Suboptimal nonlinear feedback control laws for collective dynamics

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***Abstract*— We present feedback control methodologies for the stabilization of nonlinear collective dynamics. The proposed controllers circumvent the curse of dimensionality associated to dynamic programming of large-scale nonlinear dynamics by means of different structural assumptions on the nonlinearity and the associated Hamilton-Jacobi-Bellman equation. We explore a power series expansion method, and a state-dependent Riccati equation approach. The proposed designs are studied for a relaxed optimal power flow model, and for binary interactions arising in opinion and consensus dynamics.**

I. Introduction

Collective dynamics arise in the mathematical modelling of large–scale physical and technical systems as well as in animal and human behaviour, see e.g. [2], [4], [5], [17], [20], [24], among many others. Among the different models which fall under these categories, we are particularly interested in collective dynamics which can be cast as

$$\dot{x}_i(t) = \frac{1}{n}\sum_{j=1}^{n} P(x_i,x_j)(x_j-x_i) + u_i(t), \qquad i=1,\ldots,n,$$

where $x_i(t)$ represents the state of the $i-$th agent $\in \mathbb{R}^d$, and the binary kernel $P(\cdot,\cdot)$ encompasses different interaction rules between agents, such as attraction, repulsion, or alignment. The term $u_i(t)$ corresponds to a dynamical external action. A particular feature of such models is their rich dynamical structure, which include different types of self-organization patterns, including consensus, flocking, and milling, among many others. We are interested in the design of centralized control actions which are able to steer multi–agent dynamics towards a prescribed regime by using feedback control techniques.

The topic of emergent collective behaviour in multi-agent systems has been linked the study of pattern formation and self-organization phenomena ([18], [15]). Here, we follow a different approach, enforcing consensus by optimizing the intervention of a centralized external policy maker. We do not attempt to address the decentralization of the control action, which is a fundamental topic for agent-based models in areas such as robotics or autonomous vehicles [25]. The centralized approach has been already studied for collective dynamics, with designs relating to linear feedback control [12], [21], nonlinear model predictive control [13], open-loop optimal control [6], and sparse control [14]. In this paper, we explore the synthesis of suboptimal nonlinear feedback controllers for collective dynamics, as a first approximation towards optimal feedback stabilization via dynamic programming. A similar approach to nonlinear dynamics has been previously followed in [9]; we revisit these techniques in the context of large-scale agent-based models. In general, suboptimal nonlinear feedback techniques (i.e. arising from approximations of the dynamic programming equation) are applicable to dynamics with polynomial nonlinearities. On the other hand, collective dynamics are typically characterized as multi-agent systems where agents interact with each other through convolution with a kernel encoding metric and topological interactions which are not polynomial. In this paper, we show that by a suitable parametrization of the interactions between agents, it is possible to recast the dynamics in a way that suboptimal nonlinear feedback control synthesis is possible and relates to the computation of matrix Riccati and Lyapunov equations.

This paper is organized as follows. In Section II, we state the optimal feedback stabilization problem via dynamic programming and we briefly describe different methods to develop feedback control laws approximating the original optimal feedback synthesis problem. Section III addresses the application of the aforedescribed techniques for the stabilization of collective dynamics. Section IV presents tests related to optimal power flow and consensus dynamics.

II. (Sub)Optimal feedback control laws

We begin by considering the following infinite horizon optimal control problem

$$\min_{u(\cdot)\in\mathcal{U}} \mathcal{J}(u(\cdot),x_0) := \int_0^\infty x^t(s)Qx(s) + u^t(s)Ru(s)\,ds\,, \quad (1)$$

subject to the nonlinear dynamical constraint

$$\dot{x}(t) = f(x(t)) + B(x(t))u(t)\,, \quad (2)$$

$$x(0) = x_0\,, \quad (3)$$

where we denote the state $x(t) = (x_1(t),\ldots,x_n(t))^t \in \mathbb{R}^n$, the control $u(\cdot) \in \mathcal{U}$, with $\mathcal{U} = \{u(t) : \mathbb{R}^+ \to (u_1(t),\ldots,u_m(t))^t \in \mathbb{R}^m\}$, $Q \in \mathbb{R}^{n\times n}$ symmetric positive semidefinite, and $R \in \mathbb{R}^{m\times m}$ symmetric positive definite. We assume the system dynamics $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $B(x) : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ to be $\mathcal{C}^1(\mathbb{R}^n)$, with $f(0) = 0$ and $B(0) = 0$. Our focus is therefore on asymptotic stabilization to the origin, with particular emphasis on large-scale collective dynamics, i.e. $n >> 1$. It is well-known that for this particular setting, the solution of the optimal control problem

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can be realized through Dynamic Programming [8], and that the value function

$$V(x)=\inf_{u(\cdot)\in\mathcal{U}}\mathcal{J}(u(\cdot),x) \tag{4}$$

satisfies the Hamilton-Jacobi-Bellman (HJB) equation

$$\nabla V(x)^t f(x)-\frac{1}{4}\nabla V(x)^t B(x)R^{-1}B(x)^t\nabla V(x)+x^tQx=0\,, \tag{5}$$

with the optimal control given in feedback form as

$$u(x)=-\frac{1}{2}R^{-1}B(x)^t\nabla V(x)\,. \tag{6}$$

Unfortunately, eq. (5) is a first-order fully nonlinear partial differential equation with no general explicit solution. Resorting to numerical methods limits the solvability of the equation to state space representations of low dimension (see [22] for an updated state of the art concerning computational methods for high-dimensional HJB). The so-called *curse of dimensionality* [10] is particularly critical when addressing feedback control synthesis for agent-based models, as the state space can be of dimension arbitrarily large. It is necessary then to make further structural assumptions leading to suboptimal, although feasible, feedback synthesis. In the following, we review different approximations to (5) which are tractable for large-scale collective dynamics.

A. Linear-Quadratic Control

It is well-known that for the particular case $f(x)=Ax$ with $A\in\mathbb{R}^{n\times n}$, $B(x)=B\in\mathbb{R}^m$, and by setting $V(x)=x^t\Pi x$ with $\Pi\in\mathbb{R}^{n\times n}$, the HJB equation reduces to the matrix Algebraic Riccati Equation (ARE)

$$A^t\Pi+\Pi A-\Pi BR^{-1}B^t\Pi+Q=0\,. \tag{7}$$

Assuming that the pair (A,B) is stabilizable and that (Q,A) has no unobservable modes on the imaginary axis, there exists a unique positive definite solution Π of eq. (7), which generates an asymptotically stable closed-loop. Therefore, for nonlinear dynamics $f(x)$ and $B(x)=B$, it is natural to consider as a first stabilizing candidate the feedback law

$$u(x)=-R^{-1}B^t\Pi x\,, \tag{8}$$

where Π solves eq. (7) with $A_{ij}=\frac{\partial f_i(x)}{\partial x_j}|_{x=0}$. This feedback law is clearly suboptimal compared to the fully nonlinear HJB synthesis, and can only be expected to be locally stabilizing around an equilibrium/consensus point. Such a controller is attractive for two reasons. First, there exists and advanced computational methodology which renders the solution of large-scale AREs feasible (see [11]). Second, it is often the case that for collective dynamics, equilibrium points have enhanced stability properties [1], [14].

B. Power Series Method

A second synthesis method, which is an extension of the previous linear-quadratic ansatz for nonlinear dynamics, was developed in [19], and subsequently applied in different contexts, in particular for the local stabilization of large-scale fluid flow phenomena [27]. Assuming that the dynamics can be expressed as

$$\dot{x}=A_0x+\sum_{k=2}^{N}f_k(x)+Bu(t)\,, \tag{9}$$

with $f_k(x)=\mathcal{O}(x^k)$, we can consider an expansion of the value function of the form $V(x)=\sum_{k=0}^{\infty}V_k(x)$, with $V_k(x)=\mathcal{O}(x^{k+2})$. By inserting these expression into eq. (5) and matching terms of the same order, it is possible to show that $V_0(x)$ corresponds to the solution Π_0 of the ARE associated to A_0, whereas $V_1(x)$ satisfies

$$\nabla V_1=-2(A_0^t-\Pi_0BR^{-1}B^t)^{-1}\Pi_0f_2(x)\,. \tag{10}$$

By truncating $V(x)\approx V_0(x)+V_1(x)$ and inserting into the feedback law (6), we obtain the extended, nonlinear control

$$u(x)=-R^{-1}B^t(\Pi_0x-(A_0^t-\Pi_0BR^{-1}B^t)^{-1}\Pi_0f_2(x))\,.$$

In general, such a controller improves both the total cost $\mathcal{J}(u(\cdot),x_0)$, and the closed-loop asymptotic stability region of the linear-quadratic control, as discussed in [9], [22] at the advantage of being computed as a byproduct of the same ARE of the linear-quadratic approximation.

C. State-Dependent Riccati Equation

For nonlinear systems that can be represented in the form

$$\dot{x}=A(x)x+B(x)u(t)\,, \tag{11}$$

it is possible to obtain a more complex controller by following the so-called State-dependent Riccati Equation (SDRE) approach. This method, which has been thoroughly analyzed in [7], [16], is based on the idea that infinite horizon optimal feedback control for systems of the form (11), is formally linked to the state-dependent ARE

$$A^t(x)\Pi(x)+\Pi(x)A(x)-\Pi(x)B(x)R^{-1}B^t(x)\Pi(x)+Q=0\,,$$

with a closed-loop feedback given by

$$u:=u(x(t))=-R^{-1}B^t(x(t))\Pi(x(t))x(t)\,. \tag{12}$$

We recall the following result [7] concerning asymptotic stability of the SDRE closed-loop.

Theorem 1: Assume a nonlinear system

$$\dot{x}=f(x)+B(x)u\,, \tag{13}$$

where $f(x)$ is $\mathcal{C}^1$ for $\|x\|\leq\delta$, and $B(x)$ is continuous. If $f(x)$ can be parametrized in the form $f(x)=A(x)x$, and the pair $(A(x),B(x))$ is stabilizable for every x in a nonempty neighborhood of the origin $\Omega\subset\mathcal{B}_\delta(0)$, then the closed-loop dynamics generated by the feedback law (12) are locally asymptotically stable.

In contrast with the power series method, this approach is suitable for a larger class of nonlinear dynamics. However, the sequential online solution of large-scale AREs can be computationally demanding, and unfeasible in an online setting. In the following, we recall a realization of the SDRE feedback controller that can be computed offline, in the sense that the closed-loop implementation only requires the identification of the current state of the system and no further online ARE solves. For this, we begin by expressing the operator $A(x)$ in (11) as

$$A(x) = A_0 + \sum_{k=1}^{N} f_k(x) A_k \,, \tag{14}$$

where $A_k \in R^{n\times n}, k = 0, \ldots, N$, and the state dependence is restricted to the scalar functions $f_k(x) : R^n \to \mathbb{R}$. In this case, and assuming $B(x) = B$, its is possible to show (also via a power series type of argument [7]), that an approximation to the nonlinear feedback controller (12) is given by

$$u(x) = -R^{-1}B^t \left(\Pi_0 + \sum_{k=1}^{N} \Pi_k f_k(x) \right) x \,, \tag{15}$$

where Π_0 solves the ARE (7) associated to (A_0, B), where the remaining Π_k satisfy the Lyapunov equations

$$\Pi_k C_0 + C_0^t \Pi_k + Q_k = 0 \,, \quad k = 1, \ldots, N \,, \tag{16}$$

with $C_0 = A_0 - BR^{-1}B^t\Pi_0$, and $Q_k = \Pi_0 A_k + A_k \Pi_0$. Overall, this approach requires the computation of the ARE associated to A_0 plus N Lyapunov equations, whose solution is fully parallelizable.

III. Nonlinear feedback stabilization of collective dynamics

Having described different feedback strategies, we now turn our attention towards the implementation of control laws for collective dynamics. There are two characteristic features of such systems: they are naturally cast in a high-dimensional state space setting, and many of the relevant interaction kernels correspond to dense nonlinear dynamics where each agent interacts with each other.

First-order models: It is often of interest to consider general first-order collective dynamics of the form

$$\dot{x}_i(t) = \frac{1}{n}\sum_{j=1}^{n} P(x_i, x_j)(x_j - x_i) + u_i(t), \qquad i = 1, \ldots, n, \tag{17}$$

arising in different applications such as human and animal motion, opinion formation, and autonomous vehicles, among many others. For the sake of simplicity, we omit any further parametrization of the controller, and we focus on stabilization towards the origin. In general, $P(x_i, x_j) : \mathbb{R}^2 \to \mathbb{R}$ accounts for metric and topological interactions between the agents, and therefore it is not a polynomial expression, nor something that can be conveniently cast in semilinear form. However, assuming the kernel is sufficiently regular, nonlinear feedback synthesis can be achieved by considering the Taylor approximation

$$P(x_i, x_j) \approx P(0,0) + \nabla P(0,0) \cdot (x_i, x_j) + h.o.t. \tag{18}$$

leading to polynomial dynamics. In particular, for systems of the type (17), linearization of the kernel leads to quadratic dynamics. Denoting by $P_0 := P(0,0)$ and $\nabla P(0,0) := (P_1, P_2)$, the nonlinear dynamics are approximated by the system

$$\dot{x} = Ax + F^1(x \circ x) + x \circ (F^2 x) + Bu \,, \tag{19}$$

where

$$A \in \mathbb{R}^{n\times n}, A_{ij} = \begin{cases} -P_0 + \frac{P_0}{n} & \text{if} \quad i = j \\ \frac{P_0}{n} & \text{otherwise} \end{cases}, \tag{20}$$

$$F^1 \in \mathbb{R}^{n\times n}, F^1_{ij} = \begin{cases} -P_1 + \frac{P_2}{n} & \text{if} \quad i = j \\ \frac{P_2}{n} & \text{otherwise} \end{cases}, \tag{21}$$

$$F^2 \in \mathbb{R}^{n\times n}, F^2_{ij} = P_1 - P_2 \,, \quad B = \mathbb{I}_n \,, \tag{22}$$

where $\mathbb{I}_n$ is the identity matrix in $\mathbb{R}^{n\times n}$. For such a nonlinear system, all the control techniques described in the previous section can be readily implemented.

The second-order case: Another relevant class of models in collective behaviour, especially in the modelling of human and animal crowd motion is given by second-order dynamics of the type

$$\dot{x}_i(t) = v_i(t) \tag{23}$$

$$\dot{v}_i(t) = \frac{1}{n}\sum_{j=1}^{n} P(x_i, x_j)(v_j - v_i) + u_i(t) \,, \tag{24}$$

where the micro-state (x_i, v_i) denotes position and velocity of the agent, and the interaction kernel encodes metric interactions. One of the most celebrated models of this type is the so-called Cucker-Smale model for consensus [18], where

$$P(x_i, x_j) = \frac{H}{(1 + \|x_i - x_j\|^2)^\beta} \,, \quad H > 0 \,, \beta \geq 0 \,, \tag{25}$$

i.e. the dynamics constitute an averaging of the velocities weighted by the distance between agents. Since the high-order approximation described in for the first-order case is directly applied to the interaction kernel, it can be used as well for second-order dynamics. This leads to a richer non-linear structure, however it will still be of polynomial type, and nonlinear control laws can be implemented analogously. In this case, we will focus instead on the semilinear structure of the system and its properties with respect to the SDRE feedback synthesis. Note that the system (24) can be written in semilinear form

$$\begin{bmatrix} \dot{x} \\ \dot{v} \end{bmatrix} = A(x) \begin{bmatrix} x \\ v \end{bmatrix} + Bu \,, \tag{26}$$

with

$$A(x) := \begin{bmatrix} \mathbb{O}_n & \mathbb{I}_n \\ \mathbb{O}_n & \mathbb{A}_n(x) \end{bmatrix}, \quad B := \begin{bmatrix} \mathbb{O}_n \\ \mathbb{I}_n \end{bmatrix}, \tag{27}$$

where $\mathbb{O}_n, \mathbb{I}_n, \mathbb{A}_n(x) \in \mathbb{R}^{n\times n}$, $\mathbb{O}_n$ is a zero matrix, $\mathbb{I}_n$ is the identity matrix, and $\mathbb{A}_n(x)$ is given by

$$[\mathbb{A}_n(x)]_{ij} = \begin{cases} -\frac{1}{n}\sum\limits_{k=1}^{n} P(x_i, x_k) & \text{if} \quad i = j \\ \frac{1}{n} P(x_i, x_j) & \text{otherwise.} \end{cases}, \tag{28}$$

Note that if P is symmetric and positive, then $-\mathbb{A}_n(x)$ is a Laplacian matrix. While this plays a role on the stability of the dynamics, the question of the stabilizability of the dynamics also depends on the control operator B. For our setting the following result holds.

Lemma 1: The pair $(A(x), B)$ is controllable $\forall x \in \mathbb{R}^n$.

Proof: It follows directly from the first two column blocks of the controllability matrix

$$\begin{aligned} K(A(x), B) &= [B; AB; \ldots; A^{N-1}B] \\ &= \begin{bmatrix} \mathbb{O}_n & \mathbb{I}_n & \mathbb{A}_n(x) & \ldots & \mathbb{A}_n^{n-1}(x) \\ \mathbb{I}_n & \mathbb{A}_n(x) & \mathbb{A}_n^2(x) & \ldots & \mathbb{A}_n^n(x) \end{bmatrix} \end{aligned}$$

that $rank(K(x)) = 2n, \forall x \in \mathbb{R}^n$. ∎

Corollary 1: The feedback controller from the state-dependent Riccati equation (12) applied to second-order semilinear dynamics (27) generates a locally asymptotically stable closed-loop.

Note that the previous results are only valid if we allow a control configuration in which we can act on the velocity of each agent separately. For a configuration with a single input $u_i(t) = u(t), \forall i$, it can be easily verified that K becomes rank deficient, in fact $rank(K(x)) = 2 < 2n$.

IV. Numerical tests

We address the feedback synthesis for a different applications arising in collective dynamics. The presented control strategies rely on the solution of algebraic Riccati and Lyapunov equations, for which we resort to the related `care` and `lyap` routines in MATLAB. The tests were performed in a workstation with an Intel i7 2.6GHz Processor with 16GB RAM. CPU times for the offline phase were within 10 seconds for every case with up to $n = 100$ agents.

An optimal power flow model: We apply the presented feedback methodologies on a simplified model for optimal power flow equations, see e.g. [23]. We consider only the equations for the active power x_i at nodes $i = 1, \ldots, n$ (represented as agents). The resistance at node i is denoted by the constant r_i. The voltage is kept constant and equal to 1. We also assume a grid geometry that is fully connected to simplify the presentation. We obtain then as model for the active power at each agent i

$$x_i = \sum_j (x_j - r_j x_j^2) + u_i, \tag{29}$$

where u_i is the injected power at agent i and subject to control. Such a model is stationary and different approaches have been investigated to solve the power flow equations. We are interested in a possibly fast, time–dependent feedback control for the power flow equations. Therefore, we relax the previous equations and consider a dynamical version for a small relaxation parameter $\epsilon > 0$:

$$\dot{x}_i(t) = \frac{1}{\epsilon}\left(\sum_{j=1}^n (x_j(t) - r_j x_j(t)^2) - x_i(t) + u_i(t)\right).$$

A solution to this system converges exponentially fast in time to a solution to the stationary equations. Its state space representation reads

$$\dot{x} = Ax + F(x \circ x) + Bu\,, \tag{30}$$

where

$$A \in \mathbb{R}^{n\times n}\,, A_{ij} = \begin{cases} 0 & \text{if} \quad i = j \\ \frac{1}{\epsilon} & \text{otherwise} \end{cases}, \quad B = \mathbb{I}_{n\times n}\,, \tag{31}$$

$$F \in \mathbb{R}^{n\times n}\,, F_{ij} = \frac{-r_j}{\epsilon}\,, \tag{32}$$

and the symbol $\circ$ corresponds to the component-wise Hadamard product. We cast a quadratic cost functional

$$J_{\tilde{P}}(u) := \int_0^\infty \frac{1}{n}\sum_{i=1}^n |x_i - \tilde{P}|^2 + \frac{\gamma}{2}\|u(s)\|^2 ds\,, \tag{33}$$

that describes the problem to attain certain active power flow $\tilde{P}$ at all nodes in the grid, at the expense of applying additional power $u(t) \in \mathbb{R}^n$. We set $\tilde{P} = 0$ (for other setpoints the resulting nonlinear dynamical system can be transformed into a similar structure), and therefore

$$Q = diag(1/n)\,, \quad R = diag(\gamma/2) \in \mathbb{R}^{n\times n}\,. \tag{34}$$

We implement the three different feedback controls presented in Section II. For a network of $n = 100$ agents, we randomly generate the set of coefficients r_j uniformly in $[0, 1]$. The relaxation parameter is set to $\epsilon = 1e{-}3$, and $\gamma = 1e{-}3$. The initial states are randomly generated uniformly in $[0, 0.1]$. Uncontrolled and controlled trajectories are shown in Figure 1. It can be observed that the uncontrolled dynamics converge towards a stable equilibrium point different from the origin. The system is successfully stabilized via the three proposed control designs. We omit the pictures of the ARE controller as they hardly differ from the PSE control law. However, as shown in Figure 2, there are significant differences for the control signals obtained through the SDRE approach, which makes a more parsimonious use of the control energy to steer the system towards the origin. This is reflected in Figure 3, where the running cost $x^tQx + u^tRu$ is two orders of magnitude smaller for the SDRE closed-loop.

Consensus regulation in the Cucker-Smale model: We study the second order model (24) with the Cucker-Smale kernel $P(x_i, x_j) = (1 + \|x_i - x_j\|)^{-1}$. We set the state penalization $Q = diag(1/n)$, and $\gamma = 1$. For $n = 100$ agents, initial states are randomly generated uniformly in $[-10, 10]$. In this case, we avoid linearizing the kernel and benefit from the semilinear structure (27), to implement ARE and SDRE feedback laws. The ARE feedback is computed by taking $A = A(0)$ in (27), while the SDRE is implemented in an model predictive control fashion, i.e., solving an ARE in every instance of the discrete in time trajectory. A more efficient implementation will use the offline formulas presented in Section II-C. Results are shown in Figures 4 and 5. The uncontrolled dynamics are unstable, while both ARE and SDRE feedback laws steer the system towards the origin, with a reduced running cost for the SDRE approach.

Concluding remarks

We have presented three different feedback control strategies for the stabilization of nonlinear, multi-agent dynamics. All the approaches are based on approximations of the HJB

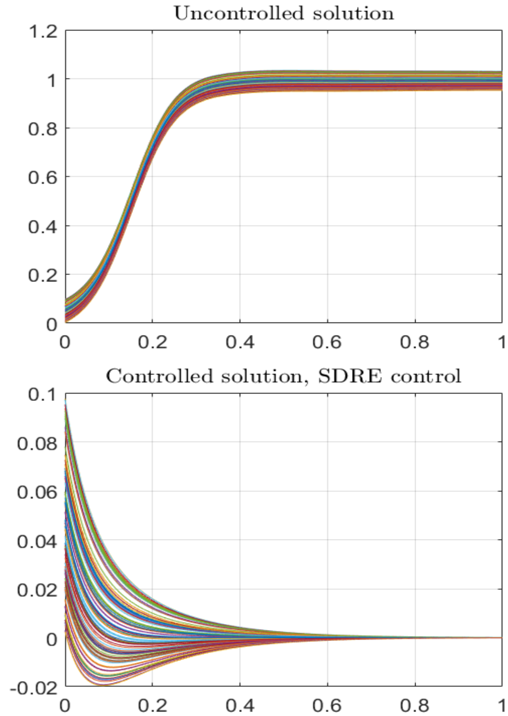

Fig. 1. Relaxed optimal power flow model, $n = 100$ agents. Uncontrolled and controlled trajectories in time. The uncontrolled dynamics converge a nontrivial equilibrium point. The SDRE control laws steer the system towards the origin.

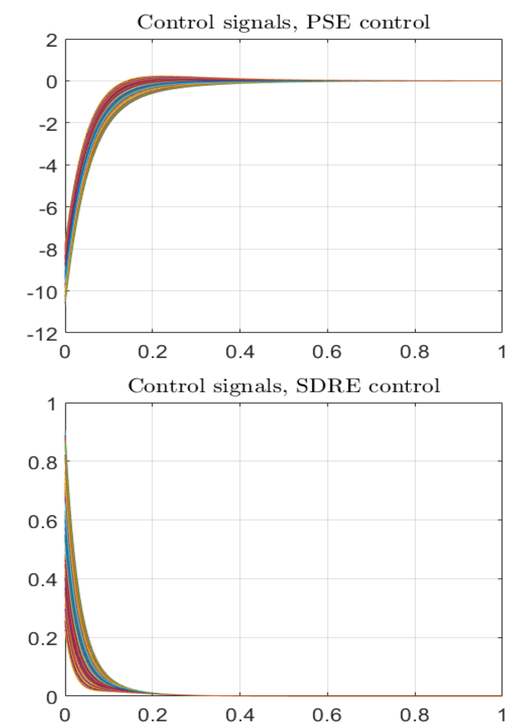

Fig. 2. Relaxed optimal power flow model, $n = 100$ agents. Control signals for the PSE and SDRE feedback synthesis. The ARE control signal is omited as it behaves similarly to the PSE control law. The SDRE control synthesis results into a parsimonius use of the control energy to generate satbilized trajectories. The transients differ both in sign and magnitude.

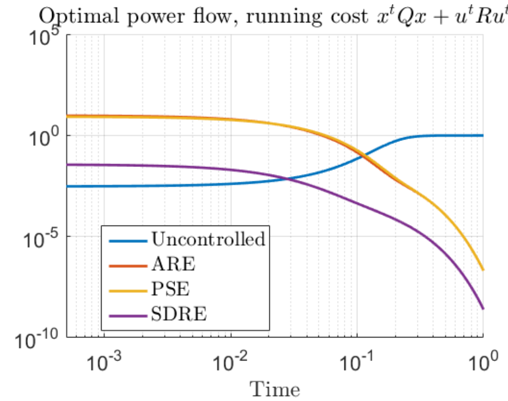

Fig. 3. Relaxed optimal power flow model, $n = 100$ agents. As a consequence of the reduced amount of control effort used by the SDRE closed-loop, the running cost is two orders of magnitude smaller than the ARE and PSE controllers. The total costs $\mathcal{J}(u(\cdot))$ are **i) ARE**: 4.99e-2, **ii) PSE**:4.47e-2, and **iii) SDRE**: 1.33e-4

equation associated to the optimal feedback control. Since the direct approximation of the HJB equation is unfeasible for high-dimensional dynamics, we resort to suboptimal control strategies originated from the solution of Riccati and Lyapunov equations. The proposed control designs successfully stabilize different nonlinear multi-agent dynamics. When comparing the different control laws, the State-Dependent Riccati Equation approach consistently yields the best results in terms of reduced total cost. However, we observe that the Power Series Expansion-based controller also performs well, with the advantage of being obtained at a considerably lower computational burden (n Lyapunov equations versus 1, respectively), and leading to a nonlinear feedback law of reduced complexity.

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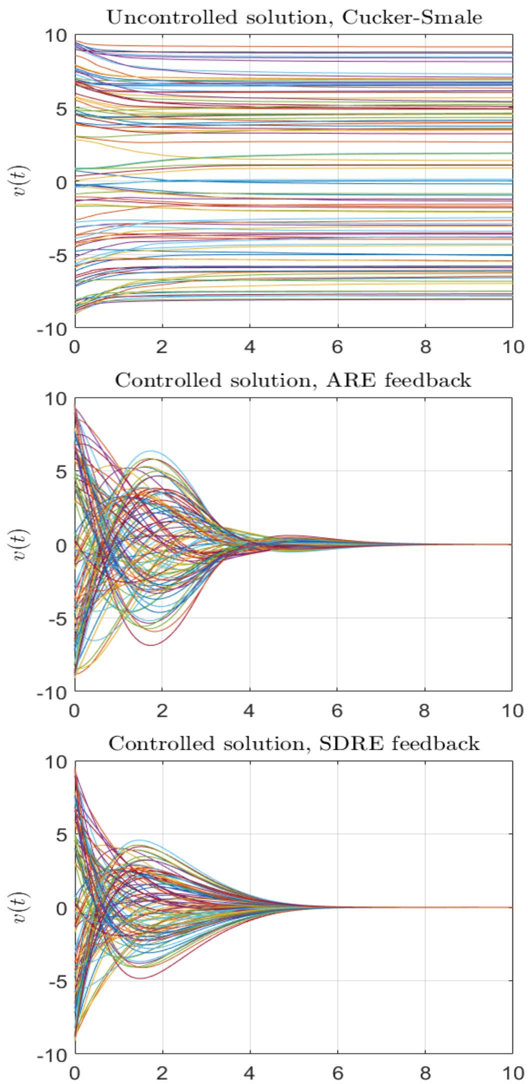

Fig. 4. Cucker-Smale dynamics. Uncontrolled and controlled trajectories in time, with $n = 100$ agents. The uncontrolled dynamics remain unstable, both ARE and SDRE closed-loops are stable.

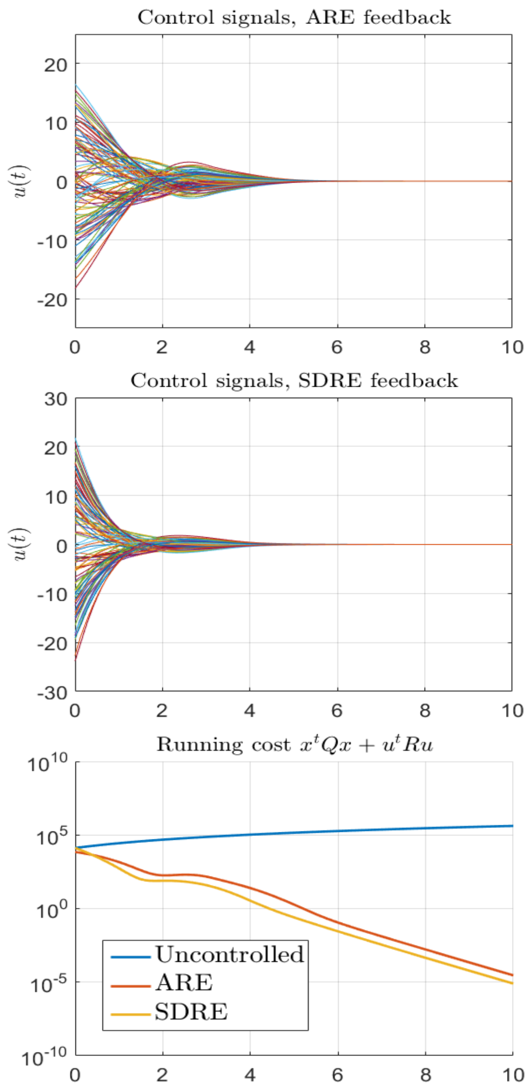

Fig. 5. Cucker-Smale dynamics. Control signals with $n = 100$ agents. The SDRE feedback stabilizes the dynamics faster, and wiith a reduced overall running cost compared to the ARE feedback (bottom).

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