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Brief paper

Time-domain moment matching for multiple-input multiple-output linear time-invariant models ☆

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ABSTRACT

Model reduction by moment matching for linear time-invariant (LTI) models is a reduction technique that has a clear interpretation in the Laplace domain. In particular, for the multiple-input multiple-output (MIMO) LTI case, Krylov subspace methods aim at matching the transfer-function matrix (and possibly its derivatives) of the reduced-order model to the transfer-function matrix of the full-order model along so-called tangential directions at desired interpolation points. A straightforward application of *time-domain* moment matching to *MIMO* LTI models does not result in such a match in the transfer-function matrix. In this paper, we derive a relation between the MIMO transfer-function matrices of the full- and the reduced-order models that follows from the application of time-domain moment matching on MIMO LTI models. This is subsequently exploited to formulate conditions on the parameters of time-domain moment matching under which the transfer-function matrix is matched along tangential directions, thus ensuring consistency with classical Krylov subspace methods.

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1. Introduction

Moment matching (MM) is a powerful and computationally efficient method to construct reduced-order dynamical models for large-scale systems. Originally, MM techniques were based on Krylov subspaces, which allow for efficient computation of reduced-order models (Antoulas, 2005a, 2005b; Gugercin, Antoulas, & Beattie, 2008). These methods have the property that in the single-input single-output (SISO) case, the transfer functions of the full- and the reduced-order model are matched at desired interpolation points in the complex plane. In the multiple-input multiple-output (MIMO) case, the transfer-function matrices are matched along so-called left and/or right tangential directions at desired interpolation points in the complex plane (Ball, Gohberg, & Rodman, 1990; Gallivan, Vandendorpe, & Van Dooren, 2004a; Gugercin et al., 2008).

In Gallivan, Vandendorpe, and Van Dooren (2004b), the notion of moments was connected to the solution of the so-called Sylvester equation, which gave rise to a novel MM method in a *time-domain* framework (Astolfi, 2007, 2010). The advantages of time-domain moment matching are twofold, namely: it facilitates parametric freedom, which can be conveniently exploited to enforce properties such as stability and passivity on the reduced-order model (Astolfi, 2010); and it has a natural extension to model order reduction for nonlinear models (Astolfi, 2010; Scarciotti & Astolfi, 2017; Shakib, Scarciotti, Pogromsky, Pavlov, & van de Wouw, 2021).

For SISO models, this *time-domain* framework preserves the property of the classical Krylov methods (Antoulas, 2005a, 2005b; Gugercin et al., 2008) that the transfer functions of the full- and the reduced-order model are matched at desired interpolation points in the complex plane (Astolfi, 2007, 2010). For MIMO models, classical Krylov subspace methods ensure that the transfer-function matrices are matched along the so-called left and/or right tangential directions at desired interpolation points in the complex plane (Ball et al., 1990; Gallivan et al., 2004a; Gugercin et al., 2008). However, the *time-domain* MM framework does not generally preserve this property in the MIMO case.

In this paper, we derive explicitly a matching property in terms of the MIMO transfer-function matrices of the full- and the reduced-order model for *time-domain* MM. Subsequently, the derived matching property is exploited to formulate additional conditions on the parameters of time-domain MM under which a

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match along the tangential directions of the transfer-function matrices is indeed ensured. Finally, a numerical example is provided that illustrates the obtained matching property. The results in our paper can be used to connect time-domain MM for MIMO models to MM based on Krylov subspace methods by specific parameter selection for time-domain MM.

To summarize, the main contributions of this paper are as follows.

- We derive a relation between the MIMO transfer-function matrices of the full- and the reduced-order models that follows from the application of time-domain MM on MIMO LTI models.
- We show under which conditions time-domain MM ensures a match of the MIMO transfer-function matrices along the tangential directions, consistent with classical Krylov subspace methods.

The remainder of the paper is outlined as follows. In Section 2, we present the main results of this paper, namely a matching property of time-domain MM together with additional conditions to ensure a match that is consistent with classical Krylov subspace methods. Section 3 applies the derived results to a numerical example. Section 4 closes with concluding remarks.

Notation: The set of real numbers is denoted by $\mathbb{R}$ and the set of complex numbers is denoted by $\mathbb{C}$. The empty set is denoted by $\emptyset$. The complex conjugate of a complex number $s \in \mathbb{C}$ is denoted by $\bar{s}$. The $n \times n$ identity matrix is denoted by I_n, the $n \times n$ zero matrix is denoted by 0_n, and the $n \times m$ zero matrix is denoted by $0_{n\times m}$. The eigenvalues of the matrix $A \in \mathbb{C}^{n\times n}$ are denoted by $\sigma(A)$. The transpose (not conjugate transpose) of $A \in \mathbb{C}^{n\times n}$ is denoted by $A^\top$. The imaginary unit $\sqrt{-1}$ is denoted by j.

2. Moment matching for MIMO models

The moment matching problem for MIMO LTI models is defined in Section 2.1, in which moments are defined through the transfer-function matrix of the LTI model. After that, Section 2.2 characterizes moments in the *time-domain* framework. Subsequently, consistent with methods for SISO models, a family of MIMO LTI models is constructed that achieves a generalized matching property. Section 2.4 proposes specific selections of *time-domain* MM parameters to ensure a match that is consistent with classical Krylov methods. Finally, Section 2.5 presents families of models that achieve zeroth-order moment matching. For brevity, we focus on moments along the right tangential directions and omit the dual of moments along the left tangential directions. The results of this paper can be readily extended to the case of matching moments along the left tangential directions as well.

2.1. The MIMO moment matching problem

Consider the MIMO LTI model

$$\Sigma: \quad \dot{x} = Ax + Bu, \quad y = Cx, \tag{1}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^p, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$. The SISO case is retrieved for $m = p = 1$. The transfer-function matrix of this model is

$$K(s) = C(sI - A)^{-1}B, \quad K : \mathbb{C} \to \mathbb{C}^{p\times m}. \tag{2}$$

Throughout the rest of this paper, we assume minimality of model (1). The moments of model (1) are defined according to Necoara and Ionescu (2020, Definition 1), which is a MIMO extension of Astolfi (2010, Definition 1).

Definition 1. The k-moment of (1) at $s \in \mathbb{C} \setminus \sigma(A)$, along tangential direction $\ell \in \mathbb{C}^m$ is:

$$\eta_k^\Sigma(s, \ell) := \frac{(-1)^k}{k!}\frac{\mathrm{d}^k K(s)}{\mathrm{d}s^k}\ell \in \mathbb{C}^p, \quad k \geq 0, \text{ integer}. \tag{3}$$

Now consider the MIMO LTI model

$$\hat{\Sigma}: \quad \dot{\xi} = F\xi + Gu, \quad \psi = H\xi, \tag{4}$$

where $\xi(t) \in \mathbb{R}^\nu, u(t) \in \mathbb{R}^m, \psi(t) \in \mathbb{R}^p, F \in \mathbb{R}^{\nu\times\nu}, G \in \mathbb{R}^{\nu\times m}$ and $H \in \mathbb{R}^{p\times\nu}$. The relation between models (4) and (1) is defined as follows.

Definition 2. Consider the interpolation point $s \in \mathbb{C} \setminus (\sigma(A) \cap \sigma(F))$, the tangential direction $\ell \in \mathbb{C}^m$, and an integer $k \geq 0$. Model (4) is called a model of (1) at (s, ℓ) if it achieves moment matching according to

$$\eta_{\tilde{k}}^\Sigma(s, \ell) = \eta_{\tilde{k}}^{\hat{\Sigma}}(s, \ell), \quad \tilde{k} = 0, \ldots, k. \tag{5}$$

In addition, if $\nu < n$, then model (4) is called a reduced-order model of (1) at (s, ℓ).

Such matching is analogue to tangential Hermite interpolation (Antoulas, 2005a) and is consistent with, e.g., Gugercin, Polyuga, Beattie, and Van Der Schaft (2012).

The problem addressed in this paper is to find a parametrization for F, G, and H, such that $\hat{\Sigma}$ in (4) is a (reduced-order) model of (1), according to Definition 2. This problem includes the case in which $\nu \geq n$.

2.2. Time-domain characterization of moments

The definition of moments in Definition 1 relies on the transfer-function matrix. Alternatively, in *time-domain* moment matching, moments are characterized through the solution $\Pi \in \mathbb{C}^{n\times\nu}$ of a Sylvester equation, namely

$$A\Pi + BL = \Pi S, \tag{6}$$

where the matrix $L \in \mathbb{C}^{m\times\nu}$ contains information about the tangential directions and the matrix $S \in \mathbb{C}^{\nu\times\nu}$ contains information about the interpolation points. The matrices S and L play a central role in *time-domain* moment matching and are further specified in the remainder of this paper. Note that under the condition $\sigma(S) \cap \sigma(A) = \emptyset$, the solution $\Pi \in \mathbb{C}^{n\times\nu}$ is unique, see de Souza and Bhattacharyya (1981).

We distinguish between two cases. The first case deals with repeated *single* complex interpolation points, being the eigenvalues of the matrix S in the *complex-valued Jordan form*. This results in a reduced-order model that consists of *complex-valued* model matrices. The second case deals with a repeated *pair* of complex conjugated interpolation points, being the eigenvalues of the matrix S in the *real-valued Jordan form*. This case is the most useful case since it results in reduced-order models that are described by *real-valued* model matrices. Hereafter, we call these cases the *complex-valued Jordan case* and the *real-valued Jordan case*, respectively. In Remark 8, we explain how the general case without any Jordan structure can be treated.

The results for the complex-valued Jordan case are summarized in the following lemma. This lemma is an extension of Lemma 2 in Astolfi (2010) from the SISO case to the MIMO case.

Lemma 3. *Consider model (1) with m inputs and p outputs, the interpolation point $s^\star \in \mathbb{C}$, the integer $k \geq 0$ and $\ell_0, \ldots, \ell_k \in \mathbb{C}^m$. Define $\nu := k + 1$. Suppose $s^\star \notin \sigma(A)$. Then*

$$(C\tilde{\Pi})^\top = \begin{bmatrix} \eta_0(s^\star, \ell_0)^\top \\ \eta_0(s^\star, \ell_1)^\top - \eta_1(s^\star, \ell_0)^\top \\ \vdots \\ \sum_{i=0}^{k-1}(-1)^i \eta_i(s^\star, \ell_{k-1-i})^\top \\ \sum_{i=0}^{k}(-1)^i \eta_i(s^\star, \ell_{k-i})^\top \end{bmatrix} \tag{7}$$

where $\eta_i(s^\star, \cdot)$, $i = 0, \dots, k$, are defined in (3) and where $\tilde{\Pi} \in \mathbb{C}^{n\times\nu}$ is the unique solution of the Sylvester equation

$$A\tilde{\Pi} + B\tilde{L} = \tilde{\Pi}\Phi_k(s^\star) \tag{8}$$

with

$$\tilde{L} := \begin{bmatrix}\ell_0, \ell_1, \dots, \ell_k\end{bmatrix} \in \mathbb{C}^{m\times\nu}$$
$$\Phi_k(s^\star) := \begin{bmatrix} s^\star & 1 & 0 & \cdots & 0 \\ 0 & s^\star & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & s^\star & 1 \\ 0 & \cdots & \cdots & 0 & s^\star \end{bmatrix} \in \mathbb{C}^{\nu\times\nu}. \tag{9}$$

Proof. The proof is analogous to the proof of Lemma 2 in Astolfi (2010) and omitted here for the sake of brevity. ■

The results for the real-valued Jordan case are summarized in the next lemma.

Lemma 4. *Consider model (1) with m inputs and p outputs, the interpolation point $s^\star = a^\star + jb^\star \in \mathbb{C}$ with $a^\star, b^\star \in \mathbb{R}$, $b^\star \neq 0$, the integer $k \geq 0$ and $\ell_0, \dots, \ell_{2k+1} \in \mathbb{R}^m$. Define $\nu := 2(k+1)$. Suppose $s^\star \notin \sigma(A)$. Then,*

$$(C\tilde{\Pi})^\top = \begin{bmatrix} Re(\eta_0(s^\star, \ell_0 + j\ell_1)^\top) \\ Im(\eta_0(s^\star, \ell_0 + j\ell_1)^\top) \\ \sum_{i=0}^{1}(-1)^i Re(\eta_i(s^\star, \ell_{2-2i} + j\ell_{3-2i})^\top) \\ \sum_{i=0}^{1}(-1)^i Im(\eta_i(s^\star, \ell_{2-2i} + j\ell_{3-2i})^\top) \\ \vdots \\ \sum_{i=0}^{k}(-1)^i Re(\eta_i(s^\star, \ell_{2k-2i} + j\ell_{2k-2i+1})^\top) \\ \sum_{i=0}^{k}(-1)^i Im(\eta_i(s^\star, \ell_{2k-2i} + j\ell_{2k-2i+1})^\top) \end{bmatrix}, \tag{10}$$

where $\eta_i(s^\star, \cdot)$, $i = 0, \dots, k$, are defined in (3) and where $\tilde{\Pi} \in \mathbb{R}^{n\times\nu}$ is the unique solution of the Sylvester equation

$$A\tilde{\Pi} + B\tilde{L} = \tilde{\Pi}\Psi_k(s^\star) \tag{11}$$

with

$$\tilde{L} := \begin{bmatrix}\ell_0, \ell_1, \dots, \ell_{2k+1}\end{bmatrix} \in \mathbb{R}^{m\times\nu}$$
$$\Psi_k(s^\star) := \begin{bmatrix} \Xi^\star & I_2 & \cdots & 0_2 & 0_2 \\ 0_2 & \Xi^\star & I_2 & \cdots & 0_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0_2 & \cdots & 0_2 & \Xi^\star & I_2 \\ 0_2 & 0_2 & \cdots & 0_2 & \Xi^\star \end{bmatrix} \in \mathbb{R}^{\nu\times\nu}, \tag{12}$$
$$\Xi^\star := \begin{bmatrix} a^\star & b^\star \\ -b^\star & a^\star \end{bmatrix} \in \mathbb{R}^{2\times 2}.$$

Proof. The proof can be found in the Appendix. ■

Lemmas 3 and 4 establish a connection between the elements of $C\tilde{\Pi}$ and the moments in Definition 1. Both lemmas carry the same information in the corresponding variable $\tilde{\Pi}$. However, in Lemma 4, this information is split into a real and imaginary part, hence, yielding a twice as large number of columns in $\tilde{\Pi} \in \mathbb{R}^{n\times 2(k+1)}$ compared to the complex-valued $\tilde{\Pi} \in \mathbb{C}^{n\times k+1}$ in Lemma 3.

Remark 5. Despite $\tilde{L}$ being a real-valued matrix in Lemma 4, (10) shows that the tangential directions are generally complex-valued.

2.3. Generalized family of reduced-order models

In the same fashion as in Astolfi (2010), this section exploits the connection between $C\Pi$ and the moments in Lemmas 3 and 4 to construct families of reduced-order models. These models, however, achieve a generalized matching property, which is exploited in Section 2.4 to formulate families of models that achieve moment matching in accordance with Definition 2. The result for the family of models in the complex-valued Jordan case is summarized in the following theorem.

Theorem 6. *Consider the model (1) with m inputs and p outputs, the integer $k \geq 0$, the constant $\nu := k+1$, the non-derogatory matrix[1] $S \in \mathbb{C}^{\nu\times\nu}$ with $det(sI - S) = (s - s^\star)^{k+1}$, where $s^\star \in \mathbb{C}$ is the interpolation point, and any matrix $L \in \mathbb{C}^{m\times\nu}$. Suppose $s^\star \notin \sigma(A)$. Then, for any $G \in \mathbb{C}^{\nu\times m}$ such that $s^\star \notin \sigma(S - GL)$, model (4) with $F := S - GL$, $H := C\Pi$, and with $\Pi \in \mathbb{C}^{n\times\nu}$ the unique solution to the Sylvester equation*

$$A\Pi + BL = \Pi S, \tag{13}$$

achieves the following match:

$$\sum_{i=0}^{\tilde{k}}(-1)^i \eta_i^{\Sigma}(s^\star, \ell_{\tilde{k}-i}) = \sum_{i=0}^{\tilde{k}}(-1)^i \eta_i^{\hat{\Sigma}}(s^\star, \ell_{\tilde{k}-i}), \tag{14}$$

for $\tilde{k} = 0, \dots, k$, and where $\ell_i \in \mathbb{C}^m$, $i = 0, \dots, k$, are the columns of the matrix LT with any $T \in \mathbb{C}^{\nu\times\nu}$ that is obtained via the Jordan decomposition $S = T\Phi_k(s^\star)T^{-1}$ with $\Phi_k(s^\star) \in \mathbb{C}^{\nu\times\nu}$ as given in (9).

Proof. Consider the class of models (4). Given that $s^\star \notin \sigma(F)$, the Sylvester equation $FP + GL = PS$ has a unique solution $P \in \mathbb{C}^{\nu\times\nu}$. Selecting $P = I_\nu$ results in $F = S - GL$ for any $G \in \mathbb{C}^{\nu\times m}$ such that $\sigma(S - GL) \cap \sigma(S) = \emptyset$. Moreover, selecting $H = C\Pi$ results in the matching $HP = C\Pi$. It is only left to show that the elements in $C\Pi$ are described by (14). Hereto, notice that the matrix $S \in \mathbb{C}^{\nu\times\nu}$ contains ν repeated eigenvalues at $s^\star$ and is non-derogatory, which guarantees that there exists a (non-unique) non-singular matrix $T \in \mathbb{C}^{\nu\times\nu}$ that brings $T^{-1}ST$ into the Jordan form as in (9), see, e.g., Weintraub (2009). Therefore, application of Lemma 3 with $\Phi_k(s^\star) = T^{-1}ST$ and $\tilde{L} = LT$, gives the relation between the moments (14) and the matrix $C\tilde{\Pi}$ with $\tilde{\Pi} \in \mathbb{C}^{n\times\nu}$ the unique solution to the Sylvester Eq. (8). The solution $\Pi \in \mathbb{C}^{n\times\nu}$ of the Sylvester equation in (13) is related to $\tilde{\Pi}$ by $\tilde{\Pi} = \Pi T$. Since T is invertible, matching the elements of the matrix $C\Pi$ by the reduced-order model (4) guarantees that the linear combination of moments in (14) are matched, which completes the proof. ■

Similarly to Theorem 6, we can construct a family of models for the real-valued Jordan case, which is presented in the following theorem.

Theorem 7. *Consider model (1) with m inputs and p outputs, the integer $k \geq 0$, the constant $\nu := 2(k+1)$, the non-derogatory matrix $S \in \mathbb{R}^{\nu\times\nu}$ with $det(sI - S) = ((s - s^\star)(s - \bar{s}^\star))^{k+1}$, where $s^\star \in \mathbb{C} \setminus \mathbb{R}$ is the interpolation point, and any matrix $L \in \mathbb{R}^{m\times\nu}$. Suppose $s^\star \notin \sigma(A)$. Then, for any $G \in \mathbb{R}^{\nu\times m}$ such that $s^\star \notin \sigma(S - GL)$, model (4) with $F := S - GL$, $H = C\Pi$, and with $\Pi \in \mathbb{R}^{n\times\nu}$ the unique solution to the Sylvester equation*

$$A\Pi + BL = \Pi S, \tag{15}$$

[1] A matrix is non-derogatory if its characteristic and minimal polynomials coincide.

achieves the following match:

$$\begin{aligned}\sum_{i=0}^{\tilde{k}}(-1)^i\eta_i^{\Sigma}(s^\star, \ell_{2\tilde{k}-2i}+j\ell_{2\tilde{k}-2i+1})\\ =\sum_{i=0}^{\tilde{k}}(-1)^i\eta_i^{\hat{\Sigma}}(s^\star, \ell_{2\tilde{k}-2i}+j\ell_{2\tilde{k}-2i+1}),\\ \sum_{i=0}^{\tilde{k}}(-1)^i\eta_i^{\Sigma}(\bar{s}^\star, \ell_{2\tilde{k}-2i}-j\ell_{2\tilde{k}-2i+1})\\ =\sum_{i=0}^{\tilde{k}}(-1)^i\eta_i^{\hat{\Sigma}}(\bar{s}^\star, \ell_{2\tilde{k}-2i}-j\ell_{2\tilde{k}-2i+1}),\end{aligned} \tag{16}$$

for $\tilde{k}=0,\dots,k$, and where $\ell_i\in\mathbb{R}^m$, $i=0,\dots,2k+1$, are the columns of the matrix LT with any $T\in\mathbb{R}^{\nu\times\nu}$ that is obtained via the Jordan decomposition $S=T\Psi_k(s^\star)T^{-1}$ with $\Psi_k(s^\star)\in\mathbb{R}^{\nu\times\nu}$ as given in (12).

Proof. The proof is analogous to the proof of Theorem 6 up to the statement that the linear combination of moments at the complex conjugate $\bar{s}^\star$ of $s^\star$ is also matched along the complex conjugate of the tangential direction. This fact is guaranteed by the following property for real rational transfer functions:

$$\overline{\eta_k(s^\star,\ell)}=\eta_k(\bar{s}^\star,\bar{\ell}),$$

for $s^\star\in\mathbb{C}$ and $\ell\in\mathbb{C}^m$. The rest of the proof is omitted for brevity. ■

The matrix S being non-derogatory in Theorems 6 and 7 is required to ensure the existence of a similarity transformation that brings S into the complex-Jordan form in (9) or the real-Jordan form in (12). The results of Theorems 6 and 7 are used in Section 2.4 to facilitate choices for the pair (S,L) such that application of Theorems 6 and 7 results in (4) being a (reduced-order) model of (1), consistent with Definition 2. Furthermore, in Section 2.5, these theorems are exploited to formulate families of models that achieve zeroth-order moment matching.

Remark 8. Lemmas 3 and 4 can be used as building blocks when multiple distinct interpolation points are considered. Any matrix $S\in\mathbb{C}^{\nu\times\nu}$ can be transformed into a block-diagonal matrix by a similarity transformation, where the block-diagonal terms are either in the complex-valued Jordan form as in (9), or in the real-valued Jordan form as in (12). Consequently, the matching property can be retrieved for each block-diagonal term by application of the results of Lemmas 3 and 4.

Remark 9. It is emphasized that the tangential direction in Theorems 6 and 7 are the columns of the matrix LT rather than the matrix L. Thus, selecting a matrix S that is not in Jordan form also results in a coordinate transformation in the tangential directions.

2.4. Family of models achieving higher-order moment matching

Theorems 6 and 7 characterize a matching property for time-domain MM for MIMO models in terms of the transfer-function matrices of the full- and of the reduced-order models. However, since there is a summation over moments in (14) and (16), in general, models found by application of these theorems do not achieve moment matching in accordance with Definition 2. In this section, we exploit the results in Theorems 6 and 7 to arrive at a family of (reduced-order) models that do achieve moment matching according to Definition 2. Hereto, we present specific selections of the tangential directions $\ell_0,\dots,\ell_k$, while restricting S to be either in the complex-valued Jordan form in Theorem 6 or in the real-valued Jordan form in Theorem 7.

Let us first turn to the case in which the reduced-order model is obtained via the result in Theorem 6. In that case, it can be shown that if S is in complex-valued Jordan form, i.e., $S=\Phi_k(s^\star)$, either of the specific choices for $L=\begin{bmatrix}\ell_0 & \ell_1 & \dots & \ell_k\end{bmatrix}\in\mathbb{C}^{m\times k+1}$:

C1 $\ell:=\ell_0=\dots=\ell_k\in\mathbb{R}^m$;
C2 $\ell:=\ell_0\in\mathbb{R}^m$ and $\ell_i=0_{m\times1}$ for $i=1,\dots,k$,

ensures the following match:

$$\eta_{\tilde{k}}^{\Sigma}(s^\star,\ell)=\eta_{\tilde{k}}^{\hat{\Sigma}}(s^\star,\ell),\ \text{for } \tilde{k}=0,\dots,k. \tag{17}$$

Consequently, model (4) is a (reduced-order) model of (1) at $(s^\star,\ell)$. The transfer-function matrices of the full- and the reduced-order models and their first k derivatives match along the direction $\ell\in\mathbb{C}^m$ at the interpolation point $s^\star$. If the matrix S is not in Jordan form, then after a similarity transformation (with matrix T, see Theorem 6), the columns of the matrix LT should satisfy **C1** or **C2** for (4) to be a model of (1) in accordance with Definition 2.

Next, we treat the case where the reduced-order model is obtained via the result in Theorem 7 and S in real-valued Jordan form, i.e., $S=\Psi_k(s^\star)$. In that case, it can be shown that either of the specific choices $L=\begin{bmatrix}\ell_0 & \ell_1 & \dots & \ell_{2k+1}\end{bmatrix}\in\mathbb{R}^{m\times2(k+1)}$:

R1 $\ell:=\ell_0=\dots=\ell_{2k+1}\in\mathbb{R}^m$;
R2 $\ell:=\ell_0\in\mathbb{R}^m$ and $\ell_i=0_{m\times1}$ for $i=1,\dots,2k+1$;
R3 $\ell:=\ell_1\in\mathbb{R}^m$ and $\ell_0=\ell_i=0_{m\times1}$ for $i=2,\dots,2k+1$;

ensures that the transfer-function matrices of the full- and the reduced-order model match as follows:

$$\eta_{\tilde{k}}^{\Sigma}(s^\star,\ell)=\eta_{\tilde{k}}^{\hat{\Sigma}}(s^\star,\ell), \tag{18a}$$
$$\eta_{\tilde{k}}^{\Sigma}(\bar{s}^\star,\ell)=\eta_{\tilde{k}}^{\hat{\Sigma}}(\bar{s}^\star,\ell), \tag{18b}$$

for $\tilde{k}=0,\dots,k$. Consequently, model (4) is a (reduced-order) model of (1) at $(s^\star,\ell)$ and $(\bar{s}^\star,\ell)$. If the matrix S is not in Jordan form, then after a similarity transformation (with matrix T, see Theorem 7), the columns of the matrix LT should satisfy **R1**, **R2**, or **R3**, for (4) to be a model of (1) in accordance with Definition 2.

The selections of ℓ_i's given above for the complex- and real-valued Jordan cases ensure that the families of models in Theorems 6 and 7 achieve moment matching as classically intended, i.e., in the sense of Definition 2. However, note that Theorem 7 can also be used to provide generalized matching conditions with a complex tangential direction. For instance, in the case

R4 $\ell_i=\ell_0$ for i even and $\ell_j=\ell_1$ for j odd, $i,j\le2k+1$, i.e., $L=\begin{bmatrix}\ell_0 & \ell_1 & \dots & \ell_0 & \ell_1\end{bmatrix}\in\mathbb{R}^{m\times2(k+1)}$,

the following match is obtained for $\ell:=\ell_0+j\ell_1$:

$$\eta_{\tilde{k}}^{\Sigma}(s^\star,\ell)=\eta_{\tilde{k}}^{\hat{\Sigma}}(s^\star,\ell), \tag{19a}$$
$$\eta_{\tilde{k}}^{\Sigma}(\bar{s}^\star,\bar{\ell})=\eta_{\tilde{k}}^{\hat{\Sigma}}(\bar{s}^\star,\bar{\ell}), \tag{19b}$$

for $\tilde{k}=0,\dots,k$. Model (4) is then a (reduced-order) model of (1) at $(s^\star,\ell)$ and $(\bar{s}^\star,\bar{\ell})$ in accordance with Definition 2. The tangential direction $\ell\in\mathbb{C}^m$, in this case, is generally complex-valued.

Remark 10. Time-domain moment matching for MIMO models was also pursued informally in Ionescu, Astolfi, and Colaneri (2014), Ionescu, Iftime, and Necoara (2020), Necoara and Ionescu (2020), and Scarciotti (2016) In those works, however, the statements for the MIMO case were ambiguously formulated without proof and only for a special case.

2.5. Family of models that achieve zeroth-order moment matching

Finally, in this section, we present a family of models that achieves moment matching for zero moments at multiple interpolation points. This family is described by real-valued matrices, which is the most practical case and may be viewed as a MIMO counterpart of the results of Astolfi (2010).

Theorem 11. *Consider model (1) with m inputs and p outputs, the non-derogatory matrix $S \in \mathbb{R}^{\nu\times\nu}$ with all distinct eigenvalues, and any matrix $L \in \mathbb{R}^{m\times\nu}$. Suppose $\sigma(A)\cap\sigma(S)=\emptyset$ and write the eigenvalues $\sigma(S)$ of the matrix S as follows:*

$$\sigma(S) := \{s_1, \bar{s}_1, \dots, s_r, \bar{s}_r, s_{r+1}, s_{r+2}, \dots, s_{r+q}\}, \tag{20}$$

consisting of r pairs of complex conjugated eigenvalues and $q = \nu - 2r$ real eigenvalues. Furthermore, consider any non-singular matrix $T \in \mathbb{R}^{\nu\times\nu}$ such that

$$\tilde{S} := T^{-1}ST = \begin{bmatrix} \Psi_0(s_1) & \cdots & 0_2 & 0_{2\times1} & \cdots & 0_{2\times1} \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0_2 & \cdots & \Psi_0(s_r) & 0_{2\times1} & \cdots & 0_{2\times1} \\ 0_{1\times2} & \cdots & 0_{1\times2} & \Phi_0(s_{r+1}) & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0_{1\times2} & \cdots & 0_{1\times2} & 0 & \cdots & \Phi_0(s_{r+q}) \end{bmatrix} \tag{21}$$

with $\Phi_0(\cdot)$ and $\Psi_0(\cdot)$ defined in Lemmas 3 and 4, respectively, Then, for any $G \in \mathbb{R}^{\nu\times m}$ such that $\sigma(S)\cap\sigma(S-GL)=\emptyset$, model (4) with $F := S - GL$, $H = C\Pi$, and with $\Pi \in \mathbb{R}^{n\times\nu}$ the unique solution of the Sylvester equation

$$A\Pi + BL = \Pi S, \tag{22}$$

achieves matching according to

$$\eta_0^{\Sigma}(s_k, \ell_{2k-1} + j\ell_{2k}) = \eta_0^{\hat{\Sigma}}(s_k, \ell_{2k-1} + j\ell_{2k}) \tag{23a}$$

$$\eta_0^{\Sigma}(\bar{s}_k, \ell_{2k-1} - j\ell_{2k}) = \eta_0^{\hat{\Sigma}}(\bar{s}_k, \ell_{2k-1} - j\ell_{2k}) \tag{23b}$$

for $k = 1, \dots, r$, and achieves zeroth-order moment matching according to

$$\eta_0^{\Sigma}(s_{r+k}, \ell_{2r+k}) = \eta_0^{\hat{\Sigma}}(s_{r+k}, \ell_{2r+k}) \tag{24}$$

for $k = 1, \dots, q$, where $\ell_1, \dots, \ell_\nu \in \mathbb{R}^m$, are the columns of the matrix LT, i.e.,

$$LT = \begin{bmatrix} \ell_1 & \cdots & \ell_\nu \end{bmatrix}. \tag{25}$$

Proof. Notice the block-diagonal structure of the matrix $\tilde{S}$ in (21). Then, by the results of Theorems 6 and 7 for zeroth moments, applied to each block-diagonal element of $\tilde{S}$ and corresponding columns of the matrix LT, the matching property as in (23) and (25) is guaranteed. ■

Application of Theorem 11 results in model (4) being a (reduced-order) model of (1) at $\sigma(S)$, in accordance with Definition 1, along the tangential directions as given in the theorem. Notice again that the tangential directions are the columns of the matrix LT rather than L, see also Remark 9. Analogue to Theorem 11, a theorem for complex-valued matrices (S, L) can be formulated.

3. Illustrative example

Consider the following fourth-order model with two inputs and one output:

$$K(s) = \frac{\begin{bmatrix} 1, & (s+5) \end{bmatrix}}{(s+1)(s+2)(s+3)(s+4)} \in \mathbb{C}^{1\times2} \tag{26}$$

with $s \in \mathbb{C}$. Denote the transfer function from input 1 to the output by $K_1(s)$ and the transfer function from input 2 to the output by $K_2(s)$. We pick the following matrix S with interpolation points $\sigma(S) = \pm j$ and matrix L:

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad L = I_2. \tag{27}$$

For this choice of (S, L), the matching property according to **R4** in Section 2.4 applies with

$$\ell := \begin{bmatrix} 1 \\ j \end{bmatrix}, \tag{28}$$

and it can be verified that $C\Pi$ is as follows:

$$C\Pi = \begin{bmatrix} \mathrm{Re}(\mathcal{K}(j)) & \mathrm{Im}(\mathcal{K}(j)) \end{bmatrix} = \begin{bmatrix} \frac{2}{17} & \frac{-1}{34} \end{bmatrix}. \tag{29}$$

with $\mathcal{K}(s) = K(s)\ell = K_1(s) + jK_2(s)$. The relation (29) holds for any minimal realization of (26).

The family of reduced-order models with state dimension $\nu = 2$, parametrized by G is described by (4) with $F = S - GL$ and $H = C\Pi$. Here, S and L are given in (27) and $C\Pi$ is given in (29). For simplicity, we select $G = I_2$, which yields

$$F = S - GL = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \tag{30}$$

and satisfies $\sigma(F)\cap\sigma(S) = \emptyset$. The transfer function of the reduced-order model is then as follows:

$$\hat{K}(s) = C\Pi(sI_2 - F)^{-1}I_2 = \frac{C\Pi}{(s+1)^2+1}\begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix}. \tag{31}$$

Using the results in Section 2.4, it is expected that the matching property (19) is satisfied for $\tilde{k} = 0$, which reads as follows:

$$K(j)\ell = \hat{K}(j)\ell, \tag{32a}$$

$$K(-j)\bar{\ell} = \hat{K}(-j)\bar{\ell}. \tag{32b}$$

It can indeed be verified that both sides of (32a) are equal to $-340 + 170j$ and both sides of (32b) are equal to $-340 - 170j$, which shows that the equalities (32a) and (32b) hold. The Bode magnitude plots of $K(j\omega)\ell$ and $\hat{K}(j\omega)\ell$ in Fig. 1 evidently show that there is a match at $\omega = 1$ rad/sec. Thus the reduced-order model achieves a generalized matching condition (corresponding to (19)) with a complex-valued tangential direction and is a model of (1) at (j, ℓ) and $(-j, \bar{\ell})$ according to Definition 2.

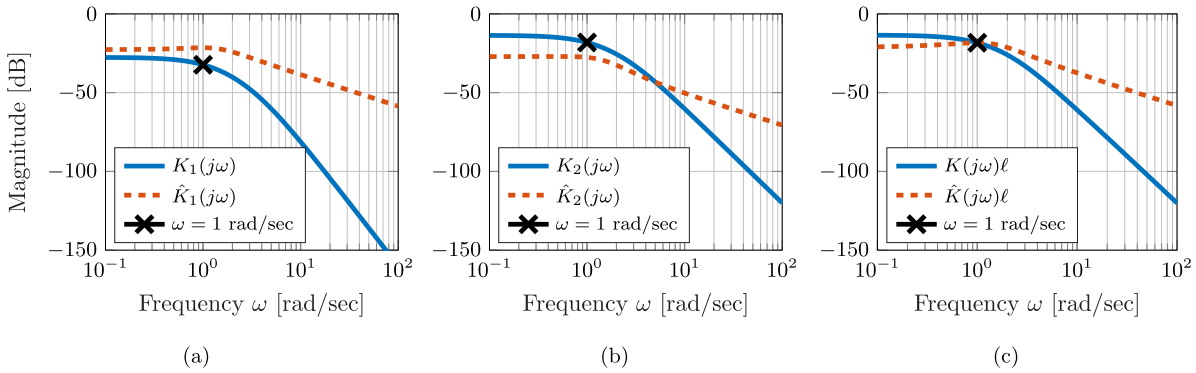


Fig. 1. (a) Bode magnitude plot of $K_1(j\omega)$ (blue/solid), $\hat{K}_1(j\omega)$ (red/dashed); (b) Bode magnitude plot of $K_2(j\omega)$ (blue/solid), $\hat{K}_2(j\omega)$ (red/dashed); (c) Bode magnitude plot of $K(j\omega)\ell$ (blue/solid), $\hat{K}(j\omega)\ell$ (red/dashed). The black crosses indicate the interpolation frequency $\omega = 1$. A mismatch is observed between $K_i(j\omega)$ and $\hat{K}_i(j\omega)$ at $\omega = 1$ for both $i = 1, 2$. A match is observed between $K(j\omega)\ell$ and $\hat{K}(j\omega)\ell$ at $\omega = 1$.

4. Conclusions

This paper points out that time-domain moment matching applied to MIMO LTI models does not always achieve matching in moments that is consistent with classical moment matching techniques based on Krylov subspaces. To achieve a match that is consistent with classical Krylov subspace methods, the parameters for time-domain moment matching should be carefully chosen. In particular, the matrix characterizing the tangential directions should be chosen with a specific structure. Furthermore, the matrix characterizing the interpolation points should be chosen either in Jordan form or the tangential directions should be corrected by means of a coordinate transformation. The results of this paper facilitate parameter selection for time-domain moment matching such that the match in the transfer-function matrix is consistent with the match that is achieved by classical Krylov subspace methods.

Appendix. Proof of Lemma 4

Proof. The goal is to prove the relation between $C\tilde{\Pi}$ and the moments as given in (10). For notational convenience, we write $s^\star = a + jb$ rather than $s^\star = a^\star + jb^\star$.

Let us first prove the case where $a = 0$ and $k = 0$. Partition $\tilde{\Pi} \in \mathbb{R}^{n\times 2}$ as follows:

$$\tilde{\Pi} = \begin{bmatrix} \tilde{\Pi}_0 & \tilde{\Pi}_1 \end{bmatrix} \tag{A.1}$$

with $\Pi_0, \Pi_1 \in \mathbb{R}^n$. Using this partitioning, we find the following two matrix equalities from the Sylvester equation (11):

$$A\tilde{\Pi}_0 + B\ell_0 = -b\tilde{\Pi}_1 \tag{A.2a}$$
$$A\tilde{\Pi}_1 + B\ell_1 = b\tilde{\Pi}_0. \tag{A.2b}$$

Multiply (A.2a) by b to obtain

$$Ab\tilde{\Pi}_0 + bB\ell_0 = -b^2\tilde{\Pi}_1. \tag{A.3}$$

Now, substitute (A.2b) into (A.3) to obtain

$$A^2\tilde{\Pi}_1 + AB\ell_1 + bB\ell_0 = -b^2\tilde{\Pi}_1. \tag{A.4}$$

Solving for $\tilde{\Pi}_1$ yields

$$\tilde{\Pi}_1 = -(A^2 + b^2 I_n)^{-1}(AB\ell_1 + bB\ell_0). \tag{A.5}$$

Similarly, one can obtain

$$\tilde{\Pi}_0 = (A^2 + b^2 I_n)^{-1}(bB\ell_1 - AB\ell_0). \tag{A.6}$$

Let us now find an expression for $\eta_0(jb, \ell_0 + j\ell_1)$ from Definition 1:

$$\begin{aligned}\eta_0(jb, \ell_0 + j\ell_1) &= C(jbI_n - A)^{-1}B(\ell_0 + j\ell_1)\\ &= -C(A - jbI_n)^{-1}(A + jbI_n)^{-1}(A + jbI_n)B(\ell_0 + j\ell_1)\\ &= -C(A^2 + b^2 I_n)^{-1}(A + jbI_n)B(\ell_0 + j\ell_1).\end{aligned} \tag{A.7}$$

It can be verified from (A.7) that $\mathrm{Re}(\eta_0(jb, \ell_0 + j\ell_1))$ is equal to $C\tilde{\Pi}_0$ in (A.6) and that $\mathrm{Im}(\eta_0(jb, \ell_0 + j\ell_1))$ is equal to $C\tilde{\Pi}_1$ in (A.5), which concludes the proof for the case $a = 0$ and $k = 0$.

For the case where $a \neq 0$ and $k = 0$, it can be shown that the same steps as taken above result in A replaced by $A - aI_n$ in (A.6) and (A.5), which gives expressions for $\tilde{\Pi}_0$ and $\tilde{\Pi}_1$, respectively. Furthermore, it can be shown that in the expression of $\eta_0(a + jb, \ell_0 + j\ell_1)$ in (A.7), again A is replaced $A - aI_n$, which concludes the proof for $a \neq 0$ and $k = 0$.

Let us now turn to the case where $a = 0$ and $k = 1$. Hereto, we partition $\tilde{\Pi} \in \mathbb{R}^{n\times 4}$ as follows:

$$\tilde{\Pi} = \begin{bmatrix} \tilde{\Pi}_0 & \tilde{\Pi}_1 & \tilde{\Pi}_2 & \tilde{\Pi}_3 \end{bmatrix} \tag{A.8}$$

with $\Pi_0, \ldots, \Pi_3 \in \mathbb{R}^n$. Using this partitioning, we find the following four matrix equalities from the Sylvester equation (11):

$$A\tilde{\Pi}_0 + B\ell_0 = -b\tilde{\Pi}_1, \tag{A.9a}$$
$$A\tilde{\Pi}_1 + B\ell_1 = b\tilde{\Pi}_0, \tag{A.9b}$$
$$A\tilde{\Pi}_2 + B\ell_2 = \tilde{\Pi}_0 - b\tilde{\Pi}_3, \tag{A.9c}$$
$$A\tilde{\Pi}_3 + B\ell_3 = \tilde{\Pi}_1 + b\tilde{\Pi}_2. \tag{A.9d}$$

The solution to the set of Eqs. (A.9a) and (A.9b) for $\tilde{\Pi}_0$ and $\tilde{\Pi}_1$ is given in (A.6) and (A.5), respectively. Let us find the solution for $\tilde{\Pi}_2$ and $\tilde{\Pi}_3$. Rewrite (A.9c) and (A.9d) as follows:

$$A\tilde{\Pi}_2 + \overline{B\ell_2} = -b\tilde{\Pi}_3, \tag{A.10a}$$
$$A\tilde{\Pi}_3 + \overline{B\ell_3} = b\tilde{\Pi}_2. \tag{A.10b}$$

with $\overline{B\ell_2} = B\ell_2 - \tilde{\Pi}_0$ and $\overline{B\ell_3} := B\ell_3 - \tilde{\Pi}_1$. Notice that (A.10a) and (A.10b) have the same form as the set of Eqs. (A.2a) and (A.2b). Therefore, their solution reads as follows (with a

pre-multiplication by C):

$$C\tilde{\Pi}_2 = C(A^2+b^2I_n)^{-1}\left(bB\ell_3 - b\tilde{\Pi}_1 - AB\ell_2 + A\tilde{\Pi}_0\right)$$
$$= Re(\eta_0(jb, \ell_2+j\ell_3)) + C(A^2+b^2I_n)^{-1}\left(-b\tilde{\Pi}_1 + A\tilde{\Pi}_0\right), \quad \text{(A.11a)}$$

$$C\tilde{\Pi}_3 = -C(A^2+b^2I_n)^{-1}\left(AB\ell_3 - A\tilde{\Pi}_1 + bB\ell_2 - b\tilde{\Pi}_0\right)$$
$$= Im(\eta_0(jb, \ell_2+j\ell_3)) + C(A^2+b^2I_n)^{-1}\left(A\tilde{\Pi}_1 + b\tilde{\Pi}_0\right). \quad \text{(A.11b)}$$

It is left to show that the following two equalities hold:

$$C(A^2+b^2I_n)^{-1}\left(-b\tilde{\Pi}_1 + A\tilde{\Pi}_0\right) = -\mathrm{Re}(\eta_1(jb, \ell_0+j\ell_1)), \quad \text{(A.12a)}$$

$$C(A^2+b^2I_n)^{-1}\left(A\tilde{\Pi}_1 + b\tilde{\Pi}_0\right) = -\mathrm{Im}(\eta_1(jb, \ell_0+j\ell_1)). \quad \text{(A.12b)}$$

Let us start with working out the left-hand side of (A.12a):

$$\begin{aligned}&(A^2+b^2I_n)^{-1}\left(-b\tilde{\Pi}_1 + A\tilde{\Pi}_0\right) \quad \text{(A.13)}\\ &= (A^2+b^2I_n)^{-1}b(A^2+b^2I_n)^{-1}(AB\ell_1 + bB\ell_0)\\ &+ (A^2+b^2I_n)^{-1}A(A^2+b^2I_n)^{-1}(bB\ell_1 - AB\ell_0)\\ &= (A^2+b^2I_n)^{-1}(A^2+b^2I_n)^{-1}b(AB\ell_1 + bB\ell_0)\\ &+ (A^2+b^2I_n)^{-1}(A^2+b^2I_n)^{-1}A(bB\ell_1 - AB\ell_0)\\ &= (A^2+b^2I_n)^{-2}\left(2bAB\ell_1 + b^2B\ell_0 - A^2B\ell_0\right),\end{aligned}$$

where the fact that any matrix A commutes with $(A^2+b^2I_n)^{-1}$ is used and the existence of $(A^2+b^2I_n)^{-1}$ is guaranteed by the assumption that $s^\star = jb \notin \sigma(A)$. Next, we derive an expression for $\eta_1(jb, \ell_0+j\ell_1)$ as follows:

$$\begin{aligned}&\eta_1(jb, \ell_0+j\ell_1) = C(jbI_n - A)^{-2}B(\ell_0+j\ell_1)\\ &= C(jbI_n - A)^{-2}(jbI_n + A)^{-2}(jbI_n + A)^2B(\ell_0+j\ell_1)\\ &= C(-b^2I_n - A^2)^{-2}(jbI_n + A)^2B(\ell_0+j\ell_1)\\ &= C(b^2I_n + A^2)^{-2}(A^2 + 2jbA - b^2I_n)B(\ell_0+j\ell_1)\\ &= C(A^2+b^2I_n)^{-2}(2jbAB + A^2B - b^2B)(\ell_0+j\ell_1).\end{aligned}$$

From here, it can be easily verified that the relation in (A.12a) holds. The proof for (A.12b) can be performed similarly and is omitted here for the sake of brevity.

The proof for $a \neq 0$ and $k = 1$ involves the same steps as given above for the case $a \neq 0$ and $k = 0$ and, therefore, omitted. The proof for $k > 1$ can be continued similarly as above and is omitted for the sake of brevity. ■

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