

Eliminating incident subtraction in diffraction tomography

P. Huthwaite

2nd November 2016

Abstract

Diffraction tomography is a powerful algorithm for producing high resolution quantitative reconstructions across a wide range of applications. A major drawback of the method is that it operates on the scattered field, which cannot generally be directly measured, but must instead be calculated by subtracting the incident field, i.e. the equivalent field with no scatterer present. Unfortunately, often the incident field is not measurable and hence must be estimated, causing errors. This paper highlights an important, but not widely recognised, result: for particular widely used formulations of the algorithm, the subtraction of the incident field is unnecessary, and the algorithm can actually be applied directly to measured signals. The theory behind this is derived, showing that the incident field will vanish under far field conditions, and the result is demonstrated in practice. Tests with subsampled arrays show that aliasing artefacts can appear, but can be removed with a filter at the expense of resolution. The incident field also has no effect for a variety of array configurations tested. Finally, the performance in the presence of both correlated and uncorrelated errors is confirmed, in all cases demonstrating that the incident field has a negligible effect on the final reconstruction.

1 Introduction

The diffraction tomography (DT) algorithm can produce quantitative images which have resolutions down to $\lambda/2$, with λ being wavelength. DT, along with variations of it, have been demonstrated across a range of areas including geophysics [1], guided wave tomography [2, 3], live cell imaging [4] and breast ultrasound tomography [5–7]. For a DT reconstruction, a wave interacts with an inhomogeneity, and the scattered field from that is passed to the imaging algorithm. However, in order to obtain the scattered field from the total (measured) field, it is necessary to subtract the incident field, i.e. the field when no scatterer is present.

When imaging with simulated data, it is typically straightforward to address this by running a second set of simulations without the scatterer present in order to obtain the incident component; while this may increase simulation time there are no other obvious disadvantages. One practical example where it is possible to do this physically is breast ultrasound tomography [6, 7], where the breast is suspended in a water bath surrounded by a circular array. A set of data can be acquired of the water bath before or after measurements have been taken with the breast in place. However, such data is susceptible to variations in the water temperature and other system errors, and therefore perfect subtraction is unachievable. In most other areas an incident data set will not be measurable, for example, in non-destructive testing one would have to have access to a second identical sample, with no defect present, and then be able to measure the incident in exactly the same way on that as on the original. Given these challenges, the practical applications of DT have been limited.

Recent work on the DT algorithm has highlighted an interesting result: the algorithm as implemented in [8] does not require the incident field to be subtracted; good images have been generated purely when imaging with the total field. Within breast ultrasound tomography, it was hypothesised that the assumed conical nature of the breast deflected the incident ultrasonic beam away from the receivers, leaving just the scattered component [9], although it was later suggested that the degree of deflection may not be sufficient in practice [10]. The same result was then demonstrated in guided wave tomography in [11] where it was suggested this was only valid for the transmission subset of measurements, however, subsequent results have shown much broader applicability, including reflection components [12].

This paper serves three purposes. Firstly, it aims to bring attention to a very convenient result which has not been widely recognised by the community. Secondly, it will provide a theoretical basis for the phenomenon. Finally the result will be demonstrated and attempts made to define the circumstances under which this can be used.

Section 2 outlines the underlying scattering theory upon which the diffraction tomography formulations are derived, then Sect. 3 provides the theory explaining why the incident component has no effect on the reconstructed image. Finally the theory is demonstrated in Sect. 4 and practical aspects evaluated, including performance with

subsampled arrays, and different array configurations (including limited view), then the sensitivity of the subtraction to transducer mispositioning, both correlated and uncorrelated is evaluated.

2 Background theory

2.1 Scattering model

While the approach in this paper may be applicable to 3D imaging, the formulation here is derived for the 2D case, which is far more widely used in practice. The scattering model used is based on the formulations of [13,14]. The acoustic wave equation will be considered here, although these results are generalisable to other modalities including electromagnetic, elastic and guided waves. It is noted that, for electromagnetic waves, polarisation effects are typically neglected [14] so the theory derived here based on the acoustic wave equation is directly applicable. In the frequency domain, this is expressed as the Helmholtz equation

$$[\nabla^2 + k(\mathbf{x})^2] \phi = 0 \quad (1)$$

where k represents the local wavenumber at position $\mathbf{x}$ and ϕ is the scalar field potential. Defining an object function

$$O(\mathbf{x}) = k_0^2 \left[\left(\frac{c_0}{c(\mathbf{x})} \right)^2 - 1 \right] \quad (2)$$

eq. (1) can be rearranged to give

$$[\nabla^2 + k_0^2] \phi = -O\phi. \quad (3)$$

Taking the free space Helmholtz equation

$$[\nabla^2 + k_0^2] \phi_0 = 0 \quad (4)$$

this can then be subtracted from (3) to give an equation in terms of the scattered field $\phi_s = \phi - \phi_0$

$$[\nabla^2 + k_0^2] \phi_s = -O\phi. \quad (5)$$

Effectively, the left hand side has become the Helmholtz equation in free space, operating on the scattered field, and the right hand side is a source of strength $-O\phi$. This equation can be reformulated into an integral by first using the Green's function solution

$$[\nabla^2 + k_0^2] G(\mathbf{y}, \mathbf{x}) = \delta(\mathbf{x} - \mathbf{y}) \quad (6)$$

where $\mathbf{x}$ and $\mathbf{y}$ are positions in space corresponding to the source and measurement locations respectively. This is solved in 2D by

$$G(\mathbf{y}, \mathbf{x}) = -\frac{i}{4} H_0^{(1)}(k_0 r) \quad (7)$$

where

$$r = |\mathbf{x} - \mathbf{y}|$$

and $H_0^{(1)}$ is the Hankel function of the first kind. Since the Green's function is an elemental solution to the wave equation in free space for a delta function source, this can be applied to eq. (5)

$$-O\phi = \int -O(\mathbf{x}) \phi(\mathbf{x}) \delta(\mathbf{x} - \mathbf{y}) d\mathbf{x} \quad (8)$$

then

$$\phi_s = - \int O(\mathbf{x}) \phi(\mathbf{x}) G(\mathbf{y}, \mathbf{x}) d\mathbf{x}. \quad (9)$$

This is the Lippmann-Schwinger equation. The presence of the scattered field, ϕ_s , on the left hand side, traditionally requires the incident field to be subtracted from the total field prior to any inversion being attempted, which is problematic in many applications of DT.

2.2 Beamforming and diffraction tomography

Inverting the Lippmann-Schwinger equation requires some form of approximation. The most widely used, and the only case considered in this paper, will be to use the Born approximation, assuming $\phi \approx \phi_0$ in the integral

$$\phi_s = - \int O(\mathbf{x}) \phi_0(\mathbf{x}) G(\mathbf{y}, \mathbf{x}) d\mathbf{x}. \quad (10)$$

A commonly used solution is to assume a far field array, allowing the incident field and Green's functions to be expressed as plane waves

$$\phi_s = - \int O(\mathbf{x}) e^{ik(\hat{\mathbf{s}} - \hat{\mathbf{s}}_0) \cdot \mathbf{x}} d\mathbf{x} \quad (11)$$

where $\hat{\mathbf{s}}$ and $\hat{\mathbf{s}}_0$ represent the scattered and incident wave directions respectively. This can be seen to represent a 2D Fourier transform

$$\phi_s(\hat{\mathbf{s}}, \hat{\mathbf{s}}_0) = \hat{O}(\mathbf{K}) \quad (12)$$

where $\mathbf{K} = k(\hat{\mathbf{s}} - \hat{\mathbf{s}}_0)$. By varying $\hat{\mathbf{s}}$ and $\hat{\mathbf{s}}_0$ through all 2π angles it is possible to extract all components within the Ewald disk of radius $2k_0$ [13]. The traditional approach to generating an image is to resample the measured data into a uniform grid in the $\mathbf{K}$ domain, then perform a Fourier transform [14]. While this is a fast approach which can exploit the speed of the Fast Fourier Transform algorithm [15], it is relatively inflexible and introduces interpolation errors so will not be considered further. Instead, the inverse Fourier transform is calculated by summing for discrete transducers, or integrating for the continuous case, around all angles in the array [16]

$$O(\mathbf{x}) = -\frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \phi_s(\hat{\mathbf{s}}, \hat{\mathbf{s}}_0) e^{-ik(\hat{\mathbf{s}} - \hat{\mathbf{s}}_0) \cdot \mathbf{x}} W(\mathbf{K}) ds_\theta dr_\theta \quad (13)$$

where s_θ and r_θ represent the angles of the incident and scattered directions respectively. Note that a weighting function

$$W(|\mathbf{K}|) = |\mathbf{K}| \sqrt{1 - \left(\frac{|\mathbf{K}|}{2k_0}\right)^2} \quad (14)$$

which can also be written as

$$W = k_0 |\sin \Delta_\theta| \quad (15)$$

where $\Delta_\theta = r_\theta - s_\theta$, is now needed because the integrals around the angles s_θ and r_θ do not evenly sample the $\mathbf{K}$ domain. This weighting is the term which essentially differentiates beamforming (also known as the total focusing method [17]), where no weighting correction is made, from DT. Rose et al. [18] have performed detailed studies comparing (amongst other methods) beamforming and diffraction tomography. Two mathematically equivalent solutions can be derived from this; one is simply to calculate a discretised sum of (13) for each value of $\mathbf{x}$ incorporating the weighting directly, and the second is to apply the weighting as a post-processing filter to the beamforming image [8]. These are mathematically equivalent in the far field, however both will be tested in this paper.

2.2.1 Near field solution

While much of the theory in this paper is derived for the far field, requiring transducers to be placed so far from the scatterer is overly restrictive in many cases of interest. Therefore, in practice, the near field scenario is more common. Equation (11) can be formulated in the near field as

$$\phi_s(\mathbf{y}, \mathbf{z}) = - \int O(\mathbf{x}) G(\mathbf{y}, \mathbf{x}) G(\mathbf{x}, \mathbf{z}) d\mathbf{x} \quad (16)$$

for a source at $\mathbf{y}$ and a receiver at $\mathbf{z}$, assuming that the sources are delta functions and hence produce incident fields corresponding to the Green's function. To perform the inversion, the plane wave components are replaced with the reciprocal of the Green's functions

$$O(\mathbf{x}) = - \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{\phi_s}{G(\mathbf{y}, \mathbf{x}) G(\mathbf{x}, \mathbf{z})} W ds_\theta dr_\theta. \quad (17)$$

As in the far field case, the weighting term can either be applied within the integral, or applied as a post-processing filter. The filtered approach has been applied to near field DT imaging in [7] and the weighted integral version applied in [18, 19].

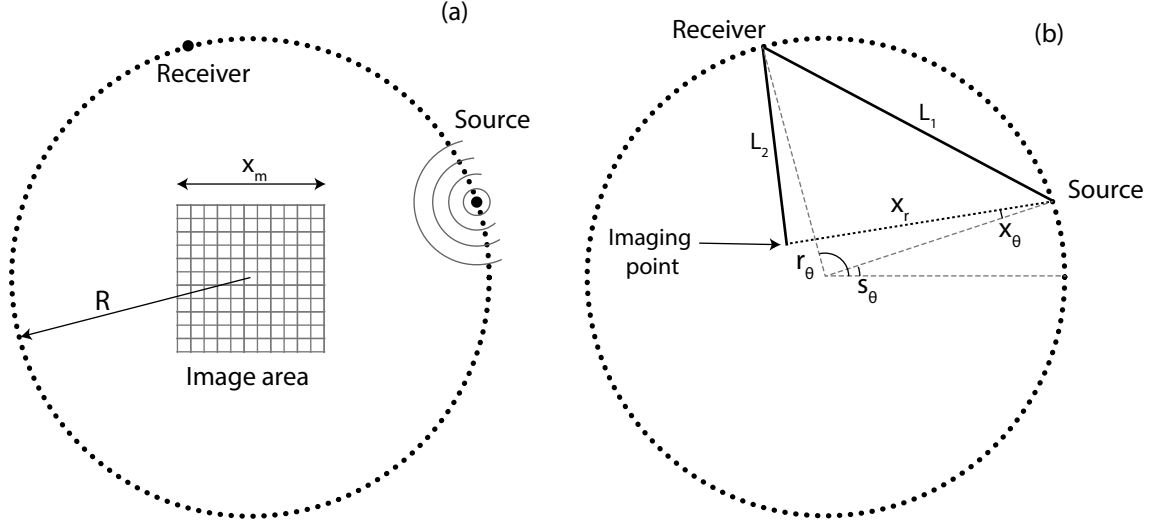


Figure 1: (a) shows a schematic layout for the imaging configuration, with a wave from the source propagating to the receiver then being imaged on the imaging area marked. A specific source-receiver pair has been marked, however, in general all combinations of sources and receivers within the array are acquired. (b) defines geometrical parameters for this source-receiver pair and an imaging point in the image area.

3 Incident field behaviour in diffraction tomography

This section will initially outline the consequences of imaging the incident field using the beamforming algorithm, since this is the basis for both DT algorithms considered in this paper. This will then be developed for DT. Figure 1(a) shows a schematic outline of the array and imaging area, and Fig. 1(b) defines various parameters which will be used in the calculation. Measurements are taken for each send-receive combination in the array, and are processed to produce pixel values across the image area.

When a measurement of the incident field is made, it is assumed that each source acts as a perfect point source with unit amplitude, with each receiver behaving in a similarly omnidirectional manner, so that the measurement of the incident field is given by the Green's function

$$\phi_0 = -\frac{i}{4} H_0^{(1)}(k_0 L_1). \quad (18)$$

This can be approximated by its asymptotic form to give

$$\phi_0 = \Pi \frac{e^{ik_0 L_1}}{\sqrt{L_1}} \quad (19)$$

where

$$\Pi = \frac{e^{i\pi/4}}{\sqrt{8\pi k_0}}.$$

It is noted that this approximation is not overly restrictive since it matches the true curve well even when $L < \lambda/2$, thus being valid even when measuring close to the source. The first stage of the beamforming algorithm is to focus the measurements onto points in the image area by applying a focusing law. For the purposes of this discussion, this focusing law will be taken as

$$f(L_2) = e^{-ik_0 L_2} \quad (20)$$

although it is recognised that variations on this exist, such as incorporating an amplitude factor depending on L_2 or the beam angle (see for example [20]). The received signal is multiplied by the focusing law for each receiver, and integrated (or summed for the discrete transducer case) around the array to give the beamforming focusing value

$$I_1(x_r, x_\theta, s_\theta) = \int_{-\pi}^{\pi} \Pi \frac{e^{ik_0 L_1}}{\sqrt{L_1}} e^{-ik_0 L_2} dr_\theta. \quad (21)$$

I_1 is the image of the first focusing stage, i.e. the beamforming image for a single source; this will subsequently be refocused and summed for each source in a similar manner to eq. (21) to give the final image. Definitions for x_r and

x_θ are given in Fig. 1(b), being the distance from the source to the imaging point and the angle that the imaging point makes relative to the diameter passing through the source respectively. Trigonometry is utilised to calculate

$$L_1 = 2R \left| \sin \frac{\Delta_\theta}{2} \right| \quad (22)$$

where $\Delta_\theta = r_\theta - s_\theta$ is the angle between the sender and receiver, and

$$L_2 = \sqrt{2R^2 - 2R^2 \cos \Delta_\theta + x_r^2 + Rx_r \cos x_\theta (2 \cos \Delta_\theta - 2) - 2Rx_r \sin \Delta_\theta \sin x_\theta}. \quad (23)$$

The assumption is now made that imaging is performed in the far field, such that R is large. The imaging area is assumed to have size in each direction x_m , and is centred in the middle of the array as shown in Fig. 1(a). Since

$$|R \sin x_\theta| < x_m$$

and R is large, x_θ must be small, so $\cos x_\theta \approx 1$ and $\sin x_\theta \approx x_\theta$, leading to

$$\left(\frac{L_2}{R} \right)^2 \approx 2 - 2 \cos \Delta_\theta + \left(\frac{x_r}{R} \right)^2 + \left(\frac{x_r}{R} \right) (2 \cos \Delta_\theta - 2) - 2 \left(\frac{x_r}{R} \right) \sin \Delta_\theta x_\theta \quad (24)$$

where a factor of R has been removed and the equation squared. This becomes

$$\left(\frac{L_2}{R} \right)^2 \approx (2 - 2 \cos \Delta_\theta) \left(1 - \frac{x_r}{R} \right) + \left(\frac{x_r}{R} \right)^2 - 2 \left(\frac{x_r}{R} \right) \sin \Delta_\theta x_\theta. \quad (25)$$

Defining a parameter $x_\Delta = R - x_r$, where $x_\Delta < x_m \ll R$, then multiplying out terms, gives

$$\left(\frac{L_2}{R} \right)^2 \approx (2 - 2 \cos \Delta_\theta) \left(\frac{x_\Delta}{R} \right) + \left(1 - \frac{x_\Delta}{R} \right)^2 - 2 \left(1 - \frac{x_\Delta}{R} \right) \sin \Delta_\theta x_\theta \quad (26)$$

$$\approx (2 - 2 \cos \Delta_\theta) \left(\frac{x_\Delta}{R} \right) + 1 - 2 \left(\frac{x_\Delta}{R} \right) + \left(\frac{x_\Delta}{R} \right)^2 - 2 \left(x_\theta - \frac{x_\Delta x_\theta}{R} \right) \sin \Delta_\theta. \quad (27)$$

Any products of small terms will be neglected, leading to

$$\left(\frac{L_2}{R} \right)^2 \approx 1 - 2 \cos \Delta_\theta \left(\frac{x_\Delta}{R} \right) - 2x_\theta \sin \Delta_\theta. \quad (28)$$

In order to take the square root of this expression, the binomial expansion

$$\sqrt{1+x} \approx 1 + \frac{x}{2} \quad (29)$$

can be used, provided x is small. In this case, all the right hand side terms except 1 are small, leading to

$$L_2 \approx R \left[1 - \cos \Delta_\theta \left(\frac{x_\Delta}{R} \right) - x_\theta \sin \Delta_\theta \right]. \quad (30)$$

The integration term of eq. (21) can be applied to these values for L_1 and L_2

$$I_1(x_r, x_\theta, s_\theta) = \int_{-\pi}^{\pi} X(\Delta_\theta) \exp \left[ik_0 \left(2R \left| \sin \frac{\Delta_\theta}{2} \right| - R + x_\Delta \cos \Delta_\theta + Rx_\theta \sin \Delta_\theta \right) \right] dr_\theta. \quad (31)$$

$$= e^{-ik_0 R} \int_{-\pi}^{\pi} X(\Delta_\theta) \exp \left(i2k_0 R \left| \sin \frac{\Delta_\theta}{2} \right| \right) \exp [ik_0 (x_\Delta \cos \Delta_\theta + Rx_\theta \sin \Delta_\theta)] dr_\theta. \quad (32)$$

where $X(\Delta_\theta) = \Pi/\sqrt{L_1}$. Considering that this integration is performed for a full 2π cycle of r_θ , and noting that the integrand has periodicity 2π , an integration across any range $a \rightarrow a + 2\pi$ will give the same result. Therefore, any constant offset can be added to, or subtracted from, Δ_θ and the result will be the same. Without loss of generality, the offset s_θ in Δ_θ can therefore be removed to give

$$I_1(x_r, x_\theta) = e^{-ik_0 R} \int_{-\pi}^{\pi} X(r_\theta) \exp \left(i2k_0 R \left| \sin \frac{r_\theta}{2} \right| \right) \exp [ik_0 (x_\Delta \cos r_\theta + Rx_\theta \sin r_\theta)] dr_\theta \quad (33)$$

indicating that the image I_1 , expressed in the local coordinates (x_r, x_θ) , will be the same for all sources. This is expected given the symmetry of the array.

The existence of analytical solutions to this integral is not apparent, however it is possible to consider the limit when $R \rightarrow \infty$ to obtain a solution. From before, it is noted that due to the image size x_m , both x_Δ and Rx_θ do not increase to infinity; therefore it is only the term $\exp[i2k_0R \sin(|r_\theta|/2)]$ which is affected as R increases. It is possible to define a linear operator based around this integral

$$L(f) = \int_{-\pi}^{\pi} \exp\left(i2k_0R \left|\sin \frac{r_\theta}{2}\right|\right) f(r_\theta) dr_\theta \quad (34)$$

when $R \rightarrow \infty$. It is assumed initially that the function f , and its gradient, are both finite throughout the range considered. Provided $\sin(|r_\theta|/2)$ is not at a stationary point, i.e.

$$\frac{d \sin(|r_\theta|/2)}{dr_\theta} = \frac{1}{2} \cos \frac{|r_\theta|}{2} \neq 0 \quad (35)$$

so that

$$r_\theta \neq \pm\pi \quad (36)$$

the frequency of the oscillations in the term $\exp[i2k_0R \sin(|r_\theta|/2)]$ must become infinite due to the presence of R . Therefore, across a single cycle, $f(r_\theta)$ can be treated as constant since the cycle is so short. Since an integration of a constant multiplied by a sinusoid across a single cycle cancels itself, the operator has no sensitivity to any value of $f(r_\theta)$ away from $r_\theta = \pm\pi$. Therefore, the integral is effectively selecting just the values at the stationary points $r_\theta = \pm\pi$ i.e.

$$L(f) = \frac{A}{2} [f(\pi) + f(-\pi)]. \quad (37)$$

where A is a constant defined as

$$A = \int_{-\pi}^{\pi} \exp\left(i2k_0R \left|\sin \frac{r_\theta}{2}\right|\right) dr_\theta. \quad (38)$$

Applying this to the imaging integral (33), and temporarily neglecting the amplitude term $X(r_\theta)$

$$I_1'(x_r, x_\theta) = e^{-ik_0R} \int_{-\pi}^{\pi} \exp\left(i2k_0R \left|\sin \frac{r_\theta}{2}\right|\right) \exp[ik_0(x_\Delta \cos r_\theta + Rx_\theta \sin r_\theta)] dr_\theta \quad (39)$$

$$= e^{-ik_0R} A \exp(-ik_0x_\Delta) \quad (40)$$

it can be seen that the initial image has the same form as a plane wave of frequency k_0 propagating away from the source (note that x_Δ increases towards the source from the above definitions). However, if the weighting $X(r_\theta)$ is taken into account, this presents a problem. $X(r_\theta) = \Pi/\sqrt{L_1}$ tends to $+\infty$ as it approaches $r_\theta = 0$ from both directions, and therefore the assumption above of a finite function is not valid. Since there will still be no response from any point where the function remains finite (apart from $r_\theta = \pm\pi$ as discussed), the only effect of the infinite value at $r_\theta = 0$ can be to introduce an additional component from the integral at $r_\theta = 0$, i.e.

$$I_1(x_r, x_\theta) = e^{-ik_0R} \int_{-\pi}^{\pi} X(r_\theta) \exp\left(i2k_0R \left|\sin \frac{r_\theta}{2}\right|\right) \exp[ik_0(x_\Delta \cos r_\theta + Rx_\theta \sin r_\theta)] dr_\theta \quad (41)$$

$$= e^{-ik_0R} [A \exp(-ik_0x_\Delta) + B \exp(ik_0x_\Delta)]. \quad (42)$$

While the value for A can be calculated (possibly numerically) as in eq. (38), B is essentially undefined. In practice, many transducer arrays are directional so the field will only be measurable on the far side of the array, and therefore the components near $\Delta_\theta = 0$ will become negligible. Also, the presence of cross-talk means that early measurements are swamped and the signals must be gated, which will remove any measurements taken near the source. However, for the purposes of this analysis, to keep it as general as possible, it will be taken that both A and B are arbitrary non-zero constants.

In the second stage of beamforming, the previous image I_1 is multiplied by a focusing term and integrated around the source array

$$I_2(\mathbf{x}) = \int_{-\pi}^{\pi} I_1(x_r, x_\theta) \exp(-ik_0x_r) ds_\theta \quad (43)$$

where the far field focusing law has been represented by a plane wave. s_θ represents the angle of the source, defined as shown in Fig. 1(b). $\mathbf{x}$ represents the imaging point in global coordinates, which is distinct from x_r and x_θ which are relative to the source location. Substituting the result of (42) gives

$$I_2(\mathbf{x}) = \int_{-\pi}^{\pi} e^{-ik_0 R} [A \exp(-ik_0 x_\Delta) + B \exp(ik_0 x_\Delta)] \exp(-ik_0 x_r) ds_\theta \quad (44)$$

$$= e^{-ik_0 2R} \int_{-\pi}^{\pi} [A \exp(-ik_0 x_\Delta) + B \exp(ik_0 x_\Delta)] \exp(ik_0 x_\Delta) ds_\theta \quad (45)$$

$$= e^{-ik_0 2R} \int_{-\pi}^{\pi} [A + B \exp(i2k_0 x_\Delta)] ds_\theta. \quad (46)$$

There are two components in the integrand, a constant term amplitude A and an oscillatory term of spatial frequency $2k_0$, amplitude B . The oscillatory term has the mathematical form of a plane wave travelling towards the source, with phase zero at the centre, so the resulting image is effectively a sum of plane waves at all angles, with a constant offset from A . The result of this is known analytically, giving

$$I_2(\mathbf{x}) = e^{ik_0 2R} [2\pi A + 2\pi B J_0(2k_0 |\mathbf{x}|)] \quad (47)$$

from the Bessel integral, where J_0 is the 0th order first kind Bessel function. This result highlights that the incident component does not equal zero when imaged with the beamforming algorithm.

The primary interest of this paper is the diffraction tomography algorithm, and the key difference as highlighted earlier between beamforming and diffraction tomography is the presence of a weighting term in the integrals around the source and receiver of $W = \sin |\Delta_\theta|$. In this case, the first focusing integral becomes

$$I_{1,dt}(x_r, x_\theta) = e^{ik_0 R} \int_{-\pi}^{\pi} X(r_\theta) \exp\left(i2k_0 R \left|\sin \frac{r_\theta}{2}\right|\right) \exp[ik_0 (-x_\Delta \cos r_\theta - R x_\theta \sin r_\theta)] |\sin r_\theta| dr_\theta \quad (48)$$

where as before, the substitution $\Delta_\theta = r_\theta$ has been used without loss of generality. Importantly, it is noted that the weighting term introduced, $|\sin r_\theta|$, disappears to zero when $r_\theta = -\pi, 0, \pi$. The results above showed that there was a non-zero response from the beamforming integral of eq. (42) at exactly the same angles $r_\theta = -\pi, 0, \pi$. Therefore, the effect of the weighting function $|\sin r_\theta|$ is to remove the only components of the incident signal which would otherwise appear, making the total response of the diffraction tomography algorithm to the incident field zero. This result indicates that the incident field has no effect on the final image, confirming what has been observed in the previous papers.

An alternative method, also discussed earlier, of generating the diffraction tomography image is by applying a filter to a beamforming image [8]; this filter is mathematically equivalent to the $|\sin \Delta_\theta|$ weighting used above, but is applied as a post-processing approach rather than incorporating it into the focusing integral. This filter has the form (from eq. (14))

$$W(|\mathbf{K}|) = |\mathbf{K}| \sqrt{1 - \left(\frac{|\mathbf{K}|}{2k_0}\right)^2}. \quad (49)$$

Examining the integrand of (46), $A + B \exp(i2k_0 x_\Delta)$, shows that there are two components, one with spatial frequency $2k_0$ and one constant, effectively with spatial frequency 0. However, when the filter of eq. (49) is applied, $W(|\mathbf{K}|) = 0$ at both of these spatial frequencies $|\mathbf{K}| = 0$ and $|\mathbf{K}| = 2k_0$, therefore removing both of these components produced by the incident field. It should be acknowledged here that the analysis has been performed on the integrand of (46) rather than the final integrated result of (47), however, the linearity of both the filter and the integration operator means that the order is irrelevant, and the conclusion is therefore that the incident field is completely removed from the final image by the application of the filter. This matches the result from before when the weighting is incorporated into the integral, confirming that under the far field assumptions used here, the incident field components will automatically completely disappear in the final images.

4 Analysis

Previous publications have already demonstrated that DT with the total field works with experimental data for breast imaging [6,21] and for guided wave tomography [11,22,23]. Therefore the approach is assumed experimentally validated, and instead the focus here is on numerically analysing the theoretical results.

In all cases in this section, unless otherwise specified, the background sound speed is 1500m/s and frequency is 1.5MHz, giving a wavelength $\lambda = 1\text{mm}$. Equation (19) is used to calculate the incident wavefield, and in all cases a full matrix of data is calculated, producing signals from all the different send-receive combinations in the array. It is noted that some form of reference is needed, against which the amplitude of any incident field residual can be judged. For these purposes the scattered field from a reference object, a cylinder radius 2mm and velocity 1650m/s (10% contrast to the background velocity), is added to the incident field prior to imaging. This field is calculated under the Born approximation, numerically performing the integration of (16) by summing across a grid with spacing 0.2mm. This same cylinder will be incorporated into all images, by adding its scattered field onto the incident field; for reference the object function within the cylinder is $-6.85 \times 10^6 \text{m}^{-2}$.

4.1 Far field behaviour

The first case is to evaluate the far field imaging behaviour, which should match the derivation from Sect. 3. Under this scenario, both the weighted integral and the filtering approach should produce identical results and perfectly remove the incident field. There is a practical limitation on how large the array can be because it must remain sampled at less than $\lambda/2$, meaning that more transducers must be used for larger array sizes. A radius $R = 40\lambda$ was chosen as a compromise for this section, and 512 transducers spaced at 0.49λ were used to sample this field. An imaging grid of size $10\lambda \times 10\lambda$ is positioned at the centre of the array, with pixel size 0.1λ .

Figure 2 shows the results when imaging in the far field. Figure 2(a) shows the true object function for the cylinder. In all images throughout this paper the object function, i.e. the fundamental quantity reconstructed by the imaging algorithm, will be plotted, although it is straightforward to convert this to the equivalent velocity via eq. (2). Figures 2(b) and (c) show the results when imaging using the filtered version and Figs. 2(d) and (e) use the weighted version of the DT algorithm. Figures 2(b) and (d) are reference images, showing the results when just the scattered field is imaged, i.e. no incident component is present. Both of these images give a good reconstruction, showing few artefacts. The size of the object and its contrast are both correct. It is noted that there is a near-constant offset in both images, which is slightly worse for the weighted DT algorithm, however this is small relative to the object contrast. Figures 2(c) and (e) are imaged from the total field, incorporating the incident component, while Fig. 2(f) plots the values on a line passing through the centre of the object. Clearly, the incident component is not completely removed in either case, and is distorting the image away from its true value. However, the effect in each case is different; the filtering of Fig. 2(c) does seem to correctly account for the low frequency component, but does not remove the high frequency component, while the weighted integral Fig. 2(e) removes the high frequency component but not the low frequency component.

Initially, the filtered beamforming image, shown in Fig. 2(c), will be examined. According to the theoretical derivation, there will be high frequency components present in the beamforming image, but these will appear at a spatial frequency of exactly $|\mathbf{K}| = 2k_0$ which will be removed by the filter. These components are caused by the amplitude of the incident field becoming very large near the source; while for the reasons mentioned (such as signal gating and directional transducers) in practice this signal component will become negligible, this does not explain why the component appears when it should be removed by the filter. The beamforming algorithm produces a sum of plane waves in all directions, which produces the result illustrated in Fig. 3(a), however, because these waves are truncated by the edge of the image, there is a degree of spectral leakage to components away from $|\mathbf{K}| = 2k_0$ and therefore the filter does not remove the entire effect of the incident field. Figure 3(b) illustrates this, with an image of a sinusoidal function at 45 degrees, described by $\exp\left[i2k_0\frac{1}{\sqrt{2}}(x+y)\right]$ being passed through the beamforming to diffraction tomography filter, showing that there is clearly still a notable residual in the reconstruction. Applying smooth windowing to the edge of the image could help with this, however there must always be some spectral leakage for any non-infinite image, so the components will always spread away from $|\mathbf{K}| = 2k_0$. One approach to this problem would be to additionally filter the image to remove high frequency components in the range $|\mathbf{K}| > k_{lim}$ where k_{lim} is sufficiently low to remove the components associated with the incident field, but still close to $2k_0$ to avoid resolution loss, which has often been used in practice (e.g. [11]). Alternatively, the approach considered here is to locally gate the incident field components to remove any values near to the source. The weighting function around the array, expressed as a function of the source to receiver distance L_1 is

$$g(L_1) = \begin{cases} 1 & L_1 < 0.1R \\ 0 & L_1 > 0.05R \\ 1 + \cos\left(\pi\frac{L_1-0.1R}{0.05R}\right) & \text{otherwise} \end{cases} \quad (50)$$

This is then multiplied by each component of the incident signal prior to adding the scattered signal. This process is very similar to what is done with experimental measurements to remove cross-talk, and it is noted that it removes

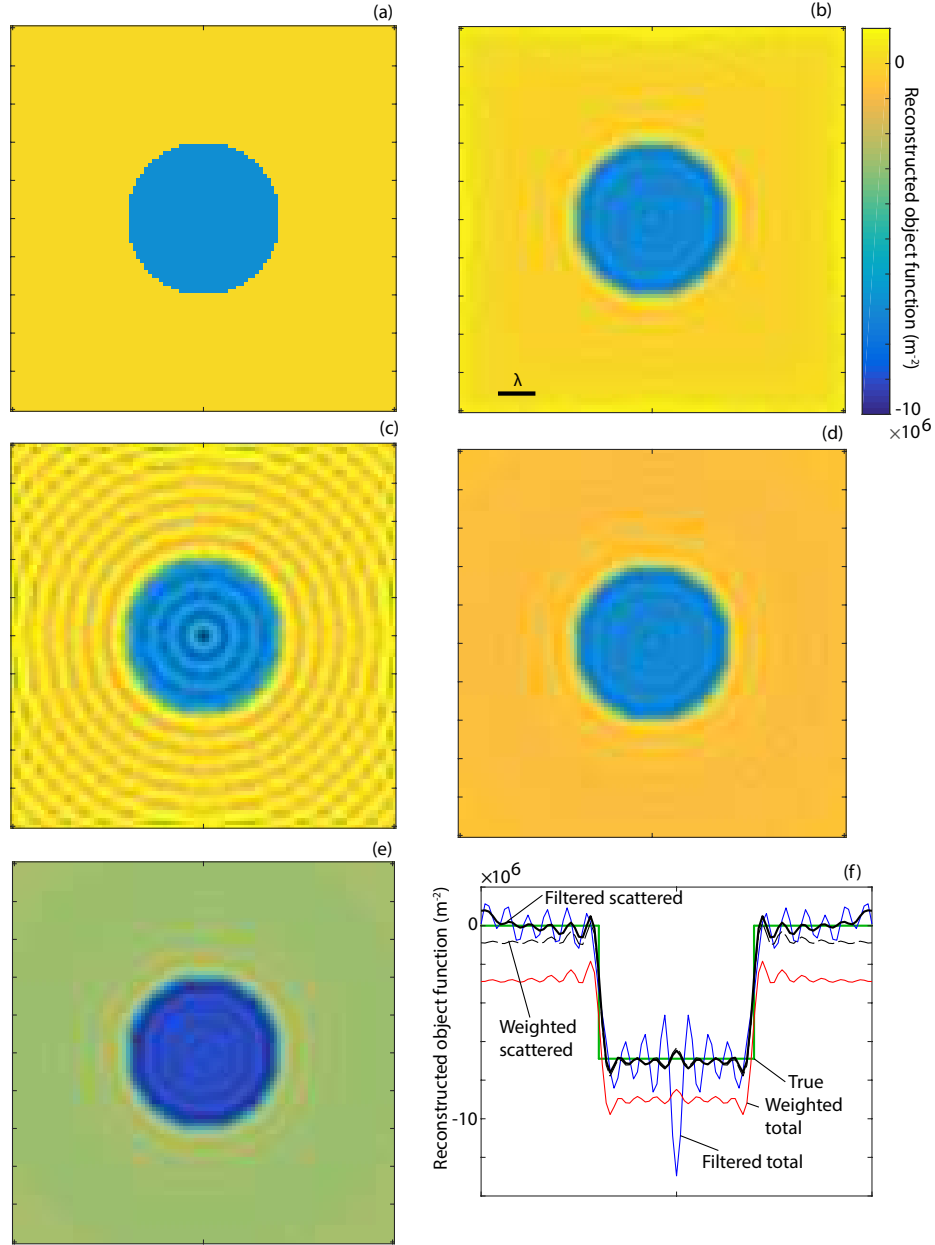


Figure 2: Results of imaging with both the scattered and total field data sets in the far field, for the scatterer shown in (a), a cylinder of velocity 1650m/s and radius 2λ . (b) and (c) give reconstructions from the filtered version of the DT algorithm, while (d) and (e) show the weighted version. (b) and (d) are reconstructed from the scattered component, while (c) and (e) are from the total field, i.e. incorporating the incident component. (f) gives a comparison of the images plotted along a horizontal diameter line through the centre of the object.

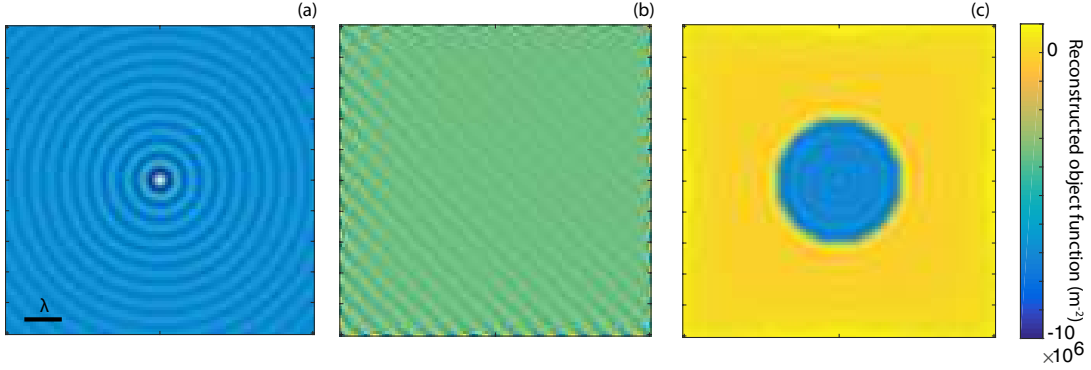


Figure 3: (a) shows the beamforming image of the incident component, a sum of plane waves which should be removed by the filter when converting to a DT image. (b) shows the result if an image of $\exp\left[i2k_0\frac{1}{\sqrt{2}}(x+y)\right]$ is passed through the beamforming to diffraction tomography filter. (c) shows the filtered image of the total field (i.e. equivalent to Fig. 2(d)) where the local values of the incident field have been removed by windowing signals closer than 10% of R according to eq. (50). The true map of this image is given in Fig. 2(a).

components which would often not be measured anyway because of directional transducers. The resulting image is shown in Fig. 3(c). This is much cleaner and essentially indistinguishable from the scattered field image of Fig. 2(b).

An analysis of the weighted DT performance is given in Appendix A; in summary the cause of the error is that the weighting term does not perform well if data is not taken sufficiently far from the scatterer and therefore a small residual in the $|\mathbf{K}| = 0$ component exists. If the data is projected into the far field, has the weighting applied, then is projected back prior to imaging, this residual is eliminated. In practice this is not a particularly practical solution, so for this reason the weighted DT algorithm is not considered further in this paper, with the focus instead being on the filtered version. However, recognising that the weighted algorithm generally eliminates the high frequency components while the filtered version is better with eliminating the low frequency components, it is also possible to derive a hybrid technique which weights the reflected (high frequency) components and filters the low frequency (transmitted) components.

This section has shown that, for the far field, the incident field vanishes in the DT images, confirming the theoretical results of Sec. 3. There are imperfections which appear in the images when implementing in practice, however, a number of practical solutions to this which are implemented in practice have been discussed.

4.2 Near field behaviour

While the far field provides idealised conditions for the theoretical calculations, simplifying much of the mathematics, placing transducers in the far field is rarely practical. Instead imaging is generally performed in a region where the transducers are closer to the scatterer. Most of the practical cases where DT imaging with the total field has been performed do not rely on the far field (i.e. plane wave illumination) assumption (e.g. [11, 24]), yet the incident component appears to have a negligible effect.

To investigate this, data is now generated for an array of radius 10λ , or 10mm for the case we are considering here. In this case, 128 transducers are used, sufficient to sample the field at just beneath $\lambda/2$. Figure 4(a) shows the true object function, and marks the array. Figures 4(b) and (c) present the filtered beamforming DT images generated with the scattered and total fields respectively. For the total field, as before, the incident components near to the source are removed via eq. (50). The image width in this case is larger than the transducer array, so the array itself lies completely within the imaging area. There will effectively be a local singularity in the image at each transducer, and therefore a ring of high values is introduced around the array, and high frequency components appear to spread from these into the main imaging area. It is noted that many of these artefacts exist when just imaging with the scattered field; Figs. 4(b) and (c) are very similar through the majority of the image which provides some indication that the incident field is having little effect for the near field configuration.

The high spatial frequency components should be minimised by the filter, however, as shown in Sect. 4.1, the performance of the filter for high frequencies is poor because of the finite size of the imaging area causing spectral leakage. Therefore the proposed hybrid approach mentioned above is implemented. Both a weighting and a filter are used, but the weighting of eq. (15) only acts on the high frequency components and the filter of eq. (14) on the

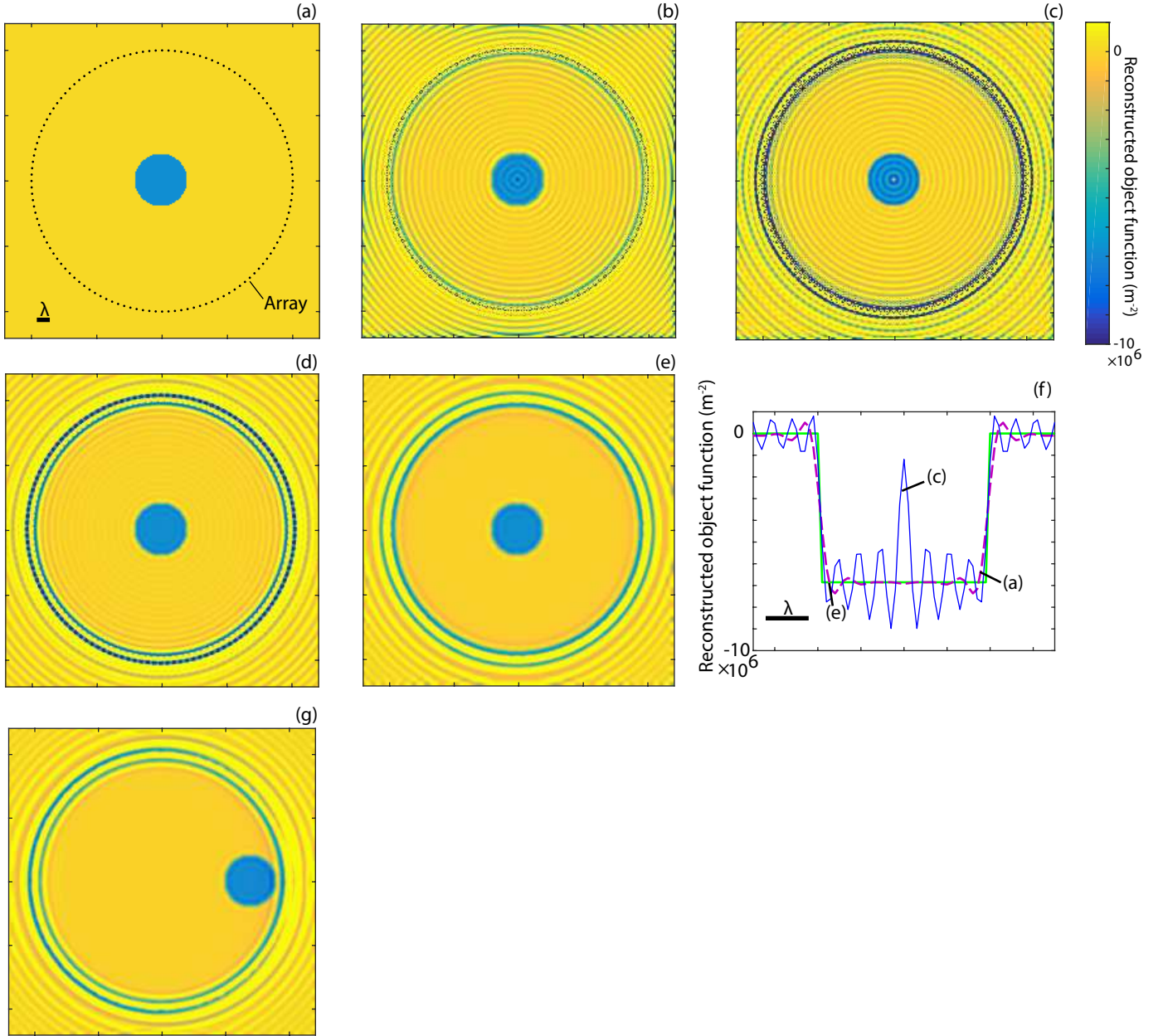


Figure 4: Imaging with near field data. In all cases the array radius is 10λ , and is marked on (a). In all cases the colours in the images have been clipped to those in the colour scale; large spikes are generally seen at the array where most of the clipping occurs. (a) shows the true object function. (b) shows the filtered beamforming image of the scattered field, and (c) shows the same image generated from the total field. In (d) the hybrid approach is used with the weighting accounting for the high spatial frequencies and the filter for the low spatial frequencies. (e) is the same as (c) except with an additional filter applied as described in eq. (53). (f) presents cross sections through the centre for images (a) (c) and (e). In (g), the image of (e) has been reproduced except with the scatterer offset to be much closer to the array.

low frequency components. The filter is described as

$$W_{filt}(|\mathbf{K}|) = \begin{cases} |\mathbf{K}| \sqrt{1 - \left(\frac{|\mathbf{K}|}{2k_0}\right)^2} & |\mathbf{K}| < \sqrt{2}k_0 \\ k_0 & otherwise \end{cases} \quad (51)$$

and the weighting in the integral as

$$W_{int}(\Delta_\theta) = \begin{cases} |\sin \Delta_\theta| & |\Delta_\theta| < \pi \\ 1 & otherwise \end{cases}. \quad (52)$$

The resulting image is shown in Fig. 4(d). The high frequency components are well suppressed in this image and the overall reconstruction is improved.

Past papers such as [8, 11] have not used this hybrid approach, instead applying an additional filter to the image. The standard DT reconstruction has a sharp boundary in its frequency domain components at $|\mathbf{K}| = 2k_0$. This introduces many high frequency ‘ringing’ components into the image, of the sort which can be seen in Figs. 4(b) and (c). The additional filter acts to smooth the sharp boundary, at the expense of resolution, and is defined as a flat function with a cosine taper to zero

$$W_{smooth}(\mathbf{K}) = \begin{cases} 1 & |\mathbf{K}|/k_0 < k_1 \\ 0 & |\mathbf{K}|/k_0 > k_2 \\ 1 + \cos\left(\pi \frac{|\mathbf{K}|/k_0 - k_1}{k_2 - k_1}\right) & otherwise \end{cases} \quad (53)$$

where k_1 and k_2 define the points between which the taper occurs. Figure 4(e) shows the DT image from the total field, generated by the filtering approach, with this additional filter incorporated where k_1 and k_2 are 1 and 2 respectively. The artefacts are suppressed here, although the resolution is worse than before. The cross sections of Fig. 4(f) highlight these differences; the artefacts are reduced, but the gradient at the edge of the cylinder is gentler for (e), which indicates worse resolution.

Finally, Fig. 4(g) confirms the performance for scatterers close to the array. This image has been generated in the same way as Fig. 4(e), except now the scatterer is offset by 7mm and is therefore much closer to the array than before, which confirms that the performance seen for the scatterers near the centre holds when the scatterer is much closer to the array. This is to be expected since the final image represents a linear superposition of images of the scattered field and the incident field; since the artefacts caused by the incident field are not significantly worse closer to the array (with the exception of the values closer than a wavelength), the scattered field image can be superposed in any location without a significant adverse effect on the reconstruction.

4.3 Sampling

The previous section utilised sufficient transducers to ensure the wavefield is fully sampled, i.e. with spacing of less than $\lambda/2$. While this $\lambda/2$ limit, based on the Shannon-Nyquist theory [25], is valid for linear arrays, this is not necessarily the case for circular configurations. This can be considered with a simple thought-experiment. In the far field, the scattered field is invariant with radius, save a factor of $e^{ik_0 r}/\sqrt{r}$ accounting for the variation in amplitude and phase. Therefore if the field is measured at a radius r_1 with sampling of $\lambda/2$ to give $4\pi r_1/\lambda$ transducers, which is sufficient to fully sample the field, the same number must be enough to sample at a larger radius too, since the field is effectively unchanged, despite the spatial sampling now being lower. As demonstrated in [26], it is possible to take the $\lambda/2$ sampling criterion and apply it to the smallest possible circle enclosing all the scatterers in order to establish the number of transducers needed. While this was developed for a circle at the centre of the array, other work has also demonstrated that lower sampling can be achieved by moving the effective sampling circle for offset scatterers [19]. This has also been studied in detail in [27]. However, all this theory was developed for sampling purely the scattered field. This section will evaluate what sampling is necessary to sample the total field instead, and what are the consequences for the resulting images.

A schematic of the sampling behaviour of the incident field is shown in Fig. 5. A wave propagates from the source and intersects two adjacent transducers, with the gap Δ marked between them. Given the oblique angle, the gap between transducers, projected parallel to the wave propagation direction is

$$\Delta_w = \Delta \sin \alpha = \Delta \cos \frac{r_\theta}{2}. \quad (54)$$

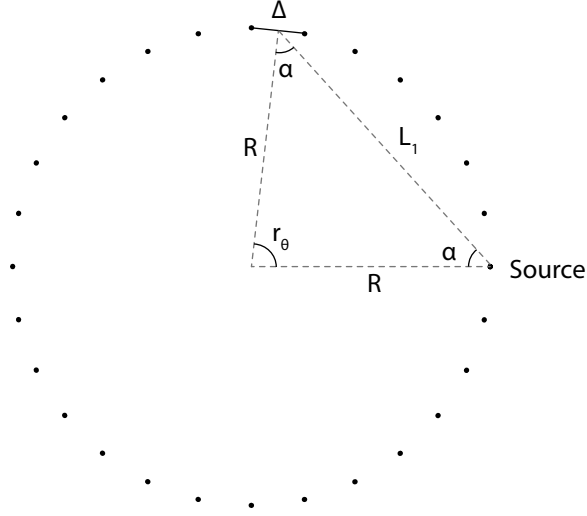


Figure 5: Schematic of the parameters for calculating the sampling criterion.

It is possible to define a limiting value for Δ as the point beyond which aliasing would occur, i.e. the Shannon-Nyquist limit

$$\Delta_{lim} \cos \frac{r_\theta}{2} = \frac{\lambda}{2}. \quad (55)$$

Recognising that the spatial frequency component $\mathbf{K}$, corresponding to a scattering angle r_θ , in the far field becomes

$$|\mathbf{K}| = 2k_0 \cos \frac{r_\theta}{2} \quad (56)$$

then

$$\frac{|\mathbf{K}|}{2k_0} = \frac{\lambda/2}{\Delta_{lim}}. \quad (57)$$

This equation defines the spatial frequency at which artefacts caused by aliasing associated with the incident field will appear in the image. When $|\mathbf{K}|$ exceeds this limit, $\cos(r_\theta/2)$ is larger so the sampling rate will be too low. Therefore this equation defines an upper limit in $|\mathbf{K}|$, with artefacts appearing at higher wavenumbers. In order to sample the full Ewald disk up to $|\mathbf{K}| = 2k_0$, the transducer spacing, as expected, becomes $\Delta_{lim} = \lambda/2$. However, this result suggests that even for subsampled cases, this subsampling should only affect a subset of the spatial frequencies which could be filtered.

Figure 6 shows the result of different levels of sampling when imaging with the total field. Figure 6(a) represents a fully sampled reconstruction, with 128 transducers in a circular array of radius 10mm, giving a spacing of 0.491mm, as used in the previous section. This image is generated through the filtered beamforming DT approach in the same way as Fig. 4(c). Figure 6(b) then halves the number of transducers, so that sampling only occurs at a spacing of around 1λ . It is clear that there are many more artefacts present in this image, which have been caused by aliasing. Figure 6(c) shows the spatial frequency components of this image, with a clear ring appearing around $|\mathbf{K}| = k_0$, which corresponds to these aliasing artefacts. These are predicted by eq. (57), and by filtering with $k_1 = 0.9$ and $k_2 = 1.0$ in eq. (53), it is possible to remove these components from the image, as shown in Figure 6(d). This can be taken to more extreme levels, as shown in Figs. 6(e) and (g), for 32 and 16 transducers respectively. Figure 6(f) then filters the 32 transducer image by halving the previous limits to $k_1 = 0.45$ and $k_2 = 0.5$, and Fig. 6(h) filters the 16 transducer image with $k_1 = 0.225$ and $k_2 = 0.25$.

This has confirmed that subsampling will generate artefacts corresponding to the incident field in the image, however as predicted these are above the limit defined by eq. (57) so can be removed with a spatial filter. However, this does inevitably limit the resolution, meaning that a final image in a subsampled scenario can be assumed to have a resolution limit proportional to the transducer spacing if this is greater than $\lambda/2$. Finally, it is noted that just a single frequency is considered; typical measurements are broadband and the use of multiple frequencies could potentially enable aliasing to be minimised and hence allow larger sampling intervals, however this is not considered in this paper.

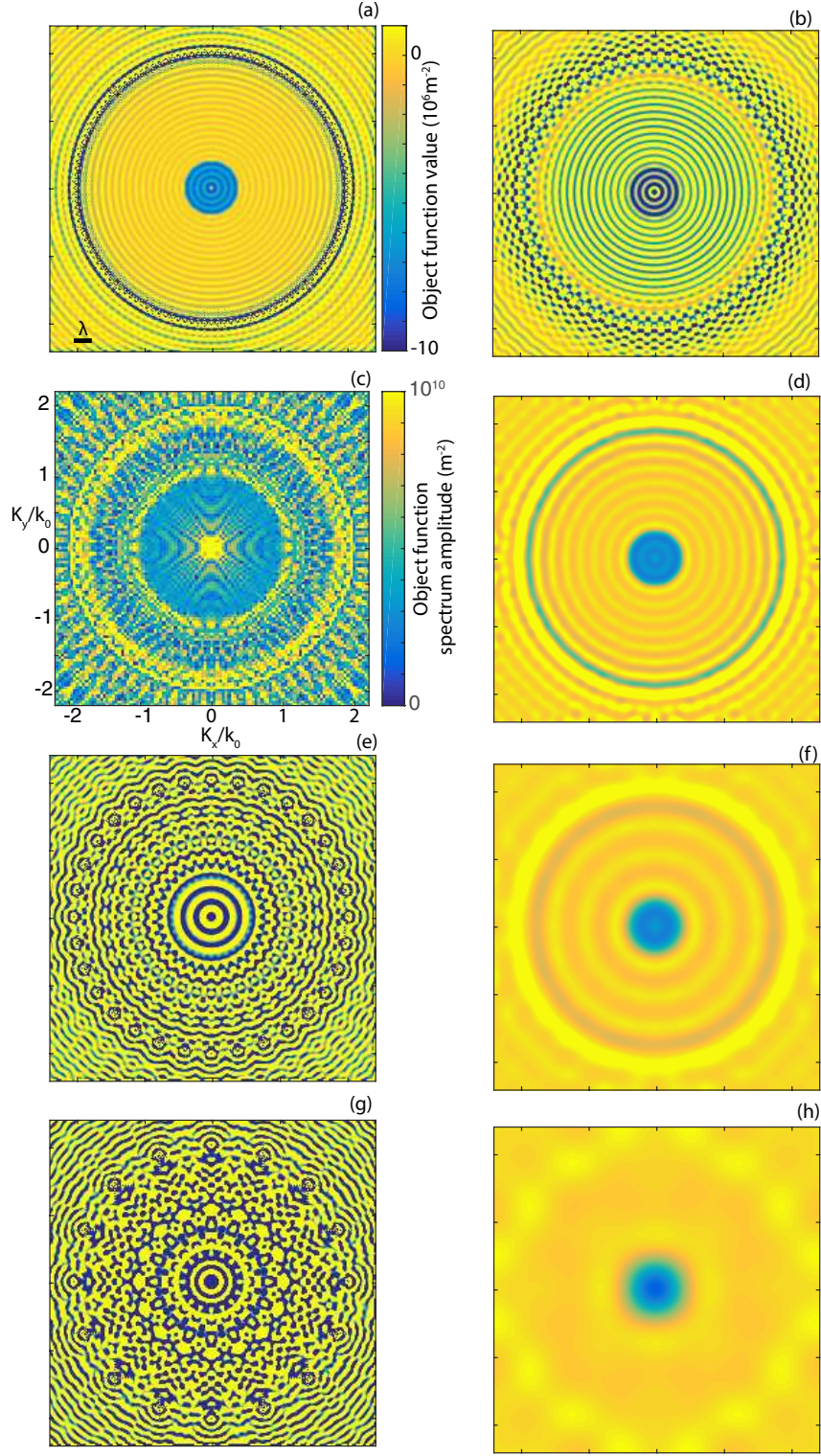


Figure 6: Imaging with different sampling. In all cases the imaging configuration corresponds to Fig. 4(a), with different numbers of transducers. All images are produced by imaging the total field. (a) shows 128 transducers, a fully sampled array with sampling just below $\lambda/2$. (b) gives the image produced with half the number of transducers, 64, and (c) gives the corresponding 2D Fourier transform of this. (d) is the same as (b) except with a filter removing components above $|\mathbf{K}| = k_0$, (e) uses 32 transducers with (f) similarly filtering off components above $|\mathbf{K}| = k_0/2$, then (g) uses 16 transducers with (h) filtering off components above $|\mathbf{K}| = k_0/4$. All images (b), (d)-(h) have colour scales corresponding to that of (a).

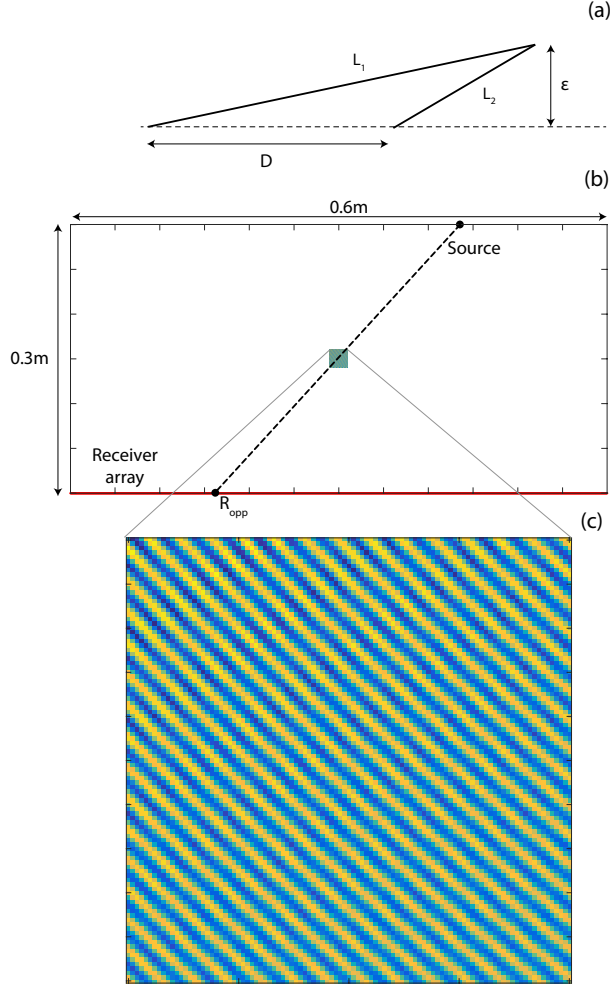


Figure 7: (a) Definitions of D and ϵ relative to L_1 and L_2 . (b) shows a setup with a source and a receiver array in the far field, with the resulting image I_1 , generated by focusing the incident field back from the receiver array, in a larger scale shown in (c).

4.4 Alternative array configurations

So far the case of ideal circular arrays has been considered. However, it is possible to image with the total field using other configurations. In the far field circular case it was demonstrated that there were two key components of the incident field to consider; one corresponded to the singularity in the source, which will not arise in practice for reasons discussed, and the other is the stationary point in the phase of the incident field $\exp[i2k_0 R \sin(|r_\theta|/2)]$. Both terms disappeared because they produce spatial frequencies which are filtered out of the final DT image.

There is now a new complexity to this problem. The circular analysis, from which the stationary point observation was made, relied on the imaging area being small relative to the ‘far field’ array radius, such that the variation of L_2 , the focusing distance from the array to the imaging point, around the array was slow compared to the variation of L_1 , the distance travelled by the incident wave from the source to the receiver. If the array is not circular, L_2 cannot be assumed to vary slowly, and therefore cannot be neglected. In this case, rather than considering just the stationary point of the phase of $\exp(ik_0 L_1)$, this must be done instead for $\exp[ik_0 (L_1 - L_2)]$.

Figure 7(a) shows a construction of L_1 and L_2 for a source and imaging point on the same horizontal line. ϵ marks the vertical distance of the receiver from that line. Considering initially the case where $\epsilon = 0$, L_1 and L_2

must align and therefore $L_1 - L_2 = D$, which remains the case for all values of L_1 and L_2 . Then if ε is nonzero

$$L_1 = \sqrt{L_{1x}^2 + \varepsilon^2} \quad (58)$$

$$L_2 = \sqrt{L_{2x}^2 + \varepsilon^2} \quad (59)$$

$$\frac{L_1}{L_{1x}} \approx 1 + \frac{\varepsilon^2}{2} \quad (60)$$

$$\frac{L_2}{L_{2x}} \approx 1 + \frac{\varepsilon^2}{2} \quad (61)$$

$$L_1 - L_2 \approx (L_{1x} - L_{2x}) \left(1 + \frac{\varepsilon^2}{2}\right) \quad (62)$$

$$\approx D \left(1 + \frac{\varepsilon^2}{2}\right) \quad (63)$$

where L_{1x} and L_{2x} are used to refer to the projections of L_1 and L_2 onto the horizontal line, and the binomial expansion has been used for the square root and truncated as in eq. (29), assuming ε is small. This demonstrates that increasing ε , in either direction (due to squaring), will increase the value $L_1 - L_2$. However, the value along the line itself will not change. Any point on the dashed line will therefore be a stationary point. It should be noted that the diagram of Fig. 7(a) can be rotated in any way without affecting the behaviour, so this is a general result. Under far field conditions, as shown before, the only contribution to the beamforming integral will come from the stationary point, because at all other points the frequency will be infinite. Therefore the only contributions will come from the point where the receiver array intersects the straight line extended from the source to the imaging point.

Figure 7(b) shows an illustration of this principle. Here the incident field has been measured across the receiver array and focused into the image area marked; this image is zoomed into for clarity in Fig. 7(c). A straight line has been marked between the source and the image centre, continuing to intersect the receiver array. Under the theory discussed above, the stationary point of the phase of $\exp[ik_0(L_1 - L_2)]$ must lie on this line, and hence the main contribution in far field conditions must be from the point where the line intersects the receiver array, marked as R_{opp} . The I_1 image of this confirms the result: a near ‘plane wave’ (although it is stressed that this is not actually a wavefield, rather an image produced by backfocusing measured wavefields) is seen with wavefronts perpendicular to the line between the source and the image point. This contribution to the image must have been produced from the point R_{opp} . Interestingly a slight curvature is visible in the wavefronts in Fig. 7(c), keeping them aligned with the source. This curvature is an indication that a true ‘far field’ situation is not possible to achieve in practice, so for each image pixel, the incident contribution is coming from its own, slightly different, R_{opp} point. In the far field, the values radiated back from the receiver to the source will have the form

$$I_1 = A \exp(ik_0 \hat{\mathbf{s}}_0 \cdot \mathbf{x}). \quad (64)$$

This equation is the same as (42), which expressed the function in a coordinate system local to each source so had no source dependence, rather than using the vector $\hat{\mathbf{s}}_0$. This has demonstrated that the form of I_1 is unchanged by the shape of the array. Therefore, the conclusions from before for the circular array will hold when accounting for the second focusing, i.e. the incident term will produce two components which are both removed by the beamforming to DT filter.

To demonstrate this, Figs. 8(b) and (c) show two DT images, generated by the beamforming filtering approach, from the scattered and the total field respectively for the configuration shown in Fig. 8(a). 40 transducers are used on each edge, separated by 0.5mm ($\lambda/2$), and it is noted that the beamforming algorithm applies a weighting when it sums around the array for each pixel to account for non-uniform angular sampling. As in the examples before, the two images are similar to each other, with only minor differences. The total field has some high frequency artefacts visible in it, but these are very small, and larger values around the transducer array. As demonstrated before, some additional image processing (e.g. filtering) to remove the high frequency components would usually be used, even just for the image from the scattered field. This is not included here since these filters can mask the effects of the incident field. In some cases, there may be a configuration where the receiver array does not lie on top of the receiver array. One envisioned example is when a pair of parallel linear arrays are used; one to transmit and the other to receive, as shown in Fig. 8(d). The result from such a configuration, with 40 transducers in each array, are shown in Figs. 8(e) and (f) for the scattered and total field respectively. In this case the image of the defect itself has suffered because of the limited view introducing artefacts, however the important result is that the incident field has very little effect on the image, just producing some additional higher frequency artefacts.

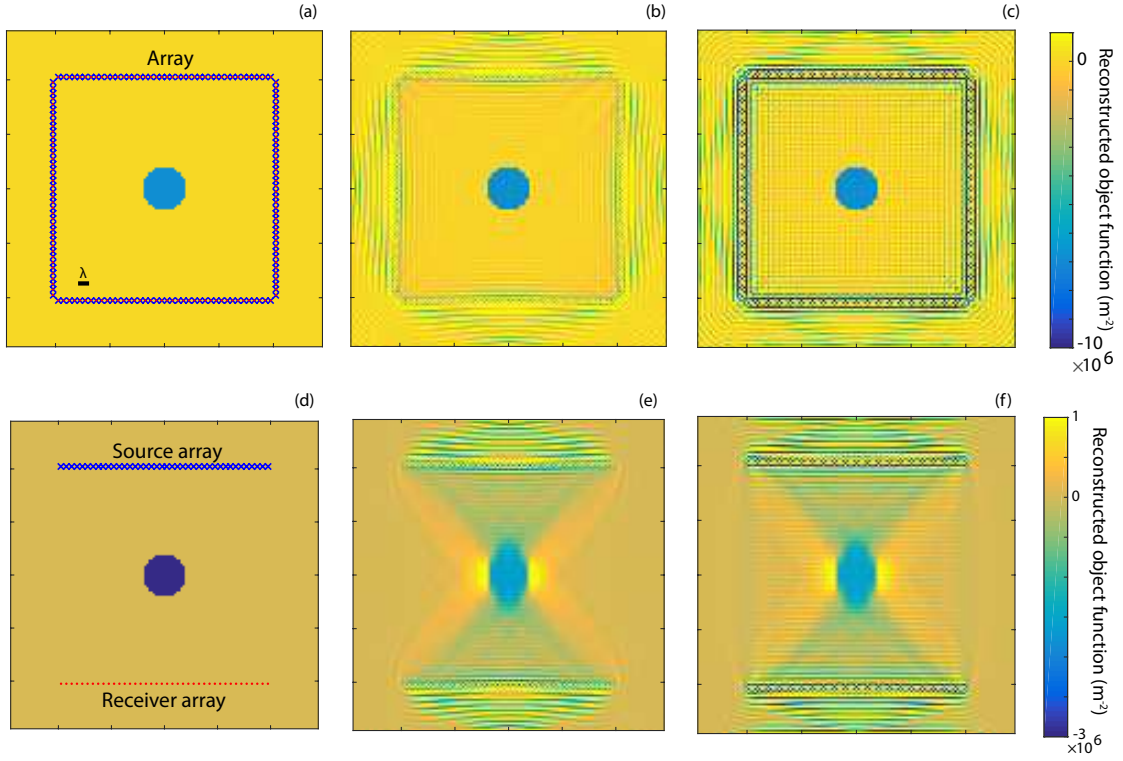


Figure 8: Images with different array configurations. (a) shows a square array around the scatterer, with each transducer functioning as both a source and a receiver. (b) shows the reconstruction from this array for the scattered field, while (c) shows the same from the total field. (d) gives a limited view configuration, with a linear source array about the scatterer and a parallel receiver array below. (e) images this using the scattered field and (f) the total field.

The previous work demonstrated that the incident field had little to no effect on images from an idealised, circular, array. This section has extended the theory, demonstrating that this is also the case with more general array configurations. Also, the case with limited view arrays has been shown, and again, the incident field has a negligible effect.

4.5 Errors

The analysis in this paper has focused on idealised data without errors present. In practice, such measurements are impossible to acquire, and in fact, if it was possible, it would be straightforward to subtract the incident field anyway. It is therefore important to assess the robustness of imaging using the total field in the presence of errors. As mentioned, previous publications have tested the performance when imaging from the total field with measured data (e.g. [11, 21, 23]), as well as in some cases looking at errors (see Chapter 3 of [28]), although the primary aim was not specifically focused on assessing the error behaviour with the incident field present. This section analyses the effect that systematic and uncorrelated random errors will have on the result. It should be noted that calibration techniques can often minimise such errors [21, 28], but these are not used here.

Systematic errors are considered initially. In this case, all measurements are distorted in some correlated manner. One of the most common scenarios for this is temperature variation, which causes the background wave speed to change. If this is the case, the background wavenumber becomes k_1 , leading to the incident field with errors in it being

$$\phi_{0es} = \Pi \frac{e^{ik_1 r}}{\sqrt{r}}. \quad (65)$$

Putting this into the integral from eq. (33), i.e. for the far field circular array, this becomes

$$I_1(x_r, x_\theta) = e^{-ik_0 R} \int_{-\pi}^{\pi} X(r_\theta) \exp\left(i2k_1 R \left|\sin \frac{r_\theta}{2}\right|\right) \exp[ik_0(x_\Delta \cos r_\theta + Rx_\theta \sin r_\theta)] dr_\theta \quad (66)$$

$$= e^{-ik_0 R} [A \exp(-ik_0 x_\Delta) + B \exp(ik_0 x_\Delta)]. \quad (67)$$

Here, since the incident field term just acts to select the point where its phase is stationary, the result is unchanged from eq. (42), under the far field assumption. It should be noted, however, that the constants A and B are likely to change, particularly in phase.

Figures 9(a) and (b) show the resulting images in the far and near fields respectively, with a 2.5% increase in velocity in the incident field; Figs. 9(c) and (d) provide respective reference images. In general, an error in velocity like this will cause an error in the scattered field, as well as the incident field, however the scattered field in this paper is only used as a reference, so the correct version is included in each case. The far field case Fig. 9(a) shows a slight difference in the background region, particularly towards the boundaries, caused by the incident field error. Towards the centre, the change does not seem to have significant effect, backing up the hypothesis that in the far field, the incident component has no effect on the resulting image when a systematic error is present. The near field example of Fig. 9(b) shows a larger distortion. It is clear that the near field does not conform as well as the far field case, particularly towards the array where the far field assumption breaks down more. However, the errors are still relatively modest, and while the incident field does not appear to completely disappear, it is still small compared to the scatterer.

Uncorrelated errors can also occur in many cases. This could be down to transducer mispositioning, or different transducers having different responses to the field. These will distort the measurements and may result in the incident field becoming significant in the images. Treating the effect of this as the combination of a small phase and amplitude errors, represented by ω and γ respectively, the response at one transducer will be

$$\phi_{0eu} = \Pi \frac{e^{i(k_0 r + \omega)}}{\sqrt{r}} (1 + \gamma). \quad (68)$$

It is known that the incident term itself has little effect on the images, so this can be subtracted

$$\phi_{0eu} - \phi_0 = \frac{\Pi}{\sqrt{r}} \left[e^{i(k_0 r + \omega)} (1 + \gamma) - e^{ik_0 r} \right] \quad (69)$$

$$= \phi_0 (e^{i\omega} + \gamma e^{i\omega} - 1). \quad (70)$$

Assuming that the phase error is small such that $e^{i\omega} = 1 + i\omega$

$$\phi_{0eu} - \phi_0 = \phi_0 (i\omega + \gamma + i\omega\gamma) \quad (71)$$

$$\approx \phi_0 (i\omega + \gamma) \quad (72)$$

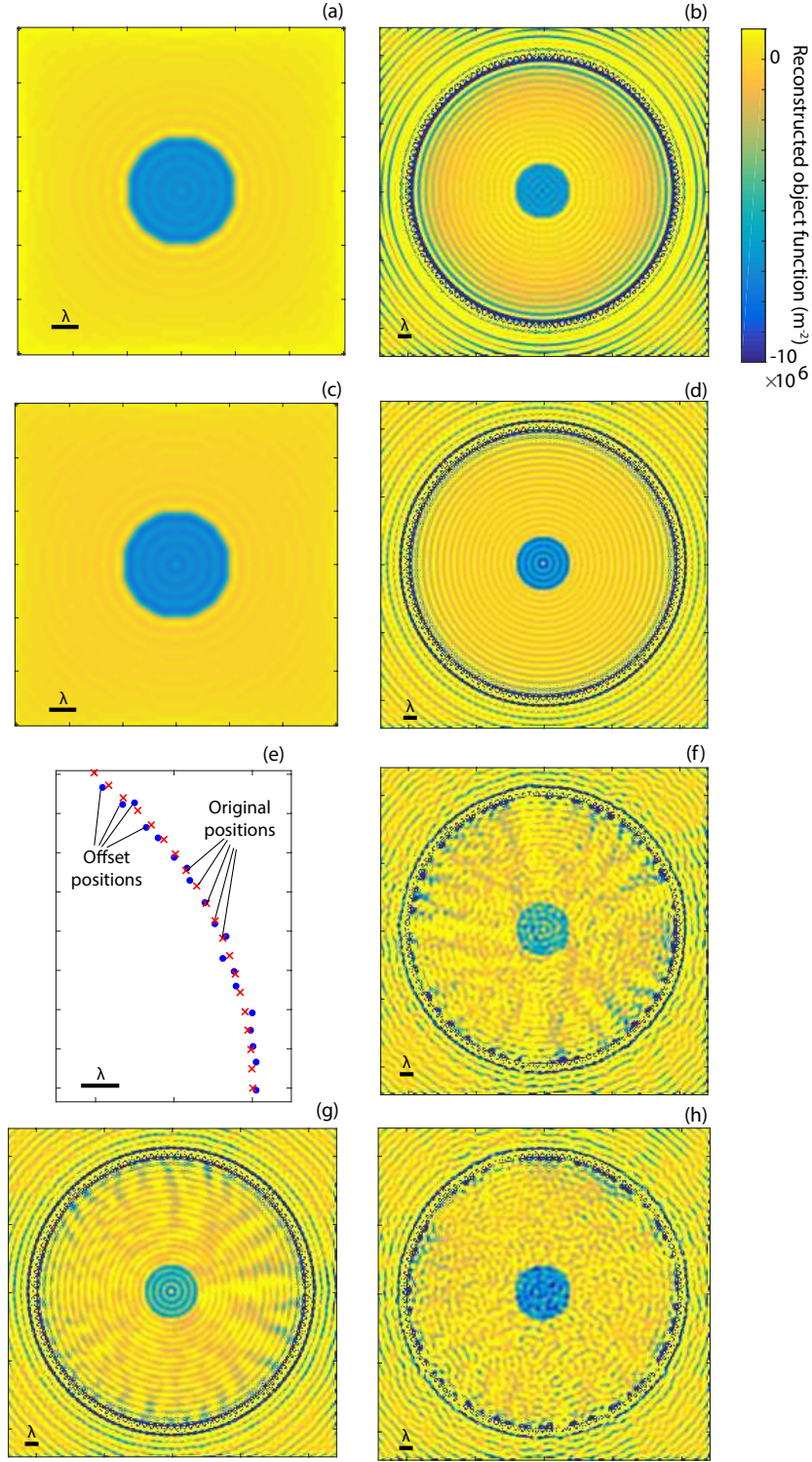


Figure 9: (a) and (b) both show the effect of imaging when the incident field is generated with velocity 2.5% greater than that assumed in the reconstruction. (a) is in the far field, with a radius of 40mm and 512 transducers, while (b) is in the near field with 128 transducers (as in 4(a)). In both cases the scattered field has no error introduced into it to maintain a consistent reference. In both cases the object is the same, but the scale is different. (c) and (d) give reference images (i.e. without errors) for the far and near field cases respectively. (e) shows a section of the circular array, illustrating the uncorrelated transducer mispositioning according to a normal distribution with standard deviation 0.1mm. (f), (g) and (h) show reconstructions with these uncorrelated errors in transducer position, all from the total field. (f) corresponds to the realistic case where errors are caused by both the incident and scattered components. In (g) the errors are just in the scattered component, while in (g) they are in the incident component.

where the second order term $i\omega\gamma$ has been dropped. Based on this, there will be an additional random component which is imaged, where the strength is proportional to the incident field. Since the DT algorithms are linear, because the errors are incoherent, they will generally average out given the many measurements taken across the array. From the result above, it is expected that the errors will be larger when ϕ_0 is large, which occurs closer to the source, i.e. closer to reflection, where the higher spatial frequencies are imaged. It is therefore expected that such errors will relatively cause more high spatial frequency errors than lower frequency errors, so again more filtering may help in practice.

Figures 9(e)-(h) test this. For each transducer, an error has been introduced by adjusting its position by a random amount in both horizontal and vertical directions. This amount is normally distributed with standard deviation of 0.1mm. Figure 9(e) shows a small section of the circular array to highlight how these errors appear; these are much larger errors than would typically be seen in practice. Figure 9(f) shows the resulting image when the offset transducer positions are used to measure the total field. There is a lot of noise in the image as expected, however the scatterer is still clearly visible. It is unclear how much of the noise comes from errors in the scattered field and how much from the incident field, so Figs. 9(g) and (h) show these respectively, although these are unphysical cases. In Fig. 9(g) the incident field has no errors in it while in Fig. 9(h) there are no errors in the scattered field. As suggested, when the errors occur in the incident field, the result is that much higher frequency noise appears in the image, while in the scattered field this tends to be a lot lower. It is clear that Fig. 9(f) is a superposition of both of these forms of noise.

Interestingly the errors in the scattered field appear to contribute to a loss in contrast of the cylinder. If the main error was in amplitude, one measurement being lower amplitude would generally average out with other measurements having higher amplitudes. However with transducer mispositioning, the primary error is in phase. A phase error in one direction cannot be cancelled out by summing a phase error in the other direction, leading to the loss in contrast in the reconstructed image. However, this is not investigated further since it is related to the scattered and not the incident field.

This section has shown that various measurement errors do have an effect on the incident component appearing in a DT image, however these are generally small for both correlated and uncorrelated errors. These errors can be reduced further by calibrating to remove the effects of transducer mispositioning and sound speed errors; this is generally necessary for the scattered component in order to generate accurate quantitative images anyway.

5 Conclusions

This paper has demonstrated that it is unnecessary to subtract the incident field in many cases when imaging with diffraction tomography, dispelling a widely held assumption which can otherwise greatly limit the method's applicability. The subtraction is unnecessary because the incident field causes artefacts clustered at spatial frequencies which are removed automatically in the diffraction tomography algorithm. This paper has proved this theoretically and demonstrated it with practical examples for a number of scenarios. This allows imaging with diffraction tomography to be performed directly using measured data, and thus avoids many of the errors associated with trying to estimate what the incident field should be and subtract it, allowing for a significantly more robust and flexible approach.

The approach has been demonstrated across many practical scenarios. While the theory is derived for far field measurements, examples in this paper also demonstrated that the principle also extends to the near field, so that measurements can be taken close to the scatterer, which is often necessary in practice. The approach was also shown to be valid for different array configurations, including a square array, and a case where the receivers were in a different location to the sources, with parallel transmitter and receiver arrays on opposite sides of the scatterer. In many cases it is impractical to acquire a fully sampled dataset, i.e. having measurements taken less than half-a-wavelength apart, due to the cost of the systems and the acquisition time. This paper has demonstrated the incident field can disappear with subsampled transducer arrays, but that filtering must be used on the final image to eliminate any arising aliasing artefacts, allowing a compromise where the resolution becomes poorer as the number of transducers is reduced.

The effect of both systematic and uncorrelated errors, such as mispositioning of transducers, on the behaviour of the incident field was investigated. This showed that, while resulting artefacts were visible in the image, these were not significantly worse than those associated with errors purely in the scattered field. This result is particularly important because it highlights that the behaviour identified is not a result of highly idealised, clean data, but rather that it is robust and practical, and therefore enables this result to be applied to the widest possible range of imaging problems.

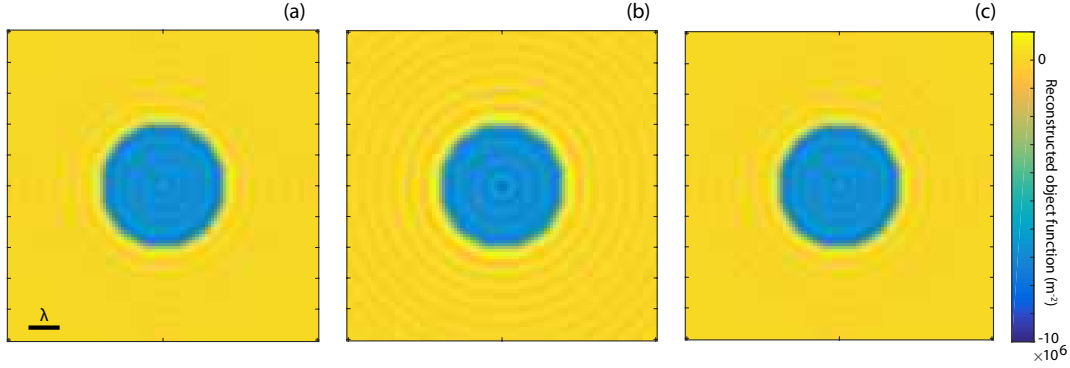


Figure 10: Weighted beamforming approach with weighting applied to data in far field form. (a) shows the image generated from the scattered field for the reference cylinder shown in Fig. 2(a), (b) the image from the total field and (c) the image from the total field with the incident components near the source removed with eq. (50).

Data

All the data sets used in this paper are available in the supplementary material.

Competing interests

I have no competing interests.

Authors' contributions

The sole author undertook all the work in this paper.

Acknowledgements

The author would like to thank Prof. Cawley from Imperial for reading the manuscript, and Dr. Velichko from Bristol University for comments on an early draft.

Funding

This work was funded by EPSRC under grant number EP/M020207/1.

Appendix A: Incident behaviour for weighted beamforming DT

While the filtered beamforming image of Fig. 2(c) showed the high frequency components associated with the incident field, but no constant components, the weighted integral version in Fig. 2(e) is the opposite, with a constant offset term present, but no high frequency components. As shown in the theoretical analysis, the incident wave should have no effect on the image when data is taken in the far field. This provides an indication that the errors in the reconstruction are caused by the array not being sufficiently large. Considering eq. (13), since the weighting term should be the component removing the incident field, it is hypothesised that it is this weighting which is not removing the incident components properly in the near field.

To test this, a process is used to project a data set into the far field. Weighting can be applied in the far field, before the data set is projected back to the near field and imaging is performed. To do this projection, the scattered field measured for a single source can be expressed as

$$\phi_s(R, r_\theta) = \sum_{n=-\infty}^{+\infty} a_n i^n H_n^{(1)}(k_0 R) e^{inr_\theta} \quad (73)$$

which, taken to its asymptotic limit, leads to the far field scattering pattern

$$f(r_\theta) = \sqrt{\frac{2}{i\pi k_0}} \sum_{n=-\infty}^{+\infty} a_n e^{inr_\theta}. \quad (74)$$

Acquiring a_n from ϕ_s is straightforward via a Fourier transform, as is the inverse. This provides a mechanism to convert the data around the receiver array to the far field. By the principle of reciprocity, the same approach can be used to convert the source locations to the far field. The weighting is applied to this far field data before being converted back via the same equations and imaged. Figure 10(a) shows the image resulting from processing the scattered field in this way. It is clear that the offset which was visible in Fig. 2(d) is now eliminated, confirming that the offset errors appear to be associated with the weighting not being performed in the far field.

It is noted that eq. (73) operates purely on the scattered field; the first kind Hankel functions are used to describe the outward propagating waves. The incident wavefield does also propagate outwards (although very near the source the wave travels almost parallel to the array), so this process can be applied to the total field too, although it is noted that the physical configuration corresponding to such far field data is unclear. Upon weighting the data in the far field, converting back to the near field and imaging, the results are shown in Fig. 10(b), which should be compared to Fig. 2(e). Again, any offset is eliminated, however, there are now more high-frequency artefacts associated with the incident field. As shown in Sec. 2, it is likely that these are caused by the large values occurring in the incident field near the source. To test this, these values are removed as for the filtered DT algorithm, using eq. (50), and the result is shown in Fig. 10(c). This is a much cleaner image which is virtually indistinguishable from the scattered field case.

This confirms that if the weighting is applied in the far field, even if the focusing itself is undertaken in the near field as is the case here, the incident field will be removed, which is not the case when weighting is applied directly to the near field. It is noted that the process to apply the weighting in the true far field is not straightforward and is likely to be challenging for practical cases with non-circular arrays.

References

- [1] A. J. Devaney, “Geophysical diffraction tomography,” *Geoscience and Remote Sensing, IEEE Transactions on*, no. 1, pp. 3–13, 1984.
- [2] A. H. Rohde, M. Veidt, L. R. F. Rose, and J. Homer, “A computer simulation study of imaging flexural inhomogeneities using plate-wave diffraction tomography,” *Ultrasonics*, vol. 48, pp. 6–15, mar 2008.
- [3] P. Huthwaite, “Evaluation of inversion approaches for guided wave thickness mapping,” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 470, pp. 20140063–20140063, apr 2014.
- [4] Y. Sung, W. Choi, C. Fang-Yen, K. Badizadegan, R. R. Dasari, and M. S. Feld, “Optical diffraction tomography for high resolution live cell imaging,” *Optics express*, vol. 17, no. 1, pp. 266–277, 2009.
- [5] M. P. Andre, H. S. Janee, P. J. Martin, G. P. Otto, B. A. Spivey, D. A. Palmer, H. S. Jane, P. J. Martin, G. P. Otto, B. A. Spivey, and D. A. Palmer, “High-Speed Data Acquisition in a Diffraction Tomography System Employing Large-Scale Toroidal Arrays,” *Int. J. Imag. Syst. Tech.*, vol. 8, no. 1, pp. 137–147, 1997.
- [6] F. Simonetti, L. Huang, N. Duric, and P. Littrup, “Diffraction and coherence in breast ultrasound tomography: A study with a toroidal array,” *Medical physics*, vol. 36, p. 2955, 2009.
- [7] P. Huthwaite and F. Simonetti, “High-resolution imaging without iteration: a fast and robust method for breast ultrasound tomography,” *The Journal of the Acoustical Society of America*, vol. 130, p. 1721, sep 2011.
- [8] F. Simonetti and L. Huang, “From beamforming to diffraction tomography,” *Journal of Applied Physics*, vol. 103, p. 103110, 2008.
- [9] F. Simonetti, L. Huang, and N. Duric, “A multiscale approach to diffraction tomography of complex three-dimensional objects,” *Appl. Phys. Lett.*, vol. 95, p. 61904, 2009.
- [10] P. Huthwaite and F. Simonetti, “Modeling the measurement of ultrasonic beams transmitted through a penetrable acoustic cone,” *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, vol. 59, no. 10, pp. 2292–2303, 2012.

- [11] P. Belanger, P. Cawley, and F. Simonetti, “Guided wave diffraction tomography within the born approximation,” *Ultrasonics, Ferroelectrics and Frequency Control, IEEE Transactions on*, vol. 57, no. 6, pp. 1405–1418, 2010.
- [12] P. Huthwaite, “Guided wave tomography with an improved scattering model,” *Submitted to Proceedings of the Royal Society A: Mathematical, Physical and Engineering Science*, 2016.
- [13] M. Born and E. Wolf, *Principles of Optics*. Cambridge: Cambridge University Press, 1999.
- [14] A. C. Kak and M. Slaney, *Principles of computerized tomographic reconstruction*. New York: IEEE Press, 1998.
- [15] J. W. Cooley and J. W. Tukey, “An Algorithm for the Machine Calculation of Complex Fourier Series,” may 1965.
- [16] A. J. Devaney, “A filtered backprojection algorithm for diffraction tomography,” *Ultrasonic Imaging*, vol. 4, pp. 336–350, oct 1982.
- [17] C. Holmes, B. Drinkwater, and P. Wilcox, “The post-processing of ultrasonic array data using the total focusing method,” *Insight - Non-Destructive Testing and Condition Monitoring*, vol. 46, pp. 677–680, nov 2004.
- [18] L. R. F. Rose, E. Chan, and C. H. Wang, “A comparison and extensions of algorithms for quantitative imaging of laminar damage in plates. {I}. Point spread functions and near field imaging,” *Wave Motion*, vol. 58, pp. 222–243, 2015.
- [19] E. Chan, L. F. Rose, and C. H. Wang, “A comparison and extensions of algorithms for quantitative imaging of laminar damage in plates. II. Non-monopole scattering and noise tolerance,” *Wave Motion*, vol. 66, pp. 220–237, 2016.
- [20] A. Velichko and P. D. Wilcox, “Post-processing of the full matrix of ultrasonic transmit-receive array data for guided wave pipe inspection,” in *AIP Conference Proceedings*, vol. 1096, pp. 137–144, 2009.
- [21] P. Huthwaite, F. Simonetti, and N. Duric, “Combining time of flight and diffraction tomography for high resolution breast imaging: initial in vivo results (L).,” *The Journal of the Acoustical Society of America*, vol. 132, p. 1249, sep 2012.
- [22] P. Huthwaite and F. Simonetti, “High-resolution guided wave tomography,” *Wave Motion*, vol. 50, no. 5, pp. 979–993, 2013.
- [23] M. Seher, P. Huthwaite, and M. Lowe, “Experimental Studies of the Inspection of Areas with Restricted Access Using A0 Lamb Wave Tomography,” *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control*, pp. 1–1, 2016.
- [24] P. Huthwaite, “Improving accuracy through density correction in guided wave tomography,” *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 472, no. 2186, 2016.
- [25] A. J. Jerri, “The Shannon Sampling Theorem – Its Various Extensions and Applications: A Tutorial Review,” *Proceedings of the IEEE*, vol. 65, no. 11, pp. 1565–1596, 1977.
- [26] F. Simonetti, L. Huang, and N. Duric, “On the sampling of the far-field operator with a circular ring array,” *J. ~Appl. ~Phys.*, vol. 101, p. 83103, 2007.
- [27] L. R. F. Rose and C. H. Wang, “Mindlin plate theory for damage detection: imaging of flexural inhomogeneities,” *The Journal of the Acoustical Society of America*, vol. 127, pp. 754–63, feb 2010.
- [28] M. Seher, *From EMAT to Image: Practical Guided Wave Tomography*. PhD thesis, Imperial College London, 2015.