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THE EXCEPTIONAL SET IN NEVANLINNA THEORY

by

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ABSTRACT

This thesis is concerned with some exceptional sets which arise in the second fundamental Theorem of Nevanlinna in the theory of value distribution of meromorphic functions.

This Theorem states that for a meromorphic function F in the plane we have the relation

$$(q-2)T(r,F) \leq \sum_{v=1}^q N(r,a_v) - N_1(r) + S(r), \quad (1)$$

for every q values $a_1, \dots, a_q$ of the extended complex plane, where $N_1(r)$ is a positive term and $S(r)$ satisfies

$$S(r) = o(T(r,F)) \quad , \quad (2)$$

as r tends to infinity outside an exceptional set E_1 .

From (1) and (2) we obtain

$$(q-2)T(r,F) \leq \left(\sum_{v=1}^q N(r,a_v) \right) (1+o(1)) \quad , \quad (3)$$

outside the exceptional set E_1 .

But the relation (3) could be true even if (1) does not hold. We can consider the exceptional set E_2 outside which (3) holds. We have

$$E_2 \subset E_1 \quad .$$

The object of this work is to study the size of both exceptional sets.

In Chapter Two I present some conditions on the size of these sets.

Specifically I show that for a meromorphic function in the plane we can find E_1, E_2 independent of λ such that

$$\int_{E_2} r^\lambda dr \leq \int_{E_1} r^\lambda dr < \infty . \quad (4)$$

Another condition is that we can enclose both sets in a sequence of intervals $[r_n, r_n + \delta_n]$ satisfying

$$\delta_n \leq \frac{1}{\Psi(n)} , \quad \Psi(1) = 1, \Psi(n) = e^{\Psi(n-1)} . \quad (5)$$

(4) and (5) are independent conditions.

In the following chapters I show that these conditions are the best that we can obtain.

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CHAPTER ONE

INTRODUCTION

The framework of this thesis is Nevanlinna's theory of the value distribution of meromorphic functions.

In this Introduction I shall give some notation and try to describe the basic facts about this theory and finally I shall explain the content of the chapters of the thesis.

The basic results in this theory are the two fundamental theorems of Nevanlinna. Next I shall describe the first fundamental Theorem with some historic remarks.

Picard [1880] had proved his famous theorem that an entire function assumes every value with at most one exception. It is quite natural to ask whether anything further can be said about the roots of the equation $f(z) = a$ for different values of a . We shall call $n(r, a)$ the number of these roots in $\{z \mid |z| \leq r\}$ and define the exponent of convergence of these roots by

$$\rho(a) = \overline{\lim}_{r \rightarrow \infty} \frac{\log n(r, a)}{\log r} .$$

Hadamard [1.893] proved that if

$$M(r) = M(r, f) = \max_{|z|=r} |f(z)| ,$$

and the order of f is defined as

$$\rho = \overline{\lim}_{r \rightarrow \infty} \frac{\log \log M(r)}{\log r} ,$$

then $\rho(a) \leq \rho$ for every a . Borel [1.897] notably extended Picard's theorem by proving that $\rho(a) = \rho$ for all a with at most one exception.

The theory obtained in this way was based on the product decomposition of entire functions of finite order due to Hadamard [1.893]. This theory lacked precision and did not work well for functions of infinite order or meromorphic functions.

For the latter $M(r, f)$ is not a good measure of the growth since f can have poles.

The situation changed with Nevanlinna in the 1.920's. He used the known formula of Jensen [1.899]

$$\log |f(0)| = \frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\vartheta})| d\vartheta + \sum_{\nu} \log \frac{r}{|b_{\nu}|} - \sum_{\mu} \log \frac{r}{|a_{\mu}|} ,$$

where a_{μ} are the zeros and b_{ν} the poles of the meromorphic function $f(z)$ in $\{z \mid |z| \leq r\}$.

Following Valiron [1.913], he wrote

$$\sum \log \frac{r}{|a_{\mu}|} = \int_0^r \log\left(\frac{r}{t}\right) dn(t, 0) = \int_0^r \frac{n(t, 0)}{t} dt = N(r, 0) ,$$

$$\sum \log \frac{r}{|b_{\nu}|} = \int_0^r \log\left(\frac{r}{t}\right) dn(t, \infty) = \int_0^r \frac{n(t, \infty)}{t} dt = N(r, \infty) .$$

He then wrote

$$\log^+ x = \max(\log x, 0) ,$$

so that

$$\log x = \log^+ x - \log^+ \frac{1}{x}, \quad x \geq 0$$

and

$$\frac{1}{2\pi} \int_0^{2\pi} \log |f(re^{i\vartheta})| d\vartheta = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\vartheta})| d\vartheta - \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\vartheta})} \right| d\vartheta.$$

This simple step is a key one for the theory. He defined then

$$m(r, \infty) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\vartheta})| d\vartheta,$$

and for any finite complex a

$$m(r, a) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| \frac{1}{f(re^{i\vartheta}) - a} \right| d\vartheta.$$

With this notation we can rewrite Jensen's formula in the form

$$m(r, \infty) + N(r, \infty) = m(r, 0) + N(r, 0) + \log |f(0)|.$$

Applying this result to $f(z)-a$ instead of $f(z)$, we obtain easily the first fundamental Theorem

$$m(r, \infty) + N(r, \infty) = m(r, a) + N(r, a) + \log |f(0)-a| + \varepsilon(a),$$

where $|\varepsilon(a)| \leq \log^+ |a| + \log 2$.

Nevanlinna wrote

$$T(r, f) = m(r, \infty) + N(r, \infty) \quad ,$$

and called $T(r, f)$ the characteristic function of $f(z)$.

Then the first fundamental Theorem shows that

$$m(r, a) + N(r, a) = T(r, f) + O(1) \quad ,$$

as r varies in the positive real axis. Here we have assumed $f(0) \neq a$.

In the general case we shall define

$$N(r, a) = \int_0^r \frac{[n(t, a) - n(0, a)]}{t} dt + n(0, a) \log r \quad .$$

The case $f(0) = \infty$ also requires some modifications.

The first fundamental theorem gives us an upper bound for $N(r, a)$, which measures how often the function f takes on the value a in $\{z \mid |z| \leq r\}$.

Nevanlinna's second fundamental Theorem deals with the more difficult question of lower bounds.

Nevanlinna proved this theorem in [1.925]. It states that if $q \geq 3$ then

$$(q-2)T(r, f) \leq \sum_{v=1}^q N(r, a_v) - N_1(r) + S(r, f) \quad , \quad (1)$$

where $N_1(r)$ measures the zeros of f' and the multiple poles of f , that is, the totality of the multiple roots of all equations $f = a$, a root of multiplicity p being counted $p-1$ times and $S(r, f)$ plays the role of an error term.

For instance, if $f(z)$ is meromorphic and not constant in the plane, then we have

$$S(r, f) = O\{\log T(r, f)\} + O(\log r) \quad , \quad (2)$$

as $r \rightarrow \infty$ through all values if $f(z)$ has finite order and, otherwise as $r \rightarrow \infty$ outside an exceptional set of finite linear measure.

Clearly (2) implies

$$S(r, f) = o(T(r, f)) \quad , \quad (3)$$

outside an exceptional set E_1 .

Using (1) and (3) we obtain

$$T(r, f) < (N(r, a_1) + N(r, a_2) + N(r, a_3)) (1 + o(1)) \quad , \quad (4)$$

outside a set of finite linear measure.

But (4) could happen even if (3) is not true. We can consider the exceptional set E_2 outside which (4) holds and we should have therefore

$$E_2 \subset E_1 \quad . \quad (5)$$

The object of this thesis is to find best possible conditions on the size of both exceptional sets.

In Chapter Two, I give some results on the size of E_1 . In particular I show that (3) holds outside a set E_1 , independent of λ , such that

$$\int_{E_1} r^\lambda dr < \infty \quad , \quad (6)$$

for every λ .

In particular (6) implies that E_1 is of finite linear measure.

I present also a condition of a different kind. The set E_1 of a meromorphic function can be enclosed in a sequence of intervals $[r_n, r_n + \delta_n]$ satisfying

$$\delta_n < \frac{1}{\Psi(n)^2} \quad \text{where} \quad \Psi(1) = 1, \quad \Psi(n) = e^{\Psi(n-1)}. \quad (7)$$

In Chapter Three, I show that these conditions are the best possible that, we can obtain. We give an example of an entire function of the form

$$F(z) = \sum_{n=1}^{\infty} \left(\frac{z}{r_n} \right)^{\lambda_n} \quad (8)$$

for which the exceptional set E_1 satisfies

$$\int_{E_1} \Phi(r) dr = \infty \quad (9)$$

where Φ is a preassigned function which grows to infinity quicker than every power r^λ .

This result is based on previous work by Hayman [3]. There he shows with a function of the form (8), an example, where the exceptional set E_1 exists. However, in order to construct an exceptional set satisfying (9), some rather detailed quantitative estimates are needed, which require fairly delicate analysis.

We also prove by means of an example of the same type that (7) cannot be improved.

Finally in Chapter Four, I prove that condition (6) is also the

best possible for the smaller set E_2 . The example is rather more complicated than in the previous case and is based on a construction of Hayman [4] of a function, defined in such a way that it approximates certain polynomials with few zeros and i -values and sufficiently large characteristic on a sequence of expanding circles.

C H A P T E R T W O

Results about the size of the exceptional set in Nevanlinna's second fundamental theorem

1. INTRODUCTION

In this chapter we shall use the notation and definitions introduced in the preceding one. First of all we shall state in detail Nevanlinna's second fundamental theorem as it can be found in Hayman [2].

Theorem 2.1: Let $F(z)$ be a nonconstant meromorphic function in $|z| \leq r$. Let $a_1, a_2, \dots, a_q$, where $q > 2$, be distinct complex numbers, $\delta > 0$, and suppose that $|a_\mu - a_\nu| \geq \delta$, $1 \leq \mu < \nu \leq q$. Then

$$m(r, \infty) + \sum_{\nu=1}^q m(r, a_\nu) \leq 2T(r, F) - N_1(r) + S(r),$$

where $N_1(r)$ is positive and is given by

$$N_1(r) = N(r, \frac{1}{F'}) + 2N(r, F) - N(r, F'),$$

and

$$S(r) = m(r, \frac{F'}{F}) + m \left\{ r, \sum_{\nu=1}^q \frac{F'}{F - a_\nu} \right\} +$$

$$+ q \log^+ \frac{3q}{\delta} + \log 2 + \log \frac{1}{|F'(0)|}, \quad (2.1)$$

with modifications if $F'(0) = 0$ or ∞ .

The quantity $S(r)$ will in general be negligible with respect to $T(r, F)$ and the combination of Theorem 2.1 and the estimation for $S(r)$ constitute Nevanlinna's second fundamental theorem.

In the next section, I present a basic result giving an estimation for $S(r)$.

2. Some conditions on the exceptional set

The following theorem is due to R. Nevanlinna

Theorem 2.2: Suppose that F is a nonconstant meromorphic function in the plane, $S(r, F)$ is defined by (2.1) and λ a fixed positive number, then we have

$$S(r, F) = O \left\{ \log T(r, F) \right\} + O(\log r) \quad , \quad (2.2)$$

as $r \rightarrow \infty$ through all values if $F(z)$ has finite order, and otherwise as $r \rightarrow \infty$ outside a set E_λ satisfying

$$\int_{E_\lambda} r^\lambda dr < \infty \quad . \quad (2.3)$$

The proof of Theorem 2.2 in the case $\lambda = 0$ can be found in [2] and as we shall show the argument extends easily to the general case. Nevanlinna gives a full but rather more complicated proof of (2.3) in [8].

(2.3) implies, in particular, that E_λ is of finite measure.

If we now impose the weaker condition,

$$S(r) = o(T(r,F)) \quad (2.4)$$

instead of (2.2) we get a slightly stronger conclusion.

Theorem 2.3: The error term $S(r,F)$ in Nevanlinna's second fundamental theorem satisfies (2.4) i.e.

$$S(r,F) = o\{T(r,F)\} ,$$

as $r \rightarrow \infty$ outside a set E , independent of λ , such that

$$\int_E r^\lambda dr < \infty \quad (2.5)$$

for every λ .

3. Some auxiliary results

To prove Theorem 2.1 we shall need the following lemmas, where we shall assume for simplicity that $F(0) \neq 0, \infty$.

Lemma 2.1 (Logarithmic derivative Lemma)

Let $F(z)$ be meromorphic in $|z| \leq R$. For $0 < r < R$, we have,

$$m(r, \frac{F'}{F}) < 4 \log^+ T(R,F) + 4 \log^+ \log^+ \frac{1}{|F(0)|} + \\ 5 \log^+ R + 6 \log^+ \frac{1}{R-r} + \log^+ \frac{1}{r} + 14.$$

The proof of Lemma 2.1 can be found in Hayman [2].

Lemma 2.2 (Suppose that $T(r)$ is a continuous increasing, real-valued function and the $T(r) \geq 1$ for $r_0 \leq r < +\infty$. Then, we have

$$T\left\{r + \frac{1}{r^{\lambda T(r)}}\right\} < 2T(r) \quad , \quad (2.6)$$

outside an exceptional set E_λ satisfying ,

$$\int_{E_\lambda} r^\lambda dr < \infty \quad . \quad (2.7)$$

The proof of Lemma 2.2 is quite a simple adaptation of that of Lemma 2.4 in [2].

Let r_1 be the lower bound of all $r > r_0$ for which (2.6) is false. If there are no such r , there is nothing to prove.

We now define a sequence of numbers r_n . Suppose that r_n has been defined and write

$$r'_n = r_n + \frac{1}{r_n^{\lambda T(r_n)}} \quad ,$$

then we define r_{n+1} as the lower bound of all $r \geq r'_n$ for which (2.6) is false. Since r_1 has already been defined, we obtain the sequence $\{r_n\}$.

By the continuity of $T(r)$, (2.6) is false for $r = r_n$, $n = 1, 2, 3, \dots$. Thus r_n belongs to the exceptional set E_λ .

Now, we have by the definition of r_{n+1} that there are no points of E_λ in (r'_n, r_{n+1}) and so E_λ is contained in the union of the intervals $[r_n, r'_n]$.

Next, we see that if there are an infinity of r_n then r_n cannot

tend to a finite limit r . For otherwise, since

$$r_n < r'_n \leq r_{n+1} \quad ,$$

also

$$r'_n \rightarrow r \quad .$$

But

$$r'_n - r_n = \frac{1}{r_n^{\lambda T(r_n)}} \geq \frac{1}{r^{\lambda T(r)}} > 0 \quad ,$$

since $T(r)$ is increasing, which gives a contradiction.

It remains only to prove that,

$$\int_{E_\lambda} r^\lambda dr < \infty \quad ,$$

and since $E_\lambda \subset E = \bigcup_{n=1}^{\infty} [r_n, r'_n]$, it will be enough that

$$\int_E r^\lambda dr < \infty \quad . \tag{2.8}$$

But (2.8) is equivalent to

$$\sum r_n^\lambda (r'_n - r_n) < \infty \quad . \tag{2.9}$$

We have, in fact,

$$T(r_{n+1}) \geq T(r'_n) \geq 2T(r_n) \quad .$$

Hence

$$T(r_{n+1}) \geq 2T(r_n) \geq \dots \geq 2^n T(r_1) \geq 2^n ,$$

since

$$T(r_1) \geq 1 .$$

Thus

$$\sum_{n=1}^{\infty} r_n^{\lambda} (r'_n - r_n) = \sum_{n=1}^{\infty} \frac{1}{T(r_n)} \leq \sum_{n=1}^{\infty} 2^{1-n} = 2 .$$

This completes the proof of Lemma 2.2.

4. Proof of Theorem 2.2

We recall that $S(r)$ is defined by (2.1), and therefore can also be written as,

$$S(r, F) = m(r, \frac{F'}{F}) + m(r, \frac{\Phi'}{\Phi}) + \text{const.}, \quad (2.10)$$

where

$$\Phi(z) = \prod_{v=1}^q \{F(z) - a_v\} .$$

By Lemma 2.1, we have for $0 \leq r < R$,

$$m(r, \frac{\Phi'}{\Phi}) < 4 \log^+ T(R, \Phi) + 5 \log^+ R + 6 \log \frac{1}{R-r} + \text{const} . \quad (2.11)$$

Now we take $R = r + \frac{1}{r^\lambda T(r,F)}$. Then if r is large we have,

$$5 \log^+ R < 5 \log r + \text{const} \quad ,$$

$$6 \log \frac{1}{R-r} = 6\lambda \log r + 6 \log T(r,F), \quad \text{and}$$

$$4 \log^+ T(R,\Phi) \leq 4 \log^+ \{qT(r,F) + \text{const}\} \leq$$

$$\leq 4 \log^+ T(r,F) + \text{const} \quad , \quad \text{outside a set } E_\lambda \text{ satisfying}$$

$$\int_{E_\lambda} r^\lambda dr < \infty \quad ,$$

by Lemma 2.2.

Thus,

$$m(r, \frac{\Phi'}{\Phi}) \leq 10 \log^+ T(r,F) + (6\lambda+5) \log r + \text{const} \quad , \quad (2.12)$$

in particular,

$$m(r, \frac{F'}{F}) \leq 10 \log^+ T(r,F) + (6\lambda+5) \log r + \text{const} \quad . \quad (2.13)$$

From (2.10), (2.12) and (2.13), we get

$$S(r,F) \leq 20 \log^+ T(r,F) + (12\lambda+10) \log r + \text{const} \quad , \quad (2.14)$$

outside a set E_λ satisfying (2.3), and (2.14) in particular implies (2.2).

5. Proof of Theorem 2.3

By (2.14) we can find for every $N \in \mathbb{N}$ a set E_N and r_N such that for $r \geq r_N$ outside E_N , we have

$$S(r, F) \leq 10 \log^+ T(r, F) + 23N \log r \quad , \quad (2.15)$$

and E_N satisfies

$$\int_{E_N \cap [r_N, \infty)} r^N dr < 2^{-N} \quad . \quad (2.16)$$

We assume without loss of generality that $r_{N+1} > r_N$ and that $r_N \rightarrow \infty$ as $N \rightarrow \infty$. Then if $N(r)$ is defined by the condition

$$r_{N(r)} \leq r < r_{N(r)+1} \quad ,$$

we further assume r_N to be so large that

$$\frac{N(r) \log r}{T(r)} \rightarrow 0 \quad . \quad (2.17)$$

This is possible, since we can assume F to be transcendental and therefore

$$\frac{\log r}{T(r)} \rightarrow 0 \quad , \quad \text{as } r \rightarrow \infty \quad .$$

Otherwise E is empty.

Now we define

$$E = \bigcup_{N=1}^{\infty} \{E_N \cap [r_N, r_{N+1}]\} \quad ,$$

and prove that

$$\int_E r^\lambda dr < \infty \quad , \quad \lambda \geq 0 \quad , \quad (2.18)$$

and

$$S(r, F) = o(T(r, F)) \quad (2.19)$$

outside E.

To prove (2.18), we note that,

$$\int_E r^\lambda dr = \sum_{N=1}^{\infty} \int_{E \cap [r_N, r_{N+1}]} r^\lambda dr ,$$

and assume $N_1 < \lambda \leq N_1 + 1$. Now using (2.16), we have,

$$\begin{aligned} \sum_{N=1}^{\infty} \int_{E \cap [r_N, r_{N+1}]} r^\lambda dr &= \sum_{N=1}^{N_1-1} \int_{E \cap [r_N, r_{N+1}]} r^\lambda dr + \sum_{N=N_1}^{\infty} \int_{E \cap [r_N, r_{N+1}]} r^\lambda dr \leq \\ &\leq \int_{E \cap [0, r_{N_1}]} r^\lambda dr + \sum_{N=N_1}^{\infty} \int_{E \cap [r_N, \infty)} r^N dr \leq K + \sum_{N=N_1}^{\infty} 2^{-N} = K_1 . \end{aligned}$$

This completes the proof of (2.18).

Next we prove (2.19). By (2.15), we have outside E,

$$S(r, F) = 10 \log^+ T(r, F) + 23 N(r) \log r$$

and using (2.17) we obtain

$$S(r, F) = o(T(r, F)) ,$$

outside E, i.e. (2.19), and the proof of Theorem 2.3 is complete.

6. Further conditions on the exceptional set

In this section, I give a new condition for the exceptional set E.

Theorem 2.4: The error term $S(r, F)$ satisfies,

$$S(r, F) = o\{T(r, F)\}$$

as $r \rightarrow \infty$ outside a set E , which can be contained in a union of intervals $[r_n, r_n + \delta_n]$, such that,

$$\delta_n < \frac{1}{\Psi(n)^2} \quad \text{where} \quad \Psi(1) = 1, \quad \Psi(n) = e^{\Psi(n-1)} \quad (2.20)$$

This condition implies, in particular, that the exceptional set has finite measure, but it is a different condition in character from (2.5) in section 2, i.e. neither implies the other for $\lambda > 0$. In fact (2.20) gives a limitation only on the size of the interval $[r_n, r_n + \delta_n]$, whereas (2.5) limits their size relative to their location.

7. An auxiliary lemma

In the proof of Theorem 2.4 we shall use the following lemma.

Lemma 2.3: Suppose that $T(r)$ is a continuous, increasing, real-valued function for $r_0 \leq r < +\infty$, and that, $T(r) \geq 1$ there.
Then we have

$$T\left(r + \frac{1}{T(r)}\right) < e^{2T(r)^{\frac{1}{2}}}, \quad (2.21)$$

outside an exceptional set contained in a union of intervals

$\bigcup_n [r_n, r_n + \delta_n]$, such that δ_n satisfies (2.20).

We write $t(r) = T(r)^{\frac{1}{2}}$, so that (2.21) becomes

$$t(r + \frac{1}{t(r)^2}) < e^{t(r)} \quad . \quad (2.22)$$

Let r_1 be the first value of r , such that $r \geq r_0$ and (2.21) is false, i.e.

$$t(r_1 + \frac{1}{t(r_1)^2}) \geq e^{t(r_1)} \quad .$$

We write $r_1' = r_1 + t(r_1)^{-2}$, and let r_2 be the lower bound of all $r \geq r_1'$ such that (2.21) is false. Then we define inductively

$$r_{n-1}' = r_{n-1} + \frac{1}{t(r_{n-1})^2} \quad ,$$

and r_n is the lower bound of all $r \geq r_n'$ for which

$$t(r + \frac{1}{t(r)^2}) \geq e^{t(r)} \quad .$$

The exceptional set is obviously contained in the union

$$\bigcup_n [r_n, r_n'] = \bigcup_n [r_n, r_n + \frac{1}{t(r_n)^2}] \quad .$$

We write $\delta_n = t(r_n)^{-2}$. Then, since

$$t(r_n) \geq t(r_n') \geq e^{t(r_{n-1}')} \quad \text{and} \quad t(r_0) \geq 1 \quad ,$$

we obtain by induction

$$t(r_n) \geq \Psi(n+1) \geq \Psi(n), \text{ and therefore}$$

$$\delta_n = \frac{1}{t(r_n)^2} \leq \frac{1}{\Psi(n)^2} .$$

This completes the proof of Lemma 2.3.

8. Proof of Theorem 2.4

We recall from (2.10) that

$$S(r, F) = m(r, \frac{F'}{F}) + m(r, \frac{\Phi'}{\Phi}) + \text{const} ,$$

where

$$\Phi(z) = \prod_{v=1}^q \{F(z) - a_v\} .$$

Also by (2.11), we have for $0 \leq r < R$

$$m(r, \frac{\Phi'}{\Phi}) \leq 4 \log^+ T(R, \Phi) + 5 \log^+ R + 6 \log \frac{1}{R-r} + \text{const}.$$

We now take $R = r + 1/T(r, F)$. Then

$$\log^+ R < \log^+ r + \text{const} ,$$

$$6 \log^+ \frac{1}{R-r} = 6 \log^+ T(r, F) ,$$

and

$$\begin{aligned} 4 \log^+ T(r, \Phi) &\leq 4 \log^+ \{qT(R, F) + \text{const}\} \\ &\leq 8T(r)^{\frac{1}{2}} + \text{const} , \end{aligned}$$

by Lemma 2.3, outside a set, which can be contained in a sequence of intervals $[r_n, r_n + \delta_n]$, such that δ_n satisfies (2.20).

Thus, for r large and outside this set and any fixed $a_1, \dots, a_q$,
we have

$$\begin{aligned} m(r, \frac{\Phi'}{\Phi}) &\leq 8T(r)^{\frac{1}{2}} + 6 \log^+ T(r, F) + 6 \log r \\ &\leq 9T(r)^{\frac{1}{2}} + 6 \log r \quad , \end{aligned}$$

and in particular

$$m(r, \frac{F'}{F}) \leq 9T(r)^{\frac{1}{2}} + 6 \log r \quad .$$

This implies (2.4) and completes the proof of Theorem 2.4.

C H A P T E R T H R E E

Some Converse Results

1. An example of function showing that the exceptional set can, in fact, occur

1.1 In this chapter we construct a general example possessing an exceptional set for $S(r, F)$ in Nevanlinna's second fundamental theorem.

We use this example to prove that (2.5) and (2.20) in the preceding chapter cannot be replaced by any stronger condition of the same type.

Theorem 3.1: Suppose that $0 < \alpha < 1$ and that $\{r_n\}$ is a sequence of positive numbers satisfying

$$r_1 = 1 \quad , \quad r_n \leq \alpha^2 r_{n+1} \quad , \quad n = 1, 2, \quad . \quad (3.1)$$

Let $\{\lambda_n\}$ be the sequence of integers defined by

$$\lambda_1 = 1 \quad , \quad \lambda_n = [r_n^{n\lambda_{n-1}}] + 1 \quad , \quad (3.2)$$

and write

$$F(z) = \sum_{n=1}^{\infty} \left(\frac{z}{r_n}\right)^{\lambda_n} . \quad (3.3)$$

Then $F(z)$ is an entire function. Further if

$$s_n = \{r_n^{\lambda_n} \cdot 2(n-1) \cdot \lambda_{n-1}\}^{\frac{1}{\lambda_n - \lambda_{n-1}}}, \quad s'_n = s_n + \delta_n, \quad \delta_n = o\left(\frac{r_n}{\lambda_n}\right), \quad (3.4)$$

and $f(z) = F'(z)$, then

$$\frac{T(r, f)}{T(r, F)} \rightarrow \infty, \quad \text{as } r \rightarrow \infty \quad \text{in } E = \bigcup_n [s_n, s'_n].$$

By (3.2), we have for n large $\lambda_n \geq n$ and by (3.1) $r_n \rightarrow \infty$ with n .

Thus $F(z)$ defined by (3.3) converges everywhere and so is entire.

We consider now

$$r_N \leq r = |z| \leq \alpha r_{N+1}, \quad (3.5)$$

and write

$$F(z) = \sum_1^{N-1} \left(\frac{z}{r_n}\right)^{\lambda_n} + \left(\frac{z}{r_N}\right)^{\lambda_N} + \sum_{N+1}^{\infty} \left(\frac{z}{r_n}\right)^{\lambda_n}.$$

Now

$$\left| \sum_1^{N-1} \right| = \left| \sum_1^{N-1} \left(\frac{z}{r_n}\right)^{\lambda_n} \right| \leq (N-1) r^{\lambda_{N-1}}, \quad (3.6)$$

and

$$\left| \sum_3^{\infty} \right| = \left| \sum_{N+1}^{\infty} \left(\frac{z}{r_n}\right)^{\lambda_n} \right| \leq \sum_{N+1}^{\infty} \alpha^{\lambda_n} \leq \sum_1^{\infty} \alpha^n < K, \quad (3.7)$$

where $K = \frac{\alpha}{1-\alpha}$.

We also have

$$z f(z) = z F'(z) = z \sum_1' + z \sum_2' + z \sum_3'$$

where

$$\left| z \sum_1' \right| \leq (N-1) \lambda_{N-1} r^{\lambda_{N-1}}, \quad (3.8)$$

and

$$\left| z \sum_3' \right| \leq \sum_{N+1}^{\infty} \lambda_n \alpha^{\lambda_n} \leq \sum_1^{\infty} n \alpha^n = K_1, \quad (3.9)$$

where $K_1 = \frac{\alpha}{(1-\alpha)^2}$.

Now, if s_n is as in (3.4), we have

$$\left(\frac{s_n}{r_n} \right)^{\lambda_n} = 2(n-1) \lambda_{n-1} s_n^{\lambda_{n-1}}. \quad (3.10)$$

We note from (3.4) that $s_n > r_n$. Next we prove that

$$s_n \sim r_n, \quad \text{as } n \rightarrow \infty, \quad (3.11)$$

i.e.

$$\lim_{n \rightarrow \infty} \frac{r_n^{\frac{\lambda_n}{\lambda_n - \lambda_{n-1}}} \cdot \{2(n-1)\}^{\frac{1}{\lambda_n - \lambda_{n-1}}} \cdot \lambda_{n-1}^{\frac{1}{\lambda_n - \lambda_{n-1}}}}{r_n} = 1. \quad (3.12)$$

To prove (3.12) we consider the logarithm of the left hand side.

This logarithm is

$$\left[\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} - 1 \right] \log r_n + \frac{\log[2(n-1)]}{\lambda_n - \lambda_{n-1}} + \frac{\log \lambda_{n-1}}{\lambda_n - \lambda_{n-1}}.$$

Since λ_n/λ_{n-1} and λ_n/n tend to infinity as n tends to infinity by (3.2), the last two terms tend to zero. It remains to show that the first term tends to zero. In fact

$$\left[\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} - 1 \right] \log r_n = \frac{\lambda_{n-1} \log r_n}{\lambda_n - \lambda_{n-1}} < \frac{\log \lambda_n}{\lambda_n - \lambda_{n-1}} \rightarrow 0,$$

as $n \rightarrow \infty$, by (3.2).

Next we see that if $r \in [s_n, s'_n]$ for N large then r satisfies (3.5),

in fact since

$$1 \leq \frac{r}{r_N} \leq \frac{s_N + \delta_n}{r_N} \leq \frac{s_N + \frac{r_N}{\lambda_N}}{r_N}, \quad s_N \sim r_N, \quad \frac{r_N}{\lambda_N} \rightarrow 0,$$

we can find $\epsilon > 0$ such that $1+\epsilon < \frac{1}{\alpha}$ and N_0 such that for $N \geq N_0$

$$1 \leq \frac{r}{r_N} \leq 1+\epsilon,$$

and therefore

$$r_N \leq r \leq (1+\epsilon) r_N \leq \frac{1}{\alpha} r_N \leq \alpha r_{N+1},$$

which is (3.5).

Now we show that as $r \rightarrow \infty$ through the intervals $[s_N, s'_N]$ we have

$$\left(\frac{s_n}{r_n} \right)^{\lambda_n} \sim \left(\frac{r}{r_n} \right)^{\lambda_n}, \quad (3.13)$$

or equivalently

$$\left(\frac{s_n + \delta_n}{s_n} \right)^{\lambda_n} \rightarrow 1.$$

But

$$\left(\frac{s_n + \delta_n}{s_n} \right)^{\lambda_n} = \left[\left(1 + \frac{1}{s_n / \delta_n} \right)^{s_n / \delta_n} \right]^{\frac{\delta_n \lambda_n}{s_n}} \rightarrow e^0 = 1$$

by (3.4) and (3.11).

Since (3.13) holds for $r \in [s_n, s'_n]$, we also have for such r

$$\left(\frac{s_n}{r_n} \right)^{\lambda_{n-1}} \sim \left(\frac{r}{r_n} \right)^{\lambda_{n-1}} . \quad (3.14)$$

Therefore we deduce from (3.10), (3.11) and (3.14)

$$\left(\frac{r}{r_n} \right)^{\lambda_n} \sim 2(n-1)\lambda_{n-1} r^{\lambda_{n-1}} .$$

Since we have already seen that for $r \in [s_n, s'_n]$ n large, r is in the range (3.5) then we can apply (3.6), (3.7), (3.8) and (3.9) to obtain

$$\begin{aligned} |F(z)| &\sim \left(\frac{r}{r_n} \right)^{\lambda_n} \sim 2n\lambda_{n-1} r^{\lambda_{n-1}} , \\ |zf(z)| &= \lambda_n \left(\frac{r}{r_n} \right)^{\lambda_n} + O\{(n-1)\lambda_{n-1} r^{\lambda_{n-1}}\} \\ &\sim \lambda_n \left(\frac{r}{r_n} \right)^{\lambda_n} \sim \lambda_n |F(z)| . \end{aligned}$$

Hence

$$T(r, F) = \lambda_{n-1} \log r + \log(2n \lambda_{n-1}) + o(1) ,$$

$$T(r, f) = T(r, F) + \log \lambda_n + o(1) - \log r .$$

From $r \in [s_n, s'_n]$ and (3.11) we deduce $r \sim r_n$ and by (3.2) we have,

$$\lambda_{n-1} \log r = o(\log \lambda_n) ,$$

therefore

$$T(r, F) = o(T(r, f)) . \quad (3.15)$$

This completes the proof of Theorem 3.1.

1.2 Next we show that the sequence of intervals $[s_n, s_n + \delta_n]$ must be exceptional for the function F .

To do this it will be enough to prove that for large n we have

$$S(r, F) \geq 2T(r, F), \quad r \in [s_n, s_n + \delta_n] \quad . \quad (3.16)$$

By (3.15), we have for r large and in $\bigcup_n [s_n, s_n + \delta_n]$

$$m(r, f) = T(r, f) \geq 3T(r, F) = 3m(r, F) \quad ,$$

therefore, since

$$S(r, F) \geq m(r, \frac{f}{F}) \quad ,$$

and

$$m(r, f) \leq m(r, \frac{f}{F}) + m(r, F) \quad ,$$

we obtain

$$S(r, F) \geq m(r, \frac{f}{F}) \geq 2m(r, F) = 2T(r, F) = 2T(r, F)$$

this proves (3.16).

In the next two sections we specialise the function (3.3) to obtain counterexamples to (2.5) and the condition given in Theorem 2.4.

2. A converse result to Theorem 2.3

Theorem 3.2: For any function $\Phi(r)$ such that

$$\frac{\Phi(r)}{r^\lambda} \rightarrow \infty \text{ as } r \rightarrow \infty \text{ for all positive } \lambda, \quad (3.17)$$

we can choose the function $F(z)$ in (3.3) such that (3.16), i.e.

$S(r, F) \geq 2T(r, F)$, holds in a set E satisfying

$$\int_E \Phi(r) dr = \infty. \quad (3.18)$$

In proving Theorem 3.2 we assume without loss of generality that $\Phi(r)$ is increasing for large r . For, by hypothesis, for every positive integer N , there exists t_N , such that

$$\Phi(r) \geq r^N, \quad r \geq t_N.$$

We suppose without loss of generality that

$$t_1 \geq 1, \quad \text{and} \quad t_{N+1} \geq 2t_N, \quad N \geq 1.$$

We then define

$$\Phi_1(r) = r^N, \quad t_N \leq r < t_{N+1}.$$

Then for $r \geq t_1$, $\Phi_1(r)$ is increasing and satisfies (3.17) and also $\Phi_1(r) \leq \Phi(r)$. Thus it is enough to prove (3.18) with $\Phi_1(r)$ replacing $\Phi(r)$.

The function $F(z)$ in (3.3) satisfies (3.16) as r tends to infinity through a sequence of intervals $\{[s_n, s_n + \delta_n]\}$ where $\{s_n\}$ and $\{\delta_n\}$ are given by (3.4). Then it will be enough that we can choose s_n, δ_n so

$$\sum_n \Phi(s_n) \delta_n = \infty \quad (3.19)$$

for (3.18) is implied by (3.19) since Φ is increasing.

We show that we can choose r_n and δ_n in Theorem 3.1 so that

$$\Phi(s_n) \cdot \delta_n \geq 1, \quad (3.20)$$

and so (3.19) holds.

We recall that by (3.2) and (3.3), we can take

$$\lambda_1 = 1, \quad \lambda_n = [r_n^{n\lambda_{n-1}}] + 1, \quad \delta_n = \frac{r_n}{n\lambda_n},$$

so that

$$\Phi(s_n) \cdot \delta_n \geq \frac{\Phi(s_n) \cdot r_n}{n r_n^{\lambda_{n-1}}}. \quad (3.21)$$

Let us assume now that $r_1, \dots, r_{n-1}, \lambda_1, \dots, \lambda_{n-1}$ and therefore $\delta_1, \dots, \delta_{n-1}$ have already been defined. Then we define r_n such that $r_n \geq 2r_{n-1}$ so that (3.1) is satisfied and in addition

$$\frac{\Phi(r)}{n r^{\lambda_{n-1}}} \geq 1, \quad r \geq r_n. \quad (3.22)$$

This is possible by (3.17).

Now having chosen r_n in this way and therefore s_n, λ_n and δ_n , (3.20) follows from (3.21) and (3.22).

This completes the proof of Theorem 3.2.

3. A converse result to Theorem 2.4

Next we construct an example to show that Theorem 2.4 cannot be sharpened.

Theorem 3.3: We define

$$r_n = \Phi(n) = \Psi(n)^{1/3} , \quad (3.23)$$

where $\Psi(n)$ is given by (2.20).

Then $F(z)$, given by (3.3), satisfies $S(r,F) \geq 2T(r,F)$ for r sufficiently large in a sequence of intervals $[t_n, t_n + \beta_n]$ with

$$\beta_n \geq \frac{1}{\Psi(n)} , \quad \text{for large } n . \quad (3.24)$$

It is clear that $\{r_n\}$ given by (3.23) satisfies (3.1), therefore $F(z)$ is well defined and (3.16) holds.

We can choose

$$\delta_n = \frac{1}{\lambda_n}$$

so that (3.4) is satisfied. It remains only to verify (3.24).

Since $\Phi(n)$ is defined by (3.23) and recalling the definition of $\Psi(n)$ in (2.20) we have

$$\Phi(n+1) = \Psi(n+1)^{1/3} = \exp \left\{ \frac{1}{3} \Psi(n) \right\} = \exp \left\{ \frac{1}{3} \Phi(n)^3 \right\} .$$

We now proceed by induction. We assume n so large that $\Phi(n)$ is much bigger than n and $\log \Phi(n)$ and we assume that for such an n , N is so large that

$$\lambda_n \leq \Phi(n+N) \leq \Psi(n+N) \quad . \quad (3.25)$$

Then we prove that this inequality holds for all the following terms.

In fact by (3.2)

$$\begin{aligned} \lambda_{n+1} &= \left[r_{n+1}^{(n+1)\lambda_n} \right] + 1 \leq r_{n+1}^{(n+2)\lambda_n} = \exp \left\{ (n+2)\lambda_n \log r_{n+1} \right\} \\ &\leq \exp \left\{ \frac{1}{3} \Phi(n+N)^3 \right\} = \Phi(n+N+1) \leq \Psi(n+N+1) \quad , \end{aligned}$$

since for large n

$$(n+2)\lambda_n \log r_{n+1} = (n+2)\lambda_n \log \Phi(n+1) \leq \frac{1}{3} \Phi(n+N)^3 \quad .$$

This proves the inductive step, so that (3.25) holds for all large n . By increasing N if necessary we ensure that (3.25) holds for all n .

Therefore we obtain

$$\delta_n = \frac{1}{\lambda_n} \geq \frac{1}{\Psi(n+N)} \quad ,$$

for some positive integer N and all n .

We conclude (3.24), writing $t_n = s_{n-N}$, $\beta_n = \delta_{n-N}$, for $n > N$.

Theorem 2.4 shows that

$$S(r, F) = o(T(r, F)) \quad (3.26)$$

outside a union of intervals $E = \bigcup_1^\infty [r_n, r_n + \delta_n]$, where

$$\delta_n \leq \frac{1}{\Psi(n)^2} \leq \frac{1}{\Psi(n)} \quad (3.27)$$

where $\Psi(n)$ is defined in (2.20). We may omit a finite number of

intervals from E without destroying (3.26) and thus replace E by

$$E' = \bigcup_{n=N+1}^{\infty} [r_n, r_n + \delta_n] = \bigcup_{n=1}^{\infty} [r_{n+N}, r_{n+N} + \delta_{n+N}], \quad \text{for some } N \in \mathbb{N}.$$

We now relabel the intervals of E' as $[r'_n, r'_n + \delta'_n]$ where $r'_n = r_{n+N}$, $\delta'_n = \delta_{n+N}$ and then (3.27) yields

$$\delta'_n < \frac{1}{\Psi(n+N)}.$$

We show that this is the best that we can expect.

Theorem 3.4: For every increasing sequence $L(n)$, such that there is no $N \in \mathbb{N}$ for which

$$\Psi(n+N) \geq L(n), \quad n \in \mathbb{N}, \quad (3.28)$$

there is an entire function F , for which we cannot find a sequence of disjoint intervals $\{[r_n, r_n + \delta_n]\}$ satisfying

$$\delta_n < \frac{1}{L(n)^2}, \quad (3.29)$$

and such that

$$S(r, F) = o(T(r, F)),$$

outside the union $\bigcup_n [r_n, r_n + \delta_n]$.

For the function F in Theorem 3.3 there is a sequence of intervals $[t_n, t_n + \beta_n]$ such that for large n we have

$$\beta_n \geq \frac{1}{\Psi(n)} , \quad (3.30)$$

and

$$S(r, F) \geq 2T(r, F) \quad \text{for } r \geq r_0 \quad \text{in } \bigcup_n [t_n, t_n + \beta_n] . \quad (3.31)$$

Therefore if we could find a sequence of intervals $\{[r_n, r_n + \delta_n]\}$ as in the statement of Theorem 3.4, we should have for a certain $N \in \mathbb{N}$

$$\bigcup_{n > N} [t_n, t_n + \beta_n] \subset \bigcup_k [r_k, r_k + \delta_k] .$$

Since we are dealing with sequences of disjoint intervals, we should conclude

$$[t_n, t_n + \beta_n] \subset [r_{k_n}, r_{k_n} + \delta_{k_n}], \quad n > N . \quad (3.32)$$

By (3.1), (3.4) and the way that we constructed the intervals $[t_n, t_n + \beta_n]$ in Theorem 3.3, we have

$$t_{n+1} - t_n \rightarrow \infty . \quad (3.33)$$

We may replace any intervals of the sequence $\{[t_n, t_n + \beta_n]\}$ by subintervals, without destroying Theorem 3.4 and so we suppose from now on that (3.30) is replaced by

$$\beta_n = \frac{1}{\Psi(n)} \leq 1 .$$

In view of (3.29) and the fact that $L(n)$ increases, we also have

$$\delta_n \leq \frac{1}{L(1)} .$$

Further, omitting a finite number of intervals $[t_n, t_n + \beta_n]$,
we suppose from now on that

$$t_{n+1} - t_n > 1 + \frac{1}{L(1)} .$$

Thus each interval $[r_k, r_k + \delta_k]$ having length at most $1/L(1)$
cannot meet more than one of the intervals $[t_n, t_n + \beta_n]$ since

$$t_{n+1} - (t_n + \beta_n) > 1 + \frac{1}{L(1)} - \beta_n \geq \frac{1}{L(1)} .$$

Since the sequence $\{r_n\}$ is assumed increasing we deduce from
here that the sequence $\{k_n\}$ is strictly increasing.

Now using (3.29), we must have for n large

$$\frac{1}{\Psi(n)} = \beta_n \leq \delta_{k_n} \leq \frac{1}{L(k_n)^2} \leq \frac{1}{L(k_n)} \leq \frac{1}{L(n-N)} ,$$

since $k_n \geq n-N$. Hence there is $n_0 \in \mathbb{N}$ such that

$$\Psi(n) \geq L(n-N) , \quad n > \max(n_0, N)$$

$$\text{i.e. } \Psi(n+N) \geq L(n) , \quad n > n_0 .$$

But then we would have

$$\Psi(n+N+n_0) \geq L(n+n_0) \geq L(n) , \quad n \in \mathbb{N}$$

in contradiction with (3.28).

This completes the proof of Theorem 3.4.

CHAPTER FOUR

A NEW EXCEPTIONAL SET

1. Introduction

In the previous chapters we have been concerned with the exceptional set outside which the error term $S(r)$, which arises in the Lemma of the logarithmic derivative in Nevanlinna's second fundamental theorem, is small compared with the characteristic function.

We have outside this set

$$S(r) = o(T(r)) , \quad (4.1)$$

and as a consequence

$$(q-2)T(r) < (N(r, a_1) + N(r, a_2) + \dots + N(r, a_q))(1+o(1)) \quad (4.2)$$

for every q complex values $a_1, a_2, \dots, a_q$.

But (4.2) could happen without (4.1) being true.

In this chapter we will deal with the exceptional set outside which (4.2) holds. By what we have said above, this set is smaller than the one outside which (4.1) holds, i.e. the one which we have already studied.

Let us denote by E_1, E_2 sets outside which (4.1) or (4.2) hold respectively. Given E_1 , we can always choose E_2 so that $E_2 \subset E_1$.

2. Statement of Theorem 4.1

By Theorem 2.3 in Chapter Two, E_1 can be chosen to satisfy

$$\int_{E_1} r^\lambda dr < \infty, \quad \forall \lambda > 0 \quad (4.3)$$

and hence we can also choose E_2 , so that

$$\int_{E_2} r^\lambda dr < \infty, \quad \forall \lambda > 0. \quad (4.4)$$

In this chapter we prove that (4.4) cannot be replaced by any stronger condition of the same type, as we did in Chapter Three with (4.3)

Theorem 4.1. For any function $\Phi(r)$ such that

$$\frac{\Phi(r)}{r^\lambda} \rightarrow \infty, \quad \forall \lambda \geq 0, \quad (4.5)$$

there exists an entire function such that the corresponding exceptional set E_2 satisfies

$$\int_{E_2} \Phi(r) dr = \infty.$$

The technique is basically that of Hayman [4] but some quantitative estimates are needed.

3. Some preliminary results

3.1 In the proof of Theorem 4.1 we shall need the following results:

Lemma 4.1. Suppose that $a_1, a_2, \dots, a_{q-1}$ are distinct complex numbers, $1 > \delta > 0$

$$|a_\mu - a_\nu| \geq \delta, \quad |a_\mu| \leq \frac{1}{\delta}, \quad \nu \neq \mu, \quad \nu, \mu = 1, \dots, q-1, \quad (4.6)$$

and that F maps conformally the unit disc onto the universal covering surface over the Riemann sphere punctured at $a_1, \dots, a_{q-1}, \infty$ and that it satisfies

$$|F(0) - a_\mu| \geq \delta, \quad |F(0)| \leq \frac{1}{\delta}, \quad \mu = 1, \dots, q-1, \quad (4.7)$$

then

$$T(r, F) \geq C_1(q) \log \frac{1}{1-r}, \quad 1 - A_0(\frac{1}{2}\delta^2)^{C_2(q)} \leq r < 1, \quad (4.8)$$

where

$$C_1(q) = \frac{1}{2(q-2)}, \quad A_0 = \frac{1}{3.000} \quad \text{and} \quad C_2(q) = 2q+3.$$

The method of the proof is essentially due to F. Nevanlinna, see Nevanlinna [6], but we obtain precise quantitative values for the constants involved, whereas he only gets a qualitative result.

In the proof of Lemma 4.1 we shall make use of the magnitude $\alpha(\omega)$ defined on the ω -plane punctured at $a_1, a_2, \dots, a_{q-1}, \infty$ by

$$\alpha(\omega) = (1 - |z|^2) |F'(z)|, \quad \omega = F(z). \quad (4.9)$$

This expression is a one-valued function of ω and if we call

$$d(\omega) = \min_{1 \leq u \leq q-1} |\omega - a_\mu| ,$$

then $\alpha(\omega)$ satisfies

- (i) $\alpha(\omega) \geq d(\omega)$
- (ii) $\alpha(\omega) \leq 2d(\omega) \left(|\log d(\omega)| + \log \frac{1}{\delta} + \frac{\Gamma(\frac{1}{4})^4}{2\pi^2} \right) .$

The first follows since the Riemann surface contains the disc $\{\xi \mid |\xi - \omega| < d(\omega)\}$ and so we can apply Schwarz's Lemma to the inverse function of

$$G(s) = F\left(\frac{s+z}{1+zs}\right)$$

in this disc.

We have

$$|(G^{-1})'(\omega)| = \left| \frac{1}{G'(0)} \right| = \left| \frac{1}{F'(z)(1-|z|^2)} \right| \leq \frac{1}{d(\omega)} , \quad (4.10)$$

and from (4.9) and (4.10) we conclude (i).

The second follows from Landau's Theorem by considering the function

$$\Phi(s) = \frac{F\left(\frac{s+z}{1+zs}\right) - a_\mu}{a_\nu - a_\mu} \quad (4.11)$$

where a_μ is the nearest omitted value and a_ν same other omitted value.

According to Landau's Theorem as in [5]

$$|\Phi'(0)| \leq 2|\Phi(0)| \left\{ |\log|\Phi(0)|| + \frac{\Gamma(\frac{1}{4})^4}{2\pi^2} \right\} . \quad (4.12)$$

Now we have

$$|\Phi'(0)| = \frac{|F'(z)(1-|z|^2)|}{|a_v - a_\mu|} = \frac{\alpha(\omega)}{|a_v - a_\mu|} , \quad (4.13)$$

and

$$|\Phi(0)| = \frac{|F(z) - a_\mu|}{|a_v - a_\mu|} = \frac{|d(\omega)|}{|a_v - a_\mu|} . \quad (4.14)$$

And from (4.12), (4.13) and (4.14), we obtain

$$\begin{aligned} \alpha(\omega) &\leq 2|d(\omega)| \left(|\log|d(\omega)|| + \log \frac{1}{|a_v - a_\mu|} + \frac{\Gamma(\frac{1}{4})^4}{2\pi^2} \right) \\ &\leq 2|d(\omega)| \left\{ |\log|d(\omega)|| + \log \frac{1}{\delta} + \frac{\Gamma(\frac{1}{4})^4}{2\pi^2} \right\} , \text{ i.e. (ii) } . \end{aligned}$$

We now define

$$\begin{aligned} E_\mu &= \left\{ \omega \mid |\omega - a_\mu| < \frac{1}{2} \delta \right\} , \quad \mu = 1, 2, \dots, q-1 , \\ E_q &= \left\{ \omega \mid |\omega| > \frac{2}{\delta} \right\} , \quad E_{q+1} \text{ elsewhere} . \end{aligned} \quad (4.15)$$

Then if $F(z)$ is in E_μ we have by (i)

$$\log((1-|z|^2)|F'(z)|) \geq \log|F(z) - a_\mu| , \quad (4.16)$$

and since, for $F(z) \in E_\mu$, we have $|F(z) - a_\mu| < \frac{1}{2} \delta < 1$, we have

$$m(r, a_\mu) = \frac{1}{2\pi} \int_{E_\mu} \log \left| \frac{1}{F(re^{i\vartheta}) - a_\mu} \right| d\vartheta + \frac{1}{2\pi} \int_{E'_\mu} \log^+ \left| \frac{1}{F(re^{i\vartheta}) - a_\mu} \right| d\vartheta, \quad (4.17)$$

where $E'_\mu = \mathbb{C} \setminus E_\mu$ is the complementary set of E_μ and where the first integral is over the set of values of ϑ such that $F(re^{i\vartheta}) \in E_\mu$ and similarly the second integral is over the set of values of ϑ such that $F(re^{i\vartheta}) \in E'_\mu$.

From (4.17) we obtain

$$\begin{aligned} \frac{1}{2\pi} \int_{E_\mu} \log |F(re^{i\vartheta}) - a_\mu| d\vartheta &= -m(r, a_\mu) + \frac{1}{2\pi} \int_{E'_\mu} \log^+ \left| \frac{1}{F(re^{i\vartheta}) - a_\mu} \right| d\vartheta \\ &\geq -m(r, a_\mu), \end{aligned} \quad 1 \leq \mu \leq q-1. \quad (4.18)$$

For $\omega = F(z) \in E_q$ we have

$$(1 - |z|^2) |F'(z)| \geq \frac{1}{2} |F(z)|. \quad (4.19)$$

In fact, in this case

$$|F(z) - a_\mu| \geq |F(z)| - |a_\mu|, \quad (4.20)$$

on the other hand

$$|a_\mu| \leq \frac{1}{\delta} \leq \frac{1}{2} |F(z)|, \quad (4.21)$$

therefore from (4.20) and (4.21) we obtain

$$|F(z) - a_\mu| \geq |F(z)| - \frac{1}{2} |F(z)| = \frac{1}{2} |F(z)|, \quad \mu = 1, \dots, q-1$$

and so by (i)

$$(1-|z|^2)|F'(z)| = \alpha(\omega) \geq d(\omega) \geq \frac{1}{2} |F(z)| ,$$

which is (4.19).

From (4.19)

$$\frac{1}{2\pi} \int_{E_q} \log((1-r^2)|F'(re^{i\vartheta})|) d\vartheta \geq \frac{1}{2\pi} \int_{E_q} \log^+ |F(re^{i\vartheta})| d\vartheta - \log 2, \quad (4.22)$$

where the integral over E_q has the same meaning as above.

On the other hand

$$m(r, F) = \frac{1}{2\pi} \int_{E_q} + \frac{1}{2\pi} \int_{E'_q} \leq \frac{1}{2\pi} \int_{E_q} \log^+ |F(re^{i\vartheta})| d\vartheta + \log \frac{2}{\delta}, \quad (4.23)$$

so that we obtain from (4.22) and (4.23)

$$\frac{1}{2\pi} \int_{E_q} \log((1-r^2)|F'(re^{i\vartheta})|) d\vartheta \geq m(r, F) - \log \frac{4}{\delta} . \quad (4.24)$$

In E_{q+1} we have by (i), $(1-|z|^2)|F'(z)| \geq d(\omega) \geq \frac{1}{2} \delta$, so that

$$\frac{1}{2\pi} \int_{E_{q+1}} \log((1-r^2)|F'(re^{i\vartheta})|) d\vartheta \geq \log \frac{\delta}{2} . \quad (4.25)$$

Now using (4.16), (4.18), (4.24) and (4.25) and using the fact that $\log|F'(z)|$ is harmonic we obtain

$$\begin{aligned} \log \left\{ (1-r^2)|F'(0)| \right\} &= \frac{1}{2\pi} \sum_{\mu=1}^{q+1} \int_{E_\mu} \log \left\{ (1-r^2)|F'(re^{i\vartheta})| \right\} d\vartheta \\ &\geq m(r, F) - \sum_{\mu=1}^{q-1} m(r, a_\mu) - \log \left(\frac{8}{\delta^2} \right) . \end{aligned} \quad (4.26)$$

Since

$$m(r, a_\mu) \leq m(r, F) - \log |F(0) - a_\mu| + \log^+ |a_\mu| + \log 2 ,$$

see Hayman [2], page 5, we deduce from (4.7) and (4.26)

$$\begin{aligned} -(q-2)m(r, F) &\leq \log(1-r^2) + \log |F'(0)| + \sum_{1}^{q-1} \log^+ |a_\nu| + \\ &+ (q-1) \log 2 + \log\left(\frac{8}{\delta^2}\right) - (q-1) \log \delta , \end{aligned}$$

i.e.

$$\begin{aligned} (q-2)T(r, F) &\geq \log \frac{1}{1-r^2} - \log |F'(0)| - \sum_{1}^{q-1} \log^+ |a_\nu| \\ &- (q-1) \log\left(\frac{2}{\delta}\right) - \log\left(\frac{8}{\delta^2}\right) . \end{aligned} \quad (4.27)$$

On the other hand, we have by hypotheses for $\mu = 1, 2, \dots, q-1$

$$\delta \leq |F(0) - a_\mu| \leq |F(0)| + |a_\mu| \leq \frac{2}{\delta} ,$$

so that

$$\delta \leq d(F(0)) \leq \frac{2}{\delta} .$$

Thus (ii) yields

$$|F'(0)| \leq \frac{4}{\delta} \left(\log \frac{2}{\delta} + \log \frac{1}{\delta} + \frac{\Gamma(\frac{1}{4})^4}{2\pi^2} \right) \leq \frac{8}{\delta} \left(\log \frac{1}{\delta} + 3 \right) ,$$

i.e.

$$\begin{aligned} \log |F'(0)| &\leq \log \frac{1}{\delta} + \log 8 + \log^+ \log \frac{1}{\delta} + \log 3 + \log 2 . \\ &< 2 \log \frac{1}{\delta} + \log 48 . \end{aligned} \quad (4.28)$$

We also have

$$\sum_{1}^{q-1} \log^+ |a_{\mu}| \leq (q-1) \log \frac{1}{\delta} . \quad (4.29)$$

From (4.27), (4.28) and (4.29), we conclude

$$\begin{aligned} (q-2)T(r, F) &\geq \log \frac{1}{1-r^2} - 2 \log \frac{1}{\delta} - \log 48 - (q-1) \log \frac{1}{\delta} \\ &- (q-1) \log \left(\frac{2}{\delta} \right) - \log \left(\frac{8}{\delta^2} \right) \geq \log \frac{1}{1-r} - (2q+3) \log \frac{1}{\delta} \\ &- (q+3) \log 2 - \log 48 , \end{aligned}$$

and so

$$(q-2)T(r, F) \geq \frac{1}{2} \log \frac{1}{1-r}$$

if

$$\frac{1}{2} \log \frac{1}{1-r} > \log \{48 \cdot 2^{q+3} \cdot \delta^{-(2q+3)}\} ,$$

i.e. if

$$r > 1 - \left\{ \frac{1}{48} \cdot 2^{-(q+3)} \delta^{2q+3} \right\}^2 ,$$

so that we get

$$T(r, F) \geq \frac{1}{2(q-2)} \log \frac{1}{1-r} ,$$

if

$$r > 1 - \frac{1}{3.000} \left(\frac{1}{2} \delta^2 \right)^{2q+3}$$

which is (4.8).

3.2

Lemma 4.2. Let p be a positive integer and x a positive number. Then there exists a_1 such that $ex < a_1 < 10ex$ and a function

$$F(z) = a_1 z + a_{p+1} z^{p+1} + \dots \quad (4.30)$$

regular in $|z| < 1$ univalent in $|z| < \sqrt{2} - 1$, omitting the values
 $\mp ex, \infty$ and assuming the values $i, 0$ not more than once in $|z| < 1$.

Furthermore for $x \geq 2$ the function F satisfies

$$T(r, F) \geq \frac{1}{192p} \log \frac{1}{1-r}, \quad R_p \leq r < 1, \quad (4.31)$$

where

$$R_p = 1 - \frac{A_0}{p} (ex)^{-26p}$$

and A_0 is a positive absolute constant.

Let E_1 be the set $\{0, \pm ex, i, \infty\}$ and let R be the universal covering surface over the complement of E_1 . Let R_1, R_2 be the surfaces obtained by cutting R from ex to $+\infty$ along the real axis and R_3, R_4 those obtained by cutting R from $-ex$ to $-\infty$. Let R_5 be the plane cut from ex to ∞ and from $-ex$ to $-\infty$ along the real axis and finally let R_0 be obtained by joining R_1, R_2 to R_5 on the segment (ex, ∞) and R_3, R_4 to R_5 along the segment $(-\infty, -ex)$.

Thus R_0 is a Riemann surface containing none of the points $\pm ex, \infty$ in any sheet and containing points over 0 and i exactly once, namely in the sheet R_5 . Further R_0 is a simply connected Riemann surface and since it does not contain points over $\pm ex, \infty$, R_0 is hyperbolic.

Therefore there is a conformal map F_0 from the unit disc

$\{z \mid |z| < 1\}$ onto R_0

$$F_0(z) = b_1 z + b_2 z^2 + \dots$$

where $b_1 > 0$.

It is clear that F_0 assumes $i, 0$ precisely once and that it never assumes $\bar{e}x, \infty$. It follows that F_0 is subordinate to the function G mapping $\{z \mid |z| < 1\}$ onto the universal covering over the plane punctured at $\bar{e}x, \infty$ and satisfying the condition

$$G(0) = 0, \quad G'(0) > 0.$$

This function G maps the sheet R_5 onto a quadrilateral Q , in the unit disc, bounded by four quarter circles orthogonal to the circumference $\{z \mid |z| = 1\}$ joining the points $z = 1, i, -1, -i$ in the form $(1, i), (i, -1), (-1, -i)$ and $(-i, 1)$.

Q contains the disc $\{z \mid |z| < \sqrt{2} - 1\}$ and since $F_0(z)$ is subordinate to $G(z)$, $F_0(z)$ maps this disc onto a subset of the sheet R_5 , so that $F_0(z)$ is univalent in $\{z \mid |z| < \sqrt{2} - 1\}$.

We obtain from Koebe's Theorem that

$$ex > \frac{b_1(\sqrt{2} - 1)}{4}.$$

On the other hand, the inverse function $z = \Phi(\omega)$ maps the disc $\{\omega \mid |\omega| < ex\}$ into the disc $\{z \mid |z| < 1\}$, so we can apply Schwarz's Lemma and obtain

$$b_1^{-1} = \Phi'(0) < (ex)^{-1}.$$

Hence we have

$$ex < b_1 < \frac{4}{\sqrt{2}-1} ex < 10ex . \quad (4.32)$$

Thus we obtain a function with a development having the required properties of the function (4.30) in the case $p = 1$.

If $p > 1$ we proceed as follows. We assume first p odd and write $E_p = \{(ex)^p, 0, i^p, \infty\}$, so that $i^p \neq \pm 1$.

Then we define $F_p(z)$ as above with E_p instead of E_1 and set

$$F(z) = \left\{ F_p(z^p) \right\}^{1/p} = b_1^{\frac{1}{p}} \left(z + \left(\frac{b_2 z^{p+1}}{b_1 p} \right) + \dots \right) . \quad (4.33)$$

Since $F_p(z) \neq 0$ for $z \neq 0$, $F(z)$ is regular. Let Δ_0 be the part of $\{z \mid |z| < 1\}$ which corresponds to the sheet R_5 by $F_p(z)$. Then $F_p(z)$ is univalent, in Δ_0 . Let Δ_p be the set of all z such that z^p is in Δ_0 .

We see that $F(z)$ is univalent in Δ_p . For if not let z_1, z_2 be two points in Δ_p such that $F(z_1) = F(z_2)$.

Then z_1^p, z_2^p are two points of Δ_0 and $F_p(z_1^p) = F_p(z_2^p)$ and therefore $z_1^p = z_2^p$. We deduce from this that there is an integer k for which

$$z_2 = z_1 \exp\left(\frac{2\pi i k}{p}\right) .$$

This implies

$$F(z_2) = F(z_1) \exp\left(\frac{2\pi i k}{p}\right) ,$$

so that $z_2 = z_1$, i.e. we have proved that $F(z)$ is univalent in Δ_p .

On the other hand Δ_p contains the disc $\{z \mid |z| < (\sqrt{2} - 1)^{1/p}\}$ and therefore the disc $\{z \mid |z| < \sqrt{2} - 1\}$.

If $F(z) = i$ or 0 , then $F_p(z^p) = i^p$ or 0 and this is possible only for z^p in Δ_0 , i.e. z in Δ_p , where F is univalent so that $F_p(z)$ assumes $i, 0$ at most once in $\{z \mid |z| < 1\}$.

Since $(ex)^p < b_1 < 10(ex)^p$ by what we proved above, one obtains that $ex < b_1^{1/p} < (10)^{1/p} ex \leq 10ex$ and so $F(z)$ has again the required development (4.30).

In the case p even, we consider the set

$$E_p' = \left\{ \pm(ex)^p, 0, -i^{p+1}, \infty \right\}$$

instead of E_p and the function $i F_p(-iz)$ instead of $F_p(z)$.

It remains to prove (4.31). For this we use Lemma 4.1.

We consider again the case p odd.

The case p even follows with small modifications. According to that Lemma, any function F_5 which maps conformally the unit disc onto the universal covering surface over the Riemann sphere punctured at the points of $E_p = \left\{ 0, \pm(ex)^p, i^p, \infty \right\}$, such that

$$|F_5(0) - a| \geq \delta, \quad \text{for every } a \in E_p, \quad |F_5(0)| \leq \frac{1}{\delta}, \quad (4.34)$$

where $\delta \leq \min\{1, (ex)^p, (ex)^p - 1\}$ and $\frac{1}{\delta} \geq \{(ex)^p, 1\}$ i.e. such that (4.6) and (4.7) hold, satisfies

$$T(r, F_5) \geq \frac{1}{6} \log \frac{1}{1-r}, \quad 1 - A_0 \left(\frac{1}{2} \delta^2\right)^{13} \leq r < 1, \quad (4.35)$$

where A_0 is an absolute constant.

For $x \geq 2$ we can take $\delta = 1/(ex)^p$.

First we prove (4.31) for a translate F_{pa} of F_p where

$$F_{pa}(z) = F_p\left(\frac{z-a}{1-az}\right),$$

for some a such that $|a| < 1$.

We fix the branch of F_5 at $z = 0$, such that $F_5(0) = 1/(ex)^p$, so that for $x \geq 2$ (4.34) is satisfied and then consider a translate F_{pa} of F_p such that $F_{pa}(0) = 1/(ex)^p$. In this way, we get a well defined function

$$\omega(z) = F_{pa}^{-1} \circ F_5(z), \quad (4.36)$$

in a neighbourhood of $z = 0$, such that $\omega(0) = 0$.

It is clear, from the definition of F_{pa} and F_5 , that ω can be continued without limit in the disc and since this is a simply connected set, by the Monodromy Theorem we obtain a function from the whole unit disc into itself such that $\omega(0) = 0$.

From (4.36) we obtain

$$F_5(z) = F_{pa}(\omega(z)),$$

i.e. we have proved that F_5 is subordinate to F_{pa} . Hence F_{pa} satisfies also (4.35)

$$T(r, F_{pa}) \geq \frac{1}{6} \log \frac{1}{1-r}, \quad 1 - A_0 \left(\frac{1}{2}\right) (ex)^{-2p} \leq r < 1. \quad (4.37)$$

Next we prove from (4.37) that F_p satisfies (4.31). To do

this, we shall use the Ahlfors-Shimizu characteristic T_0 instead of the Nevanlinna characteristic T . T_0 and T differ only by a bounded term and therefore we obtain equivalent statements to (4.31) and (4.37). In fact we recall that for a meromorphic function G , T_0 is defined by

$$T_0(r, G) = \int_0^r \frac{A(t)}{t} dt ,$$

where $\pi A(t)$ is the area, with due regard to multiplicity, of the image on the Riemann sphere of $\{z \mid |z| \leq t\}$ by G , and that if G is regular then

$$T_0(r, G) = \frac{1}{2\pi} \int_0^{2\pi} \log \sqrt{1 + |G(re^{i\vartheta})|^2} d\vartheta - \log \sqrt{1 + |G(0)|^2} .$$

Now using the inequality

$$\log^+ x \leq \log \sqrt{1+x^2} \leq \log^+ x + \frac{1}{2} \log 2$$

and recalling that for G regular

$$T(r, G) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ |G(re^{i\vartheta})| d\vartheta ,$$

we obtain

$$T(r, G) - \log \sqrt{1 + |G(0)|^2} \leq T_0(r, G) \leq T(r, G) - \log \sqrt{1 + |G(0)|^2} + \frac{1}{2} \log 2 . \quad (4.38)$$

Therefore, since in our particular case $F_{pa}(0) = F_5(0) = \frac{1}{(ex)^p}$,

and $0 \leq \frac{1}{x^p} \leq \frac{1}{2}$ for $x \geq 1$ we obtain for $G = F_{pa}, F_5$,

$$T(r, G) + A \leq T_0(r, G) \leq T(r, G) + B , \quad (4.39)$$

where A and B are absolute constants.

Next we show that

$$\left| \frac{z-a}{1-\overline{a}z} \right| \geq 1 - \frac{2}{1-|a|} (1-r) , \quad |z| = r . \quad (4.40)$$

In fact, we have

$$\left| \frac{z-a}{1-\overline{a}z} \right| \geq \frac{r-|a|}{1-|a|r} ,$$

whence

$$1 - \left| \frac{z-a}{1-\overline{a}z} \right| \leq 1 - \frac{r-|a|}{1-|a|r} = \frac{(1-r)(1+|a|)}{1-|a|r} \leq \frac{2}{1-|a|} (1-r) ,$$

which yields (4.40).

The function

$$\omega = \frac{z-a}{1-\overline{a}z} ,$$

maps the disc $\{z \mid |z| < t\}$, $t > |a|$ conformally onto a disc containing the origin and whose boundary is the image of $\{z \mid |z| = t\}$. On the other hand by (4.40)

$$|\omega(z)| \geq 1 - \frac{2}{1-|a|} (1-t) , \quad |z| = t$$

i.e. we can conclude that that domain contains the disc

$$\left\{ \omega \mid |\omega| < 1 - \frac{2}{1-|a|} (1-t) \right\} .$$

Now since the correspondence between $\{z \mid |z| \leq t\}$ and the closure of that domain is one-to-one we deduce that for every solution in

$$\left\{ \omega \mid |\omega| \leq 1 - \frac{2}{1-|a|} (1-t) \right\} \text{ of } F_{pa}(\omega) = \zeta , \text{ there is at least}$$

one solution of $F_p(z) = \zeta$ in $\{z \mid |z| \leq t\}$ and different solutions of the first equation give rise to different solutions of the last equation.

We observe now that

$$A_{F_{pa}}(t) = \int_S n_{F_{pa}}(t, \zeta) d\mu(\zeta) \quad , \quad A_{F_p}(t) = \int_S n_{F_p}(t, \zeta) d\mu(\zeta) \quad ,$$

where S is the Riemann sphere, $d\mu$ is the normalised area element in S

and $n_{F_{pa}}(t)$, $n_{F_p}(t)$ are the number of roots of $F_{pa}(\omega) = \zeta$ and

$F_p(z) = \zeta$ in $|\omega| \leq t$, $|z| \leq t$ respectively.

From the previous consideration we conclude

$$A_{F_p}(t) \geq A_{F_{pa}}\left(1 - \frac{2}{1-|a|} (1-t)\right), \quad t > |a|. \quad (4.41)$$

On the other hand F_p is univalent in $\{z \mid |z| < \sqrt{2} - 1\}$ and $F_p(0) = 0$. Thus by Koebe's Theorem the disc $\{\omega \mid |\omega| < b_1 \sqrt{2}-1)/4\}$ is covered by the image of $\{z \mid |z| < \sqrt{2} - 1\}$ by F_p . Since $F_p(-a) = F_{pa}(0) = (ex)^{-p}$, we can find a in $\{z \mid |z| < \sqrt{2} - 1\}$ provided that

$$\frac{b_1(\sqrt{2}-1)}{4} > \frac{1}{(ex)^p} \quad .$$

By (4.32), $|b_1| > (ex) > 10(ex)^{-p}$ since $x \geq 2$, so that the above condition is satisfied.

Now using an inequality for univalent functions, see Hayman [1], page 4, we obtain

$$|F_p(-a)| \geq \frac{|b_1||a|}{(1+a/\sqrt{2}-1)^2} \geq \frac{|b_1||a|}{4} .$$

Thus, since $|b_1| \geq (ex)^p$

$$|a| \leq \frac{4|F_p(-a)|}{|b_1|} \leq \frac{4}{|b_1|(ex)^p} \leq \frac{4}{(ex)^{2p}} \leq \frac{1}{e^2} \leq \frac{1}{7} ,$$

since $x \geq 2$.

Next we note that

$$1 - \frac{2}{1-|a|} (1-t) \geq t^4 , \quad \frac{2}{3} \leq t \leq 1 . \quad (4.42)$$

In fact if $|a| < 1/7$, and $2/3 \leq t < 1$,

$$\begin{aligned} 1-t^4 - \frac{2}{1-|a|} (1-t) &\geq 1-t^4 - \frac{7}{3} (1-t) \\ &= (1-t) \{1+t + t^2 + t^3 - 7/3\} \\ &\geq (1-t) \left\{ 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} - \frac{7}{3} \right\} = \frac{2}{27} (1-t) > 0 . \end{aligned}$$

Now using (4.38), (4.41) and (4.42), and the fact that

$F_p(0) = 0$, we deduce

$$\begin{aligned} T(r, F_p) + \frac{1}{2} \log 2 &\geq T_0(r, F_p) = \int_0^r A_{F_p}(t) \frac{dt}{t} \\ &\geq \int_{2/3}^r A_{F_{pa}} \left\{ 1 - \frac{2}{1-|a|} (1-t) \right\} \frac{dt}{t} \geq \int_{2/3}^r A_{F_{pa}} (t^4) \frac{dt}{t} \\ &= \frac{1}{4} \int_{(2/3)^4}^r A_{F_{pa}}(s) \frac{ds}{s} \geq \frac{1}{4} \int_{1/5}^r A_{F_{pa}}(s) \frac{ds}{s} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{4} \left\{ T_0(r^4, F_{pa}) - T_0(1/5, F_{pa}) \right\} \\
 &\geq \frac{1}{4} \left\{ T(r^4, F_{pa}) - T(1/5, F_{pa}) - \log 2 \right\} .
 \end{aligned} \tag{4.43}$$

Next we obtain a lower bound for $T(r^4, F_{pa})$ and an upper bound for $T(1/5, F_{pa})$. For the former we use (4.37), i.e.

$$\begin{aligned}
 T(r^4, F_{pa}) &\geq \frac{1}{6} \log \frac{1}{1-r^4} \geq \frac{1}{6} \log \frac{1}{4(1-r)} \\
 &= \frac{1}{6} (\log \frac{1}{1-r} - \log 4) \geq \frac{1}{12} \log \frac{1}{1-r}
 \end{aligned} \tag{4.44}$$

if

$$r^4 \geq 1 - A_0 \left(\frac{1}{2} (ex)^{-2p} \right)^{13} \tag{4.45}$$

and in addition

$$\log \frac{1}{1-r} \geq 2 \log 4, \quad \text{i.e.} \quad r \geq 1 - e^{-16},$$

both conditions can be included in (4.45) after decreasing A_0 if necessary.

Next we try to obtain an upper bound for $T(1/5, F_{pa})$.

F_p is univalent in $\{z \mid |z| < r_0 = \sqrt{2} - 1\}$, so that using a result for univalent functions, (see Hayman [1], page 4), we get

$$|F_p(z)| < \frac{|b_1| r_0^2 |z|}{(r_0 - |z|)^2} \quad |z| < r_0. \tag{4.46}$$

Also for $|\xi| \leq 1/5$, $|a| \leq 1/7$, we have

$$\left| \frac{\xi + a}{1 + a\xi} \right| \leq \frac{|\xi| + |a|}{1 + |a||\xi|} \leq \frac{1/5 + 1/7}{1 + 1/35} = \frac{1}{3}.$$

Thus for $|\xi| \leq 1/5$

$$F_{pa}(\xi) = F_p\left(\frac{\xi+a}{1+a\xi}\right) = F_p(z) ,$$

where $|z| \leq 1/3$. Hence since $r_0 = \sqrt{2}-1 > \frac{2}{5}$, we conclude from (4.46)

$$\begin{aligned} |F_p(z)| &\leq \frac{|b_1||z|}{\left(1 - \frac{|z|}{r_0}\right)^2} \leq \frac{1/3|b_1|}{(1-5/6)^2} = 12|b_1| \\ &\leq 120(ex)^p. \end{aligned} \quad (4.47)$$

From (4.47), we obtain

$$T(1/5, F_{pa}) \leq \log\{120(ex)^p\} . \quad (4.48)$$

Thus if (4.45) holds, we have by (4.43), (4.44) and (4.48)

$$\begin{aligned} T(r, F_p) &\geq \frac{1}{4} \left\{ \frac{1}{12} \log \frac{1}{1-r} - \log\{120(ex)^p\} - \log 2 \right\} \\ &\geq \frac{1}{96} \log \frac{1}{1-r} , \end{aligned} \quad (4.49)$$

if

$$\frac{1}{24} \log \frac{1}{1-r} \geq \log\{120(ex)^p\} + \log 2$$

i.e. if

$$1 > r \geq 1 - \{240(ex)^p\}^{-24} . \quad (4.50)$$

Again we can put together (4.45) and (4.50) in

$$1 > r^4 \geq 1 - A_0(ex)^{-26p} \quad (4.51)$$

with a new A_0 .

Using (4.33) and (4.49), we deduce

$$T(r, F) = \frac{1}{p} T(r^p, F_p) \geq \frac{1}{96p} \log \frac{1}{1-r^p} , \quad (4.52)$$

if

$$1 > r^{4p} \geq 1 - A_0 (\text{ex})^{-26p} . \quad (4.53)$$

This yields

$$\begin{aligned} T(r, F) &\geq \frac{1}{96p} \log \frac{1}{p(1-r)} = \frac{1}{96p} (\log \frac{1}{1-r} - \log p) \\ &\geq \frac{1}{192p} \log \frac{1}{1-r} , \end{aligned} \quad (4.54)$$

if in addition

$$\log \frac{1}{1-r} \geq 2 \log p, \quad \text{i.e. } r \geq 1 - \frac{1}{p^2} . \quad (4.55)$$

Using the fact that

$$4p(1-r) \geq 1-r^{4p} ,$$

we can put (4.53) and (4.55) together in

$$1 > r \geq 1 - \frac{A_0}{p} (\text{ex})^{-26p} \quad (4.56)$$

with A_0 absolute constant.

So we conclude that (4.54) holds in the range (4.56) which is (4.31). This completes the proof of Lemma 4.2.

3.3

Lemma 4.3. Suppose that $a_1, \dots, a_p$ are preassigned complex numbers, not all zero, and write $M = \sum_{v=1}^p |a_v|$ $\mu > eM$. Then the function

$$\omega(z) = \frac{\sum_{v=1}^p a_v z^v + \mu z^{2p}}{\mu + \sum_{v=1}^p \bar{a}_v z^{2p-v}} \quad , \quad (4.57)$$

has precisely $2p$ zeros and no poles in the disc

$$\{z \mid |z| \leq \exp(-\frac{1}{2p})\} \quad . \quad (4.58)$$

Furthermore the function ω satisfies the following inequality

$$1 - 16p^2(1 - |z|) < |\omega(z)| < 1 - \frac{1}{8p}(1 - |z|) \quad . \quad (4.59)$$

One can check that $|\omega(z)| = 1$ for $|z| = 1$ and by Rouché's Theorem $\omega(z)$ has $2p$ zeros and no poles in $\{z \mid |z| < 1\}$. Next we show that more exactly these $2p$ zeros are in the smaller disc (4.58).

We write $r_p = \exp\{-1/2p\}$. Then for $r_p < |z| < 1$, we have

$$\left| \sum_{v=1}^p a_v z^v \right| \leq M < \frac{\mu}{e} = \mu r_p^{2p} < |\mu z^{2p}| \quad .$$

Thus $\omega(z) \neq 0$ for $r_p < |z| < 1$, and so $|z_v| \leq r_p$, $1 \leq v \leq 2p$.

We consider now the function

$$\pi(z) = \prod_{v=1}^{2p} \frac{z - z_v}{1 - \bar{z}_v z} \quad .$$

Then $\Phi(z) = \omega(z)/\pi(z)$ is regular and not zero in $\{z \mid |z| \leq 1\}$ and $|\Phi(z)| = 1$ on $|z| = 1$. Thus $|\Phi(z)| \equiv 1$ and

$$\omega(z) = e^{i\vartheta_0} \pi(z) = e^{i\vartheta_0} \prod_{v=1}^{2p} \frac{z - z_v}{1 - \bar{z}_v z}, \quad (4.60)$$

where ϑ_0 is a constant such that $0 \leq \vartheta_0 \leq 2\pi$.

Using (4.58) we deduce that for $|z| = r > r_p$, we have

$$\left(\frac{r - r_p}{1 - r_p r} \right)^{2p} \leq |\pi(z)| = |\omega(z)| \leq \left(\frac{r + r_p}{1 + r_p r} \right)^{2p}. \quad (4.61)$$

We write

$$x = \frac{r - r_p}{1 - r_p r}, \quad y = \frac{r + r_p}{1 + r_p r},$$

$$\begin{aligned} 1 - x^{2p} &= (1 - x)(1 + x + \dots + x^{2p-1}) < 2p(1 - x) \\ &= 2p \frac{(1 - r)(1 + r_p)}{1 - r_p r} < \frac{4p}{1 - r_p} (1 - r). \end{aligned} \quad (4.62)$$

Also if $t = 1/2p$, so that $0 < t \leq \frac{1}{2}$,

$$1 - e^{-t} = te^{-\tau}, \quad 0 < \tau < \frac{1}{2},$$

so that

$$1 - e^{-t} > te^{-\frac{1}{2}} > \frac{1}{2}t, \quad r_p = e^{-t} < 1 - \frac{1}{2}t = 1 - \frac{1}{4p}.$$

Thus

$$1 - x^{2p} < \frac{4p}{1 - r_p} (1 - r) < 16p^2 (1 - r). \quad (4.63)$$

Now the left hand inequality of (4.59) follows from (4.61) and

(4.63). To obtain the right hand inequality we must have an upper bound for y^{2p} . We have

$$1-y^{2p} \geq 1-y = \frac{(1-r_p)(1-r)}{1+r_p r} \geq \frac{1}{2} (1-r_p)(1-r) > \frac{1-r}{8p}.$$

This yields the right hand inequality in (4.59).

3.4

Lemma 4.4. Suppose given complex numbers, $a_1, a_2, \dots, a_p$, not all zero and write $M = \sum_{v=1}^p |a_v|$. Then there exists $F_p(z)$ regular in $|z| < 1$, assuming i and 0 no more than $2p$ times and with a power series development

$$F_p(z) = a_1 z + a_2 z^2 + \dots + a_p z^p + O(z^{p+1}) \quad , \quad (4.64)$$

near $z = 0$. Further

$$|F_p(z)| < 40eM \quad \text{in} \quad |z| < \frac{\sqrt{2}-1}{2} \quad , \quad (4.65)$$

and if $M \geq 2$ we have

$$T(r, F_p) \geq \frac{1}{8.000p^3} \log \frac{1}{1-r} \quad , \quad R'_p \leq r < 1 \quad , \quad (4.66)$$

where
$$R'_p = 1 - \frac{A_0}{p^3} (16eM)^{-1.000p^4}$$

Suppose that $\mu > eM$ and consider the function

$$\omega(z) = \frac{\sum_{v=1}^p a_v z^v + \mu z^{2p}}{\mu + \sum_{v=1}^p \bar{a}_v z^{2p-v}}$$

as in Lemma 4.3.

Let $F(z)$ be the function whose existence is asserted in Lemma 4.2 with $a_1 = \mu$ and $x = M$. Then $eM < \mu < 10eM$. We now write

$$F_p(z) = F\{\omega(z)\} . \quad (4.67)$$

It is easy to see that

$$\omega(z) = \mu^{-1} \sum_{v=1}^p a_v z^v + O(z^{p+1}) ,$$

and so

$$F_p(z) = \mu \omega(z) + O(z^{p+1}) ,$$

which yields (4.64).

The equation $\omega(z) = \zeta$ has precisely $2p$ roots in $\{z \mid |z| < 1\}$ for any ζ in $\{\zeta \mid |\zeta| < 1\}$, and since the equations

$$F(z) = 0 \quad \text{and} \quad F(z) = i$$

have at most one root in the unit disc, we deduce that the equations

$$F_p(z) = 0 \quad \text{and} \quad F_p(z) = i$$

have at most $2p$ roots in $\{z \mid |z| < 1\}$.

Since $F(z)$ is univalent in $\{z \mid |z| < r_0 = \sqrt{2} - 1\}$, it follows from a classical result for univalent functions that

$$|F(z)| < \frac{\mu r_0^2 |z|}{(r_0 - |z|)^2}, \quad |z| < r_0.$$

On the other hand, by Schwarz's Lemma, $|\omega(z)| \leq |z|$ for $|z| < 1$ and so for $|z| = \frac{r_0}{2}$ we obtain

$$|F_p(z)| \leq \frac{\mu r_0^2 |\omega(z)|}{(r_0 - |\omega(z)|)^2} \leq 4\mu |\omega(z)| \leq 40eM |z| < 40eM,$$

which is (4.65).

To prove (4.66), we shall make use of (4.31) and the definition of F_p in (4.67). We shall also use the Ahlfors-Shimizu characteristic T_0 instead of the Nevanlinna characteristic T as we did in the proof of Lemma 4.2.

We first show that

$$A_{F_p}(t) \geq A_{F(1-16p^2(1-t))}, \quad t \geq r_p = \exp\{-1/2p\}. \quad (4.68)$$

This is a consequence of the definition of F_p in (4.67) and the fact that the number of roots in $\{z \mid |z| \leq t\}$ of any equation of the form

$$F_p(z) = \zeta, \quad (4.69)$$

is not less than the number of the roots in $\{\omega \mid |\omega| \leq 1-16p^2(1-t)\}$ of the corresponding equation

$$F(\omega) = \zeta. \quad (4.70)$$

In fact let ω_0 be a root of (4.70) in $\{\omega \mid |\omega| \leq 1-16p^2(1-t)\}$.

Then we assert that the equation

$$\omega(z) = \omega_0 \quad (4.71)$$

must have at least one solution z_0 in $\{z \mid |z| \leq t\}$.

To prove this, we use the left hand inequality of (4.59).

The image by ω of $\{z \mid |z| < t\}$ is a domain $D(t)$ contained in $\{\omega \mid |\omega| < 1\}$, containing the origin since $\omega(z)$ has $2p$ zeros in the smaller disc $\{z \mid |z| < \exp(-1/2p)\}$, and whose boundary is contained in the image of $\{z \mid |z| = t\}$.

By (4.59) we have

$$|\omega(z)| > 1-16p^2(1-t), \quad |z| = t, \quad t > \exp\{-1/2p\}, \quad (4.72)$$

so that we can conclude that $D(t)$ contains $\{\omega \mid |\omega| \leq 1-16p^2(1-t)\}$,

i.e. we have proved that the equation (4.69) admits at least one solution in $\{z \mid |z| \leq t\}$, say z_0 .

Now

$$F_p(z_0) = F(\omega(z_0)) = F(\omega_0) = \zeta,$$

i.e. z_0 is a solution of (4.69).

Hence for every solution of (4.70) in $\{\omega \mid |\omega| \leq 1-16p^2(1-t)\}$, we have at least one solution of (4.69) in $\{z \mid |z| \leq t\}$ and different solutions of (4.70) yield, in this way, different solutions of (4.69).

We now recall that (see Hayman [2], page 13)

$$A_{F_p}(t) = \int_S n_{F_p}(t, \zeta) d\mu(\zeta), \quad A_F(t) = \int_S n_F(t, \zeta) d\mu(\zeta)$$

where S is the Riemann sphere, $d\mu$ is the normalised area element in S and $n_{F_p}(t)$, $n_F(t)$ are the number of roots of (4.69) and (4.70) respectively and deduce (4.68).

To prove (4.66), we shall make use of the following inequality

$$3n(1-t) > 1-t^{3n} > n(1-t), \quad \frac{1}{5} \leq t^{3n} < 1, \quad n \geq 1. \quad (4.73)$$

To prove this inequality we note that the quotient

$$\frac{1-t^{3n}}{1-t} = 1 + t + t^2 + \dots + t^{3n-1}$$

increases with t , so that it is enough to consider the case $t^{3n} = 1/5$ for the right hand inequality in (4.73). The left hand inequality is obvious.

We write

$$h = \frac{1}{n}, \quad a = \frac{1}{3} \log 5,$$

and note that

$$\frac{1-t^{3n}}{1-t} = \frac{5}{4} (1-e^{-ah}) < \frac{5ah}{4} = \frac{5 \log 5}{12n} < \frac{1}{n},$$

which proves (4.73).

We apply (4.73) with $n = 16p^2$. We shall also use (4.68).

Let $r' = 5^{-1/3n}$ and observe that $r' \geq r_p$ in (4.68), therefore for $r > r'$, we obtain

$$\begin{aligned}
 T_0(r, F_p) - T_0(r', F_p) &= \int_{r'}^r A_{F_p}(t) \frac{dt}{t} \geq \int_{r'}^r A_F\{1-n(1-t)\} \frac{dt}{t} \\
 &\geq \int_{r'}^r A_F(t^{3n}) \frac{dt}{t} = \frac{1}{3n} \int_{r'^{3n}}^r A_F(s) \frac{ds}{s} \\
 &= \frac{1}{3n} \{T_0(r^{3n}, F) - T_0(\frac{1}{5}, F)\} .
 \end{aligned} \tag{4.74}$$

To obtain an upper bound for $T_0(\frac{1}{5}, F)$ we use the fact that $F(z)$ is univalent in $|z| < r_0$, where $r_0 = \sqrt{2} - 1 > 2/5$. Thus for $|z| = 1/5 \leq 1/2 r_0$, we have

$$|F(z)| \leq \frac{|F'(0)| |z| r_0^2}{(r_0 - |z|)^2} \leq 4|z| |F'(0)| \leq \frac{4}{5} \mu < 8eM .$$

Hence

$$T_0(\frac{1}{5}, F) \leq T(\frac{1}{5}, F) + \frac{1}{2} \log 2 \leq \log(8eM) + \frac{1}{2} \log 2 . \tag{4.75}$$

By (4.31) in Lemma 4.2 and (4.71) we have

$$T_0(r^{3n}, F) \geq \frac{1}{192p} \log \frac{1}{1-r^{3n}} \geq \frac{1}{192p} \left\{ \log \frac{1}{1-r} - \log(3n) \right\} , \tag{4.76}$$

if

$$r^{3n} \geq 1 - \frac{A_0}{p} (eM)^{-26p} . \tag{4.77}$$

Thus by (4.74), (4.75) and (4.76) we have

$$\begin{aligned}
 T(r, F_p) &\geq T_0(r, F_p) - \frac{1}{2} \log 2 \\
 &\geq \frac{1}{3n} \left\{ \frac{1}{192p} \left\{ \log \frac{1}{1-r} - \log 3n \right\} - \log(8eM) - \frac{1}{2} \log 2 \right\} \\
 &\quad - \frac{1}{2} \log 2 ,
 \end{aligned} \tag{4.78}$$

if (4.77) holds.

From (4.78) we obtain

$$T(r, F_p) \geq \frac{1}{8000p^3} \log \frac{1}{1-r} \quad , \quad (4.79)$$

if (4.77) holds and if addition

$$r \geq 1 - A(16eM)^{-1000p^4} \quad . \quad (4.80)$$

Using again the inequality

$$1-r^{3n} \leq 3n(1-r) = 48p^2(1-r) \quad , \quad (4.81)$$

we conclude that (4.77) will be satisfied if

$$r \geq 1 - \frac{A_0}{p^3} (eM)^{-26p} \quad . \quad (4.82)$$

Now (4.80) and (4.82) can be included in

$$r \geq 1 - \frac{A_0}{p^3} (16eM)^{-1000p^4} \quad , \quad (4.83)$$

after decreasing A_0 if necessary.

(4.79) and (4.83) yield (4.66) and the proof of Lemma 4.4 is complete.

4. Proof of Theorem 4.1

We prove Theorem 4.1 by showing that given an arbitrary function $\Phi(r)$ satisfying (4.5), there is an integral function f such that

$$N(r, 0, f) + N(r, i, f) + N(r, \infty, f) < \frac{1}{2} T(r, f) \quad , \quad (4.84)$$

$$\text{for } \rho_k^{-\lambda(\rho_k)} \leq r \leq \rho_k^{-\gamma_k} \quad , \quad k \geq k_0 \quad . \quad (4.85)$$

Here

$$\lambda(r) = \frac{\log \Phi(\frac{1}{2}r)}{\log 2r} \quad , \quad (4.86)$$

which tends to infinity, as r tends to infinity, by (4.5), $\{\rho_k\}$ is a sequence which grows to infinity very quickly and $\{\gamma_k\}$ is a sequence decreasing to zero in such a way that

$$\gamma_k \rho_k^{\lambda(\rho_k)} \rightarrow 0 \quad \text{as} \quad k \rightarrow \infty \quad . \quad (4.87)$$

Thus increasing k_0 if necessary we ensure that all the intervals (4.85) belong to E_2 .

Just as in the proof of Theorem 3.2 we assume without loss of generality that $\lambda(r)$ increases with r . We shall also assume $\rho_k \geq \rho_0 = 2$ and $\lambda(\rho_0) \geq 0$, i.e. $\Phi(1) \geq 1$. Then

$$\rho_k^{-\lambda(\rho_k)} \leq 1, \quad \text{so} \quad \rho_k^{-\lambda(\rho_k)} \geq \frac{1}{2} \rho_k \quad , \quad (4.88)$$

Then using (4.86), (4.87) and (4.88) we obtain

$$\begin{aligned} \int_{E_2} \Phi(r) dr &= \int_{E_2} (4r)^{\lambda(2r)} dr \geq \sum_{k_0} \rho_k^{\lambda(\rho_k)} \cdot \left\{ \rho_k^{-\gamma_k - (\rho_k - \rho_k)^{-\lambda(\rho_k)}} \right\} \\ &= \sum_{k_0} \rho_k^{\lambda(\rho_k)} \cdot \rho_k^{-\lambda(\rho_k)} (1 - \gamma_k \rho_k)^{\lambda(\rho_k)} \geq (1 + o(1)) \sum_{k_0} 1 = \infty . \end{aligned}$$

Next we proceed to construct the function f and the sequence $\{\rho_k\}$. We note that as soon as $\{\rho_k\}$ has been defined, we can take

$$\gamma_k = \rho_k^{-(\lambda(\rho_k)+1)} ,$$

and since ρ_k tends to infinity (4.87) holds.

We define f by

$$f(z) = \sum_{n=1}^{\infty} b_n z^n , \quad (4.89)$$

and proceed to construct its coefficients b_n successively.

We set $b_1 = 1$. Let now $\{p_k\}$ be a strictly increasing sequence of positive integers such that $p_1 = 1$. We assume that we have already defined b_n for $n \leq p_k$ and proceed to construct b_n for $p_k < n \leq p_{k+1}$.

To do this, we shall inductively define the increasing sequence of positive numbers $\{\rho_k\}$ with $\rho_0 = 2$ and tending rapidly to infinity. Let us assume that ρ_k has already been chosen and let $F_k(z)$ be the function defined in Lemma 4.4 with $p = p_k$ and for $1 \leq n \leq p_k$

$$a_n = b_n \rho_k^n . \quad (4.90)$$

Then if a_n are the coefficients of $F_k(z)$ for all n , we define b_n by (4.90) for $p_k < n \leq p_{k+1}$.

We will prove that $\{\rho_k\}$ and $\{p_k\}$ can be successively constructed so that $f(z)$ satisfies (4.84).

First of all we show that, if ρ_{k-1}, p_k have already been chosen, we can take ρ_k so large that

$$|b_n| < (2\rho_{k-1})^{-n}, \quad p_k < n \leq p_{k+1} \quad . \quad (4.91)$$

To do this, we observe from (4.65) in Lemma 4.4 and Cauchy's inequality that

$$|a_n| < 40e \left(\frac{\sqrt{2}-1}{2} \right)^{-n} M, \quad p_k < n \leq p_{k+1} \quad , \quad (4.92)$$

where

$$M = \sum_{v=1}^{p_k} |b_v| \rho_k^v < \rho_k^{p_k} \sum_{v=1}^{p_k} |b_v| \quad ,$$

which is bigger than $\rho_1 \geq 2$.

Writing

$$A_0 = \frac{\sqrt{2}-1}{2} \quad , \quad B_k = \sum_{v=1}^{p_k} |b_v| \quad ,$$

we obtain from (4.90) and (4.92) for $p_k < n \leq p_{k+1}$

$$|b_n| \leq 40e \cdot \rho_k^{p_k - n} A_0^{-n} B_k \quad .$$

Therefore (4.91) will hold if

$$\rho_k^{n-p_k} > 40.e. \left(\frac{2\rho_{k-1}}{A_0} \right)^n B_k ,$$

i.e.

$$\rho_k > (40.e.B_k)^{\frac{1}{n-p_k}} \left(\frac{2\rho_{k-1}}{A_0} \right)^{\frac{n}{n-p_k}} ,$$

and this will be satisfied for all $n > p_k$ if

$$\rho_k > 40.e.B_k \left(\frac{2\rho_{k-1}}{A_0} \right)^{p_k+1} , \quad (4.93)$$

since $B_k \geq |b_1| = 1$. We choose ρ_k to satisfy (4.93) and therefore (4.91) will also hold. From (4.91) and the fact that ρ_k tends to infinity one deduces that $f(z)$ given by (4.89) is an integral function.

Next we show that all the roots of the equations

$$F_k\left(\frac{z}{\rho_k}\right) = 0 \quad \text{and} \quad F_k\left(\frac{z}{\rho_k}\right) = i , \quad (4.94)$$

are in $\{z \mid |z| < \rho_k^{-\lambda(\rho_k)}\}$.

In fact, let us recall that $F_k(z) = F\{\omega(z)\}$, where

$$\omega(z) = \frac{\sum_{v=1}^{p_k} a_v z^v + \mu z^{2p_k}}{\mu + \sum_{v=1}^{p_k} \bar{a}_v z^{2p_k-v}} ,$$

where $eM < \mu < 10eM$ and $M = \sum_{v=1}^{p_k} |a_v| = \sum_{v=1}^{p_k} |b_v| \rho_k^v$.

The equations

$$F(\omega) = 0 \quad \text{and} \quad F(\omega) = i \quad ,$$

have at most one root in $\{\omega \mid |\omega| < 1\}$. We denote these roots if any by ω_0 and ω_1 respectively.

The roots of (4.94) are the same as those of

$$\omega\left(\frac{z}{\rho_k}\right) = \omega_0 \quad , \quad \text{and} \quad \omega\left(\frac{z}{\rho_k}\right) = \omega_1 \quad . \quad (4.95)$$

Therefore we can deal with (4.95) instead of (4.94).

The equation $\omega(z) = \zeta$ has precisely $2p_k$ roots for $|\zeta| < 1$, in particular the equations $\omega(z) = \omega_0$ and $\omega(z) = \omega_1$ have in total $4p_k$ roots. Let us call $s = \max\{|\omega_0|, |\omega_1|\}$. Then $s < 1$ and by (4.59), all the roots of the two equations above must satisfy

$$1 - 16p_k^2(1 - |z|) < s \quad , \quad \text{i.e.} \quad (4.96)$$

$$|z| < 1 - \frac{1-s}{16p_k^2} \quad .$$

We now choose ρ_k large enough to satisfy

$$\rho_k > \frac{16p_k^2}{1-s} \quad , \quad (4.97)$$

in addition to (4.93). Then

$$1 - \frac{1-s}{16p_k^2} < 1 - \frac{1}{\rho_k} < \frac{\rho_k^{-\lambda(\rho_k)} - \rho_k^{-\rho_k}}{\rho_k} \quad .$$

Thus all the roots of (4.95) are in $\{z \mid |z| < \rho_k^{-\lambda(\rho_k)}\}$.

Therefore, there is an ϵ_k such that $0 < \epsilon_k < 1$ and

$$\left| F_k\left(\frac{z}{\rho_k}\right) \right| > \epsilon_k, \quad \text{and} \quad \left| F_k\left(\frac{z}{\rho_k}\right) - i \right| > \epsilon_k, \quad (4.98)$$

in $\{\rho_k^{-\lambda(\rho_k)} \leq |z| \leq \rho_k^{-\gamma_k}\}$, since this set is compact.

Finally, we show that if p_{k+1} , which has not been determined yet, is chosen suitably, then (4.84) will hold.

We write

$$F_k\left(\frac{z}{\rho_k}\right) = \sum_{n=1}^{\infty} B_n z^n$$

and note that the series is absolutely uniformly convergent in

$\{z \mid \rho_k^{-\lambda(\rho_k)} \leq |z| \leq \rho_k^{-\gamma_k}\}$, so that we may choose p_{k+1} so large that

$$\sum_{n=p_{k+1}}^{\infty} |B_n| |z|^n < \frac{1}{2} \epsilon_k$$

in this set.

We also have by (4.91)

$$\sum_{n=p_{k+1}+1}^{\infty} |b_n| |z|^n \leq \sum_{n=p_{k+1}+1}^{\infty} 2^{-n} = 2^{-p_{k+1}},$$

for the same values of z . This is smaller than $\frac{1}{2} \epsilon_k$ if p_{k+1} is large enough. Thus we can find p_{k+1} so large that we get

$$\left| f(z) - F_k\left(\frac{z}{\rho_k}\right) \right| = \left| \sum_{p_{k+1}+1}^{\infty} (b_n - B_n) z^n \right| < \epsilon_k, \quad (4.99)$$

$$\text{in } \{z \mid \rho_k^{-\lambda(\rho_k)} \leq |z| \leq \rho_k^{-\gamma_k}\}.$$

It follows from (4.98), (4.99) and Rouché's Theorem that the equations

$$f(z) = 0 \quad \text{and} \quad F_k\left(\frac{z}{\rho_k}\right) = 0$$

have equally many roots in $\{z \mid |z| \leq r\}$, for every r such that

$$\rho_k^{-\lambda(\rho_k)} \leq r \leq \rho_k^{-\gamma_k},$$

and the same is true for

$$f(z) = i \quad \text{and} \quad F_k\left(\frac{z}{\rho_k}\right) = i.$$

By Lemma 4.4, the number of roots of all these equations is at most $2p_k$.

We note now that (4.91) implies in particular that $|b_n| \leq 1$ for all n . Thus for $|z| = \rho < \frac{1}{2}$, we have

$$|f(z)| \leq \sum_{n=1}^{\infty} \rho^n \leq \frac{\rho}{1-\rho} \leq 2\rho < 1.$$

Also for such ρ we have

$$|f(z)| \geq \rho - \sum_2^{\infty} \rho^v = \rho - \frac{\rho^2}{1-\rho} = \frac{\rho-2\rho^2}{1-\rho} > 0 .$$

Therefore we can choose a fix number δ , $0 < \delta < \frac{1}{2}$ such that the equations $f(z) = 0$ and $f(z) = i$ have no roots different from $z=0$ in $\{z \mid |z| < \delta\}$.

We conclude that for $\rho_k - \rho_k^{-\lambda(\rho_k)} \leq r \leq \rho_k - \gamma_k$ we have

$$n(t, i) = 0, \quad t < \delta, \quad \text{and} \quad n(t, i) \leq 2p_k, \quad t \leq r$$

so that

$$N(r, i) = \int_0^r \frac{n(t, i)}{t} dt \leq 2p_k \log \frac{r}{\delta} . \quad (4.100)$$

Since $f(z)$ has a simple zero at the origin and no other zeros in $\{z \mid 0 < |z| < \delta\}$, we obtain similarly

$$N(r, 0) = \int_{\delta}^r \frac{n(t, 0)}{t} dt + \log \delta \leq 2p_k \log \frac{r}{\delta} . \quad (4.101)$$

Now we deduce from Lemma 4.4 that

$$T(r, F_k(\frac{z}{\rho_k})) > \frac{1}{8.000p_k^3} \log \frac{1}{1-r/\rho_k} , \quad (4.102)$$

in

$$\rho_k - \rho_k^{-\lambda(\rho_k)} \leq r \leq \rho_k - \gamma_k ,$$

provided that

$$\frac{\rho_k - \rho_k^{-\lambda(\rho_k)}}{\rho_k} = 1 - \rho_k^{-(\lambda(\rho_k)+1)} \geq 1 - \frac{A_0}{3} (16eM)^{-1.000\rho_k^4}, \quad (4.103)$$

where M_k is given by

$$M_k = \sum_{v=1}^{p_k} |a_v|. \quad (4.104)$$

Recalling (4.90), we obtain from (4.104)

$$M_k = \sum_{v=1}^{p_k} |a_v| \leq \rho_k^{p_k} \sum_{v=1}^{p_k} |b_v| \leq c \rho_k^{p_k}, \quad (4.105)$$

since by (4.91), $\sum_{v=1}^{\infty} |b_v|$ converges.

Therefore (4.103) is satisfied provided that

$$\rho_k^{\lambda(\rho_k)+1} \geq \frac{A_0}{3} (50\rho_k^{c p_k})^{1.200\rho_k^4},$$

and since $\lambda(\rho)$ tends to infinity with ρ , this condition is satisfied if ρ_k is chosen sufficiently large.

Thus we may assume that (4.102) holds in the range

$$\rho_k - \rho_k^{-\lambda(\rho_k)} \leq r \leq \rho_k^{-\gamma_k}.$$

Finally we impose on ρ_k the condition

$$\frac{1}{8.000\rho_k^3} (\lambda(\rho_k)+1) \log \rho_k - \log 2 > 8p_k \log \frac{\rho_k}{\delta}. \quad (4.106)$$

Once we have chosen ρ_k , we choose p_{k+1} so large that (4.99) holds with $\epsilon_k = 1$. Then the inductive definition of the sequences ρ_k, p_k is complete and $f(z)$ is defined.

Now (4.102) and (4.106) yield

$$\begin{aligned} T(r, f(z)) &\geq T \left\{ r, F_k \left(\frac{z}{\rho_k} \right) \right\} - \log 2 \geq \frac{1}{8.000 p_k^3} (\lambda(\rho_k) + 1) \log \rho_k \\ &\quad - \log 2 > 8 p_k \log \frac{\rho_k}{\delta}, \quad \rho_k^{-\lambda(\rho_k)} \leq r \leq \rho_k^{-\gamma_k}. \end{aligned} \quad (4.107)$$

Thus from (4.100), (4.101) and (4.107) we obtain

$$N(r, 0) + N(r, i) + N(r, \infty) \leq \frac{1}{2} T(r, f), \quad \text{in}$$

$$\rho_k^{-\lambda(\rho_k)} \leq r \leq \rho_k^{-\gamma_k}.$$

This proves (4.84) and completes the proof of Theorem 4.1.

B I B L I O G R A P H Y

- [1] HAYMAN, W.K. Multivalent Functions. Cambridge University Press 1958.
- [2] HAYMAN, W.K. Meromorphic Functions. Oxford Clarendon Press 1964.
- [3] HAYMAN, W.K. Die Nevanlinna-Charakteristik von meromorphen Funktionen und ihren Integralen. Festband zum 70. Geburtstag von Rolf Nevanlinna. (Springer-Verlag, 1966, 16-20).
- [4] HAYMAN, W.K. On the Valiron deficiencies of integral functions of infinite order. Arkiv för matematik. 10 (1972), 163-172.
- [5] HEMPEL, J.A. Precise Bounds in the Theorems of Landau and Schottky. Aspects of Contemporary Complex Analysis. ed. D.A. Brannan and J.G. Clunie. Academic Press 1980.
- [6] NEVANLINNA, F. Über die Anwendung einer Klasse von uniformisierender Transzendenten zur Untersuchung der Wertverteilung analytischer Funktionen. Acta Math. 50 (1927).
- [7] NEVANLINNA, R. Le théoreme de Picard-Borel et la théorie des fonctions méromorphes. Gauthiers-Villars, Paris 1929.
- [8] NEVANLINNA, R. Analytic Functions. Springer-Verlag. 1970.