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Time–frequency domain local spectral analysis of seismic signals with multiple windows

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For a non-stationary seismic signal, time–frequency analysis methods often include a time window function that serves as a weighting function and by which the signal is multiplied to form a segment. The time window function often has the highest weighting coefficient for the central sample of the signal segment. For the rest of the segment, there is no adequate representation in the frequency spectrum. Here, I propose to use multiple orthogonal window functions to properly represent the local spectral property in the time–frequency plane and recover the information lost due to time windowing before applying the Fourier transform. First, I propose to construct multiple window functions directly using a stack of Gaussian functions. The weighted average spectrum of the multiple window functions has a flat passband, which is better than the conventional multiple windows. Taking advantage of the linearity of the Fourier transform, we can apply each window to the analytic signal to generate the instantaneous autocorrelation accordingly and form an averaged instantaneous autocorrelation by a weighted sum before performing the Fourier transform to generate the Wigner–Ville distribution (WVD). This multi-window WVD method successfully represents the local spectrum of the non-stationary seismic signal in the time–frequency plane.

1. Introduction

In the time–frequency representation of a non-stationary seismic signal, the frequency spectrum at a particular time position is a local property, but not exactly

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an instantaneous property [1,2]. The reason for this is that the spectral analysis at that time has to be performed over a segment of the signal and not over an isolated data sample. A segment of a seismic signal always has a certain time duration of oscillation, no matter how short the time duration is.

Time–frequency analysis methods often include a time window function that acts as a weighting function and by which the signal is multiplied to form a segment. For example, the Gabor transform-type methods, such as the S-transform [3] and the W-transform [4], which evolved from the short-time Fourier transform, make various modifications to the time window function. The continuous wavelet transform modifies the complex Gabor wavelet defined with a Gaussian function [5–7]. The Wigner–Ville distribution (WVD) also applies a time window function to the original signal to smooth out the cross-term interference between the different signal components [8–15].

The window function serves as a weighting function and the signal is multiplied by it to form a segment before the Fourier transform is applied. This multiplication in the time domain is equivalent to convolution in the frequency domain, and therefore the application of the windowing technique determines the resolution of the time–frequency spectrum. The Gabor limit [16] states that the product of time resolution and frequency resolution is at least $(4\pi)^{-1}$, which is 80 millicycles (or ms Hz). This Gabor limit is analogous to Heisenberg’s uncertainty principle, originally expressed in terms of observation and measurement [17], and characterized by a scale for the quantum dimensions, called Planck’s constant. However, the physical meaning of the Gabor limit in signal analysis is different. It is not that both time and frequency (as measurements) cannot be made arbitrarily narrow, but that time density and frequency density cannot be made narrow simultaneously [18]. For a signal $x(t)$ with a frequency spectrum $\hat{x}(f)$, the energy densities in time and frequency are $|x(t)|^2$ and $|\hat{x}(f)|^2$, respectively. The measurements for time and frequency in the signal analysis can be considered as windows on the time and frequency axis.

The window function is applied slidngly to the non-stationary signal along the time axis. The windowed signal segment is a local kernel for the Fourier transform. The time–frequency spectrum is composed of the Fourier spectra generated at different time positions, where the time position corresponds to the centre of each segment. When forming the signal segment, the time window function often has the highest weighting coefficient for the central sample of the segment. The rest of the segment is not properly represented in the frequency spectrum. Therefore, the time–frequency spectrum has a loss of information in the signal segment due to the application of a window function.

While we acknowledge that the time–frequency spectrum is a local property and not an instantaneous property at a particular time position, we will also admit that the time–frequency spectrum does not accurately represent the local property of the non-stationary seismic signal. Therefore, I propose to use multiple orthogonal window functions to recover the information lost by time windowing before the Fourier transform and to properly represent the local spectral property in the time–frequency plane.

2. Review on multiple orthogonal windows

(a) Basics of multiple orthogonal windows

The information that is lost in the conventional time–frequency spectrum can be recovered by using a multiple window scheme. In one-dimensional Fourier transform theory, this multiple window scheme is called the multi-taper method [19].

Given a complex harmonic component

$$x(t) = \hat{x}(f_0) e^{i2\pi f_0 t}, \quad (2.1)$$

where f_0 is a constant frequency, multiplication by a window function $w(t)$ produces a new signal

$$y(t) = w(t) x(t). \quad (2.2)$$

The Fourier transform of the windowed harmonic component is

$$\hat{y}(f) = \int_{-T}^T w(t) \hat{x}(f_0) e^{-i2\pi(f-f_0)t} dt, \quad (2.3)$$

where T is the (half-) length of the window function.

The window function $w(t)$ is not an arbitrary window, but an optimal window that maximizes the energy within the frequency band between $[f_0 - B]$ and $[f_0 + B]$, where B is the (half-)width of the frequency band and thus determines the spectral resolution. The energy within the frequency band is normalized to the energy in the entire frequency range

$$\lambda = \frac{\int_{f_0-B}^{f_0+B} |\hat{y}(f)|^2 df}{\int_{-\infty}^{\infty} |\hat{y}(f)|^2 df}. \quad (2.4)$$

This is the bandwidth retention factor, which is always less than one [20]. The optimal window function that is sought has a λ value close to one.

The numerator of the ratio factor can be calculated as follows:

$$\begin{aligned} \int_{f_0-B}^{f_0+B} |\hat{y}(f)|^2 df &= \int_{f_0-B}^{f_0+B} \left(\int_{-T}^T w(t) \hat{x}(f_0) e^{-i2\pi(f-f_0)t} dt \right)^* \left(\int_{-T}^T w(\tau) \hat{x}(f_0) e^{-i2\pi(f-f_0)\tau} d\tau \right) df \\ &= |\hat{x}(f_0)|^2 \int_{-T}^T w^*(t) dt \int_{-T}^T w(\tau) d\tau \int_{f_0-B}^{f_0+B} e^{i2\pi(f-f_0)(t-\tau)} df \\ &= |\hat{x}(f_0)|^2 \int_{-T}^T w^*(t) dt \int_{-T}^T w(\tau) d\tau \int_{-B}^B e^{i2\pi f(t-\tau)} df \\ &= |\hat{x}(f_0)|^2 \int_{-T}^T w^*(t) dt \int_{-T}^T b(t-\tau) w(\tau) d\tau, \end{aligned} \quad (2.5)$$

where $b(t-\tau)$ is defined in terms of a sinc function

$$b(t-\tau) = \int_{-B}^B e^{i2\pi f(t-\tau)} df = 2B \left(\frac{\sin[2\pi B(t-\tau)]}{2\pi B(t-\tau)} \right). \quad (2.6)$$

The denominator of the ratio factor can be calculated according to Parseval's theorem

$$\begin{aligned} \int_{-\infty}^{\infty} |\hat{y}(f)|^2 df &= \int_{-\infty}^{\infty} |y(t)|^2 dt = |x(t)|^2 \int_{-T}^T w^*(t) w(t) dt \\ &= |\hat{x}(f_0)|^2 \int_{-T}^T w^*(t) w(t) dt. \end{aligned} \quad (2.7)$$

Then the bandwidth retention factor λ can be expressed as follows:

$$\lambda = \frac{\int_{-T}^T w^*(t) \left[\int_{-T}^T b(t-\tau) w(\tau) d\tau \right] dt}{\int_{-T}^T w^*(t) w(t) dt}. \quad (2.8)$$

Maximizing λ is equivalent to minimizing $1-\lambda$, a quantity of spectral leakage. Therefore, equation (2.8) is an optimization problem.

Equation (2.8) can be simplified to the Fredholm integral equation [21–23] as follows:

$$\int_{-T}^T b(t - \tau)w(\tau) d\tau = \lambda w(t). \quad (2.9)$$

The Fredholm integral equation recasts the optimization problem as an eigenfunction problem, where the eigenvalue λ is the bandwidth retention factor and the corresponding eigenfunction $w(t)$ is the window function.

(b) Multiple windows defined by the prolate spheroidal wave functions

The Fredholm integral equation (2.9) defines the prolate spheroidal wave functions [24–27] that satisfy the differential equation:

$$\frac{d}{dz} \left((1 - z^2) \frac{d}{dz} U_{m,n}(c, z) \right) + \left(\chi_{m,n}(c) - c^2 z^2 - \frac{m^2}{1 - z^2} \right) U_{m,n}(c, z) = 0, \quad (2.10)$$

where z is a scaled variable within the closed interval $z \in [-1, 1]$, c is the spheroidal wave parameter, $U_{m,n}(c, z)$ are the prolate spheroidal wave functions, $\chi_{m,n}(c)$ are the eigenvalues, n and m are two integer indices and $0 \leq m \leq n$. In this differential equation, both $U_{m,n}(c, z)$ and $\chi_{m,n}(c)$ depend on the spheroidal wave parameter c , which is the product of the time limit T and the frequency band B [28].

Equation (2.10) is a scalar Helmholtz equation and is transferred from rectangular coordinates to general ellipsoidal coordinates [29]. However, the eigenvalues $\chi_{m,n}(c)$ here for the prolate spheroidal wave functions $U_{m,n}(c, z)$ are not identical to the bandwidth retention factor λ defined in equation (2.8).

The prolate spheroidal wave functions $U_{m,n}(c, z)$, which are the solutions of equation (2.10), define the window functions we are looking for:

$$w_{p=n-m} \left(c, \frac{t}{T} \right) = \frac{1}{r_{m,n}(c)} U_{m,n} \left(c, \frac{t}{T} \right), \quad (2.11)$$

where $r_{m,n}(c)$ is a normalization factor,

$$r_{m,n}(c) = \left(\int_{-T}^T \left[U_{m,n} \left(c, \frac{t}{T} \right) \right]^2 dt \right)^{1/2}. \quad (2.12)$$

The prolate spheroidal wavefunctions $U_{m,n}(c, z)$ are the associated Legendre polynomials [25], which in general are not orthogonal to each other. For example, $U_{1,1}(c, z)$ is not orthogonal to $U_{2,2}(c, z)$. However, they satisfy the orthogonality condition either for a fixed m or for a fixed n , assuming that $0 \leq m \leq n$.

When these orthogonal functions form an orthogonal basis, the seismic signal can be expressed as a linear combination of these independent window functions. Figure 1a shows five window functions $w_p(c, z)$, with $z = t/T$, $m = 0$ and $p = n - m = \{0, 1, \dots, 4\}$. The red curves correspond to the even number $p = \{0, 2, 4\}$, and the blue curves correspond to the odd number $p = \{1, 3\}$. Figure 1a shows that the number p is the number of zero crossings in the closed interval $z \in [-1, 1]$.

Figure 1b shows that the window functions $w_p(c, z)$ formed from the prolate spheroidal wave functions with different m values are similar. The red solid curves are the window functions formed with $m = 0$, and the blue dashed curves are the window functions formed with $m = 3$.

Figure 1 also shows that window $w_p(c, z)$ is an even function when $p = n - m$ is an even number, and window $w_p(c, z)$ is an odd function when $p = n - m$ is an odd number.

The spectral estimate based on $w_{p=0}$ depends mainly on the central part of the time signal, as is the case with conventional window functions. However, the spectral estimates based on higher order window functions $w_{p \geq 1}$ depend more on the energy at the beginning and end of the signal segment. Therefore, the combination of the spectral estimates produced with several window functions can largely recover the local spectrum of the signal.

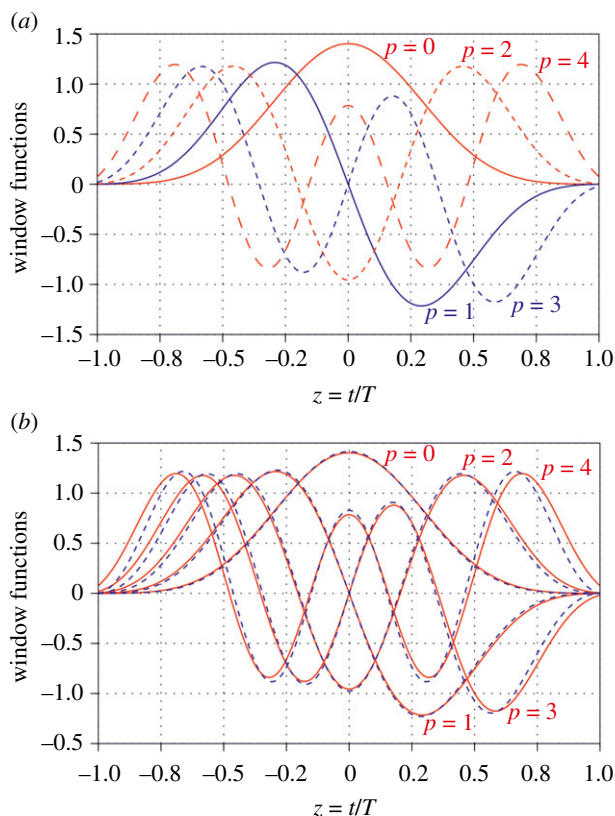


Figure 1. Multiple windows defined by the prolate spheroidal wave functions. (a) Five orthogonal window functions $w_p(c, z)$, built in terms of the prolate spheroidal wave functions $U_{m,n}(c, z)$, where $m = 0$ and $p = n - m$. The red and blue curves correspond to the even number of zero crossings $p = \{0, 2, 4\}$ and the odd number of zero crossings $p = 1, 3$. (b) Comparison of the window functions $w_p(c, z)$ built by the prolate spheroidal wave functions $U_{m,n}(c, z)$ with $m = 0$ (solid curves) and $m = 3$ (dashed curves).

(c) Multiple windows constructed by solving a discretized eigenvalue problem

Alternatively, the Fredholm integral equation (2.9) can be discretized and represented in a matrix-vector form as

$$\mathbf{B}\mathbf{w} = \lambda\mathbf{w}, \quad (2.13)$$

where $\mathbf{B}$ is the matrix with the elements defined by the sinc function $b(t - \tau)$, and $\mathbf{w}$ is the vector of a window function. The Fredholm integral equation becomes a discretized eigenvalue problem, where λ is the eigenvalue and $\mathbf{w}$ is the corresponding eigenvector.

When we solve the discretized eigenvalue problem of equation (2.13), we obtain the maximum λ values, and look for the corresponding eigenvectors $\mathbf{w}$ that attain these maximum values.

Since $\mathbf{B}$ is a positive-definite Toeplitz matrix, all eigenvalues are real and positive, and they can be listed in descending order:

$$\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \dots \quad (2.14)$$

The larger eigenvalues indicate that the corresponding eigenvectors $\mathbf{w}_p \equiv w_p(t)$ are exactly the optimal window functions we were looking for. All window functions are band-limited, and the (half-)bandwidth is B .

In constructing the Toeplitz matrix $\mathbf{B}$, we use a sinc function $b(t - \tau)$ truncated at $|2\pi B(t - \tau)| = 4\pi$, which corresponds to $c = \pi BT \leq 2\pi$ or $BT \leq 2$. Figure 2a shows the eigenvalues

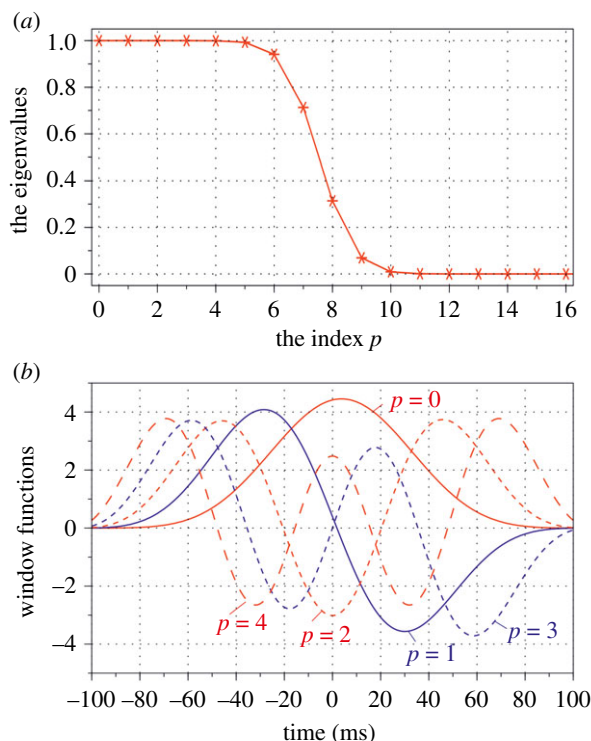


Figure 2. Multiple windows constructed by solving a discretized eigenvalue problem. (a) The eigenvalues of the Toeplitz matrix $\mathbf{B}$. The eigenvalues λ_p , where p is the index, are normalized and listed in descending order. (b) Five optimal window functions, formed by the five eigenvectors corresponding to the five highest eigenvalues.

obtained. These eigenvalues are normalized, and the five largest eigenvalues are

$$\lambda_p = \{1.0, 1.0, 0.999999, 0.999971, 0.999465\}. \quad (2.15)$$

Figure 2b shows the corresponding window functions. These five window functions are mutually orthogonal,

$$\phi_{m,n} = \int_{-T}^T w_m(t)w_n(t) dt \approx 0, \quad \text{for } m \neq n, \quad (2.16)$$

because they are the eigenvectors of the Toeplitz matrix $\mathbf{B}$.

The window function $\mathbf{w}_0 \equiv w_0(t)$, corresponding to the maximum eigenvalue λ_0 , has the least spectral leakage. The window function $\mathbf{w}_p \equiv w_p(t)$ corresponding to the eigenvalues $\lambda_p \approx 1$, for $p \leq 4$, has only a slightly higher spectral leakage. The energy of a window function was scaled to unity.

The zero-order window function $\mathbf{w}_0 \equiv w_0(t)$ resembles, but is not identical to, the truncated Gaussian window. It is almost symmetrical around the centre, peaks there and tapers to zero at the ends [30,31].

Higher order window functions are symmetric: those with even p values are symmetric about the centre, while those with odd p are antisymmetric. The window function with $p=1$ contains the energy concentrated in the early part of the signal segment, and the window functions with higher p contain the energy concentrated even earlier.

Figure 3 compares the window functions defined by the prolate spheroidal wave functions (solid red curves) with those constructed by the eigenfunctions of the Toeplitz matrix $\mathbf{B}$ (dashed blue curves). The direct comparison shows that, for low orders $p = \{0, 1\}$, the window functions $w_p(t)$ constructed by the eigenfunctions of the Toeplitz matrix $\mathbf{B}$ are skewed, while

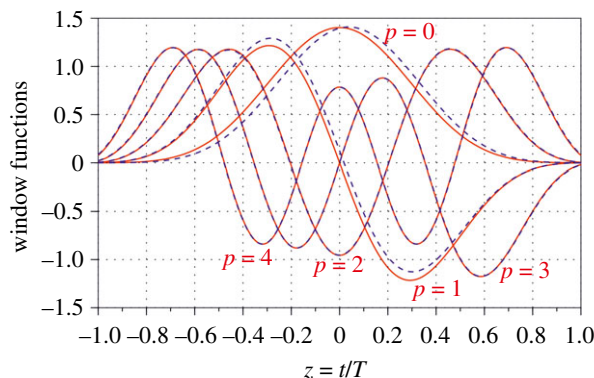


Figure 3. Comparison between the window functions defined in terms of the prolate spheroidal wave functions (solid red curves) and those constructed by the eigenfunctions of the Toeplitz matrix $\mathbf{B}$ (dashed blue curves).

the window functions defined by the prolate spheroidal wave functions are either symmetric or antisymmetric.

To allow direct comparison, we have imposed a scaling and a polarity convention in figure 3. For the scaling, the energy of each window function is set to unity,

$$\int_{-T}^T |w_p(t)|^2 dt = 1. \quad (2.17)$$

For an even-order window function, the average is set positive:

$$\int_{-T}^T w_p(t) dt > 0. \quad (2.18)$$

For an odd-order window function, the first lobe is set positive:

$$\int_{-T}^T (T - t) w_p(t) dt > 0. \quad (2.19)$$

Figure 4 shows the amplitude spectra of the window functions $|\hat{w}_p(f)|$, $p = \{0, 1, \dots, 4\}$. The smoothed amplitude spectrum is obtained by a weighted average of these power spectra

$$|\hat{w}(f)| = \left(\frac{\sum_{p=0}^P \lambda_p |\hat{w}_p(f)|^2}{\sum_{p=0}^P \lambda_p} \right)^{1/2}, \quad (2.20)$$

where $P=4$ in this example, and each eigenvalue is approximately equal to 1. The smoothed spectrum has a maximum flat spectrum within the frequency band B .

3. Multiple windows constructed by Gaussian functions

Theoretically, both methods, either the prolate spheroidal wave functions, which are the solutions of a Helmholtz equation, or the solution of the discretized eigenvalue problem, can yield a complete set of orthogonal window functions. However, if we use only five window functions, corresponding to the five largest λ values, the passband of the average spectrum is not flat enough, as can be seen in figure 4b. We need to include more window functions in the weighted average, to improve the spectral flatness within the frequency passband.

Furthermore, solving the Helmholtz equation for the prolate spheroidal wave functions, or solving the discretized eigenvalue problem for the eigenvectors, is not the efficient method we need in practice, as we often need on-the-fly calculation of the windows in signal analysis.

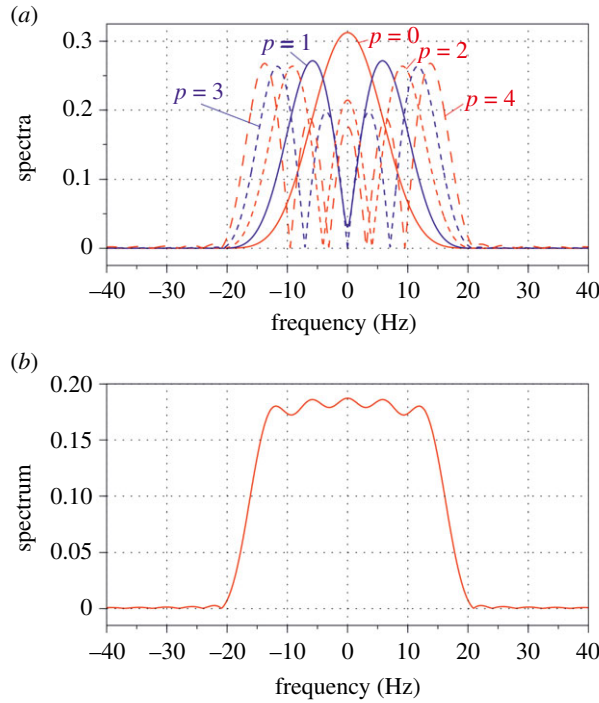


Figure 4. The amplitude spectra of multiple windows, constructed by the discretized eigenvalue problem. (a) The amplitude spectra of the five optimal window functions. The order of the eigenvalues is denoted as p . (b) The smoothed amplitude spectrum obtained by a weighted average of the five power spectra.

Therefore, instead of the sophisticated numerical calculations presented in the previous section, I propose to construct multiple window functions directly.

I construct the multiple window functions using a stack of Gaussian functions:

$$w_p(t) = \frac{1}{r_p} \sum_{m=0}^p (-1)^m e^{-((2+p)t/T) - [2(2m-p)/(BT)]^2 / (2\sigma^2)}, \quad (3.1)$$

where p is the order of the window function, T is the (half-) window time, B is the (half-)width of the frequency band, σ is the standard deviation of a Gaussian function and r_p is a normalization factor:

$$r_p = \left[\frac{\sqrt{\pi}\sigma T}{2+p} \left(1 + p + 2 \sum_{m=0}^{p-1} \sum_{n=m+1}^p (-1)^{m+n} e^{-4(m-n)^2 / (\sigma^2 B^2 T^2)} \right) \right]^{1/2}. \quad (3.2)$$

The normalization factor makes the energy of the window function $w_p(t)$ to be unity. Therefore, the energy of the signal segment is not affected when the segment is multiplied by a window function.

Here, I set the maximum order of the window function to $P=4$. For the case of $BT=2$, the standard deviation is set to be $\sigma=0.72$, and the window functions $w_p(t)$ defined within the time interval $[-T, T]$ produce a spectrum with a flat passband. Figure 5a shows five windows formed by a stack of Gaussian functions, with the (half-) bandwidth $B=20$ Hz, and $(BT, \sigma)=(2, 0.72)$. The values of the bandwidth retention factor of these five windows are close to the desirable value $\lambda=1$:

$$\lambda_p = \{0.999029, 0.999446, 0.999622, 1.000070, 1.000200\}. \quad (3.3)$$

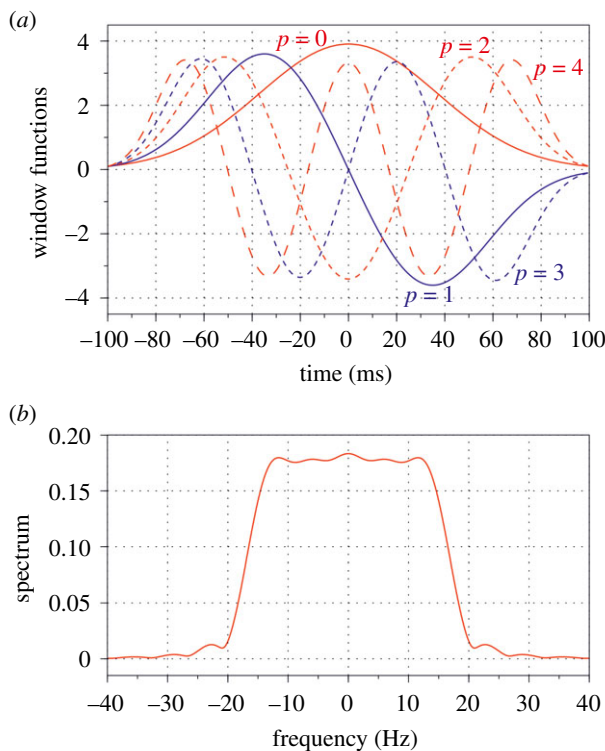


Figure 5. Multiple windows constructed with Gaussian functions, with the (half-)bandwidth $B = 20$ Hz. (a) Five windows formed by a stack of Gaussian functions, $p = \{0, 1, \dots, 4\}$. (b) The weighted average spectrum of these five window functions. Averaging the spectra produced by this set of window functions produces a spectrum with an excellently flat passband.

This shows that the factor λ does not necessarily decrease as the order of the window function increases. Therefore, the assumption contained in equation (2.15), that the window functions were formed by solving the eigenvalue problem, no longer applies.

Figure 5b shows the integrated spectrum, which is the weighted average of the spectra of these five windows. The figure clearly shows that this set of window functions can produce an average spectrum with a flat passband, as opposed to the average spectrum integrated by five eigenfunctions, as shown in figure 4b.

However, these windows formed by a stack of Gaussian functions are not strictly orthogonal to each other. The cross-correlations between any pair of windows, calculated by equation (2.16), are listed as follows:

$$\begin{bmatrix} \phi_{0,1} & \phi_{0,2} & \phi_{0,3} & \phi_{0,4} \\ & \phi_{1,2} & \phi_{1,3} & \phi_{1,4} \\ & & \phi_{2,3} & \phi_{2,4} \\ & & & \phi_{3,4} \end{bmatrix} = \begin{bmatrix} 0 & -0.043 & 0 & 0.027 \\ & 0 & -0.080 & 0 \\ & & 0 & -0.085 \\ & & & 0 \end{bmatrix}. \quad (3.4)$$

The cross-correlations $\phi_{m,n}$ indicate that these window functions are not completely orthogonal but are approximations.

Figure 6 shows another set of window functions formed by a stack of Gaussian functions, with the (half-)bandwidth of $B = 40$ Hz and $(BT, \sigma) = (2, 0.72)$. The weighted average spectrum of these five window functions also has a flat central passband, although it has small side lobes at both edges of the passband.

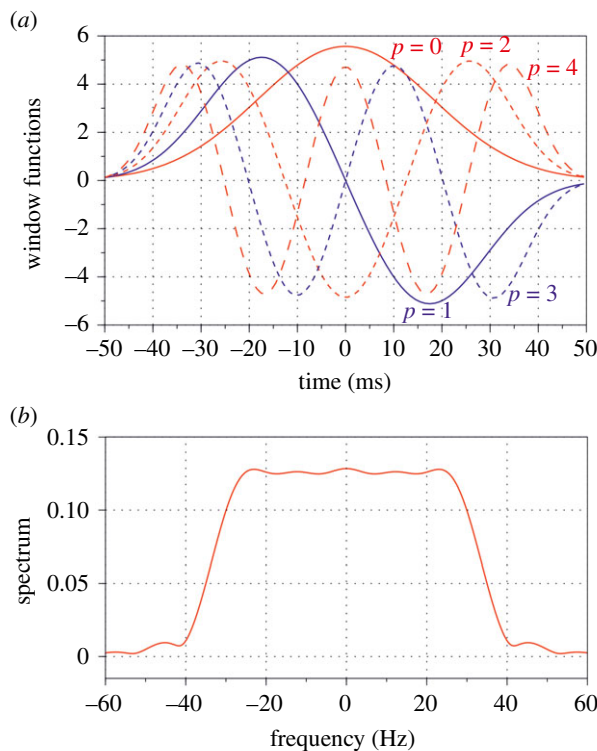


Figure 6. Multiple windows constructed by Gaussian functions, with the (half-)bandwidth $B = 40$ Hz. (a) Five windows formed by a group of Gaussian functions. (b) The weighted average spectrum of the five window functions.

While the aggregate sum of several windows gives a flat passband, the energy level of the side lobes of the spectrum is also an important property of a window function. Large sidelobes can potentially mask some of the weak events [32]. Theoretically, for a Gaussian function with infinite duration, there are no sidelobes in the frequency spectrum. Due to the finite duration of either the window function or the signal segment itself, the truncation effect leads to sidelobes in the spectrum. However, the largest sidelobe of a truncated Gaussian function is much smaller than for a rectangular window with or without tapering.

In the definition of equation (3.1) for the multiple window functions, the parameter σ can be determined by local optimization

$$J(\sigma) = \min \int_{-3B/4}^{3B/4} |\hat{w}(f, \sigma) - \tilde{w}(f, \sigma)|^2 df, \quad (3.5)$$

where $\tilde{w}(f, \sigma)$ is the local average over a small interval, around the frequency f , of the aggregate spectrum $\hat{w}(f, \sigma)$.

4. The Gabor transform with multiple windows

The Gabor transform-type method can be considered as a cross-correlation between the signal and various complex wavelets. In the classic Gabor transform, the complex wavelet, also called the Gabor wavelet, is the product of the kernel of the Fourier transform and a window function. The window function in the Gabor wavelet is defined as a Gaussian function [16,33,34], which is translated over the non-stationary signal. In the same way, a scaled complex wavelet in the continuous wavelet transform is translated across a signal. Due to the application of the window function in the formation of the complex wavelet, there is a loss of information in the frequency

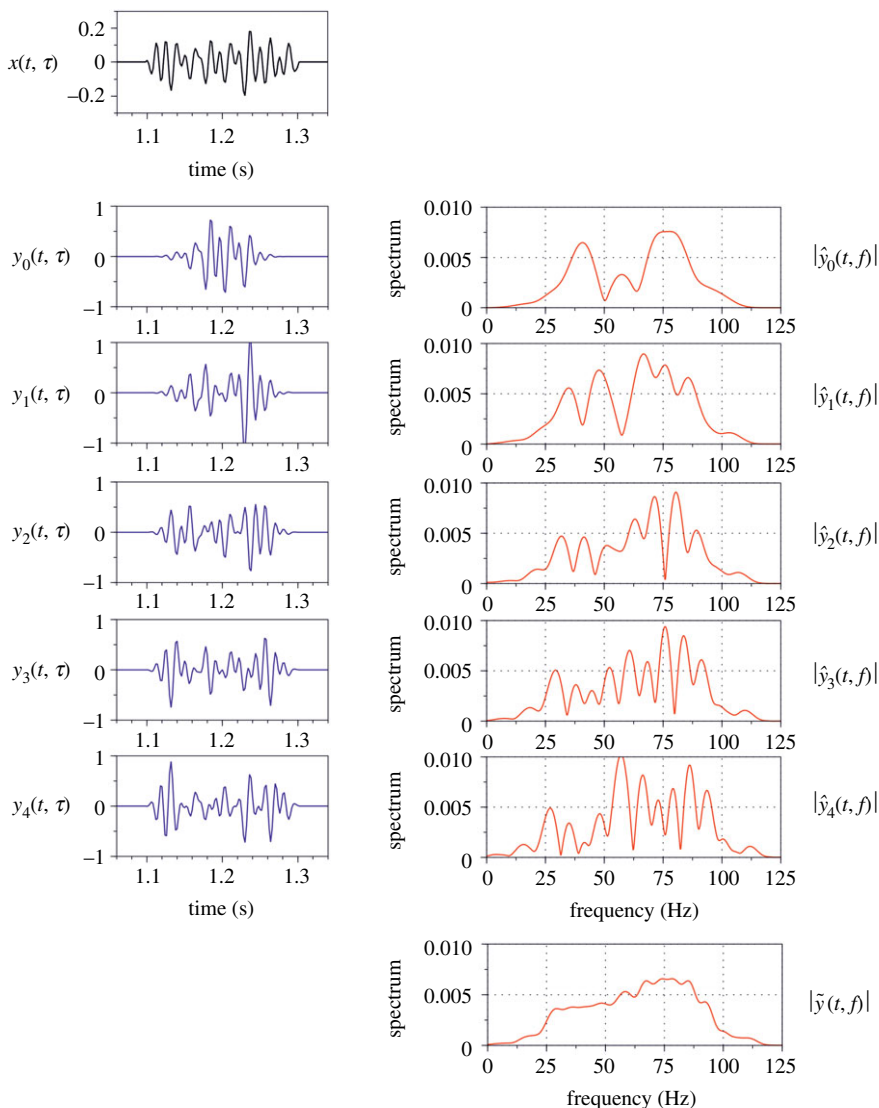


Figure 7. Spectral analysis with multiple windows, at a given time t . The original signal segment is $x(t, \tau)$ (the black signal on the top) with the reference time at $t = 1.2$ s and the (half) length of 0.1 s. The segments formed by five window functions are $y_p(t, \tau)$ (the blue signals). The average of five time–frequency spectra (in red) results in a smooth amplitude spectrum $|\tilde{y}(t, \tau)|$.

spectrum at any given time position. If we use the multiple windows, we can retain the spectral information within the frequency band.

Figure 7 illustrates the implementation of the Gabor transform with multiple orthogonal windows, at a specific point in time t . The original signal segment (the black curve in the upper-left corner) is $x(t, \tau)$ with the reference time at $t = 1.2$ s. The (half-) length of the duration of time τ is $T = 0.1$ s. A weighted segment (the blue curve) with any one of five orthogonal window functions is $y_p(t, \tau)$, where $p = \{0, 1, \dots, P\}$, and $P = 4$ is the highest order.

Recall that the bandwidth retention factor λ is defined in terms of the power spectrum. Therefore, different λ values should be applied as weighting coefficients to the corresponding power spectra $|\hat{y}_p(t, f)|^2$. The final amplitude spectrum $|\tilde{y}(t, f)|$ is a weighted average of these

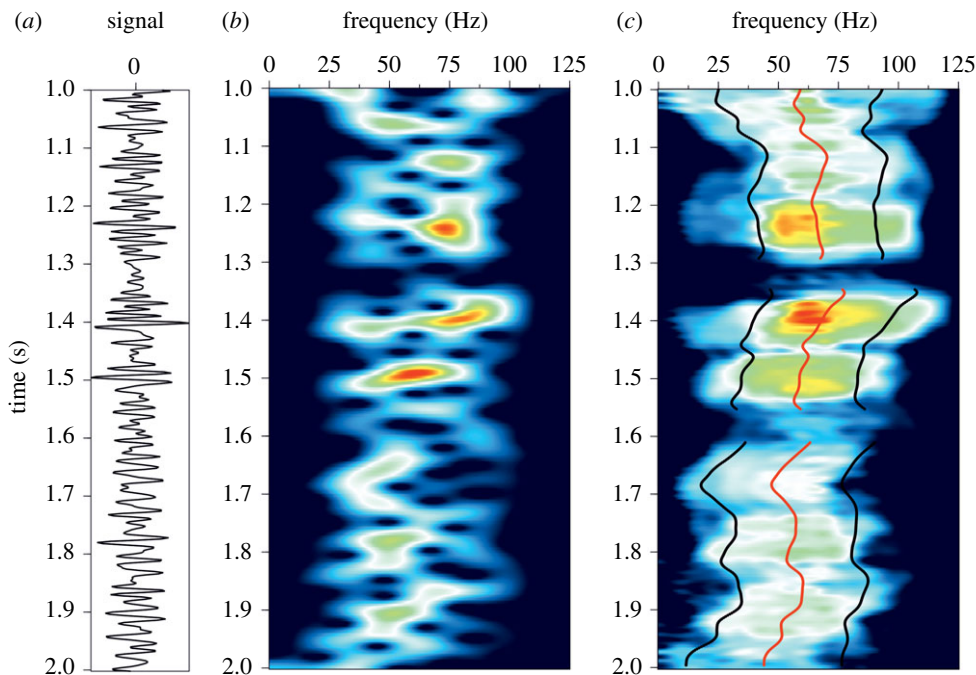


Figure 8. The Gabor transform using either a single or multiple window functions. (a) A field seismic trace. (b) Time–frequency spectrum produced by the Gabor transform using the Blackman–Harris window. (c) Local amplitude spectrum generated by the multi-window Gabor transform. The red curve is the mean frequency $f_m(t)$ and the two black curves are $f_m(t) \pm f_\sigma(t)$, respectively, where $f_\sigma(t)$ is the standard deviation.

power spectra:

$$|\tilde{y}(t, f)| = \left(\frac{\sum_{p=0}^P \lambda_p |\hat{y}_p(t, f)|^2}{\sum_{p=0}^P \lambda_p} \right)^{1/2}. \quad (4.1)$$

The final amplitude spectrum shown in figure 7 is a smooth spectrum with less spectral leakage than a conventional single window scheme.

Figure 8 compares the time–frequency spectra produced by the Gabor transform when either a single window function or multiple window functions are used. In the calculation, the same length $T = 50$ ms is used for each window function. If the Gabor transform uses only a single window function (figure 8b), the window does not properly represent the contribution of the signal segment to both ends of the time duration, and the time–frequency spectrum does not represent all the information within the signal segment. When the Gabor transform uses multiple windows (figure 8c), the information that is lost when implemented with only the single window ($p = 0$) is restored by the other window functions ($p \geq 1$).

The Gabor transform is a typical linear transformation method. One of the advantages of the linear transformation is that the time–frequency spectrum can be used to reconstruct the original signal. However, with the multi-window Gabor transform, the resultant time–frequency spectrum is a local amplitude spectrum, $|\hat{y}(t, f)|$, based on which the original signal cannot be reconstructed. Instead, the smooth time–frequency spectrum shows a clear picture of the frequency band, varying along the time axis. Based on the time–frequency spectrum, we can statistically evaluate the mean frequency and the standard deviation.

The mean frequency function $f_m(t)$ is evaluated by

$$f_m(t) = \frac{1}{E(t)} \int_0^\infty df \int_{-T}^T f |\hat{y}(t - \tau, f)|^2 L(\tau) d\tau, \quad (4.2)$$

and the standard deviation function $f_\sigma(t)$ is evaluated by

$$f_\sigma(t) = \left(\frac{1}{E(t)} \int_0^\infty df \int_{-T}^T (f - f_m)^2 |\hat{y}(t - \tau, f)|^2 L(\tau) d\tau \right)^{1/2}, \quad (4.3)$$

where $|\hat{y}(t, f)|^2$ is the power spectrum at time t , $E(t)$ is the instantaneous energy,

$$E(t) = \int_0^\infty df \int_{-T}^T |\hat{y}(t - \tau, f)|^2 L(\tau) d\tau, \quad (4.4)$$

and $L(t)$ is a low-pass filter that is applied to the time axis. For the statistical evaluations in equations (4.2) and (4.3), a pragmatic condition is set as $E(t) \geq E_\ell$, where E_ℓ is a threshold for the energy.

Figure 8c also shows a stable estimate for the mean frequency and standard deviation from the time–frequency spectrum. The red curve is the mean frequency function $f_m(t)$, and the two black curves are $f_m(t) \pm f_\sigma(t)$, respectively. According to the definition of equation (2.4), the bandwidth retention factor λ is maximized at any time position within the band $[f_0 - B, f_0 + B]$. Therefore, the evaluated standard deviation function $f_\sigma(t)$ has only a small variation with respect to time t .

Consider the multiplication of the seismic trace by window functions in the time domain as a convolution in the frequency domain. The weighted average spectrum of the multiple window functions has a wide passband, all sharp local spectra of the signal components are spread. However, the frequency resolution is not necessarily changed by multiple windows because the sampling resolution is limited by $\Delta f = (2T)^{-1}$, where T is half the length of the window [18]. This means that the frequency resolution of the spectrum at a given frequency f cannot be less than Δf . The non-smooth spectrum on the frequency axis does not mean that the resolution on the frequency axis is high.

Moreover, the spectral feature is also not localized on the time axis. The aggregate sum of multiple windows effectively has a longer observation time than a single window, and the spectrum could mix different signal components in time. Therefore, the time resolution in figure 8c appears to be lower than in figure 8b, which uses a single window and highlights the spectral feature at the central position of a signal segment.

5. The Wigner–Ville distribution with multiple windows

The WVD method analyses the instantaneous autocorrelation of the signal and generates the energy density distribution in the time–frequency plane [35,36]. Classical WVD suffers from the cross-term interference between the signal components. Since this cross-term interference has both positive and negative amplitudes, it can be suppressed by two-dimensional smoothing in the time–frequency plane.

In two-dimensional smoothing, the frequency-axis smoothing filter $\hat{h}(f)$ can be represented in the time domain as $h(\tau)$, which is equivalent to the WVD of a window function $k(t)$. This window function can be applied directly to the analytic signal $z(t)$, before the instantaneous autocorrelation $\phi(t, \tau)$ is calculated, before the Fourier transform is performed with respect to time τ . This time and frequency smoothed WVD is often referred to as ‘the smoothed pseudo-WVD’. Here, we simply call it ‘the smoothed WVD’ because, a seismic trace practically always has a finite time duration, and the WVD is always a pseudo-WVD [12].

When the analytic signal $z(t)$ is multiplied by the window function $k(t)$, the latter has the highest weight in the centre of the window, when the window function is defined with a Gaussian function. Due to the tapering effect on the analytic signal $z(t)$, there is a loss of information in the

WVD produced by the local Fourier transform. Therefore, I suggest using multiple orthogonal windows, rather than a single window, in WVD generation.

To produce the frequency spectrum at a specified time, the WVD definition in the time domain is formulated as follows:

$$\begin{aligned}\tilde{S}_{\text{WVD}}(t, f) &= \int_{-\infty}^{\infty} h(\tau) z\left(t + \frac{\tau}{2}\right) z^*\left(t - \frac{\tau}{2}\right) e^{-i2\pi f\tau} d\tau \\ &= \int_{-\infty}^{\infty} k\left(\frac{\tau}{2}\right) k^*\left(-\frac{\tau}{2}\right) z\left(t + \frac{\tau}{2}\right) z^*\left(t - \frac{\tau}{2}\right) e^{-i2\pi f\tau} d\tau.\end{aligned}\quad (5.1)$$

where $h(\tau) = k(\tau/2) k^*(-\tau/2)$ is the frequency-axis smoothing filter, which is formed as a square of the time window function $k(\tau/2)$. Considering $k(\tau/2) = k(t = \tau, \tau/2)$, equation (5.1) is thus the WVD of a windowed signal, $\tilde{z}(t) = k(t) z(t)$.

At any time position t , taking advantage of the linearity of the Fourier transform, we can apply each window directly to the analytic signal $z(t)$, to produce the instantaneous autocorrelation accordingly, and then form an averaged instantaneous autocorrelation by a weighted sum, before performing the Fourier transform to produce the final WVD spectrum.

The WVD definition in the time domain in equation (5.1) can be implemented, to generate the frequency spectrum, at any specified time. Substituting $u = t + \tau/2$, equation (5.1) can be reformulated as follows:

$$\tilde{S}_{\text{WVD}}(t, f) = 2e^{i2\pi(2f)t} \int_{-\infty}^{\infty} \tilde{z}(u) \tilde{z}^*(2t - u) e^{-i2\pi(2f)u} du, \quad (5.2)$$

where $e^{i2\pi(2f)t}$ is the effect of the time lag t . The implementation of the WVD-based local power spectrum thus consists of the following three main steps.

The first step is to prepare the analytic signal for the calculation of the half-step instantaneous autocorrelation. The resampling of the analytic signal $z(t) \rightarrow z(u)$, with half the original time interval $\Delta u = \Delta t/2$, is achieved by the fast Fourier transform. The resampled trace $z(u)$ is used for the following two steps, for any time position t .

The second step is to calculate the instantaneous autocorrelation using the multiple orthogonal windows:

- (1) Multiplication of the analytic signal $z(u)$ by a window function $w_p(u - t)$,

$$\tilde{z}_p(u; t) = w_p(u - t) z(u). \quad (5.3)$$

- (2) Corresponding calculation of the instantaneous autocorrelation,

$$\tilde{\phi}_p(u; t) = \tilde{z}_p(u; t) \tilde{z}_p^*(2t - u; t). \quad (5.4)$$

- (3) Averaging of the instantaneous autocorrelations by a weighted sum,

$$\tilde{\phi}(u; t) = \frac{\sum_{p=0}^P \lambda_p \tilde{\phi}_p(u; t)}{\sum_{p=0}^P \lambda_p}, \quad (5.5)$$

where $P = 4$, the highest order of the multiple orthogonal window functions.

The third step is to generate the WVD-based local power spectrum by Fourier transform:

- (1) Performing the Fourier transform,

$$\tilde{\phi}(2f; t) = \mathcal{F}_u[\tilde{\phi}(u; t)] = \int_{-\infty}^{\infty} \tilde{\phi}(u; t) e^{-i2\pi(2f)u} du. \quad (5.6)$$

- (2) Calculating the WVD by

$$\tilde{S}_{\text{WVD}}(t; f) = 2e^{i2\pi(2f)t} \tilde{\phi}(2f; t). \quad (5.7)$$

The second and third steps are repeated for all required time positions.

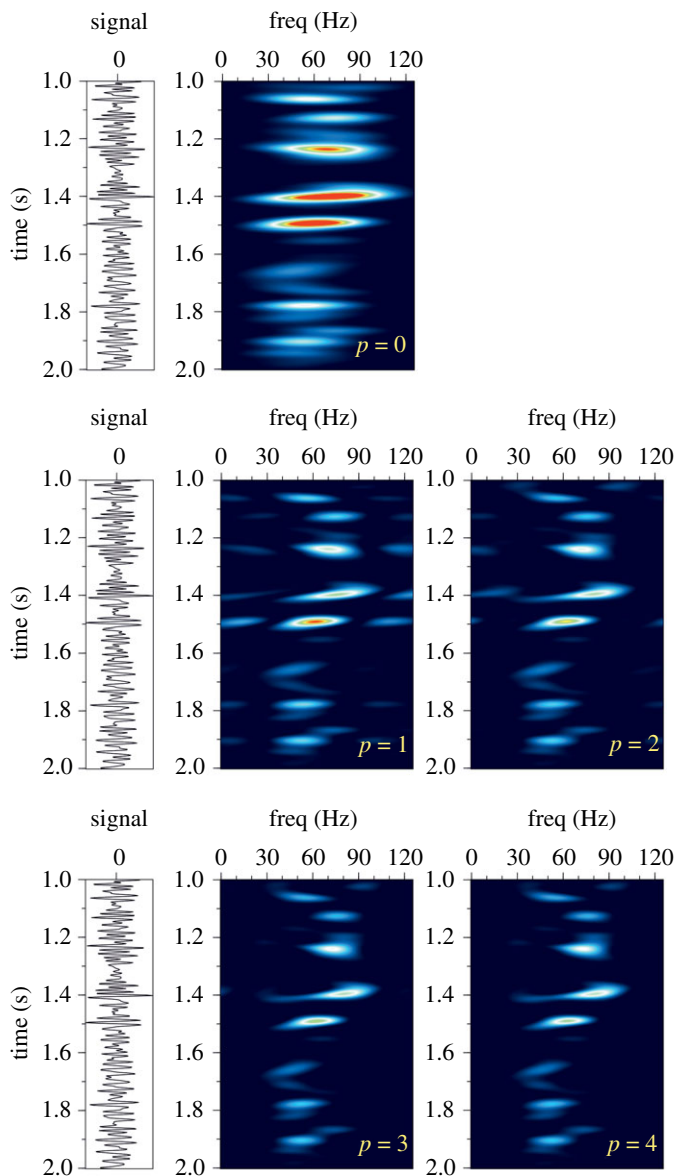


Figure 9. The multi-window WVD. Each of the multiple orthogonal windows $p = \{0, 1, \dots, 4\}$ is used to generate a WVD individually. The (half-) length of the multiple windows is $T = 12$ ms.

After completing smoothing filter for the frequency axis using the multiple orthogonal windows, we can also apply a time-axis smoothing filter $g(t)$ to the WVD, to further suppress the cross-term interference. In the following example, for the time-axis smoothing filter, the spread parameter is $\sigma_t = 0.012$ s. For the frequency smoothing filter, the (half-) length of the multiple windows is $T = 0.012$ s.

Figure 9 shows five WVDs, corresponding to the five windows with the order number $p = \{0, 1, \dots, 4\}$, respectively. When $p = 0$, the WVD clearly indicates a single energy concentration at a given time. Each energy concentration clearly corresponds to a waveform packet in the seismic signal. When $p \geq 1$, each WVD restores some small energy concentrations that were lost in the case of $p = 0$. The energy of these restored concentrations is not necessarily small, otherwise they

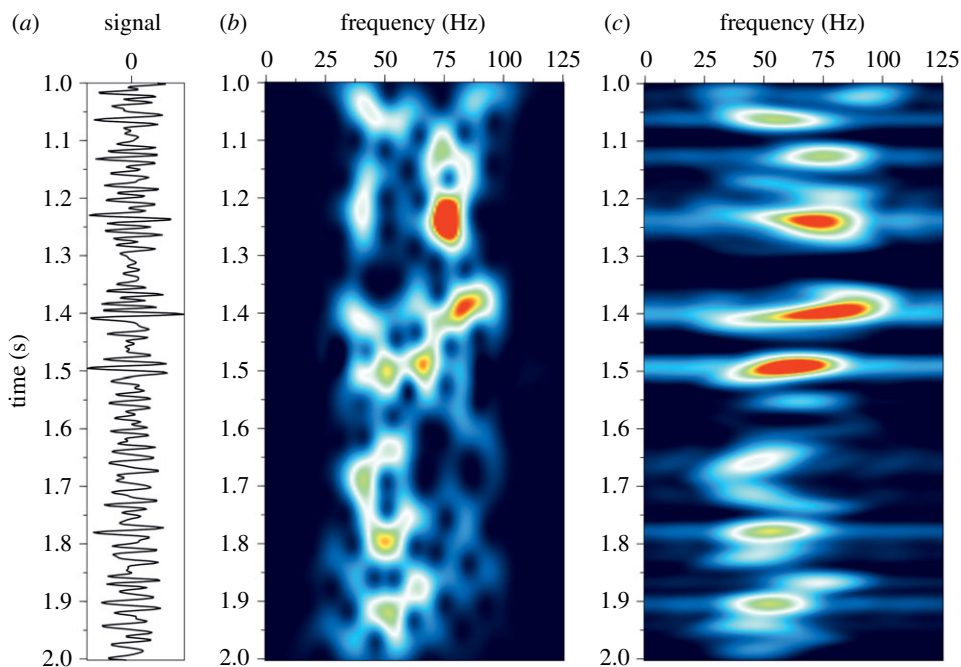


Figure 10. Comparison between the smoothed WVD and the multi-window WVD. (a) The field seismic trace. (b) The smoothed WVD, after two-dimensional filtering. (c) Local amplitude spectrum generated by the multi-window WVD.

could be important in seismic characterization. The summation of the multiple WVDs gives the local power spectrum.

Figure 10 shows that the local power spectrum produced by the multi-window WVD (figure 10c) is roughly comparable with the spectrum of the smoothed WVD (figure 10b). This similarity is due to the property of the WVD to preserve frequency support. However, the Gabor transform method does not have such a finite support property. Therefore, the local power spectrum of the multi-window Gabor transform (figure 8c) is generally different from the local power spectrum of the multi-window WVD (figure 10c).

The local power spectrum in figure 10 shows a weak energy concentration at time 1.55 s. On the contrary, in the smoothed WVD, this weak energy packet was mixed with the strong energy packet at 1.50 s. This weak energy concentration clearly corresponds to the wavelet in the seismic signal. At a time interval of 1.65–1.75 s, there are two energy concentrations next to each other. One has a frequency that gradually decreases over time, and the other has a frequency that gradually increases over time.

There are also two energy concentrations in the time interval of 1.75–1.85 s, which were not separated by the smoothed WVD but were mixed at 1.80 s. In the time interval of 1.85–1.95 s, there are three energy concentrations next to each other. The central concentration at 1.9 s has a dominant frequency of 50 Hz, the concentration at the top has a dominant frequency of 70 Hz, and the concentration at the bottom has a dominant frequency of 45 Hz. In this sense, the local power spectrum generated by the multi-window WVD is very close to the matching pursuit-based representation [37,38].

In summary, the smoothed WVD has a loss of information due to the Gaussian window used to suppress the cross-term interference. The local power spectrum generated by the multi-window WVD successfully represents the signal details, due to the application of multiple orthogonal windows.

6. Discussion

There are some methods that produce the time–frequency spectrum without windowing. One of these methods is to decompose a signal into several intrinsic mode functions (IMFs) and calculate the instantaneous frequency for each IMF [39]. Therefore, the first step is to preprocess the signal using the empirical mode decomposition (EMD) method. The IMFs are simple oscillating functions with varying amplitude and frequency and are symmetric with respect to the local mean zero. The second step is to apply the Hilbert transform to the decomposed IMFs and construct the energy distribution in the time–frequency plane.

Decomposition can be seen as an expansion of the signal in terms of IMFs, and these IMFs serve as the basis for the expansion of the signal. However, decomposition is subjective and unstable, as any small perturbation can lead to a new set of IMFs [40]. Furthermore, there are too many different instantaneous frequencies in the Hilbert spectrum at any given time position. This leads to difficulties in interpretation, in addition to the lack of mathematical theory about the decomposition.

In essence, an IMF does not necessarily satisfy physical uniqueness because it is sensitive to noise. Ensemble EMD [41] proposes to add different noise to the target signal before EMD and obtain the average of the corresponding IMFs as the final IMF.

To specifically account for the presence of noise in the seismic signal, the decomposition can be recast as an inverse problem to minimize the difference between the original signal and the reconstruction of the band-limited IMFs [42]. The constraint for the inverse problem is formed by the variational of the analytic signal with respect to time and represented in the frequency domain, so that the centroid of the power spectrum of an IMF can be iteratively re-estimated. Therefore, the variational EMD not only improves the robustness of the IMF decomposition to noisy signals, but also overcomes the apparent limitation in terms of sensitivity to sampling.

In the time–frequency representation, the complex spectrum at each time position can be modelled by a one-dimensional Fourier series. Frequency spectra at different time positions share the same set of Fourier basis functions but have different coefficients. Therefore, the coefficients are two-dimensional and vary with both frequency and time. The time–frequency representation can be reformulated as an inverse problem to estimate these coefficients [43,44]. Theoretically, with suitable regularizations, the window functions need not be used in this inverse problem. However, the number of coefficients far exceeds the number of available data samples, and the inverse problem is far underdetermined.

For WVD with multiple windows, the time and frequency resolution can be further increased by a reassignment method [14]. The reassignment method modifies the calculated time–frequency representation in two steps. First, evaluate the local centroids of the calculated two-dimensional spectrum in each point of the time–frequency representation. Second, assign the energy content to the new point within the time–frequency plane. Modifying the calculated spectrum can improve the readability of the time–frequency representation.

7. Conclusion

In this paper, I have proposed to use multiple orthogonal window functions to properly represent the local spectral property of a non-stationary seismic signal in the time–frequency plane, and recover the information lost by the time windowing before applying the Fourier transform for the frequency spectrum.

The multiple orthogonal window functions are constructed directly, using a stack of Gaussian functions. The weighted average spectrum of these multiple window functions has a flat passband, which is better than the spectrum of the classic multiple windows defined by the prolate spheroidal wave functions, and the spectrum of conventional multiple windows constructed by solving a discretized eigenvalue problem.

The WVD method analyses the instantaneous autocorrelation of the signal and generates the energy density distribution in the time–frequency plane. Taking advantage of the linearity of

the Fourier transform, we can apply each window directly to the analytic signal to generate the instantaneous autocorrelation accordingly and form an averaged instantaneous autocorrelation by a weighted sum, before performing the Fourier transform to generate the final WVD. The multi-window WVD successfully represents the local spectral details of the non-stationary seismic signal in the time–frequency plane, while satisfying the WVD property of frequency–support conservation.

Data accessibility. The data and code used in this paper are available at <https://doi.org/10.14469/hpc/10645> [45].

Conflict of interest declaration. I declare I have no competing interests.

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References

1. Cohen L. 1966 Generalized phase-space distribution functions. *J. Math. Phys.* **7**, 781–786. (doi:10.1063/1.1931206)
2. Cohen L. 1989 Time–frequency distributions: a review. *Proc. IEEE* **77**, 941–981. (doi:10.1109/5.30749)
3. Stockwell RG, Mansinha L, Lowe RP. 1996 Localization of the complex spectrum: the S transform. *IEEE Trans. Signal Process.* **44**, 998–1001. (doi:10.1109/78.492555)
4. Wang Y. 2021 The W transform. *Geophysics* **86**, V31–V39. (doi:10.1190/geo2020-0316.1)
5. Goupillaud P, Grossmann A, Morlet J. 1984 Cycle-octave and related transforms in seismic signal analysis. *Geoexploration* **23**, 85–102. (doi:10.1016/0016-7142(84)90025-5)
6. Grossmann A, Morlet J. 1984 Decomposition of Hardy functions into square integrable wavelets of constant shape. *SIAM J. Math. Anal.* **15**, 723–736. (doi:10.1137/0515056)
7. Selesnick IW, Baraniuk RG, Kingsbury NG. 2005 The dual-tree complex wavelet transform. *IEEE Signal Process Mag.* **22**, 123–151. (doi:10.1109/MSP.2005.1550194)
8. Claasen TACM, Mecklenbräuker WFG. 1980a The Wigner distribution, a tool for time–frequency signal analysis, part I: continuous-time signals. *Philips J. Res.* **35**, 217–250.
9. Claasen TACM, Mecklenbräuker WFG. 1980b The Wigner distribution, a tool for time–frequency signal analysis, part II: discrete-time signals. *Philips J. Res.* **35**, 276–300.
10. Claasen TACM, Mecklenbräuker WFG. 1980c The Wigner distribution, a tool for time–frequency signal analysis, part III: relations with other time–frequency signal transformations. *Philips J. Res.* **35**, 372–389.
11. Claasen TACM, Mecklenbräuker WFG. 1981 Time–frequency signal analysis by means of the Wigner distribution. *IEEE Int. Conf. Acoustics, Speech Signal Process.* **3**, 69–72. (doi:10.1109/ICASSP.1981.1171331)
12. Flandrin P, Escudé B. 1984 An interpretation of the Pseudo-Wigner–Ville distribution. *Signal Process.* **6**, 27–36. (doi:10.1016/0165-1684(84)90048-3)
13. Shin YS, Jeon JJ. 1993 Pseudo Wigner–Ville time–frequency distribution and its application to machinery condition monitoring. *Shock Vib.* **1**, 65–76. (doi:10.1155/1993/372086)
14. Auger F, Flandrin P. 1995 Improving the readability of time–frequency and time–scale representations by the reassignment method. *IEEE Trans. Signal Process.* **43**, 1068–1089. (doi:10.1109/78.382394)
15. Baydar N, Ball A. 2001 A comparative study of acoustic and vibration signals in detection of gear failures using Wigner–Ville distribution. *Mech. Syst. Signal Process.* **15**, 1091–1107. (doi:10.1006/mssp.2000.1338)
16. Gabor D. 1946 Theory of communication, part 1: the analysis of information. *J. Inst. Electr. Eng.* **93**, 429–441.
17. Heisenberg W. 1927 Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik (The actual content of quantum theoretical kinematics and mechanics). *Zeitschrift für Physik* **43**, 172–198. (doi:10.1007/BF01397280)
18. Hall M. 2006 Resolution and uncertainty in spectral decomposition. *First Break* **24**, 43–47. (doi:10.3997/1365-2397.2006027)
19. Thomson DJ. 1982 Spectrum estimation and harmonic analysis. *Proc. IEEE* **70**, 1055–1096. (doi:10.1109/PROC.1982.12433)
20. Park J, Lindberg CR, Thomson DJ. 1987 Multiple taper spectral analysis of terrestrial free oscillations, Part I. *Geophys. J. Int.* **91**, 755–794. (doi:10.1111/j.1365-246X.1987.tb01668.x)

21. Slepian D. 1954 Estimation of signal parameters in the presence of noise. *Trans. IRE Prof. Group Inform. Theory* **3**, 68–89. (doi:10.1109/IREPGIT.1954.6373401)
22. Slepian D. 1964 Prolate spheroidal wave functions, Fourier analysis and uncertainty, IV: extensions to many dimensions; generalized prolate spheroidal functions. *Bell Syst. Tech. J.* **43**, 3009–3057. (doi:10.1002/j.1538-7305.1964.tb01037.x)
23. Slepian D. 1978 Prolate spheroidal wave functions, Fourier analysis and uncertainty, V: the discrete case. *Bell Syst. Tech. J.* **57**, 1371–1430. (doi:10.1002/j.1538-7305.1978.tb02104.x)
24. Niven C. 1880 On the conduction of heat in ellipsoids of revolution. *Phil. Trans. R. Soc.* **171**, 117–151. (doi:10.1098/rstl.1880.0005)
25. Nicholson JW. 1925 Spheroidal wave-functions. *Proc. R. Soc. Lond. A* **107**, 43–60.
26. Van Buren AL, Boisvert JE. 2002 Accurate calculation of prolate spheroidal radial functions of the first kind and their first derivatives. *Q. Appl. Math.* **60**, 589–599. (doi:10.1090/qam/1914443)
27. Van Buren AL, Boisvert JE. 2004 Improved calculation of prolate spheroidal radial functions of the second kind and their first derivatives. *Q. Appl. Math.* **62**, 493–507. (doi:10.1090/qam/2086042)
28. Slepian D, Pollak HO. 1961 Prolate spheroidal wave functions, Fourier analysis, and uncertainty, I. *Bell Syst. Tech. J.* **40**, 43–46. (doi:10.1002/j.1538-7305.1961.tb03976.x)
29. Flammer C. 1957 *Spheroidal wave functions*. Stanford, CA: Stanford University Press.
30. Wilson R. 1987 Finite prolate spheroidal sequences and their applications I: generation and properties. *IEEE Trans. Pattern Anal. Mach. Intell.* **9**, 787–795. (doi:10.1109/TPAMI.1987.4767985)
31. Wilson R, Spann M. 1988 Finite prolate spheroidal sequences and their applications II: image feature description and segmentation. *IEEE Trans. Pattern Anal. Mach. Intell.* **10**, 193–203. (doi:10.1109/34.3882)
32. Bekara M. 2013 Taper design for block processing of seismic data. *Geophysics* **78**, A19–A22. (doi:10.1190/geo2012-0545.1)
33. Helstrom CW. 1966 An expansion of a signal in Gaussian elementary signals. *IEEE Trans. Inform. Theory* **12**, 81–82. (doi:10.1109/TIT.1966.1053847)
34. Bastiaans MJ. 1980 Gabor's expansion of a signal into Gaussian elementary signals. *Proc. IEEE* **68**, 538–539. (doi:10.1109/PROC.1980.11686)
35. Wigner E. 1932 On the quantum correction for thermodynamic equilibrium. *Phys. Rev.* **40**, 749–759. (doi:10.1103/PhysRev.40.749)
36. Ville J. 1948 Théorie et applications de la notion de signal analytique (Theory and applications of the notion of the analytic signal). *Cables et Transm.* **2**, 61–74.
37. Wang Y. 2007 Seismic time–frequency spectral decomposition by matching pursuit. *Geophysics* **72**, V13–V20. (doi:10.1190/1.2387109)
38. Wang Y. 2010 Multichannel matching pursuit for seismic trace decomposition. *Geophysics* **75**, V61–V66. (doi:10.1190/1.3462015)
39. Huang NE, Shen Z, Long SR, Wu MC, Shih HH, Zheng Q, Yen NC, Tung CC, Liu HH. 1998 The empirical mode decomposition method and the Hilbert spectrum for nonlinear and nonstationary time series analysis. *Proc. R. Soc. Lond. A* **454**, 903–995. (doi:10.1098/rspa.1998.0193)
40. Meignen S, Perrier V. 2007 A new formulation for empirical mode decomposition based on constrained optimization. *IEEE Signal Process Lett.* **14**, 932–935. (doi:10.1109/LSP.2007.904706)
41. Wu Z, Huang NE. 2009 Ensemble empirical mode decomposition: a noise-assisted data analysis method. *Adv. Adapt. Data Anal.* **1**, 1–41. (doi:10.1142/S1793536909000047)
42. Dragomiretskiy K, Zosso D. 2013 Variational mode decomposition. *IEEE Trans. Signal Process.* **62**, 531–544. (doi:10.1109/TSP.2013.2288675)
43. Liu GC, Fomel S, Chen X. 2011 Time–frequency analysis of seismic data using local attributes. *Geophysics* **76**, P23–P34. (doi: 10.1190/geo2010-0185.1)
44. Liu Y, Fomel S. 2013 Seismic data analysis using local time–frequency decomposition. *Geophys. Prospect.* **61**, 516–525. (doi:10.1111/j.1365-2478.2012.01062.x)
45. Wang Y. 2022 Time–frequency domain local spectral analysis of seismic signals with multiple windows. Imperial College London Research Data Repository. (doi:10.14469/hpc/10645)