Published for SISSA by Springer

Received: November 4, 2022
Accepted: December 23, 2022
Published: January 10, 2023

Exact strong coupling results in $\mathcal{N} = 2$ Sp($2N$) superconformal gauge theory from localization

M. Beccaria,[a] **G.P. Korchemsky**[b,c] **and A.A. Tseytlin**[d,1]

[a] *Università del Salento, Dipartimento di Matematica e Fisica* Ennio De Giorgi *and I.N.F.N. — sezione di Lecce, Via Arnesano, I-73100 Lecce, Italy*

[b] *Institut de Physique Théorique,*[2] *Université Paris Saclay, CNRS, 91191 Gif-sur-Yvette, France*

[c] *Institut des Hautes Études Scientifiques, 91440 Bures-sur-Yvette, France*

[d] *Blackett Laboratory, Imperial College London, Prince Consort Road, London, SW7 2AZ, U.K.*

E-mail: matteo.beccaria@le.infn.it, gregory.korchemsky@ipht.fr, atseytlin@gmail.com

Abstract: We apply the localization technique to compute the free energy on four-sphere and the circular BPS Wilson loop in the four-dimensional $\mathcal{N} = \in$ superconformal Sp($2N$) gauge theory containing vector multiplet coupled to four hypermultiplets in fundamental representation and one hypermultiplet in rank-2 antisymmetric representation. This theory is unique among similar $\mathcal{N} = \in$ superconformal models that are planar-equivalent to $\mathcal{N} = \triangle$ SYM in that the corresponding localization matrix model has the interaction potential containing single-trace terms only. We exploit this property to show that, to any order in large N expansion and an arbitrary 't Hooft coupling λ, the free energy and the Wilson loop satisfy Toda lattice equations. Solving these equations at strong coupling, we find remarkably simple expressions for these observables which include all corrections in $1/N$ and $1/\sqrt{\lambda}$. We also compute the leading exponentially suppressed $\mathcal{O}(e^{-\sqrt{\lambda}})$ corrections and consider a generalization to the case when the fundamental hypermultiplets have a non-zero mass. The string theory dual of this $\mathcal{N} = \in$ gauge theory should be a particular orientifold of $AdS_5 \times S^5$ type IIB string and we discuss the string theory interpretation of the obtained strong-coupling results.

Keywords: AdS-CFT Correspondence, Supersymmetric Gauge Theory, $1/N$ Expansion

ArXiv ePrint: 2210.13871

[1] Also on leave from Inst. for Theoretical and Mathematical Physics (ITMP) and Lebedev Inst., Moscow, Russia.

[2] Unité Mixte de Recherche 3681 du CNRS.

Open Access, © The Authors.
Article funded by SCOAP[3].

https://doi.org/10.1007/JHEP01(2023)037

JHEP01(2023)037

Contents

1 Introduction and summary 1

2 Partition function of $\mathcal{N} = 2$ Sp($2N$) theory 8
- 2.1 Matrix model representation 8
- 2.2 Large N expansion of free energy 11
- 2.3 Free energy from Toda lattice equation 13
- 2.4 Resummation 16
- 2.5 Leading nonperturbative correction 21

3 Circular Wilson loop 22
- 3.1 Toda equation in $\mathcal{N} = 4$ Sp($2N$) theory 23
- 3.2 Difference of Wilson loops in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories 25
- 3.3 Toda equation in $\mathcal{N} = 2$ Sp($2N$) theory 26
- 3.4 Leading nonperturbative correction 27

4 Dual string theory interpretation 28
- 4.1 Free energy in $\mathcal{N} = 4$ theory dual to IIB string on $AdS_5 \times \mathbb{RP}^5$ 30
- 4.2 Free energy in $\mathcal{N} = 2$ theory dual to IIB string on $AdS_5 \times S^5/\mathbb{Z}_{2,\text{ori}}$ 34

JHEP01(2023)037

1 Introduction and summary

It was recently appreciated that localization [1, 2] provides an important tool to study AdS/CFT duality beyond the planar limit in a class of simplest $\mathcal{N} = 2$ superconformal models that are planar-equivalent to $\mathcal{N} = 4$ SYM theory (see, in particular, [3–8]).[1] Using localization matrix model representation for some special observables like free energy on four-sphere or circular BPS Wilson loop one can find their $1/N$, large 't Hooft coupling λ expansions and interpret the resulting series as perturbative expansion in terms of the dual string theory parameters — string coupling g_s and string tension T

$$g_s = \frac{\lambda}{4\pi N}, \qquad T = \frac{L^2}{2\pi\alpha'} = \frac{\sqrt{\lambda}}{2\pi}, \qquad \lambda = g_{\rm YM}^2 N\,. \tag{1.1}$$

The conceptually simplest example is provided by the $\mathbb{Z}_2$ orbifold of the SU($2N$) $\mathcal{N} = 4$ SYM theory (i.e. the SU(N) × SU(N) $\mathcal{N} = 2$ gauge theory with bi-fundamental hypermultiplets and equal couplings) which is dual to string theory on $AdS_5 \times S^5/\mathbb{Z}_2$. However, the corresponding localization 2-matrix model is rather complicated (with the interaction potential containing double-trace terms). As a result, the strong-coupling expansion of

[1] See also [9–11] and [12–14] for related computations of special correlators of BPS operators.

only the leading $1/N^2$ term in the free energy and the Wilson loop expectation value was so far worked out explicitly [4, 7].

Here we will focus on what turns out to be the simplest representative in the family of similar $\mathcal{N}=2$ superconformal models that are planar-equivalent to $\mathcal{N}=4$ super Yang-Mills theory: the $\mathcal{N}=2$ Sp($2N$) gauge theory coupled to four hypermultiplets in the fundamental representation and one hypermultiplet is the rank 2 antisymmetric representation of Sp($2N$).

This gauge theory can be "engineered" on a collection of $2N$ D3-branes, 8 D7-branes and one O7-plane [15–19].[2] The corresponding dual string theory is then expected to be a special orientifold of type IIB superstring on $\mathrm{AdS}_5 \times S^5$ [20–24]. The dual description does not explicitly involve D7-branes, but due to orientifolding there is also an open-string sector in addition to a closed-string one (like in type I theory).

This theory (referred to as the "FA-orientifold" model in [6]) is unique in that the interaction potential in the localization matrix model contains only single-trace terms. This leads to substantial technical simplifications compared to other similar $\mathcal{N}=2$ superconformal models with localization matrix model having double-trace potentials. As a result, the large N expansion of the free energy $F_N^{\mathcal{N}=2}$ and the expectation value of the circular BPS Wilson loop $W_N^{\mathcal{N}=2}$ can be worked out rather explicitly for an arbitrary 't Hooft coupling [6].

In virtue of the planar equivalence, the large N expansion of both quantities is

$$\begin{aligned} F_N^{\mathcal{N}=2} &= F_N^{\mathcal{N}=4} + N\,\mathrm{F}_1(\lambda) + \mathrm{F}_2(\lambda) + \frac{1}{N}\mathrm{F}_3(\lambda) + \mathcal{O}\Big(\frac{1}{N^2}\Big)\,, \\ W_N^{\mathcal{N}=2} &= W_N^{\mathcal{N}=4} + \Delta W^{(1)}(\lambda) + \frac{1}{N}\Delta W^{(2)}(\lambda) + \mathcal{O}\Big(\frac{1}{N^2}\Big)\,, \end{aligned} \tag{1.2}$$

where the leading term is the $\mathcal{N}=4$ Sp($2N$) SYM result and the subleading ones are suppressed by powers of $1/N$. The free energy and the Wilson loop in $\mathcal{N}=4$ Sp($2N$) theory are given by the well-known expressions [9, 25]

$$\begin{aligned} F_N^{\mathcal{N}=4} &= -N\left(N+\frac{1}{2}\right)\log\lambda \ + C_N^{\mathcal{N}=4}\,, \\ W_N^{\mathcal{N}=4} &= 2\,e^{\frac{\lambda}{16N}}\sum_{n=0}^{N-1} L_{2n+1}\Big(-\frac{\lambda}{8N}\Big)\,, \end{aligned} \tag{1.3}$$

where the N-dependent constant $C_N^{\mathcal{N}=4}$ can be expressed in terms of Barnes $G-$function (see (2.33) below) and $L_{2n+1}(x)$ is the Laguerre polynomial.

[2] The 4 fundamental hypers are massless modes of strings stretched between the D3- and D7-branes. O7 plane is required for stability (conformal invariance) bringing in the antisymmetric hyper that arises from the action of the orientifold projection on the fields corresponding to directions transverse to the D3-branes but parallel to the D7-branes. The Coulomb branch corresponds to giving an expectation value to the complex scalar of $\mathcal{N}=2$ vector multiplet related to separation of D3-branes from D7-branes and the fixed plane. One Higgs branch is parametrized by expectation values of the scalars of the antisymmetric hyper (representing motion of D3-branes in the transverse directions within D7-branes). The second Higgs branch is parameterized by the fundamental scalars and corresponds to dissolving D3-branes inside D7-branes and may be described in terms of gauge instanton moduli space.

JHEP01(2023)037

Toda lattice equation. Using the localization matrix model representation for the free energy $F_N^{\mathcal{N}=2}$, one can show [6] that the leading non-planar correction $\mathrm{F}_1(\lambda)$ in (1.2) admits a compact integral representation in terms of Bessel function (see (2.19) below). As was found in [6], the subleading corrections in (1.2) have an interesting iterative structure. Namely, the functions $\mathrm{F}_k(\lambda)$ (with $k \geq 2$) in the free energy can be expressed in terms of the leading function $\mathrm{F}_1(\lambda)$ as

$$\begin{aligned} \mathrm{F}_2' &= \frac{1}{4}(\lambda \mathrm{F}_1)'' - \frac{1}{4}\lambda\big[(\lambda\,\mathrm{F}_1)''\big]^2\,, \\ \mathrm{F}_3 &= \frac{1}{48}\lambda^2(\lambda\mathrm{F}_1)'''' - \frac{1}{16}\lambda^2\big[(\lambda\mathrm{F}_1)''\big]^2 + \frac{1}{24}\lambda^3\big[(\lambda\mathrm{F}_1)''\big]^3\,, \qquad \dots\,, \end{aligned} \tag{1.4}$$

where prime denotes a derivative with respect to λ. For the circular Wilson loop in (1.2), the functions $\Delta W^{(k)}$ (with $k \geq 1$) satisfy similar relations

$$\begin{aligned} (\Delta W^{(1)})' &= -\frac{1}{8}\,\lambda\, W^{(0)}(\lambda)(\lambda\mathrm{F}_1)''\,, \\ \Delta W^{(2)} &= -\frac{1}{32}\lambda^2\, W^{(0)}(\lambda)\Big[(\lambda\mathrm{F}_1)'' - \lambda\big((\lambda\mathrm{F}_1)''\big)^2\Big]\,, \qquad \dots\,, \end{aligned} \tag{1.5}$$

where $W^{(0)} = \frac{4}{\sqrt{\lambda}} I_1(\sqrt{\lambda})$ is the leading term in the large N expansion of the Wilson loop in $\mathcal{N}=4$ theory, $W_N^{\mathcal{N}=4} = NW^{(0)} + O(N^0)$.

The relations (1.4) and (1.5) were derived in [6] by examining the large N expansions of the free energy and the Wilson loop in $\mathcal{N}=2$ Sp($2N$) theory at weak coupling. They are expected to hold for an arbitrary 't Hooft coupling. Being supplemented with the expression for $\mathrm{F}_1(\lambda)$, they allow one to compute subleading corrections in (1.2) for any λ.

In this paper we explain the origin of the relations (1.4) and (1.5). We exploit the fact that the localization Sp($2N$) matrix model representation here contains only a single-trace interaction potential to show that the free energy and the Wilson loop satisfy discrete Toda-like equations[3]

$$\begin{aligned} \partial_y^2 F_N &= -\exp\left(-F_{N+1} + 2F_N - F_{N-1}\right)\,, \\ \partial_y^2\, W_N &= -\left(W_{N+1} - 2W_N + W_{N-1}\right)\partial_y^2 F_N\,, \end{aligned} \tag{1.6}$$

where

$$y \equiv \frac{(4\pi)^2}{g_{\mathrm{YM}}^2}\,, \qquad\qquad \lambda = (4\pi)^2\frac{N}{y}\,. \tag{1.7}$$

The relations (1.6) are not sensitive to a detailed form of the single-trace interaction potential in the localization matrix model representation for F_N and W_N and, as a consequence, they hold both in the $\mathcal{N}=4$ and $\mathcal{N}=2$ Sp($2N$) theories. The expressions on the right-hand side of (1.6) involve functions defined for the same y and N shifted by ± 1. In terms of the 't Hooft coupling constant, this corresponds to replacing λ with $\frac{N\pm 1}{N}\lambda$.

It is straightforward to verify that the relations (1.6) are indeed satisfied in $\mathcal{N}=4$ theory (cf. (1.3)). In the $\mathcal{N}=2$ theory, we reproduce (1.4) and (1.5) by replacing $F_N \equiv F_N^{\mathcal{N}=2}$ and $W_N \equiv W_N^{\mathcal{N}=2}$ with their $1/N$ expansions (1.2) and comparing the $1/N^k$ coefficients on both sides of (1.6).

[3] Though not directly related, let us note that the strategy of using Toda-like recursions to control the full $1/N$ expansion was previously applied in [26] to compute the form factor expansions in the 2d Ising model.

JHEP01(2023)037

Strong coupling expansion. Solving the Toda equations (1.6) at strong coupling (i.e. in large λ expansion), we find that the free energy $F_N^{\mathcal{N}=2}(\lambda)$ can be naturally separated into "perturbative" and "nonperturbative" contributions

$$F_N^{\mathcal{N}=2}(\lambda) = F_{\rm p}(N,\lambda) + F_{\rm np}(N,\lambda)\,. \tag{1.8}$$

Here each term in the large N expansion of $F_{\rm p}$ is given by a series in $\frac{1}{\sqrt{\lambda}}$ (which should correspond to α'-corrections in dual string theory). In contrast, the $O(1/N^k)$ terms in $F_{\rm np}$ are given by sums of powers of exponentially small factors $\mathcal{O}(e^{-\sqrt{\lambda}})$ with coefficients depending on $\frac{1}{\sqrt{\lambda}}$.

The presence of the latter exponential corrections is reflecting an asymptotic (but, it turns out, Borel-summable) nature of the strong coupling expansion. On the dual string-theory side these corrections should correspond to "world-sheet instanton" contributions that may be non-trivial due to the presence of a non-contractible 2-cycle in the $\mathbb{Z}_2$ orbifold of S^5 which is part of the orientifold projection.

Remarkably, the perturbative contribution in the free energy (1.8) can be found in a closed form [6] (see section 2)

$$F_{\rm p}(N,\lambda) = \Big(N+\frac{3}{4}\Big)\Big(N+\frac{1}{4}\Big)\log\Big(\frac{1}{\lambda}+\frac{\log 2}{2\pi^2 N}\Big) + C_N^{\mathcal{N}=2} - \frac{\pi^2}{2}\frac{N}{\lambda}\,, \tag{1.9}$$

where the constant $C_N^{\mathcal{N}=2}$ is given in (2.67) below. The logarithmic term in (1.9) sums up an infinite series of corrections in $1/N$ proportional to powers of $\log 2$. The relation (1.9) suggests to redefine the strong-coupling expansion parameter λ as

$$\frac{1}{\lambda'} \equiv \frac{1}{\lambda} + \frac{b}{N}\,, \qquad\qquad b = \frac{\log 2}{2\pi^2}\,. \tag{1.10}$$

Similar redefinitions of gauge coupling previously appeared in the strong-coupling calculation of cusp anomalous dimension [27] and octagon correlator [28, 29] in $\mathcal{N}=4$ SYM theory.

Setting $\lambda' = g'^2_{\rm YM} N$ this redefinition can be interpreted as originating from a finite gauge coupling renormalization[4]

$$g'^2_{\rm YM} = \frac{g^2_{\rm YM}}{1+b\,g^2_{\rm YM}}\,. \tag{1.11}$$

In the localization matrix model description, the redefinition (1.10) is closely related to the asymptotic behaviour of the interaction potential at large X (see (2.68) below), namely, $S_{\rm int}(X) \to b\,{\rm tr}\,X^2$. This produces an extra contribution to the coefficient of the Gaussian term $\frac{1}{g^2_{\rm YM}}{\rm tr}\,X^2$ in the matrix model action leading to (1.11).

Rewriting $F_{\rm p}(N,\lambda)$ in terms of λ' we find that the leading $\mathcal{O}(N^2\log\lambda')$ term in (1.9) is the same as in $\mathcal{N}=4$ theory, eq. (1.3). Surprisingly, viewed as a function of N and λ',

[4] Similar 1-loop finite renormalizations of couplings appear in some supersymmetric theories: in prepotential calculation at finite N in SU(N) $\mathcal{N}=2$ theories with flavor [30, 31] and also in some superconformal models [32, 33] (cf. also [34]).

JHEP01(2023)037

the perturbative part of the $\mathcal{N}=2$ free energy (1.9) thus receives only $\mathcal{O}(N)$ and $\mathcal{O}(N^0)$ but no higher-order $\mathcal{O}(1/N)$ corrections!

At large N the leading term in the nonperturbative correction to the free energy (1.8) scales as $F_{\rm np} \sim N\lambda^{-1/4}e^{-\sqrt{\lambda}}$. The Toda equation (1.6) allows us to compute systematically subleading nonplanar corrections to $F_{\rm np}$. We find that for sufficiently large λ' these corrections are dominated by terms of the form $\mathcal{O}((\lambda'^{3/2}/N^2)^k)$. Such terms can be summed to all orders in k to give the following resummed expression

$$F_{\rm np}(N,\lambda) = \frac{8\sqrt{2}}{\pi^{3/2}}\, N\, \lambda'^{-1/4}\, e^{-\sqrt{\lambda'}-\frac{\lambda'^{3/2}}{384N^2}} + \dots\,, \tag{1.12}$$

where dots denote contributions of subleading corrections.[5]

The Toda lattice equations (1.6) can be effectively applied to derive the strong coupling expansion of the expectation value of the circular BPS Wilson loop. In the $\mathcal{N}=4$ SYM theory one finds from (1.3) that summing up the leading large λ terms at each order in $1/N$ gives [6][6]

$$W_N^{\mathcal{N}=4}(\lambda) = N\sqrt{\frac{8}{\pi}}\,\lambda^{-3/4}\left(1+\frac{\sqrt{\lambda}}{8N}\right) e^{\sqrt{\lambda}+\frac{\lambda^{3/2}}{384N^2}+\dots} + \mathcal{O}\Big(e^{-\sqrt{\lambda}-\frac{\lambda^{3/2}}{384N^2}}\Big)\,, \tag{1.13}$$

where the second (nonperturbative) term is suppressed relative to the first (perturbative) term by an exponentially small factor $e^{-2\sqrt{\lambda}}$. Here dots stand again for subleading at large λ contributions.[7] In the $\mathcal{N}=2$ theory, the Wilson loop takes the form similar to (3.12)

$$W_N^{\mathcal{N}=2}(\lambda) = W_{\rm p}(N,\lambda) + W_{\rm np}(N,\lambda)\,, \tag{1.14}$$

where the second term is exponentially small at strong coupling compared to the first one. We show below that, up to redefinition of the coupling (1.10) and an extra rescaling $\lambda' \to \lambda'\frac{N+\frac{1}{2}}{N}$ combined with the shift $N \to N+\frac{1}{2}$, the perturbative term $W_{\rm p}(N,\lambda)$ coincides with the perturbative part of the $\mathcal{N}=4$ theory result (3.12)

$$W_{\rm p}(N,\lambda) = W^{\mathcal{N}=4}_{N+\frac{1}{2}}\left(\lambda'\frac{N+\frac{1}{2}}{N}\right). \tag{1.15}$$

Recalling the definition (1.10) and replacing $\lambda' = g^2_{\rm YM}N/(1+bg^2_{\rm YM})$, the expression on the right hand side of (1.15) can be obtained from $W_N^{\mathcal{N}=4}(\lambda = g^2_{\rm YM}N)$ simply by replacing $N \to N+\frac{1}{2}$ with $g^2_{\rm YM}$ left intact.

[5]Here the iteration of the leading $1/N$ (or "open-string loop") correction is resummed by shifting $\lambda \to \lambda'$ like in (1.9) while the resummation of the iteration of the leading $1/N^2$ (or "closed string loop") correction is represented by the factor $e^{-\frac{\lambda'^{3/2}}{384N^2}}$.

[6]The factor $\exp[\frac{\lambda^{3/2}}{96(2N)^2}] = \exp(\frac{\pi}{48}\frac{g_s^2}{T})$ which is the same as in the SU($2N$) case [35, 36] may be given a string theory interpretation as a sum of separated one-handle contributions to the disc partition function.

[7]The term $\frac{\lambda^{3/2}}{384N^2}$ in exponent of (1.13) comes from resummation of $\mathcal{O}\big((\lambda^{3/2}/N^2)^k\big)$ corrections to all orders in k. Using the exact expression (1.3) for $W_N^{\mathcal{N}=4}(\lambda)$ (valid for any N and λ) one can verify that this gives a good approximation to the exact result in the formal double-scaling limit $N\to\infty$, $\lambda\to\infty$ with $\lambda^{3/2}/N^2 =$ fixed.

JHEP01(2023)037

In contrast to the $\mathcal{N}=4$ expression (1.13), the nonperturbative correction in (1.14) is suppressed only by the factor $e^{-\sqrt{\lambda}}$ as compared to $W_{\rm p}(N,\lambda)$. Explicitly, it is given by (see section 3)

$$W_{\rm np}(N,\lambda) = -\frac{1}{\pi^2}\lambda' + \mathcal{O}(\sqrt{\lambda'})\,, \tag{1.16}$$

where all $1/N$ corrections are again absorbed into the coupling λ' defined in (1.10).

Massive deformation. One interesting generalization of the $\mathcal{N}=2$ Sp($2N$) model that we consider below is to introduce a mass for the fundamental hypermultiplets (cf. [18, 19]).[8] The dependence on this mass parameter is straightforward to include in the localization matrix model potential [1]. As we find below, the free energy $F_N^{\mathcal{N}=2}$ and the Wilson loop $W_N^{\mathcal{N}=2}$ have a remarkably simple dependence on the mass parameter m at strong coupling.

In particular, the perturbative part of the free energy (1.9) becomes (see (2.67))[9]

$$\begin{aligned} F_{\rm p}(N,\lambda,m) = &-\left(N+\frac{3}{4}+2m^2\right)\left(N+\frac{1}{4}+2m^2\right)\log\lambda' + C_N^{\mathcal{N}=2}(m^2) \\ &-\left[1+\frac{32}{3}m^2(1+m^2)\right]\frac{\pi^2}{2}\frac{N}{\lambda}\,. \end{aligned} \tag{1.17}$$

Notice that the coefficient of the logarithmic term in (1.17) can be obtained from the one in (1.9) by the shift $N \to N+2m^2$.

The dependence of the perturbative part of the Wilson loop expectation value on the mass m can be obtained from (1.15) also by the same shift $N \to N+2m^2$ (with $g'_{\rm YM}$ fixed)

$$W_{\rm p}(N,\lambda,m) = W^{\mathcal{N}=4}_{N+\frac{1}{2}+2m^2}\left(\lambda'\frac{N+\frac{1}{2}+2m^2}{N}\right). \tag{1.18}$$

In the localization matrix model description, the shift $N \to N+2m^2$ naturally follows from the structure of m-dependence of the interaction potential (see section 2.4 below). The meaning of this shift on the string theory side is an open question.

For the nonperturbative part of $F_N^{\mathcal{N}=2}$ and $W_N^{\mathcal{N}=2}$, the generalization to nonzero m amounts to inserting the factor of $\cosh(2\pi m)$ into the expressions in (1.12) and (1.16)

$$\begin{aligned} F_{\rm np}(N,y,m) &= \cosh(2\pi m)\, F_{\rm np}(N,y)\,, \\ W_{\rm np}(N,y,m) &= \cosh(2\pi m)\, W_{\rm np}(N,y)\,. \end{aligned} \tag{1.19}$$

Let us note that the knowledge of the free energy in the massive $\mathcal{N}=2$ Sp($2N$) model may be useful for computing (from its derivatives over λ and m) some integrated 4-point correlators by analogy with what was done in [38–43] using the localization results for $\mathcal{N}=2^*$ models (generalizations of $\mathcal{N}=4$ theories with massive adjoint hypermultiplets) for various gauge groups. To be able to obtain in this way interesting examples of correlators

[8] On the string side this should correspond to introducing a separation between D3 and D7+O7 branes in the directions transverse to the D7-branes (cf. a related setup discussed in [37]).

[9] Note that the last $\frac{N}{\lambda} = \frac{1}{g^2_{\rm YM}}$ term here or in (1.9) may be written also as $\frac{N}{\lambda'}$ at the same time redefining the constant $C_N^{\mathcal{N}=4}$ by a $\log 2$ term (cf. (1.11), (2.67)).

JHEP01(2023)037

one should actually generalize our computation to the case when mass is given to all $\mathcal{N}=2$ hypermultiplets. This is a major complication compared to the case we treat here because the simple single trace structure of the matrix model potential is then spoiled.[10]

String theory interpretation. The fact that the free energy (1.9) admits a natural expansion in the inverse string tension, i.e. in $\frac{1}{\sqrt{\lambda}}$, is already a strong check that this $\mathcal{N}=2$ model does indeed admit a dual string theory description. The string theory interpretation of the $\log\lambda$ term in $F_N^{\mathcal{N}=4}$ in (1.3) assumes a particular choice of an IR cutoff in the volume of the AdS_5 space which represents the leading (supergravity) term in the on-shell value of the string effective action [45]. The same should apply to the log term in (1.9) (see [6] and section 4 below).

The structure of the last term in (1.9), namely $-\frac{\pi^2}{2}\frac{N}{\lambda}=-\frac{\pi}{8}\frac{1}{g_{\rm s}}$, suggests that it should come from a disc or crosscup contribution.[11] It is surprising that once the $\mathcal{N}=2$ free energy is expressed in terms of λ', there are no further non-trivial contributions of higher order in expansion in small $g_{\rm s}\sim\frac{1}{N}$. This suggests an analogy with a non-renormalization of certain protected quantities (receiving contributions only from few leading orders in perturbation theory) as it happens in some models with extended supersymmetry.

The redefinition of λ^{-1} by $1/N$ term in (1.10) may be related to the issue of how one actually compares gauge theory to dual string theory, e.g., which is the proper definition of string tension in terms of the 't Hooft coupling λ (and N). The fact that gauge-theory answer (1.9) takes a very simple form when expressed in terms of λ' rather than λ strongly suggests that it is $\frac{\sqrt{\lambda'}}{2\pi}$ that should be identified here with the string tension. The redefinition $\sqrt{\lambda}\to\sqrt{\lambda'}=\sqrt{\lambda}/\sqrt{1+b\frac{\lambda}{N}}$ may be representing a resummed contribution of (some of) the open-string sector (disc/crosscup) corrections.

A test of this would require a direct computation of the free energy on the string theory side showing the absence of all higher order $\mathcal{O}(g_{\rm s}^n)$ corrections beyond the ones given in (1.9). While this appears beyond our reach at the moment, in section 4 we will discuss the string theory derivation of the subleading in N coefficients of the $\log\lambda$ terms in the $\mathcal{N}=4$ (1.3) and the $\mathcal{N}=2$ (1.9) $\mathrm{Sp}(2N)$ models.

In the $\mathcal{N}=4$ SYM theory with a gauge group G, the coefficient of the $\log\lambda$ term in the free energy $F^{\mathcal{N}=4}$ obtained from the localization matrix model is proportional to the conformal anomaly a-coefficient given by $\frac{1}{4}\dim G$ [6]. For example, in the $\mathrm{SU}(N)$ case the latter is equal to $\frac{1}{4}(N^2-1)$. While the N^2 term comes from the classical (supergravity) part of string action evaluated on the $\mathrm{AdS}_5\times S^5$ vacuum (with all α' corrections vanishing),

[10] In particular, ref. [41] discussed integrated correlators in $\mathcal{N}=4$ $\mathrm{Sp}(2N)$ theory given by derivatives of the free energy in the $\mathcal{N}=2^*$ $\mathrm{Sp}(2N)$ model. Let us note also that linearised discrete Toda-like relations that appeared in the finite $g_{\rm YM}$ discussions in [40, 42] suggest possible connection to (1.6) and a generalization of our discussion beyond the 't Hooft expansion (i.e. including instanton effects). We thank S. Chester for a discussion of this connection.

[11] A naive expectation is that the string partition function should scale as the volume of AdS_5 and should then be always proportional to $\log\lambda$. This is indeed so in the maximally supersymmetric $\mathrm{AdS}_5\times S^5$ case (where the 1-loop or torus string correction explains the shift of N^2 term in the coefficient of the $\log\lambda$ term, cf. section 4). However, this expectation should somehow fail in the less supersymmetric cases involving singular orbifold/orientifold projections of $\mathrm{AdS}_5\times S^5$.

JHEP01(2023)037

the additional (-1) term originates from the one-loop (torus) string correction which turns out to be due to the "massless" (supergravity) modes only [46].

As we demonstrate in section 4.1, a similar argument explains the value of the subleading term in the a-anomaly or the coefficient of $\log\lambda$ in (1.3) in the Sp($2N$) $\mathcal{N}=4$ model dual to type IIB string theory on $AdS_5\times\mathbb{RP}^5$ [47]. Here $\dim G = N(2N+1) = 2(N+\frac{1}{4})^2-\frac{1}{8}$. The shift $N\to N+\frac{1}{4}$ may be interpreted as being due to the change of the D3-brane charge in the presence of O3-plane [23, 25]. Then the remaining constant shift $(-\frac{1}{8})$ comes again from the 1-loop contribution of the supergravity modes only.

In section 4.2 we present an argument that explains the origin of the order $O(N)$ term in the coefficient $N^2+N+\frac{3}{16}$ of the $\log\lambda$ term in the free energy (1.9) of the $\mathcal{N}=2$ Sp($2N$) model. However, it remains a challenge to reproduce its precise coefficient due to currently insufficient knowledge of RR 5-form dependent terms in the D7-brane action in type IIB background.

The rest of the paper is organized as follows. In section 2, we analyze the free energy of the $\mathcal{N}=2$ Sp($2N$) theory. We use localization matrix model to show that it satisfies the Toda lattice equation. We exploit this equation to derive the strong coupling expansion of the free energy and study its properties. In section 3, we repeat the same analysis for the circular BPS Wilson loop. The dual string theory interpretation of the strong-coupling expansions derived on the gauge theory side is discussed in section 4.

2 Partition function of $\mathcal{N}=2$ Sp($2N$) theory

In this section, we derive the large N expansion of the partition function of the $\mathcal{N}=2$ superconformal theory with Sp($2N$) gauge group defined on four-sphere. We also find its generalization to the case of non-zero mass of fundamental hypermultiplets.

2.1 Matrix model representation

For a generic $\mathcal{N}=2$ Sp($2N$) theory with hypermultiplets in the fundamental, adjoint, and antisymmetric representations, the localization approach can be applied to express the partition function on a unit-radius four-sphere as a matrix integral [1]

$$Z_{\mathfrak{sp}(2N)} = e^{-F_N^{\mathcal{N}=2}} = \int \mathcal{D}X\, e^{-\frac{8\pi^2}{g_{\rm YM}^2}\operatorname{tr} X^2} |Z_{\text{1-loop}}(X)|^2 |Z_{\text{inst}}(X)|^2 \,. \tag{2.1}$$

Here the integration goes over $2N\times 2N$ matrices X belonging to the Lie algebra $\mathfrak{sp}(2N)$. They describe zero modes of a scalar field (from $\mathcal{N}=2$ vector multiplet) on the sphere and have a general form

$$X = \begin{bmatrix} A & B \\ C & -A^t \end{bmatrix} = \sum_{a=1}^{2N^2+N} T^a X^a \,, \tag{2.2}$$

where A^t denotes a transposed matrix, $B^t=B$ and $C^t=C$. The second relation in (2.2) defines a decomposition of X over the generators of the fundamental representation of Sp($2N$) normalized as $\operatorname{tr}(T^aT^b)=\frac{1}{2}\delta^{ab}$. The integration measure in (2.1) is then $\mathcal{D}X = \prod_{a=1}^{2N^2+N} dX^a$.

JHEP01(2023)037

In a standard manner, the matrix integral in (2.1) can be reduced to an integral over the eigenvalues of X. By virtue of (2.2), they take the form $\{\pm x_1, \ldots, \pm x_N\}$ leading to

$$\mathcal{D}X = \frac{1}{N!}\prod_{n=1}^{N} dx_n\, x_n^2 \prod_{1\le n<m\le N} (x_n^2 - x_m^2)^2 \,. \tag{2.3}$$

The functions $Z_{\text{1-loop}}(X)$ and $Z_{\text{inst}}(X)$ in (2.1) describe the one-loop perturbative correction and the contribution of instantons, respectively. The latter runs in powers of $\exp(-\frac{8\pi^2 N}{\lambda})$ and is exponentially suppressed at large N and fixed 't Hooft coupling $\lambda = g_{\text{YM}}^2 N$. In what follows we neglect the instanton contribution and put $Z_{\text{inst}}(X) = 1$.[12]

In the $\mathcal{N} = 2$ theory with a number of hypermultiplets in the fundamental (n_F), adjoint (n_{adj}), and antisymmetric (n_A) representations, the one-loop perturbative function $Z_{\text{1-loop}}(X)$ has the following expression in terms of the eigenvalues of the matrix X [11]

$$|Z_{\text{1-loop}}|^2 = \frac{\prod_{n<m}^{N}[H(x_{nm}^+)]^2\,[H(x_{nm}^-)]^2\prod_{n=1}^{N}[H(2x_n)]^2}{\prod_{n<m}^{N}[H(x_{nm}^+)]^{2n_{\text{adj}}+2n_{\text{A}}}\,[H(x_{nm}^-)]^{2n_{\text{adj}}+2n_{\text{A}}}\,\prod_{n=1}^{N}[H(2x_n)]^{2n_{\text{adj}}}[H(x_n)]^{2n_{\text{F}}}} \tag{2.4}$$

Here $x_{nm}^{\pm} = x_n \pm x_m$ and $H(x)$ is expressed in terms of Barnes G-functions

$$\begin{aligned} H(x) &\equiv \prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2}\right)^k e^{-\frac{x^2}{k}} \\ &= e^{-(1+\gamma)x^2} G(1+ix)G(1-ix) = \exp\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}\zeta_{2n+1}x^{2(n+1)}\right], \end{aligned} \tag{2.5}$$

where $\zeta_{2n+1} \equiv \zeta(2n+1)$ is a Riemann zeta function value.

In this paper we shall consider the special $\mathcal{N} = 2$ superconformal theory with the field content

$$n_{\text{adj}} = 0, \qquad n_{\text{F}} = 4, \qquad n_{\text{A}} = 1\,, \tag{2.6}$$

for which the beta-function vanishes. Then (2.4) simplifies to[13]

$$|Z_{\text{1-loop}}(X)|^2 \equiv e^{-S_{\text{int}}(X)} = \prod_{n=1}^{N}\frac{[H(2x_n)]^2}{[H(x_n)]^8} = \prod_{n=1}^{N}\prod_{k=1}^{\infty}\frac{(1+\frac{4x_n^2}{k^2})^{2k}}{(1+\frac{x_n^2}{k^2})^{8k}}\,. \tag{2.7}$$

Substituting this expression into (2.1) and setting $Z_{\text{inst}} = 1$, we obtain an integral representation for the partition function of this $\mathcal{N} = 2$ Sp($2N$) model. We will use it to derive the $1/N$ expansion of the free energy $F_N^{\mathcal{N}=2}$.

In general, the partition function of a gauge theory on the sphere suffers contains ultraviolet divergences and requires regularization. In particular, the form in which $|Z_{\text{1-loop}}(X)|^2$

[12]The instanton contribution is important, however, when addressing the large N expansion at fixed g_{YM}^2. Notice also that the absence of instanton corrections at large N implicitly assumes that the integration over instanton moduli does not spoil the instanton action exponential suppression in the $g_{\text{YM}} \to 0$ limit. This assumption was explicitly checked in $\mathcal{N} = 2$ SQCD in [48].

[13]Note that the exponential factors $e^{-\frac{x^2}{k}}$ entering the definition (2.5) of $H(x)$ (and ensuring the convergence of the infinite product there) cancel out in the ratio of $H-$functions in (2.7).

JHEP01(2023)037

directly appears from gauge theory calculation (before the regularization introduced in [1] leading to finite $H(x)$ factors) is

$$|Z^{(\text{bare})}_{\text{1-loop}}(X)|^2 \sim \mathrm{r}^{1/6} \prod_{n=1}^{N} \prod_{k=1}^{\infty} \frac{\left(k^2 \mathrm{r}^{-2} + 4x_n^2\right)^{2k}}{\left(k^2 \mathrm{r}^{-2} + x_n^2\right)^{8k}} \,, \tag{2.8}$$

where we restored the dependence on the radius r of S^4 (which enters also the Gaussian action in (2.1) as $\frac{8\pi^2 \mathrm{r}^2}{g^2_{\text{YM}}} \mathrm{tr}\, X^2$). The factor $\mathrm{r}^{1/6}$ stands for the contribution of the "massless" multiplets (not depending on x_n) which should be taken into account in order for the r-dependence of the free energy $F_N^{\mathcal{N}=2} = -\log Z_{\mathfrak{sp}(2N)} = 4\mathrm{a} \log \mathrm{r} + \dots$ to be consistent with the value of the a-coefficient of the conformal anomaly $\mathrm{a} = \frac{1}{2}N^2 + \frac{1}{2}N - \frac{1}{24}$ of the $\mathcal{N} = 2$ Sp($2N$) theory (see appendix A in [6]).[14]

The relations (2.1) and (2.7) can be generalized to a more complicated case of the $\mathcal{N} = 2$ Sp($2N$) model with the 4 fundamental hypermultiplets having a nonzero mass m. Introducing this mass parameter amounts to the replacement the function $|Z_{\text{1-loop}}|^2$ in (2.1) by [1][15]

$$|Z_{\text{1-loop}}(X, m)|^2 = \prod_{n=1}^{N} \frac{[H(2x_n)]^2}{[H(x_n + m)\ H(x_n - m)]^4} \,. \tag{2.9}$$

Note that for $m \neq 0$ the exponential convergence factors in (2.5) that ensure the finiteness of each of the functions H no longer cancel automatically in (2.9) giving

$$\prod_{n=1}^{N} \exp \sum_{k=1}^{\infty} \left[2\frac{(2x_n)^2}{k} - 4\frac{(x_n + m)^2}{k} - 4\frac{(x_n - m)^2}{k} \right] = \exp\Big(- 8Nm^2 \sum_{k=1}^{\infty} \frac{1}{k} \Big) \,. \tag{2.10}$$

The unregularized (bare) expression $|Z^{(\text{bare})}_{\text{1-loop}}(X, m)|^2$ as it originates from the calculation of one-loop determinants in gauge theory on S^4 has the form like in (2.8). Namely, it does not have these exponential factors and thus contains the logarithmic divergence (2.10) proportional to Nm^2.

Indeed, such logarithmic divergence appears in general in the partition function of a massive hypermultiplet defined in *curved* 4-space.[16] The finite expression (2.9) obtained following [1], i.e. containing the factor (2.10), corresponds to a special choice of the UV subtraction scheme. That means, in particular, that the coefficient of the Nm^2 term in

[14] The factor of r coming from the infinite product of "massive" modes in (2.8) is $[\mathrm{r}^{12\zeta(-1)}]^N = \mathrm{r}^{-N}$. After the extraction of this factor the dependence of the integrand in (2.1) on r is only through $\mathrm{r}x_n$ and thus the extra r dependent factor comes just from the measure, i.e. is the same as in the Sp($2N$) $\mathcal{N} = 4$ SYM (Gaussian matrix model) case: $Z_{\mathcal{N}=4} \sim \mathrm{r}^{-\dim[Sp(2N)]} = \mathrm{r}^{-N(2N+1)}$. As a result, the dependences of Z on λ and on r are a priori correlated only in the Gaussian model, i.e. in the $\mathcal{N} = 4$ SYM case.

[15] Here we set again the radius r = 1; in general, the dependence on m is through the dimensionless combination $m\mathrm{r}$.

[16] One finds (using, e.g., proper-time cutoff) that $F_\infty = -\frac{1}{3(4\pi)^2} \int d^4x \sqrt{g}\, R\, m^2 \log \Lambda_{\text{UV}}$ where this divergent contribution comes only from the fermions (assuming that the scalars are conformally coupled, i.e. with $\frac{1}{6}R\phi^2$ term added). Note that the m^4 logarithmic (and all power) divergences cancel out due to supersymmetry as in flat space. In particular, in the case of S^4 of radius r (with $R = 12\mathrm{r}^{-2}$, $\mathrm{vol}(S^4) = \frac{8}{3}\pi^2\mathrm{r}^4$), this gives $F_\infty = -\frac{2}{3}(m\mathrm{r})^2 \log \Lambda_{\text{UV}}$.

JHEP01(2023)037

the resulting free energy is, in general, scheme-dependent (cf. also a discussion of scheme dependence of F in $\mathcal{N}=2^*$ SU(N) theory in [39]).[17]

To summarize, for $m \neq 0$ the partition function of the $\mathcal{N}=2$ Sp($2N$) model is

$$Z_{\mathfrak{sp}(2N)}(m) = e^{-F_N^{\mathcal{N}=2}(\lambda,m)} = \int \mathcal{D}X\, e^{-\frac{8N\pi^2}{\lambda}\mathrm{tr}\, X^2 - S_{\rm int}(X,m)}\,, \tag{2.11}$$

with the integration measure defined in (2.3) and the interaction action (expanded in mass parameter m) given by

$$S_{\rm int}(X,m) = -\log|Z_{1-\rm loop}(X,m)|^2 = S_{\rm int}^{(0)} + m^2 S_{\rm int}^{(1)} + m^4 S_{\rm int}^{(2)} + O(m^6)\,. \tag{2.12}$$

Taking into account (2.9) and (2.5) we find that

$$S_{\rm int}^{(0)} = 4\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k+1}\zeta_{2k+1}\,(4^k-1)\ \mathrm{tr}\, X^{2(k+1)}, \tag{2.13}$$

$$S_{\rm int}^{(i)} = 4\sum_{k=1}^{\infty} \frac{(-1)^{k}}{k+1}\zeta_{2k+1}\binom{2k+2}{2i}\ \mathrm{tr}\, X^{2(k+1-i)}\,, \qquad i \geq 1\,, \tag{2.14}$$

where $\mathrm{tr}\, X^{2(k+1)} = 2\sum_{n=1}^{N} x_n^{2(k+1)}$.

2.2 Large N expansion of free energy

We observe that in the planar limit, for $N \to \infty$ with $\lambda =$ fixed, the potential in the matrix integral (2.11) is dominated by a Gaussian term. Because the contribution of the interaction term $S_{\rm int}(X)$ to the partition function (2.11) is suppressed by a factor of $1/N$, the free energy of the $\mathcal{N}=2$ model $F_N^{\mathcal{N}=2}(\lambda,m) = -\log Z_{\mathfrak{sp}(2N)}(m)$ coincides in the planar limit with the free energy of the $\mathcal{N}=4$ SYM theory

$$e^{-F_N^{\mathcal{N}=4}(\lambda)} = \int \mathcal{D}X\, e^{-\frac{8N\pi^2}{\lambda}\mathrm{tr}\, X^2}\,. \tag{2.15}$$

This suggests to define the free energy difference

$$\Delta F(\lambda; N, m) = F_N^{\mathcal{N}=2}(\lambda,m) - F_N^{\mathcal{N}=4}(\lambda)\,. \tag{2.16}$$

Its large N expansion starts with order N term and runs in powers of $1/N$

$$\Delta F(\lambda; N, m) = N\mathrm{F}_1(\lambda,m) + \mathrm{F}_2(\lambda,m) + \frac{1}{N}\mathrm{F}_3(\lambda,m) + \mathcal{O}\left(\frac{1}{N^2}\right)\,. \tag{2.17}$$

Here the corresponding coefficient functions $F_n(\lambda,m)$ are given by cumulants of $S_{\rm int}(X;m)$ in the Gaussian matrix model. For example, from (2.16) the leading term in (2.17) is given by

$$\mathrm{F}_1(\lambda,m) = \lim_{N\to\infty}\frac{1}{N}\langle S_{\rm int}(X;m)\rangle = \mathrm{F}_1(\lambda) + m^2\,\mathrm{F}_1^{(1)}(\lambda) + m^4\,\mathrm{F}_1^{(2)}(\lambda) + \dots\,, \tag{2.18}$$

[17] The same applies to a constant (N-dependent) term in the free energy: for an $\mathcal{N}=2$ gauge theory defined on S^4 the free energy contains also the UV divergent term related to the conformal anomaly, $F_\infty = 4\mathrm{a}\log\Lambda_{\rm UV}$ (for a generic curved metric a is replaced by a combination of integrals of the two curvature contractions with the a- and c-anomaly coefficients). That means that comparing to string theory one would need to choose a particular IR regularization scheme that should correspond to a particular UV regularization on the gauge theory side.

JHEP01(2023)037

where $\mathrm{F}_1(\lambda) \equiv \mathrm{F}_1(\lambda, 0)$ corresponds to the massless theory and the angular brackets denote an average in the Gaussian matrix model.

Replacing $S_{\rm int}(X; m)$ in (2.18) with its small m expansion (2.14), we get an expression for $\mathrm{F}_1^{(i)}(\lambda)$ as an infinite sum of terms proportional to $\langle \mathrm{tr}\, X^{2(k+1-i)} \rangle$.[18] These expectation values can be computed in the Gaussian $\mathrm{Sp}(2N)$ model in the large N limit using the technique developed in [6]. Replacing the Riemann function value ζ_{2n+1} in (2.14) with its integral representation, one can resum the series in (2.14) to obtain the leading term of the small m expansion (2.18) as [6]

$$\mathrm{F}_1(\lambda) = \frac{4}{\sqrt{\lambda}} \int_0^\infty dt \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2} \frac{3t\sqrt{\lambda} - 8J_1(t\sqrt{\lambda}) + J_1(2t\sqrt{\lambda})}{t^2}\,, \tag{2.19}$$

where J_1 is a Bessel function.

For $m \neq 0$, the expansion in (2.18) can be summed up in a similar manner to all orders in m^2 to give

$$\mathrm{F}_1(\lambda, m) = \mathrm{F}_1(\lambda) + \frac{64}{\sqrt{\lambda}} \int_0^\infty dt\, \frac{e^{2\pi t}}{(e^{2\pi t}-1)^2} \left[\frac{J_1(t\sqrt{\lambda})}{t^2} \sin^2(m\pi t) - \frac{1}{2}\sqrt{\lambda}\, m^2 \pi^2 t \right]. \tag{2.20}$$

The relations (2.19) and (2.20) define the leading large N correction to the difference free energy (2.17) in the massless and massive $\mathcal{N} = 2$ theory, respectively. They are valid for an arbitrary 't Hooft coupling λ. At weak coupling, it is straightforward to expand $\mathrm{F}_1(\lambda)$ and $\mathrm{F}_1(\lambda, m)$ in powers of λ.

Here we will concentrate on studying the difference free energy (2.16) at strong coupling. At strong coupling, the expansion of the functions $\mathrm{F}_k(\lambda)$ in (2.17) runs in powers of the two parameters $\frac{1}{\sqrt{\lambda}}$ and $e^{-\sqrt{\lambda}}$. In particular, $\mathrm{F}_1(\lambda, m)$ can be split into the sum of the two terms

$$\mathrm{F}_1(\lambda, m) = \mathrm{F}_{1,\mathrm{p}}(\lambda, m) + \mathrm{F}_{1,\mathrm{np}}(\lambda, m)\,. \tag{2.21}$$

The first ("perturbative") term is given by a series in $\frac{1}{\sqrt{\lambda}}$ and the second ("nonperturbative") one is a sum of terms containing powers of exponentially small factors $e^{-\sqrt{\lambda}}$.

For $m = 0$ one finds from (2.18) and (2.19) (see [6])

$$\mathrm{F}_1(\lambda) \equiv \mathrm{F}_1(\lambda, 0) = \mathrm{F}_{1,\mathrm{p}}(\lambda) + \mathrm{F}_{1,\mathrm{np}}(\lambda)\,, \tag{2.22}$$

where

$$\mathrm{F}_{1,\mathrm{p}}(\lambda) = \frac{\log 2}{2\pi^2}\,\lambda - \frac{1}{2}\log\lambda + \left(\log\pi + \frac{7}{3}\log 2 + \frac{3}{2} - 12\log \mathsf{A}\right) - \frac{\pi^2}{2\lambda}\,, \tag{2.23}$$

$$\mathrm{F}_{1,\mathrm{np}}(\lambda) = \frac{8\sqrt{2}}{\pi^{3/2}} \lambda^{-1/4} e^{-\sqrt{\lambda}} \left(1 + \frac{23}{8\sqrt{\lambda}} + \frac{153}{128\lambda} - \frac{435}{1024\lambda^{3/2}} + \cdots\right) + \mathcal{O}(e^{-3\sqrt{\lambda}})\,, \tag{2.24}$$

with A being the Glaisher's constant. Remarkably, the strong coupling expansion of $F_{1,\mathrm{p}}(\lambda)$ contains only a finite number of terms and terminates at order $O(1/\lambda)$.

[18] Higher order terms in the $1/N$ expansion, cf. (2.17), involve connected correlators (cumulants) of product of traces $\langle \mathrm{tr}\, X^{k_1} \cdots \mathrm{tr}\, X^{k_n} \rangle$.

JHEP01(2023)037

In the mass-deformed $\mathcal{N}=2$ theory we found that the two terms in (2.21) are

$$\mathrm{F}_{1,\mathrm{p}}(\lambda,m)=\mathrm{F}_{1,\mathrm{p}}(\lambda)-\Big[4\log\lambda+4(1+2\gamma_{\mathrm{E}}-2\log(4\pi))+\frac{16\pi^2}{3\lambda}\Big]m^2-\frac{16\pi^2}{3\lambda}m^4\,, \tag{2.25}$$

$$\mathrm{F}_{1,\mathrm{np}}(\lambda,m)=\frac{8\sqrt{2}}{\pi^{3/2}}\,\lambda^{-1/4}\,e^{-\sqrt{\lambda}}\,\cosh(2\pi m)+\dots\,, \tag{2.26}$$

where γ_{E} is the Euler's constant[19] and dots in (2.26) denote terms suppressed by $1/\sqrt{\lambda}$ or extra $e^{-\sqrt{\lambda}}$ factors as in (2.24). Note that the small m expansion of $\mathrm{F}_{1,\mathrm{p}}(\lambda,m)$ terminates at order m^4 whereas the dependence of $\mathrm{F}_{1,\mathrm{np}}(\lambda,m)$ on m enters through the function $\cosh(2\pi m)$.[20]

The same techniques can be applied to compute subleading terms in the large N expansion of (2.17), but calculations become cumbersome as one goes to higher orders in $1/N$. It turns out, however, that the resulting expressions for the functions $\mathrm{F}_{n\geq 2}(\lambda,m)$ can all be effectively expressed in terms of the leading large N function $\mathrm{F}_1(\lambda,m)$. In particular, for $m=0$, one finds [6]

$$\begin{aligned}(\mathrm{F}_2(\lambda))'&=\frac{1}{4}(\lambda\mathrm{F}_1(\lambda))''-\frac{1}{4}\lambda\big[(\lambda\,\mathrm{F}_1(\lambda))''\big]^2\,,\\ \mathrm{F}_3(\lambda)&=\frac{1}{48}\lambda^2(\lambda\mathrm{F}_1(\lambda))''''-\frac{1}{16}\lambda^2\big[(\lambda\mathrm{F}_1(\lambda))''\big]^2+\frac{1}{24}\lambda^3\big[(\lambda\mathrm{F}_1(\lambda))''\big]^3\,,\quad \text{etc.}\,,\end{aligned} \tag{2.27}$$

where prime denotes a derivative with respect to the 't Hooft coupling λ. The origin of these relations will be explained in the next subsection.

2.3 Free energy from Toda lattice equation

As discussed above, the partition function of the $\mathcal{N}=2$ gauge theory under consideration can be represented as the partition function of the $\mathrm{Sp}(2N)$ matrix model (2.11). It is well-known that for a generic matrix model with the potential given by a sum of single trace terms with arbitrary coefficients, $V(X)=\sum_k t_k\,\mathrm{tr}\,X^k$, its partition function satisfies nontrivial relations that define an integrable Toda-like hierarchy. Such relations have been studied in past in the context of a unitary matrix model [49–52]. Similar relations apply also to the $\mathfrak{sp}(2N)$ matrix model with a generic single-trace potential.

The matrix integral (2.11) corresponds to a particular choice of the coefficients in a generic potential $V(X)$ of the $\mathfrak{sp}(2N)$ matrix model. Namely, the coefficient of the Gaussian term $\mathrm{tr}\,X^2$ is proportional to the effective coupling constant

$$y=\frac{(4\pi)^2}{g_{\mathrm{YM}}^2}=\frac{(4\pi)^2N}{\lambda}\,, \tag{2.28}$$

whereas the coefficients in front of the other single trace terms $\mathrm{tr}\,X^{2k}$ (with $k\geq 2$) are uniquely fixed by the localization representation, see eqs. (2.13) and (2.14).

[19] As was discussed above (cf. (2.10)), the constant coefficient of the m^2 term in (2.25) is renormalization scheme dependent.

[20] Equivalently, we get $\mathrm{F}_{1,\mathrm{np}}(\lambda,m)\sim\frac{1}{2}(e^{-\sqrt{\lambda}+2\pi m}+e^{-\sqrt{\lambda}-2\pi m})+\dots$ hinting at possible world-sheet instanton interpretation.

JHEP01(2023)037

To simplify the notation, let us denote the partition function (2.11) as $Z_N(y) \equiv Z_{\mathfrak{sp}(2N)}$. In general, it is a function of N and the inverse coupling constant (2.28), with the dependence on m^2 tacitly assumed. Repeating the analysis of [49–51], we find that it satisfies the following Toda lattice equation

$$\partial_y^2 \log Z_N(y) = \frac{Z_{N+1}(y)\, Z_{N-1}(y)}{Z_N^2(y)}\,. \tag{2.29}$$

The expression on the right-hand side involves the partition functions defined for the same value of y and with N shifted by ± 1.[21]

The equation (2.29) is supplemented by the boundary conditions

$$Z_{N=-1}(y) = 0\,, \qquad Z_{N=0}(y) = 1\,. \tag{2.30}$$

$Z_{N=1}(y)$ gives the partition function of the $\mathcal{N} = 2$ Sp(2) theory

$$Z_{N=1}(y) = \int_{-\infty}^{\infty} dx\, x^2\, e^{-y\,x^2 - S_{\rm int}(X,m)}\,, \tag{2.31}$$

where 2×2 matrix X has the eigenvalues $\{x, -x\}$. Here $S_{\rm int}(X, m)$ is given by eqs. (2.12) – (2.14) with $\operatorname{tr} X^{2k} = 2x^{2k}$.

Applying the Toda equation (2.29) recursively, we can express the partition function $Z_N(y)$ in terms of only *one* function $Z_1(y)$ and its derivatives. The general solution is [53]

$$Z_N(y) = \det \left\| \partial_y^{j+k} Z_1(y) \right\|_{0 \le j,k \le N-1} = \det \begin{bmatrix} Z_1 & Z_1' & \dots & Z_1^{(N-1)} \\ Z_1' & Z_1'' & \dots & Z_1^{(N)} \\ \vdots & \vdots & & \vdots \\ Z_1^{(N-1)} & Z_1^{(N)} & \dots & Z_1^{(2N-2)} \end{bmatrix}, \tag{2.32}$$

where $Z_1^{(k)} \equiv \partial_y^k Z_1(y)$.

It should be noted that the Toda equation (2.29) does not depend on particular values of the coefficients in the interaction term $S_{\rm int}(X, m)$. For example, it should hold both in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ Sp($2N$) models. In the latter case $S_{\rm int}(X, m) = 0$ and the partition function can be computed directly from the matrix integral for an arbitrary N using the orthogonal polynomial technique [54]

$$\begin{aligned} Z_N^{\mathcal{N}=4}(y) &= e^{-C_N^{\mathcal{N}=4}}\, y^{-\frac{1}{2}N(2N+1)}\,, \\ C_N^{\mathcal{N}=4} &= \log \frac{G(N+1)\, G(N+\frac{3}{2})}{G(\frac{3}{2})}\,, \end{aligned} \tag{2.33}$$

where G is Barnes function. One can verify that this $Z_N^{\mathcal{N}=4}(y)$ indeed satisfies (2.29) and (2.32). The free energy in the $\mathcal{N} = 4$ Sp($2N$) theory is thus

$$\begin{aligned} F_N^{\mathcal{N}=4}(\lambda) &\equiv -\log Z_N^{\mathcal{N}=4}(y) \\ &= \frac{1}{2} N(2N+1) \log y + C_N^{\mathcal{N}=4} \\ &= -\frac{1}{2} N(2N+1) \log \lambda + \frac{1}{2} N(2N+1) \log[(4\pi)^2 N] + C_N^{\mathcal{N}=4}\,, \end{aligned} \tag{2.34}$$

where in the last relation we replaced y with λ according to its definition in (2.28).

[21] Note that using $\lambda = \frac{(4\pi)^2 N}{y}$ instead of y as an argument of Z_N would make this equation rather cumbersome.

JHEP01(2023)037

In the $\mathcal{N}=2$ model we have from (2.16)

$$Z_N^{\mathcal{N}=2}(y) = \exp\Big(-F_N^{\mathcal{N}=4}(\lambda) - \Delta F(\lambda; N, m)\Big)\Big|_{\lambda=\frac{(4\pi)^2 N}{y}}\,, \tag{2.35}$$

where $F_N^{\mathcal{N}=4}(\lambda)$ is given by (2.34). Substitution of this expression into (2.29) leads to a nontrivial equation for the function $\Delta F(\lambda; N, m) = \Delta F(\frac{(4\pi)^2 N}{y}; N, m)$. Using the general expression (2.17) for $\Delta F(\lambda; N, m)$ and expanding both sides of (2.29) at large N with fixed λ we get to the leading order in $1/N$

$$\begin{aligned} \mathrm{F}_2'(\lambda, m) &= \frac{1}{4}\mathcal{F}(\lambda, m)\big[1 - \lambda\mathcal{F}(\lambda, m)\big]\,, \\ \mathcal{F}(\lambda, m) &\equiv \partial_\lambda^2(\lambda \mathrm{F}_1(\lambda, m))\,, \end{aligned} \tag{2.36}$$

where $\mathrm{F}_1(\lambda, m)$ is given by (2.20). At the next order in $1/N$, from (2.36) we find the following differential equation for F_3

$$\mathrm{F}_3(\lambda, m) - 2\lambda \mathrm{F}_3'(\lambda, m) = \frac{3}{16}\lambda^2\mathcal{F}^2 - \frac{5}{24}\lambda^3\mathcal{F}^3 - \frac{1}{16}\lambda^2\mathcal{F}' + \frac{1}{4}\lambda^3\mathcal{F}^2\mathcal{F}' - \frac{1}{24}\lambda^3\mathcal{F}''\,, \tag{2.37}$$

where $\mathcal{F}' = \partial_\lambda \mathcal{F}(\lambda, m)$, etc. Its general solution is

$$\mathrm{F}_3(\lambda, m) = c\sqrt{\lambda} + \frac{1}{48}\lambda^2\Big(-3\mathcal{F}^2(\lambda, m) + 2\lambda\mathcal{F}^3(\lambda, m) + \mathcal{F}'(\lambda, m)\Big)\,, \tag{2.38}$$

where c is an integration constant. The value of c can be found using the weak-coupling expansion of $\mathrm{F}_3(\lambda)$. As this expansion runs in powers of λ, it can not contain an $\mathcal{O}(\sqrt{\lambda})$ term. Using (2.38) this then leads to the conclusion that $c = 0$.

To summarize, we find that

$$\begin{aligned} \mathrm{F}_2(\lambda, m) &= \frac{1}{4}\int_0^\lambda d\lambda\, \mathcal{F}(\lambda, m)\big[1 - \lambda\,\mathcal{F}(\lambda, m)\big]\,, \\ \mathrm{F}_3(\lambda, m) &= \frac{1}{48}\lambda^2\Big[-3\mathcal{F}^2(\lambda, m) + 2\lambda\,\mathcal{F}^3(\lambda, m) + \mathcal{F}'(\lambda, m)\Big]\,, \end{aligned} \tag{2.39}$$

where $\mathcal{F}(\lambda, m)$ is given by (2.36).

Expanding both sides of (2.29) to higher orders in $1/N$ we can express all the subleading coefficient functions in (2.17) in terms of $\mathcal{F}(\lambda, m)$. In distinction to $\mathrm{F}_2(\lambda, m)$, the functions $\mathrm{F}_k(\lambda, m)$ with $k \geq 3$ depend locally on $\mathcal{F}(\lambda, m)$ and a finite number of its derivatives. These relations hold for an arbitrary mass parameter m (for $m = 0$ they coincide with the relations in (2.27)). This is again because of the universality of the Toda equation (2.29) that applies to any single-trace potential. Thus the information about the mass parameter enters only through *one* function $\mathrm{F}_1(\lambda, m)$ given by (2.20).

The expressions (2.39), etc., for $\mathrm{F}_n(\lambda, m)$ (with $n \geq 2$) are valid for an arbitrary value of 't Hooft coupling. Taking into account (2.25) and (2.26), we can then systematically work out the strong coupling expansion of the free energy (2.17) to any order in $1/N$. From (2.36), (2.25) and (2.26) we find that the strong coupling expansion of $\mathcal{F}(\lambda, m)$ is given by

$$\begin{aligned} \mathcal{F}(\lambda, m) &= \mathcal{F}_{\mathrm{p}}(\lambda, m) + \mathcal{F}_{\mathrm{np}}(\lambda, m) \\ &= \left(\frac{\log 2}{\pi^2} - \frac{\frac{1}{2} + 4m^2}{\lambda}\right) + \frac{2\sqrt{2}}{\pi^{3/2}}\lambda^{-1/4}e^{-\sqrt{\lambda}}\cosh(2\pi m)\left(1 + \mathcal{O}(\lambda^{-1/2})\right). \end{aligned} \tag{2.40}$$

JHEP01(2023)037

Substituting this expression into (2.39) we get

$$\mathrm{F}_2(\lambda,m) = -\frac{1}{2}b^2\lambda^2 + \left(4m^2+1\right)b\lambda - \left(\frac{3}{4}+2m^2\right)\left(\frac{1}{4}+2m^2\right)\log\lambda + \mathcal{O}(\lambda^0) + \frac{4\sqrt{2}}{\pi^{3/2}}\,b\,\lambda^{5/4}e^{-\sqrt{\lambda}}\cosh(2\pi m)\left(1+\mathcal{O}(\lambda^{-1/2})\right), \tag{2.41}$$

$$\mathrm{F}_3(\lambda,m) = \frac{1}{3}b^3\lambda^3 - \frac{1}{2}\left(4m^2+1\right)b^2\lambda^2 + \left(\frac{3}{4}+2m^2\right)\left(\frac{1}{4}+2m^2\right)b\lambda + \mathcal{O}(\lambda^0), + \frac{4\sqrt{2}}{\pi^{3/2}}\,b^2\,\lambda^{11/4}e^{-\sqrt{\lambda}}\cosh(2\pi m)\left(\frac{1}{4}+\mathcal{O}(\lambda^{-1/2})\right), \tag{2.42}$$

where $b = \frac{1}{2\pi^2}\log 2$ as in (1.10). The first and the second lines in (2.41) and (2.42) define the perturbative and nonperturbative corrections, respectively. The analogous expression for the leading term $\mathrm{F}_1(\lambda,m)$ is given by eqs. (2.21) – (2.25).

Using the obtained expressions for $\mathrm{F}_k(\lambda,m)$ (with $k=1,2,3$) in (2.17) we observe that the strong coupling expansion of the difference free energy has an interesting structure

$$\Delta F(\lambda;N,m) = Nb\lambda - \frac{1}{2}(b\lambda)^2 + \frac{1}{3N}(b\lambda)^3 + \ldots + N\frac{8\sqrt{2}}{\pi^{3/2}}\lambda^{-1/4}e^{-\sqrt{\lambda}}\cosh(2\pi m)\left[1+\frac{b\lambda^{3/2}}{2N}+\frac{1}{2}\left(\frac{b\lambda^{3/2}}{2N}\right)^2+\ldots\right], \tag{2.43}$$

where dots denote subleading corrections. This suggests that the strong coupling expansion can be resummed to all orders in $1/N$. Moreover, introducing the new expansion parameter

$$\lambda' = \frac{\lambda}{1+b\lambda/N}, \tag{2.44}$$

we find that the relation (2.43) leads to a remarkably simple expression for $\Delta F(\lambda;N,m)$

$$\Delta F(\lambda;N,m) = N^2\log(\lambda/\lambda') + N\frac{8\sqrt{2}}{\pi^{3/2}}\lambda'^{-1/4}e^{-\sqrt{\lambda'}}\cosh(2\pi m) + \ldots. \tag{2.45}$$

In the next subsection we explain how this relation naturally follows from the Toda equation (2.29) and also comment on the origin of the redefinition (2.44).

2.4 Resummation

Let us examine the Toda equation (2.29) at strong coupling. As was explained above, solving this equation it is advantageous to consider the free energy $F = -\log Z_N$ as a function of the inverse coupling y in (2.28) rather than of the 't Hooft coupling λ.

We have seen that the free energy of the $\mathcal{N}=4$ theory (2.34) is given by the sum of a term proportional to $\log y$ (or $\log\lambda$) and a constant. In the $\mathcal{N}=2$ theory the situation is more complicated — the difference free energy (2.17) receives corrections which are series in $\frac{1}{\sqrt{\lambda}}$ and $e^{-\sqrt{\lambda}}$. To leading order in $1/N$ they are given by (2.21). Similarly, the free energy can be in general split into the sum of perturbative and nonperturbative pieces

$$F_N^{\mathcal{N}=2}(y,m) = F_{\mathrm{p}}(N,y,m) + F_{\mathrm{np}}(N,y,m), \tag{2.46}$$

where the second term is suppressed by a factor of $e^{-\sqrt{\lambda}} = e^{-4\pi\sqrt{\frac{N}{y}}}$.

JHEP01(2023)037

Substituting $Z_N(y) = \exp(-F_N^{\mathcal{N}=2}(y))$ into (2.29) and neglecting all $\mathcal{O}(e^{-\sqrt{\lambda}})$ corrections on both sides of the equation we get the same relation just for the perturbative part of the free energy[22]

$$\partial_y^2 F_{\rm p}(N,y) = -e^{-\Delta_N F_{\rm p}(N,y)}\,, \tag{2.47}$$

where we introduced the notation for the second-order finite-difference operator

$$\Delta_N F_{\rm p}(N,y) \equiv F_{\rm p}(N+1,y) - 2F_{\rm p}(N,y) + F_{\rm p}(N-1,y)\,. \tag{2.48}$$

In a similar manner, matching the nonperturbative $O(e^{-\sqrt{\lambda}})$ terms on both sides of (2.29) we obtain

$$\partial_y^2 F_{\rm np}(N,y) = -\Delta_N F_{\rm np}(N,y)\;\partial_y^2 F_{\rm p}(N,y) + \mathcal{O}(e^{-2\sqrt{\lambda}})\,. \tag{2.49}$$

It is straightforward to extend the analysis to take into account subleading nonperturbative corrections $\mathcal{O}(e^{-n\sqrt{\lambda}})$ with $n \geq 2$. In what follows we restrict consideration to the leading $\mathcal{O}(e^{-\sqrt{\lambda}})$ nonperturbative terms.

The solutions to (2.47) and (2.49) should respect the symmetry of the Toda equation (2.29). As the equation (2.29) is invariant under an N-independent constant shift of y

$$y \to y + c_0\,, \tag{2.50}$$

the functions $F_{\rm p}(N, y+c_0)$ and $F_{\rm np}(N, y+c_0)$ satisfy (2.47) and (2.49) for an arbitrary c_0. The solutions to (2.47) and (2.49) are defined up to a contribution of the zero modes of the operators ∂_y^2 and Δ_N. The zero mode solution $\partial_y^2 F_{\rm zero} = \Delta_N F_{\rm zero} = 0$ has the form (c_i are arbitrary constants)

$$F_{\rm zero}(N,y) = c_1 + c_2 N + (c_3 + c_4 N)\, y\,. \tag{2.51}$$

Taking this into account, we look for the solution to (2.47) as

$$F_{\rm p}(N,y) = f_0(N)\log(y+\kappa) + f_1(N)\, y \;\; + f_2(N)\,, \tag{2.52}$$

where $N-$independent constant κ and the functions $f_i(N)$ are to be determined.

The motivation for choosing the ansatz (2.52) is twofold. First, it is consistent with the shift symmetry (2.50) and it takes into account the contribution of the zero mode (2.51). Second, for $\kappa = 0$ its functional dependence on y matches that of the free energy in the $\mathcal{N} = 4$ theory and the leading non-planar correction to the difference free energy in the $\mathcal{N} = 2$ theory, eqs. (2.34) and (2.25), respectively.

While we could eliminate κ in (2.52) using the shift symmetry (2.50), keeping it nonzero is important in order to correctly reproduce the λ-dependent terms in the $1/N$ expansion of the free energy. To see this, we express y in terms of λ according to its definition in (2.28) and expand the first term in (2.52) at large N and fixed λ

$$F_{\rm p}(N,y) = -f_0(N)\log\lambda + f_0(N)\log\left(1 + \frac{\lambda\,\kappa}{(4\pi)^2 N}\right) + f_1(N)\,\frac{(4\pi)^2 N}{\lambda} + \dots\,, \tag{2.53}$$

[22]For simplicity, we will not explicitly display the dependence on m in the equations below.

JHEP01(2023)037

where dots denote λ-independent terms. Due to planar equivalence between the $\mathcal{N}=4$ and $\mathcal{N}=2$ theories, the expression (2.53) should coincide in the leading large N, fixed λ limit with (2.34). This leads to the constraints

$$f_0(N) = N^2 + \mathcal{O}(N)\,, \qquad f_1(N) = \mathcal{O}(N^0)\,. \tag{2.54}$$

Then, it follows from (2.53) that all the terms in the large N expansion of $F_{\rm p}(N,y)$ which have the form λ^k/N^{k-2} (with $k \geq 1$) arise from the expansion of the logarithm in the second term. In particular, for $k=1$ we get $\lambda N\kappa/(4\pi)^2$. It should be compared with the analogous term $\lambda N \log 2/(2\pi^2)$ in the expression for the leading non-planar correction $N\mathrm{F}_{1,\rm p}$, eqs. (2.25) and (2.23). This fixes the value of κ (cf. (1.10))

$$\kappa = 8\log 2 = (4\pi)^2 b\,, \tag{2.55}$$

where b is the same as in (1.10).

To determine the functions $f_i(N)$ in (2.52) we substitute (2.52) into (2.47) and compare the y-dependence on both sides to obtain

$$\Delta_{_N} f_0(N) = 2\,, \qquad \Delta_{_N} f_1(N) = 0\,, \qquad \Delta_{_N} f_2(N) = -\log f_0(N)\,. \tag{2.56}$$

We require that the solution of these equations should admit a regular large N expansion. Then, the general solution to the first two equations in (2.56) consistent with the constraints in (2.54) is

$$\begin{aligned} f_0(N) &= (N+N_+)(N+N_-)\,,\\ f_1(N) &= c_3\,, \end{aligned} \tag{2.57}$$

where $N_\pm$ and c_3 so far are arbitrary constants. The solution of the last equation in (2.56) can be expressed in terms of the Barnes function

$$f_2(N) = -\log\left[G(N+1+N_+)G(N+1+N_-)\right] + c_1 + c_2 N\,, \tag{2.58}$$

where the last two terms represent the zero modes.

As before, we can fix the values of the parameters in (2.57) and (2.58) by comparing (2.53) with the first few terms of the large N expansion of the difference free energy (2.17). To find c_3 in (2.57), it is sufficient to compare $\mathcal{O}(\frac{N}{\lambda})$ terms in (2.53) and $N\mathrm{F}_{1,\rm p}$, eqs. (2.25) and (2.23). This leads to

$$c_3 = -\frac{1}{32} - \frac{1}{3}m^2 - \frac{1}{3}m^4\,. \tag{2.59}$$

To find $N_\pm$, we examine the coefficient in front of $\log\lambda$ in (2.53) and match it with $f_0(N)$ in (2.57). To this end, we need the expressions for the first two terms in (2.17).

We recall that the corresponding functions $\mathrm{F}_1(\lambda)$ and $\mathrm{F}_2(\lambda)$ are related to each other through (2.27). According to (2.25) and (2.23), the $\log\lambda$ term in F_1 has the coefficient

$$q \equiv -\frac{1}{2} - 4m^2\,. \tag{2.60}$$

JHEP01(2023)037

Eq. (2.27) implies that the same term in F_2 has the coefficient $\frac{1}{4}q(1-q)$. Combining this with the contribution from (2.34), we find the coefficient of $\log\lambda$ term in $F_{\rm p}$ as

$$f_0(N) = \frac{1}{2}N(2N+1) - qN - \frac{1}{4}q(1-q) = \left(N - \frac{1}{2}q\right)\left(N + \frac{1}{2}(1-q)\right). \tag{2.61}$$

Comparing this relation with (2.57) we deduce that

$$N_- = \frac{1}{4} + 2m^2\,, \qquad\qquad N_+ = \frac{3}{4} + 2m^2\,. \tag{2.62}$$

Finally, the constants c_1 and c_2 in (2.58) can be determined by matching the λ-independent terms in the expression for F_1 in (2.25) and F_2 in (2.27) with (2.52). This leads to

$$\begin{aligned} c_1 &= -\frac{1}{8}\log(16\pi) + \log G\left(\frac{3}{2}\right) - 2m^2\log(8\pi) - 8m^4\log(4\pi)\,, \\ c_2 &= -2\log\pi + 8\log G\left(\frac{3}{2}\right) - m^2(8 + 8\gamma_E)\,, \end{aligned} \tag{2.63}$$

where the Barnes function $G(\frac{3}{2})$ can be expressed in terms of the Glaisher's constant.

Combining together the above relations, we arrive at the following remarkably simple expression for the perturbative part of the free energy (2.52)

$$\begin{aligned} F_{\rm p}(N,y,m) = &\left(N + \frac{3}{4} + 2m^2\right)\left(N + \frac{1}{4} + 2m^2\right)\log\left(y + 8\log 2\right) \\ &- \log\left[G\left(N + \frac{5}{4} + 2m^2\right)G\left(N + \frac{7}{4} + 2m^2\right)\right] + c_1 + c_2 N + c_3\, y\,, \end{aligned} \tag{2.64}$$

where we explicitly indicated the dependence of $F_{\rm p}$ on m. The last three terms in (2.64) are the zero modes (2.51) of the operators ∂_y^2 and $\Delta_{_N}$. The corresponding coefficients are given by (2.59) and (2.63).

Let us now make few comments. The expression (2.64) sums up (perturbative) strong coupling corrections to the free energy to all orders in $1/N$ and $1/\sqrt{\lambda}$. Following (2.53), we can expand (2.64) in $1/N$ and determine the perturbative part of the functions $\mathrm{F}_k(\lambda, m)$ in (2.17). It is straightforward to verify that these functions satisfy (2.27).

Notice that N and m^2 enter the first two terms in (2.64) in a linear combination $N+2m^2$. This means that, up to the contribution (2.51) of the zero modes, the dependence of the free energy on m can be generated by the shift $N \to N + 2m^2$

$$F_{\rm p}(N,y,m) = F_{\rm p}(N + 2m^2, y, 0) + F_{\rm zero}(N,y)\,, \tag{2.65}$$

where $F_{\rm zero} = c_1' + c_2' N + c_3' y$ with the constants c_i' that can be read off from (2.64).

Another interesting feature of (2.64) is the appearance of the constant $8\log 2$ in the argument of the logarithm in (2.64). It follows from (2.53) that the coefficients of the strong coupling expansion of the free energy involve powers of this constant. At the same time, it is obvious from (2.64) that all such terms can be eliminated at once by a shift

JHEP01(2023)037

$y \to y - 8\log 2$. In terms of the 't Hooft coupling $\lambda = (4\pi)^2 N/y$ this amounts to changing the expansion parameter from λ to $\lambda' = (4\pi)^2 N/(y + 8\log 2)$, or, equivalently,

$$\lambda' = \frac{\lambda}{1 + \frac{\log 2}{2\pi^2 N}\lambda}\,. \tag{2.66}$$

This relation (the same as in (1.10) and (2.44)) can be interpreted as a finite renormalization of the gauge coupling constant (1.11).

Expressed in terms of the modified coupling constant (2.66), eq. (2.64) reads

$$\begin{aligned} F_{\rm p}(N,y,m) = &\left(N + \frac{3}{4} + 2m^2\right)\left(N + \frac{1}{4} + 2m^2\right)\log\frac{(4\pi)^2 N}{\lambda'} \\ &- \log\left[G\left(N + \frac{5}{4} + 2m^2\right) G\left(N + \frac{7}{4} + 2m^2\right)\right] + \tilde{c}_1 + c_2 N + c_3\,\frac{(4\pi)^2 N}{\lambda'}\,, \end{aligned} \tag{2.67}$$

where $\tilde{c}_1 = c_1 - 8\log 2\, c_3$. We verify that the leading term in this expression, $F_{\rm p} = -N^2\log(\lambda') + \ldots$, leads to $\Delta F = F^{\mathcal{N}=2} - F^{\mathcal{N}=4} = N^2\log(\lambda/\lambda') + \ldots$, in agreement with (2.45).

Let us also comment on a possible interpretation of the redefinition of the coupling constant (2.66). It follows from (2.11) that, at strong coupling $\lambda \gg 1$, the dominant contribution to the matrix integral comes from $\operatorname{tr} X^2 = 2\sum_{n=1}^N x_n^2 = O(\lambda)$ or equivalently $x_n = O(\sqrt{\lambda})$. Examining the interaction potential $S_{\rm int}$ in this limit we find from (2.9)[23]

$$\begin{aligned} S_{\rm int} &= \sum_{n=1}^N \left[4\log H(x_n + m) + 4\log H(x_n - m) - 2\log H(2x_n)\right] \\ &\overset{x_n \gg 1}{=} 8\log 2\sum_{n=1}^N x_n^2 - (1 + 8m^2)\sum_{n=1}^N \log x_n + \ldots\,. \end{aligned} \tag{2.68}$$

Combined with the Gaussian action $\frac{(4\pi)^2 N}{\lambda}\sum_{n=1}^N x_n^2$ in (2.11), the first term on the right-hand side of (2.68) implies a redefinition of the gauge coupling (2.66). The second term effectively results in an extra order N shift of the coefficient of the $\log\lambda'$ term in (2.67): $(N + \frac{3}{4} + 2m^2)(N + \frac{1}{4} + 2m^2) = N^2 + N(1 + 4m^2) + O(N^0)$. Indeed, the resulting integral over the eigenvalues x_n in (2.11) takes the form

$$Z_N = \frac{1}{N!}\int \prod_{n=1}^N dx_n\, x_n^2 \prod_{m>n=1}^N (x_n^2 - x_m^2)^2\, e^{-\frac{(4\pi)^2 N}{\lambda'}\sum_{n=1}^N x_n^2 + (1+8m^2)\sum_{n=1}^N \log x_n + \ldots}\,. \tag{2.69}$$

Rescaling the integration variables as $x_n \to x_n\sqrt{\lambda'}$ (with the measure contributing the $(\sqrt{\lambda'})^{2N^2+N}$ factor) we find that the partition function scales at large λ' as

$$Z_N \sim e^{-\left[N^2 + N(1+4m^2) + \mathcal{O}(N^0)\right]\log(1/\lambda')}\,, \tag{2.70}$$

which is in agreement with (1.9) and (2.67).

[23] Here $\log 2$ originates from the finite quantity $\eta(1) = \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k} = \log 2$ that appears in the expansion of (2.7) and is effectively due to the different arguments in the H-functions.

JHEP01(2023)037

2.5 Leading nonperturbative correction

The leading nonperturbative correction to the free energy satisfies the relation (2.49). Replacing the perturbative function $F_{\rm p}(N,y)$ with its expression (2.64), we get from (2.49)

$$\partial_y^2 F_{\rm np}(N,y,m) = \Delta_N F_{\rm np}(N,y,m)\, \frac{(N+\frac{3}{4}+2m^2)(N+\frac{1}{4}+2m^2)}{(y+8\log 2)^2}\,. \tag{2.71}$$

To leading order in $1/N$ the solution to this equation is given by (2.26)

$$F_{\rm np}(N,y,m) = N\cosh(2\pi m)\,\mathrm{F}_{1,\rm np}(\lambda) + \mathcal{O}(N^0)\,, \tag{2.72}$$

where $\mathrm{F}_{1,\rm np}(\lambda)$ is given by (2.24) with $\lambda = (4\pi)^2 N/y$. We show below that the equation (2.71) supplemented with (2.72) allows us to determine subleading corrections to (2.72) and to sum up the series in $1/N$.

Solving (2.71), it is convenient to introduce an auxiliary function

$$\widehat{F}(N,y) = F_{\rm np}\left(N-\frac{1}{2}-2m^2,\, y-8\log 2,\, m\right). \tag{2.73}$$

It follows from (2.71) that it satisfies the equation

$$\partial_y^2 \widehat{F}(N,y) = \frac{N^2-\frac{1}{16}}{y^2}\,\Delta_N \widehat{F}(N,y)\,. \tag{2.74}$$

Compared to (2.71), this relation does not involve m and $8\log 2$. In addition, the expression on the right-hand side of (2.74) is even in N and, as a consequence, the large N expansion of the function $\widehat{F}(N,y)$ runs in powers of $1/N^2$

$$\widehat{F}(N,y) = N\widehat{\mathrm{F}}_0(\lambda) + \frac{1}{N}\widehat{\mathrm{F}}_1(\lambda) + \frac{1}{N^3}\widehat{\mathrm{F}}_2(\lambda) + \mathcal{O}\left(\frac{1}{N^5}\right), \tag{2.75}$$

where again $\lambda = (4\pi)^2 N/y$. The leading term of the expansion can be obtained by matching (2.75) with (2.73) and (2.72)

$$\widehat{\mathrm{F}}_0(\lambda) = \frac{8\sqrt{2}}{\pi^{3/2}}\,\lambda^{-1/4}\,e^{-\sqrt{\lambda}}\,\cosh(2\pi m)\,\left[1+\mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right)\right]. \tag{2.76}$$

To find the subleading functions in (2.75), we substitute (2.75) into (2.74) and compare the coefficients in front of the powers of $1/N$ on both sides. This leads to

$$\begin{aligned}
\widehat{\mathrm{F}}_1(\lambda) &= \left(\frac{1}{48}\lambda^3\partial_\lambda^3 + \frac{1}{32}\lambda^2\partial_\lambda^2 - \frac{1}{16}\lambda\partial_\lambda\right)\widehat{\mathrm{F}}_0(\lambda)\,,\\
\widehat{\mathrm{F}}_2(\lambda) &= \left(\frac{1}{4608}\lambda^6\partial_\lambda^6 + \frac{11}{7680}\lambda^5\partial_\lambda^5 + \frac{1}{6144}\lambda^4\partial_\lambda^4 - \frac{1}{512}\lambda^3\partial_\lambda^3\right)\widehat{\mathrm{F}}_0(\lambda)\,, \qquad \ldots
\end{aligned} \tag{2.77}$$

As before, the solutions to (2.74) are defined up to zero modes of the operators ∂_y^2 and Δ_N. In distinction to the perturbative part (2.64), the zero modes do not contribute to (2.77). The reason for this is that $\widehat{F}(N,y)$ is exponentially small at strong coupling,

JHEP01(2023)037

or, equivalently, $\widehat{\mathrm{F}}_k(\lambda) = \mathcal{O}(e^{-\sqrt{\lambda}})$ with $k \geq 1$, and the contribution of zero modes is incompatible with this behaviour.

The relations (2.76) and (2.77) allows us to determine the coefficient functions in the large N expansion (2.75). Applying (2.73) we can then derive the large N expansion of the nonperturbative part of the free energy (for $N' = N + \frac{1}{2} + 2m^2$)

$$\begin{aligned} F_{\rm np}(N, y, m) &= \widehat{F}(N + \frac{1}{2} + 2m^2, y + 8\log 2) \\ &= N'\widehat{\mathrm{F}}_0\left(\lambda'\frac{N'}{N}\right) + \frac{1}{N'}\widehat{\mathrm{F}}_1\left(\lambda'\frac{N'}{N}\right) + \frac{1}{N'^3}\widehat{\mathrm{F}}_2\left(\lambda'\frac{N'}{N}\right) + \mathcal{O}\left(\frac{1}{N'^5}\right), \end{aligned} \tag{2.78}$$

where λ' was defined in (2.66). Here in the second line we took into account (2.75) and replaced $\lambda = (4\pi)^2 N/y$ in (2.75) with $(4\pi)^2(N + \frac{1}{2} + 2m^2)/(y + 8\log 2) = \lambda' N'/N$.

Notice that the dependence of $F_{\rm np}(N, y, m)$ on m enters through the factor of $\cosh(2\pi m)$ in (2.76) and $N' = N + \frac{1}{2} + 2m^2$. We use (2.76) to verify that, for $N \to \infty$ and fixed λ', the relation (2.78) reproduces the second, nonperturbative, term in (2.45).

As we will see in a moment, the coefficient functions in (2.78) scale at strong coupling as $\widehat{\mathrm{F}}_k/\widehat{\mathrm{F}}_0 = \mathcal{O}(\lambda^{3k/2})$ so that the expansion in (2.78) is dominated by the terms of the form $(\lambda^{3/2}/N^2)^k$. All such terms can be summed up to all orders to produce a simple expression.

To show this, we notice that the differential operators inside the brackets in (2.77) are polynomials in $\lambda\partial_\lambda$. For an arbitrary $k \geq 1$ the expression for $\widehat{\mathrm{F}}_k(\lambda)$ looks like

$$\widehat{\mathrm{F}}_k(\lambda) = \frac{1}{k!}\frac{(\lambda\partial_\lambda)^{3k}}{48^k}\widehat{\mathrm{F}}_0(\lambda) + \dots, \tag{2.79}$$

where dots denote terms involving smaller powers of $\lambda\partial_\lambda$. Using the explicit form of $\widehat{\mathrm{F}}_0(\lambda)$ in (2.76) we find (up to $\mathcal{O}(\frac{1}{\sqrt{\lambda}})$ corrections)

$$\widehat{\mathrm{F}}_k(\lambda) = \frac{1}{k!}\left(-\frac{\lambda^{3/2}}{384}\right)^k \widehat{\mathrm{F}}_0(\lambda)\,. \tag{2.80}$$

Together with (2.75) this leads to

$$F_{\rm np}(N, y, m) = \frac{8\sqrt{2}}{\pi^{3/2}} N \cosh(2\pi m)\,(\lambda')^{-1/4}\, e^{-\sqrt{\lambda'} - \frac{\lambda'^{3/2}}{384N^2}} + \dots\,. \tag{2.81}$$

To be precise, this relation holds for N and λ' going to infinity with $(\lambda')^{3/2}/N^2$ kept fixed (cf. footnote 7). In this limit, N' approaches N and (2.78) coincides with (2.75) with λ replaced by λ'. One can go beyond this approximation and systematically include subleading corrections to (2.81).

3 Circular Wilson loop

In this section, we apply localization to compute the expectation value of a circular BPS Wilson loop in the mass-deformed $\mathcal{N} = 2$ Sp($2N$) theory. It is given by a matrix integral that is similar to (2.11)

$$W_N = \frac{1}{Z_N}\int \mathcal{D}X\, \mathrm{tr}\, e^X\, e^{-\frac{8\pi^2}{g^2_{\rm YM}}\mathrm{tr}\, X^2 - S_{\rm int}(X,m)}, \tag{3.1}$$

JHEP01(2023)037

where Z_N is the partition function (2.11) and interaction action is given by (2.12). Compared to (2.11), the integral in (3.1) contains the extra factor of $\operatorname{tr} e^X$.

We have seen in the previous section that the use of the Toda equation simplifies significantly the derivation of the large N expansion of the free energy. As we find below, the same is true for the circular Wilson loop. We show that (3.1) satisfies a non-trivial finite-difference equation which allows us to calculate W_N as an expansion in large N.

To derive this equation, it is convenient to generalize (3.1) by introducing an infinite set of parameters $\boldsymbol{t} = (t_2, t_3, \dots)$ to define

$$W_N(\boldsymbol{t}) = \frac{1}{Z_N(\boldsymbol{t})} \int \mathcal{D}X \operatorname{tr} e^X \, e^{-\frac{1}{2}\sum_{n\geq 1} t_{2n} \operatorname{tr}(X^{2n})} \,. \tag{3.2}$$

Here $Z_N(\boldsymbol{t})$ is given by the same matrix integral without the $\operatorname{tr} e^X$ factor. The expression (3.2) coincides with (3.1) after one identifies the parameters t_{2n} with the coefficients in front of $\operatorname{tr}(X^{2n})$ terms in the exponent of (3.1), e.g., $t_2 = y = (4\pi)^2/g^2_{\text{YM}}$, etc.

The reason for introducing the parameters $\boldsymbol{t}$ is that, upon expanding $\operatorname{tr} e^X$ into traces of (even) powers of X, the function $W_N(\boldsymbol{t})$ can be obtained from a logarithm of the partition function by applying the following linear differential operator

$$W_N(\boldsymbol{t}) = 2N - 2\sum_{k=1}^{\infty} \frac{1}{(2k)!} \frac{\partial}{\partial t_{2k}} \log Z_N(\boldsymbol{t}) \,. \tag{3.3}$$

For $t_2 = y$ and arbitrary t_{2k} (with $k > 1$) the generalized partition function $Z_N(\boldsymbol{t})$ satisfies the Toda equation (2.29).

Applying the differential operator in (3.3) to both sides of (2.29) and setting the parameters $\boldsymbol{t}$ equal to their values corresponding to (3.1), we obtain the following finite-difference equation for W_N in (3.1)

$$\begin{aligned} \partial_y^2 W_N &= (W_{N+1} - 2W_N + W_{N-1}) \frac{Z_{N+1} Z_{N-1}}{Z_N^2} \\ &= -(W_{N+1} - 2W_N + W_{N-1}) \, \partial_y^2 F_N \,. \end{aligned} \tag{3.4}$$

Here in the second relation we applied (2.29) and used that $\log Z_N = -F_N$. Notice that the relation (3.4) is insensitive to the form of the interaction action (2.12) and, therefore, holds, in particular, in both $\mathcal{N} = 4$ and $\mathcal{N} = 2$ theories.

The relation (3.4) is remarkably similar to (2.49). The important difference between the two equations is that (3.4) is exact and holds for an arbitrary coupling. In addition, solving (2.49) we looked for a solution (nonperturbative part of the free energy) that scales at strong coupling as $\mathcal{O}(e^{-\sqrt{\lambda}})$. This boundary condition does not apply to (3.4).

3.1 Toda equation in $\mathcal{N} = 4$ Sp($2N$) theory

Let us examine the Toda equation (3.4) for the circular Wilson loop in $\mathcal{N} = 4$ theory. Replacing the free energy with its expression (2.34) we get from (3.4)

$$\partial_y^2 W_N = (W_{N+1} - 2W_N + W_{N-1}) \frac{(N + \frac{1}{4})^2 - \frac{1}{16}}{y^2} \,. \tag{3.5}$$

JHEP01(2023)037

We look for a general solution to this equation in the form of a $1/N$ expansion

$$W_N^{\mathcal{N}=4}(\lambda) = NW^{(0)}(\lambda) + W^{(1)}(\lambda) + \frac{1}{N}W^{(2)}(\lambda) + \mathcal{O}\Big(\frac{1}{N^2}\Big)\,. \tag{3.6}$$

At zero coupling we have $W_N(0) = \operatorname{tr}\mathbf{1} = 2N$, or, equivalently, $W^{(0)}(0) = 2$ and $W^{(k)}(0) = 0$ for $k \geq 1$.

Substituting (3.6) into (3.5) and matching the coefficients of $1/N$ on both sides we obtain a system of differential equations for the functions $W^{(k)}(\lambda)$. Supplemented with the boundary conditions at zero coupling, their solutions are

$$\begin{aligned} W^{(1)}(\lambda) &= \frac{1}{4}(1+\lambda\partial_\lambda)W^{(0)}(\lambda) - \frac{1}{2}\,, \\ W^{(2)}(\lambda) &= \Big(\frac{1}{16}\lambda^2\partial_\lambda^2 + \frac{1}{48}\lambda^3\partial_\lambda^3\Big)W^{(0)}(\lambda)\,, \qquad \dots\,. \end{aligned} \tag{3.7}$$

These relations are analogous to those for the free energy (2.27).

In the $\mathcal{N}=4$ SYM theory with the Sp($2N$) gauge group, the leading term of the large N expansion (3.5) is well-known [9, 25][24]

$$W^{(0)}(\lambda) = \frac{4}{\sqrt{\lambda}}I_1(\sqrt{\lambda})\,. \tag{3.8}$$

Substituting this expression into (3.7) we get

$$W^{(1)}(\lambda) = \frac{1}{2}\big[I_0(\sqrt{\lambda}) - 1\big]\,, \qquad W^{(2)}(\lambda) = \frac{1}{96}\lambda\, I_2(\sqrt{\lambda})\,, \qquad \dots\,. \tag{3.9}$$

The large N expansion (3.6) can be resumed to all orders in $1/N$ to yield the following integral representation

$$W_N^{\mathcal{N}=4} = \frac{8N}{\lambda}\oint\frac{dz}{2\pi i}\,e^{-z+\frac{\lambda}{16N}}\,\frac{1-\frac{\lambda}{8Nz}}{1-\frac{\lambda}{16Nz}}\Big[1-\Big(1-\frac{\lambda}{8Nz}\Big)^{2N}\Big]\,, \tag{3.10}$$

where the integration contour encircles the origin in anti-clockwise direction. Using that $\lambda = (4\pi)^2N/y$, it is possible to show that this expression verifies the Toda equation (3.5).

Changing the integration variable in (3.10) as $z \to 8Nz/\lambda$ we get from (3.10)

$$W_N^{\mathcal{N}=4} = 2e^{\frac{\lambda}{16N}}\sum_{i=0}^{N-1}\oint\frac{dz}{2\pi i}\;e^{-\frac{\lambda}{8N}z}\;z^{-2-2i}(z-1)^{1+2i} = 2e^{\frac{\lambda}{16N}}\sum_{i=0}^{N-1}L_{2i+1}\Big(-\frac{\lambda}{8N}\Big)\,, \tag{3.11}$$

where $L_n(x)$ is the Laguerre polynomial. This relation is exact and holds in the $\mathcal{N}=4$ Sp($2N$) theory for an arbitrary N and λ.

At strong coupling, the integral in (3.10) can be evaluated using a saddle point approximation. A close examination shows that there are two saddle points $z_\pm = \frac{1}{2}\sqrt{\lambda}\big(\frac{\sqrt{\lambda}}{8N} \mp \sqrt{\frac{\lambda}{64N^2}+1}\big)$. The integration in the vicinity of $z = z_+$ yields the contribution

[24] In our notation where the N-factor is extracted (cf. (3.6)) this is effectively the same as in the SU($2N$) case [35, 55].

JHEP01(2023)037

that scales as $\mathcal{O}(e^{\sqrt{\lambda}})$ whereas the contribution of $z = z_-$ behaves as $\mathcal{O}(e^{-\sqrt{\lambda}})$. In the double scaling limit $N \to \infty$ and $\lambda \to \infty$ with $\lambda^{3/2}/N^2$ held fixed, we find that (cf. footnote 7)

$$W_N^{\mathcal{N}=4} = N\sqrt{\frac{8}{\pi}}\,\lambda^{-3/4} e^{\sqrt{\lambda}+\frac{\lambda^{3/2}}{384N^2}} + \mathcal{O}\left(e^{-\sqrt{\lambda}-\frac{\lambda^{3/2}}{384N^2}}\right). \tag{3.12}$$

Here the second term can be formally obtained from the first one by replacing $\sqrt{\lambda} \to -\sqrt{\lambda}$. Following the terminology adopted for the free energy (2.46), the two terms on the right-hand side of (3.12) may be interpreted as representing the perturbative and nonperturbative contributions to the Wilson loop.

In contrast to the free energy (2.53), the perturbative contribution in (3.12) scales as $\mathcal{O}(e^{\sqrt{\lambda}})$ (rather than $\mathcal{O}(\log\lambda)$). At the same time, it is interesting to notice that nonperturbative corrections to (3.12) and (2.81) involve the same exponentially small factor (but here it has an imaginary prefactor, cf. [6, 56, 57]).

3.2 Difference of Wilson loops in $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories

As we have demonstrated in section 2.3, the Toda equation for the partition function (2.29) is powerful enough to predict the subleading terms in the $1/N$ expansion of the difference free energy (2.17) in terms of the leading one $\mathrm{F}_1(\lambda, m)$ (see eqs. (2.27)). We can repeat the same analysis for (3.4) to show that similar relations also hold for the coefficients in the $1/N$ expansion of the Wilson loop.

In a close analogy with the difference free energy function (2.16), we define the difference between the circular Wilson loops in the $\mathcal{N} = 4$ and $\mathcal{N} = 2$ models

$$\begin{aligned}\Delta W &= W_N^{\mathcal{N}=2} - W_N^{\mathcal{N}=4} \\ &= \Delta W^{(1)}(\lambda, m) + \frac{1}{N}\Delta W^{(2)}(\lambda, m) + \frac{1}{N^2}\Delta W^{(3)}(\lambda, m) + \mathcal{O}\left(\frac{1}{N^3}\right).\end{aligned} \tag{3.13}$$

Here we took into account that the leading $\mathcal{O}(N)$ term in the difference cancels out due to the planar equivalence of the two theories. Notice that ΔW_N vanishes at zero coupling, i.e. for $\lambda = 0$.

Substituting $W_N \to W_N^{\mathcal{N}=2} = W_N^{\mathcal{N}=4} + \Delta W$ and $F_N \to F_N^{\mathcal{N}=2} = F_N^{\mathcal{N}=4} + \Delta F$ into (3.4), using that $\lambda = (4\pi)^2 N/y$ and comparing the coefficients of powers of $1/N$ on both sides of (3.4) we get a system of linear equations for the functions $\Delta W^{(k)}(\lambda, m)$ with $k \geq 1$. These equations involve the functions F_k and $W^{(k)}$ which enter the large N expansions of ΔF and $W_N^{\mathcal{N}=4}$, respectively (see eqs. (2.17) and (3.6)). According to (2.27) and (3.7), they, in turn, can be expressed in terms of the leading functions $\mathrm{F}_1(\lambda, m)$ and $W^{(0)}(\lambda)$.

Combining these relations and supplementing them with the boundary condition at zero coupling, $\Delta W^{(k)}(\lambda = 0, m) = 0$, we get after some algebra

$$\begin{aligned}\Delta W^{(1)}(\lambda, m) &= -\frac{1}{2}\int_0^\lambda d\lambda\,\sqrt{\lambda}\, I_1(\sqrt{\lambda})\,\mathcal{F}(\lambda, m)\,, \\ \Delta W^{(2)}(\lambda, m) &= -\frac{1}{8}\,\lambda^{3/2} I_1(\sqrt{\lambda})\left[\mathcal{F}(\lambda, m) - \lambda\,\mathcal{F}^2(\lambda, m)\right],\end{aligned} \tag{3.14}$$

JHEP01(2023)037

where $\mathcal{F}(\lambda, m)$ was defined in (2.36) and (2.20). Here we replaced $W^{(0)}(\lambda)$ with its expression (3.8). Remarkably, higher terms of the $1/N$ expansion in (3.13) admit a closed form representation in terms of the functions $W^{(0)}(\lambda)$ and $\mathcal{F}(\lambda, m)$, though the corresponding expressions are lengthy, e.g.,

$$\begin{aligned}\Delta W^{(3)}(\lambda, m) = &-\frac{1}{192}\lambda^3 I_0(\sqrt{\lambda})\left(-3\mathcal{F}^2 + 2\lambda\,\mathcal{F}^3 + \mathcal{F}'\right) \\ &- \frac{1}{384}\lambda^{5/2} I_1(\sqrt{\lambda})\Big(\mathcal{F} - 12\mathcal{F}^2 + 16\lambda\mathcal{F}^3 + 4\mathcal{F}' - 24\lambda\mathcal{F}\mathcal{F}' + 24\lambda^2\mathcal{F}^2\mathcal{F}' + 4\lambda\mathcal{F}''\Big),\end{aligned} \tag{3.15}$$

where primes denote again derivatives with respect to λ. The relations (3.14) and (3.15) can be considered as the counter-parts of the analogous relations (2.36) and (2.38) for the free energy. Being combined together with (2.36) and (2.20), they allow us to compute the Wilson loop in the $\mathcal{N} = 2$ model for an arbitrary coupling.

The relations (3.14), (3.15), etc., can be used to derive the strong coupling expansion of the Wilson loop to any given order in $1/N$. They are not suitable, however, for discussing a resummation of the large N expansion. In the next subsection we apply (3.4) to compute the circular Wilson loop in $\mathcal{N} = 2$ Sp($2N$) theory at strong coupling.

3.3 Toda equation in $\mathcal{N} = 2$ Sp($2N$) theory

According to (2.46), the free energy in $\mathcal{N} = 2$ Sp($2N$) theory is given at strong coupling by the sum of perturbative and nonperturbative pieces. Correspondingly, substituting (2.46) into the Toda equation (3.4) we can look for its solution in the form

$$W_N^{\mathcal{N}=2} = W_{N,\mathrm{p}} + W_{N,\mathrm{np}}\,, \tag{3.16}$$

where the second (nonperturbative) term is exponentially small as compared to the first (perturbative) term. As we will see in a moment, the two terms on the right-hand side account for the corrections $\mathcal{O}(e^{\sqrt{\lambda}})$ and $\mathcal{O}(\lambda^0)$, respectively.

It follows from (3.4) and (2.64) that the perturbative contribution $W_{N,\mathrm{p}}$ satisfies

$$\partial_y^2 W_{N,\mathrm{p}} = (W_{N+1,\mathrm{p}} - 2W_{N,\mathrm{p}} + W_{N-1,\mathrm{p}})\,\frac{(N + \frac{1}{2} + 2m^2)^2 - \frac{1}{16}}{(y + 8\log 2)^2}\,. \tag{3.17}$$

As in the case of the free energy (2.47), it is assumed here that $W_{N,\mathrm{p}}$ is a function of y (rather than $\lambda = (4\pi)^2 N/y$). Let us compare this relation with (3.5). We observe that the two equations coincide after one applies the shift $N \to N + \frac{1}{4} + 2m^2$ and $y \to y + 8\log 2$ to (3.5). This implies that, up to the contribution of the zero modes, the solutions to (3.17) and (3.5) are related to each other through the same transformation

$$W_{N,\mathrm{p}} = W^{\mathcal{N}=4}_{N+\frac{1}{4}+2m^2}(y + 8\log 2)\,. \tag{3.18}$$

It is important to emphasize that this relation only holds at strong coupling up to exponentially small (nonperturbative) corrections. The relation (3.18) is analogous to (2.65) in that the dependence of the Wilson loop on the mass parameter can be generated by the shift $N \to N + 2m^2$. Since the perturbative part of $W_N^{\mathcal{N}=4}$ scales as $\mathcal{O}(e^{\sqrt{\lambda}})$, the contribution of the zero modes to (3.18) is exponentially small.

JHEP01(2023)037

Viewed as a function of N and $\lambda = (4\pi)^2 N/y$, $W_{N,\mathrm{p}}$ in (3.18) takes the form similar to (2.78)

$$W_{N,\mathrm{p}}(\lambda) = W_{N'}^{\mathcal{N}=4}\left(\lambda' \frac{N'}{N}\right), \tag{3.19}$$

where $N' = N + \frac{1}{2} + 2m^2$ and λ' is defined in (2.66). In the double-scaling limit (3.12), $W_{N,\mathrm{p}}$ coincides with (3.12).

3.4 Leading nonperturbative correction

According to (3.12), the leading nonperturbative correction to the Wilson loop in the $\mathcal{N} = 4$ theory is suppressed by the factor of $e^{-2\sqrt{\lambda}}$ as compared to the perturbative contribution. Interestingly, as we show below, in the $\mathcal{N} = 2$ model the situation is different in that the leading nonperturbative correction to the difference ΔW scales as $\mathcal{O}(\lambda)$, i.e. $W_{N,\mathrm{np}} = \mathcal{O}(\lambda)$ (compared to $\mathcal{O}(e^{-\sqrt{\lambda}})$ in the $\mathcal{N} = 4$ case). This correction is still exponentially suppressed relative to the perturbative one $\sim e^{\sqrt{\lambda}}$.

To demonstrate this, we apply (3.14) and replace the function $\mathcal{F}(\lambda, m)$ with its expression (2.40). At strong coupling the Bessel function in (3.14) is given by the sum of two terms that behave as $e^{\sqrt{\lambda}}$ and $e^{-\sqrt{\lambda}}$. As a consequence, the leading nonperturbative correction to (3.14) comes from the interference of the former term and the nonperturbative $\mathcal{O}(e^{-\sqrt{\lambda}})$ correction to $\mathcal{F}(\lambda, m)$. Taking into account (2.40) we get from (3.14)

$$\partial_\lambda \Delta W_{\mathrm{np}}^{(1)}(\lambda, m) = -\frac{1}{2}\sqrt{\lambda}\, I_1(\sqrt{\lambda})\, \mathcal{F}_{\mathrm{np}}(\lambda, m) = -\frac{1}{\pi^2}\cosh(2\pi m) + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right), \tag{3.20}$$

wherefrom $\Delta W_{\mathrm{np}}^{(1)}(\lambda, m) = \mathcal{O}(\lambda)$. In a similar manner, it follows from the second relation in (3.14) that

$$\Delta W_{\mathrm{np}}^{(2)}(\lambda, m) = -\frac{1}{8}\lambda^{3/2} I_1(\sqrt{\lambda})\left[\mathcal{F}_{\mathrm{np}}(\lambda, m) - 2\lambda\, \mathcal{F}_{\mathrm{np}}(\lambda, m)\mathcal{F}_{\mathrm{p}}(\lambda, m)\right] + \dots, \tag{3.21}$$

where dots denote subleading corrections.

Replacing $\mathcal{F}_{\mathrm{p}}(\lambda, m)$ in (3.21) with its expression given by the first term in (2.40) we notice that $\Delta W_{\mathrm{np}}^{(2)}(\lambda, m)$ is proportional to $\partial_\lambda \Delta W_{\mathrm{np}}^{(1)}(\lambda, m)$

$$\Delta W_{\mathrm{np}}^{(2)}(\lambda, m) = \left(\frac{1}{2} + 2m^2 - \frac{\log 2}{2\pi^2}\lambda\right)\lambda \partial_\lambda \Delta W_{\mathrm{np}}^{(1)}(\lambda, m)\,. \tag{3.22}$$

This leads to

$$\Delta W_{\mathrm{np}} = \Delta W_{\mathrm{np}}^{(1)} + \frac{1}{N}\Delta W_{\mathrm{np}}^{(2)} + \mathcal{O}\left(\frac{1}{N^2}\right) = \Delta W_{\mathrm{np}}^{(1)}\left(\lambda'\left(1 + \frac{\frac{1}{2} + 2m^2}{N}\right)\right) + \dots, \tag{3.23}$$

where λ' is as in (2.66). Although this relation holds at order $\mathcal{O}(1/N^2)$, we assumed that higher order corrections in $1/N$ only modify the argument of the leading term.

To justify (3.23) we apply the Toda equation (3.4). Substituting (3.16) and taking into account (3.17) we get from (3.4)

$$\begin{aligned}\partial_y^2 W_{N,\mathrm{np}} = (W_{N+1,\mathrm{np}} - 2W_{N,\mathrm{np}} + W_{N-1,\mathrm{np}})\,\frac{(N + \frac{1}{2} + 2m^2)^2 - \frac{1}{16}}{(y + 8\log 2)^2} \\ - (W_{N+1,\mathrm{p}} - 2W_{N,\mathrm{p}} + W_{N-1,\mathrm{p}})\partial_y^2 F_{N,\mathrm{np}}\,,\end{aligned} \tag{3.24}$$

JHEP01(2023)037

where $W_{N,\rm p}(y,m)$ and $F_{N,\rm np}(N,y,m)$ are given by (3.18) and (2.81), respectively. Here we neglected the subleading terms proportional to the product of $F_{N,\rm np}$ and $W_{N,\rm np}$.

As before, we can simplify the relation (3.24) by applying the shifts $N \to N - \frac{1}{2} - 2m^2$ and $y \to y - 8\log 2$. Introducing the function $w_N(y)$ defined by

$$W_{N,\rm np}(y) = w_{N+\frac{1}{2}+2m^2}(y + 8\log 2)\,, \tag{3.25}$$

we find from (3.24), (3.18) and (2.73) that $w_N(y)$ satisfies

$$\partial_y^2 w_N = (w_{N+1} - 2w_N + w_{N-1})\frac{N^2 - \frac{1}{16}}{y^2} - \Big(W^{\mathcal{N}=4}_{N+1/2} - 2W^{\mathcal{N}=4}_{N-1/2} + W^{\mathcal{N}=4}_{N-3/2}\Big)\partial_y^2 \widehat{F}_N\,. \tag{3.26}$$

At strong coupling, $W_N^{\mathcal{N}=4}$ is given by (3.12) whereas the function $\widehat{F}_N$ can be found from (2.81) by replacing λ' with λ. Using the large N expansion, $w_N = w^{(1)}(\lambda) + \frac{1}{N}w^{(2)}(\lambda) + \ldots$, we obtain from (3.26) that the leading term $w^{(1)}(\lambda)$ coincides with $\Delta W^{(1)}_{\rm np}(\lambda,m)$ and reads

$$w^{(1)}(\lambda) = \Delta W^{(1)}_{\rm np}(\lambda,m) = -\frac{1}{\pi^2}\lambda\,\cosh(2\pi m) + \mathcal{O}(\sqrt{\lambda})\,. \tag{3.27}$$

Combining this relation together with (3.25) we arrive at (3.23). We verified that the relations (3.23) and (3.27) correctly reproduce the leading corrections to $\Delta W_{N,\rm np}(\lambda,m)$ of the form λ^{k+1}/N^k and $m^2\lambda^k/N^k$ for $k \geq 0$. The functions $w^{(k)}$ (with $k \geq 2$) give subleading corrections to $\Delta W_{N,\rm np}(\lambda,m)$ that are suppressed by powers of $1/\sqrt{\lambda}$ as compared to those coming from (3.27).

4 Dual string theory interpretation

Let us now comment on the dual string theory interpretation of the strong-coupling expansions derived on the gauge theory side. The string theory for the Sp($2N$) FA-orientifold theory (i.e. the $\mathcal{N}=2$ Sp($2N$) superconformal model with 4 fundamental and 1 antisymmetric hypers) can be defined [20, 21, 24] using a near-horizon limit of the system of $2N$ D3-branes with 8 D7-branes stuck on one O7-plane. The effective presence of D7-branes introduces the new D3-D7 open string sector (with massless modes being related to the fundamental hypermultiplets in the corresponding gauge theory). Equivalently, it can be defined as the type IIB superstring on the orientifold $\mathrm{AdS}_5\times S'^5$ where $S'^5 = S^5/\mathbb{Z}_{2,\rm ori}$.[25] One may interpret the resulting theory as containing D7 branes wrapped on $\mathrm{AdS}_5\times S^3$ where S^3 is fixed-point locus of $\mathbb{Z}_{2,\rm ori}$.

[25]The orientifold group is defined as $\mathbb{Z}_{2,\rm ori} = \{1,\, I_{45}\,\Omega\,(-1)^{F_L}\}$. It contains the $\mathbb{Z}_2$ orbifold action: the inversion I_{45} acts on the 2-plane of $\mathbb{R}^6$ (with directions 4,...,9) transverse to the D3-branes as $x_{4,5}\to -x_{4,5}$. The fixed-point set of this $\mathbb{Z}_2$ is the hyperplane $x_{4,5}=0$, which corresponds to the position of the 8 D7-branes and O7-plane. In the near-horizon limit the $\mathbb{Z}_2$ orbifold part of $\mathbb{Z}_{2,\rm ori}$ acts on the coordinates of S^5 (with the metric $ds_5^2 = d\theta_1^2 + \sin^2\theta_1\, d\varphi_1^2 + \cos^2\theta_1\, dS_3$, $dS_3 = d\theta_2^2 + \cos^2\theta_2\, d\varphi_2^2 + \sin^2\theta_2\, d\varphi_2^2$) as $\varphi_1 \to \varphi_1 + \pi$. Then $\theta_1 = 0$ subspace is the collection of conical singularities represented by S^3.

JHEP01(2023)037

The dual string perturbation theory will then involve both closed-string and open-string world-sheet topologies, i.e. corrections of both even and odd powers in $g_{\rm s}$, corresponding to even and odd powers of $1/N$ on the gauge theory side. The orientifolding implies the presence of contributions of non-orientable surfaces with crosscups.[26]

Accounting for the open string sector term in the dual string theory effective action (that here may be interpreted as the D7+O7-brane world-volume action) allowed [23, 59] to give a holographic interpretation of the order N term in the (super)conformal anomalies of the Sp($2N$) FA-orientifold theory.

To recall, in the $\mathcal{N}=4$ and $\mathcal{N}=2$ Sp($2N$) theories the a and c conformal anomaly coefficients are given by[27]

$$\mathcal{N}=4 \text{ SYM}: \qquad \mathrm{a}=\mathrm{c}=\frac{1}{2}N^2+\frac{1}{4}N=\frac{1}{2}\left(N+\frac{1}{4}\right)^2-\frac{1}{32}\,, \tag{4.1}$$

$$\mathcal{N}=2 \text{ FA}: \qquad \mathrm{a}=\frac{1}{2}N^2+\frac{1}{2}N-\frac{1}{24}\,, \qquad \mathrm{c}=\frac{1}{2}N^2+\frac{3}{4}N-\frac{1}{12}\,. \tag{4.2}$$

On the superconformal gauge theory side the a-anomaly coefficient appears in the UV divergent part free energy on S^4 as

$$F=4\mathrm{a}\log(\Lambda_{\rm UV}\mathrm{r})+\dots \tag{4.3}$$

where $\Lambda_{\rm UV}$ is a UV cutoff and r is the radius of S^4 and dots stand for possible finite contributions. In the localization matrix model computation of F the UV divergence is automatically subtracted. In the $\mathcal{N}=4$ SYM case described by Gaussian matrix model the dependence on the radius r is correlated with dependence on λ (they enter the free part of the action in (2.1) in combination $\frac{\mathrm{r}^2}{\lambda}$). As a result,

$$F=-2\mathrm{a}\log(\lambda \mathrm{r}^{-2})+\dots\,, \tag{4.4}$$

where as in (4.3) the anomaly coefficient a controls the dependence on r. While the dependences on λ and on the S^4 radius r are a priori correlated only in the $\mathcal{N}=4$ SYM or Gaussian matrix model case (see appendix A of [6]) it turns out that in the Sp($2N$) FA model this applies also to the subleading order N term: the N^2+N combination in a$-$anomaly in (4.2) is the same as in the coefficient of the $\log\lambda$ term in (1.9).

In general, the gauge theory free energy on S^4 should be reproduced by the string partition $Z_{\rm str}$ function on $\mathrm{AdS}_5\times S'^5$ where S^4 is the boundary of AdS_5. Since this is a homogeneous space, the field theory intuition suggests that the result should be proportional to the volume of AdS_5 space. The latter is IR divergent,[28] $\mathrm{Vol}(\mathrm{AdS}_5)=\pi^2\log(\Lambda_{\rm IR}\mathrm{r})$. In [45]

[26] Note that in the Sp($2N$) $\mathcal{N}=4$ SYM case dual to string in $\mathrm{AdS}_5\times\mathbb{RP}^5$ [47] all odd-power $1/N$ contributions should come from crosscups while in the Sp($2N$) $\mathcal{N}=2$ FA-orientifold model there should be also $1/N^{2k+1}$ contributions from world sheets with boundaries reflecting the presence of D7-branes (cf. [58]).

[27] To recall, the conformal anomaly relation is $(4\pi)^2\langle T^m{}_m\rangle=-\mathrm{a}\,E_4+\mathrm{c}\,C^2=(\mathrm{c}-\mathrm{a})R^2_{mnkl}+\mathcal{O}(R^2_{mn},R^2)$, where $C^2=R^2_{mnkl}-2R^2_{mn}+\frac{1}{3}R^2$, $E_4=R^2_{mnkl}-4R^2_{mn}+R^2$. In the $\mathcal{N}=2$ superconformal models with gauge group G one has $4(2\mathrm{a}-\mathrm{c})=\dim G$ [60]. In particular, in the present case $\dim[\mathrm{Sp}(2N)]=N(2N+1)$. To capture the $\mathrm{c}-\mathrm{a}$ combination it is sufficient to assume that the boundary metric is Ricci flat, $R_{mn}=0$.

[28] The regularized volume of odd-dimensional AdS space is $\mathrm{vol}(AdS_{2n+1})=\frac{2(-\pi)^n}{\Gamma(n+1)}\log(\Lambda_{\rm IR}\mathrm{r})$ where r is the radius of the boundary sphere and $\Lambda_{\rm IR}$ is an IR cutoff.

JHEP01(2023)037

it was suggested to use a particular string tension related IR cutoff $\Lambda_{\rm IR} \sim \frac{1}{\sqrt{\lambda}}$ so that[29]

$$\mathrm{Vol}(\mathrm{AdS}_5) = \pi^2 \log(\Lambda_{\rm IR} \mathrm{r}) \quad \to \quad -\frac{1}{2}\pi^2 \log(\lambda \mathrm{r}^{-2}) \,. \tag{4.5}$$

The leading 2-sphere topology contribution to $Z_{\rm str}$ may be represented as the type IIB supergravity action (plus α'-corrections). Starting with 10d supergravity action evaluated on $\mathrm{AdS}_5 \times S^5$ one reproduces the leading N^2 term in the localization result for SU(N) $\mathcal{N} = 4$ SYM free energy,

$$\mathrm{SU}(N): \qquad F_{\mathcal{N}=4} = -\frac{1}{2}\,(N^2-1)\log(\lambda \mathrm{r}^{-2}) + C(N) \,. \tag{4.6}$$

To recall, compactifying 10d supergravity action on S^5 (of radius L) one finds (after accounting for a 5-form dependent boundary term [61]) the familiar 5d action ($\mathrm{Vol}(S^5) = \pi^3$)

$$S_{10} = -\frac{\mathrm{Vol}(S^5)}{2\kappa_{10}^2} \int d^5x\, \sqrt{-g}(R_5 + 12L^{-2}) \,, \quad 2\kappa_{10}^2 = (2\pi)^7 \alpha'^4 g_s^2 \,, \quad L^4 = 4\pi g_s N \alpha'^2 \,. \tag{4.7}$$

Using that for AdS_5 one has $R_5 = -20L^{-2}$ and (4.5) we get

$$S_{10} = \frac{1}{\pi^2}\, N^2\, \mathrm{Vol}(\mathrm{AdS}_5) = N^2\, \log(\Lambda_{\rm IR} \mathrm{r}) \,, \tag{4.8}$$

that then matches the N^2 term in (4.6).

One can also reproduce the subleading (-1) term in the a-anomaly coefficient in (4.6) by accounting for 1-loop (torus) contribution to string partition function which in the maximally supersymmetric $\mathrm{AdS}_5 \times S^5$ case happens to be given by the sum of the 1-loop contributions of just the 10d supergravity modes [46] (see also [62]).

Below we will first present the analogous dual string computation of F in the $\mathcal{N} = 4$ Sp($2N$) SYM theory and then discuss the $\mathcal{N} = 2$ Sp($2N$) FA theory.

4.1 Free energy in $\mathcal{N} = 4$ theory dual to IIB string on $\mathbf{AdS_5 \times \mathbb{RP}^5}$

The Sp($2N$) SYM theory can be realised on $2N$ D3-branes and O3 plane and is dual to the IIB string on an orientifold of $\mathrm{AdS}_5 \times S^5$ or $\mathrm{AdS}_5 \times \mathbb{RP}^5$ [47]. The dual string theory explanation of the expression for its free energy (1.3) or the conformal a-anomaly (4.1) is as follows. The N^2 term comes from the classical 10d supergravity action just as in the SU(N) case above.

The shift $N \to N + \frac{1}{4}$ in (4.1) explaining the order $O(N)$ term in the SYM conformal anomaly in (4.1) may be attributed to the redefinition of the D3 brane charge due to the presence of the O3 planes [23]: O3-planes carry fractional RR charge $\frac{1}{4}$ [63, 64]. Equivalently, from the flat-space perspective, this shift may be interpreted as being due to crosscup contributions. In view of this shift the AdS radius (given by $L^4 = 4\pi g_s \alpha'^2 N$ in the SU(N) case) is now identified as [23, 25]

$$L^4 = 4\pi g_s \alpha'^2 \left(2N + \frac{1}{2}\right) \,. \tag{4.9}$$

[29] Here we formally ignore mismatch of dimensions to indicate that the dependence on λ is correlated with that on r.

JHEP01(2023)037

	$(\Delta;\, j_1, j_2)$	SU(4)			$(\Delta;\, j_1, j_2)$	SU(4)	
$p \geq 1$	$(p;\, 0, 0)$	$(0, p, 0)$		$p \geq 3$	$(p+\frac{3}{2};\, \frac{1}{2}, 0)$	$(2, p-3, 1)_c$	
	$(p+\frac{1}{2};\, \frac{1}{2}, 0)$	$(0, p-1, 1)_c$			$(p+\frac{5}{2};\, \frac{1}{2}, 0)$	$(0, p-3, 1)_c$	
	$(p+1;\, 1, 0)$	$(0, p-1, 0)_c$	$\times$		$(p+2;\, \frac{1}{2}, \frac{1}{2})$	$(1, p-3, 1)_c$	$\times$
$p \geq 2$	$(p+1;\, 0, 0)$	$(0, p-2, 2)_c$	$\times$		$(p+2;\, 1, 0)$	$(2, p-3, 0)_c$	
	$(p+2;\, 0, 0)$	$(0, p-2, 0)_c$			$(p+3;\, 1, 0)$	$(0, p-3, 0)_c$	$\times$
	$(p+\frac{3}{2};\, \frac{1}{2}, 0)$	$(0, p-2, 1)_c$			$(p+\frac{5}{2};\, 1, \frac{1}{2})$	$(1, p-3, 0)_c$	
	$(p+1;\, \frac{1}{2}, \frac{1}{2})$	$(1, p-2, 1)$		$p \geq 4$	$(p+2;\, 0, 0)$	$(2, p-4, 2)$	
	$(p+\frac{3}{2};\, 1, \frac{1}{2})$	$(1, p-2, 0)_c$			$(p+3;\, 0, 0)$	$(0, p-4, 2)_c$	$\times$
	$(p+2;\, 1, 1)$	$(0, p-2, 0)$			$(p+4;\, 0, 0)$	$(0, p-4, 0)$	
					$(p+\frac{5}{2};\, \frac{1}{2}, 0)$	$(2, p-4, 1)_c$	
					$(p+\frac{7}{2};\, \frac{1}{2}, 0)$	$(0, p-4, 1)_c$	
					$(p+3;\, \frac{1}{2}, \frac{1}{2})$	$(1, p-4, 1)$	

Table 1. Multiplet content of compactification of type IIB supergravity on S^5. Here Δ is conformal dimension and (a, b, c) are Dynkin labels of SU(4). We use the notation $(a, b, c)_c = (a, b, c) + (c, b, a)$ with a corresponding swap of (j_1, j_2) indices. A cross means that the corresponding mode originates from the 10d tensor which is a complex combination of the NS-NS and R-R rank 2 antisymmetric tensor fields cf. [66].

This leads to $(N + \frac{1}{4})^2$ as the coefficient of the AdS_5 volume in the on-shell value of the 10d supergravity action and as a result we match the $N^2 + \frac{1}{2}N$ terms in the free energy in (1.3).

Below we will provide the explanation for the remaining $(-\frac{1}{32})$ term in a$-$anomaly coefficient in (4.1) as originating from the 1-loop contributions of the short multiplets of the supergravity modes, in full analogy with what happened for (-1) term in the $\mathcal{N} = 4$ SU(N) SYM case [46]. This demonstrates again that all long multiplets of massive string modes do not contribute to the conformal anomaly in the maximally supersymmetric case.

First, let us recall the KK spectrum of type IIB supergravity on $AdS_5 \times S^5$ [65, 66]. It is shown in table 1 where for each KK level p we list the corresponding SO(2, 4) and SU(4) representations (we use the same notation as in [46]). The dimension of SU(4) representation (a, b, c) is given by

$$\dim(a, b, c) = \frac{1}{12}(a+1)(b+1)(c+1)(a+b+2)(b+c+2)(a+b+c+3)\,. \tag{4.10}$$

The level $p = 1$ corresponds to the doubleton multiplet which is decoupled from the physical spectrum (the corresponding states are pure-gauge ones). The level $p = 2$ is the massless multiplet of gauged $\mathcal{N} = 8$ 5d supergravity. The states with $p \geq 3$ form shortened massive multiplets with spin ≤ 2.

JHEP01(2023)037

Applying the orientifold projection leading to $AdS_5\times\mathbb{RP}^5$ involves modding out by the $\mathbb{Z}_2$ subgroup of the $U(1)_R$ in the decomposition $SU(4)\supset SU(2)\times SU(2)\times U(1)_R$. In addition, the orientifold acts non-trivially on the supergravity fields changing sign of states originating from the 10d rank 2 antisymmetric tensor. This means the projection based on the value of the $U(1)_R$ charge Q_R [20]

$$(j_1,j_2)\neq(1,0),(0,1): \qquad Q_R=0\,(\mathrm{mod}\,4)\,, \tag{4.11}$$
$$(j_1,j_2)=(1,0),(0,1): \qquad Q_R=2\,(\mathrm{mod}\,4)\,. \tag{4.12}$$

For example, for the SU(4) representations of the form $(0,p,0)$ with $p=0,1,2,3$ one finds

$$\begin{aligned}
&(0,0,0)=\mathbf{1}=(\mathbf{1},\mathbf{1})_0,\\
&(0,1,0)=\mathbf{6}=(\mathbf{1},\mathbf{1})_{\pm2}+(\mathbf{2},\mathbf{2})_0,\\
&(0,2,0)=\mathbf{20'}=(\mathbf{1},\mathbf{1})_0+(\mathbf{1},\mathbf{1})_{\pm4}+(\mathbf{2},\mathbf{2})_{\pm2}+(\mathbf{3},\mathbf{3})_0,\\
&(0,3,0)=\mathbf{50}=(\mathbf{1},\mathbf{1})_{\pm2}+(\mathbf{1},\mathbf{1})_{\pm6}+(\mathbf{2},\mathbf{2})_0+(\mathbf{2},\mathbf{2})_{\pm4}+(\mathbf{3},\mathbf{3})_{\pm2}+(\mathbf{4},\mathbf{4})_0\,.
\end{aligned} \tag{4.13}$$

Here in the r.h.s. we labelled representations by the dimensions of the two SU(2) representations and indicated the R-charge ($\pm$ means the sum over the two values of the sign). Note also that since the $\mathbb{Z}_2$ action on S^5 giving $\mathbb{RP}^5$ is free so that there are no twisted-sector states.

To compute the 1-loop partition function we are to sum over the states at each KK level p and then over levels p. Let us introduce the notation:

$$\dim(a,b,c)|_q=\text{sum of dimensions of branched reps with the constraint } Q_R=q\,(\mathrm{mod}\,4)\,.$$

We have found the following explicit expressions

$$\begin{aligned}
\dim(0,p,0)|_0&=\frac{1}{16}(-1)^p+\frac{1}{48}(45+80p+52p^2+16p^3+2p^4),\\
\dim(0,p,1)|_0&=\dim(1,p,2)|_0=0,\\
\dim(0,p,0)|_2&=-\frac{1}{16}(-1)^p+\frac{1}{48}(3+32p+40p^2+16p^3+2p^4),\\
\dim(0,p,2)|_0&=-\frac{3}{16}(-1)^p+\frac{1}{16}(67+144p+96p^2+24p^3+2p^4),\\
\dim(1,p,1)|_0&=-\frac{1}{4}(-1)^p+\frac{1}{12}(87+168p+100p^2+24p^3+2p^4),\\
\dim(1,p,1)|_2&=\frac{(-1)^p}{4}+\frac{1}{12}(93+168p+100p^2+24p^3+2p^4),\\
\dim(0,p,2)|_2&=\frac{3(-1)^p}{16}+\frac{1}{16}(93+168p+100p^2+24p^3+2p^4),\\
\dim(2,p,2)|_0&=\frac{9(-1)^p}{16}+\frac{1}{16}(695+1120p+524p^2+96p^3+6p^4)\,.
\end{aligned} \tag{4.14}$$

We can then compute the total contribution a_p to the a-anomaly coefficient from all states at the level p using the expression for the a-coefficient derived in [46]. For $p\geq4$ we get

$$\mathrm{a}_p=\frac{1}{8}(-1)^p p+\frac{1}{1080}(521p-704p^3+470p^5-152p^7+12p^9)\,. \tag{4.15}$$

JHEP01(2023)037

Like what happened in the $AdS_5\times S^5$ case [46], this expression (4.15) happens to be valid for all $p\geq 1$, i.e. it actually reproduces also the results for low values of $p=1,2,3$, even though the structure of states in these cases is different from those for $p\geq 4$. In particular, due to the orientifolding and changed periodicity on the sphere, the states with $p=1$ are no longer pure-gauge doubleton ones and thus their contribution should be included in the sum [21].

As in the $\mathrm{AdS}_5\times S^5$ case, the sum representing the total 1-loop contribution to the a-anomaly coefficient is divergent and thus requires a definition that should be consistent with underlying symmetries of the 10d theory.[30] One particular regularization (used in similar context in [67, 68]) is to introduce a factor z^p with $z<1$, do the sum and then drop all terms that are singular (power-divergent) in the limit $z\to 1$. This way we get

$$\begin{aligned}\mathrm{a}_{1-\text{loop}} &= \sum_{p=1}^{\infty}\mathrm{a}_p \to \sum_{p=1}^{\infty}\mathrm{a}_p z^p\Big|_{z\to 1}\\ &= \frac{4032}{(z-1)^{10}}+\frac{20160}{(z-1)^9}+\frac{123872}{3(z-1)^8}+\frac{132608}{3(z-1)^7}+\frac{233962}{9(z-1)^6}+\frac{7950}{(z-1)^5}\\ &\quad+\frac{90277}{90(z-1)^4}+\frac{89}{15(z-1)^3}-\frac{1003}{360(z-1)^2}+\frac{49}{360(z-1)}-\frac{1}{32}+\mathcal{O}(z-1) \;\to\; -\frac{1}{32}\,.\end{aligned} \tag{4.16}$$

Keeping only the finite part of the sum we thus reproduce the 1-loop term $-\frac{1}{32}$ in the conformal a-anomaly in (4.1).[31]

The same result is found using an alternative regularization prescription [46] based on introducing an upper cutoff P in the sum over p and dropping all divergent terms that are polynomial in $P\to\infty$. Then the sum of the second $\frac{1}{1080}(521p-704p^3+470p^5-152p^7+12p^9)$ term in (4.15) gives a vanishing contribution. The remaining sum of the sign-alternating first term $\frac{1}{8}(-1)^p p$ in (4.15) is finite and is readily computed using either an analytic regularization

$$\alpha\to 1: \qquad \frac{1}{8}\sum_{p=1}^{\infty}(-1)^p p^\alpha = \frac{1}{8}(2^{\alpha+1}-1)\,\zeta(-\alpha) = -\frac{1}{32}+\mathcal{O}(\alpha-1)\,, \tag{4.17}$$

or an exponential cutoff

$$\epsilon\to 0: \qquad \frac{1}{8}\sum_{p=1}^{\infty}(-1)^p p\, e^{-p\epsilon} = -\frac{e^\epsilon}{8\,(e^\epsilon+1)^2} = -\frac{1}{32}+\mathcal{O}(\epsilon)\,, \tag{4.18}$$

in agreement with (4.16).

[30] This regularization issue appears due to the procedure of first compactifying on the 5-space and then regularizing; it would be absent if the computation were done directly in terms of the 10d determinants with a covariant regularization (see also a discussion in [46]).

[31] To compare, in the S^5 compactification case, using the same regularization one finds that $\sum_{p=1}^{\infty}\mathrm{a}_p = 0$ [46]. This implies that keeping all KK modes one gets $\mathrm{a}=\frac{1}{4}N^2$ (with no 1-loop shift) which is the result for the conformal anomaly of the $\mathcal{N}=4$ SYM with U(N) gauge group. In the $\mathrm{AdS}_5\times S^5$ case dual to SU(N) SYM theory, where the U(1) multiplet describing the D3-brane center-of-mass degrees of freedom should decouple, the $p=1$ (doubleton) contribution to the sum should not to be included and thus 1-loop correction to a should be given by $\sum_{p=2}^{\infty}\mathrm{a}_p=-1$. Once again, there is no similar decoupling of the $p=1$ level in the present orientifold case, i.e. the sum in (4.16) starts from $p=1$.

JHEP01(2023)037

4.2 Free energy in $\mathcal{N}=2$ theory dual to IIB string on $AdS_5\times S^5/\mathbb{Z}_{2,\rm ori}$

Let us now attempt to give a dual string theory understanding of the coefficient of the leading $\log\lambda$ term in the localization expression (1.9) for the free energy of the $\mathcal{N}=2$ Sp($2N$) FA theory, i.e.

$$F=-\left(N^2+N+\frac{3}{16}\right)\log\lambda+\dots\,. \tag{4.19}$$

We shall focus on the order $O(N^2)$ and $O(N)$ terms that should come from the sphere and disc/crosscup topologies. The computation of the remaining $\frac{3}{16}\log\lambda$ term (that should come from the closed-string 1-loop, i.e. torus contribution) appears to be more challenging than in the above maximally supersymmetric $\mathcal{N}=4$ SYM case and will not be attempted here.

The main idea is that as in the $\mathcal{N}=4$ SYM case [45] the $\log\lambda$ term in free energy should be associated with the regularized expression for the AdS_5 volume factor. The order $N^2\log\lambda$ term originates from the type IIB supergravity compactified on $S'^5=S^5/\mathbb{Z}_{2,\rm ori}$, while the order N term should come from the disc/crosscup contributions that may be interpreted as the action of D7+O7-branes wrapped on $AdS_5\times S^3$ where S^3 is the fixed-point locus of the orbifold action $\mathbb{Z}_{2,\rm ori}$.[32]

Let us first mention that to reproduce the conformal anomaly c-coefficient following [69, 70] one may consider the sum of the bulk 10d supergravity action and the D7+O7 action, $S\sim g_{\rm s}^{-2}\int d^{10}x\sqrt{g}(R+\dots)+g_{\rm s}^{-1}\int d^8x\sqrt{g}(RR+\dots)$ assuming that the 5d metric asymptotes to a general 4d metric at the boundary. The on-shell value of the action then contains the term $(N^2+q_1N)\int d^4x\sqrt{g}C^2_{mnkl}\log\Lambda$ where C is the Weyl tensor of the 4d boundary metric and Λ is an IR cutoff. One finds [23] that the order N term coming from the $\int d^8x\sqrt{g}RR$ action is precisely the one consistent with the value of c-a$=\frac{1}{4}N$ of the $\mathcal{N}=2$ Sp($2N$) FA theory in (4.2).[33]

To determine the a-anomaly one may chose the round 4-sphere metric at the boundary (so that the c-anomaly term proportional to the Weyl tensor squared will be vanishing). Then the on-shell value of the above action will scale as a factor of volume of AdS_5 (4.5), i.e. $(N^2+q_2N)\int d^5x\sqrt{g}\sim(N^2+q_2N)\log\Lambda_{\rm IR}$. To match the a-anomaly coefficient in (4.2) one should find that $N^2+q_2N=N^2+N$.

The computation of the $\log\lambda$ term in the free energy F on the dual string theory side is essentially equivalent to the computation of the a-anomaly term if we assume, following [45], that the $\log\lambda$ originates from a particular regularization of the AdS_5 volume factor as in (4.5). This effectively explains, on the dual string theory side, why the N^2+N terms in the a-anomaly in (4.2) happens to be the same as in the $\log\lambda$ term in the free energy F in (4.19).

[32] Here, compared to the $\mathcal{N}=4$ Sp($2N$) SYM case discussed above there will be no shift of N by $\frac{1}{4}$ so the order N term in F or in conformal anomaly will have a different interpretation.

[33] To capture the value of the coefficient c $-$ a it is sufficient to assume that $R_{mn}=0$ for the boundary metric is Ricci-flat as was effectively done in [23]. The value of the a-anomaly coefficient was not reproduced in [23] as only the R^2_{mnkl} term was included in the 8d action.

JHEP01(2023)037

In the case of the Sp($2N$) FA model dual to type IIB string on the orientifold $\mathrm{AdS}_5 \times S^5/\mathbb{Z}_{2,\mathrm{ori}}$ we start with $N_{\mathrm{cover}} = 2N$ D3-branes so that (cf. (4.9))

$$L^4 = \frac{\sqrt{\pi}}{2}\,\kappa_{10}\,\frac{N_{\mathrm{cover}}}{\mathrm{Vol}(S^5)} = 8\pi g_s N \alpha'^2\,. \tag{4.20}$$

As a result, as in the $\mathcal{N}=4$ Sp($2N$) SYM case

$$S_{10} = \frac{2}{\pi^2} N^2 \mathrm{Vol}(\mathrm{AdS}_5) = 2N^2\,\log(\Lambda_{\mathrm{IR}} \mathrm{r})\,, \qquad \mathrm{a} = \frac{1}{2}N^2 + \dots\,, \tag{4.21}$$

in agreement with (4.2).

To try to find the subleading order N term in free energy we shall as in [23] add to the bulk 10d supergravity action (2-sphere contribution) the 8d integral of the effective action S_7 of a combination of 8 D7-branes and an orientifold plane (disc+crosscup contribution). The presence of the orientifold plane cancels the tension part of S_7 (and thus there is no dilaton tadpole, implying stability consistent with supersymmetry). As a result, S_7 starts with terms quadratic in the curvature.[34] Then according to [71] (see also [72–75])

$$S_7 = k_7 \int d^8x \sqrt{g}\;\mathcal{L}_8\,, \qquad k_7 = 12\frac{(2\pi\alpha')^2}{6\times 32 g_s}\,\mu_7\,, \qquad \mu_7 = \frac{1}{2\,(2\pi)^7\alpha'^4}, \tag{4.22}$$

where μ_7 is the D7-brane tension[35] and

$$\mathcal{L}_8 = \mathcal{L}_{R^2} + \mathcal{L}_{F_5}\,, \qquad \mathcal{L}_{R^2} = (R_{\mathrm{T}})^2_{mnkl} - 2\,(R_{\mathrm{T}})^2_{mn} + 2\,\bar{R}^2_{ab}\,. \tag{4.23}$$

Here the tensor $(R_{\mathrm{T}})_{mnkl}$ is the pull-back of Riemann tensor on the brane tangent bundle, $(R_{\mathrm{T}})_{mn}$ is its Ricci tensor contraction involving only tangent indices, and $\bar{R}_{ab}$ is the tangent space contraction of the Riemann tensor with normal bundle indices (a,b) (see [71, 74, 75] for details). $\mathcal{L}_{F_5}$ in (4.23) stands for RR 5-form dependent terms that were not determined in [71].[36] We ignore other normal bundle contributions that are vanishing in the present case.

Let us note that [23] considered only the first R^2_{mnkl} term in $\mathcal{L}_{R^2}$ while ref. [76] discussed the $F^2_{mn}\mathcal{L}_{R^2}$ term in the D7-brane action in the context of investigation of the holographic dual of the Higgs branch of the $\mathcal{N}=2$ Sp($2N$) FA theory.[37]

[34] In general, for n D$_p$-branes and an orientifold p-plane we get a combination $n\,\mathrm{D}_p - 2^{p-4}\,\mathrm{O}_p$ of DBI+WZ actions while for the curvature-squared terms we get $n\,\mathrm{D}_p + 2^{p-5}\,\mathrm{O}_p$. In the present $p=7$ case we need $n=8$ to cancel the tadpole term and thus the R^2 term enters with the overall coefficient $8+4=12$.

[35] In the case of an orientifold the value of the D7 brane tension is reduced by $\frac{1}{2}$ (see, e.g., [59]).

[36] These terms could be, in principle, extracted from the six-point open string amplitudes on the disk and crosscup. The knowledge of $\mathcal{L}_{F_5}$ is not needed to compute the c-a anomaly coefficient not sensitive to the Ricci tensor of the boundary metric — as was already mentioned above, the leading order N term in $\mathrm{c}-\mathrm{a}=\frac{1}{4}N-\frac{1}{12}$ (cf. (4.2)) was reproduced in [23] just from the knowledge of the R^2_{mnkl} term in $\mathcal{L}_{R^2}$.

[37] In this case one may view D3-branes as instantons in 8d theory describing D7-branes. It was suggested in [76] that the condition of existence of uncorrected gauge-theory instanton solution imposes constraints on the structure of possible α'-corrections of the type $F^2_{mn}\mathcal{F}(R,F_5)$. The F_5 dependent terms here need not be a priori the same as appearing in $\mathcal{L}_8$ in (4.23) so the vanishing of $F^2_{mn}\mathcal{F}(R,F_5)$ on $\mathrm{AdS}_5\times S^3$ background need not imply the same for (4.22).

JHEP01(2023)037

In the present case D7+O7 branes are wrapped on $\mathrm{AdS}_5\times S^3$ where S^3 is the fixed-point locus of $\mathbb{Z}_{2,\mathrm{ori}}$. Normalizing the metric and curvature to unit scale, i.e. extracting a factor of AdS radius L in (4.20) we get for the coefficient in (4.22)

$$k_7 \;\to\; k_7' = L^4\, k_7 = 12\, \frac{(2\pi\alpha')^2}{6\times 32 g_s}\, \frac{1}{2\,(2\pi)^7\alpha'^4}\, (8\pi g_s N \alpha'^2) = \frac{N}{128\pi^4}. \tag{4.24}$$

Ignoring $\mathcal{L}_{F_5}$ in (4.23) and using that the curvature of $\mathrm{AdS}_5\times S^3$ is homogeneous we should have

$$\int d^8x\sqrt{g}\; \mathcal{L}_{R^2} = \mathrm{Vol}(\mathrm{AdS}_5)\,\mathrm{Vol}(S^3)\,\mathrm{k}_{R^2}\,. \tag{4.25}$$

Combining the S_7 term (4.22) with the bulk supergravity term (4.21) we get ($\mathrm{Vol}(S^3) = 2\pi^3$)

$$S = S_{10} + S_8 = \frac{2}{\pi^2}\mathrm{Vol}(AdS_5)\left(N^2 - \frac{1}{128}\mathrm{k}_{R^2} N\right). \tag{4.26}$$

To compute the coefficient k_{R^2} in (4.25) we need to account for the curvature of $\mathrm{AdS}_5\times S^3$. For unit-radius curvature of $AdS_5\times S^3$ we have

$$(R_{\mathrm{T}})_{mnkl} = \begin{cases} \mp(g_{mk}g_{nl} - g_{ml}g_{nk}), & \text{all indices in } AdS_5 \text{ or in } S^3 \\ 0, & \text{mixed indices.} \end{cases} \tag{4.27}$$

In d dimensions $(R_{\mathrm{T}})_{mn} = \pm(d-1)g_{mn}$ and thus

$$(R_{\mathrm{T}})^2_{mnkl} - 2\,(R_{\mathrm{T}})^2_{mn} = 2d(d-1) - 2d(d-1)^2 = -2d(d-1)(d-2) = \begin{cases} -120, & AdS_5, \\ -12, & S^3. \end{cases} \tag{4.28}$$

Finally, $\bar{R}_{ab} = g^{mn}R_{mabn} = -3g_{ab}$ with $m,n = 1,\ldots,8$ (tangent bundle) and $a,b = 9,10$ (normal bundle), so that $2\bar{R}^2_{ab} = 2\times 3^2\times 2 = 36$. The total value of the coefficient k_{R^2} in (4.25), (4.23) is then[38]

$$\mathrm{k}_{R^2} = -120 - 12 + 36 = -96\,. \tag{4.29}$$

As a result, eq. (4.26) becomes

$$S = \frac{2}{\pi^2}\mathrm{Vol}(AdS_5)\left(N^2 + \frac{3}{4}N\right). \tag{4.30}$$

This differs from the expected $N^2 + N = N^2 + \frac{3}{4}N + \frac{1}{4}N$, i.e. we are missing an extra $+\frac{1}{4}$ contribution to k_{R^2}.

This discrepancy with the expected result for the N^2+N terms in the free energy (4.19) and the conformal a-anomaly (4.2)

$$F = \frac{2}{\pi^2}\mathrm{Vol}(AdS_5)\,(N^2+N) = 2(N^2+N)\log(\Lambda_{\mathrm{IR}}\mathrm{r}) \to -(N^2+N)\log(\lambda\,\mathrm{r}^{-2}) \tag{4.31}$$

may be attributed to the fact that we did not take into account the F_5-dependent terms in (4.23). In the present case of $\mathrm{AdS}_5\times S'^5$ background possible F_5-dependent terms may

[38] This value of the R^2 coefficient is in agreement with the one found in [76].

JHEP01(2023)037

effectively contribute similarly to the Ricci-squared terms. More precisely, in the derivation of $\mathcal{L}_{R^2}$ term in [71] it was assumed that the bulk space curvature is Ricci flat, $R_{MN} = 0$, and F_5-dependent terms were ignored. The two types of terms are actually related by the 10d supergravity equations $R_{MN} = \frac{1}{4\times 4!} F_{MPQRS} F_N{}^{PQRS}$, $R^2_{MN} \sim (F_{MPQRS} F_N{}^{PQRS})^2$. Thus the missing terms may be parametrised as R^2_{MN}, implying the following possible correction on $AdS_5 \times S^3$ background

$$\mathcal{L}_{F_5} = \mathrm{k}_F \, R^2_{MN} = 160 \, \mathrm{k}_F \, . \tag{4.32}$$

Then (4.30) becomes the $N^2 + N$ combination in (4.31) if $\mathrm{k}_F = \frac{1}{5}$. Proving that this is actually the case remains an open problem.

Acknowledgments

We are thank to S. Giombi, A. Hanany, J. Minahan and A. Vainshtein for useful discussions. We are also grateful S. Chester and G. Dunne for helpful comments on the draft. MB was supported by the INFN grant GSS (Gauge Theories, Strings and Supergravity). The work of GK was supported by the French National Agency for Research grant ANR-17-CE31-0001-01. AAT was supported by the STFC grant ST/T000791/1. Part of this work was done while AAT was a participant of the "Integrability in String, Field, and Condensed Matter Theory" at KITP in Santa Barbara supported in part by the NSF under Grant No. NSF PHY-1748958.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License (CC-BY 4.0), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited. SCOAP^3 supports the goals of the International Year of Basic Sciences for Sustainable Development.

References

[1] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[2] V. Pestun et al., *Localization techniques in quantum field theories*, *J. Phys. A* **50** (2017) 440301 [arXiv:1608.02952] [INSPIRE].

[3] M. Beccaria, M. Billò, F. Galvagno, A. Hasan and A. Lerda, *$N = 2$ conformal SYM theories at large N*, *JHEP* **09** (2020) 116 [arXiv:2007.02840] [INSPIRE].

[4] M. Beccaria and A.A. Tseytlin, *$1/N$ expansion of circular Wilson loop in $N = 2$ superconformal $\mathrm{SU}(N) \times \mathrm{SU}(N)$ quiver*, *JHEP* **04** (2021) 265 [arXiv:2102.07696] [INSPIRE].

[5] M. Beccaria, G.V. Dunne and A.A. Tseytlin, *BPS Wilson loop in $N = 2$ superconformal $\mathrm{SU}(N)$ "orientifold" gauge theory and weak-strong coupling interpolation*, *JHEP* **07** (2021) 085 [arXiv:2104.12625] [INSPIRE].

[6] M. Beccaria, G.V. Dunne and A.A. Tseytlin, *Strong coupling expansion of free energy and BPS Wilson loop in $N = 2$ superconformal models with fundamental hypermultiplets*, *JHEP* **08** (2021) 102 [arXiv:2105.14729] [INSPIRE].

JHEP01(2023)037

[7] M. Beccaria, G.P. Korchemsky and A.A. Tseytlin, *Strong coupling expansion in $N = 2$ superconformal theories and the Bessel kernel*, *JHEP* **09** (2022) 226 [arXiv:2207.11475] [INSPIRE].

[8] N. Bobev, P.-J. De Smet and X. Zhang, *The planar limit of the $N = 2$ **E**-theory: numerical calculations and the large λ expansion*, arXiv:2207.12843 [INSPIRE].

[9] B. Fiol, B. Garolera and G. Torrents, *Exact probes of orientifolds*, *JHEP* **09** (2014) 169 [arXiv:1406.5129] [INSPIRE].

[10] B. Fiol, B. Garolera and G. Torrents, *Probing $N = 2$ superconformal field theories with localization*, *JHEP* **01** (2016) 168 [arXiv:1511.00616] [INSPIRE].

[11] B. Fiol, J. Martínez-Montoya and A. Rios Fukelman, *The planar limit of $N = 2$ superconformal field theories*, *JHEP* **05** (2020) 136 [arXiv:2003.02879] [INSPIRE].

[12] M. Billo, M. Frau, F. Galvagno, A. Lerda and A. Pini, *Strong-coupling results for $N = 2$ superconformal quivers and holography*, *JHEP* **10** (2021) 161 [arXiv:2109.00559] [INSPIRE].

[13] M. Billo, M. Frau, A. Lerda, A. Pini and P. Vallarino, *Three-point functions in a $N = 2$ superconformal gauge theory and their strong-coupling limit*, *JHEP* **08** (2022) 199 [arXiv:2202.06990] [INSPIRE].

[14] M. Billo, M. Frau, A. Lerda, A. Pini and P. Vallarino, *Localization vs holography in 4d $N = 2$ quiver theories*, *JHEP* **10** (2022) 020 [arXiv:2207.08846] [INSPIRE].

[15] A. Sen, *F theory and orientifolds*, *Nucl. Phys. B* **475** (1996) 562 [hep-th/9605150] [INSPIRE].

[16] K. Dasgupta and S. Mukhi, *F theory at constant coupling*, *Phys. Lett. B* **385** (1996) 125 [hep-th/9606044] [INSPIRE].

[17] T. Banks, M.R. Douglas and N. Seiberg, *Probing F theory with branes*, *Phys. Lett. B* **387** (1996) 278 [hep-th/9605199] [INSPIRE].

[18] O. Aharony, J. Sonnenschein, S. Yankielowicz and S. Theisen, *Field theory questions for string theory answers*, *Nucl. Phys. B* **493** (1997) 177 [hep-th/9611222] [INSPIRE].

[19] M.R. Douglas, D.A. Lowe and J.H. Schwarz, *Probing F theory with multiple branes*, *Phys. Lett. B* **394** (1997) 297 [hep-th/9612062] [INSPIRE].

[20] A. Fayyazuddin and M. Spalinski, *Large N superconformal gauge theories and supergravity orientifolds*, *Nucl. Phys. B* **535** (1998) 219 [hep-th/9805096] [INSPIRE].

[21] O. Aharony, A. Fayyazuddin and J.M. Maldacena, *The large N limit of $N = 2$, $N = 1$ field theories from three-branes in F theory*, *JHEP* **07** (1998) 013 [hep-th/9806159] [INSPIRE].

[22] J. Park and A.M. Uranga, *A note on superconformal $N = 2$ theories and orientifolds*, *Nucl. Phys. B* **542** (1999) 139 [hep-th/9808161] [INSPIRE].

[23] M. Blau, K.S. Narain and E. Gava, *On subleading contributions to the AdS/CFT trace anomaly*, *JHEP* **09** (1999) 018 [hep-th/9904179] [INSPIRE].

[24] I.P. Ennes, C. Lozano, S.G. Naculich and H.J. Schnitzer, *Elliptic models, type IIB orientifolds and the AdS/CFT correspondence*, *Nucl. Phys. B* **591** (2000) 195 [hep-th/0006140] [INSPIRE].

[25] S. Giombi and B. Offertaler, *Wilson loops in $N = 4$ SO(N) SYM and D-branes in $AdS_5 \times \mathbb{RP}^5$*, *JHEP* **10** (2021) 016 [arXiv:2006.10852] [INSPIRE].

JHEP01(2023)037

[26] V.V. Mangazeev and A.J. Guttmann, *Form factor expansions in the 2D Ising model and Painlevé VI*, *Nucl. Phys. B* **838** (2010) 391 [arXiv:1002.2480] [INSPIRE].

[27] B. Basso, G.P. Korchemsky and J. Kotanski, *Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling*, *Phys. Rev. Lett.* **100** (2008) 091601 [arXiv:0708.3933] [INSPIRE].

[28] A.V. Belitsky and G.P. Korchemsky, *Octagon at finite coupling*, *JHEP* **07** (2020) 219 [arXiv:2003.01121] [INSPIRE].

[29] A.V. Belitsky and G.P. Korchemsky, *Crossing bridges with strong Szegő limit theorem*, *JHEP* **04** (2021) 257 [arXiv:2006.01831] [INSPIRE].

[30] N. Dorey, V.V. Khoze and M.P. Mattis, *On $N = 2$ supersymmetric QCD with four flavors*, *Nucl. Phys. B* **492** (1997) 607 [hep-th/9611016] [INSPIRE].

[31] E. D'Hoker and D.H. Phong, *Lectures on supersymmetric Yang-Mills theory and integrable systems*, in 9th *CRM summer school: theoretical physics at the end of the* 20th *century*, (1999), p. 1 [hep-th/9912271] [INSPIRE].

[32] M. Billò, L. Gallot, A. Lerda and I. Pesando, *F-theoretic versus microscopic description of a conformal $N = 2$ SYM theory*, *JHEP* **11** (2010) 041 [arXiv:1008.5240] [INSPIRE].

[33] M. Billò, M. Frau, L. Gallot, A. Lerda and I. Pesando, *Deformed $N = 2$ theories, generalized recursion relations and S-duality*, *JHEP* **04** (2013) 039 [arXiv:1302.0686] [INSPIRE].

[34] M.A. Shifman, A.I. Vainshtein and M.B. Voloshin, *Anomaly and quantum corrections to solitons in two-dimensional theories with minimal supersymmetry*, *Phys. Rev. D* **59** (1999) 045016 [hep-th/9810068] [INSPIRE].

[35] N. Drukker and D.J. Gross, *An exact prediction of $N = 4$ SUSYM theory for string theory*, *J. Math. Phys.* **42** (2001) 2896 [hep-th/0010274] [INSPIRE].

[36] S. Giombi and A.A. Tseytlin, *Strong coupling expansion of circular Wilson loops and string theories in $AdS_5 \times S^5$ and $AdS_4 \times CP^3$*, *JHEP* **10** (2020) 130 [arXiv:2007.08512] [INSPIRE].

[37] A. Karch, B. Robinson and C.F. Uhlemann, *Precision test of gauge-gravity duality with flavor*, *Phys. Rev. Lett.* **115** (2015) 261601 [arXiv:1509.00013] [INSPIRE].

[38] S.M. Chester, M.B. Green, S.S. Pufu, Y. Wang and C. Wen, *Modular invariance in superstring theory from $N = 4$ super-Yang-Mills*, *JHEP* **11** (2020) 016 [arXiv:1912.13365] [INSPIRE].

[39] S.M. Chester, M.B. Green, S.S. Pufu, Y. Wang and C. Wen, *New modular invariants in $N = 4$ super-Yang-Mills theory*, *JHEP* **04** (2021) 212 [arXiv:2008.02713] [INSPIRE].

[40] D. Dorigoni, M.B. Green and C. Wen, *Exact properties of an integrated correlator in $N = 4$ SU(N) SYM*, *JHEP* **05** (2021) 089 [arXiv:2102.09537] [INSPIRE].

[41] L.F. Alday, S.M. Chester and T. Hansen, *Modular invariant holographic correlators for $N = 4$ SYM with general gauge group*, *JHEP* **12** (2021) 159 [arXiv:2110.13106] [INSPIRE].

[42] D. Dorigoni, M.B. Green and C. Wen, *Exact results for duality-covariant integrated correlators in $N = 4$ SYM with general classical gauge groups*, *SciPost Phys.* **13** (2022) 092 [arXiv:2202.05784] [INSPIRE].

[43] S.M. Chester, *Bootstrapping 4d $N = 2$ gauge theories: the case of SQCD*, arXiv:2205.12978 [INSPIRE].

JHEP01(2023)037

[44] Y. Hatsuda and K. Okuyama, *Large N expansion of an integrated correlator in $N = 4$ SYM*, *JHEP* **11** (2022) 086 [arXiv:2208.01891] [INSPIRE].

[45] J.G. Russo and K. Zarembo, *Large N limit of $N = 2$ SU(N) gauge theories from localization*, *JHEP* **10** (2012) 082 [arXiv:1207.3806] [INSPIRE].

[46] M. Beccaria and A.A. Tseytlin, *Higher spins in AdS_5 at one loop: vacuum energy, boundary conformal anomalies and AdS/CFT*, *JHEP* **11** (2014) 114 [arXiv:1410.3273] [INSPIRE].

[47] E. Witten, *Baryons and branes in anti-de Sitter space*, *JHEP* **07** (1998) 006 [hep-th/9805112] [INSPIRE].

[48] F. Passerini and K. Zarembo, *Wilson loops in $N = 2$ super-Yang-Mills from matrix model*, *JHEP* **09** (2011) 102 [*Erratum ibid.* **10** (2011) 065] [arXiv:1106.5763] [INSPIRE].

[49] A. Gerasimov, A. Marshakov, A. Mironov, A. Morozov and A. Orlov, *Matrix models of 2D gravity and Toda theory*, *Nucl. Phys. B* **357** (1991) 565 [INSPIRE].

[50] E.J. Martinec, *On the origin of integrability in matrix models*, *Commun. Math. Phys.* **138** (1991) 437 [INSPIRE].

[51] L. Álvarez-Gaumé, *Integrability in random matrix models*, in *Nato advanced research workshop: quantum field theory, statistical mechanics, quantum groups and topology*, (1991), p. 1.

[52] A. Morozov and S. Shakirov, *Exact 2-point function in Hermitian matrix model*, *JHEP* **12** (2009) 003 [arXiv:0906.0036] [INSPIRE].

[53] P.J. Forrester and N.S. Witte, *Application of the τ-function theory of Painlevé equations to random matrices: PIV, PII and the GUE*, *Commun. Math. Phys.* **219** (2001) 357.

[54] M.L. Mehta, *Random matrices*, Elsevier (2004).

[55] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, *Nucl. Phys. B* **582** (2000) 155 [hep-th/0003055] [INSPIRE].

[56] N. Drukker, 1/4 *BPS circular loops, unstable world-sheet instantons and the matrix model*, *JHEP* **09** (2006) 004 [hep-th/0605151] [INSPIRE].

[57] K. Zarembo, *Localization and AdS/CFT correspondence*, *J. Phys. A* **50** (2017) 443011 [arXiv:1608.02963] [INSPIRE].

[58] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183 [hep-th/9905111] [INSPIRE].

[59] O. Aharony, J. Pawelczyk, S. Theisen and S. Yankielowicz, *A note on anomalies in the AdS/CFT correspondence*, *Phys. Rev. D* **60** (1999) 066001 [hep-th/9901134] [INSPIRE].

[60] A.D. Shapere and Y. Tachikawa, *Central charges of $N = 2$ superconformal field theories in four dimensions*, *JHEP* **09** (2008) 109 [arXiv:0804.1957] [INSPIRE].

[61] S.A. Kurlyand and A.A. Tseytlin, *Type IIB supergravity action on $M^5 \times X^5$ solutions*, *Phys. Rev. D* **106** (2022) 086017 [arXiv:2206.14522] [INSPIRE].

[62] P. Mansfield, D. Nolland and T. Ueno, *Order $1/N^2$ test of the Maldacena conjecture. 2. The full bulk one loop contribution to the boundary Weyl anomaly*, *Phys. Lett. B* **565** (2003) 207 [hep-th/0208135] [INSPIRE].

JHEP01(2023)037

[63] J. Polchinski, *Tasi lectures on D-branes*, in *Theoretical Advanced Study Institute in Elementary Particle Physics (TASI 96): fields, strings, and duality*, (1996), p. 293 [hep-th/9611050] [INSPIRE].

[64] S. Mukhi, *Orientifolds: the unique personality of each space-time dimension*, in *Workshop on frontiers in field theory, quantum gravity and string theory*, (1997), p. 167 [hep-th/9710004] [INSPIRE].

[65] M. Gunaydin and N. Marcus, *The spectrum of the S^5 compactification of the chiral $N = 2$, $D = 10$ supergravity and the unitary supermultiplets of* $\mathrm{U}(2, 2/4)$, *Class. Quant. Grav.* **2** (1985) L11 [INSPIRE].

[66] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *The mass spectrum of chiral $N = 2$ $D = 10$ supergravity on S^5*, *Phys. Rev. D* **32** (1985) 389 [INSPIRE].

[67] P. Mansfield, D. Nolland and T. Ueno, *The boundary Weyl anomaly in the $N = 4$ SYM/type IIB supergravity correspondence*, *JHEP* **01** (2004) 013 [hep-th/0311021] [INSPIRE].

[68] A. Arabi Ardehali, J.T. Liu and P. Szepietowski, *$1/N^2$ corrections to the holographic Weyl anomaly*, *JHEP* **01** (2014) 002 [arXiv:1310.2611] [INSPIRE].

[69] H. Liu and A.A. Tseytlin, *$D = 4$ super Yang-Mills, $D = 5$ gauged supergravity, and $D = 4$ conformal supergravity*, *Nucl. Phys. B* **533** (1998) 88 [hep-th/9804083] [INSPIRE].

[70] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, *JHEP* **07** (1998) 023 [hep-th/9806087] [INSPIRE].

[71] C.P. Bachas, P. Bain and M.B. Green, *Curvature terms in D-brane actions and their M theory origin*, *JHEP* **05** (1999) 011 [hep-th/9903210] [INSPIRE].

[72] A. Fotopoulos, *On α'^2 corrections to the D-brane action for nongeodesic world volume embeddings*, *JHEP* **09** (2001) 005 [hep-th/0104146] [INSPIRE].

[73] H.J. Schnitzer and N. Wyllard, *An orientifold of $AdS_5 \times T^{11}$ with D7-branes, the associated α'^2 corrections and their role in the dual $N = 1$ $\mathrm{Sp}(2N + 2M) \times \mathrm{Sp}(2N)$ gauge theory*, *JHEP* **08** (2002) 012 [hep-th/0206071] [INSPIRE].

[74] D. Junghans and G. Shiu, *Brane curvature corrections to the $N = 1$ type II/F-theory effective action*, *JHEP* **03** (2015) 107 [arXiv:1407.0019] [INSPIRE].

[75] M. Weissenbacher, *On α'-effects from D-branes in 4d $N = 1$*, *JHEP* **11** (2020) 076 [arXiv:2006.15552] [INSPIRE].

[76] Z. Guralnik, S. Kovacs and B. Kulik, *Holography and the Higgs branch of $N = 2$ SYM theories*, *JHEP* **03** (2005) 063 [hep-th/0405127] [INSPIRE].

JHEP01(2023)037