

Discontinuous Galerkin Spectral/*hp* Element Modelling of Dispersive Shallow Water Systems

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Abstract

Two-dimensional shallow water systems are frequently used in engineering practice to model environmental flows. The benefit of these systems are that, by integration over the water depth, a two-dimensional system is obtained which approximates the full three-dimensional problem. Nevertheless, for most applications the need to propagate waves over many wavelengths means that the numerical solution of these equations remains particularly challenging. The requirement for an accurate discretisation in geometrically complex domains makes the use of spectral/ hp elements attractive. However, to allow for the possibility of discontinuous solutions the most natural formulation of the system is within a discontinuous Galerkin (DG) framework. In this paper we consider the unstructured spectral/ hp DG formulation of (i) weakly nonlinear dispersive Boussinesq equations and (ii) nonlinear shallow water equations (a subset of the Boussinesq equations). Discretisation of the Boussinesq equations involves resolving third order mixed derivatives. To efficiently handle these high order terms a new scalar formulation based on the divergence of the momentum equations is presented. Numerical computations illustrate the exponential convergence with regard to expansion order and finally, we simulate solitary wave solutions.

Keywords: Boussinesq equations, shallow water equations, spectral/ hp , discontinuous Galerkin method.

1 Introduction

In hydraulic, coastal, ocean and environmental engineering we frequently encounter situations where the length scale of the problem is relatively large compared to the vertical scale. Under this condition depth-integrated shallow water systems can provide good approximations to the full problem. If the waves are long enough to be considered non-dispersive, the flow can be described by the well-known shallow water equations (SWE). If it is necessary to compute shorter waves where frequency dispersion is important, dispersive shallow water systems, or Boussinesq-type equations, can be adopted. The benefit of using depth-integrated shallow water systems is that the full three-dimensional problem is reduced to a two-dimensional system, therefore allowing much larger computational domains to be modelled. Further, the depth-integrated formulations circumvent the moving boundary problem of the free water surface, greatly simplifying the solution process.

Two-dimensional shallow water systems can be derived from the three-dimensional Laplace equation together with kinematic conditions on all surfaces and a dynamic condition on the free surface. The vertical dimension of the problem is removed by integrating the Laplace equation through the depth and expanding the velocity potential in powers of the vertical coordinate. The order of approximation is then dependent on assumptions of nonlinearity and dispersion. There are therefore numerous different sets of equations presented in the literature. As mentioned above, when the dispersion is negligible we recover the SWE. If the dispersion is assumed to be small and of the same order as the nonlinearity, we arrive at the classical Boussinesq equations of Peregrine [29]. Unlike the non-dispersive SWE, the classical Boussinesq equations include third order mixed derivatives in the momentum equations. These account for the dispersive properties of the waves. The Peregrine equations are valid for weakly nonlinear and dispersive water waves in relatively shallow waters.

In the early nineties so-called enhanced or extended Boussinesq-type equations [25, 28] were introduced. These enhanced equations have improved dispersive characteristics and contain, in addition to the third order mixed derivative, third order spatial derivatives. Lately more accurate Boussinesq-type equations have been put forward, containing complex higher derivatives [26, 17, 27].

There has long been an interest in the use of spectral/ hp element methods for solving the SWE in the oceanic and atmospheric sciences, for example see [24, 18, 33, 14, 15]. The motivation has been the potentially significant saving in computational time, compared to low-order methods, for long-time integration coupled with the ability of these methods to handle geometrically

complex domains.

However, to allow for the possibility of discontinuous solutions the most natural formulation of the system is within a discontinuous Galerkin (DG) framework. The DG method, using low-order approximations in combination with slope limiters, has been applied to the SWE including shocks in [32, 23, 2]. The first application of a discontinuous Galerkin formulation of the SWE with a spectral/ hp elements discretisation was by Dupont [10]. In this work the primitive SWE was solved on unstructured triangular domains using an expansion basis made up from a product of 1D Legendre polynomials with a triangular truncation. Later, Giraldo *et al.* [16] presented a nodal high-order DG method for solving the SWE on the sphere using curvilinear quadrilaterals. In this work the elemental solution was approximated by a nodal Lagrange polynomial, constructed from a tensor product of 1D Legendre cardinal functions. This discretisation has a diagonal mass matrix through a discrete orthogonality associated with numerical under-integration. The exponential convergence of the models was confirmed by numerical tests. Recently Eskilsson and Sherwin [13] employed an orthogonal modal expansion basis on unstructured triangular elements. This approach leads to a diagonal mass matrix without resorting to under-integration. Optimal convergence of order $p + 1$ was numerically demonstrated.

There are numerous instances of the use of finite differences to solve dispersive shallow water systems. Frequently cited works include [1, 36, 31]. A number of linear finite element models also have also been put forward [20, 4, 3, 22, 35].

The application of spectral/ hp element methods to Boussinesq-type equations rather than the SWE is, however, relatively under-investigated, and has, so far, only been applied to one-dimensional problems [11, 12].

The presence of higher order derivatives in Boussinesq-type equations poses some challenging problems. In particular, there is a need to keep the numerical dispersion arising from the numerical discretisation considerably smaller than the physical dispersion. While it has been shown that for 1D extended Boussinesq-type equations numerical dispersion introduced by uniform linear elements will not contaminate the physical dispersion on normal grid sizes [35], such contamination might occur on unstructured linear triangles. For example, Walkley [35] reported that linear expansions on unstructured triangular meshes required a much finer spacing than necessary on a regular mesh, and attributed this to low order truncation terms. This suggests that the use of high-order expansions will be beneficial on unstructured triangular meshes.

For spectral/ hp methods, an explicit in time discretisation of higher order derivatives can lead to a prohibitively severe time step restriction. To

illustrate this point, consider the following 1D scalar model equations:

- The non-dispersive linear advection equation

$$\partial_t u + \partial_x u = 0.$$

- The linearised KdV equation where the dispersive term is a third order spatial derivative

$$\partial_t u + \partial_x u + \partial_{xxx} u = 0.$$

- The linearised regularised long wave (RLW) equation where the dispersion is modelled by a third order mixed derivative

$$\partial_t u + \partial_x u + 0.1 \partial_{xxt} u = 0.$$

- A modified linearised regularised long wave (RLWm) equation where we have added an extra dispersive third order spatial derivative

$$\partial_t u + \partial_x u + 0.1 \partial_{xxt} u + \partial_{xxx} u = 0.$$

These equations were discretised in space using the DG formulation with Legendre polynomials of order P as expansion basis. For the RLW and RLWm equations the ∂_{xxt} terms are combined with the time derivatives to produce a Helmholtz operator which premultiplies the time derivative [11]. We have then applied upwinded numerical fluxes to the advective terms and averaged fluxes for the dispersive terms.

Denoting the semi-discrete equations as $\mathbf{A} \partial_t \mathbf{u} + \mathbf{B} \mathbf{u} = \mathbf{0}$ we are interested in the eigenvalues of the operator $\mathbf{A}^{-1} \mathbf{B}$. In figure 1 we plot the maximum eigenvalue of $\mathbf{A}^{-1} \mathbf{B}$ against the expansion order. For the linear advection equation the growth of the maximum eigenvalue is $\mathcal{O}(P^2)$, giving an acceptable restriction of the explicit time step for higher P . It is notable that under this discretisation the growth of the maximum eigenvalue of the different dispersive models is very different. The dispersive ∂_{xxx} term in the KdV equation gives rise to a growth rate of $\mathcal{O}(P^6)$. The implicit treatment of the ∂_{xxt} term in the RLW equation leads to a maximum eigenvalue which is independent of the polynomial degree. Interestingly, for the RLWm equation we see a growth rate bounded by P^2 , indicating that the time step restriction will be no harsher than for the pure advection case. Thus an implicit treatment of the mixed derivatives facilitates the use of explicit time stepping schemes for the spatial derivatives.

This paper presents an extension of the two-dimensional spectral/*hp* DG SWE model [13] to incorporate the lowest order dispersive terms. This corresponds to solving a slightly altered version of the classical Boussinesq equations [29] containing the fundamental components of the more complicated equations. In formulating these problem we draw on the work of Cockburn and Shu on the Runge-Kutta DG method [8] as well as on the work of Bassi and Rebay [5].

The paper is organised as follows. In section 2 we outline the governing equations. The numerical models are presented in section 3, divided between the advective and the dispersive terms. We numerically demonstrate the exponential convergence of the models and model solitary waves in section 4. Finally, in section 5 we summarise the study.

2 Governing Equations

The weakly nonlinear and dispersive Boussinesq equations of Peregrine [29] are

$$\partial_t \zeta + \nabla \cdot ((\zeta + d)\mathbf{u}) = 0, \quad (1a)$$

$$\partial_t \mathbf{u} + \frac{d^2}{6} \partial_t \nabla (\nabla \cdot \mathbf{u}) - \frac{d}{2} \partial_t \nabla (\nabla \cdot (d\mathbf{u})) + (\mathbf{u} \cdot \nabla) \mathbf{u} + g \nabla \zeta = \mathbf{s}(\mathbf{u}), \quad (1b)$$

where $H(x, y, t) = \zeta(x, y, t) + d(x, y)$ is the total water depth, $d(x, y)$ the still water depth and $\zeta(x, y, t)$ the free surface elevation; $\mathbf{u} = [u(x, y, t), v(x, y, t)]^T$ denotes the depth-averaged velocities in the x - and y -direction, respectively, and g is the acceleration of gravity. We will primarily be concerned with the homogeneous equations $\mathbf{s}(\mathbf{u}) = 0$, with the exception of equatorial Rossby waves. In that case the source term accounts for forcing due to Coriolis effects; $\mathbf{s}(\mathbf{u}) = [fv, -fu]^T$. The Coriolis parameter $f(y)$ is given by the β -plane approximation, i.e. $f = f_0 + \beta y$.

Neglecting the dispersive terms, i.e. the second and third terms in equation (1b), recovers the SWE expressed in primitive variables. We note that one possibility to remove the mixed derivatives in equation (1b) is to use the zero order approximation: $\partial_t \mathbf{u} = -g \nabla \zeta$. Such an approach results in the presence of third-order spatial derivatives which can be resolved using the local DG method for KdV-type equations [37]. However, as mentioned previously the restriction on the time step for explicit schemes is very severe for this type of equation.

Considering the possibility of discontinuities, it is desirable to express the equations in the conservative variables $(H, H\mathbf{u})$. Within the order of the approximation of the Boussinesq equations, we can let $H \approx d$ to rewrite

the dispersive terms in conservative variables. The Boussinesq equations in conservation form are therefore

$$\partial_t \mathbf{U} + \partial_t \mathbf{D}(\mathbf{U}) + \nabla \cdot \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \quad (2)$$

where $\mathbf{F}(\mathbf{U}) = [\mathbf{E}(\mathbf{U}), \mathbf{G}(\mathbf{U})]^T$ and

$$\mathbf{U} = \begin{bmatrix} H \\ Hu \\ Hv \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 \\ \frac{d^3}{6} \partial_x (\nabla \cdot (\frac{H\mathbf{u}}{d})) - \frac{d^2}{2} \partial_x (\nabla \cdot (H\mathbf{u})) \\ \frac{d^3}{6} \partial_y (\nabla \cdot (\frac{H\mathbf{u}}{d})) - \frac{d^2}{2} \partial_y (\nabla \cdot (H\mathbf{u})) \end{bmatrix}, \quad (3a)$$

$$\mathbf{E} = \begin{bmatrix} Hu \\ Hu^2 + gH^2/2 \\ Huv \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} Hv \\ Huv \\ Hv^2 + gH^2/2 \end{bmatrix}. \quad (3b)$$

In addition to the Coriolis terms, the source term $\mathbf{S}(\mathbf{U})$ now contains the forcing from bed slopes

$$\mathbf{S} = \begin{bmatrix} 0 \\ gHS_{0x} + fv \\ gHS_{0y} - fu \end{bmatrix}, \quad (4)$$

where S_{0x} and S_{0y} are the bed slopes in the x - and y -direction, respectively. To recover the shallow water equations in conservative variables we set $\mathbf{D} \equiv \mathbf{0}$.

3 Discontinuous Galerkin Method

We begin by dividing equation (2) into an advective part and a dispersive part where the source term is included in the advective part, i.e.

$$\mathbf{f}(\mathbf{U}) + \nabla \cdot \mathbf{F}(\mathbf{U}) = \mathbf{S}(\mathbf{U}), \quad (5a)$$

$$\partial_t \mathbf{U} + \partial_t \mathbf{D}(\mathbf{U}) = \mathbf{f}(\mathbf{U}). \quad (5b)$$

When solving the SWE we only require the advective contribution, while for the Boussinesq equations we have to consider both the advective and dispersive parts in sequence.

Let $\mathcal{T}_h$ be a partition of the domain Ω into N triangular elements $\mathcal{T}_e$ with boundary $\partial\mathcal{T}_e$. We define the following discrete spaces

$$\mathcal{V}_\delta = \{v \in L^2(\Omega) : v|_{\mathcal{T}_e} \in \mathcal{P}^P(\mathcal{T}_e), \forall \mathcal{T}_e \in \mathcal{T}_h\}, \quad (6a)$$

$$\mathcal{W}_\delta = \{\mathbf{w} \in (L^2(\Omega))^2 : \mathbf{w}|_{\mathcal{T}_e} \in (\mathcal{P}^P(\mathcal{T}_e))^2, \forall \mathcal{T}_e \in \mathcal{T}_h\}, \quad (6b)$$

where $\mathcal{P}^P(\mathcal{T}_e)$ is the space of polynomials of degree at most P in the element $\mathcal{T}_e$.

3.1 Formulation of the Advective terms

Multiplying equation (5a) by a smooth test function $\mathbf{q}(\mathbf{x})$ and integrating over the element $\mathcal{T}_e$ we obtain:

$$\int_{\mathcal{T}_e} \mathbf{f}(\mathbf{U}) \mathbf{q} \, d\mathbf{x} + \int_{\mathcal{T}_e} (\nabla \cdot \mathbf{F}(\mathbf{U})) \mathbf{q} \, d\mathbf{x} = \int_{\mathcal{T}_e} \mathbf{S}(\mathbf{U}) \mathbf{q} \, d\mathbf{x}. \quad (7)$$

Integration by parts gives the following weak formulation

$$\begin{aligned} \int_{\mathcal{T}_e} \mathbf{f}(\mathbf{U}) \mathbf{q} \, d\mathbf{x} - \int_{\mathcal{T}_e} (\mathbf{F}(\mathbf{U}) \cdot \nabla) \mathbf{q} \, d\mathbf{x} \\ + \int_{\partial\mathcal{T}_e} (\mathbf{F}(\mathbf{U}) \cdot \mathbf{n}) \mathbf{q} \, ds = \int_{\mathcal{T}_e} \mathbf{S}(\mathbf{U}) \mathbf{q} \, d\mathbf{x}, \end{aligned} \quad (8)$$

where $\mathbf{n} = (n_x, n_y)^T$ is the outward unit normal to $\partial\mathcal{T}_e$. The discrete Galerkin approximation is then obtained by replacing $\mathbf{q}$ with functions in a discrete test space, i.e. $\mathbf{q}_\delta \in \mathcal{V}_\delta$, as well as approximating the exact solution $\mathbf{U}$ with a finite expansion $\mathbf{U}_\delta \in \mathcal{V}_\delta$. Following the standard discontinuous approach we replace the boundary flux $\mathbf{F}(\mathbf{U})$ in the third term of equation (8) with a numerical flux, denoted $\hat{\mathbf{F}}(\mathbf{U})$.

After integrating by parts once more to obtain the divergence form the DG method consists of finding $\mathbf{U}_\delta \in \mathcal{V}_\delta$ such that for all $\mathbf{q}_\delta \in \mathcal{V}_\delta$ and for all $\mathcal{T}_e \in \mathcal{T}_h$:

$$\begin{aligned} \int_{\mathcal{T}_e} \mathbf{f}(\mathbf{U}_\delta) \mathbf{q}_\delta \, d\mathbf{x} + \int_{\mathcal{T}_e} (\nabla \cdot \mathbf{F}(\mathbf{U}_\delta)) \mathbf{q}_\delta \, d\mathbf{x} \\ + \int_{\partial\mathcal{T}_e} \left((\hat{\mathbf{F}}(\mathbf{U}_\delta) - \mathbf{F}(\mathbf{U}_\delta)) \cdot \mathbf{n} \right) \mathbf{q}_\delta \, ds = \int_{\mathcal{T}_e} \mathbf{S}(\mathbf{U}_\delta) \mathbf{q}_\delta \, d\mathbf{x}. \end{aligned} \quad (9)$$

For computational convenience we prefer the divergence form (equation (9)) rather than the Green's form (equation (8)) since it involves inner products of the same form as in classical continuous Galerkin schemes. Giraldo *et al.* [16] investigated the effect of solving the SWE in divergence form rather than in Green's form. They found no significant differences in the results due to the difference in formulation.

3.1.1 Numerical flux evaluation

In order to obtain a suitable inter-elemental coupling we need to define the numerical flux $\hat{\mathbf{F}}(\mathbf{U})$. Examples of numerical fluxes used in DG SWE models are the Roe flux [2], the HLL/HLLC flux [32, 13], the Lax-Friedrich flux [23, 16] and, for smooth flows, even simple averaging [10]. In this paper

we apply the HLLC approximate Riemann solver in conjunction with the two-rarefaction assumption [34].

Introducing the rotation matrix and its inverse

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_x & n_y \\ 0 & -n_y & n_x \end{bmatrix}, \quad \mathbf{T}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & n_x & -n_y \\ 0 & n_y & n_x \end{bmatrix}, \quad (10)$$

we can define $\mathbf{Q} = \mathbf{T}\mathbf{U}_\delta = [H, H\bar{u}, H\bar{v}]^T$, where $\bar{u}$ and $\bar{v}$ are the velocities in the direction normal and tangential to the edge, respectively. The flux can then be written as

$$\hat{\mathbf{F}}(\mathbf{U}_\delta) \cdot \mathbf{n} = \mathbf{T}^{-1} \hat{\mathbf{E}}(\mathbf{Q}). \quad (11)$$

Letting the subscripts L and R denote the left- and right-hand side of the element boundary, we estimate the wave speeds as [34]:

$$S_L = \bar{u}_L - \sqrt{gH_L} s_L, \quad (12)$$

$$S_R = \bar{u}_R + \sqrt{gH_R} s_R, \quad (13)$$

$$S_* = \frac{S_L H_R (\bar{u}_R - S_R) - S_R H_L (\bar{u}_L - S_L)}{H_R (\bar{u}_R - S_R) - H_L (\bar{u}_L - S_L)}, \quad (14)$$

where

$$s_{(L,R)} = \begin{cases} \sqrt{(H_*^2 + H_* H_{(L,R)}) / (2H_{(L,R)}^2)} & \text{if } H_* > H_{(L,R)}, \\ 1 & \text{if } H_* \leq H_{(L,R)}, \end{cases} \quad (15)$$

and in which H_* is given by the two-rarefaction Riemann solver:

$$H_* = \frac{1}{g} \left(\frac{1}{2} (\sqrt{gH_L} + \sqrt{gH_R}) + \frac{1}{4} (\bar{u}_L - \bar{u}_R) \right)^2. \quad (16)$$

After the wave speeds have been computed the HLLC flux is given by:

$$\hat{\mathbf{E}}(\mathbf{Q}) = \begin{cases} \mathbf{E}(\mathbf{Q}_L) & \text{if } S_L \geq 0, \\ \mathbf{E}(\mathbf{Q}_L) + S_L (\mathbf{Q}_{*L} - \mathbf{Q}_L) & \text{if } S_L \leq 0 \leq S_*, \\ \mathbf{E}(\mathbf{Q}_R) + S_R (\mathbf{Q}_{*R} - \mathbf{Q}_R) & \text{if } S_* \leq 0 \leq S_R, \\ \mathbf{E}(\mathbf{Q}_R) & \text{if } S_R \leq 0, \end{cases} \quad (17)$$

where $\mathbf{Q}_{*L}$ and $\mathbf{Q}_{*R}$ are obtained from

$$\mathbf{Q}_{*(L,R)} = H_{(L,R)} \left(\frac{S_{(L,R)} - \bar{u}_{(L,R)}}{S_{(L,R)} - S_*} \right) \begin{bmatrix} 1 \\ S_* \\ \bar{v}_{(L,R)} \end{bmatrix}. \quad (18)$$

3.1.2 Boundary conditions

In addition to standard periodic boundaries, we also consider slip wall boundaries. At a wall boundary we assume a no-permeability condition: $\mathbf{u} \cdot \mathbf{n} = 0$. If the left state is internal to an element then the slip condition is enforced via the Riemann solver by setting the right hand state to

$$\zeta_R = \zeta_L, \quad \bar{u}_R = -\bar{u}_L, \quad \bar{v}_R = \bar{v}_L. \quad (19)$$

3.2 Formulation of the Dispersive terms

Recalling that there are no dispersive terms present in the mass equation, we are left to consider the remaining components of equation (5b), which for a constant depth, d , can be expressed as

$$\partial_t(H\mathbf{u}) - \frac{d^2}{3}\nabla(\nabla \cdot \partial_t(H\mathbf{u})) = \mathbf{a}. \quad (20)$$

In equation (20) $\mathbf{a}$ denotes the component of $\mathbf{f}(\mathbf{U})$ associated with the momentum $H\mathbf{u} = [Hu, Hv]^T$ and discretely evaluated by the solution of equation (9). Introducing a new scalar variable of the time rate of change of momentum divergence, i.e. $z \equiv \nabla \cdot \partial_t(H\mathbf{u})$, an equivalent statement to problem (20) is

$$\partial_t(H\mathbf{u}) = \frac{d^2}{3}\nabla z + \mathbf{a}, \quad (21a)$$

$$z - \nabla \cdot \partial_t(H\mathbf{u}) = 0. \quad (21b)$$

Finally, substituting (21a) into (21b) and rearranging, we arrive at the Helmholtz equation

$$\nabla^2 z + \lambda z = \lambda(\nabla \cdot \mathbf{a}), \quad (22)$$

where $\lambda = -3/d^2$. Equation (22) can be considered as the divergence of equation (20) expressed in terms of the time rate of change of momentum divergence, z . The conservative variables $H\mathbf{u}$ can be recovered by solving equation (21a) which introduces the temporal element to the problem. The solution of (22) followed by (21a) allows the solution of $H\mathbf{u}$ to be constructed in a decoupled manner as opposed to equation (20) that directly couples the components of $H\mathbf{u}$. The scalar approach therefore leads to a $N_{\text{dof}} \times N_{\text{dof}}$ Helmholtz problem as compared to solving the coupled problem of equation (20) which leads to a $2N_{\text{dof}} \times 2N_{\text{dof}}$ matrix problem. The scalar approach is also preferable to uncoupling the momentum equations by treating the cross-derivatives explicitly [36] as this still results in two $N_{\text{dof}} \times N_{\text{dof}}$ problems.

To solve equation (22) we use a discontinuous Galerkin formulation which by introducing the auxiliary variables $\mathbf{w} = \nabla z$ can be rewritten as:

$$\nabla \cdot \mathbf{w} + \lambda z = \lambda(\nabla \cdot \mathbf{a}) , \quad (23a)$$

$$\mathbf{w} - \nabla z = 0 . \quad (23b)$$

The DG method, in divergence form, for solving equations (23a)–(23b) can be stated as: find $(z_\delta, \mathbf{w}_\delta) \in \mathcal{V}_\delta \times \mathcal{W}_\delta$ such that for all $(s_\delta, \mathbf{r}_\delta) \in \mathcal{V}_\delta \times \mathcal{W}_\delta$ and for all $\mathcal{T}_e \in \mathcal{T}_h$

$$\begin{aligned} \int_{\mathcal{T}_e} (\nabla \cdot \mathbf{w}_\delta) s_\delta d\mathbf{x} + \int_{\partial\mathcal{T}_e} ((\hat{\mathbf{w}}_\delta - \mathbf{w}_\delta) \cdot \mathbf{n}) s_\delta ds + \lambda \int_{\mathcal{T}_e} z_\delta s_\delta d\mathbf{x} \\ = \lambda \int_{\mathcal{T}_e} (\nabla \cdot \mathbf{a}_\delta) s_\delta d\mathbf{x} + \lambda \int_{\partial\mathcal{T}_e} ((\hat{\mathbf{a}}_\delta - \mathbf{a}_\delta) \cdot \mathbf{n}) s_\delta ds , \end{aligned} \quad (24a)$$

$$\int_{\mathcal{T}_e} \mathbf{w}_\delta \cdot \mathbf{r}_\delta d\mathbf{x} - \int_{\mathcal{T}_e} \nabla z_\delta \cdot \mathbf{r}_\delta d\mathbf{x} - \int_{\partial\mathcal{T}_e} ((\hat{z}_\delta - z_\delta) \mathbf{r}_\delta) \cdot \mathbf{n} ds = 0 . \quad (24b)$$

We note that $\mathbf{w}_\delta$ is directly substituted into equation (24a) and not explicitly computed.

As mentioned above, after z_δ has been obtained from equations (24a)–(24b), we return to the conservative variables $H\mathbf{u}$ using equation (21a): find $(H\mathbf{u})_\delta \in \mathcal{W}_\delta$ such that for all $\mathbf{t}_\delta \in \mathcal{W}_\delta$ and for all $\mathcal{T}_e \in \mathcal{T}_h$

$$\int_{\mathcal{T}_e} \partial_t (H\mathbf{u})_\delta \cdot \mathbf{t}_\delta d\mathbf{x} = \frac{d^2}{3} \int_{\mathcal{T}_e} \nabla z_\delta \cdot \mathbf{t}_\delta d\mathbf{x} + \frac{d^2}{3} \int_{\partial\mathcal{T}_e} ((\hat{z}_\delta - z_\delta) \mathbf{t}_\delta) \cdot \mathbf{n} ds + \int_{\mathcal{T}_e} \mathbf{a}_\delta \cdot \mathbf{t}_\delta d\mathbf{x} . \quad (25)$$

where ∂_t is approximated by an appropriate time discretisation scheme, see section 3.4.

3.2.1 Numerical flux

For the dispersive fluxes we have applied a simple averaging [5]:

$$\hat{z}_\delta = \frac{1}{2} (z_L + z_R) , \quad (26a)$$

$$\hat{\mathbf{w}}_\delta = \frac{1}{2} (\mathbf{w}_L + \mathbf{w}_R) , \quad (26b)$$

$$\hat{\mathbf{a}}_\delta = \frac{1}{2} (\mathbf{a}_L + \mathbf{a}_R) . \quad (26c)$$

Currently we have only considered periodic boundaries for the dispersive terms. Further work is required to determine appropriate boundary conditions for z and $\mathbf{a}$.

3.3 Polynomial Expansion basis

In the current work we have approximated the solution using a polynomial expansion basis $\phi_{pq}(\xi_1, \xi_2)$, such that

$$\mathbf{U}_\delta(\mathbf{x}, t) = \sum_{p=0}^P \sum_{q=0}^{P-p} \tilde{\mathbf{U}}_{pq}(t) \phi_{pq}(\xi_1, \xi_2), \quad \mathbf{x} \in \mathcal{T}_e, \quad (27)$$

where $\tilde{\mathbf{U}}_{pq}(t)$ contains the local degrees of freedom of expansion coefficients.

The orthogonal hierarchical basis $\phi_{pq}(\xi_1, \xi_2)$ in a standard triangular region $\{-1 \leq \xi_1, \xi_2; \xi_1 + \xi_2 \leq 0\}$ is based on a collapsed coordinate [19] which is generated through the transformation $(\xi_1, \xi_2) \rightarrow (\eta_1, \eta_2)$ given by:

$$\eta_1 = 2 \frac{(1 + \xi_1)}{(1 - \xi_2)} - 1, \quad \eta_2 = \xi_2. \quad (28)$$

This collapsed coordinate transformation can be interpreted as a mapping to a standard quadrilateral region from the standard triangular region.

An orthogonal basis on these coordinates has been independently derived in a range of works including [30, 21, 9]. Following the formulation in [9, 19] the expansion modes ϕ_{pq} are defined in terms of principal functions $\tilde{\psi}_p^a(z)$ and $\tilde{\psi}_{pq}^b(z)$ as

$$\phi_{pq}(\xi_1, \xi_2) = \tilde{\psi}_p^a(\eta_1) \tilde{\psi}_{pq}^b(\eta_2). \quad (29)$$

The principal functions are

$$\tilde{\psi}_p^a(z) = P_p^{0,0}(z), \quad \tilde{\psi}_{pq}^b(z) = \left(\frac{1-z}{2} \right)^p P_q^{2p+1,0}(z), \quad (30)$$

where $P_p^{\alpha,\beta}(z)$ denotes the p th order Jacobi polynomial. This basis is a polynomial function in terms of both the (ξ_1, ξ_2) and (η_1, η_2) coordinate systems. In addition to giving rise to an orthogonal mass matrix, the use of the collapsed coordinate system means that integrals over the element $\mathcal{T}_e$ can be efficiently evaluated as the product of two one-dimensional integrals. Typically, we adopt Gauss-Lobatto quadrature in the η_1 -direction and Gauss-Radau quadrature in the η_2 -direction, to avoid incorporating any information from the degenerated vertex at $\xi_2 = 1$.

3.4 Time stepping

The time stepping is carried out using the explicit TVD third-order Runge-Kutta scheme of Cockburn & Shu [8]. In each of the three substeps we

- Compute the advective contribution given by equation (9).
- Solve the DG formulation of the scalar Helmholtz equations (24a)-(24b) for the auxiliary variable z .
- Recover the conservative variables using equation (25).

4 Computational Examples

4.1 *Regular linear waves*

To illustrate the previous formulation we start by demonstrating the exponential convergence of the models by considering the simple case of a sinusoidal linear wave in a periodical domain of size $L \times L$; L being the wavelength. We compute the propagation of the wave using the linearised shallow water and Boussinesq equations with the Coriolis and friction parameters set to zero. The exact solution is given by

$$H(x, y, t) = d + a \cos(kx - \omega t), \quad (31a)$$

$$u(x, y, t) = a \frac{\omega}{kd} \cos(kx - \omega t), \quad (31b)$$

$$v(x, y, t) = 0, \quad (31c)$$

in which a is the amplitude and k is the wavenumber. The frequency ω is obtained from the linear dispersion relation

$$\frac{\omega^2}{gdk^2} = \frac{1}{1 - \gamma (kd)^2}, \quad (32)$$

where $\gamma = 0$ for the SWE and $\gamma = -1/3$ for the Boussinesq equations. We use an amplitude of $a = 0.01$ and a wavelength of $L = 20$. The undisturbed water depth is set to $d = 0.5$ and $d = 5.0$ for the SWE and Boussinesq models, respectively. Thus the ratio of water depth to deep water wavelength (d/L_0) for the Boussinesq model is 0.22, right at the limit of the range of validity [25].

We consider a computational domain of 32 uniform triangles and integrate for one wave period. The time step is chosen sufficiently small so temporal error is negligible. The computational error relative to the initial condition in the L^2 and L^∞ norms is shown in figure 2. From the approximately straight lines on this semi-log plot we may conclude that the SWE and the Boussinesq models exhibit the expected exponential convergence with regard to polynomial order.

4.2 *Equatorial Rossby waves*

We now consider the governing equations, including Coriolis forces, in non-dimensional variables (denoted with asterisks):

$$\mathbf{x} = \frac{r}{E^{1/4}} \mathbf{x}^*, \quad t = \frac{E^{1/4}}{2\Omega} t^*, \quad \zeta = d_0 \zeta^*, \quad \mathbf{u} = \sqrt{gd_0} \mathbf{u}^*, \quad f = \frac{2\Omega}{E^{1/4}} y^*, \quad (33)$$

where $E = 4\Omega^2 r^2 (gd_0)^{-1}$ is the Lamb parameter, r is the radius of the earth, $\Omega = 2\pi \text{ day}^{-1}$ is the angular frequency of the earth's rotation and d_0 is the equivalent depth for a reduced gravity model. Using an equivalent depth of 0.40 m, the length scale becomes 295 km and the time scale 1.71 days [6]. For convenience we drop the asterisks for the rest of this section.

We consider the case of a westward travelling solitary Rossby wave. As this case is virtually non-dispersive we employ the SWE model. We discretise the 48×16 unit basin into 96 elements of polynomial order 8 and integrate for 40 time units using 1 000 time steps. The boundaries are treated as walls and the initial condition is given by:

$$H(x, y, 0) = d + \zeta^{(0)} + \zeta^{(1)}, \quad (34a)$$

$$\mathbf{u}(x, y, 0) = \mathbf{u}^{(0)} + \mathbf{u}^{(1)}, \quad (34b)$$

where the superscripts denote the zeroth and first order asymptotic solutions to the SWE. These solutions are given by [7]:

$$\zeta^{(0)} = \Gamma \left(\frac{3 + 6y^2}{4} \right) \exp \left(-\frac{y^2}{2} \right), \quad (35a)$$

$$u^{(0)} = \Gamma \left(\frac{-9 + 6y^2}{4} \right) \exp \left(-\frac{y^2}{2} \right), \quad (35b)$$

$$v^{(0)} = \partial_x \Gamma(2y) \exp \left(-\frac{y^2}{2} \right), \quad (35c)$$

and

$$\zeta^{(1)} = \frac{9\Gamma}{48} (5 - 2y^2) \exp \left(-\frac{y^2}{2} \right) + \Gamma^2 \bar{\zeta}^{(1)}, \quad (36a)$$

$$u^{(1)} = -\frac{9\Gamma}{48} (3 + 2y^2) \exp \left(-\frac{y^2}{2} \right) + \Gamma^2 \bar{u}^{(1)}, \quad (36b)$$

$$v^{(1)} = \Gamma \partial_x \Gamma \bar{v}^{(1)}, \quad (36c)$$

in which $\Gamma(x) = 0.771 a^2 \text{sech}^2(ax)$ and where a is the parameter determining the amplitude of the solitary wave (set to 0.395). The terms $\bar{\zeta}^{(1)}$, $\bar{u}^{(1)}$ and $\bar{v}^{(1)}$ are evaluated as a Hermite series

$$\begin{bmatrix} \bar{\zeta}^{(1)} \\ \bar{u}^{(1)} \\ \bar{v}^{(1)} \end{bmatrix} = \exp\left(-\frac{y^2}{2}\right) \sum_{n=0}^{\infty} \begin{bmatrix} \bar{\zeta}_n \\ \bar{u}_n \\ \bar{v}_n \end{bmatrix} H_n, \quad (37)$$

where $H_n(y)$ are the Hermite polynomials and $\bar{\zeta}_n$, $\bar{u}_n$ and $\bar{v}_n$ are the Hermite series coefficients given table 1 [7].

Figure 3 shows the evolution of the wave. At the outset, the non-exact initial condition causes the Rossby wave to lose mass to an eastward propagating Kelvin wave [24]. The computed phase velocity is -0.77 ms^{-1} , in good agreement with the analytical value of -0.78 ms^{-1} . The result is in accordance with results obtained by continuous SWE spectral element models [24, 18, 14].

4.3 Solitary wave

Finally we consider a solitary wave of amplitude 0.1 m propagating in a frictionless channel with an undisturbed depth of 1.0 m. For this example the dispersive effect is not negligible and therefore the Boussinesq model is required. The 100×50 m domain, with periodic boundaries, is divided into 64 similar shaped elements. The solution was approximated using a $P = 8$ order polynomial expansion and integrated for 20 s using 1 000 time steps. The solitary wave was initially centred at $x = 20$ m and the initial condition is taken to be the sech-profile solitary wave solution [36].

Figure 4(a) shows the water depths in a slice through $y = 25$ m at different times and is compared to the analytical solution. As can be seen in this plot there is a good agreement. A convergence plot was not possible since we only have an approximate initial condition and no analytic solution. Also shown in figure 4(b) is the computational mesh for this test.

5 Concluding Remarks

We have presented a two-dimensional spectral/ hp DG formulation for modelling nonlinear dispersive water waves, described by Boussinesq equations (which contains the nonlinear SWE as a sub-model). The models were applied to smooth problems where exponential convergence with respect to polynomial order was demonstrated. Computations of solitary wave solution were also seen to have good agreement with analytical results.

The discretisation of the equations we have adopted involves two steps; first an advective step is considered corresponding to the SWE (equation (9)) and subsequently a novel dispersive step which involves solving a $N_{\text{dof}} \times N_{\text{dof}}$ scalar Helmholtz problem (24a)-(24b). The scalar Helmholtz problem is constructed by introducing the time rate of change of momentum divergence as the dependent variable and reformulating the coupled momentum equations. The scalar approach notably reduces the size of the matrix problem solved in every substep of the scheme as compared to the original coupled momentum equations. However, an additional elementally decoupled equation (25) is required to recover the original variables.

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References

- [1] Abbott, M.B., McCowan, A.D. and Warren, I.R. Accuracy of short wave numerical models. *Journal of Hydraulic Engineering* 1984; **110**:1287–1301.
- [2] Aizinger, V. and Dawson, C. A discontinuous Galerkin method for two-dimensional flow and transport in shallow water. *Advances in Water Resources* 2002; **25**:67–84.
- [3] Ambrosi, D. and Quartapelle, L. A Taylor-Galerkin method for simulating nonlinear dispersive water waves. *Journal of Computational Physics* 1998; **146**:546–569.
- [4] Antunes Do Carmo, J.S. and Seabra Santos, F.J. Surface waves propagation in shallow water: a finite element model. *International Journal for Numerical Methods in Fluids* 1993; **16**:447–459.
- [5] Bassi, F. and Rebay, S. A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations. *Journal of Computational Physics* 1997; **131**:267–279.
- [6] Boyd, J.P. Equatorial solitary waves. Part 1: Rossby solitons. *Journal of Physical Oceanography* 1980; **10**:1699–1717.

- [7] Boyd, J.P. Equatorial solitary waves. Part 3: Westward-traveling mod-
ons. *Journal of Physical Oceanography* 1985; **15**:46–54.
- [8] Cockburn, B. and Shu, C.-W. Runge-Kutta discontinuous Galerkin
methods for convection-dominated problems. *Journal of Scientific Com-
puting* 2001; **16**(3):173–261.
- [9] Dubiner, M. Spectral methods on triangles and other domains. *Journal
of Scientific Computing* 1991; **6**(4):345–390.
- [10] Dupont, F. *Comparison of Numerical Methods for Modelling Ocean Cir-
culation in Basins with Irregular Coasts*. PhD thesis, McGill University,
2001.
- [11] Eskilsson, C. and Sherwin, S.J. A discontinuous spectral element model
for Boussinesq-type equations. *Journal of Scientific Computing* 2002;
17(1-4):143–152.
- [12] Eskilsson, C. and Sherwin, S.J. An *hp*/spectral element model for effi-
cient long-time integration of Boussinesq-type equations. *Coastal Engi-
neering Journal* 2003; **45**(2):295–320.
- [13] Eskilsson, C. and Sherwin, S.J. A triangular spectral/*hp* discontinu-
ous Galerkin method for modelling two-dimensional shallow water equa-
tions. *International Journal for Numerical Methods in Fluids*, in press.
- [14] Giraldo, F.X. A spectral element semi-Lagrangian method for the shal-
low water equations on unstructured grids. *Proceeding of the Fourth
World Congress on Computational Mechanics*, 1998.
- [15] Giraldo, F.X. A spectral element shallow water model on spherical
geodesic grids. *International Journal for Numerical Methods in Fluids*
2001; **35**:869–901.
- [16] Giraldo, F.X., Hesthaven, J.S. and Warburton, T. Nodal high-order dis-
continuous Galerkin methods for the spherical shallow water equations.
Journal of Computational Physics 2002; **181**:499–525.
- [17] Gobbi, M.F., Kirby, J.T. and Wei, G. A fully nonlinear Boussinesq model
for surface waves. Part 2. Extension to $\mathcal{O}(kh)^4$. *Journal of Fluid Mechan-
ics* 2000; **405**:181–210.
- [18] Iskandarani, M., Haidvogel, D.B. and Boyd, J.P. A staggered spectral
element model with application to the oceanic shallow water equations.

- International Journal for Numerical Methods in Fluids* 1995; **20**:393–414.
- [19] Karniadakis, G.Em. and Sherwin, S.J. *Spectral/hp element methods for CFD*. Oxford University Press, US, 1999.
 - [20] Katopodes, K.D. and Wu, C.T. Computation of finite amplitude dispersive waves. *Journal of Waterway, Port, Coastal and Ocean Engineering* 1987; **113**:327–346.
 - [21] Koornwinder, T. Two-variable analogues of the classical orthogonal polynomials. *Theory and Application of Special Functions*, R.A. Askey, ed., Academic Press, New York, 1975; :435-495.
 - [22] Langtangen, H.P. and Pedersen, G. Computational models for weakly dispersive nonlinear water waves. *Computer Methods in Applied Mechanics and Engineering* 1998; **160**:337–358.
 - [23] Li, H. and Liu, R.X. The discontinuous Galerkin finite element method for the 2d shallow water equations. *Mathematics and Computers in Simulation* 2001; **56**:171–184.
 - [24] Ma, H. A spectral element basin model for the shallow water equations. *Journal of Computational Physics* 1993; **109**:133–149.
 - [25] Madsen, P.A., Murray, I.R. and Sørensen, O.R. A new form of the Boussinesq equations with improved linear dispersion characteristics. *Coastal Engineering* 1991; **15**:371–388.
 - [26] Madsen, P.A. and Schäffer, H.A. Higher-order Boussinesq-type equations for surface gravity waves: derivation and analysis. *Philosophical Transactions of the Royal Society* 1998; **A356**:3123–3184.
 - [27] Madsen, P.A., Bingham, H.B. and Liu, H. A new Boussinesq method for fully nonlinear waves from shallow to deep water. *Journal of Fluid Mechanics* 2002; **462**:1–30.
 - [28] Nwogu, O. Alternative form of Boussinesq equations for nearshore wave propagation. *Journal of Waterway, Port, Coastal and Ocean Engineering* 1993; **119**:618–638.
 - [29] Peregrine, D.H. Long waves on a beach. *Journal of Fluid Mechanics* 1967; **27**:815–827.

- [30] Proriot, J. Sur une famille de polynomes á deux variables orthogonax dans un triangle. *C.R. Acad. Sci Paris* 1957; **245**:2459-2461.
- [31] Shi, F., Dalrymple, R.A., Kirby, J.T., Chen, Q. and Kennedy, A. A fully nonlinear Boussinesq model in generalized curvilinear coordintaes. *Coastal Engineering* 2001; **42**:337–358.
- [32] Schwanenberg, D. and Köngeter, J. A discontinuous Galerkin method for the shallow water equations with source terms. In *Discontinuous Galerkin Methods*, Cockburn B, Karniadakis GE, Shu C-W (eds). Springer: Heidelberg, 2000; 289–309.
- [33] Taylor, M., Tribbia, J. and Iskandarani, M. The spectral element method for the shallow water equations on a sphere. *Journal of Computational Physics* 1997; **130**:92–108.
- [34] Toro, E.F. *Shock-Capturing Methods for Free-Surface Shallow Flows*. John Wiley and Sons, 2001.
- [35] Walkley, M.A. *A Numerical Method for Extended Boussinesq Shallow-Water Wave Equations*. PhD thesis, University of Leeds, UK, 1999.
- [36] Wei, G. and Kirby, J.T. Time-dependent numerical code for extended Boussinesq equations. *Journal of Waterway, Port, Coastal and Ocean Engineering* 1995; **121**:251–261.
- [37] Yan, J. and Shu, C.-W. Local discontinuous Galerkin methods for partial differential equations with higher order derivatives. *Journal of Scientific Computing* 2002; **17**(1-4):27–47.

Table 1: The Hermite series coefficients $\bar{\zeta}_n$, $\bar{u}_n$ and $\bar{v}_n$ as given in [7].

n	$\bar{\zeta}_n$	$\bar{u}_n$	$\bar{v}_n$
0	-3.071430	1.789276	0
1	0	0	0
2	-0.3508384E-1	0.1164146	0
3	0	0	-0.6697824E-1
4	-0.1861060E-1	-0.3266961E-3	0
5	0	0	-0.2266569E-2
6	-0.2496364E-3	-0.1274022E-2	0
7	0	0	0.9228703E-4
8	0.1639537E-4	0.4762876E-4	0
9	0	0	-0.1954691E-5
10	-0.4410177E-6	-0.1120652E-5	0
11	0	0	0.2925271E-7
12	0.8354759E-9	0.1996333E-7	0
13	0	0	-0.3332983E-9
14	-0.1254222E-9	-0.2891698E-9	0
15	0	0	0.2916586E-11
16	0.1573519E-11	0.3543594E-11	0
17	0	0	-0.1824357E-13
18	-0.17023E-13	0.377013E-13	0
19	0	0	0.4920951E-16
20	0.1621976E-15	0.35476E-15	0
21	0	0	0.630264E-18
22	-0.1382304E-17	0.2994113E-17	0
23	0	0	-0.1289167E-19
24	0.1066277E-19	0.2291658E-19	0
25	0	0	0.1471189E-21
26	-0.1178252E-21	-0.1178252E-21	0

FIGURE CAPTIONS

Figure 1. Growth of the maximum eigenvalues of the scalar model equations.

Figure 2. Convergence of the total water depth H as a function of polynomial order P . (a) SWE model and (b) Boussinesq model.

Figure 3. Propagation of an equatorial Rossby solitary wave. SWE simulation. (a) $t = 10$; (b) $t = 20$; (c) $t = 30$ and (d) $t = 40$.

Figure 4. (a) Analytical (solid) and computed (dots) solitary wave propagation. (b) Computational mesh.

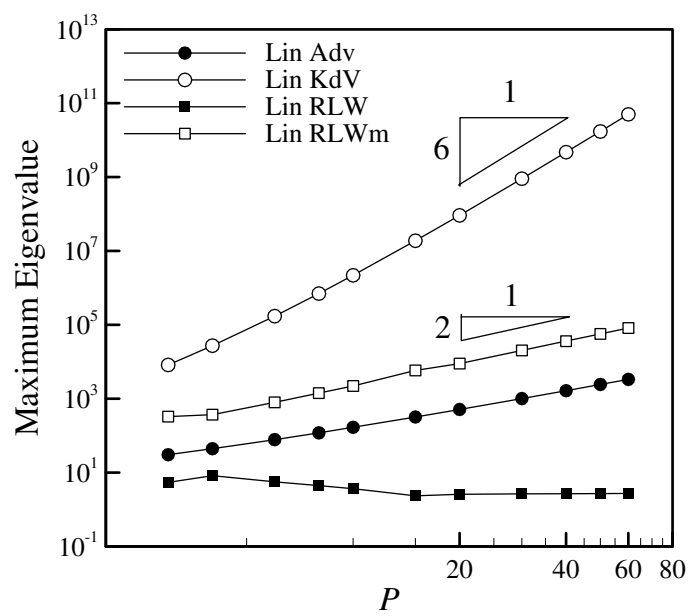
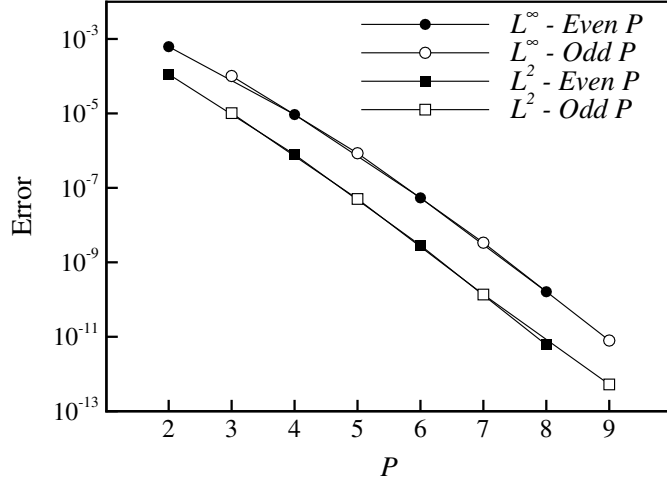


Figure 1:

(a)



(b)

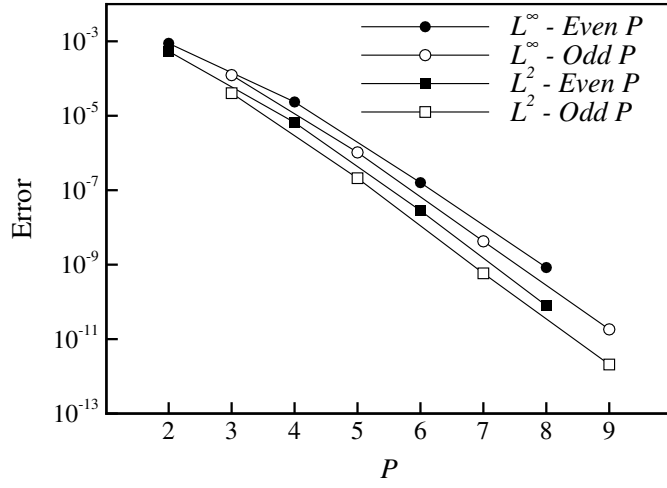


Figure 2:

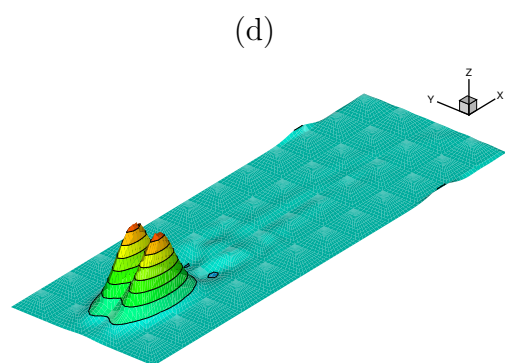
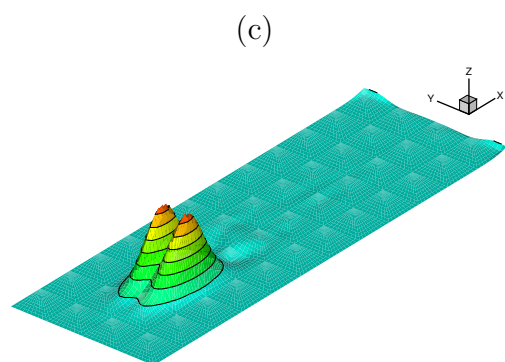
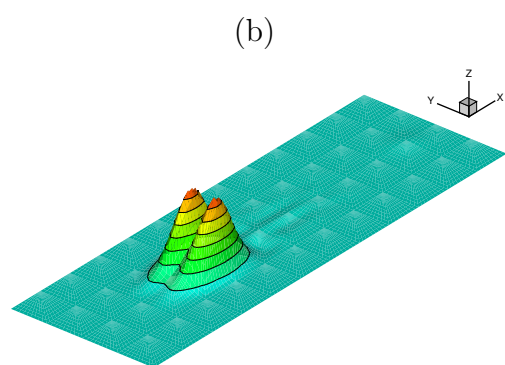
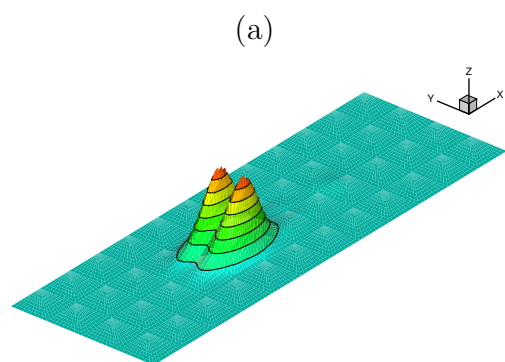


Figure 3:
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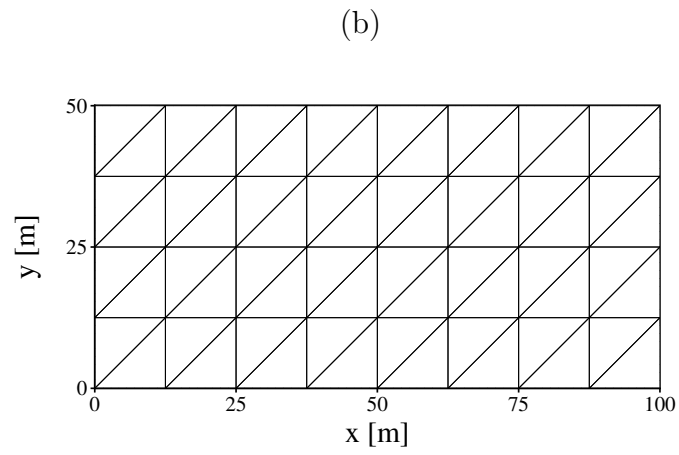
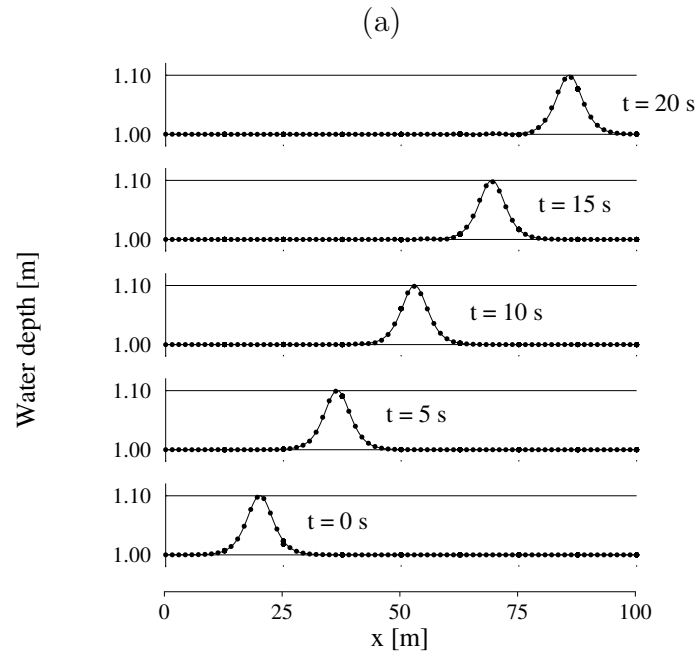


Figure 4: