

Rational Term-Structure Models and Geometric Lévy Martingales

by

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Abstract

One of the most important problems in modern finance is to understand how best to model the occurrence of jumps in asset pricing models. With this issue in mind, the main topic of this thesis is the development of a set of asset pricing models, driven by Lévy uncertainty, applicable across a wide range of asset classes. In particular, we model the term structure of interest rates in a Lévy setting, by use of the so-called positive interest models of Flesaker and Hughston. We begin with a brief review of the term-structure literature. We then introduce elements of the theory of Lévy processes and develop a rather general theory of geometric Lévy models (GLMs) for dynamic asset pricing, paying attention in particular to issues concerning the relation between risk and return for the models under consideration. The special case of a GLM with constant parameters can be regarded as a natural generalisation of the standard geometric Brownian motion used in the Black-Scholes theory. General conditions are established under which assets show a positive risk premium in such a setting. The Flesaker-Hughston approach has the advantage that positive nominal interest rates are built in from the beginning. The resulting models are rational in the sense that the price of a discount bond is given by a ratio of integrals of families of positive martingales. We develop a class of models of this type, where the martingale families are modelled by parametric families of geometric Lévy processes. Closed-form expressions are provided for the prices of discount bonds, the short rate of interest, and the prices of options on discount bonds, for various specific cases of Lévy uncertainty. In the example of the geometric Brownian motion family we include a rather detailed discussion of the behaviour of the risk premium, and establish conditions under which it is positive. We put forward a proposal for a two-stage calibration of the rational Lévy models to the market prices of options. Empirical studies are carried out on the calibration performance of (a) the rational Brownian model, and (b) the rational variance gamma model. We then develop a novel hedging strategy for a portfolio of options on discount bonds. The hedging strategy takes the form of a functional derivative of the option price with respect to the price of the underlying discount bond.

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The work presented in this thesis is my own.

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Chapter 1

Introduction

1.1 General overview

Interest rates are consistent across assets: movements in the level of interest rates result in movements in the prices of all assets. Realistic modelling of the term structure of interest rates is thus fundamentally important. The topic of this thesis is modelling the term structure of interest rates under Lévy uncertainty.

The world of Lévy models is a very rich domain, and provides what may be a satisfactory compromise between the continuous world of Brownian driven models, and the rather too broad domain of general semimartingale models. Lévy processes keep a number of the desirable features of Brownian motion, such as the independent and stationary increments, and provide a more flexible framework within which to match the stylized facts of financial markets.

With the gains that are made by moving from the Brownian to the Lévy framework there are also difficulties. Markets are generally not complete in Lévy models and notions such as “volatility” and “market price of risk” can no longer be used, at least directly. In Chapter 2 we create an arbitrage-free economy with Lévy randomness where the notion of “excess rate of return” is key and concepts such as risk aversion and volatility enter in a natural way. The resulting models are called general Lévy models. The excess rate of return is given by a nonlinear function of the risk aversion level. In Chapter 3 our contribution is to create a general framework of Lévy based term-structure models with non-negative nominal interest rates. These models are explored further in subsequent chapters.

The thesis is organised into ten chapters. The remainder of this chapter introduces the basics of interest-rate modelling. In particular, we describe in detail the modern “volatility” and “pricing kernel” approaches to interest rate modelling. We provide a clear derivation of an integral representation for the pricing kernel, and we confirm that the volatility approach is well specified under the assumptions of the pricing kernel approach. In Chapter 2 we introduce Lévy processes and their properties. We also introduce our results on general Lévy asset-pricing models. In Chapter 3 we introduce our Lévy term-structure models which we call the “rational Lévy” models. These models are rational in the sense that the price of a discount bond is given by a ratio of integrals of families of positive martingales. In Chapter 4 we study the rational Lévy models in the special case of Brownian uncertainty. Closed-form expressions for the discount-bond prices, the short rate, and the prices of European-style call options are provided. Chapter 5 discusses the observation that the rational Brownian models exhibit the unusual property that the market price of risk can potentially be calibrated to option prices. In Chapter 6 we develop further special cases of the rational Lévy models under several examples of Lévy uncertainty. In Chapter 7 we extend the rational Lévy models in such a way that the underlying stochastic process driving uncertainty has stochastic volatility. In Chapter 8 we propose a two-stage calibration routine for the rational Lévy models to the prices of interest-rate caps. Empirical studies are carried out on the calibration performance of (a) the rational Brownian model, and (b) the rational variance gamma model. In Chapter 9 we propose a strategy for the hedging of European-style options on discount bonds with respect to movements in the price of the underlying discount bond. Empirical results indicate that the hedge removes exposure to the first principle component, or changes in the “level”, of term-structure movements. In Chapter 10 we look at other asset classes, in particular foreign-exchange rates and equities, within the rational Lévy framework.

Much of the material in this thesis has benefited from conversations with other workers in the subject, especially my supervisors, to whom I would like to express my gratitude. I note in particular that the results in Chapters 1 and 2 have arisen from discussions with my supervisor L. P. Hughston, results in Chapters 5, 6, 8 and 9 from discussions with my co-supervisor D. C. Brody, and results in Chapters 3, 4, and 6 from discussions with both D. C. Brody and L. P. Hughston.

1.2 The volatility and pricing kernel approaches to modelling interest rates

1.2.1 Background and model assumptions

From a modern perspective there are broadly speaking two main approaches to modelling nominal interest rate systems, both of which have received a good deal of attention over the last two decades. The first of these uses the discount-bond volatility process, together with the initial yield curve, as primitive data. We shall label this the “volatility” approach. The setup is the usual one: we have a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the augmented filtration $\{\mathcal{F}_t\}_{t \geq 0}$ associated with a Brownian motion $\{W_t\}_{t \geq 0}$ of one or more dimensions. Here $\mathbb{P}$ denotes the “physical” measure. Expectation under $\mathbb{P}$ is denoted $\mathbb{E}[-]$, and for the conditional expectation with respect to $\mathcal{F}_t$ we often write $\mathbb{E}_t[-]$. The interest rate markets are represented by a system of discount bonds $\{P_{tT}\}_{T \geq t \geq 0}$ and a unit-initialised money-market account $\{B_t\}_{t \geq 0}$ satisfying a system of stochastic dynamical equations of the following form:

$$\frac{dP_{tT}}{P_{tT}} = r_t dt + \Omega_{tT}(dW_t + \lambda_t dt) \quad (1.1)$$

and

$$\frac{dB_t}{B_t} = r_t dt. \quad (1.2)$$

Here P_{tT} denotes the value of a discount bond at time t that matures at time T , paying the holder one unit of currency, $\{r_t\}_{t \geq 0}$ is the short rate, $\{\Omega_{tT}\}_{T \geq t \geq 0}$ is the bond volatility, and $\{\lambda_t\}_{t \geq 0}$ is the market price of risk. In the multi-dimensional setting, $\{\Omega_{tT}\}$ and $\{\lambda_t\}$ are vector-valued processes. We assume that $\{\Omega_{tT}\}$ admits a continuous first derivative with respect to the maturity T , and that for any fixed value of T we have $\lim_{t \rightarrow T} \Omega_{tT} = 0$. The latter relation ensures that the volatility of a bond dies off as the bond approaches maturity.

Given the initial term structure $\{P_{0t}\}_{t \geq 0}$, we find, under suitable technical conditions, which will be discussed below, that the solution for the discount bond system is given by

$$P_{tT} = P_{0T} B_t \exp \left(\int_0^t \Omega_{sT} (dW_s + \lambda_s ds) - \frac{1}{2} \int_0^t \Omega_{sT}^2 ds \right), \quad (1.3)$$

and that for the money-market account we have

$$B_t = \exp \left(\int_0^t r_s ds \right). \quad (1.4)$$

In particular, if we set $P_{tt} = 1$ for $t \geq 0$ we can invert equation (1.3) to obtain the following explicit formula for the money market account:

$$B_t = (P_{0t})^{-1} \exp \left(- \int_0^t \Omega_{st} (dW_s + \lambda_s ds) + \frac{1}{2} \int_0^t \Omega_{st}^2 ds \right). \quad (1.5)$$

For the short rate r_t we then have

$$r_t = -\partial_t \ln P_{0t} + \int_0^t \Omega_{st} \partial_t \Omega_{st} ds - \int_0^t \partial_t \Omega_{st} (dW_s + \lambda_s ds), \quad (1.6)$$

where ∂_t denotes differentiation with respect to t . Putting these ingredients together we are led to the following expression for the bond prices:

$$P_{tT} = P_{0tT} \frac{\exp \left(\int_0^t \Omega_{sT} (dW_s + \lambda_s ds) - \frac{1}{2} \int_0^t \Omega_{sT}^2 ds \right)}{\exp \left(\int_0^t \Omega_{st} (dW_s + \lambda_s ds) - \frac{1}{2} \int_0^t \Omega_{st}^2 ds \right)}. \quad (1.7)$$

Here

$$P_{0tT} = \frac{P_{0T}}{P_{0t}} \quad (1.8)$$

denotes the t -forward price made at time zero for a T -maturity bond. We see that $\{B_t\}$ and $\{P_{tT}\}$ are determined by the specification of the volatility $\{\Omega_{tT}\}$, the market price of risk $\{\lambda_t\}$, and the initial bond prices $\{P_{0t}\}$. There is thus no need to model the short rate separately.

Let us return now to the technical conditions required to ensure that (1.3) and (1.4) are valid. These are as follows:

$$\int_0^t \Omega_{sT}^2 ds < \infty \quad \text{a.s.}, \quad (1.9)$$

$$\int_0^t |\Omega_{sT} \lambda_s| ds < \infty \quad \text{a.s.}, \quad (1.10)$$

and

$$\int_0^t |r_s| ds < \infty \quad \text{a.s.} \quad (1.11)$$

It will be useful in what follows, to reformulate the technical conditions (1.9) and (1.10). To that end we establish the following simple but useful results.

Lemma 1.2.1. *Let $\{X_s\}_{s \geq 0}$ and $\{Y_s\}_{s \geq 0}$ be vector-processes defined on $[0, t]$. Then $\int_0^t X_s^2 ds < \infty$ a.s. and $\int_0^t Y_s^2 ds < \infty$ a.s. if and only if $\int_0^t |X_s \cdot Y_s| ds < \infty$ a.s. and $\int_0^t (X_s - Y_s)^2 ds < \infty$ a.s.*

Proof. Suppose that $\int_0^t X_s^2 ds < \infty$ a.s. and $\int_0^t Y_s^2 ds < \infty$ a.s. By the Cauchy-Schwarz inequality, for each $\omega \in \Omega$ and each $s \in [0, t]$ we have

$$(X_s \cdot Y_s)^2 \leq X_s^2 Y_s^2, \quad (1.12)$$

and hence

$$|X_s \cdot Y_s| \leq |X_s| |Y_s|. \quad (1.13)$$

We then deduce that

$$\int_0^t |X_s \cdot Y_s| ds \leq \int_0^t |X_s| |Y_s| ds \leq \sqrt{\int_0^t X_s^2 ds \int_0^t Y_s^2 ds} < \infty \quad \text{a.s.} \quad (1.14)$$

The second inequality follows because the functions

$$f_s = \frac{|X_s|}{\sqrt{\int_0^t |X_s|^2 ds}} \quad \text{and} \quad g_s = \frac{|Y_s|}{\sqrt{\int_0^t |Y_s|^2 ds}}, \quad (1.15)$$

have the property that

$$\int_0^t f_s^2 ds = 1 \quad \text{and} \quad \int_0^t g_s^2 ds = 1, \quad (1.16)$$

and

$$\int_0^t f_s g_s ds \leq 1. \quad (1.17)$$

It then follows that

$$\begin{aligned} \int_0^t (X_s - Y_s)^2 ds &= \int_0^t X_s^2 - 2X_s \cdot Y_s + Y_s^2 ds \\ &\leq \int_0^t X_s^2 + 2|X_s \cdot Y_s| + Y_s^2 ds \\ &< \infty \quad \text{a.s.} \end{aligned} \quad (1.18)$$

That establishes the “only if” part of the Lemma. To prove the “if” part of the Lemma suppose instead that $\int_0^t |X_s \cdot Y_s| ds < \infty$ and $\int_0^t (X_s - Y_s)^2 ds < \infty$. Then the inequality

$$\int_0^t (X_s - Y_s)^2 ds + 2 \int_0^t X_s \cdot Y_s ds \leq \int_0^t (X_s - Y_s)^2 ds + 2 \int_0^t |X_s \cdot Y_s| ds \quad (1.19)$$

implies that

$$\int_0^t X_s^2 ds + \int_0^t Y_s^2 ds < \infty \quad \text{a.s.}, \quad (1.20)$$

from which it follows that $\int_0^t X_s^2 ds < \infty$ a.s. and $\int_0^t Y_s^2 ds < \infty$ a.s. $\square$

Corollary 1.2.2. *With regard to the volatility process $\{\Omega_{tT}\}$ and the market price of risk process $\{\lambda_t\}$ defined on $[0, T]$, it follows that $\int_0^t \Omega_{sT}^2 ds < \infty$ a.s. and $\int_0^t \lambda_s^2 ds < \infty$ a.s. if and only if $\int_0^t |\Omega_{sT} \lambda_s| ds < \infty$ a.s. and $\int_0^t (\Omega_{sT} - \lambda_s)^2 ds < \infty$ a.s. for all $T \geq t \geq 0$.*

Corollary 1.2.2 follows from Lemma 1.2.2 if one sets $X_s = \Omega_{sT}$ and $Y_s = \lambda_s$. With this in mind we shall return again to the technical conditions required by the volatility approach when we introduce the pricing kernel method.

How do we put the volatility approach into practice? As long as we are primarily concerned with pricing and hedging, but not in asset allocation and forecasting, then a transformation to the risk-neutral measure will have the effect of removing the market price of risk from our equations. The problem of modelling the evolution of the term structure of interest rates is thus transformed into the problem of modelling the volatility structure $\{\Omega_{tT}\}$. In the methodology that has been adapted by practitioners the idea is that we specify $\{\Omega_{tT}\}$ exogenously up to some overall parametric or functional freedom. This freedom is used to calibrate the model to the prices of various derivatives, typically interest-rate caps, interest-rate floors, and swaptions. The volatility approach has been successful, and in one form or another has been widely implemented by financial institutions from the late 1980s onward.

The volatility approach does nevertheless suffer from various defects, both conceptual and practical, and it is reasonable to ask if we can do better. We shall not attempt a detailed critique of the volatility approach here, since the model has been developed in many different forms including, in particular, the Heath, Jarrow and Morton (HJM) theory (Heath *et al.* 1992) and the Libor market model (see Musiela & Rutkowski 2005, Schoenmakers 2005, Brigo & Mercurio 2006, and references cited therein). Suffice to say that it is difficult to impose a transparent condition on the discount bond volatility structure that ensures the interest rate positivity conditions

$$\partial_T P_{tT} < 0, \quad (1.21)$$

and

$$0 \leq P_{tT} \leq 1, \quad (1.22)$$

(cf. Flesaker & Hughston 1996, Brody & Hughston 2002) for all $T \geq t \geq 0$. There is no clear economic motivation as such for choosing one volatility structure over another, and the fact that the volatility is modelled in the risk-neutral measure (or some other “unnatural” measure, such as the T -forward measure) further removes the model from economic reality. In this respect the elimination of the market price of risk is ultimately a shortcoming rather than a virtue. This has arisen as a consequence of the “buy side” emphasis on pricing and hedging, as opposed to portfolio management.

An alternative to the volatility approach is to base the theory on pricing kernels. The pricing kernel method allows, in a natural way, for interest-rate positivity. It also generalises readily to models not based on Brownian motion. The connection with modern economic thinking is clearer, and the extension to other asset classes (for example, foreign exchange or inflation-linked products) is cleaner.

The setup for pricing kernel models can be summarised as follows. We model the market on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Here $\mathbb{P}$ is again assumed to be the “physical” probability measure.

Assumption 1.2.3. *We assume that the market is free of arbitrage. Market completeness is not assumed. We assume the existence of an established pricing kernel $\{\pi_t\}_{t \geq 0}$, with $\pi_0 = 1$ and $\pi_t > 0$ for $0 \leq t < \infty$ satisfying*

$$\mathbb{E}[\pi_t] < \infty. \quad (1.23)$$

The market is assumed to have the property that if $\{S_t\}_{t \geq 0}$ is the price process of any traded asset that pays a continuous dividend at the rate $\{D_t\}_{t \geq 0}$, then the process $\{M_t\}_{t \geq 0}$ defined by

$$M_t = \pi_t S_t + \int_0^t \pi_s D_s \, ds \quad (1.24)$$

is a martingale. In the case of a dividend stream that is not necessarily continuous, we let $\{\Delta_t\}_{t \geq 0}$ denote the cumulative dividend process, and we assume that the process defined by

$$M_t = \pi_t S_t + \int_0^t \pi_s \, d\Delta_s. \quad (1.25)$$

is a martingale.

Then S_t represents the “ex-dividend” price of the asset at time t . With this convention, asset price processes and cumulative dividend processes are understood to be right-continuous with left limits.

We regard Assumption 1.2.3 as being, in a certain sense, fundamental. It is sufficient to support the existence of an arbitrage-free system of discount bonds (Brody & Hughston 2004, Hughston & Rafailides 2005). Recall that a discount bond with maturity T is an asset that pays a single dividend of one unit at time T . It follows then from Assumption 1.2.3 that for each $T \geq 0$ the process

$$M_t = \pi_t P_{tT} + \pi_t \mathbb{1}\{t \geq T\} \quad (1.26)$$

is a martingale, where $\mathbb{1}\{-\}$ denotes the indicator function. The bond price drops to zero at maturity when the single dividend is paid. Often, informally, for convenience, we regard the bond price as having “value unity” at maturity. This is, of course, the value of its left limit, not its actual value, at maturity. We usually adopt this convention in what follows and when we do so it should be clear from the context. In particular we obtain

$$P_{tT} = \frac{\mathbb{E}_t[\pi_T]}{\pi_t}, \quad (1.27)$$

which should be interpreted as holding over all $0 \leq t \leq T$.

In the development of this framework it is important to ask the following question: do we lose anything by adopting the pricing kernel approach, as opposed to the volatility approach? The answer is no. In fact, in the Brownian setting the pricing kernel itself can be expressed in terms of the volatility $\{\Omega_{tT}\}$, the market price of risk $\{\lambda_t\}$, and the initial bond prices $\{P_{0t}\}$. It follows from Assumption 1.2.3 that the process $\{\Lambda_t\}_{t \geq 0}$ defined by

$$\Lambda_t = \pi_t B_t \quad (1.28)$$

is a positive martingale, and hence has dynamics of the form

$$d\Lambda_t = -\Lambda_t \lambda_t dW_t, \quad (1.29)$$

for some adapted process $\{\lambda_t\}$ which we identify as the market price of risk. Integrating (1.29) under the appropriate technical conditions we obtain

$$\Lambda_t = \exp \left(- \int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds \right). \quad (1.30)$$

Then, inserting (1.30) and the expression for the bond price under the volatility approach, (1.5), into (1.28) we deduce that

$$\pi_t = P_{0t} \exp \left(\int_0^t (\Omega_{st} - \lambda_s) dW_s - \frac{1}{2} \int_0^t (\Omega_{st} - \lambda_s)^2 ds \right). \quad (1.31)$$

Thus, any “volatility model” can be converted into a “pricing kernel model” (see also Jin & Glasserman 2001). More precisely, we can regard $\{\Omega_{tT}\}$, $\{\lambda_t\}$, and $\{P_{0t}\}$ as being specified up to some overall parametric or functional freedom, thus inducing a corresponding parameterisation of the pricing kernel, which can then be calibrated to market data and/or market forecasts. See, for example, Rogers & Kluge (2008) and Grasselli & Tsujimoto (2011).

The technical conditions needed to ensure that (1.30) and (1.31) are well defined are given by

$$\int_0^t (\Omega_{sT} - \lambda_s)^2 ds < \infty \quad \text{a.s.}, \quad (1.32)$$

and

$$\int_0^t \lambda_s^2 ds < \infty \quad \text{a.s.} \quad (1.33)$$

The volatility approach is thus well defined under the conditions of the pricing kernel approach.

Corollary 1.2.4. *For $\{\Omega_{tT}\}$ and $\{\lambda_t\}$ defined on $[0, T]$ for $T \geq t \geq 0$, it follows that if $\int_0^t \lambda_s^2 ds < \infty$ a.s. and $\int_0^t (\Omega_{sT} - \lambda_s)^2 ds < \infty$ a.s., then $\int_0^t \Omega_{sT}^2 ds < \infty$ a.s. and $\int_0^t |\Omega_{sT} \lambda_s| ds < \infty$ a.s.*

Proof. Suppose that $\int_0^t \lambda_s^2 ds < \infty$ a.s. and $\int_0^t (\Omega_{sT} - \lambda_s)^2 ds < \infty$ a.s. Let $X_s = \Omega_{sT} - \lambda_s$ and $Y_s = -\lambda_s$. Then Lemma 1.2.1 shows us that

$$\int_0^t \Omega_{sT}^2 ds < \infty \quad \text{a.s.} \quad (1.34)$$

and

$$\int_0^t |\lambda_s (\Omega_{sT} - \lambda_s)| ds < \infty \quad \text{a.s.} \quad (1.35)$$

Then, $\int_0^t \Omega_{sT}^2 ds < \infty$ a.s. and $\int_0^t \lambda_s^2 ds < \infty$ a.s., by Corollary 1.2.2, imply that

$$\int_0^t |\Omega_{sT} \lambda_s| ds < \infty \quad \text{a.s.} \quad (1.36)$$

□

The pricing kernel approach, by assumption, demands that the processes $\{\Lambda_t\}$ and $\{\pi_t P_{tT}\}_{t \geq 0}$, where

$$\pi_t P_{tT} = P_{0T} \exp \left(\int_0^t (\Omega_{sT} - \lambda_s) dW_s - \frac{1}{2} \int_0^t (\Omega_{sT} - \lambda_s)^2 ds \right), \quad (1.37)$$

are true martingales. This assumption is not typically made in the volatility approach and this can be problematic. Without Assumption 1.2.3 both $\{\Lambda_t\}$ and $\{\pi_t P_{tT}\}_{t \geq 0}$ may be strictly local martingales, and not true martingales. Various, absence of the true-martingale condition can lead to such unusual phenomena as (a) arbitrage opportunities, (b) lack of equilibrium, (c) pricing inconsistencies, and (d) bubbles. For examples of the pathologies that can arise in the strictly local-martingale situation one may consult, for example, Cox & Hobson (2005), Platen & Heath (2006, the “Benchmark Approach”), Jarrow *et al.* (2010), and references cited therein. For more information on local and true martingales one may consult Karatzas & Shreve (1991, 1998). In what follows we consider the true martingale situation exclusively, and nothing further will be said concerning the strictly local martingale case.

1.2.2 An integral representation for the pricing kernel

To proceed further we shall make two reasonable assumptions about the nature of bond markets.

Assumption 1.2.5. *Firstly, we assume the existence of a family of instantaneous forward rates $\{f_{tT}\}_{T \geq t \geq 0}$ satisfying $f_{tT} > 0$ a.s. for all $T \geq t \geq 0$, such that*

$$P_{tT} = \exp \left(- \int_t^T f_{ts} ds \right). \quad (1.38)$$

It follows that $0 \leq P_{tT} \leq 1$ for all $T \geq t \geq 0$, and that $\partial_T P_{tT} < 0$. The short rate process $\{r_t\}$ is then defined by $r_t = f_{tt}$, or equivalently

$$r_t = - \left. \frac{\partial P_{tT}}{\partial T} \right|_{T=t}. \quad (1.39)$$

Assumption 1.2.6. *Secondly, we impose the “transversality” condition*

$$\lim_{T \rightarrow \infty} P_{tT} = 0, \quad (1.40)$$

for any fixed t .

We shall regard the constraint (1.40) as part of the general requirements for arbitrage-freeness. Equivalently, we have the following condition on the forward rates:

$$\lim_{T \rightarrow \infty} \int_t^T f_{ts} ds = \infty. \quad (1.41)$$

Under the given assumptions we can derive an integral representation for the pricing kernel (see, e.g., Flesaker & Hughston 1996, Rutkowski 1997, Björk 1998, Jin & Glasserman 2001, Cairns 2004, Hunt & Kennedy 2004).

Proposition 1.2.7. *The process $\{\pi_t \partial_T P_{tT}\}_{T \geq t \geq 0}$ is a martingale.*

Proof. We observe that by (1.27) we have

$$\pi_t P_{tT} = \mathbb{E}_t [\pi_u P_{uT}] \quad (1.42)$$

for $0 \leq t \leq u < T$. Additionally, by Assumption 1.2.5 bond prices are differentiable with respect to their maturity variable. Hence, for any $T' > T$ we can write

$$\pi_t P_{tT} = \mathbb{E}_t \left[\pi_u \int_T^{T'} (-\partial_v P_{uv}) dv \right] + \mathbb{E}_t [\pi_u P_{uT'}], \quad (1.43)$$

which becomes

$$\pi_t P_{tT} = \int_T^{T'} \mathbb{E}_t [\pi_u (-\partial_v P_{uv})] dv + \mathbb{E}_t [\pi_u P_{uT'}], \quad (1.44)$$

by the Fubini-Tonelli theorem. By differentiation we deduce that

$$\pi_t \partial_T P_{tT} = \mathbb{E}_t [\pi_u \partial_T P_{uT}], \quad (1.45)$$

and therefore that $\{\pi_t \partial_T P_{tT}\}_{T \geq t \geq 0}$ is a martingale. $\square$

Following Proposition 1.2.7, we are able to define a parametric family of positive martingales $\{M_{tT}\}_{T \geq t \geq 0}$ such that $M_{0T} = 1$ and

$$\pi_t \partial_T P_{tT} = (\partial_T P_{0T}) M_{tT}. \quad (1.46)$$

We can integrate (1.46), using the boundary conditions given by $P_{tt} = 1$ and by (1.40), to obtain

$$\pi_t = \int_t^\infty (-\partial_x P_{0x}) M_{tx} dx. \quad (1.47)$$

This is the desired integral representation for the pricing kernel. Substituting (1.47) into (1.27) we deduce that the bond price admits a rational representation of the so-called Flesaker-Hughston type:

$$P_{tT} = \frac{\int_T^\infty (-\partial_x P_{0x}) M_{tx} dx}{\int_t^\infty (-\partial_x P_{0x}) M_{tx} dx}. \quad (1.48)$$

1.3 The term-structure density approach

Let us review briefly the term-structure density approach introduced in Brody & Hughston (2001). This plays a role in much of what follows. The key observation is the fact that the positivity of nominal rates along with the strong no-arbitrage condition (1.40) implies that for each fixed t , and each fixed $\omega \in \Omega$ the quantity $\{-\partial_T P_{tT}\}_{T \geq t \geq 0}$ allows us to define a probability density function over the interval $[t, \infty]$. For definiteness we switch to the so-called Musiela parameterisation, and introduce the tenor variable $x = T - t$. Then the term-structure density process $\{\rho_t(x)\}_{t \geq 0}$ is defined for $x \geq 0$ by the relation

$$\rho_t(x) = -\partial_x P_{t,t+x}. \quad (1.49)$$

The term-structure density approach is thus based on the observation that there exists an “abstract” positive random variable X whose conditional density process is given by (1.49). The bond price can then be expressed in the form

$$P_{tT} = \frac{\mathbb{E}_t^\rho[\mathbf{1}\{X > T\}]}{\mathbb{E}_t^\rho[\mathbf{1}\{X > t\}]}, \quad (1.50)$$

where $\mathbb{E}_t^\rho[-]$ denotes conditional expectation with respect to the “abstract” density defined by (1.49).

If we assume that the market filtration $\{\mathcal{F}_t\}$ is generated by a family of $\mathbb{P}$ -Brownian motions $\{W_t\}_{t \geq 0}$, then the dynamical equation satisfied by the term-structure density process is given by the so-called Brody-Hughston stochastic partial differential equation:

$$d\rho_t(x) = (r_t \rho_t(x) + \partial_x \rho_t(x)) dt + \rho_t(x)(\nu_t(x) - \bar{\nu}_t)(dW_t - \bar{\nu}_t dt), \quad (1.51)$$

where $r_t = \rho_t(0)$ is the short rate, and $\bar{\nu}_t = \mathbb{E}_t^\rho[\nu_t(x)]$, or, equivalently,

$$\bar{\nu}_t = \int_0^\infty \rho_t(x) \nu_t(x) dx. \quad (1.52)$$

The underlying volatility structure $\{\nu_t(x)\}_{t \geq 0}$ is thus specified exogenously, whereas the initial yield curve can be calibrated by the specification of

$$\rho_0(x) = -\partial_x P_{0x}. \quad (1.53)$$

The market price of risk process $\{\lambda_t\}_{t \geq 0}$ is given by $\lambda_t = -\bar{\nu}_t$. Writing $V_{tT} = \nu_t(T - t)$ to convert back from the Musiela parameter to the maturity variable, we find that the bond price admits the representation

$$P_{tT} = \frac{\int_T^\infty \rho_0(u) \exp\left(\int_{s=0}^t V_{su} dW_s - \frac{1}{2} \int_{s=0}^t V_{su}^2 ds\right) du}{\int_t^\infty \rho_0(u) \exp\left(\int_{s=0}^t V_{su} dW_s - \frac{1}{2} \int_{s=0}^t V_{su}^2 ds\right) du}, \quad (1.54)$$

which is of the Flesaker-Hughston type.

One of the advantages of the term-structure density approach over the HJM approach is that the positivity of nominal interest rates is ensured by construction. An implicit condition for interest rate positivity in the HJM-type approach has been worked out by Jin & Glasserman (2001); an explicit condition is derived in Brody & Hughston (2002). Specifically, for interest rate positivity the HJM forward rate volatility process $\{\sigma_{tT}\}_{t \geq 0}$ has to be of the following form:

$$\sigma_{tT} = f_{tT} \left(V_{tT} - \frac{\mathbb{E}_t^\rho[V_{tX} \mathbf{1}\{X > T\}]}{\mathbb{E}_t^\rho[\mathbf{1}\{X > T\}]} \right). \quad (1.55)$$

In other words, in the HJM framework, after having chosen $\{V_{tT}\}$ freely, one has to work out the term-structure density process first in order to deduce the arbitrage-free form of the forward-rate volatility via (1.55).

Another advantage of the term-structure density approach has been pointed out more recently by Filipović *et al.* (2009). In short, Filipović *et al.* argue that it is “less delicate” to add jumps to the Brody-Hughston equation (1.51) than to the familiar HJM framework in the Musiela representation. In this spirit Filipović *et al.* have extended (1.51) to incorporate the more general situation of jump-diffusion dynamics, with a possibly infinite-dimensional Wiener process driving the diffusion.

1.4 Term-structure models in the literature

Asset prices move due to changes in the expected values of cash flows, changes in investor preferences, and changes in the level of interest rates. Modelling the term

structure of interest rates is thus a fundamental problem of asset pricing. Since the history of term-structure modelling cannot be contained in a review of the volatility and pricing kernel approaches alone, we shall provide a short outline of some of the most important and novel innovations. Although by no means exhaustive, our outline should provide a sense of the variety of models that exist.

Short rate models

Term-structure modelling in general is an infinite-dimensional problem. This point was not recognised, however, until the advent of the HJM framework. Early attempts at a solution described movements in the term structure of interest rates as being driven by a single rate, usually the short rate. Examples of influential “short rate” models include those of Merton (1974), Vasicek (1977), Dothan (1978), Cox *et al.* (1985), Black *et al.* (1990), Black & Karasinski (1991), and Longstaff & Schwartz (1992).

The models of Vasicek (1977) and Cox *et al.* (1985) are perhaps the best known one-factor short rate models. They derive the equilibrium term-structure of an economy where the short rate is modelled as a Gaussian process and a squared Gaussian process respectively. These models are examples of “affine” term-structure models (see, e.g., Duffie & Kan 1996, Dai & Singleton 2002, 2003, Duffie *et al.* 2003, Piazzesi 2004). Such models are characterised by bond yields that are affine functions of one or more state variables. Richer term structures can be created by the inclusion of more state variables, and it is usual practice to make one of the state variables the short rate. Extensions to the affine class include quadratic affine models (Constantinides 1992, Leippold & Wu 2001, 2002, Ahn *et al.* 2002), and “essentially affine” models (Duffee 2002).

Models that match the initial yield curve

The idea of incorporating an arbitrary initial term structure was first mentioned by Cox *et al.* (1985). Ho & Lee (1986) created a discrete time short-rate model that could fit exactly to observed yields. It was Jamshidian (1987) who derived first the continuous version of the Ho-Lee model. In the continuous-time limit, the Ho-Lee model is a time-inhomogeneous Gaussian process (see, e.g., Rogers 1995). Arbitrary initial term structures can also be incorporated by the affine class of models if one

allows the underlying state variables to be time-inhomogeneous. Tractable examples are provided by Hull & White (1990) and Brigo & Mercurio (2001).

In the HJM framework the evolution of a family of instantaneous forward rates is modelled, where the initial forward curve $\{f_{0t}\}_{t \geq 0}$, is an input. By choosing the form of the forward-rate volatility correctly, a number of short rate models including the continuous time version of the Ho & Lee (1986), the Vasicek (1977), and the Cox *et al.* (1985) models can be recreated as special cases of the HJM framework.

Market models

The short and instantaneous forward rates of interest are useful for modelling purposes, but they are not directly observable. Market models specify the form of various observable interest rates: in particular, Libor rates and swap rates are often chosen. The benefits of this approach include (a) that the formulation of the model involves instruments which practitioners are used to dealing with, and (b) that the resulting option prices are generally of the so-called Black (1976) form, which is routinely studied in practical courses on finance and option pricing. One may consult Miltersen *et al.* (1997), Brace *et al.* (1997), Jamshidian (1997), Rebonato (2002), Glasserman & Merener (2003), Brigo & Mercurio (2003), Musiela & Rutkowski (2005) for further discussion concerning market models.

Models with jumps

It is often observed that the empirical distribution of bond returns has fatter tails than a Gaussian distribution (see, e.g., Johannes 2000, Das 2002, and Heston 2007). Many authors, for example Das & Foresi (1996), Björk *et al.* (1997), Burnetas & Ritchken (1998), Das (1998, 2002), Glasserman & Kou (2003), and Johannes (2004), among others, have created term-structure models using jump processes in an attempt to match observed data.

In addition, the HJM and market models were extended by Eberlein and co-authors to allow interest rates to be driven by a variety of non-continuous processes based on *Lévy processes* (see Eberlein & Raible 1999, Raible 2000, Eberlein *et al.* 2005, Eberlein & Özkan 2005, Eberlein & Kluge 2006a, and references cited therein).

Time-series models

In the literature, when the main concern of a term-structure model is matching an empirical time series and describing the behaviour of bond returns, affine models have proved extremely popular. Duffee (2002), Dai & Singleton (2002), Cochrane & Piazzesi (2008), and Iqbal (2010) construct affine models to fit facts about returns. In general, to preserve the affine property under more than one probability measure requires extra assumptions on the market price of risk. In contrast, when one models the pricing kernel, the market price of risk is typically endogenous. In theory the pricing kernel approach is suitable for both pricing and time-series analysis; Constantinides (1992), Flesaker & Hughston (1997), Rogers (1997) and Leippold & Wu (1999) are contributions in this direction.

More recent advances

The so-called positive interest framework of Flesaker & Hughston (1996) originated as an attempt to characterise the complete family of HJM-type models with positive interest rates and valid over an arbitrary time horizon. A generalised version of the framework was presented by Rogers (1997) and independently by Rutkowski (1997), who showed how the Flesaker-Hughston characterisation of positive-interest HJM models in terms of a rational integral representation was essentially equivalent to the specification of a positive supermartingale, and it was also clear in this later work that the framework was not tied in any special way to the use of a Brownian filtration. In the words of Rutkowski (1997), “From a theoretical viewpoint, the basic input of the generalised Flesaker-Hughston model is a strictly positive supermartingale...”. The relation of the Flesaker-Hughston models to the HJM theory and other approaches was subsequently studied in considerable detail by Jin & Glasserman (2001) and by Hunt & Kennedy (2004).

A number of other interesting and significant advances have been proposed. For discussions concerning so-called infinite dimensional models see Kennedy (1994, 1997), Filipović (2000), and Santa-Clara & Sornette (2001); for geometric approaches see Björk & Svensson (1999), Brody & Hughston (2002), Filipović & Teichmann (2004), and Tappe (2010); and for Wiener-chaos expansion techniques one may consult Brody & Hughston (2004), Hughston & Rafailides (2005), and Grasselli & Hurd (2005).

Chapter 2

Geometric Lévy Martingales and Financial Modelling

2.1 Chapter overview

In this chapter we introduce Lévy processes and recall some useful results from the literature including the Esscher transform. The definition of a Lévy process and an introduction to their properties can be found in Bertoin (1996), Sato (1999), Raible (2000), Protter (2005), Kyprianou (2006), and elsewhere.

In order to gain some understanding of the nature of the risk and risk premia associated with jumps in asset prices we propose to create a simple Lévy-driven economy. We shall derive expressions for asset prices and the term structure of interest rates. As a corollary we are also able to link the volatility and pricing kernel approaches to term-structure modelling in a Lévy setting.

2.2 Lévy processes

2.2.1 Definition and properties

We shall assume that each stochastic process that we consider is one-dimensional and is right continuous with left limits. We assume unless otherwise stated that each process begins at time 0, is initialised at 0, and takes values on the real line. All of the results in this section are stated without proof and can be found in, for example, Bertoin

(1996), Sato (1999), Protter (2005), and Kyprianou (2006).

Definition 2.2.1. A *Lévy process* $\{L_t\}_{t \geq 0}$ is a stochastic process defined on $(\Omega, \mathcal{F}, \mathbb{P})$ that has the following properties:

1. *Stationary increments:* $L_t - L_s$ has the same distribution as L_{t-s} for $t \geq s \geq 0$.
2. *Independent increments:* $L_t - L_s$ is independent of $\mathcal{F}_s^L$, where $\{\mathcal{F}_t^L\}_{t \geq 0}$ is the natural filtration associated with $\{L_t\}$.
3. *Stochastic continuity:* for every $t \geq 0$ and $\varepsilon > 0$, $\lim_{s \rightarrow t} \mathbb{P}(|L_t - L_s| > \varepsilon) = 0$.

Due to the independent and stationary increments properties of $\{L_t\}$ we have that

$$\mathbb{E}[e^{iuL_t}] = (\mathbb{E}[e^{iuL_1}])^t. \quad (2.1)$$

For each $t \geq 0$ L_t then has an infinitely-divisible distribution, i.e., L_t is equal in distribution to n independent and identically distributed random variables with the same distribution as $L_{t/n}$ for $n = 1, 2, \dots$. Let $\Psi(-)$ be the characteristic exponent of a random variable X such that

$$\Psi(u) = \ln \mathbb{E}[e^{iuX}], \quad (2.2)$$

for $u \in \mathbb{R}$. If the random variable X has an infinitely divisible distribution then $\Psi(-)$ has a known functional form in terms of a triple (z, σ, Π) .

Theorem 2.2.2. A probability law μ of a real-valued random variable is infinitely divisible with characteristic exponent Ψ , where

$$\int_{\mathbb{R}} e^{iux} \mu(dx) = e^{\Psi(u)}, \quad (2.3)$$

for $u \in \mathbb{R}$ if and only if there exists a triple (z, σ, Π) , where $z \in \mathbb{R}$, $\sigma \geq 0$ and Π is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}} \min(1, x^2) \Pi(dx) < \infty, \quad (2.4)$$

such that

$$\Psi(u) = zu - \frac{1}{2}\sigma^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux \mathbf{1}\{|x| < 1\}) \Pi(dx), \quad (2.5)$$

for all $u \in \mathbb{R}$.

Equation (2.5) is called the Lévy-Khintchine formula. The triplet (z, σ, Π) characterises X . If (z, σ, Π) is the characteristic triplet of L_1 , then by (2.1) it characterises the process $\{L_t\}$, and it is called the Lévy triplet of $\{L_t\}$. The measure Π is called the Lévy measure. In general Π does not have to be integrable.

Each Lévy process can be decomposed into a continuous process with a deterministic drift and a jump process. The jump process can then be split into a process with jump magnitudes that are greater than some number, usually unity, and a process with jump magnitudes that are less than that number. This decomposition is known as the Lévy-Itô Decomposition. The drift is specified by z , the path properties of the continuous process are governed by σ , and the properties of the jump processes are governed by the Lévy measure Π .

Definition 2.2.3. The process $\{L_t\}$ has *infinite activity* if and only if the paths of $\{L_t\}$ have an infinite number of jumps on every compact time interval almost surely.

A Lévy process $\{L_t\}$ with Lévy triplet (z, σ, Π) has infinite activity if

$$\int_{\mathbb{R}} \Pi(dx) = \infty. \quad (2.6)$$

(see Geman 2002). A Lévy process that does not have infinite activity is said to have finite activity.

Definition 2.2.4. A Lévy process $\{L_t\}$ with Lévy triplet (z, σ, Π) will have *finite (or bounded) variation* if and only if $\sigma = 0$ and

$$\int_{0 < |x| < 1} |x| \Pi(dx) < \infty. \quad (2.7)$$

The sample paths of a Brownian motion have infinite variation almost surely. The condition that $\sigma = 0$ thus ensures that the process $\{L_t\}$ does not have a continuous, or Brownian part. A process that does not have finite variation is said to have infinite variation.

Remark 2.2.5. It is helpful, but not essential, to know the Lévy measure in closed form in order to be able to quickly read off path properties of a process. The Lévy measure describes the jumps of $\{L_t\}$. For example, if $\Pi(-\infty, 0) = 0$ (*resp.* $\Pi(0, \infty) = 0$) then $\{L_t\}$ is called spectrally positive (*resp.* negative) and does not have any negative (*resp.* positive) jumps. One may consult Kyprianou (2006) for a detailed discussion.

2.2.2 Geometric Lévy martingales

Most Lévy processes admit a useful class of tractable positive martingales. The following results can be found in, for example, Raible (2000) and Cont & Tankov (2003).

Proposition 2.2.6. *Let $\{L_t\}_{t \geq 0}$ be a Lévy process with characteristic triplet (z, σ, Π) and consider any $u \in \mathbb{C}$ such that*

$$\int_{|x| > 1} e^{ux} \Pi(dx) < \infty. \quad (2.8)$$

Then the process $\{M_{tu}\}_{t \geq 0}$ defined by

$$M_{tu} = \frac{e^{uL_t}}{\mathbb{E}[e^{uL_t}]}, \quad (2.9)$$

is a martingale.

We call $\{M_{tu}\}$ the Esscher transform of $\{L_t\}$ with parameter u . Condition (2.8) is, for real values of u , equivalent to assuming that the Laplace transform of L_t exists.

The Esscher transformation has been extensively put to use in the insurance literature (see Gerber & Shui 1994, 1996) and the asset pricing literature (see Bakshi & Madan 2000) where it is used to define appropriate change of measure densities. Specifically, we may define a probability measure $\mathbb{P}^u$, such that

$$\frac{d\mathbb{P}^u}{d\mathbb{P}} = \frac{e^{uL_t}}{\mathbb{E}[e^{uL_t}]}. \quad (2.10)$$

Then under the new measure $\mathbb{P}^u$, $\{L_t\}$ is a Lévy process with characteristic triplet (z_u, σ_u, Π_u) where

$$z_u = z - u\sigma^2 + \int_{|x| < 1} (1 - e^{ux}) \Pi(dx), \quad (2.11)$$

and

$$\sigma_u = \sigma, \quad (2.12)$$

and

$$\Pi_u(dx) = e^{ux} \Pi(dx). \quad (2.13)$$

Proposition 2.2.7. *Let $\{L_t\}$ be a Lévy process. Define the Laplace exponent of L_t to be*

$$\psi(u) = \frac{1}{t} \ln \mathbb{E} [e^{uL_t}] , \quad (2.14)$$

for $u \in \mathbb{R}$. Let $\mathcal{I}$ be a subset of $\mathbb{R}$ such that $\psi(u) < \infty$ for all $u \in \mathcal{I}$. Let $\{u_t\}_{t \geq 0}$ be a predictable stochastic process that takes values in some subset of $\mathcal{I}$. Then the process $\{M_t\}_{t \geq 0}$, with

$$M_t = \exp \left(\int_0^t u_s dL_s - \int_0^t \psi(u_s) ds \right) , \quad (2.15)$$

is a martingale.

The proof of Proposition 2.2.6 follows from the stationary and independent increments of $\{L_t\}$ and can be found in, e.g., Cont & Tankov (2003). Proposition 2.2.7 defines a more general exponential martingale. A proof of Proposition 2.2.7 can be found in Raible (2000). For further discussion on the Esscher transform and associated martingales see Bingham & Kiesel (2001).

2.3 A simple Lévy-driven economy

2.3.1 The geometric Brownian motion example and one dimensional Lévy models

We shall try, in what follows, to understand something of the nature of the “risk premium” associated with jumps in asset prices. In particular we would like to understand the relation between risk, risk aversion, and the excess rate of return offered by risky assets. We wish to work in a general setting, without being tied too much to any particular model. We shall require that Assumption 1.2.3 holds. This shall be enough to characterise the existence of arbitrage-free asset prices. For mathematical simplicity, we shall assume that the dynamics of asset prices are driven by Lévy processes.

We recall what the set-up is in the geometric Brownian motion (GBM) model. The GBM model is of course a very simple model, but it captures the main features of the problem. In the one-dimensional case, we have a Brownian motion $\{W_t\}$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and the associated augmented filtration $\{\mathcal{F}_t\}$.

The GBM model is characterised by the specification of (a) a pricing kernel, and (b) a collection of one or more assets which, for simplicity, we assume to be non-dividend paying over the time-horizon considered. For the pricing kernel $\{\pi_t\}$ we write

$$\pi_t = e^{-rt} e^{-\lambda W_t - \frac{1}{2}\lambda^2 t} \quad (2.16)$$

for positive constants r and λ . Here r is the interest rate and λ is a “risk aversion” factor. For the price S_t of a typical investment asset we then write

$$S_t = S_0 e^{(r+\lambda\sigma)t} e^{\sigma W_t - \frac{1}{2}\sigma^2 t}, \quad (2.17)$$

for $\sigma > 0$, where σ is called the volatility of the asset (cf. Black & Scholes 1973, Bingham & Kiesel 2004). The term $\lambda\sigma$ is called the “risk premium” or “excess rate of return”, and is clearly positive under the assumptions made. We observe that the risk premium is monotonically increasing in both the volatility and the risk aversion factor.

Since $\lambda\sigma$ is linear in each factor we call λ the “market price of risk”. It should be evident, however, that there is no *a priori* reason why the excess rate of return should be linear in λ and σ .

We shall now attempt to construct a family of Lévy prices in the spirit of the GBM model. We shall call these geometric Lévy models (GLMs). The construction is as follows. Let $\{L_t\}$ be a Lévy process. In order for $\{L_t\}$ to give rise to a GLM we require that

$$\mathbb{E} [e^{\alpha L_t}] < \infty \quad (2.18)$$

for all $t \geq 0$ for some real interval $\alpha \in [\alpha^-, \alpha^+]$ containing the origin. It follows that there exists a function $\psi(-)$, the Laplace exponent, such that

$$\mathbb{E} [e^{\alpha L_t}] = e^{\psi(\alpha)t}, \quad (2.19)$$

is well defined for α in the range indicated. Writing $A = [\alpha^-, \alpha^+]$, and following Proposition 2.2.7, we see that for each $\alpha \in A$ the process $\{M_{t\alpha}\}_{t \geq 0}$ defined by

$$M_{t\alpha} = e^{\alpha L_t - \psi(\alpha)t} \quad (2.20)$$

is a martingale. Indeed, for $t \geq s \geq 0$ we have that

$$\begin{aligned}
 \mathbb{E}_s[M_{t\alpha}] &= \mathbb{E}_s[e^{\alpha L_t}] e^{-\psi(\alpha)t} \\
 &= \mathbb{E}_s[e^{\alpha(L_t - L_s)}] e^{\alpha L_s - \psi(\alpha)t} \\
 &= e^{\psi(\alpha)(t-s)} e^{\alpha L_s - \psi(\alpha)t} \\
 &= e^{\alpha L_s - \psi(\alpha)s} \\
 &= M_{s\alpha}.
 \end{aligned} \tag{2.21}$$

We shall construct the appropriate Lévy pricing kernel $\{\pi_t\}$. Let $\lambda > 0$ and assume that $-\lambda \in A$. Then set

$$\pi_t = e^{-rt} e^{-\lambda L_t - \psi(-\lambda)t}. \tag{2.22}$$

For a consistent pricing model we require that $\{\pi_t S_t\}_{t \geq 0}$ should be a martingale. Let us assume therefore that

$$\pi_t S_t = S_0 e^{\beta L_t - \psi(\beta)t}, \tag{2.23}$$

for some $\beta \in A$. It follows then that

$$\begin{aligned}
 S_t &= S_0 e^{rt} e^{(\beta + \lambda)L_t + (\psi(-\lambda) - \psi(\beta))t} \\
 &= S_0 e^{rt} e^{\sigma L_t + (\psi(-\lambda) - \psi(\sigma - \lambda))t},
 \end{aligned} \tag{2.24}$$

where $\sigma = \beta + \lambda$ and it is assumed that $\sigma \in A$. Thus we have obtained an asset price of the form

$$S_t = S_0 e^{(r + R(\lambda, \sigma))t} e^{\sigma L_t - \psi(\sigma)t}, \tag{2.25}$$

where

$$R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda). \tag{2.26}$$

We see immediately that

$$S_t = S_0 e^{(r + R(\lambda, \sigma))t} M_{t\sigma}, \tag{2.27}$$

and therefore that $R(\lambda, \sigma)$ is the excess rate of return. Indeed, we have

$$\mathbb{E}[S_t] = S_0 e^{(r + R(\lambda, \sigma))t}. \tag{2.28}$$

Proposition 2.3.1. *The excess rate of return, $R(\lambda, \sigma)$, is positive and monotonically increasing in λ and σ .*

Proof. We observe that

$$\psi(\alpha) = \frac{1}{t} \ln \mathbb{E} [e^{\alpha L_t}], \quad (2.29)$$

and by differentiation that

$$\partial_\alpha \psi(\alpha) = \frac{1}{t} \frac{\mathbb{E} [L_t e^{\alpha L_t}]}{\mathbb{E} [e^{\alpha L_t}]}. \quad (2.30)$$

Hence,

$$\begin{aligned} \frac{\partial^2 \psi(\alpha)}{\partial \alpha^2} &= \frac{1}{t} \frac{\mathbb{E} [L_t^2 e^{\alpha L_t}]}{\mathbb{E} [e^{\alpha L_t}]} - \frac{1}{t} \left(\frac{\mathbb{E} [L_t e^{\alpha L_t}]}{\mathbb{E} [e^{\alpha L_t}]} \right)^2 \\ &= \frac{1}{t} \frac{\mathbb{E} [(L_t - \bar{L}_t)^2 e^{\alpha L_t}]}{\mathbb{E} [e^{\alpha L_t}]}, \end{aligned} \quad (2.31)$$

where

$$\bar{L}_t = \frac{\mathbb{E} [L_t e^{\alpha L_t}]}{\mathbb{E} [e^{\alpha L_t}]}. \quad (2.32)$$

Now, recall that an Esscher transform may be used to define a change of measure. We can define a new measure $\mathbb{P}^\alpha$ by

$$\frac{d\mathbb{P}^\alpha}{d\mathbb{P}} = \frac{e^{\alpha L_t}}{\mathbb{E} [e^{\alpha L_t}]}, \quad (2.33)$$

where expectation with respect to $\mathbb{P}^\alpha$ will be denoted by $\mathbb{E}^\alpha[-]$. We therefore deduce that

$$\frac{\partial^2 \psi(\alpha)}{\partial \alpha^2} = \frac{1}{t} \left(\mathbb{E}^\alpha [(L_t - \mathbb{E}^\alpha [L_t])^2] \right). \quad (2.34)$$

Thus $\frac{\partial^2 \psi(\alpha)}{\partial \alpha^2}$ is proportional to the variance of L_t under $\mathbb{P}^\alpha$ and therefore

$$\frac{\partial^2 \psi(\alpha)}{\partial \alpha^2} > 0, \quad (2.35)$$

so $\psi(\alpha)$ is convex for $\alpha \in A$. Now, for $h > 0$ we have

$$\int_x^{x+h} \frac{\partial^2 \psi(\alpha)}{\partial \alpha^2} d\alpha = \partial_x \psi(x+h) - \partial_x \psi(x) > 0, \quad (2.36)$$

and for $b > a$ we have

$$\int_a^b \partial_x \psi(x+h) - \partial_x \psi(x) dx = \psi(b+h) - \psi(a+h) + \psi(a) - \psi(b) > 0. \quad (2.37)$$

Let $a = -\lambda$, $b = \sigma - \lambda$ and $h = \lambda$. It follows that

$$\psi(\sigma) - \psi(\sigma - \lambda) + \psi(-\lambda) > 0, \quad (2.38)$$

and from (2.26) we have that $R(\lambda, \sigma) > 0$ when $\lambda > 0$ and $\sigma > 0$. Next, we observe from (2.36) that

$$\partial_\sigma R(\lambda, \sigma) = \partial_\sigma \psi(\sigma) - \partial_\sigma \psi(\sigma - \lambda) > 0, \quad (2.39)$$

and

$$\partial_\lambda R(\lambda, \sigma) = \partial_\lambda \psi(-\lambda) - \partial_\lambda \psi(\sigma - \lambda) > 0. \quad (2.40)$$

Thus, the function $R(\lambda, \sigma)$ is monotonically increasing in both λ and σ . $\square$

In general $R(\lambda, \sigma)$ is a nonlinear function of λ and σ . This suggests that the notion of “market price of risk”, common in the finance literature, is in fact not quite right. Excess rate of return is the more correct notion. A relatively high value of σ , the volatility of the asset, indicates that the asset price process is relatively more variable. A high value of λ , the risk aversion factor of the market, indicates that investors require a high compensation for investing in risky assets.

Example 2.3.2. *The geometric Poisson model.* Let $L_t = N_t$ where $\{N_t\}_{t \geq 0}$ is a standard Poisson process with rate parameter μ , where

$$\mathbb{E} [e^{\alpha N_t}] = \exp(\mu t(e^\alpha - 1)). \quad (2.41)$$

From (2.25) the asset price is then

$$S_t = S_0 e^{(r + \mu(e^{-\lambda} - 1)(1 - e^\sigma))t + \sigma N_t - \mu t(e^\sigma - 1)}, \quad (2.42)$$

and the excess rate of return in this model is given by

$$R(\sigma, \lambda) = \mu (e^{-\lambda} - 1) (1 - e^\sigma). \quad (2.43)$$

A change of variables from σ and λ to a and b , where

$$\sigma = \log(1 + a), \quad (2.44)$$

and

$$-\lambda = \log(1 - b), \quad (2.45)$$

allows us to write excess return as μab . The excess rate of return is linear in a , b and μ . A higher value of a means that the asset is more risky and the risk premium should be higher, to compensate. The higher b is the more risk averse the market, and the risk premium increases to compensate. Separately, we have the parameter μ which also contributes to how volatile the asset is.

2.3.2 Foreign exchange

It is reasonable, in some situations, to require numeraire symmetry in an economy with multiple currencies. For example, if the dollar price of one pound sterling offers a positive excess rate of return we shall require that the sterling price of one dollar also offers a positive excess rate of return.

Let's examine the GBM case first (see also, e.g., Shreve 2004). Let the pound be the domestic currency, and the dollar the foreign currency. Let S_t denote the price of one dollar in pounds, and $\tilde{S}_t$ the price of one pound in dollars. We then have that

$$S_t = S_0 e^{(r-f+\lambda\sigma)t} e^{\sigma W_t - \frac{1}{2}\sigma^2 t} \quad (2.46)$$

for $\lambda, \sigma > 0$, where r and f are the domestic and foreign interest rates. Additionally the inverse exchange rate is given by

$$\begin{aligned} \tilde{S}_t &= \tilde{S}_0 e^{(f-r-\lambda\sigma)t} e^{-\sigma W_t + \frac{1}{2}\sigma^2 t} \\ &= \tilde{S}_0 e^{(f-r+\sigma^2-\lambda\sigma)t} e^{-\sigma W_t - \frac{1}{2}\sigma^2 t}. \end{aligned} \quad (2.47)$$

The excess rate of return on $\tilde{S}_t$ is given by $\sigma^2 - \sigma\lambda$ and is positive if and only if $\sigma > \lambda$.

In the GLM case we proceed analogously, and deduce that the exchange rate is given by

$$S_t = S_0 e^{(r-f+R(\lambda,\sigma))t} e^{\sigma L_t - \psi(\sigma)t}, \quad (2.48)$$

where $R(\lambda, \sigma) = \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda)$ is the excess rate of return. Correspondingly we may take the inverse of (2.48) to obtain

$$\begin{aligned}\tilde{S}_t &= \tilde{S}_0 e^{(f-r-R(\lambda, \sigma))t} e^{-\sigma L_t + \psi(\sigma)t} \\ &= \tilde{S}_0 e^{(f-r+\tilde{R}(\lambda, \sigma))t} e^{-\sigma L_t - \psi(-\sigma)t},\end{aligned}\tag{2.49}$$

where

$$\tilde{R}(\lambda, \sigma) = -R(\lambda, \sigma) + \psi(\sigma) + \psi(-\sigma)\tag{2.50}$$

will be called the “inverse” excess rate of return.

Proposition 2.3.3. *If $\sigma > \lambda > 0$, then both the excess rate of return, $R(\lambda, \sigma)$, and the inverse excess rate of return, $\tilde{R}(\lambda, \sigma)$, are positive.*

Proof. We observe that

$$\begin{aligned}\tilde{R}(\lambda, \sigma) &= -\psi(\sigma) - \psi(-\lambda) + \psi(\sigma - \lambda) + \psi(\sigma) + \psi(-\sigma) \\ &= \psi(-\sigma) + \psi(\sigma - \lambda) - \psi(-\lambda).\end{aligned}\tag{2.51}$$

Recall from Proposition 2.3.1 that

$$\psi(b+h) - \psi(a+h) + \psi(a) - \psi(b) > 0\tag{2.52}$$

for $b > a$ and $h > 0$. Now, suppose that $a = -\sigma$, $b = 0$ and $h = \sigma - \lambda$. Then $\tilde{R}(\lambda, \sigma) > 0$ for $\sigma > \lambda$ and $\lambda > 0$. The fact that $R(\lambda, \sigma) > 0$ follows from the result of Proposition 2.3.1 for general asset prices in a GLM. $\square$

Example 2.3.4. *Geometric Brownian motion.* The GBM model has a linear relationship between the excess return and the volatility and risk aversion parameters:

$$\begin{aligned}R(\lambda, \sigma) &= \psi(\sigma) + \psi(-\lambda) - \psi(\sigma - \lambda) \\ &= \frac{1}{2}\sigma^2 + \frac{1}{2}\lambda^2 - \frac{1}{2}(\sigma - \lambda)^2 \\ &= \sigma\lambda > 0.\end{aligned}\tag{2.53}$$

The inverse excess rate of return varies quadratically with the volatility parameter and linearly with risk aversion. Specifically,

$$\begin{aligned}\tilde{R}(\lambda, \sigma) &= \psi(-\sigma) + \psi(\sigma - \lambda) - \psi(-\lambda) \\ &= \frac{1}{2}\sigma^2 + \frac{1}{2}(\sigma - \lambda)^2 - \frac{1}{2}\lambda^2 \\ &= \sigma(\sigma - \lambda) > 0.\end{aligned}\tag{2.54}$$

Example 2.3.5. *Exponential gamma process with negative jumps.* Let $\{\gamma_t\}_{t \geq 0}$ be a standard gamma process with rate $m > 0$ such that $\mathbb{E}[\gamma_t] = mt$ and $\text{Var}[\gamma_t] = mt$. Then it is a standard result that

$$\mathbb{E} [e^{-\alpha \gamma_t}] = (1 + \alpha)^{-mt}. \quad (2.55)$$

Let $L_t = -\gamma_t$, so the jumps are “negative”. We deduce from (2.55) and the definition of $R(\lambda, \sigma)$ that

$$R(\lambda, \sigma) = m \ln \frac{1 + \sigma - \lambda}{(1 + \sigma)(1 - \lambda)}. \quad (2.56)$$

Now, observe also that $1 + \sigma - \lambda - (1 + \sigma)(1 - \lambda) = \sigma\lambda > 0$, and therefore that

$$\frac{1 + \sigma - \lambda}{(1 + \sigma)(1 - \lambda)} > 1 \quad (2.57)$$

and $R(\lambda, \sigma) > 0$. For the inverse excess rate of return we have

$$\tilde{R}(\lambda, \sigma) = m \ln \frac{1 - \lambda}{(1 - \sigma)(1 + \sigma - \lambda)}. \quad (2.58)$$

We note that $1 - \lambda - (1 - \sigma)(1 + \sigma - \lambda) = \sigma(\sigma - \lambda) > 0$ and therefore that $\tilde{R}(\lambda, \sigma) > 0$ for $\sigma > \lambda > 0$.

2.3.3 Higher dimensions

We can envisage the entire market as being driven by a single higher-dimensional Lévy process $\{\mathbf{L}_t\}_{t \geq 0}$. Thus, $\{\mathbf{L}_t\}$ is a vector Lévy process and for $\boldsymbol{\alpha}$, a vector whose elements exist within A , we may write

$$\mathbb{E} [e^{\boldsymbol{\alpha} \cdot \mathbf{L}_t}] = e^{\psi(\boldsymbol{\alpha})t}. \quad (2.59)$$

Let $\boldsymbol{\sigma}$ and $\boldsymbol{\lambda}$ be vectors of volatility and risk aversion parameters. The market is then modelled by a pricing kernel $\{\pi_t\}$ such that

$$\pi_t = e^{-rt} e^{-\boldsymbol{\lambda} \cdot \mathbf{L}_t - \psi(-\boldsymbol{\lambda})t}, \quad (2.60)$$

together with a collection of N assets $\{S_t^i\}$, for $i = 1, \dots, N$, such that

$$S_t^i = S_0^i e^{(r + R(\boldsymbol{\lambda}, \boldsymbol{\sigma}^i))t} e^{\boldsymbol{\sigma}^i \cdot \mathbf{L}_t - \psi(\boldsymbol{\sigma}^i)t}, \quad (2.61)$$

where

$$R(\boldsymbol{\lambda}, \boldsymbol{\sigma}^i) = \psi(\boldsymbol{\sigma}^i) + \psi(-\boldsymbol{\lambda}) - \psi(\boldsymbol{\sigma}^i - \boldsymbol{\lambda}). \quad (2.62)$$

2.3.4 Stochastic volatility

From Proposition 2.2.7 we know that if $\{\alpha_t\}_{t \geq 0}$ is a predictable process adapted to $\{\mathcal{F}_t\}$ such that α_t takes values on some subset of A , then the process defined by

$$M_t = e^{\int_0^t \alpha_s dL_s - \int_0^t \psi(\alpha_s) ds} \quad (2.63)$$

is a martingale.

Thus, we consider an asset pricing model of the following form: let the short rate $\{r_t\}$ be an adapted process and let $\{\lambda_t\}$ and $\{\sigma_t\}$ be predictable. Let the pricing kernel be given by

$$\pi_t = e^{-\int_0^t r_s ds} e^{-\int_0^t \lambda_s dL_s - \int_0^t \psi(-\lambda_s) ds}, \quad (2.64)$$

and for a typical asset write

$$S_t = S_0 e^{\int_0^t r_s + R(\lambda_s, \sigma_s) ds} e^{\int_0^t \sigma_s dL_s - \int_0^t \psi(\sigma_s) ds}, \quad (2.65)$$

where $R(\lambda, \sigma)$ is as before. Clearly, $\{\pi_t S_t\}_{t \geq 0}$ is a martingale. Also

$$\left\{ e^{-\int_0^t r_s ds} S_t \right\}_{t \geq 0} \quad (2.66)$$

is a submartingale and $R(\lambda_t, \sigma_t) > 0$ for $\lambda_t, \sigma_t > 0$.

2.3.5 The term structure of interest rates

Let $\{B_t\}$ denote the price process of a money market account and let $\{r_t\}$ denote the short rate, such that

$$B_t = \exp \left(\int_0^t r_s ds \right). \quad (2.67)$$

Then, using expression (2.65), we have that the price P_{tT} , of a T -maturity discount bond, is given by

$$P_{tT} = P_{0T} B_t \exp \left(\int_0^t R(\lambda_s, \sigma_{sT}) ds + \int_0^t \sigma_{sT} dL_s - \int_0^t \psi(\sigma_{sT}) ds \right), \quad (2.68)$$

where $\{\sigma_{tT}\}_{T \geq t \geq 0}$ is a predictable process with

$$\lim_{T \rightarrow t} \sigma_{tT} = 0. \quad (2.69)$$

Setting $T = t$, inserting (2.26) for $R(\lambda, \sigma)$, and rearranging in terms of the money market account we obtain

$$B_t = (P_{0t})^{-1} \exp \left(- \int_0^t \sigma_{st} dL_s - \int_0^t \psi(-\lambda_s) - \psi(\sigma_{st} - \lambda_s) ds \right). \quad (2.70)$$

For the short rate r_t we obtain the following:

$$r_t = -\partial_t \ln P_{0t} - \int_0^t \partial_t \sigma_{st} dL_s + \int_0^t \partial_t \psi(\sigma_{st} - \lambda_s) ds. \quad (2.71)$$

Putting (2.68) and (2.70) together we are left with the following novel formula for bond prices:

$$P_{tT} = P_{0tT} \frac{\exp \left(\int_0^t \sigma_{sT} dL_s + \int_0^t \psi(-\lambda_s) - \psi(\sigma_{sT} - \lambda_s) ds \right)}{\exp \left(\int_0^t \sigma_{st} dL_s + \int_0^t \psi(-\lambda_s) - \psi(\sigma_{st} - \lambda_s) ds \right)}. \quad (2.72)$$

Expression (2.72) is analogous to that of (1.7) which describes the bond price under the volatility approach to term-structure modelling outlined in Section 1.2.1.

Proposition 2.3.6. *Let the price of a T -maturity discount bond be given by (2.68). The pricing kernel $\{\pi_t\}$ then has the following representation:*

$$\pi_t = P_{0t} \exp \left(\int_0^t (\sigma_{st} - \lambda_s) dL_s - \int_0^t \psi(\sigma_{st} - \lambda_s) ds \right). \quad (2.73)$$

Proof. Observe from (2.22) that the pricing kernel may be written as

$$\pi_t = (B_t)^{-1} e^{-\int_0^t \lambda_s dL_s - \int_0^t \psi(-\lambda_s) ds}. \quad (2.74)$$

We may input expression (2.70), for the money market account, to obtain the result. $\square$

Expression (2.73) is a generalisation of (1.31). It links the volatility approach and the pricing kernel approach to term-structure modelling in a general Lévy setting.

2.4 Lévy processes in the asset-pricing literature

A good resource for jump models in finance is Cont & Tankov (2004). Jumps are important when studying data at high frequencies (see, e.g., Carr & Geman 2002, Bingham 2006, and references cited therein), but jump models are also useful at lower

frequencies as they are better equipped to model high risk events than the standard Brownian model, see also Cont (2001).

Examples of Lévy processes in the asset-pricing literature include processes of the normal inverse Gaussian type (Barndorff-Nielsen 1998), the generalized inverse Gaussian type (Barndorff-Nielsen & Shephard 2001), the variance gamma type (Madan & Seneta 1990), and the generalized hyperbolic type (Eberlein & Keller 1995). Further extensions such as Lévy driven Ornstein-Uhlenbeck processes (Barndorff-Nielsen & Shephard 2001), or stochastic volatility Lévy processes also exist (see for example Carr *et al.* 2003, Schoutens 2003). Brody *et al.* (2008) model cumulative gains processes in finance by use of a gamma process. Hoyle (2010) and Hoyle *et al.* (2011) studied *Lévy random bridges* in the context of information-based asset pricing.

Eberlein and co-authors use exponential Lévy martingales to develop a Lévy term-structure model with stochastic volatility (see Eberlein & Raible 1999, Eberlein *et al.* 2005, Eberlein & Kluge 2006a, 2006b, Eberlein *et al.* 2007 and references cited therein). Carr & Wu (2007) create Lévy option pricing models for equities and foreign exchange using the Esscher transform. In Carr & Wu (2007), stochastic volatility of foreign-exchange rates driven by Lévy processes is represented by a stochastic time-change.

The Eberlein & Raible (1999) and Carr & Wu (2007) models are formulated under an equivalent martingale measure; in other words, asset prices satisfy the requirement that $\{B_t^{-1}S_t\}$ is a martingale under this measure. The risk aversion and excess rate of return do not need to be modelled under this approach. The approach in Section 2.3.1, while producing models that are similar in some respects to those of Eberlein & Raible (1999) and Carr & Wu (2007), has the major advantage of being formulated in the physical measure with the risk aversion and excess rate of return specified in a fully natural way.

Chapter 3

Rational Term-Structure Models with Geometric Lévy Martingales

3.1 Chapter overview

In this chapter we introduce Lévy term-structure models of the Flesaker-Hughston type with the novel feature that economic uncertainty will be driven by geometric Lévy martingales. The resulting models are derived with the integral representation of the pricing kernel presented in Section 1.2.2 as the starting point.

The models under consideration are “rational” in the sense that the discount bond price is given by a ratio of integrals of positive martingales. The key ingredient is the martingale family $\{M_{tx}\}_{x \geq t \geq 0}$ present in the pricing formula (1.48). Closed-form expressions are provided for bond prices and interest rate derivatives.

3.2 An introduction to rational Lévy models

In what follows we propose to extend the methods of Flesaker and Hughston (see Flesaker & Hughston 1996, Rutkowski 1997) to incorporate Lévy randomness. We call the resulting models “rational Lévy models”. The main challenge is to include a range of possible distributions for bond prices while maintaining analytical tractability. An attractive feature of the approach is that it does not involve specifying and solving dynamics for bond prices or interest rates. Rather, we shall be modelling the solutions of these equations directly.

Our work can be compared to that of Eberlein and co-authors (see, e.g., Eberlein & Kluge 2007 and references cited therein), who set out to extend the HJM and market term-structure models to the Lévy case. One advantage our treatment has over theirs is that we are able to preserve the Flesaker-Hughston interest rate positivity conditions $\partial_T P_{tT} < 0$, and $0 \leq P_{tT} \leq 1$. Our work is motivated in part by Filipović *et al.* (2009) who extend the term-structure density approach introduced in Section 1.3 to incorporate Poisson random measure dynamics.

The rational Lévy models are derived as follows: let the assumptions of Section 1.2 hold such that the pricing kernel π_t , has the following integral representation:

$$\pi_t = \int_t^\infty \rho_x M_{tx} dx, \quad (3.1)$$

where the term-structure density ρ_x is defined by

$$\rho_x = -\partial_x P_{0x}, \quad (3.2)$$

and $\{M_{tx}\}_{x \geq t \geq 0}$ is a parametric family of positive martingales. We shall model $\{M_{tx}\}$ by use of the Esscher transform. We introduce the function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ and, given a Lévy process $\{L_t\}$ with Lévy triplet (z, σ, Π) , set

$$M_{tx} = \frac{e^{\phi_x L_t}}{\mathbb{E}[e^{\phi_x L_t}]}, \quad (3.3)$$

where

$$\int_{|z|>1} e^{\phi_x z} \Pi(dz) < \infty, \quad (3.4)$$

for all $x \geq 0$. We define the function $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}$ by setting

$$\theta_x = \frac{1}{t} \ln \mathbb{E}[e^{\phi_x L_t}]. \quad (3.5)$$

Inserting equations (3.5) and (3.3) into (3.1) we obtain a pricing kernel of the form

$$\pi_t = \int_t^\infty \rho_x \exp(\phi_x L_t - \theta_x t) dx. \quad (3.6)$$

It is implied, by the general bond pricing formula given by (1.27), that the price of a T -maturity discount bond is then

$$\begin{aligned} P_{tT} &= \frac{\mathbb{E}_t \left[\int_T^\infty \rho_x \exp(\phi_x L_T - \theta_x T) dx \right]}{\int_t^\infty \rho_x \exp(\phi_x L_t - \theta_x t) dx} \\ &= \frac{\int_T^\infty \rho_x \exp(\phi_x L_t - \theta_x t) dx}{\int_t^\infty \rho_x \exp(\phi_x L_t - \theta_x t) dx}, \end{aligned} \quad (3.7)$$

for $0 \leq t \leq T$. Recalling Assumption 1.2.5, we find that the instantaneous forward rates implied by the rational Lévy models are given by

$$f_{tT} = \frac{\rho_T \exp(\phi_T L_t - \theta_T t)}{\int_T^\infty \rho_x \exp(\phi_x L_t - \theta_x t) dx}, \quad (3.8)$$

and the corresponding short rate of interest is given by

$$r_t = \frac{\rho_t \exp(\phi_t L_t - \theta_t t)}{\int_t^\infty \rho_x \exp(\phi_x L_t - \theta_x t) dx}. \quad (3.9)$$

3.3 Introducing functional freedom

3.3.1 Markov short rates

In some circumstances it may be desirable to require that a term-structure model should have Markovian interest rates. In the rational Lévy models, even though the driving Lévy process $\{L_t\}$ is Markov, the associated rates need not be in general. The Markov property will, however, be preserved if the rates are strictly monotonic functions of L_t at time t .

Proposition 3.3.1. *If the function ϕ_x is strictly monotonic for $x \geq 0$, then the function $f : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$ defined by*

$$f(t, T, \xi) = \frac{\rho_T \exp(\phi_T \xi - \theta_T t)}{\int_T^\infty \rho_x \exp(\phi_x \xi - \theta_x t) dx}, \quad (3.10)$$

is strictly monotonic in the parameter ξ .

Proof. We differentiate expression (3.10) with respect to ξ to obtain

$$\frac{\partial f(t, T, \xi)}{\partial \xi} = f(t, T, \xi) (\phi_T - \Phi(t, T, \xi)), \quad (3.11)$$

where

$$\Phi(t, T, \xi) = \int_T^\infty \phi_x \mu(t, T, x, \xi) dx, \quad (3.12)$$

and

$$\mu(t, T, x, \xi) = \frac{\rho_x \exp(\phi_x \xi - \theta_x t)}{\int_T^\infty \rho_u \exp(\phi_u \xi - \theta_u t) du}. \quad (3.13)$$

Observe that the function

$$\Phi : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \quad (3.14)$$

is the weighted average of the function ϕ_x over values of x such that $x \geq T$. The weights $\mu(t, T, x, \xi)$ satisfy

$$\mu(t, T, x, \xi) > 0, \quad (3.15)$$

and

$$\int_T^\infty \mu(t, T, x, \xi) dx = 1, \quad (3.16)$$

for all $\xi \in \mathbb{R}$. It follows that if ϕ_x is strictly decreasing (*resp.* increasing) for $x \geq T$ then the derivative (3.11) will be strictly positive (*resp.* negative). The result thus obtained is that $f(t, T, \xi)$ is strictly increasing (*resp.* decreasing). $\square$

By use of (3.10) we can write expression (3.8) and (3.9) in the form

$$f_{tT} = f(t, T, L_t) \quad (3.17)$$

and

$$r_t = f(t, t, L_t). \quad (3.18)$$

If there is a bijective relationship between interest rates and L_t for all t , rates will preserve the Markov property of $\{L_t\}$. Specifically, let the market filtration $\{\mathcal{F}_t\}$ be generated by the observation of the instantaneous forward rates. Then, for $S \geq T \geq t \geq 0$ we have

$$\begin{aligned} \mathbb{E}[f_{TS} | \mathcal{F}_t] &= \mathbb{E}[f_{TS} | \{f_{uS}\}_{u \leq t}] \\ &= \mathbb{E}[f_{TS} | \{L_u\}_{u \leq t}] \\ &= \mathbb{E}[f_{TS} | L_t] \\ &= \mathbb{E}[f_{TS} | f_{tS}]. \end{aligned} \quad (3.19)$$

Hence we observe that $\{f_{tT}\}$ is Markov. The Markov property similarly holds for other rates and bond prices.

Remark 3.3.2. Monotonicity means that market observables are invertible in the state variable. In particular, we can obtain the value of L_t from the values of bond prices, forward rates, or even the short rate by inverting (3.7), (3.8) or (3.9). Thus the short rate is a state variable.

3.3.2 Bounded interest rates

In the literature many rational term-structure models have been developed such that they impose bounds on the short rate. These bounds stop interest rates from exploding or becoming negative, but they can also in some circumstances lead potentially to the mispricing of derivatives. One may refer to Goldberg (1997) or Cairns (2004) for further discussion of this point.

Example 3.3.3. *The rational lognormal model of Flesaker & Hughston (1996):* in the rational lognormal model the instantaneous forward rate is given by

$$f_{tT} = \frac{-\partial_T \alpha_T - \partial_T \beta_T m_t}{\alpha_T + \beta_T m_t}. \quad (3.20)$$

Here $\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and $\beta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ are positive, deterministic, strictly decreasing functions that for each $T \geq 0$ satisfy

$$\alpha_T + \beta_T = P_{0T}. \quad (3.21)$$

The process $\{m_t\}_{t \geq 0}$ is a martingale and follows a geometric Brownian motion. Let the function $f_{RLN} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be given by

$$f_{RLN}(z) = \frac{-\partial_T \alpha_T - \partial_T \beta_T z}{\alpha_T + \beta_T z}. \quad (3.22)$$

We then observe that

$$\lim_{z \rightarrow \infty} f_{RLN}(z) = -\frac{\partial_T \beta_T}{\beta_T}, \quad (3.23)$$

and

$$\lim_{z \rightarrow 0} f_{RLN}(z) = -\frac{\partial_T \alpha_T}{\alpha_T}. \quad (3.24)$$

We can see that in the rational lognormal model forward rates are bounded on account of the relation

$$-\frac{\partial_T \alpha_T}{\alpha_T} \leq f_{RLN}(z) \leq -\frac{\partial_T \beta_T}{\beta_T}. \quad (3.25)$$

In general, rational Lévy models need not have bounded rates but this can be guaranteed with some assumptions on ϕ_x .

Proposition 3.3.4. *Recall the function $f(t, T, \xi)$ given by equation (3.10). If the function ϕ_x is strictly monotonic and increasing for $x \geq 0$, then*

$$\lim_{\xi \rightarrow \infty} f(t, T, \xi) = 0, \quad (3.26)$$

and

$$\lim_{\xi \rightarrow -\infty} f(t, T, \xi) = \infty. \quad (3.27)$$

Correspondingly, if ϕ_x is strictly monotonic and decreasing, then

$$\lim_{\xi \rightarrow \infty} f(t, T, \xi) = \infty, \quad (3.28)$$

and

$$\lim_{\xi \rightarrow -\infty} f(t, T, \xi) = 0. \quad (3.29)$$

Proof. If ϕ_x is strictly monotonic and increasing then the quantity $\phi_x - \phi_T$ is positive for $x > T$. Therefore we observe that

$$\begin{aligned} \lim_{\xi \rightarrow \infty} f(t, T, \xi) &= \lim_{\xi \rightarrow \infty} \frac{\rho_T \exp(\phi_T \xi - \theta_T t)}{\int_T^\infty \rho_x \exp(\phi_x \xi - \theta_x t) \, dx} \\ &= \lim_{\xi \rightarrow \infty} \frac{\rho_T \exp(-\theta_T t)}{\int_T^\infty \rho_x \exp((\phi_x - \phi_T)\xi - \theta_x t) \, dx} \\ &= 0. \end{aligned} \quad (3.30)$$

Likewise we have

$$\lim_{\xi \rightarrow -\infty} f(t, T, \xi) = \infty. \quad (3.31)$$

The proof for the situation where ϕ_x is decreasing is similar; in that case $\phi_x - \phi_T$ will be negative for $x > T$. $\square$

It may be the case that $\{L_t\}$ takes values on some subset $\mathcal{A} \subset \mathbb{R}$. There may then be bounds on the interest rate. Suppose that $\mathcal{A} = [0, \infty)$. We observe that, for ϕ_x strictly monotonic and increasing, we have

$$\lim_{\xi \rightarrow 0} f(t, T, \xi) = \frac{\rho_T \exp(-\theta_T t)}{\int_T^\infty \rho_x \exp(-\theta_x t) \, dx}. \quad (3.32)$$

Therefore forward rates will have a strict upper bound, given by (3.32), when ϕ_x is increasing and the underlying Lévy process takes only positive values.

3.4 Infinite time horizon situation

In this section we assume that our models are defined on an infinite time horizon. We are then able to investigate their asymptotic properties.

3.4.1 Long-term maturity yields

The long-term bond yield Y_L has previously been studied in the literature by authors such as Dybvig *et al.* (1996), Hubalek *et al.* (1997), McCulloch (2000), Goldammer & Schmock (2010), and Kardaras & Platen (2011). The yield Y_L is defined by

$$Y_L = \lim_{T \rightarrow \infty} Y_{tT}, \quad (3.33)$$

where

$$Y_{tT} = \frac{-1}{T-t} \ln P_{tT} \quad (3.34)$$

is the continuously compounded yield on a T -maturity discount bond.

Following MucCulloch (2000) we know that without arbitrage, or the presence of market imperfections and frictions, Y_L should exist and should be constant. In some cases it is easier to consider the instantaneous forward rates rather than bond yields. Recall that $\lim_{T \rightarrow \infty} P_{tT} = 0$ under Assumption 1.2.6; we deduce that

$$\begin{aligned} Y_L &= \lim_{T \rightarrow \infty} \frac{-1}{T-t} \ln P_{tT} \\ &= \lim_{T \rightarrow \infty} \frac{1}{T-t} \int_t^T f_{ts} \, ds \\ &= \lim_{T \rightarrow \infty} f_{tT}. \end{aligned} \quad (3.35)$$

Expression (3.35) agrees with a similar result in Dybvig *et al.* (1996) that the long-term yield should be equal to the long-term simply-compounded forward rate.

In the case of the rational geometric Lévy models we present a sufficient condition to ensure that Y_L tends to a deterministic function. This function depends on the asymptotic behaviour of the initial term-structure density ρ_x .

Proposition 3.4.1. *Given that discount-bond prices are described by expression (3.7), let*

$$\lim_{x \rightarrow \infty} \phi_x = K, \quad (3.36)$$

where K is constant. Assume that the term-structure density ρ_x is differentiable. The long yield is then given by

$$Y_L = - \lim_{T \rightarrow \infty} \partial_T \ln \rho_T. \quad (3.37)$$

Proof. From (3.8) and (3.35) we calculate that Y_L is given by

$$Y_L = \lim_{T \rightarrow \infty} \frac{\rho_T \exp(\phi_T L_t - \theta_T t)}{\int_T^\infty \rho_x \exp(\phi_x L_t - \theta_x t) dx}. \quad (3.38)$$

Letting $K = 0$, one observes that

$$\lim_{T \rightarrow \infty} \exp(\phi_T L_t - \theta_T t) = 1. \quad (3.39)$$

Recalling that since ρ_x is a probability measure, we have

$$\lim_{x \rightarrow \infty} \rho_x = 0. \quad (3.40)$$

Hence, we are able to make the observation that

$$\begin{aligned} Y_L &= \lim_{T \rightarrow \infty} \frac{\rho_T}{\int_T^\infty \rho_x \exp(\phi_x L_t - \theta_x t) dx} \\ &= - \lim_{T \rightarrow \infty} \frac{\partial_T \rho_T}{\rho_T \exp(\phi_T L_t - \theta_T t)} \\ &= - \lim_{T \rightarrow \infty} \partial_T \ln \rho_T, \end{aligned} \quad (3.41)$$

by L'Hospital's rule, where we have implicitly assumed that the limit of $\partial_T \ln \rho_T$ exists.

For $K \neq 0$ observe that

$$\lim_{T \rightarrow \infty} \exp(\phi_T L_t - \theta_T t) = \exp(K L_t - \psi(K)t). \quad (3.42)$$

We are thus able to deduce that Y_L is given by

$$\begin{aligned} Y_L &= - \lim_{T \rightarrow \infty} \frac{\partial_T \rho_T \exp(K L_t - \psi(K)t)}{\rho_T \exp(K L_t - \psi(K)t)} \\ &= - \lim_{T \rightarrow \infty} \partial_T \ln \rho_T. \end{aligned} \quad (3.43)$$

□

It is instructive to verify Proposition 3.4.1 in the $t = 0$ case, where

$$P_{0T} = \int_T^\infty \rho_x dx. \quad (3.44)$$

In this case we have

$$\begin{aligned}
 Y_L &= - \lim_{T \rightarrow \infty} \frac{1}{T} \log \int_T^\infty \rho_x dx \\
 &= \lim_{T \rightarrow \infty} \frac{\rho_T}{\int_T^\infty \rho_x dx} \\
 &= - \lim_{T \rightarrow \infty} \partial_T \ln \rho_T.
 \end{aligned} \tag{3.45}$$

3.4.2 The limiting behaviour of the risk-free rate

Examining expressions (3.8) and (3.9) we can see that the choice $\phi_x = K$, for some constant K , gives rise to deterministic interest rates. Specifically, we obtain

$$f_{tT} = \frac{\rho_T}{\int_T^\infty \rho_x dx}, \tag{3.46}$$

and

$$r_t = \frac{\rho_t}{\int_t^\infty \rho_x dx}. \tag{3.47}$$

Interestingly, the forward rate is no longer dependent on t . Correspondingly, if ϕ_x becomes constant asymptotically, then term-structure evolution will be asymptotically deterministic. To explain we first recall a useful polynomial decomposition of the Esscher transform from the literature.

Definition 3.4.2. Let $\{L_t\}$ be a Lévy process with finite moments of all orders, and set $X_t = L_t - \mathbb{E}[L_t]$. Following Solé & Utzet (2008) we introduce a *space-time harmonic polynomial decomposition* of the Esscher transform of L_t by writing

$$\frac{e^{\phi_x L_t}}{\mathbb{E}[e^{\phi_x L_t}]} = \sum_{n=0}^{\infty} \frac{\phi_x^n}{n!} R_n(X_t, t), \tag{3.48}$$

where ϕ_x satisfies (3.4). For each $n \geq 0$ the $R_n(X_t, t)$ are space-time harmonic polynomials of order n .

For further discussion regarding stochastic processes and associated polynomials good references are Schoutens & Teugels (1998), Schoutens (2000), and Sengupta (2000).

Proposition 3.4.3. *Let economic uncertainty be driven by a Lévy process $\{L_t\}$ with finite moments of all orders. Assume that the function ϕ_t satisfies*

$$\lim_{t \rightarrow \infty} \phi_t t = 0, \quad (3.49)$$

and that $|\phi_t|$ is decreasing. Then the short rate given by (3.9) is asymptotically deterministic. In particular,

$$\lim_{t \rightarrow \infty} r_t = - \lim_{t \rightarrow \infty} \partial_t \ln \rho_t. \quad (3.50)$$

Proof. Recall that short rates in the rational Lévy models are given by

$$r_t = \frac{\rho_t \exp(\phi_t L_t - \theta_t t)}{\int_t^\infty \rho_x \exp(\phi_x L_t - \theta_x t) \, dx}. \quad (3.51)$$

By use of the decomposition (3.48) we may express the numerator of (3.51) in terms of space-time harmonic polynomials:

$$\rho_t \exp(\phi_t L_t - \theta_t t) = \rho_t \sum_{n=0}^{\infty} \frac{\phi_t^n}{n!} R_n(X_t, t). \quad (3.52)$$

Recall that the law of large numbers for Lévy processes states that,

$$\lim_{t \rightarrow \infty} \frac{X_t}{t} = 0 \quad \text{a.s.} \quad (3.53)$$

It follows that under condition (3.49) we must have that

$$\lim_{t \rightarrow \infty} \phi_t X_t = 0 \quad \text{a.s.} \quad (3.54)$$

Thus, under condition (3.49) we have the result

$$\lim_{t \rightarrow \infty} \rho_t \exp(\phi_t L_t - \theta_t t) = \lim_{t \rightarrow \infty} \rho_t. \quad (3.55)$$

This follows as, in the limit, we only consider the leading terms of the polynomials in expression (3.52) such that

$$\begin{aligned} \lim_{t \rightarrow \infty} \rho_t \sum_{n=0}^{\infty} \frac{\phi_t^n}{n!} R_n(X_t, t) &= \lim_{t \rightarrow \infty} \rho_t \sum_{n=0}^{\infty} \frac{\phi_t^n}{n!} (X_t^n + t^n) \\ &= \lim_{t \rightarrow \infty} \rho_t \exp(\phi_t X_t + \phi_t t). \end{aligned} \quad (3.56)$$

Therefore we have that

$$\lim_{t \rightarrow \infty} \rho_t \sum_{n=0}^{\infty} \frac{\phi_t^n}{n!} R_n(X_t, t) = \lim_{t \rightarrow \infty} \rho_t \quad (3.57)$$

as

$$\lim_{t \rightarrow \infty} \exp(\phi_t X_t + \phi_t t) = 1 \quad (3.58)$$

due to (3.54).

With regard to the denominator of (3.51), we wish to show that

$$\lim_{t \rightarrow \infty} \int_t^{\infty} \rho_x \exp(\phi_x L_t - \theta_x t) dx = \lim_{t \rightarrow \infty} \int_t^{\infty} \rho_x dx. \quad (3.59)$$

Equivalently, by use of the sandwich theorem we shall prove that

$$0 \leq \lim_{t \rightarrow \infty} \left| \int_t^{\infty} \rho_x \exp(\phi_x L_t - \theta_x t) dx - \int_t^{\infty} \rho_x dx \right| \leq 0. \quad (3.60)$$

Using the space-time harmonic polynomial decomposition of L_t in (3.48) and the triangle inequality, we obtain

$$\begin{aligned} \left| \int_t^{\infty} \rho_x \exp(\phi_x L_t - \theta_x t) dx - \int_t^{\infty} \rho_x dx \right| &\leq \left| \int_t^{\infty} \rho_x \exp(\phi_x L_t - \theta_x t) dx \right| + \int_t^{\infty} \rho_x dx \\ &= \left| \int_t^{\infty} \rho_x \sum_{n=0}^{\infty} \frac{\phi_x^n}{n!} R_n(X_t, t) dx \right| + \int_t^{\infty} \rho_x dx \\ &\leq \int_t^{\infty} \left| \rho_x \sum_{n=0}^{\infty} \frac{\phi_x^n}{n!} R_n(X_t, t) \right| dx + \int_t^{\infty} \rho_x dx \\ &\leq \int_t^{\infty} \rho_x \sum_{n=0}^{\infty} \frac{|\phi_t|^n}{n!} |R_n(X_t, t)| dx + \int_t^{\infty} \rho_x dx, \end{aligned} \quad (3.61)$$

where we have made use of the fact that $|\phi_t| > |\phi_s|$ for all $t > s$. Considering only the leading order terms of $R_n(X_t, t)$, we find that the first term on the right hand side of (3.61) becomes, in the limit,

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_t^{\infty} \rho_x \sum_{n=0}^{\infty} \frac{|\phi_t|^n}{n!} |R_n(X_t, t)| dx &= \lim_{t \rightarrow \infty} \int_t^{\infty} \rho_x dx \sum_{n=0}^{\infty} \frac{|\phi_t|^n}{n!} |X_t^n + t^n| \\ &\leq \lim_{t \rightarrow \infty} \int_t^{\infty} \rho_x dx \sum_{n=0}^{\infty} \frac{|\phi_t|^n}{n!} (|X_t|^n + t^n) \\ &= \lim_{t \rightarrow \infty} \int_t^{\infty} \rho_x dx \exp(|\phi_t|(|X_t| + t)) \\ &= \lim_{t \rightarrow \infty} \int_t^{\infty} \rho_x dx. \end{aligned} \quad (3.62)$$

Now, recall that $\lim_{t \rightarrow \infty} \rho_t = 0$, and therefore that

$$\lim_{t \rightarrow \infty} \int_t^\infty \rho_x \, dx = 0. \quad (3.63)$$

Combining (3.62), (3.61) and (3.63), we observe that

$$\begin{aligned} \lim_{t \rightarrow \infty} \left| \int_t^\infty \rho_x \exp(\phi_x L_t - \theta_x t) \, dx - \int_t^\infty \rho_x \, dx \right| &\leq \lim_{t \rightarrow \infty} \left(\int_t^\infty \rho_x \, dx + \int_t^\infty \rho_x \, dx \right) \\ &= 0. \end{aligned} \quad (3.64)$$

Thus, the sandwich theorem (3.60) yields

$$\lim_{t \rightarrow \infty} \int_t^\infty \rho_x \exp(\phi_x L_t - \theta_x t) \, dx = \lim_{t \rightarrow \infty} \int_t^\infty \rho_x \, dx. \quad (3.65)$$

To complete the proof we combine (3.52) and (3.65) to obtain

$$\begin{aligned} \lim_{t \rightarrow \infty} r_t &= \lim_{t \rightarrow \infty} \frac{\rho_t}{\int_t^\infty \rho_x \, dx} \\ &= - \lim_{t \rightarrow \infty} \partial_t \ln \rho_t. \end{aligned} \quad (3.66)$$

□

Example 3.4.4. *Flat initial term structure.* In the situation where the initial term structure is flat at some continuously compounded rate R we deduce that

$$\rho_t = R e^{-Rt}. \quad (3.67)$$

Choosing ϕ_x such that

$$\phi_x = e^{-\kappa x} \quad (3.68)$$

for $\kappa > 0$ we can see that

$$\lim_{t \rightarrow \infty} \phi_t t = 0 \quad (3.69)$$

and, by the law of large numbers for Lévy processes that

$$\lim_{t \rightarrow \infty} \phi_t X_t = 0 \quad \text{a.s.} \quad (3.70)$$

Thus, Proposition 3.4.3 states that

$$\begin{aligned} \lim_{t \rightarrow \infty} r_t &= - \lim_{t \rightarrow \infty} \partial_t \ln \rho_t \\ &= R. \end{aligned} \quad (3.71)$$

3.4.3 The money market account

It follows heuristically from general considerations of the requirement that markets should be free of arbitrage that the money market account $\{B_t\}_{t \geq 0}$ should eventually grow to infinity. Specifically, it is reasonable to insist that the condition

$$\lim_{t \rightarrow \infty} B_t = \infty \quad (3.72)$$

should hold. Equivalently we have the following condition on the short rate:

$$\lim_{t \rightarrow \infty} \int_0^t r_s ds = \infty. \quad (3.73)$$

It is sufficient to require the assumptions of Proposition 3.4.3 for (3.73) to hold if $-\partial_t \ln \rho_t > 0$. To explain, we consider the case where interest rates are deterministic for all t , i.e., $\phi_x = 0$ for all $x \geq 0$. The short rate is then given by

$$r_t = \frac{\rho_t}{\int_t^\infty \rho_x dx}, \quad (3.74)$$

and we can deduce that

$$\begin{aligned} \int_0^t r_s ds &= \int_0^t \frac{\rho_s}{\int_s^\infty \rho_x dx} ds \\ &= \log \left(\int_0^\infty \rho_x dx \right) - \log \left(\int_t^\infty \rho_x dx \right) \\ &= -\log \left(\int_t^\infty \rho_x dx \right). \end{aligned} \quad (3.75)$$

Taking limits, we then obtain that

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_0^t r_x dx &= -\lim_{t \rightarrow \infty} \log \left(\int_t^\infty \rho_x dx \right) \\ &= \infty. \end{aligned} \quad (3.76)$$

More generally, interest rates that asymptotically tend to a positive function $-\partial_t \ln \rho_t$ as in Proposition 3.4.3, combined with interest rate positivity, grant the result (3.72), since the integral in (3.73) will not converge.

3.5 Option pricing with rational Lévy models

Evaluating the price of an option requires calculating the expectation of its payoff. This is made considerably easier with the following simplifying assumption:

Assumption 3.5.1. *We assume that the function ϕ_x is strictly monotonic.*

3.5.1 Options on discount bonds

The price at time zero of a European-style call option expiring at time t , with strike price K , on a discount bond maturing at time T , is given in full generality by the following formula:

$$C_{0t} = \mathbb{E}[\pi_t(P_{tT} - K)^+]. \quad (3.77)$$

We assume that $0 < K < 1$. By use of (1.27) and the positivity of π_t it follows that

$$C_{0t} = \mathbb{E}[(\mathbb{E}_t[\pi_T] - K\pi_t)^+]. \quad (3.78)$$

Then, by inserting expression (3.1) for the pricing kernel we obtain

$$C_{0t} = \mathbb{E} \left[\left(\int_T^\infty \rho_x M_{tx} dx - K \int_t^\infty \rho_x M_{tx} dx \right)^+ \right]. \quad (3.79)$$

In the rational Lévy setting, (3.79) becomes

$$C_{0t} = \mathbb{E} \left[\left(\int_T^\infty \rho_x e^{\phi_x L_t - \theta_x t} dx - K \int_t^\infty \rho_x e^{\phi_x L_t - \theta_x t} dx \right)^+ \right]. \quad (3.80)$$

We thus proceed to evaluate this expectation.

Proposition 3.5.2. *We observe that if ϕ_x is decreasing (resp. increasing) in x , then the function $P : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow [0, 1]$ defined by*

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x \exp(\phi_x \xi - \theta_x t) dx}{\int_t^\infty \rho_x \exp(\phi_x \xi - \theta_x t) dx} \quad (3.81)$$

is decreasing (resp. increasing) in the variable ξ .

Proof. The argument is as follows. A short calculation shows that

$$\frac{\partial P(t, T, \xi)}{\partial \xi} = P(t, T, \xi) (\Phi(t, T, \xi) - \Phi(t, t, \xi)), \quad (3.82)$$

where the function $\Phi(t, T, \xi)$ is given by (3.12) and

$$\Phi(t, t, \xi) = \lim_{T \rightarrow t} \Phi(t, T, \xi). \quad (3.83)$$

We observe that if ϕ_x is decreasing then

$$\frac{\partial \Phi(t, T, \xi)}{\partial T} < 0. \quad (3.84)$$

This follows from the fact that

$$\frac{\partial \Phi(t, T, \xi)}{\partial T} = f(t, T, \xi) (\Phi(t, T, \xi) - \phi_T), \quad (3.85)$$

where we recall that $f(t, T, \xi)$ is given by expression (3.10). We note that $\Phi(t, T, \xi)$ is for each value of ξ a weighted average of ϕ_x for $x \geq T$. Thus, if ϕ_x is decreasing, then the right-hand side of equation (3.85) is negative. If the right-hand side of (3.85) is negative (*resp.* positive) then (3.82) is negative (*resp.* positive). It follows that if ϕ_x is decreasing in x , then $P(t, T, \xi)$ is decreasing in ξ , as claimed. A similar argument shows that if ϕ_x is increasing in x for all $x \geq 0$ then $P(t, T, \xi)$ is increasing in ξ . $\square$

Let us assume now that $P(t, T, \xi)$ is monotonic in ξ , and write $P^+(t, T)$ and $P^-(t, T)$ for the upper and lower extremal values of $P(t, T, \xi)$ as ξ varies. Then for any K in the range $[P^-(t, T), P^+(t, T)]$ we can find a number ξ^* such that

$$P(t, T, \xi^*) = K. \quad (3.86)$$

This enables us to truncate the expectation in (3.80) at the point where the maximum-function becomes non-positive. It then follows that the price of an option in the general Lévy case takes the form

$$C_{0t} = \int_T^\infty \rho_x m_{tx}(\xi^*) dx - K \int_t^\infty \rho_u m_{tu}(\xi^*) du, \quad (3.87)$$

where

$$m_{tx}(\xi^*) = \mathbb{E} \left[\mathbb{1} \left\{ \int_T^\infty \rho_u M_{tu} du > K \int_t^\infty \rho_u M_{tu} du \right\} M_{tx} \right]. \quad (3.88)$$

The function $m_{tx}(-)$ represents a family of cumulative distribution functions. When we do not have a convenient closed form we can represent $m_{tx}(-)$ in terms of the characteristic function of L_t , under the assumption that L_t has a probability density. Assume that ϕ_x is decreasing. By use of the change of measure formula (2.10) we have

$$\begin{aligned} m_{tx}(\xi^*) &= \mathbb{E}^x \left[\mathbb{1} \left\{ \int_T^\infty \rho_u M_{tu} du > K \int_t^\infty \rho_u M_{tu} du \right\} \right] \\ &= \mathbb{P}^x (P_{tT} > K) \\ &= \mathbb{P}^x (L_t \leq \xi^*). \end{aligned} \quad (3.89)$$

Here $\mathbb{E}^x[-]$ denotes expectation under the measure $\mathbb{P}^x$ which is defined by the measure change

$$\frac{d\mathbb{P}^x}{d\mathbb{P}} = \frac{e^{\phi_x L_t}}{\mathbb{E}[e^{\phi_x L_t}]} \quad (3.90)$$

Now, using the representation of cumulative distribution functions presented in Shephard (1991) (see also Bakshi & Madan 2000) we have that

$$\mathbb{P}^x(L_t \leq \xi^*) = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty \operatorname{Re} \left(\frac{\mathbb{E}[e^{(iu+\phi_x)L_t}] e^{-iu\xi^*}}{iu\mathbb{E}[e^{\phi_x L_t}]} \right) du. \quad (3.91)$$

Sometimes $m_{tx}(\xi^*)$ is available in closed form, depending on the particular underlying Lévy distribution. We shall present specific examples of Lévy processes where (3.89) is closed-form in later chapters.

3.5.2 Digital option pricing

Digital options have a payoff of unity if a reference Libor rate is in a pre-specified range at expiry. Digital options that expire at time t and pay out at time $T > t$ are said to pay “in arrears”. Following Eberlein & Kluge (2006b) we define the price at some time T , of an in-arrears European-style digital call option, with expiry time $t < T$ and strike rate L_K , to be

$$D_{Tt} = \mathbb{1} \{L_{t,t+\delta} > L_K\}, \quad (3.92)$$

where

$$L_{t,t+\delta} = \frac{1}{\delta} \left(\frac{1}{P_{t,t+\delta}} - 1 \right) \quad (3.93)$$

is the δ -maturity reference Libor rate. As $L_{t,t+\delta}$ is $\mathcal{F}_t$ -measurable the time t price of the call option is equal to

$$D_{tt} = P_{tT} \mathbb{1} \{L_{t,t+\delta} > L_K\}. \quad (3.94)$$

We thus calculate the time-zero value of the call by taking the unconditional discounted expectation of D_{tt} . Specifically, we observe that

$$D_{0t} = \mathbb{E}[\pi_t P_{tT} \mathbb{1} \{L_{t,t+\delta} > L_K\}]. \quad (3.95)$$

Information regarding digital options can be found in Nunes (2004) and in Eberlein & Kluge (2006b).

Within the rational Lévy models closed-form prices for digital options are available. Recalling the function $P(t, T, \xi)$ given by (3.81), we have

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x \exp(\phi_x \xi - \theta_x t) \, dx}{\int_t^\infty \rho_x \exp(\phi_x \xi - \theta_x t) \, dx}. \quad (3.96)$$

By Proposition 3.5.2 and Assumption 3.5.1 there exists a number ξ^D such that

$$P(t, t + \delta, \xi^D) = \frac{1}{1 + \delta L_K}. \quad (3.97)$$

Inserting (1.27) and (1.47) for π_t and P_{tT} into (3.95), we are then able to obtain a closed-form price given by

$$\begin{aligned} D_{0t} &= \mathbb{E} [\mathbb{E}_t[\pi_T] \mathbb{1} \{L_{t,t+\delta} > L_K\}] \\ &= \int_T^\infty \rho_x m_{tx}(\xi^D) \, dx, \end{aligned} \quad (3.98)$$

for the digital call option. Here

$$m_{tx}(\xi^D) = \mathbb{E} \left[\mathbb{1} \left\{ P_{t,t+\delta} < \frac{1}{1 + \delta L_K} \right\} M_{tx} \right], \quad (3.99)$$

where the condition

$$P_{t,t+\delta} < \frac{1}{1 + \delta L_K} \quad (3.100)$$

is equivalent to

$$L_{t,t+\delta} > L_K. \quad (3.101)$$

3.5.3 Swaption pricing

A swaption gives the holder the right to enter into an interest-rate swap at a pre-specified swap rate. A payer (*resp.* receiver) swap gives the holder the right to pay (*resp.* receive) a fixed rate of interest in exchange for receiving (*resp.* paying) a floating rate. We shall call a swaption that gives the holder the right to enter into a payer (*resp.* receiver) swap a “payer” (*resp.* “receiver”) swaption.

Following, e.g., Hull (2000), we interpret a swap as being equivalent to a trade in a fixed-coupon bond, and the opposite position in a floating-coupon bond where the

coupon dates coincide. The coupon rate of the fixed-coupon bond S is called the swap rate. The price of a floating bond is initially equal to its par value. The swap rate is chosen such that the initial value of the swap is equal to zero.

We thus interpret the swaption as a right to exchange a coupon bond, with its coupon equal to a pre-specified swap rate, against a floating rate note whose value is equal to one. Thus, a receiver (*resp.* payer) swaption is a call (*resp.* put) on a coupon bond with an exercise price of unity.

Following Eberlein & Kluge (2006a) we use the method of Jamshidian (1989) to value options on coupon bonds. The procedure is as follows. We denote by $B_t(c)$, the time t price of a coupon bond with maturity T_n paying its owner the amounts of $c_1, c_2, \dots, c_n$ at the dates $T_1, T_2, \dots, T_n$. For t such that $T_1 > t \geq 0$, we then have that

$$B_t(c) = c_1 P_{tT_1} + c_2 P_{tT_2} + \dots + c_n P_{tT_n}. \quad (3.102)$$

To obtain closed-form prices we assume that the function

$$g : \mathbb{R}_+^n \times \mathbb{R}_+ \times \mathbb{R}_+^n \times \mathbb{R} \rightarrow \mathbb{R}, \quad (3.103)$$

given by

$$g(\{c_i\}_{n \geq i}, t, \{T_i\}_{n \geq i}, \xi) = \sum_{i=1}^n c_i P(t, T_i, \xi), \quad (3.104)$$

is monotonic in ξ . Here the function $P(t, T_i, \xi)$ is given by (3.81). By Proposition 3.5.2, expression (3.104) is monotonic when we assume that ϕ_x is monotonic. For $t < T_1$, the arbitrage-free, time-zero price of a t -expiry call option on a coupon bond with strike price unity is then given by

$$\begin{aligned} C_{0t}^B &= \mathbb{E} \left[\pi_t \left(\sum_{i=1}^n c_i P_{tT_i} - 1 \right)^+ \right] \\ &= \mathbb{E} \left[\pi_t (g(\{c_t^i\}, t, \{T_i\}, \xi) - 1)^+ \right] \\ &= \sum_{i=1}^n c_t^i \int_{T_i}^{\infty} \rho_x m_{tx}(\xi^B) dx - \int_t^{\infty} \rho_x m_{tx}(\xi^B) dx. \end{aligned} \quad (3.105)$$

Here

$$m_{tx}(\xi^B) = \mathbb{E} [\mathbb{1} \{g(\{c_t^i\}, t, \{T_i\}, L_t) > 1\} M_{tx}], \quad (3.106)$$

where ξ^B is defined by

$$g(\{c_t^i\}, t, \{T_i\}, \xi^B) = 1. \quad (3.107)$$

In the context of a swaption, let $c_1 = c_2 = \dots = c_{n-1} = S_K$ and $c_n = 1 + S_K$, where S_K is the pre-specified swap rate of the underlying swap multiplied by 100. It must be remembered that principal payments are not exchanged in a swap contract. The price, C_{0t}^S , of a European-style call option on a receiver swaption with swap rate S_K is then given by

$$\begin{aligned} C_{0t}^S &= \mathbb{E} \left[\pi_t \left(\sum_{i=1}^n S_K P_{tT_i} - 1 \right)^+ \right] \\ &= \mathbb{E} \left[\pi_t (g(S_K, t, \{T_i\}, L_t) - 1)^+ \right] \\ &= S_K \sum_{i=1}^n \int_{T_i}^{\infty} \rho_x m_{tx}(\xi^S) dx - \int_t^{\infty} \rho_x m_{tx}(\xi^S) dx, \end{aligned} \quad (3.108)$$

where

$$m_{tx}(\xi^S) = \mathbb{E} [\mathbf{1}\{g(S_K, t, \{T_i\}, L_t) > 1\} M_{tx}], \quad (3.109)$$

and ξ^S is defined by

$$g(S_K, t, T, \xi^S) = 1. \quad (3.110)$$

Chapter 4

The Geometric Brownian Motion Martingale Family

4.1 Chapter overview

In this chapter we consider in some detail the rational Lévy models when the martingale family $\{M_{tx}\}_{x \geq t \geq 0}$ is a geometric Brownian motion. The resulting Brownian family of models has appeared in related forms in the literature (see Brody & Hughston 2001, 2002, and Brody & Friedman 2009). We derive essentially the same formula for the price of a European-style call option on a discount bond as Brody & Friedman (2009), albeit in a different setting.

One of the main contributions of this chapter is to determine restrictions on the function ϕ_x sufficient to ensure that the risk premium, which is determined by ϕ_x , will remain positive.

4.2 Brownian term-structure models

Writing $\{W_t\}_{t \geq 0}$ for a standard one-dimensional Brownian motion, we make use of the fact that $\{W_t\}$ is a Lévy process to obtain a geometric Lévy martingale family of the form

$$M_{tx} = \exp\left(\phi_x W_t - \frac{1}{2}\phi_x^2 t\right) \tag{4.1}$$

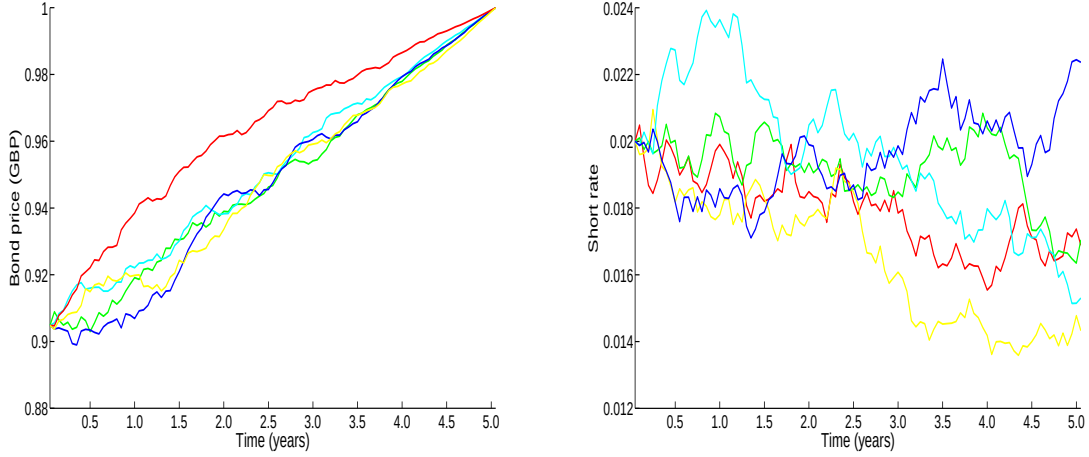


Figure 4.1: *Brownian bond-price and short-rate simulations.* This figure shows sample paths for the price of a discount bond and the associated short rate based on the Brownian family of rational Lévy models. We use expression (4.3) for the bond price and (4.4) for the short rate where $\phi_x = 0.3e^{-0.02x}$. We have used a flat initial term structure with annual rate 2% such that $P_{0T} = e^{-0.02T}$. The maturity of the bond is five years.

for $0 \leq t \leq x$. Here we have used the basic result that

$$\mathbb{E} [e^{\phi_x W_t}] = e^{\frac{1}{2}\phi_x^2 t}. \quad (4.2)$$

This leads to a bond price of the form

$$P_{tT} = \frac{\int_T^\infty \rho_x \exp(\phi_x W_t - \frac{1}{2}\phi_x^2 t) dx}{\int_t^\infty \rho_x \exp(\phi_x W_t - \frac{1}{2}\phi_x^2 t) dx} \quad (4.3)$$

for $0 \leq t \leq T < \infty$ (cf. Brody & Friedman 2009). It then follows that the corresponding short rate is of the form

$$r_t = \frac{\rho_t \exp(\phi_t W_t - \frac{1}{2}\phi_t^2 t)}{\int_t^\infty \rho_x \exp(\phi_x W_t - \frac{1}{2}\phi_x^2 t) dx}. \quad (4.4)$$

Simulations of the bond price (4.3) and the short rate (4.4) are presented in Figure 4.1.

By an application of Ito's Lemma to equation (4.3), we deduce that the dynamics of the bond price system are given by

$$\frac{dP_{tT}}{P_{tT}} = (r_t - \Phi_{tt}(\Phi_{tT} - \Phi_{tt}))dt + (\Phi_{tT} - \Phi_{tt})dW_t, \quad (4.5)$$

where

$$\Phi_{tT} = \frac{\int_T^\infty \phi_x \rho_x \exp\left(\phi_x W_t - \frac{1}{2} \phi_x^2 t\right) dx}{\int_T^\infty \rho_x \exp\left(\phi_x W_t - \frac{1}{2} \phi_x^2 t\right) dx} \quad (4.6)$$

and where $\Phi_{tt} = \Phi_{tT}|_{T=t}$. On comparison of (4.5) with equation (1.1) we observe in particular that the discount bond volatility is of the form

$$\Omega_{tT} = \Phi_{tT} - \Phi_{tt}, \quad (4.7)$$

and that the market price of risk is given by

$$\lambda_t = -\Phi_{tt}. \quad (4.8)$$

The bond volatility and the market price of risk together give us the risk premium, or excess rate of return, associated with an investment in a discount bond:

$$\lambda_t \Omega_{tT} = \Phi_{tt} (\Phi_{tt} - \Phi_{tT}). \quad (4.9)$$

It is of interest to derive the corresponding short-rate dynamics in the Brownian models. An application of Ito's Lemma to (4.4) gives

$$dr_t = \left(\frac{\partial_t \rho_t}{\rho_t} + (\partial_t \phi_t) W_t - \phi_t (\partial_t \phi_t) t + r_t - \frac{1}{2} \phi_t^2 \right) r_t dt + (\phi_t - \Phi_{tt}) r_t (dW_t - \Phi_{tt} dt). \quad (4.10)$$

Similarly, in the $\mathbb{P}$ -measure, the dynamics of the forward rates are given by

$$df_{tT} = (\phi_T - \Phi_{tT}) f_{tT} (dW_t - \Phi_{tT} dt), \quad (4.11)$$

where Φ_{tT} can now be viewed as the price of risk associated with changing to the T -forward risk neutral measure. This is the measure under which instantaneous forward rates are martingales.

Denote the forward rate volatility process by $\{\sigma_{tT}\}_{T \geq t \geq 0}$. We can compare the forward-rate volatility, which is given by

$$\sigma_{tT} = f_{tT} (\phi_T - \Phi_{tT}), \quad (4.12)$$

with the Brody-Hughston positive interest condition on HJM volatilities in (1.55).

Remark 4.2.1. Expression (4.12) describes the HJM volatility that would create a class of HJM models equivalent to the rational Brownian family of models.

If we change to the risk-neutral measure with price of risk $\{-\Phi_{tt}\}$, then our forward rate dynamics become

$$df_{tT} = (\phi_T - \Phi_{tT})(\Phi_{tt} - \Phi_{tT})f_{tT}dt + (\phi_T - \Phi_{tT})f_{tT}dW_t^*, \quad (4.13)$$

where

$$W_t^* = W_t - \int_0^t \Phi_{ss} ds \quad (4.14)$$

is a Brownian motion under the risk-neutral measure. We can compare (4.13) to the HJM forward rate dynamics implied by (4.12), which take the form

$$df_{tT} = \left((\phi_T - \Phi_{tT})f_{tT} \int_t^T (\phi_s - \Phi_{ts})f_{ts} ds \right) dt + (\phi_T - \Phi_{tT})f_{tT}dW_t^*, \quad (4.15)$$

where we have applied the standard HJM risk neutral drift condition. We then observe the relationship

$$\int_t^T (\phi_s - \Phi_{ts})f_{ts} ds = (\Phi_{tt} - \Phi_{tT}). \quad (4.16)$$

A special case of the Brownian family, the so-called “semi-linear” model, has been considered in Brody & Hughston (2002), where ϕ_x is chosen to be $\phi_x = \alpha + \beta x$. As an elementary example, if we take as initial condition a flat term structure $P_{0T} = e^{-RT}$, with yield R , then the initial term-structure density ρ_x is exponential:

$$\rho_x = R e^{-Rx}. \quad (4.17)$$

The bond price in the semi-linear model then becomes a ratio of a pair of Gaussian integrals that are readily performed to give

$$P_{tT} = \frac{\mathcal{N}\left(\pm \frac{W_t - R/\beta}{\sqrt{t}} \mp (\alpha + \beta T)\sqrt{t}\right)}{\mathcal{N}\left(\pm \frac{W_t - R/\beta}{\sqrt{t}} \mp (\alpha + \beta t)\sqrt{t}\right)}. \quad (4.18)$$

Here the $(\pm, \mp)$ signs correspond to whether β is positive or negative; the sign combination is $(+, -)$ if $\beta < 0$ and $(-, +)$ if $\beta > 0$. Although the semi-linear model admits a simple analytic expression for the bond price, as we indicate below it gives rise to a negative market price of risk. A number of resulting sample paths of (4.18) are illustrated in Figure 4.2.

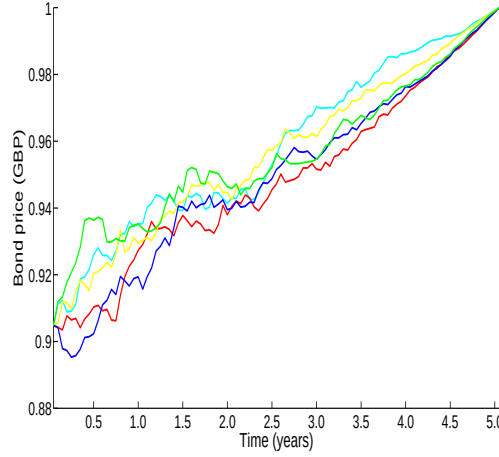


Figure 4.2: *Semi-linear Brownian bond-price simulations.* This figure shows sample paths for the bond price based on the “semi-linear” Brownian rational Lévy model. We use the pricing equation (4.18) where $R = 0.02$, $\alpha = 0.05$, and $\beta = 0.05$. The maturity of the bond is five years.

Another special case of the Gaussian term-structure model is obtained by setting

$$\phi_x = \sigma e^{-bx}, \quad (4.19)$$

where σ and b are constants (cf. Brody & Friedman 2009). In this case, if we choose the flat initial term structure (4.17), then the expression for the bond price reduces to a semi-Gaussian form:

$$P_{tT} = \frac{\int_0^{e^{-bT}} y^{R/b-1} e^{-\frac{1}{2}(\sigma\sqrt{t}y - W_t/\sqrt{t})^2} dy}{\int_0^{e^{-bt}} y^{R/b-1} e^{-\frac{1}{2}(\sigma\sqrt{t}y - W_t/\sqrt{t})^2} dy}. \quad (4.20)$$

In particular, both the denominator and the numerator can be expressed in terms of hypergeometric functions. Numerical simulations of the bond price and the associated short rate, in this model, are therefore straightforward.

4.3 Positive risk premia

4.3.1 Sufficient conditions for positive risk premia

In some contexts it is desirable to impose a condition to the effect that the discount-bond risk premium should be positive.

Proposition 4.3.1. *If we assume that either ϕ_x is positive and decreasing for all $x \geq 0$, or ϕ_x is negative and increasing for all $x \geq 0$ then the discount-bond risk premium, given by*

$$\lambda_t \Omega_{tT} = \Phi_{tt} (\Phi_{tt} - \Phi_{tT}), \quad (4.21)$$

is positive.

Proof. Suppose that ϕ_x is positive for all $x \geq 0$. Then it follows from (4.6) that Φ_{tt} is positive. Differentiating (4.6) with respect to T we obtain

$$\partial_T \Phi_{tT} = f_{tT} (\Phi_{tT} - \phi_T), \quad (4.22)$$

where

$$f_{tT} = \frac{\rho_T \exp(\phi_T W_t - \frac{1}{2} \phi_T^2 t)}{\int_T^\infty \rho_x \exp(\phi_x W_t - \frac{1}{2} \phi_x^2 t) dx} \quad (4.23)$$

is the instantaneous forward rate, defined as usual by

$$f_{tT} = -\partial_T \ln P_{tT}, \quad (4.24)$$

which is of course positive in this model. Next we observe that (4.6) can be written in the form

$$\Phi_{tT} = \int_T^\infty \phi_x \mu_{tT}(x) dx, \quad (4.25)$$

where

$$\mu_{tT}(x) = \frac{\rho_x \exp(\phi_x W_t - \frac{1}{2} \phi_x^2 t)}{\int_T^\infty \rho_u \exp(\phi_u W_t - \frac{1}{2} \phi_u^2 t) du}. \quad (4.26)$$

Note that $\mu_{tT}(x)$ is positive and that

$$\int_T^\infty \mu_{tT}(x) dx = 1. \quad (4.27)$$

Thus, according to (4.25), Φ_{tT} is a weighted average of the values of ϕ_x for x greater than or equal to T . It follows that if ϕ_x is decreasing as a function of x , then $\Phi_{tT} < \phi_T$ for $0 \leq t \leq T$. This in turn implies, by use of (4.22), that $\partial_T \Phi_{tT} < 0$, and hence by (4.21) that the risk premium is positive. A similar argument shows that if ϕ_x is negative and increasing for $x \geq 0$, then the risk premium is positive. $\square$

4.3.2 Necessary conditions for positive risk premia

We are also interested in determining the restrictions on the function ϕ_x that result if the risk premium is assumed to be positive. In particular, when $\Phi_{tt}(\Phi_{tt} - \Phi_{tT}) > 0$, we know that either

$$\Phi_{tt} > 0 \quad \text{and} \quad \Phi_{tT} - \Phi_{tt} < 0, \quad (4.28)$$

or

$$\Phi_{tt} < 0 \quad \text{and} \quad \Phi_{tT} - \Phi_{tt} > 0. \quad (4.29)$$

Proposition 4.3.2. *Assume that for any admissible initial term-structure density ρ_x either (4.28) holds or (4.29) holds. The function ϕ_x then must be either strictly positive and strictly decreasing, or strictly negative and strictly increasing.*

Proof. Assume that (4.29) holds. Let us denote by $\mu_t(x)$ a random probability measure on $[t, \infty)$ obtained by setting

$$\mu_t(x) = \frac{\rho_x \exp(\phi_x W_t - \frac{1}{2} \phi_x^2 t)}{\int_t^\infty \rho_u \exp(\phi_u W_t - \frac{1}{2} \phi_u^2 t) du}. \quad (4.30)$$

Note that $\mu_t(x)$ arises if we set T equal to t in (4.26). The measure $\mu_t(x)$ has the properties that

$$\mu_t(x) > 0, \quad (4.31)$$

and

$$\int_t^\infty \mu_t(x) dx = 1, \quad (4.32)$$

and that

$$\mu_0(x) = \rho_x. \quad (4.33)$$

We can express the condition $\Phi_{tt} < 0$ in the form

$$\int_t^\infty \phi_x \mu_t(x) dx < 0, \quad (4.34)$$

which must hold for all $0 \leq t < T \leq \infty$. Specifically, at $t = 0$ we have

$$\int_0^\infty \phi_x \rho_x dx < 0. \quad (4.35)$$

Now, condition (4.35) is required to hold for any admissible initial term-structure density. In particular we may choose

$$\rho_x = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left(-\frac{(x-t)^2}{2\varepsilon^2}\right), \quad (4.36)$$

for fixed $t \geq 0$ and $\varepsilon > 0$. From (4.35) we then obtain the relationship

$$\begin{aligned} \phi_t &= \int_0^\infty \phi_x \delta(x-t) dx \\ &= \int_0^\infty \phi_x \lim_{\varepsilon \rightarrow 0} \rho_x dx \\ &= \lim_{\varepsilon \rightarrow 0} \int_0^\infty \phi_x \rho_x dx < 0, \end{aligned} \quad (4.37)$$

where $\delta(x)$ is the Dirac delta function. Thus, for any initial term-structure density ρ_x , the function ϕ_x must be negative for $x \geq 0$. Next we observe that since $\Phi_{tT} - \Phi_{tt} > 0$ by assumption (4.29), the fact that this inequality must hold for any T slightly greater than t implies that

$$\left. \frac{\partial \Phi_{tT}}{\partial T} \right|_{T=t} > 0 \quad (4.38)$$

for all $t \geq 0$. Thus by equation (4.22), and using the fact that the instantaneous forward rates are positive,

$$\Phi_{tt} > \phi_t, \quad (4.39)$$

for all $t \geq 0$. Therefore we obtain

$$\int_t^\infty \phi_x \mu_t(x) dx > \phi_t \quad (4.40)$$

for all $t \geq 0$. Now letting

$$\rho_x = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left(-\frac{(x-u)^2}{2\varepsilon^2}\right) \quad (4.41)$$

for $u > t$, by use of (4.30) and taking the limit as ε goes to zero, we see that

$$\phi_u > \phi_t. \quad (4.42)$$

Similar arguments can be made for the proof under condition (4.28). $\square$

4.4 The limiting behaviour of the risk-free rate

The space-time harmonic polynomial representation of the Brownian martingale (4.1) is given by

$$\frac{e^{\phi_x W_t}}{\mathbb{E}[e^{\phi_x W_t}]} = \sum_{n=0}^{\infty} \frac{\phi_x^n}{n!} H_n(W_t, t). \quad (4.43)$$

Here $H_n(-, -)$ are the monic Hermite polynomials. Monic Hermite polynomials are related to classical Hermite polynomials $h_n(-)$ by the following formula:

$$H_n(W_t, t) = t^{n/2} h_n\left(\frac{W_t}{\sqrt{t}}\right). \quad (4.44)$$

The representation (4.43) is well known, as is the fact that, for each $n \geq 0$, the process $\{H_n(W_t, t)\}_{t \geq 0}$ is a martingale. For more information one can see, for example, Brody & Hughston (2004) and Solé & Utzet (2008).

Proposition 4.4.1. *Let the initial term-structure density ρ_x be given. Under the assumptions that*

$$\lim_{t \rightarrow \infty} \phi_t t = 0, \quad (4.45)$$

and $|\phi_t|$ is decreasing, the short rate r_t defined by (4.4) asymptotically tends to a deterministic function. In particular, one has

$$\lim_{t \rightarrow \infty} r_t = - \lim_{t \rightarrow \infty} \partial_t \ln \rho_t. \quad (4.46)$$

Proof. Recall the law of large numbers for Brownian motion:

$$\lim_{t \rightarrow \infty} \frac{W_t}{t} = 0 \quad \text{a.s.} \quad (4.47)$$

By Proposition 3.4.3 if we choose ϕ_t such that (4.45) holds, we have the desired result. In particular, recall that the short rate is given by

$$r_t = \frac{\rho_t e^{\phi_t W_t - \frac{1}{2} \phi_t^2 t}}{\int_t^\infty \rho_x e^{\phi_x W_t - \frac{1}{2} \phi_x^2 t} dx}. \quad (4.48)$$

If we examine the numerator we can see that

$$\begin{aligned} \lim_{t \rightarrow \infty} \rho_t e^{\phi_t W_t - \frac{1}{2} \phi_t^2 t} &= \lim_{t \rightarrow \infty} \rho_t \sum_{n=0}^{\infty} \frac{\phi_t^n}{n!} H_n(W_t, t) \\ &= \lim_{t \rightarrow \infty} \rho_t e^{\phi_t (W_t + t)} \\ &= \lim_{t \rightarrow \infty} \rho_t. \end{aligned} \quad (4.49)$$

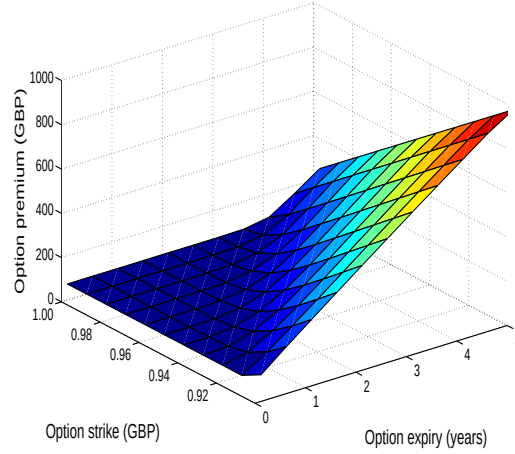


Figure 4.3: *Option pricing in the Brownian family.* This figure shows sample option prices based on the Brownian family of rational Lévy models. We use the pricing equation (4.51) where $\phi_x = 0.3e^{-0.02x}$. We have used a flat term initial structure with annual rate 2% such that $P_{0T} = e^{-0.02T}$. The maturity of the bond is five years and the strike and option expiry are varied.

Then, one may follow the proof of Proposition 3.4.3 to show that similarly, the denominator tends to $\int_t^\infty \rho_x dx$ in the limit. $\square$

4.5 Option pricing in the rational Brownian family of models

Let us consider the option pricing problem in the case of the geometric Brownian motion family. In particular, when the underlying Lévy process is a geometric Brownian motion, and ϕ_x is positive and decreasing, the option price given by (3.87) simplifies to the following expression:

$$C_{0t} = \int_T^\infty \rho_x \mathbb{E} \left[e^{\phi_x W_t - \frac{1}{2} \phi_x^2 t} \mathbf{1} \{P_{tT} > K\} \right] dx - K \int_t^\infty \rho_x \mathbb{E} \left[e^{\phi_x W_t - \frac{1}{2} \phi_x^2 t} \mathbf{1} \{P_{tT} > K\} \right] dx. \quad (4.50)$$

After a short calculation this reduces to

$$C_{0t} = \int_T^\infty \rho_x N \left(\frac{\xi^*}{\sqrt{t}} - \phi_x \sqrt{t} \right) dx - K \int_t^\infty \rho_x N \left(\frac{\xi^*}{\sqrt{t}} - \phi_x \sqrt{t} \right) dx, \quad (4.51)$$

where $N(-)$ denotes the standard normal distribution function, and the number ξ^* is defined by

$$P(t, T, \xi^*) = K, \quad (4.52)$$

where

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x \exp\left(\phi_x \xi - \frac{1}{2} \phi_x^2 t\right) dx}{\int_t^\infty \rho_x \exp\left(\phi_x \xi - \frac{1}{2} \phi_x^2 t\right) dx}. \quad (4.53)$$

The option price can therefore be computed in a manner similar to the Black-Scholes theory. It is worth remarking that an analogous formula for the option price is derived in Brody & Friedman (2009), but based on a slightly different model formulation. Figure 4.3 gives an example of a bond-option price surface generated by the Gaussian model. Prices are given for a range of strikes and maturities.

Chapter 5

Discussion concerning the Market Price of Risk

5.1 Chapter overview

For pricing, the Brownian family of models in Chapter 4 does not necessarily have to be developed under the physical probability measure. If it is, then the market price of risk is determined by the function ϕ_x which can potentially be calibrated to option prices. In general, however, this may not be true; Andruszkiewicz & Brody (2011) make a similar observation.

In this chapter we introduce a probability measure $\mathbb{P}^*$, equivalent to $\mathbb{P}$, which we call the pricing measure (cf. Cairns 2004, 2006). We then observe that bond prices remain in the same rational form as in Chapter 4. This is due to the fact that a Radon-Nikodym derivative in the numerator and the denominator cancel, although Girsanov's theorem results in the $\mathbb{P}$ -dynamics changing. The new $\mathbb{P}$ -dynamics are obtained via a measure change from $\mathbb{P}^*$ to $\mathbb{P}$. As a result the market price of risk is no longer determined only by the function ϕ_x (see expression 4.6) but by ϕ_x and the Radon-Nikodym derivative between $\mathbb{P}^*$ and $\mathbb{P}$. In particular, the market price of risk is determined by ϕ_x and an additional term that does not appear in prices and thus cannot be calibrated to option prices. However, theoretical results indicate that when one employs a one-factor model, one may be able to rank investment opportunities according to the size of their risk premiums where information regarding the market price of risk is obtained from option prices.

5.2 The choice of probability measure

We introduce a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. We let $\mathbb{P}$ be the physical measure, and let $\{W_t\}_{t \geq 0}$ be a $\mathcal{F}_t$ -Brownian motion under $\mathbb{P}$.

Definition 5.2.1. The pricing measure $\mathbb{P}^*$ is defined by

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = m_t, \quad (5.1)$$

where

$$m_t = \exp \left(- \int_0^t \alpha_s dW_s - \frac{1}{2} \int_0^t \alpha_s^2 ds \right). \quad (5.2)$$

Here $\{\alpha_t\}_{t \geq 0}$ is an adapted process satisfying conditions sufficient to ensure that $\{m_t\}_{t \geq 0}$ is a martingale under $\mathbb{P}$. Expectation and conditional expectation under $\mathbb{P}^*$ are denoted by $\mathbb{E}^*[-]$ and $\mathbb{E}_t^*[-]$ respectively.

Let the assumptions of Section 1.2 hold such that the pricing kernel π_t will have the integral representation (1.47). We define $\{m_t M_{tx}\}_{x \geq t \geq 0}$ be a family of $\mathbb{P}$ -martingales such that discount-bond prices $\{P_{tT}\}$ satisfy

$$\partial_T \pi_t P_{tT} = (\partial_T P_{0T}) m_t M_{tT}. \quad (5.3)$$

Then, under $\mathbb{P}^*$, $\{M_{tx}\}_{x \geq t \geq 0}$ is a family of martingales. We observe that for $T \geq t \geq 0$ we have that

$$\begin{aligned} \mathbb{E}^* [M_{Tx} | \mathcal{F}_t] &= \frac{1}{m_t} \mathbb{E} [m_T M_{Tx} | \mathcal{F}_t] \\ &= \frac{m_t M_{tx}}{m_t} \\ &= M_{tx}. \end{aligned} \quad (5.4)$$

The change of measure does not affect the Flesaker-Hughston form of the bond price. In particular, we have

$$\begin{aligned} P_{tT} &= \frac{\int_T^\infty \rho_x m_t M_{tx} dx}{\int_t^\infty \rho_x m_t M_{tx} dx} \\ &= \frac{\int_T^\infty \rho_x M_{tx} dx}{\int_t^\infty \rho_x M_{tx} dx}. \end{aligned} \quad (5.5)$$

Thus, specifying martingale dynamics for $\{M_{tx}\}$ is equivalent to specifying the dynamics for $\{P_{tT}\}$ under the measure $\mathbb{P}^*$. Let $\{W_t^*\}_{t \geq 0}$ be a Brownian motion under $\mathbb{P}^*$, and let the form of the martingale family $\{M_{tx}\}$ be given by

$$M_{tx} = \exp\left(\phi_x W_t^* - \frac{1}{2}\phi_x^2 t\right). \quad (5.6)$$

The price of a discount bond in this example is given by

$$P_{tT} = \frac{\int_T^\infty \rho_x \exp\left(\phi_x W_t^* - \frac{1}{2}\phi_x^2 t\right) dx}{\int_t^\infty \rho_x \exp\left(\phi_x W_t^* - \frac{1}{2}\phi_x^2 t\right) dx}, \quad (5.7)$$

and the short rate is given by

$$r_t = \frac{\rho_t \exp\left(\phi_t W_t^* - \frac{1}{2}\phi_t^2 t\right)}{\int_t^\infty \rho_x \exp\left(\phi_x W_t^* - \frac{1}{2}\phi_x^2 t\right) dx}. \quad (5.8)$$

Applying Itô's Lemma to (5.7) we deduce that the bond price is characterised by the stochastic differential equation

$$\frac{dP_{tT}}{P_{tT}} = (r_t - \Phi_{tt}^* (\Phi_{tT}^* - \Phi_{tt}^*)) dt + (\Phi_{tT}^* - \Phi_{tt}^*) dW_t^*, \quad (5.9)$$

where

$$\Phi_{tT}^* = \frac{\int_T^\infty \phi_x \rho_x \exp\left(\phi_x W_t^* - \frac{1}{2}\phi_x^2 t\right) dx}{\int_T^\infty \rho_x \exp\left(\phi_x W_t^* - \frac{1}{2}\phi_x^2 t\right) dx}, \quad (5.10)$$

and

$$\Phi_{tt}^* = \lim_{T \rightarrow t} \Phi_{tT}^*. \quad (5.11)$$

The quantities $-\Phi_{tt}^*$ and $\Phi_{tt}^* (\Phi_{tt}^* - \Phi_{tT}^*)$ do not have the interpretation of market price of risk and excess return.

Now, define the risk-neutral measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}^*} = \exp\left(\int_0^t \Phi_{ss}^* dW_s^* - \frac{1}{2} \int_0^t (\Phi_{ss}^*)^2 ds\right). \quad (5.12)$$

Then, by Girsanov's Theorem, we have that

$$W_t^\mathbb{Q} = W_t^* - \int_0^t \Phi_{ss}^* ds \quad (5.13)$$

is a Brownian motion under $\mathbb{Q}$. The dynamics of $\{P_{tT}\}$ under the risk-neutral measure are then given by

$$\frac{dP_{tT}}{P_{tT}} = r_t dt + (\Phi_{tT}^{\mathbb{Q}} - \Phi_{tt}^{\mathbb{Q}}) dW_t^{\mathbb{Q}}, \quad (5.14)$$

where

$$\Phi_{tT}^{\mathbb{Q}} = \frac{\int_T^\infty \phi_x \rho_x \exp \left(\int_0^t \phi_x + \Phi_{ss}^* dW_s^{\mathbb{Q}} - \frac{1}{2} \int_0^t (\phi_x + \Phi_{ss}^*)^2 ds \right) dx}{\int_T^\infty \rho_x \exp \left(\int_0^t \phi_x + \Phi_{ss}^* dW_s^{\mathbb{Q}} - \frac{1}{2} \int_0^t (\phi_x + \Phi_{ss}^*)^2 ds \right) dx}, \quad (5.15)$$

and

$$\Phi_{tt}^{\mathbb{Q}} = \lim_{T \rightarrow t} \Phi_{tT}^{\mathbb{Q}}. \quad (5.16)$$

Given an adapted market price of risk process $\{\lambda_t\}_{t \geq 0}$ it is possible to define a $\mathbb{P}$ -Brownian motion $\{W_t\}_{t \geq 0}$ by

$$W_t = W_t^{\mathbb{Q}} - \int_0^t \lambda_s ds, \quad (5.17)$$

and

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp \left(- \int_0^t \lambda_s dW_s - \frac{1}{2} \int_0^t \lambda_s^2 ds \right). \quad (5.18)$$

The process $\{\lambda_t\}$ is defined such that (5.18) is a martingale. Discount bond dynamics under $\mathbb{P}$ are then given by

$$\frac{dP_{tT}}{P_{tT}} = (r_t + \lambda_t (\Phi_{tT} - \Phi_{tt})) dt + (\Phi_{tT} - \Phi_{tt}) dW_t, \quad (5.19)$$

where

$$\Phi_{tT} = \frac{\int_T^\infty \phi_x \rho_x \exp \left(\int_0^t \phi_x + \Phi_{ss}^* + \lambda_s dW_s - \frac{1}{2} \int_0^t (\phi_x + \Phi_{ss}^* + \lambda_s)^2 ds \right) dx}{\int_T^\infty \rho_x \exp \left(\int_0^t \phi_x + \Phi_{ss}^* + \lambda_s dW_s - \frac{1}{2} \int_0^t (\phi_x + \Phi_{ss}^* + \lambda_s)^2 ds \right) dx}, \quad (5.20)$$

and

$$\Phi_{tt} = \lim_{T \rightarrow t} \Phi_{tT}. \quad (5.21)$$

Remark 5.2.2. It can be shown that

$$\Phi_{tT}^* = \Phi_{tT}^{\mathbb{Q}} = \Phi_{tt}. \quad (5.22)$$

However, dynamics under $\mathbb{P}^*$, $\mathbb{Q}$ and $\mathbb{P}$ are different.

We have thus specified the mechanics by which we can move from $\mathbb{P}^*$ to $\mathbb{Q}$ to $\mathbb{P}$. The corresponding measure change from $\mathbb{P}^*$ to $\mathbb{P}$ will give us an expression for the price of risk process $\{\lambda_t\}$. The change from $\mathbb{P}^*$ to $\mathbb{P}$ is described by (5.1) and (5.2). By Girsanov's Theorem and the $\mathbb{P}^*$ -dynamics given by (5.9), discount-bond dynamics under $\mathbb{P}$ are then of the form

$$\frac{dP_{tT}}{P_{tT}} = (r_t - (\Phi_{tt} - \alpha_t)(\Phi_{tT} - \Phi_{tt})) dt + (\Phi_{tT} - \Phi_{tt}) dW_t, \quad (5.23)$$

where

$$W_t = W_t^* - \int_0^t \alpha_s ds. \quad (5.24)$$

Thus, by comparing (1.1) and (5.23) we identify the price of risk process $\{\lambda_t\}$ as

$$\lambda_t = -(\Phi_{tt} - \alpha_t). \quad (5.25)$$

Remark 5.2.3. Andruszkiewicz & Brody (2011) derive the same expression for the discount-bond dynamics (5.23) through a generalisation of the function ϕ_x . They let

$$\phi_x = \psi_x - \alpha_t, \quad (5.26)$$

where $\{\alpha_t\}$ is defined above, and define the $\mathbb{P}$ -martingale family $\{M_{tx}\}$ by

$$M_{tx} = \exp \left(\int_0^t (\psi_x - \alpha_s) dW_s - \frac{1}{2} \int_0^t (\psi_x - \alpha_s)^2 ds \right). \quad (5.27)$$

Bond prices in this model are given by inserting (5.27) into (1.48) to obtain

$$\begin{aligned} P_{tT} &= \frac{\int_T^\infty \rho_x \exp \left(\int_0^t (\psi_x - \alpha_s) dW_s - \frac{1}{2} \int_0^t (\psi_x - \alpha_s)^2 ds \right) dx}{\int_t^\infty \rho_x \exp \left(\int_0^t (\psi_x - \alpha_s) dW_s - \frac{1}{2} \int_0^t (\psi_x - \alpha_s)^2 ds \right) dx} \\ &= \frac{\int_T^\infty \rho_x \exp \left(\psi_x W_t^* - \frac{1}{2} \psi_x^2 t \right) dx}{\int_t^\infty \rho_x \exp \left(\psi_x W_t^* - \frac{1}{2} \psi_x^2 t \right) dx}, \end{aligned} \quad (5.28)$$

where

$$W_t^* = W_t + \int_0^t \alpha_s ds. \quad (5.29)$$

Remark 5.2.4. Cairns (2004, 2006) also works in the pricing measure, and not in the physical measure. Cairns begins with the Flesaker-Hughston bond pricing formula and models $\{M_{tx}\}$ as a family of $\mathbb{P}^*$ martingales.

5.3 The information in asset prices and ranking investment opportunities

Recall that the present value of a European-style call option with expiry t and strike price K is given in full generality by

$$C_{0t} = \mathbb{E} [\pi_t (P_{tT} - K)^+]. \quad (5.30)$$

The addition of the process $\{\alpha_t\}$ to discount-bond dynamics does not affect the form of option prices. Following Anduszkiewicz & Brody (2011), we can define the pricing kernel $\{\pi_t\}$ to be

$$\pi_t = \int_t^\infty \rho_x m_t M_{tx} dx. \quad (5.31)$$

Inserting (5.31) and (5.5) into (5.30), and using $\{m_t\}$ to the pricing measure $\mathbb{P}^*$ we obtain

$$\begin{aligned} C_{0t} &= \mathbb{E} \left[m_t \int_t^\infty \rho_x M_{tx} dx (P_{tT} - K)^+ \right] \\ &= \mathbb{E}^* \left[\left(\int_T^\infty \rho_x M_{tx} dx - K \int_T^\infty \rho_x M_{tx} dx \right)^+ \right]. \end{aligned} \quad (5.32)$$

By Proposition 3.5.2, if we assume that ϕ_x is monotonically decreasing we can find a number ξ^* such that

$$P(t, T, \xi^*) = K, \quad (5.33)$$

where

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x \exp(\phi_x \xi - \frac{1}{2} \phi_x^2 t) dx}{\int_t^\infty \rho_x \exp(\phi_x \xi - \frac{1}{2} \phi_x^2 t) dx}. \quad (5.34)$$

The option pricing formula is then given by

$$C_{0t} = \int_T^\infty \rho_x N \left(\frac{\xi^*}{\sqrt{t}} - \phi_x \sqrt{t} \right) dx - K \int_t^\infty \rho_x N \left(\frac{\xi^*}{\sqrt{t}} - \phi_x \sqrt{t} \right) dx. \quad (5.35)$$

Thus we can make the same observation as Anduszkiewicz & Brody (2011); potentially full information about $\{-\Phi_{tt}\}_{t \geq 0}$ can be obtained from option prices once a parametric form for ϕ_x has been specified. The estimation of $\{\alpha_t\}$, and therefore the

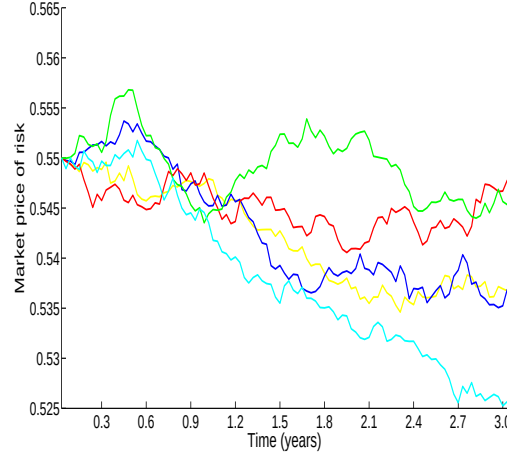


Figure 5.1: *Market price of risk simulations.* This figure shows sample paths for the market price of risk, given by expression (5.36), based on the Brownian family of rational Lévy models where $\alpha = 0.4$, and $\phi_x = -0.3e^{-0.02x}$. We have used a flat initial term structure with annual rate 2% such that $P_{0T} = e^{-0.02T}$.

market price of risk $\{-(\Phi_{tt} - \alpha_t)\}$, requires one to observe the dynamics (5.23). In the literature it is often expressed that it may be difficult to measure the drift of diffusion processes. This phenomenon is sometimes referred to as the “mean blur”. An accurate estimate of a constant drift can potentially require a data set that spans a large period (see Merton 1980, Campbell *et al.* 1997 and Bertsimas *et al.* 2000). Rogers (2001) creates a consumption-based model with parameter uncertainty regarding asset returns. Rogers then remarks that over 1500 years worth of data is required for investors to learn the value of the return with desired precision. In the rational Brownian models of Chapter 4 we are implicitly assuming that $\alpha_t = 0$ for all t . In Figure 5.1 we plot the future evolution of the market price of risk given by

$$\lambda_t = -(\Phi_{tt} - \alpha), \quad (5.36)$$

where α is a constant.

Consider now, an economy driven by a one-dimensional Brownian motion with N assets. Let $\{\mu_t^i\}_{t \geq 0}$ and $\{\sigma_t^i\}_{t \geq 0}$ denote the drift and volatility of the return on the i^{th} asset. In that case we have

$$\mu_t^i = r_t + \lambda_t \sigma_t^i, \quad (5.37)$$

for $i = 1, \dots, N$. From the analysis in Section 5.2 we know that

$$\lambda_t = -(\Phi_{tt} - \alpha_t), \quad (5.38)$$

where Φ_{tt} can be measured from the prices of options. Taking any two drifts we can then calculate the difference

$$\mu_t^i - \mu_t^j = -(\Phi_{tt} - \alpha_t)(\sigma_t^i - \sigma_t^j). \quad (5.39)$$

Since the value of each σ_t^i can be extracted from option prices, or high-frequency data, without loss of generality we can assume that $\sigma_t^1 > \sigma_t^2 > \dots > \sigma_t^N$ etc., i.e., we can label each σ_t^i in the order of size. Then, for $\sigma_t^i > \sigma_t^j$ we can see that $\mu_t^i > \mu_t^j$ if $\alpha_t > \Phi_{tt}$. We do not know α_t so it may be possible that $\alpha_t < \Phi_{tt}$ and hence $\mu_t^i < \mu_t^j$. However, in the one-factor case it may be safe to assume that $\alpha_t > \Phi_{tt}$. This is on account of the fact that if $\alpha_t < \Phi_{tt}$, then every asset would have a negative risk premium.

For each i , $\{\mu_t^i\}$ represents an investment opportunity. In the one-factor case it is thus possible to rank different opportunities. If N is very large it seems reasonable that one could discern whether $\alpha_t > \Phi_{tt}$ or $\alpha_t < \Phi_{tt}$ and hence $\mu_t^1 > \mu_t^2 > \dots > \mu_t^N$ or $\mu_t^1 < \mu_t^2 < \dots < \mu_t^N$, since one return should be clearly larger than another.

Chapter 6

Other Families of Martingales

6.1 Chapter overview

In this chapter we examine various special cases of the martingale family $\{M_{tx}\}_{x \geq t \geq 0}$ appearing in the Flesaker-Hughston bond pricing formula (1.48). In particular, we model $\{M_{tx}\}$ by taking the Esscher transform of various different Lévy processes. In certain cases we are able to provide corresponding closed-form solutions for bond prices, short rates, and option prices.

An interesting example is the zeta process (see Ehm 2007, Brody *et al.* 2010, and references cited therein). As we are able to represent the zeta process as an infinite sum of compound Poisson processes we can write down the Lévy measure of the zeta process in a simple form. We introduce two candidate financial models which we call the rational zeta process and the zeta-subordinated process.

6.2 The geometric Poisson family

6.2.1 A Poisson family of term-structure models

The Poisson process is a pure jump process. It was used early on in the asset pricing literature by Cox & Ross (1976) and by Merton (1976). We introduce a Poisson process $\{N_t\}_{t \geq 0}$, with a real parameter λ that describes the jump-arrival rate. By use of the Esscher transform (3.3) and the Laplace transform

$$\mathbb{E} [e^{\phi_x N_t}] = \exp(\lambda t (e^{\phi_x} - 1)) \quad (6.1)$$

one can construct a family of geometric Lévy martingales of the form

$$M_{tx} = \exp(\phi_x N_t - \lambda t(e^{\phi_x} - 1)). \quad (6.2)$$

Insertion of (6.2) into the bond price formula (1.48) leads to an expression of the form

$$P_{tT} = \frac{\int_T^\infty \rho_x \exp(\phi_x N_t - \lambda t(e^{\phi_x} - 1)) \, dx}{\int_t^\infty \rho_x \exp(\phi_x N_t - \lambda t(e^{\phi_x} - 1)) \, dx}, \quad (6.3)$$

and an corresponding short rate of

$$r_t = \frac{\rho_t \exp(\phi_t N_t - \lambda t(e^{\phi_t} - 1))}{\int_t^\infty \rho_x \exp(\phi_x N_t - \lambda t(e^{\phi_x} - 1)) \, dx}. \quad (6.4)$$

It may in some situations be desirable to extend the geometric Poisson model to include a second, and independent, Poisson process $\{N'_t\}_{t \geq 0}$. Let $\{N'_t\}$ also have jump rate parameter λ . We can thus propose an alternative geometric Poisson bond price of the form

$$P_{tT} = \frac{\int_T^\infty \rho_x \exp(\phi_x(N_t - N'_t) - 2\lambda t(e^{\phi_x} - 1)) \, dx}{\int_t^\infty \rho_x \exp(\phi_x(N_t - N'_t) - 2\lambda t(e^{\phi_x} - 1)) \, dx}, \quad (6.5)$$

with an associated short rate of the form

$$r_t = \frac{\rho_t \exp(\phi_t(N_t - N'_t) - 2\lambda t(e^{\phi_t} - 1))}{\int_t^\infty \rho_x \exp(\phi_x(N_t - N'_t) - 2\lambda t(e^{\phi_x} - 1)) \, dx}. \quad (6.6)$$

The new model represents an improvement since it exhibits both positive and negative jumps in the underlying state variable process $\{N_t - N'_t\}_{t \geq 0}$.

It is interesting to observe that expression (6.4) implies a strict upper (*resp.* lower) bound on interest rates in the event that ϕ_x is strictly decreasing (*resp.* increasing). In particular, if ϕ_x is decreasing the upper bound on the short rate, denoted by r_t^u , is given by

$$r_t^u = \frac{\rho_t \exp(-\lambda t(e^{\phi_t} - 1))}{\int_t^\infty \rho_x \exp(-\lambda t(e^{\phi_x} - 1)) \, dx}. \quad (6.7)$$

6.2.2 The limiting behaviour of the risk-free rate

Following Solé & Utzet (2008), we note that the space-time harmonic polynomial representation of the Poisson process is

$$\frac{e^{\phi_x N_t}}{\mathbb{E}[e^{\phi_x N_t}]} = \sum_{n=0}^{\infty} \frac{\phi_x^n}{n!} \tilde{C}_n(N_t, t), \quad (6.8)$$

where we write $\tilde{C}_n(-, t)$ for the Charlier polynomials ($n = 0, 1, 2, \dots$). Each $\tilde{C}_n(N_t, t)$ is a martingale.

Proposition 6.2.1. *Let the initial term-structure density ρ_x be given. Under the assumptions that*

$$\lim_{t \rightarrow \infty} \phi_t t = 0, \quad (6.9)$$

and $|\phi_t|$ is decreasing the short rate defined by (6.4) will tend asymptotically to a deterministic function. In particular, we have

$$\lim_{t \rightarrow \infty} r_t = - \lim_{t \rightarrow \infty} \partial_t \ln \rho_t. \quad (6.10)$$

Proof. The law of large numbers for a Poisson process is

$$\lim_{t \rightarrow \infty} \frac{N_t}{t} = \lambda \quad \text{a.s.} \quad (6.11)$$

Following the steps of Proposition 3.4.3, and by use of the fact that the leading order terms of the Charlier polynomials are N_t^n and t^n , we can see that under condition (6.9) that the function $|\phi_t|$ will dominate the Poisson process in the limit, and the result holds. $\square$

6.2.3 A jump-diffusion family of term-structure models

Merton (1976) extended the Black-Scholes option-pricing theory to include equity price processes driven by a jump-diffusion process. In term-structure modelling it may also be desirable to incorporate both jump risk and diffusion risk. For simplicity we focus on jumps that have normally distributed jump sizes.

We introduce a Poisson process $\{N_t\}_{t \geq 0}$, with rate parameter λ , to represent the average number of jumps occurring by time t . The size of the i^{th} jump is modelled by a random variable J_i . Jump sizes are independent and identically distributed random variables, each such that

$$J_i \sim N(\mu, \delta^2). \quad (6.12)$$

We also introduce a diffusion component driven by an independent Brownian motion $\{W_t\}_{t \geq 0}$. We obtain an expression for the bond price as follows. Writing $\{J_t\}_{t \geq 0}$ for

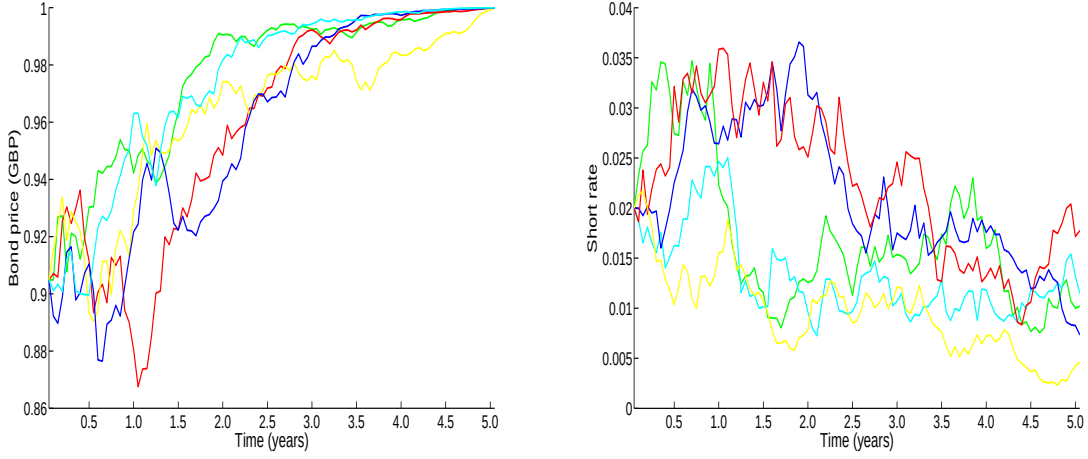


Figure 6.1: *Jump-diffusion bond-price and short-rate simulations.* This figure shows various sample paths for the price of a discount bond and the associated short rate based on the jump-diffusion family of rational Lévy models. We use expression (6.15) for the bond price and (6.16) for the short rate where $\lambda = 20$, $\mu = 0$, $\delta = 0.09$, and $\phi_x = e^{-0.02x}$. We have used a flat initial term structure with annual rate 2% such that $P_{0T} = e^{-0.02T}$. The maturity of the bond is five years.

the compound Poisson process defined by

$$J_t = \sum_{i=1}^{N_t} J_i, \quad (6.13)$$

by use of (3.3) we obtain a geometric martingale family of the form

$$M_{tx} = \exp \left(\phi_x(W_t + J_t) - \frac{1}{2}\phi_x^2 t - \lambda t(e^{\phi_x \mu + \frac{1}{2}\phi_x^2 \delta^2} - 1) \right). \quad (6.14)$$

This leads to a bond price of the form

$$P_{tT} = \frac{\int_T^\infty \rho_x \exp \left(\phi_x(W_t + J_t) - \frac{1}{2}\phi_x^2 t - \lambda t(e^{\phi_x \mu + \frac{1}{2}\phi_x^2 \delta^2} - 1) \right) dx}{\int_t^\infty \rho_x \exp \left(\phi_x(W_t + J_t) - \frac{1}{2}\phi_x^2 t - \lambda t(e^{\phi_x \mu + \frac{1}{2}\phi_x^2 \delta^2} - 1) \right) dx}, \quad (6.15)$$

and for the short rate we have

$$r_t = \frac{\rho_t \exp \left(\phi_t(W_t + J_t) - \frac{1}{2}\phi_t^2 t - \lambda t(e^{\phi_t \mu + \frac{1}{2}\phi_t^2 \delta^2} - 1) \right)}{\int_t^\infty \rho_x \exp \left(\phi_x(W_t + J_t) - \frac{1}{2}\phi_x^2 t - \lambda t(e^{\phi_x \mu + \frac{1}{2}\phi_x^2 \delta^2} - 1) \right) dx}. \quad (6.16)$$

Sample paths of the bond price process given by (6.15) and the short rate process given by (6.16) are simulated in Figure 6.1.

In Merton (1976) a key notion used to price options is the idea that so-called idiosyncratic risk (firm specific risk) can be modelled with a return equal to the risk-free rate. That is to say, there is according to Merton (1976) no excess rate of return offered by the market in connection with such risk. In Merton's model it is assumed that the jump risk is purely idiosyncratic and can be diversified away by holding a suitably broad portfolio. Merton argues that since the risk can be diversified away, the market awards no risk premium to investors who hold such assets. From a modern point of view, this assumption is clearly questionable, and it is fair to say certainly that not all jump risk is idiosyncratic; this is one of the lessons of the 2008 financial credit crisis; and perhaps the point was already even evident following the 1987 stock market crash. In our model, on the other hand, jump risk is being priced. The market price of risk is implicit in the choice of the pricing kernel. Merton's diversification hypothesis is not required.

6.2.4 Option pricing in the rational jump-diffusion family of models

To derive an expression for the price of a European call option with strike price K and expiry t , we recall again its general form

$$C_{0t} = \mathbb{E} [\pi_t (P_{tT} - K)^+] . \quad (6.17)$$

By the tower property of conditional expectation, we condition on the number of Poisson jumps to obtain

$$C_{0t} = \mathbb{E} \left[\mathbb{E} \left[\left(\int_T^\infty \rho_x M_{tx} dx - K \int_t^\infty \rho_x M_{tx} dx \right)^+ \middle| N_t \right] \right] . \quad (6.18)$$

Our task is then to compute the conditional expectation, which is essentially Gaussian, and finally the unconditional expectation, which is an expectation over the Poisson randomness. By Proposition 3.5.2, it follows that if ϕ_x is decreasing for $x \geq 0$ then we can find a number ξ^* such that

$$P(t, T, \xi^*) = K, \quad (6.19)$$

where

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x \exp\left(\phi_x \xi - \frac{1}{2} \phi_x^2 t - \lambda t (e^{\phi_x \mu + \frac{1}{2} \phi_x^2 \delta^2} - 1)\right) dx}{\int_t^\infty \rho_x \exp\left(\phi_x \xi - \frac{1}{2} \phi_x^2 t - \lambda t (e^{\phi_x \mu + \frac{1}{2} \phi_x^2 \delta^2} - 1)\right) dx}. \quad (6.20)$$

Hence after a calculation we are able to deduce that the present value of a call option, in the case of a geometric jump-diffusion martingale family, is given by the following expression:

$$C_{0t} = \sum_{n=0}^{\infty} \left[\int_T^\infty A_x \rho_x N\left(\frac{\xi^* - n\mu}{v_n(t)} - \phi_x v_n(t)\right) dx - K \int_t^\infty A_x \rho_x N\left(\frac{\xi^* - n\mu}{v_n(t)} - \phi_x v_n(t)\right) dx \right]. \quad (6.21)$$

Here we have written

$$v_n(t)^2 = t + n\delta^2, \quad (6.22)$$

and

$$A_x = \frac{e^{-\Lambda(x)t} (\Lambda(x)t)^n}{n!}, \quad (6.23)$$

where

$$\Lambda(x) = \lambda \exp\left(\phi_x \mu + \frac{1}{2} \phi_x^2 \delta^2\right). \quad (6.24)$$

6.3 The geometric gamma family

6.3.1 A gamma family of term-structure models

We begin with a brief review of the theory of the gamma process. Let α and β be positive constants. By a gamma process with growth rate α and variance rate β^2 we mean a process $\{\gamma_t\}_{t \geq 0}$ with stationary and independent increments such that $\gamma_0 = 0$ and such that γ_t has a gamma distribution with mean αt and variance $\beta^2 t$. Writing $m = \alpha^2 / \beta^2$ and $\kappa = \beta^2 / \alpha$, we have $\alpha = \kappa m$ and $\beta^2 = \kappa^2 m$. The density of γ_t is then given by

$$\mathbb{P}(\gamma_t \in du) = \frac{u^{mt-1} e^{-u/\kappa}}{\kappa^{mt} \Gamma(mt)} du \quad (6.25)$$

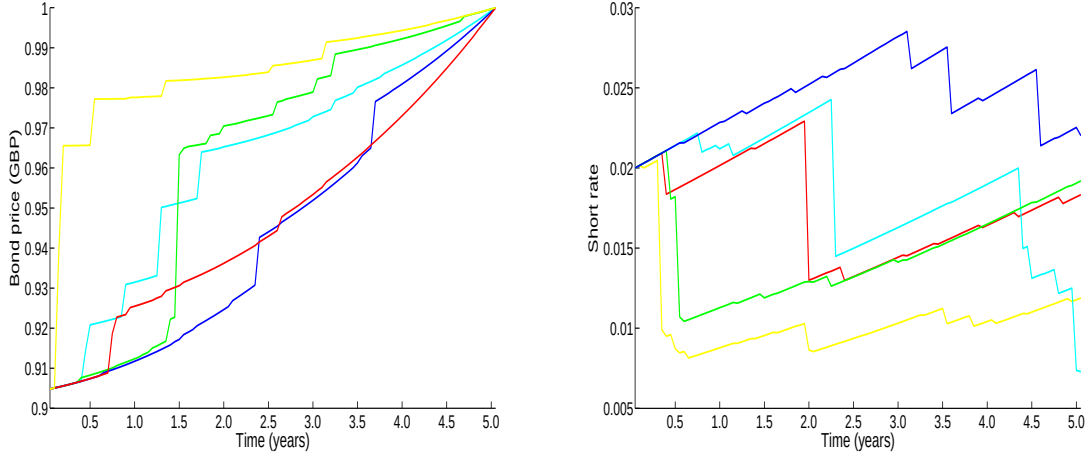


Figure 6.2: *Gamma bond-price and short-rate simulations.* This figure shows sample paths for the price of a discount bond and the associated short rate based on the gamma family of rational Lévy models. We use expression (6.29) for the bond price and (6.30) for the short rate where $m = 1$, $\kappa = 0.5$, and $\phi_x = -e^{-0.02x}$. We have used a flat initial term structure with annual rate 2% such that $P_{0T} = e^{-0.02T}$. The maturity of the bond is five years.

for $u > 0$. Here $\Gamma(a)$ is the standard gamma function, which for $a > 0$ is defined by

$$\Gamma(a) = \int_0^\infty y^{a-1} e^{-y} dy. \quad (6.26)$$

In the literature the parameter m is referred to as the “rate” parameter and κ is referred to as the “scale” parameter. A calculation shows that for $\lambda < \kappa^{-1}$ the moment generating function of γ_t exists and is given by

$$\mathbb{E}[e^{-\lambda\gamma_t}] = (1 - \kappa\lambda)^{-mt}, \quad (6.27)$$

from which it follows that $\mathbb{E}[\gamma_t] = \kappa mt$ and $\text{Var}[\gamma_t] = \kappa^2 mt$. The exponential martingale associated with $\{\gamma_t\}$ is given by $(1 - \kappa\lambda)^{mt} e^{-\lambda\gamma_t}$. See Lukacs (1955), Tsilevich *et al.* (2001), Schoutens (2003), Cont & Tankov (2004), Kyprianou (2006) and Brody *et al.* (2008) for further details of the gamma process. In particular, the parametrisation given above can be found in the latter of these papers.

Now fix m and κ , let the function ϕ_x satisfy $\phi_x < \kappa^{-1}$ for $x \geq 0$, and define a one-parameter family of positive martingales $\{M_{tx}\}$ by setting

$$M_{tx} = (1 - \kappa\phi_x)^{mt} e^{\phi_x \gamma_t}. \quad (6.28)$$

Writing ρ_x as before for the initial term-structure density, we obtain from equation (1.48) the following expression for the discount bond price:

$$P_{tT} = \frac{\int_T^\infty \rho_x (1 - \phi_x \kappa)^{mt} e^{\phi_x \gamma_t} dx}{\int_t^\infty \rho_x (1 - \phi_x \kappa)^{mt} e^{\phi_x \gamma_t} dx}, \quad (6.29)$$

and for the associated short rate process we have

$$r_t = \frac{\rho_t (1 - \phi_t \kappa)^{mt} e^{\phi_t \gamma_t}}{\int_t^\infty \rho_x (1 - \phi_x \kappa)^{mt} e^{\phi_x \gamma_t} dx}. \quad (6.30)$$

We simulate a set of sample paths for the bond price short rate in equations (6.29) and (6.30) in Figure 6.2.

For an explicit example of a rational gamma interest-rate model, we choose ρ_x and ϕ_x to be of the simple form

$$\rho_x = R e^{-Rx} \quad \text{and} \quad \phi_x = -\sigma x. \quad (6.31)$$

This “semi-linear” form of ϕ_x , as in the Brownian case of Section 4.2, leads to nice analytical results. The parameter σ has units of inverse time, and can be calibrated to option data along with R , m , and κ . Under this model choice the bond price becomes

$$\begin{aligned} P_{tT} &= \frac{\int_T^\infty (1 + x\sigma\kappa)^{mt} e^{-x\sigma(\gamma_t + R/\sigma)} dx}{\int_t^\infty (1 + x\sigma\kappa)^{mt} e^{-x\sigma(\gamma_t + R/\sigma)} dx} \\ &= \frac{\int_T^\infty (1 + x\sigma\kappa)^{mt} e^{-(1+x\sigma\kappa)(\gamma_t/\kappa + R/\sigma\kappa)} dx}{\int_t^\infty (1 + x\sigma\kappa)^{mt} e^{-(1+x\sigma\kappa)(\gamma_t/\kappa + R/\sigma\kappa)} dx}, \end{aligned} \quad (6.32)$$

after some rearrangements. By changing the integration variable according to the scheme

$$x \rightarrow y = (1 + x\sigma\kappa)(\gamma_t/\kappa + R/\sigma\kappa) \quad (6.33)$$

we obtain

$$\begin{aligned} P_{tT} &= \frac{\int_{(1+T\sigma\kappa)(\gamma_t/\kappa + R/\sigma\kappa)}^\infty y^{mt} e^{-y} dy}{\int_{(1+t\sigma\kappa)(\gamma_t/\kappa + R/\sigma\kappa)}^\infty y^{mt} e^{-y} dy} \\ &= \frac{\Gamma(mt + 1, (1 + T\sigma\kappa)(\gamma_t/\kappa + R/\sigma\kappa))}{\Gamma(mt + 1, (1 + t\sigma\kappa)(\gamma_t/\kappa + R/\sigma\kappa))}. \end{aligned} \quad (6.34)$$

Here we have written

$$\Gamma(a, x) = \int_x^\infty \frac{u^{a-1} e^{-u}}{\Gamma(a)} du \quad (6.35)$$

for the “upper” incomplete gamma function. We simulate several paths for the bond price (6.34) in Figure 6.3.

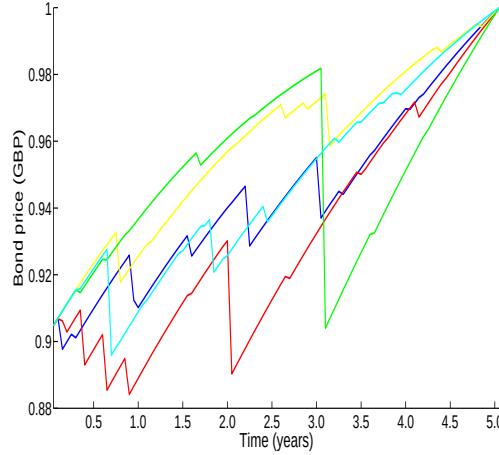


Figure 6.3: *Semi-linear gamma bond-price simulations.* This figure shows sample paths for the bond price based on the “semi-linear” gamma family of rational Lévy models. We use the pricing equation (6.34) where $R = 0.02$, $\sigma = 0.02$, $m = 0.5$ and $\kappa = 1$. The maturity of the bond is five years.

6.3.2 Option pricing in the rational gamma family of models

In the geometric gamma model we can obtain a semi-analytical expression for the price of a European-style call option with strike price K and expiry t . From Proposition 3.5.2 we know that if ϕ_x is increasing for all $x \geq 0$ then we can find ξ_1^* such that $P(t, T, \xi_1^*) = K$, and $P(t, T, \xi) > K$ for all $\xi > \xi_1^*$, where in this case we define

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x (1 - \phi_x \kappa)^{mt} e^{\phi_x \xi} dx}{\int_t^\infty \rho_x (1 - \phi_x \kappa)^{mt} e^{\phi_x \xi} dx}. \quad (6.36)$$

We then deduce, using (3.80), that the price of a call option is of the form

$$C_{0t} = \int_T^\infty \rho_x \Gamma(mt, \xi_1^* (\kappa^{-1} - \phi_x)) dx - K \int_t^\infty \rho_x \Gamma(mt, \xi_1^* (\kappa^{-1} - \phi_x)) dx. \quad (6.37)$$

Analogously, when ϕ_x is decreasing we can find ξ_2^* such that $P(t, T, \xi_2^*) = K$ and $P(t, T, \xi) < K$ for all $\xi > \xi_2^*$. We can then derive an alternate expression for the option price of the form

$$C_{0t} = \int_T^\infty \rho_x \gamma(mt, \xi_2^* (\kappa^{-1} - \phi_x)) dx - K \int_t^\infty \rho_x \gamma(mt, \xi_2^* (\kappa^{-1} - \phi_x)) dx. \quad (6.38)$$

Here we define

$$\gamma(a, x) = \int_0^x \frac{u^{a-1} e^{-u}}{\Gamma(a)} du \quad (6.39)$$

as the “lower” incomplete gamma function.

6.3.3 The limiting behaviour of the risk-free rate

Following Solé & Utzet (2008), one finds that the space-time harmonic polynomial representation of a gamma process $\{\gamma_t\}$ is

$$\frac{e^{\phi_x \gamma_t}}{\mathbb{E}[e^{\phi_x \gamma_t}]} = \sum_{n=0}^{\infty} \frac{\phi_x^n}{n!} \tilde{L}_n(\gamma_t, t), \quad (6.40)$$

where $\tilde{L}_n(\gamma_t, t)$ are the Leguerre polynomials.

Proposition 6.3.1. *Let the initial term-structure density ρ_x be given. Under the assumptions that*

$$\lim_{t \rightarrow \infty} \phi_t t = 0, \quad (6.41)$$

and $|\phi_t|$ is decreasing, the short rate defined by (6.30) will tend asymptotically to a deterministic function. In particular

$$\lim_{t \rightarrow \infty} r_t = \lim_{t \rightarrow \infty} -\partial_t \ln \rho_t. \quad (6.42)$$

Proof. The law of large numbers for a gamma process process is

$$\lim_{t \rightarrow \infty} \frac{\gamma_t}{t} = m \quad \text{a.s.} \quad (6.43)$$

By use of Proposition 3.4.3 and the fact that the leading order terms of the Leguerre polynomials are γ_t^n and t^n the result follows. In particular the function $|\phi_t|$ shall dominate γ_t and t in the limit. $\square$

6.4 The geometric zeta family

6.4.1 An introduction to the theory of zeta processes

Definition 6.4.1. The *Riemann-zeta function* is defined for real values of $s > 1$ by

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}. \quad (6.44)$$

The zeta distribution (and variants thereof) has been used to describe the law of ranked data in the field of linguistics. Zipf (1949) describes a very wide range of further applications and Mandelbrot (1967) expands on these.

Definition 6.4.2. A random variable Z is said to have a *zeta distribution* with parameter $s > 1$ if its probability mass function, for $n \in \mathbb{N}$, is given by

$$\mathbb{P}(Z = n) = \frac{n^{-s}}{\zeta(s)}. \quad (6.45)$$

The zeta distribution has many interesting properties. It is an example of a power law distribution. It is fat-tailed in the sense that, for a zeta distributed random variable Z with parameter s , we observe that

$$\mathbb{P}(Z = n) \sim n^{-s}. \quad (6.46)$$

One can readily obtain the expectations of powers of Z . Specifically, we can carry out the following calculation, valid for $s + \lambda > 1$

$$\begin{aligned} \mathbb{E}[Z^{-\lambda}] &= \sum_{n=1}^{\infty} \frac{n^{-s} n^{-\lambda}}{\zeta(s)} \\ &= \sum_{n=1}^{\infty} \frac{n^{-(s+\lambda)}}{\zeta(s)} \\ &= \frac{\zeta(s+\lambda)}{\zeta(s)}. \end{aligned} \quad (6.47)$$

Let us define the random variable $X = \ln Z$. The expectation of X is

$$\begin{aligned} \mathbb{E}[X] &= \sum_{n=1}^{\infty} \ln n \frac{n^{-s}}{\zeta(s)} \\ &= \frac{\zeta'(s)}{\zeta(s)}, \end{aligned} \quad (6.48)$$

for $s > 1$. The associated characteristic function of X is obtained immediately from (6.47). Specifically, we have

$$\mathbb{E}[e^{-i\lambda X}] = \frac{\zeta(s + i\lambda)}{\zeta(s)}, \quad (6.49)$$

where $s + \operatorname{Re}(\lambda) > 1$.

Following Ehm (2007), the random variable X can be represented as a sum of independent Poisson random variables. This is due to the *Euler product formula* representation of the Riemann-zeta function:

$$\zeta(s) = \prod_{k=1}^{\infty} (1 - p_k^{-s})^{-1}, \quad (6.50)$$

where p_k is the k^{th} prime number and $p_1 = 2$. Ehm (2007) makes the observation (6.50) and leads on to an associated exponential form of (6.49):

$$\begin{aligned} \mathbb{E} [e^{-i\lambda X}] &= \prod_{k=1}^{\infty} \frac{1 - p_k^{-s}}{1 - p_k^{-(s+i\lambda)}} \\ &= \exp \left(\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_k^{-sn}}{n} (e^{-i\lambda n \ln p_k} - 1) \right). \end{aligned} \quad (6.51)$$

It is recognised that (6.51) has the same form as the characteristic function of an infinite sum of independent Poisson random variables. It follows that

$$X = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n N^{(n,k)} \ln p_k, \quad (6.52)$$

where $N^{(n,k)}$ for each $n, k \in \mathbb{N}$ are independent Poisson random variables with rate parameters $\lambda^{(n,k)}$, given by

$$\lambda^{(n,k)} = n^{-1} p_k^{-ns}. \quad (6.53)$$

The zeta distribution has an associated multiplicative Lévy process $\{Z_t\}_{t \geq 0}$, known as the zeta process (see Alexander *et al.* 1993, Ehm 2007, 2008, and Brody *et al.* 2010). The Poisson representation (6.52) shows us that X has an infinitely divisible distribution. By the equivalence of infinitely divisible random variables and Lévy processes (see, e.g., Kyprianou 2006) this means that there exists a corresponding Lévy process $\{X_t\}_{t \geq 0}$ associated with X . Due to the infinite divisibility of X_t at t , the characteristic function of $\{X_t\}$ is

$$\mathbb{E} [e^{-i\lambda X_t}] = \left(\frac{\zeta(s + i\lambda)}{\zeta(s)} \right)^t. \quad (6.54)$$

Using (6.51), we observe that (6.54) is equivalent to

$$\mathbb{E} [e^{-i\lambda X_t}] = \exp \left(\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_k^{-sn}}{n} t (e^{-i\lambda n \ln p_k} - 1) \right). \quad (6.55)$$

We then deduce from the Lévy-Khintchine formula (2.5) that the process $\{X_t\}$ can be characterised by the triple (z, σ, Π) , where

$$z = \int_{\mathbb{R}} x \mathbf{1}\{|x| \leq 1\} \Pi(\mathrm{d}x), \quad (6.56)$$

$$\sigma = 0, \quad (6.57)$$

and

$$\Pi(x) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_k^{-sn}}{n} \delta(x - n \ln p_k). \quad (6.58)$$

We observe that $\{X_t\}$ is a subordinator. We confirm from Definition 2.2.4 that $\{X_t\}$ is of finite variation as

$$\begin{aligned} \int_{0 < |x| < 1} |x| \Pi(\mathrm{d}x) &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_{-1}^1 |x| \frac{p_k^{-sn}}{n} \delta(x - n \ln p_k) \mathrm{d}x \\ &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} p_k^{-sn} \ln p_k \mathbf{1}\{n \ln p_k \leq 1\} \\ &= 2^{-s} \ln 2 \\ &< \infty, \end{aligned} \quad (6.59)$$

and from Definition 2.2.3 that $\{X_t\}$ is of finite activity

$$\int_{\mathbb{R}} \Pi(\mathrm{d}x) = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{p_k^{-sn}}{n} < \infty, \quad (6.60)$$

for all $s > 1$.

Definition 6.4.3. Consider the Lévy process $\{X_t\}_{t \geq 0}$ with characteristic function (6.49). We define the *Zeta process* $\{Z_t\}_{t \geq 0}$ by setting

$$Z_t = \exp(X_t). \quad (6.61)$$

It is a remarkable (and, perhaps, not universally appreciated) property of the Riemann-zeta function that $\zeta(s)$ is log-convex for $s \in \mathbb{R}$ and $s > 1$. The existence of the zeta process hinges on this fact. The zeta process is Lévy multiplicative in the sense that its logarithm has the stationary, independent increments associated with

Lévy processes. The process $\{Z_t\}$ takes values in $\mathbb{N}$ and is non-decreasing. The expectation of powers of Z_t is given by

$$\mathbb{E}[Z_t^{-\lambda}] = \left(\frac{\zeta(s+\lambda)}{\zeta(s)} \right)^t, \quad (6.62)$$

where we have used expression (6.54) and the independent and stationary increments property of $\{X_t\}$.

There is a useful change of measure martingale associated with the zeta process. This martingale is given by the Esscher transform of $\{X_t\}$. If we denote $\mathbb{P}^s$ as the probability measure under which $\{Z_t\}$ is a zeta process with parameter s , and $\mathbb{E}^s[-]$ as the expectation operator under that measure then for $s + \lambda > 1$, then the process defined by

$$M_t = \frac{e^{-\lambda X_t}}{\mathbb{E}^s[e^{-\lambda X_t}]} \quad (6.63)$$

is a one-parameter family of martingales under $\mathbb{P}^s$. These martingales can be used to change measure to $\mathbb{P}^{s+\lambda}$, which is the probability measure under which $\{Z_t\}$ is a zeta process with parameter $s + \lambda$.

We can now create an arbitrage-free term-structure model in the positive interest framework using the zeta process. By use of the Esscher transform (3.3) and expression (6.62) we obtain a geometric martingale of

$$M_{tx} = Z_t^{\phi_x} \left(\frac{\zeta(s)}{\zeta(s - \phi_x)} \right)^t, \quad (6.64)$$

with the condition that $\phi_x < s - 1$, for $x \geq 0$. This leads to a bond price of the form

$$P_{tT} = \frac{\int_T^\infty \rho_x Z_t^{\phi_x} \zeta(s - \phi_x)^{-t} dx}{\int_t^\infty \rho_x Z_t^{\phi_x} \zeta(s - \phi_x)^{-t} dx}, \quad (6.65)$$

and an associated short rate of the form

$$r_t = \frac{\rho_t Z_t^{\phi_t} \zeta(s - \phi_t)^{-t}}{\int_t^\infty \rho_x Z_t^{\phi_x} \zeta(s - \phi_x)^{-t} dx}. \quad (6.66)$$

6.4.2 Option pricing in the rational zeta family of models

From Proposition 3.5.2, if ϕ_x is monotonically decreasing in x then the bond price is monotonically increasing in the state variable. We shall call $\mathbb{P}^{s-\phi_x}$ the measure under

which $\{Z_t\}$ is a zeta process with parameter $s - \phi_x$, i.e.,

$$\frac{d\mathbb{P}^{s-\phi_x}}{d\mathbb{P}} = \frac{Z_t^{\phi_x}}{\mathbb{E}[Z_t^{\phi_x}]}. \quad (6.67)$$

Using the change of measure we work out the price of a European-style call option, as defined by

$$C_{0t} = \mathbb{E} \left[\left(\int_T^\infty \rho_x Z_t^{\phi_x} \zeta(s - \phi_x)^{-t} dx - K \int_t^\infty \rho_x Z_t^{\phi_x} \zeta(s - \phi_x)^{-t} dx \right)^+ \right], \quad (6.68)$$

to be

$$C_{0t} = \int_T^\infty \rho_x \mathbb{P}^{s-\phi_x}(Z_t \leq \xi^*) dx - K \int_t^\infty \rho_x \mathbb{P}^{s-\phi_x}(Z_t \leq \xi^*) dx. \quad (6.69)$$

Here the number ξ^* is defined by

$$P(t, T, \xi^*) = K, \quad (6.70)$$

where

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x \xi^{\phi_x} \zeta(s - \phi_x)^{-t} dx}{\int_t^\infty \rho_x \xi^{\phi_x} \zeta(s - \phi_x)^{-t} dx}. \quad (6.71)$$

6.4.3 The rational zeta process

Let us define a random variable V , that takes the form of a ratio of two independent identically distributed zeta random variables, Z_1 and Z_2 , each with parameter $s > 2$. In particular we set

$$V = \frac{Z_1}{Z_2}. \quad (6.72)$$

The random variable V takes values on the set of rational numbers. It has an expectation under $\mathbb{P}$ given by

$$\begin{aligned} \mathbb{E}[V] &= \mathbb{E}[Z_1] \mathbb{E}\left[\frac{1}{Z_2}\right] \\ &= \sum_{n=1}^{\infty} \frac{n^{-(s-1)}}{\zeta(s)} \sum_{n=1}^{\infty} \frac{n^{-(s+1)}}{\zeta(s)} \\ &= \frac{\zeta(s-1)\zeta(s+1)}{\zeta(s)^2}. \end{aligned} \quad (6.73)$$

We can also define $X_1 = \ln Z_1$ and $X_2 = \ln Z_2$ so that

$$V = e^{X_1 - X_2}. \quad (6.74)$$

Definition 6.4.4. We define a new Lévy-multiplicative process $\{V_t\}_{t \geq 0}$ called the *rational zeta process*. The process $\{V_t\}$ is given by the ratio of two zeta processes. Specifically we set $V_t = Z_{1,t}/Z_{2,t}$ where $\{Z_{1,t}\}_{t \geq 0}$ and $\{Z_{2,t}\}_{t \geq 0}$ are independent zeta processes, each with parameter $s > 2$.

The process $\{V_t\}$ has both upward and downward jumps and its state space will be the set of rational numbers.

Remark 6.4.5. The process $\{V_t\}$ has a natural martingale associated with it. Let

$$\begin{aligned} N_t &= \frac{V_t^{-\lambda}}{\mathbb{E}[V_t^{-\lambda}]} \\ &= V_t \left(\frac{\zeta(s)^2}{\zeta(s-\lambda)\zeta(s+\lambda)} \right)^t, \end{aligned} \quad (6.75)$$

for $\lambda \geq 0$, where $s - \lambda > 1$. Then $\{N_t\}_{t \geq 0}$ is a martingale.

We can derive a term-structure model using the rational zeta processes. We model the martingale family $\{M_{tx}\}$, for real values of $s - \phi_x > 1$, by setting

$$\begin{aligned} M_{tx} &= \frac{V_t^{\phi_x}}{\mathbb{E}[V_t^{\phi_x}]} \\ &= \frac{V_t^{\phi_x} \zeta(s)^{2t}}{\zeta(s - \phi_x)^t \zeta(s + \phi_x)^t}. \end{aligned} \quad (6.76)$$

This leads to a bond price of

$$P_{tT} = \frac{\int_T^\infty \rho_x V_t^{\phi_x} \zeta(s - \phi_x)^{-t} \zeta(s + \phi_x)^{-t} dx}{\int_t^\infty \rho_x V_t^{\phi_x} \zeta(s - \phi_x)^{-t} \zeta(s + \phi_x)^{-t} dx}, \quad (6.77)$$

and an associated short rate of

$$r_t = \frac{\rho_t V_t^{\phi_t} \zeta(s - \phi_t)^{-t} \zeta(s + \phi_t)^{-t}}{\int_t^\infty \rho_x V_t^{\phi_x} \zeta(s - \phi_x)^{-t} \zeta(s + \phi_x)^{-t} dx}. \quad (6.78)$$

6.4.4 The zeta-subordinated process

We define a process $\{Y_t\}_{t \geq 0}$ as a Brownian motion time-changed by the subordinator $\{X_t\}$, i.e., we let

$$Y_t = B_{X_t}. \quad (6.79)$$

The characteristic function of $\{Y_t\}$ is then given by

$$\begin{aligned}\mathbb{E} [e^{i\lambda Y_t}] &= \mathbb{E} \left[e^{-\frac{1}{2}\lambda^2 X_t} \right] \\ &= \left(\frac{\zeta(s + \frac{1}{2}\lambda^2)}{\zeta(s)} \right)^t,\end{aligned}\tag{6.80}$$

for $s + \text{Re}(\frac{1}{2}\lambda^2) > 1$. The expectation of $\{Y_t\}$ is 0 and its variance is

$$\text{Var} [Y_t] = -\frac{\zeta'(s)}{\zeta(s)}t.\tag{6.81}$$

Remark 6.4.6. We note that $\{Y_t\}$ is a martingale with respect to its natural filtration.

To create a term-structure model driven by $\{Y_t\}$ we can define a geometric martingale of the form

$$M_{tx} = e^{-\phi_x Y_t} \left(\frac{\zeta(s)}{\zeta(s + \frac{1}{2}\phi_x^2)} \right)^t,\tag{6.82}$$

for $s + \frac{1}{2}\phi_x^2 > 1$. This leads to a bond price of

$$P_{tT} = \frac{\int_T^\infty \rho_x e^{-\phi_x Y_t} \zeta(s + \frac{1}{2}\phi_x^2)^{-t} dx}{\int_t^\infty \rho_x e^{-\phi_x Y_t} \zeta(s + \frac{1}{2}\phi_x^2)^{-t} dx},\tag{6.83}$$

and a corresponding short rate of the form

$$r_t = \frac{\rho_t e^{-\phi_t Y_t} \zeta(s + \frac{1}{2}\phi_t^2)^{-t}}{\int_t^\infty \rho_x e^{-\phi_x Y_t} \zeta(s + \frac{1}{2}\phi_x^2)^{-t} dx}.\tag{6.84}$$

6.5 The geometric variance gamma family

6.5.1 A variance gamma family of term-structure models

A somewhat more sophisticated model can be constructed if $\{L_t\}$ is taken to be of the variance gamma (VG) type. The VG process was introduced in the finance literature by Madan & Seneta (1990), and since then has been studied by a number of authors (see, e.g., Madan & Milne 1991 and Madan *et al.* 1998). It will be useful for our purposes to begin with a brief exposition of the theory of the VG process, treating it in a manner consistent with our earlier discussion of the gamma process.

Let $\{\gamma_{1,t}\}_{t \geq 0}$ and $\{\gamma_{2,t}\}_{t \geq 0}$ be a pair of independent gamma processes, each with scale parameter unity and rate parameter m . Thus $\mathbb{E}[\gamma_{1,t}] = mt$ and $\text{Var}[\gamma_{1,t}] = mt$,

and similarly for $\{\gamma_{2,t}\}$. Now let κ_1, κ_2 be a pair of nonnegative constants and define $\{U_t\}_{t \geq 0}$ by setting

$$U_t = \kappa_1 \gamma_{1,t} - \kappa_2 \gamma_{2,t}. \quad (6.85)$$

To investigate the properties of this process we calculate the moment generating function of U_t . The result is:

$$\begin{aligned} \mathbb{E} \left[e^{\lambda(\kappa_1 \gamma_{1,t} - \kappa_2 \gamma_{2,t})} \right] &= (1 - \kappa_1 \lambda)^{-mt} (1 + \kappa_1 \lambda)^{-mt} \\ &= \left(1 - (\kappa_1 - \kappa_2) \lambda - \kappa_1 \kappa_2 \lambda^2 \right)^{-mt}, \end{aligned} \quad (6.86)$$

valid for λ in a suitable range.

Proposition 6.5.1. *The process $\{U_t\}$ is identical in law to a process $\{V_t\}_{t \geq 0}$ of the form*

$$V_t = \mu \Gamma_t + \sigma W_{\Gamma_t}, \quad (6.87)$$

where μ and σ are constants, where $\{\Gamma_t\}_{t \geq 0}$ is a scaled gamma process satisfying

$$\mathbb{E}[\Gamma_t] = t, \quad (6.88)$$

and where $\{W_{\Gamma_t}\}_{t \geq 0}$ represents the subordination of a standard Brownian motion by $\{\Gamma_t\}$. We shall refer to $\{V_t\}$ as a drifted VG process.

Proof. To see the relation between $\{U_t\}$ and $\{V_t\}$ we calculate the moment generating function of V_t :

$$\mathbb{E} \left[e^{\lambda(\mu \Gamma_t + \sigma W_{\Gamma_t})} \right] = \left(1 - \kappa_{\Gamma} \mu \lambda - \frac{1}{2} \kappa_{\Gamma} \sigma^2 \lambda^2 \right)^{-m_{\Gamma} t}, \quad (6.89)$$

where κ_{Γ} is the scale parameter of $\{\Gamma_t\}$ and m_{Γ} is the rate parameter of $\{\Gamma_t\}$. We observe that (6.86) and (6.89) take the same form if $m_{\Gamma} = m$. The two moment generating functions can then be identified if we set

$$\kappa_{\Gamma} \mu = \kappa_1 - \kappa_2 \quad (6.90)$$

and

$$\frac{1}{2} \kappa_{\Gamma} \sigma^2 = \kappa_1 \kappa_2. \quad (6.91)$$

Next we impose the normalisation (6.88), which implies $\kappa_\Gamma m_\Gamma = 1$. This allows us to express μ and σ in terms of κ_1 , κ_2 , and m . We find that

$$\mu = (\kappa_1 - \kappa_2)m \quad \text{and} \quad \sigma^2 = 2\kappa_1\kappa_2m. \quad (6.92)$$

Inverting these relations we then deduce that

$$\kappa_1 = \frac{1}{2m} \left(\mu + \sqrt{\mu^2 + 2\sigma^2m} \right) \quad \text{and} \quad \kappa_2 = \frac{1}{2m} \left(-\mu + \sqrt{\mu^2 + 2\sigma^2m} \right). \quad (6.93)$$

Since $\{U_t\}$ and $\{V_t\}$ are Lévy processes, the fact that the moment generating functions agree is sufficient to ensure that the processes are identical in law. $\square$

The theory of the VG process outlined above is consistent with that of Madan *et al.* (1998). The parameterisation that we have chosen is for our purposes in some respects more transparent. In particular, the limiting cases where $\kappa_1 = 0$ or $\kappa_2 = 0$ are incorporated in a natural way. We are thus able to define a one-parameter family of positive martingales $\{M_{tx}\}$ by setting

$$M_{tx} = \left(1 - \frac{\mu}{m}\phi_x - \frac{\sigma^2}{2m}\phi_x^2 \right)^{mt} e^{\phi_x V_t}. \quad (6.94)$$

Then, using equation (1.27), we deduce that the bond price takes the form

$$P_{tT} = \frac{\int_T^\infty \rho_x \left(1 - \frac{\mu}{m}\phi_x - \frac{\sigma^2}{2m}\phi_x^2 \right)^{mt} e^{\phi_x V_t} dx}{\int_t^\infty \rho_x \left(1 - \frac{\mu}{m}\phi_x - \frac{\sigma^2}{2m}\phi_x^2 \right)^{mt} e^{\phi_x V_t} dx}, \quad (6.95)$$

and that the associated short rate is given by

$$r_t = \frac{\rho_t \left(1 - \frac{\mu}{m}\phi_t - \frac{\sigma^2}{2m}\phi_t^2 \right)^{mt} e^{\phi_t V_t}}{\int_t^\infty \rho_x \left(1 - \frac{\mu}{m}\phi_x - \frac{\sigma^2}{2m}\phi_x^2 \right)^{mt} e^{\phi_x V_t} dx}. \quad (6.96)$$

Sample paths of the bond price process (6.95) and the short rate process (6.96) are presented in Figure 6.4.

6.5.2 Option pricing in the rational variance gamma family of models

We proceed in this section to derive the price of a bond option in the setting of the geometric VG martingale family. We follow closely the method of Madan & Milne (1991). In the case of a call option with expiry t and strike K on a bond with maturity

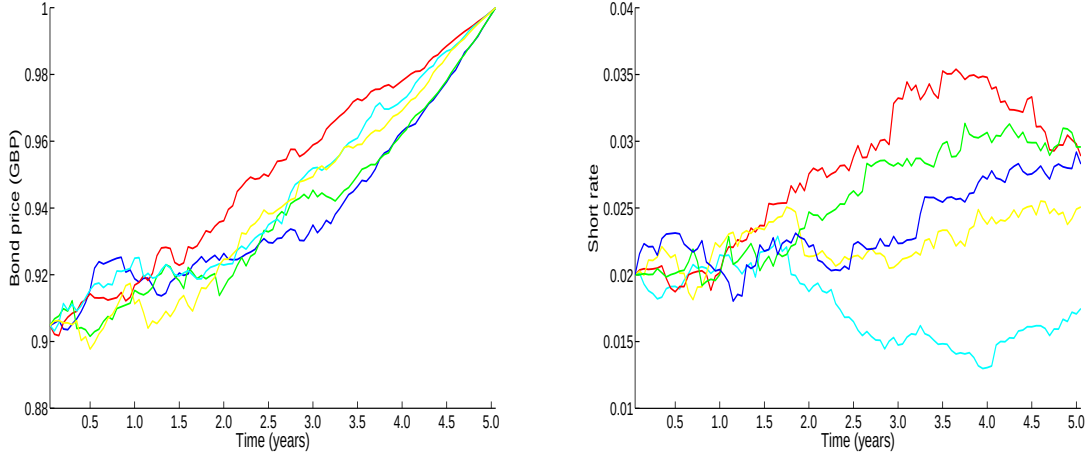


Figure 6.4: *Variance gamma-bond price and short-rate simulations.* This figure shows sample paths for the price of a discount bond and the associated short rate based on the variance gamma family of rational Lévy models. We use expression (6.95) for the bond price and (6.96) for the short rate where $\mu = 0.02$, $\sigma = 0.3$, $\nu = 0.05$, and $\phi_x = e^{-0.02x}$. We have used a flat term initial structure with annual rate 2% such that $P_{0T} = e^{-0.02T}$. The maturity of the bond is five years.

T , let us consider expression (3.79), into which we substitute (6.94). As before, when ϕ_x is decreasing for $x \geq 0$ we are able to find a number ξ^* such that $P(t, T, \xi^*) = K$, where

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x \left(1 - \frac{\mu}{m}\phi_x - \frac{\sigma^2}{2m}\phi_x^2\right)^{mt} e^{\phi_x \xi} dx}{\int_t^\infty \rho_x \left(1 - \frac{\mu}{m}\phi_x - \frac{\sigma^2}{2m}\phi_x^2\right)^{mt} e^{\phi_x \xi} dx}. \quad (6.97)$$

By the tower property of conditional expectation we obtain

$$C_{0t} = \mathbb{E} \left[\mathbb{E} \left[\int_T^\infty \rho_x \left(1 - \frac{\mu}{m}\phi_x - \frac{\sigma^2}{2m}\phi_x^2\right)^{mt} e^{\phi_x V_t} dx - K \int_t^\infty \rho_x \left(1 - \frac{\mu}{m}\phi_x - \frac{\sigma^2}{2m}\phi_x^2\right)^{mt} e^{\phi_x V_t} dx \middle| \Gamma_t \right] \right] \quad (6.98)$$

for the option price. In terms of the critical level ξ^* we then deduce that the price of the option is given by

$$C_{0t} = \int_T^\infty \rho_x \Psi \left(\frac{\xi^*}{\sigma \Phi(x)}, -\left(\frac{\mu}{\sigma} + \sigma \phi_x\right) \Phi(x), mt \right) dx - K \int_t^\infty \rho_x \Psi \left(\frac{\xi^*}{\sigma \Phi(x)}, -\left(\frac{\mu}{\sigma} + \sigma \phi_x\right) \Phi(x), mt \right) dx, \quad (6.99)$$

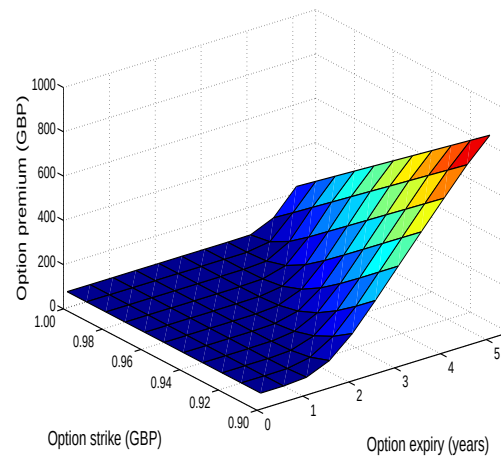


Figure 6.5: *Option pricing in the variance gamma family.* This figure shows sample option prices based on the variance gamma family of rational Lévy models. We use the pricing equation (6.99) where $\mu = 0.02$, $\sigma = 0.3$, $\nu = 0.05$, and $\phi_x = e^{-0.02x}$. We have used a flat initial term structure with annual rate 3% such that $P_{0T} = e^{-0.03T}$. The maturity of the bond is five years and the strike and option expiry are varied.

where

$$\Phi(x) = \left(m - \mu\phi_x - \frac{1}{2}\sigma^2\phi_x^2\right)^{-\frac{1}{2}} \quad (6.100)$$

and where

$$\Psi(a, b, c) = \int_0^\infty N\left(\frac{a}{\sqrt{u}} + b\sqrt{u}\right) \frac{u^{c-1}e^{-u}}{\Gamma(c)} du. \quad (6.101)$$

Figure 6.5 shows an example of an option price surface under the rational VG model.

Chapter 7

Extension to Stochastic Volatility

7.1 Chapter overview

In this chapter we extend the rational Lévy models with a view to including some form of stochastic volatility in the martingale family $\{M_{tx}\}_{x \geq t \geq 0}$. In particular, we consider uncertainty driven by a square-root process (cf. Feller 1971, Cox *et al.* 1985). Since the square-root process does not have stationary increments we can not define a martingale family via an Esscher transform. Instead we use a technique similar to that of Keller-Ressel *et al.* (2011) to create a family of exponential-affine martingales.

7.2 The square-root process

We introduce the stochastic process $\{X_t\}_{t \geq 0}$ that will drive the uncertainty in our model. We shall choose $\{X_t\}$ to be a square-root process such as the one used by Cox, Ingersoll & Ross (1985) to model the short rate of interest. In particular we let $\{X_t\}$ have the dynamics

$$dX_t = k(\theta - X_t) dt + \sigma\sqrt{X_t} dW_t, \quad (7.1)$$

where k , θ and σ are constants and $\{W_t\}_{t \geq 0}$ is a $\mathbb{P}$ -Brownian motion. Expression (7.1) has implicit stochastic volatility due to the square-root term in the coefficient of the Brownian increment. In addition, equation (7.1) has a well-defined positive solution, given that $X_0 = x_0 > 0$, if the so-called Feller condition is satisfied:

$$2k\theta > \sigma^2. \quad (7.2)$$

Definition 7.2.1. For $u \in \mathbb{R}$ such that

$$\mathbb{E} [\exp(uX_t)] < \infty, \quad (7.3)$$

the process $\{X_t\}$ is an *affine process* if its moment generating function is of the form

$$\mathbb{E} [\exp(uX_t)] = \exp(A_t(u) + B_t(u)x_0), \quad (7.4)$$

where $A_t(-)$ and $B_t(-)$ are deterministic functions.

Following Keller-Ressel *et al.* (2011), if we take an affine process and insert it into its moment generating function with inverted time, the resulting process is a family of martingales defined on $[0, T]$. Specifically, let $\mathcal{F}_t = \sigma(\{X_s\}_{0 \leq s \leq t})$ and, for u in some subset of $\mathbb{R}$ such that

$$\mathbb{E} [e^{uX_t}] < \infty, \quad (7.5)$$

define

$$M'_{Tu} = \exp(A_{T-t}(u) + B_{T-t}(u)X_t) \quad (7.6)$$

where

$$A_0(u) = 0, \quad (7.7)$$

and

$$B_0(u) = u. \quad (7.8)$$

Following Keller-Ressel *et al.* (2011) the conditional form of (7.4) is

$$\mathbb{E}_s [\exp(uX_{t+s})] = \exp(A_t(u) + B_t(u)X_s). \quad (7.9)$$

We may thus deduce that the process $\{M'_{tu}\}_{T \geq t \geq 0}$ is a martingale. In particular, we observe that

$$\mathbb{E} [M'_{Tu} | \mathcal{F}_t] = M'_{tu}. \quad (7.10)$$

Proposition 7.2.2. The functions $A_{T-t}(u)$ and $B_{T-t}(u)$ in (7.6) are given by

$$A_{T-t}(u) = -\frac{2k\theta}{\sigma^2} \log \left(1 - \left(\frac{1 - e^{k(T-t)}}{k} \right) \frac{\sigma^2 u}{2} \right), \quad (7.11)$$

and

$$B_{T-t}(u) = \frac{ue^{k(T-t)}}{1 - \left(\frac{1 - e^{k(T-t)}}{k} \right) \frac{\sigma^2 u}{2}}. \quad (7.12)$$

Proof. We apply Itô's Lemma to (7.6) to obtain

$$\begin{aligned} \frac{dM'_{tu}}{M'_{tu}} = & \left(\frac{\partial A_{T-t}(u)}{\partial t} + \frac{\partial B_{T-t}(u)}{\partial t} X_t + k(\theta - X_t) B_{T-t}(u) + \frac{\sigma^2}{2} B_{T-t}(u)^2 X_t \right) dt \\ & + \sigma B_{T-t}(u) \sqrt{X_t} dW_t. \end{aligned} \quad (7.13)$$

By use of the martingale property we note that $\{M'_{tu}\}$ is an Itô process with zero drift. Therefore, setting the drift of (7.13) to zero we see that the functions $A_{T-t}(-)$ and $B_{T-t}(-)$ must satisfy the Riccati equations:

$$\frac{\partial A_{T-t}(u)}{\partial t} + k\theta B_{T-t}(u) = 0, \quad (7.14)$$

and

$$\frac{\partial B_{T-t}(u)}{\partial t} - kB_{T-t}(u) + \frac{\sigma^2}{2} B_{T-t}(u)^2 = 0. \quad (7.15)$$

The solutions to (7.14) and (7.15) are then given by standard techniques used to solve Riccati equations where the boundary conditions are

$$A_0(u) = 0, \quad (7.16)$$

and

$$B_0(u) = u. \quad (7.17)$$

□

7.3 The geometric square-root family

Our goal in this section is to create a term-structure model driven by $\{X_t\}$. We recall that the Flesaker-Hughston bond price given by (1.48). Specifically we have that

$$P_{tT} = \frac{\int_T^\infty \rho_x M_{tx} dx}{\int_t^\infty \rho_x M_{tx} dx}. \quad (7.18)$$

We shall define the martingale family $\{M_{tx}\}_{x \geq t \geq 0}$ on the time horizon $[0, T]$ by setting

$$\begin{aligned} M_{tx} &= \frac{M'_{t\phi_x}}{M'_{0\phi_x}} \\ &= \frac{\exp(A_{T-t}(\phi_x) + B_{T-t}(\phi_x)X_t)}{\exp(A_T(\phi_x) + B_T(\phi_x)x_0)}. \end{aligned} \quad (7.19)$$

Then, $\{M_{tx}\}$ is a unit-initialised family of positive martingales whose volatility is dependent on the square root of the level of X_t . In contrast, the term-structure models developed in Chapter 3 have the usual, endogenous stochastic volatility of bond prices, but there is no stochastic volatility in the martingale family $\{M_{tx}\}$.

The price of a T -maturity discount bond in this framework is given by

$$P_{tT} = \frac{\int_T^\infty \rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)X_t - B_T(\phi_x)x_0) dx}{\int_t^\infty \rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)X_t - B_T(\phi_x)x_0) dx}, \quad (7.20)$$

and the corresponding short rate is given by

$$r_t = \frac{\rho_t \exp((A_{T-t}(\phi_t) - A_T(\phi_t)) + B_{T-t}(\phi_t)X_t - B_T(\phi_t)x_0)}{\int_t^\infty \rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)X_t - B_T(\phi_x)x_0) dx}. \quad (7.21)$$

It is useful to introduce the abstract probability density $\{\rho_t(x)\}$ where

$$\rho_t(x) = \frac{\rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)X_t - B_T(\phi_x)x_0)}{\int_t^\infty \rho_u \exp((A_{T-t}(\phi_u) - A_T(\phi_u)) + B_{T-t}(\phi_u)X_t - B_T(\phi_u)x_0) du}. \quad (7.22)$$

We denote expectation and covariance operations with respect to this density by $\mathbb{E}^\rho[-]$ and $\text{Cov}^\rho[-]$. After an application of Itô's Lemma to (7.20) we observe that

$$\begin{aligned} dP_{tT} = & (r_t P_{tT} - \sigma^2 X_t \mathbb{E}^\rho[B_{T-t}(\phi_X)] \text{Cov}^\rho[B_{T-t}(\phi_X), \mathbf{1}\{X > T\}]) dt \\ & + \sigma \sqrt{X_t} \text{Cov}^\rho[B_{T-t}(\phi_X), \mathbf{1}\{X > T\}] dW_t. \end{aligned} \quad (7.23)$$

On comparison with general Brownian discount-bond dynamics (1.1) we identify the risk premium of the discount bond in this model $\{\lambda_t \Omega_{tT}\}$ to be given as follows:

$$\frac{\lambda_t \Omega_{tT}}{P_{tT}} = -\sigma^2 X_t \mathbb{E}^\rho[B_{T-t}(\phi_X)] \text{Cov}^\rho[B_{T-t}(\phi_X), \mathbf{1}\{X > T\}]. \quad (7.24)$$

If we choose ϕ_x to be negative and increasing for all $x \geq 0$ then the function $B_{T-t}(\phi_x)$ shall be negative and strictly increasing for $x \geq 0$ and all $t < T$. The risk premium $\{\lambda_t \Omega_{tT}\}$ will then be non-negative for all $t < T$. The fact that $B_{T-t}(\phi_x)$ is increasing follows from an application of the chain rule to calculate the derivative of $B_{T-t}(\phi_x)$ with respect to x . Specifically we obtain

$$\frac{\partial B_{T-t}(\phi_x)}{\partial x} = \frac{e^{k(T-t)}}{\left(1 - \left(\frac{1-e^{k(T-t)}}{k}\right) \frac{\sigma^2 \phi_x}{2}\right)^2} \frac{\partial \phi_x}{\partial x}. \quad (7.25)$$

7.4 Option pricing in the rational square-root model

Consider a t -expiry European-style call option on a discount bond of maturity T , with strike price K . We recall that the time-zero price is given in full generality by

$$C_{0t} = \mathbb{E} [\pi_t (P_{tT} - K)^+]. \quad (7.26)$$

Proposition 7.4.1. *Define $P(t, T, \xi)$ by*

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)\xi - B_T(\phi_x)x_0) dx}{\int_t^\infty \rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)\xi - B_T(\phi_x)x_0) dx}. \quad (7.27)$$

The function $P(t, T, \xi)$ is strictly increasing with respect to ξ for $t < T$ if ϕ_x is negative, strictly increasing for all $x \geq 0$.

Proof. By differentiation of (7.27) with respect to ξ we observe that

$$\frac{\partial P(t, T, \xi)}{\partial \xi} = P(t, T, \xi) (\Phi(t, T, T, \xi) - \Phi(t, t, T, \xi)), \quad (7.28)$$

where

$$\Phi(t, T, T, \xi) = \frac{\int_T^\infty B_{T-t}(\phi_x) \rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)\xi - B_T(\phi_x)x_0) dx}{\int_T^\infty \rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)\xi - B_T(\phi_x)x_0) dx}, \quad (7.29)$$

and

$$\Phi(t, t, T, \xi) = \frac{\int_t^\infty B_{T-t}(\phi_x) \rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)\xi - B_T(\phi_x)x_0) dx}{\int_t^\infty \rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)\xi - B_T(\phi_x)x_0) dx}. \quad (7.30)$$

We observe that if ϕ_x is increasing then

$$\frac{\partial \Phi(t, y, T, \xi)}{\partial y} > 0. \quad (7.31)$$

This follows as

$$\frac{\partial \Phi(t, y, T, \xi)}{\partial y} = f(t, y, T, \xi) (\Phi(t, y, T, \xi) - B_{T-t}(\phi_y)), \quad (7.32)$$

where

$$f(t, y, T, \xi) = \frac{\rho_y \exp((A_{T-t}(\phi_y) - A_T(\phi_y)) + B_{T-t}(\phi_y)\xi - B_T(\phi_y)x_0)}{\int_y^\infty \rho_x \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) + B_{T-t}(\phi_x)\xi - B_T(\phi_x)x_0) dx}. \quad (7.33)$$

If ϕ_x is negative and strictly increasing then, as one can verify from expression (7.25),

$$\frac{\partial B_{T-t}(\phi_y)}{\partial y} > 0. \quad (7.34)$$

Then, since $\Phi(t, y, T, \xi)$ is a weighted average of $B_{T-t}(\phi_x)$ for $x \geq y$, we have that

$$\Phi(t, y, T, \xi) - B_{T-t}(\phi_x) > 0, \quad (7.35)$$

and therefore

$$\frac{\partial \Phi(t, y, T, \xi)}{\partial y} > 0. \quad (7.36)$$

Thus, we deduce that

$$\Phi(t, T, T, \xi) - \Phi(t, t, T, \xi) > 0 \quad (7.37)$$

and therefore that the derivative in (7.28) is positive. $\square$

By use of Proposition 7.4.1 there shall then exist a number ξ^* such that

$$P(t, T, \xi^*) = K, \quad (7.38)$$

and

$$P(t, T, \xi) > K \quad (7.39)$$

for all $\xi < \xi^*$.

Now, to continue, define the change of variables by setting

$$c = \frac{2k}{\sigma^2 (1 - e^{-kt})}, \quad (7.40)$$

and

$$q = \frac{2k\theta}{\sigma^2} - 1, \quad (7.41)$$

and

$$u = cx_0 e^{-kt}. \quad (7.42)$$

Remark 7.4.2. The process $2\{cX_t\}_{t \geq 0}$ has a non-central chi square distribution with degrees of freedom $2q + 2$ and non-centrality parameter $2u$ (see Cox *et al.* 1985).

By use of Remark 7.4.2 we may work out the price (7.26) of a call option on a discount bond in closed-form to be

$$\begin{aligned}
 C_{0t} = & \int_T^\infty \rho_x \vartheta_x \left(1 - \chi^2 \left(\xi^* \left(1 - \frac{B_{T-t}(\phi_x)}{c} \right); 2q + 2, \frac{2u}{1 - \frac{B_{T-t}(\phi_x)}{c}} \right) \right) dx \\
 & - K \int_t^\infty \rho_x \vartheta_x \left(1 - \chi^2 \left(\xi^* \left(1 - \frac{B_{T-t}(\phi_x)}{c} \right); 2q + 2, \frac{2u}{1 - \frac{B_{T-t}(\phi_x)}{c}} \right) \right) dx,
 \end{aligned} \tag{7.43}$$

where $\chi^2(a; b, c)$ is the non-central chi squared cumulative distribution function evaluated at a , with degrees of freedom b and non-centrality parameter c . Here we have written

$$\vartheta_x = \frac{c \exp((A_{T-t}(\phi_x) - A_T(\phi_x)) - B_T(\phi_x)x_0) \exp\left(\frac{uB_{T-t}(\phi_x)}{c - B_{T-t}(\phi_x)}\right)}{c - B_{T-t}(\phi_x)}. \tag{7.44}$$

Chapter 8

Calibration

8.1 Chapter overview

In this chapter, we carry out a calibration of the rational Brownian motion and rational variance gamma families of term-structure models using data from bond and interest-rate cap markets. Specifically, we introduce the following two-stage calibration strategy: in the first stage we obtain a smoothed initial term-structure density ρ_x from a cross-section of bond yields, and insert it into our pricing equation. In the second stage we use observed cap prices to calibrate the structural parameters of the martingale family $\{M_{tx}\}_{x \geq t \geq 0}$. We find that both models can provide a close fit to observed prices, even out of sample.

Additionally, we examine the ability of the geometric Brownian motion family to capture the dynamics of the volatility of yields. We regress realised volatility on model implied volatility; the results indicate a positive and significant relationship.

8.2 Description of the data

8.2.1 The UK data set

We make use of two UK data sets: one data set containing continuously compounded UK government bond yields, and the other data set containing premiums of interest-rate caps denominated in GBP. The caps are semiannual, they are written on six-month UK Libor with payments being made at the end of each period. For at-the-money

Table 8.1: *Summary statistics for the UK ATM cap data set.* Strikes are given as percentages and premiums are measured in GBP. The data set consists of weekly observations from January 2008 to May 2010, totalling 124 observations. Data is obtained from Bloomberg Financial Markets.

		1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Strike (%)	Mean	2.68	3.05	3.43	3.68	3.86	3.98	4.07	4.15	4.20	4.25
	Std Dev	1.94	1.57	1.32	1.14	1.01	0.90	0.81	0.74	0.69	0.64
	Min	0.89	1.36	1.71	2.06	2.36	2.61	2.84	3.02	3.17	3.28
	Max	6.38	6.35	6.28	6.16	6.04	5.93	5.82	5.73	5.66	5.59
Price (GBP)	Mean	21	84	167	249	324	394	460	523	582	639
	Std Dev	5	14	35	54	71	86	100	112	123	133
	Min	12	59	105	152	198	245	291	336	381	426
	Max	33	111	231	340	432	521	606	684	758	830

(ATM) contracts the strike rate is the forward rate for the reference Libor. In addition to ATM contracts we have price data for twelve strikes ranging from 1% to 12% for the seven-year contract. The data sets were obtained from Bloomberg Financial Markets.

Our ATM data set ranges from January 2008 to May 2010, totalling 124 observations. In particular, the data consists of 124 weekly observations of ATM cap premiums, quoted in GBP, and strikes quoted in percentage terms, for ten maturities ranging from one to ten years. Additionally, there are 124 weekly observations of UK government bond yields with seventeen maturities ranging from one month to thirty years. Table 8.1 shows summary statistics for the UK ATM cap data set.

Our data set for the seven-year maturity cap, which we call the “skew data set”, ranges from June 2005 to August 2008 totalling 130 weekly observations. Each observation consists of a cap premium for the seven-year contract at twelve strikes ranging from 1% to 12%. We also have the corresponding observations of UK government bond yields for seventeen maturities. Table 8.2 presents summary statistics for the seven-year contract, skew data set.

8.2.2 The US data set

We make use of two US data sets: one data set containing continuously compounded US treasury yields and the other data set containing premiums of interest rate caps

Table 8.2: *Summary statistics for the UK seven-year cap contract, skew data set.* Premiums are measured in GBP. The data set consists of weekly observations from June 2005 to August 2008, totalling 130 observations. All statistics are quoted in GBP. Data is obtained from Bloomberg Financial Markets.

	1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12 %
Mean	1557	1083	701	422	251	160	112	85	68	56	48	42
Std Dev	337	296	228	147	80	49	41	37	33	29	26	23
Min	977	595	350	213	121	64	38	24	17	10	6	4
Max	2431	1887	1363	881	489	249	185	149	125	107	93	82

denominated in USD. The caps are semiannual, written on six-month US Libor with the payments made at the end of each period. All option contracts are ATM, i.e., the strike rate is the forward rate for the reference Libor.

Our US data set has a sample period from April 2002 to February 2011. There are 460 weekly observations of ATM cap premiums for ten maturities from one year to ten years. There are 2300 daily observations of discount-bond yields for ten maturities from three months to twenty-five years. Summary statistics for the US ATM cap data set are presented in Table 8.3.

8.2.3 Interest-rate caps

A cap is a collection of interest-rate derivatives called caplets. Each caplet with cap rate L_K represents an option to pay the rate L_K at expiry, or a reference rate, usually Libor. Within a cap, each caplet has the same cap rate. Specifically, if we suppose that the notional amount of a purchased cap is one GBP, then each caplet expiring at t_i yields a payoff to the cap holder of $(t_{i+1} - t_i)(L_{t_i, t_{i+1}} - L_K)^+$ at time t_{i+1} . Here $L_{t_i, t_{i+1}}$ is the reference Libor rate.

The value at time t_i of a caplet is then

$$\frac{t_{i+1} - t_i}{1 + (t_{i+1} - t_i)L_{t_i, t_{i+1}}}(L_{t_i, t_{i+1}} - L_K)^+ = (1 - (1 + (t_{i+1} - t_i)L_K)P_{t_i, t_{i+1}})^+, \quad (8.1)$$

which is $(1 + (t_{i+1} - t_i)L_K)$ times the payoff of a put option on a discount bond, with strike price $1/(1 + (t_{i+1} - t_i)L_K)$. It is therefore convenient to interpret caplets as discount-bond options.

Table 8.3: *Summary statistics for the US ATM cap data set.* Strikes are given as percentages and premiums are measured in USD. The data set consists of weekly observations from April 2002 to February 2011, totalling 460 observations. Data is obtained from Bloomberg Financial Markets.

		1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Strike (%)	Mean	2.67	3.00	3.32	3.58	3.80	3.98	4.12	4.24	4.34	4.42
	Std Dev	1.67	1.44	1.26	1.11	1.00	0.92	0.86	0.81	0.78	0.76
	Min	0.38	0.52	0.75	0.11	0.14	0.17	0.20	0.22	0.24	0.57
	Max	5.68	5.63	5.61	5.62	5.63	5.65	5.66	5.67	5.68	5.73
Price (USD)	Mean	15	71	151	241	335	432	527	619	709	796
	Std Dev	5	20	43	67	90	112	133	152	170	186
	Min	7	35	81	128	179	233	287	342	397	451
	Max	44	125	238	375	498	627	757	881	995	1104

If the caplets expire at times $t_1, t_2, \dots, t_{n-1}$, where $t_n = T$ is the maturity of the cap, it is market convention that the rate of the first period is not capped so that no payment occurs at t_1 . A ten-year semiannual cap thus contains nineteen caplets.

The reader can consult Cairns (2004), Brigo & Mercurio (2004), Musiela & Rutkowski (2005), or Schoenmakers (2005) for further information regarding the payoffs of interest rate derivatives. Good resources for material concerning the calibration of financial models include Bates (2003), Chernov & Ghysels (2000), and references cited therein.

We now propose the following two-stage calibration procedure for the rational Lévy models. First we smooth the current market yield curve and use this as our observation of the initial term-structure density ρ_x . Then we calibrate the structural parameters of the martingale family $\{M_{tx}\}_{x \geq t \geq 0}$ which appears in the Flesaker-Hughston discount-bond price (see Chapter 1), given by

$$P_{tT} = \frac{\int_T^\infty \rho_x M_{tx} dx}{\int_t^\infty \rho_x M_{tx} dx}, \quad (8.2)$$

by matching observed prices of interest-rate caps.

8.3 Fitting the initial term structure

We shall impose some smoothness conditions on the shape of the yield curve, as described in Nelson & Siegel (1987). Specifically, we parameterise the initial yield Y_{0T} , of a T -maturity discount bond in the following manner:

$$Y_{0T} = \beta_1 + \beta_2 \left(\frac{1 - e^{-T/\tau}}{T/\tau} \right) + \beta_3 \left(\frac{1 - e^{-T/\tau}}{T/\tau} - e^{-T/\tau} \right). \quad (8.3)$$

The parameters β_i , $i \in \{1, 2, 3\}$, and τ are estimated by minimising squared relative differences between observed yields and the corresponding yields implied by (8.3).

Recall that the initial term-structure density ρ_x is given by

$$\rho_x = -\partial_x P_{0x}. \quad (8.4)$$

Combining (8.3) and (8.4) then gives us the following expression for ρ_x :

$$\rho_x = \left(\beta_1 + \beta_2 e^{-x/\tau} + \beta_3 \frac{x e^{-x/\tau}}{\tau} \right) e^{-Y_{0x}x}. \quad (8.5)$$

While several extensions of the Nelson & Siegel (1987) yield-curve model exist in the literature (see, e.g., Svensson 1995 and Cairns 2004), for our purposes it suffices to employ the model (8.3). In Figure 8.1 we display examples of calibrated initial term-structure densities.

8.4 Fitting option data

8.4.1 At-the-money calibration

The second stage of the calibration procedure consists of calibrating the structural parameters of the martingale family $\{M_{tx}\}_{x \geq t \geq 0}$. Recall the Esscher transform (3.3) given by

$$M_{tx} = \frac{e^{\phi_x L_t}}{\mathbb{E}[e^{\phi_x L_t}]}. \quad (8.6)$$

The calibration of the martingale family $\{M_{tx}\}$ will include determination of the parameters implicit in (a) the underlying Lévy process $\{L_t\}$, and (b) the function ϕ_x .

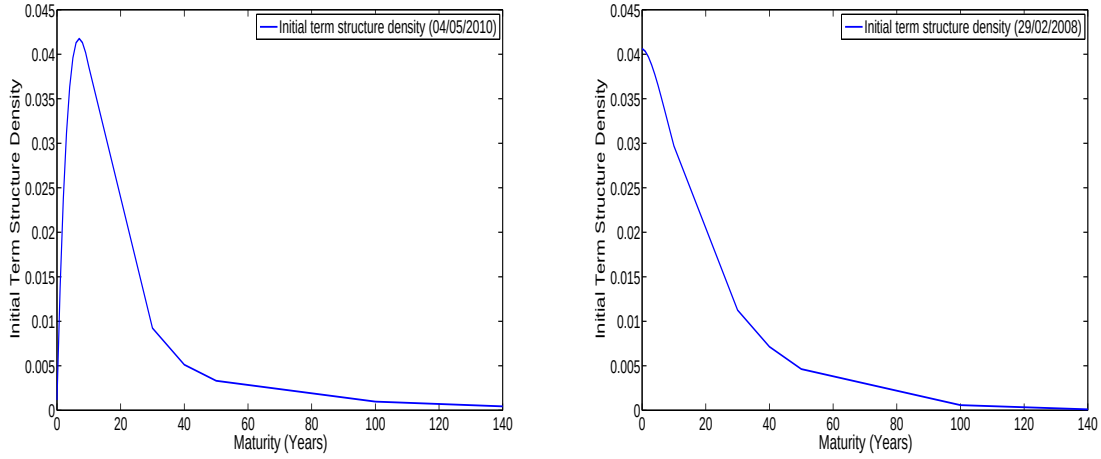


Figure 8.1: *Calibration of the initial term-structure density.* This figure shows the initial term-structure densities ρ_x , as given by expression (8.5), implied by observed yields as at the dates 4 May 2010 and 29 February 2008. The term structure is smoothed by use of the Nelson & Siegel (1987) method. Yield data was obtained from Bloomberg Financial Markets

We shall thus impose a parametric form for the function ϕ_x . In particular, we shall choose:

$$\phi_x = \sigma e^{-bx}, \quad (8.7)$$

and proceed to calibrate the parameters σ and b . The reason that we have chosen the function ϕ_x such that $|\phi_x|$ is monotonically decreasing is to ensure positive risk premium for the discount bond system (see Chapter 4). Evidently, a more general choice of ϕ_x , involving more parameters could have been made. However, our aim here is to demonstrate the effectiveness and flexibility of the geometric Lévy term-structure models by showing that even with a simple and rather restricted choice (8.7) for the function ϕ_x involving only two parameters, we are nevertheless able to fit caplet prices across different maturities relatively well. We calibrate the parameters of our model by minimising the sum of the squared differences between observed prices of caps and corresponding prices as implied by the model. This minimisation is performed for each week in our data set.

Let us denote by Θ the set of all parameter values whose i^{th} element, θ_i , represents the value of the i^{th} model parameter. For the rational Brownian family of term-structure models we thus have $\Theta = \{\sigma, b\}$ (see Chapter 4) and for the rational VG

Table 8.4: *Summary statistics for model parameters implied by weekly calibration to the UK data set.* The calibrations are performed by first minimising the squared errors between Nelson-Siegel implied yields and market yields to obtain the initial term-structure density (8.5). Then, given the term-structure density we minimise the squared errors between observed, and model implied ATM cap prices. This is carried out for the Brownian and VG families of rational Lévy models. The Brownian model has two parameters $\{\sigma, b\}$, and the VG model has three $\{\sigma, b, \nu\}$. Calibrations are performed weekly for the period January 2008 to May 2010. There are 124 observations to compute each statistic for every cap maturity. Market data was obtained from Bloomberg Financial Markets.

	Brownian		VG		
	σ	b	σ	b	ν
Mean	2.17	0.54	2.16	0.54	0.25
Std Dev	1.56	0.42	1.54	0.42	0.83
Min	0.47	0.10	0.48	0.10	0.004
Max	7.23	1.74	6.84	1.71	3.92

family we have $\Theta = \{\sigma, b, \nu, \mu\}$ (see Chapter 6). For a model with N parameters the weekly minimisation problem is then

$$\min_{\theta_i \in \Theta} \sum_{j=1, \dots, N}^M [\text{Cap}^{\text{Model}}(\Theta, T_j, L_{K,j}) - \text{Cap}^{\text{Market}}(T_j, L_{K,j})]^2. \quad (8.8)$$

Here M is the number of observable caps, $\text{Cap}^{\text{Model}}(\Theta, T_j, L_{K,j})$ is the model implied price of a T_j -maturity cap with cap rate $L_{K,j}$, and $\text{Cap}^{\text{Market}}(T_j, L_{K,j})$ is the corresponding market price.

We shall set $\mu = 0$ in the VG model. The additional freedom of the VG family over the Brownian family is then represented by the extra parameter ν , and the inclusion of jumps. Table 8.4 gives some summary statistics of the parameter sets of the Brownian family and the VG family of term-structure models after 124 weekly calibrations to the UK ATM data set. Figure 8.2 presents two graphical examples of the model's fit to the market after calibration.

Table 8.5 presents some summary statistics of the time series of pricing errors, i.e., the difference between observed and model implied cap prices. We include also results for the CIR++ model, calibrated to the same option data, for comparison. The CIR++ model is a one-factor short rate model: it is an extension of the square-root CIR model (Cox *et al.* 1985) adapted to fit arbitrary initial term structures. It is used extensively

Table 8.5: *Summary statistics for pricing errors implied by weekly calibration to the UK ATM cap data set.* Pricing errors are computed for the Brownian and VG families of rational Lévy models, and the CIR++ short rate model. Each model is calibrated on a weekly basis to ATM caps. The difference between the observed cap prices and the model implied prices are then recorded for each maturity. The procedure is repeated weekly for the period January 2008 to May 2010. There are 124 observations to compute each statistic for every cap maturity. Market data was obtained from Bloomberg Financial Markets.

Pricing Error (%)	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Brownian	Mean	-14.46	-6.62	1.79	3.54	2.53	1.15	0.12	-0.77	-1.23
	Std Dev	55.18	18.14	4.83	3.20	4.06	3.60	2.45	0.95	1.87
VG	Mean	-11.55	-3.80	3.82	5.09	3.77	2.15	0.95	-0.05	-0.60
	Std Dev	53.10	21.22	9.96	6.50	5.10	3.57	2.01	1.10	2.75
CIR++	Mean	-35.44	-8.27	3.90	6.47	4.82	2.50	0.49	-0.99	-1.85
	Std Dev	131.52	25.32	12.80	8.27	5.44	3.510	2.10	1.54	2.81

Table 8.6: *Summary statistics for out-of-sample pricing errors implied by weekly calibration to the UK ATM cap data set.* Prediction errors are computed for the Brownian and VG families of rational Lévy models, and the CIR++ short rate model. Each model is calibrated on a weekly basis to ATM caps. This calibration is then used to price new cap contracts five weeks into the future. The difference between the observed cap prices and the model implied prices are then recorded for each maturity. The procedure is repeated weekly for the period January 2008 to May 2010. There are 119 observations to compute each statistic for every cap maturity. Market data was obtained from Bloomberg Financial Markets.

Prediction Error (%)	1Y	2Y	3Y	4Y	5Y	6Y	7Y	8Y	9Y	10Y
Brownian	Mean	-30.99	-17.64	-5.88	-2.62	-2.71	-3.40	-3.83	-4.20	-4.23
	Std Dev	61.20	38.05	28.80	25.34	24.01	23.35	22.77	22.51	22.40
VG	Mean	-25.96	-14.05	-3.34	-0.70	-1.18	-2.15	-2.79	-3.31	-3.45
	Std Dev	56.87	36.62	27.67	24.23	22.92	22.43	22.07	22.06	22.17
CIR++	Mean	-71.63	-35.47	-10.12	-4.22	-4.28	-5.31	-5.84	-6.44	-6.67
	Std Dev	151.76	77.72	57.07	44.92	38.08	33.63	30.60	28.37	26.97

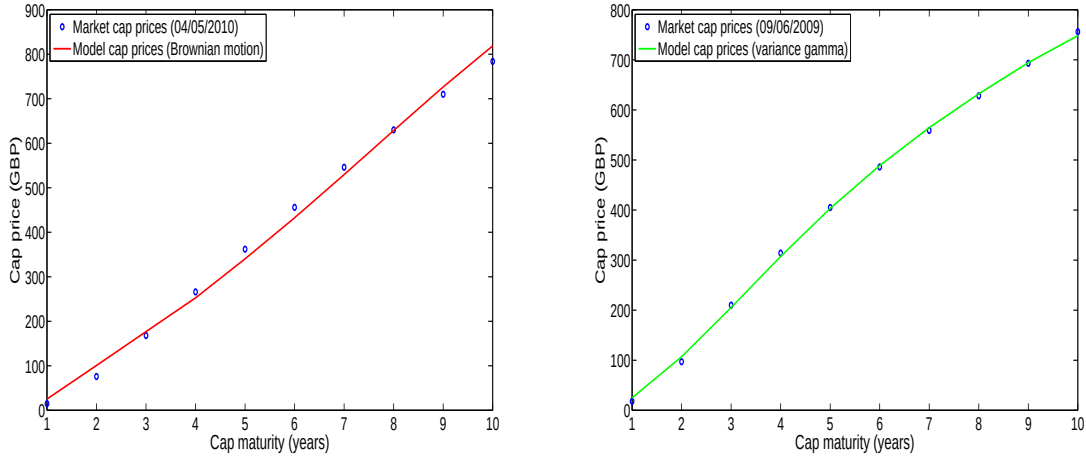


Figure 8.2: *Calibration to UK ATM cap data.* The Brownian family of models is calibrated to ten ATM caps at 4 May 2010 and the VG family at 9 June 2009. Calibrations are performed by minimising the squared errors between observed and model implied cap prices. Data was obtained from Bloomberg Financial Markets.

in the industry and the literature, so we consider it to be a fair benchmark. A brief summary of the CIR++ is presented in Appendix A (see also Brigo & Mercurio 2001, Brigo & Mercurio 2006, and references cited therein). For information regarding the calibration of the Cox *et al.* (1985) model and its extensions, one may consult, for example, Jagannathan *et al.* (2003), Yang (2006), and Mannolini *et al.* (2008), among others. In fact several extensions of the CIR++ model exist in the literature and a more fair comparison would be with the JCIR++ model recently studied in Brigo & El Bachir (2010). The JCIR++ model extends the CIR++ to include jumps.

Next, we conduct a standard out of sample pricing test as presented in Gupta *et al.* (2005). We use model parameters obtained from the calibration at time t_i to value caps at time t_{i+1} , where $t_{i+1} - t_i$ is set to be equal to five weeks. We compare the model implied cap prices to observed prices to get 119 out-of-sample pricing errors. Table 8.6 gives some summary statistics of the out of sample pricing errors for the Brownian, VG, and CIR++ models.

In general the models fit better to caps with larger maturities and have a poorer fit to caps with very low maturities. In this way the results are similar to Jagannathan *et al.* (2003). The poor fit to lower maturity caps may be because the objective function in expression (8.9) assigns importance to the caps with the highest prices. Other

Table 8.7: *Summary statistics for model parameters implied by weekly calibration to the UK seven-year cap contract, skew data set.* The calibrations are performed by first minimising the squared errors between Nelson-Siegel implied yields and market yields to obtain the initial term-structure density (8.5). Then, given the term-structure density we minimise the squared errors between observed, and model implied cap prices across several strikes for the seven-year contract. This is carried out for the Brownian family of rational Lévy models. The Brownian model has two parameters $\{\sigma, b\}$. Calibrations are performed weekly for the period June 2005 to August 2008. There are 130 observations to compute each statistic for every strike. Market data was obtained from Bloomberg Financial Markets.

	σ	b
Mean	2.05	0.38
Std Dev	1.60	0.27
Min	0.39	0.06
Max	5.07	0.75

calibration methods may exist such that the fit of each model can be improved, but for our purposes the minimisation of squared errors suffices. In general, a multi-factor model may be more appropriate to capture movements in the short end of the cap surface. Gupta *et al.* (2005) refer to evidence that short term cap prices may not be well represented by term-structure models that do not contain explicit, additional volatility factors.

8.4.2 Calibration to skew

We would now like to examine the ability of the Brownian model to capture the variation of option prices across different strikes. We keep the choice (8.7) for the function ϕ_x and calibrate the parameters of the model by minimising the sum of squared differences between observed prices of the seven-year cap contract at different strikes, and corresponding prices implied by the model. For a model with N parameters, i.e., $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$, the weekly minimisation problem is then

$$\min_{\theta_i \in \Theta} \sum_{i=1, \dots, N}^M [\text{Cap}^{\text{Model}}(\Theta, T, L_{K_j}) - \text{Cap}^{\text{Market}}(T, L_{K_j})]^2. \quad (8.9)$$

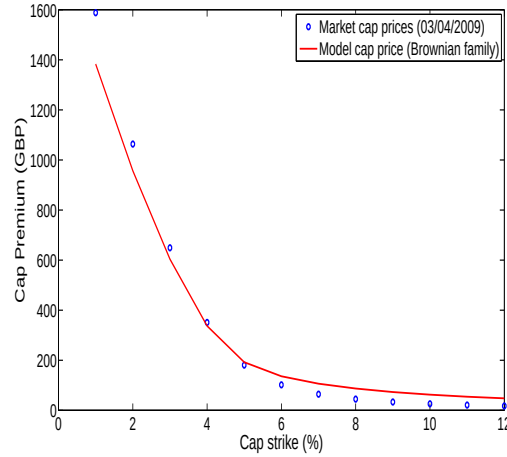


Figure 8.3: *Calibration to UK seven-year cap contract, skew data set.* The Brownian family of models is calibrated to twelve cap contracts with a seven-year maturity at 3 April 2009. Calibrations are performed by minimising the squared errors between observed and model implied cap prices. Data was obtained from Bloomberg Financial Markets.

Here M is the number of observable caps, $\text{Cap}^{\text{Model}}(\Theta, T, L_{K_j})$ is the model implied price of a T -maturity cap with cap rate L_{K_j} , and $\text{Cap}^{\text{Market}}(T, L_{K_j})$ is the corresponding market price.

Table 8.7 gives some summary statistics of the parameter sets of the Brownian family of term-structure models after 130 weekly calibrations to the UK skew data set. Figure 8.3 presents a graphical example of the rational Brownian families fit to the market. Table 8.8 presents some summary statistics of the time series of pricing errors created after calibration where we have included data for the CIR++ model for comparison.

To study out of sample pricing power we again use model parameters from the calibration at time t_i to value observed caps at time t_{i+1} , where $t_{i+1} - t_i$ is equal to five weeks. Table 8.9 gives some summary statistics of the out-of-sample pricing errors for the Brownian and CIR++ models.

8.5 Discount-bond volatility

We recall the setup of Chapter 4. With the given assumptions, in the Brownian family the discount-bond volatility can be completely determined by a cross section of option

Table 8.8: *Summary statistics for pricing errors implied by weekly calibration to the UK seven-year cap contract, skew data set.* Pricing errors are computed for the Brownian family of rational Lévy models, and the CIR++ short rate model. Each model is calibrated on a weekly basis to across several strikes for the seven-year cap contract. The difference between the observed cap prices and the model implied prices are then recorded for each strike. The procedure is repeated weekly for the period June 2005 to August 2008. There are 130 observations to compute each statistic for every strike. Market data was obtained from Bloomberg Financial Markets.

Pricing Error (%)		1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%
Brownian	Mean	3.22	2.32	-0.51	-3.28	-2.41	-3.19	-6.31	-10.62	-15.54	-23.00	-29.87	-34.79
	Std Dev	7.84	7.60	8.37	8.20	6.41	19.25	42.53	69.45	96.64	123.42	148.01	166.59
CIR++	Mean	-1.74	3.47	4.27	-0.62	-3.53	-2.29	5.61	17.81	31.12	43.03	53.60	62.54
	Std Dev	2.44	2.45	6.88	10.10	19.70	21.80	23.70	24.27	24.19	23.18	21.69	19.68

Table 8.9: *Summary statistics for out-of-sample pricing errors implied by weekly calibration to the UK seven-year cap contract, skew data set.* Prediction errors are computed for the Brownian family of rational Lévy models, and the CIR++ short rate model. Each model is calibrated on a weekly basis across several strikes for the seven-year cap contract. This calibration is then used to price new cap contracts five weeks into the future. The difference between the observed cap prices and the model implied prices are then recorded for each strike. The procedure is repeated weekly for the period January 2008 to May 2010. There are 119 observations to compute each statistic for every strike. Market data was obtained from Bloomberg Financial Markets.

Prediction Error (%)		1%	2%	3%	4%	5%	6%	7%	8%	9%	10%	11%	12%
Brownian	Mean	4.27	3.36	0.22	-3.24	-3.29	-5.15	-9.45	-15.13	-22.13	-31.10	-39.84	-46.92
	Std Dev	10.87	13.36	17.36	21.36	24.13	33.63	54.15	80.26	108.17	135.57	161.32	182.15
CIR++	Mean	-0.39	5.05	5.81	0.12	-4.06	-4.27	1.96	12.47	24.56	35.90	46.85	56.88
	Std Dev	11.06	12.69	16.38	21.63	30.25	33.10	34.76	35.16	34.59	32.55	29.02	23.58

Table 8.10: *Summary statistics for model parameters implied by weekly calibration to the US data set.* The calibrations are performed by first minimising the squared errors between Nelson-Siegel implied yields and market yields to obtain the initial term-structure density (8.5). Then, given the term-structure density we minimise the squared errors between observed, and model implied ATM cap prices. This is carried out for the Brownian family of rational Lévy models. The Brownian model has two parameters $\{\sigma, b\}$. Calibrations are performed weekly for the period January 2008 to May 2010. There are 124 observations to compute each statistic for every cap maturity. Market data was obtained from Bloomberg Financial Markets.

	σ	b
Mean	1.38	0.15
Std Dev	0.73	0.10
Min	0.57	0.00
Max	6.50	0.80

prices. We wish to examine how well the Brownian family recreates volatility dynamics. In this section we shall forecast weekly realised volatility of yields using a model calibrated to ATM option prices in our US data set. Then, following the methodology of Collins-Dufresne *et al.* (2009) we compare these forecasts to those of a random walk with GARCH(1,1) errors. Summary statistics for the parameters of the Brownian family, calibrated to the US data set, are presented in Table 8.10.

Recall that the continuously compounded T -maturity yield is given by

$$Y_{tT} = \frac{-1}{T-t} \ln P_{tT}. \quad (8.10)$$

Under the Brownian family of models of Chapter 4, expression (8.10) becomes

$$Y_{tT} = \frac{-1}{T-t} \ln \frac{\int_T^\infty \rho_x \exp(\phi_x W_t - \frac{1}{2} \phi_x^2 t) dx}{\int_t^\infty \rho_x \exp(\phi_x W_t - \frac{1}{2} \phi_x^2 t) dx}. \quad (8.11)$$

It is useful to introduce the Musiela parameterisation, $\tau = T - t$. We define

$$P_t(\tau) = P_{t,t+\tau}, \quad (8.12)$$

for the bond price, and

$$Y_t(\tau) = Y_{t,t+\tau}, \quad (8.13)$$

for the yield. Applying Itô's Lemma to expressions (8.12) and (8.13) we can obtain the following stochastic differential equations to describe bond price dynamics:

$$\frac{dP_t(\tau)}{P_t(\tau)} = (r_t - \rho_t(\tau) - \Phi_t(0) (\Phi_t(\tau) - \Phi_t(0))) dt + (\Phi_t(\tau) - \Phi_t(0)) dW_t, \quad (8.14)$$

and the dynamics of the associated yield:

$$\begin{aligned} dY_t(\tau) = & \frac{-1}{\tau} \left(r_t - \rho_t(\tau) - \frac{1}{2} (\Phi_t(\tau) - \Phi_t(0))^2 - \Phi_t(0) (\Phi_t(\tau) - \Phi_t(0)) \right) dt \\ & + \frac{-1}{\tau} (\Phi_t(\tau) - \Phi_t(0)) dW_t. \end{aligned} \quad (8.15)$$

Here we have defined

$$\rho_t(\tau) = \frac{\rho_{t+\tau} \exp \left(\phi_{t+\tau} W_t - \frac{1}{2} \phi_{t+\tau}^2 t \right)}{\int_t^\infty \rho_x \exp \left(\phi_x W_t - \frac{1}{2} \phi_x^2 t \right) dx}, \quad (8.16)$$

and

$$\Phi_t(\tau) = \frac{\int_{t+\tau}^\infty \phi_x \rho_x \exp \left(\phi_x W_t - \frac{1}{2} \phi_x^2 t \right) dx}{\int_t^\infty \rho_x \exp \left(\phi_x W_t - \frac{1}{2} \phi_x^2 t \right) dx}. \quad (8.17)$$

Following Collins-Dufresne *et al.* (2009) we estimate weekly realised volatility at time t via the proxy

$$\hat{\sigma}(\tau) = \sqrt{\sum_{n=1}^N (\Delta Y_{t_i}(\tau))^2}, \quad (8.18)$$

where $N + 1$ is the number of days in a given week, t_i is the i^{th} day in the week beginning at time t and

$$\Delta Y_{t_i}(\tau) = Y_{t_{i+1}}(\tau) - Y_{t_i}(\tau) \quad (8.19)$$

is the change in yield from t_i to t_{i+1} . We thus attempt to forecast the volatility proxy $\hat{\sigma}(\tau)$ by a Monte Carlo simulation of (8.18) where $Y_t(\tau)$ is given by (8.13). For each week beginning at time t we calculate realised volatility via Monte Carlo simulation of (8.18) with fifty-thousand paths where model parameters are implied by calibration to the US ATM data set at time t . We do this for the short end of the yield curve (the three-month maturity) and the long end (the ten-year maturity). Summary statistics for the model forecasts and realised volatility are given in Table 8.11.

Next, we fit a random walk with GARCH(1,1) errors (RWG) model to daily changes in yields, via maximum likelihood estimation. Weekly volatility forecasts may then

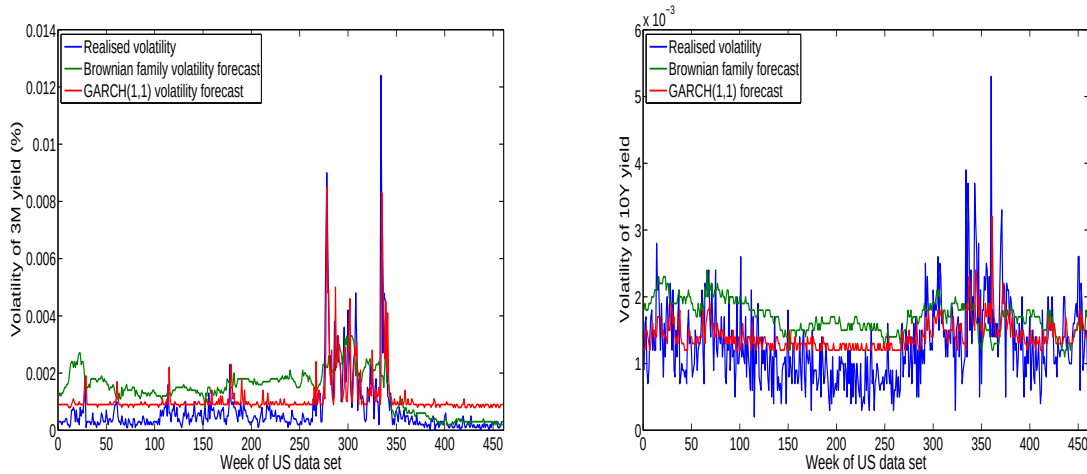


Figure 8.4: *Plots of realised and forecasted volatility.* Weekly realised volatility is measured for the three-month and ten-year yields in the US data set. Weekly volatility forecasts are computed using the Brownian family of models, which is calibrated weekly to option data, and a random walk model with GARCH(1,1) errors, which is calibrated to daily changes the yields for each maturity. There are 461 observations of realised volatility. Data was taken from Bloomberg Financial Markets.

be calculated in-sample using standard closed-form GARCH formulae (Bauwens *et al.* 2008). Figure 8.4 presents the time series of realised volatilities observed and forecasted by the Brownian and the RWG models. Summary statistics for volatility forecasts for the RWG fitted to three-month and ten-year yields are given in Table 8.11.

Finally, to test the significance of the forecasts we regress the realised volatility estimates on the volatility forecasts. The results are presented in Table 8.12. The RWG model does not take into account option prices, but it has the advantage of being fit to daily data and is generally considered to be a good benchmark in the literature (see, e.g., Collins-Dufresne *et al.* 2009). Both the Brownian model and RWG generate statistically significant forecasts with a positive relationship with realised volatility in univariate regressions. In the multivariate regression we note that the RWG model does not drive out the significance of the Brownian model.

Table 8.11: *Summary statistics for weekly realised volatility and forecasted volatility, implied by calibration to the US data set.* Realised volatility is measured for the three-month and ten-year yields using the proxy (8.18). Forecasts are computed for two models: the Brownian rational Lévy model and a random walk with GARCH(1,1) errors. The Brownian model is calibrated weekly to ATM cap prices in the US data set. The Brownian models forecasts are then computed via Monte Carlo simulations of the realised vol proxy with 50,000 paths. The random walk model is calibrated to daily changes in yields for the three-month and then the ten-year yields. The random walk model forecasts are then known in closed form via standard results (see Bauwens et al. 2008). There are 460 weekly observations to compute each statistic for both yield maturities. Statistics are quoted in percentage terms. Market data was obtained from Bloomberg Financial Markets.

	Yield:	3M	10Y
Realised	Mean	0.07	0.13
	Std Dev	0.11	0.06
	Min	0.00	0.02
	Max	1.24	0.53
Brownian	Mean	0.14	0.17
	Std Dev	0.07	0.02
	Min	0.02	0.11
	Max	0.33	0.24
Random walk GARCH	Mean	0.11	0.14
	Std Dev	0.08	0.02
	Min	0.08	0.12
	Max	0.85	0.32

Table 8.12: *Regressions of realised volatility on forecasted volatility.* Realised volatility is measured for the three-month and ten-year yields using the proxy (8.18). Forecasts are computed for two models: the Brownian rational Lévy model and a random walk with GARCH(1,1) errors. The Brownian model is calibrated weekly to ATM cap prices in the US data set. The Brownian model's forecasts are then computed via Monte Carlo simulations of the realised vol proxy with 50,000 paths. The random walk model is calibrated to daily changes in yields for the three-month and then the ten-year yields. The random walk model forecasts are then known in closed form via standard results (see Bauwens et al. 2006). Regressions are computed via ordinary least squares: first for the Brownian forecasts, then the random walk GARCH forecasts, then for both the Brownian and random walk models. Betas, t-Stats and adjusted R-squares are provided.

Yield:		3M	10Y
Brownian	Beta	0.67	0.59
		(9.98)	(5.09)
	R square	0.18	0.05
Random walk GARCH	Beta	0.98	1.30
		(20.71)	(11.17)
	R square	0.48	0.21
Combination	Brownian Beta	0.34	0.39
		(6.20)	(3.66)
	random walk GARCH Beta	0.88	1.22
		(18.18)	(10.46)
	R square	0.52	0.23

Chapter 9

Sensitivity Analysis

9.1 Chapter overview

In this chapter we present a strategy that aims to hedge movements in option prices against movements in the “level” of the underlying yield curve. We test the performance of this strategy within the rational Brownian family of models. Specifically, we use the strategy to create a hedge for each caplet in a portfolio of caps where each hedge involves trading in underlying discount bonds. Results indicate that the strategy removes the same amount of variation from the portfolio as is caused by the first principal component in the underlying yield curve.

9.2 A hedging strategy

Recall that the initial price of a European-style call option on a T -maturity discount bond with strike price K and expiry t is given by

$$C_{0t} = \mathbb{E} [\pi_t (P_{tT} - K)^+]. \quad (9.1)$$

We may ask the question: how does one determine the sensitivity of the option price C_{0t} with respect to the initial price of the underlying bond P_{0T} ? We shall present here a strategy that hedges movements in option prices against parallel shifts in the “level” of the underlying yield curve. Recall that the initial bond price is given by

$$P_{0T} = \int_T^\infty \rho_x dx. \quad (9.2)$$

The initial price of the bond is a functional of the initial term-structure density ρ_x . Likewise, the option price C_{0t} is a functional of ρ_x . Following Section 3.5, we can write C_{0t} in closed form as

$$C_{0t} = \int_T^\infty \rho_x m_{tx}(\xi^*) dx - K \int_t^\infty \rho_x m_{tx}(\xi^*) dx, \quad (9.3)$$

where

$$m_{tx}(\xi^*) = \mathbb{E} \left[\mathbf{1} \left\{ \int_T^\infty \rho_x M_{tx} dx > K \int_t^\infty \rho_x M_{tx} dx \right\} M_{tx} \right], \quad (9.4)$$

and $\{M_{tx}\}$ is a one-parameter family of martingales. To determine the option sensitivity, we are required to employ the method of *functional differentiation*. Given the bond price $P_{0T}[\rho]$, the functional derivative of P_{0T} with respect to ρ_x is

$$\frac{\delta P_{0T}}{\delta \rho} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (P_{0T}[\rho + \varepsilon \eta] - P_{0T}[\rho]), \quad (9.5)$$

where η_x must satisfy the condition

$$\int_0^\infty \eta_x dx = 0 \quad (9.6)$$

to ensure that

$$\int_0^\infty \rho_x + \varepsilon \eta_x dx = 1. \quad (9.7)$$

According to (9.5) we perturb ρ_x by an amount ε in the the direction η_x . However, there are infinitely many ways of making the perturbation

$$\rho_x \rightarrow \rho_x + \varepsilon \eta_x. \quad (9.8)$$

See for example Figure 9.1. The question of how to measure sensitivity thus becomes a question of how to choose η_x .

It has been well documented that shifts in the level of the term structure explain most of the historical variation in yields (cf. Litterman & Scheinkman 1991). We can perform a principal component analysis on the changes in bond yields over the period January 2008 to May 2010 in our UK ATM data set (see Chapter 8 for a description of the data). Results indicate that changes in the first principal component correspond to changes in the level of the yield curve, and account for 85% of the historical variance of term-structure movements.

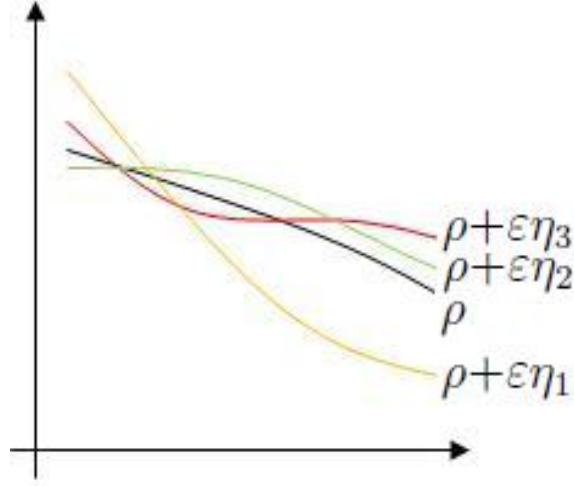


Figure 9.1: *Perturbing the initial term-structure density.* A graphical example of perturbing the initial term-structure density ρ by an amount ε in the directions η_1 , η_2 , and η_3 is shown. It is important that the direction chosen is financially intuitive.

Thus, we proceed to choose η_x as follows: in the case of a single factor model we conjecture that it is sufficient to consider a single η_x . To this end we take the perturbation to be a constant shift in the initial yield curve $\{Y_{0T}\}$ by an amount ε . Specifically, we consider movements in the initial yield curve of the form

$$Y_{0T} \rightarrow Y_{0T} + \varepsilon, \quad (9.9)$$

for all $T > 0$. Recall that the initial T -maturity discount-bond price is given by

$$P_{0T} = e^{-Y_{0T}T}. \quad (9.10)$$

Therefore the initial yield Y_{0T} is given by

$$Y_{0T} = \frac{-1}{T} \ln \int_T^\infty \rho_x dx. \quad (9.11)$$

Inserting the perturbation (9.9) into (9.10) for small ε we obtain

$$e^{Y_{0T}T} \rightarrow e^{Y_{0T}T + \varepsilon T} \approx (1 - \varepsilon T) P_{0T}. \quad (9.12)$$

By use of (9.2) and (9.8) we then have that

$$\int_T^\infty \rho_x dx \rightarrow \int_T^\infty \rho_x + \varepsilon \eta_x dx \approx (1 - \varepsilon T) P_{0T}. \quad (9.13)$$

If we accept this approximate equality as an equality we deduce that

$$\int_T^\infty \eta_x \, dx = -TP_{0T}, \quad (9.14)$$

and thus, by differentiation

$$\eta_T = P_{0T} + T\partial_T P_{0T}. \quad (9.15)$$

In other words, the perturbation that results in a constant shift of the initial yield curve is given by

$$\rho_x \rightarrow \rho_x + \varepsilon(P_{0x} - x\rho_x). \quad (9.16)$$

The functional derivative of the initial bond price under the perturbation (9.16) can then be found by inserting (9.2) and (9.15) into (9.5). After a short calculation we find that the ε disappears giving us

$$\frac{\delta P_{0T}}{\delta \rho} = \int_T^\infty \eta_x \, dx, \quad (9.17)$$

and therefore

$$\frac{\delta P_{0T}}{\delta \rho} = -TP_{0T}. \quad (9.18)$$

By use of the chain rule we obtain

$$\begin{aligned} \frac{\delta C_{0t}}{\delta P_{0T}} &= \frac{\delta \rho}{\delta P_{0T}} \frac{\delta C_{0t}}{\delta \rho} \\ &= -\frac{1}{TP_{0T}} \frac{\delta C_{0t}}{\delta \rho}. \end{aligned} \quad (9.19)$$

Now, given the call option price $C_{0t}[\rho]$ we shall consider the functional derivative of C_{0t} with respect to ρ_x , given by

$$\frac{\delta C_{0t}}{\delta \rho} = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} (C_{0t}[\rho + \varepsilon \eta] - C_{0t}[\rho]). \quad (9.20)$$

Recall that under the assumptions of Chapter 1 we are able to derive an integral expression for the pricing kernel $\{\pi_t\}$. In particular, we have

$$\pi_t = \int_t^\infty \rho_x M_{tx} \, dx. \quad (9.21)$$

Thus, by use of (9.21), the call price (9.3) becomes

$$C_{0t} = \mathbb{E} \left[\left(\int_T^\infty \rho_x M_{tx} dx - K \int_t^\infty \rho_x M_{tx} dx \right)^+ \right]. \quad (9.22)$$

By insertion of (9.22) and (9.15) into (9.20) we deduce after a short calculation that

$$\begin{aligned} \frac{\delta C_{0t}}{\delta \rho} = \mathbb{E} & \left[\mathbb{1} \left\{ \int_T^\infty \rho_x M_{tx} dx > K \int_t^\infty \rho_x M_{tx} dx \right\} \right. \\ & \times \left. \left(\int_T^\infty \eta_x M_{tx} dx - K \int_t^\infty \eta_x M_{tx} dx \right) \right]. \end{aligned} \quad (9.23)$$

Thus, by use of the functional derivatives (9.18) and (9.23), and the chain rule (9.19) we can derive a closed form for the hedging strategy $\{\Delta_{0t}(K)\}_{t \geq 0}$ that hedges out the sensitivity of option prices to constant shifts in the initial yield curve, where

$$\Delta_{0t}(K) = \frac{\delta C_{0t}}{\delta P_{0T}}. \quad (9.24)$$

Let's assume, as in Section 3.5, that the martingale family $\{M_{tx}\}$ is given by

$$M_{tx} = \frac{e^{\phi_x L_t}}{\mathbb{E}[e^{\phi_x L_t}]}, \quad (9.25)$$

and set

$$\theta_x = \frac{1}{t} \ln \mathbb{E}[e^{\phi_x L_t}]. \quad (9.26)$$

Then, we assume that ϕ_x is monotonic, and therefore by Proposition 3.5.2 that there shall exist a number ξ^* , such that $P(t, T, \xi^*) = K$, where

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x \exp(\phi_x \xi - \theta_x t) dx}{\int_t^\infty \rho_x \exp(\phi_x \xi - \theta_x t) dx} \quad (9.27)$$

and

$$P_{tT} = P(t, T, L_t). \quad (9.28)$$

Inserting (9.18) and (9.23) into (9.19) we can then calculate that the initial hedging strategy Δ_{0t} is given by

$$\Delta_{0t}(K) = \int_T^\infty \frac{(-\eta_x)}{T P_{0T}} m_{tx}(\xi^*) dx - K \int_t^\infty \frac{(-\eta_x)}{T P_{0T}} m_{tx}(\xi^*) dx, \quad (9.29)$$

where

$$m_{tx}(\xi^*) = \mathbb{E} \left[\mathbb{1} \left\{ \int_T^\infty \rho_u M_{tu} du > K \int_t^\infty \rho_u M_{tu} du \right\} M_{tx} \right]. \quad (9.30)$$

Proposition 9.2.1. *As the option expiry t tends to zero, $\Delta_{0t}(K)$ tends to the indicator function $\mathbb{1}\{P_{0T} > K\}$.*

Proof. First we observe that

$$\begin{aligned} \lim_{t \rightarrow 0} m_{tx}(\xi^*) &= \lim_{t \rightarrow 0} \mathbb{E} [\mathbb{1}\{P_{tT} > K\} M_{tx}] \\ &= \mathbb{E} \left[\lim_{t \rightarrow 0} \mathbb{1}\{P_{tT} > K\} M_{tx} \right] \\ &= \mathbb{E} [\mathbb{1}\{P_{0T} > K\} M_{0x}] \\ &= \mathbb{1}\{P_{0T} > K\}. \end{aligned} \tag{9.31}$$

Expression (9.31) then leads us to deduce that

$$\begin{aligned} \lim_{t \rightarrow 0} \Delta_{0t} &= \lim_{t \rightarrow 0} \int_T^\infty \frac{(x\rho_x - P_{0x})}{TP_{0T}} m_{tx}(\xi^*) dx - K \int_t^\infty \frac{(x\rho_x - P_{0x})}{TP_{0T}} m_{tx}(\xi^*) dx \\ &= \mathbb{1}\{P_{0T} > K\} \left(\int_T^\infty \frac{(x\rho_x - P_{0x})}{TP_{0T}} dx - K \int_0^\infty \frac{(x\rho_x - P_{0x})}{TP_{0T}} dx \right) \\ &= \mathbb{1}\{P_{0T} > K\}, \end{aligned} \tag{9.32}$$

where we have used

$$\int_T^\infty x\rho_x - P_{0x} dx = TP_{0T}, \tag{9.33}$$

and

$$\int_0^\infty x\rho_x - P_{0x} dx = 0, \tag{9.34}$$

in the third inequality. □

Proposition 9.2.2. *As the option expiry t moves closer to the maturity of the discount bond, T , we reach the limiting behaviour $\Delta_{0T}(K) = 1 - K$.*

Proof. We observe that

$$\begin{aligned} \lim_{t \rightarrow T} m_{tx}(\xi^*) &= \lim_{t \rightarrow T} \mathbb{E} [\mathbb{1}\{P_{tT} > K\} M_{tx}] \\ &= \mathbb{E} \left[\lim_{t \rightarrow T} \mathbb{1}\{P_{tT} > K\} M_{tx} \right] \\ &= \mathbb{E} [\mathbb{1}\{1 > K\} M_{Tx}] \\ &= \mathbb{1}\{1 > K\}. \end{aligned} \tag{9.35}$$

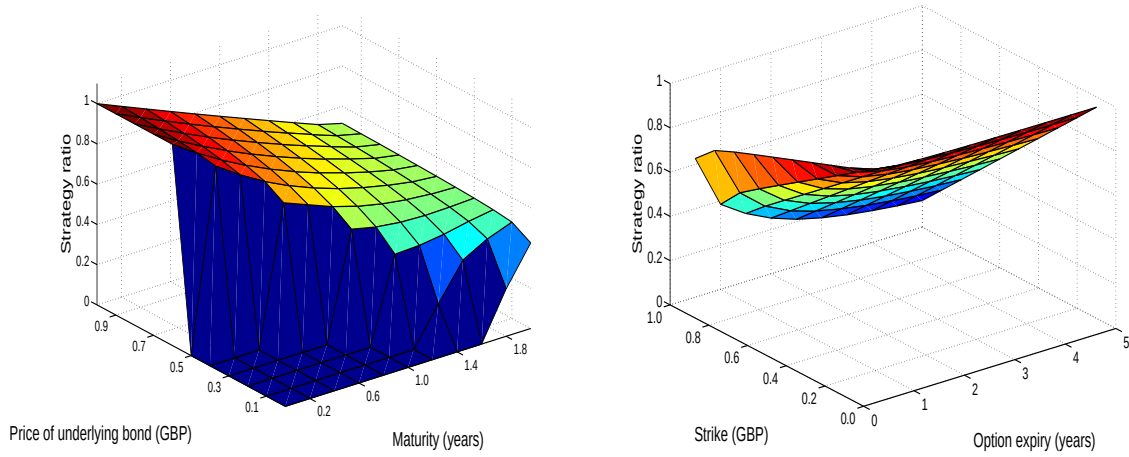


Figure 9.2: *Behaviour of the hedging strategy.* This figure shows how the hedge ratio of a bond option changes with respect to (*left*) changes in the price of the underlying bond and in the expiry date, and (*right*) the strike rate and the expiry date. The hedge ratio is calculated according to (9.37) with $\phi_x = 0.3e^{0.02x}$. In (*left*) the strike is set at 0.5 GBP and the bond maturity is two years from option valuation. In (*right*) the maturity of the bond option is always six months after option expiry, and a constant, flat initial term structure is used with an annualised rate of 2%, i.e., $P_{0T} = e^{-0.02T}$.

Expression (9.35) combined with (9.33) lead us to deduce that

$$\begin{aligned}
 \lim_{t \rightarrow T} \Delta_{0t} &= \lim_{t \rightarrow T} \int_T^\infty \frac{(x\rho_x - P_{0x})}{TP_{0T}} m_{tx}(\xi^*) dx - K \int_t^\infty \frac{(x\rho_x - P_{0x})}{TP_{0T}} m_{tx}(\xi^*) dx \\
 &= \mathbb{1}\{1 > K\} \left(\int_T^\infty \frac{(x\rho_x - P_{0x})}{TP_{0T}} dx - K \int_T^\infty \frac{(x\rho_x - P_{0x})}{TP_{0T}} dx \right) \\
 &= \mathbb{1}\{1 > K\} (1 - K) \\
 &= 1 - K.
 \end{aligned} \tag{9.36}$$

□

Expression (9.29) represents the amount of T -maturity discount bonds that one would have to sell short, at price P_{0T} , in order to hedge parallel shifts in the underlying yield curve when one has a long position in a European-style call option with expiry t and strike price K . Figure 9.2 presents the sensitivities of $\{\Delta_{0t}(K)\}$ for a range of expiries, bond maturities, bond prices, and strikes.

9.3 Hedging performance

Let economic uncertainty be driven by a Brownian motion, i.e., let $L_t = B_t$. Assume that the function ϕ_x is positive and strictly decreasing for all $x \geq 0$. The hedging strategy with respect to the rational Brownian family of models is then given by

$$\Delta_{0t} = \int_T^\infty \frac{(-\eta_x)}{TP_{0T}} N\left(\frac{\xi^*}{\sqrt{t}} - \phi_x \sqrt{t}\right) dx - K \int_t^\infty \frac{(-\eta_x)}{TP_{0T}} N\left(\frac{\xi^*}{\sqrt{t}} - \phi_x \sqrt{t}\right) dx \quad (9.37)$$

where η_x is given by (9.15) and ξ^* is defined by

$$P(t, T, \xi^*) = K, \quad (9.38)$$

where

$$P(t, T, \xi) = \frac{\int_T^\infty \rho_x \exp\left(\phi_x \xi - \frac{1}{2} \phi_x^2 t\right) dx}{\int_t^\infty \rho_x \exp\left(\phi_x \xi - \frac{1}{2} \phi_x^2 t\right) dx}. \quad (9.39)$$

We wish to study the performance of the hedging strategy (9.37) in the situation where one is holding a portfolio of options. Specifically, we consider a portfolio of the interest-rate caps present in our UK ATM data set (see Chapter 8).

The hedging strategy is implemented as follows. We construct a hedged portfolio of caps of ten different maturities and the appropriate hedges in discount bonds. For each caplet, the sensitivity to the value of the underlying bond price is calculated according to equation (9.37). We hedge against each caplet separately; thus there are as many hedging instruments as there are payoff dates. This technique is described as “bucket hedging” by Driessen *et al.* (2002). See also Bakshi *et al.* (1997) and references cited therein. The hedging interval, i.e., the amount of time we hold our hedged portfolio before we unwind, is set to be equal to one week.

Observed cap prices refer to new contracts of corresponding maturities, and not to current values of the contracts in the portfolio. However, we need to observe price changes of the same contracts in order to calculate profit and loss. To overcome this problem, we construct a price series for each cap contract as in Gupta *et al.* (2005). Specifically, we use a surrogate model price for the original cap contract at the end of the hedging interval. Similarly, for bond yields we construct a time series for particular contracts, using the Nelson & Siegel (1987) model as a surrogate.

Following Bakshi *et al.* (1997), we measure the performance of a hedging strategy by computing hedging errors. We measure hedging errors as the change in the value of

Table 9.1: *Summary statistics for hedging errors implied by weekly calibration to the UK ATM cap data set.* Hedging errors are computed for the Brownian family of models as defined by (9.40). Hedging errors are reported as an absolute percentage of the value of the hedge portfolio. The hedge portfolio consists of one cap in the UK data set per maturity and an appropriate hedge for each caplet. Each caplet is hedged by the associated underlying discount bond, according to the hedging strategy (9.37). Hedges are calculated using model parameters implied by the previous week's observed market data. The procedure is repeated weekly for the period January 2008 to May 2010. There are 123 observations to compute each statistic for every cap maturity. Market data was obtained from Bloomberg Financial Markets.

Absolute Hedging Error (%)	
Mean	0.16
Std Dev	0.15
Min	0.0006
Max	0.95

the hedged portfolio over the hedging interval. If we invest in a position in the hedged portfolio at time t , the hedging error (HE) at time $t + \delta t$ is equal to

$$HE_{t,t+\delta t} = \sum_{j=1}^M \left[\text{Cap}_{t+\delta t}^{\text{Market}}(T_j - \delta t, L_{K,j}) - \sum_{k=1}^{2j-1} \Delta_{t,t_k}(L_{K,j}) P_{t+\delta t, t_{k+1}-\delta t}^{\text{Market}} \right] - \left(\sum_{j=1}^M \left[\text{Cap}_t^{\text{Market}}(T_j, L_{K,j}) - \sum_{k=1}^{2j-1} \Delta_{t,t_k}(L_{K,j}) P_{t,t_{k+1}}^{\text{Market}} \right] \right) e^{Y_{t,t+\delta t}\delta t}, \quad (9.40)$$

where M is the number of caps we have in our portfolio, $\text{Cap}_t^{\text{Market}}(T, L_K)$ is the market price of a T -maturity cap with strike L_K , and P_{tT}^{Market} represents the market price of a T -maturity discount bond. The length of the re-balancing period δt is set to one week.

Expression (9.40) describes an investment in a portfolio of caps and the appropriate hedging instruments, with the residual cash placed in an instantaneously maturing risk free bond. This procedure is repeated for each week resulting in 123 observations. We use the current week's market data to imply parameter values and determine the current hedging position Δ_{0t} ; we then compute surrogate prices for the cap and bond contracts at the end of the week, using these parameters. We use the calibration procedure presented in Chapter 8. Table 9.1 reports summary statistics on the absolute hedging errors, as a percentage of the initial cost of the hedged portfolio.

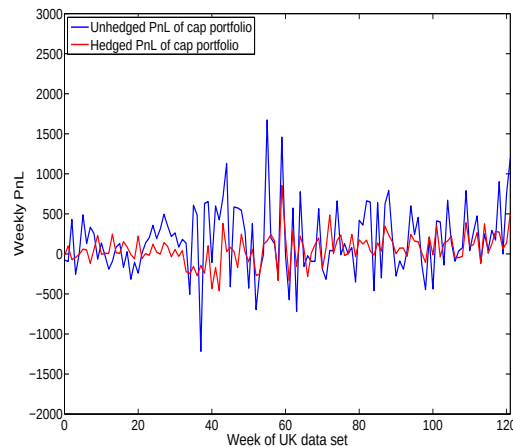


Figure 9.3: *Weekly changes in value for hedged and unhedged portfolios of caps.* Weekly profit and loss is measured for the hedged and unhedged portfolios. Each portfolio consists of ten cap contracts. In the hedged portfolio each caplet is hedged by shorting the underlying discount bond by an amount determined by the hedging strategy (9.37). The portfolios are rebalanced weekly for the period January 2008 to May 2010, totalling 123 observations. Data was taken from Bloomberg Financial Markets.

Following Driessen *et al.* (2002), we also calculate the ratio of the variance of weekly changes in the hedged portfolio and the variance of weekly changes in the portfolio unhedged. This ratio, which measures the decrease in variability provided by hedging, is calculated to be 81%. We recall that the variation in yield movements contributed by the first principal component is 85%. Figure 9.3 shows a comparison between the changes in value of the portfolios of hedged and unhedged caps.

Chapter 10

Extensions to Other Asset Classes

10.1 Chapter overview

In this chapter we extend certain aspects of the analysis of the rational Lévy term-structure models to incorporate multiple currencies and equities.

Exchange rates between the currencies of two countries are modelled as ratios of positive supermartingales. The supermartingales can be considered to be the pricing kernels associated with the respective currencies and may thus have an integral representation such as that introduced in Section 1.2.2.

We obtain an analytic expression for stock prices under stochastic interest rates by use of a linear form for the pricing kernel inspired by the rational lognormal model of Flesaker & Hughston (1996).

10.2 An introduction to foreign exchange

The problem of characterising the arbitrage-free dynamics of systems of discount bonds in different currencies and the corresponding foreign exchange rates between each currency has been developed in some generality by Flesaker & Hughston (1997), Hughston & Rafailides (2002), and Brody & Hughston (2004). We follow their characterisation closely in order to develop a rich class of Lévy-based models for foreign exchange rates.

The evolution of the exchange rate between currency i and currency j will be described by the process $\{S_t^{ij}\}_{t \geq 0}$. In particular, S_t^{ij} denotes the price of one unit of currency i in units of currency j at time t . The exchange rate must satisfy certain

requirements under the assumption of no arbitrage. In particular we have the compatibility condition

$$S_t^{ij} = S_t^{ik} S_t^{kj}, \quad (10.1)$$

and the inversion condition

$$S_t^{ij} = \frac{1}{S_t^{ji}}. \quad (10.2)$$

Following the references mentioned above we can derive an analytical expression for S_t^{ij} . Under Assumption 1.2.3 we know that the price of an asset in units of a base currency, discounted by the pricing kernel π_t , is a martingale. Letting $\{S_t^{i0}\}$ represent the exchange rate between currency i and the base currency, consider the evolution of a money market account in currency i denoted by the process $\{B_t^i\}$. By Assumption 1.2.3 we know that $\{\pi_t S_t^{i0} B_t^i\}_{t \geq 0}$ is a martingale. Under positive interest rates $\{B_t^i\}_{t \geq 0}$ is increasing and $\{\pi_t S_t^{i0}\}_{t \geq 0}$ is a supermartingale. We define, $\pi_t^i = \pi_t S_t^{i0}$, so

$$S_t^{ij} = \frac{\pi_t^i}{\pi_t^j}, \quad (10.3)$$

by use of equation (10.1).

Now consider the price process $\{P_{tT}^i\}_{T \geq t \geq 0}$, in units of currency i , of a discount bond that pays one unit of currency i at time T . By Assumption 1.2.3, $\{\pi_t S_t^{i0} P_{tT}^i\}_{T \geq t \geq 0}$ is a martingale. It follows that at time t we have that

$$\pi_t S_t^{i0} P_{tT}^i = \mathbb{E}_t [\pi_T S_T^{i0} P_{TT}^i], \quad (10.4)$$

and therefore that

$$P_{tT}^i = \frac{\mathbb{E}_t [\pi_T^i]}{\pi_t^i}, \quad (10.5)$$

where we have used (10.3) and the fact that $P_{TT}^i = 1$.

We now assume the existence of a system of strictly positive forward rates $\{f_{tT}^i\}_{T \geq t \geq 0}$ such that

$$P_{tT}^i = \exp \left(- \int_t^T f_{ts}^i ds \right). \quad (10.6)$$

We assume also the transversality condition

$$\lim_{T \rightarrow \infty} P_{tT}^i = 0. \quad (10.7)$$

It is then possible to derive the following integral representation for π_t^i , given by

$$\pi_t^i = S_0^{i0} \int_t^\infty \rho_x^i M_{tx}^i dx, \quad (10.8)$$

by following a similar procedure to that outlined in Section 1.2.2. Here $\rho_x^i = -\partial_x P_{0x}^i$ is the initial system of currency- i discount bonds and $\{M_{tx}^i\}_{x \geq t \geq 0}$ is one-parameter family of positive martingales such that

$$\pi_t S_t^{i0} \partial_T P_{tT}^i = S_0^{i0} (\partial_T P_{0T}^i) M_{tT}^i, \quad (10.9)$$

for all $T \geq t \geq 0$. Thus, inserting (10.8) into (10.3) and (10.5) we can deduce that

$$S_t^{ij} = S_0^{ij} \frac{\int_t^\infty \rho_x^i M_{tx}^i dx}{\int_t^\infty \rho_x^j M_{tx}^j dx}, \quad (10.10)$$

and that

$$P_{tT}^i = \frac{\int_T^\infty \rho_x^i M_{tx}^i dx}{\int_t^\infty \rho_x^i M_{tx}^i dx}. \quad (10.11)$$

For a model economy with N different currencies the dynamics of discount bonds and exchange rates are thus governed by the specification of the system of martingales $\{M_{tx}^i\}$, for $i = 1, \dots, N$. Following the term-structure modelling framework developed in Chapter 3 we introduce the function $\phi_x^i : \mathbb{R}_+ \rightarrow \mathbb{R}$ and we define $\{M_{tx}^i\}$ to be the Esscher transform of a Lévy process $\{L_t\}_{t \geq 0}$. Specifically, we define

$$M_{tx}^i = \exp(\phi_x^i L_t - \theta_x^i t), \quad (10.12)$$

where

$$\theta_x^i = \frac{1}{t} \ln \mathbb{E} \left[e^{\phi_x^i L_t} \right]. \quad (10.13)$$

Inserting (10.12) into (10.10) we thus obtain

$$S_t^{ij} = S_0^{ij} \frac{\int_T^\infty \rho_x^i \exp(\phi_x^i L_t - \theta_x^i t) dx}{\int_t^\infty \rho_x^j \exp(\phi_x^j L_t - \theta_x^j t) dx}. \quad (10.14)$$

10.3 Option pricing with multiple currencies

10.3.1 Options on foreign-exchange rates

The price, in the base currency, of a European-style call option on the exchange rate $\{S_t^{ij}\}$, with expiry t and strike price K , is defined to be $S_0^{j0} C_{0t}^{ij}$, where C_{0t}^{ij} is the price

of the option in currency j . By the assumption of no arbitrage we have that

$$S_0^{j0} C_{0t}^{ij} = \mathbb{E} \left[\pi_t S_t^{j0} (S_t^{ij} - K)^+ \right], \quad (10.15)$$

and by use of $\pi_t^j = \pi_t S_t^{j0}$ we obtain

$$C_{0t}^{ij} = \mathbb{E} \left[\frac{\pi_t^j}{S_0^{j0}} (S_t^{ij} - K)^+ \right]. \quad (10.16)$$

Using expressions (10.8) and (10.14) we can derive a closed-form expression for C_{0t}^{ij} by following similar steps to the option pricing framework of Section 3.5. In particular, we would like to truncate the expectation appearing in (10.16) such that it involves integration over only positive values.

Proposition 10.3.1. *If ϕ_x^i and ϕ_x^j are of opposite signs for all $x \geq 0$, the function $S^{ij} : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}_+$, defined by*

$$S^{ij}(t, \xi) = \frac{\int_t^\infty \rho_x^i \exp(\phi_x^i \xi - \theta_x^i t) \, dx}{\int_t^\infty \rho_x^j \exp(\phi_x^j \xi - \theta_x^j t) \, dx}, \quad (10.17)$$

will be strictly monotonic in the variable ξ .

Proof. Observe that the differential of (10.17) with respect to ξ is given by

$$\frac{\partial S^{ij}(t, \xi)}{\partial \xi} = S^{ij}(t, \xi) (\Phi_{tt}^i - \Phi_{tt}^j), \quad (10.18)$$

where

$$\Phi_{tt}^u = \frac{\int_t^\infty \phi_x^u \rho_x^u \exp(\phi_x^u \xi - \theta_x^u t) \, dx}{\int_t^\infty \rho_x^u \exp(\phi_x^u \xi - \theta_x^u t) \, dx} \quad (10.19)$$

for $u = i$ or $u = j$. The proof then follows as $S^{ij}(t, \xi) > 0$, and $(\Phi_{tt}^i - \Phi_{tt}^j)$ is (a) strictly positive if $\phi_x^i > 0$ and $\phi_x^j < 0$, or (b) strictly negative if $\phi_x^i < 0$ and $\phi_x^j > 0$. $\square$

Using Proposition 10.3.1 we can deduce the existence of a number ξ^* such that

$$S^{ij}(t, \xi^*) = K, \quad (10.20)$$

and

$$S^{ij}(t, \xi) > K, \quad (10.21)$$

for all $\xi > \xi^*$ when $\phi_x^i > 0$ and $\phi_x^j < 0$. Inserting (10.8) and (10.10), into (10.16) we obtain the following expression:

$$C_{0t}^{ij} = \mathbb{E} \left[\left(S_0^{ij} \int_t^\infty \rho_x^i M_{tx}^i dx - K \int_t^\infty \rho_x^j M_{tx}^j dx \right)^+ \right]. \quad (10.22)$$

If we assume that $\phi_x^i > 0$ and $\phi_x^j < 0$, expression (10.22) simplifies to

$$C_{0t}^{ij} = S_0^{ij} \int_t^\infty \rho_x^i m_{tx}^i(\xi^*) dx - K \int_t^\infty \rho_x^j m_{tx}^j(\xi^*) dx, \quad (10.23)$$

where

$$m_{tx}^u(\xi^*) = \mathbb{E} \left[\mathbf{1} \left\{ S_0^{ij} \int_t^\infty \rho_y^i M_{ty}^i dy > K \int_t^\infty \rho_y^j M_{ty}^j dy \right\} M_{tx}^u \right], \quad (10.24)$$

for $u = i$ or $u = j$.

In the example of an economy driven by a one-dimensional Brownian motion, the option price C_{0t}^{ij} is found to be

$$C_{0t}^{ij} = S_0^{ij} \int_t^\infty \rho_x^i N \left(- \left(\frac{\xi^* - \phi_x^i t}{\sqrt{t}} \right) \right) dx - K \int_t^\infty \rho_x^j N \left(- \left(\frac{\xi^* - \phi_x^j t}{\sqrt{t}} \right) \right) dx, \quad (10.25)$$

where ξ^* is defined by the relationship

$$S_0^{ij} \frac{\int_t^\infty \rho_x \exp \left(\phi_x^i \xi^* - \frac{1}{2} (\phi_x^j)^2 t \right) dx}{\int_t^\infty \rho_x \exp \left(\phi_x^i \xi^* - \frac{1}{2} (\phi_x^j)^2 t \right) dx} = K, \quad (10.26)$$

and we have implicitly assumed that $\phi_x^i > 0$ and $\phi_x^j < 0$.

10.3.2 Options on discount bonds in a foreign currency

We shall denote by C_{0t}^i the price, in currency i , of a European-style call option on a discount bond that pays one unit of currency i at time T , where t is the expiry of the option and the strike price is denoted by K^i . The price of the option in the base currency is then given by

$$S_0^{i0} C_{0t}^i = \mathbb{E} \left[\pi_t S_t^{i0} (P_{tT}^i - K^i)^+ \right], \quad (10.27)$$

and hence we deduce that

$$C_{0t}^i = \mathbb{E} \left[\frac{\pi_t^i}{S_0^{i0}} (P_{tT}^i - K^i)^+ \right]. \quad (10.28)$$

We can modify the option payoff (10.28) and express the strike in units of currency j . In particular, we denote by Q_{0t}^{ij} , the price in units of currency j , of a European-style call option, with strike price X^j in units of currency j , and expiry t , on a discount bond that pays one unit of currency i at time T . The price of this option in the base currency is given by

$$S_0^{j0} Q_{0t}^{ij} = \mathbb{E} \left[\pi_t S_t^{j0} (S_t^{ij} P_{tT}^i - X^j)^+ \right], \quad (10.29)$$

and therefore the price of the option in currency j is given by

$$Q_{0t}^{ij} = \mathbb{E} \left[\frac{\pi_t^j}{S_0^{j0}} (S_t^{ij} P_{tT}^i - X^j)^+ \right]. \quad (10.30)$$

Proposition 10.3.2. *If ϕ_x^i and ϕ_x^j are of opposite signs for all $x \geq 0$, the function $SP^{ij} : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$, defined by*

$$SP^{ij}(t, T, \xi) = \frac{\int_T^\infty \rho_x^i \exp(\phi_x^i \xi - \theta_x^i t) dx}{\int_t^\infty \rho_x^j \exp(\phi_x^j \xi - \theta_x^j t) dx}, \quad (10.31)$$

will be strictly monotonic in the variable ξ .

Proof. Observe that the differential of (10.31) with respect to ξ is

$$\frac{\partial SP^{ij}(t, T, \xi)}{\partial \xi} = S^{ij}(t, \xi) \Phi_{tT}^i - SP^{ij}(t, T, \xi) \Phi_{tt}^j, \quad (10.32)$$

where $S^{ij}(t, \xi)$ is given by (10.17), Φ_{tT}^i is given by (10.19), and

$$\Phi_{tT}^i = \frac{\int_T^\infty \phi_x^i \rho_x^i \exp(\phi_x^i \xi - \theta_x^i t) dx}{\int_t^\infty \rho_x^i \exp(\phi_x^i \xi - \theta_x^i t) dx}. \quad (10.33)$$

Expression (10.32) must be positive if $\phi_x^i > 0$ and $\phi_x^j < 0$ since we have that

$$S^{ij}(t, \xi) > 0, \quad (10.34)$$

$$\Phi_{tT}^i > 0, \quad (10.35)$$

$$SP^{ij}(t, T, \xi) \geq 0, \quad (10.36)$$

and

$$\Phi_{tt}^j < 0. \quad (10.37)$$

Alternatively if $\phi_x^j < 0$ and $\phi_x^i > 0$ expression (10.32) will always be negative via a similar argument. $\square$

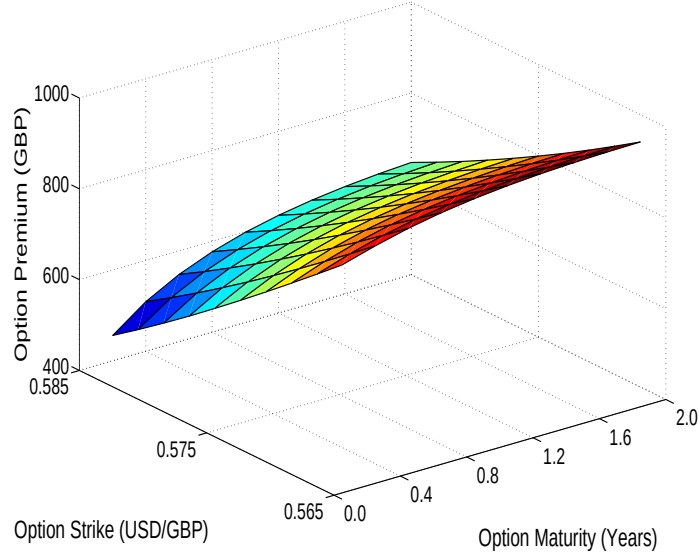


Figure 10.1: *Price surface of options on foreign-exchange rates.* Prices are calculated with the Brownian family of models model, in particular the closed-form option price (10.25) is used. Model parameters are implied from calibrating ϕ_x^i to UK and ϕ_x^j to US ATM option data as at 23 March 2010. Data is taken from Bloomberg Financial Markets.

Using Proposition 10.3.2 we may deduce the existence of a number ξ^* such that

$$SP^{ij}(t, T, \xi^*) = X^j, \quad (10.38)$$

and

$$SP^{ij}(t, T, \xi) > X^j, \quad (10.39)$$

for all $\xi > \xi^*$ when $\phi_x^i > 0$ and $\phi_x^j < 0$. The price of the foreign currency discount bond option can then be expressed as

$$Q_{0t}^{ij} = S_0^{ij} \int_T^\infty \rho_x^i m_{tx}^i(\xi^*) dx - X^j \int_t^\infty \rho_x^j m_{tx}^j(\xi^*) dx, \quad (10.40)$$

where

$$m_{tx}^u(\xi^*) = \mathbb{E} \left[\mathbf{1} \left\{ S_0^{ij} \int_t^\infty \rho_y^i M_{ty}^i dy > X^j \int_t^\infty \rho_y^j M_{ty}^j dy \right\} M_{tx}^u \right]. \quad (10.41)$$

for $u = i$ or $u = j$.

Remark 10.3.3. The multiple-currency option pricing models that we have presented can be calibrated either directly to exchange rate options, or to the interest-rate derivative markets of each currency. In Figure 10.1 we present an example of a price surface of prices of foreign-exchange options computed within the Brownian family of rational Lévy models. The parameters are implied by calibration to interest-rate derivatives in the UK and US data sets.

A popular model for pricing options on foreign exchange rates is that of Garmen & Kohlhagen (1983). The rational Lévy models represent an extension of the option pricing literature on account of: (a) the inclusion of jump process, and (b) the improved economic intuition. The rational Lévy models involve the pricing kernels associated with each currency, while other models typically begin by specifying stochastic dynamics for exchange rates.

10.4 Modelling equities

We now present a simple model for stock prices where conditions are built in to ensure (a) positive nominal interest rates and (b) a Lévy driven economy. We follow the methodology of Flesaker & Hughston (1996), Duffie (2001), and Di Graziano & Rogers (2006). Firstly, we introduce a positive pricing kernel $\{\pi_t\}_{t \geq 0}$ of the form

$$\pi_t = \alpha_t + \beta_t m_t, \quad (10.42)$$

as in Flesaker & Hughston (1996). The functions α_t and β_t are positive, deterministic, and decreasing. They are defined such that

$$\alpha_0 + \beta_0 = 1, \quad (10.43)$$

and

$$\alpha_t + \beta_t = P_{0t}. \quad (10.44)$$

The process $\{m_t\}_{t \geq 0}$ is a martingale. We shall define $\{m_t\}$ to be of the simple geometric Lévy form:

$$m_t = \exp(-\lambda L_t - \psi(-\lambda)t), \quad (10.45)$$

where $\lambda > 0$ is a constant, in a range such that $\psi(-\lambda)$ is well defined. Next, let $\{\delta_t\}_{t \geq 0}$ describe the dynamics of a continuous dividend stream and define

$$\delta_t = \delta_0 \exp((\mu - \psi(\sigma))t + \sigma L_t). \quad (10.46)$$

Here μ is the constant growth rate of the dividend stream and $\sigma > 0$ is constant that describes how much variability it has. Under no arbitrage the price S_t of an asset with dividend rate $\{\delta_t\}$ is given by

$$S_t = \frac{1}{\pi_t} \mathbb{E}_t \left[\int_t^\infty \pi_s \delta_s \, ds \right], \quad (10.47)$$

(cf. Duffie 2001, Di Graziano & Rogers 2006, and references cited therein). We can apply Fubini's theorem to (10.47). We insert (10.42) and (10.46), and calculate the conditional expectation. After a short calculation we thus obtain a closed-form stock price of the form

$$S_t = \delta_t \left(\frac{A_t + B_t m_t}{\alpha_t + \beta_t m_t} \right), \quad (10.48)$$

where

$$A_t = \int_t^\infty \alpha_s e^{\mu(t-s)} \, ds, \quad (10.49)$$

and

$$B_t = \int_t^\infty \beta_s e^{(\mu - \psi(-\lambda) - \psi(\sigma) + \psi(\sigma - \lambda))(t-s)} \, ds. \quad (10.50)$$

Appendix A

The CIR++ Model

The CIR++ model is an extension of the Cox *et al.* (1985) short rate model. The extension allows an exact fit to an arbitrary initial family of discount bonds $\{P_{0T}\}$. In what follows we shall consider only the one-factor case. We do not provide proofs or derivations; a more complete study of the CIR++ model can be found in, for example, Brigo & Mercurio (2001) and Brigo & Mercurio (2006).

We define a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Expectation with respect to $\mathbb{Q}$ is denoted by $\mathbb{E}^*[-]$ and conditional expectation is denoted by $\mathbb{E}_t^*[-]$. Let $\{x_t\}_{t \geq 0}$ be a time-homogeneous square-root process, i.e., let x_t solve

$$dx_t = k(\theta - x_t) dt + \sigma \sqrt{x_t} dW_t^*, \quad (\text{A.1})$$

where $\{W_t^*\}_{t \geq 0}$ is a $\mathbb{Q}$ -Brownian motion, and where k , θ , and σ are constants. We shall assume that $\{x_t\}$ satisfies the Feller condition:

$$2k\theta > \sigma^2, \quad (\text{A.2})$$

so that when $x_0 > 0$ the origin is inaccessible, and x_t will be positive for all $t \geq 0$. Let P_{tT}^x be defined by

$$P_{tT}^x = \mathbb{E}_t^* \left[e^{-\int_t^T x_u du} \right]. \quad (\text{A.3})$$

Since $\{x_t\}$ is an affine process, (A.3) has a closed form. In particular we observe that

$$P_{tT}^x = A_{tT} e^{-x_t B_{tT}}. \quad (\text{A.4})$$

The functions A_{tT} and B_{tT} are deterministic and are calculated to be

$$A_{tT} = \left(\frac{2he^{(k+h)(T-t)/2}}{2h + (k+h)(e^{h(T-t)} - 1)} \right)^{\frac{2k\theta}{\sigma^2}}, \quad (\text{A.5})$$

and

$$B_{tT} = \frac{2(e^{h(T-t)} - 1)}{2h + (k+h)(e^{h(T-t)} - 1)}, \quad (\text{A.6})$$

where

$$h = \sqrt{k^2 + 2\sigma^2}. \quad (\text{A.7})$$

Let $C_{tT}^x(K)$ be defined by

$$C_{tT}^x(K) = \mathbb{E}_t^* \left[e^{-\int_t^T x_u du} (P_{TS}^x - K)^+ \right], \quad (\text{A.8})$$

for $S \geq T \geq t \geq 0$. The process $\{x_t\}$ has an associated chi-squared distribution and, following Brigo & Mercurio (2001), it is possible to write (A.8) in closed form. Specifically, we have

$$\begin{aligned} C_{tT}^x(K) = P_{tS}^x \chi^2 \left(2\bar{r}(\rho + \psi + B_{TS}); \frac{4k\theta}{\sigma^2}, \frac{2\rho^2 x_t e^{h(T-t)}}{\rho + \psi + B_{TS}} \right) \\ - K P_{tT}^x \chi^2 \left(2\bar{r}(\rho + \psi); \frac{4k\theta}{\sigma^2}, \frac{2\rho^2 x_t e^{h(T-t)}}{\rho + \psi} \right), \end{aligned} \quad (\text{A.9})$$

where

$$\bar{r} = \frac{\ln(A_{tT}/K)}{B_{tT}}, \quad (\text{A.10})$$

$$\rho = \frac{2h}{\sigma^2(e^{h(T-t)} - 1)}, \quad (\text{A.11})$$

and

$$\psi = \frac{k+h}{\sigma^2}. \quad (\text{A.12})$$

We shall now extend this set-up to develop a tractable short rate model. Let the short rate $\{r_t\}_{t \geq 0}$ be given by

$$r_t = x_t + \varphi(t), \quad (\text{A.13})$$

where $\varphi(t)$ is a differentiable, deterministic function with $\varphi(0) = r_0 - x_0$. Then by Itô's Lemma we obtain

$$dr_t = \left(\frac{d\varphi(t)}{dt} + k(\theta - r_t + \varphi(t)) \right) dt + \sigma \sqrt{r_t - \varphi(t)} dW_t^*. \quad (\text{A.14})$$

The dynamics (A.14) are the CIR++ short rate dynamics.

Remark A.1. The dynamics (A.14) are time-inhomogeneous and thus consistent with the Hull & White (1990) extension of the Cox *et al.* (1985) model, which uses time-dependent parameters to match the initial term structure.

The price of a discount bond in the CIR++ model is then given by

$$\begin{aligned} P_{tT} &= \mathbb{E} \left[e^{-\int_t^T r_u du} \right] \\ &= e^{-\int_t^T \varphi(u) du} A_{tT} e^{-B_{tT} x_t}. \end{aligned} \quad (\text{A.15})$$

It is now possible to ensure that the model implied initial yield curve, $\{P_{0t}\}$, is exactly equal to that implied by the market P_{0t}^M .

Proposition A.1. (Brigo & Mercurio 2001, Corollary 3.2) *The model (A.13) fits the currently observed term structure if and only if*

$$\varphi(t) = f_{0t} - f_{0t}^x, \quad (\text{A.16})$$

where

$$f_{0t} = -\partial_t \ln P_{0t}, \quad (\text{A.17})$$

and

$$f_{0t}^x = -\partial_t \ln P_{0t}^x. \quad (\text{A.18})$$

The ability to match the initial term structure is useful for derivative pricing. Let the price, at time t , of a European-style call option on a discount bond maturing at time S be given by

$$C_{tT} = \mathbb{E}_t^* \left[e^{-\int_t^T r_u du} (P_{tT} - K)^+ \right], \quad (\text{A.19})$$

where K is the strike price, T is the expiry, and $S \geq T \geq t \geq 0$. Splitting r_t into stochastic and deterministic parts we obtain

$$C_{tT} = e^{-\int_t^S \varphi(u) du} C_{tT}^x \left(K e^{-\int_T^S \psi(u) du} \right). \quad (\text{A.20})$$

This price can be written in closed form. We note that

$$e^{-\int_t^S \varphi(u) du} = \frac{P_{0S}^M P_{0t}^x}{P_{0S}^x P_{0t}^M}. \quad (\text{A.21})$$

Then, by combining (A.21) with (A.9) and (A.20), and rearranging, we observe that the CIR++ call price is given by

$$\begin{aligned} C_{tT} = & P_{tS} \chi^2 \left(2\hat{r}(\rho + \psi + B_{TS}); \frac{4k\theta}{\sigma^2}, \frac{2\rho^2 x_t e^{h(T-t)}}{\rho + \psi + B_{TS}} \right) \\ & - K P_{tT} \chi^2 \left(2\hat{r}(\rho + \psi); \frac{4k\theta}{\sigma^2}, \frac{2\rho^2 x_t e^{h(T-t)}}{\rho + \psi} \right), \end{aligned} \quad (\text{A.22})$$

where

$$\hat{r} = \frac{\ln(A_{tT}/K) - \ln(P_{0T}^M P_{0S}^x / P_{0T}^x P_{0S}^M)}{B_{tT}}. \quad (\text{A.23})$$

For derivatives with more complex payoffs, for example path-dependent or American options, trinomial-tree methods have been proposed. One can find details in Brigo & Mercurio (2006), which also contains more information on ensuring the positivity of interest rates.

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