

VISUAL METHODS IN STELLAR PHOTOMETRY

VISUAL METHODS IN STELLAR PHOTOMETRY

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PREFACE

The main theme of this treatise is the errors, both systematic and accidental, which beset all attempts to measure the magnitudes of stars by visual methods. The sources of uncertainty in such measurements are so numerous and difficult to control that it is a tribute to the skill of the classical observers, such as Müller, Kempf, and Pickering, that their catalogues are as accurate as they are. The scope for investigation in such an apparently limited field as the errors of visual stellar photometry is large, and the author is conscious of serious gaps in the present monograph. An attempt has been made, however, to deal briefly with a few topics of special interest or importance.

The author's interests are primarily theoretical, and it may be thought that this has made itself manifest in the order of the sections, of which the first seven are of a theoretical nature, while description of the experimental work has been relegated to the last five sections. The reason for this arrangement, however, is really a desire to avoid making forward references, since the experiments can only be properly understood in the light of the preceding discussion. The same desire has dictated the order of the theoretical sections; in only one place has it been necessary to refer to a result derived in a subsequent section.

In studying errors of measurement, it is very necessary that we should have a clear idea of what we are trying to measure; no apology is

required, therefore, for the first section of this treatise. One of the chief unifying concepts of the work has been the recurrent contrast between the photometric conditions when the stars to be compared are imaged on the retina, and when a Maxwellian view is employed. An entirely different set of parameters is needed for specifying the conditions in each of the two cases. Discussion of these parameters is a surprisingly absorbing subject, and has led to a section of considerable length. Section 3 has been inserted to establish the necessary results in the theory of errors in the form needed for their application in the experimental sections.

After these preliminaries, we come to the main body of the work, in which the contrast between the modes of observation mentioned above is continually the guiding theme. In section 4, for example, it is used to illustrate a brief theory of the dependence of the observed magnitude on the spectral sensitivity, a development foreshadowed by the definition of magnitude in section 1. Only in sections 5 and 6 is this Leitmotiv inconspicuous, though the form of atmospheric extinction theory given here, with its emphasis on spectral energy distribution and spectral sensitivity, may be regarded as the natural complement of section 4. It occurred to the author that the Stiles-Crawford phenomenon might be an unsuspected source of error in visual stellar photometry; the consequences are developed mathematically in section 7; some experiments are now required to test the validity of the idea.

While the theory of errors enables us to calculate, on certain plausible assumptions, the distribution of accidental errors, it gives us no information as to the size of those errors under different conditions. This problem has been tackled, with a view to determining the optimum conditions, by means of a series of experiments based on the principle of varying one condition at a time, keeping the others constant.

The successively greater complications caused by the introduction of a difference of colour between the two half-fields, and by stellar scintillation, are not studied until the simpler problems associated with variation of the conditions in monochromatic, non-scintillant photometry have been dealt with. The work culminates in a qualitative and quantitative discussion of the relative merits of the two contrasted modes of observation, in which the results derived in the previous sections are drawn upon freely.

The work described in this treatise has all been carried out in the Technical Optics Section of Imperial College, with the aid of a Maintenance Allowance from the Department of Scientific and Industrial Research, which is gratefully acknowledged. The author wishes to express his heartfelt thanks to Professor W.D. Wright, for permission to convert part of his laboratory into a 'stellar photometry laboratory', and for his sympathetic guidance and encouragement.

Sincere thanks are offered to Professor L.C. Martin, for reading an early draft of a part of this thesis, and for making some suggestions which have enabled improvements to be made in the present version; and to Professor R.W.B. Pearse, of the Astrophysics Section, for similar kindnesses, and for eliciting some further comments from Professor R.O. Redman of Cambridge.

The author is grateful to Dr W.S. Stiles, of the National Physical Laboratory, for drawing his attention to the work done there, during the war, on the atmospheric transmission factor; and to Dr H. Buckley and Mr J.S. Preston for furnishing the author with unpublished information on this subject. Sincere thanks are due to Dr W.H. Steavenson, of Cambridge, for the benefit of his experience with the driving errors of equatorials, and for his sceptical comments on a summary of Section 7.

Gratitude to colleagues who gave up a lot of their time to act as observers, and whose initials appear in table 1, demands that their names be given in full: they are Miss J.E.M. Broughton, Dr E.G. Tadros, and Miss P.W. Trezona. Sincere thanks are due to Mlle F. Flamant, of the Institut d'Optique, for sending the author some unpublished data on the differential sensitivity for point sources.

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It is a pleasure, too, to record the stimulus which the work has derived from many telephone conversations with Mr B.R.M. Munden, and from his too rare visits to the laboratory. The care with which Mr E.R. Pearcey has read an earlier version of this thesis, and the numerous and often cogent criticisms with which he has subsequently regaled the author, have contributed very materially to any clarity which the presentation may possess, though naturally Mr Pearcey is not responsible for any of the views expressed.

James E. Gibbs.

South Kensington,
1952 September 3

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ADDENDA ET CORRIGENDA

- Page 38, line 4 from bottom, after 'systematic errors' insert 'in the observation of stellar magnitudes'
- Page 39, line 9 from top, after ' $\beta_e > 10^{-3}$ cd.cm⁻²' insert '(photopic vision)'
- Page 39, line 18 from top, after 'magnitude' insert '///'
- Page 58, after equation (48), for 'magnitude' read 'size'
- Page 76, line 25 from top, after ' ρ is uniformly distributed in' insert ' $(0, \rho_0)$ and δ is uniformly distributed in'
- Page 79, line 11 from top, for '12 and 13' read '10 and 11-12'
- Page 88, line 6 from bottom, after 'one half-field was' insert 'perceptibly brighter, to the position where it was'
- Page 95, line 10 from top, after 'rapidly' insert 'for'
- Page 133, line 2 from bottom, after 'With' insert 'the'
- Figure 4, caption, for 'Photometric probable error' read 'Punctiform photometric probable error'

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ABSTRACT

The concept of the visual stellar magnitude of a star is discussed and defined. An expression showing the dependence of the observed magnitude on the relative spectral visual sensitivity is derived. An approximate expression for the atmospheric extinction is obtained, showing its dependence on the colour-temperature of the star and on the relative spectral sensitivity; methods of measuring the extinction are reviewed. The problem of determining the illumination due to a star of zero magnitude is discussed.

Experimental studies of the precisions of the punctiform and superficial (Maxwellian view) modes of stellar photometry under various conditions are described. In the case of the punctiform mode, it is shown that the greatest precision is reached when the two stars compared are at a distance of the order of $5'$, the stellar magnitude is about 1.3, and the ambient luminance is not greater than that of the moonlight sky. It is shown, further, that, under the conditions mentioned, the precision of a photometric match is not adversely affected by the presence of a colour-temperature difference; and that the effect of scintillation of one of the two stars is to increase the probable error by a factor of the order of 1.5.

An investigation of the phenomenon of visual efficiency enhancement due to scintillation is reported, and the significance of the effect for visual stellar photometry is discussed. The relative merits of the punctiform and superficial modes are examined, attention being paid to relevant considerations such as stellar magnitude, colour-temperature difference, and the presence or absence of scintillation.

J.E.G.

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CONSPECTUS

The visual magnitude m of a star is defined and discussed (1). The punctiform and superficial (Maxwellian view) modes of observation are defined and compared. Photometric parameters, the ocular magnitude m_e and Maxwellian luminance $B_{e,b}$, suitable for characterizing the amount of light available in the respective modes, are introduced, and difficulties in defining m_e are discussed (2). The photometric probable error ρ is defined, and relevant results from the theory of errors are collected for reference (3).

An expression is derived for the dependence of the observed magnitude on the relative spectral visual sensitivity; it is applied to the Purkinje phenomenon and to the inverse Purkinje phenomenon (4). A form of atmospheric extinction theory showing the dependence of the extinction correction on the colour-temperature T of the star and on the spectral sensitivity is developed. Tabular extinction corrections

are deprecated, and reduction to outside the atmosphere is advocated. The dependence of atmospheric transmission factor on spectral type indicated by the theory is shown to agree with observation (5). The problem of measuring the illumination E_* due to a star of zero magnitude is reviewed, and the theory of a simple but correct method is given. Attention is drawn to the need for a proper determination of E_* , a need thrown into relief by Buckley and Preston's anomalous atmospheric transmission factors (6).

It is suggested that under certain conditions the Stiles-Crawford phenomenon may be an unsuspected source of error in visual stellar photometry. The order of size of the effect is estimated, and methods for eliminating it are discussed (7).

The non-scintillant, superficial photometric probable error $\epsilon_{N,s}$, for the author as subject, is measured in the laboratory. The values obtained are twice as great as those calculated from Dufay and Schwégler's formula expressing $\epsilon_{N,s}$ as a function of the luminance B_e and angular semi-diameter α of the field; possible explanations of the discrepancy are discussed (8). The author is again the subject in measurements of the non-scintillant, punctiform photometric probable error $\epsilon_{N,p}$ as a function of the angular separation σ , of m_e , and of the ambient luminance B_f . No dependence of $\epsilon_{N,p}$ on σ in (3', 20') is detected. $\epsilon_{N,p}$ is a cubic function of m_e , with a minimum of 0.030 at $m_e = 1.3$. The probable error is unaffected while $B_f < 10^{-6}$ cd. cm.⁻²; for $m_e > 0$, it increases thereafter with B_f ; for $m_e < 0$, $\epsilon_{N,p}$ is slightly depressed around $B_f = 10^{-2}$ cd. cm.⁻², but increases rapidly for greater values of B_f (9).

An approximate expression is derived for the depression of the colour-temperature T of a star due to the intervention of the atmosphere. The dependence of the photometric

precision on the difference of $1/7$ between the two half-fields in heterochromatic stellar photometry is discussed. An experiment using a Chance OB.9 glass wedge to vary τ shows that, for $m_e = 1.3$, $r_{N,P}$ is independent of $\Delta(1/7)(10)$.

The recent recalculation of interest in stellar scintillation is noted. It is suggested that the effects of twinkling in visual stellar photometry are an increased photometric probable error, and a systematic error called visual efficiency enhancement μ , which is defined as the difference of the magnitudes of a comparison star required to match a scintillating star and a non-scintillating star giving the same mean illumination. The scintillant and non-scintillant probable errors $r_{T,P}$, $r_{N,P}$ are measured by alternate sets of naked-eye observations on Sirius, brought into the field by means of a heliostat, and on an artificial non-twinkling star; it is found that $r_{T,P} = 1.45 r_{N,P}$ (11).

An artificial twinkling star is produced by rotating an opaque disk containing a transparent zone of irregular width in front of an illuminated aperture. The apparatus is used to measure μ_P , μ_S , and, as functions of magnitude level, $r_{N,P}$, $r_{N,S}$, $r_{T,P}$, $r_{T,S}$. The order of the observations is randomized, and bias is eliminated by using an automatic device for recording the readings. The results indicate significant visual efficiency enhancements $\mu_P = -0^m.075$, $\mu_S = -0^m.131$. Anomalies in the functions $r_{N,P}$, $r_{N,S}$, $r_{T,P}$, $r_{T,S}$ indicated by the observations are discussed, and definitive empirical formulae are proposed, enabling homogeneous intercomparison to be effected. It is concluded that, for $m_e > -2.5$, the precision of the punctiform mode exceeds that of the superficial mode; several other preprocessing features of the punctiform mode are enumerated (12).

The numerical references in the above are to the summaries at the beginnings of the twelve sections of the text.

VISUAL METHODS IN STELLAR PHOTOMETRY

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1. DEFINITION OF THE VISUAL STELLAR MAGNITUDE OF OF A STAR

Summary. The visual magnitude of a star is defined in the usual way:

$$m = -2.5 \log (E/E_*), \quad (A)$$

where E is the illumination due to the star, and E_* that due to a star of zero magnitude. Instead of the usual convention (reduction to the zenith), it is assumed that the effects of atmospheric extinction are completely removed from E, E_* in (A). The need for special methods in stellar photometry arises from the very small size of E_* , of the order of 2×10^{-10} lm.cm.⁻² Attention is drawn to the fact that the relative spectral visual sensitivity V_λ enters into the definition of the illumination E , and a function convenient for computation is adopted, viz.

$$V_\lambda = A\lambda^{-p} e^{-q/\lambda}. \quad (B)$$

It is noted that E_* is not a conventionally defined quantity, but must be determined by observation. The conventional nature of the visual magnitude scale adopted in practice is briefly discussed.

Because of the very low illuminations produced by the stars at the earth, the ordinary photometric units are not suitable for use in astronomy. Moreover, in view of the wide range of values taken by those illuminations for different stars, it is convenient to use a logarithmic scale to express them. It is convenient to express the illumination due to each star in terms of that due to some standard star, of which the illumination is taken as unity. These considerations explain the adoption of the definition of the visual magnitude m of a star by the equation

$$m = -2.5 \log\left(\frac{E}{E_*}\right), \quad (1)$$

where E is the illumination at the earth due to that star, and E_* is the illumination due to the standard star. ($\log$, $\ln$ denote logarithms to base 10, e , respectively).

In order to secure freedom from ambiguity in subsequent discussions, it will be necessary to specify at the outset that the illuminations E , E_* are to be freed from the effects of absorption and scattering of the light in the atmosphere. Since the thickness of the earth's atmosphere is infinitesimal in comparison with the distance of the nearest star, we may speak, informally, of E as the illumination, on a plane normal to the rays and situated just

outside the atmosphere, due to a star of magnitude m . This is contrary to the widely-accepted convention that E should be regarded as the illumination which would be produced, on a horizontal plane at the surface of the earth, if the star were in the zenith, and the atmosphere were as clear as it ever is. When it is necessary to refer to this convention, we shall use the primed symbols m' , E' , E'_* .

From the definition (1), we see that the magnitude of the hypothetical standard star is zero. Thus E_* may be described as the normal illumination outside the atmosphere due to a star of zero magnitude; and E'_* as the clear-sky zenithal illumination at the surface of the earth due to a star of zero magnitude. Since the zenithal transmission factor of the atmosphere is of the order of unity, E_* and E'_* are of the same order of size.

On the basis of a critical survey of the available data, Russell (1916) adopts $m' = -14.18$ as the most probable value of the magnitude of one metre-candle ($E' = 10^{-4}$ lm.cm⁻²). Substituting these numerical values in the equation

$$m' = -2.5 \log \left(\frac{E'}{E'_*} \right) \quad (2)$$

leads to

$$E'_* = 2.13 \times 10^{-10} \text{ lm.cm}^{-2} \quad (3)$$

This gives one an idea of the very low illuminations produced by the stars at the earth. The luminance of a surface of unit luminance factor illuminated with E'_* would be only

$$B'_* = E'_* / \pi = 6.78 \times 10^{-11} \text{ cd.cm}^{-2} \quad (3a)$$

this may be compared with the optimum luminance for optical bench photometry, which, according to Lowry (1931), is about 10^{-2} cd.cm.^{*2}, and with the threshold luminance (Dufay and Schwégler 1930) of about 10^{-10} cd. cm.^{*2}

It is important to note that m is a decreasing function of E , so that, the fainter the star, the greater the algebraic value of m . It will avoid much confusion if it is borne in mind that, when we speak of the magnitude of star A being greater than that of star B, we mean that star A is fainter than star B. The historical reason for this apparent inversion of common sense is to be found in the fact that the present exact definition of magnitude can trace back its lineage to the arbitrary division of the stars visible to the naked eye into six classes by Hipparchus and Ptolemy. The most conspicuous of these stars were said to be of the first magnitude, and the faintest, of the sixth magnitude. (There is an analogy here with the small quantities of the first, second, ... order encountered in analysis). The scale of magnitude was, even from these early beginnings, essentially logarithmic, thanks to the Weber-Fechner law. The factor 2.5 is inserted in (1) in order to make the magnitudes thus precisely defined agree approximately with those developed on the basis of Ptolemy's work. (Cf Jones 1921, 1934 pp 277-285; Russell et al. 1926, p.611-617).

We shall strive to avoid using the term brightness, or its correlatives bright, brighter, etc., in connexion with the photometric attribute of a star measurable by a terrestrial observer. This attribute can only be the areal density of luminous flux, or the normal illumination, as we have called it. The abuse referred to is deplorably widespread in the astronomical literature (cf. Fabry 1925, Brydon 1940-41).

In order that the definition (1) may have a precise meaning, it is necessary to specify what is meant by illumination. Let the spectral radiant flux density due to the star measured outside the earth's atmosphere be ϕ_λ ; then the spectral photopic illumination is $K_0 \phi_\lambda V_\lambda$, where K_0 is the photopic coefficient of maximum visibility (Le Grand c. 1949), and V_λ is the photopic relative spectral luminance. We define E to be simply

$$E = K_0 \int_0^\infty \phi_\lambda V_\lambda d\lambda. \quad (4)$$

The function V_λ has been defined by international agreement (Congrès International de l'Éclairage 1924, Bureau International des Poids et Mesures 1933) for $\lambda = 0.39$ (0.01) 0.78μ , but no standard method for interpolation has been given, so this definition is incomplete. For this reason, and also because it will facilitate certain calculations, the formula proposed by Moon and Spencer (1943) will be adopted here, viz.

$$V_\lambda = A \lambda^{-p} e^{-q/\lambda}, \quad (5)$$

where λ is in microns, and

$$\begin{aligned} A &= 10^{32.4107} \mu^{-p} \\ p &= 182.1905 \\ q &= 100.937 \mu. \end{aligned} \quad (6)$$

The definition of the visual stellar magnitude of a star given above is complete and logically satisfying, but it has never been realized in practice. There are good reasons for this.

For one thing, the illumination E_* due to a star of zero magnitude has not been specified in terms of ordinary photometric units such as

lumens per square centimetre. In practice, the zero of the visual magnitude scale has been fixed arbitrarily by specifying that the magnitudes of certain selected stars shall have given numerical values. Thus E_* is not a conventionally defined quantity, but a quantity subject to experimental determination. In principle, it can be measured by observing the illumination E due to a star of magnitude m , substituting for E and m in (1), and solving for E_* . Even if this operation could be carried out with complete accuracy, it would not yield a unique determination of E_* ; the value obtained would depend on the star selected, since the differences between the conventionally assigned m -values depend on observational data which are not free from error. The measurement of E_* will be further discussed in section 6.

Another reason why our definition of visual stellar magnitude cannot be realized in practice is inherent in the methods of visual stellar photometry. It would lead us too far afield to discuss the innumerable types of visual photometers which have been designed and constructed in order to measure the magnitudes of the stars. The classical instruments are fully described by Müller (1897) in his unique monograph on stellar photometry, and more briefly by Ambronn (1899); for more recent developments, reference must be made to the periodical literature (e.g. Maggini 1918, Danjon 1928, Dufay 1928, 1929, 1930, 1947, Steavenson 1932, Tremblot 1934, 1935, Brydon 1941, Strohmeier 1951). None of these instruments is intended to enable one to measure the illumination E due to a star. Directly or indirectly, the function of a stellar photometer is to effect a comparison of the magnitude of one star with that of another. This comparison is effected in terms of the magnitude scale of the instrument-plus-observer, a scale which, for one reason and another, must be regarded as a distortion of the true scale

defined by (1). Corrections are subsequently applied to the results to eliminate the scale distortion referred to, and to locate the zero of the scale correctly. Even when all foreseeable corrections have been applied, however, it is found that the magnitude scale varies from one combination of instrument and observer to another. Thus it becomes necessary to adopt one particular magnitude scale as standard, and to reduce all observations to this scale by the application of empirical corrections. These corrections are determined by measuring, in the system under investigation, the magnitudes of the stars whose magnitudes in the standard system are now regarded as conventionally assigned quantities.

Thus not only the zero of the magnitude scale, but the scale itself, must be regarded as conventional; and the defining equation (1) plays little or no part in the day-to-day operations of stellar photometry. More extended discussions of the conventional nature of the scale of stellar magnitudes and related topics are given by Tucker (1918), Seares (1938), Greaves (1945), Stoy and Redman (1946), Weaver (1946), and other authors.

* * * * *

2. THE PUNCTIFORM AND SUPERFICIAL MODES OF OBSERVATION

Summary. The punctiform mode of observation, in which a star is imaged on the retina, and the superficial mode, in which the light from a star is spread over a considerable area of the retina, are defined and discussed. In the punctiform mode, the amount of light available from a telescopically observed star may be characterized by the ocular magnitude m , of the star, which is defined as the magnitude of a star which, observed with the naked eye, would produce the same retinal stimulus. If the exit pupil be larger than the pupil of the eye, $m_e = m - 5 \log M$, where M is the magnification; if the exit pupil be smaller,

$$m_e = m - 5 \log (a_o/a_e), \quad (A)$$

where a_o, a_e are the respective radii of the object-glass, eye pupil. In the latter case, there are difficulties in the definition resulting from the phenomena of Le Grand-Geblewicz and Stiles-Crawford; these are discussed at length.

The effective luminance of the field in the superficial mode is determined by the values of m and a_o , and by the angular diameter 2α of the field. When m and a_o are given, however, an empirical formula of Dufay and Schwégler shows that there is a unique value of 2α , about 4° , for which the photometric precision is

greatest. The corresponding value of the effective luminance represents the most efficient use of the light available from the star, and it is proposed to distinguish it by the name of Maxwellian Luminance ($B_{e,b}$). If we assume $r_e = 0.3$ cm., then

$$m_e = m - 5 \log \epsilon_0 - 2.61, \quad (B)$$

$$\log B_{e,b} = 0.4 (m - 5 \log \epsilon_0) - 6.22. \quad (C)$$

Of the various methods which have been used for measuring stellar magnitudes, we shall regard those depending on extinction, i.e. the measured reduction of the light flux until the star disappears, as outside our scope, in view of their low accuracy. This low accuracy is due to the excessive variability of the threshold and the very fatiguing nature of the observations. The present treatise will be concerned with those methods in which two light sources are adjusted to visual equality (photometric match). It will be supposed that one of the sources produces a constant retinal illumination, while that produced by the other is under the control, through a suitable photometric device, of the observer.

The distribution on the retina of the light from the two sources being compared may take one or other of two decisively different forms. The stars themselves may be imaged on the retina, in which case the light from each is concentrated on to a minute, isolated area of the retina: these circumstances will be characterized as punctiform observation. Or the light from each star may be spread out, by means of a suitable optical system outside the eye, so as to cover a considerable area of the

retina (several square degrees of the visual field): this case will be termed the superficial mode of observation.

It will be one of the main objects of the present investigation to establish the respective merits of the two modes of observation, when due care is taken to ensure that each is employed under its peculiar optimum conditions. Discussion of this central issue is reserved for the final section, section 12, of this treatise. Meanwhile it will facilitate the intervening discussion if we make a few further remarks about the punctiform and superficial modes, and introduce one or two definitions in connexion therewith.

It appears from the work of Le Grand and Geblewicz (1939), that, owing to diffraction and aberration in the eye, the retinal images of light-sources of all angular diameters up to $1'$ are indistinguishable, and that, as a consequence, photometric matches between such sources depend only on the illuminations which they produce in the plane of the eye-pupil, and not on their angular diameters, provided the pupil be uniformly illuminated over its entire area. It will therefore be appropriate to restrict the category of punctiform observations to the comparison of objects whose angular diameters are less than $1'$. Experiments will be described in which artificial stars are employed; in each case care has been taken not to exceed an angular diameter of about $1'$. The angular subtense of the image of a star (Airy disk) seen in a telescope of semi-aperture a_0 and magnification M is roughly $1.22 \times \lambda M a_0$, where λ is the wavelength of the light. If $\lambda = 5.5 \times 10^{-5}$ cm., the diameter of the image will be $1'$ when $M \doteq 4a_0$. Much higher magnifications than this, of the order of $100a_0$, are commonly employed. Thus telescopic observations, as opposed to those performed with the naked eye, are seldom punctiform in the sense of our

definition. Reserve will therefore be necessary in applying our results for the punctiform mode of observation to telescopic observations.

The inequalities in the sensitivity of the retina from one region to another are notorious (see, e.g., Roberts 1897, Osthoff 1917, Mascart 1927, Sharples 1947); it is therefore appropriate that we should emphasize the desirability of using the substitution method in visual stellar photometry. In the simplest possible terms, the recommended procedure is as follows. If the magnitude difference between stars A, B is required, then two photometric matches are to be established in turn - first, between the light from A falling on a region 1 of the retina and that from a third source C falling on a region 2; and, secondly, between light from B falling on 1, and that from C falling on 2. Strictly speaking, in order to ensure faithful reproduction of the conditions from one match to the other, the illumination of region 2 should be kept constant; the illumination of 1 would then be the same in both matches, though not necessarily equal to that of 2, unless 1 and 2 happened to have the same sensitivities, a condition which is by no means necessary to the success of the method. It is, however, convenient from both theoretical and practical points of view to regard C as the source which gives a retinal illumination under the control of the observer, so that the illumination on 2 is changed to suit that on 1 in each of the two matches.

The auxiliary source C may be either a real star or an artificial star. In the punctiform case, the use of a real star has the advantage that the appearances of the two objects compared in each match will be closely similar, even as regards the presence of scintillation. The disadvantage, however, is the need for a complicated optical system to enable stars in different parts of the sky to be juxtaposed in

the field of view; in this respect the use of an artificial star is much more convenient. In polarizing photometers, employment of a real comparison star may cause difficulties associated with the fact that starlight may be as much as 0.12 plane polarized (Gold 1952). While it is not difficult to make an artificial star to resemble a real star seen under low magnification or with the naked eye, the difficulty increases when a high magnification is employed. Moreover, an artificial star does not twinkle, except at the expense of elaboration of the equipment which would probably not be justified by the resulting improvement, if any, in the results. Most of the above remarks apply, mutatis mutandis, to the superficial mode.

One of the chief things on which the precision of a photometric match depends is the amount of light available, which, in the laboratory, may be either too great or too small, the best results being obtained at moderate luminance levels. In stellar photometry it might be expected that the amount of light would never be too great. In the superficial mode this is probably true, but in the punctiform mode, thanks to the concentration of the light on to a very small area of the retina, it is not. (Stevenson 1928, p.10). We shall now examine what parameters are most suitable to characterize the amount of light in each of the two modes, punctiform and superficial.

In the punctiform mode, as we have seen, the retinal stimulus depends only on the illumination E_e in the plane of the eye-pupil, as long as E_e is uniform over an area sufficient to include the whole of the pupil. In the case of naked-eye observation, the condition just mentioned is abundantly satisfied; moreover, if atmospheric extinction be neglected, the illumination in question is equal to the E of 1(1) (i.e. equation 1 of section 1). The stellar magnitude m defined

by 1 (1) is a convenient measure of E , and, therefore, in the naked-eye case, of the retinal stimulus.

In the case of telescopic observation, it is convenient to measure the retinal stimulus by a quantity m_e defined as the magnitude of a star which would produce the same retinal stimulus in naked-eye observation. We shall call m_e the ocular magnitude of the star observed with the given telescope; the ocular magnitude of a star observed with the unaided eye is obviously equal to m .

The distribution of illumination in the retinal image of a star depends on the aperture $2a_o$ and the magnification M of the telescope.

If M be considerably less than a_o/a_e , where a_e is the semi-aperture of the eye pupil, then the diameter $2a_o/M$ of the exit pupil of the instrument will be appreciably greater than that of the eye pupil. The latter will therefore be completely filled by a sensibly uniform illumination, which, if we neglect light losses in the telescope, may be taken as $M^2 E$. Resolving power will be determined by diffraction at the pupil of the eye; the image will therefore be strictly punctiform, and the retinal stimulus will be identical with that due to a star of magnitude

$$m_e = -2.5 \log \left(\frac{M^2 E}{E_*} \right) = m - 5 \log M. \quad (1)$$

This is the ocular magnitude of a star of magnitude m observed with a telescope of magnification $M \ll a_o/a_e$.

If M is nearly equal to a_o/a_e , so that the exit pupil is of nearly the same size as the

eye pupil, then the flux of light entering the eye will depend critically on the relative positions of the two pupils, and the ocular magnitude has no unique value.

The case of practical interest arises when $M \gg a_o/a_e$, so that the diameter of the exit pupil is much smaller than that of the eye pupil. Under these conditions, resolving power is limited by diffraction at the object-glass, the angular diameter of the spurious disk being about $1.22 \lambda M/a_o$, or considerably greater than $1'$. Thus the image is not strictly punctiform - it may be termed pseudopunctiform -, and the retinal stimulus cannot be identical with that due to naked-eye observation of a star of any magnitude m . The ocular magnitude may still be defined, however, if we take m_e as the magnitude of a star from which an equal luminous flux would enter the eye if the eye were unobstructed by any optical system. Thus, for $M \gg a_o/a_e$,

$$\begin{aligned} m_e &= -2.5 \log \left[\left(\frac{a_o}{a_e} \right)^2 \frac{E}{E_*} \right] \\ &= m - 5 \log \left(\frac{a_o}{a_e} \right). \end{aligned} \quad (1a)$$

The value of a_e depends on many things; we cannot do better than take the value adopted by Dufay, viz.

$$a_e = 0.3 \text{ cm.} \quad (2)$$

Equation (1a) then takes the numerical form

$$m_e = m - 5 \log a_o - 2.61. \quad (3)$$

For example, in the case of a star of magnitude 5 observed with a telescope of aperture $2a_c = 32$ cm.,

$$\left. \begin{aligned} m - 5 \log a_c &= -1.02, \\ m_e &= -3.63; \end{aligned} \right\} (3a)$$

i.e. a star just visible to the naked eye appears approximately as intense in such an instrument as the planet Venus does to the naked eye.

In case $M \gg a_c/a_e$, the telescopically observed star would not, in general, match photometrically a star of equal ocular magnitude observed with the unaided eye, even if one could overcome the difficulty of comparing a punctiform image with a pseudopunctiform image. According to Le Grand and Geblewicz (1939), ceteris paribus, more luminous flux, by a factor ξ_{L-G} , say, must enter the eye from a pseudopunctiform source than from a punctiform source if the two are to appear of equal intensity; so a pseudopunctiform image would appear fainter than a punctiform image of the same ocular magnitude; this may be termed the Le Grand-Geblewicz (L-G) phenomenon.

When $m - 5 \log a_c < -3$, however, the situation is complicated by the action of the Stiles-Crawford (S-C) phenomenon. As we shall see in Section 7, this results in different mean luminous efficiencies for the two beams; the illumination E_e at the eye due to a star observed with the naked eye which would match the telescopically observed star may be calculated, ignoring the L-G effect, from the equation

$$\bar{\eta} S E_e = \bar{\eta}' S' E_e', \quad (3b)$$

where $\bar{\eta}$ is the mean luminous efficiency averaged over the area S of the eye pupil, and $\bar{\eta}'$ that

averaged over the area S' of the exit pupil of the telescope, at which the illumination is E_e' . From 7(2), if we write ρ_e for the radius a_e of the eye pupil,

$$\begin{aligned}\bar{\eta}S &= 2\pi \int_0^{\rho_e} \eta(\rho) \rho d\rho \\ &= 2\pi \int_0^{\rho_e} \exp[-(\rho \ln 10) \rho^2] \rho d\rho \\ &= \frac{\pi}{\rho \ln 10} \{1 - \exp[-(\rho \ln 10) \rho_e^2]\}; \quad (3d)\end{aligned}$$

and

$$\bar{\eta}'S' = \frac{\pi}{\rho \ln 10} \{1 - \exp[-(\rho \ln 10) \rho_e'^2]\}, \quad (3e)$$

where ρ_e' is the radius of the exit pupil. Also

$$E_e' = M^2 E = \left(\frac{a_e}{\rho_e'}\right)^2 E. \quad (3f)$$

Therefore

$$\begin{aligned}E_e &= \frac{\bar{\eta}'S'}{\bar{\eta}S} E_e' \\ &= \frac{1 - \exp[-(\rho \ln 10) \rho_e'^2]}{1 - \exp[-(\rho \ln 10) \rho_e^2]} \left(\frac{a_e}{\rho_e'}\right)^2 E \\ &= \frac{1 - \exp[-(\rho \ln 10) \rho_e'^2]}{1 - \exp[-(\rho \ln 10) a_e^2]} \left(\frac{a_e}{\rho_e'}\right)^2 \left(\frac{a_e}{a_e}\right)^2 E. \quad (3g)\end{aligned}$$

The magnitude of this star is

$$\begin{aligned}
m_e' &= -2.5 \log \left(\frac{E_e}{E_*} \right) \\
&= -2.5 \log \left[\left(\frac{a_o}{a_e} \right)^2 \frac{E}{E_*} \right] - 2.5 \log \left\{ \frac{1 - \exp[-(\rho \ln 10) \rho_o'^2]}{1 - \exp[-(\rho \ln 10) a_e^2]} \left(\frac{a_e}{\rho_o'} \right)^2 \right\} \\
&= m_e - 2.5 \log \left\{ \frac{1 - \exp[-(\rho \ln 10) (a_o/M)^2]}{1 - \exp[-(\rho \ln 10) a_e^2]} \left(\frac{a_e M}{a_o} \right)^2 \right\} \quad (3h)
\end{aligned}$$

This differs from the ocular magnitude m_e by a term whose modulus is an increasing function of M , but which tends to a finite limit when $M \rightarrow \infty$; thus

$$\begin{aligned}
(m_e' - m_e)_{S-C} &= -2.5 \log \left\{ \left(\frac{a_e M}{a_o} \right)^2 \frac{[(\rho \ln 10) (a_o/M)^2] [1 - \frac{1}{2}(\rho \ln 10) (a_o/M)^2 + \dots]}{[(\rho \ln 10) a_e^2] [1 - \frac{1}{2}(\rho \ln 10) a_e^2 + \dots]} \right\}, \\
\lim_{M \rightarrow \infty} (m_e' - m_e) &= -2.5 \log \left\{ \frac{(\rho \ln 10) a_e^2}{1 - \exp[-(\rho \ln 10) a_e^2]} \right\} \quad (3k)
\end{aligned}$$

This is negative, as we should expect from the fact that the mean luminous efficiency of the whole pupil is less than that of the centre of luminous efficiency. For $\rho = 5 \text{ cm.}^{-2}$, $a_e = 0.3 \text{ cm.}$,

$$\lim_{M \rightarrow \infty} (m_e' - m_e) = -0.51. \quad (3l)$$

Thus the maximum photometric mis-match, due to S-C phenomenon, between stars of equal ocular magnitudes is about half a magnitude. This is largely offset by the photometric mis-match due to L-G phenomenon, which is also an increasing function of M , but of opposite sign.

This may be made clear by a numerical example.

A star of magnitude $m = 6.5$ observed with a telescope of aperture $2a_0 = 32$ cm. has an ocular magnitude given by (3) as $m_e = -2.1$. The illumination due to a star of magnitude -2.1 is given by 1 (1), with $E_* = E'_* = 2.13 \times 10^{-10}$ lm. cm.⁻², as $E = 1.5 \times 10^{-9}$ lm. cm.⁻². The angular diameter of the Airy disk, for a magnification $M = 230 \gg a_0/a_e \doteq 50$, is $1.22 \lambda M/a_0 = 1.0 \times 10^{-3} = 200''$. For a pseudopunctiform source of this size and giving an illumination $E = 1.5 \times 10^{-9}$ lm. cm.⁻², LeGrand and Geblewicz (1939) find that $\zeta_{L-E} \doteq 1.4$. On the other hand, from (3h),

$$(m'_e - m_e)_{s-g} = -0.48.$$

The magnitude of the star observed with the unaided eye which would match the telescopically observed star of ocular magnitude $m_e = -2.1$ is therefore

$$\begin{aligned} m'_e &= m_e + 2.5 \log \zeta_{L-E} + (m'_e - m_e)_{s-c} \\ &= -2.2, \end{aligned} \quad (3m)$$

which differs by only $-0^m.1$ from m_e .

Equation 1 (3a) shows that the luminance of a surface of unit luminance factor illuminated by a star of zero magnitude is less than the threshold luminance of 10^{-10} cd. cm.⁻². The luminance could be increased by a factor M^2 by means of a telescope of magnification M , but the luminous area would be only M^{-2} of that of the object-glass of the telescope. Little progress can be made along these lines, because the use of a diffusing surface causes most of the light to be scattered away from the eye of the observer

(cf. Fabry and Buisson 1922). The only type of optical system which is sufficiently economical of light to enable us to realize the conditions which have been characterized as the superficial mode of observation is thus the Maxwellian view (cf., e.g., Walsh 1926). While the optical system need not be as simple as this, there is no loss in generality in supposing that the optical system consists of a single lens, which may be regarded as the object-glass of a telescope of semi-aperture α_e , and focal length f , with the eye placed at the focus.

It is convenient to calculate the effective luminance (Dufay 1947) of the lens under these conditions. The effective luminance E_e is defined as the luminance of a diffusing surface which would produce an equal retinal illumination. Thus the light flux entering the eye from unit area of the lens is equal to the light flux entering the eye from unit area of a diffusing surface of luminance B_e placed in the plane of the lens. Expressing this in symbols, and assuming no loss of light in the lens, we have at once $E = (\pi \alpha_e^2 / f^2) B_e$, or

$$B_e = \frac{f^2 E}{\pi \alpha_e^2} \quad (4)$$

While the retinal illuminations are equal in the two cases, that produced by the diffusing surface will be visually less effective, on account of the Stiles-Crawford effect, which results in the effective pupil aperture being less than the geometrical aperture (Le Grand c. 1949, p. 107); for simplicity, we shall ignore this complication.

The precision of a superficial photometric match is an increasing function both of the luminance B_e and of the angular semi-diameter

α of the field. By writing $f' = a_0/\alpha$, we can throw (4) into the form

$$B_e \alpha^2 = \frac{a_0^2 E}{\pi a_2^2} \quad (5)$$

The right-hand member of this equation is fixed by the magnitude of the star and the aperture of the telescope used to observe it. Thus $B_e \alpha^2 = \text{cons.}$ In order to obtain increased accuracy, we might attempt to increase either B_e or α ; the relation we have just derived indicates, however, that an increase in one of these two parameters is accompanied by a decrease in the other, with the result that the precision would eventually begin to fall off again.

These considerations may be rendered more precise by reference to Dufay and Schwégler's (1930) empirical formula for the precision of photometric matching of two juxtaposed semi-circular half-fields. The precision ρ is defined by Dufay and Schwégler as the reciprocal of the root-mean-square fractional deviation from the mean of $n = 10$ readings of the luminance of the comparison half-field, i.e.

$$\rho = \left\{ \frac{1}{n} \left[\left(\frac{B_e - \bar{B}_e}{\bar{B}_e} \right)^2 \right] \right\}^{-1/2}, \quad (5a)$$

where the brackets $[\]$ denote summation over n independent observations. The empirical formula for ρ as a function of B_e and α is

$$\rho = 16.4 (10 + \log B_e) - \frac{1}{2\alpha}$$

$$(0.025 < 2\alpha < 0.100, B_e < 10^{-6} \text{ cd.cm.}^{-2}, \rho > 10). \quad (6)$$

By introducing the condition (5), eliminating B_e , differentiating the resulting equation with

respect to α , and setting $dp/d\alpha$ equal to zero, it can be shown (Dufay 1947) that the value of α which makes p a maximum is

$$\alpha_b = 0''.035 \pm 2^\circ. \quad (7)$$

The important point is that this optimum angular semi-diameter of field is independent of the star magnitude and telescope aperture.

In comparing the accuracies obtainable with different apertures on different stars, we may therefore suppose that this optimum value of α is invariably adopted. The precision is then a function of B_e alone, while B_e is itself determined uniquely in terms of a_o , E ; for putting $\alpha = \alpha_b$ in (5) gives

$$B_{e,b} = \frac{a_o^2 E}{\pi \alpha_b^2 a_e^2}. \quad (8)$$

This optimum value of the effective luminance plays an important part in considerations of the precision of superficial photometry: it is the link between laboratory investigations, in which luminance is the parameter of interest, and astronomical studies, where the aperture $2a_o$ of the telescope available, and the illumination provided by the star are the important factors. It will therefore be convenient to give $B_{e,b}$ a special name: we shall call $B_{e,b}$ the Maxwellian luminance of a star of magnitude m observed superficially with a telescope of semi-aperture a_o .

The Maxwellian luminance is closely related to the ocular magnitude, which is given by (1a) as

$$m_e = m - 5 \log (a_o/a_e). \quad (9)$$

Using 1 (1), we can rewrite this

$$0.4 m_e = -\log\left(\frac{E}{E_*}\right) - 2 \log\left(\frac{a_o}{a_e}\right). \quad (9a)$$

From (8),

$$\log B_{e,b} = \log\left(\frac{E}{\pi \alpha_b^2}\right) + 2 \log\left(\frac{a_o}{a_e}\right). \quad (10)$$

and, by addition,

$$\log B_{e,b} = \log\left(\frac{E_*}{\pi \alpha_b^2}\right) - 0.4 m_e. \quad (10a)$$

If E_* be replaced by its approximate numerical equivalent E'_* , substitution from (7) and 1 (3) leads to

$$\log B_{e,b} = -0.4 m_e - 7.26. \quad (11)$$

Using (3) above, we can write this

$$\log B_{e,b} = -0.4 m + 2 \log a_o - 6.22; \quad (12)$$

$$\text{conversely, } m - 5 \log a_o = -2.5 \log B_{e,b} - 15.55. \quad (12a)$$

To return to our example of a star of $m = 5$, observed with a telescope of aperture 32 cm., for which we have already calculated (equation 3a) an ocular magnitude of -3.63 , we note from (11) or (12) that the Maxwellian luminance of this star-telescope combination is given by

$$\left. \begin{aligned} \log B_{e,b} &= -5.81, \\ B_{e,b} &= 1.5 \times 10^{-6} \text{ cd. cm.}^{-2} \end{aligned} \right\} \quad (13)$$

Thus the most favourable focal length for superficial photometry of a fifth-magnitude star with a telescope of aperture 32 cm. is

$$f' = \frac{p_0}{\alpha_b} = \frac{16}{0.035} = 450 \text{ cm.}; \quad (14)$$

the angular diameter of the photometric field is then about 4° , and its luminance about equal to that of a horizontal surface of unit luminance factor lit by the full moon (Le Grand c.1949, p. 87).

* * * * *

3. THE PHOTOMETRIC PROBABLE ERROR

Summary. It is proposed to measure the effect of accidental errors in visual stellar photometry under various conditions by the photometric probable error, which is defined as the probable error of a single reading of the scale of a photometric device used to make a comparison half-field match a half-field illuminated by the star. The photometric probable error ρ is expressed in magnitudes. The methods of calculating ρ from a set of n readings of an arbitrary scale, and of combining a number of independent determinations of ρ , are discussed. It is shown that ρ is normally distributed, with probable error

$$\rho_n = 0.4769 n^{-1/2} \rho. \quad (A)$$

Consequences of the normality of the distribution of ρ are that the method of least squares is applicable to the fitting of an empirical function to a set of observed values of ρ as a function of some parameter expressing the conditions of matching; and that two values of ρ may be considered to differ significantly if their difference exceeds three times its probable error.

The main theme of the present treatise is the various errors which beset visual measurements of stellar magnitude, and the methods available to the photometrist for eliminating or minimizing them. The chief sources of systematic error considered are deviations of the effective relative spectral visual sensitivity from the standard function $1(5)$, and atmospheric extinction; sections 4 and 5, respectively, are devoted to theoretical studies of these subjects. Other errors which follow systematic laws result from the Stiles-Crawford phenomenon, and from scintillant visual efficiency enhancement. The role played by the Stiles-Crawford phenomenon in visual stellar photometry has not been considered by previous authors; it is discussed in section 7 below. Scintillant visual efficiency enhancement also seems to have received little attention hitherto; it is dealt with in sections 11 and 12.

The remaining sections of this monograph are concerned with the accidental errors which are present in greater or less degree in all kinds of visual stellar photometry. An attempt is made to show, with the aid of laboratory experiments and otherwise, how the accuracy of a photometric match depends on the conditions as to ocular magnitude, angular dimensions, etc., the superficial case being discussed in section 8, and the punctiform in section 9. The decrease in precision resulting from the presence of a colour difference between the two objects compared is reserved for treatment in section 10. The results of sections 8 - 10 relate to artificial stars free from scintillation; sections 11 - 12 include an examination of how the accuracy is affected by the presence of scintillation.

We shall adopt, as a convenient measure of the precision of a photometric match, as affected by accidental errors, a quantity which will be called the photometric probable error. As in section 2, we have two half-fields (this term will be applied to the two objects - whether

punctiform or superficial - between which a photometric comparison is being effected), one of which is associated with a constant retinal illumination, while the other is under the control, through a suitable photometric device, of the observer. The photometric probable error is defined as the probable error of a single reading of the scale attached to the photometric device, as evaluated by the methods prescribed by the theory of errors, from the residuals of numerous readings. The probable error so defined may conveniently be expressed in magnitudes.

The choice of the probable error as a measure of dispersion of the observations is based on the simple interpretation, namely that error which is exceeded with a probability of 0.5, which it has when the observations are normally distributed. The assumption of normal distribution of a photometric quantity is probably very accurate in general, provided the quantity be expressed in logarithmic units (cf. Whittaker and Robinson 1944, p.217). This has been verified, in at least one case, by direct experiment (Dufay and Schwégler 1930). An appreciable positive excess (greater probability of small errors) has been recorded by Dufay (1947) in an analysis of some 500 superficial observations with $m = 5$, $\alpha_s = 16\text{cm}$.

Let x be a photometric quantity, such as the luminance of a comparison field, or the reading of a scale attached to a device for measurably varying the luminance. Then the probable error of x , expressed in magnitudes, may be calculated from n independent readings of x by means of the formula

$$r = 0.6745 (n-1)^{-1/2} [\sum \{m(x) - \bar{m}\}^2]^{1/2}, \quad (1)$$

where the brackets $[\]$ denote summation over n values, and $\bar{m} = n^{-1} [\sum m(x)]$. It will sometimes happen that the functional relation between the magnitude m and the scale reading x will be such as to make the calculation of the n individual values

of $m(x)$ inconvenient. In such a case, it may be permissible to approximate by writing

$$r = 0.6745 (n-1)^{-1/2} \left[\left(\left\{ m(\bar{x}) + (x - \bar{x}) \frac{dm}{d\bar{x}} + \frac{(x - \bar{x})^2}{2!} \frac{d^2m}{d\bar{x}^2} + \dots \right\} - \bar{m} \right)^2 \right]^{1/2}, \quad (2)$$

where $dm/d\bar{x} = (dm/dx)_{x=\bar{x}}$, and by neglecting $m(\bar{x}) - \bar{m}$, $\frac{1}{2}(x - \bar{x})^2 d^2m/d\bar{x}^2$, and higher terms, so that

$$r = 0.6745 (n-1)^{-1/2} \left| dm/d\bar{x} \right| [(x - \bar{x})^2]^{1/2} \quad (3)$$

When the determination of r is repeated several times under identical conditions, one obtains, say, n' independent estimates of r . These may be combined by taking the root-mean-square value

$$\sqrt{r^2} = \sqrt{\left(\frac{1}{n'} \sum_1^{n'} r^2 \right)}, \quad (4)$$

as may be shown by considering the sampling distribution of

$$\sigma_2 = n^{-1/2} [(x - \bar{x})^2]^{1/2}. \quad (5)$$

Fisher (1920) shows that the probability of σ_2 falling in the range $d\sigma_2$ is

$$\frac{n^{(n-1)/2}}{2^{(n-3)/2} \{(n-3)/2\}!} \frac{\sigma_2^{n-2}}{\sigma^{n-1}} \exp\left(-\frac{n\sigma_2^2}{2\sigma^2}\right) d\sigma_2, \quad (6)$$

where σ is the standard deviation. It follows that the mean value of σ_2 is

$$\bar{\sigma}_2 = \left(\frac{2}{n}\right)^{1/2} \frac{\{\frac{1}{2}(n-2)\}!}{\{\frac{1}{2}(n-3)\}!} \sigma, \quad (7)$$

while that of σ_2^2 is

$$\overline{\sigma_2^2} = \left(\frac{n-1}{n}\right) \sigma^2. \quad (8)$$

Thus

$$\overline{r^2} = \left\{ 0.6745 \frac{dm}{d\overline{x}} \sqrt{\left(\frac{n}{n-1}\right)} \right\}^2 \overline{\sigma^2} = r^2, \quad (9)$$

while

$$\overline{r} = 0.6745 \left| \frac{dm}{d\overline{x}} \right| \sqrt{\left(\frac{n}{n-1}\right)} \overline{\sigma} = \left(\frac{2}{n-1}\right)^{1/2} \frac{\left(\frac{1}{2}(n-2)\right)!}{\left[\frac{1}{2}(n-3)\right]!} r. \quad (10)$$

For $n = 10$, the value adopted throughout the present work, this gives

$$\overline{r} = 0.97 \sqrt{(\overline{r^2})}. \quad (11)$$

Thus the difference between $\overline{r}$ and $\sqrt{(\overline{r^2})}$ is probably of no consequence, but it is interesting to note the close agreement with (11) found when results are calculated in both ways, although n' is always only 4. This may be taken as confirmation of the normality of distribution of the observations.

As n is increased, the distribution (6) tends rapidly to the normal form, with mean $(1 - \frac{3}{4}n')\overline{\sigma}$, and standard deviation

$$\frac{\overline{\sigma}}{\sqrt{(2n)}}. \quad (12)$$

The probable error of r is thus

$$r_r = \frac{0.6745}{\sqrt{2}} \frac{r}{\sqrt{n}} = \frac{0.4769}{\sqrt{n}} r, \quad (13)$$

in agreement with the general formula due to Gauss (Whittaker and Robinson 1944, p. 201) for the probable error computed from the p^{th} powers of the deviations, viz.

$$r_r = \frac{0.4769}{p} \sqrt{\left\{ \frac{2}{n} \left[\frac{[(p-\frac{1}{2})! \sqrt{n}]}{[(\frac{1}{2}(p-1))!]}^2 - 1 \right] \right\}} r. \quad (14)$$

[$p!$ denotes $\Gamma(p+1)$; as Jeffreys (1950, p. 462) remarks, there is no reason why the latter inconvenient notation should be perpetuated].

The probable error of $\sqrt{(\overline{r^2})}$ is, of course,

$$\frac{r_r}{\sqrt{n}} = \frac{0.4769}{\sqrt{(nn')}} \sqrt{(\overline{r^2})} = 0.075 \sqrt{(\overline{r^2})}. \quad (15)$$

We make use of the fact that, for any reasonable value of n (such as 10), the distribution of r is sensibly normal, in two important ways. We are justified, in the first place, in using the method of least squares to fit an empirical function to a set of values of $\sqrt{(\overline{r^2})}$ as a function of some independent variable such as ocular magnitude. And, secondly, we may regard two values r_1, r_2 of $\sqrt{(\overline{r^2})}$ as differing significantly if their difference is greater than three times its probable error, which is

$$r_1 - r_2 = 0.075 \sqrt{(\overline{r_1^2} + \overline{r_2^2})}. \quad (16)$$

In examining whether the discrepancy of an observed point from a calculated curve is significant, we may note that the weight of the ordinate of the curve is usually much greater than the weight of the point, so that the probable error of the discrepancy can be taken as equal to that of the observed point. (See also Birge 1932).

* * * * *

4. DEPENDENCE OF THE OBSERVED MAGNITUDE ON THE RELATIVE SPECTRAL VISUAL SENSITIVITY

Summary. Equation 1(B) can be written

$$V_{\lambda} = \left\{ \frac{\lambda_m}{\lambda} \exp \left(1 - \frac{\lambda_m}{\lambda} \right) \right\}^{\rho} \quad (A)$$

where $\lambda_m (= q/p)$ is the wavelength of maximum sensitivity, and ρ is a measure of the sharpness of the maximum. By assuming that the spectral distribution of visible stellar radiation is given by Wien's formula, we can calculate the magnitude of a star of colour-temperature T as a function of λ_m, ρ . It is assumed that the differences between the relative spectral visual sensitivities of different observers are equivalent to small increments $\delta\lambda_m, \delta\rho$. Then, if the magnitude scales of two observers are made to agree for stars such that $T \rightarrow \infty$, the discrepancy in the scales for stars of temperature T will be given approximately by

$$\delta m(T) = -\frac{1}{T} \left\{ 52000 \delta\lambda_m + \left(3.4 - \frac{11000}{T} \right) \delta\rho \right\}. \quad (B)$$

This formula is applied to the Purkinje phenomenon observed in the superficial mode for $-8 < m - 5 \log a_0 < 2$, and to the inverse Purkinje phenomenon in the punctiform mode for $0 < m - 5 \log a_0 < 5$.

It is important that the definition of the visual stellar magnitude of a star should contain a clear statement, such as 1(5), of the assumed relative spectral visual sensitivity; but

it is equally important to realize that the precise form of the spectral sensitivity laid down by the definition is most unlikely to be operative in practice, since the spectral sensitivity varies from one observer to another, and, for a given observer, with the photometric conditions (see, e.g., Sloane 1938, Buckley 1941). Even under constant photometric conditions, the spectral sensitivity of a given observer varies with the season of the year (Dresler 1940).

In order to investigate the effect, on visual magnitude estimates, of these variations in spectral sensitivity, it is convenient to use Rossier's (1933, 1934, 1936, 1939) form of relative spectral sensitivity function, namely

$$V_{\lambda} = \left\{ \frac{\lambda_m}{\lambda} \exp \left(1 - \frac{\lambda_m}{\lambda} \right) \right\}^{\rho}, \quad (1)$$

which is seen to be equivalent to Moon's form 1(5) by eliminating A, q between the equations

$$\left. \begin{aligned} V_{\lambda} &= A \lambda^{-\rho} e^{-q/\lambda}, \\ \lambda_m &= q/\rho, \\ V_{\lambda_m} &= 1. \end{aligned} \right\} (2)$$

The parameters λ_m, ρ in (1) have a more obvious interpretation than β, q . λ_m is, in fact, the wavelength of maximum sensitivity, and ρ is a measure of the sharpness of the maximum. It can be shown (Rossier 1934) that, by a suitable choice of the parameters λ_m, ρ , equation (1) can be made to represent, with tolerable accuracy, the relative spectral sensitivity of any receptor whose sensitivity is unimodal; a fortiori, this form of function may be considered valid for any normal eye. In general, the appropriate values of λ_m, ρ will be obtained by applying small increments $\delta\lambda_m, \delta\rho$ to the values derived from the standard data of 1(6), viz.

$$\left. \begin{aligned} \lambda_m &= \rho / \rho = 0.554019 \mu, \\ \rho &= 182.1905. \end{aligned} \right\} (2a)$$

To determine the error δm in the observed magnitude due to small departures $\delta \lambda_m, \delta \rho$ from the standard values of λ_m, ρ , it is necessary to calculate the magnitude m as a function of the parameters λ_m, ρ and of parameters defining the stellar radiation. The latter parameters will be introduced by making a special assumption about the form of the spectral radiant flux density function φ_λ . Since the contribution to the integral $\int_0^\infty \varphi_\lambda K_\lambda d\lambda$ from those values of λ for which K_λ is sensibly zero vanishes, the form of φ_λ outside the limits of the visible spectrum does not affect the value of E , as defined by 1(4). It so happens that, for the great majority of stars, the spectral energy distribution in the visible spectrum approximates closely to that of a full radiator at some temperature T (cf., eg., Rossier 1930, Barbier 1935, Greaves 1948). Since it is only an approximation with which we are concerned here, we may assume Wien's distribution function; the more exact form of Planck has the disadvantage of leading to more difficult integrations.

Let us write, then,

$$\varphi_\lambda = C_0 C_1 \lambda^{-5} e^{-C_2/\lambda T}, \quad (3)$$

where C_1, C_2 are the radiation constants, and C_0 is a geometrical factor determined largely by the distance and radius of the star. The two parameters C_0, T may be regarded as specifying completely the radiant flux density due to the star, the first as to quantity, and the second as to quality.

In virtue of 1(5), and of (3) above, the illumination E is given by

$$E = K_0 \int_0^\infty \varphi_\lambda K_\lambda d\lambda$$

$$\begin{aligned}
 &= K_0 C_1 A C_0 \int_0^{\infty} \frac{1}{\lambda^{\rho+5}} \exp \left\{ - \left(q + \frac{c_2}{T} \right) \frac{1}{\lambda} \right\} d\lambda \\
 &= \{ K_0 C_1 A (\rho+3)! \} C_0 \left(\frac{T}{C_2 + qT} \right)^{\rho+4}.
 \end{aligned} \tag{4}$$

It follows that

$$\begin{aligned}
 m &= -2.5 \log (E/E_*) \\
 &= -2.5 \log \left\{ \frac{C_0}{C_{0*}} \left(\frac{T}{T_*} \cdot \frac{C_2 + qT_*}{C_2 + qT} \right)^{\rho+4} \right\},
 \end{aligned} \tag{5}$$

where the starred quantities refer to the standard star of zero magnitude. Since T_* is at our disposal, it is convenient to let $T_* \rightarrow \infty$; then (5) reduces to

$$m = -2.5 \log \left\{ \frac{C_0}{C_{0*}} \left(\frac{\lambda_m \rho T}{C_2 + \lambda_m \rho T} \right)^{\rho+4} \right\}, \tag{6}$$

where we have replaced q by $\lambda_m \rho$ in accordance with (2). Equation (6) is the required expression for m as a function of C_0 , T , λ_m , ρ .

For small increments $\delta \lambda_m$, $\delta \rho$, we have then

$$\begin{aligned}
 \delta m &= \frac{\delta m}{\delta \lambda_m} \delta \lambda_m + \frac{\delta m}{\delta \rho} \delta \rho = -(2.5 \log e) \frac{(\rho+4)C_2}{C_2 + \lambda_m \rho T} \times \\
 &\quad \left[\frac{\delta \lambda_m}{\lambda_m} + \left\{ 1 + \frac{\rho}{\rho+4} \cdot \frac{C_2 + \lambda_m \rho T}{C_2} \ln \left(\frac{\lambda_m \rho T}{C_2 + \lambda_m \rho T} \right) \right\} \frac{\delta \rho}{\rho} \right].
 \end{aligned} \tag{7}$$

Since T varies from about 2000°K. upwards, while $q^{-1}C_2 = 140^\circ \text{K.}$, we may neglect $q^{-1}C_2$ in comparison with T , and, upon setting $\lambda_m \rho = q$, we find that (7) reduces to

$$\begin{aligned}
 \delta m &= -(2.5 \log e) \cdot \frac{(\rho+4)C_2}{\rho q^2} \frac{1}{T} \times \\
 &\quad \left[\rho^2 \delta \lambda_m + q \left\{ 1 - \frac{\rho}{\rho+4} - \frac{\rho C_2}{2(\rho+4)q} \cdot \frac{1}{T} \delta \rho \right\} \right].
 \end{aligned} \tag{8}$$

Upon substitution for ρ, σ from 1(6), and substitution of Wensel's (1939) value

$$C_2 = 14360 \mu. ^\circ K., \quad (9)$$

(8) finally takes the numerical form

$$\delta m(\tau) = -\frac{1}{\tau} \left\{ 52000 \delta \lambda_m + \left(3.4 - \frac{11000}{\tau} \right) \delta \rho \right\}. \quad (10)$$

It will be observed that $\delta m(\infty) = 0$; this is otherwise obvious from the fact that, when $\tau = \tau_* \rightarrow \infty$, the spectral distribution of the star is identical with that of the standard star of zero magnitude. The hypothetical comparison on which the definition of m is based therefore involves two identical spectral distributions, and so is independent of the spectral sensitivity.

When observations are conducted in the punctiform mode, the stars are usually imaged on the fovea, which contains only cones. The relative spectral visual sensitivity is therefore photopic. Coblentz and Emerson (1918) measured, by a flicker method, the photopic spectral sensitivities of 125 observers having normal colour vision. The range of λ_m encountered in this investigation was about $0.02 \mu.$, from $0.549 \mu.$ to $0.570 \mu.$, with a standard deviation of about $0.004 \mu.$ Similar results were obtained by Gibson and Tyndall (1923), using the step-by-step method, with 52 subjects. Information on the variations in ρ among real observers is not readily available; for simplicity, we shall neglect such variations. Indeed, Devaux (1943) remarks that, for all practical purposes, the relative spectral sensitivity curve of any observer can be obtained from that of the standard observer by a simple translation along the wavelength axis, which suggests that the variations in ρ are not significant.

The difference between the magnitudes

m_1, m_2 assigned to a star of temperature T by two observers having identical ρ -values, but distinct λ_m -values, will be

$$\begin{aligned} m_1 - m_2 &= \delta m_1 - \delta m_2 \\ &= - \frac{52000}{T} (\delta \lambda_{m,1} - \delta \lambda_{m,2}) \\ &= - \frac{52000}{T} (\lambda_{m,1} - \lambda_{m,2}). \quad (11) \end{aligned}$$

If we take

$$\lambda_{m,1} - \lambda_{m,2} = -0.02 \mu. \quad (12)$$

as the greatest λ_m -difference likely between observers with normal colour vision, then (11) gives

$$m_1 - m_2 \approx 10^3 T^{-1}. \quad (13)$$

We observe that the discrepancy $m_1 - m_2$ between the magnitude scales of the two observers is zero for a star of infinite T , as we should expect from the reasoning mentioned after equation (10). With this convention, the greatest discrepancy will occur in the case of very red stars, for which we may take $T = 2000^\circ \text{K}$. Equation (13) then yields $m_1 - m_2 = 0.5$, which is of the right order of size to account for the actually occurring systematic differences between the magnitude estimates of different observers.

The Purkinje phenomenon gives rise to systematic errors (cf. Osthoff 1917, Tiercy 1928, 1929). The phenomenon is due to the fact that the retina contains two kinds of receptors, the rods, whose maximum sensitivity is near $\lambda = 0.51 \mu.$,

and the cones, whose maximum sensitivity is near $\lambda = 0.55\mu$. (cf., e.g., Wright 1938). The rods function at luminance levels between about 10^{-10} and 10^{-3} cd. cm. $^{-2}$, and the cones at luminances greater than about 10^{-7} cd. cm. $^{-2}$. Thus, for a photometric field whose retinal image contains both rods and cones, we shall have $\lambda_m \doteq 0.51\mu$. when $\mathcal{B}_e < 10^{-7}$ cd. cm. $^{-2}$ (scotopic vision), and $\lambda_m \doteq 0.55\mu$. when $\mathcal{B}_e > 10^{-3}$ cd. cm. $^{-2}$ while in the intervening (mesopic) range of luminances ($10^{-7} < \mathcal{B}_e < 10^{-3}$ cd. cm. $^{-2}$), there is a gradual shift of λ_m from 0.51 to 0.55 μ . as the luminance is increased.

In the superficial mode of observation, the photometric field covers an area of several square degrees, and therefore necessarily stimulates both rods and cones.

For observations of stars of moderate magnitude with telescopes of moderate semi-aperture a_o , the Maxwellian luminance will fall in the mesopic range; more precisely, in virtue of 2(12a), the Maxwellian luminance $\mathcal{B}_{e,b}$ will be mesopic if

$$-8 < m - 5 \log a_o < 2. \quad (14)$$

For $m = 5$, $a_o = 16$ cm. (our stock example), equation 2(3a) states that $m - 5 \log a_o = -1$; $\mathcal{B}_{e,b}$ is therefore near the middle of the mesopic range. From 2(13), we note that $\mathcal{B}_{e,b} = 1.5 \times 10^{-6}$ cd. cm. $^{-2}$. Weaver's measurements of the wavelength of maximum sensitivity for seven subjects (Le Grand c. 1949, p.120) indicate an average value of $\lambda_m \doteq 0.525\mu$. at this luminance.

As an example of the action of the Purkinje phenomenon in superficial stellar photometry, suppose that an observer, using our 32-cm. telescope of 450 cm. focal length, matches a red star ($T = 2000^\circ\text{K.}$) of the fifth magnitude with an artificial star having $T = \infty$.

determining the magnitude of the red star as m_1 , say. The observer now abandons the 32 - cm. instrument and sets to work to measure the magnitude of the same red star with an instrument of 8 cm. aperture and some 100 cm. focal length, using the same artificial star to illuminate the comparison half-field. The value of $m - 5 \log a_0$ is now about 2, and the Maxwellian luminance has fallen to the upper limit of the scotopic range, where $\lambda_m \doteq 0.51 \mu$. If he is acquainted with equation (11), our hypothetical observer will express no surprise when his observations with the smaller instrument lead to a value m_2 for the magnitude of the red star, greater than m_1 by an amount

$$m_2 - m_1 \doteq 0.4. \quad (15)$$

In the punctiform mode, however, the two stars being compared may be imaged on the fovea, which contains only cones. In this case, we should expect λ_m to remain constant at about 0.55μ , as the ocular magnitude of the stars is varied; and, consequently, the Purkinje effect should be absent. Examination of the principal photometric catalogues, however, indicates that there are systematic differences between them depending on the magnitude and colour of the stars. Both the Harvard and the Potsdam catalogues, for example, were compiled with the aid of punctiform photometers; yet there are systematic differences between them, of the kind mentioned, and apparently explicable only in terms of the operation of the Purkinje phenomenon in one or both cases.

We are not obliged, however, to resort to the hypothesis of extrafoveal punctiform viewing to explain these results, for there is strong evidence for the existence of an inverse Purkinje effect in the fovea. The foveal spectral sensitivity curves measured by Thomson

and Wright (1948), for example, exhibit a shift towards shorter wavelengths as the luminance is raised. The curves are not unimodal, so one would not be justified in attempting to use the simple theory developed in the present section to predict the effect of this shift on the observed magnitude of a star. Gallissot (1910) has, however, performed laboratory experiments on two artificial stars, one bluish white and one reddish yellow; these lead to the conclusion that an inverse Purkinje phenomenon exists for stars of magnitudes between 6 and 11, as seen in a 28-cm. telescope, i.e. for $0 < m - 5 \log a_0 < 5$ or $-2 < m_e < 3$. Let the magnitude of the red star, measured, when $m_e = -2$, using the white star as a comparison star, be m_1 , and the magnitude measured when $m_e = 3$ be m_2 . Then Gallissot's results indicate that $m_2 - m_1 = -0.5$. The inverse Purkinje phenomenon in the punctiform mode is thus of the same order of size as the direct Purkinje phenomenon in the superficial mode, for which, for the whole range of λ_m ($\lambda_{m,1} = 0.55 \mu$, $\lambda_{m,2} = 0.51 \mu$), and for $T = 2000^\circ \text{K}$, (11) gives $m_2 - m_1 = +1.0$. While this is twice as great as $|m_2 - m_1|$ for the inverse phenomenon, the range of $m - 5 \log a_0$ over which the effect occurs, viz. $(-8, 2)$ (cf. (14)), is also twice as great. Gallissot claims that his measurements explain satisfactorily the discrepancies between the determinations made, with different apertures $2a_0$, at Harvard.

Gallissot (1912 b) also suggests, on the basis of laboratory experiments, that differences in the effect of scintillation on estimates of the magnitudes of red and blue stars may simulate an inverse Purkinje phenomenon; but the conditions of his experiments - short illuminations separated by one second intervals - do not seem close enough to the astronomical case to lend much support to the idea.

* * * * *

5. MATHEMATICAL THEORY OF VISUAL ATMOSPHERIC EXTINCTION

Summary. On the assumption that the spectral transmission factor T_λ of the atmosphere is given by

$$\log T_\lambda = -\delta \lambda^{-2}, \quad (\text{A})$$

an atmospheric extinction theory is developed in which the dependence of the extinction correction $|\Delta m|$ on the colour-temperature T of the star and on the spectral sensitivity is clearly shown. The expression derived,

$$|\Delta m| = \mathcal{B} (1 - CT^{-1}) \operatorname{cosec} h, \quad (\text{B})$$

is shown to be correct within two per cent. for stars at an altitude $h > 15^\circ$, with $T > 2000^\circ \text{K}$. The constant $\mathcal{B}$ in equation (B) depends, through δ , on the state of the atmosphere, and, through λ_m, ρ , on the spectral sensitivity; C depends only on the latter. The use of tabular corrections for extinction is deprecated; and it is suggested that observed magnitudes should be reduced, not to the zenith, but to outside the atmosphere. An expression for the dependence of the integral atmospheric transmission factor T on T is derived; the reported dependence of T on spectral type is in agreement with the present theory.

Astronomical photometry labours under the special difficulty that the radiation from the stars is always more or less attenuated by its passage through the atmosphere to the measuring instrument, and that this extinction, as it is called, has to be allowed for (Müller 1897, Schiller 1923, Danjon and Couder 1935, Cabannes 1937). Casual observation shows that the stars appear fainter when near the horizon, and closer attention reveals that they are also redder in colour. From this it is clear that the longer wavelengths are ~~more~~^{less} attenuated than the shorter: it follows that the extinction correction depends on the spectral radiant flux density distribution, or, with the assumption of 4 (3), on the star temperature T .

At any given time and place, the extinction for stars of given T will depend primarily on the angular altitude h of the star, since, the smaller the altitude h , the greater is the path-length of the rays in the atmosphere. If, for example, a horizontally stratified atmosphere be assumed, it can be shown that the extinction in magnitudes is proportional to $\text{cosec } h$. Except for small values of h , such a law is also a good approximation for spherical strata. For many observing stations in the British Isles, however, the constant of proportionality is subject to rapid and irregular changes during the course of the night; and in some places, near large towns, a $\text{cosec } h$ law cannot be assumed at all. There are certain obvious cases where the extinction depends strongly on azimuth, as when a cloud of smoke and haze is present in a restricted range of bearings.

A useful theory will provide information about the dependence of the extinction on the star temperature T ; and will furnish means for determining rapidly the constant of proportionality mentioned above. It must be supposed,

if the theory is to be applicable, that a site has been chosen where the cosecant law holds approximately, and where changes in overall sky opacity are not too rapid and irregular. If dependence of opacity on azimuth is the only difficulty, it may be avoided by restricting observations to appropriate azimuth ranges.

As in Section 4, we shall denote by ϕ_λ the spectral radiant flux density due to the star measured outside the earth's atmosphere. Let T_λ (capital tau, suffix λ) be the instantaneous zenithal spectral transmission factor of the atmosphere, i.e. the factor by which, at any given time, the flux density of a parallel, vertical pencil of radiation of wavelength λ is reduced by passage through the whole of the atmosphere. If the atmosphere be regarded as horizontally stratified, and if refraction be neglected, the spectral radiant flux density ϕ'_λ due to the star at the surface of the earth will be given by

$$\phi'_\lambda = T_\lambda^\psi \phi_\lambda = \phi_\lambda \exp(\psi \ln T_\lambda), \quad (1)$$

where we have written, for brevity,

$$\text{cosec } h = \psi. \quad (2)$$

By analogy with 1 (4), the illumination at the earth's surface is

$$E' = K_0 \int_0^\infty \phi'_\lambda V_\lambda d\lambda, \quad (3)$$

where K_0 is the photopic coefficient of maximum visibility, and

$$V_\lambda = A \lambda^{-p} e^{-s/\lambda}. \quad (4)$$

[It should be noted that the prime notation is used in (3) in an entirely different sense from that in, for example, 1 (2), where $\lambda = \pi/2$ is implied].

As usual, we shall take

$$\varphi_\lambda = c_0 c_1 \lambda^{-5} e^{-c_2/\lambda^2}. \quad (5)$$

It will be necessary to make some assumption as to the form of the function T_λ . Müller (1897, p. 140) notes that his spectrophotometric measurements on the sun are consistent with the assumption that the negative logarithm of the spectral transmission factor is inversely proportional to the square of the wavelength; this is confirmed by the measurements of the extinction of starlight by Frl. Vocke (1938), using colour filters. We may therefore write

$$\ln T_\lambda = -\frac{\beta}{\lambda^2}, \quad (6)$$

where β is a positive constant. Substitution from (5), (6) into (1) gives

$$\varphi'_\lambda = \frac{c_0 c_1}{\lambda^5} \exp \left\{ -\left(\frac{\beta}{\lambda^2} + \frac{c_2}{\lambda^2} \right) \right\}, \quad (7)$$

where

$$\beta = b\psi. \quad (8)$$

Substituting for φ'_λ , V_λ in (3), we obtain

$$E' = K_0 c_0 c_1 A \int_0^\infty \lambda^{-n} \exp \left\{ -\left(\frac{\beta}{\lambda^2} + \frac{\gamma}{\lambda} \right) \right\} d\lambda, \quad (9)$$

where

$$n = \beta + 5, \quad \gamma = q + c_p/T. \quad (10)$$

The illumination outside the atmosphere is

$$\begin{aligned} E &= K_0 \int_0^\infty \varphi_\lambda V_\lambda d\lambda \\ &= K_0 c_0 c_1 A \int_0^\infty \lambda^{-n} \exp\left\{-\frac{\gamma}{\lambda}\right\} d\lambda. \end{aligned} \quad (11)$$

The extinction correction in magnitudes, i.e. the quantity to be added to the observed magnitude to reduce it to the magnitude which would be observed if the atmosphere were abolished, is the negative quantity

$$\Delta m = -2.5 \log\left(\frac{E}{E'}\right). \quad (12)$$

It is convenient to evaluate the positive quantity

$$\begin{aligned} |\Delta m| &= -\Delta m = -2.5 \log\left(\frac{E'}{E}\right) \\ &= -2.5 \log \left[\frac{\int_0^\infty \lambda^{-n} \exp\left\{-\left(\frac{\beta}{\lambda^2} + \frac{\gamma}{\lambda}\right)\right\} d\lambda}{\int_0^\infty \lambda^{-n} \exp\left\{-\frac{\gamma}{\lambda}\right\} d\lambda} \right]. \end{aligned} \quad (13)$$

To do this, let

$$-2.5 \log \left[\int_0^\infty \lambda^{-n} \exp\left\{-\left(\frac{\beta}{\lambda^2} + \frac{\gamma}{\lambda}\right)\right\} d\lambda \right] = F(\beta, \gamma); \quad (14)$$

then

$$|\Delta m| = F(\beta, \gamma) - F(0, \gamma) = \beta F^{(1)}(0, \gamma) + \frac{\beta^2}{2!} F^{(2)}(0, \gamma) + \dots, \quad (15)$$

where

$$F^{(r)}(0, \gamma) = \left(\frac{\partial^r F}{\partial \beta^r} \right)_{\beta=0} \quad (r=1, 2) \quad (16)$$

To obtain the successive derivatives $F^{(1)}, F^{(2)}, \dots$, we note that

$$\frac{\partial F}{\partial \beta} = - (2.5 \log e) I_n^{-1} \frac{\partial I_n}{\partial \beta}, \quad (17)$$

where

$$I_n = \int_0^\infty \lambda^{-n} e^{-\lambda} d\lambda. \quad (18)$$

Here we have written

$$\frac{\beta}{\lambda^2} + \frac{\gamma}{\lambda} = \lambda; \quad (19)$$

so

$$\begin{aligned} \frac{\partial I_n}{\partial \beta} &= \int_0^\infty \frac{\partial}{\partial \beta} (\lambda^{-n} e^{-\lambda}) d\lambda \\ &= - \int_0^\infty \lambda^{-(n+2)} e^{-\lambda} d\lambda = -I_{n+2}. \end{aligned} \quad (20)$$

Successive application of this identity enables the derivatives $F^{(1)}, F^{(2)}, \dots$ to be written down at once. Thus

$$\begin{aligned}
 F^{(1)}(0, \gamma) &= (2.5 \log e) \left(\frac{I_{n+2}}{I_n} \right)_{\beta=0}, \\
 F^{(2)}(0, \gamma) &= (2.5 \log e) \left(\frac{I_{n+2}^2}{I_n^2} - \frac{I_{n+4}}{I_n} \right)_{\beta=0}, \\
 F^{(3)}(0, \gamma) &= (2.5 \log e) \left(\frac{2 I_{n+2}^3}{I_n^3} - \frac{3 I_{n+2} I_{n+4}}{I_n^2} + \frac{I_{n+6}}{I_n} \right)_{\beta=0}, \\
 F^{(4)}(0, \gamma) &= (2.5 \log e) \left(\frac{6 I_{n+2}^4}{I_n^4} + \frac{12 I_{n+2}^2 I_{n+4}}{I_n^3} - \frac{3 I_{n+4}^2}{I_n^2} \right. \\
 &\quad \left. + \frac{4 I_{n+2} I_{n+6}}{I_n^2} - \frac{I_{n+8}}{I_n} \right)_{\beta=0},
 \end{aligned} \tag{21}$$

and so on.

Equations (15) and (21) express $|\Delta m|$ in terms of known integrals; for

$$(I_n)_{\beta=0} = \int_0^\infty \lambda^{-n} e^{-\gamma/\lambda} d\lambda = \gamma^{-(n-1)} (n-2)! \tag{22}$$

When this expression for I_n is used to express the derivatives of (21) as polynomials in n , and these are substituted in (15), the result is

$$\begin{aligned}
 |\Delta m| &= (2.5 \log e) n(n-1) \gamma^{-2} \beta \left\{ 1 - (2n+1) \gamma^{-2} \beta \right. \\
 &\quad \left. + \frac{4}{3} (5n^2 + 7n + 3) \gamma^{-4} \beta^2 - \dots \right\}.
 \end{aligned} \tag{23}$$

The series in the braces {} converges, because it is an alternating series in which the size of each term is less than that of the preceding term. Convergence is, moreover, rapid, since the ratio of successive terms is of the order of

$$n\gamma^{-2}\beta = (\rho+5)\left(\gamma + \frac{c_2}{\gamma}\right)^2 \Delta\psi + \left(\frac{\rho+5}{\gamma^2}\right)\delta\psi. \quad (24)$$

From Müller's data (1897, p.140), the value of δ is roughly $0.067 \mu^2$; and, for $h = 9.6$, we have $\psi = 6$. Substituting these values in (24), and using Moon's values [1(6)] for ρ, γ , we get

$$n\gamma^{-2}\beta = 7 \times 10^{-3}. \quad (25)$$

Noting that the coefficient of n in the second term of the series is 2, we can safely assert that, for a star of altitude 10° under Müller's conditions of transparency, the error incurred by neglecting the second and higher terms in the braces is less than two per cent. We may therefore write, with ample accuracy,

$$\begin{aligned} |\Delta m| &= (2.5 \log e) n(n-1) \gamma^{-2} \beta \\ &= (2.5 \log e) (\rho+5)(\rho+4) \left(\gamma + \frac{c_2}{\gamma}\right)^2 \Delta\psi. \end{aligned} \quad (26)$$

A simple interpretation of this result is obtained by neglecting 4 and 5 in comparison with ρ , and c_2/γ in comparison with γ , so that

$$|\Delta m| \doteq (2.5 \log e) \left(\frac{\rho}{\gamma}\right)^2 \delta\psi = (2.5 \log e) \frac{\delta}{\lambda_m^2} \cdot \psi. \quad (27)$$

In virtue of (6), (1), this reduces to

$$|\Delta m| \doteq (2.5 \log e)(-1/\tau_{\lambda_m})\psi = -2.5 \log \left(\frac{\mathcal{V}_{\lambda_m}}{\mathcal{V}_{\lambda_m}} \right), (28)$$

which is simply the monochromatic extinction for wavelength λ_m . This approximation would, in fact, be rigorous, if the eye were sensitive only to radiation of wavelength λ_m ; or if the radiation from the star were of this wavelength only.

The complete expression (26) shows that the extinction depends slightly, as we should expect, on the parameters ρ, σ of the relative spectral visual sensitivity $V_\lambda = A\lambda^{-\rho}e^{-\sigma/\lambda}$, as well as on the colour-temperature T of the star. The dependence on the sensitivity parameters, however, is almost entirely a dependence on λ_m , as is shown by writing

$$\begin{aligned} |\Delta m| &= (2.5 \log e) \left(1 + \frac{5}{\rho}\right) \left(1 + \frac{4}{\rho}\right) \left(\frac{T}{\rho}\right)^2 \left(1 + \frac{c_2}{\rho T}\right)^{-2} b \psi \\ &= (2.5 \log e) \left\{ \left(1 + \frac{5}{\rho}\right) \left(1 + \frac{4}{\rho}\right) \left(1 + \frac{c_2}{\lambda_m \rho T}\right)^2 \right\} \lambda_m^{-2} b \psi, (29) \end{aligned}$$

where it is seen that the only strongly sensitivity-dependent factor is λ_m^{-2} .

The factor

$$B = (2.5 \log e) \left(1 + \frac{5}{\rho}\right) \left(1 + \frac{4}{\rho}\right) \lambda_m^{-2} b \quad (30)$$

in the coefficient of $\psi = \operatorname{cosec} h$ depends, through b , on the instantaneous state of the atmosphere, and must be determined from observations on stars at different altitudes h . The other factor may be expanded,

$$\left(1 + \frac{c_2}{\lambda_m \rho T}\right)^{-2} = 1 - \frac{2c_2}{\lambda_m \rho T} + \dots, \quad (31)$$

and terms higher than those written may be neglected, with an error of less than two per cent for $T \geq 2000^\circ \text{K}$. Thus the formula (29) for $|\Delta m|$ may be expressed in the form

$$|\Delta m| = B(1 - CT^{-1})\psi, \quad (32)$$

where

$$C = \frac{2c_2}{\lambda_m \rho} \quad (33)$$

depends only upon the spectral sensitivity of the observer's eye, and may therefore be found by measuring λ_m, ρ in the laboratory.

ψ is simply the so-called air-mass. It is convenient to define a reduced air-mass ψ' by the equation

$$\psi' = (1 - CT^{-1})\psi; \quad (34)$$

this is equal to the ordinary air-mass ψ for stars of infinite colour-temperature T , but is less than ψ for stars of lower temperature, as we should expect. The fundamental formula (32) can now be written in its most compact form

$$|\Delta m| = B\psi'. \quad (35)$$

If one is to use the simple formula $\psi = \text{cosec } \zeta$ for the air-mass, it is important to know the order of size of the error thus introduced by neglect of the curvature of the earth

and of refraction. Since each of the two other approximations which have been made - neglect of higher terms in equations (23), (31) - has involved an error of less than two per cent, it will be convenient to calculate the altitude h at which an error of this size is present in the formula $\psi = \text{cosec } h$. An extinction formula of adequate rigour for the purpose is that of Laplace, viz.

$$\psi_L = KR \text{ cosec } z, \quad (36)$$

where z is the apparent zenith distance, R is the angle of refraction, and K is a constant (cf. Müller, 1897, p.126). The mean refraction R may be expressed in the form

$$R = A_1 \tan z + B_1 \tan^3 z; \quad (37)$$

values for the constants A_1, B_1 are quoted by Smart (1944). With this substitution, (36) becomes

$$\begin{aligned} \psi_L &= KA_1 \sec z \left(1 + \frac{B_1}{A_1} \tan^2 z\right) \\ &= \text{cosec } h \left(1 + \frac{B_1}{A_1} \cot^2 h\right), \end{aligned} \quad (38)$$

since $z = \frac{1}{2}\pi - h$, nearly, and $(\psi_L)_{z=0} = KA_1 = 1$. The fractional error in $\psi = \text{cosec } h$ is thus

$$\left| \frac{\psi - \psi_L}{\psi_L} \right| \div \left| \frac{\psi - \psi_L}{\psi} \right| = \frac{B_1}{A_1} \cot^2 h. \quad (39)$$

If we set this equal to 0.02, and substitute for A_1, B_1 , this yields $h = 13.5$. For altitudes

greater than this, the 'air-mass' error will be less than two per cent.

To sum up, we note that the formula

$$|\Delta m| = B(1 - CT^{-1}) \operatorname{cosec} \lambda \quad (40)$$

is correct within about two per cent for any star with $T > 2000$ K., $\lambda > 15$.

It should be emphasised that, throughout the above theory, the relative spectral sensitivity parameters employed may be taken to refer to the particular observer concerned. The reduction therefore converts the observed magnitude to the magnitude which would be recorded by the same observer if the atmosphere were abolished. To obtain the true visual magnitude, based on the standard values of λ_m, ρ , we have to apply a further correction for the deviations $\Delta\lambda_m, \Delta\rho$ of the observer's parameters from the standard values. This correction is equal and opposite in sign to the magnitude error displayed in 4 (10); that is, the correction is

$$-\delta m = \frac{1}{T} \left\{ 52000 \delta\lambda_m + \left(3.4 - \frac{11000}{T} \right) \delta\rho \right\}. \quad (41)$$

It may be useful to indicate in what respects the present theory differs from the theories of extinction occurring in the literature.

The chief efforts of previous theories were devoted to the accurate determination of the air-mass as a function of the zenith distance, account being taken of the curvature of the earth and of refraction; slightly different results were obtained, according to the assumed variation of transmission coefficient and refractive index with height. The formulae obtained for the

zenith reductions contained one or more arbitrary constants, of which the values would depend, of course, on the prevailing atmospheric conditions. Values for the constants under mean conditions were obtained by fitting numerous observational data, taken on different occasions, to the theoretical law of extinction. A table of extinction reductions (e.g. Müller 1897, pp. 515-516) could then be prepared, and used to correct all future photometric observations. The elaborate theory used to derive it ensures that such a table correctly represents the form of the reductions as a function of zenith distance, but the numerical values are correct only for the mean conditions.

Indeed, Danjon (1928) asserts that the use of tabular corrections for extinction is quite unjustified, in view of the large fluctuations in atmospheric transparency from one occasion to another; and claims to have shown that the practice mentioned is the chief source of accidental errors in photometric observations, particularly in the catalogues.

Thus, it is the correctness of the numerical values of the extinction reduction which is important in astronomical photometry; the precise form of the dependence on zenith distance is of academic interest only, though presenting a beautifully intricate mathematical problem. The last remark is given point by the considerations that it is only at large zenith distances that a rigorous formula differs appreciably from the secant formula, whereas photometric observations should normally be conducted well away from the horizon; and that, in any case, irregularities in extinction are likely, at most observing stations, to render futile the use of any but an approximate formula.

The most serious objection to the classical theories is, perhaps, the fact that

they treat the radiation from the stars as essentially monochromatic. At any rate, they make no attempt to allow for the dependence of the transmission coefficient on the wavelength. This may be partly justified by the assertion that, since the sensitivity of the eye is virtually confined to a relatively narrow band of wavelengths, the extinction may be computed without much error for an effective wavelength corresponding approximately to the maximum visual sensitivity. That this assertion is fallacious is shown by equation (32), which indicates a marked dependence of $|Am|$ on the star temperature T , the constant C being of the order of 300°K ., while T^{-1} ranges from 5×10^{-5} to 3×10^{-4} ($^\circ\text{K}$)⁻¹. Even the smaller of these two extremes implies a temperature correction of nearly two per cent, which is the same as the 'air-mass' correction at an altitude of about 15° .

The present theory presents an important difference from previous theories in that it does not attempt to reduce observed magnitudes to the zenith, i.e. to the values which would be observed if the star were seen in the zenith of the place of observation, but to obtain the magnitudes which would be recorded if the photometric effect of the atmosphere could be done away with completely. The necessity for this approach follows when the assumption of mean conditions is abandoned, since reduction to a local and ephemeral zenith does not provide significant data. This is clear when one reflects that the reduction from the local zenith to the mean zenith would itself be a function of T .

The formulae derived above relate to observations with the unaided eye. When a telescope is employed, the effective spectral sensitivity is the product of the relative spectral visual sensitivity and the spectral transmission factor of the instrument. This effective sensitivity will be characterized

by values of λ_m, ρ different from the naked-eye values. This must be remembered when C is calculated, for use in (32), from (33). The factor S given by (30), is also a function of λ_m, ρ ; it should be determined by observations with the same telescope as is used to observe the magnitudes which it is desired to correct by means of (32). The correction of telescopic observations to the standard relative spectral sensitivity by means of (41) involves a knowledge of the values of $\delta\lambda_m, \delta\rho$ appropriate to the combination of instrument and observer.

Equation (32), which expresses the extinction correction as a function of the star temperature T , may be used to derive the dependence of the zenithal atmospheric transmission factor T on the spectral type, and thus to effect a comparison with the experimental and theoretical results of other workers. When $\lambda = \pi/2$,

$$-2.5 \log T = |\Delta m| = B(1 - CT^{-1})$$

$$= (2.5 \log e) \left(1 + \frac{5}{\rho}\right) \left(1 + \frac{4}{\rho}\right) \frac{\delta}{\lambda_m^2} \left(1 - \frac{2c_2}{\lambda_m \rho T}\right)$$

$$\doteq (2.5 \log e) \frac{\delta}{\lambda_m^2} \left(1 - \frac{2c_2}{\lambda_m \rho T}\right); \quad (42)$$

or

$$T(T) = \exp \left\{ -\frac{b}{\lambda_m^2} \left(1 - \frac{2c_2}{\lambda_m \rho T}\right) \right\}. \quad (43)$$

If we set

$$-\frac{\delta}{\lambda_m^2} = \ln T_\lambda \approx \ln T, \quad \lambda_m/\rho = q, \quad (44)$$

in (43), it can be written

$$T(\tau) = T^{1 - 2c_2/(q\tau)} \\ \approx T \left(1 - \frac{2c_2 \ln T}{q\tau} \right). \quad (45)$$

Hence, if τ_1, τ_2 be two distinct colour temperatures,

$$\frac{T(\tau_2) - T(\tau_1)}{T} = - \frac{2c_2 \ln T}{q} \left(\frac{1}{\tau_2} - \frac{1}{\tau_1} \right). \quad (46)$$

In the paper by Fr1. Vocke (1938) already referred to, she determines, from observations using colour filters, the dependence of T on the spectral type. With an obvious notation, her results may be expressed in the forms

$$T_{G0-K5} - T_{B0-A2} = 0.023 \pm 0.016, \text{ for } T = 0.764, \quad (47)$$

$$\frac{T(\tau_2) - T(\tau_1)}{T} = \frac{T_{G0-K5} - T_{B0-A2}}{T} = 0.030.$$

It is not clear what numerical values should be assigned to τ_1, τ_2 in (47), as the ranges of spectral type concerned correspond to fairly considerable ranges of colour temperature. We shall adopt, somewhat arbitrarily, $\tau_1 = 15000$ K., $\tau_2 = 5000$ K. Substitution of numerical

values, including $T = 0.764$, in (46), then gives

$$\frac{T(\tau_2) - T(\tau_1)}{T} = 0.010, \quad (48)$$

which is of the right order of magnitude.

The following table, adapted from Frl. Vocke (1938), includes some other determinations of $\{T(\tau_1) - T(\tau_2)\} / T$, both experimental and theoretical, to be found in the literature.

Reference	$T(\tau_2) - T(\tau_1)$	Method
Müller 1913	0.005	Potsdam tables
Zipler 1921	0.021	Potsdam tables
v.d. Pahlen, n.d.	0.011	Theoretical
Vocke 1938	0.030	Colour filters
Equation (48)	0.010	Theoretical

* * * * *

6. THE ILLUMINATION DUE TO A STAR OF ZERO MAGNITUDE

Summary. The problem of determining the illumination E_* , outside the atmosphere, due to a star of zero magnitude, is regarded as that of measuring the illumination E'_* , at the surface of the earth, due to a star of magnitude m , and of simultaneously measuring the extinction correction $| \Delta m |$ for that star. The theory is given of a simple method for determining, from four sets of observations, the differences in magnitude of two stars from an artificial star giving a known illumination, as well as the extinction factors B of $5(B)$ on two occasions. The method is contrasted with two published procedures for measuring B , of which one ignores the serious changes in B over a period of hours, while the other involves the assumption that the zenithal magnitude of a star is independent of atmospheric conditions.

The published determinations of the illumination due to a zero magnitude star are criticized on the ground that only the illumination E'_* at the surface of the earth is measured, no provision being made for effecting an extinction correction appropriate to the atmospheric conditions at the time and place of the measurement. Using a published value of E'_* , and an assumed value of the atmospheric transmission factor, Buckley and Preston have estimated E_* ; variations in the transmission could then be measured by observing the illumination due to

the pole star on different occasions. Apparently impossibly high values of the transmission factor were deduced from these measurements, throwing doubt on the $\mathcal{E}_*$ obtained from the published $\mathcal{E}'_*$. It is suggested that the mystery may be resolved by a direct measurement of $\mathcal{E}_*$ by the method developed here.

The definition $l(1)$ of the visual magnitude m of a star contains an arbitrary constant $\mathcal{E}_*$, which denotes the normal illumination outside the atmosphere due to a star of zero magnitude. When one is concerned only with the relative values of the illuminations due to different stars, a knowledge of $\mathcal{E}_*$ is superfluous. For certain purposes, however, it is useful to know the value of $\mathcal{E}_*$, expressed in such units as lumens per square centimetre. It may be emphasised again that the determination of $\mathcal{E}_*$ is entirely a matter for experiment; $\mathcal{E}_*$ is not a conventionally defined quantity.

The problem of measuring $\mathcal{E}_*$ is complicated by the fact that the illumination due to any star, of magnitude m , say, can be measured only at the surface of the earth. The result of the measurement is therefore affected by the attenuation of the light from the star by absorption and scattering in its passage through the atmosphere. In order to secure freedom from ambiguity, it was proposed in Section 1 to define $\mathcal{E}_*$ as the normal illumination which would be produced at the surface of the earth by a star of zero magnitude if the atmosphere were abolished, or, more simply, as the normal illumination on a plane just outside the earth's atmosphere due to a star of zero magnitude.

The determination of E_* can then be considered in two stages: measurement of the illumination E' (not the E' of section 1) at the surface of the earth due to a star of magnitude m , and simultaneous measurement of the extinction correction for this star, defined by

$$|\Delta m| = -2.5 \log \left(\frac{E'}{E} \right), \quad (1)$$

where E is the illumination outside the atmosphere. For, by combining (1) with 1(1), we have

$$m + |\Delta m| = -2.5 \log \left(\frac{E'}{E_*} \right), \quad (2)$$

whence

$$E_* = E' \text{ antilog } \{ 0.4 (m + |\Delta m|) \}. \quad (3)$$

The theoretically simplest observational procedure is to measure the differences $m_1 - m_0$, $m_2 - m_0$ between the magnitudes m_1 , m_2 (not necessarily different) of two stars of temperatures T_1 , T_2 (not necessarily different) and the fixed magnitude m_0 of an artificial star, on two different occasions when the values of B are B_1 , B_2 (not necessarily different) and the stars are at very different altitudes. Let the observed magnitude differences be

$$\left\{ \begin{array}{l} \delta m_{11}, \delta m_{12} \\ \delta m_{21}, \delta m_{22} \end{array} \right\}, \quad (4)$$

and the corresponding reduced air-masses be

$$\left\{ \begin{array}{l} \psi'_{11}, \psi'_{12} \\ \psi'_{21}, \psi'_{22} \end{array} \right\} = \left\{ \begin{array}{l} (1 - CT_1^{-1}) \operatorname{cosec} h_{11}, (1 - CT_2^{-1}) \operatorname{cosec} h_{12} \\ (1 - CT_1^{-1}) \operatorname{cosec} h_{21}, (1 - CT_2^{-1}) \operatorname{cosec} h_{22} \end{array} \right\} \quad (5)$$

where the row-suffixes refer to the two occasions, and the column-suffixes to the two stars. By definition of $|\Delta m|$, we have

$$\delta m_{ij} = m_j + |\Delta m|_{ij} - m_0 = \delta m_j + |\Delta m|_{ij} \quad (i, j=1, 2), \quad (6)$$

where

$$\delta m_j = m_j - m_0. \quad (7)$$

Substituting for $|\Delta m|$ from 5(35), we have

$$\delta m_{ij} = \delta m_j + B_i \psi'_{ij}. \quad (8)$$

The solution of this system of four simultaneous equations for the unknowns δm_1 , δm_2 , B_1 , B_2 may be written

$$\left. \begin{aligned} D\delta m_1 &= + \psi'_{12} \psi'_{21} \delta m_{11} - \psi'_{11} \psi'_{21} \delta m_{12} \\ &\quad - \psi'_{11} \psi'_{22} \delta m_{21} + \psi'_{11} \psi'_{21} \delta m_{22}, \\ D\delta m_2 &= - \psi'_{11} \psi'_{22} \delta m_{12} + \psi'_{12} \psi'_{22} \delta m_{11} \\ &\quad + \psi'_{12} \psi'_{21} \delta m_{22} - \psi'_{12} \psi'_{22} \delta m_{21}, \\ DB_1 &= - \psi'_{22} (\delta m_{11} - \delta m_{21}) + \psi'_{21} (\delta m_{12} - \delta m_{22}), \\ DB_2 &= + \psi'_{12} (\delta m_{21} - \delta m_{11}) - \psi'_{11} (\delta m_{22} - \delta m_{12}), \end{aligned} \right\} \quad (9)$$

where

$$D = \psi'_{12} \psi'_{21} - \psi'_{11} \psi'_{22}. \quad (10)$$

The condition for accurate results is thus that $|D|$ should be large, i.e.

$$|\psi'_{12} \psi'_{21} - \psi'_{11} \psi'_{22}| \gg 0. \quad (11)$$

This will be assured if, say, ψ'_{12} , ψ'_{21} are large, and ψ'_{11} , ψ'_{22} are small. This means that, on occasion 1, star 1 must be at a great altitude, and star 2 at a small altitude, while on occasion 2 the state of affairs must be reversed. In temperate latitudes, this condition is best attained using circumpolar stars of moderate polar distance. Like all methods depending on change of altitude due to diurnal motion, the method breaks down for observatories near the poles of the earth.

It may be noted that the accuracy of the four observational data δm_{ij} can be increased by determining each as the mean of a number of readings taken nearly simultaneously. The four quantities δm_1 , δm_2 , B_1 , B_2 are obtained without any assumption being made about the magnitudes of the two stars used, except that they are constant.

In parenthesis, we may remark that other, apparently less satisfactory, methods for measuring the extinction factor B have been employed by some authors. For example, let us consider the two methods used by Erl. Vocke (1938).

In the first, the magnitude of a rising or setting star was measured when the star was at two very different altitudes. This method

is open to the grave objection that, in the two and a half to three hours elapsing between the two observations, a serious change in the extinction factor $\mathcal{B}$ is likely to occur. Indeed, Buckley and Preston (1951) found, in a long series of observations, that nights on which $\mathcal{B}$ remains sensibly constant are very rare (they encountered only one between 1942 September and 1943 March). Smart (1933) has found, in photo-electric measurements, that important changes in $\mathcal{B}$ may take place in times of the order of a few minutes.

The second method of Fr1. Vocke involved a measurement of the magnitude of a star, when near the zenith, on several occasions, and a comparison of the mean zenithal magnitude thus obtained with the magnitude observed on an occasion when the zenith distance was great. The extinction on this occasion was then calculated on the assumption that the variation in zenithal magnitude from one night to another is negligible. While this assumption is not far from the truth, it is an assumption which is unnecessary if one use the method suggested in the present treatise.

Returning to the main theme of this section, we observe that if the illumination $\mathcal{E}_a$ at the eye of the observer due to the artificial star be measured, then a knowledge of δm_1 , δm_2 and of the catalogue magnitudes m_1 , m_2 of the two stars enables two independent estimates of $\mathcal{E}_*$ to be made. For

$$\begin{aligned} -2.5 \log \left(\frac{\mathcal{E}_a}{\mathcal{E}_*} \right) &= m_c = m_j - (m_j - m_o) \\ &= m_j - \delta m_j \quad (j = 1, 2); \quad (12) \end{aligned}$$

whence

$$\mathcal{E}_* = \mathcal{E}_a \text{ antilog } \{ 0.4 (m_j - \delta m_j) \}. \quad (13)$$

The two values of E_* obtained from (13) by putting $j = 1, 2$ will not, in general, agree. This is simply a consequence of the fact that

$$m_1 - \delta m_1 \neq m_2 - \delta m_2, \quad (14)$$

or

$$m_1 - m_2 \neq \delta m_1 - \delta m_2, \quad (15)$$

which expresses the fact that the difference of the catalogue magnitudes does not agree with the observed magnitude difference. In the absence of any basis for weighting, one can adopt a geometric mean value for E_* , i.e.

$$E_* = E_a \text{ antilog } \{0.2 (m_1 + m_2 - \delta m_1 - \delta m_2)\}. \quad (16)$$

It is to be regretted that hitherto no direct determination of the illumination due to a star of zero magnitude appears to have been performed in which allowance for atmospheric extinction has been made. The only reliable direct determination of E_* is that due to Fabry (1903). A comparison star of controllable magnitude was provided in the form of the image in a microscope objective of a small source. By moving this comparison star along the line of sight, it could be matched, in turn, against a star, and against a standard lamp at a known great distance. The magnitude difference between the standard lamp and the star could be calculated from the inverse square law. Since the magnitude of the star and the illumination due to the standard lamp were both known, it was then possible to calculate the illumination due to a star of zero magnitude, which was found to be $2.09 \times 10^{-10} \text{ lm.cm.}^{-2}$. This refers to a star near the zenith, the sky being very clear,

and the observer being near sea-level. Unfortunately, this statement of conditions is not sufficiently precise to enable one to calculate E_* , the value of the illumination outside the atmosphere: the appropriate zenithal transmission factor is unknown.

Other, less direct methods have been employed to measure the illumination E_* due to a star of zero magnitude (cf. Russell 1916). These are based on measurements of both the stellar magnitude of, and the illumination due to, the sun or the moon. The values obtained are in approximate agreement with that of Fabry. Russell's mean value has been quoted in 1 (3). Dividing this value by a zenithal transmission factor of 0.766 due to Kimball (1914), Buckley and Preston (1951) obtained a value for the illumination E_* outside the atmosphere due to a star of zero magnitude. The illuminations outside the atmosphere due to the pole star and to the moon in its various phases were then calculated from their known magnitudes. Buckley and Preston used a Maxwellian-view telephotometer, modified from Collier and Taylor (1938), to measure, at different times, the illuminations at the surface of the earth due to these bodies; they hoped thus to be able to deduce the transmission factor of the atmosphere at those times. On several occasions, this method led to values of the transmission factor as great as 0.95, significantly in excess of the maximum, 0.85, allowed by fundamental considerations of the scattering properties of the atmosphere (Rayleigh 1871; Cabannes 1929; Middleton 1941, pp.4-9). Buckley and Preston do not seem to have been aware that the pole star is slightly variable (Naut. Alm., Greenw., 1951; Norton 1943), but this could not, of course, explain the discrepancy in the case of the moon. Nor do they allude to the observations of Duclaux and Gindre (1929-31; see also Duclaux 1933, 1934, Götz 1929), which suggested that the transparency of

the atmosphere might sometimes exceed the theoretical limit. The weight of evidence (e.g. King 1913, Vassy and Vassy 1939, Johnson et al. 1939), however, supports the Rayleigh-Cabannes theory; so there seems little prospect of resolving the discrepancy along these lines.

A reasonable upper limit for the atmospheric transmission factor in Buckley and Preston's investigation would be 0.80. In order to reconcile with this the maximum values of the illuminations at the surface of the earth due to the pole star and to the moon observed by Buckley and Preston, it would be necessary to increase their value of $\angle_*$ by a factor of $0.95/0.80 = 1.2$. Such an increase would be obtained if it could be shown that Kimball's transmission factor were too large; but Buckley and Preston, in a closer analysis of Kimball's data, show that the error is likely to have been in the other direction, though they throw doubt on the consistency of Kimball's measurements. The only alternative is to suppose that the illumination at the earth's surface due to a star of zero magnitude, under conditions of maximum transparency, is greater than Russell's mean value by a factor of 1.2, and thus lies outside the range of the published values. This would mean either that those values were observed under conditions less clear than those of maximum transparency, or that they were affected by some systematic error.

Whatever may be the true explanation of this puzzling discrepancy, it is clear that further measurements are needed, and that such measurements will be of little value unless they make provision, somewhat in the way outlined in the present section, for the elimination of the transmission factor of the atmosphere, and consequently for the determination of the illumination $\angle_*$ due to a star of zero magnitude, as we have defined it.

7. THE STILES-CRAWFORD PHENOMENON IN VISUAL STELLAR PHOTOMETRY

Summary. It is suggested that, when $m - 5 \log a_0 \approx -3$, the Stiles-Crawford phenomenon may, with certain optical arrangements, give rise to an appreciable spurious augmentation of the photometric probable error. If, for example, in the superficial mode, one half-field is a Maxwellian view, while the other is an illuminated diffusing surface, then random lateral displacements of the Maxwellian focus with respect to the eye pupil will generate a Stiles-Crawford probable error which is estimated, by a simple statistical treatment, as about $0^m.2$. In conjunction with a coude equatorial, a dental plate could be used to eliminate the lateral displacements of the focus; it is verified that, under this condition, the residual displacements due to rotation of the eyeball and to the driving errors of the equatorial would generate negligible probable error. Alternatively, both half-fields could be made Maxwellian: apart from driving errors, the respective foci could be made to coincide, so that their relative luminous efficiencies would be equal irrespective of eye movements. This arrangement is applicable to any equatorial; an estimate of the Stiles-Crawford probable error shows, however, that even with this precaution it may be little inferior to the photometric probable error. It is emphasized that, in the punctiform mode, irrespective of the existence of the Stiles-Crawford phenomenon, the exit pupils of the real star and of the artificial comparison star should coincide.

The Stiles-Crawford (S - C) phenomenon (Stiles and Crawford 1933, Stiles 1939) may be characterized by the statement that two Maxwellian half-fields of equal effective luminances do not, in general, appear equally luminous when the foci of the two beams are at different points in the plane of the pupil. In general, that half-field whose focus lies farther from the centre of the pupil requires a greater effective luminance if the two half-fields are to match in luminosity, and is said to have a lower luminous efficiency. Equally luminous half-fields are characterized by equal products of relative luminous efficiency η and effective luminance E_e . Thus η is defined by the equations

$$\eta B_e = \eta' B_e', \quad \max \eta = 1; \quad (1)$$

for a given luminance level, η is a function of the position of the focus in the plane of the pupil. The point of the pupil at which $\eta = 1$ may be termed the centre of luminous efficiency; it does not necessarily coincide with the geometrical centre of the pupil, and in some observers may even be near the edge of the pupil. The centre of luminous efficiency has the property that, if it be taken as the origin of polar co-ordinates (ρ, θ) in the plane of the pupil, then η is a function of ρ alone.

A close approximation to the functional dependence of η on ρ is given by the empirical formula

$$\log \eta = -\rho f^2, \quad (2)$$

where $\rho = 5 \text{ cm.}^{-2}$ for $B_e = 10^{-4} \text{ cd.cm.}^{-2}$. As B_e decreases below this limit, ρ becomes smaller, and vanishes near $B_e = 10^{-6} \text{ cd.cm.}^{-2}$. Mlle Flamant (1949) has shown that finite values of ρ , of the order of $\pm 1 \text{ cm.}^{-2}$, the sign of the ambiguity depending on the observer, may reappear around

$B_e = 10^{-7}$ cd.cm. $^{-2}$. But for the purposes of the present discussion it seems simplest to follow the suggestion of Le Grand (c.1949, p.105), and to assume that

$$\begin{aligned} \mu &= 5 \text{ cm.}^{-2} \quad (B_e > 10^{-5} \text{ cd.cm.}^{-2}), \\ \mu &= 0 \quad (B_e < 10^{-5} \text{ cd.cm.}^{-2}). \end{aligned} \quad (3)$$

Thus, in view of 2(12a), we have to reckon with the S - C phenomenon whenever

$$m - 5 \log a_o < -3, \quad (4)$$

provided, of course, that $\alpha = \alpha_h$.

In Dufay's photometer (1947), one half of the field is illuminated by light from the star under observation, forming a Maxwellian view, while the other half is filled by a diffusing surface illuminated by an artificial source. In the absence of any artificial pupil, the luminosity of the diffusing surface will remain unchanged as the eye undergoes small displacements in the focal plane; for, irrespective of these small displacements, the pupil remains filled by a cone of light of uniform angular density derived from each point of the diffusing screen. The luminosity of the Maxwellian half-field will, however, depend on the distance ρ of the focal point from the centre of luminous efficiency. If B_o be the luminance of the diffusing surface, and B_e the effective luminance of the lens, a photometric match will be observed when

$$B_o \bar{\eta} = B_e \eta(\rho), \quad (5)$$

where $\bar{\eta}$ is the mean relative luminous efficiency,

averaged over the area of the pupil. Combining (2) and (5), we obtain

$$B_0 = \bar{\eta}^{-1} B_* 10^{-\rho \rho^2}. \quad (6)$$

The luminance B_0 of the diffusing screen is under the control of the observer; we may suppose that B_0 is varied until it takes the value indicated by (6), whereupon the scale of the photometric device is used to read off the magnitude of the star, i.e.

$$m = m_0 - 2.5 \log (B_0 / \bar{\eta}) + 2.5 \rho \rho^2, \quad (7)$$

where m_0 is a constant. Thus m may be regarded as a function of the value of ρ at the instant when the photometric match is established. Unless special precautions be taken, the focus of the Maxwellian half-beam will not necessarily coincide at this instant with the centre of luminous efficiency. Let us assume, for instance, that it is equally likely to take up any position in the pupil such that

$$\rho < \rho_0 = 0.3 \text{ cm.}, \quad (8)$$

so that the probability of ρ lying in the interval $(\rho, \rho + d\rho)$ is

$$(\pi \rho_0^2)^{-1} 2\pi \rho d\rho = \frac{2\rho d\rho}{\rho_0^2}. \quad (9)$$

It is assumed that if ρ should temporarily exceed ρ_0 , the focus will approach the edge of the pupil, and the observer will instinctively adjust the relative positions of pupil and focus so as to satisfy (8).

With these assumptions we can estimate the order of size of the probable error σ_{s-c}

in m due to the assumed distribution of ρ , as follows:-

$$\begin{aligned} r_{S-C} &= 0.6745 \sqrt{(\overline{m^2} - \bar{m}^2)} \\ &= 0.6745 \times 2.5 \rho \sqrt{(\overline{\rho^4} - \bar{\rho}^2)} \\ &= 0.6745 \times 2.5 \rho \sqrt{\left[\frac{2}{\rho_0^2} \int_0^{\rho_0} \rho^5 d\rho - \left(\frac{2}{\rho_0^2} \int_0^{\rho_0} \rho^3 d\rho \right)^2 \right]} \\ &= 0.6745 \times 2.5 \rho \left(\rho_0^2 / \sqrt{12} \right). \quad (10) \end{aligned}$$

With $\rho = 5 \text{ cm.}^{-2}$, $\rho_0 = 0.3 \text{ cm.}$, this gives

$$r_{S-C} = 0^m .22. \quad (11)$$

There may well be a tendency on the part of observers to place the eye in the position where the luminosity of the Maxwellian view is greatest, i.e. where $\rho = 0$. This tendency may be represented by a distribution of ρ favouring smaller values, e.g. we may take the probability of ρ lying in $(\rho, \rho + d\rho)$ as $\rho_0^{-1} d\rho$ [cf. (9)]. A similar calculation then leads to almost the same value of r_{S-C} . Thus

$$r_{S-C} = 0.6745 \times 2.5 \rho \left(\frac{2 \rho_0^2}{3 \sqrt{5}} \right); \quad (12)$$

for $\rho = 5 \text{ cm.}^{-2}$, $\rho_0 = 0.3 \text{ cm.}$,

$$r_{S-C} = 0^m .23. \quad (13)$$

This is several times as large as the photometric probable error, and therefore by no means negligible. On the assumption of independence of r_{S-C} and r , the resultant probable error r' may be calculated from the law of compound error (Whittaker and Robinson 1944, pp. 175-176)

$$r'^2 = r_{S-C}^2 + r^2; \quad (14)$$

the S - C component predominates.

It appears, then, that the arrangement in which one half-field is Maxwellian and the other a diffusing surface is not to be recommended. It must be admitted, however, that the above analysis involves a number of arbitrary assumptions; experimental confirmation is therefore necessary before even the qualitative result of an augmentation of probable error under these conditions can be accepted. It has been suggested, for example, by Steavenson (1952) that an individual reading of m does not depend, through (7), on a unique value of ρ , but that the observer moves his eye about a good deal, and adopts a photometric match corresponding to a sort of mental average of the values of m corresponding to the various values of ρ which thus occur. Such a procedure would presumably reduce considerably the influence of the S - C phenomenon.

Using an aperture of $2a_c = 32$ cm., Dufay (1947) has determined the precision of a photometric match from the dispersion of 555 separate observations on a star of $m = 5$. The result is in close agreement with that predicted from the empirical formula 2(6). The juxtaposed semi-circular half-fields used in the investigation on which 2(6) is based were both diffusing surfaces, so the S - C phenomenon could not have any effect here. The two observers obtained results in close agreement, and one of them was Dufay. The close agreement between the laboratory and observatory results therefore indicates that the latter were not influenced by the S - C phenomenon.

One cannot conclude, however, that the phenomenon is without the effect which has been suggested, for the Maxwellian luminance of the star in Dufay's observations is given by 2(13) as $\mathcal{L}_{e,h} = 1.5 \times 10^{-6}$ cd.cm.⁻², and, for most observers, the S - C exponent ρ is nearly zero at this luminance level. In any case, the artificial pupil of diameter 0.2 cm. used by Dufay would tend to diminish the effect of the S - C phenomenon.

Our conclusion, then, is that somewhat precarious theoretical reasoning suggests that, when $m - 5 \log a_0 < -3$, the S - C phenomenon may introduce an independent source of error, of the order of one-fifth of a magnitude, into comparisons between Maxwellian and diffusing half-fields, and that experimental evidence is not available to refute this suggestion. Whether the S - C phenomenon has any serious influence may depend on the way the individual observer applies his eye to the focal plane, and on his peculiar method of arriving at the photometric match.

One way of eliminating the possibility of S - C error in matching a Maxwellian and a diffusing half-field would be to use a dental plate, upon which the observer bites, secured to the telescope tube; the position of the pupil in the focal plane could then be adjusted and kept fixed. Such an arrangement could be used only in conjunction with a telescope, e.g. a coude', in which the eyepiece is not required to execute a diurnal motion. If a dental plate could be used, there would then be only two small sources of S - C error. The first - changes in pupil diameter causing changes in luminosity of the diffuser - is probably negligible if the physiological state, in particular the state of adaptation, is constant; the second - lateral shifts of the pupil relative to the focus due to lack of complete rigidity of connexion between the dental plate and the head or to yielding of the dental plate mount, and to rotation of the eyeball in scanning the field - may be evaluated as follows. A rotation of the eyeball of 1° is accompanied by a linear movement of the pupil of about 0.02 cm. (Lord and Wright 1950, pp. 10-11). Since the optimum field diameter [equation 2(7)] is $2\alpha_b \doteq 4^\circ$, we may envisage displacements of up to ± 0.04 cm. Neglecting displacements due to lack of rigidity, we may postulate a uniform distribution of ρ in $(0, \rho_0 = 0.04 \text{ cm.})$; then (12) yields

$$\wedge_{S-C} = 0^{\text{m}}.004, \quad (15)$$

which is completely negligible.

The above treatment assumes that the image of the star under observation remains fixed in the focal plane of the telescope, i.e. that the driving errors of the equatorial motion are negligible. For an object-glass of aperture $2a_o$ used to produce, with or without the aid of an auxiliary optical system at the eye end (Dufay 1947), a Maxwellian field of angular subtense 2α , the linear separation S_o in the focal plane of the images of two stars whose angular separation in the sky is σ_o is given by

$$S_o = \frac{a_o \sigma_o}{\alpha}. \quad (16)$$

An estimate of the S - C probable error due to driving errors of the order of σ_o is obtained by putting the above expression for S_o in place of p_o in (12), so that

$$\wedge_{S-C} = 0.6745 \times 2.5 \rho \left(\frac{2}{3\sqrt{5}} \right) \left(\frac{a_o \sigma_o}{\alpha} \right)^2. \quad (17)$$

With the ordinary clock drive of the average refractor of six to ten inches aperture, the driving errors in two-minute runs are of the order of $10'' = 5'' \times 10^{-5}$, though they may be much smaller in the larger observatory instruments (Steavenson 1952). Substituting $\sigma_o = 5'' \times 10^{-5}$, $a_o = 10$ cm., $\alpha = \alpha_b = 0^{\text{r}}.035$ in (17), we obtain

$$\wedge_{S-C} = 0^{\text{m}}.0005; \quad (18)$$

thus neglect of guiding errors is justified.

In most cases, it would probably be more convenient not to use a dental impression to locate the eye of the observer in the focal plane, but,

instead, to give up the idea of using a diffusing surface in the comparison half-field. Both half-fields may then be made Maxwellian, as in Fabry and Buisson's (1922) universal photometer. By making the images of the real star and of the artificial star (illuminating the comparison half-field) coincide in the focal plane, one can ensure that, irrespective of lateral movements of the pupil, the values of ρ associated with the two half-fields are identical.

Any differences between the two values of ρ will be due to driving errors.

If B_o be the effective luminance of the comparison half-field, and B_e that due to the star under observation, we have

$$B_o \eta(\rho) = B_e \eta(\rho + s \cos \theta), \quad (19)$$

where s is the linear driving-error deviation, making an angle θ with the radius vector ρ , of the real-star image from the comparison image. Hence, as before,

$$m = m_o - 2.5 \log B_e + 5 \rho \cos \theta, \quad \dots (20)$$

where we have neglected the term $(s \cos \theta)^2$ beside $2 \rho s \cos \theta$. For simplicity, we shall set $\cos \theta = 1$, and assume that ρ is uniformly distributed in $(0, S_o)$. A calculation similar to that of (12) then leads to

$$\begin{aligned} \sigma_{s-z} &= 0.6745 \times 2.5 \rho \left(\frac{\sqrt{7}}{6} \rho \cdot S_o \right) \\ &= 0.6745 \times 2.5 \rho \left(\frac{\sqrt{7}}{6} \frac{\rho_o \omega_o \sigma_o}{\alpha} \right). \end{aligned} \quad (21)$$

Substitution of the same numerical values as before ($\rho = 5 \text{ cm.}^{-2}$, $\rho_o = 0.3 \text{ cm.}$, $\omega_o = 10 \text{ cm.}$, $\alpha = 0^{\circ} .035$,

$\sigma_0 = 5^h \times 10^{-5}$) gives

$$\sigma_{S+C} = 0^m .016. \quad (22)$$

This is of the same order of size as the photometric probable error; it may, however, be a considerable over-estimate, in view of the arbitrary assumption of $\cos \theta = 1$.

While no measurements of the Stiles-Crawford phenomenon using punctiform sources imaged on the retina appear to have been made, we may suppose that the phenomenon is quantitatively the same in this case. On the analogy of (1), we shall have

$$\bar{\eta} S E_e = \bar{\eta}' S' E_e', \quad (23)$$

where $\bar{\eta}$ is averaged over the area S of the pupil covered by the illumination E_e . The analogue of (7) is

$$m = m_0 - 2.5 \log E_e - 2.5 \log \left(\frac{S \bar{\eta}}{S' \bar{\eta}'} \right). \quad (24)$$

S - C errors are associated with inconstancy of the last term; thus they can be eliminated by making the areas S, S' identical. In other words, the real star and the artificial star must have identical exit pupils. This condition is, of course, essential, even when $m - 5 \log a_0 > -3$, and the S - C phenomenon does not exist, for then (24) reads

$$m = m_0 - 2.5 \log E_e - 2.5 \log \left(\frac{S}{S'} \right), \quad (25)$$

and it is necessary to secure the constancy of S/S' . Needless to say, there are no S - C errors in naked eye observations, the locus classicus of the punctiform mode, since here both stars illuminate the whole of the pupil.

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8. PHOTOMETRIC PRECISION IN THE SUPERFICIAL MODE OF OBSERVATION

Summary. The empirical formula of Dufay and Schwégler for the precision of superficial photometry as a function of the effective luminance B_e and the angular semi-diameter α of the field is expressed in the form

$$\tau_{N,S} = \frac{0.772}{16.4 (10 + \log B_e) - (2\alpha)^{-1}}$$

($0^\circ .025 < 2\alpha < 0^\circ .100$, $B_e < 10^{-6}$ cd.cm.⁻², $\tau_{N,S} < 0^\circ .077$)

....(A)

where $\tau_{N,S}$ is the photometric probable error for non-scintillant, homochromatic half-fields. It is assumed that, for observations by the author, the probable error is $\zeta_g \tau_{N,S}$, where ζ_g is a constant factor. ζ_g is determined by observations, with $B_e = 2 \times 10^{-2}$ cd. cm.⁻², $\alpha = 0^\circ .20$, $0^\circ .10$, $0^\circ .05$, using a Maxwellian-view apparatus in which the photometric control depends on a modified inverse-square law, which is derived. The results confirm the functional dependence of $\tau_{N,S}$ on α expressed by (A), but $\zeta_g (= 1.8)$ is surprisingly large; this may be due to the low 'gear ratio' of the photometric control, to the discomfort occasioned by the use of a dental impression, and, in particular, to the presence of a dark line, due to diffraction at the edge of the silvering in the photometer cube, separating the two half-fields. That the discrepancy is not due to observer differences is shown by the agreement of the results of three other subjects using the same apparatus.

The purpose of the remaining sections of this treatise is to effect a detailed comparison of the punctiform (P) and superficial (S) modes of observation under various conditions. For the time being, we shall simplify the problem by supposing that the two half-fields are homochromatic, and that their photometric properties are steady in time. The important questions connected with heterochromatic photometry and with the effect of stellar scintillation will be relegated to sections 12 and 13, respectively. In order to compare the precisions of the P and S modes, we shall require homogeneous data, i.e. data based on observations by the same subject, on their probable errors. The author (observer J.E.G.) will be adopted as the standard subject.

The photometric precision in the S mode depends chiefly on two parameters, the effective luminance $\mathcal{B}_e$ and the angular diameter 2α of the field, supposed bipartite and circular. We have seen in section 2, however, that, for a given telescope and star, these two parameters are not independent. It turns out, in fact, that there is a unique, fixed value α_b of α , independent of the telescope and star, which gives a minimum photometric probable error for any given telescope and star. The associated effective field luminance has been called the Maxwellian luminance $\mathcal{B}_{e,b}$; it is connected with the magnitude m of the star and the semi-aperture a_o of the telescope by equation 2(12), i.e.

$$\log \mathcal{B}_{e,b} = -0.4 (m - 5 \log a_o) - 6.22. (1)$$

The Maxwellian luminance is thus a suitable parameter in terms of which to express the probable error of the S mode.

Dufay and Schwégle's empirical formula 2(6) for the precision ρ of superficial photometry may be expressed in terms of the probable error

$r_{N,S}$ (in magnitudes) of a single observation by setting $\alpha = \beta_e$ in 3(3), so that

$$\begin{aligned} r_{N,S} &= \frac{0.6745}{\sqrt{n-1}} \left| \frac{dm}{d\beta_e} \right| [(\beta_e - \bar{\beta}_e)^2]^{1/2} \\ &= \frac{0.6745}{\sqrt{n-1}} \left(\frac{2.5 \log e}{\beta_e} \right) [(\beta_e - \bar{\beta}_e)^2]^{1/2}. \quad (2) \end{aligned}$$

[Since we shall later have to consider the effect of scintillation on the probable error, the suffix N (non-scintillant) is employed here to indicate the absence of scintillation; the suffix S, of course, denotes superficial].

From 2(5a),

$$\frac{1}{\rho} = \frac{1}{n} \cdot \frac{1}{\beta_e} [(\beta_e - \bar{\beta}_e)^2]^{1/2}; \quad (3)$$

combining this with (2), we have

$$\begin{aligned} r_{N,S} &= 0.6745 (2.5 \log e) \sqrt{\frac{n}{n-1}} \cdot \frac{1}{\rho} \\ &= \frac{0.772}{16.4 (10 + \log \beta_e) - (2\alpha)^{-1}} \end{aligned}$$

$$(0.025 < 2\alpha < 0.100, \beta_e < 10^{-5} \text{ cd.cm.}^{-2}, r_{N,S} < 0^m.077) \quad \dots (4)$$

When $\alpha = \alpha_b$, β_e can be identified with the Maxwellian luminance $\beta_{e,b}$ and we have, using 2(7),

$$\begin{aligned} r_{N,S} &= \frac{0.772}{16.4 (10 + \log \beta_{e,b}) - 14.3} \quad (-8.5 < \log \beta_{e,b} < -6) \\ &\dots (5) \end{aligned}$$

Formulae (4) and (5) are for Dufay and Schwégler as observers. For observer J.E.G., it will be assumed that, instead of (4), we may write

$$\hat{N}_{S} = \frac{0.772 \zeta_G}{16.4 (10 + \log L_e) - (2\alpha)^{-1}}, \quad (6)$$

where ζ_G is a factor independent of B_e, α .

A rough experiment was carried out to determine the constant ζ_G . Several other subjects, in addition to J.E.G., were employed, in order to gain some idea of observer differences. The luminance level used was $L_e = 2 \times 10^{-8}$ cd.cm.⁻², somewhere near the lower limit of applicability of the empirical formula, but well above the threshold (10^{-10} cd.cm.⁻²). Dufay's (1947) remark that observations become especially painful at luminances below about 5×10^{-8} cd.cm.⁻² could thus be verified.

(figure 1)

The apparatus, consisted of a photometric cube C for producing a bipartite field, and two symmetrically disposed optical systems for illuminating the two halves of the field. Both systems were illuminated by the same lamp S, in order to eliminate the effects of source fluctuations.

The right-hand half of the field, as seen in the figure, was of variable luminance, while the left-hand half, illuminated via the optics with the primed symbols, was maintained at constant luminance. In view of the symmetry of the system, only the optical system with the unprimed symbols need be described. Light from the source S is reflected on to the diffusing screen D of white card by the plane mirror M. The small circular aperture P is, in turn, illuminated by the screen D. Opaque, blackened screens B_1 , B_2 prevent light from S from reaching P, or any subsequent point in the system, otherwise than by reflexion at M followed by diffusion at D. The aperture P is at the focus of the biconvex lens L, so that a parallel beam of light enters the cube C, and is reflected through a right angle at the silvered surface covering half the diagonal plane of the cube.

The luminance of the diffusing screen D is varied by moving D along the axis of the system PL (in the direction of ξ increasing or decreasing), so as to vary the distance DS' of D from the image S' of S in the mirror M. The screen D' is kept fixed, so that its luminance remains constant.

The two parallel half-beams originating from P, P' , and emerging side-by-side from the photometer prism C, are brought to a common focus in the centre of the eye pupil by a biconvex lens L'' of focal length f'' . This lens is placed at a distance f'' from the edge of the silvering in C, so as to project the dividing line of the photometric field to infinity. An angular field diameter of approximately 2α radian is obtained by mounting a circular diaphragm of linear diameter $2\alpha f''$ almost in contact with the face of C nearest to L'' .

The luminance of 2×10^{-8} cd.cm.⁻² is well below that at which the Stiles-Crawford phenomenon probably becomes negligible [cf. 7(3)], but, in order to be on the safe side, care was taken to ensure that the images of P, P' formed at the focus of L'' should coincide. This was done by providing adjustments to enable the cube C to be tilted about horizontal and vertical axes in the plane of the silvering until the two images, observed with an eyepiece, were seen superimposed. In order to obtain as small a probable error as possible, it was desirable to work with the maximum luminous efficiency, i.e. with the foci of the two beams located at the centre of luminous efficiency. The position of the latter could not be accurately determined with the equipment available, but it seemed to be near the geometrical centre of the pupil. A wax dental impression (cf. Wright 1946, p.60) was therefore provided to enable the double image to be centred in the observer's pupil. The position of the dental impression mount was adjusted horizontally and vertically until the desired position was attained; this was recognized by the symmetrical appearances seen by the observer when the head was rocked slightly about horizontal and vertical axes parallel to the plane of the pupil, while contact was maintained with the dental plate.

The effective luminance of the field was calculated from the photometric and geometrical constants of the apparatus. If the candlepower of the source S in the direction of the mirror M be I_s , the reflexion factor of M be ρ_M , and the luminance factor of D be β_D , then the luminance of the screen D will be

$$\beta_D = \frac{\beta_D \rho_M I_s \cos \theta}{\pi s^2}, \quad (7)$$

where ι is the angle of incidence of the light falling on D, and δ is the distance of D from the image S' of S. For all points of the lens L from which the aperture P is seen filled by D, the luminance of P is uniform and equal to that of D. Hence, the illumination at L is

$$E_L = \frac{\pi a_p^2 B_D}{f_L^2} = \frac{\beta_D \rho_M I_S a_p^2 \cos \iota}{f_L^2 \delta^2}, \quad (8)$$

where a_p is the radius of P, and f_L is the focal length of L. The illumination at L' will be

$$\tau_o E_L = \tau_o \beta_D \rho_M I_S \left(\frac{a_p}{f_L \delta} \right)^2 \cos \iota, \quad (9)$$

where τ_o is the transmission factor of the optical elements L, C, L'. In accordance with (24), the effective luminance of the lens L', and therefore of the photometric field, is

$$B_e = \frac{f'^2 \tau_o E_L}{\pi a_e^2} = \frac{\tau_o \beta_D \rho_M I_S \left(\frac{f' a_p}{f_L a_e \delta} \right)^2 \cos \iota}{\pi} \quad (10)$$

We may assume $\tau_o = 0.7$, $\beta_D = 0.75$, $\rho_M = 0.88$, $R_D = 0.3$ cm., $\cos \iota = 1.0$. The candlepower I_S of the source S was measured by Lummer-Brodhun photometry as 0.087 cd. The focal lengths of the lenses used were $f_L = 40$ cm, $f' = 8.33$ cm. The radius of the aperture P was $a_p = 0.125$ cm. The mean value of δ was found to be 68.9 cm.; this was derived, in a manner described below, from the readings of the co-ordinate ξ obtained during the course of the observations. When these numerical values are substituted in (10) the result is

$$B_e = 2 \times 10^{-8} \text{ cd. cm.}^{-2}. \quad (11)$$

In terms of the co-ordinate ξ , the photometric probable error is given by 3(3) as

$$r_{N,\xi} = \frac{0.6745}{\sqrt{(n-1)}} \left| \frac{dm}{d\xi} \right| [(\xi - \bar{\xi})^2]^{1/2}. \quad (12)$$

Here

$$\left| \frac{dm}{d\xi} \right| = \left| \frac{dm}{dB_e(\xi)} \cdot \frac{dB_e}{d\xi} \right| = \frac{2.5 \log e}{B_e(\xi)} \left| \frac{dB_e}{d\xi} \right|. \quad (12a)$$

B_e may be expressed as a function of ξ with the aid of figure 2. From (10),

$$B_e = \frac{K}{f^2} \cos \epsilon = \frac{K(x+x')}{\{(x+x')^2 + y_2^2\}^{3/2}}, \quad (13)$$

where K, x', y_2 are constants, while x is related to the co-ordinate ξ by an equation of the form

$$x = x_0 - \xi, \quad (14)$$

in which x_0 is a constant. Substituting for x in (13),

and writing

$$\alpha' + \alpha_0 = \xi_0, \quad (15)$$

one obtains

$$B_e(\xi) = \frac{K(\xi_0 - \xi)}{\{(\xi_0 - \xi)^2 + y_2^2\}^{3/2}}, \quad (16)$$

whence

$$\frac{dB_e}{d\xi} = \frac{2K \left[1 - \frac{3}{2} y_2^2 \{(\xi_0 - \xi)^2 + y_2^2\}^{-1} \right]}{\{(\xi_0 - \xi)^2 + y_2^2\}^{3/2}}. \quad (17)$$

From (14), (15),

$$\xi_0 - \xi = \alpha + \alpha', \quad (18)$$

so (17) may be written

$$\frac{dB_e}{d\xi} = \frac{2K}{\delta^3} \left\{ 1 - \frac{3}{2} \left(\frac{y_2}{\delta} \right)^2 \right\}. \quad (19)$$

The expression in braces is the factor correcting for the deviation from the inverse square law due to the fact that the image S' of the source is displaced a distance y_2 from the ξ -axis. Equation (16), for B_e itself, may be written in a form similar to (19):-

$$B_e = \frac{K}{\delta^2} \left\{ 1 - \left(\frac{y_2}{\delta} \right)^2 \right\}^{1/2} \\ = \frac{K}{\delta^2} \left\{ 1 - \frac{1}{2} \left(\frac{y_2}{\delta} \right)^2 \right\}. \quad (20)$$

Substituting from (19), (20) into (12a), one obtains

$$\frac{dm}{d\xi} = (2.5 \log e) \frac{2}{\delta} \left\{ 1 - \left(\frac{y_2}{\delta} \right)^2 \right\}_{\xi = \bar{\xi}}. \quad (21)$$

To obtain the constants $\delta(\bar{\xi})$, $y_2(\bar{\xi})$ entering into (21), the screen D was set in the position corresponding to the mean of all the values of ξ observed. The distances DM, MS ($= MS'$), y_1 were measured directly; and δ , y_2 were then calculated from the geometry of figure 2. The results, expressed in inches (the units of the scale from which ξ was read) were

$$\delta = 27.8 \text{ in.}, \quad y_2 = 3.8 \text{ in.} \quad (22)$$

Putting these values in (21) enables $dm/d\xi$ to be evaluated and substituted in (12); hence, with $n = 10$,

$$\Delta'_{N,S} = 0.0172 \sqrt{[(\xi - \bar{\xi})^2]}. \quad (23)$$

The field diameters used were $2\alpha = 0''.20$, $0''.10$, and $0''.05$; the last-mentioned is the smallest field size with which reliable observations were possible. Observer J.E.G. completed three sets, of ten readings each, for each value of 2α . These observations were duplicated by observer P.W.T. Observer J.E.M.B. completed three sets for $2\alpha = 0''.20$, and observer E.G.T. completed three sets for $2\alpha = 0''.10$.

The observer was kept in the dark room for a period varying from five to fifteen minutes, and then asked to make a photometric match. This was done by rotating a drum round which passed an endless cord attached to a

carriage moving along a straight slide and bearing the diffusing screen D. The observer was encouraged to effect the match by the bracketing method, whereby the range in which the match point lies is gradually narrowed down, as quickly as possible; about thirty seconds was the usual time taken. A second person then recorded the reading E on the scale attached to the slide. A period of one minute was allowed to elapse before the next, independent photometric setting, to enable the observer to rest. A complete observing session comprised ten matches, separated by nine rest periods of one minute, or, in the case of observer J.E.M.B., of two minutes, each.

The values of the photometric probable error $\Delta_{N,s}$ calculated from these observations are presented in table 1; table 2, which is a double-entry table formed by taking the root-mean-square value [cf. 3(11)] of each column in table 1, shows the closeness of the agreement, for each value of 2α , between the different observers. While the tendency of the values of $\Delta_{N,s}$ to increase as 2α decreases is in qualitative agreement with Dufay and Schwégler's results, there is a significant difference between the numerical values and those calculated from equation (4) with $B_s = 2 \times 10^{-8}$ cd.cm.². The precision of Dufay's matches appears, in fact, to be about twice as great as that of ours.

A satisfying explanation of the discrepancy has not been found. It may be partly due, however, to the fact that several turns of the control drum were necessary to move the screen D from the position where one half-field was perceptibly fainter than the other. Le Grand (1933) has shown that with such a low 'gear ratio' the probable error may be nearly twice as great as that observed when the photometric change mentioned is effected by a rotation of

TABLE 1

Superficial photometric probable error $\sigma_{N.S.}$, with
 $E_s = 2 \times 10^{-3} \text{ cd.cm.}^2$, for four observers

J.E.M.B.		J.E.G.			
$2\alpha = 0^r.20$		$2\alpha = 0^r.20$	$2\alpha = 0^r.10$	$2\alpha = 0^r.05$	
47		37	69	63	
52		58	58	39	
51		44	47	98	
E.G.T.		P.W.T.			
$2\alpha = 0^r.10$		$2\alpha = 0^r.20$	$2\alpha = 0^r.10$	$2\alpha = 0^r.05$	
81		40	73	65	
77		46	73	85	
81		45	62	73	

The probable error is expressed in units of $0^m.001$.

TABLE 2

Summary of table 1

Observer:- J.E.M.B. J.E.G. E.G.T. P.W.T. Eqn 8(4)

2α					
$0^{\circ}.20$	50	47	-	44	24
$0^{\circ}.10$	-	59	80	70	28
$0^{\circ}.05$	-	71	-	75	43

The probable error is expressed in units of $0^{\circ}.001$.

of about half a turn, owing to the more favourable intervention of the muscular sense of the hand in the latter case.

It may not be irrelevant to point out the difference between the viewing conditions used by us and by Dufay and Schwégler, who observed diffusing surfaces. In their case, it was not necessary to clamp the head by means of a dental plate; it may be that our observers were incommoded by this device.

An even more cogent consideration is that, with diffusing half-fields, it is easy to eliminate the line of division completely, so that the field appears uniform when photometric equality is attained; but that, on account of diffraction at the edge of the photometer prism or the like, this is practically impossible with the Maxwellian view (Martens 1899, Brodhum and Schönrock 1904, Le Grand 1933, Terrien and Desvignes 1949). While a dividing line of width $1'$ is sensibly innocuous, greater widths cause a marked reduction of precision, by a factor of one half, indeed, for a width of $3'$ (Le Grand 1933).

The approximate theory given by Terrien and Desvignes (1949) indicates that, when the diameter of the pinhole image in the plane of the eye pupil is negligible by comparison with the diameter $2a_e$ of the pupil, the angular width of the dark line is approximately $\epsilon_1 = 10^{-4} (2a_e)^{-1}$. For $a_e = 0.3$ cm., this gives $\epsilon_1 = 1.7 \times 10^{-4} \approx 0'.6$, for which we should not expect any appreciable reduction of precision. Terrien and Desvignes found that the black line became visible when the calculated ϵ_1 exceeded a value of the order of $15''$ to $50''$. In the present case, the line was prominent, and it may be that, for one reason or another, its width exceeded the calculated value. Whether or not ϵ_1 was

appreciably greater than the $1'$ required by Le Grand (1933), it is tempting to postulate the line of separation as contributing at least a part of the augmentation of probable error.

It is clear that the discrepancy cannot be due to differences between observers, for it is scarcely conceivable that our four observers should all conspire to exhibit probable errors twice as great as those of Dufay and Schwégler, who were also in close mutual agreement.

Whatever the reason for the large discrepancy between the values of $\sqrt{N,s}$ for J.E.G. and those computed from equation (4), we have no alternative but to adopt a value of ζ_G in the neighbourhood of 2; we shall take

$$\zeta_G = 1.8. \quad (24)$$

The analogue of (5) for observer J.E.G. is

$$\begin{aligned} \sqrt{N,s} &= \frac{0.772 \zeta_G}{16.4 (10 + \log B_{e,b}) - 14.3} \\ & \quad (-8.5 < \log B_{e,b} < -6); \end{aligned} \quad (25)$$

substituting for $\log B_{e,b}$ from (1), and for ζ_G from (24), one obtains

$$\begin{aligned} \sqrt{N,s} &= \frac{0.085}{2.91 - 0.4 (m - 5 \log a_0)} \\ & \quad (0 < m - 5 \log a_0 < 5). \end{aligned} \quad (26)$$

From this formula one notes, for example, that the probable error of a single superficial photometric reading by observer J.E.G. with a telescope of aperture 20 cm. and focal length

$10/0.035 = 286$ cm. should be about $0^m.03$ for a star of magnitude 5, and about $0^m.09$ for a star of magnitude 10; outside these limits of magnitude, the empirical formula does not apply.

* * * * *

9. PHOTOMETRIC PRECISION IN THE PUNCTIFORM MODE OF OBSERVATION

Summary. The non-scintillant, homochromatic punctiform photometric probable error $\Delta_{N,P}$ is measured, using the author as subject, by means of an apparatus in which the two stars compared are the reduced images of illuminated apertures in an opaque screen, and the photometric control is by means of polaroid screens. The ocular magnitude of the stars is calculated from a knowledge of the geometrical-optical constants of the apparatus, of the axial candle-power of one of the illuminated apertures, and of an approximate value of $\Delta_{N,P}$. The equipment incorporates its own internal standard of magnitude to obviate the necessity to repeat this calibration after an arbitrary change in the configuration of the apparatus. A subsidiary experiment shows that the correction for unpolarized light between the polarizer and analyser is negligible.

The dependence of $\Delta_{N,P}$ on the angular separation σ , on the ocular magnitude m_e , and on the field luminance B_f is investigated. It is shown that, for $m_e = 1.5$, $B_f = 0$, there is no significant dependence of $\Delta_{N,P}$ on σ in the range ($3'$, $20'$). The advantages of foveal viewing, as adopted here, are discussed. For $\sigma = 5'$, $B_f = 0$, the dependence of $\Delta_{N,P}$ on m_e is well represented by the empirical function

$$\Delta_{N,P} = 0.0409 - 0.01423 m_e + 0.003237 m_e^2 + 0.001119 m_e^3$$

$$(-3 < m_e < 4), \quad (A)$$

which has a minimum $\gamma_{N,P} = 0.030$ at $m_e = 1.3$.

To study the effect of an illuminated field on the probable error, a circular field of diameter $3''.8$ is superposed on the star field. Values of E_f ranging from 10^{-7} to 10^{-2} cd.cm.⁻² are used in conjunction with magnitudes $m_e = -0.5$, $+1.7$, $+3.0$, and $\sigma = 5'$. It is found that $\gamma_{N,P}$ is unaffected while $E_f < 10^{-6}$ cd.cm.⁻²; for $m_e = 1.7, 3.0$, it increases indefinitely with E_f when $E_f > 10^{-6}$ cd.cm.⁻², and more rapidly the larger m_e ; while for $m_e = -0.5$, $\gamma_{N,P}$ falls slightly to a shallow minimum near $E_f = 10^{-2}$ cd.cm.⁻², after which it increases rapidly. On the basis of these results, it is concluded that the optimum conditions are $\sigma = 5'$ (not critical), $m_e \doteq 1.3$, $E_f < 10^{-6}$ cd.cm.⁻².

This section is devoted to experimental studies of the photometric probable error, for observer J.E.G., of the punctiform (P) mode. There are three important parameters on which the probable error of the P mode depends - the angular separation σ of the two punctiform images, the ocular magnitude m_e , and the luminance B_f of the field in which the images are seen. The non-scintillant punctiform photometric probable error will be denoted by $\Delta_{n,p}$. We shall continue to assume the absence of any colour difference between the two punctiform images, or stars, as we shall call them for brevity.

The apparatus used for investigating the dependence of $\Delta_{n,p}$ on σ , m_e , B_f is shown schematically in figure 3. The stars P'_1 , P'_2 are the images, in the short-focus converging lens L' , of two small, circular apertures P , P_* , illuminated by the source S via the diffusing screen D of ground glass. The stellar

images are projected to infinity by the spectacle lens L'' . The ocular magnitude m_e of P' is fixed, while that of P'_2 is controlled by the angle θ between the axes of the polaroid screens Π , Π' . Errors due to source fluctuations are eliminated by illuminating both stars from a common source.

The common radius α_s of the apertures P, P_v was fixed by the smallest hole which could conveniently be drilled in a metal plate. The minimum allowable separation s of the two apertures was then determined by the consideration that, when the angular separation σ of the two stars was as great as $20'$, the angular diameter 2ϵ of each star should not exceed $1'$. In order to reach angular separations as low as $3'$ with s thus fixed, it was necessary to use an imaging lens L' of fairly high power, and to make the distances $PL' = l_p$, $P'L'' = f''$ quite large. The illumination E at L'' could be controlled by varying the candlepower I_s of S , or by varying its distance l_s from the screen D .

The angular separation σ of the two stars was determined by measuring with a traveling microscope, the linear separation s' of the images P', P'_v , and dividing it by the focal length f'' of the lens L'' . In the same way, the angular diameter of a star was obtained as $2\epsilon = 2\alpha_p/f''$, where α_p' is the radius of the image of P .

If M denote the linear magnification s'/s between the conjugate planes $P'P'_v, PP_v$, and if the axial candlepower of the aperture P be I_p , then the axial candlepower of the image P' is

$$I_p' = \tau' M^2 I_p, \quad (1)$$

where τ' is the transmission factor of the lens L' . It follows that the ocular illumination due to P' is

$$E_s = \frac{\tau^2 \tau' M^2 I_p}{f''^2}, \quad (2)$$

where τ'' is the transmission factor of L'' ; and the ocular magnitude of the star is

$$\begin{aligned} m_e &= -2.5 \log (E_e/E_*) \\ &= -2.5 \log (\tau'' \tau' I_P/E_*) - 5 \log (M/f''). \end{aligned} \quad (3)$$

The candlepower I_P of P was measured by Lummer-Brodhun photometry. The magnification $M = S'/S$ of the lens L' was calculated from measurements, with the travelling microscope, of S, S' . The value $E_*' = 2.13 \times 10^{-10} \text{ lm.cm.}^{-2}$ of $l(3)$ was adopted for E_* .

In order to vary the angular separation σ of the stars, it is necessary to shift the illuminated apertures P, P_V along the axis of the system so as to change their distance l_P from the lens L' , and therefore the magnification M . It can be seen from (3) that, if the ocular magnitude m_e of P is to be kept constant under this change, then I_P must be altered. It would be inconvenient if every such change in I_P entailed its re-measurement by bench photometry, and the re-calculation of m_e with the aid of (3). An internal standard of ocular magnitude m_i , consisting of the image P_i' , formed by reflexion in the convex surface of L' nearer to L'' , of a small source run at constant current, is therefore incorporated in the apparatus.

The magnitude m_e of P' can be measured in terms of that of the internal standard by means of the photometric device constituted by the polaroid screens Π, Π' . The polarizer Π' is fixed, while the analyser Π can be rotated in its own plane, thus enabling the magnitude m_g of P_i' to be varied. The star P_i' is matched in turn against P' and against the

internal standard P_1' , neutral filters of densities Δ , Δ_1 being introduced, if necessary, into the beams from P_1' , P_2' , respectively. The difference $m_e - m_i$ can then be calculated from the angles between the polaroid axes for the two matches, and the values of Δ , Δ_1 . In the first instance, when m_e has been calculated from (3), knowledge of $m_e - m_i$ serves to determine m_i ; subsequently the known value of m_i is combined with the measured difference $m_e - m_i$ to furnish m_e .

The law of variation of m_θ with the angle θ between the polaroid axes is complicated by the fact that a small fraction η of the light from P_1' incident on Π is unpolarized, owing to imperfect polarization by Π' , and to scattering after passage through Π' . The illumination at the eye of the observer will be the sum of the illuminations due to the polarized and unpolarized light transmitted by Π ; it will be proportional to

$$(1-\eta) \cos^2 \theta + \frac{1}{2} \eta = \cos^2 \theta (1 - \frac{1}{2} \eta \sec^2 \theta \cos 2\theta); \quad (4)$$

and, if m_0 denote the value of m_θ when $\theta = 0$,

$$m_\theta - m_0 = -2.5 \log \cos^2 \theta - (2.5 \log e) \times$$

$$\ln \left(\frac{1 - \frac{1}{2} \eta \sec^2 \theta \cos 2\theta}{1 - \frac{1}{2} \eta} \right)$$

$$\doteq 5 \log \sec \theta + \kappa (\sec^2 \theta \cos 2\theta - 1), \quad (5)$$

where

$$\kappa = 1.25 \eta \log e. \quad (6)$$

The constant κ was determined by making matches of P_v' against P' , whose magnitude m_e was kept fixed; neutral filters of various densities $\Delta_{v,j}$ were introduced in front of P_v' so that the values θ_j of θ required for the match ranged from 25° to 80° . Thus, if $m_{0,0}$ denote the value of m_0 when $\Delta_v = 0$,

$$m_e = m_\theta = m_{0,0} + 2.5 \Delta_{v,j} + 5 \log \sec \theta_j + \kappa (\sec^2 \theta_j \cos 2\theta_j - 1). \quad (7)$$

This is a system of simultaneous linear equations for the two unknowns $(m_e - m_{0,0})$, κ , as can be seen by writing it in the form

$$(m_e - m_{0,0}) + (1 - \sec^2 \theta_j \cos 2\theta_j) \kappa = 2.5 \Delta_{v,j} + 5 \log \sec \theta_j. \quad (8)$$

Eleven different values of $\Delta_{v,j}$ were each employed twice, in random order, to eliminate the spurious effect of any secular drift in the constants of the apparatus; thus 22 values of θ_j , each the mean of ten readings, were obtained. The resulting 22 simultaneous equations were solved by least squares, with the result that $\kappa = 0.00931$; hence the fraction of unpolarized light incident on Π is

$$\eta = 0.0172. \quad (9)$$

It follows that the size of the correction term $\kappa (\sec^2 \theta \cos 2\theta - 1)$ in (5) is less than $0^m.02$ in the range $0^\circ < \theta < 55^\circ$, so that, with sufficient accuracy,

$$m_\theta - m_0 = 5 \log \sec \theta \quad (0^\circ < \theta < 55^\circ). \quad (10)$$

The observations to determine the probable error $\wedge_{N,F}$ were carried out by observer J.E.G. only. On account of the very great luminance of the stars, it was sufficient to allow a period of one minute for dark adaptation; the observer recorded his own readings with the aid of a dim light which could be switched on for the purpose. Ten readings were recorded in each observing session; the total time taken to make the ten matches and to note the readings was about ten minutes.

To test the dependence of the probable error $\wedge_{N,F}$ on the angular separation σ of the two stars, four sets of ten readings were taken for each of the separations $\sigma = 3', 5', 11', 21'$. The field in which the stars were seen was dark ($\mathcal{B}_f = 0$), and the ocular magnitude was $m_e \doteq 1.5$.

Since the zero-point of the circular scale attached to Π , at which the axes of Π , Π' were parallel, was unknown, it had to be eliminated from consideration by taking alternate readings θ_L, θ_R to the left, right of the zero-point, respectively. Thus, each set of ten readings consisted of five values of θ_L and five of θ_R .

If the mean of the five values of θ_L be $\bar{\theta}_L$, and that of the five values of θ_R be $\bar{\theta}_R$, then

$$\theta = \frac{1}{2} |\bar{\theta}_R - \bar{\theta}_L|. \quad (11)$$

The probable errors of $\bar{\theta}_R, \bar{\theta}_L$ are given by expressions of the form

$$\sigma_{\theta,R}' = \frac{0.6745}{\sqrt{\frac{1}{2}n(\frac{1}{2}n-1)}} \sqrt{\left\{ \sum (\theta_R - \bar{\theta}_R)^2 \right\}}, \quad (12)$$

where $n = 10$; the probable error r'_θ of the angle θ defined by (11) is therefore given by

$$r'^2_\theta = \left(\frac{\partial \theta}{\partial \bar{\theta}_R}\right)^2 r'^2_{\theta,R} + \left(\frac{\partial \theta}{\partial \bar{\theta}_L}\right)^2 r'^2_{\theta,L}$$

$$= \frac{1}{4} \cdot \frac{(0.6745)^2}{\frac{1}{2}n(\frac{1}{2}n-1)} \left\{ \sum (\theta_R - \bar{\theta}_R)^2 + \sum (\theta_L - \bar{\theta}_L)^2 \right\}, \quad (13)$$

whence

$$r'_\theta = \frac{0.6745}{\sqrt{\{n(n-2)\}}} \sqrt{(\sum v^2)}, \quad (14)$$

where

$$\sum v^2 = \sum (\theta_R - \bar{\theta}_R)^2 + \sum (\theta_L - \bar{\theta}_L)^2. \quad (15)$$

If the residuals v be expressed in minutes of arc, it follows that the probable error in magnitudes of the mean of ten photometric readings is

$$r_\theta = \frac{d m_\theta}{d \theta} \left(\frac{\pi r'_\theta}{10800} \right). \quad (16)$$

By differentiating (5), we obtain

$$\frac{d m_\theta}{d \theta} = (5 \log e) \tan \theta \left\{ 1 - \frac{1}{2} \eta \sec^2 \theta \right\}. \quad (17)$$

The term $\frac{1}{2} \eta \sec^2 \theta$ in the braces $\{ \}$ is the correction for the unpolarized light incident on Π ; since it is less than 0.03 for θ in $(0^\circ, 55^\circ)$, it may be neglected, so that

$$\frac{d m_\theta}{d \theta} = (5 \log e) \tan \theta \quad (0^\circ < \theta < 55^\circ). \quad (18)$$

The probable error $\sigma_{N,P}$ of a single observation is greater than σ_0 by a factor of $\sqrt{n}$; thus, from (16), (18), (14),

$$\begin{aligned}\sigma_{N,P} &= \frac{d m_e}{d \theta} \left(\frac{\pi r_0'}{10800} \right) \sqrt{n} \\ &= \frac{0.6745 \pi (5 \log e) \cdot \tan \theta}{10800 \sqrt{(n-2)}} \cdot \sqrt{(\sum v^2)},\end{aligned}\quad (19)$$

or, with $n = 10$,

$$\sigma_{N,P} = 1.51 \times 10^{-4} \tan \theta \cdot \sqrt{(\sum v^2)}.\quad (20)$$

The values of $\sigma_{N,P}$ obtained from the four sets of readings at each of the separations $\sigma = 3', 5', 11', 21'$ are recorded in table 3, together with the root-mean-square values. As mentioned above, each change in σ requires a change in σ_F if m_e is to be kept constant; the necessary change is determined by trial; consequently the values of m_e corresponding to the various values of σ differ slightly from 1.5, as can be seen by reference to the exact values of m_e in table 4. Upon inspection of table 3, it is clear that, for stars of magnitude about 1.5 seen in a dark field, there is no significant dependence of photometric precision on angular separation in the interval $(3', 20')$. There is little doubt that, for values of σ much greater than those used in this experiment, the probable error would be greater than that encountered here. This is hinted at by many authors (e.g. Müller 1897, p.194, Strohmeier 1951, p.185), though few have studied the subject systematically.

For stars of ocular magnitude - 3.4, Mlle Flamant (1950, 1951, 1952) has measured, by the method of constant stimuli (Le Grand c. 1949, pp.264-266), the differential sensitivity,

TABLE 3

Punctiform photometric probable error $\sigma_{N,P}$ as a function of angular separation σ , for $m_3 \pm 1.5$, $S_f = 0$, observer J.E.G.

σ	Set 1	Set 2	Set 3	Set 4	Root-mean-square
3'	38	42	46	27	39
5'	36	31	43	46	39
11'	44	44	47	26	41
21'	27	33	35	33	32

The probable error is expressed in units of $0^m.001$.

a function closely related to the photometric precision. Mlle Flamant found that the differential sensitivity is a maximum near $\sigma = 10'$ or $15'$, decreasing rapidly for smaller distances, where resolution becomes difficult, and more slowly for greater distances up to $30'$. Mlle Flamant also showed that, for white stars with $m_e = -3.4$, $\sigma = 10'$, the precision falls off monotonically as the retinal position of the double-star image recedes from the fovea, the probable error being more than doubled for an eccentricity of 2° . In this respect, the optimum condition, foveal viewing, has been employed throughout our experiments, though Mlle Flamant's added refinement of a separate fixation spot has been dispensed with.

For ocular magnitudes greater than about 3 or 4, however, it becomes difficult or impossible to view a white double star with the fovea (cf. e.g., Hassenstein n.d., Minnaert 1940, p.99), on account of the phototropic tendency of the eye to move so as to image the faint object on the parafovea, which is more sensitive at low luminance levels. Presumably the matching precision of the parafovea would also be greater than that of the fovea for stars of these magnitudes. The use of the parafovea for matching faint stars is attended, however, by difficulties occasioned by the differing relative spectral sensitivities of the fovea and parafovea (Wright 1949, pp. 97-103); and it seems desirable, at any rate in astronomical practice, to avoid this complication by adhering to foveal viewing, with $m_e < 3.5$. The range of magnitudes observable with a consistent receptor field could be extended to fainter stars by using a foveal fixation target to throw the double-star image on to a parafoveal zone, say 3° from the fovea, as suggested by Wright (loc. cit.) but the consequent gain would be more than offset by the serious reduction in photometric precision for the stars with $m_e < 3.5$.

The dependence of the probable error of a photometric match between two non-scintillating, homochromatic stars on the ocular magnitude was investigated with the same apparatus, observer and technique. For stars of magnitude greater than 2, however, a period of three minutes, instead of one minute, was allowed for dark adaptation. The measured probable errors $\wedge_{N,p}$ are set out in table 4, which is arranged in the same way as table 3, and contains all the results of table 3.

The root-mean-square probable errors $\wedge_{N,p}$ are plotted against the ocular magnitude m_e in figure 4. The points are clustered about a unimodal curve, with the right-hand slope rising more steeply than the left. The results may be represented closely by a cubical parabola: the curve computed by least squares is

$$\begin{aligned} \wedge_{N,p} = & 0.0409 - 0.01423 m_e + 0.003237 m_e^2 \\ & + 0.001119 m_e^3 \quad (-3 < m_e < 4); \end{aligned} \quad (21)$$

this curve is drawn in figure 4. The deviations of the points from the curve are of the order of size to be expected from equation 3(15).

The magnitude for which $\wedge_{N,p}$ is a minimum may be obtained by differentiating (21) with respect to m_e , and setting the resulting expression equal to zero. The optimum magnitude is thus found to be 1.31, and the corresponding, minimal probable error is 0^m.030. In view of the flatness of the minimum, the former figure is very ill determined; two decimal places are given merely to fix our ideas, and to avoid rounding-off errors in subsequent calculations. The reader may be struck by other cases of apparently meaningless precision in this treatise, e.g. the coefficients in (21); similar remarks apply, but will not be repeated on every occasion.

TABLE 4

Punctiform photometric probable error $\sigma_{N,F}$ as a function of ocular magnitude m_e , for $E_f = 0$, observer J.E.G.

σ	m_e	Set 1	Set 2	Set 3	Set 4	Root-mean-square
3'	1.45	38	42	46	27	39
5'	-3.85	69	101	59	71	77
	-2.34	61	117	72	61	81
	-1.32	54	121	63	54	78
	-0.31	35	42	42	61	46
	-0.10	56	34	43	52	47
	0.20	35	14	41	20	30
	1.32	38	45	27	23	34
	1.79	36	31	43	46	39
	2.39	31	53	38	22	38
	3.22	43	64	73	68	63
	3.66	91	89	76	100	89
11'	1.44	44	44	47	26	41
21'	-1.49	74	52	67	59	64
	1.60	27	33	35	33	32
	3.07	61	41	79	35	57

The probable error is expressed in units of $0^m.001$.

Mlle Flamant's value of the differential sensitivity, for $m_e = -3.4$, $\sigma = 10'$, corresponds to a probable error $\sigma_{N,P} = 0^m.069$, which is in satisfactory agreement with the value of $0^m.081$ read from the curve in figure 4.

In telescopic observations of very intense stellar objects, discomfort may be reduced by artificially illuminating the field of the instrument, so as to adapt the eye of the observer to a relatively high luminance level. The effect of such an ambient luminance on the punctiform photometric precision was studied experimentally, for non-scintillating, homochromatic stars of three different ocular magnitudes. Müller (1897, p.193) remarks vaguely that most observers agree that photometric comparison of punctiform objects can be carried out most agreeably to the eye with a weakly illuminated field, but that the precision is in no way greater than with quite bright or completely dark fields. The investigation described below shows that the situation is more complex than that.

The apparatus employed was essentially that already described, the angular separation of the two stars being $5'$. The star field was viewed in a direction at right angles to the optical axis, by reflexion in a thin piece of plate-glass mounted, at 45° to the axis, to the left, in figure 3, of the analyser II. A circular field, of diameter $3''.8$, and of uniform luminance, was seen, superposed on the star field, by transmission through this plate. The position of the reflecting plate, mentioned above, was dictated by the need to avoid any change in the field luminance when the analyser was rotated to secure a photometric match. Adjustments were provided to enable the edge of the ambient field to be focused, with the stars, at infinity; and for the stars to be centred in the field.

The ambient field luminance B_f was measured, with sufficient accuracy, by means of a Holophane lumeter which could be mounted at the eye end of the apparatus. The values of B_f which were used in the experiments ranged from 10^{-7} to 10^{-2} cd.cm.⁻²; it was therefore convenient to measure the field luminance by a new quantity

$$\angle_f = 10 + \log B_f. \quad (22)$$

The values of $\angle_f$ for the night sky, the moonlit sky, and the day sky, are about 2, 4, 10, respectively (Le Grand c. 1949, p.86; Russell et al. 1926, p.173).

The probable errors displayed in table 5 were calculated, in the usual way (equation 20), from sets of ten readings by observer J.E.G. A period of dark adaptation, lasting one minute, was allowed only for observing sessions with $\angle_f < 4.5$.

The root-mean-square values of the probable error $\angle_{N,F}$ are plotted against $\angle_f$ in figure 5, with distinct symbols for the three values of m_e used, viz. -0.5, +1.7, +3.0. For small values of $\angle_f$, corresponding to ambient luminances of negligible effect on the probable error, each curve should be asymptotic to a horizontal line of ordinate taken, for the appropriate value of m_e , from the curve in figure 4, for which $B_f = 0$. No attempt has been made to force the curves in figure 5 to satisfy this condition rigorously.

The most striking feature of figure 5 is that there are two different types of curves. For stars of all magnitudes, the precision is sensibly independent of field luminance while the latter is less than that of the moonlit sky. As $\angle_f$ is increased beyond this point, however,

TABLE 5

Photometric probable error $\Delta_{m,p}$ as a function of ambient luminance L_f and ocular magnitude m_e , for $\sigma = 5'$, observer J.E.G.

m_e	L_f	Set 1	Set 2	Set 3	Set 4	Root-mean-square
-0.5	3.90	72	50	66	60	63
	4.80	59	33	43	61	50
	5.92	64	32	22	33	41
	6.88	42	35	53	42	43
	7.88	37	47	25	30	36
1.7	3.88	38	33	44	35	38
	4.24	26	46	33	51	40
	4.99	40	26	57	33	41
	5.00	40	44	53	47	46
	5.56	50	26	40	37	39
	5.93	33	40	59	45	45
	6.48	34	61	66	48	54
	7.14	52	61	83	62	65
3.0	2.99	42	55	54	47	50
	4.09	57	50	34	19	43
	4.88	52	57	68	40	55
	4.94	29	42	47	51	43
	5.59	58	33	57	62	54
	5.92	63	32	22	33	41
	6.23	112	120	115	64	105

$$L_f = 10 + \log B_f.$$

The probable error is expressed in units of $0^m.001$.

the probable error increases indefinitely for the fainter stars ($m_s = 1.7, 3.0$), but decreases slightly for stars of magnitude -0.5 , reaching a minimum near $\angle_f = 8$, and then - though this is not shown by our observations - increasing rapidly. The change from one characteristic to the other must take place in the vicinity of $m_s = 0$. Thus we may say that the precision of matching stars of positive ocular magnitude is adversely affected by an ambient luminance greater than that of the moonlit sky, the effect being greater, the greater the magnitude; while, for stars of negative magnitude, a slight improvement results from the use of a bright field. Upon reflection, this appears to be in accord with common sense.

By plotting $m_s, \angle_f, \epsilon_{N,P}$ in three mutually perpendicular directions, a surface $\epsilon_{N,P} = \epsilon_{N,P}(m_s, \angle_f)$ is obtained. The optimum pair of values of the arguments $m_s, \angle_f$ is that which minimizes the probable error $\epsilon_{N,P}$; in other words, it defines the projection on the $(m_s, \angle_f)$ plane of the point on the surface $\epsilon_{N,P} = \epsilon_{N,P}(m_s, \angle_f)$ nearest to that plane. By building up the surface, from figures 4 and 5, in our mind's eye, we can see that the optimum values are $m_s = 1.31, \angle_f = 0$; and the corresponding, optimal value of the probable error is

$$\epsilon_{N,P}(1.31, 0) = 0^m.030. \quad (23)$$

This result refers to an angular separation of $5'$; but it is likely to be valid for θ in $(3', 20')$.

The recommended conditions for non-scintillant punctiform visual stellar photometry, then, are an angular separation of about $5'$, a field luminance of less than 10^{-6} cd.cm $^{-2}$, and an ocular magnitude $m_s \doteq 1.3$. The first condition

is readily satisfied by suitable mechanical-optical arrangements, and is, in any case, not critical. The second condition is satisfied, in the absence of artificial illumination, by the moonlit sky. The third condition may, in principle, be satisfied for any star, by introducing a neutral filter if its magnitude m be less than 1.3, or by using a telescope of adequate light-grasp if its magnitude be greater than 1.3. The necessary linear semi-aperture in the latter case may be estimated by writing $m_e = 1.31$ in equation 2(3), so that

$$\log a_s = 0.2 (m - 3.92). \quad (24)$$

* * * * *

10. HETEROCHROMATIC PHOTOMETRIC PRECISION

Summary. The effect is considered of a difference in chromaticity between the two half-fields on the photometric probable error, the discussion being simplified by the remark that the chromaticities of the stars are confined to the locus of full radiators, and are therefore characterized by a single parameter, the colour-temperature T . Using the assumption of 5(A) as to the atmospheric spectral transmission factor, the colour-temperature T' near the wavelength λ of a star at altitude h , as modified by the presence of the atmosphere, is found to be given by

$$\frac{1}{T'} - \frac{1}{T} = \frac{2b}{c_2 \lambda} \operatorname{cosec} h; \quad (\text{A})$$

or, with $\lambda = 0.55 \mu.$, $b = 0.067 \mu.^2$, $c_2 = 14360 \mu. \text{K.}$,

$$\frac{1}{T'} - \frac{1}{T} = 1.7 \times 10^{-5} \operatorname{cosec} h. \quad (\text{B})$$

The effect of a difference $\Delta(1/T)$ between the reciprocal colour-temperatures of the two half-fields on the photometric probable error, in both the punctiform and the superficial mode, is discussed. Using the optimum conditions as to σ , m_s , and B_f established in section 9, the dependence of $r_{N,F}$ on $\Delta(1/T)$ is

investigated experimentally. The apparatus is the same as in section 9, except that the photometric control is not polaroids but a wedge of Chance OB.9 blue glass, which is used simultaneously to vary the colour-temperature of one star, that of the other being kept fixed at $T_0 = 8300^\circ\text{K}$. $\Delta_{N,P}$ is observed for a number of values of T' in the comparison half-field in the range $(2500, 7300)^\circ\text{K}$, which corresponds, through (B), to the range of T' exhibited by main sequence stars at an altitude of $9^\circ.6$. To cover the whole range of T' , the range of the wedge has to be extended with the aid of fixed filters of OB.9 glass of two different thicknesses; these introduce two discontinuities in the colour-temperature calibration curve of the wedge, known from theory to be linear. The standard sources S_A , S_B , S_C are used to establish one point of each of the three sections of the curve; and the discontinuities, and hence the slope, are determined with the aid of the photometric calibration curve based on observations with rotating sectors, which turns out also to be linear.

The observations of $\Delta_{N,P}$ indicate no significant dependence of the probable error on $\Delta(1/T)$, for $m_e = 1.3$; though such a dependence would be expected, of course, for much smaller values of m_e , where the colour difference is more readily perceptible.

In general, the photometric probable error is greater when the two half-fields are of different chromaticities than when they are of the same chromaticity, i.e. homochromatic. In this section, the restriction to homochromatic

photometry will be relaxed, and we shall examine the precision of heterochromatic photometry in the P and S modes. We shall, however, continue to assume the absence of scintillation.

The problem is simplified to some extent by the fact, noted in section 4, that, for most stars, the spectral energy distribution in the visible spectrum resembles that of a full radiator at some temperature T . It follows that the chromaticities of the stars are represented by points of the locus, in the chromaticity diagram, of full radiators (cf., e.g., Wright 1944, p.169), and are fully specified by a single parameter, the colour-temperature T .

The spectral distribution of the star-light is modified, however, by its passage through the atmosphere, and through the optical system of the measuring instrument. It will be convenient to assume that the modified radiation is still that of a full radiator, but characterized by a lower colour-temperature T' , so that, with the notation of Section 5,

$$\varphi'_\lambda = \frac{C_0' C_1}{\lambda^5} \exp\left(-\frac{C_2}{\lambda T'}\right). \quad (1)$$

Let us further assume that the modification due to the instrumental spectral transmission may be neglected beside that due to the atmosphere. In section 5, it is shown that, if the zenithal spectral transmission factor T_λ of the atmosphere be given by

$$\ln T_\lambda = -\beta \lambda^{-2}, \quad (2)$$

then

$$\varphi'_\lambda = \frac{C_0' C_1}{\lambda^5} \exp\left\{-\left(\frac{\beta}{\lambda^2} + \frac{C_2}{\lambda T'}\right)\right\}, \quad (3)$$

where $\beta = b \operatorname{cosec} h$; here h is the altitude of the star.

Equating (1) and (3), we find

$$F(\lambda) = \ln\left(\frac{C_0'}{C_0}\right) - \frac{C_2}{\lambda} \left(\frac{1}{T'} - \frac{1}{T}\right) + \frac{\beta}{\lambda^2} = 0; \quad (4)$$

equating the derivatives of (1) and (3) with respect to λ , we find

$$F'(\lambda) + \ln\left(-5 + \frac{C_2}{\lambda T'}\right) - \ln\left(-5 + \frac{2\beta}{\lambda^2} + \frac{C_2}{\lambda T}\right) = 0. \quad (5)$$

For any arbitrary λ , these two simultaneous equations determine C_0' , T' , such that the functions (1) and (3) have equal ordinates and slopes at that λ . By subtraction, followed by slight rearrangement, we eliminate C_0' , and obtain

$$\frac{1}{T'} - \frac{1}{T} = \frac{2\beta}{C_2 \lambda} = \frac{2b}{C_2 \lambda} \operatorname{cosec} h. \quad (6)$$

This is the reciprocal colour-temperature step introduced by the atmosphere for wavelengths in the neighbourhood of λ . In the case of visual observations, we may adopt $\lambda = 0.55 \mu$. For very clear conditions, $b = 0.067 \mu^2$. [section 6, after (24)]; from 4(9), $C_2 = 14360 \mu^\circ \text{K}$. With these substitutions, (6) becomes

$$\frac{1}{T'} - \frac{1}{T} = 1.7 \times 10^{-5} \operatorname{cosec} h. \quad (7)$$

In the S mode, an appreciable difference $\Delta(1/T)$ between the reciprocals of the colour-temperatures of the two half-fields causes a

serious increase in the photometric probable error. How great $\Delta(1/T)$ must be before it becomes appreciable depends, however, on the effective luminance B_e of the field. When B_e is less than about 10^{-7} cd.cm⁻², vision is scotopic, differences in chromaticity are not perceived, and the precision of heterochromatic photometry is equal to that of homochromatic photometry. As B_e increases from 10^{-7} to 10^{-3} cd.cm⁻², the cones, mediating colour vision, come more and more into play, and the value of $\Delta(1/T)$ which can be allowed without causing an increase in probable error steadily diminishes. When $B_e > 10^{-3}$ cd.cm⁻², vision is photopic, and the problems of heterochromatic photometry are present in full force.

How the presence of a chromaticity difference affects the photometric probable error in the P mode is less well known. As was noted in section 4, for stars viewed foveally, vision is photopic for all ocular magnitudes. In the case of very small fields, however, the phenomena of colour vision differ essentially from those observed with large fields (cf. Thomson 1949). In particular, the just noticeable differences of chromaticity are much larger for small fields. It is therefore of interest to examine how large a value of $\Delta(1/T)$ can be tolerated, in the P mode, without detriment to the photometric precision.

This question was investigated experimentally, use being made of a modified version of the apparatus described in the previous section. The studies of that section led to the conclusion that the conditions for the greatest accuracy in homochromatic photometry can be summarized as $m_e \pm 1.3$, $\sigma \pm 5'$, $B_f < 10^{-6}$ cd.cm⁻². The values of the parameters adopted in the present investigation were $m_e = 1.2$, $\sigma = 6'$, $B_f \pm 0$, i.e. practically indistinguishable from

the optimum values. The colour-temperature of one star P' (figure 3) was kept constant at $T_0 = 8300^\circ\text{K}$. The other star P_v' was given various colour-temperatures in the range from 2500°K . to 7300°K . The photometric probable error ΔN_p could thus be observed as a function of $\Delta(1/T)$, for $\Delta(1/T)$ in the interval $(0, 300)$ mireds [$1 \text{ mired} = 1 \text{ micro-reciprocal-degree} = 10^{-6}^\circ\text{K}^{-1}$]. For $A = 9.6$, the atmospheric augmentation of reciprocal colour-temperature is given by (7) as $10^{-4}^\circ\text{K}^{-1} = 100 \text{ mireds}$. Thus, for very clear conditions ($A = 0.067 \mu^2$), a star observed, at an altitude of 9.6 , to have a colour-temperature of $T' = 2500^\circ\text{K}$. has $T = 3300^\circ\text{K}$. outside the atmosphere; similarly, if $T' = 7300^\circ\text{K}$, $T = 30,000^\circ\text{K}$. The chosen range $(2500, 7300)^\circ\text{K}$. therefore corresponds, for $A = 0.067 \mu^2$, $A = 9.6$, to the range $(3300, 30,000)^\circ\text{K}$. outside the atmosphere. This is roughly the range of colour-temperature of main sequence stars (cf. Russell *et al.*, p. 734).

The colour-temperatures of P' , P_v' were varied by means of wedges W , W_v of Chance OB.9 blue glass (Harding 1948) mounted one in front of each of the apertures P , P_v . The two wedges were practically identical; each was of length 15 cm. and width 2.5 cm., and the thickness of blue glass varied from 0.35 cm. at one end to practically zero at the other. In order to give the wedge adequate mechanical strength, it was cemented to a similar wedge of crown glass with its vertex in the opposite direction; this arrangement also eliminated prismatic refraction of light passing through the wedge. In order to secure a finite field of uniform colour-temperature, identical small compensating wedges were provided, one for each aperture. Each compensating wedge was 2.5 cm. square, and was formed of a prism of blue glass ranging from 0.058 cm. to 0.000 cm. in thickness cemented to a crown glass prism from 0.142 cm. to 0.200 cm.

thick. A pair of filters F_1 , F_2 of OB.9 glass of thicknesses 0.20 cm., 0.35 cm., respectively, were provided, to enable the range of colour-temperature given by the wedge W to be extended; an identical pair $F_{1,w}$, $F_{2,w}$ could be used in conjunction with W_w .

Each wedge was mounted in a brass frame which could be moved up and down in a vertical slide by means of an inextensible cord which was conducted, by means of pulleys, to the observing end of the apparatus. Here the cord passed over a drum to a counterpoise weight; the drum could be turned, through a backlash-free reduction gear, by a conveniently-placed control knob, thus imparting a very smooth translatory motion to the corresponding wedge. The frame of each wedge was graduated with an arbitrary linear scale, which could be read to an accuracy of 0.01 cm. by means of a telescope mounted at the eye end and provided with a projector for illuminating the scale.

In order to secure spectral distributions corresponding as closely as possible to those of full radiators at various temperatures, it was necessary to use a tungsten lamp (approximately a full radiator) as the source S , and to dispense with the polaroid screens Π , Π' , which were distinctly green. To avoid fluctuations in the colour-temperature of the lamp, it was fed at constant voltage from a Westinghouse 'Stabilistor'. A resistance Ω was connected in series with S , the size of Ω being chosen so as to give the desired range of colour-temperatures. The OB.9 glass wedge W_w was used as the photometric control; there is, of course, no reason why a photometric wedge should necessarily be neutral.

The spectral transmission factor of a thickness δ of an ideal colour-temperature transforming medium, to which OB.9 glass is a good approximation, is of the form

$$\tau_{\lambda} = \exp \left\{ -t \left(A + B/\lambda \right) \right\}, \quad (8)$$

where A, B are constants. If the spectral distribution of the radiation from S be of the form

$$\phi_{\lambda, S} = \frac{c_0 c_1}{\lambda^5} \exp \left(-\frac{c_2}{\lambda T_S} \right), \quad (9)$$

then the spectral distribution transmitted by a thickness t of the filter will be

$$\begin{aligned} \phi_{\lambda} &= \tau_{\lambda} \phi_{\lambda, S} \\ &= \frac{c_0 e^{-At} c_1}{\lambda^5} \exp \left\{ -\frac{c_2}{\lambda} \left(\frac{1}{T_S} + \frac{Bt}{c_2} \right) \right\} \\ &= \frac{c_0' c_1}{\lambda^5} \exp \left(-\frac{c_2}{\lambda T} \right), \end{aligned} \quad (10)$$

where

$$c_0' = c_0 e^{-At}, \quad \frac{1}{T} - \frac{1}{T_S} = \frac{Bt}{c_2}. \quad (11)$$

Thus the spectral distribution of the transmitted radiation is still that of a full radiator, of colour temperature T , given by (11) as a function of T_S and t .

For a uniform wedge of OB.9 glass, then, the reciprocal colour temperature T^{-1} should be very nearly a linear function of the total thickness t in the field of the compensating wedge, and therefore of the reading x of the scale attached to the wedge; thus

$$\tau^{-1} = k_0 + k_1 x. \quad (12)$$

An attempt was made to determine the constants k_0 , k_1 by colorimetric comparisons with the standard sources A, B, and C, effected in the contrast field of a Lummer-Brodhun head. The wedge W was obscured; adequate illumination was obtained from W_v by removing the diaphragm containing the apertures P, P_v . The comparison field was illuminated by a sub-standard lamp giving a colour-temperature $\tau_A = 2848^\circ \text{K}$. (source A); $\tau_B = 4800^\circ \text{K}$. and $\tau_C = 6500^\circ \text{K}$. were obtained by means of the appropriate liquid filters (Davis and Gibson 1931). The candlepower (1500 cd.) of the sub-standard lamp was so much greater than that of the illuminated field of the wedge that the former had to be reduced by means of a suitable short-focus lens as well as a rotating sector. The photometric match was finally attained by movements of the L - B head along the line joining the two sources; a simultaneous colorimetric match was achieved by adjusting the wedge.

It was found that source A could be matched in colour with the light transmitted by a moderate thickness of W_v alone; but to secure a match with source B it was necessary to introduce the OB.9 filter $F_{1,v}$ in the beam from W_v ; and to match source C, to introduce $F_{2,v}$. The constant term k_0 in (12) takes on different values k_0' , k_0'' when $F_{1,v}$, $F_{2,v}$ are introduced; thus one has

$$\left. \begin{aligned} \tau_A^{-1} &= k_0 + k_1 x_A, \\ \tau_B^{-1} &= k_0' + k_1 x_B', \\ \tau_C^{-1} &= k_0'' + k_1 x_C'', \end{aligned} \right\} (13)$$

where x , x' , x'' denote scale readings with no extra filter, with filter $F_{1,v}$, with filter $F_{2,v}$, respectively. The system (13) is insufficient to determine the four constants k_0 , k_0' , k_0'' , k_1 . This difficulty was overcome by an artifice which will be explained after the photometric calibration of the wedges has been discussed.

Throughout the photometric calibration, the wedge W was kept fixed in the position where its greatest thickness intercepted the light beam from P , and the filter F_2 was retained in position. The star P' was thus maintained at the highest possible colour temperature $T_0 = 8300$ K., as mentioned above. Smaller thicknesses of blue glass were employed in the beam from P_v ; in order to restore photometric equality between the two stars under this condition, it was necessary to introduce a photometric sector R of some fixed angular opening ϑ into the P_v beam. The change in magnitude in the star P_v caused by the introduction of the sector R , which constituted a truly neutral filter of transmission factor ϑ/π , was

$$\Delta_R m = 2.5 \log (\pi/\vartheta). \quad (14)$$

A series of values ranging from $4^{\circ}.5$ to 120° was assigned to ϑ ; with sectors of each of these angles in the P_v beam, the reading x' of the scale of W_v was observed at which a photometric match between the two stars was obtained. For the larger values of ϑ , it was necessary to introduce one or other of the filters F_1 , F_2 , in order to obtain a reading x' , x'' within the compass of the scale.

For each value of ϑ , a set of scale readings corresponding to $n = 10$ independent photometric matches was obtained; the sets were

performed in a random order determined with the aid of a table of random numbers (Fisher and Yates 1948). The whole was repeated three times in different random orders, so as to get $n' = 4$ sets of readings for each value of $\mathcal{Q}$. In table 6, the grand mean scale readings ($x, x',$ or x'') are tabulated with $\mathcal{Q}$ and $\delta_R m$ as arguments.

By plotting $\delta_R m$ against x it was found that the relation between these two quantities was very closely linear; the relations between $\delta_R m$ and x', x'' , respectively, were also linear, and all three lines had nearly the same slope. One could therefore assume

$$\left. \begin{aligned} \delta_R m(x) &= K_0 + K_1 x, \\ \delta_R m(x') &= K_0' + K_1 x', \\ \delta_R m(x'') &= K_0'' + K_1 x'', \end{aligned} \right\} (15)$$

and determine the constants K_0, K_0', K_0'', K_1 by least squares.

By eliminating x between (12) and (15), we can express $\mathcal{T}^{-1}$ as a linear function of $\delta_R m$,

$$\mathcal{T}^{-1} = \chi_0 + \chi_1 \delta_R m; \quad (16)$$

the independent variable $\delta_R m$ has the advantage that it does not present the discontinuities that x does. The constants χ_0, χ_1 were calculated by least squares from the system of equations

$$\left. \begin{aligned} \chi_0 + (\kappa_0 + \kappa_1 x_A) \chi_1 &= \tau_A^{-1}, \\ \chi_0 + (\kappa_0' + \kappa_1 x_B') \chi_1 &= \tau_B^{-1}, \\ \chi_0 + (\kappa_0'' + \kappa_1 x_C'') \chi_1 &= \tau_C^{-1}, \end{aligned} \right\} (17)$$

in which all the coefficients were known.

The colour temperature τ_0 of the star P¹ was calculated from the expression

$$\tau_0^{-1} = \chi_0 + \chi_1 (\kappa_0'' + \kappa_1 x_C''), \quad (17a)$$

where x_C'' is the fixed scale reading of W, and the constants $\kappa_0'' = \chi_0 + \chi_1 \kappa_0''$ and $\kappa_1 = \chi_1 \kappa_1$ are assumed equal to those determined for W_v. This gave the value $\tau_0 = 8300$ K. already mentioned.

The magnitude level of the experiment was determined by measuring, by Lummer-Brodhun photometry, the candlepower I_P of the aperture P when obscured by about the minimum thickness of blue glass, corresponding to a W scale reading $x = x_0$. The magnification M of the lens L¹ was also re-measured. By substituting the values thus obtained for I_P , M , in 9(3), together with the values of section 9 for τ'' , τ^1 , E_* , f'' , one got the ocular magnitude

$$m_{e,0} = -2.90 \quad (17b)$$

of P¹ under the condition stated. The value of m_e relevant to the main investigation is that for a W scale reading $x'' = x_0''$. On the assumption of identical photometric characteristics for the two wedges, it is given by

$$m_{e,0}'' = m_{e,0} + \Delta_R m(x_0) - \Delta_R m(x_0''). \quad (17c)$$

TABLE 6

Photometric calibration of OB.9 glass wedge

θ °	$\delta_p m$	x	$\Delta_{N,P}$			
			Set 1	Set 2	Set 3	Set 4
4.5	4.00	4.13	35	45	48	63
9	3.25	9.76	18	35	32	37
		x'				
22.5	2.26	3.81	55	38	32	27
30	1.94	6.49	41	30	33	44
37.5	1.70	8.43	33	43	37	40
45	1.50	9.96	13	48	58	49
		x''				
75	0.95	9.27	42	35	35	42
90	0.75	10.78	24	30	49	59
120	0.44	13.38	21	32	27	42

The scale reading (x without filter, x' with filter F_1 , x'' with F_2) is measured in cm., and the probable error $\Delta_{N,P}$ in units of 0^m.001.

Substitution from (15), (17b) gave the numerical value already mentioned, viz.

$$m_{e,0}'' = 1.20, \quad (17d)$$

For each set of readings, the probable error $r_{N,P}$ was calculated from $3(3)$, in which

$$\frac{dm}{d\bar{x}} = \frac{d\delta_F m}{d\bar{x}} = K_1 = -0.127, \quad (18)$$

so

$$r_{N,P} = 0.0285 \sqrt{[(x - \bar{x})^2]}; \quad (19)$$

the results are set out in table 6. The root-mean-square values of $r_{N,P}$ for each value of $\delta_F m$ are given in table 7, together with the arguments $\bar{T}^{-1}$, calculated from (16), and $\bar{T}$; $\Delta(\bar{T}^{-1})$ is, of course, the reciprocal colour-temperature difference $\bar{T}^{-1} - \bar{T}_0^{-1}$ between the two stars. The order of the rows is reversed between tables 6 and 7.

A glance at table 7 suggests that $r_{N,P}$ does not depend significantly on $\Delta(\bar{T}^{-1})$. The grand root-mean-square of all the values of $r_{N,P}$ in table 7, however, is

$$r_{N,P} = 0^m.040, \quad (20)$$

significantly greater than the $r_{N,P} = 0^m.030$ observed under optimum conditions in section 9 [equation (23)]. The latter consideration would, perhaps, be a more sensitive test of the existence or otherwise of an augmentation of probable error due to heterochromaticity, were it not for the facts that the mechanical arrangements of the photometric control were different

TABLE 7

Photometric probable error $\epsilon_{N,F}$ as a function of reciprocal colour-temperature difference $\Delta(T^{-1})$, for $m_e = 1.2$, $\sigma = 6^\circ$, $\Delta\epsilon_f = 0$, observer J.E.G.

$\delta_R m$	T °K.	T^{-1} mireds	$\Delta(T^{-1})$ mireds	$\epsilon_{N,F}$
0.44	7300	138	18	31
0.75	6200	161	41	43
0.95	5700	176	56	39
1.50	4600	217	97	45
1.70	4300	232	112	38
1.94	4000	250	130	37
2.26	3700	274	154	39
3.25	2900	348	228	31
4.00	2500	404	284	49

$$T_0 = 8300^\circ \text{K.}$$

$$T_0^{-1} = 120 \text{ mireds.}$$

The probable error $\epsilon_{N,F}$ is expressed in units of 0^m.001.

in the two investigations and that, after a lapse of some months, the physiological state of the observer may have changed. It may be concluded that the precision of matching photometrically two stars of ocular magnitude 1.3, 5' apart in a dark field, is but little affected by the presence of such differences of colour-temperature as exist between main-sequence stars. The conditions alluded to are therefore optimal for heterochromatic, as well as for homochromatic, punctiform photometry.

The negligible increase in photometric probable error due to heterochromaticity when $m_e = 1.2$ appears to be associated with the fact that the difference in chromaticity was scarcely perceptible to the observer over at least half the range of $\Delta(\bar{x}-')$ employed. For smaller ocular magnitudes, colour differences would be more readily perceptible. Indeed, when m_e is small enough, the number of distinguishable star colours becomes quite large (cf. Osthoff 1900, Marshall 1951) and it becomes possible to effect, in the punctiform mode, colorimetric matches of some precision (cf., e.g., Müller 1949) though, of course, under these conditions, the photometric probable error would be unduly large. Using a 68-cm. refractor, Graff (1931) has compared the colours of ninth-magnitude stars with that of an artificial star provided with a wedge of blue glass; the ocular magnitude, from 3(3), is about -1.3. This suggests that colorimetry can be extended to fainter stars in the P mode than in the S mode, for the stars in Graff's investigation have $m - 5 \log a_v = 1.3$, which places the Maxwellian luminance near the lower limit [4 (14)] of the mesopic range.

$m_e = -1.3$ is also the upper limit of magnitude for another method of punctiform stellar colorimetry, in which $\bar{x}$ is calculated

from the difference in magnitude observed when a star is viewed through specified red and blue filters (Nordmann and Le Morvan 1921); in this case the limit is set by the opacity of the necessarily dense filters used.

In order to reach the faintest possible stars, it is desirable to work with the colour-temperature modifying wedge or other device (e.g. Gauzit and Proisy 1948) in the beam from the artificial star, as in Rosenberg's stellar colorimeter (1921). On the other hand, if the wedge is intended to equalize the chromaticities with a view to precise stellar photometry, then a wedge in the real-star beam, as in Wilsing's Rotkeilkolorimeter (1920), presents the advantage that the spectral distributions of both half-fields are always that of the artificial star, so that errors due to Purkinje phenomenon and the like are eliminated.

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11. SCINTILLANT PHOTOMETRIC PRECISION IN THE PUNCTIFORM MODE

Summary. The modification in the definition of stellar magnitude necessitated by scintillation is noticed. Reference is made to the recent widespread use of photoelectric methods of studying stellar scintillation, to the observations of physiological scintillation by Hartridge, and to the diffraction theory of scintillation advanced still more recently by Little. It is noted, however, that, with the exception of Gallissot, no author has paid attention to the effects of scintillation in visual stellar photometry. It is postulated that these effects are of two kinds: an increase in the photometric probable error, due to the difficulty of matching a scintillating and a non-scintillating half-field; and a systematic error which is named scintillant visual efficiency enhancement, and defined by

$$\mu = m_T - m_N, \quad (A)$$

where m_T , m_N are the magnitudes of a non-scintillating comparison star required for a photometric match with the scintillating star, and with a non-scintillating star giving the same mean illumination, respectively. It is explained how the irregular pattern of stellar illumination, in which periods of about 10 cm. predominate, influences the appearance of scintillant half-fields in the punctiform and

superficial modes. For telescopic semi-apertures α_c much less than 10 cm., punctiform twinkling resembles that seen with the naked eye, while a superficial half-field fluctuates in the same phase all over. When $\alpha_c \gg 10$ cm., one almost recovers the non-scintillant case in both modes. The case of small apertures is studied experimentally, the scintillant and non-scintillant aspects being distinguished by suffixes T, N. $\zeta_{N,P}$, $\zeta_{T,P}$ are measured by alternate sets of observations on a non-twinkling artificial star and on an image of Sirius brought into the field by means of a Silbermann's heliostat. A procedure for locating a desired star in the field of the heliostat without using the graduated circles, and without recourse to trial and error methods, is described. The conditions of the observations are approximately the optimum conditions for non-scintillant punctiform photometry derived in section 9, viz. $\sigma = 5'$, $m_e = 1.3$, $B_f < 10^{-6}$ cd.cm⁻². It is found that, for the degree of scintillation exhibited by Sirius to the unaided eye, on this occasion, the ratio

$$\zeta_{T,P} = \zeta_{T,P} / \zeta_{N,P} = 1.45. \quad (B)$$

As a consequence of local inhomogeneities of refractive index in the atmosphere, the illumination produced by a star at the surface of the earth is very far from uniform. Owing to atmospheric turbulence, the irregular pattern of illumination is in a state of constant change; the best-known result of this is the twinkling, or scintillation, of the stars, a phenomenon

too well known to require further description (Wood 1934, Minnaert 1940, Siedentopf 1948, Zwicky 1950, Butler 1951 b). It may be noted, however, that the definition 1(2) of the magnitude of a star in terms of the illumination it gives at the surface of the earth is meaningful only if one takes $\bar{I}$ to be the mean illumination at a point over a sufficient time interval, or the mean illumination at a given time over a sufficiently large area.

It is beyond the scope of this treatise to describe all the scintillation and other phenomena associated with turbulence of an inhomogeneous atmosphere; reference may be made to the reviews of Exner (1900, 1901), Pernter and Exner (1902), Rossi (1926), Anderson (1935), Couder and Strömgren (1950), and Hoffleit (1950). Interest in the subject, which has never ceased to attract research, has recently been intensified by the widespread use of photoelectric methods for recording scintillation (Stebbins and Whitford 1936, Stebbins 1944, Goldstein 1949, 1950, Butler 1950, 1951, 1952, Ellison and Wilson 1951, Mikesell et al. 1951, Nettelblad 1951, Ellison 1952). Much of our knowledge of scintillation, however, is derived from visual observations of focal (Exner and Villiger 1902, 1904) and extrafocal images (Bruch and Clastre 1946, Coulomb and Bouttet 1949), as well as by interferometric methods (Danjon 1926). Important results have also been obtained by photographic methods (Tikhoff 1912, Schlesinger 1916, 1927, Riggs et al. 1947, Barocas and Withers 1948).

The twinkling of the stars achieved sudden topicality when Hartridge and Weale (1949) suggested that the cause of the phenomenon is purely physiological, and denied the truth of the well-established physical theory (cf. Hartridge 1950, pp. 381-384). For some weeks a controversy was maintained in the columns of

Nature [London (1950) 165, 146-147, 663-668, & 1019; 166, 151], and even found its way into the meetings of the Royal Astronomical Society [Observatory, 70, 60-62 (1950)] and the British Astronomical Association [J. Brit. astr. Ass., 60, 154-156 (1950)].

As far as the author's personal experience is concerned, a steady punctiform source of ocular magnitude greater than about 2, viewed foveally in the laboratory, usually exhibits a greater or less degree of twinkling, which must be ascribed to physiological causes; it is not observed when $m_e < 2$. The scintillation of the stars, however, is independent of the ocular magnitude; when $m_e < 2$ it is wholly due to physical causes.

Recent observations of scintillation on radio wavelengths (fluctuations in the intensity of galactic noise sources) has stimulated a new attack on the theory of scintillation. Little (1951) has shown that the observed distribution, at ground level, of the illumination due to a star can be explained in terms of Fresnel diffraction at an inhomogeneous atmospheric layer (inversion layer). The theory formerly accepted (Danjon and Couder 1935, Gaviola 1949), based on refraction at the inversion layer, demands impossibly large refractive index gradients in the upper atmosphere. This difficulty had been noticed independently by the author and Mr S.S. Blaschke; it appears from Little's paper, published shortly afterwards, that it had also been pointed out by Lord Rayleigh (1893), but subsequently forgotten. Little's theory itself appears to have been anticipated, in large measure, by a paper of Syge (1931).

With exception of Gallissot (1912 a), nobody seems to have paid any attention to the

influence of stellar scintillation in visual photometry. The mean illumination due to a scintillating star is, of course, exactly equal to the steady illumination which that star would produce if the local atmospheric inhomogeneities were miraculously smoothed out so that the twinkling ceased. It does not, follow, however, that a scintillating star would match photometrically a non-scintillating star giving the same mean illumination.

It will be observed that this statement begs the question of whether it is possible to establish a visual photometric match between two half-fields, of which one is twinkling and one is steady. The difficulty encountered in such photometry is analogous to that met with in heterochromatic photometry - the difficulty of equating two stimuli in respect of one attribute, in the presence of a difference in another attribute. In heterochromatic photometry, one seeks a match in luminosity, while striving to ignore the disparity in colour; in scintillant photometry, one seeks a match in mean-luminosity-with-respect-to-time, while striving to ignore the disparity in distribution-of-luminosity-with-respect-to-time. In either case, an observer sufficiently confident to overcome any scruples is able to surmount the difficulty; in either case, the precision of the results, as measured by the reciprocal dispersion of a number of repeated readings, is diminished by the hetero-attribute; and, in either case, the discrepancies between the measurements of different observers, or those of the same observer on different occasions, may be expected to be augmented by the hetero-attribute.

It will be seen, then, that scintillation exerts two kinds of influence in visual stellar photometry: it increases the photometric probable error; and it introduces a systematic

error, which may be termed the scintillant visual efficiency enhancement. This may conveniently be defined by the equation

$$\mu = m_T - m_N, \quad (1)$$

where m_T , m_N are the magnitudes of a non-scintillating comparison star required for a photometric match with the scintillating star, and with a non-scintillating star giving the same mean illumination, respectively. For a given state of scintillation, the size, and even the sense, of μ will depend considerably on the judgement of the observer; but we may suppose that, for a given observer in good training, μ will depend only on the state of scintillation, and on the parameters (luminance etc.) of the photometric field.

If μ were independent of the state of scintillation, but depended only on the observer and the photometric parameters, it would be of little importance, for all magnitude estimates performed by the same observer at the same magnitude level would be equally affected, and the magnitude differences determined would not be affected at all. If, however, as we shall show, μ has a finite value for some particular state of scintillation, then it must be a function of one or more parameters characterizing that state; because $\mu = 0$ for a non-scintillating star, and there must be a continuous transition between this state and that considered above. Without attempting to give any precise quantitative significance to our terms, we may say that scintillation is much more conspicuous on some occasions than on others, and that, on any given occasion, the intensity of scintillation is an increasing function of the zenith distance. It follows that a determination of the magnitude difference of two stars at different altitudes

will be affected by a systematic error equal in size to the difference between the scintillant visual efficiency enhancement values for the two stars. An analogous systematic error will be generated in working at two different magnitude levels, if μ be dependent on the photometric parameters.

In both the P and the S modes, the appearance of a scintillant half-field depends very much on the semi-aperture a_0 of the photometric telescope. The pattern of stellar illumination has an irregular periodicity in at least one direction in the horizontal plane; the pseudo-wavelength is of the order of 10 cm. Thus, if a_0 is much greater than 10 cm., fluctuations in the luminous flux entering the instrument will be small; while if a_0 is much less than 10 cm., they will be large, as in the naked-eye case. In the limit, of course, when $a_0 \rightarrow \infty$, the fluctuations will be smoothed out completely, and the mean illumination over the aperture will be constant in time and equal to the mean illumination with respect to time at any point; while, when $a_0 \rightarrow 0$, the fluctuations will be equal to the time-variation in illumination at the point considered.

In the P mode, when $a_0 \ll 10$ cm., the nature of the scintillation observed telescopically is similar to that observed with the unaided eye. This will be characterized as the scintillant aspect of the punctiform mode, and denoted by the suffix-pair T, P. When $a_0 \gg 10$ cm., the twinkling of the image is greatly reduced, and one recovers the non-scintillant aspect of the punctiform mode (N,P). In this case, however, owing to non-uniformity of the phase gradient across the aperture, the image is enlarged considerably beyond the theoretical size discussed in section 2, and it is doubtful

whether it can still be regarded as punctiform.

In the S mode, the pattern of stellar illumination on the object-glass is observed directly, the effective luminance $\mathcal{S}_e$ at each point of the lens being proportional to the illumination there $[2(4)]$. When $\alpha_0 \ll 10$ cm., however, the illumination at any instant is practically uniform over the object-glass, and one is conscious only of the fluctuations of $\mathcal{S}_e$ in time. The ordinary twinkling of a star is seen, but taking place over an extended area, instead of at a point; this will be called the scintillant aspect of the superficial mode and denoted by the suffix-pair T, S. When $\alpha_0 \gg 10$ cm., a moving and changing pattern of light and shade fills the Maxwellian view. When the optimum field diameter of $2\alpha_b = 0^r.07$ is employed, however, the individual striations subtend small angles at the eye, and, in view of the usually low effective luminance of the field, the resolving power of the eye is hardly sufficient for them to be seen (Dufay 1947). In this case, then, one recovers the non-scintillant superficial mode (N, S).

An attempt has been made to determine the photometric probable error $\wedge$ as a function of the ocular magnitude m_o for each of the four aspect-mode combinations (N,P), (N,S), (T,P), (T,S). The observer was J.E.G. throughout; in the P mode, an angular separation of $\sigma = 5'$ and a dark field ($\mathcal{B}_f \doteq 0$) were employed, and in the S mode, the field diameter was nearly equal to the optimum $2\alpha_b$. The punctiform and superficial visual efficiency enhancements (μ_P, μ_S) were also measured. The scintillant probable errors $\wedge_{T,P}, \wedge_{T,S}$ should be calculated from series of photometric matches between a real, scintillating star and an artificial star. It was not practicable to effect such matches in the S mode, as equipment was not available for

maintaining the foci of the two half-fields in sufficiently precise coincidence to eliminate the possibility of Stiles-Crawford probable error (cf. section 7). It was therefore necessary to construct an artificial scintillating star, with which both $\angle_{\tau, P}$ and $\angle_{\tau, S}$ could be measured, while only $\angle_{\tau, \mu}$ was determined also from observations of a real star.

The use of an artificial twinkling star was also essential for the measurement of μ_P and μ_S , because, given a real twinkling star, we have no direct means, independent of photometric matching, of producing a star giving a steady illumination exactly equal to the mean illumination due to the twinkling star. If we are prepared to assume the strict truth of the Talbot-Plateau law, however, this can be done for an artificial star of the type devised for this investigation, and based on the use of what may be called a twinkle disk.

The device consists of a circular glass plate T completely covered with opaque black paint except for a narrow transparent zone of irregular width, and capable of rotation in its own plane so that the transparent zone passes continuously in front of a small circular aperture P of diameter $2a_P$ (figure 6). The inner boundary of the transparent zone is circular, and appears tangential to the inner edge of the aperture P when observed in the microscope O_1 . The distance of the outer boundary from the inner boundary varies irregularly, along its length, within the limits zero and $2a_P$. A greatly reduced image of P is formed by the lens L_1 , and projected to infinity by the lenses L_1'' , L_2'' . When the disk T is rotated, then, the illuminated area of P varies irregularly, and these fluctuations in area are interpreted by an eye placed at E as fluctuations in the ocular magnitude of the punctiform source P' .

By speeding up the rotation of T sufficiently, one can make the slowest of these fluctuations inappreciable to the eye. Thus changing the angular speed of T enables one to convert a twinkling star into an apparently non-twinkling star giving the same mean illumination. By comparing each in turn with a non-twinkling comparison star, one can measure μ . It must be emphasized again, however, that this method of measuring μ involves the assumption that, when the comparison star matches the apparently non-twinkling star, it would also match a truly steady star giving the same mean illumination. This is equivalent to the Talbot-Plateau law, which appears to be well established for S mode photometry, but seems not to have been investigated in the P mode.

The apparatus shown schematically in figure 6 was designed to enable an observer, with his eye at E and biting upon a dental impression I, to effect, at will, a photometric match between an artificial non-scintillating star and either another similar star, or an artificial scintillating star, or a real star. Under these conditions, it should be possible to make a reliable comparison between, e.g. $\sqrt{I_{T,F}}$ for a real star and $\sqrt{I_{N,F}}$. Another facility provided by the apparatus is that of being able to change over, in observations of artificial stars, from the P mode to the S mode, or vice versa, without changing the ocular magnitude M_e , and, for that matter, without the observer leaving his seat.

We shall describe first how the apparatus of figure 6 may be used to compare, in the P mode, a real star with a non-scintillating artificial star. For this purpose, we must suppose that the twinkle disk T and the prisms R_1, R_2 in the figure are removed. The source S illuminates the apertures P, P_v via the

diffusing screen D, as before, a reduced image of P is formed by the lens L' at P', and, in the absence of R₁, P' is accompanied by an image P_v' of P_v. Instead of the single lens L' of figure 3, a pair of lenses L₁", L₂" is provided. One of these is movable horizontally in its own plane by a screw, and the other vertically; together they form a means of adjusting slightly the direction in which, say, P_v' is seen by reflexion in the inclined plane-parallel glass plate G. In the absence of R₃, an image of the real star in the plane mirror M₂ is seen in close proximity to that of P_v'. The light from a chosen star is thrown on to the secondary mirror M₂ by the primary mirror M₁ of a Silbermann's heliostat H, mounted in the open window of the laboratory. A photometric match between P_v' and the real star can be effected by introducing a neutral filter of suitable density at N, and by adjusting the wedge W_v of OB.9 glass. By rotating the sector B, whose plane is perpendicular to the optical axis, through a right angle, one can obscure the star P'; a further rotation through 90° causes P' to reappear. B, and B_v, which has a similar function with respect to the real star, are controlled, through endless cords, from the eye end.

Loss of light from the real star is reduced as much as possible by this design of optical system, in which there are only two reflexions from aluminized flats, and one transmission through an unsilvered plate. In case the reflexion factors of M₁, M₂, or the transmission factor of G, should not be uniform, biting upon the dental impression I ensures that constant areas of these optical elements are operative.

Silbermann's heliostat (Ambronn 1899, p. 648) consists essentially of two axes, of

which one is mounted equatorially, and is driven at the diurnal rate so that it always points towards the star under observation. The other axis is in line with the centres of the primary and secondary mirrors. The primary mirror is suspended from two forks, one attached to each axis, by means of a parallelogram linkage which ensures that the normal to the mirror always bisects the angle between the two axes. It follows that, once the moving axis has been adjusted to point to the desired celestial object, light from that object will necessarily be directed from the primary mirror on to the secondary, and continue to be so directed, provided the clock drive be accurate enough to follow the diurnal motion of the object. The secondary mirror is fixed in such an orientation that a ray coming from the primary is reflected along the axis of the receiving optical system, i.e. along M_2G in our case.

The eye at E sees an image M_1' of M in M_2 , and, within the angular field defined by this image of M_1 , an image of the star under observation. The original primary mirror of the heliostat was far too thin to be free from astigmatic distortion, and had to be replaced by a thicker optical flat of smaller area. The sides of this rectangular mirror measured 5.3 cm., 4.8 cm. The angular field of the instrument is determined by the angular subtense of the image of the primary in the secondary mirror; this image was about 220 cm. from the eye. The short side of the flat was perpendicular to the plane of incidence, and therefore subtended an angle of $75'$. The longer side was foreshortened, and subtended an angle of $83' \cos \iota$, where ι is the angle of incidence on M_1 . To fill a pupil of diameter 0.6 cm. with some safety margin requires a beam of starlight of width 1.2 cm., say; hence the greatest angle of incidence tolerable is $\cos^{-1} (1.2/5.3) = 77^\circ$, for which the angular

width of the field is only 18'. To ensure that the pupil of the eye was central in the narrow beam, it was necessary to check that the luminosity of the stellar image did not change when the eye was given slight lateral displacements in all directions from the position determined by the dental plate I.

The motions and graduated circles of the heliostat did not permit an accuracy of setting consistent with getting a desired object into a field as small as this without recourse to trial and error methods. A procedure was therefore evolved which could be relied upon to centre a star in the field without using the circles. The heliostat was mounted on a horizontal wooden shelf fitting in a unique position on to the sill of the window. The lower ends of the levelling screws rested in depressions in the upper surface of the shelf; these depressions served the double purpose of locating the instrument unambiguously with respect to the laboratory, and therefore with respect to the rest of the optical system, and of preventing its being accidentally pushed off the shelf. The heliostat could thus be levelled, and its polar axis could be orientated correctly, once and for all.

The latter adjustment was effected by observing the sun at local apparent noon. The instrument was rotated about its azimuth axis until the shadows of the trunnions supporting the polar axis appeared symmetrical, whereupon the azimuth ring was clamped. The altitude of the polar axis was then set to the latitude of the place by means of the appropriate circle; this adjustment was checked by observing the coincidence of the shadows of the sights on the moving axis when the sun's declination was set correctly on the declination arc, and the polar axis was rotated into the requisite hour

angle. The clock drive was then engaged, so as to retain the sun in the sights.

The fixed axis could then be adjusted, by means of the motions provided, so as to coincide with M_1M_2 . A low-power telescope (aperture 1 inch, magnification 7) was mounted in a vee support V , so that its axis coincided with $P''G$. With the sun screened off, M_2 was tilted until the image M_1' of M_1 was central in M_2 . The sun's light was then restored, and M_1 was tilted, by motions of the fixed axis only - the moving axis was still following the diurnal motion of the sun - , until the reflected beam fell on M_2 . Upon looking in the telescope with the aid of a dark glass, the observer was then able to continue the adjustment of the fixed axis, until the sun appeared centrally in the image M_1' of M_1 . The accuracy of the clock drive, and of all the adjustments, could be tested by returning to the telescope several minutes later and observing whether any shift of the solar image had occurred.

Once the polar axis and the fixed axis are correctly adjusted, it should be possible to bring any object into the heliostat field simply by aligning the sights on it. In practice this was found very difficult, partly owing to the non-telescopic nature of the sights, which consisted of a pair of minute apertures through which it was difficult to see anything, and partly because there was often some hard object which prevented the head being placed in the requisite position. An approximate alignment of the sights was, however, always practicable. The small telescope was used to improve this rough adjustment.

The line M_1M_2 was first defined by daylight observations in which, with the equatorial drive disengaged, the moving axis

was pointed at a small, sharp terrestrial object, and adjusted until the image of the latter was central in M_1' , as seen in the telescope resting in the vee support V_1 . The telescope was then shifted to a second vee support V_2 attached to an adjustable mounting clamped to the shelf on which the heliostat stood. This mounting was manipulated until the image of the small terrestrial object in M_1 appeared centrally in the telescopic field, whereupon V_2 was clamped. Thereafter, whenever the telescope was placed in V_2 , its axis coincided with M_1M_2 , and, moreover, any object in the centre of its field lay on the moving axis and would therefore be imaged centrally in M_1' .

To bring a star into the field of the heliostat, the driving clock was first set going, and the declination and hour angle motions were unclamped. The moving axis was directed approximately to the star, by means of the sights or otherwise. This adjustment was refined by using the same two motions to bring the star into the centre of the field of the telescope, now resting in V_2 . If it did not then appear in the centre of M_1' , when the telescope was placed in V_2 , further slight adjustments to the moving axis were made. After each successive adjustment, of course, the clock drive had to be engaged in order to compensate for the diurnal motion; finally, both motions were clamped. If it had been found necessary to adjust the moving axis after the telescope had been put in V_1 , the telescope was then returned to V_2 to enable the mounting of the latter to be re-adjusted so that the stellar image was now central in the field; this was to facilitate future settings.

In order to make the separation of the images of the real star and the comparison star $P_{V'}$ equal to $5'$, the image of the real star was made

to coincide with that of P' , whose angular distance from P_V' was known to be $5'$. This adjustment was effected approximately by tilting the plane-parallel plate G , and refined by means of the slow-motion screws attached to the lenses L_1 , L_2 . The lens moving horizontally had a power of 0.5 diopter, and that moving vertically a power of 0.25 diopter; the length of traverse of each screw being 1 cm., the angular ranges of these adjustments were therefore about $17'$ and $9'$, respectively, comparable with the minimum angular width, $18'$, of the field. When the adjustment was complete, P' was cut out by means of the opaque sector B , and photometric comparison of the real star with P_V' could proceed.

The accuracy of the equatorial drive varied from time to time, the motion being particularly erratic near the beginning of an observing session. At its best, the driving error was less than $5'$ per hour, or 1 part in 180; under these conditions, scarcely any attention to the slow motions was necessary during a set of ten matches.

It was necessary to supply the source S with current from the Westinghouse 'Stabilistor', since any fluctuations in the supply voltage, causing fluctuations in the candlepower of S , would have caused a spurious increase in the photometric probable error.

The number of nights when clear skies coincided with trouble-free operation of the apparatus was small. On only one occasion, 1952 March 14, was a satisfactory series of observations secured. Sirius was chosen as the real star; it had the advantages of giving an ocular magnitude, after reflexion from M_1 and M_2 and transmission through G , nearly equal to the punctiform optimum of 1.3; and, being low in the sky, of being easy to bring into the

field, and of exhibiting a good measure of scintillation.

A very cold east wind entering the open window caused great discomfort and seemed likely to account by itself for a considerable augmentation of the photometric probable error. In order to eliminate this disturbing effect as far as possible, alternate sets of ten readings were taken, matching P_V' with Sirius, and with the non-scintillating P' , respectively. A neutral filter N of density Δ was introduced in the P_V' beam to make the ocular magnitude of P_V' roughly equal to that of the image of Sirius, which was about 1.1. No attempt was made to equalize the colour-temperatures of Sirius and P_V' , since the experiment of section 10 had shown that, for $m_e = 1.2$, heterochromaticity causes a negligible increase in $\Delta_{N,P}$, and, presumably, in $\Delta_{T,F}$. In table 8, the time (U.T.) at which each set of readings was begun is stated to give an idea of the time taken by these observations.

The magnitudes in table 8 were calculated as follows, from the mean x of the ten W_V scale readings in each set.

Ignoring light losses by transmission through N and reflexion at G, we note, from 10 (17d), that when the W scale reading is $x'' = x_0''$, the ocular magnitude of P' is $m_{e,c}'' = 1.20$. In section 10, it was found that, when the W scale reading was x_0'' , and a rotating sector R was used to step up the m_e of P_V' by an amount $\delta_R m$, then the magnitudes of P' , P_V' were equal when the W_V scale reading was x (or x' , x''), provided [10(15)]

$$\left. \begin{aligned} \delta_R m(x) &= K_0 + K_1 x, \\ \delta_R m(x') &= K_0' + K_1 x', \\ \delta_R m(x'') &= K_0'' + K_1 x'', \end{aligned} \right\} (2)$$

TABLE 8

Photometric probable error ϵ_p for a real star (T) and an artificial star (N), for $m_e \pm 1.1$, $\sigma \pm 5'$, $E_f \pm 0$, observer J.E.G.

U.T.		m_e	$\epsilon_{T,P}$	U.T.		m_e	$\epsilon_{N,P}$
h	m			h	m		
19	42	1.17	71	19	56	1.00	33
20	17	1.17	44	20	39	0.94	45
20	52	1.11	63	21	04	0.95	47
21	22	1.09	48	21	40	0.99	31

The probable errors $\epsilon_{N,P}$, $\epsilon_{T,P}$ are expressed in units of $0^m.001$.

In the absence of R, the magnitude of P_V' , for a W_V scale reading x , would be

$$m_a = m_{a,0}'' - \delta_R m(x). \quad (3)$$

We see from (3) that the expression for the ocular magnitude m_a of P_V' , as modified by transmission of the light through N and reflexion at G, is

$$m_a = m_{a,0}'' - 2.5 \log \rho_G + 2.5 \Delta - \delta_R m(x), \quad (4)$$

where ρ_G is the reflexion factor of G, Δ is the density of N, and $\delta_R m(x)$ is given by (2). ρ_G was calculated from Fresnel's formulae, assuming, with E.C. Pickering ['Applications of Fresnel's Formula for the Reflection of Light', quoted by Fowle (1934), Table 420], that the refractive index of the glass is 1.55. The fraction of unpolarized light reflected from each surface is $\frac{1}{2}(A+B)$, where A, B are the reflexion factors for light polarized in the plane of incidence, and perpendicular thereto, respectively. Since the plate G has two surfaces, its reflexion factor is $\rho_G = A+B$, very nearly; so, for an incidence angle of 45° , we have

$$\rho_G = 0.115. \quad (4a)$$

The photometric probable errors $\epsilon_{T,P}$ (for a match between Sirius and P_V') and $\epsilon_{N,P}$ (for a match between P' and P_V') exhibited in table 8 were calculated from the formula [10(19)]

$$\epsilon_P = 0.0285 \sqrt{[(x - \bar{x})^2]}. \quad (5)$$

The root-mean-square values are, respectively,

$$\epsilon_{T,P} = 0^m.058, \quad \epsilon_{N,P} = 0^m.040. \quad (6)$$

This value of $\wedge_{N,P}$ is in (fortuitously exact) agreement with that [10(20)] recorded in the previous investigation, in which the same photometric control was employed. In order to secure homogeneity of the present results with that of 9(23), obtained with a different type of photometric control (polaroids instead of a wedge), we must multiply them by a factor of $3/4$, so that

$$\wedge_{T,P} = 0^m.044, \quad \wedge_{N,P} = 0^m.030. \quad (7)$$

This is for $m_e \doteq 1.3$, $\sigma \doteq 5'$, $B_f \doteq 0$, observer J.E.G. The value of $\wedge_{T,P}$ refers, of course, to the degree of scintillation exhibited to the unaided eye by Sirius at the time and place of these observations; it may be regarded as typical.

* * * * *

12. SCINTILLANT PHOTOMETRIC PRECISION AND VISUAL EFFICIENCY ENHANCEMENT

Summary. A universal apparatus enabling $\mu_{N,F}$, $\mu_{N,S}$, $\mu_{T,F}$, $\mu_{T,S}$, and the visual efficiency enhancements μ_p , μ_s to be measured as functions of the ocular magnitude m_e is described. A scintillating artificial star is produced by the rotation of an opaque disk containing a transparent zone of irregular width, which passes in front of an illuminated aperture and allows sometimes more, sometimes less light to pass to a reduced, punctiform image of the aperture. A non-scintillating artificial star giving the same mean illumination is obtained by speeding up the rotation of the twinkle disk until the fluctuations become too rapid to be appreciated by the eye; this enables μ to be measured. Arrangements are provided for switching quickly from the N to the T aspect, or from the punctiform to the superficial mode. The adjustments and calibration of the apparatus are described in detail. The photometric control is an OB.9 wedge. The optimum values of the parameters α , β_f in the punctiform mode, and of α in the superficial mode, are adopted.

The range of ocular magnitudes employed in the punctiform mode is from -0.5 to +3.0; in the superficial mode, $m - 5 \log \alpha_0$ ranges from -0.3 to +3.2, corresponding to Maxwellian luminances of $\log \beta_{a,b} = -6.1$ to $\log \beta_{a,b} = -7.5$. Four sets of ten readings each, in each mode, aspect and magnitude level, are completed, in five days, by the author; a random order is adopted

to eliminate any spurious trend in the results. To avoid bias in the results, the readings are noted with the aid of an automatic recorder, and remain unknown until the end of the experiment. A secular term in the readings, due to a gradual change in the configuration of the apparatus, is detected; it is removed as a quadratic function of the ordinal number of the set of observations.

The corrected data indicate statistically significant visual efficiency enhancements for this range of magnitude levels of about

$$\mu_P = -0^m.075, \mu_S = -0^m.131; \quad (A)$$

the data are not sufficient to show any significant dependence on magnitude level. The sign of μ suggests an analogy with the phenomenon observed by Brücke with sources flickering at about 10 sec.⁻¹.

As regards the probable errors, neither $\sqrt{N_P}$ nor $\sqrt{T_P}$ differs very much from the function 9(A). The fact that $\sqrt{T_P}$ is not greater than $\sqrt{N_P}$ in this investigation is attributed to the highly artificial nature of the scintillation, which repeats itself after 1.5 sec. The curves for $\sqrt{N_S}$, $\sqrt{T_S}$, however, show the expected similarity, and the expected greater ordinates of the latter. Two explanations of a curious anomaly occurring in both curves near the lower limit of mesopic Maxwellian luminances are offered. Ignoring this anomaly, it is proposed to represent $\sqrt{N_S}$ by the empirical formula

$$\sqrt{N_S} = \frac{0.085}{2.91 - 0.4 (m - 5 \log \alpha_0)}$$

$$(0 < m - 5 \log \alpha_0 < 5), \quad (B)$$

derived from 8(A) with the optimum value of α , by multiplication by ζ_0 , and

by the use of 2(C); and $\wedge_{T,S}$ by

$$\wedge_{T,S} = \sqrt{(\wedge_{N,S})^2 + 0.00145}. \quad (C)$$

Equations 9(A), 11(B), 12(B), 12(C) are used to plot $\wedge_{N,P}$, $\wedge_{T,P}$, $\wedge_{N,S}$, $\wedge_{T,S}$ against $m - 5 \log a_0$ on a common diagram, and thus to effect an intercomparison based homogeneously on observations by the author. It is deduced that, irrespective of the degree of scintillation, the superficial photometric probable error is smaller for $m - 5 \log a_0 < 2.5$, and the punctiform for $m - 5 \log a_0 > 2.5$. If one is prepared to introduce a neutral filter, however, the ocular magnitude need never be less than the punctiform optimum of 1.3; in this case the precision of the punctiform mode is superior for all values of $m - 5 \log a_0 > 0$. Quite apart from the size of $\wedge$, the punctiform mode has the advantages that the Purkinje phenomenon is smaller, that the problems of heterochromaticity are less acute, that it is easier to eliminate trouble from the Stiles-Crawford phenomenon, that there is no diffracting edge between the half-fields, that no correction for the luminance of the sky is needed, that the effect of visual efficiency enhancement is smaller, and that it is applicable to stars of 3 or 4 magnitudes fainter than the superficial mode.

We shall now indicate how the apparatus of figure 6 may be employed to measure the six functions $\wedge_{N,P}$, $\wedge_{N,S}$, $\wedge_{T,P}$, $\wedge_{T,S}$, μ_P , μ_S .

In order to ensure as much verisimilitude as possible in the scintillation produced by the twinkle disk T, the irregular boundary of its transparent zone was copied from an actual record of scintillation, made by the photoelectric method. The trace chosen was that of the scintillation of Vega, at an altitude of 33° , obtained by Butler

(1952, Plate X (f)) on 1951 April 20, with an aperture of four inches and a paper speed of 5 inches sec.⁻¹. The conditions were thick haze and a slight, cold east wind; many of the high frequencies were absent.

The abscissa of such a trace is the time, and the ordinate is a quantity proportional to the instantaneous integral with respect to wavelength of the product of the spectral sensitivity of the receiving apparatus and the spectral radiant flux density due to the star. Statistically speaking, the illumination defined by 1(4) may be expected to fluctuate in a similar way to this ordinate. If we imagine the area between the trace and the time-axis to be a transparent strip in an otherwise opaque diaphragm in front of a uniform light source, and that a narrow slit parallel to the ordinate axis moves along the time-axis in such a way that its abscissa at time t is equal to t , then the fluctuations in the luminous flux passing through the slit will exactly reproduce the fluctuations in illumination represented by the trace. If a strongly reducing optical system be employed to form a punctiform image of the slit, the illumination due to this image will vary in time in the manner prescribed by the trace.

In the twinkle disk, the t -axis is replaced by the circle $\rho = R$, in polar co-ordinates, where R is the inner radius of the transparent zone. Each unit of time is represented by an arc ωR of this circle, where ω is the angular speed of the twinkle disk. The ordinates of the original trace are replaced by the fluctuating quantity $\delta R = R' - R$, where R' is the outer radius of the transparent zone.

The twinkle disk was driven by means of an electric motor; the lowest uniform angular speed which the drive would maintain was 40 r.p.m., or $\omega = 4\pi/3$ sec.⁻¹. Thus, with $R = 6$ cm., the

linear speed of the trace was $\omega R = 25 \text{ cm. sec.}^{-1}$. Since the eye cannot appreciate fluctuations faster than about 30 sec.^{-1} , the width of the scanning slit could be of the order of 1 cm. without sensibly affecting the appearance of the scintillation. In practice, a circular aperture of radius $a_p = 0.3 \text{ cm.}$ was used. The published trace was re-drawn on rectangular co-ordinate paper, with a magnification of the abscissae by a factor of $\omega R / l_0$, where l_0 is the length of the original trace representing unit time, and of the ordinates by a factor $2a_p / y_m$, where y_m is the maximum ordinate of the original trace. The transformed trace was then transferred by hand painting to the twinkle disk.

The use of a circular scanning aperture results in non-linearity of response of the device, the rate of change of transmitted light flux with δR being much smaller when δR is near zero or $2a_p$ than when δR is near a_p . In this respect, a square aperture would be preferable. The realistic appearance of the scintillation was not impaired by the use of a circular aperture, however, though it was somewhat spoilt by the recurrence of identical fluctuations after a period of $2\pi/\omega = 1.5 \text{ sec.}$

At the beginning of each observing session, the microscope O_1 was used to check that the inner boundary of the transparent zone of T appeared tangential to the aperture P; O_1 was then removed. Before each set of observations of the scintillant aspect, the speed ω of T was checked by timing several successive passages of a mark on T through the field of a telescope O_2 . O_2 was fitted with a diagonal eyepiece for easy viewing from the observer's seat. A variable resistance, connected in series with the motor, was conveniently placed to enable ω to be adjusted, if necessary. A switch enabling a large resistance in series with the motor to be short-circuited was also within reach of the observer.

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This switch enabled one to pass at will from the scintillant to the non-scintillant aspect, and vice versa.

The light from P is rendered parallel by the pair of lenses L_1'' , L_2'' , and turned through a right angle by the plate G. In the absence of L'' , the eye of the observer, placed at E, sees an image of P' at an infinite distance along P"G produced. To avoid any vignetting effects, it is necessary for the eye of the observer to be fixed by use of the dental plate I.

In order to obtain two Maxwellian half-fields, one illuminated by P', and the other by P_V' , the optical system $L'L''$ is duplicated to deal with the light from P_V . The necessary lateral displacement of the optical axis is effected by the right-angled prisms R_1 , R_2 , and the recombination of the two light beams by reflexion through 90° at R_3 and transmission at G. The functions of L_V' , L_V'' are exactly analogous to L' , $L_1''L_2''$. A neutral filter is provided at N_V to reduce the two beams to approximate photometric equality, allowing for the fact that the beam from P is seriously attenuated by reflexion at G.

In order to make a punctiform match, the wedge W is adjusted to a fixed position $\alpha = \alpha_1$, near the middle of its range. This can be observed from the eye end by means of the telescope O_3 , which is provided with a projector for illuminating the W scale. The wedge W_V is used to control the ocular magnitude of P_4' ; it is varied in position until a photometric match is obtained. Thanks to the correct choice of the density of N_V , the scale reading of W_V for a match is near the centre of the scale; this facilitates the bracketing technique when the probable error is large.

A superficial match can be made between the same two stars by removing G and replacing it

by the prism R_4 , and by introducing the Maxwellian lens L''' in such a position that it forms coincident images of P' , P_v' at P'' , the centre of the eye pupil. The vertex X_1 of R_4 bisects the aperture of L''' as seen from P'' . The left half of the field is thus illuminated by light from P' , reflected from the face X_1, X_2 of R_4 , while the right half is illuminated by light from P_v' proceeding past the edge X_1 . To allow for the fact that there is only a single reflexion at X_1, X_2 , whereas there is a double reflexion at G in the punctiform case, a light neutral filter N (not the filter N of section 11) of density

$$\Delta = 0.30 \quad (7a)$$

is attached to the mounting of G so as to attenuate slightly the beam from P' . Once adjusted, the optical elements G, R_4, L''' can be removed and put back into operating position without loss of adjustment, since they are mounted on saddles which slide on machined triangular optical benches, and are located against abutments clamped to the benches.

The initial adjustments of G, R_4, L''' require some care, and need to be checked subsequently from time to time. Starting with G only in position, and with the observer biting upon I , we adjust G horizontally and vertically in its own plane until P_v' is seen in the middle of G ; the pupil of the eye is then central in the parallel beam whose generators meet the boundary of G . Next G is tilted about horizontal and vertical axes in its own plane until the images of P', P_v' are seen separated horizontally by $5'$. This adjustment is refined by transverse motions of L_1'', L_2'' , as described in section 11. Since the angular separation is not a critical parameter, it may be estimated, with sufficient accuracy, by a practised observer.

G is then removed, while the saddle bearing

L''' is slid into position. In the absence of R_4 , L''' is seen filled with light from P_v' . L''' is racked out of focus, and moved horizontally to and fro in its own plane, so that the Maxwellian beam is cut by the edges of the eye pupil, which give rise to characteristic shadows on the retina. By moving L''' slightly up or down, a position can be found where these shadows are symmetrical with respect to the horizontal diameter of the Maxwellian field. The image of P_v' formed by L''' then lies in the horizontal plane bisecting the pupil. By releasing the clamping screw of the L''' saddle slightly, the mounting of L''' may be rocked about a horizontal axis normal to $L''P''$, the clamping screw being used as a slow motion. This results in the cone of light from L''' being cut by the upper and lower margins of the pupil. By sliding L''' along the optical bench (perpendicular to $L''P''$), one can find a position where the shadow figures are symmetrical about the vertical diameter of the field. The image of P_v' then lies in the vertical plane bisecting the pupil. The corresponding position of the L''' saddle along the optical bench is recorded by clamping the abutment.

Foucault's method is employed to ensure that P_v' is imaged by L''' in the plane of the pupil. L''' is rocked to and fro along GP'' until the position is found where lateral movement of the lens causes the whole field to darken simultaneously when the image reaches the edge of the pupil. The saddle is then returned to contact with its abutment, whereupon the image of P_v' lies at the mid-point of the pupil.

When R_4 is replaced, half the beam from P_v' is cut out and replaced by a half-beam deriving from P' . The abutment of the saddle bearing R_4 is adjusted so that, when R_4 is in position, the edge X_1 of R_4 exactly bisects the aperture of L''' as seen from E . The image of P' formed by L''' is made to coincide with that of P_v' , already centred in the pupil, by slightly

tilting R_4 about horizontal and vertical axes in the plane of the face X_1X_2 , the images being observed by means of an eyepiece at O_2 .

In order to determine $\sqrt{N_P}, \sqrt{N_S}, \sqrt{T_P}, \sqrt{T_S}, \mu_P, \mu_S$ as functions of ocular magnitude m_e , the quantities mentioned were measured at each of six different magnitude levels, ranging from -0.5 to $+3$, which may be designated, in order of increasing m_e , by the letters A, B, C, D, E, F. One could change from one value of m_e to another by changing the density Δ_r of a neutral filter N_r introduced between PP_v and WW_v , and large enough to cover both P and P_v . In order to reach levels A, B, C, D, negative values of Δ_r would have been required. This difficulty was overcome by using a switch, conveniently placed near the observing position, to cut out the resistance Ω which had hitherto been connected permanently in series with the source S. The resulting increase in the candlepower of S, which was, of course, accompanied by a rise in colour temperature, caused a decrease, of size $\delta_R m$, in the magnitude m_e . The magnitude of P' was, of course, subject to a constant increment $\delta_T m$ due to the mean effect in time of the partial obscuration of P by T.

The scale reading of W is $x = x_1$. Since the ocular magnitude of P' for $x = x_0$ is $m_{e,0}$ [cf. 10(17b)], in the absence of G, N, N_r , T, and with Ω in series with S, the magnitude for $x = x_1$, in the presence of G, N, N_r , T, and possibly in the absence of Ω , is

$$m_e = m_{e,1} - 2.5 \log \rho_G + 2.5 (\Delta + \Delta_r') + \delta_T m, \quad (8)$$

where

$$m_{e,1} = m_{e,0} + \delta_R m(x_0) - \delta_R m(x_1), \quad (8a)$$

and

$$\left. \begin{aligned} \Delta_r' &= \Delta_r - 0.4 \Delta_\Omega \text{ for levels A, B, C, D, } (\Omega \text{ cut out}), \\ \Delta_r' &= \Delta_r \quad \text{for levels E, F } (\Omega \text{ in series}). \end{aligned} \right\} \dots\dots(9)$$

The magnitude step $\delta_T m$ introduced by the twinkle disk T was measured by observing the W_v scale reading x required for a photometric match between P_v' , P' when T was rotating rapidly in front of P. A neutral filter of density Δ_r , nearly equal to the effective density of T, was then substituted for T, and the W_v scale reading x_T for a match was observed. The magnitude of P_v' in the first case is given by (8), and in the second by

$$m_e = m_{e,1} - 2.5 \log \rho_g + 2.5 (\Delta + \Delta_r') - \delta_R m(x_T) + \delta_R m(x) + 2.5 \Delta_T. \quad (10)$$

Equating these two expressions, we find

$$\delta_T m = 2.5 \Delta_T - \delta_R m(x_T) + \delta_R m(x). \quad (11)$$

Since the sum of the last two terms is small, the accuracy of this determination depends more on knowledge of the neutral density Δ_T than on the photometric calibration of the wedge. Numerically,

$$\delta_T m = 0^m.58. \quad (12)$$

The magnitude step $-\delta_\Omega m$ caused by cutting out the resistance Ω was measured by obscuring P' by means of B, and matching P_v' against the internal standard P_i of ocular magnitude m_i , mentioned in section 9. The scale reading x was first obtained with the resistance Ω in series with the source. From (8), (9), the magnitude of P_v' was then

$$m_e = m_{e,1} - 2.5 \log \rho_g + 2.5 (\Delta + \Delta_r) + \delta_r m. \quad (13)$$

Ω was then short-circuited, and m_e was restored to roughly its previous value by introducing a neutral filter of density Δ_Ω in the beam from P_v' . An exact match with P_i was recovered by changing the scale reading from x to x_Ω , so that

$$m_e = m_{e,1} - 2.5 \log \rho_G + 2.5 (\Delta + \Delta_r) + \delta_T m - \delta_R m(x_\Omega) + \delta_R m(x) + 2.5 \Delta_\Omega - \delta_\Omega m. \quad (14)$$

By equating (13) and (14) we get

$$\delta_\Omega m = 2.5 \Delta_\Omega - \delta_R m(x_\Omega) + \delta_R m(x); \quad (15)$$

numerically,

$$\delta_\Omega m = 2^{m.56}. \quad (16)$$

In (14) above, the tacit assumption is made that the photometric characteristic of the wedge, as expressed by the function $\delta_R m$ of 10(15), is the same when Ω is short-circuited as when Ω is in series with the source S. This is not strictly true, since the wedge is not neutral, and the colour temperature, and therefore the spectral distribution, of S, is different in the two cases. It was found, however, that a repetition of the calibration with a series of rotating sectors, as described in section 10, but with Ω cut out, led to a function $\delta_R m$ indistinguishable, within the errors of observation, from 10(15). This wedge characteristic was therefore adopted throughout the work, irrespective of the colour temperature of S.

Equation (8) may be written

$$m_e = m_{e,2} + 2.5 \Delta_r', \quad (17)$$

where the numerical constant

$$m_{e,2} = m_{e,1} - 2.5 \log \rho_e + 2.5 \Delta + \delta_T m \quad (18)$$

is obtained by substitution for $m_{e,1}$, ρ_e , Δ , $\delta_T m$ from 12(8a), 11(4a), 12(7a), 12(12), respectively, and where Δ_r is given by (9) above. The densities Δ_r of the filters N_r inserted before PP_v to obtain the various levels are noted in table 9, together with the effective densities Δ_r . The expression (17) is used to compute, from these, the ocular magnitude $m_{e,p}$, for the punctiform observations, at each of the levels A, B, ... F.

In our discussion, in section 2, of the ocular magnitude and Maxwellian luminance of stars observed with telescopes, we neglected light losses in the optical system. In calculating the photometric parameters for superficial observations at the six levels, then, it will be consistent to ignore the loss of light in transmission through the lens L^{'''}. The star under consideration has an ocular magnitude, when observed with the naked eye, equal to $m_{e,p}$. From the definition of ocular magnitude, it follows that, if the star were a genuine celestial body, its magnitude would be $m = m_{e,p}$. The superficial parameter $m - 5 \log a_0$ is therefore, in our case, equal to $m_{e,p} - 5 \log a_0$, where $a_0 = 0.92$ is the linear semi-aperture of L^{'''}. This quantity

$$m - 5 \log a_0 = m_{e,p} + 0.18 \quad (19)$$

is given in the fifth column of table 9.

The focal length of L^{'''} being $f = 24$ cm., the angular semi-diameter of the field is

$$\alpha = a_0/f = 0^r.038. \quad (20)$$

TABLE 9

The adopted levels of ocular magnitude m_e , with the equivalent Maxwellian luminances $B_{e,b}$

Code	Δ_r	Δ_r'	$m_{e,p}$	$m - 5 \log a_0$	$\log B_{e,b}$	$m_{e,s}$
A	0	-1.02	-0.48	-0.30	-6.10	-2.91
B	0.39	-0.63	0.49	0.67	-6.49	-1.94
C	0.60	-0.42	1.02	1.20	-6.70	-1.41
D	0.80	-0.22	1.52	1.70	-6.90	-0.91
E	0	0.00	2.08	2.26	-7.12	-0.35
F	0.39	0.39	3.05	3.23	-7.51	0.62

$B_{e,b}$ is expressed in cd.cm^{-2} .

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Thus the photometric probable errors $\Delta_{N,S}$, $\Delta_{T,S}$ are practically indistinguishable from those which would have been observed, with the actual values of $m - 5 \log a_0$, if the value of α had been equal to the optimum α_b of 2(7). The levels may therefore be characterized, in the S mode, by the Maxwellian luminances calculated from 2(12), i.e. from

$$\log B_{e,b} = -0.4 (m - 5 \log a_0) - 6.22. \quad (21)$$

The magnitude levels in the P and S modes are not directly comparable, as they would be if the punctiform observations had been conducted by imaging on the retina the images at P", by means of an eyepiece such as O_2 . This can be seen by calculating the ocular magnitude $m_{e,s}$ for the S mode, by substituting for $m - 5 \log a_0$ in 2(3), so that

$$m_{e,s} = m - 5 \log a_0 - 2.61. \quad (22)$$

By adding (19), (22), we obtain

$$m_{e,s} = m_{e,p} - 2.43; \quad (22a)$$

the additive constant in (22a) is, of course, nothing other than $-5 \log (a_0/a_e)$, as can be easily verified.

It will be seen that the Maxwellian luminances employed range from about 8×10^{-7} cd.cm.⁻² down to 3×10^{-8} cd.cm.⁻² (cf. the effective luminance $B_e = 2 \times 10^{-8}$ cd.cm.⁻² in the investigation of section 8). These values of $B_{e,b}$ are in the region where Mlle Flamant has observed the reappearance of the Stiles-Crawford phenomenon at low luminance levels (cf. section 7). It was therefore desirable to eliminate the possibility of any spurious augmentation, due to the S-C phenomenon, of the photometric probable error. This was done by making the

images of P' , P_v' , formed by L'' , coincide at P'' , as already described.

From 4(14), we note that our values of $M - 5 \log a_0$ lie mostly in the mesopic range (-8, 2), while levels E and F are scotopic. Only at the highest levels was any perception of colour present in the S mode. Even then the difference in chromaticity between the two half-fields was not very obtrusive, since their colour temperatures were approximately equal.

Four sets of ten photometric matches each were effected in each of the four aspect-mode combinations (N,P), (N,S), (T,P), (T,S), at each of the six magnitude levels A, B, ... F. The same twinkle disk, giving the same scintillation wave-form, was used throughout; the results therefore refer to a certain degree of scintillation, which may be regarded as typical. They refer also, of course, to the particular observer J.E.G. No specially timed period of dark-adaptation was stipulated for these observations: by the time the adjustments of the apparatus had been initially checked, in almost complete darkness, the observer was at least as dark-adapted as in the previous experiments. No periods of rest were allowed; observing sessions lasted for up to two hours, and included up to ten sets of readings.

The 96 sets of readings were divided into four identical groups, each consisting of 24 different combinations of aspect, mode, and magnitude level. Each group may be written as a 4 x 6 array -

NPA,	NPB,	...	NPF	} (23)
NSA,	NSB,	...	NSF	
.....				
TSA,	TSB,	...	TSF	

- in which the first letter of each element

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represents the scintillatory attribute (aspect), the second the geometrical attribute (mode), and the third the photometric attribute (magnitude level). The elements of the array were arranged in random order, with the aid of a table of random numbers; the corresponding sets of readings were performed in the order thus prescribed. This procedure was repeated for the remaining three groups, a different randomization of the array being used each time; thus the 96 sets were completed.

The above modus operandi was adopted to eliminate as far as possible any spurious correlations, due to the order in which the observations were carried out, between the position of an element in the array (23) and the experimental results attached to that element. It is clear also that spurious values of the visual efficiency enhancements μ_p , μ_s might easily be generated if the observer had any knowledge of the scale readings corresponding to his matches in the T, N aspects, respectively. This is particularly so, in view of the great amount of judgement required in making a scintillatory match - judgement which might easily be biased by a knowledge of the results. Use of the telescope O_3 for observing the readings was therefore abandoned, and a device for recording the readings automatically, and in a permanent form, was provided.

The ends of a narrow strip of stiff, adhesive paper were attached to the top and bottom of the brass frame of the wedge W_v , so that half the width of the strip was supported by the side member of the frame, while the other half was stretched a small distance in front of the glass. The strip was not wide enough to obscure the aperture P_v . A spring-loaded gramophone needle was supported two or three millimetres in front of the paper strip,

with its axis at right angles to the plane of the paper. A cord attached to the mount of the needle passed round a pulley to the rear of the wedge mounting and returned to the eye end, where it passed over another pulley to a small weight, which served to keep the cord taut, but was insufficient to overcome the needle-loading spring. By smartly pulling this weight downwards, however, one could shift the needle axially through a short distance, whereupon the point pierced the paper strip. The size of the hole made was limited by a stop coming into contact with the wedge mount immediately the tip of the needle had penetrated.

Before each set of ten readings, the wedge was carefully set to $x = 8$ cm., the needle was depressed, and the resulting hole, which served as a calibration mark, was emphasized with pen and ink. After each photometric match, the observer pulled on the weight until the sound of the needle arrest was heard, indicating that the hole had been punched. A set of ten small levers was used as a sort of abacus to count the number of readings, one lever being switched over after each match. When all ten levers had changed into their alternative positions, the observer knew that the set was completed; he then removed the record strip to a storage book, where it was kept until the time came to reduce the observations. Care was taken, of course, to upset the match after each reading, so that the next would be independent. The observer also took care not to glance at the record strip before filing it away and replacing it with a fresh one. The automatic recorder worked very satisfactorily, except when, occasionally, two readings coincided, with the result that there was only one hole to represent them.

The mean scale reading

$$\bar{x} = n^{-1} [\sum x] \quad (24)$$

TABLE 10

Elimination of the secular term from the mean scale readings $\bar{x}$

ν	Code	$10^3 \bar{x}$	$10^3 \bar{x}_2$	$10^3 \bar{x}_c$
1	NPC	6986	7517	6214
2	TSE	3506	8696	2763
3	TSE	4872	9204	4157
4	TPE	7840	8612	7154
5	NSE	5328	8610	4670
6	NSA	5719	8635	5088
7	TSA	4843	8767	4239
8	NSC	4527	7868	3950
9	NSF	3679	7968	3128
10	NSB	5018	8545	4493
11	TPD	7837	8535	7338
12	TSB	4759	8909	4285
13	TSC	4964	9382	4514
14	TPC	7830	8210	7405
15	NPD	8145	8217	7743
16	NPB	8674	8428	8296
17	NPA	8847	8536	8492
18	TSD	3558	8956	3225
19	NPF	8438	8286	8127
20	NPE	8942	8496	8653
21	TPA	7802	8301	7535
22	TPF	7935	8083	7689
23	NSD	4672	8717	4446
24	TPB	7739	8433	7533
25	TPC	8011	8390	7825
26	NPA	8730	8419	8563
27	TSA	4330	8354	4182
28	NPB	8266	8020	8137

TABLE 10 (continued)

ν	Code	$10^3 \bar{x}$	$10^3 \bar{x}_2$	$10^3 \bar{x}_c$
29	NSA	5324	8240	5213
30	TSE	2754	7086	2660
31	NPD	8412	8484	8336
32	NSF	4742	9031	4682
33	TPE	6757	7529	6714
34	NPE	8306	7860	8279
35	NSD	4362	8407	4350
36	TPD	7582	8280	7586
37	TSF	2580	7770	2598
38	NSC	4643	7984	4676
39	TSC	3250	7668	3297
40	TSB	3341	7491	3401
41	TPB	7520	8214	7593
42	NPF	8831	8679	8917
43	NSE	4417	7699	4515
44	NPC	7670	8201	7780
45	TPA	8000	8499	8122
46	TSD	2138	7536	2271
47	NSB	4080	7607	4224
48	TPF	8023	8171	8177
49	TSA	2965	6889	3129
50	TSC	2247	6665	2420
51	NPF	7839	7687	8021
52	TSF	1723	6913	1914
53	NPC	8285	8816	8484
54	NPE	8309	7863	8516
55	TPC	7357	7736	7571
56	NPD	7989	8061	8210
57	NSA	4546	7462	4774
58	TPA	6988	7487	7222
59	TPF	7861	8009	8101

TABLE 10 (continued)

ν	Code	$10^3 \bar{x}$	$10^3 \bar{x}_2$	$10^3 \bar{x}_c$
60	NSE	4766	8048	5011
61	TSB	3466	7616	3716
62	NSB	4597	8124	4851
63	NSF	2713	7002	2971
64	TPD	7307	8005	7569
65	NPB	7793	7547	8058
66	NSC	4581	7922	4849
67	TSD	1974	7372	2245
68	NPA	7972	7661	8245
69	TPB	6864	7558	7138
70	TPE	6861	7633	7137
71	TSE	3398	7730	3674
72	NSD	3610	7655	3887
73	NSA	4745	7661	5022
74	TPE	7455	8227	7731
75	NSD	3175	7220	3451
76	TPA	7214	7713	7488
77	TPC	7287	7666	7560
78	NPD	7165	7237	7436
79	NSC	4885	8226	5153
80	NPF	7501	7349	7766
81	TSE	3647	7979	3909
82	NSF	3710	7999	3968
83	NPE	8026	7580	8280
84	NPA	7696	7385	7946
85	NPB	8250	8004	8495
86	NSE	4363	7645	4602
87	NSB	4197	7724	4431
88	TSF	3433	8623	3660
89	TSB	3835	7985	4056
90	TPF	7588	7736	7802

TABLE 10 (continued)

ν	Code	$10^3 \bar{x}$	$10^3 \bar{x}_\nu$	$10^3 \bar{x}_c$
91	TSC	3865	8283	4071
92	TPB	7102	7796	7301
93	TPD	6480	7178	6670
94	NPC	6934	7465	7116
95	TSA	4166	8090	4339
96	TSD	2738	8136	2901

The unit of $\bar{x}$ is the centimetre.

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for each of the 96 sets of $n = 10$ readings is given in table 10, together with the number n' specifying the order in which the sets were performed, and the code letters specifying the relevant element of the array (23). The determination of μ_P , μ_S depends upon a comparison of these means in the scintillant T and non-scintillant N aspects. It was by no means certain that the configuration of the apparatus was stable enough for the matching position of the W_V scale to remain fixed over the period of five days during which the observations were performed. It was envisaged, in particular, that the heating up of the lamp house when the source was switched on at the beginning of a set of readings, and its cooling down when the source was switched off at the end, might give rise to a progressive displacement of one part with respect to another, analogous to the creeping of slates down a roof exposed to alternating temperatures. If, for example, the aperture P_V moved very slightly relative to the twinkle disk, an appreciable change in the effective density of the disk might occur. No such shift obvious to inspection in the microscope O_1 did in fact occur, but it was considered desirable, in any case, to assume the existence in $\bar{x}$ of a term depending on the ordinal number ν , and to eliminate this secular term before discussing the observations further.

For each attribute (combination of aspect, mode, and magnitude level), corresponding to an element of (23), the grand mean

$$\bar{\bar{x}} = n'^{-1} \sum_1^{n'} \bar{x}, \quad (25)$$

where $n' = 4$ is the number of sets in the attribute, is given in table 11, which is arranged as the transpose of (23). The systematic differences between attributes are removed, as far as possible, by subtracting the appropriate $\bar{\bar{x}}$ from each $\bar{x}$, to form

TABLE 11

Array of uncorrected grand mean scale readings $\bar{x}$

Level	NP	NS	TP	TS
A	8311	5084	7501	4076
B	8246	4473	7306	3850
C	7469	4659	7621	3582
D	7928	3955	7302	2602
E	8446	4718	7228	3668
F	8152	3711	7852	2810

The quantity tabulated is $10^3 \bar{x}$, where $\bar{x}$ is in cm.

TABLE 12

Grand mean scale readings after removal of secular term

Level	NP	NS	TP	TS
A	8337	5024	7592	3997
B	8246	4500	7391	3864
C	7399	4657	7591	3576
D	7931	4033	7291	2660
E	8432	4700	7184	3600
F	8208	3687	7942	2734

The quantity tabulated is $10^3 \bar{x}_c$, where $\bar{x}_c$ is in cm.

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$$\bar{x}_v = \bar{x} - \bar{\bar{x}} + 8.000, \quad (26)$$

where the constant 8.000 is added to avoid negative quantities. $\bar{x}_v$ contains only the secular term and the random component in $\bar{x}$. The fourth column of table 10 reveals a steady decrease in $\bar{x}_v$ with increasing v , not entirely masked by the random variations.

The secular term is eliminated by fitting a least-squares parabola

$$\bar{x}_{v,0} = 8.802 - 0.02977 v + 0.0002054 v^2 \quad (27)$$

to the 96 values of $\bar{x}_v$. We then subtract $(\bar{x}_{v,0} - 8.000)$ from each $\bar{x}_v$, to form the corrected $\bar{x}_c$, i.e.

$$\bar{x}_c = \bar{x} - \bar{x}_{v,0} + 8.000. \quad (28)$$

The last equation can be written

$$\bar{x}_c = \bar{x} + (\bar{x}_v - \bar{x}_{v,0}) - \bar{\bar{x}}_v + 8.000, \quad (29)$$

or, in virtue of (26),

$$\bar{x}_c = \bar{\bar{x}} + (\bar{x}_v - \bar{x}_{v,0}), \quad (30)$$

which verifies that the secular term has been removed; the first term in (30) represents the systematic component, and the second the accidental errors.

Table 12 gives the corrected grand means

$$\bar{\bar{x}}_c = n'^{-1} \sum_I \bar{x}_c. \quad (31)$$

These differ little from the uncorrected grand

TABLE 13

Photometric probable error $\wedge_{N,P}$ of the non-scintillant punctiform mode as a function of ocular magnitude m_e , for $\sigma = 5'$, $B_f = 0$, observer J.E.G.

m_e	Set 1	Set 2	Set 3	Set 4	Root-mean-square
-0.48	34.7	65.7	48.6	57.9	53.0
0.49	40.7	50.1	36.0	57.2	46.7
1.02	50.4	57.5	25.6	163.3	91.1
1.52	25.8	41.4	93.0	65.0	61.8
2.08	27.4	28.8	61.1	57.0	46.3
3.05	107.3	191.4	58.7	55.8	116.9

The probable error $\wedge_{N,P}$ is expressed in units of 0^m.001.

TABLE 14

Photometric probable error $\wedge_{T,P}$ of the scintillant punctiform mode as a function of ocular magnitude m_e , for $\sigma = 5'$, $B_f = 0$, observer J.E.G.

m_e	Set 1	Set 2	Set 3	Set 4	Root-mean-square
-0.48	86.1	37.5	44.8	51.9	57.1
0.49	32.6	56.8	46.9	31.7	43.3
1.02	33.2	43.9	42.4	29.8	37.8
1.52	32.9	39.1	44.2	42.9	40.0
2.08	66.5	63.1	53.1	47.7	58.1
3.05	63.5	92.7	90.1	47.2	75.8

The probable error $\wedge_{T,P}$ is expressed in units of 0^m.001.

TABLE 15

Photometric probable error $\wedge_{N,S}$ of the non-scintillant superficial mode as a function of $m - 5 \log a_0$, for $\alpha = 0^r.038$, observer J.E.G.

$m - 5 \log a_0$	Set 1	Set 2	Set 3	Set 4	Root-mean-square
-0.30	28.0	16.4	20.4	37.1	26.7
0.67	37.5	18.8	36.1	46.5	36.1
1.20	56.7	39.3	39.6	44.0	45.5
1.70	46.8	51.5	48.4	37.0	45.2
2.26	44.8	37.4	32.5	27.6	36.1
3.23	40.2	62.6	73.1	54.1	58.7

The probable error $\wedge_{N,S}$ is expressed in units of $0^m.001$.

TABLE 16

Photometric probable error $\wedge_{T,S}$ of the scintillant superficial mode as a function of $m - 5 \log a_0$, for $\alpha = 0^r.038$, observer J.E.G.

$m - 5 \log a_0$	Set 1	Set 2	Set 3	Set 4	Root-mean-square
-0.30	42.1	42.8	57.4	55.3	49.9
0.67	37.3	54.4	73.8	58.8	57.6
1.20	61.1	47.2	36.4	88.8	61.6
1.70	53.8	47.1	44.3	54.4	50.1
2.26	69.0	39.4	32.7	31.5	45.8
3.23	105.8	66.9	47.6	59.8	73.3

The probable error $\wedge_{T,S}$ is expressed in units of $0^m.001$.

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means $\bar{x}$ in table 11, thanks to each attribute having one ν in each of the intervals (1, 24), (25, 48), (49, 72), (73, 96).

The values of $\bar{x}_c$ for the P mode are set out, with argument m_e , in table 17, in which two parallel columns refer to the non-scintillant (N) and scintillant (T) aspects, respectively. For every value of m_e but one, the N reading is greater than the T reading. This is shown graphically in figure 7, from which the existence of a small, but genuine, visual efficiency enhancement μ_P seems plausible. The data are not sufficient, however, to enable one to infer how μ_P depends on m_e .

Similar remarks apply to the data for the S mode, which are tabulated, as a function of $m - 5 \log a_0$, in table 18. The sense of μ indicated for the S mode is the same as for the P mode; in this case, there are no exceptions to the rule that the N reading is greater than the T reading. The almost exact parallelism between the N and T curves in figure 8 is striking, but probably fortuitous; otherwise one would expect it to be reproduced in figure 7. It is suggestive that the discontinuity common to both curves in figure 8 occurs between levels D and E; it may be associated with the change in colour-temperature of the source between these two levels. This change might be accompanied by a change in the angular distribution of intensity of the source, causing the magnitude increments $\delta_{\Omega m}$ to be different for P' , P_v' , respectively. Or perhaps the discontinuity is to be ascribed to a dependence, on colour-temperature, of the relative sensitivities, to full radiation, of the temporal and nasal halves of the parafovea; this would explain why the discontinuity is not observed in the P mode, with its foveal viewing.

The level of significance attained by

TABLE 17

Visual efficiency enhancement and root-mean-square photometric probable errors $\hat{r}_{N,P}$, $\hat{r}_{T,P}$ in the punctiform mode, as a function of m_e .

m_e	$\bar{x}_c$		$\hat{r}_P$	
	N	T	N	T
-0.48	8.337	7.592	53.0	57.1
0.49	8.246	7.391	46.7	43.3
1.02	7.399	7.591	91.1	37.8
1.52	7.931	7.291	61.8	40.0
2.08	8.432	7.184	46.3	58.1
3.05	8.208	7.942	116.9	75.8

$\bar{x}_c$ is expressed in cm., and $\hat{r}_P$ in units of 0^m.001.

TABLE 18

Visual efficiency enhancement and root-mean-square photometric probable errors $\hat{r}_{N,S}$, $\hat{r}_{T,S}$ in the superficial mode, as a function of $m-5 \log a_0$.

$m-5 \log a_0$	$\bar{x}_c$		$\hat{r}_S$	
	N	T	N	T
-0.30	5.024	3.997	26.7	49.9
0.67	4.500	3.864	36.1	57.6
1.20	4.657	3.576	45.5	61.6
1.70	4.033	2.660	45.2	50.1
2.26	4.700	3.600	36.1	45.8
3.23	3.687	2.734	58.7	73.3

$\bar{x}_c$ is expressed in cm., and $\hat{r}_P$ in units of 0^m.001.

the difference between the corrected grand means $\bar{x}_T, \bar{x}_N$ in the scintillant and non-scintillant aspects, respectively, may be ascertained by computing the statistic t (Chambers 1940, pp. 35-38), which is the ratio of $|\bar{x}_T - \bar{x}_N|$ to the standard error of this difference. If we calculate this standard error from the dispersion of the $n' = 4$ values of $\bar{x}_c$ contributing to each $\bar{x}_c$ (equation 31), we find

$$t = \frac{|\bar{x}_T - \bar{x}_N| n' \sqrt{n' - 1}}{\sqrt{\{n' [\sum' \bar{x}_T^2 + \sum' \bar{x}_N^2] - [(\sum' \bar{x}_T)^2 + (\sum' \bar{x}_N)^2]\}}}, \quad (32)$$

where $\sum'$ denotes summation over n' values. This expression is evaluated for each level, in both P and S modes, in table 19. Knowing t , and the number of degrees of freedom, which is $2n' - 2 = 6$, we can obtain from the tables (Fisher and Yates 1948, Table III) an indication of the quantity F , also set down in table 19; P represents the probability, on the hypothesis that $\mu = 0$, of observing a difference exceeding $|\bar{x}_T - \bar{x}_N|$.

In the P mode, $\bar{x}_T - \bar{x}_N$ is negative in five cases out of six; and of these five, four have $F < 0.1$ (the fifth relating to level F) and three $F < 0.02$. The positive value of $\bar{x}_T - \bar{x}_N$ is associated with a $F > 0.7$. In the S mode, every $\bar{x}_T - \bar{x}_N$ is negative; in five cases, $F < 0.1$, and, of these, three have $F < 0.05$. The greatest $F (< 0.2)$ relates to level F, where, as in the P mode, the low photometric precision precludes a high level of significance.

These considerations show that the observations carried out in the present investigation are sufficient to establish the existence of a small but finite visual efficiency enhancement μ , for observer J.E.G., in both the P and the S mode.

There are not enough data, however, to

TABLE 19

Significance of the observed visual efficiency enhancements in the superficial (S) and punctiform (P) modes

Level	P			S		
	$\bar{x}_T - \bar{x}_N$	t	P	$\bar{x}_T - \bar{x}_N$	t	P
A	-0.745	3.30	<0.02	-1.027	3.34	<0.02
B	-0.855	5.58	<0.01	-0.636	2.45	<0.05
C	+0.192	0.40	>0.7	-1.081	2.06	<0.1
D	-0.640	2.14	<0.1	-1.373	4.08	<0.01
E	-1.248	5.46	<0.01	-1.100	2.29	<0.1
F	-0.266	0.97	<0.4	-0.953	1.78	<0.2
Mean	-0.594			-1.029		

$\bar{x}_T - \bar{x}_N$ is expressed in cm.

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show in either mode, the dependence, if any, of μ on the magnitude level. The best that we can do is to calculate mean values of μ_P , μ_S , applicable to the whole range of levels involved in the experiment. Referring to the definition 11(1) of scintillant visual efficiency enhancement, we see that our difference of grand mean scale readings is equivalent to

$$\mu = m_T - m_N = -\delta_R m(\bar{x}_T) - [-\delta_R m(\bar{x}_N)], \quad (33)$$

where $\delta_R m(x)$ is given by 10(15). Substituting for $\delta_R m(x)$, and using 10(18), we obtain, for the mean μ , over all levels,

$$\mu = -K_1 (\overline{\bar{x}_T} - \overline{\bar{x}_N}) = 0.127 (\overline{\bar{x}_T} - \overline{\bar{x}_N}), \quad (34)$$

where $\overline{\bar{x}_T} - \overline{\bar{x}_N}$ is the mean of the values of $\bar{x}_T - \bar{x}_N$ for the six levels in the mode under consideration. The two means of this form are quoted in table 19; substituting each, in turn, in (34), we obtain the numerical values for the mean scintillant visual efficiency enhancements

$$\mu_P = -0^m.075, \mu_S = -0^m.131. \quad (35)$$

This means that, for the range of magnitude levels dealt with in our experiment, if a non-twinkling star were matched in, say, the punctiform mode, by observer J.E.G., with a comparison star of magnitude m_c , then a twinkling star would require a comparison star of magnitude $m_c - 0^m.075$ to match it. In colloquial terms, we may say that a twinkling star appears more intense to observer J.E.G. than it would if it were not twinkling. Qualitatively, the effect is in accordance with the phenomenon noticed by Brücke in 1864 (Le Grand c. 1949, p. 303) for a flickering (regularly periodic) light stimulus. For flicker frequencies considerably below the

critical frequency of fusion, the maxima are seen separately, and with a luminosity not merely greater than that associated, under steady conditions, with the mean luminance, but greater than that associated with the maximum luminance. The effect is most marked for frequencies of about 10 sec.⁻¹. It may be that something of the sort is responsible for the negative values of μ recorded above, but it must be remembered that scintillation is not by any means identical with flicker as ordinarily understood. In our investigation, for example, all frequencies between 0.67 sec.⁻¹ (the frequency of rotation of the twinkle disk) and about 30 sec.⁻¹ (the limit appreciable to the eye) were present.

The values of the photometric probable error $\sigma_{N,P}$ derived from the four sets of readings at each of the six magnitude levels are given in table 13. These were calculated from the formula 10(19). The last column contains the root-mean-square value for each level. Tables 14, 15, 16 give similar data with regard to $\sigma_{T,P}$, $\sigma_{N,S}$, $\sigma_{T,S}$, respectively.

The root-mean-square probable errors for the punctiform mode are collected in table 17, and for the superficial mode in table 18; in each case the figures for the scintillant and non-scintillant aspects are exhibited in parallel columns. The P mode probable errors are plotted against m_e in figure 9, and those for the S mode against $m - 5 \log \omega_v$ in figure 10.

The curve N_0 in figure 9 represents the empirical function 9(21) derived in the earlier investigation under non-scintillant conditions, viz.

$$\sigma_{N,P} = 0.0409 - 0.01423m_e + 0.003237m_e^2 + 0.001119m_e^3$$

$$(-3 \leq m_e < 4), \quad (36)$$

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which is also delineated in figure 4. As we should expect, the observed curve T for $\zeta_{T,P}$ follows the general course of N_0 , but with consistently greater ordinates. The observed N curve, however, has many of its ordinates greater still, which is quite contrary to expectation. If we ignore the fortuitously high value of $\zeta_{N,P}$ at level C, however, it may be argued that the N and T curves do not differ very significantly. By multiplying the ordinates of N_0 by the factor $4/3$ required for homogeneity with results from the present apparatus [cf. section 11, after equation (6)], we can make N_0 agree very roughly with both N and T. The explanation of the fact that, in the present investigation, scintillation does not seem to have caused an increase in probable error, but, if anything, a decrease, may lie in the fact that the pattern of twinkling repeats itself after 1.5 seconds. Judgement of photometric equality may, under these circumstances, be reduced to judgement of symmetry of the fluctuations of one half-field above and below the photometric level of the other, which may be an easier visual task. An analogy may be found in certain types of filar micrometer, where settings are made by symmetry of two lines with respect to a third, rather than by coincidence of one line with another.

Natural stellar scintillation, however, certainly does not do anything but increase the photometric probable error; so we are assured by the results of the experiment with Sirius, described in section 11. The ratio of the ζ_P values for the scintillant and non-scintillant aspects is given by 11(6) as

$$\zeta_{T,P} = \zeta_{T,P} / \zeta_{N,P} = 1.45. \quad (37)$$

We have no information as to the functional dependence of the factor $\zeta_{T,P}$ on m_e ; the simplest assumption in the circumstances is that $\zeta_{T,P}$ is

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independent of m_e . For scintillation such as that exhibited by Sirius on 1952 March 14, then, we may assume that

$$\begin{aligned} \gamma_{T,P} &= \zeta_{T,P} \gamma_{N,P} \\ &= 0.0593 - 0.02063m_e + 0.004694m_e^2 + 0.001623m_e^3 \quad (-3 < m_e < 4). \end{aligned}$$

....(38)

This empirical function is represented by the curve T_0 in figure 9.

The curves N and T for the S mode (figure 10) show a remarkable parallelism, the T ordinates being consistently greater than the N, in accordance with expectation. The parallelism extends to the appearance, in both curves, of an anomalous descending portion between levels C and E. If this occurred in one curve only, it could be dismissed as fortuitous, but its presence in both is suggestive of some systematic cause. From table 9, we note that the Maxwellian luminances of levels C, E are about $2 \times 10^{-7} \text{ cd.cm.}^{-2}$ and $0.9 \times 10^{-7} \text{ cd.cm.}^{-2}$, respectively. The region CE is thus the region of transition between mesopic ($\beta_e > 10^{-7} \text{ cd.cm.}^{-2}$) and scotopic ($\beta_e < 10^{-7} \text{ cd.cm.}^{-2}$) luminances. Considering curve N, for example, it may be suggested that AC represents the law of variation of $\gamma_{N,S}$ while both rods and cones are functioning, while EF represents the law for rod function alone. CE is then simply the transition zone associated with the change of law. Another possibility is that the change of law is from the effectively homochromatic photometry of the scotopic domain to the heterochromatic photometry of the mesopic region. On account of the use of an OB.9 wedge as the photometric control, slight differences in colour between the two half-fields were, in fact, perceptible at the higher luminance levels. An attempt was made, by suitably choosing the density of the neutral filter N_V , to ensure that the colour-

temperatures should be nearly equal at the W_v wedge position at which a photometric match was obtained. Owing to visual efficiency enhancement, however, this condition could not be satisfied in both the N and the T aspects, and consequently an embarrassing colour difference sometimes made itself felt. Whether this is the explanation of the anomalous curves could be settled by repeating the experiment with a neutral photometric control device; if the anomalies persisted, the first explanation would be confirmed.

If we ascribe the anomalous descending portion of the N curve to chance, and assume a single law for the whole curve, a good representation of the N observations is given by the empirical formula 8(26), viz.

$$r_{N,s} = \frac{0.085}{2.91 - 0.4(m - 5 \log a_0)} \quad (0 < m - 5 \log a_0 < 5), \quad (39)$$

derived by combining the observations of J.E.G. with Dufay and Schwégler's formula. The function (39) is represented by the curve N_0 in figure 10. Only the point C of N deviates from N_0 by more than three times its probable error, and this deviation is only slightly significant (cf. section 3).

The similarity of the curves T and N suggests that we find a similar function to (39) to represent the results for the T aspect. The fact that the two curves are further apart where their ordinates are smaller discourages us from writing $r_{T,s} = \zeta_{T,s} r_{N,s}$ and assuming $\zeta_{T,s}$ to be independent of $m - 5 \log a_0$. On the other hand, the law of compound error suggests that we should try to represent $r_{T,s}^2$ as the sum of two squares, viz. $r_{N,s}^2$ and a hypothetical scintillation component r_T^2 , the latter being assumed independent of $m - 5 \log a_0$. The mean value of $r_{T,s}^2 = r_{N,s}^2$ for the six levels is 0.00145 square

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magnitude; we therefore write

$$\angle_{T,S} = \sqrt{(\angle_{N,S})^2 + 0.00145}, \quad (40)$$

a function represented by the curve T_0 in figure 10. Here E is the only point with a deviation of more than three times its probable error.

Before we proceed to a detailed comparison of the relative merits of the P and S modes under various conditions, it is necessary to remark that the P mode has several advantages independent of the numerical sizes of the photometric probable errors of the respective modes.

Thus, the size of the inverse Purkinje phenomenon encountered in the P mode is only one half as great as that of the direct Purkinje effect in the S mode (section 4).

In the P mode, with m_e near the optimum value of 1.3, a difference in colour-temperature between the two half-fields does not cause an appreciable increase, over its homochromatic value, in the probable error. In the S mode, on the other hand, the homochromatic precision increases steadily, ceteris paribus, with the effective luminance; but, as the luminance rises into the mesopic range, the eye becomes sensitive to any heterochromaticity, and the latter itself causes an augmentation of probable error (cf. section 10).

The discussion of section 7 leads to a very simple condition for the elimination of any Stiles-Crawford probable error $\angle_{S-C}$ in the P mode. The treatment of the S-C phenomenon in the S mode is more complicated, and the conditions for eliminating $\angle_{S-C}$ are correspondingly more difficult to realize.

The unavoidable use of the Maxwellian

view in the S mode leads to two further difficulties not encountered in the P mode. One of these, the generation, by diffraction at the edge of the photometer prism, of a dark line separating the two half-fields, has been discussed in section 8. The other arises when the sky background of the star under observation has a luminance appreciable by comparison with the effective luminance due to the star. Under these conditions, the two luminances mentioned add together, and a correction is necessary to the observed magnitudes (Dufay 1947).

The work described in this section suggests that, at least for observer J.E.G., visual efficiency enhancement is smaller in the P mode than in the S mode (equations 35).

Finally, the limiting magnitude of the P mode is considerably greater than that of the S mode. To ensure foveal viewing, it has been suggested, in section 9, that observations be limited to $m_s < 3.5$; this corresponds to an upper limit of about 6 to $m - 5 \log a_0$. S mode observations are already very difficult at $B_{e,b} = 2 \times 10^{-8} \text{cd.cm.}^{-2}$ (cf. section 8), or $m - 5 \log a_0 = 3.7$; and Dufay (1947) recommends a practical limit of $B_{e,b} = 10^{-7} \text{cd.cm.}^{-2}$, which corresponds to $m - 5 \log a_0 = 2.0$.

A comparison of the numerical sizes of the P and S mode photometric probable errors can be effected with the aid of the equations (36), (38), (39), (40). The data represented by these equations are homogeneous in the sense that all refer to the same observer, J.E.G. They are inhomogeneous in the sense that, in the T aspect, for reasons which have been explained, the data for the P mode relate to natural stellar scintillation, while those for the S mode relate to artificial scintillation produced by means of a twinkle disk.

The equations show the dependence of the probable error σ on a photometric parameter, which has been loosely referred to as the magnitude level.

With regard to all other parameters specifying the conditions under which photometric matching is supposed to be performed, the equations assume the realization of the optimum values. For the P mode this means, in practice, an angular separation σ of between $3'$ and $20'$, and an ambient luminance of less than 10^{-6} cd.cm.⁻². For the S mode it means an angular semi-diameter of about 2° for the photometric field, assumed bipartite.

It is convenient to compare the probable errors of the two modes as functions of a common photometric parameter, which we shall take to be $m - 5 \log a_0$. The equations for the P mode in terms of this parameter are obtained by writing [2(3)]

$$m_e = m - 5 \log a_0 - 2.61 \quad (41)$$

in (36), (38); the range of validity of these empirical formulae then extends somewhat beyond $0 < m - 5 \log a_0 < 6$. Since (39), (40) are valid for $0 < m - 5 \log a_0 < 5$, it is appropriate to intercompare all four expressions in the interval $0 < m - 5 \log a_0 < 5$.

This intercomparison is effected graphically in figure 11, where a curve for each of the aspect-modes (N,P), (T,P), (N,S), (T,S) may be seen. The curves marked (N,S), (T,S) are identical with the curves N_0 , T_0 of figure 10, while (N,P), (T,P) are obtained from (N_0 , T_0) of figure 9 by increasing the abscissae of all points on those curves by $(m - 5 \log a_0) - m_e = 2.61$.

The precise course of the curves is, obviously, highly conjectural. This is due partly to the very nature of the observations themselves, which means that, to obtain a single point on a curve with a fractional probable error of 0.075 in the ordinate, one has to make no less than 40 tedious photometric matches. To double this accuracy would require 160 readings; and by the

time these had been carried out, the physiological and psychological conditions would probably have changed appreciably, rendering the greater precision illusory. The curves in figure 11 must also be regarded as only qualitatively correct, because of the incompleteness of the observations in several respects, which has forced us to indulge in a certain amount of guesswork. In particular, it has been wilfully assumed that $\xi_{T,P} = \sqrt{r_{T,P}} / \sqrt{r_{N,P}}$ and $\sqrt{r_T^2} = \sqrt{r_{T,S}^2} - \sqrt{r_{N,S}^2}$ are both independent of magnitude level; and the derivation of the formula for $\sqrt{r_{N,S}}$ itself leaves much to be desired.

There can be little doubt, however, of the qualitative correctness of the curves, of which the chief features are the steady increase of $\sqrt{r_{N,S}}$, $\sqrt{r_{T,S}}$ with $m - 5 \log a_0$; the minimum of $\sqrt{r_{N,P}}$, $\sqrt{r_{T,P}}$ near $m - 5 \log a_0 = 3.92$; and the higher ordinates of each T curve compared with the corresponding N curve. The general configuration of the curves is such that, both in the N and in the T aspect, the P and S curves cross in the neighbourhood of $m - 5 \log a_0 = 2.5$. We have seen in section 11 that, in telescopic observations of twinkling stars, there is a tendency for the aspect to pass from T to N as one increases the semi-aperture up to and beyond $a_0 = 10$ cm.; this is true of both P and S modes. As a_0 increases, then, the curve representing $\sqrt{r}$ as a function of $m - 5 \log a_0$ tends to move away from the T curve and towards the corresponding N curve. Since this happens both for the P and the S curves, the crossing point will remain around an abscissa of 2.5, irrespective of the degree of approximation of the aspect to the N or the T case. Thus, as far as the relative sizes of $\sqrt{r_P}$, $\sqrt{r_S}$ are concerned, the S mode is superior when $m - 5 \log a_0 < 2.5$, and the P mode when the inequality is reversed.

By means of a very obvious device, however, we can improve the accuracy of the P mode whenever m_e is less than the optimum value of 1.31. If we

had insufficient light available, the situation would be irremediable, except by obtaining a larger telescope; but when $m_e < 1.31$, it is true to say that we are suffering from a superfluity of light. The remedy is to introduce into the optical system a neutral filter of density approximately

$$0.4 \{ 3.92 - (m - 5 \log a_0) \}, \quad (42)$$

which effectively makes $m_e = 1.31$, and $m - 5 \log a_0 = 3.92$. On account of the flatness of the minimum in the $\wedge_P$ curve, the density chosen need be only very roughly that given by (42); and it is not suggested that a filter should be employed at all unless m_e be considerably less than the optimum value.

When this device is used, the left-hand parts of the $\wedge_P$ curves are flattened into the horizontal lines (N,P,0), (T,P,0) in figure 11. These are the tangents to the (N,P), (T,P) curves, respectively, at the minima, and their equations are [11(?)]

$$\left. \begin{aligned} \wedge_{N,P} &= 0^m.030, \\ \wedge_{T,P} &= 0^m.044. \end{aligned} \right\} (43)$$

With this improvement to the P curves, we see that each P curve lies below the corresponding S curve throughout the range of magnitude levels in figure 11. Only when $m - 5 \log a_0 < 0$ does the S mode once more assert its superiority. Even then, it is probably more handicapped than the P mode by any chromaticity difference between the half-fields; and when $m - 5 \log a_0 < -3$, there is the Stiles-Crawford phenomenon to contend with.

The general conclusion is that punctiform

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visual stellar photometry seems to be associated, in general, with lower values of the photometric probable error, though the superficial mode may sometimes be employed with little loss in precision. The numerical data supporting this conclusion refer, of course, only to observations by the author, who may be exceptional in being capable of a lower accuracy in the superficial than in the punctiform mode; but the size of the differences between one individual and another, recorded, for example, in table 2, do not lend much support to this hypothesis.

Even if the above conclusion should be reversed, however, observation in the punctiform mode would continue to enjoy the advantages which have been indicated in connexion with Purkinje phenomenon, heterochromaticity, Stiles-Crawford phenomenon, diffraction in the Maxwellian view, correction for sky luminance, visual efficiency enhancement, and limiting magnitude.

* * * * *

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FIGURES 1 - 11

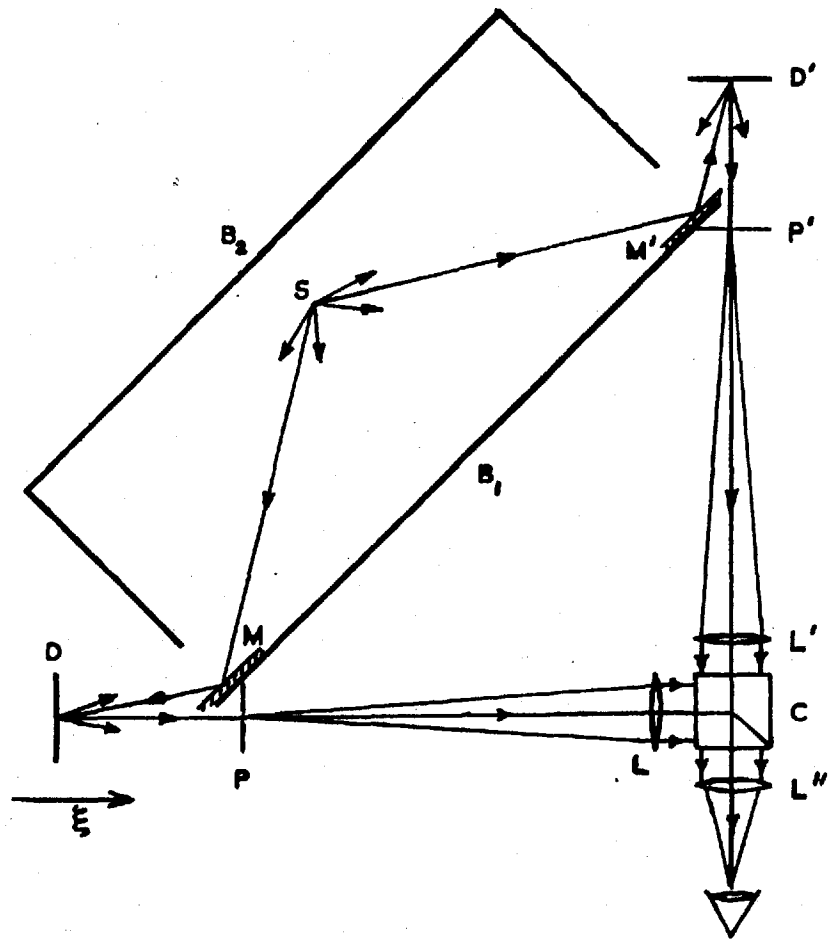


FIGURE 1

Apparatus for measuring superficial photometric precision

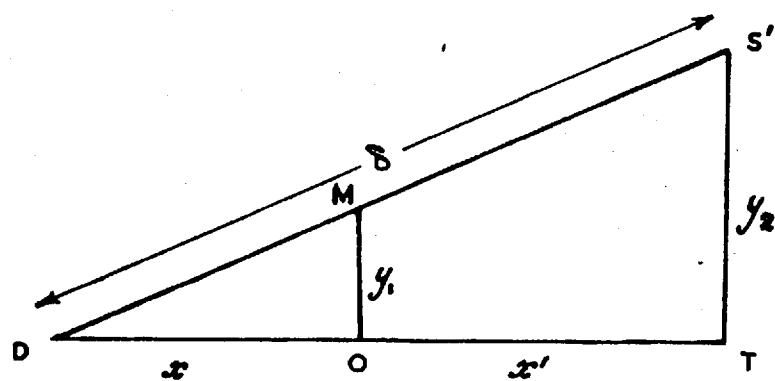


FIGURE 2
 Geometry of
 apparatus for measuring superficial photometric precision

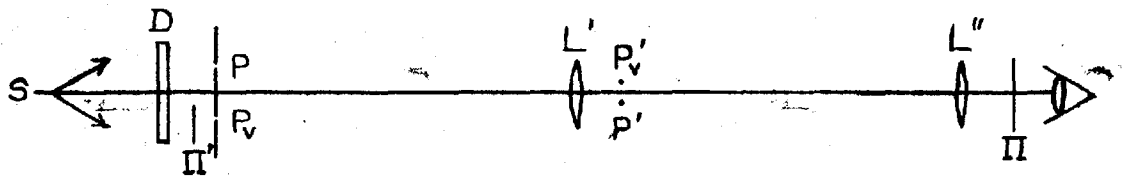


FIGURE 3

Apparatus for measuring punctiform photometric precision

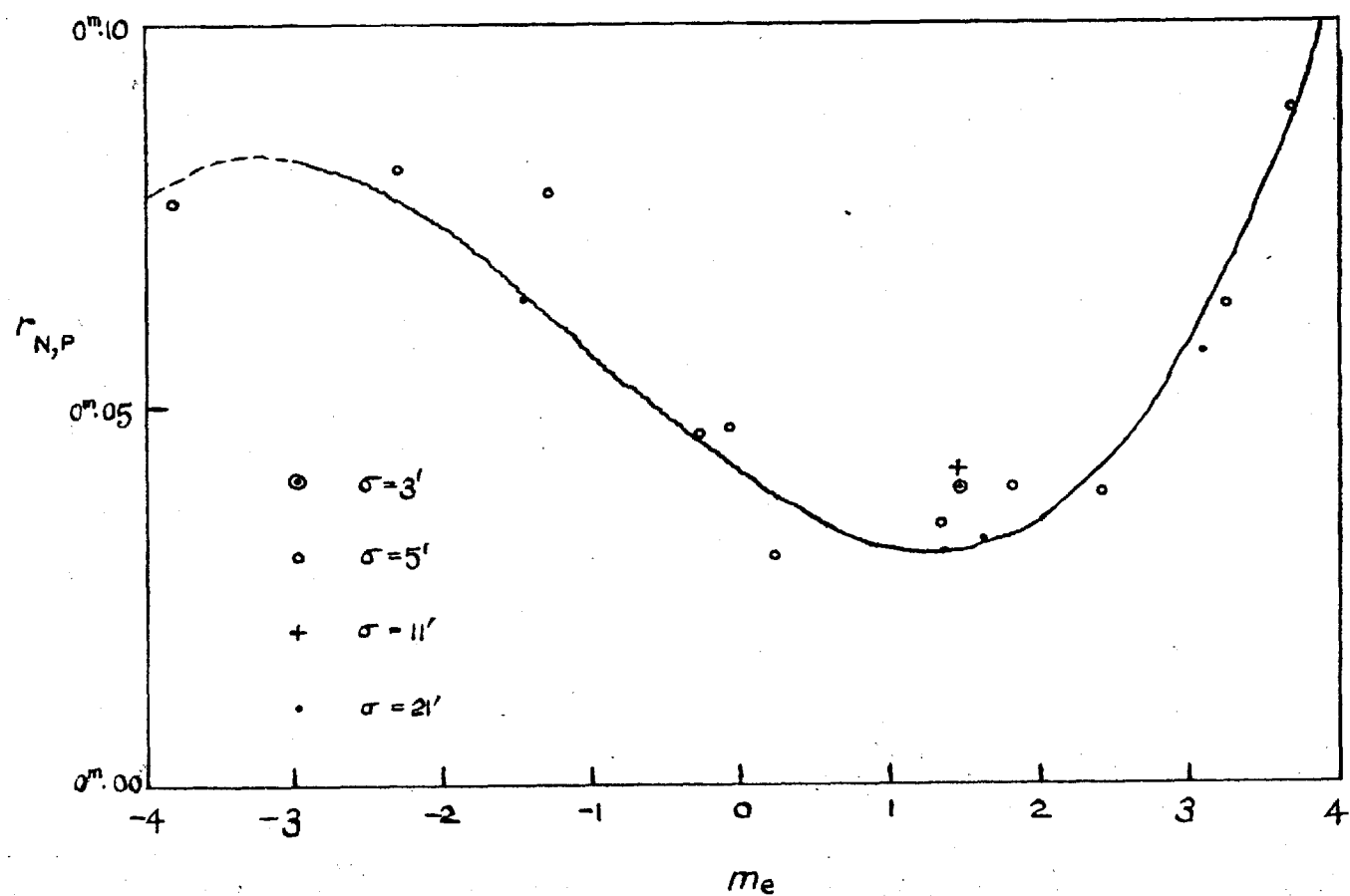


FIGURE 4: Photometric probable error $r_{N,P}$ as a function of ocular magnitude m_e

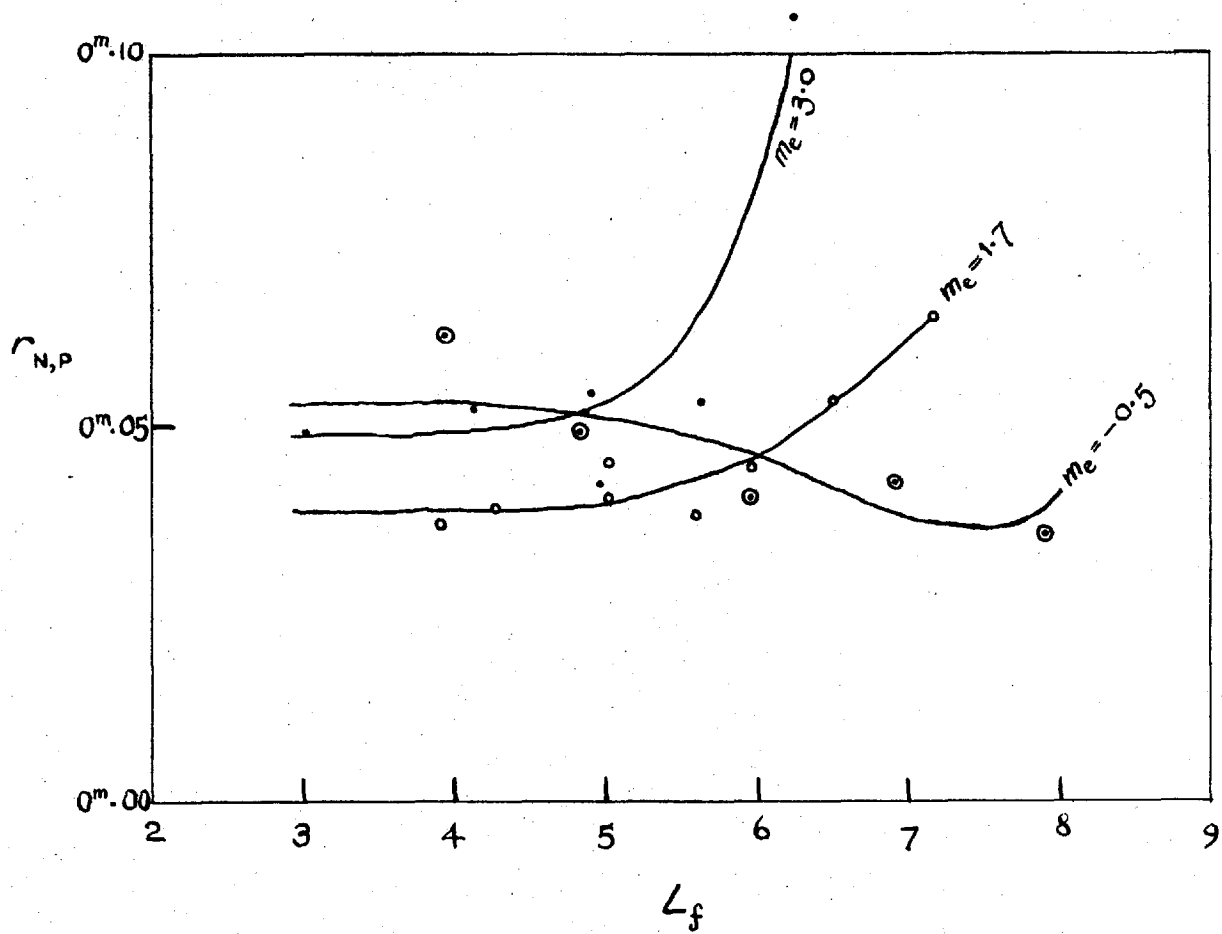


FIGURE 5

Punctiform photometric probable error $\epsilon_{N,P}$ as a function of ambient luminance L_f and ocular magnitude m_e

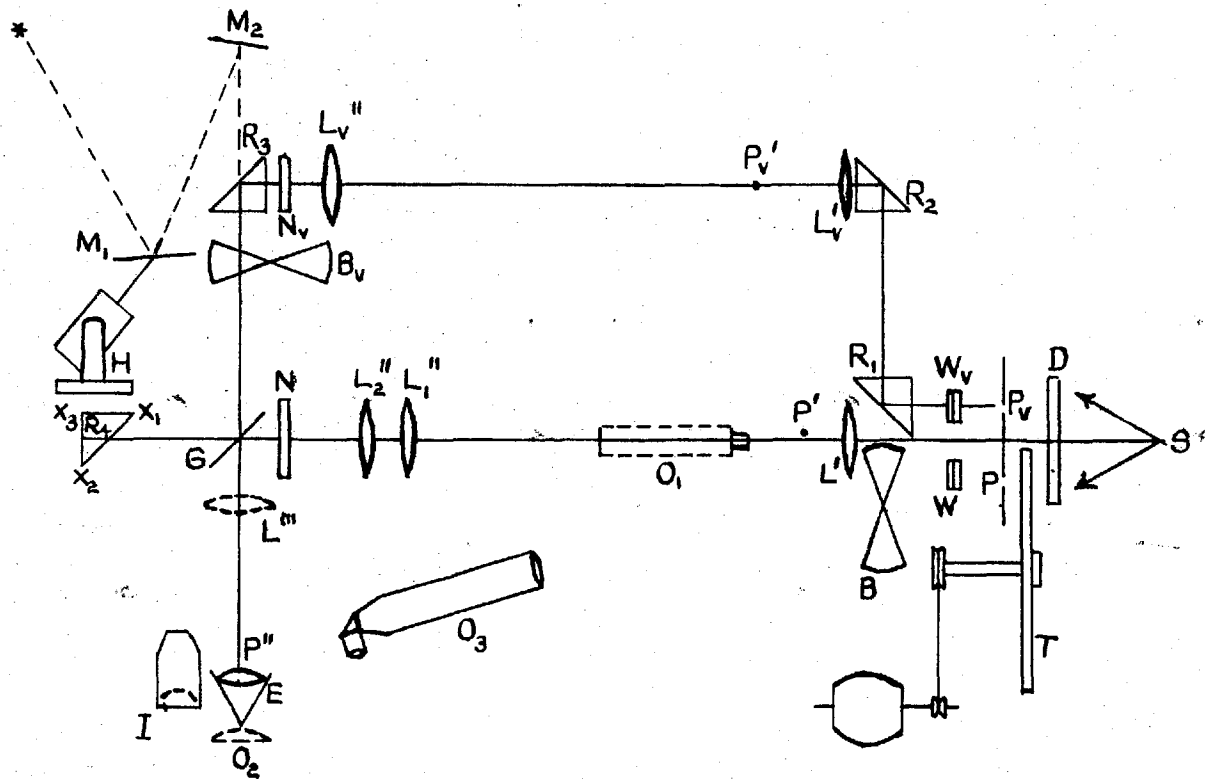


FIGURE 6

Universal apparatus for measuring photometric probable errors
and visual efficiency enhancements

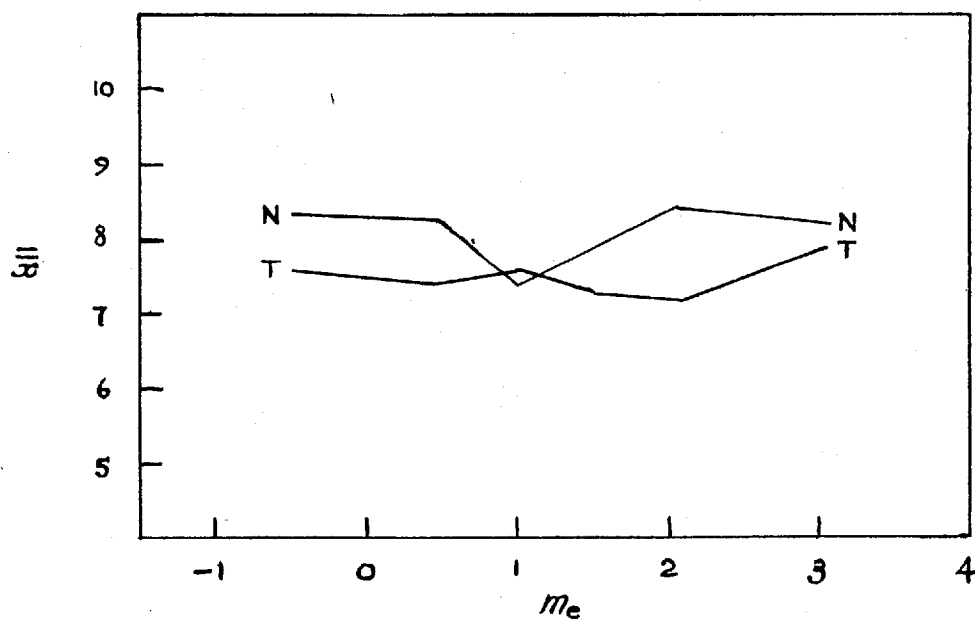


FIGURE 7

Visual efficiency enhancement in the punctiform mode

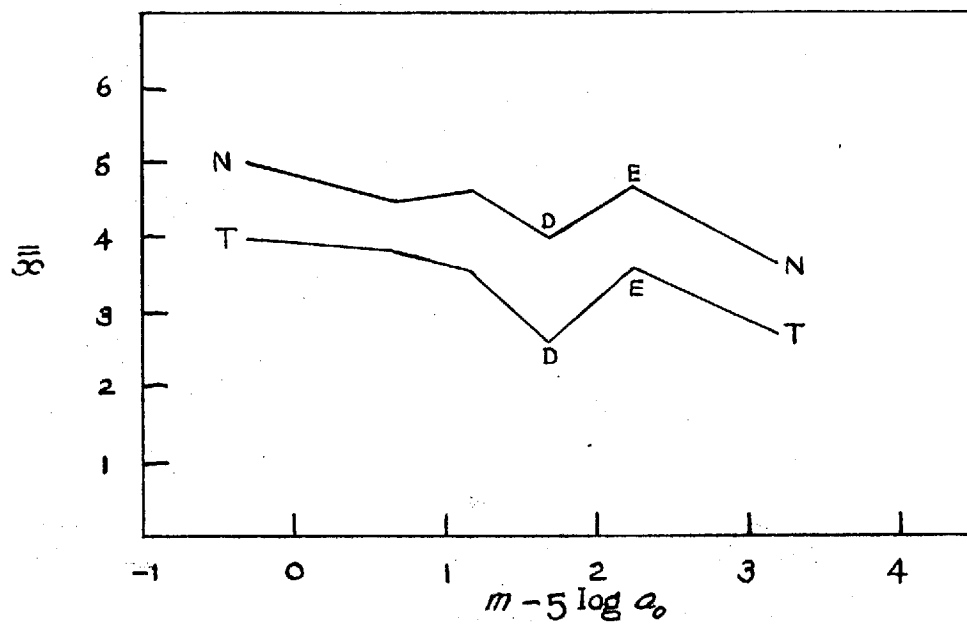


FIGURE 8

Visual efficiency enhancement in the superficial mode

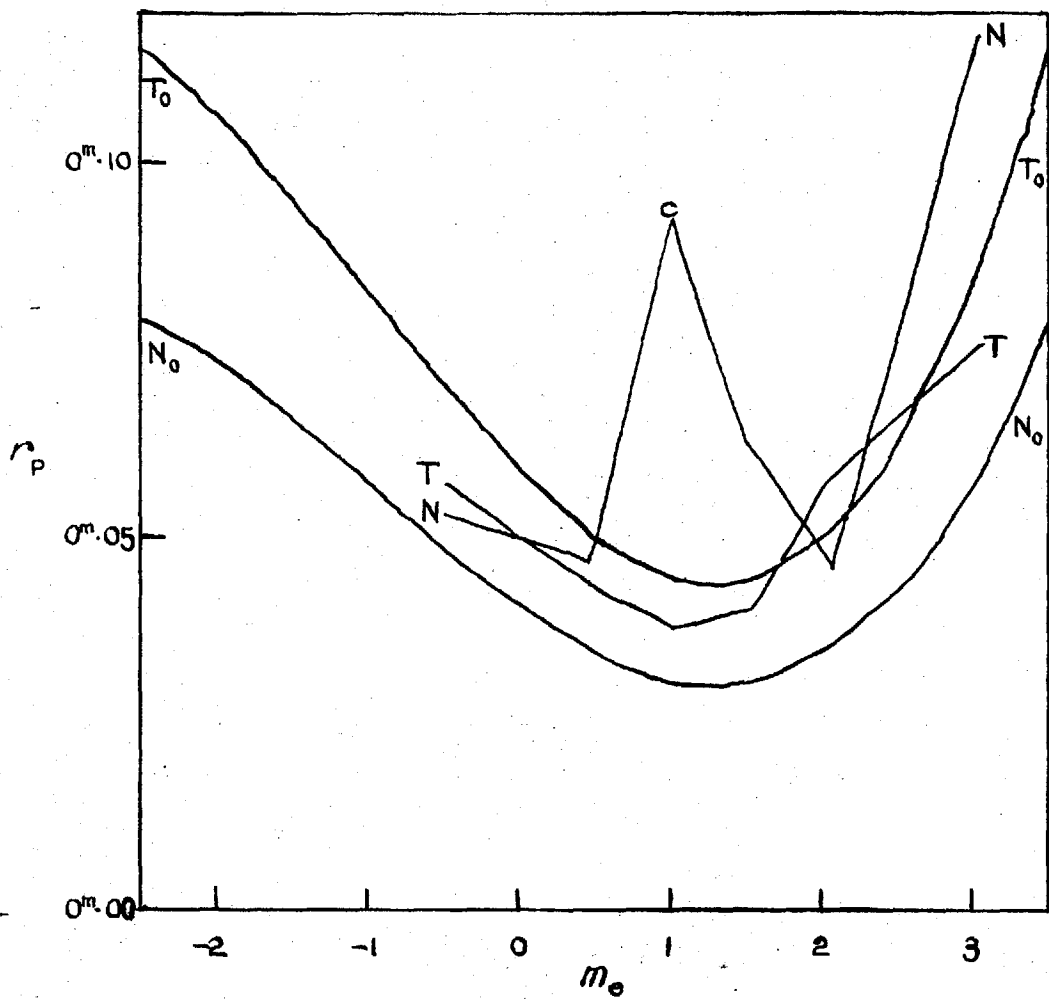


FIGURE 9

Punctiform photometric probable error, in the scintillant (T) and non-scintillant (N) aspects, as a function of ocular magnitude m_e

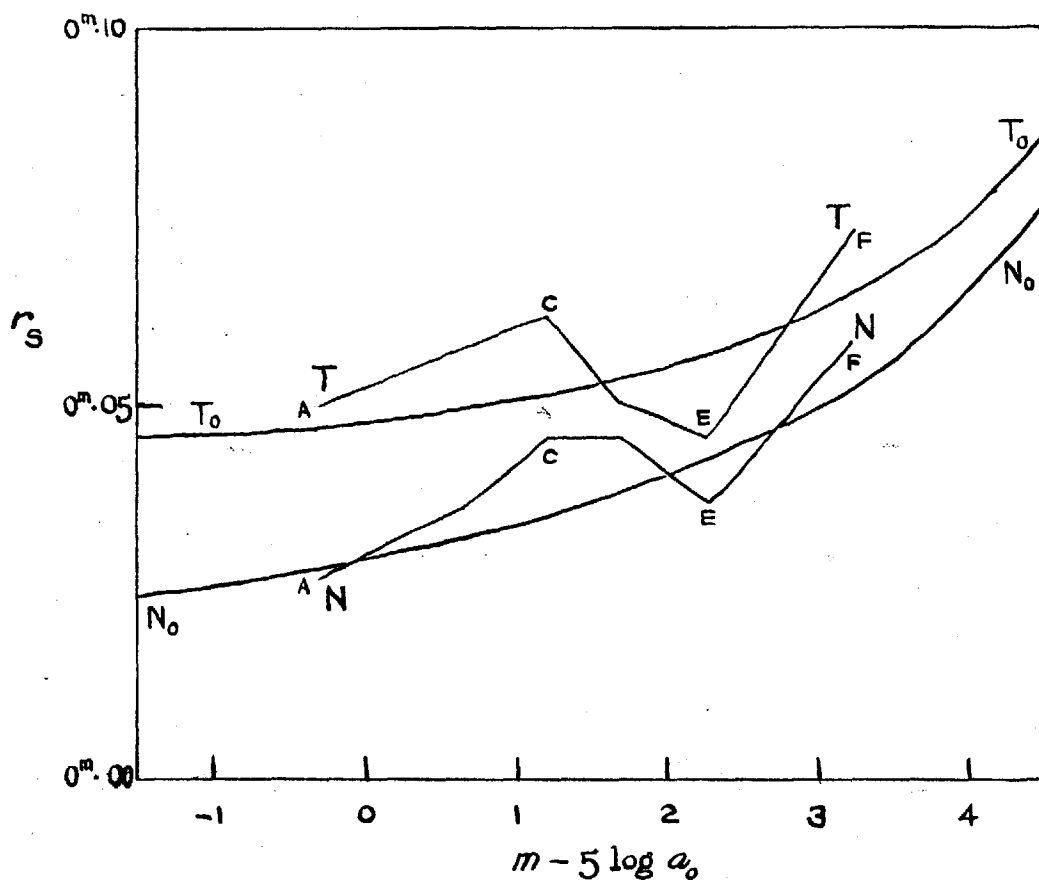


FIGURE 10

Superficial photometric probable error, in the scintillant (T) and non-scintillant (N) aspects, as a function of $m - 5 \log a_0$.

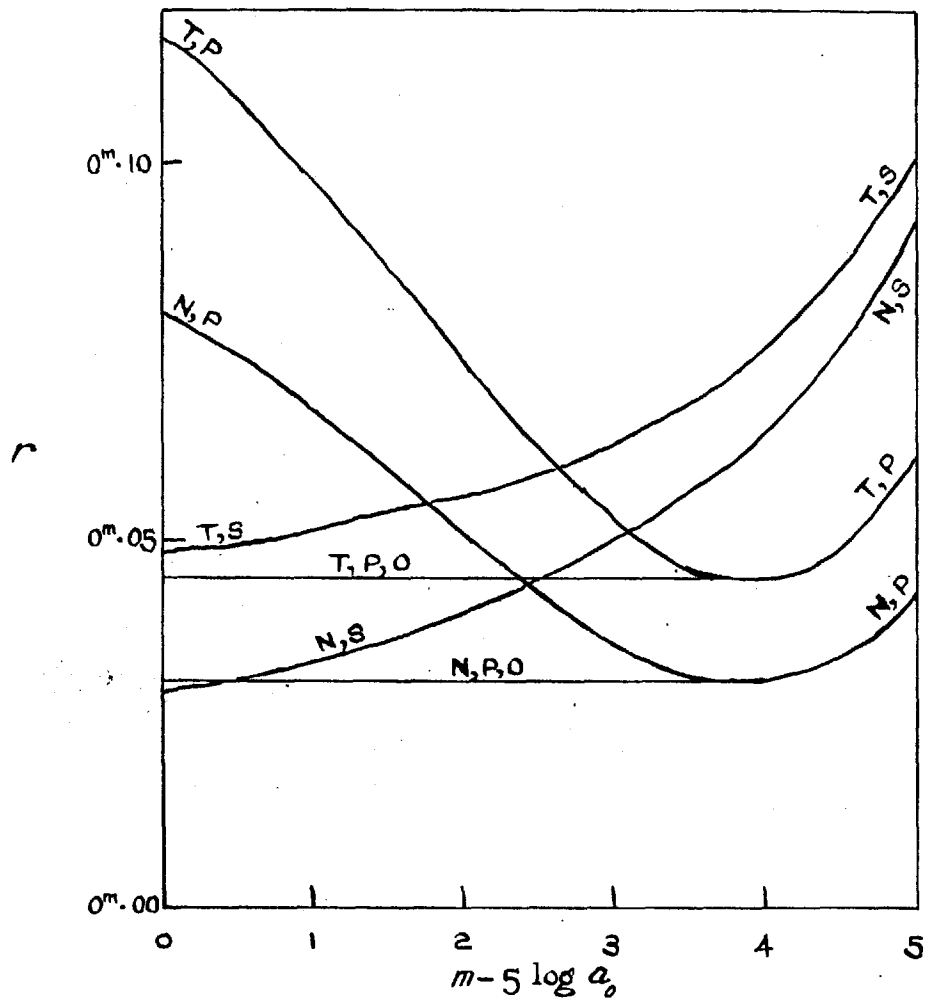


FIGURE 11

Comparison of the punctiform and superficial
photometric probable errors