

Investigation into three-dimensional dry foam modelling using the boundary integral method

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ABSTRACT

Understanding foam rheology relies on predictions of the foam structure and physical quantities associated with foam dynamics. In this work, we explore the modelling of dry foam dynamics using the boundary integral method, which can be used to predict the behaviour of dry foam in bulk or in confined geometries. The resultant equation for dry foam shows that the movement of foam films or lamellae is governed by the film curvature and the normal stress that the wall imposes on the fluid phase in bubbles and is characterized by three distinct motion modes: the background flow, the imposed flow due to the deformation of wall and the relaxation process of films. The singular integrals involved in the model are handled using a combined method of singularity subtraction and nonsingular contour integral. To demonstrate the validity of the method in potential applications, we consider two scenarios, for foam in bulk and in confined geometries, respectively. Dynamic foam motion is tracked using the model. The results show that the dry foam model has potentials applied in a variety of applications, such as foam flow in microfluidic devices and porous media.

1. Introduction

Aqueous foam is a complex non-Newtonian fluid consisting of separated bubbles suspended in a continuous liquid network [1], and has wide applications in industries, such as soil remediation [2,3] and acid diversion in well stimulation [4]. These industrial processes involve foam flow in porous media, where comprehensive descriptions of the foam behaviour are challenging due to the involved complicated multiscale processes, such as gas flow and liquid drainage [5]. Selective simplifications based on scale separations can be made to capture the main features of foam dynamics. One such treatment for a realistic foam with low liquid content is the dry foam approximation, where the liquid content is ignored while the complex structure of the liquid network maintains [6–8]. Using the dry foam assumption, we develop a tractable model, in a form of a boundary integral equation, for the dynamic motion of dry foam, aiming at addressing problems of foam flow in porous media at low Reynolds numbers. The model derivation is conducted in Section 2, followed by the numerical methods for the solution of the model in Section 3. In Section 4, we test the model validity and present the numerical results. Conclusions and future work are offered in Section 5.

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1.1. Background

Pore-scale behaviour of foam in porous media has been extensively studied in the literature [9–12]. Experimental results conclude that foam in porous media is a yield-stress fluid [12] and has varying mobility that relies on the heterogeneity in permeability of the porous media [11,13]. Specifically, a yield stress or a minimum pressure gradient needs to be overcome for the foam flow to start because of the capillary resistance induced by the structural deformation of the foam, which is confined by the local geometry [12]. Besides, for a moving foam, its apparent viscosity decreases as the flow velocity increases, and therefore foam is regarded as being shear-thinning. Bretherton [14] studied the flow of a long bubble in a capillary tube, and found that the flow of menisci, which is abundant in foams, leads to the foam's apparent viscosity scaling as $Ca^{-2/3}$. Ca is the capillary number defined as $\mu V/\gamma$, where V is the flow velocity, μ the viscosity of the continuous phase of foam, and γ the surface tension (gas-liquid interfacial tension). When applied to foam flow in porous media consisting of irregular and varying cross-sectional channels, the exponent over Ca could deviate from $-2/3$, which is affected by the cross-sectional shape of the channel that the bubble is moving through [15,16], as well as the varying cross-sectional area along the axial direction of the channel [12]. The latter is also associated with the arising of the yield stress of foam flow in porous media and contributes to a strong shear-thinning behaviour of foam at small values of Ca . This effect has been observed in “bubble trains” flow through one single channel [12] and could be more complicated for foam flow in a realistic porous medium where multiple channels exist simultaneously and are highly interconnected.

As foam flows in porous media, a major fraction of the gas phase, usually more than 70%, exists in the form of trapped bubbles, and only a small fraction of bubbles are mobile [17,18]. Moreover, foam flow is intermittent in flow channels and the preferential paths usually change with time [12], which further contributes to the complexity of foam's behaviour. For instance, the pressure drop across a foam structure varies as the foam moves, and fluctuates even when a so-called steady state of foam flow is reached, accompanied by varying foam configurations [19–22]. This pressure drop is affected by various factors, such as the density and arrangement of foam lamellae, foam quality (the gas fraction in a foam) and the flow velocity. Exploring the relation between the pressure drop and those influential factors as foam flows in porous media is the main objective in studies of foam's properties in porous media [23,24]. An approach to realizing this goal involves utilizing numerical simulations, employing tractable foam models capable of predicting the diverse arrangements of foam within porous media. Though attempts proposing or developing foam models have been made in the literature [5,25], dynamic foam models suitable in the context of foam in porous media, where the flow is usually at very low Reynolds numbers, are still in lack.

In a dynamic foam model, the effect of the viscous dissipation is incorporated. This is in contrast to the quasi-static model that assumes that foam maintains the equilibrium state under conditions that strains are slowly imposed on the foam [25]. In equilibrium, the foam structure is well described by Plateau's laws [26], as a consequence of foam's exhibiting the minimal surface, namely the configuration with the minimum total surface energy [27,28]. The quasi-static method to determine the minimal surface has been well established in Brakke's Surface Evolver [29]. In Surface Evolver, foam films are triangulated into discrete elements and the gradient descent method is used to evolve the discrete surfaces to the minimal surface, under constraints in terms of the bubble volume and the energy at contact lines between bubbles and solid surfaces. On the other hand, if strains are rapidly imposed on foams, the dynamic effects would result in that the foam configuration deviates from that in equilibrium. Moreover, the deformation of foam films becomes continuous (compared with quasi-static motion) because the characteristic time of structural relaxation is comparable with that of the strain imposed on the foam [25,30]. In this dynamic regime of movements, one simple model designed for quasi-2D dry foam is the so-called viscous froth model [25], where the pressure is assumed to be uniform in each bubble as assumed in the quasi-static model. In this model, the pressure differences across foam films, film tension forces, and viscous resistance are locally balanced for any finite segment of a foam film. The viscous resistance is modelled as a function of the normal component of velocity to the film. Once the normal velocity is calculated, the lamellae can be evolved in time. This model can qualitatively reproduce dynamic behaviour of foam confined in two closely placed flat plates [31] and has been extended to consider the effect of finite liquid content in wet foams [32] and incorporate the Marangoni effect [33,34].

More sophisticated methods applicable in the dynamic domain include those multiphase fluid models based on the governing equations of fluid dynamics [5,35]. At low Reynolds numbers, which is especially focused in our context, the fluid flow is further expected to obey the Stokes flow equations [36,37]. Based on the Stokes flow equations, the boundary integral method has been well established for multiphase fluid flow problems, such as drop suspension flow [38–40] and thus has the potential to be applied in foam-related research [41–43]. The boundary integral method incorporates solving a boundary integral equation for boundary velocities and involves the singularity and near-singularity problems. Zinchenko and Davis [42] considered drops squeezing through interparticle constrictions and deliberately handled the (near) singularities of different types, including drop-self, solid-self, solid-solid, solid-drop, drop-drop and drop-solid interactions. For foams with low liquid volumetric fraction, however, the disjoining pressure [44,45] has to be added into the interfacial force term involved in the boundary integral, which is sensitive to the varying film thickness [41]. This sensitivity could lead to numerical instability, and thus restrict the time step that can be used in simulations. As a result, the required computational cost would drastically increase if extremely thin films need to be considered, which may have confined the application of this method in large-scale foam flow problems.

Saye and Sethian [5] utilized the multiscale nature of foam dynamics and divided the dynamic process of foam flow into three stages, including the dynamic deformation of lamella, liquid drainage within films, Plateau borders, and nodes, and lamella rupture. These stages occur in sequence and are coupled through boundary conditions. They adopted the dry foam assumption in the stage of lamella deformation while the effect of film thickness is considered in the liquid drainage stage. The dry foam assumption is one key simplification, utilizing the multiscale nature of foam dynamics. Following the same philosophy, we introduce the dry foam

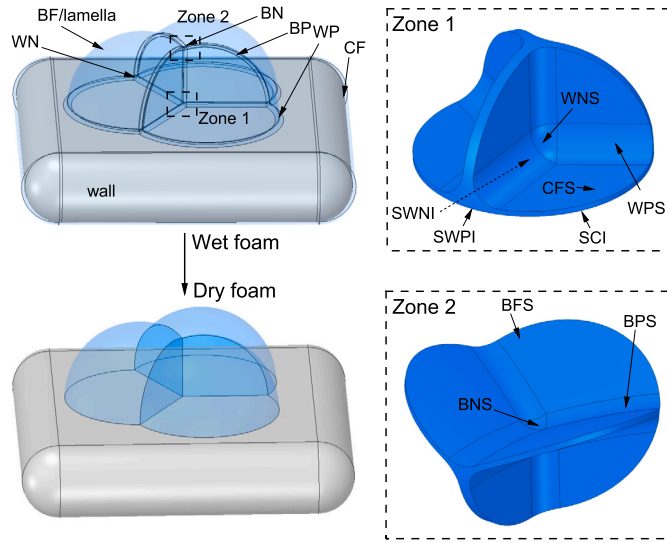


Fig. 1. Schematic of a foam consisting of three bubbles attached on the surface of a solid plate. Geometric elements of the foam include bulk films (BF) or lamellae, wetting films coating on the wall (CF), bulk Plateau borders (BP), wall Plateau borders (WP), nodes on the wall (WN), nodes in bulk (BN). The gas-liquid and solid-liquid interfaces are categorized into nine types: wall Plateau border surfaces (WPS), wall node surfaces (WNS), coating film surfaces (CFS), solid-wall node interfaces (SWNI), solid-coating film interfaces (SCI), solid-wall Plateau border interfaces (SWPI), bulk film surfaces (BFS), bulk Plateau border surfaces (BPS) and bulk node surfaces (BNS). In the limit of dry foam, the thin films reduce to non-thickness 2D surfaces embedded in 3D. The Plateau borders reduce to 1D curves embedded in 3D. The nodes reduce to four-fold points.

assumption in the boundary-integral modelling process for foam dynamics. As the model is based on the Stokes flow equations, the dynamic effect arises from the viscous dissipation, while the inertial effect has been ignored.

2. Mathematical formulation

In this section, we derive the boundary integral equation for dry foam in the presence of solid surfaces, using the boundary integral representation of the Stokes flow equations. The scenario of a foam attached on a wall is shown in Fig. 1, which involves most structural elements that could exist in a realistic 3D foam. This schematic is referenced in the following. The two key aspects involved in the formula derivation are the categorization of different types of interfaces and the boundary condition on the wall.

2.1. Stokes flow equations

As shown in Fig. 1, a foam is composed of concentrated gas bubbles separated by a continuous liquid phase (water or, more specifically, surfactant solution), placed on the upper surface of a solid plate. It has been assumed that the solid surface is water-wetting and thus the continuous liquid phase forms a wetting film coating on the solid surface. The gas-liquid surface tension is γ , and the gas viscosity is $\lambda\mu$, where μ is the viscosity of continuous liquid phase and λ is the ratio between the gas viscosity and the liquid viscosity. Assume that there are $N - 1$ bubbles and the foam as well as the solid plate are immersed in an unbounded gas domain. Use D_0 to refer to the liquid domain and D_K the K th gas domain, where $K = 1, 2, \dots, N - 1$. Use D_N to denote the outermost unbounded gas area. In the schematic diagram shown in Fig. 1, N equals 4, i.e. three bubbles and one unbounded gas area.

At small Reynolds numbers and ignoring the buoyant force, foam flow is governed by the incompressible Stokes flow equations, which can be written as [41]

$$\nabla \cdot \Pi^K = \mathbf{0}, \quad \nabla \cdot \mathbf{u}^K = 0, \quad \mathbf{x} \in D_K, \quad \text{for } K = 0, 1, \dots, N, \quad (1)$$

where Π^K is the stress tensor and K is the index of each fluid domain. The stress tensor is a term combining pressure and viscous stress

$$\Pi^K = -p^K \mathbf{I} + \lambda^K \mu (\nabla \mathbf{u}^K + (\nabla \mathbf{u}^K)^T), \quad (2)$$

where $\lambda^0 = 1.0$ and $\lambda^K = \lambda$ for $K = 1, 2, \dots, N$. The dispersed gas domains are separated from the liquid domain by the gas-liquid interfaces $\{\Gamma_K\}_{K=1}^N$.

Across each interface Γ_K , the velocity is continuous, but the stress tension undergoes a discontinuity arising from the gas-liquid interfacial tension and the disjoining pressure

$$(\Pi^0 - \Pi^K) \cdot \mathbf{n} = \Delta \mathbf{f}, \quad \mathbf{u}^0 = \mathbf{u}^K, \quad K = 1, 2, \dots, N, \quad (3)$$

where $\mathbf{n}$ is the unit vector normal to Γ_i and $\Delta \mathbf{f}$ is the interfacial force. $\Delta \mathbf{f}$ is defined as $\Delta \mathbf{f}_t + \Delta \mathbf{f}_d$, where $\Delta \mathbf{f}_t$ is the surface tension-induced component of the interfacial force and $\Delta \mathbf{f}_d$ is the disjoining pressure-induced component. $\Delta \mathbf{f}_t$ is calculated as $\Delta \mathbf{f}_t = 2\gamma k \mathbf{n}$, where k is the mean curvature at a point on the gas-liquid interfaces. The curvature k is positive when the interface is convex along the unit normal vector $\mathbf{n}$, and negative otherwise. Equivalently, one may assume that k is always positive and the value of the force is calculated as $-2\gamma k$, while the direction of the force always points into the concave side of a gas-liquid interface.

In some cases, e.g. the flow velocity being finite and uniform in the far field, the far field flow does not vanish at infinity. For these cases, the flow in the outermost domain is decomposed into two components, namely $\mathbf{u}(\mathbf{x}) = \mathbf{u}^\infty(\mathbf{x}) + \mathbf{u}^d(\mathbf{x})$, where $\mathbf{u}^\infty$ is the background flow and $\mathbf{u}^d$ is the disturbed flow. The latter vanishes at infinity while the former does not. Both flows satisfy the Stokes flow equations. In addition, the imposed flow is nonsingular in the whole space while the disturbed flow is nonsingular only in the outermost region.

2.2. Boundary integral equations

Place a point $\mathbf{x}_0$, which is called the source point, inside a bubble area D_m , $m \in \{1, 2, \dots, N-1\}$, and use the boundary integral representation of Stokes flow in fluid domains $\{D_K\}_{K=1}^{N-1}$. Depending on whether $K = m$, we have [37]

$$\begin{aligned} \int_{\Gamma_m} \Pi_{ik}(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) - \lambda \mu \int_{\Gamma_m} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) &= 8\pi \lambda \mu u_j(\mathbf{x}_0), \\ \int_{\Gamma_K} \Pi_{ik}(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) - \lambda \mu \int_{\Gamma_K} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) &= 0, \end{aligned} \quad (4)$$

with G and T the Stokeslet and stresslet, respectively, which in 3D are defined as

$$G_{ij}(\mathbf{r}) = \frac{\delta_{ij}}{|\mathbf{r}|} + \frac{r_i r_j}{|\mathbf{r}|^3}, \quad T_{ijk}(\mathbf{r}) = -6 \frac{r_i r_j r_k}{|\mathbf{r}|^5}, \quad (5)$$

where $\mathbf{r} = \mathbf{x} - \mathbf{x}_0$. The indices i and j , ranging from 1 to 3, denote the components of vectors or tensors within a 3D Cartesian coordinate framework.

Then, decompose the Stokes flow in the outermost gas domain and notice that the imposed flow is nonsingular in the whole space [37], leading to

$$\int_{\Gamma_N} \Pi_{ik}(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) - \lambda \mu \int_{\Gamma_N} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) = -8\pi \lambda \mu u_j^\infty(\mathbf{x}_0). \quad (6)$$

In addition, the boundary integral representation of the Stokes flow in the continuous liquid network is

$$\int_{\Gamma_0} \Pi_{ik}(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) - \mu \int_{\Gamma_0} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) = 0. \quad (7)$$

Subtracting Eq. (4) and Eq. (6) from Eq. (7), and substituting Eq. (3) into the resultant formula, we find a boundary integral equation for the velocity at the source point $\mathbf{x}_0$

$$\begin{aligned} u_j(\mathbf{x}_0) = u_j^\infty(\mathbf{x}_0) - \frac{1}{8\pi \lambda \mu} \int_{\Gamma_s} \Pi_{ik}(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) + \frac{1}{8\pi \lambda} \int_{\Gamma_s} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) \\ - \frac{1}{8\pi \lambda \mu} \int_{\Gamma_0 - \Gamma_s} \Delta f_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) dS(\mathbf{x}) + \frac{1 - \lambda}{8\pi \lambda} \int_{\Gamma_0 - \Gamma_s} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}), \end{aligned} \quad (8)$$

where Γ_s refers to the solid surface and $\Gamma_0 - \Gamma_s$ is the boundary of the liquid phase excluding the solid surface, i.e. all the gas-liquid interfaces. On the right-hand side of Eq. (8), the first and third integrals are called the single-layer potentials and the second and last integrals are called the double-layer potentials [37]. Each integral can be further decomposed into several terms in the form of integrals over multiple panels of the gas-liquid interfaces or the solid surface.

As shown in Fig. 1, the foam structure is composed of six different geometric elements, including the films in bulk (BF) or lamellae, coating films (CF), Plateau borders in bulk (BP), Plateau borders on the wall (WP), nodes on the wall (WN), and nodes in bulk (BN). Based on these different elements, the boundaries involved in this system can be categorized into nine types in addition to the infinity: (1) Interface between the nodes on the wall and the wall surface (SWNI); (2) Interfaces between the solid wall and the coating films (Solid-CF interface/SCI); (3) Interfaces between the solid wall and the wall Plateau borders (Solid-WPB interface/SWPI); (4) Surfaces of bulk films (BFS); (5) Surfaces of bulk Plateau borders (BPS); (6) Surfaces of nodes in bulk (BNS); (7) Surfaces of wall Plateau borders (WPS); (8) Surfaces of nodes on the wall (WNS); (9) Surfaces of coating films (CFS). With these different types of interface panels, Eq. (8) can be written as

$$u_j(\mathbf{x}_0) = u_j^\infty(\mathbf{x}_0) + I_{\text{SWNI}} + I_{\text{SCI}} + I_{\text{SWPI}} + I_{\text{BFS}} + I_{\text{BPS}} + I_{\text{BNS}} + I_{\text{WPS}} + I_{\text{WNS}} + I_{\text{CFS}}, \quad (9)$$

where I denotes the integral over each boundary section (bs). Specifically, the integrals over SWNI, SCI and SWPI are

$$I_{bs} = -\frac{1}{8\pi\lambda\mu} \int_{bs} \Pi_{ik}(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}) + \frac{1}{8\pi\lambda} \int_{bs} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}), \quad (10)$$

and, over BFS, BPS, BNS, WPS, WNS and CFS, the integrals are defined as

$$I_{bs} = -\frac{1}{8\pi\lambda\mu} \int_{bs} \Delta f_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) dS(\mathbf{x}) + \frac{1-\lambda}{8\pi\lambda} \int_{bs} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}), \quad (11)$$

where Δf_i is the interfacial force across the gas-liquid interface. As mentioned, it is calculated as $\Delta f_i = -2\gamma k(\mathbf{x})$, pointing towards the concave side of the gas-liquid interface. In Eq. (10), the single-layer potential involves an unknown stress Π_{ik} , due to the presence of walls.

2.3. The extreme case of dry foam

Under the assumption of dry foam, the double-layer potential over BFS is zero because of the opposite normal directions at the two gas-liquid interfaces of films. Besides, the infinitesimal volume of the Plateau borders and nodes lead to the double-layer potentials over BPS, WPS, SWPI, BNS, SWNI and WNS being zero. However, the double-layer potential over SCI cannot be cancelled from the double-layer potential over CFS, resulting in a remaining term $-1/(8\pi) \int_s u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x})$, where the normal vector $n_k(\mathbf{x})$ points into the solid domain.

As for the limit values of single-layer potentials, they can be considered from three aspects. Firstly, the single-layer potential over BNS is zero, because $||\Delta f|| \sim 1/R_{\text{BNS}}$ and $S_{\text{BNS}} \sim R_{\text{BNS}}^2$, where R_{BNS} and S_{BNS} are the curvature radius and the surface area of BNS, lead to $\int_{bs} \Delta f_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) dS(\mathbf{x}) \sim R_{\text{BNS}}$ that tends to zero as R_{BNS} approaches zero. Similarly, the single-layer potentials over WNS and SWNI are also zero because the areas of WNS or SWNI decreases much faster than the interfacial force across WNS or the stress on SWNI increases, respectively. Secondly, the summation of single-layer potentials over BPS is zero because the angles between the films remain 120 degrees. Otherwise, there exists singular forces act on the bulk Plateau borders, which immediately leads to the films restoring the states that forming 120-degree angles between each pair of the three meeting bulk films. Similarly, the summation of single-layer potentials over WPS and SWPI are also zero because of the force balance at wall Plateau borders. Other single-layer potentials, including those over SCI, CFS and BFS are non-zero in the dry limit. The single-layer potential over SCI and CFS can be combined as an integral over the wall, and the single-layer potential over BFS can be transferred into an integral over films with the interfacial force being doubled that of one single gas-liquid interface.

The resultant boundary integral equation for the velocity at a source point inside a gas area is written as

$$u_j(\mathbf{x}_0) = u_j^\infty(\mathbf{x}_0) - \frac{1}{8\pi\lambda\mu} \left(\int_{S_1} -\Delta F_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) dS(\mathbf{x}) + \int_{S_s} q_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) dS(\mathbf{x}) \right) - \frac{1}{8\pi} \int_{S_s} u_i(\mathbf{x}) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}), \quad (12)$$

where S_1 represents the film or lamella surface and S_s is the solid surface. $\mathbf{q}$ is the unknown normal stress acting on the separated phase of fluid by the wall. The value of the interfacial force ΔF_i is calculated as $4\gamma k$. In default, as mentioned above, the interfacial force points into the concave side of the lamella.

Denote as the characteristic length L_0 , characteristic velocity U_0 and characteristic time $T_0 = L_0/U_0$. Using these characteristic quantities, Eq. (12) can be written in the dimensionless form

$$u_j^*(\mathbf{x}_0^*) = u_j^{\infty,*}(\mathbf{x}_0^*) - \frac{1}{8\pi} \left(\int_{S_1} -\frac{1}{Ca} k^*(\mathbf{x}^*) n_i(\mathbf{x}^*) G_{ij}(\mathbf{x}^*, \mathbf{x}_0^*) dS^*(\mathbf{x}^*) + \int_{S_s} q_i^*(\mathbf{x}^*) G_{ij}(\mathbf{x}^*, \mathbf{x}_0^*) dS^*(\mathbf{x}^*) \right) - \frac{1}{8\pi} \int_{S_s} u_i^*(\mathbf{x}^*) T_{ijk}(\mathbf{x}^*, \mathbf{x}_0^*) n_k(\mathbf{x}^*) dS^*(\mathbf{x}^*), \quad (13)$$

where $Ca = \lambda\mu L_0/(4\gamma T_0)$ and $q_i^* = q_i T_0/(\lambda\mu)$. In order to simplify the notations, the superscript “*” will be ignored in the following.

The value of Ca can be estimated given that the flow is at low Reynolds numbers ($Re \ll 1$). The Reynolds number is defined as $Re = \rho L_0^2/(\lambda\mu T_0)$ and can be reformulated using Ca as $Re = 4\gamma\rho L_0 Ca/(\lambda\mu)^2$. Thus the low Reynolds number condition requires $Ca \ll (\lambda\mu)^2/(4\gamma\rho L_0)$. Apparently, Ca should be rather small if the fluid in bubbles is of low viscosity, and the characteristic length and the surface tension are large. In addition, when projecting the quantities in dimensionless form back to the realistic physical quantities, we have implicitly assumed that the realistic velocity is not much larger than the adopted characteristic velocity. In other words, the calculated dimensionless velocity should scale as $O(1)$ or smaller if the numerical results can be reproduced in the realistic world.

2.4. Boundary condition on the wall

As shown in Eq. (13), the velocity is a continuous quantity, which means that the velocity of the foam films is zero on the wall if the no-slip boundary condition is used. In other words, the no-slip boundary condition is not directly compatible with Eq. (13) since the foam films are expected to be mobile and can move on the solid surface. As an alternate choice, the slip boundary condition is considered in this work. The viscous force between the fluid and the wall is ignored under the slip boundary condition. One major effect of this simplification could be the ignorance of the viscous force imposed on the wall Plateau borders by the wall. This limitation, however, may be released through specified angles between the wall and the foam films. The slip boundary conditions requires that the fluid cannot penetrate the wall and thus the normal velocities of fluid on the wall are known quantities equal to the normal velocities of the wall itself. The shear stress on the wall is zero as the fluid is free to move tangentially along the wall. Consequently, the unknown quantities on the wall are composed of the normal stress and the two tangential velocity components.

2.5. Singularity and near-singularity subtractions

The integrals in Eq. (13) cannot be accurately computed using standard numerical methods (e.g. one-point trapezoidal rule) because the kernels, i.e. Stokeslet and stresslet, are singular and near-singular when $\mathbf{x}$ is located right at $\mathbf{x}_0$ and $\mathbf{x}$ approaches $\mathbf{x}_0$, respectively. To address the singularity or near-singularity, following properties of the Stokeslet and stresslet are utilized

$$\int_S G_{ij}(\mathbf{x}, \mathbf{x}_0) n_i dS(\mathbf{x}) = 0, \quad \int_S T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k dS(\mathbf{x}) = 0, \quad (14)$$

where the source point $\mathbf{x}_0$ is located exterior to an enclosed surface S , such as the solid surface.

2.5.1. solid-self singularity and solid-induced near-singularity

When solving Eq. (13) for the unknown pressure and tangential velocity components on solid surfaces, the source point $\mathbf{x}_0$ is placed right on the solid surfaces, leading to the singularity occurring in the single-layer and double-layer potentials over the solid surfaces. To address this singularity, we construct following identities using Eq. (14)

$$\int_{S_s} q(\mathbf{x}_0) G_{ij}(\mathbf{x}, \mathbf{x}_0) n_i dS(\mathbf{x}) = 0, \quad \int_{S_s} u_i(\mathbf{x}_0) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k dS(\mathbf{x}) = 0, \quad (15)$$

where initially the source point $\mathbf{x}_0$ is situated exterior to the solid surface S_s . $q(\mathbf{x}_0)$ gains significance through its definition employing $q_i(\mathbf{x}_0) = q(\mathbf{x}_0) n_i(\mathbf{x}_0)$ when $\mathbf{x}_0$ is precisely positioned on the solid surface or wall. Subtract Eq. (15) from Eq. (12) and then place the source point right on the wall

$$\begin{aligned} u_j(\mathbf{x}_0) = & u_j^\infty(\mathbf{x}_0) - \frac{1}{8\pi} \left(\int_{S_1} -\frac{1}{Ca} k(\mathbf{x}) n_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) dS(\mathbf{x}) \right. \\ & \left. + \int_{S_s} (q(\mathbf{x}) - q(\mathbf{x}_0)) G_{ij}(\mathbf{x}, \mathbf{x}_0) n_i(\mathbf{x}) dS(\mathbf{x}) \right) - \frac{1}{8\pi} \int_{S_s} (u_i(\mathbf{x}) - u_i(\mathbf{x}_0)) T_{ijk}(\mathbf{x}, \mathbf{x}_0) n_k(\mathbf{x}) dS(\mathbf{x}). \end{aligned} \quad (16)$$

When calculating the velocity at a field point $\mathbf{x}_f$ that is close to the wall, the single-layer and double-layer potentials associated with the flow on the wall become near singular. This near-singularity can also be cancelled using Eq. (15), in which $\mathbf{x}_0$ is selected as the point, on the wall, that is closest to the field point.

2.5.2. lamella-self singularity and lamella-induced near-singularity

When the source point is placed right on or close to a lamella, the single-layer potential, in Eq. (16), over films or lamellae become singular or near-singular. It is possible to construct identities similar to Eq. (15) to subtract this near-singularity, but may not be a convenient choice in the case of dry foam, where each film forms one side of an enclosed bubble boundary and thus is non-closed itself. Moreover, a film could be a common boundary of two neighbouring bubbles. These issues lead to the inconvenience utilizing identities as in Eq. (15) to subtract the singularity or near-singularity.

As an alternate method, we consider a non-singular contour-integral representation of single-layer potential [41], which can be used to calculate the single-layer potential over a non-closed surface

$$\begin{aligned} \int_{S'} -\frac{1}{Ca} k(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) n_i(\mathbf{x}) dS(\mathbf{x}) & \approx -\frac{1}{Ca} k_{S'}^{\text{avg}} \int_{S'} G_{ij}(\mathbf{x}, \mathbf{x}_0) n_i dS(\mathbf{x}), \\ \int_{S'} G_{ij}(\mathbf{x}, \mathbf{x}_0) n_i dS(\mathbf{x}) & = \oint_{\Gamma_{S'}} \frac{d\mathbf{r} \times \mathbf{r}}{|\mathbf{r}|}, \end{aligned} \quad (17)$$

where S' is a non-closed surface and $\Gamma_{S'}$ is the contour of S' . For S' being a small surface area, such as the area represented by a basic element that will be introduced in the next section, the averaged value $k_{S'}^{\text{avg}}$ can be approximated as $k_{\mathbf{x}_c}$, where $\mathbf{x}_c$ usually is

set to the center point of the small surface area. The direction of the curve integral is given by the unit vector $\mathbf{t}$ along the contour. $\mathbf{t}$ is defined as $\mathbf{t} = \mathbf{n} \times \mathbf{n}_{\Gamma_{S'}}$, where $\mathbf{n}_{\Gamma_{S'}}$ is the unit outward vector tangent to the surface S' and perpendicular to the contour $\Gamma_{S'}$.

2.6. Singular forces on bulk Plateau borders

In the derivation of Eq. (13), we have applied the condition of the force balance at Plateau borders on walls or in bulk. For wall Plateau borders, the no-slip boundary condition leads to that the normal gradient of the tangent velocity is zero on the wall. In other words, the velocity field maintains that the films are perpendicular to the walls. Thus, the forces are always balanced at wall Plateau borders. In contrast, the angles between films in bulk may deviate from 120 degrees due to the shearing effect of the fluid. As mentioned, once the forces acting on a bulk Plateau border leads to net non-zero force, this singular force should restore the 120-degree angles between films in bulk. For this reason, we introduce additional motion for Plateau borders in bulk and nodes to maintain the 120-degree angles between films in bulk, which is detailed in Section 3.5.

One issue arising from the additional motion of bulk Plateau borders, however, is the accumulation of errors in bubble volumes. This is because the additional motion is locally introduced to maintain the force balance at bulk Plateau borders, while its effect on the motion of other parts of the foam is not considered. To solve this problem, the contribution of the singular unbalanced force at bulk Plateau borders should be modelled in the boundary integral method. Use $\mathbf{F}$ to denote the density of singular force along the bulk Plateau borders. We approximate the contribution of the motion of bulk Plateau borders to the whole flow field as $1/(8\pi Ca) \int_{L_{BP}} \mathbf{F}_i(\mathbf{x}) G_{ij}(\mathbf{x}, \mathbf{x}_0) d\mathbf{l}(\mathbf{x})$. This integral is computed along bulk Plateau borders, denoted as L_{BP} , and should be added to the right-hand side of Eq. (13). We found this treatment can alleviate the volume deviations during simulations.

By contrast, an alternate method accounting for the singular forces along bulk Plateau borders is available in the literature [46]. Bajlekov [46] defined an “equivalent capillary pressure” in the context of three-phase flow problem, based on the equilibrium of forces including those due to capillary pressure and the tangent film tension forces, in a three-phase contact line vicinity. This method can also be applied in the present case, assuming that the singular forces at Plateau borders are balanced by the equivalent capillary pressure over film areas nearby the Plateau borders. One main advantage of this method then is that the singularity of the Plateau border’s motion is avoided, and thus the angle between films is not implemented as a boundary condition. Instead, because of no singular forces involved, we can directly use Eq. (13) to compute the velocity of any points on foam films, including those on Plateau borders. The implementation of this method is later discussed in Section 3.5.

3. Numerical methods

In this section, numerical techniques for modelling foam motion using Eq. (16) are discussed. These include the triangulation of the geometries (films and solid surfaces), the numerical approximations of the quantities involved in Eq. (16), such as the mean curvature and unit normal vector of a film, and the discretization of Eq. (16). In addition, the modelling of the motion of the bulk Plateau borders and its impact on the rest of the foam is explained.

3.1. Surface representation and mesh refinement

All surfaces, including films and walls, are triangulated in simulations. For films, an auxiliary mesh composed of so-called basic elements is constructed from the original triangle mesh, which is needed in the calculation of the single-layer potential over films [41], as shown in Fig. 2. The mesh should be sufficiently fine to resolve the details of the geometry, while the size distribution can be non-uniform in order to reduce the total number of mesh vertices and thus the computational cost [47]. Moreover, element distortion should be avoided for the benefit to the accurate discretization in terms of the curvature calculation and obtaining a well conditioned linear system. In order to maintain the favourable mesh properties during the simulations, three mesh refinement processes are applied. These operations include: (1) Element size optimization, (2) Subtraction and addition of edges and (3) Mesh topology adjustment.

Operation (1) is achieved by a physical analogy between the mesh and a truss structure [47]. Under this analogy, each mesh vertex or material point is attracted [48] or repulsed [47] by neighbouring vertices. The attractive or repulsive force is proportional or inversely proportional to the length of the edge between each vertex and its neighbours, with weighting factors accounting for the geometric properties, such as the local mean curvature. For instance, the following expression presents the tangent displacement $\mathbf{X}_t$ of a vertex $\mathbf{x}_c$ attracted by its neighbouring vertices $\mathbf{x}_{c'}$ [48]

$$\mathbf{X}_t = \phi(I - \mathbf{n}_c \mathbf{n}_c) \sum_i (k_{c'})^{3/2} \Delta S_{c'} (\mathbf{x}_{c'} - \mathbf{x}_c), \quad (18)$$

where the subscript c and c' here are indices used to denote the vertices. The summation is over all the vertices $\mathbf{x}_{c'}$ that directly connect to the vertex $\mathbf{x}_c$. The factor ϕ is chosen so that the tangential displacements of points by a time step can improve the mesh properties, instead of distorting the mesh. For vertices on wall Plateau borders, the displacement should be further projected onto the tangent plane of the wall, since these vertices consistently maintain contact with the surface of the wall.

Operation (2) enhances mesh quality by removing short edges and subdividing excessively long edges. The length of each edge between two neighbouring vertices is compared with two global length scales, $l_{\max}$ and $l_{\min}$. As the length of an edge becomes less than $l_{\min}$, the edge is deleted. In contrast, an edge, whose length is larger than $l_{\max}$, is subdivided by adding a middle vertex on the edge. For subdivided edges on the wall, the middle point is simply projected to the wall surface. For subdivided edges on foam films,

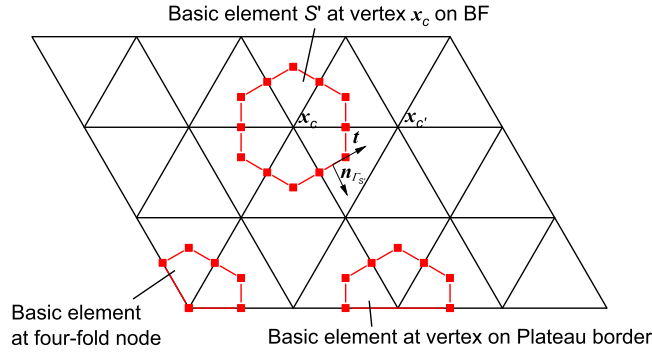


Fig. 2. Two grids are employed to discretize a foam film. The triangle elements of black contour are used to represent the film geometry. The basic elements of red contour, which are constructed from the center points of the triangle elements and the middle points of their edges, are needed in the calculation of the mean curvature and the single-layer potential over the film. There are three types of basic elements, at four-fold nodes, interior vertices on Plateau borders, and interior vertices on bulk films, respectively. The contour of the basic element S' at vertex x_c , where c is the index of the vertex, is denoted as $\Gamma_{S'}$. The neighbouring vertices of x_c are denoted by $x_{c'}$. Two distinct unit vectors are defined: the first being the unit normal vector $n_{\Gamma_{S'}}$, situated within the plane of the triangle element and extending outward from the basic element, while the second is the unit tangent vector t aligned with the contour of the element. In this schematic diagram, the unit normal vector n of the basic element is assumed to point out of the page. t is defined as $t = n \times n_{\Gamma_D}$.

the added vertex is projected onto a high-order approximation of the surface proposed by Bazhlevkov et al. [41]. Finally, operation (3) is achieved by reconnecting the vertices of two neighbouring triangles if the two triangles form a sharp quadrilateral. This operation can help to maintain favourable coordination numbers for most vertices on the mesh.

3.2. Calculation of the mean curvature and unit normal vector

There are two methods for the calculation of the mean curvature, as well as the unit normal vector, at a point on a film. The first choice is the contour integral method that utilizes the relation between the mean curvature and the gradient of the surface energy. The mean curvature at a vertex x_c on films is approximated as the magnitude of the gradient of surface energy at the vertex divided by the surface area $\Delta S_{S'}$ associated with the vertex, namely the area of the basic element S' at the vertex x_c [29]

$$k(x_c)n(x_c) \approx (1/\Delta S_{S'}) \oint_{\Gamma_{S'}} n_{\Gamma_{S'}} d\Gamma_{S'}, \quad (19)$$

where $n(x_c)$ is the unit normal vector [41]. Since k is taken positive, the normal vector always points into the concave side of the film. The other method for the computation of the mean curvature and normal vector at a vertex is through a local paraboloid fitting using the positions of the vertex and its neighbouring vertices [49].

The accuracy of each method in approximating the mean curvature of films depends on how the initial foam configuration is obtained [46]. For instance, if the relaxation process of a bubble is predicted using the boundary integral method and the mean curvature is computed using the contour integral method, the final mean curvature of the bubble can be accurately estimated by the contour integral method [46]. On the other hand, in the case of an initial bubble that conforms to a perfect sphere, with all vertices positioned on its spherical surface, utilizing paraboloid fitting presents a more accurate method for estimating the mean curvature.

Both approaches encounter difficulties in approximating the mean curvature at a vertex on the border of a film, such as when applied to basic elements at four-fold nodes or vertices on Plateau borders. The contour integral method tends to underestimate the magnitude of the mean curvature. This phenomenon is illustrated in Fig. 3, where a basic element on film border is derived by subdividing a regular basic element that encompasses its contour with six edges of equal length. The central vertex of the regular basic element is vertically offset by a small distance h from the plane defined by its contour. Using the contour integral method, the mean curvature and unit normal vector of the basic element on film border are calculated and then compared to those of the unaltered regular basic element.

It is found that the error in mean curvature maintains at a high level across the range of h , while the error in unit normal vector is relatively small, especially when the basic element approaches flat. This implies that even with the utilization of extremely refined element sizes for film discretization, the contour integral method struggles to yield a precise approximation of mean curvature at a vertex located along the film border. In addition, our further tests indicate that the error in film mean curvature tends to increase as the coordination number of the target vertex decreases. For these reasons, we instead approximate the mean curvature at vertices on film borders through interpolation. Specifically, we estimate the mean curvature at a vertex along the film border by calculating the arithmetic mean value of the mean curvature values from adjacent vertices located on the same film.

Regarding the paraboloid fitting technique for a vertex on film border, it necessitates the presence of auxiliary vertices in close proximity to the vertex. These auxiliary vertices are essentially mirror images of the interior vertices on the films, and they are positioned with respect to a specific plane that bisects the target vertex. Acquiring these auxiliary vertices for vertices located along the wall Plateau borders is straightforward, as the bisecting plane or mirror plane coincides precisely with the wall's tangent plane. Nonetheless, when addressing a vertex positioned along the bulk Plateau borders, a more complex situation emerges. In this particular context, the vertex necessitates three separate bisecting planes, all of which must be estimated.

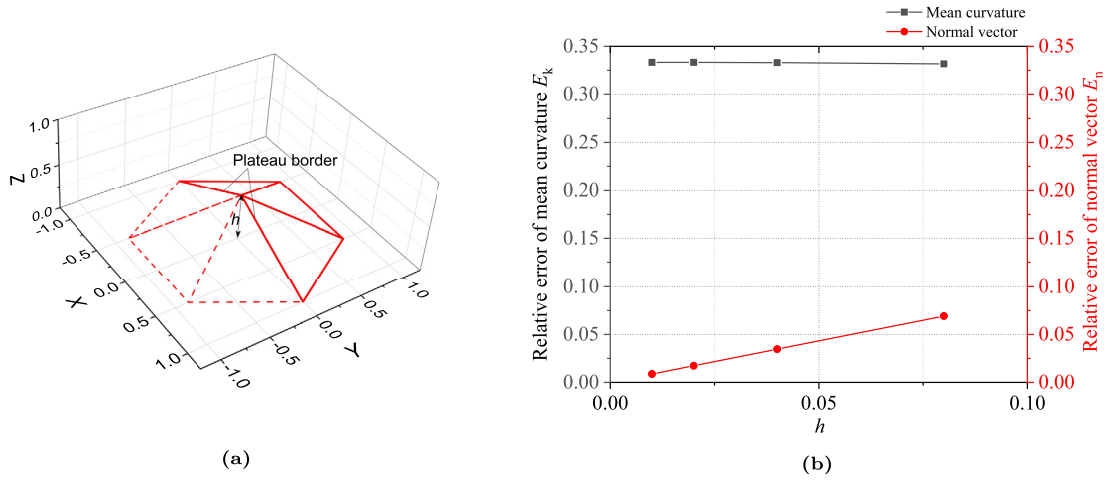


Fig. 3. The accuracy of the mean curvature and unit normal vector at the border of a film, computed by the contour integral method. (a) Schematic diagram of a basic element on the border of a film, denoted by the red solid line segments. This basic element constitutes half of a standard basic element possessing a regular hexagonal contour, with the complementary half represented by the dashed red line segments. To mimic a curved surface, the center vertex of the regular basic element is vertically offset by a small distance, h . The mean curvature and unit normal vector of the regular basic element, as computed through the contour integral method, serve as the benchmarks referred to as k_{ref} and $\mathbf{n}_{\text{ref}}$, respectively. (b) The errors in mean curvature and unit normal vector of the basic element on the film border are presented as functions of h , denoted as E_k and E_n , respectively. The errors are defined as $E_k = |k - k_{\text{ref}}|/k_{\text{ref}}$ and $E_n = \|\mathbf{n} - \mathbf{n}_{\text{ref}}\|$.

3.3. Calculation of the double-layer and single-layer potentials

The double-layer potentials are only computed over each element on walls. Since the singularity of double-layer potential in Eq. (16) has been removed, the double-layer integral over each triangle element can be calculated using the trapezoidal rule [42]

$$\int_{S''} (\phi(\mathbf{x}) - \phi(\mathbf{x}_0)) dS(\mathbf{x}) \approx (\phi(\mathbf{x}_c) - \phi(\mathbf{x}_0)) \Delta S_{S''}, \quad (20)$$

where $\mathbf{x}_c$ is the center point of each triangle element S'' and $\Delta S_{S''}$ is the area of the element. We use S'' to denote triangle elements on walls, to distinguish them from the basic elements S' on films. Note that, when $\mathbf{x}_c = \mathbf{x}_0$, namely the source point is right on the element over which the integral is calculated, the approximate integral value is zero, and thus this calculation is simply ignored in computations [42]. Similarly, the single-layer potential over each element on walls, in Eq. (16), may be also computed using Eq. (20), because the associated singularity has been removed. In the simulations, however, we instead use the nonsingular contour integral, i.e. Eq. (17), to compute the single-layer potential over each element on walls, because this ensures accurate solution of the boundary integral equation. The singularity-free form of the single-layer potential on walls remains utilized in the numerical simulations, for its facilitating the programming.

The single-layer potential over each basic element on foam films is also computed using the nonsingular contour integral namely Eq. (17). The numerical calculation of the nonsingular contour integral has been explained in detail in the original work [41]. Note that, this nonsingular contour integral can be applied no matter the source point is close to or far from the basic element. When the source point is far from the basic element, the single-layer integral is nonsingular itself. In this case, the one-point trapezoidal rule similar to Eq. (20) is more favourable due to its lower computational cost compared with that of calculating the nonsingular contour integral [41].

To establish the proper criterion for identifying the source point as being significantly distant from a basic element, a regular hexagon basic element is considered. This element is position on OXY plane and centered at the origin. Despite the element's flat nature, we make the assumption that its mean curvature is unit. We proceed to compute the velocities' magnitudes at field points positioned along the X, Y, and Z axes, induced by the basic element. Both the nonsingular contour integral method and the one-point trapezoidal rule are employed for this purpose. The outcomes obtained through the nonsingular formula serve as the exact solutions, against which we gauge the relative errors arising from the one-point trapezoidal rule. The behavior of these errors is illustrated in Fig. 4, where the relative error E_v exhibits a reduction in value as η increases. η is the ratio between the distance, from the field point to the central vertex of the basic element, and the edge size of the basic element. Our analysis reveals that when the separation between the source point and the fundamental element surpasses 20 times the size of the basic element, the error introduced by the one-point trapezoidal rule remains below 0.1%.

3.4. Discrete linear system of equations

The unknown stress exerted by the wall on the fluid in the bubbles is normal to the wall and thus can be expressed as $q_i = qn_i$ as has been utilized in Eq. (15), where q is the magnitude of the unknown normal stress to be solved. The normal velocity of fluid

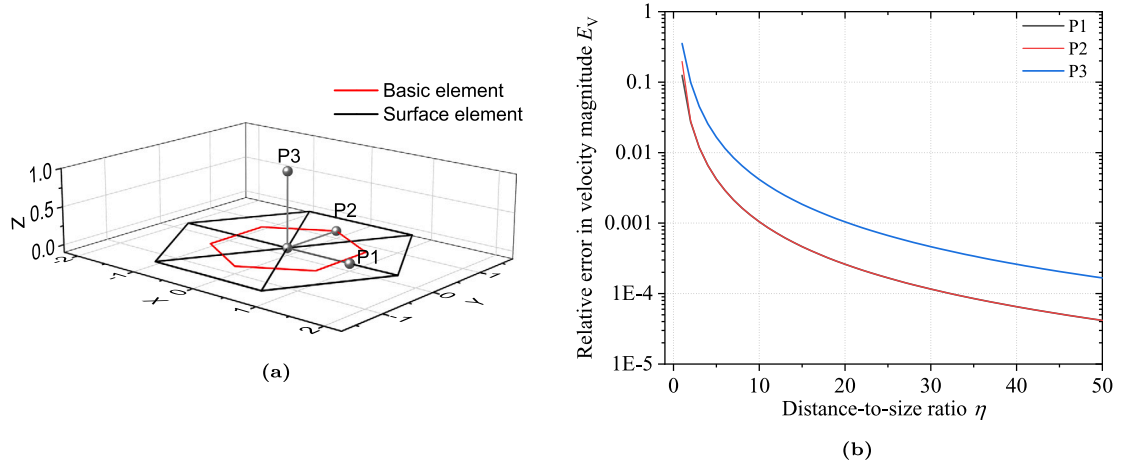


Fig. 4. Comparison between the velocity magnitudes at given field points, computed using the nonsingular contour integral method and the one-point trapezoidal rule, respectively. (a) The velocity is induced by a flat basic element whose contour is a regular hexagon on the OXY plane and centered at the origin. The edge length of the basic element is set to unit. Despite its planar nature, the basic element is assumed to possess a unit mean curvature. The field points, P1, P2, and P3, are aligned with along X, Y, and Z axis, respectively. The distance of the field points to the origin changes from 1 to 50. (b) The relative error E_v of the velocity magnitudes calculated using the one-point trapezoidal rule, compared with the result by the nonsingular formula, is presented as a function of the ratio η between the distance of field point to the origin and the edge size of the basic element.

on the wall is known to be the same as the normal velocity of the wall, while the two tangential velocity components are unknown. We utilize local unit normal vector $\mathbf{n}$ and the global unit vector $\mathbf{k}$ along Z axis to construct two unit tangent vectors $\mathbf{b}$ and $\mathbf{c}$, i.e. $\mathbf{b} = (\mathbf{n} \times \mathbf{k}) / |\mathbf{n} \times \mathbf{k}|$ and $\mathbf{c} = (\mathbf{n} \times \mathbf{b}) / |\mathbf{n} \times \mathbf{b}|$, where $\mathbf{n}$ is assumed to be unequal to $\mathbf{k}$; otherwise, we simply set $\mathbf{b}$ and $\mathbf{c}$ to the two unit vectors along X and Y axis, respectively. Using the newly defined local unit tangential vectors, the velocity of fluid on the wall can be expressed as $\mathbf{u} = A\mathbf{n} + B\mathbf{b} + C\mathbf{c}$, where A , B and C are the magnitudes of the normal and two tangential velocity components. Using this decomposition and the numerical methods for computing the integrals as outlined earlier, we obtain

$$\begin{aligned}
 & v_j \left(8\pi (B(\mathbf{x}_0)b_j(\mathbf{x}_0) + C(\mathbf{x}_0)c_j(\mathbf{x}_0)) \right. \\
 & + \sum_{\substack{s \\ \mathbf{x}_0 \notin S_s''}} (q(\mathbf{x}_c)n_i(\mathbf{x}_c) - q(\mathbf{x}_0)n_i(\mathbf{x}_0))G_{ij}(\mathbf{x}_c, \mathbf{x}_0)\Delta S_s \\
 & + \sum_{\substack{s \\ \mathbf{x}_0 \notin S_s''}} (B(\mathbf{x}_c)b_i(\mathbf{x}_c) - B(\mathbf{x}_0)b_i(\mathbf{x}_0))T_{ijk}(\mathbf{x}_c, \mathbf{x}_0)n_k(\mathbf{x}_c)\Delta S_s \\
 & \left. + \sum_{\substack{s \\ \mathbf{x}_0 \notin S_s''}} (C(\mathbf{x}_c)c_i(\mathbf{x}_c) - C(\mathbf{x}_0)c_i(\mathbf{x}_0))T_{ijk}(\mathbf{x}_c, \mathbf{x}_0)n_k(\mathbf{x}_c)\Delta S_s \right) = v_j (R_b + R_i + R_r),
 \end{aligned} \quad (21)$$

with R_b , R_i , and R_r three source terms, defined as

$$\begin{aligned}
 R_b &= 8\pi u_j^\infty(\mathbf{x}_0), \\
 R_i &= -8\pi A(\mathbf{x}_0)n_j(\mathbf{x}_0) - \sum_{\substack{s \\ \mathbf{x}_0 \notin S_s''}} (A(\mathbf{x}_c)n_i(\mathbf{x}_c) - A(\mathbf{x}_0)n_i(\mathbf{x}_0))T_{ijk}(\mathbf{x}_c, \mathbf{x}_0)n_k(\mathbf{x}_c)\Delta S_s, \\
 R_r &= -\sum_l I_l,
 \end{aligned} \quad (22)$$

where s denotes the wall and simultaneously serves as the index for each individual element S_s'' on the wall, and ΔS_s is the area of the triangle element S_s'' . v_j is a unit vector, taken as n_j , b_j and c_j , respectively. I_l is the contribution of the single-layer potential over the l th lamella or film and can be calculated using Eq. (17). In Eq. (21), we have conducted a projection to three orthogonal directions for the velocity u_j by an additional inner product with the unit vector v_j , which is found to result in a well-conditioned linear system.

Eq. (21) illustrates that the motion of foam films results from a linear combination of three distinct motions: the background flow induced by R_b , the flow stemming from wall deformation caused by R_i , and the relaxation of the film induced by R_r . Consequently, the motion within a foam flow system can be deconstructed into three separate flow systems. These include the isolated background flow when foams and solid walls are absent, the exclusive wall deformation in the absence of both background flow and foams, and the foam relaxation exclusive of wall deformation and background flow.

Depending on whether the background flow or walls are present, this decomposition can be further simplified. For instance, in the two specific applications of the method explored in this study. The first application involves examining the evolution of a

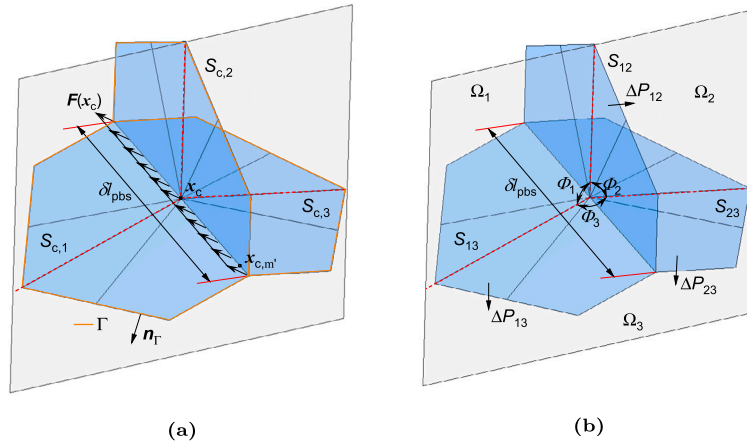


Fig. 5. Schematic diagram of three basic elements attached to a bulk Plateau border. The light gray reference plane is perpendicular to the Plateau border. (a) Reference for the method considering the singularity of the Plateau border's motion. The central vertex on the Plateau border is marked as $\mathbf{x}_c$, and the areas of its three adjacent basic elements are denoted as $S_{c,1}$, $S_{c,2}$, and $S_{c,3}$, respectively. The length of the Plateau border is indicated as δl_{pbs} . A singular force, uniformly distributed along the Plateau border, is represented as $\mathbf{F}(\mathbf{x}_c)$. The contour Γ , encompassing the combination of the three basic elements, is highlighted in orange. Its outward unit normal vector is denoted as $\mathbf{n}_\Gamma$. (b) Reference for the method involving the equivalent capillary pressure. The space is divided into three domains, denoted as Ω_1 , Ω_2 , and Ω_3 . The angles formed between each pair of the basic elements in $\{\Omega_i\}_{i=1}^3$ are indicated as $\{\Phi_i\}_{i=1}^3$. In contrast to (a), the areas of the three basic elements are now referred to as S_{12} , S_{23} , and S_{13} , respectively. The equivalent capillary pressures for the three films are denoted as ΔP_{13} , ΔP_{12} , and ΔP_{23} , respectively.

foam in a simple shear flow. In this context, the foam's motion comprises two elements: the background flow and the relaxation of the foam films. The second application centers on the motion of a foam film within confined geometries, specifically within a diverging-converging channel. For this specific scenario, the film's motion can be decomposed into two fundamental constituents: the flow induced by wall deformation and the relaxation of the foam film.

3.5. Motion of bulk Plateau borders

As detailed in Section 2.6, there exists a singular tension force exerted on bulk Plateau borders due to the departure from local equilibrium in the tension forces exerted by individual films. This singular tension force, operative on the bulk Plateau borders, shares the same dimensional characteristics as film tension, but its direction is perpendicular to the bulk Plateau borders. The disruption of local equilibrium arises from the shearing impact of the flow field, which imparts a dual influence on foam motion. Firstly, the singular tension force acts to reinstate local equilibrium, resulting in 120-degree angles formed among the three films converging at a bulk Plateau border. Besides, the restoring motion of these bulk Plateau borders contributes to the overall motion of the remaining foam structure. To account for these dual effects, we first address the estimation of the singular tension force acting on the bulk Plateau borders.

The evaluation of the total tension force exerted on a basic element is achieved through the contour integral, Eq. (19), conducted around a vertex $\mathbf{x}_c$. Similarly, by computing this contour integral across all basic elements associated with a vertex situated along the bulk Plateau borders, as shown in Fig. 5 (a), the resulting vector encapsulates the overall tension forces affecting these interconnected basic elements. It comprises two constituents of film tension forces: the components normal to each individual basic element and the singular tension force along the bulk Plateau borders. As a result, the singular tension force at the vertex $\mathbf{x}_c$ on the Plateau border, denoted as $\mathbf{F}(\mathbf{x}_c)$, can be evaluated as

$$\mathbf{F}(\mathbf{x}_c) \approx \frac{1}{\delta l_{pbs}} \left(\oint \mathbf{n}_\Gamma d\Gamma - \sum_{m=1}^M \Delta S_{c,m} k(\mathbf{x}_{c,m}) \mathbf{n}(\mathbf{x}_{c,m}) \right), \quad (23)$$

where the contour Γ refers to that of the combination of three basic elements around an interior vertex on bulk Plateau borders or four basic elements around a four-fold node. δl_{pbs} is the short length of the bulk Plateau border encompassed by the combination of the basic elements. When computed for the four-fold node, δl_{pbs} is set to unit, and $\mathbf{F}(\mathbf{x}_c)$ is interpreted as a concentrated force at the four-fold node. M is the number of neighbouring basic elements, either 3 or 4, depending on the vertex $\mathbf{x}_c$ lies within the interior of the bulk Plateau borders or precisely corresponds to a four-fold node. m is the index of the basic element around the vertex $\mathbf{x}_c$.

The effects of the singular force on bulk Plateau borders are modelled as follows. The restoring motion of bulk Plateau borders, as well as the four-fold nodes, is achieved by a series of iteration using the computed $\mathbf{F}(\mathbf{x}_c)$ that is updated at each iteration step. For each step of iteration, we move each vertex $\mathbf{x}_c$ on bulk Plateau border by a small displacement $\psi \mathbf{F}(\mathbf{x}_c)$, where ψ is small factor stabilizing the iteration process. Notably, all other vertices, except for those on bulk Plateau borders, are fixed during the iteration process.

The effect of the motion of bulk Plateau borders on the remaining part of the foam is modelled by considering the velocity induced by the singular film tension forces on bulk Plateau borders. The value of singular film tension force is set to the initial value for the

Table 1
Introduction of applications.

Application	Geometric element	Simulation platform
Five-bubble foam in a simple shear flow	BF BN BP	Surface Evolver
Foam film in a diverging-converging channel	BF CF WP	Visual Studio 19.0 (C++)

restoring motion. We use the multiple-point trapezoidal rule to compute the singular film tension force-induced velocity at a source point $\mathbf{x}_0$ not located on the Plateau borders

$$u_j(\mathbf{x}_0) \approx \frac{\delta l_{bps} F_j(\mathbf{x}_{c,m'})}{M'} \sum_{m'=1}^{M'} G_{ij}(\mathbf{x}_{c,m'}, \mathbf{x}_0), \quad (24)$$

where each short bulk Plateau border segment of length δl_{bps} is divided into M' uniform segments. m' is the index for each segment. $\mathbf{x}_{c,m'}$ refers to the center point of each short segment. $F_j(\mathbf{x}_{c,m'})$ remains equal to $F_j(\mathbf{x}_c)$ that has been computed using Eq. (23).

As mentioned in Section 2.6, there exists another method that could be used to handle the singular forces along bulk Plateau borders, as outlined in [46], which employs the concept of an equivalent capillary pressure. This method is explained as follows. The equilibrium of three forces in the vicinity of a bulk Plateau border is illustrated in Fig. 5 (b). The equivalent capillary pressure, referred to as ΔP_{eq} , acting on each basic element, is determined by various factors, including the length of the bulk Plateau border (δl_{bps}), the angles formed between each pair of basic elements ($\{\Phi_i\}_{i=1}^3$), and the areas of these basic elements, specifically S_{12} , S_{23} , and S_{13} . According to the findings in [46], the equivalent pressure can be computed as

$$\Delta P_{ij} = \frac{\delta l_{bps}}{2S_{ij}} \left(\cot\left(\frac{\Phi_i}{2}\right) - \cot\left(\frac{\Phi_j}{2}\right) \right), \quad (25)$$

where ΔP_{ij} ($i \neq j$), in the specific case illustrated in Fig. 5 (b), is the equivalent capillary pressure across the film separating domain Ω_i and Ω_j . S_{ij} denotes the area of the basic element between Ω_i and Ω_j . $\{\Phi_i\}_{i=1}^3$ refer to the angles formed between basic elements in domain $\{\Omega_i\}_{i=1}^3$.

Accordingly, an equivalent mean curvature denoted as k_{eq} can be defined, where $k_{eq} = 0.5 \Delta P_{eq}$. The sign of k_{eq} , which can be either 1 or -1, should be reversed if the default orientation across the film, as illustrated in Fig. 5 (b), does not align with the normal vector specified in Eq. (13). Substituting k in Eq. (13) with $k + k_{eq}$ allows the resulting equation to be employed at any point within the flow field. It is important to note that k_{eq} should be set to zero when the integration concerns interior basic elements on films.

When working with basic elements situated at four-fold vertices, the two approaches mentioned earlier can be easily extended and applied. It is important to observe that in the approach addressing the singular motion of bulk Plateau border, the singular force should be concentrated at a fold-fold node, as highlighted earlier. In the alternative method involving equivalent capillary pressure, each basic element surrounding a four-fold node connects to two pairs of basic elements, as opposed to just one pair (e.g. in Fig. 5 (b)).

3.6. Evolution of the foam configuration

After solving the velocities at vertices on foam films, the foam configuration is evolved by a first-order ordinary differential equation named the kinematic equation, i.e. $d\mathbf{x}/dt = \mathbf{u}(\mathbf{x})$, where $\mathbf{x}$ is the position of a vertex on films. Using the forward Euler method, the kinematic equation can be approximated as $\mathbf{x}^{k+1} - \mathbf{x}^k = \mathbf{u}(\mathbf{x}^k)\Delta t$, where k and $k+1$ denote the current and next moments of time, respectively, and Δt is the time step. Note that in modelling the singular motion of bulk Plateau borders, the positions of vertices on bulk Plateau borders are adjusted by restoring motion.

4. Numerical results

This section presents the numerical results of applying the boundary integral equation for dry foam in two distinct scenarios. These scenarios are outlined in Table 1, which provides information about the geometric elements involved and the simulation platforms employed in each specific application.

4.1. Foam evolving in a simple shear flow

Foam deformation in a simple shear flow is considered in this section to exemplify the application of the method proposed in this work to dry foams in bulk. We consider a foam comprising five bubbles, as depicted in Fig. 6. These bubbles consist of three larger ones, each with a volume of 0.2, positioned centrally and dividing two smaller bubbles with volumes of 0.1 in opposing directions. The initial configuration of the foam is established in a state of stationary equilibrium, achieved through the minimization of surface energy utilizing Surface Evolver. This initial foam formation exhibits symmetry with respect to the OYZ plane, and the alignment of the Z-axis coincides with the centers of the two smaller bubbles. Fig. 6 also shows the change in foam configuration after the initial restoring motion of bulk Plateau borders.

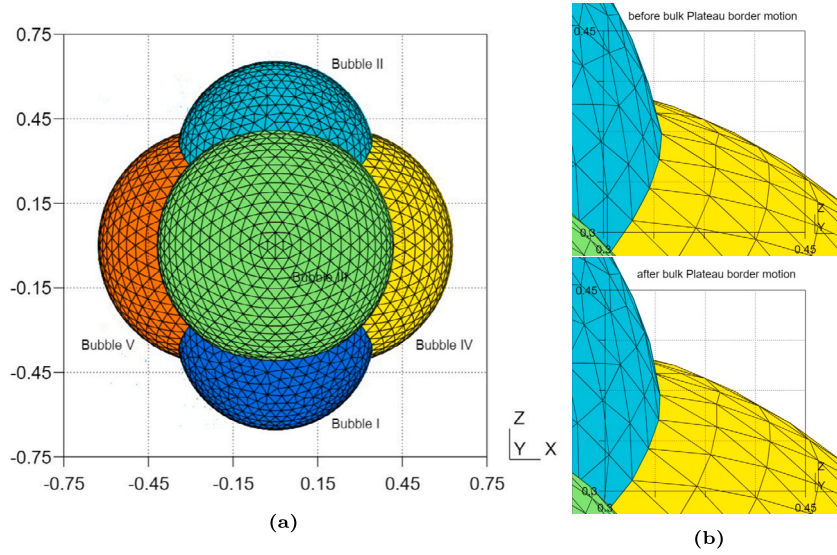


Fig. 6. The initial equilibrium configuration of a five-bubble foam. (a) The initial foam composed of five bubbles was generated by minimizing surface energy, using Surface Evolver. These bubbles are designated as I to V, with Bubble I and II having a volume of 0.1, while the remaining bubbles possess a volume of 0.2. The foam exhibits symmetry across the OYZ plane. (b) The enlarged illustrations display the vicinity of the bulk Plateau borders both before (top) and after (bottom) repositioning them to establish a local equilibrium among the three film tension forces.

A simple shear flow, $\mathbf{u}_0(x, y, z) = [0.0, 0.0, \Psi x]$, is imposed, where Ψ is the velocity gradient and is set to 0.1 in our simulations. The boundary-integral simulations of the foam evolving in the simple shear flow are conducted in Surface Evolver, and thus the associated mean curvature is computed using the contour integral method. The time step ΔT is fixed to 0.01, and the elements size ΔX is kept within 0.040 to 0.090 in the simulations. Fig. 7 presents the dynamic motion of the foam at $Ca = 1$. The foam deforms under the imposed strains by the shear flow, while attempting to restore the minimal surface due to the film tension. Therefore, the evolution of the foam configuration is in contrast to the result at $Ca \rightarrow \infty$ namely at zero film tension, where the foam configuration is purely advected by the background shear flow. Note that the impact of the motion of the bulk Plateau border has been considered in the simulations, which is important to limit the errors in bubble volumes.

During the simulations, close attention is paid to the bubble volumes, which must adhere to the principles of mass conservation. However, owing to discretization errors and, notably, the additional motion introduced to maintain force equilibrium at the bulk Plateau borders, a gradual accumulation of discrepancies in bubble volumes is observed over time. The subsequent section focuses on investigating the impact of bulk Plateau border motion on bubble volume conservation. To achieve this, two distinct sets of numerical experiments, denoted as Group 1 and Group 2, are conducted. In Group 1, the influence of bulk Plateau border motion on foam film dynamics is excluded, whereas this effect is incorporated in Group 2. Each group encompasses four varying ranges of element sizes, specifically $\Delta X \in [0.0178, 0.04]$, $[0.0267, 0.06]$, $[0.044, 0.099]$, and $[0.06, 0.135]$. Within each simulation belonging to the four cases in each group, continuous monitoring of bubble volumes occurs throughout the time interval T from 0 to 0.2.

Since the volume deviation over time exhibits a similar trend for each bubble, we will solely present the findings concerning bubble V hereafter. In Fig. 8 (a), which corresponds to Group 1, the temporal evolution of bubble volume deviation, characterized by the absolute magnitude of the relative change in actual volume relative to the initial value of 0.2, is displayed with the average element size serving as the parameter. The average element size remains relatively stable throughout the simulations, measuring 0.026, 0.046, 0.066, and 0.098 for the respective element size ranges. Fig. 8 (b) illustrates the variation in bubble V's volume concerning the average element size, with time as the parameter. The outcomes for Group 2 are depicted in Fig. 9.

In both Fig. 8 and 9, an abrupt escalation in error is observed at the initial time point, which can be attributed to several factors. Initially, the foam configuration undergoes an evolution driven by the motion of bulk Plateau borders to establish force equilibrium. This precursor phase precedes the dynamic evolution of the foam. Additionally, the force distribution in the initial foam configuration, as predicted by Surface Evolver, lacks optimal balance. This discrepancy arises because the surface energy minimization algorithm in Surface Evolver assumes that the film tension force of a vertex's basic element acts precisely on the vertex. However, for vertices situated on the bulk Plateau borders, the applied force encompasses not only the film tension force tangential to the vertex's basic element along the Plateau borders but also includes a component normal to the basic element. The initial abrupt volume deviation is alleviated as the averaged element size decreases, because the displacement of bulk Plateau borders during its motion decreases with the decreasing of the averaged element size.

After the initial sharp increment of error in bubble volume, the error still accumulates with time. However, the accumulation of the error becomes slower as the averaged element size decreases. Additionally, the incorporation of the effect of the motion of bulk Plateau borders on the film motion is found to be rather important in terms of conserving bubble volumes. The error accumulation rate with time at a given averaged element size decreases by an order of magnitude in Group 2 which involves the effect, than in Group 1 excluding the effect.

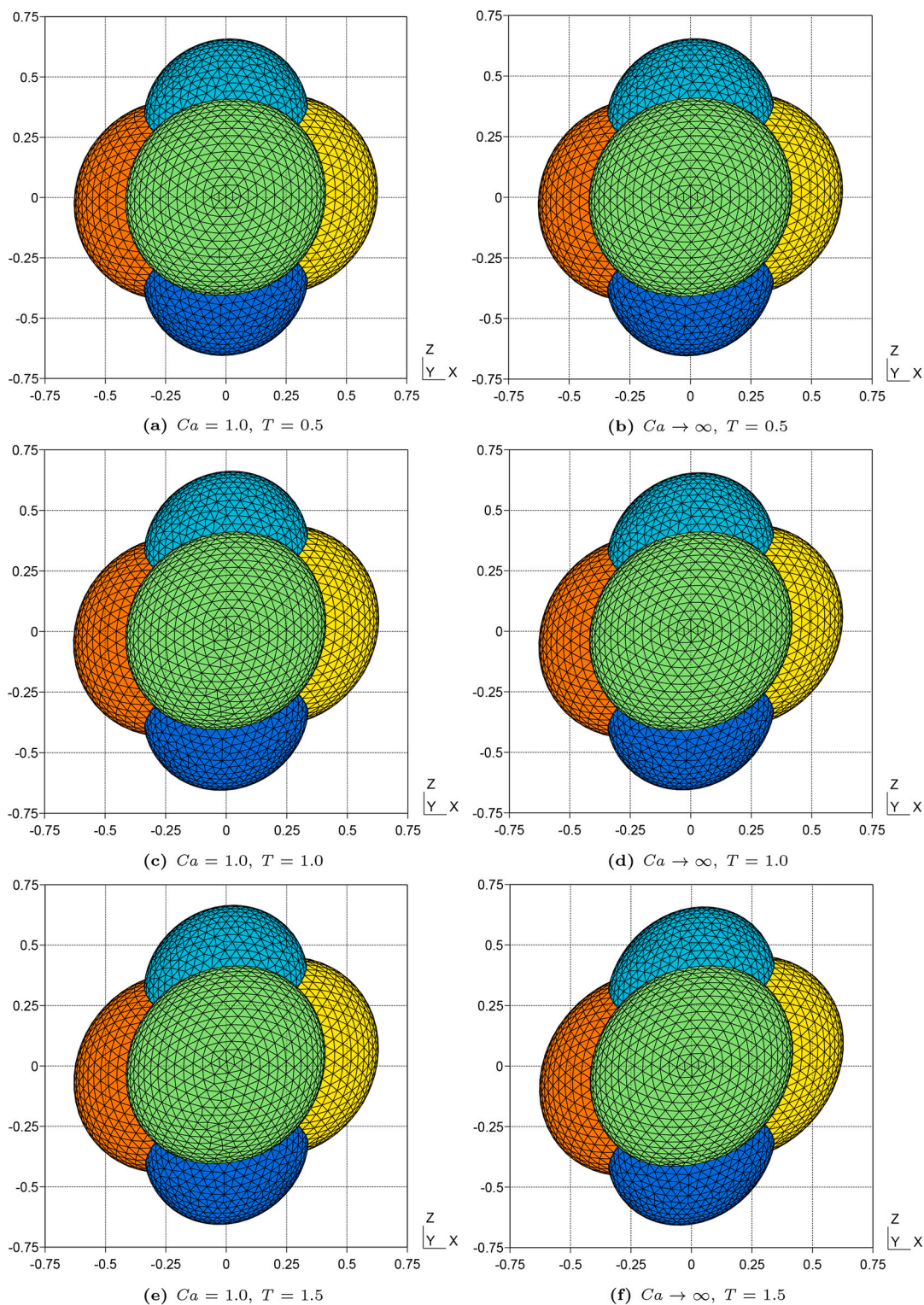


Fig. 7. The evolution of the five-bubble foam in a simple shear flow. The left column shows results at time $T = 0.5, 1.0$, and 1.5 , where Ca is set to unit. The right column shows the pure advection of the foam, namely Ca tends to infinity.

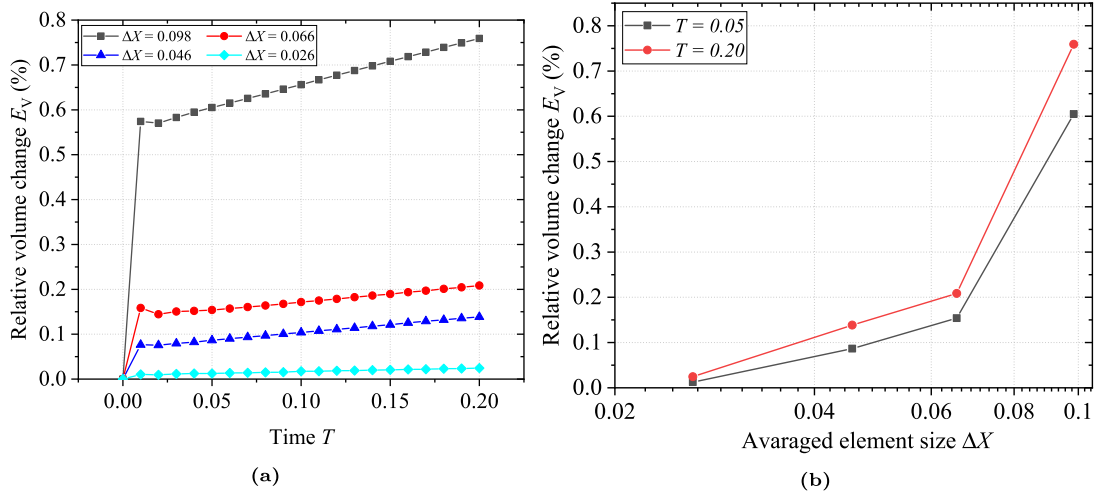


Fig. 8. The change in volume of bubble V during simulations excluding the effect of the motion of bulk Plateau borders on the motion of the remaining part of the foam. (a) The evolution of Bubble V's volume over time is showcased for four average element sizes, specifically $\Delta X = 0.098, 0.066, 0.046$, and 0.026 . (b) The variation in Bubble V's volume is explored concerning the average element size. Comparisons of volumes are made at two distinct time instances: $T = 0.05$ and 0.20 , respectively.

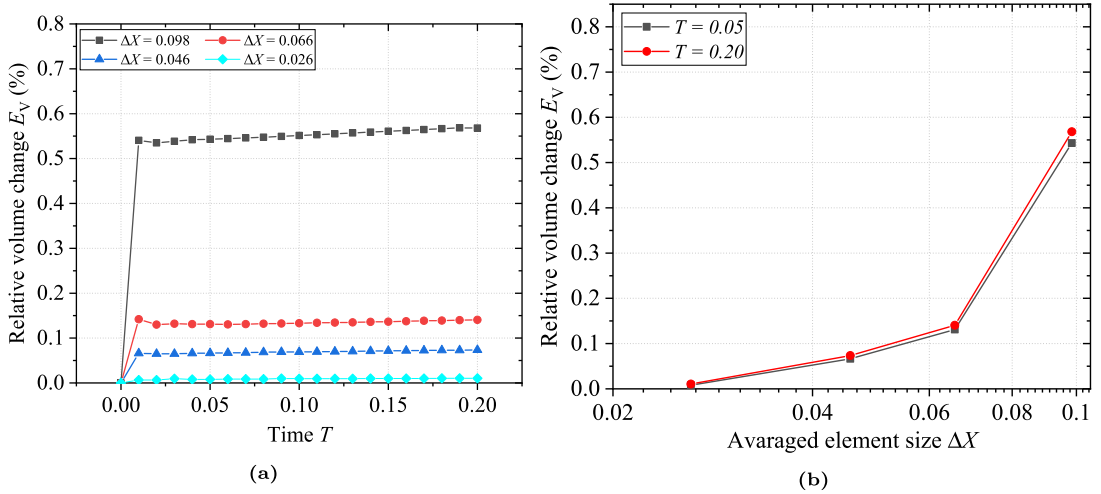


Fig. 9. The change in volume of bubble V during simulations incorporating the effect of the motion of bulk Plateau borders on the motion of the remaining part of the foam. (a) The evolution of Bubble V's volume over time is showcased for four average element sizes, specifically $\Delta X = 0.098, 0.066, 0.046$, and 0.026 . (b) The variation in Bubble V's volume is explored concerning the average element size. Comparisons of volumes are made at two distinct time instances: $T = 0.05$ and 0.20 , respectively.

In the above numerical tests, the element size is roughly uniform throughout the foam during simulations. This approach leads to a rapid escalation in the foam's element number as the element size decreases, though reducing the element size proves advantageous for minimizing errors in volume conservation. Furthermore, the computational expenditure experiences a significant rise with an increase in the number of elements, given that the computational cost typically follows a quadratic scaling in relation to the element number within the framework of the boundary integral method. A more effective approach to representing foam configurations could involve the adoption of a non-uniform mesh. This scheme would entail refining the mesh in the vicinity of Plateau borders while maintaining coarser elements in other regions, thereby reducing the overall element number in simulations.

In the subsequent part of this section, we evaluate the other method for handling the singular forces along bulk Plateau borders, incorporating the equivalent capillary pressure, by examining a scenario in which we initialize the foam configuration shown in Fig. 7 (d), employing a finer mesh with element sizes within the range of $[0.0178, 0.04]$. The foam is then allowed to relax in the absence of background flow. We calculate the velocity field within the plane at $y = 0$, passing through the geometric center of the foam configuration, using Eq. (13). In this computation, we utilize Eq. (25) for the basic elements situated on the film borders.

For comparison, we also compute the velocity field by the method where the movement of the bulk Plateau border is treated as singular. In this case, we first adjust the position of the bulk Plateau border, following the procedure outlined in Section 3.5, so that the angles between films at Plateau borders and between Plateau borders at nodes return to their static values. Subsequently, we

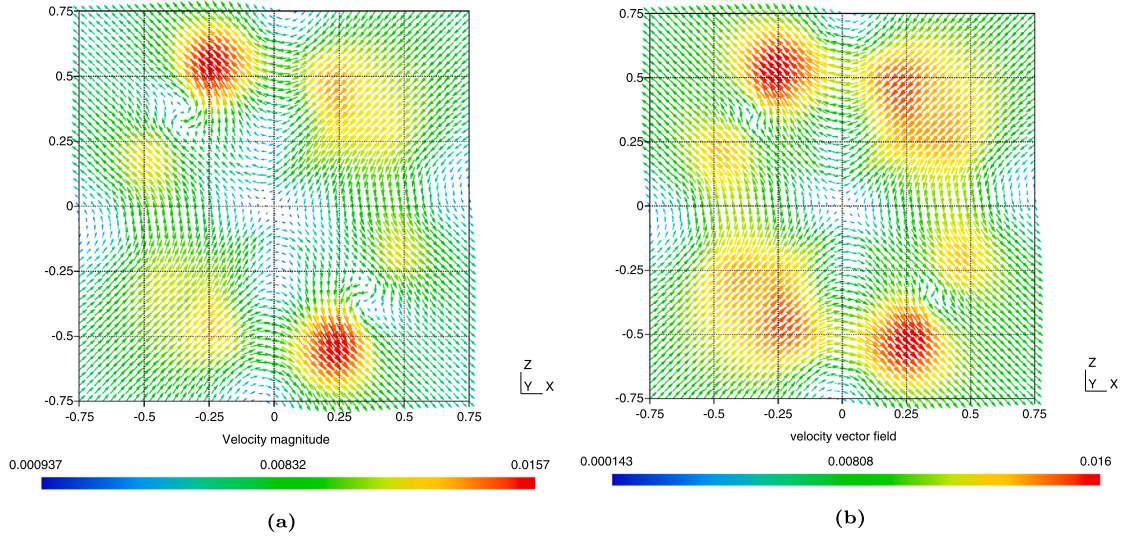


Fig. 10. Vector field of velocity within the $y = 0$ plane. (a) Result by the method involving the singular motion of the Plateau border. (b) Result obtained through the method utilizing the equivalent capillary pressure.

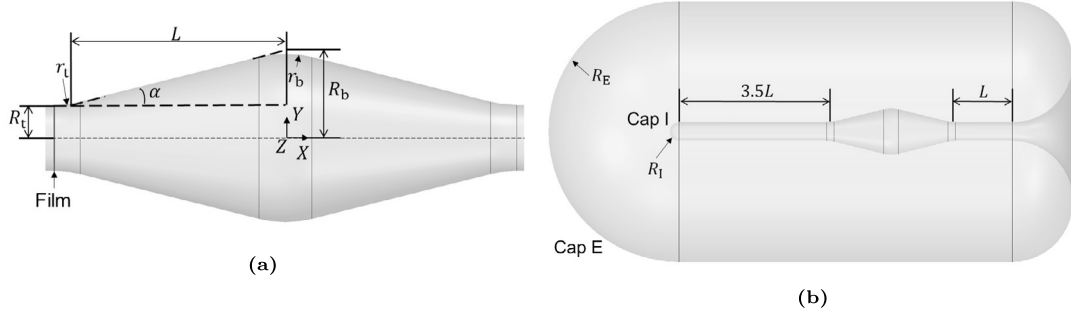


Fig. 11. Geometries of a foam film in a diverging-converging channel, in the context of the boundary integral method. (a) Geometry of the diverging-converging channel. Several dimensions are fixed, including $L = 2.0$, $R_t = 0.3$, $r_b = 0.35$ and $r_t = 1.0$ in the simulations. Other dimensions, including α and R_b , are not independent, and are related through $R_b = R_t + L \tan \alpha$. α is one half of the diverging angle of the channel. The initial position of the film is at the inlet of the pore, marked by the blue line segment. (b) The channel wall is extended to a complete enclosed surface built for boundary-integral simulations. Two hemispherical caps on the left side of the surface are denoted by Cap I and Cap E, respectively. The radius of Cap I equals the radius of the inlet of the pore, i.e. $R_I = 0.3$. The radius of Cap E is set to $R_E = 4.3$.

employ Eq. (13) to compute the velocity field within the same region. The obtained results are presented in Fig. 10. It is evident that both methods yield similar patterns in the distribution of velocity, encompassing both the direction and magnitude of the velocity.

4.2. Film motion in a diverging-converging channel

For foam flow in porous media, the geometry of flow channels plays an important role in determining foam's special properties, such as the yield stress that is reflected by the minimum pressure gradient required to initiate foam motion in porous media. To characterize foam flow in porous media, a typical scenario considered in literature is the motion of a single film in a diverging-converging channel or pore. The diverging-converging channel refers to a channel consisting of two sections, one diverging section connected to the converging section through the corner in the central pore body. A single foam film moves from one side of the channel, called the inlet port, to the other side namely the outlet port, during which the film behaviour is confined by the varying cross-sectional channel and affected by the capillary number Ca .

To explore this scenario within the framework of the boundary integral method outlined in this study, the initial step involves extending the channel surface into a complete closed surface. Moreover, this surface is characterized by a smoothly varying unit normal vector across its entirety [43], as shown in Fig. 11. The displacement of the film, initially situated at the channel inlet, is initiated through the deformation of the channel surface. This deformation specifically corresponds to the rightward motion of Cap I in Fig. 11. The dimensionless velocity of Cap I is denoted as U , which exactly equals unit as the realistic velocity of Cap I is selected as the characteristic velocity. In order to maintain the volume of the space enclosed by the surface, the large hemispherical surface, denoted as Cap E, also undergoes rightward movement. However, this motion occurs at a relatively reduced velocity of R_I^2/R_E^2 , where R_I and R_E signify the radii of Cap I and Cap E, respectively.

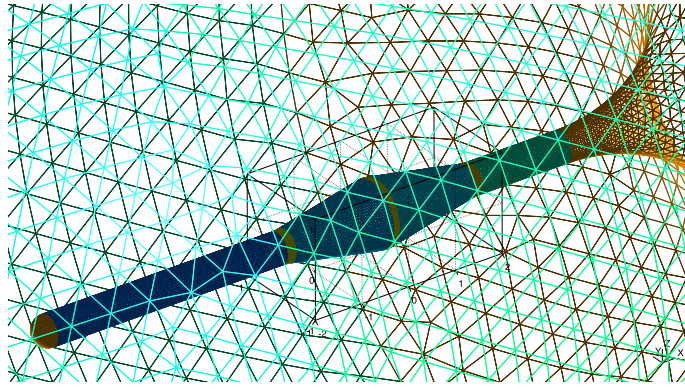


Fig. 12. Triangulated wall surface, with a mesh configuration that exhibits finer elements within the inner region of the wall surface, transitioning to coarser elements on the outer side. The entirety of this mesh comprises a total of 28642 triangle elements.

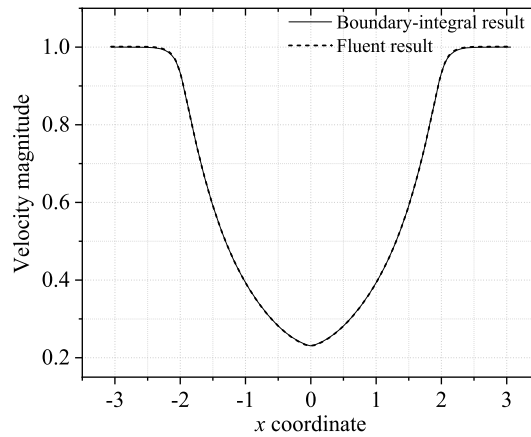


Fig. 13. Quantitative comparison of the velocities along the intersection curve of the OXY plane and the channel surface by the boundary integral method and the finite volume method.

As mentioned in Section 3.3, the film's movement in this context comprises two distinct components originating from R_l and R_r as indicated in Eq. (21). Our investigation begins by examining the induced flow resulting from the deformation of the wall, followed by an analysis of the film's relaxation motion within the channel. It is important to note that this study does not emphasize a comprehensive parameter analysis of the problem.

4.2.1. Imposed flow field due to wall deformation

The imposed flow plays a key role in the foam film motion since it acts to deviate the foam film away from the equilibrium status, which further induces the film relaxation and its interaction with the imposed flow. The imposed flow is independent of the film presence, and thus is a constant steady flow in the channel. For this reason, in simulations of film motion in the channel, the imposed flow only needs to be solved once, and, at each moment of time, the imposed component of the velocity at each field point can be simply evaluated through interpolation.

Consider the diverging-converging pore, whose diverging angle is set to 20 degrees and α is set to 10 degrees accordingly. In the simulation, the imposed flow in the pore happens because of the deformation of the wall surface. The boundary of the solid particle is triangulated into discrete elements, shown in Fig. 12, using GMSH [50], an open-source mesh generator. With the triangulated wall surface, coefficients and sources terms in Eq. (21) can be computed, which is then solved using GMRES with restart [51,52].

For comparison, we also predict the imposed flow using the finite volume method (FVM) in Fluent, a commercial software package. In this method, the Navier-Stokes equations for incompressible fluid are solved in a 3D fluid domain, where the inertial term is incorporated. In order to reproduce behavior of fluid in Stokes flow or diminish the inertial effect, the simulation parameters are carefully chosen so that the Reynolds number in the flow problem is sufficiently small. Specifically, the density, viscosity and flow rate through the channel are set to 1.0 kg/m^3 , $10^3 \text{ Pa} \cdot \text{s}$ and 1.0 m/s . Given that the characteristic length, i.e. the radius of the inlet port of the channel, is 0.3 m , the Reynolds number of the problem is evaluated to be 3.0×10^{-4} . In addition, the characteristic velocity and normal stress are computed to be 1.0 m/s and 1000.0 Pa , respectively, which are used to obtain the dimensionless counterparts of the results by the FVM.

The numerical results of the velocity magnitude on the pore surface and its intersection with OXY plane, computed using the boundary-integral model and the FVM, are compared in Fig. 13, and show good agreement. It can be found that the velocity decreases

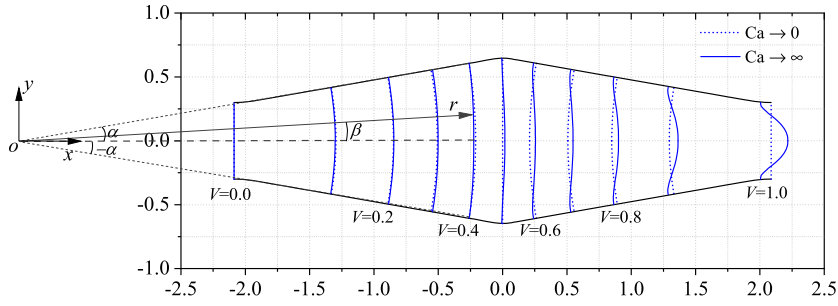


Fig. 14. Film advection by the imposed flow, in contrast to the quasi-static film motion. From the left to the right side, the film profiles correspond to the swept volume 0.0, 0.1 ... 1.0, respectively, with a constant volume gap 0.1. The volume is normalized by the pore volume namely the volume of the space between the inlet and the outlet. An auxiliary coordinate system oxy is established, based on which the polar coordinates β and r are defined. The film profile can be represented using x as a function of y , for the calculation of the mean curvature of the film. Furthermore, the mean curvature of the film can be depicted as a function of the polar coordinates β .

along the flow direction in the diverging section of the pore, while increasing in the converging section of the pore, due to the mass conservation of the flow field.

4.2.2. Advection of film by the imposed flow

With the known imposed flow field solved in the last section, the pure advection of the film by the imposed flow, corresponding to the film behaviour at Ca tending to infinity, can be simulated. A series of film profiles on the OXY plane are shown in Fig. 14. For comparison, at Ca tending to zero, the quasi-static motion of the film in the same pore is also presented. It can be found that, as the film transverses in the pore, the film configurations gradually deviate from those in quasi-static motion. This deviation becomes drastic when the film arrives in the converging section of the pore, that the film is mainly convex towards downstream, in the opposite to the orientation of quasi-static film profiles in the converging section. Consequently, there is a net increment of the film area after the film sweeps one pore volume.

For finite Ca numbers, the film profiles exhibit deviations from both those resulting purely from film advection and those seen in quasi-static states. Nonetheless, practical Ca values can often be quite small [24], implying that film profiles during motion might closely resemble those in quasi-static conditions. Within this context, the deformation of the film shape occurring within the diverging section of the pore could provide a more accurate characterization of the film's actual deformation behavior. Subsequent to the deformation initiated by film advection due to the imposed flow, immediate film relaxation takes place as the film configuration diverges from its equilibrium state.

To illustrate the perturbation effect induced by the imposed flow on the film's shape, we opt for the film configuration corresponding to the swept volume $V = 0.4$, as indicated in Fig. 14. This choice is motivated by the film's location within the diverging section of the pore, where the configuration closely approximates the equilibrium state while the accumulated strain on the film has reached a sufficient level. The film's position is denoted using a Cartesian coordinate system oxy as referenced in Fig. 14, and this system is employed in the computation of the film's mean curvature, it is important to clarify that the mean curvature of the film does not equate to the curvature of the 2D film profile.

In Fig. 15 (a), discrete vertex positions on the film profile are depicted, with the film profile being approximated through a polynomial fit. Since adding or subtracting a constant from the film's mean curvature does not affect the velocity field, we introduce the concept of the perturbation of the film's mean curvature, denoted as k' . This perturbation is defined as the actual value of the film's mean curvature k , subtracted by the area-weighted mean curvature of the film

$$k'(\mathbf{x}) = k(\mathbf{x}) - \int_{\text{film}} k dS(\mathbf{x}), \quad (26)$$

where $\mathbf{x}$ denotes the position of vertices on the film. The perturbation of the film's mean curvature is illustrated in Fig. 15 (b). To delve deeper, we proceed to perform a Fourier expansion for this perturbation with respect to the polar coordinate β . The outcome of this Fourier expansion is presented in Fig. 16. Notably, the dominant component of the mean curvature perturbation corresponds to the first-frequency component that is a cosine-wave perturbation characterized by a wavelength identical to the length of the film profile. The magnitude of the second most significant component is less than half of the dominant component's magnitude, and higher-frequency components of the perturbation exhibit considerably lower magnitudes. This observation implies that the relationship between film relaxation-induced velocity and the mean curvature of film in the film's motion is primarily influenced by the first-frequency mean curvature perturbation-induced flow.

4.2.3. Film motion under perturbed mean curvature

This section focuses on examining how the film responds to the mean curvature perturbation due to the imposed flow, where the capillary number Ca is set to unit. Given that the film's mean curvature serves as a linear source term in determining the flow field, as indicated in Eq. (16), the velocity arising from film relaxation can be expressed as a summation of velocities induced by

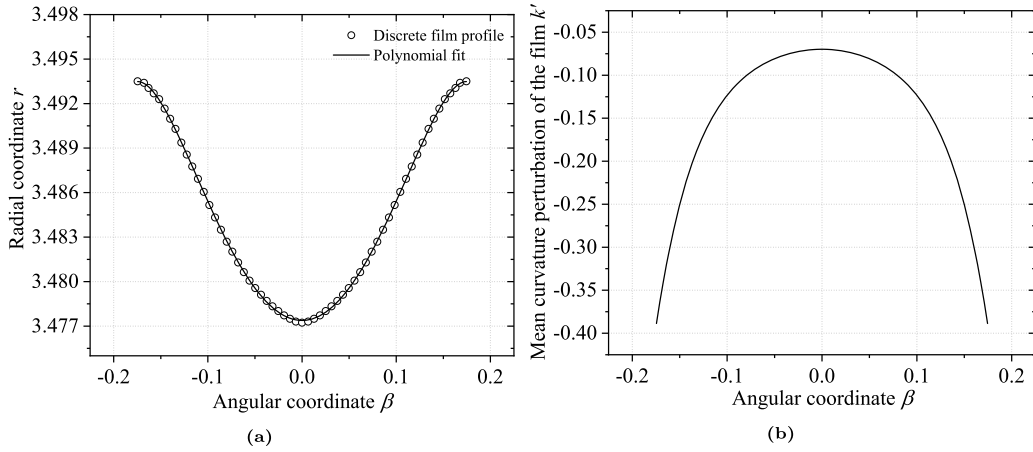


Fig. 15. The mean curvature perturbation of the film, computed with the polynomial fit of the film profile, the swept volume of which is 0.4, normalised using the pore volume. (a) The lamella profile is depicted utilizing the radial coordinate r in relation to the angular coordinate β . The original discrete data points corresponding to various vertices on the film are subject to polynomial fitting, enabling the computation of the mean curvature of film. Both radial and angular coordinates are established within a Cartesian coordinate system, where the y axis aligns with the pore axis, as indicated in Fig. 14. (b) The perturbation of the mean curvature of film, k' , is represented as a function of the angular coordinate β .

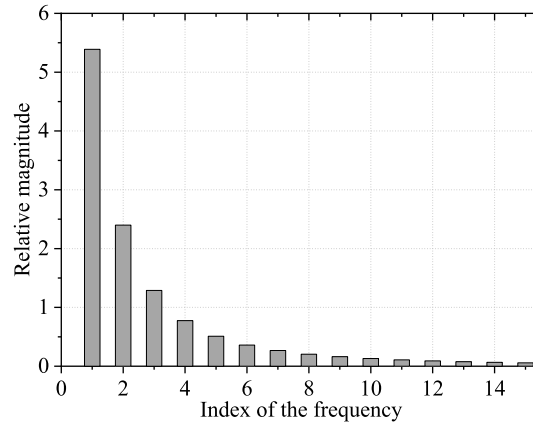


Fig. 16. The perturbation of the film's mean curvature is analyzed through Fourier expansion. The first-frequency component emerges as the prominent element within the curvature perturbation. This primary component aligns with a cosine-wave perturbation characterized by a wavelength identical to the length of the film profile.

individual frequency components derived from the Fourier expansion of the film's mean curvature perturbation. The mean curvature perturbation's frequency components, denoted as $\{k'_N\}_{N=1}^{\infty}$, can be expressed as

$$k'_N = A \cos\left(\frac{N\pi\beta}{\alpha}\right), \quad (27)$$

where A represents the relative magnitude of each film mean curvature perturbation component resulted by the Fourier expansion, and N denotes the wave number of the perturbation component. Notably, owing to the inherent linearity of the boundary integral method, the value of A solely influences the scaling of the resultant velocity magnitude, without altering the fundamental characteristics of the relationship between the film relaxation-induced velocity and the film's mean curvature. Subsequently, A is fixed to unit, and the velocity induced by film relaxation is explored, for mean curvature perturbation components corresponding to the first three frequencies namely $N = 1, 2$, and 3 .

These perturbation components of mean curvature are introduced to the film in an equilibrium state, with their horizontal positions characterized by the x coordinate of the position where the film intersects with the X -axis. Several distinct horizontal positions are tested, revealing a similar pattern in the relationship between the film velocity, particularly its normal component, and the mean curvature perturbation. As an example, consider a film that intersects with the X -axis at $x = -0.5L$. Fig. 17 shows the mean curvature perturbation imposed on the film surface and the normal component of the induced velocity, for $N = 1, 2$, and 3 , which are presented as functions of the radial coordinate calculated as $\sqrt{y^2 + z^2}$. Furthermore, Fig. 18 presents the normal velocity

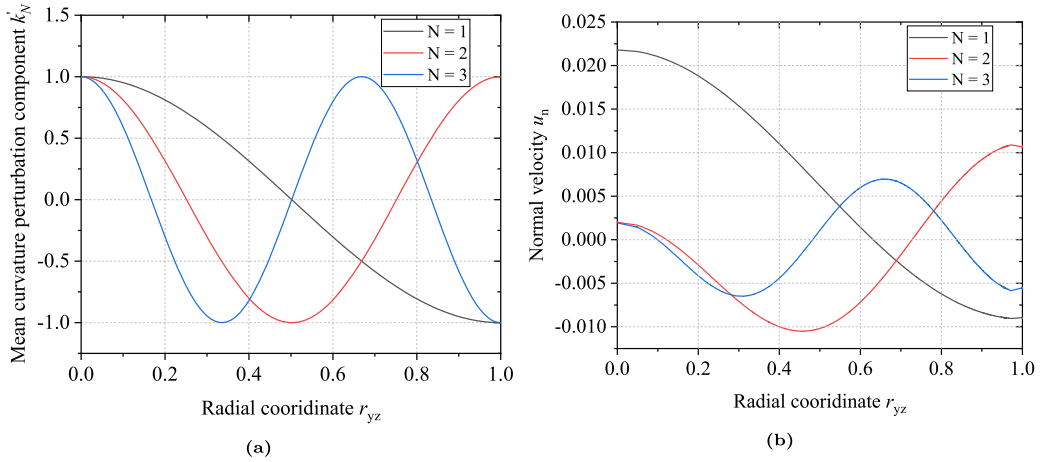


Fig. 17. The first-, second-, and third-frequency mean curvature perturbation-induced film normal velocity. (a) The mean curvature perturbation of the first three frequencies, at $N = 1, 2$, and 3 . (b) The normal velocity induced by each component of the mean curvature perturbation. In both (a) and (b), the radial coordinate is normalised by 0.4691 , the radius of the wall Plateau border.

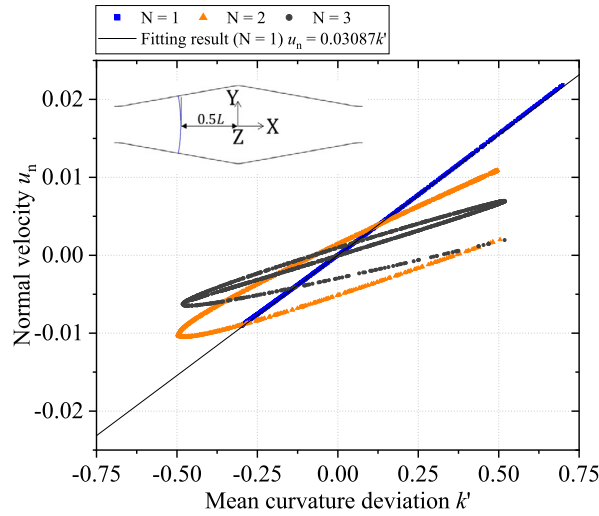


Fig. 18. The correlation between the normal velocity u_n and the mean curvature deviation k' is investigated. The curvature deviation k' is computed using Eq. (26) for perturbation components of the mean curvature, as defined in Eq. (27), for $N = 1, 2$, and 3 . The film is situated within the diverging section of the pore, with the X-coordinate of the wall Plateau border set to $-0.5L$.

component as functions of the mean curvature deviation. Here, the term “mean curvature deviation” refers to the result obtained by substituting Eq. (27) into Eq. (26).

Fig. 18 illustrates that the film motion induced by the first-frequency mean curvature perturbation results in a straightforward linear connection between the mean curvature deviation and the normal velocity. In contrast, the relationship between the mean curvature deviation and the normal velocity becomes more intricate at higher frequencies, as a single curvature value corresponds to multiple velocity values. This observation implies a lack of a direct relationship between the mean curvature and the normal velocity in general.

5. Summary and conclusions

Utilization of the boundary integral method to address the dynamic response of dry foams at low Reynolds numbers is explored in this study. The method holds applicability in contexts involving foams in bulk or in confined geometries. For the latter scenarios, the slip wall boundary condition is taken into account, resulting in the exclusion of the viscous forces exerted by walls on foams. To address the singularity challenges inherent in the resultant boundary integral equation for dry foams, two techniques are employed: the singularity subtraction method and the method of a nonsingular contour integral. These approaches are utilized to transform the boundary integral equations into a form devoid of singularities. Subsequently, in the numerical simulations, the resultant nonsingular boundary integral equation undergoes further discretization based on the mesh representation of the geometries, including both foam

films and walls. In cases involving pure foams in bulk, the discrete equation yields an explicit expression for calculating velocities at field points. Conversely, when walls are present, the discrete equation takes the form of a linear system that can be solved through GMRES with restart [51].

The method is assessed through two applications within this study. One application involves examining the motion of a foam composed of five bubbles subject to a simple shear flow. The findings reveal that film tension operates to counteract the deformation induced by the shear flow. This occurs because the film tension introduces a relaxation velocity aimed at minimizing the foam's surface energy. The singular movement of bulk Plateau borders significantly influences the overall foam motion by altering the mean curvature of films adjacent to these borders. Additionally, we highlight the importance of integrating the impact of bulk Plateau border motion on film dynamics in computations. This incorporation proves beneficial for maintaining conserved bubble volumes.

The second application examines a prototypical scenario for foam flow within confined geometries, specifically focusing on a single film traversing a diverging-converging channel. In this context, gas injection into the channel is simulated by deforming the walls. The motion of the film is deconstructed into two distinct components: one arising from the flow induced by wall deformation and the other stemming from film relaxation. The flow due to wall deformation causes deviations in the film's configuration from its equilibrium state characterized by a constant mean curvature. Employing Fourier expansion, we demonstrate that the nonuniform mean curvature predominantly exhibits a main perturbation component shaped as a cosine function. The wavelength of this perturbation, measured in terms of the polar coordinate, coincides with the diverging angle of the channel. Furthermore, our analysis reveals that the normal component of the film's relaxation velocity, induced by the primary mean curvature perturbation, exhibits a linear proportionality to the primary mean curvature perturbation itself. In contrast, higher-order mean curvature perturbations yield relation behaviors characterized by multi-value functions. The number of branches in these functions corresponds to the order of the mean curvature perturbation.

Our future work encompasses two main aspects. Firstly, we aim to extend our study to encompass scenarios akin to the configuration depicted in Fig. 1. This scenario entails the simultaneous presence of all possible geometric elements and finds relevance in practical foam flow problems, such as the flow of foam within a microfluidic porous medium model [11]. Foam films interact not only with each other but also with the confining walls. Secondly, the method for dry foams can be further improved by incorporating additional pertinent physical processes. Notably, these include accounting for the viscous force exerted on the wall Plateau borders [23,32] and the Marangoni effect arising from non-uniform surface tension [53,54]. These factors are integral to real-world foam flows.

CRediT authorship contribution statement

Haosen Zhang: Conceptualization, Data curation, Formal analysis, Methodology, Software, Visualization, Writing – original draft, Writing – review & editing. **Senyou An:** Formal analysis, Writing – review & editing. **Pablo R. Brito-Parada:** Conceptualization, Methodology, Project administration, Supervision, Writing – review & editing. **Stephen J. Neethling:** Conceptualization, Methodology, Supervision, Writing – review & editing. **Yanghua Wang:** Funding acquisition, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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