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Stably uniform affinoids are sheafy

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Abstract. We develop some of the foundations of affinoid pre-adic spaces without Noetherian or finiteness hypotheses. We give explicit examples of non-adic affinoid pre-adic spaces, and also a new condition ensuring that the structure presheaf on $\mathrm{Spa}(R, R^+)$ is a sheaf. This condition can be used to give a new proof that the spectrum of a perfectoid algebra is an adic space.

1. Introduction

Let k be a field complete with respect to a non-trivial non-archimedean norm $k \to \mathbf{R}_{\geq 0}$. If (R, R^+) is an affinoid k-algebra (in the sense of [12, Definition 2.6 (ii)]), then we can associate to it a certain topological space $X := \mathrm{Spa}(R, R^+)$ whose elements are certain valuations on R. This topological space – a so-called affinoid pre-adic space – has a natural presheaf of complete topological rings $\mathcal{O}_X$ on it. The presheaf is known to be a sheaf if R satisfies certain finiteness conditions. For example, it is a sheaf if R is a quotient of a Tate algebra $k\langle T_1, T_2, \dots, T_n\rangle$ (that is, the ring of power series which converge on the closed unit polydisc), and there are other finiteness conditions which also suffice to guarantee $\mathcal{O}_X$ is a sheaf. In [10], which is mainly concerned with the theory of étale cohomology in the context of rigid spaces, these finiteness conditions are imposed very early on (viz. in Assumption (1.1.1)). However, more recently Scholze has introduced the concept of a perfectoid k-algebra, for which these finiteness conditions essentially never hold. Scholze showed in [12, Theorem 6.3 (iii)] that for R perfectoid, $\mathcal{O}_X$ is still a sheaf. His proof is delicate, involving a direct calculation in characteristic p and then some machinery (almost mathematics, tilting) to deduce the result in characteristic zero. However, it would be technically useful in some applications to have a more general method. For example, in [13, Conjecture 2.16] Scholze asks the following question: if X can be covered by rational subsets which are perfectoid, then is X perfectoid? This unfortunately turns out not to be true in this generality, because there are examples of locally perfectoid affinoid pre-adic spaces X where $\mathcal{O}_X$ fails to be a sheaf and hence X cannot be perfectoid. This raises the general question of what extra assumptions one should put on X in order to hope that one can check that it is perfectoid via local calculations. However, except for an example by Mihara ([11]) posted to the ArXiv whilst this paper was being typed up, there seem to be no examples in the literature at all of affinoid k-algebras for which $\mathcal{O}_X$ is not a sheaf and in general the problem seems to be very poorly-understood (or at least poorly-documented).

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In this paper we give some examples of affinoid k-algebras for which $\mathcal{O}_X$ is not a sheaf, and show that the phenomenon is strongly linked to the issue that the set of power-bounded elements in an affinoid ring may not be bounded. On the other hand, we show that if every rational subset of $\mathrm{Spa}(R, R^+)$ has the property that all power-bounded elements are bounded, then $\mathcal{O}_X$ is a sheaf (with no finiteness or perfectoid assumptions). Scholze has used this latter result to verify sheafiness for certain constructions underlying his new theory of diamonds.

Note finally that if one drops the finiteness conditions that Huber imposes, then one might not expect a reasonable theory of coherent sheaves; this is an issue even in the perfectoid space setting. For example, it seems to be currently unclear whether an open immersion of affinoids induces a flat morphism on rings of global sections in this generality.

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2. Definitions

We recall from [8, Section 1] that a *Tate ring* is an f-adic ring that has a topologically nilpotent unit. More concretely, it is a topological ring that can be obtained as follows. Let R be a ring, R_0 a sub-ring of R, and $\varpi \in R_0 \cap R^\times$ such that $R = R_0[\varpi^{-1}]$. The subsets $r + \varpi^n R_0$ with $r \in R$ and $n \geq 0$ form the basis of a topology on R and the resulting topological ring is a Tate ring (see [8, Proposition 1.5]). Note that without the condition $R = R_0[\varpi^{-1}]$, multiplication in R would not be continuous. We will refer to this topology on R as the topology induced by the subring R_0 and the ideal (ϖ) of R_0. In general, there are different choices for R_0 and ϖ that lead to the same topology on R.

Here is a construction of Tate rings which will be used in the later stages of this paper. Let k be a field complete with respect to a non-trivial non-archimedean norm $|\cdot| : k \to \mathbf{R}_{\geq 0}$, let $\mathcal{O}_k$ be its integer ring, and let ϖ be any element of $k^\times$ with $0 < |\varpi| < 1$. A *Tate k-algebra* (as in [12, Definition 2.6]) is a k-algebra R equipped with an $\mathcal{O}_k$-subalgebra R_0 such that $kR_0 = R$. All the explicit examples of Tate rings appearing in section 4 in this paper are Tate k-algebras.

An element r of a Tate ring R is called *power-bounded* if there exists some $n \geq 0$ such that $r^m \in \varpi^{-n} R_0$ for all $m \geq 0$ (this property depends only on the topology on R rather than the explicit choice of R_0 and ϖ). The set R° of power-bounded elements is an open and integrally closed subring of R containing R_0. If $R^+ \subseteq R^\circ$ now denotes an arbitrary open and integrally closed subring of R (for example, $R^+ = R^\circ$), then the resulting pair (R, R^+) is called a *Tate affinoid ring*, and to this pair one can associate a topological space $X = \mathrm{Spa}(R, R^+)$ whose elements are (equivalence classes of) continuous valuations on R which are bounded by 1 on R^+. The space X is furthermore endowed with a presheaf $\mathcal{O}_X$ of complete topological rings, and the question this paper is mainly concerned with, is when this presheaf is a sheaf.

The first sections of [8] and [9] give careful definitions of X and $\mathcal{O}_X$; other good references are [14] and (for the case of k-algebras) [12]. We summarize here the facts that we

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will need. Firstly, we mention some basic results about completions for which we could find no easily-accessible reference. Let R be a Tate ring, with R_0 and ϖ as above. We can complete R with respect to the topology defined by R_0 and (ϖ); the completion, denoted $\widehat{R}$, is the limit $\varprojlim_n R/\varpi^n R_0$, endowed with the projective limit topology (the quotients $R/\varpi^n R_0$ have the discrete topology). There is a canonical map $i : R \to \widehat{R}$. If $U \subseteq R$ is an open subgroup of the group $(R, +)$, then let $\widehat{U}$ denote the closure of $i(U)$ in $\widehat{R}$ (this is just notation, but it is reasonable because the closure of $i(U)$ is isomorphic to the completion of U in the sense of [4], by [4, Chapitre II, Paragraphe 3.9, Corollaire 1]).

Lemma 1. *In the situation described above:*

(i) *The subgroups $\widehat{\varpi^n R_0} = \varpi^n \widehat{R_0}$ form a basis of open neighbourhoods of the origin of $\widehat{R}$.*

(ii) *If U is an open subgroup of R, then $i^{-1}(\widehat{U}) = U$.*

(iii) *$\widehat{R}$ naturally has the structure of a Tate ring, with topology induced from the subring $\widehat{R_0}$ and element $i(\varpi)$; moreover we have $\widehat{R^\circ} = (\widehat{R})^\circ$ (i.e., completion commutes with taking power-bounded elements).*

Proof. (i) The closure $\overline{\{0\}}$ of $\{0\} \subseteq R$ is easily checked to be $\bigcap_n \varpi^n R_0$, so the result follows by applying [4, Chapitre III, Paragraphe 3.4, Proposition 7] to R modulo this closure, noting that any subgroup containing an open subgroup is open.

(ii) Clearly, $i^{-1}(\widehat{U})$ contains U. Conversely, the universal property of the completion ([4, Chapitre III, Paragraphe 3.4, Proposition 8]) gives us a group homomorphism $\widehat{R} \to R/U$ through which the canonical map $R \to R/U$ factors. The kernel K of $\widehat{R} \to R/U$ has the property that it contains $i(U)$ and hence its closure, but also that $i^{-1}(K) = U$. This shows $i^{-1}(\widehat{U}) = U$.

(iii) Both $\widehat{R}$ and $\widehat{R_0}$ are rings by [4, Chapitre III, Paragraphe 6.5, Proposition 6, and Chapitre II, Paragraphe 3.9, Corollaire 1]. The topology on $\widehat{R}$ is induced from $\widehat{R_0}$ by [4, Chapitre III, Paragraphe 3.4, Proposition 7]. By part (i), $i(R^\circ)$ consists of power-bounded elements, and $(\widehat{R})^\circ$ is open so it contains $\widehat{R^\circ}$. Conversely, say $\hat{r} \in \widehat{R}$ is power-bounded. Because $i(R)$ is dense in $\widehat{R}$ and $(\widehat{R})^\circ$ is open, for any $n \geq 0$ we may find $r_n \in R$ such that $\hat{r} - i(r_n) \in \varpi^n \widehat{R_0}$, and one checks using the binomial theorem that $i(r_n)$ is power-bounded and hence (using (ii)) that r_n is too. Hence $\hat{r} \in \widehat{R^\circ}$. □

We now return to our description of $\mathcal{O}_X$. The ring $\mathcal{O}_X(X)$ is not R, but the completion $\widehat{R}$ of R with respect to the topology induced by R_0 and (ϖ). Let us now describe $\mathcal{O}_X$ on certain open subsets of X. Choose $t \in R$. Then we can cover X by two open subsets $U := \{x : |t(x)| \leq 1\}$ and $V := \{x : |t(x)| \geq 1\}$ (where we make the standard abuse of notation: x is a valuation on R and $|t(x)|$ is just another way of writing $x(t)$). If the (still to be defined presheaf) $\mathcal{O}_X$ is a sheaf of complete topological rings, then in particular it is a sheaf of abelian groups and so the sequence

$$(*) \qquad 0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$$

of abelian groups is exact. We will now describe these groups and homomorphims explicitly. The subsets U, V and $U \cap V$ are rational subsets of X, so it is not hard to compute $\mathcal{O}_X$ on them directly.

Set $A = R$ and $A_0 = R_0[t]$. We topologize the ring A using A_0 and ϖA_0 as above. The space $\mathcal{O}_X(U)$ is the completion $\widehat{A}$ of A with respect to this topology. Set $B = R[1/t]$, the

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localization of R at the set $\{1, t, t^2, t^3, \dots\}$ obtained by inverting t. Let $\phi : R \to B$ denote the canonical map. We set $B_0 = \phi(R_0)[1/t]$ and topologize B using B_0 and ϖB_0 as above. The space $\mathcal{O}_X(V)$ is the completion $\widehat{B}$ of B. Finally, we set $C = B$ and $C_0 = \phi(R_0)[t, 1/t]$. The space $\mathcal{O}_X(U \cap V)$ is the completion $\widehat{C}$ of C.

The abstract rings R and A coincide, but their topologies will not coincide in general. More precisely, $R_0 \subseteq A_0$ and hence the identity map $R \to A$ is continuous, but if $A_0 \not\subseteq \varpi^{-n} R_0$ for every $n \geq 0$, then the identity map $A \to R$ is not. (An example where this happens is when k is a field complete with respect to a non-trivial non-archimedean valuation, $\varpi \in k$ with $0 < |\varpi| < 1$, and $R = A = k[T]$, $R_0 = \mathcal{O}_k[\varpi T]$ and $t = T$ so $A_0 = \mathcal{O}_k[T]$.) Similarly, the identity map $B \to C$ is continuous but the identity map $C \to B$ may not be. Also, $\phi : R \to B$ is continuous as are the induced maps $\phi : R \to C$ and $\phi : A \to C$, but the induced map $A \to B$ may not be.

Define $\epsilon : R \to A \oplus B$ by $\epsilon(r) = (r, \phi(r))$, and $\delta : A \oplus B \to C$ by $\delta(a, b) = b - \phi(a)$. One checks easily that the sequence of abstract abelian groups

$$(**) \qquad 0 \to R \xrightarrow{\epsilon} A \oplus B \xrightarrow{\delta} C \to 0$$

is exact, and indeed it is naturally split, the map $C \to A \oplus B$ sending c to $(0, c)$ being a splitting. However, if we topologize R, $A \oplus B$ and C using R_0, $A_0 \oplus B_0$ and C_0 respectively, then ϵ and δ are continuous but the splitting may not be continuous.

The sequence $(*)$ whose exactness we care about consists of the first three arrows in the completion of the sequence $(**)$ with respect to the topologies defined by R_0, $A_0 \oplus B_0$ and C_0. The issue then, is whether taking completions can destroy left exactness.

Before we embark on a discussion of this, we recall the notion of strictness. A continuous map between topological groups $\psi : V \to W$ is called *strict* if the two topologies on $\psi(V)$, namely the quotient topology coming from V and the subspace topology coming from W, coincide. We see that $\delta : A \oplus B \to C$ is strict, because it is a continuous surjection and the image of $A_0 \oplus B_0$ is C_0 so δ is open. On the other hand, ϵ is strict if and only if $S_0 := A_0 \cap \phi^{-1}(B_0)$ is bounded in R, which is not always the case; we will see explicit examples of that later on.

The following lemma shows that exactness of $(*)$ is in fact equivalent to strictness of ϵ.

Lemma 2. *The following are equivalent:*

(i) *the sequence* $(*)$ *is exact,*

(ii) *the sequence* $(*)$ *is exact and furthermore the map* $\mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ *is surjective,*

(iii) *there exists some* $n \geq 0$ *such that* $\varpi^n(A_0 \cap \phi^{-1}(B_0)) \subseteq R_0$,

(iv) ϵ *is strict (and hence all maps in* $(**)$ *are strict).*

Proof. Let S denote the ring R and define $S_0 := A_0 \cap \phi^{-1}(B_0)$. Topologize S using S_0 in the usual way (note $\varpi \in S_0$). Then $\epsilon : S \to A \oplus B$ is strict and the identity map $R \to S$ is a continuous bijection. In particular, strictness of ϵ is equivalent to $R \to S$ being a homeomorphism, which is equivalent to R_0 being open in S. Hence (iii) and (iv) are equivalent.

Now (ii) implies (i) trivially. Furthermore, it is a general fact in this setting that for an exact sequence with all morphisms strict, its completion remains exact (see e.g. [3, Chapitre III, Paragraphe 2.12, Lemme 2], or [2, Corollary 1.1.9/6] for the case of k-algebras). Applying this

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to the strict surjection δ we deduce that $\mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is always surjective (in fact, this is not difficult to see directly), and so (i) implies (ii). Furthermore, if (iv) holds, then every map in $(**)$ is strict, so the completion of $(**)$ is still exact, and hence (iv) implies (ii).

It suffices to prove that (i) implies (iii). Note that the converse of the "strict implies completion exact" result used several times above is not true in general. (For example, if $R = A = k[T]$, $R_0 = \mathcal{O}_k[\varpi T]$ and $A_0 = \mathcal{O}_k[T]$, then $R \to A$ is injective and not strict, but the induced map $\widehat{R} \to \widehat{A}$ is still injective.) Clearly $0 \to S \to A \oplus B \to C \to 0$ is exact and all the maps are strict, so the sequence remains exact under completion, and if furthermore (i) holds, we deduce that the map $\widehat{R} \to \widehat{S}$ induced by the continuous map $R \to S$ must be a bijection. We now wish to invoke the open mapping theorem in this generality. One can check that the argument of [5, Chapitre I, Paragraphe 3.3, Théorème 1] holds in this slightly more general setting of Tate rings. Another reference is [7], and in the k-algebra case there is [2, Section 2.8.1]. As a consequence we deduce that $\widehat{R} \to \widehat{S}$ is open. In particular, the image of $\widehat{R}_0$ must contain $\varpi^n \widehat{S}_0$ for some $n \geq 0$. Pulling back via the natural map $R \to \widehat{R}$ and using Lemma 1 (ii) we conclude that R_0 must contain $\varpi^n S_0$, which is (iii). □

This lemma is used in two ways in the sequel. In the next section we observe that if the power-bounded elements of R are bounded, then condition (iii) of the lemma follows (Corollary 4), and hence we get a criterion for checking the sheaf axiom for the cover $X = U \cup V$, which we can turn (Theorem 7) into a criterion for checking that the presheaf $\mathcal{O}_X$ on a Tate affinoid pre-adic space is a sheaf. As a consequence (Corollary 9) we get a new proof that $\mathcal{O}_X$ is a sheaf if $X = \mathrm{Spa}(R)$ with R perfectoid.

In Section 4 we construct rings where part (iii) is violated, and use them to build examples of Tate affinoid pre-adic spaces which are not adic.

3. A criterion for $\mathcal{O}_X$ to be a sheaf on a Tate affinoid pre-adic space

Let R, R_0 and ϖ be as before. As usual we topologize R by letting $\varpi^n R_0$ for $n \geq 0$ be a basis of open neighbourhoods of zero. We recall that R° denotes the subring of power-bounded elements of R. The ring R is called *uniform* if R° is bounded, in other words if there exists some $n \geq 0$ such that $R^\circ \subseteq \varpi^{-n} R_0$. Examples of uniform rings include reduced affinoid algebras in Tate's original sense (i.e., those which are topologically of finite type over a field k), and conversely any Tate k-algebra with a non-zero nilpotent element r such that $kr \not\subseteq R_0$ would be a non-uniform ring, as $kr \subseteq R^\circ$.

The key lemma we need in this section is that if an element of R is locally in R_0, then it is globally power-bounded. This sounds geometrically reasonable, and we now give an elementary algebraic proof. We first remind the reader that every open cover of an affinoid pre-adic space can be refined to a rational cover (see Lemma 8 (i) below, and just before that lemma for the definition of a rational cover).

Lemma 3. *Let R be a Tate ring, with R_0 and ϖ as before. Let $t_1, \dots, t_n$ in R such that $t_1 R + \cdots + t_n R = R$. For each i, let $R[1/t_i]$ be the localization of R at the multiplicative set $\{1, t_i, t_i^2, \dots\}$ and $\phi_i : R \to R[1/t_i]$ the natural homomorphism. Then*

$$\bigcap_{i=1}^{n} \phi_i^{-1}(\phi_i(R_0)[t_1/t_i, \dots, t_n/t_i]) \subseteq R^\circ.$$

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Proof. Suppose $r \in \bigcap_{i=1}^{n} \phi_i^{-1}(\phi_i(R_0)[t_1/t_i, \ldots, t_n/t_i])$. For each i there is a homogeneous polynomial $f_i \in R_0[T_1, \ldots, T_n]$ such that

$$\phi_i(r) = t_i^{-\deg(f_i)} \phi_i(f_i(t_1, \ldots, t_n)).$$

Since $t_i^{\deg(f_i)} r - f_i(t_1, \ldots, t_n) \in \ker(\phi_i)$, there exists $c_i \geq 0$ such that

$$t_i^{c_i}(t_i^{\deg(f_i)} r - f_i(t_1, \ldots, t_n)) = 0.$$

So

$$t_i^{d_i} r = g_i(t_1, \ldots, t_n),$$

where $g_i = T_i^{c_i} f_i \in R_0[T_1, \ldots, T_n]$ is homogeneous of degree $d_i = c_i + \deg(f_i)$.

Set $N = d_1 + \cdots + d_n$. Take $A \geq 0$ such that $\varpi^A t_i \in R_0$ for all i. We will show, by induction on $m \geq 0$, that $\varpi^{NA} h(t_1, \ldots, t_n) r^m \in R_0$ for every $h \in R_0[T_1, \ldots, T_n]$ that is homogeneous of degree N and all $m \geq 0$. The case $m = 0$ is clear because $\varpi^A t_i \in R_0$ for all i. Induction step: $m > 0$. It is sufficient to consider the case where h is a monomial, i.e., $h = T_1^{e_1} \ldots T_n^{e_n}$. Since $e_1 + \cdots + e_n = N = d_1 + \cdots + d_n$, there is at least one i for which $e_i \geq d_i$. Without loss of generality we can assume that $i = 1$. Now

$$\varpi^{NA} t_1^{e_1} \ldots t_n^{e_n} r^m = \varpi^{NA} t_1^{e_1 - d_1} t_2^{e_2} \ldots t_n^{e_n} g_1(t_1, \ldots, t_n) r^{m-1}$$

and by the induction hypothesis this is in R_0. This concludes the induction proof.

There exist elements $a_1, \ldots, a_n \in R$ such that $a_1 t_1 + \cdots + a_n t_n = 1$. Take $B \geq 0$ such that $\varpi^B a_i \in R_0$ for all i. Applying the above result to $h = (\varpi^B a_1 T_1 + \cdots + \varpi^B a_n T_n)^N$ shows that $\varpi^{N(A+B)} r^m \in R_0$ for all $m \geq 0$, and hence $r \in R^\circ$. □

Corollary 4. *Let (R, R^+) be a uniform Tate affinoid ring, and let $X = \mathrm{Spa}(R, R^+)$ be the associated affinoid pre-adic space. Let $t \in R$ and set $U = \{x \in X : |t(x)| \leq 1\}$ and $V = \{x \in X : |t(x)| \geq 1\}$. Then the sequence*

$$0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V) \to 0$$

is exact.

Proof. The conclusion of the corollary is condition (ii) of Lemma 2, so it suffices to verify condition (iii) of that lemma. Applying Lemma 3 with $t_1 = 1$ and $t_2 = t$ (ϕ_1 is the identity, $\phi_2 = \phi$) we deduce $A_0 \cap \phi^{-1}(B_0) \subseteq R^\circ$, and we can conclude because $R^\circ \subseteq \varpi^{-n} R_0$ for some $n \geq 0$ by uniformity. □

Corollary 5. *If X is a Tate affinoid pre-adic space, $f \in \mathcal{O}_X(X)$, and X has a cover by opens U_i such that $f|U_i = 0$ for all i, then f is topologically nilpotent.*

Proof. By Lemma 8 (i) we may assume the cover is rational. By Lemma 3 any locally zero element is power-bounded. Applying this to $\varpi^{-1} f$ we see that $\varpi^{-1} f$ is power-bounded and hence f is topologically nilpotent. □

Remark 6. We will need Lemma 3 and Corollary 4 later, but Peter Scholze points out to us that Corollary 5 also follows easily from [1, Theorem 1.3.1].

We now give a new criterion for the presheaf $\mathcal{O}_X$ on $\mathrm{Spa}(R, R^+)$ to be a sheaf. Let us say that a Tate affinoid ring (R, R^+) (or the associated pre-adic space $\mathrm{Spa}(R, R^+)$) is *stably*

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uniform if every rational subset $U \subseteq \mathrm{Spa}(R, R^+)$ has the property that $\mathcal{O}_X(U)$ is uniform. We remark that a Tate ring R is uniform iff its completion is, by Lemma 1 (iii).

Theorem 7. *Let (R, R^+) be a stably uniform Tate affinoid ring. Then $X := \mathrm{Spa}(R, R^+)$ is an adic space, in other words, the presheaf $\mathcal{O}_X$ on X is a sheaf of complete topological rings.*

Note that there are no finiteness hypotheses on R whatsoever. Before we embark upon the proof, let us remark that its deduction from Corollary 4 is, to a large extent, an application of standard machinery, although unfortunately we have found no single reference in the literature that fully covers our requirements. The following sources were of great use to us: Section 2 of Huber's paper [9] (proving an analogous result for adic spaces under some Noetherian hypotheses), Chapter 8 of [2] (proving Tate's acyclicity theorem for affinoid k-algebras topologically of finite type) and finally Section 8.2 of [14]. As preparation we now consider two special types of covers of affinoid pre-adic spaces and some relationships between them.

Say (R, R^+) is a Tate affinoid ring, and $t_1, t_2, \ldots, t_n \in R$ are elements of R such that the ideal they generate is all of R. Set $X = \mathrm{Spa}(R, R^+)$, and for $1 \leq i \leq n$ define

$$U_i := \{x \in X : |t_j(x)| \leq |t_i(x)| \text{ for all } 1 \leq j \leq n\}.$$

Then each U_i is a rational subset of X and the union of the U_i is X. Such a cover is called a *rational cover*. If furthermore each $t_i \in R^\times$, the cover is called a *rational cover generated by units*.

Now say (R, R^+) is a Tate affinoid ring, and $t_1, t_2, \ldots, t_n \in R$. Set $X = \mathrm{Spa}(R, R^+)$, and for each subset I of $\{1, 2, \ldots, n\}$ define

$$U_I = \{x \in X : |t_i(x)| \leq 1 \text{ for } i \in I,\ |t_i(x)| \geq 1 \text{ for } i \notin I\}.$$

Then each U_I is a rational subset of X and the union of the 2^n sets U_I is X. Such a cover is called a *Laurent cover*.

Lemma 8 (Huber). *Let X be a Tate affinoid pre-adic space.*

(i) *For every open cover $\mathcal{U}$ of X, there exists a rational cover $\mathcal{V}$ of X which is a refinement of $\mathcal{U}$.*

(ii) *For every rational cover $\mathcal{U}$ of X, there exists a Laurent cover $\mathcal{V}$ of X such that for every $V \in \mathcal{V}$, the cover $\{U \cap V : U \in \mathcal{U}\}$ of V is a rational cover generated by units.*

(iii) *For every rational cover $\mathcal{U}$ of X generated by units, there exists a Laurent cover $\mathcal{V}$ of X which is a refinement of $\mathcal{U}$.*

Proof. (i) See [9, Lemma 2.6].

(ii) If $\mathcal{U}$ is generated by $t_1, t_2, \ldots, t_n \in R$, then by assumption there are $a_i \in R$ such that $\sum_i a_i t_i = 1$. Because R^+ is open, there exists $B \geq 0$ such that $\varpi^B a_i \in R^+$ for all i. For all $x \in X$ we have by definition that $|r(x)| \leq 1$ for all $r \in R^+$, so from $\varpi^B = \sum_i (\varpi^B a_i) t_i$ it follows that $|\varpi^B(x)| \leq \max_i |t_i(x)|$, so (since $|\varpi(x)| < 1$ by continuity of x)

$$|\varpi^{B+1}(x)| < \max_i |t_i(x)|.$$

One checks easily, see for example the proof of [2, Lemma 8.2.2/3], that the Laurent cover generated by the ct_i with $c = \varpi^{-(B+1)}$ has the desired property.

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(iii) This statement can be shown by the purely combinatorial argument in the proof of [2, Lemma 8.2.2/4]. □

Proof of Theorem 7. Note that a rational subset of a stably uniform affinoid pre-adic space is again stably uniform. First consider $\mathcal{O}_X$ as a presheaf of abelian groups on X. We claim that any Laurent cover of any rational subset of X is $\mathcal{O}_X$-acyclic. We prove this by induction on n, the number of functions t_i used to define the Laurent cover. For $n = 1$ the claim is just Corollary 4 and the inductive step is proved following [2, Corollary 8.1.4/4].

The proof of [2, Proposition 8.2.2/5], using parts (i)–(iii) of Lemma 8 above in lieu of [2, Lemmas 8.2.2/2–4], now shows that any cover by rational subsets of any rational subset of X is $\mathcal{O}_X$-acyclic. It follows that $\mathcal{O}_X$ is a sheaf of abelian groups on the site whose objects are rational subsets of $\mathrm{Spa}(R, R^+)$ and whose covers are covers of rational subsets by rational subsets.

Since $\mathcal{O}_X$ is a presheaf of rings and a sheaf of abelian groups, it is also a sheaf of rings on this site. We claim that it is even a sheaf of complete topological rings on this site. For this it suffices, by [9, first paragraph of Section 2], to check that if $U = \bigcup_i U_i$ is a cover of a rational subset U by rational subsets, then the induced map $\mathcal{O}_X(U) \to \prod_i \mathcal{O}_X(U_i)$ is strict. By Lemma 8 (i) there is a rational cover $U = \bigcup_j V_j$ that refines $U = \bigcup_i U_i$. Lemma 3 and the uniformity hypothesis imply that the induced map $\mathcal{O}_X(U) \to \prod_j \mathcal{O}_X(V_j)$ is strict and since this map factors through $\mathcal{O}_X(U) \to \prod_i \mathcal{O}_X(U_i)$ that map must be strict too.

We have established that $\mathcal{O}_X$ is a sheaf of complete topological rings on the basis of rational subsets of X and by [6, Chapitre 0, (3.2.2)] we deduce that $\mathcal{O}_X$ is a sheaf of complete topological rings on X. □

As a toy example of an application, we get a new proof of [12, Theorem 6.3 (iii)], that avoids the arguments of [12, Sections 6.10–6.14].

Corollary 9. *If k is a perfectoid field, then the affinoid pre-adic space associated to a perfectoid k-algebra is an adic space.*

Proof. Perfectoid affinoid k-algebras are uniform (by definition) and hence stably uniform (by [12, Corollary 6.8]), so the theorem directly implies that the affinoid pre-adic space associated to a perfectoid affinoid k-algebra is an adic space. □

Remark. Brian Conrad notes that Scholze has systematically removed all need for a ground field in his perfectoid theory, in his 2014 UC Berkeley course, and in particular apparently [12, Corollary 6.8] is valid more generally with a uniform and Tate condition. Hence our arguments will show that the affinoid pre-adic space associated to a perfectoid ring is an adic space. A similar comment applies to the following corollary.

We also deduce that under the stably uniform assumption, in characteristic p we can check that a ring is perfectoid locally.

Corollary 10. *If k is a perfectoid field of characteristic $p > 0$, and if A is a stably uniform complete Tate k-algebra such that* $\mathrm{Spa}(A, A^+)$ *has a rational cover by affinoids of the form* $\mathrm{Spa}(R_i, R_i^+)$ *with the R_i perfectoid k-algebras, then A is perfectoid.*

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Proof. By [12, Proposition 5.9], it suffices to show that the pth power map $A \to A$ is surjective. So say $a \in A$. Let a_i denote the restriction of a to R_i; then because R_i is perfectoid (and hence reduced) we know $a_i = (b_i)^p$ for a unique $b_i \in R_i$. A rational subspace of an affinoid perfectoid space is again perfectoid, by [12, Theorem 6.3], and hence the b_i agree on overlaps; Theorem 7 implies that the b_i glue together to give an element $b \in A$. Now $b^p - a$ is locally zero and hence zero (again by Theorem 7), and hence $b^p = a$. □

4. Counterexamples

Throughout this section, k is a field complete with respect to a non-trivial non-archimedean norm $|\cdot| : k \to \mathbf{R}_{\geq 0}$, $\mathcal{O}_k$ is its integer ring, ϖ is an element of $k^\times$ with $0 < |\varpi| < 1$, R will be a k-algebra and R_0 will be an $\mathcal{O}_k$-subalgebra of R such that $kR_0 = R$. We call such an R (equipped with the topology coming from R_0 and ϖ) a Tate k-algebra; such R are Tate rings.

In this section we give various examples of affinoid k-algebras for which the structure presheaf is not a sheaf of complete topological rings (and is not even a sheaf of abelian groups). Let us say that an affinoid k-algebra (R, R^+) is *sheafy* if $X := \mathrm{Spa}(R, R^+)$ is an adic space (that is, if $\mathcal{O}_X$ is a sheaf of complete topological rings). We remark here that as this paper was being written, a preprint of Tomoki Mihara appeared on the ArXiv [11] with another example; Mihara's work was independent of ours.

The following lemma will be helpful for us when attempting to locate the power-bounded elements in polynomial rings (which are naturally graded).

Lemma 11. *Let R be a Tate k-algebra with topology defined by an $\mathcal{O}_k$-subalgebra R_0. Say we are given a torsion-free (additive) abelian group G and a G-grading of R, that is, a decomposition*

$$R = \bigoplus_{g \in G} R^{(g)},$$

where the $R^{(g)}$ are k-subspaces of R satisfying $R^{(g)}R^{(h)} \subseteq R^{(g+h)}$. Suppose that R_0 is also graded by this grading, that is,

$$R_0 = \bigoplus_{g \in G} (R_0)^{(g)},$$

with $(R_0)^{(g)} = R_0 \cap R^{(g)}$. Then R° is also graded by this grading.

Proof. Say $r \in R^\circ$. Then $r = \sum_{i \in I} r_i$, with $I \subseteq G$ a finite subset and $r_i \in R^{(i)}$. It suffices to check that $r_i \in R^\circ$ for all i. We do this by induction on the size of I. If $|I| \leq 1$, the result is clear. For $|I| > 1$ we let H be the subgroup of G generated by I and observe that H is finitely-generated and torsion-free, and hence a free abelian group, so there is an injection $H \to \mathbf{R}$, giving us an ordering on I. Say i_0 is the smallest element of I with respect to this embedding. Write $r = r_0 + r_1$ with $r_0 = r_{i_0}$. Because $r \in R^\circ$, there is some N such that $r^n \in \varpi^{-N} R_0$ for all $n \geq 0$, and hence $r_0^n + r_2 \in \varpi^{-N} R_0$, where $r_0^n \in R^{(ni_0)}$ and r_2 is a sum of elements in $R^{(j)}$ for $j \in H$, $j > ni_0$. In particular, r_0^n must be in $\varpi^{-N}(R_0)^{(ni_0)}$ and in particular $r_0^n \in \varpi^{-N} R_0$, hence $r_0 \in R^\circ$, and hence $r_1 \in R^\circ$ and we can apply the inductive hypothesis to r_1, finishing the argument. □

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4.1. A finitely-generated non-sheafy k-algebra. Even if R is a Tate k-algebra which is finitely-generated as an abstract k-algebra, the subalgebra R_0 defining the topology might be sufficiently nasty to ensure that (R, R^+) is not sheafy. This is not surprising – indeed Rost's example of a non-sheafy ring (which is not a k-algebra) given in [9, end of Section 1] is finitely-generated over $\mathbf{Z}$. We remark here that before [11], Rost's example was the only example known to us in the literature of a non-sheafy ring. The key idea of the following counterexample is basically Rost's.

Now, let R be the ring $k[T, T^{-1}, Z]/(Z^2)$ and let R_0 denote the $\mathcal{O}_k$-submodule of R with $\mathcal{O}_k$-basis $\varpi^{|n|}T^n$ and $\varpi^{-|n|}T^nZ$ ($n \in \mathbf{Z}$). (For the avoidance of doubt, here $|\cdot|$ denotes the ordinary absolute value on $\mathbf{Z}$.) One checks easily that R_0 is an $\mathcal{O}_k$-subalgebra of R and that $kR_0 = R$. We note in passing that R_0 is not Noetherian – indeed, the ideal $ZR \cap R_0$ of R_0 is easily checked to be not finitely-generated. However, $Z \in R_0$ is nilpotent and $R_0/(Z)$ is Noetherian.

Proposition 12. *For the space $X := \mathrm{Spa}(R, R^\circ)$ the presheaf $\mathcal{O}_X$ is not a sheaf. In particular, X is covered by $U := \{x \in X : |T(x)| \leq 1\}$ and $V := \{x \in X : |T(x)| \geq 1\}$ and the map $\mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ is not injective.*

Before we begin the proof, we briefly note two consequences. Firstly, this proposition (positively) resolves [12, footnote just before Definition 2.16]. Secondly, R is Noetherian, but it cannot be strongly Noetherian because $\mathcal{O}_X$ is a sheaf for strongly Noetherian Tate k-algebras by [9, Theorem 2.2].

Proof. That X is covered by the opens U and V is obvious. By definition,

$$\mathcal{O}_X(X) = \varprojlim_n R/\varpi^n R_0,$$

the completion of R. Similarly,

$$\mathcal{O}_X(U) = \varprojlim_n R/\varpi^n R_0[T] \quad \text{and} \quad \mathcal{O}_X(V) = \varprojlim_n R/\varpi^n R_0[T^{-1}].$$

We claim that the map $\mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ is not injective, and this suffices to show that $\mathcal{O}_X$ is not even a sheaf of abelian groups on X.

More precisely, we claim that $0 \neq Z \in \mathcal{O}_X(X)$ but that Z restricts to zero in both U and V. To verify the first assertion it suffices to observe that $kZ \not\subseteq R_0$, which is clear because $kZ \cap R_0 = \mathcal{O}_k Z$. To verify the second assertion it suffices to check that $kZ \subseteq R_0[T]$ and $kZ \subseteq R_0[T^{-1}]$; but both of these are also clear because for $n \geq 0$ we have

$$\varpi^{-n}Z = \varpi^{-n}T^{-n}Z.T^n \in R_0[T]$$

and

$$\varpi^{-n}Z = \varpi^{-n}T^nZ.T^{-n} \in R_0[T^{-1}]. \qquad \square$$

Note that $\mathcal{O}_X(U)$ is the completion of $k[T, T^{-1}]$ with respect to the topology generated by the subring $\mathcal{O}_k[T, \varpi T^{-1}]$ so in fact U is isomorphic to the adic space associated to the annulus $\{|\varpi| \leq |T| \leq 1\}$. Similarly, V is isomorphic to the adic space associated to the annulus $\{1 \leq |T| \leq |\varpi|^{-1}\}$; however, X is not the adic space associated to the annulus $\{|\varpi| \leq |T| \leq |\varpi|^{-1}\}$ as $\mathcal{O}_X(X)$ contains nilpotents.

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4.2. A non-perfectoid, locally perfectoid space. In this subsection we assume the characteristic of k is p, and that k is perfectoid (or equivalently that k is perfect). In this situation we can basically "perfectify" our previous example, and in this way construct an affinoid pre-adic space which is not adic (and in particular not perfectoid), but which is locally perfectoid. In particular, we resolve [13, Conjecture 2.16] (negatively).

The details are as follows. We start by perfectifying the ring $k[T, T^{-1}]$, that is, we take the direct limit $\varinjlim_{x\mapsto x^p} k[T, T^{-1}]$; we call this ring $k[T^{1/p^\infty}, T^{-1/p^\infty}]$. We then adjoin a nilpotent by setting $R = k[T^{1/p^\infty}, T^{-1/p^\infty}][Z]/(Z^2)$. Then R has a k-basis consisting of elements of the form T^n and T^nZ for $n \in \mathbf{Z}[1/p]$. We let R_0 denote the $\mathcal{O}_k$-submodule of R with basis $\varpi^{|n|}T^n$ and $\varpi^{-|n|}T^nZ$ ($n \in \mathbf{Z}[1/p]$). Topologize R as usual by letting subsets of the form $r + aR_0$ ($r \in R, a \in k^\times$) be a basis.

Proposition 13. *The space $X := \mathrm{Spa}(R, R^\circ)$ is not an adic space, because $\mathcal{O}_X$ is not a sheaf. However, $X = U \cup V$ with $U := \{x \in X : |T(x)| \le 1\}$ and $V := \{x \in X : |T(x)| \ge 1\}$ both perfectoid spaces.*

Proof. We have $\varpi^{-1}Z \notin R_0$ and hence $Z \neq 0$ in $\mathcal{O}_X(X)$. But as before $kZ \subseteq R_0[T]$ and $kZ \subseteq R_0[T^{-1}]$, and hence Z restricts to zero on both U and V, so again $\mathcal{O}_X$ is not a sheaf.

Next observe that the completion of R with respect to the basis given by $r + aR_0[T]$, $r \in R$, $a \in k^\times$, is equal to the completion of $k[T^{1/p^\infty}, T^{-1/p^\infty}]$ with respect to the topology defined by the subring $\mathcal{O}_k[T^{1/p^\infty}, (\varpi/T)^{1/p^\infty}]$ (that is, the direct limit of $\mathcal{O}_k[T, \varpi/T]$ via $x \mapsto x^p$); from this we deduce that U is the p-finite affinoid perfectoid space associated to the annulus $\{|\varpi| \le |T| \le 1\}$; similarly, V is perfectoid. □

Scholze (personal communication) observes that R° in the lemma above is not bounded (as it contains the line kZ) and asks whether his [13, Conjecture 2.16] becomes true under the additional assumption that the ring is uniform. Explicitly, if A is uniform and complete, and $\mathrm{Spa}(A, A^+)$ has a cover by rational subsets which are perfectoid, is A perfectoid? One might also ask whether the conjecture becomes true if A is assumed stably uniform, where the question becomes more accessible – indeed we resolved this in the characteristic p case in Corollary 10, and perhaps minor modifications of these arguments will also deal with the general case.

4.3. An affinoid pre-adic space with a non-nilpotent locally zero element. We have seen examples of global sections of affinoid pre-adic k-spaces which are non-zero but locally zero. The examples we have seen so far were nilpotent, which is perhaps not surprising: by Corollary 5 any such example has to be topologically nilpotent. Here we give an example of a section which is locally zero but genuinely not nilpotent.

Set $R = k[T, T^{-1}, Z]$ and let R_0 be the $\mathcal{O}_k$-subalgebra generated by ϖT and ϖT^{-1}, and also for each $n \ge 1$ the elements $\varpi^{-n}T^{a(n)}Z$ and $\varpi^{-n}T^{-b(n)}Z$, where $a(n)$ and $b(n)$ are two sequences of positive integers both tending to infinity rapidly. More precisely, the following will suffice: set $a(1) = 1$ and then for $J \ge 1$ ensure that

$$b(J) > J^2 + J \max\{b(j) : 1 \le j < J, a(i) : 1 \le i \le J\}$$

and for $I \ge 2$ ensure that

$$a(I) > I^2 + I \max\{b(j) : 1 \le j < I, a(i) : 1 \le i < I\}.$$

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The sequence $a(1), b(1), a(2), b(2), \dots$ can be constructed recursively such that these inequalities are satisfied.

Proposition 14. *Let $X = \mathrm{Spa}(R, R^\circ)$. Then $Z \in \mathcal{O}_X(X)$ is not nilpotent but vanishes on the subsets $U := \{x : |T(x)| \leq 1\}$ and $V := \{x : |T(x)| \geq 1\}$ that cover X.*

Proof. By construction, $\varpi^{-n} Z \in R_0[T]$ and $\varpi^{-n} Z \in R_0[T^{-1}]$ for all $n \geq 1$, so Z is zero on U and V. To see that Z is not nilpotent on $\mathrm{Spa}(R, R^\circ)$ we need to show that Z^e is non-zero in the completion $\widehat{R}$ of R for any $e \geq 1$, so we need to verify that for all $e \geq 1$ there exists some $M(e) \geq 0$ such that $\varpi^{-M(e)} Z^e \notin R_0$.

The ring R_0 is graded by $\mathbf{Z} \times \mathbf{Z}$ (the powers of T and Z) and the given generators of R_0 are homogeneous. It follows from this that if $\varpi^{-M} Z^e \in R_0$, then $\varpi^{-M} Z^e$ will be an $\mathcal{O}_k$-linear sum of products of the given generators, where each of these products is of the form λZ^e (with $\lambda \in k^\times$). So it suffices to check that for any $e \geq 1$ there exists some bound $M(e) \geq 0$ such that if λZ^e is a product of the given generators of R_0 then $|\lambda| < |\varpi|^{-M(e)}$.

Set $\alpha_n = \varpi^{-n} T^{a(n)} Z$ and $\beta_n = \varpi^{-n} T^{-b(n)} Z$. Say λZ^e is a product of the given generators of R_0, and let us consider which α_i and β_j occur in this product. There are two cases. If the product mentions only α_i and β_j for $i, j \leq e$, then (because of the coefficient of Z) the product can mention only e such elements, so $|\lambda| \leq |\varpi^{-e^2}|$. If, however, the product mentions some α_i or β_i with $i > e$, then we claim that $|\lambda| \leq 1$, and it suffices to prove this claim. Let I denote the largest i such that α_i is mentioned (with $I = 0$ if no α_i are mentioned), and let J denote the largest j such that β_j is mentioned (with $J = 0$ if no β_j is mentioned). Write $\lambda Z^e = (\varpi T)^\ell (\varpi T^{-1})^m (\varpi^{-\mu} T^\nu Z^e)$, with $\varpi^{-\mu} T^\nu Z^e$ a product of αs and βs. If $I \leq J$, then because $b(J) > J^2 + J \max\{b(j) : j < J, a(i) : i \leq J\}$ we see that $|\nu| \geq b(J) - (e-1)\max\{b(j) : j < J, a(i) : i \leq J\} > J^2$ whereas $\mu \leq eJ < J^2$, and because one of ℓ and m must be at least as big as $|\nu|$ to kill the power of T, we see $|\lambda| \leq 1$. A similar argument works in the case $I > J$, this time using the defining property of $a(I)$. □

4.4. Exactness failing in the middle. Scholze (personal communication) has asked whether one could construct an example of an affinoid pre-adic k-space for which $\mathcal{O}_X$ fails to be a sheaf for a reason other than the existence of sections which are locally zero but non-zero. Here is such an example – an example where gluing fails.

If R is a Tate k-algebra that contains a k-basis $\{r_1, r_2, \dots\}$ and there exist non-negative integers $\{n_1, n_2, \dots\}$ with the property that the r_i are an $\mathcal{O}_k$-basis for R°, and $\varpi^{n_i} r_i$ are an $\mathcal{O}_k$-basis for R_0, then $(\widehat{R})^\circ$ contains no line, because $(\widehat{R})^\circ = \widehat{(R^\circ)}$ by Lemma 1 (iii). So Corollary 5 implies that if $X = \mathrm{Spa}(R, R^\circ) = U \cup V$, then the map $\mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V)$ must be injective (as the kernel is a k-vector space all of whose elements are power-bounded). Here however is an example where $\mathcal{O}_X(X) \to \mathcal{O}_X(U) \oplus \mathcal{O}_X(V) \to \mathcal{O}_X(U \cap V)$ is not exact – there are global sections of U and V which agree on $U \cap V$ but which do not glue together to give a section on $U \cup V$.

Set $R = k[T, T^{-1}, Z_1, Z_2, \dots]$ and let R_0 be the free $\mathcal{O}_k$-submodule of R generated by elements $\varpi^d T^a \prod_i Z_i^{e_i}$ with $d, a \in \mathbf{Z}$ and $e_1, e_2, \dots \in \mathbf{Z}_{\geq 0}$ satisfying

(i) if $\sum_i e_i = 0$, then $d = |a|$;

(ii) if $\sum_i e_i = 1$, then $d = |a| - 2\min\{\sum_i i e_i, |a|\}$;

(iii) if $\sum_i e_i \geq 2$, then $d = |a| - 2\sum_i i e_i$.

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It is easily checked that the product of two such generators is in R_0 and that $1 \in R_0$, so R_0 is a ring. It is clear that $kR_0 = R$. Note that ϖT and ϖT^{-1} are in R_0 but not in ϖR_0. Set $U = \{x \in X : |T(x)| \leq 1\}$ and $V = \{x \in X : |T(x)| \geq 1\}$ as usual; set $A = B = R$ and topologize them using $A_0 = R_0[T]$ and $B_0 = R_0[T^{-1}]$, so $\mathcal{O}_X(U) = \widehat{A}$ and $\mathcal{O}_X(V) = \widehat{B}$.

Proposition 15. *The ring $(\widehat{R^\circ})$ contains no line.*

Proof. By Lemma 11, R° is graded by the degrees of $T, Z_1, Z_2, \ldots$. It is easy to check that $\varpi^d T^a \prod_i Z_i^{e_i}$ is in R° if and only if

(i) if $\sum_i e_i = 0$, then $d \geq |a|$;

(ii) if $\sum_i e_i \geq 1$, then $d \geq |a| - 2\sum_i i e_i$.

This, together with the arguments above, shows that $(\widehat{R^\circ})$ contains no line. □

Note that for all $n \geq 1$ we have

$$\varpi^{-n} Z_n = \varpi^{-n} T^{-n} Z_n . T^n \in R_0[T]$$

and similarly $\varpi^{-n} Z_n \in R_0[T^{-1}]$ but $\varpi^{-1} Z_n \notin R_0$, so $\varpi^{n-1}(R_0[T] \cap R_0[T^{-1}]) \not\subseteq R_0$ for every $n \geq 1$ and hence the map $R \to A \oplus B$ is not strict.

Now because $Z_n \in \varpi^n A_0$, we have that $\sum_i Z_i$ converges in $\widehat{A}$; let a be the limit. Similarly, it converges in $\widehat{B}$; let b be the limit.

Proposition 16. *The elements $a \in \widehat{A}$ and $b \in \widehat{B}$ agree on $U \cap V$, but cannot be glued to an element of $\widehat{R}$.*

Proof. That a and b agree on $U \cap V$ is obvious, because the image of a in $\widehat{C}$ and the image of b in $\widehat{C}$ both are the limit of $\sum_i Z_i$ in $\widehat{C}$.

Let $r \in \widehat{R}$. There is a Cauchy sequence $r_1, r_2, \ldots$ in R with limit r. For each $n \geq 1$, let $\rho_n : R \to k$ be the map that sends an element of R to the coefficient of Z_n in its monomial expansion. This map is continuous and therefore factors through a unique continuous map $\widehat{\rho}_n : \widehat{R} \to k$. Similarly we define $\widehat{\alpha}_n : \widehat{A} \to k$.

We claim that $\widehat{\rho}_1(r), \widehat{\rho}_2(r), \ldots$ converges to zero. Let $M \geq 0$. There exists $I \geq 1$ such that $r_i - r_j \in \varpi^M R_0$ for all $i, j \geq I$. It follows that $\rho_n(r_i) - \rho_n(r_j) \in \varpi^M \mathcal{O}_k$ for all $i, j \geq I$ and all $n \geq 1$. Take $N \geq 1$ such that none of $Z_N, Z_{N+1}, \ldots$ occurs in r_I. For all $n \geq N$ we have $\rho_n(r_I) = 0$, so $\rho_n(r_i) \in \varpi^M \mathcal{O}_k$ for all $i \geq I$, so $\widehat{\rho}_n(r) \in \varpi^M \mathcal{O}_k$. This concludes the proof that $\widehat{\rho}_n(r)$ converges to zero.

It is easily seen that $\widehat{\alpha}_n(a) = 1$ for all $n \geq 1$. Since $\widehat{\rho}_n$ and $\widehat{\alpha}_n$ are compatible through $\widehat{R} \to \widehat{A}$, it follows that the image in $\widehat{A}$ of r cannot be a. □

4.5. A uniform space with a subspace containing a line of power-bounded elements. We now give an example of a uniform space that is not stably uniform. See also [11], which was written independently.

Consider the free $\mathcal{O}_k$-submodule R_0 of $k[T, T^{-1}, Z]$ generated by $(\varpi T)^a (\varpi Z)^b$ with $b \geq 0$ and $a \geq -b^2$. It is easily verified that R_0 is also an $\mathcal{O}_k$-subalgebra; indeed, if $a \geq -b^2$ and $a' \geq -(b')^2$, then $a + a' \geq -b^2 - (b')^2 \geq -(b + b')^2$ if $b, b' \geq 0$. Set $R = A = kR_0$ and topologize them using $R_0 \subseteq R$ and $A_0 = R_0[T] \subseteq A$.

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Proposition 17. *The affinoid k-algebra (R, R°) is uniform, but not stably uniform. More specifically, A° contains the non-zero line kZ.*

Proof. We claim that $R^\circ = R_0$. By Lemma 11 it is sufficient to check that for every $r = (\varpi T)^a(\varpi Z)^b \in R_0$ (with $b \geq 0$ and $a \geq -b^2$) and $\lambda \in k$, if $\lambda r \in R^\circ$ then $\lambda \in \mathcal{O}_k$. An elementary calculation shows that it then suffices to check that $\varpi^{-1} r^n \notin R_0$ for any $n \geq 1$, and this is easily checked.

Note that $T^{-1}Z = (\varpi T)^{-1}(\varpi Z) \in R_0$ and hence $Z \in A_0$, but $\varpi^{-1}Z \notin A_0$ (this is not hard to see, using the grading on A_0). However, for $n \geq 1$ we have

$$(\varpi^{-n}Z)^{n+1} = (\varpi T)^{-n^2-2n-1}(\varpi Z)^{n+1}T^{n^2+2n+1} \in A_0$$

and hence $\varpi^{-n}Z \in A^\circ$ for all $n \geq 1$. □

4.6. A uniform affinoid space which is non-sheafy. Finally, we give an example of a uniform affinoid X over k for which $\mathcal{O}_X$ is not a sheaf. Let R_0 be the free $\mathcal{O}_k$-submodule of $k[P, P^{-1}, Q, Q^{-1}, T, T^{-1}, Z]$ generated by elements $\varpi^d P^p Q^q T^a Z^e$ with $d, p, q, a \in \mathbf{Z}$ and $e \in \mathbf{Z}_{\geq 0}$ satisfying the following conditions:

(i) $d = \max\{p + q + a, p + q - a, p + a, q - a\}$;

(ii) if $e = 0$, then $p \geq 0$ and $q \geq 0$;

(iii) if $e = 1$, then $p \geq 0$ or $q \geq 0$.

If $\varpi^{d_1} P^{p_1} Q^{q_1} T^{a_1} Z^{e_1}$ and $\varpi^{d_2} P^{p_2} Q^{q_2} T^{a_2} Z^{e_2}$ are two such elements, then we have that $d_1 \geq p_1 + q_1 + a_1$ and $d_2 \geq p_2 + q_2 + a_2$, hence $d_1 + d_2 \geq (p_1 + p_2) + (q_1 + q_2) + (a_1 + a_2)$ and so on; from this it is not hard to see that the product of two $\mathcal{O}_k$-module generators of R_0 is in R_0; moreover $1 \in R_0$, and hence R_0 is a ring. Set $R = kR_0$.

Proposition 18. *The affinoid k-algebra (R, R°) is uniform, but for $X := \mathrm{Spa}(R, R^\circ)$ the presheaf $\mathcal{O}_X$ is not a sheaf. In particular, Z is non-zero on the subspace*

$$W := \{x \in X : |P(x)| \leq 1 \text{ and } |Q(x)| \leq 1\}$$

but vanishes on the subspaces $U := \{w \in W : |T(w)| \leq 1\}$ and $V := \{w \in W : |T(w)| \geq 1\}$ that cover W.

Proof. One verifies using Lemma 11 that $R^\circ = R_0$. Thus, R is uniform.

The subspace W has global sections given by the completion of the ring $A = R$ with respect to the topology defined by $A_0 = R_0[P, Q]$. Now we have $\varpi^{-n} Q^{-2n} T^{-n} Z \in R_0$, so $\varpi^{-n} Z = \varpi^{-n} Q^{-2n} T^{-n} Z . Q^{2n} T^n \in A_0[T]$, for all $n \geq 0$. Similarly, $\varpi^{-n} Z \in A_0[T^{-1}]$ for all $n \geq 0$ and we deduce that Z vanishes on the subspaces U and V. However, $\varpi^{-1} P^{-m} Q^{-n} Z$ is not in R_0 for any $m, n \geq 0$ (indeed for $m, n > 0$ this is not even in R), so $\varpi^{-1} Z$ is not in $A_0 = R_0[P, Q]$ and so Z is a non-zero function on W. □

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