Levels of p-adic automorphic forms and a p-adic Jacquet-Langlands correspondence

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Abstract

We investigate the arithmetic of p-adic automorphic forms for certain quaternion algebras over totally real fields (including $\mathrm{GL}_2/\mathbb{Q}$), focussing on the question of the relationships between p-adic automorphic forms with different levels at a prime l different from p, and the relation between p-adic automorphic forms for (the multiplicative groups of) different quaternion algebras (i.e. a p-adic Jacquet–Langlands correspondence). In chapters 2 and 3 we prove results of 'level raising' type, showing that certain families of p-adic automorphic forms with level prime to l (l-old forms) intersect with a family of p-adic automorphic form with Iwahori level at l, where all the classical points in this second family are l-new. Chapter 2 works with definite quaternion algebras over $\mathbb{Q}$, whilst chapter 3 works with $\mathrm{GL}_2/\mathbb{Q}$. In chapter 3 the main tool is Emerton's theory of completed cohomology. In chapter 4 we study indefinite quaternion algebras over totally real fields F, split at one infinite place, and prove level raising and lowering results. Finally, also in chapter 4, we give an example of a cohomological construction of p-adic Jacquet-Langlands functoriality, using completed cohomology.

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Chapter 1

Introduction

This thesis investigates the arithmetic of p-adic automorphic forms for certain quaternion algebras over totally real fields (including $\mathrm{GL}_2/\mathbb{Q}$). It comprises three (more or less independent) chapters, with common themes: the question of the relationships between p-adic automorphic forms with different levels at a prime $l \neq p$, and the relation between p-adic automorphic forms for (the multiplicative groups of) different quaternion algebras (i.e. a p-adic Jacquet-Langlands correspondence). Chapter 2 is based on the publication [32], chapter 3 is based on the publication [33]. In the rest of this introduction, we will take the opportunity to summarise some previous work in this area, underpinning our research, and then summarise the results contained in this thesis.

1.1 Eigenvarieties

An *eigenvariety* is a rigid analytic space, p-adically interpolating the systems of Hecke eigenvalues arising from certain automorphic forms. The first construction of an eigenvariety (which was named 'the eigencurve' since it is one-dimensional) occurs in [13], and it is constructed using Coleman's theory of overconvergent modular forms [12]. Buzzard extended this construction to automorphic forms for definite quaternion algebras over totally real fields [2, 3] and provided a 'machine' which has enabled the construction of eigenvarieties in certain other situations [9, 29]. A related construction is also carried out in [45].

A rather different approach to the study of p-adic automorphic forms can be found in [19]. Emerton's approach has recently led to a proof of many cases of the Fontaine-Mazur conjecture [20] and has therefore been the subject of much interest. Emerton's completed

cohomology and construction of eigenvarieties is the object of study in chapters 3 and 4, where we study p-adic automorphic forms for $\mathrm{GL}_2/\mathbb{Q}$ and indefinite quaternion algebras over totally real fields, split at one infinite place.

1.2 Levels of modular forms

Much of this work is inspired by work of Diamond, Ribet and Taylor from the 80s and 90s [37, 44, 38, 14, 15] together with more recent generalisations of Ribet's work due to Jarvis and Rajaei [25, 36]. Chapters 2 and 3 are influenced by work on *level raising* results for modular forms. These results describe congruences between modular forms of different levels, for example if f is a weight k cuspidal eigenform of level $\Gamma_1(N)$, with its Hecke eigenvalues at l satisfying a certain congruence condition modulo a prime p, then there is a weight k cuspidal eigenform g of level $\Gamma_1(N) \cap \Gamma_0(l)$ which is l-new and congruent to f modulo p (more precisely g is congruent to an l-old form of level $\Gamma_1(N) \cap \Gamma_0(l)$ constructed from f). In Chapters 2 and 3 we prove results in the same spirit, but for families of p-adic modular forms. This comes down to showing that a family of l-old modular forms intersects a family of l-new modular forms at a point corresponding to a non-classical p-adic modular form which is simultaneously l-old and l-new.

On the other hand one can consider *level lowering* results for modular forms. These are of the following type: given a weight k cuspidal eigenform g of level $\Gamma_1(N) \cap \Gamma_0(l)$, such that the mod p Galois representation

$$\overline{\rho}_g : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

attached to g is irreducible and *unramified* at l, then there should be a weight k cuspidal eigenform f of level $\Gamma_1(N)$ such that $\overline{\rho}_f \cong \overline{\rho}_g$. Chapter 4 investigates these questions in the setting of p-adic automorphic forms for indefinite quaternion algebras over totally real fields, split at one infinite place (so the relevant Shimura varieties are one-dimensional) — we will be interested in the situation where the characteristic 0 Galois representation attached to a p-adic automorphic form is unramified at l.

1.3 Jacquet-Langlands correspondence

The Jacquet-Langlands correspondence [24] provides a way to pass from automorphic representations of the multiplicative group of a quaternion algebra D' over a local field to automorphic representations of the multiplicative group of a different quaternion algebra D (ramified at a subset of the places where D' is ramified) — for example from automorphic representations of the multiplicative group of a definite quaternion algebra over $\mathbb{Q}$ to automorphic representations of $\mathrm{GL}_2/\mathbb{Q}$. In this case, the transfer of automorphic representations was p-adically interpolated by Chenevier [10] to produce a rigid analytic map between the eigencurve for the definite quaternion algebra over $\mathbb{Q}$ (as constructed in [2]) and the Coleman-Mazur eigencurve for $\mathrm{GL}_2/\mathbb{Q}$ [13].

In Chapter 4 we will be concerned with the Jacquet-Langlands transfer between automorphic representations of the multiplicative group of quaternion algebras over a totally real field F, which are split at one infinite place (so the attached Shimura varieties are one-dimensional). These quaternion algebras will always be split at places dividing p. For the purposes of this introduction we will suppose that $F = \mathbb{Q}$. Let D be a quaternion algebra over $\mathbb{Q}$, split at the infinite place and with discriminant N, and let D' be a quaternion algebra over $\mathbb{Q}$, split at the infinite place and with discriminant Nql for q, l distinct primes (coprime to N). In this situation there is a rather geometric description of the Jacquet-Langlands transfer from automorphic representations of $(D' \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}})^{\times}$ to automorphic representations of $(D \otimes_{\mathbb{Q}} \mathbb{A}_{\mathbb{Q}})^{\times}$, arising from the study of the reduction modulo q or l of integral models of Shimura curves attached to D and D' (where we consider Shimura curves attached to D with Iwahori level at q and l). For the case $N = 1$ and $F = \mathbb{Q}$ this is one of the key results in [38], and it was extended to general totally real fields F by Rajaei [36]. Using the techniques from these papers, together with Emerton's completed cohomology, we can then describe a p-adic Jacquet-Langlands correspondence in a geometric way — the result obtained is rather stronger than just the existence of a map between eigenvarieties, which is all that can be deduced from interpolatory techniques as in [10].

1.4 Summary of main results

In this subsection we will summarise the results contained in the following three chapters.

1.4.1 Level raising

Let $D/\mathbb{Q}$ be a definite quaternion algebra. We consider families of finite slope overconvergent p-adic automorphic forms as in [2]. We fix an integer N and a prime l, such that p, l and N are all coprime to each other and the discriminant of D. We use $U_1(N)$ and $U_0(l)$ to denote level structures as in Ch.2 §2.2.2. In Ch.2 §2.3 we prove the following result (stated here in a slightly vague form, see Ch.2 Theorem 2.3.4 for the precise formulation):

Theorem 1.4.2. *Let f be an overconvergent eigenform of finite slope and tame level $U_1(N)$, with T_l eigenvalue t_l and S_l eigenvalue s_l related by the equation $t_l^2 - (l+1)^2 s_l = 0$. The roots of the lth Hecke polynomial attached to f are equal to α and $l\alpha$ for some $\alpha \in \mathbb{C}_p$ and the l-stabilised oldform, with tame level $U_1(N) \cap U_0(l)$, corresponding to the root α lies in a one-dimensional family of l-new forms.*

In Ch.3 Theorem 3.4.2 we prove the same result for families of finite slope overconvergent p-adic modular forms [12], with the added condition that their associated Galois representations have irreducible reduction mod p. In fact we prove an equivalent formulation of these level raising statements, described in terms of the equidimensionality of a rigid analytic subspace of the appropriate eigencurve (the 'l-new' subspace). The strategies adopted in Ch.2 and Ch.3 are completely different. In Ch.2 we approach the problem similarly to the way the classical level raising problem for automorphic forms for definite quaternion algebras is solved in [15, §2]. In Ch.3 we instead use Emerton's approach to constructing the eigencurve [19] and prove an equidimensionality result using Emerton's Jacquet functor [17] and certain modules over non-commutative Iwasawa algebras.

1.4.3 Level lowering

In Ch.4 we study the completed cohomology of Shimura curves. Using techniques from [25, 36, 19, 20] we can prove an analogue of Mazur's principle for p-adic automorphic forms (see Ch.4 Theorems 4.4.26 and 4.4.29 for precise statements). Roughly speaking, we show that if a two-dimensional p-adic representation ρ of $\mathrm{Gal}(\overline{F}/F)$ (F a totally real field) is unramified at a finite place $\mathfrak{q} \nmid p$ and occurs in the completed cohomology of a system of Shimura curves with tame level equal to $U_0(\mathfrak{q})$ at the $\mathfrak{q}$-factor, then the system of Hecke eigenvalues attached to ρ occurs in the completed cohomology of Shimura curves with tame level equal to $\mathrm{GL}_2(\mathcal{O}_\mathfrak{q})$ at the $\mathfrak{q}$-factor.

One motivation for this result is its use in applying the machinery of [20] in the special case where p splits completely in F (so if $\mathfrak{p}$ is a prime of F dividing p the p-adic local Langlands correspondence is well defined for representations of $\mathrm{GL}_2(F_{\mathfrak{p}}) \cong \mathrm{GL}_2(\mathbb{Q}_p)$) — this should have applications to some cases of the Fontaine-Mazur conjecture for GL_2/F (see also [27]) and the (generalised) weight part of Serre's conjecture (see [4] and [22]).

1.4.4 A p-adic Jacquet-Langlands correspondence

Finally, also in Ch.4, we show the existence of a short exact sequence of admissible unitary Banach representations of $\mathrm{GL}_2(F \otimes_{\mathbb{Q}} \mathbb{Q}_p)$ (Ch.4 Theorem 4.4.34 — in fact the exact sequence is defined integrally) which encodes a p-adic Jacquet-Langlands correspondence from a quaternion algebra D'/F, split at one infinite place, with discriminant $\mathfrak{N}\mathfrak{q}_1\mathfrak{q}_2$, to a quaternion algebra D/F, split at the same infinite place and with discriminant $\mathfrak{N}$. Applying Emerton's eigenvariety construction to this exact sequence gives a map between eigenvarieties (Theorem 4.5.18), i.e. an overconvergent Jacquet-Langlands correspondence in the same spirit as the main theorem of [10]. As a corollary of the results in Ch.4 we also obtain level raising results (Corollary 4.5.19).

Chapter 2

Level raising for p-adic automorphic forms

2.1 Introduction

Classical level raising results typically show that if the reduction mod p of a level N modular form f has certain properties (depending on a prime $l \neq p$), then there exists a modular form g of level Nl, new at l, with $g \equiv f \mod p$. An example of a level raising result for classical modular forms is the following, due to Ribet [37]:

Theorem. *Let $f \in S_2(\Gamma_0(N))$ be an eigenform, and let $\mathfrak{p}|p$ be a finite place of $\bar{\mathbb{Q}}$ such that $p \geq 5$ and f is not congruent to an Eisenstein series modulo $\mathfrak{p}$. If $l \nmid Np$ is a prime number such that the following condition is satisfied,*

$$a_l(f)^2 \equiv (1+l)^2 \text{ (mod } \mathfrak{p}),$$

then there exists a l-new eigenform $\tilde{f} \in S_2(\Gamma_0(Nl))$ congruent to f modulo $\mathfrak{p}$.

In this chapter we prove an analogous level raising result for families of p-adic automorphic forms. In [2] and Part III of [3], Buzzard defines modules of overconvergent p-adic automorphic forms for definite quaternion algebras, and constructs from these a so-called 'eigencurve'. The eigencurve is a rigid analytic variety whose points correspond to certain systems of eigenvalues for Hecke algebras acting on these modules of automorphic forms. This space p-adically interpolates the systems of eigenvalues arising from classical automorphic forms. Emerton has constructed eigenvarieties in a cohomological framework [19], but in the following we will work with Buzzard's more concrete construction. We have also proved some cases of level raising for p-adic modular forms using the completed cohomology spaces investigated by Emerton (see Ch.3).

The first construction of an eigencurve was carried out for modular forms (automorphic forms for GL_2) in Coleman and Mazur's seminal paper [13]. An important recent result is the construction of a p-adic Jacquet-Langlands map between an eigencurve for a definite quaternion algebra and the GL_2 eigencurve (interpolating the usual Jacquet-Langlands correspondence), as carried out in [10].

We follow the general approach of the first part of Diamond and Taylor's paper [15], and our Theorem 2.2.22 is an analogue of [15, Theorem 1], but several new features appear in our work. In particular, the level raising results in [44] and [15] for definite quaternion algebras are proved by utilising a pairing on finite dimensional vector spaces of automorphic forms. In our setting, the spaces of automorphic forms are Banach modules over an affinoid

algebra, so we introduce spaces of 'dual' automorphic forms and work with the pairing between the usual space of automorphic forms and the dual space. We then prove suitable forms of Ihara's lemma, our Theorem 2.2.19 (cf. lemma 2 of [15]), for the usual and dual spaces of automorphic forms. An interesting asymmetry between the two situations can be observed.

This investigation of level raising results was motivated by a conjecture made by A. Paulin, prompted by results on local-global compatibility on the eigencurve in his thesis [35]. Paulin's conjecture was made for the GL_2-eigencurve; we may apply our theorem to the image of the p-adic Jacquet-Langlands map there to prove many cases of his conjecture. Since we have applications to the eigencurve for $\mathrm{GL}_2/\mathbb{Q}$ in mind we work with definite quaternion algebras over $\mathbb{Q}$ in this chapter, but some of the methods of section 2.2 should apply to definite quaternion algebras over any totally real number field, although we do use the fact that weight space is one-dimensional in our arguments. We end this introduction by stating the conjecture made by Paulin.

2.1.1 A geometric level raising conjecture

We fix two distinct primes p and l, and an integer N coprime to pl. Let $\mathscr{E}$ be the cuspidal Eigencurve of tame level $\Gamma_0(Nl)$, parameterising overconvergent cuspidal p-adic modular eigenforms (see [3] for its construction). If ϕ is a point of $\mathscr{E}$, corresponding to an eigenform f_ϕ, Paulin defines an associated representation of $\mathrm{GL}_2(\mathbb{Q}_l)$, denoted $\pi_{f_\phi,l}$. We call an irreducible connected component $\mathscr{Z}$ of the Eigencurve *generically special* if the $\mathrm{GL}_2(\mathbb{Q}_l)$-representations associated to the points of $\mathscr{Z}$ away from a discrete set are special. We define *generically unramified principal series* similarly. Denote by α and β the roots of the polynomial $X^2 - t_l X + l s_l$, where t_l and s_l are the T_l and S_l eigenvalues of f_ϕ. Paulin makes the conjecture:

Conjecture. *Suppose $\mathscr{Z}$ is generically unramified principal series. Suppose further that there is a point ϕ on $\mathscr{Z}$ where the ratio of α and β becomes $l^{\pm 1}$ and $\pi_{f_\phi,l}$ is special. Then there exists a generically special component $\mathscr{Z}'$ intersecting $\mathscr{Z}$ at ϕ.*

Chenevier raised the same question (in a slightly different form) in relation to the characterisation of the Zariski closure of the l-new classical forms in the eigencurve. We address this issue in section 2.3.5. Finally, in a recent preprint [34] Paulin has proved versions of his level raising (and lowering) conjectures (even for ramified principal series). His tech-

niques are completely different to ours, making use of deformation theory and requiring a recent important result of Emerton [20] showing that the space X_{fs} constructed by Kisin in [26] is equal to the GL_2-eigencurve (if one restricts to pieces of the two spaces where certain conditions are satisfied by the relevant mod p Galois representations).

2.2 Modules of p-adic overconvergent automorphic forms and Ihara's lemma

In this section we will prove the results we need about modules of p-adic overconvergent automorphic forms for quaternion algebras.

2.2.1 Banach modules

Let K be a finite extension of $\mathbb{Q}_p$. We call a normed K-algebra A a *Banach algebra* if it satisfies the following properties:

- A is Noetherian,
- the norm $|-|$ is non-Archimedean,
- A is complete with respect to $|-|$,
- for any x, y in A we have $|xy| \leq |x||y|$.

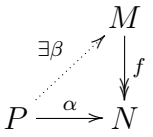
We will normally assume A is a reduced affinoid algebra with its supremum norm. A *Banach A-module* is an A-module M endowed with a norm $|-|$ such that

- for any $a \in A$, $m \in M$ we have $|am| \leq |a||m|$,
- M is complete with respect to $|-|$.

Given a set I we define the Banach A-module $c_I(A)$ to be functions $f : I \to A$ such that $lim_{i\to\infty} f(i) = 0$, with norm the supremum norm. By a finite Banach A-module we mean a Banach A-module which is finitely-generated as an abstract A-module.

Suppose M is a Banach module over a Banach algebra A. We say that M is *ONable* if it is isomorphic (as a Banach module) to some $c_I(A)$. Note that this terminology differs slightly from that of [3], where *ONable* refers to modules *isometric* to some $c_I(A)$ and *potentially ONable* replaces our notion of ONable. The Banach A-module P is said to

satisfy the universal property (Pr) if for every surjection $f : M \to N$ of Banach A-modules and continuous map $\alpha : P \to N$, α lifts to a continuous map $\beta : P \to M$ such that the below diagram commutes:



As explained in [45, 2.1.4], the universal property (Pr) is the property of being projective in the category of Banach A-modules, where the notion of projective is defined using *strict* epimorphisms of Banach modules (which are just the set-theoretically surjective epimorphisms of Banach modules). This is the correct notion of projective object, since the category of Banach A-modules is an exact category, not an Abelian catefory. A module P having property (Pr) is equivalent to P being a direct summand of an ONable module. (See the end of section 2 in [3]).

2.2.2 Some notation and definitions

Let p be a fixed prime. Let D be a definite quaternion algebra over $\mathbb{Q}$ with discriminant δ prime to p. Fix a maximal order $\mathcal{O}_D$ of D and isomorphisms $\mathcal{O}_D \otimes \mathbb{Z}_q \cong M_2(\mathbb{Z}_q)$ for primes $q \nmid \delta$. Note that these induce isomorphisms $D \otimes \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ for $q \nmid \delta$. We define $D_f = D \otimes_{\mathbb{Q}} \mathbb{A}_f$, where $\mathbb{A}_f$ denotes the finite adeles over $\mathbb{Q}$. Write Nm for the reduced norm map from D_f to $\mathbb{A}_f^\times$. Note that if $g \in D_f$ we can regard the p component of g, g_p, as an element of $M_2(\mathbb{Q}_p)$.

For an integer $\alpha \geq 1$, we let $\mathbb{M}_\alpha$ denote the monoid of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}_p)$ such that $p^\alpha | c$, $p \nmid d$ and $ad - bc \neq 0$. If U is an open compact subgroup of $D_f^\times$ and $\alpha \geq 1$ we say that U has *wild level* $\geq p^\alpha$ if the projection of U to $\mathrm{GL}_2(\mathbb{Q}_p)$ is contained in $\mathbb{M}_\alpha$.

We will be interested in two key examples of open compact subgroups of $D_f^\times$. For M any integer prime to δ, we define $U_0(M)$ (respectively $U_1(M)$) to be the subgroup of $D_f^\times$ given by the product $\prod_q U_q$, where $U_q = (\mathcal{O}_D \otimes \mathbb{Z}_q)^\times$ for primes $q | \delta$, and U_q is the matrices in $\mathrm{GL}_2(\mathbb{Z}_p)$ of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ (respectively $\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix}$) mod $q^{\mathrm{val}_q(M)}$ for all other q. We can see that if p^α divides M, then $U_1(M)$ has wild level $\geq p^\alpha$.

Suppose we have $\alpha \geq 1$, U a compact open subgroup of $D_f^\times$ of wild level $\geq p^\alpha$ and A

a module over a commutative ring R, with an R-linear right action of $\mathbb{M}_\alpha$. We define an R-module $\mathscr{L}(U, A)$ by

$$\mathscr{L}(U, A) = \{f : D_f^\times \to A : f(dgu) = f(g)u_p \text{ for all } d \in D^\times, g \in D_f^\times, u \in U\}$$

where $D^\times$ is embedded diagonally in $D_f^\times$. If we fix a set $\{d_i : 1 \leq i \leq r\}$ of double coset representatives for the finite double quotient $D^\times \backslash D_f^\times / U$, and write Γ_i for the finite group $d_i^{-1} D^\times d_i \cap U$, we have an isomorphism (see section 4 of [2])

$$\mathscr{L}(U, A) \to \bigoplus_{i=1}^{r} A^{\Gamma_i},$$

given by sending f to $(f(d_1), f(d_2), \ldots, f(d_r))$. If $U \subset U_1(N)$ for $N \geq 4$, then the groups Γ_i are trivial (this is proved in [15]).

For $f : D_f^\times \to A$, $x \in D_f^\times$ with $x_p \in \mathbb{M}_\alpha$, we define $f|x : D_f^\times \to A$ by $(f|x)(g) = f(gx^{-1})x_p$. Note that we can now also write

$$\mathscr{L}(U, A) = \{f : D^\times \backslash D_f^\times \to A : f|u = f \text{ for all } u \in U\}.$$

We can define double coset operators on the spaces $\mathscr{L}(U, A)$. If U, V are two compact open subgroups of $D_f^\times$ of wild level $\geq p^\alpha$, and A is as above, then for $\eta \in D_f^\times$ with $\eta_p \in \mathbb{M}_\alpha$ we may define an R-module map $[U\eta V] : \mathscr{L}(U, A) \to \mathscr{L}(V, A)$ as follows: we decompose $U\eta V$ into a finite union of right cosets $\coprod_i Ux_i$ and define

$$f|[U\eta V] = \sum_i f|x_i.$$

2.2.3 Overconvergent automorphic forms

Let $\mathscr{W}$ be the rigid analytic space $\mathrm{Hom}(\mathbb{Z}_p^\times, \mathbb{G}_m)$, defined over $\mathbb{Q}_p$. The reader may consult lemma 2 of [2] for details of this space's construction and properties. For example, $\mathscr{W}$ is a union of finitely many open discs. The space $\mathscr{W}$ is the *weight space* for our automorphic forms. The $\mathbb{C}_p$-points w of $\mathscr{W}$ corresponding to characters $\kappa_w : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$ with $\kappa_w(x) = x^k \varepsilon_p(x)$ for some positive integer k and finite order character ε_p are referred to as *classical* weights. Let X be a reduced connected K-affinoid subspace of $\mathscr{W}$, where $K/\mathbb{Q}_p$ is finite, and denote the ring of analytic functions on X by $\mathcal{O}(X)$. Such a space X corresponds to a character $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}(X)^\times$ induced by the inclusion $X \subset \mathscr{W}$. If we have a real number

$r = p^{-n}$ for some n, then we define $\mathbb{B}_{r,K}$ to be the rigid analytic subspace of affine 1-space over K with $\mathbb{C}_p$-points

$$\mathbb{B}_{r,K}(\mathbb{C}_p) = \{z \in \mathbb{C}_p : \exists y \in \mathbb{Z}_p \text{ such that } |z - y| \leq r\}.$$

Similarly (for $r < 1$) we define $\mathbb{B}^\times_{r,K}$ to be the rigid analytic subspace of affine 1-space over K with $\mathbb{C}_p$-points

$$\mathbb{B}^\times_{r,K}(\mathbb{C}_p) = \{z \in \mathbb{C}_p : \exists y \in \mathbb{Z}_p^\times \text{ such that } |z - y| \leq r\}.$$

A point $x \in X(\mathbb{C}_p)$ corresponds to a continuous character $\kappa_x : \mathbb{Z}_p^\times \to \mathbb{C}_p^\times$. Such maps are analytic when restricted to the set $\{z \in \mathbb{Z}_p : |1 - z| \leq r\}$ for small enough r. If κ_x and r have this property we call x an r-analytic point. A point is r-analytic if and only if its corresponding character extends to a morphism of rigid analytic varieties

$$\kappa_x : \mathbb{B}^\times_{r,K} \to \mathbb{G}_m.$$

Let X be a K-affinoid subspace of $\mathscr{W}$ as before, with associated character $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}(X)^\times$. We say that κ is r-analytic if every point in $X(\mathbb{C}_p)$ is r-analytic. Fix a real number $0 < r < 1$ and let $\mathscr{A}_{X,r}$ be the $\mathcal{O}(X)$-Banach algebra $\mathcal{O}(\mathbb{B}_{r,K} \times_K X)$, endowed with the supremum norm. If κ is $rp^{-\alpha}$-analytic we can define a right action of $\mathbb{M}_\alpha$ on $\mathscr{A}_{X,r}$ by, for $f \in \mathscr{A}_{X,r}$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_\alpha$,

$$(f \cdot \gamma)(x, z) = \frac{\kappa_x(cz + d)}{(cz + d)^2} f\left(x, \frac{az + b}{cz + d}\right).$$

where $x \in X(\mathbb{C}_p)$ (with κ_x the associated character) and $z \in \mathbb{B}_{r,K}(\mathbb{C}_p)$.

Definition 2.2.4. *Let X be a K-affinoid subspace of $\mathscr{W}$ as above, with $\kappa : \mathbb{Z}_p^\times \to \mathcal{O}(X)^\times$ the induced character. If we have a real number $r = p^{-n}$, some integer $\alpha \geq 1$ such that κ is $rp^{-\alpha}$-analytic, and U a compact open subgroup of $D_f^\times$ of wild level $\geq p^\alpha$, then define the space of r-overconvergent automorphic forms of weight X and level U to be the $\mathcal{O}(X)$-module*

$$\mathbf{S}_X^D(U; r) := \mathscr{L}(U, \mathscr{A}_{X,r}).$$

If we endow $\mathbf{S}_X^D(U;r)$ with the norm $|f| = \max_{g \in D_f^\times} |f(g)|$, then the isomorphism

$$\mathbf{S}_X^D(U;r) \cong \bigoplus_{i=1}^{r} \mathscr{A}_{X,r}^{\Gamma_i} \tag{2.1}$$

induced by fixing double coset representatives d_i is norm preserving. Since the Γ_i are finite groups, and $\mathscr{A}_{X,r}$ is an ONable Banach $\mathcal{O}(X)$-module (it is the base change to $\mathcal{O}(X)$ of $\mathcal{O}(\mathbb{B}_{r,K})$, and all Banach spaces over a discretely valued field are ONable), we see that $\mathbf{S}_X^D(U;r)$ is a Banach $\mathcal{O}(X)$-module, and satisfies property (Pr).

Note that if $\mathfrak{m}$ is a maximal ideal of $\mathcal{O}(X)$, corresponding to a point $x \in X(K')$ for K'/K finite, then taking the fibre of the module $\mathbf{S}_X^D(U;r)$ at $\mathfrak{m}$ gives the space of overconvergent forms $\mathbf{S}_x^D(U;r)$ corresponding to the point x of $\mathscr{W}(K')$ (note that a point of $\mathscr{W}(K')$ is a reduced connected K'-affinoid subspace!).

These spaces of overconvergent automorphic forms were first defined in [2], using ideas from the unpublished preprint [43].

2.2.5 Dual modules

Suppose A is a Banach algebra. Given a Banach A-module $\mathbf{M}$ we define the *dual* $\mathbf{M}^*$ to be the Banach A-module of continuous A-module morphisms from $\mathbf{M}$ to A, with the usual operator norm. We denote the $\mathcal{O}(X)$-module $\mathscr{A}_{X,r}^*$ by $\mathscr{D}_{X,r}$.

If the map κ corresponding to X is $rp^{-\alpha}$-analytic, then $\mathbb{M}_\alpha$ acts continuously on $\mathscr{A}_{X,r}$, so $\mathscr{D}_{X,r}$ has an $\mathcal{O}(X)$-linear right action of the monoid $\mathbb{M}_\alpha^{-1}$ given by $(f \cdot m^{-1})(x) := f(x \cdot m)$, for $f \in \mathscr{D}_{X,r}$, $x \in \mathscr{A}_{X,r}$ and $m \in \mathbb{M}_\alpha$. If U is as in Definition 2.2.4 then its projection to $\mathrm{GL}_2(\mathbb{Q}_p)$ is contained in $\mathbb{M}_\alpha \cap \mathbb{M}_\alpha^{-1}$, so it acts on $\mathscr{D}_{X,r}$. This allows us to make the following definition:

Definition 2.2.6. *For X, κ, r, α and U as above, we define the space of* dual *r-overconvergent automorphic forms of weight X and level U to be the $\mathcal{O}(X)$-module*

$$\mathbf{V}_X^D(U;r) := \mathscr{L}(U, \mathscr{D}_{X,r}).$$

As in 2.2.3, we have a norm preserving isomorphism

$$\mathbf{V}_X^D(U;r) \cong \bigoplus_{i=1}^{r} \mathscr{D}_{X,r}^{\Gamma_i}. \tag{2.2}$$

Thus $\mathbf{V}_X^D(U;r)$ is a Banach $\mathcal{O}(X)$-module. We note that it will not usually satisfy property (Pr), since (unless X is a point) we expect that $\mathscr{D}_{X,r}$ will not be ONable.

If U, V are two compact open subgroups of $D_f^\times$ of wild level $\geq p^\alpha$, then for $\eta \in D_f^\times$ with $\eta_p \in \mathbb{M}_\alpha^{-1}$ we get double coset operators $[U\eta V] : \mathbf{V}_X^D(U;r) \to \mathbf{V}_X^D(V;r)$.

2.2.7 Hecke operators

For an integer m, we define the *Hecke algebra away from* m, $\mathbb{T}^{(m)}$, to be the free commutative $\mathcal{O}(X)$-algebra generated by symbols T_π, S_π for π prime not dividing m. If δp divides m then we can define the usual action of $\mathbb{T}^{(m)}$ by double coset operators on $\mathbf{S}_X^D(U;r)$: for $\pi \nmid m$ define $\varpi_\pi \in \mathbb{A}_f$ to be the finite adele which is π at π and 1 at the other places. Abusing notation slightly, we also write ϖ_π for the element of $D_f^\times$ which is $\begin{pmatrix} \pi & 0 \\ 0 & \pi \end{pmatrix}$ at π and the identity elsewhere. Similarly set $\eta_\pi = \begin{pmatrix} \varpi_\pi & 0 \\ 0 & 1 \end{pmatrix}$ to be the element of $D_f^\times$ which is $\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$ at π and the identity elsewhere. On $\mathbf{S}_X^D(U;r)$ we let T_π act by $[U\eta_\pi U]$ and S_π by $[U\varpi_\pi U]$. Similarly on $\mathbf{V}_X^D(U;r)$ we define T_π to act by $[U\eta_\pi^{-1}U]$ and S_π by $[U\varpi_\pi^{-1}U]$. As usual we also have a compact operator acting on $\mathbf{S}_X^D(U;r)$, namely $U_p := [U\eta_p U]$.

2.2.8 A pairing

In this section X, κ, r, α and U will be as in Definition 2.2.4. We will denote by V another compact open subgroup of wild level $\geq p^\alpha$. We fix double coset representatives $\{d_i : 1 \leq i \leq r\}$ for the double quotient $D^\times \backslash D_f^\times / U$ and let γ_i denote the order of the finite group $d_i^{-1} D^\times d_i \cap U$. We can define an $\mathcal{O}(X)$-bilinear pairing between the spaces $\mathbf{S}_X^D(U;r)$ and $\mathbf{V}_X^D(U;r)$ by

$$\langle f, \lambda \rangle := \sum_{i=1}^{r} \gamma_i^{-1} \langle f(d_i), \lambda(d_i) \rangle,$$

where $f \in \mathbf{S}_X^D(U;r)$, $\lambda \in \mathbf{V}_X^D(U;r)$ and on the right hand side of the above definition $\langle \cdot, \cdot \rangle$ denotes the pairing between $\mathscr{A}_{X,r}$ and $\mathscr{D}_{X,r}$ given by evaluation.

This pairing is independent of the choice of the double coset representatives d_i, since for every $d \in D^\times$, $g \in D_f^\times$, $u \in U$, $f \in \mathbf{S}_X^D(U;r)$ and $\lambda \in \mathbf{V}_X^D(U;r)$ we have

$$\langle f(dgu), \lambda(dgu) \rangle = \langle f(g)u_p, \lambda(g)u_p \rangle = \langle f(g)u_p u_p^{-1}, \lambda(g) \rangle = \langle f(g), \lambda(g) \rangle.$$

Combining this observation with the isomorphisms (2.1) and (2.2) we see that our pairing identifies $\mathbf{V}_X^D(U;r)$ with $\mathbf{S}_X^D(U;r)^*$.

The following proposition summarises a standard computation [44, 15] (although these assume the level group is small enough that the finite groups Γ_i are trivial), telling us how our pairing interacts with double coset operators. In particular, it implies that $\langle T_\pi f, \lambda\rangle = \langle f, T_\pi \lambda\rangle$ for $\pi \nmid \delta p$ when T_π acts in the usual way.

Proposition 2.2.9. *Let $f \in \mathbf{S}_X^D(U;r)$ and let $\lambda \in \mathbf{V}_X^D(V;r)$. Let $g \in D_f^\times$ with $g_p \in \mathbb{M}_\alpha$. Then*

$$\langle f|[UgV], \lambda\rangle = \langle f, \lambda|[Vg^{-1}U]\rangle.$$

Proof. For $d \in D_f^\times$ set $\gamma(d) = \#(d^{-1}D^\times d \cap V)$. We have

$$f|[UgV] = \sum_{v \in (g^{-1}U\eta)\cap V \backslash V} f|(gv),$$

hence

$$\begin{aligned}
\langle f|[UgV], \lambda\rangle &= \sum_{d \in D^\times \backslash D_f^\times / V} \gamma(d)^{-1} \langle f|[UgV](d), \lambda(d)\rangle \\
&= \sum_{d \in D^\times \backslash D_f^\times / V} \sum_{v \in (g^{-1}Ug)\cap V\backslash V} \gamma(d)^{-1} \langle f|(gv)(d), \lambda(d)\rangle \\
&= \sum_{d \in D^\times \backslash D_f^\times / V} \sum_{v \in (g^{-1}Ug)\cap V\backslash V} \gamma(d)^{-1} \langle f(dv^{-1}g^{-1}) \cdot g_p v_p, \lambda(d)\rangle \\
&= \sum_{x \in D^\times \backslash D_f^\times / (g^{-1}Ug)\cap V} \langle f(xg^{-1}), \lambda(x) \cdot g_p^{-1}\rangle \\
&= \sum_{y \in D^\times \backslash D_f^\times / U\cap(gVg^{-1})} \langle f(y), \lambda(yg) \cdot g_p^{-1}\rangle \\
&= \langle f, \lambda|[Vg^{-1}U]\rangle
\end{aligned}$$

where we pass from the third line to the fourth line by counting double cosets and the final line follows by similar calculations to the first 5 lines. □

By the results of section 5 in [9] and section 3 of [3] we know that for a fixed $d \geq 0$, if X is a sufficiently small affinoid whose norm is multiplicative (with a precise bound given by Théorème 5.3.1 of [9]) then since U_p acts as a compact operator on $\mathbf{S}_X^D(U;r)$, we have

a U_p stable decomposition

$$\mathbf{S}_X^D(U;r) = \mathbf{S}_X^D(U;r)^{\leq d} \oplus \mathbf{N},$$

where $\mathbf{S}_X^D(U;r)^{\leq d}$ is the space of forms of slope $\leq d$. We need X to be small enough that the Newton polygon of the characteristic power series for U_p acting on $\mathbf{S}_X^D(U;r)$ has the same slope $\leq d$ part when specialised to any point of X.

From now on we fix d and assume that X is such that this slope decomposition exists. The key example of such an X is an open ball of small radius.

The space $\mathbf{S}_X^D(U;r)^{\leq d}$ is a finite Banach $\mathcal{O}(X)$-module with property (Pr), i.e. a projective finitely generated $\mathcal{O}(X)$-module. In fact this decomposition must be stable under the action of $\mathbb{T}^{(\delta p)}$, since the T_π and S_π operators for $\pi \neq p$ commute with U_p. We define $\mathbf{V}_X^D(U;r)^{\leq d}$ to be the maps from $\mathbf{S}_X^D(U;r)$ to $\mathcal{O}(X)$ which are 0 on $\mathbf{N}$. This space is also stable under the action of $\mathbb{T}^{(\delta p)}$ and is naturally isomorphic to the dual of $\mathbf{S}_X^D(U;r)^{\leq d}$. The following lemma implies that our pairing is perfect when restricted to $\mathbf{S}_X^D(U;r)^{\leq d} \times \mathbf{V}_X^D(U;r)^{\leq d}$.

Lemma 2.2.10. *Let $\mathbf{M}$ be a finite Banach $\mathcal{O}(X)$-module with property (Pr). Then the usual natural map $\mathbf{M} \to (\mathbf{M}^*)^*$ is an isomorphism. In other words, the $\mathcal{O}(X)$-module $\mathbf{M}$ is* reflexive.

Proof. Since $\mathbf{M}$ is finite we have a surjection of Banach $\mathcal{O}(X)$-modules $\mathcal{O}(X)^{\oplus n} \to \mathbf{M}$ for some n. Applying the universal property (Pr) to this surjection shows that we have a Banach $\mathcal{O}(X)$-isomorphism $\mathbf{M} \oplus \mathbf{N} \cong \mathcal{O}(X)^{\oplus n}$ for some module $\mathbf{N}$, so $\mathbf{M}$ is a projective $\mathcal{O}(X)$-module. Proposition 2.1 of [3] states that the category of finite Banach $\mathcal{O}(X)$-modules, with continuous $\mathcal{O}(X)$-linear maps as morphisms, is equivalent to the category of finite $\mathcal{O}(X)$-modules so we can just compute duals module-theoretically. We have exact sequences

$$0 \longrightarrow \mathbf{M} \longrightarrow \mathcal{O}(X)^{\oplus n} \longrightarrow \mathbf{N} \longrightarrow 0$$

$$0 \longrightarrow \mathbf{N} \longrightarrow \mathcal{O}(X)^{\oplus n} \longrightarrow \mathbf{M} \longrightarrow 0$$

and since $\mathbf{M}$, $\mathbf{N}$, $\mathbf{M}^*$ and $\mathbf{N}^*$ are all projective as $\mathcal{O}(X)$-modules we can take the dual

of these exact sequences twice to get commutative diagrams with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{M} & \longrightarrow & \mathcal{O}(X)^{\oplus n} & \longrightarrow & \mathbf{N} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{M}^{**} & \longrightarrow & \mathcal{O}(X)^{\oplus n} & \longrightarrow & \mathbf{N}^{**} & \longrightarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbf{N} & \longrightarrow & \mathcal{O}(X)^{\oplus n} & \longrightarrow & \mathbf{M} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbf{N}^{**} & \longrightarrow & \mathcal{O}(X)^{\oplus n} & \longrightarrow & \mathbf{M}^{**} & \longrightarrow & 0 \end{array}$$

where the vertical maps are the natural maps from a module to its double dual. Since the central maps are isomorphisms, we conclude that the outer maps are too. □

Direct limits and Fréchet spaces

We should remark here that our use of the dual Banach modules $\mathbf{V}_X^D(U;r)$ is slightly unsatisfactory. For example, the modules do not satisfy property (Pr), and we must restrict to 'slope $\leq d$' subspaces to get a perfect pairing. One could alternatively work with modules of *all* overconvergent automorphic forms, rather than imposing r-overconvergence for a particular r. One defines

$$\mathbf{S}_X^D(U)^\dagger := \varinjlim_r \mathbf{S}_X^D(U;r),$$

where the (compact) transition maps in the direct system are induced by the inclusions $\mathbb{B}_{s,K} \subset \mathbb{B}_{r,K}$ for $s < r$. If X is a point (so $\mathcal{O}(X)$ is a field) then it is a standard result that the vector space $\mathbf{S}_X^D(U)^\dagger$ is reflexive (see Proposition 16.10 of [40]). Using this fact it is fairly straightforward to show that for any X, $\mathbf{S}_X^D(U)^\dagger$ is a reflexive $\mathcal{O}(X)$-module, with dual the Fréchet space

$$\mathbf{V}_X^D(U)^\dagger := \varprojlim_r \mathbf{V}_X^D(U;r).$$

2.2.11 Old and new

Fix an integer $N \geq 1$ (the tame level) coprime to p and fix an auxiliary prime $l \nmid Np\delta$. Let X be an affinoid subspace of weight space with associated character κ which is $rp^{-\alpha}$ analytic, for some integer $\alpha \geq 1$. Set $U = U_1(Np^\alpha)$, $V = U_1(Np^\alpha) \cap U_0(l)$. To simplify

notation we set

$$\begin{aligned} L &:= \mathbf{S}_X^D(U;r)^{\leq d} \\ L^* &:= \mathbf{V}_X^D(U;r)^{\leq d} \\ M &:= \mathbf{S}_X^D(V;r)^{\leq d} \\ M^* &:= \mathbf{V}_X^D(V;r)^{\leq d}. \end{aligned}$$

We define a map $i : L \times L \to M$ by

$$i(f,g) := f|[U1V] + g|[U\eta_l V].$$

Since the map i is defined by double coset operators with trivial component at p it commutes with U_p and thus gives a well defined map between these spaces of bounded slope forms. A simple calculation shows that these double coset operators act very simply. Regarding f and g as functions on $D_f^\times$ we have $f|[U1V] = f$, $g|[U\eta_l V] = g|\eta_l$. The image of i inside M will be referred to as the space of *oldforms*.

We also define a map $i^\dagger : M \to L \times L$ by

$$i^\dagger(f) := (f|[V1U], f|[V\eta_l^{-1}U]).$$

The kernel of $i^\dagger$ is the space of *newforms*. The maps i and $i^\dagger$ commute with Hecke operators T_q, S_q, where $q \nmid Npl\delta$.

The same double coset operators give maps

$$j : L^* \times L^* \to M^*,$$

$$j^\dagger : M^* \to L^* \times L^*.$$

Using Proposition 2.2.9 we have

$$\langle i(f,g), \lambda \rangle = \langle (f,g), j^\dagger \lambda \rangle$$

for $f, g \in L$, $\lambda \in M^*$. Similarly

$$\langle f, j(\lambda,\mu) \rangle = \langle i^\dagger f, (\lambda,\mu) \rangle$$

for $d \in M$, $\lambda, \mu \in L^*$.

An easy calculation shows that $i^\dagger i$ acts on the product $L \times L = L^2$ by the matrix (acting on the right)

$$\begin{pmatrix} l+1 & [U\varpi_l^{-1}U][U\eta_l U] \\ [U\eta_l U] & l+1 \end{pmatrix} = \begin{pmatrix} l+1 & S_l^{-1}T_l \\ T_l & l+1 \end{pmatrix}.$$

We have exactly the same double coset operator formula for the action of $j^\dagger j$ on the product $L^* \times L^* = L^{*2}$. The Hecke operators S_l, T_l act by $[U\varpi_l^{-1}U], [U\eta_l^{-1}U]$ respectively on L^*. Also, the double coset $U\eta_l U$ is the same as $U\varpi_l\eta_l^{-1}U$, since the matrix which is the identity at every factor except l and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ at l is in U. From these two facts we deduce that, in terms of Hecke operators, $j^\dagger j$ acts on $L^* \times L^*$ by the matrix (again acting on the right)

$$\begin{pmatrix} l+1 & T_l \\ S_l^{-1}T_l & l+1 \end{pmatrix}.$$

If the affinoid X is sufficiently nice, then we can show that the map $i^\dagger i$ is injective. Before we prove this, we note that in our setting a family of p-adic automorphic eigenforms over an affinoid $X \subset \mathscr{W}$ is just a Hecke eigenform f in $\mathbf{S}_X^D(U; r)$.

Proposition 2.2.12. *If X is a one dimensional irreducible connected smooth affinoid, then the map $i^\dagger i$ is injective.*

Proof. Let L_0 be the projective (since $\mathcal{O}(X)$ is a Dedekind domain) finite Banach $\mathcal{O}(X)$-module $\ker(i^\dagger i)$, and note that $L_0 \subset L^2$ is stable under the action of all the Hecke operators, since they all commute with $i^\dagger i$. Suppose L_0 is not zero. For (f, g) in L_0 we have $(l+1)f + S_l^{-1}T_l g = T_l f + (l+1)g = 0$. Eliminating g we get $T_l^2 f - (l+1)^2 S_l f = 0$, so projecting L_0 down to L (taking either the first or the second factor) we see that the Hecke operator $T_l^2 - (l+1)^2 S_l$ acts as 0 on a non-zero projective submodule of L. This (applying the local eigenvariety construction as described in section 6.2 of [9]) implies that there is a family of eigenforms over some one dimensional sub-affinoid of X, all with the eigenvalue of $T_l^2 - (l+1)^2 S_l$ equal to 0. Now the Hecke algebra element $T_l^2 - (l+1)^2 S_l$ induces a rigid analytic function on the tame level N eigencurve for D (by taking the appropriate eigenvalue associated to a point), so this function must vanish on the whole irreducible component containing the one dimensional family constructed above. However, every irreducible component contains a classical point, and these cannot be contained in the

kernel of $T_l^2 - (l+1)^2 S_l$ since this would contradict the Hecke eigenvalue bounds given by the Ramanujan-Petersson conjecture. □

Note that the injectivity of $i^\dagger i$ implies the injectivity of i. The above shows that if X is as in the statement of Proposition 2.2.12, we have $\ker(i^\dagger) \cap \mathrm{im}(i) = 0$ so our families in M are not both old and new at l. However, if X is just a point, then $i^\dagger i$ may have a kernel - this corresponds to p-adic automorphic forms which are both old and new at l.

2.2.13 Some modules

We denote the fraction field of $\mathcal{O}(X)$ by F. If A is an $\mathcal{O}(X)$-module we write A_F for the F-vector space $A \otimes_{\mathcal{O}(X)} F$.

We begin this section by noting that the injectivity of $i^\dagger i$ implies the injectivity of $j^\dagger j$:

Suppose $j^\dagger j(\lambda, \mu) = 0$. Then $\langle (f,g), j^\dagger j(\lambda,\mu)\rangle = 0$ for all $(f,g) \in L_F$, so (by Proposition 2.2.9) $\langle i^\dagger i(f,g), (\lambda,\mu)\rangle = 0$ for all $(f,g) \in L_F$. Now since $i^\dagger i : L_F \to L_F$ is an injective endomorphism of a finite dimensional vector space, it is an isomorphism, so we see that $\lambda = \mu = 0$. Hence $j^\dagger j$ (thus a fortiori j) is injective.

We now define two chains of modules which will prove useful:

$$\begin{aligned}
\Lambda_0 &:= L^2 & \Lambda_0^* &:= L^{*2} \\
\Lambda_1 &:= i^\dagger M & \Lambda_1^* &:= j^\dagger M^* \\
\Lambda_2 &:= i^\dagger(M \cap i(L_F^2)) & \Lambda_2^* &:= j^\dagger(M^* \cap j((L_F^*)^2)) \\
\Lambda_3 &:= i^\dagger i L^2 & \Lambda_3^* &:= j^\dagger j L^{*2}.
\end{aligned}$$

We note that $\Lambda_0 \supset \Lambda_1 \supset \Lambda_2 \supset \Lambda_3$, and that

$$\Lambda_2/\Lambda_3 = i^\dagger(M \cap i(L_F^2)/iL^2) = i^\dagger((M/iL^2)^{\mathrm{tors}}),$$

with analogous statements for the starred modules.

We fix the usual action of $\mathbb{T}^{(N\delta pl)}$ on all these modules. We can now describe some pairings between them which will be equivariant under the $\mathbb{T}^{(N\delta pl)}$ action. They will not all be equivariant with respect to the action of T_l.

We have a (perfect) pairing $\langle , \rangle : L_F^2 \times (L_F^*)^2 \to F$ which, since j is injective, induces a pairing

$$\Lambda_0 \times (M^* \cap j((L_F^*)^2)) \to F/\mathcal{O}(X),$$

which in turn induces a pairing

$$P_1 : \Lambda_0/\Lambda_1 \times (M^* \cap j((L_F^*)^2)/j(L^{*2})) \to F/\mathcal{O}(X).$$

The fact that this pairing is perfect follows from the following lemma:

Lemma 2.2.14. *The pairing on $L_F^2 \times (L_F^*)^2$ induces isomorphisms*

$$\mathrm{Hom}_{\mathcal{O}(X)}(\Lambda_1, \mathcal{O}(X)) \cong M^* \cap j((L_F^*)^2)$$

and

$$\mathrm{Hom}_{\mathcal{O}(X)}(\Lambda_0, \mathcal{O}(X)) \cong j(L^{*2}).$$

Proof. For the first isomorphism, elements of the module $\mathrm{Hom}_{\mathcal{O}(X)}(\Lambda_1, \mathcal{O}(X))$ correspond to $l \in (L_F^*)^2$ such that $\langle i^\dagger m, l\rangle \in \mathcal{O}(X)$ for all $m \in M$. We have $\langle i^\dagger m, l\rangle = \langle m, jl\rangle$ so $\langle i^\dagger m, l\rangle \in \mathcal{O}(X)$ for all $m \in M$ if and only if $\langle m, jl\rangle \in \mathcal{O}(X)$ for all $m \in M$, i.e. if and only if $jl \in M^*$.

The second isomorphism is obvious, since j is injective. □

In exactly the same way, we have a perfect pairing

$$P_2 : (M \cap i(L_F^2))/i(L^2) \times \Lambda_0^*/\Lambda_1^* \to F/\mathcal{O}(X).$$

The final pairing we will need is induced by the pairing between M and M^*. It is straightforward to check that this gives a perfect pairing:

$$P_3 : \ker(i^\dagger) \times M^*/(M^* \cap j((L_F^*)^2)) \to \mathcal{O}(X).$$

2.2.15 An analogue of Ihara's lemma

In classical level raising results (such as [15, 37, 44]) analogues of 'Ihara's lemma' (Lemma 3.2 in [23]) are used to show that prime ideals of a Hecke algebra containing the annihilators of certain modules of automorphic forms are in some sense 'uninteresting', or even to show that these modules are trivial. In this section we prove the appropriate analogue of Ihara's lemma in our setting.

From this section onwards we will assume that X is a one dimensional irreducible connected smooth affinoid in weight space $\mathscr{W}$, so we can apply Proposition 2.2.12. We want

to obtain information about the $\mathbb{T}^{(N\delta pl)}$ action on the quotients $\Lambda_2/\Lambda_3 \cong i^\dagger(M/iL^2)^{\mathrm{tors}}$, $\Lambda_2^*/\Lambda_3^* \cong j^\dagger(M^*/jL^{*2})^{\mathrm{tors}}$, Λ_0/Λ_1 and Λ_0^*/Λ_1^*. The pairings P_1 and P_2 allow us to use an analogue of Ihara's lemma (the following two propositions and theorem) to obtain crucial information about all four quotients. Recall that the radius of overconvergence r equals p^{-n} for some positive integer n. Fix a positive integer c such that $Nm(U_1(Np^{\alpha+n}))$ contains all elements of $\hat{\mathbb{Z}}^\times$ congruent to 1 modulo c. We first need a lemma allowing us to control certain forms with weight a point in weight space.

Lemma 2.2.16. *Let $x \in \mathscr{W}(K')$ for some K' a finite extension of $\mathbb{Q}_p$.*

1. *Let $y \in \mathbf{S}_x^D(U;r)$ be non-zero. Suppose y factors through Nm, that is $y(g) = y(h)$ for all $g, h \in D_f^\times$ with $Nm(g) = Nm(h)$. Then κ_x is a classical weight $z \mapsto z^2\varepsilon_p(z)$, and for all but finitely many primes $q \equiv 1$ mod c, (where c is the fixed integer chosen above), $(T_q - q - 1)y = 0$.*

2. *Let $y \in \mathbf{V}_x^D(U;r)$. If y factors through Nm, that is $y(g) = y(h)$ for all $g, h \in D_f^\times$ with $Nm(g) = Nm(h)$, then y is zero.*

Proof. We first prove part (i). Suppose y is as in the statement of that part. For $u_p \in \mathrm{SL}_2(\mathbb{Q}_p) \cap U$ we have $y(g) = y(gu_p) = y(g) \cdot u_p$ for all $g \in D_f^\times$. Noting that $\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Q}_p) \cap U$ for all $a \in \mathbb{Z}_p$, we see that $y(g)(z+a) = y(g)(z)$ for all $a \in \mathbb{Z}_p$, $z \in \mathbb{B}_{r,K'}$ so $y(g)(z)$ is constant in z, since non-constant rigid analytic functions have discrete zero sets. Recall that $U = U_1(Np^\alpha)$, so $u_0 := \begin{pmatrix} 1 & 0 \\ p^\alpha & 1 \end{pmatrix}$ is in $\mathrm{SL}_2(\mathbb{Q}_p) \cap U$, and for $z \in \mathbb{B}_{r,K'}$ we have

$$y(g)(z) = (y(g)u_0)(z) = \frac{\kappa_x(p^\alpha z + 1)}{(p^\alpha z + 1)^2} y(g),$$

so κ_x must correspond to the classical weight given by $z \mapsto z^2\epsilon_p(z)$ for some character ϵ_p trivial on $1 + p^{\alpha+n}\mathbb{Z}_p$, where $r = p^{-n}$. This now implies that for each $g \in D_f^\times$ we have $y(g)\gamma = y(g)$ for all γ in the projection of $U_1(Np^{\alpha+n})$ to $\mathbb{M}_{\alpha+n}$, since these matrices all have bottom right hand entry congruent to $1 \mod p^{\alpha+n}$.

We now follow [15] to complete the proof of the first part of the lemma. There is a $d_0 \in D^\times$ with $Nm(d_0) = q$, so $Nm(d_0^{-1}\eta_q) \in \mathbb{A}_f^\times$ is actually in $\hat{\mathbb{Z}}^\times$ and is congruent to 1 mod c. Thus (by the way we picked c) there is $u_0 \in U_1(Np^{\alpha+n})$ such that $Nm(u_0) = Nm(d_0^{-1}\eta_q)$. Now we have

$$\begin{aligned}
T_q(y)(g) &= \sum_{u \in (\eta_q^{-1} U \eta_q) \cap U \backslash U} y(gu^{-1}\eta_q^{-1}) \cdot u_p \\
&= \sum_{u \in (\eta_q^{-1} U \eta_q) \cap U \backslash U} y(g\eta_q^{-1}u^{-1}) \cdot u_p \\
&= \sum_{u \in (\eta_q^{-1} U \eta_q) \cap U \backslash U} y(g\eta_q^{-1}) = (q+1)y(g\eta_q^{-1}) \\
&= (q+1)y(g\eta_q^{-1}d_0d_0^{-1}) = (q+1)y(gu_0^{-1}d_0^{-1}) = (q+1)y(gu_0^{-1}) = (q+1)y(g),
\end{aligned}$$

where to pass from the first line to the second we use the fact that y factors through Nm to commute y's arguments, from the second to the third we use that y is modular of level U and in the final line we first substitute u_0 for $d_0^{-1}\eta_q$ (since they have the same reduced norm), then commute y's arguments and use the left invariance of y under $D^\times$ followed by the fact that $u_0^{-1} \in U_1(Np^{\alpha+n})$ implies that $y(gu_0^{-1}) = y(g)u_{0,p}^{-1} = y(g)$.

We now give a proof of the second part of the lemma. First we perform a formal calculation. Fix an isomorphism

$$\mathscr{A}_{x,r} \cong \prod_{\alpha=1}^{n} K'\langle T \rangle$$

where $K'\langle T \rangle$ is the ring of power series with coefficients in K' tending to zero (T a formal variable), and n is a positive integer depending on r. Such an isomorphism exists since $\mathbb{B}_{r,K'}$ is just a disjoint union of finitely many affinoid discs. We then have an identification of $\mathscr{D}_{x,r}$ with $\prod_{\alpha=1}^{n} K'\langle [T] \rangle$, where $K'\langle [T] \rangle$ denotes the ring of power series with bounded coefficients in K', and the pairing between $\mathscr{A}_{x,r}$ and $\mathscr{D}_{x,r}$ is given on each component by $\langle \sum a_i T^i, \sum b_j T^j \rangle = \sum a_i b_i$. Now we can compute the action of $\gamma = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ on $\mathscr{D}_{x,r}$. Let $f = (f_\alpha)_{\alpha=1,\ldots,n}$ be an element of $\mathscr{D}_{x,r}$, with $f_\alpha = \sum b_{j,\alpha} T^j$. For each $\alpha = 1, \ldots, n$ and $i \geq 0$ fix $e_{i,\alpha}$ to be the element of $\mathscr{A}_{x,r}$ which is T^i at the α component, and zero elsewhere.

We have

$$\begin{aligned}\langle e_{i,\alpha}, f\cdot\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\rangle = \langle T^i, (\sum b_{j,\alpha}T^j)\cdot\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}\rangle &= \langle T^i\cdot\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \sum b_{j,\alpha}T^j\rangle\\ &= \langle (T+1)^i, \sum b_{j,\alpha}T^j\rangle\\ &= \langle \sum_{k=0}^{i}\binom{i}{k}T^k, \sum b_{j,\alpha}T^j\rangle\\ &= \sum_{j=0}^{i}\binom{i}{j}b_{j,\alpha}.\end{aligned}$$

We can now see that if we have $f = f\cdot\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}$ we get $\sum_{j=0}^{i}\binom{i}{j}b_{j,\alpha} = b_{i,\alpha}$ for all i and α, which implies that $f = 0$.

Now we return to the statement in the lemma and suppose $y \in \mathbf{V}_x^D(U;r)$ factors through Nm. Let u_1 be the element of $U \subset D_f^\times$ with p component equal to γ and all other components the identity. For all $g \in D_f^\times$, $y(gu_1) = y(g)\gamma$ since $u_1 \in U$ and $y(gu_1) = y(g)$ since $Nm(u_1) = 1$. Hence $y(g) = y(g)\gamma$ and the above calculation shows that $y(g) = 0$ for all g.

□

The following two propositions apply the preceding lemma to give a form of Ihara's lemma for modules of overconvergent automorphic forms and dual overconvergent forms.

Proposition 2.2.17. *For all but finitely many primes $q \equiv 1$ mod c, $T_q - q - 1$ annihilates the module $\mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/\mathfrak{m})$ for each maximal ideal $\mathfrak{m}$ of $\mathcal{O}(X)$.*

Proof. Fix $q \equiv 1$ mod c with $q \nmid Np\delta l$ and set $H_q := T_q - q - 1$. Let $\mathfrak{m}$ be a maximal ideal of $\mathcal{O}(X)$ and set $K' = \mathcal{O}(X)/\mathfrak{m}$. The maximal ideal $\mathfrak{m}$ corresponds to a point x of $X(K')$, with corresponding weight $\kappa_x : \mathbb{Z}_p^\times \to K'^\times$ the specialisation of κ at $\mathfrak{m}$.

We have a short exact sequence

$$0 \longrightarrow L^2 \xrightarrow{i} M \longrightarrow M/iL^2 \longrightarrow 0\,.$$

Noting that L^2 and M are $\mathcal{O}(X)$-torsion free, hence flat, and taking derived functors of

$- \otimes_{\mathcal{O}(X)} K'$ gives an exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, K') & \xrightarrow{\delta} & L^2 \otimes_{\mathcal{O}(X)} K' & & \\ & & & & & \searrow^{i} & \\ 0 & \longleftarrow & M/iL^2 \otimes_{\mathcal{O}(X)} K' & \longleftarrow & M \otimes_{\mathcal{O}(X)} K' & & \end{array}.$$

We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^2 & \xrightarrow{i} & M & \longrightarrow & M/iL^2 & \longrightarrow & 0\,, \\ & & \downarrow{\scriptstyle H_q} & & \downarrow{\scriptstyle H_q} & & \downarrow{\scriptstyle H_q} & & \\ 0 & \longrightarrow & L^2 & \xrightarrow{i} & M & \longrightarrow & M/iL^2 & \longrightarrow & 0 \end{array}$$

so by the naturality of the long exact sequence for Tor the diagram

$$\begin{array}{ccc} \mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, K') & \xrightarrow{\delta} & L^2 \otimes_{\mathcal{O}(X)} K' \\ \downarrow{\scriptstyle H_q} & & \downarrow{\scriptstyle H_q} \\ \mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, K') & \xrightarrow{\delta} & L^2 \otimes_{\mathcal{O}(X)} K' \end{array}$$

commutes. To complete the proof it suffices to prove that H_q annihilates the kernel of

$$i : L^2 \otimes_{\mathcal{O}(X)} K' \to M \otimes_{\mathcal{O}(X)} K'.$$

We proceed by viewing these modules as spaces of automorphic forms with weight x (a single point in weight space). We define two finite dimensional K'-vector spaces:

$$\begin{aligned} L_x &:= \mathbf{S}_x^D(U; r)^{\leq d} \\ M_x &:= \mathbf{S}_x^D(V; r)^{\leq d}. \end{aligned}$$

There are maps $i_x : L_x^2 \to M_x$ and $i_x^\dagger : M_x \to L_x$ as defined in section 2.2.11 (taking the weight X in that section to be the point x), but note that as now the weight is just a point in weight space, Proposition 2.2.12 does not apply. In particular the map i_x might not be injective.

Recall that L and M are finitely generated $\mathcal{O}(X)$ modules, whence $L \otimes_{\mathcal{O}(X)} K'$ and $M \otimes_{\mathcal{O}(X)} K'$ are finite dimensional K'-vector spaces. Since the Newton polygon of the characteristic power series for U_p acting on $\mathbf{S}_X^D(U; r)$ has the same slope $\leq d$ part when spe-

cialised to any point of X, and the specialisation of the Banach $\mathcal{O}(X)$-modules $\mathbf{S}_X^D(U;r)$ and $\mathbf{S}_X^D(V;r)$ at $\mathfrak{m}$ gives $\mathbf{S}_x^D(U;r)$ and $\mathbf{S}_x^D(V;r)$ respectively, we have isomorphisms $L \otimes_{\mathcal{O}(X)} K' \to L_x$ and $M \otimes_{\mathcal{O}(X)} K' \to M_x$ which commute suitably with $i : L^2 \otimes_{\mathcal{O}(X)} K' \to M \otimes_{\mathcal{O}(X)} K'$ and $i_x : L_x^2 \to M_x$. These isomorphisms also commute with double coset operators, so to prove the proposition it suffices show that H_q annihilates the kernel of i_x.

Suppose $i_x(y_1, y_2) = 0$. Then $y_1 = -y_2|\eta_l$, so we have $y_2 \in \mathbf{S}_x^D(U;r)$, $y_2|\eta_l \in \mathbf{S}_x^D(U;r)$. Therefore y_2 and $y_2|\eta_l$ are both invariant under the action of the group U, so y_2 is invariant under the action of the group generated by U and $\eta_l U \eta_l^{-1}$ in $D_f^\times$. (Note that every element of this group has projection to its pth component lying in $\mathbb{M}_\alpha$.) Since by II.1.4, Corollary 1 of [42]

$$\mathrm{SL}_2(\mathbb{Q}_l) = \langle \mathrm{SL}_2(\mathbb{Z}_l), \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \mathrm{SL}_2(\mathbb{Z}_l) \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}^{-1} \rangle,$$

and the l-factor of U is $\mathrm{GL}_2(\mathbb{Z}_l)$, we have that y_2 is invariant under $\mathrm{SL}_2(\mathbb{Q}_l)$, where we embed $\mathrm{SL}_2(\mathbb{Q}_l)$ into $D_f^\times$ in the obvious way.

Denote by $D^{Nm=1}$ the algebraic subgroup of $D^\times$ whose elements are of reduced norm 1. We have $D^{Nm=1}(\mathbb{Q}_l) \cong \mathrm{SL}_2(\mathbb{Q}_l)$, since D is split at l. The strong approximation theorem applied to $D^{Nm=1}$ implies that $D^{Nm=1}(\mathbb{Q}) \cdot \mathrm{SL}_2(\mathbb{Q}_l)$ is dense in $D_f^{Nm=1} := D^{Nm=1}(\mathbb{A}_f)$, where $D^{Nm=1}(\mathbb{Q})$ is embedded diagonally in $D_f^{Nm=1}$. For each $g \in D_f^\times$ we define

$$X^g := \{h \in D_f^{Nm=1} : y_2(gh) = y_2(g)\}.$$

Since y_2 is continuous, X^g is closed, and for $\delta \in D^{Nm=1}(\mathbb{Q})$, $\gamma \in \mathrm{SL}_2(\mathbb{Q}_l)$ we have $y_2(gg^{-1}\delta\gamma g) = y_2(\delta\gamma g) = y_2(\gamma g) = y_2(gg^{-1}\gamma g) = y_2(g)$, since $g^{-1}\gamma g \in \mathrm{SL}_2(\mathbb{Q}_l)$. Therefore X^g contains the dense set $g^{-1}D^{Nm=1}(\mathbb{Q})\mathrm{SL}_2(\mathbb{Q}_l)g$, so X^g is the whole of $D_f^{Nm=1}$. This shows that y_2 factors through Nm. Now the first part of Lemma 2.2.16 applies. □

Proposition 2.2.18. *The module* $\mathrm{Tor}_1^{\mathcal{O}(X)}(M^*/jL^{*2}, \mathcal{O}(X)/\mathfrak{m})$ *is* 0 *for all maximal ideals* $\mathfrak{m}$ *of* $\mathcal{O}(X)$

Proof. We again set $K' = \mathcal{O}(X)/\mathfrak{m}$, and let $x \in X(K')$ be the point corresponding to $\mathfrak{m}$. Proceeding as at the beginning of the proof of Proposition 2.2.17 we see that we must show that the map

$$j : L^{*2} \otimes_{\mathcal{O}(X)} K' \to M^* \otimes_{\mathcal{O}(X)} K'$$

is injective. We define

$$\begin{aligned} L_x^* &:= \mathbf{V}_x^D(U;r)^{\leq d} \\ M_x^* &:= \mathbf{V}_x^D(V;r)^{\leq d}. \end{aligned}$$

As in the previous proposition, it is sufficient to show that the map $j_x : L_x^{*2} \to M_x^*$ is injective. Now we continue as in the proof of Proposition 2.2.17, and finally apply the second part of Lemma 2.2.16. □

The following consequence of the preceding two propositions will be the most convenient analogue of Ihara's lemma for our applications.

Theorem 2.2.19. *1. There is a positive integer e such that for all but finitely many primes $q \equiv 1 \mod c$, $(T_q - q - 1)^e$ annihilates $(M/iL^2)^{\mathrm{tors}}$. Therefore these Hecke operators annihilate the modules Λ_2/Λ_3 and, by consideration of the pairing P_2, Λ_0^*/Λ_1^*.*

2. The module $(M^*/jL^{*2})^{\mathrm{tors}}$ is equal to 0. Therefore the modules Λ_2^*/Λ_3^* and, by consideration of the pairing P_1, Λ_0/Λ_1 are also equal to 0.

Proof. We first prove the first part of the theorem. Fix a $q \equiv 1 \mod c$ with $q \nmid Np\delta l$. The module $(M/iL^2)^{\mathrm{tors}}$ is finitely generated (and torsion) over the Dedekind domain $\mathcal{O}(X)$, so it is isomorphic as an $\mathcal{O}(X)$-module to $\bigoplus_i \mathcal{O}(X)/\mathfrak{m}_i^{e_i}$ for some finite set of maximal ideals $\mathfrak{m}_i$ in $\mathcal{O}(X)$. We set e to be the maximum of the e_i. Set $H_q := T_q - q - 1$ as before. We will show that H_q^e annihilates $(M/iL^2)^{\mathrm{tors}}$.

Indeed, suppose that $m \in M$ represents a nonzero torsion class in M/iL^2. Thus, there exists a nonzero $\alpha \in \mathcal{O}(X)$ such that $\alpha m \in iL^2$. We have

$$\mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/(\alpha)) = \{m \in M/iL^2 : \alpha m = 0\}.$$

So we are required to prove that H_q^e annihilates

$$\mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/(\alpha)).$$

Since

$$(M/iL^2)^{\mathrm{tors}} \cong \bigoplus_i \mathcal{O}(X)/\mathfrak{m}_i^{e_i},$$

we can assume that $(\alpha) \supset \prod_i \mathfrak{m}_i^{e_i}$, so it is enough to show that H_q^e annihilates

$$\bigoplus_i \mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/\mathfrak{m}_i^{e_i}).$$

Taking derived functors of $- \otimes_{\mathcal{O}(X)} M/iL^2$ of the short exact sequence

$$0 \longrightarrow \mathfrak{m}_i/\mathfrak{m}_i^{e_i} \longrightarrow \mathcal{O}(X)/\mathfrak{m}_i^{e_i} \longrightarrow \mathcal{O}(X)/\mathfrak{m}_i \longrightarrow 0$$

and applying Proposition 2.2.17 and induction on e_i (note that $\mathfrak{m}_i/\mathfrak{m}_i^{e_i}$ is isomorphic to $\mathcal{O}(X)/\mathfrak{m}_i^{e_i-1}$), we see that for each i, $H_q^{e_i}$ annihilates $\mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/\mathfrak{m}_i^{e_i})$. Now $e_i \leq e$ for all i, so H_q^e annihilates all of $\mathrm{Tor}_1^{\mathcal{O}(X)}(M/iL^2, \mathcal{O}(X)/(\alpha))$. Since α was arbitrary, H_q^e annihilates $(M/iL^2)^{\mathrm{tors}}$.

The second part of the theorem follows easily from Proposition 2.2.18. □

2.2.20 Supporting Hecke ideals

We will call a maximal ideal $\mathfrak{M}$ in the Hecke algebra $\mathbb{T}^{(N\delta p)}$ *Eisenstein* (compare [30]) if there is some positive integer c such that for all but finitely many primes $q \equiv 1 \mod c$, $T_q - q - 1 \in \mathfrak{M}$. The motivation for this definition is the following well-known lemma:

Lemma 2.2.21. *Suppose we have a Galois representation $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$, continuous and unramified at all but finitely many primes, and a positive integer c such that for all but finitely many primes $q \equiv 1 \mod c$, ρ is unramified at q and the trace of Frobenius at q, $\mathrm{Tr}(\mathrm{Frob}_q) = q + 1$. Then ρ is reducible.*

Proof. The formula for the traces implies that ρ restricted to the absolute Galois group of the cyclotomic field $\mathbb{Q}(\zeta_c)$ has semisimplification isomorphic to $\mathbf{1} \oplus \chi$, where $\mathbf{1}$ is the trivial one dimensional representation, and χ is the p-adic cyclotomic character. Denoting $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by G and $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_c))$ by H and applying Frobenius reciprocity we conclude that

$$\mathrm{Hom}(\mathrm{Ind}_H^G(\mathbf{1} \oplus \chi), \rho)$$

is non-zero. $\mathrm{Ind}_H^G(\mathbf{1} \oplus \chi)$ is just a direct sum of one dimensional representations, so ρ is reducible. □

We recall that for an arbitrary (commutative) ring R the *support* of an R-module A is the set of prime ideals $\mathfrak{p} \lhd R$ such that the localisation $A_\mathfrak{p}$ is non-zero. If A is finitely

generated as an R-module then the support of A is equal to the set of prime ideals in R containing the annihilator of A. We write $\mathbb{T}_L$ for the image of $\mathbb{T}^{(N\delta p)}$ in $\mathrm{End}_{\mathcal{O}(X)}(L)$ and similarly $\mathbb{T}_M$ for the image of $\mathbb{T}^{(N\delta pl)}$ in $\mathrm{End}_{\mathcal{O}(X)}(M)$. Analogously we define $\mathbb{T}_{L^*}$ and $\mathbb{T}_{M^*}$ to be the image of the Hecke algebra in the endomorphism rings of the relevant dual modules. Note that there are natural maps $\mathbb{T}_M \to \mathbb{T}_L$ and $\mathbb{T}_{M^*} \to \mathbb{T}_{L^*}$. If I is an ideal of $\mathbb{T}^{(N\delta p)}$ we write I_L for the image of I in $\mathbb{T}_L$ and I'_M for the image of $I \cap \mathbb{T}^{(N\delta pl)}$ in $\mathbb{T}_M$

We can now state and prove the main theorem of this section.

Theorem 2.2.22. *Suppose $\mathfrak{M}$ is a non-Eisenstein maximal ideal of $\mathbb{T}^{(N\delta p)}$ containing $T_l^2 - (l+1)^2 S_l$. Further suppose that $\mathfrak{M}_L$ is in the support of the $\mathbb{T}_L$-module L. Then $\mathfrak{M}'_M$ is in the support of the $\mathbb{T}_M$-module $\ker(i^\dagger) \subset M$.*

Proof. We write $\mathfrak{M}_{L^*}$ for the image of $\mathfrak{M}$ in $\mathbb{T}_{L^*}$ and $\mathfrak{M}'_{M^*}$ for the image of $\mathfrak{M} \cap \mathbb{T}^{(N\delta pl)}$ in $\mathbb{T}_{M^*}$. Since $\mathfrak{M}_L$ is in the support of L, and the perfect pairing between L and L^* is equivariant with respect to all of $\mathbb{T}^{(N\delta p)}$ (including T_l), we know that $\mathfrak{M}_{L^*}$ is in the support of L^*. Consider the module

$$Q := \Lambda_0^*/\Lambda_3^* = L^{*2} \Big/ L^{*2} \begin{pmatrix} l+1 & T_l \\ S_l^{-1}T_l & l+1 \end{pmatrix},$$

Since $L^{*2}_{\mathfrak{M}_{L^*}} \neq 0$ and $\det \begin{pmatrix} l+1 & T_l \\ S_l^{-1}T_l & l+1 \end{pmatrix} \in \mathfrak{M}$ we know that $Q_{\mathfrak{M}_{L^*}} \neq 0$, i.e. $\mathfrak{M}_{L^*}$ is in the support of Q. We can view Q as a $\mathbb{T}_{M^*}$-module, with $\mathfrak{M}'_{M^*}$ in its support.

Theorem 2.2.19 implies that if $\mathfrak{M}'_{M^*}$ is in the support of Λ_0^*/Λ_1^* or Λ_2^*/Λ_3^* then it is Eisenstein, so it must be in the support of Λ_1^*/Λ_2^*. This quotient is a homomorphic image of $M^*/(M^* \cap j(L_F^{*2}))$, so $\mathfrak{M}'_{M^*}$ is in the support of $M^*/(M^* \cap j(L_F^{*2}))$. Finally we can apply pairing P_3 (which is equivariant with respect to $\mathbb{T}^{(N\delta pl)}$) to conclude that $\mathfrak{M}'_M$ is in the support of $\ker(i^\dagger)$. □

2.3 Applications

In this section we explain some applications of the preceding results, including the proof of some cases of the conjecture mentioned in the introduction.

2.3.1 Geometric level raising for p-adic modular forms

We firstly describe the application of Theorem 2.2.22 to the conjecture of our Introduction. Chenevier [10] extended the classical Jacquet-Langlands correspondence to a rigid analytic embedding from the eigencurve for a definite quaternion algebra to some part of the GL_2 eigencurve. We may use this to translate the results of the previous section to the GL_2 eigencurve.

We state the case of Theorem 3 in [10] that we will use. We have primes p, l and a coprime integer $N \geq 1$. Fix another prime q and a character ε of $\mathbb{Z}/Np\mathbb{Z}$. Let $\mathscr{E}$ be the tame level $\Gamma_1(N) \cap \Gamma_0(q)$ and character ε reduced cuspidal eigencurve. Let $D/\mathbb{Q}$ be a quaternion algebra ramified at the infinite place and at q, and let $\mathscr{E}^D$ be the corresponding reduced eigencurve of tame level $U_1(N)$ and character ε.

Theorem 2.3.2. *There is a closed rigid analytic immersion*

$$JL_p : \mathscr{E}^D \hookrightarrow \mathscr{E}$$

whose image is the Zariski closure of the classical points of $\mathscr{E}$ that are new at q. This map is defined over weight space and is Hecke equivariant.

We get the same result if we change the level of $\mathscr{E}$ to $\Gamma_1(N) \cap \Gamma_0(ql)$ (call this eigencurve $\mathscr{E}'$) and change the level of $\mathscr{E}^D$ to $U_1(N) \cap U_0(l)$ (call this eigencurve $\mathscr{E}^{D'}$), where we construct these eigencurves using the Hecke operators at l in addition to the usual Hecke operators away from the level. This allows us to relate $\mathscr{E}'$ and the two-covering $\mathscr{E}^{\mathrm{old}}$ of $\mathscr{E}$ corresponding to taking roots of lth Hecke polynomial.

Lemma 2.3.3. *There is a closed embedding $\mathscr{E}^{\mathrm{old}} \hookrightarrow \mathscr{E}'$, with image the Zariski closure of the classical l-old points in $\mathscr{E}'$.*

Proof. Given X an affinoid subdomain of $\mathscr{W}$, let M_X and M'_X denote the Banach $\mathcal{O}(X)$-modules of families (weight varying over X) of overconvergent modular forms of tame levels $\Gamma_1(N) \cap \Gamma_0(q)$ and $\Gamma_1(N) \cap \Gamma_0(ql)$ respectively, as defined in section 7 of [3]. The two degeneracy maps from level $\Gamma_1(N) \cap \Gamma_0(ql)$ to level $\Gamma_1(N) \cap \Gamma_0(q)$ give a natural embedding $M_X^2 \to M'_X$. Denote the image of this map by M_X^{old} - it is stable under all Hecke operators (including at l). The lemma follows from the observation that applying the eigenvariety machine of [3] to the Banach modules M_X^{old} (with X varying) gives the space $\mathscr{E}^{\mathrm{old}}$. □

Let $\mathscr{Z} \subset \mathscr{E}'$ be the Zariski closure of the classical points in $\mathscr{E}'$ corresponding to forms new at l. Proposition 4.7 of [10] shows that $\mathscr{Z}$ can be identified with the points of $\mathscr{E}'$ lying in a one dimensional family of l-*new* points, where l-new means they come from overconvergent modular forms in the kernel of the map analogous to $i^\dagger$ in the GL_2 setting. The following theorem corresponds to the conjecture in the introduction for points of $\mathscr{E}'$ in the image of JL_p.

Theorem 2.3.4. *Suppose we have a point $\phi \in \mathscr{E}$ lying in the image of JL_p, with $T_l^2(\phi) - (l+1)^2 S_l(\phi) = 0$. Let the roots of the lth Hecke polynomial corresponding to ϕ be α and $l\alpha$ where $\alpha \in \mathbb{C}_p$. Then the point over ϕ of $\mathscr{E}^{\mathrm{old}}$ corresponding to α lies in $\mathscr{Z}$.*

Proof. We pick d, r and α such that the automorphic form corresponding to the preimage of ϕ under JL_p is r-overconvergent of slope $\leq d$ and level $U_1(Np^\alpha)$. Now fix a closed ball in $\mathscr{W}$, containing the weight of ϕ, which is small enough (note that 'small enough' depends on d, r and α) to apply the local eigenvariety construction described in section 6.2 of [9]. Denote this ball by X. As usual the $rp^{-\alpha}$-analytic character $\mathbb{Z}_p^\times \to \mathcal{O}(X)$ induced by the embedding $X \hookrightarrow \mathscr{W}$ is denoted by κ.

The system of Hecke eigenvalues given by ϕ corresponds to a maximal ideal $\mathfrak{M}$ in the $\mathcal{O}(X)$ Hecke algebra $\mathbb{T}^{(Npq)}$. We know that $T_l^2 - (l+1)^2 S_l \in \mathfrak{M}$. If we set $L = \mathbf{S}_X^D(U_1(Np^\alpha); r)^{\leq d}$ as before, and use the notation of the previous section, then $\mathfrak{M}_L$ is in the support of L. At this stage Theorem 2.2.22 applies to the ideal $\mathfrak{M}$, which is not Eisenstein since the Galois representation attached to ϕ is irreducible (recall that we are working on the cuspidal part of the eigencurve). Therefore we know that $\mathfrak{M}'_M$ is in the support of $\ker(i^\dagger) \subset M$. We can then take a height one prime ideal $\mathfrak{p} \subset \mathfrak{M}'_M$ in the support of $\ker(i^\dagger) \subset M$, and then this corresponds (by proposition 6.2.4 of [9]) to a p-adic family of automorphic forms, new at l, passing through a point ϕ' with system of Hecke eigenvalues the same as those for ϕ away from l. Now applying the map JL_p we see that one of the points over ϕ must lie in $\mathscr{Z}$. A calculation using the fact that ϕ' comes from an eigenform in the kernel of $i^\dagger$ shows that the point corresponds to the root α. □

To translate this theorem into the language of the introduction, note that $\mathscr{E}^{\mathrm{old}}$ corresponds to the generically unramified principal series components of $\mathscr{E}'$, whilst $\mathscr{Z}$ corresponds to the generically special or supercuspidal components. We may identify the point over ϕ lying in $\mathscr{Z}$ as the one whose attached $\mathrm{GL}_2(\mathbb{Q}_l)$-representation is special.

2.3.5 Eigenvarieties of newforms

We return to the situation of a definite quaternion algebra D over $\mathbb{Q}$ with arbitrary discriminant δ prime to p. Denote the levels $U_1(Np^\alpha)$ by U and $U_1(Np^\alpha) \cap U_0(l)$ by V, as before. We denote by $\mathscr{E}^D$ the tame level $U_1(N) \cap U_0(l)$ reduced eigencurve for D. Suppose ϕ is a point of $\mathscr{E}^D$, with weight x. We say that ϕ is *l-new* if it corresponds to a Hecke eigenform in the kernel of the map $i_x^\dagger : \mathbf{S}_x^D(V;r) \to \mathbf{S}_x^D(U;r)$, where this is defined as in section 2.2.11 by $i_x^\dagger(f) := (f|[V1U], f|[V\eta_l^{-1}U])$. Denote by $\mathscr{Z}$ the Zariski closure of the points in $\mathscr{E}^D$ arising from *classical* l-new forms. We have the following proposition, due to Chenevier.

Proposition 2.3.6. *1. The set of x in $\mathscr{E}^D$ that are l-new is the set of points of a closed reduced analytic subspace $\mathscr{E}^D_{\mathrm{new}} \subset \mathscr{E}^D$.*

2. $\mathscr{Z}$ is a closed subspace of $\mathscr{E}^D_{\mathrm{new}}$, and its complement is the union of irreducible components of dimension 0 in $\mathscr{E}^D_{\mathrm{new}}$.

3. A point of $\mathscr{E}^D_{\mathrm{new}}$ lies in $\mathscr{Z}$ if and only if it lies in a one-dimensional family of points in $\mathscr{E}^D_{\mathrm{new}}$.

Proof. Exactly as for Proposition 4.7 of [10]. □

We now apply the results of section 2.2 to show that a point of $\mathscr{E}^D_{\mathrm{new}} \setminus \mathscr{Z}$ lies in a one-dimensional family of points in $\mathscr{E}^D_{\mathrm{new}}$, so by contradiction we can conclude that $\mathscr{E}^D_{\mathrm{new}}$ is equal to $\mathscr{Z}$.

Theorem 2.3.7. *$\mathscr{E}^D_{\mathrm{new}}$ is equal to $\mathscr{Z}$. In particular, $\mathscr{E}^D_{\mathrm{new}}$ is equidimensional of dimension* 1.

Proof. Let ϕ be a point of $\mathscr{E}^D_{\mathrm{new}} \setminus \mathscr{Z}$ with weight x. The proof will proceed by showing that ϕ is also l-old, then raising the level at ϕ, as in the previous theorem, to show that it lies in a family of l-new points.

We pick d, r and α such that ϕ comes from a r-overconvergent automorphic form of slope d and level $U = U_1(Np^\alpha)$. Now fix a closed ball in $\mathscr{W}$, containing the weight of ϕ, which is small enough (note that 'small enough' depends on d, r and α) to apply the local eigenvariety construction described in section 6.2 of [9]. Denote this ball by X. The point x in X corresponds to a maximal ideal $\mathfrak{m}$ of $\mathcal{O}(X)$. If we set $M = \mathbf{S}_X^D(V;r)^{\leq d}$ as before, then ϕ lies in a family corresponding to a Hecke eigenvector in M.

As in the previous subsection, we have a closed embedding $\mathscr{E}^D_{\mathrm{old}} \hookrightarrow \mathscr{E}^D$, where $\mathscr{E}^D_{\mathrm{old}}$ is a two-covering of the tame level U reduced eigencurve for D, and the image of this

embedding is the Zariski closure of the classical l-old points in $\mathscr{E}^D$, which also equals the space of l-old points in $\mathscr{E}^D$ (as is clear from applying the proof of Lemma 2.3.3 to modules of overconvergent automorphic forms for D). Since the space $\mathscr{E}^D$ is equidimensional of dimension 1, but ϕ does not lie in a family of points in $\mathscr{E}^D_{\mathrm{new}}$, ϕ must lie in a family of points in $\mathscr{E}^D_{\mathrm{old}}$. So ϕ is l-old and l-new.

We now 'raise the level' at ϕ. As in the proof of Proposition 2.2.17 we specialise the $\mathcal{O}(X)$ modules $L = \mathbf{S}^D_X(U;r)^{\leq d}$ and M at the maximal ideal $\mathfrak{m}$ to give vector spaces L_x and M_x. Since ϕ is l-new and l-old, it arises from an eigenform g in $\mathrm{im}(i_x) \cap \ker(i_x^\dagger)$. Now a calculation using the explicit matrix for the map $i_x^\dagger i_x$ shows that g is of the form $i_x(\alpha f, -f)$, where f is an eigenform in L_x with $(T_l^2 - (l+1)^2 S_l)f = 0$, and the roots of the lth Hecke polynomial for f are α and $l\alpha$ (note that with our normalisations g is the l-stabilisation of f corresponding to the root α). Now applying the proof of Theorem 2.3.4 we see that ϕ lies in $\mathscr{Z}$, so we have a contradiction and therefore must have $\mathscr{E}^D_{\mathrm{new}} = \mathscr{Z}$. □

Modules of newforms and the eigenvariety machine

Define the Banach $\mathcal{O}(X)$-module $M^{\mathrm{new}}_{X,r}$ for varying affinoid subdomains $X \subset \mathscr{W}$ (with corresponding character κ) to be the kernel of the map

$$i^\dagger : \mathbf{S}^D_X(U_1(Np^\alpha) \cap U_0(l); r) \to \mathbf{S}^D_X(U_1(Np^\alpha); r) \times \mathbf{S}^D_X(U_1(Np^\alpha); r).$$

One might wish to construct the rigid analytic space $\mathscr{E}^D_{\mathrm{new}}$ from these modules using Buzzard's eigenvariety machine [3]. The key issue is to show that the modules $M^{\mathrm{new}}_{X,r}$ behave well under base change between affinoid subdomains. The author is not sure whether this should be true or not - the results in this chapter can be viewed as showing that these modules behave well under base change from a sufficiently small affinoid to a point and it is not clear that one can conclude something about base change between open affinoids from this.

Chapter 3

Completed cohomology and level raising for p-adic modular forms

3.1 Introduction

In this note we describe an application of Poincaré duality for completed homology spaces (as defined by Emerton) to level raising for p-adic modular forms. Fix a prime $l \neq p$ and a positive integer N coprime to pl. Let $D(N)$ and $D(N,l)$ be the reduced eigencurves of tame level $\Gamma_1(N)$ and $\Gamma_1(N) \cap \Gamma_0(l)$ respectively, constructed using the Hecke operators away from N (we *do* use the Hecke operators at l). Let $D(N,l)_c$ be the equidimensional closed rigid analytic subvariety of $D(N,l)$ whose points are those with irreducible attached residual Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, and denote by $D(N)_c$ the analogous subvariety of $D(N)$. Define $D(N,l)_c^{\mathrm{new}}$ to be the the closed rigid analytic subvariety of $D(N,l)_c$ whose points come from overconvergent modular forms which vanish under the two natural trace maps from forms of level $\Gamma_1(N) \cap \Gamma_0(l)$ to forms of level $\Gamma_1(N)$. Our main result is the following, proved in section 3.4:

Theorem. *$D(N,l)_c^{\mathrm{new}}$ is equidimensional of dimension 1. It is equal to the Zariski closure of the classical l-new points in $D(N,l)_c$.*

This answers a question raised by the main theorem of [10], where the image of a p-adic Jacquet-Langlands map is identified as the Zariski closure of the classical l-new points in $D(N,l)$. At least in the subspace $D(N,l)_c$, the above theorem allows a point in this image to be identified by a natural condition on the corresponding overconvergent modular form.

This theorem is equivalent to a level raising result on the eigencurve. For each point x of $D(N)_c$ there are 2 points x_1, x_2 of $D(N,l)_c$ corresponding to the two roots of the lth Hecke polynomial for x. Suppose $T_l^2(x) - (l+1)^2 S_l(x) = 0$. Equivalently, one of the two points x_1, x_2, say x_1, is in $D(N,l)_c^{\mathrm{new}}$. The above theorem implies that there is a 1-dimensional irreducible component of $D(N,l)_c^{\mathrm{new}}$ passing through x_1. This irreducible component is an irreducible component of $D(N,l)_c$, with the property that every classical point on it is l-new. The point x_1 is not smooth, since two irreducible components (an old one and a new one) intersect there.

This short chapter builds on Ch.2 above, where an analogous theorem is proved for the eigencurve corresponding to overconvergent automorphic forms on a definite quaternion algebra - in fact in that setting our results include the points where the attached residual Galois representation is reducible. This difference arises because in Ch.2 a form of Ihara's lemma in characteristic 0 is used, whereas in this chapter we make use of the classical Ihara's lemma to show the injectivity of a level raising map between completed cohomol-

ogy spaces. We make use of completed cohomology (and homology) because Poincaré duality provides us with a natural pairing between completed homology spaces, which are finitely generated modules under a non-commutative local ring. Constructing a suitable pairing between spaces of overconvergent modular forms, the approach taken in Ch.2, seems to be rather more difficult in the more geometric setting of overconvergent modular forms.

3.2 Completed cohomology

We begin by defining completed homology and cohomology spaces, following Emerton's seminal paper [19] and Calegari and Emerton's survey [5]. Fix a finite extension E of $\mathbb{Q}_p$, with ring of integers $\mathcal{O}$ and uniformiser ϖ. Let K^p be a compact open subgroup of $\mathrm{GL}_2(\mathbb{A}_f^p)$. If K_p is a compact open subgroup of $G := \mathrm{GL}_2(\mathbb{Q}_p)$ then we denote by $Y(K_pK^p)$ the open modular curve (over $\mathbb{C}$) of level K_pK^p and write

$$H^i(K^p, A) := \varinjlim_{K_p} H^i(Y(K_pK^p), A),$$

where the limit is over all compact open K_p, and A is either E, $\mathcal{O}$ or $\mathcal{O}/\varpi^s$ for some $s > 0$. Note that there is a natural G action on $H^i(K^p, A)$, and a natural G-equivariant isomorphism

$$H^i(K^p, \mathcal{O})/\varpi^s H^i(K^p, \mathcal{O}) \cong H^i(K^p, \mathcal{O}/\varpi^s).$$

Definition 3.2.1. *We define the completed ith cohomology space of tame level K^p to be*

$$\widehat{H}^i(K^p, \mathcal{O}) := \varprojlim_s H^i(K^p, \mathcal{O})/\varpi^s H^i(K^p, \mathcal{O}) \cong \varprojlim_s H^i(K^p, \mathcal{O}/\varpi^s).$$

Applying the analogue of this construction to compactly supported cohomology, we obtain the space $\widehat{H}^i_c(K^p, \mathcal{O})$. We can also form completed homology spaces in a dual manner:

Definition 3.2.2. *The completed ith homology space of tame level K^p is*

$$\widehat{H}_i(K^p, \mathcal{O}) := \varprojlim_{K_p} H_i(Y(K_pK^p), \mathcal{O}).$$

The same construction with Borel-Moore homology gives $\widehat{H}_i^{\mathrm{BM}}(K^p, \mathcal{O})$. We write G_0

for $\mathrm{GL}_2(\mathbb{Z}_p) \subset G$. The key finiteness property for the spaces we have defined above is contained in the following result:

Proposition 3.2.3. *1. $\widehat{H}_i(K^p, \mathcal{O})$ and $\widehat{H}_i^{\mathrm{BM}}(K^p, \mathcal{O})$ are finitely generated left modules under the natural action of $\mathcal{O}[[G_0]]$. The topology induced by the $\mathcal{O}[[G_0]]$-module structure is equivalent to the projective limit topology on these spaces.*

2. We have $\mathrm{Hom}_{\mathrm{cts}}(\widehat{H}_1^{\mathrm{BM}}(K^p, \mathcal{O}), \mathcal{O}) \cong \widehat{H}_c^1(K^p, \mathcal{O})$ and $\mathrm{Hom}_{\mathrm{cts}}(\widehat{H}_c^1(K^p, \mathcal{O}), \mathcal{O}) \cong \widehat{H}_1^{\mathrm{BM}}(K^p, \mathcal{O})$.

Proof. The first part of the proposition follows from the first part of Theorem 1.1 in [5]. As for the second part, Proposition 4.3.6 of [19] shows that $\widehat{H}_c^2(K^p, \mathcal{O}) = 0$, whilst viewing Borel-Moore homology as the homology of the compactified modular curve relative to the cusps, it is easy to see that $\widehat{H}_0^{\mathrm{BM}}(K^p, \mathcal{O}) = 0$. Our proposition now follows from the third part of Theorem 1.1 in [5]. □

Remark. *The statement of Theorem 1.1 in [5] also describes the connection between $\widehat{H}_i(K^p, \mathcal{O})$ and $\widehat{H}^i(K^p, \mathcal{O})$, which is slightly more complicated than in the compactly supported case, since $\widehat{H}^0(K^p, \mathcal{O})$ is non-zero.*

Definition 3.2.4. *The Hecke algebra $\mathbb{T}(K^p)$ is defined to be the weakly closed $\mathcal{O}$-algebra of G-equivariant endomorphisms of $\widehat{H}^1(K^p, \mathcal{O})$ topologically generated by Hecke operators T_q, S_q for primes $q \nmid p$ that are unramified in K^p.*

3.3 Non-optimal levels

We now work with two fixed tame levels,

$$K(N) := \{g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}^p) : g \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \mod N\}$$

and

$$K(N, l) := K(N) \cap \{g \in \mathrm{GL}_2(\widehat{\mathbb{Z}}^p) : g \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod l\}.$$

We have a continuous morphism $\mathbb{T}(K(N, l)) \to \mathbb{T}(K(N))$, with image $\mathbb{T}(K(N))^{(l)}$ the subalgebra of $\mathbb{T}(K(N))$ generated by the Hecke operators away from l. The following definition can be found in section 1.2 of [1].

Definition 3.3.1. *An ideal I of $\mathbb{T}(K(N))^{(l)}$ is* Eisenstein *if the map $\lambda : T(K(N))^{(l)} \to T(K(N))^{(l)}/I$ satisfies $\lambda(T_q) = \epsilon_1(q) + \epsilon_2(q)$ and $\lambda(qS_q) = \epsilon_1(q)\epsilon_2(q)$ for all $q \nmid plN$, for some characters $\epsilon_1, \epsilon_2 : \mathbb{Z}_{plN}^{\times} \to T(K(N))^{(l)}/I$.*

Fix a maximal ideal $\mathfrak{m}$ of $\mathbb{T}(K(N))^{(l)}$ which is not Eisenstein. We can pull back $\mathfrak{m}$ to a maximal ideal $\overline{\mathfrak{m}}$ of $\mathbb{T}(K(N,l))$ (maximal since $\overline{\mathfrak{m}}$ will certainly contain p).

Proposition 3.3.2. *The natural maps $\alpha : \widehat{H}^1_c(K(N), \mathcal{O}) \to \widehat{H}^1(K(N), \mathcal{O})$ and $\beta : \widehat{H}^1_c(K(N,l), \mathcal{O}) \to \widehat{H}^1(K(N,l), \mathcal{O})$ become isomorphisms after localising at the non-Eisenstein maximal ideals $\mathfrak{m}$ and $\overline{\mathfrak{m}}$ respectively. The same holds for the maps from usual to Borel-Moore homology.*

Proof. Proposition 4.3.9 of [19] shows that the maps α and β are surjective. Let $\widehat{M}$ denote the kernel of α. Note that $\widehat{M}$ is a torsion free $\mathcal{O}$-module, since $\widehat{H}^1_c(K(N), \mathcal{O})$ is a torsion free $\mathcal{O}$-module. Corollaire 3.1.3 of [1] shows that $\widehat{M}$ is 'Eisenstein', i.e. if there is a system of Hecke eigenvalues $\lambda : T(K(N))^{(l)} \to L$, L a finite extension of E, with $M_L^{\lambda} \neq 0$, then $\ker(\lambda)$ is an Eisenstein ideal.

Note that $\widehat{M}$ contains a dense subspace $M := \ker(H^1_c(K(N), \mathcal{O}) \to H^1(K(N), \mathcal{O}))$. In fact $\widehat{M}$ is equal to the ϖ-adic completion of M, one way to see this is using the explicit description of $\widehat{M}$ (and M) given by Proposition 3.1.1 of [1]. The space $M \otimes_{\mathcal{O}} \overline{E}$ is spanned by Hecke eigenvectors, which have Eisenstein systems of eigenvalues, so $(M \otimes_{\mathcal{O}} \overline{E})_{\mathfrak{m}} = 0$ and hence $M_{\mathfrak{m}} = 0$, since M is $\mathcal{O}$-torsion free. Now $\widehat{M}_{\mathfrak{m}}$ is a direct summand of $\widehat{M}$, and the projection $\widehat{M} \to \widehat{M}_{\mathfrak{m}}$ is ϖ-adically continuous, with M in the kernel, hence this projection is the zero map and $\widehat{M}_{\mathfrak{m}} = 0$. So α does give an isomorphism after localising at $\mathfrak{m}$.

The same argument applies to the map β. The statement for homology follows from the duality of the second part of Proposition 3.2.3 (note that this duality holds for usual homology and cohomology after localising at a non-Eisenstein maximal ideal, since this kills $\widehat{H}^0(K^p, \mathcal{O})$ by the same argument we used to show that $\widehat{M}_{\mathfrak{m}} = 0$). □

Remark. *In the proof of Proposition 7.7.13 of [18] the fact that localising at non-Eisenstein maximal ideals induces an isomorphism between compactly supported completed cohomology and completed cohomology is used. The author thanks Matthew Emerton for communicating the above argument for why this follows from Corollaire 3.1.3 of [1].*

Since the Hecke algebras are semilocal, localising at the maximal ideals $\mathfrak{m}$ and $\overline{\mathfrak{m}}$ gives us direct summands of the original (co)homology spaces. Note that the second

part of Proposition 3.2.3 now gives us dualities between $\widehat{H}^1(K(N),\mathcal{O})_{\mathfrak{m}}$ (respectively $\widehat{H}^1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}}$) and $\widehat{H}_1(K(N),\mathcal{O})_{\mathfrak{m}}$ (respectively $\widehat{H}_1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}}$).

For each compact open $K_p \subset G$ there are two degeneracy maps from $Y(K_pK(N,l))$ to $Y(K_pK(N))$, which in the limit give rise to a natural level raising map

$$i : \widehat{H}^1(K(N),\mathcal{O})^2_{\mathfrak{m}} \to \widehat{H}^1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}}.$$

There is a map in the other direction

$$i^\dagger : \widehat{H}^1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}} \to \widehat{H}^1(K(N),\mathcal{O})^2_{\mathfrak{m}}$$

given by the inverse system of maps

$$i^\dagger_s : H^1_c(K_pK(N,l),\mathcal{O}/\varpi^s) \to H^1_c(K_pK(N),\mathcal{O}/\varpi^s)^2$$

where $i^\dagger_s$ is the adjoint under Poicaré duality of the usual level raising map (as described in section 3 of [15] for the Shimura curve case). We can form the $\mathcal{O}$-duals of these maps to get maps $i^*, i^{\dagger *}$ between homology spaces.

A standard calculation shows that the composition $i^\dagger \cdot i$ acts by the matrix

$$\begin{pmatrix} l+1 & T_l \\ S_l^{-1}T_l & l+1 \end{pmatrix}$$

on $\widehat{H}^1(K(N),\mathcal{O})^2_{\mathfrak{m}}$.

Definition 3.3.3. *We define the l-new space to be*

$$\widehat{H}^1_{\mathrm{new}}(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}} := \ker(i^\dagger).$$

By duality, we can identify the dual $\mathrm{Hom}_{\mathrm{cts}}(\widehat{H}^1_{\mathrm{new}}(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}},\mathcal{O})$ with

$$\widehat{H}_1^{\mathrm{new}}(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}} := \widehat{H}_1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}}/i^{\dagger *}(\widehat{H}_1(K(N),\mathcal{O})^2_{\mathfrak{m}}).$$

There is a form of Ihara's lemma in this situation, which follows easily from the classical Ihara's lemma:

Lemma 3.3.4. *The map i is an injection. Dually, the map i^* is a surjection.*

Proof. It is enough to show that the induced map

$$i : \widehat{H}^1(K(N), \mathcal{O})^2_{\mathfrak{m}}/\varpi^s \to \widehat{H}^1(K(N,l), \mathcal{O})_{\overline{\mathfrak{m}}}/\varpi^s$$

is an injection for all s. Let $\mathbb{T}$ be the abstract Hecke algebra over $\mathcal{O}$ generated by T_q, S_q for primes $q \nmid plN$. It is clear that $\mathfrak{m}$ and $\overline{\mathfrak{m}}$ pull back to the same non-Eisenstein maximal ideal $\mathfrak{M}$ of $\mathbb{T}$. Since $\widehat{H}^1(K(N), \mathcal{O})_{\mathfrak{m}}/\varpi^s \cong H^1(K(N), \mathcal{O})_{\mathfrak{M}}/\varpi^s$ and $\widehat{H}^1(K(N,l), \mathcal{O})_{\overline{\mathfrak{m}}}/\varpi^s \cong H^1(K(N,l), \mathcal{O})_{\mathfrak{M}}/\varpi^s$ we just need to show that the map

$$i : H^1(K(N), \mathcal{O})^2_{\mathfrak{M}} \to H^1(K(N,l), \mathcal{O})_{\mathfrak{M}}$$

is injective with torsion free cokernel, which is the usual Ihara's lemma as in [37]. □

Poincaré duality at finite levels also gives rise to a duality between completed homology spaces:

Proposition 3.3.5. *There are $\mathcal{O}[[K_p]]$-module isomorphisms*

$$\begin{aligned} \delta_1 : \mathrm{Hom}(\widehat{H}_1(K(N), \mathcal{O})_{\mathfrak{m}}, \mathcal{O}[[K_p]]) &\cong \widehat{H}_1(K(N), \mathcal{O})_{\mathfrak{m}} \\ \delta_2 : \mathrm{Hom}(\widehat{H}_1(K(N,l), \mathcal{O})_{\overline{\mathfrak{m}}}, \mathcal{O}[[K_p]]) &\cong \widehat{H}_1(K(N,l), \mathcal{O})_{\overline{\mathfrak{m}}}, \end{aligned}$$

where K_p is any open subgroup of G_0.

Proof. This follows from localising the Poincaré duality spectral sequence that can be found in [5]. □

Lemma 3.3.6. *We have a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}(\widehat{H}_1(K(N), \mathcal{O})^2_{\mathfrak{m}}, \mathcal{O}[[K_p]]) & \xrightarrow{{}^t(i^*)} & \mathrm{Hom}(\widehat{H}_1(K(N,l), \mathcal{O})_{\overline{\mathfrak{m}}}, \mathcal{O}[[K_p]]) \\ \sim\downarrow{\scriptstyle \delta_1 \oplus \delta_1} & & \sim\downarrow{\scriptstyle \delta_2} \\ \widehat{H}_1(K(N), \mathcal{O})^2_{\mathfrak{m}} & \xrightarrow{i^{\dagger *}} & \widehat{H}_1(K(N,l), \mathcal{O})_{\overline{\mathfrak{m}}} \end{array},$$

where the map ${}^t(i^*)$ is the $\mathcal{O}[[K_p]]$-dual of i^*.

Proof. It suffices to check that for each K'_p an open normal subgroup of K_p, the map induced by ${}^t(i^*)$ on the finite level quotient $H_1(K(N)K'_p, \mathcal{O})^2_{\mathfrak{m}}$ is equal to the map induced by $i^{\dagger *}$. This follows from the adjointness of i and $i^\dagger$ under classical Poincaré duality, and

the fact that the duality of Proposition 3.3.5 is induced by the maps

$$\mathrm{Hom}_{\mathcal{O}[K_p/K_p']}(H_1(K(N)K_p',\mathcal{O}),\mathcal{O}[K_p/K_p']) \cong \mathrm{Hom}_{\mathcal{O}}(H_1(K(N)K_p',\mathcal{O}),\mathcal{O}) \to H_1^{\mathrm{BM}}(K(N)K_p',\mathcal{O}),$$

where the first map is a canonical isomorphism of $\mathcal{O}[K_p/K_p']$-modules (as in Lemma 6.1 of [16]) and the second map is given by classical Poincaré duality. □

Say that a compact open subgroup K_f of $\mathrm{GL}_2(\mathbb{A}_f)$ is *neat* if $\mathrm{GL}_2(\mathbb{Q})$ acts on $(\mathbb{C}\backslash\mathbb{R}) \times \mathrm{GL}_2(\mathbb{A}_f)/K_f$ without fixed points. The following proposition is due to Emerton [20, Corollary 5.3.19], see also Proposition 4.5.6 and Corollary 4.5.8 below for the proof of a similar result:

Proposition 3.3.7. *For a compact open $K_p \subset G$ which is pro-p, such that $K_pK(N)$ and $K_pK(N,l)$ are neat (equivalently $K_pK(N)$ is neat), $\widehat{H}_1(K(N),\mathcal{O})_{\mathfrak{m}}$ and $\widehat{H}_1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}}$ are free $\mathcal{O}[[K_p]]$-modules.*

Theorem 3.3.8. *If K_p satisfies the conditions of the above proposition, then $\widehat{H}_1^{\mathrm{new}}(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}}$ is also a free $\mathcal{O}[[K_p]]$-module.*

Proof. By lemma 3.3.4, there is a short exact sequence

$$0 \longrightarrow \ker(i^*) \longrightarrow \widehat{H}_1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}} \xrightarrow{i^*} \widehat{H}_1(K(N),\mathcal{O})^2_{\mathfrak{m}} \longrightarrow 0\,.$$

Since $\widehat{H}_1(K(N),\mathcal{O})^2_{\mathfrak{m}}$ is free, there is a section to i^*, and so $\ker(i^*)$ is a direct summand of the free module $\widehat{H}_1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}}$. Hence $\ker(i^*)$ is projective. But we assumed K_p pro-p, so $\mathcal{O}[[K_p]]$ is local implying (by the noncommutative version of Nakayama's lemma) that $\ker(i^*)$ is free. Now applying the functor $\mathrm{Hom}(-,\mathcal{O}[[K_p]])$ to the above short exact sequence of free modules, we get a diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Hom}(\widehat{H}_1(K(N),\mathcal{O})^2_{\mathfrak{m}},\mathcal{O}[[K_p]]) & \to & \mathrm{Hom}(\widehat{H}_1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}},\mathcal{O}[[K_p]]) & \to & \mathrm{Hom}(\ker(i^*),\mathcal{O}[[K_p]]) & \to & 0 \\ & & \sim\downarrow & & \sim\downarrow & & & & \\ & & \widehat{H}_1(K(N),\mathcal{O})^2_{\mathfrak{m}} & \xrightarrow{i^{\dagger *}} & \widehat{H}_1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}}, & & & & \end{array}$$

where the first row is short exact, the vertical maps are the isomorphisms provided by Proposition 3.3.5 and the horizontal map on the second row is identified as $i^{\dagger *}$ by Lemma 3.3.6. Hence this diagram gives an isomorphism of $\mathcal{O}[[K_p]]$-modules between the free module $\mathrm{Hom}(\ker(i^*),\mathcal{O}[[K_p]])$ and $\mathrm{coker}(i^{\dagger *}) = \widehat{H}_1^{\mathrm{new}}(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}}$. □

3.4 Eigencurves of newforms

We now recall Emerton's eigenvariety construction (see sections 2.3 and 4 of [19]), which we will apply to the space $\widehat{H}^1_{\mathrm{new}}(K(N,l),E)_{\overline{\mathfrak{m}}}$, with the slight variation that we will use Hecke operators at l as well. Applying the Jacquet functor (see [17]) to the locally analytic vectors in $\widehat{H}^1_{\mathrm{new}}(K(N,l),E)_{\overline{\mathfrak{m}}}$ we get an essentially admissible $T(\mathbb{Q}_p)$ representation, where T is a maximal torus of GL_2. This corresponds to a coherent sheaf $\mathscr{E}$ on the rigid analytic space $\widehat{T}$ which classifies continuous characters of $T(\mathbb{Q}_p)$. Let $\mathbb{T}_l$ be the abstract Hecke algebra over $\mathcal{O}$ generated by T_q, S_q for primes $q \nmid plN$ and the operator U_l. The algebra $\mathbb{T}_l$ acts on $\mathscr{E}$, generating a coherent sheaf of algebras $\mathscr{A}$ on $\widehat{T}$. We have the relative spectrum $\underline{\mathrm{Spec}}(\mathscr{A})$ of $\mathscr{A}$ over $\widehat{T}$, a rigid analytic space over E. We define $\widetilde{D}^{\mathrm{new}}$ to be the reduced rigid analytic space $\underline{\mathrm{Spec}}(\mathscr{A})^{\mathrm{red}}$

We have the following corollary of Theorem 3.3.8:

Corollary 3.4.1. *The space $\widetilde{D}^{\mathrm{new}}$ is equidimensional of dimension 2.*

Proof. Fix a pro-p compact open subgroup of G, which is small enough so that $K_pK(N)$ is neat. Now Theorem 3.3.8 implies that $\widehat{H}^{\mathrm{new}}_1(K(N,l),\mathcal{O})_{\overline{\mathfrak{m}}}$ is a free $\mathcal{O}[[K_p]]$-module, i.e. isomorphic to $\mathcal{O}[[K_p]]^r$ as an $\mathcal{O}[[K_p]]$-module for some integer r. For ease of notation, let V denote the G-representation $\widehat{H}^1_{\mathrm{new}}(K(N,l),E)_{\overline{\mathfrak{m}}}$. By duality (and then inverting p), V is isomorphic to $\mathscr{C}(K_p,E)^r$ as a K_p-representation, where $\mathscr{C}(K_p,E)$ is the space of continuous functions from K_p to E, viewed as a representation of K_p by letting K_p act by right translation on functions. This shows that the space of locally analytic vectors V_{la} is isomorphic to $\mathscr{C}^{la}(K_p,E)^r$ as a K_p representation, where $\mathscr{C}^{la}(K_p,E)$ is the space of locally analytic functions from K_p to E. We let T_0 denote the compact subgroup $T \cap K_p$ of T, and write M for the strong dual of the Jacquet module $J_B(V_{la})'_b$, noting that M is the space of global sections of the sheaf $\mathscr{E}$ over $\widehat{T}$. The rigid analytic variety $\widehat{T}_0$ parameterises the characters of T_0, and there is a natural map $\widehat{T} \to \widehat{T}_0$ induced by restriction of characters.

The proof of Proposition 4.2.36 in [17] shows that, writing $\widehat{T}_0$ as a union of admissible affinoid subdomains $\mathrm{MaxSpec}(A_n)$, there is an isomorphism of $E\{\{z,z^{-1}\}\}\hat{\otimes}_E A_n$-modules

$$M_n := M\hat{\otimes}_{\mathscr{C}^{an}(\widehat{T}_0,E)}A_n \cong E\{\{z,z^{-1}\}\}\hat{\otimes}_{E[z]}(W_n\hat{\otimes}_E A_n),$$

where W_n is an E-Banach space and z is a certain element of T acting as a compact operator on $W_n\hat{\otimes}_E A_n$. Let Y be the subgroup of T generated by z, with corresponding rigid analytic character variety $\widehat{Y}$. We can now describe how to apply the machinery of [3, 9] to get our

desired equidimensionality result. Let $\mathscr{E}_n$ be the pullback of $\mathscr{E}$ to $\widehat{T} \times_{\widehat{T}_0} A_n$, let $\mathscr{A}_n$ be the pullback of $\mathscr{A}$ to $\widehat{T} \times_{\widehat{T}_0} A_n$, and let $Z_n \hookrightarrow \widehat{Y} \times_E A_n$ be the Fredholm variety cut out by the characteristic power series of z acting on $W_n \hat{\otimes}_E A_n$. Since $\mathscr{E}_n$ is the sheaf associated to M_n, there is a finite map $\underline{\mathrm{Spec}}(\mathscr{A}_n) \to Z_n$. In the language of [3], Z_n is the 'spectral variety', and $\mathrm{MaxSpec}(A_n)$ is 'weight space'.

An admissible cover $\mathscr{C}$ of Z_n is constructed in section 4 of [3], so that for $X \in \mathscr{C}$, with image V in $\mathrm{MaxSpec}(A_n)$ under projection, the pullback of $\underline{\mathrm{Spec}}(\mathscr{A}_n)$ to Y is given by the spectrum of a commutative algebra of endomorphisms (coming from the Hecke operators and the action of T) of a locally free, finite type $\mathcal{O}(V)$-module. Now Lemme 6.2.10 of [9] applies, in the same way as the proof of Proposition 6.4.2 in [9], to deduce that $\underline{\mathrm{Spec}}(\mathscr{A}_n)$ is equidimensional of dimension 2, for all n, hence $\underline{\mathrm{Spec}}(\mathscr{A})$ is equidimensional of dimension 2. □

It follows from Proposition 4.4.6 in [19] that $\widetilde{D}^{\mathrm{new}}$ is a product of an equidimension 1 space D^{new} and weight space, by left exactness of the Jacquet functor and the fact that $\widehat{H}^1_{\mathrm{new}}(K(N,l),E)_{\overline{\mathfrak{m}}}$ is stable under the action of $\mathscr{C}(T_0,E)$ induced by the map labelled 4.2.5 in [19]. This fact holds since this twisting action is induced from a $\mathrm{GL}_2(\mathbb{A}_f)$-equivariant action on the direct limit of completed cohomology spaces over all tame levels, so it will preserve the kernel of $i^\dagger$ (which was defined by means of degeneracy maps). Note that we can also apply the same construction to $\widehat{H}^1(K(N,l),E)_{\overline{\mathfrak{m}}}$ to get an equidimension 1 space D.

We now return to the notation of the introduction. For each non-Eisenstein maximal ideal $\overline{\mathfrak{m}}$ of $\mathbb{T}(K(N,l))$ let $D(N,l)_{\overline{\mathfrak{m}}}$ be the equidimensional closed rigid subspace of $D(N,l)_c$ whose points have residual Galois representation corresponding to the Hecke character induced by $\overline{\mathfrak{m}}$. Set $D(N,l)^{\mathrm{new}}_{\overline{\mathfrak{m}}} = D(N,l)^{\mathrm{new}}_c \cap D(N,l)_{\overline{\mathfrak{m}}}$. There is a natural isomorphism between $D(N,l)_{\overline{\mathfrak{m}}}$ and D, since both spaces are reduced, equidimensional and contain a Zariski dense set of points corresponding to the systems of Hecke eigenvalues arising from classical modular forms of tame level $\Gamma_1(N) \cap \Gamma_0(l)$ (see the proof of Theorem 7.5.8 in [18]). Similarly, since D^{new} is equidimensional, it is isomorphic to the Zariski closure of the classical l-new points in $D(N,l)_{\overline{\mathfrak{m}}}$, so there is a closed embedding $D^{\mathrm{new}} \hookrightarrow D(N,l)^{\mathrm{new}}_{\overline{\mathfrak{m}}}$. We can now prove the theorem stated in the introduction.

Theorem 3.4.2. *$D(N,l)^{\mathrm{new}}_c$ is equidimensional of dimension 1. It is equal to the Zariski closure of the classical l-new points in $D(N,l)_c$.*

Proof. It is enough to show that the closed embedding $D^{\mathrm{new}} \hookrightarrow D(N,l)^{\mathrm{new}}_{\overline{\mathfrak{m}}}$ is an isomorphism (for each non-Eisenstein $\overline{\mathfrak{m}}$). As in Proposition 4.7 of [10] the complement of the image of this embedding is a union of irreducible components of dimension 0. Suppose the complement $\mathscr{Z}$ is non-empty, containing a point x_1. This point must lie over a point x of $D(N)_{\mathfrak{m}}$, since in $D(N,l)_{\overline{\mathfrak{m}}}$ it lies in an irreducible component of dimension 1 which only contains l-old classical points, since $x_1 \in \mathscr{Z}$. We can now compare the completed cohomology side and the overconvergent modular form side, since we can characterise 'old and new' points like x_1 using Hecke operators - they come from points x satisfying $T_l(x)^2-(l+1)^2S_l(x)=0$. On the cohomology side this corresponds to an eigenclass in the kernel of $i^\dagger \cdot i$ (recall that this composition can be expressed in terms of Hecke operators), so in particular x_1 corresponds to an eigenclass in the kernel of $i^\dagger$ - therefore it lies in D^{new}, contradicting the assumption that x_1 lies in the complement. □

3.5 Addendum: a remark on the level at p

This chapter has concentrated on the local behaviour at a prime $l \neq p$ of families of (finite slope, overconvergent) p-adic modular forms. Of course, one is also interested in the behaviour at the prime p. A serious study of this question really requires the consideration of p-adic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to p-adic modular forms, and the p-adic local Langlands correspondence (see [20]). In this section we will limit ourselves to making a few remarks about the 'level' at p of finite slope overconvergent p-adic modular forms.

Lemma 3.5.1. *Let $f \in S_k(\Gamma_1(Np^r),\epsilon,\mathbb{C})$ be a newform, for some nebentypus ϵ, $r \geq 1$, $p \nmid N$ and suppose that f is a Hecke eigenform (for all the Hecke operators, including U_q for primes q dividing Np). Suppose that $U_p f = \alpha f$, with $\alpha \neq 0$. Let π denote the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by f. Then there are two distinct possibilities for the local factor of π at p:*

1. *The local factor π_p is a (ramified) principal series representation $PS(\mu_1,\mu_2)$ (the normalised induction of $\mu_1 \times \mu_2$) with μ_1 unramified and μ_2 conductor p^r. In particular, the p-power part, ϵ_p, of ϵ has conductor p^r.*

2. *The local factor π_p is a special representation of conductor p. In particular, ϵ has conductor prime to p and $r=1$. The eigenvalue α satisfies $\alpha^2 = \epsilon_N(p)p^{k-2}$.*

Proof. This follows from analysing the possibilities for the conductor $\mathfrak{c}$ of ϵ_p. If $\mathfrak{c}$ equals p^r then we are in the situation of case 1 (see the example in [8, pg. 119]). If $\mathfrak{c}$ is smaller than p^r, then we are in the situation of case 2 by the third part of [28, Theorem 3]. □

For an integer N, coprime to p, let $D(N)_0$ denote the (reduced) cuspidal tame level $\Gamma_1(N)$ eigencurve. We define two subsets of the classical points of $D(N)_0$. Firstly, denote by $\mathscr{Z}^{p-\mathrm{PS}}$ the points whose associated automorphic representation have local factor at p a principal series representation (unramified or ramified). Secondly, denote by $\mathscr{Z}^{p-\mathrm{Sp}}$ the points whose associated automorphic representation have local factor at p a special representation (which is then necessarily of conductor p, by lemma 3.5.1). Note that the union $\mathscr{Z}^{p-\mathrm{PS}} \cup \mathscr{Z}^{p-\mathrm{Sp}}$ is the set of classical points of $D(N)_0$, by lemma 3.5.1.

Proposition 3.5.2. *The set of points $\mathscr{Z}^{p-\mathrm{PS}}$ is Zariski dense in $D(N)_0$.*

Proof. We claim that any affinoid open neighbourhood U of a point $z \in \mathscr{Z}^{p-\mathrm{Sp}}$ contains a point in $\mathscr{Z}^{p-\mathrm{PS}}$. This claim establishes the proposition, since the classical points are Zariski dense in $D(N)_0$. To prove the claim, first shrink U so that the slope and nebentypus (with conductor prime to p) of points in U are constant. For $x \in U$ let $\alpha(x)$ denote the U_p eigenvalue of the overconvergent modular form corresponding to the point x. By Lemma 3.5.1 we have $v_p(\alpha(z)) = (k-2)/2$, where k is the weight of z. The neighbourhood U contains points z' with arbitrarily large classical weights k', but $v_p(\alpha(z')) = (k-2)/2$ by our assumption on constancy of slope. Pick z' so that $v_p(\alpha(z')) < k' - 1$ (i.e. $k' > k/2$) and $k' \neq k$. Then Coleman's classicality criterion implies that z' is a classical point, whilst $v_p(\alpha(z')) \neq (k'-2)/2$ so by Lemma 3.5.1 z' must lie in $\mathscr{Z}^{p-\mathrm{PS}}$. □

Note that the proof of the above proposition in fact shows that given a sufficiently small neighbourhood U of a point $z \in \mathscr{Z}^{p-\mathrm{Sp}}$, all the classical points in U lie in $\mathscr{Z}^{p-\mathrm{PS}}$. In other words, $\mathscr{Z}^{p-\mathrm{Sp}}$ is closed in the p-adic topology. One might conjecture that it is also Zariski closed. The following corollary of some results due to Chenevier [11] gives some evidence towards this conjecture.

Proposition 3.5.3. *Suppose $N = 1$ and $p = 2, 3, 5$ or 7. Then the set of p-special points $\mathscr{Z}^{p-\mathrm{Sp}}$ is Zariski closed.*

Proof. Let π be the map $D(N)_0 \to \mathscr{W} \times \mathbb{G}_m$ sending a point x to the pair $(\kappa(x), U_p(x)^{-2})$, where $\kappa(x)$ is the weight of x and $U_p(x)$ is the U_p-eigenvalue of x. The image of π is a reduced curve $\mathscr{X}$ (note that this is not quite the spectral curve for U_p since we took

$U_p(x)^{-2}$ as the second factor of the map). Let $\mathscr{K}$ be the set of integers k such that $\dim_{\mathbb{C}}(S_k(\Gamma_0(p),\mathbb{C})^{p-\mathrm{new}}) \geq 2$. Let $\mathscr{Z}_{\mathscr{K}}^{p-\mathrm{Sp}} \subset \mathscr{Z}^{p-\mathrm{Sp}}$ be the subset consisting of points with weight in $\mathscr{K}$. Its complement is a finite set, since the dimension of $S_k(\Gamma_0(p),\mathbb{C})^{p-\mathrm{new}}$ grows with (even) k. It follows from the main theorem of [11] that $\pi(\mathscr{Z}_{\mathscr{K}}^{p-\mathrm{Sp}})$ is precisely the set of non-smooth points in $\mathscr{X}$, whence we deduce that $\mathscr{Z}_{\mathscr{K}}^{p-\mathrm{Sp}}$ is Zariski closed, and therefore adding a finite number of additional points we see that $\mathscr{Z}^{p-\mathrm{Sp}}$ is also Zariski closed. □

Chapter 4

Completed cohomology of Shimura curves and a p-adic Jacquet-Langlands correspondence

4.1 Introduction

In an important paper [38], Ribet proves the 'ϵ conjecture' of Serre, on optimising the level of mod p Galois representations. A key result of Ribet's paper establishes a relationship between the arithmetic of Shimura curves and modular curves (Theorem 4.1 in [38]), which provides a geometric realisation of an integral Jacquet-Langlands correspondence. More recently, Rajaei [36] extended Ribet's techniques to the case of Shimura curves over totally real fields.

In this chapter we use Rajaei's results together with Emerton's completed cohomology [19] to construct an integral p-adic Jacquet-Langlands map between completed cohomology spaces for Shimura curves of different discriminant. Applying Emerton's eigenvariety construction to these spaces gives a Jacquet-Langlands map between eigenvarieties - we note that our approach to proving this kind of 'p-adic functoriality' is rather different to that followed in [10].

The techniques of this chapter also allow us to prove a p-adic analogue of Mazur's principle (as extended to the case of totally real fields by Jarvis, [25]), which should have applications to questions of local-global compatibility at $l \neq p$ in the p-adic setting - this generalises Theorem 6.2.4 of [20] to totally real fields. Having proved a level lowering result for completed cohomology, we can deduce a level lowering result for eigenvarieties. For $F = \mathbb{Q}$ these level lowering results have already been obtained by Paulin [34], using the results of [20] (in an indirect way, rather than by applying Emerton's construction of the eigencurve).

4.2 Preliminaries

4.2.1 Vanishing cycles

In this section we will follow the exposition of vanishing cycles given in the first chapter of [36] - we give some details to fix ideas and notation. Let $\mathcal{O}$ be a characteristic zero Henselian discrete valuation ring, residue field k of characteristic l, fraction field L. Let $\mathscr{X} \to S = \mathrm{Spec}(\mathcal{O})$ be a proper, generically smooth curve with reduced special fibre, and define Σ to be the set of singular points on $\mathscr{X} \otimes \overline{k}$. Moreover assume that a neighbourhood of each point $x \in \Sigma$ is étale locally isomorphic over S to $\mathrm{Spec}(\mathcal{O}[t_1, t_2]/(t_1 t_2 - a_x))$ with a_x a non-zero element of $\mathcal{O}$ with valuation $e_x = v(a_x) > 0$. Define morphisms i and j to

be the inclusions:

$$i : \mathscr{X} \otimes \overline{k} \to \mathscr{X}, j : \mathscr{X} \otimes \overline{L} \to \mathscr{X}.$$

For $\mathscr{F}$ a constructible torsion sheaf on $\mathscr{X}$, with torsion prime to l, the Grothendieck (or Leray) spectral sequence for the composition of the two functors $\Gamma(\mathscr{X}, -)$ and j_* gives an identification

$$R\Gamma(\mathscr{X} \otimes \overline{L}, j^*\mathscr{F}) = R\Gamma(\mathscr{X}, R(j)_* j^*\mathscr{F}).$$

Applying the proper base change theorem we also have

$$R\Gamma(\mathscr{X}, R(j)_* j^*\mathscr{F}) \simeq R\Gamma(\mathscr{X} \otimes \overline{k}, i^* R(j)_* j^*\mathscr{F}).$$

The natural adjunction $id \implies R(j)_* j^*$ gives a morphism $i^*\mathscr{F} \to i^* R(j)_* j^*\mathscr{F}$ of complexes on $\mathscr{X} \otimes \overline{k}$. We denote by $R\Phi(\mathscr{F})$ the mapping cone of this morphism (the complex of *vanishing cycles*), and denote by $R\Psi(\mathscr{F})$ the complex $i^* R(j)_* j^*\mathscr{F}$ (the complex of *nearby cycles*). There is a distinguished triangle

$$i^*\mathscr{F} \to R\Psi(\mathscr{F}) \to R\Phi(\mathscr{F}) \to,$$

whence a long exact sequence of cohomology

$$\cdots \to H^i(\mathscr{X} \otimes \overline{k}, i^*\mathscr{F}) \to H^i(\mathscr{X} \otimes \overline{k}, R\Psi(\mathscr{F})) \to H^i(\mathscr{X} \otimes \overline{k}, R\Phi(\mathscr{F})) \to \cdots$$

As on page 36 of [36] there is a *specialisation exact sequence*, with '(1)' denoting a Tate twist of the $\mathrm{Gal}(\overline{L}/L)$ action:

$$0 \longrightarrow H^1(\mathscr{X} \otimes \overline{k}, i^*\mathscr{F})(1) \longrightarrow H^1(\mathscr{X} \otimes \overline{L}, j^*\mathscr{F})(1) \xrightarrow{\beta} \bigoplus_{x \in \Sigma} R^1\Phi(\mathscr{F})_x(1)$$

$$\xrightarrow{\gamma} H^2(\mathscr{X} \otimes \overline{k}, i^*\mathscr{F})(1) \xrightarrow{sp(1)} H^2(\mathscr{X} \otimes \overline{L}, j^*\mathscr{F})(1) \longrightarrow 0. \tag{4.1}$$

We can extend the above results to lisse étale $\mathbb{Z}_p$-sheaves ($p \neq l$), by taking inverse limits of the exact sequences for the sheaves $\mathscr{F}/p^i\mathscr{F}$. Since our groups satisfy the Mittag-Leffler condition we preserve exactness. In particular we obtain the specialisation exact sequence (4.1) for $\mathscr{F}$. From now on $\mathscr{F}$ will denote a $\mathbb{Z}_p$-sheaf, and we define

$$X(\mathscr{F}) := \ker(\gamma) = \mathrm{im}(\beta) \subset \bigoplus_{x\in\Sigma} R^1\Phi(\mathscr{F})_x(1),$$

so there is a short exact sequence

$$0 \longrightarrow H^1(\mathscr{X}\otimes\overline{k}, i^*\mathscr{F})(1) \longrightarrow H^1(\mathscr{X}\otimes\overline{L}, j^*\mathscr{F})(1) \longrightarrow X(\mathscr{F}) \longrightarrow 0.$$

As in section 1.3 of [36] there is also a cospecialisation sequence

$$0 \longrightarrow H^0(\widetilde{\mathscr{X}\otimes\overline{k}}, R\Psi(\mathscr{F})) \longrightarrow H^0(\widetilde{\mathscr{X}\otimes\overline{k}}, \mathscr{F}) \xrightarrow{\gamma'} \textstyle\bigoplus_{x\in\Sigma} H^1_x(\mathscr{X}\otimes\overline{k}, R\Psi(\mathscr{F}))$$

$$\xrightarrow{\beta'} H^1(\mathscr{X}\otimes\overline{k}, R\Psi(\mathscr{F})) \longrightarrow H^1(\mathscr{X}\otimes\overline{k}, \mathscr{F}) \longrightarrow 0, \tag{4.2}$$

where we write $\widetilde{\mathscr{X}\otimes\overline{k}}$ for the normalisation of $\mathscr{X}\otimes\overline{k}$, and (in the above, and from now on) we also denote by $\mathscr{F}$ the appropriate pullbacks of $\mathscr{F}$ and $R\Psi(\mathscr{F})$ (e.g. $i^*\mathscr{F}$). The normalisation map is denoted $r : \widetilde{\mathscr{X}\otimes\overline{k}} \to \mathscr{X}\otimes\overline{k}$. We can now define

$$\check{X}(\mathscr{F}) := \mathrm{im}(\beta'),$$

and apply Corollary 1 of [36] to get the following:

Proposition 4.2.2. *We have the following diagram made up of two short exact sequences:*

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \check{X}(\mathscr{F}) & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & H^1(\mathscr{X}\otimes\overline{k}, \mathscr{F}) & \longrightarrow & H^1(\mathscr{X}\otimes\overline{L}, \mathscr{F}) & \longrightarrow & X(\mathscr{F})(-1) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & H^1(\mathscr{X}\otimes\overline{k}, r_*r^*\mathscr{F}) & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Note that the map $\check{X}(\mathscr{F}) \to H^1(\mathscr{X} \otimes \overline{k}, \mathscr{F})$ is the one induced by the fact that

$$\check{X}(\mathscr{F}) \subset H^1(\mathscr{X} \otimes \overline{k}, R\Psi(\mathscr{F}))$$

is in the image of the injective map

$$H^1(\mathscr{X} \otimes \overline{k}, \mathscr{F}) \to H^1(\mathscr{X} \otimes \overline{k}, R\Psi(\mathscr{F})).$$

Finally, as in (1.13) of [36] there is an injective map

$$\lambda : X(\mathscr{F}) \to \check{X}(\mathscr{F})$$

coming from the monodromy pairing.

4.3 Shimura curves and their bad reduction

4.3.1 Notation

Let $F/\mathbb{Q}$ be a totally real number field of degree d. We denote the infinite places of F by $\tau_1, ..., \tau_d$. We will be comparing the arithmetic of two quaternion algebras over F. The first quaternion algebra will be denoted by D, and we assume that it is split at τ_1 and at all places in F dividing p, and non-split at τ_i for $i \neq 1$ together with a finite set S of finite places. If $d = 1$ then we furthermore assume that D is split at some finite place (so we avoid working with non-compact modular curves). Fix two finite places $\mathfrak{q}_1$ and $\mathfrak{q}_2$ of F, which do not divide p and where D is split, and denote by D' a quaternion algebra over F which is split at τ_1 and non-split at τ_i for $i \neq 1$ together with each finite place in $S \cup \{\mathfrak{q}_1, \mathfrak{q}_2\}$. We fix maximal orders $\mathcal{O}_D$ and $\mathcal{O}_{D'}$ of D and D' respectively, and an isomorphism $D \otimes_F \mathbb{A}_F^{(\mathfrak{q}_1\mathfrak{q}_2)} \cong D' \otimes_F \mathbb{A}_F^{(\mathfrak{q}_1\mathfrak{q}_2)}$ compatible with the choice of maximal orders in D and D'.

We write $\mathfrak{q}$ to denote one of the places $\mathfrak{q}_1, \mathfrak{q}_2$, and denote the completion $F_\mathfrak{q}$ of F by L, with ring of integers $\mathcal{O}_\mathfrak{q}$ and residue field $k_\mathfrak{q}$. Denote $F \cap \mathcal{O}_\mathfrak{q}$ by $\mathcal{O}_{(\mathfrak{q})}$. We have the absolute Galois group $G_\mathfrak{q} = \mathrm{Gal}(\overline{L}/L)$ with its inertia subgroup $I_\mathfrak{q}$.

Let G and G' denote the reductive groups over $\mathbb{Q}$ arising from the unit groups of D and D' respectively. Note that G and G' are both forms of $\mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_2/F)$. For U a compact open subgroup of $G(\mathbb{A}_f)$ and V a compact open subgroup of $G'(\mathbb{A}_f)$ we have complex

(disconnected) Shimura curves

$$M(U)(\mathbb{C}) = G(\mathbb{Q})\backslash G(\mathbb{A}_f) \times (\mathbb{C} - \mathbb{R})/U$$

$$M'(U)(\mathbb{C}) = G'(\mathbb{Q})\backslash G'(\mathbb{A}_f) \times (\mathbb{C} - \mathbb{R})/V,$$

where $G(\mathbb{Q})$ and $G'(\mathbb{Q})$ act on $\mathbb{C} - \mathbb{R}$ via the τ_1 factor of $G(\mathbb{R})$ and $G'(\mathbb{R})$ respectively.

Both these curves have canonical models over F, which we denote by $M(U)$ and $M'(U)$.

4.3.2 Integral models and coefficient sheaves

Let $U \subset G(\mathbb{A}_f)$ be a compact open subgroup (we assume U is a product of local factors over the places of F), with $U_\mathfrak{q} = U_0(\mathfrak{q})$ (the matrices in $\mathrm{GL}_2(\mathcal{O}_\mathfrak{q})$ which have upper triangular image in $\mathrm{GL}_2(k_\mathfrak{q})$), and U unramified at all the finite places v where D is non-split (i.e. $U_v = \mathcal{O}_{D,v}^\times$). Let $\Sigma(U)$ denote the set of finite places where U is ramified. In Theorem 8.9 of [25], Jarvis constructs an integral model for $M(U)$ over $\mathcal{O}_\mathfrak{q}$ which we will denote by $\mathbb{M}_\mathfrak{q}(U)$. In fact this integral model only exists when U is sufficiently small (a criterion for this is given in [25, Lemma 12.1]), but this will not cause us any difficulties (see the remark at the end of this subsection). Section 10 of [25] shows that $\mathbb{M}_\mathfrak{q}(U)$ satisfies the conditions imposed on $\mathscr{X}$ in section 4.2.1 above.

We also have integral models for places dividing the discriminant of our quaternion algebra. Let $V \subset G'(\mathbb{A}_f)$ be a compact open subgroup with $V_\mathfrak{q} = \mathcal{O}_{D',\mathfrak{q}}^\times$. We let $\mathbb{M}'_\mathfrak{q}(V)$ denote the integral model for $M'(V)$ over $\mathcal{O}_\mathfrak{q}$ coming from Theorem 5.3 of [46] (this is denoted $\mathfrak{C}$ in [36]). This arises from the $\mathfrak{q}$-adic uniformisation of $M'(V)$ by a formal $\mathcal{O}_\mathfrak{q}$-scheme, and gives integral models for varying V which are compatible under the natural projections, and again satisfy the conditions necessary to apply section 4.2.1.

We now want to construct sheaves on $\mathbb{M}_\mathfrak{q}(U)$ and $\mathbb{M}'_\mathfrak{q}(V)$ corresponding to the possible (cohomological) weights of Hilbert modular forms. Let $k = (k_1, \ldots k_d)$ be a d-tuple of integers (indexed by the embeddings τ_i of F into $\mathbb{R}$ if the reader prefers), all ≥ 2 and of the same parity. The d-tuple of integers v is then characterised by having non-negative entries, at least one of which is zero, with $k + 2v = (w, ..., w)$ a parallel vector. We will now define p-adic lisse étale sheaves $\mathscr{F}_k$ on $\mathbb{M}_\mathfrak{q}(U)$, as in [7, 31, 39]. Denote by Z the centre of the algebraic group G. Note that $Z \cong \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_{m,F})$. We let Z_s denote the maximal subtorus of Z that is split over $\mathbb{R}$ but which has no subtorus split over $\mathbb{Q}$. It is straightforward to see

that Z_s is the subtorus given by the kernel of the norm map

$$N_{F/\mathbb{Q}} : \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_{m,F}) \to \mathbb{G}_{m,\mathbb{Q}}.$$

In the notation of [31, Ch. III] the algebraic group G^c is the quotient of G by Z_s. Take $F' \subset \mathbb{C}$ a number field of finite degree, Galois over $\mathbb{Q}$ and containing F such that F' splits both D and D'. Since $F'/\mathbb{Q}$ is normal, F' contains all the Galois conjugates of F and we can identify the embeddings $\{\tau_i : F \to \mathbb{C}\}$ with the embedding $\{\tau_i : F \to F'\}$, via the inclusion $F' \subset \mathbb{C}$. Since $D \otimes_{F,\tau_i} F' \cong M_2(F')$ for each i, we obtain representations ξ_i of $D^\times = G(\mathbb{Q})$ on the space $W_i = F'^2$. Let $\nu : G(\mathbb{Q}) \to F^\times$ denote the reduced norm. Then

$$\xi^{(k)} = \otimes_{i=1}^d (\tau_i \circ \nu)^{v_i} \otimes \mathrm{Sym}^{k_i-2}(\xi_i)$$

is a representation of $G(\mathbb{Q})$ acting on $W_k = \otimes_{i=1}^d \mathrm{Sym}^{k_i-2}(W_i)$. In fact $\xi^{(k)}$ arises from an algebraic representation of G on the $\mathbb{Q}$-vector space underlying W_k (or rather the associated $\mathbb{Q}$-vector space scheme). For $z \in Z(\mathbb{Q}) = F^\times$ the action of $\xi^{(k)}(z)$ is multiplication by $N_{F/\mathbb{Q}}(z)^{w-2}$, so $\xi^{(k)}$ factors through G^c. The representation $\xi^{(k)}$ gives rise to a representation of $G(\mathbb{Q}_p)$ on $W_k \otimes_{\mathbb{Q}} \mathbb{Q}_p = W_k \otimes_{F'} (F' \otimes_{\mathbb{Q}} \mathbb{Q}_p)$, so taking a prime $\mathfrak{p}$ in F' above p and projecting to the relevant factor of $F' \otimes_{\mathbb{Q}} \mathbb{Q}_p$ we obtain an $F'_{\mathfrak{p}}$-linear representation $W_{k,\mathfrak{p}}$ of $G(\mathbb{Q}_p)$.

Now we may define a $\mathbb{Q}_p$-sheaf on $M(U)(\mathbb{C})$:

$$\mathscr{F}_{k,\mathbb{Q}_p,\mathbb{C}} = G(\mathbb{Q})\backslash(G(\mathbb{A}_f) \times (\mathbb{C} - \mathbb{R}) \times W_{k,\mathfrak{p}}/U,$$

where $G(\mathbb{Q})$ acts on $W_{k,\mathfrak{p}}$ by the representation described above (via its embedding into $G(\mathbb{Q}_p)$), and U acts only on the $G(\mathbb{A}_f)$ factor, by right multiplication. Note that this definition only works because $\xi^{(k)}$ factors through G^c.

Choose an $\mathcal{O}_{F'_{\mathfrak{p}}}$ lattice $W^0_{k,\mathfrak{p}}$ in $W_{k,\mathfrak{p}}$. After projection to $G(\mathbb{Q}_p)$, U acts on $W_{k,\mathfrak{p}}$, and if U is sufficiently small it stabilises $W^0_{k,\mathfrak{p}}$. Pick a normal open subgroup $U_1 \subset U$ acting trivially on $W^0_{k,\mathfrak{p}}/p^n$. We now define a $\mathbb{Z}/p^n\mathbb{Z}$-sheaf on $M(U)(\mathbb{C})$ by

$$\mathscr{F}_{k,\mathbb{Z}/p^n\mathbb{Z},\mathbb{C}} = (M(U_1)(\mathbb{C}) \times (W^0_{k,\mathfrak{p}}/p^n))/(U/U_1).$$

Now on the integral model $\mathbb{M}_{\mathfrak{q}}(U)$ we can define an étale sheaf of $\mathbb{Z}/p^n\mathbb{Z}$-modules by

$$\mathscr{F}_{k,\mathbb{Z}/p^n\mathbb{Z}} = (\mathbb{M}_{\mathfrak{q}}(U_1) \times (W^0_{k,\mathfrak{p}}/p^n))/(U/U_1).$$

By section 2.1 of [7] this construction is independent of the choice of lattice $W^0_{k,\mathfrak{p}}$. Taking inverse limits of these $\mathbb{Z}/p^n\mathbb{Z}$-sheaves we get a $\mathbb{Z}_p$- (in fact $\mathcal{O}_{F'_{\mathfrak{p}}}$-) lisse étale sheaf on $\mathbb{M}_{\mathfrak{q}}(U)$ which will be denoted by $\mathscr{F}_k$.

The same construction applies to $\mathbb{M}'_{\mathfrak{q}}(V)$, and we also denote the resulting étale sheaves on $\mathbb{M}'_{\mathfrak{q}}(V)$ by $\mathscr{F}_k$. Note that we chose F' so that it split both D and D' - this allows $F'_{\mathfrak{p}}$ to serve as a field of coefficients for both $\mathbb{M}'_{\mathfrak{q}}(U)$ and $\mathbb{M}'_{\mathfrak{q}}(V)$.

From now on we work with a coefficient field E. This can be any finite extension of $F'_{\mathfrak{p}}$. Denote the ring of integers in E by $\mathcal{O}_{\mathfrak{p}}$, and fix a uniformiser ϖ. The sheaf $\mathscr{F}_k$ on the curves $\mathbb{M}_{\mathfrak{q}}(U)$ and $\mathbb{M}'_{\mathfrak{q}}(V)$ will now denote the sheaves constructed above with coefficients extended to $\mathcal{O}_{\mathfrak{p}}$.

Remark. *Note that in the above discussion we assumed that the level subgroup U was 'small enough'. In the applications below we will be taking direct limits of cohomology spaces as the level at p, U_p, varies over all compact open subgroups of $G(\mathbb{Q}_p)$, so we can always work with a sufficiently small level subgroup by passing far enough through the direct limit.*

4.3.3 Hecke algebras

Definition 4.3.4. *The Hecke algebra $\mathbb{T}_k(U)$ is defined to be the $\mathcal{O}_{\mathfrak{p}}$-algebra of endomorphisms of $H^1(M(U)_{\overline{F}}, \mathscr{F}_k)_E$ generated by Hecke operators T_v, S_v for primes $v \nmid p\mathfrak{q}_1\mathfrak{q}_2$ such that U and D are both unramified at v. The Hecke algebra $\mathbb{T}'_k(V)$ is defined to be the $\mathcal{O}_{\mathfrak{p}}$-algebra of endomorphisms of $H^1(M'(V)_{\overline{F}}, \mathscr{F}_k)_E$ generated by Hecke operators T_v, S_v for primes $v \nmid p$ such that V and D' are both unramified at v.*

The Hecke algebras $\mathbb{T}_k(U)$ and $\mathbb{T}'_k(V)$ are semilocal, and their maximal ideals correspond to mod p Galois representations arising from Hecke eigenforms occuring in $H^1(M(U)_{\overline{F}}, \mathscr{F}_k)_{\overline{E}}$ and $H^1(M'(V)_{\overline{F}}, \mathscr{F}_k)_{\overline{E}}$ respectively. Given a maximal ideal $\mathfrak{m}$ of $\mathbb{T}_k(U)$ we denote the Hecke algebra's localisation at $\mathfrak{m}$ by $\mathbb{T}_k(U)_{\overline{\rho}}$, where $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is the mod p Galois representation attached to $\mathfrak{m}$, and more generally for any $\mathbb{T}_k(U)$-module M, denote its localisation at $\mathfrak{m}$ by $M_{\overline{\rho}}$. We apply the same notational convention to $\mathbb{T}'_k(V)$ and modules for this Hecke algebra.

The Hecke algebra $\mathbb{T}_k(U)$ acts faithfully on the $\mathcal{O}_\mathfrak{p}$-torsion free quotient $H^1(M(U)_{\overline{F}}, \mathscr{F}_k)^{\mathrm{tf}}$ of $H^1(M(U)_{\overline{F}}, \mathscr{F}_k)$. Moreover, since the action of the Hecke operators on $H^1(M(U)_{\overline{F}}, \mathscr{F}_k)^{\mathrm{tors}}$ is Eisenstein (see [15, Lemma 4]), if $\overline{\rho}$ as above is irreducible then $H^1(M(U)_{\overline{F}}, \mathscr{F}_k)^{\mathrm{tf}}_{\overline{\rho}}$ can be naturally viewed as an $\mathcal{O}_\mathfrak{p}$-direct summand of $H^1(M(U)_{\overline{F}}, \mathscr{F}_k)$, and we denote it by $H^1(M(U)_{\overline{F}}, \mathscr{F}_k)_{\overline{\rho}}$. The same consideration applies to any subquotient M of $H^1(M(U)_{\overline{F}}, \mathscr{F}_k)$, and for the Hecke algebra $\mathbb{T}'_k(V)$. Note that if $k = (2, ..., 2)$ then $H^1(M(U)_{\overline{F}}, \mathscr{F}_k) = H^1(M(U)_{\overline{F}}, \mathcal{O}_\mathfrak{p})$ is already $\mathcal{O}_\mathfrak{p}$-torsion free.

4.3.5 Mazur's principle

In the next two subsections we summarise the results of [25] and [36]. First we fix a (sufficiently small) compact open subgroup $U \subset G(\mathbb{A}_f)$ such that $U_\mathfrak{q} = \mathrm{GL}_2(\mathcal{O}_\mathfrak{q})$. For brevity we denote the compact open subgroup $U \cap U_0(\mathfrak{q})$ by $U(\mathfrak{q})$.

We apply the theory of Section 4.2.1 to the curve $\mathbb{M}_\mathfrak{q}(U(\mathfrak{q}))$ and the $\mathbb{Z}_p$-sheaf given by $\mathscr{F}_k$, for some weight vector k, to obtain $\mathcal{O}_\mathfrak{p}$-modules $X_\mathfrak{q}(U(\mathfrak{q}), \mathscr{F}_k)$ and $\check{X}_\mathfrak{q}(U(\mathfrak{q}), \mathscr{F}_k)$. We have short exact sequences

$$0 \to H^1(\mathbb{M}_\mathfrak{q}(U(\mathfrak{q})) \otimes \overline{k_\mathfrak{q}}, \mathscr{F}_k) \to H^1(\mathbb{M}_\mathfrak{q}(U(\mathfrak{q})) \otimes \overline{L}, \mathscr{F}_k) \to X_\mathfrak{q}(U(\mathfrak{q}), \mathscr{F}_k)(-1) \to 0, \tag{4.3}$$

$$0 \to \check{X}_\mathfrak{q}(U(\mathfrak{q}), \mathscr{F}_k) \to H^1(\mathbb{M}_\mathfrak{q}(U(\mathfrak{q})) \otimes \overline{k_\mathfrak{q}}, \mathscr{F}_k) \to H^1(\mathbb{M}_\mathfrak{q}(U(\mathfrak{q})) \otimes \overline{k_\mathfrak{q}}, r_* r^* \mathscr{F}_k) \to 0. \tag{4.4}$$

The above short exact sequences are equivariant with respect to the $G_\mathfrak{q}$ action, as well as the action of $\mathbb{T}_k(U(\mathfrak{q}))$ and the Hecke operator $\mathsf{U}_\mathfrak{q}$. The sequence 4.4 gives rise to Mazur's principle (see [25] and Theorem 4.4.26 below).

4.3.6 Ribet-Rajaei exact sequence

We also apply the theory of Section 4.2.1 to the curve $\mathbb{M}'_\mathfrak{q}(V)$ and the sheaf $\mathscr{F}_k$, where $V \subset G'(\mathbb{A}_f)$ is a (sufficiently small) compact open subgroup which is unramified at $\mathfrak{q}$. We obtain $\mathcal{O}_\mathfrak{p}$-modules $Y_\mathfrak{q}(V, \mathscr{F}_k)$ and $\check{Y}_\mathfrak{q}(V, \mathscr{F}_k)$, where Y and $\check{Y}$ correspond to X and $\check{X}$ respectively. As in equation 3.2 of [36] these modules satisfy the ($G_\mathfrak{q}$ and $\mathbb{T}'_k(V)$ equivariant) short exact sequences

$$0 \longrightarrow H^1(\mathbb{M}'_\mathfrak{q}(V) \otimes \overline{k_\mathfrak{q}}, \mathscr{F}_k) \longrightarrow H^1(\mathbb{M}'_\mathfrak{q}(V) \otimes \overline{L}, \mathscr{F}_k) \longrightarrow Y_\mathfrak{q}(V, \mathscr{F}_k)(-1) \longrightarrow 0,$$

$$0 \longrightarrow \check{Y}_{\mathfrak{q}}(V, \mathscr{F}_k) \longrightarrow H^1(\mathbb{M}'_{\mathfrak{q}}(V) \otimes \overline{k_{\mathfrak{q}}}, \mathscr{F}_k) \longrightarrow H^1(\mathbb{M}'_{\mathfrak{q}}(V) \otimes \overline{k_{\mathfrak{q}}}, r_* r^* \mathscr{F}_k) \longrightarrow 0.$$

It is remarked in section 3 of [36] that in fact $H^1(\mathbb{M}'_{\mathfrak{q}}(V) \otimes \overline{k_{\mathfrak{q}}}, r_* r^* \mathscr{F}_k) = 0$, since the irreducible components of the special fibre of $\mathbb{M}'_{\mathfrak{q}}(V)$ are rational curves, so we have

$$\check{Y}_{\mathfrak{q}}(V, \mathscr{F}_k) \cong H^1(\mathbb{M}'_{\mathfrak{q}}(V) \otimes \overline{k_{\mathfrak{q}}}, \mathscr{F}_k).$$

We can now relate our constructions for the two quaternion algebras D and D'. This requires us to now distinguish between the two places $\mathfrak{q}_1$ and $\mathfrak{q}_2$. We make some further assumptions on the levels U and V. First we assume that $U_{\mathfrak{q}_1} = \mathcal{O}_{D,\mathfrak{q}_1} \cong \mathrm{GL}_2(\mathcal{O}_{\mathfrak{q}_1})$ and $U_{\mathfrak{q}_2} = \mathcal{O}_{D,\mathfrak{q}_2} \cong \mathrm{GL}_2(\mathcal{O}_{\mathfrak{q}_2})$, and that V is unramified at both $\mathfrak{q}_1$ and $\mathfrak{q}_2$. We furthermore assume that the factors of U away from $\mathfrak{q}_1$ and $\mathfrak{q}_2$, $U^{\mathfrak{q}_1\mathfrak{q}_2}$, match with $V^{\mathfrak{q}_1\mathfrak{q}_2}$ under the fixed isomorphism $D \otimes_F \mathbb{A}_{F,f}^{(\mathfrak{q}_1\mathfrak{q}_2)} \cong D' \otimes_F \mathbb{A}_{F,f}^{(\mathfrak{q}_1\mathfrak{q}_2)}$. We will use the notation $U(\mathfrak{q}_1), U(\mathfrak{q}_2), U(\mathfrak{q}_1\mathfrak{q}_2)$ to denote $U \cap U_0(\mathfrak{q}_1), U \cap U_0(\mathfrak{q}_2), U \cap U_0(\mathfrak{q}_1) \cap U_0(\mathfrak{q}_2)$ respectively.

We fix a Galois representation $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$, satisfying two assumptions. Firstly, we assume that $\overline{\rho}$ arises from a Hecke eigenform in $H^1(M'(V)_{\overline{F}}, \mathscr{F}_k)$. Secondly, we assume that $\overline{\rho}$ is irreducible. Equivalently, the corresponding maximal ideal $\mathfrak{m}'$ in $\mathbb{T}'_k(V)$ is not Eisenstein.

There is a surjection $\mathbb{T}_k(U(\mathfrak{q}_1\mathfrak{q}_2)) \to \mathbb{T}'_k(V)$ arising from the classical Jacquet–Langlands correspondence (this identifies $\mathbb{T}'_k(V)$ as the $\mathfrak{q}_1\mathfrak{q}_2$-new quotient of $\mathbb{T}_k(U(\mathfrak{q}_1\mathfrak{q}_2))$), so there is a (non-Eisenstein) maximal ideal of $\mathbb{T}_k(U(\mathfrak{q}_1\mathfrak{q}_2))$ corresponding to $\overline{\rho}$ (in fact this can be established directly from the results contained on pg. 55 of [36], but for expository purposes it is easier to rely on the existence of the Jacquet–Langlands correspondence). We can therefore regard all $\mathbb{T}'_k(V)_{\overline{\rho}}$-modules as $\mathbb{T}_k(U(\mathfrak{q}_1\mathfrak{q}_2))_{\overline{\rho}}$-modules. Now [36, Theorem 3] implies the following:

Theorem 4.3.7. *There are $\mathbb{T}_k(U(\mathfrak{q}_1\mathfrak{q}_2))_{\overline{\rho}}$-equivariant short exact sequences*

$$0 \longrightarrow Y_{\mathfrak{q}_1}(V, \mathscr{F}_k)_{\overline{\rho}} \longrightarrow X_{\mathfrak{q}_2}(U(\mathfrak{q}_1\mathfrak{q}_2), \mathscr{F}_k)_{\overline{\rho}} \xrightarrow{i^\dagger} X_{\mathfrak{q}_2}(U(\mathfrak{q}_2), \mathscr{F}_k)^2_{\overline{\rho}} \longrightarrow 0,$$

$$0 \longrightarrow \check{X}_{\mathfrak{q}_2}(U(\mathfrak{q}_2), \mathscr{F}_k)^2_{\overline{\rho}} \xrightarrow{i} \check{X}_{\mathfrak{q}_2}(U(\mathfrak{q}_1\mathfrak{q}_2), \mathscr{F}_k)_{\overline{\rho}} \longrightarrow \check{Y}_{\mathfrak{q}_1}(V, \mathscr{F}_k)_{\overline{\rho}} \longrightarrow 0.$$

In the theorem, the maps labelled i and $i^\dagger$ are the natural 'level raising' map and its adjoint, as in the previous chapters (they should be defined using double coset operators here, since there is no direct moduli theoretic description of our Shimura curves allowing a definition via degeneracy maps as in chapter 3). We think of this short exact sequence

as a geometric realisation of the Jacquet–Langlands correspondence between automorphic forms for G' and for G. The proof of the above theorem relies on relating the spaces involved to the arithmetic of a third quaternion algebra $\overline{D}$ (which associated reductive algebraic group $\overline{G}/\mathbb{Q}$) which is non-split at all the infinite places of F and at the places in $S \cup \{\mathfrak{q}_2\}$. However, we will just focus on the groups G' and G, and view $\overline{G}$ as playing an auxiliary role.

4.4 Completed cohomology of Shimura curves

4.4.1 Completed cohomology

We now consider compact open subgroups U^p of $G(\mathbb{A}_f^{(p)})$, with factor at v equal to $\mathcal{O}_{D_v}^\times$ at all places v of F where D is non-split. Then we have the following definitions, as in [19, 20]:

Definition 4.4.2. *We write $H^n_D(U^p, \mathscr{F}_k/\varpi^s)$ for the $\mathcal{O}_\mathfrak{p}$ module*

$$\varinjlim_{U_p} H^n(M(U_pU^p)_{\overline{F}}, \mathscr{F}_k/\varpi^s),$$

which has a smooth action of $G(\mathbb{Q}_p) = \prod_{v|p} \mathrm{GL}_2(F_v)$ and a continuous action of G_F.

We then define a ϖ-adically complete $\mathcal{O}_\mathfrak{p}$-module

$$\widetilde{H}^n(U^p, \mathscr{F}_k) := \varprojlim_s H^n(U^p, \mathscr{F}_k/\varpi^s),$$

which has a continuous action of $G(\mathbb{Q}_p)$ and G_F.

Lemma 4.4.3. *The $\mathcal{O}_\mathfrak{p}[G(\mathbb{Q}_p)]$-module $\widetilde{H}^n_D(U^p, \mathscr{F}_k)$ is a ϖ-adically admissible $G(\mathbb{Q}_p)$-representation over $\mathcal{O}_\mathfrak{p}$ (in the sense of [21, Definition 2.4.7]).*

Proof. This follows from [19, Theorem 2.2.11], taking care to remember the $\mathcal{O}_\mathfrak{p}$-integral structure. □

Write $\widetilde{H}^n(U^p, \mathscr{F}_k)_E$ for the admissible continuous Banach $E[G(\mathbb{Q}_p)]$-representation $\widetilde{H}^n(U^p, \mathscr{F}_k) \otimes_{\mathcal{O}_\mathfrak{p}} E$.

Lemma 4.4.4. *There is a canonical isomorphism*

$$\widetilde{H}^n(U^p, \mathcal{O}_\mathfrak{p}) = \varprojlim_s H^n(U^p, \mathcal{O}_\mathfrak{p})/\varpi^s H^n(U^p, \mathcal{O}_\mathfrak{p}),$$

compatible with G_F and $G(\mathbb{Q}_p)$ actions.

Proof. Since $M(U_pU^p)_{\overline{F}}$ is proper of dimension 1, this follows from Corollary 2.2.27 of [19] □

Remark. *Note that in Emerton's notation the above lemma says that $\widetilde{H}^n(U^p) = \widehat{H}^n(U^p)$.*

The following lemma allows us to restrict to considering completed cohomology with trivial coefficients.

Lemma 4.4.5. *There is a canonical $G(\mathbb{Q}_p)$, G_F equivariant isomorphism*

$$\widetilde{H}^n(U^p, \mathscr{F}_k) \cong \widetilde{H}^n(U^p, \mathcal{O}_\mathfrak{p}) \otimes_{\mathcal{O}_\mathfrak{p}} W^0_{k,\mathfrak{p}}.$$

Proof. This follows from Theorem 2.2.17 of [19]. □

Proposition 4.4.6. *Let $\overline{F^\times}$ be the closure of $F^\times$ in $\mathbb{A}_F^{f,\times}$. There is an isomorphism between $\widetilde{H}^0(U^p, \mathcal{O}_\mathfrak{p})$ and the space of continuous functions*

$$\mathscr{C}(\overline{F^\times}\backslash\mathbb{A}_F^{f,\times}/\det(U^p), \mathcal{O}_\mathfrak{p}),$$

with $\mathrm{GL}_2(\mathbb{Q}_p)$ action on the right hand side induced by g acting on $\mathbb{A}_F^{f,\times}$ by multiplication by $\det(g)$.

Proof. This follows from the fact that the product of the sign map on $(\mathbb{C} - \mathbb{R})$ and the reduced norm map on $G(\mathbb{A}_f)$ induces a map

$$G(\mathbb{Q})\backslash G(\mathbb{A}_f) \times (\mathbb{C} - \mathbb{R})/U \to F^\times\backslash\mathbb{A}_{F,f}^\times \times \{\pm\}/\det(U) = F^\times\backslash\mathbb{A}_{F,f}^\times/\det(U)$$

with connected fibres. □

Proposition 4.4.7. *The space $\widetilde{H}^2(U^p, \mathcal{O}_\mathfrak{p})$ is equal to 0.*

Proof. Exactly as for Proposition 4.3.6 of [19] □

If v is a prime in F such that U^p and D are both unramified at v then there are Hecke operators T_v, S_v defined as usual on the spaces $H^1(U^p, \mathcal{O}_\mathfrak{p}/\varpi^s)$ (i.e. by double coset operators, as in [36]). By continuity this extends to an action of these Hecke operators on the space $\widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})$, so we can make the following definition:

Definition 4.4.8. *The Hecke algebra $\mathbb{T}(U^p)$ is defined to be $\varprojlim_{U_p} \mathbb{T}_{(2,\dots,2)}(U^pU_p)$. We endow it with the projective limit topology given by taking the ϖ-adic topology on each finitely generated $\mathcal{O}_\mathfrak{p}$-algebra $\mathbb{T}_{(2,\dots,2)}(U^pU_p)$.*

The topological $\mathcal{O}_\mathfrak{p}$-algebra $\mathbb{T}(U^p)$ acts faithfully on $\widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})$, and is topologically generated by Hecke operators T_v, S_v for primes $v \nmid p$ such that U^p and D are both unramified at v. Note that the isomorphisms of lemmas 4.4.4 and 4.4.5 are equivariant with respect to the Hecke action, so we will often just work with the completed cohomology spaces defined with trivial coefficients.

Definition 4.4.9. *An ideal I of $\mathbb{T}(U^p)$ is* Eisenstein *if the map $\lambda : T(U^p) \to T(U^p)/I$ satisfies $\lambda(T_q) = \epsilon_1(q) + \epsilon_2(q)$ and $\lambda(qS_q) = \epsilon_1(q)\epsilon_2(q)$ for all $q \notin \{p\} \cup \Sigma$, for some characters $\epsilon_1, \epsilon_2 : \mathbb{Z}^\times_{p\Sigma} \to T(U^p)/I$.*

The Hecke algebra $\mathbb{T}(U^p)$ is semilocal and Noetherian, with maximal ideals corresponding to mod p Galois representations arising from Hecke eigenforms in (an extension of scalars of) $H^1_D(U^p, \mathcal{O}_\mathfrak{p}/\varpi)$. Given a maximal ideal $\mathfrak{m}$ of $\mathbb{T}(U^p)$ we denote the Hecke algebra's localisation at $\mathfrak{m}$ by $\mathbb{T}(U^p)_{\overline{\rho}}$, where $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is the mod p Galois representation attached to $\mathfrak{m}$, and more generally for any $\mathbb{T}(U^p)$-module M, denote its localisation at $\mathfrak{m}$ by $M_{\overline{\rho}}$.

We now fix such a $\overline{\rho}$, and assume that it is irreducible. There is a deformation $\rho^m_{\overline{\rho},U^p}$ of $\overline{\rho}$ to $\mathbb{T}(U^p)_{\overline{\rho}}$ (taking a projective limit, over compact open subgroups U_p of $G(\mathbb{Q}_p)$, of the deformations to $\mathbb{T}_{(2,\dots,2)}(U^pU_p)_{\overline{\rho}}$). For a closed point $\mathscr{P} \in \mathrm{Spec}(\mathbb{T}(U^p)_{\overline{\rho}}[1/p])$ we denote by $\rho(\mathscr{P})$ the attached Galois representation, defined over the characteristic 0 field $k(\mathscr{P})$.

Definition 4.4.10. *Given a $\mathbb{T}(U^p)_{\overline{\rho}}$ module M and a closed point $\mathscr{P} \in \mathrm{Spec}(\mathbb{T}(U^p)_{\overline{\rho}}[1/p])$, we denote by $M[\mathscr{P}]$ the $k(\mathscr{P})$-vector space $\{m \in M \otimes_{\mathcal{O}_\mathfrak{p}} k(\mathscr{P}) : Tm = 0 \forall T \in \mathscr{P}\}$. Suppose we have a homomorphism $\lambda : \mathbb{T}(U^p)_{\overline{\rho}} \to E'$, where E'/E is a finite field extension. We say that the system of Hecke eigenvalues λ* occurs in *a $\mathbb{T}(U^p)_{\overline{\rho}}$-module M if $M \otimes_{\mathcal{O}_\mathfrak{p}} E'$ contains a non-zero element where $\mathbb{T}(U^p)$ acts via the character λ. In other words, λ occurs in M if and only if $M[\ker(\lambda)] \neq 0$. We say that λ is Eisenstein whenever $\ker\lambda$ is.*

We have the following result, generalising [20, Proposition 5.5.3]. Denote by X the $\mathcal{O}_\mathfrak{p}$-module $\mathrm{Hom}_{\mathbb{T}(U^p)_{\overline{\rho}}[G_F]}(\rho^m_{\overline{\rho},U^p}, \widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho}})$.

Proposition 4.4.11. *The natural evaluation map*

$$ev : \rho^m_{\overline{\rho},U^p} \otimes_{\mathbb{T}(U^p)_{\overline{\rho}}} X \to \widetilde{H}^1_D(U^p, \mathcal{O}_{\mathfrak{p}})_{\overline{\rho}}$$

is an isomorphism.

Proof. First let $\mathscr{P} \in \mathrm{Spec}(\mathbb{T}(U^p)_{\overline{\rho}}[1/p])$ be the prime ideal of the Hecke algebra corresponding to a classical point coming from a Hecke eigenform in $H^1(M(U^pU_p), \mathcal{O}_{\mathfrak{p}})_{\overline{\rho}} \otimes_{\mathcal{O}_{\mathfrak{p}}} k(\mathscr{P})$. Then the image of the map

$$\rho(\mathscr{P}) \otimes_{k(\mathscr{P})} X[\mathscr{P}] \to \widetilde{H}^1_D(U^p, \mathcal{O}_{\mathfrak{p}})_{\overline{\rho}} \otimes_{\mathcal{O}_{\mathfrak{p}}} k(\mathscr{P})$$

induced by ev contains $H^1(M(U^pU_p)_{\overline{\rho}}, \mathcal{O}_{\mathfrak{p}})[\mathscr{P}]$. From this, it is straightforward to to see that the image of the map

$$ev_E : E \otimes_{\mathcal{O}_{\mathfrak{p}}} \rho^m_{\overline{\rho},U^p} \otimes_{\mathbb{T}(U^p)_{\overline{\rho}}} X \to \widetilde{H}^1_D(U^p, \mathcal{O}_{\mathfrak{p}})_{\overline{\rho},E}$$

is dense, since it will contain the dense subspace $H^1_D(U^p, \mathcal{O}_{\mathfrak{p}})_{\overline{\rho},E}$. Now X is a ϖ-adically admissible $G(Q_p)$-representation over $\mathbb{T}(U^p)_{\overline{\rho}}$, hence by [20, Proposition 3.1.3] the image of ev_E is closed, therefore ev_E is surjective. We now note that the evaluation map

$$\overline{\rho} \otimes_{k(\mathfrak{p})} (\mathrm{Hom}_{G_F}(\overline{\rho}, H^1_D(U^p, \mathcal{O}_{\mathfrak{p}}/\varpi)_{\overline{\rho}}) \to H^1_D(U^p, \mathcal{O}_{\mathfrak{p}}/\varpi)_{\overline{\rho}}$$

is injective, by the irreducibility of $\overline{\rho}$. We conclude the proof of the proposition by applying the following lemma (see [20, Lemma 3.1.6]). □

Lemma 4.4.12. *Let π_1 and π_2 be two ϖ-adically admissible $G(\mathbb{Q}_p)$ representations over a complete local Noetherian $\mathcal{O}_{\mathfrak{p}}$-algebra A (with maximal ideal $\mathfrak{m}$), which are torsion free as $\mathcal{O}_{\mathfrak{p}}$-modules. Suppose $f : \pi_1 \to \pi_2$ is a continuous $A[G(\mathbb{Q}_p)]$-linear morphism, satisfying*

1. *The induced map $\pi_1 \otimes_{\mathcal{O}_{\mathfrak{p}}} E \to \pi_2 \otimes_{\mathcal{O}_{\mathfrak{p}}} E$ is a surjection*
2. *The induced map $\pi_1/\varpi[\mathfrak{m}] \to \pi_2/\varpi[\mathfrak{m}]$ is an injection.*

Then f is an isomorphism.

Proof. First we show that f is an injection. Denote the kernel of f by K. It is a ϖ-adically admissible $G(\mathbb{Q}_p)$ representation over A, so in particular $K/\varpi = \bigcup_{i\geq 1} K/\varpi[\mathfrak{m}^i]$. Our

second assumption on f then implies that $K/\varpi = 0$, which implies $K = 0$, since K is $\varpi-$adically separated. We now know that f is an injection, and out first assumption on f implies that it has $\mathcal{O}_\mathfrak{p}$-torsion cokernel C. Since π_1 is $\mathcal{O}_\mathfrak{p}$-torsion free there is an exact sequence

$$0 \to C[\varpi] \to \pi_1/\varpi \to \pi_2/\varpi.$$

Since $\pi_1/\varpi = \bigcup_{i\geq 1} \pi_1/\varpi[\mathfrak{m}^i]$ we have $C[\varpi] = \bigcup_{i\geq 1} C[\varpi][\mathfrak{m}^i]$, but the second assumption on f implies that $C[\varpi][\mathfrak{m}] = 0$. Therefore $C[\varpi] = 0$, and since C is $\mathcal{O}_\mathfrak{p}$-torsion we have $C = 0$. Therefore f is an isomorphism. □

4.4.13 Completed vanishing cycles

We now assume that our tame level U^p is unramified at the prime $\mathfrak{q}$, and let $U^p(\mathfrak{q})$ denote the tame level $U^p \cap U_0(\mathfrak{q})$. Fix a non-Eisenstein maximal ideal $\mathfrak{m}$ of $\mathbb{T}(U^p(\mathfrak{q}))$. We are going to '$\varpi$-adically complete' the constructions of subsection 4.3.5.

Definition 4.4.14. *For $\mathscr{F}$ equal to either $\mathscr{F}_k$ or $r_*r^*\mathscr{F}_k$ we define $\mathcal{O}_\mathfrak{p}$-modules with smooth $G(\mathbb{Q}_p)$ actions*

$$H^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F}) = \varinjlim_{U_p} H^1(\mathbb{M}_\mathfrak{q}(U_pU^p(\mathfrak{q})) \otimes \overline{k_\mathfrak{q}}, \mathscr{F}),$$

$$X_\mathfrak{q}(U^p(\mathfrak{q}), \mathscr{F}_k) = \varinjlim_{U_p} X_\mathfrak{q}(U_pU^p(\mathfrak{q}), \mathscr{F}_k),$$

and

$$\check{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathscr{F}_k) = \varinjlim_{U_p} \check{X}_\mathfrak{q}(U_pU^p(\mathfrak{q}), \mathscr{F}_k).$$

Proposition 4.4.15. *Equation 4.3 induces a short exact sequence, equivariant with respect to Hecke, $G(\mathbb{Q}_p)$ and $G_\mathfrak{q}$ actions:*

$$0 \longrightarrow H^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F}_k) \longrightarrow H^1(U^p(\mathfrak{q}), \mathscr{F}_k) \longrightarrow X_\mathfrak{q}(U^p(\mathfrak{q}), \mathscr{F}_k)(-1) \longrightarrow 0.$$

Similarly, Equation 4.4 induces

$$0 \longrightarrow \check{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathscr{F}_k) \longrightarrow H^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F}_k) \longrightarrow H^1_{\mathrm{red}}(U^p(\mathfrak{q}), r_*r^*\mathscr{F}_k) \longrightarrow 0.$$

Proof. This just follows from taking the direct limit of the relevant exact sequences. □

Definition 4.4.16. *We define $\mathcal{O}_\mathfrak{p}$-modules with continuous $G(\mathbb{Q}_p)$ actions (again $\mathscr{F}$ equals either $\mathscr{F}_k$ or $r_*r^*\mathscr{F}_k$)*

$$\widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}),\mathscr{F}) = \varprojlim_s H^1_{\mathrm{red}}(U^p(\mathfrak{q}),\mathscr{F})/\varpi^s H^1_{\mathrm{red}}(U^p(\mathfrak{q}),\mathscr{F}),$$

$$\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathscr{F}_k) = \varprojlim_s X_\mathfrak{q}(U^p(\mathfrak{q}),\mathscr{F}_k)/\varpi^s X_\mathfrak{q}(U^p(\mathfrak{q}),\mathscr{F}_k)$$

and

$$\widetilde{\check{X}}_\mathfrak{q}(U^p(\mathfrak{q}),\mathscr{F}_k) = \varprojlim_s \check{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathscr{F}_k)/\varpi^s \check{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathscr{F}_k).$$

Now ϖ-adically completing the short exact sequences of Proposition 4.4.15, we obtain

Proposition 4.4.17. *We have short exact sequences of ϖ-adically admissible $G(\mathbb{Q}_p)$-representations over $\mathcal{O}_\mathfrak{p}$, equivariant with respect to Hecke, $G(\mathbb{Q}_p)$ and $G_\mathfrak{q}$ actions:*

$$0 \longrightarrow \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}),\mathscr{F}_k)_\mathfrak{m} \longrightarrow \widetilde{H}^1_D(U^p(\mathfrak{q}),\mathscr{F}_k)_\mathfrak{m} \longrightarrow \widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathscr{F}_k)_\mathfrak{m}(-1) \longrightarrow 0,$$

$$0 \longrightarrow \widetilde{\check{X}}_\mathfrak{q}(U^p(\mathfrak{q}),\mathscr{F}_k)_\mathfrak{m} \longrightarrow \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}),\mathscr{F}_k)_\mathfrak{m} \longrightarrow \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}),r_*r^*\mathscr{F}_k)_\mathfrak{m} \longrightarrow 0.$$

Proof. The existence of a short exact sequence of $\mathcal{O}_\mathfrak{p}$-modules follows from the fact that $X_\mathfrak{q}(U_pU^p(\mathfrak{q}),\mathscr{F}_k)_\mathfrak{m}$ and $H^1_{\mathrm{red}}(U^p(\mathfrak{q}),r_*r^*\mathscr{F}_k)_\mathfrak{m}$ is always $\mathcal{O}_\mathfrak{p}$-torsion free. The module $X_\mathfrak{q}(U_pU^p(\mathfrak{q}),\mathscr{F}_k)$ is in fact torsion free even without localising at $\mathfrak{m}$ (see the remark following [36, Corollary 1]), whilst for $H^1_{\mathrm{red}}(U^p(\mathfrak{q}),r_*r^*\mathscr{F}_k)_\mathfrak{m}$ this fact follows as in [15, Lemma 4]. The fact that we obtain a short exact sequence of ϖ-adically admissible $\mathcal{O}_\mathfrak{p}[G(\mathbb{Q}_p)]$-modules follows from Proposition 2.4.4 of [21]. □

It is now straightforward to deduce the analogue of Lemma 4.4.5:

Lemma 4.4.18. *There are canonical $G(\mathbb{Q}_p)$, $G_\mathfrak{q}$ and Hecke equivariant isomorphisms*

$$\widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}),\mathscr{F}_k)_\mathfrak{m} \cong \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_\mathfrak{m} \otimes_{\mathcal{O}_\mathfrak{p}} W^0_{k,\mathfrak{p}},$$

$$\widetilde{\check{X}}_\mathfrak{q}(U^p(\mathfrak{q}),\mathscr{F}_k)_\mathfrak{m} \cong \widetilde{\check{X}}_\mathfrak{q}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_\mathfrak{m} \otimes_{\mathcal{O}_\mathfrak{p}} W^0_{k,\mathfrak{p}},$$

$$\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathscr{F}_k)_\mathfrak{m} \cong \widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_\mathfrak{m} \otimes_{\mathcal{O}_\mathfrak{p}} W^0_{k,\mathfrak{p}}.$$

4.4.19 Mazur's principle

We can now proceed as in Theorem 6.2.4 of [20], following Jarvis's approach to Mazur's principle over totally real fields [25]. First we fix an absolutely irreducible mod p Galois representation $\overline{\rho}$, coming from a maximal ideal of $\mathbb{T}(U^p(\mathfrak{q}))$. We have the following proposition

Proposition 4.4.20. *The injection*

$$\widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p}) \hookrightarrow \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$$

induces an isomorphism

$$\widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}} \cong (\widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}})^{I_\mathfrak{q}}.$$

Proof. This follows from Proposition 4 and (2.2) in [36]. □

There is a Hecke operator $\mathsf{U}_\mathfrak{q}$ acting on $H^1(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$, which induces an action on the spaces $\check{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$ and $X_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$. This extends by continuity to give an action of $\mathsf{U}_\mathfrak{q}$ on $\widetilde{\check{X}}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$ and $\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$.

Lemma 4.4.21. *The Frobenius element of* $\mathrm{Gal}(\overline{k}_\mathfrak{q}/k_\mathfrak{q})$ *acts on* $\widetilde{\check{X}}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$ *by* $(\mathbf{N}\mathfrak{q})\mathsf{U}_\mathfrak{q}$ *and on* $\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$ *by* $\mathsf{U}_\mathfrak{q}$.

Proof. Section 6 of [7] tells us that Frobenius acts as required on the dense subspaces $\check{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$ and $X_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$, so by continuity we are done. □

Lemma 4.4.22. *There is a natural isomorphism* $\widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), r_* r^* \mathcal{O}_\mathfrak{p}) \cong \widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})^{\oplus 2}$.

Proof. This follows from Lemma 16.1 in [25]. □

Lemma 4.4.23. *The monodromy pairing map* $\lambda : X_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p}) \to \check{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$ *induces a* $G_\mathfrak{q}$*- and* $\mathbb{T}(U^p(\mathfrak{q}))$*-equivariant isomorphism* $\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}} \cong \widetilde{\check{X}}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}}$.

Proof. This follows from Proposition 5 in [36]. □

Remark. *Note that the isomorphism in the above lemma is not* $\mathsf{U}_\mathfrak{q}$ *equivariant.*

Before giving an analogue of Mazur's principle, we will make explicit the connection between the modules $\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}}$ and a space of newforms. There are natural level raising maps $i : \widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})^{\oplus 2}_{\overline{\rho}} \to \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}}$, and their adjoints $i^\dagger : \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}} \to \widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})^{\oplus 2}_{\overline{\rho}}$, defined as in the previous chapter by taking the limit of maps at finite level.

Definition 4.4.24. *The space of* $\mathfrak{q}$*-newforms*

$$\widetilde{H}^1_D(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}-\mathrm{new}}_{\overline{\rho}}$$

is defined to be the kernel of the map

$$i^\dagger : \widetilde{H}^1_D(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_{\overline{\rho}} \to \widetilde{H}^1_D(U^p,\mathcal{O}_\mathfrak{p})^{\oplus 2}_{\overline{\rho}}.$$

Proposition 4.4.25. *A system of Hecke eigenvalues* $\lambda : \mathbb{T}(U^p(\mathfrak{q})) \to E'$ *(*E'/E *finite) occurs in* $\widetilde{H}^1_D(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}-\mathrm{new}}_{\overline{\rho}}$ *if and only if it occurs in* $\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_{\overline{\rho}}$*.*

Proof. Proposition 4.4.17 and Lemma 4.4.22, together with Definition 4.4.24 imply that we have the following commutative diagram of ϖ-adically admissible $G(\mathbb{Q}_p)$-representations over $\mathcal{O}_\mathfrak{p}$:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\widetilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_{\overline{\rho}} & \longrightarrow & \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_{\overline{\rho}} & \longrightarrow & \widetilde{H}^1_D(U^p,\mathcal{O}_\mathfrak{p})^{\oplus 2}_{\overline{\rho}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \cong} & & \\
0 & \longrightarrow & \widetilde{H}^1_D(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}-\mathrm{new}}_{\overline{\rho}} & \longrightarrow & \widetilde{H}^1_D(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_{\overline{\rho}} & \longrightarrow & \widetilde{H}^1_D(U^p,\mathcal{O}_\mathfrak{p})^{\oplus 2}_{\overline{\rho}} & \longrightarrow & 0,
\end{array}$$

where the two rows are exact, the maps α and β are injections and the final vertical map is the identity. Moreover $\mathrm{coker}(\beta) = \widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_{\overline{\rho}}(-1) \cong \widetilde{\widetilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_{\overline{\rho}}(-1)$, where the second isomorphism comes from the monodromy pairing (see Lemma 4.4.23). Applying the snake lemma we see that $\mathrm{coker}(\alpha) = \mathrm{coker}(\beta)$, so (applying Lemma 4.4.23 once more to the source of the map α) we have a short exact sequence

$$0 \longrightarrow \widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_{\overline{\rho}} \xrightarrow{\alpha} \widetilde{H}^1_D(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}-\mathrm{new}}_{\overline{\rho}} \longrightarrow \widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_{\overline{\rho}}(-1) \longrightarrow 0.$$

From this short exact sequence it is easy to deduce that the systems of Hecke eigenvalues occurring in $\widetilde{H}^1_D(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}-\mathrm{new}}_{\overline{\rho}}$ and $\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})_{\overline{\rho}}$ are the same. □

Theorem 4.4.26. *Let* $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ *be a Galois representation, with irreducible reduction* $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$*. Suppose the following two conditions are verified:*

1. ρ *is unramified at the prime* $\mathfrak{q}$
2. $\rho(\mathrm{Frob}_\mathfrak{q})$ *is not a scalar*

3. *There is a system of Hecke eigenvalues* $\lambda : \mathbb{T}(U^p)_{\overline{\rho}} \to E'$ *attached to* ρ *(i.e.* $\rho \cong \rho(\ker(\lambda))$*), with* λ *occurring in* $\widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$.

Then λ *occurs in* $\widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})$.

Proof. We take ρ as in the statement of the theorem. The third assumption, combined with Proposition 4.4.11 implies that there is a prime ideal $\mathscr{P} = \ker(\lambda)$ of $\mathbb{T}(U^p(\mathfrak{q}))$, and an embedding

$$\rho \hookrightarrow \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})[\mathscr{P}].$$

Our aim is to show that $\widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})[\mathscr{P}] \neq 0$.

Let $\mathbb{T}(U^p(\mathfrak{q}))^*$ denote the finite ring extension of $\mathbb{T}(U^p(\mathfrak{q}))$ obtained by adjoining the operator $\mathsf{U}_\mathfrak{q}$. Since ρ is irreducible there is a prime $\mathscr{P}^*$ of $\mathbb{T}(U^p(\mathfrak{q}))^*$ such that there is an embedding

$$\rho \hookrightarrow \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})[\mathscr{P}^*].$$

Recall that by Propositions 4.4.17 and 4.4.20 and Lemma 4.4.22 there is a $G_\mathfrak{q}$ and $\mathbb{T}(U^p(\mathfrak{q}))$-equivariant short exact sequence

$$0 \longrightarrow \widetilde{\tilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}} \longrightarrow \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})^{I_\mathfrak{q}}_{\overline{\rho}} \longrightarrow \widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})^{\oplus 2}_{\overline{\rho}} \longrightarrow 0.$$

Suppose for a contradiction that $\widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho}}[\mathscr{P}] = 0$

Let V denote the $G_\mathfrak{q}$-representation obtained from ρ by restriction. Since ρ is unramified at $\mathfrak{q}$, there is an embedding $V \hookrightarrow \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})^{I_\mathfrak{q}}_{\overline{\rho}}[\mathscr{P}^*]$. The above short exact sequence then implies that the embedding

$$\widetilde{\tilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}}[\mathscr{P}] \hookrightarrow \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})^{I_\mathfrak{q}}_{\overline{\rho}}[\mathscr{P}]$$

is an isomorphism, and hence the embedding

$$\widetilde{\tilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}}[\mathscr{P}^*] \hookrightarrow \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})^{I_\mathfrak{q}}_{\overline{\rho}}[\mathscr{P}^*]$$

is also an isomorphism.

This implies that there is an embedding $V \hookrightarrow \widetilde{\tilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_{\overline{\rho}}[\mathscr{P}^*]$, so by Lemma 4.4.21 $\rho(\mathrm{Frob}_\mathfrak{q})$ is a scalar $(\mathbf{N}\mathfrak{q})\alpha$ where α is the $\mathsf{U}_\mathfrak{q}$ eigenvalue coming from $\mathscr{P}^*$, contradicting our second assumption. □

4.4.27 Mazur's principle and the Jacquet functor

After taking locally analytic vectors and applying Emerton's locally analytic Jacquet functor (see [17]) we can prove a variant of Theorem 4.4.26. We let B denote the Borel subgroup of $G(\mathbb{Q}_p)$ consisting of upper triangular matrices, and let T denote the maximal torus contained in B. We then have a locally analytic Jacquet functor J_B as defined in [17], which can be applied to the space of locally analytic vectors $(V)^{an}$ of an admissible continuous Banach $E[G(\mathbb{Q}_p)]$-representation V. The following lemma follows from Proposition 4.4.11, but we give a more elementary separate proof.

Lemma 4.4.28. *Let $\lambda : \mathbb{T}(U^p(\mathfrak{q})) \to E$ be a system of Hecke eigenvalues such that λ occurs in $J_B(\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{an})$. Suppose the attached Galois representation ρ is absolutely irreducible.Then there is a non-zero $\mathrm{Gal}(\overline{F}/F)$-equivariant map (necessarily an embedding, since ρ is irreducible)*

$$\rho \hookrightarrow J_B(\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{an}).$$

Proof. To abbreviate notation we let M denote $J_B(\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{an})$. Since M is an essentially admissible T representation, the system of Hecke eigenvalues λ occurs in the χ-isotypic subspace M^χ for some continuous character χ of T. So (again by essential admissibility) $M^{\chi,\lambda}$ is a non-zero finite dimensional $\mathrm{Gal}(\overline{F}/F)$-representation over E, and therefore by the Eichler-Shimura relations (as in section 10.3 of [6]) and the irreducibility of ρ the desired embedding exists. □

Theorem 4.4.29. *Suppose that the system of Hecke eigenvalues $\lambda : \mathbb{T}(U^p(\mathfrak{q})) \to E$ occurs in $J_B(\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{an})$, that the attached Galois representation $\rho : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(E)$ is unramified at $\mathfrak{q}$, with $\rho(Frob_\mathfrak{q})$ not a scalar, and that the reduction $\overline{\rho} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is irreducible.*
Then λ occurs in $J_B(\widetilde{H}^1_D(U^p, E)^{an})$.

Proof. We take ρ as in the statement of the theorem. Let $\mathfrak{m}$ be the maximal ideal of $\mathbb{T}(U^p(\mathfrak{q}))$ attached to $\overline{\rho}$, and let $\mathscr{P}$ be the prime ideal of $\mathbb{T}(U^p(\mathfrak{q}))$ attached to ρ, so applying Lemma 4.4.28 gives an embedding

$$\rho \hookrightarrow J_B(\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{an}_{\overline{\rho}})[\mathscr{P}].$$

Our aim is to show that $J_B(\widetilde{H}^1_D(U^p, E)^{an}_{\overline{\rho}})[\mathscr{P}] \neq 0$. Let $\mathbb{T}(U^p(\mathfrak{q}))^*$ denote the finite ring

extension of $\mathbb{T}(U^p(\mathfrak{q}))$ obtained by adjoining the operator $\mathsf{U}_\mathfrak{q}$. Since ρ is irreducible there is a prime $\mathscr{P}^*$ of $\mathbb{T}(U^p(\mathfrak{q}))^*$ such that there is an embedding

$$\rho \hookrightarrow J_B(\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{an}_{\overline{\rho}})[\mathscr{P}^*].$$

By Propositions 4.4.17 and 4.4.20 and Lemma 4.4.22 there is a short exact sequence

$$0 \longrightarrow \widetilde{\widetilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}), E)_{\overline{\rho}} \longrightarrow \widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{I_\mathfrak{q}}_{\overline{\rho}} \longrightarrow \widetilde{H}^1_D(U^p, E)^2_{\overline{\rho}} \longrightarrow 0.$$

Taking locally analytic vectors (which is exact) and applying the Jacquet functor (which is left exact), we get an exact sequence

$$0 \longrightarrow J_B(\widetilde{\widetilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}), E)^{an}_{\overline{\rho}}) \longrightarrow J_B((\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{I_\mathfrak{q}}_{\overline{\rho}})^{an}) \longrightarrow J_B(\widetilde{H}^1_D(U^p, E)^{an}_{\overline{\rho}})^2.$$

Note that $J_B((\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{I_\mathfrak{q}}_{\overline{\rho}})^{an})$ is equal to $J_B(\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{an}_{\overline{\rho}})^{I_\mathfrak{q}}$, since under the isomorphism of Proposition 4.4.11 they both correspond to $(\rho^m_{\overline{\rho},U^p(\mathfrak{q})})^{I_\mathfrak{q}} \otimes_{\mathbb{T}(U^p(\mathfrak{q}))_{\overline{\rho}}} J_B(X^{an}_E)$. Let V denote the $G_\mathfrak{q}$-representation obtained from ρ by restriction. Since ρ is unramified at $\mathfrak{q}$, there is an embedding $V \hookrightarrow J_B((\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{I_\mathfrak{q}}_{\overline{\rho}})^{an})[\mathscr{P}^*]$. Suppose for a contradiction that $J_B(\widetilde{H}^1_D(U^p, E)^{an}_{\overline{\rho}})[\mathscr{P}] = 0$. The above exact sequence then implies that the embedding

$$J_B(\widetilde{\widetilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}), E)^{an}_{\overline{\rho}})[\mathscr{P}] \hookrightarrow J_B((\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{I_\mathfrak{q}}_{\overline{\rho}})^{an})[\mathscr{P}]$$

is an isomorphism, and hence the embedding

$$J_B(\widetilde{\widetilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}), E)^{an}_{\overline{\rho}})[\mathscr{P}^*] \hookrightarrow J_B((\widetilde{H}^1_D(U^p(\mathfrak{q}), E)^{I_\mathfrak{q}}_{\overline{\rho}})^{an})[\mathscr{P}^*]$$

is also an isomorphism.

This implies that there is an embedding $V \hookrightarrow J_B(\widetilde{\widetilde{X}}_\mathfrak{q}(U^p(\mathfrak{q}), E)^{an}_{\overline{\rho}})[\mathscr{P}^*]$, so by Lemma 4.4.22 $\rho(\mathrm{Frob}_\mathfrak{q})$ is a scalar $(\mathbf{N}\mathfrak{q})\alpha$ corresponding to the $(\mathbf{N}\mathfrak{q})\mathsf{U}_\mathfrak{q}$ eigenvalue in $\mathscr{P}^*$. We then obtain a contradiction exactly as in the proof of Theorem 4.4.26. □

4.4.30 A p-adic Jacquet–Langlands correspondence

We now wish to relate the arithmetic of p-adic automorphic forms for G and G'. The definitions and results of subection 4.4.1 apply to G'. We denote the associated completed

cohomology spaces of tame level V^p by

$$\widetilde{H}^1_{D'}(V^p, \mathscr{F}_k),$$

and its Hecke algebra by $\mathbb{T}'(V^p)$. The integral models $\mathbb{M}'_{\mathfrak{q}}$ and the vanishing cycle formalism also allows us to define some more topological $\mathcal{O}_{\mathfrak{p}}$-modules:

Definition 4.4.31. *We define $\mathcal{O}_{\mathfrak{p}}$-modules with smooth $G(\mathbb{Q}_p)$ actions*

$$Y_{\mathfrak{q}}(V^p, \mathscr{F}_k) = \varinjlim_{V_p} Y_{\mathfrak{q}}(V_p V^p, \mathscr{F}_k),$$

and

$$\check{Y}_{\mathfrak{q}}(V^p, \mathscr{F}_k) = \varinjlim_{V_p} \check{Y}_{\mathfrak{q}}(V_p V^p, \mathscr{F}_k).$$

We define $\mathcal{O}_{\mathfrak{p}}$-modules with continuous $G(\mathbb{Q}_p)$ actions

$$\widetilde{Y}_{\mathfrak{q}}(V^p, \mathscr{F}_k) = \varprojlim_{s} Y(V^p, \mathscr{F}_k)/\varpi^s Y(V^p, \mathscr{F}_k)$$

and

$$\widetilde{\check{Y}}_{\mathfrak{q}}(V^p, \mathscr{F}_k) = \varprojlim_{s} \check{Y}_{\mathfrak{q}}(V^p, \mathscr{F}_k)/\varpi^s \check{Y}_{\mathfrak{q}}(V^p, \mathscr{F}_k).$$

Fixing a non-Eisenstein maximal ideal $\mathfrak{m}'$ of $\mathbb{T}'(V^p)$, then in exactly the same way as we obtained Proposition 4.4.17 we get:

Proposition 4.4.32. *We have a short exact sequence of ϖ-adically admissible $G(\mathbb{Q}_p)$-representations over $\mathcal{O}_{\mathfrak{p}}$, equivariant with respect to $\mathbb{T}'(V^p)_{\mathfrak{m}'}$, $G(\mathbb{Q}_p)$ and $G_{\mathfrak{q}}$ actions:*

$$0 \longrightarrow \widetilde{\check{Y}}_{\mathfrak{q}}(V^p, \mathscr{F}_k)_{\mathfrak{m}'} \longrightarrow \widetilde{H}^1_{D'}(V^p, \mathscr{F}_k)_{\mathfrak{m}'} \longrightarrow \widetilde{Y}_{\mathfrak{q}}(V^p, \mathscr{F}_k)_{\mathfrak{m}'}(-1) \longrightarrow 0.$$

Fix $U^p \subset G(\mathbb{A}_f^{(p)})$ such that U^p is unramified at the places $\mathfrak{q}_1, \mathfrak{q}_2$, and matches with V^p at all other places. We now fix an irreducible mod p Galois representation $\overline{\rho}$, arising from a (necessarily non-Eisenstein) maximal ideal $\mathfrak{m}'$ of $\mathbb{T}'(V^p)$. There is a corresponding maximal ideal $\mathfrak{m}'_0$ of $\mathbb{T}'_{(2,\dots,2)}(V^p V_p)$ for V_p a small enough compact open subgroup of $G'(\mathbb{Q}_p)$, and this pulls back by the map $\mathbb{T}_{(2,\dots,2)}(U^p(\mathfrak{q}_1\mathfrak{q}_2)U_p) \to \mathbb{T}'_{(2,\dots,2)}(V^p V_p)$ to a maximal ideal $\mathfrak{m}_0$ of $\mathbb{T}_{(2,\dots,2)}(U^p(\mathfrak{q}_1\mathfrak{q}_2)U_p)$ for U_p the compact open subgroup of $G(\mathbb{Q}_p)$ which is identified with V_p under the isomorphism $G(\mathbb{Q}_p) \cong G'(\mathbb{Q}_p)$. Hence there is a map $\mathbb{T}(U^p(\mathfrak{q}_1\mathfrak{q}_2))_{\overline{\rho}} \to \mathbb{T}'(V^p)_{\overline{\rho}}$.

We can give a refinement of Proposition 4.4.11 in this situation. Firstly we note that the Galois representation $\rho^m_{\overline{\rho},V^p} : G_F \to \mathrm{GL}_2(\mathbb{T}'(V^p)_{\overline{\rho}} \otimes_{\mathcal{O}_\mathfrak{p}} E)$ has an unramified sub-$G_\mathfrak{q}$ representation $V_{0,E}$ (which is a free $\mathbb{T}'(V^p)_{\overline{\rho}} \otimes_{\mathcal{O}_\mathfrak{p}} E$-module of rank one), with unramified quotient $V_{1,E} \cong V_{0,E}(-1)$ (also a free $\mathbb{T}'(V^p)_{\overline{\rho}} \otimes_{\mathcal{O}_\mathfrak{p}} E$-module of rank one). This follows from the fact that each representation $\rho : G_F \to \mathrm{GL}_2(\mathbb{T}'_{(2,\dots,2)}(V^pV_p)_{\overline{\rho}} \otimes_{\mathcal{O}_\mathfrak{p}} E)$ has a unique unramified subrepresentation of the right kind. Denote by Y the $\mathcal{O}_\mathfrak{p}$-module $\mathrm{Hom}_{\mathbb{T}'(V^p)_{\overline{\rho}}[G_F]}(\rho^m_{\overline{\rho},V^p}, \widetilde{H}^1_{D'}(V^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho}})$. By proposition 4.4.11 we have an isomorphism

$$ev_E : E \otimes_{\mathcal{O}_\mathfrak{p}} \rho^m_{\overline{\rho},V^p} \otimes_{\mathbb{T}'(V^p)_{\overline{\rho}}} Y \cong \widetilde{H}^1_{D'}(V^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho}}.$$

Proposition 4.4.33. *The isomorphism ev_E identifies the image of the embedding*

$$\widetilde{\check{Y}}_\mathfrak{q}(V^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho},E} \to \widetilde{H}^1_{D'}(V^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho},E}$$

with $V_{0,E} \otimes_{\mathbb{T}'(V^p)_{\overline{\rho}}} Y$. It follows that the quotient $\widetilde{Y}_\mathfrak{q}(V^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho},E}(-1)$ is identified with $V_{1,E} \otimes_{\mathbb{T}'(V^p)_{\overline{\rho}}} Y$.

Proof. We proceed as in the proof of proposition 4.4.11. First let $\mathscr{P} \in \mathrm{Spec}(\mathbb{T}'(V^p)_{\overline{\rho}} \otimes_{\mathcal{O}_\mathfrak{p}} E)$ be the prime ideal of the Hecke algebra corresponding to a classical point coming from a Hecke eigenform in $H^1(M'(V^pV_p), \mathcal{O}_\mathfrak{p})_{\overline{\rho}} \otimes_{\mathcal{O}_\mathfrak{p}} k(\mathscr{P})$. Taking V_p invariants, extending scalars to $k(\mathscr{P})$ and taking the $\mathscr{P}$ torsion parts, the isomorphism ev_E induces an isomorphism

$$\rho(\mathscr{P}) \otimes_{k(\mathscr{P})} \mathrm{Hom}_{k(\mathscr{P})[G_F]}(\rho(\mathscr{P}), H^1(M'(V^pV_p), \mathcal{O}_\mathfrak{p})_{\overline{\rho},E}[\mathscr{P}]) \cong H^1(M'(V^pV_p), \mathcal{O}_\mathfrak{p})_{\overline{\rho},E}[\mathscr{P}],$$

under which $\check{Y}_\mathfrak{q}(V^pV_p, \mathcal{O}_\mathfrak{p})_{\overline{\rho},E}[\mathscr{P}]$ is identified with

$$V_{0,E}(\mathscr{P}) \otimes_{k(\mathscr{P})} \mathrm{Hom}_{k(\mathscr{P})[G_F]}(\rho(\mathscr{P}), H^1(M'(V^pV_p), \mathcal{O}_\mathfrak{p})_{\overline{\rho},E}[\mathscr{P}]).$$

A density argument now completes the proof. □

Passing to the direct limit over compact open subgroups U_p of $G(\mathbb{Q}_p)$ and ϖ-adically completing the short exact sequences of Theorem 4.3.7, we obtain

Theorem 4.4.34. *We have the following short exact sequences of ϖ-adically admissible $G(\mathbb{Q}_p)$-representations over $\mathcal{O}_\mathfrak{p}$, equivariant with respect to $\mathbb{T}(U^p(\mathfrak{q}_1\mathfrak{q}_2))_{\overline{\rho}}$ and $G(\mathbb{Q}_p) \cong$*

$G'(\mathbb{Q}_p)$ actions:

$$0 \longrightarrow \widetilde{Y}_{\mathfrak{q}_1}(V^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho}} \longrightarrow \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})_{\overline{\rho}} \xrightarrow{i^\dagger} \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^2_{\overline{\rho}} \longrightarrow 0,$$

$$0 \longrightarrow \widetilde{\check{X}}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^2_{\overline{\rho}} \longrightarrow \widetilde{\check{X}}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})_{\overline{\rho}} \longrightarrow \widetilde{\check{Y}}_{\mathfrak{q}_1}(V^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho}} \longrightarrow 0.$$

Proof. The proof is as for Proposition 4.4.17. We just need to check that the short exact sequences of Theorem 4.3.7 are compatible as U_p and V_p vary, but this is clear from the proof of [36, Theorem 3], since the descriptions of the dual graph of the special fibres of $\mathbb{M}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2)U_p)$ and $\mathbb{M}'_{\mathfrak{q}_1}(V^pV_p)$ are compatible as U_p and V_p vary. □

We regard Theorem 4.4.34 (especially the first exact sequence therein) as a geometric realisation of a p-adic Jacquet–Langlands correspondence. The following propositions explain why this is reasonable. We let $\widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1-\mathrm{new}}_{\overline{\rho}}$ denote the kernel of the map

$$i^\dagger : \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})_{\overline{\rho}} \to \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\oplus 2}_{\overline{\rho}}.$$

Proposition 4.4.35. *A system of Hecke eigenvalues $\lambda : \mathbb{T}(U^p(\mathfrak{q}_1\mathfrak{q}_2)) \to E'$ occurs in $\widetilde{H}^1_D(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1\mathfrak{q}_2-\mathrm{new}}_{\overline{\rho}}$ if and only if it occurs in $\widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1-\mathrm{new}}_{\overline{\rho}}$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1-\mathrm{new}}_{\overline{\rho}} & \longrightarrow & \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})_{\overline{\rho}} & \longrightarrow & \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\oplus 2}_{\overline{\rho}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \widetilde{H}^1_D(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1\mathfrak{q}_2-\mathrm{new}}_{\overline{\rho}} & \longrightarrow & \widetilde{H}^1_D(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_2-\mathrm{new}}_{\overline{\rho}} & \longrightarrow & (\widetilde{H}^1_D(U^p(\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_2-\mathrm{new}}_{\overline{\rho}})^{\oplus 2} & & ,
\end{array}$$

where the two rows are exact, and the maps α, β and γ are injections. Note that we are applying Lemma 4.4.23. By the snake lemma and the proof of Proposition 4.4.25 we have an exact sequence

$$0 \longrightarrow \mathrm{coker}(\alpha) \longrightarrow \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})_{\overline{\rho}}(-1) \longrightarrow \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\oplus 2}_{\overline{\rho}}(-1),$$

whence a short exact sequence

$$0 \to \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1-\mathrm{new}}_{\overline{\rho}} \xrightarrow{\alpha} \widetilde{H}^1_D(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1\mathfrak{q}_2-\mathrm{new}}_{\overline{\rho}} \to \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1-\mathrm{new}}_{\overline{\rho}}(-1) \to 0.$$

From this last short exact sequence it is easy to deduce that the systems of Hecke eigenvalues occurring in $\widetilde{H}^1_D(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1\mathfrak{q}_2-\mathrm{new}}_{\overline{\rho}}$ and $\widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1-\mathrm{new}}_{\overline{\rho}}$ are the same.

□

Proposition 4.4.36. *A system of Hecke eigenvalues $\lambda : \mathbb{T}'(V^p) \to E'$ occurs in $\widetilde{H}^1_{D'}(V^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho}}$ if and only if it occurs in $\widetilde{Y}_{\mathfrak{q}_1}(V^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho}}$.*

Proof. This follows from proposition 4.4.33. □

Combined with Theorem 4.4.34 this has as a consequence

Corollary 4.4.37. *The first exact sequence of Theorem 4.4.34 induces a bijection between the systems of Hecke eigenvalues (for $\mathbb{T}'(V^p)$) appearing in $\widetilde{H}^1_{D'}(V^p, \mathcal{O}_\mathfrak{p})_{\overline{\rho}}$ and the systems of Hecke eigenvalues (for $\mathbb{T}(U^p(\mathfrak{q}_1\mathfrak{q}_2))$) appearing in $\widetilde{H}^1_D(U^p(\mathfrak{q}_1\mathfrak{q}_2), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1\mathfrak{q}_2-\mathrm{new}}_{\overline{\rho}}$.*

4.5 Eigenvarieties and an overconvergent Jacquet–Langlands correspondence

In this section we will apply Emerton's construction of eigenvarieties to deduce an overconvergent Jacquet–Langlands correspondence from Theorem 4.4.34.

4.5.1 Locally algebraic vectors

Fix a maximal ideal $\mathfrak{m}$ of $\mathbb{T}(U^p(\mathfrak{q}))$ which is not Eisenstein. Recall that given an admissible $G(\mathbb{Q}_p)$-representation V over E, Schneider and Teitelbaum [41] define an exact functor by passing to the locally $\mathbb{Q}_p$-analytic vectors V^{an}, and the $G(\mathbb{Q}_p)$-action on V^{an} differentiates to give an action of the Lie algebra $\mathfrak{g}$ of $G(\mathbb{Q}_p)$.

Proposition 4.5.2. *There are natural isomorphisms (for $\mathscr{F} = \mathscr{F}_k$ or $r_*r^*\mathscr{F}_k$)*

1. $H^0(\mathfrak{g}, \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathscr{F}_k)^{an}_{\mathfrak{m},E}) \cong H^1_D(U^p(\mathfrak{q}), \mathscr{F}_k)_{\mathfrak{m},E}$,
2. $H^1(\mathfrak{g}, \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathscr{F}_k)^{an}_{\mathfrak{m},E}) \cong 0$,
3. $H^0(\mathfrak{g}, \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F})^{an}_{\mathfrak{m},E}) \cong H^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F})_{\mathfrak{m},E}$,
4. $H^1(\mathfrak{g}, \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F})^{an}_{\mathfrak{m},E}) \cong 0$,
5. $H^0(\mathfrak{g}, \widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathscr{F}_k)^{an}_{\mathfrak{m},E}) \cong X_\mathfrak{q}(U^p(\mathfrak{q}), \mathscr{F}_k)_{\mathfrak{m},E}$,
6. $H^0(\mathfrak{g}, \widetilde{\check{X}}_\mathfrak{q}(U^p(\mathfrak{q}), \mathscr{F}_k)^{an}_{\mathfrak{m},E}) \cong \check{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathscr{F}_k)_{\mathfrak{m},E}$,

Proof. The first and second natural isomorphisms comes from the spectral sequence (for completed étale cohomology) discussed in Proposition 2.4.1 of [19], since localising at $\mathfrak{m}$ kills $H^i_D(U^p(\mathfrak{q}), \mathscr{F}_k)$ and $\widetilde{H}^i_D(U^p(\mathfrak{q}), \mathscr{F}_k)$ for $i = 0, 2$.

The same argument works for the third and fourth isomorphisms, since as noted in the proof of [19, Proposition 2.4.1] the spectral sequence of [19, Corollary 2.2.18] can be obtained using the Hochschild-Serre spectral sequence for étale cohomology, which applies equally well to the étale cohomology of the special fibre of $\mathbb{M}_{\mathfrak{q}}(U_pU^p(\mathfrak{q}))$.

We now turn to the fifth isomorphism. By the first short exact sequence of Proposition 4.4.17 and the exactness of localising at $\mathfrak{m}$, inverting ϖ and then passing to locally analytic vectors, there is a short exact sequence

$$0 \longrightarrow \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F}_k)^{an}_{\mathfrak{m},E} \longrightarrow \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathscr{F}_k)^{an}_{\mathfrak{m},E} \longrightarrow \widetilde{X}_{\mathfrak{q}}(U^p(\mathfrak{q}), \mathscr{F}_k)(-1)^{an}_{\mathfrak{m},E} \longrightarrow 0.$$

Taking the long exact sequence of $\mathfrak{g}$-cohomology then gives an exact sequence

$$0 \longrightarrow H^0(\mathfrak{g}, \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F}_k)^{an}_{\mathfrak{m},E}) \longrightarrow H^0(\mathfrak{g}, \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathscr{F}_k)^{an}_{\mathfrak{m},E})$$

$$\longrightarrow H^0(\mathfrak{g}, \widetilde{X}_{\mathfrak{q}}(U^p(\mathfrak{q}), \mathscr{F}_k)(-1)^{an}_{\mathfrak{m},E}) \longrightarrow H^1(\mathfrak{g}, \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F}_k)^{an}_{\mathfrak{m},E}),$$

so applying the third, first and fourth part of the proposition to the first, second and fourth terms in this sequence respectively we get a short exact sequence

$$0 \longrightarrow H^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F}_k)_{\mathfrak{m},E} \longrightarrow H^1_D(U^p(\mathfrak{q}), \mathscr{F}_k)_{\mathfrak{m},E} \longrightarrow H^0(\mathfrak{g}, \widetilde{X}_{\mathfrak{q}}(U^p(\mathfrak{q}), \mathscr{F}_k)(-1)^{an}_{\mathfrak{m},E}) \longrightarrow 0,$$

so identifying this with the short exact sequence of Proposition 2.3.2 gives the desired isomorphism

$$H^0(\mathfrak{g}, \widetilde{X}_{\mathfrak{q}}(U^p(\mathfrak{q}), \mathscr{F}_k)^{an}_{\mathfrak{m},E}) \cong X_{\mathfrak{q}}(U^p(\mathfrak{q}), \mathscr{F}_k)_{\mathfrak{m},E}.$$

Finally the sixth isomorphism follows from the first four in the same way as the fifth (using the second short exact sequence of Proposition 4.4.17. □

Given a continuous $G(\mathbb{Q}_p)$-representation V we denote by V^{alg} the space of locally algebraic vectors in V. Let $E(1)$ denote a one dimensional E-vector space on which $G(\mathbb{Q}_p) \times \mathrm{Gal}(\overline{L}/L)$ acts by $\prod_{v|p} N_{F_v/\mathbb{Q}_p} \circ \det \otimes \epsilon$, where ϵ is the unramified character of $\mathrm{Gal}(\overline{L}/L)$ sending $Frob_\lambda$ to $N_{F/\mathbb{Q}}(\lambda)$. For any $n \in \mathbb{Z}$ let $E(n) = E(1)^{\otimes n}$. We can now proceed as in Theorem 7.4.2 of [18] to deduce

Theorem 4.5.3. *There are natural $G(\mathbb{Q}_p) \times G_{\mathfrak{q}} \times \mathbb{T}(U^p(\mathfrak{q}))$-equivariant isomorphisms*

1. $\bigoplus_{k,n\in\mathbb{Z}} H^1_D(U^p(\mathfrak{q}), \mathscr{F}_k)_{\mathfrak{m},E} \otimes_E W^\vee_{k,\mathfrak{p}} \otimes_E E(n) \cong \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_{\mathfrak{p}})^{alg}_{\mathfrak{m},E}$.
2. $\bigoplus_{k,n\in\mathbb{Z}} H^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathscr{F}_k)_{\mathfrak{m},E} \otimes_E W^\vee_{k,\mathfrak{p}} \otimes_E E(n) \cong \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathcal{O}_{\mathfrak{p}})^{alg}_{\mathfrak{m},E}$.
3. $\bigoplus_{k,n\in\mathbb{Z}} H^1_{\mathrm{red}}(U^p(\mathfrak{q}), r_*r^*\mathscr{F}_k)_{\mathfrak{m},E} \otimes_E W^\vee_{k,\mathfrak{p}} \otimes_E E(n) \cong \widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), r_*r^*\mathcal{O}_{\mathfrak{p}})^{alg}_{\mathfrak{m},E}$.
4. $\bigoplus_{k,n\in\mathbb{Z}} X_{\mathfrak{q}}(U^p(\mathfrak{q}), \mathscr{F}_k)_{\mathfrak{m},E} \otimes_E W^\vee_{k,\mathfrak{p}} \otimes_E E(n) \cong \widetilde{X}_{\mathfrak{q}}(U^p(\mathfrak{q}), \mathcal{O}_{\mathfrak{p}})^{alg}_{\mathfrak{m},E}$.
5. $\bigoplus_{k,n\in\mathbb{Z}} \check{X}_{\mathfrak{q}}(U^p(\mathfrak{q}), \mathscr{F}_k)_{\mathfrak{m},E} \otimes_E W^\vee_{k,\mathfrak{p}} \otimes_E E(n) \cong \widetilde{\check{X}}_{\mathfrak{q}}(U^p(\mathfrak{q}), \mathcal{O}_{\mathfrak{p}})^{alg}_{\mathfrak{m},E}$.

In all the above direct sums, the index k runs over d-tuples with all entries the same parity and ≥ 2.

Proof. The proof is as for Theorem 7.4.2 of [18], since

$$\{(\xi^{(k)})^\vee \otimes (\otimes_{i=1}^d \det \circ \xi_i)^n\}_{k,n}$$

is a complete set of isomorphism class representatives of irreducible algebraic representations of G which factor through G^c. □

Applying the same arguments to G', with $\mathfrak{m}'$ a non-Eisenstein maximal ideal of $\mathbb{T}'(V^p)$ we also obtain

Theorem 4.5.4. *There are natural $G'(\mathbb{Q}_p) \times G_{\mathfrak{q}} \times \mathbb{T}'(V^p)$-equivariant isomorphisms*

1. $\bigoplus_{k,n\in\mathbb{Z}} H^1_{D'}(V^p, \mathscr{F}_k)_{\mathfrak{m}',E} \otimes_E W^\vee_{k,\mathfrak{p}} \otimes_E E(n) \cong \widetilde{H}^1_{D'}(V^p, \mathcal{O}_{\mathfrak{p}})^{alg}_{\mathfrak{m}',E}$.
2. $\bigoplus_{k,n\in\mathbb{Z}} Y_{\mathfrak{q}}(V^p, \mathscr{F}_k)_{\mathfrak{m}',E} \otimes_E W^\vee_{k,\mathfrak{p}} \otimes_E E(n) \cong \widetilde{Y}_{\mathfrak{q}}(V^p, \mathcal{O}_{\mathfrak{p}})^{alg}_{\mathfrak{m}',E}$.
3. $\bigoplus_{k,n\in\mathbb{Z}} \check{Y}_{\mathfrak{q}}(V^p, \mathscr{F}_k)_{\mathfrak{m}',E} \otimes_E W^\vee_{k,\mathfrak{p}} \otimes_E E(n) \cong \widetilde{\check{Y}}_{\mathfrak{q}}(V^p, \mathcal{O}_{\mathfrak{p}})^{alg}_{\mathfrak{m}',E}$.

4.5.5 Cofreeness results

We now give analogues of Corollary 5.3.19 in [20].

Proposition 4.5.6. *Choose U_p small enough so that U_p is pro-p and $U_pU^p(\mathfrak{q})$ is neat. Then for each $s > 0$, $H^1_D(U^p(\mathfrak{q}), \mathcal{O}_{\mathfrak{p}}/\varpi^s)_{\mathfrak{m}}$, $H^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathcal{O}_{\mathfrak{p}}/\varpi^s)_{\mathfrak{m}}$ and $H^1_{\mathrm{red}}(U^p(\mathfrak{q}), r_*r^*\mathcal{O}_{\mathfrak{p}})_{\mathfrak{m}}$ are injective as smooth representations of $U_p/\overline{F^\times \cap U_pU^p}$ over $\mathcal{O}_{\mathfrak{p}}/\varpi^s$.*

In the statement of the proposition, we write $U_p/\overline{F^\times \cap U_pU^p}$ to denote the quotient of U_p by the closure in U_p of the projection of $F^\times \cap U_pU^p$ to its factor at the place p.

Proof. We follow the proof of Proposition 5.3.15 in [20], with modifications due to the presence of infinitely many global units $\mathcal{O}_F^\times$. Let L be any finitely generated smooth representation of $U_p/\overline{F^\times \cap U_pU^p} = U_p/\overline{F^\times \cap U_pU^p(\mathfrak{q})}$ over $\mathcal{O}_\mathfrak{p}/\varpi^s$, with Pontriagin dual $L^\vee := \mathrm{Hom}_{\mathcal{O}_\mathfrak{p}/\varpi^s}(L, \mathcal{O}_\mathfrak{p}/\varpi^s)$. There is an induced local system $\mathscr{L}^\vee$ on each of the Shimura curves $M(U_p'U^p(\mathfrak{q}))$ as U_p' varies over the open normal subgroups of U_p. As in Proposition 5.3.15 of [20] there is a natural isomorphism

$$H^1(M(U_pU^p(\mathfrak{q})), \mathscr{L}^\vee)_\mathfrak{m} \cong \mathrm{Hom}_{U_p/\overline{F^\times \cap U_pU^p}}(L, H^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p}/\varpi^s)_\mathfrak{m}).$$

Now starting from a short exact sequence $0 \to L_0 \to L_1 \to L_2 \to 0$ of finitely generated smooth $U_p/\overline{F^\times \cap U_pU^p}$-representations over $\mathcal{O}_\mathfrak{p}/\varpi^s$ we obtain a short exact sequence of sheaves on $M(U_pU^p(\mathfrak{q}))$:

$$0 \to \mathscr{L}_2^\vee \to \mathscr{L}_1^\vee \to \mathscr{L}_0^\vee \to 0.$$

Taking the associated long exact cohomology sequence and localising at $\mathfrak{m}$ gives another short exact sequence

$$0 \to H^1(M(U_pU^p(\mathfrak{q})), \mathscr{L}_2^\vee)_\mathfrak{m} \to H^1(M(U_pU^p(\mathfrak{q})), \mathscr{L}_1^\vee)_\mathfrak{m} \to H^1(M(U_pU^p(\mathfrak{q})), \mathscr{L}_0^\vee)_\mathfrak{m} \to 0,$$

or equivalently

$$0 \to \mathrm{Hom}_{U_p/\overline{F^\times \cap U_pU^p}}(L_2, H^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p}/\varpi^s)_\mathfrak{m}) \to \mathrm{Hom}_{U_p/\overline{F^\times \cap U_pU^p}}(L_1, H^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p}/\varpi^s)_\mathfrak{m}) \to$$
$$\to \mathrm{Hom}_{U_p/\overline{F^\times \cap U_pU^p}}(L_0, H^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p}/\varpi^s)_\mathfrak{m}) \to 0.$$

Therefore we conclude that $H^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p}/\varpi^s)_\mathfrak{m}$ is injective as a smooth representation of $U_p/\overline{F^\times \cap U_pU^p}$ over $\mathcal{O}_\mathfrak{p}/\varpi^s$. The same argument applies to the other cohomology spaces (using Lemma 4.4.22 for the final case). □

Definition 4.5.7. *We say that a representation V of a topological group Γ over $\mathcal{O}_\mathfrak{p}$ is* cofree *if $V \cong \mathscr{C}(\Gamma, \mathcal{O}_\mathfrak{p})^r$ for some integer r.*

Corollary 4.5.8. *If U_p is small enough so that U_p is pro-p and $U_pU^p(\mathfrak{q})$ is neat, then $\widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_\mathfrak{m}$, $\widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_\mathfrak{m}$ and $\widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}), r_*r^*\mathcal{O}_\mathfrak{p})_\mathfrak{m}$ are all cofree representations of $U_p/\overline{F^\times \cap U_pU^p}$.*

Proof. The proof of Corollary 5.3.19 in [20] goes through. □

Corollary 4.5.9. *If U_p is small enough so that U_p is pro-p and $U_pU^p(\mathfrak{q})$ is neat, then $\widetilde{X}_{\mathfrak{q}}(U^p(\mathfrak{q}),\mathcal{O}_{\mathfrak{p}})_{\mathfrak{m}}$ is a cofree representation of $U_p/\overline{F^\times\cap U_pU^p}$.*

Proof. Taking the first exact sequence of Proposition 4.4.17 and applying the functor $\mathrm{Hom}_{cts}(-,\mathcal{O}_{\mathfrak{p}})$ gives a short exact sequence (we can ignore the Tate twist for the purposes of this Corollary)

$$0\to \mathrm{Hom}_{cts}(\widetilde{X}_{\mathfrak{q}}(U^p(\mathfrak{q}),\mathcal{O}_{\mathfrak{p}})_{\mathfrak{m}},\mathcal{O}_{\mathfrak{p}})\to \mathrm{Hom}_{cts}(\widetilde{H}^1_D(U^p(\mathfrak{q}),\mathcal{O}_{\mathfrak{p}})_{\mathfrak{m}},\mathcal{O}_{\mathfrak{p}})$$
$$\to \mathrm{Hom}_{cts}(\widetilde{H}^1_{\mathrm{red}}(U^p(\mathfrak{q}),\mathcal{O}_{\mathfrak{p}})_{\mathfrak{m}},\mathcal{O}_{\mathfrak{p}})\to 0.$$

Corollary 4.5.8 implies that the second and third non-zero terms in this sequence are free $\mathcal{O}_{\mathfrak{p}}[[U_p/\overline{F^\times\cap U_pU^p}]]$-modules, so $\mathrm{Hom}_{cts}(\widetilde{X}_{\mathfrak{q}}(U^p(\mathfrak{q}),\mathcal{O}_{\mathfrak{p}})_{\mathfrak{m}},\mathcal{O}_{\mathfrak{p}})$ is projective, hence free since $\mathcal{O}_{\mathfrak{p}}[[U_p/\overline{F^\times\cap U_pU^p}]]$ is local. Applying the functor $\mathrm{Hom}_{cts}(-,\mathcal{O}_{\mathfrak{p}})$ again gives the desired result. □

Similarly we obtain

Corollary 4.5.10. *If V_p is small enough so that V_p is pro-p and V_pV^p is neat, then $\widetilde{H}^1_{D'}(V^p,\mathcal{O}_{\mathfrak{p}})_{\mathfrak{m}'}$, $\widetilde{Y}_{\mathfrak{q}}(V^p,\mathcal{O}_{\mathfrak{p}})_{\mathfrak{m}'}$ and $\widetilde{\widetilde{Y}}_{\mathfrak{q}}(V^p,\mathcal{O}_{\mathfrak{p}})_{\mathfrak{m}'}$ are cofree representations of $V_p/\overline{F^\times\cap V_pV^p}$.*

4.5.11 Jacquet-Langlands maps between eigenvarieties

We can now apply the results of section 4.4.30 to deduce some cases of *overconvergent* Jacquet-Langlands functoriality, or in other words maps between eigenvarieties interpolating the classical Jacquet-Langlands correspondence. Let $\widehat{T}$ denote the rigid analytic variety parametrising continuous characters $\chi: T\to\mathbb{C}_p^\times$.

Definition 4.5.12. *Suppose V is a ϖ-adically admissible $G(\mathbb{Q}_p)$-representation over $\mathcal{O}_{\mathfrak{p}}$, with a commuting $\mathcal{O}_{\mathfrak{p}}$-linear action of a commutative Noetherian $\mathcal{O}_{\mathfrak{p}}$-algebra $\mathbf{A}$. The essentially admissible locally analytic T-representation $J_B(V_E^{an})$ gives rise (by duality) to a coherent sheaf $\mathscr{M}$ on $\widehat{T}$, and the action of $\mathbf{A}$ gives rise to a coherent sheaf on $\widehat{T}$ of $\mathcal{O}_{\mathfrak{p}}$-algebras $\mathscr{A}\hookrightarrow End_{\widehat{T}}(\mathscr{M})$. Define the* Eigenvariety

$$\mathscr{E}(V,\mathbf{A})$$

to be the rigid analytic space given by taking the relative spectrum of $\mathscr{A}$ over $\widehat{T}$.

Lemma 4.5.13. *Suppose V and $\mathbf{A}$ are as in Definition 4.5.12, further suppose that V is a (non-zero) cofree representation of U_p/X for some compact open subgroup U_p of $G(\mathbb{Q}_p)$, X some closed subgroup of $Z(\mathbb{Q}_p) \cap U_p$. Then $\mathscr{E}(V, \mathbf{A})$ is equidimensional of dimension equal to that of U_p/X.*

Proof. This is a combination of a mild generalisation of the proof of Proposition 4.2.36 in [17] and Corollary 3.4.1 in Chapter 3 above. □

In our applications we will have $U_p/X = U_p/\overline{F^\times \cap U_p U^p}$ for some tame level U^p. The dimension of U_p/X will therefore depend on the Leopoldt conjecture for F. Applying the above lemma to the cofreeness results of the previous section, we have the following:

Corollary 4.5.14. *The following eigenvarieties are all equidimensional:*

- $\mathscr{E}(\widetilde{H}^1_D(U^p, \mathcal{O}_\mathfrak{p})_\mathfrak{m}, \mathbb{T}(U^p)_\mathfrak{m})$
- $\mathscr{E}(\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_\mathfrak{m}, \mathbb{T}(U^p(\mathfrak{q}))_\mathfrak{m})$
- $\mathscr{E}(\widetilde{H}^1_{D'}(V^p, \mathcal{O}_\mathfrak{p})_{\mathfrak{m}'}, \mathbb{T}'(V^p)_{\mathfrak{m}'})$
- $\mathscr{E}(\widetilde{Y}_\mathfrak{q}(V^p, \mathcal{O}_\mathfrak{p})_{\mathfrak{m}'}, \mathbb{T}'(V^p)_{\mathfrak{m}'})$
- $\mathscr{E}(\widetilde{\check{Y}}_\mathfrak{q}(V^p, \mathcal{O}_\mathfrak{p})_{\mathfrak{m}'}, \mathbb{T}'(V^p)_{\mathfrak{m}'})$

Lemma 4.5.15. *The embedding $\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p}) \to \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$ induced by composing the (monodromy pairing) embedding $\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p}) \hookrightarrow \widetilde{\check{X}}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$ with the embedding $\widetilde{\check{X}}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p}) \to \widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})$ provided by Proposition 4.4.17 induces an isomorphism of eigenvarieties*

$$\mathscr{E}(\widetilde{X}_\mathfrak{q}(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})_\mathfrak{m}, \mathbb{T}(U^p(\mathfrak{q}))_\mathfrak{m}) \cong \mathscr{E}(\widetilde{H}^1_D(U^p(\mathfrak{q}), \mathcal{O}_\mathfrak{p})^{\mathfrak{q}-\mathrm{new}}_\mathfrak{m}, \mathbb{T}(U^p(\mathfrak{q}))_\mathfrak{m}).$$

Proof. This follows from Proposition 4.4.25. □

We have a similar result for the group G'.

Lemma 4.5.16. *The embedding $\widetilde{Y}_\mathfrak{q}(V^p, \mathcal{O}_\mathfrak{p}) \to \widetilde{H}^1_{D'}(V^p, \mathcal{O}_\mathfrak{p})$induced by composing the (monodromy pairing) embedding $\widetilde{Y}_\mathfrak{q}(V^p, \mathcal{O}_\mathfrak{p}) \hookrightarrow \widetilde{\check{Y}}_\mathfrak{q}(V^p, \mathcal{O}_\mathfrak{p})$with the embedding $\widetilde{\check{Y}}_\mathfrak{q}(V^p, \mathcal{O}_\mathfrak{p}) \hookrightarrow \widetilde{H}^1_{D'}(V^p, \mathcal{O}_\mathfrak{p})$provided by Proposition 4.4.32 induces an isomorphism between eigenvarieties*

$$\mathscr{E}(\widetilde{Y}_\mathfrak{q}(V^p, \mathcal{O}_\mathfrak{p})_{\mathfrak{m}'}, \mathbb{T}'(V^p)_{\mathfrak{m}'}) \cong \mathscr{E}(\widetilde{H}^1_{D'}(V^p, \mathcal{O}_\mathfrak{p})_{\mathfrak{m}'}, \mathbb{T}'(V^p)_{\mathfrak{m}'}).$$

Proof. This follows from Proposition 4.4.33. □

Lemma 4.5.17. *The natural embedding* $\widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1-\mathrm{new}}_\mathfrak{m} \to \widetilde{H}^1_D(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1\mathfrak{q}_2-\mathrm{new}}_\mathfrak{m}$ *induces an isomorphism between eigenvarieties*

$$\mathscr{E}(\widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1-\mathrm{new}}_\mathfrak{m},\mathbb{T}(U^p(\mathfrak{q}_1\mathfrak{q}_2))_\mathfrak{m}) \cong \mathscr{E}(\widetilde{H}^1_D(U^p(\mathfrak{q}_1\mathfrak{q}_2),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1\mathfrak{q}_2-\mathrm{new}}_\mathfrak{m},\mathbb{T}(U^p(\mathfrak{q}_1\mathfrak{q}_2))_\mathfrak{m}).$$

Proof. This follows from Proposition 4.4.35. □

We now put ourselves in the situation of 4.4.30, so U^p and V^p are isomorphic at places away from $\mathfrak{q}_1$ and $\mathfrak{q}_2$, and $\mathfrak{m}$ and $\mathfrak{m}'$ give rise to the same irreducible mod p Galois representation $\overline{\rho}$.

Theorem 4.5.18. (Overconvergent Jacquet-Langlands correspondence) *The embedding*

$$\widetilde{Y}_{\mathfrak{q}_1}(V^p,\mathcal{O}_\mathfrak{p})_{\overline{\rho}} \to \widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2),\mathcal{O}_\mathfrak{p})_{\overline{\rho}}$$

given by Theorem 4.4.34 induces an isomorphism between eigenvarieties

$$\mathscr{E}(\widetilde{H}^1_{D'}(V^p,\mathcal{O}_\mathfrak{p})_{\overline{\rho}},\mathbb{T}'(V^p)_{\overline{\rho}}) \cong \mathscr{E}(\widetilde{H}^1_D(U^p(\mathfrak{q}_1\mathfrak{q}_2),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1\mathfrak{q}_2-\mathrm{new}}_{\overline{\rho}},\mathbb{T}(U^p(\mathfrak{q}_1\mathfrak{q}_2))_{\overline{\rho}}).$$

Proof. It is clear that Theorem 4.4.34 induces an isomorphism $\mathscr{E}(\widetilde{Y}_{\mathfrak{q}_1}(V^p,\mathcal{O}_\mathfrak{p})_{\overline{\rho}},\mathbb{T}'(V^p)_{\overline{\rho}}) \cong \mathscr{E}(\widetilde{X}_{\mathfrak{q}_2}(U^p(\mathfrak{q}_1\mathfrak{q}_2),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1-\mathrm{new}}_{\overline{\rho}},\mathbb{T}(U^p(\mathfrak{q}_1\mathfrak{q}_2))_{\overline{\rho}})$. Our theorem now follows by applying Lemmas 4.5.16 and 4.5.17. □

Finally we note that level raising results in the same spirit as those in the first two chapters follow from the equidimensionality of the eigenvarieties described in this section. In particular we have

Corollary 4.5.19. *The following eigenvarieties are equidimensional:*

- $\mathscr{E}(\widetilde{H}^1_D(U^p(\mathfrak{q}),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}-\mathrm{new}}_{\overline{\rho}},\mathbb{T}(U^p(\mathfrak{q}))_{\overline{\rho}})$
- $\mathscr{E}(\widetilde{H}^1_D(U^p(\mathfrak{q}_1\mathfrak{q}_2),\mathcal{O}_\mathfrak{p})^{\mathfrak{q}_1\mathfrak{q}_2-\mathrm{new}}_{\overline{\rho}},\mathbb{T}(U^p(\mathfrak{q}_1\mathfrak{q}_2))_{\overline{\rho}})$

Proof. The first claim follows from the second part of Corollary 4.5.14 and Lemma 4.5.15. The second claim follows from the fourth part of Corollary 4.5.14 and Theorem 4.5.18. □

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