

**A state-space analysis for a class of
 H^∞ optimal control problems**

by

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to my parents

Abstract

This thesis presents a state-space analysis for a class of $\mathcal{H}^\infty$ -optimal control problems. The $\mathcal{H}^\infty$ approach to design provides a framework within which multivariable frequency response characteristics may be shaped in an optimal manner. This is achieved by modelling noise/disturbance signals and plant uncertainties in a set framework and subsequently minimising their effect in the closed loop for the worst case in the set, under the constraint of closed loop stability for the nominal plant. Formally, any $\mathcal{H}^\infty$ -optimal control problem may be described by the constrained minimisation,

$$\gamma_0 = \inf_{K \in \Xi} \|P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}\|_\infty \quad (1)$$

where Ξ is taken to be the set of all (internally) stabilising compensators. Using the Youla parametrisation of this set in terms of a free parameter $Q(s) \in \mathcal{H}_+^\infty$, we may transform (1) to

$$\gamma_0 = \inf_{Q(s) \in \mathcal{H}_+^\infty} \|\mathcal{R}(Q)\|_\infty \quad (2)$$

where $\mathcal{R}(Q)$ is an affine function of $Q(s)$, or, alternatively to

$$\gamma_0 = \inf_{Q(s) \in \mathcal{H}_+^\infty} \left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty \quad (3)$$

in which $R(s) = (R_{ij}(s)) \in \mathcal{H}_+^\infty$.

Allowing $P_{12}(s)$ to have more rows than columns (with $P_{21}(s)$ square) or, alternatively, $P_{21}(s)$ to have more columns than rows (with $P_{12}(s)$ square), results in a minimisation in (3) in which $R_{12}=0$ and $R_{22}=0$. These problems have been termed “problems of the second kind” to distinguish them from the case when both $P_{12}(s)$ and $P_{21}(s)$ are square (problems of the first kind) and the general case in which both $P_{12}(s)$ and $P_{21}(s)$ are non-square (problems of the third kind). A detailed state-space analysis is performed for the $\mathcal{H}^\infty$ control problems that belong to this class. It is shown that for every $\mathcal{H}^\infty$ problem of the second kind a continuum of optimal controllers exists requiring no more than n states (where $n = \deg(P)$). A parametrisation of the corresponding interpolation functions $\mathcal{R}(Q)$ is also obtained and the results are extended to the suboptimal ($\gamma > \gamma_0$) case.

Seeking to use the degrees of freedom that are available in the solution set of (3) for problems of the first kind in order to minimise all the singular values of $(R_{11} - Q)(s)$ in turn, leads to the formulation of the so-called “super-optimal” Nehari approximation problem. Formally, this may be stated as follows: Given a rational matrix $R_{11}(s) \in \mathcal{H}_+^\infty$, find a matrix function $Q(s) \in \mathcal{H}_+^\infty$ which minimises the sequence $[s_1^\infty, s_2^\infty, \dots]$ with respect to lexicographic ordering, where $s_j^\infty(R_{11} - Q) = \sup_{\omega \in \mathbb{R}} s_j(R_{11} - Q)(j\omega)$ and $s_j(\cdot)$ denotes the j -th singular value of a matrix (in decreasing order of magnitude).

This is a stronger version of Nehari’s theorem for multi-input/output systems which restores

uniqueness to the optimal solution and provides a finer measure of approximation. A state-space algorithm is derived for the solution of this problem. The procedure is to use a set of all-pass transformations (determined by the maximal Schmidt pairs of a sequence of Hankel operators) to sequentially minimise each frequency-dependent singular value in a dimension-peeling algorithm. Bounds on the minimal achievable values of $s_j^\infty(R_{11} - Q)$ are derived along with the MacMillan degree of the super-optimal extension and the corresponding $\mathcal{H}^\infty$ controller. The construction is finally extended to super-optimal Hankel-norm approximations and to other mixed-type ($\mathcal{L}^\infty/\mathcal{L}^2$, $\mathcal{L}^\infty$ /Maximum entropy) problems.

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Abbreviations

RHP	Right half plane
LHP	Left half plane
CRHP, ORHP	Closed right half plane, open right half plane
CLHP, OLHP	Closed left half plane, open left half plane
LQG	Linear-Quadratic Gaussian
SISO	Single input single output
MIMO	Multi-input multi-output

General Introduction

During the last decade $\mathcal{H}^\infty$ theory has been developing as a new methodology for multivariable feedback design. This method provides a framework within which multivariable closed-loop frequency response characteristics may be optimally shaped. In this way, a number of important issues related to the stability and the performance properties of the system may be addressed. The particular importance of $\mathcal{H}^\infty$, however, is its ability to treat the issue of uncertainty in a systematic way.

Uncertainty in control systems may be attributed to a variety of factors. For the purposes of analysing its effects in design (for example its effects in causing instability), it may be distinguished to exogenous signals and model uncertainty. The first type of uncertainty arises due to the presence of disturbances, process noise and measurement noise signals. Model uncertainty (taken to describe any mismatch between the model and the actual system) typically arises due to neglected dynamics, uncertain coefficients in differential equations and the effects of linearisation. To describe uncertainty of this type, the true plant is assumed to belong to a family of which the model is also a member. Since any design based on a model is required to work for the actual system when it is implemented, the issue of *robust control* arises, i.e. the problem of achieving the goals of the design not only for the nominal plant but also for every other plant in the family.

In the $\mathcal{H}^\infty$ formulation both plant perturbations and the exogenous signals are assumed to belong to norm-bounded sets. In particular, model uncertainties are assumed to be bounded in the $\mathcal{L}^\infty$ norm, giving rise to robust-stability criteria mathematically expressed as infinity-norm constraints on certain closed-loop transfer functions. In the case of 'unstructured' perturbations (i.e. if no information is available on how the uncertainty is distributed among the elements of the matrix function representing the plant) this choice of norm is the most appropriate one since it provides both necessary and sufficient conditions for robust stability, under mild assumptions on the perturbed system's stability properties. Exogenous signals, on the other hand, are assumed to be unknown but bounded in energy, their spectral characteristics being reflected in an associated frequency weight acting as a weighting factor on their spectral density. With this type of uncertainty characterisation enforced, $\mathcal{H}^\infty$ problems may be formulated as constrained minimax optimisations: A typical $\mathcal{H}^\infty$ problem seeks to minimise the effects of the exogenous signals/plant perturbations in the closed loop under the constraint of internal stability for the nominal plant. Since the induced norm of an operator acting on $\mathcal{L}^2$ -bounded signals is the $\mathcal{L}^\infty$ -norm of its transfer function, both types of objectives (exogenous signal rejection and robust stabilisation) may be expressed as infinity-norm minimisation problems. In this way, mixed-type problems incorporating multiple performance and robust stability objectives may be formulated and solved within the $\mathcal{H}^\infty$ framework.

The reason why $\mathcal{H}^\infty$ is particularly suitable in dealing with uncertainty is thus twofold: First, the characterisation of exogenous signals implicit in the $\mathcal{H}^\infty$ formulation is more realistic than the ones offered by other design methods: Algebraic control methods, for example, assume perfect knowledge of

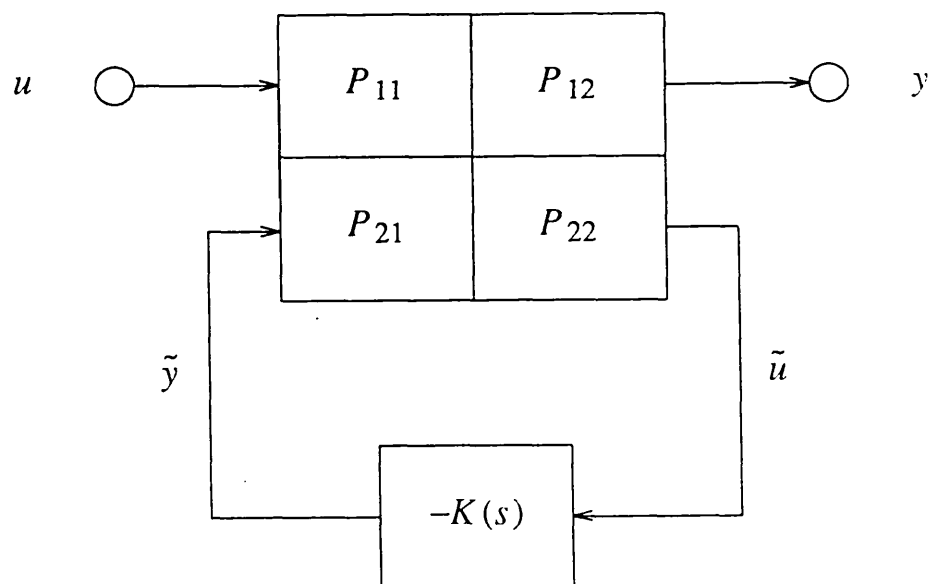


Figure 1: Generalised-regulator configuration

signal uncertainty, while LQG, though modelling process and measurement noise as stochastic signals, assumes perfect knowledge of their spectral characteristics. The second reason is to be found in the ability of $\mathcal{H}^\infty$ to deal directly with robust-stability objectives. In this respect, other methodologies offer very poor results: Pole-placement and geometric methods, for example, seem to be incapable of extending their results when model uncertainty is taken into account. A simple example by J. Doyle [D5] has demonstrated that the stability margins guaranteed by LQG theory in the perfect-state information case are lost when incorrect models are implemented in the Kalman filter employed within the LQG configuration to reconstruct the state of the system. Loop transfer recovery techniques have been suggested to 'robustify' the design, but these are only guaranteed to work under a number of restrictive assumptions.

Formally, any $\mathcal{H}^\infty$ optimal control problem may be embedded in the generalised regulator configuration of Figure 1. In this diagram, $v(s)$ is the vector of all external (or fictitious) inputs, $z(s)$ contains the error signals to be regulated, $u(s)$ are the control inputs and $y(s)$ the measured outputs. $K(s)$ represents the controller to be designed, while $P(s)$ is a transfer matrix whose structure is defined by the particular problem to be solved, incorporating the model of the nominal plant together with all scalings and frequency-dependent weights employed by the designer to shift the emphasis with frequency between different design objectives. Throughout the thesis attention is restricted to linear time-invariant finite-dimensional systems so that both $P(s)$ and $K(s)$ are assumed to be real-rational matrix functions.

With $P(s)$ in Figure 1 partitioned as

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad (1)$$

the general $\mathcal{H}^\infty$ problem may be expressed as the constrained minimisation,

$$\gamma_{opt} = \inf_{K \in \Xi} \| \mathcal{F}_\ell(P, K) \|_\infty := \inf_{K \in \Xi} \| P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \|_\infty \quad (2)$$

in which $K(s)$ is allowed to range over the set Ξ of all internally stabilising compensators. In other words, we seek to minimise the worst-case energy transfer between $v(s)$ and $z(s)$ over all energy-bounded input signals, using compensators that do not generate any unstable hidden modes in the feedback configuration of Figure 1.

The type of solution of the general $\mathcal{H}^\infty$ problem (2) strongly depends on the input/output dimensions of the off-diagonal blocks of $P(s)$. Three general problem-categories will be distinguished: $\mathcal{H}^\infty$ problems of the *first kind* are characterised by the assumption that both $P_{12}(s)$ and $P_{21}(s)$ are square (though not necessarily of the same size); the condition that just one of the two blocks is square defines a problem of the *second kind*, while the condition that both blocks are non-square a problem of the *third kind*. The reason behind this distinction will become apparent in Chapter 1 when a solution to (2) is outlined.

The optimisation problem in (2) is a constrained minimisation of a bilinear function in $K(s)$ over an implicitly-defined set. The problem may be simplified considerably by invoking the Youla parametrisation of internally stabilising compensators,

$$\Xi = \{ \mathcal{F}_\ell(K_0, Q) \mid Q \in \mathcal{H}_+^\infty \} \quad (3)$$

in which $Q(s)$ is allowed to vary over $\mathcal{H}_+^\infty$ and $K_0(s)$ (the generator of the family of all stabilising controllers) is obtained by performing a coprime factorisation on $P(s)$ over $\mathcal{H}_+^\infty$ and solving a double Bezout identity defined by the coprime factors. Substituting $\mathcal{F}_\ell(K_0, Q)$ for $K \in \Xi$ allows (2) to be written as

$$\gamma_{opt} = \inf_{Q \in \mathcal{H}_+^\infty} \| \mathcal{F}_\ell(P, \mathcal{F}_\ell(K_0, Q)) \|_\infty := \inf_{Q \in \mathcal{H}_+^\infty} \| T_{11} - T_{12}QT_{21} \|_\infty \quad (4)$$

in which $T_{ij}(s) \in \mathcal{H}_+^\infty$. Since the problem has now been expressed as a *free* minimisation of an *affine* function, it is much easier to tackle than in its original form.

Equation (4) is often referred to as the “model-matching” problem. Since the T_{ij} ’s and $Q(s)$ are all stable functions, the norm of $\mathcal{R}(s) = T_{11} - T_{12}QT_{21}$ (in the case when T_{12} and T_{21} are square) is constrained only by the RHP transmission zeros of T_{12} and T_{21} (and in the MIMO case by the corresponding zero directions as well). $\mathcal{H}^\infty$ problems of the first kind, therefore, reduce to the construction of a stable matrix function of minimum norm satisfying a finite number of interpolation constraints in the RHP. Problems of this type may be solved via classical Nevanlinna-Pick interpolation theory [D1]. γ_{opt} is obtained by solving a generalised eigenvalue problem and a generator of all optimal interpolating functions may be constructed as a “chain” of linear-fractional compositions of degree-one functions, each associated with a RHP interpolation constraint [L5], [K10].

The interpolation approach in solving $\mathcal{H}^\infty$ control problems is particularly appealing from an intuitive point of view since it provides insight to the constraints on the magnitude of the closed-loop functions on the $j\omega$ -axis due to the presence of RHP constraints. Unstable zeros and poles of the plant, for example, may be easily seen to impose pointwise interpolation constraints on the sensitivity and the complementary-sensitivity functions, if stability in the closed loop is to be attained; these are in fact equivalent (via Poisson’s integral formula) to integral relations that the two functions must satisfy on the $j\omega$ axis [F7], [F9]. $\mathcal{H}^\infty$ design may be viewed, in this way, as a method of shaping the closed-loop frequency-response characteristics by trading the magnitude of the target closed-loop functions at different frequency intervals in a way which keeps all integral constraints in force. Unfortunately, the application of Nevanlinna-Pick interpolation theory is only restricted to $\mathcal{H}^\infty$ problems of the first kind (although a generalised interpolation procedure for $\mathcal{H}^\infty$ problems of the second and third kind involving matrix-like interpolation constraints of the form $A(s)\mathcal{R}(s)=B(s)$ has been proposed [H3], [H4]). In addition, the presence of interpolation constraints defined at zeros having multiplicities greater than one makes the application of the theory awkward, even for problems of the first kind. To deal with such difficulties, an alternative method for solving (4) has been developed.

The model-matching problem (4) is first transformed to

$$\gamma_{opt} = \inf_{Q \in \mathcal{H}_+^\infty} \| T_{11} - [T_{12} \ T_\perp] \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T_{21} \\ \tilde{T}_\perp \end{bmatrix} \|_\infty \quad (5)$$

in which $[T_{12} \ T_\perp]$ and $[T_{21}^* \ \tilde{T}_\perp^*]^*$ are inner matrices. (Note that this is in general possible under mild assumptions on the structure of $P(s)$: First, inner-outer factorisations may be performed on T_{12} and T_{21} , the outer factors being absorbed in $Q(s)$; matrices $T_\perp$ and $\tilde{T}_\perp$ may next be constructed as inner complements of T_{12} and T_{21} [D4]). The norm-preserving properties of the multiplication by all-pass matrices may be next be employed to transform (5) to

$$\gamma_{opt} = \inf_{Q(s) \in \mathcal{H}_+^\infty} \left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty \quad (6)$$

in which

$$R(s) = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} = \begin{bmatrix} T_{12}^* \\ T_\perp^* \end{bmatrix} T_{11} \begin{bmatrix} T_{21}^* & \tilde{T}_\perp^* \end{bmatrix} \in \mathcal{H}^\infty \quad (7)$$

(6) is referred as the four-block general distance problem. In the case of $\mathcal{H}^\infty$ problems of the first kind, it reduces to the approximation

$$\gamma_{opt} = \inf_{Q(s) \in \mathcal{H}_+^\infty} \| R_{11} - Q \|_\infty \quad (8)$$

known as the Nehari-extension problem. By Nehari's theorem [G1], [F2], the infimum in (8) is achieved and is equal to the norm of the Hankel operator induced by $R_{11}(-s)$ (definition given in section 1.5), while a simple state-space characterisation of all $Q(s)$'s which attain this value has been derived by Glover [G1]. The case of $\mathcal{H}^\infty$ problems of the second and third kind is considerably harder: In fact, the presence of matrix blocks in $R(s)$ which cannot be directly approximated (see (6)) makes only iterative solutions possible. Such a solution, based on the work of Davis, Kahan and Weinberger [D10], has been suggested by Doyle [D4]. The method consists of a bisection algorithm at each stage of which an inequality of the form

$$\left\| \begin{bmatrix} R_{11} - Q & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \right\|_\infty \leq \gamma \quad (9)$$

is checked by transforming (9) to a two-block and a Nehari-extension problem in turn, thus reducing it to the calculation of a Hankel norm. This two-stage transformation requires the performance of one inner-outer and two spectral factorisations and, in practical situations, has to make use of model-reduction routines at each stage to suppress state-inflation effects.

$\mathcal{H}^\infty$ originated from the work of Zames in the late seventies who first suggested the use of infinity-norm minimisations in dealing with plant uncertainty and shaping the sensitivity function. The scalar sensitivity problem was solved by Zames and Francis [Z1], [F4] using interpolation theory. The multivariable sensitivity minimisation problem was treated by Francis, Helton and Zames [F5], Chang

and Pearson [C1] and Savonov and Verma [S1], while Savonov and Chen have obtained a solution in the case that sensitivity is constrained to be decoupled [S6]. Robust stabilisation problems in the context of $\mathcal{H}^\infty$ optimisation were treated by Kimura [K6], Glover [G2] and Glover and McFarlane [G6], while mixed-type problems were addressed by Kwakernaak [K9], Verma and Johnckhere [V1] and Foo and Postlethwaite [F1]. Extensions of $\mathcal{H}^\infty$ theory to address robust performance and structured perturbation problems are mainly due to Doyle et al [D1] and Savonov [S5]. Investigations of analytic tradeoffs in closed loop functions due to the presence of unstable poles and non-minimum phase zeros (in the context of $\mathcal{H}^\infty$ optimisation) are due to Freudenberg and Looze [F8], [F9] in the scalar case and O'Young and Francis in the multivariable case [O1], [O2]. Most of the state-space advances in the $\mathcal{H}^\infty$ and model-reduction areas are due to Glover [G1], [G3], [G4], Ball and Ran [B5] and Doyle [D4]. Among recent developments in this area the solution of the general distance problem by all-pass embedding techniques [G5], a generalised interpolation approach [H3], [H4] and a J-factorisation approach [G8] are of particular importance. A polynomial-based approach for solving several $\mathcal{H}^\infty$ problems has been developed by Kwakernaak [K12]. His approach requires the solution of systems of matrix polynomial equations and implicitly treats the ensuing Nehari-extension problem as a Wiener-Hopf minimisation weighted by a strictly-proper function. (This function in fact corresponds to the maximal Schmidt vector of the corresponding Hankel operator [S7], [S8]). The polynomial approach heavily depends in keeping track of polynomial degrees and for this reason, despite recent advances in this area, it is rather awkward. From a numerical point of view it seems to be inferior to state-space methods.

Although $\mathcal{H}^\infty$ is predominantly a frequency-domain method, some very interesting time-domain interpretations have recently emerged. Khargonegar et al [K5] have shown that if all states are accessible, the minimum-norm interpolating function may be constructed using a constant state feedback. This result suggested the possibility of a separation principle between estimation and regulation in $\mathcal{H}^\infty$ compensators, similar to the one obtained in LQG theory. Such a separation principle has in fact been established by Doyle et al [D9] for a class of $\mathcal{H}^\infty$ controllers; their analysis has also produced considerable computational simplifications resulting in a generator of all $\mathcal{H}^\infty$ controllers based on the solution of two indefinite Riccati equations. An interpretation of $\mathcal{H}^\infty$ optimisation in game-theory terms has been developed by Limebeer et al [L10]. Their analysis is based on an interpretation of the $\mathcal{H}^\infty$ minimax-type cost as a linear-quadratic differential game between the exogenous signal ('trying' to maximise the overall cost) and the controlled signal ('trying' to minimise it). The description of the problem in these terms illustrates the significance of the indefinite Riccati equations obtained in [D9], [G5] and has led to extensions to the time-varying case. Such extensions have also been obtained by Tadmor [T1] using the maximum principle.

Considerable interest has also been devoted in using the degrees of freedom that are available to $\mathcal{H}^\infty$ interpolating functions in the suboptimal ($\|R(s)\|_\infty \leq \gamma$, $\gamma_{opt} < \gamma$) and the optimal case. *Maximum entropy* extensions (in the context of $\mathcal{H}^\infty$ control) have been considered by Limebeer and Hung [L6] for the one-block case; their results have been extended to the four block case by Glover and Mustafa [G7] and Glover et al [G5] using all-pass embedding techniques. In [G5] particular attention is

given in maximising entropy at infinity establishing interesting links with risk-sensitive LQG theory. Finally, *superoptimal* extensions proposed by Young [Y3] have been considered by Limebeer et al [L9], Postlethwaite et al [P4] and Jaimouka [J1].

Thesis layout and contribution:

This thesis presents a state-space analysis for a class of $\mathcal{H}^\infty$ optimal control problems. As has been mentioned in previous paragraphs the general $\mathcal{H}^\infty$ problem may be distinguished to problems of the first, second and third kinds according to a dimensionality condition. Problems of the first kind admit closed form solutions, the optimal cost being obtained by solving a Nehari or “free” approximation problem. Problems in the last two categories may be solved via an iterative procedure which involves a number of state-dimension inflation steps.

In Chapter 1 some background theory for $\mathcal{H}^\infty$ control is outlined. The general problem is formally defined as an optimisation problem incorporating internal stability in the closed loop as a constraint. Using the Youla parametrisation of internally stabilising compensators allows this problem to be transformed to a general-distance approximation in terms of a free matrix parameter in $\mathcal{H}^\infty$. In the one block case this may be solved using the state-space methods in [G1] while a bisection algorithm is outlined for solving the approximation problem in the general case. A number of control problems of engineering interest (sensitivity minimisation, robust stabilisation and the mixed-sensitivity minimisation) are fitted in the configuration of Figure 1. A class of problems treated in [L11] using the results in [D9] (parametric mixed-sensitivity problems) which nominally belong to the family of problems of the second kind are shown to reduce to one-block problems and a closed-form solution of the optimal cost is obtained. In the last section of the chapter links between RHP interpolation constraints and the magnitude of certain closed-loop functions are described.

Chapter 2 presents a detailed state-space analysis of $\mathcal{H}^\infty$ problems of the second kind which leads to the identification of degree-inflation phenomena in the interpolating function and the controller. Riccati equation balancing techniques are employed to establish links between the interpolation constraints of the problem, the rank of the solution of the two Riccati equations associated with this family of problems and the degree of $R(s)$ in the two-block general-distance problem. A detailed state-space analysis is performed yielding a realisation and a degree bound for the family of all optimal interpolating functions. This leads to the derivation of the degree bound for the $\mathcal{H}^\infty$ -optimal controller by identifying the possible pole/zero cancellations which can occur in the feedback configuration of Figure 1. It is shown that for every problem of the second kind a continuum of $\mathcal{H}^\infty$ controllers exists requiring no more than $n-1$ states in the optimal case and n states in the suboptimal case ($\gamma > \gamma_{opt}$), where $n = \deg(P)$. Finally, the last two sections include a mixed sensitivity/additive robustness example illustrating the results together with a discussion of the computational improvements in $\mathcal{H}^\infty$ software resulting from the cancellation analysis of the Chapter.

Seeking to use the degrees of freedom that are available in the solution set of (6) for problems

of the first kind in order to minimise all the singular values of $(R_{11}-Q)(s)$ in turn, leads to the formulation of the so-called “super-optimal” Nehari extension problem, treated in Chapter 3. Formally, this may be stated as follows: Given a rational matrix $R_{11}(s) \in \mathcal{H}_+^\infty$, find a matrix function $Q(s) \in \mathcal{H}_+^\infty$ which minimises the sequence $[s_1^\infty, s_2^\infty, \dots]$ with respect to lexicographic ordering, where $s_j^\infty(R_{11}-Q) = \sup_{\omega \in \mathbb{R}} s_j(R_{11}-Q)(j\omega)$ and $s_j(\cdot)$ denotes the j -th singular value of a matrix (in decreasing order of magnitude). A state-space algorithm is derived for the solution of this problem. The procedure is to use a set of all-pass transformations (determined by the maximal Schmidt pairs of a sequence of Hankel operators) to sequentially minimise each frequency-dependent singular value in a dimension-peeling algorithm. Bounds on the minimal achievable values of $s_j^\infty(R_{11}-Q)$ are derived along with the McMillan degree of the super-optimal extension and the corresponding $\mathcal{H}^\infty$ controller. The construction is extended to super-optimal Hankel-norm approximations and to other mixed-type $(\mathcal{L}^\infty/\mathcal{L}^2, \mathcal{L}^\infty/\text{Maximum entropy})$ problems. Finally, a number of examples are included to illustrate the results of the Chapter and superoptimal robust stabilisation designs are discussed in the context of a-posteriori error characterisation tests.

Chapter 1

1.0 Introduction

This thesis presents a state-space analysis for a class of $\mathcal{H}^\infty$ -optimal control problems. The purpose of this Chapter is threefold: First, it defines the notation and gives some preliminary results used in the following Chapters. Secondly, it formally defines the general optimisation problem to be considered, links it to a number of issues arising in multivariable design and outlines a number of methods for its solution. The general problem is distinguished to problems of the first, second and third kind according to a dimensionality condition: The first of these reduces to a “free” approximation for which closed-form solutions exist; for the other two, however, only iterative solutions are in general possible (although section 1.6 describes a class of problems of the second kind for which closed-form solutions exist). Finally, the last section of the Chapter relates the minimum cost which can be achieved in the $\mathcal{H}^\infty$ problem to a number of interpolation constraints which must be satisfied in the RHP; this illustrates the dependence of the optimal cost on a number of structural constraints due to the plant such as the position of unstable poles/non-minimum phase zeros and approximate pole/zero cancellations in the RHP.

1.1 Notation and preliminaries

$\mathbb{R}, \mathbb{R}_+, \mathbb{C}$	real, non-negative and complex numbers
$\mathbb{R}(s)$	field of rational functions in s with real coefficients
$\mathbb{F}^{m \times l}$	set of $m \times l$ matrices with elements in $\mathbb{F} (= \mathbb{R}, \mathbb{C}, \mathbb{R}(s) \text{ etc.})$
$\mathbb{C}_+, \bar{\mathbb{C}}_+$	open (resp. closed) right half plane
$\mathbb{C}_-, \bar{\mathbb{C}}_-$	open (resp. closed) left half plane
$\lambda(A), \lambda_{\max}(A)$	eigenvalue of a square matrix A , largest eigenvalue of A
A^*	complex conjugate transpose of $A \in \mathbb{C}^{m \times l}$ (transpose if $A \in \mathbb{R}^{m \times l}$)
$A \geq 0, A > 0$	A is positive semidefinite (resp. positive definite)
$A \leq 0, A < 0$	A is negative semidefinite (resp. negative definite)
$\mathbb{RL}^\infty$	space of matrices in $\mathbb{R}(s)^{m \times l}$ which have no poles on the $j\omega$ axis (including the point at ∞)
$\ \cdot\ _\infty$	$\mathcal{L}^\infty$ -norm of matrices in $\mathbb{RL}^\infty$

$\mathcal{RH}_+^\infty, \mathcal{RH}_-^\infty$	subspaces of $\mathcal{RL}^\infty$; matrices which have no poles in $\bar{\mathbb{C}}_+$ (resp. $\bar{\mathbb{C}}_-$)
Γ_G	Hankel operator associated with $G(s) \in \mathcal{RH}_+^\infty$
$\sigma_i(G(s))$	i -th Hankel singular value of $G(s)$ (i.e. of Γ_G) in decreasing order of magnitude
$\ G(s)\ _H$	$= \sigma_1(G(s))$, the Hankel norm of $G(s)$
$\operatorname{Re}(s), \bar{s}, s $	the real part, complex conjugate and modulus of $s \in \mathbb{C}$
$G^*(s)$	$= G(-\bar{s})^*$, the para-hermitian conjugate of $G(s)$

Associated with a transfer function matrix $G(s) \in \mathbb{R}(s)^{m \times l}$ of MacMillan degree n is a state-space realisation

$$G(s) = D + C(sI - A)^{-1}B \quad (1.1.1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times l}$, $C \in \mathbb{R}^{m \times n}$ and $D \in \mathbb{R}^{m \times l}$. The alternative notation $G(s) \triangleq (A, B, C, D)$ or

$$G(s) \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (1.1.2)$$

will also be used.

In the above notation, we have $G^*(s) \triangleq (-A^*, C^*, -B^*, D^*)$ and in the case that D is non-singular, $G^{-1}(s) \triangleq (A - BD^{-1}C, BD^{-1}, -D^{-1}C, D^{-1})$. If $G^{-1}(s) = G^*(s)$, then $G(s)$ is *all-pass*. $G(s)$ is called *stable* (*asymptotically stable*) if it has no poles in $\mathbb{C}_+$ (resp. $\bar{\mathbb{C}}_+$). If $G(s)$ has no pole in $\mathbb{C}_-$ it is called *unstable*. If $G(s)$ is stable and all-pass it is called *inner*. Unstable and all-pass matrices are called *anti-inner*. If $G(s)$ is unstable and $\|G(s)\|_\infty \leq 1$ it is called a *matrix contraction*. The set of all matrix contractions is called the *unit ball* and is denoted by $\mathcal{BK}_-^\infty$.

If $G(s) \triangleq (A, B, C, D)$, the system matrix of the given realisation is defined as

$$\begin{bmatrix} sI - A & -B \\ C & D \end{bmatrix} \quad (1.1.3)$$

and the system zeros are defined to be the points at which the system matrix loses normal rank. In the case when D is non-singular, the system zeros are also given by $\lambda(A - BD^{-1}C)$. The input decoupling zeros (uncontrollable modes) are points at which $[sI - A \mid B]$ loses rank. The output decoupling zeros (unobservable modes) are the points at which $[sI - A^* \mid C^*]$ loses rank. In the sequel, the term “zero” refers to “system zero” unless stated otherwise. Obviously, $\{\text{input decoupling zeros}\} \cup \{\text{output decoupling zeros}\}$ are a subset of both $\lambda(A)$ and the set of system zeros. The realisation (A, B, C, D) is minimal if it has no input/output decoupling zeros. A sufficient condition for this is that all system zeros are distinct from $\lambda(A)$.

If $G_1(s) \triangleq (A_1, B_1, C_1, D_1)$ and $G_2(s) \triangleq (A_2, B_2, C_2, D_2)$ then the cascade system $G_1 G_2(s)$ has a realisation given by

$$\begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} * \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} = \left[\begin{array}{cc|c} A_1 & B_1 C_2 & B_1 D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1 C_2 & D_1 D_2 \end{array} \right] \quad (1.1.4)$$

where we have taken the “multiplication” of two realisations to mean cascading the two systems. This is not to be confused with ordinary matrix multiplication. The context will always make the distinction between these two possible interpretations clear.

If a basis change T is introduced into the state-space of $G(s)$, this will be taken to mean the transformation $(A, B, C, D) \xrightarrow{T} (TAT^{-1}, TB, CT^{-1}, D)$. The MacMillan degree of $G(s)$ will be written as $\deg(G)$ and the set of poles (zeros) of $G(s)$ will be denoted $\{\text{poles of } G\}$ ($\{\text{zeros of } G\}$).

Linear fractional maps and compositions

Let $P(s)$ be a partitioned matrix with a state space realisation given by

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} (s) \triangleq \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (1.1.5)$$

then

$$P_{ij}(s) = C_i(sI - A)^{-1}B_j + D_{ij} \quad (1.1.6)$$

is a state-space realisation of $P_{ij}(s)$. A (lower) linear fractional transformation for the partitioned matrix P and a matrix K is defined as

$$\mathcal{F}_\ell(P, K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21} \quad (1.1.7)$$

where K is of dimension $\ell \times m$ if P_{22} has dimension $m \times \ell$. Occasionally, the upper linear fractional transformation will also be used. This is defined as

$$\mathcal{F}_u(P, K) = P_{22} + P_{21}K(I - P_{11}K)^{-1}P_{12} \quad (1.1.8)$$

for a suitable partitioning of $P(s)$. A state space realisation for $\mathcal{F}_\ell(P, K)$ is given by the next lemma.

Lemma 1.1.1

If $P(s) \in \mathbb{R}^{(p+q) \times (m+r)}(s)$ and $K(s) \in \mathbb{R}^{r \times q}(s)$ have state-space realisations

$$P(s) \triangleq \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad K(s) \triangleq \left[\begin{array}{cc} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{array} \right] \quad (1.1.9)$$

Then if the well-posedness condition $\det(I - D_{22}\hat{D}) \neq 0$ is satisfied,

$$\mathcal{F}_\ell(P, K) \triangleq \left[\begin{array}{cc|c} A+B_2\hat{D}L_1C_2 & B_2L_2\hat{C} & B_1+B_2\hat{D}L_1D_{21} \\ \hline \hat{B}L_1C_2 & \hat{A}+\hat{B}L_1D_{22}\hat{C} & \hat{B}L_1D_{21} \\ \hline C_1+D_{12}L_2\hat{D}C_2 & D_{12}L_2\hat{C} & D_{11}+D_{12}\hat{D}L_1D_{21} \end{array} \right] \quad (1.1.10)$$

where

$$L_1 = (I - D_{22}\hat{D})^{-1} \quad ; \quad L_2 = (I - \hat{D}D_{22})^{-1} \quad (1.1.11)$$

Proof:

This result follows by routine computation. $\square$

The next result says that if $H(s)$ and $U(s)$ are all-pass, then $\mathcal{F}_\ell(H, U)$ is all-pass also. (See also [G4], [L5] and [L9]).

Lemma 1.1.2

Suppose

$$H(s) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \quad (1.1.12)$$

satisfies $H(s)H^*(s) = \sigma^2 I$, that $H_{12}(j\omega)$ has full row rank for all ω and that

$$R(s) = \mathcal{F}_\ell(H(s), U(s)) \quad (1.1.13)$$

Then if $(I - UH_{22})^{-1}(j\omega)$ exists for all ω ,

$$\sigma^2 I - R(j\omega)R^*(j\omega) = 0 \Leftrightarrow I - \sigma^2 U(j\omega)U^*(j\omega) = 0 \quad (1.1.14)$$

Proof:

From the assumed properties of $H(s)$ we get (suppressing the s -dependence):

$$H_{11}H_{11}^* + H_{12}H_{12}^* = \sigma^2 I \quad (1.1.15)$$

$$H_{11}H_{21}^* + H_{12}H_{22}^* = 0 \quad (1.1.16)$$

$$H_{21}H_{21}^* + H_{22}H_{22}^* = \sigma^2 I \quad (1.1.17)$$

Using equations (1.1.15) to (1.1.17) gives,

$$\sigma^2 I - RR^* = \sigma^2 I - \{H_{11} + H_{12}U(I - H_{22}U)^{-1}H_{21}\} \{H_{11} + H_{12}U(I - H_{22}U)^{-1}H_{21}\}^*$$

$$\begin{aligned}
 &= (\sigma^2 I - H_{11} H_{11}^*) + H_{12} H_{22}^* (I - U^* H_{22}^*)^{-1} U^* H_{12}^* + H_{12} U (I - H_{22} U)^{-1} H_{22} H_{12}^* \\
 &- H_{12} U (I - H_{22} U)^{-1} \{ \sigma^2 I - H_{22} H_{22}^* \} (I - U^* H_{22}^*)^{-1} U^* H_{12}^* \\
 &= H_{12} (I - U H_{22})^{-1} \{ I - \sigma^2 U U^* \} (I - H_{22}^* U^*)^{-1} H_{12}^*
 \end{aligned} \tag{1.1.18}$$

which proves the lemma. $\square$

For two systems $H(s) \in \mathbb{R}^{(p+m) \times (p+m)}(s)$ and $\tilde{H}(s) \in \mathbb{R}^{(p+m), (p+m)}(s)$ partitioned as

$$H(s) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \quad \tilde{H}(s) = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{bmatrix} \tag{1.1.19}$$

in which $H_{11}(s) \in \mathbb{R}^{p \times m}(s)$ and $\tilde{H}_{11}(s) \in \mathbb{R}^{p \times m}(s)$ such that $(I - H_{22} \tilde{H}_{11})^{-1}$ exists, the *linear fractional composition* of $H(s)$ and $\tilde{H}(s)$ is defined as

$$H(s) \odot \tilde{H}(s) = \begin{bmatrix} \mathcal{F}_\ell(H, \tilde{H}_{11}) & H_{12} (I - \tilde{H}_{11} H_{22})^{-1} \tilde{H}_{12} \\ \tilde{H}_{21} (I - H_{22} \tilde{H}_{11})^{-1} H_{21} & \mathcal{F}_u(\tilde{H}, H_{22}) \end{bmatrix} \in \mathbb{R}^{(p+m) \times (p+m)}(s) \tag{1.1.20}$$

We finally associate with $P(s)$ a *substitution* matrix

$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} P_{21} - P_{22} P_{12}^{-1} P_{11} & P_{22} P_{12}^{-1} \\ -P_{12}^{-1} P_{11} & P_{12}^{-1} \end{bmatrix} \tag{1.1.21}$$

From Fig. 1 we see that

$$y(s) = \mathcal{F}_\ell(P, K) u(s) \tag{1.1.22}$$

and

$$\begin{bmatrix} \tilde{u}(s) \\ \tilde{y}(s) \end{bmatrix} = \tilde{S}(s) \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} \tag{1.1.23}$$

Algebraic Riccati equations

The algebraic Riccati equation

$$A^* P + P A - P B B^* P + Q^* Q = 0 \tag{1.1.24}$$

is of considerable importance in the state-space analysis of Chapters 2 and 3. Among the possible multitude of solutions to (1.1.24) stabilising solutions will refer to those for which $A - B B^* P$ is stable. Certain properties of (1.1.24) are summarised below [W2], [W1], [P3], [K7]:

- (1) If $Q^* Q > 0$, the real symmetric solution $P > 0$. (2) If the pair $[Q, A]$ is observable, then $P > 0$.
- (2) The stabilisability of $[A, B]$ and the condition that the Hamiltonian

$$H = \begin{bmatrix} A & -BB^* \\ -Q^*Q & -A^* \end{bmatrix} \quad (1.1.25)$$

has no eigenvalues on the imaginary axis is necessary and sufficient for the existence of a stabilising solution.

(3) If a stabilising solution exists it is unique.

1.2 Control design objectives

Consider the feedback configuration of Figure 2 in which $G(s)$ denotes the plant which has to be controlled by a compensator $K(s)$. Both the plant and the compensator are assumed to be linear finite-dimensional time-invariant MIMO systems. $r(s)$ is a reference or command input, $d(s)$ a disturbance signal entering at the plant's output, $y(s)$ is the system's output and $n(s)$ denotes sensor noise. Designing an appropriate compensator for a given $G(s)$ should address a number of issues:

(1) Stability considerations: $K(s)$ should stabilise the closed loop system and provide adequate stability margins.

(2) Performance considerations: The output $y(s)$ should track the reference signal $r(s)$ fast and accurately. Further, it should do so in the presence of $d(s)$ and $n(s)$. The linearity assumption implies that the individual effects of these two signals at the output should be small.

(3) Control effort considerations: The control signal $u(s)$ should remain within acceptable bounds to avoid actuator saturation. Finally,

(4) Robust stability considerations: The above objectives should be maintained as far as possible in the presence of plant uncertainty.

It is shown in this section that design goals of this type may be achieved by appropriately shaping a number of closed-loop transfer functions in certain frequency intervals. In particular, attaining a small magnitude in the sensitivity function at an interval, say, $[0, \omega_L]$ will result, among other things, in good attenuation of disturbances entering at the plant output (typically having their power concentrated in a low frequency interval) and in the accurate following of set input changes; on the other hand, the magnitude of the complementary sensitivity in a high-frequency interval $[\omega_H, \infty]$ governs wide-band measurement noise attenuation and the sensitivity of the design to multiplicative model errors in the plant.

The system's output $y(s)$ in Figure 2 may be written as

$$y(s) = y_r(s) + y_n(s) + y_d(s) = T(s)(r(s) - n(s)) + S(s)d(s) \quad (1.2.1)$$

where $S(s)$ and $T(s)$ denote the system's sensitivity and complementary sensitivity functions

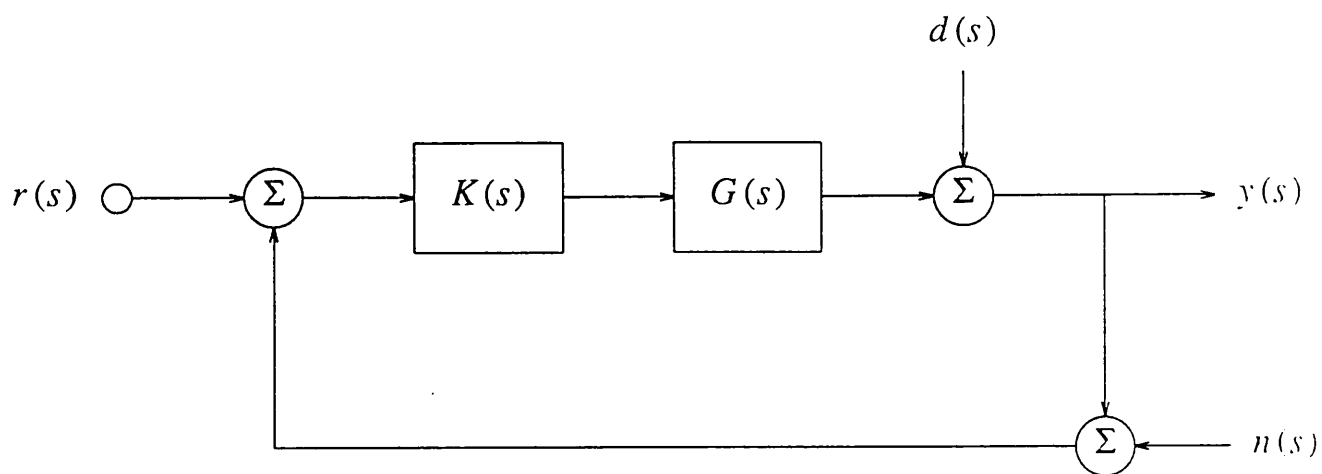


Figure 2: Control system feedback configuration

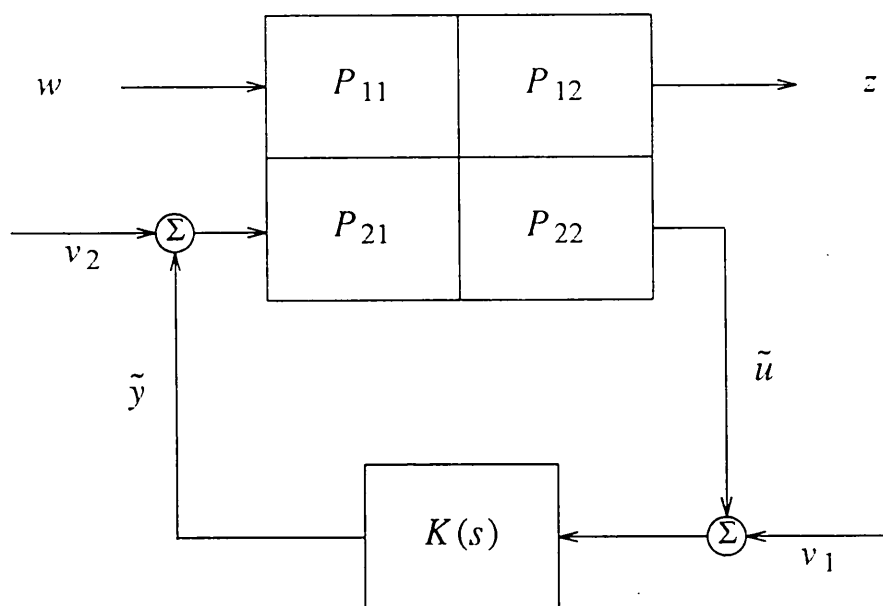


Figure 3: Generalised regulator. Definition of Internal stability

$$S(s)=(I+Q)^{-1}(s)=(I+GK)^{-1}(s) \quad (1.2.2)$$

and

$$T(s)=(I+Q^{-1})^{-1}(s)=GK(I+GK)^{-1}(s)=I-S(s) \quad (1.2.3)$$

(1.2.1) shows that the steady-state response of the system output to a sinusoid disturbance signal $d(t)=d_0 e^{j\omega t}$ is governed by the sensitivity function as $y_d(s) \rightarrow S(j\omega)d_0 e^{j\omega t}$. The effect of $d(t)$ at the output, therefore, can be made small by requiring that

$$\frac{\|S(j\omega)d_0\|_2}{\|d_0\|_2} \ll 1 \quad (1.2.4)$$

so that disturbance attenuation independently of the direction d_0 requires

$$\sup_{\|d_0\|_2=1} \|S(j\omega)d_0\|_2 = \bar{\sigma}(S(j\omega)) \ll 1. \quad (1.2.5)$$

at the disturbance frequency ω . Thus, assuming that $d(t)$ is a random signal having a power spectrum concentrated in the interval $[0, \omega_1]$, its effect at the output will be negligible provided that $\bar{\sigma}(S(j\omega)) \ll 1 \forall \omega \in [0, \omega_1]$. The tracking error $e(s)=r(s)-y(s)$ can be expressed as

$$e(s)=(I+Q(s))^{-1}r(s):=S(s)r(s) \quad (1.2.6)$$

and hence accurate following of command inputs also requires $\bar{\sigma}(S(j\omega)) \ll 1$ for frequencies ω belonging to an interval $[0, \omega_1]$. In particular, zero steady-state error to step commands requires $\bar{\sigma}(S(0))=0$ (this is achievable if there is at least one integrator at each path of the feedback loop), while fast following of the command signal can be attained by making the interval over which sensitivity is small wide enough. The upper bound of this interval, however, cannot be pushed arbitrarily high (partly) due to the effects of sensor noise amplification at the output: (1.2.1) implies that the attenuation of sensor noise in the interval $[\omega_2, \omega_3]$ requires

$$\bar{\sigma}(T(j\omega)) \ll 1 \quad (1.2.7)$$

for $\omega \in [\omega_2, \omega_3]$ while the algebraic identity $S(s)+T(s)=I$ indicates that conditions (1.2.5) and (1.2.7) cannot be satisfied simultaneously at a single frequency. Since sensor noise normally has wide-band spectral characteristics it is typical to require that

$$\begin{cases} \bar{\sigma}(S(j\omega)) \ll 1 & \omega \in [0, \omega_1] \\ \bar{\sigma}(T(j\omega)) \ll 1 & \omega > \omega_2 \end{cases} \quad (1.2.8)$$

for frequencies ω_1 and ω_2 chosen so that $\omega_2 > \omega_1$.

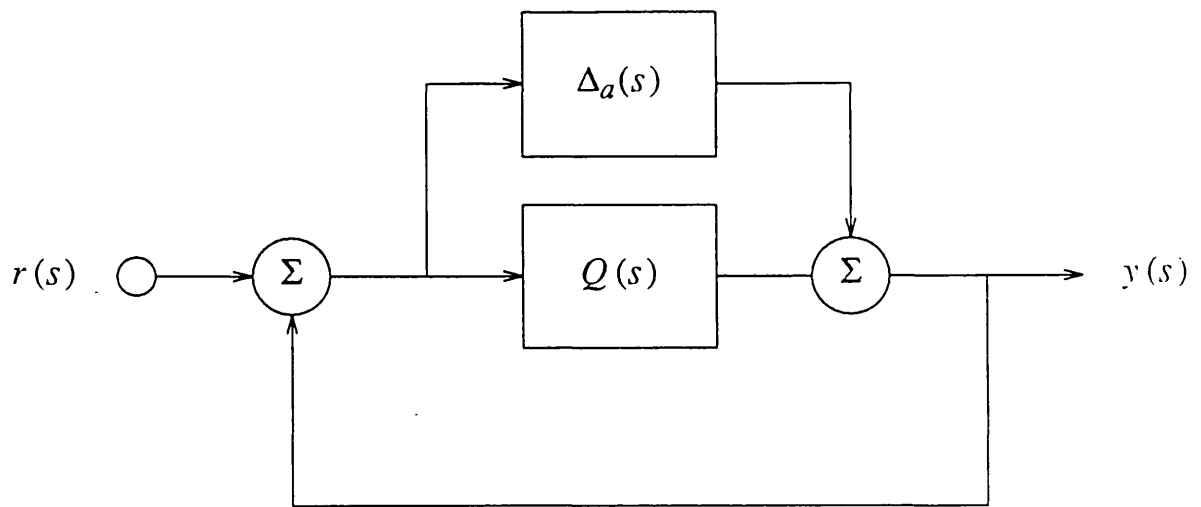


Figure 4: Additive model uncertainty

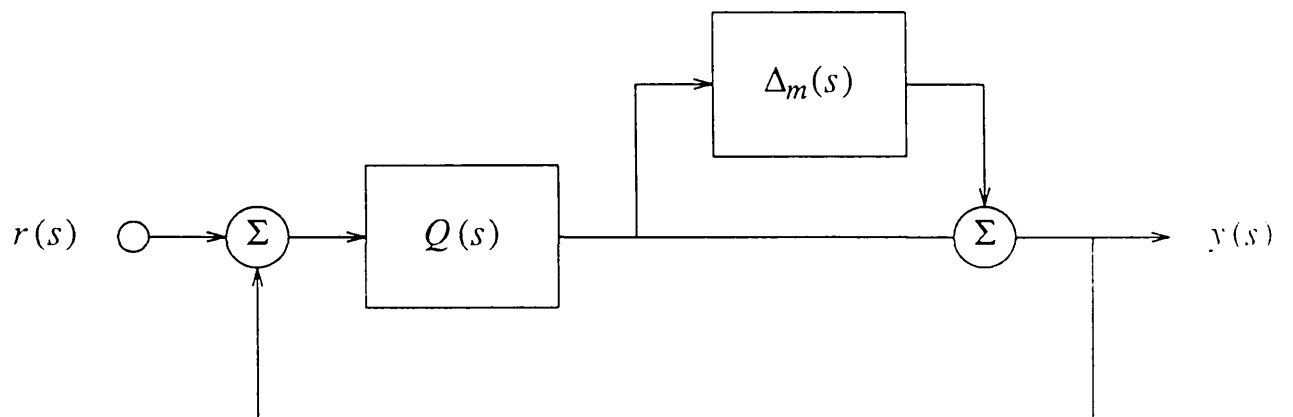


Figure 5: Multiplicative model uncertainty

An important issue in control system design is the effect of model uncertainty. In particular, it is important to ensure that a compensator design based on a nominal model $G_0(s)$ will retain its closed-loop stability properties if “reasonable” plant perturbations are taken into account. The term “plant perturbation” is taken to describe any mismatch between the model and the actual system. Such a mismatch may arise due to neglected (usually fast) dynamics, plant parameter variations, “drifting” of the operating point at which the system has been linearised, or the modelling of time-delays by finite-dimensional models.

Model uncertainty may be distinguished in *structured* and *unstructured* uncertainty: Structured uncertainty is used to describe situations in which spatial information about the perturbation is available; it normally arises due to uncertain coefficients in differential equations which model the underlying physical system [D4]. In contrast, by “unstructured uncertainty” (frequency-dependent) norm-bounded variations are described. These typically arise due to unmodelled dynamics: In the MIMO case, for example, high-frequency uncertainty is typically present due to the loss of phase information at high frequencies. This may be represented as

$$G(s) = (I + \Delta(s)) G_0(s) \quad (1.2.9)$$

or

$$G(s) = G_0(s) (I + \Delta(s)) \quad (1.2.10)$$

depending on whether it arises at the output or the input of the plant. (Uncertainty, for example, arising due to unmodelled actuator dynamics would be represented by (1.2.9), while (1.2.10) would more accurately represent sensor uncertainty). In either case, $\Delta(s)$ is assumed to be an arbitrary unknown transfer function (subject to certain stability requirements) which satisfies $\|W\Delta\| \leq 1$ for a suitable norm $\|\cdot\|$ and a frequency weighting $W(s)$. Alternative models of unstructured perturbation in the plant include additive perturbations (Figure 4), inverse-additive/inverse multiplicative perturbations, perturbations in the coprime factors of the plant etc.

The following two Theorems establish closed-loop robust-stability conditions in the case of additive and multiplicative uncertainty in the plant dynamics:

Theorem 1.2.1

The additively perturbed system of Figure 4 is closed-loop stable if the following conditions are satisfied:

- (i) $Q_0(s)$ is closed-loop stable
- (ii) $Q_0(s)$ and $Q(s) = Q_0(s) + \Delta_a(s)$ have the same number of poles in the CRHP.
- (iii) $\bar{\sigma}(\Delta_a(s)) < \underline{\sigma}(I + Q_0(s)) \quad \forall s \in D_R$.

where D_R is a Nyquist contour with radius R large enough to contain all the RHP poles of $Q_0(s)$ and $\Delta_a(s)$. (Indentations are made into the LHP whenever Q_0 or Δ_0 have imaginary axis poles.)

Proof: See [H6]. □

A similar condition may be established in the case of multiplicative uncertainty.

Theorem 1.2.2

The multiplicatively perturbed system of Figure 5 is closed-loop stable if the following conditions are satisfied:

- (i) $Q_0(s)$ is closed-loop stable
- (ii) $Q_0(s)$ and $(I + \Delta_m(s))Q_0(s)$ have the same number of poles in the CRHP.
- (iii) $\bar{\sigma}(\Delta_m(s)) < \underline{\sigma}(I + Q_0^{-1}(s)) \quad \forall s \in D_R$.

Proof: See [H6]. □

Theorem 1.2.2 shows that the smallest singular value of $I + Q_0^{-1}$ defines a (frequency-dependent) bound that the largest singular of the uncertainty must satisfy for robust stability in the closed-loop to be maintained. This bound, may be written as

$$\bar{\sigma}(\Delta_m) < \underline{\sigma}(I + Q_0^{-1}) = \frac{1}{\bar{\sigma}(I + Q_0^{-1})^{-1}} = \frac{1}{\bar{\sigma}(I - (I + Q_0)^{-1})} = \frac{1}{\bar{\sigma}(Q_0(I + Q_0)^{-1})} = \frac{1}{\bar{\sigma}(T)} \quad (1.2.11)$$

from which it follows that the system will tolerate large high-frequency uncertainties if $\bar{\sigma}(T) \ll 1$ at high frequencies. This requirement, therefore, is consistent with (1.2.8). It is shown next that the infimum of $\underline{\sigma}(I + Q_0^{-1}(j\omega))$ is also related to the gain and phase margins of the system. Gain and phase margins in SISO systems denote the maximum amounts of pure gain and phase shifts that the (stable) closed-loop system can tolerate without inducing instability. The next definitions generalise these concepts to the MIMO case.

Definition 1.2.1

Suppose that $Q(s) \in \mathbb{R}^{m \times m}(s)$ is closed-loop stable. Let $M = \text{diag}(k_1, k_2, \dots, k_m)$, $k_i \in \mathbb{R}$. Then the gain margin interval (GMI) is defined to be the largest interval (a, b) such that $k_i \in (a, b) \quad \forall i=1, 2, \dots, m$ implies that $MQ(s)$ is closed-loop stable.

Definition 1.2.2

Suppose that $Q(s) \in \mathbb{R}^{m \times m}(s)$ is closed-loop stable. Let $M = \text{diag}(e^{j\theta_1}, e^{j\theta_2}, \dots, e^{j\theta_m})$, $\theta_i \in [-\pi, \pi]$. Then the phase margin interval (PMI) is defined to be the largest interval (α, β) such that $\theta_i \in (\alpha, \beta) \quad \forall i=1, 2, \dots, m$ implies that $MQ(s)$ is closed-loop stable.

The following Theorem applies the result of Theorem 1.2.2 to the special structures of (multiplicative) uncertainty which is implicit in the definitions of the gain and phase margins. It shows

that the stability margins of the system can be extended by maximising $\inf_{\omega} \underline{\sigma}(I+Q^{-1}(j\omega)) = \|T\|_{\infty}^{-1}$.

Theorem 1.2.3

Let $Q(s) \in \mathbb{R}^{m \times m}(s)$ be strictly proper and closed-loop stable and let α be given by

$$\alpha = \inf_{\omega \in \mathbb{R}_+} \underline{\sigma}(I+Q^{-1}(j\omega)) \quad (1.2.12)$$

Then the following intervals are estimates of the Gain Margin Interval and the Phase Margin Interval:

$$(i) (1-\alpha, 1+\alpha) \subseteq GMI \quad (1.2.13)$$

$$(ii) \left(-\cos^{-1}\left(1-\frac{\alpha^2}{2}\right), \cos^{-1}\left(1-\frac{\alpha^2}{2}\right) \right) \subseteq PMI \quad (1.2.14)$$

Proof: [H6]

Follows by using the special multiplicative uncertainty structures

$$\Delta_m(s) = \text{diag}(\delta_i) \quad \text{with } \delta_i \in (-\alpha, \alpha) \quad (1.2.15)$$

and

$$\Delta_m(s) = \text{diag}(e^{j\theta_i}) = \text{diag}\left(2j \cdot \sin\frac{\theta_i}{2} \exp\left(\frac{j\theta_i}{2}\right)\right) \quad \text{with } \cos\theta_i \in \left(-\left(1-\frac{\alpha^2}{2}\right), \left(1-\frac{\alpha^2}{2}\right)\right) \quad (1.2.16)$$

in Theorem 1.2.2. See [H6] for details. $\square$

The above analysis illustrates some of the desirable properties that certain closed-loop functions must satisfy in order to achieve objectives (1) to (4). Different design methodologies may be assessed with respect to how easily their design parameters can be used to shape the appropriate closed-loop functions.

Classical SISO design methods use the open-loop gain and phase as design tools: Large gains at low frequencies result in low sensitivity ($|S(j\omega)| \ll 1$) and complementary-sensitivity near unity ($|T(j\omega)| \simeq 1$) while low gains at high frequencies have the opposite effect. In addition, at the “cross-over frequencies” (open-loop gain close to unity) feedback properties are sensitive to the phase characteristics: bounding the system’s phase away from $\pm 180^\circ$ gives the closed-loop system adequate stability margins; most of classical design methodologies, therefore, concentrate in “bending” the Nyquist plots in the critical region (by employing, for example lead-lag compensators) to achieve this effect. One of the limitations of this method is that the two design parameters cannot be independently manipulated as is illustrated by the Bode gain-phase relation [F7], [D8]: For any proper rational function with real coefficients which is both stable and minimum phase, the following integral relation must hold at each frequency ω_0 :

$$\angle Q(j\omega_0) - \angle Q(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{d \log |Q|}{d\nu} \left\{ \log \coth \frac{|\nu|}{2} \right\} d\nu \quad (1.2.17)$$

where

$$\nu = \log\left(\frac{\omega}{\omega_0}\right) \quad (1.2.18)$$

In effect, this relation provides a limit on how fast the open-loop gain can be rolled-off in a frequency interval for a given phase variation. Thus, (1.2.17) demonstrates a fundamental analytic tradeoff between performance on the one hand and stability margins/robustness on the other [F7]. This tradeoff manifests itself clearly in the critical region: To extend the benefits of feedback in terms of performance, the frequency below which sensitivity is small should approach the frequency above which the complementary sensitivity is small; this, however, would result in a steep gain roll-off near crossover which by the the Bode relation would produce large phase variations thus affecting the stability margins of the system.

The tradeoff exhibited by the Bode relation continuous to operate in multivariable systems. In fact, the integral relation in (1.2.17) may be generalised by replacing $Q(s)$ by its algebraic function [D8]. In addition, for non-minimum phase systems the gain-rate/phase-variation tradeoff is aggravated due to the additional phase lag contributed by each RHP transmission zero. (The presence of non-minimum phase transmission zeros produces by itself analytic tradeoffs in the magnitude of the sensitivity function while unstable poles have a dual effect for complementary sensitivity. These tradeoffs are described in section 1.7). The increased complication in linking the open-loop gain and phase characteristics in multivariable systems to the attainable shape of its closed loop functions (sensitivity, complementary sensitivity, etc.) has given rise to methodologies which try to manipulate *directly* the frequency characteristics of these functions. (Such methods include $\mathcal{H}^\infty$ -control described in section 1.5 and the Fractional representation approach [D1]). In opposition to these methods, various attempts have been made to reduce MIMO design problems to a sequence of SISO designs. The most important of these include the Inverse Nyquist Array, the Characteristic Loci and a number of Sequential-Design methods. Although these methods have proved to be effective and reliable in a number of cases, their (implicit or explicit) use of eigenvalues as the basic tool of analysis prevents them from dealing systematically with the issue of uncertainty. (See the example given in [D8]).

The first systematic optimisation-based approach in multivariable design which arose out of Kalman's work is the LQG methodology [Y2]. This method assumes stochastic models for the disturbance and sensor noise signals and minimises a weighted mean square error criterion. In the ensuing design, closed loop stability is automatically enforced by the optimality condition. In addition, as a result of the separation principle, the optimal controller consists of the standard interconnection of a Kalman-Bucy filter (used to track the states of the plant) and a pure gain factor K_{LQ} arising from the solution of an infinite-horizon LQ regulator problem based on the state estimate $\hat{x}$. The control signal which drives the plant is the product $K_{LQ}\hat{x}$.

LQG originated as a time-domain method. This made the method in its original form difficult to apply to real design situations since it was impossible to relate the frequency-domain properties of the closed loop functions to the time-domain penalty-weighting on the control and state energy. An additional criticism consisted of the design's apparent lack of integrity [R3]. Interest in the frequency domain properties of LQG controllers started to arise when the excellent gain/phase margins of LQ

designs were demonstrated [L3], [S4]. Although these properties are lost with the loss of perfect state information [D5], modern LQG approaches concentrate on shaping the open loop function using “full state transfer recovery” techniques [K11], [D7], [D8]. These techniques, however, require a number of restrictive assumptions on the input/output dimensions of the plant and cannot be applied at all when the plant has transmission zeros in the RHP [D8].

The next section outlines the second major optimisation-based approach to multivariable design: $\mathcal{H}^\infty$ control.

1.3 The $\mathcal{H}^\infty$ -optimal control problem

$\mathcal{H}^\infty$ control originated from the work of G. Zames [Z1], [Z2] during the early 80's. It is an optimisation-based method for shaping directly the frequency characteristics of the relevant closed loop functions (sensitivity, complementary sensitivity etc.). In addition to being a synthesis tool for multivariable design it may also be used to exhibit possible fundamental limitations imposed by the structure of the plant. $\mathcal{H}^\infty$ is predominantly a frequency-domain method although certain interesting time-domain interpretations have recently emerged [D9], [L10], [T1].

Rather than assuming a fixed and known power spectrum for the exogenous signals, $\mathcal{H}^\infty$ models them as being bounded in power, their spectral characteristics being reflected in an associated frequency weight. In particular, disturbance/noise signals $d(s)$, are assumed to belong to a set

$$\mathcal{D} = \{d(s): d = W_1 x \text{ for } x \in \mathcal{H}_2: \|x\|_2 \leq 1\} \quad (1.3.1)$$

or, equivalently,

$$\mathcal{D} = \{d(s) \in \mathcal{H}_2: \|W_1^{-1} d\|_2 \leq 1\} \quad (1.3.2)$$

in which $W_1(s) \in \mathcal{H}_+^\infty$ and $W_1^{-1}(s) \in \mathcal{H}_+^\infty$. $W_1(s)$ may be viewed as a frequency weight which determines the spectral characteristics of the signals in $\mathcal{D}$: For example, the inequality in (1.3.2) for a scalar-valued $d(s)$ may be written as

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} |d(j\omega)|^2 |W_1(j\omega)|^{-2} d\omega \leq 1 \quad (1.3.3)$$

which in turn may be interpreted as a constraint on the weighted energy of the signal $d(s)$, $|W_1(j\omega)|^{-2}$ being a weighting factor on the energy spectral density $|d(j\omega)|^2$. Thus, a signal having its power (uniformly) concentrated in a frequency interval $[\omega_1, \omega_2]$ would correspond to a band-pass filter $W_1(s)$ having this interval as its band-pass.

Suppose next that $d(s)$ enters the feedback loop at a point A and that it is required to minimise its effect at a point B . The corresponding $\mathcal{H}^\infty$ optimal control problem is to find a compensator $K(s)$ such that (i) the feedback loop is internally stable and (ii) the output signal power

(weighted by a matrix function $W_2(s) \in \mathcal{H}_+^\infty$ such that $W_2^{-1}(s) \in \mathcal{H}_+^\infty$) at the point B due to the worst-case $d(s) \in \mathcal{D}$ is minimised. If $S_{AB}(G, K)$ denotes the transfer function between points A and B , the optimisation problem may be formally expressed as the constrained minimisation

$$\inf_{K \in \Xi} \sup_{d \in \mathcal{D}} \|S_{AB}(G, K)d\|_2 = \inf_{K \in \Xi} \sup_{x \in \mathcal{H}_+^2, \|x\|_2 \leq 1} \|W_2 S_{AB}(G, K) W_1 x\|_2 \quad (1.3.4)$$

in which Ξ denotes the set of all internally stabilising compensators of $G(s)$ in the configuration of Figure 1. Since $K(s)$ in (1.3.4) is internally stabilising, $W_2 S_{AB}(G, K) W_1 \in \mathcal{H}_+^\infty$ and (1.3.4) may be written as

$$\inf_{K \in \Xi} \|W_2 S_{AB}(G, K) W_1\|_\infty \quad (1.3.5)$$

Robust stabilisation problems may be also formulated as infinity-norm minimisations. In fact, the choice of this norm is the most appropriate in the case of unstructured perturbations since it provides both necessary and sufficient conditions for robust stability under mild assumptions on the perturbed system's stability properties. (In contrast the $\|\cdot\|_2$ norm does not even give sufficient conditions for robust stability [D4].) Consider for example the additive perturbation case in which the nominal plant $G_0(s)$ is perturbed to $G_0 + \Delta(s)$. Assume further that $G_0 + \Delta(s)$ has the same number of poles in $\mathbb{C}_+$ as $G_0(s)$ and let $r(s)$ be a scalar-valued function in $\mathcal{H}_+^\infty$ such that $\bar{\sigma}(\Delta(j\omega)) < |r(j\omega)|$ for all ω . It is shown in [C2] that in this case the condition $\|rK(I + PK)^{-1}\|_\infty \leq 1$ is both sufficient and necessary for the stabilisation of all plants in this family.

The important fact in the $\mathcal{H}^\infty$ formulation is that both noise/disturbance attenuation and robust stability requirements may be formulated as infinity-norm inequalities for certain transfer functions which may be represented as linear fractional transformations of the compensator and the plant. Thus "mixed-type problems" may be defined which incorporate a number of design objectives. Particular examples of such problems are given later in the section. Before that, however, the general problem is defined:

The $\mathcal{H}^\infty$ -optimal control problem

Let $P(s)$ in the generalised regulator configuration of Figure 1 be partitioned as

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \in \mathbb{R}^{(m+p) \times (r+q)}(s) \quad (1.3.6)$$

in which $P_{11}(s) \in \mathbb{R}^{m \times r}$ with $m \geq q$ and $p \geq r$. Suppose also that $P(s)$ has a minimal realisation

$$P(s) \triangleq \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (1.3.7)$$

partitioned conformally with (1.3.6). Suppose further that

- (i) (A, B_2) is stabilisable and (A, C_2) is detectable.
- (ii) D_{12} has full column rank and $\text{Re}(\lambda(A - B_2(D_{12}^* D_{12})^{-1} D_{12} C_1)) \neq 0$
- (iii) D_{21} has full row rank and $\text{Re}(\lambda(A - B_1(D_{21} D_{21}^*)^{-1} C_2)) \neq 0$

A compensator $K_0(s)$ is $\mathcal{H}^\infty$ -optimal if (i) the feedback configuration in Figure 1 is internally stable and (ii) $K_0(s)$ achieves the infimum $\inf_{K \in \Xi} \|\mathcal{F}_\ell(P, K)\|_\infty$ where Ξ denotes the set of all internally stabilising compensators.

In the above formulation of the problem, condition (i) is included to ensure the existence of stabilising compensators and thus to guarantee that the optimisation problem is not vacuous (see section 1.4). Conditions (ii) and (iii) are included to ensure that $\inf_{K \in \Xi} \|\mathcal{F}_\ell(P, K)\|_\infty$ can be achieved for some $K(s) \in \Xi$. (Internal stability in the context of Figure 1 is formally defined in section 1.4).

In Chapter 2 a detailed state-space analysis is carried out for a special class of $\mathcal{H}^\infty$ -control problems. This class is characterised by the assumption that $P_{21}(s)$ is square while $P_{12}(s)$ has more rows than columns. (In this case D_{21} is assumed to be non-singular and $\text{Re}(\lambda(A - B_1 D_{21}^{-1} C_2)) \neq 0$). If one considers $\mathcal{R}^T(s) = \mathcal{F}_\ell^T(P, K)$, it is easy to see that an equivalent characterisation is given by the assumptions that $P_{12}(s)$ is square while $P_{21}(s)$ has more columns than rows. Any problem fitting either of these alternative descriptions will be called a problem of the second kind. In contrast, the assumption that both the off-diagonal blocks of $P(s)$ are square defines a problem of the first kind whereas the general case a problem of the third kind. The state-space analysis in Chapter 2 shows that for the class under consideration certain cancellations are a direct consequence of $\mathcal{H}^\infty$ optimality. Chapter 3 deals with a strengthened optimality condition for $\mathcal{H}^\infty$ problems of the first kind.

In the remaining paragraphs of this section a number of $\mathcal{H}^\infty$ problems will be presented: The weighted sensitivity problem [C1], [F2], [F3], [F5], [F6], [S1], [Z1], [Z2] is defined as

$$\inf_{K \in \Xi} \|W_2(I + GK)^{-1} W_1\|_\infty = \inf_{K \in \Xi} \|\mathcal{R}_s(s)\|_\infty \quad (1.3.8)$$

where Ξ is the set of stabilising compensators. By choosing

$$P_s(s) = \begin{bmatrix} W_2 W_1 & W_2 G \\ W_1 & G \end{bmatrix} \quad (1.3.9)$$

this problem may be embedded in the generalised regulator configuration of Fig. 1. Since,

$$\mathcal{R}_s(s) = \mathcal{F}_\ell(P, -K) = P_{11} - P_{12}K(I + P_{22}K)^{-1}P_{21} \quad (1.3.10)$$

direct comparison with (1.1.7) reveals that

$$\mathcal{R}_s(s) = \mathcal{F}_\ell(P_s(s), -K(s)) \quad (1.3.11)$$

In the case that $G(s)$ is square, this problem falls into the class of problems of the first kind analysed in [L5], [L6]. If, on the other hand, $G(s)$ has more outputs than inputs and $W_2(s)$ is square, $P_{12}(s) = W_2 G(s)$ will have more rows than columns giving rise to a problem of the second kind.

Another problem which has received attention is the mixed sensitivity problem [F1], [F2], [F3], [F6], [V1], [V2]. In this case we seek

$$\inf_{K \in \Xi} \left\| \begin{bmatrix} W_2 G K (I + G K)^{-1} W_1 \\ W_3 (I + G K)^{-1} W_1 \end{bmatrix} \right\|_\infty = \inf_{K \in \Xi} \|\mathfrak{R}_{ms}(s)\|_\infty \quad (1.3.12)$$

Setting

$$P_{ms}(s) = \left[\begin{array}{c|c} 0 & W_2 G \\ \hline W_3 W_1 & W_3 G \\ \hline W_1 & G \end{array} \right] \quad (1.3.13)$$

gives

$$\mathfrak{R}_{ms}(s) = \mathfrak{F}_\ell(P_{ms}(s), -K(s)) \quad (1.3.14)$$

which is also an example of a problem of the second kind.

Several other problems of the second kind may be found in the literature. See for example [F2], [F6] and the numerous references therein. Rather than analyse these problems individually, the analysis in Chapter 2 identifies the common characteristics shared by all $\mathcal{H}^\infty$ control problems of the second kind.

In the next two sections a few methods for solving the $\mathcal{H}^\infty$ problem will be outlined. Section 1.4 summarises the Youla parametrisation [D3], [Y2] which is used to characterise the class of all stabilising compensators and the corresponding closed loop transfer functions $\mathfrak{R}(s)$ in Fig. 1. Following that, the closed loop transfer functions which have minimum $\mathcal{L}^\infty$ norm are identified.

1.4 Parametrisation of all internally stabilising compensators

Consider the feedback system in Figure 2 in which $K(s)$ denotes the compensator, $G(s)$ the plant, u_1 and u_2 denote externally applied inputs, e_1 and e_2 the inputs to the compensator and the plant, while y_1 and y_2 the outputs to the compensator and the plant respectively. It is assumed that $K(s)$ and $G(s)$ represent lumped linear time-invariant systems of compatible dimensions.

The equations describing the various interconnections of the feedback system in Figure 2 are

$$\begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} - \begin{bmatrix} 0 & G \\ -K & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (1.4.1)$$

and

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} K & 0 \\ 0 & G \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad (1.4.2)$$

which may be written more compactly as

$$e = u - F\tilde{G}e \quad (1.4.3)$$

and

$$y = \tilde{G}e \quad (1.4.4)$$

in obviously-defined notation. If the determinant $\det(I+GK)$ is not identically zero for all $s \in \mathbb{C}$, the system is *well-posed* and it is possible to solve (1.4.1) and (1.4.2) for e ; this gives

$$e = (I + F\tilde{G})^{-1}u := H(G, K)u \quad (1.4.5)$$

where

$$H(G, K) = \begin{bmatrix} (I+GK)^{-1} & -G(I+KG)^{-1} \\ K(I+GK)^{-1} & (I+KG)^{-1} \end{bmatrix} \quad (1.4.6)$$

Stability in the feedback loop in Figure 2 is defined by the condition $H(G, K) \in \mathcal{H}_+^\infty$ (provided that the well-posedness condition $\det(I+GK) \neq 0$ is satisfied). This is a generalisation of the usual condition for SISO stability given as $G(I+KG)^{-1} \in \mathcal{H}_+^\infty$; in fact it may be easily verified by considering the four blocks of (1.4.6) that the two conditions are equivalent in the case that a stable compensator is used [V3]. It is shown in the next Lemma that the condition $H(G, K) \in \mathcal{H}_+^\infty$ is equivalent to *internal stability* for the feedback interconnection in Figure 2. (Internal (or asymptotic) stability implies that the state vector decays to zero with time when released from an arbitrary initial condition, provided that all external inputs have been set to zero). Thus Lemma 1.4.1 guarantees that no unstable hidden modes are present in a state-space realisation which describes the system in Figure 2 provided that no such modes are present in the realisations of the plant and the compensator.

Lemma 1.4.1

Let $K(s)$ and $G(s)$ in Figure 2 be described by the state-space equations

$$\dot{x}_c = A_c x_c + B_c u_c, \quad y_c = C_c x_c + D_c u_c \quad (1.4.7)$$

$$\dot{x}_p = A_p x_p + B_p u_p, \quad y_p = C_p x_p + D_p u_p \quad (1.4.8)$$

Let also (A_c, B_c, C_c) and (A_p, B_p, C_p) be stabilisable and detectable triples and assume that the well-posedness condition

$$\det(I+K(\infty)G(\infty))=\det(I+D_cD_p)\neq 0 \quad (1.4.9)$$

is satisfied. Then the system in Figure 2 is internally stable if and only if $H(G, K) \in \mathcal{H}_+^\infty$.

Proof: [V3]

Let the composite state vector of the interconnected system be $x=[x_c^T \ x_p^T]^T$. Also let $y=[y_1^T \ y_2^T]^T$, $e=[e_1^T \ e_2^T]^T$, $u=[u_1^T \ u_2^T]^T$ and define the block diagonal matrices $A=A_c \oplus A_p$, $B=B_c \oplus B_p$, $C=C_c \oplus C_p$ and $D=D_c \oplus D_p$. Further define

$$F = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \quad (1.4.10)$$

Then,

$$\dot{x} = Ax + Be \quad (1.4.11)$$

and

$$y = Cx + De \quad (1.4.12)$$

In addition,

$$e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} u_1 - y_2 \\ u_2 + y_1 \end{bmatrix} = u - F(Cx + De) \Rightarrow e = (I + FD)^{-1}u - (I + FD)^{-1}FCx \quad (1.4.13)$$

Substituting (1.4.13) in (1.4.11) and (1.4.12) gives a state-space realisation of the interconnected system $W(G, K)$ with input $u(s)$ and output $y(s)$ as

$$\dot{x} = \tilde{A}x + \tilde{B}u \quad (1.4.14)$$

and

$$y = Cx + Du \quad (1.4.15)$$

where

$$\tilde{A} = A - B(I + FD)^{-1}FC \quad (1.4.16)$$

$$\tilde{B} = B(I + FD)^{-1} \quad (1.4.17)$$

$$\tilde{C} = (I + DF)^{-1}C \quad (1.4.18)$$

and

$$\tilde{D} = (I + DF)^{-1}D \quad (1.4.19)$$

In addition, since (A_c, B_c) and (A_p, B_p) are stabilisable so is (A, B) which implies that there exists a state feedback matrix L such that $A - BL$ is asymptotically stable. Defining

$$\tilde{L} = (I + FD) \left\{ L - (I + FD)^{-1} FC \right\} \quad (1.4.20)$$

gives

$$\tilde{A} - \tilde{B}\tilde{L} = A - B(I + FD)^{-1}FC - B \left\{ L - (I + FD)^{-1}FC \right\} = A - BL \quad (1.4.21)$$

which shows that $(\tilde{A}, \tilde{B})$ is stabilisable; similarly $(\tilde{A}, \tilde{C})$ can be shown to be detectable which implies by [K1] that the interconnected system is asymptotically stable (“internally stable”) if and only if $W(G, K) \in \mathcal{H}_+^\infty$. The result follows on noting that

$$H(s) = I - FW(s) \Leftrightarrow W(s) = F(H(s) - I) \quad (1.4.22)$$

so that

$$H(G, K) \in \mathcal{H}_+^\infty \Leftrightarrow W(G, K) \in \mathcal{H}_+^\infty \quad (1.4.23)$$

This concludes the proof. $\square$

The set of all internally stabilising compensators for the plant $G(s)$ may be parametrised in term of a free parameter $Q(s) \in \mathcal{H}_+^\infty$ via a right and left coprime factorisation of $G(s)$. Before stating this Theorem, however, we include a formal definition of coprime factorisations. Since a real-rational plant is assumed, the definition is restricted to factorisation over $\mathbb{R}\mathcal{H}_+^\infty$:

Definition:

(1) Two matrices F and G in $\mathbb{R}\mathcal{H}_+^\infty$ are right coprime (over $\mathbb{R}\mathcal{H}_+^\infty$) if they have equal number of columns and there exist matrices X and Y in $\mathbb{R}\mathcal{H}_+^\infty$ such that

$$\begin{bmatrix} X & Y \end{bmatrix} \begin{bmatrix} F \\ G \end{bmatrix} = XF + YG = I \quad (1.4.24)$$

This is equivalent to saying that $\begin{bmatrix} F^T & G^T \end{bmatrix}^T$ is left invertible in $\mathbb{R}\mathcal{H}_+^\infty$.

(2) Two matrices F and G in $\mathcal{H}_+^\infty$ are left coprime (over $\mathbb{R}\mathcal{H}_+^\infty$) if they have equal number of rows and there exist matrices X and Y in $\mathbb{R}\mathcal{H}_+^\infty$ such that

$$\begin{bmatrix} F & G \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = FX + GY = I \quad (1.4.25)$$

This is equivalent to saying that $\begin{bmatrix} F & G \end{bmatrix}$ is right invertible in $\mathbb{R}\mathcal{H}_+^\infty$.

(3) Let G be a proper real rational matrix. A right coprime factorisation is a factorisation $G = NM^{-1}$ where N and M are right coprime $\mathbb{R}\mathcal{H}_+^\infty$ matrices. A left coprime factorisation has the form $G = \tilde{M}^{-1}\tilde{N}$ where $\tilde{N}$ and $\tilde{M}$ are left coprime over $\mathbb{R}\mathcal{H}_+^\infty$. $\square$

Theorem 1.4.2

Let $G(s)$ admit right and left coprime factorisations

$$G(s) = N_r D_r^{-1}(s) = D_l^{-1}(s) N_l(s) \quad (1.4.26)$$

over $\mathcal{H}_+^\infty$ and let V_r , V_l , U_r and U_l solve the corresponding Bezout identity

$$\begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (1.4.27)$$

Then the family of all internally stabilising compensators for the feedback loop in Figure 2 is parametrised by

$$\begin{aligned} \mathcal{K}(s) &= \{ (U_l + D_r Q)(V_l - N_r Q)^{-1} \mid Q \in \mathcal{H}_+^\infty, \det(V_l - N_r Q) \neq 0 \} \\ &= \{ (V_r - Q N_l)^{-1}(U_r + Q D_l) \mid Q \in \mathcal{H}_+^\infty, \det(V_r - Q N_l) \neq 0 \} \end{aligned} \quad (1.4.28)$$

Proof:

See [V3]. □

The generalised regulator configuration of Figure 1 will be considered next. Internal stability in this context may be defined by introducing the additional inputs v_1 and v_2 (Figure 3) and requiring that the nine transfer functions from $[w^T \ v_1^T \ v_2^T]^T$ to $[z^T \ u^T \ y^T]^T$ all belong to $\mathcal{H}_+^\infty$. Restricting $P(s)$ by setting P_{11} , P_{12} and P_{21} to the zero matrices (of the appropriate dimension) and the input $w=0$, shows that this definition of internal stability is a generalisation of the definition given in the context of Figure 2.

The equations corresponding to Figure 3 may be written as

$$\begin{bmatrix} I & -P_{12} & 0 \\ 0 & I & -K \\ 0 & -P_{22} & I \end{bmatrix} \begin{bmatrix} z \\ u \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & 0 & 0 \\ 0 & I & 0 \\ P_{21} & 0 & I \end{bmatrix} \begin{bmatrix} w \\ v_1 \\ v_2 \end{bmatrix} \quad (1.4.29)$$

These may be solved for the output vector $[z^T \ u^T \ y^T]^T$ provided that the well-posedness condition

$$\left| \begin{bmatrix} I & -P_{12} & 0 \\ 0 & I & -K \\ 0 & -P_{22} & I \end{bmatrix} \right| \neq 0 \Leftrightarrow \det(I - P_{22}K) \neq 0 \quad (1.4.30)$$

is satisfied.

For an arbitrary plant $G(s)$ there is always a compensator $K(s)$ such that the feedback system

in Figure 2 is internally stable. This follows from Theorem 1.4.2 since coprime factorisations of $G(s)$ are always possible. The existence of stabilising compensators, however, is not guaranteed in the case of the generalised regulator: Consider for example the case $P_{12}=0$, $P_{21}=0$, $P_{22}=0$ and $P_{11} \in \mathcal{H}_+^\infty$; since P_{11} is disconnected from control action by the zero off-diagonal blocks of $P(s)$ this system is not stabilisable. In terms of a minimal state-space realisation of $P(s)$

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (1.4.31)$$

existence of an internally stabilising controller is equivalent to the condition that that (A, B_2, C_2) is stabilisable and detectable. Intuitively, this implies that $P(s)$ and $P_{22}(s)$ must share all unstable poles so that stabilising $P_{22}(s)$ is sufficient to stabilise $P(s)$. Under the condition that $P(s)$ is stabilisable, a compensator $K(s)$ stabilises the feedback system in Fig. 2 if and only if it stabilises $P_{22}(s)$ [F3]. This implies that Theorem 1.4.2 may be applied to $P_{22}(s)$ to parametrise all stabilising compensators for the configuration of Figure 3. This parametrisation is summarised in the following theorem.

Theorem 1.4.3

(1) Let $P(s)$ have a minimal state-space realisation given in 1.4.31. Then an internally stabilising compensator in Figure 3 exists, if and only if (A, B_2) is stabilisable and (A, C_2) is detectable.

(2) Assume that $P(s)$ is stabilisable. Let $P_{22}(s)$ admit right and left coprime factorisations

$$P_{22}(s) = N_r D_r^{-1}(s) = D_l^{-1}(s) N_l(s) \quad (1.4.32)$$

over $\mathcal{H}_+^\infty$ and let V_r, V_l, U_r and U_l solve the corresponding Bezout identity

$$\begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (1.4.33)$$

Then the family of all internally stabilising compensators for the feedback loop in Figure 3 is parametrised by

$$\begin{aligned} \mathcal{K}(s) &= \{ (U_l + D_r Q)(V_l - N_r Q)^{-1} \mid Q \in \mathcal{H}_+^\infty, \det(V_l - N_r Q) \neq 0 \} \\ &= \{ (V_r - Q N_l)^{-1}(U_r + Q D_l) \mid Q \in \mathcal{H}_+^\infty, \det(V_r - Q N_l) \neq 0 \} \end{aligned} \quad (1.4.34)$$

Proof:

See [F3].

□

Applying the controller parametrisation of Theorem 1.4.3 it is easy to verify that for every stabilising

controller $K(s)$,

$$K(I + P_{22}K)^{-1}(s) = (U_l + D_r Q)D_l(s) \quad (1.4.35)$$

Hence

$$\begin{aligned} \mathfrak{R}(s) &= [P_{11} - P_{12}K(I + P_{22}K)^{-1}P_{21}](s) \\ &= [(P_{11} - P_{12}U_l D_l P_{21}) - (P_{12}D_r)Q(D_l P_{21})](s) \\ &= [T_{11} - T_{12}Q T_{21}](s) \end{aligned} \quad (1.4.36)$$

where the $T_{ij}(s)$'s are defined in an obvious way. Equation (1.4.36) shows that $\mathfrak{R}(s)$ is parametrised *linearly* in terms of a *free* matrix $Q(s) \in \mathbb{RH}_+^\infty$. Since $\mathfrak{R}(s) \in \mathbb{RH}_+^\infty$ if and only if $Q(s) \in \mathbb{RH}_+^\infty$, we would expect that T_{11} , T_{12} and T_{22} all belong to $\mathbb{RH}_+^\infty$.

Next, state space realisations will be given for the construction outlined in the previous paragraphs. Since (A, B_2) is stabilisable, there exists a state feedback matrix F such that $A - B_2 F$ is stable. Similarly, since (A, C_2) is detectable there exists an output injection matrix H such that $A - H C_2$ is stable. Given any such pair of stabilising matrices F and H , right and left coprime factorisations of $P_{22}(s)$ together with the solutions of the Bezout identities are given by [D4]

$$\begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - B_2 F & B_2 & H \\ \hline -F & I & 0 \\ C_2 - D_{22} F & D_{22} & I \end{array} \right] \quad (1.4.37)$$

and

$$\begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} A - H C_2 & B_2 - H D_{22} & H \\ \hline F & I & 0 \\ -C_2 & -D_{22} & I \end{array} \right] \quad (1.4.38)$$

Substituting (1.4.37) and (1.4.38) into (1.4.36) gives after minor calculation [D4], [L5], [L6],

$$\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & 0 \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|cc} A - B_2 F & B_2 F & B_1 & B_2 \\ 0 & A - H C_2 & B_1 - H D_{21} & 0 \\ \hline C_1 - D_{12} F & D_{12} F & D_{11} & D_{12} \\ 0 & C_2 & D_{21} & 0 \end{array} \right] \quad (1.4.39)$$

Note that the $T_{ij}(s)$'s all belong to $\mathbb{RH}_+^\infty$ as expected.

Remark:

In the special case that $P_{22}(s)$ is stable, we may set $F=0$ and $H=0$. This gives $U_l=0$, $U_r=0$, $D_r=I$, $D_l=I$, $V_l=I$, $V_r=I$ and $N_r=N_l=P_{22}(s)$. Substituting these into (2.13) gives

$$Q(s) = K(I + P_{22}K)^{-1}(s)$$

which is the Q-parametrisation of Zames [Z2].

1.5 Solving the model-matching problem

It is shown in the previous section that using the controller parametrisation of Theorem 1.4.3 it is possible to transform the problem

$$[\mathbf{P1}]: \gamma_{opt} = \inf_{K \in \Xi} \|\mathfrak{F}_\ell(P, K)\|_\infty = \inf_{K \in \Xi} \|P_{11} - P_{12}K(I + P_{22}K)^{-1}P_{21}\|_\infty \quad (1.5.1)$$

to

$$[\mathbf{P2}]: \gamma_{opt} = \inf_{Q \in \mathcal{H}_+^\infty} \|\Re(Q)\|_\infty = \inf_{Q \in \mathcal{H}_+^\infty} \|T_{11} - T_{12}QT_{21}\|_\infty \quad (1.5.2)$$

The advantage of this transformation is clear: whereas [P1] describes a minimisation of a complex objective function over an implicitly defined set, [P2] involves a cost function which is *affine* in a *free* parameter $Q \in \mathcal{H}_+^\infty$. Since by assumption $P_{12}(j\omega)$ and $P_{21}(j\omega)$ are full row and column rank, so are $T_{12}(j\omega)$ and $T_{21}(j\omega)$ (see (1.4.36)) and hence by Theorem 5.1 in [F6], the infimum in [P2] is achieved.

In the next sections a number of methods for solving [P2] will be outlined. For this task the two types of problem introduced in section 1.3 will be distinguished. Problems of the first kind will be treated by transforming (1.5.2) to a Nehari approximation problem

$$\gamma_{opt} = \inf_{Q \in \mathcal{H}_+^\infty} \|R - Q\|_\infty \quad (1.5.3)$$

This may be achieved by making a specific choice for the state feedback and output injection matrices F and H which define the coprime factors of $P_{22}(s)$ in (1.4.37) and (1.4.38) so as to make $T_{12}(s)$ and $T_{21}(s)$ inner. (This is always possible after scaling the off-diagonal blocks of $P(s)$ at infinity to orthogonal matrices of the appropriate dimensions). Alternatively, if arbitrary stabilising matrices F and H have been chosen in the state-space realisation of (1.4.39), T_{12} and T_{21} may be made inner by performing inner-outer factorisations on them and absorbing the outer factors in the free parameter $Q(s)$. In either case, this step involves the solution of two algebraic Riccati equations. Multiplication from left and right by T_{12}^* and T_{21}^* is then a norm-preserving operation giving rise to a problem having the form of (1.5.3). The infimum in (1.5.3) is known to be the Hankel norm of $R(-s)$, while a

complete parametrisation of all $Q(s)$'s which *achieve* this value has been described in [G1].

Alternatively, the model-matching problem for the class of $\mathcal{H}^\infty$ problems of the first kind may be solved via classical Nevanlinna-Pick interpolation theory. This approach is outlined later in the section; γ_{opt} is obtained by solving a generalised eigenvalue problem, while the family of all optimal interpolating functions may be obtained as a “chain” of linear fractional compositions of degree-one matrix functions, each associated with a RHP interpolation constraint. [L5]. An advantage of the interpolation approach in solving the model-matching problem is that no special choice of F and H is necessary (and thus no need arises for solving the two Riccati equations), while the solution of an eigenvalue problem is (computationally) equivalent to the calculation of a Hankel norm. In addition, this method gives insight to the constraints on the magnitude of $\mathcal{R}(s)$ on the imaginary axis, arising due to the interpolation constraints in the RHP; in contrast, the approximation solution conceals the interpolation constraints on $\mathcal{R}(s)$ although these are still implicit in the rank of the solution of the two Riccati equations and the MacMillan degree of $R(s)$. A disadvantage of the interpolation-based approach is the awkward treatment of constraints defined at zeros having multiplicities greater than one.

For problems of the second kind a special choice of matrices F and H makes $T_{21}(s)$ and $T_{12}(s)$ (square) inner and part of a (square) inner matrix respectively. Multiplication from left and right by $[T_{12} \ T_\perp]^*$ and T_{21}^* (where $T_\perp$ is the inner complement of T_{12}) is again norm preserving and thus (1.5.2) may be transformed to

$$\gamma_{opt} = \inf_{Q \in \mathcal{H}_+^\infty} \left\| \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} - \begin{bmatrix} Q \\ 0 \end{bmatrix} \right\|_\infty \quad (1.5.4)$$

where $R(s) = [R_1^T \ R_2^T]^T \in \mathcal{H}^\infty$. This is referred as the two-block general distance problem. If $Q(s)$ was allowed to “cover” the whole of R , the minimum distance would be obtained from Nehari’s Theorem as $\|R(-s)\|_H$, from which we conclude that $\gamma_{opt} \geq \|R(-s)\|_H$. In addition, since $Q(s)=0$ is admissible, $\gamma_{opt} \leq \|R(-s)\|_\infty$ which may be bounded, in turn, by the more easily computable value $\bar{\sigma}(R(\infty)) + 2\sum \sigma_i(R(-s))$, where the σ_i denote the Hankel singular values of $R(-s)$ [G1]. Later in this section an iterative algorithm is described which solves (1.5.4). This is essentially a binary search procedure between the upper and lower bounds given above.

A generalised interpolation procedure for solving problems of the second and third kind is given in [H3], [H4]. This involves the characterisation of all $\mathcal{R}(s)$ of minimum norm which satisfy matrix-like interpolation constraints of the form $\mathcal{A}(s)\mathcal{R}(s)=\mathcal{B}(s)$.

Solution of the model-matching problems for problems of the second kind

Let $P(s)$ in Fig. 1 be given by

$$P(s) = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} (s) \triangleq \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & I & D_{22} \end{array} \right] \quad (1.5.5)$$

and suppose that (A, B_2, C_2) is stabilisable and detectable. Note that there is no loss of generality in assuming that $D_{21}=I$ and that D_{12} is part of an orthogonal matrix [L6], [S2]. It is always possible to achieve this by a constant rescaling of the problem. Under these assumptions there exist unique stabilising positive semi-definite solutions to the algebraic Riccati equations [D4], [S2]

$$\mathfrak{X}(A - B_2 D_{12}^* C_1) + (A - B_2 D_{12}^* C_1)^* \mathfrak{X} - \mathfrak{X} B_2 B_2^* \mathfrak{X} + C_1^* D_{\perp} D_{\perp}^* C_1 = 0 \quad (1.5.6)$$

and

$$\mathfrak{Y}(A - B_1 C_2)^* + (A - B_1 C_2) \mathfrak{Y} - \mathfrak{Y} C_2^* C_2 \mathfrak{Y} = 0 \quad (1.5.7)$$

$D_{\perp}$ has been chosen to make the augmented matrix $[D_{12} \mid D_{\perp}]$ orthogonal [D4], [S2]. Associated with these two Riccati equations we have the stabilising matrices F and H given by [D4], [L6], [S2]

$$F = D_{12}^* C_1 + B_2^* \mathfrak{X} \quad (1.5.8)$$

and

$$H = B_1 + \mathfrak{Y} C_2^* \quad (1.5.9)$$

The Youla parametrisation theory allows us to write

$$\mathfrak{R}(s) = [T_{11} - T_{12} X T_{21}](s) \quad (1.5.10)$$

and

$$K(s) = \mathfrak{T}_{\ell}(K_0(s), X(s)) \quad (1.5.11)$$

in which

$$K_0(s) \triangleq \left[\begin{array}{c|cc} A - B_2 F - H C_2 + H D_{22} F & -H & B_2 - H D_{22} \\ \hline -F & 0 & I \\ C_2 - D_{22} F & I & D_{22} \end{array} \right] \quad (1.5.12)$$

and

$$\begin{bmatrix} T_{11} & T_{12} & T_{\perp} \\ T_{21} & 0 & 0 \end{bmatrix} \triangleq \left[\begin{array}{cc|ccc} A - B_2 F & B_2 F & B_1 & B_2 - \mathfrak{X}^{\#} C_1^* D_{\perp} & \\ 0 & A - H C_2 & -\mathfrak{Y} C_2^* & 0 & 0 \\ \hline C_1 - D_{12} F & D_{12} F & D_{11} & D_{12} & D_{\perp} \\ 0 & C_2 & I & 0 & 0 \end{array} \right] \quad (1.5.13)$$

$\mathfrak{X}^\#$ is the Moore–Penrose generalised inverse of $\mathfrak{X}$. With the particular choice of the matrices F and H given in (1.5.8) and (1.5.9), T_{21} and $[T_{12} \mid T_\perp]$ are inner [D4], [S2]. We call $T_\perp(s)$ an inner or all-pass extension of $T_{12}(s)$.

The γ -iteration

In this subsection we will outline an algorithm [D4], [F1], [F2], [F3], [F6], [S2], [V1], which reduces the problem of finding an upper bound for

$$\inf_{X(s) \in \mathbb{RH}_+^\infty} \| (T_{11} - T_{12}X T_{21})(s) \|_\infty = \gamma_{opt} \quad (1.5.14)$$

to the Nehari problem [N1] of identifying those matrices in $\mathbb{RH}_+^\infty$ which are closest (in the sense of the $\mathcal{L}^\infty$ -norm) to a given point (matrix) in $\mathbb{RL}^\infty$. The approach will be to generate a sequence $\gamma_i = \|R_i(s)\|_\infty = \|T_{11} - T_{12}X_i T_{21}\|_\infty$ which converges on γ_{opt} (from above). In applications, the calculation of this sequence is terminated when an upper bound α for γ_{opt} has been found such that $(\alpha - \gamma_{opt}) < \epsilon$ for some sufficiently small $\epsilon > 0$.

It may be seen from (1.5.10) that

$$\mathfrak{R}(s) = (T_{11} - [T_{12} \mid T_\perp] \begin{bmatrix} X \\ 0 \end{bmatrix} T_{21})(s) \quad (1.5.15)$$

Since $[T_{12} \mid T_\perp](s)$ and $T_{21}(s)$ are inner and thus norm-preserving, we can write

$$\|\mathfrak{R}(s)\|_\infty = \left\| \begin{bmatrix} (T_{12}^* T_{11} T_{21} - X)(s) \\ T_\perp^* T_{11} T_{21}(s) \end{bmatrix} \right\|_\infty \quad (1.5.16)$$

$$= \left\| \begin{bmatrix} (R_1 - X)(s) \\ R_2(s) \end{bmatrix} \right\|_\infty \quad (1.5.17)$$

and a routine direct calculation from (1.5.13) shows that

$$R(s) = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \stackrel{!}{=} \left[\begin{array}{cc|c} -(A - B_2 F)^* & -\{(C_1 - D_{12} F)^* D_{11} + \mathfrak{X} B_1\} C_2 \mathfrak{Y} & (C_1 - D_{12} F)^* D_{11} + \mathfrak{X} B_1 \\ 0 & -(A - H C_2)^* & -C_2^* \\ \hline -B_2^* & F \mathfrak{Y} - D_{12}^* D_{11} C_2 \mathfrak{Y} & D_{12}^* D_{11} \\ D_\perp^* C_1 \mathfrak{X}^\# & -D_\perp^* D_{11} C_2 \mathfrak{Y} & D_\perp^* D_{11} \end{array} \right]$$

which implies that $R(s)$ is completely unstable. We also observe that if

$$\|\mathfrak{R}(s)\|_\infty \leq \gamma \quad (1.5.18)$$

then

$$(1.5.17) \Rightarrow (R_1 - X)^*(s)(R_1 - X)(s) \leq \gamma^2 I - R_2^*(s)R_2(s) = M^*(s)M(s) \quad (1.5.19)$$

where $M(s)$ is a stable and minimum-phase spectral factor of $\gamma^2 I - R_2^* R_2(s)$ (the existence of which requires $\gamma \geq \|R_2(s)\|_\infty$). Continuing, we deduce from (1.5.19) that

$$\|R_1 M^{-1}(s) - X M^{-1}(s)\|_\infty = \mu \leq 1 \quad (1.5.20)$$

Decomposing $R_1 M^{-1}(s)$ into stable and unstable parts gives

$$\|[R_1 M^{-1}(s)]_- - \tilde{X}(s)\|_\infty = \mu \leq 1 \quad (1.5.21)$$

in which

$$\tilde{X}(s) = [R_1 M^{-1}(s)]_+ + X M^{-1}(s) \quad (1.5.22)$$

Rearranging leads to

$$X(s) = (\tilde{X} - [R_1 M^{-1}]_+) M(s) \quad (1.5.23)$$

(1.5.21) is a Nehari or optimal approximation problem. The characterisation of all solutions to this problem is given in the next section.

The specific algorithm for the γ -iteration is essentially a binary search procedure described in [D4], [F1], [F2], [F3], [F6], [S2], [V1].

Characterisation of all solutions to the Nehari approximation problem

The purpose of the γ -iteration was to reduce the problem in (1.5.17) to a Nehari-extension problem. That is, the problem of finding all $X(s)$'s which achieve

$$\inf_{X \in \mathcal{RH}_+^\infty} \|H(s) - X^*(s)\|_\infty \quad (1.5.24)$$

or else which satisfy

$$\|H(s) - X^*(s)\|_\infty \leq \rho \quad (1.5.25)$$

By Nehari's theorem the infimum in (1.5.24) is achieved and is equal to the norm of the Hankel operator induced by $H(s)$. (For $H(s) \in \mathcal{H}_+^\infty$, the Hankel operator induced by $H(s)$ is defined by $\Gamma_H : \mathcal{H}_2^\perp \rightarrow \mathcal{H}_2$, $\Gamma_H u(s) = [Hu(s)]_+$, where $[\cdot]_+$ denotes the operation of taking stable projection). In addition, Glover [G1] has shown that all the solutions to problems (1.5.24) and (1.5.25) may be characterised in terms of a balanced realisation of $H(s)$. In [G1], this characterisation is in terms of a linear fractional

transformation parametrised by a constrained unstable matrix contraction for the optimal case (1.5.24) and a free unstable contraction for the suboptimal case (1.5.25). The result for the former case adapted from Theorem 8.7 in [G1] appears next:

Theorem 1.5.1

Let $G(s)$ be a stable, rational, pxm transfer function with Hankel singular values

$$\sigma_1 = \sigma_2 \dots = \sigma_r \geq \sigma_{r+1} \geq \sigma_{r+2} \geq \dots \geq \sigma_n > 0 \quad (1.5.26)$$

Suppose that a balanced realisation of $G(s) \triangleq (A, B, C, 0)$ having controllability and observability grammians

$$P = Q = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \sigma_1 I_r \end{bmatrix} \quad (1.5.27)$$

in which $\Sigma_1 = \text{diag}(\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_n)$ is partitioned (conformally with 1.5.27) as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C^T = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} \quad (1.5.28)$$

and let Γ be defined as

$$\Gamma = \Sigma_1^2 - \sigma_1^2 I_{n-r} \quad (1.5.29)$$

Then all $F(s) \in \mathcal{H}^\infty$ for which

$$\|G - F\|_\infty = \sigma_1 \quad (1.5.30)$$

are generated by the linear fractional map

$$F = H_{11} - H_{12}K(I + H_{22}K)^{-1}H_{21} \quad (1.5.31)$$

in which

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \triangleq \left[\begin{array}{c|cc} \Gamma^{-1}(\sigma_1^2 A_{11}^* + \Sigma_1 A_{11} \Sigma_1) & \Gamma^{-1} \Sigma_1 B_1 & -\Gamma^{-1} C_1^* \\ \hline C_1 \Sigma & 0 & I \\ -\sigma_1 B_1^* & \sigma_1 I & 0 \end{array} \right] \quad (1.5.32)$$

and $K(s)$ is any element of $\mathcal{BH}^\infty$ which satisfies $C_2 + K(s)B_2^* = 0$.

Proof:

See Theorem 8.7 in [G1]. □

In [L6], a different version of this result is given which characterises all solutions in terms of a bounded real type condition. This characterisation will be used in the proof of Theorem 2.2.1 in Chapter 2; it will also be employed in Lemma 3.8.2 to establish the degree bound for super-optimal Nehari extensions. The theorem (adapted to the Nehari case) appears next:

Theorem 1.5.2

Let $G(s)$ be a stable, minimal and balanced realisation with Hankel singular values

$$\sigma_1 = \sigma_2 \dots = \sigma_r \geq \sigma_{r+1} \geq \sigma_{r+2} \geq \dots \geq \sigma_n > 0 \quad (1.5.33)$$

Assume that the Hankel singular values have been arranged so that the grammians are given by

$$\text{diag}(\Sigma_1, \sigma_1 I_r) \quad (1.5.34)$$

where

$$\Sigma_1 = \text{diag}(\sigma_{r+1}, \sigma_{r+2}, \dots, \sigma_n) \quad (1.5.35)$$

and let (A, B, C) be partitioned conformally with (1.5.34) as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad C^T = \begin{bmatrix} C_1^T \\ C_2^T \end{bmatrix} \quad (1.5.36)$$

Also let

$$\Gamma = \Sigma^2 - \sigma_1^2 I_r \quad (1.5.37)$$

Then for every error system $E(s) = G(s) - F(s)$ with

$$(a) \|E(j\omega)\|_\infty \leq \sigma_1 \quad (1.5.38)$$

(b) $F(s)$ totally unstable

there exist $\hat{A}$, $\hat{B}$, $\hat{C}$ and $\hat{D}$ such that $E(s)$ has a realisation (A_e, B_e, C_e, D_e)

$$\left[\begin{array}{ccc|cc} A_{11} & A_{12} & 0 & 0 & B_1 \\ A_{21} & A_{22} & 0 & 0 & B_2 \\ 0 & 0 & \Gamma^{-1}(\sigma_1^2 A_{11}^* + \Sigma_1 A_{11} \Sigma_1 - \sigma_1 C_1^* \hat{D} B_1^*) & \Gamma^{-1} C_1^* \hat{C} & \Gamma^{-1}(\Sigma B_1 + \sigma_1 C_1^* \hat{D}) \\ 0 & 0 & -\sigma_1 \hat{B} B_1^* & \hat{A} & \sigma_1 \hat{B} \\ \hline C_1 & C_2 & -(C_1 \Sigma + \sigma_1 \hat{D} B_1^*) & \hat{C} & \sigma_1 \hat{D} \end{array} \right]$$

(1.5.39)

which satisfies both

$$\begin{bmatrix} -(A_e P_e^* + P_e A_e^* + B_e B_e^*) & -(B_e D_e^* + P_e C_e^*) \\ -(B_e D_e + P_e C_e^*)^* & \sigma_1^2 I - D_e D_e^* \end{bmatrix} = \begin{bmatrix} L_e \\ W_e \end{bmatrix} \begin{bmatrix} L_e^* & W_e^* \end{bmatrix} \quad (1.5.40)$$

and

$$\begin{bmatrix} -(Q_e A_e + A_e^* Q_e + C_e^* C_e) & -(C_e^* D_e + Q_e B_e) \\ -(C_e^* D_e + Q_e B_e)^* & \sigma_1^2 I - D_e^* D_e \end{bmatrix} = \begin{bmatrix} L_{ed} \\ W_{ed} \end{bmatrix} \begin{bmatrix} L_{ed}^* & W_{ed}^* \end{bmatrix} \quad (1.5.41)$$

where in the above equations

$$P_e = \begin{bmatrix} \Sigma & 0 & I & 0 \\ 0 & \sigma_1 I_r & 0 & 0 \\ I & 0 & \Sigma \Gamma^{-1} & 0 \\ 0 & 0 & 0 & \sigma_1^2 \hat{P} \end{bmatrix}, \quad Q_e = \begin{bmatrix} \Sigma & 0 & -\Gamma & 0 \\ 0 & \sigma_1 I_r & 0 & 0 \\ -\Gamma & 0 & \Sigma \Gamma & 0 \\ 0 & 0 & 0 & \hat{Q} \end{bmatrix} \quad (1.5.42)$$

and

$$L_e^* = [0 \ 0 \ \sigma_1 \hat{W}^* C_1 \Gamma^{-1} \ \sigma_1 \hat{L}] \quad , \quad L_{ed}^* = [0 \ 0 \ -\sigma_1 \hat{W}_d B_d^* \ \hat{L}_d] \quad (1.5.43)$$

for some $\hat{L}$, $\hat{L}_d$, $\hat{W}$, $\hat{W}_d$, $\hat{P} = \hat{P}^* < 0$, $\hat{Q} = \hat{Q}^* < 0$, and further

$$W_e = \sigma_1 \hat{W}, \quad W_{ed} = \sigma_1 \hat{W}_d \quad \text{and} \quad P_e Q_e = \sigma_1^2 I. \quad (1.5.44)$$

Proof:

See [L6].

□

Interpolation approach

In section 1.4 it was shown that all internally stable closed loops may be parametrised as

$$\mathcal{R}(s) = (T_{11} - T_{12} Q T_{21})(s) \quad (1.5.45)$$

in which $T_{12}(s)$ and $T_{21}(s)$ are stable and $Q(s)$ is a free element of $\mathcal{H}_+^\infty$. (Assumptions (ii) and (iii) in the definition of the $\mathcal{H}^\infty$ problem also imply that $T_{12}(s)$ and $T_{21}(s)$ have no transmission zeros anywhere in the imaginary axis including the point at infinity). When both $T_{12}(s)$ and $T_{21}(s)$ are square (problems of the first kind) it has been shown that it is possible to transform 1.5.45 to a Nehari extension problem. In this section an alternative way of solving the model-matching problem is outlined arising from classical Nevanlinna-Pick-Schur interpolation theory.

Since $Q(s) \in \mathcal{RH}_+^\infty$, no pole/zero cancellation may occur in the product $T_{12}QT_{21}(s)$ in $\bar{\mathbb{C}}_+$, and thus every right half plane zero of either $T_{12}(s)$ or $T_{21}(s)$ is also a zero of $T_{12}QT_{21}(s)$. In the SISO case, therefore,

$$r(s_i) = t_{11}(s_i) = b_i \quad (1.5.46)$$

where s_i is any right half plane zero of either $t_{12}(s)$ or $t_{21}(s)$. The pairs (s_i, b_i) are the interpolation constraints to be satisfied by any $r(s)$ which corresponds to an internally stable closed loop. From [D1] it is known that there exists an interpolating function $r(s) \in \mathcal{RH}_+^\infty$, with $\|r(s)\|_\infty \leq \rho$, if and only if the hermitian Pick matrix

$$\Pi = \left\{ \frac{\rho^2 - b_i \bar{b}_j}{s_i + \bar{s}_j} \right\}_{i,j=1,z}^{i=1,z} \quad (1.5.47)$$

is positive semi-definite. Also,

(i) If $\Pi \geq 0$ and is not definite, the interpolating function is unique with $\deg(r) = \text{rank}(\Pi)$

(ii) The minimum norm taken by any interpolating function is given by the solution of a generalised eigenvalue problem: Find the maximum value of ρ such that $\det(\Pi)$ is zero.

In the MIMO case if $T_{21}(s)$ has a transmission zero at s_i , it must lose rank at this point; this implies that there exists a vector $a_i \neq 0$ such that

$$\mathcal{R}(s_i)a_i = T_{11}(s_i)a_i = b_i \quad i=1, 2, \dots, z_{21} \quad (1.5.48)$$

Similarly, if s_i are the right half plane zeros of $T_{12}(s)$, there exist vectors $a_i^* \neq 0$ such that

$$a_i^* \mathcal{R}(s_i) = a_i^* T_{11}(s_i) = b_i^* \quad i=z_{21}+1, \dots, z \quad (1.5.49)$$

Equations (1.5.48) and (1.5.49) taken together describe the MIMO interpolation constraints associated with all internally stable closed loops in the case of simple zeros. (z_{21} denotes the number of the RHP transmission zeros of $T_{21}(s)$ and z the total number of such zeros contained in both $T_{12}(s)$ and $T_{21}(s)$). The Nevanlinna-Pick tangent problem may be formulated as follows:

[P1] Tangent interpolation problem

Given a (finite) sequence of points in the open right half plane

$$\left\{ s_i \right\}, \quad i=1, 2, \dots, n_r \quad (1.5.50)$$

together with the sequences of vectors

$$a_i \in \mathbb{C}^p \quad \text{and} \quad b_i \in \mathbb{C}^m \quad i=1, 2, \dots, n_r \quad (1.5.51)$$

and a second sequence of points (also in the right half plane)

$$\left\{ s_i \right\}, i=n_r+1, \dots, n_r+n_i=n \quad (1.5.52)$$

together with two further sequences of vectors

$$a_i \in \mathbb{C}^m \quad \text{and} \quad b_i \in \mathbb{C}^p \quad i=n_r+1, \dots, n \quad (1.5.53)$$

For a given positive number ρ find necessary and sufficient conditions for the existence of a stable interpolating matrix function $\mathcal{R}(s) \in \mathbb{R}^{m \times p}(s)$ with $\|\mathcal{R}\|_\infty < \rho$ which satisfies both

$$\mathcal{R}(s_i) a_i = b_i \quad i=1, 2, \dots, n_r \quad (1.5.54)$$

and

$$a_i^* \mathcal{R}(s_i) = b_i^* \quad i=n_r+1, \dots, n \quad (1.5.55)$$

□

A necessary and sufficient condition for the existence of an interpolating function bounded in norm by ρ is provided by the following Theorem [L5]. In addition, [L5] characterises all (admissible) interpolating functions in terms of a Blaschke product of MacMillan degree one factors.

Theorem 1.5.3

There exist a stable $m \times p$ matrix function $R(s)$ with $\|R(s)\|_\infty \leq \rho$ satisfying the interpolation constraints (1.5.54) and (1.5.55) if and only if the matrix

$$\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix} \quad (1.5.56)$$

is positive semi-definite, where

$$\Pi_{11} = \left\{ \frac{\rho^2 a_i^* a_k - b_i^* b_k}{\bar{s}_i + s_k} \right\}_{k=1, n_r}^{i=1, n_r} \quad \Pi_{12} = \left\{ \frac{\rho a_i^* b_k - \rho b_i^* a_k}{\bar{s}_i - \bar{s}_k} \right\}_{k=n_r+1, n}^{i=1, n_r} \quad (1.5.57)$$

and

$$\Pi_{12}^* = \left\{ \frac{\rho b_i^* a_k - \rho a_i^* b_k}{s_k - s_i} \right\}_{k=1, n_r}^{i=n_r+1, n} \quad \Pi_{22} = \left\{ \frac{\rho^2 a_i^* a_k - b_i^* b_k}{\bar{s}_k + s_i} \right\}_{k=n_r+1, n}^{i=n_r+1, n} \quad (1.5.58)$$

Proof:

See [L5]

1.6 Problems of the second kind which admit closed-form solutions

It has been shown in a previous section that allowing one of the off-diagonal blocks of $P(s)$ to

be non-square leads to the approximation problem

$$\gamma_0 = \inf_{X \in \mathbb{RH}_+^\infty} \left\| \begin{bmatrix} R_1 - X \\ R_2 \end{bmatrix} \right\|_\infty \quad (1.6.1)$$

in which $[R_1^T \ R_2^T]^T \in \mathbb{RH}_+^\infty$. This minimisation may be solved via the computationally expensive binary search procedure (known as the γ -iteration) described in section 1.5. It is clear, however, that special cases exist for which (1.6.1) may be reduced directly to a Nehari approximation problem: If $R_2(s)=0$, for example, (1.6.1) simply becomes

$$\gamma_0 = \inf_{X \in \mathbb{RH}_+^\infty} \|R_1 - X\|_\infty = \|R_1\|_H \quad (1.6.2)$$

Similarly, following the sequence of arguments in section 1.5, if

$$R_2^* R_2(s) = \delta^2 I \quad (1.6.3)$$

then

$$\gamma_{opt} = \left\| \begin{bmatrix} R_1 - X_{opt} \\ R_2 \end{bmatrix} \right\|_\infty \Leftrightarrow \left\| \frac{R_1(-s)}{\sqrt{\gamma_{opt}^2 - \delta^2}} \right\|_H = 1 \quad (1.6.4)$$

which implies that in this case,

$$\gamma_{opt} = \sqrt{\|R_1(-s)\|_H^2 + \delta^2} \quad (1.6.5)$$

In this section the so-called parametric mixed-sensitivity problem will be considered. Although this nominally belongs to the class of $\mathcal{H}^\infty$ -problems of the second kind, a closed formula has been derived for the optimal cost in [L11] via the indefinite Riccati equation solution to the general $\mathcal{H}^\infty$ problem given in [D9]. Here the optimal cost will be derived by transforming the parametric mixed-sensitivity problem to one belonging to the class of problems of the second kind which satisfy (1.6.3); in this way the optimal cost formula reduces to the calculation of a Hankel norm.

The parametric mixed-sensitivity problem may be defined as

$$\gamma_{opt} = \inf_{K \in \Xi} \left\| \begin{bmatrix} (1-\epsilon)(I_m - S(s)) \\ \epsilon S(s) \end{bmatrix} \right\|_\infty := \inf_{K \in \Xi} \|\Phi(S)\|_\infty \quad (1.6.6)$$

where $S(s) \in \mathbb{R}^{m \times m}(s)$ denotes the sensitivity function

$$S(s) = (I + GK)^{-1}(s) \quad (1.6.7)$$

and ϵ belongs to the interval $[0, 1]$. Defining the ϵ -dependent orthogonal transformation

$$\Theta_\epsilon = \begin{bmatrix} I_m \cdot \cos\left\{\tan^{-1}\left(\frac{-\epsilon}{1-\epsilon}\right)\right\} & I_m \cdot \sin\left\{\tan^{-1}\left(\frac{-\epsilon}{1-\epsilon}\right)\right\} \\ -I_m \cdot \sin\left\{\tan^{-1}\left(\frac{-\epsilon}{1-\epsilon}\right)\right\} & I_m \cdot \cos\left\{\tan^{-1}\left(\frac{-\epsilon}{1-\epsilon}\right)\right\} \end{bmatrix} \quad (1.6.8)$$

(1.6.6) may be transformed to

$$\begin{aligned} \gamma_{opt} &= \inf_{K \in \Xi} \|\Phi(S)\|_\infty = \inf_{K \in \Xi} \|\Theta_\epsilon \Phi(S)\|_\infty \\ &= \inf_{K \in \Xi} \left\| \begin{bmatrix} \alpha I_m + \beta S(s) \\ \delta I_m \end{bmatrix} \right\|_\infty \end{aligned} \quad (1.6.9)$$

where we have defined

$$\alpha := (1-\epsilon) \cdot \cos\left\{\tan^{-1}\left(\frac{-\epsilon}{1-\epsilon}\right)\right\} \quad (1.6.10)$$

$$\beta := \epsilon \cdot \sin\left\{\tan^{-1}\left(\frac{-\epsilon}{1-\epsilon}\right)\right\} - (1-\epsilon) \cdot \cos\left\{\tan^{-1}\left(\frac{-\epsilon}{1-\epsilon}\right)\right\} \quad (1.6.11)$$

$$\delta := -(1-\epsilon) \cdot \sin\left\{\tan^{-1}\left(\frac{-\epsilon}{1-\epsilon}\right)\right\} \quad (1.6.12)$$

(1.6.9) may be reformulated along the lines given in section 1.5 to

$$\inf_{K \in \Xi} \left\| \mathcal{F}_\ell \left(\begin{bmatrix} (\alpha+\beta)I_m & \beta G \\ \delta I_m & 0 \\ \hline I_m & G \end{bmatrix}, K(s) \right) \right\|_\infty := \inf_{K \in \Xi} \|\mathcal{F}_\ell(P, K)\|_\infty \quad (1.6.13)$$

Using the constant output scaling $\beta^{-1}G^{-1}(\infty)$ this may be written as

$$\inf_{K \in \tilde{\Xi}} \left\| \mathcal{F}_\ell \left(\begin{bmatrix} (\alpha+\beta)I_m & GG^{-1}(\infty) \\ \delta I_m & 0 \\ \hline I_m & \beta^{-1}GG^{-1}(\infty) \end{bmatrix}, K(s) \right) \right\|_\infty := \inf_{K \in \tilde{\Xi}} \|\mathcal{F}_\ell(P_s, K)\|_\infty \quad (1.6.14)$$

where $\tilde{\Xi} = \beta G(\infty)\Xi$ denotes the set of internally stabilising compensators for $\beta^{-1}GG^{-1}(\infty)$. If $G(s) \triangleq (A, B, C, D)$, $P_s(s)$ in (1.6.14) may be written as

$$P_s(s) \triangleq \begin{bmatrix} A & 0 & \beta^{-1}BD^{-1} \\ \beta C & (\alpha+\beta)I_m & I_m \\ 0 & \delta I_m & 0 \\ C & I_m & \beta^{-1}I_m \end{bmatrix} \quad (1.6.15)$$

Hence, since

$$\begin{bmatrix} 0_m & I_m \end{bmatrix} \mathcal{F}_\ell(P, K) = \delta I_m \quad \forall \quad K(s) \in \tilde{\Xi} \quad (1.6.16)$$

(1.6.9) may be transformed to

$$\gamma_{opt} = \inf_{Q \in \mathcal{H}_+^\infty} \left\| \begin{bmatrix} T_{11} - T_{12} Q T_{21} \\ \delta I_m \end{bmatrix} \right\|_\infty \quad (1.6.17)$$

in which $T_{12}(s)$ and $T_{21}(s)$ may be assumed inner with no loss of generality. Hence,

$$\begin{aligned} \gamma_{opt} &= \inf_{Q \in \mathcal{H}_+^\infty} \left\| \begin{bmatrix} T_{12}^* & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} T_{11} - T_{12} Q T_{21} \\ \delta I_m \end{bmatrix} T_{21}^* \right\|_\infty \\ &= \inf_{Q \in \mathcal{H}_+^\infty} \left\| \begin{bmatrix} T_{12}^* T_{11} T_{21}^* - Q \\ \delta T_{21}^* \end{bmatrix} \right\|_\infty \end{aligned} \quad (1.6.18)$$

Since $T_{21}^*(s)$ is antiinner,

$$\gamma_{opt} = \sqrt{\|T_{12}^* T_{11} T_{21}^*(-s)\|_H^2 + \delta^2} \quad (1.6.19)$$

The closed formula for γ_{opt} may now be obtained via a state space realisation of $T_{12}^* T_{11} T_{21}^*(s)$. Choosing the state feedback matrix F and the output injection matrix H as

$$F = \beta C + \frac{1}{\beta} D^{-*} B^* \mathfrak{X} \quad (20)$$

and

$$H = \mathfrak{Y} C^* \quad (1.6.21)$$

where $\mathfrak{X}$ and $\mathfrak{Y}$ denote the stabilising solutions to the algebraic Riccati equations

$$\mathfrak{X}(A - BD^{-1}C) + (A - BD^{-1}C)^* \mathfrak{X} - \frac{1}{\beta^2} \mathfrak{X} B D^{-1} D^{-*} B^* \mathfrak{X} = 0 \quad (1.6.22)$$

and

$$\mathfrak{Y} A^* + A \mathfrak{Y} - \mathfrak{Y} C^* C \mathfrak{Y} = 0 \quad (1.6.23)$$

makes $T_{12}(s)$ and $T_{21}(s)$ in (1.6.17) inner. Using (1.6.20), (1.6.21), (1.6.22), (1.6.23) and (1.6.15) in (1.4.39) gives

$$T_{12}^* T_{11} T_{21}^* \triangleq \left[\begin{array}{cc|c} \mathcal{A}_{11} & \mathcal{A}_{12} & \mathfrak{B}_1 \\ 0 & \mathcal{A}_{22} & \mathfrak{B}_2 \\ \hline \mathfrak{C}_1 & \mathfrak{C}_2 & \mathfrak{D} \end{array} \right]$$

$$:= \left[\begin{array}{cc|c} -\left\{A - BD^{-1}\left(C + \frac{1}{\beta^2}D^{-*}B^*\mathfrak{X}\right)\right\}^* & \frac{\alpha+\beta}{\beta}\mathfrak{X}BD^{-1}C\mathfrak{Y} & -\frac{\alpha+\beta}{\beta}\mathfrak{X}BD^{-1} \\ 0 & -(A - \mathfrak{Y}C^*C)^* & -C^* \\ \hline -\frac{1}{\beta}D^{-*}B^* & (\frac{1}{\beta}D^{-*}B^*\mathfrak{X} - \alpha C)\mathfrak{Y} & (\alpha+\beta)I_m \end{array} \right] \quad (1.6.24)$$

The Hankel norm of $T_{12}^* T_{11} T_{21}^*(s)$ will be calculated as $\lambda_{max}(PQ)$ where P and Q are the controllability and observability grammians of the realisation in (1.6.24). At this point it will be assumed that $\mathfrak{X}$ and $\mathfrak{Y}$ are positive definite. (This is equivalent to requiring that $G(s)$ is completely unstable and non-minimum phase). This assumption is made only for notational convenience and involves no loss of generality in the derived formula for γ_{opt} . The Riccati equation balancing techniques developed in [L6] may be employed to establish the formula for γ_{opt} for the general case.

Lemma 1.6.1

Let $\mathfrak{X}$ and $\mathfrak{Y}$ defined in (1.6.22) and (1.6.23) be non-singular. Then the controllability and observability grammians corresponding to the realisation in (1.6.24) are

$$P = \left[\begin{array}{cc} (\alpha+\beta)^2\mathfrak{X} & 0 \\ 0 & -\mathfrak{Y}^{-1} \end{array} \right] \quad (1.6.25)$$

and

$$Q = \left[\begin{array}{cc} -\mathfrak{X}^{-1} & \mathfrak{Y} \\ \mathfrak{Y} & -(\alpha^2 I + \mathfrak{Y}\mathfrak{X})\mathfrak{Y} \end{array} \right] \quad (1.6.26)$$

respectively.

Proof:

The two grammians should satisfy

$$\left[\begin{array}{cc} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{array} \right] \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{array} \right] + \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{array} \right] \left[\begin{array}{cc} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{array} \right]^* + \left[\begin{array}{c} \mathfrak{B}_1 \\ \mathfrak{B}_2 \end{array} \right] \left[\begin{array}{cc} \mathfrak{B}_1^* & \mathfrak{B}_2^* \end{array} \right] = 0 \quad (1.6.27)$$

and

$$\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{bmatrix}^* \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{bmatrix} + \begin{bmatrix} \mathcal{C}_1^* \\ \mathcal{C}_2^* \end{bmatrix} [\mathcal{C}_1 \ \mathcal{C}_2] = 0 \quad (1.6.28)$$

Comparing (1.6.23) with the (2,2) block of (1.6.27) shows that $P_{22} = -\mathfrak{Y}^{-1}$; substituting this to the (1,2) block of (1.6.27) gives, after cancelling terms,

$$\left\{ -A + BD^{-1} \left(C + \frac{1}{\beta^2} D^{-*} B^* \mathfrak{X} \right)^* P_{12} + P_{12} \left\{ -A + \mathfrak{Y} C^* C \right\} \right\} = 0 \quad (1.6.29)$$

from which, since both terms inside the brackets are completely unstable, $P_{12} = 0$. Using this in the (1,1) block of (1.6.28) gives, after direct comparison with (1.6.22) that $P_{11} = -(\alpha + \beta)\mathfrak{X}$.

The substitution $\mathfrak{X}^{-1}(22)\mathfrak{X}^{-1} = 0$ into the (1,1) block of (1.6.28) verifies that $Q_{11} = -\mathfrak{X}^{-1}$. Using this in the (1,2) block of (1.6.28) shows that $Q_{12} = \mathfrak{Y}$. Finally, the (2,2) block of (1.6.28) may be written as

$$\begin{aligned} & \left(\frac{\alpha + \beta}{\beta} \right) \mathfrak{Y} \mathfrak{X} B D^{-1} C \mathfrak{Y} + Q_{22} (-A^* + C^* C \mathfrak{Y}) + \left(\frac{\alpha + \beta}{\beta} \right) \mathfrak{Y} C^* D^{-*} B^* \mathfrak{X} \mathfrak{Y} + (-A + \mathfrak{Y} C^* C) Q_{22} + \\ & \mathfrak{Y} \left(\frac{1}{\beta} \mathfrak{X} B D^{-1} - \alpha C^* \right) \left(\frac{1}{\beta} \mathfrak{X} B D^{-1} - \alpha C^* \right)^* \mathfrak{Y} = 0 \end{aligned} \quad (1.6.30)$$

which becomes after cancelling terms

$$Q_{22} (-A^* + C^* C \mathfrak{Y}) + (-A + \mathfrak{Y} C^* C) Q_{22} + \mathfrak{Y} \mathfrak{X} A \mathfrak{Y} + \mathfrak{Y} A^* \mathfrak{X} \mathfrak{Y} + \alpha^2 \mathfrak{Y} C^* C \mathfrak{Y} = 0 \quad (1.6.31)$$

Substituting from $\mathfrak{Y}^{-1}(1.6.23)\mathfrak{Y}^{-1}$ gives

$$Q_{22} \mathfrak{Y}^{-1} A \mathfrak{Y} + \mathfrak{Y} A^* \mathfrak{Y}^{-1} Q_{22} + \mathfrak{Y} \mathfrak{X} A \mathfrak{Y} + \mathfrak{Y} A^* \mathfrak{X} \mathfrak{Y} + \alpha^2 \mathfrak{Y} C^* C \mathfrak{Y} = 0 \quad (1.6.32)$$

from which the (2,2) block of (1.6.26) may be verified. $\square$

Using the derived grammians, γ_{opt} may be written as

$$\gamma_{opt} = \delta^2 + \lambda_{max}(PQ) = \delta^2 + \lambda_{max} \left\{ \begin{bmatrix} (\alpha + \beta)^2 I & -(\alpha + \beta) \mathfrak{X} \mathfrak{Y} \\ -I & \alpha^2 I + \mathfrak{X} \mathfrak{Y} \end{bmatrix} \right\} \quad (1.6.33)$$

so that

$$\gamma_{opt}^2 = \delta^2 + \frac{1}{2} \left\{ (\alpha + \beta)^2 + \alpha^2 + \lambda_{max}(\mathfrak{X} \mathfrak{Y}) \right\} + \frac{1}{2} \sqrt{\left\{ \beta^2 + \lambda_{max}(\mathfrak{X} \mathfrak{Y}) \right\} \left\{ (2\alpha + \beta)^2 + \lambda_{max}(\mathfrak{X} \mathfrak{Y}) \right\}}$$

Substituting from (1.6.10), (1.6.11) and (1.6.12) gives

$$\gamma_{opt}^2 = \frac{1}{2} \left\{ \epsilon^2 + (1-\epsilon)^2 + \lambda_{max}(\mathfrak{X}\mathfrak{Y}) \right\} + \frac{1}{2} \sqrt{\left\{ \epsilon^2 + (1-\epsilon)^2 + \lambda_{max}(\mathfrak{X}\mathfrak{Y}) \right\} \left\{ \frac{(2\epsilon-1)^2}{\epsilon^2 + (1-\epsilon)^2} + \lambda_{max}(\mathfrak{X}\mathfrak{Y}) \right\}}$$

in agreement with [L11].

1.7 Interpolation constraints on the Sensitivity and Complementary Sensitivity functions

It has been shown in sections 1.4 and 1.5 that a class of $\mathcal{H}^\infty$ control problems may be transformed to the model-matching problem

$$\gamma_{opt} = \inf_{Q \in \mathcal{H}_+^\infty} \|T_{11} - T_{12}Q T_{21}\|_\infty \quad (1.7.1)$$

in which $T_{12}(s)$ and $T_{21}(s)$ are square. It is clear that the presence of $T_{12}(s)$ and $T_{21}(s)$ in section 1.7.1 constrains the value γ_{opt} can attain. If both these functions were equal to the unit matrix, for example, no interpolation constraints would be present and it would be possible to achieve $\gamma_{opt}=0$ by taking $Q_{opt}=T_{11} \in \mathcal{H}_+^\infty$. In section 1.5 the constraints on the optimal cost were more precisely formulated in terms of certain interpolation constraints that the function

$$\mathfrak{R}(s) = T_{11} - T_{12}Q T_{21}(s) \quad (1.7.2)$$

must satisfy at each RHP transmission zero of $T_{12}(s)$ and $T_{21}(s)$ and along certain zero directions.

A natural question arising is whether these interpolation constraints in the RHP impose any restrictions on the magnitude of $\mathfrak{R}(s)$ on the $j\omega$ axis. The interpolation-approach solution of the model-matching problem in section 1.5 suggests that they do, the minimum norm $\|\mathfrak{R}\|_\infty$ being defined in terms of the solution to an eigenvalue problem constructed from the interpolation data. In this section certain constraints on the magnitude of the sensitivity and complementary sensitivity functions will be described. In this case, the unstable poles and the non-minimum phase zeros of the open-loop transfer function define the relevant interpolation constraints; these may be translated to integral relations on the $j\omega$ axis using Poisson's integral formula.

A number of integral constraints on the sensitivity function for certain classes of systems have been known for a long time. Consider first the open-loop system

$$q(s) = \frac{k}{(s+1)^2} \quad (1.7.3)$$

where k is an arbitrary gain factor. Since the sensitivity of the system is the reciprocal distance of the Nyquist plot from the critical point -1 , it may be seen that in this case there is always a frequency $s=j\omega_0$ above which the sensitivity exceeds one; moreover this transition frequency ω_0 may be pushed

arbitrarily high by increasing the gain. In fact, this tradeoff in sensitivity is a special case of a general class of transfer functions as it may be seen from Bode's sensitivity integral [F7]: For every open loop rational transfer function $Q(s)$ which is stable, minimum phase, has at least two more poles than zeros and stabilises the closed-loop, the sensitivity function $S(s)$ must satisfy

$$\int_0^\infty \log |S(j\omega)| d\omega = 0 \quad (1.7.4)$$

so that requiring $|S(j\omega)| < 1$ at some frequency implies that $|S(j\omega)| > 1$ at another. It is clear, however, that this type of constraint does not imply by itself (i.e. unless additional bandwidth constraints are imposed) any real design limitation: The sensitivity can be made arbitrarily small for an arbitrarily large frequency interval $[0, \omega]$ by making the sensitivity marginally larger than one for the upper tail of the frequency axis (ω, ∞) .

As the interpolation formulation in section 1.5 suggests, a more serious constraint on the magnitude of the sensitivity function on the $j\omega$ axis is due to the presence of non-minimum phase zeros in the open-loop transfer function. Intuitively this is expected since such zeros provide additional phase lag and thus have the effect of "bending" the Nyquist plot towards the critical point. Note first that if $Q(s)$ has a zero at $s=z$ of multiplicity m , $S(s)=(1+Q(s))^{-1}$ is constrained at this point by

$$S(z)=1 \quad , \quad \frac{d^i}{ds^i} S|_{s=z}=0 \text{ for } i=1, 2, \dots, m-1 \quad (1.7.5)$$

Similarly, at any pole at $s=p$ of $Q(s)$ of multiplicity n ,

$$S(p)=0 \quad , \quad \frac{d^i}{ds^i} S|_{s=p}=0 \text{ for } i=1, 2, \dots, n-1 \quad (1.7.6)$$

The Poisson integral formula may be used to translate the RHP zero constraints to a weighted integral relation that $S(s)$ must satisfy along the $j\omega$ axis. Noting that any RHP zero of $S(s)$ is due to a RHP pole of $Q(s)$, these can be factored out as

$$\tilde{S}(s)=S(s)B_p^{-1}(s) \quad (1.7.7)$$

where $B_p(s)$ is the Blaschke product

$$B_p(s)=\prod_{i=1}^{N_p} \frac{p_i-s}{p_i+s} \quad (1.7.8)$$

which contains all RHP poles of $Q(s)$ counted according to their multiplicity. Applying Poisson's integral formula to $\log[\tilde{S}(s)]$ ($\tilde{S}$ is by construction analytic and free of zeros in the RHP) may be used to derive the following Theorem given in [F7], [F9]

Theorem 1.7.1 [F7]

Suppose that the open-loop transfer function $Q(s)$ has a zero, $z=x+jy$, with $x>0$. Assume that the associated feedback system is stable. Then the sensitivity function must satisfy

$$\int_0^\infty \log |S(j\omega)| W(z, \omega) d\omega = \pi \log |B_p^{-1}(z)| \quad (1.7.9)$$

where $W(z, \omega)$ is a weighting function. For a real zero, $z=x$,

$$W(x, \omega) = \frac{2x}{x^2 + \omega^2} \quad (1.7.10)$$

and for a complex zero,

$$W(z, \omega) = \frac{x}{x^2 + (y - \omega)^2} + \frac{x}{x^2 + (y + \omega)^2} \quad (1.7.11)$$

In addition, for a complex zero, the sensitivity function must satisfy,

$$\int_0^\infty \log |S(j\omega)| V(z, \omega) d\omega = \pi \angle B_p^{-1}(z) \quad (1.7.12)$$

where

$$V(z, \omega) = \frac{\omega - y}{x^2 + (y - \omega)^2} - \frac{\omega + y}{x^2 + (y + \omega)^2} \quad (1.7.13)$$

Proof:

See [F7], [F9]

□

The integral relation (1.7.9) is much more severe in constraining sensitivity than the one given in (1.7.4). This is due to the type of weighting function $W(z, \omega)$. Since the weighted length of the $j\omega$ axis in this case is *finite*, the amount by which the sensitivity can exceed one at high frequencies by keeping its value small at low frequencies cannot be made arbitrarily small; in fact it is shown in [F7] that constraining $S(s)$ in a low-frequency interval as

$$|S(j\omega)| \leq \alpha < 1, \quad \forall \omega \in \Omega = [0, \omega_1] \quad (1.7.14)$$

results in a peak in the magnitude of sensitivity at a frequency outside the interval, bounded from below as

$$\|S\|_\infty \geq \left(\frac{1}{\alpha}\right)^{\frac{W(z, \Omega)}{\pi - W(z, \Omega)}} |B_p^{-1}(z)|^{\frac{\pi}{\pi - W(z, \Omega)}} \quad (1.7.15)$$

This shows that $\|S\|_\infty$ becomes unbounded as $\alpha \rightarrow 0$ which may imply poor stability margins for the closed loop system. z in (1.7.15) denotes a non-minimum phase zero of $Q(s)$ and

$$W(z, \Omega) = \int_0^{\omega_1} W(z, \omega) d\omega \quad (1.7.16)$$

with $W(z, \omega)$ being defined in Theorem 1.7.1; $W(z, \Omega)$ may be interpreted (in the real zero case) as minus the phase lag contributed by the zero at the upper endpoint of the frequency interval Ω . Note in this respect that $j\omega$ -axis zeros (let $x \rightarrow 0$ in (1.7.10) and (1.7.11)) do not impose integral constraints on $S(j\omega)$ but only a point discontinuity at $S(jy)$ which does not affect $\|S\|_\infty$. (See [S3] for a treatment of the technical issues arising due to the presence of imaginary axis zeros in the context of $\mathcal{H}^\infty$ control).

The effect of the RHP transmission zeros of the plant on the magnitude of the sensitivity function may be generalised for the MIMO case. (Note that by the internal stability requirement, any RHP transmission zero of $G(s)$ must appear as a zero of the open loop transfer function $Q(s)$ with at least the same multiplicity.) If the plant $G(s)$ is strictly proper and there is a point $s_0 \in \mathbb{C}_+$ such that the rank of $G(s_0)$ is less than the number of its rows, then there exists a positive real number α such that for every stabilising compensator $K(s)$

$$\|\chi(I+GK)^{-1}\|_\infty \|(I+GK)^{-1}\|_\infty^\alpha \geq 1 \quad (1.7.17)$$

where $\chi(\cdot)$ denotes the characteristic function

$$\chi(j\omega) = \begin{cases} 1 & \text{if } |\omega| \leq \omega_1 \\ 0 & \text{if } |\omega| > \omega_1 \end{cases} \quad (1.7.18)$$

for some frequency ω_1 [F3]. On the other hand, if $G(s)$ has a right inverse which is analytic in $\mathbb{C}_+$, bounds

$$\|\chi(I+GK)^{-1}\|_\infty < \epsilon \quad (1.7.19)$$

and

$$\|(I+GK)^{-1}\|_\infty < \delta \quad (1.7.20)$$

may be achieved for arbitrary $\epsilon > 0$ and $\delta > 1$. [F3]

It is also interesting to note that Theorem 1.7.1 illustrates the dependence of the optimal cost in the $\mathcal{H}^\infty$ sensitivity minimisation problem to approximate pole/zero cancellations in the RHP. Consider as an example the unstable and non-minimum phase plant

$$g(s) = \frac{s-1+\epsilon}{s-1} \quad (1.7.21)$$

for $0 < \epsilon < 1$. The optimal cost is given as [L6]

$$\gamma_{opt} = \inf \|S\|_\infty = \sqrt{1 + \lambda_{\max}(\mathfrak{X}\mathfrak{Y})} \quad (1.7.22)$$

where $\mathfrak{X}$ and $\mathfrak{Y}$ are the stabilising solutions to the Riccati equations (1.5.6) and (1.5.7) respectively. It may be readily verified that in this case

$$\mathfrak{X} = \frac{2(1-\epsilon)}{\epsilon}, \quad \mathfrak{Y} = \frac{2}{\epsilon} \quad (1.7.23)$$

so that

$$\gamma_{opt} = \sqrt{1 + \frac{4(1-\epsilon)}{\epsilon^2}} = \frac{2-\epsilon}{\epsilon} \rightarrow \infty \quad \text{as } \epsilon \rightarrow 0 \quad (1.7.24)$$

On noting that the magnitude of the optimal sensitivity is constant over all frequencies and also that a *constant* optimal controller exists in this case,

$$\begin{aligned} \log(\gamma_{opt}) \int_0^\infty \frac{2(1-\epsilon)}{(1-\epsilon)^2 + \omega^2} d\omega &= \pi \log |B_p^{-1}(1-\epsilon)| \\ \Rightarrow \gamma_{opt} &= \left| \left(\frac{1+s}{1-s} \right)_{s=1-\epsilon} \right| = \frac{2-\epsilon}{\epsilon} \end{aligned} \quad (1.7.25)$$

in agreement with (1.7.24).

A similar analysis may be performed for the complementary-sensitivity function $T=1-S(s)$, the unstable poles of the open-loop transfer function now defining the appropriate interpolation constraints. The (dual) integral relation which the complementary-sensitivity function must satisfy is described by the following Theorem [F7].

Theorem 1.7.2

Suppose that the open-loop transfer function $Q(s)$ has a pole, $p=x+jy$, with $x>0$. Assume that the associated feedback system is stable. Let $B_z(s)$ be the Blaschke product

$$B_z(s) = \prod_{i=1}^{N_z} \frac{z_i - s}{\bar{z}_i + s} \quad (1.7.26)$$

containing all the RHP zeros of $Q(s)$ (N_z in number) counted according to multiplicity. Then the complementary sensitivity function must satisfy

$$\int_0^\infty \log |T(j\omega)| W(p, \omega) d\omega = \pi \log |B_z^{-1}(p)| \quad (1.7.27)$$

where $W(p, \omega)$ is a weighting function. For a real pole, $p=x$,

$$W(x, \omega) = \frac{2x}{x^2 + \omega^2} \quad (1.7.28)$$

and for a complex pole,

$$W(p, \omega) = \frac{x}{x^2 + (y - \omega)^2} + \frac{x}{x^2 + (y + \omega)^2} \quad (1.7.29)$$

In addition, for a complex pole, the complementary sensitivity function must satisfy,

$$\int_0^\infty \log |T(j\omega)| V(p, \omega) d\omega = \pi \angle B_z^{-1}(p) \quad (1.7.30)$$

where

$$V(p, \omega) = \frac{\omega - y}{x^2 + (y - \omega)^2} - \frac{\omega + y}{x^2 + (y + \omega)^2} \quad (1.7.31)$$

Proof:

See [F7]

□

Chapter 2

2.0. Introduction

The generalised regulator in Fig. 1 has been adopted as the standard configuration on which $\mathcal{H}^\infty$ -optimal control studies are based. By appropriately choosing the four partitions of $P(s)$, most design examples of engineering interest may be embedded in this diagram. Early $\mathcal{H}^\infty$ studies were special in the sense that $P_{12}(s)$ and $P_{21}(s)$ could be chosen square. Examples of this case (for which the term “*problems of the first kind*” or “*one-block problems*” has been adopted) are the optimal sensitivity [C1], [F4], [F5], [S1], [Z1], [Z2] and the robust stabilisation problem [G2], [K6]. Problems of the first kind have been fully analysed and a controller degree bound has been derived [L5], [L6]. Although this class of problems admits a particularly elegant and simple solution, it is too special for most practical design applications since it typically formulates single-target optimisation objectives. This limits their application to:

(1) Investigating *analytic* tradeoffs of a single feedback property (ie. tradeoffs of the property in different frequency ranges). These may be dealt by including a frequency weighting in the cost function of the optimisation. (Directionality properties in the multivariable case may also be dealt by exploiting the spatial nature of the weighting function).

(2) Evaluating control designs by comparing the properties of the closed-loop system with the optimal levels obtained by solving a relevant $\mathcal{H}^\infty$ problem. This may help to give a qualitative account of possible *inherent* limitations which may be due to structural properties of the plant such as the position of its poles and its transmission zeros.

Control system design should attain a number of different objectives. These include good performance for the closed-loop system (accurate and fast following of set-point changes), good stability margins, the attenuation of disturbances and measurement noise, moderate control action in order to avoid actuator saturation, etc. In addition, these characteristics should be maintained as far as possible in the presence of plant uncertainty. It was shown in the previous chapter that it is possible to translate most of these objectives to bounds on the magnitude of certain *closed-loop transfer functions* governing the response of signals within the feedback loop to exogenous inputs. Minimising, for example, the *sensitivity function* at low frequencies results in good tracking and in the rejection of disturbances (which typically have their power concentrated in this frequency range) entering at the plant output. Similarly, limiting the magnitude of the *complementary-sensitivity function* at high frequencies has the effect of measurement-noise attenuation (typically having wide-band spectral

characteristics) and minimises the effects of (multiplicative) plant uncertainty on the stability margins in the closed loop. (Model uncertainty is again dominant in the high-frequency region partly due to the loss of phase information caused by unmodelled fast dynamics and the approximation of time delays by finite-order LTI models).

It is clear that the two functions mentioned in the previous paragraph (sensitivity and complementary sensitivity) cannot be manipulated independently. In fact, since their sum is the unit matrix, it follows that it is not possible to achieve, for example, *both* disturbance rejection *and* measurement noise attenuation over all frequencies and directions. This illustrates one of the basic “*algebraic*” constraints in control system design [F7]. In the context of optimisation-based design methodologies (which include $\mathcal{H}^\infty$ methods) the tradeoffs which must be performed between conflicting design goals suggest the need to include all such “conflicting” functions (appropriately weighted in frequency) in formulating the optimisation problem.

By allowing one of the off-diagonal blocks of $P(s)$ to be non-square, the range of problems which one may study within the $\mathcal{H}^\infty$ framework is considerably enlarged. For this type of problems the term “*problems of the second kind*” has been adopted. This family is now sufficiently large to include multitarget objectives; an important example is the so-called mixed-sensitivity problem [F1], [F2], [F3], [F6], [V1]. As one would expect, this enlarged class is more difficult to analyse as well as being computationally more demanding; in general the optimal solution must be determined via an iterative procedure which involves a sequence of (apparent) state-dimension inflating operations.

In this chapter a detailed state-space analysis of $\mathcal{H}^\infty$ -optimal control problems of the second kind is carried out. It is shown that the transition from problems of the first kind to problems of the second kind is free in terms of the controller state dimension. In particular it is shown that for every $\mathcal{H}^\infty$ problem of the second kind a continuum of optimal controllers exists requiring no more than n states (where $n = \deg(P)$). A parametrisation of the optimal interpolating functions is also obtained and the results are subsequently extended to the suboptimal case.

The layout of this chapter is as follows: Section 2.1 describes some Riccati equation balancing techniques; this simplifies the exposition by making transparent the interpolation constraints of the problem. In section 2.2 a detailed state-space analysis is performed and a degree bound is derived for the optimal interpolating function under a few simplifying assumptions. Section 2.3 deals with the possible pole/zero cancellations which may occur in the linear fractional map of Figure 1 under the constraint of internal stability, while section 2.4 removes the assumptions made in section 2.2. In section 2.5 the results are combined to derive the controller degree bounds both for the optimal and the sub-optimal cases. Finally sections 2.6 and 2.7 include an example and some computational remarks.

2.1 Balancing the Riccati equations

The state space analysis of $\mathcal{H}^\infty$ problems of the second kind will be greatly simplified if the Riccati equations (1.5.6) and (1.5.7) have a diagonal balanced structure. Balancing of the Riccati equations in problems of the first kind has the effect of removing the right half plane zeros of the square off-diagonal blocks of $P(s)$ in the early stages of the analysis; in effect this makes transparent the interpolation constraints defined at each right half plane zero of $P_{12}(s)$ and $P_{21}(s)$.

Problems of the second kind have a mixed type of interpolation constraints. 'Right' interpolation constraints (zeros of the square $P_{21}(s)$ block) will be dealt with identically as in problems of the first kind: Balancing of the Riccati equation (1.5.7) will have the effect of identifying the right half plane zeros of $P_{21}(s)$ with the full-rank subblock of its solution. A similar dimension deflation corresponding to the non-square off-diagonal block and its interpretation in terms of the left interpolation constraints is also possible. In addition, the balanced Riccati equations lead directly to a minimal realisation for

$$\begin{bmatrix} T_{11} & T_{12} & T_{\perp} \\ T_{21} & 0 & 0 \end{bmatrix}$$

We begin by considering any basis change T in the state-space of $P(s)$ in (1.5.5). In the new basis, the realisation of $P(s)$ becomes

$$P(s) \triangleq \left[\begin{array}{c|cc} TAT^{-1} & TB_1 & TB_2 \\ \hline C_1T^{-1} & D_{11} & D_{12} \\ C_2T^{-1} & D_{21} & D_{22} \end{array} \right] \quad (2.1.1)$$

and the algebraic Riccati equation (1.5.6) becomes

$$\mathfrak{X}(TAT^{-1} - TB_2D_{12}^*C_1T^{-1}) + (TAT^{-1} - TB_2D_{12}^*C_1T^{-1})^*\mathfrak{X} - \mathfrak{X}TB_2B_2^*T^*\mathfrak{X} + T^*C_1D_{\perp}D_{\perp}^*C_1T^{-1} = 0 \quad (2.1.2a)$$

where T^{**} denotes $(T^{-1})^*$. After pre-multiplication by T^* and post-multiplication by T we get

$$T^*\mathfrak{X}T(A - B_2D_{12}^*C_1) + (A - B_2D_{12}^*C_1)^*T^*\mathfrak{X}T - T^*\mathfrak{X}TB_2B_2^*T^*\mathfrak{X}T + C_1^*D_{\perp}D_{\perp}^*C_1 = 0 \quad (2.1.2b)$$

which shows that $\mathfrak{X}$ undergoes the congruence transformation

$$\mathfrak{X} \rightarrow T^{**}\mathfrak{X}T^{-1} \quad (2.1.3)$$

Similarly, in the new basis, (1.5.7) becomes

$$T^{-1}\mathfrak{Y}T^{**}(A - B_1C_2)^* + (A - B_1C_2)T^{-1}\mathfrak{Y}T^{**} - T^{-1}\mathfrak{Y}T^{**}C_2^*C_2T^{-1}\mathfrak{Y}T^{**} = 0 \quad (2.1.4)$$

and hence

$$\mathfrak{Y} \rightarrow T\mathfrak{Y}T^* \quad (2.1.5)$$

(2.1.3) and (2.1.5), together with $\mathfrak{X}=\mathfrak{X}^* \geq 0$ and $\mathfrak{Y}=\mathfrak{Y}^* \geq 0$ show that the construction in [G1, Appendix B] may be used to select a T which diagonalises both $\mathfrak{X}$ and $\mathfrak{Y}$. This construction is reproduced in the next Lemma for completeness.

Lemma 2.1.1:

Let $\mathfrak{X}$ and $\mathfrak{Y}$ be the solutions to the Riccati equations (1.5.6) and (1.5.7) respectively. Then there exists a change of basis T in the state-space of (2.1.1) such that

$$\mathfrak{X} = \begin{bmatrix} \tilde{\Sigma}_1 & & \\ & \tilde{\Sigma}_2 & \\ & & 0 \\ & & & 0 \end{bmatrix} = \begin{bmatrix} \Sigma_1 & \\ & 0 \end{bmatrix} \quad (2.1.6)$$

and

$$\mathfrak{Y} = \begin{bmatrix} \tilde{\Sigma}_1 & & \\ & 0 & \\ & & \tilde{\Sigma}_3 \\ & & & 0 \end{bmatrix} \quad (2.1.7)$$

with $\tilde{\Sigma}_1$, $\tilde{\Sigma}_2$ and $\tilde{\Sigma}_3$ positive definite and diagonal.

Proof: [G1]

The proof essentially depends on simultaneously diagonalising the two solutions of the Riccati equations, using congruence transformations on $\mathfrak{X}$ and $\mathfrak{Y}$, $\tilde{\mathfrak{X}}=T\mathfrak{X}T^*$ and $\mathfrak{Y}=T^{-*}\mathfrak{Y}T^{-1}$. Since $\mathfrak{X} \geq 0$ there exists a congruence transformation T_1 such that

$$T_1\mathfrak{X}T_1^* = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \quad (2.1.8)$$

Let

$$T_1^{-*}\mathfrak{Y}T_1^{-*} = \begin{bmatrix} \mathfrak{Y}_{11} & \mathfrak{Y}_{12} \\ \mathfrak{Y}_{12}^* & \mathfrak{Y}_{22} \end{bmatrix} \quad (2.1.9)$$

Let U_1 be the unitary matrix such that

$$U_1\mathfrak{Y}_{11}U_1^* = \begin{bmatrix} \tilde{\Sigma}_1 & 0 \\ 0 & 0 \end{bmatrix} \quad (2.1.10)$$

with $\tilde{\Sigma}_1 > 0$. Writing

$$T_2^{-*}=T_2=\begin{bmatrix} U_1^* & 0 \\ 0 & I \end{bmatrix} \quad (2.1.11)$$

gives

$$(T_2 T_1)^{-*} \mathfrak{Y} (T_2 T_1)^{-1} = \begin{bmatrix} \tilde{\Sigma}_1 & 0 & \mathfrak{Y}_{13} \\ 0 & 0 & \mathfrak{Y}_{23} \\ \mathfrak{Y}_{13}^* & \mathfrak{Y}_{23}^* & \mathfrak{Y}_{33} \end{bmatrix} \quad (2.1.12)$$

and since $\mathfrak{Y} \geq 0 \Rightarrow \mathfrak{Y}_{23} = 0$. Now let

$$T_3^* = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -\mathfrak{Y}_{13}^* \tilde{\Sigma}_1^{-1} & 0 & I \end{bmatrix} \quad (2.1.13)$$

which gives that

$$(T_3 T_2 T_1)^{-*} \mathfrak{Y} (T_3 T_2 T_1)^{-1} = \begin{bmatrix} \tilde{\Sigma}_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathfrak{Y}_{33} - \mathfrak{Y}_{13}^* \tilde{\Sigma}_1^{-1} \mathfrak{Y}_{13} \end{bmatrix} \quad (2.1.14)$$

and since congruence transformations do not affect the inertia of a matrix, $\mathfrak{Y}_{33} - \mathfrak{Y}_{13}^* \tilde{\Sigma}_1^{-1} \mathfrak{Y}_{13} \geq 0$, so that there exists a unitary transformation U_2 such that

$$U_2^* (\mathfrak{Y}_{33} - \mathfrak{Y}_{13}^* \tilde{\Sigma}_1^{-1} \mathfrak{Y}_{13}) U_2 = \begin{bmatrix} \tilde{\Sigma}_3^{1/2} & 0 \\ 0 & 0 \end{bmatrix} \quad (2.1.15)$$

with $\tilde{\Sigma}_3 > 0$. Finally, defining

$$T_4^* = \begin{bmatrix} \tilde{\Sigma}_1^{-1/4} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & U_2^* \end{bmatrix} \quad (2.1.16)$$

and letting $T = T_4 T_3 T_2 T_1$ proves the result for the transformed $\mathfrak{X}$ and $\mathfrak{Y}$. $\square$

It is convenient at this point to introduce a permutation matrix J so that

$$J \mathfrak{Y} J^* = \begin{bmatrix} \tilde{\Sigma}_1 & & & \\ & \tilde{\Sigma}_3 & & \\ & & 0 & \\ & & & 0 \end{bmatrix} = \begin{bmatrix} \Sigma_2 & & \\ & 0 & \end{bmatrix} \quad (2.1.17)$$

Clearly, $\Sigma_1 > 0$ and $\Sigma_2 > 0$.

In the rest of the analysis it will be assumed that the realisation in (1.5.5) has been put into a

basis corresponding to balanced Riccati equations. We continue by defining

$$M := J(A - B_1 C_2) J^* \quad (2.1.18)$$

$$Z := A - B_2 D_{12}^* C_1 \quad (2.1.19)$$

and introduce the partitioning

$$C_1 = [C_{11} \mid C_{12}] \quad (2.1.20)$$

$$[B_1 \mid B_2] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad (2.1.21)$$

which is consistent with that in (2.1.6). We will also make use of the partitioning

$$C_2 J^* = [\hat{C}_{21} \mid \hat{C}_{22}] \quad (2.1.22)$$

and

$$F J^* = [\hat{F}_1 \mid \hat{F}_2] \quad (2.1.23)$$

which is consistent with that in (2.1.17). Allowing (2.1.6) to induce a partitioning on Z and substituting into (1.5.6) gives

$$\begin{aligned} & \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} + \begin{bmatrix} Z_{11}^* & Z_{21}^* \\ Z_{12}^* & Z_{22}^* \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \\ & - \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} [B_{12}^* \ B_{22}^*] \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} C_{11}^* \\ C_{12}^* \end{bmatrix} D_\perp D_\perp^* [C_{11} \ C_{12}] = 0 \end{aligned} \quad (2.1.24)$$

From the (2,2) block of (2.1.24) we get

$$C_{12}^* D_\perp D_\perp^* C_{12} = 0 \Rightarrow D_\perp^* C_{12} = 0 \Rightarrow C_{11}^* D_\perp D_\perp^* C_{12} = 0 \quad (2.1.25)$$

Consequently, from the (1,2) block of (2.1.24) one obtains

$$\Sigma_1 Z_{12} = 0 \Rightarrow Z_{12} = 0 \quad (\text{Since } \Sigma_1 > 0) \quad (2.1.26)$$

Finally, the (1,1) block gives

$$\Sigma_1 Z_{11} + Z_{11}^* \Sigma_1 - \Sigma_1 B_{12} B_{12}^* \Sigma_1 + C_{11}^* D_\perp D_\perp^* C_{11} = 0 \quad (2.1.27)$$

which is a deflated Riccati equation with a positive definite solution Σ_1 . Since

$$A - B_2 F = Z - B_2 B_2^* \mathfrak{F} \quad (2.1.28)$$

we get from (2.1.26) that

$$\begin{aligned} A - B_2 F &= \begin{bmatrix} Z_{11} & 0 \\ Z_{21} & Z_{22} \end{bmatrix} - \begin{bmatrix} B_{12} \\ B_{22} \end{bmatrix} \begin{bmatrix} B_{12}^* & B_{22}^* \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} Z_{11} - B_{12} B_{12}^* \Sigma_1 & 0 \\ Z_{21} - B_{22} B_{12}^* \Sigma_1 & Z_{22} \end{bmatrix} \end{aligned} \quad (2.1.29)$$

Since $A - B_2 F$ is asymptotically stable, so too is Z_{22} . One concludes also that each eigenvalue of Z_{22} corresponds to a stable mode of $A - B_2 D_{12}^* C_1$ which is undetectable through $D_\perp^* C_1$. In fact Z_{22} consists precisely of these modes [P2]; this is shown next.

Lemma 2.1.2:

If z_i is a stable mode of $A - B_2 D_{12}^* C_1$ which is undetectable through $D_\perp^* C_1$ then $z_i \in \lambda(Z_{22})$.

Proof:

Suppose for contradiction that Z_{11} contains some stable modes of Z which are undetectable through $D_\perp^* C_{11}$. Define a basis change U such that

$$\begin{bmatrix} sI - Z_{11} \\ D_\perp^* C_{11} \end{bmatrix} \xrightarrow{U} \begin{bmatrix} sI - \tilde{Z}_{11} & -\tilde{Z}_{12} \\ 0 & sI - \tilde{Z}_{22} \\ 0 & \Phi_2 \end{bmatrix} := \begin{bmatrix} sI - \bar{Z}_{11} \\ \Phi \end{bmatrix} \quad (2.1.30)$$

in which $\tilde{Z}_{11}$ contains all the stable modes of Z_{11} which are undetectable through $D_\perp^* C_{11}$. Then (2.1.27) may be transformed to

$$\bar{\Sigma}_1 \bar{Z}_{11} + \bar{Z}_{11}^* \bar{\Sigma}_1 - \bar{\Sigma}_1 B_{12} B_{12}^* \bar{\Sigma}_1 + \Phi^* \Phi = 0 \quad (2.1.31)$$

where we have defined

$$\bar{\Sigma}_1 := U^{-*} \Sigma_1 U^{-1} \quad (2.1.32)$$

and

$$\bar{B}_{12} := U B_{12} \quad (2.1.33)$$

Since Σ_1 is the stabilising solution to the Riccati equation (2.1.27)

$$Z_{11} - B_{12}B_{12}^*\Sigma_1 = U^{-1}(Z_{11} - \bar{B}_{12}\bar{B}_{12}^*\bar{\Sigma}_1)U \quad (2.1.34)$$

is asymptotically stable. Let

$$\bar{\Sigma}_1 = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \quad (2.1.35)$$

partitioned conformally with (2.1.30). The (transformed) Riccati equation (2.1.31) may be written out in full as

$$\begin{aligned} & \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ 0 & \tilde{Z}_{22} \end{bmatrix} + \begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ 0 & \tilde{Z}_{22} \end{bmatrix}^* \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \\ & - \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \begin{bmatrix} \tilde{B}_{12} \\ \tilde{B}_{22} \end{bmatrix} [\tilde{B}_{12}^* \ \tilde{B}_{22}^*] \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} + \begin{bmatrix} 0 \\ \Phi_2 \end{bmatrix} [0 \ \Phi_2] = 0 \end{aligned} \quad (2.1.36)$$

where

$$\bar{B}_{12} := \begin{bmatrix} \tilde{B}_{12} \\ \tilde{B}_{22} \end{bmatrix} \quad (2.1.37)$$

partitioned conformally with (2.1.30). Expanding, we get

$$X_{11}\tilde{Z}_{11} + \tilde{Z}_{11}^*X_{11} - (X_{11}\tilde{B}_{11} + X_{12}\tilde{B}_{12})(X_{11}\tilde{B}_{11} + X_{12}\tilde{B}_{12})^* = 0 \quad (2.1.38)$$

$$X_{11}\tilde{Z}_{12} + \tilde{Z}_{11}^*X_{12} - (X_{11}\tilde{B}_{11} + X_{12}\tilde{B}_{12})(\tilde{B}_{11}^*X_{12} + \tilde{B}_{12}^*X_{22}) = 0 \quad (2.1.39)$$

and

$$X_{12}^*\tilde{Z}_{12} + X_{22}\tilde{Z}_{22} + \tilde{Z}_{12}^*X_{12} + \tilde{Z}_{22}^*X_{22} - (X_{12}^*\tilde{B}_{11} + X_{22}\tilde{B}_{12}^*)(\tilde{B}_{11}^*X_{12} + \tilde{B}_{12}^*X_{22}) + \Phi_2^*\Phi_2 = 0 \quad (2.1.40)$$

Now, since $\bar{\Sigma}_1$ is the stabilising solution of the Riccati equation (2.1.31), $\bar{\Sigma}_1 \geq 0$ which implies that $X_{11} \geq 0$. From (2.1.38), however, $\tilde{Z}_{11}$ is stable which implies that $X_{11} \leq 0$. Hence $X_{11} = 0$, which when substituted in (2.1.38) gives

$$X_{12}\tilde{B}_{12} = 0 \quad (2.1.41)$$

so that (2.1.39) becomes

$$X_{12}\tilde{Z}_{22} + \tilde{Z}_{11}^*X_{12} = 0 \quad (2.1.42)$$

However, since $\bar{\Sigma}_1$ is the stabilising solution to (2.1.31) by (2.1.34), we have that

$$\begin{bmatrix} \tilde{Z}_{11} & \tilde{Z}_{12} \\ 0 & \tilde{Z}_{22} \end{bmatrix} - \begin{bmatrix} \tilde{B}_{12} \\ \tilde{B}_{22} \end{bmatrix} [\tilde{B}_{12}^* \ \tilde{B}_{22}^*] \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^* & X_{22} \end{bmatrix} \quad (2.1.43)$$

is asymptotically stable, or (using (2.1.41)) that

$$\begin{bmatrix} \tilde{Z}_{11} - \tilde{B}_{11}\tilde{B}_{12}^*X_{12}^* & \tilde{Z}_{12} - \tilde{B}_{11}(\tilde{B}_{11}^*X_{12} + \tilde{B}_{12}^*X_{22}) \\ 0 & \tilde{Z}_{22} - \tilde{B}_{12}(\tilde{B}_{11}^*X_{12} + \tilde{B}_{12}^*X_{22}) \end{bmatrix} \quad (2.1.44)$$

is asymptotically stable, which implies that $\tilde{Z}_{22} - \tilde{B}_{12}(\tilde{B}_{11}^*X_{12} + \tilde{B}_{12}^*X_{22})$ is also asymptotically stable. On using (2.1.41), (2.1.44) may be rewritten as

$$X_{12}[\tilde{Z}_{22} - \tilde{B}_{12}(\tilde{B}_{11}^*X_{12} + \tilde{B}_{12}^*X_{22})] + \tilde{Z}_{11}^*X_{12} = 0 \quad (2.1.45)$$

with the term in the square bracket being stable as a diagonal subblock of (2.1.44). Since $\tilde{Z}_{11}$ is also asymptotically stable, equation (2.1.45) admits the unique solution $X_{12}=0$, and

$$\bar{\Sigma}_1 = U^{-*}\Sigma_1 U^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & X_{22} \end{bmatrix} \quad (2.1.46)$$

Noting that Σ_1 is full rank (by construction) gives the required contradiction. $\square$

A similar procedure has already been applied to (1.5.7) [L6] to obtain

$$J(A - HC_2)J^* = \begin{bmatrix} -\Sigma_2 M_{11}^* \Sigma_2^{-1} & M_{12} - \Sigma_2 \hat{C}_{21}^* \hat{C}_{22} \\ 0 & M_{22} \end{bmatrix} \quad (2.1.47)$$

in which $\{\lambda(M_{11})\} = \{\text{right half plane zeros of } P_{21}\}$.

Making use of (2.1.21), (2.1.23), (2.1.25), (2.1.29), (2.1.47) and (1.5.6), we can rewrite (1.5.13) as

$$\begin{bmatrix} T_{11} & T_{12} & T_{\perp} \\ T_{21} & 0 & 0 \end{bmatrix} \stackrel{s}{=} \quad (2.1.48)$$

$$\left[\begin{array}{cc|ccc} Z_{11} - B_{12}B_{12}^*\Sigma_1 & B_{12}\hat{F}_1 & B_{11} & B_{12} & -\Sigma_1^{-1}C_{11}D_{\perp} \\ 0 & -\Sigma_2 M_{11}^* \Sigma_2^{-1} & -\Sigma_2 \hat{C}_{21}^* & 0 & 0 \\ \hline D_{\perp} D_{\perp}^* C_{11} - D_{12}B_{12}^*\Sigma_1 & D_{12}\hat{F}_1 & D_{11} & D_{12} & D_{\perp} \\ 0 & \hat{C}_{21} & I & 0 & 0 \end{array} \right]$$

We also note that the change of basis

$$T = \begin{bmatrix} \Sigma_1^{\frac{1}{2}} & 0 \\ 0 & \Sigma_2^{-\frac{1}{2}} \end{bmatrix} \quad (2.1.49)$$

will balance the realisation in (2.1.48). From the balanced version of (2.1.48) we obtain

$$T_{21}(s) \triangleq \begin{bmatrix} -\Sigma_2^{\frac{1}{2}} M_{11} \Sigma_2^{-\frac{1}{2}} & -\Sigma_2^{\frac{1}{2}} \hat{C}_{21}^* \\ \hat{C}_{21} \Sigma_2^{-\frac{1}{2}} & I \end{bmatrix} \quad (2.1.50)$$

which has both its controllability and observability grammians the identity. We conclude, therefore, that (2.1.50) is a minimal and balanced realisation of $T_{21}(s)$. We also deduce that

$$[T_{12}|T_\perp] \triangleq \left[\begin{array}{c|cc} \Sigma_1^{\frac{1}{2}}(Z_{11}-B_{12}B_{12}^*\Sigma_1)\Sigma_1^{-\frac{1}{2}} & \Sigma_1^{\frac{1}{2}}B_{12} & -\Sigma_1^{-\frac{1}{2}}C_{11}^*D_\perp \\ \hline D_\perp D_\perp^* C_{11}\Sigma_1^{-\frac{1}{2}} - D_{12}B_{12}^*\Sigma_1^{\frac{1}{2}} & D_{12} & D_\perp \end{array} \right] \quad (2.1.51)$$

is balanced with controllability and observability grammians again the identity ; this realisation is thus also minimal.

We conclude this section by pointing out that the realisation in (2.1.48) is also minimal. This may be established by proving that all the system zeros in (2.1.48) lie in the open right half plane and consequently cannot cancel any of the poles of this realisation which lie in the left half plane. An almost identical argument may be found in section 3 in [L6].

As one would expect, replacing the realisation (1.5.13) with (2.1.48), allows the realisation of $R(s)$ to be reduced to

$$R(s) \triangleq \left[\begin{array}{c|cc} -(Z_{11}-B_{12}B_{12}^*\Sigma_1)^* & A_R(1,2) & B_R(1,1) \\ 0 & M_{11} & \Sigma_2 \hat{C}_{21}^* \\ \hline -B_{12}^* & D_{12}^* D_{11} \hat{C}_{21} - \hat{F}_1 & D_{12}^* D_{11} \\ D_\perp^* C_1 \Sigma_1^{-1} & D_\perp^* D_{11} \hat{C}_{21} & D_\perp^* D_{11} \end{array} \right] \quad (2.1.52)$$

where

$$A_R(1,2) = C_{11}^* D_\perp D_\perp^* D_{11} \hat{C}_{21} + \Sigma_1 B_{11} \hat{C}_{21} - \Sigma_1 B_{12} D_{12}^* D_{11} \hat{C}_{21} \quad (2.1.53)$$

and

$$B_R(1,1) = C_{11}^* D_\perp D_\perp^* D_{11} - \Sigma_1 B_{12} D_{12}^* D_{11} + \Sigma_1 B_{11} \quad (2.1.54)$$

The realisation in (2.1.52) need not be minimal. The results of the analysis of this section are summarised in the following lemma.

Lemma 2.1.3

- [i] (a) The number of zeros of $P_{21}(s)$ in $\mathbb{C}_+$ = $\text{rank}(\mathfrak{Y})$
- (b) The number of stable modes of $A - B_2 D_{12}^* C_1$ which are undetectable through $D_\perp^* C_1$
= $\text{rank defect}(\mathfrak{X})$
- [ii] The realisation (3.22) is minimal with MacMillan degree = $\text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y})$
- [iii] The realisation (3.24) is minimal and $\deg(T_{21}) = \text{rank}(\mathfrak{Y})$
- [iv] The realisation (3.25) is minimal and $\deg([T_{12} \mid T_\perp]) = \text{rank}(\mathfrak{X})$
- [v] $\deg(R) \leq \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y})$

Proof:

[i] to [iv] follow from the discussion of this section. [v] follows by inspecting the realisation in (2.1.52). $\square$

Remark 2.1.1:

Lemma 2.1.3i(b) shows that any stable mode of $A - B_2 D_{12}^* C_1$ which is undetectable through $D_\perp^* C_1$ will cause a drop in the rank of the stabilising solution $\mathfrak{X}$ to the Riccati equation (1.5.6). A future Lemma (2.3.2) will identify these modes with the minimum-phase Smith zeros of the non-square off-diagonal block $P_{12}(s)$. We can associate, therefore, the drop in the rank of the solutions to both Riccati equations with the minimum phase zeros of the off-diagonal blocks of $P(s)$. Although, however, for the square block we can associate directly the rank of the corresponding Riccati equation solution ($\mathfrak{Y}$) with the non-minimum phase zeros of $P_{21}(s)$ (and hence with the right interpolation constraints), this is no longer the case for the non-square block.

In order to give an interpretation of the rank-defect of the Riccati solution corresponding to the non-square block in interpolation terms, the model matching problem may be reformulated in terms of matrix-like (rather than pointwise) interpolation constraints, along the lines given in [H3], [H4]. Assuming for simplicity that $T_{21}(s) = I$, the model-matching problem

$$E(s) = T_{11} - T_{12}Q(s) \tag{2.1.55}$$

may be transformed on using $T_\perp^* T_{12}(s) = 0$ to

$$T_\perp^* E(s) = T_\perp^* T_{11}(s) \tag{2.1.56}$$

Now, the RHS of (2.1.56) is an element of $\mathcal{H}_-^\infty$ by a standard state-space calculation. In addition (2.1.55) and (2.1.56) are equivalent since $T_\perp^*(E - T_{11})(s) = 0$ with $T_\perp^*(s) \in \mathcal{H}_-^\infty$ and $(E - T_{11})(s) \in \mathcal{H}_+^\infty$ forces $(E - T_{11})(s)$ to admit a factorisation $E - T_{11} = T_{12}Q(s)$ for some $Q(s) \in \mathcal{H}_-^\infty$ [H3]. Equation (2.1.56) may now be interpreted as a set of (interpolation) constraints for the minimisation

$$\min \|E(s)\|_\infty ; E(s) \in \mathcal{H}_+^\infty \quad (2.1.57)$$

since it forces the residues of $T_\perp^* E(s)$ and $T_\perp^* T_{11}(s)$ to be identical at any (r.h.p.) pole of $T_\perp^*$. Consequently, the number of the left interpolation constraints will depend on the MacMillan degree of the all-pass extension $T_\perp(s)$. It may be seen, however, from the state-space realisation in (2.1.48) that all the modes of Z_{22} (which correspond to the stable modes of $A - B_2 D_{12}^* C_1$ which are undetectable through $D_\perp^* C_1$) are not contained in a minimal realisation of $T_\perp(s)$ and consequently do not correspond to active interpolation constraints.

2.2 The H^∞ - optimal interpolation function for problems of the second kind.

In the following sections the results obtained so far are combined with a new theorem to obtain the controller degree bound for all problems of the second kind. Both the optimal and suboptimal cases will be treated.

Suppose $n = \deg(P)$, $t = \deg(\mathcal{P})$ and let c be the number of cancellations which occur between $P(s)$ and $K(s)$ as a result of closing the feedback loop in Fig. 1. Then $t = n + \deg(K) - c$, that is $\deg(K) \leq t_b + c_b - n$ where t_b and c_b are upper bounds on t and c respectively. The upper bound t_b will be derived in section 2.4 while c_b will be found in section 2.3. These results will be combined in section 2.5 to give the controller bound.

The derivation of the bound t_b for the degree of the closed loop requires several steps. Before stating and proving the main theorem of this section a brief sketch will be given of the route which will be taken:

(a) It may be recalled from Lemma 2.1.3(ii) that

$$\deg \begin{bmatrix} T_{11} & T_{12} & T_\perp \\ T_{21} & 0 & 0 \end{bmatrix} = \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) \quad (2.2.1)$$

(b) If T_{11} and T_{21} are all-pass right coprime, and T_{11} and $[T_{12} \mid T_\perp]$ are all pass-left coprime, [L6, Theorem 4.1] shows that

$$\deg \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \deg \begin{bmatrix} T_{12}^* T_{11}^* T_{21}^* \\ T_\perp^* T_{11}^* T_{21}^* \end{bmatrix} = \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) \quad (2.2.2)$$

T_{11} and T_{21} will be called all-pass right coprime if in

$$\begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} = \begin{bmatrix} \tilde{T}_{11} \\ \tilde{T}_{21} \end{bmatrix} A(s) \quad (2.2.3)$$

all all-pass common right divisors $A(s)$ of $T_{11}(s)$ and $T_{12}(s)$ are constant orthogonal matrices. All-pass

left coprimeness is defined in a similar way.

(c) It will be assumed at this point (and in Theorem 2.2.1), that the all-pass coprimeness condition is satisfied. If this is not the case, the factorisation

$$\begin{bmatrix} T_{11} & T_{12} & T_{\perp} \\ T_{21} & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_l(s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} & \tilde{T}_{\perp} \\ \tilde{T}_{21} & 0 & 0 \end{bmatrix} \begin{bmatrix} A_r(s) & 0 \\ 0 & I \end{bmatrix} \quad (2.2.4)$$

is always possible in which $\tilde{T}_{11}$ and $\tilde{T}_{21}$ will be all-pass right coprime, and $\tilde{T}_{11}$ and $[\tilde{T}_{12} \mid \tilde{T}_{\perp}]$ will be all-pass left coprime [L6, Theorem 4.1]. It will be shown in a later section that for reasons almost identical to those given in [L6], the existence or otherwise of these all-pass common factors makes no difference to the bound t_b we seek.

(d) Next, it will be proved that

$$\deg([R_1 M^{-1}]_-) \leq \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) \quad (2.2.5)$$

(the definition of $M(s)$ may be found in (1.5.19)). In the hypothesis of Theorem 2.2.1 it will be assumed that (2.2.5) is met with equality since this assumption simplifies one of the later calculations.

(e) (d) together with work by Glover [G1] ensures us that

$$\deg(\tilde{X}) \leq \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) + \deg(U) - 1 \quad (2.2.6a)$$

in the optimal case, and

$$\deg(\tilde{X}) \leq \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) + \deg(U) \quad (2.2.6b)$$

in the suboptimal case. In the above $U(s) \in \mathbb{RH}_-^\infty$ is a free matrix contraction to be chosen by the designer and $\tilde{X}(s)$ is defined in (1.5.22).

(f) It will be shown that

$$\deg(X) \leq \deg(\tilde{X}) \quad (2.2.7)$$

and that

$$\deg(\mathfrak{R}) \leq \deg(\tilde{X}) \quad (2.2.8)$$

$X(s)$ and $\tilde{X}(s)$ are related in (1.5.23). Consequently,

$$t_b = \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) + \deg(U) - 1 \quad (2.2.9a)$$

in the optimal case, or

$$t_b = \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) + \deg(U) \quad (2.2.9b)$$

in the suboptimal case. In the first instance the reader may wish to skip to sections 2.3, 2.4 and 2.5. In this way one may get an initial overview without getting swamped in the details surrounding the proofs of claims (d), (e) and (f) above; these details are considerable.

The general form of the state space model for

$$\begin{bmatrix} T_{11} & T_{12} & T_{\perp} \\ T_{21} & 0 & 0 \end{bmatrix}$$

(in (2.1.48)) is the basis of the hypothesis for the next theorem which establishes an inclusion result of the poles of the optimal interpolating function, along with explicit state-space formulae of its state-space realisation under the simplifying assumptions [iii] and [iv] given below; these will be removed at a later point (section 2.4). In the interest of clarity, long calculations and certain technical points are treated in a number of Lemmas which are proved in a sequence of Appendices at the end of this chapter.

Theorem 2.2.1

Let

$$\begin{bmatrix} T_{11} & T_{12} & T_{\perp} \\ T_{21} & 0 & 0 \end{bmatrix} \stackrel{\triangle}{=} \left[\begin{array}{cc|ccc} A_{11} & A_{12} & B_{11} & B_{12} & B_{13} \\ 0 & A_{22} & B_{21} & 0 & 0 \\ \hline C_{11} & C_{12} & D_{11} & D_{12} & D_{13} \\ 0 & C_{22} & I & 0 & 0 \end{array} \right] \quad (2.2.10)$$

be asymptotically stable and suppose also that

$$i] \quad T_{21}(s) \stackrel{\triangle}{=} \begin{bmatrix} A_{22} & B_{21} \\ C_{22} & I \end{bmatrix}$$

is all-pass, minimal and balanced

$$ii] \quad [T_{12} \mid T_{\perp}](s) \stackrel{\triangle}{=} \left[\begin{array}{c|cc} A_{11} & B_{12} & B_{13} \\ \hline C_{11} & D_{12} & D_{13} \end{array} \right]$$

is all-pass, minimal and balanced

iii] $T_{11}(s)$ and $T_{22}(s)$ are all-pass right coprime

iv] $T_{11}(s)$ and $[T_{12} \mid T_{\perp}]$ are all-pass left coprime

$$v] \quad D_{13}^* C_{12} = 0 \quad (2.2.11)$$

$$vi] \quad A_{12} + C_{11}^* C_{12} = 0 \quad (2.2.12)$$

Then

$$a) \quad \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|c} -A_{11}^* & -C_{11}^* D_{11} B_{21}^* - B_{11} B_{21}^* & C_{11}^* D_{11} + B_{11} \\ 0 & -A_{22}^* & C_{22}^* \\ \hline -B_{12}^* & -D_{12}^* (D_{11} B_{21}^* + C_{12}) & D_{12}^* D_{11} \\ -B_{13}^* & -D_{13}^* D_{11} B_{21}^* & D_{13}^* D_{11} \end{array} \right] \quad (2.2.13)$$

$$\stackrel{s}{=} \left[\begin{array}{c|c} -\bar{A}^* & \bar{C}^* \\ \hline -\bar{B}_1^* & \bar{D}_1^* \\ -\bar{B}_2^* & \bar{D}_2^* \end{array} \right]$$

is a minimal balanced realisation.

$$b) \quad (R_1 M^{-1})_- \stackrel{s}{=} \begin{bmatrix} -\bar{A}^* & F^* E^{\frac{1}{2}} \\ -\bar{B}_1^* & 0 \end{bmatrix} \quad (2.2.14)$$

where the matrices E and F are defined in the proof, see (2.2.41) and (2.2.45) respectively.

$$c) \quad \text{If } (R_1 M^{-1})_- \stackrel{s}{=} \begin{bmatrix} -\bar{A}^* & F^* E^{\frac{1}{2}} \\ -\bar{B}_1^* & 0 \end{bmatrix}$$

is minimal and

$$\tilde{X}(s) \stackrel{s}{=} \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix}$$

is chosen so that

$$[(R_1 M^{-1})_- + \tilde{X}](s) \stackrel{s}{=} \left[\begin{array}{cc|c} -\bar{A}^* & 0 & F^* E^{\frac{1}{2}} \\ 0 & \hat{A} & \hat{B} \\ \hline -\bar{B}_1^* & \hat{C} & \hat{D} \end{array} \right] = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \quad (2.2.15)$$

satisfies the bounded real type equations

$$\begin{bmatrix} -(\tilde{A}P + P\tilde{A}^* + \tilde{B}\tilde{B}^*) & -(\tilde{B}\tilde{D}^* + P\tilde{C}^*) \\ -(\tilde{D}\tilde{B}^* + \tilde{C}P) & I - \tilde{D}\tilde{D}^* \end{bmatrix} = \begin{bmatrix} L \\ W \end{bmatrix} \begin{bmatrix} L^* & W^* \end{bmatrix} \quad (2.2.16)$$

and their dual

$$\begin{bmatrix} -(\tilde{A}^*Q + Q\tilde{A} + \tilde{C}^*\tilde{C}) & -(\tilde{C}^*\tilde{D} + Q\tilde{B}) \\ -(\tilde{D}^*\tilde{C} + \tilde{B}^*Q) & I - \tilde{D}^*\tilde{D} \end{bmatrix} = \begin{bmatrix} L_d^* \\ W_d^* \end{bmatrix} \begin{bmatrix} L_d & W_d \end{bmatrix} \quad (2.2.17)$$

in which

$$i) \quad QP = I \quad (2.2.18)$$

and

$$ii) \quad L^* = \begin{bmatrix} 0 & L_{21}^* \end{bmatrix} \quad ; \quad L_d = \begin{bmatrix} 0 & L_{21d} \end{bmatrix} \quad (2.2.19)$$

where the partitioning in (2.2.19) is conformal with that in (2.2.15),

Then:

$$1.] \quad \{ \text{poles of } X \} \subseteq \{ \text{poles of } \tilde{X} \} \quad (2.2.20)$$

$$2.] \quad \{ \text{poles of } T_{11} + T_{12}XT_{21} \} \subseteq \{ \text{poles of } \tilde{X} \} \quad (2.2.21)$$

Explicit formulae for $X(s)$ and $[T_{11} + T_{12}XT_{21}](s)$ are given in equations (2.2.59) and (2.2.71) below.

Remark 2.2.1

The validity of the bounds in (2.2.9) is proven by applying Theorem 2.2.1 to a balanced version of (2.1.48). In this regard we should note the following :

(a) The general form of the realisations in (2.1.48) and (2.2.10) is the same.

(b) T_{21} and $[T_{12} \mid T_{\perp}]$ are both inner and their realisations are minimal and balanced.

(c) Equations (2.2.11) and (2.2.12) are easily seen to be satisfied after balancing (2.1.48) since (in the notation of section 2.1)

$$(i) \quad D_{13}^* C_{12} = D_{\perp}^* D_{12} \hat{F}_{12} \Sigma_2^{1/2} = 0$$

and

$$(ii) \quad A_{12} + C_{11}^* C_{12} = \Sigma_1^{1/2} B_{12} \hat{F}_{12} \Sigma_2^{1/2} + [(D_{\perp} D_{\perp}^* C_{11} - D_{12} B_{12}^* \Sigma_1) \Sigma_1^{-1/2}]^* D_{12} \hat{F}_{12} \Sigma_2^{1/2} = 0$$

on using the orthogonality of $[D_{12} \mid D_{\perp}]$. It will be shown at a later point that these equations remain valid even if it is necessary to extract all-pass common factors.

(d) Theorem 2.1 and Corollary 2.2 in [20] ensures us that the error systems corresponding to any Nehari or suboptimal extension of $[R_1 M^{-1}]_-$ will satisfy the bounded real type equations in (2.2.16) and (2.2.17)

(e) Theorem 2.2.1 and Lemma 2.1.3(ii) thus establish the validity of (2.2.5), (2.2.6), (2.2.7), (2.2.8) and (2.2.9).

Remark 2.2.2

In the case that the final value of $\gamma > \gamma_{opt}$, the cancellation phenomena predicted by Theorem 2.2.1 can only be guaranteed if $\tilde{X}(s)$ is a suboptimal extension of $[R_1 M^{-1}]_-$ corresponding to an error system with an infinity norm of one.

Proof:

The assumed properties of the realisation of $T_{21}(s)$ in (i) enforces

$$A_{22} + A_{22}^* + B_{21} B_{21}^* = 0 \quad (2.2.22)$$

and

$$C_{22} = -B_{21}^* \quad (2.2.23)$$

Similarly, the realisation of $[T_{12} \mid T_{12}^\perp]$ must satisfy

$$A_{11} + A_{11}^* + B_{12} B_{12}^* + B_{13} B_{13}^* = 0 \quad (2.2.24)$$

$$A_{11} + A_{11}^* + C_{11} C_{11}^* = 0 \quad (2.2.25)$$

$$D_{12}^* C_{11} + B_{12}^* = 0 \quad (2.2.26)$$

$$D_{13}^* C_{11} + B_{13}^* = 0 \quad (2.2.27)$$

$$D_{12} B_{12}^* + D_{13} B_{13}^* + C_{11} = 0 \quad (2.2.28)$$

$$[D_{12} \mid D_{13}] [D_{12} \mid D_{13}]^* = I \quad (2.2.29)$$

(Equations (2.2.22) to (2.2.29) follow from a state-space characterisation of all-pass matrices given in [G1]).

From (2.2.10) and (2.2.23) we get

$$\begin{aligned}
 T_{11}T_{21}^* &\stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12} & B_{11} \\ 0 & A_{22} & B_{21} \\ \hline C_{11} & C_{12} & D_{11} \end{array} \right] * \left[\begin{array}{cc} -A_{22}^* & B_{21} \\ B_{21}^* & I \end{array} \right] \\
 &\stackrel{s}{=} \left[\begin{array}{ccc|c} A_{11} & A_{12} & B_{11}B_{21}^* & B_{11} \\ 0 & A_{22} & B_{21}B_{21}^* & B_{21} \\ 0 & 0 & -A_{22}^* & B_{21} \\ \hline C_{11} & C_{12} & D_{11}B_{21}^* & D_{11} \end{array} \right]
 \end{aligned} \tag{2.2.30}$$

Introducing the change of basis

$$T = \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & I & -I \\ 0 & 0 & I \end{array} \right] \tag{2.2.31}$$

and making use of (2.2.22) gives

$$\begin{aligned}
 T_{11}T_{21}^* &\stackrel{s}{=} \left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{12}+B_{11}B_{21}^* & B_{11} \\ 0 & A_{22} & 0 & 0 \\ 0 & 0 & -A_{22}^* & B_{21} \\ \hline C_{11} & C_{12} & D_{11}B_{21}^*+C_{12} & D_{11} \end{array} \right] \\
 &\stackrel{s}{=} \left[\begin{array}{cc|c} A_{11} & A_{12}+B_{11}B_{21}^* & B_{11} \\ 0 & -A_{22}^* & B_{21} \\ \hline C_{11} & D_{11}B_{21}^*+C_{12} & D_{11} \end{array} \right]
 \end{aligned} \tag{2.2.32}$$

Next,

$$\begin{aligned}
 \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}(s) &= \begin{bmatrix} T_{12}^* \\ T_{12}^{\perp *} \end{bmatrix} T_{11}T_{21}^*(s) \stackrel{s}{=} \left[\begin{array}{c|c} -A_{11}^* & C_{11}^* \\ \hline -B_{12}^* & D_{12}^* \\ -B_{13}^* & D_{13}^* \end{array} \right] * \left[\begin{array}{cc|c} A_{11} & A_{12}+B_{11}B_{21}^* & B_{11} \\ 0 & -A_{22}^* & B_{21} \\ \hline C_{11} & D_{11}B_{21}^*+C_{12} & D_{11} \end{array} \right] \\
 &\stackrel{s}{=} \left[\begin{array}{ccc|c} -A_{11}^* & C_{11}^*C_{11} & C_{11}^*(D_{11}B_{21}^*+C_{12}) & C_{11}^*D_{11} \\ 0 & A_{11} & A_{12}+B_{11}B_{21}^* & B_{11} \\ 0 & 0 & -A_{22}^* & B_{21} \\ \hline -B_{12}^* & D_{12}^*C_{11} & D_{12}^*(D_{11}B_{21}^*+C_{12}) & D_{12}^*D_{11} \\ -B_{13}^* & D_{13}^*C_{11} & D_{13}^*D_{11}B_{21}^* & D_{13}^*D_{11} \end{array} \right]
 \end{aligned}$$

by (2.2.11). Introducing the change of basis

$$T = \begin{bmatrix} I & I & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

and invoking (2.2.12), (2.2.25), (2.2.26) and (2.2.27) gives

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{cc|c} -A_{11}^* & -C_{11}^* D_{11} B_{21}^* - B_{11} B_{21}^* & C_{11}^* D_{11} + B_{11} \\ 0 & -A_{22}^* & -B_{21} \\ \hline -B_{12}^* & -D_{12}^* (D_{11} B_{21}^* + C_{12}) & D_{12}^* D_{11} \\ -B_{13}^* & -D_{13}^* D_{11} B_{21}^* & D_{13}^* D_{11} \end{array} \right] \quad (2.2.33)$$

$$\stackrel{s}{=} \left[\begin{array}{c|c} -\bar{A}^* & \bar{C}^* \\ \hline -\bar{B}_1^* & \bar{D}_1^* \\ -\bar{B}_2^* & \bar{D}_2^* \end{array} \right]$$

which is the same as (2.2.13). The minimality of this realisation follows from assumptions (i), (ii), (iii) and (iv) together with Limebeer and Hung [L6, Theorem 4.1]. This completes the proof of (a).

We begin the proof of the remainder of the theorem by writing down the equations describing the spectral factorisation of $(\gamma^2 I - R_2^* R_2)(s)$. Clearly,

$$(\gamma^2 I - R_2^* R_2)(s) \stackrel{s}{=} \left[\begin{array}{cc|c} \bar{A} & \bar{B}_2 \bar{B}_2^* & \bar{B}_2 \bar{D}_2^* \\ 0 & -\bar{A}^* & -\bar{C}^* \\ \hline -\bar{C} & -\bar{D}_2 \bar{B}_2^* & \gamma^2 I - \bar{D}_2 \bar{D}_2^* \end{array} \right] \quad (2.2.34)$$

The next step is to carry out the decomposition

$$(\gamma^2 I - R_2^* R_2)(s) = Z(s) + Z^*(s) \quad (2.2.35)$$

in which $Z(s)$ is positive real. Since $\bar{A}$ is asymptotically stable there exists a unique $\Theta = \Theta^* \geq 0$ which satisfies the Lyapunov equation

$$\bar{A} \Theta + \Theta \bar{A}^* + \bar{B}_2 \bar{B}_2^* = 0 \quad (2.2.36)$$

For our later convenience we will also introduce the partitioning

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \Theta_{12}^* & \Theta_{22} \end{bmatrix} \quad (2.2.37)$$

which is conformable with that of $\bar{A}$ in (2.2.33). The change of basis

$$T = \begin{bmatrix} I & -\Theta \\ 0 & I \end{bmatrix} \quad (2.2.38)$$

in the state-space of (2.2.34) gives

$$Z(s) \triangleq [\bar{A}, N, -\bar{C}, \frac{1}{2}E] \quad (2.2.39)$$

in which

$$N = \bar{B}_2 \bar{D}_2^* + \Theta \bar{C}^* = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \quad (2.2.40)$$

where the partitioning is also induced by that in (2.2.33). Finally,

$$E = \gamma^2 I - \bar{D}_2 \bar{D}_2^* \quad (2.2.41)$$

The spectral factor $M(s)$ may now be expressed in terms of the unique positive definite solution of the Riccati equation [A2]

$$Y(\bar{A} + NE^{-1}\bar{C}) + (\bar{A} + NE^{-1}\bar{C})^* Y + YNE^{-1}N^* Y + \bar{C}^* E^{-1}\bar{C} = 0 \quad (2.2.42)$$

The next lemma shows that the required solution to (4.42) always exists.

Lemma 2.2.2

The Riccati equation (2.2.42) has a unique positive definite stabilising solution.

Proof:

See Appendix 2.A at the end of this chapter. □

As with (2.2.37), the solution to (2.2.42) has a partitioning induced by (2.2.33)

$$Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{bmatrix} \quad (2.2.43)$$

Since Y is stabilising,

$$M(s) \triangleq [\bar{A}, N, -E^{\frac{1}{2}}F, E^{\frac{1}{2}}] \quad (2.2.44)$$

in which

$$F = E^{-1}(\bar{C} + N^* Y) = [F_1 \mid F_2] \quad (2.2.45)$$

is a minimum phase spectral factor (it is easy to check that $M^*(s)M(s)=Z(s)+Z^*(s)$). $\gamma>||R_2||_\infty$ ensures the positive definiteness of E and therefore that

$$M^{-1}(s) \triangleq \begin{bmatrix} \bar{A}+NF & NE^{-\frac{1}{2}} \\ F & E^{-\frac{1}{2}} \end{bmatrix} \quad (2.2.46)$$

is proper.

We will now carry out the calculations which lead to state space realisations of $R_1 M^{-1}(s)$, $[R_1 M^{-1}(s)]_+$, and $[R_1 M^{-1}(s)]_-$.

$$\begin{aligned} R_1 M^{-1}(s) &\triangleq \begin{bmatrix} -\bar{A}^* & \bar{C}^* \\ -\bar{B}_1^* & \bar{D}_1^* \end{bmatrix}^* \begin{bmatrix} \bar{A}+NF & NE^{-\frac{1}{2}} \\ F & E^{-\frac{1}{2}} \end{bmatrix} \\ &\triangleq \left[\begin{array}{cc|c} -\bar{A}^* & \bar{C}^* F & \bar{C}^* E^{-\frac{1}{2}} \\ 0 & \bar{A}+NF & NE^{-\frac{1}{2}} \\ -\bar{B}_1^* & \bar{D}_1^* F & \bar{D}_1^* E^{-1/2} \end{array} \right] \end{aligned} \quad (2.2.47)$$

The change of basis

$$T = \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}$$

in the state space of (2.2.47) together with (2.2.42) and (2.2.45) yields

$$R_1 M^{-1}(s) \triangleq \left[\begin{array}{cc|c} -\bar{A}^* & 0 & F^* E^{\frac{1}{2}} \\ 0 & \bar{A}+NF & NE^{-\frac{1}{2}} \\ -\bar{B}_1^* & \bar{D}_1^* F + \bar{B}_1^* Y & \bar{D}_1^* E^{-1/2} \end{array} \right] \quad (2.2.48)$$

In (2.2.48) we note that $-\bar{A}^*$ is completely unstable while $\bar{A}+NF$ is asymptotically stable. Thus

$$[R_1 M^{-1}(s)]_- \triangleq \begin{bmatrix} -\bar{A}^* & F^* E^{\frac{1}{2}} & -\bar{B}_1^* & 0 \end{bmatrix} \quad (2.2.49)$$

and

$$[R_1 M^{-1}(s)]_+ \triangleq \begin{bmatrix} \bar{A}+NF & NE^{-\frac{1}{2}} & \bar{D}_1^* F + \bar{B}_1^* Y & \bar{D}_1^* E^{-\frac{1}{2}} \end{bmatrix} \quad (2.2.50)$$

The remainder of the proof is based on detailed manipulations requiring various partitions of the bounded real type equations (2.2.16) and (2.2.17). The (1,1) and (2,1) blocks of (2.2.16) may be written out in full as

$$\begin{aligned}
 & \begin{bmatrix} -A_{11}^* & -(C_{11}^* D_{11} + B_{11}) B_{21}^* & 0 \\ 0 & -A_{22}^* & 0 \\ 0 & 0 & \hat{A} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^* & P_{22} & P_{23} \\ P_{13}^* & P_{23}^* & P_{33} \end{bmatrix} + \\
 & \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^* & P_{22} & P_{23} \\ P_{13}^* & P_{23}^* & P_{33} \end{bmatrix} \begin{bmatrix} -A_{11} & 0 & 0 \\ -B_{21}(D_{11}^* C_{11} + B_{11}^*) & -A_{22} & 0 \\ 0 & 0 & \hat{A}^* \end{bmatrix} + \\
 & \begin{bmatrix} F_1^* E^{\frac{1}{2}} \\ F_2^* E^{\frac{1}{2}} \\ \hat{B} \end{bmatrix} \begin{bmatrix} E^{\frac{1}{2}} F_1 & E^{\frac{1}{2}} F_2 & \hat{B}^* \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{21} \end{bmatrix} \begin{bmatrix} 0 & 0 & L_{21}^* \end{bmatrix} = 0 \quad (2.2.51)
 \end{aligned}$$

and

$$\hat{D} \begin{bmatrix} E^{\frac{1}{2}} F_1 & E^{\frac{1}{2}} F_2 & \hat{B}^* \end{bmatrix} + \begin{bmatrix} -\bar{B}_{11}^* & -\bar{B}_{21}^* & \hat{C} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{12}^* & P_{22} & P_{23} \\ P_{13}^* & P_{23}^* & P_{33} \end{bmatrix} + W \begin{bmatrix} 0 & 0 & L_{21}^* \end{bmatrix} = 0 \quad (2.2.52)$$

where the partitioning of $\bar{B}_1^* = [\bar{B}_{11}^* \mid \bar{B}_{21}^*]$ is defined in (2.2.33). In the same way the (1,1) and (2,1) blocks of (2.2.17) may be written out as

$$\begin{aligned}
 & \begin{bmatrix} -A_{11} & 0 & 0 \\ -B_{21}(D_{11}^* C_{11} + B_{11}^*) & -A_{22} & 0 \\ 0 & 0 & \hat{A}^* \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^* & Q_{22} & Q_{23} \\ Q_{13}^* & Q_{23}^* & Q_{33} \end{bmatrix} + \\
 & \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^* & Q_{22} & Q_{23} \\ Q_{13}^* & Q_{23}^* & Q_{33} \end{bmatrix} \begin{bmatrix} -A_{11}^* & -(C_{11}^* D_{11} + B_{11}) B_{21}^* & 0 \\ 0 & -A_{22}^* & 0 \\ 0 & 0 & \hat{A} \end{bmatrix} + \\
 & \begin{bmatrix} -\bar{B}_{11} \\ -\bar{B}_{21} \\ \hat{C}^* \end{bmatrix} \begin{bmatrix} -\bar{B}_{11}^* & -\bar{B}_{21}^* & \hat{C} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ L_{21d}^* \end{bmatrix} \begin{bmatrix} 0 & 0 & L_{21d} \end{bmatrix} = 0 \quad (2.2.53)
 \end{aligned}$$

and

$$\hat{D}^* \begin{bmatrix} -\bar{B}_{11}^* & -\bar{B}_{21}^* & \hat{C} \end{bmatrix} + \begin{bmatrix} E^{\frac{1}{2}} F_1 & E^{\frac{1}{2}} F_2 & \hat{B}^* \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12}^* & Q_{22} & Q_{23} \\ Q_{13}^* & Q_{23}^* & Q_{33} \end{bmatrix} + W_d^* \begin{bmatrix} 0 & 0 & L_{21d} \end{bmatrix} = 0 \quad (2.2.54)$$

An easy rearrangement shows that (2.2.42) may be written in the alternative form

$$Y\bar{A} + \bar{A}^* Y + F^* E F = 0 \quad (2.2.55)$$

which together with (2.2.51) yields

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{bmatrix} = - \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \quad (2.2.56)$$

The (1,1) block of (2.2.53) together with (2.2.24), (2.2.36) and (2.2.37) yields

$$I + Q_{11} = \Theta_{11} \quad (2.2.57)$$

Next, we have

$$\begin{aligned} X(s) &= (\tilde{X} - (R_1 M^{-1})_+) M(s) \\ &\stackrel{s}{=} \left[\begin{array}{cc|c} \hat{A} & 0 & \hat{B} \\ 0 & \bar{A} + NF & -NE^{-\frac{1}{2}} \\ \hline \hat{C} & \bar{D}_1^* F + \bar{B}_1^* Y & -\bar{D}_1^* E^{-\frac{1}{2}} + \hat{D} \end{array} \right] \begin{bmatrix} \bar{A} & N \\ -E^{\frac{1}{2}} F & E^{\frac{1}{2}} \end{bmatrix} \\ &\stackrel{s}{=} \left[\begin{array}{ccc|c} \hat{A} & 0 & -\hat{B} E^{\frac{1}{2}} F & \hat{B} E^{\frac{1}{2}} \\ 0 & \bar{A} + NF & NF & -N \\ 0 & 0 & \bar{A} & N \\ \hline \hat{C} & \bar{D}_1^* F + \bar{B}_1^* Y & (\bar{D}_1^* E^{-1/2} - \hat{D}) E^{1/2} F & \hat{D} E^{1/2} - \bar{D}_1^* \end{array} \right] \end{aligned}$$

which after the change of basis

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & I \\ 0 & 0 & I \end{bmatrix}$$

becomes

$$X(s) \stackrel{s}{=} \left[\begin{array}{cc|c} \hat{A} & -\hat{B} E^{\frac{1}{2}} F & \hat{B} E^{\frac{1}{2}} \\ 0 & \bar{A} & N \\ \hline \hat{C} & -\hat{D} E^{1/2} F - \bar{B}_1^* Y & \hat{D} E^{1/2} - \bar{D}_1^* \end{array} \right] \quad (2.2.58)$$

The change of basis

$$T = \begin{bmatrix} I & P_{13}^* & P_{23}^* \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

together with the (3,1) and (3,2) blocks of (2.2.51), and the (1,1) and (1,2) blocks of (2.2.52) gives

$$X(s) \stackrel{\triangle}{=} \begin{bmatrix} \hat{A} & \hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2 \\ \hat{C} & \hat{D}E^{\frac{1}{2}} - \bar{D}_1^* \end{bmatrix} \quad (2.2.59)$$

thereby proving (2.2.20).

The last part of the proof is concerned with showing that (2.2.21) is true. This will be established by a chain of intricate manipulations of the state-space realisation of $T_{12}XT_{21}(s)$. From (2.2.10) and (2.2.59) we have

$$\begin{aligned} T_{12}XT_{21}(s) &\stackrel{\triangle}{=} \left[\begin{array}{c|c} A_{11} & B_{12} \\ \hline C_{11} & D_{12} \end{array} \right] \cdot \left[\begin{array}{c|c} \hat{A} & \hat{B}E^{1/2} + P_{13}^*N_1 + P_{23}^*N_2 \\ \hline \hat{C} & \hat{D}E^{1/2} - \bar{D}_1^* \end{array} \right] \cdot \left[\begin{array}{c|c} A_{22} & B_{21} \\ \hline C_{22} & I \end{array} \right] \\ &\stackrel{\triangle}{=} \left[\begin{array}{ccc|c} A_{11} & B_{12}(\hat{D}E^{\frac{1}{2}} - \bar{D}_1^*)C_{22} & B_{12}\hat{C} & B_{12}(\hat{D}E^{\frac{1}{2}} - \bar{D}_1^*) \\ 0 & A_{22} & 0 & B_{21} \\ 0 & (\hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2)C_{22} & \hat{A} & \hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2 \\ \hline C_{11} & D_{12}(\hat{D}E^{1/2} - \bar{D}_1^*)C_{22} & D_{12}\hat{C} & D_{12}(\hat{D}E^{1/2} - \bar{D}_1^*) \end{array} \right] \end{aligned} \quad (2.2.60)$$

The first change of basis in the state space of (2.2.60) is given by

$$T = \begin{bmatrix} I & 0 & -Q_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad (2.2.61)$$

This together with the (1,3) block of (2.2.53) gives

$$\begin{aligned} T_{12}XT_{21}(s) &\stackrel{\triangle}{=} \left[\begin{array}{ccc|c} A_{11} & B_{12}(\hat{D}E^{\frac{1}{2}} - \bar{D}_1^*)C_{22} - Q_{13}(\hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2)C_{22} & 0 & \Pi \\ 0 & A_{22} & 0 & B_{21} \\ 0 & (\hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2)C_{22} & \hat{A} & \hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2 \\ \hline C_{11} & D_{12}(\hat{D}E^{1/2} - \bar{D}_1^*)C_{22} & D_{12}\hat{C} + C_{11}Q_{13} & D_{12}(\hat{D}E^{1/2} - \bar{D}_1^*) \end{array} \right] \\ &\quad (2.2.62) \end{aligned}$$

Our next lemma gives a simplified expression for Π in (2.2.62)

Lemma 2.2.3

In the state-space realisation of $T_{12}XT_{21}(s)$ in (2.2.62) above,

$$\Pi = (\Theta_{12} - Q_{12})B_{21} - B_{11}$$

Proof:

See Appendix 2.B at the end of this chapter. $\square$

A second change of basis

$$T = \begin{bmatrix} I & Q_{12} - \Theta_{12} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}$$

gives

$$T_{12}XT_{21}(s) \stackrel{\triangle}{=} \left[\begin{array}{ccc|c} A_{11} & \Sigma & 0 & -B_{11} \\ 0 & A_{22} & 0 & B_{21} \\ 0 & (\hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2)C_{22} & \hat{A} & \hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2 \\ \hline C_{11} & D_{12}(\hat{D}E^{1/2} - \bar{D}_1^*)C_{22} - C_{11}(Q_{12} - \Theta_{12}) & D_{12}\hat{C} + C_{11}Q_{13} & D_{12}(\hat{D}E^{1/2} - \bar{D}_1^*) \end{array} \right] \quad (2.2.63)$$

Lemma 2.2.4

In the state space realisation of $T_{12}XT_{21}(s)$ in (2.2.63) above, $\Sigma = -A_{12}$.

Proof:

See Appendix 2.C at the end of this chapter. $\square$

Making use of Lemma 2.2.4 yields

$$T_{12}XT_{21}(s) \stackrel{\triangle}{=} \left[\begin{array}{ccc|c} A_{11} & A_{12} & 0 & -B_{11} \\ 0 & A_{22} & 0 & -B_{21} \\ 0 & -(\hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2)C_{22} & \hat{A} & \hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2 \\ \hline C_{11} & -D_{12}(\hat{D}E^{1/2} - \bar{D}_1^*)C_{22} + C_{11}(Q_{12} - \Theta_{12}) & D_{12}\hat{C} + C_{11}Q_{13} & D_{12}(\hat{D}E^{1/2} - \bar{D}_1^*) \end{array} \right] \quad (2.2.64)$$

Before continuing further, we need to link Θ and Y (the solutions to (2.2.36) and (2.2.42) respectively). This connection is provided in Lemma 2.2.6. The proof of this Lemma, however, requires an

intermediate result that is established next.

Lemma 2.2.5:

There exists a nonsingular matrix T_1 transforming the Hamiltonian matrix corresponding to the Riccati equation in Lemma 2.2.2 to

$$T_1 H T_1^{-1} = - \begin{bmatrix} -\tilde{H}_{11}^* & -\mathcal{A}^* & \tilde{H}_{21} & 0 \\ 0 & A_{22}^* & 0 & 0 \\ \tilde{H}_{12} & \mathfrak{B}^* & \tilde{H}_{11} & 0 \\ \mathfrak{B} & \mathfrak{D} & \mathcal{A} & -A_{22} \end{bmatrix} \quad (2.2.65)$$

where $\tilde{H}$ is the Hamiltonian matrix associated with the Riccati equation (2.2.65) partitioned as

$$\tilde{H} = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{12}^* & \tilde{H}_{22} \end{bmatrix} \quad (2.2.66)$$

In particular, the similarity transformation T_1 can be chosen as

$$T_1 = \begin{bmatrix} I & \Theta - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \\ 0 & I \end{bmatrix} \quad (2.2.67)$$

in which case,

$$\mathcal{A} := C_{22}^* E^{-1} D_{11}^* D_{13} \quad (2.2.68a)$$

$$\mathfrak{B} := C_{22}^* E^{-1} (D_{11}^* C_{11} + B_{11}^*) \quad (2.2.68b)$$

$$\mathfrak{D} := C_{22}^* E^{-1} C_{22} \quad (2.2.68c)$$

in previously defined notation.

Proof:

See Appendix 2.D.1 at the end of this chapter. □

Lemma 2.2.6

(a) There exists a unique “largest” solution Z_0 to the Riccati equation

$$\begin{aligned} & \{A_{11} + B_{13} D_{13}^* D_{11} E^{-1} (D_{11}^* C_{11} + B_{11}^*)\} Z + Z \{A_{11} + B_{13} D_{13}^* D_{11} E^{-1} (D_{11}^* C_{11} + B_{11}^*)\}^* + \\ & Z (C_{11}^* D_{11} + B_{11}) E^{-1} (D_{11}^* C_{11} + B_{11}^*) Z + B_{13} [I + D_{13}^* D_{11} E^{-1} D_{11}^* D_{13}] B_{13} = 0 \end{aligned} \quad (2.2.69)$$

Further,

$$H_0 = (B_{13} D_{13}^* D_{11} + Z_0 (C_{11}^* D_{11} + B_{11})) E^{-1} \quad (2.2.70)$$

is a destabilising output injection for $[A_{11}, (D_{11}^* C_{11} + B_{11}^*)]$ (i.e. $\text{Re}[\lambda(A_{11} + H_0(D_{11}^* C_{11} + B_{11}^*))] > 0$)

(b) The stabilising solution to (2.2.42) is related to Z_0 and Θ by

$$Y^{-1} + \Theta = \begin{bmatrix} Z_0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \quad (2.2.71)$$

where the partitioning is induced by that in (2.2.37) and (2.2.43).

Proof:

See Appendix 2.D at the end of this chapter. □

For convenience we introduce the notation

$$Y^{-1} = \begin{bmatrix} \hat{Y}_{11} & \hat{Y}_{12} \\ \hat{Y}_{12}^* & \hat{Y}_{22} \end{bmatrix} \quad (2.2.72)$$

and the last transformation we require is

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & P_{13}^* \hat{Y}_{12} + P_{23}^* \hat{Y}_{22} & I \end{bmatrix} \quad (2.2.73)$$

Applying this to (2.2.64) gives

$$T_{12} X T_{21} \stackrel{s}{=} \left[\begin{array}{ccc|c} A_{11} & A_{12} & 0 & -B_{11} \\ 0 & A_{22} & 0 & -B_{21} \\ 0 & \Phi & \hat{A} & \Delta \\ \hline C_{11} & \Psi & D_{12} \hat{C} + C_{11} Q_{13} & D_{12}(\hat{D} E^{1/2} - \bar{D}_1^*) \end{array} \right] \quad (2.2.74)$$

Lemma 2.2.7

In the state space realisation of $T_{12} X T_{21}(s)$ in (2.2.74) above,

$$\Psi = C_{12} \quad (2.2.75a)$$

$$\Phi = 0 \quad (2.2.75b)$$

$$\Delta = \hat{B} E^{\frac{1}{2}} + P_{13}^* B_{13} D_{13}^* D_{11} + P_{23}^* C_{22} E + (P_{13}^* \Theta_{11} + P_{23}^* \Theta_{12}^*)(C_{11}^* D_{11} + B_{11}) \quad (2.2.75c)$$

Proof:

See Appendix 2.E at the end of this chapter. $\square$

Invoking Lemma 2.2.7 gives

$$(T_{11} + T_{12}XT_{21})(s) \triangleq \begin{bmatrix} \hat{A} & \Delta \\ D_{12}\hat{C} + C_{11}Q_{13} & D_{12}(\hat{D}E^{\frac{1}{2}} - \bar{D}_1^*) \end{bmatrix} \quad (2.2.76)$$

which verifies (2.2.21) and concludes our proof. $\square$

2.3 An upper bound for the pole/zero cancellations in $F_\ell(P, K)$

The bound c_b will be established by counting those points in the complex plane (including multiplicities) at which cancellations may occur in $\mathcal{F}_\ell(P(s), -K(s))$ in the case that $K(s)$ is (internally) stabilising.

Theorem 2.3.1 below shows that every uncontrollable mode in $\mathcal{F}_\ell(P(s), -K(s))$ is due to a cancellation at a zero of $P_{21}(s)$. In the case that $K(s)$ is internally stabilising, the number of uncontrollable modes is bounded above by the number of zeros of $P_{21}(s)$ in $\mathbb{C}_-$ (counting multiplicities). In the same way, the number of unobservable modes is bounded above by the number of zeros of $P_{12}(s)$ in $\mathbb{C}_-$ (counting multiplicities).

Theorem 2.3.1 [L6], [L7]

Let

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \triangleq \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (4.72)$$

in which $P_{12}(s) \in \mathbb{R}^{p_1 \times m_2}(s)$ with $p_1 \geq m_2$ and $P_{21}(s) \in \mathbb{R}^{p_2 \times m_1}(s)$ with $m_1 \geq p_2$. Suppose also that

$$K(s) \triangleq \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \quad (4.73)$$

is a minimal realisation and that the well posedness condition $\det(I - D_{22}\hat{D}) \neq 0$ is satisfied. Then in the

closed loop of Fig.1

(a) every unobservable mode (from y_1) is a Smith zero of

$$\begin{bmatrix} sI-A & -B_2 \\ C_1 & D_{12} \end{bmatrix} \quad (4.74)$$

(b) every uncontrollable mode (from u_1) is a Smith zero of

$$\begin{bmatrix} sI-A & -B_1 \\ C_2 & D_{21} \end{bmatrix} \quad (4.75)$$

Proof:

The equations describing the closed loop in Figure 1 are

$$\dot{x} = Ax + B_1 u + B_2 \tilde{y} \quad (2.3.5)$$

$$y = C_1 x + D_{11} u + D_{12} \tilde{y} \quad (2.3.6)$$

$$\tilde{u} = C_2 x + D_{21} u + D_{22} \tilde{y} \quad (2.3.7)$$

$$\dot{\hat{x}} = \hat{A}x + \hat{B}\tilde{u} \quad (2.3.8)$$

$$\tilde{y} = \hat{C}\hat{x} + \hat{D}\tilde{u} \quad (2.3.9)$$

Eliminating the variables $\tilde{u}$ and $\tilde{y}$ leads to the following state-space model for the closed loop

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A + B_2 \hat{D} M C_2 & B_2 [I + \hat{D} M D_{22}] \hat{C} \\ \hat{B} M C_2 & \hat{A} + \hat{B} M D_{22} \hat{C} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B_1 + B_2 \hat{D} M D_{21} \\ \hat{B} M D_{21} \end{bmatrix} \begin{bmatrix} u \end{bmatrix} \quad (2.3.10)$$

in which

$$M := (I - D_{22} \hat{D})^{-1} \quad (2.3.11)$$

and

$$\begin{bmatrix} y \\ \hat{x} \end{bmatrix} = \begin{bmatrix} C_1 + D_{12} \hat{D} M C_2 & D_{12} [I + \hat{D} M D_{22}] \hat{C} \\ D_{11} + D_{12} \hat{D} M D_{21} \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} D_{11} + D_{12} \hat{D} M D_{21} \end{bmatrix} \begin{bmatrix} u \end{bmatrix} \quad (2.3.12)$$

If s_0 is an unobservable mode of the closed loop state-space model (2.3.10) and (2.3.12), then there exists a vector $[w_1^* \ w_2^*]^* \neq 0$ such that

$$\begin{bmatrix} s_0 I - A - B_2 \hat{D} M C_2 & -B_2 [I + \hat{D} M D_{22}] \hat{C} \\ -\hat{B} M C_2 & s_0 I - \hat{A} - \hat{B} M D_{22} \hat{C} \\ C_1 + D_{12} \hat{D} M C_2 & D_{12} [I + \hat{D} M D_{22}] \hat{C} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = 0 \quad (2.3.13)$$

Defining

$$z_2 := \hat{D}MC_2w_1 + [I + \hat{D}MD_{22}]\hat{C}w_2 \quad (2.3.14)$$

implies that

$$\begin{bmatrix} s_0I - A & -B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} w_1 \\ z_2 \end{bmatrix} = 0 \quad (2.3.15)$$

The proof of the (a) part is concluded by establishing that $[w_1^* \ w_2^*] \neq 0 \Rightarrow [w_1^* \ z_2^*] \neq 0$. Suppose for contradiction that $[w_1^* \ z_2^*] = 0$. This implies that

$$\begin{aligned} & [I + \hat{D}MD_{22}]\hat{C}w_2 = 0 \\ \Leftrightarrow & (I - \hat{D}D_{22})^{-1}\hat{C}w_2 = 0 \\ \Leftrightarrow & \hat{C}w_2 = 0 \end{aligned} \quad (2.3.16)$$

We also have from (2.3.13) that

$$(s_0I - \hat{A})w_2 = 0 \quad (2.3.17)$$

(2.3.16) and (2.3.17) taken together contradict the assumed minimality of (2.3.2) which proves the (a) condition. The (b) part may be established by a parallel sequence of arguments. $\square$

The next lemma provides a link between the Smith zeros of $P_{12} \triangleq (A, B_2, C_1, D_{12})$ and the modes of $(A - B_2D_{12}^*C_1)$ which are undetectable through $D_{\perp}^*C_1$.

Lemma 2.3.2

Suppose that

$$P(s) = \begin{bmatrix} sI - A & -B_2 \\ C_1 & D_{12} \end{bmatrix} \quad (2.3.18)$$

is a polynomial matrix of dimension $(n+p) \times (n+m)$ with $p > m$. If D_{12} is part of an orthogonal matrix and $D_{\perp}$ is its orthogonal completion, then every Smith zero of $P(s)$ is an unobservable mode of $[A - B_2D_{12}^*C_1, \ D_{\perp}^*C_1]$ and vice versa.

Proof

If s_0 is a Smith zero of $P(s)$, there exists a vector $[w^* \mid v^*] \neq 0$ such that

$$\begin{bmatrix} s_0 I - A & -B_2 \\ C_1 & D_{12} \end{bmatrix} \begin{bmatrix} w \\ v \end{bmatrix} = 0 \quad (2.3.19)$$

We note also that $w \neq 0$ since if this were not the case, we would have $D_{12}v = 0$ which is impossible. From (2.3.19) we get

$$(s_0 I - A)w - B_2 v = 0 \quad (2.3.20)$$

and

$$C_1 w + [D_{12} \mid D_\perp] \begin{bmatrix} v \\ 0 \end{bmatrix} = 0 \quad (2.3.21)$$

Multiplying (2.3.21) on the left by $[D_{12} \mid D_\perp]^*$ gives

$$D_{12}^* C_1 w + v = 0 \quad (2.3.22a)$$

$$D_\perp^* C_1 w = 0 \quad (2.3.22b)$$

Substituting (2.3.22a) into (2.3.20) and combining the result with (2.3.22b) yields

$$\begin{bmatrix} s_0 I - A + B_2 D_{12}^* C_1 \\ D_\perp C_1 \end{bmatrix} w = 0 \quad (2.3.24)$$

which completes the proof in one direction.

If (2.3.24) is satisfied, we may write

$$(s_0 I - A)w - B_2 v = 0 \quad (2.3.25)$$

where

$$v := -D_{12}^* C_1 w \quad (2.3.26)$$

Combining (2.3.26) with the (2,1) block of (2.3.24) gives

$$\begin{bmatrix} D_{12}^* \\ D_\perp^* \end{bmatrix} C_1 w + \begin{bmatrix} v \\ 0 \end{bmatrix} = 0$$

and hence also

$$\Rightarrow C_1 w + D_{12} v = 0 \quad (2.3.27)$$

Finally, we note that (2.3.27) combined with (2.3.25) gives (2.3.19) thereby establishing the result. $\square$

In the next lemma, an upper bound c_b is established on the number of cancellations that can occur in the linear fractional map $\mathcal{F}_\ell(P(s), K(s))$, with $K(s)$ chosen as any internally stabilising controller.

Lemma 2.3.3:

If $K(s)$ is any internally stabilising compensator in Figure 1, the number of pole/zero cancellations that can occur in the linear fractional map $\mathcal{F}_\ell(P(s), K(s))$, is bounded from above by $c_b = 2n - \text{Rank}(\mathfrak{X}) - \text{Rank}(\mathfrak{Y})$, where $n = \deg(P(s))$ and $\mathfrak{X}$ and $\mathfrak{Y}$ are the stabilising solutions to the Riccati equations (1.5.6) and (1.5.7) respectively.

Proof:

Since $K(s)$ is an internally stabilising compensator, no pole/zero cancellations in $\mathcal{F}_\ell(P, K)$ can occur in $\mathbb{C}_+$. Theorem 2.3.1 shows that every uncontrollable mode in a realisation of $\mathcal{F}_\ell(P, K)$ is due to a cancellation with a zero of $P_{21}(s)$ and every unobservable mode is due to a cancellation with a zero of $P_{12}(s)$. Consequently, an upper bound on the number of cancellations is given by

$$c_b = (\text{number of zeros of } P_{12}(s) \text{ in } \mathbb{C}_-) + (\text{number of zeros of } P_{21}(s) \text{ in } \mathbb{C}_-) \quad (2.3.28)$$

Now, Lemma 2.1.3.ia] gives that the number of minimum phase zeros of $P_{21}(s)$ is equal to $n - \text{Rank}(\mathfrak{Y})$, while Lemma 2.3.2 identifies the minimum phase zeros of $P_{12}(s)$ with the stable modes of $A - B_2 D_{12}^* C_1$ which are undetectable through $D_{12}^* C_1$; then Lemma 2.1.3ib] gives their number as $n - \text{Rank}(\mathfrak{Y})$.

Combining these results we obtain that

$$c_b = 2n - \text{Rank}(\mathfrak{X}) - \text{Rank}(\mathfrak{Y}) \quad (2.3.29)$$

as required. $\square$

2.4 The bound t_b on the degree of the optimal interpolating function.

In this section we will establish an upper bound for the MacMillan degree of the $\mathcal{H}^\infty$ -optimal interpolating function $\mathfrak{R}(s)$. Theorem 2.2.1 establishes such a bound under the special assumptions (iii)–(vi) for the realisation of $T(s)$. Here these assumptions will be removed.

In Lemma 2.1.3v] it was shown that the MacMillan degree of $R(s)$ is bounded from above by

the sum of the rank of the two Riccati solutions, $\text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y})$. It turns out that this bound is tight only under the condition that (T_{11}, T_{21}) are all-pass right coprime and $(T_{11}, [T_{12} \ T_\perp])$ are all-pass left coprime (the notions of all-pass right and left coprimeness being defined in section 1.4). This is not just a technical point; in fact, a whole class of $\mathcal{H}^\infty$ optimisation problems does not satisfy these conditions (see the example at the end of this chapter). The next theorem shows that if these coprimeness conditions fail, a factorisation of

$$\begin{bmatrix} T_{11} & T_{12} & T_\perp \\ T_{21} & 0 & 0 \end{bmatrix} = \begin{bmatrix} A_l(s) & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} & \tilde{T}_\perp \\ \tilde{T}_{21} & 0 & 0 \end{bmatrix} \begin{bmatrix} A_r(s) & 0 \\ 0 & I \end{bmatrix} \quad (2.4.1)$$

is always possible, in which $A_r(s)$ and $A_l(s)$ are the maximal left and right all-pass common factors respectively. In addition, the corresponding realisation of

$$\begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \begin{bmatrix} \tilde{T}_{12}^* \\ \tilde{T}_\perp \end{bmatrix} \tilde{T}_{11} \tilde{T}_{21}^* \quad (2.4.2)$$

is minimal of degree $\deg(R) = \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) - \deg(A_r) - \deg(A_l)$. This theorem is proved in [Limebeer and Hung]. In the subsequent Lemma, it is shown that after the extraction of the all-pass common factors $A_r \oplus I$ and $A_l \oplus I$, a state-space realisation of $\tilde{T}(s)$ retains the structure required for the application of Theorem 2.2.1. In particular:

- (a) The A , B and C matrices of a state space realisation of $\tilde{T}(s)$ can be assumed to be block-upper triangular while the resulting realisations of $[T_{12} \ T_\perp]$ and $T_{21}(s)$ are still minimal and balanced.
- (b) Equations (2.2.11) and (2.2.12) are still satisfied for the state space realisations of $\tilde{T}(s)$.

Consequently, Theorem 2.4.1 and Lemma 2.4.2 ensure that if the all-pass coprimeness conditions are not satisfied, Theorem 2.2.1 can be applied after extracting all inner common factors $A_r \oplus I$ and $A_l \oplus I$ in $T(s)$.

Theorem 2.4.1: [L6]

Let

$$\begin{bmatrix} G & A_1 \\ A_2 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} & B_{21} & 0 \\ \hline C_{11} & C_{12} & D & I \\ 0 & C_{22} & I & 0 \end{array} \right] \quad (2.4.3)$$

in which $A_1(s)$ and $A_2(s)$ are assumed inner. Suppose also that the realisations for $A_1(s)$ and $A_2(s)$ given in (2.4.3) are minimal. Then there exists a change of basis such that (2.4.3) can be put to the form

$$\begin{bmatrix} G & A_1 \\ A_2 & 0 \end{bmatrix}(s) = \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix}$$

in which,

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{00} & \tilde{A}_{01} & \tilde{A}_{02} & \tilde{A}_{03} \\ 0 & \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ 0 & 0 & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & 0 & \tilde{A}_{33} \end{bmatrix} \quad \tilde{B}^T = \begin{bmatrix} \tilde{B}_{01}^T & \tilde{B}_{11}^T & \tilde{B}_{21}^T & \tilde{B}_{31}^T \\ \tilde{B}_{02}^T & \tilde{B}_{12}^T & 0 & 0 \end{bmatrix}$$

$$\tilde{C} = \begin{bmatrix} \tilde{C}_{10} & \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{13} \\ 0 & 0 & \tilde{C}_{22} & \tilde{C}_{23} \end{bmatrix}$$

and

$$\tilde{D} = \begin{bmatrix} D & I \\ I & 0 \end{bmatrix} \quad (2.4.4)$$

which admits the following factorisations

$$A_1(s) = A_l(s)\tilde{A}_1(s) = \begin{bmatrix} \tilde{A}_{00} & \tilde{B}_{02} \\ \tilde{C}_{10} & I \end{bmatrix} \begin{bmatrix} \tilde{A}_{11} & \tilde{B}_{12} \\ \tilde{C}_{11} & I \end{bmatrix} \quad (2.4.5)$$

$$A_2(s) = \tilde{A}_2(s)A_r(s) = \begin{bmatrix} \tilde{A}_{22} & \tilde{B}_{21} \\ \tilde{C}_{10} & I \end{bmatrix} \begin{bmatrix} \tilde{A}_{33} & \tilde{B}_{31} \\ \tilde{C}_{23} & I \end{bmatrix} \quad (2.4.6)$$

$$G(s) = A_l(s)\tilde{G}A_r(s) = \begin{bmatrix} \tilde{A}_{00} & \tilde{B}_{02} \\ \tilde{C}_{10} & I \end{bmatrix} \left[\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_{11} \\ 0 & \tilde{A}_{22} & \tilde{B}_{21} \\ \hline \tilde{C}_{11} & \tilde{C}_{12} & D \end{array} \right] \begin{bmatrix} \tilde{A}_{33} & \tilde{B}_{31} \\ \tilde{C}_{23} & I \end{bmatrix} \quad (2.4.7)$$

further

(a) $\tilde{A}_1(s)$, $\tilde{A}_2(s)$, $A_l(s)$, $A_r(s)$ are inner

(b) The factorisation in (2.4.5), (2.4.6) and (2.4.7) are minimal in the sense that

$$\deg(A_1) = \deg(A_l) + \deg(\tilde{A}_1) \quad (2.4.8)$$

$$\deg(A_2) = \deg(A_r) + \deg(\tilde{A}_2) \quad (2.4.9)$$

$$\deg(G) = \deg(A_l) + \deg(\tilde{G}) + \deg(A_r) \quad (2.4.10)$$

(c) $A_1^*GA_1^* = \tilde{A}_1^*\tilde{G}\tilde{A}_1^*$ has a minimal realisation given by

Errata

1. Figure 1, next to page 2: “ $-K(s)$ ” should read “ $K(s)$ ”
2. Page 5, lines 1 and 6: “Savonov” should read “Safonov”.
3. Page 6, line 3: “Postlethwaite et al [P4]” should read “Tsai et al [T2]”.
4. Page 9, line 16: “unstable” should read “anti-stable”.
5. Page 12, line 22: phrase “(1) If $Q^*Q > 0$, the real symmetric solution $P > 0$ ” should be omitted.
6. Page 21, lines 3 and 4: Conditions (ii) and (iii) should be replaced by:

(ii) D_{12} is full column rank and D_{21} is full row rank

$$(iii) \begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix} \text{ is full column rank } \forall \omega \in \mathbb{R}$$

$$(iv) \begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} \text{ is full row rank } \forall \omega \in \mathbb{R}$$

7. Page 27, lines 23 and 24: Conditions “ $\det(V_l - N_r Q) \neq 0$ ” and “ $\det(V_r - Q N_l) \neq 0$ ” should read “ $\det[(V_l - N_r Q)(\infty)] \neq 0$ ” and “ $\det[(V_r - Q N_l)(\infty)] \neq 0$ ” respectively.
8. Page 54, line 9: “ $\tilde{\Sigma}_3^{1/2}$ ” should read “ $\tilde{\Sigma}_3$ ”.
9. Page 56, lines 10, 12, 15; page 60, line 3: “undetectable” should read “unobservable”.
10. Page 60, lines 5, 6 and 7: “(3.22)”, “(3.24)” and “(3.25)” should read “(2.1.48)”, “(2.1.50)” and “(2.1.51)”.
11. Page 63, line 22: “ $T_{22}(s)$ ” should read “ $T_{21}(s)$ ”
12. Page 65, line 12: “(2.2.71)” should read “(2.2.76)”.
13. Page 66, line 1: “[20]” should read “[L6]”.
14. Page 69, line 11: “(4.42)” should read “2.2.42”.
15. Page 109, line 20: “the standard $\|G_0\|_H$ ” should read “the maximum level”.

16. Page 116, line 11: “ $(\Gamma_h u)(t) = \int_0^\infty h(t+s)u(s)ds \in \mathcal{L}_2(0, \infty; \mathbb{C}^p)$ ” should be replaced by

$$“(\Gamma_h u)(t) = \left[\int_0^\infty h(t+s)u(s)ds \right]_+ \in \mathcal{L}_2(0, \infty; \mathbb{C}^p), \text{ where } [\cdot]_+ \text{ denotes causal projection.}”$$

17. Page 118, bottom line, page 119, lines 1 and 2: Expression (3.3.4) should be replaced by:

$$\begin{aligned} &= \int_0^\infty \frac{1}{\sqrt{\sigma_i \sigma_j}} \epsilon_i^* e^{A^* t} C^* C e^{A t} \epsilon_j dt \\ &= \frac{1}{\sqrt{\sigma_i \sigma_j}} \epsilon_i^* \left(\int_0^\infty e^{A^* t} C^* C e^{A t} dt \right) \epsilon_j \\ &= \frac{1}{\sqrt{\sigma_i \sigma_j}} \epsilon_i^* \Sigma \epsilon_j = \delta_{ij} \end{aligned}$$

18. Page 129, line 18: Expression (3.4.12) should be replaced by:

$$\epsilon_n^* (\Sigma \tilde{A} + \tilde{A}^* \Sigma) \epsilon_n + \epsilon_n^* (j\alpha + A^*)^{-1} C^* C (A - j\alpha)^{-1} \epsilon_n = 0$$

19. Page 131, line 6: Expression (3.4.21) should be replaced by:

$$(sI - A_{11})^{-1} A_{12} = (\bar{A}_{22} + \bar{A}_{23} \bar{T}_2, \bar{A}_{23}, I, O)$$

20. Page 147, line 19: "(See _)" should read "See section 3.4".

21. Page 155, line 3: Equation number (4.16) should be omitted.

22. Page 164, line 19: "uniqueness" should read "multiplicity".

23. Page 164, line 20: Phrase "no finite cost solution exists" should be replaced by "a finite cost solution may fail to exist".

24. Page 167, line 1: "the largest Hankel singular value of $G_1(s)$ is equal to one" should read "the $(k+1)$ th Hankel singular value of $G_1(s)$ is strictly less than one".

25. Page 172, line 18: "[4]" should read "[G3]".

26. Page 196, line 14: ")" should read "3)"

27. Page 196, line 18: " Q_+ " should read " $[Q_{i+1}]_+$ ".

28. Page 197, line 10: Expression (3.13.1) should be replaced by:

$$\bar{\sigma}[\Delta P(j\omega)] \leq |W(j\omega)| \quad \text{for all } 0 \leq \omega < \infty$$

29. Page 203, line 3: "3.4.35" and "3.4.36" should read "3.4.31" and "3.4.41" respectively.

References corrections and updatings

[B1] has appeared in the Int. J. Control, Vol. 46, pp. 785-816, 1987.

[B2] has appeared in Systems and Control Letters, 9, (1987), pp. 105-115.

[B5] has appeared in Integral Equations and Operator Theory, 10, (1987), pp. 416-436.

[F6] has appeared in SIAM J. Control and Optimization, Vol. 25, No. 4, July 1987.

[G5] has been accepted for publication in SIAM J. Control and Optimization.

[G7] has appeared in the Int. J. Control, 1989, Vol. 50, No 3, pp-899-916.

[G8] has been accepted for publication in SIAM J. Control and Optimization.

[H2] is under preparation.

[L1] Certain results have also appeared in Latham G. A. and B.D.O. Anderson, "*Frequency weighted optimal Hankel-norm approximations of stable transfer functions*", Systems and Control Letters, 5, (1985), pp. 229-236.

[M1] has also appeared in Systems and Control Letters, Vol.12, No.3, April 1989.

[N1] "1984" should read "1957".

[O2] Correct reference is: S.D. O' Young and B. A. Francis, "*Optimal performance and robust stabilization*", Automatica, Vol. 22, No. 2, 1986, pp. 171-183.

[P2] has also appeared in the IEEE Trans. Auto. Control, Vol. 35, No 2, February 1988, pp. 177-85.

[S2] has been published in the Int. J. Control, Vol 45, pp. 817-842, 1987.

[S7] has appeared in the Proc. IEEE CDC, Los Angeles, CA, Dec. 1987.

[S8] has appeared in the IEEE Trans. Auto. Control, Vol. 35, No 3, March 1990.

[T1] has appeared in Mathematics of Control, Signals and Systems, 1988.

Reference "[P2] I. Postlethwaite, D.W. Gu and S.D. O'Young "*A state-space approach to super-optimal $\mathcal{H}^\infty$ Control problems*", submitted for publication, 1988" should be replaced by "[T2] Tsai M.C., D.W. Gu and I. Postlethwaite, "*A state-space approach to super-optimal $\mathcal{H}^\infty$ Control problems*", IEEE Trans. Auto. Control, Vol. 33, No. 9, September 1988, pp. 833-43".

$$\left[\begin{array}{cc|c} -\tilde{A}_{11}^* & \tilde{A}_{12} + \tilde{B}_{11}\tilde{B}_{21}^* + \tilde{C}_{11}^*(\tilde{C}_{12} + D\tilde{B}_{21}^*) & \tilde{C}_{11}^*D + \tilde{B}_{11} \\ 0 & -\tilde{A}_{22}^* & -\tilde{C}_{22}^* \\ \hline -\tilde{B}_{12}^* & \tilde{C}_{12} + D\tilde{B}_{21}^* & D \end{array} \right] \quad (2.4.11)$$

Proof:

See [L6]

□

Lemma 2.4.2

Suppose that

$$\left[\begin{array}{cc} G & A_1 \\ A_2 & 0 \end{array} \right] = \left[\begin{array}{cc|cc} A_{11} & A_{12} & B_{11} & B_{12} \\ 0 & A_{22} & B_{21} & 0 \\ \hline C_{11} & C_{12} & D & I \\ 0 & C_{22} & I & 0 \end{array} \right] \quad (2.4.12)$$

in which $A_1(s)$ and $A_2(s)$ are assumed inner, with their corresponding realisations $A_1(s) = C_{11}(sI - A_{11})^{-1}B_{12} + I$ and $C_{22}(sI - A_{22})^{-1}B_{21} + I$ balanced, is factored as

$$\left[\begin{array}{cc} G & A_1 \\ A_2 & 0 \end{array} \right] = \left[\begin{array}{cc} A_l & 0 \\ 0 & I \end{array} \right] \left[\begin{array}{cc} \tilde{G} & \tilde{A}_1 \\ \tilde{A}_2 & 0 \end{array} \right] \left[\begin{array}{cc} A_r & 0 \\ 0 & I \end{array} \right] \quad (2.4.13)$$

in which $A_r(s)$ and $A_l(s)$ are inner, $(\tilde{G}, \tilde{A}_1)$ is an all-pass left coprime pair and $(\tilde{G}, \tilde{A}_2)$ an all-pass right coprime pair. Suppose also that

$$\begin{bmatrix} I & 0 \end{bmatrix} C_{12} = 0 \quad (2.4.14)$$

and

$$A_{12} + C_{11}^* C_{12} = 0 \quad (2.4.15)$$

Then, there exists a state space realisation for

$$\left[\begin{array}{cc} \tilde{G} & \tilde{A}_1 \\ \tilde{A}_2 & 0 \end{array} \right] = \left[\begin{array}{cc|cc} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_{11} & \tilde{B}_{12} \\ 0 & \tilde{A}_{22} & \tilde{B}_{21} & 0 \\ \hline \tilde{C}_{11} & \tilde{C}_{12} & D & I \\ 0 & \tilde{C}_{22} & I & 0 \end{array} \right] \quad (2.4.16)$$

such that the realisations for $\tilde{A}_1(s) = \tilde{C}_{11}(sI - \tilde{A}_{11})^{-1}\tilde{B}_{12} + I$ and $\tilde{C}_{22}(sI - \tilde{A}_{22})^{-1}\tilde{B}_{21} + I$ are minimal and balanced; in addition

$$(i) [0 \ I] \tilde{C}_{12} = 0 \quad (2.4.17)$$

$$(ii) \tilde{A}_{12} + \tilde{C}_{11}^* \tilde{C}_{12} = 0 \quad (2.4.18)$$

Proof:

Consider a realisation of

$$\begin{bmatrix} G & A_1 \\ A_2 & 0 \end{bmatrix} \quad (2.4.19)$$

in the state space of (2.4.12). Conditions (2.4.14) and (2.4.15) imply that in the notation of Theorem 2.4.1,

$$[0 \ I] [\tilde{C}_{12} \ \tilde{C}_{13}] = 0 \quad (2.4.20)$$

and

$$\begin{bmatrix} \tilde{A}_{01} & \tilde{A}_{03} \\ \tilde{A}_{12} & \tilde{A}_{13} \end{bmatrix} + \begin{bmatrix} \tilde{C}_{10}^* \\ \tilde{C}_{11}^* \end{bmatrix} \begin{bmatrix} \tilde{C}_{12} & \tilde{C}_{13} \end{bmatrix} = 0 \quad (2.4.21)$$

Since

$$\begin{bmatrix} \tilde{G} & \tilde{A}_1 \\ \tilde{A}_2 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{B}_{11} & \tilde{B}_{12} \\ 0 & \tilde{A}_{22} & \tilde{B}_{21} & 0 \\ \hline \tilde{C}_{11} & \tilde{C}_{12} & D & I \\ 0 & \tilde{C}_{22} & I & 0 \end{array} \right] \quad (2.4.23)$$

the required relations

$$[0 \ I] \tilde{C}_{12} = 0 \quad (2.4.24)$$

and

$$\tilde{A}_{12} + \tilde{C}_{11}^* \tilde{C}_{12} = 0 \quad (2.4.25)$$

may be obtained from the (1,1) block of (2.4.20) and the (2,1) block of (2.4.21) respectively. Note also that the (2,2) block of (4.36) in the proof of Theorem 4.1 given in [L6] implies that the realisation of

$$\tilde{A}_1(s) = \begin{bmatrix} \tilde{A}_{11} & \tilde{B}_{12} \\ \tilde{C}_{11} & I \end{bmatrix} \quad (2.4.26)$$

is minimal and balanced. Similarly for the realisation of $\tilde{A}_2(s)$ given in (2.4.23). $\square$

We conclude this section by establishing an upper bound t_b on the MacMillan degree of the optimal interpolation function. This shows that there is always such a function requiring no more than $\text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) - 1$ states. Since by Lemma 2.1.3(a) the number of zeros of $P_{21}(s)$ in $\mathbb{C}_+$ (the number of right interpolation constraints) is equal to $\text{rank}(\mathfrak{Y})$ while the number of the left interpolation constraints (in the sense given in Remark 2.1.1) is equal to $\text{rank}(\mathfrak{X})$, this Lemma is a generalisation of the result which bounds the degree of the optimal interpolating function in problems of the first kind by the total number of interpolation constraints (counted according to multiplicity) minus one [L5], [L6].

Lemma 2.4.3:

An upper bound on the MacMillan degree of any $\mathcal{H}^\infty$ -optimal interpolation function $\mathfrak{R}(s)$, is given by $t_b = \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) + \deg(U) - 1$, where $\mathfrak{X}$ and $\mathfrak{Y}$ are the stabilising solutions of the Riccati equations (2.1.2a) and (2.1.2b) respectively and U is an arbitrary contraction (which may be chosen as constant).

Proof:

From the factorisation given in (2.4.1), the interpolating function $\mathfrak{R}(s)$ can be written as

$$\mathfrak{R}(s) = A_l(s) \left(\tilde{T}_{11} + [\tilde{T}_{12} \ \tilde{T}_{1\perp}] \begin{bmatrix} Q \\ 0 \end{bmatrix} \tilde{T}_{21} \right) A_r(s) \quad (2.4.27)$$

$$=: A_l(s) \tilde{\mathfrak{R}}(s) A_r(s) \quad (2.4.28)$$

Consequently,

$$\{\text{poles } \mathfrak{R}(s)\} \subseteq \{\text{poles } A_l(s)\} \cup \{\text{poles } \tilde{\mathfrak{R}}(s)\} \cup \{\text{poles } A_r(s)\} \quad (2.4.29)$$

After extracting the maximal all-pass common factors $A_l(s)$ and $A_r(s)$, we can ensure from Lemma 2.4.2 that a state space realisation of

$$\begin{bmatrix} \tilde{T}_{11} & \tilde{T}_{12} & \tilde{T}_{1\perp} \\ \tilde{T}_{21} & 0 & 0 \end{bmatrix} \quad (2.4.30)$$

satisfies the conditions of Theorem 2.2.1. Consequently, we conclude that

$$\{\text{poles } \tilde{\mathfrak{R}}(s)\} \subseteq \{\text{poles } \tilde{X}(s)\} \quad (2.4.31)$$

where $\tilde{X}(s)$ is any optimal Nehari extension of $[R_1 M^{-1}]_-$, with

$$\deg(R_1 M^{-1})_- \leq \deg \begin{bmatrix} R_1 \\ R_2 \end{bmatrix} = \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) - \deg(A_r) - \deg(A_l) \quad (2.4.32)$$

Hence from [G1],

$$\deg[\tilde{\mathcal{R}}(s)] \leq \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) + \deg(U) - 1 \quad (2.4.33)$$

where U is an arbitrary unstable contraction describing all the degrees of freedom in the approximation problem $\inf \|R_1 M^{-1} + \tilde{X}\|_\infty$ (which can be chosen constant if a minimum-degree extension is required). Inequality (2.4.33) establishes the upper bound t_b on the degree of the optimal interpolating function. $\square$

2.5 The Controller degree bound

A bound on the MacMillan degree of the controller can now be proved by combining the bounds t_b and c_b .

Theorem 2.5.1

For any $\mathcal{H}^\infty$ -optimal control problem of the second kind, every $\mathcal{H}^\infty$ -optimal controller satisfies

$$(1) \deg(K) \leq n + \deg(U) - 1 \quad (2.5.1)$$

and every suboptimal controller ($\|\mathcal{R}(s)\|_\infty > \gamma_{opt}$) satisfies

$$(2) \deg(K) \leq n + \deg(U) \quad (2.5.2)$$

In (2.5.1) and (2.5.2) $U(s) \in \mathbb{R}\mathcal{H}_-^\infty$ is an arbitrary matrix contraction of specified dimensions, which may be chosen constant (or even zero).

Proof:

Follows directly from Lemmas 2.3.3 and 2.4.3. $\square$

2.6 Example

In this section we will illustrate the cancellation results of this chapter by means of a small computer example. Rather than working in an arbitrary generalised regulator setting (for which the cancellation analysis would also be valid), a two-objective sensitivity/robustness problem will be described.

Let a single input/output plant $g(s)$ be described by the model

$$g(s) = \frac{s^2+7s+22}{s^2-2s-15} = \frac{s^2+7s+22}{(s+3)(s-5)} \quad (2.6.1)$$

which will be assumed to be accurate at low frequencies only (say below 10 rads/s); above this frequency model uncertainty will be represented by an unknown additive component $\delta(s)$. Suppose that it is required to design a compensator $k(s)$ that minimises the system's sensitivity at frequencies below 1 rad/s (in order to achieve, for example, good tracking for command inputs and a high disturbance rejection of signals at the output of the plant in this frequency range); in addition, suppose that the controller is required to achieve closed-loop stability not only for the nominal plant $g(s)$, but also for a whole family of 'perturbed' plants $\{g(s)+\delta(s) \mid \|\delta(j\omega)\|_\infty \leq \mu\}$ for an (unspecified) uncertainty level μ .

An $\mathcal{H}^\infty$ optimisation problem corresponding to the two design objectives may be formulated as

$$\inf_{K \in \Xi} \left\| \begin{bmatrix} \frac{\rho w_1 k}{1+gk} \\ \frac{w_2}{1+gk} \end{bmatrix} \right\|_\infty = \gamma(\rho) \quad (2.6.2)$$

This is a combined weighted sensitivity/additive-robustness minimisation over the set Ξ of all (internally) stabilising compensators $k(s)$. ρ is a scalar non-dynamic parameter which describes the relevant emphasis between the two design objectives in the minimisation. $w_1(s)$ and $w_2(s)$ are weighting functions which will be chosen to reflect the relevant frequency ranges over which the two objective functions will be minimised. We will choose $w_1(s)$ as the high-pass filter

$$w_1(s) = \frac{s+1}{s+10} \quad (2.6.3)$$

achieving a relative penalty ratio of 1/10 between low and high frequencies, with the 10 rad/s frequency (above which the high-frequency model uncertainty is concentrated) being 3db only below the maximum penalty level. For analogous reasons $w_2(s)$ will be chosen as the low pass filter

$$w_2(s) = \frac{0.1s+1}{s+1} \quad (2.6.4)$$

It can be easily verified that the optimisation problem (2.6.2) may be reformulated in the generalised regulator configuration as

$$\inf_{K \in \Xi} \|\mathcal{F}_\ell(P, K)\|_\infty = \inf_{K \in \Xi} \left\| \mathcal{F}_\ell \left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, K \right) \right\|_\infty \quad (2.6.5)$$

where

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \left[\begin{array}{c|c} w_2 & w_2 g \\ \hline 0 & w_1 \\ \hline -1 & -g \end{array} \right] \quad (2.6.6)$$

Since $P_{21}(s)=-1$ and $P_{12}(s) = [w_2 g \ w_1]^T \in \mathbb{R}^{2 \times 1}(s)$ this problem belongs to the class of $\mathcal{H}^\infty$ control problems of the second kind.

The first part of the algorithm described in Chapter 1, uses constant input and output scalings on $P(s)$ to make the D-matrices of $P_{12}(s)$ and $P_{21}(s)$ orthogonal and part of an orthogonal matrix respectively [D4]. In our case, an input scaling

$$P_{sc}(s) = P(s) \begin{bmatrix} 1 & 0 \\ 0 & s_1 \end{bmatrix} \quad (2.6.7)$$

with $s_1=0.995037$ achieves this effect. A state-space realisation of $P_{sc}(s)$ (for $\rho=1$) may be obtained as

$$\begin{aligned} P_{sc}(s) &= \begin{bmatrix} P_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 1 & 0.9 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & -10 & -9 & 0 \end{bmatrix} \begin{bmatrix} s+1 & -3 & -2.7 & 0 \\ 0 & s+3 & -1 & 0 \\ 0 & 0 & s-5 & 0 \\ 0 & 0 & 0 & s+10 \end{bmatrix}^{-1} \begin{bmatrix} 0.3 & 0.2985 \\ 0 & 0 \\ 0 & 0.9950 \\ 0 & -2.9851 \end{bmatrix} \\ &+ \begin{bmatrix} 0.1 & 0.09950 \\ 0 & 0.99504 \\ 1 & -0.99504 \end{bmatrix} := \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} (sI-A)^{-1} \begin{bmatrix} B_1 & B_2 \end{bmatrix} + \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \end{aligned} \quad (2.6.8)$$

The optimal controller will be calculated for $P_{sc}(s)$ and subsequently the scaling will be reversed to obtain the optimal controller for the original optimisation problem; clearly, no MacMillan degree inflation or deflation is involved in this process.

The solutions to the two Riccati equations (1.5.6) and (1.5.7) were obtained as

$$\mathfrak{X} = \begin{bmatrix} 1.9629 & 1.2713 & 0.9253 & -0.1499 \\ 1.2713 & 0.9627 & 0.8267 & -0.1297 \\ 0.9253 & 0.8267 & 2.5705 & -0.9498 \\ -0.1499 & -0.1297 & -0.9498 & 0.4232 \end{bmatrix} = \mathfrak{X}^T \quad (2.6.9)$$

and

$$\mathfrak{Y} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.1487 \cdot 10^{-2} & 0.1190 \cdot 10^{-1} & 0 \\ 0 & 0.1190 \cdot 10^{-1} & 0.9518 \cdot 10^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathfrak{Y}^T \quad (2.6.10)$$

$\mathfrak{X}$ is a full rank matrix (eigenvalues 0.0108, 0.1109, 1.6859 and 4.1117). From Lemma 2.1.3ib], therefore, we expect all the stable modes of $A - B_2 D_{12}^* C_1$ to be detectable through $D_\perp^* C_1$. Indeed, $A - B_2 D_{12}^* C_1$ can be shown to have two stable and two unstable eigenvalues while the matrix

$$\begin{bmatrix} \lambda_i - A + B_2 D_{12}^* C_1 \\ D_\perp^* C_1 \end{bmatrix} \quad (2.6.11)$$

to have full column rank for either stable eigenvalue λ_i of $A - B_2 D_{12}^* C_1$. The rank of matrix $\mathfrak{Y}$ was found to be one (eigenvalues 0.0967, 0, 0, 0) and consequently we expect from Lemma 2.1.3.ia] a single non-minimum phase zero in the state-space realisation of $P_{21}(s)$. This is in fact the case for $s=5$ as can be verified from the state space realisation in (2.6.8). Lemma 2.1.3 also shows that minimal realisations of $T(s)$, $[T_{12} \ T_\perp](s)$ and $T_{21}(s)$ have MacMillan degrees $\text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y})$, $\text{rank}(\mathfrak{X})$ and $\text{rank}(\mathfrak{Y})$ respectively, while $R(s)$ has a minimal realisation of degree bounded by $\text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y})$. From minimal realisations of the various partitions of $T(s)$ we obtain for our example:

(a) $T_{11}(s)$ has five stable poles. $\{-5, -4.65 \pm j 0.7613, -1.959 \pm j 2.458\}$ with corresponding Hankel singular values $\{0.857, 0.5815, 0.3134, 0.803 \cdot 10^{-1}, 0.2623 \cdot 10^{-2}\}$

(b) $[T_{12} \ T_\perp](s)$ has four stable poles $\{-4.65 \pm j 0.7613, -1.959 \pm j 2.458\}$ with Hankel singular values $\{1, 1, 1, 1\}$. (This also verifies that $[T_{12} \ T_\perp]$ is all-pass).

(c) $T_{21}(s)$ has a single stable pole at $s=-5$ with a corresponding Hankel singular value equal to one.

(d) Forming the product

$$R(s) = \begin{bmatrix} T_{12}^* \\ T_\perp^* \end{bmatrix} T_{11} T_{21}^*(s) \quad (2.6.12)$$

we obtain the Hankel singular values of $R(s)$ as $\{0.53868, 0.21339, 0.15706, 0.11873, 0.26 \cdot 10^{-8}, 0, 0, 0\}$ implying that $\deg(R(s))=4 < \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y})=5$ in agreement with the bound in Lemma 2.1.3v].

Lemma 2.1.3 has therefore been verified in its entirety. It can be also seen from (d) above that the bound on $\deg(R)$ is not tight; this indicates that an all-pass common factor exists in one of the pairs $(T_{11}, [T_{12} \ T_\perp])$ or (T_{11}, T_{21}) causing the cancellation of the corresponding mode either in the

product

$$\begin{bmatrix} T_{12}^* \\ T_{\perp}^* \end{bmatrix} T_{11} \quad (2.6.13)$$

or in $T_{11}T_{21}^*(s)$. Indeed it may be verified from the realisations of $T_{11}(s)$ and $T_{21}(s)$ that the scalar all pass common factor $(s-5)/(s+5)$ may be extracted from both $T_{11}(s)$ and $T_{21}(s)$; this accounts for the extra cancellation in $R(s)$.

In fact, it may be shown that for the class of $\mathcal{H}^\infty$ problems considered in this example, the bound on the degree of $R(s)$ given in Lemma 2.1.3v] as $\deg(R) \leq \text{rank}(\mathcal{Y}) + \text{rank}(\mathcal{X})$ may be tightened to $\deg(R) \leq \text{rank}(\mathcal{X})$. This is due to the fact that an all-pass right common factor can always be extracted from $T_{11}(s)$ and $T_{21}(s)$ (containing the modes of W_1 and the unstable poles of G). To show why this is the case consider the general weighted sensitivity/additive robustness $\mathcal{H}^\infty$ problem

$$\inf_{K \in \Xi} \left\| \begin{bmatrix} W_2(I+GK)^{-1}W_1 \\ W_3K(I+GK)^{-1}W_1 \end{bmatrix} \right\|_\infty \quad (2.6.14)$$

in the usual notation. (2.6.14) may be transformed to the standard regulator setting as

$$\inf_{K \in \Xi} \|\mathcal{F}_\ell(P, K)\|_\infty = \inf_{K \in \Xi} \left\| \mathcal{F}_\ell \left(\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, K \right) \right\|_\infty \quad (2.6.15)$$

where

$$\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} W_2W_1 & W_2G \\ 0 & W_3 \\ W_1 & G \end{bmatrix} \quad (2.6.16)$$

(2.6.16) is equivalent to

$$\inf_{K \in \Xi} \|\mathcal{F}_\ell(P, K)\|_\infty = \inf_{Q \in \mathcal{H}^\infty} \|T_{11} - T_{12}QT_{21}\|_\infty \quad (2.6.17)$$

in which

$$T_{11} = P_{11} - P_{12}U_lD_lP_{21} \quad (2.6.18)$$

$$T_{12} = P_{12}D_r \quad (2.6.19)$$

and

$$T_{21} = D_lP_{21} \quad (2.6.20)$$

where G admits left and right coprime factorisations $G(s) = N_rD_r^{-1} = D_l^{-1}N_l$. Also, the Bezout identity

$$\begin{bmatrix} V_r & U_r \\ -N_l & D_l \end{bmatrix} \begin{bmatrix} D_r & -U_l \\ N_r & V_l \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (2.6.21)$$

is satisfied for some matrices $U_r(s)$, $U_l(s)$, $V_r(s)$ and $V_l(s)$ in $\mathcal{H}_+^\infty$. In our case, therefore,

$$T_{11} - T_{12}Q T_{21}(s) = \begin{bmatrix} W_2 & 0 \\ 0 & W_3 \end{bmatrix} \left\{ \begin{bmatrix} I - G U_l D_l \\ -U_l D_l \end{bmatrix} - \begin{bmatrix} G \\ I \end{bmatrix} D_r Q D_l \right\} W_1 \quad (2.6.22)$$

$$= \begin{bmatrix} W_2 & 0 \\ 0 & W_3 \end{bmatrix} \left\{ \begin{bmatrix} D_l^{-1}(I - N_l U_l) D_l \\ -U_l D_l \end{bmatrix} - \begin{bmatrix} N_r \\ D_r \end{bmatrix} Q D_l \right\} W_1 \quad (2.6.23)$$

$$= \begin{bmatrix} W_2 & 0 \\ 0 & W_3 \end{bmatrix} \begin{bmatrix} V_l \\ -U_l \end{bmatrix} D_l W_1 - \begin{bmatrix} W_2 & 0 \\ 0 & W_3 \end{bmatrix} \begin{bmatrix} N_r \\ D_r \end{bmatrix} Q D_l W_1 \quad (2.6.24)$$

on using the (2,2) block of (2.6.21). Consequently,

$$T_{11}(s) = \begin{bmatrix} W_2 & 0 \\ 0 & W_3 \end{bmatrix} \begin{bmatrix} V_l \\ -U_l \end{bmatrix} D_l W_1 \quad (2.6.25)$$

$$T_{12}(s) = \begin{bmatrix} W_2 & 0 \\ 0 & W_3 \end{bmatrix} \begin{bmatrix} N_r \\ D_r \end{bmatrix} \quad (2.6.26)$$

and

$$T_{21}(s) = D_l W_1 \quad (2.6.27)$$

In addition, the left and right coprime factorisations may be arranged so that $T_{21}(s) = D_l W_1$ is inner and $T_{12}(s)$ is part of an inner matrix function [D4]. This implies, however, that the whole of $T_{21}(s)$ can be extracted as a (maximal) right all-pass common factor in a factorisation

$$\begin{bmatrix} T_{11} \\ T_{21} \end{bmatrix} = \begin{bmatrix} \tilde{T}_{11} \\ I \end{bmatrix} D_l W_1 \quad (2.6.28)$$

where

$$\tilde{T}_{11}(s) = \begin{bmatrix} W_2 & 0 \\ 0 & W_3 \end{bmatrix} \begin{bmatrix} V_l \\ -U_l \end{bmatrix} \quad (2.6.29)$$

Consequently, for $A_l(s)$ and $A_r(s)$ chosen as the maximal all-pass left and right common factors of $(T_{11}, [T_{12} \ T_{21}])$ and (T_{11}, T_{21}) respectively, Theorem 2.4.1 shows that

$$\begin{aligned}
 \deg(R) &= \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) - \text{rank}(A_l) - \text{rank}(A_r) \\
 &\leq \text{rank}(\mathfrak{X}) + \text{rank}(\mathfrak{Y}) - \deg(D_l W_1) \\
 &= \text{rank}(\mathfrak{X})
 \end{aligned} \tag{2.6.30}$$

since, from Lemma 2.1.3[iii], $\deg(T_{21}) = \deg(D_l W_1) = \text{rank}(\mathfrak{Y})$. This establishes the bound for $\deg(R)$ in the special class of $\mathfrak{H}^\infty$ sensitivity/additive robustness problems and verifies the drop in the MacMillan degree in $R(s)$ by the number of unstable poles of $G(s)$ in the unweighted problem; this is due to the fact that the rank of the Riccati solution $\mathfrak{Y}$ is equal to the number of non minimum phase zeros of $D_l(s)$ which is equal to the number of unstable poles of $G(s)$.

The γ -iteration for the problem we are considering has been performed with an accuracy level for $\mu = \|R_1 M^{-1}\|_H$ equal to 10^{-7} . The Hankel singular values of $R_1 M^{-1}(\gamma_0)$ for the optimal $\gamma_0 = 0.5387$ are given by $\{1.0, 0.21897, 0.02876 \cdot 10^{-1}, 0.11038 \cdot 10^{-2}\}$ indicating that $\deg[(R_1 M^{-1}(\gamma_0))_-] = 4$. (Note that after the extraction of the all-pass common factor in

$$T(s) = \begin{bmatrix} \frac{s-5}{s+5} & 0 \\ 0 & 1 \end{bmatrix} \tilde{T}(s) \tag{2.6.31}$$

the MacMillan degree of $\tilde{T}(s)$ (for which Theorem 2.2.1 is applied) is equal to 4). The eigenvalues of the (unique in this case) Nehari extension of $[R_1 M^{-1}(\gamma_0)]_-$, $Q(s)$ are given by $\{-3.4607, -1.899 \pm j 1.7468\}$ with corresponding Hankel singular values $\{0.1341, 0.1929 \cdot 10^{-1}, 0.62106 \cdot 10^{-3}\}$ indicating a MacMillan degree $\deg(Q) = 3$ in agreement with minimal-order Nehari extension results [G1]. Finally, the Hankel singular values of the $\mathfrak{H}^\infty$ optimal controller (obtained by taking $Q(s) + [R_1 M^{-1}(\gamma_0)]_+ := \tilde{Q}(s)$ as the feedback term in the linear fractional map defined in Figure 1 and subsequently reversing the effects of scaling) were obtained as

$$\left\{ 0.33682, 0.13264, 0.60817 \cdot 10^{-1}, 0.6988 \cdot 10^{-9}, 0.15316 \cdot 10^{-11}, 0, 0 \right\}$$

indicating a MacMillan degree for the controller $\deg[k_0(s)] = 3$ in agreement with Theorem 2.5.1. It may be verified, in addition, that

$$f(k_0(s)) := \left\| \begin{bmatrix} w_1 k_0(1+gk_0)^{-1} \\ w_2(1+gk_0)^{-1} \end{bmatrix} \right\|_\infty = \gamma_0 = 0.5387 \tag{2.6.32}$$

with

$$\begin{bmatrix} w_1 k_0(1+gk_0)^{-1} \\ w_2(1+gk_0)^{-1} \end{bmatrix} \tag{2.6.33}$$

all-pass at that level.

2.7 Computation time trials and conclusions

In this section some quantitative data are presented which demonstrate the importance of removing cancellation phenomena from $\mathcal{H}^\infty$ computer software. A number of test computations have been performed on both the original software, and an improved program which takes into account the cancellation phenomena predicted in this chapter. Both programs were run on a VAX750 computer under UNIX. The timing data was obtained using the UNIX routine *dtime*. In every case the γ -iteration was allowed to run until the solution was almost optimal: The computation exited from the iterative loop when μ was in the interval $1 - 0.5 \cdot 10^{-5} \leq \mu \leq 1$.

The results given in Table 2.1 show the state dimension of $P(s)$, the execution time of the original program and the execution time of the improved program (both in seconds). Apart from a marked reduction in computation time, the improved program demonstrated improved robustness properties.

<i>No of states</i>	<i>Program I</i>	<i>Program II</i>
1	4.517s	18.23s
2	11.617s	67.95s
3	53.417s	406.98s
4	102.470s	442.32s
6	141.430s	692.98s
8	215.770s	1611.42s
14	1128.480s	25273.70s
17	1703.470s	terminated after 12 hours of processor time

Table 2.1

Experience suggests that the solution of large $\mathcal{H}^\infty$ control problems is time consuming especially when several iterations corresponding to various weight selections are required. Further, long calculations of this type are susceptible to severe numerical difficulties. The results in this chapter suggest that there is considerable scope for reducing these problems by using cancellation theory to remove state inflation effects from computer code. Although the level of benefit varies from problem to problem, a cpu time reduction of between 5 and 10 times is easy to achieve. The modified code is also considerably more robust from a numerical point of view.

Appendix 2.A: Proof of Lemma 2.2.2

Lemma 2.2.2

The Riccati equation (2.2.42) has a unique positive definite stabilising solution.

Proof:

Since $\bar{A}$ is asymptotically stable, $[\bar{A}, N]$ is stabilisable. $\gamma > \|R_2(s)\|_\infty \Rightarrow \gamma^2 I - \bar{D}_2 \bar{D}_2^* = E$ is positive definite and

$$[sI - \bar{A} \mid N] \begin{bmatrix} I & 0 \\ -E^{-1}\bar{C} & E^{-\frac{1}{2}} \end{bmatrix} = [sI - \bar{A} - NE^{-1}\bar{C} \mid NE^{-\frac{1}{2}}]$$

ensures that $[\bar{A} + NE^{-1}\bar{C}, NE^{-\frac{1}{2}}]$ is also stabilisable. The Hamiltonian matrix associated with (2.2.42) is

$$H = \begin{bmatrix} \bar{A} + NE^{-1}\bar{C} & NE^{-1}N^* \\ -\bar{C}^* E^{-1}\bar{C} & -(\bar{A} + NE^{-1}\bar{C})^* \end{bmatrix}$$

and we note that

$$\{\lambda(H)\} \subseteq \{\text{Smith-MacMillan zeros of } (\gamma^2 I - R_2^* R_2)(s)\} \cup \{\lambda_i(\bar{A})\} \cup \{\lambda_i(-\bar{A})\} \quad (A.1)$$

Since $\gamma > \|R_2(s)\|_\infty$ no Smith-MacMillan zero of $(\gamma^2 I - R_2^* R_2)(s)$ lies on the imaginary axis. This together with the asymptotic stability of $\bar{A}$ ensures that H is free of imaginary axis eigenvalues. Finally, the stabilisability of $[\bar{A} + NE^{-1}\bar{C}, NE^{-1/2}]$ ensures the existence of a unique stabilising solution $Y = Y^* \geq 0$ to (2.2.42). Part (a) in the theorem statement ensures that $[\bar{A}, E^{-1/2}\bar{C}]$ is observable $\Rightarrow Y$ is nonsingular or else that the unique stabilising solution satisfies $Y = Y^* > 0$. $\square$

Appendix 2.B: Proof of Lemma 2.2.3

Lemma 2.2.3

In the state-space realisation of $T_{12}XT_{21}(s)$ in (2.2.62),

$$\Pi = (\Theta_{12} - Q_{12})B_{21} - B_{11}$$

Proof:

The change of basis (2.2.61) in the state-space of (2.2.60) gives

$$\Pi = B_{12}(\hat{D}E^{\frac{1}{2}} - \bar{D}_1^*) - Q_{13}(\hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2) \quad (B.1)$$

Making use of the (1,1) block of (2.2.54) yields

$$\Pi = Q_{11}F_1^*E + Q_{12}F_2^*E - B_{12}D_{12}^*D_{11} - Q_{13}P_{13}^*N_1 - Q_{13}P_{23}^*N_2 \quad (B.2)$$

From the (1,1) and (1,2) blocks of (2.2.18), and (2.2.56) along with the defining equations of $\bar{C}$ (2.2.13) and F (2.2.45), we get

$$\Pi = -N_1 + Q_{11}(C_{11}^*D_{11} + B_{11}) + Q_{12}C_{22}^* - B_{12}D_{12}^*D_{11} \quad (B.3)$$

(2.2.57) gives

$$\Pi = -C_{11}^*D_{11} - B_{11} + (Q_{12} - \Theta_{12})C_{22}^* - B_{13}D_{13}^*D_{11} - B_{12}D_{12}^*D_{11} \quad (B.4)$$

Finally, from (2.2.28) we get

$$\Pi = -B_{11} + (Q_{12} - \Theta_{12})C_{22}^* \quad (B.5)$$

as required. $\square$

Appendix 2.C: Proof of Lemma 2.2.4

Lemma 2.2.4:

In the state space realisation of $T_{12}XT_{21}(s)$ in (2.2.63), $\Sigma = -A_{12}$.

Proof:

After the coordinate change (2.2.61) in (2.2.60) we get

$$\Sigma = (Q_{12} - \Theta_{12})A_{22} - A_{11}(Q_{12} - \Theta_{12}) + B_{12}(\hat{D}E^{\frac{1}{2}} - \bar{D}_1^*)C_{22} - Q_{13}(\hat{B}E^{\frac{1}{2}} + P_{13}^*N_1 + P_{23}^*N_2)C_{22} \quad (C.1)$$

Making use of (2.2.13) and the (1,1) block of (2.2.54) gives

$$\Sigma = (Q_{12} - \Theta_{12})A_{22} - A_{11}(Q_{12} - \Theta_{12}) - B_{12}D_{12}^*D_{11}C_{22} - Q_{13}(P_{13}^*N_1 + P_{23}^*N_2)C_{22} + (Q_{11}F_1^*E + Q_{12}F_2^*E)C_{22} \quad (C.2)$$

which can be expanded, using the defining equations (2.2.45) and (2.2.13) to

$$\begin{aligned} \Sigma = & (Q_{12} - \Theta_{12})A_{22} - A_{11}(Q_{12} - \Theta_{12}) + Q_{11}[C_{11}^*D_{11} + B_{11} + Y_{11}N_1 + Y_{12}N_2]C_{22} + \\ & Q_{12}[-B_{21} + Y_{12}^*N_1 + Y_{22}^*N_2]C_{22} - B_{21}D_{12}^*D_{11}C_{22} - Q_{13}(P_{13}^*N_1 + P_{23}^*N_2)C_{22} \end{aligned} \quad (C.3)$$

Next, using (2.2.56) to substitute the various partitions of Y by the corresponding partitions of P and simplifying by the (1,1) and (1,2) blocks of equation (2.2.18) we get,

$$\Sigma = (Q_{12} - \Theta_{12})A_{22} - A_{11}(Q_{12} - \Theta_{12}) - N_1C_{22} - Q_{12}B_{21}C_{22} + Q_{11}(C_{11}^*D_{11} + B_{11})C_{22} \quad (C.4)$$

which, on substituting the defining equation (2.2.40) for N_1 and N_2 and using (2.2.22) can be expanded to

$$\begin{aligned} \Sigma = & (Q_{12} - \Theta_{12})A_{22} - A_{11}(Q_{12} - \Theta_{12}) - \{B_{13}D_{13}^*D_{11} + \Theta_{11}(C_{11}^*D_{11} + B_{11}) + \Theta_{12}C_{22}^*\}C_{22} \\ & + Q_{11}(B_{11} + C_{11}^*D_{11})C_{22} - B_{12}D_{12}^*D_{11}C_{22} + Q_{12}(-A_{22} - A_{22}^*) \end{aligned} \quad (C.5)$$

Invoking (2.2.22) and (2.2.23) gives

$$\begin{aligned} \Sigma = & (Q_{12} - \Theta_{12})A_{22} - A_{11}(Q_{12} - \Theta_{12}) - (B_{13}D_{13}^* + B_{12}D_{12}^*)D_{11}C_{22} + (Q_{11} - \Theta_{11})(B_{11} + C_{11}^*D_{11})C_{22} + \\ & (Q_{12} - \Theta_{12})(-A_{22} - A_{22}^*) \end{aligned} \quad (C.6)$$

which, on cancelling terms and using (2.2.28) and (2.2.57) becomes

$$\Sigma = A_{11}(\Theta_{12} - Q_{12}) - (Q_{12} - \Theta_{12})A_{22}^* - B_{11}C_{22} \quad (C.7)$$

Finally, on substituting the (1,2) block of (2.236) and the (1,2) block of (2.253) and simplifying with (2.223) and (2.257) we obtain

$$\Sigma = B_{12}D_{12}^*C_{12} = C_{11}^*D_{12}D_{12}^*C_{12} = C_{11}^*C_{12} = -A_{12} \quad (C.8)$$

as required on using (2.226), (2.211), (2.229) and (2.212). $\square$

Appendix 2.D1: Proof of Lemma 2.2.5

Lemma 2.2.5:

There exists a nonsingular matrix T_1 transforming the Hamiltonian matrix corresponding to the Riccati equation in Lemma 2.A to

$$T_1 H T_1^{-1} = - \begin{bmatrix} -\tilde{H}_{11}^* & -\mathcal{A}^* & \tilde{H}_{21} & 0 \\ 0 & A_{22}^* & 0 & 0 \\ \tilde{H}_{12} & \mathfrak{B}^* & \tilde{H}_{11} & 0 \\ \mathfrak{B} & \mathfrak{D} & \mathcal{A} & -A_{22} \end{bmatrix}$$

where $\tilde{H}$ is the Hamiltonian matrix associated with the Riccati equation (4.65) partitioned as

$$\tilde{H} = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{12}^* & \tilde{H}_{22} \end{bmatrix}$$

In particular, the similarity transformation T_1 can be chosen as

$$T_1 = \begin{bmatrix} I & \Theta - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \\ 0 & I \end{bmatrix}$$

in which case,

$$\mathcal{A} := C_{22}^* E^{-1} D_{11}^* D_{13}$$

$$\mathfrak{B} := C_{22}^* E^{-1} (D_{11}^* C_{11} + B_{11}^*)$$

$$\mathfrak{D} := C_{22}^* E^{-1} C_{22}$$

in previously defined notation.

Proof:

Using the notation of Theorem 2.2.1, it suffices to show that:

$$\begin{aligned}
 & - \begin{bmatrix} I & \Theta - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{A} + NE^{-1}\bar{C} & NE^{-1}N^* \\ -\bar{C}^*E^{-1}\bar{C} & -(\bar{A} + NE\bar{C})^* \end{bmatrix} = \\
 & \begin{bmatrix} -\tilde{H}_{11}^* & -\mathcal{A}^* & \tilde{H}_{21} & 0 \\ 0 & A_{22}^* & 0 & 0 \\ \tilde{H}_{12} & \mathfrak{B}^* & \tilde{H}_{11} & 0 \\ \mathfrak{B} & \mathfrak{D} & \mathcal{A} & -A_{22} \end{bmatrix} \begin{bmatrix} I & \Theta - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \\ 0 & I \end{bmatrix} \quad (D1.1)
 \end{aligned}$$

This will be verified by treating each block separately. Equality in the (1,1) block of (D1.1) is equivalent to

$$\bar{A} + NE^{-1}\bar{C} - \left\{ \Theta - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right\} \bar{C}^*E^{-1}\bar{C} = \begin{bmatrix} \tilde{H}_{11}^* & \mathcal{A}^* \\ 0 & -A_{22}^* \end{bmatrix} \quad (D1.2)$$

or, using the defining equations for N in (2.2.40) to

$$\bar{A} + \bar{B}_2\bar{D}_2^*E^{-1}\bar{C} + \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \bar{C}^*E^{-1}\bar{C} = \begin{bmatrix} \tilde{H}_{11}^* & \mathcal{A}^* \\ 0 & -A_{22}^* \end{bmatrix} \quad (D1.3)$$

The (1,1) and (1,2) blocks of (D1.3) may be verified directly; (2.2.41) and (2.2.22) may be used to establish equality in the (2,2) block and while equation (4.41) will verify the equality in the (2,1) block.

For the (2,1) block of (D1.1) we note that

$$\bar{C}^*E^{-1}\bar{C} = \begin{bmatrix} C_{11}^*D_{11} + B_{11} \\ -B_{21} \end{bmatrix} E^{-1} \begin{bmatrix} D_{11}^*C_{11} + B_{11}^* & -B_{21}^* \end{bmatrix} = \begin{bmatrix} \tilde{H}_{12} & \mathfrak{B}^* \\ \mathfrak{B} & \mathfrak{D} \end{bmatrix} \quad (D1.4)$$

directly from the definitions of $\mathfrak{B}$ (2.2.68b) and $\tilde{H}_{12}$ using (2.2.23). This in turn implies that

$$(\bar{A} + NE^{-1}\bar{C})^* - \begin{bmatrix} \tilde{H}_{12} & \mathfrak{B}^* \\ \mathfrak{B} & \mathfrak{D} \end{bmatrix} \left\{ \Theta - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right\} - \begin{bmatrix} \tilde{H}_{11} & 0 \\ \mathcal{A} & -A_{22} \end{bmatrix} = \quad (D1.5)$$

$$(\bar{A} + NE^{-1}\bar{C})^* - \bar{C}^*E^{-1}\bar{C} \left\{ \Theta - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right\} - \begin{bmatrix} \bar{H}_{11} & 0 \\ \mathcal{A} & -A_{22} \end{bmatrix} = 0$$

by taking complex conjugate transposes in (D1.3) and using the defining equation (2.2.40), thus verifying the (2,2) block of (D1.1).

It remains to show that

$$-NE^{-1}N^* + \left\{ \Theta - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right\} (\bar{A} + NE^{-1}\bar{C})^* = \begin{bmatrix} \bar{H}_{11}^* & \mathcal{A}^* \\ 0 & -A_{22}^* \end{bmatrix} \left\{ \Theta - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \right\} + \begin{bmatrix} \bar{H}_{21} & 0 \\ 0 & 0 \end{bmatrix} \quad (D1.6)$$

The L.H.S. of this equation can be rewritten, using (2.2.40) and (2.2.36) as

$$-\bar{B}_2 \bar{D}_2^* E^{-1} N^* - \bar{A} \Theta - \bar{B}_2 \bar{B}_2^* - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} (\bar{A} + NE^{-1}\bar{C})^* \quad (D1.7)$$

It suffices, therefore, to show that

$$\begin{aligned} -\left\{ \bar{B}_2 \bar{D}_2^* E^{-1} \bar{C} + \bar{A} + \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \bar{C}^* E^{-1} \bar{C} \right\} \Theta - \bar{B}_2 [I + \bar{D}_2^* E^{-1} \bar{D}_2] \bar{B}_2^* - \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \bar{A}^* - \\ \begin{bmatrix} 0 & 0 \\ 0 & \gamma^2 I \end{bmatrix} \bar{C}^* E^{-1} \bar{D}_2 \bar{B}_2^* - \begin{bmatrix} \bar{H}_{21} & \gamma^2 \mathcal{A}^* \\ 0 & -\gamma^2 A_{22}^* \end{bmatrix} = 0 \end{aligned} \quad (D1.8)$$

The (1,1) block of (D1.8) is verified directly from the definition of $\bar{H}_{21}$. Finally equations block(1,2)=0, block(2,1)=0 and block(2,2)=0 follow on using the defining equations of $\mathcal{A}$ (2.2.68a) and (2.2.41).

□

Appendix 2.D: Proof of Lemma 2.2.6

Lemma 2.2.6

(a) There exists a unique "largest" solution Z_0 to the Riccati equation

$$\begin{aligned} \{A_{11} + B_{13} D_{13}^* D_{11} E^{-1} (D_{11}^* C_{11} + B_{11}^*)\} Z + Z \{A_{11} + B_{13} D_{13}^* D_{11} E^{-1} (D_{11}^* C_{11} + B_{11}^*)\}^* + \\ Z (C_{11}^* D_{11} + B_{11}) E^{-1} (D_{11}^* C_{11} + B_{11}^*) Z + B_{13} [I + D_{13}^* D_{11} E^{-1} D_{11}^* D_{13}] B_{13}^* = 0 \end{aligned}$$

Further,

$$H_0 = (B_{13}D_{13}^*D_{11} + Z_0(C_{11}^*D_{11} + B_{11}))E^{-1}$$

is a destabilising output injection for $[A_{11}, (D_{11}^*C_{11} + B_{11}^*)]$ (i.e. $\text{Re}[\lambda(A_{11} + H_0(D_{11}^*C_{11} + B_{11}^*))] > 0$)

(b) The stabilising solution to (4.42) is related to Z_0 and Θ by

$$Y^{-1} + \Theta = \begin{bmatrix} Z_0 & 0 \\ 0 & \gamma^2 I \end{bmatrix}$$

where the partitioning is induced by that in (2.2.37) and (2.2.43).

Proof:

The existence of a destabilising solution Z_0 is established by proving that

(i) $[A_{11}^*, C_{11}^*D_{11} + B_{11}]$ is controllable

(ii) The Hamiltonian matrix corresponding to (2.2.69) is free of imaginary axis eigenvalues.

The fact that Z_0 is the largest solution (ie $Z_0 - Z \geq 0$ for any other solution) is only of peripheral interest and consequently will not be proved here. In fact, all that is needed is a minor modification of an argument in [W1, Lemma 3].

We begin by showing that (i) is true. We know from part (a) of the theorem that

$$\left\{ \begin{bmatrix} -A_{11}^* & (C_{11}^*D_{11} + B_{11})C_{22} \\ 0 & -A_{22}^* \end{bmatrix}, \begin{bmatrix} C_{11}^*D_{11} + B_{11} \\ C_{22} \end{bmatrix} \right\}$$

is controllable. Next, we suppose for contradiction that (i) is not satisfied. That is there exists a vector $w \neq 0$ such that

$$w^* A_{11}^* = \lambda w^*$$

$$w^* (C_{11}^* D_{11} + B_{11}) = 0$$

Whence

$$[w^* \mid 0] \begin{bmatrix} -A_{11}^* & (C_{11}^* D_{11} + B_{11}) C_{22} \\ 0 & -A_{22}^* \end{bmatrix} = [-\lambda w^* \mid 0] \quad (D.1)$$

and

$$[w^* \mid 0] \begin{bmatrix} C_{11}^* D_{11} + B_{11} \\ C_{22} \end{bmatrix} = [0 \mid 0] \quad (D.2)$$

which contradicts part (a) of the theorem (which is already proved). This contradiction establishes (i).

As will be now shown, (ii) is in fact a consequence of Lemma 2.2.2. Direct substitution into the

Hamiltonian in Lemma 2.2.2 gives

$$H = \begin{bmatrix} A_{11} + N_1 E^{-1} (D_{11}^* C_{11} + B_{11}^*) & N_1 E^{-1} C_{22} \\ (-C_{22}^* + N_2 E^{-1}) (D_{11}^* C_{11} + B_{11}^*) & A_{22} + N_2 E^{-1} C_{22} \\ -(C_{11}^* D_{11} + B_{11}) E^{-1} (D_{11}^* C_{11} + B_{11}^*) & -(C_{11}^* D_{11} + B_{11}) E^{-1} C_{22} \\ -C_{22}^* E^{-1} (D_{11}^* C_{11} + B_{11}^*) & -C_{22}^* E^{-1} C_{22} \end{bmatrix} \quad (D.3)$$

$$\begin{bmatrix} N_1 E^{-1} N_1^* & N_1 E^{-1} N_2^* \\ N_2 E^{-1} N_1^* & N_2 E^{-1} N_2^* \\ -A_{11}^* - (C_{11}^* D_{11} + B_{11}) E^{-1} N_1^* & (B_{11} + C_{11}^* D_{11}) (E^{-1} N_2^* - C_{22}) \\ -C_{22}^* E^{-1} N_1^* & -A_{22}^* - C_{22}^* E^{-1} N_2^* \end{bmatrix}$$

A simple rearrangement of the rows and columns of $-T_1 H T_1^{-1}$ in Lemma 2.2.5 above, establishes the existence of a non-singular matrix T such that $-T H T^{-1}$ is equal to

$$\begin{bmatrix} A_{11}^* + (C_{11}^* D_{11} + B_{11}) E^{-1} D_{11}^* D_{13} B_{13}^* & (C_{11}^* D_{11} + B_{11}) E^{-1} (D_{11}^* C_{11} + B_{11}^*) & (C_{11}^* D_{11} + B_{11}) E^{-1} C_{22} & 0 \\ -B_{13} D_{13}^* [I + D_{11} E^{-1} D_{11}^*] D_{13} B_{13}^* & -A_{11} - B_{13} D_{13}^* D_{11} E^{-1} (D_{11}^* C_{11} + B_{11}^*) & -B_{13} D_{13}^* D_{11} E^{-1} C_{22} & 0 \\ 0 & 0 & A_{22}^* & 0 \\ C_{22}^* E^{-1} D_{11}^* D_{13} B_{13}^* & C_{22}^* E^{-1} (D_{11}^* C_{11} + B_{11}^*) & C_{22}^* E^{-1} C_{22} & -A_{22} \end{bmatrix} \quad (D.4)$$

where

$$T = \begin{bmatrix} 0 & 0 & I & 0 \\ I & 0 & \Theta_{11} & \Theta_{12} \\ 0 & I & \Theta_{12}^* & \Theta_{22} - \gamma^2 I \\ 0 & 0 & 0 & I \end{bmatrix} \quad (D.5)$$

Since H in (D.3) is free of imaginary axis eigenvalues (Lemma 2.2.2), so too is the matrix in (D.4). Suppose

$$S = \begin{bmatrix} S(1,1) & S(1,2) \\ S(2,1) & S(2,2) \end{bmatrix} \quad (D.6)$$

where $S(i,j)$ represents the (i,j) th block of (D.4). Then it is easy to see that

- (a) S is the Hamiltonian associated with (4.65), and
- (b) $\lambda(S) \subseteq \lambda(H) \Rightarrow$ (ii)

This concludes the proof of the existence part of (a)

Next, let us suppose that the orthogonal matrix W transforms S into an ordered upper Schur

form [L2], specifically

$$S \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \quad (D.7)$$

in which T_{11} is completely unstable and T_{22} is asymptotically stable. Substituting from (D.4) into (D.6) gives

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} = \begin{bmatrix} W_{11} \\ W_{21} \end{bmatrix} \begin{bmatrix} T_{11} \end{bmatrix} \quad (D.8)$$

where

$$S_{11} = \{A_{11} + B_{13}D_{13}^*D_{11}E^{-1}(D_{11}^*C_{11} + B_{11})\}^* \quad (D.8a)$$

$$S_{12} = (C_{11}^*D_{11} + B_{11})E^{-1}(D_{11}^*C_{11} + B_{11}) \quad (D.8b)$$

$$S_{21} = -B_{13}(I + D_{13}^*D_{11}E^{-1}D_{11}^*D_{13})B_{13}^* \quad (D.8c)$$

and

$$S_{22} = -S_{11}^* \quad (D.8d)$$

The Riccati equation solution is given by [L2]

$$Z_0 = W_{21}W_{11}^{-1} \quad (D.9)$$

Conjugating the (1,1) block of (D.8) and multiplying on the left by W_{11}^{-*} gives

$$A_{11} + \{(B_{13}D_{13}^*D_{11} + Z_0(C_{11}^*D_{11} + B_{11}))E^{-1}\}(D_{11}^*C_{11} + B_{11}) = W_{11}^{-*}T_{11}^*W_{11} \quad (D.10)$$

which proves that H_0 in (2.2.70) is a destabilising output injection. Thus (a) is proved.

Part (b) of the lemma will be established in two steps. Firstly, it will be proved that *any* solution to (2.2.69) will generate a solution to (2.2.42) via (2.2.71). Following that, we show that the largest (destabilising) solution to (2.2.69) generates the stabilising solution to (2.2.42).

From (2.2.55) we get

$$Y^{-1}\bar{A}^* + \bar{A}Y^{-1} + Y^{-1}F^*EFY^{-1} = 0 \quad (D.11)$$

Adding this to (2.2.36) gives

$$(Y^{-1} + \Theta)\bar{A}^* + \bar{A}(Y^{-1} + \Theta) + Y^{-1}F^*EFY^{-1} + \bar{B}_2\bar{B}_2^* = 0 \quad (D.12)$$

Substituting from (2.2.71) and writing (D.12) out in full gives

$$\begin{aligned}
 & \begin{bmatrix} Z & 0 \\ 0 & \gamma^2 I \end{bmatrix} \begin{bmatrix} A_{11}^* & \bar{A}_{12}^* \\ 0 & A_{22}^* \end{bmatrix} + \begin{bmatrix} A_{11} & 0 \\ \bar{A}_{12} & A_{22} \end{bmatrix} \begin{bmatrix} Z & 0 \\ 0 & \gamma^2 I \end{bmatrix} + \\
 & \begin{bmatrix} Z-\Theta_{11} & -\Theta_{12} \\ -\Theta_{12}^* & \gamma^2 I - \Theta_{22} \end{bmatrix} \begin{bmatrix} C_{11}^* D_{11} + B_{11} \\ C_{22}^* \end{bmatrix} E^{-1} [D_{11}^* C_{11} + B_{11}^* \mid C_{22}] \begin{bmatrix} Z-\Theta_{11} & -\Theta_{12} \\ -\Theta_{12}^* & \gamma^2 I - \Theta_{22} \end{bmatrix} + \\
 & \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} E^{-1} [D_{11}^* C_{11} \mid C_{22}] \begin{bmatrix} Z-\Theta_{11} & -\Theta_{12} \\ -\Theta_{12}^* & \gamma^2 I - \Theta_{22} \end{bmatrix} + \begin{bmatrix} B_{13} \\ B_{21} D_{11}^* D_{13} \end{bmatrix} [B_{13}^* \mid D_{13}^* D_{11} B_{21}^*] + \\
 & \begin{bmatrix} Z-\Theta_{11} & -\Theta_{12} \\ -\Theta_{12}^* & \gamma^2 I - \Theta_{22} \end{bmatrix} \begin{bmatrix} C_{11}^* D_{11} + B_{11} \\ C_{22}^* \end{bmatrix} E^{-1} [N_1 \mid N_2] + \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} E^{-1} \begin{bmatrix} N_1^* & N_2^* \end{bmatrix} = 0 \quad (D.13)
 \end{aligned}$$

in which

$$\bar{A}_{12} = B_{21}(B_{11}^* + D_{11}^* C_{11}) \quad (D.14)$$

(2.2.40) and (2.2.23) allow the (2,2) block (denoted (2,2)) of (D.13) to be written out in full as:

$$\begin{aligned}
 (2,2) &= \gamma^2 A_{22} + \gamma^2 A_{22}^* + [B_{21} E + N_2] E^{-1} [E B_{21}^* + N_2^*] - N_2 [B_{21}^* + E^{-1} N_2^*] + \\
 & N_2 E^{-1} N_2^* - [B_{21} + N_2 E^{-1}] N_2^* + B_{21} (\gamma^2 I - E) B_{21}^*
 \end{aligned}$$

after cancelling terms this becomes

$$\begin{aligned}
 (2,2) &= \gamma^2 (A_{22} + A_{22}^* + B_{21} B_{21}^*) \\
 &= 0 \quad \text{by (2.2.22)} \quad (D.15)
 \end{aligned}$$

The definitions of N and E in (2.2.40) and (2.2.41) allow the (1,2) block of (D.13) (denoted (1,2)) to be written out in full as

$$\begin{aligned}
 (1,2) &= Z \bar{A}_{12}^* + [Z(C_{11}^* D_{11} + B_{11}) + B_{13} D_{13}^* D_{11} - N_1] E^{-1} [-E B_{21}^* - N_2^*] \\
 & - N_1 [B_{21}^* + E^{-1} N_2^*] + [Z(C_{11}^* D_{11} + B_{11}) + B_{13} D_{13}^* D_{11} - N_1] E^{-1} N_2^* \\
 & + N_1 E^{-1} N_2^* + B_{13} D_{13}^* D_{11} B_{21}^*
 \end{aligned}$$

After cancelling terms this becomes

$$\begin{aligned} (1,2) &= Z(\bar{A}_{12}^* - (C_{11}^* D_{11} + B_{11}) B_{21}^*) \\ &= 0 \quad \text{by (D.14)} \end{aligned} \tag{D.16}$$

Clearly, $(2,1)=0$ follows by symmetry.

As with (D.15) and (D.16), it is easy to show that the $(1,1)$ block of (D.13) is zero provided Z is a solution of (2.2.69). This verification only requires the definitions of N_1 and N_2 in (2.2.40).

The sequence of arguments which follows proves that the largest solution Z_0 to (2.2.69) generates the stabilising solution to (2.2.42). Since H_0 in (2.2.70) is a destabilising output injection, and since A_{22} is stable, the matrix

$$\begin{bmatrix} A_{11}^* + (C_{11}^* D_{11} + B_{11}) H_0^* & 0 \\ C_{22}^* E^{-1} [D_{11}^* D_{13} B_{13}^* + (D_{11}^* C_{11} + B_{11}^*) Z_0] & -A_{22} \end{bmatrix} \tag{D.17}$$

is completely unstable which implies that

$$\begin{bmatrix} A_{11}^* + (C_{11}^* D_{11} + B_{11}) H_0^* & \bar{A}_{12}^* + (C_{11}^* D_{11} + B_{11}) C_{22} \\ C_{22}^* E^{-1} [D_{11}^* D_{13} B_{13}^* + (D_{11}^* C_{11} + B_{11}^*) Z_0] & A_{22}^* + C_{22}^* C_{22} \end{bmatrix} \tag{D.18}$$

is completely unstable by (D.14), (2.2.22) and (2.2.23). This implies by (2.2.70) that

$$\begin{bmatrix} A_{11}^* & \bar{A}_{12}^* \\ 0 & A_{22}^* \end{bmatrix} + \begin{bmatrix} C_{11}^* D_{11} + B_{11} \\ C_{22}^* \end{bmatrix} E^{-1} [D_{11}^* D_{13} B_{13}^* + (D_{11}^* C_{11} + B_{11}^*) Z_0 \mid EC_{22}] \tag{D.19}$$

is completely unstable. Using the notation

$$Y^{-1} = \begin{bmatrix} \hat{Y}_{11} & \hat{Y}_{12} \\ \hat{Y}_{12}^* & \hat{Y}_{22} \end{bmatrix} \tag{D.20}$$

together with (2.2.71), (2.2.40) and (D.19) establishes that

$$\begin{bmatrix} A_{11}^* & \bar{A}_{12}^* \\ 0 & A_{22}^* \end{bmatrix} + \begin{bmatrix} C_{11}^* D_{11} + B_{11} \\ C_{22}^* \end{bmatrix} E^{-1} [\mathcal{A} \mid \mathfrak{B}] \tag{D.21}$$

is completely unstable, where we have defined

$$\begin{aligned} \mathcal{A} &= N_1^* + (D_{11}^* C_{11} + B_{11}^*) \hat{Y}_{11} + C_{22} \hat{Y}_{12}^* \\ \mathfrak{B} &= N_2^* + (D_{11}^* C_{11} + B_{11}^*) \hat{Y}_{12} + C_{22} \hat{Y}_{22} \end{aligned}$$

Substituting from (2.2.13) shows that

$$\bar{A}^* + \bar{C}^* E^{-1} (N^* + \bar{C}^* \hat{Y}) \quad (D.22)$$

is completely unstable. From (2.2.42) we get

$$\bar{A}^* + \bar{C}^* E^{-1} (N^* + \bar{C}^* \hat{Y}) = -Y \{ \bar{A} + N E^{-1} \bar{C} + N E^{-1} N^* Y \} \hat{Y} \quad (D.23)$$

Whence by (2.2.45)

$$(D.22) \Rightarrow \bar{A} + N E^{-1} (\bar{C} + N^* Y) = \bar{A} + N F \quad (D.24)$$

is completely stable. This thus proves that Z_0 generates the required solution to (2.2.42). $\square$

Appendix 2.E: Proof of Lemma 2.2.7

Lemma 2.2.7

In the state space realisation of $T_{12} X T_{21}(s)$ in (4.70),

$$\Psi = C_{12}$$

$$\Phi = 0$$

$$\Delta = \hat{B} E^{\frac{1}{2}} + P_{13}^* B_{13} D_{13}^* D_{11} + P_{23}^* C_{22}^* E + (P_{13}^* \Theta_{11} + P_{23}^* \Theta_{12}^*)(C_{11}^* D_{11} + B_{11})$$

Proof:

We begin by proving that

$$\Psi = C_{11}(Q_{12} - \Theta_{12}) - D_{12}(\hat{D} E^{\frac{1}{2}} - D_{12}^* D_{11}) C_{22} - (D_{12} \hat{C} + C_{11} Q_{13})(P_{13}^* \hat{Y}_{12} + P_{23}^* \hat{Y}_{22}) = C_{12} \quad (E.1)$$

From the (1,1) and (1,2) blocks of (2.2.52), (2.2.56) and (2.2.13) we get

$$-\hat{C} P_{13}^* = \hat{D} E^{\frac{1}{2}} F_1 + B_{12}^* Y_{11} + D_{12}^* (D_{11} B_{21}^* + C_{12}) Y_{12} \quad (E.2)$$

$$-\hat{C} P_{23}^* = \hat{D} E^{\frac{1}{2}} F_2 + B_{12}^* Y_{12} + D_{12}^* (D_{11} B_{21}^* + C_{12}) Y_{22} \quad (E.3)$$

We will also use the three equations from (see (2.2.72))

$$\begin{bmatrix} Y_{11} & Y_{12} \\ Y_{12}^* & Y_{22} \end{bmatrix} \begin{bmatrix} \hat{Y}_{11} & \hat{Y}_{12} \\ \hat{Y}_{12}^* & \hat{Y}_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \quad (E.4)$$

Substituting (E.2), (E.3) and (E.4) into (E.1) gives

$$\begin{aligned} \Psi = & C_{11}(Q_{12}-\Theta_{12})-D_{12}\hat{D}E^{\frac{1}{2}}C_{22}+D_{12}\hat{D}E^{\frac{1}{2}}F_1\hat{Y}_{12}+D_{12}D_{12}^*C_{12}-C_{11}Q_{13}P_{13}^*\hat{Y}_{12} \\ & +D_{12}\hat{D}E^{\frac{1}{2}}F_2\hat{Y}_{22}-C_{11}Q_{13}P_{23}^*\hat{Y}_{22} \end{aligned} \quad (E.5)$$

Using (2.2.11), (2.2.29), (2.2.45), (2.2.23) and (E.4) gives

$$\begin{aligned} \Psi = & C_{11}(Q_{12}-\Theta_{12}-Q_{13}P_{13}^*\hat{Y}_{12}-Q_{13}P_{23}^*\hat{Y}_{22})+C_{12}+ \\ & D_{12}\hat{D}E^{-\frac{1}{2}}[(D_{11}^*C_{11}+B_{11}^*)\hat{Y}_{12}+C_{22}\hat{P}_{22}+N_2^*+EC_{22}] \end{aligned} \quad (E.6)$$

Using (2.2.11), (2.2.23), (2.2.71) and (2.2.41) we get from (E.6)

$$\Psi = C_{11}(Q_{12}-\Theta_{12}-Q_{13}P_{13}^*\hat{Y}_{12}-Q_{13}P_{23}^*\hat{Y}_{22})+C_{12}$$

Using (2.2.18) and

$$\hat{Y}_{22} = (Y_{22}-Y_{12}^*Y_{11}^{-1}Y_{12})^{-1} \quad (Y>0 \Rightarrow Y_{11}>0 \Rightarrow Y_{11}^{-1} \text{ exists}) \quad (E.7)$$

$$\hat{Y}_{12} = Y_{11}^{-1}Y_{12}(Y_{22}-Y_{12}^*Y_{11}^{-1}Y_{12})^{-1} \quad (E.8)$$

gives (E.1) as required. Next,

$$\Phi = (P_{13}^*\hat{Y}_{12}+P_{23}^*\hat{Y}_{22})A_{22}-\hat{A}(P_{13}^*\hat{Y}_{12}+P_{23}^*\hat{Y}_{22})-(\hat{B}E^{\frac{1}{2}}+P_{13}^*N_1+P_{23}^*N_2)C_{22}=0 \quad (E.9)$$

will be established. Substituting the (3,1) and (3,2) blocks of (2.2.51), (2.2.45) and (E.4) gives

$$\begin{aligned} \Phi = & \hat{B}E^{-\frac{1}{2}}\{D_{11}^*C_{11}\hat{Y}_{12}+B_{11}^*\hat{Y}_{12}+N_2^*+C_{22}\hat{Y}_{22}-EC_{22}\}- \\ & P_{23}^*\{\bar{A}_{12}\hat{Y}_{12}+A_{22}\hat{Y}_{22}+N_2C_{22}-\hat{Y}_{22}A_{22}\}-P_{13}^*\{A_{11}\hat{Y}_{12}+N_1C_{22}-\hat{Y}_{12}A_{22}\} \end{aligned} \quad (E.10)$$

By (2.2.40), (2.2.41) and (2.2.71) we get

$$\Phi = -P_{23}^*\{\bar{A}_{12}\hat{Y}_{12}+A_{22}\hat{Y}_{22}+N_2C_{22}-\hat{Y}_{22}A_{22}\}-P_{13}^*\{A_{11}\hat{Y}_{12}+N_1C_{22}-\hat{Y}_{12}A_{22}\} \quad (E.11)$$

Making use of (D.14), (2.2.11), (2.2.22), (2.2.40), (2.2.71), and the (2,2) block of (2.2.36) gives

$$\Phi = -P_{13}^* \{ A_{11} \hat{Y}_{12} + B_{13} D_{13}^* D_{11} C_{22} + \Theta_{11} (C_{11}^* D_{11} + B_{11}) C_{22} - \Theta_{12} A_{22}^* \} \quad (E.12)$$

From the (1,2) block of (2.2.36) we obtain

$$A_{11} \Theta_{12} - \Theta_{11} (C_{11}^* D_{11} + B_{11}) C_{22} + \Theta_{12} A_{22}^* = B_{13} D_{13}^* D_{11} C_{22} \quad (E.13)$$

and this together with the (1,2) block of (2.2.71) gives

$$\Phi = 0 \quad (E.14)$$

as required.

Finally, we have from (2.2.64), (2.2.73) and (2.2.74) that

$$\Delta = \hat{B} E^{\frac{1}{2}} + P_{13}^* N_1 + P_{23}^* N_2 - (P_{13}^* \hat{Y}_{12} + P_{23}^* \hat{Y}_{22}) B_{21} \quad (E.15)$$

From (2.2.40) this becomes

$$\begin{aligned} \Delta = & \hat{B} E^{\frac{1}{2}} + P_{13}^* \{ B_{13} D_{13}^* D_{11} + \Theta_{11} (C_{11}^* D_{11} + B_{11}) + \Theta_{12} C_{22}^* - \hat{Y}_{12} B_{21} \} + \\ & P_{23}^* \{ B_{21} D_{11}^* D_{13} D_{13}^* D_{11} + \Theta_{12}^* (C_{11}^* D_{11} + B_{11}) + \Theta_{22} C_{22}^* - \hat{Y}_{22} B_{21} \} \end{aligned}$$

Using (2.2.23), (2.2.41) and (2.2.71) to cancel terms gives

$$\begin{aligned} \Delta = & \hat{B} E^{\frac{1}{2}} + P_{13}^* (B_{13} D_{13}^* D_{11} + \Theta_{11} (C_{11}^* D_{11} + B_{11})) + P_{23}^* (C_{22}^* E + \Theta_{12}^* (C_{11}^* D_{11} + B_{11})) \\ = & \hat{B} E^{\frac{1}{2}} + P_{13}^* B_{13} D_{13}^* D_{11} + P_{23}^* C_{22}^* E + (P_{13}^* \Theta_{11} + P_{23}^* \Theta_{12}^*) (C_{11}^* D_{11} + B_{11}) \end{aligned}$$

as required. □

Chapter 3

3.0 Introduction

As we have seen in Chapter 1, a large class of $\mathcal{H}^\infty$ control problems may be expressed as Nevanlinna-Pick interpolation problems, in particular, the minimisation of the supremum norm over the set of $\mathcal{H}^\infty$ functions subject to a finite set of interpolation constraints. In the case of scalar problems, the required interpolating function is unique and may be characterised in terms of a maximal Schmidt pair of a Hankel operator [F2, pg.71]. In the case of matrix valued-problems, however, this interpolating matrix function is hardly ever unique. In order to restore uniqueness, N. Young [Y3] proposed tightening the minimising condition by also considering the second and subsequent singular values of the interpolating function. For $G_0(s) \in \mathbb{R}\mathcal{H}_+^{\infty, (p \times m)}$ let

$$s_j^\infty(G_0) = \sup_{\omega \in \mathbb{R}} s_j(G_0(j\omega)) \quad j=1, 2, \dots \quad (3.0.1)$$

and denote

$$s^\infty(G_0) = (s_1^\infty(G_0), s_2^\infty(G_0), \dots) \quad (3.0.2)$$

In this notation,

$$\|G_0\|_\infty = s_1^\infty(G_0) \quad (3.0.3)$$

The standard Nevanlinna-Pick type problem is to find an interpolating function for which $s_1^\infty(\cdot)$ is minimised. In the superoptimal interpolation problem we seek to minimise each $s_j^\infty(\cdot)$, $j=1, 2, \dots$, in turn rather than just $s_1^\infty(\cdot)$. Apart from offering a way of using up redundant degrees of freedom, superoptimal interpolating functions may have an $\mathcal{L}^2$ gain in certain singular subspace directions well below the standard $\|G_0\|_H$. The main problem which will be considered in this chapter is the following:

Problem [P1]: Suppose we are given a rational matrix $G_0(s) \in \mathbb{R}\mathcal{H}_+^{\infty, (p \times m)}$. Find a rational $F(s) \in \mathbb{R}\mathcal{H}_-^{\infty, (p \times m)}$ such that the (finite) sequence,

$$\{s_j^\infty(G_0 + F)\} \quad j=1, 2, \dots, \text{rank}(G_0) \leq \min(p, m) \quad (3.0.4)$$

is minimised with respect to lexicographic ordering. Rank is taken to mean the rank for almost all s .

□

More specifically a concrete state-space procedure which leads to the superoptimal

approximation $F_{so}(s)$ to $G_0(s)$ will be derived. This algorithm requires only standard linear algebraic type library routines and is, therefore, easy to implement. The procedure minimises sequentially each frequency-dependent singular value of the interpolating function in a dimension peeling algorithm. For the purpose of clarity three parts in this algorithm will be distinguished: In the first, a pair of all-pass matrix functions is constructed based on the maximal Schmidt pairs of a sequence of Hankel operators which correspond to a sequence of recursively-defined matrix functions $G_i(s)$. These are employed in the second part to diagonalise *all* Nehari extensions of $G_i(s)$. The minimisation of the $(i+1)$ th largest singular value of the interpolating function (over all frequencies) is formulated as a constrained optimisation problem and solved as a 'free' Nehari extension problem for $G_i(s)$. This process terminates when the original problem is reduced to one of unit rank; at this stage all the available degrees of freedom have been exhausted. Finally, in the third phase, the super-optimal extension is reconstructed from the solutions to optimisation problems arising in phase 2.

In addition to deriving and proving the algorithm for the construction of the super-optimal Nehari extension,

[i] It will be proved that

$$(a) \ s_j^\infty(G_0 - F_{so}) = \sigma_1(G_{j-1}) \leq \sigma_j(G_0) \quad j=1, 2, \dots, n \quad (3.0.5)$$

$$(b) \ \sigma_j(G_{i+1}) \leq \sigma_{j+1}(G_i) \quad i=0, 1, \dots, \text{rank}(G_0)-1 \quad j=1, 2, \dots, n-1 \quad (3.0.6)$$

where $n = \deg(G_0)$. The $G_i(s)$'s $i=1, 2, \dots, \text{rank}(G_0)-1$ will be defined later and are generated as the algorithm progresses. By (1.6), we see that they provide progressively tighter bounds on $s_j^\infty(G_0 - F_{so})$.

[ii] It will be shown that under certain assumptions

$$\deg(F_{so}) \leq \sum_{i=1}^{\min(n,r)} (n-i) \quad (3.0.7)$$

where $r = \text{rank}(G_0)$ and $n = \deg(G_0)$. If the algorithm is terminated after $\ell < \min(r, n)$ steps, (thus minimising the first ℓ levels) there exist extensions for which

$$\deg(F) \leq \sum_{i=1}^{\ell} (n-i) \quad (3.0.8)$$

is satisfied.

[iii] A number of alternative uses of the degrees of freedom in the solution set of the multivariable Nehari extension problem (closely related to the super-optimal construction) will be discussed. These include the minimisation of an $\mathcal{L}^2$ -type cost and an "entropy"-like cost associated with the $r-1$ smallest singular values of the error system. In particular the minimisations

$$\mu_1 = \inf_{F \in \mathfrak{F}} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=2}^r s_i^2 (G_0 - F)(j\omega) d\omega \right\} \quad (3.0.9)$$

and

$$\mu_2 = \inf_{F \in \mathfrak{F}} \left\{ \frac{\|G_0\|_H}{2\pi} \int_{-\infty}^{\infty} \sum_{i=2}^r \log \left| 1 - \frac{s_i^2 (G_0 - F)(j\omega)}{\|G_0\|_H} \right| \left(\frac{\operatorname{Re}(s_0)}{|s_0 - j\omega|} \right)^2 d\omega \right\} \quad (3.0.10)$$

will be considered where $\mathfrak{F}$ denotes the family of all Nehari extensions of $G_0(s)$ and s_0 in (3.0.10) is an arbitrary point in the RHP.

As a generalisation of the super-optimal Nehari extension problem the super-optimal Hankel-norm approximation problem may be defined similarly to [P1], the difference being that now a number of stable poles is allowed in the approximant $F(s)$. Formally the super-optimal Hankel norm approximation problem may be formulated as following:

Problem [P2]: Suppose that we are given a rational matrix $G_0(s) \in \mathbb{RH}_+^{\infty, (p \times m)}$. Find a rational $F(s) \in \mathbb{RH}_-^{\infty, (p \times m)}(k)$ such that the (finite) sequence, $\{s_j^\infty(G_0 - F)\}$, $j=1, 2, \dots$, $\operatorname{rank}(G_0) \leq \min(m, p)$ is minimised with respect to lexicographic ordering. $\square$

Apart from being an obvious generalisation to the super-optimal Nehari extension problem, [P2] has an interest in its own right. There are numerous occasions on which reliable low-order approximations to high-order models are required. For this reason, the model order reduction problem has been the subject of numerous theoretical investigations, and several different approaches have been developed. Following the paper of Glover [G1], the optimal Hankel-norm approach has emerged as the front runner. This is largely due to the fact that the method offers good guaranteed performance characteristics which are close to verifiable low bounds.

In matrix-valued case the optimal Hankel-norm approximation problem will also typically have a continuum of solutions. The question then arises as to which one (if there is one) is best. A partial answer to this question is implicit in [G1], in that $\mathcal{L}^\infty$ -error bounds are available for only certain optimal Hankel-norm reduced-order models. In [Y4] Young introduced an alternative approach to the uniqueness question by proposing again the lexicographic minimisation for the sequence $s^\infty(E) = (s_1^\infty(E), s_2^\infty(E), \dots)$ rather than just $s_1^\infty(E)$ where the $s_i^\infty(E)$ now correspond to the error arising from the approximation. The reduced-order model which minimises $s^\infty(E)$ has been shown to exist and to be unique [Y4].

In this chapter an algorithmic solution to the super-optimal Hankel-norm problem will be derived in state-space terms. Although the first part of the algorithm is almost identical for both problems, a number of technical details are involved at the later stages of [P2]. For the purposes of clarity, the two cases will be clearly distinguished.

The idea behind super-optimality is easily illustrated by way of a simple 2x2 example. Suppose

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} = \text{diag}\{g_1(s), g_2(s)\} \quad (3.0.11)$$

in which $\sigma_i(G)=1, \frac{1}{2}$. Suppose also that a reduced order approximation in $\mathcal{H}_\infty(1)$ is required. Any solution of the form

$$F(s) = \text{diag}\{f_1(s), \frac{1}{2}\} \quad (3.0.12)$$

is an optimal approximation to $G(s)$, provided that $f_1(s)$ is chosen to be in $\mathcal{H}_\infty(1)$ such that

$$\left\| \frac{2}{s+1} - f_1 \right\|_\infty \leq \frac{1}{2} \quad (3.0.13)$$

For example, the solution

$$F_{ap}(s) := \text{diag}\left\{\frac{s+3}{2(s+1)}, \frac{1}{2}\right\} \quad (3.0.14)$$

results in the all-pass error system,

$$G - F_{ap}(s) = \begin{bmatrix} -\frac{1}{2}\left(\frac{s-1}{s+1}\right) & 0 \\ 0 & -\frac{1}{2}\left(\frac{s-1}{s+1}\right) \end{bmatrix} \quad (3.0.15)$$

and in this case $(s_1^\infty(G - F_{ap}), s_2^\infty(G - F_{ap})) = (\frac{1}{2}, \frac{1}{2})$. It is clear, however, that we can do better than this: Suppose we use our one stable pole to cancel $g_1(s)$. This gives

$$F_{so}(s) = \begin{bmatrix} \frac{2}{s+1} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad (3.0.16)$$

resulting in an error system

$$G - F_{so}(s) = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{2}\left(\frac{s-1}{s+1}\right) \end{bmatrix} \quad (3.0.17)$$

for which $(s_1^\infty(G - F_{so}), s_2^\infty(G - F_{so})) = (\frac{1}{2}, 0)$. This is the super-optimal approximation to $G(s)$. The case in which the (1, 2) and (2, 1) elements are present in (3.0.11) is harder, and a formal algorithmic procedure is required.

3.1 Preliminaries

This section provides a description of all k -th order optimal approximations of a rational transfer function $G_0(s) \in \mathbb{RH}_+^{\infty, (p \times m)}$. The description is in terms of a balanced realisation of $G_0(s)$ and is based on [G1, Theorem 8.7]. If $G_0(s) \triangleq (A, B, C, D)$ is balanced and minimal, the following Lyapunov equations are satisfied:

$$AP + PA^* + BB^* = 0 \quad (3.1.1)$$

$$A^*Q + QA + C^*C = 0 \quad (3.1.2)$$

in which

$$P = Q = \text{diag}(\Sigma, \sigma_{k+1}) \quad (3.1.3)$$

where

$$\Sigma = \text{diag}(\sigma_1, \dots, \sigma_k, \sigma_{k+2}, \dots, \sigma_n) \quad (3.1.4)$$

Remark 3.1.1

In the interest of a clear presentation, it is assumed that the $(k+1)$ th Hankel singular value is unique. Matrices A , B and C are partitioned conformally with P and Q in (3.1.3) as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} ; \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} ; \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \quad (3.1.5)$$

In a further attempt to keep the notation simple, it will be assumed that $G_0(s)$ has been scaled to give $C_2^*C_2 = B_2B_2^* = 1$. This involves no loss of generality, since, by the (2, 2) blocks of equations 3.1.1 and 3.1.2 the (column) vectors C_2 and B_2^* have equal lengths.

The next Lemma parametrises the family of all optimal Hankel-norm approximations and their corresponding extensions in terms of an (arbitrary) unstable contraction.

Lemma 3.1.1

Let the transfer function $G_0(s) \in \mathbb{RH}_+^{\infty, (p \times m)}$ have a stable, minimal and balanced realisation $G_0(s) \triangleq (A, B, C, D)$ with Hankel singular values $\sigma_1 \geq \dots \geq \sigma_k > \sigma_{k+1} > \sigma_{k+2} \geq \dots \geq \sigma_n > 0$. Suppose also that

$$\Gamma = \Sigma^2 - \sigma_{k+1}^2 I_{n-1} \quad (3.1.6)$$

Then all error systems $E(s) = G_0(s) - F(s)$ with

$$\|E(s)\|_\infty = \sigma_{k+1} =: \sigma \quad (3.1.7)$$

and $F(s) \in \mathcal{H}_-^\infty(k)$ are generated by

$$E(s) = \mathcal{F}_\ell \left(H(s), \frac{1}{\sigma} U(s) \right) \quad U(s) \in \mathcal{B}\mathcal{H}_-^\infty \quad (3.1.8)$$

in which $H(s)H^*(s) = \sigma^2 I$, and

$$H(s) = \begin{bmatrix} G_0 + F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \quad (3.1.9)$$

where

$$F(s) = \left[\begin{array}{cc|cc} \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^*) & & \Gamma^{-1}(\Sigma B_1 + \sigma C_1^* U) & -\sigma \Gamma^{-1} C_1^* C_\perp^* \\ \hline -(C_1 \Sigma + \sigma U B_1^*) & & \sigma U & -\sigma C_\perp^* \\ -\sigma B_\perp B_1^* & & \sigma B_\perp & 0 \end{array} \right] \quad (3.1.10)$$

Also $F(s) \in \mathbb{R}\mathcal{H}_-^{\infty, (p+m-1) \times (p+m-1)}(k)$. The matrices U , $B_\perp$ and $C_\perp$ are defined in the proof.

Proof:

This Lemma is a minor variation of [G1, Theorem 8.7] and [G4]. Since

$$B_2 B_2^* = C_2^* C_2 = 1 \quad (3.1.11)$$

there exist orthogonal extensions to C_2^* and B_2 such that

$$\begin{bmatrix} C_2^* \\ C_\perp^* \end{bmatrix} C_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} ; \quad \begin{bmatrix} B_2 \\ B_\perp \end{bmatrix} B_2^* = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad (3.1.12)$$

We know from [G1, Theorem 8.7] that all error systems may be characterised in terms of a matrix $K(s) \in \mathcal{B}\mathcal{H}_-^\infty$ which must satisfy

$$C_2 + K(s) B_2^* = 0 \quad (3.1.13)$$

Next, (3.1.13) implies that

$$\begin{bmatrix} C_2^* \\ C_\perp^* \end{bmatrix} C_2 + \begin{bmatrix} C_2^* \\ C_\perp^* \end{bmatrix} K(s) \begin{bmatrix} B_2^* & B_\perp^* \end{bmatrix} \begin{bmatrix} B_2 \\ B_\perp \end{bmatrix} B_2^* = 0 \quad (3.1.14)$$

or

$$\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} C_2^* \\ C_\perp \end{bmatrix} K(s) \begin{bmatrix} B_2^* & B_\perp^* \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0 \quad (3.1.15)$$

(3.1.15) together with the fact that

$$\left\| \begin{bmatrix} C_2^* \\ C_\perp \end{bmatrix} K(s) \begin{bmatrix} B_2^* & B_\perp^* \end{bmatrix} \right\|_\infty \leq 1 \quad (3.1.16)$$

implies

$$\begin{bmatrix} C_2^* \\ C_\perp \end{bmatrix} K(s) \begin{bmatrix} B_2^* & B_\perp^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & U(s) & \\ 0 & & & \end{bmatrix} \quad (3.1.17)$$

Consequently,

$$\begin{aligned} -K(s) &= - \begin{bmatrix} C_2 & C_\perp \end{bmatrix} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & & & \\ \vdots & & U(s) & \\ 0 & & & \end{bmatrix} \begin{bmatrix} B_2 \\ B_\perp \end{bmatrix} \\ &= U - C_\perp^* U(s) B_\perp \quad ; \quad U := -C_2 B_2 \end{aligned} \quad (3.1.18)$$

Substituting the characterisation of $K(s)$ given in (3.1.18) into [G1, Theorem 8.7] yield the result. The all-pass character of $H(s)$ follows from a calculation almost identical to the one given in the proof of [G1, Theorem 6.3]. $\square$

3.2 Definition of the Hankel operator

For a linear, time-invariant and causal system with input $u(t) \in \mathcal{L}_2(0, \infty; \mathbb{C}^m)$ and output $y(t) \in \mathcal{L}_2(0, \infty; \mathbb{C}^p)$, the input/output map may be expressed as the convolution,

$$y(t) = \int_0^\infty h(t-s) u(s) ds = h(t) * u(t) \quad (3.1.1)$$

where $h(t)$ is the system's impulse response. The Hankel operator Γ_h in the time domain is defined by,

$$\begin{aligned} \Gamma_h : \mathcal{L}_2(0, \infty; \mathbb{C}^m) &\rightarrow \mathcal{L}_2(0, \infty; \mathbb{C}^p) \\ (\Gamma_h u)(t) &= \int_0^\infty h(t+s) u(s) ds \end{aligned} \quad (3.2.2)$$

Defining $v(t) = u(-t)$, we see that (3.1.1) may be rewritten as

$$(\Gamma_h u)(t) = \int_0^\infty h(t+s)v(-s)ds = \int_{-\infty}^0 h(t-s)v(s)ds = h(t) \star v(t) \quad (3.2.3)$$

and so, the Hankel operator may be interpreted as the input-output map between past square-integrable inputs to future square-integrable outputs via the state $x(0)$ at $t=0$. In the s -domain, the Hankel operator may be defined as

$$\begin{aligned} \Gamma_G: \mathcal{H}_2^\perp &\rightarrow \mathcal{H}_2 \\ \Gamma_G u(s) &= [Gu(s)]_+ \end{aligned} \quad (3.2.4)$$

where $[\cdot]_+$ denotes the operation of taking stable projection. The equivalence of the two definitions follows by considering the multiplication/convolution duality of the Laplace transform in the time and frequency domains. The following diagram summarises the definition of the Hankel operator in the two domains:

$$\begin{array}{ccc} u(t) \in \mathcal{L}_2(0, \infty; \mathbb{C}^m) & \xrightarrow{\Gamma_h} & (\Gamma_h u)(t) = \int_0^\infty h(t+s)u(s)ds \in \mathcal{L}_2(0, \infty; \mathbb{C}^p) \\ \downarrow & / & \downarrow L\text{-transform} \\ u(-t) \in \mathcal{L}_2(-\infty, 0; \mathbb{C}^m) & \xrightarrow{h \star u(-t)} & \\ \downarrow L\text{-transform} & & \\ u(-s) \in \mathcal{H}_2^{\perp, m} & \xrightarrow{\Gamma_G} & [Gu(-s)]_+ \in \mathcal{H}_2^p \end{array}$$

In discrete-time, the Hankel operator may be defined in a similar way. Consider the asymptotically stable system,

$$\{x_{k+1} = Ax_k + Bu_k, y_k = Cx_k\} \quad (3.2.5)$$

and suppose that its transfer function $G(z^{-1}) = C(zI - A)^{-1}B$ is expanded as the infinite series,

$$G(z^{-1}) = \sum_{k=1}^{\infty} CA^{k-1}Bz^{-k} := \sum_{k=1}^{\infty} H_k z^{-k} \quad (3.2.6)$$

The coefficients of this series expansion $H_k = CA^{k-1}B$ are known as the *Markov parameters*. They are the Fourier coefficients of $G(z^{-1})$ and may be evaluated via the integrals

$$H_k = \int_{\partial \mathbb{D}} G(z^{-1}) z^k d\theta \quad ; \quad z = e^{j\theta} \quad (3.2.7)$$

defined around the unit circle. The doubly-infinite (block) Hankel matrix is defined as

$$H = \left\{ H_{i+j-1} \right\}_{i=1, j=1}^{\infty, \infty} \quad (3.2.8)$$

This may be decomposed as

$$H = \begin{bmatrix} H_1 & H_2 & H_3 & \cdots \\ H_2 & H_3 & \ddots & \\ H_3 & \ddots & \ddots & \\ \vdots & & & \end{bmatrix} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \end{bmatrix} \begin{bmatrix} B & AB & A^2B & \cdots \end{bmatrix} = W_o W_c \quad (3.2.9)$$

Considering the composition of the two mappings

$$W_c : \begin{bmatrix} u_{-1} \\ u_{-2} \\ \vdots \end{bmatrix} \rightarrow x_0 = W_c \begin{bmatrix} u_{-1} \\ u_{-2} \\ \vdots \end{bmatrix} \quad W_o : x_0 \rightarrow \begin{bmatrix} y_0 \\ y_1 \\ \vdots \end{bmatrix} = W_o x_0 \quad (3.2.10)$$

the Hankel matrix may again be interpreted as the map between the past (square sumable) input sequence to the future (square sumable) output sequence via the state at $t=0$. Note also that since $\text{rank}(W_c)$ and $\text{rank}(W_o)$ are equal to the number of controllable and observable states respectively, the rank of the Hankel matrix is finite and bounded by the MacMillan degree of $G(z^{-1})$. In fact, due to Kronecker's theorem (see e.g. [G1]), this bound is tight.

The Hankel singular values $\{\sigma_i\}$ are the square-roots of the eigenvalues of $\Gamma_h^* \Gamma_h$ ($H^* H$ in discrete-time), where Γ_h^* is the adjoint operator,

$$(\Gamma_h^* y)(t) = \int_0^\infty h^*(t+s) y(s) ds \quad (3.2.11)$$

and $y \in \mathcal{L}_2(0, \infty; \mathbb{C}^p)$. Since $\text{rank}(\Gamma_h) = \deg(G)$ the number of Hankel singular values is also equal to this number. It may be shown [G1] that $\sigma_i^2 = \lambda_i(PQ)$ where P and Q are the controllability and observability Grammians of the realisation of $G(s)$, i.e.

$$P = \int_0^\infty e^{At} B B^* e^{A^* t} dt \quad (3.2.12)$$

and

$$Q = \int_0^\infty e^{A^* t} C^* C e^{At} dt \quad (3.2.13)$$

or, equivalently as the solutions to the Lyapunov equations

$$AP + PA^* + BB^* = 0 \quad (3.2.14)$$

and

$$A^* Q + QA + C^* C = 0 \quad (3.2.15)$$

Note that if (A, B, C) undergoes a general basis change, $(A, B, C) \xrightarrow{T} (TAT^{-1}, TB, CT^{-1})$ P and Q undergo the congruence transformations $P \rightarrow TPT^*$ and $Q \rightarrow T^*QT^{-1}$, so that $PQ \rightarrow T(PQ)T^{-1}$. This

implies that the Hankel singular values are invariant under state-space transformations as is expected from the definition of the Hankel operator purely in input/output transfer terms.

A special basis exists [G1], [M2] for which

$$P = \int_0^\infty e^{At} B B^* e^{A^* t} dt = Q = \int_0^\infty e^{A^* t} C^* C e^{At} dt = \Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_n) \quad (3.2.16)$$

where $\{\sigma_i\}$ are the Hankel singular values of $G(s)$. Such a realisation is called "*balanced*". The existence of balanced realisations may be verified by introducing a transformation $T = \Sigma^{-1/2} U^* R$ in an (arbitrary) minimal realisation (A, B, C) of $G(s)$ with corresponding controllability and observability grammians P and Q , where R is a Cholesky factor of Q (i.e. $Q = R^* R$) and $R P R^*$ admits the singular value decomposition $R P R^* = U \Sigma^2 U^*$. Balanced realisations for a particular singular value ordering are essentially unique (up to an arbitrary unitary transformation T such that $T \Sigma = \Sigma T$) [G1].

3.3 Singular vectors of the Hankel operator

Let (A, B, C) be a minimal and balanced realisation of $G(s)$. The impulse response of the system, $h(\cdot)$, may be decomposed as

$$\begin{aligned} h(t+\tau) &= C e^{A(t+\tau)} B = C e^{At} \left\{ \sum_{i=1}^n \epsilon_i \epsilon_i^* \right\} e^{A\tau} B \\ &= \sum_{i=1}^n \sigma_i \left(\frac{1}{\sqrt{\sigma_i}} C e^{At} \epsilon_i \right) \left(\frac{1}{\sqrt{\sigma_i}} \epsilon_i^* e^{A\tau} B \right) \\ &= \sum_{i=1}^n \sigma_i f_i(t) g_i^*(\tau) \end{aligned} \quad (3.3.1)$$

where we have denoted by ϵ_i the i -th unit (column) vector and defined

$$f_i(t) = \frac{1}{\sqrt{\sigma_i}} C e^{At} \epsilon_i \quad (3.3.2)$$

and

$$g_i^*(t) = \frac{1}{\sqrt{\sigma_i}} \epsilon_i^* e^{At} B \quad (3.3.3)$$

for $i=1, 2, \dots, n$. Note that since A is asymptotically stable, $\{f_i\}$ and $\{g_i\}$ are $\mathcal{L}_2(0, \infty)$ functions. In addition the $\{f_i\}$ and $\{g_i\}$ are orthonormal. For example, for the $\{f_i\}$'s,

$$\begin{aligned} \langle f_i, f_j \rangle_{\mathcal{L}_2(0, \infty)} &= \int_0^\infty f_i^*(t) f_j(t) dt \\ &= \int_0^\infty \frac{1}{\sqrt{\sigma_i \sigma_j}} \epsilon_i^* e^{A^* t} C^* C e^{At} \epsilon_j dt \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{\sigma_i \sigma_j}} \epsilon_i^* \left(\int_0^\infty e^{A^* t} C^* C e^{A t} dt \right) \epsilon_i \\
 &= \frac{1}{\sqrt{\sigma_i \sigma_j}} \epsilon_i^* \Sigma \epsilon_i = \delta_{ij}
 \end{aligned} \tag{3.3.4}$$

where we have made use of the assumption that the realisation is balanced, so that the observability grammian is equal to Σ . Similarly, on using the fact that the controllability grammian is also equal to Σ , $\{g_i\}$ can also be seen to be a collection of orthonormal functions in $\mathcal{L}_2(0, \infty)$. Next consider the definition of the Hankel operator in the time-domain:

$$\begin{aligned}
 (\Gamma_h u)(t) &= \int_0^\infty h(t+s) u(s) ds \\
 &= \int_0^\infty \sum_{i=1}^n \sigma_i f_i(t) g_i^*(\tau) u(\tau) d\tau \\
 &= \sum_{i=1}^n \sigma_i f_i(t) \int_0^\infty g_i^*(\tau) u(\tau) d\tau \\
 &= \sum_{i=1}^n \sigma_i f_i(t) \langle g_i, u \rangle_{L_2(0, \infty)}
 \end{aligned} \tag{3.3.5}$$

This is the polar decomposition of Γ_G ; in particular, choosing as input vectors

$$u(t) = g_j(t) \quad j=1, 2, \dots, n \tag{3.3.6}$$

we obtain that

$$(\Gamma_h g_j)(t) = \sum_{i=1}^n \sigma_i f_i(t) \langle g_i, g_j \rangle_{L_2(0, \infty)} = \sigma_j f_j(t) \tag{3.3.7}$$

since $\{g_j\}$ are orthonormal in $\mathcal{L}_2(0, \infty)$. Hence, for the $\{f_i\}$'s and $\{g_i\}$'s defined in (3.3.2) and (3.3.3) we may write that

$$\Gamma_h g_j = \sigma_j f_j \tag{3.3.8}$$

and

$$\Gamma_h^* f_j = \sigma_j g_j \tag{3.3.9}$$

for $j=1, 2, \dots, n$, where the second set of relations follows by duality. (f_i, g_i) is called a *Schmidt pair* of Γ_h corresponding to σ_i (equivalently, f_j is an eigenvector of $\Gamma_h \Gamma_h^*$ and g_j an eigenvector of $\Gamma_h^* \Gamma_h$ corresponding to σ_j). In the s-domain the Schmidt pair of Γ_G corresponding to σ_j is given by the

Laplace transforms of $f_j(t)$ and $g_j(-t)$ and will be denoted by $f_j(s)$ and $g_j(-s)$ where $f_j(s) = C(sI - A)^{-1}\epsilon_n$ and $g_j(-s) = B^*(sI + A^*)^{-1}\epsilon_n$.

The next result demonstrates that any Schmidt pair of Γ_{G_0} has singular vector type properties for $(G_0 - F)$ i.e. it shows that if $F(s)$ is *any optimal approximation* of $G_0(s) \in \mathcal{H}_+^\infty$, then

$$(G_0 - F)g(-s) = \sigma_{k+1}f(s) \quad (3.3.10)$$

Thus, modulo scaling, $f(s)$ and $g(s)$ are singular vectors of the error system $E(s) = (G_0 - F)(s)$ corresponding to the $(k+1)$ th Hankel singular value $\sigma_{k+1}(G_0)$.

First a proof for the Nehari case is given and subsequently this is extended to the $k \neq 0$ case.

Theorem 3.3.1

Suppose $G_0(s) \in \mathcal{H}_+^{\infty, (m \times p)}$ and suppose also that $F(s) \in \mathcal{H}_-^{\infty, (m \times p)}$ is any matrix function satisfying $\|G_0 - F\|_\infty = \sigma_1(G_0)$. Then $(G_0 - F)g(-s) = \Gamma_G g(-s) = \sigma_1(G_0)f(s)$.

Proof:

A proof in the scalar case is given in [F2, pgs.70-71]. The multivariable extension is trivial and is reproduced here for easy reference.

As in [F2], we look at the $\mathcal{L}_2$ norm of $(G_0 - F)g - \Gamma_{G_0}g$:

$$\begin{aligned} \|(G_0 - F)g - \Gamma_{G_0}g\|_2^2 &= \langle (G_0 - F)g - [G_0g]_+, (G_0 - F)g - [G_0g]_+ \rangle \\ &= \langle (G_0 - F)g, (G_0 - F)g \rangle - \langle [G_0g]_+, (G_0 - F)g \rangle \\ &\quad - \langle (G_0 - F)g, [G_0g]_+ \rangle + \langle [G_0g]_+, [G_0g]_+ \rangle \end{aligned} \quad (3.3.11)$$

Since $[G_0g]_+$ is orthogonal to the $\mathcal{H}_2^\perp$ component of $(G_0 - F)g$, we get

$$\langle (G_0 - F)g, [G_0g]_+ \rangle = \langle [G_0g]_+, [G_0g]_+ \rangle \quad (3.3.12)$$

Consequently,

$$\begin{aligned} \|(G_0 - F)g - \Gamma_{G_0}g\|_2^2 &= \langle (G_0 - F)g, (G_0 - F)g \rangle - \langle [G_0g]_+, [G_0g]_+ \rangle \\ &= \langle (G_0 - F)g, (G_0 - F)g \rangle - \langle f, \sigma_1^2 f \rangle \\ &\leq \|G_0 - F\|_\infty^2 \|g\|_2^2 - \sigma_1^2 \\ &= \|G_0 - F\|_\infty^2 - \sigma_1^2 \\ &= 0 \end{aligned} \quad (3.3.13)$$

This together with Nehari's theorem completes the proof. $\square$

Theorem 3.3.2 [P1]

Every $F(s) \in \mathcal{H}_-^\infty(k)$ which achieves the infimum

$$\inf_{F(s) \in \mathcal{H}_-^\infty(k)} \|G_0 - F\|_\infty = \sigma_{k+1}(G_0) < \sigma_k(G_0) \quad (3.3.14)$$

satisfies

$$(G_0 - F)g_{k+1}(-s) = \Gamma_{G_0} g_{k+1}(-s) = \sigma_{k+1}(G_0) f_{k+1}(s) \quad (3.3.15)$$

where (g_{k+1}, f_{k+1}) is a Schmidt pair of Γ_{G_0} corresponding to the $(k+1)$ th singular value of Γ_{G_0} .

Proof:

Let $F(s)$ be any (matrix) function which achieves the infimum in (3.3.14) and let Γ_F be the Hankel operator which it induces. Since $F(s) \in \mathcal{H}_-^\infty(k)$, $\text{rank}(\Gamma_F) \leq k$. Suppose (g_i, f_i) $i=1, 2, \dots$ are the Schmidt pair of Γ_{G_0} associated with σ_i and define P to be the orthogonal projection onto $\text{Span}(f_1, f_2, \dots, f_{k+1})$. Then $\|\Gamma_{G_0} - \Gamma_F\| = \sigma_{k+1}(G_0)$ implies that

$$\|P(\Gamma_{G_0} - \Gamma_F)\| \leq \sigma_{k+1}(G_0) \quad (3.3.16)$$

The operator $P\Gamma_F$ mapping $\text{Span}(g_1, g_2, \dots, g_{k+1}) \rightarrow \text{Span}(f_1, f_2, \dots, f_{k+1})$ has rank at most k and therefore there exists a function of norm equal to one such that

$$x = \sum_{i=1}^{k+1} \alpha_i g_i \in \text{Ker}(P\Gamma_F) \quad (3.3.17)$$

Consequently,

$$\|P(\Gamma_{G_0} - \Gamma_F)x\|_2 = \|P\Gamma_{G_0}x\|_2 \leq \sigma_{k+1}(G_0) \quad (3.3.18)$$

Also,

$$\begin{aligned} \|P\Gamma_{G_0}(\sum_{i=1}^{k+1} \alpha_i g_i)\|_2 &= \|P(\sum_{i=1}^{k+1} \alpha_i \Gamma_{G_0} g_i)\|_2 = \|P\sum_{i=1}^{k+1} \sigma_i \alpha_i f_i\|_2 \\ &= \|\sum_{i=1}^{k+1} \sigma_i \alpha_i f_i\|_2 \\ &= \sqrt{\sum_{i=1}^{k+1} \sigma_i^2 |\alpha_i|^2} \end{aligned} \quad (3.3.19)$$

since the f_i 's are orthonormal. This implies that $\sum_{i=1}^{k+1} \sigma_i^2 |\alpha_i|^2 \leq \sigma_{k+1}^2$ and since $\sigma_{k+1} < \sigma_k \leq \dots \leq \sigma_1$ together with $\sum_{i=1}^{k+1} |\alpha_i|^2 = 1$, we conclude that $\alpha_i = 0$ for $i=1, 2, \dots, k$, so that x must be a multiple of

g_{k+1} , say $x = \beta g_{k+1}$ with $|\beta| = 1$. Now, since $x \in \text{Ker}(P\Gamma_F)$, we have that $\Gamma_F x \perp \text{Span}(f_1, f_2, \dots, f_{k+1})$ and in particular $\langle \Gamma_F x, f_{k+1} \rangle_2 = 0$. Consequently,

$$\begin{aligned} \|\Gamma_{G_0} x - \Gamma_F x\|_2^2 &= \|\beta \Gamma_{G_0} g_{k+1} - \Gamma_F x\|_2^2 \\ &= \|\sigma_{k+1} f_{k+1}\|_2^2 + \|\Gamma_F x\|_2^2 \\ &= \sigma_{k+1}^2 + \|\Gamma_F x\|_2^2 \end{aligned} \quad (3.3.20)$$

But since $\|\Gamma_{G_0} x - \Gamma_F x\|_2^2 \leq \|\Gamma_{G_0} - \Gamma_F\|^2 \|x\|_2^2 = \sigma_{k+1}^2$, we conclude that

$$\sigma_{k+1}^2 + \|\Gamma_F x\|_2^2 \leq \sigma_{k+1}^2 \Rightarrow \Gamma_F x = 0 \Rightarrow \Gamma_F g_{k+1} = 0 \quad (3.3.21)$$

so that

$$(\Gamma_{G_0} - \Gamma_F) g_{k+1} = \sigma_{k+1} f_{k+1} \quad (3.3.22)$$

Also, defining Π_+ to be the stable projection operator,

$$\begin{aligned} \sigma_{k+1} = \|\sigma_{k+1} f_{k+1}\|_2 &= \|\Pi_+(G_0 - F) g_{k+1}\|_2 \\ &\leq \|(G_0 - F) g_{k+1}\|_2 \\ &\leq \|G_0 - F\|_\infty \|g_{k+1}\|_2 \\ &= \|G_0 - F\|_\infty = \sigma_{k+1} \end{aligned} \quad (3.3.23)$$

since $F(s)$ achieves the infimum in (3.3.14). This shows that $\Pi_+(G_0 - F) g_{k+1} = (G_0 - F) g_{k+1} = \sigma_{k+1} f_{k+1}$ as required. $\square$

Theorems 3.3.1 and 3.3.2 above define “constraining equations” for the set of all admissible approximations (extensions) of $G_0(s)$. For example, in the scalar case f_j and g_j are scalar functions and $F(s)$ is determined *uniquely* as,

$$F(s) = G_0(s) - \sigma_{k+1} \frac{f_{k+1}(s)}{g_{k+1}(-s)} \in \mathbb{RH}_-^\infty \quad (3.3.24)$$

(In addition, in this case, $|f(j\omega)| = |g(-j\omega)| \quad \forall \omega \in \mathbb{R}$, and the resulting error system $G_0 - F(s)$ is all pass.) In fact, for this special case, the realisations of the Schmidt vectors together with the result from Lemma 3.4.1 may be used to obtain the state-space formula of the (unique) optimal approximation $F(s)$ given in [G1] directly from (3.3.15). This derivation is made in the following Lemma. For simplicity it is assumed that the $(k+1)$ th Hankel singular value has multiplicity one and also that the realisation of $G(s)$ is real.

Lemma 3.3.3

Let $g(s) \in \mathbb{RH}_+^{\infty, (1 \times 1)}$ have a real balanced realisation (A, B, C) partitioned conformally with the grammians $P=Q=\tilde{\Sigma}=\text{diag}(\Sigma, \sigma)$ as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}; \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}; \quad C = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \quad (3.3.25)$$

where σ denotes the $(k+1)$ th Hankel singular value of $g(s)$. Then, the (unique) optimal Hankel-norm approximation $F(s) \in \mathbb{RH}_+^{\infty, (1 \times 1)}(k)$ is given by $(\hat{A}, \hat{b}, \hat{c}, \hat{d})$ as

$$\hat{A} := \Gamma^{-1}(\sigma^2 A_{11}^T + \Sigma A_{11} \Sigma - \sigma u C_1^T B_1^T) \quad (3.3.26)$$

$$\hat{b} = \Gamma^{-1}(\Sigma B_1 + \sigma C_1^T u) \quad (3.3.27)$$

$$\hat{c} = C_1 \Sigma + \sigma u B_1^T \quad (3.3.28)$$

$$\hat{d} = \sigma u \quad (3.3.29)$$

where $\Gamma = \Sigma^2 - \sigma^2 I$ and u satisfies $B_2 = -u C_2^T$.

Proof:

From Lemma 3.4.1, the Schmidt vectors $f(s)$ and $g(-s)$ corresponding to σ may be scaled to $\xi(s)$ and $\psi(s)$ respectively as $f(s) = \xi(s)\phi(s)$ and $g(-s) = -\psi(s)\phi(-s)$ where,

$$\xi(s) = C_1(sI - A_{11})^{-1}A_{12} + C_2, \quad \psi(s) = -B_1^T(sI + A_{11}^T)^{-1}A_{21}^T + B_2^T \quad (3.3.30ab)$$

and

$$\phi(s) = [s - A_{22} - A_{21}(sI - A_{11})^{-1}A_{12}]^{-1} = \det(sI - A_{11}) / \det(sI - A) \quad (3.3.31)$$

From the $(2, 2)$ blocks of equations 3.1.1 and 3.1.2 (with the partitioning induced by (3.3.25)) we obtain that $B_2^T B_2 = C_2 C_2^T = -2\sigma A_{22}$ with B_2 , C_2 and A_{22} scalar. Assume next without loss of generality that $B_2 > 0$ (otherwise the realisation resulting from the transformation $T = \text{diag}(I_{n-1}, -1)$ is still balanced); further assume that $C_2 > 0$. (3.1.1) and (3.1.2) now imply that

$$C_2 = B_2 = \sqrt{-2\sigma A_{22}} \quad (3.3.32)$$

$$A_{12} = -\Gamma^{-1}(\Sigma C_1^T - \sigma B_1^T) \sqrt{-2\sigma A_{22}} \quad (3.3.33)$$

$$A_{21}^T = -\Gamma^{-1}(\Sigma B_1 - \sigma C_1^T) \sqrt{-2\sigma A_{22}} \quad (3.3.34)$$

The (unique) optimal error system $G - F$ may be written as

$$G - F = \sigma \frac{f(s)}{g(-s)} = -\sigma \frac{\det(sI + A)}{\det(sI - A)} \frac{\det(sI - A_{11})}{\det(sI + A_{11})} \xi(s) \psi^{-1}(s) \quad (3.3.35)$$

$\psi^{-1}(s)$ may be written as

$$\psi^{-1}(s)=[1+B_1^T(sI-\hat{A})^{-1}\hat{B}]\frac{1}{\sqrt{-2\sigma A_{22}}} \quad (3.3.36)$$

where we have defined

$$\hat{A}=\Gamma^{-1}(\sigma^2 A_{11}^T+\Sigma A_{11}\Sigma+\sigma C_1^T B_1^T) \quad (3.3.37)$$

$$\hat{b}=\Gamma^{-1}(\Sigma B_1-\sigma C_1^T) \quad (3.3.38)$$

and have substituted from (3.3.32), (3.3.34) and (3.1.1). Also define

$$\delta(s)=\det(sI-A_{11})/\det(sI+A_{11}) \quad (3.3.39)$$

Since $\delta(s)$ is all-pass with $\delta(\infty)=1$ and $\deg(\delta)=n-1$, (3.3.39) may be written in a balanced state-space as

$$\delta(s)=1+C_1\Sigma^{-1}(sI+A_{11}^T)^{-1}C_1^T \quad (3.3.40)$$

Cascading $\delta(s)$ with $\xi(s)$ and using equations from various sub-blocks of (3.1.2) to remove the (unobservable) modes of A_{11} results in

$$\delta(s)\xi(s)=C_2-C_1\Sigma^{-1}(sI+A_{11}^T)^{-1}\sigma A_{21}^T \quad (3.3.41)$$

Similarly with (3.3.39) define,

$$\Delta(s):=\det(sI+A)/\det(sI-A)=1-C(sI-A)^{-1}\tilde{\Sigma}^{-1}C^T \quad (3.3.42)$$

Introducing a finer partitioning in (3.3.42) which is conformal with (3.1.3) gives after cascading with $\delta(s)$,

$$\Delta(s)\delta(s)\xi(s)=\left[\begin{array}{cc|c} A & \Phi & \Xi \\ 0 & -A_{11}^T & -\sigma A_{21}^T \\ \hline C & 0 & C_2 \end{array}\right] \quad (3.3.43)$$

where we defined

$$\Phi=\begin{bmatrix} 0 \\ \sigma^{-1}A_{12}^T \end{bmatrix} ; \quad \Xi=\begin{bmatrix} A_{12} \\ -\sigma^{-1}C_2^T C_2 \end{bmatrix} \quad (3.3.44)$$

and used the state-space transformation

$$T = \begin{bmatrix} I & 0 & \Sigma^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad (3.3.45)$$

Finally, cascading (3.3.43) with (3.3.36) and introducing the transformation

$$T = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & \sigma^2 I \\ 0 & 0 & 0 & I \end{bmatrix} \quad (3.3.46)$$

to remove the uncontrollable modes of $-A_{11}^T$ together with

$$T = \begin{bmatrix} I & 0 & \Sigma \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad (3.3.47)$$

in the resulting realisation, gives the decomposition

$$\Delta(s)\delta(s)\xi(s)\psi^{-1}(s) = -C(sI-A)^{-1}B + \left[\begin{array}{c|c} \hat{A} & \Gamma^{-1}(-\Sigma B_1 + \sigma C_1^T) \\ \hline -C_1 \Sigma + \sigma B_1^T & \sigma \end{array} \right] \quad (3.3.48)$$

$$\Rightarrow F(s) = \sigma + (C_1 \Sigma - \sigma B_1^T) \left\{ sI - \Gamma^{-1}(\sigma^2 A_{11}^T + \Sigma A_{11} \Sigma - \sigma C_1^T B_1^T) \right\}^{-1} \Gamma^{-1}(\Sigma B_1 - \sigma C_1^T) \quad (3.3.49)$$

This is in agreement with [G1, Theorem 6.3] on noting that $u=-1$ due to the choice in the sign of C_2 . (Choosing $C_2 < 0$ gives an identical result). $\square$

In general, the relation $(G_0 - F)g(-s) = \sigma_{k+1}f(s)$ does not constrain completely the admissible approximations as is shown in the following example:

Example 3.3.1

Again let

$$G(s) = \begin{bmatrix} \frac{2}{s+1} & 0 \\ 0 & \frac{1}{s+1} \end{bmatrix} = \text{diag}\{g_1, g_2\} \quad (3.3.50)$$

and consider equation $(G - F)g(-s) = \sigma_{k+1}f(s)$ as constraining the elements of an optimal Hankel-norm approximation $F(s) \in \mathcal{H}_-^\infty(k)$. The Schmidt vectors corresponding to $\sigma_2=1$ may be written as

$$g_2(-s) = \frac{1}{\sqrt{\sigma_2}} B^* (sI + A^*)^{-1} \epsilon_2 = \begin{bmatrix} 0 & -\frac{\sqrt{2}}{s-1} \end{bmatrix}^T \quad (3.3.51)$$

and

$$f_2(s) = \frac{1}{\sqrt{\sigma_2}} (sI - A)^{-1} \epsilon_2 = \begin{bmatrix} 0 & \frac{\sqrt{2}}{s-1} \end{bmatrix}^T \quad (3.3.52)$$

Hence, $(G - F)g(-s) = \sigma_{k+1} f(s)$ implies that

$$\begin{bmatrix} \frac{2}{s+1} - f_{11} & -f_{12} \\ -f_{21} & \frac{1}{s+1} - f_{22} \end{bmatrix} \begin{bmatrix} 0 \\ -\frac{\sqrt{2}}{s-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\sqrt{2}}{s-1} \end{bmatrix} \quad (3.3.53)$$

From the first block of this equation we conclude that $f_{12}(s) = 0$ while the second block fixes $f_{22}(s)$ as

$$f_{22}(s) = \frac{1}{s+1} + \frac{1}{2} \left(\frac{s-1}{s+1} \right) \Rightarrow f_{22}(s) = \frac{1}{2} \quad (3.3.54)$$

Hence, the family of admissible error systems is constrained to

$$G(s) - \mathcal{F}(s) = \begin{bmatrix} \frac{2}{s+1} - f_{11} & 0 \\ -f_{21} & -\frac{1}{2} \left(\frac{s-1}{s+1} \right) \end{bmatrix} \quad (3.3.55)$$

Next note that the norm constraint $\|G - \mathcal{F}\|_\infty = \frac{1}{2}$ implies that $f_{21}(s) = 0$ and $\|\frac{2}{s+1} - f_{11}\|_\infty \leq \frac{1}{2}$ as before. Formally, minimising $s_2^\infty(\cdot)$ in the family of all optimal Hankel-norm approximations, may be expressed in this case (since no stable pole has been used in f_{22}) as

$$\inf_{f_{11} \in \mathcal{H}_+^\infty(1)} \left\| \frac{2}{s+1} - f_{11} \right\|_\infty \quad (3.3.56)$$

This has the trivial solution $f_{11} = \frac{2}{s+1}$, resulting in the super-optimal approximation

$$F_{so}(s) = \begin{bmatrix} \frac{2}{s+1} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \quad (3.3.57)$$

as before. $\square$

Remark 3.3.1

The constraining equations $(G_0 - F)g(-s) = \sigma_{k+1} f(s)$ may be verified directly from the state-space realisations describing the family of all optimal approximations given in [G1] or [L6]: Following

the proof of Lemma 3.3.3, (3.3.15) may be written in state-space terms as

$$\mathcal{T}(s)[B_2^* - B_1^*(sI + A_{11}^*)^{-1}A_{21}^*] = \sigma C_2 - C_1\Sigma(sI + A_{11}^*)^{-1}A_{21}^* \quad (3.3.58)$$

or, equivalently, as

$$\left[\begin{array}{cc|c} \hat{A} & \hat{B}B_1^* & \hat{B}B_2^* \\ 0 & -A_{11}^* & -A_{21}^* \\ \hline \hat{C} & \hat{D}B_1^* - C_1\Sigma & \hat{D}B_2^* - \sigma C_2 \end{array} \right] = 0 \quad (3.3.59)$$

where we have defined $\mathcal{T}(s) = (\hat{A}, \hat{B}, \hat{C}, \hat{D})$. Using the state-space parametrisation in [L6], any optimal error system may be written as

$$\mathcal{S}(s) = G - \mathcal{T}(s) = \left[\begin{array}{cc|c} A & 0 & B \\ 0 & \hat{A} & \hat{B} \\ \hline C & -\hat{C} & -\hat{D} \end{array} \right] = \begin{bmatrix} A_e & B_e \\ C_e & D_e \end{bmatrix} \quad (3.3.60)$$

which satisfies the bounded-real type equations,

$$\left[\begin{array}{cc} -(A_e P_e + P_e A_e^* + B_e B_e^*) & -(B_e D_e^* + P_e C_e^*) \\ -(B_e D_e^* + P_e C_e^*)^* & I - D_e D_e^* \end{array} \right] = \begin{bmatrix} L_e \\ W_e \end{bmatrix} \begin{bmatrix} L_e^* & W_e^* \end{bmatrix} \quad (3.3.61)$$

with

$$P_e = \left[\begin{array}{cccc} \Sigma & 0 & I & 0 \\ 0 & \sigma & 0 & 0 \\ I & 0 & \Sigma\Gamma^{-1} & 0 \\ 0 & 0 & 0 & \sigma^2 \hat{P} \end{array} \right] = \begin{bmatrix} \Sigma & 0 & P_{13} \\ 0 & \sigma & 0 \\ P_{13}^* & 0 & P_{33} \end{bmatrix} \quad (3.3.62)$$

and L_e of the form $L_e^* = [0 \mid \cdot]$ partitioned conformally with the A matrix in $\mathcal{S}$. Invoking the transformation

$$T = \begin{bmatrix} I & -P_{13}^* \\ 0 & I \end{bmatrix} \quad (3.3.63)$$

in the state-space of (3.3.59), gives

$$\left[\begin{array}{cc|c} \hat{A} & P_{13}^* A_{11}^* + \hat{A} P_{13} + \hat{B} B_1^* & P_{13}^* A_{21}^* + \hat{B} B_2^* \\ 0 & -A_{11}^* & -A_{21}^* \\ \hline \hat{C} & \hat{C} P_{13}^* + \hat{D} B_1^* - C_1 \Sigma & \hat{D} B_2^* - \sigma C_2 \end{array} \right] = 0 \quad (3.3.64)$$

The constraining equations (3.3.58) may now be verified: The $A(1, 2)$ and the $B(1)$ blocks in (3.3.64) vanish on using the (1,3) and (2,3) block equations which correspond to the (1, 1) block of (3.3.61). Similarly the $C(2)$ term and the D -matrix may also be shown to be identically zero on using the (1,1) and the (1,2) block equations which correspond to the (1, 2) block of (3.3.61).

3.4 Scaling of Schmidt vectors and construction of all-pass completions

In our application, we use $f(s)$ and $g(s)$ as a basis for constructing two all-pass transformations to be used in a diagonalisation procedure. The next Lemma represents the first step in a two stage scaling process. The aim is to find vectors $\xi(s)$ and $\psi(s)$ which have full-rank at infinity, but which retain the singular vector properties of the Schmidt pair.

Lemma 3.4.1

Let $G_0(s) \triangleq (A, B, C, D)$ be the minimal and balanced realisation referred to in (3.1.1) to (3.1.3). Then

$$a) f(s) \triangleq (A, \epsilon_n, C, 0) \quad (3.4.1a)$$

$$g(s) \triangleq (-A^*, \epsilon_n, B^*, 0) \quad (3.4.1b)$$

are a Schmidt pair for the Hankel operator Γ_{G_0} corresponding to the singular value σ_{k+1} . Also

$$b) f(s) = \phi(s)\xi(s) \quad (3.4.2a)$$

$$g(s) = \phi^*(s)\psi(s) \quad (3.4.2b)$$

in which

$$\xi(s) \triangleq \begin{bmatrix} A_{11} & A_{12} \\ C_1 & C_2 \end{bmatrix} ; \xi(j\omega) \neq 0, \omega \in \mathbb{R} \cup \{\infty\} \quad (3.4.3a)$$

$$\psi(s) \triangleq \begin{bmatrix} -A_{11}^* & -A_{21}^* \\ B_1^* & B_2^* \end{bmatrix} ; \psi(j\omega) \neq 0, \omega \in \mathbb{R} \cup \{\infty\} \quad (3.4.3b)$$

$$\phi(s) = \det(sI - A_{11}) / \det(sI - A) \quad (3.4.4)$$

Proof:

A proof of the (a) part is given in [F2, pg.69]. This proof is adapted by assuming that the controllability and observability grammians are balanced. That is

$$P=Q=diag(\sigma_1, \dots, \sigma_k, \sigma_{k+2}, \dots, \sigma_n, \sigma_{k+1}) \quad (3.4.5)$$

To prove (b) note that

$$f(s)=C(sI-A)^{-1}\epsilon_n = [C_1 \ C_2] \begin{bmatrix} sI-A_{11} & -A_{12} \\ -A_{21} & sI-A_{22} \end{bmatrix}^{-1} \epsilon_n \quad (3.4.6)$$

Invoking a result on the inverse of a partitioned matrix [K1, pg 656] we get

$$\begin{aligned} f(s) &= [C_1 \ C_2] \begin{bmatrix} (sI-A_{11})^{-1}A_{12} \\ 1 \end{bmatrix} [sI-A_{22}-A_{21}(sI-A_{11})^{-1}A_{12}]^{-1} \\ &= \xi(s)\phi(s) \end{aligned} \quad (3.4.7)$$

by the Schur formulae for partitioned determinants.

When proving that $\xi(j\omega) \neq 0$, we deal with infinity first. Clearly, $\xi(\infty)=C_2 \neq 0$ by standard balanced realisation results. Next, observe that

$$(j\alpha+A^*)\Sigma+\Sigma(A-j\alpha)+C^*C=0 \quad (3.4.8)$$

Since no eigenvalue of A is on the imaginary axis, we can write

$$(j\alpha+A^*)^{-1}\Sigma+\Sigma(A-j\alpha)^{-1}+(j\alpha+A^*)^{-1}C^*C(A-j\alpha)^{-1}=0 \quad (3.4.9)$$

or more compactly

$$\tilde{A}^*\Sigma+\Sigma\tilde{A}+\tilde{C}^*\tilde{C}=0 \quad (3.4.10)$$

in which

$$\tilde{A}:= (A-j\alpha)^{-1} \quad \tilde{C}:= C(A-j\alpha)^{-1} \quad (3.4.11)$$

Finally,

$$2\epsilon_n^*\Sigma\tilde{A}\epsilon_n+\epsilon_n^*(j\alpha+A^*)^{-1}C^*C(A-j\alpha)^{-1}\epsilon_n=0 \quad (3.4.12)$$

Since $[\tilde{A}, \tilde{C}]$ is balanced by (3.4.10), and since $\tilde{A}$ has no eigenvalues on the imaginary axis, $\epsilon_n^*\Sigma\tilde{A}\epsilon_n \neq 0$ and so $f(j\alpha)f(j\alpha) \neq 0 \Rightarrow f(j\alpha) \neq 0$.

The results pertaining to $g(s)$ and $\psi(s)$ are proved in the same way. $\square$

The following Lemma shows that the realisations of $f(s)$ and $\xi(s)$ given above have the same number of uncontrollable modes. Since both (A, C) and (A_{11}, C_1) are observable pairs, $\deg(\xi) = \deg(f) - 1$. A dual argument shows that $\deg(\psi) = \deg(g) - 1$.

Lemma 3.4.2

The pairs (A, ϵ_n) and (A_{11}, A_{12}) have the same number of uncontrollable modes.

Proof:

Assume first that (A, ϵ_n) is controllable and suppose for contradiction that there exist a $\lambda \in \mathbb{C}$ and a $\beta \neq 0$ such that

$$\beta^*[\lambda I - A_{11} \mid A_{12}] = 0 \quad (3.4.13)$$

This implies that

$$[\beta^* \mid 0] \begin{bmatrix} \lambda I - A_{11} & -A_{12} & 0_{n-1} \\ -A_{21} & \lambda - A_{22} & 1 \end{bmatrix} = 0 \quad (3.4.14)$$

which contradicts the assumed controllability of (A, ϵ_n) .

Next, assume that k modes of the pair (A, ϵ_n) are uncontrollable. Then there exists a non-singular transformation T such that

$$TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} \quad (3.4.15)$$

and

$$T\epsilon_n = \epsilon_n \quad (3.4.16)$$

in which $\tilde{A}_{11} \in \mathbb{C}^{k \times k}$; it has also been assumed without loss of generality that the pair $(\tilde{A}_{22}, \epsilon_{n-k})$ has been put into controllable canonical form. Now (3.4.16) implies that T has the form

$$T = \begin{bmatrix} T_{11} & 0 \\ T_{21} & 1 \end{bmatrix} \quad (3.4.17)$$

in which T_{11} is non-singular. Further let TAT^{-1} in (3.4.15) be partitioned finer as

$$TAT^{-1} = \begin{bmatrix} \tilde{A}_{11} & 0_{1,n-k-1} & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{31} & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix} \quad (3.4.18)$$

in which $\bar{A}_{33}$ is a scalar. Then,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} T_{11}^{-1} & 0_{n-1} \\ T_{21} T_{11}^{-1} & 1 \end{bmatrix} \left[\begin{array}{cc|c} \bar{A}_{11} & 0_{1,n-k-1} & 0 \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} \\ \hline \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} \end{array} \right] \begin{bmatrix} T_{11} & 0 \\ T_{21} & 1 \end{bmatrix} \quad (3.4.19)$$

which implies that

$$[A_{11} \mid A_{12}] = \left[T_{11}^{-1} \begin{bmatrix} \bar{A}_{11} & 0 \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} T_{11} + T_{11}^{-1} \begin{bmatrix} 0 \\ \bar{A}_{23} \end{bmatrix} T_{21} \mid T_{11}^{-1} \begin{bmatrix} 0 \\ \bar{A}_{23} \end{bmatrix} \right] \quad (3.4.20)$$

so that

$$(sI - A_{11})^{-1} A_{12} \triangleq [\bar{A}_{22} + \bar{A}_{23} \bar{T}_2 \mid \bar{A}_{23}] \quad (3.4.21)$$

where the uncontrollable modes of $\bar{A}_{11}$ in (3.4.20) have been removed by introducing the transformation T_{11} and have defined

$$T_{21} T_{11}^{-1} = [\bar{T}_1 \mid \bar{T}_2] \quad (3.4.22)$$

with $\bar{T}_1 \in \mathbb{C}^{1 \times k}$. Since

$$\left(\begin{bmatrix} \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{32} & \bar{A}_{33} \end{bmatrix}, \epsilon_{n-k} \right) \quad (3.4.23)$$

is controllable by construction so is $(\bar{A}_{22}, \bar{A}_{23})$ and hence (3.4.21) is minimal. $\square$

To scale $\xi(s)$ to be of unit length, note that the scalar function $\xi^*(s)\xi(s)$ is positive on the imaginary axis (by 3.16a) and hence can be spectrally factorised. In particular, let $\xi = nd^{-1}$ be a coprime factorisation over the polynomials with $\xi^* \xi = d^{-*} n^* n d^{-1}$. Factoring $n^* n = \tilde{n}^* \tilde{n}$ where $\tilde{n}$ is scalar with its zeros in $\mathbb{C}_+$, we obtain

$$v := \xi(\tilde{n} d^{-1})^{-1} = n \tilde{n}^{-1} \in \mathcal{RH}_-^{\infty, p \times 1} \quad (3.4.24)$$

with $v^* v(s) = 1$.

The construction of $V_\perp(s)$ such that $V = [v \ V_\perp](s)$ is all-pass, is given in [G1, Theorem 5.2] and is possible without increasing the MacMillan degree (i.e. $\deg(V) = \deg(v)$). Explicit state-space formulae for $V(s)$ and $W(s)$ may be obtained in terms of the solutions of Riccati equations. The details appear later in the section.

The construction given above may be summarised by the following Lemma.

Lemma 3.4.3

There exist all-pass matrices $V \in \mathcal{RH}_-^{\infty, p \times p}$ and $W \in \mathcal{RH}_+^{\infty, m \times m}$ given by

$$V = [v \ V_\perp](s) \quad \text{with } v = f(v^* f)^{-1} \quad (3.4.25)$$

and

$$W = [w \ W_\perp](s) \quad \text{with } w = g(w^* g)^{-1} \quad (3.4.26)$$

Furthermore, minimal realisations of $V(s)$ and $W(s)$ are controllable from the first input.

Proof:

Follows from the discussion in this section. □

Next, we give a concrete state-space construction of the scaled Schmidt vectors $v(s)$ and $w(s)$ along with their all-pass completions which are derived from standard spectral factorisation theory.

Scaling $\xi(s) = C_2 + C_1(sI - A_{11})^{-1}A_{12}$ to unit length as $v(s) = \xi(s)m^{-1}(s)$, requires the spectral factorisation $\xi^*(s)\xi(s) = m^*(s)m(s)$. Since $v(s)$ is required to be completely unstable (for reasons that will become apparent later), the spectral factor $m(s)$ must be non-minimum phase. This is achieved by choosing the appropriate solution to the corresponding Riccati equation [A2]. The construction of the all-pass completion $V_\perp(s)$ may be achieved in a standard manner [G1] and requires no increase in degree; this accounts for the fact that minimal realisations of $V(s)$ and $W(s)$ are controllable from the first input.

Forming the product $\xi^*(s)\xi(s)$ and decomposing into stable and unstable parts gives

$$\begin{aligned} \xi^*(s)\xi(s) &= \left[\begin{array}{cc|c} -A_{11}^* & C_1^* C_1 & C_1^* C_2 \\ 0 & A_{11} & A_{12} \\ \hline -A_{12}^* & C_2^* C_1 & 1 \end{array} \right] \\ &= \left[\begin{array}{cc|c} -A_{11}^* & 0 & C_1^* C_2 + \Sigma A_{12} \\ 0 & A_{11} & A_{12} \\ \hline -A_{12}^* & A_{12}^* \Sigma + C_2^* C_1 & 1 \end{array} \right] \end{aligned} \quad (3.4.27)$$

Thus

$$\xi^*(s)\xi(s) = \zeta(s) + \zeta^*(s) \quad (3.4.28)$$

where

$$\zeta(s) := \frac{1}{2} - \sigma A_{21}(sI - A_{11})^{-1}A_{12} \quad (3.4.29)$$

on substituting from (3.1.1). Since $\zeta(s)$ is asymptotically stable and $\zeta(s)+\zeta^*(s)=\xi^*(s)\xi(s)$, $\zeta(s)$ is positive-real. The spectral factor $m(s)$ and the scaled Schmidt vector $v(s)$ may be constructed via a controllable realisation of $\zeta(s)$.

If (A_{11}, A_{12}) is completely controllable, $v(s)$ is given by

$$v(s) = \left[\begin{array}{c|c} A_{11}+A_{12}(A_{12}^*\Omega - C_2^*C_1) & A_{12} \\ \hline C_\perp^*C_\perp C_1 + C_2A_{12}^*\Omega & C_2 \end{array} \right] \quad (3.4.30)$$

where Ω is the destabilising solution to the Riccati equation

$$\Omega(A_{11}-A_{12}C_2^*C_1)+(A_{11}-A_{12}C_2^*C_1)^*\Omega+\Omega A_{12}A_{12}^*\Omega - C_1^*C_\perp^*C_\perp C_1 = 0 \quad (3.4.31)$$

The corresponding Hamiltonian matrix for this equation is given by

$$H_1 = \left[\begin{array}{cc} A_{11}-A_{12}C_2^*C_1 & A_{12}A_{12}^* \\ C_1^*C_\perp^*C_\perp C_1 & -(A_{11}-A_{12}C_2^*C_1)^* \end{array} \right] \quad (3.4.32)$$

which is free of imaginary axis eigenvalues since $\xi(s)$ is full rank everywhere on the $j\omega$ -axis. In the case that (A_{11}, A_{12}) is not completely controllable, a transformation T_1 in the state space of (3.4.3a) gives

$$\xi(s) \stackrel{\Delta}{=} \left[\begin{array}{cc} A_{11} & A_{12} \\ C_1 & C_2 \end{array} \right] \xrightarrow{T_1} \left[\begin{array}{cc|c} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ 0 & \tilde{A}_{22} & 0 \\ \hline \tilde{C}_1 & \tilde{C}_2 & C_2 \end{array} \right] \quad (3.4.33)$$

in which $(\tilde{A}_{11}, \tilde{A}_{13})$ is completely controllable. In this basis, $v(s) \in \mathcal{H}_-^\infty$ may be constructed as

$$v(s) = \left[\begin{array}{c|c} \tilde{A}_{11}+\tilde{A}_{13}(\tilde{A}_{13}^*\Omega_r - C_2^*\tilde{C}_1) & \tilde{A}_{13} \\ \hline C_\perp^*C_\perp \tilde{C}_1 + C_2\tilde{A}_{13}^*\Omega_r & C_2 \end{array} \right] \quad (3.4.34)$$

where Ω_r is the destabilising solution to the Riccati equation

$$\Omega_r(\tilde{A}_{11}-\tilde{A}_{13}C_2^*\tilde{C}_1)+(\tilde{A}_{11}-\tilde{A}_{13}C_2^*\tilde{C}_1)^*\Omega_r+\Omega_r\tilde{A}_{13}\tilde{A}_{13}^*\Omega_r - \tilde{C}_1^*C_\perp^*C_\perp \tilde{C}_1 = 0 \quad (3.4.35)$$

In this case, since $(\tilde{A}_{11}, \tilde{A}_{13})$ is controllable by construction while the corresponding Hamiltonian is again free of $j\omega$ -axis eigenvalues, the destabilising solution Ω_r is guaranteed to exist.

An all-pass completion to $v(s)$ can be constructed as

$$V_\perp(s) \stackrel{\Delta}{=} \left[\begin{array}{c|c} A_{11}+A_{12}(A_{12}^*\Omega - C_2^*C_1) & \Omega^\# C_1^*C_\perp^* \\ \hline C_\perp^*C_\perp C_1 + C_2A_{12}^*\Omega & C_\perp^* \end{array} \right] \quad (3.4.36)$$

Note that even if (A_{11}, A_{12}) is not completely controllable, $[v \mid V_\perp]$ may be described by the (non-minimal) realisation,

$$V(s) = [v \mid V_\perp] \triangleq \left[\begin{array}{c|c} A_{11} + A_{12}(A_{12}^* \Omega - C_2^* C_1) & A_{12} \quad \Omega^\# C_1^* C_\perp^* \\ \hline C_\perp^* C_\perp C_1 + C_2 A_{12}^* \Omega & C_2 \quad C_\perp^* \end{array} \right] \quad (3.4.37)$$

in which

$$\Omega := T_1^* \begin{bmatrix} \Omega_r & 0 \\ 0 & 0 \end{bmatrix} T_1 \quad (3.4.38)$$

This may be verified by augmenting the state space realisation of $[v \mid V_\perp]$ corresponding to (3.4.34) and (3.4.36) by the uncontrollable modes of $\tilde{A}_{22}$ as

$$\left[\begin{array}{cc|cc} \tilde{A}_{11} + \tilde{A}_{13}(\tilde{A}_{13}^* \Omega_r - C_2^* \tilde{C}_1) & \tilde{A}_{12} - \tilde{A}_{13} C_2^* \tilde{C}_2 & \tilde{A}_{13} & \Omega_r^\# \tilde{C}_1^* C_\perp^* \\ 0 & \tilde{A}_{22} & 0 & 0 \\ \hline C_\perp^* C_\perp \tilde{C}_1 + C_2 \tilde{A}_{13}^* \Omega_r & C_\perp^* C_\perp \tilde{C}_2 & C_2 & C_\perp^* \end{array} \right] \quad (3.4.39)$$

and subsequently applying the state-space transformation T_1^{-1} .

A similar treatment may be used for the scaling of $\psi(s)$ to $w(s)$ and the construction of $[w \mid W_\perp](s)$. Assuming that (A_{11}, A_{21}) is an observable pair,

$$W(s) = [w \mid W_\perp] \triangleq \left[\begin{array}{c|c} -A_{11}^* - A_{21}^*(A_{21} \tilde{\Omega} - B_2 B_1^*) & -A_{21}^* \quad -\tilde{\Omega}^\# B_1 B_\perp^* \\ \hline B_\perp^* B_\perp B_1^* + B_2^* A_{21} \tilde{\Omega} & B_2^* \quad B_\perp^* \end{array} \right] \quad (3.4.40)$$

where $\tilde{\Omega}$ is the destabilising solution to the Riccati equation

$$\tilde{\Omega}(A_{11} - B_1 B_2^* A_{21})^* + (A_{11} - B_1 B_2^* A_{21}) \tilde{\Omega} + \tilde{\Omega} A_{21}^* A_{21} \tilde{\Omega} - B_1 B_\perp^* B_\perp B_1^* = 0 \quad (3.4.41)$$

Again, if (A_{11}, A_{21}) is not completely observable a state-space transformation $\tilde{T}_1$ may be defined in the state-space of $\psi(s)$,

$$\psi(s) \triangleq \begin{bmatrix} -A_{11}^* & -A_{12}^* \\ B_1^* & B_2^* \end{bmatrix} \xrightarrow{\tilde{T}_1} \left[\begin{array}{cc|c} -\bar{A}_{11}^* & -\bar{A}_{21}^* & -\bar{A}_{31}^* \\ 0 & -\bar{A}_{22}^* & 0 \\ \hline \bar{B}_1^* & \bar{B}_2^* & B_2^* \end{array} \right] \quad (3.4.42)$$

in which $(\bar{A}_{11}, \bar{A}_{31})$ is an observable pair. In this basis $W = [w \mid W_\perp]$ may be constructed as

$$W(s) = [w \mid W_\perp] \triangleq \left[\begin{array}{c|c} -\bar{A}_{11}^* - \bar{A}_{31}^*(\bar{A}_{31}\tilde{\Omega}_r - B_2\bar{B}_1^*) & -\bar{A}_{21}^* \quad -\tilde{\Omega}^\# \bar{B}_1 B_\perp^* \\ \hline B_\perp^* B_\perp \bar{B}_1^* + B_2^* \bar{A}_{31} \tilde{\Omega}_r & B_2^* \quad B_\perp^* \end{array} \right] \quad (3.4.43)$$

where $\tilde{\Omega}_r$ is defined as the destabilising solution to the (reduced) Riccati equation

$$\tilde{\Omega}_r(\bar{A}_{11} - \bar{B}_1 B_2^* \bar{A}_{31})^* + (\bar{A}_{11} - \bar{B}_1 B_2^* \bar{A}_{31})\tilde{\Omega}_r + \tilde{\Omega}_r \bar{A}_{21}^* \bar{A}_{21} \tilde{\Omega}_r - \bar{B}_1 B_\perp^* B_\perp \bar{B}_1^* = 0 \quad (3.4.44)$$

which now is guaranteed to exist. Again, by defining

$$\tilde{\Omega} := \tilde{T}_1^* \begin{bmatrix} \tilde{\Omega}_r & 0 \\ 0 & 0 \end{bmatrix} \tilde{T}_1 \quad (3.4.45)$$

it may be verified that (3.4.40) still defines a (non-minimal) realisation of $W(s)$ even in the case that (A_{11}, A_{21}) is not observable.

The next Lemma shows that the reduced-order Riccati equations Ω_r and $\tilde{\Omega}_r$ defined in the construction of $V(s)$ and $W(s)$ exist and are positive definite.

Lemma 3.4.4

In the Nehari-extension case the destabilising solutions Ω_r and $\tilde{\Omega}_r$ are positive definite.

Proof:

Consider first the algebraic Riccati equation

$$\Omega_r(-\tilde{A}_{11} + \tilde{A}_{13} C_2^* \tilde{C}_1) + (-\tilde{A}_{11} + \tilde{A}_{13} C_2^* \tilde{C}_1)^* \Omega_r - \Omega_r \tilde{A}_{13} \tilde{A}_{13}^* \Omega_r + \tilde{C}_1^* C_\perp^* C_\perp \tilde{C}_1 = 0 \quad (3.4.46)$$

By construction, $(\tilde{A}_{11}, \tilde{A}_{13})$ controllable $\Rightarrow (-\tilde{A}_{11} + \tilde{A}_{13} C_2^* \tilde{C}_1, \tilde{A}_{13})$ controllable. In addition,

$$H = \begin{bmatrix} -\tilde{A}_{11} + \tilde{A}_{13} C_2^* \tilde{C}_1 & -\tilde{A}_{13} \tilde{A}_{13}^* \\ -\tilde{C}_1 C_\perp^* C_\perp \tilde{C}_1 & (\tilde{A}_{11} - \tilde{A}_{13} C_2^* \tilde{C}_1)^* \end{bmatrix} \quad (3.4.47)$$

is free of $j\omega$ -axis eigenvalues. To see why this must be so, note that if H had a $j\omega$ -eigenvalue this would had to be either an uncontrollable mode of

$$[-\tilde{A}_{11} + \tilde{A}_{13} C_2^* \tilde{C}_1, \tilde{A}_{13}] \quad (3.4.48)$$

or an unobservable mode of

$$\begin{bmatrix} -\tilde{A}_{11} + \tilde{A}_{13} C_2^* \tilde{C}_1 \\ C_\perp \tilde{C}_1 \end{bmatrix} \quad (3.4.49)$$

Again (3.4.48) is controllable by construction. The fact that (3.4.49) has no unobservable modes on the $j\omega$ axis (in fact nowhere in the CLHP) follows from the second part of the proof.

From Lemma 2.1.3,

$$\text{Rank}(\Omega_r) = \{\# \text{ unstable eigenvalues of } -\tilde{A}_{11} + \tilde{A}_{13}C_2^* \tilde{C}_1\} + \\ \{\# \text{ stable eigenvalues of } -\tilde{A}_{11} + \tilde{A}_{13}C_2^* \tilde{C}_1 \text{ which are observable through } C_\perp \tilde{C}_1\}$$

To prove that Ω_r is non-singular, therefore, it suffices to show that all stable eigenvalues of $-A_{11} + A_{12}C_2^* C_1$ are observable through $C_\perp C_1$. Next, suppose for contradiction that λ is a stable, unobservable mode of this pair. Then $\exists \xi \neq 0$ such that

$$\begin{bmatrix} \lambda I + A_{11} - A_{12}C_2^* C_1 \\ C_\perp C_1 \end{bmatrix} \xi = 0 \quad (3.4.50)$$

which implies that

$$\begin{bmatrix} \lambda I - \Gamma^{-1} \hat{A}^* \Gamma - \Sigma \Gamma^{-1} C_1^* C_\perp C_1 \\ C_\perp C_1 \end{bmatrix} \xi = 0 \quad (3.4.51)$$

where we have defined $\hat{A} = \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^*)$ and substituted from (3.1.1), (3.1.2), (3.1.3), (3.1.18) and (3.1.11). (3.4.51) implies that

$$(\lambda I - \hat{A}^*)(\Gamma \xi) = 0 \quad (3.4.52)$$

which is impossible since $\hat{A}$ is completely unstable and Γ is non-singular. The non-singularity of $\tilde{\Omega}$ follows by a dual argument. $\square$

The state-space construction of $V(s)$ and $W(s)$ when the Riccati equations (3.4.31) and (3.4.41) are solvable is summarised in the following Theorem.

Theorem 3.4.5

Suppose that Ω and $\tilde{\Omega}$ satisfy the algebraic Riccati equations

$$\Omega(A_{11} - A_{12}C_2^* C_1) + (A_{11} - A_{12}C_2^* C_1)^* \Omega + \Omega A_{12} A_{12}^* \Omega - C_1^* C_\perp C_\perp C_1 = 0 \quad (3.4.53)$$

and

$$\tilde{\Omega}(A_{11} - B_1 B_2^* A_{21})^* + (A_{11} - B_1 B_2^* A_{21}) \tilde{\Omega} + \tilde{\Omega} A_{21}^* A_{21} \tilde{\Omega} - B_1 B_\perp^* B_\perp B_1^* = 0 \quad (3.4.54)$$

Then

$$V(s) = [v \mid V_\perp] \triangleq \left[\begin{array}{c|c} A_{11} + A_{12}(A_{12}^* \Omega - C_2^* C_1) & A_{12} \quad \Omega^\# C_1^* C_\perp^* \\ \hline C_\perp^* C_\perp C_1 + C_2 A_{12}^* \Omega & C_2 \quad C_\perp^* \end{array} \right] \quad (3.4.55)$$

and

$$W(s) = [w \mid W_\perp] \triangleq \left[\begin{array}{c|c} -A_{11}^* - A_{21}^*(A_{21} \tilde{\Omega} - B_2 B_1^*) & -A_{21}^* \quad -\tilde{\Omega}^\# B_1 B_\perp^* \\ \hline B_\perp^* B_\perp B_1^* + B_2^* A_{21} \Omega & B_2^* \quad B_\perp^* \end{array} \right] \quad (3.4.56)$$

are all-pass.

Proof:

The proof follows from the discussion of the previous section together with standard spectral factorisation results [A2]. The method of converting a spectral factorisation problem into the problem of solving an algebraic Riccati equation may be found in [A4]. The construction of the all-pass completion is given in the proof of [G1, Theorem 5.2]. $\square$

Remark 3.4.1

A direct computation will verify that (3.4.55) and (3.4.56) are all-pass for any solutions to (3.4.53) and (3.4.54). In our application, it is important that $V(s)$ and $W(s)$ have the correct stability properties. For this reason we will select the solutions to (3.4.53) and (3.4.54) which make $V(s)$ and $W(s)$ completely unstable, or, in general, if no such solutions exist, we follow the construction given previously. In the future, the solutions to the Riccati equations (3.4.55) and (3.4.56) (or (3.4.35) and (3.4.44)) will refer to the destabilising solutions unless otherwise stated.

The next two results establish a number of connections between (3.4.53) and (3.4.54) and their associated Hamiltonian matrices. It is first shown that the two Hamiltonians have the same spectrum and that they are related in a surprisingly simple way (see (3.4.59)).

Lemma 3.4.6

The Hamiltonians associated with the two Riccati equations (3.4.53) and (3.4.54) are

$$H_1 = \left[\begin{array}{cc} A_{11} - A_{12} C_2^* C_1 & A_{12} A_{12}^* \\ C_1^* C_\perp^* C_\perp C_1 & -(A_{11} - A_{12} C_2^* C_1)^* \end{array} \right] \quad (3.4.57)$$

and

$$H_2 = \left[\begin{array}{cc} A_{11}^* - A_{21}^* B_2 B_1^* & A_{21}^* A_{21} \\ B_1 B_\perp^* B_\perp B_1^* & -(A_{11}^* - A_{21}^* B_2 B_1^*)^* \end{array} \right] \quad (3.4.58)$$

Further

$$\begin{bmatrix} I & -\Sigma \\ -\Sigma & \Gamma \end{bmatrix}^{-1} H_1 \begin{bmatrix} I & -\Sigma \\ -\Sigma & \Gamma \end{bmatrix} = H_2^* \quad (3.4.59)$$

Proof:

See Appendix 3.A at the end of this Chapter. $\square$

In the first part of the next Lemma a linear fractional connection is established between the solutions of the two Riccati equations. This connection is a direct consequence of (3.4.59). In the second part we prove a technical result which is required later.

Lemma 3.4.7

Suppose that Ω and $\tilde{\Omega}$ are solutions to the Riccati equations (3.4.53) and (3.4.54) respectively, and suppose also that $[\cdot]_+$ and $[\cdot]_-$ denote the stabilising and destabilising solutions to these equations. Then if $\tilde{\Omega}_+^{-1}$ exists,

$$(i) \quad \Omega_- = \mathcal{F}_\ell(\Xi, \tilde{\Omega}_+^{-1}) \quad (3.4.60)$$

where

$$\Xi := \begin{bmatrix} -\Sigma & \sigma I \\ \sigma I & -\Sigma \end{bmatrix} \quad (3.4.61)$$

(ii) If

$$\Delta(\Omega, \tilde{\Omega}) := \tilde{\Omega}\Omega + \Sigma\Omega + \tilde{\Omega}\Sigma + \Gamma \quad (3.4.62)$$

Then

$$\Delta(\Omega_-, \tilde{\Omega}_+) = 0 \quad (3.4.63)$$

and

$$\Delta(\Omega_-, \tilde{\Omega}_-) \quad (3.4.64)$$

is non-singular.

Proof:

See Appendix 3.B at the end of this Chapter. $\square$

3.5 Diagonalisation of all-optimal Hankel-norm approximations (extensions)

The construction of section 3.5 together with Lemmas 3.3.1 and 3.3.2 imply that $V(s)$ and $W(s)$ will block-diagonalise all the optimal error systems $(G_0 + F)(s)$. Moreover, the augmented system

$$G_a(s) = \begin{bmatrix} G_0(s) & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{RH}_-^\infty, (p+m) \times (p+m) \quad (3.5.1)$$

will have Schmidt vectors $[f^T(s) \ 0]^T$ and $[g^T(s) \ 0]^T$ and hence is block-diagonalised by

$$V_a(s) := \begin{bmatrix} V(s) & 0 \\ 0 & I_m \end{bmatrix} \quad W_a(s) := \begin{bmatrix} W(s) & 0 \\ 0 & I_p \end{bmatrix} \quad (3.5.2)$$

The Nehari case will first be considered. $F(s)$ in Lemma 3.3.1 is an optimal Nehari extension of $G_a(s)$, so that

$$V_a^*(G_a + F)W_a(s) = \begin{bmatrix} \sigma_1(G_0)a(s) & 0 & 0 \\ 0 & G_1 + Q_{11} & Q_{12} \\ 0 & Q_{21} & Q_{22} \end{bmatrix} \quad (3.5.3)$$

where $a(s)$ is scalar and all-pass (in fact inner) and $Q_{22}(s) = F_{22}(s)$. Also, the (1,3) and (3,1) blocks of (3.5.3) are given by

$$v^* F_{12}(s) = 0 \quad F_{21}w(s) = 0 \quad (3.5.4)$$

Suppose that

$$V^* \triangleq \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{bmatrix} \quad v^* \triangleq \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{c}_1 & \tilde{d}_1 \end{bmatrix} \quad F_{12} \triangleq \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \quad (3.5.5)$$

Then (3.5.4) implies that

$$v^* F_{12} = 0 \triangleq \left[\begin{array}{cc|c} \tilde{A} & \tilde{B}\hat{C} & \tilde{B}\hat{D} \\ 0 & \hat{A} & \hat{B} \\ \hline \tilde{c}_1 & \tilde{d}_1\hat{C} & \tilde{d}_1\hat{D} \end{array} \right] \quad (3.5.6)$$

Since $\operatorname{Re}(\lambda_i(\tilde{A})) < 0$ and $\operatorname{Re}(\lambda_i(\hat{A})) > 0$ the equation

$$X\hat{A} - \tilde{A}X + \tilde{B}\hat{C} = 0 \quad (3.5.7)$$

has a unique solution X . Introducing the basis change

$$T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \quad (3.5.8)$$

in the state-space of (3.5.6) gives

$$0 \triangleq \left[\begin{array}{cc|c} \tilde{A} & 0 & \tilde{B}\hat{D}+X\hat{B} \\ 0 & \hat{A} & \hat{B} \\ \hline \tilde{c}_1 & \tilde{d}_1\hat{C}-\tilde{c}_1X & \tilde{d}_1\hat{D} \end{array} \right] \quad (3.5.9)$$

Since $[\tilde{A}, \tilde{c}_1]$ is completely observable by Lemma 3.4.2,

$$\tilde{B}\hat{D}+X\hat{B}=0, \quad \tilde{d}_1\hat{D}=0 \quad (3.5.10)$$

This now implies that

$$\begin{aligned} V^*F_{12}(s) &\triangleq \left[\begin{array}{cc|c} \tilde{A} & 0 & 0 \\ 0 & \hat{A} & \hat{B} \\ \hline \tilde{C} & \tilde{D}\hat{C}-\tilde{C}X & \tilde{D}\hat{D} \end{array} \right] = \left[\begin{array}{cc} \hat{A} & \hat{B} \\ \tilde{D}\hat{C}-\tilde{C}X & \tilde{D}\hat{D} \end{array} \right] \\ &= \begin{bmatrix} 0 \\ Q_{12} \end{bmatrix} \in \mathbb{RH}_-^\infty \end{aligned} \quad (3.5.11)$$

Repeating these arguments for $F_{21}W(s)$ completes the proof of the following Lemma.

Lemma 3.5.1

(3.5.3) holds for $a(s)$ all-pass, $G_1(s) \in \mathbb{RH}_+^{\infty, (p-1) \times (m-1)}$ and $Q(s) \in \mathbb{RH}_-^{\infty, (p+m-1) \times (p+m-1)}$. Further,

$$\frac{1}{\sigma_1(G)} \begin{bmatrix} G_1+Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad (3.5.12)$$

is all-pass.

Proof:

Follows from the discussion of this section. □

The next result is a state-space version of Lemma 3.5.1; it demonstrates that $V(s)$ and $W(s)$ will block-diagonalise the entire set of error systems generated by any $U(s) \in \mathbb{BH}_-^\infty$; even non-rational ones. (Recall that $E(s) = \mathcal{F}_\ell(H(s), \frac{1}{\sigma}U(s))$ $U(s) \in \mathbb{BH}_-^\infty$ where $H(s)$ is given in (3.1.9)). In addition, it gives a simple state-space characterisation of $G_1(s)$ and $Q(s)$ in the case that the Riccati equations

(3.4.53) and (3.4.54) exist. These realisations will be employed at a later point of this chapter to establish MacMillan degree bounds for the super-optimal extension.

Theorem 3.5.2

Let $H(s)$ be the generator of all Nehari extensions given in Lemma 3.1.1. Then, assuming that the Riccati equations 3.4.35 and 3.4.36 have positive-definite destabilising solutions Ω and $\tilde{\Omega}$,

$$\begin{bmatrix} v^* & 0 \\ V_\perp^* & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} w & W_\perp & 0 \\ 0 & 0 & I_p \end{bmatrix} = \begin{bmatrix} \sigma a(s) & 0 & 0 \\ 0 & G_1 + Q_{11} & Q_{12} \\ 0 & Q_{21} & Q_{22} \end{bmatrix} \quad (3.5.13)$$

in which

$$(i) \ a(s) \in \mathbb{RH}_+^\infty \text{ and is all-pass, } G_1 \in \mathbb{RH}_+^{\infty, (m-1) \times (p-1)}, \ Q \in \mathbb{RH}_-^{\infty, (m+p-2) \times (m+p-2)}$$

$$(ii) \ G_1(s) \triangleq \left[\begin{array}{c|c} -A_{11}^* - A_{21}^*(A_{21}\tilde{\Omega} - B_2B_1^*) & \tilde{\Omega}^{-1}B_1B_\perp^* \\ \hline C_\perp C_1 \Omega^{-1}(\Omega\Sigma + \Gamma + \Omega\tilde{\Omega} + \Sigma\tilde{\Omega}) & 0 \end{array} \right] \quad (3.5.14)$$

$$(iii) \ Q(s) \triangleq \left[\begin{array}{cc|cc} \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^*) & (\tilde{\Omega}^{-1} + \Gamma^{-1} \Sigma) B_1 B_\perp^* & -\sigma \Gamma^{-1} C_1^* C_\perp^* & \\ \hline -C_\perp C_1(\Sigma + \Omega^{-1} \Gamma) & 0 & -\sigma I & \\ -\sigma B_\perp B_1^* & \sigma I & 0 & \end{array} \right] \quad (3.5.15)$$

$\sigma_1(G_0)$ has been abbreviated to just σ .

Proof:

See Appendix 3.C at the end of this Chapter. □

Remark 3.5.1:

It may be seen from the proof of Theorem 3.5.2 that the decomposition

$$\begin{bmatrix} V_\perp^* & 0 \\ 0 & I_m \end{bmatrix} H(s) \begin{bmatrix} W_\perp & 0 \\ 0 & I_p \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} + Q(s) \quad (3.5.16)$$

with $G_1(s) \in \mathbb{RH}_+^\infty$ and $Q(s) \in \mathbb{RH}_-^\infty$ is independent of the assumption on the existence and non-singularity of the solutions to the two Riccati equations; the state-space realisation, however, of $G_1(s)$ depends on this assumption. Note also that Theorem 3.5.2 remains valid for the Hankel-norm approximation case the only change of notation being the substitution of $Q(s) \in \mathbb{RH}_-^{\infty, (m+p-2) \times (m+p-2)}$ by $Q(s) \in \mathbb{RH}_-^{\infty, (m+p-2) \times (m+p-2)}(k)$. □

Next, Lemma 3.5.1 is generalised to the AAK case.

Lemma 3.5.3

The generator of all k -th order optimal error systems $G_a - F(s)$ can be diagonalised as

$$V_a^*(G_a - F)W_a(s) = \begin{bmatrix} \sigma_{k+1}a(s) & 0 & 0 \\ 0 & G_1 + Q_{11} & Q_{12} \\ 0 & Q_{21} & Q_{22} \end{bmatrix} \quad (3.5.17)$$

in which

(a) $a(s)$ is inner

(b) $G_1(s) \in \mathcal{H}_+^\infty$ and $Q(s) \in \mathcal{H}_-^\infty(k)$, with $Q(s)$ containing modes of $F(s)$ only.

$$(c) \frac{1}{\sigma_{k+1}(G_0)} \begin{bmatrix} G_1 + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \text{ is all-pass.} \quad (3.5.18)$$

Proof:

The augmented system $G_a(s)$ has Schmidt vectors $[f^T(s) \ 0]^T$ and $[g^T(s) \ 0]^T$ corresponding to the $(k+1)$ th Hankel singular value of $G_0(s)$ and therefore is block-diagonalised by $V_a(s)$ and $W_a(s)$. The fact that $V_a(s)$, $W_a(s)$ and $(G_a - F)(s)$ are all-pass, implies that

$$\frac{1}{\sigma_{k+1}(G_0)} \begin{bmatrix} G_1 + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad (3.5.19)$$

and $a(s)$ are all-pass.

To show that $a(s)$ is in fact inner, we consider the product

$$v^*(G - F_{11})w(s) = m^{-*}\xi^*(G - F_{11})w(s) \quad (3.5.20)$$

where $m(s) = \tilde{n}d^{-1}(s) \Rightarrow m^{-*}(s) = \tilde{n}^{-*}(s)d^*(s) = \frac{d(-s)}{\tilde{n}(-s)} \in \mathbb{RH}_+^\infty$. The only unstable modes in this product must be contained in

$$\frac{d(-s)}{\tilde{n}(-s)} \xi^*(s) F_{11}(s) w(s) \quad (3.5.21)$$

as a subset of the unstable poles of $\xi^* F_{11}(s)$. A straightforward state-space calculation based on a concrete realisation of $F(s)$ in (3.1.10) shows that all the poles of $F_{11}(s)$ cancel. In fact,

$$\xi^* F_{11}(s) = \sigma B_2 - A_{12}^*(sI + A_{11}^*)^{-1} \Sigma B_1 \quad (3.5.22)$$

On noting that all the modes of a minimal realisation of $\xi^* F_{11}(s)$ (resulting from removing all the

unobservable modes of the pair (A_{11}^*, A_{12}^*) are contained in the zeros of $d(-s)$ accounts for the remaining cancellations of all unstable modes in $a(s)$ thereby proving that $a(s)$ is inner.

Finally, to show that the decomposition in (b) is valid, assume that $V_a^*(s)$, $W_a(s)$ and $F(s)$ have realisations

$$V_a^*(s) = \begin{bmatrix} v^* \\ V_\perp^* \\ I_m \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c|c} \tilde{A} & \tilde{B} & 0 \\ \hline \tilde{C}_1 & \tilde{d}_1 & 0 \\ \tilde{C}_2 & \tilde{D}_2 & 0 \\ 0 & 0 & I_m \end{array} \right] \quad (3.5.23)$$

$$W_a(s) = \begin{bmatrix} w & W_\perp \\ I_p \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c|c|c} \check{A} & \check{B}_1 & \check{B}_2 & 0 \\ \hline \check{C} & \check{d}_1 & \check{D}_2 & 0 \\ 0 & 0 & 0 & I_p \end{array} \right] \quad (3.5.24)$$

and

$$F(s) \stackrel{s}{=} \left[\begin{array}{c|c|c} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & \hat{D}_{11} & \hat{D}_{12} \\ \hat{C}_2 & \hat{D}_{21} & 0 \end{array} \right] \quad (3.5.25)$$

with $V_a^*(s), W_a(s) \in \mathcal{RH}_+^\infty$ and $F(s) \in \mathcal{RH}_-^\infty(k)$. Next, let $F_{12}(s)$ admit a right factorisation

$$F_{12}(s) := NM^{-1}(s) \stackrel{s}{=} \begin{bmatrix} \hat{A} + \hat{B}_2 F & \hat{B}_2 \\ \hat{C}_1 + \hat{D}_{12} F & \hat{D}_{12} \end{bmatrix} \begin{bmatrix} \hat{A} + \hat{B}_2 F & \hat{B}_2 \\ F & I \end{bmatrix}^{-1} \quad (3.5.26)$$

in which F is chosen so that $\lambda(\hat{A} + \hat{B}_2 F) \cap \lambda(\tilde{A}) = \emptyset$. This can be done if no stable mode of $\hat{A}$ which is uncontrollable through $\hat{B}_2$ belongs to the spectrum $\lambda(\tilde{A})$; in fact no such assumption is required (see Remark 3.5.1).

$v^* F_{12}(s) = 0 \Rightarrow v^* N(s) = 0$ and we may write

$$0 \stackrel{s}{=} \begin{bmatrix} \tilde{A} & \tilde{B} \\ \tilde{C}_1 & \tilde{d}_1 \end{bmatrix} \begin{bmatrix} \hat{A} + \hat{B}_2 F & \hat{B}_2 \\ \hat{C}_1 + \hat{D}_{12} F & \hat{D}_{12} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|c} \tilde{A} & \tilde{B}(\hat{C}_1 + \hat{D}_{12} F) \\ \hline 0 & \hat{A} + \hat{B}_2 F \end{array} \middle| \begin{array}{c} \tilde{B} \hat{D}_{12} \\ \hat{B}_2 \end{array} \right] \quad (3.5.27)$$

Now let X be the (unique) solution to the linear matrix equation

$$X(\hat{A} + \hat{B}_2 F) - \tilde{A}X + \tilde{B}(\hat{C}_1 + \hat{D}_{12} F) = 0 \quad (3.5.28)$$

Since $V_a^*(s)$ is observable through its first output, the basis change

$$T = \begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \quad (3.5.29)$$

in (3.5.27) establishes that

$$\bar{B}\hat{D}_{12} + X\hat{B}_2 = 0 \quad (3.5.30)$$

which when substituted back into (3.5.28) gives

$$X\hat{A} - \bar{A}X + \bar{B}\hat{C}_1 = 0 \quad (3.5.31)$$

A similar argument based on $F_{21}w(s)=0$ will establish the existence of a matrix Y such that

$$Y\check{A} - \hat{A}Y + \hat{B}_1\check{C} = 0 \quad (3.5.32)$$

and

$$\hat{D}_{21}\check{C} - \hat{C}_2Y = 0 \quad (3.5.33)$$

In fact, a long but straightforward calculation based on the proof of Theorem 3.5.2, shows that (possibly non-minimal) realisations of $V_a(s)$ and $W_a(s)$ exist for which equations (3.5.30) to (3.5.33) admit the solutions $X = -\Gamma$ and $Y = -I$. No a priori spectrum assumptions are required.

Forming the product

$$\begin{bmatrix} V_\perp & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G - F_{11} & -F_{12} \\ -F_{21} & -F_{22} \end{bmatrix} \begin{bmatrix} W_\perp & 0 \\ 0 & I \end{bmatrix} \stackrel{s}{=} \begin{bmatrix} \bar{A} & \bar{B}C & -\bar{B}\hat{C}_1 & -\bar{B}\hat{D}_{11}\check{C} & -\bar{B}\hat{D}_{11}\check{D}_2 & -\bar{B}\hat{D}_{12} \\ 0 & A & 0 & B\check{C} & B\check{D}_2 & 0 \\ 0 & 0 & \hat{A} & \hat{B}_1\check{C} & \hat{B}_1\check{D}_2 & \hat{B}_2 \\ 0 & 0 & 0 & \check{A} & \check{B}_2 & 0 \\ \hline \check{C}_2 & \check{D}_2C & -\check{D}_2\hat{C}_1 & -\check{D}_2\hat{D}_{11}\check{C} & -\check{D}_2\hat{D}_{11}\check{D}_2 & -\check{D}_2\hat{D}_{12} \\ 0 & 0 & -\hat{C}_2 & -\hat{D}_{21}\check{C} & -\hat{D}_{21}\check{D}_2 & 0 \end{bmatrix} \quad (3.5.34)$$

and introducing the state-space transformations

$$T_1 = \begin{bmatrix} I & 0 & -X & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \quad T_2 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & Y \\ 0 & 0 & 0 & I \end{bmatrix} \quad (3.5.35)$$

together with equations (3.5.30) to (3.5.33) establishes the decomposition

$$\begin{bmatrix} V_\perp & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G-F_{11} & -F_{12} \\ -F_{21} & -F_{22} \end{bmatrix} \begin{bmatrix} W_\perp & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} G_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad (3.5.36)$$

in which $G_1(s) \in \mathcal{RH}_+^\infty$ and

$$Q(s) = \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 \check{D}_2 + Y \check{B}_2 & \hat{B}_2 \\ \hline -\check{D}_2 \hat{C}_1 + \check{C}_2 X & 0 & -\check{D}_2 \hat{D}_{12} \\ -\hat{C}_2 & -\hat{D}_{21} \check{D}_2 & 0 \end{array} \right] \in \mathcal{RH}_-^\infty(k) \quad (3.5.37)$$

This proves the result. $\square$

Remark 3.5.2

A direct calculation from the state-space formula for the scaled Schmidt vectors gives the following concrete realisation for $Q(s)$ in terms of Ω and $\tilde{\Omega}$ defined in 3.4.27 and 3.4.34:

$$Q(s) \triangleq \left[\begin{array}{c|cc} \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^*) & (\tilde{\Omega}^\# + \Gamma^{-1} \Sigma) B_1 B_\perp^* & -\sigma \Gamma^{-1} C_1^* C_\perp^* \\ \hline -C_\perp C_1 (\Sigma + \Omega^\# \Gamma) & 0 & -\sigma I \\ -\sigma B_\perp B_1^* & \sigma I & 0 \end{array} \right] \quad (3.5.38)$$

3.6 Construction of Super-optimal Nehari extensions

In this section a solution to the super-optimal Nehari extension problem is derived. Theorem 3.6.1 below characterises all Nehari extensions which minimise $s_2^\infty(\cdot)$; its proof also shows that the lexicographic minimisation of $s^\infty(\cdot)$ reduces to a sequence of problems of this type. The construction of the super-optimal extension is given in Algorithm 3.6.1 and the results of the section are summarised in Theorem 3.6.2.

Theorem 3.6.1

The family $\tilde{\mathcal{F}}(s)$ of all Nehari extensions $F(s) \in \mathcal{H}_-^\infty$ which minimise the pair

$$\left(s_1^\infty(G_0 + F), s_2^\infty(G_0 + F) \right) \quad (3.6.1)$$

with respect to lexicographic ordering is parametrised by

$$\tilde{\mathcal{F}}(s) = F_{11}(s) + V_\perp (-Q_{11} + \tilde{\mathcal{F}}_1) W_\perp^*(s) \quad (3.6.2)$$

where $\tilde{\mathcal{F}}_1$ denotes the family of Nehari extensions to $G_1(s)$.

Proof:

The parametrisation of all Nehari extensions is given by

$$\begin{aligned} V^*(G_0 + \mathcal{F})W &= V^* \mathcal{F}_\ell(G_a + F, U)W(s) \\ &= \mathcal{F}_\ell(V_a^*(G_a + F)W_a, U)(s) \\ &= \begin{bmatrix} \sigma_1(G_0)a(s) & 0 \\ 0 & G_1 + \mathcal{F}_\ell(Q, U) \end{bmatrix} U \in \frac{1}{\sigma_1} \mathcal{B}\mathcal{H}_-^\infty \end{aligned} \quad (3.6.3)$$

The family of error systems is hence diagonalised with all the degrees of freedom in $U(s) \in \sigma_1^{-1} \mathcal{B}\mathcal{H}_-^\infty$ only affecting the (2,2) block. This block can then be used to minimise the second singular value $s_2^\infty(G_0 + F) = \inf \{ \|G_1 + \mathcal{F}_\ell(Q, U)\|_\infty : U \in \sigma_1^{-1} \mathcal{B}\mathcal{H}_-^\infty \}$. Since $Q \in \mathcal{RH}_-^\infty$, $U \in \sigma_1^{-1} \mathcal{B}\mathcal{H}_-^\infty$ and $\|Q_{22}U\|_\infty < 1$, the small-gain theorem ensures that $\mathcal{F}_\ell(Q, U) \in \mathcal{H}_-^\infty$.

As will be shown next, replacing $\mathcal{F}_\ell(Q, U)$ by a free parameter in $\mathcal{H}_-^\infty$ does not change the optimisation problem. Given any $\mathcal{F}_1 \in \mathcal{H}_-^\infty$ we set

$$\begin{aligned} G_0 + \mathcal{F} &= V \begin{bmatrix} \sigma_1(G_0)a(s) & 0 \\ 0 & G_1 + \mathcal{F}_1 \end{bmatrix} W^* \\ &= V \begin{bmatrix} \sigma_1(G_0)a(s) & 0 \\ 0 & G_1 + Q_{11} \end{bmatrix} W^* + V \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{F}_1 - Q_{11} \end{bmatrix} W^* \\ &= G_0 + \mathcal{F}_\ell(F, 0) + V \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{F}_1 - Q_{11} \end{bmatrix} W^* \end{aligned} \quad (3.6.4)$$

gives

$$\mathcal{F}(s) = F_{11} + V_\perp (\mathcal{F}_1 - Q_{11}) W_\perp^* \in \mathcal{H}_-^\infty \quad (3.6.5)$$

Since $\mathcal{F}_\ell(Q, U) \in \mathcal{H}_-^\infty$, $s_2^\infty(G_0 + \mathcal{F})$ is bounded from above by $\|G_1\|_H$ by Nehari's theorem. Since the minimisation of $\|G_1 + F\|_\infty$ is achieved precisely by the Nehari extensions of $G_1(s)$, $\mathcal{F}_1$, while the substitution of $\mathcal{F}_\ell(Q, U)$ by any $F(s) \in \mathcal{F}_1$ gives rise to an element of $\mathcal{F}$, we conclude that (3.6.5) generates the extensions in $\mathcal{F}$ which minimise the pair $(s_1^\infty(G_0 + F), s_2^\infty(G_0 + F))$ with respect to lexicographic ordering. $\square$

As a consequence of Theorem 3.6.1, all solutions that minimise $s_2^\infty(G_0 + \mathcal{F})$ can be

parametrised in terms of all solutions that minimise $s_1^\infty(G_1 + \mathcal{F}_1)$ which is a problem of dimension $(p-1) \times (m-1)$. The procedure can continue until one of the dimensions is unity. At this point there is a unique Nehari extension and the process stops.

Remark 3.6.1

The proof of Theorem 3.6.1 shows that $\mathcal{F}_\ell(Q, U)$, $U \in \sigma_1^{-1} \mathcal{BK}^\infty$, generates all (optimal) Nehari extensions of $G_1(s)$. In fact it is shown in Lemma 3.10.5 that $\mathcal{F}_\ell(Q, U)$ generates all $\sigma_1(G_0)$ sub-optimal extensions of $G_1(s)$; this is a (strictly) larger set by the result of Theorem 2.7.2.

Algorithm for super-optimal Nehari extension

The following algorithm will generate the unique superoptimal Nehari extension.

Algorithm 3.6.1

Given any $G_0(s) \in \mathcal{RH}_+^{\infty, p \times m}$, this algorithm finds $F_{so}(s) \in \mathcal{RH}_-^{\infty, p \times m}$ which is a super-optimal extension of $G_0(s)$.

- 1) $r = \text{rank}(G_0)$
- 2) Construct F_{11} for G_0 (see 3.1.10)
- 3) $E_0 = G_0 + F_{11}$
- 4) $F_{so} = F_{11}$
- 5) $V = I_m$ and $W = I_p$
- 6) For $i=0$ to $r-1$
 - 1) Find V_i and W_i (See —)
 - 2) $V = VV_{\perp i}(s)$ and $W = WW_{\perp i}(s)$
 - 3) $G_{i+1} := [V_{\perp i}^* E_i W_{\perp i}]_+$
 - 4) $Q_{i+1} := [V_{\perp i}^* E_i W_{\perp i}]_-$
 - 5) Find F_{i+1} corresponding to G_{i+1} (see 3.1.10)
 - 6) $E_{i+1} = G_{i+1} + F_{i+1}$
 - 7) $F_{so} = F_{so} + V(F_{i+1} - Q_{i+1})W^*$

Note that if in the above algorithm the inner loop [6] was terminated after $\ell < r$ iterations, the first $\ell+1$ singular values would be minimised.

The results of this section can be summarised in the following theorem.

Theorem 3.6.2

Suppose we are given a rational $G_0(s) \in \mathbb{RH}_+^{\infty, (p \times m)}$ of rank r , where $1 < r \leq \min(p, m)$. Then its (unique) super-optimal extension $F_{so}(s)$ is given by

$$F_{so}(s) = F_{11}(s) + \sum_{k=0}^{r-2} \left\{ \prod_{i=0}^k V_{\perp i} (-Q_{k+1} + F_{k+1}) \left(\prod_{i=0}^k W_{\perp i} \right)^* \right\}(s) \quad (3.6.6)$$

where $F_{11}(s)$ corresponds to $G_0(s)$ and the matrices $V_{\perp i}(s)$, $W_{\perp i}(s)$, $F_k(s)$, $Q_k(s)$ in (3.6.6) are (recursively) defined as

$$\begin{aligned} V_{\perp k}(s), W_{\perp k}(s) & \quad \text{all-pass completions to the scaled maximal Schmidt pairs} \\ & \quad \text{of } \Gamma_{G_k} \text{ (Theorems 3.4.3 and 3.4.5)} \\ G_{k+1}(s) & \quad [V_{\perp k}^*(G_k(s) + F_k)W_{\perp k}]_+(s) \\ Q_{k+1}(s) & \quad [V_{\perp k}^*(G_k(s) + F_k)W_{\perp k}]_-(s) \\ F_{k+1}(s) & \quad \text{the rational Nehari extension of } G_{k+1}(s) \text{ corresponding to} \\ & \quad U(s)=0 \text{ in (3.1.8).} \end{aligned}$$

where $k=0, 1, \dots, r-2$.

Proof:

Follows from the repeated application of Theorem 3.6.1. □

3.7 Optimal level bounds

In this section the bounds (3.0.5) and (3.0.6) on the levels of the singular values of the super-optimal interpolating function will be established. The proof is given in Theorem 3.7.2 below. Before proving this result, however, we need to relate the Hankel singular values of matrices which sum to form all-pass matrices. The following Lemma due to K. Glover is a matrix generalisation of a result in [L1] which relates the number of the Hankel singular values of an all-pass matrix which are equal to one to the number of stable and unstable poles.

Lemma 3.7.1

Let $G(s) = G_1(s) + G_2(s)$ satisfy $G(s)G^*(s) = \sigma^2 I$ with $G_1(s) \in \mathbb{RH}_+^{\infty}$ and $G_2(s) \in \mathbb{RH}_-^{\infty}$. Suppose also that $\deg(G_0) = n_1 + n_2$ where $n_1 = \deg(G_1)$ and $n_2 = \deg(G_2)$. Then, if $n_1 \geq n_2$

$$\sigma_i(G_1) = \begin{cases} \sigma & i=1, 2, \dots, n_1 - n_2 \\ \sigma_{i-n_1+n_2}(G_2^*) \leq \sigma & i=n_1 - n_2 + 1, \dots, n_1 \end{cases} \quad (3.7.1)$$

Proof:

Let

$$G_1(s) \triangleq \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix} ; \quad G_2(s) \triangleq \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix} \quad (3.7.2)$$

be minimal realisations. By the assumptions of the Lemma,

$$G(s) \triangleq \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \left[\begin{array}{cc|c} A_1 & 0 & B_1 \\ 0 & A_2 & B_2 \\ \hline C_1 & C_2 & D_1+D_2 \end{array} \right] \quad (3.7.3)$$

is minimal also. It then follows from properties of minimal realisations of all-pass systems [G1] that there exist $P=P^*$ and $Q=Q^*$ such that

$$AP+PA^*+BB^*=0 \quad (3.7.4)$$

and

$$A^*Q+QA+C^*C=0 \quad (3.7.5)$$

with

$$PQ=\sigma^2 I \quad (3.7.6)$$

Let

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^* & P_3 \end{bmatrix} ; \quad Q = \begin{bmatrix} Q_1 & Q_2^* \\ Q_2 & Q_3 \end{bmatrix} \quad (3.7.7)$$

in which

$$P_1, Q_1 > 0 \quad \text{and} \quad P_3, Q_3 < 0.$$

From the (1,1) block of $PQ=\sigma^2 I$ we get,

$$\begin{aligned} \det(\lambda I_{n_1} - P_1 Q_1) &= \det[\lambda I_{n_1} - (\sigma^2 I + P_2 Q_2)] \\ &= \det[(\lambda - \sigma^2) I_{n_1} + P_2 Q_2] \\ &= (\lambda - \sigma^2)^{n_1 - n_2} \det[\lambda I_{n_2} - (\sigma^2 I - Q_2 P_2)] \\ &= (\lambda - \sigma^2)^{n_1 - n_2} \det[\lambda I_{n_2} - P_3 Q_3] \end{aligned} \quad (2.7.8)$$

(by (2, 2) block of $PQ=\sigma^2 I$). This proves that

$$\sigma_i(G_1) = \begin{cases} \sigma & i=1, 2, \dots, n_1-n_2 \\ \sigma_{i-n_1+n_2}(G_2^*) & i=n_1-n_2+1, \dots, n_1 \end{cases} \quad (3.7.9)$$

Since $\sigma_1(G_1) = \|G_1\|_H \leq \|G_1 + G_2\|_\infty = \sigma$ by Nehari's theorem, (3.7.1) is verified. $\square$

The following inequality may now be established:

Theorem 3.7.2

Suppose $G_1(s) \in \mathcal{RH}_+^\infty$ is defined in (3.5.12). Then

$$\sigma_{i+1}(G_0) \geq \sigma_i(G_1) \quad i=1, 2, \dots, n-1 \quad (3.7.10)$$

Proof:

For G_a and F defined in (3.5.1) and Lemma 3.1.1, $\frac{1}{\sigma_1}(G_a + F)$ is all-pass with $\deg(G_a) = n > \deg(F) = n-1$. Hence, by Lemma 3.7.1

$$\begin{aligned} \sigma_{i+1}(G_0) &= \sigma_i(F^*) \quad i=1, 2, \dots, n-1 \\ &= \inf_{\Phi \in \mathcal{H}_-^\infty(i-1)} \|F^* + \Phi\|_\infty \\ &= \inf_{\Phi \in \mathcal{H}_-^\infty(i-1)} \|F^* + G_a^* + \Phi\|_\infty \\ &\geq \inf_{\Phi \in \mathcal{H}_-^\infty(i-1)} \left\| \begin{bmatrix} W_\perp^* & 0 \\ 0 & I_p \end{bmatrix} (F^* + G_a^* + \Phi) \begin{bmatrix} V_\perp & 0 \\ 0 & I_m \end{bmatrix} \right\|_\infty \\ &\geq \inf_{\Phi \in \mathcal{H}_-^\infty(i-1)} \left\| \begin{bmatrix} G_1^* + Q_{11}^* & Q_{12}^* \\ Q_{21}^* & Q_{22}^* \end{bmatrix} + \Phi \right\|_\infty \\ &= \sigma_i(Q(-s)) \end{aligned} \quad (3.7.11)$$

Lemma 3.5.1 shows that

$$\frac{1}{\sigma_1(G_0)} \begin{bmatrix} G_1 + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad (3.7.12)$$

is all-pass. Moreover, it is easily verified that Q_{12} (or alternatively Q_{21}) is full rank on the imaginary axis, whence $\|G_1 + Q_{11}\|_\infty < \sigma_1 \Rightarrow \sigma_1(G_1) < \sigma_1$ (by Nehari's theorem). Therefore by Lemma 3.7.1

$$\sigma_i(G_1) \leq \sigma_i(Q(-s)) \leq \sigma_{i+1}(G_0) \quad i=1, 2, \dots, n-1 \quad (3.7.13)$$

Propagating this inequality through the subsequent steps of the algorithm gives (3.0.6). $\square$

3.8 MacMillan degree bounds for super-optimal extensions

The purpose of this section is to carry out a detailed cancellation analysis for Algorithm 3.6.1. The results show that several cancellation phenomena are quite general and lead to a superoptimal Nehari extension of surprisingly low order. If $\deg(G_0)=n$, there exist Nehari extensions of degree $n-1$; these are optimal in the supremum norm sense. It will be shown that $s_2^\infty(E)$ may be optimised with an extension of degree $2n-3=\sum_{i=1}^2(n-i)$. Following that, a simple induction leads to (3.0.7), and (3.0.8) if the algorithm is terminated after minimising only $\ell < \text{rank}(G_0)$ levels. The results of this section will be proved under the following assumption:

Assumption 3.8.1

It is assumed that the required solutions to (3.4.53) and (3.4.54) exist.

We begin by illustrating one of the cancellation phenomena which occurs in (3.6.5) in the $r=2$ problem. From Theorems 3.4.5 and 3.5.2 we see that

$$\deg(V_\perp), \deg(W_\perp^*), \deg(Q_{11}) \leq n-1 \quad (3.8.1a)$$

which implies that

$$\deg(V_\perp Q_{11} W_\perp^*) \leq 3(n-1) \quad (3.8.1b)$$

The next result shows that

$$V_\perp Q_{11} W_\perp^* = J(s) + F_{11} \quad (3.8.2a)$$

in which $\deg(J) \leq 2(n-1)$, whence

$$\deg(F_{11} - V_\perp Q_{11} W_\perp^*) \leq 2(n-1) \quad (3.8.2b)$$

Lemma 3.8.1

$V_\perp Q_{11} W_\perp^*(s)$ may be decomposed as $V_\perp Q_{11} W_\perp^*(s) = J(s) + F_{11}(s)$ in which

$$J(s) \triangleq \begin{bmatrix} A_J & B_J \\ C_J & -\sigma U \end{bmatrix} \quad (3.8.3)$$

where

$$A_J = \begin{bmatrix} A_{11} + A_{12}(A_{12}^* \Omega - C_2^* C_1) & -\Omega^{-1} \Gamma (\tilde{\Omega}^{-1} + \Gamma^{-1} \Sigma) B_1 B_\perp^* B_\perp \\ 0 & A_{11} + (\tilde{\Omega} A_{21}^* - B_1 B_2^*) A_{21} \end{bmatrix} \quad (3.8.4)$$

$$B_J = \begin{bmatrix} -\Omega^{-1} \Gamma (\tilde{\Omega}^{-1} + \Gamma^{-1} \Sigma) B_1 B_\perp^* B_\perp B_1^* \tilde{\Omega}^{-1} \\ B_1 B_\perp^* B_\perp + \tilde{\Omega} A_{21}^* B_2 \end{bmatrix} \quad (3.8.5)$$

$$C_J = \begin{bmatrix} C_\perp^* C_\perp C_1 + C_2 A_{12}^* \Omega & -(C_1 \Sigma + \sigma U B_1^*) \tilde{\Omega}^{-1} \end{bmatrix} \quad (3.8.6)$$

Proof:

From (3.4.55) and (3.5.15) we see that

$$\begin{aligned} V_\perp Q_{11} &\triangleq \begin{bmatrix} A_{11} + A_{12}(A_{12}^* \Omega - C_2^* C_1) & \Omega^{-1} C_1 C_\perp^* \\ C_\perp^* C_\perp C_1 + C_2 A_{12}^* \Omega & C_\perp^* \end{bmatrix} \\ &\begin{bmatrix} \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^*) & (\tilde{\Omega}^{-1} + \Gamma^{-1} \Sigma) B_1 B_\perp^* \\ -C_\perp^* C_1 (\Sigma + \Omega^{-1} \Gamma) & 0 \end{bmatrix} \\ &\triangleq \left[\begin{array}{cc|c} A_{11} + A_{12}(A_{12}^* \Omega - C_2^* C_1) & \Omega^{-1} C_1 C_\perp^* C_\perp C_1 (\Sigma + \Omega^{-1} \Gamma) & 0 \\ 0 & \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^*) & (\tilde{\Omega}^{-1} + \Gamma^{-1} \Sigma) B_1 B_\perp^* \\ \hline C_\perp^* C_\perp C_1 + C_2 A_{12}^* \Omega & -C_\perp^* C_\perp C_1 (\Sigma + \Omega^{-1} \Gamma) & 0 \end{array} \right] \end{aligned} \quad (3.8.7)$$

Introducing the change of basis

$$T = \begin{bmatrix} I & -\Omega^{-1} \Gamma \\ 0 & I \end{bmatrix} \quad (3.8.8)$$

shows that $V_\perp Q_{11}(s)$ is equal to

$$\left[\begin{array}{cc|c} A_{11}+A_{12}(A_{12}^*\Omega-C_2^*C_1) & A(1,2) & -\Omega^{-1}\Gamma(\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^* \\ 0 & \Gamma^{-1}(\sigma^2A_{11}^*+\Sigma A_{11}\Sigma-\sigma C_1^*UB_1^*) & (\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^* \\ \hline C_\perp^*C_\perp C_1+C_2A_{12}^*\Omega & C(1,2) & 0 \end{array} \right] \quad (3.8.9)$$

in which

$$\begin{aligned} C(1,2) &= -C_\perp^*C_\perp C_1\Sigma+C_2A_{12}^*\Gamma \\ &= -C_\perp^*C_\perp C_1\Sigma-C_2[-(A_{12}^*\Sigma+\sigma A_{21})\Sigma+\sigma(A_{21}\Sigma+\sigma A_{12}^*)] \\ &= -C_\perp^*C_\perp C_1\Sigma-C_2C_2^*C_1\Sigma+\sigma C_2B_2B_1^* \\ &= -C_1\Sigma-\sigma UB_1^* \end{aligned} \quad (3.8.10)$$

by (3.1.1), (3.1.2) and (3.1.5); also

$$\begin{aligned} A(1,2) &= -\Omega^{-1}C_1^*C_\perp^*C_\perp C_1(\Sigma+\Omega^{-1}\Gamma)+[A_{11}+A_{12}(A_{12}^*\Omega-C_2^*C_1)]\Omega^{-1}\Gamma \\ &\quad -\Omega^{-1}(\sigma^2A_{11}^*+\Sigma A_{11}\Sigma-\sigma C_1^*UB_1^*) \end{aligned}$$

Invoking the Riccati equation (3.4.53) gives

$$A(1,2) = -\Omega^{-1}\{C_1^*C_\perp^*C_\perp C_1\Sigma+\sigma^2A_{11}^*+\Sigma A_{11}\Sigma-\sigma C_1^*UB_1^*+(A_{11}^*-C_1^*C_2A_{12}^*)\Gamma\}$$

Substituting from (3.1.1), (3.1.2) and (3.1.5), and cancelling terms gives

$$A(1,2) = -\Omega^{-1}\{C_1^*C_\perp^*C_\perp C_1\Sigma+C_1^*C_2C_2^*C_1\Sigma+(A_{11}^*\Sigma+\Sigma A_{11})\Sigma\} = 0 \quad (3.8.11)$$

by (3.1.5) and the orthogonality of $[C_2 \ C_\perp^*]$. Hence $V_\perp Q_{11} W_\perp^*$ may be written as

$$\begin{aligned} &\left[\begin{array}{cc|c} A_{11}+A_{12}(A_{12}^*\Omega-C_2^*C_1) & 0 & -\Omega^{-1}\Gamma(\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^* \\ 0 & \Gamma^{-1}(\sigma^2A_{11}^*+\Sigma A_{11}\Sigma-\sigma C_1^*UB_1^*) & (\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^* \\ \hline C_\perp^*C_\perp C_1+C_2A_{12}^*\Omega & -(C_1\Sigma+\sigma UB_1^*) & 0 \end{array} \right] \cdot \\ &\left[\begin{array}{cc} A_{11}+(\tilde{\Omega}A_{21}^*-B_1B_2^*)A_{21} & B_1B_\perp^*B_\perp+\tilde{\Omega}A_{21}^*B_2 \\ B_\perp B_1^*\tilde{\Omega}^{-1} & B_\perp \end{array} \right] = \\ &\left[\begin{array}{ccc|c} A_{11}+A_{12}(A_{12}^*\Omega-C_2^*C_1) & 0 & -\Omega^{-1}\Gamma(\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^*B_\perp B_1^*\tilde{\Omega}^{-1} & B(1,1) \\ 0 & \Gamma^{-1}(\sigma^2A_{11}^*+\Sigma A_{11}\Sigma-\sigma C_1^*UB_1^*) & (\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^*B_\perp B_1^*\tilde{\Omega}^{-1} & B(2,1) \\ 0 & 0 & A_{11}+(\tilde{\Omega}A_{21}^*-B_1B_2^*)A_{21} & B(3,1) \\ \hline C_\perp^*C_\perp C_1+C_2A_{12}^*\Omega & -(C_1\Sigma+\sigma UB_1^*) & 0 & 0 \end{array} \right] \quad (3.8.12) \end{aligned}$$

where we have defined,

$$B(1,1) = -\Omega^{-1}\Gamma(\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^*B_\perp \quad (3.8.13)$$

$$B(2,1) = (\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^*B_\perp \quad (3.8.14)$$

$$B(3,1) = B_1B_\perp^*B_\perp + \tilde{\Omega}A_{21}^*B_2 \quad (3.8.15)$$

The change of basis

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & -\tilde{\Omega}^{-1} \\ 0 & 0 & I \end{bmatrix} \quad (3.8.16)$$

shows that $V_\perp Q_{11} W_\perp^*$ is equal to

$$\left[\begin{array}{ccc|c} A_{11}+A_{12}(A_{12}^*\Omega-C_2^*C_1) & 0 & -\Omega^{-1}\Gamma(\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^*B_\perp B_1^*\tilde{\Omega}^{-1} & B(1,1) \\ 0 & \Gamma^{-1}(\sigma^2 A_{11}^*+\Sigma A_{11}\Sigma-\sigma C_1^*UB_1^*) & A(2,3) & B(2,1) \\ 0 & 0 & A_{11}+(\tilde{\Omega}A_{21}^*-B_1B_2^*)A_{21} & B(3,1) \\ \hline C_\perp^*C_\perp C_1+C_2A_{12}^*\Omega & -(C_1\Sigma+\sigma UB_1^*) & -(C_1\Sigma+\sigma UB_1^*)\tilde{\Omega}^{-1} & 0 \end{array} \right] \quad (3.8.17)$$

where

$$B(1,1) = -\Omega^{-1}\Gamma(\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^*B_\perp \quad (3.8.18)$$

$$B(3,1) = B_1B_\perp^*B_\perp + \tilde{\Omega}A_{21}^*B_2 \quad (3.8.19)$$

and

$$\begin{aligned} B(2,1) &= \Gamma^{-1}\Sigma B_1B_\perp^*B_\perp - A_{21}^*B_2 \\ &= \Gamma^{-1}\{\Sigma B_1B_\perp^*B_\perp + [-\Sigma(\sigma A_{12}+\Sigma A_{21}^*)+\sigma(A_{21}^*\sigma+\Sigma A_{12})]B_2\} \end{aligned} \quad (3.8.20)$$

which by (3.1.1) and (3.1.2) becomes

$$\begin{aligned} B(2,1) &= \Gamma^{-1}\{\Sigma B_1B_\perp^*B_\perp + \Sigma B_1B_2^*B_2 - \sigma C_1^*C_2B_2\} \\ &= \Gamma^{-1}\{\Sigma B_1 + \sigma C_1^*U\} \end{aligned} \quad (3.8.21)$$

Also

$$\begin{aligned} A(2,3) &= (\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^*B_\perp B_1^*\tilde{\Omega}^{-1} - \tilde{\Omega}^{-1}(A_{11}-B_1B_2^*A_{21}) - A_{21}^*A_{21} \\ &\quad + \Gamma^{-1}(\sigma^2 A_{11}^*+\Sigma A_{11}\Sigma - \sigma C_1^*UB_1^*)\tilde{\Omega}^{-1} \end{aligned} \quad (3.8.22)$$

Substituting the Riccati equation (3.4.54) gives

$$A(2,3) = (A_{11}-B_1B_2^*A_{21})^*\tilde{\Omega}^{-1} + \Gamma^{-1}\Sigma B_1B_\perp^*B_\perp B_1^*\tilde{\Omega}^{-1} + \Gamma^{-1}(\sigma^2 A_{11}^*+\Sigma A_{11}\Sigma - \sigma C_1^*UB_1^*)\tilde{\Omega}^{-1}$$

Invoking (3.1.1), (3.1.2) and (3.1.18) and simplifying gives

$$A(2,3) = \Gamma^{-1} \{ \Sigma^2 A_{11}^* + \Sigma A_{11} \Sigma + \Sigma B_1 B_1^* B_\perp B_\perp^* + \Sigma B_1 B_2^* B_2 B_1^* \} \tilde{\Omega}^{-1}$$

which by (3.1.1) and the orthogonality of $[B_2^* \ B_\perp^*]$ gives

$$A(2,3) = 0 \quad (4.16)$$

This last equation (ie (3.8.23)), together with (3.8.17) produces the decomposition in (3.8.2a). $\square$

Remark 3.8.1

Should the assumption on the existence of the Riccati solutions Ω and $\tilde{\Omega}$ fail, the proof of Lemma 3.8.1 would still be valid by defining Ω and $\tilde{\Omega}$ via (3.4.38) and (3.4.43) and replacing proper inverses by pseudo-inverses throughout. Consider for example the term $C(1,2)$ defined in (3.8.10); this becomes

$$C(1,2) = -C_\perp^* C_\perp C_1 \Sigma + C_2 A_{12}^* \Omega \Omega^\# \Gamma$$

Invoking the transformation T_1 defined in (3.4.33),

$$A_{12}^* \Omega \Omega^\# = A_{12}^* T_1^* \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_1^{-*} = [\tilde{A}_{13}^* \ 0] T_1^{-*} = A_{12}^*$$

so that $C(1,2)$ in (3.8.10) remains unchanged. Further, the nonsingularity of the reduced Riccati equation solution Ω_r (Lemma 3.4.4) shows that

$$(A_{11} - A_{12} C_2^* C_1) \Omega^\# + \Omega^\# (A_{11} - A_{12} C_2^* C_1) \Omega^\# + A_{12} A_{12}^* - \Omega^\# C_1^* C_\perp^* C_\perp C_1 \Omega^\# = 0$$

which accounts for the output decomposition ($A(1,2)=0$) of the Lemma.

Removing the uncontrollable and unobservable modes associated with the rank defect of $\Omega^\#$ and $\tilde{\Omega}^\#$ in the state-space of (3.8.4) to (3.8.6) shows that in the general case,

$$\deg(J) \leq \deg(f) + \deg(g) - 2 \leq 2(n-1)$$

where f and g denote the maximal Schmidt vectors of Γ_{G_0} . $\square$

Remark 3.8.2

Algorithm 3.6.1 is a sequential updating scheme of the super-optimal extension $F_{s,o}(s)$, of the form $F_{s,o}^{i+1} := \phi(F_{s,o}^i, F_i)$, where each $F_i(s)$ is defined as a particular Nehari extension of $G_i(s)$. The choice of $F_i(s)$ made at step 6.5 of the algorithm, however, is far from being the only possible one. In fact, the algorithm would remain valid if $F_i(s)$ at this step is chosen as *any* Nehari extension $\mathfrak{T}_i(F, U)$ of $G_i(s)$.

Using the equations arising from the (1,3) and (3,1) blocks of 3.5.3 shows that the decomposition in Lemma 3.8.1 may be generalised to

$$V_\perp \mathfrak{F}_\ell(Q, U) W_\perp^* = J(s) + \mathfrak{F}_\ell(F, U) \quad (3.8.23)$$

for all $U \in \sigma^{-1} \mathfrak{B} \mathcal{H}_\perp^\infty$. Since

$$\begin{aligned} F_{s,o}(s) &= \mathfrak{F}_\ell(F, U) + V_\perp \mathfrak{F}_\ell(Q, U) W_\perp^* + V_\perp \mathfrak{F}_1 W_\perp^* \\ &= -J(s) + V_\perp \mathfrak{F}_1 W_\perp^* \end{aligned} \quad (3.8.24)$$

the “updated” super-optimal extension constructed at step 6.7 depends *only* on the choice of F_{i+1} made at the current step. Since finally a rank one Nehari extension problem is encountered, the corresponding extension F_{r-1} at the last step is unique and therefore so is $F_{s,o}$. $\square$

Substituting (3.8.2a) into (3.6.5) gives

$$F_{s,o}(s) = V_\perp \mathfrak{F}_1 W_\perp^* - J(s) \quad (3.8.25)$$

in which $\mathfrak{F}_1$ represents any rational Nehari extension of $G_1(s)$. In the next result it is shown that

$$\deg(F_{s,o}) \leq n-1 + \deg(\mathfrak{F}_1) \quad (3.8.26)$$

Since $\deg(G_1) \leq n-1$, there exist $\mathfrak{F}_1(s)$ such that

$$\deg(F_{s,o}) \leq 2n-3 \quad (3.8.27)$$

Lemma 3.8.2

Suppose that

$$\mathfrak{F}_1(s) \triangleq \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \quad (3.8.28)$$

is any rational Nehari extension of $G_1(s)$. Then

$$F_{s,o}(s) = -J(s) + V_\perp \mathfrak{F}_1 W_\perp^* \triangleq \begin{bmatrix} A_F & B_F \\ C_F & D_F \end{bmatrix} \quad (3.8.29)$$

where

$$A_F = \begin{bmatrix} A_{11} + A_{12}(A_{12}^* \Omega - C_2^* C_1) & \Omega^{-1} C_1^* C_\perp^* \hat{C} \\ 0 & \hat{A} \end{bmatrix} \quad (3.8.30)$$

$$B_F = \begin{bmatrix} \Omega^{-1}(\Sigma B_1 B_\perp^* B_\perp + C_1^* C_\perp^* \hat{D} B_\perp - \Gamma A_{21}^* B_2) \\ \hat{B} B_\perp - P_{12}^*(B_1 B_\perp^* B_\perp + \tilde{\Omega} A_{12}^* B_2) \end{bmatrix} \quad (3.8.31)$$

$$C_F = \begin{bmatrix} C_\perp^* C_\perp C_1 + C_2 A_{12}^* \Omega & C_\perp^* \hat{C} \end{bmatrix} \quad (3.8.32)$$

and

$$D_F = C_\perp^* \hat{D} B_\perp + \sigma U \quad (3.8.33)$$

Proof:

We begin by noting that

$$\begin{bmatrix} -J(s) & V_\perp(s) \\ W_\perp(s) & 0 \end{bmatrix} \stackrel{s}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (3.8.34)$$

with

$$A = \begin{bmatrix} A_{11} + A_{12}(A_{12}^* \Omega - C_2^* C_1) & -\Omega^{-1} \Gamma (\tilde{\Omega}^{-1} + \Gamma^{-1} \Sigma) B_1 B_\perp^* B_\perp B_1^* \tilde{\Omega}^{-1} \\ 0 & A_{11} + (\tilde{\Omega} A_{21}^* - B_1 B_2^*) A_{21} \end{bmatrix} \quad (3.8.35i)$$

$$B = \begin{bmatrix} \Omega^{-1} \Gamma (\tilde{\Omega}^{-1} + \Gamma^{-1} \Sigma) B_1 B_\perp^* B_\perp & \Omega^{-1} C_1^* C_\perp^* \\ -B_1 B_\perp^* B_\perp - \tilde{\Omega} A_{21}^* B_2 & 0 \end{bmatrix} \quad (3.8.35ii)$$

$$C = \begin{bmatrix} C_\perp^* C_\perp C_1 + C_2 A_{12}^* \Omega & -(C_1 \Sigma + \sigma U B_1^*) \tilde{\Omega}^{-1} \\ 0 & -B_\perp B_1^* \tilde{\Omega}^{-1} \end{bmatrix} \quad (3.8.35iii)$$

and

$$D = \begin{bmatrix} \sigma U & C_\perp^* \\ B_\perp & 0 \end{bmatrix} \quad (3.8.35iv)$$

Applying Lemma 1.1.1 to (3.8.28) and (3.8.34) gives that $F_{s,o}(s)$ has a state-space realisation

$$\left[\begin{array}{ccc|c} A_{11}+A_{12}(A_{12}^*\Omega-C_2^*C_1) & A(1,2) & \Omega^{-1}C_1^*C_\perp^*\hat{C} & B(1,1) \\ 0 & A_{11}+(\tilde{\Omega}A_{21}^*-B_1B_2^*)A_{21} & 0 & -B_1B_\perp^*B_\perp-\tilde{\Omega}A_{21}^*B_2 \\ 0 & \hat{B}B_\perp B_1^*\tilde{\Omega}^{-1} & \hat{A} & \hat{B}B_\perp \\ \hline C_\perp^*C_\perp C_1+C_2A_{12}^*\Omega & C(1,2) & C_\perp^*\hat{C} & \sigma U+C_\perp^*\hat{D}B_\perp \end{array} \right] \quad (3.8.36)$$

where

$$\begin{aligned} A(1,2) &= -\Omega^{-1}\Gamma(\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^*B_\perp B_1^*\tilde{\Omega}^{-1}-\Omega^{-1}C_1^*C_\perp^*\hat{D}B_\perp B_1^*\tilde{\Omega}^{-1} \\ B(1,1) &= \Omega^{-1}\{\Gamma(\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^*B_\perp+C_1^*C_\perp^*\hat{D}B_\perp\} \\ C(1,2) &= -(C_1\Sigma+\sigma UB_1^*)\tilde{\Omega}^{-1}-C_\perp^*\hat{D}B_\perp B_1^*\tilde{\Omega}^{-1} \end{aligned}$$

Next, we put (3.8.36) into a more convenient form by introducing the basis change

$$T = \begin{bmatrix} I & \Omega^{-1}\Gamma\tilde{\Omega}^{-1} & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \quad (3.8.37)$$

so that $F_{s,o}(s)$ becomes:

$$\left[\begin{array}{ccc|c} A_{11}+A_{12}(A_{12}^*\Omega-C_2^*C_1) & A(1,2) & \Omega^{-1}C_1^*C_\perp^*\hat{C} & B(1,1) \\ 0 & A_{11}+(\tilde{\Omega}A_{21}^*-B_1B_2^*)A_{21} & 0 & -B_1B_\perp^*B_\perp-\tilde{\Omega}A_{21}^*B_2 \\ 0 & -\hat{B}B_\perp B_1^*\tilde{\Omega}^{-1} & \hat{A} & \hat{B}B_\perp \\ \hline C_\perp^*C_\perp C_1+C_2A_{12}^*\Omega & C(1,2) & C_\perp^*\hat{C} & C_\perp^*\hat{D}B_\perp \end{array} \right] \quad (3.8.38)$$

where

$$B(1,1) = \Omega^{-1}\{\Sigma B_1B_\perp^*B_\perp+C_1^*C_\perp^*\hat{D}B_\perp-\Gamma A_{21}^*B_2\}$$

The terms $A(1,2)$ and $C(1,2)$ can be written as

$$\begin{aligned} C(1,2) &= -\{(C_1\Sigma+\sigma UB_1^*)+C_\perp^*\hat{D}B_\perp B_1^*+C_\perp^*C_\perp C_1\Omega^{-1}\Gamma+C_2A_{12}^*\Gamma\}\tilde{\Omega}^{-1} \\ &= \{-C_1\Sigma-C_\perp^*\hat{D}B_\perp B_1^*-C_\perp^*C_\perp C_1\Omega^{-1}\Gamma+C_2C_2^*C_1\Sigma\}\tilde{\Omega}^{-1} \quad \text{by (3.1.1)} \\ &= -\{C_\perp^*\hat{D}B_\perp B_1^*-C_\perp^*C_\perp C_1(\Omega^{-1}\Gamma+\Sigma)\}\tilde{\Omega}^{-1} \end{aligned} \quad (3.8.39)$$

$$\begin{aligned} A(1,2) &= -\Omega^{-1}\Gamma(\tilde{\Omega}^{-1}+\Gamma^{-1}\Sigma)B_1B_\perp^*B_\perp B_1^*\tilde{\Omega}^{-1}-\Omega^{-1}C_1^*C_\perp^*\hat{D}B_\perp B_1^*\tilde{\Omega}^{-1} \\ &\quad +\Omega^{-1}\Gamma\tilde{\Omega}^{-1}[A_{11}+(\tilde{\Omega}A_{21}^*-B_1B_2^*)A_{21}]-[A_{11}+A_{12}(A_{12}^*\Omega-C_2^*C_1)]\Omega^{-1}\Gamma\tilde{\Omega}^{-1} \end{aligned}$$

Invoking the Riccati equations (3.4.53) and (3.4.54) gives

$$\begin{aligned} A(1,2) &= -\Omega^{-1}\{\Gamma(A_{11}-B_1B_2^*A_{21})^*+\Sigma B_1B_\perp^*B_\perp B_1^*-(A_{11}-A_{12}C_2^*C_1)^*\Gamma\}\tilde{\Omega}^{-1} \\ &\quad -\Omega^{-1}C_1^*C_\perp^*\hat{D}B_\perp B_1^*\tilde{\Omega}^{-1}-\Omega^{-1}C_1^*C_\perp^*C_\perp C_1\Omega^{-1}\Gamma\tilde{\Omega}^{-1} \end{aligned}$$

Several substitutions from (3.1.1), (3.1.2) and (3.1.5) yields after cancelling terms

$$A(1,2) = -\Omega^{-1}C_1^*C_\perp^*C_\perp C_1(\Sigma+\Omega^{-1}\Gamma)\tilde{\Omega}^{-1}-\Omega^{-1}C_1^*C_\perp^*\hat{D}B_\perp B_1^*\tilde{\Omega}^{-1} \quad (3.8.40)$$

The cancellations predicted by this lemma rely on the fact that every member of the class $\mathcal{F}_1(s)$ must satisfy certain bounded real type equations [L6]. Since $\mathcal{F}_1(s)$ is the class of all (rational) Nehari extensions of $G_1(s)$, the corresponding error systems ($\mathcal{E}(s)$) are given by

$$\mathcal{E}(s)=G_1(s)+\mathcal{F}_1(s) \doteq \left[\begin{array}{cc|c} -A_{11}^*-A_{21}^*(A_{21}\tilde{\Omega}-B_2B_1^*) & 0 & \tilde{\Omega}^{-1}B_1B_\perp^* \\ 0 & \hat{A} & \hat{B} \\ \hline C_\perp C_1\Omega^{-1}(\Omega\Sigma+\Gamma+\Omega\tilde{\Omega}+\Sigma\tilde{\Omega}) & \hat{C} & \hat{D} \end{array} \right] \quad (3.8.41)$$

and the required bounded real-type equations are [L6]:

$$\begin{aligned} &\left[\begin{array}{cc} -A_{11}^*-A_{21}^*(A_{21}\tilde{\Omega}-B_2B_1^*) & 0 \\ 0 & \hat{A} \end{array} \right] \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{array} \right] \\ &+ \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{array} \right] \left[\begin{array}{cc} A_{11}+(\tilde{\Omega}A_{21}^*-B_1B_2^*)A_{21} & 0 \\ 0 & \hat{A}^* \end{array} \right] \\ &+ \left[\begin{array}{c} \tilde{\Omega}^{-1}B_1B_\perp^* \\ \hat{B} \end{array} \right] \left[\begin{array}{cc} B_\perp B_1^*\tilde{\Omega}^{-1} & \hat{B}^* \end{array} \right] + \left[\begin{array}{c} 0 \\ \hat{C} \end{array} \right] \left[\begin{array}{cc} 0 & \hat{C} \end{array} \right] = 0 \end{aligned} \quad (3.8.42)$$

and

$$\begin{aligned} &\left[\begin{array}{cc} C_\perp C_1\Omega^{-1}(\Omega\Sigma+\Gamma+\Omega\tilde{\Omega}+\Sigma\tilde{\Omega}) & \hat{C} \end{array} \right] \left[\begin{array}{cc} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{array} \right] \\ &+ \hat{D} \left[\begin{array}{cc} B_\perp B_1^*\tilde{\Omega}^{-1} & \hat{B}^* \end{array} \right] + \left[\begin{array}{cc} 0 & \hat{C} \end{array} \right] = 0 \end{aligned} \quad (3.8.43)$$

A long and rather tedious calculation based on (3.4.53), (3.4.54) and (3.4.59) verifies that

$$P_{11}=(\Omega\Sigma+\Gamma+\Omega\tilde{\Omega}+\Sigma\tilde{\Omega})^{-1}(\Gamma+\Omega\Sigma)\tilde{\Omega}^{-1} \quad (3.8.44)$$

in which the indicated inverse exists by (3.4.64). The last transformation we require is

$$T = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & P_{12}^* & I \end{bmatrix} \quad (3.8.45)$$

which is applied to (3.8.38). This shows that $F_{s,o}(s)$ is equal to

$$\left[\begin{array}{ccc|c} A_{11}+A_{12}(A_{12}^*\Omega-C_2^*C_1) & A(1,2) & \Omega^{-1}C_1^*C_\perp^*\hat{C} & B(1,1) \\ 0 & A_{11}+(\tilde{\Omega}A_{21}^*-B_1B_2^*)A_{21} & 0 & -B_1B_\perp^*B_\perp-\tilde{\Omega}A_{21}^*B_2 \\ 0 & A(3,2) & \hat{A} & B(3,1) \\ \hline C_\perp^*C_\perp C_1+C_2A_{12}^*\Omega & C(1,2) & C_\perp^*\hat{C} & \sigma U+C_\perp^*\hat{D}B_\perp \end{array} \right] \quad (3.8.46)$$

where

$$\begin{aligned} A(1,2) &= -\Omega^{-1}C_1^*C_\perp^*\{\hat{D}B_\perp B_1^*\tilde{\Omega}^{-1}+C_\perp C_1(\Sigma+\Omega^{-1}\Gamma)\tilde{\Omega}^{-1}+\hat{C}P_{12}^*\} \\ &= -\Omega^{-1}C_1^*C_\perp^*\{C_\perp C_1(\Sigma+\Omega^{-1}\Gamma)\tilde{\Omega}^{-1}-C_\perp C_1\Omega^{-1}(\Omega\Sigma+\Gamma+\Omega\tilde{\Omega}+\Sigma\tilde{\Omega})P_{11} \end{aligned}$$

by the (1,1) block of (3.8.43). Therefore,

$$A(1,2) = 0 \quad \text{by (3.8.44)}$$

Also

$$A(3,2) = -\hat{B}B_\perp B_1^*\tilde{\Omega}^{-1}-\hat{A}P_{12}^*+P_{12}^*[A_{11}+(\tilde{\Omega}A_{21}^*-B_1B_2^*)A_{21}]=0 \quad \text{by (2,1) block of (3.8.42)}$$

$$\begin{aligned} C(1,2) &= -C_\perp^*\hat{D}B_\perp B_1^*\tilde{\Omega}^{-1}-C_\perp^*C_\perp C_1(\Omega^{-1}\Gamma+\Sigma)\tilde{\Omega}^{-1}-C_\perp^*\hat{C}P_{12}^* \\ &= -C_\perp^*C_\perp C_1\{(\Omega^{-1}\Gamma+\Sigma)\tilde{\Omega}^{-1}-\Omega^{-1}(\Omega\Sigma+\Gamma+\Omega\tilde{\Omega}+\Sigma\tilde{\Omega})P_{11} \} \\ &= 0 \end{aligned}$$

on using the (1,1) block of (3.8.43) and (3.8.44). Finally,

$$\begin{aligned} B(3,1) &= \hat{B}B_\perp P_{12}^*(B_1B_\perp^*B_\perp+\tilde{\Omega}A_{21}^*B_2) \\ B(1,1) &= \Omega^{-1}\Sigma B_1B_\perp^*B_\perp+\Omega^{-1}C_1^*C_\perp^*\hat{D}B_\perp-\Omega^{-1}\Gamma A_{21}^*B_2 \end{aligned}$$

Removing the unobservable modes associated with the second block row and column in (3.8.46) yields (3.8.29) and completes the proof. $\square$

The next theorem summarises the results of this section.

Theorem 3.8.3

Under Assumption 3.8.1, any rational transfer function $G_0(s) \in \mathbb{RH}_+^{\infty, (p \times m)}$ has a (unique) superoptimal extension $F_{s,o}(s)$ with a degree bounded by

$$\deg(F_{so}(s)) \leq \sum_{k=1}^{\min(n, r)} n - k \quad (3.8.47)$$

where $r = \text{rank}(G_0(s)) \leq \min(p, m)$ and $n = \deg(G_0(s))$.

Proof:

Suppose first that $n \geq r$. Referring to Theorem 3.6.2 we recall that the super-optimal extension is generated by

$$F_{so}(s) = F_0(s) + \sum_{k=0}^{r-2} \left[\prod_{i=0}^k V_{\perp i} (-Q_{k+1} + F_{k+1}) \left(\prod_{i=0}^k W_{\perp i} \right)^* \right](s) \quad (3.8.48)$$

or, equivalently, by the recursion

$$F_{so}(s) = S_0 \circ S_1 \circ \cdots \circ S_{r-2}(F_{r-1}(s)) \quad (3.8.49)$$

where

$$S_k(\cdot) = F_k(s) + V_{\perp k} [-Q_{k+1} + (\cdot)] W_{\perp k}^* \quad k=0, 1, \dots, r-2 \quad (3.8.50)$$

$F_{r-1}(s)$ is the unique extension to the rank 1 transfer function $G_{r-1}(s)$, and the matrices $V_{\perp k}$, $W_{\perp k}$, Q_k , G_k and F_k are generated recursively (See Theorem 3.6.2).

Applying Lemma 3.8.1 to each of the S_k 's gives

$$S_k(\cdot) = -J_k(s) + V_{\perp k}(\cdot) W_{\perp k}^*(s) \quad k=0, 1, \dots, r-2 \quad (3.8.51)$$

We also recall that

$$\left. \begin{aligned} \deg(G_k) \leq n - k \Rightarrow \deg(V_{\perp k}) \\ \deg(W_{\perp k}) \end{aligned} \right\} \leq n - k - 1 \quad (3.8.52)$$

and

$$\deg(J_k) \leq 2(n - k - 1) \quad (3.8.53)$$

We complete the cancellation analysis by applying Lemma 3.8.2 to (3.8.51). This result says that if $(\cdot)$ is any rational Nehari extension to G_{k+1} , then

$$\deg(S_k) = n - k - 1 + \deg(\cdot) \quad (3.8.54)$$

Noting that $S_k(s)$ is a rational Nehari extension to $G_k(s)$ allows us to use Lemma 3.8.2 iteratively. In the case that $r \leq n$ this gives

$$\deg(F_{so}(s)) \leq \sum_{k=1}^r n-k \quad (3.8.55)$$

In the case that $r > n$ we get

$$\deg(F_{so}(s)) \leq \sum_{k=1}^n n-k \quad (3.8.56)$$

Combining (3.8.55) and (3.8.56) gives (3.8.47). $\square$

3.9 Example

In this section we demonstrate the algorithm and illustrate the results with the aid of an example. Let us suppose that $G_0(s) \in \mathbb{RH}_+^\infty$, (3×3) is given by

$$G_0(s) \triangleq \left[\begin{array}{ccc|ccc} -1 & 1 & 0 & 1 & 2 & 0 \\ 0 & -2 & 0 & 1 & 1 & 1 \\ 0 & 0 & -3 & 2 & 2 & 1 \\ \hline 1 & -1 & 2 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 \\ 1 & -5 & -1 & 0 & 0 & 0 \end{array} \right] \quad (3.9.1)$$

Inequalities (3.0.5) and (3.0.6) are illustrated first. Table 3.9.1 contains a list of the $\sigma_i(G_k)$'s which are seen to satisfy both (3.0.5) and (3.0.6).

	$G_0(s)$	$G_1(s)$	$G_2(s)$
$\sigma_1(\cdot)$	4.6925	1.0442	0.12908
$\sigma_2(\cdot)$	1.4132	0.1294	0
$\sigma_3(\cdot)$	0.3097	0	0

Table 3.9.1 Hankel singular values of the $G_i(s)$'s

Figure 6 shows how each of the frequency response singular values of the error system are minimised in turn. Fig. 6(A) shows that $s_1^\infty(\cdot) = 4.6925$ and is consequently optimal. The other two singular values do not exhibit an all-pass characteristic. In Fig. 6(B), $s_2^\infty(\cdot)$ has been optimised and is given by $\|G_1\|_H = 1.0442$. As one would expect, $s_1^\infty(\cdot)$ remains unaltered. Finally, in Fig. 6(C) the super-optimal error system is illustrated. In this figure we note that all the singular values show all-pass characteristics with

$$s_1^\infty(G_0 + F_{so}) = \sigma_1(G_i) \quad i=0, 1, 2 \quad (3.9.2)$$

The MacMillan degree of the superoptimal extension is 3 which is in accordance with (3.0.7).

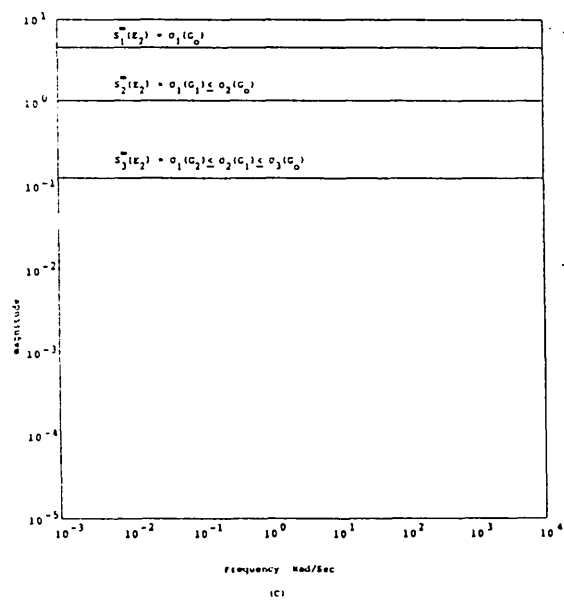
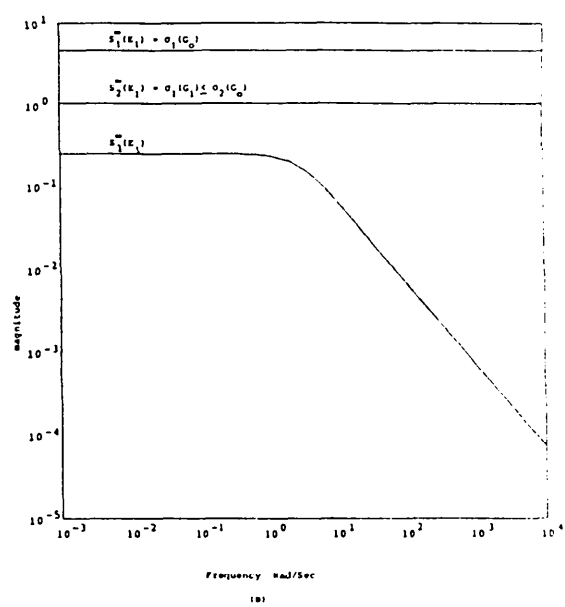
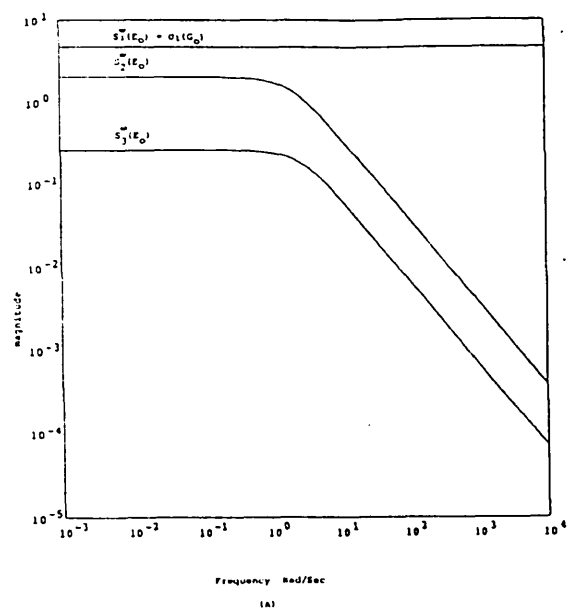


Figure 6: Singular value levels in super-optimal error system

3.10 L^∞/L^2 and maximum-entropy optimal Nehari extensions

In this section a number of mixed-type problems will be considered. For $G(s) \in \mathbb{R}\mathcal{L}^2$, the $\mathcal{L}^2$ and the $\mathcal{L}^\infty$ norms have been defined as

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[G^*(j\omega)G(j\omega)] d\omega \quad (3.10.1)$$

and

$$\|G\|_\infty = \sup_{\omega} \bar{\sigma}(G(j\omega)) \quad (3.10.2)$$

respectively. Consider the minimisation

$$\min_{F \in \mathcal{H}_-^\infty} \|G+F\|_\alpha \quad (3.10.3)$$

for $\alpha=2$ or ∞ : For the case $\alpha=2$ the optimal solution is simply obtained by projection as $F_{opt}(s) = -G(\infty) = 0$. The case $\alpha=\infty$ corresponds to the Nehari extension problem which has been considered in detail in previous sections. For scalar problems the optimal solution $\tilde{F}(s)$ is unique and the optimal error $G+\tilde{F}$ may be characterised in terms of (the maximal) Schmidt pair of the Hankel operator whose symbol is $G(s)$. In the case of matrix-valued problems, however, the solution is hardly ever unique and in general may be parametrised in terms of a linear fractional map of an (arbitrary) unstable contraction (see Lemma 3.1.1). In either case, imposing an $\mathcal{L}^\infty$ -norm in the minimisation (3.10.3) makes the largest singular value of the optimal error system constant over all frequencies. Writing the definition of the $\mathcal{L}^2$ norm as

$$\begin{aligned} \|G+F\|_2^2 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}[(G+F)^*(j\omega)(G+F)(j\omega)] d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=1}^r \lambda_i[(G+F)^*(G+F)](j\omega) d\omega \\ &= \frac{1}{2\pi} \int_0^\infty \sum_{i=1}^r s_i^2(G+F)(j\omega) d\omega \end{aligned} \quad (3.10.4)$$

we see that the optimal $\mathcal{L}^\infty$ solution corresponds to an infinite $\mathcal{L}^2$ cost. It follows, that if we wished to use the degrees of freedom that are available in the solution set of the multivariable Nehari extension problem in order to minimise some measure of an $\mathcal{L}^2$ -type cost, we would have to restrict our attention to the smallest $r-1$ singular values. The first problem that will be considered in this section may be formally expressed as follows:

[P3] : Let $G(s) \in \mathbb{R}\mathcal{H}^{2, (m \times p)}$. Find an $F(s) \in \mathbb{R}\mathcal{H}^\infty$ such the pair

$$\left(\|G+F\|_\infty, \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=2}^r s_i^2(G+F)(j\omega) d\omega \right) \quad (3.10.5)$$

is minimised with respect to lexicographic ordering. ($r \leq \min(p, m)$ is the (normal) rank of $G(s)$ in $\mathbb{C}$).

□

This is equivalent to minimising

$$\mu_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=2}^r s_i^2 (G+F)(j\omega) d\omega \quad (3.10.6)$$

in the family of Nehari extensions of $G(s)$. Intuitively, (3.10.6) represents the output energy $G+F(s)$ when it is driven by white noise of unit intensity constrained to lie in directions orthogonal to the maximal Schmidt vector of Γ_G .

For $G(s)$ such that $\|G\|_\infty \leq \gamma$ and $s_0 \in \mathbb{C}_+$, entropy at s_0 is defined [L6], [M1], [G7] as

$$I(G; \gamma; s_0) = \frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln \left| \det \left(I - \frac{1}{\gamma^2} G^*(j\omega) G(j\omega) \right) \right| \left(\frac{\operatorname{Re}(s_0)}{|s_0 - j\omega|} \right)^2 d\omega \quad (3.10.7)$$

Note that I is non-positive and that $I(G; \gamma; s_0) = 0$ if and only if $G = 0$. We also note that if $\mathcal{F}$ denotes the family of Nehari extensions of $G \in \mathcal{RH}_+^\infty$ and $\sigma = \|G\|_H$, then

$$I(G+F; \sigma; s_0) = -\infty \quad \text{for all } F \in \mathcal{F}, s_0 \in \mathbb{C}_+ \quad (3.10.8)$$

Rewriting the definition of the entropy function as

$$\begin{aligned} I(G; \gamma; s_0) &= \frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \sum_{i=1}^r \ln \left| 1 - \frac{1}{\gamma^2} s_i^2 (G(j\omega)) \right| \left(\frac{\operatorname{Re}(s_0)}{|s_0 - j\omega|} \right)^2 d\omega \\ &= \frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \ln \left| 1 - \frac{1}{\gamma^2} s_1^2 (G(j\omega)) \right| \left(\frac{\operatorname{Re}(s_0)}{|s_0 - j\omega|} \right)^2 d\omega \\ &\quad + \frac{\gamma^2}{2\pi} \int_{-\infty}^{\infty} \sum_{i=2}^r \ln \left| 1 - \frac{1}{\gamma^2} s_i^2 (G(j\omega)) \right| \left(\frac{\operatorname{Re}(s_0)}{|s_0 - j\omega|} \right)^2 d\omega \\ &:= I_1(s_1) + \tilde{I}(G; \sigma; s_0) \end{aligned} \quad (3.10.9)$$

we define the second optimisation problem to be addressed in this section:

$$[\mathcal{P}4]: \max_{F \in \mathcal{F}} \tilde{I}(G+F; \sigma; s_0) \quad (3.10.10)$$

It is emphasised that the assumption on the uniqueness of the largest singular value (Remark 3.1.1) remains in force throughout this section. If this assumption fail, no finite cost solution exists for either $[\mathcal{P}3]$ or $[\mathcal{P}4]$. In this case $[\mathcal{P}3]$ and $[\mathcal{P}4]$ may be redefined by considering the sum of the $r-\ell$ smallest singular values of the error system, where ℓ denotes the multiplicity of σ_1 .

Generators of sub-optimal Hankel-norm approximations/ Nehari extensions

In this section it is shown that $Q(s)$ defined in Lemma 3.5.1 acts as a generator of all $\sigma_1(G_0)$ -suboptimal extensions to $G_1(s)$. The proof relies on a characterisation of sub-optimal generators given in Lemmas 3.10.1 to 3.10.4 together with a result on the position of the system zeros of $Q_{12}(s)$ and $Q_{21}(s)$. The characterisation of sub-optimal generators is given in terms of Hankel-norm approximations which contain the Nehari case as a special case.

Lemma 3.10.1

Let

$$E(s) = \frac{1}{\sigma} \begin{bmatrix} G_1 + F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} = \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline C & \hat{C}_1 & 0 & I \\ 0 & \hat{C}_2 & I & 0 \end{array} \right] \quad (3.10.11)$$

a minimal realisation of an all-pass matrix function with $G_1(s) \in \mathcal{RH}_+^\infty$ and $Q(s) \in \mathcal{RH}_-^\infty(k)$. Let the grammians of this realisation be

$$P_\epsilon = \begin{bmatrix} \Sigma & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} \quad Q_\epsilon = \begin{bmatrix} \Sigma & Q_{12} \\ Q_{12}^* & Q_{22} \end{bmatrix} \quad (3.10.12)$$

where we have assumed without loss of generality that (A, B, C) is balanced. Then, all solutions to the equation $P_\epsilon Q_\epsilon = \sigma^2 I$ subject to the constraint $\text{In}(P_{22}) = \text{In}(Q_{22}) = (k, \ell, 0)$ are given by

$$P_{22} = T \begin{bmatrix} \Sigma \Gamma^{-1} & 0 \\ 0 & -\sigma I_\ell \end{bmatrix} T^* \quad Q_{22} = T^{*-1} \begin{bmatrix} \Sigma \Gamma & 0 \\ 0 & -\sigma I_\ell \end{bmatrix} T^{-1} \quad (3.10.13)$$

$$P_{12} = [I_n \mid 0] T^* \quad Q_{12} = [-\Gamma \mid 0] T^{-1} \quad (3.10.14)$$

where $\Gamma = \Sigma^2 - \sigma^2 I$. Further, $\ell \geq n-k$ and the zero columns of $P_{12} T^{*-1}$ and $Q_{12} T$ are present only if $\ell > n-k$.

Proof:

The proof follows by a slight modification to the proof of Lemmas 8.2 and 8.3 in [G1].

Next the definition of the linear fractional composition of two systems is given.

Definition 3.10.1

For $H(s) \in \mathbb{R}^{(p+m) \times (p+m)}(s)$ and $\tilde{H}(s) \in \mathbb{R}^{(p+m), (p+m)}(s)$ partitioned as

$$H(s) = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \quad \tilde{H}(s) = \begin{bmatrix} \tilde{H}_{11} & \tilde{H}_{12} \\ \tilde{H}_{21} & \tilde{H}_{22} \end{bmatrix} \quad (3.10.15)$$

in which $H_{11}(s) \in \mathbb{R}^{p \times m}(s)$ and $\tilde{H}_{11}(s) \in \mathbb{R}^{p \times m}(s)$ such that $(I - H_{22}\tilde{H}_{11})^{-1}$ exists, the linear fractional composition of $H(s)$ and $\tilde{H}(s)$ is defined as

$$H(s) \otimes \tilde{H}(s) = \begin{bmatrix} \mathfrak{F}_\ell(H, \tilde{H}_{11}) & H_{12}(I - \tilde{H}_{11}H_{22})^{-1}\tilde{H}_{12} \\ \tilde{H}_{21}(I - H_{22}\tilde{H}_{11})^{-1}H_{21} & \mathfrak{F}_u(\tilde{H}, H_{22}) \end{bmatrix} \in \mathbb{R}^{(p+m) \times (p+m)}(s)$$

where $\mathfrak{F}_\ell(\cdot, \cdot)$ and $\mathfrak{F}_u(\cdot, \cdot)$ denote the lower and upper linear fractional maps respectively. $\square$

Note that if

$$H(s) : \begin{bmatrix} u_1 \\ \xi \end{bmatrix} \rightarrow \begin{bmatrix} y_1 \\ \eta \end{bmatrix}, \quad \tilde{H}(s) : \begin{bmatrix} \eta \\ y_2 \end{bmatrix} \rightarrow \begin{bmatrix} \xi \\ y_2 \end{bmatrix} \quad (3.10.16)$$

then

$$H \otimes \tilde{H}(s) : \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \rightarrow \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad (3.10.17)$$

From the definition of the linear fractional composition the following properties may be easily established whenever the compositions are well-posed:

- (i) $(H_1 \otimes H_2) \otimes H_3 = H_1 \otimes (H_2 \otimes H_3)$
- (ii) $\mathfrak{F}_\ell(H \otimes \tilde{H}, U) = \mathfrak{F}_\ell(H, \mathfrak{F}_\ell(\tilde{H}, U))$
- (iii) If $H(s)$ and $\tilde{H}(s)$ are all pass such that

$$\tilde{H} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} H^*(s) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

then,

$$H(s) \otimes \tilde{H}(s) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad \text{and} \quad \tilde{H}(s) \otimes H(s) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

The next Lemma characterises matrix functions in $\mathbb{RH}^\infty(k)$ which are all-pass completions to (an augmented) stable matrix function, in terms of a linear fractional composition of two systems; the first of these is an element of $\mathbb{RH}^\infty(k)$, has minimal degree, and all system zeros of its off-diagonal blocks lie in the ORHP (so that it acts as the generator of all sub-optimal extension constructed in [G3]) while the second is anti-inner. It is assumed for simplicity (and without loss of generality) that

the largest Hankel singular value of $G_1(s)$ is equal to one.

Lemma 3.10.2

Let

$$\begin{bmatrix} G_1+Q_{12} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \stackrel{\Delta}{=} \left[\begin{array}{cc|cc} A & 0 & B & 0 \\ 0 & \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline C & \hat{C}_1 & 0 & I \\ 0 & \hat{C}_2 & I & 0 \end{array} \right] \quad (3.10.18)$$

be a minimal realisation of an all-pass matrix function with $G_1(s) \in \mathcal{RH}_+^\infty$ and $Q(s) \in \mathcal{RH}_-^\infty(k)$. Suppose also that $\dim(A)=n$ and $\dim(\hat{A})=\ell+k$ with $\ell \geq n-k$. Then,

(i) $Q(s)$ decomposes as $Q(s)=H(s) \otimes \tilde{H}(s)$ in which

(a) $\tilde{H}(s)$ is completely unstable and all-pass of degree $\ell+k-n$.

(b) $\begin{bmatrix} G_1+H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$ is all-pass; further $H(s) \in \mathcal{RH}_-^\infty(k)$ with all the zeros of its off-diagonal blocks $H_{12}(s)$ and $H_{21}(s)$ in the open right half plane.

(ii) Every system zero of $Q_{12}(s)$ ($Q_{21}(s)$) that lies in the LHP is a system zero of $\tilde{H}_{12}(s)$ ($\tilde{H}_{21}(s)$)

Proof:

Applying Lemma 3.10.1, we may partition the state-space of (3.10.18) as

$$A_e = \begin{bmatrix} A & 0 & 0 \\ 0 & \hat{A}_{11} & \hat{A}_{12} \\ 0 & \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \quad B_e = \begin{bmatrix} B & 0 \\ \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix} \quad (3.10.19)$$

$$C_e = \begin{bmatrix} C & \hat{C}_{11} & \hat{C}_{12} \\ 0 & \hat{C}_{21} & \hat{C}_{22} \end{bmatrix} \quad D_e = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

with corresponding controllability and observability grammians

$$P_e = \begin{bmatrix} \Sigma & I & 0 \\ I & \Sigma\Gamma^{-1} & 0 \\ 0 & 0 & -I_l \end{bmatrix} \quad Q_e = \begin{bmatrix} \Sigma & -\Gamma & 0 \\ -\Gamma & \Sigma\Gamma & 0 \\ 0 & 0 & -I_l \end{bmatrix} \quad (3.10.20)$$

(Note that without loss of generality we have assumed that (A, B, C) is balanced and have chosen the basis $T=I$ in Lemma 3.10.1). The all-pass equations corresponding to (3.10.18) are

$$A_e P_e + P_e A_e^* + B_e B_e^* = 0 \quad (3.10.21)$$

and

$$C_e P_e + D_e B_e^* = 0 \quad (3.10.22)$$

which may be expanded using (3.10.19) and (3.10.20) to yield

$$A\Sigma + \Sigma A^* + BB^* = 0 \quad (3.10.23)$$

$$A + \hat{A}_{11}^* + B\hat{B}_{11}^* = 0 \quad (3.10.24)$$

$$\hat{A}_{21}^* + B\hat{B}_{21}^* = 0 \quad (3.10.25)$$

$$\hat{A}_{11}\Sigma\Gamma^{-1} + \Sigma\Gamma^{-1}\hat{A}_{11}^* + \hat{B}_{11}\hat{B}_{11}^* + \hat{B}_{12}\hat{B}_{12}^* = 0 \quad (3.10.26)$$

$$\hat{A}_{12} - \Sigma\Gamma^{-1}\hat{A}_{21}^* - \hat{B}_{11}\hat{B}_{21}^* - \hat{B}_{12}\hat{B}_{22}^* = 0 \quad (3.10.27)$$

$$\hat{A}_{22} + \hat{A}_{22}^* - \hat{B}_{21}\hat{B}_{21}^* - \hat{B}_{22}\hat{B}_{22}^* = 0 \quad (3.10.28)$$

and

$$C\Sigma + \hat{C}_{11} = 0 \quad (3.10.29)$$

$$C + \hat{C}_{11}\Sigma\Gamma^{-1} + \hat{B}_{12}^* = 0 \quad (3.10.30)$$

$$\hat{B}_{22}^* - \hat{C}_{12} = 0 \quad (3.10.31)$$

$$\hat{C}_{21} + B^* = 0 \quad (3.10.32)$$

$$\hat{C}_{21}\Sigma\Gamma^{-1} + \hat{B}_{11}^* = 0 \quad (3.10.33)$$

$$\hat{B}_{21}^* - \hat{C}_{22} = 0 \quad (3.10.34)$$

respectively.

Now suppose that $H(s)$ and $\tilde{H}(s)$ are defined as

$$H(s) \triangleq \left[\begin{array}{c|cc} \hat{A}_{11} & \hat{B}_{11} & \hat{B}_{12} \\ \hline \hat{C}_{11} & 0 & I \\ \hat{C}_{21} & I & 0 \end{array} \right] \quad \tilde{H}(s) \triangleq \left[\begin{array}{c|cc} \hat{A}_{22} & \hat{B}_{21} & \hat{B}_{22} \\ \hline \hat{C}_{12} & 0 & I \\ \hat{C}_{22} & I & 0 \end{array} \right] \quad (3.10.35)$$

Suppose

$$H(s) : \begin{bmatrix} u_1 \\ \phi \end{bmatrix} \rightarrow \begin{bmatrix} y_1 \\ \xi \end{bmatrix} \quad \text{and} \quad \tilde{H}(s) = \begin{bmatrix} \xi \\ u_2 \end{bmatrix} \rightarrow \begin{bmatrix} \phi \\ y_2 \end{bmatrix} \quad (3.10.36)$$

Eliminating ξ and ϕ shows that

$$\begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{bmatrix} \left(sI - \begin{bmatrix} \hat{A}_{11} & \hat{B}_{12}\hat{C}_{12} \\ \hat{B}_{21}\hat{C}_{21} & \hat{A}_{22} \end{bmatrix} \right)^{-1} \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix} + \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

maps $[u_1^T \ u_2^T]^T \rightarrow [y_1^T \ y_2^T]^T$. The all-pass equations (3.10.25), (3.10.33), (3.10.32) and (3.10.31) prove that

$$\begin{aligned}
 \hat{A}_{12} &= \Sigma \Gamma^{-1} \hat{A}_{21}^* + \hat{B}_{11} \hat{B}_{21}^* + \hat{B}_{12} \hat{B}_{22}^* \\
 &= \Sigma \Gamma^{-1} (-B \hat{B}_{21}^*) + \hat{B}_{11} \hat{B}_{21}^* + \hat{B}_{12} \hat{B}_{22}^* \\
 &= -\Sigma \Gamma^{-1} (B + \hat{C}_{21}^*) \hat{B}_{21} + \hat{B}_{12} \hat{B}_{22}^* \\
 &= \hat{B}_{12} \hat{C}_{12}
 \end{aligned} \tag{3.10.37}$$

while (3.10.25) and (3.10.32) show that

$$\hat{A}_{21} = -\hat{B}_{21} B^* = \hat{B}_{21} \hat{C}_{21} \tag{3.10.38}$$

which establishes the required fractional decomposition of $Q(s)$.

To verify i(a) and i(b) for this decomposition, we note using (3.10.28) that $\tilde{H}(s)$ is completely unstable, while (3.10.28), (3.10.31) and (3.10.34) establish its all-pass character. The inertia properties of $\hat{A}_{11}$ follow by (3.10.26), while the positions of the zeros of $H_{12}(s)$ are established by noting that

$$\begin{aligned}
 \hat{A}_{11} - \hat{B}_{12} \hat{C}_{11} &= -A^* - \hat{B}_{11} B^* + \Gamma^{-1} C^* C \Sigma \\
 &= -A^* + \Gamma^{-1} \Sigma (A \Sigma + \Sigma A^*) - \Gamma^{-1} (\Sigma A + A^* \Sigma) \Sigma \\
 &= -\Gamma^{-1} A^* \Gamma
 \end{aligned} \tag{3.10.39}$$

on using (3.10.24), (3.10.29), (3.10.30), (3.10.32), (3.10.34), the definition of Γ and the (1,1) blocks of (3.10.21) and its dual. A similar calculation proves that the zeros of $H_{21}(s)$ are also located at $-\lambda_i(A)$. Finally,

$$\det(H_{12}(j\omega)) \neq 0 \Rightarrow \|H_{22}(j\omega)\|_\infty < 1 \Rightarrow \|H_{22}(j\omega) \tilde{H}_{11}(j\omega)\|_\infty < 1 \tag{3.10.40}$$

and the fractional decomposition is well-posed.

To show (ii) note that

$$\begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} - \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{22} \end{bmatrix} \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \end{bmatrix} = \begin{bmatrix} \hat{A}_{11} - \hat{B}_{12} \hat{C}_{11} & 0 \\ \hat{A}_{21} - \hat{B}_{22} \hat{C}_{11} & \hat{A}_{22} - \hat{B}_{22} \hat{C}_{12} \end{bmatrix}$$

on using (3.10.37). The result follows on noting that by (3.10.39) all system zeros of $H_{12}(s)$ lie in the ORHP. This concludes the proof. $\square$

The next Lemma characterises all matrix functions which act as generators of Nehari extensions for $G_1 \in \mathbb{R}\mathcal{H}_+^\infty$.

Lemma 3.10.3

Let

$$\begin{bmatrix} G_1 + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

be all-pass with $G_1(s) \in \mathbb{RH}_+^{\infty, (p \times m)}$ and $Q(s) \in \mathbb{RH}_-^{\infty, (p+m) \times (p+m)}(k)$ and suppose that all system zeros of $Q_{12}(s)$ and $Q_{21}(s)$ in a minimal realisation of $Q(s)$ lie in the ORHP.

Then:

- (i) $\mathcal{F}_\ell(Q, U)$ generates all approximations $F(s) \in \mathcal{H}_-^\infty(k)$ such that $\|G_1 + F\|_\infty \leq 1$
- (ii) If $F(s)$ denotes the generator of all sub-optimal approximations given in [G3] for which

$$(a) \deg(F) = \deg(G_1)$$

$$(b) F(\infty) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

then,

$$F(s) = Q(s) \otimes \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} Q^*(\infty) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad (3.10.41)$$

Proof:

Since all system zeros of $Q_{12}(s)$ and $Q_{21}(s)$ lie in the ORHP, $\det(Q_{12}(\infty)) \neq 0$ and $\det(Q_{21}(\infty)) \neq 0$. Since

$$Q(\infty) = \begin{bmatrix} Q_{11}(\infty) & Q_{12}(\infty) \\ Q_{21}(\infty) & Q_{22}(\infty) \end{bmatrix} \quad (3.10.42)$$

is unitary and its off-diagonal blocks are invertible,

$$\bar{\sigma}(Q_{22}(\infty)) < 1 \quad (3.10.43)$$

Defining

$$\bar{Q}(s) = Q(s) \otimes \bar{Q}^*(\infty) = \begin{bmatrix} \mathcal{F}_\ell(Q, Q_{22}^*(\infty)) & Q_{12}(I - Q_{22}^*(\infty)Q_{22})^{-1}Q_{12}^*(\infty) \\ Q_{21}^*(\infty)(I - Q_{22}Q_{22}^*(\infty))^{-1}Q_{21} & \mathcal{F}_u\left[\begin{bmatrix} Q_{22}^* & Q_{12}^* \\ Q_{22} & Q_{22}^* \end{bmatrix}(\infty), Q_{22}\right] \end{bmatrix}$$

where

$$\bar{Q}^*(\infty) := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} Q^*(\infty) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad (3.10.44)$$

and noting that due to (3.10.43) the composition is well-posed, we conclude that

$$\bar{Q}(\infty) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad (3.10.45)$$

In addition, the system zeros of $Q_{12}(s)$ ($Q_{21}(s)$) coincide with those of $\bar{Q}_{12}(s)$ ($\bar{Q}_{21}(s)$): If, for example, $Q(s)$ has a state-space realisation

$$Q(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array} \right] \quad (3.10.46)$$

it may be shown by a straightforward calculation that

$$\bar{Q}_{12}(s) = \left[\begin{array}{cc} A + B_2(I - D_{22}^* D_{22})^{-1} D_{12}^* C_2 & B_2(I - D_{22}^* D_{22})^{-1} D_{12}^* \\ C_1 + D_{12}(I - D_{22}^* D_{22})^{-1} D_{12}^* C_2 & D_{12}(I - D_{22}^* D_{22})^{-1} D_{12}^* \end{array} \right] \quad (3.10.47)$$

with $\bar{Q}(s)$ and $\bar{Q}_{12}(s)$ sharing the same A-matrix. The system zeros of $\bar{Q}_{12}(s)$ are located to the eigenvalues of

$$\begin{aligned} & A + B_2(I - D_{22}^* D_{22})^{-1} D_{12}^* C_2 - B_2(I - D_{22}^* D_{22})^{-1} D_{12}^* \left\{ D_{12}(I - D_{22}^* D_{22})^{-1} D_{12}^* \right\}^{-1} \left\{ C_1 + D_{12}(I - D_{22}^* D_{22})^{-1} D_{12}^* C_2 \right\} \\ &= A - B_2 D_{12}^{-1} C_2 \end{aligned} \quad (3.10.48)$$

so that $\{\text{system zeros of } Q_{12}\} = \{\text{system zeros of } \bar{Q}_{12}\}$. A similar calculation shows that the system zeros of $Q_{21}(s)$ are identical with those of $\bar{Q}_{21}(s)$.

Next, since

$$\left[\begin{array}{cc} G_1 + \bar{Q}_{12} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{array} \right] = \left[\begin{array}{cc} G_1 + Q_{12} & Q_{12} \\ Q_{21} & Q_{22} \end{array} \right] \otimes \tilde{Q}^*(\infty) \quad (3.10.49)$$

with $G_1 + \bar{Q}$ all-pass and $\tilde{Q}^*(\infty)$ unitary, we conclude that $G_1 + Q(s)$ is all-pass also. This implies that we may obtain a decomposition

$$\bar{Q}(s) = H(s) \otimes \tilde{H}(s) \quad (3.10.50)$$

along the lines given in Lemma 3.10.2. Next note that since, by assumption, all system zeros of Q_{12} and Q_{21} (and hence also of $\bar{Q}_{12}$ and $\bar{Q}_{21}$) lie in the ORHP, $\tilde{H}(s)$ (being completely unstable) must be the constant matrix

$$\tilde{H}(s) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad (3.10.51)$$

To establish this fact, suppose for contradiction that $\tilde{H}(s)$ has a (dynamic) minimal balanced realisation

$$\tilde{H}(s) = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & I \\ C_2 & I & 0 \end{array} \right] \quad (3.10.52)$$

The all-pass character of $\tilde{H}(s)$ shows that the equations

$$A + A^* - B_1 B_1^* - B_2 B_2^* = 0 \quad (3.10.53)$$

$$C_1 = B_2^* \quad (3.10.54)$$

and

$$C_2 = B_1^* \quad (3.10.55)$$

are satisfied. Hence, since the system zeros of $\tilde{H}_{12}(s)$ and $\tilde{H}_{21}(s)$ lie in the ORHP,

$$\lambda_i(A - B_2 B_2^*) < 0 \quad (3.10.56)$$

and

$$\lambda_i(A - B_1 B_1^*) < 0 \Rightarrow \lambda_i(A^* - B_1 B_1^*) < 0 \quad (3.10.57)$$

and in view of (3.10.53), (3.10.56) and (3.10.57) cannot be satisfied simultaneously.

Since by the proof of Theorem 4.1 in [G3] the conditions

(i) $G_{1a} + Q(s)$ all pass with $G_{1a} \in \mathcal{H}_+^\infty$ and $Q \in \mathcal{H}_-^\infty$

(ii) $\deg(G_1) = \deg(Q)$

$$(iii) \quad Q(\infty) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$$

determine $Q(s)$ uniquely, we conclude that $\bar{Q}(s) (= H(s))$ is the generator constructed in [4] and consequently $\mathcal{F}_\ell(\bar{Q}, \mathcal{B}\mathcal{H}_-^\infty)$ generates all $F(s) \in \mathcal{H}_-^\infty(k)$ such that $\|G_1 + F\|_\infty \leq 1$. To show that $Q(s)$ is also a generator, note that

$$\begin{aligned} \mathcal{F}_\ell(\bar{Q}, \mathcal{B}\mathcal{H}_-^\infty) &= \mathcal{F}_\ell(Q \otimes \bar{Q}^*(\infty), \mathcal{B}\mathcal{H}_-^\infty) \\ &= \mathcal{F}_\ell(\bar{Q}, \mathcal{F}_\ell(\bar{Q}^*(\infty), \mathcal{B}\mathcal{H}_-^\infty)) \end{aligned} \quad (3.10.58)$$

Since $\bar{Q}^*(\infty)$ is unitary with non-singular off-diagonal blocks the map

$$X \in \mathcal{BK}^\infty \rightarrow \mathcal{F}_\ell(\tilde{Q}^*(\infty), X) \in \mathcal{BK}^\infty \quad (3.10.59)$$

is a bijection and consequently

$$\mathcal{F}_\ell(\bar{Q}, \mathcal{F}_\ell(\tilde{Q}^*(\infty), \mathcal{BK}^\infty)) = \mathcal{F}_\ell(\bar{Q}, \mathcal{BK}^\infty) \quad (3.10.60)$$

This concludes the proof. $\square$

In the case of sub-optimal Nehari extensions the conditions of Lemma 3.10.3 may be simplified by replacing system zeros of the off-diagonal blocks with MacMillan zeros. This is shown next.

Corollary 3.4:

Let

$$\begin{bmatrix} G_1 + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$$

be all-pass with $G_1(s) \in \mathcal{RH}_+^{\infty, (p \times m)}$ and $Q(s) \in \mathcal{RH}_-^{\infty, (p+m) \times (p+m)}$ and suppose that all MacMillan zeros of $Q_{12}(s)$ and $Q_{21}(s)$ in a minimal realisation of $Q(s)$ lie in the ORHP. Then $\mathcal{F}_\ell(Q, U)$ generates all extensions $F(s) \in \mathcal{K}^\infty$ such that $\|G_1 + F\|_\infty \leq 1$.

Proof:

Using the decomposition of $Q(s) = H(s) \otimes \tilde{H}(s)$ in Lemma 3.10.2,

$$Q_{12}(s) = H_{12}(I - \tilde{H}_{11}H_{22})^{-1}\tilde{H}_{12}$$

in which $H_{12} \in \mathcal{K}^\infty$ has all its MacMillan zeros in the ORHP while $\tilde{H}_{11}H_{22} \in \mathcal{BK}^\infty$ and $\tilde{H}_{12} \in \mathcal{K}^\infty$. This implies that every MacMillan zero of $\tilde{H}_{12}$ which lies in the LHP must be a MacMillan zero of Q_{12} . By assumption, therefore, $\tilde{H}_{12}$ is free of any such zero (and similarly for $\tilde{H}_{21}$).

Next, following the proof of Lemma 3.10.3 we see using (3.10.53) to (3.10.57) that if $\tilde{H}$ is dynamic there exists at least one system zero of $\tilde{H}_{12}$ ($\tilde{H}_{21}$) in the LHP; since this cannot be a MacMillan zero it must be a non-minimal (uncontrollable/unobservable) mode. This however is impossible since by (3.10.53) the A-matrix of $\tilde{H}(s)$ is completely unstable. $\square$

Remark 3.10.1

Lemmas 3.10.2 and 3.10.3 show that the generator of all sub-optimal extensions constructed in [G3, Theorem 4.1] is essentially unique (up to a linear fractional composition with a constant unitary matrix). All-pass completions $F(s) \in \mathcal{K}^\infty(k)$ to $G_{1a}(s) := G_1 \oplus O$, with $\deg(F) > \deg(G_1)$ introduce minimum-phase system zeros to the off-diagonal blocks $F_{12}(s)$ and $F_{21}(s)$. Using the decomposition $F(s) = H \otimes \tilde{H}$ established in Lemma 3.10.2, we see that all these system zeros are inherited by the off-

diagonal blocks of $\tilde{H}(s)$. Note also that if $\tilde{H}(s)$ is dynamic $\mathcal{F}_\ell(\tilde{H}, \mathcal{BK}_\infty^-)$ reproduces, in general, only a strict subset of $\mathcal{BK}_\infty^-$. This may be seen by considering the example

$$\mathcal{F}_\ell(\tilde{H}, \mathcal{BK}_\infty^-) = \mathcal{F}_\ell\left(\begin{bmatrix} 0 & \frac{s+1}{s-1} \\ 1 & 0 \end{bmatrix}, \mathcal{BK}_\infty^-\right) = \left(\frac{s+1}{s-1}\right) \mathcal{BK}_\infty^- \quad (3.10.61)$$

which is a strict subset of $\mathcal{BK}_\infty^-$ since the zero at $s=-1$ can not be cancelled. The 'unnecessary' unstable dynamics in $F(s)$, therefore, have the effect of 'filtering' the unstable contractions $U(s) \in \mathcal{BK}_\infty^-$ in the 'feedback loop' of the linear fractional map $\mathcal{F}_\ell(H, U)$, $H(s)$ being the minimal degree generator constructed in [G3]). In this case, therefore, $\mathcal{F}_\ell(F, \mathcal{BK}_\infty^-)$, will generate only a strict subset of all $F(s) \in \mathcal{RK}_\infty^-(k)$ which satisfy $\|G_1 + F\|_\infty \leq 1$. $\square$

Remark 3.10.2

The relaxation of the assumptions of Lemma 3.10.3 for the Nehari case is not possible in the general AAK case. The point to note is that stable pole/zero cancellations may occur in the product $Q_{12} = H_{12}(I - \tilde{H}_{11}H_{22})^{-1}\tilde{H}_{12}$ in the general case. (Consider for example the $\tilde{H}$ given in (3.10.61) and assume that H_{12} has a pole at $s=-1$). $\square$

The next Lemma shows that $Q(s)$ defined in 3.5.1 acts as a generator of all $\sigma_1(G_0)$ sub-optimal Nehari extensions of $G_1(s)$. (Note that by Theorem 3.7.2, $\sigma_1(G_1) \leq \sigma_2(G_0) < \sigma_1(G_0)$).

Lemma 3.10.5

$\mathcal{F}_\ell(Q, \sigma_1^{-1}(G_0)\mathcal{BK}_\infty^-)$ generates all $F(s) \in \mathcal{K}_\infty^-$ such that $\|G_1 + F\|_\infty \leq \sigma_1(G)$.

Proof:

From the diagonalisation Lemma 3.5.1 we may write that

$$\begin{bmatrix} v^* \\ V_\perp^* \end{bmatrix} \tilde{F}_{12} C_\perp^* = \begin{bmatrix} 0 \\ Q_{12} \end{bmatrix} \quad (3.10.62)$$

where we have factored

$$F_{12} := \tilde{F}_{12} C_\perp^* \quad (3.10.63)$$

and defined

$$\tilde{F}_{12}(s) = \sigma(C_1 \Sigma + \sigma U B_1^*) \left\{ sI - \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^*) \right\}^{-1} \Gamma^{-1} C_1^* - \sigma I_p$$

A simple calculation [G3] shows that the system zeros of $\tilde{F}_{12}$ are located at the eigenvalues of $-A_{11}$.

Since in addition $V^*(s)$ is inner, the LHS of (3.10.62) is a non-singular transformation of a full (column) rank matrix for every point in the LHP. This shows that Q_{12} has no MacMillan zero in the LHP which, by direct application of Corollary 3.10.4, proves the result. $\square$

Next, [3] and [P4] will be transformed to (sub-optimally) constrained $\mathcal{L}^2/\mathcal{L}^\infty$ and maximum entropy/ $\mathcal{L}^\infty$ problems respectively. In a number of papers [M1], [B4] mixed $\mathcal{L}^2/\mathcal{L}^\infty$ problems have been tackled by minimising an *upper bound* of the $\mathcal{L}^2$ cost. In [M1] it is shown that imposing this bound is equivalent to minimising the entropy function at infinity. Similarly, imposing the corresponding bound in (1.6) will reduce [P3] to a special case of [P4]. The next theorem gives an exact solution to [P3] for the special class of matrix functions $G_0(s)$ for which $\|G_1\|_\infty \leq \sigma$. A sufficient condition for this inequality to hold is given in the remark following the proof.

Theorem 3.10.6

If $\|G_1\|_\infty \leq \sigma$, $F_{opt}(s) = F_{11} - V_\perp Q_{11} W_\perp^*(s) \in \mathcal{RH}_-^\infty$ minimises (3.10.6) uniquely in the family of Nehari extensions of $G(s)$.

Proof:

Consider the parametrisation of all Nehari-optimal error systems given in Lemma 3.6.1 as

$$\begin{aligned} V^*(G + \mathcal{F})W &= V^* \mathcal{F}_\ell(G_a + F, U)W(s) = \mathcal{F}_\ell(V_a^*(G_a + F)W_a, U)(s) \\ &= \begin{bmatrix} \sigma_1(G)a(s) & 0 \\ 0 & G_1 + \mathcal{F}_\ell(Q, U) \end{bmatrix} U(s) \in \frac{1}{\sigma} \mathcal{BH}_-^\infty \end{aligned} \quad (3.10.64)$$

Since the cost in (3.10.6) is unitarily invariant,

$$\begin{aligned} \inf_{F \in \mathcal{F}} \frac{1}{2\pi} \sum_{i=2}^r \int_{-\infty}^{\infty} s_i^2(G + \mathcal{F})(j\omega) d\omega &= \inf_{\frac{1}{\sigma} U \in \mathcal{BH}_-^\infty} \frac{1}{2\pi} \sum_{i=1}^{r-1} \int_{-\infty}^{\infty} s_i^2(G_1 + \mathcal{F}_\ell(Q, U))(j\omega) d\omega \\ &\geq \inf_{F_1 \in \mathcal{H}_-^\infty} \frac{1}{2\pi} \sum_{i=1}^{r-1} \int_{-\infty}^{\infty} s_i^2(G_1 + F_1)(j\omega) d\omega = \inf_{F_1 \in \mathcal{H}_-^\infty} \|G_1 + F_1\|_2^2 = \|G_1\|_2^2 \end{aligned} \quad (3.10.65)$$

with the (unique) $F_1(s)$ achieving this infimum equal to $-G_1(\infty)=0$. The result follows by back-substitution on noting that $\|G_1\|_\infty \leq \sigma \Rightarrow 0 \in \{\mathcal{F}_\ell(Q, U) \mid U(s) \in \frac{1}{\sigma} \mathcal{BH}_-^\infty\}$ and the bound in (3.10.65) is attained. $\square$

Remark 3.10.3

A sufficient condition for $\|G_1\|_\infty \leq \sigma$ may be established in terms of the sequence of Hankel

singular values of $G(s)$: Since $G_1(s)$ is strictly proper,

$$2\left(\sum_i \sigma_i(G_1)\right) \leq 2\left(\sum_i \sigma_{i+1}(G)\right) \quad (3.10.66)$$

on using a bound for the infinity norm of a matrix given in [G1] and the bound

$$\sigma_i(G_1) \leq \sigma_{i+1}(G) \quad (3.10.67)$$

for each i which follows from Lemma 3.7.2. Consequently $\|G_1\|_\infty \leq \sigma$ will be satisfied whenever

$$\sigma_1(G) \geq 2\left(\sum_{i=1}^{n-1} \sigma_{i+1}(G)\right) \quad (3.10.68)$$

Since the bound on the infinity norm can be arbitrarily weak, the class of G 's for which $\|G_1\|_\infty \leq \sigma$ may be considerably larger than (3.10.68) indicates.

The next Theorem gives a lower bound for the optimal cost in terms of the solutions to the two Ricatti equations (3.4.53) and (3.4.54)

Theorem 3.10.7

$$\mu_0 = \min_{F \in \mathcal{F}} \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{i=2}^r s_i^2 (G_0 - F)(j\omega) d\omega \right\} \geq \|G_1\|_2^2 \quad (3.10.69)$$

the bound being tight if $\|G_1\|_\infty \leq \sigma$. Provided that the solutions of the Ricatti equations (3.4.53) and (3.4.54) exist the bound (3.10.69) may be written as

$$\|G_1\|_2^2 = \text{trace} \left\{ C_\perp C_1 \Omega^{-1} (\Gamma + \Omega \Sigma) \tilde{\Omega}^{-1} (\tilde{\Omega} \Omega + \tilde{\Omega} \Sigma + \Sigma \Omega + \Gamma) \Omega^{-1} C_1^* C_\perp^* \right\} \quad (3.10.70)$$

Proof:

From Theorem 3.5.2, a state space realisation of $G_1(s)$ when Ω and $\tilde{\Omega}$ exist, is given by

$$G_1(s) = C_\perp C_1 \Omega^{-1} \{ sI + A_{11}^* + (\Omega A_{12} - C_1^* C_2) A_{12}^* \}^{-1} (\Omega \tilde{\Omega} + \Sigma \tilde{\Omega} + \Omega \Sigma + \Gamma) \tilde{\Omega}^{-1} B_1 B_\perp^* \quad (3.10.71)$$

The corresponding controllability grammian may be calculated as

$$\tilde{P} = (\Gamma + \Omega \Sigma) \tilde{\Omega}^{-1} (\tilde{\Omega} \Omega + \tilde{\Omega} \Sigma + \Sigma \Omega + \Gamma) \Omega^{-1} \quad (3.10.72)$$

The result follows on noting that for $G(s) = \tilde{C}(sI - \tilde{A})^{-1} \tilde{B}$ with controllability Grammian $\tilde{P}$, $\|G_1\|_2^2 = \text{trace} \{ \tilde{C} \tilde{P} \tilde{C}^* \}$. $\square$

The next theorem gives a degree bound of $F_{opt}(s)$ under the assumption on the existence and non-singularity of the solutions to the Riccati equations (3.10.35) and (3.10.36). The proof follows immediately from Lemma 3.8.1.

Theorem 3.10.8

Let $f(s)$ and $g(s)$ be the Schmidt vectors of Γ_G . Then $\deg(F_{opt}) \leq \deg(f) + \deg(g) - 2 \leq 2(n-1)$. In particular, $F_{opt}(s) = \sigma U - \tilde{C}(sI - \tilde{A})^{-1} \tilde{B}$ where,

$$\tilde{A} = \begin{bmatrix} A_{11} + A_{12}(A_{12}^* \Omega - C_2 C_1) & -\Omega^\# \Gamma(\tilde{\Omega}^\# + \Gamma^{-1} \Sigma) B_1 B_\perp^* B_\perp B_1^* \tilde{\Omega}^\# \\ 0 & A_{11} + (\tilde{\Omega} A_{21}^* - B_1 B_2^*) A_{21} \end{bmatrix} \quad (3.10.73)$$

$$\tilde{B} = \begin{bmatrix} -\Omega \Gamma(\tilde{\Omega}^\# + \Gamma^{-1} \Sigma) B_1 B_\perp^* B_\perp \\ B_1 B_\perp^* B_\perp + \tilde{\Omega} A_{21}^* B_2 \end{bmatrix} \quad (3.10.74)$$

$$\tilde{C} = \begin{bmatrix} C_\perp^* C_\perp C_1 + C_2 A_{12}^* \Omega & -(C_1 \Sigma + \sigma U B_1^*) \tilde{\Omega}^\# \end{bmatrix} \quad (3.10.75)$$

Proof:

Follows directly from Lemma 3.8.1 and Remark 3.8.1. $\square$

Remark 3.10.4

The proof of Lemma 3.8.1 shows that the two diagonal sub-blocks in the $\tilde{A}$ are similar. This shows that, in general, the optimal extension $F_{opt}(s)$ has repeated poles.

Theorem 3.10.9

$[\mathcal{P}4]$ is solved uniquely by $F_{opt}(s) = \mathcal{F}_\ell(F, \sigma^{-1} F_{22}^*(s_0))$.

Proof:

Throughout the proof it is assumed that σ has been normalised to 1. The family of all Nehari extensions $F \in \mathcal{F}$ of $G(s)$ is parametrised as

$$G + F(s) = V(s) \text{diag}(a(s), G_1 + \mathcal{F}_\ell(Q, U)) W^*(s) \quad ; \quad U \in \mathcal{BK}_\infty \quad (3.10.76)$$

and consequently,

$$s_{i+1}(G+F)(j\omega) = s_i(G_1 + \mathcal{F}_\ell(Q, U))(j\omega) \quad \text{for all } \omega \in \mathbb{R}, F \in \mathcal{F}, i=1, 2, \dots \quad (4.10.77)$$

Since the entropy function is unitarily invariant, and since $\mathcal{F}_\ell(Q, U)$ generates all the suboptimal

extensions of $G_1(s)$, [P4] is equivalent to

$$\max_{U \in \mathcal{B}\mathcal{H}_\infty^-} \tilde{I}(G_1 + \mathcal{F}_\ell(Q, U); 1; s_0) = \max I(G_1 + F_1; 1; s_0) \quad (4.10.78)$$

where the maximum in the RHS is taken over all $F_1 \in \mathcal{RH}_\infty^-$ such that $\|G_1 + F_1\|_\infty \leq 1$. From [L6] this is solved uniquely by

$$F_1^{opt}(s) = \mathcal{F}_\ell(H, H_{22}^*(s_0)) \quad (3.10.79)$$

in which $H(s)$ is the generator of -suboptimal extensions of $G_1(s)$ given in [G3]. From Lemma 3.10.3,

$$H(s) = Q(s) \otimes \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} Q^*(\infty) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \quad (4.10.80)$$

Next note that $\mathcal{F}_\ell(Q, Q_{22}^*(s_0))$ may be written as

$$\begin{aligned} \mathcal{F}_\ell(Q, Q_{22}^*(s_0)) &= \mathcal{F}_\ell\left(Q(s) \otimes \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} Q^*(\infty) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \otimes Q(\infty), Q_{22}^*(s_0)\right) \\ &= \mathcal{F}_\ell\left(Q(s) \otimes \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} Q^*(\infty) \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \mathcal{F}_\ell(Q(\infty), Q_{22}^*(s_0))\right) \\ &= \mathcal{F}_\ell(H(s), \mathcal{F}_\ell(Q(\infty), Q_{22}^*(s_0))) \end{aligned} \quad (3.10.81)$$

where

$$H(s) = \begin{bmatrix} \mathcal{F}_\ell(Q, Q_{22}^*(\infty)) & Q_{12}(I - Q_{22}^*(\infty)Q_{22})^{-1}Q_{12}^*(\infty) \\ Q_{21}^*(\infty)(I - Q_{22}Q_{22}^*(\infty))^{-1}Q_{21} & \mathcal{F}_u\left(\begin{bmatrix} Q_{22}^* & Q_{12}^* \\ Q_{22} & Q_{22}^* \end{bmatrix}(\infty), Q_{22}\right) \end{bmatrix}$$

On noting that

$$H_{22}(s_0) = \mathcal{F}_u\left(\begin{bmatrix} Q_{22}^*(\infty) & Q_{12}^*(\infty) \\ Q_{22}^*(\infty) & Q_{22}^*(\infty) \end{bmatrix}, Q_{22}(s_0)\right) = \mathcal{F}_\ell^*(Q(\infty), Q_{22}^*(s_0))$$

we conclude that

$$\mathcal{F}_\ell(Q, F_{22}^*(s_0)) = \mathcal{F}_\ell(Q, Q_{22}^*(s_0)) = \mathcal{F}_\ell(H, H_{22}^*(s_0)) = F_{opt}(s) \quad (3.10.82)$$

(Note that since $F_{22}(s)$ is completely unstable, by the maximum modulus theorem

$$\bar{\sigma}[F_{22}^*(s_0)] \leq \|F_{22}^*(j\omega)\|_\infty < 1 \quad (3.10.83)$$

so that this choice of feedback term is admissible). The optimal extension is given by back-substitution as

$$\begin{aligned} G + F_{opt} &= V \begin{bmatrix} a(s) & 0 \\ 0 & G_1 + \mathcal{F}_\ell(Q, F_{22}(s_0)) \end{bmatrix} W^* \\ &= V \begin{bmatrix} a(s) & 0 \\ 0 & G_1 + Q_{11} \end{bmatrix} W^* + V \begin{bmatrix} 0 & 0 \\ 0 & -Q_{11} + \mathcal{F}_\ell(Q, F_{22}(s_0)) \end{bmatrix} W^* \\ &= G + \mathcal{F}_\ell(F, 0) + V_\perp Q_{12} (I - F_{22} F_{22}^*(s_0))^{-1} Q_{21} W_\perp^*(s) \end{aligned} \quad (3.10.84)$$

Further,

$$V_\perp Q_{12} = V_\perp V_\perp^* F_{12} = (I - vv^*) F_{12} = F_{12}(s) \quad (3.10.85)$$

on using the (1,3) blocks of (3.5.3). Similarly, $Q_{21} W_\perp^* = F_{21}$, so that,

$$F_{opt}(s) = \mathcal{F}_\ell(F, F_{22}^*(s_0)) \quad (3.10.86)$$

as required. $\square$

3.11 Example

In this example, we work through the various steps of the algorithm for calculating super-optimal and $\mathcal{L}^\infty/\mathcal{L}^2$ -optimal Nehari extensions. This will verify the results presented in the previous sections and give an overview of the complete algorithmic procedure. Since the lexicographic minimisation of the complete sequence of singular values essentially involves solving a recursively-defined optimisation problem, a two input/output, two state example will be considered; this captures the essential features of the algorithm without any unnecessary complications.

Let $G(s) \in \mathcal{H}_+^{\infty, (2 \times 2)}$ be defined as:

$$G(s) = \frac{3}{\phi(s)} \begin{bmatrix} 12s+3 & 12s-1 \\ 12s-1 & 24s+7 \end{bmatrix} \quad (3.11.1)$$

where $\phi(s) := 36s^2 + 45s + 5$. $G(s)$ may be expressed in terms of the balanced realisation

$$G(s) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1/3 \\ 1/3 & s+1/4 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (3.11.2)$$

with corresponding observability and controllability grammians given by $P=Q=\text{diag}(1, 2)$

(a) Maximising singular vectors and their all-pass completions

From Theorem 3.3 it follows that the scaled singular vectors corresponding to the Hankel norm of $G(s)$ $\|G(s)\|_H = \sigma_1 = 2$ can be calculated via the solution to the two (identical in this case) Riccati equations:

$$\frac{1}{9}\Omega^2 - \frac{4}{3}\Omega - 1 = 0 \Rightarrow \Omega = 6 \pm 3\sqrt{5}, \tilde{\Omega} = \Omega \text{ from symmetry.} \quad (3.11.3)$$

It is also easily verified that the destabilising solutions correspond to the largest solution $6+3\sqrt{5}$; we also note in passing that the relations predicted in Lemma 3.6 can be easily seen to be satisfied, namely

$$\begin{aligned} \Delta(\Omega_-, \tilde{\Omega}_+) &= \tilde{\Omega}_+ \Omega_- + \Sigma \Omega_- + \tilde{\Omega}_+ \Sigma + \Gamma = \\ &= (6+3\sqrt{5})(6+3\sqrt{5}) + (6+3\sqrt{5}) - (6+3\sqrt{5}) - 3 \\ &= 0 \end{aligned} \quad (3.11.4)$$

and

$$\Delta(\Omega_-, \tilde{\Omega}_-) < 0 \quad (3.11.5)$$

Using the state-space formulae of section 3.4, the (scaled) maximising vectors of the Hankel operator Γ_G can be found to be

$$v_1(s) = \begin{bmatrix} -1 \\ 3s+2 \end{bmatrix} \frac{1}{3s-\sqrt{5}} \quad w_1(s) = \begin{bmatrix} 1 \\ 3s-2 \end{bmatrix} \frac{1}{3s+\sqrt{5}} \quad (3.11.6)$$

and their corresponding all-pass extensions

$$V_\perp(s) = \begin{bmatrix} 3s-2 \\ -1 \end{bmatrix} \frac{1}{3s-\sqrt{5}} \quad W_\perp(s) = \begin{bmatrix} 3s+2 \\ 1 \end{bmatrix} \frac{1}{3s+\sqrt{5}} \quad (3.11.7)$$

(b) Diagonalisation

From Glover [G1] we can express the complete family of error systems $\mathcal{S} = G + \mathcal{F}$ with $\mathcal{F} \in \mathcal{H}_\infty^-$ and $\|\mathcal{S}\|_\infty \leq 2$ in terms of the parametrisation $\mathcal{S} = G + \mathcal{F}_\ell(F_a, U)$ where

$$\mathcal{F}_\ell(\mathcal{F}_a, U) = \begin{bmatrix} \frac{1}{3(s-1)} & \frac{-1}{3(s-1)} \\ \frac{-1}{3(s-1)} & \frac{7-6s}{3(s-1)} \end{bmatrix} + \begin{bmatrix} \frac{4-6s}{3(s-1)} \\ \frac{2}{3(s-1)} \end{bmatrix} U \left(1 - \frac{2}{3(s-1)} U\right)^{-1} \begin{bmatrix} \frac{6s-4}{3(s-1)} & \frac{-2}{3(s-1)} \end{bmatrix}$$

with $U(s) \in \mathcal{H}_\infty^-$ such that $\|U(s)\|_\infty \leq \frac{1}{2}$.

Let us apply, next, transformations $V_a^*(s)$ from the left and $W_a(s)$ from the right to the generator $H(s)$ of all error systems $\mathcal{S}(s)$

$$V_a^* H(s) W_a = \left(\frac{1}{3s+\sqrt{5}}\right)^2 \begin{bmatrix} 1 & 3s-2 & 0 \\ 3s+2 & 1 & 0 \\ 0 & 0 & 3s+\sqrt{5} \end{bmatrix} H(s) \begin{bmatrix} 1 & 3s+2 & 0 \\ 3s-2 & 1 & 0 \\ 0 & 0 & 3s+\sqrt{5} \end{bmatrix} \quad (3.11.8)$$

where V_a and W_a denote the augmented matrices $V(s)$ and $W(s)$ and $H(s)$ is given by

$$H(s) = \begin{bmatrix} \frac{1}{3(s-1)} + G_{11} & \frac{-1}{3(s-1)} + G_{12} & \frac{4-6s}{3(s-1)} \\ \frac{-1}{3(s-1)} + G_{21} & \frac{7-6s}{3(s-1)} + G_{22} & \frac{2}{3(s-1)} \\ \frac{6s-4}{3(s-1)} & \frac{-2}{3(s-1)} & \frac{-4}{3(s-1)} \end{bmatrix} \quad (3.11.9)$$

After a number of calculations we arrive at the result,

$$V_a^* H(s) W_a = \begin{bmatrix} \sigma_1 a(s) & 0 & 0 \\ 0 & G_1 + Q_{11} & Q_{12} \\ 0 & Q_{21} & Q_{22} \end{bmatrix} \quad (3.11.10)$$

where

$$\sigma_1 a(s) = 2 \frac{(-3s+\sqrt{5})(36s^2-45s+5)}{(3s+\sqrt{5})(36s^2+45s+5)}, \quad G_1 + Q_{11}(s) = \frac{4(3s-\sqrt{5})}{3(s-1)(3s+\sqrt{5})} \quad (3.11.11)$$

$$Q_{21} = -Q_{12} = \frac{2(3s-\sqrt{5})}{3(s-1)} \quad \text{and} \quad Q_{22} = \frac{-4}{3(s-1)} \quad (3.11.12)$$

Note:

(i) The (1,1) block is indeed all-pass at a level equal to the Hankel norm of $G(s)$, $\|G(s)\|_H=2$. The poles of $a(s)$ consist of the poles of $G(s)$ together with either the poles of $V^*(s)$ or $W(s)$, in agreement with the results of the diagonalisation theorem 3.5.2.

(ii) All the terms (1,2), (2,1), (1,3) and (2,3) cancel identically as expected.

(iii) Q_{12} , Q_{21} , $Q_{22} \in \mathcal{H}_-^\infty$ and the (2, 2) block of $V_a^* H(s) W_a$ has both a stable and an unstable part consisting of the poles of $V^*(s)$ (or $W(s)$) together with the poles of the initial extension $F_0(s)$ (here the unstable poles of the generator) respectively.

Next, the (2,2) block will be decomposed into its stable and unstable parts $G_1(s) \in \mathcal{H}_+^\infty$ and $Q_{11}(s) \in \mathcal{H}_-^\infty$; this gives

$$\frac{4(3s-\sqrt{5})}{3(s-1)(3s+\sqrt{5})} = \frac{6\sqrt{5}-10}{3s+\sqrt{5}} + \frac{14-6\sqrt{5}}{3(s-1)} = G_1(s) + Q_{11}(s) \quad (3.11.13)$$

Minimising the maximum level of the second singular value (super-optimal solution) or its integral over all frequencies ($\mathcal{L}^\infty/\mathcal{L}^2$ solution) in the family of Nehari extensions of $G(s)$, $\mathcal{N}_G \subseteq \mathcal{H}_-^\infty$ can be formulated as

$$\begin{aligned} \inf_{F \in \mathcal{N}_G} s_2^\infty(G+F) &= \inf_{2U \in \mathfrak{B}\mathfrak{H}_-^\infty} \left\| \frac{6\sqrt{5}-10}{3s+\sqrt{5}} + \mathcal{F}_\ell \left(\begin{bmatrix} \frac{14-6\sqrt{5}}{3(s-1)} & \frac{-2(3s-\sqrt{5})}{3(s-1)} \\ \frac{2(3s-\sqrt{5})}{3(s-1)} & \frac{-4}{3(s-1)} \end{bmatrix}, U \right) \right\|_\alpha \\ &= \inf_{2U \in \mathfrak{B}\mathfrak{H}_-^\infty} \left\| \frac{4(3s-\sqrt{5})}{3(s-1)(3s+\sqrt{5})} - \frac{4(3s-\sqrt{5})^2}{9(s-1)^2} U (1 + \frac{4}{3(s-1)} U)^{-1} \right\|_\alpha \end{aligned} \quad (3.11.14)$$

where $\alpha=\infty$ or 2. Applying Theorems 3.6.1 and 3.10.6 (while noting that for the case $\alpha=2$, $\sigma_1(G)=2\sigma_2(G) \Rightarrow \|G_1\|_\infty \leq 2$) the two constrained minimisations can be replaced by a 'free' approximation on the stable part of the (2, 2) block, $G_1(s)$,

$$\inf_{F_1 \in \mathfrak{H}_-^\infty} \left\| \frac{6\sqrt{5}-10}{3s+\sqrt{5}} + F_1 \right\|_\alpha \quad (3.11.15)$$

For $\alpha=2$, the optimum choice of F_1 is equal to zero. In the case $\alpha=\infty$, the (unique) $F_1(s)$ which achieves the infimum in (3.11.15) is the constant that makes the sum all-pass at $\|G_1(s)\|_H$. This is given by $F_1=\sqrt{5}-3$, corresponding to an all-pass error system,

$$G_1(s)+F_1 = (\sqrt{5}-3) \frac{3s-\sqrt{5}}{3s+\sqrt{5}} \quad (3.11.16)$$

(c) Construction of the $\mathcal{L}^\infty/\mathcal{L}^2$ -optimal extension.

The $\mathcal{L}^\infty/\mathcal{L}^2$ optimal extension is given by,

$$\begin{aligned} F_{opt}(s) &= F_{11} - V_\perp Q_{11} W_\perp^*(s) \\ &= \begin{bmatrix} \frac{3(6\sqrt{5}-13)s+17-8\sqrt{5}}{(3s-\sqrt{5})^2} & \frac{-3s+(11-4\sqrt{5})}{(3s-\sqrt{5})^2} \\ \frac{-3s+(11-4\sqrt{5})}{(3s-\sqrt{5})^2} & \frac{-18s^2+(3+12\sqrt{5})s-(7+2\sqrt{5})}{(3s-\sqrt{5})^2} \end{bmatrix} \end{aligned} \quad (3.11.17)$$

The $\mathcal{L}^2$ -optimal cost may be evaluated as

$$\mu_0 = \|G_1\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(6\sqrt{5}-10)^2}{5+9\omega^2} d\omega = \frac{1}{3\pi} \frac{(6\sqrt{5}-10)^2}{\sqrt{5}} \left\{ \tan^{-1} \left(\frac{3\omega}{\sqrt{5}} \right) \right\}_0^\infty = \frac{(6\sqrt{5}-10)^2}{6\sqrt{5}} \quad (3.11.18)$$

which achieves the bound given in Theorem 3.10.7.

(d) *Recovery of the superoptimal extension.*

The super-optimal extension is recovered by the 'unwrapping part of the algorithm as

$$F_{so}(s) = (F_0 - V_\perp Q_{11} W_\perp^*)(s) + V_\perp F_1 W_\perp^*(s) = F_{opt} + V_\perp F_1 W_\perp^*(s)$$

$$= \begin{bmatrix} \frac{(\sqrt{5}-3)s+4-\sqrt{5}}{3s+\sqrt{5}} & \frac{2-\sqrt{5}}{3s+\sqrt{5}} \\ \frac{2-\sqrt{5}}{3s+\sqrt{5}} & \frac{1+2\sqrt{5}-6s}{3s+\sqrt{5}} \end{bmatrix} \quad (3.11.19)$$

We note:

(i) The level of $s_2^\infty(G+F_{so})(s) = \|G_1\|_H = \sqrt{\lambda(\Sigma+\Gamma\Omega^{-1})(\Sigma+\Gamma\bar{\Omega}^{-1})} = 3-\sqrt{5} < 1 = \sigma_2(G)$ in agreement with Theorem 3.7.2.

(ii) The cancellation results in Lemmas 3.8.1 and 3.8.2 are verified. In particular, the residue of the pole at $s=1$ is zero independently of F_1 ; this implies a cancellation of the poles of the extension $F_0(s)$ (and in fact of any initial extension) in the term $F_0 - V_\perp Q_{11} W_\perp^*(s)$ (see Lemma 3.8.1 and Remark 3.8.1). The choice of F_1 as the (unique in this case) extension of $G_1(s)$ results in the remaining cancellation (of the poles of $V_\perp(s)$ or $W_\perp^*(s)$) in agreement with Lemma 3.8.2.

Figure 7 shows the singular values of $G+F(s)$ with $F(s)$ taken as the super-optimal extension, the $\mathcal{L}^2$ -optimal extension and the extension which minimises the entropy cost (at infinity). The largest singular value of all three error systems is constant over all frequencies at 2. Note also that since [P4] involves a logarithmic cost function while [P3] a linear one, the maximum-entropy solution emphasises the approximation at low frequencies relative to the $\mathcal{L}^2$ solution.

It may be seen from (3.11.19) that we end up with a real-symmetric super-optimal extension $F_{so}(s)$ corresponding to a real-symmetric $G(s)$. That this is true in general is shown in the next lemma:

Lemma 3.11.1:

The superoptimal extensions of real-symmetric rational matrix functions are real-symmetric.

Proof:

The proof will be by induction on the input/output dimension of $G(s) = G^T(s)$. Clearly, for $p=1$ the result holds trivially. Suppose it is true for all real-symmetric rational $(p-1) \times (p-1)$ matrix functions, and consider a real-symmetric rational $p \times p$ matrix function $G(s)$. Since $G(s)$ is real symmetric, $\Gamma_G = \Gamma_{G^T} = \Gamma_G^T$; this in turn implies that the maximising vectors of Γ_G satisfy $f(s) = g(-s) \forall s$. Scaling $f(s)$ and $g(-s)$ does not affect their symmetry, so that $v(s) = w(-s)$. It follows that the all-pass matrices based on $v(s)$ and $w(s)$ satisfy, $V^*(s) = V^T(-s) = W^T(-s)$ and in particular,

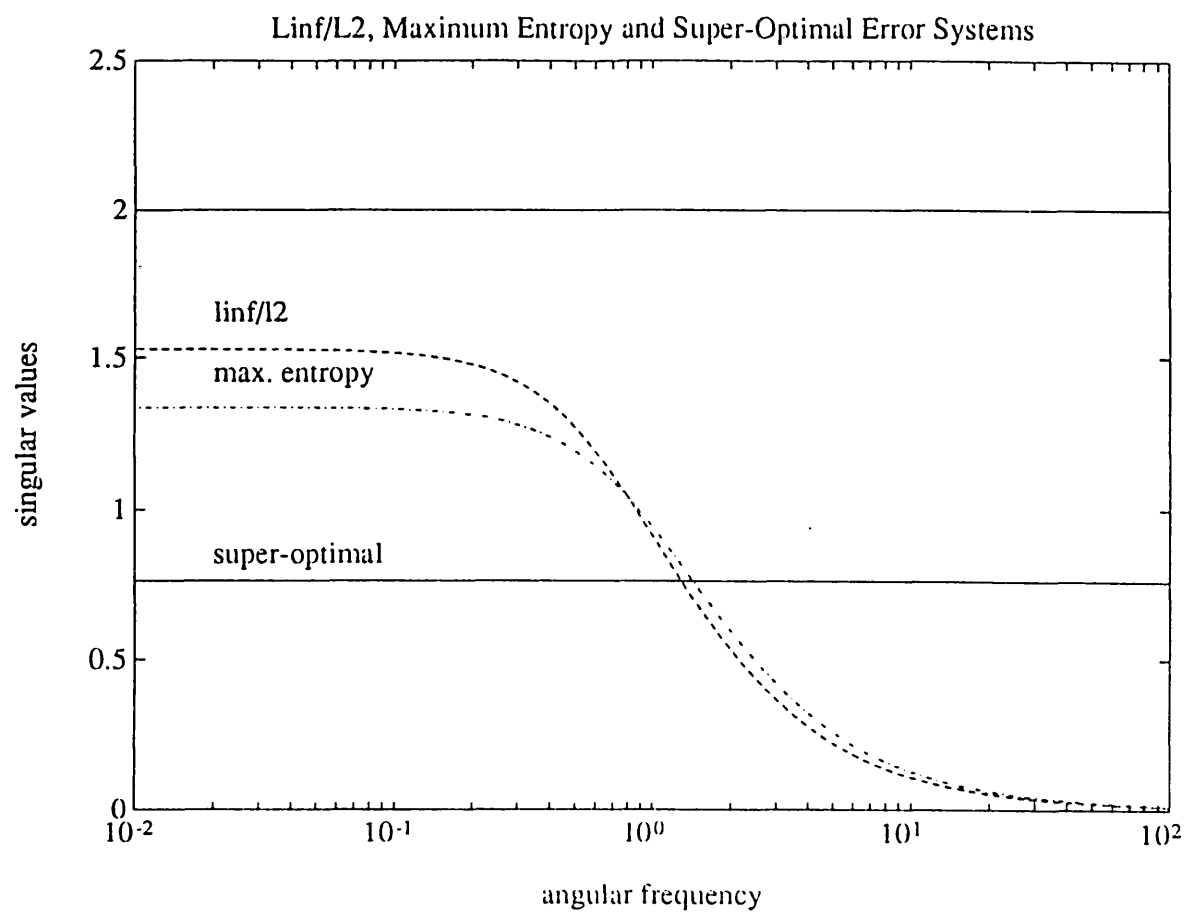


Figure 7: Singular values for super-optimal, $\mathcal{H}^\infty/\mathcal{H}^\infty$ and maximum entropy error systems

$V_\perp^*(s) = V_\perp^T(-s) = W_\perp^T(s)$. Consider next the generator of all Nehari extensions $\mathcal{T}_\ell(F, U)$ (defined in Lemma 3.1.1), corresponding to $G(s) \in \mathcal{H}_+^{\infty, (p \times p)}$. $F_{11} = \mathcal{T}_\ell(F, 0)$ is an admissible extension since it is generated by $U=0$; in addition, since for real symmetric rational matrices $G(s)$, real symmetric Nehari extensions correspond precisely to real symmetric 'feedback' terms $U(s)$ [B2], $F_{11}^T(s) = F_{11}(s)$. Hence,

$$G_1(s) + Q_{11}(s) = V_\perp^*(s) H_{11}(s) W_\perp(s) = W_\perp^T(s) H_{11}(s) W_\perp(s) \quad (3.11.20)$$

is real symmetric, and, therefore, since $G_1(s) \in \mathcal{H}_+^\infty$, $Q_{11}(s) \in \mathcal{H}_-^\infty$ with both G_1 and Q_{11} strictly proper, $G_1(s) = G_1^T(s)$ and $Q_{11}(s) = Q_{11}^T(s)$. Now, the super-optimal extension for $G(s)$ can be expressed as

$$F_{s,o}(s) = F_0(s) - V_\perp(\tilde{F} + Q_{11}) W_\perp^*(s) \quad (3.11.21)$$

where $F_0(s) = H_{11}(s)$ and $\tilde{F}(s)$ is the super-optimal extension to $G_1(s) \in \mathcal{H}_+^{\infty, (p-1) \times (p-1)}$; from the induction hypothesis and the symmetry of G_1 it follows that $\tilde{F}^T(s) = \tilde{F}(s)$. The result follows on noting that $F_0(s) = H_{11}(s) = F_0^T(s)$ and that $W_\perp^*(s) = V_\perp^T(s)$ so that

$$F_{s,o}(s) = F_0(s) - V_\perp(\tilde{F} + Q_{11}) V_\perp^T(s) = F_{s,o}^T(s) \quad (3.11.22)$$

as required. □

3.12 Construction of Super-optimal Hankel-norm approximations

In this section the construction of super-optimal Hankel-norm approximations is derived. Before stating and proving the main Theorem of the section a preliminary result is required concerning the zeros of the off-diagonal blocks of $Q(s)$ defined in Lemma 3.5.3. This is given next:

Lemma 3.12.1

Let

$$Q(s) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & -\sigma I \\ \hat{C}_2 & \sigma I & 0 \end{array} \right] \quad (3.12.1)$$

be defined as in (3.5.38). Then

- (1) All the MacMillan zeros of $Q_{12}(s)$ and $Q_{21}(s)$ lie in the open right half plane.
- (2) Any system zero of the realisation of $Q_{12}(s)$ that lies in the left half plane is an uncontrollable mode of the pair $(\hat{A}, \hat{B}_2)$.
- (3) Any system zero of the realisation of $Q_{21}(s)$ that lies in the left half plane is an

unobservable mode of the pair $(\hat{A}, \hat{C}_2)$.

(4) If λ is a stable system zero of $Q_{12}(s)$ then $-\bar{\lambda}$ is a MacMillan zero of $\xi(s)$.

(5) If λ is a stable system zero of $Q_{21}(s)$ then it is also a MacMillan zero of $\psi(s)$.

Proof:

The position of the zeros of $Q_{12}(s)$ will be established directly from the state-space realisation (3.5.38). From this realisation it is clear that the system zeros of $Q_{12}(s)$ are located at the eigenvalues of

$$\Phi = \hat{A} + \Gamma^{-1} C_1^* C_\perp C_\perp C_1 (\Sigma + \Omega^\# \Gamma) \quad (3.12.2)$$

in which

$$\hat{A} = \Gamma^{-1} (\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^*) \quad (3.12.3)$$

Using the (1,1) block of (3.1.2) and the (1,2) blocks of (3.1.1) and (3.1.2) gives

$$\hat{A} = -\Gamma^{-1} C_1^* C_\perp C_\perp C_1 \Sigma - \Gamma^{-1} (A_{11} - A_{12} C_2^* C_1)^* \Gamma \quad (3.12.4)$$

Provided that the desired solution to (3.4.31) exists and is invertible, substituting (3.12.4) into (3.12.2) gives

$$\Phi = \Gamma^{-1} \Omega \{ A_{11} + A_{12} (A_{12}^* \Omega - C_2^* C_1) \} \Omega^{-1} \Gamma \quad (3.12.5)$$

which is completely unstable because Ω is chosen to be destabilising.

In the event that (A_{11}, A_{12}) is not controllable, the required solution to (3.4.31) will not exist. In this case we substitute (3.12.4) into (3.12.2) and make the transformation T_1 in (3.4.33) to obtain

$$\begin{aligned} T_1^{-*} \Gamma \Phi \Gamma^{-1} T_1^* &= T_1^{-*} [C_1^* C_\perp C_\perp C_1 \Omega^\# - A_{11}^* + C_1^* C_2 A_{12}^*] T_1^* \\ &= \begin{bmatrix} \tilde{C}_1^* C_\perp C_\perp \tilde{C}_1 \Omega_r^\# - \tilde{A}_{11}^* + \tilde{C}_1^* C_2 \tilde{A}_{13}^* & 0 \\ \cdot & -\tilde{A}_{22}^* \end{bmatrix} \end{aligned} \quad (3.12.6)$$

in which $\lambda(-\tilde{A}_{22}^*) \subset \lambda(-A_{11}^*)$ are completely unstable and Ω_r is the appropriate solution to the reduced dimension Riccati equation (3.4.24). As explained in section 3.4, (3.4.24) will always have the required destabilising solution. In the case that Ω_r is invertible, we may substitute $(3.4.24)\Omega_r^{-1}$ into (3.12.6) to obtain a (1,1) block of

$$\tilde{C}_1^* C_\perp C_\perp \tilde{C}_1 \Omega_r^{-1} - \tilde{A}_{11}^* + \tilde{C}_1^* C_2 \tilde{A}_{13}^* = \Omega_r [\tilde{A}_{11} + \tilde{A}_{13} (\tilde{A}_{13}^* \Omega_r - C_2^* \tilde{C}_1)] \Omega_r^{-1} \quad (3.12.7)$$

which is completely unstable due to our choice of solution to (3.4.24). Thus, in this case also, the system zeros of $Q_{12}(s)$ are completely unstable.

In the remaining of this proof we will consider the third possibility that Ω_r is singular. We begin by introducing the notation

$$\mathcal{A} := \tilde{A}_{11} - \tilde{A}_{13} \tilde{C}_2^* \tilde{C}_1 \quad (3.12.8)$$

$$\mathcal{B} := \tilde{A}_{13} \quad (3.12.9)$$

$$\mathcal{C} := C_\perp \tilde{C}_1 \quad (3.12.10)$$

In this notation (3.4.24) becomes

$$\Omega_r \mathcal{A} + \mathcal{A}^* \Omega_r + \Omega_r \mathcal{B} \mathcal{B}^* \Omega_r - \mathcal{C} \mathcal{C}^* = 0 \quad (3.12.11)$$

Next we note that if $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ undergoes a general basis change T we may write,

$$T^{-*} (\Omega_r T^{-1} \cdot T \mathcal{A} + \mathcal{A}^* T^* \cdot T^{-*} \Omega_r + \Omega_r T^{-1} \cdot T \mathcal{B} \mathcal{B}^* T^* \cdot T^{-*} \Omega_r - \mathcal{C} \mathcal{C}^*) T^{-1} = 0$$

so that $\Omega_r \rightarrow T^{-*} \Omega_r T^{-1}$. Now following the construction of section 2.1 we can find a special basis T for which

$$\Omega_r = \begin{bmatrix} \Omega_{11} & 0 \\ 0 & 0 \end{bmatrix} \quad (3.12.12)$$

with $\Omega_{11} > 0$. Note also that with no loss of generality we can absorb this special basis change T into T_1 (defined in section 3.4) by redefining

$$T_1 := \begin{bmatrix} T & 0 \\ 0 & I \end{bmatrix} T_1$$

Hence, redefining $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ in this basis and partitioning conformally with (3.12.12), we get

$$\begin{aligned} & \begin{bmatrix} \Omega_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} + \begin{bmatrix} \mathcal{A}_{11}^* & \mathcal{A}_{21}^* \\ \mathcal{A}_{12}^* & \mathcal{A}_{22}^* \end{bmatrix} \begin{bmatrix} \Omega_{11} & 0 \\ 0 & 0 \end{bmatrix} + \\ & \begin{bmatrix} \Omega_{11} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} \begin{bmatrix} \mathcal{B}_1^* & \mathcal{B}_2^* \end{bmatrix} \begin{bmatrix} \Omega_{11} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathcal{C}_1^* \\ \mathcal{C}_2^* \end{bmatrix} \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \end{bmatrix} = 0 \end{aligned} \quad (3.12.13)$$

From the (2,2) block of this equation we get that

$$\mathcal{C}_2^* \mathcal{C}_2 = 0 \Rightarrow \mathcal{C}_2 = 0 \quad (3.12.14)$$

while the (1,2) block and (3.12.13) gives

$$\Omega_{11} \mathcal{A}_{12} = 0 \Rightarrow \mathcal{A}_{12} = 0 \quad (3.12.15)$$

since Ω_{11} is invertible. We conclude that $\mathcal{A}_{22}$ contains the modes of $\tilde{A}_{11} - \tilde{A}_{13}C_2^* \tilde{C}_1$ which are unobservable through $C_\perp \tilde{C}_1$. In addition, since $\mathcal{A} + \mathcal{B}\mathcal{B}^* \Omega_r$ is completely unstable, we observe that

$$\begin{bmatrix} \mathcal{A}_{11} & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{bmatrix} + \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{bmatrix} \begin{bmatrix} \mathcal{B}_1^* & \mathcal{B}_2^* \end{bmatrix} \begin{bmatrix} \Omega_{11} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_{11} + \mathcal{B}_1 \mathcal{B}_1^* \Omega_{11} & 0 \\ * & \mathcal{A}_{22} \end{bmatrix} \quad (3.12.16)$$

is completely unstable, whence

$$Re[\lambda_i(\mathcal{A}_{11} + \mathcal{B}_1 \mathcal{B}_1^* \Omega_{11})] > 0 \quad (3.12.17)$$

and

$$Re[\lambda_i(\mathcal{A}_{22})] > 0 \quad (3.12.18)$$

The (1,1) block of $T^{-*} \Gamma \Phi \Gamma^{-1} T^*$, $\tilde{\Phi}_{11}$, given in (3.12.6) becomes (in the new notation),

$$\begin{aligned} \tilde{\Phi}_{11} &= \begin{bmatrix} \mathcal{C}_1^* \\ 0 \end{bmatrix} \begin{bmatrix} \mathcal{C}_1 & \mathcal{C}_2 \end{bmatrix} \begin{bmatrix} \Omega_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathcal{A}_{11}^* & \mathcal{A}_{21}^* \\ 0 & \mathcal{A}_{22}^* \end{bmatrix} \\ &= \begin{bmatrix} \mathcal{C}_1^* \mathcal{C}_1 \Omega_{11}^{-1} - \mathcal{A}_{11}^* & -\mathcal{A}_{21}^* \\ 0 & -\mathcal{A}_{22}^* \end{bmatrix} \end{aligned} \quad (3.12.19)$$

The (1,1) block of $\tilde{\Phi}_{11}$ is completely unstable by the (1,1) block of the transformed Riccati equation (3.12.13) since,

$$\mathcal{C}_1^* \mathcal{C}_1 \Omega_{11}^{-1} - \mathcal{A}_{11}^* = \Omega_{11} [\mathcal{A}_{11} + \mathcal{B}_1 \mathcal{B}_1^* \Omega_{11}] \Omega_{11}^{-1} \quad (3.12.20)$$

with $\mathcal{A}_{11} + \mathcal{B}_1 \mathcal{B}_1^* \Omega_{11}$ completely unstable by (3.12.17). We also note that all the modes of $-\mathcal{A}_{22}^*$ are stable (by (3.12.18)) and appear as eigenvalues of the zero matrix Φ . It will next be shown that all the stable system zeros of $Q_{12}(s)$ which are associated with $-\mathcal{A}_{22}^*$ are uncontrollable modes in the realisation of $Q_{12}(s)$ given in (3.5.38).

Suppose that s_0 is any (unstable) mode of $\tilde{A}_{11} - \tilde{A}_{13}C_2^* \tilde{C}_1$ which is unobservable through $C_\perp \tilde{C}_1$, so that $-\bar{s}_0 \in \lambda(-\mathcal{A}_{22}^*)$. Then there exists a vector $\xi \neq 0$ such that

$$\begin{bmatrix} s_0 I - A_{11} + A_{12} C_2^* C_1 \\ C_1^* C_\perp C_\perp C_1 \end{bmatrix} \xi = 0 \quad (3.12.21)$$

However,

$$\Delta = \Gamma \dot{A} \Gamma^{-1} + (A_{11} - A_{12} C_2^* C_1)^* + C_1^* C_\perp C_\perp C_1 \Gamma^{-1} \Sigma$$

$$\begin{aligned}
 &= [\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^* + (A_{11}^* - C_1^* C_2 A_{12}^*) \Gamma + C_1^* C_\perp C_\perp C_1 \Sigma] \Gamma^{-1} \\
 &= -C_1^* C_2 [C_2^* C_1 \Sigma - \sigma B_2 B_1^* + A_{12}^* \Gamma] \Gamma^{-1} \\
 &= 0
 \end{aligned} \tag{3.12.22}$$

on substituting from (3.1.1), (3.1.2), (3.1.3), (3.1.18) and (3.1.11); Using this in (3.12.21) we get

$$\xi^* [\bar{s}_0 I - A_{11}^* + C_1^* C_2 A_{12}^* \mid C_1^* C_\perp C_\perp C_1] = 0 \tag{3.12.23}$$

$$\Leftrightarrow \xi^* [\bar{s}_0 I + \Gamma \hat{A} \Gamma^{-1} + C_1^* C_\perp C_\perp C_1 \Gamma^{-1} \Sigma \mid C_1^* C_\perp C_\perp C_1] = 0 \tag{3.12.24}$$

$$\Leftrightarrow \xi^* [\bar{s}_0 I + \Gamma \hat{A} \Gamma^{-1} \mid C_1^* C_\perp C_\perp C_1] = 0 \tag{3.12.25}$$

so that

$$(\xi^* \Gamma) [-\bar{s}_0 I - \hat{A} \mid \Gamma^{-1} C_1^* C_\perp] = 0 \tag{3.12.26}$$

with $\xi^* \Gamma \neq 0$ since Γ is non-singular. Inspecting the realisation of $Q_{12}(s)$ given in (3.5.38) we conclude that every eigenvalue of $-\mathcal{A}_{22}^*$ in the zero matrix Φ corresponds to an uncontrollable mode in the realisation given in (3.5.38). This shows that:

- (1) All the MacMillan zeros of $Q_{12}(s)$ lie in the open right half plane.
- (2) Every system zero of $Q_{12}(s)$ that lies in the left half plane is an uncontrollable mode in (3.5.38).

The link between the stable system zeros of $Q_{12}(s)$ and the unstable MacMillan zeros of $\xi(s)$ may be established using Lemma 2.3.2: If s_0 is a stable system zero of $Q_{12}(s)$ then it must belong to the spectrum of $-\mathcal{A}_{22}^*$, where $\mathcal{A}_{22}$ contains the unstable modes of $\tilde{A}_{11} - \tilde{A}_{13} C_2^* \tilde{C}_1$ which are unobservable through $C_\perp \tilde{C}_1$. However, from Lemma 2.3.2 these modes coincide with the unstable system zeros of

$$\begin{bmatrix} sI - \tilde{A}_{11} & -\tilde{A}_{13} \\ \tilde{C}_1 & C_2 \end{bmatrix} \tag{3.12.27}$$

or, equivalently, (since $(\tilde{A}_{11}, \tilde{A}_{13}, \tilde{C}_1, C_2)$ is a minimal realisation), with the unstable MacMillan zeros of $\xi(s)$.

A dual argument will establish the position of the zeros of $Q_{21}(s)$. □

Theorem 3.12.2

Assume that the Schmidt vectors $\xi(s)$ and $\psi^*(s)$ given in (3.16) are free of right half plane MacMillan zeros. Then the family $\tilde{\mathcal{F}}(s)$ of all approximations $F \in \mathcal{H}_-^\infty(k)$ that minimise the pair

$$(s_1^\infty(G_0 - F), s_2^\infty(G_0 - F)) \tag{3.12.28}$$

lexicographically is parametrised by

$$\tilde{\mathcal{F}}(s) = F_{11} + V_\perp (Q_{11} + \tilde{\mathcal{F}}_1) W_\perp^*(s) \quad (3.12.29)$$

in which $\tilde{\mathcal{F}}_1$ denotes the family of all optimal ℓ -th order extensions to $G_1(s)$. That is all $F_1(s) \in \mathcal{H}_-^\infty(\ell)$ that satisfy $\|G_1 - F_1\|_\infty = \sigma_{\ell+1}(G_1)$ where $\ell = \deg(Q_+) \leq k$.

Note: The case in which either $\xi(s)$ or $\psi^*(s)$ or both have right half plane MacMillan zeros will be dealt with later in Remark 3.12.4.

Proof:

Using the results of Lemma 3.5.3, the family of all ‘optimal’ error systems (with respect to $s_1^\infty(\cdot)$) may be parametrised by

$$G_0 + \mathcal{F}(s) = \left\{ V(s) [\sigma_{k+1}(G_0) a(s) \oplus G_1 + \mathcal{F}_\ell(Q, U)] W^*(s) \mid U(s) \in \frac{1}{\sigma_{k+1}(G_0)} \mathcal{B}\mathcal{H}_-^\infty \right\} \quad (3.12.30)$$

Since $V(s)$, $W^*(s)$ and $a(s)$ are all-pass, the family of all extensions which minimise $s_2(G_0 + \mathcal{F})$ are generated by

$$\inf_{U \in 1/\sigma_{k+1}(G_0) \mathcal{B}\mathcal{H}_-^\infty} \|G_1 + \mathcal{F}_\ell(Q, U)\|_\infty = s_2^\infty(G_0 + \mathcal{F}) \quad (3.12.31)$$

It will now be shown that provided $\xi(s)$ and $\psi^*(s)$ have no MacMillan zeros in the right half plane, the minimisation problem in (3.12.31) is equivalent to the unconstrained problem

$$\inf_{\tilde{F} \in \mathcal{H}_-^\infty(\ell)} \|G_1 + \tilde{F}\|_\infty = \sigma_{\ell+1}(G_1) \quad (3.12.32)$$

where $\ell \leq k$ is the MacMillan degree of the stable part of $Q(s)$. Since $Q(s) \in \mathcal{H}_-^\infty(\ell)$ and $\|Q_{22} U\|_\infty = \|F_{22} U\|_\infty < 1$, a ‘small-gain’ argument [6, 10] shows that $\mathcal{F}_\ell(Q, U) \in \mathcal{H}_-^\infty(\ell)$, for all $U(s) \in 1/\sigma_{k+1}(G_0) \mathcal{B}\mathcal{H}_-^\infty$. By comparing (3.12.31) and (3.12.32) we may write

$$\inf_{U(s) \in 1/\sigma_{k+1}(G_0) \mathcal{B}\mathcal{H}_-^\infty} \|G_1 + \mathcal{F}_\ell(Q, U)\|_\infty \geq \inf_{\tilde{F} \in \mathcal{H}_-^\infty(\ell)} \|G_1 + \tilde{F}\|_\infty = \sigma_{\ell+1}(G_1) \quad (3.12.33)$$

It will be shown that every $\tilde{F} \in \mathcal{H}_-^\infty(\ell)$ which achieves the minimum in (3.12.32) is generated by some $U(s) \in 1/\sigma_{k+1}(G_0) \mathcal{B}\mathcal{H}_-^\infty$ via $\mathcal{F}_\ell(Q, U)$. This is based on an argument given in [G3].

Since all the MacMillan zeros of $Q_{12}(s)$ lie in the open right half plane (Lemma 3.12.1), $Q_{12}(j\omega)$ is non-singular for every $\omega \in \mathbb{R}$. This together with the all-pass character of (Lemma 3.5.3)

$$\frac{1}{\sigma_{k+1}(G_0)} \begin{bmatrix} G_1 + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad (3.12.34)$$

gives

$$\|G_1 + Q_{11}\|_\infty < \sigma_{k+1}(G_0) \quad (3.12.35)$$

Further,

$$\sigma_{\ell+1}(G_1) \leq \|G_1 + Q_{11}\|_\infty \quad (3.12.36)$$

so that

$$\sigma_{\ell+1}(G_1) < \sigma_{k+1}(G_0) \quad (3.12.37)$$

In addition, we will demonstrate that

$$\left\{ \mathcal{F}_\ell(Q, U) : \|U(s)\|_\infty < 1/\sigma_{k+1}, U \in \mathcal{H}_-^\infty \right\} \quad (3.12.38)$$

generates every $\tilde{F}(s) \in \mathcal{H}_-^\infty(\ell)$ with the property $\|G_1 - \tilde{F}\|_\infty < \sigma_{k+1}(G_0)$. Suppose that some $\tilde{F}(s)$ satisfies $\|G_1 + \tilde{F}\|_\infty < \sigma_{k+1}(G_0)$. Then

$$\tilde{U}(s) = \mathcal{F}_\ell \left(\frac{1}{\sigma_{k+1}^2(G_0)} \begin{bmatrix} Q_{22}^* & Q_{12}^* \\ Q_{21}^* & G_1 + Q_{11}^* \end{bmatrix}, G_1 + \tilde{F}(s) \right) \quad (3.55)$$

gives

$$\tilde{F}(s) = \mathcal{F}_\ell \left(\begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \tilde{U}(s) \right) \quad (3.56)$$

Since $\|G_1 + Q_{11}\|_\infty < \sigma_{k+1}(G_0)$ and $\|G_1 + \tilde{F}\|_\infty < \sigma_{k+1}(G_0)$, the linear fractional map defining $\tilde{U}(s)$ is well-posed and further $\|\tilde{U}(s)\|_\infty < \sigma_{k+1}^{-1}(G_0)$. It remains to show that $\tilde{U}(s) \in \mathcal{H}_-^\infty$. Suppose contrary to what will be proved that $\tilde{U}(s)$ has r poles in the left half plane. This implies that the A-matrix for $\mathcal{F}_\ell(Q, \tilde{U})$ has $\ell + r$ eigenvalues in the left half plane by a Nyquist-type argument [G4], [R1]. Since $\tilde{F} \in \mathcal{H}_-^\infty(\ell)$ by assumption, there must be at least r cancellations in the closed-loop. Since any cancellation is constrained to occur at a system zero of $Q_{12}(s)$ or $Q_{21}(s)$ (see Theorem 2.3.1), we obtain the required contradiction since by Lemma 3.12.1 and the Theorem's hypothesis $Q_{12}(s)$ and $Q_{21}(s)$ are free of any such zeros.

It follows, therefore, that all optimal ℓ -th order extensions of $G_1(s)$ are generated via $\mathcal{F}_\ell(Q, U)$ as claimed. As a consequence, the set of all extensions $F(s)$ which minimise $(s_1^\infty(G_0 - F), s_2^\infty(G_0 - F))$ lexicographically, is parametrised by

$$\tilde{\mathcal{F}}(s) = G_0 - V \begin{bmatrix} \sigma_{k+1}(G_0)a(s) & 0 \\ 0 & G_1(s) - \mathcal{F}_1(s) \end{bmatrix} W^*$$

$$\begin{aligned}
 &= G_0 - V \begin{bmatrix} \sigma_{k+1}(G_0)a(s) & 0 \\ 0 & G_1 + Q_{11} \end{bmatrix} W^* + V \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{F}_1 + Q_{11} \end{bmatrix} W^* \\
 &= \mathcal{F}_\ell(F, 0) + V_\perp(\tilde{\mathcal{F}}_1 + Q_{11})W_\perp^* = F_{11}(s) + V_\perp(\tilde{\mathcal{F}}_1 + Q_{11})W_\perp^*(s) \quad (3.12.41)
 \end{aligned}$$

where $\tilde{\mathcal{F}}_1(s)$ denotes the family of all ℓ -th order optimal approximations of $G_1(s)$. This completes the proof of the theorem. $\square$

Remark 3.12.1

Theorem 3.12.2 establishes that all solutions which minimise $s_2^\infty(G_0 - F)$ can be parametrised in terms of all solutions that minimise $s_1^\infty(G_0 - F)$ which is a problem of dimension $(p-1) \times (m-1)$. Similarly to the Nehari super-optimal case, the procedure can now be continued until a unity rank problem is encountered (or until $\deg(G_{i+1}) \leq \deg[(Q_{i+1})_+]$). At this point there is a unique optimal approximation and the process stops. $\square$

Remark 3.12.2

Theorem 3.12.2 gives $\tilde{\mathcal{F}}(s) = F_{11}(s) + V_\perp(\tilde{\mathcal{F}}_1 + Q_{11})W_\perp^*(s)$ as the family of all optimal approximations with respect to the first two singular values. Since $F_{11}(s)$, $Q_{11}(s)$ and any member of $\tilde{\mathcal{F}}_1(s)$ all belong to $\mathcal{H}_-^\infty(k)$, we expect a number of cancellations to occur in this sum since the members of $\tilde{\mathcal{F}}(s)$ are k -th order approximations of $G_0(s)$. A simple calculation based on (3.5.30) to (3.5.33) and the all-pass equations corresponding to $V(s)$ and $W(s)$, establishes that in the case when $\deg(v) = \deg(w) = n-1$, $V_\perp Q_{11} W_\perp^*(s)$ may be decomposed as $V_\perp Q_{11} W_\perp^*(s) = -F_{11}(s) + J(s)$, where

$$J(s) = \left[\begin{array}{cc|c} -\tilde{A}^* & -X\Theta\check{B}_2^* & X\Theta\check{D}_2^* \\ 0 & -\check{A}^* & \check{C}^* \\ \hline -\tilde{B}^* & \hat{C}_1 Y & \hat{D}_{11} \end{array} \right] \in \mathbb{RH}_-^\infty \quad (3.12.42)$$

(Here we have defined $\Theta = \hat{B}_1 \check{D}_2 + Y \check{B}_2$ and assumed that $W(s)$ and $V^*(s)$ are described by the minimal and balanced realisations (3.5.23) and (3.5.24) while X and Y are the solutions to the linear equations (3.5.31) and (3.5.32). This decomposition verifies that $\tilde{\mathcal{F}}(s) \in \mathcal{H}_-^\infty(k)$ and shows that all its stable poles are those of $\tilde{\mathcal{F}}_1(s)$. $\square$

Remark 3.12.3 [H2]

It is interesting to note that the construction given in Theorem 3.12.2 is particularly simple in the special case that $k=n-1$. To see this we begin by noting that all error systems are given by

$$\mathfrak{S}(s) = G + F_{11} + \frac{1}{\sigma_n} F_{12} U (I - \frac{1}{\sigma_n} F_{22} U)^{-1} F_{21} \quad (3.12.43)$$

in which $F(s)$ is stable. Next suppose we set $U := \frac{1}{\sigma_n} F_{22}^*(s)$ which we note has the characteristic $\|U\|_\infty < 1$ by the all-pass property $H(s)H^*(s) = \sigma_n^2 I$ and since $\det(F_{12}^* F_{12}(j\omega)) \neq 0$ for all real ω . In addition, $\frac{1}{\sigma_{k+1}} F_{22}^*$ is completely unstable. By exploiting the all-pass character of (3.1.9), the expression for the error system may be rearranged as

$$\begin{aligned} \mathfrak{S}(s) &= G + F_{11} + \frac{1}{\sigma_n^2} F_{12} F_{22}^* (I - \frac{1}{\sigma_n^2} F_{22} F_{22}^*)^{-1} F_{21} \\ &= G + F_{11} + F_{12} F_{22}^* (F_{21} F_{21}^*)^{-1} F_{21} \\ &= G + F_{11} - (G + F_{11}) F_{21}^* (F_{21} F_{21}^*)^{-1} F_{21} \\ &= (G + F_{11}) (I_m - F_{21}^* (F_{21} F_{21}^*)^{-1} F_{21}) \end{aligned} \quad (3.12.44)$$

Since $F_{21}(j\omega)$ has rank $m-1$, the error system has rank 1 and satisfies $\|\mathfrak{S}\|_\infty = \sigma_n$ which is super-optimal. $\square$

Remark 3.12.4

The following construction gives the super-optimal approximation in the case when the vectors $\psi^*(s)$ and $\xi(s)$ have right half plane MacMillan zeros:

From the definition of $Q(s)$,

$$\begin{bmatrix} V_\perp & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} G + F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} W_\perp & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} G_1 + Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \quad (3.12.45)$$

We may also write

$$\begin{aligned} Q_{12}^* Q_{12}(s) &= F_{12}^* V_\perp V_\perp^* F_{12}(s) = F_{12}^* (I - vv^*) F_{12}(s) \\ &= F_{12}^* F_{12}(s) = \sigma^2 I - F_{22}^* F_{22}(s) \end{aligned} \quad (3.12.46)$$

since $v^* F_{12} = 0$ and $G_a + F$ is all-pass. Similarly,

$$Q_{21} Q_{21}^*(s) = F_{21} W_\perp W_\perp^* F_{21}^*(s) = \sigma^2 I - F_{22} F_{22}^*(s) \quad (3.12.47)$$

Let

$$F_{22} \triangleq \begin{bmatrix} \hat{A} & \hat{B}_2 \\ \hat{C}_2 & 0 \end{bmatrix} \quad (3.12.48)$$

then

$$Q_{12}^* Q_{12} \triangleq \left[\begin{array}{cc|c} -\hat{A}^* & \hat{C}_2^* \hat{C}_2 & 0 \\ 0 & \hat{A} & \hat{B}_2 \\ \hline \hat{B}_2^* & 0 & \sigma^2 I \end{array} \right] \quad (3.12.49)$$

and

$$Q_{21} Q_{21}^* \triangleq \left[\begin{array}{cc|c} \hat{A} & \hat{B}_2 \hat{B}_2^* & 0 \\ 0 & -\hat{A}^* & \hat{C}_2^* \\ \hline \hat{C}_2 & 0 & \sigma^2 I \end{array} \right] \quad (3.12.50)$$

Next suppose without loss of generality that $Q_{22}(=F_{22})$ has been put in the Kalman canonical form $(\hat{A}, \hat{B}_2, \hat{C}_2, 0)$ in which

$$\hat{A} = \begin{bmatrix} \hat{A}_{11} & 0 & \hat{A}_{13} & 0 \\ \hat{A}_{21} & \hat{A}_{22} & \hat{A}_{23} & \hat{A}_{24} \\ 0 & 0 & \hat{A}_{33} & 0 \\ 0 & 0 & \hat{A}_{43} & \hat{A}_{44} \end{bmatrix} \quad \hat{B} = \begin{bmatrix} \hat{B}_{12} \\ \hat{B}_{22} \\ 0 \\ 0 \end{bmatrix} \quad (3.12.51a,b)$$

and

$$\hat{C} = \begin{bmatrix} \hat{C}_{21} & 0 & \hat{C}_{23} & 0 \end{bmatrix} \quad (3.12.51c)$$

where $(\hat{A}_{11}, \hat{B}_{12})$ is completely controllable and $(\hat{A}_{11}, \hat{C}_{21})$ is completely observable. Substituting into (3.12.50) we see that

$$Q_{21} Q_{21}^* \triangleq \left[\begin{array}{cc|c} \hat{A}_{11} & \hat{B}_{12} \hat{B}_{12}^* & 0 \\ 0 & -\hat{A}_{11}^* & \hat{C}_{21}^* \\ \hline \hat{C}_{21} & 0 & \sigma^2 I \end{array} \right] \quad (3.12.52)$$

where we have written

$$Q(s) = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \triangleq \left[\begin{array}{c|cc} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ \hline \hat{C}_1 & 0 & \sigma I \\ \hat{C}_2 & \sigma I & 0 \end{array} \right] \quad (3.12.53)$$

with $(\hat{A}, \hat{B}_2, \hat{C}_2)$ as in (3.12.48) and $\hat{B}_1, \hat{C}_1$ partitioned conformally as

$$\hat{B}_1^T = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{21} & \hat{B}_{31} & \hat{B}_{41} \end{bmatrix}^T \quad (3.12.54a)$$

and

$$\hat{C}_1 = \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} & \hat{C}_{13} & \hat{C}_{14} \end{bmatrix} \quad (3.12.54b)$$

Similarly, substituting the minimal part of (3.12.51) into (3.12.49) gives

$$Q_{12}^* Q_{12} \stackrel{\triangle}{=} \left[\begin{array}{cc|c} -\hat{A}_{11}^* & \hat{C}_{21}^* \hat{C}_{21} & 0 \\ 0 & \hat{A}_{11} & \hat{B}_{12} \\ \hline \hat{B}_{12}^* & 0 & \sigma^2 I \end{array} \right] \quad (3.12.55)$$

We note that all the modes of $\hat{A}_{22}$ in the realisation of $Q_{12}(s)$ given by

$$Q_{12} \stackrel{\triangle}{=} \left[\begin{array}{cc|c} \hat{A}_{11} & 0 & \hat{B}_{12} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_{22} \\ \hline \hat{C}_{11} & \hat{C}_{12} & \sigma I \end{array} \right] \quad (3.12.56)$$

cancel in the product $Q_{12}^* Q_{12}(s)$. The stable modes of $\hat{A}_{22}$ are controllable by construction and are also observable since by Lemma 3.12.1 all system zeros of $Q_{12}(s)$ that lie in the left half plane are due to uncontrollable modes. Hence, there exists a (left) all-pass factor $\mathfrak{B}(s)$, i.e. $Q_{12}(s) = \mathfrak{B}(s) \bar{Q}(s)$ with $\deg(\mathfrak{B}_+)$ given by the number of stable modes of $\hat{A}_{22}$. This means that without loss of generality the realisation of $Q_{12}(s)$ in (3.12.56) may be chosen such that

$$Q_{12} \stackrel{\triangle}{=} \left[\begin{array}{cc|c} \hat{A}_{11} & 0 & \hat{B}_{12} \\ \hat{A}_{21} & \hat{A}_{22} & \hat{B}_{22} \\ \hline \hat{C}_{11} & \hat{C}_{12} & \sigma I \end{array} \right] = \left[\begin{array}{cc} \hat{A}_{22} & \sigma^{-1} \hat{B}_{22} \\ \hat{C}_{12} & I \end{array} \right] \left[\begin{array}{cc} \hat{A}_{11} & \hat{B}_{12} \\ \hat{C}_{11} & \sigma I \end{array} \right] \quad (3.12.57)$$

in which $\mathfrak{B}(s) = I + \sigma^{-1} \hat{C}_{12}(sI - \hat{A}_{22})^{-1} \hat{B}_{22}$ and $\hat{A}_{21} = \frac{1}{\sigma} \hat{B}_{22} \hat{C}_{11}$. Similarly, considering

$$Q_{21} \stackrel{\triangle}{=} \left[\begin{array}{cc|c} \hat{A}_{11} & \hat{A}_{13} & \hat{B}_{11} \\ 0 & \hat{A}_{33} & \hat{B}_{31} \\ \hline \hat{C}_{21} & \hat{C}_{23} & \sigma I \end{array} \right] \quad (3.12.58)$$

in the product $Q_{21} Q_{21}^*(s)$ we conclude that there exists a (right) all-pass factor $\mathcal{A}(s)$ containing the modes of $\hat{A}_{33}$, i.e.

$$Q_{21} = \left[\begin{array}{cc|c} \hat{A}_{11} & \hat{A}_{13} & \hat{B}_{11} \\ 0 & \hat{A}_{33} & \hat{B}_{31} \\ \hline \hat{C}_{21} & \hat{C}_{23} & \sigma I \end{array} \right] = \left[\begin{array}{cc} \hat{A}_{11} & \hat{B}_{11} \\ \hat{C}_{21} & \sigma I \end{array} \right] \left[\begin{array}{cc} \hat{A}_{33} & \hat{B}_{31} \\ \sigma^{-1} \hat{C}_{23} & I \end{array} \right] \quad (3.12.59)$$

with $\mathcal{A}(s) = I + \sigma^{-1} \hat{C}_{23}(sI - \hat{A}_{33})^{-1} \hat{B}_{31}$ and $\hat{A}_{13} = \sigma^{-1} \hat{B}_{11} \hat{C}_{23}$. Hence,

$$\begin{aligned}
 \inf_{\sigma U \in \mathcal{B}\mathcal{H}_-^\infty} \|G_1 + \mathcal{F}_\ell(Q, U)\|_\infty &= \inf_{\sigma U \in \mathcal{B}\mathcal{H}_-^\infty} \|G_1 + Q_{11} + Q_{12} U(I - Q_{22} U)^{-1} Q_{21}\|_\infty \\
 &= \inf_{\sigma U \in \mathcal{B}\mathcal{H}_-^\infty} \|G_1 + Q_{11} + \mathcal{B}(s) \bar{Q}_{12} U(I - Q_{22} U)^{-1} \bar{Q}_{21} \mathcal{A}(s)\|_\infty \\
 &= \inf_{\sigma U \in \mathcal{B}\mathcal{H}_-^\infty} \|\mathcal{B}^*(G_1 + Q_{11}) \mathcal{A}^* - \bar{Q}_{11} + \mathcal{F}_\ell(\bar{Q}, U)\|_\infty \quad (3.12.60)
 \end{aligned}$$

where

$$\bar{Q}(s) = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{21} & \bar{Q}_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc} \hat{A}_{11} & \hat{B}_{11} & \hat{B}_{12} \\ \hline \hat{C}_{11} & 0 & \sigma I \\ \hat{C}_{21} & \sigma I & 0 \end{array} \right] \quad (3.12.61)$$

Since all the stable modes of $\hat{A}_{11}$ are controllable through $\hat{B}_{12}$ (observable through $\hat{C}_{21}$) by construction, all the system zeros of $\bar{Q}_{12}$ ($\bar{Q}_{21}$) in the above realisation lie in the open right half plane.

Hence, following the proof of Theorem 3.12.2 we see that in this (the general) case, the family of all extensions that minimise the pair $(s_1^\infty(G_0 - F), s_2^\infty(G_0 - F))$ with respect to lexicographic ordering may be parametrised by

$$\begin{aligned}
 \tilde{\mathcal{F}}(s) &= G_0 - V \begin{bmatrix} I & 0 \\ 0 & \mathcal{B}(s) \end{bmatrix} \left[\begin{array}{cc} \sigma a(s) & 0 \\ 0 & \mathcal{B}^*(G_1 + Q_{11}) \mathcal{A}^* - \bar{Q}_{11} - \mathcal{F}_1 \end{array} \right] \begin{bmatrix} I & 0 \\ 0 & \mathcal{A}(s) \end{bmatrix} W^* \\
 &= G_0 - V \begin{bmatrix} \sigma_{k+1} a(s) & 0 \\ 0 & G_1 + Q_{11}(s) \end{bmatrix} W^* + V \begin{bmatrix} I & 0 \\ 0 & \mathcal{B}(s) \end{bmatrix} \left[\begin{array}{cc} 0 & 0 \\ 0 & \mathcal{F}_1 + \bar{Q}_{11} \end{array} \right] \begin{bmatrix} I & 0 \\ 0 & \mathcal{A}(s) \end{bmatrix} W^* \\
 &= \mathcal{F}_\ell(F, 0) + V_\perp \mathcal{B}(s) (\mathcal{F}_1 + Q_{11}) \mathcal{A}(s) W_\perp^* \\
 &= F_{11}(s) + V_\perp \mathcal{B}(s) (\mathcal{F}_1 + Q_{11}) \mathcal{A}(s) W_\perp^*(s) \quad (3.12.62)
 \end{aligned}$$

where $\mathcal{F}_1$ denotes the family of all optimal Hankel-norm approximations of $\mathcal{B}^*(G_1 + Q_{11}) \mathcal{A}^* - \bar{Q}_{11}$ of degree $\ell = \deg(\bar{Q}_+) = \deg([F_{22}]_+)$. $\square$

A simple inductive generalisation of Theorem 3.12.2 will establish the following algorithm for calculating the (unique) super-optimal approximation.

Algorithm 3.12.1

Given any $G_0(s) \in \mathbb{RH}_+^{\infty, p \times m}$, this algorithm finds $F_{so}(s) \in \mathbb{H}_-^{\infty, p \times m}(k)$ which is the superoptimal k -th order approximation of $G_0(s)$.

- 1) $r = \text{rank}(G_0) \leq \min(p, m)$
- 2) Find $F(s)$ for $G_0(s)$ (the generator of all optimal k -th order approximations — use state-space formulae in (3.1.9))
- 3) $E_0 = G_0 - F_{11}$
- 4) $F_{so} = F_{11}$
- 5) $V = I_m$ and $W = I_p$
- 6) For $i=0$ to $r-1$
 - 1) Find $\xi_i(s)$ and $\psi_i(s)$ corresponding to $G_i(s)$ and construct $v_i(s)$, $w_i(s)$, $V_{\perp i}(s)$ and $W_{\perp i}(s)$ using state-space formulae in section 3.4.
 - 2) Define $G_{i+1}(s) \in \mathbb{H}_+^\infty$ and $Q_{i+1}(s) \in \mathbb{H}_-^\infty(k)$ via decompositions given in Lemma 3.4.3
 - IF $\xi_i(s)$ and $\psi_i^*(s)$ have RHP zeros THEN
 - 1) Extract all-pass common factors $\mathcal{A}(s)$ and $\mathcal{B}(s)$ (see Remark (3.12.4)).
 - 2) Redefine: $V_{\perp i}(s) := V_{\perp i}(s)\mathcal{B}(s)$, $W_{\perp i}^*(s) := \mathcal{A}(s)W_{\perp i}^*(s)$,
 $Q_{i+1}(s) := \bar{Q}_{i+1}(s)$, $G_{i+1}(s) := \mathcal{B}^*(G_{i+1} + Q_{i+1}^{11})\mathcal{A}^* - Q_{i+1}^{11}$
 - 4) $\ell = \deg(Q_+)$
 - 5) Find F_{i+1} , the generator of all ℓ -th optimal approximations to G_{i+1} (using (3.1.9))
 - 6) $E_{i+1} = G_{i+1} - F_{i+1}$
 - 7) $V = VV_{\perp i}$ and $W = WW_{\perp i}$
 - 8) $F_{so}(s) = F_{so} + V(F_{i+1}^{11} + Q_{i+1}^{11})W^*(s) \in \mathbb{H}_-^\infty(k)$

Note that if the above algorithm were terminated after $\ell < r$ steps, the first $\ell+1$ singular values will be minimised and

$$F_{so}^{(l)}(s) = F_{11}(s) + \sum_{j=0}^l \left\{ \prod_{i=0}^j V_{\perp i} (Q_{j+1} + F_{j+1}) \left(\prod_{i=0}^j W_{\perp i} \right)^* \right\}(s) \quad (3.12.63)$$

will have no more than k poles in the open left half plane.

3.13 A-posteriori tests in robust stability analysis

In this section the use of super-optimal Nehari extensions in the approximation stage of $\mathbb{H}^\infty$ algorithms is discussed. In [P4] super-optimality was justified for the case of multi-objective disturbance rejection (for example the rejection of the combined effect of a disturbance signal at the plant output and a measurement noise signal in the feedback path). It was argued that if the combined signal is constrained not to lie along a particular singular vector direction, the level of the second

singular value describes the new worst case effect. Since $\mathcal{L}^\infty$ -norm optimisation is concerned precisely with the worst case effect of sets of signals, lexicographic (or “nested”) minimisation was seen as capturing the essence of the $\mathcal{H}^\infty$ -design philosophy in using the available degrees of freedom in multi-input/output problems.

Here super-optimal extensions will be discussed in the context of robust-stability $\mathcal{H}^\infty$ design. Consider the additive perturbation case, i.e. a family of plants $\{P+\Delta P\}$ where P is assumed to be the nominal plant and ΔP represents an unknown perturbation, arising, for example, due to unmodelled dynamics or parameter variations. Suppose also that ΔP is bounded on the imaginary axis by the ‘radius function’ $W(jw)$

$$\|\Delta P(jw)\|_\infty \leq \|W(jw)\|_\infty \quad \text{for all } 0 \leq w \leq \infty \quad (3.13.1)$$

or, equivalently,

$$\|W^{-1}\Delta P(jw)\|_\infty \leq 1 \quad (3.13.1)$$

It was shown in Chapter 1 that the *robust stabilisation problem* for this type of uncertainty, i.e. the problem of designing a controller to maximise $\mu = \|\Delta P\|_\infty$ such that $P+\Delta P$ is internally stable in the closed loop, may be recast as the $\mathcal{L}^\infty$ optimisation

$$\delta = \inf_{K \in \Xi} \|W(s)K(I-PK)^{-1}\|_\infty \quad (3.13.3)$$

where Ξ is the set of all (internally) stabilising compensators. Provided that an infimum $\delta \leq 1$ can be achieved, it is guaranteed that all plants in the family

$$\left\{ P+\Delta P : \|\Delta P\|_\infty \leq \|W(jw)\|_\infty \quad P(s) \text{ and } P+\Delta P(s) \text{ have the same number of } \mathbb{C}_+ \text{ poles} \right\}$$

are internally stable in the closed loop. For a known ‘radius function’ $W(jw)$, therefore, the condition that $\bar{\sigma}\{WK(I-PK)^{-1}(j\omega)\} \leq 1$ for all frequencies guarantees robust stability irrespective of the behaviour of the remaining singular values. In what sense, therefore, can the super-optimal solution to (3.13.3) be considered as a ‘better’ solution? In answering this question particular emphasis will be given to the relative improvement of the robustness test after imposing some ‘a-posteriori structure’ on the admissible set of errors. By ‘error structure’ spatial information about the uncertainty is implied (similar to the one arising due to uncertain coefficients in the differential equations describing the model) the term ‘unstructured’ being retained to describe frequency weighted bounds on a particular *norm* of the uncertainty.

A robust-stability analysis in control system design must provide a *simple* characterisation of a model error set which contains a *sufficiently large* subclass of model errors which do not destabilise the feedback system. Without this simple characterisation, “even the best engineering judgment may not

be adequate to answer the question on whether the class of model errors for which the feedback system is stable actually contains these particular errors that can be reasonably expected to occur" [L3]. A compromise, therefore, between simplicity of characterisation and the extend of the subclass of non-destabilising errors that can be considered is necessary [L3].

In [L3], a-posteriori robustness tests are employed to assess the increase in the magnitude of the error that can cause instability, by imposing 'structure' on the model error set. This is achieved by a partial characterisation of the modelling error, in particular, its projection onto a subspace. In essence, the idea is the following (consider the case of additive perturbations): From Nyquist theory, the stability properties of a closed-loop system can be inferred by the multiplicity of the encirclements of a critical point on the negative real axis. Suppose that from such a test closed-loop stability can be inferred for a nominal open loop transfer function $Q_0(s)$. A perturbed system $\tilde{Q}(s)$ will have identical stability properties if the Nyquist loci of the nominal plant can be deformed in such a way that we avoid crossing the critical point, that is, if there exists a homotopy $Q(s, \epsilon) = Q_0(s) + \epsilon \Delta(s)$ for $0 \leq \epsilon \leq 1$ and with $Q(s, 1) = \tilde{Q}(s)$, such that

$$\det[I + Q_0(s) + \epsilon \Delta(s)] \neq 0 \quad \text{for all } s \in \mathfrak{D}_R, 0 \leq \epsilon \leq 1 \quad (3.13.4)$$

where $\mathfrak{D}_R$ is a suitably large D -contour in the right-half of the complex plane and Δ is the additive error $Q(s) - \tilde{Q}(s)$. (We must also ensure that Q and $\tilde{Q}$ have the same number of closed-loop right-half plane poles, that the closed loop corresponding to the nominal plant is stable and that provided the perturbed system $\tilde{Q}(s)$ has a pole on the imaginary axis, so does the nominal system at the same location).

The range of robust stability to additive perturbations, therefore, essentially depends on the extend by which the matrix $A(s) = I + Q_0(s)$ can be deformed by an error matrix $E(s)$ without making the sum $A + E$ singular. Now, if $A + E$ is singular and $\bar{\sigma}(E) = \underline{\sigma}(A)$, then $\langle u_n v_n^*, E \rangle = -\sigma_n = -\underline{\sigma}(A)$ where A has a singular value decomposition $A = U \Sigma V^* = \sum_{i=1}^k \sigma_i u_i v_i^*$ ($\sigma_i \geq \sigma_{i+1}$) and the matrix 'inner product' $\langle \cdot, \cdot \rangle$ is defined by $\langle A, B \rangle = \text{trace}(A^* B)$. In addition, if $\bar{\sigma}(E) < \underline{\sigma}(A)$ then $A + E$ is nonsingular. Constructing a constraint set so that E cannot have a projection of magnitude σ_n in the most critical direction $u_n v_n^*$, we can see that $\bar{\sigma}(E)$ must increase for $A + E$ to lose rank [L3]. A solution to the problem "by how much $\bar{\sigma}(E)$ has to increase for this to happen?", allows the formulation of the main theorem in [L3]. The crucial result is that if $A = G^{-1} + K$ has a singular value decomposition $\sum_{i=1}^k \sigma_i u_i v_i^*$, then robust-stability for all additive errors $E(s)$ bounded by

$$\mu = \sqrt{\sigma_n \sigma_{n-1} + c(\sigma_n - \sigma_{n-1})} \quad (3.13.5)$$

is guaranteed provided a bound c of the 'structured' error

$$|\langle u_n v_n^*, E \rangle| \leq c \leq \sigma_n \quad (3.13.6)$$

can be found whenever $\bar{\sigma}(E) \geq \sigma_n$. Similar results follow for other types of uncertainty (multiplicative, inverse additive, inverse multiplicative) by an almost identical analysis. Note, that since u_n and v_n are not specified a-priori (i.e. they are not defined until the compensator $K(s)$ has been designed), any structural information about the error $E(s)$ is utilised only indirectly in determining a bound for $|\langle u_n v_n^*, E \rangle|$.

Consider next the robust stabilisation problem

$$\inf_{K \in \Xi} \|K(I+GK)^{-1}\|_\infty = \inf_{K \in \Xi} \|\mathcal{F}_\ell \left(\begin{bmatrix} 0 & I \\ I & G \end{bmatrix}, K \right)\|_\infty \quad (3.13.7)$$

Let $G(s)$ admit right and left coprime factorisations $G = N_r D_r^{-1} = D_l^{-1} N_l$ with inner denominators D_r and D_l . Next, (3.13.7) is equivalent to

$$\inf_{Q \in \mathcal{H}_+^\infty} \|-U_l D_l - D_r Q D_l\|_\infty = \inf_{Q \in \mathcal{H}_+^\infty} \|-D_r(D_r^{-1} U_l + Q) D_l\|_\infty = \inf_{Q \in \mathcal{H}_+^\infty} \|D_r^{-1} U_l + Q\|_\infty \quad (3.13.8)$$

on using (1.4.33). Let Γ denote the Hankel operator with symbol $D_r^{-1} U_l(-s)$ and denote its sequence of singular values by $\{\sigma_i\}$ with $\sigma_1 = 1/\sigma_{\min}(\Gamma_{G(-s)})$ by [G2], [F2]. Let also $H(s)$ denote the generator of the family of all Nehari extensions of $D_r^{-1} U_l$. Then, all $\mathcal{H}^\infty$ -optimal interpolating functions are generated by

$$\begin{aligned} \mathcal{R}(s) &= \left\{ K(I+GK)^{-1} \mid K \in \Xi \text{ achieving the infimum in (3.13.7)} \right\} \\ &= \left\{ -D_r [D_r^{-1} U_l + \mathcal{F}_\ell(H, U)] D_l \mid U \in \frac{1}{\sigma_1} \mathcal{B}\mathcal{H}_+^\infty \right\} \end{aligned}$$

Using Lemma 3.5.1, $\mathcal{R}(s)$ may be parametrised as

$$\mathcal{R}(s) = -D_r V \mathcal{F}_\ell([\sigma_1 a \oplus (G_{1a} + Q)], U) W^* D_l \quad (3.13.9)$$

where $V = [v_1 \ V_\perp]$ and $W^* = [w_1^* \ W_\perp^*]$ correspond to the maximal Schmidt vectors of Γ . It follows that

$$K_o + G^{-1}(j\omega) = \mathcal{R}^{-1}(j\omega) = \frac{1}{\sigma_1} D_l^* w_1^* v_1^* D_r^*(j\omega) a(-j\omega) - D_l^* W_\perp \left\{ \mathcal{F}_\ell(Q, U) + G_1 \right\} V_\perp^* D_r^*(j\omega)$$

and since $\|\mathcal{F}_\ell(Q, U)\|_\infty \leq \sigma_1$, $D_l^* w_1(j\omega)$ and $v_1^* D_r^*(j\omega)$ are singular vectors for every element of $\mathcal{R}^{-1}(j\omega)$ corresponding to the smallest singular value of $\sigma_{\min}(\mathcal{R}^{-1}(j\omega)) = \sigma_1$. Consequently, if a (uniform) bound c may be established on the 'structured' error

$$\begin{aligned} |\langle u_n(\mathcal{R}^{-1}) v_n^*(\mathcal{R}^{-1}), E \rangle| &= \left| \text{trace} \left\{ v_n(\mathcal{R}^{-1}) u_n^*(\mathcal{R}^{-1}) E(j\omega) \right\} \right| \\ &= \left| \sum_i \lambda_i \left\{ D_l^* w_1 v_1^* D_r^* E(j\omega) \right\} \right| \\ &= |v_1^* D_r^* E D_l^* w_1(j\omega)| \leq c \end{aligned} \quad (3.13.10)$$

the magnitude $\bar{\sigma}(E(j\omega))$ of the nearest destabilising error (for the 'structured' error set) is increased by a factor

$$\begin{aligned}
 \xi &= \sqrt{\frac{\sigma_{n-1}(\mathcal{P}^{-1}(j\omega))}{\sigma_n(\mathcal{P}^{-1}(j\omega))} \left\{ 1 - \frac{c}{\sigma_n(\mathcal{P}^{-1}(j\omega))} \right\} + \frac{c}{\sigma_n(\mathcal{P}^{-1}(j\omega))}} \\
 &= \sqrt{\sigma_{n-1}(\mathcal{P}^{-1}(j\omega))\sigma_1(\Gamma) + c\sigma_1(\Gamma)} \\
 &= \sqrt{\frac{\sigma_1(\Gamma)}{\sigma_2(\mathcal{P}(j\omega))} (1 - c\sigma_1(\Gamma)) + c\sigma_1(\Gamma)} \tag{3.13.11}
 \end{aligned}$$

Maximising ξ for the worst ω is achieved by the super-optimal solution,

$$\xi_{so} = \sqrt{\frac{\sigma_1(\Gamma)}{\sigma_2(\mathcal{P}(j\omega))} (1 - c\sigma_1(\Gamma)) + c\sigma_1(\Gamma)} \geq \sqrt{\frac{\sigma_1(\Gamma)}{\sigma_2(\Gamma)} (1 - c\sigma_1(\Gamma)) + c\sigma_1(\Gamma)} \tag{3.13.12}$$

by Theorem (2.7.2). Note that when the bound in (3.13.12) is tight, the ratio of the two largest Hankel singular values of $D_r^* U_l(-s)$ is an indicator of the level of improvement that is likely to be achieved by imposing structural constraints on the uncertainty. Figure 8 gives a plot of $20 \cdot \log(\xi)$ versus $c\sigma_1$ for various ratios σ_1/σ_2 .

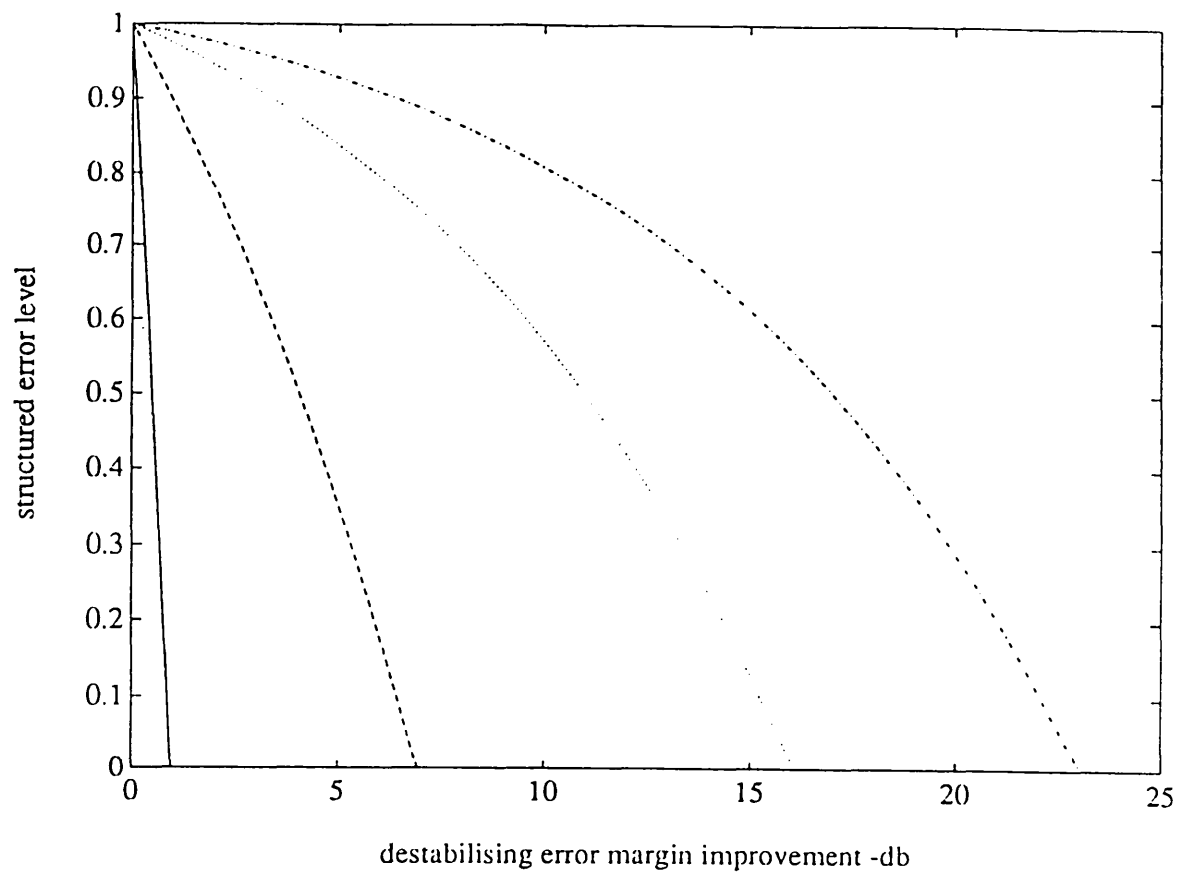


Figure 8: Error margin improvements in a-posteriori stability tests

Appendix 3.A: Proof of Lemma 3.4.6

Lemma 3.4.6

The Hamiltonians associated with the two Riccati equations (3.4.35) and (3.4.36) are

$$H_1 = \begin{bmatrix} A_{11} - A_{12}C_2^*C_1 & A_{12}A_{12}^* \\ C_1^*C_1^* C_1^* C_1 & -(A_{11} - A_{12}C_2^*C_1)^* \end{bmatrix} \quad (A.1)$$

and

$$H_2 = \begin{bmatrix} A_{11}^* - A_{21}^*B_2B_1^* & A_{21}^*A_{21} \\ B_1B_1^* B_1^* B_1^* & -(A_{11}^* - A_{21}^*B_2B_1^*)^* \end{bmatrix} \quad (A.2)$$

Further

$$\begin{bmatrix} I & -\Sigma \\ -\Sigma & \Gamma \end{bmatrix}^{-1} H_1 \begin{bmatrix} I & -\Sigma \\ -\Sigma & \Gamma \end{bmatrix} = H_2^* \quad (A.3)$$

Proof:

(A.1) and (A.2) follow from standard theory on the algebraic Riccati equation [A2]. We will verify (A.3) by showing that each of the four partitions of

$$H_1 \begin{bmatrix} I & -\Sigma \\ -\Sigma & \Gamma \end{bmatrix} - \begin{bmatrix} I & -\Sigma \\ -\Sigma & \Gamma \end{bmatrix} H_2^* \quad (A.4)$$

is zero.

Block (1,1) is equal to

$$\begin{aligned} (1,1) &= A_{11} - A_{12}C_2^*C_1 - A_{12}A_{12}^*\Sigma - A_{11} + B_1B_2^*A_{21} \\ &= A_{12}(A_{12}^*\Sigma + \sigma A_{21}) - A_{12}A_{12}^*\Sigma - (\sigma A_{12} + \Sigma A_{21}^*)A_{21} + \Sigma A_{21}^*A_{21} \\ &= 0 \end{aligned} \quad (A.5)$$

using (3.1.1), (3.1.2), (3.1.3) and (3.1.8). Similarly,

$$\begin{aligned} (1,2) &= -(A_{11} - A_{12}C_2^*C_1)\Sigma + A_{12}A_{12}^*\Gamma - B_1B_1^*B_1^*B_1^* - \Sigma(A_{11} + B_1B_2^*A_{21})^* \\ &= -A_{11}\Sigma + A_{12}(A_{12}^*\Sigma + \sigma A_{21})\Sigma + A_{12}A_{12}^*\Sigma^2 - \sigma^2 A_{12}A_{12}^* - B_1B_1^*B_1^*B_1^* \\ &\quad - \Sigma A_{11}^* + \Sigma A_{21}^*B_2B_1^* \\ &= -\sigma A_{12}A_{21}\Sigma - \sigma^2 A_{12}A_{12}^* + \sigma A_{12}(\sigma A_{12}^* + A_{21}\Sigma) \\ &= 0 \end{aligned} \quad (A.6)$$

$$\begin{aligned}
 (2,1) &= C_1^* C_\perp^* C_\perp C_1 + (A_{11}^* - C_1^* C_2 A_{12}^*) \Sigma + \Sigma (A_{11} - B_1 B_2^* A_{21}) - \Sigma^2 A_{21}^* A_{21} + \sigma^2 A_{21}^* A_{21} \\
 &= C_1^* C_\perp^* C_\perp C_1 + (A_{11}^* \Sigma + \Sigma A_{11}) + C_1^* C_2 C_2^* C_1 + \sigma C_1^* C_2 A_{21} + \sigma \Sigma A_{12} A_{21} + \sigma^2 A_{21}^* A_{21} \\
 &= -(\Sigma A_{12} + \sigma A_{21}^*) \sigma A_{21} + \sigma \Sigma A_{12} A_{21} + \sigma^2 A_{21}^* A_{21} \\
 &= 0
 \end{aligned} \tag{A.7}$$

Finally,

$$\begin{aligned}
 (2,2) &= -C_1^* C_1 \Sigma + C_1^* C_2 (C_2^* C_1 \Sigma + A_{12}^* \Gamma) - A_{11}^* \Gamma + \Sigma B_1 B_1^* - (\Sigma B_1 B_2^* + \Gamma A_{21}^*) B_2 B_1^* + \Gamma A_{11}^* \\
 &= -C_1^* C_1 \Sigma + C_1^* C_2 [A_{12}^* \Sigma^2 - \sigma^2 A_{12}^* - (A_{12}^* \Sigma + \sigma A_{21}) \Sigma] - A_{11}^* \Gamma + \Gamma A_{11}^* + \Sigma B_1 B_1^* \\
 &\quad - [\Sigma^2 A_{21}^* - \sigma^2 A_{21}^* - \Sigma (\sigma A_{12} + \Sigma A_{21}^*)] B_2 B_1^* \\
 &= -C_1^* C_1 \Sigma - A_{11}^* \Sigma^2 + \Sigma^2 A_{11}^* + \Sigma B_1 B_1^* \\
 &= 0
 \end{aligned} \tag{A.8}$$

where (3.1.1), (3.1.2), (3.1.3) and (3.1.8) have been used. Finally, the fact that

$$\det \begin{bmatrix} I & -\Sigma \\ -\Sigma & \Gamma \end{bmatrix} = \det(\Gamma - \Sigma^2) = \det(-\sigma^2 I) \neq 0 \tag{A.9}$$

establishes the non-singularity of the transformation and hence proves the result. $\square$

Appendix 3.B: Proof of Lemma 3.4.7

Lemma 3.4.7

Suppose that Ω and $\tilde{\Omega}$ are solutions to the Riccati equations (3.4.35) and (3.4.36) respectively, and suppose also that $[\cdot]_+$ and $[\cdot]_-$ denote the stabilising and destabilising solutions to these equations. Then if $\tilde{\Omega}_+^{-1}$ exists,

$$(i) \quad \Omega_- = \mathcal{F}_\ell(\Xi, \tilde{\Omega}_+^{-1}) \quad (B.1)$$

where

$$\Xi := \begin{bmatrix} -\Sigma & \sigma I \\ \sigma I & -\Sigma \end{bmatrix} \quad (B.2)$$

(ii) If

$$\Delta(\Omega, \tilde{\Omega}) := \tilde{\Omega}\Omega + \Sigma\Omega + \tilde{\Omega}\Sigma + \Gamma \quad (B.3)$$

Then

$$\Delta(\Omega_-, \tilde{\Omega}_+) = 0 \quad (B.4)$$

and

$$\Delta(\Omega_-, \tilde{\Omega}_-) \quad (B.5)$$

is non-singular.

Proof:

Suppose that

$$\begin{bmatrix} X_1^T & X_2^T \end{bmatrix}^T \quad (B.6)$$

is a basis for the completely unstable eigenspace of H_2^* . Since

$$H_1 T = T H_2^* \quad (B.7)$$

where

$$T = \begin{bmatrix} I & -\Sigma \\ -\Sigma & \Gamma \end{bmatrix} \quad (B.8)$$

it is easy to see that

$$\begin{bmatrix} I & -\Sigma \\ -\Sigma & \Gamma \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 - \Sigma X_2 \\ -\Sigma X_1 + \Gamma X_2 \end{bmatrix} \quad (B.9)$$

is a basis for the completely unstable eigenspace of H_1 . Consequently

$$\begin{aligned} \Omega_- &= (\Gamma X_2 - \Sigma X_1)(X_1 \Sigma X_2)^{-1} \\ &= (\Gamma X_2 X_1^{-1} - \Sigma)(I - \Sigma X_2 X_1^{-1})^{-1} \\ &= -\Sigma - \sigma X_2 X_1^{-1}(I - \Sigma X_2 X_1^{-1})^{-1} \sigma I \end{aligned} \quad (B.10)$$

By assumption

$$H_2^* \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} [\Lambda] \quad (B.11)$$

in which Λ is completely unstable. If

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \quad (B.12)$$

we can write

$$J H_2^* J \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = -J \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} [\Lambda] \quad (B.13)$$

Since H_2^* is Hamiltonian, this gives

$$H_2 \begin{bmatrix} -X_2 \\ X_1 \end{bmatrix} = \begin{bmatrix} -X_2 \\ X_1 \end{bmatrix} [-\Lambda] \quad (B.14)$$

which implies that $[-X_2^T \ X_1^T]^T$ is a basis for the asymptotically stable eigenspace of H_2 , so that

$$-\tilde{\Omega}_+^{-1} = X_2 X_1^{-1} \quad (B.15)$$

whence

$$\begin{aligned} \Omega_- &= -\Sigma + \sigma \tilde{\Omega}_+^{-1} (I + \Sigma \tilde{\Omega}_+^{-1})^{-1} \sigma I \\ &= \mathfrak{F}_\ell(\Xi, \tilde{\Omega}_+^{-1}) \end{aligned} \quad (B.16)$$

which proves the first part. From (B.16) we get

$$(I + \tilde{\Omega}_+^{-1} \Sigma) \Omega_- = -\tilde{\Omega}_+^{-1} \Gamma - \Sigma$$

$$\begin{aligned} \Rightarrow \tilde{\Omega}_+ \Omega_- + \Sigma \Omega_- + \Gamma + \tilde{\Omega}_+ \Sigma &= 0 \\ \Rightarrow \Delta(\Omega_-, \tilde{\Omega}_+) &= 0 \end{aligned} \tag{B.17}$$

which verifies (B.4). Finally,

$$\begin{aligned} \Delta(\Omega_-, \tilde{\Omega}_-) &= \Delta(\Omega_-, \tilde{\Omega}_-) - \Delta(\Omega_-, \tilde{\Omega}_+) \\ &= (\tilde{\Omega}_- - \tilde{\Omega}_+)(\Omega_- + \Sigma) \end{aligned} \tag{B.18}$$

Since $\Sigma > 0$ and $\Omega_- \geq 0$, $(\Omega_- + \Sigma)$ is non-singular. The non-singularity of $(\tilde{\Omega}_- - \tilde{\Omega}_+)$ follows from a simple Lyapunov equation type argument [W1, pgs 626-627]. $\square$

Appendix 3.C: Proof of Theorem 3.5.2

Theorem B.1

Let $H(s)$ be the generator of all Nehari extensions given in Lemma 3.1.1. Then, assuming that the Riccati equations (3.4.35) and (3.4.36) have positive-definite destabilising solutions Ω and $\tilde{\Omega}$,

$$\begin{bmatrix} v^* & 0 \\ V_\perp^* & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} w & W_\perp & 0 \\ 0 & 0 & I_p \end{bmatrix} = \begin{bmatrix} \sigma a(s) & 0 & 0 \\ 0 & G_1 + Q_{11} & Q_{12} \\ 0 & Q_{21} & Q_{22} \end{bmatrix} \quad (C.1)$$

in which

(i) $a(s) \in \mathbb{RH}_+^\infty$ and is all-pass, $G_1 \in \mathbb{RH}_+^{\infty, (m-1) \times (p-1)}$, $Q \in \mathbb{RH}_-^{\infty, (m+p-2) \times (m+p-2)}$

$$(ii) \ G_1(s) \triangleq \left[\begin{array}{c|c} -A_{11}^* - A_{21}^*(A_{21}\tilde{\Omega} - B_2B_1^*) & \tilde{\Omega}^{-1}B_1B_\perp^* \\ \hline C_\perp C_1 \Omega^{-1}(\Omega\Sigma + \Gamma + \Omega\tilde{\Omega} + \Sigma\tilde{\Omega}) & 0 \end{array} \right] \quad (C.2)$$

$$(iii) \ Q(s) \triangleq \left[\begin{array}{cc|cc} \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11}\Sigma - \sigma C_1^* U B_1^*) & (\tilde{\Omega}^{-1} + \Gamma^{-1}\Sigma)B_1B_\perp^* & -\sigma\Gamma^{-1}C_1^* C_\perp^* \\ \hline -C_\perp C_1(\Sigma + \Omega^{-1}\Gamma) & 0 & -\sigma I \\ -\sigma B_\perp B_1^* & \sigma I & 0 \end{array} \right] \quad (C.3)$$

We have abbreviated $\sigma_1(G_0)$ to just σ .

Proof:

From Lemmas 3.1.1 and Theorem 3.4.5, we observe that $H(s)$ is given by the state-space realisation,

$$\left[\begin{array}{ccc|cc} A_{11} & A_{12} & 0 & B_1 & 0 \\ A_{21} & A_{22} & 0 & B_2 & 0 \\ 0 & 0 & \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11}\Sigma - \sigma C_1^* U B_1^*) & \Gamma^{-1}(\Sigma B_1 + \sigma C_1^* U) & -\sigma\Gamma^{-1}C_1^* C_\perp^* \\ \hline C_1 & C_2 & -(C_1\Sigma + \sigma U B_1^*) & \sigma U & -\sigma C_\perp^* \\ 0 & 0 & -\sigma B_\perp B_1^* & \sigma B_\perp & 0 \end{array} \right] \quad (C.4)$$

Also $V_a^*(s)$ may be written as

$$\begin{bmatrix} v^* & 0 \\ V_\perp^* & 0 \\ 0 & I_m \end{bmatrix} (s) \doteq \left[\begin{array}{c|ccc} -A_{11}^* - (\Omega A_{12} - C_1^* C_2) A_{12}^* & C_1^* C_\perp^* C_\perp + \Omega A_{12} C_2^* & 0 \\ \hline -A_{12}^* & C_2^* & 0 \\ -C_\perp^* C_1 \Omega^{-1} & C_\perp & 0 \\ 0 & 0 & I_m \end{array} \right] \quad (C.5)$$

In the first part of the proof we cascade the realisations in (C.4) and (C.5), and then implement the basis changes

$$T_1 = \begin{bmatrix} I & -\Omega & & \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} ; \quad T_2 = \begin{bmatrix} I & & & -\Gamma \\ & I & & \\ & & I & \\ & & & I \end{bmatrix} \quad (C.7)$$

Simplifying calculations which make use of the Riccati equations (3.4.35) and (3.4.36), and also various partitions of (3.1.1) and (3.1.2) gives that $V_a^* H(s)$ is equal to

$$\left[\begin{array}{cccc|cc} -A_{11}^* - (\Omega A_{12} - C_1^* C_2) A_{12}^* & 0 & 0 & 0 & \bar{B}_{11} & 0 \\ 0 & A_{11} & A_{12} & 0 & B_1 & 0 \\ 0 & A_{21} & A_{22} & 0 & B_2 & 0 \\ 0 & 0 & 0 & \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^*) & \bar{B}_{41} & \bar{B}_{42} \\ \hline -A_{12}^* & C_2^* C_1 - A_{12}^* \Omega & 1 & 0 & -\sigma B_2 & 0 \\ -C_\perp^* C_1 \Omega^{-1} & 0 & 0 & -C_\perp^* C_1 (\Sigma + \Omega^{-1} \Gamma) & 0 & -\sigma I \\ 0 & 0 & 0 & -\sigma B_\perp B_1^* & \sigma B_\perp & 0 \end{array} \right] \quad (C.8)$$

where we have defined,

$$\begin{aligned} \bar{B}_{11} &= -\sigma \Omega A_{12} B_2 - \Omega B_1 - \Sigma B_1 - \sigma C_1^* U \\ \bar{B}_{41} &= \Gamma^{-1}(\Sigma B_1 + \sigma C_1^* U) \\ \bar{B}_{42} &= -\sigma \Gamma^{-1} C_1^* C_\perp \end{aligned}$$

Next, we multiply (C.8) on the right by

$$\begin{bmatrix} w & W_\perp & 0 \\ 0 & 0 & I_p \end{bmatrix} \doteq \left[\begin{array}{c|ccc} -A_{11}^* - A_{21}^* (A_{21} \tilde{\Omega} - B_2 B_1^*) & -A_{21}^* & -\tilde{\Omega}^{-1} B_1 B_\perp^* & 0 \\ \hline B_\perp^* B_\perp B_1 + B_2^* A_{21} \tilde{\Omega} & B_2^* & B_\perp^* & 0 \\ 0 & 0 & 0 & I \end{array} \right] \quad (C.9)$$

which is taken from Theorem 3.4.5. Following that, we make use of the change of basis:

$$T = \begin{bmatrix} & I & \begin{bmatrix} -(\Omega + \Sigma)\tilde{\Omega} \\ \tilde{\Omega} \\ 0 \\ -I \end{bmatrix} \\ 0 & & I \end{bmatrix} \quad (C.10)$$

Simplifications based on the Riccati equation (3.4.36) and the partitions of (3.1.1) and (3.1.2) yield

$$\begin{bmatrix} v^* & 0 \\ V_\perp^* & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} w & W_\perp & 0 \\ 0 & 0 & I_p \end{bmatrix} \stackrel{s}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad (C.11)$$

in which

$$A = \begin{bmatrix} -A_{11}^* - (\Omega A_{12} - C_1^* C_2) A_{12}^* & 0 & 0 & 0 & 0 \\ 0 & A_{11} & A_{12} & 0 & 0 \\ 0 & A_{21} & A_{22} & 0 & 0 \\ 0 & 0 & 0 & \Gamma^{-1}(\sigma^2 A_{11}^* + \Sigma A_{11} \Sigma - \sigma C_1^* U B_1^*) & 0 \\ 0 & 0 & 0 & 0 & -A_{11}^* - A_{21}^*(A_{21} \tilde{\Omega} - B_2 B_1^*) \end{bmatrix} \quad (C.12)$$

$$B = \begin{bmatrix} (\Omega \Sigma + \Gamma + \Omega \tilde{\Omega} + \Sigma \tilde{\Omega}) A_{21}^* & 0 & 0 \\ B_1 B_2^* - \Omega A_{21}^* & 0 & 0 \\ 1 & 0 & 0 \\ 0 & (\tilde{\Omega}^{-1} - \Gamma^{-1} \Sigma) B_1 B_\perp^* & -\sigma \Gamma^{-1} C_1^* C_\perp^* \\ -A_{21}^* & -\tilde{\Omega}^{-1} B_1 B_\perp^* & 0 \end{bmatrix} \quad (C.13)$$

$$C = \begin{bmatrix} -A_{12}^* & C_2^* C_1 - A_{12}^* \Omega & 1 & 0 & 0 \\ -C_\perp C_1 \Omega^{-1} & 0 & 0 & -C_\perp C_1 (\Sigma + \Omega^{-1} \Gamma) & -C_\perp C_1 \Omega^{-1} (\Omega \Sigma + \Gamma + \Omega \tilde{\Omega} + \Sigma \tilde{\Omega}) \\ 0 & 0 & 0 & -\sigma B_\perp B_1^* & 0 \end{bmatrix} \quad (C.14)$$

$$D = \begin{bmatrix} -\sigma & 0 & 0 \\ 0 & 0 & -\sigma I \\ 0 & \sigma I & 0 \end{bmatrix} \quad (C.15)$$

An inspection of the first row of C in conjunction with A establishes that $a(s) \in \mathbb{RH}_+^\infty$. Since the three matrices on the left of (C.1) are all-pass, it follows that $a(s)$ is necessarily all-pass also. The second and third columns of B , together with the second and third rows of C , and A verify (C.2) and (C.3). The fact that $Q(s) \in \mathbb{RH}_-^{\infty, (m+p-2) \times (m+p-2)}$ follows from [G1, (6.35) pg.1140]. That $G_1(s) \in \mathbb{RH}_+^{\infty, (m-1, p-1)}$ is immediate; review Remark 3.4.1. $\square$

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