

Dissertation

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Three Nucleon Problem

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Introduction

In this dissertation we review the work of Faddeev on non-relativistic three-particle systems. The approximation scheme applied to the Faddeev equations, developed by Lovelace, which uses the non local separable potential, for the practical study of the three body system is studied. We include the application of the Lovelace theory to the low energy three-nucleon problem, with the inclusion of the three body force, work of A.C. Phillips, which represents the effects on the three-nucleon system of the nucleon-nucleon tensor and short range forces. We give a brief account of the formalism for the three-nucleon scattering state, bound state, disintegration process, and bound state wave function. We also present the results of the numerical calculation carried out by A.C. Phillips for the triton binding energy, quartet, and the doublet scattering length for $n - d$ scattering.

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Part I. Some General Aspects of Three-particle Scattering

1. Preliminaries.

Consider particles α , β , and γ forming a non relativistic three-particle system. m_α , k_α , and $\mathbf{r}_\alpha$ represent mass, momentum, and position of particle α etc. The total momentum is

$$\mathbf{K} = \mathbf{k}_\alpha + \mathbf{k}_\beta + \mathbf{k}_\gamma$$

we can use another set of coordinates, to describe the momenta of the particles, namely,

$$\mathbf{p}_\alpha = \frac{\mathbf{k}_\beta m_\gamma - \mathbf{k}_\gamma m_\beta}{m_\beta + m_\gamma} \quad (1.1)$$

which is the relative momentum of the (β, γ) subsystem. And

$$\mathbf{q}_\alpha = \frac{\mathbf{k}_\alpha (m_\beta + m_\gamma) - (\mathbf{k}_\beta + \mathbf{k}_\gamma) m_\alpha}{m_\alpha + m_\beta + m_\gamma} \quad (1.2)$$

which is the momentum of particle α relative to the centre-of-mass of the (β, γ) subsystem. α , β , and γ are taken cyclically. Thus altogether we have six momenta

$$\mathbf{p}_\alpha, \mathbf{q}_\alpha, \mathbf{p}_\beta, \mathbf{q}_\beta, \mathbf{p}_\gamma, \mathbf{q}_\gamma$$

only two of which are linearly independent, and the relation between them can be obtained from (1.1) and (1.2). e.g.

$$\mathbf{p}_\alpha = \mathbf{q}_\beta + \frac{m_\beta}{m_\beta + m_\gamma} \mathbf{q}_\alpha \quad (1.1a)$$

$$\mathbf{p}_\beta = \mathbf{q}_\alpha \left(1 - \frac{m_\beta m_\alpha}{(m_\beta + m_\gamma)(m_\alpha + m_\gamma)} \right) + \frac{m_\alpha}{m_\alpha + m_\gamma} \mathbf{p}_\alpha \quad (1.2a)$$

It should be mentioned that the Jacobian of $p_\alpha q_\alpha \rightarrow p_\beta q_\beta$ is equal to 1, i.e.

$$\begin{vmatrix} \frac{\partial p_\beta}{\partial p_\alpha} & \frac{\partial q_\beta}{\partial p_\alpha} \\ \frac{\partial p_\beta}{\partial q_\alpha} & \frac{\partial q_\beta}{\partial q_\alpha} \end{vmatrix} = 1$$

We make the convention that whenever we use two of these momenta in a formula, they are linearly independent. if they have different number of primes. We will omit the subscript α on p_α , and q_α when it is not relevant. For the kinetic energy we have

$$E_0 = \frac{k^2}{2M} + \frac{p_\alpha^2}{2\mu_\alpha} + \frac{q_\alpha^2}{2\mu_\alpha} \quad (1.3)$$

where M is a total mass, μ_α the reduced mass of (β, γ) subsystem, and M_α reduced mass of $(\alpha, (\beta, \gamma))$ scattering system.

i.e.

$$\frac{1}{M_\alpha} = \frac{1}{m_\beta} + \frac{1}{m_\gamma} \quad (1.4)$$

and

$$\frac{1}{\mu_\alpha} = \frac{1}{m_\alpha} + \frac{1}{m_\beta + m_\gamma}$$

In three nucleon systems, we have equality of masses, and we shall work in units in which nucleon mass is unity. Thus, we have

$$\mu_\alpha = \mu = \frac{2}{3} \quad (1.5)$$

For the three particles, there are two kinds of asymptotic basis states, or channel states. The first is the eigenstates of the free

Hamiltonian, describing the free motion of three unbound particles.

Their momentum space representation is given by

$$\langle \underline{k}'_{\alpha} \underline{k}'_{\beta} \underline{k}'_{\gamma} | \phi_0 \rangle = \delta_3(\underline{K} - \underline{K}') \delta_3(\underline{p}_{\alpha} - \underline{p}'_{\alpha}) \delta_3(\underline{q}_{\alpha} - \underline{q}'_{\alpha}). \quad (1.6)$$

The second is the eigenstates of the free motion of one particle with respect to the other two. For illustration suppose that particle α is free, β and γ are in a bound subsystem $\phi_{an}(\underline{p}_{\alpha})$, then we have

$$\langle \underline{k}'_{\alpha} \underline{k}'_{\beta} \underline{k}'_{\gamma} | \phi_{an} \rangle = \delta_3(\underline{K} - \underline{K}') \delta_3(\underline{q}_{\alpha} - \underline{q}'_{\alpha}) \phi_{an}(\underline{p}'_{\alpha}). \quad (1.7)$$

We will refer to this state as an channel state. If ϵ_{an} denotes the binding energy of the (β, γ) bound state, the kinetic energy is given by

$$E_{an} = \frac{K^2}{2M} + \frac{q_{\alpha}^2}{2\mu_{\alpha}} - \epsilon_{an}. \quad (1.8)$$

The delta-function arising from the conservation of total momentum will be factored out, so we work in three-particle centre of mass system.

The Hilbert space which is being considered is $L_2(p, q)$ - the space of square - integrable functions of two momentum vectors $[p, q]$, or of six real variables. The operators will be written in this Hilbert space without hats. The free particle Hamiltonian operator H_0 is given by,

$$\langle \underline{p}_{\alpha} \underline{q}_{\alpha} | H_0 | \underline{p}'_{\alpha} \underline{q}'_{\alpha} \rangle = \left(\frac{p_{\alpha}^2}{2M_{\alpha}} + \frac{q_{\alpha}^2}{2\mu_{\alpha}} \right) \delta_3(\underline{p}_{\alpha} - \underline{p}'_{\alpha}) \delta_3(\underline{q}_{\alpha} - \underline{q}'_{\alpha}) \quad (1.9)$$

and its resolvent is

$$G_0(s) = (H_0 - s\mathbb{1})^{-1} \quad (1.10)$$

where $\mathbb{1}$ indicates a unit operator and s is a complex variable outside the spectrum of H_0 . For the matrix element of the $G_0(s)$ we have

$$\langle p_\alpha q_\alpha | G_0(s) | p'_\alpha q'_\alpha \rangle = \left(\frac{p_\alpha^2}{2m_\alpha} + \frac{q_\alpha^2}{2\mu_\alpha} - s \right)^{-1} \delta_3(q_\alpha - q'_\alpha) \delta_3(p_\alpha - p'_\alpha) \quad (1.11)$$

Next the $(\alpha, (\beta, \gamma))$ scattering system is considered:

The Hamiltonian of this system is

$$H_\alpha = H_0 + V_\alpha$$

and for the corresponding resolvent operator

$$G_\alpha(s) = (H_\alpha - s\mathbb{1})^{-1}. \quad (1.12)$$

In addition to H_α and $G_\alpha(s)$, one also needs the total Hamiltonian and its resolvent operator. They are given by the following expressions

$$\begin{aligned} H &= H_0 + V \\ G(s) &= (H - s\mathbb{1})^{-1} \end{aligned} \quad (1.13)$$

where $V = \sum_{\alpha=1}^3 V_\alpha$, V_α being the two body interaction potential. By using the following identity

$$\frac{1}{(H_0 + V - s\mathbb{1})} = \frac{1}{(H_0 - s\mathbb{1})} - \frac{V}{(H_0 - s\mathbb{1})(H_0 + V - s\mathbb{1})}$$

and equation (1.10), one can express the $G(s)$ in Lippman-Schwinger form

$$G(s) = G_0(s) - G_0(s)VG(s) \quad (1.14)$$

The solution of the (1.14) will give all the interesting information about the physical system under consideration. This can be done by applying Fredholm methods if the iterated kernel, $K(s) = G_0(s)V$ is a

square-integrable or "Hilbert-Schmidt" operator i.e.

$$\int d^3p \int d^3p' |K(p, p')|^2 < \infty$$

which reduced to $\text{Tr}(KK^\dagger) < \infty$.

In the case of the two body problem, the required condition can be satisfied under reasonable restriction on two body interaction. But for the three body interaction this is not so. The reason being that one of the particles may be too far from the other two to be able to interact with them. In order to make this point clear, one can write the explicit form of the matrix element of $K(s)$ as:

$$\langle p_\alpha q_\alpha | G_0(s) V | p'_\alpha q'_\alpha \rangle = \frac{1}{\left(\frac{p_\alpha^2}{2M_\alpha} + \frac{q_\alpha^2}{2\mu_\alpha} - s \right)}$$

$$\sum_{\alpha=1}^3 \langle p_\alpha | \hat{V}_\alpha | p'_\alpha \rangle$$

$$\delta_3(q_\alpha - q'_\alpha).$$

Here the term $\langle p_\alpha | \hat{V}_\alpha | p'_\alpha \rangle \delta_3(q_\alpha - q'_\alpha)$ represents the situation where the interaction does not occur, so the momentum of the particle α is conserved. But for the complete interaction, the momentum of α particle is not conserved and consequently the related δ -function cannot be factored out. So the $\text{Tr}(KK^\dagger)$ will contain the terms of the form $[\delta_3(q - q')]^2$. Furthermore, by iterating the kernel we will not be able to get rid of these terms. So the presence of the δ function prevents us to use methods which worked well for two-body problem. In the elegant theory of Faddeev¹, this difficulty has been considered. The kernels in

this theory depend on the two body scattering amplitude off the energy shell. A decided advantage which ^{enables the} ~~makes~~ theory to be generalized in ^{the} relativistic region where ^{the} potential model fails.⁵

2. The Lovelace-Faddeev Equations

The S matrix for a process describing the scattering from ^{any} asymptotic state $|\phi_{\beta m}\rangle$ to the state $|\phi_{\alpha n}\rangle$, is defined by

$$S_{\alpha\beta} = \delta_{\alpha\beta} - 2\pi i \delta(E_{\alpha} - E_{\beta}) \langle \phi_{\alpha n} | U_{\alpha\beta}^+(s) | \phi_{\beta m} \rangle \quad (2.1)$$

The state $\langle \phi_{\alpha n} |$, refers to a free motion of particle α , relative to the bound state of (β, γ) subsystem. $U_{\alpha\beta}^+$ is the transition operator, from an incoming state with free Hamiltonian H_{α} , to an outgoing state with free Hamiltonian H_{β} . For a physical process $s \rightarrow E + i\epsilon = E + i\epsilon'$.

An attempt can be made now to get an expression for the transition amplitudes $U_{\alpha\beta}^+(s)$. The outgoing scattering state in βm channel can be represented by $|\psi_{\beta m}\rangle_+$. It is given by the following Lippman-Schwinger equation

$$|\psi_{\beta m}\rangle_+ = |\phi_{\beta m}\rangle - G(E + i\epsilon) \sum_{\substack{\delta \neq \beta \\ \delta=1}}^3 V_{\delta} |\phi_{\delta m}\rangle. \quad (2.2)$$

A similar relation holds for the incoming scattering state, $|\psi_{\alpha n}\rangle_-$

$$|\psi_{\alpha n}\rangle_- = |\phi_{\alpha n}\rangle - G(E - i\epsilon) \sum_{\substack{\delta \neq \alpha \\ \delta=1}}^3 V_{\delta} |\phi_{\delta n}\rangle. \quad (2.3)$$

For the S. matrix elements we have

$$\langle \phi_{\alpha n} | S | \phi_{\beta m} \rangle = \langle \psi_{\alpha n} | \psi_{\beta m} \rangle + \quad (2.4)$$

By using the following identities

$$G(E \pm i\epsilon) = \frac{1}{H - (E \pm i\epsilon)} = \frac{p}{H - E} \pm i\pi\delta(H - E) \quad (2.5)$$

where p stands for Cauchy principal value, and relations (2.2) and (2.3), we get

$$|\psi_{\beta m} \rangle_+ - |\psi_{\beta m} \rangle_- = -2\pi i\delta(H - E) \sum_{\substack{\delta \neq \beta \\ \delta=1}}^3 V_{\delta} |\phi_{\beta m} \rangle. \quad (2.6)$$

Insertion of equation (2.6) into equation (2.4) gives

$$\begin{aligned} \langle \phi_{\alpha n} | S | \phi_{\beta m} \rangle &= \delta_{\alpha\beta} - 2\pi i\delta(E_{\alpha} - E_{\beta}) \\ &\quad \langle \phi_{\alpha n} | \sum_{\substack{\gamma \neq \alpha \\ \gamma=1}}^3 V_{\gamma} | \psi_{\beta m} \rangle_+ \end{aligned} \quad (2.7)$$

The relation (2.7) is then compared to the relation (2.1) in order to get the boundary value of the $U_{\alpha\beta}^+$:

$$\langle \phi_{\alpha n} | U_{\alpha\beta}^+(E + i\epsilon) | \phi_{\beta m} \rangle = \langle \phi_{\alpha n} | \sum_{\substack{\gamma=1 \\ \gamma \neq \alpha}}^3 V_{\gamma} | \psi_{\beta m} \rangle_+ \quad (2.8)$$

Replacing the value of $|\psi_{\beta m} \rangle_+$ in (2.8) from (2.2), one gets the following relation for $U_{\alpha\beta}^+(s)$

$$U_{\alpha\beta}^+(s) = \sum_{\gamma \neq \alpha} V_{\gamma} - \sum_{\gamma \neq \alpha} \sum_{\delta \neq \beta} V_{\gamma} G(s) V_{\delta}. \quad (2.9)$$

Similarly one can express the transition operator $U_{\alpha\beta}^-(s)$, which is defined in terms of the incoming scattering state $|\psi_{\alpha n}^>_-$, i.e.

$$\langle \phi_{\alpha n} | U_{\alpha\beta}^-(E + i\epsilon) | \phi_{\beta m} \rangle = \langle \psi_{\alpha n} | \sum_{\delta \neq \beta} V_{\delta} | \phi_{\beta m} \rangle$$

thus

$$U_{\alpha\beta}^-(s) = \sum_{\delta \neq \beta} V_{\delta} - \sum_{\gamma \neq \alpha} \sum_{\delta \neq \beta} V_{\gamma} G(s) V_{\delta}. \quad (2.10)$$

Considering the fact that $|\phi_{\alpha n}^>$ and $|\phi_{\beta m}^>$ are eigenstates of $H_0 + V_{\alpha}$ and $H_0 + V_{\beta}$ respectively, one must have

$$\langle \phi_{\alpha n} | U_{\alpha\beta}^+(s) | \phi_{\beta m} \rangle = \langle \phi_{\alpha n} | U_{\alpha\beta}^-(s) | \phi_{\beta m} \rangle$$

on the energy shell

$$E_{\alpha} = \frac{q_{\alpha}^2}{2\mu_{\alpha}} - \epsilon_{\alpha n} = E_{\beta} = \frac{q_{\beta}^2}{2\mu_{\beta}} - \epsilon_{\beta m} = s.$$

Therefore the $\pm$ sign in $U_{\alpha\beta}(s)$ represents the extensions to off-the-energy shell. In order to accommodate the disintegration transition operator into equation (2.9), V_0 is put equal to zero and α, β, γ and δ are allowed to run from 0 to 3, so the matrix element from $|\phi_{\beta m}^>$ to $\langle \phi_0 |$ (the final state of three particles) will be

$$\langle \phi_0 | U_{\alpha\beta}^+(s) | \phi_{\beta m} \rangle = \langle \phi_0 | \sum_{\gamma=1}^3 V_{\gamma} | \psi_{\beta m}^+ \rangle. \quad (2.11)$$

For the calculation of the wave function $|\psi_{\alpha n}^+>$ and transition operators $U_{\alpha\beta}^{\pm}$ we need to solve equation (1.14), whose kernel is not square-integrable. The underlying principle of Faddeev equations is to

solve exactly two particle subsystems which in terms of the resolvent $G_\alpha(s)$ are given by

$$T_\alpha(s) = V_\alpha - V_\alpha G_\alpha(s) V_\alpha \quad (2.12)$$

(remember discussion leading to the derivation of the (2.9) and (2.10)).

Here $T_\alpha(s)$ and $G_\alpha(s)$ are operators in the three particle Hilbert space $L_2(p, q)$. The differences between them and $\hat{T}(s)$, $\hat{G}(s)$ are of two kinds.

First, one must multiply the matrix elements of $\langle p_\alpha | \hat{T}(s) | p'_\alpha \rangle$ and $\langle p_\alpha | \hat{G}(s) | p'_\alpha \rangle$ by δ -function in other momentum $\delta_3(q_\alpha - q'_\alpha)$.

Second, the total energy of the two body subsystem in this case is not s but $s - \frac{q_\alpha^2}{2\mu_\alpha}$. Thus we have

$$\begin{aligned} \langle p_\alpha q_\alpha | T_\alpha(s) | p'_\alpha q'_\alpha \rangle &= \langle p_\alpha | \hat{T}_\alpha(s - \frac{q_\alpha^2}{2\mu_\alpha}) | p'_\alpha \rangle \\ &\quad \delta_3(q_\alpha - q'_\alpha). \end{aligned} \quad (2.13)$$

Here the matrix elements $\langle p_\alpha q_\alpha | T_\alpha(s) | p'_\alpha q'_\alpha \rangle$ corresponds to a situation where particle α does not interact. A similar formula holds for $G_\alpha(s)$.

It is now attempted to get an expression for three body wave function in terms of the solution of our two-particle subsystem. The three body wave function $|\psi_{\alpha n}\rangle$ can be defined by

$$|\psi_{\alpha n}\rangle = |\phi_{\alpha n}\rangle - \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^3 G_\beta(s) V_\beta |\phi_{\alpha n}\rangle. \quad (2.14)$$

From the definitions (1.11), (1.12), and (1.13), one gets the following identities

$$\begin{aligned}
 G_{\alpha}(s) &= G_0(s) - G_0(s)V_{\alpha}G_{\alpha}(s) \\
 &= G_0(s) - G_{\alpha}(s)V_{\alpha}G_0(s)
 \end{aligned}
 \tag{2.15}$$

$$\begin{aligned}
 G(s) &= G_{\alpha}(s) - \sum_{\beta \neq \alpha} G_{\alpha}(s)V_{\beta}G(s) \\
 &= G_{\alpha}(s) - \sum_{\beta \neq \alpha} G(s)V_{\beta}G_{\alpha}(s).
 \end{aligned}
 \tag{2.16}$$

Furthermore, comparison of (2.15) and (2.12) gives the following relation

$$G_{\alpha}(s)V_{\alpha} = G_0(s)T_{\alpha}(s). \tag{2.17}$$

Now from (2.17) and (2.16)

$$G(s)V_{\alpha} = G_0(s)T_{\alpha}(s) - \sum_{\beta \neq \alpha} G(s)V_{\beta}G_0(s)T_{\alpha}(s) \tag{2.18}$$

Successive approximation to $|\Psi_{\alpha n}\rangle$ can be obtained by inserting the value of $G(s)V_{\alpha}$ from relation (2.18) into (2.14) and iterating. For example

$$\begin{aligned}
 |\Psi_{1m}\rangle &= \{ |\phi_{1m}\rangle - G_0(s)T_2(s)|\phi_{1m}\rangle \\
 &\quad - G_0(s)T_3(s)|\phi_{1m}\rangle \\
 &\quad + G_0(s)T_3(s)G_0(s)T_2(s)|\phi_{1m}\rangle \\
 &\quad + G_0(s)T_1(s)G_0(s)T_2(s)|\phi_{1m}\rangle \\
 &\quad + G_0(s)T_1(s)G_0(s)T_3(s)|\phi_{1m}\rangle \\
 &\quad + G_0(s)T_2(s)G_0(s)T_3(s)|\phi_{1m}\rangle \\
 &\quad - \dots \dots \dots \}
 \end{aligned}
 \tag{2.19}$$

By defining

$$|\phi_{\alpha n}^{(\beta)}\rangle = \delta\alpha\beta |\phi_{\alpha n}\rangle$$

the equation (2.19) can be written in the following form

$$|\Psi_{lm}\rangle = |\Psi_{lm}^{(1)}\rangle + |\Psi_{lm}^{(2)}\rangle + |\Psi_{lm}^{(2)}\rangle$$

where

$$|\Psi_{lm}^{(\alpha)}\rangle = |\Phi_{lm}^{(\alpha)}\rangle \quad (2.20)$$

$$\begin{aligned} & - G_0(s)T_\alpha(s)\{|\Phi_{lm}^{(\beta)}\rangle + |\Phi_{lm}^{(\gamma)}\rangle\} \\ & + G_0(s)T_\alpha(s)G_0(s)T_\beta(s)\{|\Phi_{lm}^{(\gamma)}\rangle + |\Phi_{lm}^{(\eta)}\rangle\} \\ & + G_0(s)T_\alpha(s)G_0(s)T_\gamma(s)\{|\Phi_{lm}^{(\alpha)}\rangle + |\Phi_{lm}^{(\beta)}\rangle\} \\ & - \dots \dots \dots \} \end{aligned}$$

It is easy to see that the iteration of the following relation produces the equation (2.20)

$$|\Psi_{lm}^{(\alpha)}\rangle = |\Phi_{lm}^{(\alpha)}\rangle - G_0(s)T_\alpha(s) \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^3 |\Psi_{lm}^{(\beta)}\rangle$$

Therefore the scattering state is given by

$$|\Psi_{an}\rangle = \sum_{\beta=1}^3 |\Psi_{an}^{(\beta)}\rangle \quad (2.21)$$

where $|\Psi_{an}^{(\beta)}\rangle$ satisfies the linked set of integral equation

$$|\Psi_{an}^{(\beta)}\rangle = |\Phi_{an}^{(\beta)}\rangle - G_0(s)T_\beta(s) \sum_{\substack{\gamma=1 \\ \gamma \neq \beta}}^3 |\Psi_{an}^{(\gamma)}\rangle \quad (2.22)$$

An interesting feature of the equation (2.22) is that although its kernel

$$K(s) = \begin{pmatrix} 0 & G_o T_1 & G_o T_1 \\ G_o T_2 & 0 & G_o T_2 \\ G_o T_3 & G_o T_3 & 0 \end{pmatrix}$$

has in its matrix elements the δ -function, they do not appear in the matrix elements of the $K^2(s)$, i.e.

$$\int \langle pq | K(s) | p'q' \rangle d^3p' d^3q' \langle p'q' | K(s) | p'q' \rangle$$

In fact, it has been proved by Faddeev¹ and Lovelace² that $K^2(s)$ is completely continuous for a wide class of potentials, i.e:

$$\text{Trace } K^2(s) K^2(s)^\dagger < \infty$$

In particular if the potentials V_α are superposition of Yukawa potentials,

$$V_\alpha(p) = \int_{\mu_\alpha}^{\infty} d\xi \varphi_\alpha(\xi) / (p^2 + \xi^2)$$

and the nondecreasing function (ξ) is such that the integral

$$I_\alpha(s) = \int_{\mu_\alpha}^{\infty} \varphi_\alpha(\eta) d\eta \text{ is of the bounded variation on } (\mu_\alpha, +\infty).$$

Then this condition is true for all values of the s which is not equal to the energy of a physical state. The integral equations having the kernel form like (2.22) will be referred as the Faddeev equations.

By inserting equation (2.21) into equation (2.8), on iterating equation (2.21) and rearranging terms, we obtain

$$U_{\alpha\beta}^+(s) = \sum_{\gamma \neq \alpha} V_{\gamma} - \sum_{\delta \neq \beta} U_{\alpha\delta}^+(s) G_O(s) T_{\delta}(s) \quad (2.23)$$

similarly $U_{\alpha\beta}^-(s)$ is given by

$$U_{\alpha\beta}^-(s) = \sum_{\delta \neq \beta} V_{\delta} - \sum_{\gamma \neq \alpha} T_{\gamma}(s) G_O(s) U_{\gamma\beta}^-(s) \quad (2.24)$$

There are two kinds of contributions for the imaginary parts of $U_{\alpha\beta}^{\pm}(s)$. First, by the three particle scattering cut, starting from $s = 0$. Second, from the bound state scattering cuts, starting from the branch points below the three particle threshold in on channel, i.e. $s = -\epsilon_{\alpha n}$.

The unitarity relation for the $U_{\alpha\beta}^{\pm}(s)$ has the following expression²

$$\begin{aligned} U_{\alpha\beta}^{\pm}(E + i\epsilon) - U_{\alpha\beta}^{\pm}(E - i\epsilon) &= U_{\alpha o}^+(E + i\epsilon) \Delta_o(E) \\ \times \quad U_{o\beta}^-(E - i\epsilon) + \sum_{\gamma=1,2,3} \sum_n U_{\alpha\gamma}^+(E + i\epsilon) \Delta_{\gamma n}(E) U_{\gamma n}^-(E - i\epsilon) \end{aligned} \quad (2.25)$$

where $\Delta_o(E)$ indicates the projection operator for three free particles with energy E , and $\Delta_{\gamma n}(E)$ is the projection operator for the γn bound state plus a free particle γ . i.e.

$$\begin{aligned} \langle p q | \Delta_o(E) | p' q' \rangle &= \delta\left(\frac{p^2}{2M} + \frac{q^2}{2\mu} - E\right) \\ &\delta_3(p - p') \delta_3(q - q') \end{aligned} \quad (2.26)$$

and

$$\begin{aligned}
 \langle p q | \Delta_{\gamma n}(E) | p' q' \rangle &= \delta\left(\frac{p_Y^2}{2\mu_Y} - \varepsilon_{\gamma n} - E\right) \\
 &\quad \delta_3(q_Y - q'_Y) \phi_{\gamma n}(p_\delta) \\
 &\quad \phi_{\gamma n}^*(p_Y)
 \end{aligned} \tag{2.27}$$

Consider now the three particle bound state. If E_t is the binding energy of the three particles, then the wave function satisfies the following Schrodinger equation

$$|\Psi\rangle = -G_O(-E_t) \sum_{\alpha=1}^3 V_\alpha |\Psi\rangle.$$

The procedures similar to those used in derivation of (2.20) and (2.21) will give us

$$|\Psi\rangle = \sum_{\alpha=1}^3 |\Psi^{(\alpha)}\rangle$$

where $|\Psi^{(\alpha)}\rangle$ has the following forms

$$|\Psi^{(\alpha)}\rangle = -G_O(-E_t) T_\alpha(-E_t) \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^3 |\Psi^{(\beta)}\rangle$$

The three body force V_4 can be accommodated in the above formulation.

As a result the bound state or scattering state is given by

$$|\Psi_{\alpha n}\rangle = |\Phi_{\alpha n}\rangle - \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^4 G(\varepsilon) V_\beta |\Phi_{\alpha n}\rangle$$

$$|\psi_{\alpha n}\rangle = |\phi_{\alpha n}\rangle - \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^4 G(s) V_{\beta} |\phi_{\alpha n}\rangle$$

By defining $T_h(s)$ as

$$T_h(s) = V_h - V_h G(s) V_h \quad (2.28)$$

the generalized form of the equations (2.21) and (2.22) can be obtained:

Thus

$$|\psi_{\alpha n}\rangle = \sum_{\beta=1}^4 |\psi_{\alpha n}^{(\beta)}\rangle \quad (2.29)$$

where

$$|\psi_{\alpha n}^{(\beta)}\rangle = |\phi_{\alpha n}^{(\beta)}\rangle - G_0(s) \sum_{\substack{\gamma=1 \\ \gamma \neq \beta}}^4 T_{\beta}(s) |\psi_{\alpha n}^{(\gamma)}\rangle \quad (2.30)$$

Now the relation for $U_{\alpha\beta}^+(s)$ is

$$U_{\alpha\beta}^+(s) = \sum_{\substack{\gamma=0 \\ \gamma \neq \alpha}}^4 V_{\gamma} - \sum_{\substack{\delta=0 \\ \delta \neq \beta}}^4 U_{\alpha\delta}(s) G_0(s) T_{\delta}(s) \quad (2.31)$$

The bound state wave function becomes

$$|\psi\rangle = \sum_{\alpha=1}^4 |\psi^{(\alpha)}\rangle \quad (2.32)$$

where

$$|\psi^{(\alpha)}\rangle = -G_0(-E_t) T_{\alpha}(-E_t) \sum_{\substack{\beta=1 \\ \beta \neq \alpha}}^4 |\psi^{(\beta)}\rangle \quad (2.33)$$

Part II. Approximation Method

3. Introduction

In chapters I and II the formal aspects of three-particle scattering have been considered. In particular it was pointed out that from equations (2.30), (2.31) and (2.33) one can derive properties of the three-particle system, as long as one knows the exact scattering amplitudes (off-the-energy shell) for the two-particle subsystem, plus information on the form of the three-particle force.

In the case of the two-particle interaction, the scattering data determines the scattering amplitude on the energy shell. One may attempt to use the Faddeev equations as the basis for the study of the off-shell behaviour and its effects on the three-particle system. However, this does not seem desirable as the number of degrees of freedom involved in three body problems makes any attempt for numerical solution impossible, therefore one is forced to approximate the two body interaction in order to treat the three body problem exactly. There has been several attempts to deal with three-body problems using this form of approximation^{1,3,2}. However the most comprehensive study is due to Lovelace².

The Lovelace theory deals with three-particle systems, when the two particle subsystems can form the bound states and resonances. In this theory one takes into account the fact that the energy and form factor of the bound and resonance states can determine the behaviour of the two-particle amplitude (off and on the energy shell) in the vicinity of the

poles associated with these states. The assumption that the two particle scattering amplitude is dominated by the bound states and resonances, gives rise to an approximation of the two-particle amplitude which is separable in the initial and final variables. Using this type of interaction in the three-body problem, reduces the Faddeev equations which in general are coupled integral equations in two variables, to coupled equations in a single variable.

Yamaguchi⁶ first used the separable approximation of the two nucleon system. Since then it has been used by many people⁷. The approximation gives a good representation of the low energy two-nucleon data. The inclusion of the nucleon-nucleon short range and tensor forces, and the effects of the three-nucleon force if taken into theory would greatly increase the numerical difficulties involved in the solution of the three nucleon problem. Assuming that their effects are small we represent them by a single adjustable parameter¹⁷.

4. The separable approximation of the two-nucleon system

(a) The Separable Interaction

Consider the two nucleon system interacting through potential $\hat{V}$.

The total Hamiltonian has the form

$$\hat{H} = \hat{H}_0 + \hat{V}$$

$\hat{H}$ is self-adjoint and its eigenstates form a complete set. Let $\Lambda(E)$ with $E > 0$ be the projection operators for the continuous eigenstates of $\hat{H}$ on the positive real axis, and $\hat{J}_n$ the projection operators for the bound

states of binding energies E_n . The fact that the eigenstates of $\hat{H}$ form a complete set means

$$\sum_n \hat{J}_n + \int_0^\infty dE \hat{\Delta}(E) = 1 \quad (4.1)$$

For the $\hat{H}$ and $\hat{G}(s)$ one can construct the following spectral representation:

$$\hat{H} = \int_0^\infty dE E \hat{\Delta}(E) - \sum_n E_n \hat{J}_n \quad (4.2)$$

$$\hat{G}(s) = \int_0^\infty dE (E - s)^{-1} \hat{\Delta}(E) - \sum_n (E_n + s)^{-1} \hat{J}_n \quad (4.3)$$

The equation (2.12) gives the relation between the $\hat{T}$ matrix and the resolvent of $\hat{H}$ in the following form:

$$\hat{T}(s) = \hat{V} - \hat{V} \hat{G}(s) \hat{V}.$$

Thus, whenever $\hat{H}$ gives rise to a bound state ϕ_n with binding energy ϵ_n , $\hat{G}(s)$ and consequently $\hat{T}(s)$, will have a pole at $s = -\epsilon_n$. From the relation (4.3) one can obtain

$$\begin{aligned} \langle p | \hat{G}(s) | p' \rangle = & - \sum_n \frac{\langle p | \phi_n \rangle \langle \phi_n | p' \rangle}{E_n + s} \\ & + \int_0^\infty dE \frac{\langle p | \hat{\Delta}(E) | p' \rangle}{E - s} \end{aligned} \quad (4.4)$$

Now the assumption that the two-body interaction is dominated by bound states, makes it possible to write the following approximate relation

$$\langle \underline{p} | \hat{G}(s) | \underline{p}' \rangle = - \sum_n \frac{\langle \underline{p} | \phi_n \rangle \langle \phi_n | \underline{p}' \rangle}{s + \epsilon_n} \quad (4.5)$$

then from relation (2.12)

$$\langle \underline{p} | \hat{T}(s) | \underline{p}' \rangle = \sum_n \frac{\langle \underline{p} | n \rangle \langle n | \underline{p}' \rangle}{s + \epsilon_n} \quad (4.6)$$

where

$$\langle \underline{p} | n \rangle = \langle \underline{p} | \hat{V} | \phi_n \rangle = -(p^2 + \epsilon_n) \langle \underline{p} | \phi_n \rangle.$$

$\langle \underline{p} | n \rangle$ is called the form factor of the bound state. The relation (4.6) shows that the scattering amplitude is separable in off shell variables $\underline{p}$ and $\underline{p}'$. The property which is going to cause a considerable simplification in the three-body problem.

The inspection of the relations (4.5), and (4.6) reveals that they do not contain the unitarity cut due to our approximation. In order to include this cut, these equations can be rewritten in the following form²

$$\langle \underline{p} | \hat{G}(s) | \underline{p}' \rangle = - \sum_n \frac{\langle \underline{p} | \phi_n \rangle S_n(s) \langle \phi_n | \underline{p}' \rangle}{s + \epsilon_n}$$

and

$$\langle \underline{p} | \hat{T}(s) | \underline{p}' \rangle = \sum_n \frac{\langle \underline{p} | n \rangle S_n(s) \langle n | \underline{p}' \rangle}{s + \epsilon_n}$$

where $S_n(s)$ contains the branch cut demanded by unitarity. By setting $S_n(-\epsilon_n) = 1$ the correct residue of the bound state pole is obtained. If there is one bound state per partial wave, we have

$$\hat{T}_l(p', p', s) = \frac{\langle p | nl \rangle G_{nl}(s) \langle nl | p' \rangle}{s + \epsilon_{nl}} \quad (4.8)$$

The relation (4.8) can be achieved by approximating the potential by the sum of the separable potential adjusted to give a good representation of the bound state, $|\phi_{nl}\rangle^2$. One can write

$$\langle p | \hat{V} | p' \rangle = \sum_l (2l + 1) p_l(p, p') \hat{V}_l(p, p') \quad (4.9)$$

where $\hat{V}_l(p, p')$ is given by

$$\hat{V}_l(p, p') = \lambda_{nl} \langle p | nl \rangle \langle nl | p' \rangle \quad (4.10)$$

In the case of the bound state, one has

$$(\hat{H}_0 + \epsilon_{nl}) |\phi_{nl}\rangle = -\hat{V}_l |\phi_{nl}\rangle = -\lambda_{nl} \langle nl \rangle \langle nl | \phi_{nl} \rangle \quad (4.11)$$

The equation (4.11) accepts a solution in the following form:

$$|\phi_{nl}\rangle = -\hat{G}_0(-\epsilon_{nl}) |nl\rangle \quad (4.12)$$

if and only if, there is a positive ϵ_{nl} such that

$$\lambda_{nl} \langle nl | \phi_{nl} \rangle = 1$$

$$\text{i.e.} \quad \lambda_{nl} \langle nl | \hat{G}_0(-\epsilon_{nl}) |nl\rangle = -1 \quad (4.13)$$

Due to positive definiteness of $G_0(s)$ for $s < 0$ equation (4.13) has no solution for $\lambda_{nl} > 0$.

For the continuum states, the substitution of equation (4.10) in the two particle version of (2.12) gives

$$\hat{T}_l(s) = \lambda_{nl} |nl\rangle \langle nl| \{1 - \lambda_{nl} \langle nl| G(s) |nl\rangle\}^{-1} \quad (4.14)$$

Equation (1.14) in the case of the two particle interaction becomes

$$\langle nl | \hat{G}(s) | nl \rangle = \frac{\langle nl | \hat{G}_0(s) | nl \rangle}{1 + \lambda_{nl} \langle nl | \hat{G}_0(s) | nl \rangle} \quad (4.15)$$

Insertion of (4.15) into (4.14) gives

$$\hat{T}_l(s) = |nl\rangle t_{nl}(s) \langle nl| \quad (4.16)$$

where

$$t_{nl}(s) = \left\{ \frac{1}{\lambda_{nl}} + \langle nl | \hat{G}_0(s) | nl \rangle \right\}^{-1} \quad (4.17)$$

In the base of a bound state, equation (4.13) holds and from equation (4.17) one can obtain

$$t_{nl}(s) = \frac{1}{s + \epsilon_{nl}} \left\{ 4\pi \int_0^\infty \frac{dp p^2 |\langle nl | p \rangle|^2}{(p^2 + \epsilon_{nl})(p^2 - s - i\epsilon)} \right\}^{-1} \quad (4.18)$$

The $\hat{T}(s)$ will be given by

$$\hat{T}_l(s) = \frac{|nl\rangle S_{nl}(s) \langle nl|}{s + \epsilon_{nl}}$$

where

$$S_{nl}(s) = \left\{ 4\pi \int \frac{dp p^2 |\langle nl | p \rangle|^2}{(p^2 + \epsilon_{nl})(p^2 - s - i\epsilon)} \right\}^{-1} \quad (4.19)$$

Now the relation for $\hat{T}_l(s)$ has the unitarity cut and a pole at $s = -\epsilon_{nl}$.

From (4.12) the bound state pole has the residue

$$S_{nl}(-\epsilon_{nl}) = \{4\pi \int dp p^2 |\phi_{nl}(p)|^2\}^{-1} \quad (4.20)$$

where the bound state wave function $\phi_{nl}(p)$ is given by

$$\phi_{nl}(p) = \langle p | \phi_{nl} \rangle = \frac{\langle p | nl \rangle}{-G_0(-\epsilon_{nl})}$$

one can see that, by normalizing the bound state wave function to 1, it is possible to get the separable potential scattering amplitude in the desired form of (4.8).

Writing the separable amplitude as the off-shell form of the N/D equations,^{2,17} the equation (4.16) gives

$$\hat{T}_l(p, p'; s) = \frac{N_{nl}^{1/2}(p^2) N_{nl}^{1/2}(p'^2)}{D_{nl}(s)} \quad (4.21)$$

where

$$N_{nl}^{1/2}(p^2) = \langle p | nl \rangle \quad (4.22)$$

and

$$D_{nl}(s) = \frac{1}{\lambda_{nl}} + \langle nl | \hat{G}_0(s) | nl \rangle \quad (4.23)$$

Inserting (4.22) into (4.23) gives

$$D_{nl}(s) = \frac{1}{\lambda_{nl}} + \frac{1}{\pi} \int_0^\infty \frac{2\pi^2 \sqrt{s'} N_{nl}(s') ds'}{s' - s - i\epsilon} \quad (4.24)$$

and by using (4.13), we have

$$D_{nl}(s) = \frac{(s + \epsilon_{nl})}{\pi} \int_0^\infty \frac{2\pi^2 \sqrt{s'} N_{nl}(s') ds'}{(s' + \epsilon_{nl})(s' - s - i\epsilon)} \quad (4.25)$$

The equation (4.22) shows that the function $N_{nl}(s)$, can be evaluated by the form factor of the bound state. Thus it contains the left hand cut. The form factor which is going to be used corresponds to the pole or double pole approximation to the left hand cut. The condition for no bound state from (4.13) is

$$\lim_{E \rightarrow 0^-} \langle nl | \hat{G}_0(E) | nl \rangle < -\frac{1}{\lambda_{nl}} \quad (4.26)$$

because $\langle nl | \hat{G}_0(-X) | nl \rangle < 0$ decreases with increasing X . By equation (4.26) $D_{nl}(s)$ will have no zero on the physical sheet, and $D_{nl}(s)$ can be continued through the two particle unitarity cut by using (4.24).

The relation for the analytic continuation of $D_{nl}(s)$ onto the second sheet¹⁷ is

$$\begin{aligned} D_{nl}^{II}(s + i\epsilon) &= D_{nl}(s - i\epsilon) \\ &= D_{nl}(s + i\epsilon) - 2i \text{Im} D_{nl}(s + i\epsilon) \end{aligned} \quad (4.27)$$

By using the relation

$$\frac{1}{X - X_0 \pm i\epsilon} = \frac{P}{X - X_0} \mp i\pi \delta(X - X_0)$$

and equation (4.24), the equation (4.27) reduces to

$$D_{nl}^{II}(s + i\epsilon) = D_{nl}(s + i\epsilon) - 4\pi^2 \sqrt{s} N_{nl}(s) \quad (4.28)$$

Let ϵ have the negative value then we get from (4.27)

$$D_{nl}^{II}(s - i\epsilon) = D_{nl}(s + i\epsilon)$$

Thus unitarity cut connects just two Riemann sheets through above relation.

For $s = -X$ one gets

$$D_{nl}^{II}(-X) = D_{nl}(-X) + 4\pi^2 \sqrt{\lambda_{nl}}(-X) \quad (4.29)$$

Inserting equation (4.23) in (4.29), the condition for having zero for $D_{nl}^{II}(s)$ at $s = -X$ is

$$-\frac{1}{\lambda_{nl}} = \langle nl | \hat{G}_0(-X) | nl \rangle + 4\pi^2 \sqrt{\lambda_{nl}}(-X) \quad (4.30)$$

Thus, if (4.26) is true, then $D_{nl}^{II}(s)$ will have a zero for real s below the elastic threshold. Moreover if this zero is near to the physical region, it will have a considerable effect in the low energy scattering. This zero corresponds to an anti-bound state. Our spectral representation (4.4) does not give the behaviour of the off-the-energy shell amplitude in the neighbourhood of the anti-bound state. If by relation (4.26) λ_{nl} is made more negative, the anti-bound state goes over to a zero energy bound state. So equation (4.4) can give the residue of the associated pole in the off-shell variables. Assuming that this behaviour is independent of slight variations in λ_{nl} , the use of separable potential, when an amplitude is dominated by anti bound state, is justified.

(b) The Two-Nucleon System

The assumption is that the low energy two-nucleon scattering amplitude is dominated by the deuteron bound state d , and the singlet anti-bound state s . If the small D wave component of the deuteron is ignored, it has spin 1, isospin 0 and orbital angular momentum 0. For the anti-bound state s , isospin is 1 spin and orbital angular momentum both are 0. We take our scattering amplitude pure S -wave, and with the assumption of charge independence, (4.16) gives

$$\begin{aligned} \hat{T}_0(p, p', s) &= \langle p | d \rangle t_d(s) \langle d | p' \rangle \\ &+ \langle p | s \rangle t_s(s) \langle s | p' \rangle \end{aligned} \quad (4.31)$$

Here

$$t_n(s) = D_n(s)^{-1} = \left\{ \frac{1}{\lambda n} + \langle n | \hat{G}_0(s) | n \rangle \right\}^{-1} \quad (4.32)$$

and

$$\langle p | n \rangle \langle n | p' \rangle = p_n^{\sigma\tau} g_n(p) g_n(p') \quad (4.33)$$

$p_n^{\sigma\tau}$ corresponds to the spin-isospin ^{projection} operators and $g_n(p)$ is the radial part of the form factor for the bound state n . For the deuteron wave function the Hulthén wave function has been adopted

$$\phi_d(p) = \frac{-\eta_d}{(p^2 + \mu_d^2)(p^2 + \alpha^2)} \quad (4.34)$$

where α is the binding energy of the deuteron in units $m = \text{nucleon mass} = 1$. μ_d is the average mass of the particles whose exchange produces the potential, and η_d can be determined by the normalization of deuteron wave

function, i.e.

$$4\pi \int_0^\infty dp p^2 |\phi_d(p)|^2 = 1$$

using (4.34)

$$\eta_d = \{\mu_d \alpha (\mu_d + \alpha)^2 / \mu\}^{1/2}$$

The parameters μ_d , λ_d , μ_s , λ_s and η_d are determined by the deuteron binding energy, the triplet and singlet scattering lengths, and the singlet effective range.⁶ From (4.7) and (4.34) one can obtain the s and d form factors

$$g_n(p) = \frac{\eta_d}{p^2 + \mu_n^2} \quad (4.35)$$

Equation (4.22) shows that the $N_n(p^2)$ can be obtained by the form factor of the bound state. So from the N/D point of view equation (4.35) is a double pole approximation to the left hand cut. The relation between the on shell scattering amplitude and the effective range approximation is given by the following equations

$$2\pi^2 T_\ell(k, k, k^2) = \frac{1}{-k \cot \delta_\ell + ik} \quad (4.36)$$

here $k^2 = s$

so

$$-2\pi^2 \hat{T}(k, k, k^2) = \sum_{n=s,d} \frac{p_n^{\sigma\tau}}{a_n - \frac{1}{2} r_n k^2 - p_n r_n^3 - ik} \quad (4.37)$$

where a_n , r_n , and p_n are scattering length, effective range and shape dependence corresponding to the state n , and have simple relations with the parameters μ_n and λ_n .⁸ One can choose an alternative form for the form factor of our s and d states^{1D} i.e.

$$g_n(p) = \frac{\eta_d}{\sqrt{(p^2 + \mu_n^2)}} \quad (4.38)$$

which corresponds to a single pole approximation for the left hand cut.

Consider now the data for the low-energy neutron-neutron and neutron-proton systems. The pp system is excluded because it is not desired to consider the coulomb effect. The triplet parameters are known accurately but there is no such accuracy for the singlet parameters. By the assumption of charge independence one must consider an average neutron-neutron and neutron-proton interaction. Taking the average of

$$\frac{1}{k \cot \delta_0} = \frac{1}{-\frac{1}{a} + \frac{1}{2} rk^2 - pr^3}$$

For the singlet scattering length and effective range one obtains the following relations¹⁷

$$a_s = \frac{1}{2}(a_s^{np} + a_s^{nn}) \quad (4.39)$$

$$r_s = \frac{2}{a_s^{np} + a_s^{nn}} \left[\frac{a_s^{np} a_s^{np} + a_s^{nn} a_s^{nn}}{a_s^{np} a_s^{nn} (r_s^{np} + r_s^{nn})} \right] \quad (4.40)$$

From the experiment, the singlet neutron-proton scattering length is -23.68F ,¹² and for the neutron-neutron system it has the value $-17 \pm 1\text{F}$,⁸⁻¹⁰ which is consistent with the charge symmetric value of -16.96F .¹¹ There are no experimental measurements of the neutron-neutron effective range, so the charge symmetric value of $r_s^{nn} = 2.8455\text{F}$ ¹¹ has been adopted in the present work.

It will not be attempted to take into account effects due to nucleon-nucleon tensor and short range forces as they complicate the calculations. However for the following reasons they should be taken to be small. First, it has been shown by the Takahin that the effect of the high energy nucleon-nucleon repulsion, usually represented by hard core, on the binding energy of the triton is of the order of 5%.¹³ Second, the deuteron is known to have a small D state component, and since the Faddeev equations depend on the solutions of the two-body systems, and not on the original forces between them, it is reasonable to assume that the effect of the tensor force is small. Third, it has been shown by the Gruber¹⁴ that, with respect to three nucleon force, the effect of a meson field-theoretic force of this kind on the binding energy of the triton is less than 2%. Thus, it is reasonable to represent these effects on the three nucleon system by a single adjustable parameter.¹⁷ It is preferable to introduce an additional interaction, for the effects due to tensor, short range, and three body forces by a phenomenological three-body force. One can assume that this force is separable in the initial and final variables, i.e.

$$V_h = \lambda_h |h\rangle\langle h| \quad (4.41)$$

The corresponding form factor will be

$$\langle p_\alpha q_\alpha | h \rangle = \frac{1}{\frac{p_\alpha^2}{2\mu_\alpha} + \frac{q_\alpha^2}{2M_\alpha} + \mu_d^2} \quad (4.42)$$

in (4.41) λ_h represents the single adjustable parameter. Using (2.28) one gets

$$T_h(s) = |h\rangle \tau_h(s) \langle h| \quad (4.43)$$

$$\text{where } \tau_h(s) = \left\{ \frac{1}{\lambda_h} + \langle h | G_O(s) | h \rangle \right\}^{-1} \quad (4.44)$$

5. The Separable Approximation of the Lovelace-Faddeev Equations

It is now attempted to find out what the Lovelace-Faddeev equations (2.23) and (2.24) reduce to, when one assumes that each of the two particle sub-systems, are dominated by a finite number of bound states. In section 4 it was shown that the two particle amplitude in (β, γ) subsystem has the following form

$$\hat{T}_\alpha(s) = \sum_n |an\rangle t_{an}(s) \langle an| \quad (5.1)$$

where n stands for different bound states within one subsystem. In the three particle Hilbert space, (5.1) becomes

$$T_\alpha(s) = \sum_n |an\rangle \tau_{an}(s) \langle an| \quad (5.2)$$

where $|an\rangle$ related to the asymptotic state $|\phi_{an}\rangle$ by (4.12)

$$-G_0(s)|\alpha n\rangle = |\phi_{\alpha n}\rangle \quad (5.3)$$

and $\tau_{\alpha n}(s)$ is an operator in the Hilbert space, corresponding to bound-state scattering by the third particle, and has the following form

$$\langle q_\alpha | \tau_{\alpha n}(s) | q'_\alpha \rangle = \delta_3(q_\alpha - q'_\alpha) t_{\alpha n}(s - \frac{q_\alpha^2}{2\mu_\alpha}) \quad (5.4)$$

By inserting (5.3) into (2.8), the amplitude for elastic and rearrangement, bound state scattering is given by

$$X_{\alpha n, \beta m}(s) = \langle \alpha n | G_0(s) U_{\alpha\beta}^+(s) G_0(s) | \beta m \rangle \quad (5.5)$$

Here α and β can also take the value 4, so that the three body force can also be included. Inserting the $U_{\alpha\beta}^+(s)$ from (2.23) in (5.5) gives

$$\begin{aligned} X_{\alpha n, \beta m}(s) &= \sum_{\gamma r} \langle \alpha n | G_0(s) | \gamma r \rangle \lambda_{\gamma r} \langle \gamma r | G_0(s) | \beta m \rangle \\ &- \sum_{\delta \neq \beta} \int_r X_{\alpha n, \delta r}(s) \tau_{\delta r}(s) \langle \delta r | G_0(s) | \beta m \rangle \end{aligned} \quad (5.6)$$

Equation (5.5) can be redefined by adding an additional term to it in such a way that it does not affect any physical observables. As a result one can obtain a compact form for relation (5.6). Thus:

$$\begin{aligned} X_{\alpha n, \beta m}(s) &= \langle \alpha n | G_0(s) U_{\alpha\beta}^+(s) G_0(s) | \beta m \rangle \\ &- (1 - \delta_{\alpha\beta}) \langle \alpha n | G_0(s) | \beta m \rangle \\ &\{1 + \lambda_{\beta m} \langle \beta m | G_0(s) | \beta m \rangle\} \end{aligned} \quad (5.7)$$

The additional term in relation (5.7) modifies the unphysical amplitudes $X_{4,\beta m}(s)$ or $X_{\alpha n,4}(s)$, but when α and β run from 1 to 3 it cancels; since according to (4.13)

$$\lambda_{\beta m} \langle \beta m | G_0(s) | \beta m \rangle = -1.$$

Now the relation (5.6) takes the following form

$$X_{\alpha n, \beta m}(s) = -Z_{\alpha n, \beta m}(s) - \sum_{\gamma r} X_{\alpha n, \gamma r}(s) \tau_{\gamma r}(s) Z_{\gamma r, \beta m}(s) \quad (5.8)$$

where

$$Z_{\alpha n, \beta m}(s) = (1 - \delta_{\alpha \beta}) \langle \alpha n | G_0(s) | \beta m \rangle. \quad (5.9)$$

Solution of equation (5.8) for the three-body force leads to

$$X_{\alpha n, 4}(s) = -Z_{\alpha n, 4}(s) - \sum_{\gamma r}^3 X_{\alpha n, \gamma r}(s) \tau_{\gamma r}(s) Z_{\gamma r, 4}(s) \quad (5.10)$$

Here α runs from 1 to 3 since according to (5.9), $Z_{4,4}$ and thence $X_{4,4}$ vanishes. Inserting (5.10) into (5.8) one gets the following relation for $X_{\alpha n, \beta m}$ with α and β running from 1 to 3

$$\begin{aligned} X_{\alpha n, \beta m}(s) = & - \{Z_{\alpha n, \beta m}(s) + Z'_{\alpha n, \beta m}(s)\} \\ & - \sum_{\gamma r} X_{\alpha n, \gamma r}(s) \tau_{\gamma r}(s) \{Z_{\gamma r, \beta m}(s) + Z'_{\gamma r, \beta m}(s)\} \end{aligned} \quad (5.11)$$

where

$$Z'_{\alpha n, \beta m}(s) = -Z_{\alpha n, 4}(s) \tau_4(s) Z_{4, \beta m}(s) \quad (5.12)$$

(5.11) is the equation for the transition to and from the bound state scattering channels only. The terms $Z_{\alpha n, \beta m}(s)$, $Z'_{\alpha n, \beta m}(s)$, $\tau_{\alpha n}(s)$ and

$X_{\alpha n, \beta m}(s)$ are operators in the bound state scattering Hilbert space $L_2(q)$ i.e. when a bound state is scattered by a third particle which corresponds to effective two-body system. Hence equation (5.11) is an integral equation in a single momentum variable. Thus, if the two particle subsystem is dominated by a finite number of bound states, the three particle equations reduce to many channel two particle Lippman-Schwinger equations. The effective "potentials" and propagators are $-(Z_{\alpha n, \beta m}(s) + Z'_{\alpha n, \beta m}(s))$ and $-\tau_{\alpha n}(s)$ respectively.

Writing an expression for the disintegration processes, to describe the transition from α channel into the state of three final particles, the scattering amplitude is given from (5.5), (remembering (5.3)) as

$$F_{\alpha n}(s) = -\langle \alpha n | G_0(s) U_{\alpha o}^+(s) \rangle \quad (5.13)$$

Inserting $U_{\alpha o}^+(s)$ from (2.31) into (5.13), and using (5.2), (5.9) and (4.10) we get

$$F_{\alpha n}(s) = \sum_{\beta m} X_{\alpha n, \beta m}(s) \tau_{\beta m}(s) \langle \beta m |$$

after eliminating $X_{\alpha n, 4}$, $F_{\alpha n}(s)$ reduces to

$$\begin{aligned} F_{\alpha n}(s) = & -Z_{\alpha n, 4}(s) \tau_4(s) \langle 4 | \\ & + \sum_{\substack{\beta m \\ \beta \neq 4}} X_{\alpha n, \beta m}(s) \tau_{\beta m}(s) \{ \langle \beta m | -Z_{\beta m, 4}(s) \tau_4 \langle 4 | \} \end{aligned} \quad (5.14)$$

The equations (5.11) and (5.14) are the exact solutions of three-body problem for separable potentials.

It is now attempted to obtain a relation for the bound state wave function. By inserting (5.2) into (2.33), we get

$$|\Psi\rangle = \sum_{\alpha=1}^4 |\Psi^{(\alpha)}\rangle$$

and

$$|\Psi^{(\alpha)}\rangle = -G_0(-E_t) \sum_m |\alpha m\rangle |Q_{\alpha m}\rangle \quad (5.16)$$

where

$$\mu \quad |Q_{\alpha m}\rangle = -\tau_{\alpha m}(-E_t) \sum_{\gamma r} Z_{\alpha m, \gamma r}^{(-E_t)} |Q_{\gamma r}\rangle \quad (5.17)$$

solving (5.17), for the three body force by eliminating $|Q_h\rangle$ one gets

$$|Q_{\alpha m}\rangle = -\tau_{\alpha m}(-E_t) \sum_{\substack{\gamma r \\ \gamma \neq h}} \{Z_{\alpha m, \gamma r}^{(-E_t)} + Z'_{\alpha m, \gamma r}^{(-E_t)}\} |Q_{\gamma r}\rangle \quad (5.18)$$

These sets of homogeneous integral equations have solutions only at an energy equal to the binding energy of the three-particle system. As mentioned previously, the terms involved in these equations are operators in $L_2(q)$. Thus these equations have a single momentum variable due to separable character of T_α and elimination of $|Q_h\rangle$.

6. The Three-Nucleon System

(a) The Elastic scattering Amplitudes

In the following section the Lovelace-Faddeev equations are applied to three nucleon system, with the consideration of the approximation method developed in the previous sections. The omission of tensor forces and the assumption of charge independence implies that the isospin, spin, and orbital angular momentum are conserved. Therefore, it is convenient to choose L.S. representation. One is interested in the quartet and the doublet states with $S = 3/2$, $I = 1/2$ and $S = 1/2$, $I = 1/2$ respectively. The bound state wave function is associated with the doublet state.

Consider now the αn channel state, represented by the total spin and component S and s and isospin and component I, i . Likewise I_α, i_α , and $I_{\beta\gamma}, i_{\beta\gamma}$, denote the isospins and components of the α -nucleon, and (β, γ) subsystem of two nucleons respectively. With S_α, s_α and $S_{\beta\gamma}, s_{\beta\gamma}$, the corresponding spin quantum numbers, the channel form factor given by (5.3) is

$$\begin{aligned} \langle p_\alpha q_\alpha | \alpha n \rangle &= |(I_\beta I_\gamma) I_{\beta\gamma}, I_\alpha, I, i\rangle \\ &\quad |(S_\beta S_\gamma) S_{\beta\gamma}, S_\alpha, S, s\rangle \\ &\quad \delta_3(q_\alpha - q_0) \varepsilon_n(p_\alpha) \end{aligned} \quad (6.1)$$

where q_0 is the relative momentum of α particle with respect to the (β, γ) subsystem. For $n = s$, the singlet anti bound state, $I_{\beta\gamma} = 1$, and $S_{\beta\gamma} = 0$. For $n = d$, the deuteron bound state, $I_{\beta\gamma} = 0$ and $S_{\beta\gamma} = 1$. (6.1) is the

expression for the form factor of total spin S and isospin I . In all the expressions used one is concerned with a single, general spin-isospin state. Thus omission of labels S and I causes no confusion. But for the amplitudes associated with a particular S and I , these labels will be used. As three identical particles are being considered (three nucleon system), the channel states will be labelled by n only. Therefore the completely antisymmetric channel state of three identical nucleons is given by

$$|n\rangle = \sum_{\alpha=1}^3 |\alpha n\rangle \quad (6.2)$$

In all relations particles are ordered cyclically, i.e.

$\alpha = 1$ corresponds to $(2,3) + 1$

$\alpha = 2$ corresponds to $(3,1) + 2$

$\alpha = 3$ corresponds to $(1,2) + 3$

Due to the identity of the particles, the bound state form factor $g_{n\alpha}(p_\alpha)$ is independent of α , but not of n which denotes the different bound states within one subsystem. Therefore, the nondiagonal elements ($\alpha \neq \beta$) of $Z_{\alpha n, \beta m}(s)$, in relation (5.9) are independent of α and β . They will be represented by $Z_{n,m}(s)$. For the diagonal elements, one has, from (5.9), $Z_{\alpha n, \beta m}(s) = 0$ for $\alpha \neq \beta$. Similarly $\tau_{\alpha m}(s)$, which is independent of α , will be represented as $\tau_m(s)$. In the expression for the $X_{\alpha n, \beta m}(s)$, (5.11), all the non diagonal elements ($\alpha \neq \beta$), as well as diagonal elements ($\alpha = \beta$), are independent of α and β . However, the diagonal elements are not necessarily equal to the nondiagonal ones.

Therefore they are represented as

$$X_{\alpha n, \alpha m}(s) = X_{nm}^D(s) \quad \text{for } \alpha = \beta \quad (6.3)$$

$$X_{\alpha n, \beta m}(s) = X_{nm}^N(s) \quad \text{for } \alpha \neq \beta \quad (6.4)$$

Then from the equation (5.11) the following relations are obtained:

$$\begin{aligned} X_{n,m}^D(s) = & -\{Z_{n,m}'^D(s)\} - 2 \sum_r X_{n,r}^N(s) \tau_r(s) \\ & \{Z_{r,m}^N(s) + Z_{r,m}'^N(s)\} \\ & - \sum_r X_{n,r}^D(s) \tau_r(s) Z_{r,m}^D(s) \end{aligned} \quad (6.5)$$

and

$$\begin{aligned} X_{n,m}^N(s) = & -\{Z_{n,m}^N(s) + Z_{n,m}'^N(s)\} \\ & - \sum_r X_{n,r}^N(s) \tau_r(s) \{Z_{r,m}^N(s) + Z_{r,m}'^D(s) + Z_{r,m}'^N(s)\} \\ & - \sum_r X_{n,r}^D(s) \tau_r(s) \{Z_{r,m}^N(s) + Z_{r,m}'^N(s)\} \end{aligned} \quad (6.6)$$

Here the γ summation is carried out in equation (5.11). The $Z_{n,m}^N(s)$ represents the non diagonal elements of $Z_{\alpha n, \beta m}(s)$ and $Z_{n,m}'^N(s)$, $Z_{n,m}'^D$, are the non-diagonal and diagonal elements of $Z_{\alpha n, \beta m}'(s)$ respectively. The suitable combination of the $X_{n,m}^D(s)$ and $X_{n,m}^N(s)$ could couple the equations (6.5) and (6.6).by writing

$$X_{n,m}(s) = X_{n,m}^D(s) + 2X_{n,m}^N(s) \quad (6.7)$$

$$Y_{n,m}(s) = X_{n,m}^D(s) - X_{n,m}^N(s) \quad (6.8)$$

The term $X_{n,m}(s)$ represents the observed bound state scattering amplitude, because the final pairings of the particles ($\alpha = 1, 2, 3$) are indistinguishable and summing must be carried out over them. The term $Y_{n,m}(s)$ does not represent anything observable, and there is no need to calculate it. Insertion of the equations (6.5) and (6.6) into equations (6.7) and (6.8) give:

$$\begin{aligned} X_{n,m}(s) = & -\{2Z_{n,m}^N(s) + 2Z_{n,m}'^N(s) + Z_{n,m}'^D(s)\} \\ & - \sum_r X_{n,r}(s) \tau_r(s) \{2Z_{r,m}^N(s) + 2Z_{r,m}'^N(s) + Z_{r,m}'^D(s)\} \end{aligned} \quad (6.9a)$$

and

$$\begin{aligned} Y_{n,m}(s) = & \{Z_{n,m}^N(s) + Z_{n,m}'^N(s) - Z_{n,m}'^D(s)\} \\ & + \sum_r Y_{n,r}(s) \tau_r(s) \{Z_{r,m}^N(s) + Z_{r,m}'^N(s) - Z_{r,m}'^D(s)\} \end{aligned} \quad (6.9b)$$

it can be seen that the equations for $X_{n,m}(s)$ are independent of those for $Y_{n,m}(s)$. With the aid of (5.4), the equation (6.9a) can be written in the following form

$$\begin{aligned} \langle q | X_{n,m}(s) | q' \rangle = & \langle q | U_{n,m}(s) | q' \rangle \\ & + \sum_r \int dq'' \langle q | X_{n,r}(s) | q'' \rangle \tau_r(s - \frac{q''^2}{2\mu}) \\ & \langle q'' | U_{r,m}(s) | q' \rangle \end{aligned} \quad (6.9)$$

where

$$U_{n,m}(s) = -\{2Z_{n,m}^N(s) + 2Z_{n,m}'^N(s) + Z_{n,m}'^D(s)\}$$

and $t_r(s - \frac{q^2}{2\mu})$ is given by (4.32). By using equations (5.9) and (6.1) one obtains the following relation for the dominant contribution to the effective neutron-deuteron potential

$$\begin{aligned} -2\langle q | Z_{n,m}^N(s) | q' \rangle &= -2S_{Rn,m} I_{Rn,m} g_n(p_1) \\ \int \delta_3(q - q_1) d^3p_1 d^3q_1 \langle p_1 q_1 | G_0(s) | p'_2 q'_2 \rangle d^3p'_2 d^3q'_2 g_m(p'_2) \delta_3(q' - q'_2) \end{aligned} \quad (6.10)$$

where $S_{Rn,m}$ stands for the three-particle spin recoupling coefficient and has the following expression

$$S_{Rn,m} = \langle (S_2 S_3) S_{23}, S_1, S | (S_3 S_1) S_{31}, S_2, S \rangle$$

$I_{Rn,m}$ is the corresponding isospin recoupling coefficient. Inserting (1.1a) and (1.2b) in (6.10) with $m_\alpha = m_\beta = m = 1$, and after carrying out integration one gets:

$$\begin{aligned} -2\langle q | Z_{n,m}^N(s) | q' \rangle &= -2S_{Rn,m} I_{Rn,m} \\ &\frac{g_n(\frac{1}{2}q+q') g_m(q+\frac{1}{2}q')}{q^2+q'^2+qq'-s-i\epsilon} \end{aligned} \quad (6.11)$$

Similarly, for the terms $Z_{n,m}'^D(s)$ and $Z_{n,m}'^N(s)$, using the equations (5.12) and (6.1) the following relation is obtained:

$$\begin{aligned}
-\langle q | Z'_{n,m}(s) | q' \rangle &= - \int \frac{d^3 p g_n(p) \langle p q | t_4(s) \rangle}{p^2 + \frac{3}{4} q^2 - s - i\epsilon} \\
&\quad - \int \frac{d^3 p' \langle 4 | p' q' \rangle g_m(p')}{p'^2 + \frac{3}{4} q'^2 - s - i\epsilon}
\end{aligned} \tag{6.12}$$

and

$$-2 \langle q | Z'_{n,m}(s) | q' \rangle = -2 S_{Rn, m Rn, m} \langle q | Z'_{n,m}(s) | q' \rangle \tag{6.13}$$

Using the well known formula

$$\frac{1}{p^2 - s - i\epsilon} = \frac{p}{p^2 - s} + i\pi \delta(p^2 - s)$$

and (4.35) from the equations (4.17) and (4.18) one gets

$$t_n^{-1}(k^2) = \frac{1}{\lambda_n} + \frac{\pi^2 \eta_d^2}{\mu_n} \frac{(\mu_n^2 - k^2)}{\mu_n^2} + \frac{2\pi^2 i k \eta_d^2}{(\mu_n^2 + k^2)^2} \tag{6.14}$$

above the elastic threshold, and

$$t_n^{-1}(-\gamma^2) = \frac{1}{\lambda_n} + \frac{\pi^2 \eta_d^2}{\mu_n (\mu_n + \gamma)^2} \tag{6.15}$$

for energies below. For the deuteron-nucleon propagator from equation

(4.18) one obtains

$$\begin{aligned}
t_d(s - \frac{q^2}{2\mu}) &= \frac{-2\mu}{q^2 - q_0^2 - i\epsilon} \left\{ \frac{4\pi \int d^3 p g_d(p)^2}{(p^2 + \alpha^2)(p^2 + \frac{q^2}{2\mu} - s - i\epsilon)} \right\}^{-1} \\
&= \frac{-2\mu S_d(s - \frac{q^2}{2\mu})}{q^2 - q_0^2 - i\epsilon}
\end{aligned} \tag{6.16}$$

where

$$s = \frac{q_0^2}{2\mu} - \alpha^2 + , \quad (6.17)$$

and $S_d(s - \frac{q^2}{2\mu})$ is given by (4.19)

The partial wave analysis reduces equation (6.9) to an integral equation in a single variable. The partial wave bound state scattering amplitude is given by

$$\begin{aligned} X_{n,ml}(q, q'; s) &= U_{n,ml}(q, q'; s) \\ &- \sum_r \int 4\pi dq'' q''^2 X_{n,rl}(q, q''; s) t_r(s - \frac{q''^2}{2\mu}) \\ &U_{r,ml}(q'', q'; s) \end{aligned} \quad (6.18)$$

Inserting (4.35) in (6.11) and introducing the variables

$$Y_1 = (q^2 + q'^2 - s - i\epsilon)/qq' \quad (6.19)$$

$$Y_2 = (\frac{q^2}{4} + q'^2 + \mu_n^2)/qq' \quad (6.20)$$

$$Y_3 = (q^2 + \frac{q'^2}{4} + \mu_m^2)/qq' \quad (6.21)$$

the $Z_{n,m}^N(q, q'; s)$ is given by

$$Z_{n,m}^N(q, q'; s) = S_{Rn,m} I_{Rn,m} \frac{(n_d)^2}{(qq')^3 (Y_1 + \cos\theta)(Y_2 + \cos\theta)(Y_3 + \cos\theta)}$$

from which:

$$Z_{n,ml}(q,q';s) = S_{Rn,m} I_{Rn,m} (-1)^{l+1} \frac{(n_d)^2}{(qq')^3} \left\{ \frac{Q_2(Y_1)}{(Y_2 - Y_1)(Y_3 - Y_1)} + \frac{Q_2(Y_2)}{(Y_3 - Y_2)(Y_1 - Y_2)} + \frac{Q_2(Y_3)}{(Y_1 - Y_3)(Y_2 - Y_3)} \right\} \quad (6.22)$$

where $Q_2(Y)$ is the Legendre function of the second kind and is given by

$$Q_l(Y) = \frac{(-1)^{l+1}}{2} \int_{-1}^{+1} \frac{d(\cos \theta) P_l(\cos \theta)}{Y + \cos \theta}$$

and for $q = 0$ or $q' = 0$

$$Z_{n,ml}^H(q,q';s) = \frac{S_{Rn,m} I_{Rn,m} n_d^2 \delta_{l0}}{(qq')^3 Y_1 Y_2 Y_3} \quad (6.23)$$

With regard to the contribution for the partial wave potential of three-body force (4.41), the terms $\langle q | Z_{n,m}^D(s) | q' \rangle$ and $\langle q | Z_{n,m}^N(s) | q' \rangle$ are pure s wave. Because the term $q \cdot q'$ does not appear in the equations (6.12) and (6.13).

Equation (6.18) forms the basis of the predictions for n-d scattering. The spin-isospin recoupling coefficients used in these calculations are given by the notation of Edmonds¹⁸ as

$$\begin{aligned} & \langle (S_\alpha S_\beta) S_{\alpha\beta}, S_\gamma, s | (S_\beta S_\gamma) S_{\beta\gamma}, S_\alpha, s \rangle \\ &= (-1)^{2S_\alpha + S_\beta + S_\gamma} \{ (2S_{\alpha\beta} + 1)(2S_{\beta\gamma} + 1) \}^{1/2} \\ & \quad \begin{pmatrix} S_\alpha & S_\beta & S_{\alpha\beta} \\ S_\gamma & S & S_{\beta\gamma} \end{pmatrix} \end{aligned} \quad (6.24)$$

a similar relation holds for the isospin recoupling coefficients. In the problem under consideration, $S_\alpha = S_\beta = S_\gamma = \frac{1}{2}$, while $S_{\alpha\beta}$ and $S_{\beta\gamma}$ are 1 for the d particle, or 0 for the s particle. The isospin recoupling coefficients are exactly similar, except that the roles of the s and d particle are now interchanged. From the equation (6.24), for the three nucleon system:

$$S_{Rn,m} = \begin{matrix} s & d \\ s & \frac{1}{2} & -\sqrt{3/2} \\ d\sqrt{3/2} & -\frac{1}{2} \end{matrix} \quad (6.25)$$

and

$$I_{Rn,m} = \begin{matrix} s & d \\ s & \frac{1}{2} & \sqrt{3/2} \\ d & -\sqrt{3/2} & -\frac{1}{2} \end{matrix} \quad (6.26)$$

For the quartet state, from (6.24)

$$-2S_{Rn,m} I_{Rn,m} = 1 \quad (6.27)$$

and for the doublet state

$$-2S_{Rn,m} I_{Rn,m} = \begin{matrix} s & d \\ s & \frac{1}{2} & 3/2 \\ d & 3/2 & -\frac{1}{2} \end{matrix} \quad (6.28)$$

Thus the quartet state is independent of the singlet two-nucleon interaction, and equation (6.18) reduces to a single channel Lippmann-Schwinger equation. The correction s wave terms $-2Z_{n,m}^N(s)$ and $-Z_{n,m}^D(s)$ which arise from the three-body force in equation (6.18) cancel, because

$-2S_{Rn,m}^I R_{n,m} = 1$ (see equation (6.13)). The doublet partial wave amplitude, on the other hand, satisfies a two-channel Lippman-Schwinger equation related by (6.25). Here the triplet and singlet two-nucleon interactions contribute, and the effect of the three-body force is to weaken the overall attractive forces in the s wave state.

(b) The Disintegration Amplitudes

If the small three-body force contribution in (5.14) is neglected, then the quartet and doublet neutron-deuteron disintegration amplitudes will be given by^{2,17}

$$\begin{aligned}
 \langle i | F_{ad}^s(s) | f \rangle &= \sum_{\substack{m \\ \beta \neq 4}} \langle i | X_{ad,\beta m}^s(s) \tau_{\beta m}(s) | \beta m \rangle \langle \beta m | f \rangle \\
 &= \sum_m \langle i | X_{d,m}^{Ds}(s) \tau_{1m}(s) | 1m \rangle \langle 1m | f \rangle \\
 &\quad + \sum_m \langle i | X_{d,m}^{Ns}(s) \tau_{2m}(s) | 2m \rangle \langle 2m | f \rangle \\
 &\quad + \sum_m \langle i | X_{d,m}^{Is}(s) \tau_{3m}(s) | 3m \rangle \langle 3m | f \rangle
 \end{aligned} \tag{6.29}$$

By using the equations (6.7) and (6.8)

$$\begin{aligned}
 \frac{1}{3} X_{n,m}(s) &= \frac{1}{3} X_{n,m}^D(s) + \frac{2}{3} X_{n,m}^N(s) \\
 \frac{1}{3} Y_{n,m}(s) &= \frac{1}{3} X_{n,m}^D(s) - \frac{1}{3} X_{n,m}^N(s)
 \end{aligned}$$

the relation (6.29) reduces to

$$\begin{aligned}
\langle i | F_{\alpha d}^S(s) | f \rangle &= \frac{1}{3} \sum_m \sum_{\beta \neq 4} \langle i | X_{d,m}^S(s) \tau_{\beta m}(s) \langle \beta m | f \rangle \\
&+ \frac{1}{3} \sum_m \{ 2 \langle i | Y_{d,m}^S(s) \tau_{1m}(s) \langle 1m | f \rangle \\
&- \langle i | Y_{d,m}^S(s) \tau_{2m}(s) \langle 2m | f \rangle \\
&- \langle i | Y_{d,m}^S(s) \tau_{3m}(s) \langle 3m | f \rangle \}
\end{aligned} \tag{6.30}$$

If the forces are charge independent, then the contributions depending on $Y_{n,m}$ cancel out because the final pairings of the particles cannot be distinguished.

(c) The Three-Nucleon Bound state (Triton)

Consider (nnp) bound state (triton) with $I = S = \frac{1}{2}$. The bound state wave function is given by (2.32), (2.33), (5.16) and (5.18). In the case of the identical particles the bound state wave function is

$$|\Psi\rangle = \sum_{\alpha=1}^4 |\Psi(\alpha)\rangle$$

where

$$|\Psi(\alpha)\rangle = -G_0(-E_t) T_\alpha(-E_t) \sum_{\beta=\alpha}^4 |\Psi(\beta)\rangle$$

For $\alpha \neq 4$ we get from (5.16)

$$\langle p_\alpha q_\alpha | \Psi(\alpha) \rangle = \frac{-1}{p_\alpha^2 + \frac{3}{4} q_\alpha^2 + E_t} \sum_{m=s, d} \langle p_\alpha | \alpha m \rangle Q_m(q_\alpha) \tag{6.31}$$

where $Q_m(q_\alpha)$ is given for the identical particles, by using (5.18), through the following relation

$$Q_m(q_\alpha) = -t_m(-E_t - \frac{3}{4}q^2) \int_r \int_{U_{m,r_0}} (q, q' - E_t) 4\pi q'^2 dq' Q_r(q') \quad (6.32)$$

If $\alpha = 4$, then we have

$$\langle p_\alpha q_\alpha | \psi^{(\alpha)} \rangle = \frac{-1}{p_\alpha^2 + \frac{3}{4}q_\alpha^2 + E_t} \langle p_\alpha q_\alpha | 4 \rangle Q_4(p_\alpha q_\alpha) \quad (6.39)$$

Now, it can be seen that if the two nucleon system is dominated by d bound state and singlet anti-bound state s, then the three nucleon wave function is represented as the sum of products of d or s wave function

$$\frac{-1}{p_\alpha^2 + \frac{3}{4}q_\alpha^2 + E_t} \langle p_\alpha | \alpha m \rangle$$

and the corresponding relative wave function of the spectator nucleon, $Q_m(q_\alpha)$. Those parts of the wave function that cannot be represented in this way are crudely accounted for by the three-body force term $|\psi^{(4)}\rangle$. So by knowing the s and d form factors, the three-nucleon wave function can be found by solving equation (6.32) for the spectator wave function $Q_m(q)$ and inserting it into (6.31). The equation (6.32) also gives the binding energy of the triton if it is written in the following form¹⁷

$$K_{ij}(E_t) X_j(E_t) = \eta X_j(E_t) \quad (6.34)$$

If E_t is the binding energy of the triton, $\eta_{\max} = 1$.

If the bound state wave function is studied in more detail, the spin-isospin structure of $\psi^{(\alpha)}$, is included in the term $|\alpha m\rangle$. e.g.

$$\langle p_1 | 1m \rangle = g_m(p_1) X_{1m} \eta_{1m} \quad (6.35)$$

where

$$\eta_{1m} = |(I_2 I_3) I_{23}, I_1, I, i\rangle \quad (6.36)$$

$$X_{1m} = |(S_2 S_3) S_{23}, S_1, S, s\rangle \quad (6.37)$$

If $\psi^{(4)}$ is neglected, by assuming that it is small for all reasonable values of V_h , from equation (6.31) one can write:

$$\langle p_\alpha, q_\alpha | \psi \rangle = \sum_{\alpha=1}^3 \sum_{m=s,d} X_{\alpha m} \eta_{\alpha m} f_{\alpha m}(p_\alpha, q_\alpha) \quad (6.38)$$

where

$$f_{\alpha m}(p_\alpha, q_\alpha) = \frac{-1}{p_\alpha^2 + \frac{3}{4} q_\alpha^2 + E_t} g_m(p_\alpha) q_m(q_\alpha) \quad (6.39)$$

Of six spin states $X_{\alpha m}$ only two are linearly independent, e.g. $X_{\alpha s}$, $X_{\alpha d}$ similar situation hold for isospin states $\eta_{\alpha m}$. The combination of spin and isospin states is formed in the following manner:

$$\phi_0 = \frac{1}{\sqrt{2}} (X_{1s}\eta_{1s} - X_{1d}\eta_{1d}) \quad (6.40)$$

$$\phi_1 = \frac{1}{\sqrt{2}} (X_{1s}\eta_{1d} - X_{1d}\eta_{1s}) \quad (6.41)$$

$$\phi_2 = \frac{-1}{\sqrt{2}} (X_{1s}\eta_{1s} + X_{1d}\eta_{1d}) \quad (6.42)$$

The function ϕ_0 is fully antisymmetric for the interchange of nucleons 2 and 3. Whereas the functions ϕ_1 and ϕ_2 are symmetric and antisymmetric respectively. By introducing the following function

$$g_\alpha(p_\alpha, q_\alpha) = \frac{-\sqrt{3}}{2} \{f_{ad}(p_\alpha, q_\alpha) + f_{as}(p_\alpha, q_\alpha)\} \quad (6.43)$$

the triton wave function (6.38) becomes

$$\langle p_\alpha, q_\alpha | \psi \rangle = \phi_0 u + (\phi_1 v_2 - \phi_2 v_1)$$

where

$$u = \frac{1}{\sqrt{2}} \sum_{\alpha=1}^3 \{f_{ad} - f_{as}\}$$

$$v_1 = \frac{1}{\sqrt{6}} \{g_2 + g_3 - 2g_1\}$$

$$v_2 = \frac{1}{\sqrt{2}} \{g_3 - g_2\}$$

The term $\phi_0 u$, which is fully symmetric in the space coordinates of all three nucleons, is dominant state. The term $(\phi_1 v_2 - \phi_2 v_1)$ corresponds to the state of mixed symmetry.

7. The Triton Binding Energy and the Neutron-Deuteron scattering length

The doublet and quartet scattering length, and the triton binding energy are of most theoretical interest. Their experimental values can be read in the first row of table 1.² There are two sets of scattering lengths, set 1 and set 2, both of which are consistent with the low-energy neutron-deuteron data. The solution of the s wave equation (6.18) at the elastic threshold gives the scattering lengths. The Triton binding energy is determined by (6.32).

The quartet scattering length from table 1 is 6.28F. which agrees with previous theoretical values^{4,16,15} and with the experimental results associated with set 1 (table 1). The triton binding energy and the doublet scattering length have been calculated for the charge symmetric value of np singlet effective range of 2.73F, are 11.1 Mev and -0.79F. Now it can be seen that the low energy two-nucleon interaction while giving a reasonable description of the quartet state, gives too much attraction in the doublet state.

Consider the results when the effects of the nucleon-nucleon tensor and short range forces on (nmp) system are represented by the three body force V_h (equation (4.41)). For the quartet state the terms $2Z'_{n,m}^N(s)$ and $Z'_{n,m}^D(s)$ in (6.9) cancel and so this state is independent of the three-body force. The doublet state seems to be sensitive to V_h . However, it appears that one cannot obtain the set 2 doublet scattering length for any reasonable value of V_h , nevertheless a good value for the triton binding energy and set 1 doublet scattering length has been obtained.

In particular for a singlet effective range, $r_s^{np} = 2.73F$, the value of V_h which corresponds to a doublet scattering length of $0.7F$ is repulsive and is associated with $\sim 4.3\%$ effect, i.e. the value of $\langle q | Z'_{n,m}(s) | q' \rangle$ in equation (5.11) is 0.043 of $\langle q | Z_{n,m}(s) | q' \rangle$ for $q = q' = 0$ at the elastic threshold. It seems reasonable to consider 4% as a good measure for effects due to nucleon-nucleon tensor and short range forces.

Table 1 contains the values of scattering lengths, and triton binding energy, both for $V_h = 0$ and $V_h \neq 0$. These values are compared with similar theories, i.e. theories by Mitra and Bhasin (MB)¹⁵, Sitenko and Kharenchenko (SK)¹⁶, and Aaron, Amado and Yam (AAY)⁴. The parameter Z introduced in (AAY)'s theory corresponds to the deuteron wave function renormalization. They have taken into account the effects of tensor forces by increasing Z , from $Z = 0$ which corresponds to the separable potential limit of their theory. The table 1 shows that the results obtained with $V_h = 0$ agree with the results of (AAY) with $Z = 0$. The calculation with $V_h \neq 0$ gives the set 1 scattering lengths and a reasonable value of the triton binding energy.

Triton binding energy (MeV)		Doublet scattering length (F)	Quartet scattering length (F)
Experimental ²¹			
8.49		Set 1, 0.7 ± 0.3	6.38 ± 0.06
		Set 2, 8.26 ± 0.12	2.6 ± 0.2
Phillips (this work)			
$V_h = 0$	11.1	-0.79	6.28
$V_h \neq 0$	9.1	0.7	6.28
MB		9.25	5.91
SK	12.5	-2.76	6.28
AAY			
$Z = 0$	11.0	-1.01	6.28
$Z = 0.0488$	8.76	0.7	6.14

Conclusions

Recalling the ideas behind the Lovell's approximation method presented in the previous chapters, the main philosophy is to approximate the nucleon-nucleon interaction, and to treat the three-body aspect of the problem exactly. The advantage is that the approximation is applied to that part of the problem which is relatively better known. Particularly, having chosen the two-nucleon scattering amplitudes, one can predict several aspects of the three-nucleon problem. For example, the binding energy of the triton, the values of the s-wave doublet and quartet scattering lengths, and the structure of the three nucleon bound state wave function. With the help of the low energy two-nucleon parameters, and a small contribution due to phenomenological three-body force which represents the two-nucleon tensor and short range forces, quantitative agreement with the experimental data has been obtained for the triton binding energy, doublet and quartet scattering lengths. So it could be argued that this agreement supports the basic assumptions of the approximation scheme.

Criticism

The standard theory of three-body problem these days is the Faddeev equations or one of its variants. In order to obtain any of these systems of equations, the Lippmann-Schwinger equations are rearranged in such a way that one avoids the δ -function singularities which are present in the kernel of the Lippmann-Schwinger equations and accounts for the disconnected diagrams indicating that one of the particles is disconnected while the other two interacting. In the Faddeev equations the kernels depend on the off-shell two body T-matrix and as shown by Faddeev¹ that the kernels are mathematically well defined and square integrable, so that the equations admit solution by the Fredholm method. But to carry out such a solution with a full nucleon-nucleon potential is a tremendously difficult job and with present day computational facilities almost impossible. So when one speaks about the "exact" solution of the three-body problem one must bear in mind its limitations. However, one tries to find some simplified form of two-body interaction which would facilitate a three body treatment in an exact fashion.

Apparently the choice of the non-local separable potential in the Lovelace's² approximation method was in response to such a motive. In fact, Yamaguchi⁶ earlier had discovered that the two nucleon problem happened to have an explicit algebraic solution with separable potentials which fitted the two-nucleon data in sufficient details through the inclusion of the tensor forces. But such an artificial choice only could be defended on the ground that our ignorance about the exact form of the two body interaction might well be turned to our advantage in

treating the three-body problem.

Let us look at the shortcomings of such an approach. First of all one does not solve the three-body problem exactly and by choosing such a potential one reduces the magnitude of the three body problem in a way that no further approximation in its final solution is necessary. Obviously this is not a good substitute for a true potential and the only justification seems to be that the true potential is not known. On the other hand, the separable potentials are highly non-local which is not desirable due to our intuition and experience. Furthermore, it is now well established that the longest-range part of the nucleon force - the one pion exchange region - can be characterized by a Yukawa type potential which is local. Thus the essential physics behind the Lovelace's approximation scheme has been confined to very simple (rather idealized) forms of the input potentials which are easy to manipulate mathematically. It seems that one is not justified to claim that such an approach could provide new physical information over what can be done by less sophisticated techniques, e.g. Schrodinger equation with the same input information. In other words, one can get more output information by feeding in more input information. For instance, adding terms like tensor force, short range repulsion etc., parameterized in a suitable manner and inserted in three-body framework. The A.C. Phillips's calculation of Lovelace's theory has taken into account such effects by introducing a phenomenological three-body force in the doublet channel with a single adjustable parameter in order to get an agreement with the triton binding energy and the n - d scattering length. But because tensor force and hard

core are typically potential concepts, it has its natural place in Schrodinger equations rather than Faddeev formalism, where the emphasis is on scattering amplitudes. As shown in previous chapters, the separable interaction is not justified if the condition of pole dominance of two body interaction is not satisfied. Indeed, the original work of Lovelace² suggested that the approximation is not valid for the p -wave interaction and tensor forces. So the inclusion of tensor forces in a separable form by A.C. Phillips can be thought of as disregarding this mathematical criterion.

Finally if we take p and p' as the centre of mass momenta of two nucleons before and after interaction respectively, then as demonstrated in previous chapters, the central separable potential can be written:

$$\langle p|V|p' \rangle = -\lambda g(p)g(p')$$

where λ is a positive constant for attractive interactions and $g(p)$ is given by the equation

$$g(p) = (2\pi)^{-3/2} \int d\mathbf{r} e^{i\mathbf{p} \cdot \mathbf{r}} V(\mathbf{r}).$$

Subsequently the Schrodinger equation in momentum space assumes the following form:

$$(p^2 - E)\psi(p) = \lambda \int dp' g(p)g(p')\psi(p')$$

where E is the centre of mass energy of the system. For the bound state (deuteron) one puts

$$E = +E_B = -\alpha^2,$$

and one gets the eigenvalue equation

$$\lambda^{-1} = \int dq g^2(q)(q^2 + \alpha^2)^{-1},$$

and the (normalized) wave function $\psi(p) = Hg(p)(\alpha^2 + q^2)^{-1}$, where

$$N^{-2} = \int dq g^2(q)(\alpha^2 + q^2)^{-2}.$$

It appears that in a separable interaction treatment one tries to get the eigenvalue of the strength parameter, λ , rather than of the binding parameter, α , although the physical problem is in reality the opposite. This sort of structure is characteristic of treatment with separable potential.

Appendix A

Numerical Methoda. Quadrature Rules

In this appendix the numerical method for solving a one-variable integral equation is given⁽²⁰⁾. The aim is to approximate an integral equation by a finite linear system which can be solved with a computer. For illustration, suppose an integral equation in the following form:

$$f(x) = g(x) + \int_a^b K(x,y)f(y)dy \quad a < x < b \quad (A.1)$$

where $f(x)$ is an unknown function, and $g(x)$, $K(x,y)$ are known functions. Furthermore it is assumed that a unique solution $f(x)$ exists. In order to transform eq. (A.1) into a finite matrix problem, the integral should be replaced with a sum.

The techniques which allow us to approximate integrals by finite sums are called quadrature rules. Let the set

$$D(N) = \{Y_i : Y_i \in (a,b) \quad i \leq N\}$$

be a discrete set of N points on the interval (a,b) of integration. The integrand can be split to a product of two factors, $W(x)$, $h(x)$, where $W(x)$ contains all the singularities and rapidly varying portion of the integrand, and $h(x)$ is a smooth, singularity-free function with a rapidly converging Taylor series. With these requirements the general form of a quadrature rule is

$$\int_a^b h(y)W(y)dy = \sum_{i=1}^N h(y_i)W_i + \epsilon_N(h) \quad (A.2)$$

Here $W(y)$ is called the weighting function, W_i are the weights, and $\epsilon_N(h)$ is the error associated with this technique. $\{W_i\}$ are a set of N numbers determined by the condition that $\epsilon_N(h)$ be small. To be specific, one can calculate the weights by requiring that $\epsilon_N(Y^k)$ be zero for all values of

$k = 0, 1, \dots, N-1$. That is, the following integral:

$$I_k = \int_a^b Y^k W(Y) dY$$

can be calculated either analytically or by some numerical scheme for each integer value of k , so that the W_i are the solution of linear $N \times N$ matrix problem given by

$$I_k = Y_1^k W_1 + Y_2^k W_2 + \dots + Y_N^k W_N, \quad (A.3)$$

$$k = 0, \dots, N-1$$

There exists a simple relation between such a technique and the Taylor series expansion of $h(Y)$. Let the series be written

$$h(Y) = \sum_{n=0}^{\infty} a_n Y^n \quad Y \in (a, b) \quad (A.4)$$

$$= \sum_{n=0}^{N-1} a_n Y^n + R_N(Y)$$

where $R_N(Y)$ is the remainder term. The definition of the W_i implies that the error term is just

$$\epsilon_N(h) = \int_a^b R_N(Y) W(Y) dY. \quad (A.5)$$

So the first N terms in the power series for $h(y)$ can be integrated by

the quadrature rule, eq. (A.2). The quadrature rule is valid if $|\epsilon_N(h)| < \delta$, where δ is some small positive number. For a given δ , N can be increased in such a manner as to satisfy the above inequality.

Now the equation (A.1) can be transformed into a matrix equation by using a quadrature rule to replace the integral term. Thus $f(x)$ is given by:

$$f(x) = g(x) + \sum_{i=1}^N K(x, y_i) f(y_i) W_i + \epsilon_N(K(x, a) f(o)). \quad (A.6)$$

This relation is true for all x (a,b). If x is considered as N point subset $D(N)$ of (a,b), then eq. (A.6) becomes N equations for N unknown $f(y_i)$. The assumption that the quadrature rule is accurate enough implies that $\epsilon_N(K(x, o) f(o)) < \delta$, where δ is small enough so that $\epsilon_N(K(x, o) f(o))$ may be neglected from the right hand side of eq. (A.6). The resulting $N \times N$ matrix equation is then

$$\tilde{f}(x_i) = g(x_i) + \sum_{j=1}^N K(x_i, y_j) \tilde{f}(y_j) W_j. \quad (A.7)$$

The tilde over f has been used to indicate that this equation is different from eq. (A.6) because the term $\epsilon_N(K(x, o) f(o))$ is neglected. The linear system (A.7) can be solved on a computer if N is not too large.

b. Error Analysis

The function $\epsilon_N(K(x_i, o)f(o))$ is not known unless the form of $f(x)$ is given exactly. However, it can be small enough when the integration rules are adequate. We now examine the consequences of neglecting $\epsilon_N(K(x_i, o)f(o))$. Let $\tilde{f}(x_i)$ be the solution of eq. (A.7), and $f(x_i)$ the solution of following equation

$$f(x_i) = g(x_i) + \sum_{j=1}^N K(x_i, y_j) f(y_j) W_j + \epsilon_N(K(x_i, o)f(o)) \quad (A.8)$$

The $E(x_i)$ is defined as

$$E(x_i) = f(x_i) - \tilde{f}(x_i)$$

and satisfies

$$E(x_i) = \epsilon_N(K(x_i, o)f(o)) + \sum_{j=1}^N K(x_i, y_j) E(y_j) W_j. \quad (A.9)$$

Thus the error term is given by

$$E(x_i) = \sum_{j=1}^N [1 - KW]_{ij}^{-1} \epsilon_N(K(x_j, o)f(o)) \quad (A.10)$$

where $[1 - KW]_{ij}^{-1}$ is the inverse of the matrix determined by

$\delta_{ij} - K(x_i, y_j) W_j$. So it can be concluded that if

$$|\epsilon_N(K(x_i, o)f(o))| \ll |g(x_i)|,$$

a small percentage error is expected in replacing $f(x_i)$ by $\tilde{f}(x_i)$. Once the $f(x_i)$ has been obtained, then $f(x)$ can be found for all x by using

eq. (A.5) and neglecting the error term.

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