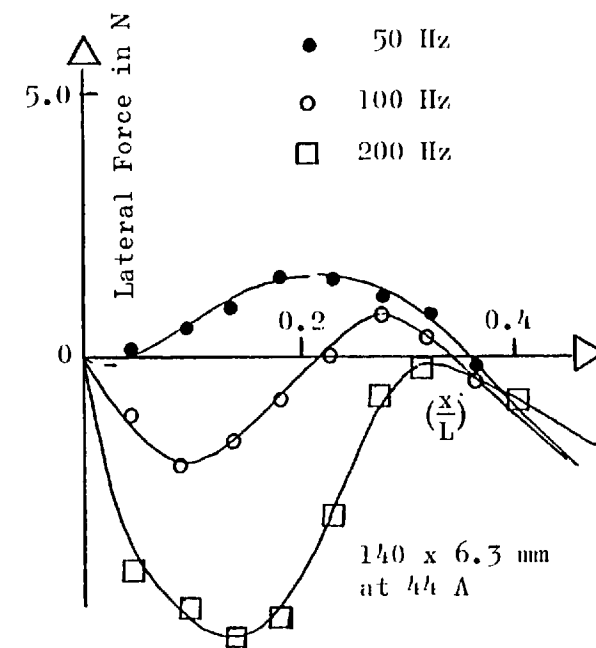
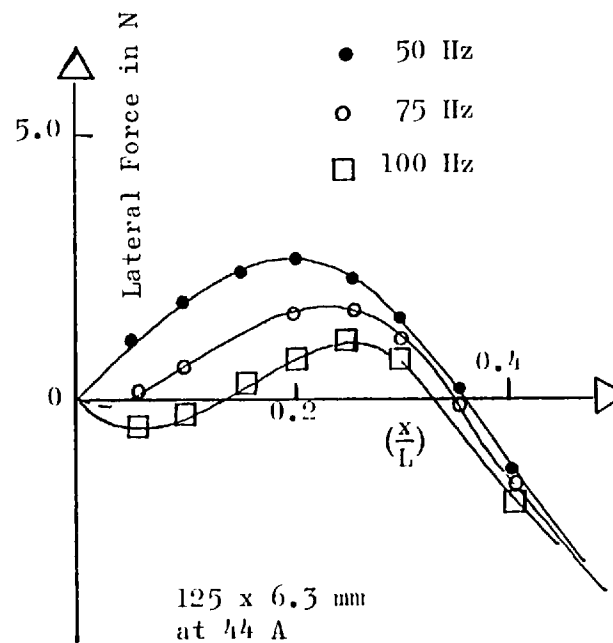
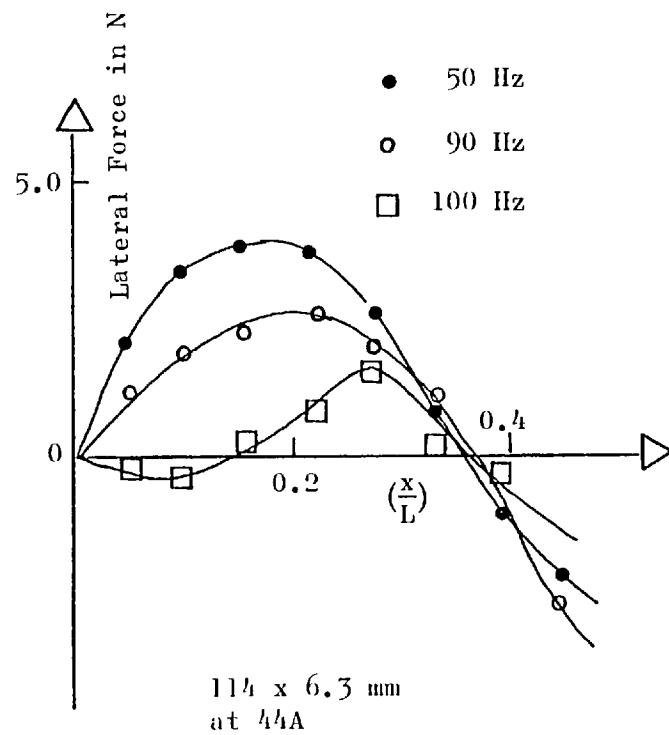


Figure 2.21:

Lateral stability variation with frequency.
 All plates at the same airgap of 19 mm.

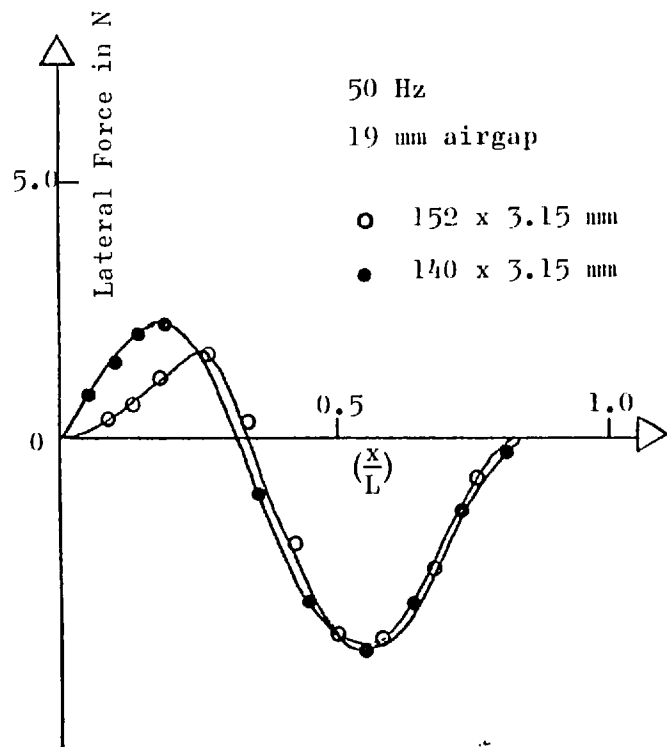


Figure 2.22: Lateral Stability Characteristics.

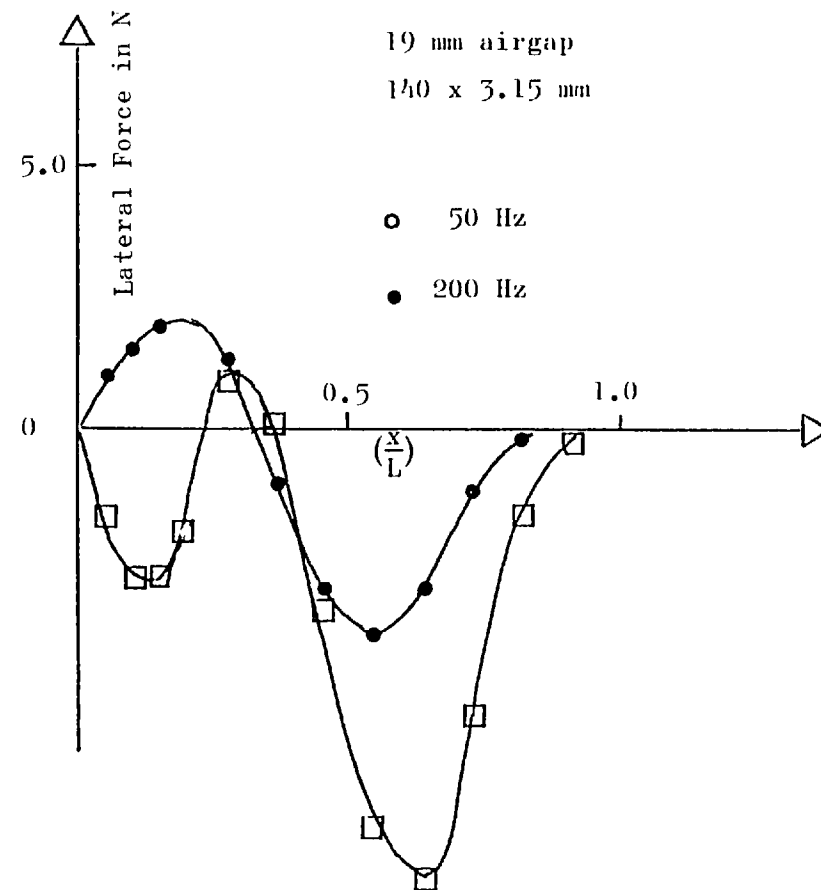


Figure 2.23: Lateral Stability Characteristics.

The results of experiments (a) and (b) are shown in Fig. 2.17 and 2.18. These show that increase of plate width causes the stabilising lateral force to reverse into a destabilising one. A reduction of airgap causes the same tendency as the increase of plate width at a larger airgap. Before examining the results of the other experiments, the effect of expressing the lateral displacement as a fraction of the plate width is considered. Fig. 2.19 and Fig. 2.20 show the same curves as Fig. 2.17 and 2.18, with the displacement expressed in this manner. The usefulness of the resulting curves can be seen clearly. We see that all the curves have the same cross-over point, which is approximately 0.35. The value of the airgap is seen to have no effect upon the location of the cross-over point.

The results of the remaining experiments (c), (d) and (e) are shown in Figs. 2.21, 2.22 and 2.23, with displacement expressed as a fraction. These show, firstly, that the cross-over point is always 0.35, irrespective of frequency, and plate thickness. Secondly, they show that increasing frequency produces the same destabilising effect as increasing plate width, while maintaining the frequency constant. Also evident is the fact that reducing the plate thickness brings about lateral stability, as happens when the plate thickness is held constant and the width reduced.

2.5.1 The Significance of the Cross-Over Point

In view of the fact that the cross-over point of 0.35 is independent of the rotor dimensions, frequency and airgap, the logical conclusion is that it depends only on the stator dimensions. For each of the plates the displacement $x = 0.35 L$ was calculated,

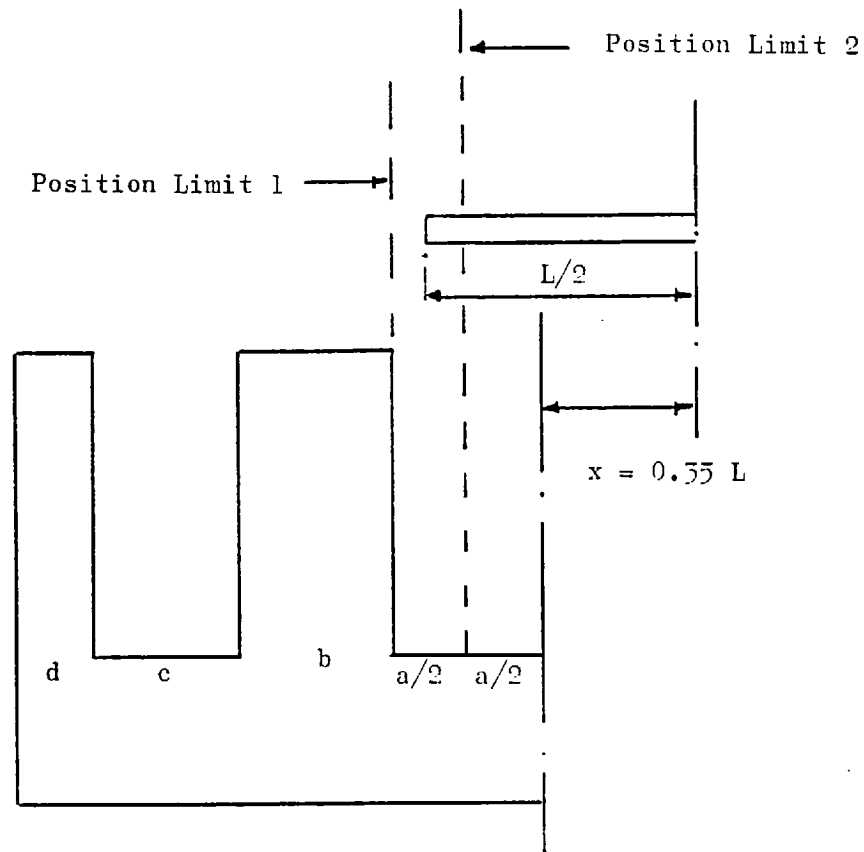


Figure 2.24: Lateral displacement at which induced current distinction changes from two loops into one.

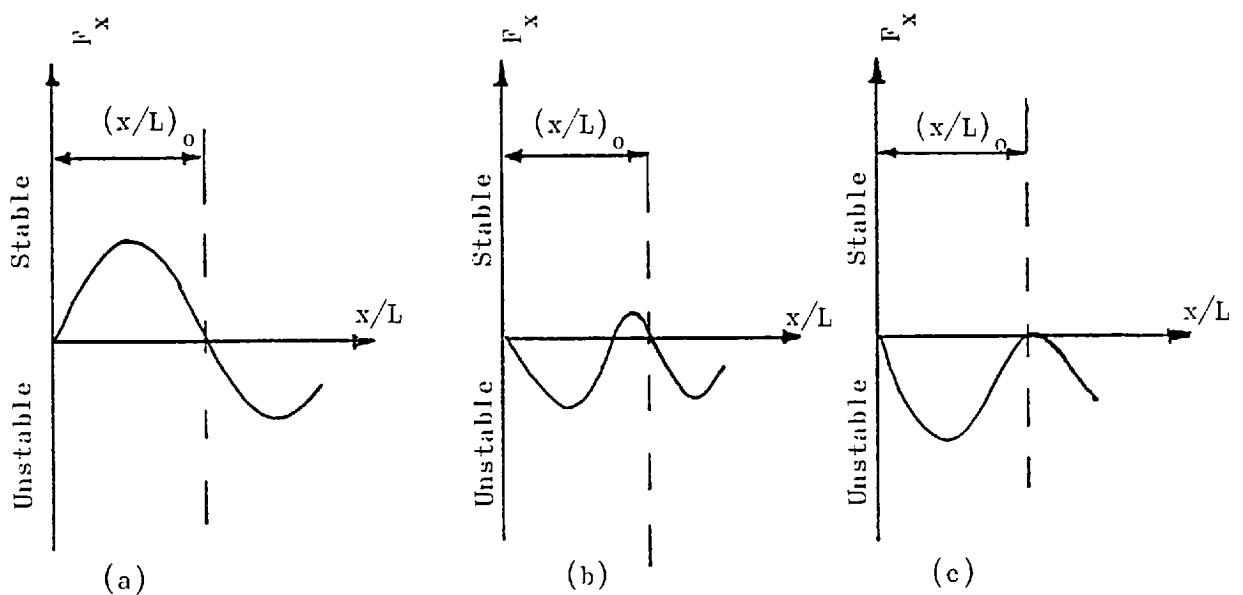


Figure 2.25: The general forms of lateral force/displacement characteristics.

and it was found that the calculated displacement x corresponds to the displacement when the trailing plate edge is between the position limits shown in Fig. 2.24.

The significance of the cross-over point emerges now. It corresponds to the displacement at which the induced current distribution in the rotor changes from two closed loops into a single loop. Though this can be deduced from physical considerations, a more formal confirmation is obtained in a later chapter, using a finite difference numerical approach for a plate of width $L = 2a + 2ab + c$.

2.5.2 General Form of the Lateral Force Variation

From the experimental results, it is seen that the general shape of the force/displacement curves is one of the three forms shown in Fig. 2.25. The forms (a) and (c) are the two extreme forms, while (b) is the transition stage from (a) to (c). Provided the plate width L is greater than $2a+2b$, a plate which is completely stable and therefore showing form (a) transforms into (b) and (c) by:

- (A) Decreasing the airgap, while maintaining all other parameters such as frequency, plate thickness and plate width constant.
- (B) Increasing plate thickness, while maintaining frequency, airgap and plate width constant.
- (C) Increasing plate width, while maintaining frequency, airgap and plate thickness constant.
- (D) Increasing frequency, while maintaining plate width, airgap and plate thickness constant.

Recalling the characteristics of the Goodness Factor G discussed earlier, we can replace the above conditions by a single condition as follows. A plate showing characteristic (a) in Fig. 2.25 transforms into (b) and (c) if its Goodness Factor is increased, provided the plate is wider than $2a+2b$.

It is interesting to observe that the reverse transition from form (c) to (a) via (b) is possible by a reduction in the value of G . On a constant frequency supply for a given plate of fixed dimensions, the only method of reducing G is by an increase of airgap. Therefore with increasing airgaps, plates of progressively increasing widths can be levitated stably. This has been described by Eastham and Laithwaite^{2,3} who say that stable rotors of transverse flux levitators are those that can be wedged into an expanding trough. For plates of 6.3 mm thick, at 50 Hz, such a trough constructed from the results shown in Fig. 2.16 and Fig. 2.17, is shown in Fig. 2.26. The effect of increasing the goodness factor is to decrease the base width of the trough and draw its sides closer together. The limiting shape is as shown in Fig. 2.26. The effect of decreasing the goodness factor is to widen the base of the trough, also as shown in Fig. 2.26.

2.6 LATERAL STABILITY AND STORED ENERGY

In order to investigate the manner in which stored magnetic energy varies with lateral displacement, the effective inductance at the input terminals was measured for several lateral displacements of the plate. The variation of the effective inductance obtained in this manner is shown in Fig. 2.27, for two plates at two airgaps, together with their lateral force/lateral displacement characteristics.

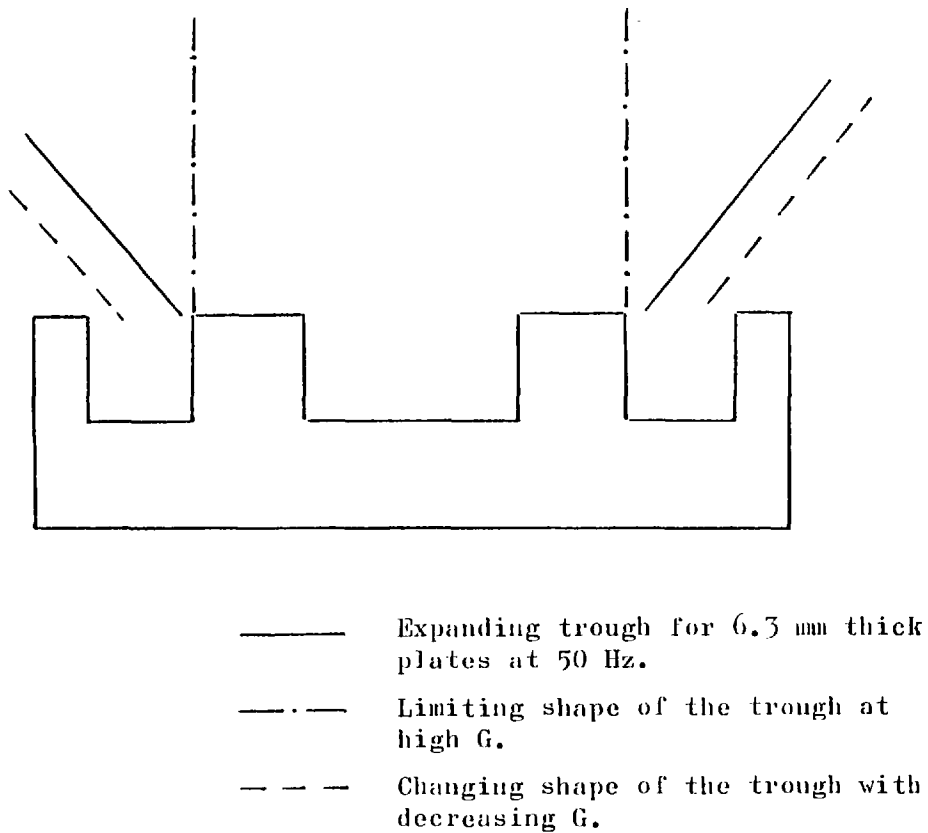


Figure 2.26: The expanding trough, for stable rotors of the Xi-core.

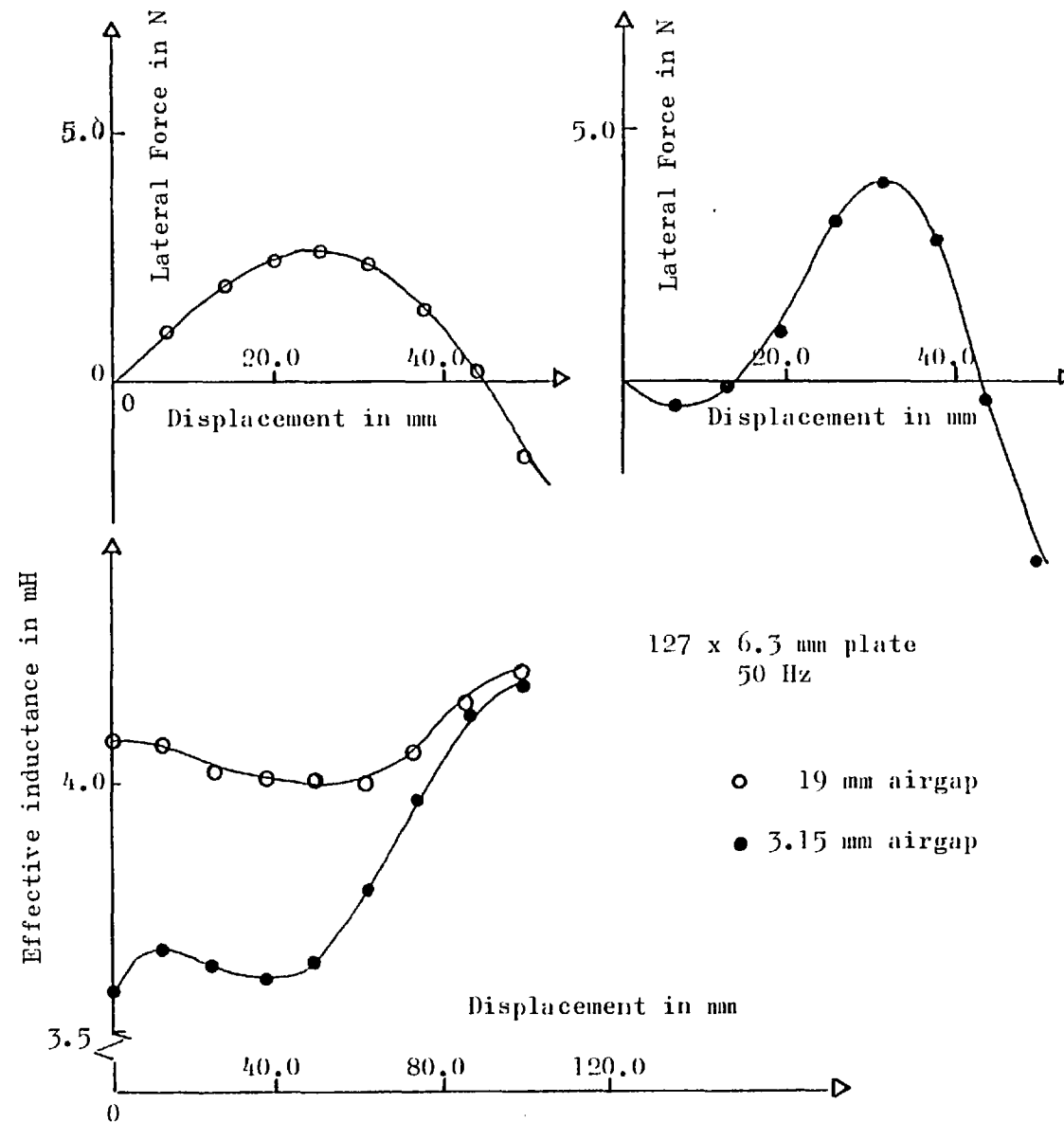


Figure 2.27: Lateral stability dependence on stored magnetic energy.

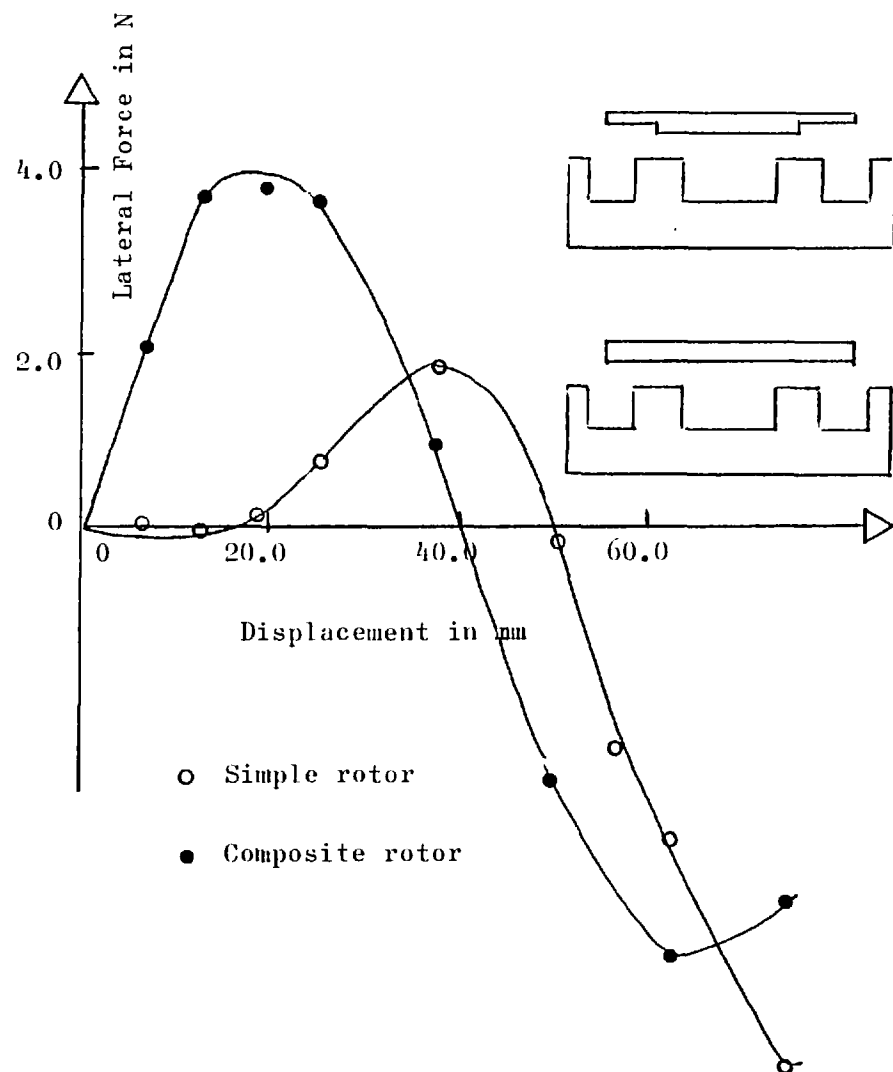


Figure 2.28: Improvement of lateral stability by change of rotor geometry.

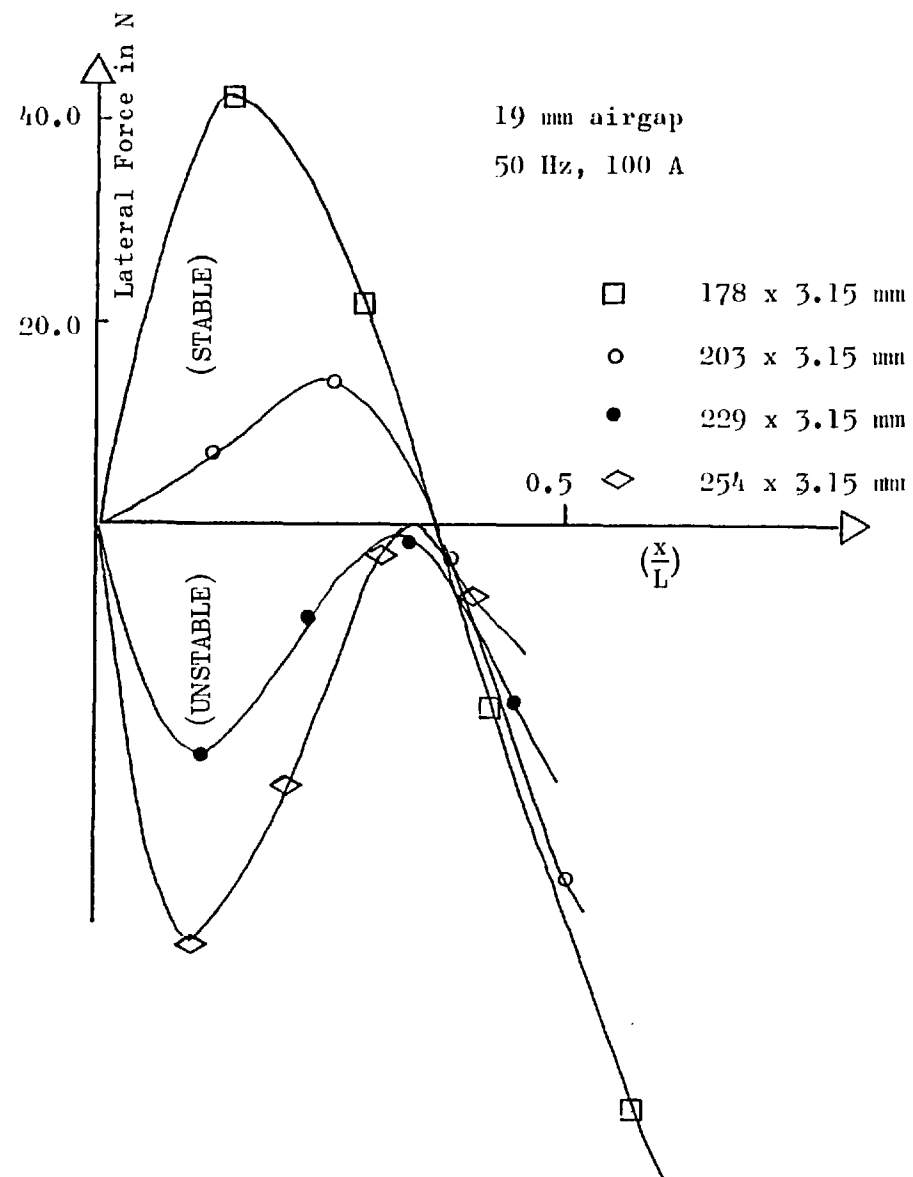


Figure 2.29: Lateral stability of the three-phase Xi-core.

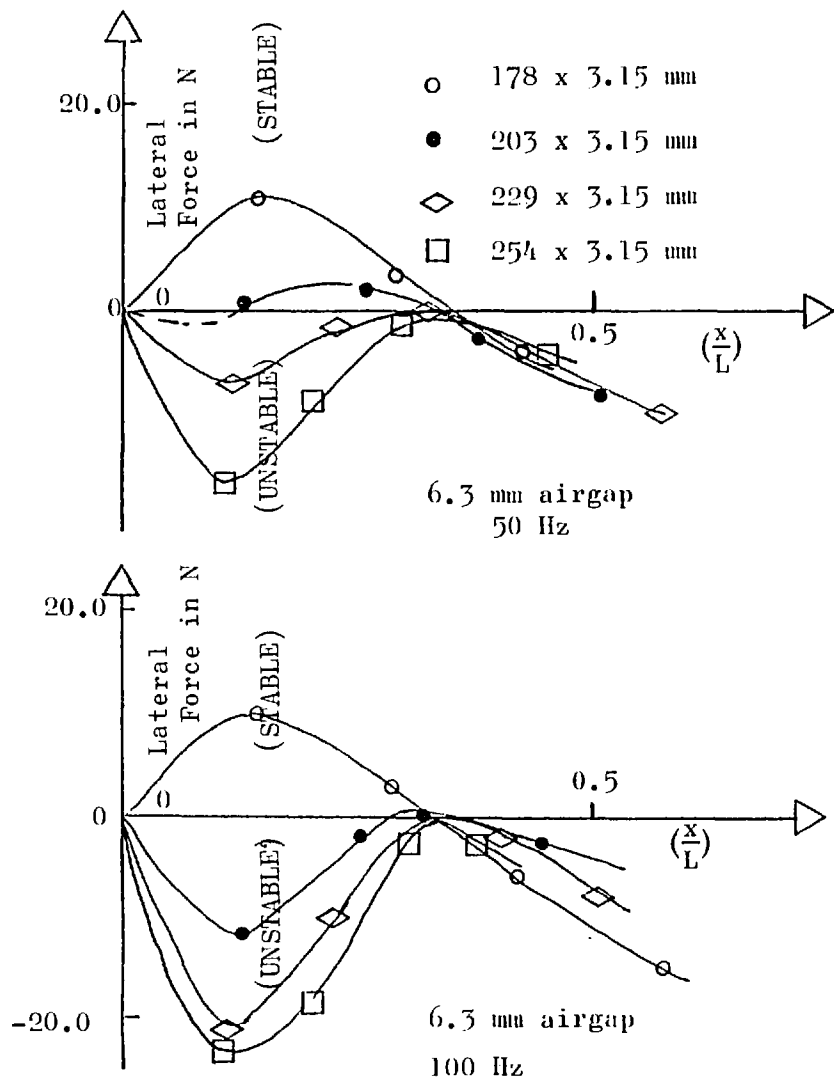


Figure 2.30: Lateral stability of the three-phase Xi-core.

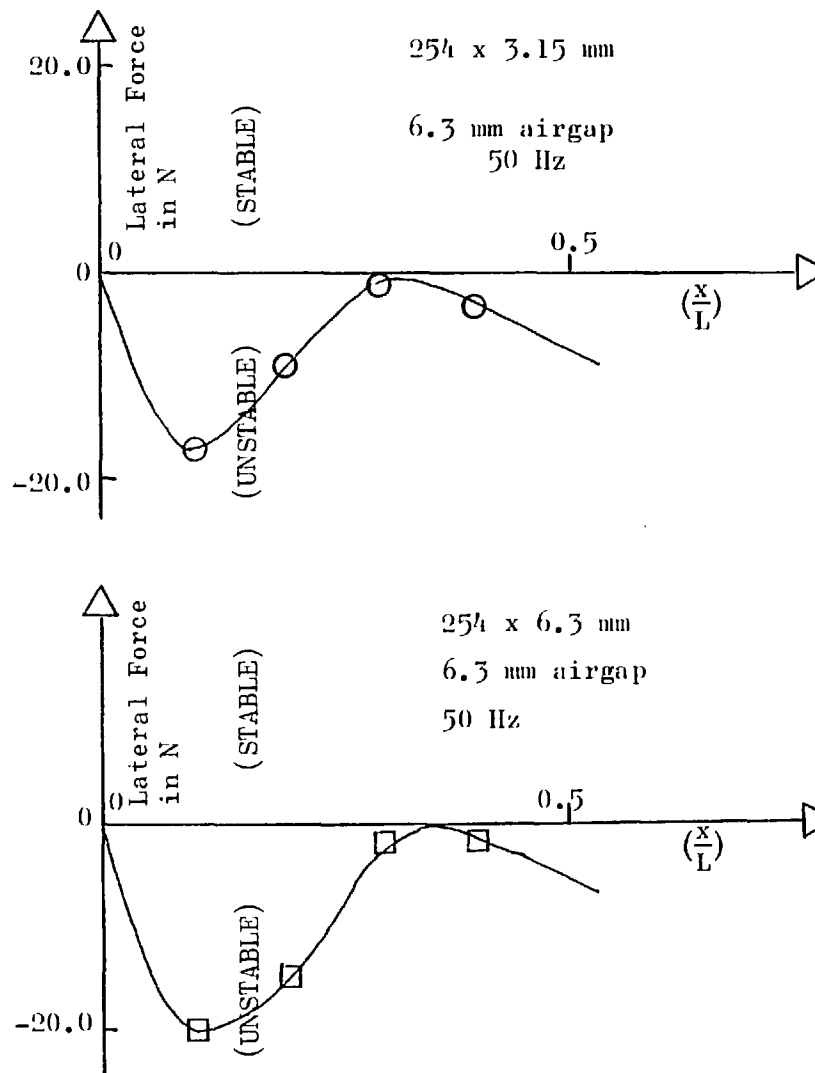


Figure 2.31: Lateral stability of the three-phase Xi-core.

From these results, it is seen that a plate is stable because the symmetrical position is one of maximum stored energy, for small lateral perturbations. Decrease of airgap changes this situation and the symmetrical position becomes one of minimum stored energy for small perturbations, making the plate unstable.

2.7 A METHOD OF INCREASING LATERAL STABILITY

From the experimental results presented so far it emerges that the wider plates which are desirable from the point of view of normal force magnitude are laterally unstable. Since narrow plates (of widths less than $2(a+b)$) are highly stable, it was decided to clamp a narrow plate and a wide plate rigidly and measure the lateral force variation of the combination. This arrangement can also be considered as a wider plate in which the conducting material near the plate edges has been carved out. The lateral force variation obtained experimentally is shown in Fig. 2.28. It is seen that the removal of material in this manner increases the lateral stability dramatically. However, this form of rotor configuration when examined theoretically in a later chapter, will be shown to possess a reduced rolling stability. This is not important if a catamaran arrangement is used.

2.8 EXPERIMENTS ON THREE-PHASE Xi-CORE

In the introduction it was stated, by physical reasoning, that for lateral stability investigation one can apply the conclusions of the single-phase machine to a three-phase Xi-core as well. In order to obtain experimental confirmation of this, the lateral forces of the three-phase Xi-core (Fig. 2.2) were measured. The experiments carried out are:

- 1) Measurement of lateral force with displacement of several plate widths, all of the same thickness and at the same airgap at 50 Hz.
- 2) Measurement of lateral force with displacement for two plates of equal widths at 50 Hz, at the same airgap, but differing only in their thicknesses.
- 3) Measurement of lateral forces at 50 Hz, for all plates used in (1) but at a smaller airgap.
- 4) Repetition of (3), with all parameters remaining the same but the frequency being increased to 100 Hz.

These results are presented in Figs. 2.29, 2.30 and 2.31, with the horizontal displacement expressed as a fraction of the plate width. The cross-over point of the curves is seen to be 0.35 for the 19 mm airgap and 0.325 for the 6.3 mm airgap. Disregarding this minor variation, the characteristics of the curves are the same as those for the single-phase Xi-core.

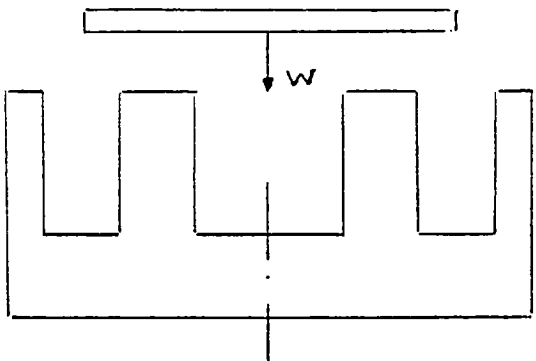
2.9 CONCLUSIONS

The experimental results presented in this chapter have shown the practical possibility of levitating conducting plates stably over the Xi-core. The kVA requirements from the supply, however, have been seen to be very large, showing that in any full scale machine reactive power compensation is essential. A favourable characteristic of the machine, from the design point of view, is the fact that when a particular rotor is levitated at several airgaps, the rotor power loss remains substantially constant. Therefore, in

levitators that are required to support a constant weight at variable airgaps, the large power fluctuations at the input terminals are mainly due to the ohmic loss variations of the stator windings. As a consequence of the nearly constant plate power loss, it was found that one can approximately assume direct proportionality between levitation force and plate power loss.

The factor determining whether a particular rotor is stable or not has been seen to be the Goodness Factor. While the precise relationship could not be found, it has been established that for a stable rotor, any change which increases its goodness factor leads to instability. The following table is an overall indication of the order of magnitude of the various quantities of the single-phase Xi-core:

kVA	kW	Plate Loss kW	P.F. (Input)	P.F. (Shadow)
2600	360	170	0.138	0.065



$W = 1 \text{ tonne}$
 Airgap = 20 mm
 Plate; 127 x 6.3 mm
 Maximum Stabilising Force = 0.25 Tonne
 Maximum Permissible Displacement = 25 mm

Table 2.2

Considering the lateral force/lateral displacement characteristics of the stable plates, they can be approximately described by a sinusoidal variation as:

$$F_x = F_{x,\max} \sin\left(\frac{2\pi}{0.35} \cdot \frac{x}{L}\right)$$

The maximum permissible lateral displacement for the stable restoration of these plates is therefore 0.17L. Finally, it has been shown that the lateral stability of a three-phase machine is similar to that of the single-phase machine. In the following chapter, theoretical investigations carried out are described, which confirm the experimentally found characteristics of the single-phase Xi-core described in this chapter.

SIMPLE MODELS USING COILS3.1 INTRODUCTION

This chapter explores the possibility of using a circuit approach for the analysis of axially symmetric levitation systems. The analysis is not general, but confined to two classes of levitators: the E-core and the Xi-core. Analytical expressions are derived for the levitation force of these two machines, in terms of rates of change of measurable quantities at the input terminals. The dimensionless ratio G introduced in Chapter 2 is shown to be the goodness factor. In the case of the E-core levitator, lateral stability is considered and it is shown that $G > 1$ is a necessary but not sufficient condition for stability. Experimental results are included to show the limitations of the derived force expressions in the case of the Xi-core.

3.2 REVIEW OF PUBLISHED WORK

Smith^{3.1,3.2,3.3,3.4,3.5} has investigated the relationship between the force acting on a group of short-circuited coils, which are inductively coupled to a set of primary coils, and the rate of change of effective inductance measured at the input terminals of the primary coils. A most important assumption in his theoretical model is that the circuit self-parameters (self-inductance, resistance) of the short-circuited coils are independent of their position with respect to the fixed primary coils. Therefore his model excludes the presence of iron, and the final result obtained is:

$$F_x = \frac{1}{2} I_1^2 \frac{\partial L_e}{\partial x} \quad (5.1)$$

where, F_x = time average levitation force acting in the direction of displacement x ,

L_e = the effective inductance measured at the input terminals.

This result has been critically examined by Harris^{5,6}, for the case of one primary exciting coil and an arbitrary number of secondary short-circuited coils. In this examination a "universally valid" force expression is assumed:

$$F = \frac{1}{2} [I^*] \cdot \left[\frac{\partial L}{\partial x} \right] \cdot [I] \quad (5.2)$$

where, $[I]$ is a column matrix of all the currents, including the single primary current,

$[I^*]$ is a row vector whose elements are the complex conjugates of $[I]$,

$[L]$ is the complete inductance matrix containing the self and mutual inductances of all the coils.

After expanding equation (5.2), the difference between F and $\frac{1}{2} \frac{I_1^2}{\omega} \cdot \frac{\partial X_e}{\partial x}$ is obtained as:

$$F - \frac{1}{2} \frac{I_1^2}{\omega} \cdot \frac{\partial X_e}{\partial x} = \frac{1}{2} I_1^2 \left\{ \omega^2 [M] \cdot [Z_2^*]^{-1} \left[\frac{\partial L_2}{\partial x} \right] [Z_2]^{-1} [M] \right\} \\ - \frac{1}{2} I_1^2 \left\{ \omega [M] \operatorname{Im} \left(\frac{\partial}{\partial x} [Z_2]^{-1} \right) [M] \right\} \quad (5.3)$$

where, $[M]$ = primary/secondary mutual inductance matrix,

$[Z_2]$ = secondary complex impedance matrix,

$[L_2]$ = secondary inductance matrix,

I_1 = primary phasor current of r.m.s. value I_1

Harris proceeds to examine the conditions under which the right-hand side (R.H.S.) of equation (5.3) is zero, thus establishing the

conditions under which one may use the force equation (3.1).

3.3 EXTENSION OF THEORY FOR TWO COILS

One can adopt an alternative approach, and instead of finding the conditions under which the R.H.S. of equation (3.3) is zero, explore the possibility of expressing it in terms of measurable parameters at the input terminals. If the effort is successful, then one can write:

$$F = \frac{1}{2} \frac{I_1^2}{\omega} \cdot \frac{\partial X_e}{\partial x} + \text{R.H.S. of equation (3.2) expressed in terms of measurable parameters.}$$

Such an investigation carried out for the general case proved unsuccessful, largely due to the presence of the inverse of the secondary impedance matrix, $[Z_2]^{-1}$. However, for the special case of one primary and one secondary coil, it was found possible to express the R.H.S. of equation (3.3) in terms of measurable parameters.

3.3.1 The Model and the Assumptions

Fig. 3.1 shows the model which can also be considered as a simplified model of the air-backed E-core levitator.

The assumptions are:

R_1 - primary resistance;	$\frac{\partial R_1}{\partial x} = 0$
X_1 - primary reactance;	$\frac{\partial X_1}{\partial x} = 0$
M - mutual inductance;	$\frac{\partial M}{\partial x} \neq 0$

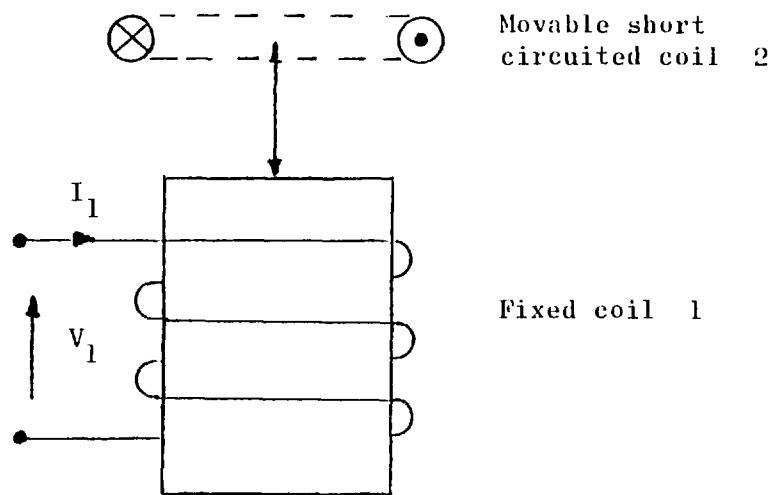


Figure 3.1: The two-coil model.

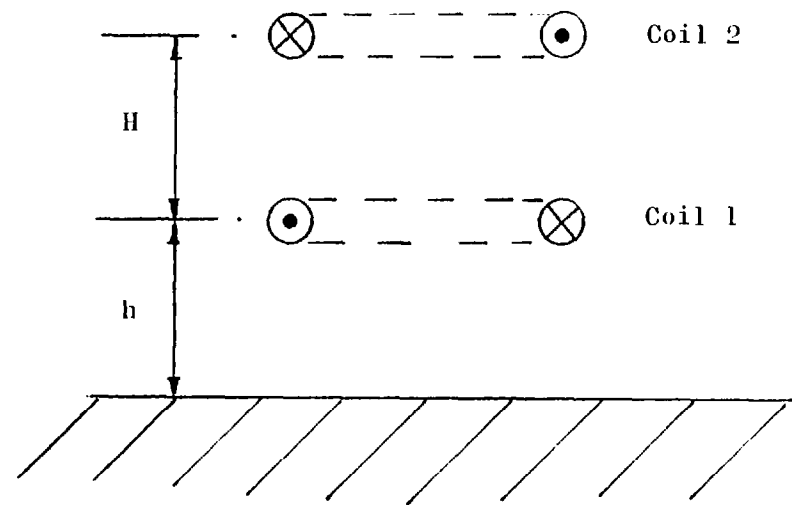


Figure 3.2: The model used to obtain analytical proof.

X_2 - secondary reactance; $\frac{\partial X_2}{\partial x} \neq 0$

R_2 - secondary resistance; $\frac{\partial R_2}{\partial x} = 0$

Substituting these circuit parameters in the equation (3.3) yields:

$$F = \frac{1}{2} \frac{I_1^2}{\omega} \cdot \frac{\partial X_e}{\partial x} = \frac{1}{2} I_1^2 \omega M^2 \left\{ \frac{1}{R_2^2 + X_2^2} \cdot \frac{\partial X_2}{\partial x} - \operatorname{Im} \left[\frac{\partial}{\partial x} \left(\frac{1}{R_2 + jX_2} \right) \right] \right\}$$

i.e.

$$F = \frac{1}{2} \frac{I_1^2}{\omega} \cdot \frac{\partial X_e}{\partial x} = \frac{\omega M^2 R_2^2 I_1^2}{(R_2^2 + X_2^2)^2} \cdot \frac{\partial X_2}{\partial x} \quad (3.4)$$

Since the objective is to express the R.H.S. of equation (3.4) in terms of the measurable parameters at the input terminals, these will be examined next.

3.3.2 The Parameters at the Input Terminals

By writing the circuit equations for the two coils and eliminating the secondary coil current, we arrive at the equation:

$$V_1 = \left[R_1 + jX_1 + \frac{\omega^2 M^2 (R_2 - jX_2)}{(R_2^2 + X_2^2)} \right] I_1 \quad (3.5a)$$

Let the effective resistance and effective reactance at the input terminals be R_e and X_e respectively. Then:

$$R_e = R_1 + \frac{\omega^2 M^2 R_2}{R_2^2 + X_2^2} \quad (3.5b)$$

$$X_e = X_1 - \frac{\omega^2 M^2 X_2}{R_2^2 + X_2^2} \quad (3.5c)$$

$$\text{Therefore } \frac{(X_1 - X_e)}{(R_e - R_1)} = \frac{X_2}{R_2} = G \quad (3.5d)$$

The dimensionless ratio G is the secondary circuit time constant multiplied by the angular frequency ω and is therefore the Goodness Factor of the secondary circuit and can be measured at the terminals using equation (3.5d). From equations (3.5b), (3.5c) and (3.5d), the following relationship, which will be required later, is obtained:

$$\frac{(R_e - R_1)^3}{[(R_e - R_1)^2 + (X_e - X_1)^2]} = \frac{\omega^2 M^2 R_2^3}{(R_2^2 + X_2^2)^2} \quad (3.6)$$

3.3.3 Force under all Conditions

Since $X_2 = R_2 G$, equation (3.4) can be written as:

$$F = \frac{1}{2} \cdot \frac{I_1^2}{\omega} \cdot \frac{\partial X_e}{\partial x} = \frac{I_1^2}{\omega} \cdot \left[\frac{\omega^2 M^2 R_2^3}{(R_2^2 + X_2^2)^2} \right] \cdot \frac{\partial G}{\partial x}$$

Substituting equation (3.6):

$$F = \frac{1}{2} \cdot \frac{1}{(G^2 + 1)} \cdot \frac{I_1^2}{\omega} \cdot \left\{ (G^2 - 1) \frac{\partial X_e}{\partial x} - 2G \cdot \frac{\partial R_e}{\partial x} \right\} \quad (3.7)$$

Thus equation (3.7) gives the levitation force, in terms of the measurable parameters at the input, for levitation systems that can be approximated by two simple coils.

3.3.4 Stability of the Two-Coil Model

Consider the stored magnetic energy of the system. If E is the stored energy, then:

$$E = \frac{1}{\omega} \cdot X_e \cdot I_1^2$$

Under constant current excitation, let the secondary coil be free to move horizontally (in the x direction). Then:

$$\frac{\partial E}{\partial x} = \frac{1}{\omega} \cdot I_1^2 \cdot \frac{\partial X_e}{\partial x}.$$

Hence:

$$\frac{\partial E}{\partial x} = \left[-\frac{2I_1^2 \omega X_2^M}{(R_2^2 - X_2^2)} \cdot \frac{\partial M}{\partial x} \right] + \left[\frac{I_1^2 \omega M^2}{(R_2^2 + X_2^2)^2} (R_2^2 - X_2^2) \cdot \frac{\partial X_2}{\partial x} \right] \quad (3.8)$$

Making the assumptions that $\frac{\partial M}{\partial x}$ and $\frac{\partial X_2}{\partial x}$ are always negative, for small displacements around the symmetrical position, we examine the sign of $\frac{\partial E}{\partial x}$.

Case 1: $G < 1$

If $G < 1$, then $R_2 > X_2$, and both terms in square brackets in the R.H.S. of equation (3.8) are positive, and therefore their sum is also positive.

$$\therefore \quad \text{if } G < 1, \quad \frac{\partial E}{\partial x} > 0,$$

i.e. if $G < 1$, then the symmetrical position is one of minimum stored energy, for small perturbations and therefore the coil is unstable.

Case 2: $G > 1$

If $G > 1$, $R_2 < X_2$, and the first term in square brackets is always positive as before, but the second term in square brackets is now always negative. Thus the first requirement for the sum to be negative is fulfilled. Consequently we see that if $G > 1$, stable levitation is possible but not guaranteed (see Appendix A).

3.4 VERIFICATION OF THE FORCE EXPRESSION

The direct application of the derived force equation (3.7) would be in the evaluation of the levitation force of the E-core levitator. Since all experimental research by the author was restricted to the Xi-core machine only, a theoretical model was chosen for the verification of the force equation (3.7).

Fig. 3.2 shows the model chosen, in which an infinitely long rectangular coil carrying an alternating current is present above an infinitely permeable iron surface.

In a second coil, which is also an infinitely long rectangular coil, but short circuited, a secondary complex current is induced. For different values of the coil separation H , the parameters G , R_e , X_e , $\frac{\partial R_e}{\partial H}$, $\frac{\partial X_e}{\partial H}$ were evaluated allowing for the presence of the iron surface by means of images.

Thus the levitation force on the secondary can be evaluated using the force equation (3.7). The levitation force for each of the coil separation values H was also calculated using Ampere's force expression between two current -carrying conductors. For this calculation too, the self and mutual inductances, and hence the complex secondary current, was evaluated, by the method of images. Finally the levitation force arising from the use of the Smith equation (3.1) was computed as well. The results are shown in Fig. 3.3, for several values of coil separation.

It is seen that complete agreement obtained between the derived force expression and Ampere's force values, at all values of coil separation. The force evaluated by using Smith's

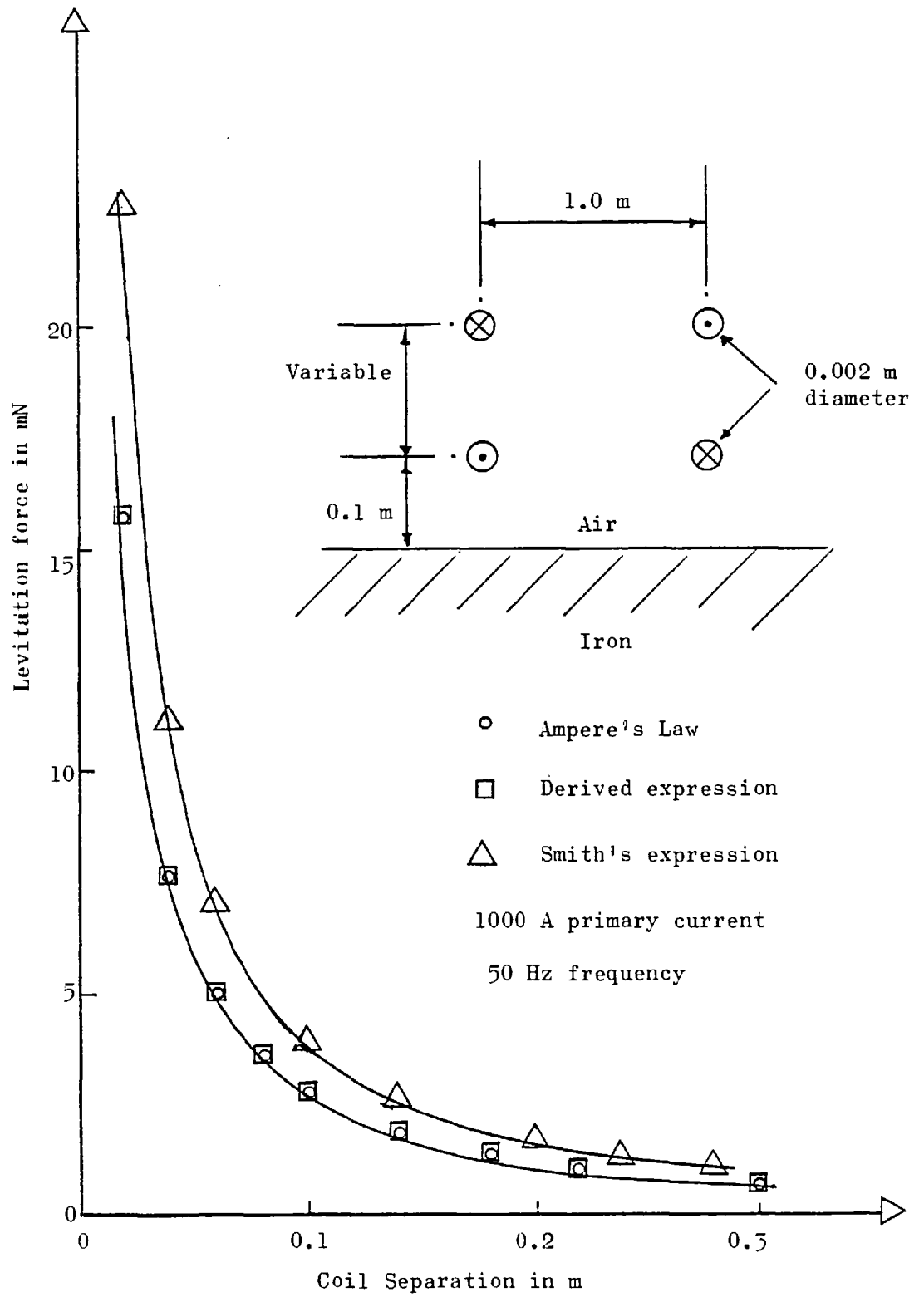


Figure 3.3: Analytical Verification of the Derived Force Expression.

expression approaches the values obtained by the other two methods only at large values of coil separation. The reason becomes clear when the derived force equation (3.7) is examined. As G tends to zero (as happens at large coil separation), equation (3.7) reduces to the Smith equation (3.1).

3.5 A LUMPED CIRCUIT MODEL OF THE Xi-CORE

The theory developed in the preceding sections cannot be applied to the Xi-core, because for an approximate description of the Xi-core at least four coils are required, as shown in Fig. 3.4. The coils are assumed to move in the vertical direction only, so that the currents in coils (3) and (4) are equal. The assumptions on the variations of coil parameters are as follows:

$$X_1 = X_2; \quad \frac{\partial X_1}{\partial y} = \frac{\partial X_2}{\partial y} = 0$$

$$R_1 = R_2; \quad \frac{\partial R_1}{\partial y} = \frac{\partial R_2}{\partial y} = 0$$

$$X_3 = X_4; \quad \frac{\partial X_3}{\partial y} = \frac{\partial X_4}{\partial y} \neq 0$$

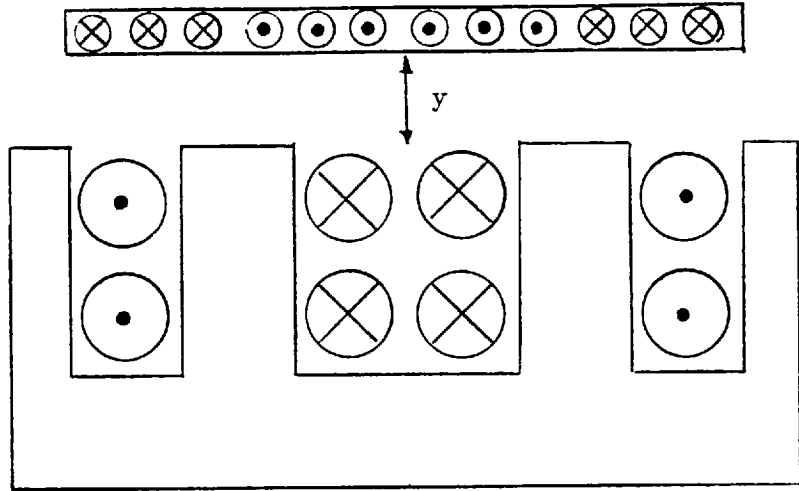
$$R_3 = R_4; \quad \frac{\partial R_3}{\partial y} = \frac{\partial R_4}{\partial y} = 0$$

$$\frac{\partial M_{13}}{\partial y} = 0; \quad \frac{\partial M_{14}}{\partial y} \neq 0; \quad \frac{\partial M_{34}}{\partial y} \neq 0; \quad \frac{\partial M_{12}}{\partial y} = 0$$

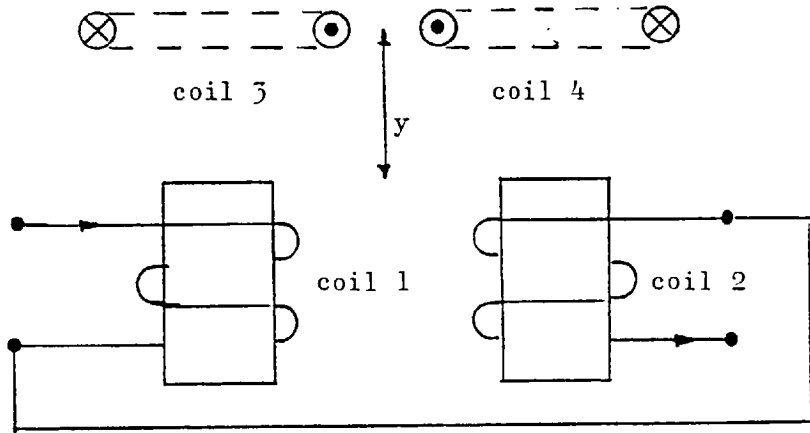
$$M_{13} = M_{24}; \quad M_{14} = M_{23}$$

$$I_3 = I_4 = I_s \text{ (say)}$$

The total time average force acting on the rigidly connected coils (3) and (4) is twice the time average force acting on coil (1).



The Xi-core



The Model

Figure 3.4: The lumped circuit model of the Xi-core, with symmetrical rotor.

The force acting on coil (1) is:

$$F = \operatorname{Re} \left[\frac{1}{2} I I^* \frac{\partial L_{11}}{\partial y} + I I_s^* \frac{\partial L_{13}}{\partial y} + I I_s^* \frac{\partial L_{14}}{\partial y} \right]$$

$$\frac{\partial L_{11}}{\partial y} = 0$$

Therefore:

$$F = \operatorname{Re} \left[I I_s^* \cdot \frac{\partial}{\partial y} (L_{13} + L_{14}) \right]$$

Hence the total force on coils (3) and (4) is:

$$F = \operatorname{Re} \left[2 I I_s^* \frac{\partial}{\partial y} (L_{13} + L_{14}) \right] \quad (3.9)$$

3.5.1 The Parameters at the Input Terminals

Applying the circuit equations for coils (1) and (2) we obtain:

$$V = 2 \{ (R_1 + jX_1 + jX_{12})I + j(X_{13} + X_{14})I_s \}$$

For either of coils (3) or (4):

$$0 = \{ R_s + j(X_s + X_{34}) \} I_s + j(X_{32} + X_{31})I$$

$$\therefore I_s = - \frac{j(X_{14} + X_{13})}{[R_s + j(X_s + X_{34})]} \cdot I \quad (3.10)$$

and:

$$V = 2 \left\{ (R_1 + jX_1 + jX_{12}) + \frac{(X_{13} + X_{14})^2}{R_s + j(X_s + X_{34})} \right\} I$$

From this equation, the effective resistance R_e , and the effective reactance X_e can be deduced as follows:

$$R_e = 2R_1 + 2 \cdot \frac{(X_{13} + X_{14})^2 \cdot R_s}{[R_s^2 + (X_s + X_{34})^2]} \quad (3.11a)$$

$$X_e = 2(X_1 + X_{12}) - 2 \frac{(X_s + X_{34})(X_{13} + X_{14})^2}{[R_s^2 + (X_s + X_{34})^2]} \quad (3.11b)$$

$$\text{Hence } G = \frac{(X_o - X_e)}{(R_e - 2R_1)} = \frac{(X_s + X_{34})}{R_s} \quad (3.11c)$$

Thus G is the angular frequency of supply, multiplied by the time constant of decay of secondary current, when the primary circuit is open circuited. It is important to note that the algebraic signs of X_s and X_{34} depend on the manner of arranging the primary excitation. For the Xi-core excitation shown in Fig. 3.4, they are opposite in sign, leading to lower values of G , than would obtain if the excitation is such that the inner slot currents are in opposite directions. The latter arrangement has been found to give rise to total instability by Laithwaite and Eastham^{3.7}. Therefore this is further evidence that stability of the Xi-core is associated with low values of G .

From equations (3.11a) and (3.11b), the following relationships, to be used later, are obtained:

$$(R_e - 2R_1) = 2 \cdot \frac{(X_{13} + X_{14})^2 \cdot R_s}{[R_s^2 + (X_s + X_{34})^2]} \quad (3.12a)$$

$$(X_e - X_o) = -2 \cdot \frac{(X_{13} + X_{14})^2 \cdot [X_s + X_{34}]}{[R_s^2 + (X_s + X_{34})^2]} \quad (3.12b)$$

$$(R_e - 2R_1)^2 + (X_e - X_o)^2 = 4 \cdot \frac{(X_{13} + X_{14})^4}{[R_s^2 + (X_s + X_{34})^2]} \quad (3.12c)$$

3.5.2 The Levitation Force of the Xi-Core

The levitation force on the two coils (3) and (4), in Fig. 3.4, was shown to be (equation (3.9)):

$$F = \operatorname{Re} \left[2 I I_s^* \frac{\partial}{\partial y} (L_{13} + L_{14}) \right]$$

and I_s was shown to be (equation (3.10)):

$$I_s = \frac{-j(X_{14} + X_{13})}{[R_s + j(X_s + X_{34})]} \cdot I$$

.. substituting for I_s ,

$$F = -\left(\frac{I^2}{\omega}\right) \cdot \frac{2(X_{14} + X_{13})(X_s + X_{34})}{[R_s^2 + (X_s + X_{34})^2]} \cdot \frac{\partial}{\partial y} [X_{14} + X_{13}]$$

Substituting equation (3.12b):

$$F = \left(\frac{I^2}{\omega}\right) \cdot \frac{(X_e - X_o)}{(X_{14} + X_{13})} \cdot \frac{\partial}{\partial y} (X_{14} + X_{14})$$

From equations (3.12a) and (3.12c):

$$(X_{13} + X_{14})^2 = \frac{[(R_e - 2R_1)^2 + (X_e - X_o)^2]}{(R_e - 2R_1)} \cdot \frac{R_s}{2}$$

Differentiating both sides with respect to y and dividing the result by $(X_{13} + X_{14})^2$:

$$\frac{2}{(X_{13} + X_{14})} \cdot \frac{\partial}{\partial y} [X_{13} + X_{14}] = \frac{(R_e - 2R_1)}{[(R_e - 2R_1)^2 + (X_e - X_o)^2]} \cdot \frac{\partial}{\partial y} \left[\frac{(R_e - 2R_1)^2 + (X_e - X_o)^2}{(R_e - 2R_1)} \right]$$

Substituting the result in the force equation:

$$F = \frac{1}{2} \cdot \frac{I^2}{\omega} \cdot \frac{(R_e - 2R_1)(X_e - X_0)}{[(R_e - 2R_1)^2 + (X_e - X_0)^2]} \cdot \frac{\partial}{\partial y} \left[\frac{(R_e - 2R_1)^2 + (X_e - X_0)^2}{(R_e - 2R_1)} \right] \quad (3.13)$$

Equation (3.13) gives the levitation force in terms of measurable parameters at the input terminals. The same equation can be expressed in terms of G as:

$$F = \frac{1}{2} \left(\frac{I^2}{\omega} \right) \cdot \frac{G}{(1 + G^2)} \cdot \frac{\partial}{\partial y} [(1 + G^2)R_e] \quad (3.14)$$

3.6 VERIFICATION OF THE FORCE EXPRESSION

The derived force equation (3.14) was verified experimentally, by measuring the input impedance variation with airgap of a secondary plate situated symmetrically above the Xi-core. The result (Figure 3.5) shows that the simplified circuit model represents the real machine only at large values of airgap. The major assumption which is not satisfied by the real machine is the assumed constancy of the secondary resistance R_2 . That is $\frac{\partial R_2}{\partial y} \neq 0$ at small airgaps. This is so, because the secondary induced current distribution pattern is subject to rapid change at small airgaps due to the concentration of flux pattern due to the iron corners of the stator teeth.

3.7 CONCLUSION

The major outcome of the simplified analysis described in this chapter is the quantitative proof that the dimensionless ratio G derives from the input impedance, is the secondary circuit time constant multiplied by the angular frequency of supply. G therefore is the Goodness Factor of the secondary circuit. The analysis has shown that the value of G for the E-core represented by the single

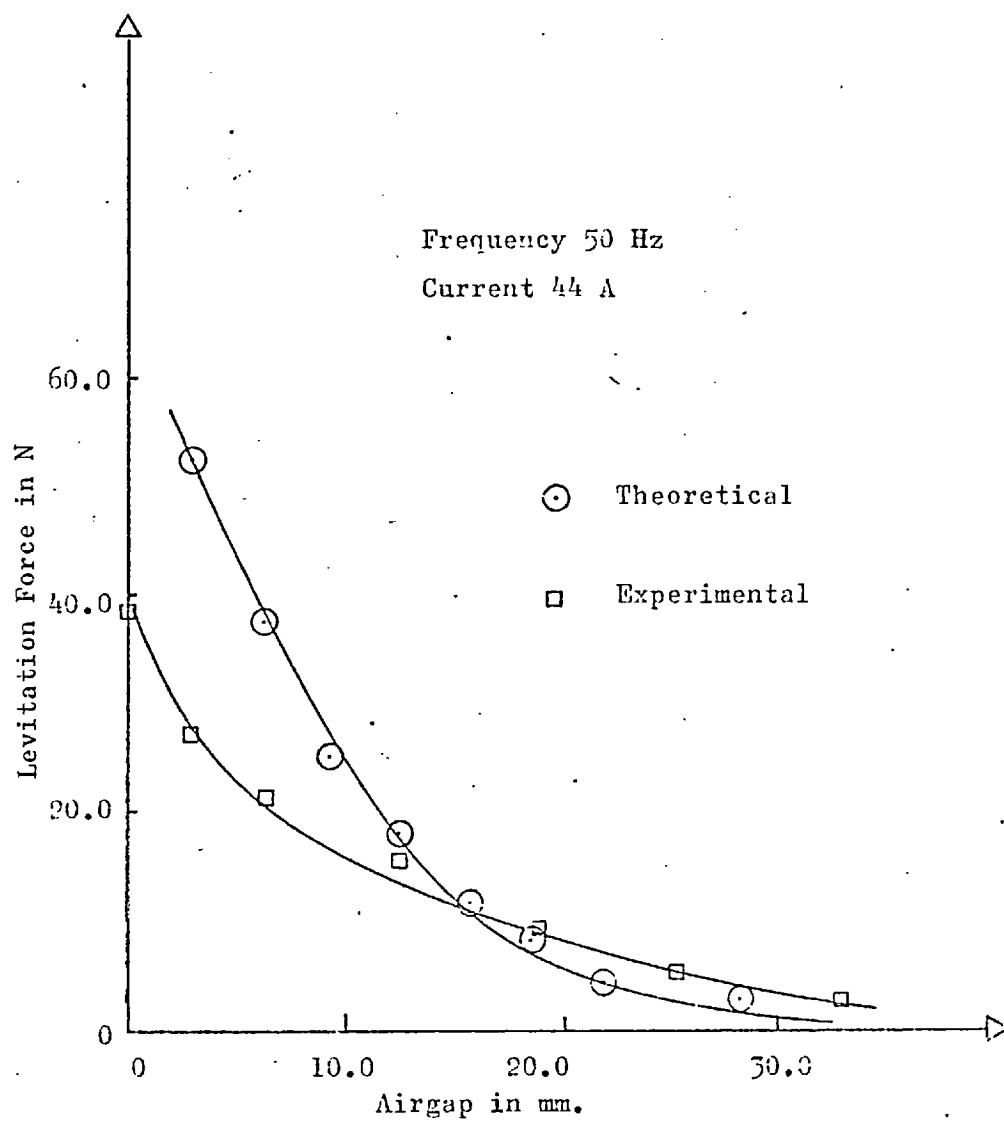


Figure 3.5: Experimental Verification of the Derived Force Expression for the Xi-Core.

coil model is always higher than that of the Xi-core, as represented by the two-coil model. If the primary coils of the Xi-core are connected so as to have currents flowing in opposite directions in the centre slot, then the value of G will always be higher than that of the E-core. This, however, leads to instability of rotor plates. Thus we see indirect evidence that low Goodness Factor is associated with lateral stability, in the case of the Xi-core. This approximate analysis also enables one to examine the nature of the input impedance variations, and this is given in Appendix B. The limitations of circuital approaches which require assumptions of one kind or another have been clearly brought out in this chapter. Therefore the need arises for a more rigorous method of analysis which requires the smallest number of assumptions, if any. In the following chapters, two different numerical methods employing field theory are presented.

THEORY OF FINITE DIFFERENCE ANALYSIS4.1 INTRODUCTION

The solution of electromagnetic field problems by finite difference method, using a digital computer, was pioneered by Mamak and Laithwaite^{4.1}, who employed a matrix inversion method. This was followed by Roberts^{4.2}, who introduced a method of solving eddy current problems by means of R-C network analogues experimentally. After extending the concept of magnetic equivalent circuits^{4.3}, Carpenter has carried out extensive research^{4.4}, formulating low frequency electromagnetic field problems in terms of networks, which can be solved on a digital computer, using iterative methods. This network approach is adopted in this chapter for the description of the finite difference method. The potential function chosen to describe the two-dimensional field is the magnetic vector potential. It is shown that the required network analogue of the field can be obtained directly by the application of Ampere's circuital law, without invoking the duality principle, as is customarily done^{4.5}, when the magnetic vector potential is used. The open nature of the boundaries necessitated the use of a special technique called the "Picture Frame Transformation". This technique has been used recently by Wyatt^{4.6}, but the derivation of the theory in network terms is original. The factors influencing the convergence of the iterative solution are discussed. The methods used to calculate forces, power loss, etc. from the computed field solution, are given.

4.2 THE ELECTROMAGNETIC FIELD EQUATIONS

The steady state, low frequency, phasor form of Maxwell's equations, together with two other constitutive relations, are:

$$\text{Curl } H = J_s + J_e \quad (4.1a)$$

$$\text{Curl } E = -j\omega B \quad (4.1b)$$

$$\text{Div}(J_s + J_e) = 0 \quad (4.1c)$$

$$\text{Div } B = 0 \quad (4.1d)$$

$$J_e = \sigma E \quad (4.1e)$$

$$B = \mu_0 \mu_r H \quad (4.1f)$$

In the above equations, the symbols represent the following complex quantities:

J_s : Applied source current density,

J_e : Induced current density,

H : Intensity of magnetic field,

B : Magnetic flux density,

E : Intensity of electric field,

σ : Electrical conductivity,

μ_0 : Permeability of free space,

μ_r : Relative permeability,

ω : Angular frequency of supply.

4.3 PROBLEM DESCRIPTION BY VECTOR POTENTIAL

A Xi-core machine which is infinitely long, is chosen for analysis. Fig. 4.1 shows the co-ordinate axes chosen and is used as a reference in the rest of this chapter. The field variables H and B have spatial components H_x, H_y, B_x, B_y , while the current densities

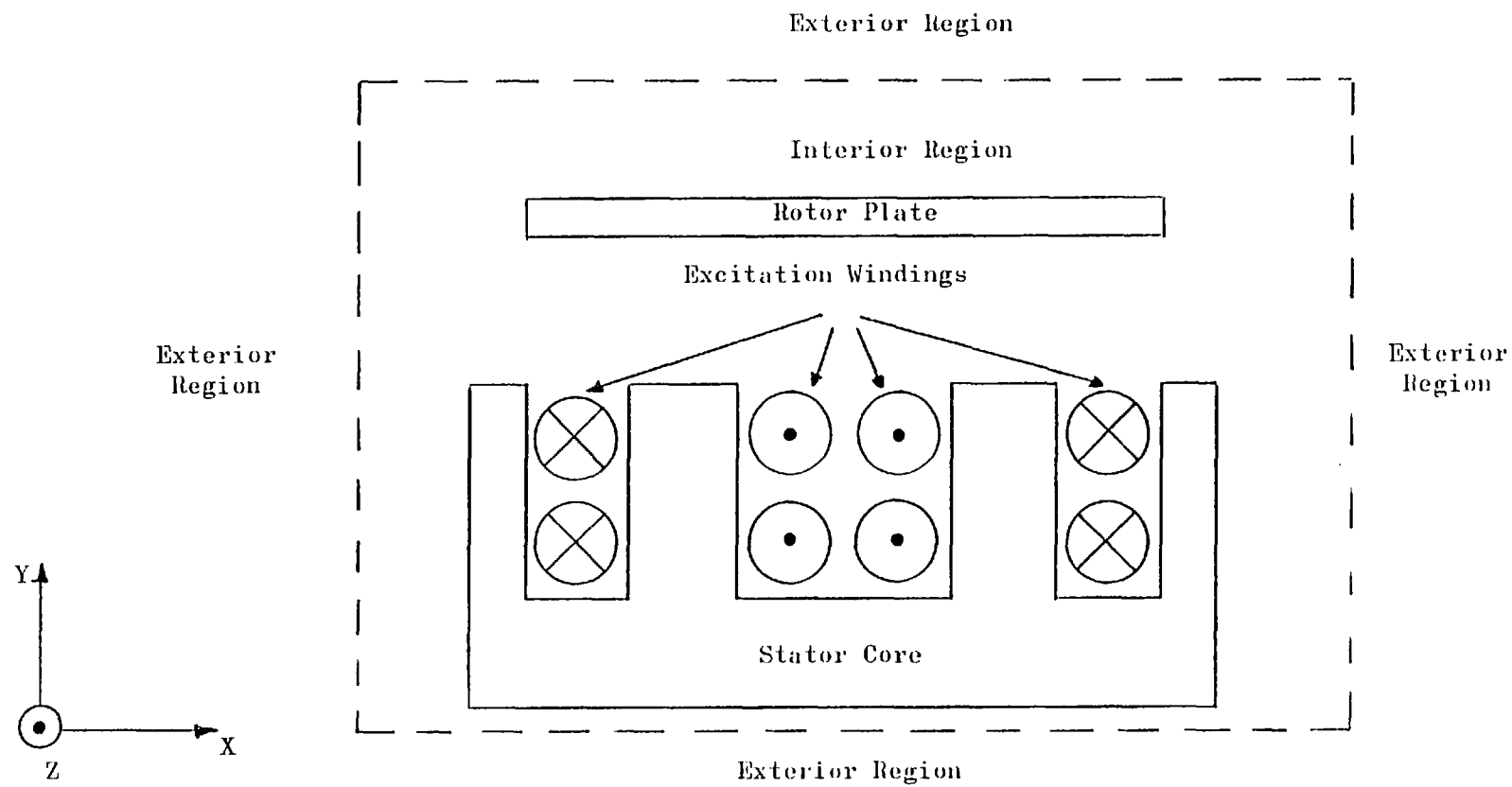


Figure 4.1: The two-dimensional model for solution by the finite difference method.

J_e and J_s have only Z-directed components. The analysis will be vastly simplified if one adopts a potential function whose one component enables the derivation of the two spatial components of the field variables. The choice of a potential function for three-dimensional eddy current problems is a difficult one, as discussed by Sarma^{4.7} and Carpenter^{4.8}. However, for two-dimensional field problems, the magnetic vector potential A is sufficient. It is defined as:

$$\text{Curl } A = B \quad (4.2)$$

Since the divergence of the curl of a vector is zero, the requirement $\text{Div } B = 0$ is satisfied immediately. However, by Helmholtz's uniqueness theorem, the vector A is not unique until its divergence is specified as well. Adopting the commonly used value,

$$\text{Div } A = 0 \quad (4.3)$$

Substituting for B in terms of A, in equation (4.1b), one obtains:

$$\text{Curl } E = -j\omega \cdot \text{Curl } A \quad (4.4)$$

Since curl is a differentiation with respect to the space variables, integration of both sides of equation (4.4) yields:

$$E = -j\omega A + \text{constant of integration.}$$

The curl of a gradient is always zero. Therefore if the constant of integration is expressed as the gradient $-\Delta\phi$ of a scalar function ϕ , then equation (4.4) is satisfied always. The electric field in terms of A is now given by:

$$E = -j\omega A - \text{Grad } \phi \quad (4.5)$$

4.3.1 Significance of the Grad ϕ Component

For the flow of current in a conductor, there must exist a distribution of charges on its surface, as discussed by Hammond^{4.9}. With reference to the Xi-core, the direction of the total electric field E within the conducting plate in Fig. 4.1 is governed by the direction of flow of the induced current density J_e , since $J_e = \sigma E$. Since J_e is constrained to be Z-directed, the electric field also needs to be Z-directed, within the conductor, while outside it can have any direction in the X-Y plane. Due to the resulting discontinuity of the total electric field at the conductor/air interface, free charges appear to form a surface distribution on the plate. The grad ϕ term represents the conservative field due to these charges. Therefore the scalar function ϕ needs to be computed only in the conducting plate region.

4.3.2 Field Equations in Terms of A

$$\text{Curl } A = B \quad \text{by definition}$$

$$\therefore \text{Curl.Curl } A = \mu_0 \mu_r \text{ Curl } H$$

$$\text{i.e. Grad.Div } A - \text{Div.Grad } A = \mu_0 \mu_r (J_s + J_e)$$

$$\text{But Div } A = 0 \quad \text{by definition}$$

$$\therefore \nabla^2 A = -\mu_0 \mu_r (J_s + J_e) \quad (4.6)$$

Since the excitation current is only in the Z-direction, only one component of A (i.e. A_z) is non-zero. For the rest of this chapter, therefore, A is to be regarded as representing the A_z component only. The three different forms of equation (4.6) that apply to the Xi-core are:

Stator Winding:

$$\nabla^2 A = -\mu_o \mu_r J_s \quad (4.6a)$$

Current Free Regions:

$$\nabla^2 A = 0 \quad (4.6b)$$

Plate Region:

$$\nabla^2 A = -\mu_o \mu_r J_e$$

Since $J_e = \sigma E$ and $E = -(j\omega A + \Delta\phi)$,

$$\nabla^2 A = \mu_o \mu_r \sigma (j\omega A + \Delta\phi)$$

i.e.
$$\nabla^2 A = j\omega \mu_o \mu_r \sigma (A - \frac{j}{\omega} \cdot \Delta\phi)$$

The term $\frac{j}{\omega} \cdot \Delta\phi$ can be written as an equivalent Z-directed component of the vector potential A_s . Therefore the final form of the equation is:

$$\nabla^2 A = j\omega \mu_o \mu_r \sigma (A - A_s) \quad (4.6c)$$

4.4 FINITE DIFFERENCE EQUATIONS FROM AMPERE'S LAW

To derive an admittance network, which is a physical analogue of the electromagnetic field described by equations (4.6a), (4.6b) and (4.6c), the cross-section of the machine shown in Figure 4.1 is filled with a set of regularly spaced nodes. In Figure 4.2, which shows a small area of such a distribution of nodes, KLMN is a square contour enclosing a source current of $h^2 J_s$ and eddy current of $h^2 J_e$. Applying Ampere's circuital law to the contour KLMN:

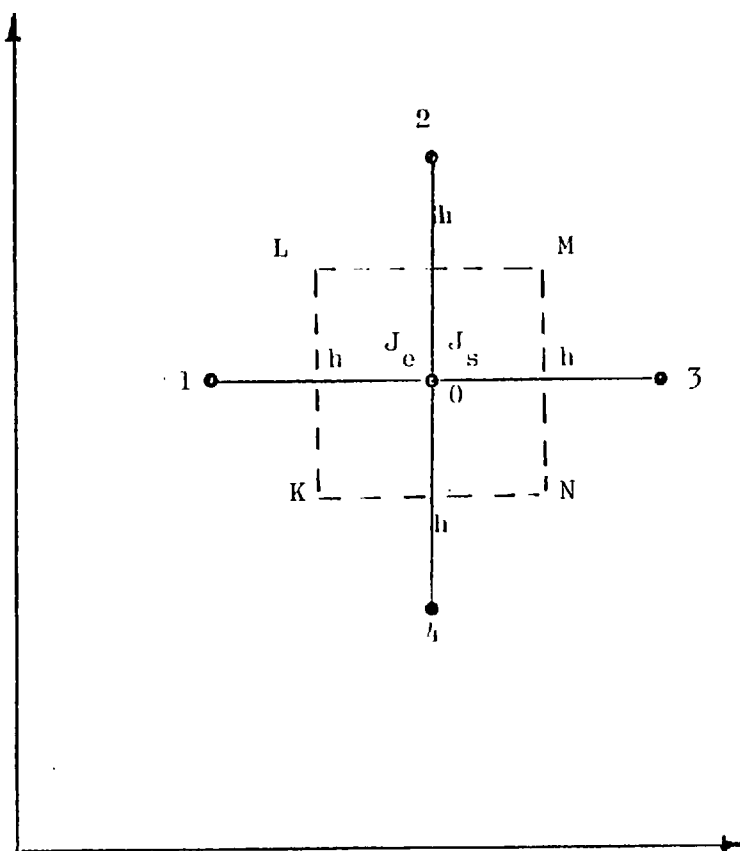


Figure 4.2: A region of the complete model.

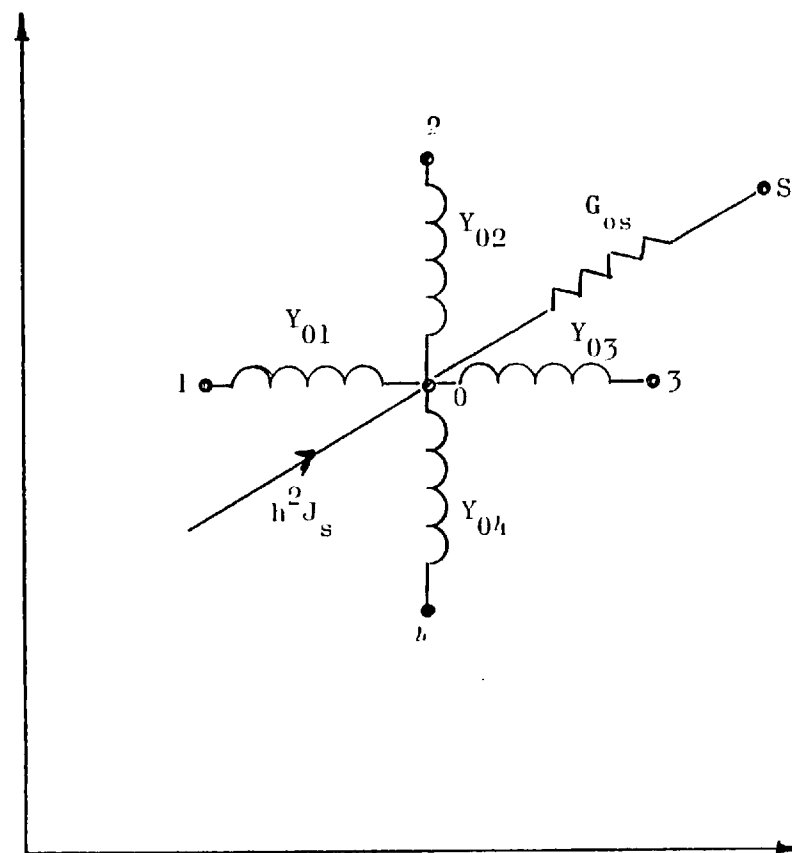


Figure 4.3: The admittance network analogue of the region.

$$\int_{KL} H \cdot dl + \int_{LM} H \cdot dl + \int_{MN} H \cdot dl + \int_{NK} H \cdot dl = h^2 J_s + h^2 J_e$$

Since $\int H \cdot dl = \text{Flux}(\phi) \times \text{Reluctance (S)}$,

$$\phi_{KL} \cdot S_{KL} + \phi_{LM} \cdot S_{LM} + \phi_{MN} \cdot S_{MN} + \phi_{NK} \cdot S_{NK} = h^2 J_s + h^2 J_e \quad (4.7)$$

Considering unit length in the Z-direction, the flux values in equation (4.7) are:

$$\phi_{KL} = B_{KL} \cdot h \quad ; \quad \phi_{LM} = B_{LM} \cdot h \quad ; \quad \text{etc.}$$

The flux densities are derived from the vector potential values as follows:

$$\begin{aligned} B_{KL} &= \frac{A_0 - A_1}{h} & : & & B_{LM} &= \frac{A_0 - A_2}{h} \\ B_{MN} &= \frac{A_0 - A_3}{h} & : & & B_{NK} &= \frac{A_0 - A_4}{h} \end{aligned}$$

Substituting for the flux values in equation (4.7), one obtains:

$$\begin{aligned} (A_0 - A_1) \cdot S_{KL} + (A_0 - A_2) \cdot S_{LM} + (A_0 - A_3) \cdot S_{MN} + (A_0 - A_4) \cdot S_{NK} \\ = h^2 J_e + h^2 J_s \end{aligned}$$

The induced current density J_e was shown to be $-j\omega\sigma(A_0 - A_s)$.

Substituting this in the above equation, the final form of the equation that describes Ampere's law around node 0 in Fig. 4.2 is:

$$\begin{aligned} (A_0 - A_1) \cdot S_{KL} + (A_0 - A_2) \cdot S_{LM} + (A_0 - A_3) \cdot S_{MN} + (A_0 - A_4) \cdot S_{NK} \\ + j h^2 \omega \sigma (A_0 - A_s) = h^2 J_s \end{aligned} \quad (4.8)$$

Now consider the admittance network shown in Figure 4.3. By applying the continuity equation to node 0 in Figure 4.3:

$$\begin{aligned}
(A_0 - A_1)Y_{01} + (A_0 - A_2)Y_{02} + (A_0 - A_3)Y_{03} + (A_0 - A_4)Y_{04} + \\
+ (A_0 - A_s)G_{os} = h^2 J_s
\end{aligned} \tag{4.9}$$

Thus it is seen that equation (4.8) is correctly modelled by the admittance network of Fig. 4.3, if the admittance and conductance values are chosen to be:

$$\begin{aligned}
Y_{01} = S_{KL} ; \quad Y_{02} = S_{LM} ; \quad Y_{03} = S_{MN} ; \quad Y_{04} = S_{NK} ; \\
G_{os} = j\omega h^2 \sigma
\end{aligned}$$

By constructing a map of all the admittance and conductance values, at every node, a complete network model of the field problem is arrived at. An important aspect of such a network is its topology. All the admittances are in the X-Y plane and represent the magnetic field, while all the conductances are in the Z-X plane and represent the electric field. The flow in an admittance between any two horizontal nodes is the vertical magnetic field H_y , and the flow in an admittance between any two vertical nodes is the horizontal magnetic field H_x . The flow in the conductance directly gives the eddy current at that node.

4.4.1 Calculation of Admittance, Conductance Values

Fig. 4.4a shows a uniform Poissonian region, which is readily changed into Laplacian by making the current density J_s zero. The node spacing is h . When the region is Poissonian, the current at node 0 is $J_s h^2$, otherwise it is zero. The magnitude of Y_{01} is the magnetic reluctance of the region abcd, from ab to cd. Since the region is uniform, $Y_{01} = Y_{02} = Y_{03} = Y_{04}$, and $G_{os} = 0$.

$$\therefore Y_{01} = Y_{02} = Y_{03} = Y_{04} = \frac{1}{\mu_o \mu_r} .$$

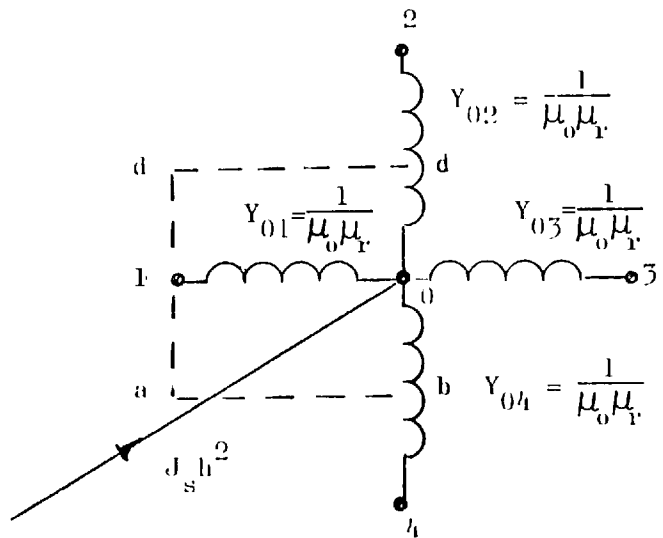


Figure 4.4a: Admittance network for the Poisson's region.

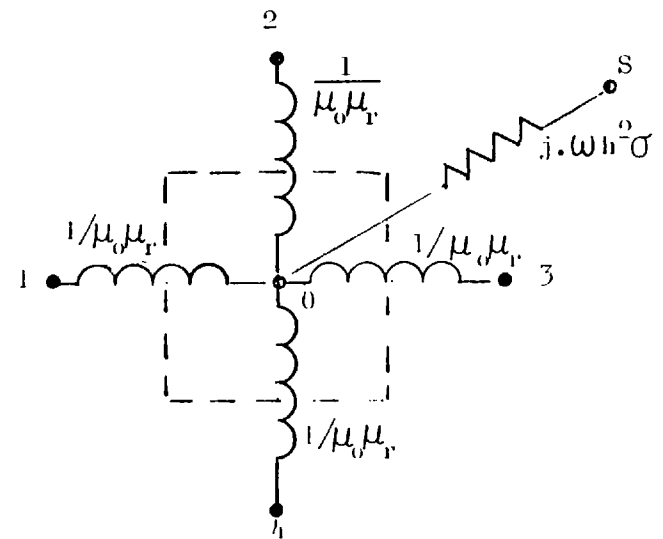


Figure 4.4b: Admittance network for the diffusion region.

Substituting these values in equation (4.9), and re-arranging the variables, one obtains:

$$A_1 = A_2 + A_3 + A_4 - 4A_0 = \mu_o \mu_r h^2 J_s \quad (4.10a)$$

Equation (4.10a) is seen to be identical to the finite difference equations derived in standard texts (e.g. Binns and Lawrenson^{4.10}) for a Poissonian field using Taylor series method.

Consider now the diffusion region. The conductance G_{os} is given by $G_{os} = j\omega h^2 \sigma$. The admittance value remains the same as before. Substitution of the G_{os} , and the admittance values in equation (4.9), yields:

$$A_1 + A_2 + A_3 + A_4 - 4A_0 = j\omega \mu_o \mu_r h^2 \sigma (A_0 - A_s) \quad (4.10b)$$

Again it is seen that equation (4.10b) is identical to that derived by the Taylor series method. Let us now examine the iron/air interface shown in Fig. 4.5a. The reluctances corresponding to Y_{01} and Y_{03} consist partly of air and partly of iron and are connected in series. The reluctance corresponding to Y_{04} is entirely in air, and that corresponding to Y_{04} is entirely in iron. The admittance and the conductance values are:

$$Y_{01} = \frac{(1 + \mu_r)}{2\mu_o \mu_r} \quad ; \quad Y_{02} = \frac{1}{\mu_o} \quad ; \quad Y_{03} = \frac{(1 + \mu_r)}{2\mu_o \mu_r}$$

$$Y_{04} = \frac{1}{\mu_o \mu_r} \quad ; \quad G_{os} = 0 \quad ; \quad J_s = 0$$

Substitution of these values in equation (4.9) yields:

$$A_1 + \frac{2\mu_r}{(1 + \mu_r)} \cdot A_2 + A_3 + \frac{2}{(1 + \mu_r)} \cdot A_4 - 4A_0 = 0 \quad (4.10c)$$

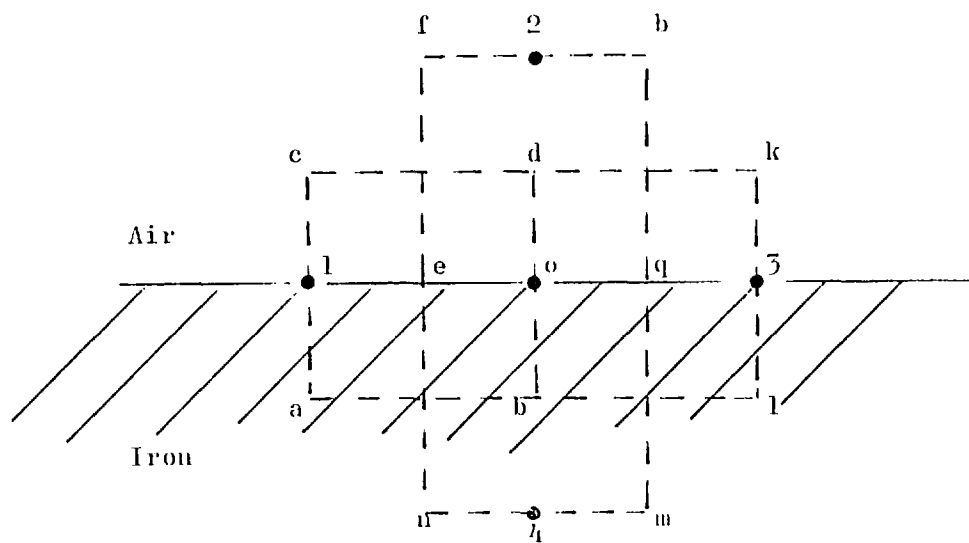


Figure 4.5a: A region with an iron/air interface.

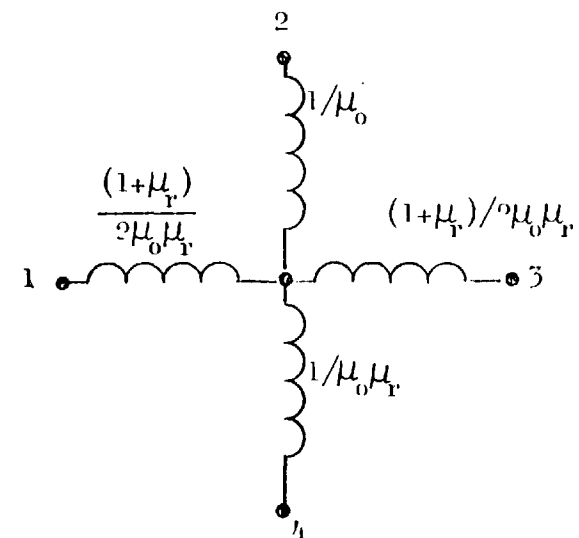


Figure 4.5b: The admittance network of the interface.

In the conventional Taylor series method, in order to satisfy the boundary conditions at the iron/air interface, elaborate mathematical manipulations are done and equation (4.10c) is finally arrived at. In the network approach, it is seen that once the admittance/conductance map is created, the admittance values automatically satisfy the boundary conditions at all interfaces. The admittance network corresponding to Fig. 4.5a is shown in Fig. 4.5b.

4.4.2 Evaluation of Datum Node Potential

The datum node potential A_s needs to be expressed in terms of the node potentials in the X-Y plane. This is achieved by making use of the requirement that the algebraic sum of the currents in the plate must be zero. Expressed mathematically:

$$\sum (A_o - A_s) \cdot G_{os} = 0$$

$$\therefore A_s = \frac{\sum A_o G_{os}}{\sum G_{os}} \quad (4.11)$$

4.5 METHOD OF REPRESENTING AN OPEN BOUNDARY

In the method used for the representation of the open boundary of the Xi-core, the space continuum is first divided into two regions. A rectangular region enclosing the machine is marked out and is termed the interior region. The remaining Laplacian space surrounding the interior region and extending to infinity in all directions is termed the exterior region. A co-ordinate transformation is now introduced, which transforms the unbounded exterior region into a bounded strip which surrounds the interior region after

the manner of a picture frame. For this reason the name "Picture Frame Transformation" has been given to this method.

4.5.1 The Mathematical Transformation

Fig. 4.6a shows the unbounded real space in the first quadrant of the U-V plane. Consider the effect of the co-ordinate transformation given by:

$$x = \frac{u \cdot \Delta_u}{\Delta_u + u} \quad (4.12a)$$

$$y = \frac{v \cdot \Delta_v}{\Delta_v + v} \quad (4.12b)$$

The unbounded space in the U-V plane is transformed into a bounded rectangular space of dimensions $\Delta_u \times \Delta_v$ in the X-Y plane of Fig. 4.6b. The constants Δ_u and Δ_v are arbitrary, which gives one great freedom in the choice of the dimensions of the rectangular transformed space. By inspection, the inverse transformation is:

$$u = \frac{\Delta_u \cdot x}{\Delta_u - x} \quad (4.13a)$$

$$v = \frac{\Delta_v \cdot y}{\Delta_v - y} \quad (4.13b)$$

4.5.2 Network Interpretation of the Transformation

Fig. 4.7b shows a set of uniformly spaced nodes in the transformed X-Y plane, with a node spacing of h . Consider the node (m,n) and its four neighbours $(m+1, n)$, $(m-1, n)$, $(m, n+1)$, $(m, n-1)$ which are designated 0, 3, 1, 2 and 4 respectively. The co-ordinates of these nodes are:

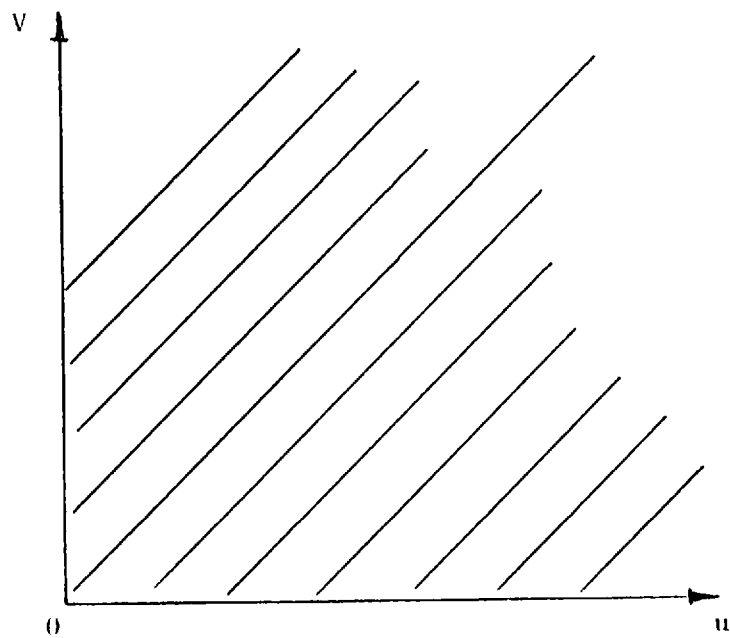


Figure 4.6a: Unbounded real space.

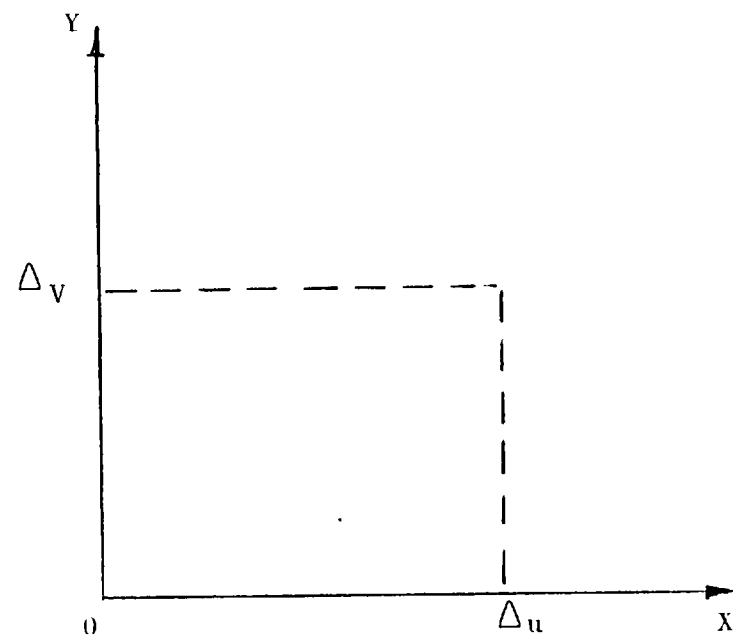


Figure 4.6b: Transformed bounded space.

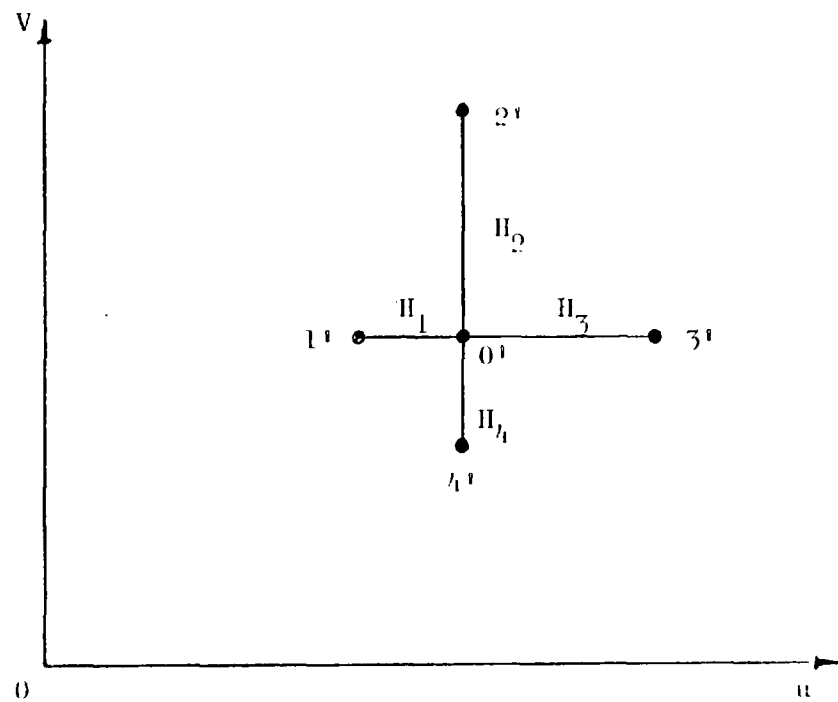


Figure 4.7a: Real space with irregular node spacing.

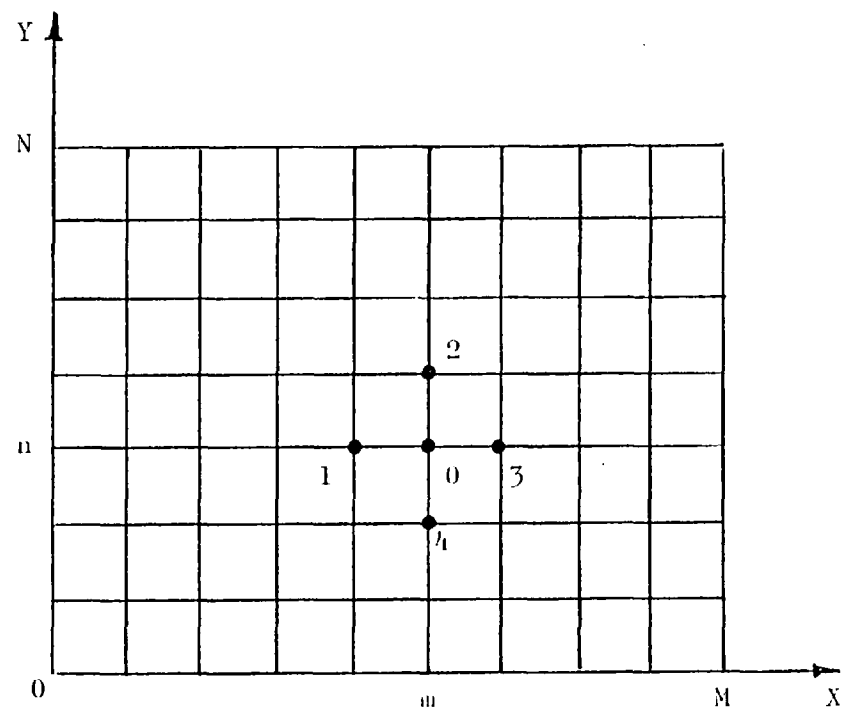


Figure 4.7b: Transformed space with regular node spacing.

$$\begin{aligned}
x_0 &= (m-1)h & : & & y_0 &= (n-1)h \\
x_1 &= (m-2)h & : & & y_1 &= (n-1)h \\
x_2 &= (m-1)h & : & & y_2 &= nh \\
x_3 &= mh & : & & y_3 &= (n-1)h \\
x_4 &= (m-1)h & : & & y_4 &= (n-2)h
\end{aligned}$$

The dimensions of the rectangle are:

$$\Delta_u = (M-1)h \quad \text{and} \quad \Delta_v = (N-1)h$$

The positions of the nodes 1, 2, 3, 4 and 0 in the real U-V plane are 1', 2', 3', 4' and 0' as shown in Fig. 4.7a, with unequal node spacings. By using the inverse transformation given by equation (4.13), these can be evaluated as:

$$\begin{aligned}
H_1 &= \frac{\Delta_u^2 h}{[\Delta_u - (m-1)h][\Delta_u - (m-2)h]} \\
H_2 &= \frac{\Delta_v^2 h}{[\Delta_v - nh][\Delta_v - (n-1)h]} \\
H_3 &= \frac{\Delta_u^2 h}{[\Delta_u - mh][\Delta_u - (m-1)h]} \\
H_4 &= \frac{\Delta_v^2 h}{[\Delta_v - (n-1)h][\Delta_v - (n-2)h]}
\end{aligned}$$

The field equation that applies to the node 0' in the real space in the U-V plane in network terms is:

$$\begin{aligned}
(A_{1'} - A_{0'}) \cdot Y_{1'0'} + (A_{2'} - A_{3'}) \cdot Y_{2'0'} + (A_{3'} - A_{0'}) \cdot Y_{3'0'} \\
+ (A_{4'} - A_{0'}) \cdot Y_{4'0'} = 0
\end{aligned} \tag{4.14}$$

The admittance values are:

$$Y_{1,0} = \frac{1}{\mu_o} \cdot \left(\frac{\Delta_v}{\Delta_u} \right)^2 \cdot \frac{[\Delta_u - (m-1)h] [\Delta_u - (m-2)h]}{[\Delta_v - nh] [\Delta_v - (n-2)h]}$$

$$Y_{2,0} = \frac{1}{\mu_o} \cdot \left(\frac{\Delta_u}{\Delta_v} \right)^2 \cdot \frac{[\Delta_v - nh] [\Delta_v - (n-1)h]}{[\Delta_u - (m-2)h] [\Delta_u - mh]}$$

$$Y_{3,0} = \frac{1}{\mu_o} \cdot \left(\frac{\Delta_v}{\Delta_u} \right)^2 \cdot \frac{[\Delta_u - mh] [\Delta_u - (m-1)h]}{[\Delta_v - nh] [\Delta_v - (n-2)h]}$$

$$Y_{4,0} = \frac{1}{\mu_o} \cdot \left(\frac{\Delta_u}{\Delta_v} \right)^2 \cdot \frac{[\Delta_v - (n-1)h] [\Delta_v - (n-2)h]}{[\Delta_u - (m-2)h] [\Delta_u - mh]}$$

If, however, the above admittances are connected to the nodes 1, 2, 3, 4 and 0 in the transformed X-Y plane, then:

$$(A_1 - A_0) \cdot Y_{1,0} + (A_2 - A_0) \cdot Y_{2,0} + (A_3 - A_0) \cdot Y_{3,0} + (A_4 - A_0) \cdot Y_{4,0} = 0 \quad (4.15)$$

Comparison of equations (4.14) and (4.15) shows that $A_1 = A_1$, $A_2 = A_2$, etc. Thus in network terms, the node admittances in the transformed space are functions of the node locations, even though the node spacing is uniform. The admittance decreases rapidly to zero, as the nodes approach the outer edges which represents the transformed infinity.

4.6 SOLUTION OF THE ASSEMBLED NETWORK

After creating a map of all the admittance and conductance values at each node, a complete network can be assembled. Consider the most general part of the network shown in Fig. 4.3. The equation that applies to the centre node 0 is:

$$(A_0 - A_1)Y_{01} + (A_0 - A_2)Y_{02} + (A_0 - A_3)Y_{03} + (A_0 - A_4)Y_{04} \\ + (A_0 - A_s)G_{os} = h^2 J_s$$

Substituting $Y_0 = Y_{01} + Y_{02} + Y_{03} + Y_{04} + G_{os}$,

$$Y_0 A_0 - A_1 \cdot Y_{01} - A_2 \cdot Y_{02} - A_3 \cdot Y_{03} - A_4 \cdot Y_{04} - A_s \cdot G_{os} = h^2 J_s \quad (4.16)$$

At every node an equation similar to (4.16) obtains and the resulting equations form a set of simultaneous equations. These are often large in number and when expressed in matrix form are found to be diagonally dominant. Therefore the cheapest method of solution is the iterative successive-over-relaxation (S.O.R.) method.

4.6.1 The S.O.R. Method in Network Terms

Consider part of the admittance network shown in Fig. 4.8, in which the potentials at the nodes are not of the magnitudes required corresponding to the injected source currents. Then in order to satisfy the continuity law at each node, one needs to inject currents $J_0^i, J_1^i, J_2^i, J_3^i, J_4^i$ and J_s^i as shown. These injected currents are called the residuals. The S.O.R. method of solution, interpreted in network terms, systematically reduces the magnitude of these residual currents to a negligible value. The residual at the node 0 is:

$$J_0^i = -Y_0 A_0 + Y_{01} A_1 + Y_{02} A_2 + Y_{03} A_3 + Y_{04} A_4 + G_{0s} A_s - J_0 \quad (4.17)$$

Now consider the effect of increasing the node potential A_0 by

$\alpha \cdot \frac{1}{Y_0} \cdot J_0^i$, while maintaining the potentials at the other five nodes constant. The constant α is known as the acceleration factor and

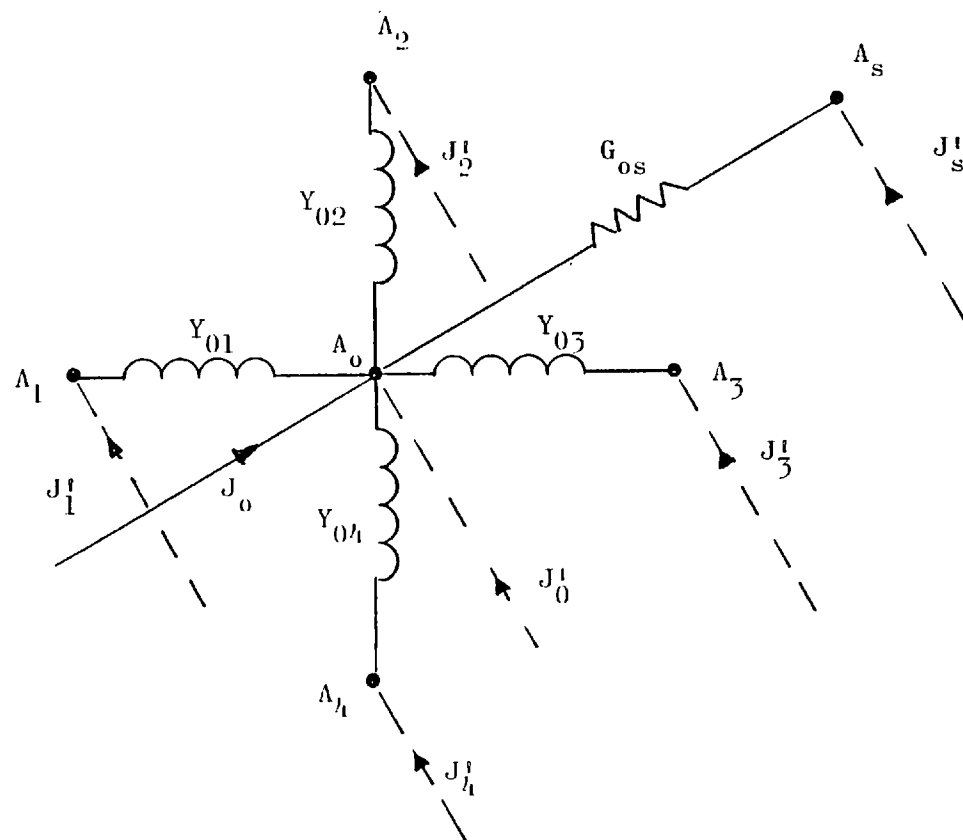


Figure 4.8: Admittance network with residual currents.

$1 \leq \alpha \leq 2$. The new residual J_0'' at node 0 is given by:

$$\begin{aligned} J_0'' &= -Y_0(A_0 + \frac{\alpha}{Y_0} \cdot J_0^i) + A_1 \cdot Y_{01} + A_2 \cdot Y_{02} + A_3 \cdot Y_{03} + A_4 Y_{03} \\ &\quad + A_5 \cdot G_{05} - J_0 \\ &= J_0^i - \alpha J_0^i = (1 - \alpha) J_0^i \end{aligned}$$

Therefore increasing the potential of node 0 by $\frac{\alpha}{Y_0} \cdot J_0^i$ reduces the residual at node 0 from J_0^i to $(1 - \alpha)J_0^i$ and reverses the direction of the residual current. The change in residual at node 0 is:

$$\delta J_0^i = (1 - \alpha)J_0^i - J_0^i = -\alpha J_0^i$$

Now consider node 1. The potential has been kept constant at A.

By increasing the potential at node 0, an additional outflow $(\alpha \cdot \frac{Y_{01}}{Y_0} \cdot J_0^i)$ through Y_{01} results. To maintain the continuity requirement at node 1, while keeping the potential constant at A_1 , a current equal and opposite to $(\alpha \cdot \frac{Y_{01}}{Y_0} \cdot J_0^i)$ needs to be injected at 1. Therefore the residual change at node 1 is:

$$\delta J_1^i = +\alpha \cdot \frac{Y_{01}}{Y_0} \cdot J_0^i$$

By identical reasoning, the residual changes at the six nodes are:

$$\delta J_1^i = +\alpha \cdot \frac{Y_{01}}{Y_0} \cdot J_0^i \quad (4.18a)$$

$$\delta J_2^i = +\alpha \cdot \frac{Y_{02}}{Y_0} \cdot J_0^i \quad (4.18b)$$

$$\delta J_3^i = +\alpha \cdot \frac{Y_{03}}{Y_0} \cdot J_0^i \quad (4.18c)$$

$$\delta J_4^i = +\alpha \cdot \frac{Y_{04}}{Y_0} \cdot J_0^i \quad (4.18d)$$

$$\delta J_5^i = +\alpha \cdot \frac{G_{05}}{Y_0} \cdot J_0^i \quad (4.18e)$$

$$\delta J_0^i = -\alpha J_0^i \quad (4.18f)$$

By adding the above equations, one notes that the sum of the residual changes is zero, which is an important feature of the S.O.R. method. We thus see that the S.O.R. method is one of increasing the potential of each node and dispersing the residual at that node to the neighbouring five nodes. By repeatedly doing this, the residuals are moved towards the specified potential nodes, where they are absorbed.

4.6.2 Influence of Material Properties on Convergence

It has been shown that when the residual at the centre node is dispersed to its neighbours, the amount dispersed to any one node is proportional to the admittance between that node and the centre node (equation (4.18)). Therefore the material properties of the various regions will have an influence on the convergence of the solution. Let us examine this in more detail. For this examination, the acceleration factor α is assumed to be unity for the sake of simplicity. Two regions are of most interest. One is the stator winding slot, the other is the secondary plate region, the reasons becoming clearer later.

Consider the stator winding slot, and assume the iron to be infinitely permeable. Fig. 4.9 shows the admittance values. While the node scanning is in progress, when the node 2 is reached, let its residual be R . When the potential A_2 is modified, the residual R is dispersed to three nodes only, since the admittance into the iron is zero. Therefore $\frac{1}{2}R$ reaches node 5, $\frac{1}{4}R$ each reaches nodes 1 and 3. Thus a large proportion of the residuals is reflected away from the iron towards node 5. When node 5 is reached next in

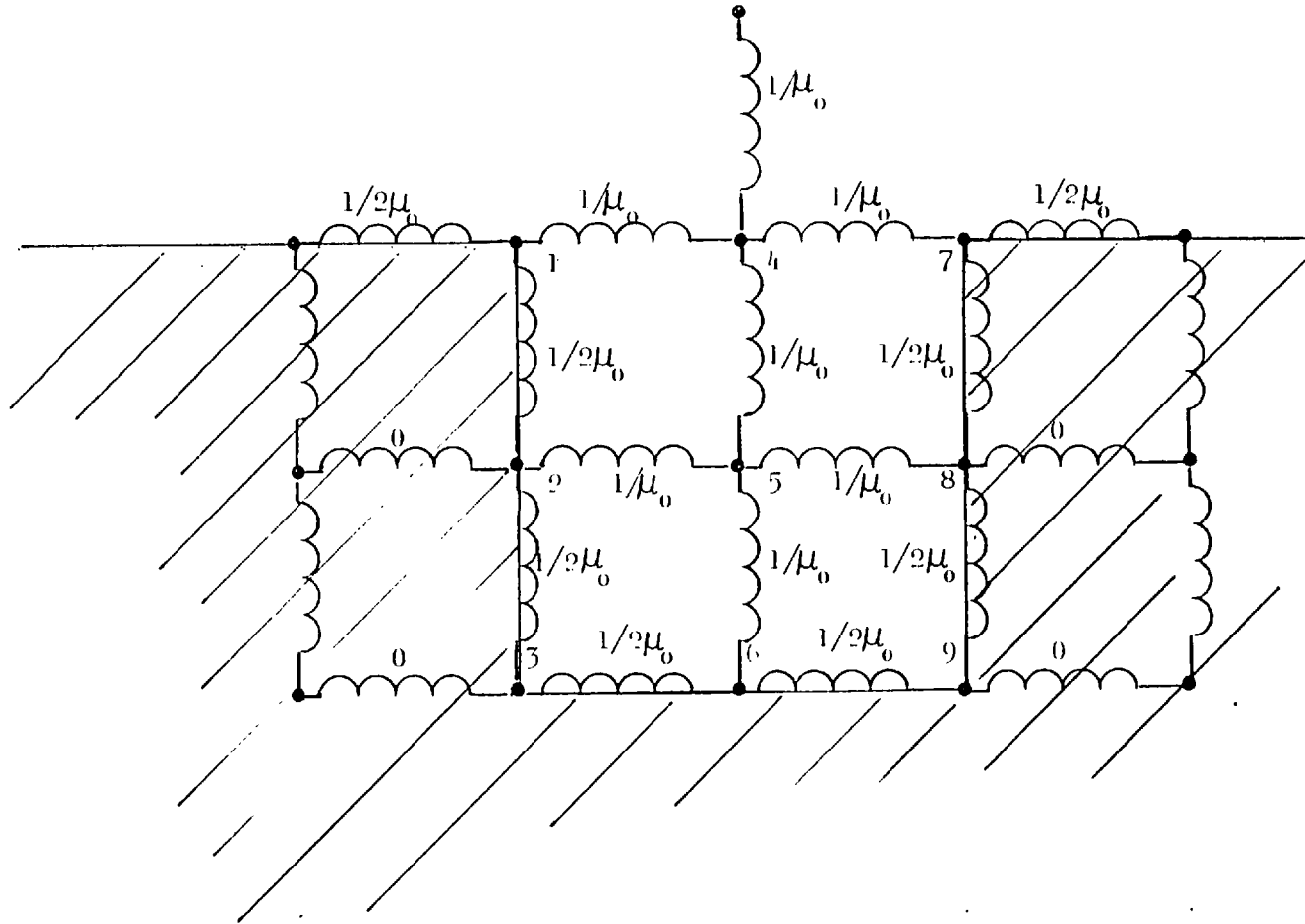


Figure 4.9: Admittance network of the stator slot.

the scan, the new residual at 5 is dispersed equally to nodes 2, 4, 8 and 6. Therefore a large part of the residual that was originally dispersed from the node 2 is sent back to it. When the scan reaches node 8, half the new residual at 8 is reflected away from the iron and sent back to node 5. It is seen thus, in the space between the iron in the slot, the residuals are reflected back and forth. The only exit for the bulk movement of residuals is through the nodes 1, 4 and 7 in the slot opening. If the slot opening is faced by another iron surface, the residuals are reflected back into the slot and convergence may never be achieved.

The next region of interest is the plate region. Detailed analysis is not necessary because the analysis of the conductor in slot has shown that poor convergence is due to the presence of the low admittance branches. In the plate region, however, convergence is affected by the presence of high conductance paths. The proportion of the residuals dispersed to the datum node S is proportional to G_{os} . If G_{os} is much larger than the Y admittance values, a large proportion of the residual is drawn out of the X-Y plane. Since, for the duration of each complete iteration scan, the datum node potential is treated as a constant, the large proportion of the residual dispersed to the datum node S is absorbed there. Remembering that the object of the iteration scheme is to push the residuals to the specified potential boundaries in the X-Y plane, the requirement that G_{os} must satisfy for good convergence emerges. Considering uniform regions, $Y_{01} = Y_{02} = Y_{03} = Y_{04} = 1/\mu_o$ and $|G_{os}| = \omega h^2 \sigma$. Therefore for good convergence the requirement is:

$$|G_{os}| < \frac{1}{\mu_o}$$

$$\therefore \omega h^2 \sigma < \frac{1}{\mu_o}$$

i.e. $\frac{h^2}{d} < 1,$

where h = depth of penetration.

This requirement (i.e. $h^2/d < 1$) for the convergence of eddy current problems is well-known (e.g. Stoll^{4.11}). However, it is seen that one can deduce the same result using the network approach as well.

4.7 FINE MESH EXPANSION

When the finite difference method of analysis was implemented on a digital computer, it was soon realized that for the node spacing used (6.35 mm), the accuracy of solution was not of an acceptable order. Therefore it was decided to make use of the solution obtained thus, and expand a small rectangular region surrounding the plate. The node spacing for this region was half the value for the full model. The boundary potentials required at the edges of the rectangular fine mesh region were obtained by linear interpolation of the values from the coarse mesh solution.

4.8 METHODS OF CALCULATING FORCES AND IMPEDANCE

Two methods are available for the calculation of forces from the solution of an electromagnetic field. The first is the $J \times B$ method, which gives the total mechanical force and the volume distribution of stresses in a non-magnetic body. The second method, due to Carpenter^{4.12}, is the Maxwell stress method. The latter is the only method available for the calculation of the total force on a body which is a combination of non-magnetic and magnetic materials.

However, in such instances the Maxwell stress method does not provide the stress distributions. The distributions of Maxwell stress have no physical meaning. For both methods calculation of the flux densities is essential.

4.8.1 Calculation of Flux Densities

When the computed vector potential values are available, the flux densities are obtained from $\text{Curl } A = B$:

$$B_y = \frac{\partial A}{\partial x}$$

$$B_x = -\frac{\partial A}{\partial y}$$

Therefore at any node (m,n) the flux densities $B_y(m,n)$ and $B_x(m,n)$ are:

$$B_y(m,n) = \frac{A(m+1,n) - A(m-1,n)}{2h}$$

$$B_x(m,n) = \frac{A(m,n+1) - A(m,n-1)}{2h}$$

It is seen that significant errors can occur if the node spacing is not small enough. In methods of calculating forces which involve squaring the flux densities evaluated thus, the inaccurate flux densities lead to large errors in the calculated force values. This applies to the Maxwell stress method. In the $J \times B$ method, however, the error in the flux density does not lead to comparable errors in force values.

4.8.2 Force from Maxwell Stress Method

In Fig. 4.10, PQRS is a rectangular contour enclosing the plate and lying in the air space within the interior region. At each

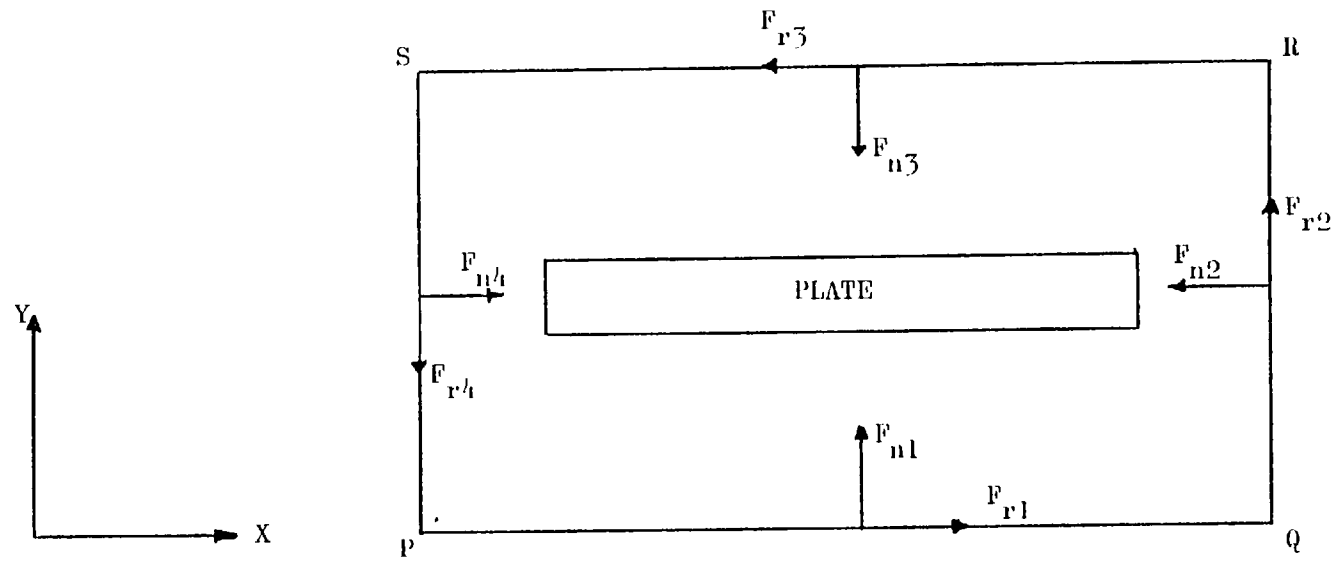


Figure 4.10: Closed Maxwell stress contour.

of the sides of the rectangle, at each point the normal and tangential force densities are given by:

$$f_n = \operatorname{Re} \frac{1}{2} \mu_o [H_n \cdot H_n^* - H_t \cdot H_t^*]$$

$$f_t = \operatorname{Re} \mu_o H_n \cdot H_t^*$$

where:

H_n = R.M.S. complex normal magnetic field,

H_n^* = The complex conjugate of H_n ,

H_t = R.M.S. complex tangential magnetic field,

H_t^* = The complex conjugate of H_t .

The total force on each of the four sides are:

$$\text{Side PQ} \quad F_{n1} = \int_P^Q \operatorname{Re} \cdot \frac{1}{2} \mu_o [H_y \cdot H_y^* - H_x \cdot H_x^*] dx$$

$$F_{t1} = \int_P^Q \operatorname{Re} \cdot \mu_o H_y \cdot H_x^* dx$$

$$\text{Side QR} \quad F_{n2} = \int_Q^R \operatorname{Re} \cdot \frac{1}{2} \mu_o [H_x \cdot H_x^* - H_y \cdot H_y^*] dy$$

$$F_{t2} = \int_Q^R \operatorname{Re} \cdot \mu_o \cdot (-H_x) \cdot H_y^* dy$$

$$\text{Side RS} \quad F_{n3} = \int_R^S \operatorname{Re} \cdot \frac{1}{2} \mu_o [H_y \cdot H_y^* - H_x \cdot H_x^*] dx$$

$$F_{t3} = \int_R^S \operatorname{Re} \cdot \mu_o \cdot (-H_y) \cdot (-H_x) \cdot dx$$

$$\text{Side SP} \quad F_{n4} = \int_S^P \operatorname{Re} \cdot \frac{1}{2} \mu_o [H_x \cdot H_x^* - H_y \cdot H_y^*] dy$$

$$F_{t4} = \int_S^P \operatorname{Re} \cdot \mu_o \cdot H_x \cdot (-H_y^*) dy$$

The total vertical and horizontal forces are:

$$F_y = F_{n1} + F_{t2} - F_{n3} - F_{t4}$$

$$F_x = F_{t1} - F_{n2} - F_{t3} + F_{n4}$$

4.8.3 Force from JxB Method

In this well-known method, the horizontal and vertical forces are given by:

$$F_y = \int \text{Re} \cdot J_e \cdot B_x^* \cdot dx dy$$

$$F_x = \int \text{Re} \cdot J_e \cdot B_y^* \cdot dx dy$$

The integration in the above two equations is carried out over the plate region only. At each node the induced current density J_e is given by:

$$J_e = -j\omega\sigma(A - A_s)$$

The advantages of this method over the Maxwell stress method is that it gives the actual volume distribution of the forces on the plate. Knowledge of such stress distribution is essential in the study of the stability of the plate, as will be seen in the next chapter.

4.8.4 Calculation of Input Impedance

Consider the excitation winding area of the machine. The complex power coming out of the winding is:

$$P_{out} = \int E^* \cdot J_s \cdot dx dy$$

Where E^* is the complex conjugate of the electric field in the winding area and J_s is the applied source current density. Since

$$E = -j\omega A$$

$$P_{out} = j\omega \int A^* J \, dx dy$$

If the input impedance is $Z_{in} = R_{in} + jX_{in}$ and if the r.m.s. input current is I_{in} , then:

$$|I_{in}|^2 \cdot R_{in} = \omega I_m \int A^* J \, dx dy$$

$$|I_{in}|^2 \cdot X_{in} = \omega R_e \int A^* J \, dx dy$$

The above equations enable the evaluation of input impedance.

4.9 CONCLUSION

The theory of the finite difference method has been shown to be readily derivable by the application of Ampere's circuital law. Thus one obtains an admittance network whose solution gives the electromagnetic field of the Xi-core. The results of the finite difference analysis carried out in this manner are given in the following chapter.

RESULTS OF FINITE DIFFERENCE ANALYSIS5.1 INTRODUCTION

In this chapter, the results of the finite difference analysis, applied to the Xi-core, carried out on a digital computer are presented. Initially the convergence characteristics of the numerical algorithm are examined by solving a few problems of interest and which have a bearing on the Xi-core. The rest of the chapter falls into five main sections, dealing with the open circuit Xi-core, the Xi-core with symmetrical plates, the Xi-core with asymmetrical plates, factors determining lateral stability, and the conclusions reached by the analysis. Of these sections, the first three begin with the presentation of the agreement between experimental and computed results, before examining the solutions in more detail. The conclusions reached are summarised, but it is stressed that these are not necessarily valid for any Xi-core of any stator dimensions. The drawbacks encountered in the analysis of the Xi-core using the finite difference method are discussed.

5.1.1 Verification of Picture Frame Transformation

In order to verify the picture frame transformation, the magnetostatic field of four busbars in air was solved. For the experimental simulation of the busbars, the copper windings of the Xi-core were taken out of the iron and placed in a wooden former of identical shape and size as the iron core. Fig. 5.1 shows the agreement obtained between computed, measured and analytical flux densities.

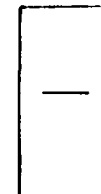
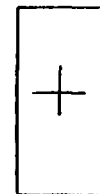
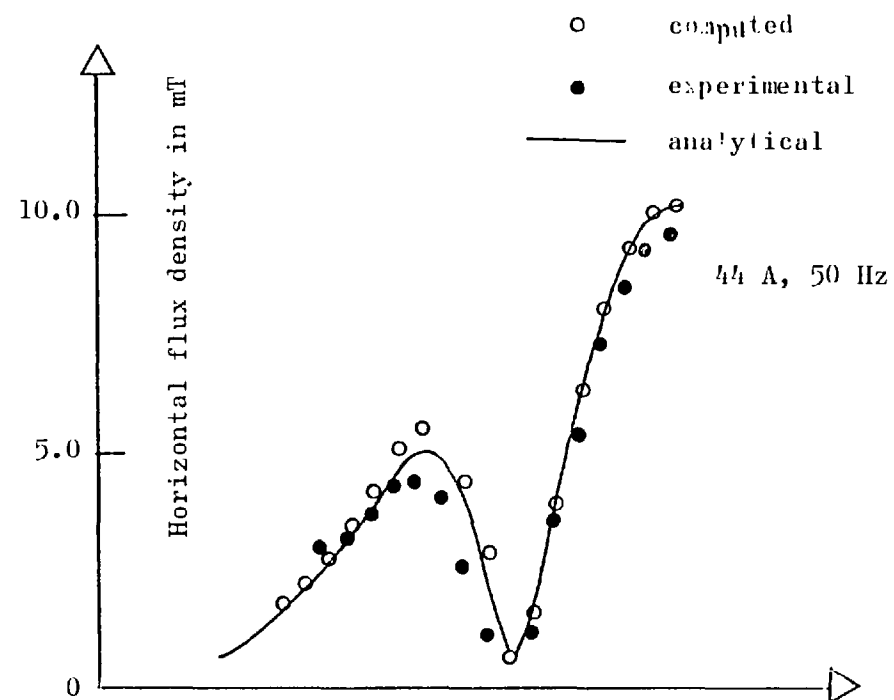
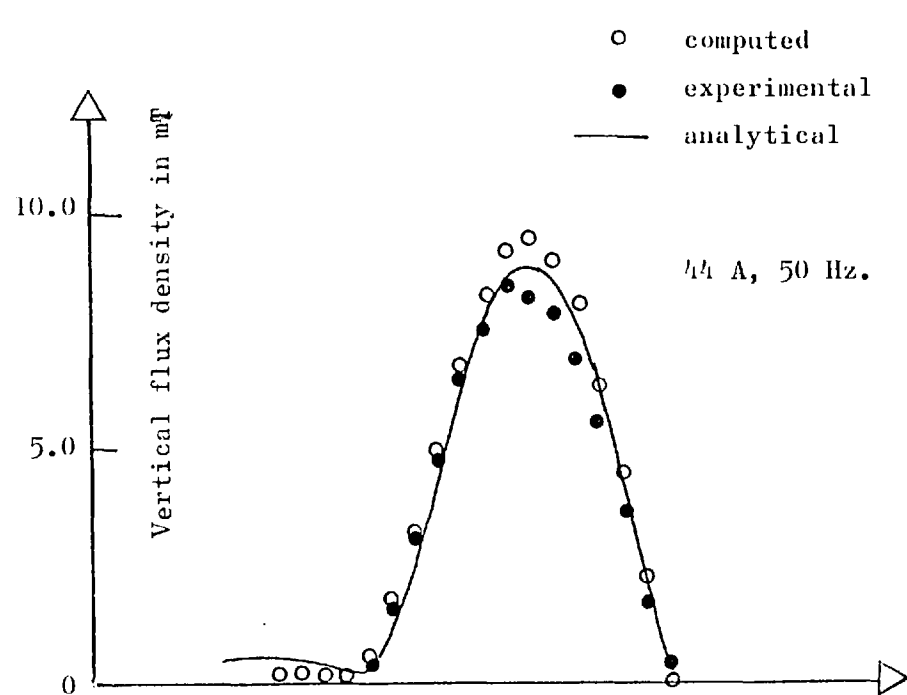


Figure 5.1: Comparison of flux densities to verify the picture frame transformation.

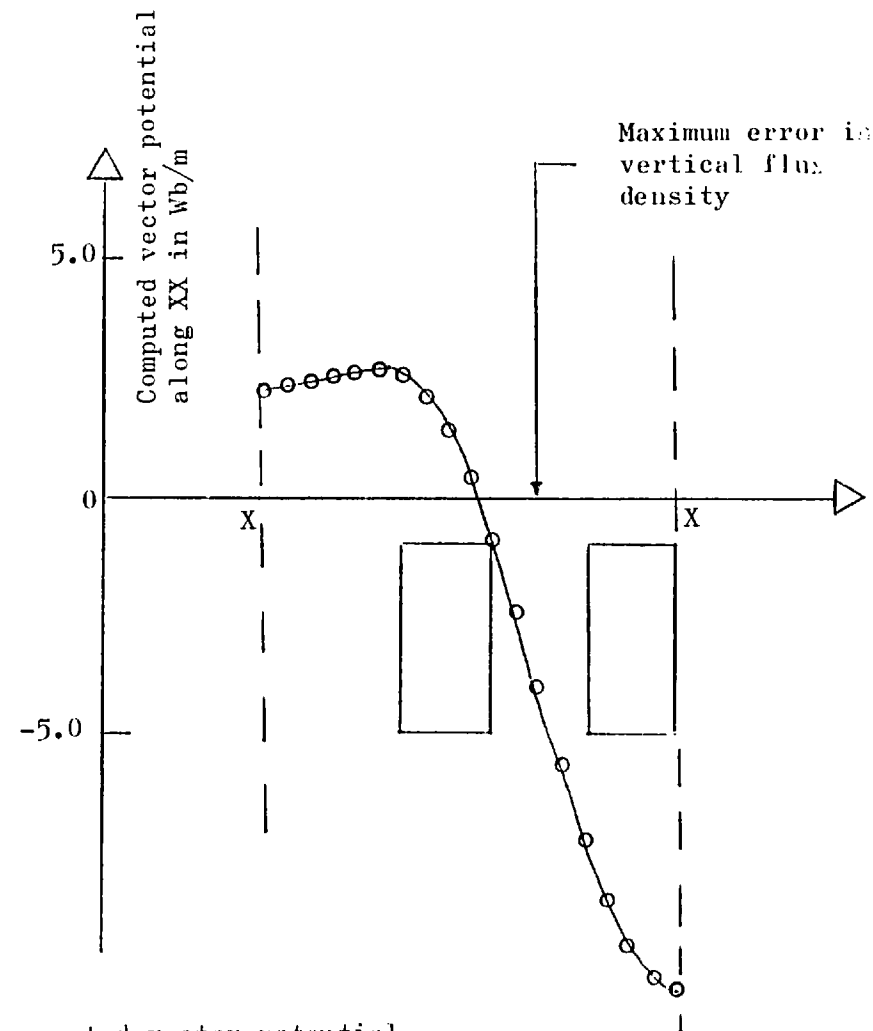
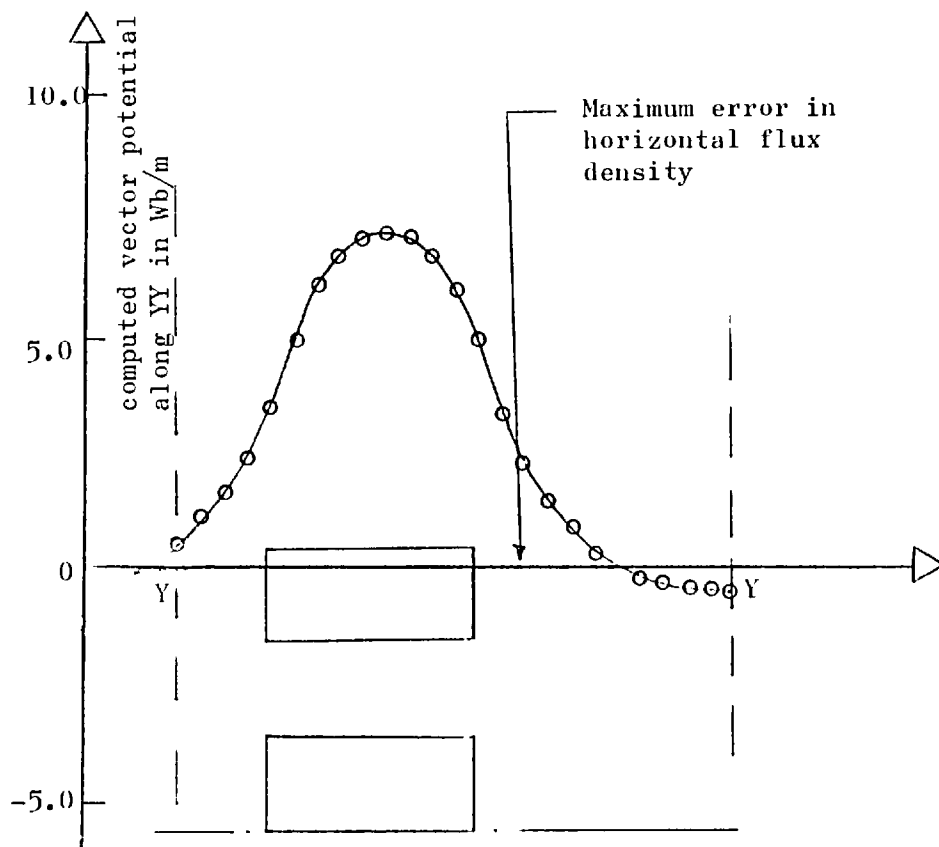


Figure 5.2: The space variation of the computed vector potential.

Examination of the curves of Fig. 5.1 shows that the largest errors in the horizontal and vertical flux densities are 10% and 8% respectively. Since the computed flux densities are obtained from the space derivatives of the vector potential, its spatial variation along a horizontal and vertical line through the nodes at which these errors occur is shown in Fig. 5.2. It is seen that largest errors in flux densities occur in the regions of steepest space variation of the vector potential.

5.1.2 Material Properties and Convergence Time

In Section 4.6.3, the influence of material properties on convergence was discussed. To test the conclusions reached in that discussion, the following problems were solved, using the same mesh model and convergence criterion.

(a) Four Bus-bars on O.C.

This is the problem already solved for the verification of the picture frame. All the "Y" admittances are equal at every node in the interior region. Therefore, as shown earlier in Section 4.6.2, the residuals are equally dispersed everywhere at nodes within the interior region, until they are absorbed at the boundaries of specified potentials.

(b) Four Bus-bars with a rotor plate

In this problem, the node admittances are everywhere equal as in (a), but now there are conductances in the plate region extending out of the X-Y plane. The residuals are equally dispersed everywhere, except in the plate region. For the values of node spacing, frequency and plate conductivity

used, $|G_{os}| = 0.498|Y|$. Using equations (4.18a) to (4.18f), of Section 4.6.1, 88% of the residual at any node in the plate is equally dispersed in the X-Y plane, while 11.11% is absorbed at the datum node. The consequence of this is a faster convergence than in problem (a).

(c) The Xi-Core on open circuit

No conductances are present as in problem (a), but now the "Y" admittances are different in the interior region. As shown in section 4.6.2, due to the reflection of residuals at the iron surfaces, the solution is slower than in problem (a).

(d) The Xi-core with a rotor plate

The convergence is faster than in problem (c), due to the absorption of residuals at the datum node, but slower than in problem (b) due to the presence of iron.

The above reasoning for the four cases is confirmed when a quantitative comparison is made of the convergence times of the four problems:

Problem	Number of Iterations	Solution Time in Secs.
O.C. busbars (a)	137	288
Busbar with plate (b)	133	280
O.C. Xi-core (c)	165	342
Xi-core with plate (d)	145	300

TABLE 5.1

5.2 THE Xi-CORE ON OPEN CIRCUIT

The electromagnetic field of the Xi-core on open circuit has already been introduced briefly in the previous section. This solution is examined in more detail now. In order to compare the magnetic circuit of the Xi-core with that of the U-core from which it evolved, the field of the Xi-core with the relative permeability of the outermost limbs set to unity, is also shown alongside the field plot of the Xi-core proper in Fig. 5.3.

5.2.1 Computed and Experimental Results

The only experimental measurements that can be made on the open circuit machine, to test the computed solution, are the flux densities and the input reactance. Fig. 5.4(a) shows the agreement obtained between theoretical and measured flux densities.

In order to examine the effect of the right-angled corners of the stator teeth, the vertical flux density distribution across a section just on top of the iron core was measured. For this measurement a flat rectangular Hall probe was used. The overall measured distribution is found to be closely similar to the computed values in Fig. 5.4(b). As should be expected, due to the magnetic saturation at the corners in the real machine, the measured values are lower than the computed ones.

The self inductance at the input terminals was computed by evaluating the stored energy. The computed value is 3.72 mH, while the experimental value is 4.2 mH.

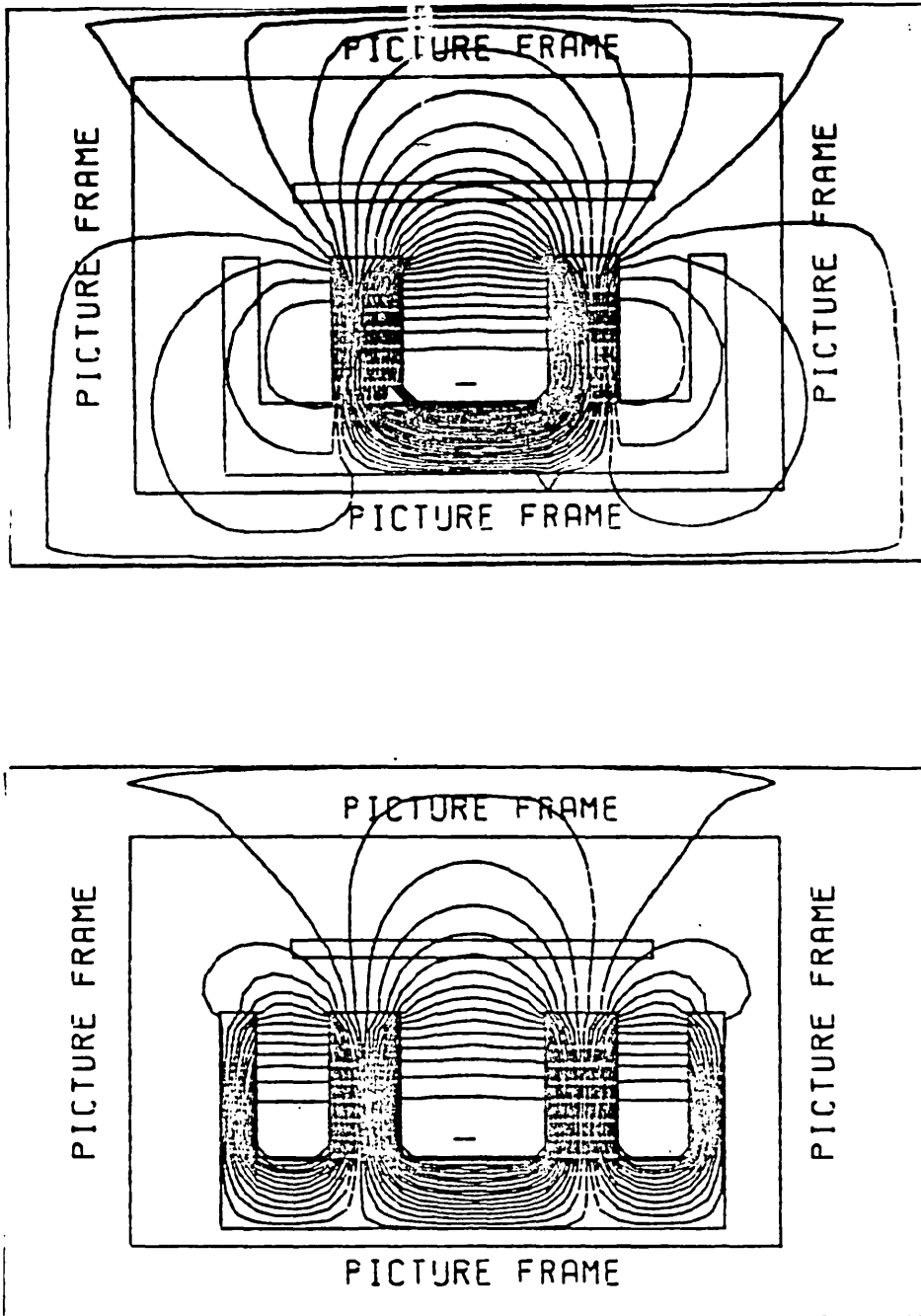


Figure 5.3:

The open circuit field plot of the U-core and the Xi-core.

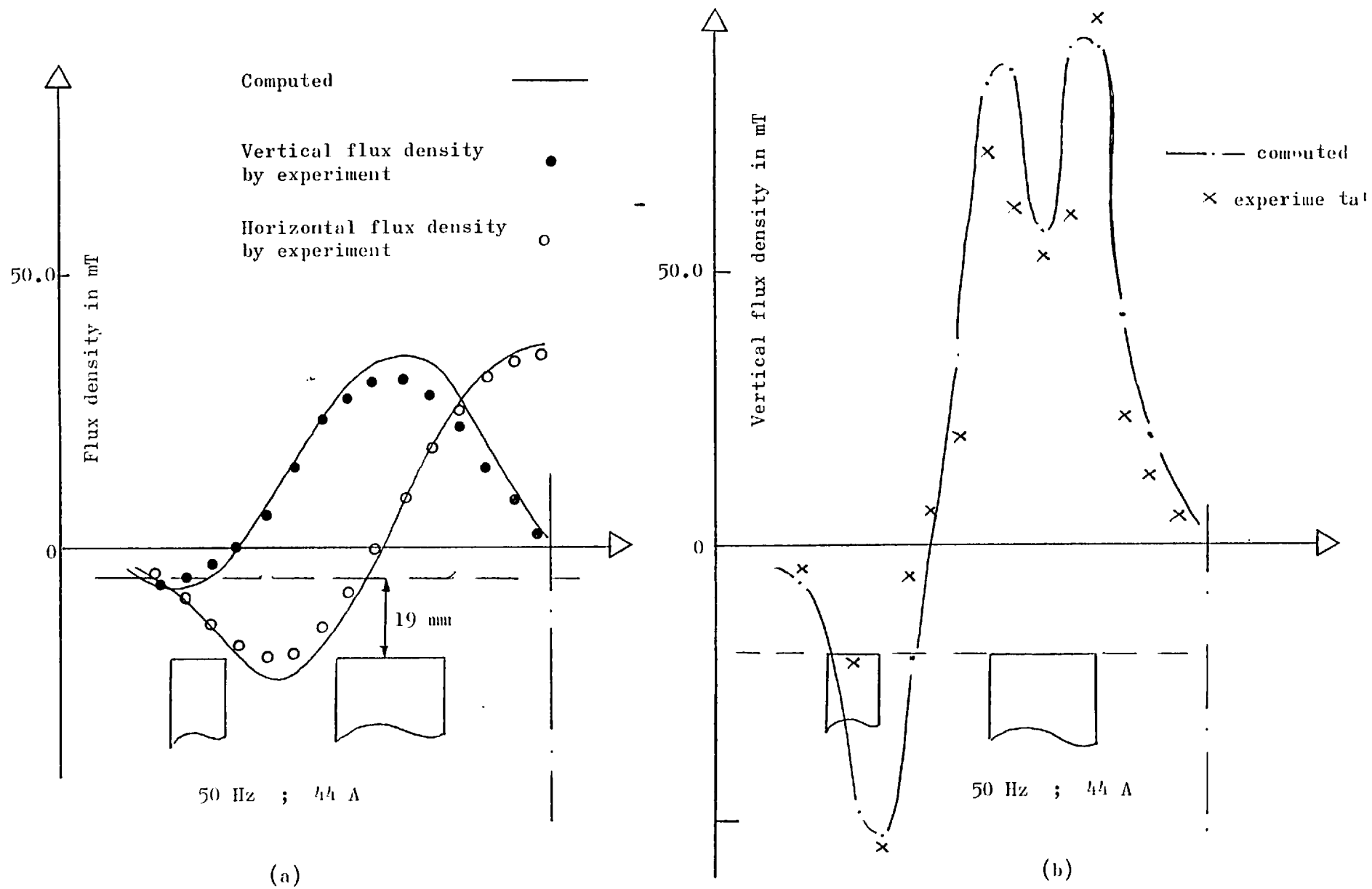


Figure 5.4: Flux density comparisons.

5.2.2 The Core Flux Distributions

When the total flux carried by each of the iron limbs was computed, it was found that the maximum flux was carried by the inner-tooth of the Xi-core at its root. For an excitation current of 44 Amps, this value is 6.3 mWb. Taking 6.3 mWb as 100 units, the core flux distributions are as shown in Fig. 5.5(a).

In order to examine the effect of the two outermost iron limbs of the Xi-core, the core flux distribution of the U-core was computed and is shown in Fig. 5.6(a). In comparing the magnetic circuit of the two machines, we can adopt either of two viewpoints. In the first, we consider the function of the magnetic circuit of the core to be one of containing all of the flux below the tooth tops, in the iron limbs, and bring out the maximum possible flux above the tooth tops. On this basis the U-core, i.e. the Xi-core without the outermost iron limbs, emerges superior because 74% as opposed to the 53% in the Xi-core is available above the iron. Examination of the leakage flux values shows that this is due to the high slot leakage on either side of the inner limb in the Xi-core. The alternative viewpoint is to consider the function of the core to be one of providing maximum magnetising flux, on a given constant supply voltage. The magnetising flux values shown in Figs. 5.5(a) and 5.6(a) are for the same current in both machines. The computed input reactances, however, are 2.55 ohms for the Xi-core and 1.93 ohms for the U-core. Therefore, disregarding the voltage drop due to winding resistance, the current in the U-core is 1.3 times that in the Xi-core, on a constant voltage supply. Hence when operated on constant voltage supply, the Xi-core produces 3.34 mWb at 44 Amps, while the U-core produces 3.77 mWb at 57 Amps. Thus we see that on

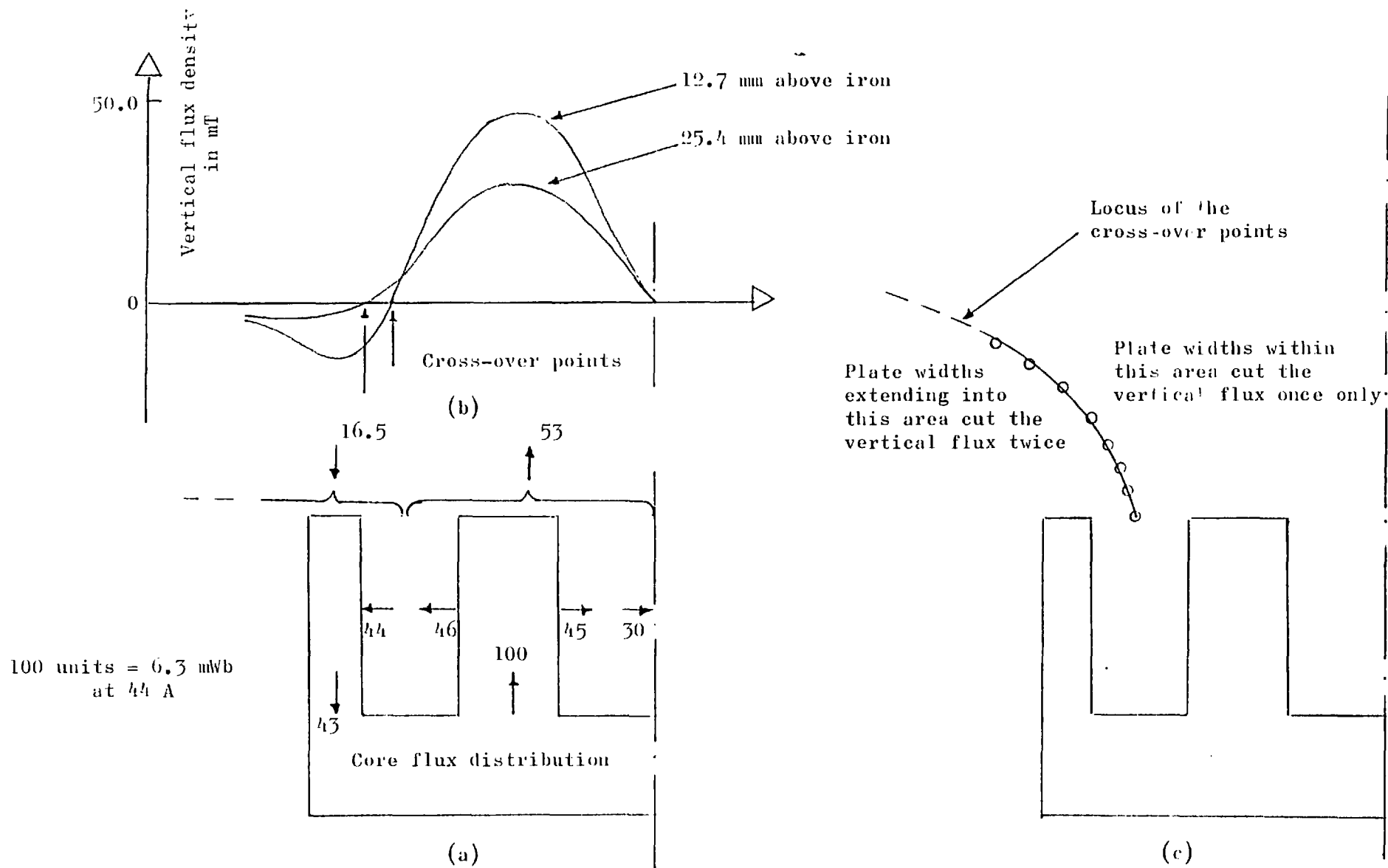
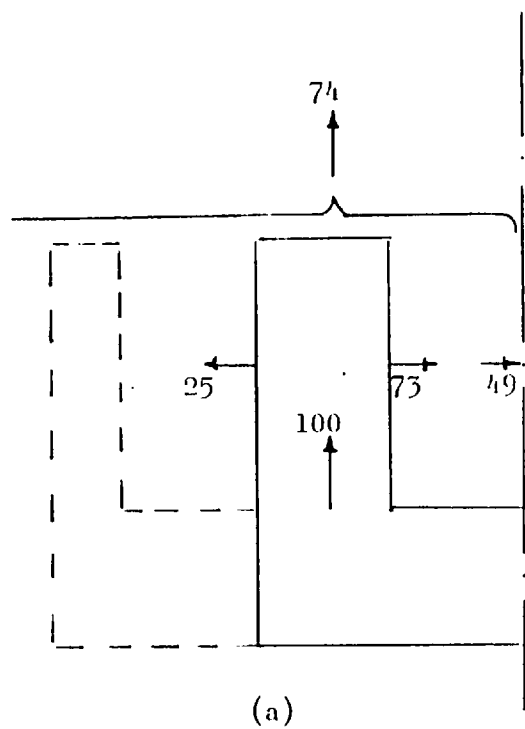
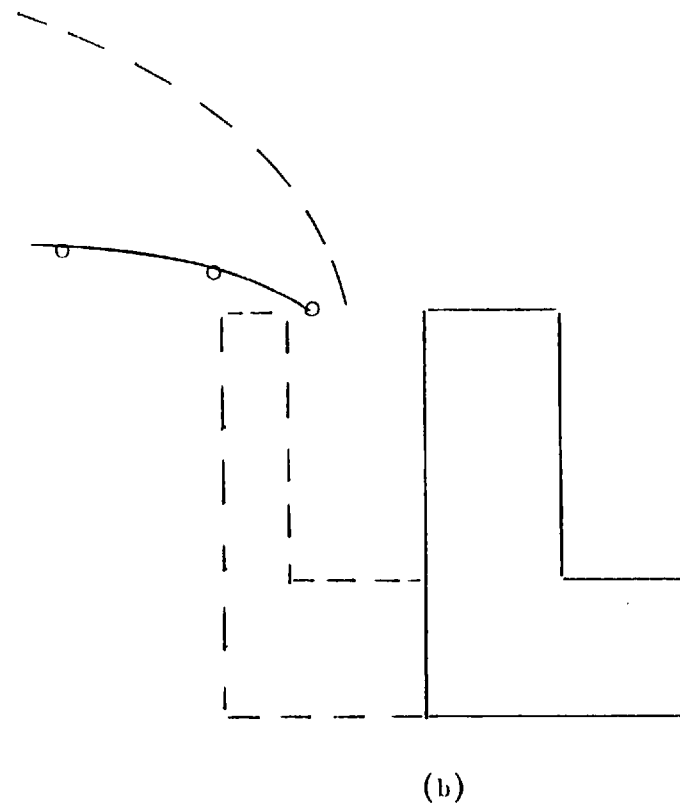


Figure 5.5:

The 0/C magnetic field of the Xi-core.



Change of core flux distribution
after removal of outermost iron limbs.



Change of cross-over locus after
removal of outermost iron limbs.

Figure 5.6:

The O/C magnetic field of the U-core.

this basis too the U-core emerges as superior. It is pointed out, however, that the above argument applies to the O/C case only.

5.2.5 The Nature of the Airgap Field

Attention is now focussed on the nature of the field distributions above the iron core in the airspace. The vertical flux density distribution across a horizontal section at two different heights above the tooth tops of the Xi-core is shown in Fig. 5.5(b). The distributions show that there are two distinct regions, one in which the flux enters the section and one where it leaves, in the same half of the machine. The widths of these regions marked by the cross-over points shown are dependent on the height of the section above the iron core. For successive heights these cross-over points were computed and their locus is shown in Fig. 5.5(c). We see that all symmetrical plates contained within this locus are magnetised by a two-pole distribution, while plates extending through the locus are magnetised by a four-pole distribution. When a plate is present, however, the resultant magnetising flux will be different and less than the open circuit value. It will be shown in a later section that the effect of the plate currents is to leave the locus unchanged, if the plate is contained within the O/C locus, but draw the locus inwards if the plate extends through the O/C locus. The same locus, constructed for the U-core and shown in Fig. 5.6(b), reveals that the removal of the outer limbs in the Xi-core makes the loci move rapidly away from each other, becoming nearly horizontal. We see that in the Xi-core, the magnetising flux distribution could be either four-pole or two-pole, depending on the plate width and its airgap value, but in the U-core, except at very small airgaps, it is always a two-pole distribution.

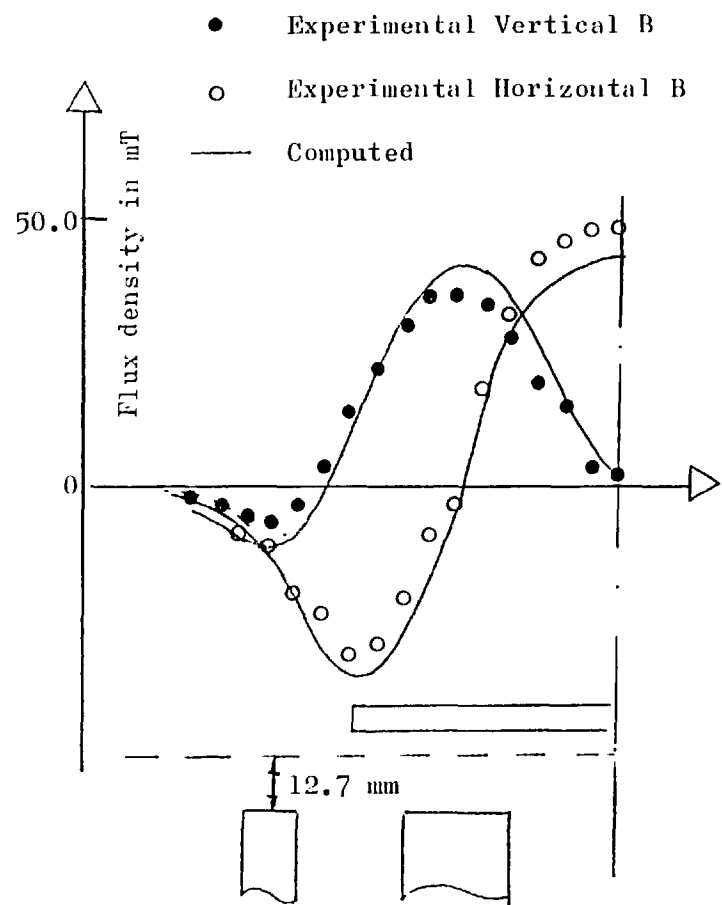
5.3 THE Xi-CORE WITH SYMMETRICAL PLATES

We begin a study of the the analysis of the Xi-core with rotor plates, symmetrically located, by first presenting the agreement achieved between theoretical and experimental results. Two cases are considered; one in which a particular plate is situated at several airgaps, and one in which several plates of variable widths situated at one airgap value. For both cases constant current at constant frequency applies. For one particular plate, a case of variable current excitation is considered.

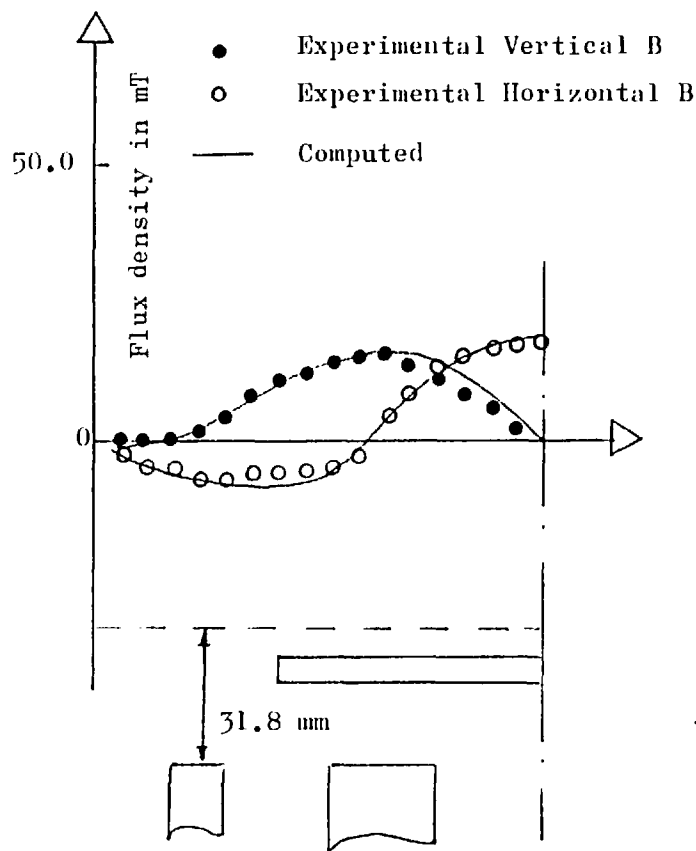
5.3.1 Computed and Experimental Results

Figs. 5.7(a) and (b) show the degree of agreement reached between computed and measured flux densities at a section below the plate and above the plate. The largest deviations are seen to occur at the peaks, which correspond to regions with the steepest space variation of vector-potential, as in the case of the busbar problem.

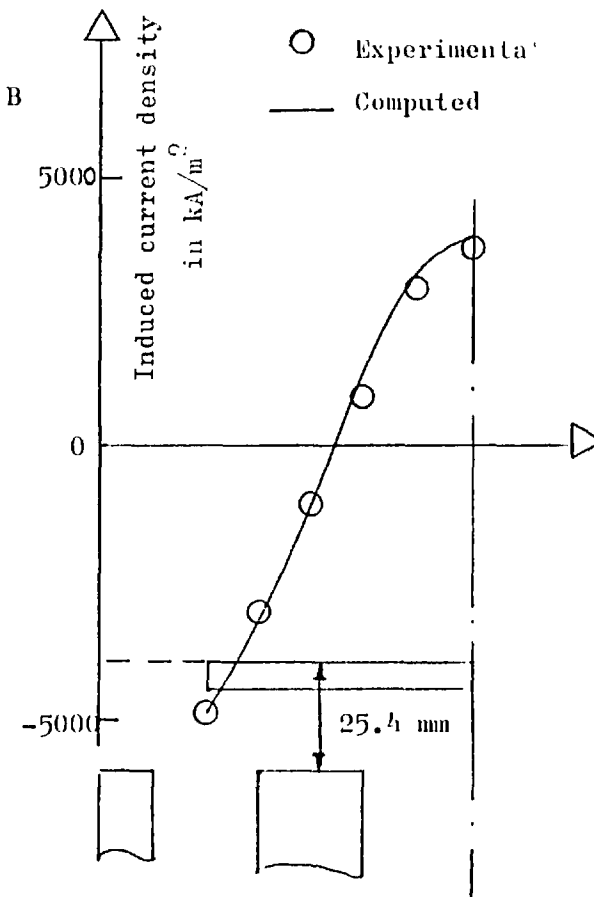
The induced current density on the top surface of the plate was measured, by measuring the electric field. This was done by using a sensitive digital voltmeter to measure the voltage drop between two points lying in a straight line along the Z-axis (see Fig. 4.1 of Chapter 4) of the machine. The leads from these points were finely twisted together to avoid picking up any transformer voltage. By dividing this voltage by the distance between the two points, the electric field E was obtained, and hence the current density $J = \sigma E$. The comparison with the computed value is shown in Fig. 5.7(c).



(a)



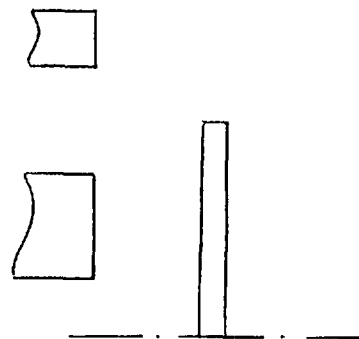
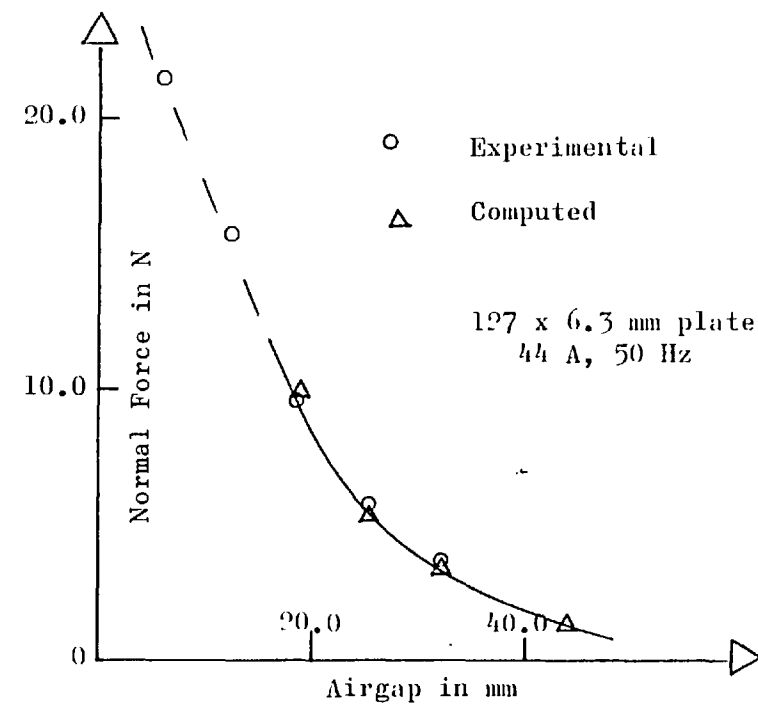
(b)



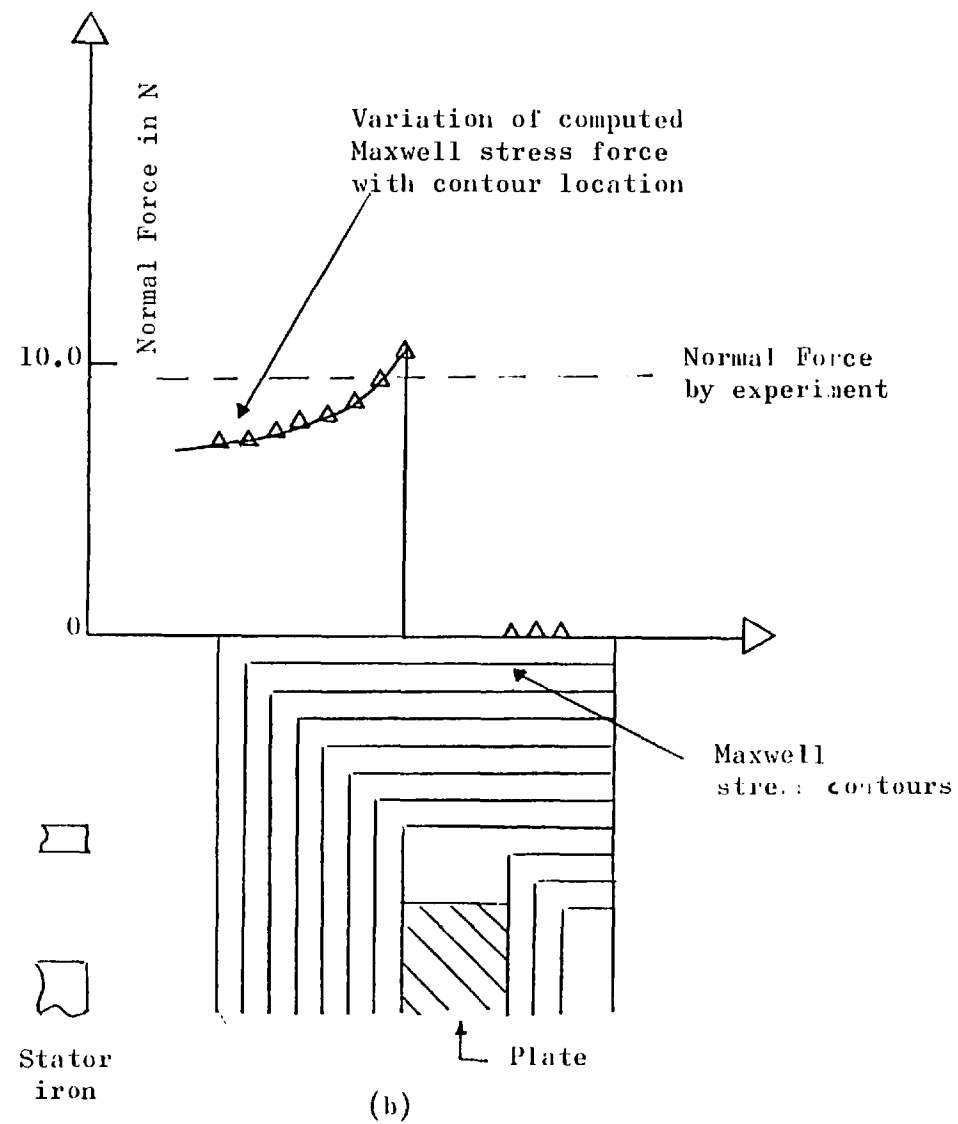
(c)

Figure 5.7: Agreement between experimental and computed results.

For one particular plate symmetrically located at several airgaps, the normal (levitation) force was computed using the $\mathbf{J} \times \mathbf{B}$ method, and measured using the weighing method. The latter is a method by which one measures the weight of the plate, before and after energising the machine, and noting the difference, which gives the normal force. The computed and measured values are shown in Fig. 5.8(a). For the same plate width, at one airgap, the normal force was also computed using the closed contour Maxwell stress method explained in Section 4.8.2 of Chapter 4. Since theoretically, if the field solution is correct, the total force is independent of the location of the contour, the force was calculated for several contours, some enclosing and some not enclosing the plate. The result shown in Fig. 5.8(b) shows that for contours not enclosing the plate, the total force is zero, as it should be. But for other contours enclosing the plate, the computed total force is seen to vary with contour location. Furthermore, the errors are seen to increase, with the Maxwell stress force becoming smaller than the $\mathbf{J} \times \mathbf{B}$ force and the measured value, as the contour approaches the iron surface. The explanation for this behaviour is as follows. In the expression, $\int (B_n^2 - B_t^2) dx$, for the normal force calculation by Maxwell stress method, $\int B_n^2 dx$ gives the attractive part of the normal force, while $\int B_t^2 dx$ gives the levitation part of the normal force. The fact that the total levitation force is less than the measured value for contours closer to the iron surface suggests that, near the iron surfaces $\int B_n^2 dx$ is larger than it should be. In Section 5.2.1, and in Fig. 5.4(b), it was shown that due to the right-angled corners of the iron, near the iron surface, the computed flux density is higher than the measured values. Hence $\int B_n^2 dx$ is larger than it should be near the iron surface, giving a smaller levitation force.



(a)



(b)

Figure 5.8:

Normal Force Comparison

As a further test on the method of calculating forces by $J \times B$ method, the levitation forces on several plate widths, all at the same airgap, were measured and are compared with the computed values in Fig. 5.9. In this instance, the effect of the fine mesh expansion is also shown, by presenting the force values obtained after the first coarse mesh solution.

For the plate used in comparing the forces in Fig. 5.8(a), the input inductance and power losses were evaluated by the method described in Section 4.8.4 of Chapter 4. Experimentally, these were obtained by simple, shadow voltage, input current, and power intake measurements, at the input terminals. Fig. 5.10 shows the comparison with computed values.

So far we have considered the Xi-core under constant current excitation. Before concluding the presentation of experimental/computed results, a case of variable excitation is considered. At 50 Hz the levitated height of a plate, at different input currents, achieved by changing the applied voltage, was measured along with the kVA input and secondary plate power loss. Under these circumstances, the normal force is constant and is equal to the weight of the plate. Because of the large excitation currents involved, and due to the attendant rotor heating, the input power measurements are subject to errors. The comparisons of levitated height vs input current, and the kVA power loss vs levitated height, are shown in Fig. 5.11.

5.3.2 Space Distribution of Flux Densities

Having presented the theoretical/computed results, we proceed to examine the nature of the space distributions of the various quantities of interest.

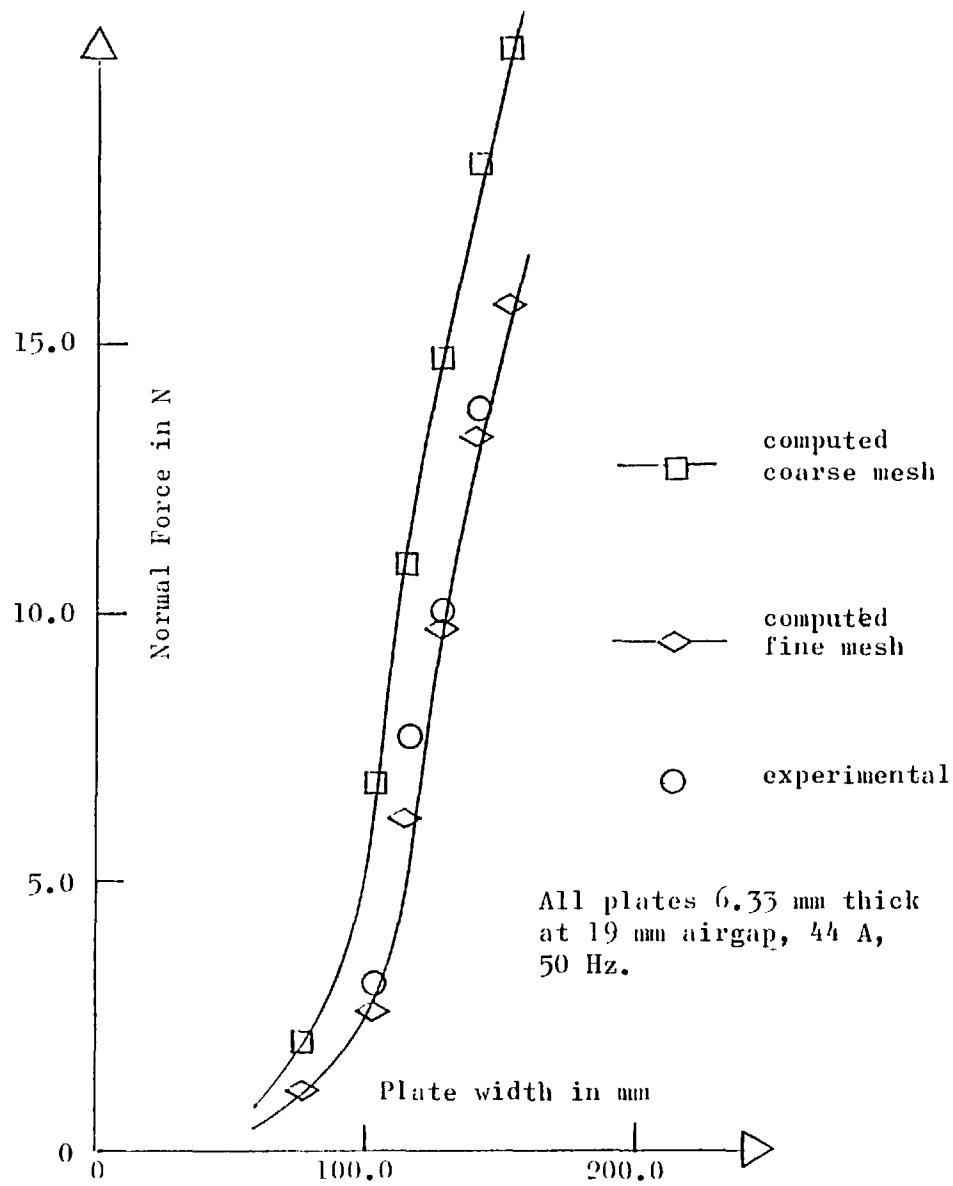


Figure 5.9: Normal force comparison.

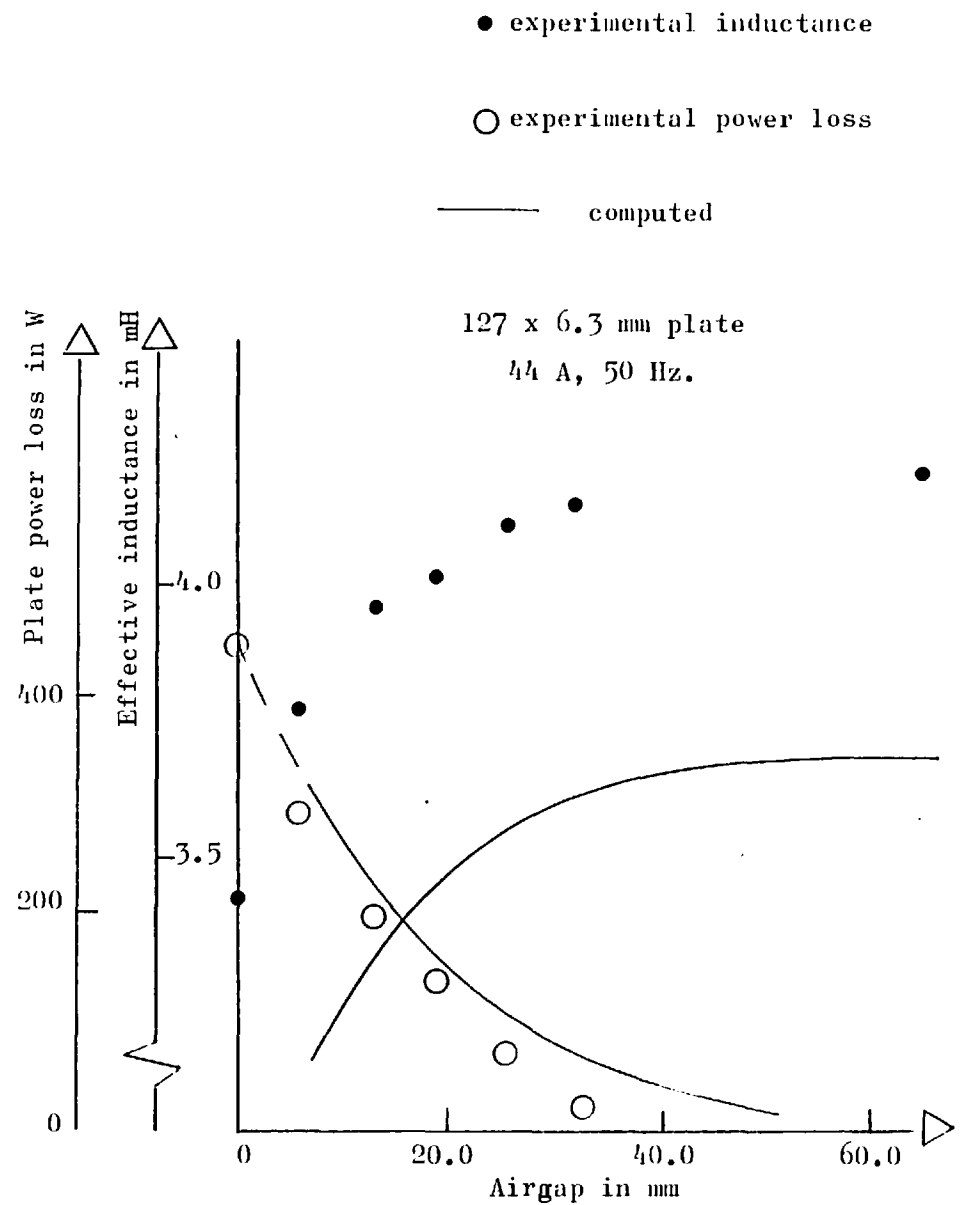


Figure 5.10: Input impedance, plate loss comparison.

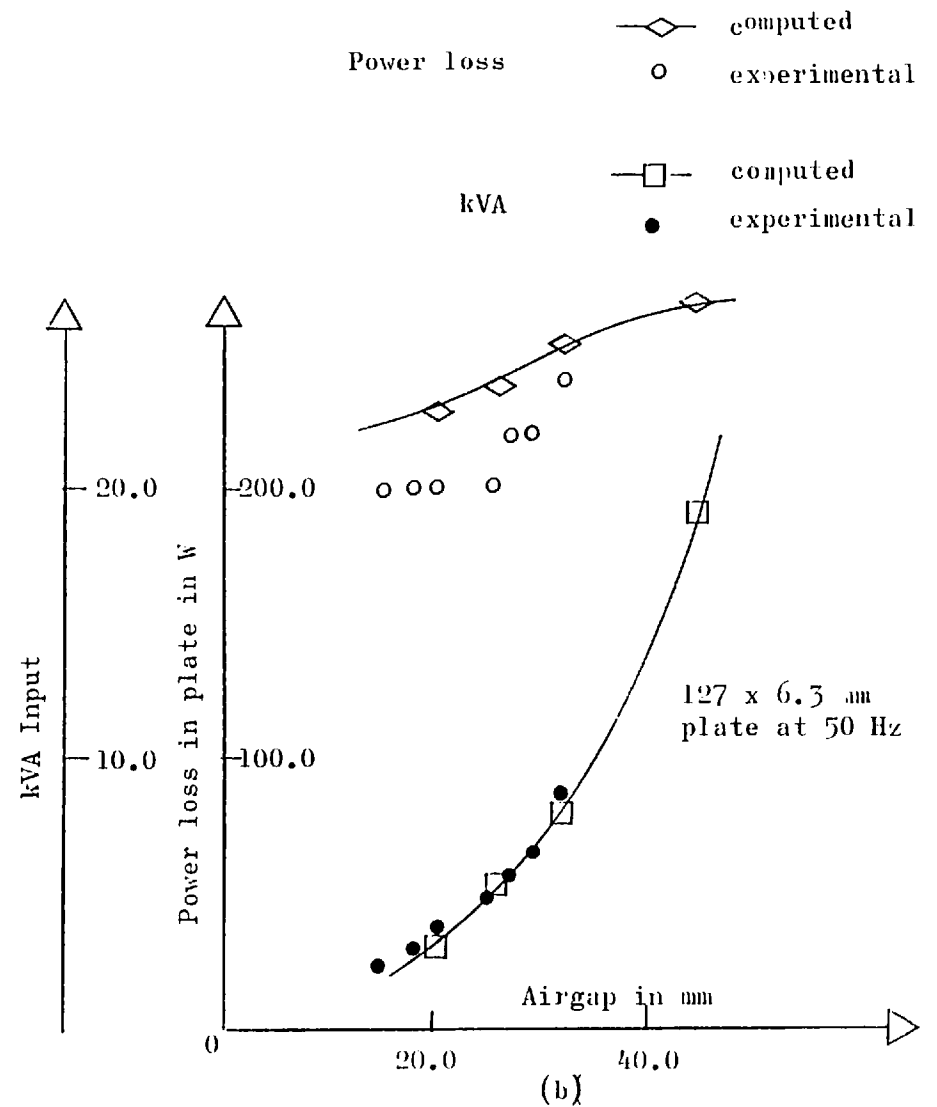
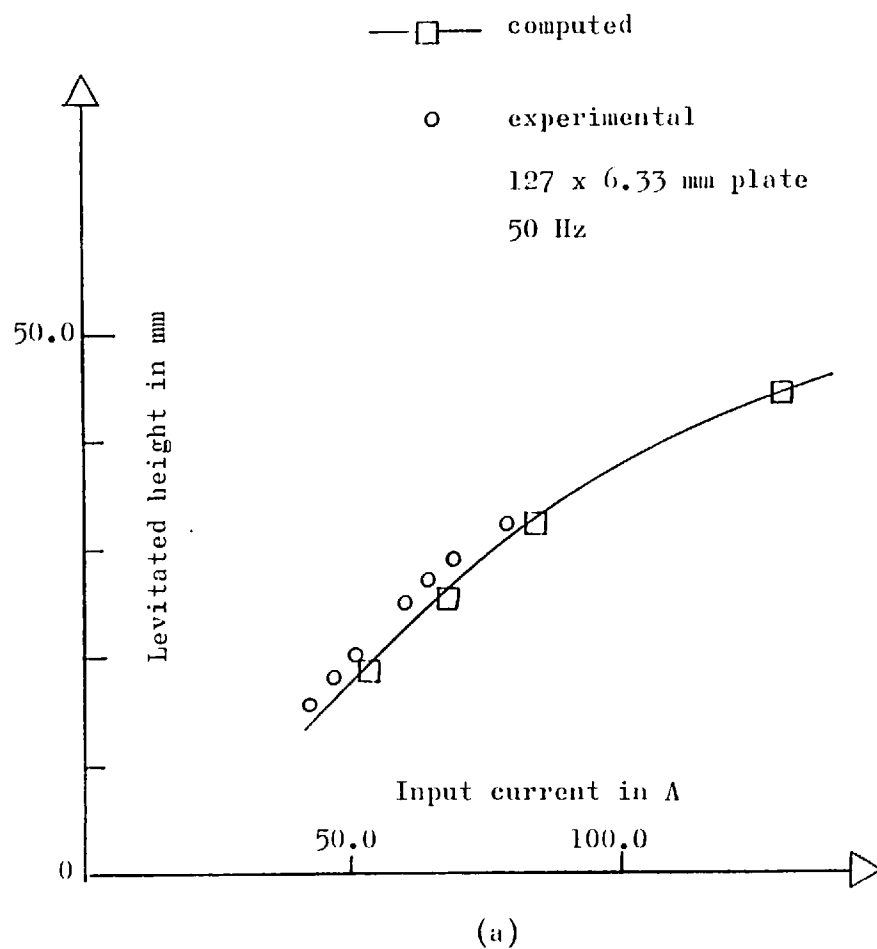


Figure 5.11: Comparison of computed and experimental results under variable excitation currents.

The foremost among these is the vertical flux density distribution under the plate, since it provides the magnetising flux for the rotor plate. For the purpose of this investigation, three plate widths, falling into the narrow, intermediate and wide categories are considered. Fig. 5.12 shows the flux density distribution, along with the O.C. distribution for the three plates. We observe that for the two plates contained within the locus of the cross-over points referred to earlier, the in-phase magnetising flux is altered from the O.C. value only under the plate. Hence the locus remains unchanged by the introduction of the two plates. For the wide plate extending into the O.C. locus, the magnetising flux distribution is completely altered both under and outside the plate. Besides, the locus of the cross-over points is seen to have been drawn in towards the centre-line of the machine. In all three cases, a reduction of the vertical flux density from the O.C. value results. This alters the proportion of $\int B_n^2 dx$ and $\int B_t^2 dx$ from the O.C. case, and since B_n is smaller than the O.C. case, while B_t remains virtually unaltered, a net levitation force results.

Fig. 5.13 shows the phase angle variation of the vertical flux density under the plates. For the two narrow plates contained within the locus, the phase angle variation is seen to lag progressively towards the centre-line of the machine, while for the wide plate it lags progressively away from the centre-line. Therefore a variable amplitude travelling wave travelling inwards exists for plates contained within the O.C. locus, while for the wider plates outward travelling waves exist. These travelling waves, cutting the conducting plate, produce induction forces directed horizontally in the two halves. Therefore one would expect a compressive lateral

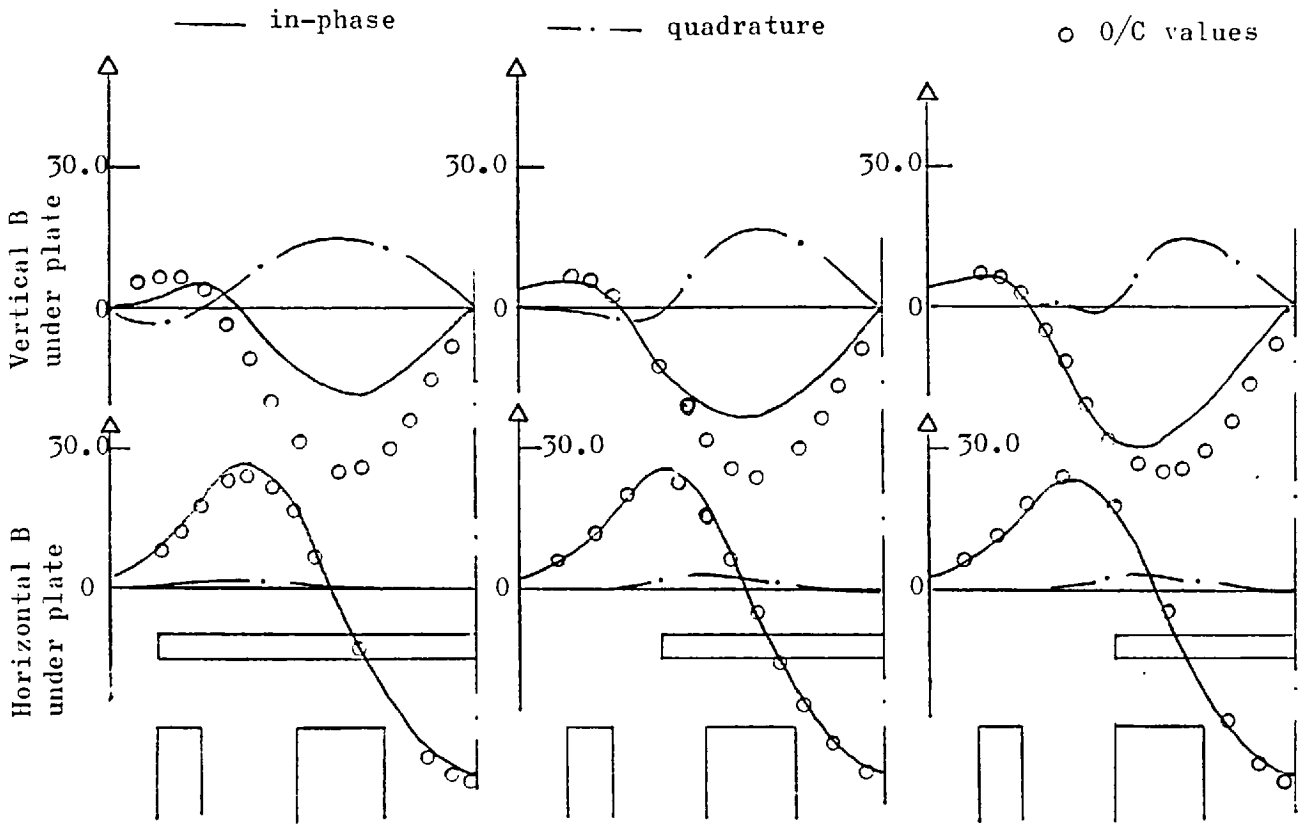


Figure 5.12: Space distribution of the magnetic field under the plate.

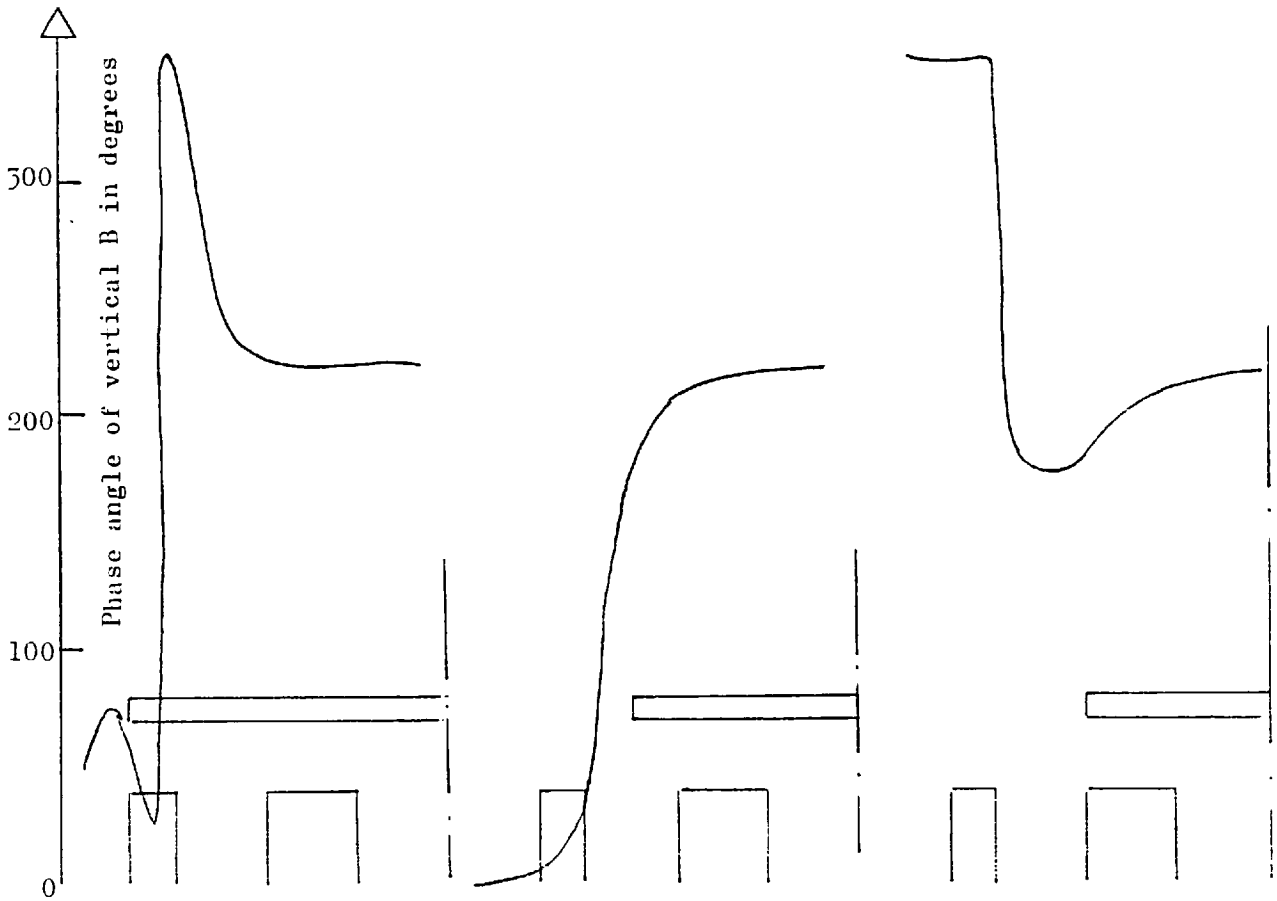


Figure 5.13: Phase angle variation of the vertical flux density.

force distribution for the narrow plates, and a tensile lateral force distribution for the wide plates. These observations will be shown to be true in a subsequent section dealing with stresses.

Fig. 5.12 also shows the horizontal flux density distribution under the plate. Little quadrature flux is observed, and the in-phase values are very little different from the O.C. value.

5.3.3 Space Distribution of Induced Current

Fig. 5.14 shows the computed induced current density distributions at the lower edge of the plate for the three plate widths considered in the previous section. The phase angle variation is shown in Fig. 5.15. Examination of the in-phase and quadrature distribution shows that during one cycle of the alternating current excitation, the induced current distribution remains nearly the same, unlike the magnetising vertical flux density distribution. But the phase changes follow the pattern of the vertical flux density phase variations.

For all three plates, the computed distributions show that in one half of the plate there is only one loop of induced currents, forming an image of the primary current. Since it has already been shown that plates extending into the O.C. cross-over locus are magnetised by a two-pole distribution in one half of the plate, one might have expected a two-loop induced current distribution for these plates. That this is not the case is due to the fact that the magnetising pole outside the locus is much weaker than the opposite pole within it. Considering zero airgap, the strength of these poles is 16.5 units and 53 units respectively. Thus even for an infinitely wide plate, the resultant magnetising flux in one half of the plate cannot reverse sign.

—— in-phase

— · — quadrature

123.

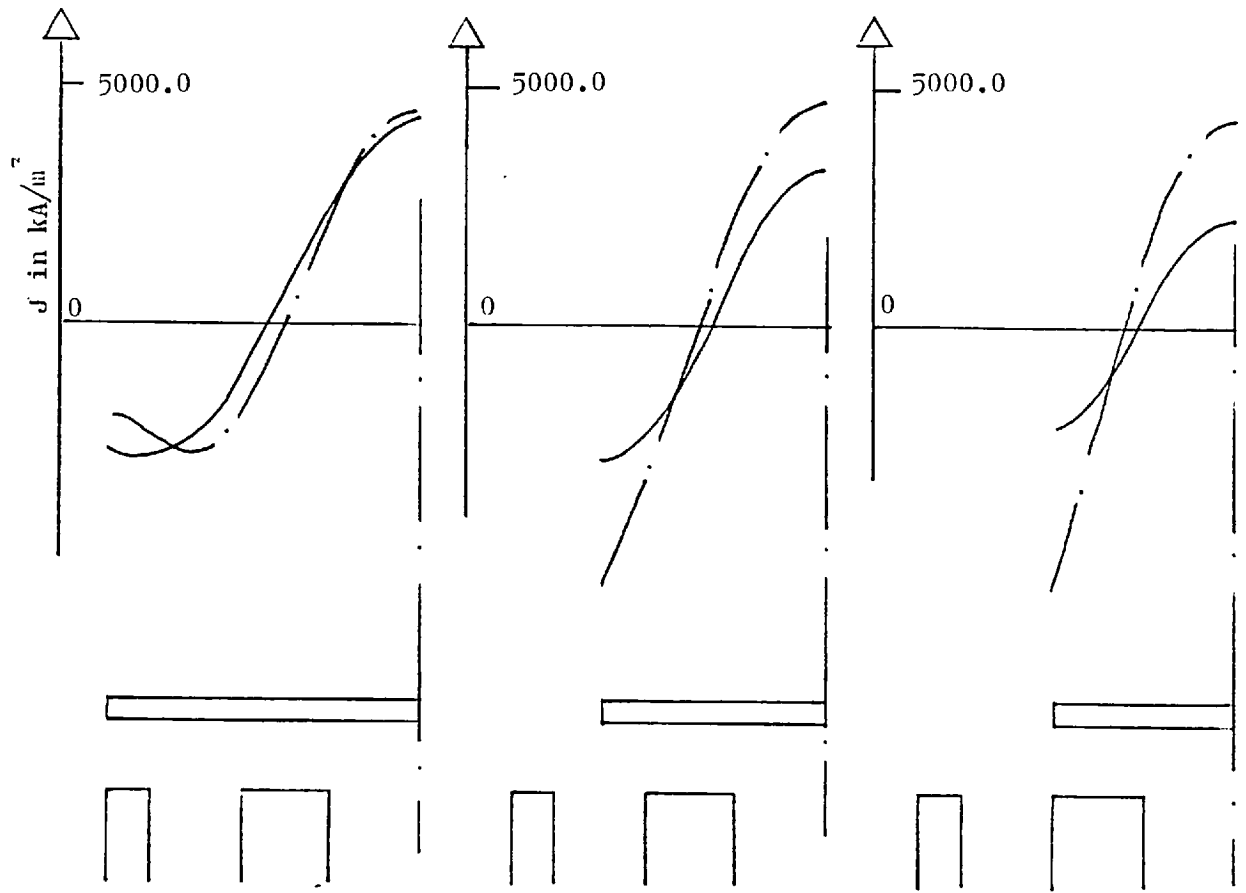


Figure 5.14: Space distribution of induced current density.

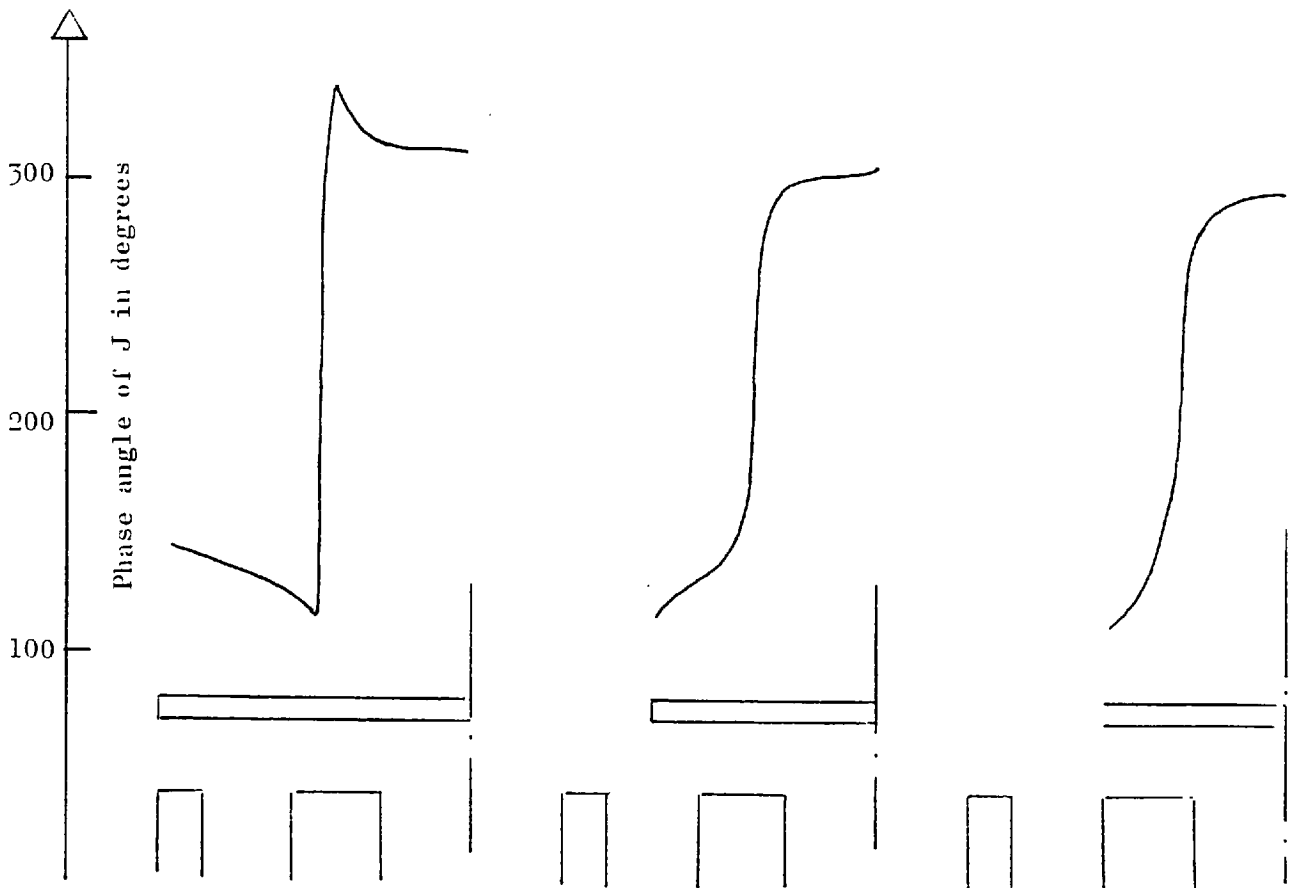


Figure 5.15: Phase angle variation of the induced current density.

When plate width is increased gradually, the induced current distribution remains a single loop centred roughly along the centre-line of the inner tooth, while the vertical flux density distribution changes from one to two poles. This shows that increasing the plate width gradually results in increasing magnetising flux, until the plate width meets the cross-over locus. Beyond this limit, any increase in plate width brings a reduction in magnetising flux. A parallel situation is possible in a rotary electrical machine, if the coil pitch is progressively increased. So long as the increasing coil pitch is less than the pole pitch, increase of coil pitch results in an increase of magnetising flux. But when the coil pitch exceeds the pole pitch, further increase of coil pitch will bring about a reduction of magnetising flux.

A possible situation for two loops of induced current to occur in one half of the Xi-core is when the inner slot width is very large in comparison with the outer slot width (Fig. 5.16). Under these conditions one half of the Xi-core assumes a geometry approaching that of the U-core, and the cross-over locus will be a straight line through the centre of the outer slot.

5.3.4 Computed Stress Distributions

The volume force density $J \times B$ was computed for several plate widths at a constant airgap, and the stress distributions obtained are shown in Figs. 5.17 and 5.18.

Considering the normal force distribution first, we see that there is a negligible amount of attractive force in the plate region directly above the inner teeth, while the major proportion of

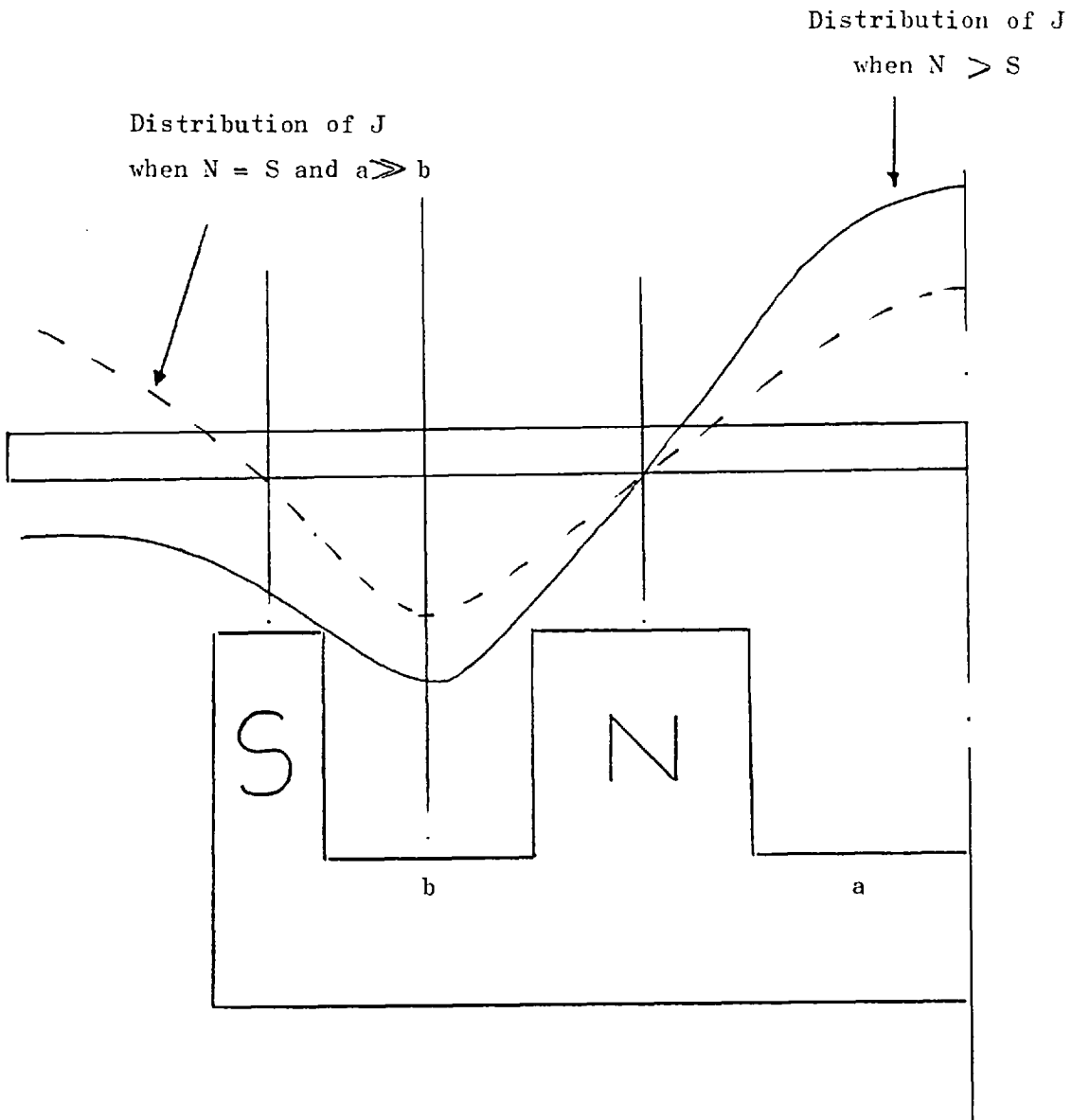


Figure 5.16: Xi-core stator for which induced currents can form four closed loops.

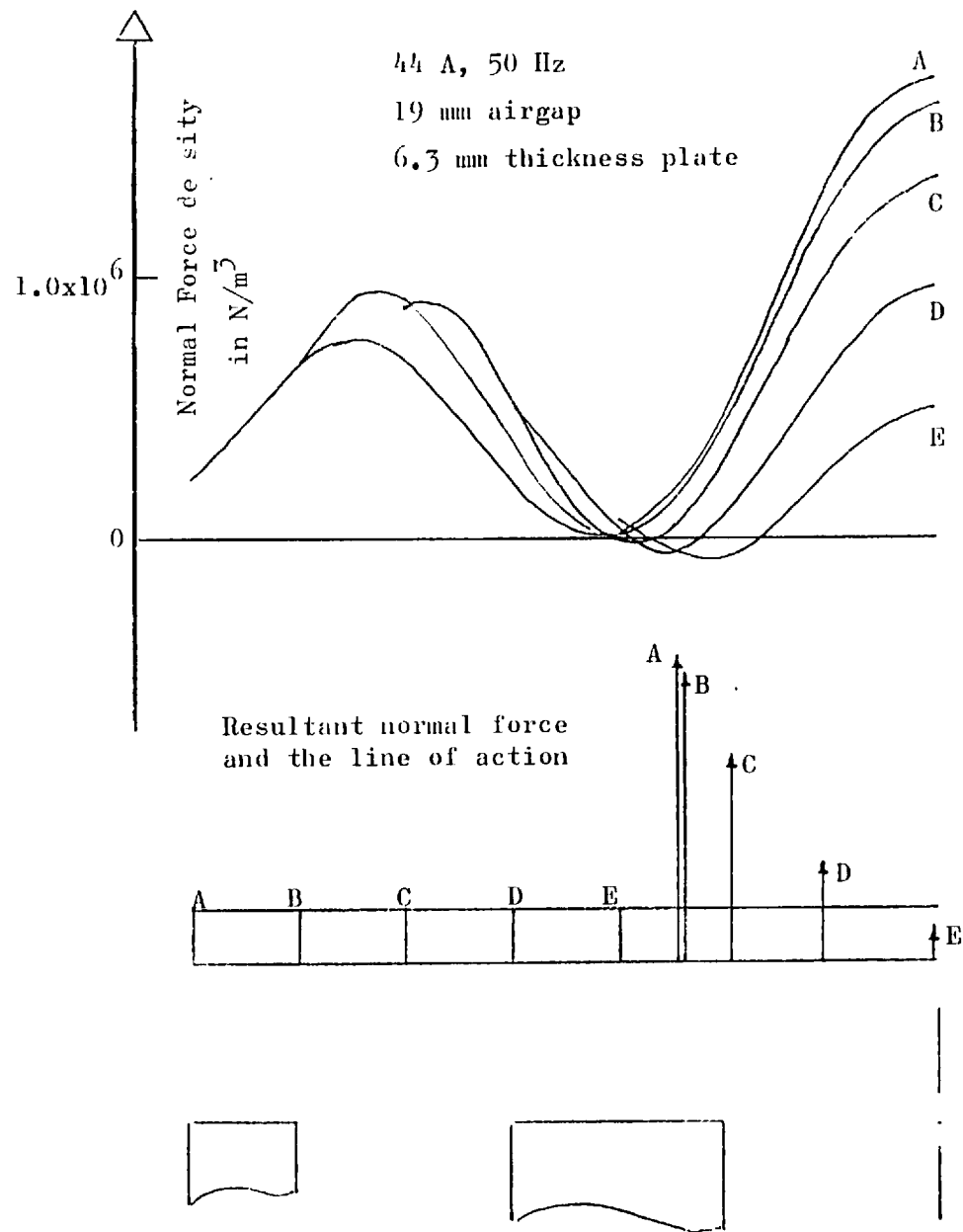


Figure 5.17: Normal force distribution.

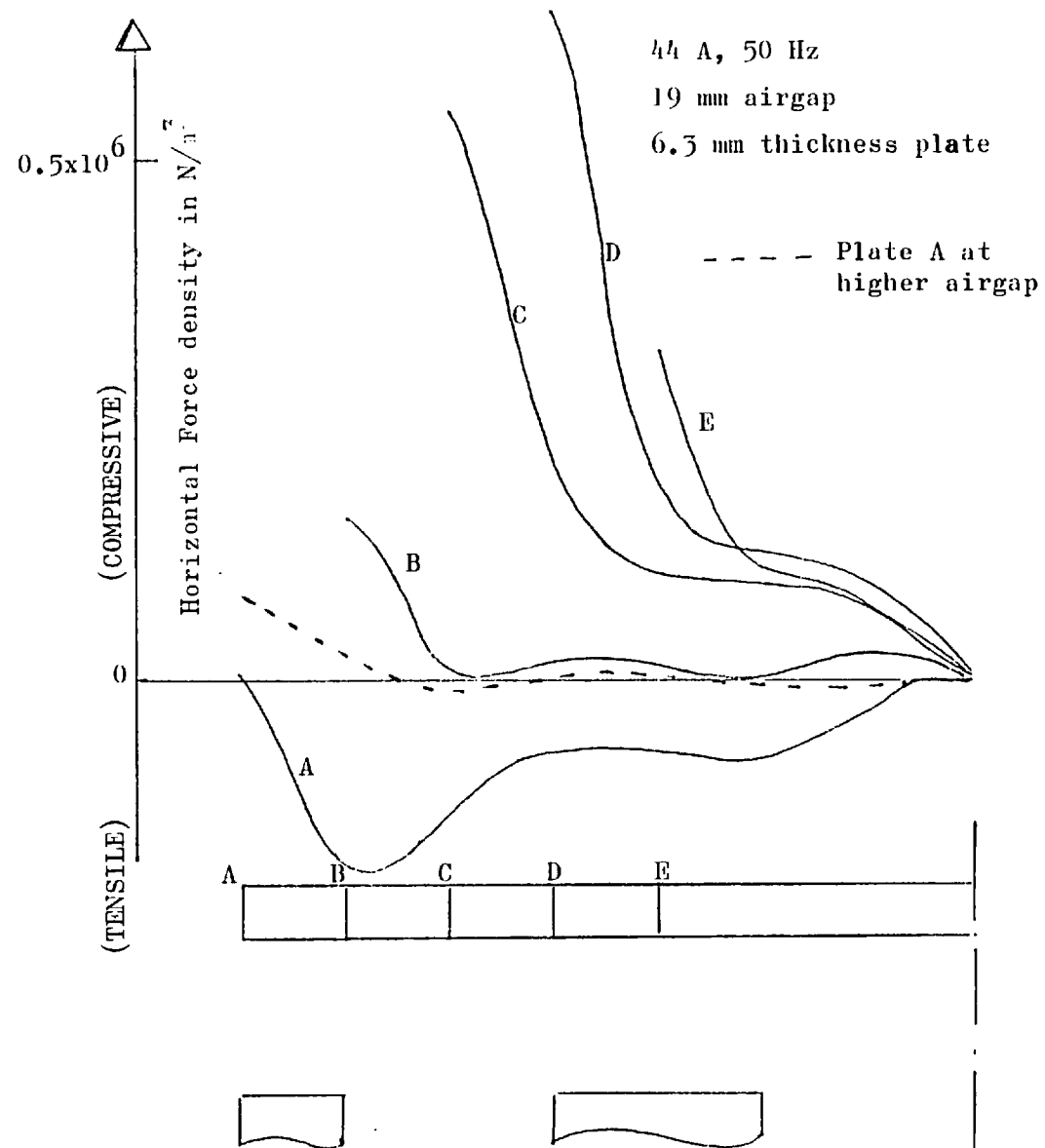


Figure 5.18: Horizontal force distribution.

the lift forces is concentrated in the high leakage flux regions above the slots. No pronounced changes in the stress distribution pattern are noticeable when plate widths are increased. By taking moments about the centre of gravity of the plates, the magnitude and location of the resultant vertical force in one half of the plate were computed. This enables one to view the levitated plate in mechanical terms as a plate resting on two knife edges. The positions of the knife edges are seen in Fig. 5.17 to rapidly approach the centre of gravity as the plate width is reduced below that of plate D. Thus for levitated plates smaller in width than plate D, the equilibrium is identical to the mechanical case of balancing a plate on one knife edge placed under its centre of gravity. This explains the experimentally observed, wobbling behaviour of plates that are narrower than plate D.

Next we examine the lateral stress distribution shown in Fig. 5.18. It is evident that the pattern of distribution is subject to changes depending on plate width. As predicted from the distribution of vertical flux density in Section 5.3.2, the stresses are compressive for plates contained within the O.C. cross-over locus, and tensile for those outside it. In Fig. 5.18 at the airgap considered, plate A extends through the O.C. cross-over locus. In order to test the observations made earlier with regard to crossing or not crossing the locus of the cross-over points, the airgap of this plate A was increased to such a level as to bring the whole plate within the area of two-pole magnetising flux distribution. The new stress distribution at this airgap for plate A is shown in broken lines in Fig. 5.18. We see that removal of the plate into the region of two-pole magnetisation results in a compressive distribution as in the case if plates B, C, D and E, which are at a smaller airgap, but within the limits of the two-pole area.

5.4 THE Xi-CORE WITH ASYMMETRICAL PLATES

The numerical solutions obtained with asymmetrical rotor plates present above the Xi-core are presented in this section. As in the previous main sections the agreement reached between computed and experimental values are presented first.

5.4.1 Computed and Experimental Results

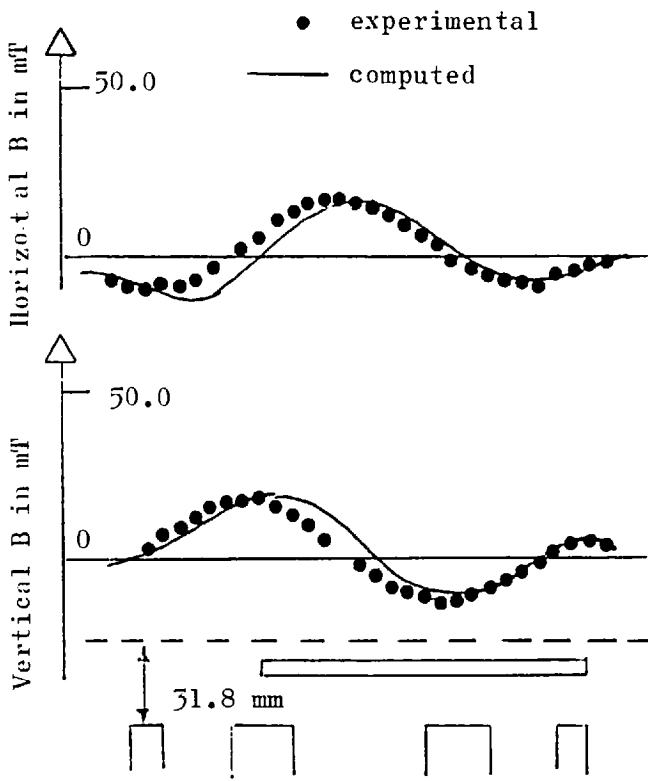
Figs. 5.19(a) and (b) show the degree of accuracy obtained in the computed and measured flux densities at two sections, one below and one above the asymmetrical plate. The largest errors are again seen to occur around the peaks as in the case of the O.C. and symmetrical cases.

For the same plate and stator core configuration the induced current density distribution measured at the top surface is compared with the computed values in Fig. 5.20(a). For this particular case, the phase angle of the induced current density was measured as well, using a phase shifting transformer/C.R.O. arrangement. The comparison with computed values is shown in Fig. 5.20(b).

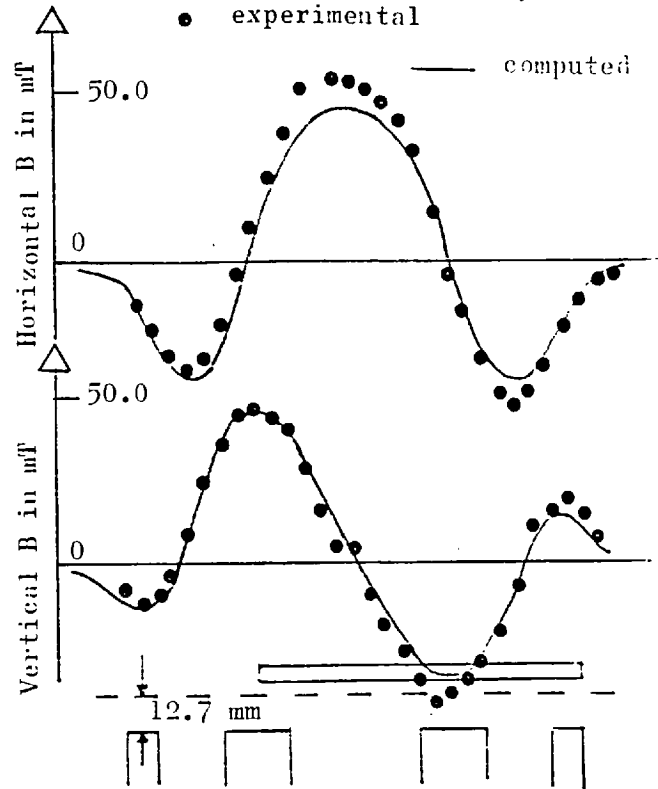
The comparison of measured normal and lateral forces with computed values is shown in Figs. 5.21(a) and (b). The normal force was measured using a weighing technique, while the lateral force was measured using the static rig described in Chapter 2.

5.4.2 Space Distribution of Flux Densities

The changes occurring in the distribution of the magnetising vertical flux density, directly underneath the plate, are shown in

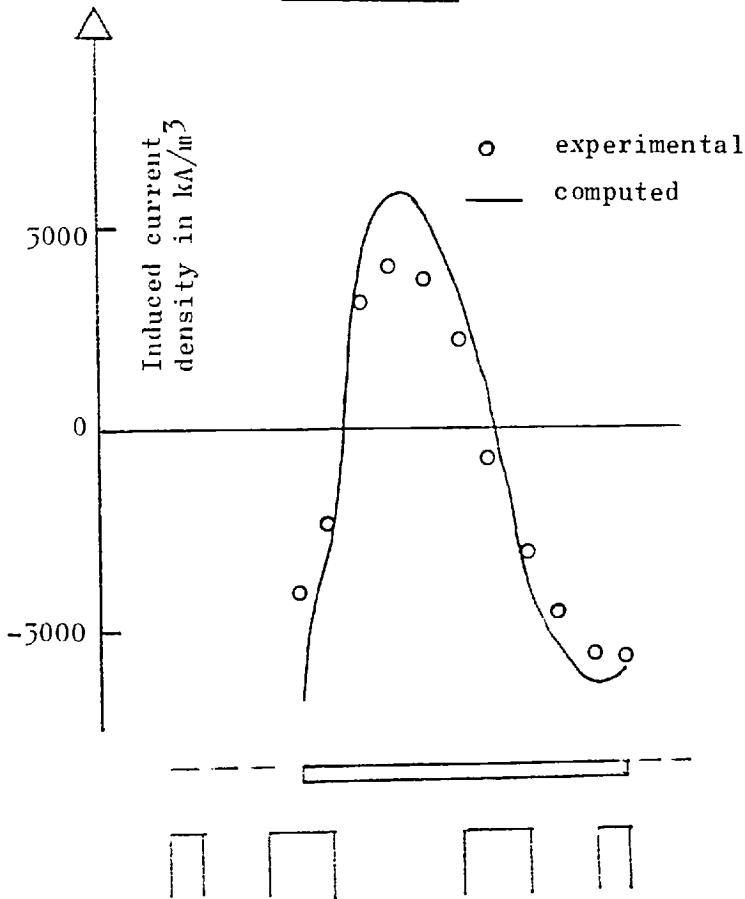


(a)

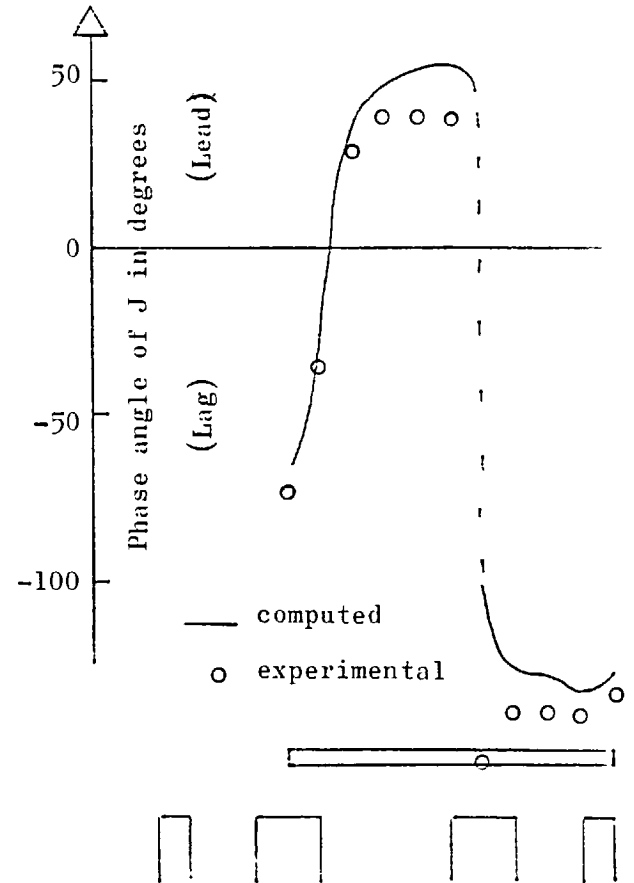


(b)

Figure 5.19: Flux density comparisons.



(a)



(b)

Figure 5.20: Comparisons of magnitude and phase angle of induced currents.

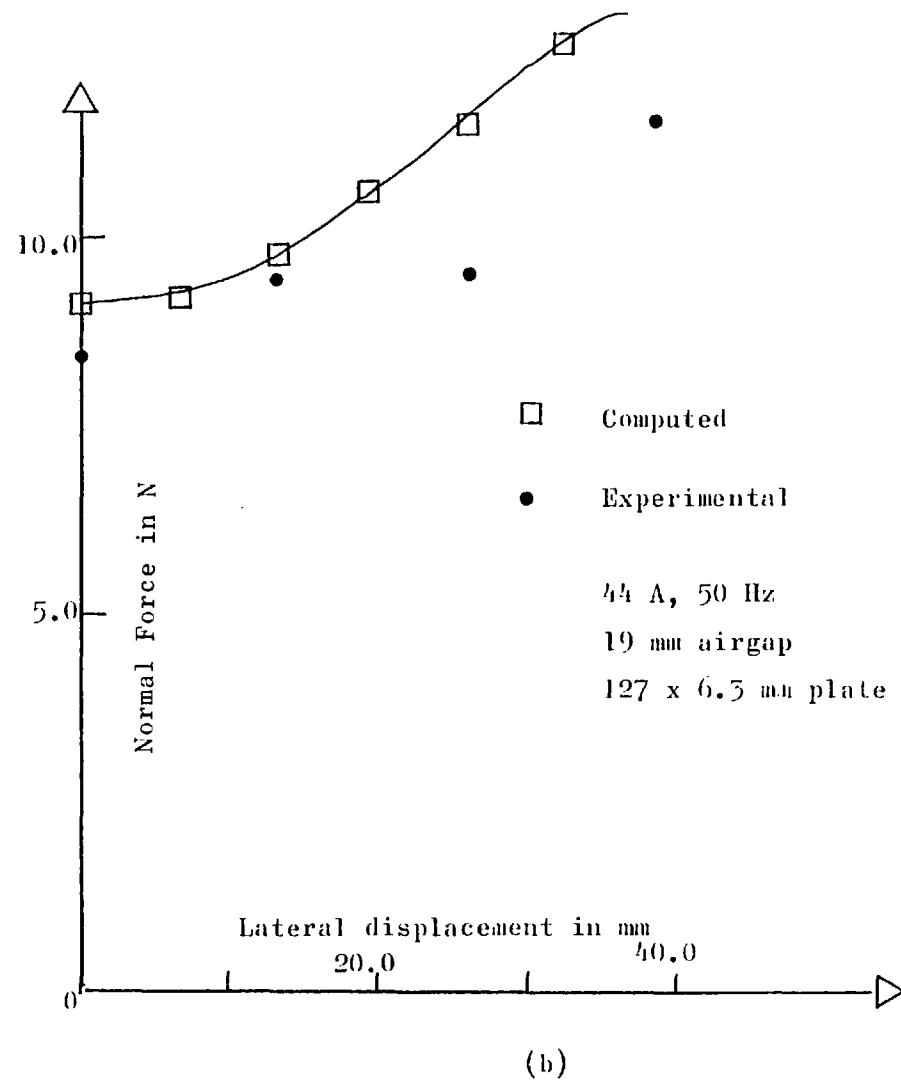
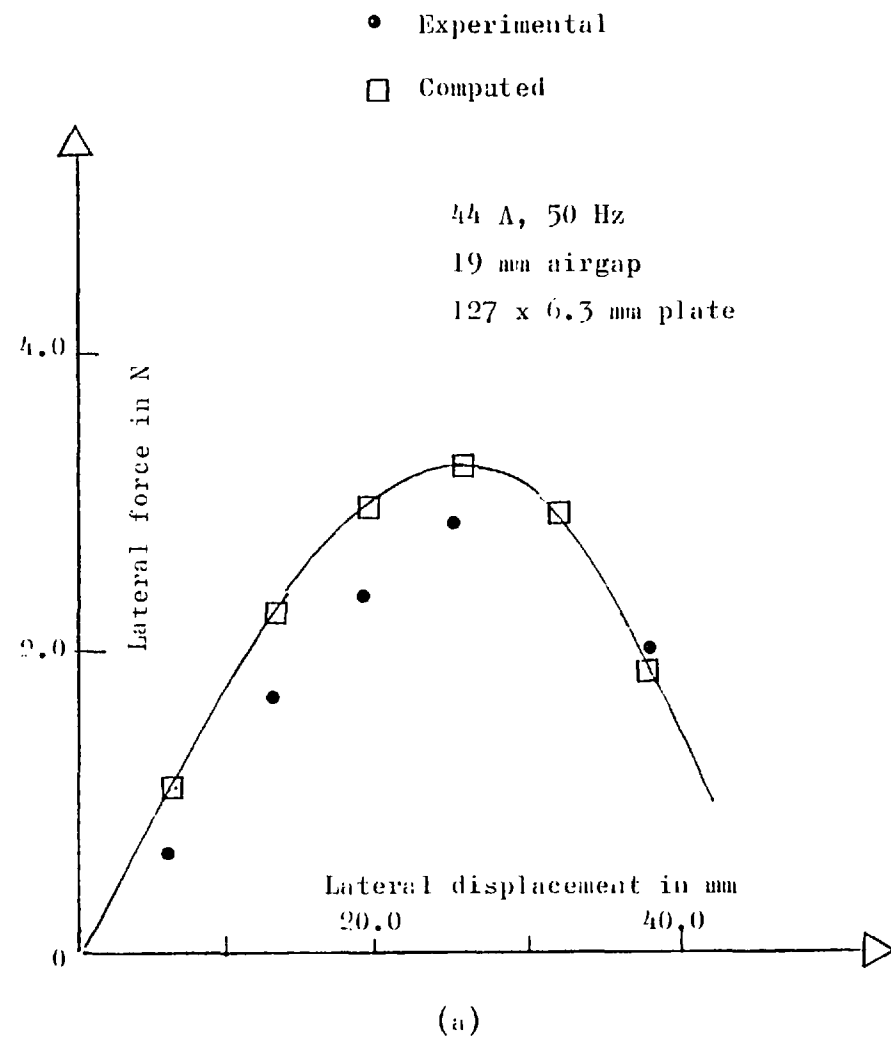


Figure 5.21: Comparison of forces.

Fig. 5.23(a), for the symmetrical and three displaced positions.

These four positions of the plate will be referred to as:

displacement 0, displacement 1, displacement 2 and displacement 3.

Fig. 5.22(a) illustrates the significance of these plate displacements.

At displacement 0, both the in-phase and quadrature flux density distributions in Fig. 5.23(a) can be replaced by a two-pole magnetisation. For displacement 1 and displacement 2, the two-pole distribution dominates, although a weak third pole has appeared in the leading edge (see Fig. 5.22(b)) at displacement 2. This is a consequence of the leading edge entering the O.C. cross-over locus. At displacement 3, the strength of the third pole in the leading half is increased. At this position the plate is magnetised by a three-pole distribution, with a strong pole around the centre-line of the plate flanked by two weaker poles. The computed quadrature field plot shown in Fig. 5.24 reveals this pictorially.

5.4.3 Space Distribution of Induced Current Densities

The induced current density distribution shown in Fig. 5.25(b) serves to confirm the conclusion reached in the previous section. The current distribution pattern forming two loops centred along the centre-line of the inner teeth transforms into a single loop, as the lateral displacement of the plate increases. In addition, Fig. 5.25(b) provides quantitative confirmation of the interpretation of the cross-over point $(\frac{x}{L})_0$ made in Section 2.5.1 of Chapter 2.

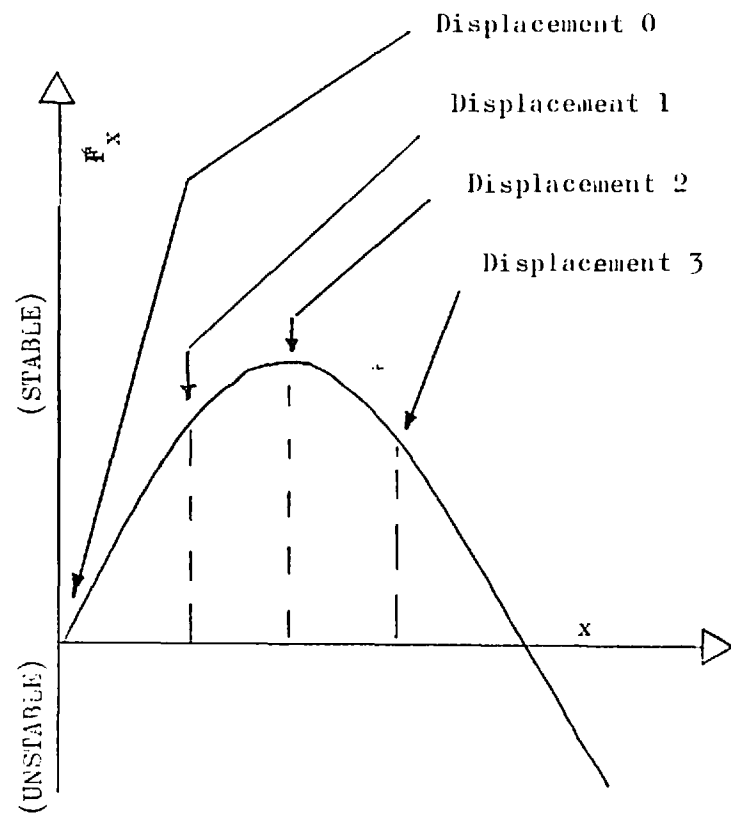


Figure 5.22(a): The four displacements considered in Section 5.4.2.

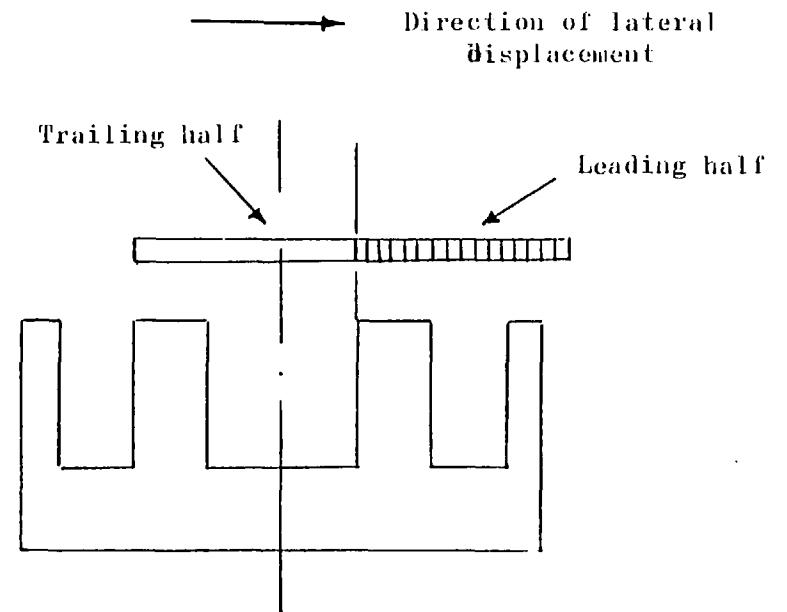


Figure 5.22(b): The meaning of the terms leading and trailing halves.

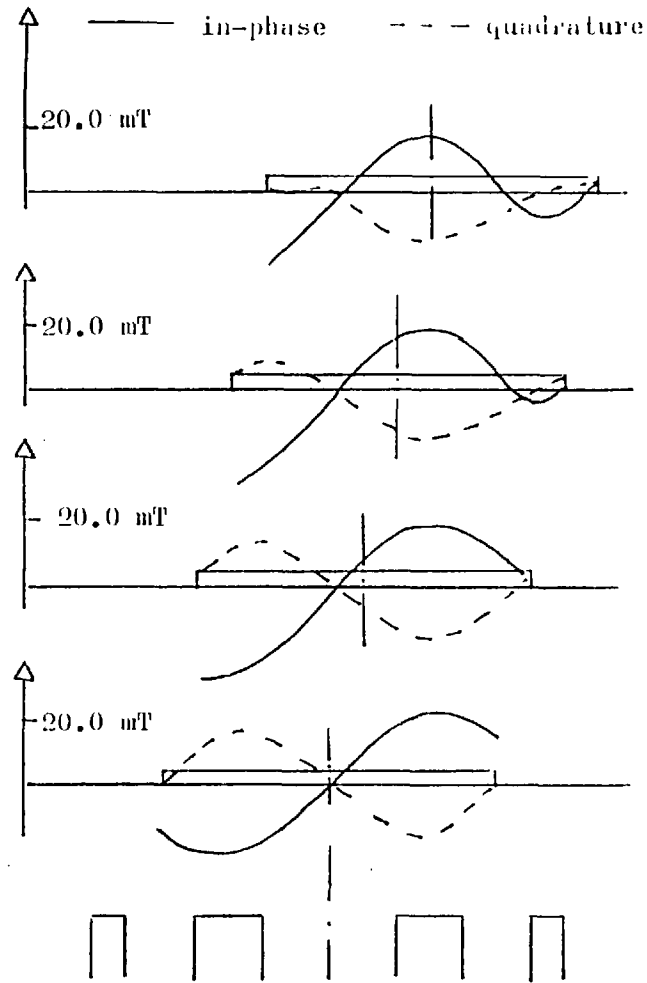


Figure 5.23(a): Space distribution of vertical flux density at the four displacements in Figure 5.22(a).

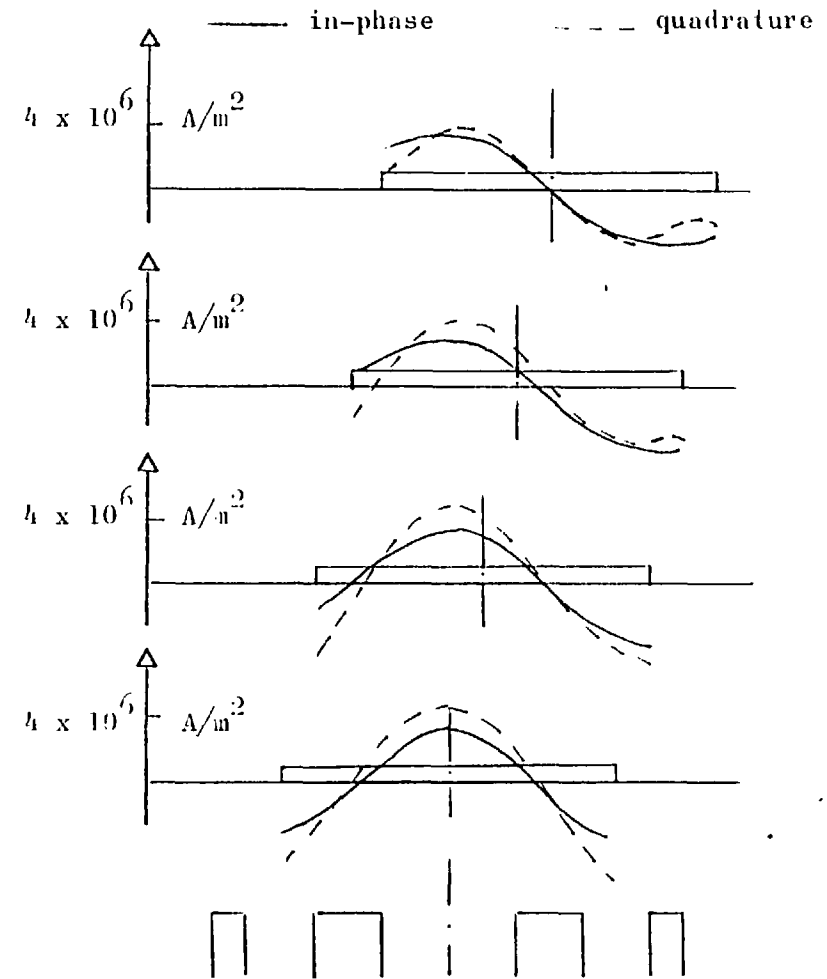


Figure 5.23(b): Space distribution of induced current density at the four displacements of Figure 5.22(a).

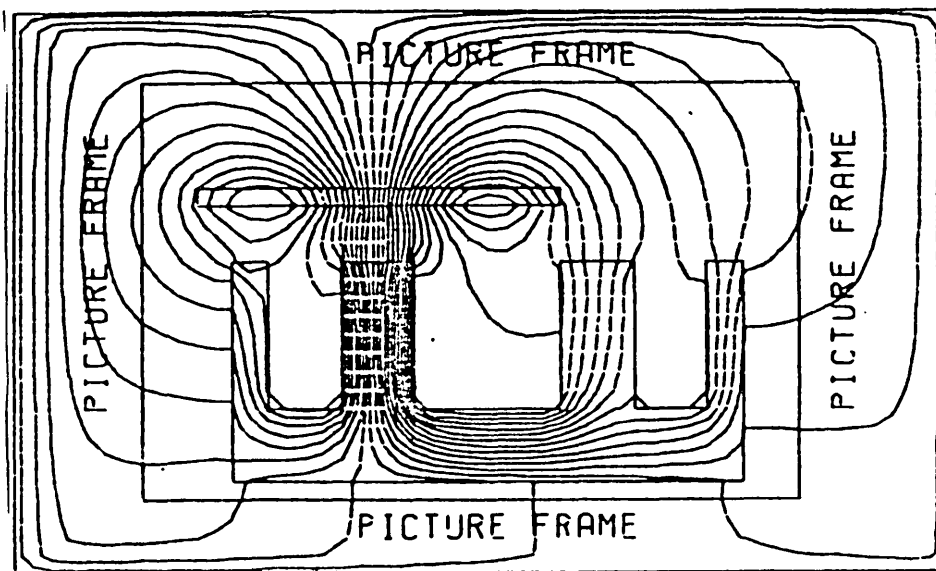
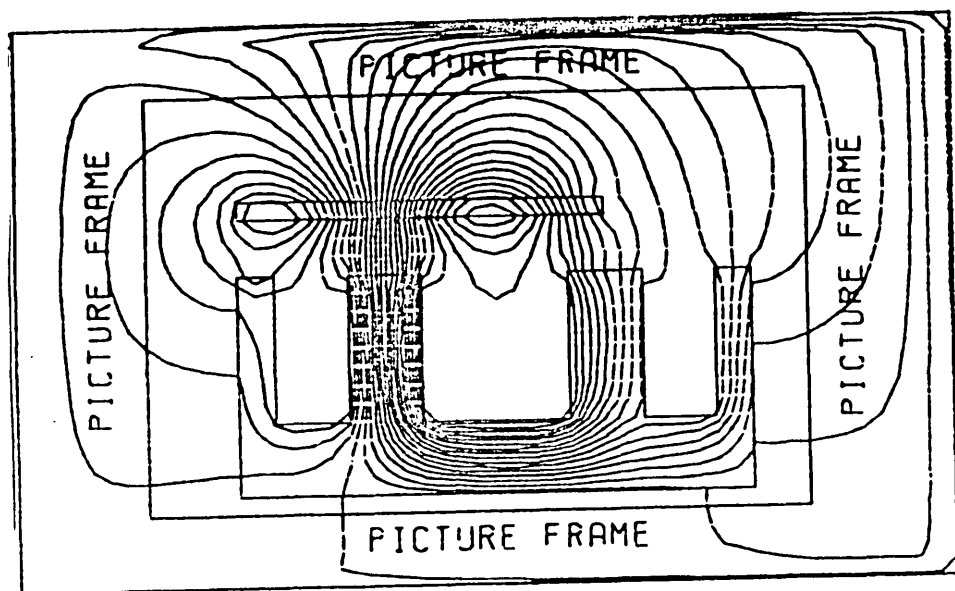
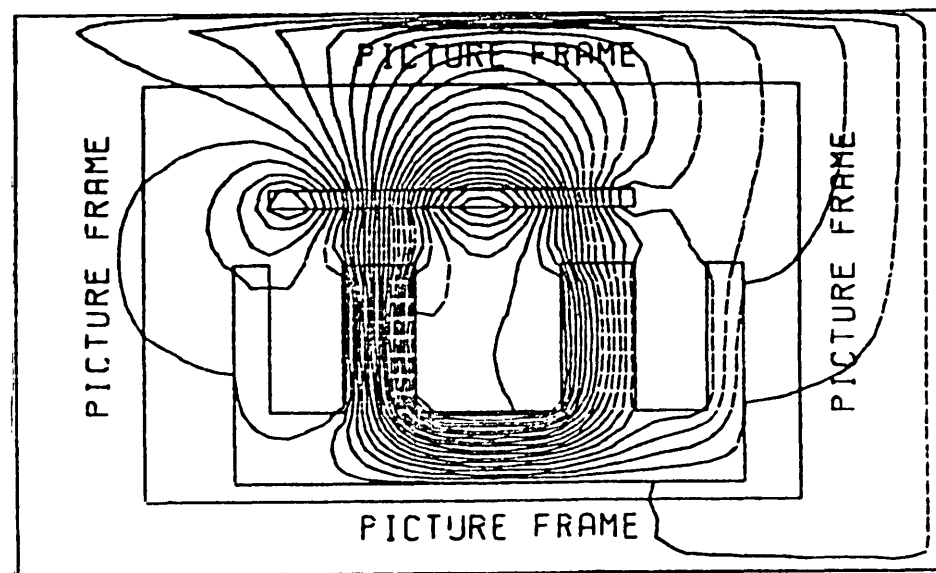
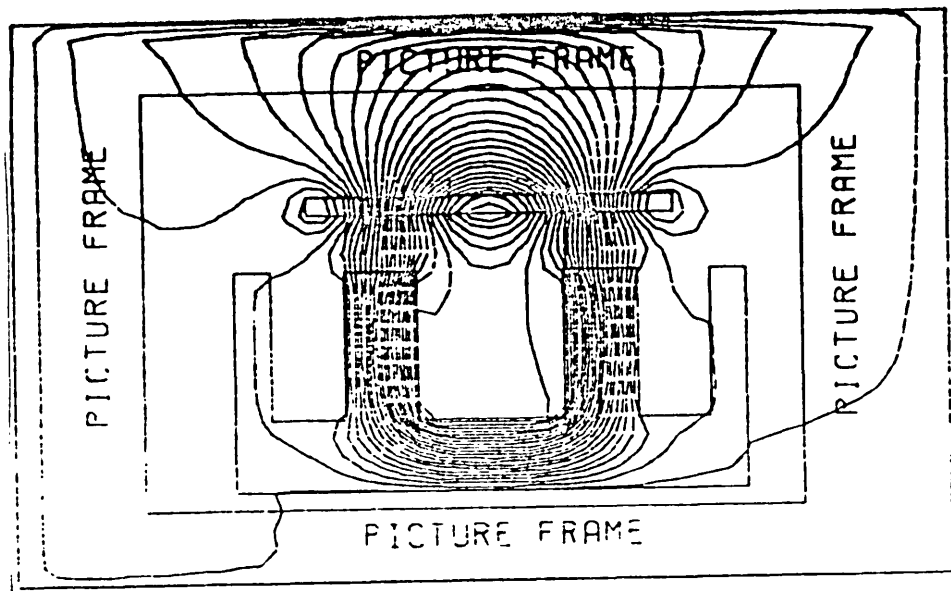


Figure 5.24: The computed quadrature field plot at the four displacements of Figure 5.22(a).

5.4.4 Stress Distributions

The normal force stress distribution is examined first. The distribution of this stress with plate displacement is shown in Fig. 5.25 for the displacements considered in the preceding sections for the same plate. An important effect of the normal force distribution is the resulting rolling torque. In order to examine this, the magnitude and location of the resultant normal force in each geometrical half of the plate were computed and are shown beneath the stress distributions in Fig. 5.25. We see that initially for displacement 1, when the plate is on the stable part of the lateral force/displacement curve, the forces are distributed in such a way as to restore it to the central position, with the leading edge tilted away from the stator. For the displacement corresponding to the peak lateral force (i.e. displacement 2), the direction of the rolling torque reverses, showing that for further displacement the plate is ejected, with the trailing edge tilted away from the stator. This is illustrated in Fig. 5.26.

The computed lateral force distributions for the same four positions of the secondary plate are shown in Fig. 5.27. The total lateral forces in the two halves of the plate were evaluated and are also shown. These enable one to see at a glance that with increasing lateral displacement, the forces in the two halves of the plate reverse directions. However, in order to examine the changes at every node in the plate in more detail, a phasor diagram of the lateral force producing magnetising flux density and the induced current density was drawn and is shown in Fig. 5.28

Consider nodes 1 to 5 in the trailing half of Fig. 5.28. After the first displacement the phase lag between the current and

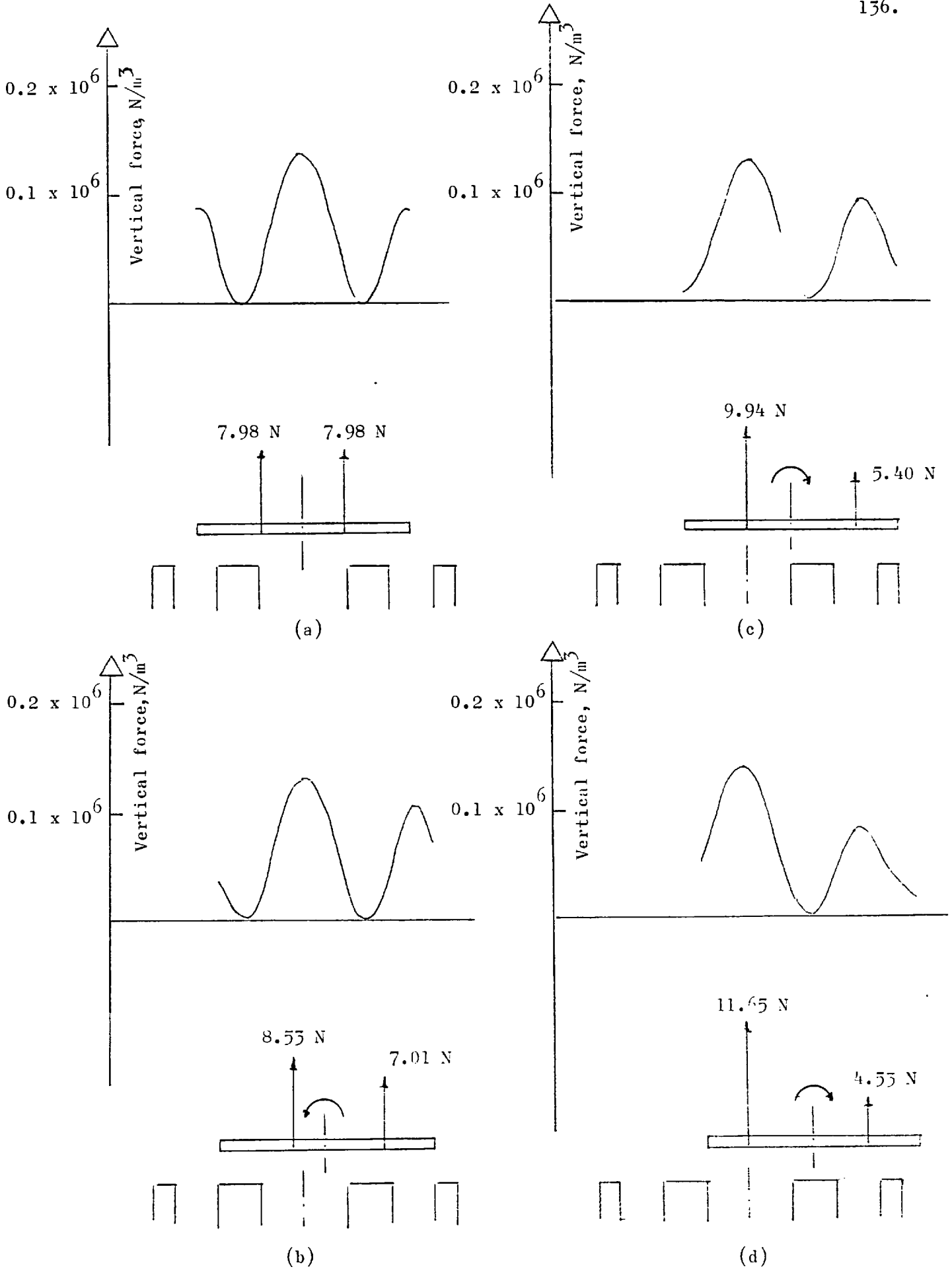
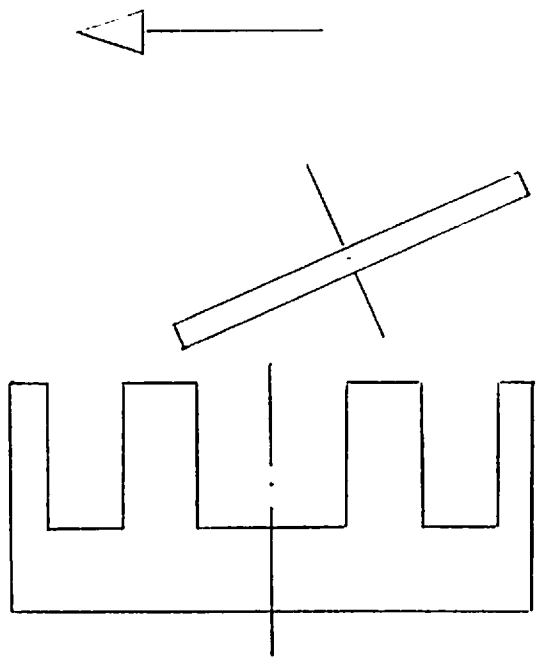
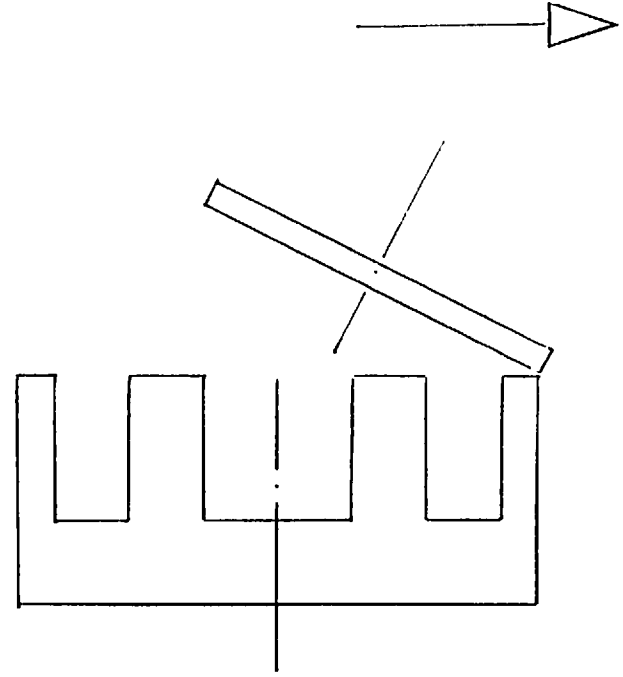


Figure 5.25: Change of vertical force distribution with lateral displacement.



Stable restoration with the
leading half turned away from the stator



Unstable ejection with the trailing
half turned away from the stator

Figure 5.26: The two modes of plate behaviour due to the normal force distribution.

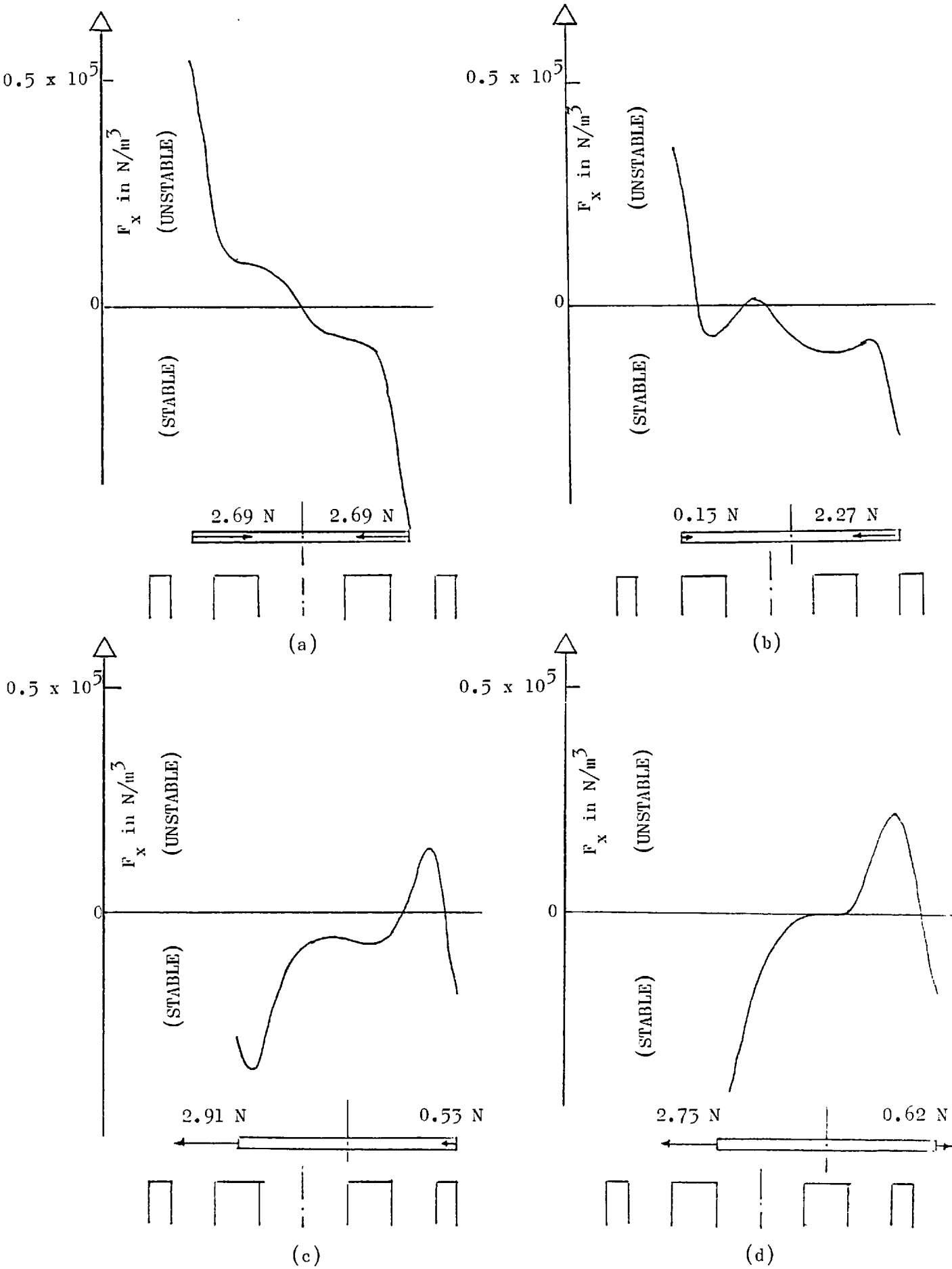


Figure 5.27: Change of lateral force distribution with lateral displacement.

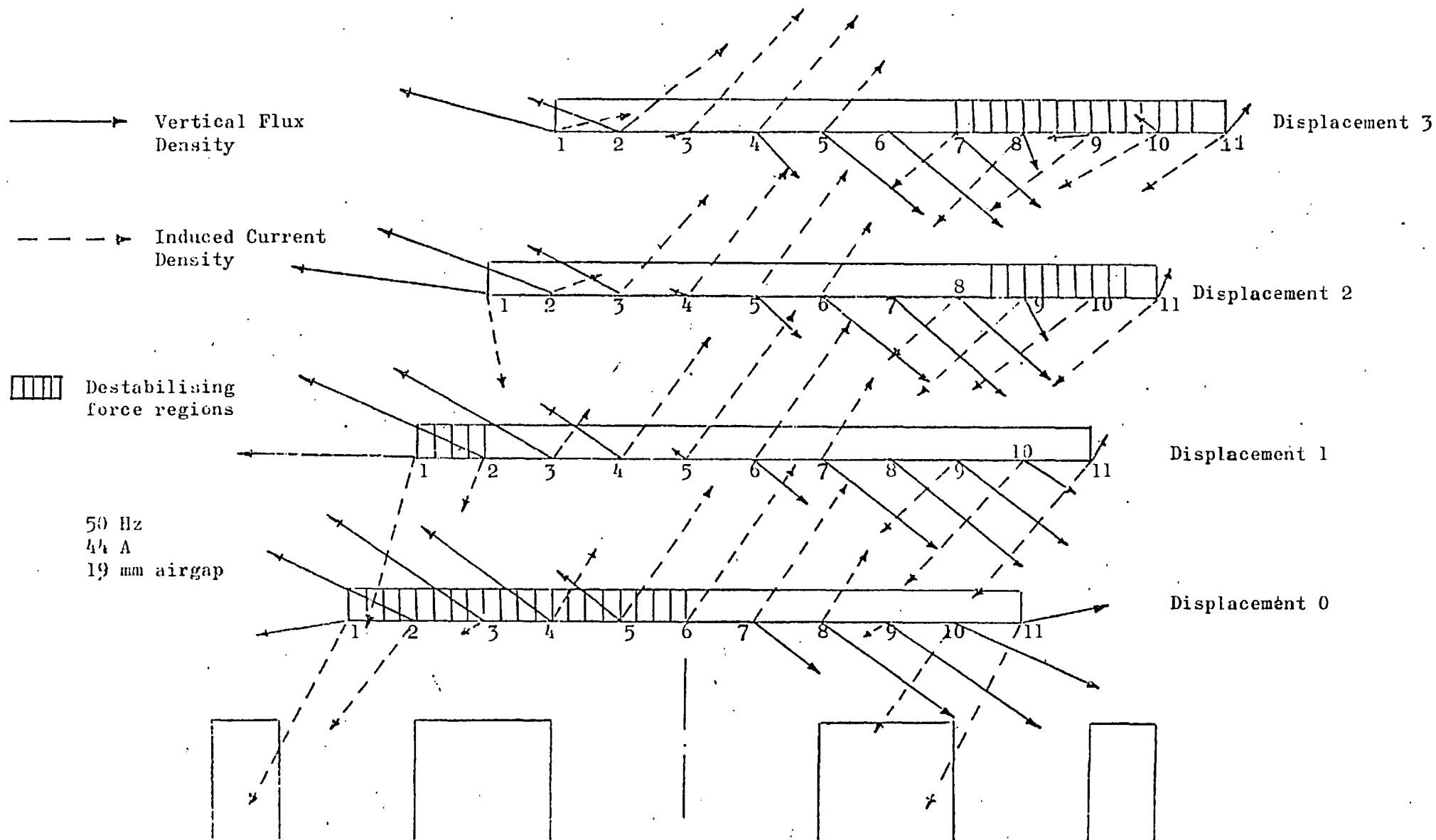


Figure 5.28: The phasor diagram of induced current density and vertical flux density at the nodes in the plate.

flux density phasors in these nodes has diverged from their values in the symmetrical position. For nodes 2 to 5, this divergence has transitioned through 90° and therefore the direction of the lateral forces has reversed, producing a stabilising force at these nodes. At node 1, though the phase lag has diverged from 52° to 76° , the transition has not occurred and therefore the force remains a destabilising one, but has the potential to become a stabilising force. After the second displacement, this force reversal has occurred at node 1 and from node 2 to 5, the increasingly diverging phase angle maintains the stabilising force. We thus see that, at the nodes in the trailing half, stabilising forces are produced and maintained by the tendency of the phase angle between the current and flux density phasors to be out of phase by as large an angle as possible. Let us now examine the leading half. For stabilising forces to appear, after the first displacement, in nodes 7 to 11, the lateral forces at these nodes should not reverse sign. The phase angle between current and flux density in the symmetrical position is already greater than 90° . Therefore the requirement for maintaining stabilising forces is that these phase angles should diverge with displacement. But the phasors at these nodes after displacement show that the opposite tendency for the phasors to be in-phase occurs. Stabilising forces are only present because the transition through 90° has not occurred yet. Therefore eventually, with increasing displacement, the forces in the leading half must reverse direction and produce destabilising forces. This is confirmed by observing the phase changes at these nodes, for the other two displacements.

Therefore we conclude this section by drawing the inference that any parameter change that causes the induced current and the

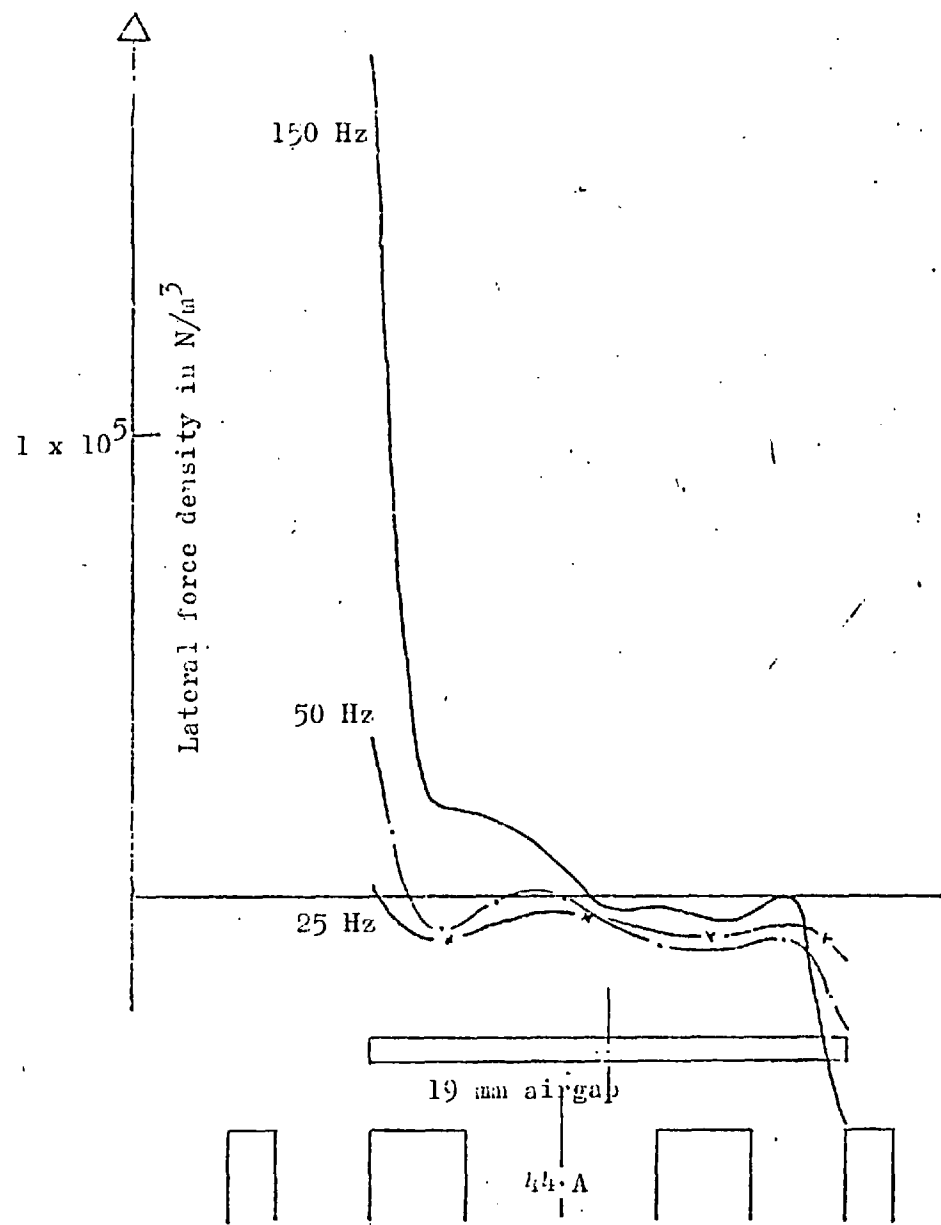
magnetising flux density to be in-phase will contribute to lateral instability. In the following sections, it will be shown that these parameters are: increasing frequency, increasing plate width, decreasing airgap, and in general changing any parameters leading to a high Goodness Factor.

5.5 FACTORS DETERMINING LATERAL STABILITY

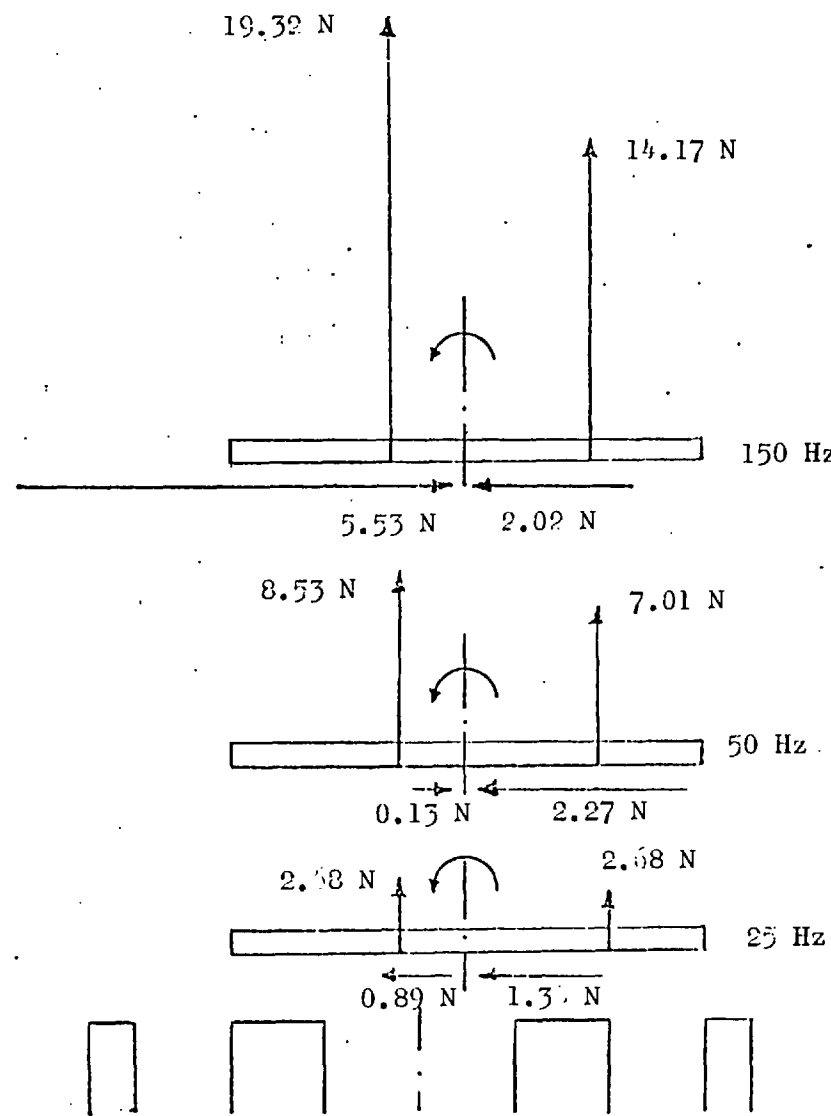
In this section, certain aspects of lateral stability, such as changes of frequency, airgap and plate width are examined. In the preceding section one plate at several displacements was considered, but here major consideration is given to plates that are just displaced from their symmetrical positions.

5.5.1 Effect of Frequency on Lateral Stability

It was shown in Chapter 2 that a stable plate becomes unstable if the supply frequency is increased while maintaining all other parameters constant. Let us examine the changes in lateral stress distribution, with increase of frequency, for a slightly displaced plate. These are shown in Fig. 5.29. At the lowest frequency considered, namely 25 Hz, the stabilising forces are distributed throughout the width of the plate. The total magnitude of the lateral force is 2.25 N, of which 60% arises from the leading half and 40% from the trailing half. This stabilising force, in conjunction with the rolling torque, returns the plate to its symmetrical position, with the leading edges lifted away from the stator. Increasing the frequency to 50 Hz causes the stabilising forces distributed in the leading half to increase everywhere, but causes the direction reversal of the stresses in the trailing half.



(a)



(b)

Figure 5.29:

The lateral force distribution at three frequencies.

The total effect is to produce a smaller stabilising force of 2.14 N, greater lift force, and a greater rolling torque which restores the plate in the same manner as at 25 Hz. Increasing the frequency further to 150 Hz reveals that the direction reversal tendency continues in the trailing half and begins to appear in the leading half. A net destabilising force of 3.51 N results, with an increased lift force, but a diminished rolling torque acting in the same direction as at 50 and 25 Hz. Therefore the plate is ejected, but with the trailing half away from the stator.

In order to examine the changes taking place at every node, at two different frequencies (50 Hz and 100 Hz), a phasor diagram is drawn in Fig. 5.30. Let us examine the changes at a few nodes. At node 1, at 50 Hz a destabilising force is present and the phase lag is 74.6° , and is already past the 90° transition. Therefore, further increase of frequency causes the phase lag to decrease, contributing to a larger force in the same direction as at 50 Hz. At nodes 2 and 3 at 50 Hz, the forces are stabilising and the angle of lag greater than 90° . Therefore a change of direction of force is possible when the increasing frequency causes the phase lag to decrease through 90° , and it does occur at 100 Hz. At node 4, the phase lag is already past 90° , and therefore increasing frequency can only increase the already destabilising force present. From nodes 6 to 11, all the phase angles are greater than 90° , therefore a reversal of direction of the forces is possible, if the frequency is high enough.

5.5.2 Effect of Plate Width on Lateral Stability

It was shown in Chapter 2 that as the plate width is increased, while keeping other parameters constant, the force changes

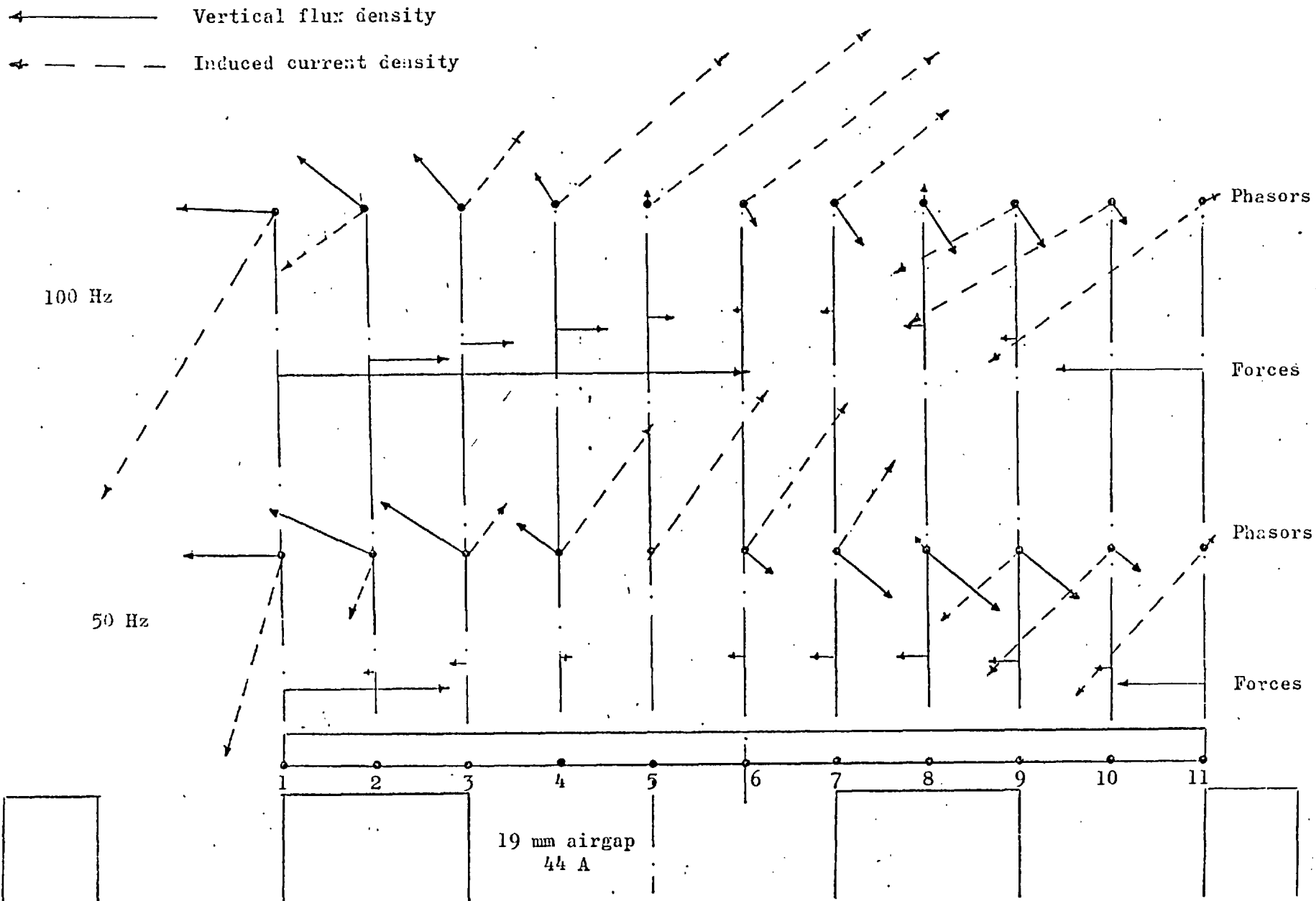


Figure 5.30: Phasor diagram for one plate at two frequencies.

from stabilising to destabilising. In order to obtain theoretical confirmation of this, the lateral force/displacement curves were computed for several plate widths at constant frequency and constant airgap. Fig. 5.31 shows the results obtained and clearly provides the theoretical confirmation.

Considering the lateral force distributions in Fig. 5.32, we note that the stabilising forces appear everywhere throughout the narrowest plate, similar to the case of the wider plate at a lower frequency discussed in the preceding section (Fig. 5.29, the distribution at 25 Hz). For the widest plate equal to the stator width, destabilising forces are seen to occur in most parts of the plate. In order to investigate this particular plate further, a phasor diagram at every node is drawn and shown in Fig. 5.33.

Consider nodes 1 to 7, of the symmetrical plate. For the plate to be stable, after displacement the direction of the forces at these nodes should not reverse. This means that the phase lag between the current density and flux density after displacement, should diverge from the already greater than 90° lag in the symmetrical position. But the actual change is seen to be the opposite, by the tendency of the flux density and current density phasors to be in-phase, exactly as happens when frequency is increased gradually. Considering nodes 9 to 15 in the symmetrical position, for stabilising forces to appear after displacement, at these nodes the force direction should reverse. This requires that as in nodes 1 to 7, the displacement should cause a diverging phase lag between current density and flux density. But the actual changes are again seen to be in the opposite way, by the tendency of the flux density to be in-phase.

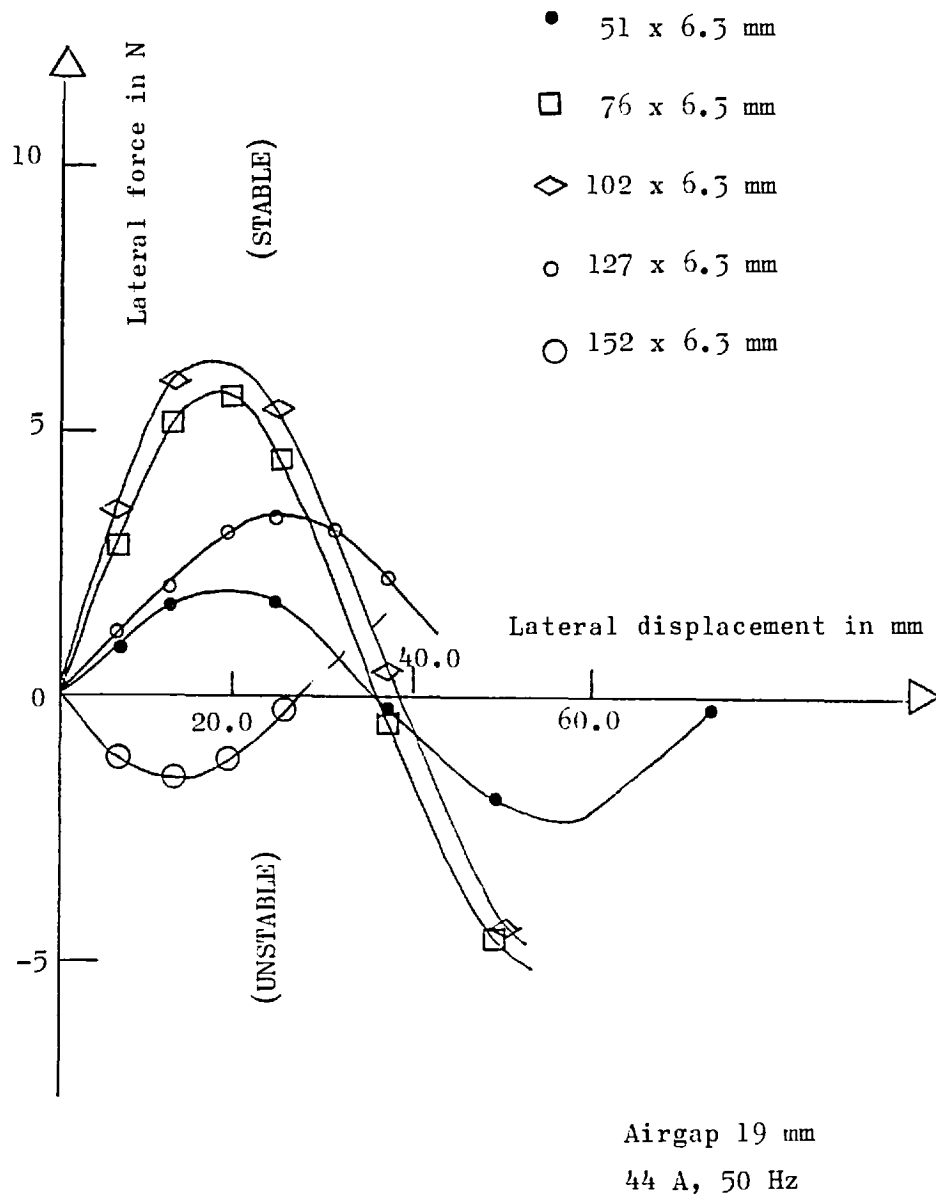


Figure 5.31: Theoretical confirmation of the lateral instability of wide plate.

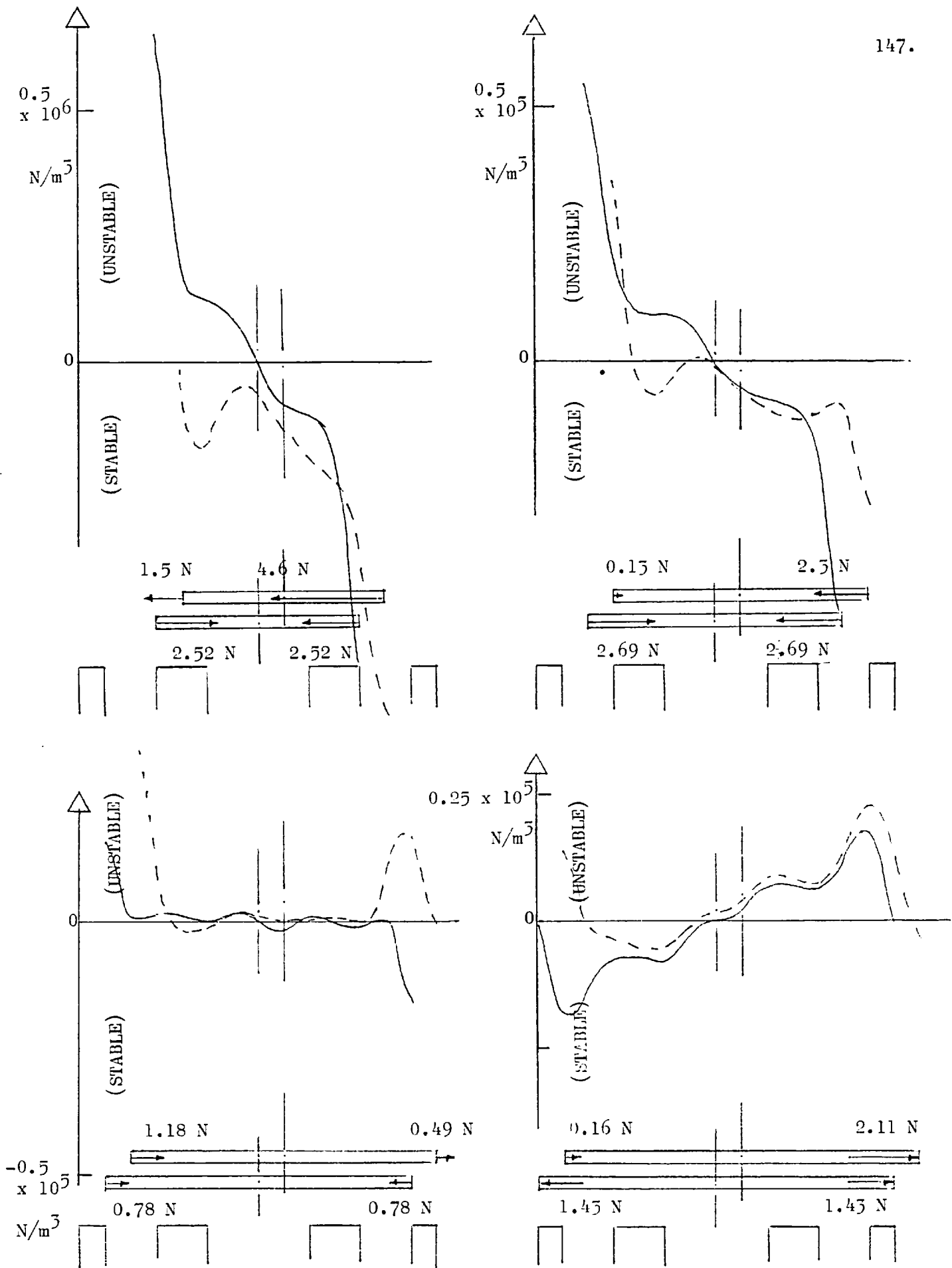


Figure 5.32:

The horizontal force density distribution before and after displacement, for four plate widths.

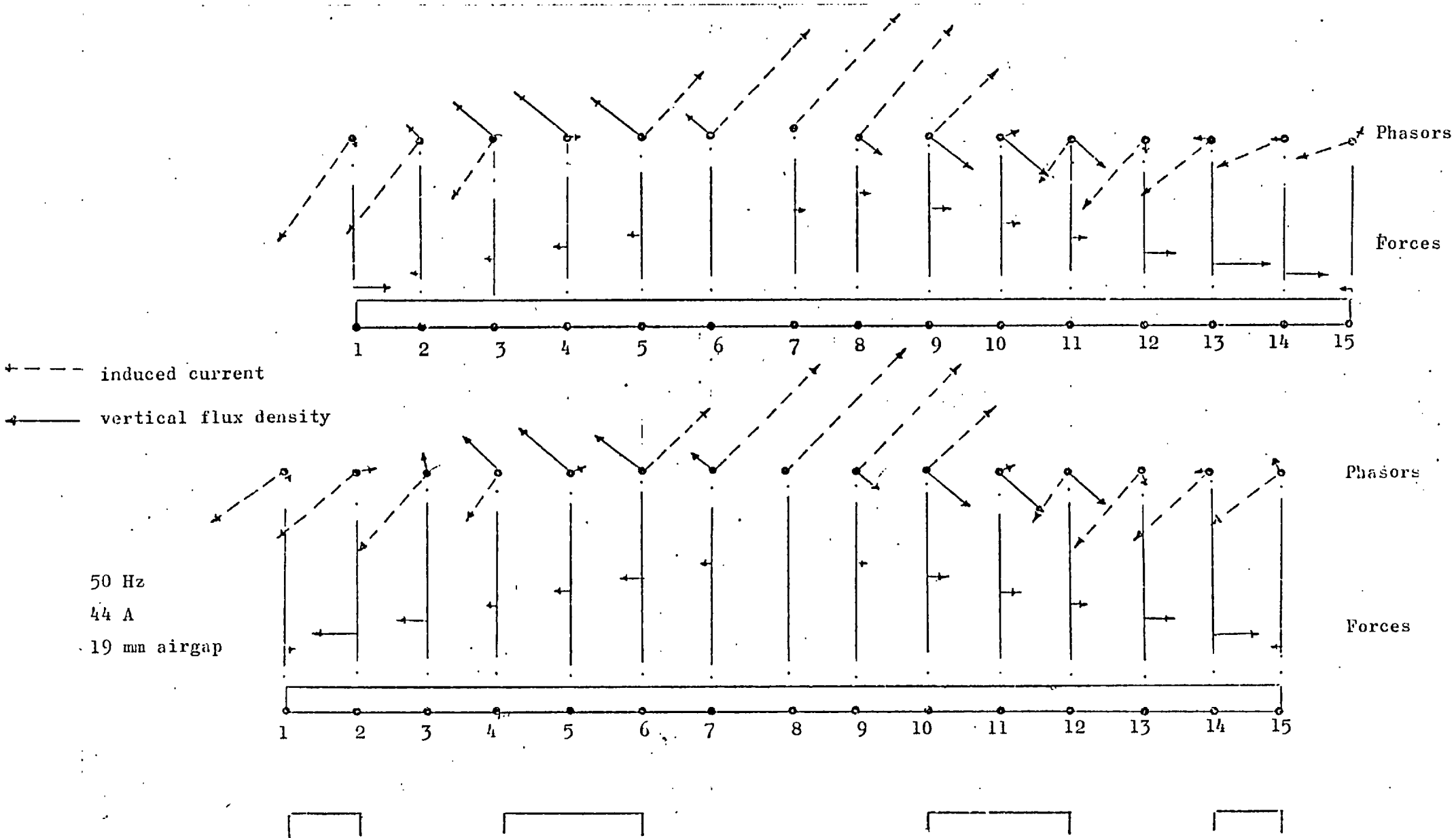


Figure 5.53: Phasor diagram for one plate before and after displacement.

Therefore we see that a wider plate after a small displacement behaves like a narrower plate at higher frequency, and the common feature between the two being a tendency of the magnetising flux density to be in-phase with the induced current density at every point.

5.5.3 Effect of Airgap on Lateral Stability

The experimental research presented in Chapter 2 revealed that a plate in stable levitation would become unstable if the airgap was reduced. In order to investigate this, the numerical solutions of one plate slightly displaced, situated at two different airgaps was obtained. The airgap values used were 6.5 mm and 19 mm, and the plate width was 127 mm.

The lateral force distributions at the two airgaps are shown in Fig. 5.34. As in the earlier cases, the phase changes occurring when the airgap is reduced, is shown by means of a phasor diagram in Fig. 5.35. It can be seen that, similar to the case of increasing frequency and increasing plate width, decreasing airgap is seen to produce the effect of making the vertical flux density and current density become in-phase and cause lateral instability.

5.5.4 Effect of Stored Energy on Lateral Stability

In this section the variation of the stored magnetic energy of the field with the position of the rotor plate is examined. For all the plates of differing widths considered in Section 5.5.2 in Fig. 5.31, the stored magnetic energy variation with lateral displacement is shown in Fig. 5.36. It is seen that for all stable plates, the symmetrical position is one of maximum stored energy,

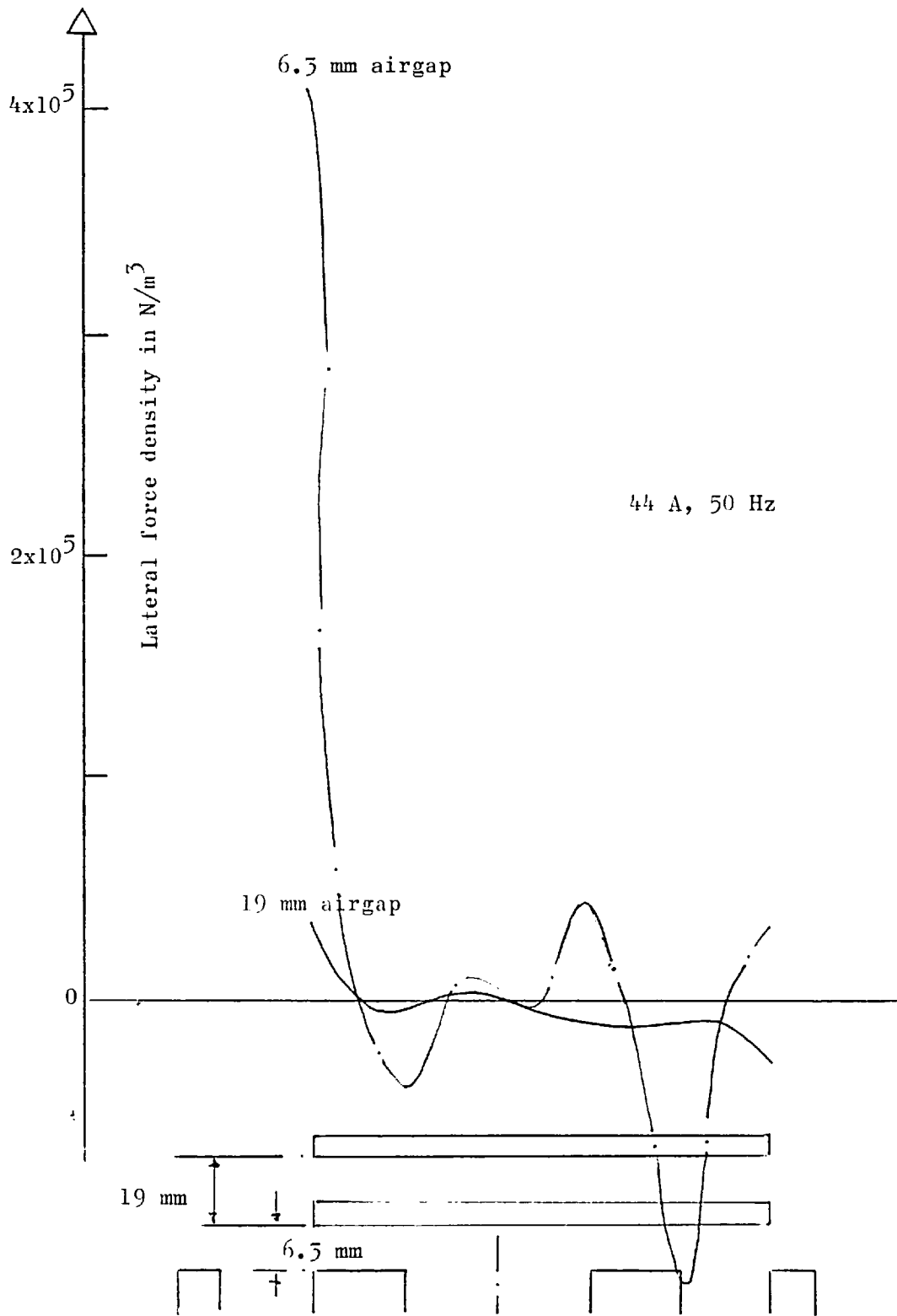


Figure 5.54:

The lateral force density distribution in a plate at two different airgaps.

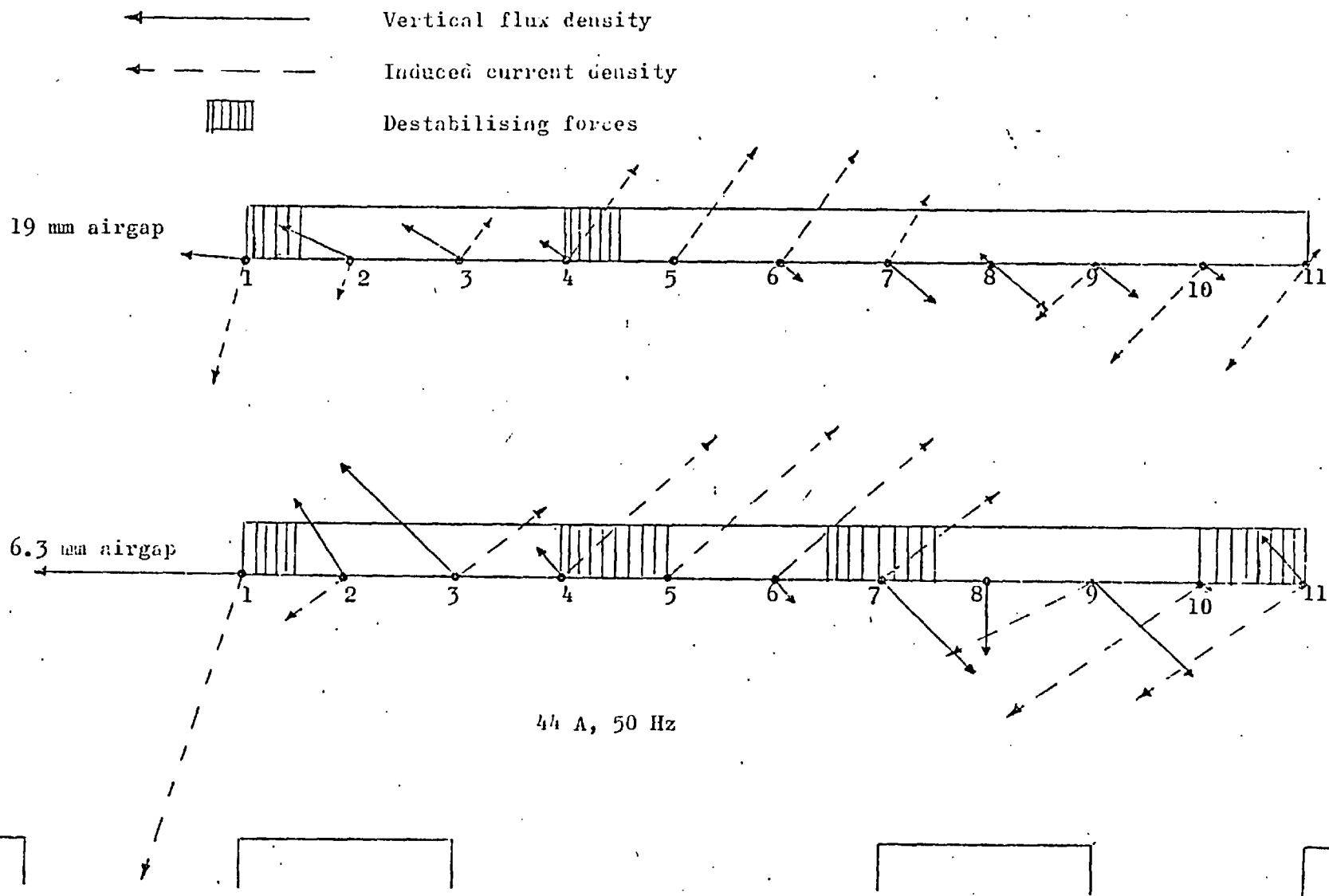


Figure 5.35: Phasor diagram for one plate at two airgaps.

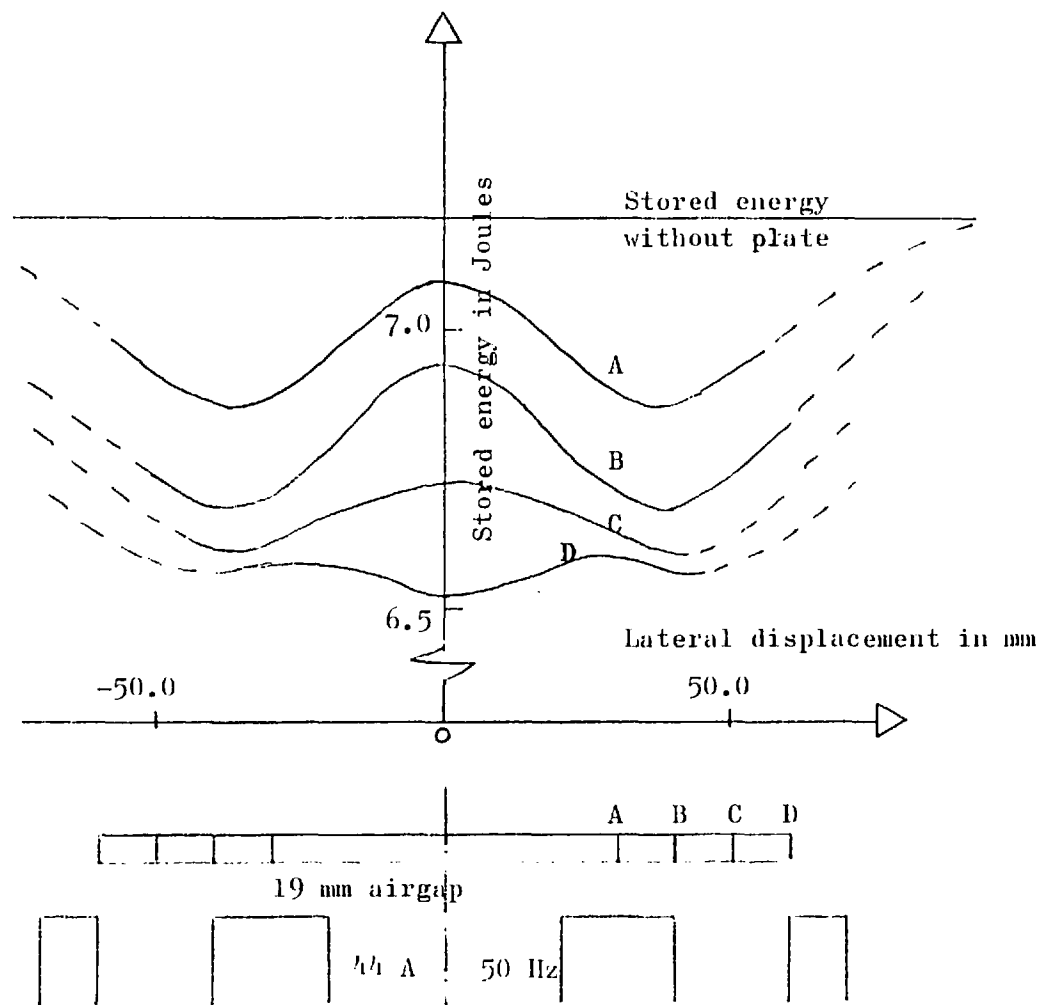


Figure 5.36: Magnetic stored energy variation with plate position.

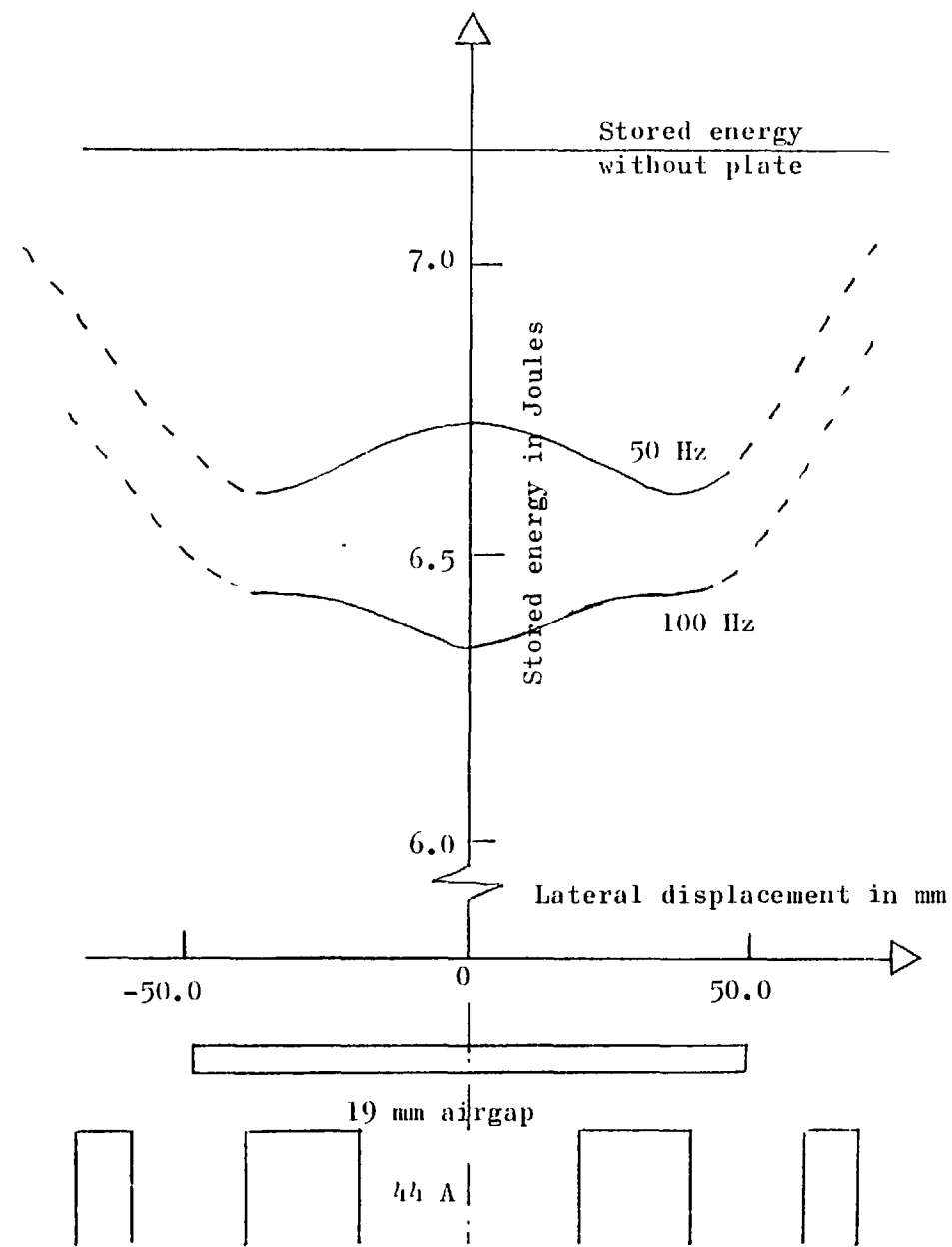


Figure 5.37: Magnetic stored energy variation with plate position.

while for the unstable plates it is one of minimum stored energy. Therefore for small perturbations about the symmetrical position, the stable plates are restored to the initial position, enabling the system to maximise the stored energy. For the unstable plate after a small perturbation, a position of maximum stored energy is either slightly displaced from the symmetrical position, or corresponds to infinite displacement. Therefore the plate either reaches a slightly displaced position, or is ejected completely.

We thus reach the following interpretation of the stabilising mechanism in terms of stored energy. For all plates, when they are symmetrically situated, the stored magnetic energy is less than the open circuit value. One position for all the plates, in order that the stored energy may be maximum, is after their complete ejection. Therefore all the plates must necessarily be unstable after a certain displacement, causing their complete ejection. However, the presence for certain plates, of a position of maximum stored energy other than that corresponding to infinite displacement, provides a stable position for these plates. Of these plates, the stable plates are those for which this position of maximum stored energy coincides with the symmetrical position. This view of lateral stability in terms of maximum stored energy is further reinforced, when the stored energy variation with position, of a plate at 50 Hz and 100 Hz, shown in Fig. 5.37, is examined. Increase of frequency from 50 Hz to 100 Hz alters the shape of the variation to that corresponding to a wider plate of 50 Hz. The symmetrical position changes into a position of minimum stored energy making the plate unstable.

5.5.5 Lateral Force Stiffness

Consider a plate, under constant current excitation, at constant frequency, the variable parameter being the airgap value. Fig. 5.38(a) shows the variation of the lateral stiffness with airgap value. It is seen that for this particular plate, under the assumed supply conditions, the minimum airgap required for stability is approximately 10 mm. Between the airgap limits of 10 and 20 mm, the stiffness varies between 0 and 0.15 N/mm, at a constant current of 44 A, at 50 Hz. In contrast, when the plate is unstable at an airgap of 6.3 mm, the stiffness is 0.39 N/mm.

Fig. 5.38(b) shows the lateral stiffness variation at a constant airgap, under constant current, constant frequency excitation, with the variable parameter being the plate width. At this airgap, the plate width giving the maximum stiffness of 0.55 N/mm is about 90 mm, which is approximately the distance between the outer edges of the two inner teeth of the stator. Increase of plate width beyond 90 mm, at this airgap, causes a steady decrease in lateral stiffness. The limiting plate width for stable levitation, under the assumed operating conditions, is seen to be 140 mm, which is approximately the width between the inner edges of the two outermost stator teeth.

Fig. 5.38(c) shows the lateral stiffness variation of one plate, at constant airgap, the variable parameter being the frequency of the constant excitation current. The maximum stiffness of 0.2 N/mm is achieved at a frequency of about 35 Hz. Beyond this limit, increase of frequency leads to a steady decrease in the lateral stiffness. For frequencies higher than 85 Hz, the plate becomes unstable.

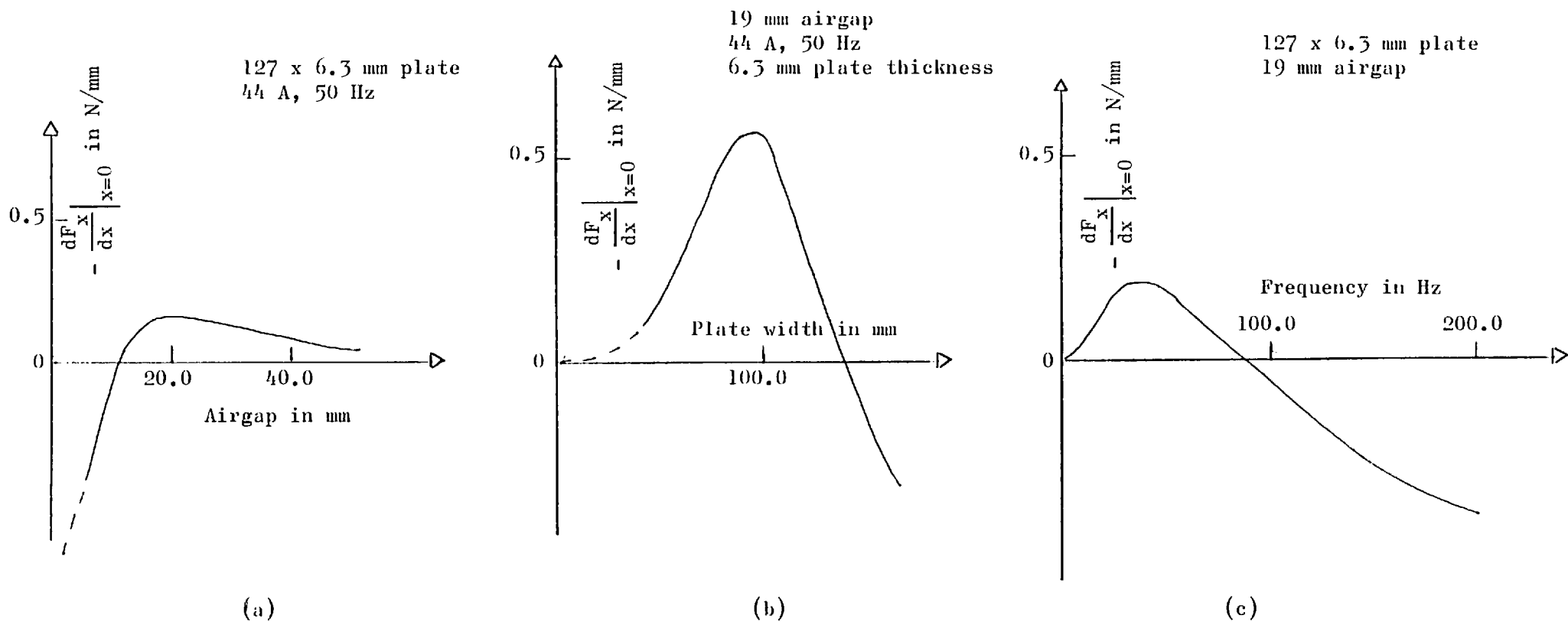


Figure 5.38: The lateral force stiffness variations.

Finally, the equivalence of increasing plate width, decreasing airgap and increasing frequency, on producing instability is illustrated with numerical values, as follows. Consider the 127 mm wide plate in Fig. 5.38(a). At 19 mm airgap, at 50 Hz, it has a stiffness of +0.15 N/mm. While maintaining all other parameters constant, a reduction of airgap from 19 mm to 8 mm changes the stiffness from +0.15 to -0.20. The same change in stiffness can also be achieved by maintaining the airgap and frequency constant, and increasing the plate width from 127 mm to 155 mm. Yet another way of achieving the same change in stiffness is to increase the frequency from 50 Hz to 130 Hz, while maintaining the airgap and plate width constant. Since all those changes are associated with increasing Goodness, we once again see the evidence for the conclusion that high Goodness leads to instability.

5.6 CONCLUSIONS

The open circuit magnetic circuit of the Xi-core has been shown to be inferior to the machine without the outermost iron limbs. The magnetising vertical flux density distribution above the stator, on open circuit, has been shown to be such that there is a region of two-pole magnetisation and a region of four-pole magnetisation. When a rotor plate is present symmetrically, they are magnetised by either the two-pole or the four-pole distribution, depending on the plate width and the airgap values. All symmetrical plates, however, have only a two-loop induced current density distribution. Stable plates, before displacement, have been shown to have a two-pole magnetisation and a two-loop induced current density distribution. Lateral displacement causes a gradual transformation of the two-pole magnetisation into a three-pole magnetisation and the induced current

density from two loops into a single loop. The plate is stable before this transition occurs and unstable after it has occurred. For plates that are magnetised by the four-pole distribution, the change from four-pole to three-pole occurs immediately after a displacement from the symmetrical position. Therefore these plates are totally unstable. While these plates are definitely unstable, all plates magnetised by a two-pole distribution are not necessarily stable. This is due to the dependency of stability for these plates on frequency and airgap. However, these plates can be stabilised by a reduction of frequency and increase of airgap.

The production of destabilising forces has been shown to be due to the tendency of the magnetising flux density and induced current density to become in-phase after a small displacement. This tendency is enhanced by an increase of frequency, reduction of airgap, and increase of plate width. Since all these changes are associated with increasing Goodness Factor, we can conclude that high Goodness Factor favours the production of destabilising forces. Lateral stability has also been interpreted in terms of the magnetic stored energy. Stable plates have been shown to be those that make the stored energy maximum in the symmetrical position.

On rolling stability, it has been shown that narrow plates, exhibiting strong lateral stability, are inherently unstable in roll. This was seen to be a consequence of the normal force distribution, making the plate identical to the mechanical example of a plate resting on a knife edge at its centre of gravity.

Considering the finite difference method itself, we see that it yields satisfactory results, as judged by the agreement between

experimental and computed results. The major disadvantage, however, is the long and expensive computing time involved due to the iterative method of solution. In order to improve the description of the space variation of the vector-potential and thus obtain more accurate results, one has to reduce the node spacing. Since the machine dimensions remain the same, this results in an increase in the number of nodes in the model. For instance halving the node spacing will increase the total number of nodes four times, which will increase the solution time dramatically. One other drawback encountered was due to the small picture frame employed. It limited the lateral displacement for the plates in the interior region. This prevented the author from obtaining complete lateral force/displacement curves.

Therefore in the next chapter we look for an alternative cheaper method, which overcomes these drawbacks.

THEORY OF FINITE ELEMENT ANALYSIS6.1 INTRODUCTION

This chapter describes the theory of the finite element numerical method, which is rapidly gaining popularity in many research areas. The potential function chosen is again the magnetic vector potential. The derivation of the various field equations in terms of the vector potential has already been shown in Chapter 4 and is not repeated. The theory of the finite element method is explained with reference to the diffusion equation, from which Laplace's and Poisson's equations can be derived as particular cases.

To allow for the open boundary of the Xi-core, the "ballooning" technique referred to in Chapter 1 is used. It is shown that a digital computer can be programmed to generate the infinite boundary from a knowledge of the element layout in the region containing the device under study.

To solve the large number of simultaneous equations that result in a finite element analysis, a numerical algorithm was written that exploits the sparse structure of the equations. From the numerical solutions obtained, the methods of calculating the quantities of interest, such as forces, flux densities, etc. are described.

6.2 THE EULER-LAGRANGE EQUATION

A mathematical result, from the calculus of variations, which forms the basis of the finite element analysis, is introduced first. Only the result is quoted, the full detailed proof being

found in standard texts (e.g. Hildebrand^{6.1}).

Consider the equation:

$$L(A) = \iint_R F(A, \frac{\partial A}{\partial x}, \frac{\partial A}{\partial y}, x, y) dx dy \quad (6.1)$$

where: A is a function of x and y ,

R is the region in the x - y plane over which A is defined,

F is a function of x , y , A and its derivatives $\frac{\partial A}{\partial x}$, $\frac{\partial A}{\partial y}$.

Since F is a function of A , which itself is another function of x and y , F is called a functional. Therefore $L(A)$ is also another functional. For each value of the function $A(x,y)$; $A_1(x,y)$, $A_2(x,y)$, etc., the functional $L(A)$ takes the values; $L(A_1)$, $L(A_2)$, etc.

If it is required to find the function $A_0(x,y)$, which makes the functional $L(A)$ assume a minimum value $L(A_0)$, then the required function $A_0(x,y)$ is given by:

$$\frac{\partial}{\partial x} \left\{ \frac{\partial F}{\partial (\frac{\partial A}{\partial x})} \right\} + \frac{\partial}{\partial y} \left\{ \frac{\partial F}{\partial (\frac{\partial A}{\partial y})} \right\} = \frac{\partial F}{\partial A} \quad (6.2)$$

That is, $A_0(x,y)$ is the function satisfying the partial differential equation (P.D.E.) (6.2). Equation (6.2) is called the Euler-Lagrange equation of the original functional F . Thus the functional minimising problem is reduced to one of solving a P.D.E.

In the finite element analysis, one seeks to accomplish the reverse process of going from equation (6.2) to (6.1). That is, a functional is sought whose Euler-Lagrange equation is the

same as the P.D.E. describing the electromagnetic field. Having found such a functional, one proceeds to find by direct methods the variable $(A_0(x,y))$ which makes the functional assume a minimum value. This value $A_0(x,y)$ of the variable $A(x,y)$ is then the solution of the original P.D.E. describing the field problem. While one can readily obtain the P.D.E. from a given functional, one cannot readily obtain the functional corresponding to a given P.D.E. This is a consequence of the fact that while one can always differentiate any function, the same is not true of integration. Therefore the methods of obtaining the required functional from a given P.D.E. are examined next.

6.2.1 The Functional of a given P.D.E.

The procedure commonly adopted is to first guess a functional, then apply the Euler-Lagrange equation to it. If the resulting P.D.E. is the same as the original equation, then the assumed functional is the required one. This method is made easier by the fact that the functional $L(A)$ of equation (6.1) often has a physical meaning.

6.2.1.1 Magnetostatic problems (the energy functional)

In this type of problem, the functional $L(A)$ is the difference between the stored energy in the field and the energy input from the excitation sources. Since no energy is dissipated, by the conservation of energy principle, this difference should be zero. In a numerical approach to the problem, one seeks a field solution, which will make this difference as small as possible, i.e. a solution is sought which makes $L(A)$ a minimum.

Consider then the O.CXi-core. The input energy to the field has been shown to be:

$$E_{in} = \iiint_{\text{all-space}} A \cdot J \cdot dx dy$$

The integration over all space does not affect the value of E_{in} , because J is zero outside the windings. Now consider the energy stored in the field:

$$\begin{aligned} E_{stored} &= \iiint_{\text{all-space}} \frac{1}{2} B H \, dx dy \\ &= \iiint_{\text{all-space}} \frac{1}{2\mu} B^2 \, dx dy \\ &= \iiint_{\text{all-space}} \frac{1}{2\mu} \{ B_x^2 + B_y^2 \} \, dx dy \end{aligned}$$

It has already been shown that:

$$B_x = - \frac{\partial A}{\partial y}$$

and
$$B_y = \frac{\partial A}{\partial x}$$

$$\therefore E_{stored} = \iint \frac{1}{2\mu} \left\{ \left(\frac{\partial A}{\partial y} \right)^2 + \left(\frac{\partial A}{\partial x} \right)^2 \right\} dx dy$$

Let
$$L(A) = E_{stored} - E_{in}$$

Then:

$$L(A) = \iint_{\text{all-space}} \left\{ \frac{1}{2\mu} \left(\frac{\partial A}{\partial x} \right)^2 + \frac{1}{2\mu} \left(\frac{\partial A}{\partial y} \right)^2 \right\} dx dy - \iint_{\text{all-space}} A \cdot J \cdot dx dy$$

(6.3)

The functional F is therefore:

$$F = \frac{1}{2\mu} \left(\frac{\partial A}{\partial x} \right)^2 + \frac{1}{2\mu} \left(\frac{\partial A}{\partial y} \right)^2 - AJ$$

On applying the Euler-Lagrange equation to this functional, one obtains:

$$\frac{\partial}{\partial x} \left(\frac{1}{\mu} \cdot \frac{\partial A}{\partial x} \right) + \frac{\partial}{\partial y} \left(\frac{1}{\mu} \cdot \frac{\partial A}{\partial y} \right) = -J$$

Thus it is proved that minimising the functional $L(A)$ given by equation (6.3) is equivalent to solving Poisson's equation. This method of deriving the required functional also reveals that the F.E. method seeks to minimise the difference in stored energy and input energy, which should ideally be zero.

6.2.1.2 Eddy current problems

The method of obtaining the functional by the application of the principle of conservation of energy poses difficulties in eddy current problems in which energy is dissipated. The major difficulty is that of minimising a complex functional, since steady state eddy current problems are described by complex P.D.E. Carpenter^{6.2} while deriving the finite element network models of the diffusion equation discusses the methods currently in use to circumvent this difficulty. Ramsden^{6.3} presents a method in which the complex P.D.E. is separated into two, by grouping the real and imaginary parts. The functional minimising principle is applied to each of these separately, assuming that the other is constant and the results are recombined finally. Silvester and Haslam^{6.4} altogether depart from the direct functional minimising

approach, and adopt a Bubnov-Galerkin projective approach formulating a higher order finite element method of solving the two-dimensional linear diffusion equation which occurs in magnetotelluric geophysical prospecting. Disregarding the lack of rigorous mathematical validity, Chari^{6.5} has applied the functional minimising procedure directly to the complex diffusion equation. Although lacking mathematical basis, the finite element equations obtained in this manner by Chari are identical to those derived by Carpenter using network models, and Silvester and Haslam by Bubnov-Galerkin method. In this research, the complex functional used by Chari, together with a modification to include the Grad ∇ term in the diffusion equation, is taken as the starting point.

Consider then a complex functional F given by:

$$F = \frac{1}{2\mu} \left\{ \left(\frac{\partial A}{\partial x} \right)^2 + \left(\frac{\partial A}{\partial y} \right)^2 \right\} + \frac{j\omega\sigma}{2} A^2 - j\omega\sigma A A_s - JA \quad (6.4)$$

Applying the Euler-Lagrange equation to F one obtains:

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} = -j\omega\mu\sigma (A - A_s) + \mu J$$

Therefore we can conclude that the assumed functional corresponds to the diffusion equation.

6.3 APPROXIMATION NEEDED FOR FUNCTIONAL MINIMISATION

In this section, the problems arising in the direct minimisation of a functional are highlighted and the methods used to overcome them described. The energy functional to be minimised is:

$$L = \iint \frac{1}{2\mu} \left\{ \left(\frac{\partial A}{\partial x} \right)^2 + \left(\frac{\partial A}{\partial y} \right)^2 \right\} dx dy + \frac{j\omega\sigma}{2} \iint A^2 dx dy - j\omega\sigma A_s \iint A dx dy - \iint A J dx dy$$

The integration is required to be carried out over all space. In order to minimise the functional L , the space variation of A needs to be known. But if this is known, then the solution one seeks is already available, and therefore there is no need for an analysis. Thus the need arises for approximations, which are described next.

6.3.1 Space Discretisation

The entire region in the X-Y plane, over which the unknown A is defined, is divided into a finite number of subregions. Each of these subregions is called a finite element. These finite elements can have any shape whatever, but have the following restrictive properties:

- (a) The element sides must coincide with material interfaces.
- (b) Within any element, the material properties must not change.

In this research, triangular elements are adopted. Therefore the first approximation required is that of dividing the space continuum into a finite number of triangular elements. When this is accomplished, the integration of the functional L over all space can be achieved by integrating the functional L over a finite number of triangles.

6.3.2 Local Space Variation

The second approximation required relates to the manner of the local space variation of the unknown vector potential, within any one element. The simplest variation that one can postulate is a linear variation:

$$\text{i.e.} \quad A = k_1 x + k_2 y + k_3 \quad (6.5)$$

Under these circumstances, the method is called first order finite element analysis. Higher order space variation can always be postulated (Silvester^{6.6}), at the expense of increasing complexity of programming. First order finite element analysis is undertaken in this thesis.

6.3.2.1 Local space variation in Cartesian co-ordinates

Consider any element whose vertices are ℓ , m and n , and whose vertex potentials are A_ℓ , A_m , A_n . If the locations of the vertices are referred to a Cartesian reference frame, then:

$$A_\ell = k_1 x_\ell + k_2 y_\ell + k_3$$

$$A_m = k_1 x_m + k_2 y_m + k_3$$

$$A_n = k_1 x_n + k_2 y_n + k_3$$

Solution of these three equations gives the constants k_1 , k_2 , k_3 for that particular element. These are given in matrix form below.

$$\begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} b_\ell & b_m & b_n \\ c_\ell & c_m & c_n \\ a_\ell & a_m & a_n \end{bmatrix} \begin{bmatrix} A_\ell \\ A_m \\ A_n \end{bmatrix} \quad (6.6)$$

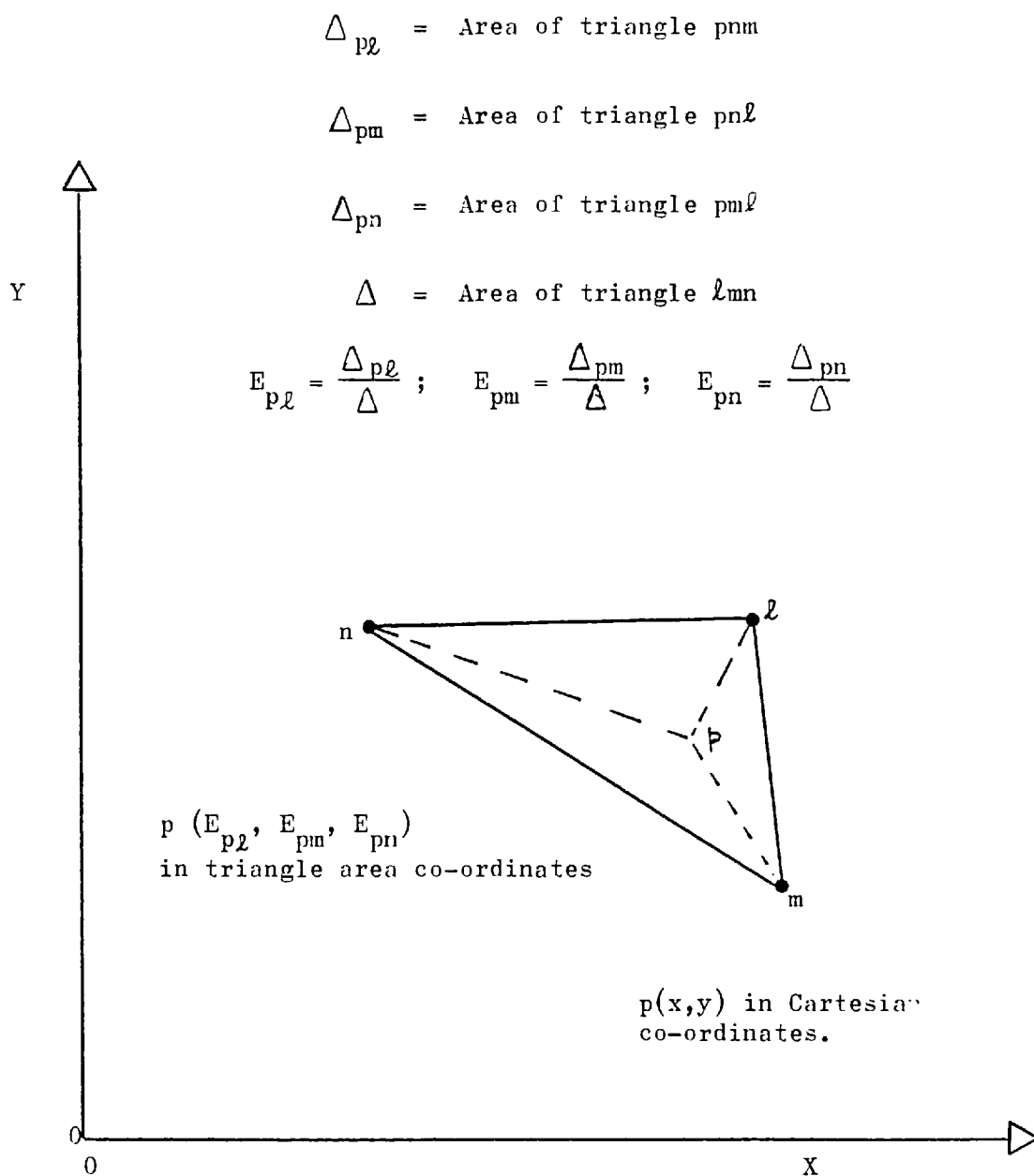


Figure 6.1: The triangle area co-ordinate system.

where: Δ is the area of the element,

$$\begin{aligned} b_\ell &= (y_m - y_n); \quad b_m = (y_n - y_\ell); \quad b_n = (y_\ell - y_m), \\ c_\ell &= (x_n - x_m); \quad c_m = (x_\ell - x_n); \quad c_n = (x_m - x_\ell), \\ a_\ell &= (x_m y_n - x_n y_m); \quad a_m = (x_n y_\ell - x_\ell y_n); \quad a_n = (x_\ell y_m - x_m y_\ell). \end{aligned}$$

Therefore the space variation within the element is given by:

$$\begin{aligned} A &= \frac{1}{2\Delta} \{a_\ell + b_\ell x + c_\ell y\} A_\ell + \frac{1}{2\Delta} \{a_m + b_m x + c_m y\} A_m \\ &\quad + \frac{1}{2\Delta} \{a_n + b_n x + c_n y\} A_n \quad (6.7) \end{aligned}$$

6.3.2.2 Local space variation in triangle area co-ordinates

It will be seen in later sections that one often requires polynomials of the form $(ax + by + c)^n$, to be integrated over the surface of a triangular element. This is a cumbersome procedure in the Cartesian frame. Therefore an alternative co-ordinate system called the triangle area co-ordinates can be used. The salient features of this co-ordinate system are explained here, the full details being found in standard textbooks (e.g. Zienkiewicz^{6,7}).

Consider a point p within the element ℓ, m, n (Fig. 6.1). If the location of the point p is known with respect to the vertices ℓ, m and n , then its location with respect to the X-Y co-ordinates can be deduced. The triangle area co-ordinates of the point p ($E_{p\ell}, E_{pm}, E_{pn}$) which fulfils this requirement are given by:

$$E_{p\ell} = \frac{\Delta_{p\ell}}{\Delta} \quad (0 \leq E_{p\ell} \leq 1)$$

$$E_{pm} = \frac{\Delta_{pm}}{\Delta} \quad (0 \leq E_{pm} \leq 1)$$

$$E_{pn} = \frac{\Delta_{pn}}{\Delta} \quad (0 \leq E_{pn} \leq 1)$$

$$\therefore E_{pl} + E_{pm} + E_{pn} = 1 \text{ always.}$$

When the areas Δ_{pl} , Δ_{pm} , and Δ_{pn} , Δ are computed in terms of the Cartesian co-ordinates of l , m , n and p , the transformation equation from Cartesian to triangle area co-ordinates is obtained as:

$$\begin{bmatrix} E_{pl} \\ E_{pm} \\ E_{pn} \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} a_l & b_l & c_l \\ a_m & b_m & c_m \\ a_n & b_n & c_n \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix} \quad (6.8)$$

The coefficients a_l , a_m , etc. are the same as those in equation (6.6). Comparing equations (6.8) and (6.7), we obtain the local space variation in triangle area co-ordinates as follows:

$$A_p = E_{pl} \cdot A_l + E_{pm} \cdot A_m + E_{pn} A_n \quad (6.9)$$

The principal advantages of triangle area co-ordinates lies in the in the following very useful result:

$$\iint_{\text{over element area}} E_{pl}^p \cdot E_{pm}^q \cdot E_{pn}^r \, dx dy = \frac{p!q!r!}{(p+q+r+2)!} 2\Delta \quad (6.10)$$

6.4 DIRECT MINIMISATION OF FUNCTIONALS

Consider a model for which space discretisation has been done, resulting in N first order finite elements in the model. The functional L to be minimised is written in terms of the contributions from each element as:

$$L = \sum_{i=1}^N L_i$$

When each of the contributions is minimum, the total is also a minimum:

$$\text{i.e.} \quad L_{\min} = \sum_{i=1}^N (L_i)_{\min}$$

Therefore the functional L is minimised, by minimising each of the contributions individually. Thus we examine the conditions necessary for the functional contribution from a typical element to be minimum.

6.4.1 Functional Minimisation for One Element

The contribution from a typical element (r^{th} , say) to the total functional is L_r , given by:

$$\begin{aligned} L_r = \iint_{\Delta_r} \frac{1}{2\mu} \left\{ \left(\frac{\partial A}{\partial x} \right)^2 + \left(\frac{\partial A}{\partial y} \right)^2 \right\} dx dy + j \frac{\omega \sigma}{2} \iint_{\Delta_r} A^2 dx dy \\ - j\omega \sigma A_s \iint_{\Delta_r} A dx dy - \iint_{\Delta_r} A J dx dy \end{aligned} \quad (6.11)$$

The conditions that the vertex potentials A_ℓ , A_m , A_n must satisfy for L_r to be minimum are:

$$\frac{\partial L_r}{\partial A_\ell} = 0 \quad (6.12a)$$

$$\frac{\partial L_r}{\partial A_m} = 0 \quad (6.12b)$$

$$\frac{\partial L_r}{\partial A_n} = 0 \quad (6.12c)$$

When the conditions (6.12a), (6.12b) and (6.12c) are imposed on equation (6.11), the following result is obtained (full derivation is given in Appendix C):

$$g = -4G$$

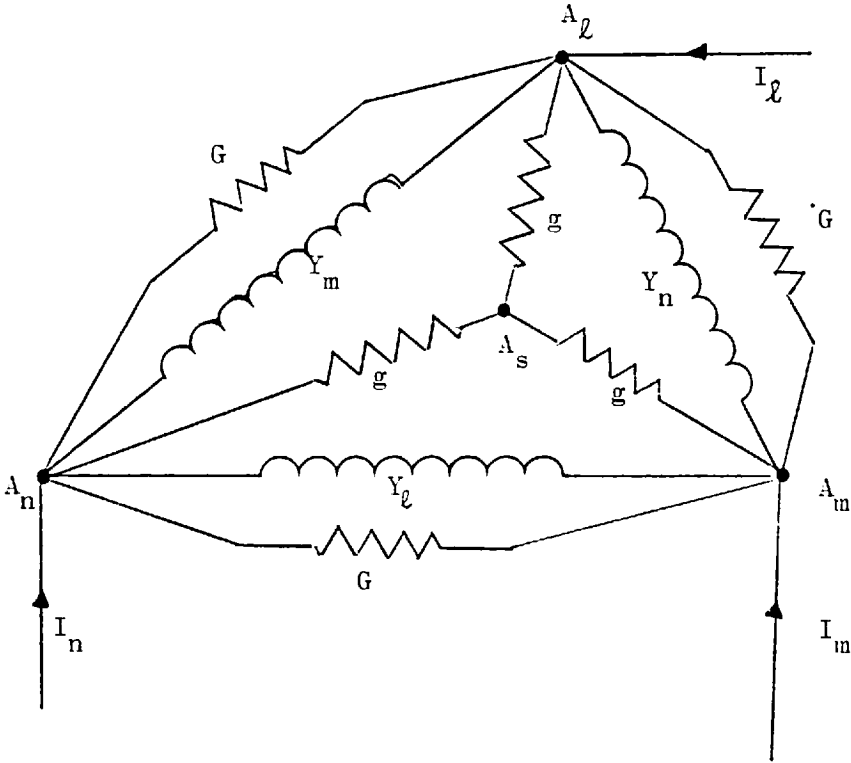


Figure 6.2: The admittance network model due to Carpenter.

$$\begin{aligned}
& \frac{1}{4\mu\Delta} \begin{bmatrix} (b_\ell b_\ell + c_\ell c_\ell) & (b_\ell b_m + c_\ell c_m) & (b_\ell b_n + c_\ell c_n) \\ (b_m b_\ell + c_m c_\ell) & (b_m b_m + c_m c_m) & (b_m b_n + c_m c_n) \\ (b_n b_\ell + c_n c_\ell) & (b_n b_m + c_n c_m) & (b_n b_n + c_n c_n) \end{bmatrix} \begin{bmatrix} A_\ell \\ A_m \\ A_n \end{bmatrix} + \\
& + \frac{j\omega\sigma\Delta}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} A_\ell \\ A_m \\ A_n \end{bmatrix} - j\frac{\omega\sigma}{3}\Delta \begin{bmatrix} A_s \\ A_s \\ A_s \end{bmatrix} = \frac{1}{3}\Delta \begin{bmatrix} J_\ell \\ J_m \\ J_n \end{bmatrix} \quad (6.13)
\end{aligned}$$

Thus equation (6.13) expresses the constraints on the vertex potentials of an element, so that its contribution to the total functional shall be minimum.

6.4.1.1 Network analogue of an element

The finite element equations for the diffusion equation have also been derived by Carpenter, as stated earlier, using a network approach. The admittance network derived by him is shown in Fig. 6.2. By applying the continuity equations to each of the vertex nodes in the network, the following equation is obtained:

$$\begin{aligned}
& \begin{bmatrix} (Y_m + Y_n) & -Y_n & -Y_m \\ -Y_n & (Y_n + Y_\ell) & -Y_\ell \\ -Y_m & -Y_\ell & (Y_m + Y_\ell) \end{bmatrix} \begin{bmatrix} A_\ell \\ A_m \\ A_n \end{bmatrix} - G \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} A_\ell \\ A_m \\ A_n \end{bmatrix} \\
& - g \begin{bmatrix} A_s \\ A_s \\ A_s \end{bmatrix} = \begin{bmatrix} I_\ell \\ I_m \\ I_n \end{bmatrix} \quad (6.14)
\end{aligned}$$

$$\text{where: } G = - \frac{j\omega\sigma\Delta}{12}$$

$$Y = - \frac{(b_m b_n + c_m c_n)}{4\mu\Delta}$$

$$Y_m = - \frac{(b_\ell b_n + c_\ell c_n)}{4\mu\Delta}$$

$$Y_n = - \frac{(b_\ell b_m + c_\ell c_m)}{4\mu\Delta}$$

$$\text{and } I_\ell = \frac{1}{3}\Delta J_\ell$$

$$I_m = \frac{1}{3}\Delta J_m$$

$$I_n = \frac{1}{3}\Delta J_n$$

When these values are substituted, the network becomes the analogue of equation (6.13).

6.4.2 Functional Minimisation of the Full Model

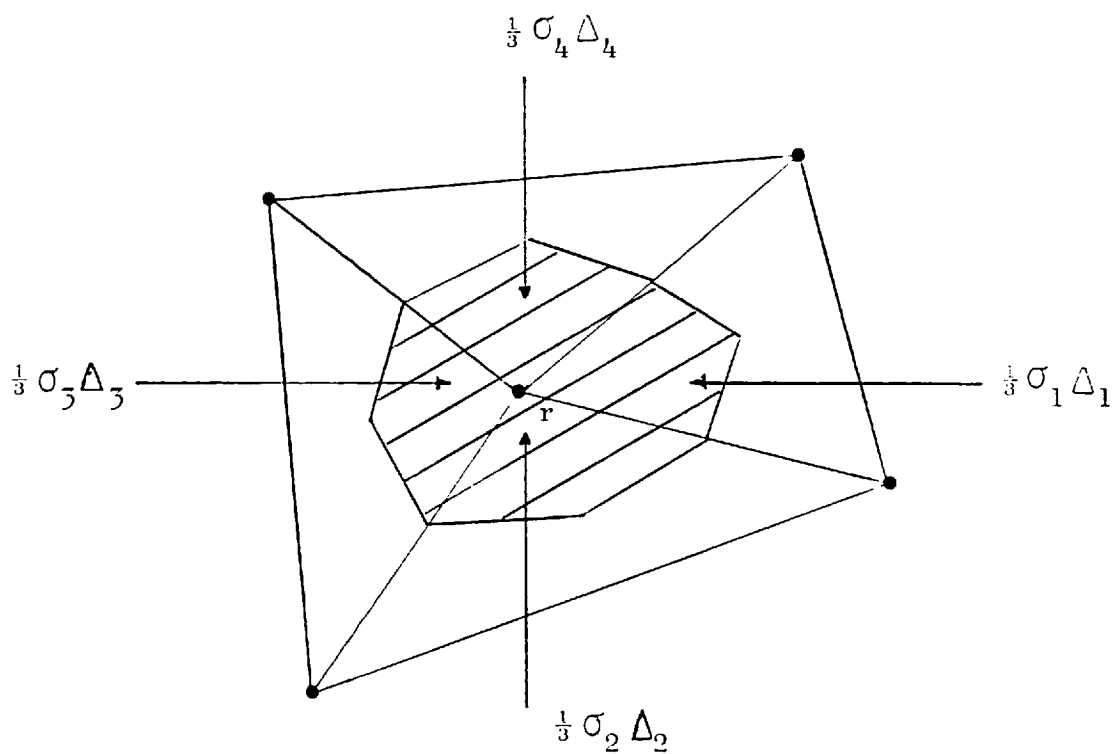
For the total functional $L = \sum_{r=1}^N L_r$, to be minimum, the three node potentials of every element should satisfy a set of three simultaneous equations similar to equation (6.13). If there are N' nodes, one obtains N' simultaneous equations which can be expressed in matrix form as follows:

$$[S'] [A] - [D] \cdot j\omega A_s = [J] \quad (6.15)$$

In equation (6.15), $[S']$ is a symmetrical square complex matrix of order $N' \times N'$. $[D]$ and $[J]$ are column matrices.

After the equations are assembled in this matrix form, a typical term D_r of the column matrix $[D]$ has the following value:

$$D_r = \sum \frac{1}{3} \sigma \Delta$$



$$D_r = \frac{1}{3} \sigma_1 \Delta_1 + \frac{1}{3} \sigma_2 \Delta_2 + \frac{1}{3} \sigma_3 \Delta_3 + \frac{1}{3} \sigma_4 \Delta_4$$

Figure 6.5: The meaning of the r^{th} element of the $[D]$ matrix.

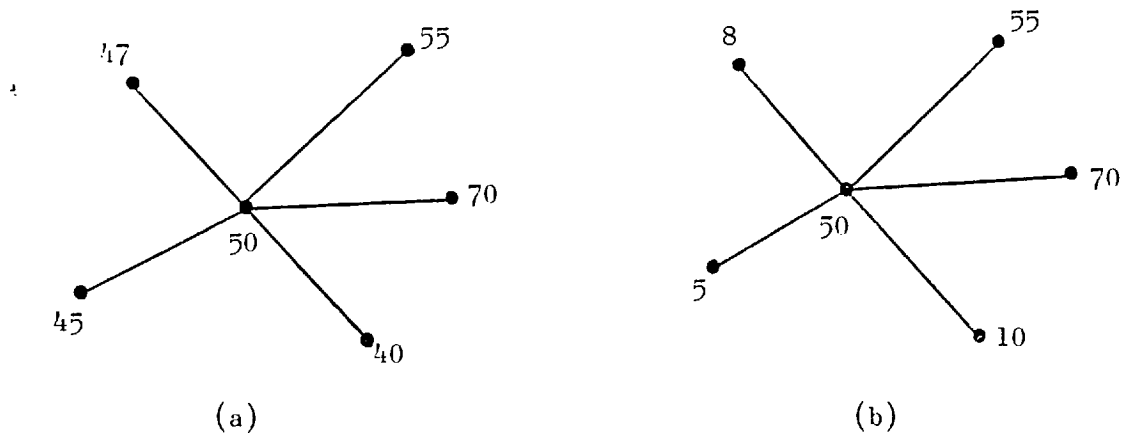


Figure 6.4: The same set of nodes numbered in two different ways.

where the $\sum$ summation extends over all the elements sharing node r as one of its vertices. Fig. 6.3 illustrates this.

6.4.2.1 Elimination of the datum node

The matrix equation (6.15) cannot be solved in its present form, due to the presence of the datum node potential A_s . In order to express this in terms of the other node potentials, we make use of the requirement that the sum of the induced currents should be zero. In Fig. 6.3, the total induced current at the node r is:

$$I_r = \sum -j\omega C(A_r - A_s) \cdot \frac{1}{3} \Delta$$

$$\text{i.e.} \quad I_r = -j\omega(A_r - A_s) \cdot D_r$$

$$\text{Since} \quad \sum_{r=1}^{N^t} I_r = 0,$$

$$\sum_{r=1}^{N^t} (A_r - A_s) \cdot D_r = 0$$

$$\text{i.e.} \quad A_s = \frac{1}{\sum_{r=1}^{N^t} D_r} \cdot \sum_{r=1}^{N^t} A_r \cdot D_r$$

$$\text{Let} \quad \square = \sum_{r=1}^{N^t} D_r$$

$$\text{then} \quad A_s = \frac{1}{\square} \cdot [D]^T \cdot [A]$$

$$\therefore -[D] \cdot j\omega A_s = -\frac{j\omega}{\square} \cdot [D][D]^T [A] = [D'] [A]$$

Therefore equation (6.15) can now be reduced to its final form:

$$[S][A] = [J] \quad (6.16)$$

where $[S] = [S^i] + [D^i]$

The matrix $[S]$ is commonly referred to as the S-matrix, and the matrix $[J]$ as the source matrix.

6.4.2.2 The sparsity structure of the S-matrix

It has already been seen that the S-matrix is square and symmetrical. This enables one to halve the memory storage required in a digital computer, by storing only the lower triangular half of the S matrix. One can reduce the storage requirement further, by exploiting the sparse nature of the S matrix. The method of such a storage scheme and the solution of the equations will be described in a later section. As a preliminary, however, the sparsity structure of the S matrix is examined.

Consider a part of the finite element model shown in Fig. 6.4a. Since only the lower triangular half of the S matrix needs to be examined, only the columns 1 to 50, of the 50th row, need be considered. Hence of all the nodes connected to the node 50, those with node numbers greater than 50 are disregarded. Therefore, according to the numbering scheme of Fig. 6.4a, of the 50 columns in the 50th row, only the following are non-zeros:

$$S(50,40), \quad S(50,45), \quad S(50,47), \quad S(50,50)$$

Thus the 50th row has the following sparsity structure. As the columns are scanned from 1 to 50, only zeros are encountered in the first 39 columns. The first non-zero element is met in the 40th column. Between the 40th and 50th column there are four non-zeros interspersed by 7 zeros.

Now consider the same set of nodes re-numbered as in Fig. 6.4b. The 50th row of the S-matrix now assumes a different sparsity structure as follows. The first non-zero element is encountered in the 5th column, and between the 5th and 50th column there are four non-zero elements and 42 zeros.

Let us now generalise the specific example of Figs. 6.4a and 6.4b. Fig. 6.5 shows the general case of a node I connected to nodes $K_1, K_2, K_3, \dots, K_n$. As in the example of Figs. 6.4a and 6.4b, node numbers greater than I are disregarded from $K_1, K_2, \dots$, etc. Let K_r be the minimum value of the remaining node numbers which are m in total. Now the sparsity structure of the I^{th} row can be deduced as follows. From column 1 to (K_r-1) one encounters zeros. Between the columns K_r to I, there are $(m+1)$ non-zeros and $(I-m-K_r)$ zeros. We thus see, in order to achieve a sparsity structure similar to that corresponding to Fig. 6.4a, that $(I-m-K_r)$ should be as small as possible. In order to achieve this, the requirement is that the node numbers K_1, K_2 , etc. should be as close to the node number I as is possible.

6.4.2.3 Problems posed by the open boundary

Consider the potentials of the nodes at the boundary which defines the solution domain. The constraints on these boundary potentials are the only boundary conditions to be satisfied in the solution of the matrix equation:

$$[S][A] = [J] \quad (6.17)$$

These constants are of two types only. In the first type, the boundary potentials have to satisfy the condition that $\frac{\partial A}{\partial n} = 0$, and

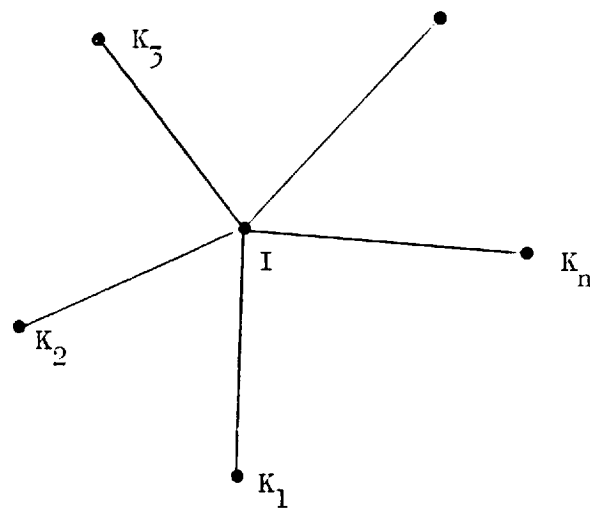


Figure 6.5: The general case of node I connected to n nodes.

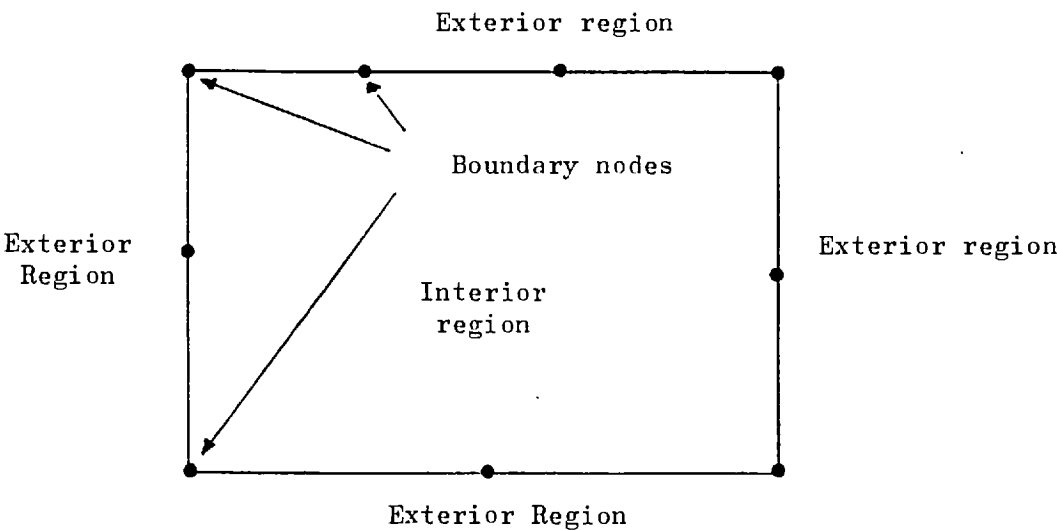


Figure 6.6: Division of space into interior and exterior regions.

in the second the boundary potentials assume certain specified values. For the first type of constraints, one simply solves the equation (6.17), and obtains all the node potentials including those at the boundary. For the second type, the following procedure is adopted. The matrix equation $[S][A] = [J]$ is re-arranged so that the specified potentials appear as the last entries in the column matrix $[A]$. Next the matrix equation is partitioned as follows:

$$\begin{bmatrix} S_{11}^i & S_{12}^i \\ S_{21}^i & S_{22}^i \end{bmatrix} \begin{bmatrix} A^i \\ A'' \end{bmatrix} = \begin{bmatrix} J^i \\ J'' \end{bmatrix}$$

$$\therefore [S_{11}^i][A^i] = [J^i] - [S_{12}^i][A''] \quad (6.18)$$

Thus the effect of the specified boundary potentials is to act as an additional source, and the unknown potentials $[A^i]$ are obtained by solving equation (6.18). A problem arises, however, when the solution domain extends to infinity, and therefore specified potentials occur at infinity. This is overcome by a special technique to be described in the next section.

6.5 THE THEORY OF THE INFINITE BOUNDARY

Silvester, Lowther, Carpenter and Wyatt^{6.8} have described the theory of a recursive technique which enables the accurate representation of the open boundaries of certain types of electromagnetic devices such as SLIMS and busbars in air. The method

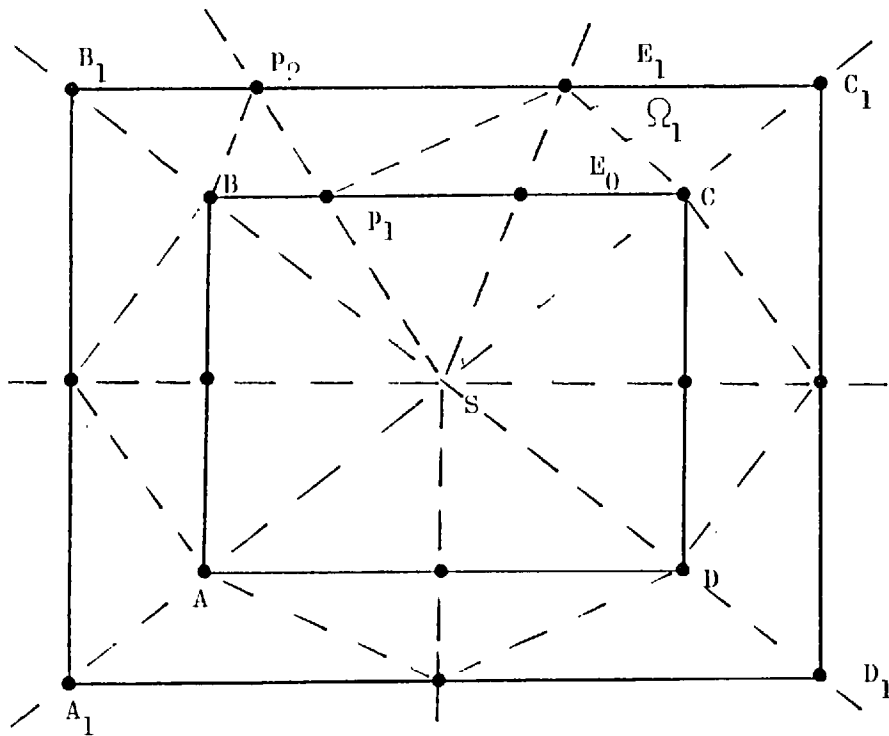


Figure 6.7: Creation of the first exterior region.

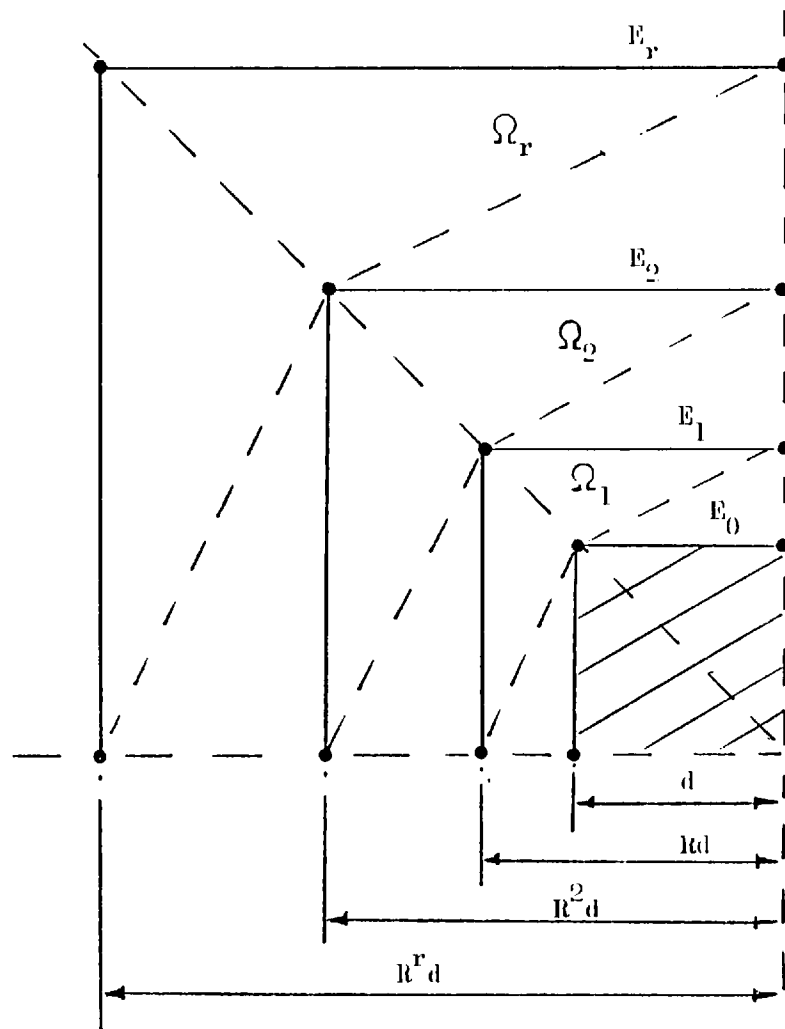


Figure 6.8: Successive exterior regions.

places no restriction on the interior boundary shape, other than that it shall be of a non-re-entrant type. The same theory is presented here, but restricting the interior region boundary to a rectangular shape. This restriction, which can be easily met in most problems, has the following advantages:

- (a) The simpler boundary shape provides a clearer understanding of the theory of the method.
- (b) It enables one to program a digital computer to do the automatic generation of the exterior elements (to be described) from a knowledge of the interior mesh layout.

6.5.1 Division into Interior and Exterior Regions

The device, which has its boundaries at infinity, is first enclosed in a rectangle. This rectangular boundary, and the space enclosed by it, are termed the inner boundary and interior region respectively. The space outside the inner boundary is termed the exterior region. It is assumed that the exterior region is Laplacian. The interior region is discretised in any convenient manner, as a result of which a certain number of boundary nodes is present on the inner boundary. Fig. 6.6 shows the steps involved so far.

6.5.2 Subdivision of Exterior Region

(a) First Exterior Region

A rectangular boundary is now drawn within the exterior region, completely enclosing the interior region symmetrically.

This boundary is termed the first outer boundary. The region bounded by the first outer boundary and the inner boundary will be called the first exterior region (Fig. 6.7).

From the point of intersection of the diagonals, straight line rays, passing through the interior nodes are drawn. New nodes are thus formed at the points of intersection of the outer boundary and the rays originating from S. The number of nodes on the outer boundary obtained is thus the same as that on the inner boundary. Consider any two nodes p_1, q_1 on the inner boundary and their corresponding nodes p_2, q_2 on the outer boundary:

$$\frac{p_2 S}{p_1 S} = \frac{B_1 S}{BS}$$

and
$$\frac{q_2 S}{q_1 S} = \frac{B_1 S}{BS}$$

$$\therefore \frac{p_2 S}{p_1 S} = \frac{q_2 S}{q_1 S} = R \quad \text{where } R \text{ is a constant } > 1 \quad (6.19)$$

Since p_1, q_1 are any two arbitrary points on the inner boundary, it is proved thus that the distance of any exterior node from the star-point bears a constant ratio to the distance of its corresponding interior node from the star-point. This ratio $R > 1$ is called the mapping ratio. From this result the following alternative method of constructing the first outer boundary can be deduced.

Choose a point within the interior region as the star-point. Choose a mapping ratio $R > 1$. Create the exterior nodes satisfying equation (6.19) and draw a contour connecting all of them. It stands to reason that, unless the star-point is chosen at the point of intersection of the diagonals, the outer boundary cannot

be a rectangle. This method of constructing the first outer boundary is made use of in the automatic generation algorithm, to be described later.

After the generation of the first outer boundary, the nodes are inter-connected to form triangular elements.

(b) Second and Successive Exterior Regions

Rectangles progressively increasing in size are now drawn in the remaining exterior region, subject to the following restriction. The ratio of the distances of the corresponding sides from the star-point, of any two consecutive rectangles shall always be R (the mapping ratio). When this restriction is met, and nodes are formed at the points of intersection of the sides of the rectangles and the rays originating from the star-point, one obtains the same number of nodes in any outer boundary. In the region bounded by any two consecutive boundaries, elements are formed in the same manner as in the first exterior region. Fig. 6.8 illustrates this. It will be observed that the r^{th} outer boundary is distant $r \cdot d$ from the star-point S .

6.5.3 The S-Matrix for any Exterior Region

Consider the exterior region Ω_1 in Fig. 6.9. For every element in Ω_1 , there is a corresponding one in $\Omega_2, \Omega_3, \Omega_4, \dots, \Omega_r$, etc. Besides these corresponding elements are all similar triangles, progressively increasing in size but not in shape. Therefore for any exterior region Ω_r , the S matrix relating the potentials on the $(r-1)^{\text{th}}$ and r^{th} outer boundaries is the same as

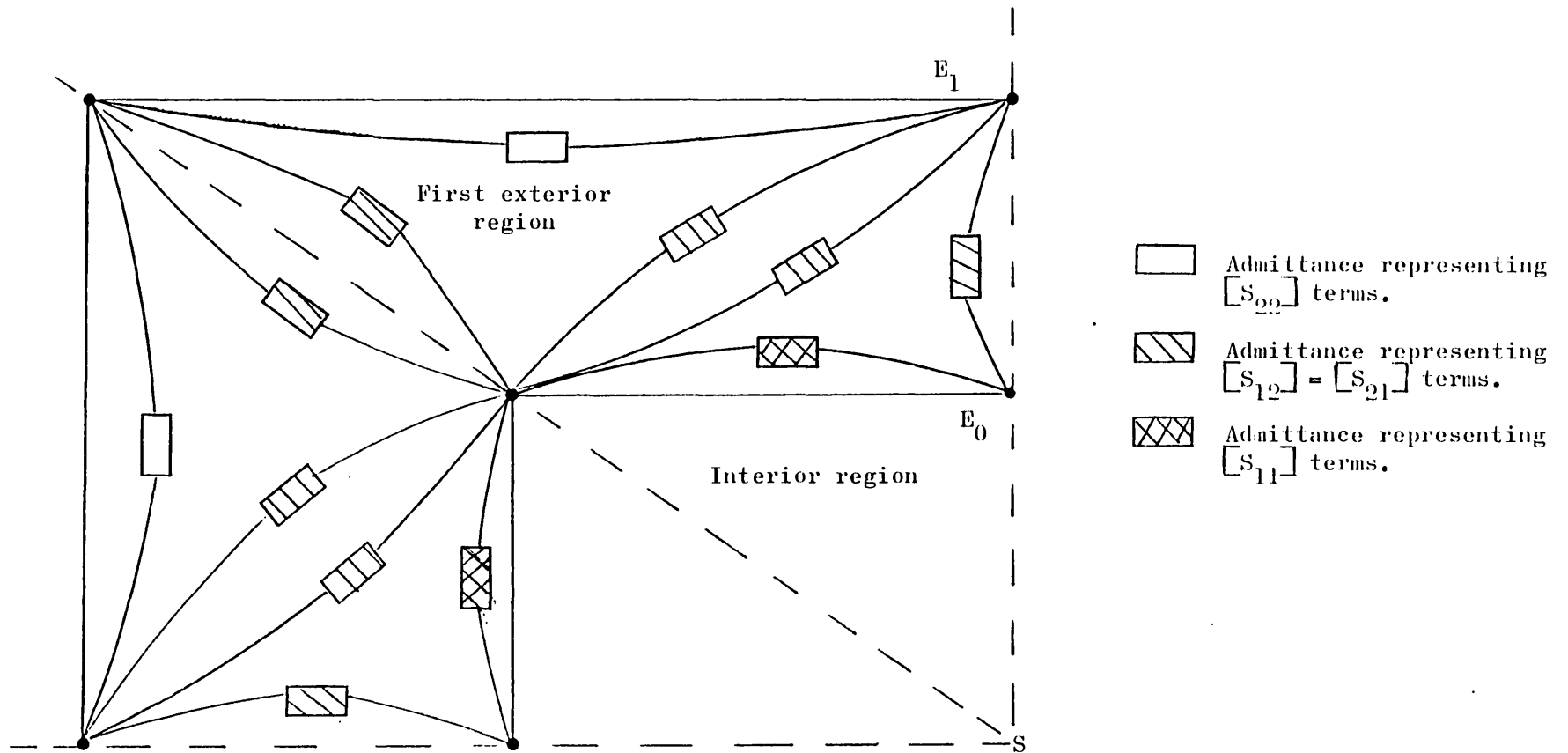


Figure 6.9: Admittance network for the first exterior region.

the S-matrix for the first exterior region Ω_1 .

Consider the S-matrix for the first exterior region Ω_1 . Let there be N nodes on this inner boundary E_0 , and let:

$[\phi_0]$ = a column matrix containing all the N potentials on the inner boundary E_0 ,

$[\phi_1]$ = a column matrix containing all the N potentials on the first outer boundary E_1 .

Since the field is Laplacian, the $2N \times 2N$ S-matrix for the first exterior region can be partitioned and written as:

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = 0$$

Each of the partitioned matrices S_{11} , S_{22} , S_{12} and S_{21} is square and of the order $N \times N$. Referring to Fig. 6.9 their physical interpretation is as follows:

- (a) The entries of the $[S_{11}]$ matrix represent the admittances connected between the nodes on the inner boundary. Any one row of the matrix has only three non-zero elements.
- (b) The entries of the $[S_{22}]$ matrix represent the admittances connected between the nodes on the outer boundary. Any one row has only three non-zero elements.
- (c) The entries of the $[S_{12}]$ and $[S_{21}]$ matrices represent the admittances connecting the nodes on the inner and outer boundaries. The number of non-zeros on any row is variable, being dependent upon element layout.

6.5.4 Successive Combination of Exterior Regions

Consider the first two exterior regions Ω_1 and Ω_2 in Fig. 6.8. The potentials on the three boundaries E_0 , E_1 and E_2 satisfy the following equations:

$$[S_{11}] [\phi_0] + [S_{12}] [\phi_1] = 0 \quad (6.20)$$

$$[S_{21}] [\phi_0] + [S_{22}] [\phi_1] = 0 \quad (6.21)$$

$$[S_{11}] [\phi_1] + [S_{12}] [\phi_2] = 0 \quad (6.22)$$

$$[S_{21}] [\phi_1] + [S_{22}] [\phi_2] = 0 \quad (6.23)$$

Adding (6.21) and (6.22):

$$[S_{21}] [\phi_0] + [S_{22} + S_{11}] [\phi_1] + [S_{12}] [\phi_2] = 0$$

and let $[B] = [S_{22} + S_{11}]^{-1}$

then:

$$[\phi_1] = -[B] [S_{21}] [\phi_0] - [B] [S_{12}] [\phi_2]$$

Substituting for $[\phi_1]$ in equations (6.20) and (6.23):

$$\begin{bmatrix} [S_{11}] - [S_{12}] [B] [S_{21}] & - [S_{21}] [B] [S_{21}] \\ - [S_{21}] [B] [S_{21}] & [S_{22}] - [S_{21}] [B] [S_{12}] \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_2 \end{bmatrix} = 0$$

$$\text{i.e.} \quad \begin{bmatrix} 1 & 1 \\ S_{11} & S_{12} \\ 1 & 1 \\ S_{21} & S_{22} \end{bmatrix} \cdot \begin{bmatrix} \phi_0 \\ \phi_2 \end{bmatrix} = 0 \quad (6.24)$$

where:

$$S^1 = \begin{bmatrix} 1 & 1 \\ S_{11} & S_{12} \\ 1 & 1 \\ S_{21} & S_{22} \end{bmatrix}$$

The matrix equation (6.24) relates the potentials on the inner boundary E_0 to those on the second outer boundary E_2 , which is distant R^2d from the star-point. This has been achieved by combining the exterior regions Ω_1 and Ω_2 , into a single region. The elements of $[S_{11}^1] = [S_{21}^1]$ matrices are now smaller, since they represent the admittances connecting the nodes on the E_0 boundary and the E_2 boundary which is R^2d distant from the star-point. The matrices $[S_{11}^1]$ and $[S_{22}^1]$ have fewer non-zero elements now, because each node on the boundaries is connected to more than two nodes on the same boundary. To carry out the exterior region combination process further, after this step, we seek out a boundary from the remaining outer boundaries so that its boundary potentials are related to the second boundary potentials $[\phi_2]$ by the same matrix as in equation (6.24), i.e. we seek a boundary E_k such that:

$$\begin{bmatrix} S_{11}^1 & S_{12}^1 \\ S_{21}^1 & S_{22}^1 \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_k \end{bmatrix} = 0$$

The fact that the S-matrix is the same for all exterior regions enables one to find the required boundary as follows. Since two regions have been combined to obtain equation (6.24), we skip the same number of regions that have been combined to obtain the required boundary. In this instance two regions are skipped and the required boundary is E_4 . In the general case, when m regions

have been combined, the potentials already related are $[\phi_0]$ and $[\phi_m]$, where the m^{th} boundary is distant $R^m d$. The next boundary required is E_{2m} , which is distant $R^{2m} d$. Therefore after the first step, $m = 2$, and the boundary required $E_{2m} = E_4$.

$$\begin{bmatrix} S_{11}^1 & S_{12}^1 \\ S_{21}^1 & S_{22}^1 \end{bmatrix} \begin{bmatrix} \phi_2 \\ \phi_4 \end{bmatrix} = 0 \quad (6.25)$$

From equations (6.24) and (6.25), the intermediate boundary potential $[\phi_2]$ can be eliminated, exactly as before, and the following new matrix equation is obtained:

$$\begin{bmatrix} S_{11}^2 & S_{12}^2 \\ S_{21}^2 & S_{22}^2 \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_4 \end{bmatrix} = 0 \quad (6.26)$$

Now four regions have been combined, and the inner boundary potentials $[\phi_0]$ are related to the potentials of a boundary which is distant $R^4 d$ from the star-point. Therefore the elements of $[S_{12}^2] = [S_{21}^2]$ have become even more diminished in magnitude, while the $[S_{11}^2]$ and $[S_{22}^2]$ matrices have become more dense.

As the procedure is repeated, the manner in which the exterior regions are combined and intermediate boundaries eliminated is shown in Table 6.1. The table shows that in only a few steps, one can relate the potentials on the inner boundary to those on a boundary which is very far distant from the inner boundary. For example, a square of 10 mm x 10 mm will be related to a square boundary of 36623 x 36623 light years in only seven steps, using

Step Number	Number of regions combined	Number of intermediate boundaries eliminated	Distance of the boundary to which the inner boundary potentials are related
1	2	1	R^2d
2	4	3	R^4d
3	8	7	R^8d
-	-	-	-
-	-	-	-
n	2^n	(2^n-1)	$R^{2^n}d$
-	-	-	-

TABLE 6.1

a mapping ratio of 1.5. Therefore after only a few steps (k, say) equation (6.24) becomes:

$$\begin{bmatrix} S_{11}^k & S_{12}^k \\ S_{21}^k & S_{22}^k \end{bmatrix} \begin{bmatrix} \phi_0 \\ \phi_2^k \end{bmatrix} = 0$$

The matrix $[S_{12}^k] = [S_{21}^k]$ will be very nearly a matrix whose elements are zeros. This is so, because they represent the admittances connecting the inner boundary and an outer boundary which is light years away. At such distances the potentials are very small.

$$\therefore \begin{bmatrix} S_{11}^k & 0 \\ 0 & S_{22}^k \end{bmatrix} \begin{bmatrix} \phi_0 \\ 0 \end{bmatrix} = 0 \tag{6.27}$$

$\therefore$ The required equation is $[S_{11}^k][\phi_0] = 0$.

This is the constraint that the inner boundary potentials

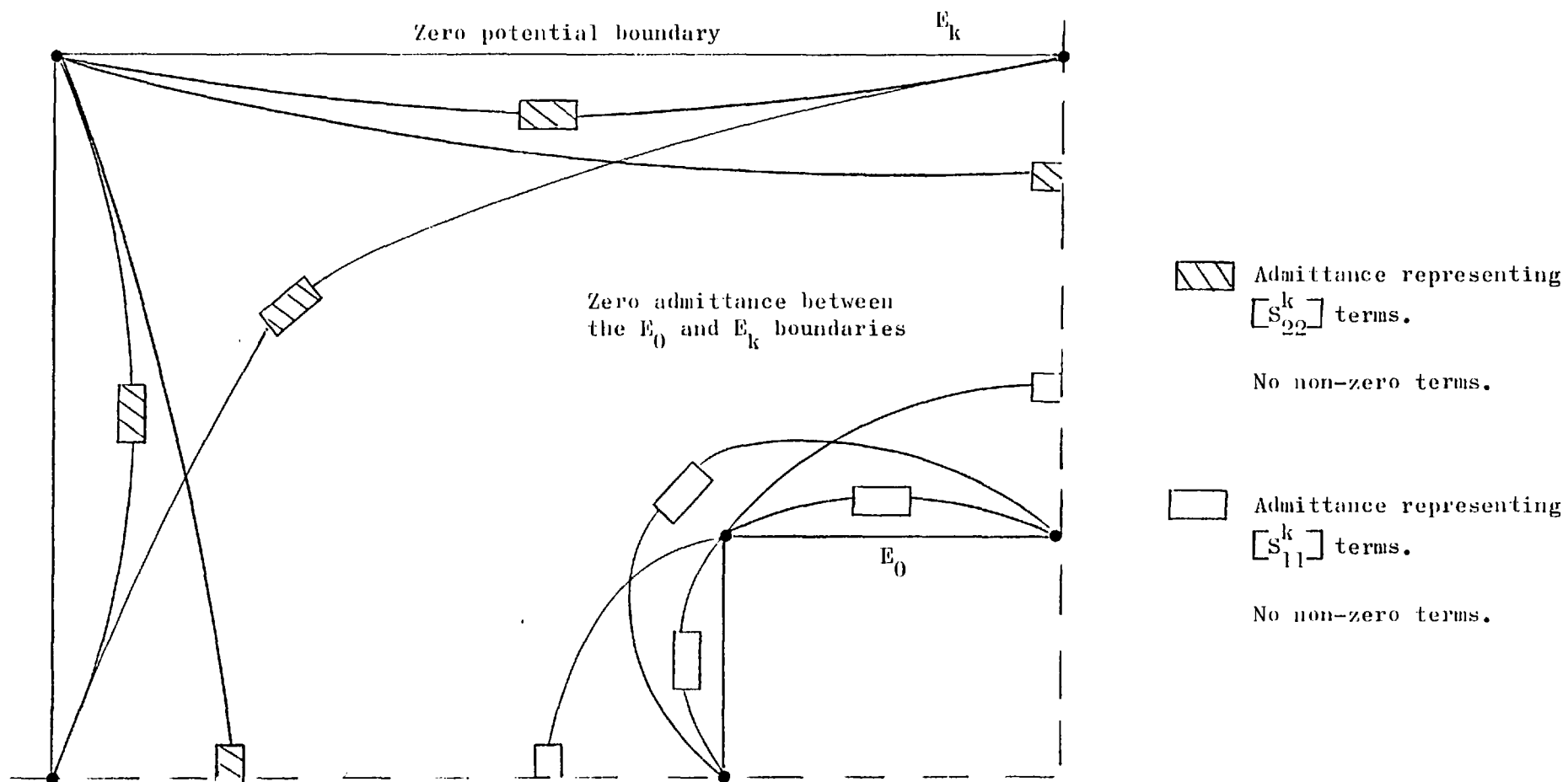


Figure 6.10: The admittance network of the full exterior region after recursion.

must satisfy to allow for the open boundary. The physical interpretation of equation (6.27) is shown in Fig. 6.10 in terms of admittances.

6.5.5 Generating the $[S_{11}]$ matrix by Recursion

The theory described so far reveals that in order to obtain the required final result $[S_{11}][\phi_2] = 0$, one has to construct only the first exterior region from a given interior region. After construction of the first exterior region, the required final matrix is obtained by a recursion method as follows.

Consider the k^{th} step in the exterior region combining process:

$$\begin{bmatrix} S_{11}^k & S_{12}^k \\ S_{21}^k & S_{22}^k \end{bmatrix} \begin{bmatrix} \phi_0' \\ \phi_{2^k}' \end{bmatrix} = 0$$

and

$$\begin{bmatrix} S_{11}^k & S_{12}^k \\ S_{21}^k & S_{22}^k \end{bmatrix} \begin{bmatrix} \phi_{2^k}' \\ \phi_{2^{k+1}}' \end{bmatrix} = 0$$

Eliminating $[\phi_{2^k}']$:

$$\begin{bmatrix} [S_{11}^k] - [S_{12}^k][B^k][S_{21}^k] & -[S_{21}^k][B^k][S_{12}^k] \\ -[S_{21}^k][B^k][S_{21}^k] & [S_{22}^k] - [S_{21}^k][B^k][S_{12}^k] \end{bmatrix} \begin{bmatrix} \phi_0' \\ \phi_{2^{k+1}}' \end{bmatrix} = 0$$

where $[B^k] = [S_{11}^k + S_{22}^k]^{-1}$

$$\text{i.e.} \quad S^{k+1} = \begin{bmatrix} S_{11}^k & 0 \\ 0 & S_{22}^k \end{bmatrix} - \begin{bmatrix} S_{12}^k \cdot B^k \cdot S_{21}^k & S_{21}^k \cdot B^k \cdot S_{12}^k \\ S_{21}^k \cdot B^k \cdot S_{21}^k & S_{21}^k \cdot B^k \cdot S_{12}^k \end{bmatrix}$$

$$\text{where} \quad B^k = [S_{11}^k + S_{22}^k]^{-1} \quad (6.28)$$

Starting with $k = 0$, repeated computation of (6.28) yields the required final matrix $[S_{11}^k]$, in only a few steps, as the terms of $[S_{12}]$ matrix rapidly diminish to zero. It is to be noted that when $k = 0$, S_{11}^0 , S_{22}^0 , S_{12}^0 , S_{21}^0 are the partitioned sections of the S-matrix for the first exterior region. No additional computer storage is required for the recursion since the S-matrix can be repeatedly overwritten.

6.5.6 Attaching the $[S_{11}]$ matrix to the Interior S-Matrix

The final equation to be solved is obtained by adding the $[S_{11}^k]$ matrix to the S-matrix of the interior region. In order to facilitate this addition, it is convenient to re-number the boundary nodes on which the infinite boundary is required, starting with 1 and in ascending order. The original node numbers of the re-numbered boundary nodes can be recovered by storing the following table in the computer memory:

Re-numbered nodes in ascending order	New Number	Old Number	Node number as appearing in the input data
	1	5	
	2	20	
	3	31	
	4	51	
	j	j''	
	m'	m''	
	N'	N''	

TABLE 6.2

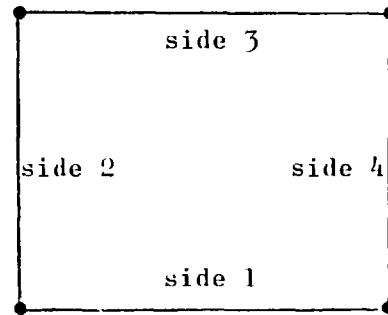


Figure 6.11: The figure used to define the problem.

d = average node spacing on sides 3 and 4

$$R = \frac{1}{8} \left\{ \frac{\frac{\sqrt{3}}{2}d + y_A}{y_A} + \frac{\frac{\sqrt{3}}{2} + x_B}{x_B} \right\}$$

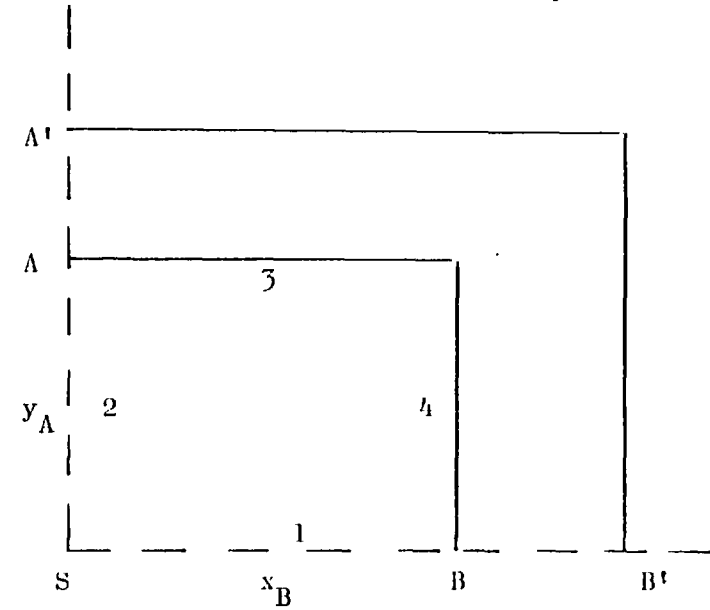


Figure 6.12: The averaging method used to determine the mapping ratio.

If there are N^i nodes on the sides on which an infinite boundary is required, the $[S_{11}^k]$ matrix is of order $N^i \times N^i$. The addition to the interior S-matrix is done as follows.

Consider a typical element $S_{11}^k (m^i, j^i)$ of the $[S_{11}^k]$ matrix. This element is added to the element $S (m'', j'')$ of the interior region S-matrix, when m'', j'' are the old numbers of the boundary nodes, whose new node numbers are m^i and j^i . Thus all the elements of the $[S_{11}^k]$ matrix can be added to the main S-matrix of the interior region.

6.6 AUTOMATIC GENERATION OF INFINITE BOUNDARY

A computer package was developed (see Lowther, Rajanathan and Silvester^{6.9}) to implement all the steps explained in the previous section for any given rectangular interior region mesh layout. The developed algorithm generates the infinite boundary for the following cases:

- 1) Problems requiring an infinite boundary on all four sides of the interior region.
- 2) Problems requiring an infinite boundary on any three sides of the interior region.
- 3) Problems requiring an infinite boundary on any two sides of the interior region.

In Fig. 6.11 the sides of the inner rectangular boundary are designated 1, 2, 3 and 4. The problem is defined by stating on how many sides the infinite boundary is not required, and their

identifying side numbers. With the problem defined thus, the following steps are implemented on the computer.

6.6.1 Location of the Star-Point

Case 1: When the infinite boundary is required on all four sides, the star-point is at the point of intersection of the diagonals.

Case 2: When the infinite boundary is required on three sides, the star-point is at the mid-point of the omitted side.

Case 3: When the infinite boundary is required on two sides only, the star-point is at the intersection of the omitted sides.

6.6.2 Determination of the Mapping Ratio

Consider the problem when the infinite boundary is required on sides 3 and 4 in Fig. 6.12. If there are N_T nodes on these sides, then the average node spacing (d) on these sides is:

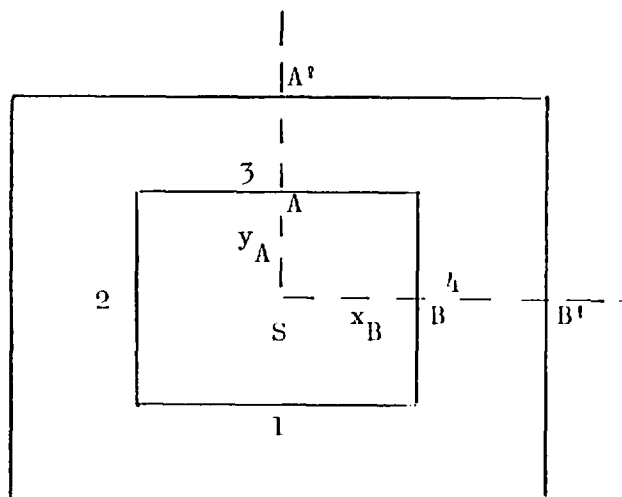
$$d = \frac{\text{Length of side 3} + \text{Length of side 4}}{N_T}$$

In order for the elements in the exterior region to be as nearly an equilateral shape as possible the width AA' of the exterior region should be $\frac{\sqrt{3}}{2} d$. This requirement determines the mapping ratio (R) as:

$$R = \frac{\frac{\sqrt{3}}{2} d + y_A}{y_A}$$

However, when another ray such as SBB' is considered, it is clear that one would obtain a different value for R , unless the rectangle

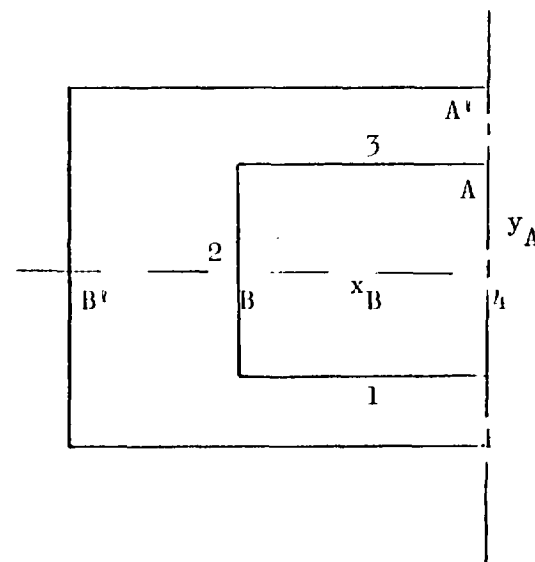
d = average node spacing on sides 1, 2, 3 and 4



$$R = \frac{1}{2} \left\{ -\frac{\frac{\sqrt{3}}{2}d + y_A}{y_A} + \frac{\frac{\sqrt{3}}{2}d + x_A}{x_A} \right\}$$

Figure 6.13: The averaging method used to determine the mapping ratio.

d = average node spacing on sides 1, 2 and 3



$$R = \frac{1}{2} \left\{ \frac{\frac{\sqrt{3}}{2}d + y_A}{y_A} + \frac{\frac{\sqrt{3}}{2}d + x_B}{x_B} \right\}$$

Figure 6.14: The averaging method used to determine the mapping ratio.

is also a square. Therefore the different mapping ratios are calculated and their average value used as the actual mapping ratio. For the other two types of problems this averaging method is also used, as illustrated in Figs. 6.13 and 6.14.

6.6.3 Re-numbering Inner Boundary Nodes

Before the generation of the exterior nodes, it is necessary to re-number the nodes on the sides on which the infinite boundary is required, in sequence starting with 1. If there are N_T nodes on these sides, the re-numbering will range from 1 to N_T . A table such as 6.2 is created and stored in the computer memory.

6.6.4 Generation of Exterior Nodes

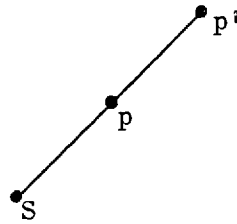


Figure 6.15: An interior node and its corresponding exterior node.

Consider any boundary node p . The exterior node to be generated is p' . Since:

$$\frac{p'S}{pS} = R,$$

$$\frac{(x_{p'} - x_S)}{(x_p - x_S)} = R \quad \text{and} \quad \frac{(y_{p'} - y_S)}{(y_p - y_S)} = R$$

$\therefore$

$$x_{p'} = Rx_p + (1 - R)x_S$$

$$y_{p'} = Ry_p + (1 - R)y_S$$

Thus from a knowledge of the interior boundary node co-ordinates, the mapping ratio and the star-point co-ordinates, the exterior nodes are generated.

6.6.5 Formation of Exterior Elements

Consider any two consecutive nodes on the inner boundary, i and $(i+1)$. Their corresponding exterior nodes are $(i+N_T)$ and $(i+1+N_T)$, where N_T is the total number of nodes on the sides on which the infinite boundary is required. These four nodes form a quadrilateral which is divided into two triangles along the shorter of the two diagonals. The exterior elements are formed in this manner, by scanning the nodes 1 to (N_T-1) . For the special case of needing the infinite boundary on all four sides, the scanning is from node 1 to N_T . The resulting node number (N_T+1) is made equal to 1.

6.6.6 Obtaining the $[S_{11}]$ Matrix after Recursion

Since the co-ordinates and the node numbers of all three vertices of every exterior element are known, the S-matrix for the exterior region can be assembled. The S-matrix thus obtained is partitioned as required for the recursion and the recursion equation (6.28) is computed for a pre-determined number of steps, often not exceeding five or six. The partitioned section $[S_{11}]$ after recursion is now available, for incorporation into the interior region S-matrix, as explained in Section 6.5.6.

6.7 SPARSE MATRIX SOLUTION

The final form of the matrix equation which requires solution is now examined. The structure of the S-matrix is such that it is sparse and symmetrical. The solution of large numbers of simultaneous equations, whose coefficient matrix is sparsely populated, is a problem encountered in many branches of engineering. Consequently there is considerable interest in sparse matrix research. In a survey of sparse matrix research, Duff^{6.10} quotes as many as 604 references. Of these, a method called "compact storage solution" due to Jennings^{6.11} has been successfully used for the solution of finite element equation by Preston^{6.12}. This method is used in this research. A suitable numerical algorithm was written, implementing the Jennings's method of compact storage solution.

6.7.1 Compact Storage of S-Matrix

In this storage scheme, it is first necessary to form a column matrix known as the bandwidth matrix. In any row (I) of the S-matrix, when the columns are scanned from 1 to I, let the column in which the first non-zero element is encountered be k. Then the bandwidth of the Ith row is defined as:

$$\text{BANDWIDTH (I)} = (I - k + 1)$$

Starting from the first row, all the elements from the first non-zero column to the leading diagonal column are stored in a column matrix SMAT. This is illustrated by the following example:

5	0	5	0	0	1	5	First Row
0	14	0	0	5	1	<u>14</u>	Second Row
5	0	23	50	0	2	5	Third Row
0	0	50	32	0	2	<u>23</u>	Fourth Row
0	5	0	0	41	4	50	Fifth Row
						<u>32</u>	
						5	
						0	
						0	
						4	

The sparse matrix S

The band-width matrix, BANDWIDTH

Compact stored matrix, SMAT

6.7.1.1 Storage in general case

The preceding example shows the essential features of the storage scheme. Now consider a general sparse matrix S . It is required to find the storage location of an element $S(M,I)$, in the column matrix $SMAT$.

$$\begin{aligned} \text{Let } K &= \text{BANDWIDTH}(M) \\ \text{INDEX} &= (M - I + 1) \\ \text{NDIAG} &= \sum_{j=1}^M \text{BANDWIDTH}(j) \end{aligned}$$

If $\text{INDEX} > k$, then the element occurs before the first non-zero column is encountered in the M^{th} row. Therefore the element $S(M,I)$ is not stored. If $\text{INDEX} \leq k$, then the required storage location J is given by, $J = (\text{NDIAG} - \text{INDEX} + 1)$

$$\therefore \text{SMAT}(J) = S(M,I)$$

6.7.1.2 Retrieval in the general case

Consider the general S-matrix which has been stored as a column matrix SMAT. Any complete row of the original S-matrix can be retrieved as follows. It is required to retrieve the elements $S(M, i)$, $i = 1, 2, \dots, M$.

$$\text{Let } \text{NDIAG} = \sum_{j=1}^M \text{BANDWIDTH}(j)$$

$$K = \text{BANDWIDTH}(M)$$

$$\text{INDEX} = M - k + 1$$

$$S(M, i) = 0 \quad i = 1, 2, \dots, (\text{INDEX}-1).$$

$$S(M, i) = \text{SMAT}(\ell), \quad i = (\text{INDEX}), (\text{INDEX}+1), \dots, M$$

as

$$\ell = (\text{NDIAG}-k+1), \text{NDIAG}$$

6.7.2 Solution of Compact Stored Equations

After storing the S-matrix by compact storage method, one requires a method of obtaining the solution of the original matrix equations, without re-creating the original full S-matrix. This is achieved by using a modified Gaussian elimination due to Jennings. In this method, the matrix equation is scanned row by row. When the m^{th} row is reached, all the columns from 1 to m are eliminated, by making use of the previous $(m-1)$ equations.

6.7.2.1 An example of modified Gaussian elimination

The method is described first, with reference to a simple example, before generalising it. The theory is described in the

thesis by Preston with reference to a simple example, without generalising. Therefore this subsection will be found to be similar to his description of the method. Consider a set of simultaneous equations:

$$S_{11} \cdot A_1 + S_{12} \cdot A_2 + S_{13} \cdot A_3 + S_{14} \cdot A_4 = J_1$$

$$S_{21} \cdot A_1 + S_{22} \cdot A_2 + S_{23} \cdot A_3 + S_{24} \cdot A_4 = J_2$$

$$S_{31} \cdot A_1 + S_{32} \cdot A_2 + S_{33} \cdot A_3 + S_{34} \cdot A_4 = J_3$$

$$S_{41} \cdot A_1 + S_{42} \cdot A_2 + S_{43} \cdot A_3 + S_{44} \cdot A_4 = J_4$$

The coefficient matrix and the source matrix are stored in column matrices as follows:

S_{11}	S_{21}	S_{22}	S_{31}	S_{32}	S_{33}	S_{41}	S_{42}	S_{43}	S_{44}
----------	----------	----------	----------	----------	----------	----------	----------	----------	----------

J_1	J_2	J_3	J_4
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The final form in which the four original equations are required is as follows:

$$A_1 + S_{12}^{**} A_2 + S_{13}^{**} A_3 + S_{14}^{**} A_4 = J_1^{**}$$

$$A_2 + S_{23}^{**} A_3 + S_{24}^{**} A_4 = J_2^{**}$$

$$A_3 + S_{34}^{**} A_4 = J_3^{**}$$

$$A_4 = J_4^{**}$$

The coefficients of these equations are stored as:

1	S_{12}^{**}	1	S_{13}^{**}	S_{23}^{**}	1	S_{14}^{**}	S_{24}^{**}	S_{34}^{**}	1
---	---------------	---	---------------	---------------	---	---------------	---------------	---------------	---

J_1^{**}	J_2^{**}	J_3^{**}	J_4^{**}
------------	------------	------------	------------

The coefficients with double asterisks in the final form are called the reduced coefficients and reduced sources.

Consider now the state of the original four equations, after the first two equations have been reduced:

$$A_1 + S_{12}^{**}A_2 + S_{13}^{**}A_3 + S_{14}^{**}A_4 = J_1^{**}$$

$$A_2 + S_{23}^{**}A_3 + S_{24}^{**}A_4 = J_2^{**}$$

$$S_{31}A_1 + S_{32}A_2 + S_{33}A_3 + S_{34}A_4 = J_3$$

$$S_{41}A_1 + S_{42}A_2 + S_{43}A_3 + S_{44}A_4 = J_4$$

At this stage, three storage positions have been vacated from the main store by the eliminated elements S_{11} , S_{21} and S_{22} . Therefore of the five reduced L.H.S. coefficients, only three can be stored in the main column matrix. The actual number stored is still less, because the quantities $\frac{1}{S_{11}}$ and $\frac{1}{(S_{22} - S_{12}^{**} \cdot S_{21})}$ are required later, and therefore need to be stored. Hence only one reduced coefficient S_{12}^{**} is stored in the location vacated by S_{21} , and $\frac{1}{S_{11}}$ is stored in the location vacated by S_{11} , and $\frac{1}{S_{22}^{*}} = \frac{1}{(S_{22} - S_{12}^{**} \cdot S_{21})}$ is stored in the location vacated by S_{22} .

Therefore the state of the main and source storage is as follows:

$\frac{1}{S_{11}}$	S_{12}^{**}	$\frac{1}{S_{22}^*}$	S_{31}	S_{32}	S_{33}	S_{41}	S_{42}	S_{43}	S_{44}
--------------------	---------------	----------------------	----------	----------	----------	----------	----------	----------	----------

J_1^{**}	J_2^{**}	J_3	J_4
------------	------------	-------	-------

Let us examine the steps in the reduction of the third equation. A temporary column matrix is created first. All temporary storage will be denoted with a single asterisk. The first coefficient S_{31} is put in temporary storage first, i.e. $S_{31}^* = S_{31}$. The storage thus released in the main store is now used to store the S_{13}^{**} reduced coefficient, i.e. $S_{13}^{**} = \frac{S_{31}^*}{S_{11}}$. Next $S_{32}^* = (S_{32} - S_{31}^* \cdot S_{12}^*)$ is put in temporary storage. Since S_{32} is not needed in any further steps, its location in the main store is used to store $S_{23}^{**} = \frac{S_{32}^*}{S_{22}^*}$. The quantity $1/S_{22}^*$ is the one stored in the main array, in the reduction of the previous row. Lastly, the quantity $1/S_{33}^* = 1/(S_{33} - S_{31}^* S_{13}^{**} - S_{32}^* S_{23}^{**})$ which will be needed in the reduction of the following equations is stored in the location of S_{33} , in the main array.

The source matrix J_3 is reduced to J_3^{**} by:

$$J_3^{**} = \frac{1}{S_{33}^*} [J_3 - S_{31}^* \cdot J_1^{**} - S_{32}^* \cdot J_2^{**}]$$

At the completion of the third row reduction, the original equations have the following form:

$$\begin{aligned} A_1 + S_{12}^{**} A_2 + S_{13}^{**} A_3 + S_{14}^{**} A_4 &= J_1^{**} \\ A_2 + S_{23}^{**} A_3 + S_{24}^{**} A_4 &= J_2^{**} \\ A_3 + S_{34}^{**} A_4 &= J_3^{**} \\ S_{41} A_1 + S_{42} A_2 + S_{43} A_3 + S_{44} A_4 &= J_4 \end{aligned}$$

Of the six reduced coefficients, however, only three can be stored in the main storage, as shown below:

$\frac{1}{S_{11}}$	S_{12}^{**}	$\frac{1}{S_{22}^{**}}$	S_{13}^{**}	S_{23}^{**}	$\frac{1}{S_{33}^{**}}$	S_{41}	S_{42}	S_{43}	S_{44}
--------------------	---------------	-------------------------	---------------	---------------	-------------------------	----------	----------	----------	----------

J_1^{**}	J_2^{**}	J_3^{**}	J_4
------------	------------	------------	-------

6.7.2.2 General case of modified Gaussian elimination

Consider now the general case of N simultaneous equations. When the m^{th} row is being reduced, the storable reduced coefficients obtained are, $S^{**}(k,m)$, with $k = 1, 2, \dots, (m-1)$. These are stored in the main array in the positions vacated by $S(m,k)$, with $k = 1, 2, \dots, (m-1)$. The steps involved in this procedure are as follows (see Fig. 6.16):

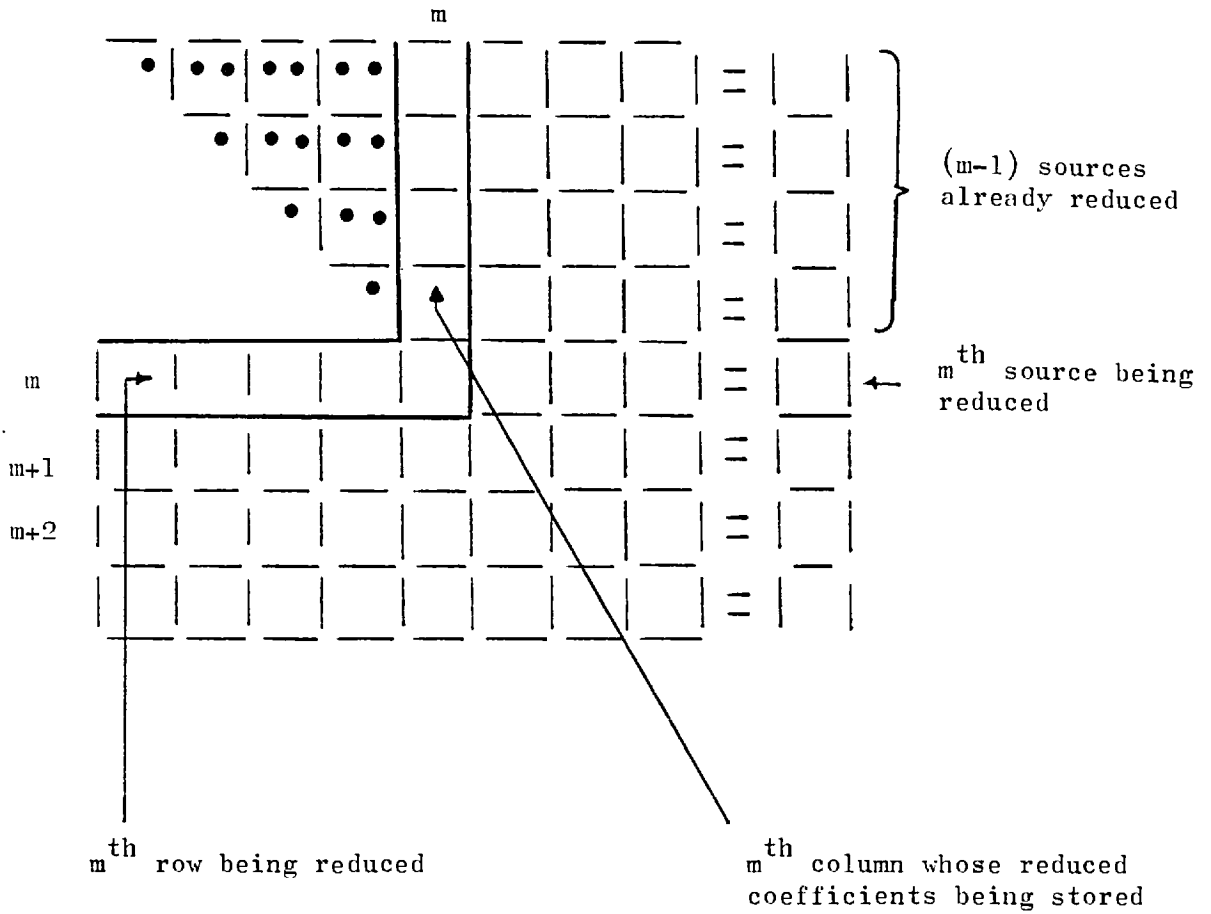
Step 1:

Create a temporary column array of $(m-1)$ storage locations. Let this array be denoted T . The array T is filled as follows:

$$\begin{aligned}
 T(1) &= S(m,1) \\
 T(k) &= S(m,k) - \sum_{j=1}^{k-1} T(j) \cdot S^{**}(j,k) \\
 &\quad \text{with } k = 2, 3, \dots, (m-1).
 \end{aligned}$$

The entries in the array T have the following meaning:

$$T(1) = S^*(m,1), \quad T(2) = S^*(m,2), \quad \text{etc.}$$



- $(m-1)$ diagonal positions used as temporary storage.
- • Reduced coefficients from column 1 to $(m-1)$ already stored.

Figure 6.16: The general case of modified Gaussian Elimination.

Step 2:

The storage space of $S(m,m)$ in the main array is replaced with $1/(S^*(m,m))$, where:

$$\frac{1}{S^*(m,m)} = \frac{1}{\left[S(m,m) - \sum_{j=1}^{(m-1)} T(j) \cdot S^{**}(j,m) \right]}$$

Step 3:

The reduced coefficients are stored in the main array as follows:

$$S^{**}(k,m) = T(k)/S^*(k,k)$$

with $k = 1, 2, \dots, (m-1)$

Step 4:

The source matrix element J_m is reduced as follows:

$$J_m^{**} = \frac{1}{S^*(m,m)} \left\{ J(m) - \sum_{k=1}^{(m-1)} T(k) \cdot J_k^{**} \right\}$$

Step 5:

The temporary array T is cleared, for use in the reduction of the $(m+1)^{th}$ row, which is ready for commencement now.

6.7.2.3 Back substitution

After the reduction of all the N equations, the unknowns are obtained starting from the N^{th} reduced equation, as follows:

$$A(n) = J^{**}(n)$$

$$A(n-1) = J^{**}(n-1) - S^{**}(n-1, n).A(n)$$

$$A(k) = J^{**}(k) - \sum_{I=(k+1)}^N S^{**}(k, I).A(I)$$

This method, which is readily implemented on a computer, is commonly called back substitution.

6.8 CALCULATION OF FLUX DENSITIES, FORCE AND INPUT IMPEDANCE

After obtaining a numerical solution by the finite element method, one needs to evaluate the flux densities, forces, etc. These are examined here.

6.8.1 Flux Densities

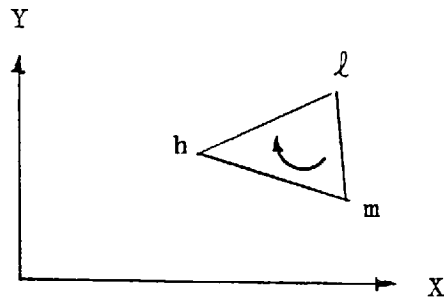


Figure 6.17: A typical element.

The local space variation of the vector potential within an element is given by (Fig. 6.17):

$$A = \frac{1}{2\Delta} (a_l + b_l x + c_l y).A_l + \frac{1}{2\Delta} (a_m + b_m x + c_m y).A_m + \frac{1}{2\Delta} (a_n + b_n x + c_n y).A_n$$

From the definition of the vector potential, the flux densities are derived as:

$$B_x = - \frac{\partial A}{\partial y}$$

$$B_y = \frac{\partial A}{\partial x}$$

$$\text{i.e.} \quad B_x = \frac{1}{2\Delta} (-c_2 A_2 - c_m A_m - c_n A_n)$$

$$B_y = \frac{1}{2\Delta} (b_2 A_2 + b_m A_m + b_n A_n)$$

$$\text{Since} \quad c_2 = (x_n - x_m), \quad b_2 = (y_m - y_n), \text{ etc.}$$

$$B_x = \frac{1}{2\Delta} [(x_m - x_n)A_2 + (x_n - x_2)A_m + (x_2 - x_m)A_n]$$

$$B_y = \frac{1}{2\Delta} [(y_m - y_n)A_2 + (y_n - y_2)A_m + (y_2 - y_m)A_n]$$

These two expressions enable the calculation of the flux density at any point within an element. They also reveal that the flux density over any element is constant, and that its algebraic sign depends on whether the element vertices are scanned clockwise or anti-clockwise. Therefore in forming the finite element model one should avoid large size elements in regions of rapidly changing field. In evaluating flux densities at several points, one must consistently scan the element vertices either in a clockwise or anti-clockwise sense.

6.8.2 Forces

The normal and horizontal forces are calculated by using the $\mathbf{J} \times \mathbf{B}$ method, as in the finite difference analysis. Consider the normal force first:

$$F_y = \int \text{Re} \mathbf{J}_e \cdot \mathbf{B}_x^* dx dy$$

where: J_e = induced eddy current,
 B_x^* = complex conjugate of the horizontal flux density.

The normal force contribution from a typical element is given by:

$$(F_y)_{\text{elem.}} = \int_{\Delta} \text{Re } J_e \cdot B_x^* \, dx dy$$

The integration is carried out over the surface of the element only. Besides, since B_x is constant within any element:

$$\begin{aligned} (F_y)_{\text{elem.}} &= \text{Re} \left[B_x^* \cdot \int_{\Delta} J_e \, dx dy \right] \\ &= \text{Re} \left[B_x^* \cdot I_e \right] \end{aligned}$$

Thus only I_e , which is the total induced current over the element area needs to be calculated, since the method of evaluating B_x is already known.

$$J_e = -j\omega\sigma(A - A_s)$$

$$\therefore I_e = j\omega\sigma\Delta \cdot A_s - j\omega\sigma \int_{\Delta} A \, dx dy$$

$$\begin{aligned} \int_{\Delta} A \, dx dy &= \int_{\Delta} \frac{A_2}{2\Delta} (a_2 + b_2 x + c_2 y) \, dx dy \\ &\quad + \int_{\Delta} \frac{A_m}{2\Delta} (a_m + b_m x + c_m y) \, dx dy \\ &\quad + \int_{\Delta} \frac{A_n}{2\Delta} (a_n + b_n x + c_n y) \, dx dy \end{aligned}$$

$$\begin{aligned} \text{i.e.} \quad \int_{\Delta} A \, dx dy &= \frac{\Delta}{3} \cdot A + \frac{\Delta}{3} \cdot A_m + \frac{\Delta}{3} \cdot A_n \\ &= \frac{\Delta}{3} [A_2 + A_m + A_n] \end{aligned}$$

$$\therefore I_e = j\omega\sigma\Delta A_s - \frac{j\omega\sigma\Delta}{3} [A_2 + A_m + A_n]$$

To proceed further, the datum node potential A_s needs to be evaluated explicitly. This is easily found by making use of the condition that the sum of the induced currents should be zero.

$$\text{i.e.} \quad \sum I_e = 0$$

$$\therefore A_s = \frac{\sum \frac{1}{3} \Delta \cdot (A_\ell + A_m + A_n)}{\sum \Delta}$$

The $\sum$ summation is carried out only for the elements in the plate region. Now the steps in the evaluation of the normal force are:

- (1) Obtain the datum node potential A_s from:

$$A_s = \frac{\sum \frac{1}{3} \Delta \cdot (A_\ell + A_m + A_n)}{\sum \Delta}$$

- (2) Obtain the element current I_e , from:

$$I_e = J\omega\sigma\Delta \cdot A_s - \frac{j\omega\sigma\Delta}{3} [A_\ell + A_m + A_n]$$

- (3) Obtain the horizontal flux density B_x from:

$$B_x = \frac{1}{2\Delta} \left\{ (x_m - x_n)A_\ell + (x_n - x_\ell)A_m + (x_\ell - x_m)A_n \right\}$$

- (4) Obtain the contribution to the normal force from:

$$(F_y)_{\text{elem.}} = \text{Re} [B_x^* I_e]$$

The force contributions from every element can be computed thus, and the total obtained by summation. The horizontal force is calculated in exactly the same manner, but substituting the vertical flux density in place of the horizontal flux density.

6.8.3 Input Impedance

As in the finite-difference analysis, the input impedance and hence the power losses, is found by evaluating the complex energy input to the device. Let the input impedance be Z_{in} , where:

$$Z_{in} = R_{in} + j X_{in}$$

If I_{in} is the r.m.s. input current corresponding to the source current density j ,

$$|I_{in}|^2 \cdot R_{in} = \omega I_m \int A^* J \, dx dy$$

$$|I_{in}|^2 \cdot X_{in} = \omega \operatorname{Re} \int A^* J \, dx dy$$

$$\therefore |I_{in}|^2 \cdot R_{in} = \omega \operatorname{Im} \left\{ \sum J \int_{\Delta} A^* \, dx dy \right\}$$

$$\text{i.e.} \quad |I_{in}|^2 \cdot R_{in} = \omega \operatorname{Im} \left\{ \sum \frac{1}{3} J (A_l^* + A_m^* + A_n^*) \right\}$$

The $\sum$ summation extends over all elements containing source current densities. A_l , A_m and A_n are the vertex potentials of a typical element, and J is the current density in the element. Thus R_{in} can be found. The input reactance X_{in} is obtained in a similar way from:

$$|I_{in}|^2 \cdot X_{in} = \omega \operatorname{Re} \left\{ \sum \frac{1}{3} J (A_{\ell}^* + A_m^* + A_n^*) \right\}$$

From the resistive part of the input impedance, the plate power loss is obtained as:

$$P_{loss} = |I_{in}|^2 \cdot R_{in}$$

RESULTS OF FINITE ELEMENT ANALYSIS7.1 INTRODUCTION

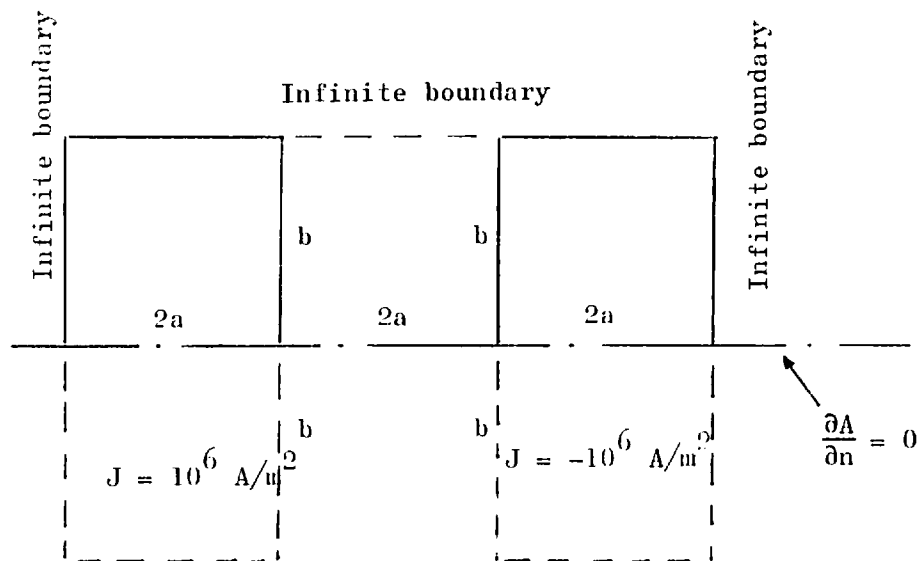
The results of the finite element analysis of the Xi-core are presented in this chapter. The method of setting up the finite element model of the Xi-core is briefly described and the verification of the theory and the automatic generation of the infinite boundary are presented. After these sections, to avoid repetition of those features of the Xi-core already established by the finite difference method, a different manner of presentation is adopted. The agreement reached between experiment and theory is presented for the symmetrical and asymmetrical plates, with comments where appropriate. With the accuracy of the finite element analysis thus established, the forces and their distributions are investigated. The lateral force stiffness is investigated in more detail than in the finite difference analysis, paying closer attention to the effect of plate thickness and airgap values. The rolling stability of the plate is examined by actually inclining the plates, since this presents no problem in a finite element analysis. The effect of altering the rotor geometry is investigated and it is shown that the composite double rotor geometry (briefly investigated experimentally in Chapter 2) is superior to the simple rotor form. Finally, the usefulness of adding the extra iron limbs to the U-core which leads to the Xi-core geometry is questioned and it is shown that this can introduce unfavourable performance qualities. The chapter is concluded by first comparing the finite difference method with finite element method in terms of computing costs and the degree of agreement of forces with experiment. This is followed by a final discussion of the Xi-core performance in the light of the results achieved by the finite element analysis.

7.1.1 Method of Setting up the Model

In the finite element analysis one of the major problems encountered is in the setting up of the model on a computer. This is a consequence of the irregular disposition of the nodes in the model, which requires that every node be addressed separately. Thus one has to feed in a large amount of numerical data (containing the following information) to the computer:

- (a) The x and y co-ordinates of every node in the model with respect to the chosen Cartesian axes.
- (b) The node numbers of the three vertices of every element.
- (c) The material properties of every element. This is usually achieved by attaching a label to every element.
- (d) The node numbers of the boundary nodes which require the satisfaction of Dirichlet or infinite boundary conditions.

A computer package developed by fellow research workers (at Imperial College and McGill University) which enables one to set up the model on a VDU in a short span of time, was made available for this research. The input data required in the above form is obtained as the output of this package, on completion of the model defining process. Since the development of this package was not a part of this research, it is not described in more detail. The operation of this package is fully described in the MAGNET reference manual^{7.1}.



$$2a = 25.4 \text{ mm}$$

$$b = 25.4 \text{ mm}$$

Figure 7.1: The busbar problem used to verify the infinite boundary algorithm.

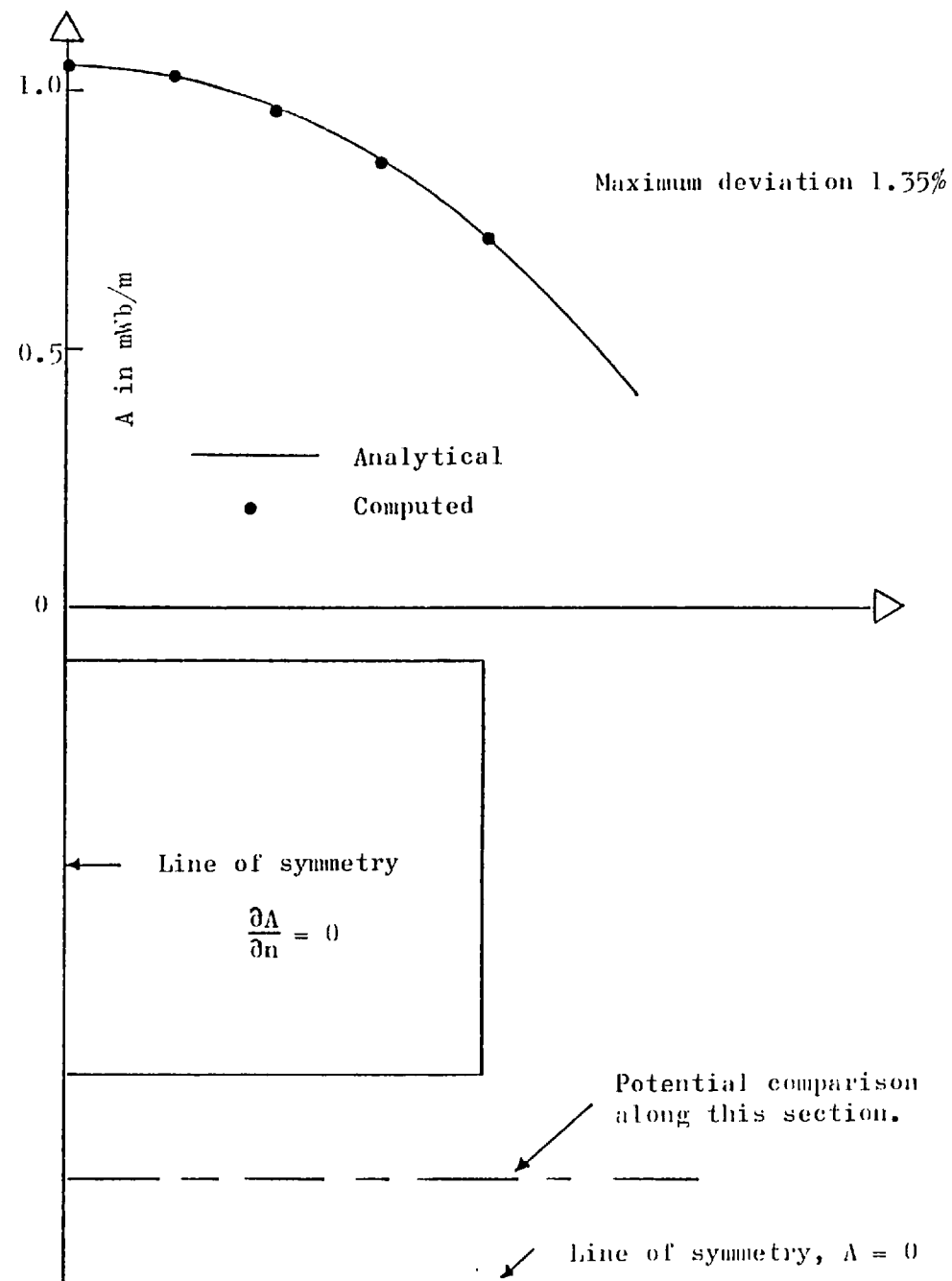


Figure 7.2: Comparison of computed/analytical results.

7.1.2 Verification of Infinite Boundary

Both the theory of the infinite boundary and the automatic generation routine were tested first by solving the magnetostatic field of two busbars in air. To impose the most stringent test on the theory, the infinite boundary was brought to the edges of the busbars. The model consisted of 101 nodes and 180 elements. Only one half of the arrangement (Fig. 7.1) was solved, requiring the infinite boundary on three sides only. The computed vector potential values are compared with the analytical values in Fig. 7.2. The maximum deviation can be seen to be less than 1.5%.

7.2 COMPARISON OF COMPUTED AND EXPERIMENTAL RESULTS

The degree of accuracy reached between computed and experimental values of such quantities as flux densities, forces, etc. are presented now, to establish the reliability of the developed algorithm. The case of the Xi-core with symmetrical plates is considered first and is followed by consideration of asymmetrical plates.

7.2.1 Xi-Core with Symmetrical Plates

Only one half the machine was considered, imposing the Neuman boundary conditions along the centre line of the machine. The infinite boundary was therefore required on the remaining three sides. In addition it was decided to have the infinite boundary "hugging" the contours of the stator iron and the top edge of the plate. This choice enabled the use of large numbers of elements and nodes in the crucial plate and airgap regions. The model itself

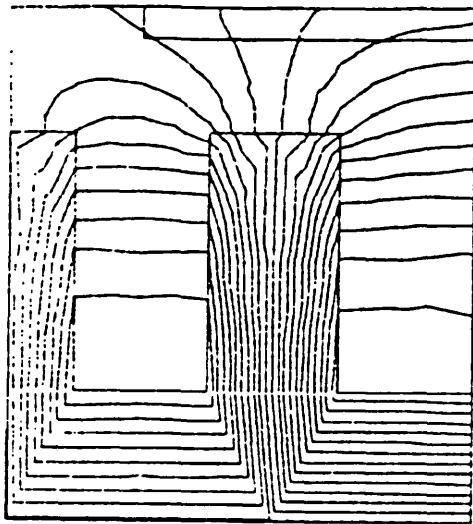
consisted of 446 elements and 240 nodes. Fig. 7.3 shows the computed field solution for one plate along with the model. The field plots serve to illustrate the difference in philosophy between the picture frame transformation and the infinite boundary. In the former, the full information is available after the field computation, while in the latter information is available only within a window formed by the interior region.

Flux Densities:

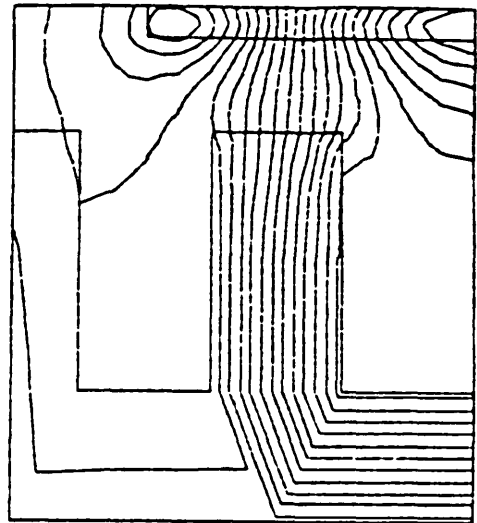
Fig. 7.4 shows the comparison of computed and experimental flux densities at 50 Hz for one plate at a section between the plate and the stator top surface. Due to the fact that the flux density within any first order finite element is constant, in most first order finite element solutions it is common practice to obtain flux density distributions that are erratic. In Fig. 7.4, however, no smoothing techniques have been used and the very close agreement and smooth variation is a result of the use of large numbers of elements in the airgap, made possible by the infinite boundary. This close agreement also serves to bring out the advantages of the finite element method over the finite difference method, in that one can choose fewer elements in the regions that are not of interest, concentrating most of the nodes in the regions of interest and yet achieve a high degree of accuracy in solution.

Induced Current Density:

The induced current density comparison at the top edge of the plate is shown in Fig. 7.5. The agreement is close despite the



IN PHASE FIELD



QUADRATURE FIELD

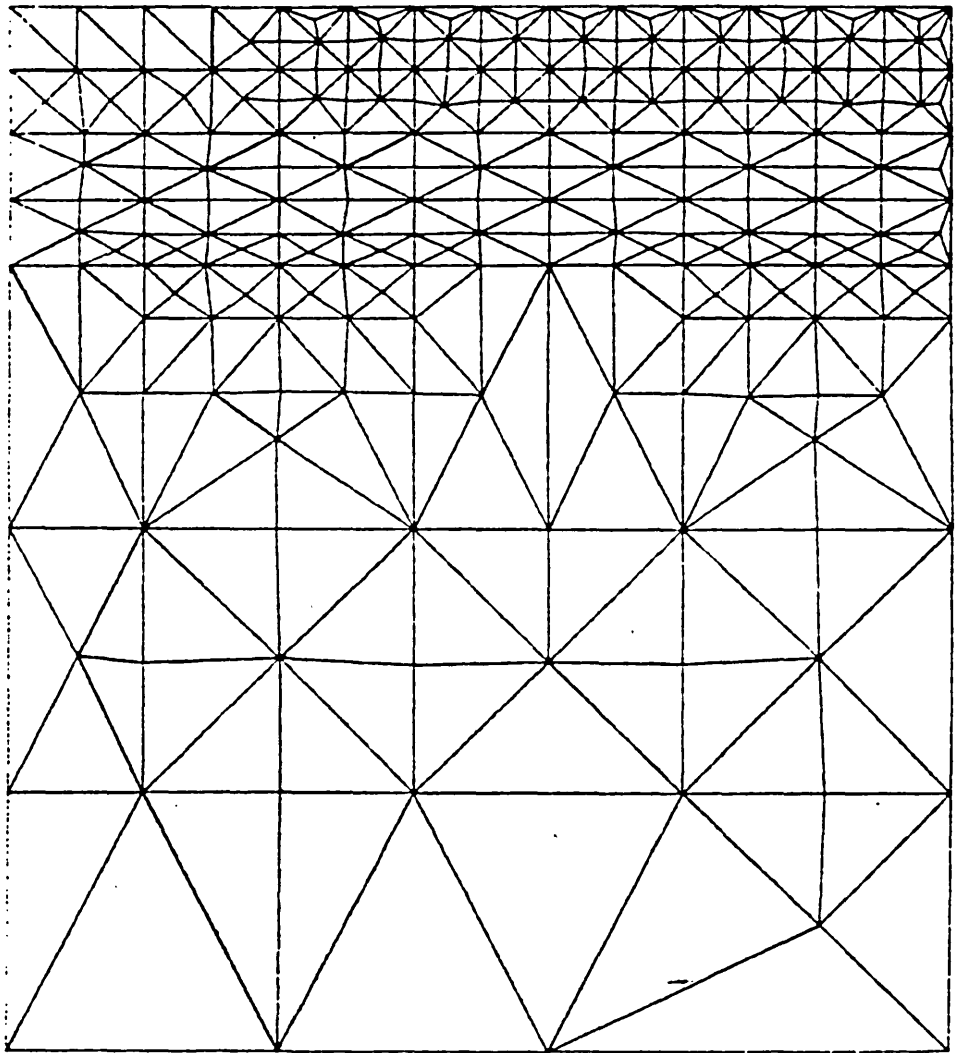


Figure 7.5: The finite element mesh and field plots for one half of the Xi-core.

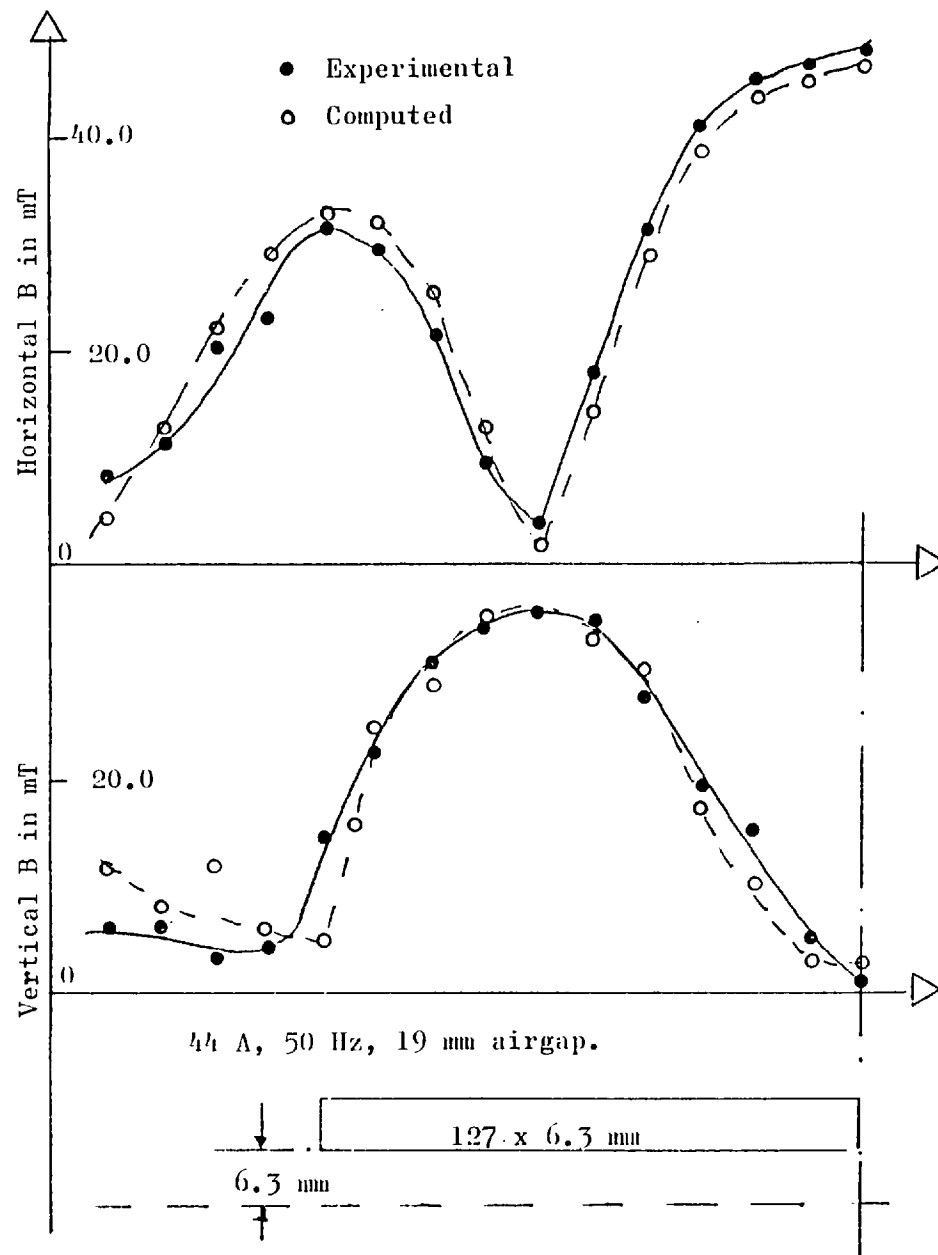


Figure 7.4: Comparison of flux densities.

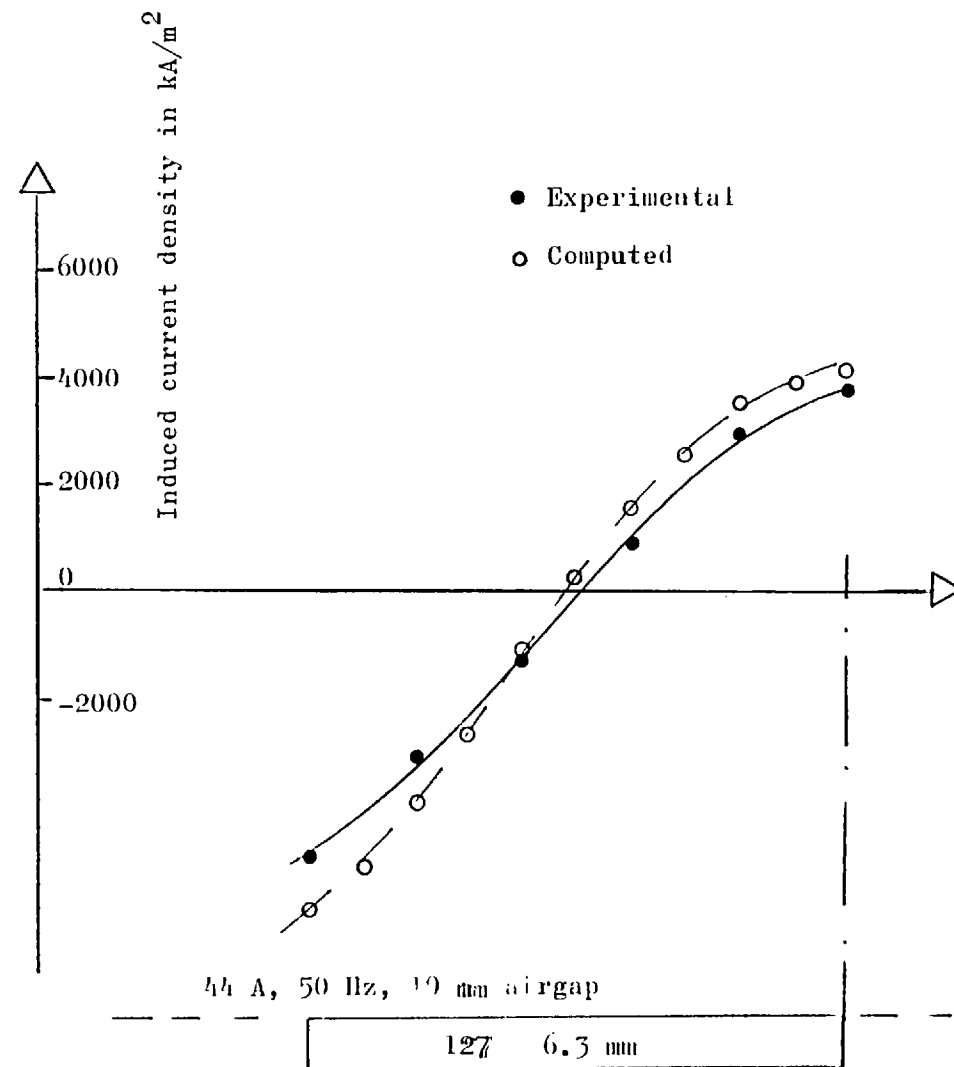


Figure 7.5: Comparison of current densities.

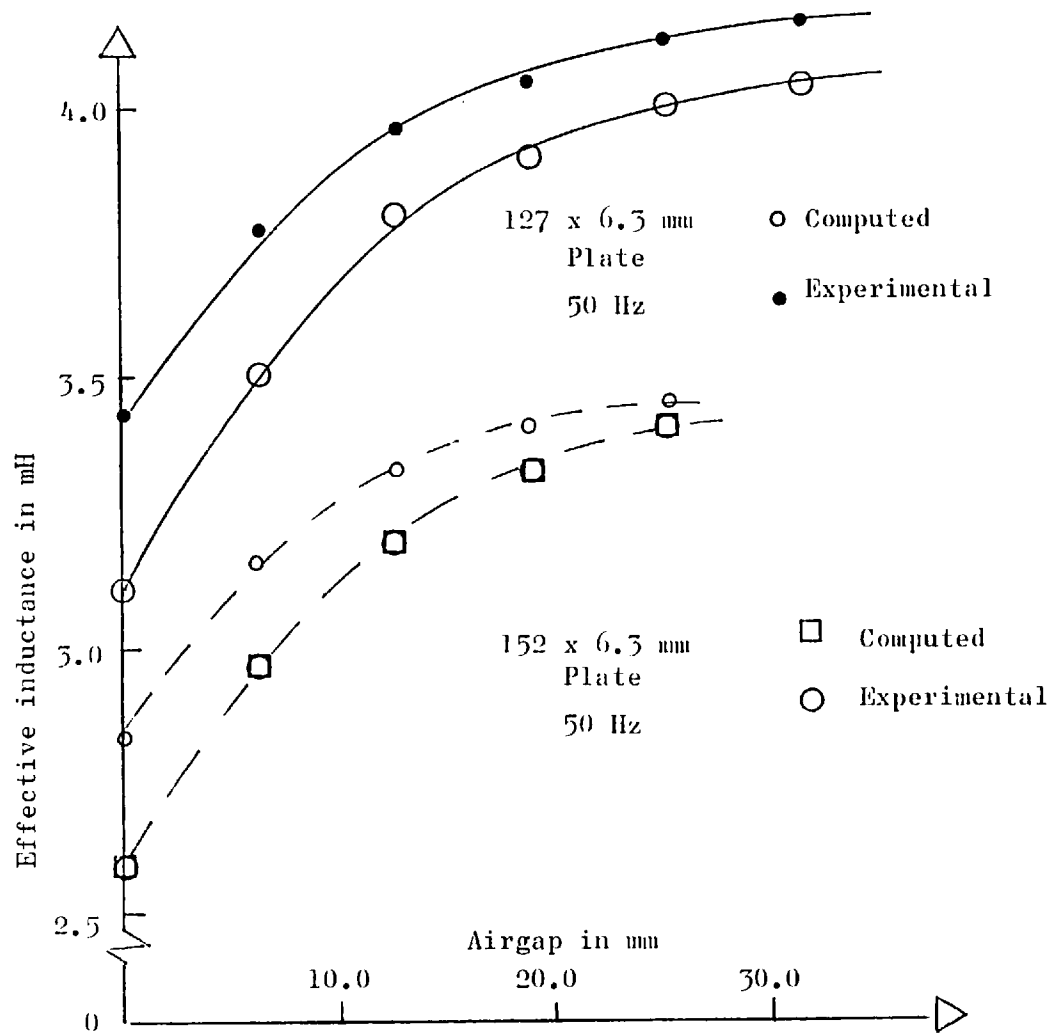


Figure 7.6: Comparison of input impedance.

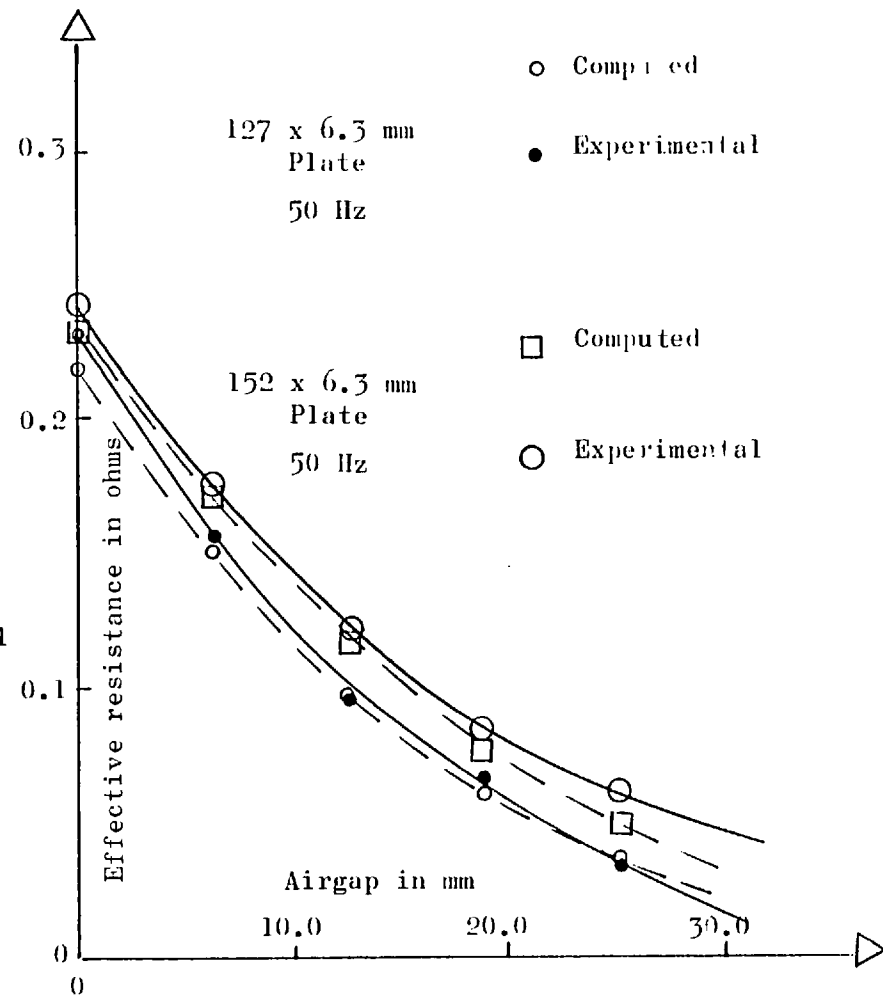


Figure 7.7: Comparison of input impedance.

fact that the infinite boundary borders a diffusion region. This, in conjunction with the busbar problem, shows that the infinite boundary can be confidently relied upon, whether it borders a Poissonian or a diffusion region.

Input Impedance:

The comparison of input impedance variation with airgap for two plates is shown in Figs. 7.6 and 7.7. While the resistive part of the input impedance shows good agreement, the reactive part shows a maximum discrepancy of about 16%. This is believed to be a consequence of the fact that in the finite element formulation used here one seeks to minimise the energy difference:

$$\Delta E = \text{Energy Stored} - \text{Energy Input},$$

$$\text{i.e.} \quad \text{Energy Input} = \text{Energy Stored} - \Delta E.$$

We first note that unless the residual energy ΔE is zero, the energy input evaluated by $\int A.J. \, dx dy$ is less than the energy stored, leading to a lesser value of input impedance. Secondly, the proportion of the residual energy that represents the dissipated energy residual will in general be small, since the plate region is the only area where energy is dissipated and this is well represented by a large number of elements and nodes. The proportion of the residual energy representing the residual stored energy, however, will be large, since energy is stored in all space. In poorly described regions, the flux density is assumed constant over large areas, leading to errors in the stored magnetic energy which depend on the square of the flux density. These observations partly explain the lower computed reactance.

Power Factor:

From a knowledge of the input impedance, the power factor was computed. The comparison with measured values, for two plates, is shown in Fig. 7.8.

Normal Force:

From the finite element solution of three plates at several airgaps, the normal force was computed using $\int J_x B \, dx dy$. The comparison with measured values is shown in Fig. 7.9. To obtain the three curves of Fig. 7.9, the same model was used; by changing the material properties of the elements in the airgap, the correct plate width and airgap values could be obtained. The number of elements in the plate region which is adequate at large airgaps, becomes inadequate at and near zero airgaps, contributing to the discrepancies observed at such airgap values in Fig. 7.9.

7.2.2 The Xi-Core with Asymmetrical Plates

The field of the Xi-core with asymmetrical rotors was solved using the finite element technique. For these solutions the full model was needed with sufficient space for the displacement of the plate. In order to reduce the number of nodes on the infinite boundary it was pushed slightly above the top surface of the plate. In the case of the Xi-core with symmetrical plates, 240 nodes and 440 elements were used for the half model. Thus for the full model, one would require at least 500 nodes in order to achieve a solution of comparable accuracy. However, because of the rapid increase in the size of the stiffness matrix with increasing number of nodes

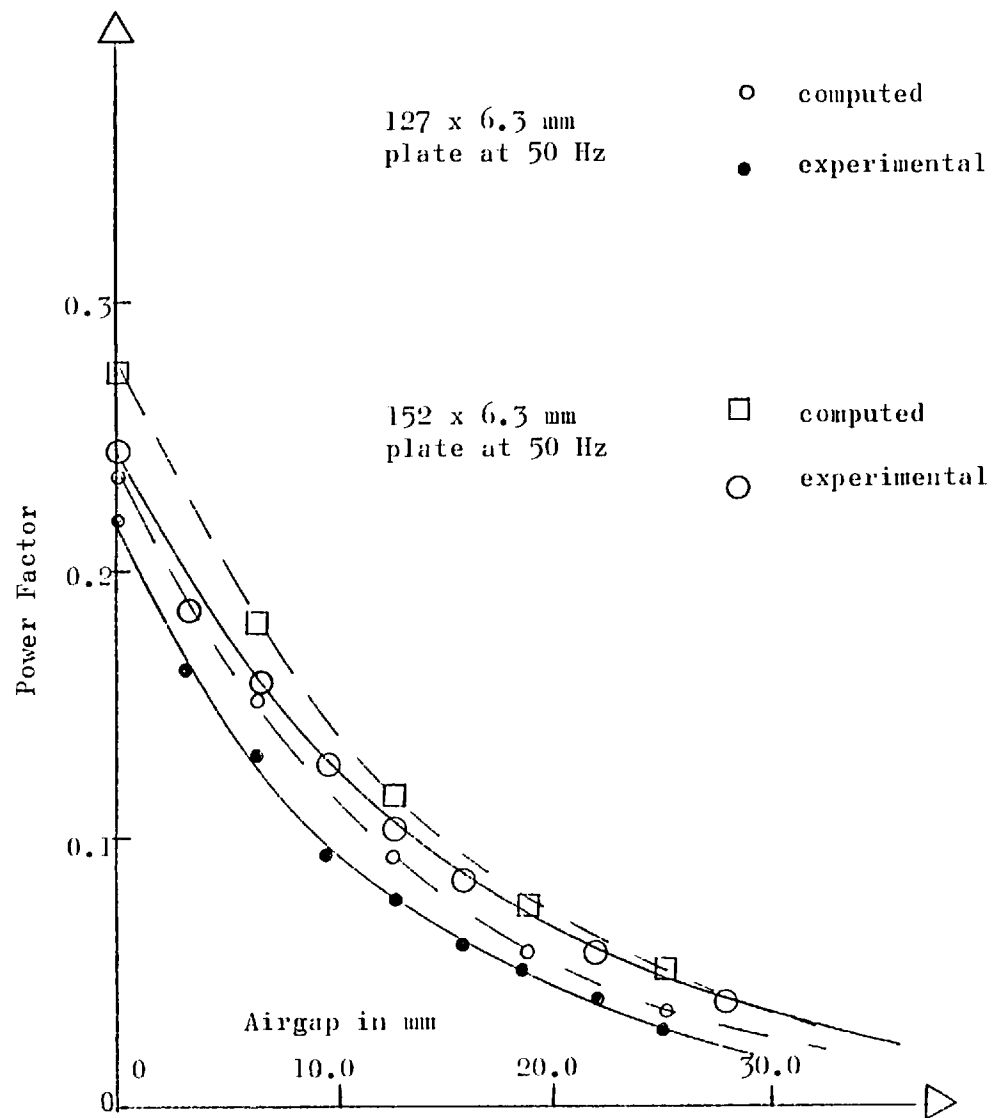


Figure 7.8: Power factor comparison.

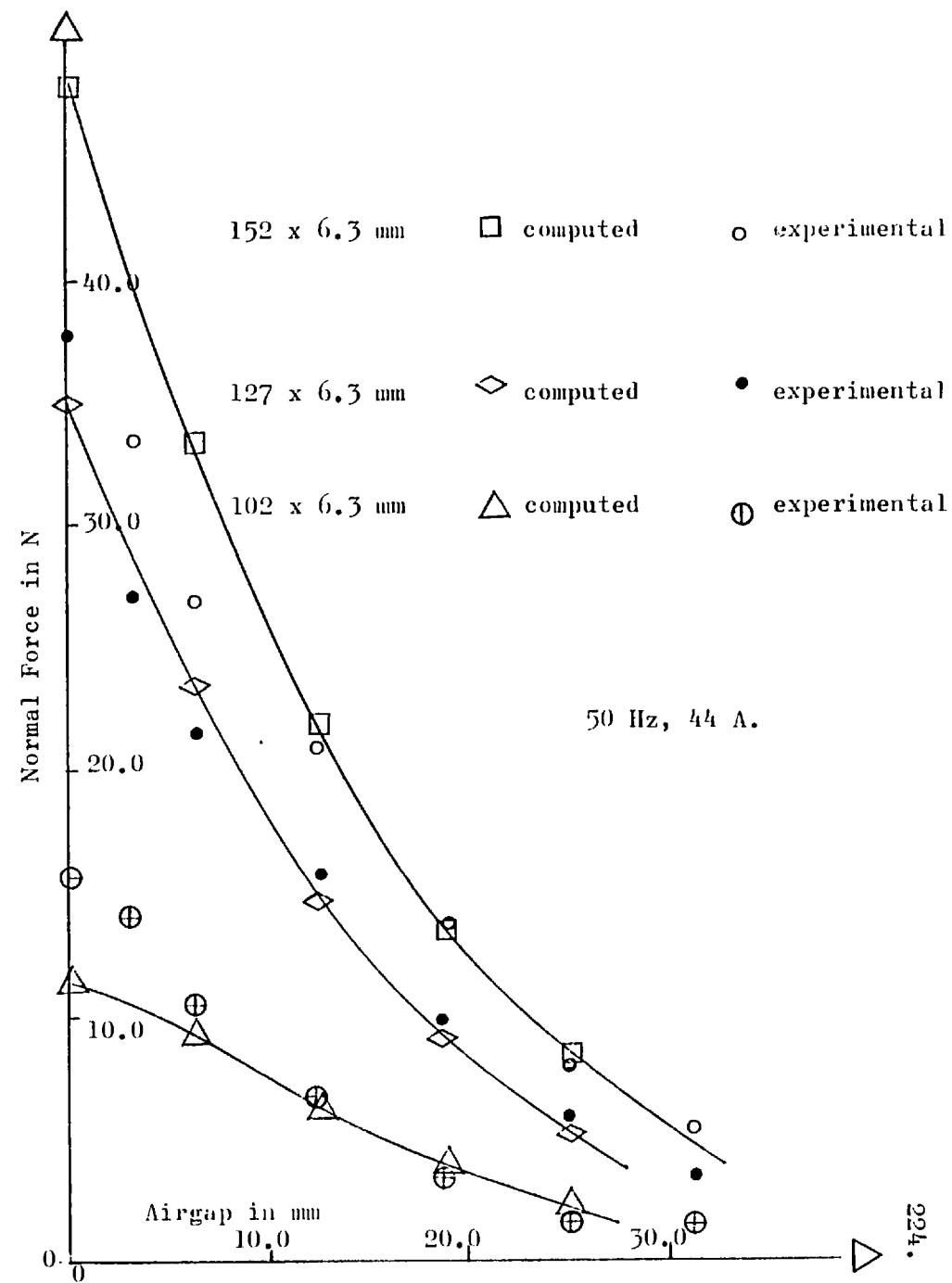


Figure 7.9: Normal force comparison.

(size increases with the square of the number of nodes), the maximum number of nodes that could be used within the memory limits of the computer was in the region of 400. This should be borne in mind, in judging the agreement between theory and experiment, which is given next.

Flux Densities:

Fig. 7.10 shows the comparison of computed and experimental values of flux density at a section in the airgap with a displaced plate. In the finite difference analysis, it was shown that when a plate is displaced, the magnetic field distribution under the plate changes from a two-pole into a one-pole distribution. This is further confirmed by the field plots shown in Fig. 7.11, which shows the field distribution at four successive displacements.

Induced Current Density:

The variation of the induced current density at the top edge of the plate and its phase angle with respect to the excitation current are compared in Figs. 7.12 and 7.13.

Lateral Forces:

For the three plates considered in the symmetrical position in Section 7.2.1, the horizontal forces were computed at several displacements. These are compared with the measured values in Fig. 7.14. In Chapter 2, it was shown experimentally that if the lateral displacement was expressed as a fraction of the plate width, all the curves of lateral force vs. the fractional displacement have the same

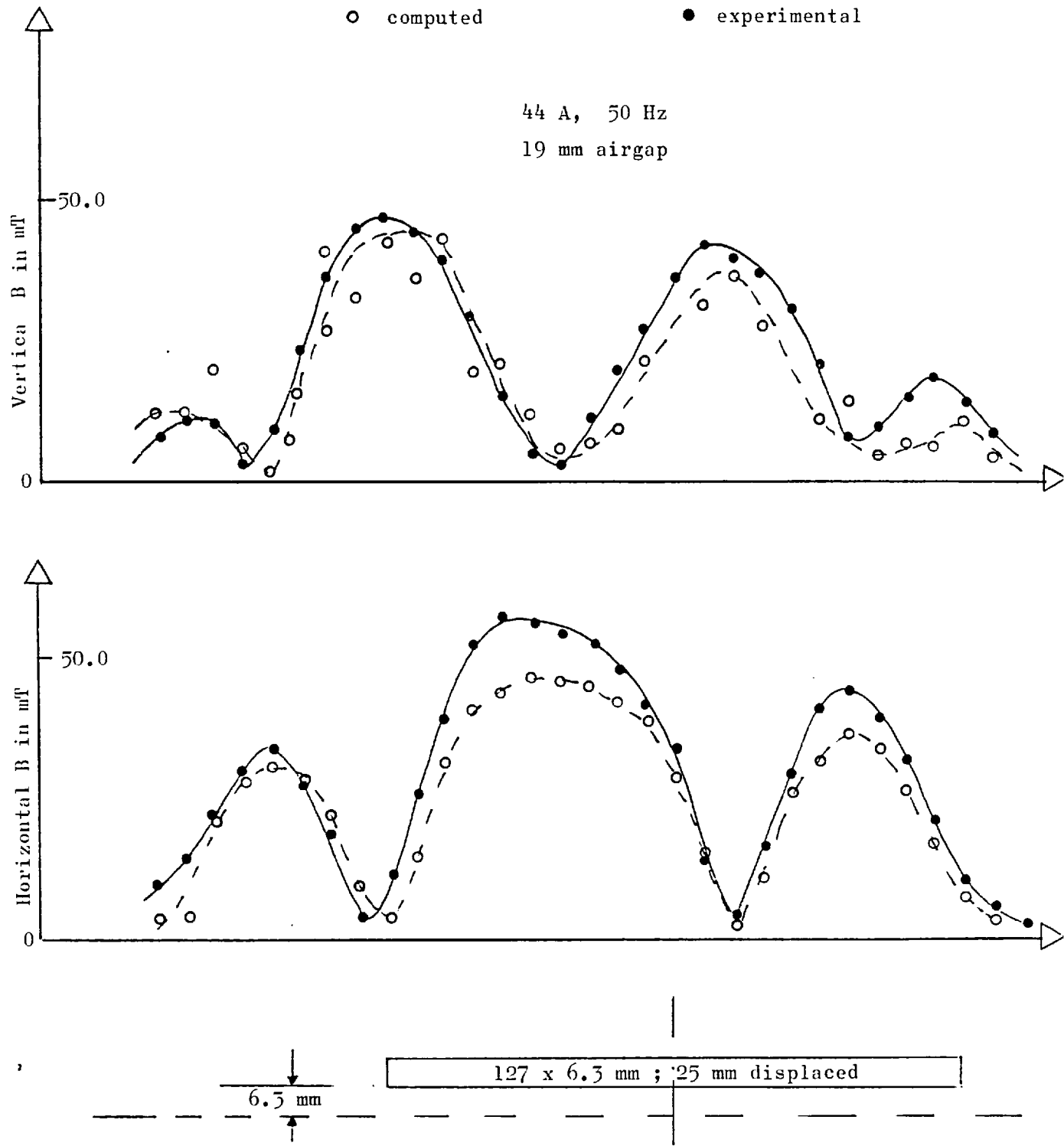


Figure 7.10: Comparison of flux densities.

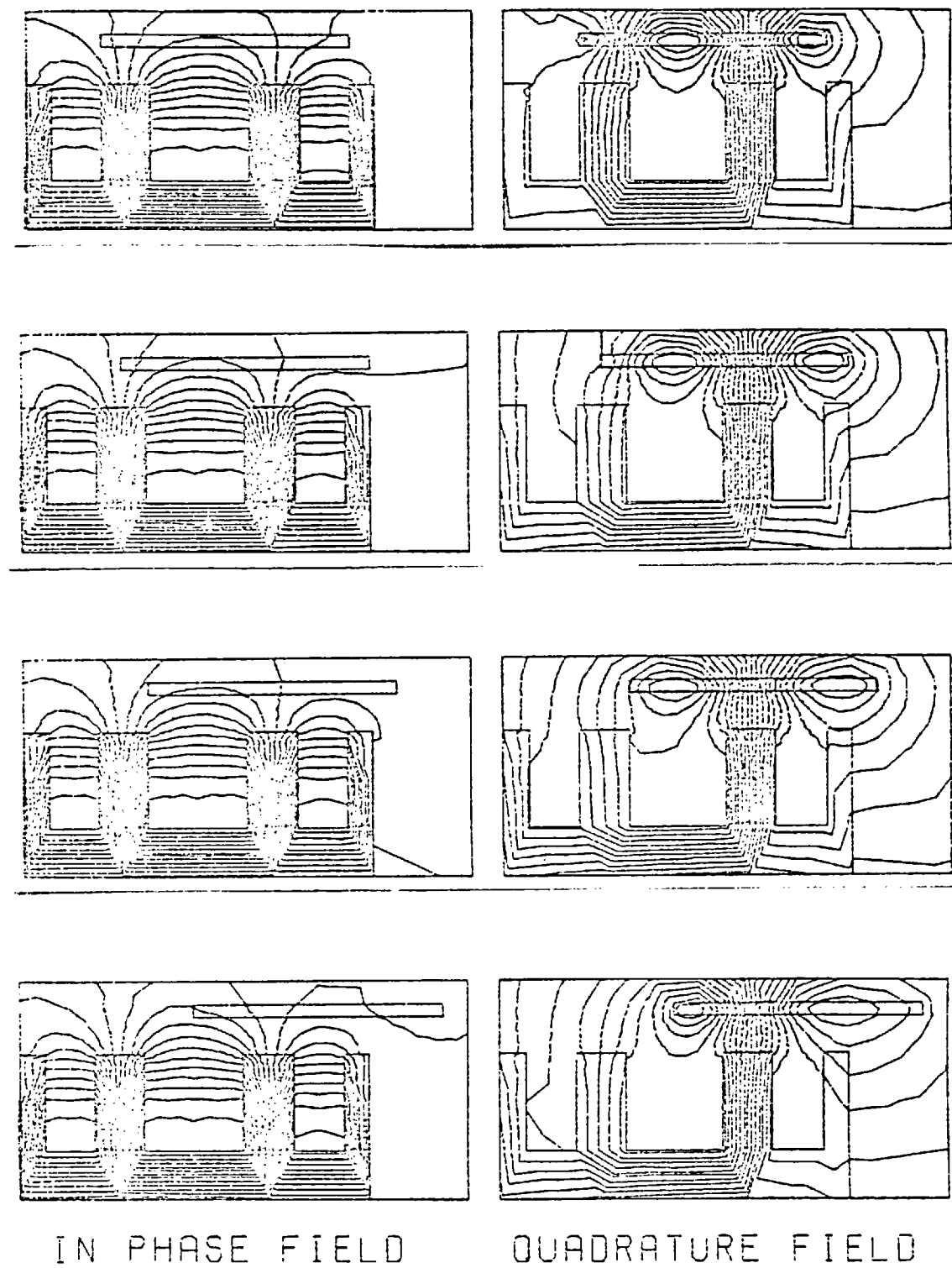


Figure 7.11: Computed field plots of one plate at successive lateral displacements.

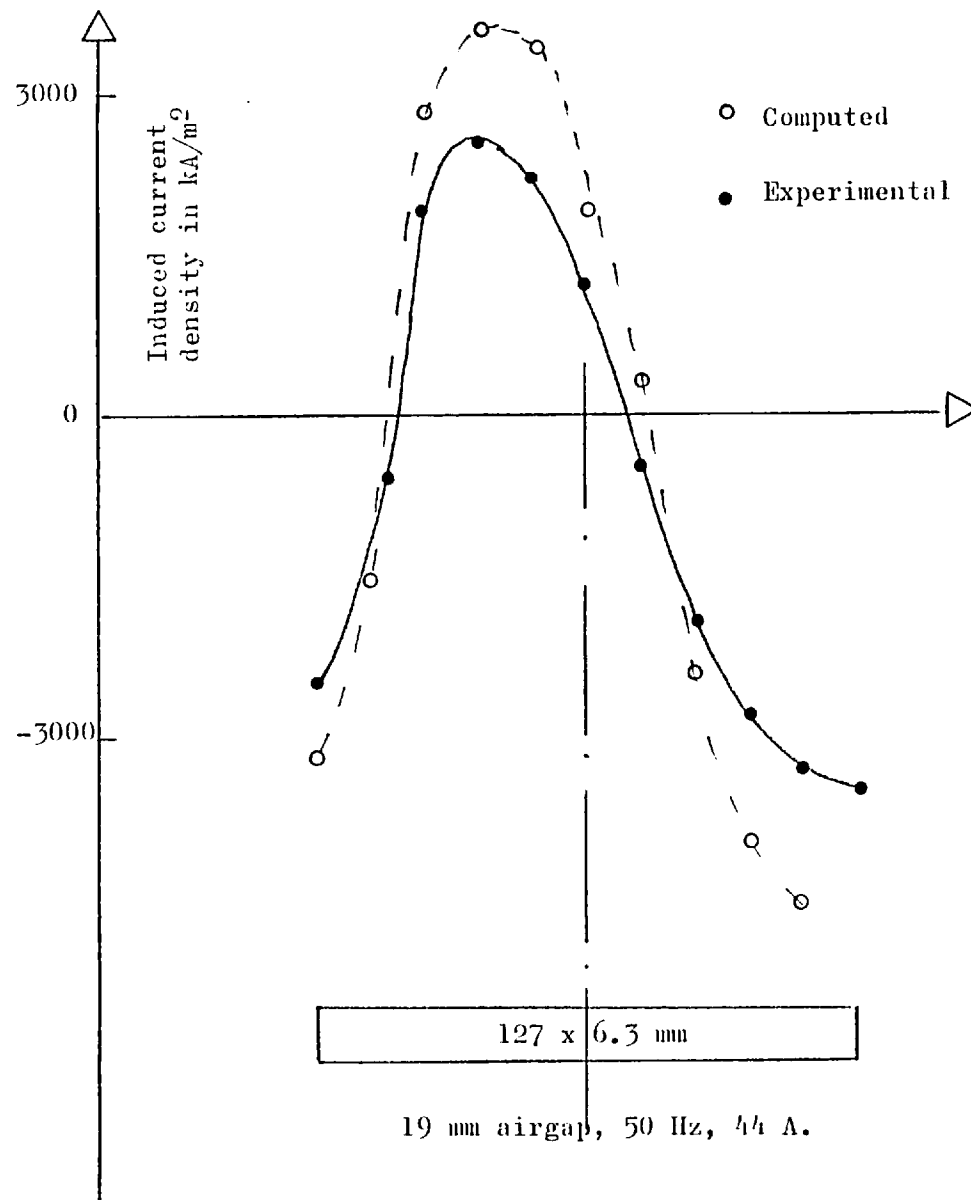


Figure 7.12: Induced current density comparison.

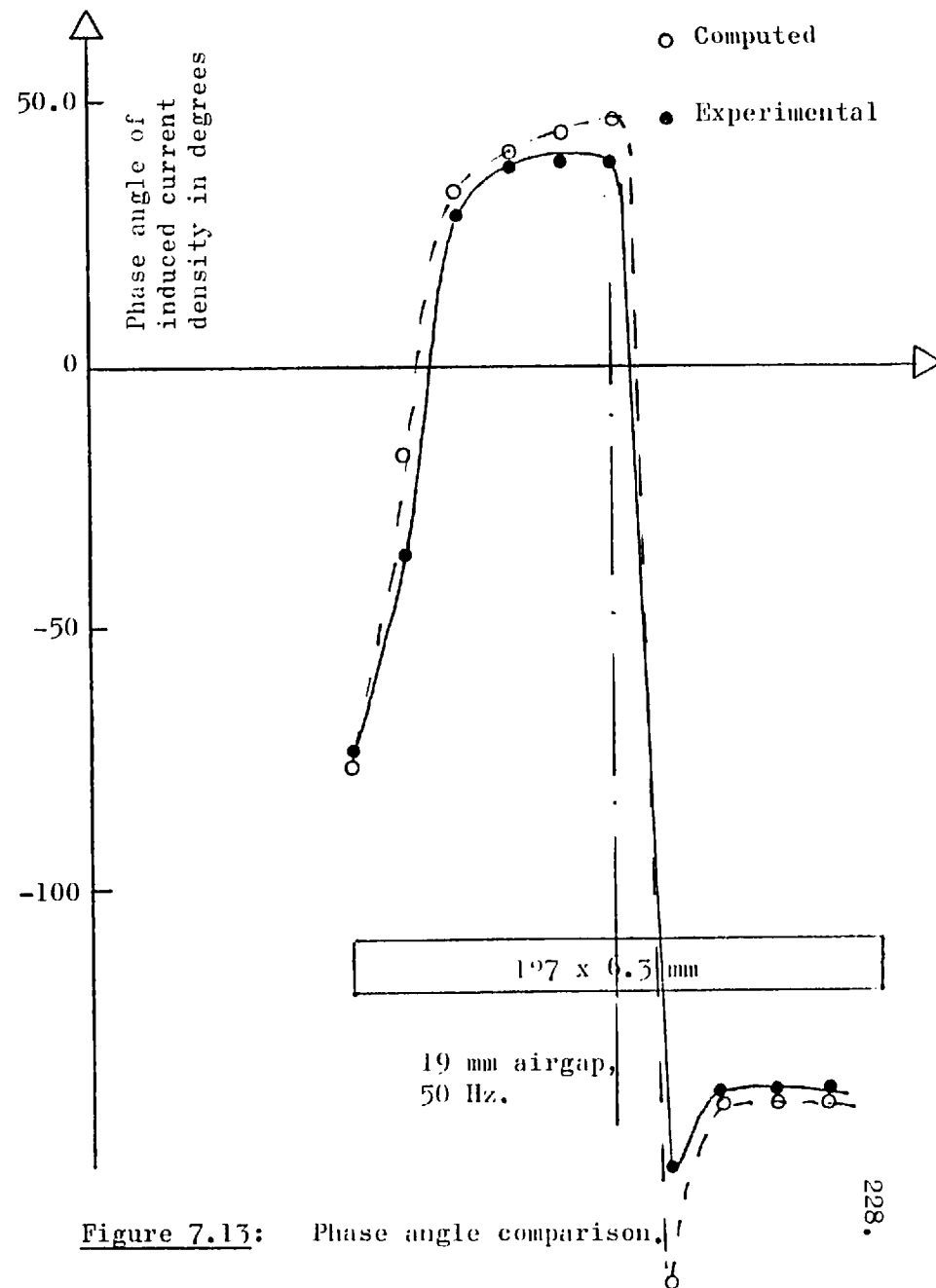


Figure 7.13: Phase angle comparison.

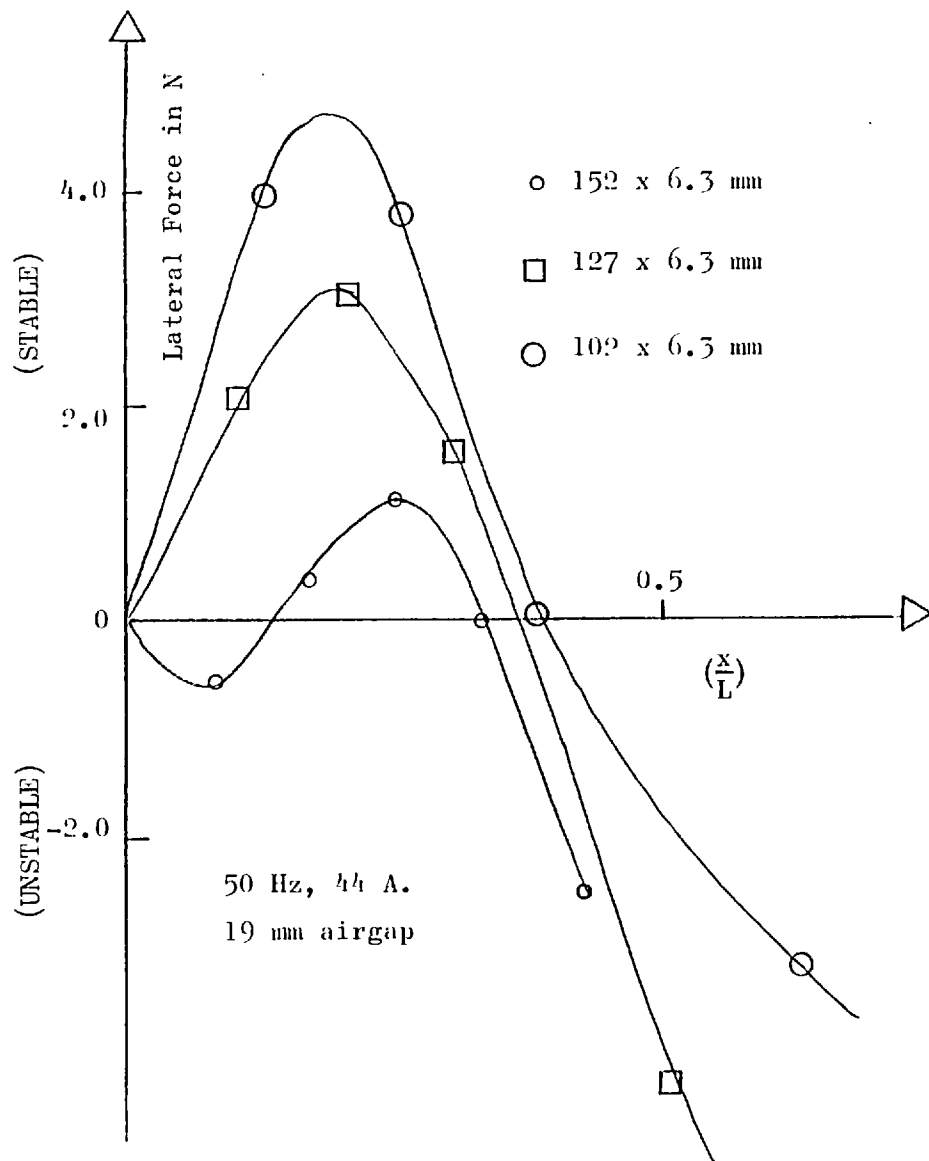


Figure 7.15: Theoretical confirmation of the cross-over point.

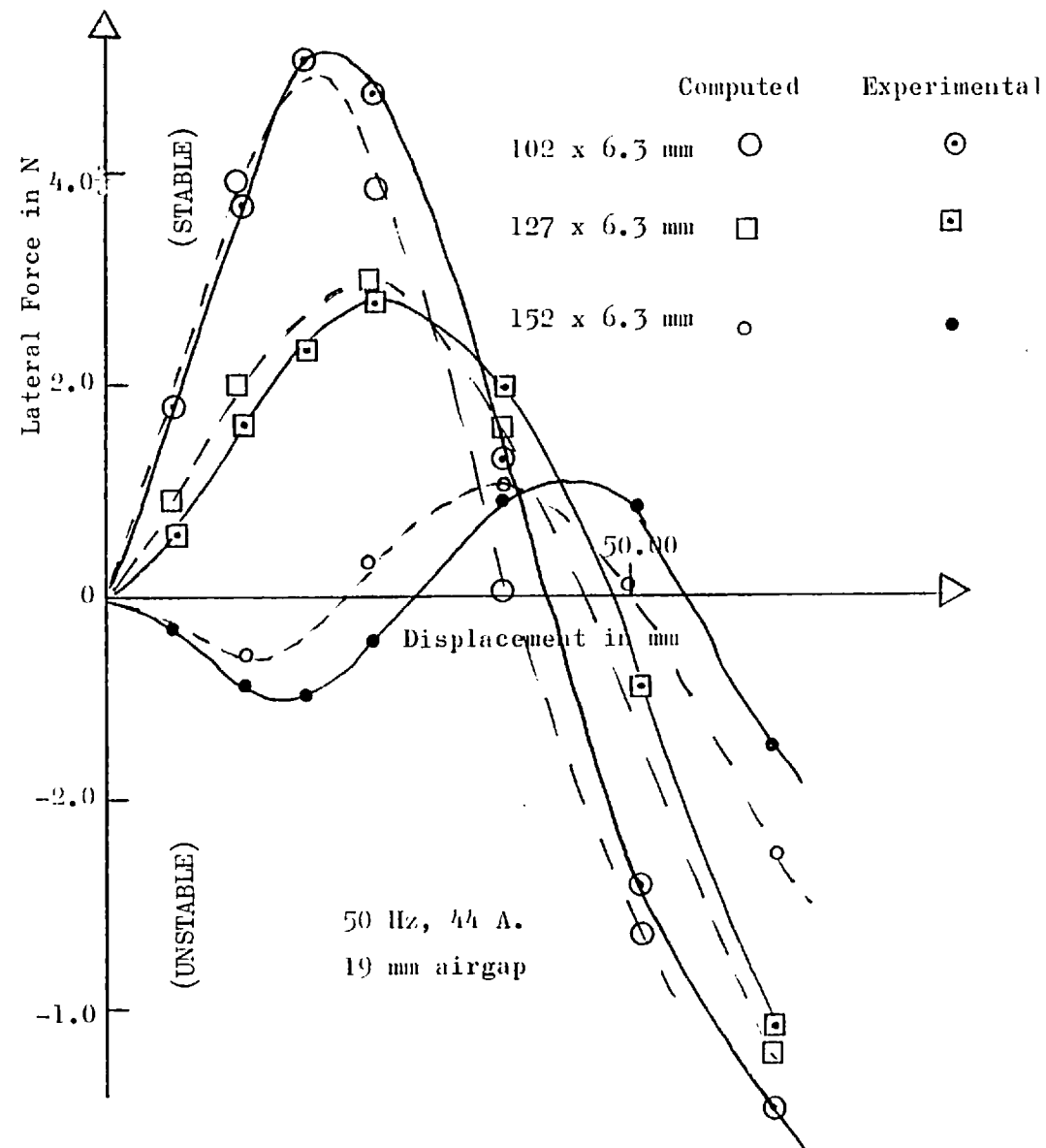


Figure 7.14: Lateral force comparison.

cross-over point. To obtain theoretical confirmation of this, the curves of Fig. 7.14 are redrawn with displacement expressed as a fraction of the plate width and shown in Fig. 7.15.

The computed cross-over points are seen to be within the range 0.33 and 0.37, whereas in Chapter 2 experimentally it was found to be 0.35.

7.3 STRESS DISTRIBUTIONS

Having presented the computed/experimental agreement, we examine the nature of the stress distributions in rotor plates. The objective is partly to confirm the conclusions reached in the finite difference analysis and partly to reveal further information, if evident.

7.3.1 Symmetrical Plates

The magnitude and direction of the total force acting on each element in the plate region of the model was computed. The stress distribution was obtained by drawing vectors from the centroid of each element to represent the force acting on that element. The stress distribution obtained by such vectors is shown in Figs. 7.16 and 7.17 for two plates, one narrow and one wide. The distributions firstly confirm the conclusion that narrow plates are in a state of compression while the wider plates are in tension. Secondly, the distributions reveal the presence of attractive forces for the narrow plates. To examine this more quantitatively the total attractive and lift forces were computed separately, for the two plates shown in Figs. 7.16 and 7.17, and one other whose width is intermediate.

JXB TOTAL FORCES
FREQUENCY 50.00 HZ

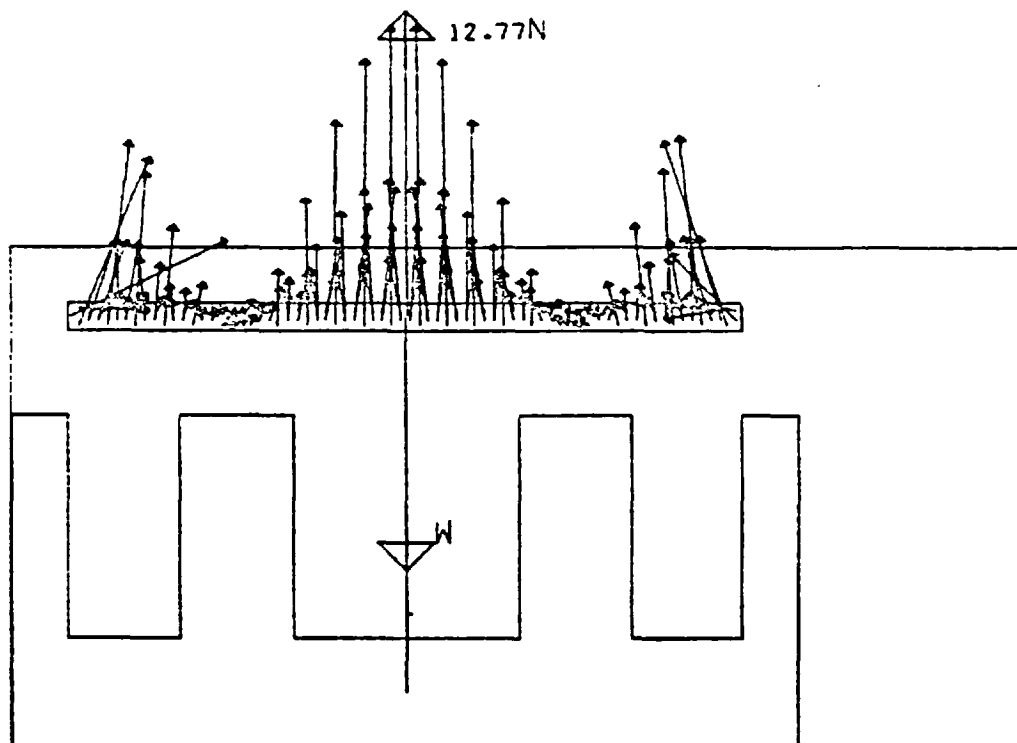


Figure 7.16: The computed stress distribution.

JXB TOTAL FORCES
FREQUENCY 50.00 HZ

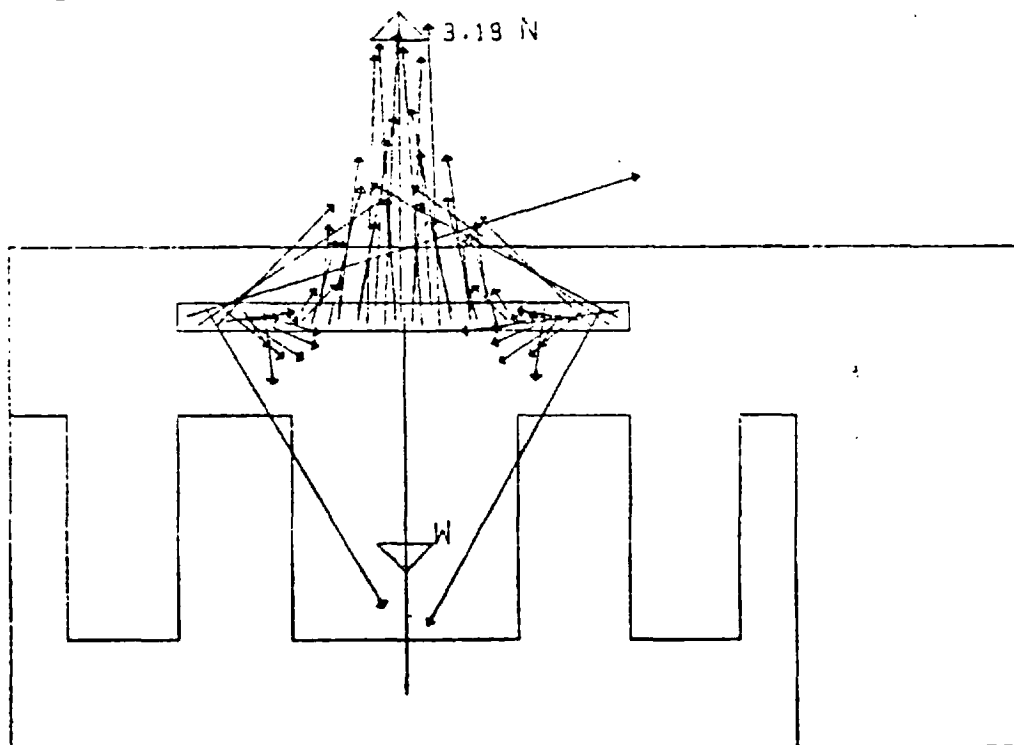


Figure 7.17: The computed stress distribution

Taking the nett lift force as 100 units for each of the plates, these computed components are as follows:

Plate Width	Nett Lift Force	Attractive Force	Repulsion Force
102 mm.	100	28.4	128.4
127 mm.	100	9.06	109.06
152 mm.	100	0.25	100.25

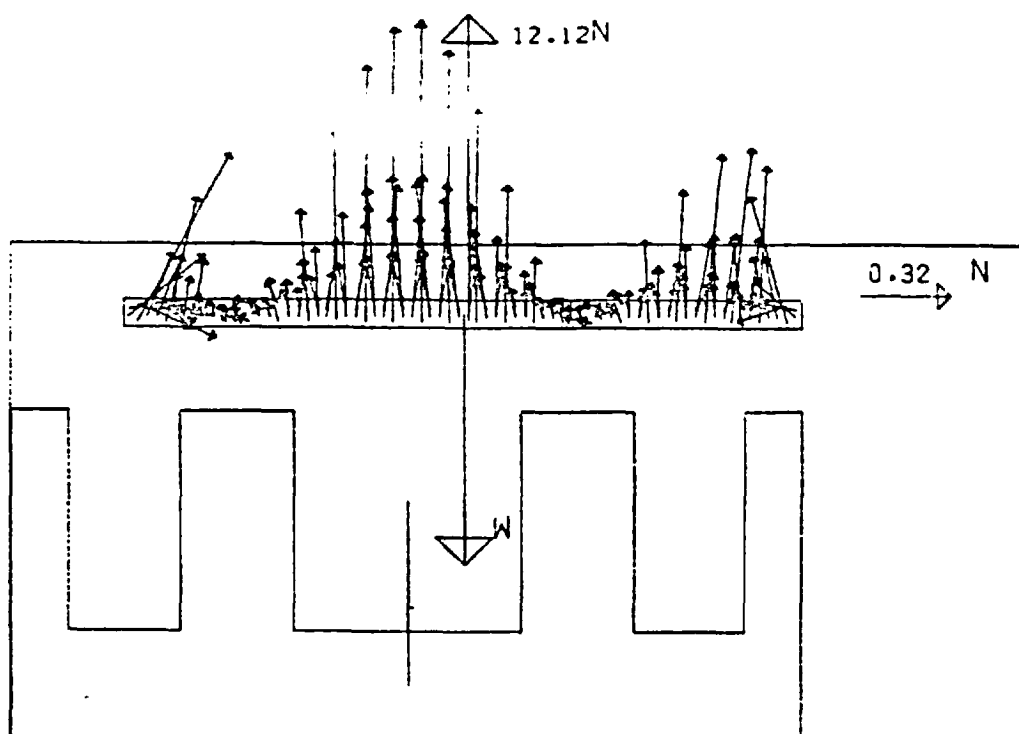
TABLE 7.1

The table confirms that for narrow plates strong attractive forces comparable to the lift forces are present, while for the wider plates they are negligible. The presence of the attractive forces in the narrow plates, explains their high lateral stability, since attractive systems are inherently stable.

7.3.2 Asymmetrical Plates

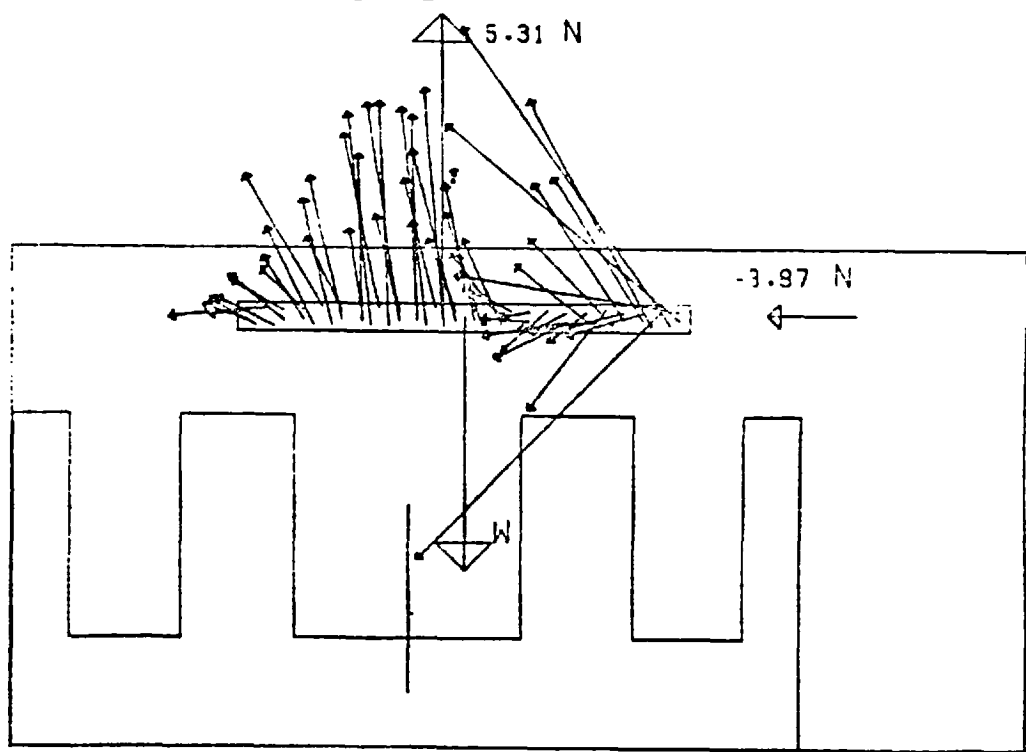
The stress distribution after a small displacement in a narrow and wide plate are shown in Fig. 7.18. These are the same plates which when they are symmetrical are shown in Figs. 7.16 and 7.18. After the displacement, for the narrow plate we observe the complete absence of attractive forces in the trailing half and an increased attractive force in the leading half. The total attractive forces computed for the two halves showed a 45% increase for the leading half and a 97% decrease for the trailing half. We thus see that the change in the distribution of the attractive forces is such as to increase the stabilising force and decrease the rolling stability. The total stabilising force due to both the attractive and repulsive forces was computed as 1.66 N. Of this 0.72 N (44%) is due to the attractive forces. The figures provide quantitative

UXB TOTAL FORCES
FREQUENCY 50.00 HZ



(b)

UXB TOTAL FORCES
FREQUENCY 50.00 HZ



(a)

Figure 7.18:

Computed stress distribution.

proof that attractive forces play a major role in the strong lateral stability of narrow plates. For the wider plate a similar computation showed that, of the total destabilising force of 0.23 N, less than 10% is due to the small attractive forces.

7.4 LATERAL FORCE STIFFNESS

In Chapter 5, using the finite difference method, the lateral force stiffness of one plate at several frequencies was investigated. Due to the expensive computation involved, this aspect of the Xi-core was not examined for several plates. Therefore using the cheaper finite element method (see later) this discussion is re-opened now. Three plates falling into the narrow, wide and intermediate categories are considered. Each of the plates was displaced slightly and the lateral force computed. The lateral force stiffness variation obtained in this manner with frequency for the three plates is shown in Fig. 7.19. For the same plates, the same variation is shown in Fig. 7.20, but with the plate thickness halved. From Figs. 7.19 and 7.20 two important facts emerge:

- (a) For a plate whose width is equal to the distance between the outer corners of the inner teeth, the lateral stiffness is high and is always stabilising whatever the frequency. Decreasing the plate thickness only has the effect of increasing the frequency at which the peak stiffness occurs. Therefore for these plates, however high the goodness factor, the lateral force is stabilising.
- (b) For a very wide plate width (infinitely wide) the lateral force stiffness is one of destabilising, however small

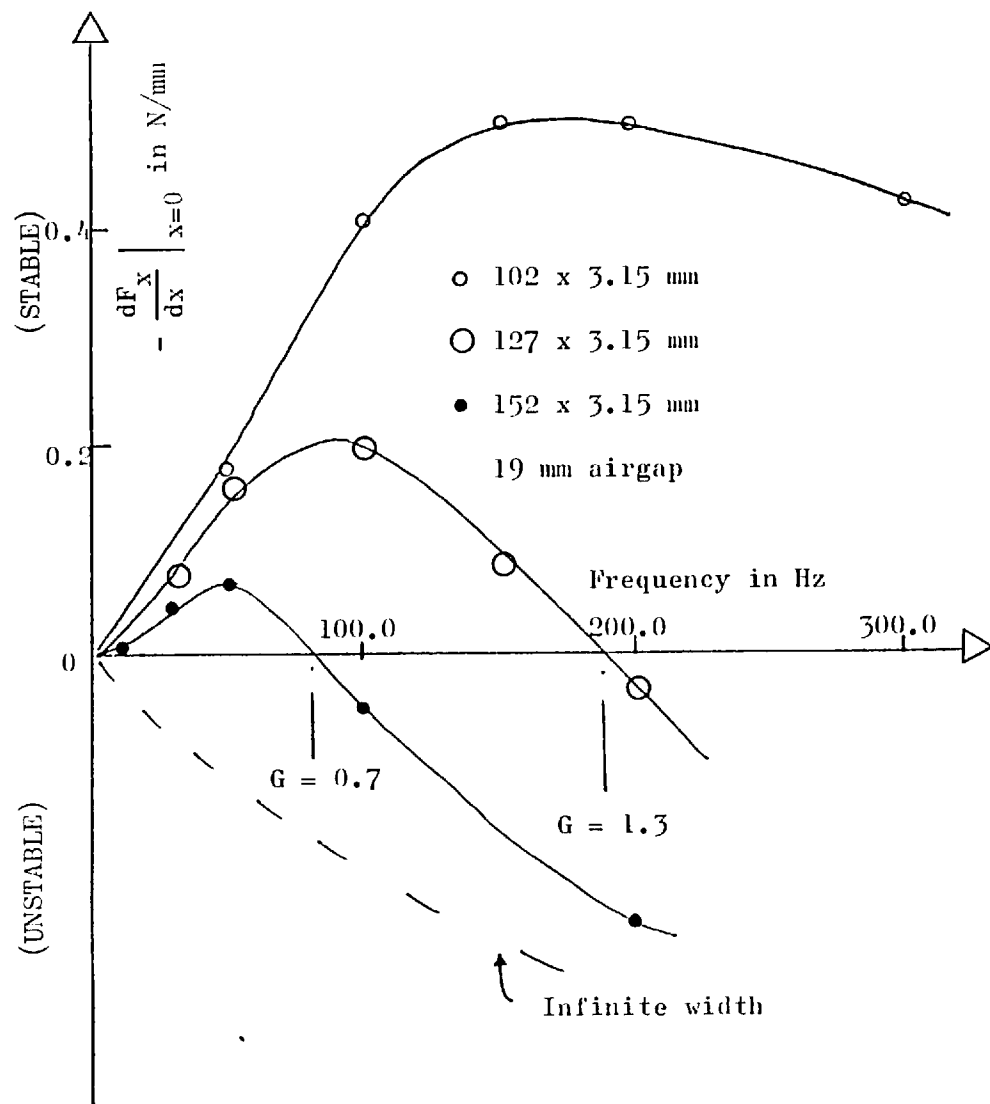


Figure 7.20: The lateral stiffness variations.

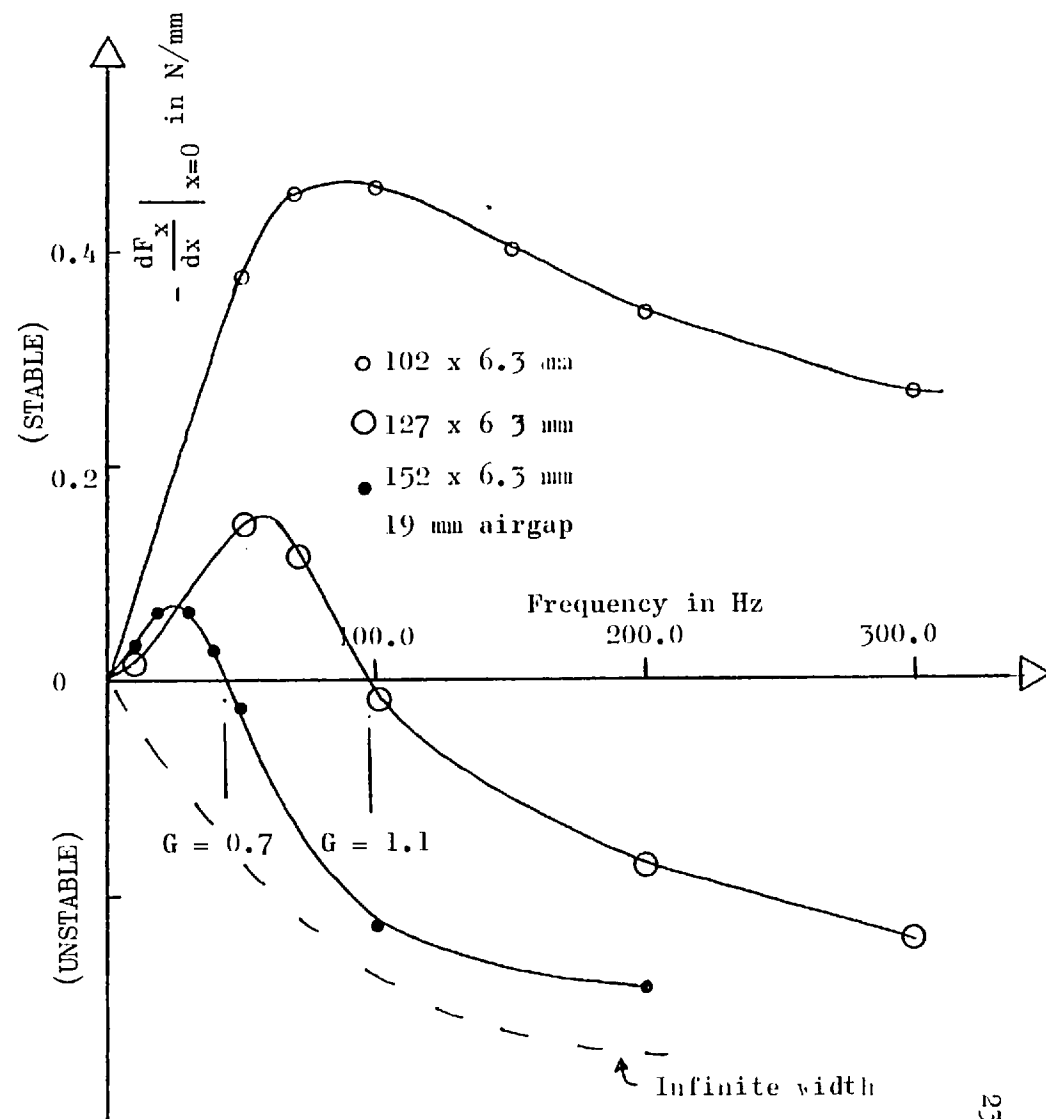


Figure 7.19: Lateral stiffness variations.

the plate thickness and frequency are. Therefore for these plates, however low the goodness factor, the lateral force is destabilising.

For the intermediate plate widths, the curves show that, in the low frequency region, the stiffness is a stabilising one while at higher frequency it is destabilising. Decreasing the plate thickness increases the range of frequency over which the plate is stable.. For these plates, low goodness factor is associated with stability and high goodness factor with instability. The value of G at the frequency at which the changeover from stability to instability occurs was computed and is marked in Figs. 7.18 and 7.20. The values of G show that for narrow plates the changeover occurs at infinite goodness, while for the very wide plate it occurs at zero goodness. In order to investigate the effect of the airgap, the fields of the same three plates were computed at zero airgap, with the plates slightly displaced. The lateral force stiffness variation is shown in Fig. 7.21 which confirms the stability of the narrow plate even at zero airgap.

7.5 ROLLING STABILITY

In the finite difference analysis of Chapter 5, the rolling stability of the plates was investigated from the nature of the normal force distributions. It was established that narrow plates are unstable in roll while wider plates are stable. The finite element method provides the opportunity to examine this further, by inclining the plates and solving the field. Such an exercise was carried out for three plates at two frequencies. Fig. 7.22 shows the computed stabilising rolling torque variation with angle of roll. Increase of

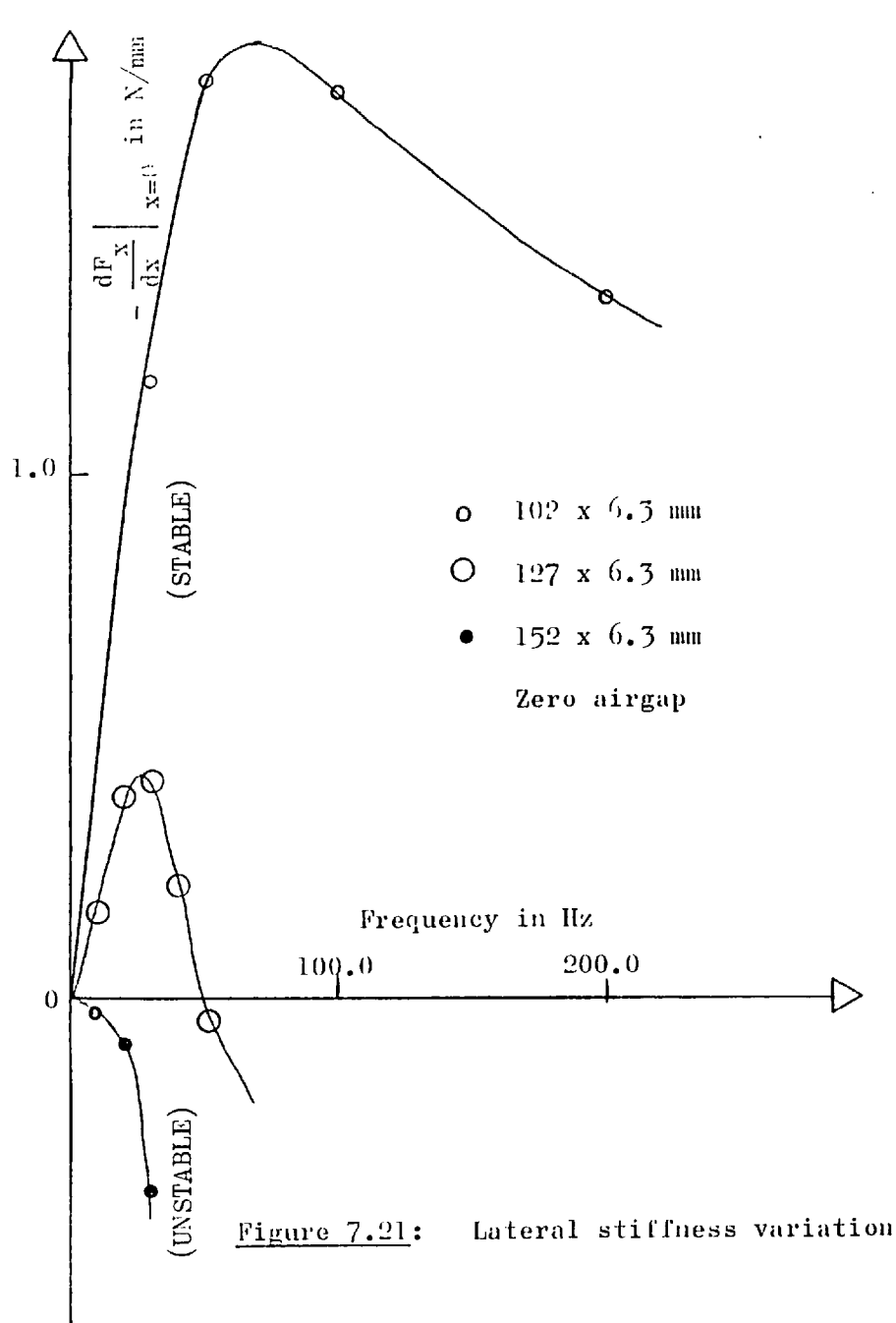


Figure 7.21: Lateral stiffness variation

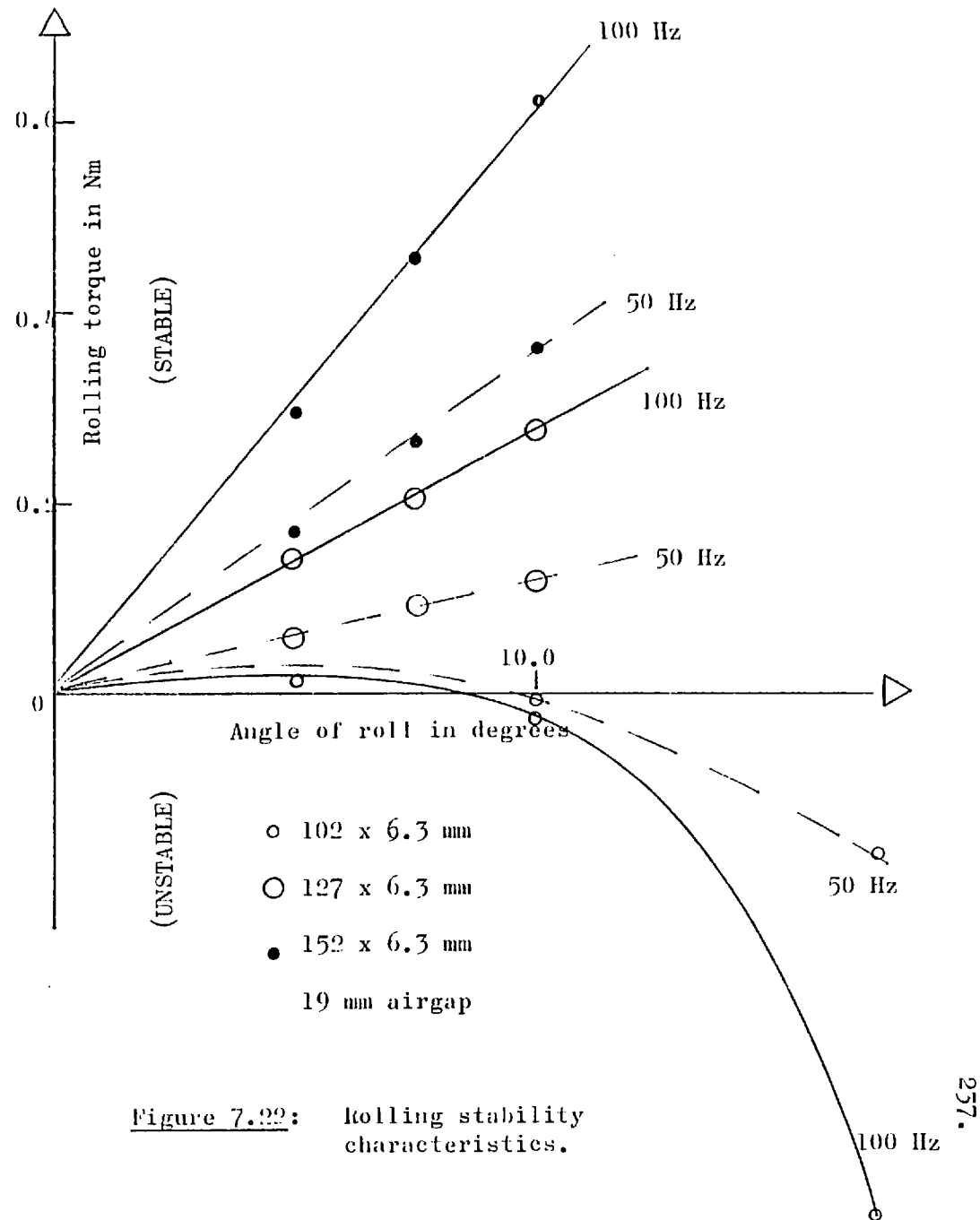


Figure 7.22: Rolling stability characteristics.

frequency, while causing lateral instability for wider plates, however is seen to increase their rolling stability. The computed stress distributions for two of the plates is shown in Fig. 7.23. The distributions show the presence of attractive forces in the narrow plate which is a contributory factor in its rolling instability.

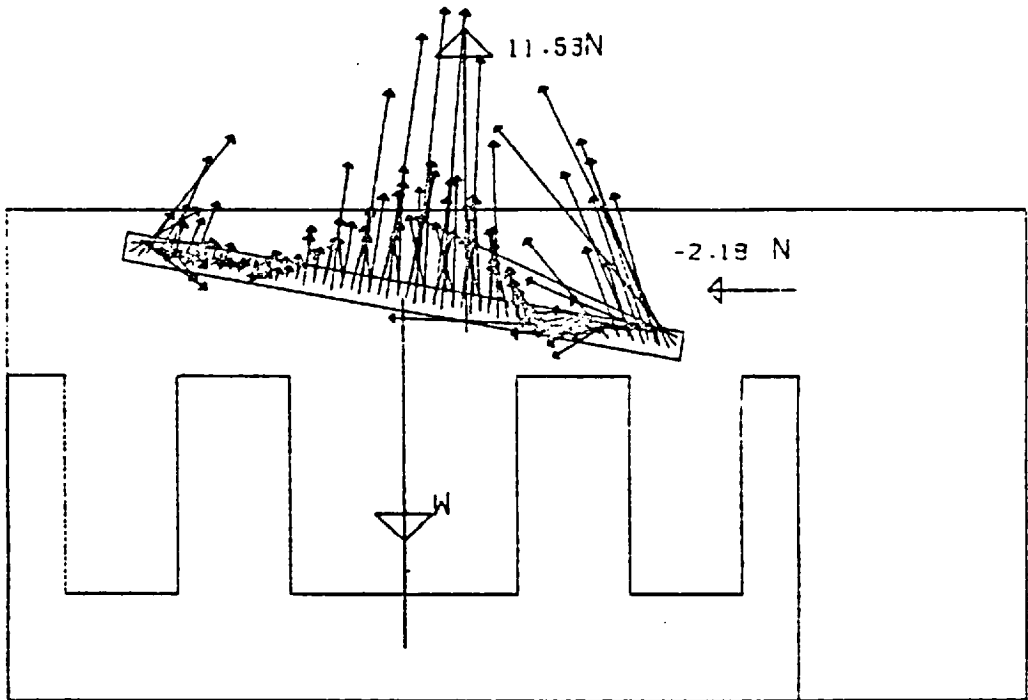
7.6 EFFECT OF CHANGES IN Xi-CORE GEOMETRY

The performance characteristics of the Xi-core have been examined so far in terms of the four limbed stator and secondary plates of rectangular cross-section. In this section the effect of changes in this geometry are examined. Such a study leads to numerous possibilities all of which could not be examined due to the lack of time. However, two important cases are considered, one being a change in the rotor geometry and the other in stator geometry.

7.6.1 A Change in Rotor Geometry

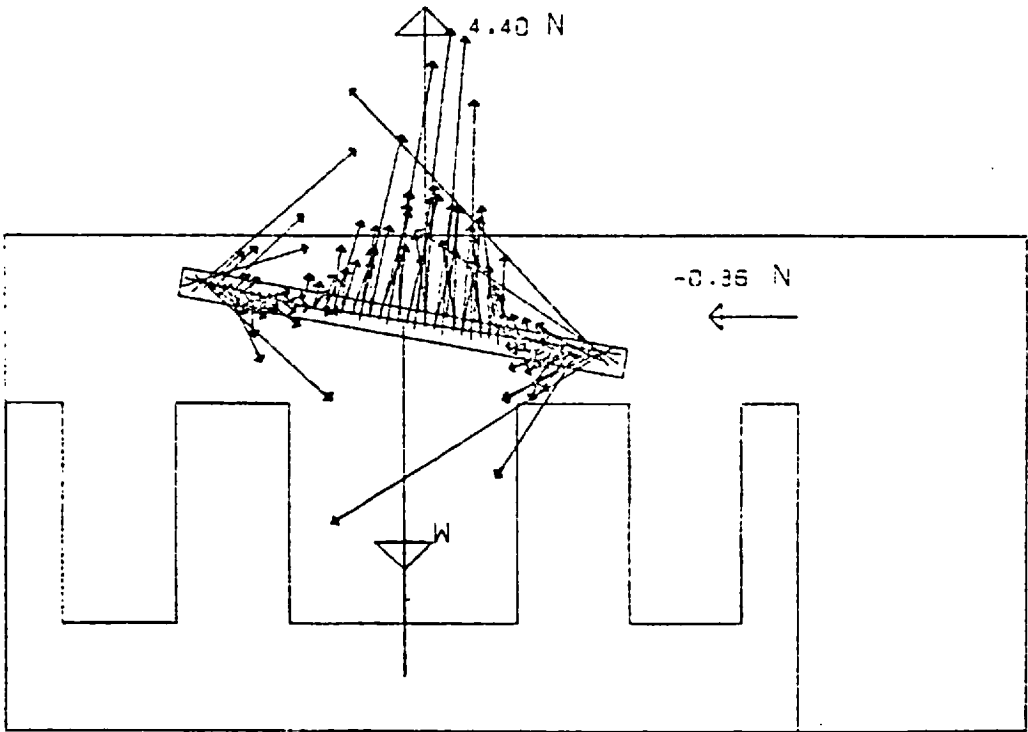
The fact that the narrow plate exhibits lateral stability under all conditions is by itself of little consequence because it has been shown that they possess little rolling stability and lift force. On the other hand, the wider plates show high rolling stability and lift forces, but they are unstable in lateral displacement. Therefore the answer is to combine the features of the narrow plate and the wider plate, in a single rotor. In practice this is achieved by rigidly clamping a narrow and a wide plate. In Chapter 2 it was experimentally shown that such an arrangement indeed provides high lateral stability. Using the finite element method, this arrangement (which will be referred to as double rotor hereafter in the text) was investigated theoretically. Since zero airgap produces the highest destabilising force for the wider plates, the

UXB TOTAL FORCES
FREQUENCY 50.00 HZ



(b)

UXB TOTAL FORCES
FREQUENCY 50.00 HZ



(a)

Figure 7.23: Computed stress distribution.

lateral force stiffness variation of the double rotor at zero airgap with frequency was computed and is shown in Fig. 7.24. The changes taking place in the force magnitudes at 50 Hz are shown alongside in Fig. 7.25. The lateral stiffness variation with frequency reveals that, while the conventional rotor geometry is unstable at all frequencies, the double rotor is stable from zero frequency up to 65 Hz. The changes in force magnitudes at 50 Hz shown, reveal that the loss in normal force is only 26%, while the change in lateral force is as high as 122%, taking the direction reversal into account. The removal of the two flaps of the double rotor, while increasing the stabilising force tenfold, reduces the lift force by more than half.

The effect of a finite airgap is considered next. At 50 Hz, the complete lateral force/lateral displacement curves were computed for the double rotor and a single rotor. This is shown in Fig. 7.26, the airgap being 19 mm. The curves provide further evidence that the double rotor provides high lateral stability even when the plate width approaches the width of the stator. For one of the displacements of the curves of Fig. 7.26, the stress distributions were obtained and shown in Fig. 7.27. The distributions reveal that the changeover from instability to stability is brought about by the large changes in direction of the forces at the leading edge.

To examine the stability in roll of the double rotor, the double rotor was inclined at 7.50 degrees at 19 mm airgap, and the field computed. The rolling torque was computed before and after removing material from the edges of the wider plate, which leads to the double rotor configuration. The values obtained are:

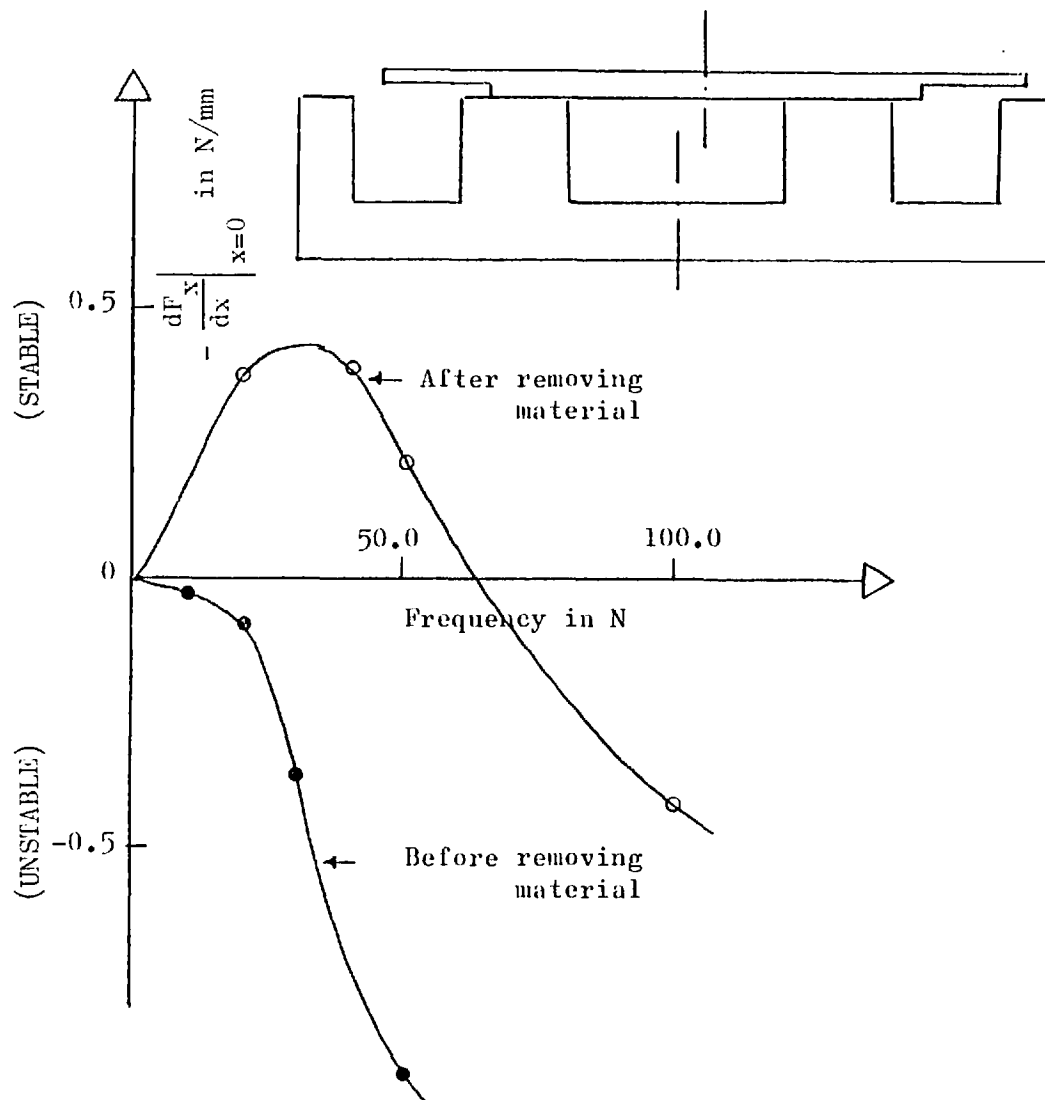


Figure 7.24: Lateral stiffness of the double rotor.

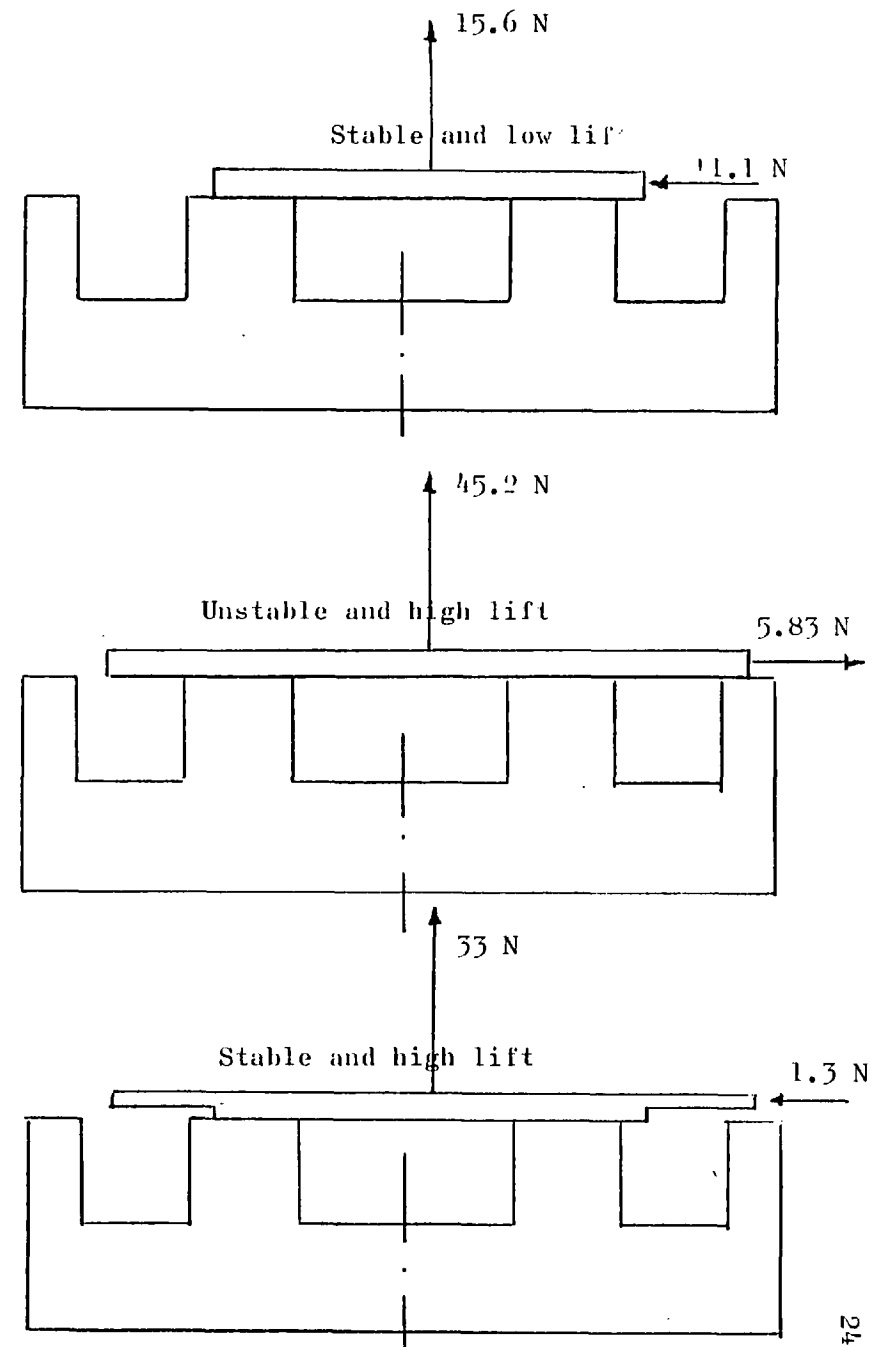


Figure 7.25: Changes in forces when rotor geometry is altered.

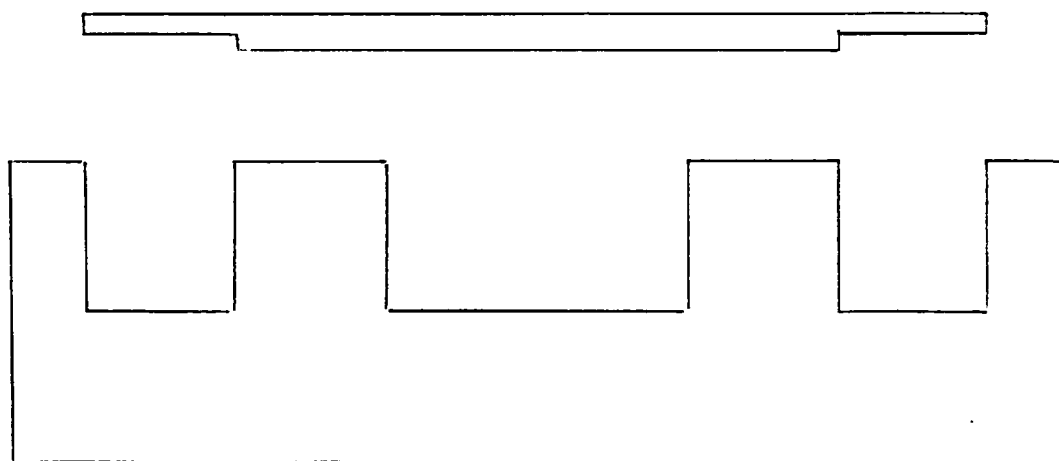
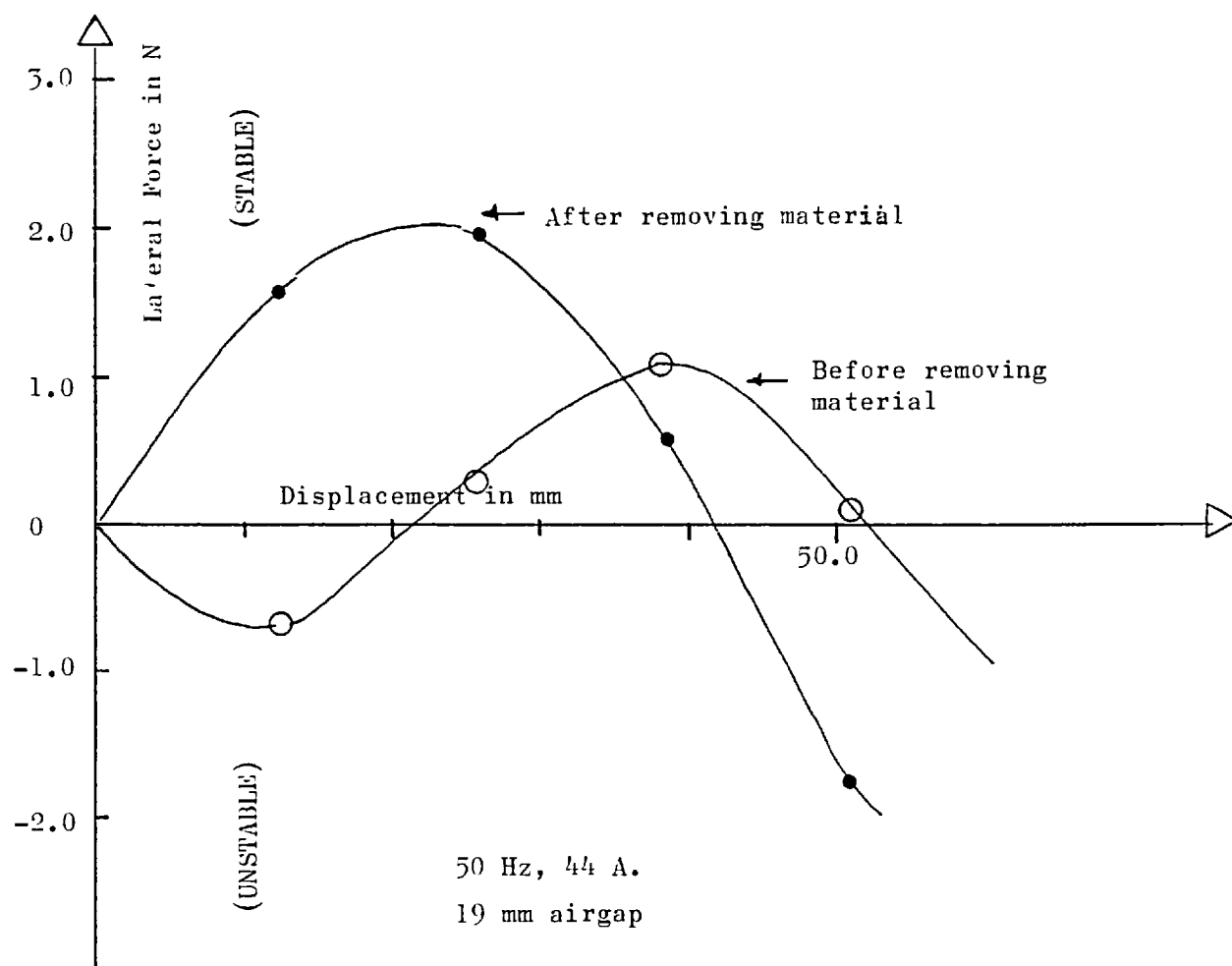
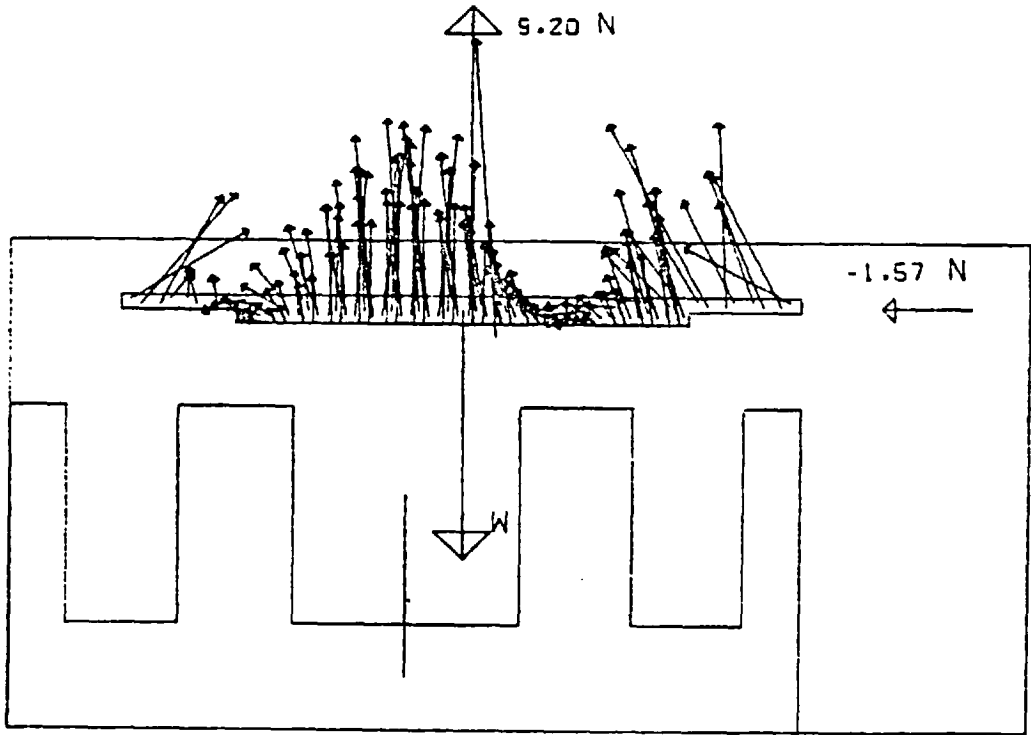


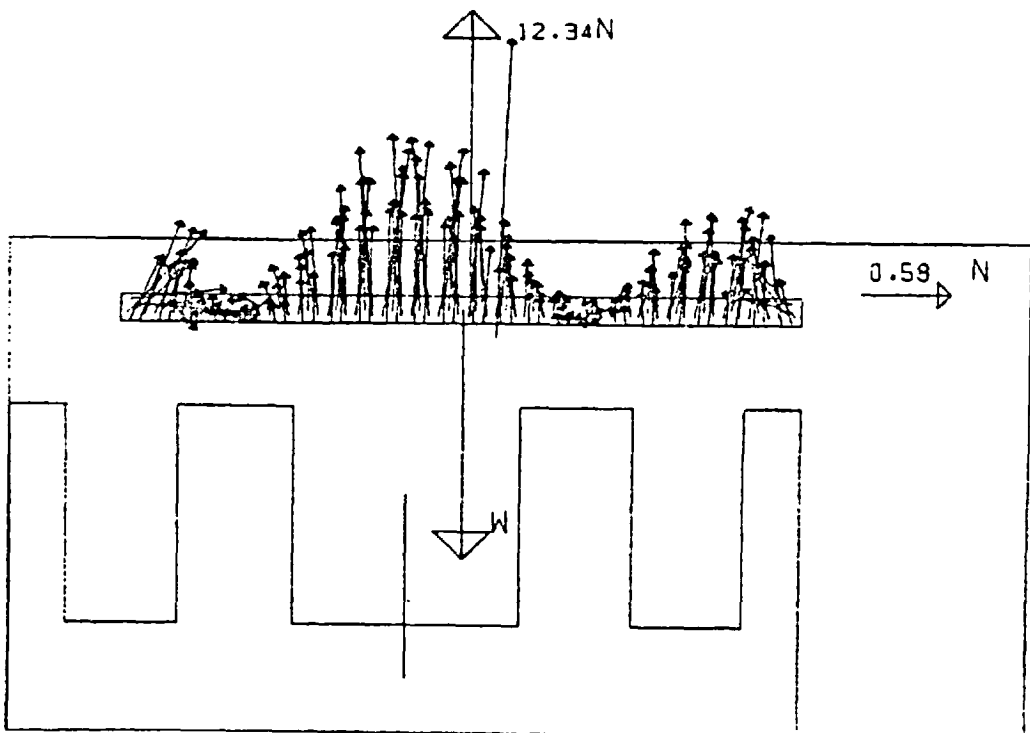
Figure 7.26: Lateral stability characteristics of the double rotor geometry.

UXB TOTAL FORCES
FREQUENCY 50.00 HZ



(b)

UXB TOTAL FORCES
FREQUENCY 50.00 HZ



(a)

Figure 7.27: The computed stress distribution.

	Single Rotor	Double Rotor
Rolling Torque per degree of Roll	100% (stable)	46% (stable)

TABLE 7.2

The values show that the double rotor has only about half the rolling stability of the single rotor. We thus see that the double rotor geometry produces larger stabilising lateral force, with reduced but nevertheless stabilising rolling torque.

7.6.2 A Change in Stator Geometry

As mentioned earlier, while many changes are possible, the investigation here will be confined to one change only. This change is the removal of the outermost stator teeth from the Xi-core geometry. When this change is made, the stator assumes the U-core geometry, and therefore such an investigation also becomes a comparison of the Xi-core with U-core. The two machines have already been compared on O.C. using the finite difference method in Chapter 5. For the study of the two machines with a rotor present, the finite element models were set up as follows. A model was set up to represent the Xi-core first. To obtain the U-core, the material properties of the elements in the outermost limbs of the Xi-core model were made equal to that of air. In this way, the uncertainties such as whether one machine is better modelled than the other are eliminated.

Considering the symmetrical plates first, two cases were investigated, with three plates at zero airgap and at a finite non-zero airgap. From the computed field solution, the power factor variations

with plate width was evaluated and is shown in Fig. 7.28. The normal force computed is marked alongside each of the computed points in the curves of Fig. 7.28. It is seen that the U-core power factor is always higher than that of the Xi-core, its (U-core) lift force being slightly smaller. Considering the widest plate of the three, at the non-zero airgap, the U-core power factor is 128% of that of the Xi-core but with a lift force of 85% of that of the Xi-core. These figures for comparison were obtained under constant current excitation for the two machines. The input impedance computed showed that the Xi-core input impedance is 1.18 times that of the U-core. Therefore on a constant voltage supply (neglecting primary resistance), if the excitation current of the Xi-core is 44 A giving a lift force of 13.4 N, on the same supply voltage the U-core current is 52 A, which gives a lift force of 15.8 N. The power factor magnitudes, however, remain the same as on constant current supply. Thus it is seen that on constant voltage operation, the U-core has 28% higher p.f. than the Xi-core as well as a 18% higher lift force.

The case of asymmetrical plates is now considered. The lateral force/lateral displacement curves were computed for one plate, for the two machines. These are shown in Fig. 7.29. The most striking feature of these curves is that the removal of the outermost iron limbs of the Xi-core leads to vastly increased lateral stability even for a plate which is 1.5 times the width of the U-core at 50 Hz. Besides, the maximum permissible lateral displacement for stable restoration of the plate is as large as 25% of the width of the plate in Fig. 7.8. In the case of the Xi-core it has been shown both experimentally and theoretically that all the lateral force/lateral displacement curves have a cross-over point at a displacement

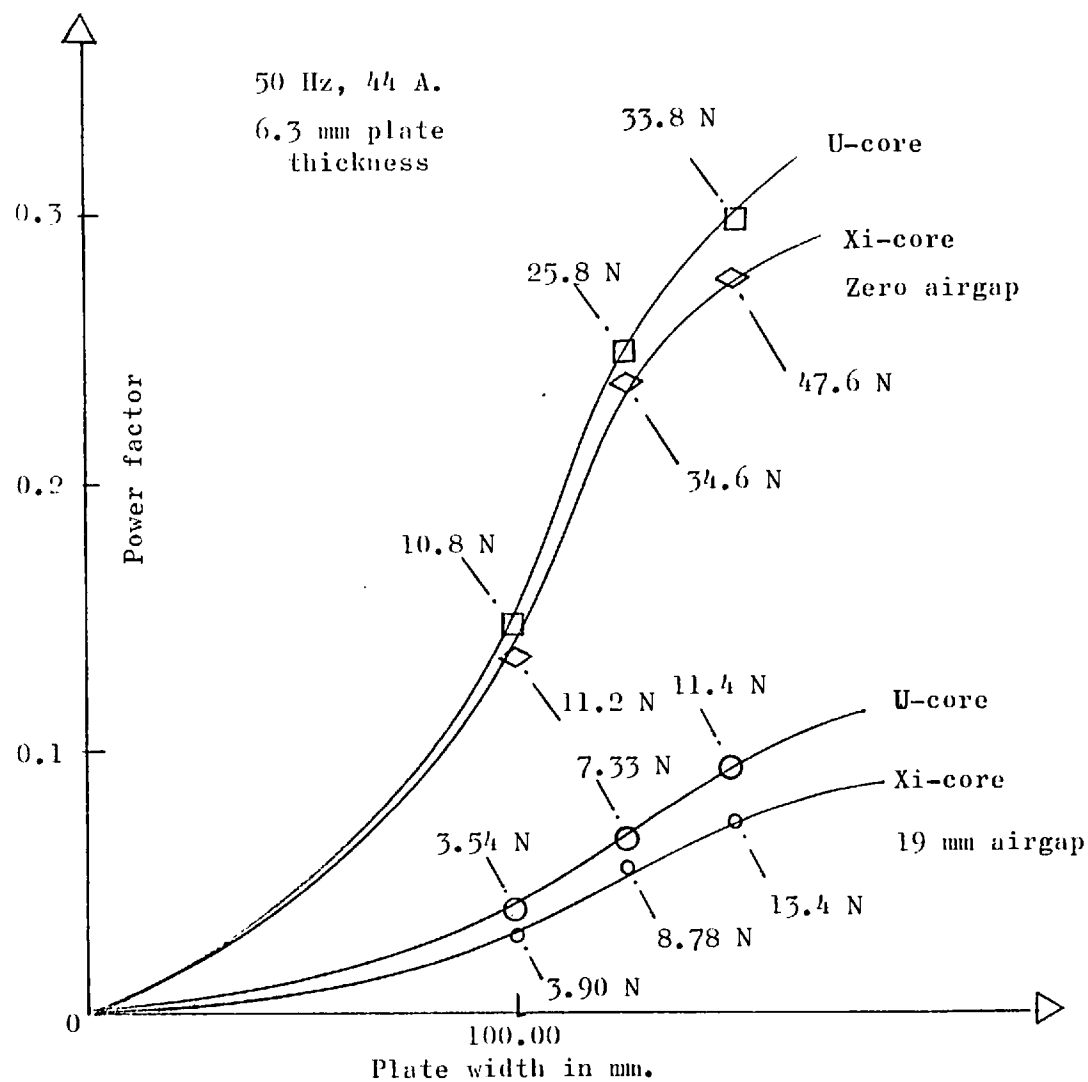


Figure 7.28: Comparison of the p.f. at the Xi-core and the U-core.

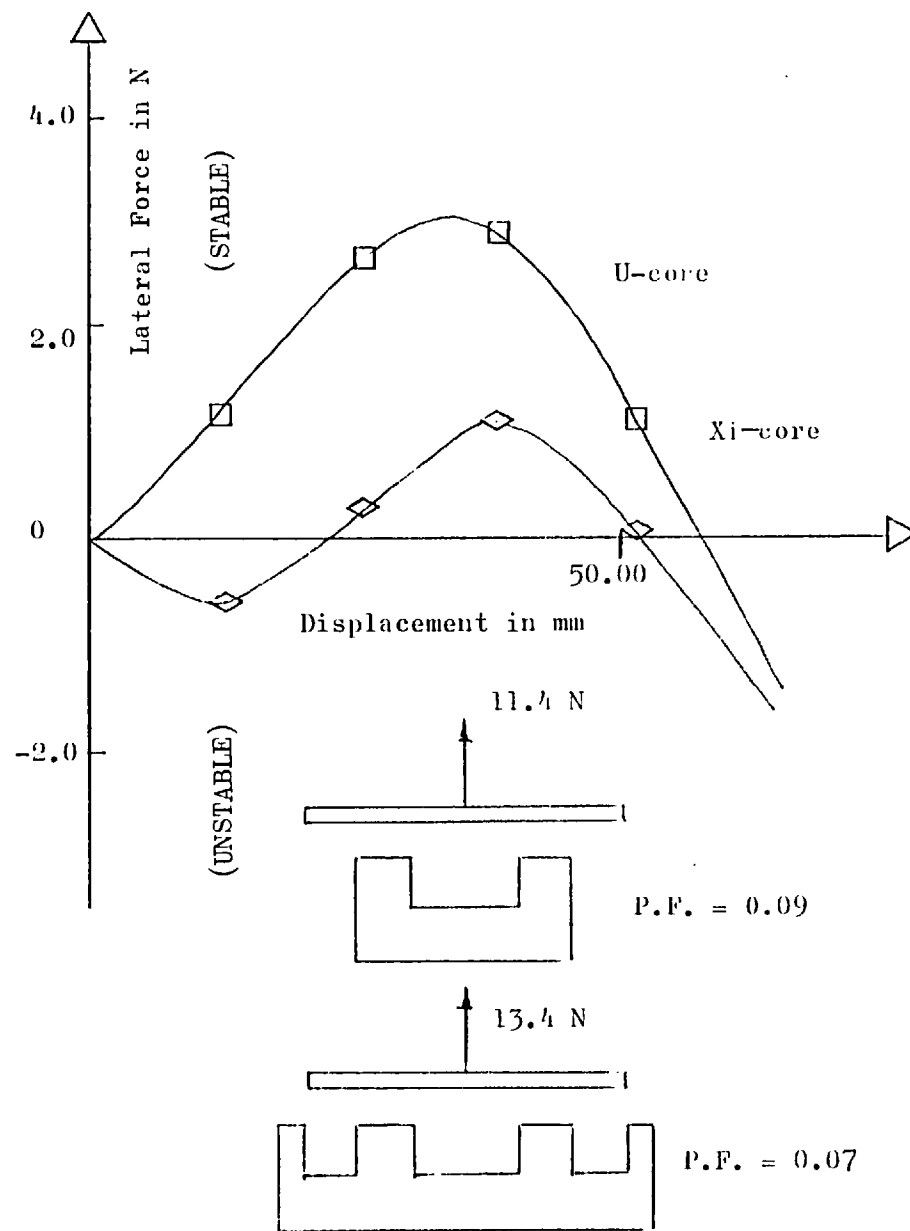


Figure 7.29: Comparison of the lateral stability of the Xi-core and the U-core.

of approximately 0.55 times their width. Furthermore, if a plate is stable, between zero displacement and the cross-over point referred to, the curves are nearly sinusoidal. Hence for stable plates, in the case of the Xi-core the maximum stable displacement is only about 17% of their width. In order to show the changes taking place, the stress distributions for an asymmetrical plate before and after removal of the outer limbs of the Xi-core are shown in Fig. 7.30. The most noticeable change is the appearance of the attractive forces in the edges of the plate after removal of the iron limbs.

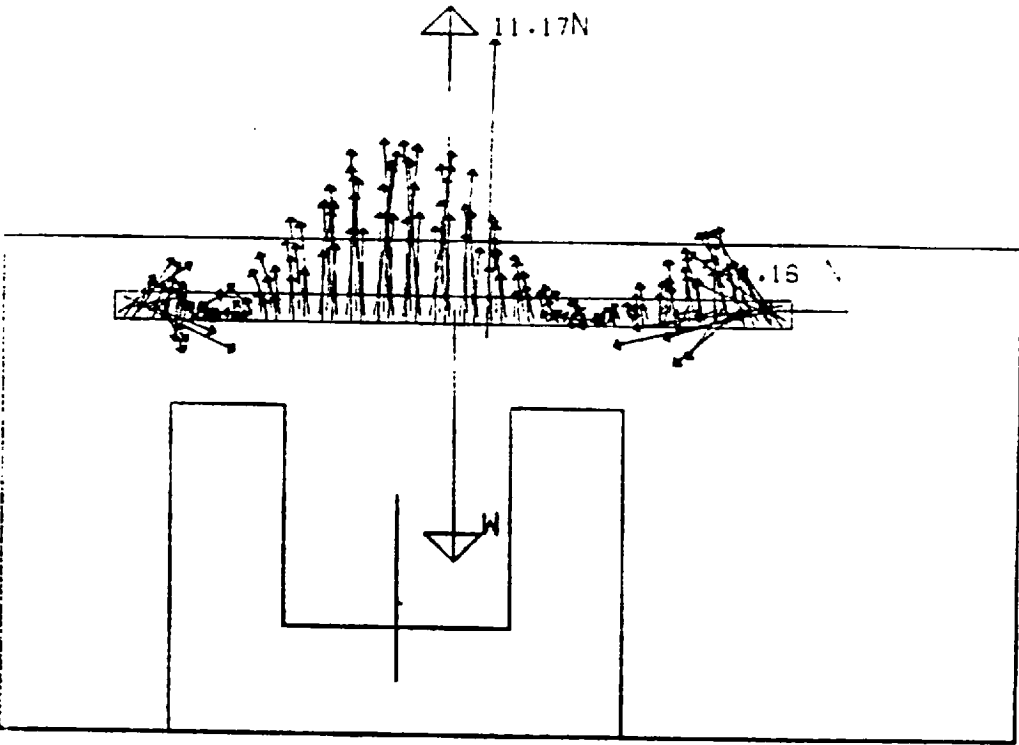
Though the U-core appears to be superior to the Xi-core from the investigations so far, the rolling stability has not been taken into account. Therefore the field of an inclined plate (the same plate as in Fig. 7.30) was computed. The magnitude of the rolling torque obtained is:

	Xi-Core	U-Core
Rolling Torque per degree of Roll	100% (stable)	41% (stable)

TABLE 7.3

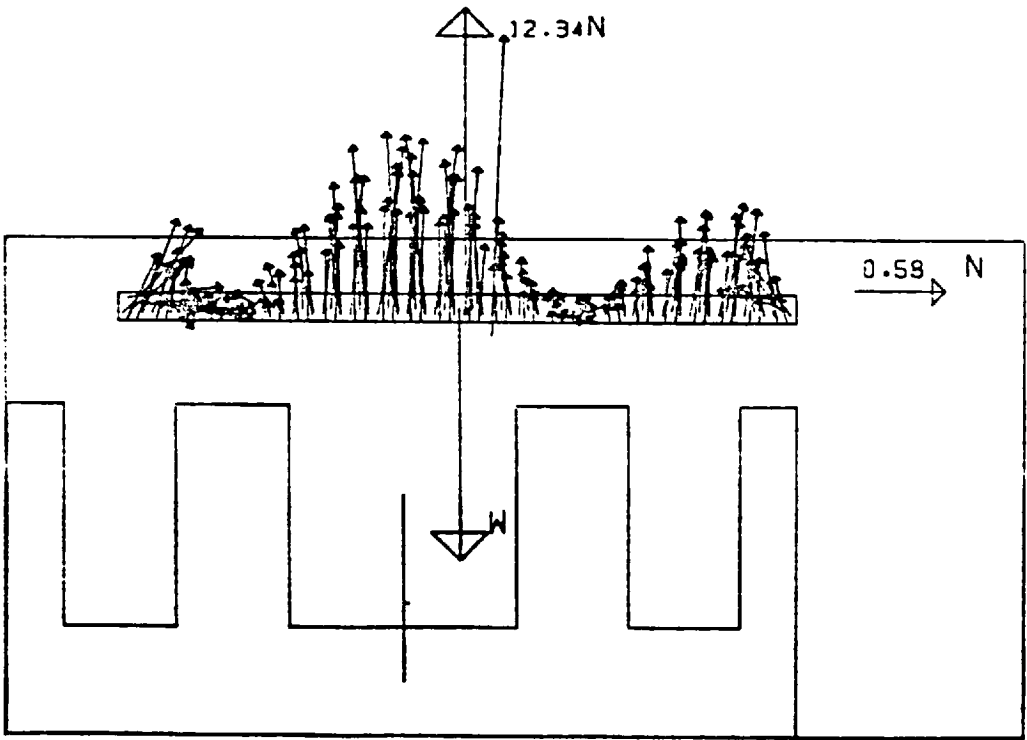
The figures show that the removal of the iron limbs of the Xi-core reduces the rolling stability to the same extent as does the double rotor without the removal of the iron limbs. But it is worth noting that the rolling torque is stabilising nevertheless. However, if a catamaran arrangement is to be used, then of course the rolling torque becomes a less significant factor.

JXB STRESS DISTRIBUTION
JXB TOTAL FORCES
FREQUENCY 50.00 HZ



(b)

JXB STRESS DISTRIBUTION
JXB TOTAL FORCES
FREQUENCY 50.00 HZ



(a)

Figure 7.50: Computed stress distributions.

7.7 CONCLUSION

In the conclusion of the chapter dealing with the finite difference analysis, the need for a cheaper numerical method was stressed. The extent to which the finite element method fulfils this requirement is illustrated in Fig. 7.31. This shows the lateral force/displacement curves obtained by the two numerical methods along with the experimental values. The curves show the closer agreement of the finite element analysis with experiment. Besides the finite difference model used had a regularly spaced node totalling 1664, while the finite element model consisted of 396 nodes and 759 elements. Expressing the computation times for a typical displacement as percentages, if the finite difference method requires 100 units, then the finite element method requires 45 units. The methods of solution of the equations, however, are different in the two cases. The finite difference equations were solved iteratively, while the finite element equations were solved by a direct method. Nevertheless, with the quoted computation times as a criterion of cost, the finite element method fulfils the original purpose, which was to seek a cheaper numerical method.

Another positive advantage of the finite element method is its versatility in assisting the researcher in an initial exploration. With reference to the Xi-core, for example, if it is required to find whether a plate of certain dimensions at a certain airgap, under a certain supply conditions, is stable or not, one can set up a model with only a relatively few elements and obtain the answer. If it is desired to explore the model thus obtained in more detail, one can refine the model (with the model-setting

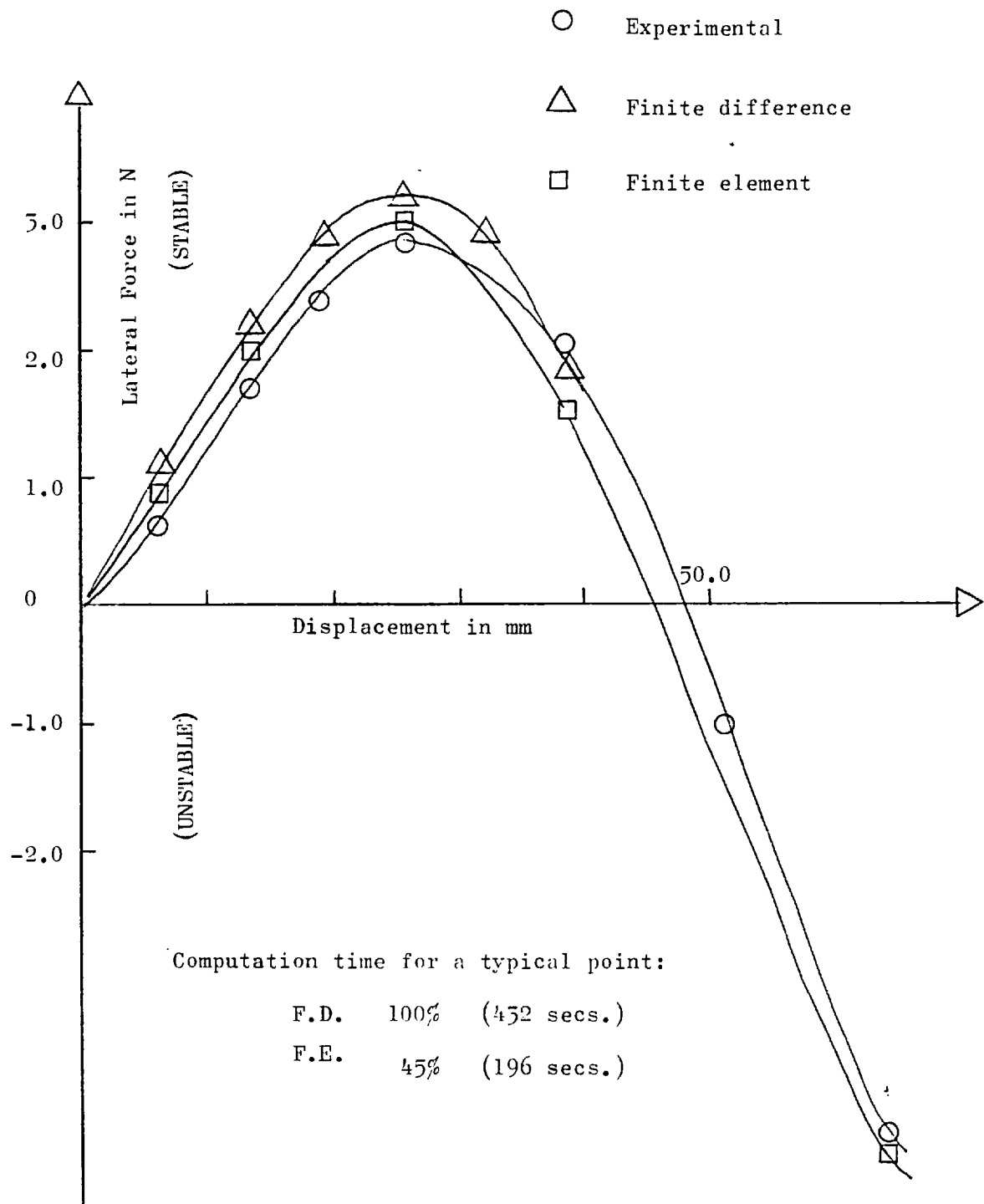


Figure 7.51: Comparison of the finite difference and finite element method.

computer package referred to), using the already obtained field solution as a guide to decide where a large number of elements should be placed.

Focussing attention on the results obtained, the important fact emerging is that, for the arrangement examined, the presence of the outermost iron limbs in the Xi-core leads to undesirable performance characteristics. For a constant voltage supply, the normal force magnitudes, lateral stability, and power factor have been seen to increase when these iron limbs are removed. The rolling stability is reduced by more than one half, but it still remains stabilising. Further work is necessary to find out if this is generally true for all Xi-core models and for three-phase as well as single-phase machines.

CHAPTER 8

DISCUSSION, FURTHER WORK AND CONCLUSIONS

8.1 THE PERFORMANCE OF THE Xi-CORE

The single-phase Xi-core, with airbacked rotors, has been seen to be less promising in its performance than would have been expected. The kVA requirement in particular, is of such magnitude that reactive power compensation is essential in any practical full scale machine. Another notable drawback is the impossibility of stably levitating plates that are wider than the stator. This, however, is a vital requirement in any practical machine, in order to provide low resistance return paths for the induced rotor currents as in the extended rotor end rings of the rotary squirrel cage machine. Considering rotor plates that are stable laterally, the maximum permissible displacement has been seen to be approximately 0.17 times their plate width, for the particular Xi-core investigated. The rolling stability has been shown to be high for all plate widths except the narrowest. A method of levitating plates approaching the stator widths was shown to be possible, by the removal of rotor material from the edges of the wider plate. However, the price to be paid is a loss of rolling stability, but this is of no consequence if a catamaran type arrangement is envisaged.

8.1.1 The Effect of the Goodness Factor

A method of measuring the Goodness Factor (G) of the Xi-core, which is also applicable to the other two types of T.F.M., has been presented. This enabled the experimental examination of the

lateral stability in terms of G . In the case of the Xi-core, it has shown that a plate in stable levitation tends to become unstable if any change in the airgap or supply frequency occurs so as to increase the prevailing value of G . The simplified lumped circuit models have shown that the manner of arranging the primary excitation in the stator has a pronounced effect on the value of G . For the arrangement corresponding to the Xi-core, G must necessarily be low, since X_s and X_{34} ($G = \frac{X_s + X_{34}}{R_2}$) are opposite in sign in equation (3.11c) of Chapter 3. In theory therefore, a Xi-core and a U-core can be designed to have an extremely low value of G at a high frequency, if the design makes $|X_s|$ approach $|X_{34}|$. For the E-core machine, however, the value of G ($G = \frac{X_2}{R_2}$; equation (3.5d) of Chapter 3) can approach zero only at zero frequency. If the excitation of the Xi-core and U-core machines is re-arranged so that currents flow in opposite directions in the centre slot, X_s and X_{34} are always of the same sign and zero G is possible only at zero frequency as in the E-core. Furthermore, at the same frequency this arrangement, which is unstable, has a higher G than the Xi-core. We see that the Xi-core and U-core are inherently low Goodness machines. The simplified analysis also showed that a necessary but not sufficient condition for the E-core to be stable is $G > 1$. The finite element solution of the E-core given in Appendix A confirms this. Therefore we thus see that increasing the Goodness of the E-core brings it within the "Magnetic River" class, while increasing the Goodness of the Xi-core removes it from it. This casts a serious doubt on the hypothesis that T.F.M. geometries with even numbers of stator limbs are the only machines falling into the "Magnetic River" class.

8.1.2 The Merits of the Finite Difference Method

The finite difference method of analysis suffers from two major drawbacks. Firstly, it can model only machines with simple rectangular geometries. This, however, is not a major problem in the analysis of linear machines. Secondly, long expensive computation times are involved in the solution of equations by iteration. This, the author believes, is a problem of the present since with the advent of mini- and micro-computers, the computation time may not be the first criterion of cost in the future. The attractiveness of the finite difference method, in contrast to the finite element method, is the ease of handling the input data defining the model. Therefore the finite difference method may still be popular in the future.

8.1.3 The Merits of the Finite Element Method

The finite element method emerges superior to the finite difference method when the accuracy and computation costs are compared. The comparison of the computation time is, strictly speaking, not valid, because the two methods require different computer software and methods of solution of equations. However, if the large number of equations resulting from finite difference analysis is also solved by direct methods, it is likely that the computation time will be many times that of the finite element equations for the same accuracy. The finite element method can readily accept complicated geometries in all cases and magnetic non-linearities in magnetostatic cases. To the author's best knowledge no-one has yet succeeded in solving the saturated eddy current problem. This is due to the fact that one can no longer

replace the time derivative in the diffusion equation by a single frequency $j\omega$ term. With regard to data generation, although it was not a problem in this research due to the model setting package referred to in Chapter 7, it remains a major obstacle for many research workers. The "ballooning" technique used in this research is believed to be the first practical application in the study of new machines, although van Rij and Hofstra^{8.1} have also reported the implementation of the same technique on a digital computer. The saving in computer storage due to this technique is considerable, which enables the use of large number of elements in the regions of interest leading to field solutions of greater accuracy. Therefore this technique, particularly the automatic generation version of it, is likely to become a popular and powerful aid in the analysis of field problems with open boundaries.

8.1.4 The Stator Geometry of the Single-Phase Xi-Core

The effect of the removal of the outermost iron limbs of the Xi-core was investigated with and without secondary plates. This investigation therefore also becomes a comparison of the Xi-core and U-core. The numerical analysis has shown that the removal of the outermost iron limbs of the Xi-core leads to higher power factor, higher lift forces and stronger lateral stability on constant voltage operation. In addition, plates that are wider than the stator have been shown to be stable. The only loss in the desirable properties is that in rolling stability. This, however, is unimportant if two machines are used in a catamaran arrangement. Thus the conclusion is that the original U-core T.F.M. is superior to the Xi-core in most respects.

8.2 FURTHER WORK

Due to the lack of time many promising aspects could not be pursued. The most immediate extension of the numerical analysis presented in this thesis will be the investigation of the damping of the single-phase machine. In order to investigate the damping in lateral movement while retaining the two-dimensional model the following steps are suggested:

- (a) Include a transverse velocity term in the two-dimensional diffusion equation (see Nasar and Elcid^{8.2}, who employ the finite difference method to analyse an A.F.M. with velocity included).
- (b) Obtain the lateral force/lateral displacement curves with zero and a non-zero velocity.
- (c) Obtain the difference between the two lateral forces at each displacement obtained thus, which represents the damping corresponding to the value of velocity used.
- (d) By repeating for several velocity values, the damping coefficient at each displacement may be estimated.

A further step is the inclusion of a travelling wave excitation in the two-dimensional model. However, as discussed by Carpenter^{8.3}, the magnetic vector potential is not suitable for this, and the future researcher needs to formulate the problem in $T-\Omega$ (see Wyatt^{8.4}, who analyses the E-core using $T-\Omega$) terms.

The circle diagram work as presented in this thesis will be of real value only when one is able to obtain the input impedance of the Xi-core using algebraic analysis. To achieve this, in the author's opinion, the most promising method is due to Hammond^{8.5}, who has presented a method based on variational principles but employing algebraic analysis. This method in its present form enables the calculation of input impedance of simple geometries. With further research it may become more general, enabling the derivation of the input impedance of complicated geometries such as that of the Xi-core.

8.3 CONCLUSIONS

The research presented in this thesis examined the single-phase Xi-core both experimentally and analytically using two different numerical methods. It is hoped that this thesis will prove a useful addition to the growing literature on T.F.M.s and numerical methods of analysing field problems.

THE LATERAL STABILITY OF THE E-CORE

In Chapter 3 it was shown by using a coil model that a necessary for ~~but~~ sufficient condition for the stability of the single-phase E-core is that $G > 1$. This implies that, at increasing frequency values the rotor plates may become stable. With a view to verifying this conclusion, the finite element solution of an E-core, using the infinite boundary, was obtained. The lateral force/lateral displacement characteristics at several frequencies are shown in Fig. A.1. These show that with increasing frequency (therefore increasing G) the rotor plate does become stable, and the permissible lateral displacement increases.

As a matter of interest, the stress distribution for one displacement at one frequency is included (Fig. A.2). The constructional details of the E-core machine are:

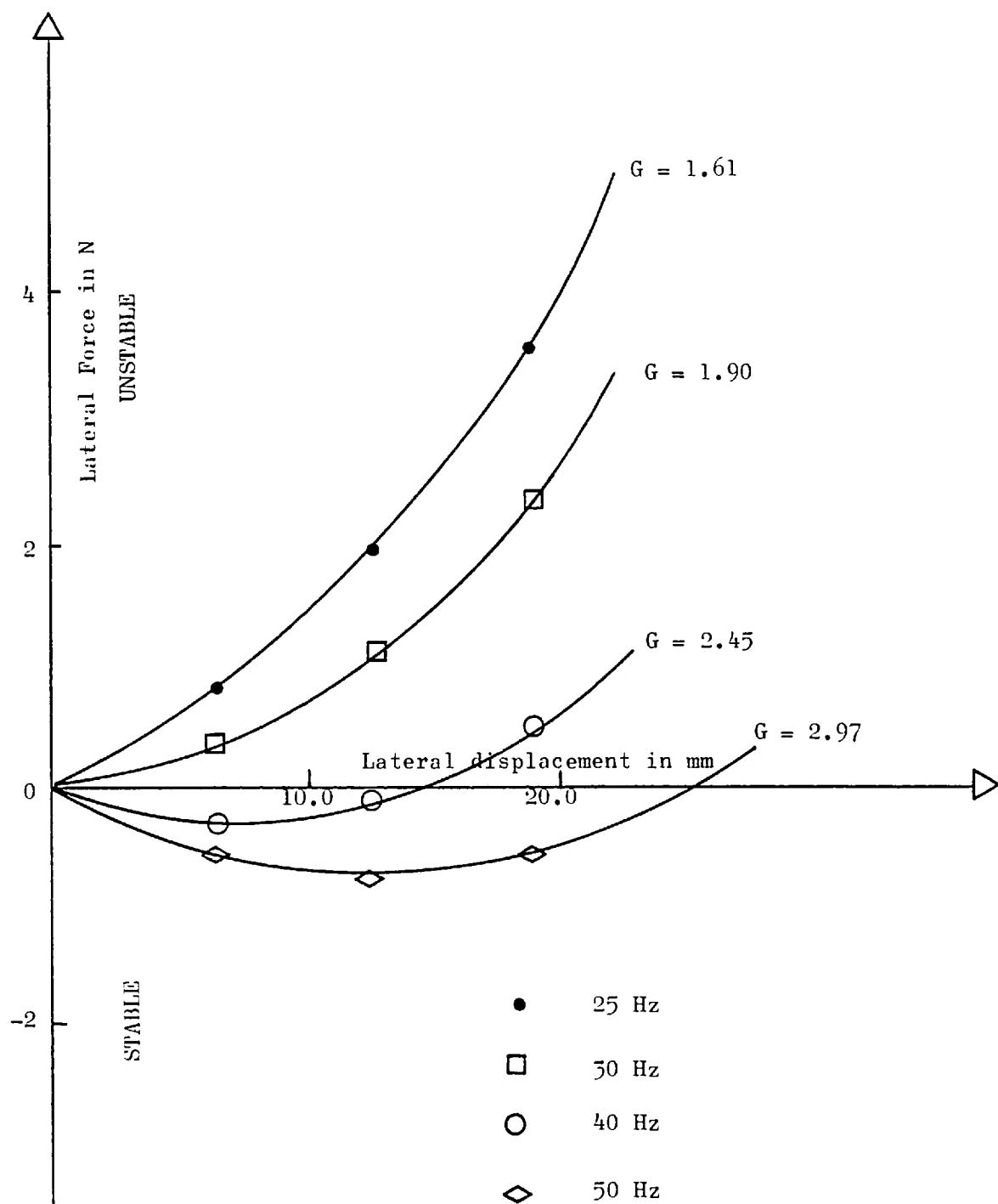
Width	0.267 m
Depth	0.089 m
Length	0.29 m

Two square slots of side 0.044 m are positioned symmetrically between the centre line and the edges. The coil positioned in the slots has a total of 80 turns which can be split into four equal coils.

Aluminium reaction plate of dimensions

0.267 m x 0.00635 m x 0.508 m

Airgap at 0.01 m



Constant current of 44 A

Figure A.1: The lateral stability characteristics of the E-core.

JXB STRESS DISTRIBUTION
JXB TOTAL FORCES
FREQUENCY 30.00 HZ

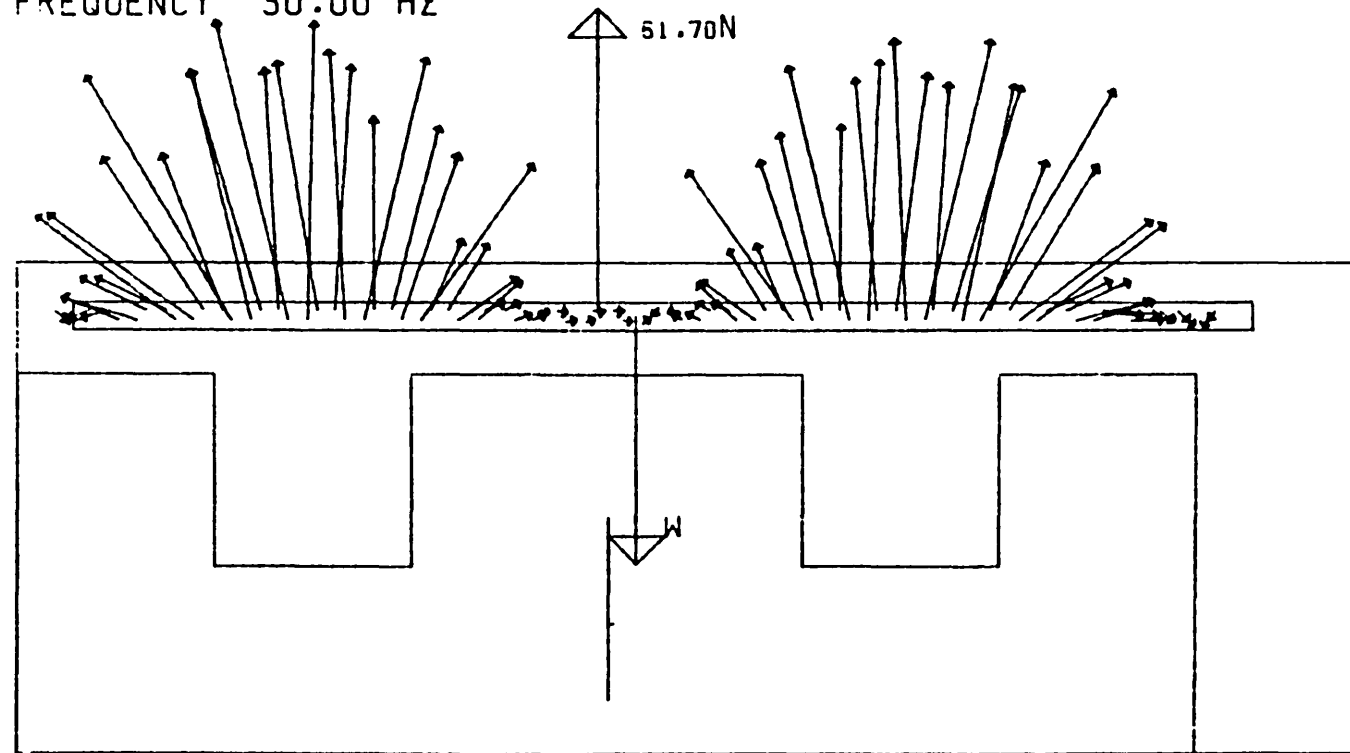


Figure A.2: The computed stress distribution in an asymmetrical E-core rotor.

APPENDIX BTHE CIRCLE DIAGRAMB.1 THEORY

From the lumped circuit model of the Xi-core introduced in Chapter 3, let us examine the nature of the impedance loci with frequency.

From equation (3.11a):

$$\frac{R_e - 2R_1}{X_o} = \frac{(X_{13} + X_{14})^2 \cdot R_s}{(X_1 + X_{12}) [R_s^2 + (X_s + X_{34})^2]}$$

Let $R' = \frac{R_e - 2R_1}{X_o}$

$$A_1 = \frac{(L_{13} + L_{14})^2}{(L_1 + L_{12})}$$

$$A_2 = (L_s + L_{34})^2$$

Let us assume the variation of the secondary resistance R_s with frequency to be given by:

$$R_s = k \omega^n, \quad 0 \leq n < 1$$

Then $R' = \frac{A_1 k}{(k^2 / \omega^{1-n}) + (\omega^{1-n} \cdot A_2)}$

As $\omega \rightarrow 0$, $R' \rightarrow 0$.

As $\omega \rightarrow \infty$, $R' \rightarrow 0$.

The frequency of $\omega_{\max}$ at which R' is maximum is the value of ω , when $\frac{dR'}{d\omega} = 0$:

Hence: $\omega_{\max} = \left[\frac{k}{(L_s + L_{34})} \right]^{\frac{1}{1-n}}$

And the maximum value of $R^1 = R_{\max}^1$, at this frequency is:

$$R_{\max}^1 = \frac{1}{2} \frac{(L_{13} + L_{14})^2}{(L_1 + L_{12})(L_s + L_{34})}$$

Consider now the equation (3.11b) of Chapter 3:

$$\frac{X_e}{X_o} = 1 - \frac{[X_{13} + X_{14}]^2 [X_s + X_{34}]}{(X_1 + X_{12}) [R_s^2 + (X_s + X_{34})^2]}$$

i.e. $X^1 = 1 - \frac{A_1 \sqrt{A_2}}{[k^2/\omega^{2(1-n)}] + [A_2]}$

As $\omega \rightarrow 0$, $X^1 \rightarrow 1$.

As $\omega \rightarrow \infty$, $X^1 \rightarrow (1 - \frac{A_1}{\sqrt{A_2}})$.

i.e. $X_{\infty}^1 = 1 - \frac{(L_{13} + L_{14})^2}{(L_1 + L_{12})(L_s + L_{34})}$

Therefore the locus of X^1 and R^1 with ω is a circle, as shown in Fig. B.1.

B.2 SOME QUALITATIVE DEDUCTIONS

A. The diameter of the circle is independent of the secondary resistance.

B. Since L_1 and L_{12} are independent of the movable coil height, the circle diameter is proportional to $\frac{(L_{13} + L_{14})^2}{(L_s + L_{34})}$. Therefore as the airgap increases, L_{13} and L_{14} diminish and tend to zero. But L_s and L_{34} remain finite, even at infinite airgap. Hence, as the airgap is increased, the circle

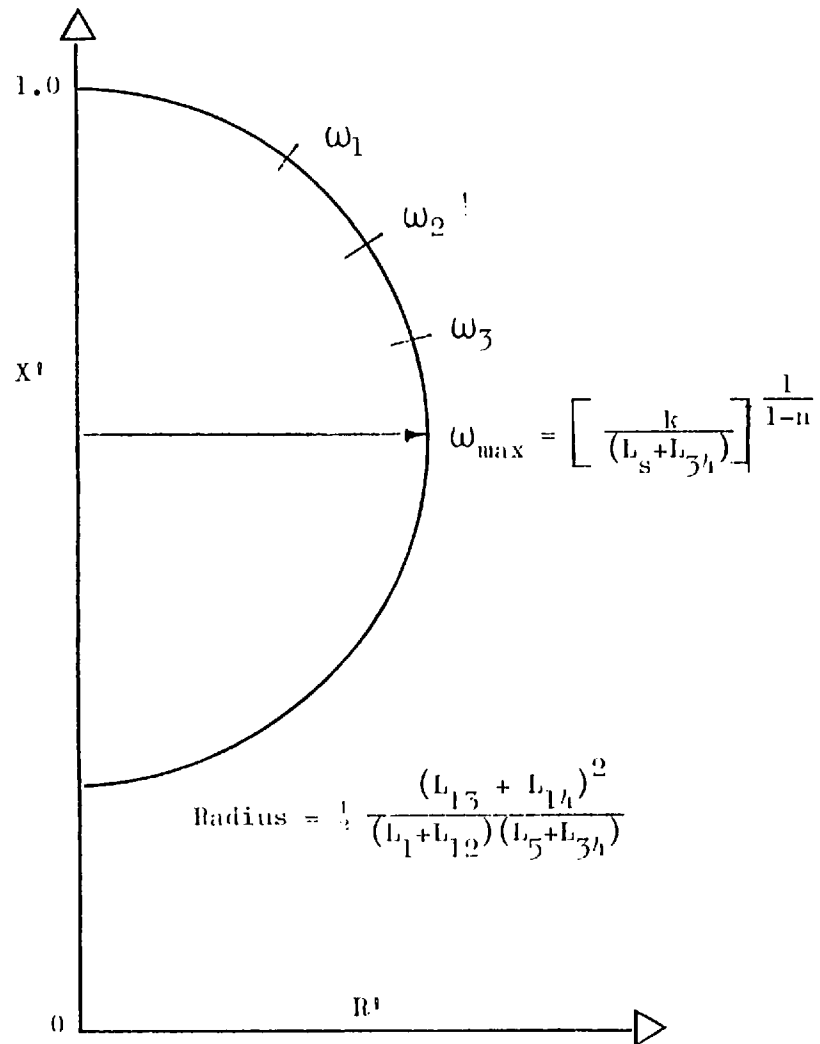


Figure B.1: The circular locus of X' and R' .

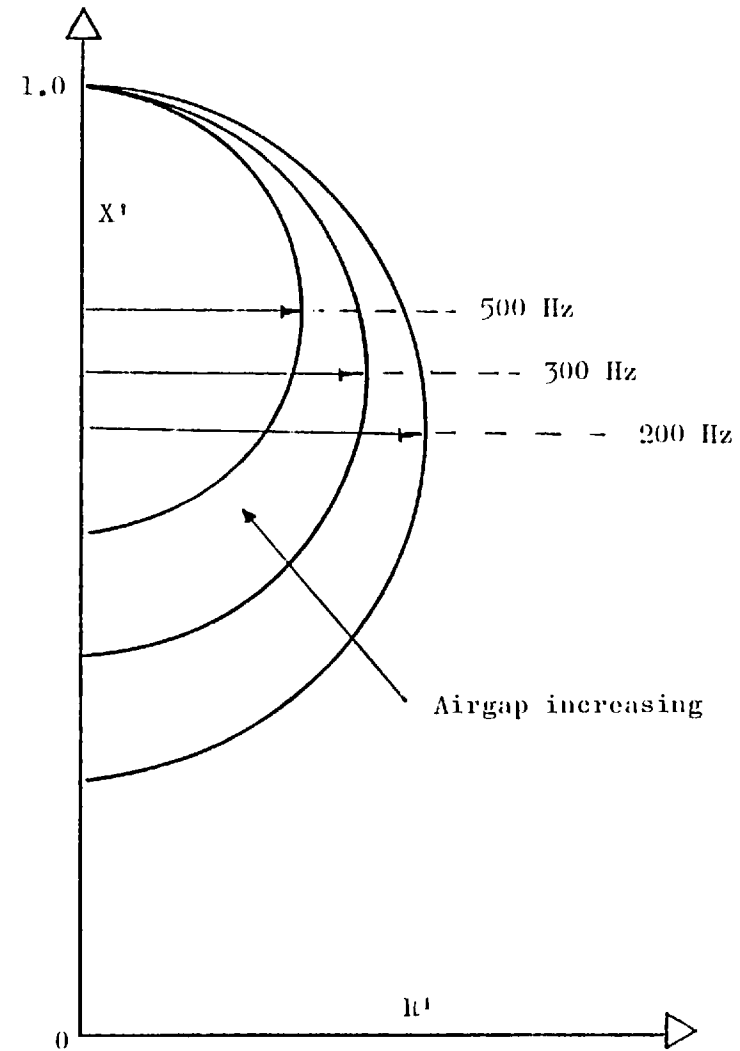


Figure B.2: Effect of airgap on the loci.

rapidly (since the numerator is squared) collapses into a point, i.e. a circle of zero radius.

C. The frequency at which the normalised resistance (ie. R^1) becomes maximum was shown to be:

$$\omega_{\max} = \left[\frac{k}{(L_s + L_{34})} \right]^{\frac{1}{1-n}}$$

As the airgap is increased, the secondary coils "see" less and less of the primary iron core. Therefore L_s and L_{34} diminish, but will attain a finite value. Hence $\omega_{\max}$ increases, with increasing airgap.

These deductions are illustrated in Fig. B.2.

B.3 EXPERIMENTAL VERIFICATIONS

Experiments were carried out, with a view to obtaining confirmation of the circular loci of the impedance variation with frequency. Although the theory developed is strictly valid only for one of the following cases of experimental investigations, the other cases are included since the results suggest that with further work the simple theory may be extended.

B.3.1 Single-Phase Xi-Core

Case 1: Symmetrical Rotor:

Fig. B.3 shows the experimentally obtained impedance loci for one symmetrical plate. The near-circular shape provides confirmation that the locus is a circle.

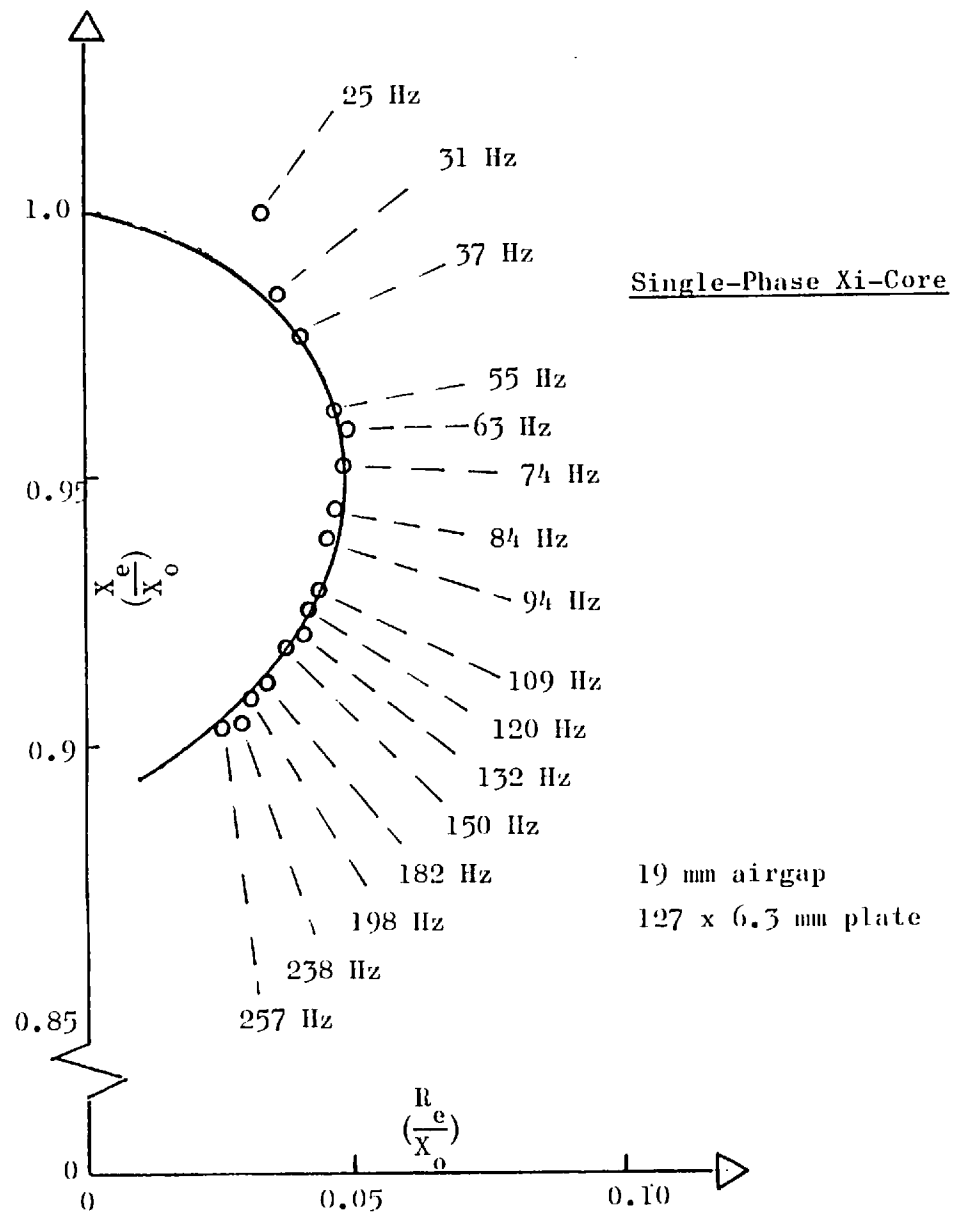


Figure B.3: The impedance loci of a symmetrical plate.

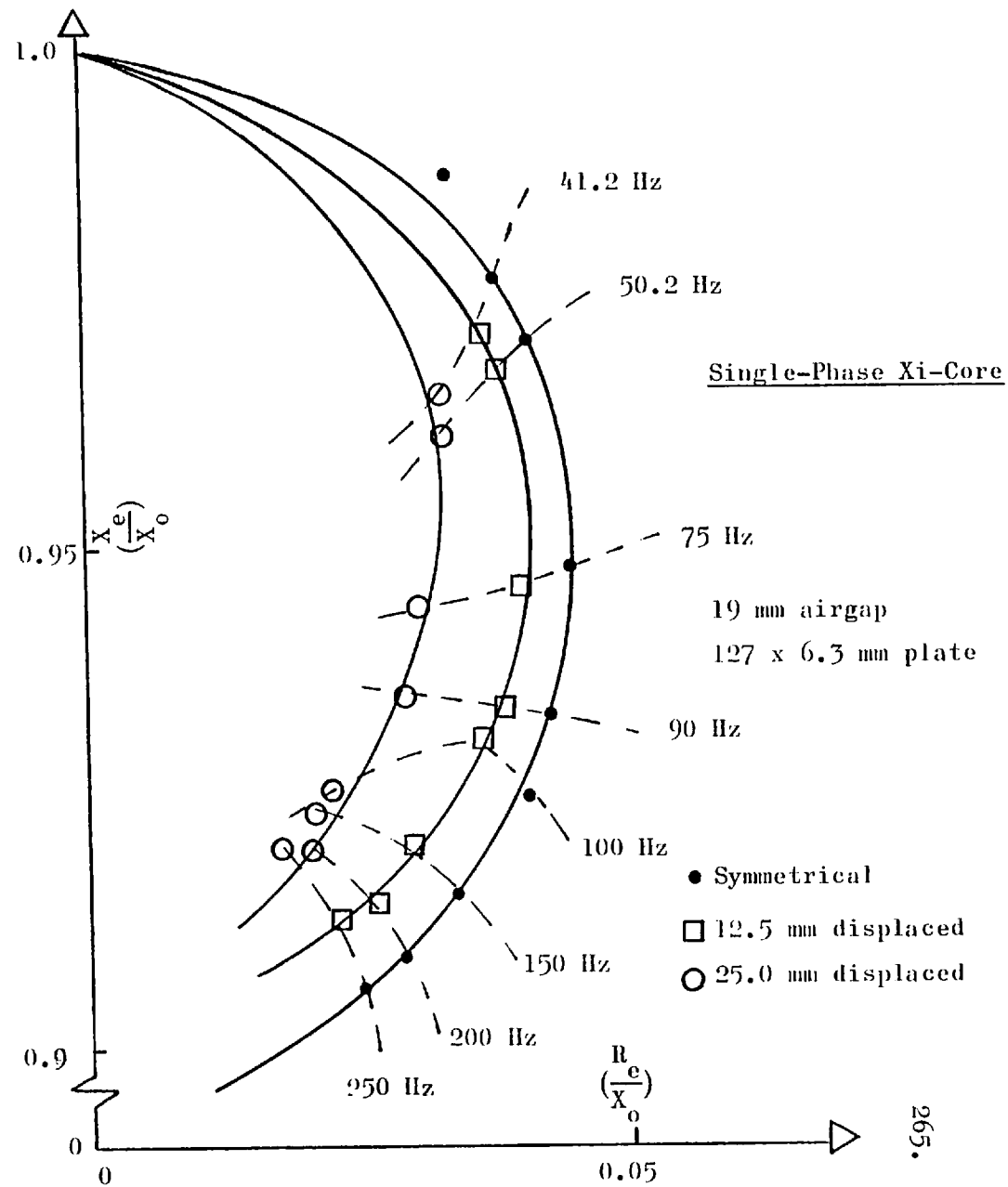


Figure B.4: The impedance loci of one plate at three displacements.

Case 2: Asymmetrical Rotor:

Fig. B.4 shows the experimentally obtained loci for one plate at a constant airgap at the symmetrical and two displaced positions. An interesting feature of the loci is that with increasing frequency the change of the symmetrical position from being one of maximum stored energy to minimum stored energy can be seen. Thus indirectly the circle diagram can be used to predict the lateral stability of rotor plates.

B.3.2 Three-Phase Xi-Core

Case 1: Air-backed Rotors:

The impedance loci of an air-backed rotor of the three-phase Xi-core were obtained experimentally and shown in Fig. B.5 for two airgap values. The variation shows a much more marked departure from a semi-circle than that observed for the single-phase machine.

Case 2: Iron-backed Rotors:

The experiments were repeated for an iron-backed rotor at several airgaps. The obtained impedance loci are shown in Fig. B.6. Again, the marked departure from a semi-circle shape can be observed.

B.4 CONCLUSIONS

Experimental results suggest that the impedance loci of the single-phase Xi-core is a semi-circle as predicted by the theory using simplified lumped circuit models.

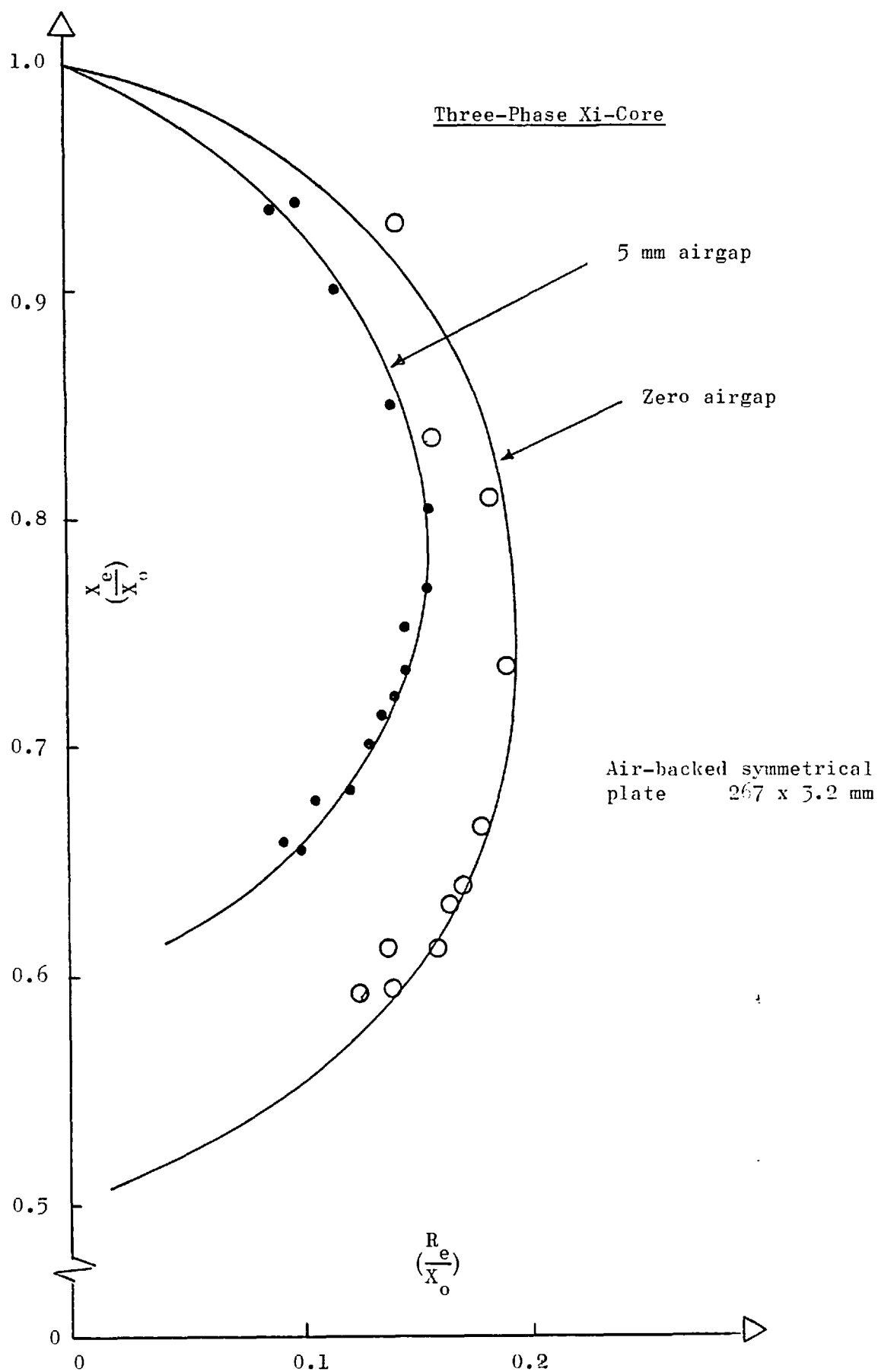


Figure B.5: The impedance loci of the three-phase Xi-core.

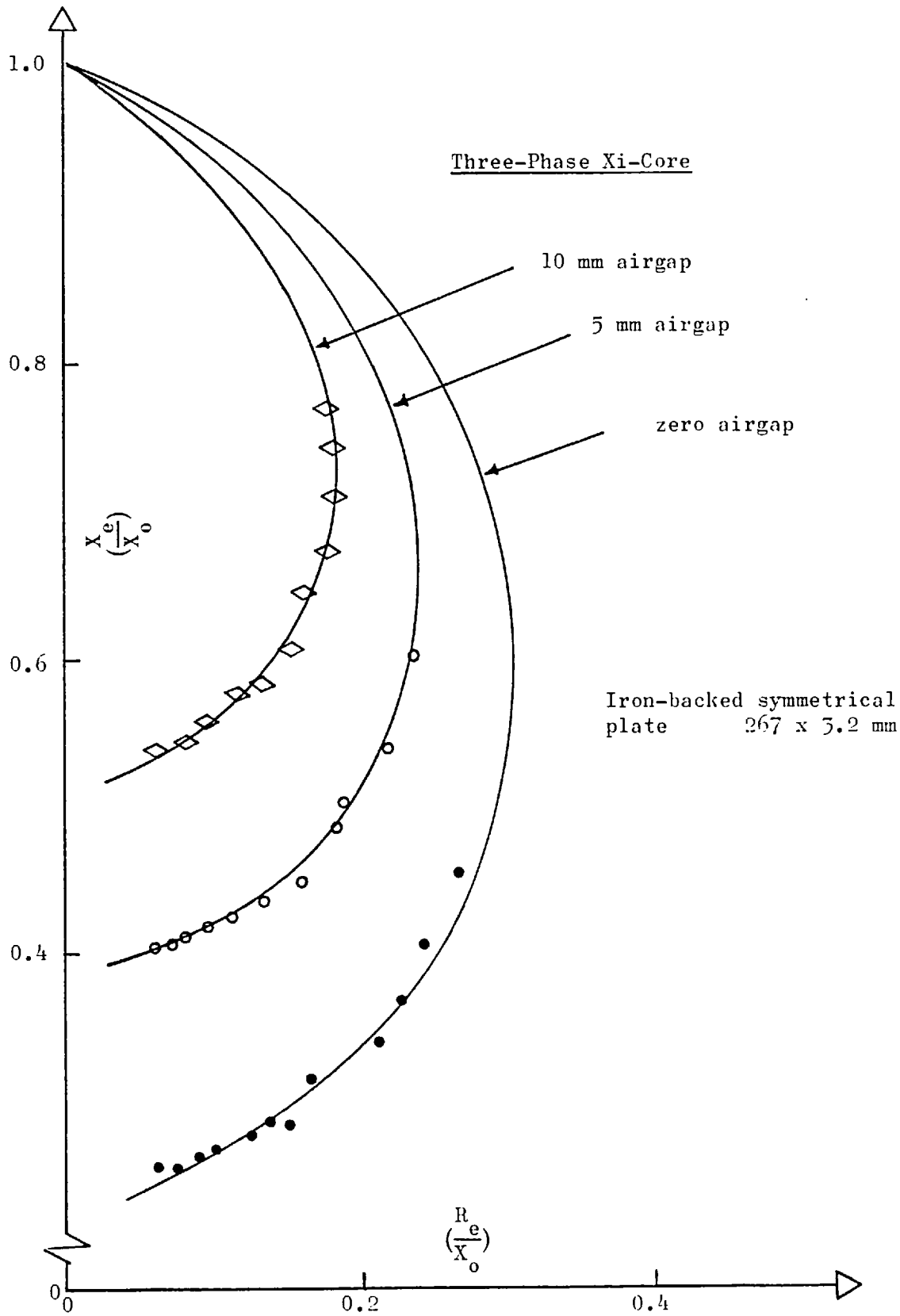


Figure B.6: The impedance loci of the three-phase Xi-core.

In the case of the single-phase Xi-core, when a family of circular loci at several lateral displacements of a plate (similar to Fig. B.4) is available, it is possible to predict the lateral stability of the plate. However, this will be of real value only when one can theoretically predict the input impedance rather than obtain it experimentally. We thus see that if one can develop a theory employing algebraic methods, and derive the input impedance in terms of airgap, lateral displacement of rotor, and rotor/stator dimensions, then the circle diagram technique will prove very useful indeed.

APPENDIX CC.1 FUNCTIONAL MINIMISATION OF ONE ELEMENT

In equation (6.11) of Chapter 6 which expresses the contribution to the total functional, of a typical element, let us denote the four terms as follows:

$$I_1 = \iint_{\Delta_r} \frac{1}{2\mu\mu_0} \left\{ \left(\frac{\partial A}{\partial x} \right)^2 + \left(\frac{\partial A}{\partial y} \right)^2 \right\} dx dy$$

$$I_2 = j \frac{\omega \sigma}{2} \iint_{\Delta_r} A^2 dx dy$$

$$I_3 = - j \omega \sigma \iint_{\Delta_r} A_s \cdot A dx dy$$

$$I_4 = - \iint_{\Delta_r} A \cdot J dx dy$$

Therefore equation (6.11) can be written as:

$$L_r = I_1 + I_2 + I_3 + I_4$$

The functional contribution L_r is minimum when the following three equations are simultaneously satisfied by the vertex potentials A_l , A_m and A_n :

$$\frac{\partial I_1}{\partial A_l} + \frac{\partial I_2}{\partial A_l} + \frac{\partial I_3}{\partial A_l} + \frac{\partial I_4}{\partial A_l} = 0 \quad (C.1)$$

$$\frac{\partial I_1}{\partial A_m} + \frac{\partial I_2}{\partial A_m} + \frac{\partial I_3}{\partial A_m} + \frac{\partial I_4}{\partial A_m} = 0 \quad (C.2)$$

$$\frac{\partial I_1}{\partial A_n} + \frac{\partial I_2}{\partial A_n} + \frac{\partial I_3}{\partial A_n} + \frac{\partial I_4}{\partial A_n} = 0 \quad (C.3)$$

C.2 EVALUATION OF $\frac{\partial I_1}{\partial A_\ell}$, $\frac{\partial I_1}{\partial A_m}$ AND $\frac{\partial I_1}{\partial A_n}$

The potential at any point within the element is given by:

$$A = \frac{1}{2\Delta} [a_\ell + b_\ell x + c_\ell y] A + \frac{1}{2\Delta} [a_m + b_m x + c_m y] \\ + \frac{1}{2\Delta} [a_n + b_n x + c_n y]$$

$$\therefore \frac{\partial A}{\partial x} = \frac{1}{2\Delta} [b_\ell A_\ell + b_m A_m + b_n A_n]$$

$$\text{and} \quad \frac{\partial A}{\partial y} = \frac{1}{2\Delta} [c_\ell A_\ell + c_m A_m + c_n A_n]$$

$$\text{Thus:} \quad I_1 = \frac{1}{2\mu\mu_0} \iint \left\{ \frac{[b_\ell A_\ell + b_m A_m + b_n A_n]^2}{4\Delta^2} + \frac{[c_\ell A_\ell + c_m A_m + c_n A_n]^2}{4\Delta^2} \right\} dx dy$$

Hence:

$$\frac{\partial I_1}{\partial A} = \frac{1}{4\mu\mu_0} \iint \sum_{k=\ell}^n \frac{(b_k A_k + c_k A_k)}{\Delta^2} dx dy$$

$$\frac{\partial I_1}{\partial A_m} = \frac{1}{4\mu\mu_0} \iint \sum_{k=\ell}^n \frac{(b_m A_k + c_m A_k)}{\Delta^2} dx dy$$

$$\frac{\partial I_1}{\partial A_n} = \frac{1}{4\mu\mu_0} \iint \sum_{k=\ell}^n \frac{(b_n A_k + c_n A_k)}{\Delta^2} dx dy$$

In the above three equations, the integral on the left-hand side is independent of x and y. Besides, $\iint_{\Delta_r} dx dy = \Delta$.

$$\therefore \frac{\partial I_1}{\partial A} = \frac{1}{4\mu_0\mu\Delta} \sum_{k=\ell}^n (b_\ell A_k + c_\ell A_k) \quad (C.4)$$

$$\frac{\partial I_1}{\partial A_m} = \frac{1}{4\mu_o\mu\Delta} \sum_{k=\ell}^n (b_m b_k A_k + c_m c_k A_k) \quad (C.5)$$

$$\frac{\partial I_1}{\partial A_n} = \frac{1}{4\mu_o\mu\Delta} \sum_{k=\ell}^n (b_n b_k A_k + c_n c_k A_k) \quad (C.6)$$

C.3 EVALUATION OF $\frac{\partial I_2}{\partial A_\ell}$, $\frac{\partial I_2}{\partial A_m}$ AND $\frac{\partial I_2}{\partial A_n}$

Consider $\frac{\partial I_2}{\partial A_\ell}$ expressed in Cartesian co-ordinates:

$$\frac{\partial I_2}{\partial A_\ell} = \frac{j\omega\sigma}{(2\Delta)^2} \iint \sum_{k=\ell}^m (a_\ell + b_\ell x + c_\ell y) \cdot (a_k + b_k x + c_k y) \cdot (A_k) \cdot dx dy$$

This integral in its present form cannot be easily integrated because it requires the surface integration of polynomials of the form $(ax + bx + c)^n$. This difficulty is overcome by the use of triangle area co-ordinates referred to in Chapter 6. The potential of a point within an element, in terms of triangle area co-ordinates, is:

$$A = E_{p\ell} \cdot A_\ell + E_{pm} \cdot A_m + E_{pn} \cdot A_n$$

$$\therefore \frac{\partial I_2}{\partial A_\ell} = j\omega\sigma \iint E_{p\ell}^2 \cdot A_\ell \cdot dx dy + j\omega\sigma \iint E_{p\ell} \cdot E_{pm} \cdot A_m \cdot dx dy + j\omega\sigma \iint E_{pn} \cdot E_p \cdot dx dy$$

$$\text{Since } \iint E_{p\ell}^p \cdot E_{pm}^q \cdot E_{pn}^r \cdot dx dy = \frac{p!q!r!}{(p+q+r+2)!} 2\Delta,$$

$$\frac{\partial I_2}{\partial A_\ell} = \frac{j\omega\sigma\Delta}{12} [A_\ell + 2A_m + 2A_n] \quad (C.7)$$

By identical procedure:

$$\frac{\partial I_2}{\partial A_m} = \frac{j\omega\sigma\Delta}{12} [A_\ell + 2A_m + A_n] \quad (C.8)$$

$$\frac{\partial I_2}{\partial A_n} = \frac{j\omega\sigma\Delta}{12} [A_\ell + A_m + 2A_n] \quad (C.9)$$

C.4 EVALUATION OF $\frac{\partial I_3}{\partial A_\ell}$, $\frac{\partial I_3}{\partial A_m}$ AND $\frac{\partial I_3}{\partial A_n}$

Consider $\frac{\partial I_3}{\partial A_\ell}$ expressed in triangle area co-ordinates:

$$\begin{aligned} \frac{\partial I_3}{\partial A_\ell} &= -j\omega\sigma A_s \iint_{\Delta_r} E_{p\ell} \cdot dxdy \\ &= -j \frac{\omega\sigma\Delta}{3} \cdot A_s \end{aligned} \quad (C.10)$$

Similarly:

$$\frac{\partial I_3}{\partial A_m} = -j \frac{\omega\sigma\Delta}{3} \cdot A_s \quad (C.11)$$

and $\frac{\partial I_3}{\partial A_n} = -j \frac{\omega\sigma\Delta}{3} \cdot A_s \quad (C.12)$

C.5 EVALUATION OF $\frac{\partial I_4}{\partial A_\ell}$, $\frac{\partial I_4}{\partial A_m}$ AND $\frac{\partial I_4}{\partial A_n}$

Again we use triangle area co-ordinates as follows:

$$\begin{aligned} \frac{\partial I_4}{\partial A_\ell} &= -J \iint_{\Delta_r} E_{p\ell} \cdot dxdy \\ &= -\frac{1}{3} \Delta J \end{aligned} \quad (C.15)$$

Similarly:

$$\frac{\partial I_4}{\partial A_m} = -\frac{1}{3} \Delta J \quad (C.14)$$

and
$$\frac{\partial I_4}{\partial A_n} = -\frac{1}{3} \Delta J \quad (C.15)$$

C.6 THE COMPLETE S-MATRIX OF ONE ELEMENT

Combining equations (C.4) to (C.15) one obtains:

$$\begin{aligned} & \frac{1}{4\mu\mu_o \Delta} \begin{bmatrix} (b_\ell b_\ell + c_\ell c_\ell) & (b_\ell b_m + c_\ell c_m) & (b_\ell b_n + c_\ell c_n) \\ (b_m b_\ell + c_m c_\ell) & (b_m b_m + c_m c_m) & (b_m b_n + c_m c_n) \\ (b_n b_\ell + c_n c_\ell) & (b_n b_m + c_n c_m) & (b_n b_n + c_n c_n) \end{bmatrix} \begin{bmatrix} A_\ell \\ A_m \\ A_n \end{bmatrix} \\ & + j \frac{\omega \sigma \Delta}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} A \\ A_m \\ A_n \end{bmatrix} - j \frac{\omega \sigma \Delta}{3} \begin{bmatrix} A_s \\ A_s \\ A_s \end{bmatrix} \\ & = \frac{1}{3} \Delta \begin{bmatrix} J \\ J_m \\ J_n \end{bmatrix} \quad (C.16) \end{aligned}$$

Equation (C.16) is the S-matrix for one element.

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