

**IMPERIAL**

CONTRIBUTIONS TO NEOSTABILITY  
THEORY

Author

ALBERTO MIGUEL GÓMEZ

Supervised by

PROF DAVID EVANS

DR CHARLOTTE KESTNER

A Thesis submitted in fulfillment of requirements for the degree of  
**Doctor of Philosophy in Mathematics**

Department of Mathematics  
Imperial College London  
2026

# Abstract

This thesis makes several contributions to the study of structure theory for  $\text{NSOP}_1$  and  $\text{NSOP}_4$  theories, as well as the theories of abstract independence relations and patterns in neostability theory. Firstly, we provide a partial answer to a question asked independently by d’Elbée and Kim and show that, under the assumption of the stable Kim-forking conjecture, every  $\text{NSOP}_1$  rosy theory must be simple. We also prove that the theory of a Frobenius field has stable Kim-forking.

Secondly, we provide two proofs, one combinatorial and another using abstract independence relations, of the model-theoretic classification of the countable homogeneous  $\mathbf{H}_4$ -free 3-hypertournament studied by Cherlin, Hubička, Konečný, and Nešetřil. Our main result regarding this structure is that its theory is  $\text{SOP}_3$ ,  $\text{TP}_2$ , and  $\text{NSOP}_4$ .

Thirdly, to contextualise the phenomena observed previously more broadly, we develop a framework, in the style of Adler, for studying the notion of “witnessing” (appearing as variants of Kim’s Lemma in the literature) as a binary relation between abstract independence relations. This involves the relativisation of the notions of Kim- and Conant-independence. Our notion of witnessing allows us to obtain several results from simplicity,  $\text{NSOP}_1$ ,  $\text{NTP}_2$ , etc., as instances of general theorems in this abstract framework.

We finish this thesis by systematically exploring the notions of coding and realising graphs appearing in the work of Bailetti, Laskowski, and Shelah, and provide new characterisations of stability, NIP,  $\text{NSOP}_3$ ,  $\text{NPM}^{(2)}$ ,  $\text{NIP}_n$ , and  $\text{NFOP}_n$  using these tools. Alongside this, we further develop the theory of positive patterns in connection with realisability, and introduce the notion of duality of patterns to formally capture the relationship between syntactic properties such as  $\text{SOP}_2$  and ATP.

# Statement of Originality

I declare that this thesis, and the research to which it refers, are the product of my own work, except where I have quoted and referenced the work of other people in accordance with the standard referencing practices of the discipline. Several results in this thesis have been published in [86], [87], [88], which I completed during my PhD.

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# Acknowledgments

Even if theorems do live in a Platonic realm of truth after all, the mathematician does not. Whatever narrow slice of truth I have managed to grasp during these years would not have been attainable without the invaluable support and the patient trust of the people around me.

First and foremost, I would like to heartily thank my supervisors, David Evans and Charlotte Kestner, for their guidance and support throughout my PhD. Thank you for the freedom you have given me and the confidence that you have shown in me, even with some dead ends along the way, and for your many suggestions and careful revisions of my writing, which have enormously improved it. I could not have hoped for better supervisors.

Thanks to Paolo for guiding me through my first year with lots of invaluable advice which shaped and improved my PhD journey, and for hosting me in Vienna and sharing some wonderful coffees with me. Thanks also to Mark for all the time he spent teaching me categorical logic during my first and second years, and for hosting me in Brno during my third year.

A significant portion of my PhD was completed at the University of Notre Dame, and I have very fond memories of the almost three months I spent there. I would like to sincerely thank Anand Pillay for organising this visit, for the pleasure of working together, and for his great support both during this time and afterwards; and Nick Ramsey for his wonderful lectures on model-theoretic tree properties, which have deeply influenced my own research, and for being so generous with his time. Thanks to Atticus, who became a true friend and not just a colleague, as well as all the wonderful people I met at Notre Dame, including David, Gabriel, Tan, Yuyan, Sulin, Tom, Katie, Gavin, and Jenny.

I also want to thank everyone who has made my academic life at Imperial fun and memorable. I will always fondly remember being part of such an incredible office as Huxley 613 and all those who made it so. Thanks to Bernat, Diego, Chingun, Jed, Emre, Amir, Vincent, Emilia and Jamie for building a little home within Imperial. Thanks also to Sabrina, Alice, Naomi, William, and all other friends beyond the office who, at lunches, concerts or coffee breaks, have truly lit up my days.

There are also other friends from outside Imperial who have continuously reminded me of how much life there is beyond mathematics and shared the pleasure of their presence with me. Thanks to all those friends back home who have been there through the years, to Manuel, David, Marina, Javi, Jaime, Fer, and Charlie, who, with our meetings and long video calls, have kept our connection alive and provided the best antidote against loneliness. Thanks to Aditya and Brynn for countless hours at the cinema; to Philip for his constant encouragement and friendship across the years; to Mateusz for deep and insightful conversations about life, politics, philosophy, and beyond; and to Jesús for endless fun facts and his passion for science and life in all of its complexity and its beauty.

I am also grateful to the beautiful community at Bloomsbury Central Baptist Church for opening their arms to welcome me, making me part of their wonderful family, and showing an uncompromising commitment to the “voice that calls us to justice when the world favours oppression, to courage when the world favours convenience, and to love when the world favours fear,” to use Simon’s words. Through you, I have come to experience the profound truth in the quote of Paul Tillich that “ultimately knowledge is an act of love.”

أتوجه بخالص شكري وعميق امتناني أيضاً إلى إنجي، التي كانت حقاً سندي الدائم طوال فترة دراستي للدكتوراه. أشكرك على مودتك واهتمامك الصادق، وعلى دعمك اللامحدود وثقتك بي، وعلى وقوفك بجاني لرفع همتي كلما ساورني الشك في قدراتي، وعلى كل اللحظات الجميلة والأوقات الطيبة التي أمضيها معاً. ما كان لهذه الأطروحة أن تكتمل لولاك. والشكر موصول أيضاً إلى هدى وسارة على كرم الضيافة وحسن الاستقبال مراراً وتكراراً في منزلهما، وعلى معاملتهما لي كفرد من أفراد العائلة.

Finalmente, y por encima de todo lo demás, quiero dar mis más sinceras y profundas gracias a mi familia. Gracias a mis padres por su constante esfuerzo, apoyo y sacrificio, por luchar para ofrecerme las mejores oportunidades posibles, por confiar en mí en esta aventura y por animarme siempre a apuntar al Sol (sí, al Sol), por si alcanzaba alguna estrella. Gracias a mi hermana Noelia por ser el mejor regalo que mis padres jamás me hayan dado, por el amor incomparable que me ha ofrecido siempre, por llevarme constantemente con ella en su cuerpo y en su corazón, por estar ahí y ayudarme en los momentos difíciles, y por visitarme y explorar Londres en su compañía. Este y todos los logros que consiga en mi vida os pertenecen tanto como a mí, e incluso más.

The several epigraphs included in the thesis are meant to be personal snapshots of these three and a half years. I can't think of a better way to close these acknowledgements than by offering a final tribute to the voice that has accompanied me during many hours of writing this thesis, Robe Iniesta, by quoting his profound (and profoundly Augustinian) reflection on the mystery of time:

Que el tiempo pasa siempre a nuestro lado  
Y ahí no se queda  
Y ahora se ha convertido en pasado  
Y no nos espera  
Cabalga desbocado al infinito  
Y yo, yo me quedo contigo aquí.

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# 1

## Introduction

The purpose of this thesis is to contribute to different directions of the *neostability programme*, a central area of contemporary model theory focused on the classification of mathematical structures in terms of properties of their first-order theories. The problems addressed in this thesis reflect the current diversity of the subject and include open questions about  $\text{NSOP}_1$  theories, new examples and structure theory for  $\text{NSOP}_4$  theories, the development of a novel abstract framework to study witnessing for independence relations, and the systematic application of coding and realising to characterise dividing lines.

To understand and motivate the neostability programme, we first need to go back to the development of *stability theory*. This area originates with Morley’s Theorem [90], showing that a theory that is  $\lambda$ -categorical for *some* uncountable  $\lambda$  is  $\lambda$ -categorical for *every* uncountable  $\lambda$ . This is an example of one of several influential results in model theory having the structure of an equivalence between a universal and an existential condition. The proof also reveals the unity of several syntactic and semantic notions which would significantly shape the following research in model theory.

The methods introduced by Morley in his proof were widely generalised by Shelah to the context of all stable theories, culminating in his monumental work [105]. The underlying methodology pursued in this work is the search for *dividing lines*, “properties  $P$  of a theory  $T$  such that we have relevant information on both sides about those theories which have property  $P$  and about those which do not have it” [110, Thesis 1.1]. Shelah identifies several conditions which a property that aspires to be a dividing line can satisfy, including two conditions which he claims to be necessary [110, Informal Definition 1.2]. On one hand, such a property must be *robust*, meaning “it has an internal definition [...] and an external definition.” The split between “internal” and “external” definitions corresponds to what we here refer

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to as “syntactic” and “semantic” characterisations, and illustrations of this will be provided later in this introduction. On the other hand, such a property must be *successful*, meaning “there is a serious structure theory on the positive side.”

*Stability* is one of the most influential dividing lines that Shelah identifies in [105]. A fundamental theorem by Shelah proves that stable theories are robust, as they can be equivalently characterised either syntactically (i.e., in terms of the order property) or semantically (i.e., in terms of the size of type spaces of models of a given cardinality), this being the exemplar of his distinction between “internal” and “external” definitions [110]. The argument for the success of stable theories lies in the development of a *structure theory* which classifies them and proves their usefulness. This included the introduction of influential technical machinery, such as the notion of *forking* as a measure of the “smallness” of a definable set. Later work by Hrushovski, Pillay, Zilber and others on stability theory also offered a range of groundbreaking applications to other areas of mathematics, such as algebraic geometry (e.g., [57]), differential algebra (e.g., [101]), transcendental number theory (e.g., [116]), or combinatorics (e.g., [16]).

The term “neostability,” although not formally defined anywhere in the literature, has been widely used to refer to the extension of methods from stability theory to classes of (potentially) unstable first-order theories. While the significant developments in the study of NIP and specifically o-minimal theories form a substantial part of neostability theory (see, e.g., [111]), we will not discuss them here, as they are unrelated to the contents of the thesis. The origins of the direction of neostability theory considered here can be traced back to the work on *simplicity*, a notion introduced by Shelah in his work in stability theory [106]. “After a period of neglect” [65, p. 822], these theories started gathering new attention in the work of several authors, and the general theory was developed by Kim and Pillay in several papers [64], [65], [70]. Central to their work was the observation that forking actually retained most of its useful properties in the more general context of simple theories. This gives rise to an independence relation which can be used to provide geometric tools in the study of models of simple theories. The culmination of their work was the Kim-Pillay Theorem [70, Theorem 4.2], which provides a characterisation of (the syntactically defined) simplicity in terms of the existence of an independence relation satisfying a number of properties (a semantic concept), which is yet another example of a universal-existential equivalence.

The success of the study of simple theories inspired a whole generation of model-theorists to investigate larger classes of first-order theories to which the methods

developed by Kim and Pillay could be generalised. Notable attempts include the work on rosy [97],  $\text{NTP}_2$  [26], [29], and  $\text{NSOP}_1$  theories [32], [59], [60], [61]. The structure theory for each of these classes involved generalisations of “Kim’s lemma”, originally isolated by Kim in [64] and playing a key role in the study of simplicity. Specifically in the case of the latter two classes, this provided a connection between the syntactically defined notions of  $\text{NTP}_2$  and  $\text{NSOP}_1$ , and semantic properties about the geometric structure of types in models of such theories, establishing them as new examples of dividing lines. Moreover, within the development of  $\text{NSOP}_1$  theories, the notion of *Kim-independence* was developed under the oft-repeated motto of “non-forking at a generic scale” [59, p. 1423], which will play an important role in subsequent chapters.

The study of  $\text{NSOP}_1$  theories is currently far from completed, as demonstrated by the work of several authors on extending Kim-independence to arbitrary bases [30], [45], [69], [93] or developing a theory of definable groups in  $\text{NSOP}_1$  theories [15], [42], among other problems. In chapter 3, we consider one of these open questions about  $\text{NSOP}_1$  theories, which concerns its relationship to rosy theories, and was posed explicitly by d’Elbée and Kim [37], [67]. We provide a partial answer to this question that relies on Bossut’s *stable Kim-forking conjecture* [14]:

**Corollary 3.3.8.** *Suppose that the stable Kim-forking conjecture holds. If  $T$  is  $\text{NSOP}_1$  and rosy, then  $T$  is simple.*

In light of the development of structure theory for  $\text{NSOP}_1$  theories by Chernikov, Kaplan, Ramsey and Shelah, a natural question is whether we can keep extending these techniques further down the  $\text{NSOP}_n$  hierarchy introduced by Shelah in [107]. Recently, Mutchnik has shown in [95] that the classes of  $\text{NSOP}_2$  and  $\text{NSOP}_1$  coincide, and it remains open whether there are any  $\text{NSOP}_3$   $\text{SOP}_1$  theories. In contrast, there have been many known examples of natural  $\text{NSOP}_4$   $\text{SOP}_3$  theories. The first such examples, already appearing in [107], were of a combinatorial nature.

The question regarding the development of structure theory for  $\text{NSOP}_4$  theories underlies much of the work in this thesis, either explicitly (as in chapter 4) or implicitly (as in chapter 5). A contribution to the study of  $\text{NSOP}_4$  theories was Conant’s study of free amalgamation theories in [34], which include many previously known examples such as Henson’s generic digraphs [53]. In a different direction, Evans and Wong observed that certain Hrushovski constructions provided new examples of strictly  $\text{NSOP}_4$  structures [47]. More recently, new algebraic examples of strictly  $\text{NSOP}_4$  structures have been found, including d’Elbée, Müller, Ramsey

and Siniora's generic  $c$ -nilpotent Lie algebras over  $\mathbb{F}_p$  for  $c > 2$  and  $p$  prime [39] and Johnson and Ye's curve-excluding fields [58]. A common feature of all these examples is the presence of an invariant independence relation defined over models satisfying full existence, symmetry, and stationarity.

A more recent contribution towards a systematic theory of independence in the context of  $\text{NSOP}_4$  theories is due to Mutchnik. In [94], the notion of *Conant-dividing* (also appearing in the literature as *strong Kim-dividing* in [61] and, later, as *universally Kim-dividing* in [76]) is studied in depth. It aims to capture the notion of dividing at a maximally generic scale. A connection is achieved between the related notion of Conant-independence and  $\text{NSOP}_4$  by showing that the symmetry of the former implies the latter. Furthermore, in the same paper, Mutchnik generalises the structure theory for  $\text{NSOP}_1$  theories relative to a choice of an invariant independence relation satisfying full existence, monotonicity, and stationarity over models, and gives criteria for identifying Conant-independence as one of these relative notions of Kim-independence.

In chapter 4, we generalise Mutchnik's work further in the study of a new example of an  $\text{NSOP}_4$   $\text{SOP}_3$  theory for which no stationary independence relation (in the sense of the previous paragraph) has been found. This structure comes from the work of Cherlin, Hubička, Konečný and Nešetřil on homogeneous 3-hypertournaments and their connections to structural Ramsey theory [25]. We provide a full characterisation in terms of the main current neostability-theoretic properties:

**Theorem 4.0.1.** *The theory of the countable homogeneous  $\mathbf{H}_4$ -free 3-hypertournament is  $\text{SOP}_3$ ,  $\text{TP}_2$ , and  $\text{NSOP}_4$ . Moreover, it is  $\text{IP}_2$  and  $\text{NFOP}_3$ .*

One of our proofs of  $\text{NSOP}_4$  in that chapter includes a characterisation of Conant-independence in this theory and applies Mutchnik's machinery to a non-stationary independence relation. This exhibits some new behaviour within the class of  $\text{NSOP}_4$  theories.

All these developments testify to the enduring predominance that independence relations have enjoyed in the structure theory of dividing lines in recent decades. This prevalence has motivated treating abstract independence relations as objects of study on their own. The first explicit appearance of this notion in the model-theoretic literature is due to Makkai [83], and had the aim of re-elaborating the results of Shelah on forking in stable theories from [105, Chapter III]. There are several results in the area that, viewed from our contemporary standpoint, can be

restated as results about independence relations; an important example is Harnik and Harrington’s characterisation of stable theories in terms of properties of certain extensions of types [51]. This theorem was generalised by Kim and Pillay’s work on simple theories (see [64], [65], [70]), which, as we previously discussed, motivated the study of classes of theories with a nice independence relation, such as rosy theories.

As a generalisation of this work, Adler [3] introduced the notion of an *abstract independence relation* as a ternary relation on small subsets of a monster model of a fixed theory satisfying a number of properties. This treated independence relations as a “semantic” notion that abstracts from the more “syntactic” properties that characterise the traditional dividing lines, such as the order property in stable theories. It was also an early systematic attempt to ground many results from stable and simple theories on properties of these abstract independence relations. Moreover, [3] shows that central results about simple theories (for instance, the fact that non-forking independence is symmetric) can be derived purely from properties of abstract independence relations. This also provided the framework in which the study of  $\text{NTP}_2$  theories took place (see [26], [29]).

Recent developments in neostability have shown the limitations of Adler’s original definition. The ternary relations that we have previously mentioned and play a key role in the study of  $\text{NSOP}_1$  and  $\text{NSOP}_4$  theories, namely Kim-independence and Conant-independence, are generally far from satisfying all the properties that Adler required of an “independence relation”. Moreover, there are several results across the different dividing lines (such as the equality of the relevant notions of forking and dividing) that point to an underlying unity not captured by Adler’s framework.

A large focus of this thesis, particularly in chapter 5, will be the extension of Adler’s framework to account for the more recent developments in neostability theory. The key idea behind this extension is the formalisation of “Kim’s lemma”, in its various forms across neostability theory, as a binary relation between two abstract independence relations. This notion, together with the abstract relativisations of Kim- and Conant-independence to a choice of independence relation  $\perp$ , allows us to recover many results appearing in simple,  $\text{NTP}_2$ ,  $\text{NSOP}_1$ , and other classes of theories as instances of general theorems in this framework. This provides a new perspective on Adler’s formal notion of an *independence relation* and captures (versions of) the characterisation of several dividing lines in terms of symmetry and the independence theorem for a “canonical” independence relation. We also apply this abstract framework to obtain semantically a new dichotomy for a class of  $\text{NSOP}_4$  theories between the syntactically defined properties  $\text{NSOP}_1$  and  $\text{NBTP}$ :

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**Proposition 5.7.5.** *Let  $T$  be a complete theory. Let  $\perp \implies \perp^i$  be an independence relation over small subsets of a monster model of  $T$  satisfying full existence, monotonicity, and quasi-strong finite character such that  $\perp^i$  satisfies GUWP w.r.t.  $\perp$ . Then  $T$  is either  $\text{NSOP}_1$  or BTP.*

Finally, another current research direction in neostability theory, which motivates our results in chapter 6 and the appendix, concerns the search for a “good” dividing line that extends both  $\text{NTP}_2$  and  $\text{NSOP}_1$  theories and preserves some of the structural results that characterise both theories. Three different proposals have recently appeared in the literature: NATP [5], [6], [72], NBTP [76], and NCTP [50], [95]. While each of these classes generalises different properties from  $\text{NTP}_2$  and  $\text{NSOP}_1$ , the precise relation between the three remains open, and it is currently unknown whether (at least) some of them will collapse into one class of theories. Our results from chapter 6 suggest a possible fourth candidate for this question (or the analogous one in the case that  $\text{NSOP}_1 \neq \text{NSOP}_3$ ) coming out of Bailetti’s work on positive maximality, namely, the class of  $\text{NPM}^{(2)}$  theories [8]. These results stem from a systematic application of the notions of coding and realising as developed in the work of Bailetti, Laskowski, and Shelah [8], [79] to characterise several dividing lines. Some of the new characterisations in terms of realisability that we prove in chapter 6 are the following:

**Theorem 6.0.3.** *(i)  $T$  is unstable iff it realises all half-graphs.  
(ii)  $T$  is IP iff it realises all bipartite graphs, iff it realises all triangle-free graphs.  
(iii)  $T$  is  $\text{SOP}_3$  iff it realises all complements of half-graphs.  
(iv)  $T$  is  $\text{PM}^{(2)}$  iff it realises all co-bipartite graphs.*

This result establishes a combinatorial analogy between the pair “stable or NIP”, on one hand, and the pair “ $\text{NSOP}_3$  or  $\text{NPM}^{(2)}$ ”, on the other. The thesis thus ends with this syntactic result that motivates the semantic study of  $\text{NPM}^{(2)}$  theories and the generalisation of properties of  $\text{NTP}_2$  theories to this wider context, thus exhibiting, once again, the unity of syntax and semantics permeating model theory.

## Summary of chapters

A summary of the contents of the thesis is as follows. In chapter 2, “Preliminaries”, we collect those standard results and notions from the existing literature on neostability theory that will appear throughout this thesis.

The two chapters that follow these preliminaries focus on concrete existing properties in the neostability literature that exhibit a unity of syntactic and semantic features. In chapter 3, “Rosiness”, we consider a question of d’Elbée and Kim regarding the interaction between  $\text{NSOP}_1$  and rosy theories. We relate this question to the stable Kim-forking property recently introduced by Bossut, and prove Corollary 3.3.8 mentioned above. The chapter also contains an application of this result to pregeometric theories (Corollary 3.3.11) as well as a proof that the theory of any Frobenius field satisfies the stable Kim-forking property (Proposition 3.2.10).

In chapter 4, “Hypertournaments”, we turn toward some combinatorial structures introduced by Cherlin in [24] and studied in detail in [25]: the Fraïssé limits of 4-constrained classes of finite 3-hypertournaments. After a brief description of the classification properties of the remaining cases, we focus on the generic  $\mathbf{H}_4$ -free 3-hypertournament and prove several neostability-theoretic properties about it (Theorem 4.0.1, previously referenced).

We offer two proofs of the classification of the generic  $\mathbf{H}_4$ -free 3-hypertournament, mirroring the division between syntax and semantics. In §4.2, we give a direct proof of these properties using combinatorial arguments to exhibit, or show the lack of, formulas corresponding to each property. After this, in §§4.3-4.4, we develop semantic technology involving independence relations to provide an alternative proof of  $\text{TP}_2$  and  $\text{NSOP}_4$ . This characterises Conant-independence in the theory of the generic  $\mathbf{H}_4$ -free 3-hypertournament while generalising Mutchnik’s machinery to the non-stationary context. We end the chapter by considering in §4.5 Cherlin’s  $\mathbf{H}_{n+1}$ -free  $n$ -hypertournaments for  $n > 3$  and prove that they are supersimple with trivial forking (Corollary 4.5.5).

The strategy for the remaining chapters of the thesis is different. The core idea consists of the development of the syntactic and semantic aspects of dividing lines separately, so as to delineate more precisely what each technology allows us to prove and from which assumptions, and find points in each development that connect with the other side. The basic underlying motto for each of these chapters is that the thorough development of semantic properties eventually generates syntactic conclusions, and vice versa. Examples of this connection appear in the new dichotomy between syntactic properties obtained via the study of witnessing for abstract independence relations (Proposition 5.7.5) or the combinatorial analogy between  $\text{NPM}^{(2)}$  and  $\text{NIP}$  which motivates the semantic study of this class of theories (Theorem 6.0.3).

Thus, in chapter 5, “Independence”, we focus on the semantic aspect of structure

theory and develop a framework *à la* Adler for the study of witnessing as a binary relation between abstract independence relations. To do this, we first continue Adler’s programme in §5.1 with a new property of independence relations that forces strong finite character, and the generalisation of “ $\perp$ -freeness” which unifies several versions of forking. Using these tools, we develop in §5.2 the notions of  $\perp$ -Kim-independence and  $\perp$ -Conant-independence for an abstract independence relation  $\perp$ , and prove some basic properties. This sets the stage for the main object of study of this chapter: the notion of witnessing as elaborated in §5.3. The rest of the chapter employs this formalisation of witnessing to generalise several results across neostability-theoretic properties (§§5.4-5.6) and prove a new dichotomy for a subclass of NSOP<sub>4</sub> theories which includes all NSOP<sub>1</sub> theories (Proposition 5.7.5).

We then move from semantic to syntactic considerations in chapter 6, “Realisability”. We focus on the notions of coding and realising graphs which have appeared in the work on Bailetti, Laskowski, and Shelah [8], [79]. After clarifying the precise relationship between these two notions in §6.1, we prove some general results about coding in §6.2, which set the stage for new characterisations of stability and NIP in terms of coding several classes of graphs in §6.3. We shift our focus toward realisability in §6.4 with some initial results that illustrate the differences with their analogues for coding. These results prepare us for §6.5, where we provide new characterisations of stability, NIP, NTP<sub>2</sub>, NSOP<sub>3</sub>, and NPM<sup>(2)</sup> in terms of realising different classes of graphs, as given in Theorem 6.0.3. We also prove some results on some higher-arity versions of stability (NFOP<sub>n</sub>) and NIP (NIP<sub>n</sub>).

After this, the conclusion to this thesis collects and discusses several open questions and future research directions inspired or suggested by the contributions from previous chapters.

We finish this thesis with an appendix which deepens the exploration of syntactic properties by moving from graphs to patterns. The appendix consists of two main sections: §A.1 clarifies and formalises the relationship between realising graphs and exhibiting patterns, whereas §A.2 develops a notion of duality of patterns that captures the interaction of several neostability-theoretic properties (such as SOP<sub>2</sub> and ATP) and proves that this notion does not preserve implications of properties at the level of theories (Corollary A.2.23).

# 2

## Preliminaries

Così le idee, che io usavo prima per  
figurarmi un cavallo che non avevo  
ancora visto, erano puri segni,  
come erano segni dell'idea di  
cavallo le impronte sulla neve: e si  
usano segni e segni di segni solo  
quando ci fanno difetto le cose.

---

*Umberto Eco*

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The purpose of this chapter is to introduce the notions that are presupposed throughout the thesis. Although we will explicitly provide the necessary definitions in this chapter, a working knowledge of model theory is assumed.

## 2.1 Model-theoretic notions

The reference for all these definitions is [113]. We write  $a$  indiscriminately for elements and tuples (with the only exception of chapter 4, where we indicate this explicitly). For sets  $A, B$ , we write  $AB$  for  $A \cup B$ . Let  $\mathcal{L}$  be a first-order language and  $T$  be a complete  $\mathcal{L}$ -theory. Given models  $M, N \models T$ , we say a map  $f: M \rightarrow N$

is **elementary** if, for all  $\mathcal{L}$ -formulas  $\varphi(x)$  and tuples  $a$  in  $M$ , if  $M \models \varphi(a)$ , then  $N \models \varphi(f(a))$ . A bijection  $f: M \rightarrow M$  is an **automorphism** if it is elementary.

Model theory is often described as the study of definable sets of the models of a complete theory (e.g., see [73]). Given an  $\mathcal{L}$ -structure  $M$ , we say a set  $X \subseteq M^n$  is **definable** if there is an  $\mathcal{L}$ -formula  $\varphi(x)$  such that  $X = \{a \in M^n : M \models \varphi(a)\}$ . We extend this terminology to elements of  $M$ . We also say  $b \in M^n$  is **algebraic** if there is an  $\mathcal{L}$ -formula  $\varphi(x)$  with finitely many realisations in  $M$  such that  $M \models \varphi(b)$ . We can allow parameters in the formula as well: For a set  $A$ , we denote by  $\mathcal{L}(A)$  the set of all  $\mathcal{L}$ -formulas containing parameters from  $A$ . Thus, for  $A \subseteq M$  and  $b \in M^n$ , we say  $b$  is **definable** (resp., **algebraic**) **over**  $A$  exactly as before, but taking  $\varphi(x) \in \mathcal{L}(A)$ . We can also extend the notion of definability to sets of formulas and declare that a set  $X$  is **type-definable over**  $A$  if it is the set of realisations in  $M$  of a consistent set of  $\mathcal{L}(A)$ -formulas  $\Sigma(x)$ .

Much of contemporary model theory translates the problem of studying the definable sets of models of a theory, via Stone's duality, into an analysis of its *types*. A (**complete**)  $n$ -**type** over a set  $A$  is a maximal consistent set of  $\mathcal{L}(A)$ -formulas with  $n$  free variables. We denote by  $S_n(A)$  the space of complete  $n$ -types over  $A$ , and write  $S(A) := \bigcup_{n < \omega} S_n(A)$ . Recall that  $S(A)$  is a compact Hausdorff space. If  $p \in S(B)$  and  $A \subseteq B$ , we write  $p|_A$  for the type  $q \in S(A)$  consisting of those formulas in  $p$  with parameters in  $A$ .

We are oftentimes interested in choosing and working with realisations of certain partial or complete types. Rigorously, this requires moving from the current model of the theory we are studying to an elementary extension, and this procedure might need to be repeated several times in the course of a single proof. To avoid these complications (which can often prevent us from seeing the forest for the trees), we usually work in a **monster model**  $\mathbb{M} \models T$ , i.e., a model  $\mathbb{M}$  which, for a very large<sup>1</sup> cardinal  $\bar{\kappa}$ , is:

- **$\bar{\kappa}$ -saturated**, i.e., for all  $A \subset \mathbb{M}$  of cardinality  $< \bar{\kappa}$  and every type  $p \in S(A)$ , there is a realisation of  $p$  in  $\mathbb{M}$ ; and
- **strongly  $\bar{\kappa}$ -homogeneous**, i.e., for all subsets  $A, B \subset \mathbb{M}$  both of cardinality  $< \bar{\kappa}$  and elementary bijections  $f: A \rightarrow B$ , there is an automorphism  $\sigma$  of  $\mathbb{M}$  that extends  $f$ .

As it is standard in the literature, we assume that all elements, tuples, sets, and

<sup>1</sup>By this, we do not necessarily mean “large” in the set-theoretic sense of “not provable in ZFC”. For the subtleties of finding monster models of arbitrary first-order theories, see [49].

models we work with are small (i.e., of cardinality  $< \bar{\kappa}$ ) and contained in  $\mathbb{M}$ . For any formula  $\varphi(x)$  and tuple  $c$ , we may write  $\models \varphi(c)$  to mean  $\mathbb{M} \models \varphi(c)$ . We write  $\text{Aut}(\mathbb{M})$  for the group of automorphisms of  $\mathbb{M}$ , and for a set  $A$ , we write  $\text{Aut}(\mathbb{M}/A)$  for the subgroup of  $\text{Aut}(\mathbb{M})$  of those  $\sigma \in \text{Aut}(\mathbb{M})$  that fix  $A$  pointwise (that is, the pointwise stabiliser of  $A$  in  $\text{Aut}(\mathbb{M})$ ).

With a fixed theory  $T$  and monster model  $\mathbb{M}$ , we now say that a set of formulas is **(in)consistent (modulo  $T$ )** if it does (not) have a realisation in  $\mathbb{M}$ . Types defined over  $\mathbb{M}$  are called **global**. For a tuple  $a$  and a set  $B$ , we define  $\text{tp}(a/B) := \{\varphi(x) \in \mathcal{L}(B) : \mathbb{M} \models \varphi(a)\}$ , and we write  $a \equiv_C b$  if  $\text{tp}(a/C) = \text{tp}(b/C)$ , or equivalently, if there is some  $\sigma \in \text{Aut}(\mathbb{M}/C)$  such that  $\sigma(a) = b$ . Similarly, for sets  $A, B$ , we write  $A \equiv_C B$  if there is some  $\sigma \in \text{Aut}(\mathbb{M}/C)$  such that  $\sigma(A) = B$  (as sets). It is useful in several contexts to distinguish certain special kinds of global types. For a small set  $A$ , we say a global type  $p$  is:

- **$A$ -invariant** if, for all  $\mathcal{L}$ -formulas  $\varphi(x, y)$  and  $b \equiv_A c$ , we have  $\varphi(x, b) \in p$  iff  $\varphi(x, c) \in p$ ;
- an **heir over  $A$**  if, for all  $b$  and each  $\mathcal{L}(A)$ -formula  $\varphi(x, y)$  such that  $\varphi(x, b) \in p$ , there is some  $a \in A$  such that  $\varphi(x, a) \in p$ ;
- a **coheir over  $A$** , or **finitely satisfiable in  $A$** , if for all  $b$  and every  $\mathcal{L}(A)$ -formula  $\varphi(x, y)$  such that  $\varphi(x, b) \in p$ , there is  $a \in A$  such that  $\models \varphi(a, b)$ .

An **imaginary** is an equivalence class of a definable equivalence relation  $E$  over  $\emptyset$ . We can extend  $\text{Aut}(\mathbb{M}/A)$  to  $A$  a set of imaginaries by declaring that  $\sigma \in \text{Aut}(\mathbb{M})$  fixes  $a_E$  (for  $E$  a definable equivalence relation over  $\emptyset$  and  $a_E$  the imaginary coding the  $E$ -class of  $a \in \mathbb{M}$ ) if  $\models E(a, \sigma(a))$ .

Let us recall the standard way of adding imaginaries to an  $\mathcal{L}$ -theory  $T$ : let  $(E_i)_{i \in I}$  enumerate all definable equivalence relations over  $\emptyset$ , where  $E_i$  is a relation on  $\mathbb{M}^{n_i}$ , and define the language  $\mathcal{L}^{\text{eq}}$  by letting  $S_=_$  be the sort generated by  $=$  and, for each  $i \in I$ , adding a new sort  $S_i$ . The symbols of  $\mathcal{L}$  are now of sort  $S_=_$ , and for each  $i \in I$ , we add a function symbol  $\pi_i$  with domain sort  $S_=_^{n_i}$  and codomain sort  $S_i$ . We then define the  $\mathcal{L}^{\text{eq}}$ -theory  $T^{\text{eq}}$  by (i) the axioms of  $T$  with quantifiers now ranging over  $S_=_$ ; (ii) for all  $i \in I$ , the sentence  $\forall y \exists x_0, \dots, x_{n_i-1} \pi_i(x_0, \dots, x_{n_i-1}) = y$ , where  $y$  is of sort  $S_i$  and  $x_0, \dots, x_{n_i-1}$  of sort  $S_=_$ ; and (iii) for all  $i$ , the sentence  $\forall x x' (\pi_i(x) = \pi_i(x') \leftrightarrow E_i(x, x'))$ , where  $x, x'$  are  $n_i$ -tuples of sort  $S_=_$ . We use the notation  $\text{dcl}^{\text{eq}}$  and  $\text{acl}^{\text{eq}}$  to refer to  $\text{dcl}$  and  $\text{acl}$  in  $T^{\text{eq}}$ .

This allows us to strengthen the notion of two tuples having the same type. For tuples  $a, b$  and a set  $A$ , we say  $a$  and  $b$  have the same **strong type** over  $A$ , and



To prove that a class has AP, it is enough to show it for  $|B_i \setminus f_i(A)| = 1$ ,  $i = 1, 2$ .

Let us note in passing that many model-theorists call a structure as in (i) above **ultrahomogeneous** instead. However, in what follows, we adopt the convention from [25] and use “homogeneous” as above.

**Definition 2.2.2.** Let  $\mathcal{L}$  be a relational language. We say a class  $\mathcal{C}$  of finite  $\mathcal{L}$ -structures is a (resp., **strong**) **amalgamation class** if it is closed under substructures and isomorphisms, has countably many isomorphism classes, and has AP (resp., SAP).

**Fact 2.2.3** (Fraïssé’s Theorem). *Let  $\mathcal{L}$  be a relational language and  $\mathcal{C}$  be a class of finite  $\mathcal{L}$ -structures. Then  $\mathcal{C}$  is an amalgamation class iff there is a countable homogeneous  $\mathcal{L}$ -structure  $M$  such that  $\mathcal{C}$  is the age of  $M$ . This  $M$  is unique up to isomorphism. (We call  $M$  the **Fraïssé limit** of  $\mathcal{C}$ .)*

**Fact 2.2.4** (cf. [18, p. 2.22]). *Let  $M$  be a countable homogeneous structure and let  $T = \text{Th}(M)$ . Then  $T$  is  $\omega$ -categorical and has quantifier elimination.*

**Fact 2.2.5** (cf. [18, p. 2.15]). *Let  $M$  be a countable homogeneous structure in a relational language. Then the age of  $M$  is a strong amalgamation class iff  $\text{acl}(A) = A$  for all finite  $A \subset M$ .*

## 2.3 Classification theory

Recall that we say a formula  $\varphi(x, y)$  is **stable** if there are no sequences  $(a_i)_{i < \omega}$  and  $(b_i)_{i < \omega}$  such that, for all  $i, j < \omega$ , we have  $\models \varphi(a_i, b_j)$  iff  $i < j$ ; and a theory is **stable** if every formula is. After the crowning achievement and completion of the stability programme during the 1970s, much of contemporary model theory has been characterised by the search for new dividing lines that exhibit some of the good behaviour that stable theories do. In this section, we will introduce the definitions of the main dividing lines studied in the literature, to which we will refer repeatedly in the upcoming chapters.

Given an ordinal  $\alpha$ , we denote by  $\alpha^{<\omega}$  the tree of sequences of elements of  $\alpha$  of finite length, and we denote its partial order by  $\leq$ , which is defined by  $\eta \leq \nu$  iff  $\nu|_{\text{dom}(\eta)} = \eta$  (thinking about these sequences as functions). Given  $\eta \in \alpha^\omega$  and  $i \in \omega$ , we write  $\eta|_i$  for the restriction of  $\eta$  to the first  $i$  entries. Given  $\eta, \nu \in \alpha^{<\omega}$ , we write

$\eta \smallfrown \nu$  to denote the concatenation of  $\eta$  and  $\nu$  as sequences, and we write  $\eta \perp \nu$  if they are incomparable with respect to  $\trianglelefteq$ .

We now define some special subsets of trees:

**Definition 2.3.1.** Let  $X \subseteq 2^{<\omega}$ .

- (i) We say  $X$  is an **antichain** if  $\eta \perp \nu$  for all  $\eta, \nu \in X$ .
- (ii) We say  $X$  is a **right-comb** if it is an antichain and, for any  $\eta \in 2^{<\omega}$ , if there is some  $\nu \in X$  such that  $\nu \geq \eta \smallfrown \langle 1 \rangle$ , then there is at most one  $\nu' \in X$  such that  $\nu' \geq \eta \smallfrown \langle 0 \rangle$ . (We illustrate this later.)

Let us denote by  $\alpha_*^{<\omega}$  the forest obtained from  $\alpha^{<\omega}$  by removing the root. Let  $(\eta_n)_{n < \omega}$  be a sequence in  $\alpha_*^{<\omega}$ . The following definitions come from [76]:

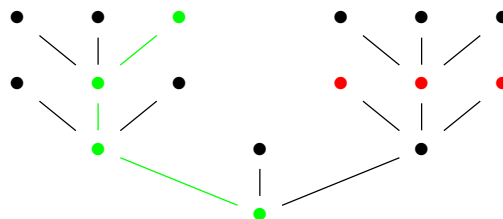
- (iii) We say  $(\eta_n)_{n \in \omega}$  is a **left-leaning path** if, whenever  $\eta_n = \nu \smallfrown \langle i \rangle$ , there is some  $j \leq i$  such that  $\nu \smallfrown \langle j \rangle \triangleleft \eta_{n+1}$ .
- (iv) We say  $(\eta_n)_{n \in \omega}$  is a **right-veering path** if, whenever  $\eta_n = \nu \smallfrown \langle i \rangle$ , there is some  $j > i$  such that  $\nu \smallfrown \langle j \rangle \trianglelefteq \eta_{n+1}$ .

Fix a complete theory  $T$  and a monster model  $\mathbb{M} \models T$ . We now recall the relevant definitions of the classification properties we use. The properties in (i)-(iii) are due to Shelah and all appear in [105] under different names; (iv) and (v) are due to Shelah and Džamonja and appear in [40]; (vi) was introduced by Ahn and Kim in [5]; (vii) was introduced by Mutchnik in [95]; and finally (viii) is due to Kruckman and Ramsey and appears in [76].

**Definition 2.3.2.** Let  $\varphi(x, y)$  be a formula, and  $k \in \omega$ .

- (i) We say  $\varphi(x, y)$  has the  **$k$ -tree property** (or is  **$k$ -TP**) if there is a tree  $(a_\eta)_{\eta \in \omega^{<\omega}}$  such that:
  - for all  $\sigma \in \omega^\omega$ , the set  $\{\varphi(x, a_{\sigma \smallfrown i}) : i \in \omega\}$  is consistent;
  - for all  $\eta \in \omega^{<\omega}$ , any  $k$ -subset of  $\{\varphi(x, a_{\eta \smallfrown i}) : i \in \omega\}$  is inconsistent.

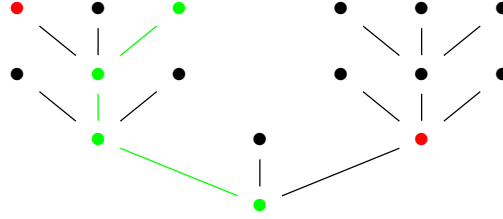
As an example, in the picture, instances of  $\varphi$  along the tuples on the green path give a consistent set of formulas, whereas those along the tuples on red give a  $k$ -inconsistent set.



(ii) We say  $\varphi(x, y)$  has the **tree property of the first kind** (or is **TP<sub>1</sub>**) if there is a tree  $(a_\eta)_{\eta \in \omega^{<\omega}}$  such that:

- for all  $\sigma \in \omega^\omega$ , the set  $\{\varphi(x, a_{\sigma|_i}) : i \in \omega\}$  is consistent;
- for all  $\eta, \nu \in \omega^{<\omega}$  such that  $\eta \perp \nu$ , the set  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$  is inconsistent.

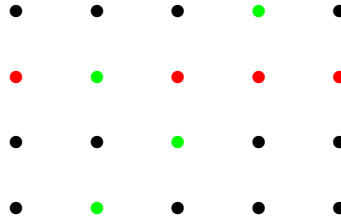
As an example, in the picture, the green path gives, again, a consistent set of formulas, whereas the red tuples give an inconsistent set.



(iii) We say  $\varphi(x, y)$  has the  **$k$ -tree property of the second kind** (or is  **$k$ -TP<sub>2</sub>**) if there is an array  $(a_{i,j})_{i,j \in \omega}$  such that:

- for all  $f: \omega \rightarrow \omega$ , the set  $\{\varphi(x, a_{i,f(i)}) : i \in \omega\}$  is consistent;
- for all  $i \in \omega$ , any  $k$ -subset of  $\{\varphi(x, a_{i,j}) : j \in \omega\}$  is inconsistent.

As an example, in the picture, think of each row as  $b_i := (a_{i,j})_{j < \omega}$ . Then the instances of  $\varphi$  along the tuples on green give a consistent set, whereas the tuples in a fixed row, given by the red points, form a  $k$ -inconsistent set.

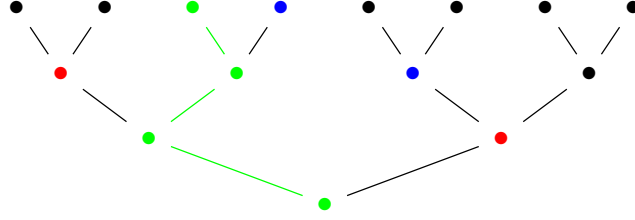


(iv) We say  $\varphi(x, y)$  has the **1-strong order property** (or is **SOP<sub>1</sub>**) if there is a tree  $(a_\eta)_{\eta \in 2^{<\omega}}$  such that:

- for all  $\sigma \in 2^\omega$ , the set  $\{\varphi(x, a_{\sigma|_i}) : i \in \omega\}$  is consistent;
- for all  $\eta, \nu \in 2^{<\omega}$  with  $\nu \frown \langle 0 \rangle \sqsubseteq \eta$ , the set  $\{\varphi(x, a_\eta), \varphi(x, a_{\nu \frown \langle 1 \rangle})\}$  is inconsistent.

As an example, in the picture, the green path is a consistent set of formulas, the red tuples give an inconsistent set, and the blue tuples could either form

a consistent or an inconsistent set—this remains undecided by  $\text{SOP}_1$ .



(v) We say  $\varphi(x, y)$  has the **2-strong order property** (or is **SOP<sub>2</sub>**) if there is a tree  $(a_\eta)_{\eta \in 2^{<\omega}}$  such that:

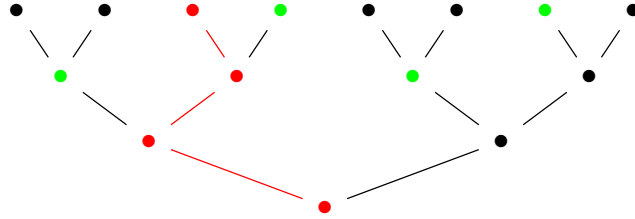
- for all  $\sigma \in 2^\omega$ , the set  $\{\varphi(x, a_{\sigma|_i}) : i \in \omega\}$  is consistent;
- for all  $\eta, \nu \in 2^{<\omega}$  with  $\nu \perp \eta$ , the set  $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$  is inconsistent.

This has the same picture as for  $\text{SOP}_1$ , except that this time we also know that the blue tuples form an inconsistent set.

(vi) We say  $\varphi(x, y)$  has the  **$k$ -antichain tree property** (or is  **$k$ -ATP**) if there is a tree  $(a_\eta)_{\eta \in 2^{<\omega}}$  such that:

- for all antichains  $X \subseteq 2^{<\omega}$ , the set  $\{\varphi(x, a_\eta) : \eta \in X\}$  is consistent;
- for all  $\sigma \in 2^\omega$ , any  $k$ -subset of  $\{\varphi(x, a_{\sigma|_i}) : i < \omega\}$  is inconsistent.

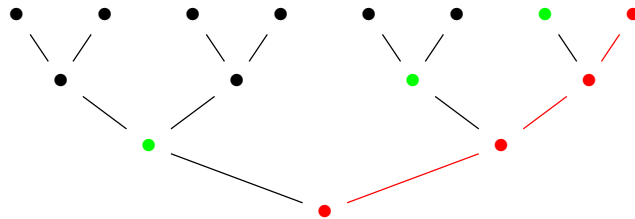
As an example, in the picture, the green tuples give a consistent set of formulas, whereas the red path is a  $k$ -inconsistent set.



(vii) We say  $\varphi(x, y)$  has the  **$k$ -comb tree property** (or is  **$k$ -CTP**) if there is a tree  $(a_\eta)_{\eta \in 2^{<\omega}}$  such that:

- for all right-combs  $X \subseteq 2^{<\omega}$ , the set  $\{\varphi(x, a_\eta) : \eta \in X\}$  is consistent;
- for all  $\sigma \in 2^\omega$ , any  $k$ -subset of  $\{\varphi(x, a_{\sigma|_i}) : i < \omega\}$  is inconsistent.

As an example, in the picture, the green tuples give a consistent set of formulas, whereas the red path is a  $k$ -inconsistent set.



(viii) We say  $\varphi(x, y)$  has the  **$k$ -bizarre tree property** (or is  **$k$ -BTP**) if there is a forest  $(a_\eta)_{\eta \in \omega_*^{<\omega}}$  such that:

- for all left-leaning paths  $(\eta_n)_{n \in \omega} \subseteq \omega_*^{<\omega}$ , the set  $\{\varphi(x, a_{\eta_n}) : n < \omega\}$  is consistent;
- for all right-veering paths  $(\eta_n)_{n \in \omega} \subseteq \omega_*^{<\omega}$ , any  $k$ -subset of  $\{\varphi(x, a_{\eta_n}) : n < \omega\}$  is inconsistent.

For  $P \in \{\text{TP}, \text{TP}_2, \text{ATP}, \text{CTP}, \text{BTP}\}$ , we say  $\varphi(x, y)$  is **P** if it is  $k$ -P for some  $k \in \omega$ .

For  $P \in \{\text{TP}, \text{TP}_1, \text{TP}_2, \text{SOP}_1, \text{SOP}_2, \text{ATP}, \text{CTP}, \text{BTP}\}$ , we say  $T$  is **NP** if no formula has  $P$ . As usual, we call an NTP theory **simple**.

Other classification properties are not defined in terms of trees. An early example, due to Shelah, is NSOP [104]: a formula  $\varphi(x, y)$  is **SOP** if there is an infinite sequence  $(a_i)_{i < \omega}$  such that  $\models \forall x(\varphi(x, a_i) \rightarrow \varphi(x, a_j))$  iff  $i \leq j$ , and a theory  $T$  is **NSOP** if no formula is SOP. In [107], Shelah introduced a hierarchy of properties conceived as “approximations” to NSOP:

**Definition 2.3.3.** Let  $3 \leq n < \omega$ . We say  $T$  has the  **$n$ -strong order property** (or is **SOP $_n$** ) if there are a formula  $\varphi(x, y)$  and a sequence  $(a_i)_{i \in \omega}$  such that:

- $\models \varphi(a_i, a_j)$  for all  $i < j$ ;
- the set  $\{\varphi(x_0, x_1), \varphi(x_1, x_2), \dots, \varphi(x_{n-2}, x_{n-1}), \varphi(x_{n-1}, x_0)\}$  is inconsistent.

Otherwise, we say  $T$  is **NSOP $_n$** .

Let us note here the following useful characterisation of NSOP $_3$ :

**Fact 2.3.4** ([107, Claim 2.19]). *A theory  $T$  is SOP $_3$  iff there exist formulas  $\psi_0(x, y)$  and  $\psi_1(x, y)$  and a sequence  $(a_i)_{i < \omega}$  such that:*

- for all  $k < \omega$ , the set  $\{\psi_0(x, a_i) : i \leq k\} \cup \{\psi_1(x, a_i) : i > k\}$  is consistent;
- for every  $i < j < \omega$ , the set  $\{\psi_1(x, a_i), \psi_0(x, a_j)\}$  is inconsistent.

Another recent programme in model theory consists in extending the traditional binary classification-theoretic properties to higher arities. Two of the most fruitful notions in this direction are given in the following definition. The first was introduced by Shelah in [109] and developed further by Chernikov, Hempel, Palacín and Takeuchi in several papers [27], [28], [31], [52]. The second was proposed by Terry and Wolf in [115] and extended by Abd-Aldaim, Conant, and Terry in [1].

**Definition 2.3.5.** Let  $T$  be a complete theory.

- (i) Let  $0 < n < \omega$ . We say  $T$  has the  $n$ -**independence property** (or is **IP<sub>n</sub>**) if there are a formula  $\varphi(x_0, \dots, x_{n-1}, y)$  and sequences  $(a_{0,i})_{i \in \omega}, \dots, (a_{n-1,i})_{i \in \omega}$  and  $(b_I)_{I \subseteq \omega^n}$  such that

$$\models \varphi(a_{0,i_0}, \dots, a_{n-1,i_{n-1}}, b_I) \iff (i_0, \dots, i_{n-1}) \in I$$

for all  $i_0, \dots, i_{n-1} \in \omega$  and  $I \subseteq \omega^n$ . Otherwise, we say  $T$  is **NIP<sub>n</sub>**. When  $n = 1$ , we just say  $T$  is **NIP**.

- (ii) Let  $2 \leq n < \omega$ . We say that  $T$  has the  $n$ -**functional order property** (or is **FOP<sub>n</sub>**) if there are a formula  $\varphi(x_0, \dots, x_n)$  and sequences  $(a_f)_{f: \omega^{n-1} \rightarrow \omega}$ ,  $(b_{0,i})_{i \in \omega}, \dots, (b_{n-1,i})_{i \in \omega}$  such that

$$\models \varphi(a_f, b_{1,i_1}, \dots, b_{n,i_n}) \iff i_n \leq f(i_1, \dots, i_{n-1})$$

for all  $i_1, \dots, i_n \in \omega$  and  $f: \omega^{n-1} \rightarrow \omega$ . Otherwise, we say  $T$  is **NFOP<sub>n</sub>**.

The following diagram represents the (known) implications between the above notions at the level of theories. It combines many results from the literature; a recopilation of most of them containing all the relevant references is [33] (the remaining ones can be found in [1], [50], [95]).

$$\begin{array}{ccccccc} \text{NFOP}_2 & \implies & \text{NIP}_2 & \implies & \text{NFOP}_3 & \implies & \text{NIP}_3 \implies \dots \\ \uparrow & & & & & & \\ \text{NIP} & \implies & \text{NTP}_2 & \implies & \text{NBTP} & \implies & \text{NCTP} \implies \text{NATP} \\ \uparrow & & \uparrow & & \uparrow & & \\ \text{stable} & \implies & \text{simple} & \implies & \text{NSOP}_1/\text{NSOP}_2 & \implies & \text{NSOP}_3 \implies \text{NSOP}_4 \implies \dots \end{array}$$

It is open whether there exists an  $\text{NSOP}_n$  theory for any  $n \geq 3$  which is  $\text{NTP}_2$  but not simple, and also whether (any of) the implications  $\text{NBTP} \implies \text{NCTP} \implies \text{NATP}$  and  $\text{NSOP}_1 \implies \text{NSOP}_3$  are strict.

## 2.4 Abstract independence relations

In the previous section, we introduced dividing lines as combinatorial configurations that a formula and a sequence of tuples might exhibit. However, as explained in the introduction, dividing lines capture a close interaction between syntactic, or combinatorial, and semantic properties. The development on the “semantic” side has been characterised, since Kim and Pillay’s work on simple theories (see [70]), by

the study of *independence relations*. To this notion we turn now.

Let  $T$  be a fixed complete theory with a monster model  $\mathbb{M} \models T$ .

**Definition 2.4.1.** An **independence relation**  $\perp$  is an  $\text{Aut}(\mathbb{M})$ -invariant ternary relation on small subsets of  $\mathbb{M}$ . We write  $A \perp_C B$  if the triple  $(A, B, C)$  lies in this relation, and say that  $A$  is independent from  $B$  over  $C$ . We say  $\perp$  is an independence relation **over models** if  $C$  is only allowed to be models of  $T$ .

**Remark 2.4.2.** Our definition of the informal notion of an independence relation is extremely general, and does not correspond to most formal uses of the notion previously appearing in the literature (e.g., [3]). Its immediate precedent is found in what Chernikov and Kaplan call a “preindependence relation” in [29]. We favour this lax use of the term in this thesis, since our purpose is to be explicit about the properties used in each result.

**Convention.** Let  $a, b$  be tuples enumerating the sets  $A, B$ , respectively. We often-times write  $a \perp_C b$  to mean  $A \perp_C B$ .

**Definition 2.4.3.** The following is a list of common properties that independence relations may exhibit (where all sets and models are small):

- (i) *Left monotonicity*: if  $A \perp_C B$  and  $A' \subseteq A$ , then  $A' \perp_C B$ .
- (ii) *Right monotonicity*: if  $A \perp_C B$  and  $B' \subseteq B$ , then  $A \perp_C B'$ .
- (iii) *Existence*: for any  $A$  and  $C$ ,  $A \perp_C C$ .
- (iv) *Left extension*: if  $A \perp_C B$  and  $A \subseteq A'$ , then there is some  $B' \equiv_{CA} B$  such that  $A' \perp_C B'$ .
- (v) *Right extension*: if  $A \perp_C B$  and  $B \subseteq B'$ , then there is some  $A' \equiv_{CB} A$  such that  $A' \perp_C B'$ .
- (vi) *Left normality*: if  $A \perp_C B$ , then  $AC \perp_C B$ .
- (vii) *Right normality*: if  $A \perp_C B$ , then  $A \perp_C BC$ .
- (viii) *Left base monotonicity*: if  $C \subseteq D \subseteq A$  and  $A \perp_C B$ , then  $A \perp_D B$ .
- (ix) *Right base monotonicity*: if  $C \subseteq D \subseteq B$  and  $A \perp_C B$ , then  $A \perp_D B$ .
- (x) *Left transitivity*: Given  $C \subseteq D$ , if  $D \perp_C A$  and  $B \perp_D A$ , then  $BD \perp_C A$ .
- (xi) *Right transitivity*: Given  $C \subseteq D$ , if  $A \perp_C D$  and  $A \perp_D B$ , then  $A \perp_C BD$ .
- (xii) *Strong finite character*: if  $A \not\perp_C B$ , then there are finite tuples  $a \in A, b \in B$  and some  $\varphi(x, b) \in \text{tp}(a/BC)$  such that, for all  $a' \models \varphi(x, b)$ , we have  $a' \not\perp_C b$ .
- (xiii) *Quasi-strong finite character*: for all types  $p(x)$  and  $q(y)$  over  $C$ , there is a set of  $\mathcal{L}(C)$ -formulas  $\Sigma_{p,q}(x, y)$  such that

$$(a, b) \models \Sigma_{p,q}(x, y) \iff a \perp_C b \quad \& \quad \models p(a) \wedge q(b).$$

- (xiv) *Local character*: For all  $A$ , there is a cardinal  $\kappa = \kappa(A)$  such that, for all  $B$ , there is a set  $C \subseteq B$  with  $|C| < \kappa$  such that  $A \downarrow_C B$ .
- (xv) *Symmetry*: if  $A \downarrow_C B$ , then  $B \downarrow_C A$ .
- (xvi) *Stationarity over models*: if  $M \models T$ ,  $A \downarrow_M B$ ,  $A' \downarrow_M B$ , and  $A \equiv_M A'$ , then  $A \equiv_{MB} A'$ .
- (xvii) *Independence Theorem over Models*: if  $M \models T$ ,  $A \equiv_M A'$ ,  $A \downarrow_M B$ ,  $A' \downarrow_M C$ , and  $B \downarrow_M C$ , then there exists some  $A''$  such that  $A'' \equiv_{MB} A$ ,  $A'' \equiv_{MC} A'$ , and  $A'' \downarrow_M BC$ .

**Remark 2.4.4.** (i) We say  $\downarrow$  has **monotonicity** if it has both left and right monotonicity. Similarly with all other “left” and “right” properties.

(ii) We say  $\downarrow$  satisfies existence **over models** if, for any  $A$  and  $M \models T$ , we have  $A \downarrow_M M$ .

(iii) Note that existence, right extension, and right monotonicity jointly imply the following property:

(\*) *Full existence*: For all  $A, B, C$ , there is  $A' \equiv_C A$  such that  $A' \downarrow_C B$ .

Conversely, full existence implies existence.

**Remark 2.4.5.** Several of the above properties already appear, in some form, in [105]. Their explicit versions in this abstract setting can be traced back to [10]. The modern notation is due to [3]. Some more recent properties appear in [95], [114].

**Example 2.4.6** (cf. [19]). Some common independence relations in the literature are the following:

- $A \downarrow_C^a B$  iff  $\text{acl}(AC) \cap \text{acl}(BC) \subseteq \text{acl}(C)$ . In any theory,  $\downarrow^a$  satisfies monotonicity, existence, left and right extension, normality, transitivity, strong finite character, local character, and symmetry.
- $A \downarrow_C^i B$  iff, for some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/BC)$  extends to a global type which is Lascar-invariant over  $C$ . In any theory,  $\downarrow^i$  satisfies monotonicity, existence over models, right extension, normality, right base monotonicity, left transitivity, and strong finite character.
- $A \downarrow_C^u B$  iff, for some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/BC)$  extends to a global type finitely satisfiable over  $C$  (also known as a *coheir extension*). In any theory,  $\downarrow^u$  satisfies monotonicity, existence over models, left and right extension, normality, right base monotonicity, and strong finite character.

The contemporary focus on independence relations came to prominence with Kim and Pillay’s study of simple theories [64], [70]. A fundamental tool in their analysis was the following notion, first introduced by Shelah (see, e.g., [105]):

**Definition 2.4.7.** Let  $\varphi(x, y)$  be a formula,  $b$  a tuple, and  $C$  a set.

- (i) We say that  $\varphi(x, b)$  **divides over**  $C$  if there is a  $C$ -indiscernible sequence  $(b_i)_{i < \omega}$  with  $b_0 = b$  such that the set  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent.
- (ii) We say that  $\varphi(x, b)$  **forks over**  $C$  if there exist formulas  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$  and each  $\psi_i(x, d_i)$  divides over  $C$ .
- (iii) We say  $A$  is **nonforking independent** from  $B$  over  $C$  if, for some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/BC)$  does not contain any formula that forks over  $C$ . We denote this by  $A \downarrow_C^f B$ .

**Fact 2.4.8** (cf., [19]). *In any theory,  $\downarrow^f$  satisfies monotonicity, existence over models, right extension, normality, right base monotonicity, left transitivity, and strong finite character.*

**Definition 2.4.9.** Given two independence relations  $\downarrow^1$  and  $\downarrow^2$  on small subsets of  $\mathbb{M} \models T$ , we write  $\downarrow^1 \implies \downarrow^2$  if, whenever  $A \downarrow_C^1 B$ , we have  $A \downarrow_C^2 B$ .

**Remark 2.4.10.** The following are all the implications between the independence relations from Example 2.4.6 and Definition 2.4.7 in a general theory:

$$\downarrow^u \implies \downarrow^i \implies \downarrow^f \implies \downarrow^a.$$

Adler observed in [3] that several constructions in model theory can be obtained as the result of operations on abstract independence relations. We can find in the literature some common examples of unary and binary operations that allow us to build new independence relations out of old ones.

**Definition 2.4.11.** Given an independence relation  $\downarrow$ , we define  $\downarrow^*$  by

$$A \downarrow_C^* B \iff \text{for all } B' \supseteq B, \text{ there exists } A' \equiv_{BC} A \text{ such that } A' \downarrow_C B'.$$

**Fact 2.4.12** ([3, Lemma 3.1]). *Let  $\downarrow$  be an independence relation.*

- (i)  $\downarrow^*$  is an independence relation and  $\downarrow^* \implies \downarrow$ .
- (ii) If  $\downarrow^1 \implies \downarrow^2$  and  $\downarrow^1$  satisfies right extension, then  $\downarrow^1 \implies (\downarrow^2)^*$ . In particular, if  $\downarrow^1 \implies \downarrow^2$ , then  $(\downarrow^1)^* \implies (\downarrow^2)^*$ .
- (iii) If  $\downarrow$  satisfies monotonicity/right base monotonicity/left transitivity/left normality, then so does  $\downarrow^*$ .
- (iv) If  $\downarrow$  satisfies monotonicity, then  $\downarrow = \downarrow^*$  iff  $\downarrow$  satisfies right extension.
- (v) If  $\downarrow$  satisfies monotonicity and strong finite character, then  $\downarrow^*$  satisfies strong finite character.

**Example 2.4.13.** As it is standard in the literature, let us write  $A \downarrow_C^d B$  if, for some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/BC)$  does not contain any formula that divides over  $C$ . Then  $\downarrow^f = (\downarrow^d)^*$  (see [3, §5]).

The next operation has been explicitly defined by d’Elbée ([38, Definition 3.2.9]):

**Definition 2.4.14.** Given an independence relation  $\downarrow$ , we define  $\downarrow^{\text{opp}}$  by

$$A \downarrow_C^{\text{opp}} B \iff B \downarrow_C A.$$

**Fact 2.4.15.** Let  $\downarrow$  be an independence relation.

- (i)  $\downarrow^{\text{opp}}$  is an independence relation.
- (ii) If  $\downarrow^1 \implies \downarrow^2$  and  $\downarrow^1$  satisfies symmetry, then  $\downarrow^1 \implies (\downarrow^2)^{\text{opp}}$ . In particular, if  $\downarrow^1 \implies \downarrow^2$ , then  $(\downarrow^1)^{\text{opp}} \implies (\downarrow^2)^{\text{opp}}$ .
- (iii) If  $\downarrow$  satisfies the left version of any axiom, then  $\downarrow^{\text{opp}}$  satisfies the right version.
- (iv)  $\downarrow = \downarrow^{\text{opp}}$  iff  $\downarrow$  satisfies symmetry.
- (v) If  $\downarrow$  satisfies full existence, then so does  $\downarrow^{\text{opp}}$ .

**Example 2.4.16.** Again, using the standard notation and terminology from the literature, we write  $A \downarrow_C^h B$  if, for some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/BC)$  is an heir over  $C$  (see §2.1). Then  $\downarrow^u = (\downarrow^h)^{\text{opp}}$ .

We now introduce the “left-handed” version of Adler’s star operation:

**Definition 2.4.17.** Given an independence relation  $\downarrow$ , we define  $\downarrow^{\text{le}}$  by

$$A \downarrow_C^{\text{le}} B \iff \text{for all } D \supseteq A, \text{ there exists } B' \equiv_{AC} B \text{ with } D \downarrow_C B'.$$

**Lemma 2.4.18.**  $\downarrow^{\text{le}} = ((\downarrow^{\text{opp}})^*)^{\text{opp}}$ .

*Proof.* Simply unfold the definitions:

$$\begin{aligned} (A \downarrow_C^{\text{opp}*} B)^{\text{opp}} &\iff B \downarrow_C^{\text{opp}*} A \\ &\iff \text{for all } D \supseteq A, \text{ there is } B' \equiv_{AC} B \text{ such that } B' \downarrow_C^{\text{opp}} D \\ &\iff \text{for all } D \supseteq A, \text{ there is } B' \equiv_{AC} B \text{ such that } D \downarrow_C B' \\ &\iff A \downarrow_C^{\text{le}} B. \end{aligned}$$

□

**Corollary 2.4.19.** *Let  $\perp$  be an independence relation.*

- (i)  $\perp^{\text{le}}$  is an independence relation and  $\perp^{\text{le}} \implies \perp$ .
- (ii) If  $\perp^1 \implies \perp^2$  and  $\perp^1$  satisfies left extension, then  $\perp^1 \implies (\perp^2)^{\text{le}}$ . In particular, if  $\perp^1 \implies \perp^2$ , then  $(\perp^1)^{\text{le}} \implies (\perp^2)^{\text{le}}$ .
- (iii) If  $\perp$  satisfies monotonicity/left base monotonicity/right transitivity/right normality, then so does  $\perp^{\text{le}}$ .
- (iv) If  $\perp$  satisfies monotonicity, then  $\perp = \perp^{\text{le}}$  iff  $\perp$  satisfies left extension.

We can also define higher-arity operations on several independence relations, such as binary operations that, out of two given independence relations, define a new one. Here, we develop some basic operations that capture most of the kinds of relations which have been defined in the literature on neostability theory.

**Definition 2.4.20.** Let  $\perp^1$  and  $\perp^2$  be two independence relations. We define  $\perp^{1\wedge 2}$  by:

$$A \perp_C^{1\wedge 2} B \iff A \perp_C^1 B \text{ and } A \perp_C^2 B.$$

For notational convenience, we may also write  $\perp^1 \wedge \perp^2$  instead of  $\perp^{1\wedge 2}$ .

**Lemma 2.4.21.** *Let  $\perp^1$  and  $\perp^2$  be independence relations.*

- (i)  $\perp^{1\wedge 2}$  is an independence relation and  $\perp^{1\wedge 2} \implies \perp^1$  (as well as  $\perp^{1\wedge 2} \implies \perp^2$ ).
- (ii) If both  $\perp^1$  and  $\perp^2$  satisfy (strong) finite character/symmetry/normality/base monotonicity/monotonicity/transitivity, then so does  $\perp^{1\wedge 2}$ .
- (iii)  $\perp^{1\wedge 2}$  satisfies existence iff both  $\perp^1$  and  $\perp^2$  do.
- (iv) If at least one of  $\perp^1$  and  $\perp^2$  satisfies stationarity over  $C$ , then so does  $\perp^{1\wedge 2}$ .
- (v) If  $\perp^{1\wedge 2}$  satisfies local character, then so do  $\perp^1$  and  $\perp^2$ . Conversely, if  $\perp^1$  and  $\perp^2$  satisfy local character and right base monotonicity, then  $\perp^{1\wedge 2}$  satisfies local character.

*Proof.* It is immediate to check (i)-(iv) separately. We prove (v). The first part of the statement is clear. Suppose now that  $\perp^1$  and  $\perp^2$  both satisfy local character and right base monotonicity. Take  $A$ . By local character, there exist cardinals  $\kappa_1, \kappa_2$  depending on  $A$  such that, for any  $B$ , we can find  $C_1, C_2 \subseteq B$  such that  $|C_1| < \kappa_1$ ,  $|C_2| < \kappa_2$ ,  $A \perp_{C_1}^1 B$  and  $A \perp_{C_2}^2 B$ . Now,  $|C_1 \cup C_2| < \kappa_1 + \kappa_2$  and  $C_1 \cup C_2 \subseteq B$ . Thus, by right base monotonicity, we have  $A \perp_{C_1 \cup C_2}^1 B$  and  $A \perp_{C_1 \cup C_2}^2 B$ , and hence  $A \perp_{C_1 \cup C_2}^{1\wedge 2} B$ . This proves that  $\kappa_1 + \kappa_2$  witnesses local character for  $\perp^{1\wedge 2}$ .  $\square$

**Example 2.4.22.** (i) We will use this operation in chapter 4 to define  $\perp^{\text{hti}}$ .

- (ii) We can extract from the definition of *bi-invariant* types in [50] a new independence relation  $\perp^{\text{bi}}$  defined by  $\perp^{\text{i}} \wedge (\perp^{\text{i}})^{\text{opp}}$ .

We often make use of the following terminology:

**Definition 2.4.23.** We say that independence relations  $\perp^1$  and  $\perp^2$  are **compatible** if  $\perp^{1 \wedge 2}$  satisfies full existence.

For instance, whenever  $\perp$  satisfies full existence and symmetry,  $\perp$  and  $\perp^{\text{opp}}$  are trivially compatible (e.g.,  $\perp^{\text{f}}$  in simple theories, or  $\perp^{\text{i}}$  in NIP theories). We will see a new example of compatible independence relations in chapter 4 (see Lemma 4.4.10).

Inspired by the presentation of strict coheir independence from [72], we introduce the following abstract operation:

**Definition 2.4.24.** Given two independence relations  $\perp^1$  and  $\perp^2$ , we define  $\perp^2$ -**strict**  $\perp^1$  to be  $(\perp^1 \wedge (\perp^2)^{\text{opp}})^*$ . That is,  $a$  is  $\perp^2$ -strict  $\perp^1$  from  $b$  over  $D$  if, for all  $c$ , there is  $c' \equiv_{Db} c$  such that  $a \perp_D^1 bc'$  and  $bc' \perp_D^2 a$ .

**Lemma 2.4.25.** Let  $\perp^1$  and  $\perp^2$  be independence relations.

- (i)  $\perp^2$ -strict  $\perp^1$  is an independence relation stronger than  $\perp^1$  and  $(\perp^2)^{\text{opp}}$ .
- (ii) If  $\perp^1 = \perp^2$  and it satisfies symmetry, then  $\perp^2$ -strict  $\perp^1 = (\perp^1)^*$ .
- (iii) If  $\perp^1$  has right (resp., left) monotonicity and  $\perp^2$  has left (resp., right) monotonicity, then  $\perp^2$ -strict  $\perp^1$  has right (resp., left) monotonicity. If both sides hold, then  $\perp^2$ -strict  $\perp^1$  has right extension and right normality.
- (iv) If  $\perp^1$  has right base monotonicity and  $\perp^2$  has left base monotonicity, then  $\perp^2$ -strict  $\perp^1$  has right base monotonicity.
- (v) If  $\perp^1$  has left transitivity and  $\perp^2$  has right transitivity, then  $\perp^2$ -strict  $\perp^1$  has left transitivity.
- (vi) If  $\perp^1 \implies \perp^3$  or  $\perp^2 \implies \perp^4$ , then  $\perp^2$ -strict  $\perp^1 \implies \perp^4$ -strict  $\perp^3$ .

*Proof.* Combine all the different parts we have proven so far for each of the items in the above statement.  $\square$

**Example 2.4.26.** Several notions from the literature follow under this schema ( $\perp^{\text{K}}$  is defined in chapter 3):

- $\perp^{\text{st}}$  is defined by choosing both  $\perp^1$  and  $\perp^2$  to be  $\perp^{\text{f}}$ .
- $\perp^{\text{ist}}$  is defined by choosing  $\perp^1$  to be  $\perp^{\text{i}}$  and  $\perp^2$  to be  $\perp^{\text{f}}$ .

- $\perp^{\text{Kst}}$  is defined by choosing  $\perp^1$  to be  $\perp^i$  and  $\perp^2$  to be  $\perp^K$ .
- $\perp^{\text{ust}}$  is defined by choosing  $\perp^1$  to be  $\perp^u$  and  $\perp^2$  to be  $\perp^K$ .

All of the above relations satisfy monotonicity, right extension, and right normality.

Following the convention from the literature [76], we often say **Kim-strict**  $\perp^1$  instead of  $\perp^K$ -strict  $\perp^1$ .

Finally, let us introduce the following notion that originates with the work of Kim on simple theories [64] and is generalised by Adler in [3]:

**Definition 2.4.27.** Let  $\perp$  be an independence relation.

- (i) We say  $(a_i)_{i \in \omega}$  is an  $\perp$ -**independent sequence over**  $C$  if  $a_i \perp_C a_{<i}$  for all  $i \in \omega$ .
- (ii) We say  $(a_i)_{i \in \omega}$  is an  $\perp$ -**Morley sequence over**  $C$  if it is  $C$ -indiscernible and  $\perp$ -independent over  $C$ .
- (iii) Let  $p$  be a global type. We say  $(a_i)_{i \in \omega}$  is a **Morley sequence in**  $p$  **over**  $C$  if  $a_i \models p|_{C a_{<i}}$  for all  $i \in \omega$ .

There are many situations in model theory where it is convenient to replace a sequence we start with by an indiscernible one that “looks” like it. In order to do this, we will repeatedly appeal to the following two well-known results coming from infinite combinatorics (see, e.g., [38]). For an infinite sequence  $I = (a_i)_{i < \omega}$  and set of parameters  $A$ , we denote by  $\text{EM}(I/A)$  the set of all  $\mathcal{L}(A)$ -formulas  $\varphi(x_0, \dots, x_{n-1})$  such that  $\models \varphi(a_{i_0}, \dots, a_{i_{n-1}})$  for all  $i_0 < \dots < i_{n-1}$ . The following result tells us that we can replace our initial sequence  $I$  by another indiscernible one while preserving the truth of certain formulas:

**Fact 2.4.28** (Ramsey and compactness, cf. [38, Lemma 3.2.5]). *If  $I = (a_i)_{i < \omega}$  is a sequence of same-length tuples and  $A$  is a set of parameters, then there exists an  $A$ -indiscernible sequence  $J$  such that  $J \models \text{EM}(I/A)$ .*

This is enough for several purposes in model theory. Often, however, we will want to preserve full types, and not simply formulas. This is particularly the case when we start with an  $\perp$ -independent sequence and want to replace it by an  $\perp$ -Morley sequence. The following theorem tells us that, if we can make our sequence long enough, such a replacement is possible:

**Fact 2.4.29** (Erdős-Rado and compactness, cf. [38, Lemma 3.2.6]). *Let  $\kappa > |T| + |C|$  and  $\lambda = \beth_{(2^\kappa)^+}$ . If  $(a_i)_{i < \lambda}$  is a sequence of tuples such that  $|a_i| \leq \kappa$ , then*

---

there is a  $C$ -indiscernible sequence  $(b_i)_{i < \omega}$  such that, for any  $n < \omega$ , there exist  $j_0 < \dots < j_{n-1} < \lambda$  such that  $b_0 \dots b_{n-1} \equiv_C a_{j_0} \dots a_{j_{n-1}}$ .

This technical theorem allows us to prove the following folklore result, which will be used repeatedly in the upcoming chapters without explicit mention:

**Fact 2.4.30.** *Suppose  $\perp$  satisfies full existence and right monotonicity. For any tuple  $a$  and set  $C$ , there exists an  $\perp$ -Morley sequence  $(a_i)_{i < \omega}$  over  $C$  with  $a_0 = a$ .*

# 3

## Rosiness

Die Rose ist ohne Warum.  
Sie blühet, weil sie blühet.  
Sie achtet nicht ihrer selbst,  
fragt nicht, ob man sie siehet.

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*Angelus Silesius*

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Two intensely studied generalisations of the class of simple theories are that of rosy theories (introduced by Onshuus in [97] and developed further by Onshuus and Ealy in [96] and [46]) and that of  $\text{NSOP}_1$  theories (introduced by Džamonja and Shelah in [40]). A common characteristic of both theories is the presence of a “canonical” independence relation whose properties determine the class. The connection of these classes with notions of independence was something present in the study of rosy theories from the start, but appeared in the context of  $\text{NSOP}_1$  theories more recently, first in the work of Chernikov and Ramsey in [32], and then explicitly in terms of the notion of Kim-independence in several papers of Kaplan, Shelah, and Ramsey in [59], [60], and [61].

The following natural question regarding the relationship between these two classes was posed by Christian d’Elbée and Byungham Kim, independently of each other, in 2021:

**Question 3.0.1** ([37], [67]). Is there an  $\text{NSOP}_1$  rosy theory which is not simple?

In this chapter, we show that an answer to this question can be obtained in connection with another conjecture coming out of the study of simple theories.

In analogy to the stable forking conjecture proposed by Hart, Kim, and Pillay (see [65]), and which also relates to the dependent dividing conjecture stated by Chernikov (see [26, Proposition 4.14]), Bossut has recently introduced, in [14], what we call here the *stable Kim-forking conjecture*. The main result of this chapter (Theorem 3.3.7) says that, if the stable Kim-forking conjecture holds, every NSOP<sub>1</sub> rosy theory  $T$  must be simple. In fact, we prove something slightly stronger, since we only require that  $T^{\text{eq}}$  has stable Kim-forking over models.

The contents of this chapter have been published in [88]. We thank Anand Pillay, Nicholas Ramsey, and Atticus Stonestrom for several useful comments.

### 3.1 Thorn- and Kim-independence

In [97], Onshuus introduced the notions of  $\mathfrak{p}$ -forking (read as “thorn-forking”) and rosy theories, a class that generalises both simple and o-minimal theories. We start by recalling all the relevant definitions from that paper (following [55]).

**Definition 3.1.1.** Let  $\varphi(x, y)$  be a formula,  $b$  a tuple, and  $C$  a set.

- (i) We say that  $\varphi(x, b)$  **strongly divides over**  $C$  if the set of images of  $\varphi(\mathbb{M}, b)$  under elements of  $\text{Aut}(\mathbb{M}/C)$  is infinite and  $k$ -inconsistent for some  $k < \omega$ .
- (ii) We say that  $\varphi(x, b)$   **$\mathfrak{p}$ -divides over**  $C$  if there is a tuple  $d$  such that  $\varphi(x, b)$  strongly divides over  $Cd$ .
- (iii) We say that  $\varphi(x, b)$   **$\mathfrak{p}$ -forks over**  $C$  if there exist formulas  $\psi_i(x, c_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, c_i)$  and each  $\psi_i(x, c_i)$   $\mathfrak{p}$ -divides over  $C$ .
- (iv) We say  $A$  is  **$\mathfrak{p}$ -independent** (or **thorn-independent**) from  $B$  over  $C$  if, for some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/CB)$  does not contain a formula that  $\mathfrak{p}$ -forks over  $C$ . We denote this directly for tuples by  $a \perp_C^{\mathfrak{p}} B$ .

**Definition 3.1.2.** We say a theory  $T$  is **rosy** if  $\perp^{\mathfrak{p}}$  in  $T^{\text{eq}}$  satisfies local character.

The following result of Onshuus tells us which of the properties from Definition 2.4.3 are satisfied by  $\mathfrak{p}$ -independence:

**Fact 3.1.3** ([97, §2]). *In any theory,  $\perp^{\mathfrak{p}}$  satisfies monotonicity, existence over models, right extension, normality, right base monotonicity, left transitivity, and strong finite character. If  $T$  is rosy, then, in  $T^{\text{eq}}$ ,  $\perp^{\mathfrak{p}}$  also satisfies symmetry.*

Another independence relation that has become widespread in the literature appears in the work of Kaplan and Ramsey in [59], who introduced the notion of *Kim-independence*. This notion has been central to the development of structure theory of  $\text{NSOP}_1$ . Much of this thesis will employ this notion in different forms and generalisations.

**Definition 3.1.4.** Let  $\varphi(x, y)$  be a formula,  $b$  a tuple, and  $M \models T$ .

- (i) We say a sequence  $(b_i)_{i < \omega}$  is  **$M$ -invariant Morley** if it is a Morley sequence over  $M$  in a global  $M$ -invariant type (see §2.1). In this case,  $(b_i)_{i < \omega}$  is  $M$ -indiscernible. It can be shown (and we will do this in a more general context in chapter 5) that this is equivalent to being  $\downarrow^i$ -Morley over  $M$  (see Definition 2.4.27).
- (ii) We say that  $\varphi(x, b)$  **Kim-divides over**  $M$  if there is some  $M$ -invariant Morley sequence  $(b_i)_{i < \omega}$  with  $b_0 = b$  such that  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent.
- (iii) We say that  $\varphi(x, b)$  **Kim-forks over**  $M$  if there exist formulas  $\psi_i(x, c_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, c_i)$  and each  $\psi_i(x, c_i)$  Kim-divides over  $M$ .
- (iv) We say  $A$  is **Kim-independent** from  $B$  over  $M$  if, for some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/MB)$  does not contain any formula that Kim-forks over  $M$ . We denote this by  $a \downarrow_M^K B$ .

One of the main results from [59] characterises the class of  $\text{NSOP}_1$  theories in terms of Kim-independence. We will not need the combinatorial definition of  $\text{NSOP}_1$  given in Definition 2.3.2 for this chapter, only its semantic characterisations in terms of Kim-independence. As an illustration of this approach, we mention the following result, which will also motivate our results in §5.5:

**Theorem 3.1.5** ([59, Theorem 5.16]). *A theory  $T$  is  $\text{NSOP}_1$  iff Kim-independence is symmetric.*

**Remark 3.1.6.** Let us note that, in any theory, we have  $\downarrow^i \implies \downarrow^K$  (over models) and  $\downarrow^i \implies \downarrow^b$  (see Example 2.4.6).

## 3.2 The stable Kim-forking conjecture

The stable forking conjecture offers an intuitive picture of simple theories. If true, it would tell us that forking in a simple theory is always witnessed by a stable formula. In this way, we can view a simple theory as a stable reduct to which

we add some “noise” which does not affect the structure of forking (other than blocking stationarity). This question remains open, but a recent observation by Bossut indicates that a similar stable witnessing behaviour holds in several  $\text{NSOP}_1$  theories. In [14], he introduces the following version of this notion:

**Definition 3.2.1.** Let  $T$  be a complete  $\mathcal{L}$ -theory. We say  $T$  has the **stable Kim-forking property over models** if, for any model  $M \models T$  and  $B \supseteq M$ , whenever  $p(x) \in S(B)$  Kim-forks over  $M$ , there is some stable formula  $\varphi(x, y) \in \mathcal{L}(M)$  and some  $b \in B$  such that  $\varphi(x, b) \in p(x)$  Kim-forks over  $M$ .

**Conjecture 3.2.2** (Stable Kim-forking, cf., [9, Question 7.3]). *Every  $\text{NSOP}_1$  theory has the stable Kim-forking property over models.*

If the above conjecture is true, this would provide further evidence that  $\text{NSOP}_1$  theories are simple theories with even more “noise.”

**Remark 3.2.3.** Bossut calls the above property the “weak stable Kim-forking property,” in order to distinguish it from a stronger version. However, he shows in [14, Corollary 3.8] that having the stronger version implies that the theory is simple. Since we focus on the version of the conjecture that applies to (strictly)  $\text{NSOP}_1$  theories, we omit this stronger version and follow the terminology in [9].

In what follows, we discuss three of the most studied  $\text{NSOP}_1$  examples, which we will see all have the stable Kim-forking property. This provides some evidence toward Conjecture 3.2.2.

**Example 3.2.4.** Let  $\mathcal{L}$  be a two-sorted language with sorts  $P$  and  $O$ , and a ternary relation  $E \subseteq P \times O^2$ . Let  $T_{feq}^*$  be the model completion of the theory saying:

- $P$  and  $O$  partition the structure.
- For all  $p \in P$ ,  $E(p, x, y)$  defines an equivalence relation on  $O$ .

Chernikov and Ramsey show in [32, Corollary 6.4] that  $T_{feq}^*$  is  $\text{NSOP}_1$ . Adler proves directly in [3, Example 2.6] that  $T_{feq}^*$  is not rosy. Bossut explains that  $T_{feq}^*$  has stable Kim-forking in [14, discussion after Corollary 4.11].

**Example 3.2.5.** Let  $T_\infty$  be the model companion of the theory of an infinite-dimensional vector space over an algebraically closed field with an alternating or symmetric bilinear form, in the two-sorted language with a sort for the vector space and another sort for the underlying field.

Chernikov and Ramsey show that  $T_\infty$  is NSOP<sub>1</sub> in [32, Corollary 6.1], and Granger shows it is not simple in [48, Proposition 7.4.1]. In [68], a proof of non-rosiness is suggested. Finally, by [59, Proposition 9.37],  $\downarrow^K$  coincides with algebraic independence in  $T_\infty$ , that is, linear independence of the algebraic closures in the vector space sort, and algebraic freeness in the field sort. This implies that  $T_\infty$  has the stable Kim-forking property.

**Example 3.2.6.** We say a field  $F$  is **pseudo-algebraically closed** (or **PAC**) if every absolutely irreducible variety defined over  $F$  has an  $F$ -rational point.

Let us denote by  $\mathcal{G}(F)$  the absolute Galois group of  $F$ , viewed as a topological group in the usual fashion. We say  $F$  is **Frobenius** if  $\mathcal{G}(F)$  satisfies the *Iwasawa property*, i.e., if  $\varphi: \mathcal{G}(F) \rightarrow A$  and  $\theta: B \rightarrow A$  are continuous epimorphisms to a finite topological group  $A$ , and  $B$  is a finite quotient of  $\mathcal{G}(F)$ , then there is a continuous epimorphism  $\psi: \mathcal{G}(F) \rightarrow B$  making the following diagram commute:

$$\begin{array}{ccc} & \mathcal{G}(F) & \\ \psi \swarrow & & \searrow \varphi \\ B & \xrightarrow{\theta} & A \end{array}$$

In particular, whenever  $F$  is  $e$ -free (i.e.,  $\mathcal{G}(F) \cong \hat{\mathbb{F}}_e$ , the free profinite group on  $e$  many generators) for some  $e \in \omega$ , or  $\omega$ -free (i.e.,  $\mathcal{G}(F) \cong \hat{\mathbb{F}}_\omega$ ), it is Frobenius.

The theory of a Frobenius field  $F$  in  $\mathcal{L} = \{+, -, \cdot, 0, 1\}$  is NSOP<sub>1</sub> (see [59, §9.3]). If  $F$  is an  $e$ -free PAC field, it is simple (see [56]), and hence rosy, whereas if it is  $\omega$ -free it is NSOP<sub>1</sub> (see [32, Corollary 6.2]) and not simple (see [22, Theorem 3.9]). A proof that the theory of an  $\omega$ -free PAC field is not rosy is hinted at in [23, p. 3.5] and written explicitly (and in more generality) in [89].

In what follows, we will show that the theory of a Frobenius field has the stable Kim-forking property over models (even over algebraically closed sets). We use in our proof the notion of an *equation*, first introduced by Srour in [100]. We take as a definition an equivalent characterisation of equations which follows by compactness (see [85]):

**Definition 3.2.7.** A formula  $\varphi(x, y)$  is an **equation** if, for every indiscernible sequence  $(a_i, b_i)_{i < \omega}$ , if  $\models \varphi(a_i, b_j)$  for all  $i < j$ , then  $\models \varphi(a_i, b_i)$  for all  $i < \omega$ .

**Fact 3.2.8** ([100, Proposition 2.6]). *If  $\varphi(x, y)$  is an equation, then it is stable.*

**Remark 3.2.9.** We use the characterisation of Kim-independence for the theory

$T$  of a Frobenius field, which comes from [59, Theorem 9.32]: for any acl-closed subfields  $C \subseteq A, B$  of a sufficiently saturated field  $F \models T$ ,  $A \downarrow_C^K B$  iff  $A \downarrow_C^{\text{SCF}} B$  and  $S\mathcal{G}(A) \downarrow_{S\mathcal{G}(C)}^f S\mathcal{G}(B)$ , where  $\downarrow^f$  is nonforking independence in the  $\omega$ -sorted theory of the inverse system of  $\mathcal{G}(F)$  (for more details, see [21]).

In [32, Lemma 6.2], Chernikov and Ramsey explicitly give the formula witnessing strong finite character for Kim-independence in the theory of an  $\omega$ -free PAC field, which they attribute to Zoé Chatzidakis. We note here that the same formula proves strong finite character in the theory of any Frobenius field. In what follows, we briefly describe this formula.

Suppose, towards proving stable Kim-forking, that  $A \not\downarrow_C^K B$  with  $A, B, C$  acl-closed and  $C \subseteq A, B$  contained in a large saturated field  $F \models T$ . If  $A \not\downarrow_C^{\text{SCF}} B$ , this is witnessed by an equation and an inequation, which are always stable formulas. So it suffices to consider the case where  $A \downarrow_C^{\text{SCF}} B$  and  $S\mathcal{G}(A) \not\downarrow_{S\mathcal{G}(C)}^f S\mathcal{G}(B)$ . Using finite character and [21, Proposition 4.1], it follows that we can find open normal subgroups  $M_1 \triangleleft \mathcal{G}(A)$  and  $M_2 \triangleleft \mathcal{G}(B)$  such that, if  $N$  is the smallest open normal subgroup of  $\mathcal{G}(F)$  containing  $M_1$  and  $M_2$ , then  $N \not\subseteq \mathcal{G}(C)$ .

Using some Galois theory and the fact that  $F$  is a regular extension of  $A$  and  $B$ , this means that there are some  $\alpha \in \langle AC \rangle^{\text{alg}}$  and  $\beta \in \langle BC \rangle^{\text{alg}}$  not in  $F$  such that  $F(\alpha) = F(\beta)$  is Galois over  $F$  and  $\beta \notin F \langle C \rangle^{\text{alg}}$ . We may choose some  $b' \in \text{acl}(CB)$  such that  $\langle CB\beta \rangle \cap F \subseteq \langle CBb' \rangle$  and  $\langle CBb' \rangle$  is closed under  $\text{Aut}(\text{acl}(CB)/\langle CB \rangle)$ . Similarly, we can pick  $a' \in \text{acl}(AC)$  such that  $\langle CA\alpha \rangle \cap F \subseteq \langle CAa' \rangle$  and  $\langle CAa' \rangle$  is closed under  $\text{Aut}(\text{acl}(CA)/\langle CA \rangle)$ . (Note:  $\alpha, \beta, a', b'$  can be taken as singletons by the Primitive Element Theorem.)

Fix some enumerations  $a, b, c$  of  $A, B, C$ , respectively. The idea now is to find a formula that captures the behaviour of  $a, b, \alpha, \beta$  as described above. To do so, let:

- $P(Y, b, c)$  be the minimal polynomial of  $b'$  over  $\langle BC \rangle$ .
- $Q(Z, Y, b, c)$  be such that  $Q(Z, b', b, c)$  is the minimal polynomial of  $\beta$  over  $\langle CBb' \rangle$ .
- $R(W, a, c)$  be the minimal polynomial of  $a'$  over  $\langle AC \rangle$ .
- $S(W, T, a, c)$  be such that  $S(W, a', a, c)$  is the minimal polynomial of  $\alpha$  over  $\langle CAa' \rangle$ .

We now take  $\varphi(t, b, c)$  (with variable  $t$ ) to be the formula saying: “there exists some  $y \equiv_{bc} b'$  and a solution  $w$  of  $R(W, t, c) = 0$  such that (i)  $S(X, w, t, c)$  is irreducible over  $F$  of degree  $[\langle CA\alpha \rangle : \langle CAa' \rangle]$ , and (ii) for every solution  $z$  of  $Q(Z, y, b, c) = 0$ ,

$F(z)$  contains a solution of  $S(X, w, t, c) = 0$ ." This can indeed be expressed using a first-order formula (for more details, see [21], [32]). In [32, Lemma 6.2], it is shown that  $\varphi(t, b, c)$  Kim-forks over  $C$ .

**Proposition 3.2.10.** *The theory of a Frobenius field has the stable Kim-forking property over algebraically closed sets.*

*Proof.* It is enough to show that the formula  $\varphi(t, s, c)$  (in variables  $t, s$ ) from the previous remark is stable. We prove something stronger (cf., Fact 3.2.8): namely, we will show that  $\varphi(t, s, c)$  is an equation in Srour's sense.

Suppose  $(a_i, b_i)_{i < \omega}$  is a  $C$ -indiscernible sequence such that  $\models \varphi(a_i, b_j, c)$  for all  $i < j$ . Let us choose, for each  $i < j$ ,  $d_{ij}$  and  $e_{ij}$  to be witnesses to  $y$  and  $w$  as stipulated by each appropriate instance of  $\varphi$ , and also pick  $f_{ij}$  to be a solution to  $Q(Z, d_{ij}, b_j, c) = 0$ . Since by choice they are all roots of  $Q$ , it follows that, for all  $i < j$ ,  $F(f_{ij})$  is a proper Galois extension of  $F$  of degree  $[\langle CB\beta \rangle : \langle CBB' \rangle]$  which is not contained in  $F\langle C \rangle^{\text{alg}}$ . Note that, by definition of  $\varphi$ , for each  $i < j$ ,  $F(f_{ij})$  is precisely the field generated by any of its roots of  $S(X, e_{ij}, a_i, c) = 0$  over  $F$ .

**Claim 1.** There exist  $k < i < j$  such that  $e_{ki} = e_{kj}$  and  $d_{kj} = d_{ij}$ .

*Proof of Claim 1.* Suppose that  $P(Y, b_0, c) = 0$  has  $m$  solutions and  $R(W, a_0, c) = 0$  has  $n$  solutions. By  $C$ -indiscernibility, we get the same number of solutions for any indices  $i, j < \omega$ . Thus, our choice of  $d_{ij}$  and  $e_{ij}$  defines an  $mn$ -colouring on  $\omega^{(2)}$ . Hence, by Ramsey's Theorem, there is an infinite subset  $X \subseteq \omega$  such that  $X^{(2)}$  is monochromatic. Pick  $k, i, j \in X$  with  $k < i < j$ . Then  $\{k, i\}, \{k, j\}, \{i, j\} \in X^{(2)}$ , so it follows that  $e_{ki} = e_{kj}$  and  $d_{kj} = d_{ij}$ , as required.  $\square_{\text{Claim 1}}$

Let  $d_j := d_{kj} = d_{ij}$  and  $e_k := e_{ki} = e_{kj}$ . Then  $F(f_{ki})$  and  $F(f_{kj})$  both contain solutions to  $S(X, e_k, a_k, c) = 0$ , and since they are Galois, by our earlier remarks we get  $F(f_{ki}) = F(f_{kj})$ . Moreover,  $F(f_{kj})$  and  $F(f_{ij})$  also contain all solutions of  $Q(Z, d_j, b_j, c) = 0$ , so we get  $F(f_{kj}) = F(f_{ij})$ . Therefore,  $F(f_{ki}) = F(f_{ij})$ .

**Claim 2.**  $\models \varphi(a_i, b_i, c)$ .

*Proof.* Assume not, for contradiction. So we can find a solution  $f_{ii}$  of  $Q(Z, d_{ki}, b_i, c) = 0$  such that  $F(f_{ii})$  contains no solutions of  $S(X, e_{ij}, a_i, c) = 0$ . But note that, by this choice of  $f_{ii}$  and the above remarks, it follows that  $f_{ii} \in F(f_{ki}) = F(f_{ij})$ , and hence,  $F(f_{ii})$  contains a solution to  $S(X, e_{ij}, a_i, c) = 0$ , a contradiction.  $\square_{\text{Claim 2}}$

By  $C$ -indiscernibility of  $(a_i, b_i)_{i < \omega}$ , we conclude that  $\models \varphi(a_i, b_i, c)$  for all  $i < \omega$ .  $\square$

### 3.3 A dichotomy between NSOP<sub>1</sub> and rosy theories

A frequent question in neostability theory concerns the relationship between the “canonical” independence relations arising from different classes of theories (a topic which will be explored further in chapter 5). One example is how  $\perp^f$  relates to  $\perp^b$  in simple theories. A partial answer to this question appears in [97, Corollary 5.1.3]:

**Theorem 3.3.1.** *Suppose that the stable forking conjecture holds. If  $T$  is simple, then  $\perp^f = \perp^b$ .*

A similar question can then be asked about the relationship between  $\perp^K$  and  $\perp^b$ , which is closely related to d’Elbée and Kim’s Question 3.0.1. In what follows, we give a partial answer to this question under the assumption of the stable Kim-forking conjecture. This resembles Onshuus’ result for simple theories (cf., Theorem 3.3.1). But first, we need to collect some results from the literature.

A close inspection of the proof appearing in [55, Proposition 3.3] shows the following result, first proved by Onshuus in [97, Theorem 5.1.1]:

**Fact 3.3.2.** *Let  $M \prec N$  be models of  $T = T^{\text{eq}}$ , and  $p(x) \in S(N)$ . If there is some  $\varphi(x, n) \in p(x)$  that forks over  $M$  such that  $\varphi(x, y) \in \mathcal{L}(M)$  is stable, then  $p(x)$   $\text{b-forks}$  over  $M$ .*

Our proof will also require a modified version of local character (see Definition 2.4.3(xiv)) that applies when the independence relation is only defined over models. We use the notion as in [43], although it originates in [61], and the terminology from [38]. This notion will also play a role in §5.4. Recall that  $(M_i)_{i < \kappa}$  is a **continuous chain** of models if  $M_i \models T$  for all  $i$ ,  $M_i \preceq M_j$  for all  $i < j$ , and  $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$  for  $\alpha$  limit. We assume our continuous chains are strictly increasing.

**Lemma 3.3.3.** *Suppose  $T = T^{\text{eq}}$  is rosy. Then  $\perp^b$  satisfies **chain local character**, that is: For any finite tuple  $a$ , there is a regular  $\kappa > |T|^+$  such that, for every continuous chain of models  $(M_i)_{i < \kappa}$  of  $T$  such that  $|M_i| < \kappa$  for all  $i$ , there is  $i < \kappa$  such that  $a \perp_{M_i}^b M$ , where  $M := \bigcup_{i < \kappa} M_i$ .*

*Proof.* For an abstract proof of this result at the level of independence relations, see, e.g., [43, Lemma 9.6] or alternatively Fact 5.4.1 (both using the properties listed in Fact 3.1.3).  $\square$

We also require some important facts about NSOP<sub>1</sub> theories:

**Fact 3.3.4.** (i) [59, Proposition 8.8] An  $\text{NSOP}_1$  theory  $T$  is simple iff  $\perp^K$  satisfies base monotonicity over models.

(ii) (This is a corollary to [59, Theorem 6.5]) Let  $T$  be  $\text{NSOP}_1$  and  $M \models T$ . If  $a \perp_M^K b$  and  $I = (b_i)_{i < \omega}$  is a  $\perp^K$ -Morley sequence over  $M$  with  $b_0 = b$ , then there is  $a' \equiv_{Mb} a$  such that  $a'b_i \equiv_M ab$  for all  $i < \omega$ .

We also adapt the terminology from [43]:

**Definition 3.3.5.** Let  $\perp$  be an independence relation,  $M \models T$ , and  $(a_i)_{i < \kappa}$  an  $\perp$ -independent sequence. We say that a continuous chain of models  $(M_i)_{i < \kappa}$  is an  $\perp$ -**independence chain** for  $(a_i)_{i < \kappa}$  over  $M$  if  $M \preceq M_0$ ,  $a_{<i} \subset M_i$ , and  $a_i \perp_{M_i} M_i$  for all  $i < \kappa$ .

**Remark 3.3.6.** It is an easy observation that, if  $\perp$  satisfies existence and right extension, then we can find an  $\perp$ -independence chain of models for any  $\perp$ -independent sequence  $(a_i)_{i < \omega}$  over a model  $M \models T$ .

**Theorem 3.3.7.** If  $T$  is  $\text{NSOP}_1$  and rosy, and  $T^{\text{eq}}$  has the stable Kim-forking property over models, then  $T$  is simple.

*Proof.* Let  $T$  be  $\text{NSOP}_1$  and rosy. Recall (see, e.g., [66, Remark & Example 2.3.8], which can also be adapted for  $\text{NSOP}_1$ ) that  $T$  is  $\text{NSOP}_1$  (resp., simple) iff  $T^{\text{eq}}$  is  $\text{NSOP}_1$  (resp., simple), so without loss of generality we may assume  $T = T^{\text{eq}}$ . In particular, by assumption,  $T$  has the stable Kim-forking property over models.

**Claim.**  $\perp^K = \perp^b$  over models.

*Proof.* ( $\Leftarrow$ ) Suppose that  $a \perp_M^b b$  but  $a \not\perp_M^K b$ . By the stable Kim-forking property, there is some  $\mathcal{L}(M)$ -formula  $\varphi(x, y)$  that is stable such that  $\varphi(x, b) \in \text{tp}(a/Mb)$  Kim-forks over  $M$ . By right extension, we can find some  $N \supset M \cup \{b\}$  such that  $a \perp_M^b N$  and  $N \models T$ . Then  $\varphi(x, b) \in \text{tp}(a/N)$ , and in particular  $\varphi(x, b)$  forks over  $M$ . Thus, by Fact 3.3.2,  $\text{tp}(a/N)$   $b$ -forks over  $M$ , a contradiction.

( $\Rightarrow$ ) We adapt the proof from [43, Theorem 9.1] (which is in turn based on the original proof from [30]). So suppose that  $a \not\perp_M^K b$ . By the above remark, we can find a long  $M$ -indiscernible sequence  $(b_i)_{i < \kappa}$  with  $b_i \equiv_M b$  for all  $i < \kappa$  and an  $\perp^i$ -independence chain  $(M_i)_{i < \kappa}$  for  $(b_i)_{i < \kappa}$  over  $M$ . In particular, we have  $b_i \perp_M^K M_i$  for all  $i < \kappa$ , and so, by monotonicity,  $(b_i)_{i < \kappa}$  is  $\perp^K$ -Morley over  $M$ . Hence, since  $T$  is  $\text{NSOP}_1$ , we can apply Fact 3.3.4(ii) to find some  $a' \equiv_{Mb} a$  such that  $a'b_i \equiv_M ab$  for all  $i < \omega$ .

Since  $T$  is rosy, by Lemma 3.3.3,  $\downarrow^b$  satisfies chain local character. By monotonicity and downwards Löwenheim-Skolem, we may assume  $\kappa$  is the cardinal coming out of chain local character for  $\downarrow^b$ , and  $(M_i)_{i \in \kappa}$  is a continuous chain with  $|M_i| < \kappa$ . Let  $M_\kappa := \bigcup_{i \in \kappa} M_i$ . Therefore, there is  $i_0 < \kappa$  such that  $a' \downarrow_{M_{i_0}}^b M_\kappa$ , and so by monotonicity,  $a' \downarrow_{M_{i_0}}^b b_{i_0}$ . But also  $b_{i_0} \downarrow_M^b M_{i_0}$ . Hence, by symmetry and transitivity,  $a' \downarrow_M^b b_{i_0}$ , so  $a \downarrow_M^b b$  by invariance, as required.  $\square_{\text{Claim}}$

This equality implies that  $\downarrow^K$  satisfies base monotonicity over models. Therefore, by Fact 3.3.4(i),  $T$  is simple. This concludes the proof.  $\square$

**Corollary 3.3.8.** *Suppose that the stable Kim-forking conjecture holds. If  $T$  is NSOP<sub>1</sub> and rosy, then  $T$  is simple.*

**Remark 3.3.9.** It is an open problem whether, in order to conclude that  $T^{\text{eq}}$  has stable Kim-forking over models, it suffices to prove stable Kim-forking for types over real parameters (see [13, §5] for more details).

Let us note that this pattern fails for higher levels of the SOP <sub>$n$</sub>  hierarchy: for instance, the generic triangle-free graph, which is NSOP<sub>4</sub> but SOP<sub>3</sub>, is rosy.

A corollary of this result emphasises the distinction between simple and NSOP<sub>1</sub> theories. Recall that  $T$  is said to be **pregeometric** if  $\text{acl}$  satisfies exchange, i.e., whenever  $a \in \text{acl}(Ab) \setminus \text{acl}(A)$ , we have  $b \in \text{acl}(Aa)$ . This large class of theories contains those of strongly minimal and o-minimal theories, plus a plethora of other NIP examples, such as the theory of  $p$ -adically closed fields. We can also find several examples of simple pregeometric theories, such as simple free amalgamation theories (see [34]) and pseudofinite fields (see [56]).

Generally speaking, if  $T$  is pregeometric,  $T^{\text{eq}}$  need not be so. However, all that is required to extend the notion of dimension given by algebraic closure to the imaginary case is the weakest form of elimination of imaginaries, namely, **geometric elimination of imaginaries**, i.e., for any imaginary  $e$ , there is some real tuple  $a$  such that  $e \in \text{acl}^{\text{eq}}(a)$  and  $a \in \text{acl}(e)$ . This is the main idea underlying the proof of the following result, due to Ealy and Onshuus:

**Fact 3.3.10** ([46, Theorem 4.12]). *If  $T$  is pregeometric and has geometric elimination of imaginaries, then  $T$  is rosy.*

Thus, together with Theorem 3.3.7, this implies the following:

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**Corollary 3.3.11.** *Suppose the stable Kim-forking conjecture holds. Then there is no strictly NSOP<sub>1</sub> pregeometric theory with geometric elimination of imaginaries.*

**Example 3.3.12.** Let us note that there are known strictly NSOP<sub>1</sub> examples with exactly one of each of the two properties in the conclusion. We thank Christian d’Elbée for suggesting the following example: Consider ACFG, the theory of an algebraically closed field of characteristic  $p > 0$  expanded by a predicate for an additive subgroup. This theory is strictly NSOP<sub>1</sub> and pregeometric, but does not have pregeometric elimination of imaginaries (see [37]). We can (weakly) eliminate imaginaries by adding a new sort for the quotient group  $K/G$  and a function symbol for the quotient map  $\pi: K \rightarrow K/G$ , but the resulting two-sorted theory is not pregeometric.

# 4

## Hypertournaments

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In this chapter, we present the first example of a strictly NSOP<sub>4</sub> structure with no known invariant independent relations satisfying full existence, monotonicity, and stationarity, namely, the  $\mathbf{H}_4$ -free 3-hypertournament. More precisely, we prove:

**Theorem 4.0.1.** *The theory of the countable homogeneous  $\mathbf{H}_4$ -free 3-hypertournament is SOP<sub>3</sub>, TP<sub>2</sub>, and NSOP<sub>4</sub>. Moreover, it is IP<sub>2</sub> and NFOP<sub>3</sub>.*

This contrasts with the situation for the other three known homogeneous 3-hypertournaments, all of which are NTP<sub>2</sub>.

There is a long history of interactions between model theory and the combinatorial study of tournaments, i.e., directed graphs  $(V, E)$  such that, for all distinct  $a, b \in V$ , exactly one of  $E(a, b)$  and  $E(b, a)$  holds. As a result of this investigation,

Lachlan famously classified in [77] the countable homogeneous tournaments into three structures up to isomorphism: the generic tournament, the countable dense linear order without endpoints, and the homogeneous local order.

Around the same time, a combinatorial generalisation of tournaments to higher arities was introduced by Assous in [7], which was soon slightly modified into the notion of an *n-hypertournament*. In an attempt to extend Lachlan’s classification of homogeneous tournaments, Cherlin studied the homogeneous *4-constrained* 3-hypertournaments (see [24, §23A.4.1]), showing that there exist only four up to isomorphism. It remains open whether there exist any other countable homogeneous 3-hypertournaments beyond these four cases. Here, we focus on one of these four structures, namely, the  $\mathbf{H}_4$ -free 3-hypertournament. We introduce its definition in §4.1, and show some of its basic model-theoretic properties, which include weak elimination of imaginaries (Lemma 4.1.8) and the existence of global types Lascar-invariant over  $A$  for  $A$  a non-empty set (Proposition 4.1.13).

Cherlin’s four homogeneous 3-hypertournaments have recently appeared in connection with some fundamental questions in structural Ramsey theory. One of the major open questions in this area is that posed by Bodirsky, Pinsker, and Tsankov in [12]: Does every homogeneous structure in a finite relational language have a homogeneous expansion by finitely many relations which is Ramsey? In [25], Cherlin, Hubička, Konečný, and Nešetřil set out to find Ramsey expansions of each of the four homogeneous 4-constrained 3-hypertournaments in finite relational languages, and were able to find them for all except one of the four, namely, the  $\mathbf{H}_4$ -free 3-hypertournament. Thus, this example serves as motivation for extending the traditional techniques of structural Ramsey theory.

In §4.2, we offer the first proof of Theorem 4.0.1 using combinatorial constructions. We offer a second proof of this result in §4.4, this time using the machinery of independence relations. As preliminary work towards this proof, in §4.3, we adapt the notion of strong Lascar independence from Tartarotti’s master’s thesis [112] to the context of the Kim-dividing order introduced by Mutchnik in [94]. Tartarotti defines strong Lascar independence in terms of minimal extensions of types with respect to the fundamental order, and uses this towards a proof of Lascar’s Reconstruction Theorem. Using analogous ideas, we obtain an independence relation we can use to characterise, abstractly, the notions of relative Kim’s lemma and strong witnessing property, which play an important role in [94].

In contrast to most known  $\text{NSOP}_4$  examples, including all of those we have

mentioned before, there cannot exist any independence relation over models of the theory of the  $\mathbf{H}_4$ -free 3-hypertournament satisfying full existence, symmetry, and stationarity. In our second proof of NSOP<sub>4</sub> (§4.4), we show that:

**Theorem 4.0.2.** *In the  $\mathbf{H}_4$ -free 3-hypertournament, there is a non-stationary independence relation  $\perp^{\text{hti}}$  satisfying full existence and the relative Kim's lemma. In particular, Conant-independence coincides with  $\perp^{\text{a}}$ .*

Thus, although our proof uses some of the concepts and results that Mutchnik introduces in [94], the specific methods applied to examples in that paper use the existence of an independence relation satisfying monotonicity, full existence, and stationarity over models, which we have not been able to find in the  $\mathbf{H}_4$ -free 3-hypertournament. We conjecture that, in fact, there is no such independence relation defined over models of this theory. However, regardless of the outcome of this conjecture, the present example remains a theoretical novelty in the context of NSOP<sub>4</sub> as the first application of Mutchnik's concepts with a non-stationary independence relation.

The contents of this chapter have been published in [87]. We thank Christian d'Elbée, Paolo Marimon, Scott Mutchnik and Nicholas Ramsey for helpful discussions about the results in this chapter.

## 4.1 Basic properties

Since we need to keep track of elements and tuples in many of the proofs concerning 3-hypertournaments, throughout this chapter, we use  $a$  to denote an element, the bar notation  $\bar{a}$  to denote a tuple, and  $A$  to denote a set. For an  $n$ -tuple  $\bar{a}$ , we write  $\bar{a} = (a_0, \dots, a_{n-1})$ . The same conventions apply to variables and tuples of variables.

### 4.1.1 Definition of the $\mathbf{H}_4$ -free 3-hypertournament

The definitions and results included in this subsection can be found, explicitly or implicitly, in [24, Appendix B, §B1.4.1] and [25]. For the remainder of this section, let  $\mathcal{L} = \{R\}$ , where  $R$  is a ternary relation symbol.

**Definition 4.1.1.** Let  $T$  be the  $\mathcal{L}$ -theory axiomatised by the following formulas:

$$(i) \quad \forall x_1 x_2 x_3 (R(x_1, x_2, x_3) \rightarrow \bigwedge_{i \neq j} x_i \neq x_j).$$

- (ii)  $\forall x_1 x_2 x_3 (R(x_1, x_2, x_3) \leftrightarrow R(x_2, x_3, x_1) \leftrightarrow R(x_3, x_1, x_2)).$
- (iii)  $\forall x_1 x_2 x_3 (\bigwedge_{i \neq j} x_i \neq x_j \rightarrow (\neg R(x_1, x_2, x_3) \leftrightarrow R(x_3, x_2, x_1))).$

We call any model  $M \models T$  a **3-hypertournament**.

The general classification of all countable homogeneous 3-hypertournaments is still open. As a partial answer to this problem, Cherlin's result classifies those homogeneous 3-hypertournaments that arise from looking at the possible structures that can occur on any subset of four points contained in them.

**Remark 4.1.2.** Let  $M$  be a 3-hypertournament. Given a linear order  $\leq$  on  $M$ , define a 3-uniform hypergraph  $\widehat{M}$  on  $M$  in  $\mathcal{L}$  by declaring, for  $a \leq b \leq c$ ,  $\{a, b, c\} \in R^{\widehat{M}}$  iff  $(a, b, c) \in R^M$ . Formally, a linear order on  $M$  determines an isomorphism of categories between the age of  $M$  with embeddings and the category of 3-uniform hypergraphs on finite subsets of  $M$  with embeddings.

There are three 4-point substructures of a 3-hypertournament up to isomorphism:

- **C<sub>4</sub>**: there is an order  $\leq$  on  $C_4$  (the underlying set of four points) such that  $\widehat{C}_4$  is a complete 3-uniform hypergraph. In different orders,  $\widehat{C}_4$  might have no hyperedges at all, or exactly two hyperedges which intersect at two vertices that are adjacent relative to the ordering on the four points.
- **O<sub>4</sub>**: for any linear order  $\leq$  on  $O_4$ ,  $\widehat{O}_4$  has an odd number of hyperedges.
- **H<sub>4</sub>**: for any linear order  $\leq$  on  $H_4$ ,  $\widehat{H}_4$  has exactly two hyperedges intersecting in vertices  $a < b$  such that there is a unique  $c \in H_4$  with  $a < c < b$ .

For any  $\mathbf{A} \in \{\mathbf{C}_4, \mathbf{O}_4, \mathbf{H}_4\}$ , given a 3-hypertournament  $M$  and a substructure  $B \subseteq M$ , we say  $B$  is an **A-structure** if it is isomorphic to  $\mathbf{A}$ , and we call  $M$  **A-free** if it contains no **A**-structures.

**Definition 4.1.3.** Let  $\mathcal{C}$  be a class of finite 3-hypertournaments. We say  $\mathcal{C}$  is **4-constrained** if there is  $S \subseteq \{\mathbf{C}_4, \mathbf{O}_4, \mathbf{H}_4\}$  such that a finite 3-hypertournament  $M \in \mathcal{C}$  iff any substructure  $B \subseteq M$  with  $|B| = 4$  is an **A**-structure for some  $\mathbf{A} \in S$ .

As alluded to in [25], Cherlin earlier proved that there are exactly four 4-constrained classes of 3-hypertournaments that are (strong) amalgamation classes (Definition 2.2.1(iii)). Here, we focus on just one of these classes, namely, that of finite 3-hypertournaments omitting  $\mathbf{H}_4$ .

**Remark 4.1.4.** Let us note that we may also define  $\mathbf{H}_4$ , up to isomorphism, as a structure on four points  $\{a, b, c, d\}$  such that

$$\models R(a, b, c) \wedge R(a, c, d) \wedge R(a, d, b) \wedge R(b, d, c).$$

We often use without mention the following easy observation: if  $a, b, c, d \in M$  are elements of a 3-hypertournament such that  $M \models R(a, b, c) \wedge R(a, b, d)$ , then  $\{a, b, c, d\}$  is  $\mathbf{H}_4$ -free.

**Lemma 4.1.5.** *The class of finite  $\mathbf{H}_4$ -free 3-hypertournaments is a strong amalgamation class.*

*Proof.* We may assume that our amalgamation problem is of the form  $Ab_1 \leftarrow A \rightarrow Ab_2$  for  $b_1, b_2 \notin A$ . Let  $\leq$  be a linear order on  $Ab_1b_2$  such that  $b_1 < A < b_2$ . This generates an amalgamation problem in the category of finite 3-uniform hypergraphs, and so we can freely amalgamate to obtain a solution  $\widehat{C}$ . Applying the inverse functor, we obtain a finite 3-hypertournament  $C$  which is in our class. Indeed, by construction, the only potential  $\mathbf{H}_4$ -structure must be of the form  $\{a_1, a_2, b_1, b_2\}$  with  $a_1, a_2 \in A$ . Since  $\widehat{C}$  is obtained by free amalgamation, no hyperedges involving both  $b_1$  and  $b_2$  appear in  $\widehat{C}$ , and so it follows that  $C \models R(b_1, b_2, a_1) \wedge R(b_1, b_2, a_2)$ , which implies that  $\{a_1, a_2, b_1, b_2\}$  is  $\mathbf{H}_4$ -free.  $\square$

By Fact 2.2.3, the Fraïssé limit of the class in Lemma 4.1.5 exists, and we denote its theory by  $T_{\mathbf{H}_4\text{-free}}$ . By Facts 2.2.4 and 2.2.5,  $T_{\mathbf{H}_4\text{-free}}$  is  $\omega$ -categorical and has quantifier elimination and trivial algebraic closure. Often, we refer to the countable model of  $T_{\mathbf{H}_4\text{-free}}$  as *the* (generic)  $\mathbf{H}_4$ -free 3-hypertournament.

The three remaining countable homogeneous 3-hypertournaments are determined by the constrained classes they come from. These are:

- The **generic 3-hypertournament**, which is the Fraïssé limit of the 4-constrained class determined by  $S = \{\mathbf{C}_4, \mathbf{O}_4, \mathbf{H}_4\}$ . Using, e.g., disjoint 3-amalgamation, one can show that the generic 3-hypertournament is supersimple with trivial forking (see [74, Theorem 3.14]).
- The **even 3-hypertournament**, which is the Fraïssé limit of the 4-constrained class determined by  $S = \{\mathbf{C}_4, \mathbf{H}_4\}$ . As before, one can show that the even 3-hypertournament is supersimple with trivial forking.
- The **cyclic 3-hypertournament**, which is the Fraïssé limit of the 4-constrained class determined by  $S = \{\mathbf{C}_4\}$ . One can show that the cyclic 3-hypertournament is bi-interpretable with  $(\mathbb{Q}, \text{cyc})$ , and hence it is distal and dp-minimal.

The goal of this chapter is to provide similar classification results for the  $\mathbf{H}_4$ -free 3-hypertournament.

### 4.1.2 Existence of invariant extensions

In this section, we prove that  $T_{\mathbf{H}_4\text{-free}}$  is locally  $\perp^i$ -extensible but not  $\perp^i$ -extensible, properties which we define below (recall that  $\perp^i$  was defined in Example 2.4.6). During the course of the proof, some other basic model-theoretic properties of  $T_{\mathbf{H}_4\text{-free}}$  are established, such as weak elimination of imaginaries. From now on, let  $\mathbb{M} \models T_{\mathbf{H}_4\text{-free}}$  be a monster model.

**Definition 4.1.6** (cf. [29]). Let  $T$  be a complete theory.

- (i) We say  $T$  is  $\perp^i$ -**extensible** if  $\perp^i$  satisfies existence, i.e., every type  $p \in S(A)$  has a global extension  $q$  which is Lascar-invariant over  $A$ .
- (ii) We say  $T$  is **locally**  $\perp^i$ -**extensible** if it is  $\perp^i$ -extensible after adding finitely many parameters.

**Remark 4.1.7.** Suppose that we have proven  $\perp^i$ -extensibility for 1-types, i.e., for every singleton  $a$  and set  $B$ , there is a global extension of  $\text{tp}(a/B)$  Lascar-invariant over  $B$ . Take two singletons  $a, b$ , and fix a set  $C$ . We freely use the properties listed in Example 2.4.6. By assumption,  $a \perp_C^i C$  and  $b \perp_{Ca}^i Ca$ , hence  $b \perp_{Ca}^i C$  by right monotonicity. But then, by left transitivity and left monotonicity,  $ab \perp_C^i C$ , i.e.,  $\text{tp}(ab/C)$  also has a global Lascar-invariant extension. Using induction and compactness, this shows that (local)  $\perp^i$ -extensibility for 1-types suffices to prove the general case.

**Lemma 4.1.8.**  $T_{\mathbf{H}_4\text{-free}}$  has weak elimination of imaginaries, i.e., for every imaginary  $e$ , there is some real  $\bar{c}$  such that  $e \in \text{dcl}^{\text{eq}}(\bar{c})$  and  $\bar{c} \in \text{acl}(e)$ .

*Proof.* Let  $M \models T_{\mathbf{H}_4\text{-free}}$  be countable. By [102, Lemma 16.17], it suffices to show that

$$H := \langle \text{Aut}(M/A), \text{Aut}(M/B) \rangle = \text{Aut}(M/A \cap B) \quad (4.1.1)$$

for finite algebraically closed  $A, B \subset M$ . Since  $M$  has trivial algebraic closure, by [80, Proposition 4.3], it is enough to prove (4.1.1) when  $A = Ca$ ,  $B = Cb$  for some finite set  $C \subseteq M$  and distinct elements  $a, b \in M$ . We may assume  $a, b \notin C$ .

**Claim 1.** If  $b' \equiv_C b$ , then there is  $a' \models \text{tp}(a/Cb)$  such that  $b' \equiv_{Ca'} b$ .

*Proof of Claim 1.* Write  $C = \{c_1, \dots, c_n\}$  and  $p(x, b, \bar{c}) := \text{tp}(a/b\bar{c})$ , and let  $b' \models \text{tp}(b/\bar{c})$ . First note that, if  $b' = b$ , then we can pick  $a' = a$ . Hence, assuming that  $b' \neq b$ , we need to show that  $p(x, b, \bar{c}) \cup \{R(x, b, c_i) \leftrightarrow R(x, b', c_i) : i \in [n]\}$  is

consistent. We split the proof of this claim into two cases. In both, our strategy is to build a finite structure that satisfies the formulas in the set.

Case 1:  $b' = a$ .

Let  $\Sigma(x, b, a, \bar{c})$  be the extension of  $p(x, b, \bar{c})$  obtained by adding the following formulas:

- (i)  $R(x, a, c_i) \leftrightarrow R(a, b, c_i)$  for all  $i \in [n]$ .
- (ii)  $R(x, a, b)$ .

We claim that  $\Sigma$  is consistent modulo  $T_{\mathbf{H}_4\text{-free}}$ . Since  $\Sigma$  extends  $p$ , it suffices to show that 4-point sets containing  $x$  and  $a$  are  $\mathbf{H}_4$ -free. But this also follows immediately from the fact that  $\Sigma$  extends  $p$ , using Remark 4.1.4.

Case 2:  $b' \neq a$ .

This time, let  $\Sigma(x, b, b', a, \bar{c})$  be the extension of  $p(x, b, \bar{c})$  obtained by adding the following formulas:

- (i)  $R(x, b', c_i) \leftrightarrow R(a, b, c_i)$  for all  $i \in [n]$ .
- (ii)  $R(x, a, b) \wedge R(x, a, b')$ .
- (iii)  $R(x, a, c_i)$  for all  $i \in [n]$ .
- (iv)  $R(x, b, b') \leftrightarrow R(a, b, b')$ .

As in Case 1, we show that  $\Sigma$  is consistent modulo  $T_{\mathbf{H}_4\text{-free}}$  by showing that 4-point sets containing  $x$  and at least one of  $a$  or  $b'$  are  $\mathbf{H}_4$ -free. By (iii),  $\Sigma$  implies that any set of the form  $\{x, a, c_i, c_j\}$  is  $\mathbf{H}_4$ -free, and by (ii) and (iii), so is  $\{x, a, b, c_i\}$ . Moreover, by (i) and since  $b \equiv_C b'$ ,  $\Sigma$  implies that any set of the form  $\{x, b', c_i, c_j\}$  or  $\{x, b, b', c_i\}$  is  $\mathbf{H}_4$ -free. Finally, by (ii),  $\Sigma$  implies that  $\{x, a, b, b'\}$  is also  $\mathbf{H}_4$ -free, and (ii) and (iii) combined imply that so is  $\{x, a, b', c_i\}$ .

Therefore, in either case,  $\Sigma$  defines a type over  $a, b, b', \bar{c}$ , and so we can choose  $a' \models \Sigma$  in  $M$ . By (i) in both cases, this  $a'$  works.  $\square_{\text{Claim 1}}$

We now show (4.1.1). Note that  $\leq$  is clear. For  $\geq$ , let  $g \in \text{Aut}(M/C)$ , and let  $b' := g(b)$ . Then, by Claim 1, there is  $a' \equiv_B a$  such that  $b \equiv_{Ca'} b'$ .

**Claim 2.**  $\text{Aut}(M/Ca') \leq H$ .

*Proof of Claim 2.* Since  $a' \equiv_B a$ , there is some  $h \in \text{Aut}(M/B)$  sending  $a'$  to  $a$ . Hence, for any  $k \in \text{Aut}(M/Ca')$ , we have that  $hkh^{-1} \in \text{Aut}(M/A) \leq H$ . As  $h \in \text{Aut}(M/B) \leq H$ , it follows that  $k \in H$ .  $\square_{\text{Claim 2}}$

Thus, since  $b \equiv_{Ca'} b'$ , there is some  $k \in \text{Aut}(M/Ca')$  such that  $k(b) = b'$ , that is,  $k(b) = g(b)$ . Therefore,  $k^{-1}g \in H$ , and so, since  $k \in H$  by Claim 2,  $g \in H$ .  $\square$

**Remark 4.1.9.** Claim 1 in the previous proof is a strong property for which we need to restrict ourselves to both  $a$  and  $b$  being singletons. See Remark 4.2.4 for a counterexample to the version of the Claim obtained by replacing  $b$  with a pair of elements.

**Corollary 4.1.10.** *For all  $\bar{a}, \bar{b}$  and  $A$ ,  $\bar{a} \equiv_{\text{acl}^{\text{eq}}(A)} \bar{b}$  iff  $\bar{a} \equiv_{\text{acl}(A)} \bar{b}$ .*

*Proof.* This follows directly from [20, Proposition 3.4].  $\square$

**Proposition 4.1.11.**  $T_{\mathbf{H}_4\text{-free}}$  is not  $\downarrow^i$ -extensible.

*Proof.* Note that, for any two pairs of distinct points  $ab$  and  $cd$ , we have  $ab \equiv cd$  in  $T_{\mathbf{H}_4\text{-free}}$ . Since  $T_{\mathbf{H}_4\text{-free}}$  has trivial algebraic closure, we get  $ab \equiv_{\text{acl}(\emptyset)} cd$ , and so, by Corollary 4.1.10, it follows that  $ab \equiv_{\text{acl}^{\text{eq}}(\emptyset)} cd$ . Finally, since  $T_{\mathbf{H}_4\text{-free}}$  is  $\omega$ -categorical, it follows that  $ab \equiv^{\text{Ls}} cd$ .

Now assume, for contradiction, that there is some global 1-type  $p$  which is Lascar-invariant over  $\emptyset$ . Then, for any  $a \neq b$ ,  $ab \equiv^{\text{Ls}} ba$ , and thus,  $R(x, a, b) \leftrightarrow R(x, b, a) \in p(x)$ . Hence,  $p$  is inconsistent modulo  $T_{\mathbf{H}_4\text{-free}}$ , a contradiction.  $\square$

**Remark 4.1.12.** It follows that there cannot exist any canonical independence relation on the finite subsets of the countable homogeneous  $\mathbf{H}_4$ -free 3-hypertournament in the sense of Kaplan and Simon [62], since they all satisfy stationarity over  $\emptyset$ , and thus give rise to global invariant types. This contrasts with the situation for the countable homogeneous tournaments (see [62, Example 4.4]). The question of whether a canonical independence relation exists *over models* remains open and is related to our Question 4.4.19.

**Proposition 4.1.13.**  $T_{\mathbf{H}_4\text{-free}}$  is locally  $\downarrow^i$ -extensible.

*Proof.* By Remark 4.1.7, it suffices to show local  $\downarrow^i$ -extensibility for 1-types. Let  $p(x) \in S(A)$  with  $A \neq \emptyset$ . Fix a singleton  $a^* \in A$ . Let  $q(x)$  be the global expansion of  $p(x)$  obtained by adding the following formulas:

- (i)  $R(x, a, b)$  for all  $a \in A$  and  $b \notin A$ .
- (ii)  $R(x, b_1, b_2) \leftrightarrow R(a^*, b_1, b_2)$  for all  $b_1, b_2 \notin A$ .

We claim that this is consistent. Indeed, since  $\Sigma$  extends  $p$ , any potential  $\mathbf{H}_4$ -structure must contain elements outside of  $A$ , and by (i),  $q$  implies that any set of

the form  $\{x, a, a', b\}$  or  $\{x, a, b, b'\}$  with  $a, a' \in A$  and  $b, b' \notin A$  is  $\mathbf{H}_4$ -free. Also, by (ii),  $q$  implies that 4-point sets of the form  $\{x, b, b', b''\}$  with  $b, b', b'' \notin A$  are  $\mathbf{H}_4$ -free. Therefore,  $q$  is a global type.

Furthermore, suppose that  $b_1 b_2 \equiv_A b'_1 b'_2$ . Since  $a^* \in A$ , this means that  $\models R(a^*, b_1, b_2) \leftrightarrow R(a^*, b'_1, b'_2)$ , and thus, by (ii),  $R(x, b_1, b_2) \leftrightarrow R(x, b'_1, b'_2) \in q$ . Therefore,  $q$  is a global  $A$ -invariant extension of  $p$ , and so, by the argument from Proposition 4.1.11,  $q$  is Lascar-invariant over  $A$ .  $\square$

**Corollary 4.1.14.** *In  $T_{\mathbf{H}_4\text{-free}}$ ,  $\bigcup^i$  satisfies left extension over non-empty bases.*

*Proof.* This is an immediate application of [29, Lemma 3.23].  $\square$

## 4.2 First proof of NSOP<sub>4</sub>

In this section, we prove the main classification results for  $T_{\mathbf{H}_4\text{-free}}$  using concrete constructions instead of the more abstract machinery that will be used in upcoming sections. In what follows, we work in a monster model  $\mathbb{M} \models T_{\mathbf{H}_4\text{-free}}$ , and as usual, assume that all sets and tuples are in  $\mathbb{M}$  and all models elementarily embed in  $\mathbb{M}$ . Recall that all the relevant definitions of the classification properties in here were introduced in §2.3.

**Proposition 4.2.1.**  *$T_{\mathbf{H}_4\text{-free}}$  is NFOP<sub>3</sub> and IP<sub>2</sub>.*

*Proof.* NFOP<sub>3</sub> follows directly from [1, Proposition 3.23].

We claim that the formula  $R(x_1, x_2; y)$  has IP<sub>2</sub>. Fix a total order  $<^*$  on  $\mathcal{P}(\omega^2)$ . Let  $\Sigma(x_{0,i}, x_{1,j}, y_I)_{i,j \in \omega, I \subseteq \omega^2}$  be the following set of formulas:

- (i)  $R(x_{0,i}, x_{1,j}, y_I)$  iff  $(i, j) \in I$ .
- (ii)  $R(x_{n,i}, x_{n,j}, x_{n,k})$  iff  $i < j < k$ , for any  $n \in \{0, 1\}$ .
- (iii)  $R(x_{n,i}, x_{n,j}, x_{n',l})$  iff  $i < j$ , for any  $l$  and  $\{n, n'\} = \{0, 1\}$ .
- (iv)  $R(x_{n,i}, x_{n,j}, y_I)$  iff  $i < j$ , for any  $I \subseteq \omega^2$  and  $n \in \{0, 1\}$ .
- (v)  $R(x_{n,i}, y_I, y_J)$  iff  $I <^* J$ , for any  $i$  and  $n \in \{0, 1\}$ .
- (vi)  $R(y_I, y_J, y_K)$  iff  $I <^* J <^* K$ .

We claim that this is a complete type. The only thing that needs to be checked is that it does not generate any  $\mathbf{H}_4$ -structures. By (i) and (ii),  $\Sigma$  implies that any 4-point set containing only elements from  $\bar{x}_0$  and  $\bar{x}_1$  is  $\mathbf{H}_4$ -free, and, by (vi), it also

implies that those containing only  $y_I$ 's are  $\mathbf{H}_4$ -free. By (iv),  $\Sigma \vdash R(x_{n,i}, x_{n,j}, y_I) \leftrightarrow R(x_{n,i}, x_{n,j}, y_J)$  for all  $I, J \subseteq \omega^2$ , so that  $\{x_{n,i}, x_{n,j}, y_I, y_J\}$  is  $\mathbf{H}_4$ -free, and by (v),  $\Sigma \vdash R(x_{n,i}, y_I, y_J) \leftrightarrow R(x_{n',j}, y_I, y_J)$  for  $\{n, n'\} = \{0, 1\}$ , so that  $\{x_{n,i}, x_{n',j}, y_I, y_J\}$  is  $\mathbf{H}_4$ -free. Moreover, also by (iv),  $\Sigma$  implies that  $\{x_{n,i}, x_{n,j}, x_{n',l}, y_I\}$  is  $\mathbf{H}_4$ -free. Finally, since  $<^*$  is a total order, by (v)  $\Sigma$  implies that  $\{x_{n,i}, y_I, y_J, y_K\}$  is  $\mathbf{H}_4$ -free.

Since these are all possible  $\mathbf{H}_4$ -structures, it follows that  $\Sigma$  is consistent, and thus, we can choose a realisation  $(a_{0,i}, a_{1,j}, b_I)_{i,j \in \omega, I \subseteq \omega^2} \models \Sigma$ . By (i), this realisation witnesses  $\text{IP}_2$  for the formula  $R(x_1, x_2, y)$ .  $\square$

**Proposition 4.2.2.**  $T_{\mathbf{H}_4\text{-free}}$  is  $\text{SOP}_3$ .

*Proof.* Fix some elements  $a, b$  from the monster model. We prove that the formula  $\varphi(x, y) := R(x, y, a) \wedge R(x, b, y)$  witnesses  $\text{SOP}_3$ . First, assume  $\varphi(x_0, x_1) \wedge \varphi(x_1, x_2) \wedge \varphi(x_2, x_0)$  is consistent. Let  $(c_0, c_1, c_2)$  be a realisation. Then the substructure on  $\{c_0, c_1, c_2, a\}$  implies that  $\models R(c_0, c_1, c_2)$ , since otherwise we obtain an  $\mathbf{H}_4$ -structure. Similarly, the substructure on  $\{c_0, c_1, c_2, b\}$  implies that  $\models R(c_0, c_2, c_1)$ , so we get a contradiction. Hence,  $\varphi(x_0, x_1) \wedge \varphi(x_1, x_2) \wedge \varphi(x_2, x_0)$  is inconsistent.

Now let  $\Sigma(x_i)_{i \in \omega}$  be the set containing the following formulas over  $ab$ :

- (i)  $R(x_i, x_j, a) \wedge R(x_i, b, x_j)$  for all  $i < j$ .
- (ii)  $R(x_i, x_j, x_k)$  iff  $i < j < k$ .
- (iii)  $R(x_i, b, a)$  for all  $i$ .

By (ii),  $\Sigma$  implies that any 4-point set containing only  $x_i$ 's is  $\mathbf{H}_4$ -free, and by (i), so is any 4-point set of the form  $\{x_i, x_j, x_k, a\}$  or  $\{x_i, x_j, x_k, b\}$ . Thus, any potential  $\mathbf{H}_4$ -structure must be on a set of the form  $\{x_i, x_j, a, b\}$ . But  $\Sigma$  implies that this is  $\mathbf{H}_4$ -free by (iii).

Therefore,  $\Sigma(x_i)_{i \in \omega}$  is consistent, and so we can choose a realisation  $(c_i)_{i \in \omega} \models \Sigma$ . By stipulation,  $\models \varphi(c_i, c_j)$  for all  $i < j$ .  $\square$

**Proposition 4.2.3.**  $T_{\mathbf{H}_4\text{-free}}$  is  $\text{TP}_2$ .

*Proof of Proposition 4.2.3 (v.1).* For convenience, let us expand  $\mathcal{L}$  by adding two constant symbols  $e$  and  $f$ . Let  $\bar{b} = (c, d)$  be a pair such that

$$\models R(c, d, e) \wedge R(d, e, f) \wedge R(c, e, f) \wedge R(c, f, d).$$

Now let  $\varphi(x, \bar{b})$  be the following  $\mathcal{L}(e, f)$ -formula:

$$R(x, c, d) \wedge R(x, d, e) \wedge R(x, e, f) \wedge R(x, c, f) \wedge R(x, e, c) \wedge R(x, f, d).$$

It is easy to see from the definition that, for any  $a \models \varphi(x, \bar{b})$ , the set  $\{a, c, d, e, f\}$  is  $\mathbf{H}_4$ -free, and therefore,  $\varphi(x, \bar{b})$  isolates a complete type in  $\mathcal{L}(e, f)$ .

Now let  $\Sigma(\bar{v}_i)_{i \in \omega}$  with  $\bar{v}_i := (x_i, y_i)$  be the following set of formulas:

- (i)  $\bar{v}_i \equiv_{ef} \bar{b}$  for all  $i$ .
- (ii)  $R(e, y_j, x_i) \wedge R(f, x_i, y_j)$  iff  $i < j$ .
- (iii)  $R(e, x_i, x_j) \wedge R(f, x_i, x_j) \wedge R(e, y_i, y_j) \wedge R(f, y_i, y_j)$  iff  $i < j$ .
- (iv)  $R(x_i, x_j, x_k) \wedge R(y_i, y_j, y_k)$  iff  $i < j < k$ .
- (v)  $R(x_i, x_j, y_n)$  iff  $i < j$  and  $R(x_m, y_k, y_l)$  iff  $k < l$ , for all  $m$  and  $n$ .

We claim that  $\Sigma(\bar{v}_i)_{i \in \omega}$  is consistent. Indeed, by (i), it suffices to consider those 4-sets containing elements from different  $\bar{v}_i$ 's. It is straightforward using (iv) and (v) that 4-sets containing only  $x_i$ 's and  $y_j$ 's are  $\mathbf{H}_4$ -free. Similarly, by (ii) and (iii), 4-sets containing only  $x_i$ 's,  $y_j$ 's and exactly one of  $e$  or  $f$  are also  $\mathbf{H}_4$ -free, and similarly for those 4-sets containing both  $e$  and  $f$  and elements from exactly one of  $\bar{x}$  or  $\bar{y}$ . So we just need to check that sets of the form  $\{e, f, x_i, y_j\}$  are  $\mathbf{H}_4$ -free. But these are  $\mathbf{H}_4$ -free by (i). Note that a realisation will give an  $ef$ -indiscernible sequence.

Hence,  $\Sigma(\bar{v}_i)_{i \in \omega}$  defines a type in  $\mathcal{L}(e, f)$ , so we can choose a realisation  $\bar{b}_i := (c_i, d_i)_{i \in \omega} \models \Sigma$ . Then, by (ii), the substructure on  $\{x, d_j, e, c_i\}$  implies that  $\varphi(x, \bar{b}_i) \wedge \varphi(x, \bar{b}_j) \vdash R(x, d_j, c_i)$  for  $i < j$ , since otherwise we obtain an  $\mathbf{H}_4$ -structure. Similarly, the substructure on  $\{x, c_i, f, d_j\}$  implies that  $\varphi(x, \bar{b}_i) \wedge \varphi(x, \bar{b}_j) \vdash R(x, c_i, d_j)$  for  $i < j$ . Therefore, the set  $\{\varphi(x, \bar{b}_i) : i \in \omega\}$  is 2-inconsistent.

To make this into an instance of TP<sub>2</sub>, we just take copies of this indiscernible sequence. Formally, we choose an array  $(\bar{b}_{i,j})_{i,j \in \omega} = (c_{i,j}, d_{i,j})_{i,j \in \omega}$  such that:

- (i') For all  $i \in \omega$ ,  $(\bar{b}_{i,j})_{j \in \omega} \equiv_{ef} (\bar{b}_j)_{j \in \omega}$ .
- (ii') For any  $i \neq j$ ,  $l$  and  $m$ , we have

$$\models R(z, c_{j,m}, d_{j,m}) \wedge R(z, c_{j,m}, e) \wedge R(z, c_{j,m}, f) \wedge R(z, d_{j,m}, e) \wedge R(z, d_{j,m}, f)$$

for any  $z \in \bar{b}_{i,l}$ .

- (iii') For any  $i < j < k$ , any  $l, m, n$ , and  $z_\ell \in \bar{b}_{\ell,u}$  for  $\ell \in \{i, j, k\}$  and  $u \in \{l, m, n\}$ , we have  $\models R(z_i, z_j, z_k)$ .

It is then easy to see that (i') implies that  $\{\varphi(x, \bar{b}_{i,j}) : j \in \omega\}$  is 2-inconsistent for each  $i \in \omega$  by the argument from the previous paragraph, and that (ii') and (iii') jointly imply that  $\{\varphi(x, \bar{b}_{i,f(i)}) : i \in \omega\}$  is consistent for any  $f : \omega \rightarrow \omega$ .  $\square$

**Remark 4.2.4.** Let us note, using the choice of elements from the previous proof, that the inconsistency of  $\varphi(x, \bar{b}_0) \wedge \varphi(x, \bar{b}_1)$  implies that, for any  $a \models \varphi(x, \bar{b}_0)$ , we cannot find any  $a' \equiv_{ef\bar{b}_0} a$  such that  $\bar{b}_1 \equiv_{ef a'} \bar{b}_0$ . This shows that we cannot extend Claim 1 from the proof of Lemma 4.1.8 to tuples of finite length greater than 1.

**Proposition 4.2.5.**  $T_{\mathbf{H}_4\text{-free}}$  is NSOP<sub>4</sub>.

*Proof of Proposition 4.2.5 (v.1).* It suffices to show that, for any indiscernible sequence  $(\bar{b}_i)_{i < \omega}$  with common intersection  $\bar{c}$  (possibly empty), and letting  $p(\bar{x}, \bar{y}) := \text{tp}(\bar{b}_0 \bar{b}_1)$ ,  $p(\bar{x}_0, \bar{x}_1) \wedge p(\bar{x}_1, \bar{x}_2) \wedge p(\bar{x}_2, \bar{x}_3) \wedge p(\bar{x}_3, \bar{x}_0)$  is consistent. By indiscernibility, we can equivalently take  $(\bar{b}_i)_{i < \omega}$  to be  $\bar{c}$ -indiscernible and consist only of those elements outside of  $\bar{c}$  from each tuple.

So let  $(\bar{b}_i)_{i < \omega}$  be a  $\bar{c}$ -indiscernible sequence as above, and let  $p(\bar{x}, \bar{y}) := \text{tp}(\bar{b}_0 \bar{b}_1 / \bar{c})$ . Let  $n := |\bar{b}_0|$ . Notice that  $p$  is of the following form:

$$p(\bar{x}, \bar{y}) = \text{tp}(\bar{x} / \bar{c}) \wedge \text{tp}(\bar{y} / \bar{c}) \wedge \text{tp}(\bar{x} \bar{y}) \wedge \bigwedge_{j,k < n} R(c_i, x_j, y_k)^{\varepsilon(i,j,k)},$$

where  $\varepsilon(i, j, k) \in \{0, 1\}$ . (Note that  $j, k$  can be equal.)

Define  $\Sigma(\bar{x})$  with  $|\bar{x}| = n$  as the set containing the following formulas:

- (i)  $\bar{b}_0 \bar{b}_1 \equiv_{\bar{c}} \bar{b}_1 \bar{x}$ .
- (ii)  $R(c_i, b_{0,j}, x_k)$  for all  $i$  and  $j, k < n$  (not necessarily distinct).
- (iii)  $R(b_{0,i}, x_j, b_{1,k})$  for all  $i, j, k < n$ .

We claim that this is  $\mathbf{H}_4$ -free. Indeed, by (i), any potential  $\mathbf{H}_4$ -structures involving  $\bar{b}_0, \bar{b}_1$  and  $\bar{x}$  arise from the last big conjunction in  $p(\bar{b}_0, \bar{b}_1)$  and  $p(\bar{b}_1, \bar{x})$ . It also follows from (i) and (iii) that there cannot be an  $\mathbf{H}_4$ -structure which does not contain any element from  $\bar{c}$ . Thus, it suffices to consider possible  $\mathbf{H}_4$ -structures on sets of the form  $\{c_i, b_{0,j}, b_{1,k}, x_l\}$  for some fixed  $i, j, k$  and  $l$ .

- If  $\varepsilon(i, j, k) = \varepsilon(i, k, l) = 1$ , then  $\models R(c_i, b_{0,j}, b_{1,k})$  and  $\Sigma(\bar{x}) \vdash R(c_i, b_{0,j}, x_l)$  by (ii). Therefore,  $\Sigma(\bar{x})$  implies that  $\{c_i, b_{0,j}, b_{1,k}, x_l\}$  is  $\mathbf{H}_4$ -free.
- If  $\varepsilon(i, j, k) = \varepsilon(i, k, l) = 0$ , then  $\models R(b_{0,i}, c_i, b_{1,k})$  and  $\Sigma(\bar{x}) \vdash R(b_{0,i}, x_l, b_{1,k})$  by (iii). Hence, once again,  $\Sigma(\bar{x})$  implies that  $\{c_i, b_{0,j}, b_{1,k}, x_l\}$  is  $\mathbf{H}_4$ -free.

- If  $\varepsilon(i, j, k) = 1$  but  $\varepsilon(i, k, l) = 0$ , then we have  $\models R(c_i, b_{0,j}, b_{1,k})$  and  $\Sigma(\bar{x}) \vdash R(c_i, x_l, b_{1,k})$  by (i), and so  $\Sigma(\bar{x})$  implies that  $\{c_i, b_{0,j}, b_{1,k}, x_l\}$  is  $\mathbf{H}_4$ -free. A similar argument works if  $\varepsilon(i, j, k) = 0$  but  $\varepsilon(i, k, l) = 1$ .

So  $\Sigma(\bar{x})$  defines a partial type over  $\bar{b}_0\bar{b}_1\bar{c}$  (note that we do not specify  $\text{tp}(\bar{b}_0\bar{x})$  since it is not needed for our purposes), and so we choose a realisation  $\bar{b}_2^* \models \Sigma$ .

Now define  $\Gamma(\bar{y})$  with  $|\bar{y}| = n$  as the set containing the following formulas:

- (i')  $\bar{y}\bar{b}_0 \equiv_{\bar{c}} \bar{b}_2^*\bar{y} \equiv_{\bar{c}} \bar{b}_0\bar{b}_1$ .
- (ii')  $R(b_{0,i}, b_{2,j}^*, y_k)$  for all  $i, j, k < n$ .

Once again, by the two stipulations, it suffices to show that no set of the form  $\{c_i, b_{2,j}^*, y_k, b_{0,l}\}$  gives rise to an  $\mathbf{H}_4$ -structure for some fixed  $i, j, k$  and  $l$ .

- If  $\varepsilon(i, j, k) = \varepsilon(i, k, l) = 1$ , then  $\Gamma(\bar{y}) \vdash R(c_i, b_{2,j}^*, y_k) \wedge R(b_{0,l}, b_{2,j}^*, y_k)$ , and thus  $\Gamma(\bar{y})$  implies  $\{c_i, b_{2,j}^*, y_k, b_{0,l}\}$  is  $\mathbf{H}_4$ -free.
- If  $\varepsilon(i, j, k) = \varepsilon(i, k, l) = 0$ , then  $\models R(c_i, b_{0,l}, b_{2,j}^*)$  and  $\Gamma(\bar{y}) \vdash R(c_i, b_{0,l}, y_k)$ . Hence, once again,  $\Gamma(\bar{y})$  implies that  $\{c_i, b_{2,j}^*, y_k, b_{0,l}\}$  is  $\mathbf{H}_4$ -free.
- If  $\varepsilon(i, j, k) = 1$  but  $\varepsilon(i, k, l) = 0$ , then  $\Gamma(\bar{y}) \vdash R(c_i, b_{2,j}^*, y_k) \wedge R(c_i, b_{0,l}, y_k)$ , and so  $\Gamma(\bar{y})$  implies that  $\{c_i, b_{2,j}^*, y_k, b_{0,l}\}$  is  $\mathbf{H}_4$ -free. A similar argument works if  $\varepsilon(i, j, k) = 0$  but  $\varepsilon(i, k, l) = 1$ .

Hence, again,  $\Gamma(\bar{y})$  defines a type over  $\bar{b}_0\bar{b}_2^*\bar{c}$ , and so by  $\omega$ -saturation we can choose a realisation  $\bar{b}_3^* \models \Gamma$ . Then  $\models p(\bar{b}_0, \bar{b}_1) \wedge p(\bar{b}_1, \bar{b}_2^*) \wedge p(\bar{b}_2^*, \bar{b}_3^*) \wedge p(\bar{b}_3^*, \bar{b}_0)$ .  $\square$

This completes our first proof of Theorem 4.0.1.

## 4.3 $\leq_K$ -independence

In order to apply Mutchnik's framework from [94] to offer a different proof of some of the above classification results for the  $\mathbf{H}_4$ -free 3-hypertournament, we need to make a slight generalisation of his results. In their current form, the results from that paper apply whenever we have an independence relation satisfying, among other properties, stationarity over models. However, as will become clear in the next section, we want to apply them in the context of a non-stationary independence relation. To achieve this, in this section, we introduce an independence relation having a close relationship with *strong Lascar independence* as defined in [112].

### 4.3.1 Relative Kim's lemma

We start with the following notion from the theory of abstract independence relations introduced by Chernikov and Kaplan in [29, Definition 2.26]. We will generalise this notion to non-global types and study it in more detail in §5.1.

**Definition 4.3.1.** Let  $T$  be a complete theory, and  $\perp$  be an independence relation over subsets of  $\mathbb{M} \models T$ . We say a global type  $q$  is  $\perp$ -**free** over  $M$  if, for all  $B \supset M$  and  $\bar{a} \models q|_B$ , we have  $\bar{a} \perp_M B$ .

**Example 4.3.2.** A global type is  $\perp^i$ -free over  $M$  iff it is  $M$ -invariant (see §2.1).

Recently, Mutchnik in [94] has generalised the fundamental order introduced by Poizat [78] to Kim-dividing, extending the work of Ben Yaacov and Chernikov in [11], who already generalised it to dividing. Our presentation follows that of the fundamental order given by Pillay in [99]. This will be generalised in §5.3.

**Definition 4.3.3.** Let  $T$  be a complete theory,  $M \models T$ , and  $r \in S(M)$ .

- (i) Let  $p$  be a global  $M$ -invariant extension of  $r$ . The **Kim-dividing class** of  $p$ , denoted  $\text{cl}_K(p)$ , is defined to be the set of formulas  $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}(M)$  such that  $\varphi(\bar{x}, \bar{a})$  is consistent for some (equiv., any) realisation  $\bar{a} \models p|_M$  and there is  $(\bar{a}_i)_{i \in \omega}$  with  $\bar{a}_i \models p|_{M\bar{a}_{<i}}$  for all  $i \in \omega$  and  $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$  inconsistent. We may equivalently ask that  $\{\varphi(\bar{x}, \bar{a}_i) : i \in \omega\}$  is inconsistent for all  $(\bar{a}_i)_{i \in \omega}$  as above.
- (ii) Given global  $M$ -invariant extensions  $p, q$  of  $r$ , we write  $p \leq_K q$  if  $\text{cl}_K(p) \subseteq \text{cl}_K(q)$ . We say  $p$  is **least in the Kim-dividing order**, or  $\leq_K$ -**least**, if  $p \leq_K q$  for all global  $M$ -invariant extensions  $q$  of  $r$ .

Kim's Lemma was originally proven in [64, Proposition 2.1] and constitutes a key result in the study of simple theories. Due to its central role, many authors have proven variants of this original result when extending tools and techniques from the simple context to larger ones, such as NTP<sub>2</sub> [29, Lemma 3.14], NSOP<sub>1</sub> [59, Theorem 3.16], NBTP [76, Theorem 5.2], NCTP and NATP theories [50]. To find such variants in the study of strictly NSOP<sub>4</sub> theories, in [94] Mutchnik relativises Kim's lemma to a choice of independence relation:

**Definition 4.3.4.** We say an independence relation  $\perp$  defined over models satisfies the **relative Kim's lemma** if, for all global types  $q$  and models  $M \models T$ , if  $q$  is  $\perp$ -free over  $M$ , then  $q$  is a  $\leq_K$ -least extension of  $q|_M$ .

**Remark 4.3.5.** Note that, by the definition of the Kim-dividing order, if  $\perp$  satisfies the relative Kim's lemma, then every global type that is  $\perp$ -free over  $M$  must be  $M$ -invariant.

**Example 4.3.6.** Another way of restating Kaplan and Ramsey's version of Kim's lemma for NSOP<sub>1</sub> theories [59, Theorem 3.16] is the following (and hence the terminology): a theory  $T$  is NSOP<sub>1</sub> iff  $\perp^i$  satisfies the relative Kim's lemma.

We will significantly generalise this framework in chapter 5.

**Remark 4.3.7.** In the special case where  $\perp$  satisfies stationarity over models, some of the previous definitions may be simplified. This is due to the following easy observations (using the properties defined in §2.4):

- If  $\perp$  satisfies full existence and stationarity over models, then every type  $p \in S(M)$  has a unique global extension that is  $\perp$ -free over  $M$ . Such an extension is additionally  $M$ -invariant.
- If  $\perp$  satisfies stationarity over models, then every  $\perp$ -independent sequence over  $M$  is  $M$ -indiscernible, and hence  $\perp$ -Morley over  $M$ .

In the context of Mutchnik's work on relative Kim-independence (see [94]), the relevant independence relation satisfies stationarity over models by assumption, and hence the above remarks apply. As we will see in §4.4, we want to use this machinery in the context of a non-stationary independence relation. The observations in this remark will not be used later in this chapter; they serve as a point of comparison between these two approaches.

### 4.3.2 Defining $\leq_K$ -independence

In this section, we work in a general complete theory  $T$  with monster model  $\mathbb{M}$ .

**Definition 4.3.8.** For  $M \models T$ , we define  $\bar{a} \perp_M^{\leq_K} \bar{b}$  if  $\text{tp}(\bar{a}/M\bar{b})$  extends to a  $\leq_K$ -least extension of  $\text{tp}(\bar{a}/M)$ .

**Lemma 4.3.9.**  $\perp^{\leq_K}$  is an independence relation (in the sense of Definition 2.4.1).

*Proof.* Simply note that  $q$  is a  $\leq_K$ -least extension of  $q|_M$  iff  $\sigma(q)$  is a  $\leq_K$ -least extension of  $\sigma(q)|_{\sigma(M)}$  for any  $\sigma \in \text{Aut}(\mathbb{M})$ , using the fact that the types involved are all invariant over the appropriate models of  $T$ .  $\square$

**Remark 4.3.10.** It is immediate from the definition that  $\leq_K$ -independence satisfies right monotonicity and right extension. In particular, it has existence iff it has full existence. Moreover,  $M$ - $\perp^{\leq_K}$ -free types are precisely the global types which are  $\leq_K$ -least over their restrictions to  $M$ .

**Lemma 4.3.11.** *If  $\perp^i$  satisfies left extension over models, then  $\perp^{\leq_K}$  satisfies left monotonicity.*

*Proof.* Given a type  $p(\bar{x}, \bar{y}) \in S(M)$  and a global  $M$ -invariant extension  $q(\bar{x}, \bar{y})$ , let us write  $\tilde{p}(\bar{y})$  and  $\tilde{q}(\bar{y})$  for their respective restrictions to complete types in the variables  $\bar{y}$ . It suffices to show that, if  $q(\bar{x}, \bar{y})$  is a  $\leq_K$ -least extension of  $p(\bar{x}, \bar{y}) \in S(M)$ , then  $\tilde{q}(\bar{y})$  is a  $\leq_K$ -least extension of  $\tilde{p}(\bar{y})$ .

Suppose  $\varphi(\bar{z}, \bar{y}) \in \text{cl}_K(\tilde{q}(\bar{y}))$ . Then  $\varphi(\bar{z}, \bar{y}) \in \text{cl}_K(q(\bar{x}, \bar{y}))$ , and so, since  $q(\bar{x}, \bar{y})$  is a  $\leq_K$ -least extension of  $p(\bar{x}, \bar{y})$ , it follows that  $\varphi(\bar{z}, \bar{y}) \in \text{cl}_K(r(\bar{x}, \bar{y}))$  for all global  $M$ -invariant extensions  $r(\bar{x}, \bar{y})$  of  $p(\bar{x}, \bar{y})$ . Now, for any global  $M$ -invariant extension  $\tilde{r}(\bar{y})$  of  $\tilde{p}(\bar{y})$ , we can find, by left extension for  $\perp^i$ , a global  $M$ -invariant completion of  $p(\bar{x}, \bar{y}) \cup \tilde{r}(\bar{y})$ , and thus it follows that  $\varphi(\bar{z}, \bar{y}) \in \text{cl}_K(\tilde{r}(\bar{y}))$ . So  $\tilde{q}(\bar{y})$  is a  $\leq_K$ -least extension of  $\tilde{p}(\bar{y})$ .  $\square$

Beyond satisfying the relative Kim's lemma,  $\leq_K$ -independence also provides a criterion for this property:

**Proposition 4.3.12.** *Suppose  $\perp$  satisfies full existence and right extension. Then  $\perp$  satisfies the relative Kim's lemma iff  $\perp \implies \perp^{\leq_K}$ .*

*Proof.*  $(\implies)$  Suppose  $\bar{a} \perp_M \bar{b}$ . By right extension,  $\text{tp}(\bar{a}/M\bar{b})$  extends to a global  $M$ - $\perp$ -free type  $q$  which, by the relative Kim's lemma, is  $\leq_K$ -least. So  $\bar{a} \perp_M^{\leq_K} \bar{b}$ .

$(\impliedby)$  Suppose  $q$  is a global type that is  $\perp$ -free over  $M \models T$ , which exists by full existence. Since  $\perp \implies \perp^{\leq_K}$ , it follows that  $q$  is also  $\perp^{\leq_K}$ -free over  $M$ , and thus, by Remark 4.3.10,  $q$  is a  $\leq_K$ -least extension of  $q|_M$ .  $\square$

### 4.3.3 A criterion for the strong witnessing property

The following property was introduced by Mutchnik in [94, Definition 3.23]:

**Definition 4.3.13.** Let  $T$  be a complete theory.

- (i) We say a global type  $q$  is a **strong witnessing extension** of  $q|_M$  for  $M \models T$  if, for all  $\bar{b}$  in  $\mathbb{M}$ , all  $\bar{a} \models q|_{M\bar{b}}$ , and all  $\bar{c}$ , there is some  $\bar{c}' \equiv_{M\bar{a}} \bar{c}$  such that  $\text{tp}(\bar{a}\bar{c}'/M\bar{b})$  extends to a  $\leq_K$ -least extension of  $\text{tp}(\bar{a}\bar{c}/M)$ .
- (ii) We say  $T$  has the **strong witnessing property** if, for all small  $M \models T$ , every  $p \in S(M)$  has a global strong witnessing extension.

There are no known NSOP<sub>4</sub> counterexamples to the above property. The main application is given in [94, Theorem 3.25], where it is shown that any theory with the strong witnessing property is either simple or TP<sub>2</sub>.

**Lemma 4.3.14.** *Let  $M \models T$ , and let  $q$  be a global type. The following are equivalent:*

- (i)  $q$  is a strong witnessing extension of  $q|_M$ .
- (ii)  $q$  is  $(\perp^{\leq_K})^{\text{le}}$ -free over  $M$ .

*Proof.* ((i)  $\Rightarrow$  (ii)) Suppose  $q$  is a strong witnessing extension of  $q|_M$ . Let  $\bar{a} \models q|_{M\bar{b}}$ . Take some  $\bar{c}$ . Then, by strong witnessing, there is  $\bar{c}' \equiv_{M\bar{a}} \bar{c}$  such that  $\bar{a}\bar{c}' \perp_M^{\leq_K} \bar{b}$ , and thus, by invariance, there is  $\bar{b}' \equiv_{M\bar{a}} \bar{b}$  such that  $\bar{a}\bar{c} \perp_M^{\leq_K} \bar{b}'$ . Therefore, we have  $\bar{a} (\perp_M^{\leq_K})^{\text{le}} \bar{b}$ , as required.

((ii)  $\Rightarrow$  (i)) Suppose  $q$  is  $(\perp^{\leq_K})^{\text{le}}$ -free over  $M$ . Take  $\bar{a} \models q|_{M\bar{b}}$ , and some  $\bar{c}$ . Then, by assumption and invariance, there is  $\bar{c}' \equiv_{M\bar{a}} \bar{c}$  such that  $\bar{a}\bar{c}' \perp_M^{\leq_K} \bar{b}$ . Therefore,  $q$  is a strong witnessing extension of  $q|_M$ , as required.  $\square$

**Theorem 4.3.15.** (i) *If there is an independence relation  $\perp$  over models of  $T$  satisfying full existence, left and right extension, and the relative Kim's lemma, then  $T$  satisfies the strong witnessing property.*

- (ii) *If  $\perp^{\text{i}}$  satisfies left extension over models and  $T$  satisfies the strong witnessing property, then there is an independence relation  $\perp$  over models of  $T$  satisfying full existence, left extension, and the relative Kim's lemma.*

*Proof.* (i) Suppose that such an independence relation  $\perp$  exists. By right extension and Proposition 4.3.12, we have  $\perp \Rightarrow \perp^{\leq_K}$ . Since  $\perp$  satisfies left extension, it follows from Corollary 2.4.19(ii) that  $\perp \Rightarrow (\perp^{\leq_K})^{\text{le}}$ . Hence, by full existence, every type over a model  $M \models T$  has a global  $M$ -( $\perp^{\leq_K}$ )<sup>le</sup>-free extension. Thus, by Lemma 4.3.14,  $T$  satisfies the strong witnessing property.

(ii) Suppose that  $T$  satisfies the strong witnessing property. In particular, by Lemma 4.3.14, this entails that every type  $p \in S(M)$  has a global  $M$ -( $\perp^{\leq_K}$ )<sup>le</sup>-free

extension, which implies that  $(\perp^{\leq \kappa})^{\text{le}}$  satisfies full existence. Thus, by Proposition 4.3.12,  $(\perp^{\leq \kappa})^{\text{le}}$  satisfies the relative Kim's lemma. Finally, as  $\perp^{\text{i}}$  satisfies left extension by assumption,  $\perp^{\leq \kappa}$  satisfies left monotonicity by Lemma 4.3.11, and thus  $(\perp^{\leq \kappa})^{\text{le}}$  satisfies left extension by Corollary 2.4.19(iv).  $\square$

**Corollary 4.3.16.** *Suppose there is an independence relation  $\perp$  over models satisfying full existence, left and right extension, and the relative Kim's lemma. Then  $T$  is either simple or  $\text{TP}_2$ .*

*Proof.* This follows from Theorem 4.3.15 and [94, Theorem 3.25].  $\square$

## 4.4 Second proof of $\text{NSOP}_4$

### 4.4.1 Triviality of Conant-independence

We now return to the  $\mathbf{H}_4$ -free 3-hypertournament and provide a new, abstract proof of some of the classification properties from §4.2. Our second proof of Theorem 4.0.1 goes via properties of independence relations, using several results from [94]. First, we recall the key result we employ in this section:

**Definition 4.4.1.** Let  $T$  be a complete theory and  $M \models T$ .

- (i) We say  $\varphi(\bar{x}, \bar{b})$  **Conant-divides over**  $M$  if, for all  $M$ -invariant Morley sequences  $(\bar{b}_i)_{i \in \omega}$  with  $\bar{b}_0 = \bar{b}$ , the set  $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$  is inconsistent.
- (ii) We say  $\varphi(\bar{x}, \bar{b})$  **Conant-forks over**  $M$  if there exist formulas  $\psi_i(\bar{x}, \bar{c}_i)$  for  $i < n$  such that  $\varphi(\bar{x}, \bar{b}) \vdash \bigvee_{i < n} \psi_i(\bar{x}, \bar{c}_i)$  and each  $\psi_i(\bar{x}, \bar{c}_i)$  Conant-divides over  $M$ .
- (iii) We say  $\bar{a}$  is **Conant-independent** from  $\bar{b}$  over  $M$  if  $\text{tp}(\bar{a}/M\bar{b})$  does not contain a formula that Conant-forks over  $M$ .

**Remark 4.4.2.** In  $T_{\mathbf{H}_4\text{-free}}$ , since  $\perp^{\text{i}}$  satisfies left extension over models by Corollary 4.1.14, we have that  $\varphi(\bar{x}, \bar{b})$  Conant-forks over  $M$  iff it Conant-divides over  $M$ . The proof of this fact will be given, in a more general setting, in Lemma 5.2.33.

**Fact 4.4.3** ([94, Theorem 6.2]). *If Conant-independence is symmetric, then  $T$  is  $\text{NSOP}_4$ .*

Now work in  $\mathbb{M} \models T_{\mathbf{H}_4\text{-free}}$ . Let us define an independence relation  $\perp^{\text{ht}}$  over

subsets of  $\mathbb{M}$  as follows:  $A \downarrow_C^{\text{ht}} B$  iff  $A \cap B \subseteq C$  and  $\models R(a, c, b)$  for all  $a \in A \setminus C$ ,  $c \in C$ , and  $b \in B \setminus C$ .

**Remark 4.4.4.** For any sets  $A, B, C, D$ , if  $A \downarrow_D^{\text{ht}} B$  and  $A \downarrow_D^{\text{ht}} C$ , then  $A \downarrow_D^{\text{ht}} BC$ .

**Lemma 4.4.5.**  $\downarrow^{\text{ht}}$  is an independence relation satisfying existence, monotonicity, left and right extension, normality, transitivity, quasi-strong finite character, the Independence Theorem over Models, and freedom.

*Proof.* Checking that  $\downarrow^{\text{ht}}$  is an independence relation, and that it satisfies existence, monotonicity, and normality is immediate.

(Right extension) Suppose that  $A \downarrow_C^{\text{ht}} B$  and  $B' \supseteq B$ . Without loss of generality, we may assume  $AB' \cap C = \emptyset$ . Fix enumerations  $\bar{a}, \bar{b}$  of  $A, B$ , respectively, and let  $p(\bar{x}, \bar{b}) := \text{tp}(\bar{a}/C\bar{b})$ . Fix some  $c^* \in C$ . Let us define the type  $\Sigma(\bar{x})$  over  $CB'$  as the union of the following formulas:

- (i)  $p(\bar{x}, \bar{b})$ .
- (ii)  $R(x_i, b, b')$  for all  $b \in BC$  and  $b' \in B' \setminus BC$ .
- (iii)  $R(x_i, x_j, b')$  whenever  $i < j$ .
- (iv)  $R(x_i, b', b'') \leftrightarrow R(c^*, b', b'')$  for all  $b', b'' \in B' \setminus BC$ .

Just as we have done previously in this chapter, it suffices to show that  $\Sigma$  does not imply the existence of any  $\mathbf{H}_4$ -structures. By (i),  $\Sigma$  implies that any sets only containing variables from  $\bar{x}$  and elements from  $BC$  are  $\mathbf{H}_4$ -free. By (ii), (i) and  $A \downarrow_C^{\text{ht}} B$ ,  $\Sigma$  implies that any sets containing at most two variables from  $\bar{x}$  and elements from  $BC$  and  $B'$  are  $\mathbf{H}_4$ -free. By (iii), sets of the form  $\{x_i, x_j, x_k, b'\}$  with  $b' \in B' \setminus BC$  are also consistent according to  $\Sigma$ , and by (iv), so are those of the form  $\{x_i, b', b'', b'''\}$  whenever  $b', b'', b''' \in B' \setminus BC$ . This covers all the possibilities.

Let  $\bar{a}' \models \Sigma(\bar{x})$ . By (i),  $\bar{a}' \equiv_{BC} \bar{a}$ , and by (i) and (ii) it follows that  $\bar{a}' \downarrow_C^{\text{ht}} B'$ . This completes the proof.

(Left extension) This is analogous to the proof for right extension.

(Transitivity) We show right transitivity; the proof of left transitivity is analogous. So suppose  $C \subseteq D \subseteq B$ ,  $A \downarrow_C^{\text{ht}} D$  and  $A \downarrow_D^{\text{ht}} B$ . First, observe that  $A \cap D \subseteq C$  and  $A \cap B \subseteq D$ , and so  $A \cap B = A \cap (A \cap B) \subseteq A \cap D \subseteq C$ . Now let  $a \in A \setminus C$ ,  $b \in B \setminus C$ , and  $c \in C$ . If  $a \in D$ , then  $a \in A \cap D$  and thus  $a \in C$ , which is a contradiction. So  $a \in A \setminus D$ . Now we have two cases. If  $b \in B \setminus D$ , then by  $A \downarrow_D^{\text{ht}} B$  we have  $\models R(a, c, b)$ . If  $b \notin B \setminus D$ , then  $b \in D \setminus C$ , and so by  $A \downarrow_C^{\text{ht}} D$  we have  $\models R(a, c, b)$ . Therefore,  $A \downarrow_C^{\text{ht}} B$ .

(Quasi-strong finite character) Clearly, it is enough to show this for the case where  $\bar{a}\bar{b} \cap C = \emptyset$ . For complete types  $p(\bar{x}), q(\bar{y}) \in S(C)$ , let

$$\Sigma_{p,q}(\bar{x}, \bar{y}) := p(\bar{x}) \cup q(\bar{y}) \cup \{x_i \neq y_j : i, j\} \cup \{R(x_i, c, y_j) : c \in C\}.$$

Clearly  $\bar{a}, \bar{b} \models \Sigma_{p,q}(\bar{x}, \bar{y})$  iff  $\models p(\bar{a}) \wedge q(\bar{b})$  and  $\bar{a} \perp_C^{\text{ht}} \bar{b}$ .

(Independence Theorem) Suppose that  $M \models T_{\mathbf{H}_4\text{-free}}$ ,  $A \perp_M^{\text{ht}} B$ ,  $A' \perp_M^{\text{ht}} C$ , and  $B \perp_M^{\text{ht}} C$ . Fix enumerations  $\bar{a}$  and  $\bar{a}'$  of  $A$  and  $A'$  respectively such that  $\text{tp}(\bar{a}/M) = \text{tp}(\bar{a}'/M)$ , which can be done by assumption. We claim that

$$q(\bar{x}) := \text{tp}(\bar{a}/MB) \cup \text{tp}(\bar{a}/MC) \cup \{R(x_i, b, c) : b \in B \setminus M, c \in C \setminus M\}$$

is consistent. As usual, we only need to check that it does not generate any  $\mathbf{H}_4$ -structures. Sets only containing variables and elements from  $M$  are  $\mathbf{H}_4$ -free. Similarly, those containing only elements from  $B$  or only elements from  $C$  are consistent (and well-defined since  $BM \cap CM \subseteq M$ ). Sets only containing elements from  $BM$ , those only containing elements from  $CM$ , or those of the form  $\{x_i, m, b, c\}$  for  $m \in M$ ,  $b \in B \setminus M$  and  $c \in C \setminus M$  are all consistent by  $A \perp_M^{\text{ht}} B$  and  $A' \perp_M^{\text{ht}} C$ . Finally, if we have a set containing only elements from  $BC \setminus M$ , then this is  $\mathbf{H}_4$ -free by the last set in the definition of  $q$ .

Let  $A''$  be the set enumerated by a realisation of  $q(x)$ . Then, by definition,  $A'' \equiv_{MB} A$  and  $A'' \equiv_{MC} A'$ . Finally, by invariance and Remark 4.4.4, it follows that  $A'' \perp_M^{\text{ht}} BC$ , as required.

(Freedom) Suppose that  $A \perp_C^{\text{ht}} B$  and  $AB \cap C \subseteq D \subseteq C$ . Let  $a \in A \setminus D$ ,  $b \in B \setminus D$ , and  $d \in D$ . In particular,  $d \in C$ . Note that, if  $a \in C$ , then  $a \in A \cap C$  and so  $a \in D$ , a contradiction. So  $a \in A \setminus C$ , and by a similar argument,  $b \in B \setminus C$ . Therefore, by  $A \perp_C^{\text{ht}} B$ , we have  $\models R(a, d, b)$ . Hence,  $A \perp_D^{\text{ht}} B$ .  $\square$

**Remark 4.4.6.**  $\perp^{\text{ht}}$  does not satisfy stationarity (over any subset) nor symmetry. In fact, if  $A \perp_C^{\text{ht}} B$  and  $\emptyset \neq C \subset A, B$ , then, for all  $a \in A \setminus C$ ,  $c \in C$ , and  $b \in B \setminus C$ ,  $\models R(a, c, b)$ , and thus,  $\not\models R(b, c, a)$ . Hence,  $B \not\perp_C^{\text{ht}} A$ . It is also easy to construct a counterexample to both left and right base monotonicity.

So  $\perp^{\text{ht}}$  is neither a stationary weak independence relation in the sense of [81] nor a free amalgamation relation in the sense of [34]. It is also different from the independence relations studied in [94].

**Remark 4.4.7.** We can characterise  $\perp^{\text{ht}}$ -Morley sequences over some  $M \models T_{\mathbf{H}_4\text{-free}}$

starting at some  $\bar{b}_0$  disjoint from  $M$  as precisely those  $M$ -indiscernible sequences  $(\bar{b}_i)_{i \in \omega}$  such that  $\models R(m, b_{i,k}, b_{j,l})$  for all  $i < j$ , any indices  $k, l$  and all  $m \in M$ .

We need to introduce here a relativisation of Kim-independence from Definition 3.1.4 to  $\downarrow^{\text{ht}}$ -Morley sequences. This is a particular instance of the much more general relativisation carried out in chapter 5.

**Definition 4.4.8.** Let  $M \models T_{\mathbf{H}_4\text{-free}}$ , and  $\bar{a}, \bar{b}$  be tuples.

- (i) We say a formula  $\varphi(\bar{x}, \bar{b}) \downarrow^{\text{ht}}$ -**Kim-divides** over  $M$  if there is some global extension  $q \supset \text{tp}(\bar{b}/M) \downarrow^{\text{ht}}$ -free over  $M$  and a Morley sequence  $(\bar{b}_i)_{i \in \omega}$  in  $q$  over  $M$  with  $\bar{b}_0 = \bar{b}$  such that the set  $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$  is inconsistent.
- (ii) We say a formula  $\varphi(\bar{x}, \bar{b}) \downarrow^{\text{ht}}$ -**Kim-forks** over  $M$  if there exist formulas  $\psi_i(\bar{x}, \bar{c}_i)$  for  $i < n$  such that  $\varphi(\bar{x}, \bar{b}) \vdash \bigvee_{i < n} \psi_i(\bar{x}, \bar{c}_i)$  and each  $\psi_i(\bar{x}, \bar{c}_i) \downarrow^{\text{ht}}$ -Kim-divides over  $M$ .
- (iii) We say  $\bar{a}$  is  $\downarrow^{\text{ht}}$ -**Kim-independent** from  $\bar{b}$  over  $M$  if  $\text{tp}(\bar{a}/M\bar{b})$  does not contain any formula that  $\downarrow^{\text{ht}}$ -Kim-forks over  $M$ .

**Proposition 4.4.9.** For (possibly infinite) tuples  $\bar{a}$  and  $\bar{b}$ , and a model  $M \models T_{\mathbf{H}_4\text{-free}}$ ,  $\text{tp}(\bar{a}/M\bar{b})$  does not  $\downarrow^{\text{ht}}$ -Kim-divide over  $M$  iff  $\bar{a} \cap \bar{b} \subseteq M$ .

*Proof.* It is enough to prove the right-to-left direction. Suppose  $\bar{a} \cap \bar{b} \subseteq M$ . Note that we may assume that  $\bar{a}$  and  $\bar{b}$  are both finite tuples. Also note that, if either  $\bar{a} \subseteq \bar{b}$  or  $\bar{b} \subseteq \bar{a}$  holds, then there is nothing to do. So, without loss of generality, suppose that  $\bar{a} \cap \bar{b} = \emptyset$  (in particular, they are disjoint from  $M$ ). Let  $(\bar{b}_i)_{i \in \omega}$  be an  $\downarrow^{\text{ht}}$ -Morley sequence over  $M$  starting with  $\bar{b}$ . Let  $\varphi(\bar{x}, \bar{b}) \in \text{tp}(\bar{a}/M\bar{b})$ , which, since  $T_{\mathbf{H}_4\text{-free}}$  is  $\omega$ -categorical, we may assume is a formula isolating a complete type over  $M\bar{b}$  for some finite  $M' \subset M$ . We claim that  $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$  is consistent.

We show that

$$\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\} \cup \{R(x_k, b_{i,l}, b_{j,l'}) : i < j \in \omega\}$$

is consistent. Note that, by the condition on elements from different indexed tuples from the  $\downarrow^{\text{ht}}$ -Morley sequence and the fact that  $\varphi$  isolates a complete type, all 4-substructures containing only variables and elements from the  $\bar{b}_i$ 's are  $\mathbf{H}_4$ -free. Moreover, since  $\varphi$  isolates a type, we have, for any  $i, j \in \omega$ , that  $R(x_k, m, b_{i,l}) \in \varphi(\bar{x}, \bar{b}_i)$  iff  $R(x_k, m, b_{j,l}) \in \varphi(\bar{x}, \bar{b}_j)$ , and thus the set  $\{x_k, m, b_{i,l}, b_{j,l'}\}$  is  $\mathbf{H}_4$ -free. Finally, since  $(\bar{b}_i)_{i \in \omega}$  is an  $\downarrow^{\text{ht}}$ -Morley sequence over  $M$ , we have  $\models R(m, b_{i,l}, b_{j,l'})$  whenever

$i < j$ , and so it follows that  $\{x, m, b_{i,l}, b_{j,l'}\}$  is  $\mathbf{H}_4$ -free. These are all the possible  $\mathbf{H}_4$ -substructures.

Therefore,  $\text{tp}(\bar{a}/M\bar{b})$  does not  $\downarrow^{\text{ht}}$ -Kim-divide over  $M$ , as required.  $\square$

**Lemma 4.4.10.** *Every type  $p \in S(M)$  has a global  $M$ -invariant extension that is  $\downarrow^{\text{ht}}$ -free over  $M$ .*

*Proof.* It suffices to show that  $p$  has a global  $M$ -invariant extension  $q$  such that  $R(x, m, a) \in q$  for all  $m \in M$  and  $a \notin M$ . Let us write  $p(\bar{x}) = \text{tp}(\bar{b}/M)$  for some realisation  $\bar{b}$  in the monster. We may assume, without loss of generality, that  $\bar{b}$  is disjoint from  $M$ . Fix some tuple  $\bar{m}$  from  $M$  of the same length as  $\bar{b}$ . Then it is enough to show that the extension of  $\text{tp}(\bar{b}/M)$  obtained by adding the formulas:

- (i)  $R(x_i, m, a)$  for all  $m \in M$  and  $a \notin M$ ;
- (ii)  $\varphi(\bar{x}, \bar{a})$  whenever  $\bar{a} \cap M = \emptyset$  and  $\models \varphi(\bar{m}, \bar{a})$

is consistent. (Note that  $M$ -invariance is clear.)

As usual, it suffices to check all possible  $\mathbf{H}_4$ -substructures. Note that, by (ii), no 4-point set containing only variables and elements outside of  $M$  can be  $\mathbf{H}_4$ . So the only possibilities are  $\{x_i, x_j, m, a\}$ ,  $\{x_i, m, m', a\}$  and  $\{x_i, m, a, a'\}$  for  $m, m' \in M$  and  $a, a' \notin M$ . But all these are  $\mathbf{H}_4$ -free by (i).  $\square$

*Proof of Proposition 4.2.5 (v.2).* We claim that  $\bar{a}$  is Conant-independent from  $\bar{b}$  over  $M$  iff  $\bar{a} \cap \bar{b} \subseteq M$ . The left-to-right direction is always true. For the converse, suppose that  $\bar{a} \cap \bar{b} \subseteq M$ . By Proposition 4.4.9,  $\text{tp}(\bar{a}/M\bar{b})$  does not  $\downarrow^{\text{ht}}$ -Kim-divide over  $M$ . Moreover, by Lemma 4.4.10, there exists an  $M$ -invariant  $\downarrow^{\text{ht}}$ -Morley sequence  $(\bar{b}_i)_{i \in \omega}$  over  $M$  with  $\bar{b}_0 = \bar{b}$ . So, for all  $\varphi(\bar{x}, \bar{b}) \in \text{tp}(\bar{a}/M\bar{b})$ ,  $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$  is consistent, and thus,  $\varphi(\bar{x}, \bar{b})$  does not Conant-divide over  $M$ . Therefore, by Remark 4.4.2,  $\bar{a}$  is Conant-independent from  $\bar{b}$  over  $M$ .

In particular, it follows that Conant-independence is symmetric. Thus, by Fact 4.4.3,  $T_{\mathbf{H}_4\text{-free}}$  is NSOP<sub>4</sub>.  $\square$

## 4.4.2 Proof of TP<sub>2</sub>

In this subsection, we focus on another one of the notions introduced by Mutchnik, namely, the relative Kim's lemma. In view of Remark 4.3.5, we immediately notice

that  $\downarrow^{\text{ht}}$  cannot satisfy the relative Kim's lemma, as global  $M$ - $\downarrow^{\text{ht}}$ -free types might fail to be  $M$ -invariant.

To fix this, let us define  $\bar{a} \downarrow_M^{\text{hti}} B$  iff  $\bar{a} \downarrow_M^i B$  and  $\bar{a} \downarrow_M^{\text{ht}} B$ .

**Lemma 4.4.11.**  $\downarrow^{\text{hti}}$  is an independence relation satisfying existence over models, monotonicity, left and right extension, normality, left transitivity, and quasi-strong finite character.

*Proof.* That  $\downarrow^{\text{hti}}$  is an independence relation follows from Lemma 2.4.21(i).

(Existence, monotonicity, normality, left transitivity, quasi-strong finite character) These follow by definition from Lemma 2.4.21, using Example 2.4.6 and Lemma 4.4.5.

(Full existence) Let  $\bar{a}$  be a tuple,  $M \models T$ , and  $B$  a set. Let  $p := \text{tp}(\bar{a}/M)$ . By Lemma 4.4.10, there is a global  $M$ -invariant extension  $q$  of  $p$  which is  $\downarrow^{\text{ht}}$ -free over  $M$ . Let  $\bar{a}' \models q|_{MB}$ , so that  $\bar{a}' \equiv_M \bar{a}$ . Then  $\bar{a}' \downarrow_M^i B$  by  $M$ -invariance of  $q$ , and  $\bar{a}' \downarrow_M^{\text{ht}} B$ , since  $q$  is  $\downarrow^{\text{ht}}$ -free over  $M$ . Hence,  $\bar{a}' \downarrow_M^{\text{hti}} B$ .

(Left extension) This proof is analogous to that of Lemma 4.4.10: Suppose that  $\bar{a}$ ,  $M \models T$ , and  $B$  are given such that  $\bar{a} \downarrow_M^{\text{hti}} B$ , and  $\bar{a}'$  is a tuple extending  $\bar{a}$ . We may write  $\bar{a}'$  as the concatenation of  $\bar{a}$  with some other tuple  $\bar{c}$ . Let  $\bar{b}$  be a (possibly infinite) tuple enumerating  $B$ . Fix a tuple  $\bar{m}$  in  $M$  of the same length as  $\bar{c}$ . We now want to show the consistency of the extension of  $\text{tp}(\bar{b}/M\bar{a})$  by the following set of formulas:

- (i)  $R(c_i, m, x_j)$  for all indices  $i, j$  of the appropriate length and  $m \in M$ .
- (ii)  $\varphi(\bar{c}, \bar{x}) \leftrightarrow \varphi(\bar{m}, \bar{x})$  for any  $\mathcal{L}$ -formula  $\varphi$ .

As usual, it suffices to show that this does not generate any  $\mathbf{H}_4$ -structures. In this case, the proof is analogous to that of Lemma 4.4.10, so we omit it. Pick a realisation  $\bar{b}'$  of this type. By (i), we have  $\bar{a}' \downarrow_M^{\text{ht}} \bar{b}'$ , and by (ii),  $\text{tp}(\bar{a}'/M\bar{b}')$  is  $M$ -invariant. Therefore, letting  $B'$  be the set enumerated by  $\bar{b}'$ , it follows that  $B' \equiv_{M\bar{a}} B$  and  $\bar{a}' \downarrow_M^{\text{hti}} B'$ .

(Right extension) This can be proved analogously to left extension. □

We can now also see that the Morley sequences appearing in Lemma 4.4.10 are precisely  $\downarrow^{\text{hti}}$ -Morley sequences. So, from Proposition 4.4.9, we immediately obtain the following:

**Corollary 4.4.12.** *For (possibly infinite) tuples  $\bar{a}$  and  $\bar{b}$ , and a model  $M \models T_{\mathbf{H}_4\text{-free}}$ ,  $\text{tp}(\bar{a}/M\bar{b})$  does not  $\downarrow^{\text{hti}}$ -Kim-divide over  $M$  iff  $\bar{a} \cap \bar{b} \subseteq M$ .*

Our goal now is to prove that  $\downarrow^{\text{hti}}$ -free global types are  $\leq_K$ -least. At this point,  $\omega$ -categoricity is particularly useful:

**Lemma 4.4.13.** *Let  $T$  be  $\omega$ -categorical,  $M \models T$ , and  $q, r$  be global  $M$ -invariant types with  $q|_M = r|_M$ . Assume that, for all formulas  $\varphi(\bar{x}, \bar{y})$  isolating a complete type over some finite subset of  $M$ , if  $\varphi(\bar{x}, \bar{y}) \in \text{cl}_K(q)$ , then  $\varphi(\bar{x}, \bar{y}) \in \text{cl}_K(r)$ . Then  $\text{cl}_K(q) \subseteq \text{cl}_K(r)$ .*

*Proof.* Let  $\psi(\bar{x}, \bar{y}) \in \text{cl}_K(q)$ . Since  $T$  is  $\omega$ -categorical, there are finitely many types over a finite set of parameters and every such type is principal, so we can write  $\psi(\bar{x}, \bar{y}) := \bigvee_{i < l} \varphi_i(\bar{x}, \bar{y})$ , where each  $\varphi_i$  isolates a complete type over the set of parameters in  $\psi$ . Since  $\varphi_i(\bar{x}, \bar{y}) \vdash \psi(\bar{x}, \bar{y})$  for each  $i$ , it follows that  $\varphi_i(\bar{x}, \bar{y}) \in \text{cl}_K(q)$ , and thus, by assumption,  $\varphi_i(\bar{x}, \bar{y}) \in \text{cl}_K(r)$  for all  $i < l$ . Thus, for all  $M$ -invariant Morley sequences  $(\bar{b}_j)_{j \in \omega}$  in  $r$ , the set  $\{\varphi_i(\bar{x}, \bar{b}_j) : j \in \omega\}$  is inconsistent for all  $i < l$ .

Assume, for contradiction, that  $\psi(\bar{x}, \bar{y}) \notin \text{cl}_K(r)$ . So there exists some  $M$ -invariant Morley sequence  $(\bar{c}_i)_{i \in \omega}$  in  $r$  such that  $\{\psi(\bar{x}, \bar{c}_i) : i \in \omega\}$  is consistent. Let  $\bar{a}$  be a realisation. Then, by the pigeonhole principle, we can find an infinite subsequence  $(\bar{c}_{i_j})_{j \in \omega}$  and some  $k$  such that  $\bar{a} \models \{\varphi_k(\bar{x}, \bar{c}_{i_j}) : j \in \omega\}$ . But note that  $(\bar{c}_{i_j})_{j \in \omega}$  is also an  $M$ -invariant Morley sequence in  $r$ , a contradiction.  $\square$

This result tells us that it suffices to look at types over finite parameter sets to determine  $\leq_K$ -minimality.

**Remark 4.4.14.** Note that, for a model  $M \models T_{\mathbf{H}_4\text{-free}}$  and a tuple  $\bar{b}$ , if  $\varphi(\bar{x}, \bar{b}) \vdash \bar{x} \cap \bar{b} \subseteq M$ , then  $\varphi(\bar{x}, \bar{b})$  Conant-divides over  $M$ . In other words, for every global  $M$ -invariant extension  $q$  of  $\text{tp}(\bar{b}/M)$ ,  $\varphi(\bar{x}, \bar{y}) \in \text{cl}_K(q)$ .

**Lemma 4.4.15.** *Let  $q$  be a global type  $\downarrow^{\text{hti}}$ -free over  $M \models T_{\mathbf{H}_4\text{-free}}$ . For all  $\bar{b} \models q|_M$  and all formulas  $\varphi(\bar{x}, \bar{b})$  isolating a complete type over  $M'\bar{b}$  with  $M' \subset M$  finite, if  $\varphi(\bar{x}, \bar{y}) \in \text{cl}_K(q)$ , then  $\varphi(\bar{x}, \bar{b}) \vdash \bar{x} \cap \bar{b} \subseteq M$ .*

*Proof.* Suppose that  $\varphi(\bar{x}, \bar{b}) \vdash \bar{x} \cap \bar{b} \subseteq M$ . Since  $q$  is a global  $M$ - $\downarrow^{\text{hti}}$ -free type, by Corollary 4.4.12,  $\{\varphi(\bar{x}, \bar{b}_i) : i \in \omega\}$  is consistent for any Morley sequence  $(\bar{b}_i)_{i \in \omega}$  in  $q$  over  $M$ . This means that  $\varphi(\bar{x}, \bar{y}) \notin \text{cl}_K(q)$ .  $\square$

**Proposition 4.4.16.**  $\downarrow^{\text{hti}}$  satisfies the relative Kim's lemma in  $T_{\mathbf{H}_4\text{-free}}$ .

*Proof.* Let  $p \in S(M)$  and let  $q$  be a global extension of  $p \downarrow^{\text{hti}}$ -free over  $M$ . We want to show that  $q$  is a  $\leq_K$ -least extension of  $p$ . By Lemma 4.4.13, it is enough to show that, if  $\varphi(\bar{x}, \bar{y})$  isolates a complete type over a finite set, then  $\varphi(\bar{x}, \bar{y}) \in \text{cl}_K(q)$  implies  $\varphi(\bar{x}, \bar{y}) \in \text{cl}_K(r)$  for all global  $M$ -invariant extensions  $r$  of  $p$ . But notice that, by Lemma 4.4.15,  $\varphi(\bar{x}, \bar{y}) \in \text{cl}_K(q)$  implies that  $\varphi(\bar{x}, \bar{b}) \vdash \bar{x} \cap \bar{b} \not\subseteq M$  for any  $\bar{b} \models q|_M$ . Hence, by Remark 4.4.14, it follows that  $\varphi(\bar{x}, \bar{y}) \in \text{cl}_K(r)$ .  $\square$

*Proof of Proposition 4.2.3 (v.2).* By Proposition 4.2.2,  $T_{\mathbf{H}_4\text{-free}}$  is  $\text{SOP}_3$ , and so, in particular, it is not simple. Since  $\downarrow^{\text{hti}}$  satisfies full existence and left and right extension by Lemma 4.4.11 and the relative Kim's lemma by Proposition 4.4.16, it follows by Corollary 4.3.16 that  $T_{\mathbf{H}_4\text{-free}}$  is  $\text{TP}_2$ .  $\square$

This completes our second proof of Theorem 4.0.1.

**Remark 4.4.17.** Note that, by [34, Lemma 4.3], there does not exist an independence relation  $\downarrow$  over subsets of  $\mathbb{M} \models T_{\mathbf{H}_4\text{-free}}$  satisfying full existence, symmetry, and stationarity over models, because for any pair  $ab \in \mathbb{M}$  of distinct elements and  $M \models T_{\mathbf{H}_4\text{-free}}$ , if  $\models R(m, a, b)$  for some  $m \in M$ , then  $\models \neg R(m, b, a)$ , and so  $ab \not\equiv_M ba$ . This means that  $T_{\mathbf{H}_4\text{-free}}$  is not a free amalgamation theory, so we cannot apply Conant's methods to show  $\text{NSOP}_4$ . The same argument also shows that we cannot use the criterion from [39, Theorem 5.10] either.

**Remark 4.4.18.** Mutchnik uses [94, Theorem 3.25] (cf., Corollary 4.3.16) in the context of a theory with an independence relation  $\downarrow$  that satisfies monotonicity, full existence, and stationarity over models. It is shown there that, if  $\downarrow$  satisfies a generalisation of freedom, known as generalised freedom, then satisfying the relative Kim's lemma and the symmetry of  $\downarrow$ -Kim-independence are equivalent, which means that we do not need to check them separately as we did here. However, it is unclear whether one can find such an independence relation in  $T_{\mathbf{H}_4\text{-free}}$ .

**Question 4.4.19.** Is there an independence relation  $\downarrow$  in  $T_{\mathbf{H}_4\text{-free}}$  satisfying monotonicity, full existence, and stationarity over models?

## 4.5 Higher arity

In this chapter so far, we have exhibited some model-theoretic complexity for  $\mathbf{H}_4$ -free 3-hypertournaments from the point of view of neostability theory, as well as briefly discussing in what senses the other known homogeneous 3-hypertournaments are

“tamer” from the classification point of view. It is thus natural to consider the general case. In [25], Cherlin, Hubička, Konečný, and Nešetřil define  $n$ -hypertournaments for arbitrary  $n$ , and indicate how to define  $\mathbf{H}_{n+1}$  as a certain  $n$ -hypertournament. In this closing section, we consider the resulting structures and show that they do not exhibit the same model-theoretic complexity that the  $\mathbf{H}_4$ -free 3-hypertournament does.

Let  $n \geq 2$ . We define an  $n$ -**hypertournament** to be an  $n$ -ary relational structure  $M$  such that, for any  $S \subset M$  with  $|S| = n$ , the automorphism group of the induced structure on  $S$  is isomorphic to the alternating group  $\text{Alt}(n)$ .

Let  $S \subset M$  be a substructure on  $n + 1$  vertices. We say  $S$  is an  $\mathbf{H}_{n+1}$ -**structure** if  $\text{Aut}(S) \cong \text{Alt}(n + 1)$ . We say an  $n$ -hypertournament is  $\mathbf{H}_{n+1}$ -**free** if it contains no  $\mathbf{H}_{n+1}$ -structures. It is not hard to show that, for  $n = 3$ , we recover the definitions from §4.1. The proof of the following fact is analogous to Lemma 4.1.5:

**Fact 4.5.1.** *The class of finite  $\mathbf{H}_{n+1}$ -free  $n$ -hypertournaments is a strong amalgamation class.*

Adapting the notation employed in previous sections, we denote by  $T_{\mathbf{H}_{n+1}\text{-free}}$  the theory of the Fraïssé limit of this class. Note that, by Fact 2.2.5, we have  $\text{acl}(A) = A$  for any subset  $A$  of this Fraïssé limit.

**Lemma 4.5.2.** *Suppose that  $M$  is an  $n$ -hypertournament and  $S$  is a substructure of  $M$  of cardinality  $n + 1$ . If there are distinct  $a_0, \dots, a_{n-2}, b, c \in S$  and  $0 \leq i < n - 1$  such that  $M \models R(a_0, \dots, a_i, b, a_{i+1}, \dots, a_{n-2}) \wedge R(a_0, \dots, a_i, c, a_{i+1}, \dots, a_{n-2})$ , then  $S$  is  $\mathbf{H}_{n+1}$ -free.*

*Proof.* By definition, if  $S$  is an  $\mathbf{H}_{n+1}$ -structure and  $\bar{e}, \bar{f} \in R(S)$ , then there is some  $\sigma \in \text{Alt}(n + 1)$  such that  $\sigma(\bar{e}) = \bar{f}$ . But the unique permutation sending the tuple  $(a_0, \dots, a_{i-1}, b, a_i, \dots, a_{n-2})$  to  $(a_0, \dots, a_{i-1}, c, a_i, \dots, a_{n-2})$  has sign  $-1$ . Therefore,  $S$  is not an  $\mathbf{H}_{n+1}$ -structure.  $\square$

**Remark 4.5.3.** (i) Note that it is not necessary to add the case where  $b, c$  come first in the tuple.

(ii) Lemma 4.5.2 is the  $n$ -ary version of Remark 4.1.4.

**Theorem 4.5.4.** *Let  $n > 3$ . Then  $T_{\mathbf{H}_{n+1}\text{-free}}$  has disjoint 3-amalgamation, i.e., for any types  $p(\bar{x}, \bar{y}), q(\bar{y}, \bar{z}), r(\bar{x}, \bar{z}) \in S_{2r}(A)$  such that  $p|_{\bar{x}} = r|_{\bar{x}}, p|_{\bar{y}} = q|_{\bar{y}}, q|_{\bar{z}} = r|_{\bar{z}}$ , and each type implies that the variables are all distinct and any realisation is disjoint from  $A$ , there is a type  $t(\bar{x}, \bar{y}, \bar{z}) \in S_{3r}(A)$  extending  $p(\bar{x}, \bar{y}) \cup q(\bar{y}, \bar{z}) \cup r(\bar{x}, \bar{z})$ .*

*Proof.* Let  $p(\bar{x}, \bar{y}), q(\bar{y}, \bar{z}), r(\bar{x}, \bar{z}) \in S_{2r}(A)$  be as above. Let  $(\bar{a}_i)_{i \in I}$  be an arbitrary enumeration of  $A$ . Let us define the type  $t(\bar{x}, \bar{y}, \bar{z})$  as the type given by:

- (i)  $p(\bar{x}, \bar{y}) \wedge q(\bar{y}, \bar{z}) \wedge r(\bar{x}, \bar{z})$ .
- (ii) For any  $\ell_0, \ell_1, \ell_2 > 0$  and  $\ell_3 \geq 0$  such that  $\ell_0 + \ell_1 + \ell_2 + \ell_3 = n$ ,

$$R(x_{i_0}, \dots, x_{i_{\ell_0-1}}, y_{j_0}, \dots, y_{j_{\ell_1-1}}, z_{k_0}, \dots, z_{k_{\ell_2-1}}, a_{m_0}, \dots, a_{m_{\ell_3-1}}),$$

where  $\bar{i}, \bar{j}, \bar{k} \subset [n]$  and  $\bar{m} \in I$  are such that  $i_0 < \dots < i_{\ell_0}, j_0 < \dots < j_{\ell_1}, k_0 < \dots < k_{\ell_2}$  and  $m_0 < \dots < m_{\ell_3}$ . (Observe that this determines all instances of  $R$  by applying  $\text{Alt}(n)$ .)

It is enough to show that  $t(\bar{x}, \bar{y}, \bar{z})$  does not generate any  $\mathbf{H}_{n+1}$ -structures. Any set of cardinality  $n + 1$  containing only variables from at most two of  $\bar{x}, \bar{y}, \bar{z}$  together with elements from  $A$  is consistent by assumption. In any other  $(n + 1)$ -set, there must be either at least two variables from one of the  $\bar{x}, \bar{y}, \bar{z}$ , or exactly  $(n + 1) - 3 \geq 2$  elements from  $A$ . In either case, we can use (ii) together with Lemma 4.5.2 to show consistency.  $\square$

**Corollary 4.5.5.**  *$T_{\mathbf{H}_{n+1}\text{-free}}$  is supersimple with  $SU$ -rank 1. Moreover, for a monster model  $\mathbb{M} \models T_{\mathbf{H}_{n+1}\text{-free}}$  and any small  $A, B, C \subset \mathbb{M}$ , we have  $A \downarrow_C^d B$  iff  $A \downarrow_C^f B$  iff  $A \cap B \subseteq C$ .*

*Proof.* Combine Fact 4.5.1 and Theorem 4.5.4 with [74, Theorem 3.14].  $\square$

**Example 4.5.6.** While the failure of disjoint 3-amalgamation for the  $\mathbf{H}_4$ -free 3-hypertournament follows from results in previous sections (e.g., Proposition 4.2.3), let us give a concrete counterexample here. Pick two elements  $a_0, a_1$ . We define the type  $p(x, y, a_0, a_1)$  by  $R(x, a_0, y) \wedge R(x, y, a_1) \wedge R(x, a_1, a_0) \wedge R(y, a_1, a_0)$ , the type  $q(y, z, a_0, a_1)$  by  $R(y, a_0, z) \wedge R(y, z, a_1) \wedge R(y, a_1, a_0) \wedge R(z, a_1, a_0)$ , and the type  $r(x, z, a_0, a_1)$  by  $R(x, z, a_0) \wedge R(x, a_1, z) \wedge R(x, a_1, a_0) \wedge R(z, a_1, a_0)$ . It is clear that each of the types is consistent, and that  $p|_x = r|_x$ ,  $p|_y = q|_y$ , and  $q|_z = r|_z$ .

Assume now, for contradiction, that  $p(x, y) \cup q(y, z) \cup r(x, z)$  is consistent. Let  $b_0, b_1, b_2$  be a realisation. Then the set  $\{b_0, b_1, b_2, a_0\}$  implies that  $\models R(b_0, b_2, b_1)$ , whereas the set  $\{b_0, b_1, b_2, a_1\}$  implies that  $\models R(b_0, b_1, b_2)$ , a contradiction. Therefore, disjoint 3-amalgamation fails.

**Remark 4.5.7.** As noted in [25], in the  $n = 2$  case, finite  $\mathbf{H}_3$ -free tournaments are exactly the finite linear orders. Thus, the theory of the generic structure is precisely DLO, and since it is o-minimal we also get  $A \downarrow_C^d B$  iff  $A \downarrow_C^f B$ .

**Question 4.5.8.** Is it true that  $A \downarrow_C^d B$  iff  $A \downarrow_C^f B$  in  $T_{\mathbf{H}_4\text{-free}}$ ?

Observe that, since algebraic closure is trivial in this theory, a negative answer to the above question would also answer [36, Question 2.20].

# 5

## Independence

Haabet er en ny Klædning, stiv og stram  
og glimrende, dog har man aldrig havt  
den paa, og veed derfor ikke, hvorledes  
den vil klæde En, eller hvorledes den  
sidder. Erindringen er en aflagt  
Klædning, som, hvor skøn den end er,  
dog ikke passer, da man er voxet fra den.  
Gjentagelsen er en uopslidelig Klædning,  
der slutter fast og ømt, hverken trykker  
eller flagrer.

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*Søren Kierkegaard*

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The previous chapters have focused on the interaction between syntactic properties, such as  $\text{NSOP}_1$  and  $\text{NSOP}_4$ , and semantic notions of independence such as Kim- and Conant-independence. The approach taken in this and the next chapters,

as well as the appendix, will be different. We will focus on each of the two sides, syntax and semantics, separately. The goal is, on the one hand, to capture at a general level many common phenomena which have appeared in the literature only in their particular instances for certain classes such as  $\text{NSOP}_1$  or  $\text{NTP}_2$ . This will also contribute to determining more precisely where the interaction between syntax and semantics comes in, and which properties can be obtained purely in terms of one side or the other. On the other hand, we aim to contribute to the currently open questions about extending the neostability programme beyond  $\text{NSOP}_1$  theories.

The present chapter focuses on the semantic side, and its goal is twofold. First, we develop a framework that formalises the different notions of “witnessing” which have appeared across several papers in the neostability literature during the last few decades. These have traditionally been referred to as “Kim’s Lemma”, after the original version of this result in the context of simple theories (see [64]). In this chapter, we treat witnessing, or Kim’s Lemma, as a relation between two abstract independence relations. Equipped with this, we then show that many results from the theories of simplicity,  $\text{NTP}_2$ ,  $\text{NSOP}_1$  and beyond can be obtained as particular instances of general theorems for abstract independence relations.

This basic idea for the formalisation of the concept of witnessing, although original in its explicit formulation, is far from being so in its content. Many authors in recent years have encountered the issue of extending notions of witnessing beyond simple theories, and in doing so, have provided the building blocks for the theory presented here. Thus, this chapter also aims to be a compilation of many results that neostability theory has established in recent years and a general reference to different areas of interest ( $\text{NTP}_2$ ,  $\text{NSOP}_1$ , etc.).

A limitation of the present study is that we have restricted ourselves to working over models. It is an ongoing area of research how to extend the known results from this case in particular contexts (most notably  $\text{NSOP}_1$ ) to arbitrary bases (see, e.g., [45], [93]). The project of extending the results contained in this chapter to arbitrary bases over which  $\perp$  satisfies full existence is left for future work. Let us note on this point that some results from §§5.5-5.6 have been independently obtained by Itay Kaplan and Nicholas Ramsey in unpublished work on defining Kim-independence over arbitrary bases in  $\text{NSOP}_1$  theories. The setting, purpose and extent of the two projects, however, are very different.

The structure of this chapter is as follows. We start by extending the notion of *relative Kim-independence* introduced by Mutchnik in [94] to a more general

framework. This is done in two parts. In §5.1, we introduce a new operation on abstract independence relations, in the style of Adler, that corresponds to “forcing strong finite character”. With this operation in the background, we construct in §5.2 the relativisation of Kim-independence to an independence relation  $\perp$  from the more primitive relation of lifting the indiscernibility of Morley sequences (based on [106]). In this section, we also adapt the universal notion of Conant-independence appearing, beyond Mutchnik’s work, in [61], [72], [76].

In §5.3 we introduce three different formulations of witnessing stemming from the literature, which we prove to be equivalent in most cases. We also include some basic results about witnessing here. The notion of witnessing we use for the rest of the chapter is called GUWP (for *generalised universal witnessing property*; see Definition 5.3.7). Essentially,  $\perp^1$  has GUWP w.r.t.  $\perp^2$  if  $\perp^2$ -Kim-dividing is witnessed by all  $\perp^1$ -Morley sequences.

In §5.4, we move towards adapting more general results about the equivalence of witnessing to other traditional properties of abstract independence relations. We start with *chain local character* and show the following:

**Proposition 5.4.5.** *Let  $\perp$  be an independence relation that satisfies full existence, monotonicity, strong finite character, and right extension. Then  $\perp$ -Conant-independence satisfies chain local character.*

As a corollary, we deduce that Conant-independence (in the sense of [94]) satisfies chain local character in any theory.

In §5.5, we recover the equivalence with symmetry:

**Theorem 5.5.1.** *Let  $\perp$  be an independence relation satisfying full existence and monotonicity. Suppose that  $\perp$ -Kim-independence is compatible with  $\perp$  and satisfies left transitivity. The following are equivalent:*

- (i)  $\perp$ -Kim-independence satisfies GUWP w.r.t.  $\perp$ .
- (ii)  $\perp$ -Kim-independence is symmetric.

We also show in this section how this characterisation of symmetry allows us to recover one direction of [94, Theorem 3.12] from purely semantic arguments, as well as relate witnessing to Adler’s notion of an independence relation.

In §5.6, we consider relative versions of the Independence Theorem. We show that a general version of the “weak independence theorem” appearing in [59] and [95]

follows from the assumption of witnessing and another property which we call “weak transitivity” (Theorem 5.6.7). We also show that a slight strengthening of these axioms provides a new sufficient condition for NSOP<sub>1</sub> theories (Corollary 5.6.15). We then prove the following partial converse to the weak independence theorem:

**Theorem 5.6.16.** *Let  $\perp^1 \implies \perp^2$  be independence relations satisfying full existence and monotonicity such that  $\perp^1$  also satisfies left and right extension. If  $\perp^2$ -Conant-independence satisfies the  $\perp^2$ -independence theorem over models, then  $\perp^1$  satisfies GUWP w.r.t.  $\perp^2$ .*

This result allows us to show, in a purely semantic fashion, that if Conant-independence satisfies the independence theorem over models, then  $T$  is NSOP<sub>1</sub> (Proposition 5.6.17). This proof thus avoids the tree construction from [32, Proposition 5.2].

Finally, in §5.7, we prove a technical result that allows us to prove a dichotomy between NSOP<sub>1</sub> and BTP for the class of theories with an independence relation  $\perp$  satisfying full existence, monotonicity, and quasi-strong finite character, and such that  $\perp^i$  satisfies GUWP w.r.t.  $\perp$ . In particular, this shows that most known strictly NSOP<sub>4</sub> examples have BTP.

The contents of this chapter have appeared in preprint form in [86]. We thank Christian d’Elbée for helpful suggestions.

## 5.1 Strong finite character

In this section, we extend Adler’s original programme by introducing a new operation to be understood as “forcing strong finite character” as given in Definition 2.4.3(xii). As we will see, this allows us to formalise several intuitions about what occurs when forcing right extension, and provides the formal general framework for building relative Kim- and Conant-independence in the next section.

Before doing this, let us introduce some terminology that will be useful. This expands the use of the term in [29] and unifies many properties of formulas which have appeared throughout neostability theory. For the rest of this section, let  $\perp$  be an independence relation.

**Definition 5.1.1.** We say a partial type  $\pi(x)$  over  $C$  is  $\perp$ -free over  $B$  if there is some  $a \models \pi(x)$  such that  $a \perp_B C$ .

**Lemma 5.1.2.** *Suppose  $B \subseteq C$ . A type  $p(x) \in S(C)$  is  $\perp$ -free over  $B$  iff, for all  $a \models p$ , we have  $a \perp_B C$ .*

*Proof.* ( $\Rightarrow$ ) Suppose  $p$  is  $\perp$ -free over  $B$ . Then there is some  $a \models p$  such that  $a \perp_B C$ . Since  $p$  is complete, for any  $a' \models p$ ,  $a' \equiv_C a$ . By invariance,  $a' \perp_B C$ .

( $\Leftarrow$ ) This follows immediately from the definition and the consistency of  $p$ .  $\square$

We will later see (Lemma 5.1.12) that, if  $\perp$  satisfies strong finite character (Definition 2.4.3(xii)), then the notion defined in Definition 5.1.1 coincides with what Chernikov and Kaplan call “ $\perp$ -free” in [29].

**Example 5.1.3.** The following characterisations are easy to check for a type  $p(x) \in S(C)$  and a set  $B \subseteq C$ :

- (i)  $p(x)$  is  $\perp^u$ -free over  $B$  iff it is finitely satisfiable in  $B$ . Moreover, a formula  $\varphi(x, c)$  is  $\perp^u$ -free over  $B$  iff there is  $b \in B$  such that  $b \models \varphi(x, c)$  (for the non-trivial direction, use the argument from, e.g., [113, Lemma 8.1.3]).
- (ii)  $p(x)$  is  $\perp^i$ -free over  $B$  iff it has a global extension Lascar-invariant over  $B$ . Moreover, it follows from [76, Remark 2.19] that a formula  $\varphi(x, c)$  is  $\perp^i$ -free over  $B$  iff it does not quasi-fork over  $B$ .
- (iii)  $p(x)$  is  $\perp^f$ -free over  $B$  iff it does not fork over  $B$ . Moreover, a formula  $\varphi(x, c)$  is  $\perp^f$ -free over  $B$  iff it does not fork over  $B$  (see, e.g., [113, Lemma 7.1.11]).

Let us make a small digression to prove a new fact about  $\perp^a$ . For this, we need to introduce some new terminology extending in a natural fashion Definition 3.1.1:

**Definition 5.1.4.** Let  $\varphi(x, y)$  be a formula,  $b$  a tuple, and  $C$  a set of parameters. We say  $\varphi(x, b)$  **strongly forks over**  $C$  if there exist formulas  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$  and each  $\psi_i(x, d_i)$  strongly divides over  $C$ .

**Example 5.1.5.** Let  $T$  be the theory of infinite sets in the empty language. Pick two singletons  $a, b$ , and let  $\varphi(x, a, b)$  be the formula  $x = a \vee x = b$ . It is clear that each one of  $x = a$  and  $x = b$  strongly divides over  $\emptyset$ , and so by definition  $\varphi(x, a, b)$  strongly forks over  $\emptyset$ . Pick now a sequence of automorphisms  $(\sigma_i)_{i \in \omega} \in \text{Aut}(\mathbb{M})$  such that  $\sigma_i(ab) = ab_i$  where  $b_i \neq b_j$  for all  $i \neq j$ . Then clearly  $\bigcap_{i < \omega} \sigma_i(\varphi(\mathbb{M}, a, b)) = \{a\}$ , which shows that  $\varphi(x, a, b)$  does not strongly divide over  $\emptyset$ .

With this, we can provide a characterisation of  $\perp^a$ -free formulas, which is somewhat buried in Adler’s proof of [2, Proposition 4.7].

**Proposition 5.1.6.** *Let  $C$  be a set,  $b$  be a tuple, and  $\varphi(x, y) \in \mathcal{L}(C)$ . Then  $\varphi(x, b)$  is  $\perp^a$ -free over  $C$  iff it does not strongly fork over  $C$ .*

*Proof.* ( $\Rightarrow$ ) Let us first note that, by [3, Proposition 4.7(3) $\Rightarrow$ (4)], if a formula strongly divides over  $C$ , then it is not  $\perp^a$ -free over  $C$ . Suppose now that  $\varphi(x, b)$  strongly forks over  $C$ . So there are formulas  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$  and each  $\psi_i(x, d_i)$  strongly divides over  $C$ . Take  $a \models \varphi(x, b)$ . By assumption,  $a \models \psi_i(x, d_i)$  for some  $i < n$ , and by the above, this means  $a \not\perp_C^a d_i$ . So, by monotonicity,  $a \not\perp_C^a b d_{<n}$ . Since  $a \models \varphi(x, b)$  was arbitrary, it follows that, for any such  $a$ , we have  $a (\not\perp_C^a)^* b$ , and thus  $a \not\perp_C^a b$  by right extension. Therefore,  $\varphi(x, b)$  is not  $\perp^a$ -free over  $C$ .

( $\Leftarrow$ ) We adapt the proof of [2, Proposition 4.7(2) $\Rightarrow$ (6)]. Suppose  $\varphi(x, b)$  is not  $\perp^a$ -free over  $C$ . This means that, for every  $d \models \varphi(x, b)$ , we have  $d \not\perp_C^a b$ .

**Claim.** There is a formula  $\psi_d(x_d, b_d) \in \text{tp}(d/Cb)$  that strongly divides over  $C$ .

*Proof of Claim.* Since  $d \not\perp_C^a b$ , we can find some  $e \notin \text{acl}(C)$  such that  $e \in \text{acl}(Cd) \cap \text{acl}(Cb)$ . In particular, there is  $\theta(x_d, y_d, z_d) \in \mathcal{L}$ , a finite tuple  $d' \subseteq d$ , and a finite tuple  $c \in C$  such that  $\models \theta(e, d', c)$  and  $\theta(x_d, d', c)$  is algebraic with  $k$  many solutions.

Let  $\psi_d(x_d, y_d, z_d) := \theta(x_d, y_d, z_d) \wedge \exists \leq^k w \theta(w, y_d, z_d)$ . Then  $\models \psi_d(e, d', b)$ , so that  $\psi_d(e, y_d, c) \in \text{tp}(d/Cb)$ . Now observe that, since  $e \notin \text{acl}(C)$ , the set of  $C$ -conjugates of  $\psi_d(e, y_d, c)$  is infinite, and by definition, it is  $k$ -inconsistent. Therefore,  $\psi_d(e, y_d, c)$  strongly divides over  $C$ . Since by monotonicity we may assume that  $e \in Cb$ , this completes the proof.  $\square_{\text{Claim}}$

Now, since  $\{\varphi(x, b)\} \cup \{\neg\psi_d(x_d, b_d) : d \models \varphi(x, b)\}$  is inconsistent, there exist realisations  $d_i$  of  $\varphi(x, b)$  for  $i < n$  such that, after rewriting the variables,

$$\varphi(x, b) \vdash \bigvee_{i < n} \psi_{d_i}(x, b_{d_i}).$$

Therefore,  $\varphi(x, b)$  strongly forks over  $C$ .  $\square$

Let us recall at this point that we extend the terminology of  $\perp$ -freeness to global types as follows (as in Definition 4.3.1): for a small set  $B$ , we say a global type  $p(x)$  is  **$\perp$ -free over  $B$**  if, for all small  $B \subseteq C$  and  $a \models p|_C$ , we have  $a \perp_B C$ .

**Definition 5.1.7.** Given an independence relation  $\perp$ , we define  $\perp^{\text{sfc}}$  as follows:

$$A \perp_C^{\text{sfc}} B \iff \text{for all } \varphi(x, b) \in \text{tp}(A/BC), \text{ there is } a' \models \varphi(x, b) \text{ such that } a' \perp_C b.$$

**Lemma 5.1.8.** *Let  $\perp$  be an independence relation.*

- (i)  $\perp^{\text{sfc}}$  is an independence relation satisfying strong finite character and monotonicity.
- (ii) If  $\perp$  satisfies monotonicity, then  $\perp \implies \perp^{\text{sfc}}$ .
- (iii) If  $\perp^1 \implies \perp^2$  and  $\perp^2$  satisfies strong finite character, then  $(\perp^1)^{\text{sfc}} \implies \perp^2$ . In particular, if  $\perp^1 \implies \perp^2$ , then  $(\perp^1)^{\text{sfc}} \implies (\perp^2)^{\text{sfc}}$ .
- (iv) If  $\perp$  satisfies monotonicity, then  $\perp = \perp^{\text{sfc}}$  iff  $\perp$  satisfies strong finite character.

*Proof.* (i) Assume  $A \perp_C^{\text{sfc}} B$  and  $\sigma \in \text{Aut}(\mathbb{M})$ . Let  $\varphi(x, \sigma(b)) \in \text{tp}(\sigma(A)/\sigma(B)\sigma(C))$ . Then  $\varphi(x, b) \in \text{tp}(A/BC)$ , and thus, by assumption, there is  $a' \models \varphi(x, b)$  such that  $a' \perp_C b$ . But then  $\sigma(a') \models \varphi(x, \sigma(b))$ , and since  $\perp$  is an independence relation,  $\sigma(a') \perp_{\sigma(C)} \sigma(b)$ . Therefore,  $\sigma(A) \perp_{\sigma(C)}^{\text{sfc}} \sigma(B)$ , as claimed.

(Strong finite character) Suppose that  $A \not\perp_C^{\text{sfc}} B$ . By definition, there exists some finite  $b$  in  $B$  and  $\varphi(x, b) \in \text{tp}(A/BC)$  such that, for all  $a \models \varphi(x, b)$ , we have  $a \not\perp_C b$ . But notice that  $\varphi(x, b) \in \text{tp}(a/Cb)$  for each such  $a \models \varphi(x, b)$ . So, in fact,  $a \not\perp_C^{\text{sfc}} b$ .

(Monotonicity) Suppose  $A \perp_C^{\text{sfc}} B$  and  $A' \subseteq A$ ,  $B' \subseteq B$ . Then  $\text{tp}(A'/B'C)$  is contained in the restriction of  $\text{tp}(A/BC)$  to the appropriate variables. In particular, if  $\varphi(x, b') \in \text{tp}(A'/B'C)$ , then  $\varphi(x, b') \in \text{tp}(A/BC)$  and thus by assumption there is  $a' \models \varphi(x, b')$  such that  $a' \perp_C b'$ . Therefore,  $A' \perp_C^{\text{sfc}} B'$ .

(ii) Suppose that  $A \not\perp_C^{\text{sfc}} B$ . So there exists a formula  $\varphi(x, b) \in \text{tp}(A/BC)$  such that, for all  $a' \models \varphi(x, b)$ , we have  $a' \not\perp_C b$ . Picking a tuple  $a$  from  $A$  such that  $a \models \varphi(x, b)$  and using monotonicity, it follows that  $A \not\perp_C B$ .

(iii) Assume that  $A \not\perp_C^2 B$ . By strong finite character, there is a formula  $\varphi(x, b) \in \text{tp}(A/BC)$  such that, for all  $a' \models \varphi(x, b)$ ,  $a' \not\perp_C^2 b$ . Thus, by assumption,  $a' \not\perp_C^1 b$ . Hence, by definition,  $A (\not\perp_C^1)^{\text{sfc}} B$ .

(iv) From right to left, note that  $\perp \implies \perp^{\text{sfc}}$  follows from (ii) and  $\perp^{\text{sfc}} \implies \perp$  follows from (iii). The converse follows from (i).  $\square$

The following formulation can be useful:

**Lemma 5.1.9.** *Let  $\perp$  be an independence relation. Then  $A \perp_C^{\text{sfc}} B$  iff every formula  $\varphi(x, b) \in \text{tp}(A/BC)$  is  $\perp$ -free over  $C$ .*

*Proof.* Simply note that  $A \not\perp_C^{\text{sfc}} B$  iff there is a formula  $\varphi(x, b) \in \text{tp}(A/BC)$  such that  $a \not\perp_C b$  for all  $a \models \varphi(x, b)$ , iff there is  $\varphi(x, b) \in \text{tp}(A/BC)$  which is not  $\perp$ -free

over  $C$ . □

This allows us to generalise [55, Remark 2.3]:

**Corollary 5.1.10.** *In any theory, the following are equivalent:*

- (i)  $A \downarrow_C^a B$ .
- (ii) For some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/BC)$  does not contain a formula that strongly divides over  $C$ .
- (iii) For some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/BC)$  does not contain a formula that strongly forks over  $C$ .

*Proof.* First, observe that  $\downarrow^a = (\downarrow^a)^{\text{sfc}}$  by Example 2.4.6(i) and Lemma 5.1.8(iv). Thus, by Lemma 5.1.9,  $A \downarrow_C^a B$  iff every formula in  $\text{tp}(a/BC)$  (for  $a$  an enumeration of  $A$ ) is  $\downarrow^a$ -free over  $C$ . By Proposition 5.1.6, this is equivalent to no formula in  $\text{tp}(a/BC)$  strongly forking over  $C$ . The equivalence with no formula strongly dividing over  $C$  follows from the Claim in the proof of Proposition 5.1.6. □

There are two immediate and useful applications of Lemma 5.1.9 for the theory of abstract independence relations. The first one formalises the fact that strong finite character only affects types:

**Lemma 5.1.11.** *Suppose  $\downarrow$  satisfies monotonicity. A formula  $\varphi(x, b)$  is  $\downarrow$ -free over  $C$  iff it is  $(\downarrow)^{\text{sfc}}$ -free over  $C$ .*

*Proof.*  $(\Rightarrow)$  This follows directly from Lemma 5.1.8(ii).

$(\Leftarrow)$  Suppose  $\varphi(x, b)$  is  $\downarrow^{\text{sfc}}$ -free over  $C$ . This means there is some  $a \models \varphi(x, b)$  such that  $a \downarrow_C^{\text{sfc}} b$ . By Lemma 5.1.9, every formula in  $\text{tp}(a/Cb)$  is  $\downarrow$ -free over  $C$ . Therefore,  $\varphi(x, b)$  is  $\downarrow$ -free over  $C$ . □

The second one shows the equivalence of our notion of “being  $\downarrow$ -free” with that of Chernikov and Kaplan [29] under the assumption of strong finite character:

**Lemma 5.1.12.** *Suppose  $\downarrow$  satisfies strong finite character. A type  $p(x) \in S(C)$  is  $\downarrow$ -free over  $B \subseteq C$  iff, for every  $B \subseteq D \subseteq C$  and every  $a \models p|_D$ , we have  $a \downarrow_B D$ .*

*Proof.* Assume, for contradiction, that  $p(x)$  is  $\downarrow$ -free over  $B$  but there is some  $B \subseteq D \subseteq C$  and  $a \models p|_D$  such that  $a \not\downarrow_B D$ . By strong finite character, there is  $\varphi(x', d) \in \text{tp}(a/D)$  with  $x' \subseteq x$  and  $d \in D$  finite such that  $\varphi(x', d)$  is not  $\downarrow$ -free over  $B$ . But  $\varphi(x', d) \in p(x)$ , and so by Lemma 5.1.9 we obtain a contradiction. □

Another useful result about sfc comes from its interaction with the star operation from Definition 2.4.11, which implicitly appears in [4, Lemma 1.2]. The core of the argument can be traced back to a general characterisation of the result of forcing strong finite character and extension:

**Lemma 5.1.13.** *Suppose that  $\perp$  satisfies monotonicity. Then  $A(\perp^{\text{sfc}})_C^* B$  iff, for every formula  $\varphi(x, b) \in \text{tp}(A/BC)$ , if  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$ , then there is some  $i < n$  such that  $\psi_i(x, d_i)$  is  $\perp$ -free over  $C$ .*

*Proof.* ( $\Rightarrow$ ) Suppose there is some  $\varphi(x, b) \in \text{tp}(A/BC)$  and formulas  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$  and no  $\psi_i(x, d_i)$  is  $\perp$ -free over  $C$ . Pick  $A' \equiv_{BC} A$ . Then, if  $a'$  is the corresponding subtuple of  $A'$ , we have  $a' \models \varphi(x, b)$ , and so  $a' \models \psi_i(x, d_i)$  for some  $i < n$ . So  $\psi_i(x, d_i) \in \text{tp}(A'/CBd_0 \dots d_{n-1})$ , and so by Lemma 5.1.9  $A' \not\perp_C^{\text{sfc}} Bd_0 \dots d_{n-1}$ . Therefore,  $A(\perp^{\text{sfc}})_C^* B$ .

( $\Leftarrow$ ) Suppose that  $A(\perp^{\text{sfc}})_C^* B$ . This means there is some  $D \supseteq B$  such that, for all  $A' \equiv_{BC} A$ , we have  $A' \perp_C^{\text{sfc}} D$ . By Lemma 5.1.9, for each  $A' \equiv_{BC} A$ , we can find some  $\psi_{A'}(x_{A'}, d_{A'}) \in \text{tp}(A'/CD)$  that is not  $\perp$ -free over  $C$ .

Now, the set  $\text{tp}(A/BC) \cup \{\neg\psi_{A'}(x_{A'}, d_{A'}) : A' \models \text{tp}(A/BC)\}$  is inconsistent. So, by compactness, we can find a formula  $\varphi(x, b) \in \text{tp}(A/BC)$  and realisations  $a_0, \dots, a_{n-1} \models \text{tp}(A/BC)$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_{a_i}(x_{a_i}, d_{a_i})$ . Rewriting the variables in each formula, this gives  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$ , and each  $\psi_i(x, d_i)$  is not  $\perp$ -free over  $C$  by monotonicity. This proves the claim.  $\square$

**Lemma 5.1.14.** *Suppose that  $\perp$  satisfies monotonicity. Then  $(\perp^*)^{\text{sfc}} = (\perp^{\text{sfc}})^*$ .*

*Proof.* ( $\Rightarrow$ ) Suppose that  $A(\perp^*)_C^{\text{sfc}} B$ . We will use Lemma 5.1.13. So assume that  $\varphi(x, b) \in \text{tp}(A/BC)$  and there are formulae  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$ . By assumption and Lemma 5.1.9,  $\varphi(x, b)$  is  $\perp^*$ -free over  $C$ . Thus, there is some  $a' \models \varphi(x, b)$  such that  $a' \perp_C bd_0 \dots d_{n-1}$ . But, by choice of formulae, we have  $a' \models \psi_i(x, d_i)$  for some  $i < n$ . So, by monotonicity,  $\psi_i(x, d_i)$  is  $\perp$ -free over  $C$ . Hence, by Lemma 5.1.13,  $A(\perp^{\text{sfc}})_C^* B$ .

( $\Leftarrow$ ) Suppose that  $A(\perp^{\text{sfc}})_C^* B$ . So there is a formula  $\varphi(x, b) \in \text{tp}(A/BC)$  such that, for all  $a' \models \varphi(x, b)$ , we have  $a' \not\perp_C^* b$ . In particular, for  $a$  the corresponding subtuple of  $A$ , we have  $a \not\perp_C^* b$ . So there is some  $D \supseteq b$  such that, for all  $a' \equiv_{Cb} a$ , we have  $a' \not\perp_C D$ , and so this holds in particular for all  $a' \models \varphi(x, b)$ . Thus, for any such  $a'$ ,  $a' \not\perp_C^{\text{sfc}} D$ . Hence,  $a(\perp^{\text{sfc}})_C^* b$ . Therefore, by monotonicity,  $A(\perp^{\text{sfc}})_C^* B$ , as claimed.  $\square$

With this, we can recover Adler's result (cf., Fact 2.4.12(v)) as a special case:

**Proposition 5.1.15** ([4, Lemma 1.2]). *Suppose that  $\perp$  satisfies monotonicity. If  $\perp = \perp^{\text{sfc}}$ , then  $\perp^* = (\perp^*)^{\text{sfc}}$ .*

But we also get more than Adler:

**Proposition 5.1.16.** *Suppose that  $\perp$  satisfies monotonicity. If  $\perp = \perp^*$ , then  $\perp^{\text{sfc}} = (\perp^{\text{sfc}})^*$ .*

*Equivalently, if  $\perp$  satisfies monotonicity and right extension, then  $\perp^{\text{sfc}}$  also satisfies right extension.*

We can also provide a concrete description of how adding strong finite character and extension works on formulas:

**Proposition 5.1.17.** *Suppose  $\perp$  satisfies monotonicity. A formula  $\varphi(x, b)$  is  $(\perp^{\text{sfc}})^*$ -free over  $C$  iff there are no formulae  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$  and no  $\psi_i(x, d_i)$  is  $\perp$ -free over  $C$ .*

*Proof.* ( $\Rightarrow$ ) Suppose  $\varphi(x, b)$  is  $(\perp^{\text{sfc}})^*$ -free over  $C$ . Let  $a \models \varphi(x, b)$  be such that  $a (\perp^{\text{sfc}})^*_C b$ . By Lemma 5.1.13, as  $\varphi(x, b) \in \text{tp}(a/Cb)$ , we conclude that, whenever  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$ , there is some  $i < n$  such that  $\psi_i(x, d_i)$  is  $\perp$ -free over  $C$ .

( $\Leftarrow$ ) We adapt the classical proof for non-forking (cf., [113, Lemma 7.1.11]). Suppose that the right-hand side holds. Let  $\pi(x, b)$  be a maximal partial type over  $Cb$  containing  $\varphi(x, b)$  such that every formula in  $\pi$  satisfies the right-hand side.

We claim that  $\pi(x, b)$  is complete. Indeed, assume, for contradiction, that it is not. Then there is a formula  $\psi(x, b') \in \mathcal{L}(Cb)$  such that  $\psi(x, b'), \neg\psi(x, b') \notin \pi(x, b)$ . By definition of  $\pi$ , this means that there exist formulae  $\theta_i(x, d_i)$  for  $i < n$  and  $\rho_j(x, e_j)$  for  $j < m$  such that  $\psi(x, b') \vdash \bigvee_{i < n} \theta_i(x, d_i)$ ,  $\neg\psi(x, b') \vdash \bigvee_{j < m} \rho_j(x, e_j)$ , and no  $\theta_i, \rho_j$  is  $\perp$ -free over  $C$ . Now,  $\pi(x, b) \vdash \psi(x, b') \vee \neg\psi(x, b')$ . By compactness, there is some  $\eta(x, b'') \in \pi(x, b)$  such that  $\eta(x, b'') \vdash \psi(x, b') \vee \neg\psi(x, b')$ . But then  $\eta(x, b'') \vdash \bigvee_{i < n} \theta_i(x, d_i) \vee \bigvee_{j < m} \rho_j(x, e_j)$ , contradicting our definition of  $\pi$ .

Since  $\pi(x, b)$  is complete, by Lemma 5.1.13, for any  $a \models \pi(x, b)$  we have  $a (\perp^{\text{sfc}})^*_C b$ . Since  $\varphi(x, b) \in \pi(x, b)$ , this means that  $\varphi(x, b)$  is  $(\perp^{\text{sfc}})^*$ -free over  $C$ , as claimed.  $\square$

**Corollary 5.1.18.** *Suppose  $\perp$  satisfies monotonicity. A formula  $\varphi(x, b)$  is  $\perp^*$ -free over  $C$  iff there are no formulae  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$*

and no  $\psi_i(x, d_i)$  is  $\downarrow$ -free over  $C$ .

*Proof.* By Proposition 5.1.17, it suffices to show that a formula is  $\downarrow^*$ -free over  $C$  iff it is  $(\downarrow^*)^{\text{sfc}}$ -free over  $C$ . But this follows from Lemma 5.1.11.  $\square$

## 5.2 Relative Kim- and Conant-independence

We now move on to the main objects of study in this chapter: the relativised versions of Kim- and Conant-independence. Both originate in the work of Mutchnik for more restricted cases: in [94],  $\downarrow$ -Kim-independence is defined whenever  $\downarrow$  is an independence relation satisfying full existence, monotonicity, and stationarity over models, and [94], [95] introduce  $\downarrow$ -Conant-independence for  $\downarrow = \downarrow^i, \downarrow^u$ . In this section, we want to generalise these notions to any choice of independence relation.

However, our presentation will be different to what is usual in the current literature on Kim-independence (cf., [59], [94]). Instead of defining, e.g.,  $\downarrow$ -Kim-independence as a primitive notion and then noting that the “basic characterisation” of  $\downarrow$ -Kim-dividing holds (cf., [59, Lemma 3.18]), we will start with the lifting of indiscernibility for  $\downarrow$ -Morley sequences as an independence relation and build out of this  $\downarrow$ -Kim-independence. This procedure is structurally analogous to Adler’s abstract definition of  $\downarrow^d$  and  $\downarrow^f$  and his clarification of their relationship in [3].

For the benefit of the reader, let us note that this level of detail is presented for conceptual clarification; for the rest of this chapter, it is enough for the reader to take Lemmas 5.2.9 and 5.2.28 (paired with Definition 5.2.8(ii)) as definitions instead of results.

**Remark 5.2.1.** All the independence relations we will introduce from this point onwards will be defined only over models (see Definition 2.4.1). This requires a suitable modification of the properties listed in Definition 2.4.3. This is standard, but we will explicitly describe it. For monotonicity, extension, normality, strong finite character, quasi-strong finite character, and symmetry, the version over models is clear. For (left) transitivity, we need to choose models  $M \prec N \models T$ , and we recast the property as “if  $N \downarrow_M A$  and  $B \downarrow_N A$ , then  $NB \downarrow_M A$ ”. For (left) base monotonicity, we again need to take models  $M \prec N$  which are contained in a set  $A$ , and redefine this as “if  $A \downarrow_M B$ , then  $A \downarrow_N B$ ”. The case of local character is more subtle; we introduce in §5.4 the appropriate version of local character over models for our work.

### 5.2.1 Relative Kim-independence

We begin by introducing an independence relation inspired by [106, Lemma 1.4] (observe the similarity with [2, §5]):

**Definition 5.2.2.** Let  $\perp$  be an independence relation. We define  $\perp^+$  as follows:  $A \perp_M^+ B$  iff, for every  $\perp$ -Morley sequence  $I$  over  $M$  starting at  $B$ , there is some  $I' \equiv_{MB} I$  which is  $MA$ -indiscernible.

Observe that this definition implicitly assumes the (easy) fact that the choice of enumeration for  $B$  does not matter. From now on, we will omit explicit mention of underlying enumerations whenever there is no confusion.

**Lemma 5.2.3.** *Let  $\perp$  be an independence relation. Then  $\perp^+$  is an independence relation satisfying existence and left normality.*

*Proof.* We first check that  $\perp^+$  is an independence relation. Suppose that  $A \perp_M^+ B$  and  $\sigma \in \text{Aut}(\mathbb{M})$ . Let  $(B'_i)_{i < \omega}$  be an  $\perp$ -Morley sequence over  $\sigma(M)$  with  $B'_0 = \sigma(B)$ . Then  $(\sigma^{-1}(B'_i))_{i < \omega}$  is  $\perp$ -Morley over  $M$  and  $\sigma^{-1}(B'_0) = B$ . Since  $A \perp_M^+ B$ , we can find some  $A' \equiv_{MB} A$  such that  $(\sigma^{-1}(B'_i))_{i < \omega}$  is  $MA'$ -indiscernible. Hence,  $\sigma(A') \equiv_{\sigma(M)\sigma(B)} \sigma(A)$  and  $(B'_i)_{i < \omega}$  is  $\sigma(M)\sigma(A')$ -indiscernible. Therefore, we get  $\sigma(A) \perp_{\sigma(M)}^+ \sigma(B)$ .

(Existence) Suppose  $(M_i)_{i < \omega}$  is  $\perp$ -Morley over  $M$  and  $M_0 = M$ . By indiscernibility, it follows that  $M_i = M$  for all  $i < \omega$ . So clearly, for any  $A$ , the sequence  $(M_i)_{i < \omega}$  is  $MA$ -indiscernible. Therefore,  $A \perp_M^+ M$ .

(Left normality) This is immediate.  $\square$

Of course, if  $\perp$ -Morley sequences do not exist, then the above independence relation is trivial. We often add the assumptions of full existence and monotonicity to ensure non-triviality.

**Lemma 5.2.4.** *If  $\pi(x, b)$  is  $\perp^+$ -free over  $M$ , then, for every  $\perp$ -Morley sequence  $(b_i)_{i < \omega}$  over  $M$  with  $b_0 = b$ , the set  $\bigcup_{i < \omega} \pi(x, b_i)$  is consistent.*

*Proof.* Suppose  $\pi(x, b)$  is  $\perp^+$ -free over  $M$ . Let  $I = (b_i)_{i < \omega}$  be  $\perp$ -Morley over  $M$  with  $b_0 = b$ . Since  $\pi(x, b)$  is  $\perp^+$ -free, we can find some  $c \models \pi(x, b)$  such that  $c \perp_M^+ b$ . So, by an automorphism, there is some  $c' \equiv_{Mb} c$  such that  $I$  is  $Mc'$ -indiscernible. But then  $c' \models \pi(x, b)$ , and thus  $c' \models \bigcup_{i < \omega} \pi(x, b_i)$  by indiscernibility.  $\square$

**Lemma 5.2.5.** *A type  $p(x, b) \in S(Mb)$  is  $\perp^+$ -free over  $M$  iff, for every  $\perp$ -Morley sequence  $(b_i)_{i < \omega}$  over  $M$  with  $b_0 = b$ , the set  $\bigcup_{i < \omega} p(x, b_i)$  is consistent.*

*Proof.*  $(\Rightarrow)$  This is Lemma 5.2.4.

$(\Leftarrow)$  Let  $I = (b_i)_{i < \omega}$  be a  $\perp$ -Morley sequence over  $M$  with  $b_0 = b$ . By assumption,  $\bigcup_{i < \omega} p(x, b_i)$  is consistent. Let  $a' \models \bigcup_{i < \omega} p(x, b_i)$ . By Ramsey, compactness, an automorphism and the completeness of  $p$ , we may assume  $I$  is  $Ma'$ -indiscernible. Moreover, by assumption,  $a' \models p(x, b)$ . So  $p$  is  $\perp^+$ -free over  $M$ .  $\square$

**Proposition 5.2.6.** *If  $\perp$  satisfies monotonicity, then  $\perp^+$  satisfies strong finite character.*

*Proof.* We use Lemma 5.1.9. Suppose that  $A \not\perp_M^+ B$ . Fix an enumeration  $a$  of  $A$  and  $b$  of  $B$  and let  $p(x, y) := \text{tp}(ab/M)$ . Then  $p(x, b)$  is not  $\perp^+$ -free over  $M$ . By Lemma 5.2.5, there is some  $\perp$ -Morley sequence  $(b_i)_{i < \omega}$  over  $M$  with  $b_0 = b$  such that  $\bigcup_{i < \omega} p(x, b_i)$  is inconsistent. By compactness, there is a formula  $\varphi(x', b') \in p(x, b)$  with  $x' \subseteq x$ ,  $b' \subseteq b$  finite such that  $\bigcup_{i < \omega} \{\varphi(x', b'_i)\}$  is inconsistent, where  $b'_i$  denotes the restriction of  $b_i$  to the appropriate variables. Therefore, by monotonicity and Lemma 5.2.4,  $\varphi(x', b')$  is not  $\perp^+$ -free over  $M$ . Therefore,  $A (\perp^+)^{\text{sfc}}_M B$ .  $\square$

**Lemma 5.2.7.** *Suppose that  $\perp$  satisfies monotonicity. A formula  $\varphi(x, b)$  is  $(\perp^+)^*$ -free over  $M$  iff, for every  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$ , there is some  $i < n$  such that, for every  $\perp$ -Morley sequence  $(d_{i,j})_{j < \omega}$  over  $M$  with  $d_{i,0} = d_i$ , the set  $\{\psi_i(x, d_{i,j}) : j < \omega\}$  is consistent.*

*Proof.*  $(\Rightarrow)$  By Propositions 5.2.6 and 5.1.17, we have that  $\varphi(x, b)$  is  $(\perp^+)^*$ -free over  $M$  iff, for every  $\psi_i(x, d_i)$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$ , there is  $i < n$  such that  $\psi_i(x, d_i)$  is  $\perp^+$ -free over  $C$ . By Lemma 5.2.4, it follows that, for every  $\perp$ -Morley sequence  $(d_{i,j})_{j < \omega}$  over  $M$  with  $d_{i,0} = d_i$ , the set  $\{\psi_i(x, d_{i,j}) : j < \omega\}$  is consistent.

$(\Leftarrow)$  Suppose that  $\varphi(x, b)$  is not  $(\perp^+)^*$ -free over  $M$ . Let  $a \models \varphi(x, b)$ . Then  $a (\perp^+)^*_M b$ , so there is some  $d$  such that  $a \not\perp_M^+ bd$ . Let  $p_a(x, b, d) := \text{tp}(a/Mbd)$ . By Lemma 5.2.5, there is an  $\perp$ -Morley sequence  $(b_i d_i)_{i < \omega}$  over  $M$  with  $b_0 = b$  such that  $\bigcup_{i < \omega} p_a(x, b_i, d_i)$  is inconsistent. By compactness, we can find some formula  $\psi_a(x, b_a, d_a) \in p_a(x, b, d)$  with  $b_a \subseteq b$ ,  $d_a \subseteq d$  such that  $\{\psi_a(x, b_{a,i}, d_{a,i}) : i < \omega\}$  is inconsistent, where  $b_{a,i} d_{a,i}$  denotes the restriction of  $b_i d_i$  to the appropriate variables.

Now, the set  $\{\varphi(x, b)\} \cup \{\neg \psi_a(x, b_a, d_a) : a \models \varphi(x, b)\}$  is inconsistent. By com-

pactness, we can thus find realisations  $a_0, \dots, a_{n-1} \models \varphi(x, b)$  such that

$$\varphi(x, b) \vdash \bigvee_{i < n} \psi_{a_i}(x, b_{a_i}, d_{a_i}).$$

This proves the claim.  $\square$

**Definition 5.2.8.** Let  $\perp$  be an independence relation.

- (i) We define  $\perp$ -**Kim-independence** to be  $(\perp^+)^*$ .
- (ii) We say  $\varphi(x, b)$   $\perp$ -**Kim-divides over**  $M$  if there is an  $\perp$ -Morley sequence  $(b_i)_{i < \omega}$  over  $M$  with  $b_0 = b$  such that  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent.
- (iii) We say  $\varphi(x, b)$  **weakly**  $\perp$ -**Kim-divides over**  $M$  if it is not  $\perp^+$ -free over  $M$ .
- (iv) We say  $\varphi(x, b)$   $\perp$ -**Kim-forks over**  $M$  if it is not  $(\perp^+)^*$ -free over  $M$ .

**Lemma 5.2.9.** Let  $\perp$  be an independence relation satisfying monotonicity.

- (i) If  $\varphi(x, b)$   $\perp$ -Kim-divides over  $M$ , then it weakly  $\perp$ -Kim-divides over  $M$ .
- (ii) A formula  $\varphi(x, b)$   $\perp$ -Kim-forks over  $M$  iff there exist formulae  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$  and each  $\psi_i(x, d_i)$   $\perp$ -Kim-divides over  $M$ .
- (iii) We have  $A$  is  $\perp$ -Kim-independent from  $B$  over  $M$  iff, for some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/MB)$  does not contain any formula that  $\perp$ -Kim-forks over  $M$ .

*Proof.* (i) This follows directly from Lemma 5.2.4.

(ii) This is a rephrasing of Lemma 5.2.7.

(iii) This follows from strong finite character for  $(\perp^+)^*$  by Lemma 5.1.9.  $\square$

**Example 5.2.10.** We refer to  $\perp^i$ -Kim-independence as **Kim-independence**, and denote it by  $\perp^K$ . We will later show that this notion coincides with that first introduced in [59] and which we already defined in Definition 3.1.4. Note:

$$\perp^f \implies \perp^K \implies \perp^a.$$

**Remark 5.2.11.** We extend the terminology above for complete types as follows: we say  $p(x) \in S(Mb)$   $\perp$ -**Kim-divides over**  $M$  (resp.,  $\perp$ -**Kim-forks over**  $M$ ) if it is not  $\perp^+$ -free (resp.,  $(\perp^+)^*$ -free) over  $M$ . Using Lemma 5.2.5 and monotonicity of  $\perp$ , it is easy to show that  $p(x)$   $\perp$ -Kim-divides over  $M$  iff there is a formula  $\varphi(x, b') \in p(x)$  such that  $\varphi(x, b')$   $\perp$ -Kim-divides over  $M$ .

The notion of “weak  $\perp$ -Kim-dividing” is new and arises from the attempt to build  $\perp$ -Kim-independence directly from the condition of lifting indiscernibility. It would be very satisfactory if the notions of  $\perp$ -Kim-dividing and weak  $\perp$ -Kim-dividing coincided. However, this only happens in the limit case when  $\perp$ -Kim-dividing coincides with  $\perp$ -Kim-forking:

**Proposition 5.2.12.** *Let  $\perp$  be an independence relation satisfying monotonicity. The following are equivalent:*

- (i) *If  $\pi(x, b)$  weakly  $\perp$ -Kim-divides over  $M$ , then it  $\perp$ -Kim-divides over  $M$ .*
- (ii) *There is an independence relation  $\perp^\oplus$  satisfying monotonicity such that  $\pi(x, b)$  does not  $\perp$ -Kim-divide over  $M$  iff it is  $\perp^\oplus$ -free over  $M$ .*
- (iii) *If  $\pi(x, b)$  does not  $\perp$ -Kim-divide over  $M$ , then there is a complete type  $p(x) \in S(Mb)$  containing  $\pi(x, b)$  such that  $p(x)$  does not  $\perp$ -Kim-divide over  $M$ .*
- (iv) *If  $\pi(x, b)$   $\perp$ -Kim-forks over  $M$ , then it  $\perp$ -Kim-divides over  $M$ .*

*Proof.* ((i)  $\Rightarrow$  (ii)) By (i) and Lemma 5.2.4, we can take  $\perp^\oplus = \perp^+$ .

((ii)  $\Rightarrow$  (iii)) Suppose such an independence relation exists.

**Claim.**  $(\perp^\oplus)^{\text{sfc}} = \perp^+$ .

*Proof of Claim.* ( $\Rightarrow$ ) Suppose that  $A \not\perp_M^+ B$ . Let  $p(x, b) := \text{tp}(A/MB)$  (where  $b$  enumerates  $B$ ). By Lemma 5.2.5, there is an  $\perp$ -Morley sequence  $(b_i)_{i < \omega}$  over  $M$  with  $b_0 = b$  such that  $\bigcup_{i < \omega} p(x, b_i)$  is inconsistent. By compactness, there is a formula  $\varphi(x', b') \in p(x, b)$  with  $x' \subseteq x$ ,  $b' \subseteq b$  finite such that  $\{\varphi(x', b'_i) : i < \omega\}$  is inconsistent. Thus, by monotonicity,  $\varphi(x', b')$   $\perp$ -Kim-divides over  $M$ . By our choice of independence relation,  $\varphi(x', b')$  is not  $\perp^\oplus$ -free over  $M$ . Hence  $A (\not\perp_M^\oplus)^{\text{sfc}} B$ .

( $\Leftarrow$ ) Suppose  $A (\not\perp_M^\oplus)^{\text{sfc}} B$ . By Lemma 5.1.9, there is some  $\varphi(x, b) \in \text{tp}(A/MB)$  which is not  $\perp^\oplus$ -free over  $M$ . By definition,  $\varphi(x, b)$   $\perp$ -Kim-divides over  $M$ . Hence, by Lemma 5.2.4,  $\varphi(x, b)$  is not  $\perp^+$ -free over  $M$ . Therefore, by Lemma 5.1.9 and Proposition 5.2.6,  $A \not\perp_M^+ B$ .  $\square_{\text{Claim}}$

Therefore, by Lemma 5.1.11, if  $\pi(x, b)$  does not  $\perp$ -Kim-divide over  $M$ , then it is  $\perp^\oplus$ -free over  $M$ , hence also  $(\perp^\oplus)^{\text{sfc}}$ -free over  $M$  by Lemma 5.1.8(ii). Thus, there is  $a \models \pi(x, b)$  such that  $a (\perp^\oplus)^{\text{sfc}} b$ . By the Claim, this means that  $a \perp_M^+ b$ . By Lemma 5.2.5,  $\text{tp}(a/Mb)$  is the type we wanted.

((iii)  $\Rightarrow$  (iv)) Assume, for contradiction, that  $\pi(x, b)$   $\perp$ -Kim-forks but does not  $\perp$ -Kim-divide over  $M$ . By Lemma 5.2.9(ii), there are formulae  $\psi_i(x, d_i)$  for  $i < n$

such that  $\pi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$  and each  $\psi_i(x, d_i) \perp\!\!\!\downarrow$ -Kim-divides over  $M$ .

Now, observe there is a partial type  $\rho(x, b, d_{<n})$  such that  $\pi(x, b) \leftrightarrow \rho(x, b, d_{<n})$  holds. Let  $(b_i d_{0,i} \dots d_{n-1,i})_{i < \omega}$  be an  $\perp\!\!\!\downarrow$ -Morley sequence over  $M$  starting at  $bd_{<n}$ . By monotonicity,  $(b_i)_{i < \omega}$  is an  $\perp\!\!\!\downarrow$ -Morley sequence over  $M$  starting at  $b$ , and thus, by assumption,  $\bigcup_{i < \omega} \pi(x, b_i)$  is consistent. Since  $b_i d_{<n,i} \equiv_M bd_{<n}$ , it follows that  $\pi(x, b_i) \leftrightarrow \rho(x, b_i, d_{<n,i})$  for each  $i$ . Therefore,  $\bigcup_{i < \omega} \rho(x, b_i, d_{<n,i})$  is consistent.

By (iii), there is a complete type  $q(x) \in S(Mbd_0 \dots d_{n-1})$  such that  $\rho(x, b, d_{<n}) \subseteq q(x)$  and  $q$  does not  $\perp\!\!\!\downarrow$ -Kim-divide over  $M$ . By choice of formulas, there is some  $i < n$  such that  $\psi_i(x, d_i) \in q(x)$ . But then  $q(x)$  does  $\perp\!\!\!\downarrow$ -Kim-divide over  $M$ , a contradiction.

((iv)  $\Rightarrow$  (i)) Suppose  $\pi(x, b)$  does not  $\perp\!\!\!\downarrow$ -Kim-divide over  $M$ . By (iv), it does not  $\perp\!\!\!\downarrow$ -Kim-fork over  $M$ , and so by definition it is  $(\perp\!\!\!\downarrow^+)^*$ -free over  $M$ . In particular,  $\pi(x, b)$  is  $\perp\!\!\!\downarrow^+$ -free over  $M$ .  $\square$

Although the notions of  $\perp\!\!\!\downarrow$ -Kim-dividing and weak  $\perp\!\!\!\downarrow$ -Kim-dividing have different properties (cf., Lemma 5.2.17), one of the goals of this chapter is to show that both notions can equivalently be used in developing the structure theory of several neostability-theoretic classes via independence relations. We will make this point explicitly in the next section (Lemmas 5.3.10 and 5.3.11).

We can also provide an explicit criterion for the equivalence of weak  $\perp\!\!\!\downarrow$ -Kim-dividing and  $\perp\!\!\!\downarrow$ -Kim-forking at the level of formulas:

**Lemma 5.2.13.** *Let  $\perp\!\!\!\downarrow$  be an independence relation satisfying monotonicity. The following are equivalent:*

- (i) *For any  $\varphi(x, y) \in \mathcal{L}(M)$ , if  $\varphi(x, b) \perp\!\!\!\downarrow$ -Kim-forks over  $M$ , then it weakly  $\perp\!\!\!\downarrow$ -Kim-divides over  $M$ .*
- (ii) *If  $p(x) \in S(Mb)$  and  $p(x) \perp\!\!\!\downarrow$ -Kim-forks over  $M$ , then  $p(x) \perp\!\!\!\downarrow$ -Kim-divides over  $M$ .*

*Proof.* ((i)  $\Rightarrow$  (ii)) Suppose  $p(x) \perp\!\!\!\downarrow$ -Kim-forks over  $M$ . By Proposition 5.2.6 and definition, there is some  $\varphi(x, b) \in p(x)$  that  $\perp\!\!\!\downarrow$ -Kim-forks over  $M$ . By (i),  $\varphi(x, b)$  weakly  $\perp\!\!\!\downarrow$ -Kim-divides over  $M$ . Since  $\varphi(x, b) \in p(x)$ , this means that  $p(x) \perp\!\!\!\downarrow$ -Kim-divides over  $M$ .

((ii)  $\Rightarrow$  (i)) Assume  $\varphi(x, b)$  does not weakly  $\perp\!\!\!\downarrow$ -Kim-divide over  $M$ . Then there is some  $a \models \varphi(x, b)$  such that  $a \perp\!\!\!\downarrow_M^+ b$ , i.e.,  $\text{tp}(a/Mb)$  does not  $\perp\!\!\!\downarrow$ -Kim-divide over

$M$ . By (ii),  $\text{tp}(a/Mb)$  does not  $\perp$ -Kim-fork over  $M$ . Hence, by Lemmas 5.2.5 and 5.1.14,  $\varphi(x, b)$  does not  $\perp$ -Kim-fork over  $M$ .  $\square$

We will discuss examples of this later on (see Example 5.2.20). First, we will establish several basic properties of  $\perp$ -Kim-independence as defined here.

**Lemma 5.2.14.** *For any independence relation  $\perp$ ,  $\perp$ -Kim-independence satisfies monotonicity, strong finite character, existence, right extension, and normality.*

*Proof.* This follows directly from Lemma 5.2.3, Proposition 5.2.6, Lemma 5.1.8, and Fact 2.4.12. Existence will follow from Lemma 5.2.18.  $\square$

**Lemma 5.2.15.** *Let  $\perp^1$  and  $\perp^2$  be two independence relations and  $M \models T$ . If  $\perp^1 \implies \perp^2$  and  $\varphi(x, b)$   $\perp^1$ -Kim-divides over  $M$ , then it  $\perp^2$ -Kim-divides over  $M$ . In particular,  $\perp^2$ -Kim-independence implies  $\perp^1$ -Kim-independence.*

*Proof.* Suppose  $\varphi(x, b)$   $\perp^1$ -Kim-divides over  $M$ . Let  $(b_i)_{i < \omega}$  be an  $\perp^1$ -Morley sequence over  $M$  such that  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent. As  $\perp^1 \implies \perp^2$ , it follows that  $(b_i)_{i < \omega}$  is  $\perp^2$ -Morley over  $M$ , and thus  $\varphi(x, b)$  also  $\perp^2$ -Kim-divides over  $M$ . This proof can be adapted for the case of weak  $\perp$ -Kim-dividing. For a more abstract proof, observe that, since  $\perp^1 \implies \perp^2$ , we have  $(\perp^2)^+ \implies (\perp^1)^+$ .

For the “in particular”, note that, by the previous paragraph and Fact 2.4.12(ii),  $((\perp^2)^+)^* \implies ((\perp^1)^+)^*$ , which means that  $\perp^2$ -Kim-independence implies  $\perp^1$ -Kim-independence.  $\square$

**Example 5.2.16.**  $\perp^f$ -Kim-independence  $\implies$  Kim-independence  $\implies \perp^u$ -Kim-independence.

The following is an example of a result that seems to require the assumption that “ $\perp$ -Kim-forking =  $\perp$ -Kim-dividing” and not just the weaker “ $\perp$ -Kim-forking = weakly  $\perp$ -Kim-dividing”:

**Lemma 5.2.17.** *Let  $\perp$  be an independence relation satisfying full existence and monotonicity. Suppose that, for any formula  $\varphi(x, b)$  and any  $M \models T$ ,  $\varphi(x, b)$   $\perp$ -Kim-forks over  $M$  iff it  $\perp$ -Kim-divides over  $M$ . Then  $\perp$ -Kim-independence satisfies left extension.*

*Proof.* We adapt the proof of [29, Claim 3.20]. Suppose that  $a$  is  $\perp$ -Kim-independent from  $b$  over  $M$ , and let  $c$  be a tuple. By Lemma 5.2.9(iii), it suffices to show

that there is  $c' \equiv_{Ma} c$  such that  $\text{tp}(c'a/Mb)$  does not  $\downarrow$ -Kim-divide over  $M$ . Let  $p(x, a) := \text{tp}(c/Ma)$ . We will show that the set

$$p(x, a) \cup \{\neg\varphi(x, a', b') : \varphi \in \mathcal{L}(M), a' \subseteq a, b' \subseteq b, \varphi(x, y, b') \downarrow\text{-Kim-divides over } M\}$$

is consistent.

Assume not, for contradiction. By compactness,  $p(x, a) \vdash \bigvee_{i < \ell} \varphi_i(x, a', b')$  and each  $\varphi_i(x, y, b') \downarrow$ -Kim-divides over  $M$ . Let  $\psi(x, y, b) := \bigvee_{i < \ell} \varphi_i(x, y, b')$ . Then  $\psi(x, y, b) \downarrow$ -Kim-forks over  $M$ , so by our assumption it  $\downarrow$ -Kim-divides over  $M$ .

Let  $I = (b_i)_{i < \omega}$  be an  $\downarrow$ -Morley sequence over  $M$  such that  $\{\psi(x, y, b_i) : i < \omega\}$  is inconsistent. Since  $a$  is  $\downarrow$ -Kim-independent from  $b$  over  $M$ , in particular we have  $a \downarrow_M^+ b$ . So there exists a sequence  $I'' = (b''_i)_{i < \omega}$  with  $I'' \equiv_{Mb} I$  such that  $I''$  is  $Ma$ -indiscernible. But  $p(x, a) \vdash \psi(x, a, b)$ . So, by indiscernibility,  $p(x, a) \vdash \{\psi(x, a, b''_i) : i < \omega\}$ , and thus  $p(x, a)$  is inconsistent, a contradiction.  $\square$

**Lemma 5.2.18.** *For any independence relation  $\downarrow$ , we have  $\downarrow^f \implies \downarrow$ -Kim-independence.*

*Proof.* By Fact 2.4.12(v), it suffices to show that  $\downarrow^d \implies \downarrow^+$ . But this follows from [106, Lemma 1.4] (the basic characterisation of dividing) and the fact that every  $\downarrow$ -Morley sequence over  $M$  (if it exists) is  $M$ -indiscernible by definition.  $\square$

Let us note that, despite the close relation between forking and the relative notions of Kim-forking introduced above, in general, we cannot fit forking within this framework. There is, however, a large class of theories in which this is possible:

**Lemma 5.2.19.** *If  $T$  is  $\text{NTP}_2$ , then, for all  $M \models T$ ,  $\varphi(x, b)$  divides over  $M$  iff it  $\downarrow^a$ -Kim-divides over  $M$ . In particular,  $\downarrow^f$  and  $\downarrow^a$ -Kim-independence coincide over models.*

*Proof.* Suppose  $\varphi(x, b)$  divides over  $M \models T$ . Since  $T$  is  $\text{NTP}_2$ ,  $\varphi(x, b)$  divides along every strictly  $M$ -invariant Morley sequence over  $M$ . Since such sequences exist and are in particular  $\downarrow^a$ -Morley, it follows that  $\varphi(x, b) \downarrow^a$ -Kim-divides over  $M$ .  $\square$

**Example 5.2.20.** Let  $T$  be the theory of the generic triangle-free graph. In [35], Conant characterises dividing in this theory. His proof (more specifically, the choice of indiscernible sequence in Construction 4.3) shows that  $\varphi(x, b)$  divides over  $C$  iff it  $\downarrow^a$ -Kim-divides over  $C$ , for any set of parameters  $C$ . In particular, the converse to the above lemma does not hold.

This example also illustrates the distinction between  $\perp$ -Kim-dividing and weak  $\perp$ -Kim-dividing in general. By [35, Theorem 5.3], forking and dividing agree for complete types in this theory, and thus, by the previous paragraph and Lemma 5.2.13, weak  $\perp^a$ -Kim-dividing and  $\perp^a$ -Kim-forking agree for formulas. [35, Theorem 6.2] shows that  $\perp^a$ -Kim-dividing and weak  $\perp^a$ -Kim-dividing do not agree.

**Example 5.2.21.** Let  $T = T_L^\emptyset$  be the model completion of the empty  $L$ -theory, where  $L = \{f\}$  consists of a binary function symbol. This theory is known to be strictly NSOP<sub>1</sub> (see [75]). Let  $\varphi(x, a, b) := f(x, a) = b$ , and let  $M \models T$  be a model such that  $a, b \notin M$ .

Let  $(a_i b_i)_{i < \omega}$  be an  $M$ -indiscernible sequence with  $a_0 = a$  and  $b_0 = b$  such that  $b_i \neq b_j$  for all  $i \neq j$  and  $a_i = a$  for all  $i < \omega$ . (Note that we can indeed find such a sequence.) In this case,  $\{\varphi(x, a_i, b_i) : i < \omega\} = \{f(x, a) = b_i : i < \omega\}$  is 2-inconsistent. Now let  $(a_i^* b_i^*)_{i < \omega}$  be an  $M$ -indiscernible  $\perp^a$ -Morley sequence over  $M$  with  $a_0^* = a$  and  $b_0^* = b$ . Since  $a \notin M$ , we have that  $a_i^* \neq a_j^*$  for all  $i \neq j$ . In particular,  $\{\varphi(x, a_i^*, b_i^*) : i < \omega\} = \{f(x, a_i^*) = b_i^* : i < \omega\}$  is consistent. This shows that  $\varphi(x, a, b)$  divides but does not  $\perp^a$ -Kim-divide over  $M$ .

## 5.2.2 Relative Conant-independence

We now introduce a new independence relation, which can be seen as an “existential version” of  $\perp^+$  from the previous subsection:

**Definition 5.2.22.** Let  $\perp$  be an independence relation. We define  $\perp^\times$  as follows:  $A \perp_M^\times B$  iff there is an  $\perp$ -Morley sequence  $I$  over  $M$  starting at  $B$  which is  $MA$ -indiscernible.

Observe that concrete instances of this relation have previously appeared in the literature, e.g., [92].

**Lemma 5.2.23.** *Let  $\perp$  be an independence relation. Then  $\perp^\times$  is an independence relation satisfying left normality. If, in addition,  $\perp$  satisfies full existence and monotonicity, then  $\perp^\times$  also satisfies existence.*

*Proof.* We first check that  $\perp^\times$  is an independence relation. Suppose that  $A \perp_M^\times B$  and  $\sigma \in \text{Aut}(M)$ . Let  $(B_i)_{i < \omega}$  be an  $\perp$ -Morley sequence over  $M$  with  $B_0 = B$  which is  $MA$ -indiscernible. Then  $(\sigma(B_i))_{i < \omega}$  is  $\perp$ -Morley over  $\sigma(M)$ ,  $\sigma(B_0) = \sigma(B)$ , and the sequence is  $\sigma(M)\sigma(A)$ -indiscernible. Therefore,  $\sigma(A) \perp_{\sigma(M)}^\times \sigma(B)$ .

(Existence) By full existence and monotonicity, there is an  $\perp$ -Morley sequence  $(M_i)_{i<\omega}$  over  $M$  with  $M_0 = M$ . But then, by indiscernibility,  $M_i = M$  for all  $i < \omega$ . So clearly, for any  $A$ , we have  $A \perp_M^\times M$ .

(Left normality) This is again immediate.  $\square$

**Lemma 5.2.24.** *If  $\pi(x, b)$  is  $\perp^\times$ -free over  $M$ , then there is an  $\perp$ -Morley sequence  $(b_i)_{i<\omega}$  over  $M$  such that  $\bigcup_{i<\omega} \pi(x, b_i)$  is consistent.*

*Proof.* Suppose  $\pi(x, b)$  is  $\perp^\times$ -free over  $M$ . By definition, there is some  $c \models \pi(x, b)$  such that  $c \perp_M^\times b$ . Thus, there exists an  $\perp$ -Morley sequence  $(b_i)_{i<\omega}$  over  $M$  with  $b_0 = b$  that is  $Mc$ -indiscernible. In particular,  $c \models \bigcup_{i<\omega} \pi(x, b_i)$ .  $\square$

**Lemma 5.2.25.** *A type  $p(x, b) \in S(Mb)$  is  $\perp^\times$ -free over  $M$  iff there exists an  $\perp$ -Morley sequence  $(b_i)_{i<\omega}$  over  $M$  with  $b_0 = b$  such that  $\bigcup_{i<\omega} p(x, b_i)$  is consistent.*

*Proof.* ( $\Rightarrow$ ) This is the previous lemma.

( $\Leftarrow$ ) Suppose there is an  $\perp$ -Morley sequence  $I = (b_i)_{i<\omega}$  over  $M$  with  $b_0 = b$  such that  $\bigcup_{i<\omega} p(x, b_i)$  is consistent. Let  $a' \models \bigcup_{i<\omega} p(x, b_i)$ . By Ramsey, compactness, an automorphism and the completeness of  $p$ , we may assume  $I$  is  $Ma'$ -indiscernible. We can then use automorphisms to find such a sequence for any  $a \models p$ , which implies that  $p$  is  $\perp^\times$ -free over  $M$ .  $\square$

But note that the proof of strong finite character in Proposition 5.2.6 does not extend to this case. By applying compactness, we obtain, for each  $\perp$ -Morley sequence, a formula that divides along it. But there is no guarantee here that we can find a single formula that works for all of those sequences.

However, in this case, we can characterise  $(\perp^\times)^{\text{sfc}}$  in a very concrete fashion:

**Theorem 5.2.26.** *Suppose  $\perp$  satisfies monotonicity. If there is an  $\perp$ -Morley sequence  $(b_i)_{i<\omega}$  over  $M$  such that the set  $\{\varphi(x, b_i) : i < \omega\}$  is consistent, then  $\varphi(x, b)$  is  $\perp^\times$ -free over  $M$ .*

*Proof.* Suppose that  $\varphi(x, b)$  is not  $\perp^\times$ -free over  $M$ . This means that, for all  $a \models \varphi(x, b)$ , the type  $\text{tp}(a/Mb)$  is not  $\perp^\times$ -free over  $M$ .

Fix an  $\perp$ -Morley sequence  $(b_i)_{i<\omega}$  over  $M$  with  $b_0 = b$ . Let  $a \models \varphi(x, b)$  and denote  $p_a(x, y) := \text{tp}(ab/M)$ . Then, by Lemma 5.2.25, the set  $\bigcup_{i<\omega} p_a(x, b_i)$  is inconsistent. So, by compactness, there is a formula  $\psi_a(x, b_a) \in \text{tp}(a/Mb)$  with

$b_a \subseteq b$  such that  $\{\psi_a(x, b_{a,i}) : i < \omega\}$  is inconsistent, where as usual  $b_{a,i}$  denotes the restriction of  $b_i$  to the appropriate variables. Now, the set  $\{\varphi(x, b)\} \cup \{\neg\psi_a(x, b_a) : a \models \varphi(x, b)\}$  is inconsistent by our choice of  $\psi_a$ 's. Thus, by compactness, there exist realisations  $a_0, \dots, a_{n-1} \models \varphi(x, b)$  such that

$$\varphi(x, b) \vdash \bigvee_{j < n} \psi_{a_j}(x, b_{a_j}).$$

Assume now, for contradiction, that  $\{\varphi(x, b_i) : i < \omega\}$  is consistent. By the above and pigeonhole, there is an infinite subset  $I \subseteq \omega$  and some  $j < n$  such that  $\{\psi_{a_j}(x, b_{a_j,i}) : i \in I\}$  is consistent. So, by  $M$ -indiscernibility,  $\{\psi_{a_j}(x, b_{a_j,i}) : i < \omega\}$  is consistent, which contradicts our choice of  $\psi_{a_j}$ .

Since the sequence was arbitrary, we have shown that  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent for all  $\downarrow$ -Morley sequences  $(b_i)_{i < \omega}$  over  $M$  with  $b_0 = b$ .  $\square$

**Definition 5.2.27.** Let  $\downarrow$  be an independence relation.

- (i) We define  $\downarrow$ -**Conant-independence** to be  $((\downarrow^\times)^{\text{sfc}})^*$ .
- (ii) We say  $\varphi(x, b) \downarrow$ -**Conant-divides over**  $M$  if it is not  $\downarrow^\times$ -free over  $M$ .
- (iii) We say  $\varphi(x, b) \downarrow$ -**Conant-forks over**  $M$  if it is not  $(\downarrow^\times)^*$ -free over  $M$ .

**Lemma 5.2.28.** Let  $\downarrow$  be an independence relation satisfying monotonicity.

- (i) A formula  $\varphi(x, b) \downarrow$ -Conant-divides over  $M$  iff, for every  $\downarrow$ -Morley sequence  $(b_i)_{i < \omega}$  over  $M$  with  $b_0 = b$ , the set  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent.
- (ii) A formula  $\varphi(x, b) \downarrow$ -Conant-forks over  $M$  iff there exist formulae  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$  and each  $\psi_i(x, d_i) \downarrow$ -Conant-divides over  $M$ .
- (iii) We have  $A$  is  $\downarrow$ -Conant-independent from  $B$  over  $M$  iff, for some (equiv., any) enumeration  $a$  of  $A$ ,  $\text{tp}(a/MB)$  does not contain any formulas  $\downarrow$ -Conant-forking over  $M$ .

*Proof.* (i) This restates Lemma 5.2.24 and Theorem 5.2.26.

(ii) We have that  $\varphi(x, b) \downarrow$ -Conant-forks over  $M$  iff  $\varphi(x, b)$  is not  $((\downarrow^\times)^{\text{sfc}})^*$ -free over  $M$ . By Proposition 5.1.17, this holds iff there are formulae  $\psi_i(x, d_i)$  for  $i < n$  such that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, d_i)$  and no  $\psi_i(x, d_i)$  is  $\downarrow^\times$ -free over  $M$ . But by definition  $\psi_i(x, d_i)$  is not  $\downarrow^\times$ -free over  $M$  iff it  $\downarrow$ -Conant-divides over  $M$ . This proves the result.

(iii) This is Lemma 5.1.13 and (i).  $\square$

- Example 5.2.29.** (i) We refer to  $\perp^i$ -Conant-independence simply as **Conant-independence**. As before, we will later see that this notion coincides with that introduced by Mutchnik with the same name in [94].
- (ii) Following [72], we sometimes refer to  $\perp^u$ -Conant-independence as **coheir-Conant-independence**.

**Remark 5.2.30.** As before, we extend the terminology to complete types as follows: we say  $p(x) \in S(Mb) \perp$ -**Conant-divides** (resp.,  $\perp$ -**Conant-forks**) over  $M$  if it is not  $(\perp^\times)^{\text{sfc}}$ -free (resp.,  $((\perp^\times)^{\text{sfc}})^*$ -free) over  $M$ . It is immediate from the definition that  $p(x) \perp$ -Conant-divides over  $M$  iff there is a formula  $\varphi(x, b') \in p(x)$  such that  $\varphi(x, b') \perp$ -Conant-divides over  $M$ .

We have observed in the previous subsection that  $\perp$ -Kim-forking and  $\perp$ -Kim-dividing might coincide for complete types without doing so for formulas (see Example 5.2.20). An analogous argument to Lemma 5.2.13 shows that this cannot happen for  $\perp$ -Conant-dividing:

**Proposition 5.2.31.** *Let  $\perp$  be an independence relation satisfying full existence and monotonicity. Then  $\perp$ -Conant-forking and  $\perp$ -Conant-dividing agree for complete types iff they agree for formulas.*

*Proof.* Left-to-right is clear by definition. For the converse, suppose that  $\varphi(x, b) \perp$ -Conant-forks over  $M$ . By definition, it is not  $(\perp^\times)^*$ -free over  $M$ , so by Lemma 5.1.14, for every  $a \models \varphi(x, b)$ ,  $\text{tp}(a/Mb) \perp$ -Conant-forks over  $M$ . By assumption,  $\text{tp}(a/Mb) \perp$ -Conant-divides over  $M$ . Hence, for every  $a \models \varphi(x, b)$ , we have  $a(\perp^\times)_M^{\text{sfc}} b$ . Therefore,  $\varphi(x, b)$  is not  $\perp^\times$ -free over  $M$ , i.e.,  $\varphi(x, b) \perp$ -Conant-divides over  $M$ .  $\square$

For the rest of this subsection, we will focus on proving some preliminary properties of  $\perp$ -Conant-independence.

**Lemma 5.2.32.** *For any independence relation  $\perp$ ,  $\perp$ -Conant-independence satisfies monotonicity, strong finite character, right extension, and normality. Moreover, if  $\perp$  satisfies full existence and monotonicity, then  $\perp$ -Conant-independence satisfies existence, and the following holds:*

**Weak left extension:** *If  $a$  is  $\perp$ -Conant-independent from  $b$  over  $M$  and  $c$  is a tuple, then there is some  $c'a' \equiv_M ca$  such that  $\text{tp}(c'a'/Mb)$  does not  $\perp$ -Conant-divide over  $M$ .*

*Proof.* The first four properties follow directly from Lemmas 5.2.23 and 5.1.8 and

Fact 2.4.12. For the “moreover” part, we note that existence will follow from Lemma 5.2.36 and prove weak left extension here.

(Weak left extension) We adapt the proof from [91, Lemma 2.1(2)]. Suppose that  $a$  is  $\perp$ -Conant-independent from  $b$  over  $M$ . Choose  $M' \succ M$  to be a sufficiently saturated extension. By right extension, we can find some  $a' \equiv_{Mb} a$  such that  $a'$  is  $\perp$ -Conant-independent from  $M'$  over  $M$ . So we may replace  $a$  by  $a'$  and assume that  $b = M'$ .

**Claim.** For any  $c$  such that  $\text{tp}(c/M')$  does not  $\perp$ -Conant-divide over  $M$ , we have that  $c$  is  $\perp$ -Conant-independent from  $M'$  over  $M$ .

*Proof.* Suppose otherwise. Let  $\varphi(x, d) \in \text{tp}(c/M')$  be a formula which  $\perp$ -Conant-forks over  $M$ . Thus, there exist  $\psi_i(x, e_i)$   $\perp$ -Conant-dividing over  $M$  such that  $\varphi(x, d) \vdash \bigvee_{i < n} \psi_i(x, e_i)$ . By saturation of  $M'$ , we can find some  $e'_0, \dots, e'_{n-1}$  in  $M'$  such that  $e'_0 \dots e'_{n-1} \equiv_{Md} e_0 \dots e_{n-1}$ . Hence,  $\psi_i(x, e'_i)$   $\perp$ -Conant-divides over  $M$ , and  $\varphi(x, d) \vdash \bigvee_{i < n} \psi_i(x, e'_i)$  by invariance. But then, as  $\text{tp}(c/M')$  is a complete type over  $M'$ , it follows that there is some  $i < n$  with  $\psi_i(x, e'_i) \in \text{tp}(c/M')$ , contradicting our initial assumption. □<sub>Claim</sub>

Take some  $c$ . We want to show that there is  $c'a' \equiv_M ca$  with  $a' \equiv_{M'} a$  and  $\text{tp}(c'a'/M')$  not  $\perp$ -Conant-dividing over  $M$ . By compactness, for  $\psi(y, x) \in \text{tp}(ca/M)$  and  $\varphi(x, d) \in \text{tp}(a/M')$  with  $d \subseteq M'$ , it is enough to find some  $c'a'$  such that  $\models \psi(c', a') \wedge \varphi(a', d)$ , so that  $\text{tp}(c'a'/Md)$  does not  $\perp$ -Conant-divide over  $M$ .

Now, the formula  $\exists y(\psi(y, x) \wedge \varphi(x, d)) \in \text{tp}(a/M')$ , and so in particular it does not  $\perp$ -Conant-divide over  $M$ . Hence, there is some  $\perp$ -Morley sequence  $I = (d_i)_{i < \omega}$  over  $M$  starting at  $d$  such that  $\{\exists y(\psi(y, x) \wedge \varphi(x, d_i)) : i < \omega\}$  is consistent. Let  $a'$  be a realisation. By Ramsey, compactness, and an automorphism, we may assume  $a'$  is such that  $I$  is  $Ma'$ -indiscernible. In particular,  $\models \exists y \psi(y, a')$ , so let  $c'$  be a realisation. Another application of Ramsey and compactness allows us to assume  $I$  is  $Ma'c'$ -indiscernible.

So it suffices to show that  $\text{tp}(c'a'/Md)$  does not  $\perp$ -Conant-divide over  $M$ . Let  $e \subseteq d$  be finite. For  $i < \omega$ , we can find  $e_i \subseteq d_i$  such that  $(e_i)_{i < \omega}$  starts with  $e$  and is  $Ma'c'$ -indiscernible. Now, as  $I$  is  $\perp$ -Morley, we have

$$d_i \underset{M}{\perp} d_0 \dots d_{i-1}.$$

By monotonicity of  $\perp$ , it follows that

$$e_i \underset{M}{\perp} e_0 \dots e_{i-1}.$$

Hence,  $(e_i)_{i < \omega}$  is  $\perp$ -Morley.

Now take  $\theta(x, y, e) \in \text{tp}(c'a'/Md)$ . By  $Ma'c'$ -indiscernibility,  $\{\theta(x, y, e_i) : i < \omega\}$  is consistent. Therefore,  $\theta(x, y, e)$  does not  $\perp$ -Conant-divide over  $M$ .  $\square$

**Lemma 5.2.33.** *Suppose  $\perp$  satisfies full existence, monotonicity, and left and right extension. Then  $\varphi(x, b)$   $\perp$ -Conant-forks over  $M$  iff it  $\perp$ -Conant-divides over  $M$ .*

*Proof.* Again, we adapt the standard proof from the literature (cf., [95, Proposition 5.2]). Let us first begin with a claim:

**Claim.** If  $\models \varphi(x, a) \leftrightarrow \psi(x, a, b)$ , then  $\varphi(x, a)$   $\perp$ -Conant-divides over  $M$  iff so does  $\psi(x, a, b)$ .

*Proof of Claim.* ( $\Rightarrow$ ) Suppose  $\psi(x, a, b)$  does not  $\perp$ -Conant-divide over  $M$ ; let  $(a_i b_i)_{i < \omega}$  be an  $\perp$ -Morley sequence over  $M$  witnessing this. By monotonicity,  $(a_i)_{i < \omega}$  is still  $\perp$ -Morley over  $M$ . By assumption,  $\{\varphi(x, a_i) : i < \omega\}$  is consistent. So  $\varphi(x, a)$  does not  $\perp$ -Conant-divide over  $M$ .

( $\Leftarrow$ ) Suppose  $(a_i)_{i < \omega}$  is an  $\perp$ -Morley sequence over  $M$  with  $a_0 = a$  such that  $\{\varphi(x, a_i) : i < \omega\}$  is consistent. By right and left extension for  $\perp$ , there is an  $\perp$ -Morley sequence  $(a_i b_i)_{i < \omega}$  over  $M$  such that  $a_0 b_0 \equiv_M ab$ . Therefore,  $\{\psi(x, a_i, b_i) : i < \omega\}$  is consistent, as required.  $\square_{\text{Claim}}$

Now suppose that  $\varphi(x, b)$   $\perp$ -Conant-forks but does not  $\perp$ -Conant-divide over  $M$ . By the claim, this means that  $\varphi(x, b) \vdash \bigvee_{i < n} \psi_i(x, b)$  and each  $\psi_i(x, b)$   $\perp$ -Conant-divides over  $M$ . Let  $(b_i)_{i \in \omega}$  be an  $\perp$ -Morley sequence over  $M$  with  $b_0 = b$  such that  $\{\varphi(x, b_i) : i < \omega\}$  is consistent. Then, by the pigeonhole principle, there is an infinite  $I \subseteq \omega$  and some  $k < n$  such that  $\{\psi_k(x, b_i) : i < \omega\}$  is consistent, a contradiction.  $\square$

**Lemma 5.2.34.** *Let  $\perp^1$  and  $\perp^2$  be two independence relations satisfying full existence and monotonicity, and let  $M \models T$ . If  $\perp^1 \Rightarrow \perp^2$  and  $\varphi(x, b)$   $\perp^2$ -Conant-divides over  $M$ , then it  $\perp^1$ -Conant-divides over  $M$ . In particular,  $\perp^1$ -Conant-independence implies  $\perp^2$ -Conant-independence.*

*Proof.* Simply observe that, since  $\perp^1 \Rightarrow \perp^2$ , we have  $(\perp^1)^\times \Rightarrow (\perp^2)^\times$ , and

thus we get  $((\downarrow^1)^\times)^{\text{sfc}} \implies ((\downarrow^2)^\times)^{\text{sfc}}$  by Lemma 5.1.8(iii) and Fact 2.4.12(ii), i.e.,  $\downarrow^1$ -Conant-independence implies  $\downarrow^2$ -Conant-independence.  $\square$

**Example 5.2.35.**  $\downarrow^u$ -Conant-independence (what is called “Conant-independence” in [95]) implies  $\downarrow^i$ -Conant-independence (what is called “Conant-independence” in [94] and Example 5.2.29). In particular, combining Lemmas 5.2.33 and 5.2.34, we recover [94, Fact 6.1].

**Lemma 5.2.36.** *Let  $\downarrow$  be an independence relation satisfying full existence and monotonicity. Then  $\downarrow$ -Kim-independence implies  $\downarrow$ -Conant-independence.*

*Proof.* This is clear by the definition of  $\downarrow$ -Kim-dividing and Lemma 5.2.28.  $\square$

**Lemma 5.2.37.** *In any theory,  $\downarrow^a = \downarrow^a$ -Conant-independence.*

*Proof.* By [75, Lemma 2.7], we know that, in any theory, if  $A \not\downarrow_C^a B$ , then there is a formula  $\varphi(x, b) \in \text{tp}(A/BC)$  that  $\downarrow^a$ -Conant-divides over  $C$ . This proves the result.  $\square$

**Corollary 5.2.38.** *For any independence relation  $\downarrow$  satisfying full existence and monotonicity such that  $\downarrow \implies \downarrow^a$ , we have  $\downarrow$ -Conant-independence  $\implies \downarrow^a$ .*

*Proof.* Combine Lemmas 5.2.34 and 5.2.37.  $\square$

## 5.3 Notions of witnessing

After introducing formally the relativised notions of Kim- and Conant-independence, we can consider the main topic of this chapter: the notion of *witnessing* that has become central to the development of neostability theory outside of NIP.<sup>1</sup> The literature on neostability theory (e.g., [29], [59], [60]) seems to introduce slightly different formalisations of this notion, meant to capture the idea that an instance of (appropriate) dividing is witnessed by a whole class of “generic” sequences. In this section, we will introduce three such formalisations that make explicit those notions one can find in the literature.

The first approach adapts the notion of *relative Kim’s lemma* from [94], which we already saw in §4.3. We first need to generalise the concept of *class* in the

<sup>1</sup>This does not mean that the notion of witnessing has been absent from the development of NIP theories; see [63] for important results in this regard.

definitions of the dividing order in [11] and the Kim-dividing order in [94] (which was introduced in Definition 4.3.3):

**Definition 5.3.1.** Fix  $M \models T$ .

- (i) For a global type  $p$ , we define the **class of  $p$** , denoted  $\text{cl}(p)$ , to be the class of  $\mathcal{L}(M)$ -formulas  $\varphi(x, y)$  such that, for any  $M$ -indiscernible Morley sequence  $(a_i)_{i \in \omega}$  in  $p$  over  $M$ , the set  $\{\varphi(x, a_i) : i \in \omega\}$  is inconsistent.
- (ii) Let  $\perp^0$  be an independence relation and fix  $r \in S(M)$ . We denote by  $C_0^r$  the class of global  $M$ - $\perp^0$ -free extensions of  $r$ . Given  $M \models T$  and  $r \in S(M)$ , we say that a global extension  $p$  of  $r$  is  $\leq_0$ -**greatest** if, for all global  $q \in C_0^r$ , we have  $\text{cl}(q) \subseteq \text{cl}(p)$ .

**Remark 5.3.2.** In general, despite what the name may suggest,  $\leq_0$ -greatest extensions of types need not be unique, as different global types may share the same class over  $M$ .

**Example 5.3.3.** (i)  $\perp^f$ :  $C_f^r$  is the class of global non-forking extensions of  $r$ . This is closely related to the **dividing order** from [11].  
(ii)  $\perp^i$ :  $C_K^r$  is the class of global  $M$ -invariant extensions of  $r$ . This is closely related to the **Kim-dividing order** from [94].

If we have independence relations  $\perp^1$  and  $\perp^2$ , we use  $C_1^r$  and  $\leq_1$  (resp.,  $C_2^r$  and  $\leq_2$ ) to refer to the previous notions as they apply to  $\perp^1$  (resp.,  $\perp^2$ ).

**Example 5.3.4.** Observe that, if  $M \models T$  and  $p$  is  $M$ -invariant, then for any  $\varphi(x, y) \in \mathcal{L}(M)$ , we have that  $\varphi(x, y) \in \text{cl}(p)$  iff there is some  $M$ -indiscernible Morley sequence  $(a_i)_{i \in \omega}$  in  $p$  over  $M$  such that  $\{\varphi(x, a_i) : i \in \omega\}$  is inconsistent.

**Definition 5.3.5.** Let  $\perp^1$  and  $\perp^2$  be independence relations satisfying full existence and monotonicity. We say that  $\perp^1$  satisfies the **universal witnessing property (UWP)** with respect to  $\perp^2$  if  $\perp^1$ -free types are  $\leq_2$ -greatest, i.e., for all  $M \models T$  and  $r \in S(M)$ , if  $p$  is a global  $M$ - $\perp^1$ -free extension of  $r$ , then  $p$  is  $\leq_2$ -greatest.

We can render the above definition more explicitly (without appealing to classes of types) as follows:  $\perp^1$  satisfies UWP w.r.t.  $\perp^2$  iff, whenever  $\varphi(x, b) \perp^2$ -Kim-divides over  $M$ , we have that  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent for all global  $\perp^1$ - $M$ -free extensions  $q$  of  $\text{tp}(b/M)$  and every  $M$ -indiscernible Morley sequence  $(b_i)_{i < \omega}$  in  $q$  over  $M$  with  $b_0 = b$ .

**Terminology.** Given an independence relation  $\perp^1$ , sometimes we say that  $\leq_1$  is **trivial** if  $\perp^1$  satisfies UWP w.r.t.  $\perp^1$ . This is equivalent to saying that, for all  $r \in S(M)$  and all  $p, q \in C_1^r$ ,  $\text{cl}(p) = \text{cl}(q)$ .

**Example 5.3.6.** (i) ([64, Proposition 2.1], [65, Theorem 2.4])  $T$  is simple iff  $\leq_f$  is trivial. In particular,  $\perp^f$  satisfies UWP w.r.t.  $\perp^a$ .  
(ii) ([29, Lemma 3.14]) If  $T$  is NTP<sub>2</sub>, then  $\perp^{\text{ist}}$  satisfies UWP w.r.t.  $\perp^a$ .  
(iii) ([59, Theorem 3.16])  $T$  is NSOP<sub>1</sub> iff  $\perp^i$  satisfies UWP w.r.t.  $\perp^i$ .  
(iv) ([75, Theorem 5.2]) If  $T$  is NBTP, then  $\perp^{\text{Kst}}$  satisfies UWP w.r.t.  $\perp^i$ .  
(v) ([50, Theorem 1.8])  $T$  is NCTP iff  $\perp^{\text{bi}}$  satisfies UWP w.r.t.  $\perp^i$ .  
(vi) ([95, Theorem 6.2]) If there is an independence relation  $\perp$  satisfying full existence such that  $\perp^i$  satisfies UWP w.r.t.  $\perp$  and  $\perp$ -Kim-independence is symmetric, then  $T$  is NSOP<sub>4</sub>.

Another possibility for formalising the notion of witnessing is to forgo the restriction to global  $\perp$ -free extensions and talk directly about  $\perp$ -Morley sequences. This is closer to the original role that Morley sequences played in simple theories, and also aligns better with the phrasing of results in several contexts. For instance:

- In simple theories, one can show that, whenever  $a \perp_M^f b$ , there is a  $\perp^f$ -Morley sequence  $(a_i)_{i \in \omega}$  over  $M$  starting at  $a$  that is  $Mb$ -indiscernible, and from this one deduces symmetry for  $\perp^f$ .
- In NSOP<sub>1</sub> theories, one can show that, whenever  $a \perp_M^K b$ , there is a tree Morley sequence  $(a_i)_{i \in \omega}$  over  $M$  starting at  $a$  that is  $Mb$ -indiscernible (see [59, Lemma 5.12]), and from this one deduces symmetry. It was later realised (see [60]) that it is enough to find a  $\perp^K$ -Morley sequence over  $M$  with the same properties.

*A priori*, there is no explicit appeal to global extensions of types in either of these two cases. This motivates the following notion of witnessing:

**Definition 5.3.7.** Let  $\perp^1$  and  $\perp^2$  be independence relations satisfying full existence and monotonicity. We say that  $\perp^1$  satisfies the **generalised universal witnessing property (GUWP)** with respect to  $\perp^2$  if, whenever  $\varphi(x, b)$   $\perp^2$ -Kim-divides over  $M$ , it also  $\perp^1$ -Conant-divides over  $M$ .

**Remark 5.3.8.** GUWP  $\implies$  UWP.

Finally, a different notion of witnessing that has appeared in the literature can be found, e.g., in the Kim-Pillay-style theorem for NSOP<sub>1</sub> theories in [59]. The noteworthy modification is that we begin with a *type* that  $\perp$ -Kim-divides, instead of a formula.

**Definition 5.3.9.** Let  $\perp^1$  and  $\perp^2$  be independence relations satisfying full existence and monotonicity. We say that  $\perp^1$  satisfies the **type universal witnessing property (TUWP)** with respect to  $\perp^2$  if, whenever  $M \subset B$  and  $p(x) \in S(B)$   $\perp^2$ -Kim-divides over  $M$ , there is a formula  $\varphi(x, b) \in p(x)$  that  $\perp^1$ -Conant-divides over  $M$ .

TUWP admits two other important interpretations: as the analogue of GUWP for the notion of weak  $\perp$ -Kim-dividing; and as an implication between independence relations.

**Lemma 5.3.10.** *Suppose that  $\perp^1$  and  $\perp^2$  are independence relations satisfying full existence and monotonicity. The following are equivalent:*

- (i)  $\perp^1$  satisfies TUWP w.r.t.  $\perp^2$ .
- (ii) If  $\varphi(x, b)$  weakly  $\perp^2$ -Kim-divides over  $M$ , then it  $\perp^1$ -Conant-divides over  $M$ .
- (iii)  $((\perp^1)^\times)^{\text{sfc}} \implies (\perp^2)^+$ .

*Proof.* ((i)  $\implies$  (iii)) Suppose that  $A (\perp^2)_M^+ B$ . Let  $p(x) := \text{tp}(a/MB)$ , where  $a$  enumerates  $A$ . By Lemma 5.1.2,  $p(x)$  is not  $(\perp^2)^+$ -free over  $M$ , and thus, by Lemma 5.2.25,  $p(x)$   $\perp^2$ -Kim-divides over  $M$ . By TUWP, there is some formula  $\varphi(x', b) \in p(x)$  with  $x' \subseteq x$  finite that  $\perp^1$ -Conant-divides over  $M$ . By definition,  $\varphi(x', b)$  is not  $(\perp^1)^\times$ -free over  $M$ . Therefore,  $A ((\perp^1)^\times)_M^{\text{sfc}} B$ .

((iii)  $\implies$  (ii)) Suppose  $\varphi(x, b)$  weakly  $\perp^2$ -Kim-divides over  $M$ . By definition, for each  $a \models \varphi(x, b)$ , we must have  $a (\perp^2)_M^+ b$ . By our assumption,  $a ((\perp^1)^\times)_M^{\text{sfc}} b$ . Thus, there is some  $\psi_a(x_a, b_a) \in \text{tp}(a/Mb)$  with  $x_a \subseteq x$ ,  $b_a \subseteq b$  that is not  $(\perp^1)^\times$ -free over  $M$ . Hence,  $\psi_a(x_a, b_a) \perp^1$ -Conant-divides over  $M$ .

Now, the set  $\{\varphi(x, b)\} \cup \{\neg\psi_a(x_a, b_a) : a \models \varphi(x, b)\}$  is inconsistent. Hence, by compactness, there exist realisations  $a_0, \dots, a_{n-1} \models \varphi(x, b)$  such that

$$\varphi(x, b) \vdash \bigvee_{i < n} \psi_{a_i}(x_{a_i}, b_{a_i}).$$

We now use Lemma 5.2.28(i). Let  $(b_j)_{j < \omega}$  be an  $\perp^1$ -Morley sequence over  $M$  with  $b_0 = b$ . Assume, for contradiction, that  $\{\varphi(x, b_j) : j < \omega\}$  is consistent. Let  $c$  be a realisation. By the above and pigeonhole, there is some  $i < n$  an infinite  $I \subseteq \omega$  such that  $c \models \{\psi_{a_i}(x_{a_i}, b_{a_i, j}) : j \in I\}$ , denoting as usual by  $b_{a_i, j}$  the restriction of  $b_j$  to the appropriate variables. By  $M$ -indiscernibility and monotonicity of  $\perp^1$ , this contradicts the fact that  $\psi_{a_i}(x_{a_i}, b_{a_i}) \perp^1$ -Conant-divides over  $M$ .

((ii)  $\Rightarrow$  (i)) Suppose that  $\text{tp}(a/Mb) \downarrow^2$ -Kim-divides over  $M$ . By Lemma 5.2.5,  $\text{tp}(a/Mb)$  is not  $(\downarrow^2)^+$ -free over  $M$ . Hence, by Proposition 5.2.6 and Lemma 5.1.9, there is a formula  $\varphi(x, b) \in \text{tp}(a/Mb)$  which is not  $(\downarrow^2)^+$ -free over  $M$ , i.e., which weakly  $\downarrow^2$ -Kim-divides over  $M$ . Thus, by our assumption,  $\varphi(x, b) \downarrow^1$ -Conant-divides over  $M$ , as required.  $\square$

For the remainder of this section, we will describe the relationships between these witnessing notions. We start with noting that, for the purposes of witnessing, the distinction between formulas and types is not important:

**Lemma 5.3.11.** *Suppose that  $\downarrow^1$  and  $\downarrow^2$  are independence relations satisfying full existence and monotonicity. Then  $\downarrow^1$  satisfies GUWP w.r.t.  $\downarrow^2$  iff it satisfies TUWP w.r.t.  $\downarrow^2$ .*

*Proof.* ( $\Rightarrow$ ) Immediate.

( $\Leftarrow$ ) If  $\varphi(x, b) \downarrow^2$ -Kim-divides over  $M$ , then by Lemma 5.2.9(i),  $\varphi(x, b)$  weakly  $\downarrow^2$ -Kim-divides over  $M$ . Hence, by TUWP and Lemma 5.3.10,  $\varphi(x, b) \downarrow^1$ -Conant-divides over  $M$ , as required.  $\square$

These results make precise the claim made in the previous section that, for the purposes of neostability theory (centred on consequences of Kim’s lemma), we can essentially sidestep the distinction between  $\downarrow$ -Kim-dividing and weak  $\downarrow$ -Kim-dividing. This makes it possible to fully treat the upcoming results of this paper in terms of  $\downarrow^+$ -freeness. It also proves formally that Kim’s lemma can be viewed, at its core, as a relationship at the level of abstract independence relations.

In contrast, it is not immediate that the notions of UWP and GUWP need to coincide for arbitrary independence relations. We will now show that, in fact, for most independence relations considered in the literature, they do coincide. A suggestive remark regarding the relationship between GUWP and UWP comes from the proof of [59, Proposition 5.13]: if  $(a_i)_{i \in \omega}$  is an  $\downarrow^1$ -Morley sequence over  $M$ , then by “the compactness of the space of  $M$ -invariant types,” we can find a global  $M$ -invariant extension  $q \supset \text{tp}(a_0/M)$  such that  $(a_i)_{i \in \omega}$  is Morley in  $q$  over  $M$ . The purpose of this section is to employ the notion of *quasi-strong finite character*, introduced by Mutchnik in [95], developed further in [72], and defined here in Definition 2.4.3(xiii), to generalise these results.

We start by adapting some arguments from [72]. As in that paper, assuming  $\downarrow$  has full existence, monotonicity and quasi-strong finite character, we can define

$\Sigma_{a \perp_M b}(x, y)$  to be the  $\subseteq$ -maximal set of formulas  $\Sigma(x, y)$  such that  $a', b' \models \Sigma(x, y)$  iff  $a \equiv_M a'$ ,  $b \equiv_M b'$ , and  $a' \perp_M b'$ . The following weakening of [72, Corollary 3.16], with a similar proof, does not require NATP (we will see in §5.7 that this applies to the full result too):

**Lemma 5.3.12.** *Suppose  $\perp$  satisfies full existence, monotonicity, and quasi-strong finite character. Let  $M \models T$ ,  $a \models p(x) \in S(M)$ , and  $\Sigma(x) := \{\psi(x', d') \in \mathcal{L}(Md) : x' \subseteq x, z' \subseteq z \text{ such that } \psi(x', z') \in \Sigma_{a \perp_M d}(x, z) \text{ and } d' \subseteq d \text{ is a tuple corresponding to } z'\}\}$ . Then  $p(x) \cup \Sigma(x)$  is consistent, and any completion is  $\perp$ -free over  $M$ .*

*Proof.* Assume, for contradiction, that  $p(x) \cup \Sigma(x)$  is not consistent. So

$$p(x) \vdash \bigvee_{i < n} \psi_i(x'_i, d'_i),$$

where  $\neg\psi_i(x'_i, d'_i) \in \Sigma(x)$  for all  $i < n$ . Let  $a \models p(x)$ . By full existence, there is  $a^* = (a_i^*)$  with  $a^* \equiv_M a$  such that  $a^* \perp_M d'_{<n}$ . Since  $a^* \models p(x)$ , there is some  $i$  such that  $\models \psi_i(a_i^*, d'_i)$ . Thus,  $(a_i^*, d'_i) \not\models \Sigma_{a \perp_M d'_i}(x, z_i)$ , and so, it follows that  $a^* \not\perp_M d'_i$ . This contradicts monotonicity.

Now let  $q(x)$  be a completion of  $p(x) \cup \Sigma(x)$ . Let  $a \models q|_{Mb}$ . Then, in particular,  $a \models \Sigma(x)|_{Mb}$ , and thus,  $a \models \Sigma_{a \perp_M b}(x, b)$ . Therefore,  $a \perp_M b$ , as required.  $\square$

**Proposition 5.3.13.** *Suppose  $\perp$  satisfies existence, monotonicity, right extension, and quasi-strong finite character, and let  $M \models T$ . If  $(a_i)_{i < \omega}$  is an  $\perp$ -Morley sequence over  $M$ , then there is a global  $M$ - $\perp$ -free extension  $q \supset \text{tp}(a_0/M)$  such that  $(a_i)_{i < \omega}$  is Morley in  $q$  over  $M$ .*

*Proof.* We adapt the proof from [44, Proposition 2.4]. By compactness, we find  $a_\omega$  such that  $(a_i)_{i \leq \omega}$  is  $M$ -indiscernible. Let  $p(x) := \text{tp}(a_\omega/Ma_{<\omega})$  and let  $\Sigma(x)$  be the partial global type as in Lemma 5.3.12.

**Claim.**  $p(x) \cup \Sigma(x)$  is consistent.

*Proof of Claim.* For any finite  $p'(x) \subseteq p(x)$ , there is some  $i < \omega$  such that  $p'(x)$  contains only parameters from  $Ma_{<i}$ , and so  $a_i \models p'$  by  $M$ -indiscernibility. Since  $a_i \perp_M a_{<i}$ , by right extension  $p'(x)$  extends to a global  $M$ - $\perp$ -free type  $q'(x)$ , and any realisation of  $q'(x)$  will then be a realisation of  $p'(x) \cup \Sigma(x)$ . Thus,  $p'(x) \cup \Sigma(x)$  is finitely consistent.  $\square_{\text{Claim}}$

Let  $a' \models p(x) \cup \Sigma(x)$ , and set  $q'(x) := \text{tp}(a'/M)$ . Hence,  $q'(x)$  is a completion of  $\text{tp}(a_0/M) \cup \Sigma(x)$ , and thus, by Lemma 5.3.12, it is  $\perp$ -free over  $M$ . Now, if we

fix some  $a'' \equiv_{Ma_{<\omega}} a'$ , then there is some  $\sigma \in \text{Aut}(\mathbb{M}/Ma_{<\omega})$  such that  $\sigma(a'') = a_\omega$ . Set  $q := \sigma(q')$ , so that, by invariance for  $\perp$ ,  $q$  is also a global  $M$ - $\perp$ -free extension of  $p$  and, for any  $a \models q$ , we have  $a \equiv_{Ma_{<\omega}} a_\omega$ .

For any  $i < \omega$  and  $a \models q$ , we thus have  $a_i \equiv_{Ma_{<i}} a_\omega \equiv_{Ma_{<i}} a$ . We therefore have  $a_i \models q|_{Ma_{<i}}$  for all  $i < \omega$ , and so  $(a_i)_{i < \omega}$  is a Morley sequence in  $q$  over  $M$ .  $\square$

**Corollary 5.3.14.** *Suppose  $\perp$  satisfies existence, monotonicity, right extension, and quasi-strong finite character. Then  $\varphi(x, b) \perp$ -Kim-divides over  $M$  iff there is some global  $M$ - $\perp$ -free extension  $q$  of  $\text{tp}(b/M)$  and some  $M$ -indiscernible Morley sequence  $(b_i)_{i < \omega}$  in  $q$  over  $M$  with  $b_0 = b$  such that  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent.*

*Proof.*  $(\Leftarrow)$  Clear, and requires no assumptions.

$(\Rightarrow)$  Suppose  $\varphi(x, b) \perp$ -Kim-divides over  $M$ . So there is some  $\perp$ -Morley sequence  $(b_i)_{i \in \omega}$  over  $M$  such that  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent. By Proposition 5.3.13, there is a global  $M$ - $\perp$ -free type  $q$  extending  $\text{tp}(b/M)$  such that  $(b_i)_{i \in \omega}$  is a Morley sequence in  $q$  over  $M$ . This completes the proof.  $\square$

**Example 5.3.15.** In particular, this shows that our definition of Kim-independence in Example 5.2.10 does, indeed, coincide with the definition in [59] which we gave in Definition 3.1.4.

**Corollary 5.3.16.** *Suppose  $\perp^1$  satisfies existence, monotonicity, right extension, and quasi-strong finite character, and  $\perp^2$  satisfies full existence. Then  $\perp^1$  satisfies UWP w.r.t.  $\perp^2$  iff it satisfies GUWP w.r.t.  $\perp^2$ .*

*Proof.*  $(\Leftarrow)$  Clear, since every  $M$ -indiscernible Morley sequence over  $M$  in a global  $M$ - $\perp$ -free extension of  $\text{tp}(b/M)$  is in particular  $\perp$ -Morley over  $M$ .

$(\Rightarrow)$  Suppose  $\varphi(x, b) \perp^2$ -Kim-divides over  $M$ . Let  $(b_i)_{i \in \omega}$  be an  $\perp^1$ -Morley sequence over  $M$  with  $b_0 = b$ . By Proposition 5.3.13, we can find some global  $M$ - $\perp^1$ -free extension  $q \supset \text{tp}(b/M)$  such that  $(b_i)_{i \in \omega}$  is Morley in  $q$  over  $M$ . By UWP,  $\varphi(x, y) \in \text{cl}(q)$ , which means that  $\{\varphi(x, b_i) : i \in \omega\}$  is inconsistent, as required.  $\square$

In particular, since strong finite character implies quasi-strong finite character,  $\perp^u$ ,  $\perp^i$ ,  $\perp^f$ , and  $\perp^a$  all have quasi-strong finite character, and by Lemmas 5.2.14 and 5.2.32, this also holds for  $\perp$ -Kim-independence and  $\perp$ -Conant-independence for any  $\perp$  with full existence and monotonicity. This means that all the results on witnessing in the context of simple, NTP<sub>2</sub>, NSOP<sub>1</sub>, etc., theories from Example 5.3.6 can be equivalently formulated in terms of GUWP or TUWP.

Let us finish this section with some basic results about GUWP, which, by the above, can be easily reformulated in terms of UWP and TUWP (potentially under additional assumptions):

**Lemma 5.3.17.** *Let  $\perp^1 \implies \perp^2$  and  $\perp^3 \implies \perp^4$  be independence relations satisfying full existence and monotonicity. If  $\perp^2$  satisfies GUWP w.r.t.  $\perp^4$ , then  $\perp^1$  satisfies GUWP w.r.t.  $\perp^3$ .*

*Proof.* Suppose that  $\varphi(x, b) \perp^3$ -Kim-divides over  $M$ . By Lemma 5.2.15,  $\varphi(x, b)$  also  $\perp^4$ -Kim-divides over  $M$ . So, as  $\perp^2$  satisfies GUWP w.r.t.  $\perp^4$ , it follows that  $\varphi(x, b) \perp^2$ -Conant-divides over  $M$ , and thus, by Lemma 5.2.34, it also  $\perp^1$ -Conant-divides over  $M$ , as required.  $\square$

**Example 5.3.18.** Using Example 5.3.6, the above lemmas allow us to recover several implications from the classification picture abstractly (references can be found in chapter 2):

- (i) If  $T$  is simple, then it is  $\text{NTP}_2$ .
- (ii) If  $T$  is simple, then it is  $\text{NSOP}_1$ .
- (iii) If  $T$  is  $\text{NSOP}_1$ , then it is  $\text{NBTP}$ .
- (iv) If  $T$  is  $\text{NTP}_2$ , then it is  $\text{NBTP}$ .
- (v) If  $T$  is  $\text{NBTP}$ , then it is  $\text{NCTP}$ .

**Lemma 5.3.19.** *Let  $\perp^1$  and  $\perp^2$  be compatible independence relations satisfying full existence and monotonicity such that  $\perp^1$  satisfies GUWP w.r.t.  $\perp^2$ , and let  $M \models T$ . Then  $\varphi(x, b) \perp^2$ -Kim-divides over  $M$  iff it  $\perp^1$ -Conant-divides over  $M$ . In particular,  $\perp^2$ -Kim-independence and  $\perp^1$ -Conant-independence coincide.*

*Proof.* ( $\Rightarrow$ ) This follows directly from the definition of GUWP.

( $\Leftarrow$ ) Assume that  $\varphi(x, b) \perp^1$ -Conant-divides over  $M$ . By compatibility, there is a global  $M$ - $\perp^{1 \wedge 2}$ -free extension  $q \supset \text{tp}(b/M)$ . Let  $(b_i)_{i \in \omega}$  be an  $M$ -indiscernible Morley sequence in  $q$  over  $M$  with  $b_0 = b$ . Since  $(b_i)_{i \in \omega}$  is  $\perp^1$ -Morley, by assumption  $\{\varphi(x, b_i) : i \in \omega\}$  is inconsistent. Since  $(b_i)_{i \in \omega}$  is also  $\perp^2$ -Morley, this means that  $\varphi(x, b) \perp^2$ -Kim-divides over  $M$ .  $\square$

**Example 5.3.20.** Using again Example 5.3.6, the previous lemma recovers the following results from the literature:

- (i) ([64, Proposition 2.1]) If  $T$  is simple, then  $\varphi(x, b)$  divides over  $M$  iff it  $\perp^f$ -Conant-divides over  $M$ .

- (ii) ([59, Theorem 3.16]) If  $T$  is  $\text{NSOP}_1$ , then  $\varphi(x, b)$  Kim-divides over  $M$  iff it Conant-divides over  $M$ .
- (iii) ([29, Lemma 3.14]) If  $T$  is  $\text{NTP}_2$ , then  $\varphi(x, b)$  divides over  $M$  iff it  $\downarrow^{\text{ist}}$ -Conant-divides over  $M$ .
- (iv) ([76]) If  $T$  is  $\text{NBTP}$ , then  $\varphi(x, b)$  Kim-divides over  $M$  iff it  $\downarrow^{\text{Kst}}$ -Conant-divides over  $M$ .

**Proposition 5.3.21.** *Let  $\downarrow^1$  and  $\downarrow^2$  be compatible independence relations satisfying full existence and monotonicity such that  $\downarrow^1$  also satisfies GUWP w.r.t.  $\downarrow^2$ , and let  $M \models T$ . Then  $\varphi(x, b) \downarrow^2$ -Kim-forks over  $M$  iff it  $\downarrow^2$ -Kim-divides over  $M$ .*

*Proof.* Suppose that  $\varphi(x, b) \downarrow^2$ -Kim-forks over  $M$ . By Lemma 5.2.9(ii), we have that  $\varphi(x, b) \vdash \bigvee_{j < n} \psi_j(x, c_j)$  for some  $\psi_j(x, c_j)$  for  $j < n$  where each  $\psi_j(x, c_j) \downarrow^2$ -Kim-divides over  $M$ . By compatibility, we can find some  $\downarrow^{1 \wedge 2}$ -Morley sequence  $(b_i, c_{0,i}, \dots, c_{n-1,i})_{i \in \omega}$  over  $M$  starting at  $(b, c_0, \dots, c_{n-1})$ .

Assume, for contradiction, that  $\varphi(x, b)$  does not divide along  $I_* := (b_i)_{i \in \omega}$ . Then there is some  $a \models \{\varphi(x, b_i) : i \in \omega\}$ , and thus, by the pigeonhole principle, we have  $a \models \psi_j(x, c_{j,i})$  for some  $j < n$  and infinitely many  $i \in \omega$ . But, by monotonicity and Lemma 5.3.19,  $\{\psi_j(x, c_{j,i}) : i \in I\}$  is inconsistent for any infinite  $I \subseteq \omega$ . This gives us the required contradiction. Therefore,  $\varphi(x, b) \downarrow^2$ -Kim-divides over  $M$ .  $\square$

**Example 5.3.22.** This allows us to recover the following results from the literature abstractly:

- (i) ([29, Corollary 3.22]) If  $T$  is  $\text{NTP}_2$ , then  $\varphi(x, b)$  forks over  $M$  iff it divides over  $M$ .
- (ii) ([95, Corollary 4.9.1]) If  $T$  is  $\text{NCTP}$ , then  $\varphi(x, b)$  Kim-forks over  $M$  iff it Kim-divides over  $M$ .

In particular, by Lemma 5.2.17,  $\downarrow^f$  satisfies left extension over models in all  $\text{NTP}_2$  theories (cf., [29]), and  $\downarrow^K$  also satisfies left extension in all  $\text{NCTP}$  theories.

**Lemma 5.3.23.** *Suppose that  $\downarrow^1$ ,  $\downarrow^2$  and  $\downarrow^3$  are pairwise compatible independence relations satisfying full existence and monotonicity such that  $\downarrow^1$  satisfies GUWP w.r.t.  $\downarrow^3$  and  $\downarrow^2 \implies \downarrow^3$ , and let  $M \models T$ . Then  $\varphi(x, b) \downarrow^2$ -Kim-divides over  $M$  iff it  $\downarrow^3$ -Kim-divides over  $M$ .*

*Proof.* By Lemma 5.3.17,  $\downarrow^1$  also satisfies GUWP w.r.t.  $\downarrow^2$ . Therefore, by Lemma 5.3.19, we have that  $\varphi(x, b) \downarrow^2$ -Kim-divides over  $M$  iff it  $\downarrow^1$ -Conant-divides over  $M$  iff it  $\downarrow^3$ -Kim-divides over  $M$ .  $\square$

**Example 5.3.24.** For instance, this lemma recovers the fact that, if  $T$  is  $\text{NTP}_2$ , then  $\varphi(x, b)$  divides over  $M$  iff it Kim-divides over  $M$ .

We include the explicit proof of the following result as an illustration of how we can combine all that has been developed so far. The result itself has been previously generalised to  $\text{NATP}$  theories in [72, Theorem 3.10], but the proof in that case is purely combinatorial. Recall that  $\perp^{\text{Kst}}$  and  $\perp^{\text{ust}}$  were introduced in Example 2.4.26.

**Corollary 5.3.25.** *If  $T$  is  $\text{NBTP}$ , then  $\varphi(x, b)$  Kim-divides over  $M$  iff it  $\perp^{\text{u}}$ -Kim-divides over  $M$ .*

*Proof.* Since  $\perp^{\text{Kst}}$  satisfies GUWP w.r.t.  $\perp^{\text{i}}$  and  $\perp^{\text{ust}} \implies \perp^{\text{Kst}}$ , by Lemma 5.3.17 it follows that  $\perp^{\text{ust}}$  also satisfies GUWP w.r.t.  $\perp^{\text{i}}$ . (Note that  $\perp^{\text{ust}}$  has full existence by [72, Corollary 3.16].) Hence, by Lemma 5.3.19, if  $\varphi(x, b)$  Kim-divides over  $M$ , then it  $\perp^{\text{ust}}$ -Kim-divides over  $M$ , and since  $\perp^{\text{ust}} \implies \perp^{\text{u}}$ , it follows that  $\varphi(x, b)$   $\perp^{\text{u}}$ -Kim-divides over  $M$ .  $\square$

Finally, let us establish a connection between witnessing and the notion of *weight* (see [98] for a treatment of this concept outside of stability). The following proof is based on [26, Theorem 4.9]:

**Lemma 5.3.26.** *Let  $\perp^1$  and  $\perp^2$  be independence relations satisfying full existence and monotonicity. The following are equivalent:*

- (i)  $\perp^1$  satisfies GUWP w.r.t.  $\perp^2$ .
- (ii) For every  $M \models T$ , every  $\perp^1$ -independent sequence  $(a_i)_{i < |T|^+}$  over  $M$ , and any finite tuple  $b$ , there is some  $i < |T|^+$  such that  $\text{tp}(b/Ma_i)$  does not  $\perp^2$ -Kim-divide over  $M$ .

*Proof.* ((i)  $\implies$  (ii)) Suppose that there exist  $M \models T$ ,  $b$ , and  $(a_i)_{i < |T|^+}$  such that  $(a_i)_{i < |T|^+}$  is  $\perp^1$ -independent over  $M$  but  $\text{tp}(b/Ma_i)$   $\perp^2$ -Kim-divides over  $M$  for all  $i < |T|^+$ . By pigeonhole, we may assume that the same formula  $\varphi(x, y)$  witnesses  $\perp^2$ -Kim-dividing for all  $i < |T|^+$ . By Erdős-Rado and compactness, we can find an  $Mb$ -indiscernible sequence  $(a'_i)_{i < \omega}$  based on  $(a_i)_{i < |T|^+}$ . In particular, by monotonicity,  $(a'_i)_{i < \omega}$  is  $\perp^1$ -Morley over  $M$  and  $\{\varphi(x, a'_i) : i < \omega\}$  is consistent, but  $\varphi(x, a'_0)$   $\perp^2$ -Kim-divides over  $M$ . So  $\perp^1$  does not satisfy GUWP w.r.t.  $\perp^2$ .

((ii)  $\implies$  (i)) Suppose that  $\perp^1$  does not satisfy GUWP w.r.t.  $\perp^2$ . So there is a formula  $\varphi(x, a)$  and an  $\perp^1$ -Morley sequence  $(a_i)_{i < \omega}$  over  $M$  with  $a_0 = a$  such that

$\varphi(x, a) \perp^2$ -Kim-divides over  $M$  but  $\{\varphi(x, a_i) : i < \omega\}$  is consistent. Stretch the sequence  $(a_i)_{i < \omega}$  to an  $\perp^1$ -independent sequence of length  $|T|^+$ . Take  $b \models \{\varphi(x, a_i) : i < |T|^+\}$ . Then clearly  $\text{tp}(b/Ma_i) \perp^2$ -Kim-divides over  $M$  for all  $i < |T|^+$ , as it contains  $\varphi(x, a_i)$  and  $a_i \equiv_M a$ . In particular, (ii) fails.  $\square$

Let us note for the benefit of the reader that an application of the results on witnessing from this section is given in §5.7 which does not depend on the contents of §§5.4-5.6.

## 5.4 Chain local character

The goal for the upcoming sections of this chapter is to use the technology of witnessing developed in the previous section to generalise the proofs of traditional properties of Kim- and Conant-independence in tame contexts, such as symmetry and transitivity, to their relativised versions. As we will see, although the original proofs tend to be done in concrete contexts (such as  $\text{NSOP}_1$ ), for several traditional properties they can be carried out under minimal assumptions on  $\perp$  as long as some form of witnessing is in place.

To achieve this, we first need to recall a slight variation of local character from Definition 2.4.3(xiv), which we already used in Lemma 3.3.3. Recall that we say  $(M_i)_{i < \kappa}$  is a **continuous chain** of models if  $M_i \models T$  for all  $i$ ,  $M_i \preceq M_j$  for all  $i \leq j$ , and  $M_\alpha = \bigcup_{\beta < \alpha} M_\beta$  for every limit ordinal  $\alpha$ .

(\*) *Chain local character*: For all finite  $A$ , there is a regular cardinal  $\kappa$  such that, for all continuous chains of models  $(M_i)_{i < \kappa}$  of  $T$  such that  $|M_i| < \kappa$  for all  $i < \kappa$ , we have  $A \perp_{M_i} M$  for some  $i < \kappa$ , where  $M := \bigcup_{i < \kappa} M_i$ .

The following result is folklore:

**Fact 5.4.1.** *Let  $\perp$  be an independence relation. If  $\perp$  satisfies local character and right base monotonicity, then it also satisfies chain local character.*

*Proof.* Let  $A$  be finite. Let  $\kappa_0 = \kappa_0(A)$  be the cardinal coming out of local character. Take  $\kappa := (\kappa_0 + |T|)^+$  and pick  $(M_i)_{i < \kappa}$  a continuous chain of models of  $T$  with  $|M_i| < \kappa$  for all  $i < \kappa$ . Let  $M := \bigcup_{i < \kappa} M_i$ . By local character, there is some  $C \subseteq M$  with  $|C| < \kappa$  such that  $A \perp_C M$ . By definition of  $M$ , we can find some  $i < \kappa$  such that  $C \subseteq M_i$ . Hence, by right base monotonicity, it follows that  $A \perp_{M_i} M$ , as required.  $\square$

We also have the following result for  $\perp^h$ , defined in Example 2.4.16:

**Fact 5.4.2** ([38, Theorem 3.3.7]).  $\perp^h$  satisfies chain local character.

More generally, it satisfies local character and right base monotonicity, among other properties, so a quick application of Fact 5.4.1 yields this fact.

**Lemma 5.4.3.** *Let  $\perp$  be an independence relation satisfying full existence, monotonicity, and right extension. If  $a \perp_M b$ , then  $\text{tp}(b/Ma)$  does not  $\perp$ -Conant-divide over  $M$ .*

*Proof.* We adapt the proof from [59, Proposition 3.22]. So suppose that  $a \perp_M b$ . By right extension, we can find an arbitrarily long sequence  $(a_i)_{i < \kappa}$  such that  $a_i \perp_M ba_{<i}$  for all  $i < \kappa$ . Applying Erdős-Rado, compactness, monotonicity and an automorphism, we obtain an  $Mb$ -indiscernible  $\perp$ -Morley sequence  $(a'_i)_{i < \omega}$  with  $a'_0 = a$ . Therefore,  $\text{tp}(b/Ma)$  does not  $\perp$ -Conant-divide over  $M$ .  $\square$

**Proposition 5.4.4.** *Let  $\perp$  be an independence relation satisfying existence, monotonicity, and right extension. Suppose, in addition, that  $\perp^u \implies \perp$ . Then  $\perp$ -Conant-independence satisfies chain local character.*

*Proof.* Let us first note that, if  $a \perp_M^h b$ , then  $b \perp_M^u a$  by Example 2.4.16, so  $b \perp_M a$  by assumption, hence  $\text{tp}(a/Mb)$  does not  $\perp$ -Conant-divide over  $M$  by Lemma 5.4.3. So  $\perp^h$  implies non- $\perp$ -Conant-dividing. Since  $\perp^h$  satisfies right extension, by Fact 2.4.12(ii) it follows that  $a$  is  $\perp$ -Conant-independent from  $b$  over  $M$ .

Now, let  $\kappa$ ,  $(M_i)_{i < \kappa}$ , and  $a$  be as in the statement of chain local character, and let  $M_\kappa := \bigcup_{i < \kappa} M_i$ . By Fact 5.4.2, there is  $i < \kappa$  such that  $a \perp_{M_i}^h M_\kappa$ . Hence, by the previous paragraph,  $a$  is  $\perp$ -Conant-independent from  $M_\kappa$  over  $M_i$ . Therefore,  $\perp$ -Conant-independence satisfies chain local character.  $\square$

It is well-known that we can remove the assumption of  $\perp^u \implies \perp$  by using abstract properties of independence relations. Using [43, Lemma 9.5], we obtain:

**Proposition 5.4.5.** *Let  $\perp$  be an independence relation satisfying existence, monotonicity, strong finite character, and right extension. Then  $\perp$ -Conant-independence satisfies chain local character.*

**Example 5.4.6.** In any theory, Conant-independence satisfies chain local character.

**Remark 5.4.7.** More abstractly, we can use Lemma 5.1.8 and Proposition 5.1.16

to rephrase the above result as follows: if  $\perp$  satisfies existence, monotonicity, and right extension, then  $\perp^{\text{sfc}}$ -Conant-independence satisfies chain local character.

While chain local character, as we have seen above, is a pervasive property among relative Conant-independence, it is less common for relative Kim-independence. A way in which we can obtain it is via a witnessing property:

**Lemma 5.4.8.** *Let  $\perp^1$  and  $\perp^2$  be compatible independence relations satisfying full existence and monotonicity such that  $\perp^1$  also satisfies strong finite character, right extension, and GUWP w.r.t.  $\perp^2$ . Then  $\perp^2$ -Kim-independence satisfies chain local character.*

*Proof.* Combine Proposition 5.4.5 and Lemma 5.3.19.  $\square$

**Example 5.4.9.** With this and Example 5.3.6, we recover the following results from the literature:

- (i) (Corollary to [65, Theorem 2.4]) If  $T$  is simple, then  $\perp^f$  satisfies chain local character.
- (ii) ([61, Corollary 4.6]) If  $T$  is NSOP<sub>1</sub>, then Kim-independence satisfies chain local character.

**Remark 5.4.10.** Note that the above lemma does not extend to  $\perp^f$  and  $\perp^K$  in NTP<sub>2</sub> and NBTP theories, respectively, because outside of simple and NSOP<sub>1</sub> theories,  $(\perp^f)^{\text{opp}}$  and  $(\perp^K)^{\text{opp}}$  do not satisfy strong finite character.

A different way to obtain chain local character for relative Kim-independence which avoids witnessing is via symmetry.

**Lemma 5.4.11.** *Let  $\perp$  be an independence relation satisfying full existence and monotonicity. If  $\perp$ -Kim-independence is symmetric, then it satisfies chain local character.*

*Proof.* Suppose  $\perp$ -Kim-independence is symmetric. Observe that, by Lemma 5.2.18, we have  $\perp^u \implies \perp$ -Kim-independence. (Alternatively, we can use Lemma 5.2.14 and [43, Lemma 9.5].) Thus, by symmetry,  $\perp^h \implies \perp$ -Kim-independence. Hence, as before,  $\perp$ -Kim-independence satisfies chain local character.  $\square$

## 5.5 Symmetry

In this section, we will make explicit the formal connection between witnessing and symmetry that appears in contexts such as simplicity and  $\text{NSOP}_1$ . This result forms the basis for more complicated properties, such as relativisations of the Independence Theorem.

**Theorem 5.5.1.** *Let  $\perp$  be an independence relation satisfying full existence and monotonicity. Suppose that  $\perp$ -Kim-independence is compatible with  $\perp$  and satisfies left transitivity. The following are equivalent:*

- (i)  $\perp$ -Kim-independence satisfies GUWP w.r.t.  $\perp$ .
- (ii)  $\perp$ -Kim-independence is symmetric.

*Proof.* ((i)  $\Rightarrow$  (ii)) Suppose that  $a$  is  $\perp$ -Kim-independent from  $b$  over  $M$ . By Lemma 5.4.3,  $\text{tp}(b/Ma)$  does not  $\perp^{\text{K}\perp}$ -Conant-divide over  $M$ . Thus, by GUWP,  $\text{tp}(b/Ma)$  does not  $\perp$ -Kim-divide over  $M$ , and hence, by compatibility and Proposition 5.3.21,  $b$  is  $\perp$ -Kim-independent from  $a$  over  $M$ .

((ii)  $\Rightarrow$  (i)) We adapt the argument from [60]. We will make use of the notion of  $\perp$ -independence chain of models introduced in Definition 3.3.5. Let  $\varphi(x, b)$  be a formula that  $\perp$ -Kim-divides over  $M$ . Let  $(b_i)_{i < \omega}$  be an  $\perp^{\text{K}\perp}$ -Morley sequence over  $M$  with  $b_0 = b$ . Using right extension, we may extend the sequence to length  $\kappa$  for a sufficiently large  $\kappa$ .

Assume, for contradiction, that  $\{\varphi(x, b_i) : i < \kappa\}$  is consistent. Let  $a$  be a realisation. Since it is symmetric, by Lemma 5.4.11  $\perp$ -Kim-independence satisfies chain local character; we may assume  $\kappa$  is the cardinal coming out of this property for the finite tuple  $a$ . Let  $(M_i)_{i < \kappa}$  be an  $\perp^{\text{K}\perp}$ -independence chain of models for  $(b_i)_{i < \kappa}$  over  $M$  such that  $|M_i| < \kappa$ , which exists by Remark 3.3.6. By chain local character, if we let  $M_\kappa := \bigcup_{i < \kappa} M_i$ , then  $a$  is  $\perp$ -Kim-independent from  $M_\kappa$  over  $M_i$  for some  $i < \kappa$ . So, by monotonicity,  $a$  is  $\perp$ -Kim-independent from  $b_i$  over  $M_i$ . But, by choice of the continuous chain of models,  $b_i$  is  $\perp$ -Kim-independent from  $M_i$  over  $M$ , so by symmetry and transitivity it follows that  $M_i a$  is  $\perp$ -Kim-independent from  $b_i$  over  $M$ . This contradicts the fact that  $\varphi(x, b_i) \in \text{tp}(a/Mb_i)$  and it  $\perp$ -Kim-forks over  $M$  by assumption.

Hence,  $\{\varphi(x, b_i) : i < \kappa\}$  is inconsistent, and thus, by indiscernibility, it is  $k$ -inconsistent for some  $k < \omega$ . Therefore,  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent, which

means that  $\varphi(x, b) \perp^{\text{K}\perp}$ -Conant-divides over  $M$ .  $\square$

**Remark 5.5.2.** The condition of compatibility between  $\perp$  and  $\perp$ -Kim-independence is a technical assumption that is easy to prove in most concrete cases. For example, one can easily see from Lemma 5.2.18 that it suffices to have compatibility between  $\perp$  and  $\perp^{\text{f}}$ . In particular,  $\perp^{\text{u}}$ ,  $\perp^{\text{i}}$ ,  $\perp^{\text{f}}$ ,  $\perp^{\text{b}}$  and  $\perp^{\text{a}}$  all satisfy this condition.

**Example 5.5.3.** By Kim's Lemma for simple theories [64, Proposition 2.1], if  $T$  is simple, then  $\perp^{\text{f}}$  satisfies GUWP w.r.t.  $\perp^{\text{a}}$  and  $\perp^{\text{f}} = \perp^{\text{a}}$ -Kim-independence. Since  $\perp^{\text{f}}$  always satisfies left transitivity (Fact 2.4.8), the above result recovers symmetry of  $\perp^{\text{f}}$  [64, Theorem 2.5].

One immediate consequence of the above result is a new characterisation of simplicity within  $\text{NTP}_2$  theories. This result is the analogue of [59, Proposition 8.7], which similarly characterises simplicity within  $\text{NSOP}_1$  theories.

**Corollary 5.5.4.** *An  $\text{NTP}_2$  theory is simple iff  $\perp^{\text{ist}}$ -Kim-independence satisfies GUWP w.r.t.  $\perp^{\text{ist}}$ .*

*Proof.* ( $\Leftarrow$ ) By Example 5.3.6 and Lemmas 5.2.18 and 5.2.36, if  $T$  is  $\text{NTP}_2$ , then  $\perp^{\text{f}} = \perp^{\text{ist}}$ -Kim-independence and so it satisfies left transitivity. Thus, if  $\perp^{\text{ist}}$ -Kim-independence satisfies GUWP w.r.t.  $\perp^{\text{ist}}$ , then by Theorem 5.5.1  $\perp^{\text{f}}$  is symmetric, and thus  $T$  is simple.

( $\Rightarrow$ ) If  $T$  is simple, then  $\perp^{\text{f}} = \perp^{\text{ist}}$ -Kim-independence is symmetric, and therefore by Theorem 5.5.1  $\perp^{\text{ist}}$ -Kim-independence satisfies GUWP w.r.t.  $\perp^{\text{ist}}$ .  $\square$

Another application of our result provides an abstract, purely semantic proof of one of the implications of [94, Theorem 3.12], without the need to use the technical tree machinery of the original proof. Let us recall from [94, Definition 3.3] that  $\perp$  satisfies the **generalised freedom axiom** if, whenever  $M \prec N \models T$  and there is an  $N$ -indiscernible  $\perp$ -Morley sequence over  $M$  starting at  $a$ , then every  $\perp$ -Morley sequence over  $N$  starting at  $a$  is also  $\perp$ -Morley over  $M$ .

**Lemma 5.5.5.** *Suppose  $\perp$  satisfies full existence, monotonicity, and stationarity over models. Then  $\perp$ -Kim-forking and  $\perp$ -Kim-dividing coincide.*

*Proof.* This is [94, Proposition 3.10]. Within our framework, this result can be obtained from two facts: (i) given our assumptions,  $\perp$  also satisfies left and right extension, which means that (ii) by Lemma 5.2.33,  $\perp$ -Conant-forking and  $\perp$ -

Conant-dividing coincide. Since stationarity implies that  $\perp$ -Kim-independence and  $\perp$ -Conant-independence agree, the result follows.  $\square$

**Lemma 5.5.6.** *Suppose  $\perp$  satisfies full existence, monotonicity, stationarity over models, and the generalised freedom axiom. Then  $\perp$ -Kim-independence satisfies left transitivity.*

*Proof.* Suppose that  $M \prec N$  are models of  $T$ ,  $N$  is  $\perp$ -Kim-independent from  $a$  over  $M$ , and  $b$  is Kim-independent from  $a$  over  $N$ . In particular, by full existence and monotonicity for  $\perp$ , there exists an  $N$ -indiscernible  $\perp$ -Morley sequence over  $M$  starting at  $a$ . Similarly, there exists an  $Nb$ -indiscernible  $\perp$ -Morley sequence  $I$  over  $N$  starting at  $a$ . In particular, by the generalised freedom axiom,  $I$  is also  $\perp$ -Morley over  $M$ , and since it is  $Nb$ -indiscernible, it is also  $Mb$ -indiscernible. Since, by stationarity,  $\text{tp}(a/M)$  has a unique global  $M$ - $\perp$ -free extension  $q$  and  $I$  is Morley in  $q$  over  $M$  by Proposition 5.3.13, it follows by definition and Lemma 5.5.5 that  $b$  is  $\perp$ -Kim-independent from  $a$  over  $M$ , as claimed.  $\square$

**Corollary 5.5.7.** *Let  $\perp$  be an independence relation satisfying full existence, monotonicity, stationarity over models, and the generalised freedom axiom. If  $\perp$ -Kim-independence is symmetric, then  $\perp^i$  satisfies GUWP w.r.t.  $\perp$ .*

*Proof.* By Lemma 5.5.6,  $\perp$ -Kim-independence satisfies left transitivity. So, by Theorem 5.5.1 (observe that the compatibility condition holds in this case), if  $\perp$ -Kim-independence is symmetric, then  $\perp$ -Kim-independence satisfies GUWP w.r.t.  $\perp$ . As  $\perp^i \implies \perp$ -Kim-independence, it follows from Lemma 5.3.17 that  $\perp^i$  satisfies GUWP w.r.t.  $\perp$ .  $\square$

In this way, we recover half of [94, Theorem 3.12]. The other direction requires further tools; Theorem 5.5.1 can only prove the weakening that involves  $\perp$ -Kim-independence instead of  $\perp^i$ .

Let us finish this section by discussing further the relationship between Adler's notion of an *independence relation* and the notion of witnessing that we have developed here. Let us start with the following definition (we adapt the terminology from [2], [19], [38]):

**Definition 5.5.8.** We say  $\perp$  is an **Adler preindependence relation** if it satisfies monotonicity, right base monotonicity, left transitivity, left normality, and strong finite character.

We say  $\perp$  is an **Adler independence relation** if it satisfies monotonicity, right

base monotonicity, left transitivity, left normality, finite character, local character, and right extension.

**Fact 5.5.9** ([3, Theorem 2.5]). *Every Adler independence relation is symmetric.*

The proof of this result sheds some light on the relationship of this notion with relative Kim- and Conant-independence:

**Lemma 5.5.10.** *Suppose  $\perp$  is an Adler independence relation. Then  $\perp = \perp^\times$  over models, i.e.,  $a \perp_M b$  iff there exists an  $Ma$ -indiscernible  $\perp$ -Morley sequence over  $M$  that starts at  $b$ .*

*Proof.* ( $\Rightarrow$ ) Suppose  $a \perp_M b$ . By Fact 5.5.9,  $\perp$  satisfies left extension, so we can build inductively a sufficiently long sequence  $(b_i)_{i < \kappa}$  with  $b_0 = b$  such that  $b_{<i}a \perp_M b_i$  for all  $i < \kappa$ . By Erdős-Rado, compactness, monotonicity and an automorphism, there is an  $\perp^{\text{opp}}$ -Morley sequence  $(b'_i)_{i < \omega}$  based on  $(b_i)_{i < \kappa}$  and starting at  $b$  which is  $Ma$ -indiscernible. By symmetry,  $(b'_i)_{i < \omega}$  is the sequence we wanted.

( $\Leftarrow$ ) Suppose that there exists an  $Ma$ -indiscernible  $\perp$ -Morley sequence  $(b_i)_{i < \omega}$  with  $b_0 = b$ . Then it follows directly from [3, Proposition 2.4] and symmetry that  $a \perp_M b$ , as required.  $\square$

**Example 5.5.11.** The above lemma strengthens [97, Theorem 4.1.7] for thorn-forking in the context of rosy theories.

**Corollary 5.5.12.** *Suppose that  $\perp$  is an Adler independence relation satisfying strong finite character. Then  $\perp = \perp$ -Conant-independence over models.*

*Proof.* By Lemmas 5.5.10 and 5.1.14, working over models, we have  $\perp = \perp^* = (\perp^*)^{\text{sfc}} = (\perp^{\text{sfc}})^* = ((\perp^\times)^{\text{sfc}})^* = \perp$ -Conant-independence.  $\square$

It remains an open question (see [4, Question 1.1]) whether every Adler independence relation satisfies strong finite character. The above result shows that this condition is equivalent to the property called “witnessing” in [59].

**Theorem 5.5.13.** *Let  $\perp$  be an Adler independence relation. The following are equivalent:*

- (i)  $\perp = \perp$ -Kim-independence over models.
- (ii)  $\perp$  satisfies GUWP w.r.t.  $\perp$ .

*Proof.* ((ii)  $\Rightarrow$  (i)) If  $\perp$  satisfies GUWP w.r.t.  $\perp$ , then by Lemma 5.3.19  $\perp$ -Kim-independence and  $\perp$ -Conant-independence coincide, and by Proposition 5.3.21  $\perp$ -Kim-dividing and  $\perp$ -Kim-forking also agree. Thus, by Lemma 5.5.10,  $\perp = \perp$ -Kim-independence.

((i)  $\Rightarrow$  (ii)) Suppose that  $\perp = \perp$ -Kim-independence. Then  $\perp$ -Kim-independence satisfies left transitivity and symmetry by Fact 5.5.9. Therefore, by Theorem 5.5.1,  $\perp$ -Kim-independence satisfies GUWP w.r.t.  $\perp$ . As  $\perp = \perp$ -Kim-independence, this means that  $\perp$  satisfies GUWP w.r.t.  $\perp$ .  $\square$

We thus recover the implication that, if  $\perp^d$  is an Adler independence relation, then  $T$  is simple: if  $\perp^d$  is an Adler independence relation, then  $\perp^d = (\perp^d)^* = \perp^f$  and so by Lemma 5.5.10  $\perp^f$ -Kim-independence implies  $\perp^f$ . Hence, by Lemma 5.2.18,  $\perp^f = \perp^f$ -Kim-independence. Therefore, by Theorem 5.5.13,  $\perp^f$  satisfies GUWP w.r.t.  $\perp^f$ , i.e., Kim's Lemma for simple theories holds. So  $T$  is simple.

## 5.6 Independence Theorem

In this section, we want to use the notion of witnessing developed previously to prove a generalised version of the independence theorem. This result has been key to the study of simple and NSOP<sub>1</sub> theories. Our version of the Independence Theorem for relative Kim-independence will require us to briefly explore a property which appears already in [3] and also, in passing, in [59, §9.2], where it is not given an explicit name but only described as “a weak form of transitivity.” We will name it accordingly:

( $\star$ ) *Weak transitivity.* For all  $M \models T$  and  $A, B, C \subset \mathbb{M}$ , if  $A \perp_M BC$  and  $C \perp_M B$ , then  $AC \perp_M B$ .

This is a relatively common property of traditional independence relations.

**Remark 5.6.1.** Let us note that, by [38, Proposition 3.1.3], if  $\perp$  is an Adler preindependence relation, then  $\perp^*$  is also an Adler preindependence relation which, in addition, satisfies right normality.

**Lemma 5.6.2.** *Adler preindependence relations satisfying right normality also satisfy weak transitivity.*

*Proof.* This is [3, Remark 2.1].  $\square$

**Remark 5.6.3.** (i) The above does not require strong finite character.

(ii) The use of right normality can be avoided if we redefine weak transitivity by:  $A \downarrow_M BCM$  and  $C \downarrow_M BM$  implies  $AC \downarrow_M B$ . However, the definition given above, as well as being the original formulation in [59], is also more naturally related to transitivity in the usual sense.

**Example 5.6.4.**  $\downarrow^u$ ,  $\downarrow^i$ ,  $\downarrow^f$ , and  $\downarrow^d$  are all Adler preindependence relations satisfying right normality (see [19]), so by the above, they all satisfy weak transitivity.

The importance of weak transitivity lies in the fact that it allows us to find new Morley sequences out of old ones:

**Lemma 5.6.5.** *Let  $\downarrow$  be an independence relation satisfying weak transitivity and  $n < \omega$ . If  $(b_i)_{i < \omega}$  is  $\downarrow$ -independent over  $M$ , then so is  $(b_{ni}, b_{ni+1}, \dots, b_{(i+1)n-1})_{i < \omega}$ .*

*Proof.* Fix  $n < \omega$ . Let  $\bar{b}_i := (b_{in}, b_{in+1}, \dots, b_{(i+1)n-1})$ . We prove by induction on  $i$  that  $\bar{b}_i \downarrow_M \bar{b}_{<i}$ .

First, note that, by  $\downarrow$ -independence,  $b_n \downarrow_M b_0 \dots b_{n-1}$  and  $b_{n+1} \downarrow_M b_0 \dots b_n$ , so by weak transitivity  $b_n b_{n+1} \downarrow_M b_0 \dots b_{n-1}$ . Inductively, for any  $1 \leq k \leq n-1$ , we have  $b_n b_{n+1} \dots b_{n+k} \downarrow_M b_0 \dots b_{n-1}$ , and again by  $\downarrow$ -independence, we also have  $b_{n+k+1} \downarrow_M b_0 \dots b_{n+k}$ . So, by weak transitivity again,  $b_n \dots b_{n+k+1} \downarrow_M b_0 \dots b_{n-1}$ . This proves that  $b_n \dots b_{2n-1} \downarrow_M b_0 \dots b_{n-1}$ .

Suppose now that we have shown  $\bar{b}_i \downarrow_M \bar{b}_{<i}$  for some  $i < \omega$ . Then it is easy to adapt the argument from the previous paragraph to show that  $\bar{b}_{i+1} \downarrow_M \bar{b}_{\leq i}$ . This concludes the induction.  $\square$

Employing this notion, we can turn to the main topic of this section, which is a generalisation of the independence theorem over models in terms of a given independence relation  $\downarrow'$ :

(†)  $\downarrow'$ -independence theorem over models: If  $M \models T$ ,  $a_1 \equiv_M a_2$ ,  $a_1 \downarrow_M b_1$ ,  $a_2 \downarrow_M b_2$ , and  $b_1 \downarrow'_M b_2$ , then there is some  $a_*$  such that  $a_* \equiv_{Mb_1} a_1$ ,  $a_* \equiv_{Mb_2} a_2$ , and  $a_* \downarrow_M b_1 b_2$ .

This corresponds precisely to what is called “ $\downarrow^0$ -amalgamation over models” in [38]. It is also clear that we recover Definition 2.4.3(xvii) when we set  $\downarrow' = \downarrow$ . With this notion, we can adapt several results found in, e.g., [59, §6]. The main inspiration for

this comes from [95], which constitutes the first generalisation of the independence theorem to a more general version of witnessing outside  $\text{NSOP}_1$  theories.

**Lemma 5.6.6** (Chain condition). *Let  $\perp$  be an independence relation satisfying full existence, monotonicity, and weak transitivity. If  $\text{tp}(a/Mb)$  does not  $\perp$ -Kim-divide over  $M$  and  $I$  is an  $\perp$ -Morley sequence over  $M$  starting at  $b$ , there is  $a' \equiv_{Mb} a$  such that  $\text{tp}(a'/MI)$  does not  $\perp$ -Conant-divide over  $M$  and  $I$  is  $Ma'$ -indiscernible.*

*Proof.* Note that, by definition, there is  $a' \equiv_{Mb} a$  such that  $I = (b_i)_{i < \omega}$  is  $Ma'$ -indiscernible. By strong finite character, it suffices to show that  $\text{tp}(a'/Mb_{<n})$  does not  $\perp$ -Conant-divide over  $M$  for all  $n < \omega$ . So fix some  $n$ . By Lemma 5.6.5, the sequence  $(b_{kn}, b_{kn+1}, \dots, b_{(k+1)n-1})_{k < \omega}$  is  $\perp$ -Morley over  $M$  and clearly  $Ma'$ -indiscernible. This implies (by definition and Lemma 5.1.8(ii)) that  $\text{tp}(a'/Mb_0 \dots b_{n-1})$  does not  $\perp$ -Conant-divide over  $M$ . This completes the induction.  $\square$

**Theorem 5.6.7.** *Let  $\perp^1 \implies \perp^2$  be independence relations satisfying existence, monotonicity, and right extension, such that  $\perp^1$  satisfies left extension and  $\perp^2$  satisfies weak transitivity. Suppose that  $\perp^2 \implies \perp^2$ -Kim-independence and the latter satisfies symmetry and GUWP w.r.t.  $\perp^2$ . Then  $\perp^2$ -Kim-independence satisfies the  $\perp^1$ -independence theorem over models.*

*Proof.* We adapt the argument from [95, Proposition 5.8], which is itself based on the original version of the weak independence theorem in [59, Proposition 6.1]. So let  $M \models T$  and suppose that:

- (i)  $a_1 \equiv_M a_2$ .
- (ii)  $a_1$  is  $\perp^2$ -Kim-independent from  $b_1$  over  $M$ .
- (iii)  $a_2$  is  $\perp^2$ -Kim-independent from  $b_2$  over  $M$ .
- (iv)  $b_1 \perp_M^1 b_2$ .

Let us first note that, by Lemmas 5.2.34 and 5.3.19 (where we use GUWP to apply the latter lemma) together with  $\perp^2 \implies \perp^2$ -Kim-independence, a formula  $\perp^2$ -Kim-divides over  $M$  iff it  $\perp^2$ -Conant-divides over  $M$ . We can also use Proposition 5.3.21 to show that  $\perp^2$ -Kim-forking and  $\perp^2$ -Kim-dividing coincide for formulas.

**Claim.** There is some  $b'_1$  such that  $a_1 b_1 \equiv_M a_2 b'_1$  and  $a_2$  is  $\perp^2$ -Kim-independent from  $b'_1 b_2$  over  $M$ .

*Proof of Claim.* Since  $\perp^2$ -Kim-independence is symmetric, it suffices to find  $b'_1$  such that  $a_1 b_1 \equiv_M a_2 b'_1$  and  $b'_1 b_2$  is  $\perp^2$ -Kim-independent from  $a_2$  over  $M$ . Letting

$p(x, a_1) := \text{tp}(b_1/Ma_1)$ , by (ii) and symmetry  $b_1$  is  $\perp$ -Kim-independent from  $a_1$  over  $M$ , and so, by (i) and Lemma 5.2.9(iii),  $p(x, a_2)$  does not contain any formulas  $\perp$ -Kim-forking over  $M$ . Hence, it suffices to show the consistency of

$$p(x, a_2) \cup \{\neg\varphi(x, b_2^*, a_2^*) : b_2^* \subseteq b_2, a_2^* \subseteq a_2, \varphi(x, y, a_2^*) \perp^2\text{-Kim-forks over } M\}.$$

So assume, for contradiction, that the above set is inconsistent. By compactness, we have that  $p(x, a_2) \vdash \varphi(x, b_2^*, a_2^*)$  for some finite  $b_2^* \subseteq b_2$ ,  $a_2^* \subseteq a_2$  and a formula  $\varphi(x, y, a_2^*)$  that  $\perp^2$ -Kim-forks over  $M$ , and hence  $\perp^2$ -Kim-divides over  $M$ .

By (iii) and symmetry,  $b_2$  is  $\perp^2$ -Kim-independent from  $a_2$  over  $M$ . So, by definition, full existence and monotonicity for  $\perp^2$ , there is an  $\perp^2$ -Morley sequence  $(a_{2,i})_{i \in \omega}$  over  $M$  starting at  $a_2$  which is  $Mb_2$ -indiscernible. Hence,

$$\bigcup_{i < \omega} p(x, a_{2,i}) \vdash \{\varphi(x, b_2^*, a_{2,i}^*) : i < \omega\}.$$

(As usual,  $a_{2,i}^*$  denotes the restriction of  $a_{2,i}$  to the appropriate variables.) Since  $p(x, a_2)$  does not contain any formulas  $\perp^2$ -Kim-dividing over  $M$ , by compactness and our choice of  $(a_{2,i})_{i < \omega}$ , we know  $\bigcup_{i < \omega} p(x, a_{2,i})$  is consistent. Therefore,  $\{\varphi(x, y, a_{2,i}^*) : i < \omega\}$  is consistent. But this is a contradiction, since by assumption and the paragraph before the claim  $\varphi(x, y, a_2^*) \perp^2$ -Conant-divides over  $M$ .  $\square$

Let  $p_1(x, b_1) := \text{tp}(a_1/Mb_1)$ . We want to show there is  $a \models p_1(x, b_1) \cup \text{tp}(a_2/Mb_2)$  such that  $a$  is  $\perp^2$ -Kim-independent from  $b_1b_2$  over  $M$ . By (iv) and right extension for  $\perp^1$ , there is  $b_1'' \equiv_{Mb_2} b_1$  such that  $b_1'' \perp_M^1 b_1'b_2$ . It thus suffices to show that  $p_1(x, b_1'') \cup \text{tp}(a_2/Mb_2)$  has a realisation  $a$  that is  $\perp^2$ -Kim-independent from  $b_1''b_2$  over  $M$ .

Since  $b_1'' \equiv_M b_1 \equiv_M b_1'$ , by (iv) and left extension for  $\perp^1$ , we can find some  $b_2'$  such that  $b_1''b_2' \equiv_M b_1'b_2$  and  $b_1''b_2' \perp_M^2 b_1'b_2$  (since  $\perp^1 \implies \perp^2$ ). By right extension for  $\perp^2$ , it follows that  $b_1'b_2, b_1''b_2'$  begin an  $\perp^2$ -Morley sequence over  $M$ . So by Lemma 5.6.6, monotonicity and an automorphism, there is some  $a \equiv_{Mb_1b_2} a_2$  such that  $a$  is  $\perp^2$ -Conant-independent from  $b_1''b_2'$  over  $M$ . Hence, by the first paragraph of the proof,  $a$  is  $\perp^2$ -Kim-independent from  $b_1''b_2'$  over  $M$ . Since  $a \equiv_{Mb_2} a_2$ , we have that  $a \models \text{tp}(a_2/Mb_2)$ , and since  $ab_1'' \equiv_M ab_1' \equiv_M a_2b_1' \equiv_M a_1b_1$  by the Claim, we have  $a \models p_1(x, b_1'')$ .  $\square$

**Remark 5.6.8.** Since the compatibility condition is clear in this case, note that, by Theorem 5.5.1, we can slightly weaken the above result by replacing “ $\perp^2$ -Kim-independence satisfies GUWP w.r.t.  $\perp^2$ ” in the statement with (the oftentimes

easier-to-check) “ $\perp^2$ -Kim-independence satisfies left transitivity”.

- Example 5.6.9.** (i) If  $T$  is simple, then by Example 5.5.3  $\perp^f$  is symmetric and thus satisfies left and right extension. Since  $\perp^f$  always satisfies weak transitivity, the above recovers the independence theorem over models in simple theories [70, Theorem 3.5].
- (ii) If  $T$  is NSOP<sub>1</sub> and we take  $\perp^1 = \perp^u$  and  $\perp^2 = \perp^i$ , this recovers [59, Proposition 6.1].
- (iii) If  $T$  is NSOP<sub>2</sub> (= NSOP<sub>1</sub> by [95]) and we take  $\perp^1 = \perp^2$  to be canonical coheir independence, this recovers [95, Proposition 5.7].
- (iv) If  $T$  is a free amalgamation theory, there is a “free amalgamation relation”  $\perp$  in the sense of Conant [34]. By definition, such a relation always satisfies existence, monotonicity, left and right extension, weak transitivity, and symmetry (among other properties). Moreover, since it is stationary,  $\perp$  satisfies GUWP w.r.t.  $\perp$ . Therefore, the above recovers the version of the independence theorem for free amalgamation theories that appears in the introduction of [95].

Observe that, in (ii) and (iii), we need to assume symmetry of Kim-independence [59, Theorem 5.16] and  $\perp^u$ -Conant-independence [95, Theorem 5.4] in their respective contexts, which we cannot (yet) recover abstractly only from Theorem 5.5.1 and Example 5.3.6.

There is a slight modification of the assumptions of the theorem above that provides a sufficient condition for NSOP<sub>1</sub> using the results from [60]:

**Proposition 5.6.10.** *Let  $\perp$  be an independence relation satisfying full existence and monotonicity. Suppose that  $\perp$ -Kim-independence satisfies left and weak transitivity and GUWP w.r.t.  $\perp$ . Suppose further that  $\perp \implies \perp$ -Kim-independence. Then  $T$  is NSOP<sub>1</sub> and  $\perp$ -Kim-independence =  $\perp^K$ .*

*Proof.* First, we claim that, given our assumptions,  $\perp$ -Kim-independence satisfies the independence theorem over models (as in Definition 2.4.3(xvii)). Note that the first part of the previous proof (including the claim) still goes through. For the second part, simply note that  $\perp$ -Kim-independence satisfies left (and right) extension. To show this, we can either use symmetry (which follows from Theorem 5.5.1) and Lemma 5.2.14, or GUWP and Lemma 5.2.32. Moreover, since we assumed weak transitivity for  $\perp$ -Kim-independence, we can still apply Lemma 5.6.6. So the argument works.

Now, observe that  $\perp$ -Kim-independence satisfies strong finite character, existence, and monotonicity by Lemma 5.2.14, symmetry by Theorem 5.5.1, and the independence theorem over models by the previous paragraph. Hence, together with GUWP w.r.t  $\perp$ , [60, Theorem 6.11] implies that  $T$  is NSOP<sub>1</sub> and  $\perp$ -Kim-independence and  $\perp^K$  coincide.  $\square$

With this proposition, we can also devise a new criterion for a certain subclass of NSOP<sub>1</sub> theories that does not explicitly invoke any witnessing properties. For this, we need a few intermediate lemmas that use weak transitivity.

**Lemma 5.6.11.** *Let  $\perp$  be an independence relation satisfying full existence, monotonicity, and weak transitivity. If  $\varphi(x, b)$   $\perp$ -Kim-divides over  $M$ , then it  $\perp^{\text{opp}}$ -Kim-divides over  $M$ . In particular,  $\perp^{\text{opp}}$ -Kim-independence implies  $\perp$ -Kim-independence.*

*Proof.* Suppose there is  $\varphi(x, b)$  and an  $\perp$ -Morley sequence  $(b_i)_{i < \omega}$  over  $M$  with  $b_0 = b$  such that  $\{\varphi(x, b_i) : i < \omega\}$  is inconsistent, and so  $k$ -inconsistent for some  $k$  by  $M$ -indiscernibility. By [38, Claim 7], there exists some  $M$ -indiscernible sequence  $(b'_i)_{i < \omega}$  such that  $b'_0 \dots b'_n \equiv_M b_n \dots b_0$  for all  $n < \omega$ . In particular, the set  $\{\varphi(x, b'_i) : i < \omega\}$  is also  $k$ -inconsistent.

It remains to show that  $(b'_i)_{i < \omega}$  is  $\perp^{\text{opp}}$ -independent over  $M$ . For any  $n < \omega$ , we have  $b'_0 \perp_M b'_1 \dots b'_n$ , so by  $M$ -indiscernibility  $b'_1 \perp_M b'_2 \dots b'_{n+1}$  and thus by right monotonicity  $b'_1 \perp_M b'_2 \dots b'_n$ . Hence, by weak transitivity,  $b'_0 b'_1 \perp_M b'_2 \dots b'_n$ . Proceeding inductively, we obtain that  $b'_0 \dots b'_{n-1} \perp_M b'_n$ , as required.  $\square$

**Example 5.6.12.** If a formula divides along a  $\perp^u$ -Morley sequence over a model  $M \models T$ , then it also divides along an  $\perp^h$ -Morley sequence over  $M$ .

**Lemma 5.6.13.** *Let  $\perp^1$  and  $\perp^2$  be independence relations satisfying full existence and monotonicity. Assume that  $\perp^1$  also satisfies weak transitivity. Suppose that, for every  $M \models T$ , every  $(\perp^1)^{\text{opp}}$ -independent sequence  $(a_i)_{i < |T|^+}$  over  $M$ , and any finite tuple  $b$ , there is some  $i < |T|^+$  such that  $\text{tp}(b/Ma_i)$  does not  $\perp^2$ -Kim-divide over  $M$ . Then  $\perp^1$  satisfies GUWP w.r.t.  $\perp^2$ .*

*Proof.* Same proof as in Lemma 5.3.26((ii)  $\Rightarrow$  (i)), noting that we may extract and extend an  $(\perp^1)^{\text{opp}}$ -Morley sequence using the claim from the previous proof.  $\square$

**Proposition 5.6.14.** *Let  $\perp$  be an independence relation satisfying full existence and monotonicity. Suppose that  $\perp$ -Kim-independence is compatible with  $\perp$  and satisfies weak and left transitivity, and left extension. The following are equivalent:*

- (i)  $\perp$ -Kim-independence satisfies GUWP w.r.t.  $\perp$ .
- (ii)  $\perp$ -Kim-independence satisfies chain local character.

*Proof.* ((i)  $\Rightarrow$  (ii)) This follows by left transitivity, Theorem 5.5.1 and Lemma 5.4.11.

((ii)  $\Rightarrow$  (i)) We use Lemma 5.6.13. So let  $M \models T$  be a model and  $b$  be a finite tuple. Let  $(a_i)_{i < |T|^+}$  be such that  $a_{<i}$  is  $\perp$ -Kim-independent from  $a_i$  over  $M$ . Observe that  $M$  is  $\perp$ -Kim-independent from  $a$  over  $M$ . Thus, by left extension and Remark 3.3.6 (applied to  $(\perp\text{-Kim-independence})^{\text{opp}}$ ), we can find a continuous chain of models  $(M_i)_{i < |T|^+}$  such that  $M \preceq M_i$ ,  $a_{<i} \subset M_i$ , and  $M_i$  is  $\perp$ -Kim-independent from  $a_i$  over  $M$  for all  $i < |T|^+$ .

By chain local character, we can find some  $i < |T|^+$  such that  $b$  is  $\perp$ -Kim-independent from  $M_{|T|^+} := \bigcup_{j < |T|^+} M_j$  over  $M_i$ . Thus, by right monotonicity,  $b$  is  $\perp$ -Kim-independent from  $a_i$  over  $M_i$ . Therefore, by left transitivity, it follows that  $b$  is  $\perp$ -Kim-independent from  $a_i$  over  $M$ .  $\square$

**Corollary 5.6.15.** *Let  $\perp$  be an independence relation satisfying full existence and monotonicity. Suppose that  $\perp$ -Kim-independence satisfies weak and left transitivity, left extension, and chain local character. Suppose further that  $\perp \implies \perp\text{-Kim-independence}$ . Then  $T$  is NSOP<sub>1</sub> and  $\perp\text{-Kim-independence} = \perp^K$ .*

*Proof.* Combine Propositions 5.6.10 and 5.6.14.  $\square$

This imposes some restrictions on any potential converse to Lemma 5.4.11.

Returning to the  $\perp'$ -independence theorem over models, we can adapt the argument in [45, Proposition 6.3] to obtain a partial converse to Theorem 5.6.7:

**Theorem 5.6.16.** *Let  $\perp^1 \implies \perp^2$  be independence relations satisfying full existence and monotonicity such that  $\perp^1$  also satisfies left and right extension. If  $\perp^2$ -Conant-independence satisfies the  $\perp^2$ -independence theorem over models, then  $\perp^1$  satisfies GUWP w.r.t.  $\perp^2$ .*

*Proof.* Assume, for contradiction, that  $\perp^1$  does not satisfy GUWP w.r.t.  $\perp^2$ . So there is a formula  $\varphi(x, b)$ , an  $\perp^1$ -Morley sequence  $(b_i)_{i < \omega}$  over  $M$  and an  $\perp^2$ -Morley sequence  $(b'_i)_{i < \omega}$  over  $M$  such that  $b_0 = b'_0 = b$ ,  $\{\varphi(x, b_i) : i < \omega\}$  is consistent, and  $\{\varphi(x, b'_i) : i < \omega\}$  is inconsistent. Pick  $a \models \{\varphi(x, b_i) : i < \omega\}$  such that  $(b_i)_{i < \omega}$  is  $Ma$ -indiscernible, which implies using  $b_0 = b'_0$  and Lemmas 5.2.33 and 5.2.34 that  $a$  is  $\perp^2$ -Conant-independent from  $b'_0$  over  $M$ .

By indiscernibility, there is some  $N < \omega$  such that  $\{\varphi(x, b'_i) : i \leq N\}$  is inconsistent. We will find by induction on  $k \leq N$  some  $a_k \models \{\varphi(x, b'_i) : 0 \leq i \leq k\}$  such that  $a_i$  is  $\downarrow^2$ -Conant-independent from  $b'_{\leq k}$  over  $M$ , which will give us the required contradiction.

For the base case, we can take  $a_0 = a$ . For the inductive step, suppose that we have found  $a_k \models \{\varphi(x, b'_i) : i \leq k\}$  such that  $a_k$  is  $\downarrow^2$ -Conant-independent from  $b'_{\leq k}$  over  $M$ . Since  $b'_{k+1} \equiv_M b'_k$ , we can pick  $\sigma \in \text{Aut}(M/M)$  such that  $\sigma(b'_k) = b'_{k+1}$  and let  $a' := \sigma(a)$ , so that  $a' \equiv_M a_k$ . Hence, by invariance,  $a'$  is  $\downarrow^2$ -Conant-independent from  $b'_{k+1}$  over  $M$ . Moreover, by choice,  $b'_{k+1} \downarrow_M^2 b'_{\leq k}$ , and by induction hypothesis,  $a_k$  is  $\downarrow^2$ -Conant-independent from  $b'_{\leq k}$  over  $M$ . Therefore, by the  $\downarrow^2$ -independence theorem over models, there exists some  $a_*$  such that  $a_* \equiv_{Mb'_{\leq k}} a_k$ ,  $a_* \equiv_{Mb'_{k+1}} a'$ , and  $a_*$  is  $\downarrow^2$ -Conant-independent from  $b'_{\leq k+1}$  over  $M$ . Moreover,  $a_* \models \{\varphi(x, b'_i) : i \leq k+1\}$ . So we can let  $a_{k+1} := a_*$ . This completes the induction.  $\square$

With this lemma, we can obtain an analogue to [32, Proposition 5.2]. The innovation here is that, while Chernikov and Ramsey's proof involves the construction of a tree and the syntactic properties of  $\text{SOP}_1$ , our proof is completely “semantic” and only involves properties of abstract independence relations (modulo the equivalence of  $\text{NSOP}_1$  and Kim's Lemma for coheir Morley sequences):

**Proposition 5.6.17.** *Let  $\downarrow^u \implies \downarrow$  be an independence relation satisfying full existence and monotonicity. If  $\downarrow$ -Conant-independence satisfies the  $\downarrow$ -independence theorem over models, then  $T$  is  $\text{NSOP}_1$ .*

*Proof.* By Theorem 5.6.16,  $\downarrow^u$  satisfies GUWP w.r.t.  $\downarrow$ . As  $\downarrow^u \implies \downarrow$ , by Lemma 5.3.17  $\downarrow^u$  satisfies GUWP w.r.t.  $\downarrow^u$ . Therefore, by [59, Theorem 3.16],  $T$  is  $\text{NSOP}_1$ .  $\square$

**Corollary 5.6.18.** *The following are equivalent:*

- (i)  $T$  is  $\text{NSOP}_1$ .
- (ii) Coheir-Conant-independence satisfies the independence theorem over models.
- (iii) Conant-independence satisfies the independence theorem over models.

*In this case, coheir-Conant-independence, Conant-independence, and  $\downarrow^K$  coincide.*

**Remark 5.6.19.** Let us note that, in the proof of Theorem 5.6.16, we only need left extension of  $\downarrow^1$  to obtain the equivalence between  $\downarrow^1$ -Conant-forking and  $\downarrow^1$ -Conant-dividing. Thus, if we redo the proof only working with non- $\downarrow^2$ -Conant-dividing, we can show the equivalence of the following two semantically:

- (i)  $\perp^i$  satisfies GUWP w.r.t.  $\perp^i$ .
- (ii) Non-Conant-dividing independence satisfies the independence theorem over models.

Of course, once we can appeal to the syntactic features of  $\text{NSOP}_1$  theories, this is again equivalent to the conditions from the above results.

**Example 5.6.20.** We have seen before that, if  $T$  is a free amalgamation theory with relation  $\perp$ , then Conant-independence satisfies the  $\perp$ -independence theorem over models, and there are strictly  $\text{NSOP}_4$  such theories. In particular, as any free amalgamation relation satisfies left extension, Conant-independence fails the independence theorem in any strictly  $\text{NSOP}_4$  free amalgamation theory.

## 5.7 Dichotomies

In this section, we will prove a result that illustrates how we can use the witnessing framework developed in §5.3 to prove some old and new dichotomies across dividing lines. A precedent to this type of result can be found in the proofs in [41] of the facts that simple theories are either stable or IP, and  $\text{NSOP}_1$  theories are either simple or  $\text{TP}_2$ . Although these results, in their original form, trace back to Shelah, the proofs in [41] use semantic arguments involving Kim-independence to show this, in contrast to the original, combinatorial proofs. In what follows, we will prove an extension of these dichotomies that has new applications within the context of  $\text{NSOP}_4$  theories.

The key lemma is the following technical result, which uses the notion of Kim-strictness introduced in Example 2.4.26:

**Lemma 5.7.1.** *Let  $\perp^1$ ,  $\perp^2$  and  $\perp^3$  be independence relations satisfying full existence and monotonicity. Suppose that:*

- (i) *Kim-strict  $\perp^2$  satisfies full existence.*
- (ii)  *$\perp^2 \implies \perp^3$ .*
- (iii)  *$\perp^1$  satisfies GUWP w.r.t.  $\perp^2$ .*
- (iv) *Kim-strict  $\perp^3$  satisfies GUWP w.r.t.  $\perp^1$ .*

*Then  $\leq_1$  is trivial.*

*Proof.* Assume  $\varphi(x, b) \perp^1$ -Kim-divides over  $M$ . By (iv), it follows that  $\varphi(x, b)$  Kim-strict  $\perp^3$ -Conant-divides over  $M$ . By (i),  $\text{tp}(b/M)$  has a global Kim-strict

$M$ - $\downarrow^2$ -free extension, which by (ii) is also Kim-strict  $M$ - $\downarrow^3$ -free. Therefore,  $\varphi(x, b)$   $\downarrow^2$ -Kim-divides over  $M$ . Hence, by (iii), it follows that  $\varphi(x, b)$   $\downarrow^1$ -Conant-divides over  $M$ , as required.  $\square$

**Remark 5.7.2.** It is clear that the use of Kim-independence in the previous proof is not necessary. We could have stated the more general result using another independence relation  $\downarrow^4$  and the corresponding  $\downarrow^4$ -strict versions of  $\downarrow^2$  and  $\downarrow^3$ . However, we have preferred to avoid this since Kim-independence is all we will use later on, and also to make it less cumbersome.

As we will see in what follows, the above result has important applications in neostability theory. We need the following definition from [29, Definition 3.2]:

**Definition 5.7.3.** Let  $A$  be a set and  $b$  a tuple. We say  $\varphi(x, b)$  **quasi-divides over**  $A$  if there exist  $m < \omega$  and  $(b_i)_{i < m}$  with  $b_i \equiv_A b$  for all  $i < m$  such that the set  $\{\varphi(x, b_i) : i < m\}$  is inconsistent.

It is clear that, in order to apply Lemma 5.7.1, we need to find instances of independence relations  $\downarrow$  for which Kim-strict  $\downarrow$  has full existence. The key is to make the following observation: the proof of [72, Corollary 3.16] does not really use any specific features of NATP theories; rather, it just requires the assumption (which is now a theorem; see [76, Corollary 2.25]) that every formula that Kim-forks over  $M$  also quasi-divides over  $M$ . Thus, this gives us a general theorem:

**Lemma 5.7.4.** *Let  $\downarrow$  be an independence relation satisfying full existence, monotonicity, and quasi-strong finite character. Let  $M \models T$ . Then every type over  $M$  has a global Kim-strict  $\downarrow$ -free extension over  $M$ . In other words, Kim-strict  $\downarrow$  satisfies full existence.*

Combining this result with the previous lemma, we obtain the following dichotomy:

**Proposition 5.7.5.** *Let  $\downarrow \implies \downarrow^i$  be an independence relation satisfying full existence, monotonicity, and quasi-strong finite character such that  $\downarrow^i$  satisfies GUWP w.r.t.  $\downarrow$ . Then  $T$  is either NSOP<sub>1</sub> or BTP.*

*Proof.* Suppose that  $T$  is NBTP. By the New Kim's Lemma ([76, Theorem 5.2]), Kim-strict  $\downarrow^i$  satisfies GUWP w.r.t.  $\downarrow^i$ . Moreover, by assumption,  $\downarrow^i$  satisfies GUWP w.r.t.  $\downarrow$ , and by quasi-strong finite character and Lemma 5.7.4, Kim-strict  $\downarrow$  satisfies full existence. Therefore, by Lemma 5.7.1, it follows that  $\leq_i$  is trivial. Hence,  $T$  is NSOP<sub>1</sub>.  $\square$

Note that, from this, we can use Shelah's dichotomies to show that, in this context,  $T$  is either simple or  $\text{TP}_2$ , and stable or IP. Moreover, note that the quasi-strong finite character condition means that the class of theories where an independence relation as in the hypothesis of Proposition 5.7.5 contains the class of  $\text{NSOP}_1$  theories, since we can take  $\perp = \perp^i$ .

The main application of the above result shows that most known  $\text{NSOP}_4$  theories from the literature are BTP. To see this, let us note that the independence relations considered by Mutchnik in [94] satisfy the assumptions of our theorem:

**Lemma 5.7.6.** *If  $\perp$  satisfies full existence, monotonicity, and stationarity over a set  $C$ , then it also satisfies quasi-strong finite character over  $C$ .*

*Proof.* Let  $C$  be a set of parameters such that  $\perp$  satisfies stationarity and full existence over  $C$ , and let  $p(x), q(y) \in S(C)$ . Pick  $b_0 \models q$ . By full existence, there is  $a_0 \models p$  such that  $a_0 \perp_C b_0$ . Let  $\Sigma_{p,q}(x, y) := \text{tp}(a_0 b_0 / C)$ . We claim that this  $\Sigma_{p,q}$  works.

Indeed, if  $ab \models \Sigma_{p,q}(x, y)$ , then  $ab \equiv_C a_0 b_0$  by definition, and thus by invariance  $a \perp_C b$ . Conversely, suppose that  $a \models p(x)$  and  $b \models q(y)$  are such that  $a \perp_C b$ . As  $b \equiv_C b_0$ , there is some  $\sigma \in \text{Aut}(\mathbb{M}/C)$  such that  $\sigma(b) = b_0$ . Let  $a' := \sigma(a)$ . Then  $a' b_0 \equiv_C ab$ , so  $a' \perp_C b_0$ . By assumption,  $a_0 \perp_C b_0$ . Thus, as  $a' \equiv_C a_0$ , it follows from stationarity that  $a' \equiv_{Cb_0} a_0$ . But then  $a_0 b_0 \equiv_C a' b_0 \equiv_C ab$ , as required.  $\square$

**Corollary 5.7.7.** (i) *Strictly  $\text{NSOP}_4$  free amalgamation theories are BTP.*

(ii) *The model companion of the theory of graphs without cycles of length  $\leq n$  is BTP.*

(iii) *Strictly  $\text{NSOP}_4$  Hrushovski constructions with a “good” control function are BTP.*

(iv) *The generic  $\mathbf{H}_4$ -free 3-hypertournament is BTP.*

*Proof.* In all cases, we use Proposition 5.7.5. For (i)-(iii), we can use the stationary independence relations from [94] together with Lemma 5.7.6. For (iv), we can use  $\perp^{\text{hti}}$  from chapter 4.  $\square$

**Remark 5.7.8.** Let us mention in passing that the result for  $T_{\mathbf{H}_4\text{-free}}$  can be improved in a similarly “semantic” fashion, although in a way that does not seem to directly generalise to cover other cases. The idea is to adapt the notion of reliable extensions introduced by Hanson in [50] to  $\perp^{\text{hti}}$ -free extensions of types, and use [50, Proposition 2.6] to conclude that  $T_{\mathbf{H}_4\text{-free}}$  must also be CTP.

# 6

## Realisability

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In this final chapter, after the semantic incursion into independence relations from the last chapter, we will consider more syntactic questions about dividing lines. We will focus on the combinatorial properties which are used to define such dividing lines, such as the order property or the independence property. It has already been noted in the literature that there is some correspondence between these properties of formulas and the encoding of certain graphs. To fix our terminology, all the graphs considered in this chapter are assumed to be simple (i.e., no loops) and not to have parallel edges.

The encoding of graphs by formulas can be formalised in different ways. Two prominent proposals, one due to Laskowski and Shelah in [79] and the other more recently introduced by Bailetti in [8], are the following:

**Definition 6.0.1.** Let  $T$  be a complete theory,  $\mathbb{M} \models T$  a monster model,  $\varphi(x, y)$  a partitioned formula, and  $(V, R)$  be a graph.

- (i) We say  $\varphi(x, y)$  **codes**  $(V, R)$  if there exists a tuple  $(b_v)_{v \in V}$  such that  $\models \varphi(b_v, b_w)$  iff  $\{v, w\} \in R$ .
- (ii) We say  $\varphi(x, y)$  **realises**  $(V, R)$  if there exists a tuple  $(b_v)_{v \in V}$  such that:

- For every clique  $I \subseteq V$  such that  $|I| \geq 2$ , the set  $\{\varphi(x, b_v) : v \in I\}$  is consistent.
- For any  $v \neq w \in V$  such that  $\{v, w\} \notin R$ , the set  $\{\varphi(x, b_v), \varphi(x, b_w)\}$  is inconsistent.

Although the precise relationship between these two notions has not yet been explored in the literature (a task which we undertake in §6.1), it is clear that these notions are useful in characterising several classes of complete first-order theories from neostability theory. Some examples of this include the following:

**Fact 6.0.2.** *Let  $T$  be a complete theory.*

- (i) (Folklore, cf. [54, §6.7])  $T$  is unstable iff there is a formula  $\varphi(x, y)$  such that, for every  $n < \omega$ , there is a sequence  $(a_i)_{i < n}$  such that  $\models \varphi(a_i, a_j)$  iff  $i \leq j$ .
- (ii) [79, Lemma 2.2]  $T$  is IP iff there is a formula that codes all finite graphs.
- (iii) [8, Proposition 6.3]  $T$  is  $\text{PM}^{(2)}$  iff there is a formula that realises all finite graphs.

We will not provide an explicit definition of  $\text{PM}^{(2)}$  in this chapter; for the purposes of what follows, the reader may take (iii) above as a definition of this property. To give some intuition of where this property falls in the classification picture, observe that  $\text{PM}^{(2)} \implies \text{ATP}$  [8]. Let us also note that we have included (i), although it does not use the concept of coding from Definition 6.0.1, because it can be informally read as “coding all finite linear orders” (and, as we will see, this result leads to genuine coding characterisations of stability).

In this chapter, we carry out a systematic study of the notion of realising and provide several new characterisations of stability and NIP in terms of the realisability of various classes of finite graphs. We also provide some new characterisations of classes at the edge of the current classification picture, such as  $\text{NPM}^{(2)}$  and  $\text{NSOP}_3$  theories. A summary of the main results in this direction is the following:

**Theorem 6.0.3.** *Let  $T$  be a complete theory.*

- (i)  $T$  is unstable iff it realises all half-graphs.
- (ii)  $T$  is IP iff it realises all bipartite graphs, iff it realises all triangle-free graphs.
- (iii)  $T$  is  $\text{SOP}_3$  iff it realises all complements of half-graphs.
- (iv)  $T$  is  $\text{PM}^{(2)}$  iff it realises all co-bipartite graphs.

One of the main outcomes of this result is the establishment of a certain com-

binatorial correspondence between the pair of stable and NIP theories, on the one hand, and the pair of NSOP<sub>3</sub> and NPM<sup>(2)</sup> theories, on the other. This suggests a potential dichotomy, at the level of NSOP theories, between NSOP<sub>3</sub> and PM<sup>(2)</sup> theories, which we aim to explore in future work.

In the final two sections of this chapter, we turn to some of the higher-arity versions of stability and NIP that have appeared in the literature. In particular, we prove some basic characterisations of NFOP<sub>*n*</sub> and NIP<sub>*n*</sub> in terms of coding, which generalise the results for binary classes, and in the process, introduce a combinatorial structure that sharpens a necessary and sufficient condition for FOP<sub>*n*</sub> from [1].

As previously mentioned, in this chapter, we will not give the original definition of PM<sup>(2)</sup> from [8], as this requires the technology of patterns, which is beyond the scope of the present chapter. More details about patterns, and in particular the definition of PM<sup>(2)</sup>, can be found in the appendix. In that appendix, we also consider the natural question about the precise relationship between the realisability of graphs, as studied in this chapter, and the exhibitability of patterns. These results shed some light on the possibilities of the realisability approach, as several model-theoretic properties, such as simplicity, do not seem to be captured by this framework. The appendix will also include some further technical results about patterns and the notion of duality.

From now on, given a class  $\mathcal{K}$  of finite graphs, we say a theory  $T$  **codes** (resp., **realises**) all graphs in  $\mathcal{K}$  if there is a formula  $\varphi(x, y)$  that codes (resp., realises) all graphs in  $\mathcal{K}$ . Thus, e.g., we can reformulate Fact 6.0.2(iii) as:  $T$  is PM<sup>(2)</sup> iff it realises all finite graphs.

## 6.1 Realisability and coding

We begin with a preliminary section that recasts the notion of realisability in terms of a strengthened form of coding graphs. This will allow us to compare the relative force of realisability and coding, and also move between the two notions later on. For the rest of this chapter, let us fix a complete first-order theory  $T$  but not necessarily with infinite models (this will be justified shortly), and with a monster model  $\mathbb{M}$ . As we begin our enumerations with 0, we will use  $[n]$  to denote the set  $\{0, 1, \dots, n-1\}$ .

**Definition 6.1.1.** Let  $k < \omega$  and  $\Sigma = \{\varphi_2, \varphi_3, \dots\}$  be a (potentially infinite) set of formulas such that  $\varphi_i$  has arity  $ki$  for all  $i$ . Let  $(V, E)$  be a hypergraph, where

$E \subseteq \mathcal{P}(V)$ . We say that  $\Sigma$  **codes**  $(V, E)$  if there is a sequence of  $k$ -tuples  $(a_v)_{v \in V}$  such that, for all  $2 \leq j \leq |V|$  and all distinct  $v_0, \dots, v_{j-1} \in V$ ,

$$\{v_0, \dots, v_{j-1}\} \in E \iff \models \varphi_j(a_{v_0}, \dots, a_{v_{j-1}}).$$

Observe that, if  $(V, R)$  is a graph and  $\varphi(x, y)$  codes  $(V, R)$  in the sense of Definition 6.0.1(i), then  $\Sigma = \{\varphi(x, y)\}$  codes  $(V, R)$  in the sense of Definition 6.1.1.

We now define a way of converting graphs into hypergraphs and formulas into sets of formulas so that we can move between realising and coding. The results of this section will show that there are subtleties in the relationship between these two concepts.

**Definition 6.1.2.** Let  $(V, R)$  be a graph. We define its **associated hypergraph**  $(V, E_R)$  as follows: for all  $2 \leq j \leq |V|$  and distinct  $v_0, \dots, v_{j-1} \in V$ , we declare  $\{v_0, \dots, v_{j-1}\} \in E_R$  iff  $\{v_0, \dots, v_{j-1}\}$  is a clique in  $(V, R)$ .

Let  $\varphi(x, y)$  be a formula. We define its **associated set of formulas**  $\Sigma_\varphi$  to be  $\{\psi_i : i > 1\}$ , where  $\psi_i(y_0, \dots, y_{i-1}) := \exists x \bigwedge_{j < i} \varphi(x, y_j)$  for all  $i > 1$ .

**Remark 6.1.3.** If  $(V, R)$  is a triangle-free graph, then  $E_R = R$ , since any clique in  $(V, R)$  has size 2.

**Lemma 6.1.4.** Let  $(V, R)$  be a finite graph. A formula  $\varphi(x, y)$  realises  $(V, R)$  iff  $\Sigma_\varphi$  codes  $(V, E_R)$ .

*Proof.*  $(\Rightarrow)$  Suppose  $\varphi(x, y)$  realises  $(V, R)$  with a sequence  $(a_v)_{v \in V}$ . Let  $v_0, \dots, v_{j-1} \in V$  be distinct for some  $2 \leq j \leq |V|$ . Then

$$\begin{aligned} \{v_0, \dots, v_{j-1}\} \in E_R &\iff \{v_0, \dots, v_{j-1}\} \text{ is a clique in } (V, R) \\ &\iff \{\varphi(x, a_{v_0}), \dots, \varphi(x, a_{v_{j-1}})\} \text{ is consistent} \\ &\iff \models \psi_j(a_{v_0}, \dots, a_{v_{j-1}}). \end{aligned}$$

Therefore,  $\Sigma_\varphi$  codes  $(V, E_R)$  with  $(a_v)_{v \in V}$ .

$(\Leftarrow)$  Suppose  $\Sigma_\varphi$  codes  $(V, E_R)$  with a sequence  $(a_v)_{v \in V}$ . First, take a clique  $\{v_0, \dots, v_{j-1}\} \in E_R$ . Then  $\{v_0, \dots, v_{j-1}\} \in E_R$ , so, by assumption,  $\models \psi_j(a_{v_0}, \dots, a_{v_{j-1}})$ . By definition of  $\psi_j$ , this means that  $\{\varphi(x, a_{v_0}), \dots, \varphi(x, a_{v_{j-1}})\}$  is consistent.

On the other hand, suppose that  $v, w \in V$  are distinct and  $\{v, w\} \notin R$ . Then,

by assumption,  $\models \neg\psi_2(a_v, a_w)$ . By the definition of  $\psi_2$ ,  $\{\varphi(x, a_v), \varphi(x, a_w)\}$  is inconsistent. Therefore,  $\varphi(x, y)$  realises  $(V, R)$  with  $(a_v)_{v \in V}$ .  $\square$

**Remark 6.1.5.** In a general complete theory  $T$  (possibly with finite models), given a graph  $(V, R)$ , we can only say that, if  $\Sigma_\varphi$  codes  $(V, E_R)$  and  $\theta(y_0, y_1) := \psi_2(y_0, y_1) \wedge y_0 \neq y_1$  is consistent, then  $\theta(y_0, y_1)$  codes  $(V, R)$ . We will see an example later where the implication from realising to coding graphs fails.

## 6.2 Some preliminary results on coding

In this section, we present a preliminary result about coding that illustrates some initial ideas of later results, as well as some of the main kinds of graphs that we will study. The main result of this subsection will be a characterisation of complete theories with infinite models in terms of coding several classes of graphs. We have emphasised in the previous section that, for formulas in arbitrary complete theories (possibly with finite models), the relation between coding and realising graphs is not immediate. Let us start with an example of the potential gap we may find between these two notions:

**Lemma 6.2.1.** *Every theory  $T$  realises all finite complete graphs.*

*Proof.* Let  $M \models T$  and  $b \in M$ . Let  $(b_i)_{i < \omega}$  be identically  $b$ , and  $\varphi(x, y) := x = y$ . Then the set  $\{\varphi(x, b_i) : i < \omega\}$  is consistent. Therefore,  $\varphi(x, y)$  and  $(b_i)_{i < \omega}$  realise the countable complete graph.  $\square$

Nonetheless, observe that the formula  $\varphi(x, y)$  realising complete graphs is not consistent with  $x \neq y$ , so we cannot apply Remark 6.1.5. In fact, we will soon prove that *not* all theories code all finite complete graphs. We do this by observing a more general fact, namely, that if  $T$  has infinite models, then realising does imply coding:

**Lemma 6.2.2.** *Suppose  $T$  is a complete theory with infinite models, and let  $(V, R)$  be a finite graph. If  $T$  realises  $(V, R)$ , then it also codes  $(V, R)$ .*

*Proof.* Let  $\varphi(x_0, x_1)$  and  $(b_v)_{v \in V}$  realise  $(V, R)$ . Since  $T$  has infinite models, we can find, in any model containing  $(b_v)_{v \in V}$ , a sequence of singletons  $(c_v)_{v \in V}$  such that  $c_v \neq c_{v'}$  whenever  $v \neq v'$ . We then define a new formula  $\psi(x_0 y_0, x_1 y_1)$  by  $\exists z(\varphi(z, x_0) \wedge \varphi(z, x_1)) \wedge y_0 \neq y_1$ , and new tuples  $d_v := b_v c_v$  for  $v \in V$ .

Then, for  $v, w \in V$ , we have  $\{v, w\} \in R$  iff  $\{\varphi(x, b_v), \varphi(x, b_w)\}$  is consistent, iff

$\models \exists z(\varphi(z, b_v) \wedge \varphi(z, b_w))$ , iff  $\models \psi(d_v, d_w)$  (as clearly  $\models c_v \neq c_w$  by our choice of tuples). Therefore,  $\psi(x_0y_0, x_1y_1)$  and  $(d_v)_{d \in V}$  code  $(V, R)$ .  $\square$

In particular, we obtain the following equivalence:

**Proposition 6.2.3.** *Let  $T$  be a complete theory. Then  $T$  has infinite models iff it codes all finite complete graphs.*

*Proof.*  $(\Rightarrow)$  By Lemma 6.2.1,  $T$  realises all finite complete graphs. Since  $T$  has infinite models, by Lemma 6.2.2  $T$  also codes all finite complete graphs.

$(\Leftarrow)$  Suppose that there is  $M \models T$  such that  $|M| = n$ . Consider the complete graph  $(V, R)$  with  $n + 1$  vertices. Assume, for contradiction, that there is a formula  $\varphi(x, y)$  and  $(a_i)_{i < n+1}$  in  $M$  coding  $(V, R)$ . By the pigeonhole principle, there are  $i \neq j$  such that  $a_i = a_j$ . But we must also have  $M \models \neg\varphi(a_i, a_i)$ , which entails that  $M \models \neg\varphi(a_i, a_j)$ , a contradiction. Therefore,  $T$  does not code  $(V, R)$ .  $\square$

This first example shows that realising need not imply coding. At the same time, Lemma 6.2.2 ensures that we can assume that the implication holds in most model-theoretic applications.

**Remark 6.2.4.** Let us observe that Proposition 6.2.3 implies that the converse of Lemma 6.2.2 is also true: namely, if  $T$  is a complete theory such that, for all graphs  $(V, R)$ , if  $T$  realises  $(V, R)$  then  $T$  also codes  $(V, R)$ , then  $T$  has infinite models.

Another useful consequence of assuming infinite models is that we can identify a class of graphs for which coding and realising are equivalent at the level of theories:

**Lemma 6.2.5.** *Let  $T$  be a complete theory with infinite models, and let  $(V, R)$  be a triangle-free graph. Then  $\varphi(x, y)$  realises  $(V, R)$  iff the formula  $\psi(y_0z_0, y_1z_1) := \exists x(\varphi(x, y_0) \wedge \varphi(x, y_1)) \wedge z_0 \neq z_1$  codes it. In particular,  $T$  realises  $(V, R)$  iff it codes  $(V, R)$ .*

*Proof.* Left-to-right follows from the proof of Lemma 6.2.2. The converse is clear since  $E_R = R$ .  $\square$

Let us note that, in general, it is not possible to obtain a single formula that both codes and realises triangle-free graphs:

**Example 6.2.6.** Let  $T$  be the theory of the generic bipartite graph, which is simple unstable. One can check that  $\varphi(x, y) := R(x, y)$  codes all bipartite graphs. We claim

that  $\varphi(x, y)$  cannot realise most bipartite graphs. Suppose  $(V, R)$  is a bipartite graph with one independent set consisting of three elements, and  $(a_0, a_1, a_2)$  are elements in the realisation of  $(V, R)$  by  $\varphi(x, y)$  standing for this independent set. In particular,  $\{R(x, a_0), R(x, a_1)\}$  is inconsistent, which implies that  $a_0$  and  $a_1$  must belong to different parts in the generic bipartite graph. But then the same argument implies that  $a_0, a_1$  and  $a_2$  must all lie in different independent sets, which is impossible.

We conclude with a further characterisation of the existence of infinite models. We need to introduce a certain class of graphs commonly studied in combinatorics. We use the standard terminology from the literature (cf., [82, ch. 5]):

**Definition 6.2.7.** We say a graph  $(V, R)$  is **split** if there is a partition of  $V$ , say  $\{v_i : i < m\} \cup \{w_i : i < n\}$ , such that  $\{v_i : i < m\}$  is a clique and  $\{w_i : i < n\}$  is an independent set.

Given a split graph  $(V, R)$  with partition  $V = \{v_i : i < m\} \cup \{w_i : i < n\}$  as above, we say  $(V, R)$  is **complete** if  $\{v_i, w_j\} \in R$  for all  $i < m$  and  $j < n$ .

**Proposition 6.2.8.** *The following are equivalent:*

- (i)  $T$  has infinite models.
- (ii)  $T$  codes all finite complete multipartite graphs.
- (iii)  $T$  codes all finite complete split graphs.

*Proof.* ((i)  $\Rightarrow$  (ii)) Let  $M \models T$  and  $(a_i)_{i < \omega}$  be such that  $a_i \neq a_j$  for all  $i \neq j$ . Let  $(V, R)$  be the complete multipartite graph with  $n$  parts, each of which has size  $m_i$  (for  $i < n$ ). Let  $\varphi(x, y) := x \neq y$ , and let  $(a_{ij})_{i < n, j < m_i}$  be the sequence given by  $a_{ij} = a_i$  for all  $i < n$  and  $j < m_i$ . Then  $\models \varphi(a_{ij}, a_{kl})$  iff  $i \neq k$ . In particular,  $\varphi(x, y)$  and  $(a_{ij})_{i < n, j < m_i}$  code  $(V, R)$ . Hence,  $T$  codes all finite complete multipartite graphs.

((ii)  $\Rightarrow$  (iii)) Suppose  $\varphi(x, y)$  codes all finite complete multipartite graphs. Without loss of generality, let  $(V, R)$  be the complete split graph with parts having  $n$  vertices each. By assumption, there are tuples  $(a_{ij})_{i < n+1, j < n}$  witnessing that  $\varphi(x, y)$  realises the complete multipartite graph with  $n+1$  parts of  $n$  vertices each. Define  $c_i := a_{i+1,0}$  and  $d_i := a_{0,i}$  for all  $i < n$ . Then  $\varphi(x, y)$  and  $(c_i d_i)_{i < \omega}$  code all finite complete split graphs.

((iii)  $\Rightarrow$  (i)) Suppose that  $\varphi(x, y)$  codes all finite complete split graphs. If we just take the clique row in each case, it follows that  $\varphi(x, y)$  codes all finite complete graphs. Therefore, by Proposition 6.2.3,  $T$  has infinite models.  $\square$

### 6.3 Stability and NIP

The purpose of this section is to extend the characterisations of stability and NIP from Fact 6.0.2 in terms of coding other natural classes of finite graphs. This will provide more tools to understand these classes and allow us to reformulate the following well-known theorem due to Shelah in “purely combinatorial” terms:

**Fact 6.3.1** ([104, Theorem 4.1]). *A theory is stable iff it is NSOP and NIP.*

(Note that all of these properties have been defined in §2.3.) This will also set the stage for our work on realisability in the upcoming sections.

To carry out this programme, we need to appeal to several classes of finite graphs from combinatorics. As perhaps expected, we will work with half-graphs and also bipartite graphs. But, for reasons that will become clearer later on, we also want to introduce “counterparts” of half-graphs when we add structure to each of the two parts. In the case of split graphs, these “counterparts” have been studied in the literature as examples of so-called threshold graphs (see again [82]), also known as nested split graphs (see, e.g., [103]). However, we have not found a specific name for the graphs we want to consider here. Therefore, we introduce the following terminology:

**Definition 6.3.2.** We say a graph  $(V, R)$  is a **split half-graph** if there is a partition  $\{v_i : i < n\} \cup \{w_i : i < n\}$  of  $V$  such that (i)  $\{v_i : i < n\}$  is a clique, (ii)  $\{w_i : i < n\}$  is an independent set, and (iii)  $\{v_i, w_j\} \in R$  iff  $i \leq j$  for all  $i, j < n$ .

If we have a half-graph structure between two cliques instead of two independent sets, then this graph comes from the complement of a usual half-graph. Such complements have been called **co-half-graphs** in the literature. Analogously, we will refer to complements of bipartite graphs as **co-bipartite graphs**.

**Remark 6.3.3.** Note that, if  $\varphi(x, y)$  codes/realises all finite split half-graphs, then it also codes/realises all finite complete split graphs. Indeed, if we want to code/realise the complete split graph on  $2n$  vertices, we may take a tuple  $(a_i b_i)_{i < 2n}$  coding/realising the finite split graph on  $4n$  vertices, and take  $c_i := a_i$ ,  $d_i := b_{n+i}$  for all  $i < n$ . It is easy to check that  $\varphi$  and  $(c_i d_i)_{i < n}$  now code/realise the complete split graph that we wanted.

With these preliminary combinatorial remarks in place, we now provide our characterisation of stability and NIP in terms of coding, extending Facts 6.0.2(i)&(ii):

**Proposition 6.3.4.** *The following are equivalent:*

- (i)  $T$  is unstable.
- (ii)  $T$  codes all finite half-graphs.
- (iii)  $T$  codes all finite split half-graphs.
- (iv)  $T$  codes all finite co-half-graphs.

*Proof.* ((i)  $\Rightarrow$  (iii)) Suppose  $T$  is unstable. By Fact 6.0.2(i), there is a formula  $\psi(x, y)$  that “codes all finite linear orders”. Let  $\varphi(x, x'; y, y') := (\psi(x, y') \vee \psi(y, x')) \wedge (x \neq y \vee x' \neq y')$ . We claim that  $\varphi$  codes all finite split half-graphs.

So let  $(V, R)$  be such a graph, say  $|V| = 2n$ . Let us write  $\{v_i : i < n\}$  and  $\{w_i : i < n\}$  for the clique and the independent set, respectively. By our choice of formula, we can find tuples  $c_0, \dots, c_{n+1}$  such that  $\models \psi(c_i, c_j)$  iff  $i \leq j$  for all  $i, j < n+2$  (which, in particular, means that  $\models c_i \neq c_j$  if  $i \neq j$ ). For  $i < n$ , define  $a_i := c_i c_n$  and  $b_i := c_{n+1} c_i$ . We claim that  $\varphi$  and  $(a_i b_i)_{i < n}$  code  $(V, R)$ . Indeed:

- $\models \neg\varphi(a_i, a_i) \wedge \neg\varphi(b_i, b_i)$  for all  $i < n$ , since the last conjunct fails.
- $\models \varphi(a_i, a_j)$  for all  $i \neq j < n$ , since

$$\models \varphi(a_i, a_j) \iff \models \psi(c_i, c_n) \vee \psi(c_j, c_n) \iff i \leq n \text{ or } j \leq n,$$

which is always true.

- $\models \neg\varphi(b_i, b_j)$  for all  $i \neq j < n$ , since

$$\models \varphi(b_i, b_j) \iff \models \psi(c_{n+1}, c_j) \vee \psi(c_{n+1}, c_i) \iff n < i \text{ or } n < j,$$

which is false for all  $i, j < n$ .

- Finally,

$$\models \varphi(a_i, b_j) \iff \models \psi(c_i, c_j) \iff i \leq j \iff \{v_i, w_j\} \in R.$$

Therefore,  $\varphi$  codes  $(V, R)$ , as required.

((iii)  $\Rightarrow$  (ii)) Suppose  $\varphi(x, y)$  codes all finite split half-graphs. Let  $\psi(x, x'; y, y') := \varphi(x, y) \wedge \varphi(x', y')$ . We claim that  $\psi$  codes all finite half-graphs.

Let  $(V, R)$  be the finite half-graph on  $2n$  vertices. By assumption, we can find a tuple  $(a_i b_i)_{i < n}$  coding the finite split half-graph on  $2n$  vertices with  $\varphi$ , and, by Remark 6.3.3, a tuple  $(c_i d_i)_{i < n}$  coding the finite complete split graph with two parts of  $n$  vertices each. Let  $e_i := a_i d_i$  and  $f_i := b_i c_i$  for all  $i < n$ . We claim that  $\psi$  and

$(e_i f_i)_{i < n}$  code  $(V, R)$ . Indeed:

- $\models \neg\psi(e_i, e_i) \wedge \neg\psi(f_i, f_i)$  for all  $i < n$ , since  $\models \neg\varphi(a_i, a_i) \wedge \neg\varphi(c_i, c_i)$  for all  $i < n$ .
- $\models \neg\psi(e_i, e_j)$  for all  $i \neq j < n$ , since  $\models \neg\varphi(d_i, d_j)$ .
- $\models \neg\psi(f_i, f_j)$  for all  $i \neq j < n$ , since  $\models \neg\varphi(b_i, b_j)$ .
- Finally,

$$\models \psi(e_i, f_j) \iff \models \varphi(a_i, b_j) \wedge \varphi(d_i, c_j) \iff \{i, j\} \in R.$$

Therefore,  $\psi$  codes  $(V, R)$ , as required.

((ii)  $\Rightarrow$  (iv)) Suppose  $\varphi(x_0, x_1)$  codes all finite half-graphs. In particular, this implies that  $T$  has a model with at least two elements. Let  $y_0, y_1$  be new variables, and let  $\psi(x_0 y_0, x_1 y_1) := \varphi(x_0, x_1) \vee (x_0 \neq x_1 \wedge y_0 = y_1)$ . We claim that  $\psi(x_0 y_0, x_1 y_1)$  codes all finite co-half-graphs.

Let  $(V, R)$  be such a graph. By assumption, we can find tuples  $(a_v)_{v \in V}$  witnessing that  $\varphi(x_0, x_1)$  codes the half-graph obtained by forgetting the edges within the same part (which in particular implies that the  $a_v$ 's are pairwise distinct). Let  $P_0$  and  $P_1$  denote the two parts of the half-graph. Pick two elements  $b_0 \neq b_1$  living in a model of  $T$  that contains  $(a_v)_{v \in V}$ . Define

$$c_v := \begin{cases} a_v b_0 & \text{if } v \in P_0, \\ a_v b_1 & \text{if } v \in P_1. \end{cases}$$

We claim that  $\psi(x_0 y_0, x_1 y_1)$  and  $(c_v)_{v \in V}$  code  $(V, R)$ . Indeed:

- $\models \neg\psi(c_v, c_v)$  for all  $v \in V$ , since  $\models \neg\varphi(a_v, a_v) \wedge a_v \neq a_v$ .
- If  $v \neq w \in P_i$  for some  $i \in \{0, 1\}$ , then  $\models \psi(c_v, c_w)$ , since  $\models a_v \neq a_w \wedge b_0 = b_0$ .
- If  $v \in P_0$  and  $w \in P_1$ , then  $\models \psi(c_v, c_w)$  iff  $\models \varphi(a_v, a_w)$  iff  $\{v, w\} \in R$ .

Therefore,  $\varphi(x, y)$  and  $(c_v)_{v \in V}$  code  $(V, R)$ , as required.

((iv)  $\Rightarrow$  (i)) If  $\varphi(x, y)$  codes all finite co-half-graphs, then by compactness there is a sequence  $(a_i b_i)_{i < \omega}$  that codes the countable co-half-graph. In particular,  $\models \varphi(a_i, b_j)$  iff  $i \leq j$ . So  $T$  is unstable.  $\square$

**Proposition 6.3.5.** *The following are equivalent:*

- (i)  $T$  has IP.

- (ii)  $T$  codes all finite bipartite graphs.
- (iii)  $T$  codes all finite split graphs.
- (iv)  $T$  codes all finite co-bipartite graphs.

*Proof.* ((i)  $\Rightarrow$  (iii)) If  $T$  has IP, then by Fact 6.0.2(ii) there is a formula  $\varphi(x, y)$  that codes all finite graphs, and thus in particular all finite split graphs.

((iii)  $\Rightarrow$  (ii)) We use the same trick as in the proof of Proposition 6.3.4((ii) $\Rightarrow$ (iii)) with complete split graphs to note that, if  $\varphi(x, y)$  codes all finite split graphs, then it also codes all finite bipartite graphs.

((ii)  $\Rightarrow$  (iv)) Once again, the same technique used in the proof of Proposition 6.3.4((ii) $\Rightarrow$ (iv)) of adding new variables shows that, if  $\varphi(x, y)$  codes all finite bipartite graphs, then the formula  $\psi(x_0y_0, x_1y_1) := \varphi(x_0, x_1) \vee (x_0 \neq x_1 \wedge y_0 = y_1)$  codes all finite co-bipartite graphs.

((iv)  $\Rightarrow$  (i)) Suppose that  $\varphi(x, y)$  codes all finite co-bipartite graphs. Fix  $n < \omega$ . Consider the co-bipartite graph  $G = (V, R)$  with parts given by  $\{v_i : i < n\}$  and  $\{w_J : J \subseteq [n]\}$  and such that  $\{v_i, w_J\} \in R$  iff  $i \in J$ . By assumption, we can find tuples  $(a_i)_{i < n}$  and  $(b_J)_{J \subseteq [n]}$  witnessing that  $\varphi(x, y)$  codes  $G$ . By definition, this means that  $\models \varphi(a_i, b_J)$  iff  $i \in J$ . Since  $n$  was arbitrary, this shows that  $\varphi(x, y)$  has IP, as required.  $\square$

One of the main motivations of developing these coding characterisations of stability and NIP is to find some alternative versions of Fact 6.3.1. Before proving the main such version, let us note that something quite special happens with bipartite graphs:

**Lemma 6.3.6.** *A theory  $T$  codes all finite bipartite graphs iff it “ $R$ -codes” them, i.e., there is a formula  $\varphi(x, y)$  and, for each finite bipartite graph  $G = (V, R)$  with parts  $P_0, P_1$ , a sequence  $(a_v)_{v \in V}$  such that  $\models \varphi(a_{v_0}, a_{v_1})$  iff  $\{v_0, v_1\} \in R$  for all  $v_i \in P_i$ .*

*Proof.* ( $\Rightarrow$ ) The formula that codes all finite bipartite graphs works.

( $\Leftarrow$ ) Let  $\varphi(x_0, x_1)$  be the formula in the statement. Let  $y_0, y_1$  be fresh new variables, and define

$$\psi(x_0y_0, x_1y_1) := \varphi(x_0, x_1) \wedge y_0 \neq y_1.$$

We claim that  $\psi$  codes all finite bipartite graphs.

Let  $G = (V, R)$  be a finite bipartite graph with parts  $P_0, P_1$ . By assumption, we can find a tuple  $(a_v)_{v \in V}$  such that, for all  $v_i \in P_i$ ,  $\models \varphi(a_{v_0}, a_{v_1})$  iff  $\{v_0, v_1\} \in R$ . For  $i = 0, 1$ , choose some  $b_i$  from a model containing the  $a_v$ 's such that  $b_0 \neq b_1$ . Let  $c_v := a_v b_i$ , where  $v \in P_i$ . Then:

- If  $v, v' \in V$  belong to the same  $P_i$ , then  $\models b_i = b_i$ , and thus  $\models \neg\psi(c_v, c_{v'})$ .
- If  $v_i \in P_i$ , then  $\models \varphi(a_{v_0}, a_{v_1})$  by assumption, and  $\models b_0 \neq b_1$ , which means that  $\models \psi(c_{v_0}, c_{v_1})$ .

Therefore,  $\psi$  and  $(c_v)_{v \in V}$  code  $G$ . □

Let us note that the same trick works for finite half-graphs too. With this, we are ready to reprove Shelah's dichotomy:

**Theorem 6.3.7.** *If  $T$  codes all finite half-graphs but not all finite bipartite graphs, then there is a formula  $\psi(x', y')$  with the strict order property.*

*Proof.* This adapts the argument in [104, Theorem 4.1]. Suppose  $\varphi(x, y)$  codes all finite half-graphs but not all finite bipartite graphs. Thus, by Lemma 6.3.6,  $\varphi(x, y)$   $R$ -codes all finite half-graphs, but not all finite bipartite graphs. (This is why we need to make the assumption on the *theory* and not just on  $\varphi$ .) In particular, by compactness,  $\varphi(x, y)$   $R$ -codes the countable half-graph. We may choose a sequence  $(d_i e_i)_{i \in \mathbb{Q}}$  witnessing this such that  $(d_i)_{i \in \mathbb{Q}}$  and  $(e_i)_{i \in \mathbb{Q}}$  are indiscernible sequences.

Since  $\varphi(x, y)$  does not  $R$ -code all finite bipartite graphs, there is a finite bipartite  $(V, R)$  which is not  $R$ -coded by  $\varphi(x, y)$ .

**Claim.** There exists a finite bipartite graph  $(\bar{V}, \bar{R})$  such that (i)  $|\bar{V}| = 2n$  for some  $n < \omega$ , (ii)  $(\bar{V}, \bar{R})$  has degree sequence  $[n, n-1, \dots, 1]$  in the top row, and (iii)  $(\bar{V}, \bar{R})$  contains  $(V, R)$  as an induced subgraph.

*Proof of Claim.* Note that, without loss of generality, we may assume that  $(V, R)$  itself is such that  $|V| = 2k$  for some  $k < \omega$ , since we can add vertices wherever it is needed. The claim then can be rephrased in terms of biadjacency matrices as follows: Given an  $m \times m$   $(0, 1)$ -matrix  $B$ , we can find an  $n \times n$   $(0, 1)$ -matrix  $\bar{B} = (b_{ij})_{i, j < n}$  that contains  $B$  as an  $m \times m$ -minor such that

$$\left\{ \sum_{i=1}^n b_{ij} : j < n \right\} = \{1, 2, \dots, n\}.$$

This is easy to check. □<sub>Claim</sub>

Since  $(V, R)$  is an induced subgraph of  $(\bar{V}, \bar{R})$ ,  $\varphi(x, y)$  also does not  $R$ -code  $(\bar{V}, \bar{R})$ . We may permute the vertices on the “top” row of  $\bar{V}$  so that their order agrees with the degree sequence of the graph; since this describes a graph isomorphism, the resulting graph is also not  $R$ -coded by  $\varphi(x, y)$ . Hence, without loss, we may assume  $(\bar{V}, \bar{R})$  is already of this form.

Let us write  $\bar{V}$  as  $\{v_i : i < n\} \cup \{w_i : i < n\}$ , where the  $v_i$ ’s form the “top” row and the  $w_i$ ’s the “bottom” row. For each  $i < n$ , let us define  $U_i := \{j < n : (v_i, w_j) \in \bar{R}\}$ . For  $i < n$ , we define a sequence  $(U_i^\ell)_{\ell \leq \alpha_i}$  of sets such that:

- $U_i^0 = U_i$  and  $U_i^{\alpha_i} = \{j < n : i \leq j\}$ .
- If  $U_i^\ell \neq U_i^{\alpha_i}$ , then there is some  $k_\ell \in U_i^\ell$  such that  $k_\ell + 1 \notin U_i^\ell$  and  $U_i^{\ell+1} = (U_i^\ell \setminus \{k_\ell\}) \cup \{k_\ell + 1\}$ .

Let us denote, for  $i < n$  and  $\ell \leq \alpha_i$ , by  $V_{i,\ell}$  the graph such that (i) for  $h < i$ , the neighbourhood of  $v_h$  is  $U_h^{\alpha_h}$ , (ii) the neighbourhood of  $v_i$  is  $U_i^\ell$ , and (iii) for  $i < j$ , the neighbourhood of  $v_j$  is  $U_j^0$  (i.e., the same as in  $(\bar{V}, \bar{R})$ ). In particular,  $V_{i,\alpha_i} = V_{i+1,0}$  for all  $i < n - 1$ . Moreover,  $V_{0,0} = (\bar{V}, \bar{R})$ , whereas  $V_{n-1,\alpha_{n-1}}$  is a half-graph.

Therefore, since  $\varphi(x, y)$   $R$ -codes  $V_{n-1,\alpha_{n-1}}$  but not  $V_{0,0}$ , we can find some  $i < n$  and  $\ell < \alpha_i$  such that  $\varphi(x, y)$   $R$ -codes  $V_{i,\ell+1}$  with one of the parts represented by a finite subsequence of  $(e_i)_{i \in \mathbb{Q}}$ , but not  $V_{i,\ell}$ . Without loss, we may assume  $(d'_i, e_i)_{i < n}$  is the tuple witnessing that  $\varphi(x, y)$   $R$ -codes  $V_{i,\ell+1}$ . Let us define the formula

$$\psi(x, y) := \varphi(x, y) \wedge \bigwedge_{\substack{j < n \\ j \neq k_\ell, k_\ell + 1}} \varphi(x, e_j)^{(j \in U_i^\ell)}.$$

Then  $\models \exists x(\psi(x, e_{k_\ell+1}) \wedge \neg\psi(x, e_{k_\ell}))$ , since  $\varphi(x, y)$   $R$ -codes  $V_{i,\ell+1}$ , and furthermore  $\models \neg\exists x(\psi(x, e_{k_\ell}) \wedge \neg\psi(x, e_{k_\ell+1}))$ , since  $\varphi(x, y)$  does not  $R$ -code  $V_{i,\ell}$  with a finite subsequence of  $(e_i)_{i \in \mathbb{Q}}$ . In particular, by indiscernibility of  $(e_i)_{i \in \mathbb{Q}}$ , for all  $q, q' \in (k_\ell, k_\ell + 1) \cap \mathbb{Q}$ , we have

$$\models \exists x(\neg\psi(x, e_q) \wedge \psi(x, e_{q'})) \iff q < q'.$$

Therefore,  $\psi(x, y)$  has the strict order property.  $\square$

Let us note that the version of the above proposition which replaces finite bipartite graphs by all finite graphs (in the line of Fact 6.0.2) is easily seen to fail: e.g., take  $R(x, y)$  in the generic triangle-free graph.

There are two main corollaries of the above theorem. The first is that it implies

(in fact, it is equivalent to) Fact 6.3.1:

**Corollary 6.3.8.** *If  $T$  is NIP and NSOP, then it is stable.*

*Proof.* Suppose  $T$  is NIP but unstable. Then, by Proposition 6.3.4, there is a formula  $\varphi(x, y)$  that codes all finite half-graphs. Since  $T$  is NIP, by Proposition 6.3.5,  $\varphi(x, y)$  does not code all finite bipartite graphs. Therefore, by Theorem 6.3.7,  $T$  is SOP.  $\square$

In other words, we have shown that Shelah's argument for Fact 6.3.1 is, at its core, combinatorial. Observe, moreover, that our characterisations in Propositions 6.3.4 and 6.3.5 also show that every stable theory is NIP. We will see later on that a combinatorial proof in this sense also exists for the fact that every stable theory is NSOP.

The second corollary is that, since we work with triangle-free graphs, we automatically obtain a new dichotomy at the level of realising:

**Corollary 6.3.9.** *If a theory  $T$  realises all finite half-graphs but not all finite bipartite graphs, then there is a formula  $\psi(x', y')$  with the strict order property.*

*Proof.* Let  $\varphi(x, y)$  be as described. Then, by Lemma 6.2.5, there is a formula  $\theta(w, z)$  coding all finite half-graphs. Moreover, since bipartite graphs are triangle-free, it follows by Lemma 6.2.5 that  $\theta(w, z)$  does not code all finite bipartite graphs. Therefore, we can apply Theorem 6.3.7 to find a formula  $\psi(x', y')$  with the strict order property.  $\square$

We will see in the upcoming sections that this is another rephrasing of Fact 6.3.1.

## 6.4 Some general results on realising

We now turn our attention to realisability. The added complexity of the notion of realisability compared to coding allows us to capture neostability-theoretic classification properties beyond IP. In this section, we want to show that some results and techniques from the study of coding graphs generalise to the level of realisability, and offer us a picture of the relations between realising the different classes of graphs we have introduced previously.

Let us begin by noticing an interesting property of our characterisation of IP (Proposition 6.3.5). One consequence of this result is that coding all graphs (as was

shown to be equivalent to IP in [79]) turns out to be equivalent to coding a smaller class of “essentially binary” graphs, such as bipartite ones. By “essentially binary,” we mean that we have graphs that we can partition into two “generic” parts, i.e., either cliques or independent sets.

This “reduction to the binary” finds new instances at the level of realising graphs:

**Lemma 6.4.1.** *A theory  $T$  realises all finite bipartite graphs iff it realises all finite triangle-free graphs.*

*Proof.* Right-to-left is clear. For the converse, suppose that  $\varphi(x, y)$  realises all finite bipartite graphs. Let  $(V, R)$  be a finite triangle-free graph. Write  $V = \{v_i : i < n\}$  for some  $n < \omega$ . We define a new graph  $(\bar{V}, \bar{R})$  by letting  $\{b_i : i < n\} \cup \{c_i : i < n\}$  be a partition of  $\bar{V}$  and defining  $\bar{R}$  by the following properties:

- $\{b_i, b_j\}, \{c_i, c_j\} \notin \bar{R}$  for all  $i \neq j < n$ .
- $\{b_i, c_i\} \notin \bar{R}$  for all  $i < n$ .
- $\{b_i, c_j\} \in \bar{R}$  iff  $\{v_i, v_j\} \in R$  for all  $i \neq j$ .

It is clear that  $(\bar{V}, \bar{R})$  is bipartite. Thus, there is a sequence  $(d_{b_i}, d_{c_i})_{i < n}$  that realises  $(\bar{V}, \bar{R})$  with  $\varphi$ . Let  $e_i := d_{b_i} d_{c_i}$  and  $\psi(x; y, z) := \varphi(x, y) \vee \varphi(x, z)$ .

We claim that  $\psi(x; y, z)$  and  $(e_i)_{i < n}$  realise  $(V, R)$ . Indeed, let us note that, for any  $i, j < n$ ,

$$\psi(x; e_i) \wedge \psi(x; e_j) \leftrightarrow (\varphi(x, d_{b_i}) \wedge \varphi(x, d_{c_j})) \vee (\varphi(x, d_{b_j}) \wedge \varphi(x, d_{c_i})),$$

so that, by definition of  $(\bar{V}, \bar{R})$ ,  $\{\psi(x; e_i), \psi(x; e_j)\}$  is consistent iff  $\{v_i, v_j\} \in R$ . Since  $V$  is triangle-free, all cliques in  $V$  are of size 2. This proves our claim.  $\square$

**Lemma 6.4.2.** *A theory  $T$  realises all finite co-bipartite graphs iff it realises all finite graphs.*

*Proof.* Right-to-left is clear. For the converse, suppose that  $\varphi(x, y)$  realises all finite co-bipartite graphs. Let  $(V, R)$  be a finite graph. Enumerate  $V = \{0, \dots, n-1\}$ . We define a new graph  $(\tilde{V}, \tilde{R})$  by letting  $\{b_i : i < n\} \cup \{c_i : i < n\}$  be a partition of  $\tilde{V}$  and  $\tilde{R}$  be given by the following conditions:

- $\{b_i, c_i\} \in \tilde{R}$  for all  $i < n$ .
- $\{b_i, b_j\}, \{c_i, c_j\} \in \tilde{R}$  for all  $i \neq j$ .
- $\{b_i, c_j\} \in \tilde{R}$  iff  $\{i, j\} \in R$  for all  $i \neq j$ .

It is easy to see that  $(\tilde{V}, \tilde{R})$  is co-bipartite. Let  $(d_v)_{v \in \tilde{V}}$  be the tuple witnessing that  $\varphi(x, y)$  realises  $(\tilde{V}, \tilde{R})$ . Note that  $\{d_v : v \in \tilde{V}\} = \{d_{b_i} : i < n\} \cup \{d_{c_i} : i < n\}$ . Let  $e_i := d_{b_i}d_{c_i}$ , and let  $\psi(x; y, z) := \varphi(x, y) \wedge \varphi(x, z)$ .

We claim that  $\psi(x; y, z)$  and  $(e_i)_{i < n}$  realise  $(V, R)$ . Let  $I \subseteq V$  be a clique. Then  $\{b_i : i \in I\} \cup \{c_i : i \in I\}$  forms a clique in  $\tilde{V}$ . In particular,  $\{\psi(x; e_i) : i \in I\} = \{\varphi(x, d_{b_i}) \wedge \varphi(x, d_{c_i}) : i \in I\}$  is consistent. On the other hand, if  $\{i, j\} \notin R$ , then  $\{\psi(x; e_i), \psi(x; e_j)\} \vdash \varphi(x, d_{b_i}) \wedge \varphi(x, d_{c_j})$ , which is inconsistent.  $\square$

This result has two immediate consequences:

**Corollary 6.4.3.** *A theory  $T$  has  $\text{PM}^{(2)}$  iff it realises all finite co-bipartite graphs.*

*Proof.* Combine Fact 6.0.2(iii) and Lemma 6.4.2.  $\square$

**Corollary 6.4.4.** *If  $T$  realises all finite co-bipartite graphs, then it also realises all finite split graphs.*

*Proof.* This follows directly from Lemma 6.4.2.  $\square$

In a different direction, we can also notice that some of the constructions we have used in proving implications between coding properties can also be adapted to prove implications between realisability properties.

**Lemma 6.4.5.** *If  $T$  realises all finite split graphs, then it also realises all finite bipartite graphs.*

*Proof.* Suppose  $\varphi(x, y)$  realises all finite split graphs. Let  $\psi(x, x'; y, y') := \varphi(x, y) \wedge \varphi(x', y')$ . We claim that  $\psi$  realises all finite bipartite graphs.

Let  $(V, R)$  be a finite bipartite graph. Without loss of generality, we may assume the two parts of  $V$ ,  $V_1$  and  $V_2$ , have size  $n$ . Define  $(V, \tilde{R})$  to be the split graph on the vertex set  $V$  such that  $\tilde{R} = R \cup \{\{v, v'\} : v, v' \in V_1\}$ . Pick a realisation  $(a_i b_i)$  of  $(V, \tilde{R})$  with  $\varphi$ , and a realisation  $(c_i d_i)_{i < n}$  of the complete split graph with  $n$  vertices in each part with  $\varphi$ . Let  $a'_i := d_i$  and  $b'_i := c_i$ . Then:

- $\{\psi(x, x'; a_i, a'_i), \psi(x, x'; a_j, a'_j)\} \vdash \varphi(x', d_i) \wedge \varphi(x', d_j)$ , which by assumption is inconsistent for all  $i \neq j$ .
- $\{\psi(x, x'; b_i, b'_i), \psi(x, x'; b_j, b'_j)\} \vdash \varphi(x, b_i) \wedge \varphi(x, b_j)$ , which by assumption is inconsistent for all  $i \neq j$ .
- $\{\psi(x, x'; a_i, a'_i), \psi(x, x'; b_j, b'_j)\} = \{\varphi(x, a_i), \varphi(x, b_j)\} \cup \{\varphi(x', c_j), \varphi(x', d_i)\}$ , which is consistent iff  $\{i, j\} \in R$ .

Therefore,  $\psi(x, x'; y, y')$  and  $(a_i a'_i, b_i b'_i)_{i < n}$  realise  $(V, R)$ .  $\square$

In particular, just as with bipartite and co-bipartite graphs, the above result shows that, when  $T$  realises all finite split graphs, there are also other, non-split graphs which  $T$  also realises (note that all these statements are only valid for *theories*). Thus, a natural question is the following:

**Question 6.4.6.** Can we identify the largest class  $\mathcal{K}$  of finite graphs such that a theory  $T$  realises all finite split graphs iff it realises all graphs in  $\mathcal{K}$ ?

Lemma 6.4.2 solves the above question for co-bipartite graphs, but Lemma 6.4.1 does not; it is still possible that we can realise some graphs with triangles in this case, perhaps with a different formula.

A similar trick, also using the same formula, shows the analogous statement for the corresponding half-graphs:

**Lemma 6.4.7.** *If  $T$  realises all finite split half-graphs, then it also realises all finite half-graphs.*

Combining these results with Remark 6.1.5 and our characterisations of stability and NIP (i.e., Propositions 6.3.4 and 6.3.5), we obtain the following picture for realisability properties at the level of theories:

$$\begin{array}{ccccccc}
 \text{co-bipartite graphs (PM}^{(2)}) & \implies & \text{split graphs} & \implies & \text{bipartite graphs} & \implies & \text{IP} \\
 \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
 \text{co-half-graphs} & \implies & \text{split half-graphs} & \implies & \text{half-graphs} & \implies & \text{unstable}
 \end{array}$$

The implication from realising co-half-graphs to realising split half-graphs has not been proved yet. It can be proved directly (meaning: as in the proofs of this section), but we will defer this to the following section, where we provide an indirect proof.

To finish this section, we can consider the natural question as to where the analogues of the properties in Proposition 6.2.8 for realisability lie in this picture. As expected from comparing, e.g., Propositions 6.2.8 and 6.3.4, the property corresponding to the “essentially binary” graph in that result is relatively weak:

**Lemma 6.4.8.** *If  $T$  realises all finite half-graphs, then it realises all finite complete split graphs.*

*Proof.* Suppose  $\varphi(x, y)$  realises all finite half-graphs. Let  $\theta(x; y, y')$  be the formula  $\varphi(x, y) \vee \varphi(x, y')$ . We claim that  $\theta$  realises all finite complete split graphs.

Without loss of generality, let  $(V, R)$  be the complete split graph with two parts of  $n$  vertices each. By assumption, we can find a sequence  $(a_i b_i)_{i < 2n+2}$  realising the half-graph on  $4n + 4$  vertices with  $\varphi$ . For  $i < n$ , let us define:

$$\begin{aligned} c_i &:= a_{n+1+i}, & c'_i &:= a_n, \\ d_i &:= b_{n+1+i}, & d'_i &:= b_i. \end{aligned}$$

Then we have:

- For  $j < n$ , the set  $\{\theta(x; c_i, c'_i) : i < n\} \cup \{\theta(x; d_j, d'_j)\}$  is consistent, since the disjunct  $\bigwedge_{i < n} \varphi(x, c'_i) \wedge \varphi(x, d_j)$  is equivalent to  $\varphi(x, a_n) \wedge \varphi(x, b_{n+1+j})$  and thus consistent by assumption.
- For  $i \neq j$ , the formula  $\theta(x; d_i, d'_i) \wedge \theta(x; d_j, d'_j)$  is equivalent to

$$\begin{aligned} &(\varphi(x, b_{n+1+i}) \wedge \varphi(x, b_{n+1+j})) \vee (\varphi(x, b_{n+1+i}) \wedge \varphi(x, b_j)) \\ &\vee (\varphi(x, b_i) \wedge \varphi(x, b_{n+j+1})) \vee (\varphi(x, b_i) \wedge \varphi(x, b_j)), \end{aligned}$$

and by assumption, all disjuncts are inconsistent. So  $\{\theta(x; d_i, d'_i), \theta(x; d_j, d'_j)\}$  is inconsistent for all  $i \neq j$ .

Therefore,  $\theta(x; y, y')$  and  $(c_i c'_i, d_i d'_i)_{i < n}$  realise  $(V, R)$ , as required.  $\square$

In contrast, we will see in the following section that realising all finite complete multipartite graphs is a significantly stronger property.

## 6.5 Neostability and realisability

Our goal in this section is to relate the realisability properties considered in the previous section to neostability-theoretic classification properties. We will provide new characterisations of stability and NIP that extend our results from §6.3 and show the equivalence of realising different classes of finite graphs. We will also provide new realisability characterisations of  $\text{TP}_2$ ,  $\text{SOP}_3$ , and  $\text{PM}^{(2)}$ .

**Lemma 6.5.1.** *If  $T$  has IP, then it realises all finite split graphs.*

*Proof.* Suppose  $T$  has IP. By Proposition 6.3.5, there is a formula  $\varphi(x, y)$  that

codes all finite bipartite graphs. Let  $(V, R)$  be a finite split graph with partition  $V = \{v_i : i < n\} \cup \{w_i : i < m\}$ . For each  $i < m$ , let  $N_i := \{j < n : \{v_i, w_j\} \in R\}$ . We define a bipartite graph  $(\tilde{V}, \tilde{R})$  with parts  $\{\tilde{v}_i : i < m\}$  and  $\{\tilde{w}_i : i < 2n + m\}$  as follows:

- For  $i < m$  and  $j < n$ , we have  $\{\tilde{v}_i, \tilde{w}_j\} \in \tilde{R}$  iff  $j \in N_i$ .
- For all  $i < m$  and  $n \leq j < 2n$ , we have  $\{\tilde{v}_i, \tilde{w}_j\} \notin \tilde{R}$ .
- For all  $i, j < m$ , we have  $\{\tilde{v}_i, \tilde{w}_{2n+j}\} \in \tilde{R}$  iff  $i = j$ .

By assumption, we can find tuples  $(d_i)_{i < m}$  and  $(b_i)_{i < n} + (c_i)_{i < n} + (f_i)_{i < m}$  witnessing that  $\varphi(x, y)$  codes  $(\tilde{V}, \tilde{R})$ . In particular, by the third condition, the  $d_i$ 's are pairwise distinct. As  $T$  has IP, it has infinite models, so we can also pick  $a$  and  $e$  to be any elements from a model of  $T$  distinct from all the  $d_i$ 's,  $b_i$ 's and  $c_i$ 's.

Let  $\psi(x; wyz) := x = w \vee (\varphi(x, y) \wedge \neg\varphi(x, z))$ . We claim that  $\psi$  and  $(ab_i c_i)_{i < n}$ ,  $(d_i e e)_{i < m}$  realise  $(V, R)$ . Indeed:

- The set  $\{\psi(x; ab_i c_i) : i < n\}$  is consistent since  $x = a$  is consistent.
- For  $i \neq j < m$ , the set  $\{\psi(x; d_i e e), \psi(x; d_j e e)\}$  is equivalent to

$$(x = d_i \wedge x = d_j) \vee (\varphi(x, e) \wedge \neg\varphi(x, e)),$$

which is inconsistent.

- If  $I \subseteq V$  is a clique such that  $|I| \geq 2$  and  $I \not\subseteq \{v_i : i < n\}$ , we can write  $I$  as a disjoint union  $I^+ \cup I^-$ , where  $I^+ = \{v_{i_j} : j < \ell\} \subseteq \{v_i : i < n\}$  and  $I^- = \{w_i\}$  for some  $i < m$ . In particular,  $I^+ \subseteq N_i$ . Hence  $\{\psi(x; ab_{i_j} c_{i_j}) : j < \ell\} \cup \{\psi(x; d_i e e)\}$  is consistent, since the disjunct  $x = d_i \wedge \bigwedge_{j < \ell} (\varphi(x, b_{i_j}) \wedge \neg\varphi(x, c_{i_j}))$  is consistent.
- If  $\{v_i, w_j\} \notin R$ , then, since  $\{\psi(x; ab_i c_i), \psi(x; d_j e e)\}$  is equivalent to

$$\begin{aligned} (x = a \wedge x = d_j) \vee (x = a \wedge \varphi(x, e) \wedge \neg\varphi(x, e)) \\ \vee (\varphi(x, b_i) \wedge \varphi(x, c_i) \wedge x = d_j) \\ \vee (\varphi(x, b_i) \wedge \varphi(x, c_i) \wedge \varphi(x, e) \wedge \neg\varphi(x, e)), \end{aligned}$$

it follows from our choice of  $d_j$  that the set is inconsistent.

Therefore,  $\psi$  and  $(ab_i c_i)_{i < n}, (d_j e e)_{j < m}$  realises  $(V, R)$ .  $\square$

We can thus combine our results so far to provide several equivalent realisability characterisations of IP:

**Theorem 6.5.2.** *The following are equivalent:*

- (i)  $T$  has IP.
- (ii)  $T$  realises all finite split graphs.
- (iii)  $T$  realises all finite triangle-free graphs.
- (iv)  $T$  realises all finite bipartite graphs.

*Proof.* (i)  $\Rightarrow$  (ii) is Lemma 6.5.1. (ii)  $\Rightarrow$  (iv) is Lemma 6.4.5. (iv)  $\Rightarrow$  (i) is Remark 6.1.5 and Proposition 6.3.5. Finally, (iii)  $\Leftrightarrow$  (iv) is Lemma 6.4.1.  $\square$

Towards proving a similar characterisation of stability, let us first observe the following result about SOP theories:

**Lemma 6.5.3.** *If  $T$  has SOP, then it realises all finite split half-graphs.*

*Proof.* Suppose  $T$  has SOP. By definition, this means there exist a formula  $\varphi(x, y)$  and a sequence  $(a_i)_{i < \omega}$  such that, defining  $\psi(y, z) := \forall x(\varphi(x, y) \rightarrow \varphi(x, z))$ , we have

$$\models \psi(a_i, a_j) \iff i \leq j.$$

Let  $(V, R)$  be the split half-graph on  $n$  vertices in each part. Let us define, for  $i < n$ ,

$$\begin{aligned} b_i &:= a_{6i}a_{8n-2-i}, \\ c_i &:= a_{6i+2}a_{6i+4}. \end{aligned}$$

Let  $\theta(x; yz) := \psi(y, x) \wedge \psi(x, z)$ . We claim that  $\theta$  and  $(b_i c_i)_{i < n}$  realise  $(V, R)$ . Indeed:

- For  $i < n$ ,  $\{\theta(x, b_j) : j \leq i\} \cup \{\theta(x, c_i)\}$  is consistent, since it is witnessed by  $a_{6i+3}$ .
- For  $j < i$ ,  $\{\theta(x, b_i), \theta(x, c_j)\} \vdash \psi(x, a_{6j+4}) \wedge \psi(a_{6i}, x)$ , which is inconsistent since  $6i \geq 6j + 6 > 6j + 4$ .
- $\{\theta(x, c_i), \theta(x, c_j)\} \vdash (\psi(x, a_{6i+4}) \wedge \psi(a_{6j+2}, x)) \wedge (\psi(x, a_{6j+4}) \wedge \psi(a_{6i+2}, x))$ , which is inconsistent since we have either  $6j \geq 6i + 6$  or  $6i \geq 6j + 6$ .

Therefore,  $T$  realises all finite split half-graphs.  $\square$

Hence, using Fact 6.3.1 and our previous results, we obtain:

**Theorem 6.5.4.** *The following are equivalent:*

- (i)  $T$  is unstable.
- (ii)  $T$  realises all finite split half-graphs.

(iii)  $T$  realises all finite half-graphs.

*Proof.* ((i)  $\Rightarrow$  (ii)) Suppose  $T$  is unstable. By Fact 6.3.1 (reproven in Theorem 6.3.7),  $T$  is either IP or SOP. If  $T$  is IP, then by Lemma 6.5.1 it realises all finite split graphs, and so in particular all finite split half-graphs. If  $T$  is SOP, then by Lemma 6.5.3 it also realises all finite split half-graphs. Either way, the result follows.

((ii)  $\Rightarrow$  (iii)) This is Lemma 6.4.7.

((iii)  $\Rightarrow$  (i)) This is Remark 6.1.5 and Proposition 6.3.4.  $\square$

We may note that, with this characterisation, we obtain new proofs between neostability-theoretic classification properties:

**Lemma 6.5.5.** *If a theory  $T$  is stable, then it is  $\text{NSOP}_1$ .*

*Proof.* Suppose  $T$  is  $\text{SOP}_1$ . So there is a formula  $\varphi(x, y)$  and a tree of parameters  $(c_\eta)_{\eta \in 2^{<\omega}}$  that witness  $\text{SOP}_1$ . Let  $(V, R)$  be a split half-graph on  $2n$  vertices, so that we can write  $V = \{a_i : i < n\} \cup \{b_i : i < n\}$ . Define  $d_{a_i} := c_{0^i}$  and  $d_{b_i} := c_{0^i \smallfrown 1}$ . Then:

- $\{\varphi(x, d_{a_i}) : i < n\} = \{\varphi(x, c_{0^i}) : i < n\}$  is consistent, since it is contained in a path of the tree.
- $\{\varphi(x, d_{b_i}), \varphi(x, d_{b_j})\} = \{\varphi(x, c_{0^i \smallfrown 1}), \varphi(x, c_{0^j \smallfrown 1})\}$  is inconsistent for all  $i < j < n$ , as  $0^i \smallfrown 1 \wedge 0^j \smallfrown 1 = 0^i$ .
- $\{\varphi(x, d_{a_i}), \varphi(x, d_{b_j})\} = \{\varphi(x, c_{0^i}), \varphi(x, c_{0^j \smallfrown 1})\}$  is consistent iff  $0^i \not\smallfrown 0^j \smallfrown 1$  iff  $i \leq j$ .

Therefore,  $\varphi(x, y)$  realises  $(V, R)$ . Thus,  $T$  realises all finite split half-graphs, and so, by Theorem 6.5.4,  $T$  is unstable.  $\square$

In a different direction, we can also prove that other properties that we have previously introduced are *not* equivalent to stability:

**Example 6.5.6.** Let  $T$  be the theory of an equivalence relation with infinitely many infinite classes. Let  $(M, E) \models T$  be a model. Pick a sequence  $(b_i)_{i < \omega}$  of distinct elements such that  $\models E(b_i, b_j)$  for all  $i \neq j$ , and  $(d_i)_{i < \omega}$  a sequence of distinct elements such that  $\models \neg E(d_i, d_j) \wedge \neg E(d_i, b_0)$  for all  $i \neq j$ . Finally, pick two more disjoint sequences  $(a_i)_{i < \omega}$  and  $(c_i)_{i < \omega}$  of distinct elements, all belonging to the  $E$ -class of  $b_0$ .

Let  $\varphi(x; yz) := x = y \vee E(x, z)$ . We claim that  $\varphi$  and  $(a_i b_i, c_i d_i)_{i < \omega}$  realise the countable complete split graph. Indeed:

- For any  $j < \omega$ ,  $\{\varphi(x; a_i b_i) : i < \omega\} \cup \{\varphi(x; c_j d_j)\}$  is consistent, since the set  $\{x = c_j\} \cup \{E(x, b_i) : i < \omega\}$  is consistent.
- $\{\varphi(x; c_i d_i), \varphi(x; c_j d_j)\}$  is equivalent to

$$\begin{aligned} (x = c_i \wedge x = c_j) \vee (x = c_i \wedge E(x, d_j)) \\ \vee (E(x, d_i) \wedge x = c_j) \\ \vee (E(x, d_i) \wedge E(x, d_j)), \end{aligned}$$

and none of the disjuncts is consistent for  $i \neq j$ .

Therefore,  $T$  realises all finite complete split graphs.

Theorem 6.5.4, Lemma 6.4.8, Proposition 6.2.8, and Example 6.5.6 jointly imply that the class of theories which realise all finite complete split graphs lies somewhere between the class of unstable theories and the class of all theories with infinite models.

We can also find realisability equivalences for other classification properties. The following characterisation is mostly a reformulation of the traditional definition:

**Lemma 6.5.7.** *A theory  $T$  is  $\text{TP}_2$  iff it realises all complete multipartite graphs.*

*Proof.* ( $\Rightarrow$ ) Suppose that  $T$  has  $\text{TP}_2$ . Then, by [26, Lemma 3.2], there are a formula  $\varphi(x, y)$  and an array  $(a_{ij})_{i,j < \omega}$  that realise  $2\text{-TP}_2$ . We claim that  $\varphi$  and  $(a_{ij})$  realise the countable complete multipartite graph  $(V, R)$  with infinitely many parts, each of which is infinite. This can be easily seen. Since every finite complete multipartite graph embeds as an induced subgraph into  $(V, R)$ , we are done.

( $\Leftarrow$ ) Suppose that there is a formula  $\varphi(x, y)$  that realises all finite complete multipartite graphs. By compactness, there is a sequence  $(d_v)_{v \in V}$  that realises the complete  $\infty$ -partite graph  $(V, R)$ . Let us write  $V = \{V_i : i < \omega\}$ , where  $V_i := \{v_{ij} : j < \omega\}$  is an independent part. Let  $a_{ij} := d_{v_{ij}}$ . Then it is easy to check that  $\varphi(x, y)$  and  $(a_{ij})_{i,j < \omega}$  witness  $2\text{-TP}_2$ .  $\square$

This shows some interesting behaviour: whereas, by Propositions 6.2.8 and 6.3.5, every theory coding all finite split graphs also codes all finite complete multipartite graphs, by Theorem 6.5.2 and Lemma 6.5.7, every theory realising all finite complete multipartite graphs also realises all finite split graphs.

**Lemma 6.5.8.** *A theory  $T$  has  $\text{SOP}_3$  iff it realises all finite co-half-graphs.*

*Proof.* ( $\Leftarrow$ ) Suppose that  $\varphi(x, y)$  realises all finite co-half-graphs. By compactness, it follows that  $\varphi(x, y)$  realises the countable co-half-graph, i.e., the countable graph  $(V, R)$  with  $V = \{v_i : i \in \omega\}$  and  $R$  determined, for all  $i < j$ , by

$$R(v_{2i}, v_{2j}) \wedge R(v_{2i+1}, v_{2j+1}) \wedge R(v_{2i}, v_{2j+1}) \wedge \neg R(v_{2i+1}, v_{2j}),$$

as well as  $\neg R(v_{2i}, v_{2i+1})$  for all  $i$ . This is realised by  $\varphi$  with some sequence  $(b_i)_{i < \omega}$ . Let  $c_i := (b_{2i}, b_{2i+1})$ . Let  $\psi_0(x; y, z) := \varphi(x, y)$  and  $\psi_1(x; y, z) := \varphi(x, z)$ . Then, for each  $k \in \omega$ ,

$$\{\psi_0(x, c_i) : i \leq k\} \cup \{\psi_1(x, c_i) : i > k\} = \{\varphi(x, b_{2i}) : i \leq k\} \cup \{\varphi(x, b_{2i+1}) : i > k\},$$

and this is consistent as  $\{v_{2i} : i \leq k\} \cup \{v_{2i+1} : i > k\}$  forms a clique on  $V$ . On the other hand, for  $i < j$ ,

$$\{\psi_0(x, c_j), \psi_1(x, c_i)\} = \{\varphi(x, b_{2j}), \varphi(x, b_{2i+1})\},$$

which is inconsistent since  $\neg R(v_{2i+1}, v_{2j})$  holds. Hence, by Fact 2.3.4  $T$  has  $\text{SOP}_3$ .

( $\Rightarrow$ ) We adapt the argument from [84, Lemma 6.6]. Suppose  $T$  has  $\text{SOP}_3$ . By Fact 2.3.4, we can find formulae  $\psi_0(x, y)$ ,  $\psi_1(x, y)$ , and  $(a_i)_{i \in \mathbb{Z}}$  such that:

- For all  $k \in \mathbb{Z}$ ,  $\{\psi_0(x, a_i) : i \leq k\} \cup \{\psi_1(x, a_i) : i > k\}$  is consistent.
- For all  $i < j$ ,  $\{\psi_0(x, a_j), \psi_1(x, a_i)\}$  is inconsistent.

Let  $\chi(x; y, z) := \psi_0(x, y) \wedge \psi_1(x, z) \wedge y \neq z$ . We claim that  $\chi(x; y, z)$  realises all finite co-half-graphs. So let  $(V, R)$  be one such. Without loss of generality, we may assume that  $V = \{v_i : i < n + 1\} \cup \{w_i : i < n + 1\}$  for some  $n < \omega$ . Furthermore, we may assume that we also have  $\{v_i, w_i\} \in R$  for  $i < n + 1$ ; realising all graphs with this additional condition is easily seen to be equivalent to realising all co-half-graphs.

Following Malliaris, for  $0 \leq i \leq n$ , we define

$$\begin{aligned} c_i &:= a_{2i-1} a_{4n-2i+1}, \\ d_i &:= a_{-2i} a_{2i}. \end{aligned}$$

Let  $I \subseteq V$  be a clique. Then  $I \subseteq \{v_j : j \leq i\} \cup \{w_k : i \leq k\}$  for some  $i \leq n + 1$ . We have

$$\{\chi(x, c_j) : j \leq i\} \cup \{\chi(x, d_k) : i \leq k\} = \{\psi_0(x, a_{2j-1}) : j \leq i\} \cup \{\psi_0(x, a_{-2k}) : i \leq k\}$$

$$\cup \{\psi_1(x, a_{4n-2j+1}) : j \leq i\} \cup \{\psi_1(x, a_{2k}) : i \leq k\}.$$

But note that we have  $\max\{\{2j - 1 : j \leq i\} \cup \{-2k : i \leq k\}\} = \max\{2i - 1, 0\}$ , whereas  $\min\{\{4n - 2j + 1 : j \leq i\} \cup \{2k \geq i\}\} = 2i$ . Therefore, by the choice of  $a_i$ 's, the set  $\{\chi(x, c_j) : j \leq i\} \cup \{\chi(x, d_k) : i \leq k\}$  is consistent.

On the other hand, if  $\{v_i, w_j\} \notin R$ , it follows that  $j < i$ . But then we have  $\{\chi(x, c_i), \chi(x, d_j)\} \vdash \psi_0(x, a_{2i-1}) \wedge \psi_1(x, a_{2j})$  and this is inconsistent since  $2j \leq 2i - 2 < 2i - 1$ .  $\square$

Theorems 6.5.2 and 6.5.4, Corollary 6.4.3, and the previous lemma offer us a new combinatorial connection between stable and NIP theories, on the one hand, and NSOP<sub>3</sub> and NPM<sup>(2)</sup> theories, on the other. This suggests that perhaps a similar dichotomy to that of Shelah's holds in NSOP theories:

**Question 6.5.9.** Is every NSOP theory either NSOP<sub>3</sub> or PM<sup>(2)</sup>?

Some positive evidence toward this conjecture comes from Bailetti's many examples of strictly NSOP<sub>4</sub> examples with PM<sup>(2)</sup> [8]. This motivates the further study of properties of realisability. However, there are technical reasons that prevent our proof idea for Theorem 6.3.7 from extending to co-bipartite and co-half-graphs. Moreover, observe that this suggested dichotomy is a significant generalisation of Proposition 5.7.5. We believe it is plausible that such a generalisation might require the further development of semantic properties for both NPM<sup>(2)</sup> and NSOP<sub>3</sub> theories.

To finish this section, let us note that these realisability characterisations, in general, do not have “one-variable theorems.” That is, in general, we cannot reduce the process of checking whether there is a formula realising a class of finite graphs to formulas  $\varphi(x, y)$  with  $|x| = 1$ . As an example, consider DLO. It clearly has SOP witnessed by  $x < y$ . Hence, by Lemma 6.5.3, we can find some formula  $\psi(x, y)$  that realises all finite half-graphs. However, no formula in a single  $x$ -variable works:

**Lemma 6.5.10.** *Let  $T = \text{DLO}$ . No formula in a single variable realises all finite half-graphs.*

*Proof.* Assume, for contradiction, that there is a formula  $\varphi(x, y)$  and a sequence  $(a_i b_i)_{i < \omega}$  that realise the countable half-graph, where  $|x| = 1$ . Since DLO is o-minimal, we know that  $\varphi$  is a union of points and open intervals. We may assume no interval is of the form  $(-\infty, y_i)$  or  $(y_j, +\infty)$ . Let us assume that  $\varphi(x, y)$  is a union

of  $m$  points and  $n$  bounded open intervals. In what follows, we abuse notation and write the tuple  $a_i$  to refer to the definable set  $\varphi(\mathbb{M}, a_i)$ .

**Claim.** For any set  $A = \{a_\ell : \ell \in I\}$  of size  $\geq 2n^2 + m^2 + 3mn + 1$  and any  $j \geq \max I$ , there is some  $i \leq \max I$  such that  $a_i \subseteq \bigcup_{k < n+m} b_{j+k}$ .

*Proof of Claim.* Fix  $n$  and  $m$ , a set  $A := \{a_\ell : \ell \in I\}$  of size  $\geq 2n^2 + m^2 + 3mn + 1$ , and some  $j \geq \max I$ . Observe that, by this choice,  $\{\varphi(x, a_i), \varphi(x, b_j)\}$  is consistent for every  $a_i \in A$ . We show, by induction on  $1 \leq k \leq n + m$ , that there exists a subset  $A_k \subseteq A$  of size  $\geq 2n^2 + m^2 + 3mn - k(2n + m) + 1$  such that every  $a_i$  in  $A_k$  has a point or an interval contained in  $b_{j+h}$  for each  $0 \leq h \leq k - 1$ . For the base case, observe that  $b_j$  has  $n$  intervals and thus  $2n$  endpoints. It also has  $m$  isolated points. So there is  $A_1 \subseteq A$  of size  $\geq 2n^2 + m^2 + 3mn + 1 - (2n + m)$  such that every  $a_i$  in  $A_1$  has a point or an interval contained in  $b_j$ .

For the inductive step, assume we have found  $A_k \subseteq A$  for  $1 \leq k < n + m$  of size  $\geq 2n^2 + m^2 + 3mn - k(2n + m) + 1$  satisfying the required condition. Once again,  $b_{j+k}$  has  $2n$  endpoints and  $m$  isolated points. So there is a subset  $A_{k+1} \subseteq A_k \subseteq A$  of size  $\geq 2n^2 + m^2 + 3mn - k(2n + m) + 1 - (2n + m) = 2n^2 + m^2 + 3mn - (k+1)(2n + m) + 1$  such that every  $a_i$  in  $A_{k+1}$  has a point or an interval contained in  $b_{j+k}$ . As these are disjoint, for each  $a_i$ , such point or interval will be different from each of those contained in  $b_j, \dots, b_{j+k-1}$ . This completes the induction.

Let  $a_i \in A_{n+m}$ , which exists as  $|A_{n+m}| \geq 2n^2 + m^2 + 3mn - (n+m)(2n+m) + 1 = 1$  by induction. Then each isolated point and interval of  $a_i$  is contained in one of  $b_j, b_{j+1}, \dots, b_{j+n+m-1}$ , as required.  $\square_{\text{Claim}}$

Pick some  $i \leq 2n^2 + m^2 + 3mn =: N$  such that each point and interval of  $a_i$  is contained in one of  $b_N, b_{1+N}, \dots, b_{n+m-1+N}$ . Now,  $\{\varphi(x, a_i), \varphi(x, b_{n+m+N})\}$  is also consistent. But then  $b_{j+N} \cap b_{n+m+N} \neq \emptyset$  for some  $j < n + m$ , which implies that the set  $\{\varphi(x, b_{j+N}), \varphi(x, b_{n+m+N})\}$  is consistent, a contradiction.  $\square$

## 6.6 Higher arity: $\text{IP}_n$ and $\text{PM}^{(n)}$

In the following two sections, we want to consider extensions of the results from the previous sections to the higher-arity context. As we will see, many results lift automatically to some of the developed higher-arity versions of classification properties. In what follows, we fix a complete theory  $T$  with infinite models and

a monster model  $\mathbb{M} \models T$ . For convenience, we restrict ourselves to *n-uniform hypergraphs* for  $n < \omega$  for the rest of the chapter, i.e., hypergraphs  $(V, R)$  where  $R \subseteq \mathcal{P}(V)$  such that every subset in  $R$  has size  $n$ . In particular, 2-uniform hypergraphs are precisely graphs.

The notions of coding and realisability can be lifted straightforwardly to hypergraphs. Recall that, for an  $n$ -uniform hypergraph  $(V, R)$ , we say  $I \subseteq V$  is a **clique** if every set of  $n$  distinct elements from  $I$  lies in  $R$ .

**Definition 6.6.1.** Let  $\varphi(x_0, \dots, x_{n-1})$  be a formula and  $(V, R)$  be an  $n$ -uniform hypergraph. We say  $\varphi(x_0, \dots, x_{n-1})$  **codes**  $(V, R)$  if there is some  $(a_v)_{v \in V}$  such that  $\models \varphi(a_{v_0}, \dots, a_{v_{n-1}})$  iff  $\{v_0, \dots, v_{n-1}\} \in R$ .

**Definition 6.6.2.** Let  $\varphi(x, y)$  be a partitioned formula and  $(V, R)$  an  $n$ -uniform hypergraph. We say  $\varphi(x, y)$  **realises**  $(V, R)$  if there is some  $(a_v)_{v \in V}$  such that:

- For every clique  $I \subseteq V$  with  $|I| \geq n$ , the set  $\{\varphi(x, a_v) : v \in I\}$  is consistent.
- For any distinct  $v_0, \dots, v_{n-1} \in V$  such that  $\{v_0, \dots, v_{n-1}\} \notin E$ , the set  $\{\varphi(x, a_{v_0}), \dots, \varphi(x, a_{v_{n-1}})\}$  is inconsistent.

This is Bailetti's definition in [8, Proposition 6.3] and straightforwardly generalises Definition 6.0.1.

We can also adapt the machinery developed in §6.1 to formalise the interaction of these two notions in the higher-arity context:

**Definition 6.6.3.** Let  $(V, R)$  be a finite  $n$ -uniform hypergraph. We define its **associated hypergraph**  $(V, E_R)$  as follows: for all  $n \leq j \leq |V|$  and distinct  $v_0, \dots, v_{j-1} \in V$ ,  $\{v_0, \dots, v_{j-1}\} \in E_R$  iff  $\{v_0, \dots, v_{j-1}\}$  is a clique in  $(V, R)$ .

Let  $\Sigma := \{\varphi_n, \varphi_{n+1}, \dots\}$  be an infinite set of formulas such that  $\varphi_i$  has arity  $li$  for all  $i < \omega$  and a fixed  $l$ . Let  $(V, E)$  be a hypergraph. We say that  $\Sigma$  **codes**  $(V, E)$  if there is a sequence of  $l$ -tuples  $(a_v)_{v \in V}$  such that, for all  $n \leq j \leq |V|$  and distinct  $v_0, \dots, v_{j-1} \in V$ ,

$$\{v_0, \dots, v_{j-1}\} \in E \iff \models \varphi_j(a_{v_0}, \dots, a_{v_{j-1}}).$$

We already saw that, in the case of graphs, we can start with a particular formula  $\varphi$ , define its associated set of formulas  $\Sigma_\varphi$ , and then show that  $\varphi$  realises  $(V, R)$  iff  $\Sigma_\varphi$  realises  $(V, E_R)$ . It is straightforward to generalise  $\Sigma_\varphi$  to the higher-arity case:

**Definition 6.6.4.** Let  $n < \omega$  and let  $\varphi(x, y)$  be a partitioned formula. We define its **associated set of formulas**  $\Sigma_\varphi^n$  to be  $\{\psi_i : i \geq n\}$ , where we let  $\psi_i(y_0, \dots, y_{i-1}) := \exists x \bigwedge_{\ell < i} \varphi(x, y_\ell)$ .

In other words, this is just the restriction of  $\Sigma_\varphi$  as previously defined to those  $i \geq n$  (and, conversely,  $\Sigma_\varphi$  from the foregoing sections is simply  $\Sigma_\varphi^2$ ). Using the same argument as in Lemma 6.1.4, we are able to show the following:

**Lemma 6.6.5.** *Let  $(V, R)$  be an  $n$ -uniform hypergraph. A formula  $\varphi(x, y)$  realises  $(V, R)$  iff  $\Sigma_\varphi^n$  codes  $(V, E_R)$ . In particular, if  $T$  realises  $(V, R)$ , then it codes  $(V, R)$ .*

*Proof.* The first part is proven in the same way as Lemma 6.1.4. For the second part, we can use the same trick as in the previous sections: if  $\varphi(x, y)$  realises  $(V, R)$ , then  $\Sigma_\varphi^n = \{\psi_i : i \geq n\}$  codes  $(V, E_R)$ , and thus the formula  $\theta(y_0 z_0, \dots, y_{n-1} z_{n-1}) := \psi_n(y_0, \dots, y_{n-1}) \wedge \bigwedge_{i \neq j} z_i \neq z_j$  codes  $(V, R)$ .  $\square$

Recall that  $K_m^n$  denotes the complete  $n$ -uniform hypergraph on  $m$  vertices. We say a hypergraph is  $K_m^n$ -**free** if it does not contain  $K_m^n$  as an induced subgraph. With this notion, we can adapt Lemma 6.2.5 to show the following:

**Lemma 6.6.6.** *Let  $(V, R)$  be a  $K_{n+1}^n$ -free  $n$ -uniform hypergraph. A formula  $\varphi(x, y)$  realises  $(V, R)$  iff  $\theta(y_0 z_0, \dots, y_{n-1} z_{n-1}) := \psi_n(y_0, \dots, y_{n-1}) \wedge \bigwedge_{i \neq j} z_i \neq z_j$  codes it, where  $\psi(y_0, \dots, y_{n-1}) := \exists x \bigwedge_{i < n} \varphi(x, y_i)$ .*

Extending the terminology from the previous sections, we say an  $n$ -uniform hypergraph  $(V, R)$  is  $n$ -**partite** if there is a partition  $P_0, \dots, P_{n-1}$  of  $V$  such that, for all  $v_0, \dots, v_{n-1} \in V$ ,  $\{v_0, \dots, v_{n-1}\} \in R$  implies that there is some  $\sigma \in \text{Sym}(n)$  such that  $v_{\sigma(i)} \in P_i$  for all  $i < n$ . In particular, a 2-partite 2-uniform hypergraph is precisely a bipartite graph. We also say  $(V, R)$  is **co- $n$ -partite** if it is the complement of an  $n$ -partite  $n$ -uniform hypergraph.

It is known from the literature that one equivalence does generalise:

**Fact 6.6.7** ([31, Proposition 5.2]). *A theory  $T$  has  $\text{IP}_n$  iff there is a formula  $\varphi(z_0, \dots, z_n)$  and, for each finite  $(n+1)$ -partite  $(n+1)$ -uniform hypergraph  $(V, R)$  with parts  $P_0, \dots, P_n$ , a sequence  $(a_v)_{v \in V}$  such that  $\models \varphi(a_{v_0}, \dots, a_{v_n})$  iff  $\{v_0, \dots, v_n\} \in R$  for all  $v_i \in P_i$ .*

We can use this result to provide a characterisation of  $\text{IP}_n$  in terms of coding, which simultaneously generalises the equivalence of  $\text{IP}$  and coding graphs (Fact

6.0.2(ii)) and the equivalence between coding and “ $R$ -coding” (Lemma 6.3.6):

**Proposition 6.6.8.** *The following are equivalent:*

- (i)  $T$  has  $\text{IP}_n$ .
- (ii)  $T$  codes all  $(n+1)$ -uniform hypergraphs.
- (iii)  $T$  codes all  $(n+1)$ -partite  $(n+1)$ -uniform hypergraphs.

*Proof.* ((i)  $\Rightarrow$  (ii)) We follow the usual proof for  $\text{IP}$ . Suppose that  $\varphi(x_0, \dots, x_{n-1})$  witnesses  $\text{IP}_n$ . Let  $\bar{x}_i := (x_{0,i}, \dots, x_{n-1,i}, x_{n,i})$ , and let us define  $\psi(\bar{x}_0, \dots, \bar{x}_n) := \varphi(x_{0,0}, x_{1,1}, \dots, x_{n-1,n-1}, x_{n,n})$ . We claim that  $\psi$  codes all  $(n+1)$ -uniform hypergraphs. Indeed, let  $(V, R)$  be a finite  $(n+1)$ -uniform hypergraph, and write  $V = \{v_i : i < m\}$ . For each  $i < m$ , let  $w_i := \{(j_0, \dots, j_{n-1}) \in [m]^n : \{i, j_0, \dots, j_{n-1}\} \in R\}$ . Since  $\varphi$  has  $\text{IP}_n$ , there exist sequences  $(a_{0,i})_{i < m}, \dots, (a_{n-1,i})_{i < m}$  and  $(b_w)_{w \subseteq [m]^n}$  such that

$$\models \varphi(a_{0,i_0}, \dots, a_{n-1,i_{n-1}}, b_w) \iff (i_0, \dots, i_{n-1}) \in w.$$

For each  $i < m$ , let  $c_{v_i} := (a_{0,i}, a_{1,i}, \dots, a_{n-1,i}, b_{w_{i_n}})$ . Then it follows that, for all  $i_0, \dots, i_n < m$ ,

$$\begin{aligned} \models \psi(c_{v_{i_0}}, \dots, c_{v_{i_{n-1}}}, c_{v_{i_n}}) &\iff \models \varphi(a_{0,i_0}, \dots, a_{n-1,i_{n-1}}, b_{w_{i_n}}) \\ &\iff (i_0, \dots, i_{n-1}) \in w_{i_n} \\ &\iff \{i_0, \dots, i_{n-1}, i_n\} \in R. \end{aligned}$$

((ii)  $\Rightarrow$  (iii)) Clear.

((iii)  $\Rightarrow$  (i)) If  $\varphi(x_0, \dots, x_{n-1})$  codes all  $(n+1)$ -partite  $(n+1)$ -uniform hypergraphs, then this formula satisfies the condition from Fact 6.6.7, and thus  $T$  has  $\text{IP}_n$ , as required.  $\square$

On the other hand, we also have Bailetti’s result:

**Fact 6.6.9** ([8, Proposition 6.3]). *A theory  $T$  is  $\text{PM}^{(n)}$  iff it realises all  $n$ -uniform hypergraphs.*

Thus, we obtain an independent proof of [8, Theorem 6.11]:

**Corollary 6.6.10.**  $\text{PM}^{(n+1)} \implies \text{IP}_n$ .

*Proof.* If  $T$  is  $\text{PM}^{(n+1)}$ , then by Fact 6.6.9 it realises all  $(n+1)$ -uniform hypergraphs, and thus, by Lemma 6.6.5, it codes all  $(n+1)$ -uniform hypergraphs. Therefore, by

Proposition 6.6.8,  $T$  has  $\text{IP}_n$ . □

Before moving on to the higher-arity versions of stability, let us briefly consider the question of how  $\text{IP}_n$  relates to realisability. Theorem 6.5.2 suggests us, by analogy, that  $\text{IP}_n$  might be equivalent to the realisability of all  $(n+1)$ -partite  $(n+1)$ -uniform hypergraphs. By Proposition 6.6.8, we know that the right-to-left direction holds. However, it is unclear whether the converse is also true. The proofs of the analogous result for  $\text{IP}$  in Theorem 6.5.2 do not seem to extend to the higher arity context.

Nonetheless, we can still adapt some of our results about realising bipartite graphs to the higher-arity case:

**Lemma 6.6.11.** *A theory  $T$  realises all finite  $n$ -partite  $n$ -uniform hypergraphs iff it realises all finite  $K_{n+1}^n$ -free  $n$ -uniform hypergraphs.*

*Proof.* Right-to-left is clear. For the converse, suppose that  $\varphi(x, y)$  realises all finite  $n$ -partite  $n$ -uniform hypergraphs. Let  $(V, R)$  be a finite  $K_{n+1}^n$ -free  $n$ -uniform hypergraph. Write  $V = \{v_i : i < m\}$  for some  $m < \omega$ . We define a new  $n$ -partite  $n$ -uniform hypergraph  $(\bar{V}, \bar{R})$  with each part of order  $m$  by letting  $\bar{V} := \{b_{0,i}, \dots, b_{n-1,i} : i < m\}$  and defining  $\bar{R}$  by the following properties:

- For any  $I \subseteq \bar{V}$  such that  $|I| = n$  and  $b_{j,k}, b_{j,l} \in I$  with  $k \neq l < m$  and  $j < n$ ,  $I \notin \bar{R}$ .
- For any  $I \subseteq \bar{V}$  such that  $|I| = n$  and  $b_{k,i}, b_{l,i} \in I$  with  $i < m$  and  $k \neq l < n$ ,  $I \notin \bar{R}$ .
- $\{b_{0,i_0}, \dots, b_{n-1,i_{n-1}}\} \in \bar{R}$  iff  $\{i_0, \dots, i_{n-1}\} \in R$  for all distinct  $i_0, \dots, i_{n-1} < m$ .

By assumption, we can find a sequence  $(d_{b_{0,i}}, \dots, d_{b_{n-1,i}})_{i < m}$  that realises  $(\bar{V}, \bar{R})$  with  $\varphi$ ; to simplify our notation, we will write  $d_{i,j}$  instead of  $d_{b_{i,j}}$ . Let us define  $e_i := d_{0,i}d_{1,i} \dots d_{n-1,i}$  for all  $i < m$ , and  $\psi(x; y_0, \dots, y_{n-1}) := \bigvee_{j < n} \varphi(x, y_j)$ .

We claim that  $\psi(x; y_0, \dots, y_{n-1})$  and  $(e_i)_{i < m}$  realise  $(V, R)$ . Indeed, observe first that, for any  $i_0, \dots, i_{n-1} < m$ , by the first two conditions in the definition of  $\bar{R}$ , the set  $\{\psi(x; e_{i_0}), \dots, \psi(x; e_{i_{n-1}})\}$  is consistent iff  $\{\varphi(x, d_{0,i_{\sigma(0)}}), \dots, \varphi(x, d_{n-1,i_{\sigma(n-1)}})\}$  is consistent for some  $\sigma \in \text{Sym}(n)$ . Therefore, for any  $i_0, \dots, i_{n-1} < m$ :

- If  $\{i_0, \dots, i_{n-1}\} \in R$ , then the conjunct  $\bigwedge_{\ell < n} \varphi(x, d_{\ell, i_\ell})$  is consistent, and by definition  $\bigwedge_{\ell < n} \varphi(x, d_{\ell, i_\ell}) \vdash \bigwedge_{\ell < n} \psi(x; e_{i_\ell})$ . Thus,  $\{\psi(x; e_{i_0}), \dots, \psi(x; e_{i_{n-1}})\}$  is also consistent. Since  $(V, R)$  is  $K_{n+1}^n$ -free, this covers all cliques of the hypergraph.

- If  $\{i_0, \dots, i_{n-1}\} \notin R$ , then none of the conjuncts  $\bigwedge_{\ell < n} \varphi(x, d_{\ell, i_{\sigma(\ell)}})$  for any  $\sigma \in \text{Sym}(n)$  is consistent, and therefore, by the previous paragraph, the set  $\{\psi(x; e_{i_0}), \dots, \psi(x; e_{i_{n-1}})\}$  is inconsistent.

This completes the proof.  $\square$

**Lemma 6.6.12.** *A theory  $T$  realises all finite co- $n$ -partite  $n$ -uniform hypergraphs iff it realises all finite  $n$ -uniform hypergraphs.*

*Proof.* Right-to-left is clear. For the converse, suppose that  $\varphi(x, y)$  realises all finite co- $n$ -partite  $n$ -uniform hypergraphs. Let  $(V, R)$  be a finite  $n$ -uniform hypergraph. Write  $V = \{v_i : i < m\}$  for some  $m < \omega$ . We define a new co- $n$ -partite  $n$ -uniform hypergraph  $(\bar{V}, \bar{R})$  by letting  $\bar{V} := \{b_{0,i}, \dots, b_{n-1,i} : i < m\}$  and defining  $\bar{R}$  by the following properties:

- For any  $I \subseteq \bar{V}$  such that  $|I| = n$  and  $b_{j,k}, b_{j,l} \in I$  with  $k \neq l < m$  and  $j < n$ ,  $I \in \bar{R}$ .
- For any  $I \subseteq \bar{V}$  such that  $|I| = n$  and  $b_{k,i}, b_{l,i} \in I$  with  $i < m$  and  $k \neq l < n$ ,  $I \in \bar{R}$ .
- $\{b_{0,i_0}, \dots, b_{n-1,i_{n-1}}\} \in \bar{R}$  iff  $\{i_0, \dots, i_{n-1}\} \in R$  for all distinct  $i_0, \dots, i_{n-1} < m$ .

By assumption, we can find a sequence  $(d_{b_{0,i}}, \dots, d_{b_{n-1,i}})_{i < m}$  that realises  $(\bar{V}, \bar{R})$  with  $\varphi$ ; as in the previous proof, we write  $d_{i,j}$  instead of  $b_{i,j}$ . Let  $e_i := d_{0,i}d_{1,i} \dots d_{n-1,i}$  for all  $i < m$ , and  $\psi(x; y_0, \dots, y_{n-1}) := \bigwedge_{j < n} \varphi(x, y_j)$ .

We claim that  $\psi(x; y_0, \dots, y_{n-1})$  and  $(e_i)_{i < m}$  realise  $(V, R)$ . Indeed:

- If  $I \subseteq V$  is a clique, then  $\{b_{0,i}, \dots, b_{n-1,i} : i \in I\}$  forms a clique in  $\bar{V}$  by construction. By choice of formula, the set  $\{\varphi(x, d_{0,i}), \dots, \varphi(x, d_{n-1,i}) : i \in I\}$  is consistent. In particular,  $\{\psi(x; e_i) : i \in I\}$  is consistent.
- If  $\{i_0, \dots, i_{n-1}\} \notin R$ , then  $\{b_{0,i_0}, \dots, b_{n-1,i_{n-1}}\} \notin \bar{R}$  by construction, so that  $\bigwedge_{\ell < n} \varphi(x, d_{\ell, i_\ell})$  is inconsistent. In particular, the set  $\{\psi(x; e_{i_0}), \dots, \psi(x; e_{i_{n-1}})\}$  is inconsistent.

This completes the proof.  $\square$

**Corollary 6.6.13.** *A theory  $T$  is  $\text{PM}^{(n)}$  iff it realises all finite co- $n$ -partite  $n$ -uniform hypergraphs.*

*Proof.* Combine Fact 6.6.9 and Lemma 6.6.12.  $\square$

**Question 6.6.14.** Is a theory  $\text{IP}_n$  iff it realises all finite  $(n+1)$ -partite  $(n+1)$ -uniform hypergraphs?

## 6.7 Higher arity: $\text{FOP}_n$

In this section, we will consider in detail the  $\text{FOP}_n$  hierarchy introduced by Terry and Wolf [115] and developed by Abd Aldaim, Conant and Terry [1]. The latter paper provides a coding criterion for  $\text{FOP}_n$  in terms of the following combinatorial structures:

**Definition 6.7.1** ([1, Definition 3.12]). Let  $\mathcal{L}_n := \{P_1, \dots, P_{n+1}, <, <_n, R\}$ , where the  $P_i$ 's are unary,  $<$  is binary,  $<_n$  is  $2n$ -ary, and  $R$  is  $(n+1)$ -ary. Let  $Q = P_1 \times \dots \times P_n$ . Define  $T_n$  to be the set of  $\mathcal{L}_n$ -formulas saying the following:

- (i)  $P_1, \dots, P_{n+1}$  is a partition.
- (ii)  $<$  is a linear order with  $P_1 < \dots < P_{n+1}$ .
- (iii)  $R$  only holds on  $P_1 \times \dots \times P_{n+1}$ .
- (iv)  $<_n$  only holds on  $Q \times Q$  and is a linear order on  $Q$ .
- (v) For any  $\bar{x}, \bar{y} \in Q$  and  $w, z \in P_{n+1}$ ,

$$\bar{x} \leq_n \bar{y} \wedge R(\bar{y}, w) \wedge w \leq z \implies R(\bar{x}, z).$$

We let  $\mathcal{H}_n$  be the Fraïssé limit of the class of finite models of  $T_n$ , which exists by [1].

**Fact 6.7.2** ([1, Proposition 3.20]). *A theory  $T$  has  $\text{FOP}_n$  iff there is a formula  $\varphi(z_0, \dots, z_n)$  and, for each finite model  $(V, P_1(V), \dots, P_{n+1}(V), <, <_n, R) \models T_n$ , a sequence  $(a_v)_{v \in V}$  in a model of  $T$  such that  $\models \varphi(a_{v_0}, \dots, a_{v_n})$  iff  $\{v_0, \dots, v_n\} \in R$  for all  $v_i \in P_i(V)$ .*

In order to connect this result with the notion of coding, we need to extract an  $(n+1)$ -uniform hypergraph out of a model of  $T_n$ . We use the following notation:

**Definition 6.7.3.** Let  $G = (V, P_1(V), \dots, P_{n+1}(V), <, <_n, R)$  be an  $\mathcal{L}_n$ -structure. We denote by  $G^\natural$  the  $(n+1)$ -uniform hypergraph  $(V, \bar{R})$  given by  $\bar{R} := \{\{v_0, \dots, v_n\} : \text{there is } \sigma \in \text{Sym}(n+1) \text{ such that } (v_{\sigma(0)}, \dots, v_{\sigma(n)}) \in R\}$ .

**Lemma 6.7.4.** *A theory  $T$  has  $\text{FOP}_n$  iff there is a formula in  $T$  that codes  $G^\natural$  for all finite  $G \models T_n$ .*

*Proof.* ( $\Rightarrow$ ) Suppose  $T$  has  $\text{FOP}_n$ . Let  $\varphi(x_0, \dots, x_n)$  be the formula from Fact 6.7.2. Pick  $G = (V, P_1(V), \dots, P_{n+1}(V), <, <_n, R) \models T_n$  finite; let  $(a_v)_{v \in V}$  be a sequence witnessing the condition from Fact 6.7.2. Let us choose distinct elements  $d_0, \dots, d_n$  that are different from all the  $a_v$ 's. Let us define the formula

$$\psi(x_0 y_0, \dots, x_n y_n) \leftrightarrow \varphi(x_0, \dots, x_n) \wedge \bigwedge_{i \leq n} y_i = d_i,$$

and, for  $v \in V$ , let us define  $c_v = a_v d_i$ , where  $i \leq n$  is such that  $v \in P_i(V)$ . Finally, let us define  $\theta(x_0 y_0, \dots, x_n y_n)$  by  $\bigvee_{\sigma \in \text{Sym}(n+1)} \psi(x_{\sigma(0)} y_{\sigma(0)}, \dots, x_{\sigma(n)} y_{\sigma(n)})$ . It is now easy to check that  $\theta$  and  $(c_v)_{v \in V}$  code  $G^\natural = (V, \bar{R})$ :

- If  $v_0, \dots, v_n \in V$  and there are  $0 \leq i < j \leq n$  and  $k \leq n$  with  $v_i, v_j \in P_k(V)$ , then  $\bigwedge_{i \leq n} y_{\sigma(i)} = d_i$  fails for any  $\sigma \in \text{Sym}(n+1)$ , and therefore it follows that  $\models \neg \theta(c_{v_0}, \dots, c_{v_n})$ .
- If  $v_0, \dots, v_n \in V$  are vertices in distinct parts, then there is exactly one  $\sigma \in \text{Sym}(n+1)$  such that  $c_{v_0} \dots c_{v_n} \models \bigwedge_{i \leq n} y_{\sigma(i)} = d_i$ . Therefore, we have:

$$\begin{aligned} \models \theta(c_{v_0}, \dots, c_{v_n}) &\iff \models \psi(c_{v_{\sigma(0)}}, \dots, c_{v_{\sigma(n)}}) \\ &\iff \models \varphi(a_{v_{\sigma(0)}}, \dots, a_{v_{\sigma(n)}}) \\ &\iff \{v_0, \dots, v_n\} \in \bar{R}. \end{aligned}$$

Therefore,  $\theta$  codes  $G^\natural$ , as claimed.

( $\Leftarrow$ ) The formula coding  $G^\natural$  for all finite  $G \models T_n$  will satisfy the condition in Fact 6.7.2, and therefore  $T$  is  $\text{FOP}_n$ .  $\square$

Our goal is to understand the class of models of  $T_n$  better to clarify the relation between  $\text{IP}_n$  and  $\text{FOP}_n$ . Our strategy begins with the observation that  $G \models T_1$  iff  $G^\natural$  is an induced subgraph of some half-graph. In this sense, coding/realising half-graphs gives us enough information to code/realise all other “reducts” of models of  $T_1$  (in the sense of Definition 6.7.3). We will define an analogous notion of half-graphs for hypergraphs that suffices to code/realise all other such “reducts” of models of  $T_n$ . Beyond providing conceptual clarity on these  $\mathcal{H}_n$ , this should also help us relate this to  $\text{IP}_n$ . From now on, let us denote by  $T_{0,n}$  the set of  $\mathcal{L}_n$ -formulas comprising Definition 6.7.1(i)-(iv).

**Definition 6.7.5.** Let  $G = (V, P_1(V), \dots, P_{n+1}(V), <, <_n, R) \models T_{0,n}$  be finite. We define the **closure** of  $G$ , denoted  $\bar{G}$ , to be the  $\mathcal{L}_n$ -structure  $(V, P_1(V), \dots, P_{n+1}(V), <, <_n, \tilde{R})$ , where  $\tilde{R}$  is defined recursively as follows:

- $R_0 := R$ .
- $R_{k+1} := R_k \cup \{\{\bar{x}, z\} : \text{there are } \bar{y} \in Q \text{ and } w \in P_{n+1} \text{ such that } \bar{x} \leq_n \bar{y}, w \leq z, \text{ and } \{\bar{y}, w\} \in R_k\}$ .
- $\tilde{R} := \bigcup_{k < \omega} R_k$ .

In particular,  $\overline{G} \models T_n$ .

**Remark 6.7.6.** The closure can also be defined via a universal property: if  $G \models T_{0,n}$  is finite and  $H \models T_n$  is such that there is an  $\mathcal{L}_n$ -homomorphism  $G \rightarrow H$ , there is a unique  $\mathcal{L}_n$ -homomorphism  $\overline{G} \rightarrow H$  such that the following diagram commutes:

$$\begin{array}{ccc} G & \hookrightarrow & \overline{G} \\ & \searrow & \downarrow \\ & & H \end{array}$$

**Definition 6.7.7.** Fix  $n \geq 2$ ,  $m < \omega$ , and  $<_{n-1}^*$  a well-ordering on  $\omega^{n-1}$ . Let us define an  $\mathcal{L}_{n-1}$ -structure  $H_{m,0}^n = (V, P_1(V), \dots, P_n(V), <, <_{n-1}, R)$  on  $nm$  vertices, after enumerating each  $P_i(V)$  as  $(a_{i,j})_{j < m}$ , as follows:

1. For  $0 < i, j < n$  and  $k, \ell \leq m$ , we have  $a_{i,k} < a_{j,\ell}$  iff  $(i, k) <_{\text{lex}} (j, \ell)$ , the lexicographic ordering on  $\mathbb{N} \times \mathbb{N}$ .
2.  $<_{n-1}$  is the ordering induced by the restriction of  $<_{n-1}^*$  to  $[m]^{n-1}$ .
3.  $R = \{(a_{1,i_1}, \dots, a_{n,i_n}) : (i_1, \dots, i_n) \text{ is the entry at index } i \sum_{k=0}^{n-2} m^k \text{ in the sequence of elements of } Q(V) \text{ determined by } <_{n-1}^*\}$ .

We define  $H_m^n := \overline{H_{m,0}^n}$ , i.e., the closure of  $H_{m,0}^n$ .

We define the  $(n, <_{n-1}^*)$ -**half-graph** on  $nm$  vertices to be  $\mathcal{H}_n^m(<_{n-1}^*) := (H_m^n)^\natural$ .

**Example 6.7.8.** The previous definition can appear very abstract, but there is a simple “canonical” example that provides much of the intuition. Let us take as  $<_{n-1}^*$  the lexicographic ordering on  $\omega^{n-1}$  induced by the usual ordering on  $\omega$ .

**Claim.** There are  $i \sum_{k=0}^{n-2} m^k$  tuples coming before  $(i, \dots, i)$  in the lexicographic ordering on  $[m]^{n-1}$ , where  $[m] = \{0, 1, \dots, m-1\}$ .

*Proof of Claim.* This result is folklore; we will add the proof for completeness. We show this, for a fixed  $m$ , by induction on  $n$ . Base case is clear. Suppose that we know this holds for  $(n-1)$  tuples. Fix  $i < m$ . There are  $im^{n-1}$   $n$ -tuples which start with some  $j < i$ . So it is enough to count how many  $n$ -tuples starting with  $i$  appear before  $(i, \dots, i)$ . But, as the starting  $i$  is fixed, this quantity will be exactly the same as that of  $(n-1)$ -tuples coming before the  $(n-1)$ -tuple  $(i, \dots, i)$ . By

induction hypothesis, there are  $i \sum_{k=0}^{n-2} m^k$  such tuples. Therefore, the number of  $n$ -tuples in  $Q(H_m^{n+1})$  coming before  $(i, \dots, i)$  is

$$im^{n-1} + i \sum_{k=0}^{n-2} m^k = i \sum_{k=0}^{n-1} m^k.$$

This completes the induction. □<sub>Claim</sub>

Hence, in this case,  $R$  from (3) above simply becomes  $\{(a_{1,i}, \dots, a_{n,i}) : i < \omega\}$ . This is then analogous to how we can construct a half-graph: given parts of vertices  $(v_i)_{i < n}$  and  $(w_i)_{i < n}$  (enumerated according to the underlying orderings), we can start with the graph whose only edges are those of the form  $\{v_i, w_i\}$ , and then take the closure of this graph in the sense of Definition 6.7.5.

**Remark 6.7.9.** It is straightforward to check that the  $n$ -half-graph is well-defined (i.e., does not depend on our initial choice of enumeration for  $H_m^n$ ), and that it agrees with the usual half-graph on  $2m$  vertices when  $n = 2$ .

By construction,  $H_m^{n+1} \models T_n$  for all  $m, n < \omega$  and any enumeration of the parts.

In order to establish some basic combinatorial properties about this object, as well as to prove coding results later on, the following auxiliary notion will be useful:

**Definition 6.7.10.** Let  $G = (V, P_1(V), \dots, P_{n+1}(V), <, <_n, R) \models T_{0,n}$  with  $|P_i(V)| = m$  for all  $1 \leq i \leq n+1$ . Enumerate  $P_1(V) \times \dots \times P_n(V)$  as  $\bar{a}_0, \bar{a}_1, \dots, \bar{a}_{m^n-1}$  (writing  $\bar{a}_i = (a_{i,0}, \dots, a_{i,n-1})$  for each  $i < m^n$ ), and  $P_{n+1}(V)$  as  $b_0, \dots, b_{m-1}$ . We define  $G^\flat$  to be the  $\mathcal{L}_1$ -structure  $(V^*, P_1(V^*), P_2(V^*), <^*, <_1^*, R^*)$  with  $P_1(V^*) := \{c_i : i < m^n\}$  and  $P_2(V^*) := \{d_i : i < m\}$  such that (i)  $V^* \models c_i <_1^* c_j$  iff  $V \models \bar{a}_i <_n \bar{a}_j$ , (ii)  $<^* \upharpoonright_{\{c_i : i < m^n\}} = <_1^*$ , (iii)  $V^* \models d_i <^* d_j$  iff  $V \models b_i < b_j$ , and (iv)  $(c_i, d_j) \in R^*$  iff  $(a_{i,0}, \dots, a_{i,n-1}, b_j) \in R$ .

**Remark 6.7.11.** By construction,  $|R(G)| = |R^*(G^\flat)|$ . It is also clear that  $|R(G^\natural)| = |R((G^\flat)^\natural)|$ .

**Lemma 6.7.12.** Let  $G \models T_{0,n}$ . Then  $G \models T_n$  iff  $G^\flat \models T_1$ .

*Proof.* By construction, it is clear that  $G^\flat \models T_{0,1}$ , so we only need to check the final axiom. But note that, by definition, we have  $G \models R(\bar{a}_j, b_k) \wedge \bar{a}_i <_n \bar{a}_j \wedge b_k < b_l$  iff  $G^\flat \models R^*(c_j, d_k) \wedge c_i <_1^* c_j \wedge d_i <^* d_j$ , so the result follows. □

**Lemma 6.7.13.** *Fix  $n \geq 2$  and  $m < \omega$ . Let  $<_{n-1}^1$  and  $<_{n-1}^2$  be well-orderings on  $\omega^{n-1}$ . Then  $\mathcal{H}_n^m(<_{n-1}^1)^\flat = \mathcal{H}_n^m(<_{n-1}^2)^\flat$ . In other words, all  $(n, <_{n-1}^*)$ -half-graphs for different choices of  $<_{n-1}^*$  induce the same bipartite graph (even as an  $\mathcal{L}_1$ -structure).*

*Proof.* This follows immediately from the construction of the induced bipartite graph and the definition of  $(n, <_{n-1}^*)$ -half-graphs.  $\square$

**Lemma 6.7.14.** *For  $n \geq 2$ ,  $m < \omega$ , and  $<_{n-1}^*$  a well-ordering on  $\omega^n$ , the  $(n, <_{n-1}^*)$ -half-graph on  $mn$  vertices has  $(m^n + m)/2$  hyperedges.*

*Proof.* By Lemma 6.7.13 and Remark 6.7.11, for any choice of orderings  $<_{n-1}^1$  and  $<_{n-1}^2$ ,  $\mathcal{H}_n^m(<_{n-1}^1)$  and  $\mathcal{H}_n^m(<_{n-1}^2)$  have the same number of hyperedges. Thus, it is enough to prove this for the  $(n, <_{n-1}^*)$ -half-graph where  $<_{n-1}^*$  is the lexicographic ordering on  $\omega^{n-1}$ .

Observe that, in the  $H_m^n$  for this choice of ordering, if we enumerate  $P_i(H_m^n) = \{a_{i,j} : j < m\}$  for  $i < n$  and  $P_n(H_m^n) = \{b_i : i < m\}$ , then by definition of closure we know that the tuple  $(a_{0,j_0}, \dots, a_{n-1,j_{n-1}}, b_i) \in R(H_m^n)$  iff the  $(n-1)$ -tuple  $(j_0, \dots, j_{n-2})$  comes before the  $(n-1)$ -tuple  $(i, \dots, i)$  in the lexicographic order on  $[m]^{n-1}$ . In particular, this means that the claim from Example 6.7.8 (or, equiv., (3) from the definition) gives us the number of  $n$ -tuples in  $R(H_m^n)$  containing  $b_i$  for each  $i < m$ , minus 1 (as we did not count the tuple  $(i, \dots, i)$  itself). Therefore,

$$|R(H_m^n)| = \sum_{j=0}^{m-1} \left( 1 + j \sum_{k=0}^{n-2} m^k \right) = m + \left( \frac{(m-1)m}{2} \right) \left( \frac{m^{n-1} - 1}{m-1} \right) = \frac{m^n + m}{2}. \quad \square$$

**Corollary 6.7.15.** *For  $n < \omega$ , letting  $K_{m,\dots,m}^n$  denote the complete  $n$ -partite  $n$ -uniform hypergraph with parts of order  $m$ , we have:*

$$\lim_{m \rightarrow \infty} \frac{|R((H_m^n)^\flat)|}{|R(K_{m,\dots,m}^n)|} = \frac{1}{2}.$$

This justifies the terminology of “ $n$ -half-graphs.”

We now want to use this framework to show the existence of finite models of  $T_n$  of arbitrary size. This technical result is expected to help us towards answering the question of whether every NIP<sub>n</sub> NSOP theory is NFOP<sub>n</sub>.

**Remark 6.7.16.** In what follows, we make use of the observation that, given an induced subgraph  $G = (V, R)$  of some finite half-graph, we can extend it to an  $\mathcal{L}_1$ -structure  $G^+$  with the same underlying set such that  $G^+ \models T_1$ . Indeed, we can

enumerate the parts of  $V$  as  $P_1(V) = \{a_i : i < m_1\}$  and  $P_2(V) = \{b_i : i < m_2\}$  in such a way that  $R(a_i, b_j)$  implies  $R(a_k, b_l)$  for all  $k \leq i$  and  $l \geq j$  (this condition is equivalent to being an induced subgraph of some half-graph). We can thus define  $<$  on  $V$  by  $a_0 < a_1 < \dots < a_{m_1-1} < b_0 < \dots < b_{m_2-1}$ ;  $<_1$  by  $<|_{\{a_i : i < m_1\}}$ ; and  $\tilde{R}$  by choosing, for every pair in  $R$ , the ordered pair that puts the element from  $P_1(V)$  first (and note that  $|\tilde{R}| = |R|$ ). It is then easy to check that  $G^+ := (V, P_1(V), P_2(V), <, <_1, \tilde{R}) \models T_1$ .

**Lemma 6.7.17.** *Fix  $m_1, m_2 < \omega$ , and  $k \leq m_1 m_2$ . Then there exists a model  $G = (V, P_1(V), P_2(V), <, <_1, R) \models T_1$  such that  $|P_1(V)| = m_1$ ,  $|P_2(V)| = m_2$ , and  $|R| = k$ .*

*Proof.* By Remark 6.7.16, it suffices to find a graph with parts of size  $m_1$  and  $m_2$  and  $k$  edges that is the induced subgraph of some half-graph. Let us define write  $<_*$  for the reverse lexicographic ordering on  $([m_1], \geq) \times ([m_2], \leq)$ , i.e.:

- If  $i < j < m_2$ , then  $(k, j) <_* (\ell, i)$  for any  $k, \ell < m_1$ .
- If  $i < j < m_1$ , then  $(i, k) <_* (j, k)$  for any  $k < m_2$ .

(For instance,  $<_*$  on  $[3] \times [2]$  gives the ordering  $(0, 1) <_* (1, 1) <_* (2, 1) <_* (0, 0) <_* (1, 0) <_* (2, 0)$ .)

We define a sequence of finite bipartite graphs  $(G_i)_{i < m_1 m_2}$ , with  $V(G_i) := \{v_j : j < m_1\} \cup \{w_\ell : \ell < m_2\}$  for all  $i < m_1 m_2$ , recursively as follows:

- $G_0$  is the empty bipartite graph on  $m_1 + m_2$  vertices.
- $G_{i+1}$  is the bipartite graph defined by  $V(G_{i+1}) = V(G_i)$  and  $R(G_{i+1}) = R(G_i) \cup \{\{v_{j_*}, w_{\ell_*}\} : (j_*, \ell_*) = \min_{(j, \ell) \in [m_1] \times [m_2]} (\{v_j, w_\ell\} \notin R(G_i))\}$ , where the minimum is taken with respect to  $<_*$ .

Let  $G := G_k$ . By construction,  $G$  is bipartite with parts containing  $m_1$  and  $m_2$  vertices, and, since we are only adding one edge at each step,  $|R(G_k)| = k$ .

It remains to show that  $G_k$  is the induced subgraph of some half-graph. So suppose there are  $i, j < m_1$  and  $k, \ell < m_2$  such that  $i < j$ ,  $k < \ell$ , and  $\{v_j, w_k\} \in R(G_k)$ . Then, by definition,  $(i, \ell) <_* (j, k)$ , which means that  $\{v_i, w_\ell\} \in R(G_k)$ . This concludes the proof.  $\square$

**Lemma 6.7.18.** *Let  $G \models T_1$  such that  $|P_1(G)| = m^n$  and  $|P_2(G)| = m$  for some  $m, n < \omega$ , and take a well-ordering  $<^*_n$  on  $\omega^n$ . Then there is a model  $H \models T_n$  with*

$|P_i(H)| = m$  for all  $1 \leq i \leq n+1$  such that  $H^b = G$  and the initial segment of  $<_n^*$  of size  $m^n$  induces  $<_n^H$ .

*Proof.* Write  $G = (V, P_1(V), P_2(V), <_1, <, R)$ , say with  $P_1(V) = \{a_i : i < m^n\}$  and  $P_2(V) := \{b_i : i < m\}$ . Let  $H = (V', R')$  be the graph containing  $mn + m$  vertices split into  $n+1$  parts  $P'_1(V'), \dots, P'_{n+1}(V')$  of  $m$  vertices each. Write  $P'_i(V') := \{c_{i,j} : j < m\}$ . Let  $<_n$  be the order on  $P'_1(V') \times \dots \times P'_n(V')$  induced by the initial segment of  $<_n^*$  of size  $m^n$ .

We can then find a one-to-one correspondence between  $P_1(V)$  and  $P'_1(V') \times \dots \times P'_n(V')$  that sends  $a_i$  to some tuple  $\bar{c}_i := (c_{0,i_0}, \dots, c_{n-1,i_{n-1}})$  and such that  $\bar{c}_i <_n \bar{c}_j$  iff  $G \models a_i <_1 a_j$ . So we may let  $<'_n := <_n$ . Moreover, we have  $H \models R'(c_{0,i_0}, \dots, c_{n-1,i_{n-1}}, c_{n,i_n})$  iff  $G \models R(a_i, b_{i_n})$ . On the other hand, we can pick another one-to-one correspondence sending, e.g.,  $b_i$  to  $c_{n,i}$ . Choose an ordering  $<'$  on  $V'$  such that  $P'_1(V') <' \dots <' P'_{n+1}(V')$  and  $H \models c_{n,i} <' c_{n,j}$  iff  $G \models b_i < b_j$ . This defines an  $\mathcal{L}_n$ -structure  $H = (V', P'_1(V'), \dots, P'_{n+1}(V'), <', <'_n, R')$ , and by construction  $H^b = G$ . Hence, by Lemma 6.7.12,  $H \models T_n$ .  $\square$

**Corollary 6.7.19.** *Fix  $m, n < \omega$ , and  $k \leq m^{n+1}$ . Then there exists a model  $G = (V, P_1(V), \dots, P_{n+1}(V), <, <_n, R) \models T_n$  such that  $|P_i(V)| = m$  for all  $1 \leq i \leq n+1$  and  $|R| = k$ .*

*Proof.* By Lemma 6.7.17, there is some  $H \models T_1$  such that  $|P_1(H)| = m^n$ ,  $|P_2(H)| = m$ , and  $|R(H)| = k$ . By Lemma 6.7.18, there is some  $G \models T_n$  such that  $|P_i(G)| = m$  for all  $1 \leq i \leq n+1$  and  $G^b = H$ , and by Remark 6.7.11,  $|R(G)| = k$ .  $\square$

**Lemma 6.7.20.** *For each finite model  $G \models T_n$ , there exists a well-ordering  $<_n^*$  on  $\omega^n$  and a finite  $(n+1, <_n^*)$ -half-graph  $H$  such that  $G^\natural$  is an induced subgraph of  $H$ .*

*Proof.* Let  $G = (V, P_1(V), \dots, P_{n+1}(V), <, <_n) \models T_n$  be finite; without loss of generality, we may assume  $|P_i(G)| = m$  for all  $1 \leq i \leq n+1$ . By Lemma 6.7.12,  $G^b$  is an  $\mathcal{L}_1$ -structure with  $|P_1(G^b)| = m^n$  and  $|P_2(G^b)| = m$  such that  $G^b \models T_1$ . By previous results, there is a half-graph, say  $H' = \{v_i : i < h\} \cup \{w_i : i < h\}$  for some  $h \geq m^n$ , such that the  $(G^b)^\natural$  is an induced subgraph of  $H'$ .

Let us extend  $H'$  to a new bipartite graph  $H^*$  with parts of order  $h^n$  and  $h$ , by adding  $(h^n - h)/(h - 1) = \sum_{\ell=1}^{n-1} h^\ell$  new vertices  $v_0^*, \dots, v_{h^{n-1}+\dots+h-1}^*$  between  $v_i$  and  $v_{i+1}$  for  $i < h - 1$ , and adding edges  $\{v_\ell^*, w_j\}$  for all  $i < j$ . Since  $H^*$  is also an induced subgraph of some half-graph, by Remark 6.7.16 we may assume  $H^* \models T_1$  (in other words, that it is an  $\mathcal{L}_1$ -structure). Moreover,  $(G^b)^\natural$  is still an

induced subgraph of  $(H^*)^\natural$ ; let us treat the corresponding vertices as distinguished. Finally, pick an ordering  $<_n^*$  on  $\omega^n$  whose restriction to the corresponding vertices from  $(G^\flat)^\natural$  induces  $<_n$ .

Thus, using the strategy from Lemma 6.7.18, we can construct a model  $H \models T_n$  such that  $H^\flat = H^*$  and the ordering  $<_n$  on tuples from the first  $n$  parts that correspond to distinguished vertices agrees with that in  $G$ . The same argument applies, but we need to ensure that the one-to-one correspondence preserves the relations between tuples from  $G$  (i.e., the positions at which they agree). It thus follows that  $G^\natural$  is an induced subgraph of  $H^\natural$ . Moreover, by definition of  $H^*$ , we may assume  $H$  is an  $(n+1, <_n^*)$ -half-graph.  $\square$

**Corollary 6.7.21.** *A theory  $T$  is  $\text{FOP}_n$  iff it codes all  $(n+1, <_n^*)$ -half-graphs for any well-ordering  $<_n^*$  on  $\omega^n$ .*

*Proof.* This follows from Lemmas 6.7.4 and 6.7.20.  $\square$

We can put forward an analogous question to that at the end of the previous section:

**Question 6.7.22.** Is a theory  $\text{FOP}_n$  iff it realises all  $(n+1, <_n^*)$ -half-graphs for any well-ordering  $<_n^*$  on  $\omega^n$ ?

Nonetheless, the results from these last two sections point to certain similarities in characterising the binary and higher-arity versions of stability and NIP. Moreover, what seemingly distinguishes an  $n$ -half-graph from an arbitrary  $n$ -partite  $n$ -uniform hypergraph is the presence of an order. Given the combinatorial proof of Theorem 6.3.7, the main question now is whether we can employ the characterisations from these two sections to answer the following question:

**Question 6.7.23.** Is every  $\text{NIP}_n$  NSOP theory also  $\text{NFOP}_n$ ?

# Conclusion

Je veux bien qu'on sache que le peu  
que j'ai appris jusqu'ici n'est  
presque rien à comparaison de  
ce que j'ignore et que je ne  
désespère pas de pouvoir apprendre

---

*René Descartes*

A good conclusion in a mathematical work points the way to questions and gaps which have remained after, or have been opened by, its developments. Thus, the purpose of this conclusion, as the final word of this thesis, is to collect and connect the open questions for future work that have been sprinkled throughout the previous chapters, as well as discuss possible research directions inspired by our results.

## Hypertournaments

We proved in chapter 4 several model-theoretic results concerning the generic  $\mathbf{H}_4$ -free 3-hypertournament, which notably provide a classification of this structure according to currently known dividing lines.

One general open question concerns the notion of  $\leq_K$ -independence, introduced in §4.3. While some properties are proved in there, together with a useful characterisation of the strong witnessing property, there is still much structure theory on Lascar independence (as elaborated in [112]) that one could try to lift to this general notion. Moreover, after the abstract framework developed in chapter 5, a natural project would be to redefine  $\leq_K$  for any choice of initial independence relation  $\downarrow$  using GUWP. It is, however, at the moment unclear whether this approach can yield new results for the theory of witnessing for independence relations.

**Problem 1.** Develop further the theory of  $\leq_K$ -independence and its generalisation to arbitrary independence relations and GUWP.

Returning to the 3-hypertournament, a question concerning the first proof of its classification properties is how much of our techniques, which are combinatorial in nature, can be recast as combinatorial properties of the  $\mathbf{H}_4$ -free 3-hypertournaments. In other words:

**Question 2.** Can useful combinatorial information be extracted from the proof of the classification properties of  $T_{\mathbf{H}_4\text{-free}}$  in §4.2?

This could potentially be useful if model-theoretic results can be shown to bear a relationship to the structural Ramsey-theoretic properties of the  $\mathbf{H}_4$ -free 3-hypertournament.

The main open question about this structure, as posed at the end of §4.4, asks whether this is an example that genuinely falls outside of Mutchnik’s framework:

**Question 3** (4.4.19). Is there an independence relation  $\perp$  in  $T_{\mathbf{H}_4\text{-free}}$  satisfying monotonicity, full existence, and stationarity over models?

So far, several attempts to define such a relation have been unsuccessful. However, at the moment, trying to prove that no such relation exists seems to require a better understanding of the properties of the generic  $\mathbf{H}_4$ -free 3-hypertournament than is currently available.

The final question on the  $\mathbf{H}_4$ -free 3-hypertournament concerns the description of forking, which relates to its model-theoretic similarity to the triangle-free graph and might provide a key example in the study of [36, Question 2.20]:

**Question 4** (4.5.8). Is it true that  $A \perp_C^d B$  iff  $A \perp_C^f B$  in  $T_{\mathbf{H}_4\text{-free}}$ ? Can a concrete combinatorial characterisation be provided for non-forking?

As a final word here on  $n$ -hypertournaments, we have established in §4.5 that, for all  $n < \omega$ , the generic  $\mathbf{H}_{n+1}$ -free  $n$ -hypertournament is supersimple. However, we have not examined which higher amalgamation properties it can exhibit. This would be particularly interesting to study given the behaviour of the  $n = 3$  case.

## Stable Kim-forking

The main accomplishment of chapter 3 consists in linking d’Elbée and Kim’s question on rosy and NSOP<sub>1</sub> theories with the stable Kim-forking conjecture. This provides a novel approach to the solution of either of the two open questions. If

an example of an NSOP<sub>1</sub>, rosy, and non-simple theory is found, the failure of stable Kim-forking could signal the potential failure of the stable forking conjecture in simple theories.

If we instead attempt to approach d’Elbée and Kim’s question via the stable Kim-forking conjecture, one way would be to look at what we can adapt from the stable forking property in simple theories to this NSOP<sub>1</sub> version. There is, in particular, an improvement of Onshuus’ Theorem 3.3.1 using the notion of *weak elimination of hyperimaginaries* from [17]. More concretely, [17, Corollary 2.6] shows that, if  $T$  is simple and has weak elimination of hyperimaginaries, then  $\perp^f = \perp^b$ . However, an inspection of the proof of this result reveals the pervasive use of base monotonicity for  $\perp^f$ , which by Fact 3.3.4(i) cannot be used for  $\perp^K$  in strictly NSOP<sub>1</sub> theories.

**Question 5.** Suppose that  $T$  is NSOP<sub>1</sub> and rosy, and has weak elimination of hyperimaginaries. Must  $T$  be simple?

Of course, the reason this is a strengthening of Onshuus’ Theorem 3.3.1 is that stable forking implies weak elimination of hyperimaginaries, as can be deduced from the proof of [71, Proposition 2.3]. Thus, we would also need the analogue of this result for stable Kim-forking in order for the answer to the above question to strengthen our Theorem 3.3.7.

An inspection of Kim and Pillay’s proof of [71, Proposition 2.3] reveals that it can be split into two steps: (i) reducing weak elimination of hyperimaginaries to weakly eliminating the canonical bases of types over real sets of parameters, and (ii) showing, from the stable forking property, that we can indeed weakly eliminate canonical bases. Without entering into further details, as we require a different notion of “canonical bases for NSOP<sub>1</sub> theories (such as that from [67]), neither of these two steps is immediately generalisable.

**Question 6.** Suppose  $T$  has the stable Kim-forking property. Does  $T$  have weak elimination of hyperimaginaries?

## Abstract independence relations

The framework for studying independence relations, developed in Chapter 5, represents a first generalisation of Adler’s work to account for the contemporary directions that neostability theory is taking. As such, deepening this work requires two things:

(i) improving the technical machinery developed in that chapter, and (ii) extending its results to other important theorems in neostability theory which have not yet been generalised.

As to (i), a problem that was already evident from chapter 5 was the presumably undesired reliance on weak transitivity for  $\perp$ -Kim-independence. As we noted in the chapter, Kim-independence does not generally satisfy that property even in the context of NSOP<sub>1</sub> theories, which suggests that a more useful generalisation would dispense with this property and replace it with other, more common ones.

**Question 7.** Can we obtain Theorem 5.6.7 without assuming weak transitivity (that is, from a “weaker” set of assumptions)?

Another important question concerns those equivalences from the literature which we have not yet been able to establish. These mainly concern two properties: chain local character and right transitivity. Local character (over arbitrary sets) for  $\perp^f$  is shown to be equivalent to simplicity in [65, Theorem 2.4]; it can be shown that it is enough to require chain local character for  $\perp^f$ . It is also known that NSOP<sub>1</sub> is equivalent to what we call here chain local character [59, Theorem 8.1]. Our Lemma 5.4.11 shows that symmetry (and/or GUWP) implies chain local character, but the converse remains open.

**Question 8.** Can we fit chain local character into the equivalence in Theorem 5.5.1?

We can also consider right transitivity. It is known that simplicity is equivalent to right transitivity for  $\perp^f$  [65, Theorem 2.4] and NSOP<sub>1</sub> to right transitivity for  $\perp^K$  [60, Corollary 4.4]. However, neither of these equivalences is yet captured by our framework. As before, it is clear from Theorem 5.5.1 that right transitivity follows from symmetry (and/or GUWP), but the converse has not yet been established.

**Question 9.** Can we fit right transitivity into the equivalence in Theorem 5.5.1? Is left transitivity required?

With respect to (ii), our results from chapter 5 are particularly related to the behaviour observed in simple and NSOP<sub>1</sub> theories. The next step of this programme concerns the generalisation of important results from the theory of NTP<sub>2</sub>. A possible starting point is suggested by Kruckman and Ramsey’s observation [76, Remark 2.19] that  $\perp^i$  coincides with non-quasi-forking. A rigorous reconstruction of this equivalence for abstract independence relations could plausibly explain the apparently surprising fact that, while the implication from forking to quasi-dividing requires NTP<sub>2</sub> [29, Corollary 3.13], the implication from Kim-forking to quasi-dividing

always holds [76, Corollary 2.25]. We conjecture that the latter is the primitive fact from the viewpoint of abstract independence relations, whereas the former is a consequence of Example 5.3.24 (which itself is a particular instance of the more general Lemma 5.3.23).

**Problem 10.** Develop the notion of quasi- $\perp$ -forking, elucidate the role of a generalised version of the Broom Lemma in studying this notion, and generalise results from the  $\text{NTP}_2$  context using this and GUWP.

## Coding and realising patterns

Chapter 6 provides the first systematic account of coding and realising characterisations of classification properties in neostability theory. While we have succeeded in providing such characterisations for some properties, others (importantly  $\text{SOP}_2$ ) have so far remained outside of the picture.

**Question 11.** Can we find realisability characterisations for  $\text{SOP}_2$ , or one of  $\text{ATP}$ ,  $\text{BTP}$ , and  $\text{CTP}$ ?

Eventually, this might generate a new classification map of combinatorial structures in terms of their realisability properties. At the moment, explicit connections between this approach and combinatorial results are yet unknown.

Separately, our partial results on the higher arity notions of  $\text{FOP}_n$  and  $\text{IP}_n$  characterise them in terms of coding, extending the results of Shelah and Laskowski. However, as indicated in the chapter, we have not yet been successful in providing characterisations in terms of realisability. Nonetheless, Bailetti's work on the  $\text{PM}^{(k)}$  hierarchy suggests that realisability is still a useful notion to study higher arity classification properties (see [8, Proposition 6.3]).

**Question 12.** Are there useful realisability characterisations of  $\text{FOP}_n$  and  $\text{IP}_n$ ?

# A Patterns

El orden inferior es un espejo del orden superior; las formas de la tierra corresponden a las formas del cielo; las manchas de la piel son un mapa de las incorruptibles constelaciones; Judas refleja de algún modo a Jesús.

---

*Jorge Luis Borges*

## Contents

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We have seen in chapter 6 that the realisability of graphs captures several important dividing lines from neostability theory. While the results mentioned previously suggest that this approach can be useful when dealing with questions about, e.g., the relationship between  $\text{NPM}^{(2)}$  and  $\text{NSOP}_3$ , it is less clear that this technology can be adapted to study other well-known dividing lines, such as simplicity or  $\text{NSOP}_1$ .

There is a more general “combinatorial” framework to study the syntactic properties that often appear in neostability theory. This framework goes back to Shelah [108], who termed it “having the weak  $\Gamma$ -property”. More recently, the terminology has been simplified by Bailetti, Kruckman and Ramsey [8], [76] and is referred to as *exhibiting patterns*. This general notion captures most definitions of dividing lines in

neostability theory, and leads to classification properties that, as we saw in chapter 6, might lead to new dichotomies, such as  $\text{PM}^{(2)}$ .

In this appendix, we collect a few technical results on patterns that relate to some questions arising from the contents of this thesis. In §A.1, we consider the relationship between “realising graphs” (in the sense of chapter 6) and “exhibiting patterns,” which has not yet been explored in the literature. We establish a formal correspondence between realising graphs and exhibiting certain sets of finite patterns. To prove this, we introduce several technical notions about sets of patterns which could be useful in future systematic studies of this notion.

In §A.2, we consider a different question about patterns, namely, duality. We have discussed in previous chapters of this thesis the currently open question about what could be a well-behaved class of first-order theories extending those of  $\text{NTP}_2$  and  $\text{NSOP}_1$  theories. Two of these proposals, namely, CTP and ATP, have explicitly appeared in the literature as “duals” of  $k\text{-DCTP}_1$  (see [95]) and  $\text{SOP}_2$ , respectively. Although the term “dual” has been used in the literature on these properties (see, e.g., [50]), this notion has not yet been formalised. The motivation to pursue this formalisation comes from the fact (proved in [95]) that, at the level of theories,  $k\text{-DCTP}_1$  and  $\text{SOP}_2$  are equivalent. Thus, investigating which properties of patterns are stable under taking duals could help answer whether ATP and CTP coincide.

Thus, we develop a formal construction of sets of patterns which allows for the definition of “duals” and investigate a number of basic properties about this construction. The main result here, however, is essentially negative: whereas the question about whether implications between properties of formulas are preserved under taking duals remains open, the analogous question about properties of theories receives a negative answer, as we find out that there is an implication between the dual properties of  $\text{TP}_1$  and  $\text{TP}_2$  (Corollary A.2.23). Nonetheless, we believe the technical notions required for this result could be of use in the development of the theory of positive patterns.

## A.1 Graphs and patterns

Let us start by recalling several definitions about patterns as they appear in [76]:

**Definition A.1.1** ([76, Definition 6.2]). (i) For  $n < \omega$ , an  $n$ -**code** (for a partial type) is a pair  $A = (A_+, A_-)$  of disjoint subsets of  $[n]$ . Given a formula  $\varphi(x, y)$ ,

$\mathbb{M} \models T$  a monster model, and tuples  $a_0, \dots, a_{n-1} \in \mathbb{M}^y$ , the partial type coded by  $A = (A_+, A_-)$  is

$$q_A(x) := \{\varphi(x, a_i) : i \in A_+\} \cup \{\neg\varphi(x, a_i) : i \in A_-\}.$$

- (ii) For  $n < \omega$ , an  **$n$ -pattern** is a pair  $(C, I)$  of disjoint sets of  $n$ -codes. We say an  $n$ -pattern is **positive** if, whenever  $(A_+, A_-) \in C \cup I$ , we have  $A_- = \emptyset$ . A **finite pattern** is an  $n$ -pattern for some  $n < \omega$ . We say a formula  $\varphi(x, y)$  **exhibits** the  $n$ -pattern  $(C, I)$  in a theory  $T$  if there are tuples  $a_0, \dots, a_{n-1} \in \mathbb{M}^y$  such that, for every code  $A \in C$ ,  $q_A(x)$  is consistent, and for every code  $A \in I$ ,  $q_A(x)$  is inconsistent. We say  $(C, I)$  is **exhibitable** if there is some theory  $T$  and some formula  $\varphi(x, y)$  that exhibits  $(C, I)$  in  $T$ .
- (iii) We say a set  $\mathcal{F}$  of finite patterns is **exhibitable** if there is a theory  $T$  and a formula  $\varphi(x, y)$  that exhibits  $(C, I)$  in  $T$  for every  $(C, I) \in \mathcal{F}$ .
- (iv) A property of formulas  $P$  is **definable by patterns** if there is a set  $\mathcal{F}$  of finite patterns such that  $\varphi(x, y)$  has property  $P$  iff  $\varphi(x, y)$  exhibits every pattern in  $\mathcal{F}$ .
- (v) A property of theories  $Q$  is **definable by patterns** if there is a property  $P$  of formulas definable by patterns such that  $T$  has property  $Q$  iff there is some formula  $\varphi(x, y)$  which has property  $P$ .

This definition captures abstractly the way that many classification-theoretic properties are defined. In particular, TP, TP<sub>1</sub>, TP<sub>2</sub>, SOP<sub>1</sub>, SOP<sub>2</sub>, ATP, CTP, and BTP are all easily seen to be definable by positive patterns (and we will also show this explicitly in Example A.2.9).

A natural question at this point is whether we can set down some conditions that ensure the exhibitability of a pattern. To do this, we now appeal to a notion introduced by Bailetti in [8]:

**Definition A.1.2.** Let  $n < \omega$ . We say an  $n$ -pattern  $(C, I)$  is **reasonable** if it satisfies the following conditions:

- (i) For all  $(A_+, A_-) \in C$  and  $(B_+, B_-) \in I$ , we have  $B_\varepsilon \not\subseteq A_\varepsilon$  for some  $\varepsilon \in \{+, -\}$ .
- (ii) For all  $(A_+, A_-) \in C$ ,  $A_+ \cap A_- = \emptyset$ .
- (iii) For all  $(A_+, A_-) \in I$ ,  $A_+ \cap A_- = \emptyset$ .

We say a set of finite patterns  $\mathcal{F}$  is **reasonable** if every  $(C, I) \in \mathcal{F}$  is reasonable.

**Remark A.1.3.** In [8, Remark 5.3], Bailetti shows that every reasonable finite positive pattern is exhibitable by finding a formula  $\varphi(x, y)$  in the theory of the

countable atomless Boolean algebra that exhibits every such pattern. In particular, if  $\mathcal{F}$  is a reasonable set of finite positive patterns, then  $\varphi(x, y)$  exhibits every  $(C, I) \in \mathcal{F}$ . Therefore, every reasonable set of finite positive patterns is exhibitable.

This suggests the definition of a property of theories introduced by Bailetti [8]:

**Definition A.1.4.** A formula  $\varphi(x, y)$  is **2-positive maximal**, or has **PM**<sup>(2)</sup>, if, for every  $n < \omega$  and every reasonable positive  $n$ -pattern  $(C, I)$  such that  $|A| = 2$  whenever  $(A, \emptyset) \in I$ ,  $\varphi(x, y)$  exhibits  $(C, I)$ . We say a theory  $T$  has **PM**<sup>(2)</sup> if there is a formula with **PM**<sup>(2)</sup>, and otherwise we say  $T$  is **NPM**<sup>(2)</sup>.

Let us formalise the intuition that we are only interested in patterns up to exhibitability with the following terminology:

**Definition A.1.5.** We say two  $n$ -patterns  $(C_1, I_1)$  and  $(C_2, I_2)$  are **equivalent** if, for any theory  $T$  and any formula  $\varphi(x, y)$ , we have  $\varphi(x, y)$  exhibits  $(C_1, I_1)$  in  $T$  iff it exhibits  $(C_2, I_2)$  in  $T$ .

We say two sets of finite patterns  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are **equivalent** if, for any theory  $T$  and any formula  $\varphi(x, y)$ , we have  $\varphi(x, y)$  exhibits  $\mathcal{F}_1$  in  $T$  iff it exhibits  $\mathcal{F}_2$  in  $T$ .

**Remark A.1.6.** Observe that, if there is a bijection  $\sigma: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  such that  $(C, I)$  and  $\sigma(C, I)$  are equivalent for all  $(C, I) \in \mathcal{F}_1$ , then  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are equivalent. We will use this observation implicitly in what follows. We will also see later (Lemma A.1.8) that this is not a necessary condition for the equivalence of sets of finite patterns.

We now want to introduce some basic machinery that will allow us to make explicit the connection between realising graphs and exhibiting patterns. Although we do not use the concepts introduced in this subsection anywhere else, they may end up being useful in further research about patterns.

**Definition A.1.7.** Let  $(C, I)$  be an  $n$ -pattern. We define:

$$\begin{aligned} C^* &:= \{(A'_+, A'_-) \text{ } n\text{-code} : A'_+ \subseteq A_+ \text{ and } A'_- \subseteq A_- \text{ for some } (A_+, A_-) \in C\}, \\ I^* &:= \{(A'_+, A'_-) \text{ } n\text{-code} : A'_+ \supseteq A_+ \text{ and } A'_- \supseteq A_- \text{ for some } (A_+, A_-) \in I\}. \end{aligned}$$

(Note that  $C \subseteq C^*$  and  $I \subseteq I^*$ .) Let  $\mathcal{F}$  be a set of finite patterns. We define the **closure of  $\mathcal{F}$** , denoted  $\text{cl}(\mathcal{F})$ , to be  $\{(C^*, I^*) : n < \omega, (C, I) \text{ } n\text{-pattern in } \mathcal{F}\}$ .

For sets of finite patterns  $\mathcal{F}_1, \mathcal{F}_2$ , let us write  $\mathcal{F}_1 \leq \mathcal{F}_2$  if, for all  $n < \omega$  and every  $n$ -pattern  $(C_1, I_1)$  in  $\mathcal{F}_1$ , there is some  $n$ -pattern  $(C_2, I_2)$  in  $\mathcal{F}_2$  such that  $C_1 \subseteq C_2$  and  $I_1 \subseteq I_2$ . It is not hard to check that  $\text{cl}$  is a closure operator with respect to  $\leq$ .

**Lemma A.1.8.**  *$\mathcal{F}$  and  $\text{cl}(\mathcal{F})$  are equivalent.*

*Proof.* Fix a formula  $\varphi(x, y)$ . We want to show that  $\varphi(x, y)$  exhibits every pattern in  $\mathcal{F}$  iff it exhibits every pattern in  $\text{cl}(\mathcal{F})$ . Observe that the right-to-left direction is trivial since  $\mathcal{F} \leq \text{cl}(\mathcal{F})$ .

So suppose  $\varphi(x, y)$  exhibits every pattern in  $\mathcal{F}$  with a sequence  $(a_i)_{i \in J}$ . Let  $(C^*, I^*)$  be an  $n$ -code in  $\text{cl}(\mathcal{F})$ , and let  $(A'_+, A'_-) \in C^*$ . By definition, this means that there are  $A_+ \supseteq A'_+$  and  $A_- \supseteq A'_-$  such that  $(A_+, A_-) \in C$  for some  $n$ -code  $(C, I)$  in  $\mathcal{F}$ . Since  $\varphi(x, y)$  exhibits  $(C, I)$ , it follows that  $\{\varphi(x, a_i) : i \in A_+\} \cup \{\neg\varphi(x, a_i) : i \in A_-\}$  is consistent for some tuples  $(a_i)_{i < n}$ . But now, by the above, the set  $\{\varphi(x, a_i) : i \in A'_+\} \cup \{\neg\varphi(x, a_i) : i \in A'_-\}$  is contained in the set above, so that it is also consistent.

A similar argument works for the  $n$ -codes coming from  $I^*$ . Therefore,  $\varphi(x, y)$  exhibits  $(C^*, I^*)$ . Since  $n$  was arbitrary, it follows that  $\varphi(x, y)$  exhibits every pattern in  $\text{cl}(\mathcal{F})$ .  $\square$

In particular, the above result means that it suffices to determine sets of finite patterns up to their closure.

Our main focus for the rest of this chapter will lie on *positive* patterns, i.e., those where  $A_- = \emptyset$  whenever  $(A_+, A_-) \in C \cup I$ . From now on, we restrict  $\leq$  and  $\text{cl}$  to the class of reasonable positive patterns. We start by singling out a special class of positive patterns:

**Definition A.1.9.** We say a positive  $n$ -pattern  $(C, I)$  is **p-complete** if, for all  $X \subseteq [n]$ , the code  $(X, \emptyset)$  lies in exactly one of  $C$  and  $I$ .

We can alternatively characterise p-complete reasonable positive patterns using our notions of  $\text{cl}$  and  $\leq$ :

**Lemma A.1.10.** *A reasonable positive pattern  $(C, I)$  is p-complete iff it is  $\leq$ -maximal.*

*Proof.*  $(\Rightarrow)$  Suppose  $\mathcal{F}_0$  is a p-complete reasonable pattern. Let  $\mathcal{F}_1$  be a reasonable pattern such that  $\mathcal{F}_0 \leq \mathcal{F}_1$ . Let  $(X, \emptyset) \in C(\mathcal{F}_1)$ . If  $(X, \emptyset) \notin C(\mathcal{F}_0)$ , then by

p-completeness  $(X, \emptyset) \in I(\mathcal{F}_0)$ , and since  $\mathcal{F}_0 \leq \mathcal{F}_1$ , this means  $(X, \emptyset) \in I(\mathcal{F}_1)$ , contradicting that  $\mathcal{F}_1$  is reasonable. The case for  $\mathcal{I}$  is similar. This shows  $\mathcal{F}_1 \leq \mathcal{F}_0$ , and so  $\mathcal{F}_0 = \mathcal{F}_1$ .

( $\Leftarrow$ ) Suppose  $\mathcal{F}$  is a reasonable positive pattern which is not p-complete. So we can find some  $X \subseteq [n]$  such that  $(X, \emptyset) \notin C(\mathcal{F}) \cup I(\mathcal{F})$ . We define a new reasonable positive pattern  $\mathcal{F}^*$  as follows:

- If there is some  $A \subseteq X$  such that  $(A, \emptyset) \in I(\mathcal{F})$ , then let  $C(\mathcal{F}^*) := C(\mathcal{F})$  and  $I(\mathcal{F}^*) := I(\mathcal{F}) \cup \{(X, \emptyset)\}$ .
- Otherwise, let  $C(\mathcal{F}^*) := C(\mathcal{F}) \cup \{(X, \emptyset)\}$  and  $I(\mathcal{F}^*) := I(\mathcal{F})$ .

In both cases, it is clear that  $\mathcal{F}^*$  is a reasonable pattern and  $\mathcal{F} < \mathcal{F}^*$ . This shows that  $\mathcal{F}$  is not  $\leq$ -maximal.  $\square$

**Corollary A.1.11.** *Every p-complete reasonable positive pattern is closed.*

*Proof.* This is immediate from the above since  $\mathcal{F} \leq \text{cl}(\mathcal{F})$  always.  $\square$

**Definition A.1.12.** We say  $\mathcal{F}_1$  is a **p-completion** of  $\mathcal{F}_0$  if it is a set of finite p-complete reasonable positive patterns such that  $\mathcal{F}_0 \leq \mathcal{F}_1$ .

**Lemma A.1.13.** *If  $\mathcal{F}_1$  is a p-completion of  $\mathcal{F}_0$ , then  $\text{cl}(\mathcal{F}_0) \leq \mathcal{F}_1$ .*

*Proof.* This is by Corollary A.1.11 and the universal property of  $\text{cl}$ .  $\square$

**Example A.1.14.** Both 2-TP<sub>1</sub> and 2-TP<sub>2</sub> are p-completions of 2-TP.

Let us, at least momentarily, call each sequence  $a_0, \dots, a_{n-1}$  that witnesses the exhibition of a pattern  $\mathcal{F}$  by a formula  $\varphi(x, y)$  a **display**.

**Proposition A.1.15.** *If  $\varphi(x, y)$  exhibits a reasonable positive pattern  $\mathcal{F}$ , then, for every display  $(a_i)_{i < n}$  of  $\mathcal{F}$  with  $\varphi$ , there is a p-completion  $\mathcal{F}^*$  of  $\mathcal{F}$  such that  $(a_i)_{i < n}$  is also a display of  $\mathcal{F}^*$  with  $\varphi$ .*

*Proof.* Let  $(a_i)_{i < n}$  be a display of  $(C, I)$  with  $\varphi(x, y)$ . Let us define a new pattern  $(C^\dagger, I^\dagger)$  as follows: for any  $A \subseteq [n]$ ,

- $(A, \emptyset) \in C^\dagger$  iff  $\{\varphi(x, a_i) : i \in A\}$  is consistent;
- $(A, \emptyset) \in I^\dagger$  iff  $\{\varphi(x, a_i) : i \in A\}$  is inconsistent.

It is immediate to see that  $(C^\dagger, I^\dagger)$  is a reasonable positive pattern, and that it is additionally p-complete. Moreover, as  $(a_i)_{i < n}$  is a display of  $(C, I)$  with  $\varphi(x, y)$ , it follows that  $(C, I) \leq (C^\dagger, I^\dagger)$ . This completes the proof.  $\square$

**Remark A.1.16.** We could perhaps call  $(C^\dagger, I^\dagger)$  as constructed above the *positive n-pattern generated by  $\varphi(x, y)$  and  $(a_i)_{i < n}$* , but we will not make use of this notion in what follows. We simply observe that p-complete reasonable positive patterns are exactly those which can arise as patterns generated by some formula and some tuple of the appropriate length.

**Remark A.1.17.** Note that this does not prove Shelah's theorem [105, Theorem 7.11] that every TP theory is either  $TP_1$  or  $TP_2$ . *A priori*, there are many other p-completions of TP which are possibly exhibited. Shelah's result is significantly stronger.

We need a final piece of terminology (this will be related to a more general application of the name in subsequent sections):

**Definition A.1.18.** We say a positive pattern  $(C, I)$  is **2-constrained** if, whenever  $(A, \emptyset) \in I$ , there is some  $A' \subseteq A$  with  $|A'| = 2$  such that  $(A', \emptyset) \in I$ .

We can then extract the core of Bailetti's theorem [8, Proposition 6.3] as a correspondence between graphs and patterns. This will be the key technical result allowing us to make the explicit connection between the notions in chapter 6 and the exhibitability of patterns:

**Proposition A.1.19.** *There exists a one-to-one correspondence between finite graphs and 2-constrained p-complete reasonable positive patterns.*

*Proof.* Let  $G = (n, E)$  be a finite graph. We define a positive  $n$ -pattern  $\mathcal{F}'$  by

$$\begin{aligned} C &:= \{(\{i_0, \dots, i_{\ell-1}\}, \emptyset) : \{i_0, \dots, i_{\ell-1}\} \text{ is a clique of size } \ell\}, \\ I &:= \{(\{i_0, i_1\}, \emptyset) : i_0 \neq i_1 \text{ and } \neg E(i_0, i_1)\}. \end{aligned}$$

Let  $\mathcal{F} = \text{cl}(\mathcal{F}')$ . It is clear that  $\mathcal{F}$  is reasonable and 2-constrained. We claim that  $\mathcal{F}$  is p-complete. Let  $X \subseteq [n]$ . Then  $X$  is either a clique or not. If it is, then by definition  $(X, \emptyset) \in C(\mathcal{F})$ . If it is not, then there are  $i, j \in X$  such that  $\neg E(i, j)$ . By definition,  $(\{i, j\}, \emptyset) \in I(\mathcal{F})$ . Since  $\mathcal{F}$  is closed and  $\{i, j\} \subseteq X$ , it follows that  $(X, \emptyset) \in I(\mathcal{F})$ . This completes the proof.

Now suppose that  $\mathcal{F}$  is a 2-constrained p-complete reasonable positive pattern. Then we can define a graph  $G = (n, E)$  by declaring  $E(i, j)$  iff  $(\{i, j\}, \emptyset) \in C(\mathcal{F})$  for all  $i \neq j < n$ . Since  $\mathcal{F}$  is reasonable, we cannot have  $(\{i, j\}, \emptyset) \in C(\mathcal{F}) \cap I(\mathcal{F})$ ; and by p-completeness, for every pair  $\{i, j\} \in X$ , we have  $(\{i, j\}, \emptyset) \in C(\mathcal{F}) \cup I(\mathcal{F})$ . So  $G$  is well-defined.  $\square$

To make the connection with graphs even more explicit, let us introduce some *ad hoc* notation (which will not be used in further sections). Given a graph  $G$ , we denote by  $G^\sharp$  the 2-constrained p-complete reasonable positive pattern associated to it. Similarly, given a 2-constrained p-complete reasonable positive pattern  $\mathcal{F}$ , we denote by  $\mathcal{F}^\flat$  the graph associated to it.

**Remark A.1.20.**  $\sharp$  and  $\flat$  are inverse operations, i.e.,  $G^\sharp = \mathcal{F}$  iff  $G = \mathcal{F}^\flat$ .

**Lemma A.1.21.** A formula  $\varphi(x, y)$  realises a graph  $G$  iff it exhibits  $G^\sharp$ .

*Proof.* It is clear that the same sequence plays both roles.  $\square$

Combining this lemma with Proposition A.1.15, we recover [8, Proposition 6.3].

## A.2 Duality of patterns

We now move to consider the notion of duality in this section. As explained at the beginning of this appendix, the motivation is to formally capture the relationship between  $k$ -DCTP<sub>1</sub> and  $k$ -CTP [95], or between SOP<sub>2</sub> and ATP [5].

We start by introducing a special kind of set of finite positive patterns:

**Definition A.2.1.** Let  $\mathcal{J}$  be an indexing structure of cardinality  $n$ ,  $X, Y \subseteq \mathcal{P}(\mathcal{J})$ , and  $k < \omega$ . Fix an enumeration  $h: [n] \rightarrow \mathcal{J}$ . We define an  $n$ -pattern  $(C, I)$  as follows: for  $A \subseteq [n]$ ,

- $(A, \emptyset) \in C$  iff  $h(A) \in X$ .
- $(A, \emptyset) \in I$  iff  $|A| = k$  and  $h(A) \in Y$ .

We call  $(C, I)$  the  **$k$ -constrained  $n$ -pattern induced by  $\mathcal{J}$  with  $(X, Y)$** .

**Lemma A.2.2.** The above is well-defined up to equivalence.

*Proof.* Let  $\mathcal{J}$ ,  $X$ ,  $Y$ , and  $k$  be as above. Consider two enumerations  $h, k: [n] \rightarrow \mathcal{J}$ . Let us write  $(C_h, I_h)$  and  $(C_k, I_k)$  for the corresponding  $n$ -patterns as in the previous

definition. Let  $A \subseteq [n]$ . Define  $A' := k^{-1} \circ h(A)$ . Then, by definition,

$$(A', \emptyset) \in C_k \iff k(A') \in X \iff h(A) \in X \iff (A, \emptyset) \in C_h,$$

and similarly for  $I_h$  and  $I_k$ . It is also clear that every  $n$ -code in  $C_k$  arises in this form, and similarly for  $I_k$ . Therefore, if  $\varphi(x, y)$  exhibits  $(C_h, I_h)$  with a sequence of tuples  $(a_i)_{i < n}$ , we can define  $a'_i := a_{h^{-1} \circ k(i)}$  and check that  $\varphi(x, y)$  exhibits  $(C_k, I_k)$  with  $(a'_i)_{i < n}$ .  $\square$

In order to extend this construction to define a set of finite patterns, we need to break  $\mathcal{J}$  down into finite parts for which we can obtain a unique  $n$ -pattern for each  $n < \omega$ .

**Definition A.2.3.** We call a chain  $(\mathcal{J}_i)_{i < \omega}$  of finite subsets of  $\mathcal{J}$  an **approximation** of  $\mathcal{J}$  if  $\mathcal{J} = \bigcup_{i < \omega} \mathcal{J}_i$ .

**Definition A.2.4.** Let  $\mathcal{J}$  be an indexing structure with an approximation  $(\mathcal{J}_i)_{i < \omega}$  and  $X, Y \subseteq \mathcal{P}(\mathcal{J})$  be sets of subsets of  $\mathcal{J}$  containing no singletons. For each  $n < \omega$ , let  $X_n := X \cap \mathcal{J}_n$ ,  $Y_n := Y \cap \mathcal{J}_n$ , and let us denote by  $(C_n, I_n)$  the  $k$ -constrained  $n$ -pattern induced by  $\mathcal{J}_n$  with  $(X_n, Y_n)$ . We define the  **$k$ -constrained  $(X, Y)$ -describable** set of finite patterns to be  $\mathcal{F} := \{(C_n, I_n) : n < \omega\}$ .

**Lemma A.2.5.** *The above is well-defined up to equivalence.*

*Proof.* We need to show that the above construction does not depend on the enumerations or the approximation used. First, fix an approximation  $(\mathcal{J}_i)_{i < \omega}$ . If we make different choices for the enumeration of each finite pattern, we can define a permutation of  $\mathcal{J}$  where each pattern in  $\mathcal{F}$  and its image are equivalent by Lemma A.2.2. Therefore, the two resulting sets of finite patterns are equivalent.

It remains to show that the definition does not depend on our choice of approximation. So let  $(\mathcal{J}_i)_{i < \omega}$  and  $(\mathcal{J}'_i)_{i < \omega}$  be two approximations of  $\mathcal{J}$ . Let  $\mathcal{F}$  and  $\mathcal{F}'$  be the respective  $k$ -constrained  $(X, Y)$ -describable sets of finite patterns. It is enough to show one direction of the equivalence. So suppose  $\varphi(x, y)$  and  $(a_i)_{i < \omega}$  exhibit  $\mathcal{F}$ . Let  $(A, \emptyset) \in C'_n$  for some  $n < \omega$ . By definition, there is some enumeration  $h: [n] \rightarrow \mathcal{J}$  such that  $h(A) \in X$ , hence  $h(A) \in X_m$  for some  $m < \omega$ , and thus  $(A, \emptyset) \in C_m$ . Hence, up to replacing the sequence,  $\{\varphi(x, a_i) : i \in A\}$  is consistent.  $\square$

**Definition A.2.6.** Let  $k < \omega$ . We call  $(X, Y)$  a  **$(k)$ -description** of a set  $\mathcal{F}$  of finite patterns if  $\mathcal{F}$  and the  $k$ -constrained  $(X, Y)$ -describable set of finite patterns are equivalent.

We now address the question of the exhibitability of describable patterns.

**Definition A.2.7.** Let  $\mathcal{J}$  be an indexing structure and  $X, Y \subseteq \mathcal{P}(\mathcal{J})$  be sets of subsets of  $\mathcal{J}$  containing no singletons. We say the pair  $(X, Y)$  is **compatible** if:

- (i)  $X \cap Y = \emptyset$ .
- (ii) For all  $A \in X$  and  $B \in Y$ , we have  $B \not\subseteq A$ .

We say  $(X, Y)$  is **strongly compatible** if we replace (ii) above by:

- (ii') For all  $A \in X$  and  $B \in Y$ , we have  $A \not\subseteq B$  and  $B \not\subseteq A$ .

**Lemma A.2.8.** *Let  $(X, Y)$  be a compatible pair of subsets of  $\mathcal{J}$  and  $k < \omega$ . Then the  $k$ -constrained  $(X, Y)$ -describable set of finite patterns  $\mathcal{F}$  is reasonable.*

*Proof.* Since  $\mathcal{F}$  is a set of positive patterns, conditions (ii) and (iii) from the definition of reasonability are trivial. So assume, for contradiction, that there exist some  $n < \omega$  and  $A_1, A_2 \subseteq [n]$  such that  $(A_1, \emptyset) \in C_n$ ,  $(A_2, \emptyset) \in I_n$ , and  $A_2 \subseteq A_1$ . Let  $h: [n] \rightarrow \mathcal{J}_n$  be any enumeration. Then, by definition of  $\mathcal{F}$ , we have that  $h(A_1) \in X$  and  $h(A_2) \in Y$ . Since  $h(A_2) \subseteq h(A_1)$ , we contradict the assumption that  $(X, Y)$  forms a compatible pair.  $\square$

**Example A.2.9.** The following gives the standard generators for the usual classification properties:

- $k$ -TP is generated by the  $k$ -constrained  $(X, Y)$ -describable set of finite patterns with  $X$  the set of paths in  $\omega^{<\omega}$  and  $Y$  the set of children of a common node in  $\omega^{<\omega}$ .
- $k$ -TP<sub>1</sub> is generated by the  $k$ -constrained  $(X, Y)$ -describable set of finite patterns with  $X$  the set of paths in  $\omega^{<\omega}$  and  $Y$  the set of sets of pairwise incomparable elements in  $\omega^{<\omega}$ .
- $k$ -TP<sub>2</sub> is generated by the  $k$ -constrained  $(X, Y)$ -describable set of finite patterns with  $X$  the set of “paths” through the array  $\omega \times \omega$  and  $Y$  the set of “rows” in the array  $\omega \times \omega$ .
- SOP<sub>1</sub> is generated by the 2-constrained  $(X, Y)$ -describable set of finite patterns with  $X$  the set of paths in  $2^{<\omega}$  and  $Y$  the set of pairs satisfying the usual condition.
- SOP<sub>2</sub> is generated by the 2-constrained  $(X, Y)$ -describable set of finite patterns with  $X$  the set of paths in  $2^{<\omega}$  and  $Y$  the set of antichains in  $2^{<\omega}$ .
- $k$ -ATP is generated by the  $k$ -constrained  $(X, Y)$ -describable set of finite patterns with  $X$  the set of antichains in  $2^{<\omega}$  and  $Y$  the set of paths in  $2^{<\omega}$ .

- $k$ -BTP is generated by the  $k$ -constrained  $(X, Y)$ -describable set of finite patterns with  $X$  the set of left-leaning paths in  $\omega_*^{<\omega}$  and  $Y$  the set of right-veering paths in  $\omega_*^{<\omega}$ .
- $k$ -CTP is generated by the  $k$ -constrained  $(X, Y)$ -describable set of finite patterns with  $X$  the set of right-combs in  $2^{<\omega}$  and  $Y$  the set of paths in  $2^{<\omega}$ .

It is important to note that, in all of the above cases, the pair  $(X, Y)$  is strongly compatible. We always take duals with respect to these ‘canonical’ descriptions.

**Definition A.2.10.** Let  $P$  be a property of formulas defined by a set of finite patterns  $\mathcal{F}$ . Let  $(X, Y)$  be a description of  $\mathcal{F}$ . Then the **dual** of  $P$  relative to  $(X, Y)$ , denoted  $P_{(X, Y)}^*$ , is the property of formulas defined by the  $k$ -constrained  $(Y, X)$ -describable set of finite patterns. We write  $P^*$  if the underlying description of  $\mathcal{F}$  is clear.

The following lemma guarantees that we can take duals as described above for all of the traditional neostability-theoretic properties listed in Example A.2.9:

**Lemma A.2.11.** *Let  $(X, Y)$  be a strongly compatible pair of subsets of  $\mathcal{J}$  and  $k < \omega$ . Then the  $k$ -constrained  $(Y, X)$ -describable set of finite patterns  $\mathcal{F}^*$  is reasonable.*

*Proof.* Since  $(X, Y)$  is strongly compatible,  $(Y, X)$  is compatible. So we can directly apply Lemma A.2.8.  $\square$

**Example A.2.12.** For any  $k < \omega$ ,  $k$ -CTP is precisely  $(k\text{-DCTP}_1)^*$  (cf., [95]). We will see later on that our construction also captures the relationship between ATP and  $\text{SOP}_2$ .

**Remark A.2.13.** For any compatible pair  $(X, Y)$  and any property  $P$  of formulas definable by the  $k$ -constrained  $(X, Y)$ -describable set of finite patterns, if  $\varphi(x, y)$  is  $\text{PM}^{(k)}$ , then  $\varphi(x, y)$  has  $P$ . In particular, using Example A.2.9 and [6, Lemma 3.20], it follows that all the classes from Definition 2.3.2, as well as  $\text{NSOP}_3$  by chapter 6, are subclasses of the class of  $\text{NPM}^{(2)}$  theories.

### A.2.1 Dual properties of formulas

We are now interested in identifying those proofs from the literature that are “dualisable,” i.e., those allowing us to transform an implication between two properties definable by positive patterns into an implication between their respective duals. To

restrict ourselves to meaningful results, we will say that two descriptions  $(X_1, Y_1)$  and  $(X_2, Y_2)$  are **comparable** if, for all  $n < \omega$ ,  $X_1$  (resp.,  $Y_1$ ) contains a set of size  $n$  iff  $X_2$  (resp.,  $Y_2$ ) does. The most general class of dualisable proofs we have been able to find so far is described in the following result:

**Proposition A.2.14.** *Let  $k < \omega$ . For  $i = 1, 2$ , let  $\mathcal{J}_i$  be an indexing set, let  $(X_i, Y_i)$  be comparable pairs, let  $\mathcal{F}_i$  be the  $k$ -constrained  $(X_i, Y_i)$ -describable set of finite patterns, and let  $P_i$  be the property of formulas defined by  $\mathcal{F}_i$ . Let  $h: \mathcal{J}_2 \rightarrow \mathcal{J}_1$  be an embedding such that:*

- (i) *whenever  $\{j_1, \dots, j_n\} \in X_2$ , we have  $\{h(j_1), \dots, h(j_n)\} \in X_1$ , and*
- (ii) *whenever  $\{j_1, \dots, j_n\} \in Y_2$ , we have  $\{h(j_1), \dots, h(j_n)\} \in Y_1$ .*

*If  $\varphi(x, y)$  has  $P_1$ , then it also has  $P_2$ .*

*In particular, if  $\varphi(x, y)$  has  $P_1^*$ , then it also has  $P_2^*$ .*

*Proof.* Suppose  $\varphi(x, y)$  has  $P_1$ , which by definition means that it exhibits every pattern in  $\mathcal{F}_1$ . Let  $(C_n^2, I_n^2)$  be an  $n$ -pattern in  $\mathcal{F}_2$ . We want to find a sequence  $(a_i)_{i < n}$  that exhibits every  $n$ -code in  $(C_n^2, I_n^2)$ .

By definition, there is an enumeration  $k: [n] \rightarrow \mathcal{J}_2$  that induces  $(C_n^2, I_n^2)$ . Consider the composition  $h \circ k: [n] \rightarrow \mathcal{J}_1$ . This picks out a finite subset of  $\mathcal{J}_1$ . So we may pick an enumeration  $l: [m] \rightarrow \mathcal{J}_1$  for some  $m \geq n$  which extends  $h \circ k$  and induces an  $m$ -pattern  $(C_m^1, I_m^1) \in \mathcal{F}_1$ . By assumption,  $\varphi(x, y)$  exhibits  $(C_m^1, I_m^1)$ , so there is a sequence  $(b_i)_{i < m}$  that witnesses this. Let us define  $a_i := b_i$  for  $i < n$ . We claim that  $\varphi(x, y)$  and  $(a_i)_{i < n}$  exhibit  $(C_n^2, I_n^2)$ .

Let  $(A, \emptyset) \in C_n^2$ . By definition,  $k(A) \in X_2$ , and thus, by (i),  $h \circ k(A) \in X_1$ . Since  $l$  extends  $h \circ k$ , we get  $l(A) \in X_1$ . Hence, by definition,  $(A, \emptyset) \in C_m^1$ . Because  $(b_i)_{i < m}$  exhibits  $C_m^1$  and  $A \subseteq [n]$ , we get that  $\{\varphi(x, b_i) : i \in A\} = \{\varphi(x, a_i) : i \in A\}$  is consistent. A similar argument works for  $n$ -codes in  $I_n^2$ . As this pattern was arbitrary, it follows that  $\varphi(x, y)$  exhibits  $\mathcal{F}_2$ , and thus has  $P_2$ .

For the “in particular” part, note that  $P_1^*$  and  $P_2^*$  are determined by comparable descriptions  $(Y_1, X_1)$  and  $(Y_2, X_2)$ , respectively. Since the embedding  $h$  still satisfies conditions (i) and (ii) with the appropriate sets, the same proof as in the previous paragraph goes through.  $\square$

**Corollary A.2.15.** *For every  $k < \omega$ :*

- (i)  $k\text{-TP}_2^* \implies k\text{-TP}^*$ .

- (ii)  $k\text{-ATP}^* \iff k\text{-TP}_1$ .
- (iii)  $k\text{-ATP}^* \implies k\text{-BTP}^*$ .
- (iv)  $k\text{-BTP}^* \implies k\text{-TP}_2^*$ .

*Proof.* (i) By Proposition A.2.14, it suffices to find an embedding  $h: \omega^{<\omega} \rightarrow \omega \times \omega$  such that for any path  $(\eta_i)_{i \in \omega}$  in  $\omega^{<\omega}$ ,  $(h(\eta_i))_{i \in \omega}$  forms a “path” through the array  $\omega \times \omega$ , and for any  $\eta_1, \dots, \eta_k \in \omega^{<\omega}$  which are children of a common node,  $h(\eta_1), \dots, h(\eta_k)$  form part of the same row in the array  $\omega \times \omega$ . This is how the classical proof goes.

(ii) ( $\implies$ ) By Proposition A.2.14, it suffices to find an embedding  $h: \omega^{<\omega} \rightarrow 2^{<\omega}$  such that  $h(\eta_i) \perp h(\eta_j)$  iff  $\eta_i \perp \eta_j$ . This is done in, e.g., [32, Fact 4.2].

( $\Leftarrow$ ) Simply note that the inclusion map  $h: 2^{<\omega} \rightarrow \omega^{<\omega}$  satisfies the appropriate conditions.

(iii) By Proposition A.2.14, it suffices to find an embedding  $h: \omega_*^{<\omega} \rightarrow 2^{<\omega}$  such that, whenever  $(\lambda_n)_{n \in \omega}$  forms a left-leaning path in  $\omega_*^{<\omega}$ ,  $(h(\lambda_n))_{n \in \omega}$  forms an antichain in  $2^{<\omega}$ ; and whenever  $(\rho_n)_{n \in \omega}$  forms a right-veering path in  $\omega_*^{<\omega}$ ,  $h(\rho_n)_{n \in \omega}$  forms a chain in  $2^{<\omega}$ . This is done in [76, Proposition 5.6].

(iv) By Proposition A.2.14, it suffices to find an embedding  $h: \omega^2 \rightarrow \omega_*^{<\omega}$  such that, for  $f: \omega \rightarrow \omega$ ,  $(h(i, f(i)))_{i \in \omega}$  forms a left-leaning path, and for a fixed  $i \in \omega$ ,  $(h(i, j))_{j \in \omega}$  forms a right-veering path. This is done in [76, Proposition 5.3].  $\square$

A special case of the Proposition A.2.14 is the following:

**Lemma A.2.16.** *Let  $P_1$  and  $P_2$  be properties of formulas definable by positive patterns with comparable descriptions  $(X_1, Y_1)$  and  $(X_2, Y_2)$ , respectively, but with the same indexing set  $\mathcal{J}$ . Suppose that  $X_2 \subseteq X_1$  and  $Y_2 \subseteq Y_1$ . Then, if  $\varphi(x, y)$  has  $P_1$ , then it also has  $P_2$ .*

*In particular, if  $\varphi(x, y)$  has  $P_1^*$ , then it also has  $P_2^*$ .*

*Proof.* Simply check that, under the above conditions, the identity map  $h: \mathcal{J} \rightarrow \mathcal{J}$  satisfies the requirements of Proposition A.2.14.  $\square$

**Corollary A.2.17.** *For every  $k < \omega$ :*

- (i)  $k\text{-TP}_1^* \implies k\text{-TP}^*$ .
- (ii)  $k\text{-ATP}^* \implies k\text{-CTP}^*$ .

**Remark A.2.18.** There are also results in the literature regarding  $\text{SOP}_1$  and  $\text{SOP}_2$  that follow similar proof strategies as the ones considered above. However, a problem with the usual description of either of the two (as per Example A.2.9) is that  $Y$  contains only pairs. Thus, when we take the dual, we have very few consistent sets; this disparity in the cardinalities of the sets in  $Y$  explains why we cannot use Proposition A.2.14 for these properties (and also why we introduce the *ad hoc* notion of comparability).

**Question A.2.19.** Is logical equivalence of properties of formulas preserved under duals? That is, let  $P_1, P_2$  be properties of formulas defined by patterns with comparable descriptions, and suppose that  $\varphi(x, y)$  has  $P_1$  iff it has  $P_2$ . Does it follow that  $\varphi(x, y)$  has  $P_1^*$  iff it has  $P_2^*$ ?

### A.2.2 Dual properties of theories

Let us move now to consider what happens at the level of theories. Of course, all of the implications from the above diagram are inherited in this context. However, there is a recent result due to Mutchnik (see [95]) which holds at the level of theories but not formula-by-formula (see [5, §6]), namely,  $\text{SOP}_1 \implies \text{SOP}_2$ . This has some important consequences that greatly simplify the diagram of implications. From now on, we take our properties to be of theories instead of formulas.

We can gather several results from the literature to collapse some of the classes considered in the previous section:

**Lemma A.2.20.** *For all  $k$ ,  $k\text{-ATP}^* = k\text{-CTP}^* = k\text{-TP}_1 = \text{SOP}_2 = \text{SOP}_1$ .*

*Proof.* By Corollary A.2.15(ii) and [68, Proposition 6.1],  $k\text{-ATP}^* = k\text{-TP}_1 = \text{SOP}_2$ . Moreover, as in Example A.2.12,  $k\text{-CTP}^*$  coincides with Mutchnik's  $k\text{-DCTP}_1$ , and so by [95, Lemma 2.8]  $k\text{-CTP}^* = \text{SOP}_2$ . Finally, by [95, Theorem 1.6],  $\text{SOP}_2 = \text{SOP}_1$ .  $\square$

**Corollary A.2.21.**  $\text{SOP}_1 \implies k\text{-BTP}^*$  for all  $k$ .

*Proof.* Apply Lemma A.2.20 and Corollary A.2.15(iii).  $\square$

**Remark A.2.22.** It is yet open whether the converse holds. The natural attempt would be to dualise the proof of [76, Proposition 5.5]; however, that does not seem to give us the appropriate parallel sequences required to conclude  $\text{SOP}_1$ .

**Corollary A.2.23.** *Implications between properties of theories are not preserved under taking duals.*

*Proof.* Assume otherwise, i.e., for all properties  $P_1, P_2$  of theories definable by patterns with comparable descriptions such that  $P_1 \implies P_2$ , we have  $P_1^* \implies P_2^*$ . Suppose that  $T$  is  $2\text{-TP}_1^*$ . By assumption and Corollary A.2.15(ii),  $T$  is  $2\text{-ATP}$ . By [5, Proposition 4.4],  $T$  is  $\text{SOP}_1$ , and so by Corollary A.2.21,  $T$  is  $2\text{-BTP}^*$ . Therefore, by Corollary A.2.15(iv),  $T$  is  $2\text{-TP}_2^*$ . But then, by our assumption,  $\text{TP}_1 \implies \text{TP}_2$ , which is false.  $\square$

While this is an important restriction, on its own this result does not suffice to show that the stronger condition of *equivalence* is not preserved when taking duals. If true, such a claim offers a potential strategy suggested by what we have proven so far for proving that ATP and BTP are not equivalent, namely, trying to show that their duals (one of which is  $\text{SOP}_2$ ) are not equivalent, or in other words, finding an  $\text{NSOP}_1$  BTP\* theory. It would also provide us with a way to tackle the relationship between ATP and CTP:

**Question A.2.24.** Is logical equivalence of properties of theories preserved under duals? That is, let  $P_1, P_2$  be properties of theories definable by patterns with comparable descriptions, and suppose that a theory  $T$  has  $P_1$  iff it has  $P_2$ . Does it follow that  $T$  has  $P_1^*$  iff it has  $P_2^*$ ?

**Remark A.2.25.** Let us observe that, if there is a dualisable proof of  $k\text{-BTP} \implies \text{SOP}_1$ , then we can answer the above question negatively. Indeed, suppose such a proof exists, and that logical equivalence is preserved under duals. By Lemma A.2.20, we have that  $\text{SOP}_1^* = k\text{-ATP}$ , and thus  $\text{SOP}_1^* \implies k\text{-BTP}$ . But we have  $k\text{-BTP} \implies \text{SOP}_1$  by [76, Proposition 5.5],  $\text{SOP}_1 \implies k\text{-BTP}^*$  by Corollary A.2.21, and  $k\text{-BTP}^* \implies \text{SOP}_1^*$  by our hypothesis. This all implies that  $k\text{-BTP} = \text{SOP}_1$ , which is a contradiction.

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# Acronyms

- $k$ -ATP.**  $k$ -antichain tree property. 25  
 **$k$ -BTP.**  $k$ -bizarre tree property. 26  
 **$k$ -CTP.**  $k$ -comb tree property. 25  
 **$k$ -TP.**  $k$ -tree property. 23  
 **$k$ -TP<sub>2</sub>.**  $k$ -tree property of the second kind. 24
- AP.** amalgamation property. 21
- FOP <sub>$n$</sub> .**  $n$ -functional order property. 27
- GUWP.** generalised universal witnessing property. 101
- IP <sub>$n$</sub> .**  $n$ -independence property. 27
- NATP.** not the antichain tree property. 27  
**NBTP.** not the bizarre tree property. 27  
**NCTP.** not the comb tree property. 27  
**NFOP <sub>$n$</sub> .** not the  $n$ -functional order property. 27  
**NIP <sub>$n$</sub> .** not the  $n$ -independence property. 27  
**NPM<sup>(2)</sup>.** not 2-positive maximal. 173  
**NSOP.** not the strict order property. 26  
**NSOP<sub>1</sub>.** not the 1-strong order property. 27  
**NSOP<sub>2</sub>.** not the 2-strong order property. 27  
**NSOP <sub>$n$</sub> .** not the  $n$ -strong order property. 26  
**NTP<sub>2</sub>.** not the tree property of the second kind. 27
- PAC.** pseudo-algebraically closed. 40  
**PM<sup>(2)</sup>.** 2-positive maximal. 173
- SAP.** strong amalgamation property. 21  
**SOP<sub>1</sub>.** 1-strong order property. 24  
**SOP<sub>2</sub>.** 2-strong order property. 25  
**SOP <sub>$n$</sub> .**  $n$ -strong order property. 26
- TP<sub>1</sub>.** tree property of the first kind. 24  
**TUWP.** type universal witnessing property. 102

**UWP.** universal witnessing property. 100

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