

ASYMPTOTIC ANALYSIS OF NEW STOCHASTIC VOLATILITY MODELS

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Asymptotic analysis of new stochastic volatility models

ABSTRACT

A good options pricing model should be able to fit the market volatility surface with high accuracy. While the standard continuous stochastic volatility models can generate volatility smiles consistent with market data for relatively larger maturities, these models cannot reproduce market smiles for small maturities, which have the well-observed ‘small-time explosion’ feature.

In this thesis we propose three new types of stochastic volatility models, and we focus on the small-time asymptotic behaviour of the implied volatility in these models. We show that these models can generate implied volatilities with explosion, hence they can theoretically provide a better fit to the market data. The thesis is organised as follows.

Chapter 0 is the introduction. We briefly discuss the development and performance of standard continuous stochastic volatility models, and raise the small-time fitness issue of these traditional models.

In Chapter 1 we propose the randomised Heston model and analyse its small and large time asymptotic behaviours. In particular, we show that any small-time explosion rate in between of $[0, 1/2]$ for the implied variance can be captured by a suitable choice of the initial randomisation.

In Chapter 2 we propose a fractional version of the Heston model and detail the small-time asymptotic behaviour of the implied volatility in this setting. We precise the link between the explosion rate and the Hurst parameter.

Finally, in Chapter 3 we propose a new stochastic volatility model based on the recent work by Conus and Wildman [28] in which the stock price can have past dependency. We show that in the case of a CIR variance process this model has similar behaviours to a fractional Heston environment.

To my parents, and in memory of my grandmother.

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Twenty years ago I was a seven-year-old schoolboy living in a remote small town in Northwest China, and now I am approaching the end of my PhD study. It would not be possible without my parents' emphasis on education, their constant support and unconditional love. I also thank my wife Xiao for her love, encouragement and great patience throughout these years.

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NOTATIONS

$\mathbb{R}^*$	$\mathbb{R} \setminus \{0\}$
$\mathbb{R}_+$	$[0, \infty)$
$\mathbb{R}_+^*$	$(0, \infty)$
$\mathcal{S}^o$	Interior of a given set $\mathcal{S}$
$\overline{\mathcal{S}}$	Closure of a given set $\mathcal{S}$
$f \sim g$	$\lim_{x \rightarrow x_0} f(x)/g(x) = 1$ for some $x_0 \in \mathbb{R} \cup \{\infty\}$
$f^\leftarrow(x)$	Generalised inverse function of f
$\text{sgn}(u)$	$1_{\{u \geq 0\}} - 1_{\{u < 0\}}$
$\mathbb{P}$	Physical probability measure
$\mathbb{Q}$	Risk-neutral pricing measure
$\sigma_t(x)$	Implied volatility of a European call option with time to maturity t and log-strike x
$Z \sim \text{LDP}_0(g(t), \Lambda_Z^*)$	Sequence $(Z_t)_{t \geq 0}$ satisfies a large deviations principle as t tends to zero with speed $g(t)$ and good rate function Λ_Z^*
$Z \sim \text{LDP}_\infty(g(t), \Lambda_Z^*)$	Sequence $(Z_t)_{t \geq 0}$ satisfies a large deviations principle as t tends to infinity with speed $g(t)$ and good rate function Λ_Z^*



INTRODUCTION

Throughout this thesis we assume a normalised market in which the underlying stock price process $(S_t)_{t \geq 0}$ satisfies $S_0 = 1$ and the constant risk-free interest rate is $r = 0$. These assumptions should not cause a loss of generality.

Over the past two decades, a number of stochastic volatility models have been proposed in order to understand the dynamics and reproduce the features of the implied volatility. In general, in a continuous stochastic volatility model the instantaneous volatility of the stock price is governed by a second SDE, so that we can write

$$\begin{aligned} dS_t &= \sigma_t S_t \left(\rho dW_t^{(1)} + \bar{\rho} dW_t^{(2)} \right), \quad S_0 = 1, \\ d\sigma_t &= a(t, \sigma_t) dt + b(t, \sigma_t) dW_t^{(1)}, \quad \sigma_0 > 0, \end{aligned} \tag{1}$$

where the process $(W_t^{(1)}, W_t^{(2)})_{t \geq 0}$ is a standard two-dimensional Brownian motion supported on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$, $\rho \in [-1, 1]$, $\bar{\rho} := \sqrt{1 - \rho^2}$, and functions a and b satisfy certain regularity conditions such that the solution (S, σ) to Equation (1) exists. By definition, stochastic volatility models naturally provide the desirable stochasticity for the volatility of the underlier, and the correlation parameter ρ captures the well-

observed leverage effect in equity markets. For these reasons, stochastic volatility models quickly became a favourable choice in the industry, and practitioners and academic researchers have proposed various dynamics for the volatility process [92, 93, 98]. These models generally provide a better fit to the market volatility surfaces compared with the Black-Scholes framework; however, as we will see in the example of the Heston model, they also have a common shortcoming.

In his seminal paper [62] Heston proposed a stochastic volatility model in which the instantaneous variance of the underlying is a Cox-Ingersoll-Ross process. Specifically, we have

$$\begin{aligned} dS_t &= S_t \sqrt{V_t} \left(\rho dW_t^{(1)} + \bar{\rho} dW_t^{(2)} \right), \quad S_0 = 1, \\ dV_t &= \kappa(\theta - V_t)dt + \xi \sqrt{V_t} dW_t^{(1)}, \quad V_0 = v_0 > 0, \end{aligned} \quad (2)$$

where the positive constants κ, θ and ξ are usually required to satisfy the Feller condition $2\kappa\theta > \xi^2$ such that the zero boundary is unattainable. The Heston model quickly gained its popularity due to an explicit representation for the characteristic function of the log-price (the model is briefly reviewed in Appendix B), so that a Fourier Transform method for options pricing can be applied. In Figure 1 we calibrate the Heston model to the S&P options data on Sept. 20th, 2016. The calibration results are accurate for a wide range of maturities. However, as the time to maturity becomes smaller, the model-generated smile is not steep enough to match the market data, which has the so-called ‘small-time explosion’ feature. This phenomenon is described in detail by Gatheral [52, Chapters 3 and 5]. As a side note, we comment that a local volatility setting does not have the small-time fitness issue since the Dupire formula [37] can be applied to back out the volatility directly from market option prices. However, the local volatility model underestimates the randomness of the volatility, thus provides unrealistic dynamics for the volatility and tends to misprice exotic products such as forward starting options and barrier options.

To improve the small-time fitness of stochastic volatility models, Gatheral (among others) comments that jumps should be added to model the stock

price dynamics. The literature on the influence of jumps is vast: a clear review of exponential Lévy models is provided in [94], where the small-time implied volatility explodes at a rate of $|t \log t|$ for small t . To observe non-trivial convergence, Mijatović and Tankov [85] introduced strikes that are maturity-dependent, and they studied the behaviour of the smile in this regime.

Alternatively, a portion of the mathematical finance community has recently been advocating the use of fractional Brownian motion (where the Hurst parameter H is in $(0, 1/2)$) as the driving process of the volatility. Alòs, León and Vives [3] first showed that such a model is indeed capable of generating steep volatility smiles for small maturities (see also the recent work by Fukasawa [49]), and Gatheral, Jaisson and Rosenbaum [54] recently showed that financial data exhibits strong evidence that volatility is rough (an estimate for SPX volatility actually gives $H \approx 0.14$). This is currently a very active research area, and the reader is invited to consult [10, 38, 39, 40, 46, 81] for further developments. However, a severe concern for the fractional models is its significant increase of the computational cost.

In the following chapters we provide three new types of pricing models, namely randomised Heston, fractional Heston, and Conus-Wildman with stochastic volatility, and we focus on their asymptotic behaviours at the short end. We show that these models are capable of generating implied volatilities ‘with explosion’, so that they have better small-time fitness to the market data.

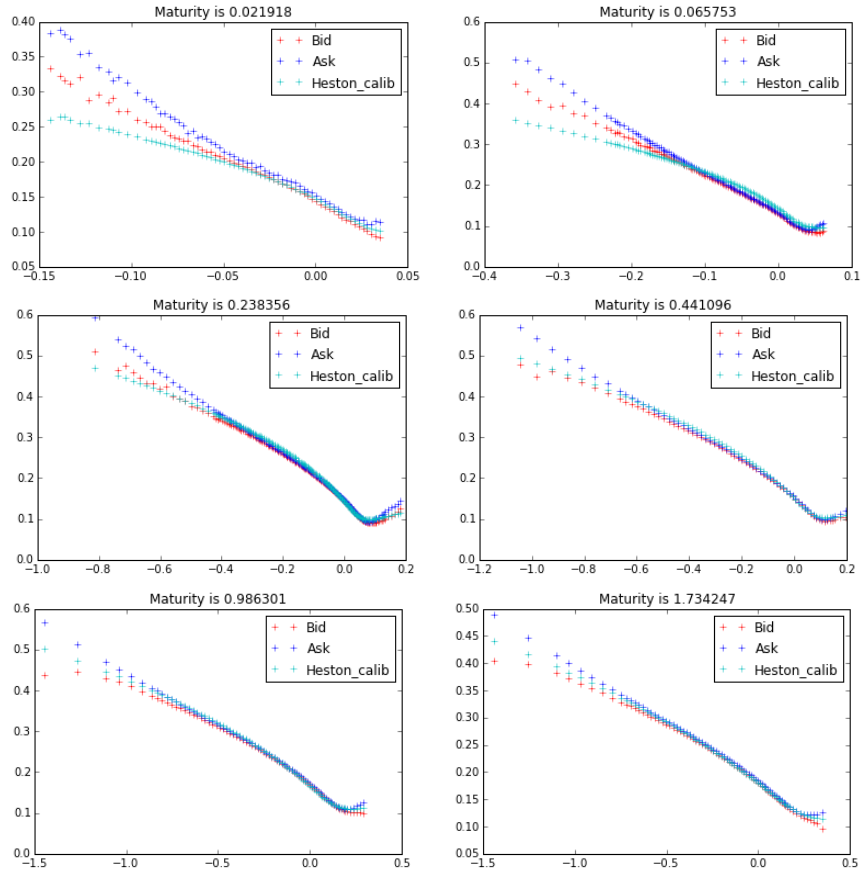


Figure 1: We calibrate the standard Heston model using the SPXW data on September 20th, 2016 with six different time to maturities (represented in years). As the expiration tends to zero the market volatility smile is much steeper than the model-generated smile.

1

RANDOMISED HESTON MODEL

In this chapter we propose and investigate the idea of initial randomisation. We focus on the Heston case, and analyse the small/large-time asymptotic behaviours of the implied volatility in the randomised Heston environment. We show that as the time to maturity tends to zero, our model generates a much steeper volatility smile compared with standard SV models. *

1.1 INTRODUCTION

We propose a new class of models, namely standard stochastic volatility models (driven by standard Brownian motions) where the initial value of the variance is a random variable. The motivation for this approach originates from the analysis of forward-start smiles by Jacquier and Roome [70, 71], who proved that the forward implied volatility explodes at a rate of $t^{1/4}$ as t tends to zero. A simple version of our current study is the ‘CEV-randomised Black-Scholes model’ introduced in [72], where the Black-Scholes volatility

*Major results and numerical examples presented in Sections 1.1-1.6 and their proofs in Section 1.9 have previously appeared in the submitted paper ‘the randomised Heston model’ [74].

has a distribution generated by an independent CEV process; in this work, the authors proved that this simplified model generates the desired explosion of the smile. The Black-Scholes randomised setting where the volatility has a discrete distribution corresponds to the lognormal mixture dynamics studied in [21, 22]. Here we push this analysis further; our intuition behind this new type of models is that the starting point of the volatility process is actually not observed accurately, but only to some degree of uncertainty. Traders, for example, might take it as the smallest (maturity-wise) observed at-the-money implied volatility. Our initial randomisation aims at capturing this uncertainty. This approach was recently taken by Mechkov [84], who considers the ergodic distribution of the CIR process as starting distribution. Mechkov argues that by randomising the starting point we are able to capture potential hidden variables.

The main result of this chapter is to provide the connection between the explosion rate of the small-time implied volatility and the choice of the initial distribution of the variance process. The following table (a more complete version with more examples can be found in Table 1.1) gives an idea of the range of explosion rates that can be achieved through our procedure; for each suggested distribution of the initial variance, we indicate the asymptotic behaviour (up to a constant multiplier) of the square of the out-of-the-money implied volatility smile. In the first row, the function f will be determined precisely later, but the absence of time-dependence is synonymous with absence of explosion. This result might seem undesirable at first, but in Sections 1.6 and 1.7 we show that the bounded support case can generate a steepened small-time volatility smile even without an explosion factor. Hence this special case could still be of importance from a practitioner's point of view.

Name	Behaviours of $\sigma_t^2(x)$ ($x \neq 0$)	Reference
Uniform	$f(x)$	Equation 1.6
Exponential(λ)	$ x t^{-1/2}$	Theorem 13
χ -squared	$ x t^{-1/2}$	Theorem 13
Rayleigh	$x^{2/3}t^{-1/3}$	Theorem 7
Weibull ($k > 1$)	$(x^2/t)^{1/(1+k)}$	Theorem 7

The rest of this chapter is structured as follows: Section 1.2 includes a discussion on the connection between our model and the study on model uncertainty. A simplified numerical example is provided to give a flavour of the quality of such a randomisation. We formally introduce the randomised Heston model in Section 1.3, and discuss its main properties. Section 1.4 is the main part of this chapter, in which we prove large deviations principles for the log-price process, and translate them into small- and large-time behaviours of the implied volatility. In particular, we prove the claimed relation between the explosion rate of the small-time smile and the tail behaviour of the initial distribution. The small-time limit of the at-the-money implied volatility is, as usual in this literature, treated separately in Section 1.4.6. Section 1.5 includes a dynamic pricing framework: based on the distribution at time zero and the evolution of the variance process, we discuss how to re-price the option during the life of the contract. Section 1.6 presents examples of common initial distributions, together with numerical examples. In Section 1.7 we calibrate the randomised Heston model using the USD/JPY market data. The result shows that our model substantially improves the calibration accuracy. In Section 1.8 we discuss the hedging problem in the randomised Heston framework. A martingale representation theorem in a randomised environment is included in this section. Finally, Section 1.9 gathers the proofs of the main theorems.

1.2 CONNECTION TO MODEL UNCERTAINTY AND A NUMERICAL APPE-TISER

1.2.1 MODEL UNCERTAINTY

Our framework is closely related to model uncertainty, a detailed review of which can be found in [60], and our assumption of an initial distribution of the variance directly corresponds to calibration and parameter uncertainties: as mentioned, in a standard stochastic volatility model, the starting point v_0 is not observed precisely, and is usually approximated by at-the-money (ATM) implied volatilities with the smallest maturities. As a result, the noises of market data contribute to the parameter uncertainty, and the

Bayesian method immediately provides a posterior distribution for v_0 . Consider for example a standard Heston environment, in which a trader wants to calibrate v_0 . From [44, Theorem 3.1], the small-time asymptotic expansion of European option prices is given by

$$\frac{C_{\mathcal{H}}(0, t)}{S_0} = \sqrt{\frac{v_0 t}{2\pi}} - \mathfrak{b} t^{3/2} + \mathcal{O}(t^{5/2}), \quad (1.1)$$

for some explicit constant $\mathfrak{b}$. Then the map $(v_0, t) \mapsto \sqrt{v_0 t / (2\pi)}$, which only depends on v_0 and t , is an approximation to the (Heston) ATM option price with small maturities. Denote by $\mathcal{P} := (\mathcal{P}^{(i)})_{i=1}^n$ a sequence of observed market ATM mid-prices with one-week maturity ($t \approx 0.019$) on n different dates, and assume that the approximation noises $\varepsilon^{(i)} := \mathcal{P}^{(i)} - \left(\frac{\tilde{v}_0 t}{2\pi}\right)^{1/2}$, for $i = 1, \dots, n$ form an independent sequence satisfying

$$\mathbb{Q}(\varepsilon^{(i)} \in A | \tilde{v}_0) = \frac{1}{\delta_i \sqrt{2\pi}} \int_A \exp\left(-\frac{x^2}{2\delta_i^2}\right) dx, \quad \text{for any Borel subset } A \subseteq \mathbb{R},$$

where $\tilde{v}_0$ is the ‘true’ value of v_0 and $(\delta_i)_{i=1}^n$ are positive constants. In this simplified setting, the errors $(\varepsilon^{(i)})_{i=1}^n$ summarise all microstructure effects that can potentially generate market noises. Finally, assume that the prior is a Gamma distribution $\Gamma(\alpha, \beta)$ that approximates the invariant distribution of the variance process. Then the posterior of v_0 is given by

$$\begin{aligned} p(v_0 | \mathcal{P}) &= \mathcal{C} p(v_0) p(\mathcal{P} | v_0) \\ &= \mathcal{C} v_0^{\alpha-1} \exp \left\{ -\beta v_0 - \sum_{i=1}^n \frac{\left(\sqrt{v_0 t / (2\pi)} - \mathcal{P}^{(i)} \right)^2}{2\delta_i^2} \right\} \\ &= \mathcal{C} v_0^{\alpha-1} \exp \left\{ - \left(\beta + \frac{t}{4\pi} \sum_{i=1}^n \delta_i^{-2} \right) v_0 + \sqrt{\frac{t v_0}{2\pi}} \sum_{i=1}^n \frac{\mathcal{P}^{(i)}}{\delta_i^2} \right\}, \end{aligned}$$

for some normalising factor $\mathcal{C}$ changing from line to line. With the posterior provided, the pricing scheme in our randomised setting can be seen as the partial Bayesian method [19, 60], in which prices (and hedging ratios) are averaged according to the posterior.

1.2.2 THE BOUNDED SUPPORT CASE: A PRACTICAL APPETISER

As mentioned in the introduction, the main drawback of classical continuous-path stochastic volatility models (driven by standard Brownian motions and without randomisation), is that the small-maturity smile they generate is not steep enough to reflect the reality of the market. Figure 1.1 below represents a comparison of the implied volatility surface generated by the standard Heston model with

$$\kappa = 2.1, \quad \theta = 0.05, \quad V_0 = 0.06, \quad \rho = -0.6, \quad \xi = 0.1,$$

and that of the Heston model randomised by a uniform distribution with $\mathbf{v}_- = 0.04$ and $\mathbf{v}_+ = 0.082$. Clearly, the randomisation steepens the smile for small maturities, while its effect fades away as the time to maturity increases. This numerical example intuitively yields the following informal conjecture:

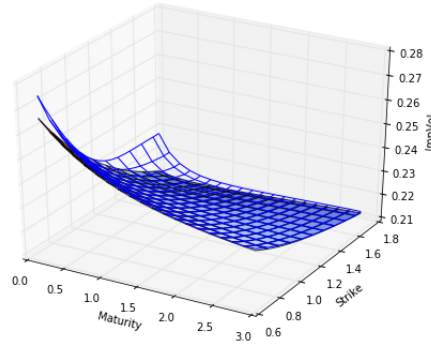


Figure 1.1: Volatility surfaces of standard Heston (coloured) and with randomisation $\mathcal{V}^{(\text{Law})} = \mathcal{U}(4\%, 8.2\%)$.

Conjecture 1. *Under randomisation of the initial volatility, the smile ‘explodes’ for small maturities.*

The next section provides a precise formulation and exact statements of this conjecture. In fact, despite the appearances in Figure 1.1, the conjecture is actually false when the initial distribution has bounded support, such as in the uniform case here. However, as will be detailed in Section 1.6.1—where

small-time expansions of the option price are proved—greater steepness of the smile (compared to the standard Heston model) does appear for a wide range of strikes, but not in the far tails (the at-the-money curvature in the uncorrelated case is discussed in Section 1.6.1). This leads us to believe that, even if ‘explosion’ does not actually occur in the bounded support case, this assumption may still be of practical relevance given the range of traded strikes.

1.3 MODEL AND MAIN PROPERTIES

On a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ supporting two Brownian motions $W^{(1)}$ and $W^{(2)}$, we consider a market with no interest rates, and propose the following dynamics for the log-price process:

$$\begin{aligned} dX_t &= -\frac{1}{2}V_t dt + \sqrt{V_t} \left(\rho dW_t^{(1)} + \bar{\rho} dW_t^{(2)} \right), & X_0 &= 0, \\ dV_t &= \kappa(\theta - V_t)dt + \xi \sqrt{V_t} dW_t^{(1)}, & V_0 &= \mathcal{V}, \end{aligned} \quad (1.2)$$

where $\rho \in [-1, 1]$, $\bar{\rho} := \sqrt{1 - \rho^2}$, and κ, θ, ξ are strictly positive real numbers. Here $\mathcal{V}$ is a continuous random variable that is independent of the filtration $(\mathcal{F}_t)_{t \geq 0}$, supported on $(\mathbf{v}_-, \mathbf{v}_+)$ for some $0 \leq \mathbf{v}_- \leq \mathbf{v}_+ \leq \infty$. Denote the moment generating function of $\mathcal{V}$ as $M_{\mathcal{V}}(u) := \mathbb{E}(e^{u\mathcal{V}})$, for all $u \in \mathcal{D}_{\mathcal{V}} := \{u \in \mathbb{R} : \mathbb{E}(e^{u\mathcal{V}}) < \infty\} \supset (-\infty, 0]$, and we further assume that $\mathcal{D}_{\mathcal{V}}$ contains at least an open neighbourhood of the origin, namely that $\mathfrak{m} := \sup\{u \in \mathbb{R} : M_{\mathcal{V}}(u) < \infty\}$ belongs to $(0, \infty]$. Then clearly all positive moments of $\mathcal{V}$ exist. We define the solution to the SDE (1.2) in the sense of [78, Chapter 5, Definition 2.1]: the process (X, V) is adapted to the augmented filtration $\sigma(\mathcal{F}_t \vee \sigma(\mathcal{V}))_{t \geq 0}$, it satisfies (1.2) in the integral form, and $\mathbb{Q}((X_0, V_0) = (0, \mathcal{V})) = 1$. Since $\mathcal{V}$ admits a second moment, [78, Chapter 5, Theorem 2.9] guarantees the existence and uniqueness of the solution to this stochastic system. Also notice that the process (X, V) is not adapted to the filtration $(\mathcal{F}_t)_{t \geq 0}$ due to the lack of information on $\mathcal{V}$ in $\mathcal{F}_t$. The process is Markovian, however, with respect to the augmented filtration $\sigma(\mathcal{F}_t \vee \sigma(\mathcal{V}))_{t \geq 0}$.

When $\mathcal{V}$ is a Dirac distribution ($\mathbf{v}_- = \mathbf{v}_+$), we recover the standard Hes-

ton model [62], and it is well known that the stock price process $\exp(X)$ is a $\mathbb{Q}$ -martingale. Behaviour [97], asymptotics [43, 44, 45], estimation and calibration [6, 97] of the standard Heston model have been treated at length in several papers, and we refer the interested reader to this literature for more details about it; we shall therefore assume that $\mathbf{v}_- < \mathbf{v}_+$. Throughout this chapter we always deal with a risk-neutral pricing framework, hence when taking expectations we omit the notation $\mathbb{Q}$ for brevity.

Remark 2. *For any $t \geq 0$, the tower property for conditional expectations yields*

$$\begin{aligned}\mathbb{E}(V_t) &= \mathbb{E}[\mathbb{E}(V_t|\mathcal{V})] = \theta(1 - e^{-\kappa t}) + e^{-\kappa t}\mathbb{E}(\mathcal{V}), \\ \mathbb{V}(V_t) &= e^{-2\kappa t} \left(\mathbb{V}(\mathcal{V}) + \frac{\xi^2}{\kappa} (e^{\kappa t} - 1) \mathbb{E}(\mathcal{V}) \right) + \frac{\xi^2 \theta}{2\kappa} (1 - e^{-\kappa t})^2.\end{aligned}$$

Consider the standard Heston model ($\mathbf{v}_- = \mathbf{v}_+ =: V_0$), and construct $\mathcal{V}$ such that $\mathbb{E}(\mathcal{V}) = V_0$. Then, for any time $t \geq 0$, both random variables V_t (in (1.2) and V_0 in the standard Heston model) have the same expectations; however, the randomisation of the initial variance increases the variance by $e^{-2\kappa t}\mathbb{V}(\mathcal{V})$. As time tends to infinity, it is straightforward to show that the randomisation preserves the ergodicity of the variance process, with a Gamma distribution as invariant measure, with mean and variance:

$$\lim_{t \uparrow \infty} \mathbb{E}(V_t) = \theta \quad \text{and} \quad \lim_{t \uparrow \infty} \mathbb{V}(V_t) = \frac{\xi^2 \theta}{2\kappa}.$$

For any $t \geq 0$, let $M(t, u)$ denote the moment generating function (mgf) of X_t :

$$M(t, u) := \mathbb{E}(e^{uX_t}), \quad \text{for all } u \in \mathcal{D}_M^t := \{u \in \mathbb{R} : \mathbb{E}(e^{uX_t}) < \infty\}. \quad (1.3)$$

The tower property then yields

$$M(t, u) = \mathbb{E}[\mathbb{E}[e^{uX_t}|\mathcal{V}]] = \mathbb{E}[e^{C(t,u)+D(t,u)\mathcal{V}}] = e^{C(t,u)}M_{\mathcal{V}}(D(t, u)), \quad (1.4)$$

where the functions C and D arise directly from the (affine) representation of the moment generating function of the standard Heston model, recalled

in Equation (B.1).

1.4 ASYMPTOTIC BEHAVIOURS

This section is the core of this chapter, and relates the explosion of the implied volatility in small times to the tail behaviour of the randomised initial variance. Section 1.4.1 (Proposition 3) provides the small-time limiting behaviour of the cumulant generating function (cgf) of the random sequence $(X_t)_{t \geq 0}$, and relates it to the choice of the initial distribution $\mathcal{V}$. With this result we can prove the large deviations principle for the sequence $(X_t)_{t \geq 0}$. Section 1.4.2 concentrates on the case where both $\mathfrak{m}$ and $\mathfrak{v}_+$ are infinite: Theorem 7 indicates that the squared implied volatility has an explosion rate of t^γ with $\gamma \in (0, 1/2)$. The case where $\mathfrak{m} < \mathfrak{v}_+ = +\infty$ is covered in Section 1.4.3, where an explosion rate of $\sqrt{t}$ is obtained for the implied variance. Results of moderate deviations are stated in Section 1.4.4. In Section 1.4.5 we provide the large-time asymptotic behaviour of the implied volatility in our randomised setting. In particular, the long-term similarities between standard and randomised Heston models are presented in this section. Section 1.4.6 covers the singular case of the small-time at-the-money implied volatility.

1.4.1 PRELIMINARIES

As a first step in understanding the behaviour of the implied volatility, we analyse the short-time limit of the rescaled cgf of the sequence $(X_t)_{t \geq 0}$. To do so, let $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a smooth function, which can be extended at zero by continuity such that $h(0) := \lim_{t \downarrow 0} h(t) = 0$. Therefore, in light of (1.4), for any $t \geq 0$, we introduce the effective domain of the moment generating function of the rescaled random variable $X_t/h(t)$:

$$\mathcal{D}_t := \left\{ u \in \mathbb{R} : M \left(t, \frac{u}{h(t)} \right) < \infty \right\},$$

as well as the following sets, for any $t > 0$:

$$\mathcal{D}_{\mathcal{V}}^t := \left\{ u \in \mathbb{R} : M_{\mathcal{V}} \circ D \left(t, \frac{u}{h(t)} \right) < \infty \right\}, \quad \mathcal{D}^* := \liminf_{t \downarrow 0} \mathcal{D}_t = \bigcup_{t > 0} \bigcap_{s \leq t} \mathcal{D}_s,$$

$$\mathcal{D}_{\mathcal{V}}^* := \liminf_{t \downarrow 0} \mathcal{D}_{\mathcal{V}}^t = \bigcup_{t > 0} \bigcap_{s \leq t} \mathcal{D}_{\mathcal{V}}^s.$$

We now denote the pointwise limit $\Lambda_h(u) := \lim_{t \downarrow 0} \Lambda_h(t, u/h(t))$, where

$$\Lambda_h \left(t, \frac{u}{h(t)} \right) := h(t) \log M \left(t, \frac{u}{h(t)} \right). \quad (1.5)$$

The seemingly identical notations for the function and its pointwise limit should not create any confusion. The following theorem, whose proof is postponed to Section 1.9.1, summarises the limiting behaviour of $\Lambda_h(\cdot, \cdot)$ as t tends to zero. The real numbers u_- and u_+ and the function Λ are defined in (B.3)-(B.4). In view of Remark 4(ii) below, we shall only consider power functions of the type $h(t) \equiv ct^\gamma$. Without loss of generality we take $c = 1$, as it only acts as a space-scaling factor. We shall therefore replace the notation Λ_h by Λ_γ to highlight the power in action.

Proposition 3. *Let $h(t) \equiv t^\gamma$, with $\gamma \in (0, 1]$. As t tends to zero, the following pointwise limits hold:*

$$\Lambda_\gamma(u) := \lim_{t \downarrow 0} \Lambda_\gamma \left(t, \frac{u}{t^\gamma} \right)$$

$$= \begin{cases} 0, & u \in \mathbb{R}, & \text{if } \gamma \in (0, 1/2), \text{ for any } \mathcal{V}, \\ 0, & u \in \mathbb{R}, & \text{if } \gamma \in [1/2, 1), \quad \mathbf{v}_+ < \infty, \\ \Lambda(u)\mathbf{v}_+, & u \in (u_-, u_+), & \text{if } \gamma = 1, \quad \mathbf{v}_+ < \infty, \\ L_{\pm} 1_{\{u = \pm \sqrt{2\mathbf{m}}\}}, & u \in [-\sqrt{2\mathbf{m}}, \sqrt{2\mathbf{m}}], & \text{if } \gamma = 1/2, \quad \mathbf{v}_+ = \infty, \quad \mathbf{m} < \infty, \end{cases}$$

and is infinite elsewhere. Whenever $\gamma > 1$ (for any $\mathcal{V}$), or $\mathbf{m} < \infty$ and $\gamma > 1/2$, the pointwise limit is infinite everywhere except at the origin.

We shall call the (pointwise) limit ‘degenerate’ if it is equal to zero everywhere or if it is zero at the origin and infinity everywhere else. In Proposition 3 only the last two cases are not degenerate.

Remark 4.

- (i) *The case where $\mathbf{v}_+$ and $\mathbf{m}$ are both infinite is treated separately in Section 1.4.2, as more assumptions are needed on the tail behaviour of the distribution of $\mathcal{V}$.*
- (ii) *If h is not a power function, the proofs of Proposition 3 and Lemma 8 indicate that we need to compare the order of h with orders of $t^{1/2}$ and t . The asymptotic results then depend on the explicit representation of h . The methodology for this slightly more general case is the same as the case where $h(t) \equiv t^\gamma$, hence we focus on the case where h is a power function.*
- (iii) *In the last case, $L_\pm$ depend on the explicit form of the mgf of $\mathcal{V}$. Example 1.6.2 illustrates this.*

If the initial distribution $\mathcal{V}$ has bounded support ($\mathbf{v}_+ < \infty$), Proposition 3 indicates that the only possible speed factor is $h(t) \equiv t$, and a direct application of the Gärtner-Ellis theorem (Theorem 61) implies a large deviations principle for the sequence $(X_t)_{t \geq 0}$; adapting directly the methodology from [42], we obtain the small-time asymptotics of the implied volatility:

Corollary 5. *If $\mathbf{v}_+ < \infty$, then $X \sim \text{LDP}_0(t, \Lambda_{\mathbf{v}_+}^*)$ with $\Lambda_{\mathbf{v}_+}^*$ defined as the Fenchel-Legendre transform of Λ : $\Lambda_{\mathbf{v}_+}^*(x) := \sup_{u \in (u_-, u_+)} \{ux - \Lambda(u)\mathbf{v}_+\}$, and*

$$\lim_{t \downarrow 0} \sigma_t^2(x) = \frac{x^2}{2\Lambda_{\mathbf{v}_+}^*(x)}, \quad \text{for all } x \neq 0. \quad (1.6)$$

Approximations of the rate function $\Lambda_{\mathbf{v}_+}^*$, in particular around the at-the-money point $x = 0$, can also be found in [42, Theorem 3.2] and therefore can be applied here directly.

1.4.2 THE THIN-TAIL CASE

In the case where both $\mathbf{m}$ and $\mathbf{v}_+$ are infinite, Proposition 3 is not sufficient as several different behaviours can occur. In this case, which we naturally coin

as the ‘thin-tail’ case, a more refined analysis is needed, and the following assumption shall be of uttermost importance:

Assumption 6 (Thin-tail). $\mathfrak{m} = \mathfrak{v}_+ = \infty$ and $\mathcal{V}$ admits a smooth density f with $\log f(v) \sim -l_1 v^{l_2}$ as v tends to infinity, for some $(l_1, l_2) \in \mathbb{R}_+^* \times (1, \infty)$.

For notational convenience, we introduce the following two special rates of convergence $\frac{1}{2} < \underline{\gamma} < 1 < \overline{\gamma}$, and two positive constants $\underline{\mathfrak{c}}, \overline{\mathfrak{c}}$:

$$\underline{\gamma} := \frac{l_2}{l_2 + 1}, \quad \overline{\gamma} := \frac{l_2}{l_2 - 1}, \quad \underline{\mathfrak{c}} := (2l_1 l_2)^{\frac{1}{1+l_2}}, \quad \overline{\mathfrak{c}} := (2l_1 l_2)^{\frac{1}{1-l_2}}. \quad (1.7)$$

The following theorem is the main result of this thin-tail section, and provides both a large deviations principle for the log-price process as well as its implications on the small-maturity behaviour of the implied volatility. Define the function $\underline{\Lambda}^* : \mathbb{R} \rightarrow \mathbb{R}_+$ by

$$\underline{\Lambda}^*(x) := \frac{\underline{\mathfrak{c}}}{2\underline{\gamma}} x^{2\underline{\gamma}}, \quad \text{for any } x \text{ in } \mathbb{R}. \quad (1.8)$$

Theorem 7. *Under Assumption 6, $X \sim \text{LDP}_0(t^{\underline{\gamma}}, \underline{\Lambda}^*)$ with $\underline{\Lambda}^*$ given in (1.8), and, for any $x \neq 0$,*

$$\lim_{t \downarrow 0} t^{1-\underline{\gamma}} \sigma_t^2(x) = \underline{\mathfrak{c}}^{-1} \underline{\gamma} x^{2(1-\underline{\gamma})}.$$

In exponential Lévy models, [94, Proposition 4] indicates that the implied variance $\sigma_t^2(x)$ for non-zero x explodes at a rate $|t \log t|$ as t tends to zero. Theorem 7 implies that in a thin-tail randomised Heston environment we have a much slower explosion rate of t^η with $\eta \in (0, 1/2)$. In [85] the authors commented that market data suggests that implied volatility with decreasing maturity still has a reasonable range of values and does not explode significantly; this might provide empirical grounds that justify the potential value of this randomised model as an alternative to the exponential Lévy models. The theorem relies on the study of the asymptotic behaviour of the rescaled mgf of X_t :

Lemma 8. *Under Assumption 6, the only non-degenerate speed factor is*

$\gamma = \underline{\gamma}$, and

$$\Lambda_{\underline{\gamma}}(u) = \frac{\bar{c}}{2^{\underline{\gamma}}} u^{2^{\underline{\gamma}}}, \quad \text{for any } u \text{ in } \mathbb{R}. \quad (1.9)$$

Assumption 6 in particular implies that the function $\log f$ is regularly varying with index l_2 (which we denote $|\log f| \in \mathcal{R}_{l_2}$, see also Appendix A.1 for a review of and useful results on regularly variation). Without this slightly stronger assumption, however, the constant in (1.9)—essential to compute precisely the rate function governing the corresponding large deviations principle (Theorem 7)—would not be available. In order to prove the lemma and hence the theorem, let us first state and prove the following result:

Lemma 9. *If $|\log f| \in \mathcal{R}_l$ ($l > 1$), then $\log M_{\mathcal{V}}(z) \sim (l-1) \left(\frac{z}{l}\right)^{\frac{l}{l-1}} \psi(z)$ at infinity, with $\psi \in \mathcal{R}_0$ defined as*

$$\psi(z) := \left(\frac{z}{|\log f|^{\leftarrow}(z)} \right)^{\leftarrow} z^{\frac{l}{1-l}}.$$

Proof. Since $|\log f| \in \mathcal{R}_l$, Bingham's Lemma (Lemma 58) implies $\log(\mathbb{Q}(\mathcal{V} \geq x)) = \log \int_x^\infty e^{\log f(y)} dy \sim \log f(x)$, as x tends to infinity, and the result follows from Kasahara's Tauberian theorem [18, Theorem 4.12.7]. $\square$

Proof of Lemma 8 and of Theorem 7. Lemma 59 implies that the mgf of $\mathcal{V}$ is well-defined on $\mathbb{R}_+$. Lemmas 9 and 63 imply that as t tends to zero,

$$\begin{aligned} & t^\gamma \log M_{\mathcal{V}} \left(D \left(t, \frac{u}{t^\gamma} \right) \right) \\ &= \begin{cases} t^\gamma \log M_{\mathcal{V}} \left(\frac{u^2}{2} t^{1-2\gamma} (1 + \mathcal{O}(t^{1-\gamma})) \right) & \sim \frac{\bar{c}}{2^{\underline{\gamma}}} u^{2^{\underline{\gamma}}} t^{\bar{\gamma}(1-\gamma/\underline{\gamma})}, \quad \text{when } \gamma \in (\tfrac{1}{2}, 1), \\ t \log M_{\mathcal{V}} \left(\frac{\Lambda(u)}{t} (1 + \mathcal{O}(t)) \right) & \sim \frac{\bar{c}}{2^{\underline{\gamma}}} 2^{\bar{\gamma}} \Lambda(u)^{\bar{\gamma}} t^{1-\bar{\gamma}}, \quad \text{when } \gamma = 1. \end{cases} \end{aligned}$$

For $u \neq 0$ the right-hand side is well defined with non-zero limit if and only if $\gamma = \underline{\gamma} \in (1/2, 1)$; the case $\gamma = 1$ does not yield any non-degenerate behaviour, and the lemma follows.

The large deviations principle stated in Theorem 7 is a direct consequence of Lemma 8 and the Gärtner-Ellis theorem (Theorem 61), noting that the

function $\Lambda_{\underline{\gamma}}$ in (1.9) satisfies all the required conditions and its Fenchel-Legendre transform is $\underline{\Lambda}^*$. The translation of this asymptotic behaviour into implied volatility follows the same lines as in [42]. $\square$

1.4.3 THE FAT-TAIL CASE

If $\mathbf{v}_+$ is infinite and $\mathbf{m}$ is finite, Proposition 3 states that the only choice for the rescaling factor is $h(t) = t^{1/2}$, but the form of the limiting rescaled cumulant generating function does not yield any immediate asymptotic estimates for the probabilities. In this case, we impose the following assumption on the mgf of $\mathcal{V}$ in the vicinity of the upper bound $\mathbf{m}$ of its effective domain:

Assumption 10. *There exists $(\gamma_0, \gamma_1, \gamma_2, \omega) \in \mathbb{R}^* \times \mathbb{R} \times \mathbb{R} \times \mathbb{N}_+^*$, such that the following asymptotics hold for the cgf of $\mathcal{V}$ as u tends to $\mathbf{m}$ from below:*

$$\begin{aligned} \log M_{\mathcal{V}}(u) & \quad (1.10) \\ = & \begin{cases} \gamma_0 \log(\mathbf{m} - u) + \gamma_1 + o(1), & \omega = 1, \gamma_0 < 0, \\ \frac{\gamma_0 \{1 + \gamma_1(\mathbf{m} - u) \log(\mathbf{m} - u) + \gamma_2(\mathbf{m} - u) + o(\mathbf{m} - u)\}}{(\mathbf{m} - u)^{\omega-1}}, & \omega \geq 2, \gamma_0 > 0, \end{cases} \end{aligned}$$

and the derivative of the cgf satisfies

$$\frac{M'_{\mathcal{V}}(u)}{M_{\mathcal{V}}(u)} = \begin{cases} \frac{|\gamma_0|}{\mathbf{m} - u} (1 + o(1)), & \omega = 1, \gamma_0 < 0, \\ \frac{(\omega - 1)\gamma_0}{(\mathbf{m} - u)^{\omega}} (1 + \mathbf{a}(\mathbf{m} - u) \log(\mathbf{m} - u) + \mathbf{b}(\mathbf{m} - u) (1 + o(1))), & \omega \geq 2, \gamma_0 > 0, \end{cases} \quad (1.11)$$

with $\mathbf{a} := \gamma_1(\omega - 2)(\omega - 1)^{-1}$ and $\mathbf{b} := (\gamma_2(\omega - 2) - \gamma_1)(\omega - 1)^{-1}$.

Remark 11. *Condition (1.11) together with the expressions of $\mathbf{a}$ and $\mathbf{b}$ imply that the asymptotics of $(\log(M_{\mathcal{V}}))'$ can be obtained by differentiating (1.10) term by term. This is of course not always true; however, Condition (1.11) is rather mild, and we shall check it directly in several cases where $M_{\mathcal{V}}$ is known in closed form.*

Example 12.

- For the Exponential distribution with parameter $\mathbf{m}$, $(\gamma_0, \gamma_1, \omega) = (-1, \log \mathbf{m}, 1)$.

- For the non-central χ -squared distribution as in Example 1.6.2, $(\gamma_0, \gamma_1, \gamma_2, \omega) = \left(\frac{\lambda}{4}, -\frac{2q}{\lambda}, -2\left(1 + \frac{q}{\lambda} \log 2\right), 2\right)$.

For $\mathbf{m} \in (0, \infty)$, introduce the function $\Lambda^* : \mathbb{R} \rightarrow \mathbb{R}_+$ as

$$\Lambda^*(x) := \sqrt{2\mathbf{m}}|x|, \quad (1.12)$$

as well as, for any $t > 0$ the functions $\mathcal{E}_t, \mathcal{C}_t : \mathbb{R}^* \rightarrow \mathbb{R}_+^*$ by $\mathcal{E}_t(x) := \mathbf{1}_{\{\omega=1\}} + \exp\left(\frac{c_1(x)}{t^{1/4}}\right) \mathbf{1}_{\{\omega=2\}}$ and

$$\mathcal{C}_t(x) := \begin{cases} \exp\left(\frac{1}{2}(\rho\xi\mathbf{m} + 1)x + \gamma_1\right) \frac{|x|^{|\gamma_0|-1}t^{1-|\gamma_0|/2}}{\Gamma(|\gamma_0|)(2\mathbf{m})^{1+|\gamma_0|/2}}, & \text{for } \omega = 1, \\ \exp\left(\frac{1}{2}(\rho\xi\mathbf{m} + 1)x + \gamma_0\gamma_2 + \frac{\gamma_0}{4\mathbf{m}}\right) \frac{t^{7/8+\gamma_0\gamma_1/4}}{2\mathbf{m}\sqrt{2\pi}\zeta(x)}, & \text{for } \omega = 2, \end{cases} \quad (1.13)$$

where the functions c_1 and ζ are defined in Lemmas 41-42 respectively. Then the following behaviour, proved in Section 1.9.2, holds for European option prices:

Theorem 13. *Under Assumption 10, European Call options with strike e^x have the following expansion:*

$$\mathbb{E}(e^{X_t} - e^x)^+ = (1 - e^x)^+ + \exp\left(-\frac{\Lambda^*(x)}{\sqrt{t}}\right) \mathcal{E}_t(x) \mathcal{C}_t(x) (1 + o(1)), \quad \text{for } x \neq 0,$$

as t tends to zero. Moreover, the small-time implied volatility behaves as follows whenever $x \neq 0$:

$$\sigma_t^2(x) = \frac{|x|}{2\sqrt{2\mathbf{m}t}} + \begin{cases} h_1^{(1)}(x) + h_2^{(1)} \log(t) + o(1), & \text{for } \omega = 1, \\ \frac{c_1(x)}{4\mathbf{m}t^{1/4}} + h_1^{(2)}(x) + h_2^{(2)} \log(t) + o(1), & \text{for } \omega = 2, \end{cases} \quad (1.14)$$

where

$$\begin{aligned} h_1^{(1)}(x) &:= \frac{1}{4\mathbf{m}} \left\{ \frac{2|\gamma_0| - 1}{2} \log |x| + \log \left(\frac{4e^{\gamma_1} \sqrt{\pi}}{\Gamma(|\gamma_0|)} \right) - \frac{2|\gamma_0| + 1}{4} \log(2\mathbf{m}) \right\} + \frac{\rho \xi x}{8}, \\ h_1^{(2)}(x) &:= \frac{1}{4\mathbf{m}} \left\{ \frac{\rho \xi \mathbf{m}}{2} x + \gamma_0 \gamma_2 + \frac{9\gamma_0}{4\mathbf{m}} + \frac{5}{8} \log 2 - \frac{3}{8} \log \mathbf{m} + \frac{1}{4} \log \gamma_0 - \frac{1}{4} \log |x| \right\}, \\ h_2^{(1)} &:= \frac{1}{8\mathbf{m}} \left(\frac{1}{2} - |\gamma_0| \right), \quad h_2^{(2)} := \frac{1}{16\mathbf{m}} \left(\frac{1}{2} + \gamma_0 \gamma_1 \right). \end{aligned}$$

A particular example of a randomisation satisfying Assumption 10 is the non-central Chi-squared distribution. This case was the central focus of [70], where the small-time behaviour of the forward smile in the Heston model was analysed. As a sanity check, our Theorem 13 corresponds to [70, Theorem 4.1].

Corollary 14. *Under Assumption 10, for $\omega \leq 2$, $X \sim \text{LDP}_0(\sqrt{t}, \Lambda^*)$.*

Remark 15. *Even though the leading order in the expansion is symmetric, Theorem 13 explains how the asymmetry in the volatility smile is generated. In particular, the term $\rho \xi x / 8$ immediately shows how the leverage effect can be produced with $\rho < 0$.*

1.4.4 MODERATE DEVIATIONS

The asymptotic behaviours we have studied so far belonged to the realm of large deviations. We temporarily leave this territory and enter that of moderate deviations; mathematically, moderate deviations can be achieved from large deviations via space rescaling as follows: for $\alpha \neq 0$, define the process $X^{(\alpha)}$ pathwise via $X_t^{(\alpha)} := t^{-\alpha} X_t$. The moderate deviations principle for the sequence $(X_t)_{t \geq 0}$ (equivalently a large deviations principle for $(X_t^{(\alpha)})_{t \geq 0}$) as t tends to zero can be derived similarly to Corollary 5 and Theorem 7.

Theorem 16. *Assume that $\mathbf{v}_+$ is finite. For any $\gamma \in (0, 1)$, set $\alpha := \frac{1}{2}(1 - \gamma)$; then $X^{(\alpha)} \sim \text{LDP}_0 \left(t^\gamma, \frac{x^2}{2\mathbf{v}_+} \right)$.*

Theorem 16 presents a moderate deviations principle in the bounded support case. One of the striking feature of moderate deviations is that, contrary to

classical large deviations, the rate function is usually available analytically, and is often of quadratic form [48, 58, 59].

Proof. Let $\alpha, \gamma \in (0, 1)$. From (1.4) and (1.5), for any $t \geq 0$, the rescaled cgf of $X_t^{(\alpha)}$ reads

$$\begin{aligned}\Lambda_\gamma^{(\alpha)}\left(t, \frac{u}{t^\gamma}\right) &:= t^\gamma \log \mathbb{E} \left[\exp \left(\frac{u X_t^{(\alpha)}}{t^\gamma} \right) \right] = t^\gamma \log \mathbb{E} \left[\exp \left(\frac{u X_t}{t^{\gamma+\alpha}} \right) \right] \\ &= t^\gamma C\left(t, \frac{u}{t^{\gamma+\alpha}}\right) + t^\gamma \log M_\nu\left(D\left(t, \frac{u}{t^{\gamma+\alpha}}\right)\right),\end{aligned}$$

for all $u \in \mathbb{R}$ such that the left-hand side exists. Lemma 63 and Lemma 65 imply that the term $(\gamma + \alpha)$ has to be less than one. Moreover, as t tends to zero, the following asymptotic behaviours hold:

$$\Lambda_\gamma^{(\alpha)}\left(t, \frac{u}{t^\gamma}\right) = \begin{cases} \mathcal{O}(t^\gamma) + \mathbf{v}_+ \Lambda(u) t^{\gamma-1}, & \text{if } \gamma + \alpha = 1, \text{ for all } u \in (u_-, u_+), \\ o(t^\gamma) + \frac{\mathbf{v}_+}{2} u^2 t^{1-\gamma-2\alpha}, & \text{if } \gamma + \alpha < 1, \text{ for all } u \in \mathbb{R}. \end{cases}$$

Since $\alpha \neq 0$, the non-degenerate result is obtained if and only if $\gamma + 2\alpha = 1$, i.e. $\alpha = \frac{1-\gamma}{2}$, and the proof follows from the Gärtner-Ellis theorem (Theorem 61). $\square$

Theorem 17 (the thin-tail case). *Under Assumption 6, the following statements hold as t tends to zero:*

- (i) for any $\gamma \in (0, \bar{\gamma})$, set $\alpha = \frac{1}{2}(1 - \gamma/\underline{\gamma})$; then $X^{(\alpha)} \sim \text{LDP}_0(t^\gamma, \underline{\Lambda}^*)$ with $\underline{\Lambda}^*$ given in (1.8);
- (ii) if $\gamma = \bar{\gamma}$, set $\alpha = 1 - \bar{\gamma}$, then $X^{(\alpha)} \sim \text{LDP}_0(t^{\bar{\gamma}}, \bar{\Lambda}^*)$ where (with Λ and $u_\pm$ defined in (B.3)-(B.4))

$$\bar{\Lambda}^*(x) := \sup_{u \in (u_-, u_+)} \left\{ ux - \frac{\bar{\mathbf{c}}}{\bar{\gamma}} 2^{\bar{\gamma}-1} \Lambda(u)^{\bar{\gamma}} \right\}, \quad \text{for all } x \in \mathbb{R}.$$

Proof. Following the proof of Lemma 8, if $\gamma + \alpha < 1$, then

$$\Lambda_\gamma^{(\alpha)}\left(t, \frac{u}{t^\gamma}\right) \sim \frac{\bar{\mathbf{c}}}{2\bar{\gamma}} u^{2\bar{\gamma}} t^{\gamma+[1-2(\alpha+\gamma)]\bar{\gamma}}, \quad \text{as } t \text{ tends to zero, for all } u \in \mathbb{R}.$$

The only non-degenerate result is obtained when $\alpha = \frac{1}{2}(1 - \gamma/\underline{\gamma})$, and $\gamma + \alpha < 1$ implies that $\gamma < \bar{\gamma}$. The rest follows directly from the Gärtner-Ellis theorem (Theorem 61). If $\gamma + \alpha = 1$, then

$$\Lambda_\gamma^{(\alpha)}\left(t, \frac{u}{t^\gamma}\right) \sim \frac{\bar{\mathfrak{c}}}{\bar{\gamma}} 2^{\bar{\gamma}-1} \Lambda(u)^{\bar{\gamma}} t^{\gamma-\bar{\gamma}}, \quad \text{as } t \text{ tends to zero, for all } u \in (u_-, u_+),$$

which imposes $\gamma = \bar{\gamma}$. Define now the function $f(u) \equiv \bar{\gamma}^{-1} \bar{\mathfrak{c}} 2^{\bar{\gamma}-1} \Lambda(u)^{\bar{\gamma}}$ on (u_-, u_+) , then

$$f'(u) = 2^{\bar{\gamma}-1} \bar{\mathfrak{c}} \Lambda(u)^{\bar{\gamma}-1} \Lambda'(u), \quad f''(u) = 2^{\bar{\gamma}-1} \bar{\mathfrak{c}} \Lambda(u)^{\bar{\gamma}-2} [(\bar{\gamma} - 1) \Lambda'(u)^2 + \Lambda(u) \Lambda''(u)].$$

Since $\bar{\gamma} > 1$, and since Λ is strictly convex and tends to infinity at $u_\pm$, then so does f . Consequently, for any $x \in \mathbb{R}$ the equation $x = f'(u)$ admits a unique solution in (u_-, u_+) , hence the function $\bar{\Lambda}^*$ is well defined on $\mathbb{R}$ and is a good rate function. The large deviations principle then follows from the Gärtner-Ellis theorem (Theorem 61). $\square$

Remark 18. *The case where $\gamma < \underline{\gamma}$ belongs to the regime of moderately out-of-the-money as in [48, 85], with the time-dependent log-strike $x_t = xt^\alpha$ with $x \in \mathbb{R}_+^*$ and $\alpha \in (0, 1/2)$. In a thin-tail randomised environment, the rescaled limiting cgf does not satisfy [48, Assumption 6.1] in which the limit is assumed to have a quadratic form. Moreover, Theorem 17 implies that for the original process $(X_t)_{t \geq 0}$,*

$$\mathbb{Q}(X_t \geq x_t) = \mathbb{Q}\left(X_t^{(\alpha)} \geq x\right) = \exp\left(-\frac{\bar{\Lambda}^*(x)}{t^\gamma}(1 + o(1))\right), \quad \text{as } t \text{ tends to zero.}$$

Corollary 19. *Recall the following two different regimes:*

- *Moderately out-of-the-money (MOTM):* $(\alpha, x) \in (0, 1/2) \times \mathbb{R}^*$;
- *Small time and large strike:* $(\alpha, x) \in (1 - \bar{\gamma}, 0) \times \mathbb{R}^*$.

Under Assumption 6, define the time-dependent log-strike $x_t := xt^\alpha$, and set $\hat{\gamma} := (1 - 2\alpha)(1 - \underline{\gamma}) > 0$. Then in each of the two regimes, the implied

volatility has the short-time behaviour

$$\lim_{t \downarrow 0} t^{\widehat{\gamma}} \sigma_t^2(x_t) = \underline{\mathfrak{c}}^{-1} \underline{\gamma} x^{2(1-\underline{\gamma})}.$$

Proof. We only prove the large-strike regime with $x > 0$, the other cases being analogous. For $\gamma := \underline{\gamma}(1 - 2\alpha) > 0$, Remark 18 implies that as t tends to zero, $-\log(\mathbb{Q}(X_t \geq x_t)) \sim t^{-\gamma} \underline{\Lambda}^*(x)$. It is easy to check that the sequence $((t^\gamma, x_t))_{t \geq 0}$ satisfies [24, Hypothesis 2.2], so that as t tends to zero, [24, Theorem 2.3] implies

$$\begin{aligned} \sigma_t^2(x_t) &\sim \frac{2xt^\alpha}{t} \left(\sqrt{\frac{\underline{\mathfrak{c}}}{2\underline{\gamma}} \frac{x^{2\underline{\gamma}-1}}{t^{\gamma+\alpha}}} - \sqrt{\frac{\underline{\mathfrak{c}}}{2\underline{\gamma}} \frac{x^{2\underline{\gamma}-1}}{t^{\gamma+\alpha}} - 1} \right)^2 \\ &\sim \frac{2x}{t^{1-\alpha}} \left(\sqrt{\frac{\underline{\mathfrak{c}}}{2\underline{\gamma}} \frac{x^{2\underline{\gamma}-1}}{t^{\gamma+\alpha}}} + \sqrt{\frac{\underline{\mathfrak{c}}}{2\underline{\gamma}} \frac{x^{2\underline{\gamma}-1}}{t^{\gamma+\alpha}} - 1} \right)^{-2} \\ &\sim \frac{\gamma x^{2(1-\underline{\gamma})}}{\underline{\mathfrak{c}} t^{\widehat{\gamma}}}, \end{aligned}$$

and the corollary follows. $\square$

Corollary 20. *Under Assumption 6, for any $\gamma \in (0, \underline{\gamma})$ and any slowly varying (at zero) function $s : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$, let $\alpha := \frac{1}{2}(1 - \gamma/\underline{\gamma})$ and define the time-dependent log-strike $x_t := t^\alpha s(t)$. Then*

$$\mathbb{Q}(X_t \geq x_t) = \exp \left(-\frac{\underline{\Lambda}^*(s(t))}{t^\gamma} (1 + o(1)) \right).$$

Proof. The function $q : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ defined by $q(t) := s(t)^{-2\underline{\gamma}}$ is slowly varying at zero, and $\lim_{t \downarrow 0} t^\gamma q(t) = \lim_{t \downarrow 0} \left(t^{\frac{\gamma}{2\underline{\gamma}}} / s(t) \right)^{2\underline{\gamma}} = 0$. Notice that $\gamma + \alpha \in (1/2, 1)$, so that $t = o(t^{\gamma+\alpha} q(t) s(t))$, and Lemmas 63-65 imply

that the rescaled cgf of the process $(X_t/(s(t)t^\alpha))_{t \geq 0}$ is given by

$$\begin{aligned} & t^\gamma q(t) \log M \left(t, \frac{u}{t^{\gamma+\alpha} q(t) s(t)} \right) \\ &= t^\gamma q(t) C \left(t, \frac{u}{t^{\gamma+\alpha} q(t) s(t)} \right) + t^\gamma q(t) \log M_\nu \left(D \left(t, \frac{u}{t^{\gamma+\alpha} q(t) s(t)} \right) \right) \\ &= \mathcal{O} \left(t^{1+2\gamma+\alpha} q(t)^2 s(t) \right) + t^\gamma q(t) \log M_\nu \left(\frac{u^2 t (1 + o(1))}{2 t^{2(\gamma+\alpha)} (q(t) s(t))^2} \right). \end{aligned}$$

Then from Lemma 9, plugging in the expressions for α and the function q , the limit of the rescaled cgf reads

$$\lim_{t \downarrow 0} t^\gamma q(t) \log M \left(t, \frac{u}{t^{\alpha+\gamma} q(t) s(t)} \right) = \frac{\bar{c}}{2\bar{\gamma}} u^{2\bar{\gamma}} \lim_{t \downarrow 0} t^\gamma q(t) \left(\frac{1 + o(1)}{t^{\bar{\gamma}/\bar{\gamma}} (q(t) s(t))^2} \right)^{\bar{\gamma}} = \frac{\bar{c}}{2\bar{\gamma}} u^{2\bar{\gamma}}.$$

The Gärtner-Ellis theorem (Theorem 61) implies that $\left(\frac{X_t}{s(t)t^\alpha} \right)_{t \geq 0} \sim \text{LDP}_0(t^\gamma q(t), \underline{\Lambda}^*)$ with $\underline{\Lambda}^*$ in (1.8). Consequently,

$$\begin{aligned} - \inf_{x \in (1, \infty)} \underline{\Lambda}^*(x) &\leq \lim_{t \downarrow 0} t^\gamma q(t) \log \mathbb{Q}(X_t \geq x_t) \\ &= \lim_{t \downarrow 0} t^\gamma q(t) \log \mathbb{Q} \left(\frac{X_t}{t^\alpha s(t)} \geq 1 \right) \\ &\leq - \inf_{x \in [1, \infty)} \underline{\Lambda}^*(x). \end{aligned}$$

The proof follows by noticing that $\frac{\underline{\Lambda}^*(1)}{q(t)} = \frac{\underline{c}}{2\underline{\gamma}} s(t)^{2\underline{\gamma}} = \underline{\Lambda}^*(s(t))$ for all $t > 0$. $\square$

Finally, in the fat-tail case we apply the sharp large deviations technique to the rescaled process $(X_t^g)_{t \geq 0}$ defined by $X_t^g := X_t g(t)^{-1}$, for $t \geq 0$, where the function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfies $g(t) = o(1)$ and $\sqrt{t} = o(g)$, as t tends to zero. For any rescaling function $h(t) := \sqrt{t}/g(t)$ ($= o(1)$), denote the rescaled cumulant generating function as

$$\Lambda_t^g(u) := h(t) \log \mathbb{E} \left[e^{u X_t^g / h(t)} \right].$$

We provide a full asymptotic expansion for the European call option price

with a time-dependent log-strike $x_t := xg(t)$, for any fixed $x \neq 0$, and translate this into small-time asymptotic behaviour of the implied variance $\sigma_t^2(x_t)$. We discuss the case where the initial randomisation satisfies Assumption 10 with $\omega = 1$. The case where $\omega = 2$ can be processed in a similar fashion.

Remark 21. *The two rescaling factors g and h have to satisfy $g(t)h(t) = \sqrt{t}$, otherwise we obtain a degenerate small-time limit of Λ_t^g , as implied by Lemma 63 (also see Figure 1.2).*

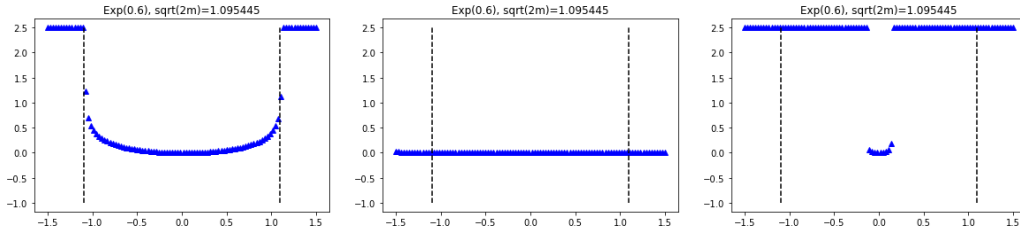


Figure 1.2: Function Λ_t^g with different rescaling factors g and h . The initial randomisation is an exponential distribution with $\mathbf{m} = 0.6$, and $t = 10^{-3}$. From left to right we have $gh = \sqrt{t}$, $\sqrt{t} = o(gh)$, and $gh = o(\sqrt{t})$. Vertical dashed lines represent the boundaries: $x = \pm\sqrt{2\mathbf{m}} = \pm 1.095$. The function's value is set to 2.5 when the cgf explodes.

Theorem 22. *For any $x \neq 0$, as the time-to-maturity t tends to zero, a European call option with strike x_t satisfies*

$$\begin{aligned} \mathbb{E}(e^{X_t} - e^{x_t})^+ &= (1 - e^{x_t})^+ \\ &+ \exp\left(-\sqrt{\frac{2\mathbf{m}}{t}}|x_t| + \gamma_1 + x_t\right) \frac{|x_t|^{|\gamma_0|-1}}{\Gamma(|\gamma_0|)(2\mathbf{m})^{1-\gamma_0/2}} t^{1+\gamma_0/2} g(t) (1 + o(1)). \end{aligned} \quad (1.15)$$

Moreover, the implied variance satisfies

$$\sigma_t^2(x_t) = \frac{|x_t|}{2\sqrt{2\mathbf{m}t}} + h_1^{(1)}(x) + h_2^{(1)} \log(t) + \frac{1}{4\mathbf{m}} \log(g(t)) + o(1), \quad \text{as } t \downarrow 0,$$

where

$$\begin{aligned} h_1^{(1)}(x) &:= \frac{1}{8\mathbf{m}} \left\{ x_t - (2\gamma_0 + 1) \log|x_t| + \log\left(\frac{16\pi e^{2\gamma_1}}{\Gamma(|\gamma_0|)^2}\right) - \left(|\gamma_0| + \frac{1}{2}\right) \log(2\mathbf{m}) \right\}, \\ h_2^{(1)} &:= \frac{1}{8\mathbf{m}} \left(\frac{1}{2} - |\gamma_0|\right). \end{aligned}$$

Proof. The proof is similar to the one of Theorem 13 provided in Section 1.9.2, thus we only highlight the main differences. Notice that $\sqrt{t} = o(h(t))$.

Lemma 23. *For any $x \neq 0$ and small $t > 0$, the equation $\partial_u \Lambda_t^g(u) = x$ admits a unique solution $u_t^*(x)$ satisfying $u_t^*(x) = \text{sgn}(x)\sqrt{2\mathfrak{m}} - \frac{|\gamma_0|}{x}h(t) + \mathcal{O}(h(t)^2 + \sqrt{t})$.*

The proof of the lemma is similar to that of Lemma 40. As t tends to zero, direct computations yield

$$\begin{aligned} & \exp \left\{ \frac{-xu_t^*(x) + \Lambda_t^g(u_t^*(x))}{h(t)} \right\} \\ &= \exp \left\{ -\sqrt{\frac{2\mathfrak{m}}{t}}|x_t| - \gamma_0 + \gamma_1 \right\} \left(\frac{|\gamma_0|\sqrt{2\mathfrak{m}t}}{|x_t|} \right)^{\gamma_0} (1 + o(1)). \end{aligned}$$

For fixed $x \neq 0$ and small $t > 0$, define the time-dependent measure $\mathbb{Q}_t$ by

$$\frac{d\mathbb{Q}_t}{d\mathbb{Q}} := \exp \left\{ \frac{u_t^*(x)X_t^g - \Lambda_t^g(u_t^*(x))}{h(t)} \right\},$$

then for $x > 0$, the call option price is given by

$$\begin{aligned} & \mathbb{E} (e^{X_t} - e^{x_t})^+ = \mathbb{E}^{\mathbb{Q}_t} \left[e^{x_t} \left(e^{g(t)(X_t^g - x)} - 1 \right)^+ \frac{d\mathbb{Q}}{d\mathbb{Q}_t} \right] \\ &= \exp \left\{ \frac{-xu_t^*(x) + \Lambda^g(u_t^*(x))}{h(t)} \right\} e^{x_t} \mathbb{E}^{\mathbb{Q}_t} \left[\exp \left(\frac{-u_t^*(x)Z_t}{h(t)} \right) (e^{g(t)Z_t} - 1)^+ \right], \end{aligned} \tag{1.16}$$

where $Z_t := X_t^g - x$, for $t \geq 0$. The rest of the proof is identical to the proof of Theorem 13. $\square$

1.4.5 LARGE-TIME ASYMPTOTICS

As observed in Figure 1.1, and further explained in Remark 2, the effect of initial randomness decays exponentially fast when the maturity becomes large, hence the large-time behaviour of the randomised Heston model should be similar to that of the standard Heston model, which has been discussed in

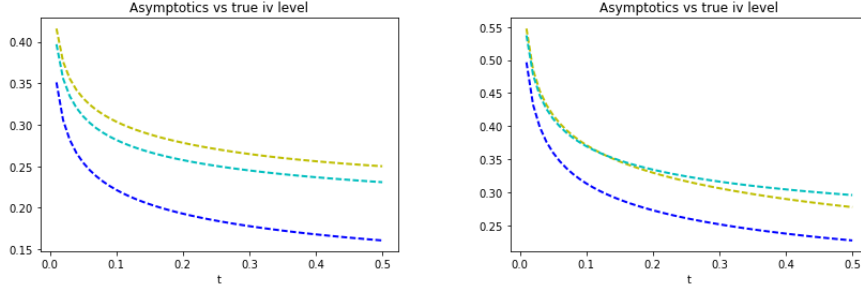


Figure 1.3: In an exponential randomisation scheme ($\mathfrak{m} = 13.09$) we plot the implied volatility $\sigma_t(x_t)$ with $x_t := xt^{0.1}$, $x = 0.2$ (left) and $x = 0.4$ (right). Blue and cyan dashed lines are first- and second- order asymptotics, and the yellow dashed line represents the true implied volatility.

detail in [43, 45, 68]. In the particular example of the forward Heston model—which coincides with randomising with a non-central χ -squared distribution—such a large-time behaviour has been analysed in [71]. Throughout this section we assume that $|\rho| < 1$ and $\kappa > \rho\xi$ (this condition usually holds on equity markets, where the instantaneous correlation ρ is negative—the so-called leverage effect), which guarantees the essential smoothness of the limiting cgf in a standard Heston as t tends to infinity, and define the function $\mathfrak{L}$ on $\mathbb{R}$ by:

$$\mathfrak{L}(u) := \begin{cases} \frac{\kappa\theta}{\xi^2} (\kappa - \rho\xi u - d(u)), & \text{for } u \in [\bar{u}_-, \bar{u}_+], \\ +\infty, & \text{for } u \in \mathbb{R} \setminus [\bar{u}_-, \bar{u}_+], \end{cases} \quad (1.17)$$

where $\bar{u}_\pm := \frac{1}{2\rho^2\xi} \left(\xi - 2\kappa\rho \pm \sqrt{(\xi - 2\kappa\rho)^2 + 4\kappa^2\rho^2} \right)$, and where the function d is given in (B.1). We further denote $\mathfrak{L}^*(x) := \sup_{u \in \mathbb{R}} \{ux - \mathfrak{L}(u)\}$, the convex conjugate of $\mathfrak{L}$. Forde and Jacquier [43, Theorem 2.1] proved that $\bar{u}_- < 0$ and $\bar{u}_+ > 1$. Consider now the following assumption:

Assumption 24. $\max\{\bar{u}_-(\bar{u}_- - 1), \bar{u}_+(\bar{u}_+ - 1)\} < \mathfrak{m}\xi^2 \leq \infty$.

Remark 25. *Assumption 24 is a technical one, needed to ensure that the limiting cgf of the randomised model is essentially smooth. Should it break down, a more refined analysis, similar to the one in [71] could be carried out to prove large deviations.*

Theorem 26. *Under Assumption 24, $(t^{-1}X_t) \sim \text{LDP}_\infty(t^{-1}, \mathfrak{L}^*)$ and*

$$\lim_{t \uparrow \infty} \sigma_t^2(xt) = \begin{cases} 2 \left(2\mathfrak{L}^*(x) - x + 2\sqrt{\mathfrak{L}^*(x)(\mathfrak{L}^*(x) - x)} \right), & \text{for } x \in \left(-\frac{\theta}{2}, \frac{\bar{\theta}}{2} \right), \\ 2 \left(2\mathfrak{L}^*(x) - x - 2\sqrt{\mathfrak{L}^*(x)(\mathfrak{L}^*(x) - x)} \right), & \text{for } x \in \mathbb{R} \setminus \left[-\frac{\theta}{2}, \frac{\bar{\theta}}{2} \right], \end{cases}$$

where $\bar{\theta} := \frac{\kappa\theta}{\kappa - \rho\xi} > 0$. If $x \in \left\{ -\frac{\theta}{2}, \frac{\bar{\theta}}{2} \right\}$, then $\lim_{t \uparrow \infty} \sigma_t^2 \left(\frac{\bar{\theta}t}{2} \right) = \bar{\theta}$, and $\lim_{t \uparrow \infty} \sigma_t^2 \left(-\frac{\theta t}{2} \right) = \theta$.

Remark 27.

- As proved in [43], the map $x \mapsto \mathfrak{L}^*(x) - x$ is smooth, strictly convex, attains its minimum at the point $\bar{\theta}/2$, and $\mathfrak{L}^*(\bar{\theta}/2) - \bar{\theta}/2 = \mathfrak{L}^*(\bar{\theta}/2)' - 1 = 0$.
- Theorem 26 has the same form as [43, Corollary 2.4], hence confirms the similar large-time behaviours of the classical and the randomised Heston models.
- Higher-order terms can be derived similarly to Theorem 13 (see also [71, Proposition 2.12]).

Theorem 26 provides the large-time behaviour of the implied volatility smile with a time-dependent strike. In the fixed strike case we recover the so-called flattening effect of the smile, which has the same form as [43, Corollary 2.17]:

Corollary 28 (fixed strike). *Under Assumption 24, for any $x \in \mathbb{R}$,*

$$\lim_{t \uparrow \infty} \sigma_t^2(x) = 8\mathfrak{L}^*(0) = \frac{4\kappa\theta}{\xi^2(1 - \rho^2)} \left(-2\kappa + \rho\xi + \sqrt{\xi^2 + 4\kappa^2 - 4\kappa\rho\xi} \right).$$

1.4.6 AT-THE-MONEY (ATM) CASE

So far our small-maturity results have been focusing on the out-of-the-money case $x \neq 0$. As usual in the literature on implied volatility asymptotics, the

at-the-money behaviour of the implied volatility is radically different, and a separate analysis is necessary.

Theorem 29. *In the randomised Heston model the at-the-money implied volatility satisfies $\lim_{t \downarrow 0} \sigma_t(0) = \mathbb{E}(\sqrt{\mathcal{V}})$.*

Proof. Since $\mathbf{m} \in (0, \infty]$ then $\mathbb{E}(\sqrt{\mathcal{V}})$ is finite. Denote by $C_{\text{BS}}(t, x, \Sigma)$ the European Call option price in the Black-Scholes model with maturity t , strike e^x and volatility Σ . Further denote $C_{\text{H}}(t, x, v)$ as its price in the standard Heston model with $V_0 = v$, and $C_{\mathcal{H}}(t, x)$ as its price in the randomised Heston setting. Using the tower property,

$$C_{\mathcal{H}}(t, 0) = \mathbb{E} \left[(e^{X_t} - 1)^+ \right] = \mathbb{E} \left[\mathbb{E} \left[(e^{X_t} - 1)^+ \mid \mathcal{V} \right] \right] = \mathbb{E}[C_{\text{H}}(t, 0, \mathcal{V})]. \quad (1.18)$$

[44, Theorem 3.1] states that the following equation holds almost surely:

$$C_{\text{H}}(t, 0, \mathcal{V}) = \sqrt{\frac{\mathcal{V}t}{2\pi}} - Bt^{3/2} + \mathcal{O}_P(t^{5/2}),$$

where $B = \frac{1}{48} \sqrt{\frac{2}{\pi \mathcal{V}}} (\xi^2(1 - \rho^2/4) + \mathcal{V}(\mathcal{V} - 3\rho\xi) - 6\kappa(\theta - \mathcal{V}))$. Notice that the terms in the expansion are random, hence we use $\mathcal{O}_P$ instead of $\mathcal{O}$ to highlight the difference. The sequence of random variables $(C_{\text{H}}(t, 0, \mathcal{V})/\sqrt{t} - \sqrt{\mathcal{V}/(2\pi)})_{t \geq 0}$ converges to zero almost surely as t tends to zero. Since $\left| C_{\text{H}}(t, 0, \mathcal{V})/\sqrt{t} - \sqrt{\mathcal{V}/(2\pi)} \right| \leq C_{\text{H}}(t, 0, \mathcal{V})/\sqrt{t} + \sqrt{\mathcal{V}/(2\pi)}$, the Dominated Convergence Theorem implies that

$$\lim_{t \downarrow 0} \mathbb{E} \left[\frac{1}{\sqrt{t}} C_{\text{H}}(t, 0, \mathcal{V}) - \sqrt{\frac{\mathcal{V}}{2\pi}} \right] = \mathbb{E} \left[\lim_{t \downarrow 0} \left(C_{\text{H}}(t, 0, \mathcal{V})/\sqrt{t} - \sqrt{\mathcal{V}/(2\pi)} \right) \right] = 0,$$

from which we have $C_{\mathcal{H}}(t, 0) = \mathbb{E}(\sqrt{\mathcal{V}}) \sqrt{t/(2\pi)} + o(\sqrt{t})$, where the little- o term is now deterministic. [44, Proposition 3.4] states that

$$C_{\text{BS}}(t, 0, \Sigma) = \frac{1}{\sqrt{2\pi}} \left(\Sigma t^{1/2} - \frac{\Sigma^3}{24} t^{3/2} + \mathcal{O}(t^{5/2}) \right).$$

Equate it with the asymptotics of $C_{\mathcal{H}}(t, 0)$ and compare the leading order we immediately have the result. $\square$

Remark 30. If $\mathbb{E}(\mathcal{V}^{-1/2})$ is finite then following a similar procedure we obtain higher order terms of $\sigma_t(0)$,

$$\sigma_t(0) = \mathbb{E}(\sqrt{\mathcal{V}}) + \left\{ \mathfrak{c}_1 \mathbb{E}(\mathcal{V}^{-1/2}) + \mathfrak{c}_2 \mathbb{E}(\sqrt{\mathcal{V}}) + \mathfrak{c}_3 \left(\mathbb{E}(\sqrt{\mathcal{V}})^3 - \mathbb{E}(\mathcal{V}^{3/2}) \right) \right\} t + o(t),$$

where $\mathfrak{c}_1 := \frac{1}{4}(\kappa\theta + \xi^2(\rho^2 - 4)/24)$, $\mathfrak{c}_2 := \frac{1}{8}(\rho\xi - 2\kappa)$, and $\mathfrak{c}_3 := \frac{1}{24}$. In the non-central chi-squared case we recover the result of [70, Theorem 4.4].

1.5 A DYNAMIC PRICING FRAMEWORK

So far we have been studying the randomised Heston model in a static way, namely from the inception time of the (European) contract, with a view towards calibration of the implied volatility surface. While the randomised Heston model provides a better fit to small-maturity options by steepening the volatility smile, it is not obvious, however, how to use the model dynamically; in particular, it is unclear how to choose the initial distribution of the variance process during the life of the contract, should one be wishing to sell or buy the option, or for hedging purposes. Mathematically, assume that at time zero the trader chooses an initial randomisation $\mathcal{V}$ (or classically a Dirac mass at some positive point), and suppose that at some later time $t > 0$ she will need to reprice the option (with remaining maturity $\tau := T - t$). How should she estimate the choice of the initial random variable $\mathcal{V}_t$? Since the variance process has continuous paths, a suitable candidate that is consistent with the dynamics of the variance is obviously V_t , the solution of the SDE (1.2), after running it from time zero to time t . With an initial guess $\mathcal{V}$ at time zero, V_t conditional on $\mathcal{V}$ is distributed as $\beta_t \chi^2(q, \lambda)$, where $\beta_t := \xi^2(1 - e^{-\kappa t})/(4\kappa)$, and $\chi^2(q, \lambda)$ is a non-central Chi-squared distribution with $q := 4\kappa\theta/\xi^2$ degrees of freedom and non-centrality parameter $\lambda := 4\kappa\mathcal{V}/(\xi^2(e^{\kappa t} - 1))$. From the tower property, the moment generating function of $\mathcal{V}_t$ then reads

$$\mathbb{M}_{\mathcal{V}_t}(u) = \mathbb{E} \left[\mathbb{E} (e^{uV_t} | \mathcal{V}) \right] = (1 - 2\beta_t u)^{-q/2} \mathbb{M}_{\mathcal{V}} \left(\frac{\exp(-\kappa t)u}{1 - 2\beta_t u} \right), \quad (1.19)$$

for all $u \in \mathcal{D}_t^H := \{u \in \mathbb{R} : M_{\mathcal{V}_t}(u) < \infty\}$. Setting $\mathbf{b}_t := 1/(2\beta_t)$, we have

$$\begin{aligned} \mathcal{D}_t^H &= (-\infty, \mathbf{b}_t) \cap \left\{ u \in \mathbb{R} : \frac{\exp(-\kappa t)u}{1 - 2\beta_t u} \in \mathcal{D}_{\mathcal{V}} \right\} \\ &= \begin{cases} (-\infty, \mathbf{b}_t), & \text{if } \mathbf{m} = \infty, \\ (-\infty, \mathbf{b}_t) \cap \left(-\infty, \frac{\mathbf{m}}{e^{-\kappa t} + 2\beta_t \mathbf{m}} \right) = (-\infty, \mathbf{b}_t^*), & \text{if } \mathbf{m} < \infty. \end{cases} \end{aligned}$$

where $\mathbf{b}_t^* := \frac{\mathbf{m}\mathbf{b}_t}{\mathbf{m} + \mathbf{b}_t e^{-\kappa t}}$. We now discuss the impact of different choices of $\mathcal{V}$ at time zero on the distribution of $\mathcal{V}_t$ and on the implied variance $\sigma_\tau^2(x, t)$ at time t for a remaining maturity τ . We keep here the terminology introduced in Section 1.4 regarding the tail behaviour of $\mathcal{V}$.

Before diving into the detailed analysis, we argue that $\mathcal{V}_t$ chosen this way should only serve as a candidate for the initial distribution at time t and in practice it should be recalibrated according to updated (noisy) market observations at time t . A Bayesian approach which was briefly discussed in Section 1.2.1 can be applied. Market noises explain how the distribution of $\mathcal{V}_t$ at a given time t can deviate from the ergodic distribution: the impact of the (instantaneous) noises can change the shape and parameterisation of the initial randomisation. Finally, we comment that understanding the choice of $\mathcal{V}_t$ is also useful from a model risk point of view: at time zero, it is important to understand and simulate the behaviours of model parameters at any given future time.

1.5.1 THE BOUNDED-SUPPORT CASE

In this case, $\mathcal{D}_t^H = (-\infty, \mathbf{b}_t)$. In Section 1.9.1 we showed that $\lim_{u \uparrow \infty} \log M_{\mathcal{V}}(u)/u = \mathbf{v}_+$. Combining this with (1.19), then as u tends to $\mathbf{b}_t$ from below,

$$\begin{aligned} \log M_{\mathcal{V}_t}(u) &= -\frac{q}{2} \log(1 - 2\beta_t u) + \frac{e^{-\kappa t} \mathbf{v}_+ u}{1 - 2\beta_t u} (1 + o(1)) \\ &= \frac{q}{2} \log \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - u} \right) + \frac{e^{-\kappa t} \mathbf{v}_+ \mathbf{b}_t u}{\mathbf{b}_t - u} (1 + o(1)) \\ &= \frac{e^{-\kappa t} \mathbf{v}_+ \mathbf{b}_t^2}{\mathbf{b}_t - u} (1 + o(1)). \end{aligned}$$

Consequently, at leading order $\mathcal{V}_t$ behaves asymptotically as a fat-tail distribution as in Assumption 10 with $\omega = 2$. In the particular case of a uniform distribution on $[\mathbf{v}_-, \mathbf{v}_+] \subset [0, \infty)$, as u tends to $\mathbf{b}_t$ from below, we obtain

$$\begin{aligned} \log M_{\mathcal{V}_t}(u) &= \frac{q}{2} \log \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - u} \right) + \mathbf{b}_t \mathbf{v}_+ e^{-\kappa t} \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - u} - 1 \right) \\ &\quad + \log \left(\frac{1 - \exp\{(\mathbf{v}_- - \mathbf{v}_+)e^{-\kappa t} \mathbf{b}_t u / (\mathbf{b}_t - u)\}}{e^{-\kappa t} \mathbf{b}_t u (\mathbf{v}_+ - \mathbf{v}_-)} (\mathbf{b}_t - u) \right) \\ &= \frac{e^{-\kappa t} \mathbf{b}_t^2 \mathbf{v}_+}{\mathbf{b}_t - u} \left(1 + \frac{e^{\kappa t} (2 - q)}{2 \mathbf{b}_t^2 \mathbf{v}_+} (\mathbf{b}_t - u) \log(\mathbf{b}_t - u) \right) \\ &\quad + \frac{e^{-\kappa t} \mathbf{b}_t^2 \mathbf{v}_+}{\mathbf{b}_t - u} \left[\left(\frac{e^{\kappa t}}{\mathbf{b}_t \mathbf{v}_+} \log \left(\frac{e^{\kappa t} \mathbf{b}_t^{q/2-2}}{\mathbf{v}_+ - \mathbf{v}_-} \right) - 1 \right) \frac{\mathbf{b}_t - u}{\mathbf{b}_t} + o(\mathbf{b}_t - u) \right]. \end{aligned}$$

Hence in a uniform randomisation environment, at future time t the shape of the distribution of $\mathcal{V}_t$ depends both on $\mathcal{V}$ and on the parameters κ, θ, ξ that control the dynamics of the variance process. Moreover from Theorem 13, the implied variance $\sigma_\tau^2(x, t)$ at time t has an explosion rate of $\sqrt{\tau}$:

$$\sigma_\tau^2(x, t) = \frac{|x| \tau^{-1/2}}{2\sqrt{2} \mathbf{b}_t} + \frac{\sqrt{\mathbf{v}_+ |x|}}{2e^{\kappa t/2}} (2\mathbf{b}_t \tau)^{-1/4} + o(\tau^{-1/4}), \quad \text{for } x \neq 0, \text{ as } \tau \downarrow 0.$$

1.5.2 THE THIN-TAIL CASE (ASSUMPTION 6)

Here again, $\mathcal{D}_{\mathcal{V}} = \mathbb{R}$ implies that $\mathcal{D}_t^H = (-\infty, \mathbf{b}_t)$. Apply Lemma 9 with $\log f \sim -l_1 v^{l_2}$, we have

$$\begin{aligned} \log M_{\mathcal{V}_t}(u) &= \frac{q}{2} \log \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - u} \right) + l_1(l_2 - 1) \left(\frac{1}{l_1 l_2} \right)^{\frac{l_2}{l_2-1}} \left(\frac{e^{-\kappa t} \mathbf{b}_t^2}{\mathbf{b}_t - u} \right)^{\frac{l_2}{l_2-1}} (1 + o(1)) \\ &= \frac{q}{2} \log \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - u} \right) + \frac{2^{\bar{\gamma}-1} \bar{\mathbf{c}}}{\bar{\gamma}} \left(\frac{e^{-\kappa t} \mathbf{b}_t^2}{\mathbf{b}_t - u} \right)^{\bar{\gamma}} (1 + o(1)), \end{aligned} \quad (1.20)$$

as u tends to $\mathbf{b}_t$ from below, so that a thin-tail initial randomisation generates a fat-tail distribution for $\mathcal{V}_t$ at time t . In light of (1.20), Assumption 10 does not hold, hence $\mathcal{V}_t$ is of neither Gamma nor non-central Chi-squared type. A case-by-case analysis depending on the distribution of $\mathcal{V}$ is therefore needed in order to make the $o(\cdot)$ term in (1.20) more precise.

Example 31 (Folded-Gaussian randomisation). When $f(v) \equiv \mathbf{c}e^{-l_1 v^2}$, straightforward computations yield

$$\begin{aligned} \log M_{\mathcal{V}_t}(u) &= \frac{q}{2} \log \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - u} \right) + \log \left(\frac{1}{2} + \Phi \left(\frac{e^{-\kappa t} \mathbf{b}_t u}{\sqrt{2l_1}(\mathbf{b}_t - u)} \right) \right) \\ &\quad + \frac{1}{4l_1} \left(\frac{e^{-\kappa t} \mathbf{b}_t u}{\mathbf{b}_t - u} \right)^2 + \log \left(\mathbf{c} \sqrt{\frac{\pi}{l_1}} \right) \\ &=: \frac{\mathbf{c}_0}{(\mathbf{b}_t - u)^2} + \frac{\mathbf{c}_1}{\mathbf{b}_t - u} + \mathbf{c}_2 - \frac{q}{2} \log(\mathbf{b}_t - u) + o(1). \end{aligned}$$

We can obtain the small-time asymptotic expansion of the option price using an approach similar to the proof of Theorem 13. Specifically, only Lemma 42 needs to be adjusted, in which the rescaling factor is now $\vartheta(\tau) = \tau^{1/6}$; the main contribution to the asymptotics of out-of-the-money option prices is still given in Lemma 41. Translating this into the asymptotics of the implied variance, we obtain, for small τ ,

$$\sigma_\tau^2(x, t) = \frac{|x|}{2\sqrt{2\mathbf{b}_t\tau}} + \frac{3|x|^{2/3}}{8(4l_1)^{1/3}} \frac{\exp\left(-\frac{2\kappa t}{3}\right)}{\tau^{1/3}} + o(\tau^{-1/3}).$$

1.5.3 THE FAT-TAIL CASE

In this case $\mathcal{D}_t^H = (-\infty, \mathbf{b}_t^*)$. Here we only discuss two special cases for $\mathcal{V}$: the Gamma distribution, and the (scaled) non-central χ -squared distribution.

Example 32 (Gamma randomisation). If $\mathcal{V} \stackrel{(Law)}{=} \Gamma(\alpha, \mathbf{m})$, then from (1.19), we have

$$\begin{aligned} \log M_{\mathcal{V}_t}(u) &= \frac{q}{2} \log \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - u} \right) - \alpha \log \left(1 - \frac{e^{-\kappa t} u \mathbf{b}_t}{\mathbf{m}(\mathbf{b}_t - u)} \right) \\ &= -\alpha \log \left(\frac{\mathbf{m}\mathbf{b}_t - (\mathbf{m} + \exp(-\kappa t)\mathbf{b}_t)u}{\mathbf{m}(\mathbf{b}_t - u)} \right) \\ &= -\alpha \log(\mathbf{b}_t^* - u) + \alpha \log \left(\frac{\mathbf{m}(\mathbf{b}_t - u)}{\mathbf{m} + \exp(-\kappa t)\mathbf{b}_t} \right) + \frac{q}{2} \log \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - u} \right). \end{aligned}$$

Consequently $\mathcal{V}_t$ is still a fat-tail distribution satisfying Assumption 10 with $\omega = 1$, while the upper-bound of the support of the mgf now depends on both the

initial distribution $\mathcal{V}$ and on the evolution of the process through $\mathbf{b}_t^*$. A direct application of Theorem 13 further suggests that, for small enough $\tau > 0$,

$$\sigma_\tau^2(x, t) = \frac{|x|}{2\sqrt{2\mathbf{b}_t^*\tau}} + o(\tau^{-1/2}).$$

Example 33 (Non-central χ^2 randomisation). If $\mathcal{V} \stackrel{(Law)}{=} \alpha\chi^2(\mathbf{a}, \mathbf{b})$ then $\mathbf{m} = 1/(2\alpha)$, and

$$\begin{aligned} \log M_{\mathcal{V}_t}(u) &= \frac{q}{2} \log \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - u} \right) + \left(\frac{\alpha \mathbf{b} z}{1 - 2\alpha z} - \frac{\mathbf{a}}{2} \log(1 - 2\alpha z) \right) \Big|_{z = \frac{u}{1 - 2\beta_t u} e^{-\kappa t}} \\ &= \frac{e^{-\kappa t} \mathbf{b} \mathbf{b}_t^* u}{2\mathbf{m}(\mathbf{b}_t^* - u)} - \frac{\mathbf{a}}{2} \log \left(\frac{\mathbf{b}_t(\mathbf{b}_t^* - u)}{\mathbf{b}_t^*(\mathbf{b}_t - u)} \right) + \frac{q}{2} \log \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - u} \right) \\ &= \frac{e^{-\kappa t} \mathbf{b} \mathbf{b}_t^{*2}}{2\mathbf{m}(\mathbf{b}_t^* - u)} - \frac{\mathbf{a}}{2} \log(\mathbf{b}_t^* - u) + \frac{q - \mathbf{a}}{2} \log \left(\frac{\mathbf{b}_t}{\mathbf{b}_t - \mathbf{b}_t^*} \right) \\ &\quad + \frac{\mathbf{a}}{2} \log \mathbf{b}_t^* - \frac{e^{-\kappa t} \mathbf{b} \mathbf{b}_t^*}{2\mathbf{m}} + \mathcal{O}(\mathbf{b}_t^* - u), \end{aligned}$$

which satisfies (1.10) in Assumption 10 with $\omega = 2$ as u tends to $\mathbf{b}_t^*$. As a result, the implied variance $\sigma_\tau^2(x, t)$ has an explosion rate of $\sqrt{\tau}$ as τ tends to zero.

This analysis shows that a suitable choice for $\mathcal{V}_t$, consistent with the dynamics of the variance process, can actually depend on the initial randomisation at time zero, as well as the evolution of the variance. Even though all three types of initial randomisation imply a fat-tail initial distribution at future time, the generated small remaining-maturity implied volatility smiles are very different. The folded-Gaussian (thin-tail) generates a steeper smile compared to the bounded support case; a fat-tail distribution for $\mathcal{V}$ generate an even steeper volatility smile at t , since the coefficient of the leading order is $\mathbf{b}_t^*$, which is strictly less than $\mathbf{b}_t$.

1.5.4 NUMERICS

We now illustrate numerically the results in Section 1.5. In Figure 1.4 we price the option in three different randomisation schemes after one month

($t = 1/12$) into the life of the contract. To compare different schemes, at time zero we again match the parameters of $\mathcal{V}$ with different distributions according to Theorem 29. We see that the higher-order term in Theorem 13 is quite accurate even for relatively large remaining time to maturity. Not surprisingly (especially in the folded-Gaussian case) the leading order is insufficient, and higher orders are needed for reliable approximations.

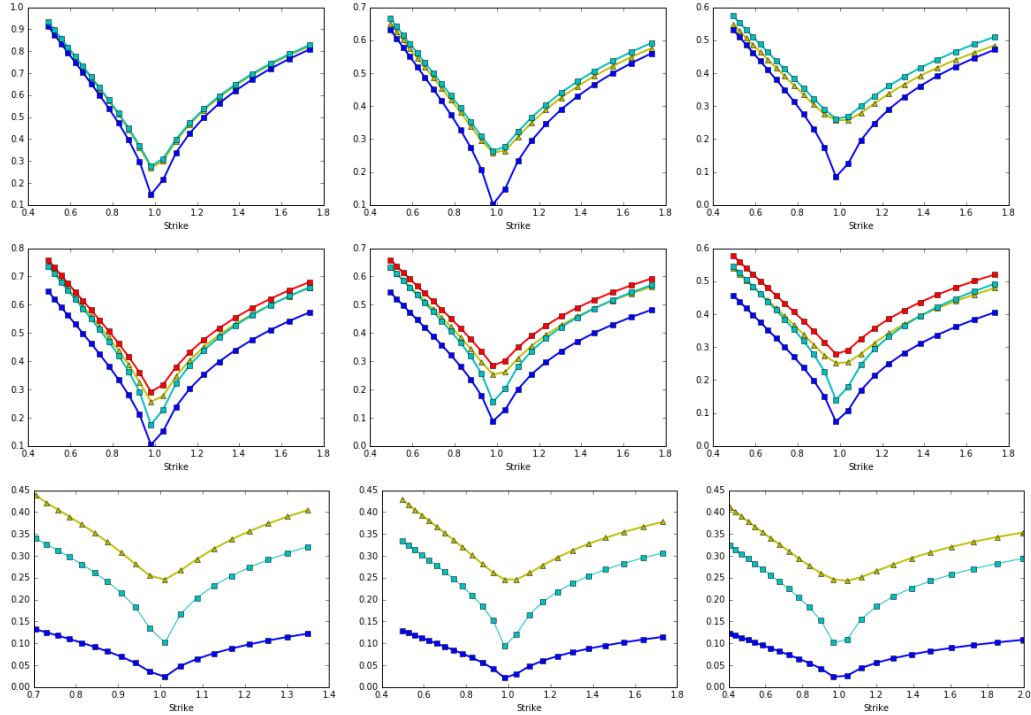


Figure 1.4: We price the option at $t = 1/12$ in three different randomisation schemes: Gamma ($\Gamma(0.4, 3.868)$, top), non-central Chi-squared ($0.07\chi^2(0.23, 1.25)$, middle), folded-Gaussian ($l_2 = 2, l_1 = 63.46$, bottom). Remaining time to maturities τ are one week, one month, and two months (left to right). Blue and cyan squares are first- and second-order asymptotics, red squares (in the second row) are third-order asymptotics; yellow triangles are true implied volatilities.

1.6 EXAMPLES AND NUMERICS

In this section we choose some common distributions supported on a subset of $[0, \infty)$ for the initial randomisation to illustrate the results in Section 1.4. We start with the bounded support case, and provide rigorous justifications to the statements in Section 1.2. In Section 1.6.1 we consider a uniformly distributed initial variance, and provide full asymptotics of European Call prices. The remaining examples are devoted to the unbounded support case; specifically, Sections 1.6.2-1.6.3 correspond to the fat-tail case, so that Theorem 13 can be applied. The thin-tail environment is illustrated in Section 1.6.4 where the initial distribution satisfies Assumption 6 with $l_2 = 2$. We provide thoughts on how to deal with this special case.

1.6.1 UNIFORM RANDOMISATION

Assume that $\mathcal{V}$ is uniformly distributed on $[\mathbf{v}_-, \mathbf{v}_+]$ with $0 \leq \mathbf{v}_- < \mathbf{v}_+ < \infty$, then Corollary 5 provides the small-time limit of the implied volatility. However, as will be shown in Section 1.6.6, the true volatility smile for small t is much steeper compared with the leading order, hence higher-order terms shall be considered. For any $x \neq 0$, denote by $u_{\mathbf{v}_+}^*(x)$ the unique solution in (u_-, u_+) to the equation $x = \Lambda'(u)\mathbf{v}_+$ where the function Λ is described as in (B.4). From [44], existence and uniqueness of such a solution are straightforward, and $u_{\mathbf{v}_+}^*(x) \neq 0$ holds for any non-zero x . Introduce the function $U_{\mathbf{v}_+} : \mathbb{R}^* \rightarrow \mathbb{R}_+^*$ by

$$U_{\mathbf{v}_+}(x) := \exp \left\{ D_0^0(u_{\mathbf{v}_+}^*(x))\mathbf{v}_+ + C_0(u_{\mathbf{v}_+}^*(x)) + x \right\}, \quad (1.21)$$

where the functions D_0^0 and C_0 are provided in (B.5)-(B.6). From [44, Remark 3.2], the function $U_{\mathbf{v}_+}$ is well defined on $\mathbb{R}^*$. The following theorem provides a full asymptotic expansion of call option prices as the maturity becomes small:

Theorem 34. *Under uniform randomisation, for any $x \neq 0$, the small-time*

European Call option price behaves as

$$\mathbb{E} (e^{X_t} - e^x)^+ = (1 - e^x)^+ + \frac{\exp\left(-\frac{\Lambda_{\mathbf{v}_+}^*(x)}{t}\right) U_{\mathbf{v}_+}(x) t^{5/2} (1 + o(1))}{(\mathbf{v}_+ - \mathbf{v}_-) \Lambda(u_{\mathbf{v}_+}^*(x)) u_{\mathbf{v}_+}^*(x)^2 \sqrt{2\pi \mathbf{v}_+ \Lambda''(u_{\mathbf{v}_+}^*(x))}},$$

where the function $\Lambda_{\mathbf{v}_+}^*$ was introduced in Corollary 5.

Remark 35.

- The higher order term is of order $t^{5/2}$, instead of $t^{3/2}$ as in both standard Heston and Black-Scholes models [44]. This can also be seen at the level of the (asymptotics of) corresponding densities, as noted in Remark 36 below.
- Theorem 34 holds locally for any fixed log-strike $x \neq 0$. However, the numerics indicate that for fixed $t > 0$, as x tends to zero the provided asymptotics of option prices explode to infinity. This is in contrast with the standard Heston case as in [44, Section 5].
- Since the function Λ is strictly positive and strictly convex on $(u_-, u_+) \setminus \{0\}$ and $u_{\mathbf{v}_+}^*(x) \in (u_-, u_+) \setminus \{0\}$ for any $x \neq 0$, the quotient on the right-hand side is well defined.

Proof. The procedure is essentially the same as the proof of Theorem 13. Applying Lemmas 63 and 65, the rescaled cgf of X_t for each t is given by (with the same notations as in (1.5))

$$\begin{aligned} \Lambda_t(u) &:= \Lambda_1\left(t, \frac{u}{t}\right) = tC\left(t, \frac{u}{t}\right) + t \log M_{\mathcal{V}}\left(D\left(t, \frac{u}{t}\right)\right) \\ &= tC\left(t, \frac{u}{t}\right) + t \log \left(\frac{e^{\mathbf{v}_+ D(t, u/t)} - e^{\mathbf{v}_- D(t, u/t)}}{(\mathbf{v}_+ - \mathbf{v}_-) D(t, u/t)} \right) \\ &= \mathbf{v}_+ \Lambda(u) + t \left(C_0(u) + \mathbf{v}_+ D_0^0(u) - \log((\mathbf{v}_+ - \mathbf{v}_-) \Lambda(u)) \right) + t \log t + \mathcal{O}(t^2). \end{aligned} \tag{1.22}$$

For fixed $x > 0$ and small enough $t > 0$, introduce the time-dependent probability measure $\mathbb{Q}_t$ by

$$\frac{d\mathbb{Q}_t}{d\mathbb{Q}} := \exp\left(\frac{u_{\mathbf{v}_+}^*(x) X_t - \Lambda_t(u_{\mathbf{v}_+}^*(x))}{t}\right).$$

Changing the measure, applying (1.22) and rearranging terms yield the following expression for the Call option price with strike e^x :

$$\begin{aligned} & \mathbb{E} (e^{X_t} - e^x)^+ \\ &= \frac{e^{\frac{-\Lambda_{\mathbf{v}_+}^*(x)}{t}} U_{\mathbf{v}_+}(x) t (1 + \mathcal{O}(t))}{(\mathbf{v}_+ - \mathbf{v}_-) \Lambda(u_{\mathbf{v}_+}^*(x))} \mathbb{E}^{\mathbb{Q}_t} \left[\exp \left(\frac{-u_{\mathbf{v}_+}^*(x)(X_t - x)}{t} \right) (e^{X_t - x} - 1)^+ \right]. \end{aligned}$$

It is easy to show that under $\mathbb{Q}_t$ the random variable $\left(\frac{X_t - x}{\sqrt{t}}\right)$ converges weakly to a Gaussian distribution for small t . The rest of the proof is similar to Section 1.9.2 and we omit the details. $\square$

We now explain the steepness of the volatility smile in the uncorrelated case $\rho = 0$. Using the at-the-money curvature formula for the implied volatility (for general uncorrelated stochastic volatility models) proved by De Marco and Martini [32, Equation (2.9)], we can write, for any $t > 0$,

$$\partial_x^2 \sigma_t^2(x) \big|_{x=0} = \frac{2}{t} \left\{ \sigma_t(0) \sqrt{2\pi t} \exp \left(\frac{\sigma_t^2(0)t}{8} \right) p_t(0) - 1 \right\}, \quad (1.23)$$

where p_t is the density of the log-price at time t . In the standard Heston model with the initial condition $V_0 = v_0 \in (\mathbf{v}_-, \mathbf{v}_+)$ satisfying $\mathbb{E}(\sqrt{\mathcal{V}}) = \sqrt{v_0}$, the small-time asymptotics of the density reads [48, Section 5.3]

$$p_t(x) = \exp \left(-\frac{\Lambda_{v_0}^*(x)}{t} \right) \frac{U_{v_0}(x)}{\sqrt{2\pi v_0 \Lambda''(x)}} t^{-1/2} (1 + o(1)), \quad \text{for any } x \neq 0,$$

with the function U defined in (1.21). Applying the saddle point method similar to the proof of [44, Theorem 3.1], the small-time asymptotics of the density in a randomised setting, denoted as $\tilde{p}_t$, has the expression

$$\tilde{p}_t(x) = \exp \left(-\frac{\Lambda_{\mathbf{v}_+}^*(x)}{t} \right) \frac{U_{\mathbf{v}_+}(x)}{\sqrt{2\pi \mathbf{v}_+ \Lambda''(x)}} \frac{t^{1/2} (1 + o(1))}{(\mathbf{v}_+ - \mathbf{v}_-) \Lambda(u_{\mathbf{v}_+}^*(x))}, \quad x \neq 0.$$

Remark 36. Note the difference between the powers $t^{1/2}$ and $t^{-1/2}$ in the expressions for p_t and $\tilde{p}_t$ above. Even if in the bounded support case the leading-order term is not affected by the randomisation, the latter does act at

higher order. We leave a precise study of this issue to further research.

For any $x \neq 0$, the ratio $p_t(x)/\tilde{p}_t(x)$ then reads

$$\frac{p_t(x)}{\tilde{p}_t(x)} = \frac{1}{t} e^{-\frac{\Lambda_{v_0}^*(x) - \Lambda_{v_+}^*(x)}{t}} \left(\frac{v_+}{v_0} \right)^{\frac{1}{2}} \frac{U_{v_0}(x)}{U_{v_+}(x)} (v_+ - v_-) \Lambda(u_{v_+}^*(x)) (1 + o(1)).$$

It is easy to verify that $\lim_{x \downarrow 0} U_{v_0}(x) = \lim_{x \downarrow 0} U_{v_+}(x) = 1$ and $\lim_{x \downarrow 0} \Lambda(u_{v_+}^*(x)) = 0$. Moreover, for any fixed $x \neq 0$,

$$\begin{aligned} \partial_v \Lambda_v^*(x) &= \partial_v [u_v^*(x)x - v\Lambda(u_v^*(x))] \\ &= [x - v\Lambda'(u_v^*(x))] \frac{\partial u_v^*(x)}{\partial v} - \Lambda(u_v^*(x)) \\ &= -\Lambda(u_v^*(x)) < 0. \end{aligned}$$

Combining these results, assume that the density at $x = 0$ can be approximated by $p_t(x)$ for small $x > 0$, then there exists $t^* > 0$ small enough such that $p_t(x)/\tilde{p}_t(x) \ll 1$ for all $t \in (0, t^*)$. Plugging it back to (1.23), and noticing that (Section 1.4.6) $\sigma_t(0) \sim \mathbb{E}[\sqrt{\mathcal{V}}] = \sqrt{v_0} \sim \sigma_t(0, v_0)$ holds as t tends to zero, then the small-time curvature in a uniform randomisation scheme is much larger compared with that of a standard Heston, implying a much steeper smile around the at-the-money. We provide a visual explanation in Figure 1.6, and this steepening effect in a uniform randomisation scheme is further tested in Section 1.7.

Finally, we mention that the tail behaviour of the implied volatility in a uniformly randomised Heston model is similar to that of the standard Heston. To see this, notice that the moment explosion property in the standard Heston setting is described in [5, Proposition 3.1]. Specifically, the explosion of the mgf of X_t is equivalent to the explosion of the function D provided in (B.1). Since $\mathbf{m}$ is infinity, Equation (1.4) suggests that it is still the case in the uniform randomised setting. Then the similarity of the tail behaviours follows from [80] (see also [14, 24]).

1.6.2 NON-CENTRAL χ -SQUARED

Assume that $\mathcal{V}$ is non-central χ -squared distributed with $q > 0$ degrees of freedom and non-centrality parameter $\lambda > 0$, then its moment generating function reads

$$M_{\mathcal{V}}(u) = \frac{1}{(1 - 2u)^{q/2}} \exp\left(\frac{\lambda u}{1 - 2u}\right), \quad \text{for all } u \in \mathcal{D}_{\mathcal{V}} = (-\infty, 1/2).$$

In this case $\mathfrak{v}_+$ is infinite and $\mathfrak{m} = 1/2$. By Proposition 3, the only suitable scale function is $h(t) \equiv \sqrt{t}$ —corresponding exactly to the forward-start Heston model—the asymptotics of which have been studied thoroughly in [70]. Applying Equation (1.26) in Section 1.9.1 and L'Hôpital's rule with $\frac{M'_{\mathcal{V}}(u)}{M_{\mathcal{V}}(u)} = \lambda(1 - 2u)^{-2} + q(1 - 2u)^{-1}$ implies that, at the right endpoint $u = \sqrt{2\mathfrak{m}} = 1$, as t tends to zero, the pointwise limit

$$\lim_{t \downarrow 0} \Lambda_{1/2}\left(t, \frac{1}{\sqrt{t}}\right) = \frac{4}{2 - \rho\xi} \lim_{s \downarrow 0} \frac{s^2 M'_{\mathcal{V}}(1/2 - s)}{M_{\mathcal{V}}(1/2 - s)} = \frac{\lambda}{2 - \rho\xi}$$

can be either finite or infinite. In particular, since $\lambda > 0$, the limiting rescaled cumulant generating function is not continuous at the right boundary of its effective domain. Finally, the cgf of $\mathcal{V}$ satisfies Assumption 10 with $\omega = 2$, then Theorem 13 implies that we can recover [70, Theorem 4.1]:

$$\sigma_t^2(x) = \frac{|x|}{2} t^{-1/2} + \frac{\sqrt{\lambda|x|}}{2} t^{-1/4} + o(t^{-1/4}), \quad \text{as } t \text{ tends to zero.}$$

1.6.3 STARTING FROM THE ERGODIC DISTRIBUTION

Remark 2 shows that the stationary distribution of a randomised Heston model has the density

$$f_{\infty}(v) = \frac{a^b}{\Gamma(b)} v^{b-1} e^{-av}, \quad \text{for } v > 0,$$

where $a := \frac{2\kappa}{\xi^2}$ and $b := a\theta$. Assume now that the initial randomisation has the density f_{∞} , so that the cgf of $\mathcal{V}$ is given by $\log M_{\mathcal{V}}(u) = -b \log(a - u) +$

$b \log a$, with $u < a = \mathbf{m}$. Then Assumption 10 is satisfied with $\omega = 1$. A direct application of Theorem 13 implies that

$$\sigma_t^2(x) = \frac{\xi|x|}{4\sqrt{\kappa t}} + o(t^{-1/2}), \quad \text{for any } x \neq 0.$$

1.6.4 FOLDED GAUSSIAN RANDOMISATION

Assume that $\mathcal{V} \stackrel{(\text{Law})}{=} |\mathcal{N}(0, 1)|$, then the density of $\mathcal{V}$ reads

$$f(v) = \sqrt{\frac{2}{\pi}} \exp\left(-\frac{1}{2}v^2\right), \quad \text{for all } v \in \mathcal{D}_{\mathcal{V}} = \mathbb{R}_+,$$

which satisfies Assumption 6. Simple computations yield $M_{\mathcal{V}}(z) = 2e^{z^2/2}\Phi(z)$, for any $z \in \mathbb{R}$, where Φ denotes the Gaussian cumulative distribution function. Lemma 63 implies that for $\gamma \in (0, 1)$,

$$t^\gamma \log M_{\mathcal{V}}\left(D\left(t, \frac{u}{t^\gamma}\right)\right) = \frac{u^4}{8}t^{2-3\gamma} + \frac{\rho\xi u^5}{8}t^{3-4\gamma} - \frac{u^3}{4}t^{2-2\gamma} + \mathcal{O}(t^{4-5\gamma}) + \mathcal{O}(t^\gamma).$$

If $\gamma = 1$, then $M_{\mathcal{V}}(x\Lambda(u)) = 2 \exp\left(\frac{1}{2}\Lambda^2(u)x^2\right)\Phi(x\Lambda(u))$, and hence

$$t \log M_{\mathcal{V}}\left(D\left(t, \frac{u}{t}\right)\right) = \frac{\Lambda^2(u)}{2t} + D_0^0(u) + \mathcal{O}(t).$$

The limit is therefore non-degenerate if and only if $h(t) = t^{2/3}$, in which case $\Lambda_{2/3}(u) = \frac{1}{8}u^4$ for all $u \in \mathbb{R}$, and Theorem 7 implies

$$\lim_{t \downarrow 0} t^{1/3} \sigma_t^2(x) = \frac{(2x)^{2/3}}{3}, \quad \text{for all } x \neq 0.$$

1.6.5 OTHER DISTRIBUTIONS

The following table presents some common continuous distributions for the initial variance, together with the corresponding parameters $\mathbf{v}_+$, $\mathbf{m}$, and l_2 . In each case, we indicate (up to a constant multiplier) the short-time behaviour of the smile.

Table 1.1: Some continuous distributions with support in $\mathbb{R}_+$

Name	$\mathbf{v}_+$	$\mathbf{m}$	l_2	Behaviour of $\sigma_t^2(x)$ ($x \neq 0$)	Reference
Beta	1	∞		$x^2/\Lambda_1^*(x)$	Equation 1.6
Exponential(λ)	∞	$\lambda < \infty$	1	$ x t^{-1/2}$	Theorem 13
χ -squared	∞	1/2	1	$ x t^{-1/2}$	Theorem 13
Rayleigh	∞	∞	2	$x^{2/3}t^{-1/3}$	Theorem 7
Weibull ($k > 1$)	∞	∞	k	$(x^2/t)^{1/(1+k)}$	Theorem 7

1.6.6 NUMERICS

We present numerical examples for the volatility smiles and surfaces for four types of initial randomisation: uniform ($\mathbf{v}_+ < \infty$), folded-Gaussian ($\mathbf{m} = \mathbf{v}_+ = \infty$), exponential ($\mathbf{m} < \mathbf{v}_+ = \infty$), and Gamma distribution ($\mathbf{m} < \mathbf{v}_+ = \infty$). To generate volatility surfaces, we apply the Fast Fourier Transform (FFT) method [25] to derive a matrix of option prices, and then compute the corresponding implied volatilities using a root-finding algorithm. The Heston parameters are given by $(\kappa, \theta, \xi, V_0, \rho) = (2.1, 0.05, 0.1, 0.06, -0.6)$, which corresponds to a realistic data set calibrated on the S&P options data. In view of Theorem 29, parameters of $\mathcal{V}$ are chosen to satisfy $\mathbb{E}(\sqrt{\mathcal{V}}) = \sqrt{V_0}$, so that results of standard and randomised Heston models can be compared.

The numerics show that the randomised Heston model provides a much steeper small-time volatility smile compared with the standard Heston model, while this difference tends to fade away as the time to maturity increases. The exponential randomisation in Figure 1.9 provides the steepest short-time volatility smile, reflecting that the largest explosion rate of $\sqrt{t}$ (Theorem 13) occurs when $\mathbf{m} < \mathbf{v}_+ = \infty$. In the uniform case, Figure 1.5 and Corollary 5 may seem contradictory at first, since the former indicates steepness and the latter excludes explosion. There is no issue here, and in fact suggests that even though there is no proper explosion, it is still possible to generate steep short-time volatility smiles in a randomised setting.

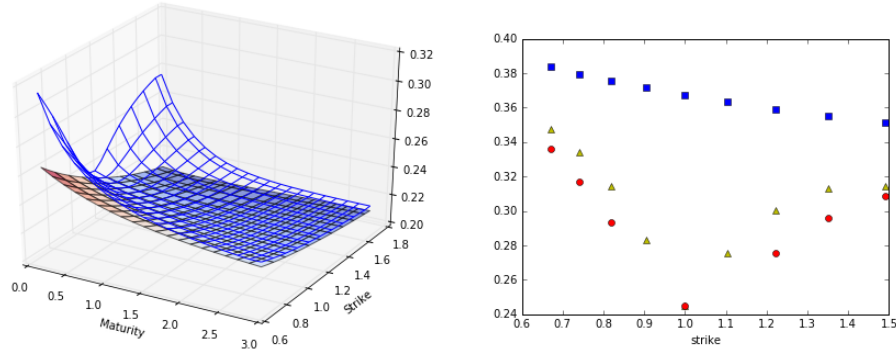


Figure 1.5: Uniform randomisation with $(v_-, v_+) = (0, 0.135)$. Left: implied volatility surfaces. Right: triangles, squares and circles represent implied volatility by FFT, leading-, and second-order asymptotics. Time to maturity is $t = 1/24$. Higher-order terms are obtained by inverting the asymptotic formula in Theorem 34.

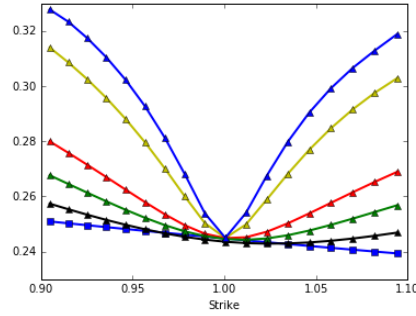


Figure 1.6: Uniform randomisation with $(v_-, v_+) = (0, 0.135)$. Blue squares is the implied volatility obtained from the standard Heston model, maturity $t = 5 \times 10^{-3}$. Blue, yellow, red, green and black triangles represent the implied volatilities computed from the randomised Heston model, with the maturities ($\times 10^{-3}$) equal to 5, 10, 50, 100 and 200 respectively. The graph illustrates the increase of steepness in a randomised Heston setting as the maturity tends to zero.

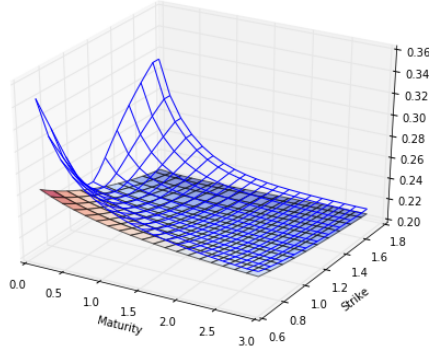


Figure 1.7: Implied volatility surfaces of folded-Gaussian randomisation and standard Heston.

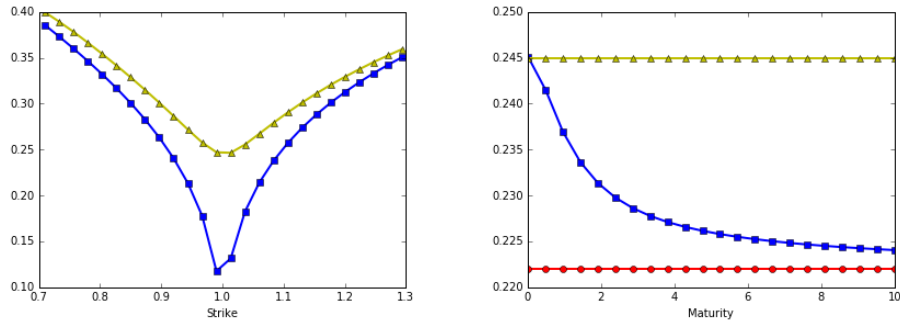


Figure 1.8: $\mathcal{V}^{(\text{Law})}$ Folded-Gaussian. Left: implied volatility by FFT (triangles) and the leading order (squares), $t = 1/24$. Right: triangles, squares and circles represent $\sqrt{V_0}$, ATM implied volatility $\sigma_t(0)$ by FFT, and large-time limit. The parameter l_1 in Assumption 6 is 63.46.

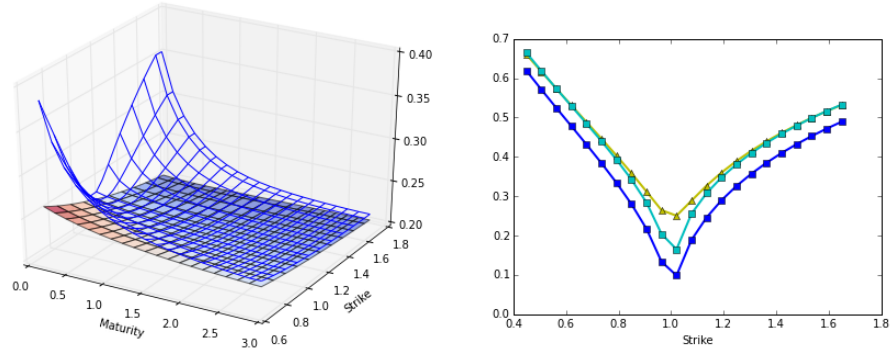


Figure 1.9: $\mathcal{V}^{(\text{Law})} = \text{Exponential distribution with } m = 13.1$. Left: volatility surfaces of exponential randomisation and standard Heston. Right: implied volatility computed by FFT (triangles) and leading order described in Theorem 13 (squares). Time to maturity is $t = 1/24$.

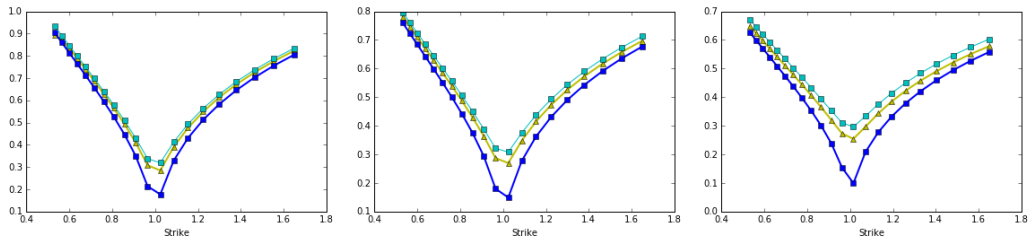


Figure 1.10: $\mathcal{V}^{(\text{Law})} = \Gamma(0.4, 3.868)$. Blue and cyan squares are first- and second-order asymptotics, yellow triangles are true implied volatilities. From left to right maturities are one week, two weeks and one month.

1.7 CALIBRATION RESULTS USING FX MARKET DATA

In this section we test the calibration accuracy of the randomised Heston model using the USD/JPY FX market data[†]. Notice that in the FX market the implied volatility still has the small-time explosion feature: in Figure 1.11 the standard Heston is calibrated to the market data with 6 different time to maturities, and the result shows that the model-generated volatility smile is too flat compared with the market data for small maturities. This finding agrees with the existing literature. For instance, in [75] the authors fixed κ and v_0 , and calibrated the remaining 3 parameters (θ, ξ, ρ) in a standard Heston environment to the EUR/USD market data. They selected time to maturities ranging from one week to 2 years, then calibrated the Heston model for each fixed maturity. Even with this ‘slice-by-slice’ calibration procedure, the authors still observed a poor fitness of the standard Heston to the market data when the time to maturity becomes smaller. Unsurprisingly, they commented that time-dependent parameters, or a ‘stochastic volatility plus jumps’ setting (SVJ) as appeared in [7, 8], are needed to improve calibration accuracy.

Before processing the data, we propose a ‘mixed’ distribution for $\mathcal{V}$:

$$\mathcal{V} \stackrel{(\text{Law})}{=} pU(a, b) + (1 - p)\Gamma(\alpha, \beta), \quad 0 < p < 1,$$

where the random variables $U(a, b)$ and $\Gamma(\alpha, \beta)$ are independent. We have $M_{\mathcal{V}}(u) = M_U(pu)M_{\Gamma}((1 - p)u)$, indicating that $\mathcal{D}_{\mathcal{V}} \supset \mathcal{D}_{\Gamma}$. Theorem 13 then implies that the mixed distribution generates volatility smiles with the steepness in between of uniform and Gamma randomisations, thus it is similar to a thin-tail environment. The advantage of this scheme is that the computational cost is less than that of a thin-tail distribution in which case we have to perform a numerical integration to obtain the pair (l_1, l_2) .

[†]Market prices are obtained from Bloomberg Terminal.

1.7.1 ASK PRICES ON JANUARY 20TH, 2017

We compare the calibration accuracy of standard and randomised Heston using market ask prices on January 20th, 2017. The results are presented in Figure 1.12 and Table 1.2: notice that all three different randomisation schemes have substantial improvement over the standard Heston model. A volatility surface comparison is provided in Figure 1.13.

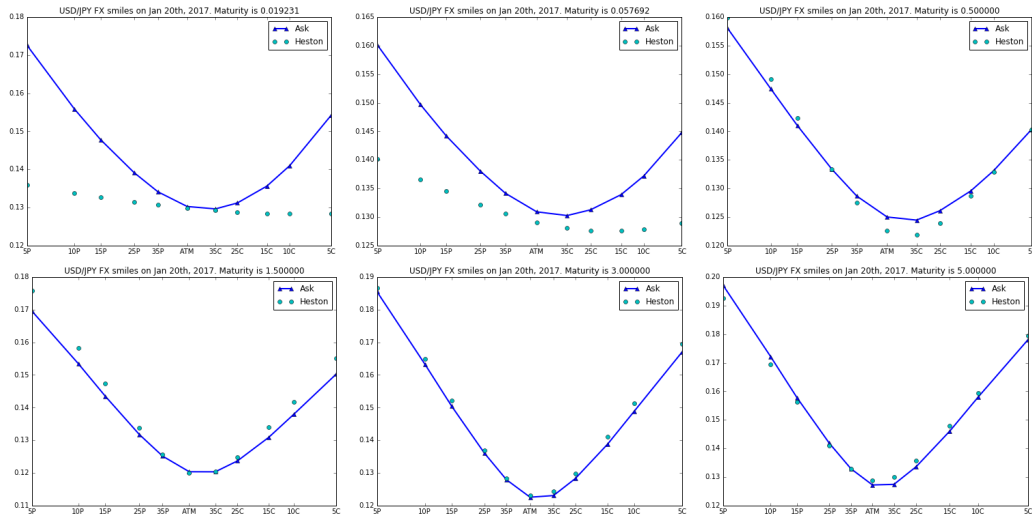


Figure 1.11: Calibration results of the standard Heston model. Blue triangles are market ask prices and the cyan circles are Heston volatility smiles. The time to maturity ranges from one week to 5 years.

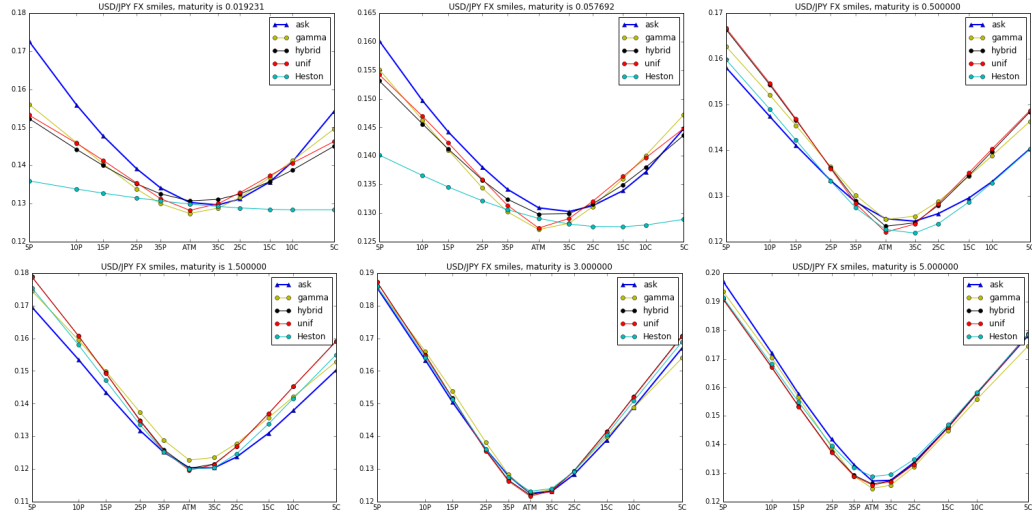


Figure 1.12: Calibration results of standard and randomised Heston models using the market ask prices on January 20th, 2017. We present three different randomisation schemes: uniform, mixed, and Gamma randomisations.

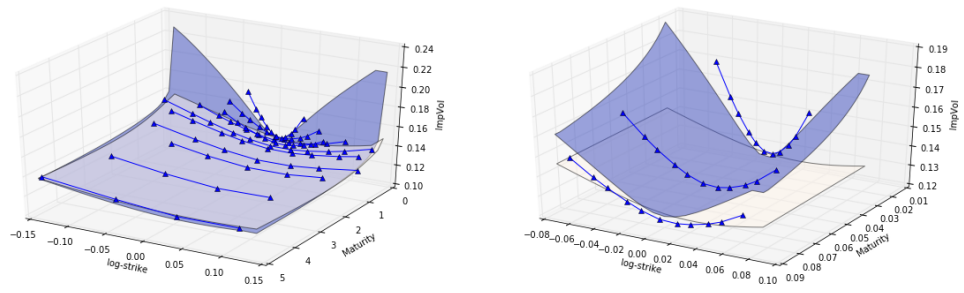


Figure 1.13: The volatility surface comparison of standard and randomised Heston. Blue lines with triangles are USD/JPY market data. The coloured surface is generated by the Gamma randomisation, and the transparent surface is the standard Heston. On the right we have the small-time zoomed-in version.

Model	Small maturities	Less than one year	Total
Heston	11.356	8.260	6.894
Gamma	5.568	5.239	4.655
Uniform	5.888	6.014	5.424
Mixed distribution	5.977	5.936	5.363

Table 1.2: The root-mean-square deviation (RMSD) of standard and randomised Heston models ($\times 10^{-3}$), calibrated to the USD/JPY ask prices on January 20th, 2017. Small maturities are those less than one month; the total RMSD is calculated over all maturities up to five years.

1.7.2 MID PRICES ON AUGUST 26TH, 2016

We analyse market mid prices on August 26th, 2016. Figure 1.14 shows that the randomised Heston is capable of providing an improved calibration result across all maturities. The mean squared errors of different schemes are listed in Table 1.3.

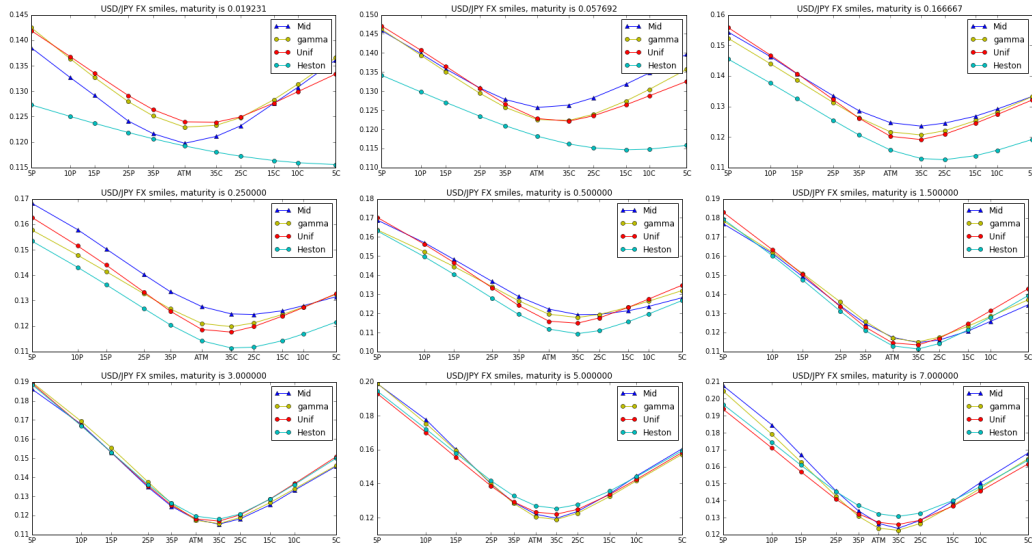


Figure 1.14: Calibration results of standard and randomised Heston settings using USD/JPY mid prices on August 26th, 2016.

Model	Small maturities	Less than one year	Total
Heston	17.200	13.343	10.916
Gamma	8.650	6.398	5.278
Uniform	9.027	6.748	6.074

Table 1.3: The RMSD of standard and randomised Heston models ($\times 10^{-3}$), calibrated to the USD/JPY mid price data on August 26th, 2016. The total RMSD is calculated for maturities up to 7 years.

1.8 A MARTINGALE REPRESENTATION THEOREM

We briefly discuss the hedging problem in the randomised Heston model. The main result of this section is a martingale representation theorem with an augmented filtration (Proposition 39). We define the enlarged filtration $\mathcal{H} := (\mathcal{H}_t)_{t \geq 0}$ by $\mathcal{H}_t := \cap_{s > t} (\mathcal{F}_s \vee \sigma(\mathcal{V}))$ for any $t \geq 0$. Also define the filtration $\mathcal{G} := (\mathcal{G}_t)_{t \geq 0}$ by $\mathcal{G}_t \equiv \mathcal{H}_t$ for $t > 0$, and $\mathcal{G}_0 = \mathcal{F}_0$ as the trivial σ -field up to completion. Then $\mathcal{F}_t \subset \mathcal{G}_t \subset \mathcal{H}_t$ holds for all $t \geq 0$.

The setting of two filtrations $\mathcal{F}$ and its augmentation $\mathcal{H}$ connects our model to the literature on initial enlargement of filtrations and insiders' trading, see [4, 41, 51] for more details. However, we require it that the information on $\mathcal{V}$ should not be available at time zero, hence the market obtainable information should be represented by the filtration $\mathcal{G}$ instead of $\mathcal{H}$.

Remark 37. *The filtration $\mathcal{G}$ is not right-continuous at time zero. One way to avoid potential issues with this setting is to assume that the information on $\mathcal{V}$ is obtained at an infinitesimal ‘observation time’ $\varepsilon > 0$. This assumption is similar to the one in [23] where the authors proposed a discretely randomised version of the Black-Scholes model. We refer the interested reader to [86, Section 1.2] for an interpretation of this setting and its connection with model uncertainty. For the remainder of this section we stick with our original definition of $\mathcal{G}$, as it will not affect the major result of this section which is based on the filtration $\mathcal{H}$.*

Proposition 38 (Martingale preservation). *The price process $(e^{X_t})_{t \geq 0}$ is a martingale with the filtration $\mathcal{G}$.*

Proof. Since the filtrations $\mathcal{G}$ and $\mathcal{H}$ only differ at time zero, we only need

to show that the proposition holds with the filtration $\mathcal{H}$. The independence of $\sigma(\mathcal{V})$ and $\mathcal{F}$ validates Hypothesis (ii) in [76, Proposition 5.9.1.1], which is equivalent to our statement. $\square$

Proposition 39 (Martingale representation). *Fix $T > 0$, then for any $H \in L^2(\Omega, \mathcal{H}_T, \mathbb{Q})$, there exists a process $(g_t)_{t \in [0, T]}$ adapted to $(\mathcal{H}_t)_{t \in [0, T]}$, such that*

$$H = \mathbb{E}(H|\mathcal{V}) + \int_0^T g_t^\top \cdot dW_t$$

with $W_t := (W_t^{(1)}, W_t^{(2)})^\top$ for $t \geq 0$.

Proof. This result can be seen as a special case of [4, Theorem 4.2]. Here we provide an alternative proof based on the systematic approach given in [31, 87]. As shown in [87, Lemma 4.3.1], by applying the martingale convergence theorem we can prove that the set

$$\mathcal{D} := \{h(\mathcal{V}, W_{t_1}^{(1)}, W_{t_1}^{(2)}, \dots, W_{t_n}^{(1)}, W_{t_n}^{(2)}) | h \in C_0^\infty(\mathbb{R}^{2n+1}), 0 < t_1 < \dots < t_n \leq T\}$$

is dense in $L^2(\Omega, \mathcal{H}_T, \mathbb{Q})$. Then for $n = 2$, and for any $0 < t_1 \leq t \leq t_2 \leq T$,

$$\mathbb{E} \left[h(\mathcal{V}, W_{t_1}^{(1)}, W_{t_1}^{(2)}, W_{t_2}^{(1)}, W_{t_2}^{(2)}) | \mathcal{H}_t \right] = v_1 \left(t, \mathcal{V}, W_{t_1}^{(1)}, W_{t_1}^{(2)}, W_t^{(1)}, W_t^{(2)} \right), \quad \text{with}$$

$$v_1(t, v, x, y, u, v) := \frac{1}{2\pi(t_2-t)} \iint h(v, x, y, s, l) \exp \left\{ \frac{-1}{2(t_2-t)} [(s-u)^2 + (l-v)^2] \right\} ds dl$$

for $t < t_2$. Also for $t \leq t_1$, define

$$v_0 \left(t, \mathcal{V}, W_t^{(1)}, W_t^{(2)} \right) := \mathbb{E} \left[v_1(t_1, \mathcal{V}, W_{t_1}^{(1)}, W_{t_1}^{(2)}, W_t^{(1)}, W_t^{(2)}) | \mathcal{H}_t \right],$$

where $v_0(t, v, x, y) = \frac{1}{2\pi(t_1-t)} \iint v_0(t_1, v, s, l) \exp \left\{ \frac{-1}{2(t_1-t)} [(s-x)^2 + (l-y)^2] \right\} ds dl$ for $0 \leq t < t_1$. It can be easily checked that

$$\frac{\partial v_1}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 v_1}{\partial u^2} + \frac{\partial^2 v_1}{\partial v^2} \right) = 0, \quad \text{and} \quad \frac{\partial v_0}{\partial t} + \frac{1}{2} \left(\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} \right) = 0.$$

Applying the Itô formula and the tower property, and notice that $W^{(1)}$

and $W^{(2)}$ are independent Brownian motions, then

$$\begin{aligned} h\left(\mathcal{V}, W_{t_1}^{(1)}, W_{t_1}^{(2)}, W_{t_2}^{(1)}, W_{t_2}^{(2)}\right) &= \mathbb{E}\left[h\left(\mathcal{V}, W_{t_1}^{(1)}, W_{t_1}^{(2)}, W_{t_2}^{(1)}, W_{t_2}^{(2)}\right) | \mathcal{V}\right] \\ &\quad + \int_0^{t_2} g_s^\top(\mathcal{V}, W_s^{(1)}, W_s^{(2)}) \cdot dW_s, \end{aligned}$$

where

$$g_s(\mathcal{V}, W_s^{(1)}, W_s^{(2)}) := \begin{cases} \left(\frac{\partial v_0}{\partial x}, \frac{\partial v_0}{\partial y}\right)^\top \left(s, \mathcal{V}, W_s^{(1)}, W_s^{(2)}\right), & s < t_1, \\ \left(\frac{\partial v_1}{\partial u}, \frac{\partial v_1}{\partial v}\right)^\top \left(s, \mathcal{V}, W_{t_1}^{(1)}, W_{t_1}^{(2)}, W_s^{(1)}, W_s^{(2)}\right), & t_1 \leq s < t_2. \end{cases}$$

The result can be generalised to the case where $n \geq 3$, indicating that the martingale representation property holds on the set $\mathcal{D}$. For any $H \in L^2(\Omega, \mathcal{H}_T, \mathbb{Q})$, assume that the sequence $\{H_n\}_{n=1}^\infty \subset \mathcal{D}$ has the limit H . Write $H_n = \mathbb{E}(H_n | \mathcal{V}) + \int_0^T g_s^{(n)\top} \cdot dW_s$, and define $Y_n := H_n - \mathbb{E}(H_n | \mathcal{V})$, then the sequence $\{Y_n\}_{n=1}^\infty$ tends to $H - \mathbb{E}(H | \mathcal{V})$, as n tends to infinity. For any $n \neq m$,

$$\begin{aligned} \mathbb{E}[(Y_n - Y_m)^2] &= \mathbb{E}\left[\left(\int_0^T (g_s^{(n)} - g_s^{(m)})^\top \cdot dW_s\right)^2\right] \\ &= \mathbb{E}\left[\int_0^T (g_s^{(n)} - g_s^{(m)})^\top (g_s^{(n)} - g_s^{(m)}) dt\right]. \end{aligned}$$

Then $\{Y_n\}_{n=1}^\infty$ being a Cauchy sequence implies that $\{g^{(n)}\}_{n=1}^\infty$ is a Cauchy sequence in $L^2(\Omega \times [0, T])$. Hence there exists an adapted process g such that $g_n \rightarrow g$, as n tends to infinity. Consequently,

$$H = \lim_{n \uparrow \infty} \left(\mathbb{E}(H_n | \mathcal{V}) + \int_0^T g_s^{(n)\top} \cdot dW_s \right) = \mathbb{E}(H | \mathcal{V}) + \int_0^T g_s^\top \cdot dW_s.$$

□

The quantity $\mathbb{E}[H | \mathcal{V}]$ in Proposition 39 requires full information on $\mathcal{V}$ at time zero, which is not available in our setting and can only be approximated. Since the model is also incomplete by nature, Proposition 39 serves as a first step towards constructing a hedging strategy. We leave the study on the

hedging problem in the initially randomised environment for future research.

1.9 PROOFS OF THE MAIN RESULTS

1.9.1 PROOF OF PROPOSITION 3

Jacquier and Roome [70, Section 6] proved that $\mathcal{D}^* = \mathbb{R}$ whenever $\gamma < 1$, and $\mathcal{D}^* = (u_-, u_+)$ if $\gamma = 1$. Throughout the proof we keep the notation h , emphasising that the statement still holds for function h with a general form, not only polynomials.

Case $\gamma \in (0, 1/2)$. We need to analyse the behaviour of $\log M(z)$ as z approaches zero. Since $\mathbf{m}$ is strictly positive, by continuity of the mgf around the origin, $M_{\mathcal{V}}(u^2 t (2h^2(t))^{-1} (1 + \mathcal{O}(h(t))))$ converges to $M_{\mathcal{V}}(0) = 1$ as t tends to zero for any u in $\mathbb{R}$, which implies that $\mathcal{D}_{\mathcal{V}}^* = \mathbb{R}$. For small t , a Taylor expansion indicates that

$$\begin{aligned} & \log M_{\mathcal{V}} \left(D \left(t, \frac{u}{h(t)} \right) \right) \\ &= \log \mathbb{E} \left(\exp \left\{ \frac{u^2 t \mathcal{V}}{2h^2(t)} \left[1 - \frac{h(t)}{u} + \mathcal{O} \left(\frac{t}{h(t)} \right) + \mathcal{O}(h^2(t)) \right] \right\} \right) \\ &= \log \left\{ 1 + \frac{u^2 \mathbb{E}(\mathcal{V}) t}{2h^2(t)} \left[1 - \frac{h(t)}{u} + \mathcal{O} \left(\frac{t}{h(t)} + h^2(t) \right) \right] + \frac{u^4 \mathbb{E}(\mathcal{V}^2) t^2}{8h^4(t)} + \mathcal{O} \left(\frac{t^3}{h^6(t)} \right) \right\} \\ &= \frac{u^2 \mathbb{E}(\mathcal{V}) t}{2h^2(t)} \left(1 + \mathcal{O} \left(h(t) + \frac{t}{h^2(t)} \right) \right). \end{aligned}$$

Since $h(t)C(t, u/h(t))$ is of order $\mathcal{O}(t^2/h(t) + h^4(t))$, then

$$\Lambda_{\gamma} \left(t, \frac{u}{h(t)} \right) = \frac{u^2 \mathbb{E}(\mathcal{V}) t}{2h(t)} \left\{ 1 + \mathcal{O} \left(h(t) + \frac{t}{h^2(t)} + h^4(t) \right) \right\}, \quad (1.24)$$

and therefore $\lim_{t \downarrow 0} \Lambda_{\gamma}(t, u/h(t)) = 0$, for all $u \in \mathbb{R}$.

Case $\gamma \in (1/2, 1]$. We need to evaluate $M_{\mathcal{V}}$ at infinity. If $\mathbf{m}$ is finite, for t sufficiently small, the term $M_{\mathcal{V}}(\frac{1}{2} u^2 t h^{-2}(t) (1 + \mathcal{O}(t/h(t))))$ is infinite for any non-zero u , hence $\mathcal{D}_{\mathcal{V}}^* = \{0\}$, and $\Lambda_{\gamma}(u)$ is null at $u = 0$, and infinite elsewhere. If $\mathbf{m}$ is infinite, then $\mathcal{D}_{\mathcal{V}}^* = \mathbb{R}$. Assume first that $\mathbf{v}_+$ is finite;

we claim that $\lim_{u \uparrow \infty} (\mathbf{v}_+ u)^{-1} \log M_{\mathcal{V}}(u) = 1$. In fact, let $F_{\mathcal{V}}$ be the cumulative distribution function of $\mathcal{V}$, then

$$M_{\mathcal{V}}(u) = \mathbb{E}(e^{u\mathcal{V}}) \leq \exp(u\mathbf{v}_+) \int_{[\mathbf{v}_-, \mathbf{v}_+]} F_{\mathcal{V}}(dv) = \exp(u\mathbf{v}_+).$$

For any small $\varepsilon > 0$, fix $\delta \in (0, \varepsilon\mathbf{v}_+/2)$, so that

$$\begin{aligned} \frac{\log M_{\mathcal{V}}(u)}{u\mathbf{v}_+} &\geq \frac{1}{u\mathbf{v}_+} \log \left(\int_{\mathbf{v}_+ - \delta}^{\mathbf{v}_+} e^{uv} F_{\mathcal{V}}(dv) \right) \\ &\geq \frac{1}{u\mathbf{v}_+} \log (e^{u(\mathbf{v}_+ - \delta)} \mathbb{Q}(\mathcal{V} \geq \mathbf{v}_+ - \delta)) \\ &= 1 - \frac{\delta}{\mathbf{v}_+} + \frac{1}{u\mathbf{v}_+} \log \mathbb{Q}(\mathcal{V} \geq \mathbf{v}_+ - \delta), \end{aligned}$$

since $\mathbf{v}_+$ is the upper bound of the support; therefore $\mathbb{Q}(\mathcal{V} \geq \mathbf{v}_+ - \delta)$ is strictly positive, and the result follows. If $\gamma \in (1/2, 1)$, notice that $h(t)C(t, u/h(t))$ is of order $t^{2-\gamma}$ from Lemma 65, hence

$$\begin{aligned} \lim_{t \downarrow 0} \Lambda_{\gamma} \left(t, \frac{u}{h(t)} \right) &= \lim_{t \downarrow 0} \mathcal{O}(t^{2-\gamma}) + \lim_{t \downarrow 0} t^{\gamma} \log M_{\mathcal{V}} \left(\frac{u^2 t^{1-2\gamma}}{2} (1 + \mathcal{O}(t^{1-\gamma})) \right) \\ &= \frac{u^2 \mathbf{v}_+}{2} \lim_{t \downarrow 0} t^{1-\gamma} = 0, \quad \text{for any } u \text{ in } \mathbb{R}. \end{aligned}$$

When $\gamma = 1$, $\Lambda(u)$ is positive whenever $u \in (u_-, u_+) \setminus \{0\}$. Therefore,

$$\lim_{t \downarrow 0} \Lambda_{\gamma} \left(t, \frac{u}{h(t)} \right) = \lim_{t \downarrow 0} \mathcal{O}(t) + \lim_{t \downarrow 0} t \left(\frac{\mathbf{v}_+ \Lambda(u)}{t} (1 + \mathcal{O}(t)) \right) = \Lambda(u) \mathbf{v}_+,$$

for any $u \in \mathcal{D}^* = (u_-, u_+)$.

Case $\gamma = 1/2$. If $\mathbf{v}_+$ is finite then the pointwise limit is null on the whole real line. Assume now that $\mathbf{v}_+$ is infinite and $\mathbf{m}$ is finite. Following Remark 64(iii), $\frac{u^2}{2} + \left(\frac{\rho \xi u^3}{4} - \frac{u}{2} \right) t^{1/2} + \mathcal{O}(t) < \mathbf{m}$ implies $\mathcal{D}_{\mathcal{V}}^{*o} = (-\sqrt{2\mathbf{m}}, \sqrt{2\mathbf{m}}) \subseteq \mathcal{D}_{\mathcal{V}}^* \subseteq [-\sqrt{2\mathbf{m}}, \sqrt{2\mathbf{m}}] = \overline{\mathcal{D}_{\mathcal{V}}^*}$. For sufficiently small t ,

$$\Lambda_{1/2} \left(t, \frac{u}{\sqrt{t}} \right) = \frac{\kappa \theta u^2}{4} t^{3/2} + \mathcal{O}(t^2) + t^{1/2} \log M_{\mathcal{V}} \left(\frac{u^2}{2} + \left(\frac{\rho \xi u^3}{4} - \frac{u}{2} \right) t^{1/2} + \mathcal{O}(t) \right).$$

For any fixed u in $\mathcal{D}_{\mathcal{V}}^{*o}$, by definition there exists a positive t_0 such that u is in $\mathcal{D}_{\mathcal{V}}^t$ for all t less than t_0 . Then the mgf of $\mathcal{V}$ is infinitely differentiable around the point $u^2/2$, and the n -th order derivative at this point is $M_{\mathcal{V}}^{(n)}(\frac{1}{2}u^2) = \mathbb{E}[\mathcal{V}^n \exp(\frac{1}{2}u^2\mathcal{V})]$. Denote now $a_n(u) := M_{\mathcal{V}}^{(n)}(\frac{1}{2}u^2) M_{\mathcal{V}}^{-1}(\frac{1}{2}u^2)$, for $n \in \mathbb{N}_+$, and $a_0(u) := \log M_{\mathcal{V}}(\frac{1}{2}u^2)$. A Taylor expansion of the function $M_{\mathcal{V}}$ around the point $\frac{1}{2}u^2$ yields

$$\begin{aligned} \Lambda_{1/2}\left(t, \frac{u}{\sqrt{t}}\right) &= \sqrt{t} \log \left\{ M_{\mathcal{V}}\left(\frac{u^2}{2}\right) \left[1 + a_1(u) \left(\frac{\rho \xi u^2}{2} - 1 \right) \frac{u\sqrt{t}}{2} + \mathcal{O}(t) \right] \right\} \\ &\quad + \frac{\kappa \theta u^2}{4} t^{3/2} + \mathcal{O}(t^2) \\ &= a_0(u) \sqrt{t} + a_1(u) \left(\frac{\rho \xi u^2}{2} - 1 \right) \frac{ut}{2} + \mathcal{O}(t^{3/2}). \end{aligned} \quad (1.25)$$

Letting t tend to zero, we finally obtain

$$\Lambda_{1/2}(u) = \begin{cases} 0, & \text{when } u \in \mathcal{D}_{\mathcal{V}}^{*o}, \\ \infty, & \text{when } u \in \mathbb{R} \setminus \overline{\mathcal{D}_{\mathcal{V}}^*}. \end{cases}$$

However, the limit at the boundaries, i.e. $\Lambda_{1/2}(t, \pm\sqrt{2\mathbf{m}/t})$, depends on the explicit form of $M_{\mathcal{V}}$. To see this we take the example where $\rho \xi \mathbf{m} < 1$, which is guaranteed in particular when $\rho \leq 0$, and compute the limit when $u = \sqrt{2\mathbf{m}}$. The L'Hôpital's rule implies

$$\lim_{t \downarrow 0} \sqrt{t} \log M_{\mathcal{V}}\left(\mathbf{m} + \sqrt{\frac{\mathbf{m}}{2}} (\rho \xi \mathbf{m} - 1) \sqrt{t} + \mathcal{O}(t)\right) = \frac{\sqrt{2/\mathbf{m}}}{1 - \rho \xi \mathbf{m}} \lim_{s \downarrow 0} \frac{s^2 M'_{\mathcal{V}}(\mathbf{m} - s)}{M_{\mathcal{V}}(\mathbf{m} - s)}, \quad (1.26)$$

and the right-hand-side is dependent on both $M_{\mathcal{V}}$ and its first order derivative.

1.9.2 PROOF OF THEOREM 13

The systematic procedure is similar to the proof of [70, Theorem 3.1]. To simplify notations, write $\tilde{\Lambda}_t(u) := \Lambda_{1/2}(t, u/\sqrt{t})$, $\tilde{C}_t(u) := C(t, u/\sqrt{t})$ and $\tilde{D}_t(u) := D(t, u/\sqrt{t})$, whenever these quantities are well defined. We shall prove the theorem in several steps: in Lemma 40 we show that a saddle-point analysis is feasible; by taking the expectation under a new probability

measure, the main contribution of the option price arises and its asymptotic expansion is provided in Lemma 41; in Lemma 42 we prove the convergence (with rescaling) of the sequence $(X_t - x)_{t \geq 0}$ under this new measure; finally, the full asymptotics of the Call option price is obtained via inverse Fourier transform.

Lemma 40. *Under Assumption 10, for any $x \neq 0$, $t > 0$ small enough, the equation $\partial_u \tilde{\Lambda}_t(u) = x$ admits a unique solution $u_t^*(x)$ such that $\tilde{D}_t(u_t^*(x)) \in \mathcal{D}_{\mathcal{V}}^t$, and the following holds as t tends to zero:*

$$u_t^*(x) = \begin{cases} \operatorname{sgn}(x)\sqrt{2\mathbf{m}} + b_1(x)t^{\frac{1}{2\omega}} + o\left(t^{\frac{1}{2\omega}}\right), & \text{for } \omega = 1, \\ \operatorname{sgn}(x)\sqrt{2\mathbf{m}} + b_1(x)t^{\frac{1}{2\omega}} + b_2(x)t^{\frac{1}{\omega}} \log t + b_3(x)t^{\frac{1}{\omega}} + o\left(t^{\frac{1}{\omega}}\right), & \text{for } \omega \geq 2, \end{cases}$$

where

$$\begin{aligned} b_1(x) &:= -\operatorname{sgn}(x)(2\mathbf{m})^{\frac{1-\omega}{2\omega}} \left(\frac{1_{\{\omega=1\}}|\gamma_0| + 1_{\{\omega \geq 2\}}(\omega-1)\gamma_0}{|x|} \right)^{1/\omega} + \frac{1 - \rho\xi\mathbf{m}}{2} 1_{\{\omega=1\}}, \\ b_2(x) &:= -\operatorname{sgn}(x) \frac{\mathbf{a}\sqrt{2\mathbf{m}}}{2\omega^2} b_1^2(x), \\ b_3(x) &:= \operatorname{sgn}(x) \left\{ \frac{b_1^2(x)}{\sqrt{2\mathbf{m}\omega}} \left(1 - \frac{\omega}{2} - 2\mathbf{a}\mathbf{m} \log \left(\sqrt{2\mathbf{m}}|b_1(x)| \right) - 2\mathbf{b}\mathbf{m} \right) \right\} + \frac{1 - \rho\xi\mathbf{m}}{2} 1_{\{\omega=2\}}. \end{aligned}$$

If $x = 0$, then $u_t^*(0)$ defined as the solution to $\partial_u \tilde{\Lambda}_t(u) = 0$ satisfies $u_t^*(0) = \frac{1}{2}\sqrt{t} + o(\sqrt{t})$.

Proof of Lemma 40. Assume that $x > 0$, the case when $x < 0$ being analogous. Equation (1.4) implies that for any $u \in \mathbb{R}$, the equation $\partial_u \tilde{\Lambda}_t(u) = x$ reads

$$x = \partial_u \tilde{\Lambda}_t(u) = \sqrt{t} \left(\log M \left(t, \frac{u}{\sqrt{t}} \right) \right)' = \sqrt{t} \tilde{C}'_t(u) + \sqrt{t} \frac{M'_{\mathcal{V}} \left(\tilde{D}_t(u) \right)}{M_{\mathcal{V}} \left(\tilde{D}_t(u) \right)} \tilde{D}'_t(u). \quad (1.27)$$

The existence and uniqueness of the solution to (1.27) are guaranteed by the strict convexity of the rescaled cgf $\tilde{\Lambda}_t$ for each t [77, Theorem 2.3] and (1.11), in which the denominator tends to zero as u tends to the boundary of $\mathcal{D}_{\mathcal{V}}^t$. Denote now the unique solution by $u_t^*(x)$. Applying Lemma 63 and Lemma 65

with $h(t) \equiv t^{1/2}$,

$$\tilde{C}'_t(u) = \frac{u\kappa\theta}{2}t + \mathcal{O}\left(t^{\frac{3}{2}}\right), \quad \text{and} \quad \tilde{D}'_t(u) = u + \left(\frac{3\rho\xi u^2}{4} - \frac{1}{2}\right)\sqrt{t} + \mathcal{O}(t).$$

We first prove that $\lim_{t \downarrow 0} u_t^*(x) = \sqrt{2\mathbf{m}}$. If $\lim_{t \downarrow 0} u_t^*(x) \neq \sqrt{2\mathbf{m}}$, there exists a sequence $\{t_n\}_{n=1}^\infty$ and (small enough) $\varepsilon_0 > 0$, satisfying $\lim_{n \uparrow \infty} t_n = 0$ and $|u_{t_n}^*(x) - \sqrt{2\mathbf{m}}| \geq \varepsilon_0$ for any $n \geq 1$. In Section 1.9.1 it is proved that $\lim_{t \downarrow 0} \mathcal{D}_V^t \subseteq \limsup_{t \downarrow 0} \mathcal{D}_V^t \subseteq \overline{\mathcal{D}_V^*} = [-\sqrt{2\mathbf{m}}, \sqrt{2\mathbf{m}}]$. Also notice that for any fixed t small enough the map $\partial_u \tilde{\Lambda}_t : \mathcal{D}_V^t \rightarrow \mathbb{R}$ is continuous and strictly increasing. Hence for fixed positive ε_0 there are at most finitely many t_i in the sequence such that $u_{t_i}^*(x) \geq \sqrt{2\mathbf{m}} + \varepsilon_0$.

Equation (1.27) implies that for fixed $x > 0$ the limit of $t^{-1/2} \partial_u \tilde{\Lambda}_t(u_t^*(x))$ is infinity as t tends to zero. Taking a subsequence of $\{t_n\}_{n \geq 1}$ if necessary, assume now that $u_{t_n}^*(x) \leq \sqrt{2\mathbf{m}} - \varepsilon_0$, for any $n \geq 1$. Since $\tilde{D}_t(\sqrt{2\mathbf{m}} - \varepsilon) = \mathbf{m} - \sqrt{2\mathbf{m}}\varepsilon + \varepsilon^2/2 + \mathcal{O}(\sqrt{t})$, then for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that

$$\left| \tilde{D}_{t_n}(\sqrt{2\mathbf{m}} - \varepsilon) - \mathbf{m} + \sqrt{2\mathbf{m}}\varepsilon - \frac{\varepsilon^2}{2} \right| < \frac{\sqrt{2\mathbf{m}}\varepsilon}{2}, \quad \text{for any } n \geq N(\varepsilon).$$

Fix $0 < \varepsilon_1 < \min(\varepsilon_0, \sqrt{2\mathbf{m}})$ small enough so that $\mathbf{m} - 3\sqrt{2\mathbf{m}}\varepsilon_1/2 + \varepsilon_1^2/2 > \mathbf{m} - \delta_0$, where $\delta_0 > 0$ is chosen to satisfy that for any $\mathbf{m} - \delta_0 < u < \mathbf{m}$, the higher-order term in (1.11) is bounded above by one. Then for such ε_1 and for any $n \geq N(\varepsilon_1)$ we have $\mathbf{m} - \delta_0 < \tilde{D}_{t_n}(\sqrt{2\mathbf{m}} - \varepsilon_1) < \mathbf{m} - \sqrt{2\mathbf{m}}\varepsilon_1/2 + \varepsilon_1^2/2 < \mathbf{m}$. The function $\partial_u \tilde{\Lambda}_t$ is strictly increasing, implying

$$\lim_{n \uparrow \infty} \frac{\partial_u \tilde{\Lambda}_{t_n}(u_{t_n}^*(x))}{\sqrt{t_n}} \leq \lim_{n \uparrow \infty} \frac{\partial_u \tilde{\Lambda}_{t_n}(\sqrt{2\mathbf{m}} - \varepsilon_1)}{\sqrt{t_n}} \leq \frac{2^{\omega+1}\delta_1}{\varepsilon_1^\omega(\sqrt{2\mathbf{m}} - \varepsilon_1)^{\omega-1}} < \infty,$$

where $\delta_1 := \mathbf{1}_{\{\omega=1\}}|\gamma_0| + \mathbf{1}_{\{\omega \geq 2\}}(\omega-1)\gamma_0$, hence the contradiction. Therefore $\lim_{t \downarrow 0} u_t^*(x) = \sqrt{2\mathbf{m}}$. Analogously we can prove that $\lim_{t \downarrow 0} u_t^*(0) = 0$.

Case $\omega = 1$. Assume that $u_t^*(x) = \sqrt{2\mathbf{m}} + h_x(t)$, where $h_x(t) = o(1)$. Equation (1.27) implies that $h_x(t) = \mathcal{O}(\sqrt{t})$, hence all the terms of order $\mathcal{O}(\sqrt{t})$

in the expansion of $\tilde{D}_t(u_t^*(x))$ should be included. More specifically,

$$\tilde{D}_t(u_t^*(x)) = \mathbf{m} + \sqrt{2\mathbf{m}}h_x(t) + \frac{\sqrt{2\mathbf{m}}}{2}(\rho\xi\mathbf{m} - 1)\sqrt{t} + o(\sqrt{t}).$$

Plugging this back into (1.27) and solving at the leading order yield the desired result.

Case $\omega \geq 2$. In this case $h_x(t) = \mathcal{O}(t^{1/(2\omega)})$. Equation (1.27) now reads

$$\begin{aligned} \frac{x}{\sqrt{t}} &\equiv \frac{\kappa\theta\sqrt{2\mathbf{m}}}{2}t + \frac{\kappa\theta}{2}th_x(t) + \mathcal{O}(t^{3/2}) \\ &+ \left(\frac{\delta_0(1+o(1))}{(-\sqrt{2\mathbf{m}}h_x(t) + \mathcal{O}(h_x^2 + \sqrt{t}))^\omega} \right) \left(\sqrt{2\mathbf{m}} + h_x(t) + \mathcal{O}(\sqrt{t}) \right). \end{aligned} \quad (1.28)$$

Denote by h_x^* as the leading order of the function h_x . Solving (1.28) at the leading order, then $\delta_0\sqrt{2\mathbf{m}}t \equiv x(-\sqrt{2\mathbf{m}}h_x^*(t))^\omega$, from which $h_x^*(t) = -(2\mathbf{m})^{\frac{1-\omega}{2\omega}}(\delta_0/x)^{\frac{1}{\omega}}t^{\frac{1}{2\omega}}$. Higher orders in the expansion of $u_t^*(x)$ can be derived similarly, simply by replacing the little-o term in (1.28) with precise higher-order terms provided in (1.11). We omit the details.

Finally, when $x = 0$, write $u_t^*(0) = h(t)$ with $h(t) = o(1)$. As t tends to zero, $M_{\mathcal{V}}(u_t^*(0)) \sim 1$, $M'_{\mathcal{V}}(u_t^*(0)) \sim \mathbb{E}(\mathcal{V})$, and $\tilde{D}'_t(u_t^*(0)) = h(t) - \frac{1}{2}\sqrt{t} + \mathcal{O}(t + h^2(t)\sqrt{t})$. Plugging these into (1.27) with $x = 0$ proves the lemma. $\square$

Lemma 41.

1. When $\gamma_0 > 0$ and $\omega \geq 2$, as t tends to zero,

$$\exp\left(\frac{-xu_t^*(x) + \tilde{\Lambda}_t(u_t^*(x))}{\sqrt{t}}\right) = \exp\left(-\frac{\Lambda^*(x)}{\sqrt{t}} + c_1(x)t^{\frac{1-\omega}{2\omega}} + o\left(t^{\frac{1-\omega}{2\omega}}\right)\right),$$

for any $x \neq 0$, where $c_1(x) := \omega\gamma_0^{1/\omega} \left(\frac{|x|}{\sqrt{2\mathbf{m}(\omega-1)}}\right)^{1-1/\omega}$, and the function Λ^* is defined in (1.12);

2. If $\gamma_0 < 0$ and $\omega = 1$, then for any $x \neq 0$, as t tends to zero,

$$\begin{aligned} & \exp \left(\frac{-xu_t^*(x) + \tilde{\Lambda}_t(u_t^*(x))}{\sqrt{t}} \right) \\ &= \exp \left(-\frac{\Lambda^*(x)}{\sqrt{t}} + c_2(x) + \gamma_1 \right) \left(\frac{|x|}{|\gamma_0| \sqrt{2\mathbf{m}t}} \right)^{|\gamma_0|} (1 + o(1)), \end{aligned}$$

where $c_2(x) := \frac{1}{2} (\rho\xi\mathbf{m} - 1)x - \gamma_0$.

Proof of Lemma 41. Case $\omega \geq 2$. Assumption 10 and Lemma 40 imply

$$\begin{aligned} \exp \left(-\frac{xu_t^*(x)}{\sqrt{t}} \right) &= \exp \left\{ -\frac{x}{\sqrt{t}} \left[\sqrt{2\mathbf{m}} + b_1(x)t^{\frac{1}{2\omega}} + o(t^{\frac{1}{2\omega}}) \right] \right\} \\ &= \exp \left\{ -\frac{\Lambda^*(x)}{\sqrt{t}} - b_1(x)xt^{\frac{1-\omega}{2\omega}} + o\left(t^{\frac{1-\omega}{2\omega}}\right) \right\}, \\ \exp \left(\frac{\tilde{\Lambda}_t(u_t^*(x))}{\sqrt{t}} \right) &= \exp \left\{ \tilde{C}_t(u_t^*) + \log \circ M_V(\tilde{D}_t(u_t^*)) \right\} \\ &= \exp \left\{ \frac{\gamma_0}{(\sqrt{2\mathbf{m}}|b_1(x)|)^{\omega-1}} t^{\frac{1-\omega}{2\omega}} + o\left(t^{\frac{1-\omega}{2\omega}}\right) \right\}. \end{aligned}$$

Using the expression of $b_1(\cdot)$ provided in Lemma 40, the coefficient of the term of order $t^{\frac{1-\omega}{2\omega}}$ is given by

$$\begin{aligned} -b_1(x)x + \frac{\gamma_0}{(\sqrt{2\mathbf{m}}|b_1|)^{\omega-1}} &= \left\{ [(\omega-1)\gamma_0]^{\frac{1}{\omega}} + \gamma_0[(\omega-1)\gamma_0]^{\frac{1-\omega}{\omega}} \right\} (\sqrt{2\mathbf{m}})^{\frac{1-\omega}{\omega}} |x|^{1-\frac{1}{\omega}} \\ &= \omega\gamma_0^{\frac{1}{\omega}} \left(\frac{|x|}{\sqrt{2\mathbf{m}}(\omega-1)} \right)^{1-\frac{1}{\omega}} = c_1(x). \end{aligned}$$

Case $\omega = 1$. This case follows straightforward computations after noticing that

$$\begin{aligned} \exp \left\{ \frac{\tilde{\Lambda}_t(u_t^*(x))}{\sqrt{t}} \right\} &= \exp \left\{ \mathcal{O}(t) + \gamma_0 \log \left(\mathbf{m} - \tilde{D}_t(u_t^*(x)) \right) + \gamma_1 + o(1) \right\} \\ &= e^{\gamma_1} \left(\frac{|\gamma_0| \sqrt{2\mathbf{m}t}}{|x|} \right)^{\gamma_0} (1 + o(1)). \end{aligned}$$

□

For each $x \neq 0$ and $t > 0$ small enough, define the time-dependent measure $\mathbb{Q}_t$ by

$$\frac{d\mathbb{Q}_t}{d\mathbb{Q}} := \exp \left\{ \frac{u_t^*(x)X_t - \tilde{\Lambda}_t(u_t^*(x))}{t^{1/2}} \right\}.$$

Lemma 40 implies that $\tilde{\Lambda}_t(u_t^*(x))$ is finite for small t . Also by definition it is obvious that $\mathbb{E}[d\mathbb{Q}_t/d\mathbb{Q}] = 1$, then $\mathbb{Q}_t$ is a well-defined probability measure for each t .

Lemma 42. *For any $x \neq 0$, let $Z_t := (X_t - x)/\vartheta(t)$, where $\vartheta(t) := 1_{\{\omega=1\}} + 1_{\{\omega=2\}}t^{1/8}$. Under Assumption 10, as t tends to zero, the characteristic function of Z_t under $\mathbb{Q}_t$ is*

$$\Psi_t(u) := \mathbb{E}^{\mathbb{Q}_t} (e^{iuZ_t}) = \begin{cases} e^{-iux} \left(1 - \frac{iux}{|\gamma_0|}\right)^{\gamma_0} (1 + o(1)), & \text{for } \omega = 1, \\ \exp\left(\frac{-u^2 \zeta^2(x)}{2}\right) (1 + o(1)), & \text{for } \omega = 2, \end{cases}$$

where $\zeta(x) := \sqrt{2} \left(\frac{2\mathfrak{m}}{\gamma_0^2}\right)^{1/8} |x|^{3/4}$.

Remark 43. *Lemma 42 and Lévy's Convergence Theorem [96, Theorem 18.1] imply that under $\mathbb{Q}_t$ the process $(Z_t)_{t \geq 0}$ converges weakly to a Gamma distribution (or a Gamma distribution mirrored to the negative real half line) if $x > 0$ (or $x < 0$) minus the constant x when $\omega = 1$, and to a Gaussian distribution when $\omega = 2$.*

Remark 44. *Intuitively, the case $\omega \geq 3$ should be similar to the case $\omega = 2$, so that a suitable candidate for the function ϑ can be found. However, in such scenario more information on the asymptotics of $\log M_Y$ and its derivative are required in order to obtain the suitable (non-constant) characteristic function. These extra assumptions turn out to be very restrictive and of little practical use, and are thus omitted.*

Proof of Lemma 42. Assume that $x > 0$, with $x < 0$ being analogous. Func-

tion $\log \Psi_t$ can be written as

$$\begin{aligned}
\log \Psi_t(u) &= \log \mathbb{E} \left[\exp \left(\frac{\mathbf{i}u(X_t - x)}{\vartheta(t)} + \frac{u_t^*(x)X_t - \tilde{\Lambda}_t(u_t^*(x))}{\sqrt{t}} \right) \right] \\
&= -\frac{\mathbf{i}ux}{\vartheta(t)} + \log \mathbb{E} \left[\exp \left(\frac{(\mathbf{i}u\sqrt{t}/\vartheta(t) + u_t^*(x)) X_t}{\sqrt{t}} \right) \right] - \frac{\tilde{\Lambda}_t(u_t^*(x))}{\sqrt{t}} \\
&= -\frac{\mathbf{i}ux}{\vartheta(t)} + \frac{1}{\sqrt{t}} \left(\tilde{\Lambda}_t \left(u_t^*(x) + \mathbf{i}u\sqrt{t}/\vartheta(t) \right) - \tilde{\Lambda}_t(u_t^*(x)) \right).
\end{aligned} \tag{1.29}$$

Case $\omega = 1$. Lemma 40 implies that

$$\begin{aligned}
\tilde{D}_1(u) &:= \tilde{D}_t \left(u_t^*(x) + \frac{\mathbf{i}u\sqrt{t}}{\vartheta(t)} \right) = \mathbf{m} + \frac{\gamma_0\sqrt{2\mathbf{m}t}}{x} + \mathbf{i}u\sqrt{2\mathbf{m}t} + o(\sqrt{t}), \\
\tilde{D}_2 &:= \tilde{D}(u_t^*(x)) = \mathbf{m} + \frac{\gamma_0\sqrt{2\mathbf{m}t}}{x} + o(\sqrt{t}), \\
\tilde{C}_1(u) &:= \tilde{C} \left(u_t^*(x) + \frac{\mathbf{i}u\sqrt{t}}{\vartheta(t)} \right) = \mathbf{m}\kappa\theta t/2 + \mathcal{O}(t^{3/2}), \\
\tilde{C}_2 &:= \tilde{C}(u_t^*(x)) = \mathbf{m}\kappa\theta t/2 + \mathcal{O}(t^{3/2}).
\end{aligned}$$

As a result, the lemma follows in this case from the following computations:

$$\begin{aligned}
\log \Psi_t(u) &= -\mathbf{i}ux + \tilde{C}_1(u) - \tilde{C}_2 + \log M_{\mathbf{V}} \left(\tilde{D}_1(u) \right) - \log M_{\mathbf{V}} \left(\tilde{D}_2 \right) \\
&= -\mathbf{i}ux + \gamma_0 \log \left(\frac{\mathbf{m} - \tilde{D}_1(u)}{\mathbf{m} - \tilde{D}_2} \right) + o(1) \\
&= -\mathbf{i}ux + \gamma_0 \log \left(1 - \frac{\mathbf{i}ux}{|\gamma_0|} + o(1) \right) + o(1).
\end{aligned}$$

Case $\omega = 2$. Denote $\theta := 1/8$, then $\frac{1}{2} - \theta > \frac{1}{4} = \frac{1}{2\omega}$. Lemma 40 implies

$$\begin{aligned}
\tilde{D}_1(u) &= \mathbf{m} + \sqrt{2\mathbf{m}b_1}t^{\frac{1}{4}} + \mathbf{i}u\sqrt{2\mathbf{m}t}^{\frac{1}{2}-\theta} + \left(\frac{b_1^2}{2} + \sqrt{2\mathbf{m}b_3} + \sqrt{\frac{\mathbf{m}}{2}}(\mathbf{m}\rho\xi - 1) \right) t^{\frac{1}{2}} + o(\sqrt{t}), \\
\tilde{D}_2 &= \mathbf{m} + \sqrt{2\mathbf{m}b_1}t^{\frac{1}{4}} + \left(\frac{b_1^2}{2} + \sqrt{2\mathbf{m}b_3} + \sqrt{\frac{\mathbf{m}}{2}}(\mathbf{m}\rho\xi - 1) \right) t^{\frac{1}{2}} + o(\sqrt{t}), \\
\tilde{C}_1(u) &= \frac{\mathbf{m}\kappa\theta t}{2} + \frac{\kappa\theta\sqrt{2\mathbf{m}b_1}}{2}t^{5/4} + \mathcal{O}(t^{11/8}), \quad \text{and} \\
\tilde{C}_2 &= \frac{\mathbf{m}\kappa\theta t}{2} + \frac{\kappa\theta\sqrt{2\mathbf{m}b_1}}{2}t^{5/4} + \mathcal{O}(t^{3/2}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{\tilde{\Lambda}_t(u_t^* + iut^{1/2-\theta}) - \tilde{\Lambda}_t(u_t^*)}{\sqrt{t}} &= \tilde{C}_1(u) - \tilde{C}_2 + \log M_{\mathcal{V}}(\tilde{D}_1(u)) - \log M_{\mathcal{V}}(\tilde{D}_2) \\
&= \frac{\gamma_0 (\tilde{D}_1(u) - \tilde{D}_2)}{(\mathbf{m} - \tilde{D}_1(u)) (\mathbf{m} - \tilde{D}_2)} + \gamma_0 \gamma_1 \left(\log (\mathbf{m} - \tilde{D}_1(u)) - \log (\mathbf{m} - \tilde{D}_2) \right) + o(1) \\
&= \frac{\gamma_0 (iu\sqrt{2\mathbf{m}}t^{1/2-\theta} + o(\sqrt{t}))}{2\mathbf{m}b_1^2t^{1/2}} \left[1 - \frac{iut^{1/4-\theta}}{b_1} + \mathcal{O}(t^{1/4}) \right] + \frac{i u \gamma_0 \gamma_1 t^{1/4-\theta}}{b_1} + o(1) \\
&= \frac{i\gamma_0 u}{\sqrt{2\mathbf{m}b_1^2}} t^{-\theta} + \frac{\gamma_0 u^2}{\sqrt{2\mathbf{m}b_1^3}} + o(1),
\end{aligned}$$

and the proof follows by noticing that $b_1 < 0$ and $\gamma_0 = x\sqrt{2\mathbf{m}b_1^2}$, from Lemma 40. $\square$

We finally prove the main theorem, when $x > 0$. The price of a European Call option with strike e^x is

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}(e^{X_t} - e^x)^+ &= \mathbb{E}^{\mathbb{Q}_t} \left[(e^{X_t} - e^x)^+ \frac{d\mathbb{Q}}{d\mathbb{Q}_t} \right] \\
&= \mathbb{E}^{\mathbb{Q}_t} \left[\exp \left(\frac{-u_t^*(x)X_t + \tilde{\Lambda}_t(u_t^*(x))}{t^{1/2}} \right) (e^{X_t} - e^x)^+ \right] \\
&= \exp \left(\frac{-xu_t^*(x) + \tilde{\Lambda}_t(u_t^*(x))}{\sqrt{t}} \right) e^x \mathbb{E}^{\mathbb{Q}_t} \left[\exp \left(\frac{-u_t^*(x)(X_t - x)}{\sqrt{t}} \right) (e^{X_t - x} - 1)^+ \right] \\
&= \exp \left(\frac{-xu_t^*(x) + \tilde{\Lambda}_t(u_t^*(x))}{\sqrt{t}} \right) e^x \mathbb{E}^{\mathbb{Q}_t} \left[\exp \left(\frac{-u_t^*(x)Z_t}{\sqrt{t}/\vartheta(t)} \right) (e^{Z_t\vartheta(t)} - 1)^+ \right].
\end{aligned}$$

Case $\omega = 2$. The proof is identical to [70, Theorem 3.1] and is therefore omitted.

Case $\omega = 1$. The Fourier transform of the modified payoff $\exp \left(-\frac{u_t^*(x)Z_t}{\sqrt{t}} \right) (e^{Z_t} - 1)^+$

under $\mathbb{Q}_t$ is

$$\begin{aligned}
& \int_0^\infty \exp\left(-\frac{u_t^*(x)z}{\sqrt{t}}\right) (e^z - 1) e^{iuz} dz \\
&= \left[\frac{e^{(1+iu-u_t^*(x)t^{-1/2})z}}{1+iu-u_t^*(x)t^{-1/2}} \right]_0^\infty - \left[\frac{e^{(iu-u_t^*(x)t^{-1/2})z}}{iu-u_t^*(x)t^{-1/2}} \right]_0^\infty \\
&= \frac{1}{iu-u_t^*(x)t^{-1/2}} - \frac{1}{1+iu-u_t^*(x)t^{-1/2}} \\
&= \frac{t}{(u_t^*(x) - (1+iu)\sqrt{t})(u_t^*(x) - iu\sqrt{t})},
\end{aligned}$$

where in the second line we use the fact that $\lim_{t \downarrow 0} u_t^*(x)t^{-1/2} = +\infty$. Recall that the Gamma distribution with shape $|\gamma_0|$ and scale $|\frac{x}{\gamma_0}|$ has density $f_\Gamma \in L^2(\mathbb{R})$ given by

$$f_\Gamma(y) = \frac{y^{|\gamma_0|-1}}{\Gamma(|\gamma_0|)} \exp\left(-\left|\frac{\gamma_0}{x}\right|y\right) \left(\left|\frac{\gamma_0}{x}\right|\right)^{|\gamma_0|}, \quad \text{for } y > 0. \quad (1.30)$$

Applying [55, Theorem 13.E] and Lemma 42,

$$\begin{aligned}
& \mathbb{E}^{\mathbb{Q}_t} \left[\exp\left(\frac{-u_t^*(x)Z_t}{\sqrt{t}}\right) (e^{Z_t} - 1)^+ \right] \\
&= \frac{t}{2\pi} \int_{-\infty}^\infty \frac{\Psi_t(u) du}{(u_t^*(x) - (1-iu)\sqrt{t})(u_t^*(x) + iu\sqrt{t})} \\
&= \frac{t}{4\pi\mathbf{m}} \int_{-\infty}^\infty e^{-iux} \left(1 - \frac{iux}{|\gamma_0|}\right)^{\gamma_0} (1 + o(1)) du \\
&= \frac{tf_\Gamma(x)}{2\mathbf{m}} (1 + o(1)), \quad (1.31)
\end{aligned}$$

where the last line follows from Fourier inversion. Combining Lemma 41 and (1.31), the Call price reads

$$\begin{aligned}
\mathbb{E}(e^{X_t} - e^x)^+ &= \exp\left(-\frac{\Lambda^*(x)}{\sqrt{t}} + x + c_2(x) + \gamma_1\right) \left(\frac{x}{|\gamma_0|\sqrt{2\mathbf{m}}}\right)^{|\gamma_0|} \\
&\quad \frac{f_\Gamma(x)}{2\mathbf{m}} t^{1-\frac{|\gamma_0|}{2}} (1 + o(1)), \quad \text{for } x > 0.
\end{aligned}$$

Assume now that $x < 0$, the price of a European Put option with strike e^x is

$$\mathbb{E} (e^x - e^{X_t})^+ = \exp \left(\frac{-xu_t^*(x) + \tilde{\Lambda}_t(u_t^*(x))}{\sqrt{t}} \right) e^x \mathbb{E}^{\mathbb{Q}_t} \left[\exp \left(\frac{-u_t^*(x)Z_t}{\sqrt{t}} \right) (1 - e^{Z_t})^+ \right],$$

and the Fourier transform of the modified payoff function $\exp \left(\frac{-u_t^*(x)Z_t}{\sqrt{t}} \right) (1 - e^{Z_t})^+$ is

$$\int_{-\infty}^0 \exp \left(-\frac{u_t^*(x)z}{\sqrt{t}} \right) (1 - e^z) e^{iuz} dz = \frac{t}{(u_t^*(x) - (1 + iu)\sqrt{t})(u_t^*(x) - iu\sqrt{t})}.$$

Following a similar procedure, and noticing that $(e^{X_t})_{t \geq 0}$ is a $\mathbb{Q}$ -martingale, the Put-Call parity implies

$$\begin{aligned} \mathbb{E} (e^{X_t} - e^x)^+ &= (1 - e^x) + \exp \left(-\frac{\Lambda^*(x)}{\sqrt{t}} + x + c_2(x) + \gamma_1 \right) \left(\frac{|x|}{|\gamma_0|\sqrt{2\mathbf{m}}} \right)^{|\gamma_0|} \\ &\quad \frac{f_\Gamma(|x|)}{2\mathbf{m}} t^{1-\frac{|\gamma_0|}{2}} (1 + o(1)), \quad \text{for } x < 0. \end{aligned}$$

In the standard Black-Scholes model with volatility $\Sigma > 0$, the short-time asymptotics of the Call option price reads [44, Corollary 3.5] $\mathbb{E} (e^{X_t} - e^x)^+ = (1 - e^x)^+ + \frac{1}{\sqrt{2\pi x^2}} \exp \left(-\frac{x^2}{2\Sigma^2 t} + \frac{x}{2} \right) (\Sigma^2 t)^{3/2} (1 + \mathcal{O}(t))$. Then the asymptotics of implied volatility can be derived following the systematic approach provided in [50, Corollary 7.1].

1.9.3 PROOF OF THEOREM 26

We first prove the large deviations statement, which we then translate into the large-maturity behaviour of the implied volatility. Andersen and Piterbarg [5, Proposition 3.1] analysed moment explosions in the standard Heston model, and proved that for any $u > 1$, the quantity $\mathbb{E}(e^{uX_t})$ always exists as long as

$$\kappa > \rho\xi u \quad \text{and} \quad d(u) \geq 0. \quad (1.32)$$

Moreover, the assumption $\kappa > \rho\xi$ implies (see [43]) that (1.32) holds for any $u \in [\bar{u}_-, \bar{u}_+]$, so that $\mathbb{E}(e^{uX_t})$ is well-defined for $u \in [\bar{u}_-, \bar{u}_+]$ and any

(large) t in the standard Heston model. The tower property then yields

$$M(t, u) = \mathbb{E} [\mathbb{E}(e^{uX_t} | \mathcal{V})] = C(t, u) (M_{\mathcal{V}} \circ D(t, u)).$$

Consequently, for any large t , $M(t, u)$ is well-defined for $u \in \mathcal{S} := [\bar{u}_-, \bar{u}_+] \cap \mathcal{S}_{\mathcal{V}}$, where the set $\mathcal{S}_{\mathcal{V}}$ is defined by

$$\mathcal{S}_{\mathcal{V}} := \bigcup_{t>0} \bigcap_{s \geq t} \{u : D(s, u) < \mathfrak{m}\}.$$

Using the expressions of functions C and D in (B.1), the rescaled cgf of the process $(t^{-1}X_t)_{t \geq 0}$ reads

$$\begin{aligned} \Xi(t, u) &:= \frac{1}{t} \log \mathbb{E}(e^{uX_t}) = \frac{1}{t} C(t, u) + \frac{1}{t} \log M_{\mathcal{V}}(D(t, u)) \\ &= \frac{\kappa\theta(\kappa - \rho\xi u - d(u))}{\xi^2} - \frac{2\kappa\theta}{\xi^2 t} \log \left(\frac{1 - g(u)e^{-d(u)t}}{1 - g(u)} \right) \\ &\quad + \frac{1}{t} \log M_{\mathcal{V}} \left(\frac{\kappa - \rho\xi u - d(u)}{\xi^2} \frac{1 - e^{-d(u)t}}{1 - g(u)e^{-d(u)t}} \right). \end{aligned} \quad (1.33)$$

For any $u \in (\bar{u}_-, \bar{u}_+)$, since the quantity $d(u)$ is strictly positive, then

$$\begin{aligned} \lim_{t \uparrow \infty} \frac{1}{t} \log \left(\frac{1 - g(u)e^{-d(u)t}}{1 - g(u)} \right) &= 0, \quad \text{and} \\ \lim_{t \uparrow \infty} \frac{\kappa - \rho\xi u - d(u)}{\xi^2} \frac{1 - e^{-d(u)t}}{1 - g(u)e^{-d(u)t}} &= \frac{\kappa - \rho\xi u - d(u)}{\xi^2}. \end{aligned} \quad (1.34)$$

Since $u \mapsto \Xi(t, u)$ is continuous for each $t > 0$, L'Hôpital's rule implies that $\lim_{t \uparrow \infty} \Xi(t, \bar{u}_{\pm}) = \kappa\theta(\kappa - \rho\xi\bar{u}_{\pm})/\xi^2$.

Case $\mathfrak{m} = \infty$. In this case $\mathcal{S}_{\mathcal{V}} = \mathbb{R}$, implying that $\mathcal{S} = [\bar{u}_-, \bar{u}_+]$. Equation (1.33) shows that

$$\Xi(u) := \lim_{t \uparrow \infty} \Xi(t, u) \equiv \mathfrak{L}(u), \quad \text{for any } u \in \mathbb{R},$$

with $\mathfrak{L}$ provided in (1.17). In [43, Theorem 2.1], it is proved that the limiting function Ξ and its effective domain $\mathcal{S}$ satisfy all the assumptions of the Gärtner-Ellis theorem (Theorem 61), and hence the large deviations principle

for the sequence $(t^{-1}X_t)_{t \geq 0}$ follows.

Case $\mathfrak{m} < \infty$. Equation (1.34) implies that

$$\left\{ u : \frac{\kappa - \rho\xi u - d(u)}{\xi^2} < \mathfrak{m} \right\} \subset \mathcal{S}_V \subset \left\{ u : \frac{\kappa - \rho\xi u - d(u)}{\xi^2} \leq \mathfrak{m} \right\}.$$

As a result, the essential smoothness of function Ξ is guaranteed if

$$\begin{aligned} [\bar{u}_-, \bar{u}_+] &\subset \left\{ u : \frac{\kappa - \rho\xi u - d(u)}{\xi^2} < \mathfrak{m} \right\} \\ &= \left\{ u : \kappa - \rho\xi u < \xi^2 \mathfrak{m} + \sqrt{(\kappa - \rho\xi u)^2 + \xi^2 u(1-u)} \right\}. \end{aligned}$$

Since $\kappa - \rho\xi u > 0$ holds for any $u \in [\bar{u}_-, \bar{u}_+]$,

$$\begin{aligned} \kappa - \rho\xi u &< \xi^2 \mathfrak{m} + \sqrt{(\kappa - \rho\xi u)^2 + \xi^2 u(1-u)} \\ \iff 0 &< u(1-u) + \xi^2 \mathfrak{m}^2 + 2\mathfrak{m} \sqrt{(\kappa - \rho\xi u)^2 + \xi^2 u(1-u)} \\ \iff \frac{u(u-1)}{\xi^2} &< \mathfrak{m}^2 + \frac{2\mathfrak{m}}{\xi^2} \sqrt{(\kappa - \rho\xi u)^2 + \xi^2 u(1-u)}. \end{aligned} \quad (1.35)$$

Since $[0, 1] \subset (\bar{u}_-, \bar{u}_+)$, Condition (1.35) holds for any $u \in [0, 1]$. Whenever $u > 1$ or $u < 0$, as functions of u , the left-hand-side is strictly increasing while the right-hand-side is strictly decreasing. Therefore, (1.35) holds for any $u \in [\bar{u}_-, \bar{u}_+]$ if and only if $\max\{\bar{u}_-(\bar{u}_- - 1), \bar{u}_+(\bar{u}_+ - 1)\} < \mathfrak{m}^2 \xi^2$. Consequently, Assumption 24 ensures that $\mathcal{S} = [\bar{u}_-, \bar{u}_+]$, and the proof follows from the Gärtner-Ellis theorem (Theorem 61).

We now prove the asymptotic behaviour for the implied volatility. We claim that in a randomised Heston setting the European option price has the following limiting behaviour:

$$\begin{aligned} -\lim_{t \uparrow \infty} \frac{1}{t} \log \left(\mathbb{E} \left(e^{X_t} - e^{xt} \right)^+ \right) &= \mathfrak{L}^*(x) - x, & \text{for } x \geq \frac{\bar{\theta}}{2}, \\ -\lim_{t \uparrow \infty} \frac{1}{t} \log \left(1 - \mathbb{E} \left(e^{X_t} - e^{xt} \right)^+ \right) &= \mathfrak{L}^*(x) - x, & \text{for } -\frac{\theta}{2} \leq x \leq \frac{\bar{\theta}}{2}, \\ -\lim_{t \uparrow \infty} \frac{1}{t} \log \left(\mathbb{E} \left(e^{xt} - e^{X_t} \right)^+ \right) &= \mathfrak{L}^*(x) - x, & \text{for } x \leq -\frac{\theta}{2}. \end{aligned}$$

The proof is covered in detail in [71, Section 5.2.2], and we therefore highlight the main ideas for completeness. From Theorem 26, define a time-dependent probability measure $\mathbb{Q}_t$:

$$\frac{d\mathbb{Q}_t}{d\mathbb{Q}} := \exp \{ u^*(x) X_t - \Xi(t, u^*(x)) t \},$$

where $u^*(x)$ is the solution to the equation $x = \Xi'(u)$. The option price is then expressed as the expectation under $\mathbb{Q}_t$ of a modified payoff, and can be computed by (inverse) Fourier transform with the main contribution equal to $\exp \{ -(\mathfrak{L}^*(x) - x) t \}$. It is also known (see [42, Corollary 2.12] for instance) that in the Black-Scholes model with volatility Σ , the asymptotics of European option prices with strike e^{xt} are given by

$$\begin{aligned} -\lim_{t \uparrow \infty} \frac{1}{t} \log \left(\mathbb{E} (e^{X_t} - e^{xt})^+ \right) &= \Lambda_{\text{BS}}^*(x, \Sigma) - x, & \text{for } x \geq \frac{\Sigma^2}{2}, \\ -\lim_{t \uparrow \infty} \frac{1}{t} \log \left(1 - \mathbb{E} (e^{X_t} - e^{xt})^+ \right) &= \Lambda_{\text{BS}}^*(x, \Sigma) - x, & \text{for } -\frac{\Sigma^2}{2} \leq x \leq \frac{\Sigma^2}{2}, \\ -\lim_{t \uparrow \infty} \frac{1}{t} \log \left(\mathbb{E} (e^{xt} - e^{X_t})^+ \right) &= \Lambda_{\text{BS}}^*(x, \Sigma) - x, & \text{for } x \leq -\frac{\Sigma^2}{2}, \end{aligned}$$

where $\Lambda_{\text{BS}}^*(x, \Sigma) := \frac{(x + \Sigma^2/2)^2}{2\Sigma^2}$. Then the leading order of the large-time implied variance is obtained by solving

$$\mathfrak{L}^*(x) - x = \Lambda_{\text{BS}}^*(x, \Sigma) - x = \frac{(-x + \Sigma^2/2)^2}{2\Sigma^2}.$$

We omit the details of the proof which can be found in [43, 45].

2

FRACTIONAL HESTON: ROUGH VOLATILITY

In this chapter we discuss the asymptotic behaviour of a fractional version of the Heston model. * Comte, Coutin and Renault [27] firstly proposed this model and analysed the ‘smooth’ case where the Hurst parameter H is greater than $1/2$. Here we focus on the ‘rough’ case with $H \in (0, 1/2)$, and we show that the small-time implied variance has an explosion rate of $t^{H-1/2}$.

2.1 INTRODUCTION

An increasing amount of literature on volatility modelling has been focusing on the rough volatility environment. In this section we briefly review some recent development. Firstly, it is worth noting that there are other ways to include rough volatility in a Heston setting: a different fractional Heston model was proposed and analysed in [38, 39, 40], in which the authors defined a dif-

*This chapter is based on the submitted paper ‘Asymptotic behaviour of the fractional Heston model’ [57], in which I contribute to the numerics and a complete proof of Proposition 46. These results are included in this chapter.

ferent structure for the variance process through the fractional integration. In particular, El Euch, Fukasawa and Rosenbaum [40] bridge the connection between market microstructure and rough volatility by proposing a microscopic price model and by showing that it converges to a rough Heston setting in the long term. Another rough volatility setting, i.e. the rough Bergomi model, has been extensively analysed: Bayer, Friz and Gatheral [10] argued that the model is capable of providing a much improved calibration accuracy compared with standard stochastic volatility models. Jacquier, Pakkanen and Stone [73] proved the large deviations principle for the underlying process in the rough Bergomi model, and provided the small-time asymptotic behaviour of the implied volatility in this setting. McCrickerd and Pakkanen [83] proposed a novel simulation method which significantly reduces the computational cost of log-normal rough volatility models including the rough Bergomi. Finally, we comment that in [65] the authors combined the ideas of initial randomisation and rough volatility by proposing and analysing a general randomised fractional volatility setting. For other recent development we refer interested readers to [2, 9, 11, 16, 46, 69, 81].

2.2 MODEL SETUP

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$ be a given filtered probability space supporting two independent standard Brownian motions B and W . We denote by $(S_t)_{t \geq 0}$ the stock price process, and let $X_t := \log(S_t)$ be the unique strong solution to the stochastic differential equation

$$\begin{aligned} dX_t &= -\frac{1}{2}V_t^d dt + \sqrt{V_t^d} dB_t, & X_0 &= 0, \\ dV_t &= \kappa(\theta - V_t)dt + \xi\sqrt{V_t}dW_t, & V_0 &= v_0 > 0, \\ V_t^d &:= \eta + I_{0+}^d V_t, \end{aligned} \tag{2.1}$$

where $d \in (-1/2, 1/2)$ and the coefficients satisfy $\kappa, \theta, \xi > 0$. The operator I_{0+}^d is the classical left fractional Riemann-Liouville integral of order d :

$$I_{0+}^d \phi(t) := \begin{cases} \int_0^t \frac{(t-s)^{d-1}}{\Gamma(d)} \phi(s) ds, & \text{for } d \in (0, 1/2), \\ \frac{d}{dt} I_{0+}^{d+1} \phi(t), & \text{for } d \in (-1/2, 0), \end{cases}$$

valid for any function $\phi \in L^1([0, t])$, where Γ is the standard Gamma function. If the function ϕ is $1/2$ -Hölder continuous, which is the case for the path of the CIR process V in Equation (2.1), then [67, Proposition 1.2] states that V^d is $(1/2 + d)$ -Hölder continuous. For more details on these integrals, we refer the interested reader to the monograph [91, Chapter 1, Section 2], and for the application to discretisation schemes we refer to [27, Section 5].

Finally, notice that the fair strike of a continuously monitored variance swap with maturity T reads

$$\begin{aligned} \frac{1}{T} \mathbb{E} \left(\int_0^T V_t^d dt \right) &= \frac{1}{T} \int_0^T \{ \eta + I_{0+}^d [v_0 e^{-\kappa t} + \theta(1 - e^{-\kappa t})] \} dt \\ &= \eta + \frac{v_0 T^d}{\Gamma(d+2)} + \frac{\kappa(\theta - v_0) e^{-\kappa T}}{T \Gamma(d+2)} \int_0^T t^{d+1} e^{\kappa t} dt. \end{aligned} \quad (2.2)$$

Note in particular that, when $\eta = d = 0$, this representation coincides with the expected variance formula in the standard Heston case, provided in [52, Chapter 11]. Furthermore, for small $T > 0$, Equation (2.2) reads

$$\frac{1}{T} \mathbb{E} \left(\int_0^T V_t^d dt \right) = \frac{v_0 T^d}{\Gamma(d+2)} + \eta + \mathcal{O}(T^{d+1}),$$

so that in the rough Heston case ($d < 0$) the variance swap rate explodes with a rate of T^d .

2.3 MAIN RESULTS

In this section we summarise the main results. Recall that the mgf of X_t is denoted as $M(t, u) := \mathbb{E}(e^{uX_t})$, for $t \geq 0$, and we define the effective domain

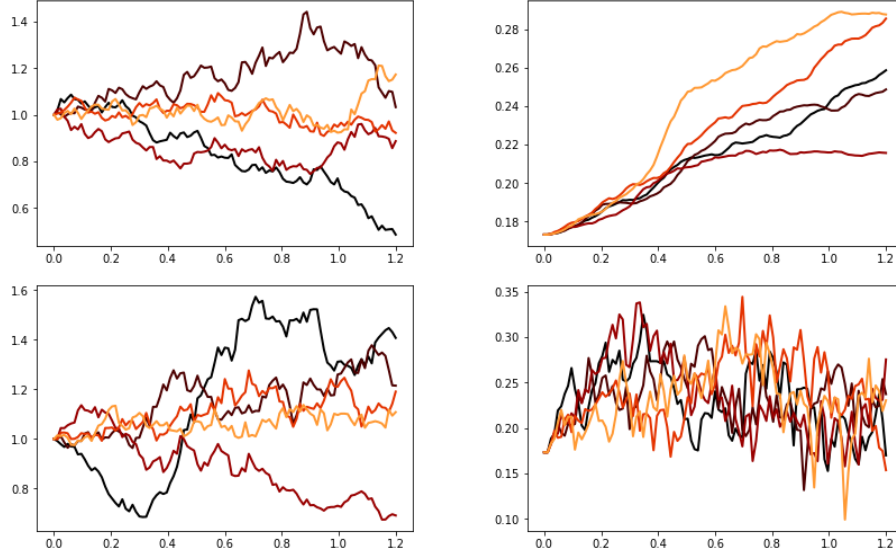


Figure 2.1: We simulate five smooth ($d = 0.4$, above) and rough ($d = -0.3$, bottom) sample paths for the stock prices (left) and their volatility movements (right). Other parameters for the variance process are $(\kappa, \theta, \xi, v_0, \eta) = (2.1, 0.05, .2, 0, 0.03)$.

of the (rescaled) mgf as $\mathcal{D}_t := \left\{ u \in \mathbb{R} : M\left(t, \frac{u}{h(t)}\right) < \infty \right\}$. The following theorem shows that (at least in the symmetric case) the mgf remains affine in the fractional Heston case.

Theorem 45. *For any $t \geq 0$,*

$$\log M(t, u) = \frac{u(u-1)\eta t}{2} - B(t, u)v_0 + A(t, u), \quad \text{for all } u \in \mathcal{D}_t, \quad (2.3)$$

where the functions A and B satisfy the ordinary differential equations

$$\begin{aligned} \partial_s A(s, u) + \kappa \theta B(s, u) &= 0, \\ \partial_s B(s, u) + \kappa B(s, u) + \frac{\xi^2}{2} B(s, u)^2 + \frac{u(u-1)}{2\Gamma(d+1)} s^d &= 0, \end{aligned} \quad (2.4)$$

for $0 \leq s \leq t$, with the initial conditions $A(0) = B(0) = 0$.

Proof. The proof is an application of the Feynman-Kac formula and the details can be found in [57]. $\square$

Introduce functions $\lambda_{\pm}^* : \mathbb{R} \rightarrow \mathbb{R}_+$,

$$\lambda_+^*(x) := \frac{x^2}{2\eta}, \quad \lambda_-^*(x) := \frac{\Gamma(2+d)x^2}{2v_0}, \quad (2.5)$$

Proposition 46. *The following hold as t tends to zero:*

- (i) *if $d \in (0, 1/2)$, let $h(t) = t$, then $\lim_{t \downarrow 0} \mathcal{D}_t = \mathbb{R}$ and $\lim_{t \downarrow 0} t \log M(t, u/t) = \frac{\eta u^2}{2}$, for any $u \in \mathbb{R}$;*
- (ii) *if $d \in (-1/2, 0)$, let $h(t) = t^{1+d}$, then $\lim_{t \downarrow 0} \mathcal{D}_t = \mathbb{R}$ and $\lim_{t \downarrow 0} t^{1+d} \log M\left(t, \frac{u}{t^{1+d}}\right) = \frac{v_0 u^2}{2\Gamma(2+d)}$, for any $u \in \mathbb{R}$.*

Theorem 47. *As t tends to zero,*

- (a) *if $d \in (0, 1/2)$, then $X \sim \text{LDP}_0(t, \lambda_+^*)$. Moreover, for any $x \neq 0$, $\sigma_t^2(x)$ converges to η ;*
- (b) *if $d \in (-1/2, 0)$, then $X \sim \text{LDP}_0(t^{1+d}, \lambda_-^*)$. As a result, for any $x \neq 0$, $t^{-d} \sigma_t^2(x)$ converges to $v_0/\Gamma(d+2)$.*

Proof. The proof of the large deviations principle is nothing more than applying the Gärtner-Ellis theorem (Theorem 61) and Proposition 46. Translating this into the small-time asymptotic behaviour of the implied volatility is standard in the literature (see [50, Corollary 7.1] for example), hence we omit the details. $\square$

2.4 NUMERICS

In this section we provide numerical examples describing the behaviour of our model, with parameters $(\kappa, \theta, \xi) = (2.1, 0.06, 0.2)$. In Figure 2.3 we provide a comparison of volatility surfaces generated by fractional and standard Heston models. We observe that in the case where the Hurst parameter is less than $1/2$ ($d < 0$), a larger value of v_0 pushes up the small-time volatility

smile, resonating our small-time analysis in Theorem 47. Finally, the term structure of the expected annualised variance is given in Figure 2.4.

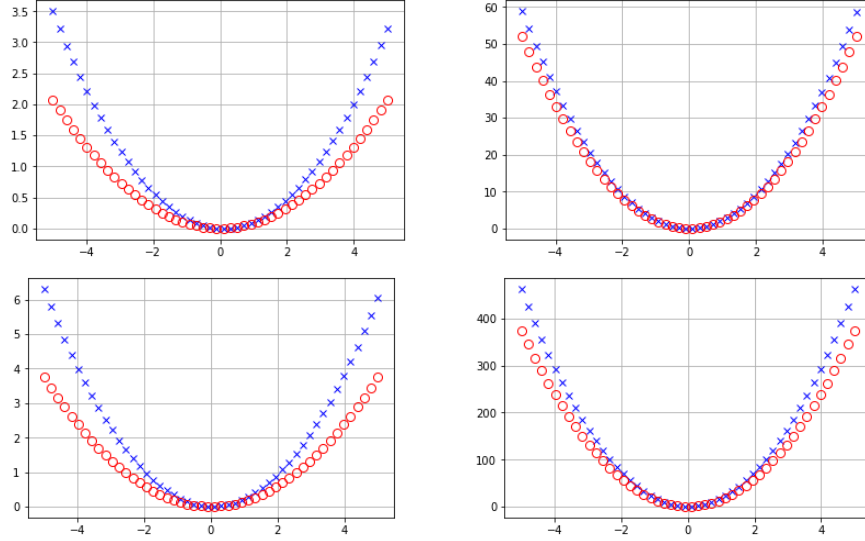


Figure 2.2: The rescaled cgf of X_t (blue) and its small-time limits (red) given in Proposition 46. The time parameter is $t = 1/10$ on the left column, and $t = 10^{-3}$ on the right. The first row is the rough case where $d = -0.3$, and we have $d = 0.2$ at the bottom. $(\kappa, \theta, \xi, v_0, \eta) = (2.1, 0.06, 0.2, 0.03, 0.03)$.

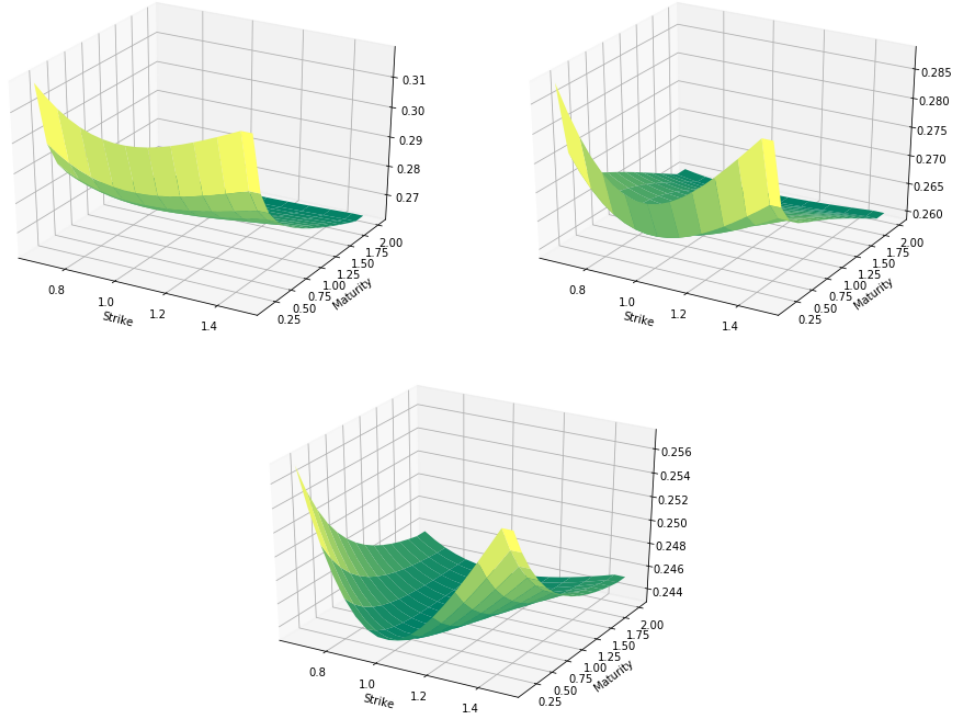


Figure 2.3: Comparison of volatility surfaces. The first row represents the fractional case with $(v_0, \eta, d) = (0.03, 0.02, -0.3)$ on the left and $(v_0, \eta, d) = (0.02, 0.02, -0.3)$ on the right. The bottom plot is the standard Heston case ($d = 0$) with $v_0 = 0.06$.

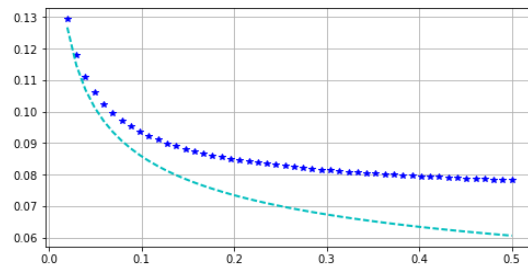


Figure 2.4: Term structure of the expected annualised variance (blue) in (2.2) where $(v_0, \eta, d) = (0.03, 0.02, -0.3)$. Cyan dashed lines represent its small-time approximation $\left(\frac{v_0 T^d}{\Gamma(d+2)} + \eta \right)$.

2.5 PROOF OF PROPOSITION 46

We separate the case for $\kappa = 0$ and $\kappa > 0$.

Case $\kappa = 0$ This special case was proved by Guennoun, Jacquier and Roome [57]. We sketch the essence of the proof here for completeness. The function B as the solution of the ODE

$$B'(t) = -\frac{\xi^2}{2}B^2(t) + \frac{u(1-u)}{2\Gamma(d+1)}t^d, \quad B(0) = 0$$

can be written as $B(t) = \sum_{i \geq 1} \alpha_i \zeta^i t^{id+2i-1}$, where $\zeta := \frac{u(1-u)}{2\Gamma(d+2)}$. The sequence of real numbers $\{\alpha_i\}_{i \geq 1}$ is determined by matching orders of t , and is given by $\alpha_1 = 1$, $\alpha_i = \frac{-\xi^2}{2(i(d+2)-1)} \sum_{k=1}^{i-1} \alpha_k \alpha_{i-k}$, for $i \geq 2$. The authors proved that the infinite sum $\sum_{i \geq 1} \alpha_i x^i$ has a radius of convergence greater than $1/4$, so that the representation of B is well-defined for small $t > 0$.

As a result, in the case where $d \in (0, 1/2)$ the function B satisfies

$$\begin{aligned} t^{1-d}B\left(\frac{u}{t}, t\right) &= t^{-d} \sum_{i \geq 1} \alpha_i \left(\frac{ut^{d+1}}{2\Gamma(d+2)} - \frac{u^2 t^d}{2\Gamma(d+2)} \right)^i \\ &= t^{-d} \sum_{i \geq 1} \alpha_i t^{id} \left(\frac{ut}{2\Gamma(d+2)} - \frac{u^2}{2\Gamma(d+2)} \right)^i \\ &= \frac{ut}{2\Gamma(d+2)} - \frac{u^2}{2\Gamma(d+2)} + \sum_{i \geq 1} \alpha_{i+1} t^{id} \left(\frac{ut}{2\Gamma(d+2)} - \frac{u^2}{2\Gamma(d+2)} \right)^{i+1}, \end{aligned}$$

and the sum of higher order terms converges to zero as t tends to zero, so $t^{1-d}B(u/t, t)$ must converge to $-\frac{u^2}{2\Gamma(d+2)}$. Then the small-time limit follows directly from Equation (2.3).

In the case where $d \in (-1/2, 0)$, the function B satisfies

$$\begin{aligned} t^{1+d}B(u/t^{1+d}, t) &= t^d \sum_{i \geq 1} \alpha_i \left(\frac{ut}{2\Gamma(d+2)} - \frac{u^2 t^{-d}}{2\Gamma(d+2)} \right)^i \\ &= \left(\frac{ut^{1+d}}{2\Gamma(d+2)} - \frac{u^2}{2\Gamma(d+2)} \right) \sum_{i \geq 1} \alpha_{i+1} \left(\frac{ut}{2\Gamma(d+2)} - \frac{u^2 t^{-d}}{2\Gamma(d+2)} \right)^i \\ &\quad + \frac{ut^{1+d}}{2\Gamma(d+2)} - \frac{u^2}{2\Gamma(d+2)}. \end{aligned}$$

Applying the same approach as in the previous case we can show that the first term above tends to zero as t tends to zero, so $t^{1+d}B(u/t^{1+d}, t)$ converges to $-\frac{u^2}{2\Gamma(d+2)}$, which proves the proposition.

Case $\kappa > 0$ Plug $B(t) := \sum_{i,j \geq 1} \beta_{i,j} t^{id+j}$ into (2.4) yields

$$\begin{aligned} &\left(\beta_{1,1}(d+1) + \frac{u(u-1)}{2\Gamma(d+1)} \right) t^d + \frac{\xi^2}{2} \left(\sum_{i,j \geq 1} \beta_{i,j} t^{id+j} \right)^2 + \kappa \left(\sum_{i,j \geq 1} \beta_{i,j} t^{id+j} \right) \\ &+ \sum_{(i,j) \neq (1,1)} \beta_{i,j} (id+j) t^{id+j-1} \equiv 0. \end{aligned}$$

The following statements hold, and are related to the case $\kappa = 0$ above:

1. $\beta_{i,j} = 0$ for any $i \geq 2$ and for $1 \leq j \leq 2i - 2$;
2. $\beta_{i,2i-1} = \alpha_i \zeta^i$ for any $i \geq 1$.

Combining terms of the same order yields $\beta_{1,1} = \alpha_1 \zeta$. Moreover, for any $(i, j) \in \mathbb{N}_+^* \times (\mathbb{N}_+^* \setminus \{1\})$,

$$\left(\beta_{i,j}(id+j) + \kappa \beta_{i,j-1} + \frac{\xi^2}{2} \left(\sum_{p=1}^{i-1} \sum_{q=1}^{j-2} \beta_{p,q} \beta_{i-p,j-1-q} \right) \right) t^{id+j-1} \equiv 0.$$

Then the two statements above can be easily verified by induction, from

which $B(t) = \sum_{i \geq 1} \sum_{j \geq 2i-1} \beta_{i,j} t^{id+j}$, with

$$\beta_{i,j} = \left[-\kappa \beta_{i,j-1} - \frac{\xi^2}{2} \left(\sum_{p=1}^{i-1} \sum_{q=2p-1}^{j-2i+2p} \beta_{p,q} \beta_{i-p,j-1-q} \right) \right] \frac{1}{id+j}. \quad (2.6)$$

We show that the series is absolutely convergent. Notice that

$$\begin{aligned} \sum_{i \geq 1} \sum_{j \geq 2i-1} |\beta_{i,j}| t^{id+j} &= \sum_{i \geq 1} t^{id+2i-1} \left(\sum_{j \geq 2i-1} |\beta_{i,j}| t^{j-2i+1} \right) \\ &= \sum_{i \geq 1} |\beta_{i,2i-1}| t^{id+2i-1} \left(1 + \sum_{j=2i}^{\infty} \left| \frac{\beta_{i,j}}{\beta_{i,2i-1}} \right| t^{j-2i+1} \right). \end{aligned} \quad (2.7)$$

Following Lemma 48, $1 + \sum_{j=2i}^{\infty} \left| \frac{\beta_{i,j}}{\beta_{i,2i-1}} \right| t^{j-2i+1} < 2^{i-1} e^{i\kappa t}$ holds for any $i \geq 1$, and (2.7) then reads

$$\sum_{i \geq 1} \sum_{j \geq 2i-1} |\beta_{i,j}| t^{id+j} < t^{-1} \sum_{i \geq 1} 2^{i-1} |\alpha_i| (|\zeta| t^{d+2} e^{\kappa t})^i. \quad (2.8)$$

Since the infinite sum $\sum_{i \geq 1} \alpha_i x^i$ has a radius of convergence no less than $1/4$, then the series $\sum_{i \geq 1} 2^{i-1} |\alpha_i| x^i$ has a radius of convergence no less than $1/8$. Notice that $(d+2)$ is strictly positive, then (2.8) implies that the series $B(t) = \sum_{i \geq 1} \sum_{j \geq 2i-1} \beta_{i,j} t^{id+j}$ is absolutely convergent for small t such that $|\zeta| t^{d+2} e^{\kappa t} < 1/8$ holds. Therefore the previous ansatz $B(t)$ is the solution to the Riccati equation for small t . The rest of the proof is essentially the same as the case where $\kappa = 0$, and we therefore omit the details.

Lemma 48. *For any $i \geq 1$, the following estimation for $\beta_{i,j}$ defined in (2.6) holds for any $j \geq 2i$:*

$$|\beta_{i,j}| < \frac{2^{i-1} (i\kappa)^{j-2i+1}}{(j-2i+1)!} |\beta_{i,2i-1}|.$$

Proof. We prove by induction. In the case where $i = 1$, direct computations yield

$$\left| \frac{\beta_{1,j}}{\beta_{1,1}} \right| = \frac{\Gamma(d+2)}{\Gamma(d+1+j)} \kappa^{j-1} \leq \frac{\kappa^{j-1}}{(j-1)!}, \quad \text{for any } j \geq 1.$$

Assume that the upper bound holds for $|\beta_{p,q}|$ for any $p \leq i-1$. Then we have

$$\begin{aligned} |\beta_{p,q}| &\leq \frac{2^{p-1}(p\kappa)^{q-2p+1}|\beta_{p,2p-1}|}{(q-2p+1)!}, \\ |\beta_{i-p,j-1-q}| &\leq \frac{2^{i-p-1}((i-p)\kappa)^{j-q-2i+2p}|\beta_{i-p,2(i-p)-1}|}{(j-q-2i+2p)!}. \end{aligned}$$

As a result, for any fixed $1 \leq p \leq i-1$,

$$\begin{aligned} &\sum_{q=2p-1}^{j-2i+2p} |\beta_{p,q}\beta_{i-p,j-1-q}| \\ &\leq \sum_{q=2p-1}^{j-2i+2p} \frac{2^{i-2}(p\kappa)^{q-2p+1}((i-p)\kappa)^{j-q-2i+2p}|\beta_{p,2p-1}\beta_{i-p,2(i-p)-1}|}{(q-2p+1)!(j-q-2i+2p)!} \\ &= \frac{2^{i-2}|\beta_{p,2p-1}\beta_{i-p,2(i-p)-1}|}{(j-2i+1)!} \sum_{q=2p-1}^{j-2i+2p} \frac{(j-2i+1)!(p\kappa)^{q-2p+1}((i-p)\kappa)^{j-q-2i+2p}}{(q-2p+1)!(j-q-2i+2p)!} \\ &= \frac{2^{i-2}(i\kappa)^{j-2i+1}}{(j-2i+1)!} |\beta_{p,2p-1}\beta_{i-p,2(i-p)-1}|. \end{aligned}$$

Plug it into (2.6), and notice that $|\beta_{i,2i-1}| = \frac{\xi^2}{2(id+2i-1)} \sum_{k=1}^{i-1} |\beta_{k,2k-1}\beta_{i-k,2(i-k)-1}|$ which follows from the fact that all the terms $\beta_{k,2k-1}\beta_{i-k,2(i-k)-1}$ have the same sign for any k , then

$$\begin{aligned} |\beta_{i,j}| &\leq \frac{\kappa|\beta_{i,j-1}|}{id+j} + \frac{\xi^2}{2(id+j)} \sum_{p=1}^{i-1} \sum_{q=2p-1}^{j-2i+2p} |\beta_{p,q}\beta_{i-p,j-1-q}| \\ &\leq \frac{\kappa|\beta_{i,j-1}|}{id+j} + \frac{\xi^2(id+2i-1)2^{i-2}(i\kappa)^{j-2i+1}}{2(id+j)(id+2i-1)(j-2i+1)!} \sum_{p=1}^{i-1} |\beta_{p,2p-1}\beta_{i-p,2(i-p)-1}| \\ &\leq \frac{\kappa|\beta_{i,j-1}|}{id+j} + \frac{2^{i-2}(i\kappa)^{j-2i+1}(id+2i-1)}{(j-2i+1)!(id+j)} |\beta_{i,2i-1}|. \end{aligned}$$

Apply the inequality iteratively, we finally obtain

$$\begin{aligned}
|\beta_{i,j}| &\leq \frac{\kappa^{j-2i+1}|\beta_{i,2i-1}|}{\prod_{k=2i}^j(id+k)} + 2^{i-2}\kappa^{j-2i+1}|\beta_{i,2i-1}| \sum_{k=2i}^j \frac{i^{k-2i+1}(id+2i-1)}{(k-2i+1)!\prod_{m=k}^j(id+m)} \\
&\leq \frac{2^{i-2}\kappa^{j-2i+1}|\beta_{i,2i-1}|}{(j-2i+1)!} \sum_{k=2i-1}^j i^{k-2i+1} \\
&= \frac{2^{i-2}\kappa^{j-2i+1}|\beta_{i,2i-1}|(i^{j-2i+2}-1)}{(j-2i+1)!(i-1)} \\
&< \frac{2^{i-2}(i\kappa)^{j-2i+1}|\beta_{i,2i-1}|i}{(j-2i+1)!(i-1)} \\
&\leq \frac{2^{i-1}(i\kappa)^{j-2i+1}}{(j-2i+1)!} |\beta_{i,2i-1}|,
\end{aligned}$$

where the second line follows from the trivial estimation $i(d+2) \geq 1$, for any $d \in (-1/2, 1/2)$. $\square$

3

CONUS-WILDMAN: PAST-DEPENDENT PRICE PROCESS

In this chapter we investigate a new type of stochastic volatility model in which the driving process of the underlier is a modified Dobrić-Ojeda process instead of a standard Brownian motion. The new driving process allows us to model the past dependency of the stock. We focus on the case where the variance is described by a CIR process, and we prove the existence of a risk-neutral pricing measure in this setting. We also provide an affine representation of the moment generating function of the log-price. Similarities to a fractional Heston model can be found therein.

3.1 INTRODUCTION

Several empirical studies have been carried out to examine the past dependency of stock returns [56, 66, 82]. These studies are not only important from a statistical point of view, but they also have potential applications in derivatives pricing: the path-dependent volatility model (see [61, 63] for example), the one that describes the instantaneous volatility as a function of

the historical performance of the underlier (for instance its maximum and/or its average level over a fixed amount of time), is promoted as an alternative to local volatility and stochastic volatility models due to the fact that it is a complete model with satisfying volatility skews for both vanilla options and forward starting options [61]. For these reasons, modelling of the past-dependency of the underlier looks necessary.

In the literature some researchers suggested to use a fractional Brownian motion as the driving process for the stock price: under the physical measure the price process $(S_t)_{t \geq 0}$ satisfies

$$dS_t = \mu S_t dt + \sigma S_t dW_t^H,$$

where the process $(W_t^H)_{t \geq 0}$ is a centred Gaussian process with the covariance structure $\mathbb{E}[W_t^H W_s^H] = \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H})$, for $H \in (0, 1)$. Bayraktar, Poor and Sircar [12] estimated the Hurst parameter using the S&P 500 index and showed that $H \approx 0.6$. However, as explained in [26, 90] and summarised in detail in [17, Chapter 7], it is now clear that the introduction of a fractional Brownian motion with $H \neq 1/2$ includes arbitrage opportunities. Hence a different driving process is needed in order to model the past dependency.

Dobrić and Ojeda [35] proposed an approximation of the fractional Brownian motion which remains a semi-martingale. We briefly review this approximation process. Consider a fractional Brownian field $(W_t^H)_{(t,H) \in [0,\infty) \times (0,1)}$ with the covariance structure $\mathbb{E}[W_t^H W_s^{H'}] = a_{H,H'} (t^{H+H'} + s^{H+H'} - |t - s|^{H+H'})/2$, for some positive constant $a_{H,H'}$ precised in [35]. In the special case where $H + H' = 1$ the constant $a_{H,H'}$ has the form

$$a_H := a_{H,1-H} = \sqrt{\Gamma(2H+1)\Gamma(3-2H)} \sin^2(\pi H).$$

Define the filtration $(\mathcal{F}_t^H)_{t \geq 0}$ by $\mathcal{F}_t^H := \sigma(W_s^H, 0 \leq s \leq t)$, and further define the process $(M_t)_{t \geq 0}$ by $M_t := \mathbb{E}(W_t^{1-H} | \mathcal{F}_t^H)$, for $t \geq 0$. The process M is a martingale with the filtration $(\mathcal{F}_t^H)_{t \geq 0}$, and Dobrić and Ojeda [35] further showed that

$$\mathbb{E}[M_t M_s] = c_M (s \wedge t)^{2-2H}, \quad c_M := \frac{a_H^2 \Gamma(3/2 - H)}{2H \Gamma(H + 1/2) \Gamma(3 - 2H)}.$$

Finally, the approximation of a fractional Brownian motion $(W_t^H)_{t \geq 0}$ with the filtration $(\mathcal{F}_t^H)_{t \geq 0}$ is the process $(Z_t)_{t \geq 0} := (\Psi(t)M_t)_{t \geq 0}$ where the (deterministic) function $\Psi(\cdot)$ is chosen to minimise the difference $\mathbb{E}(W_t^H - \Psi(t)M_t)^2$ for each $t \geq 0$. It has the form

$$\Psi(t) = \frac{2H\Gamma(3-2H)\Gamma(H+1/2)}{a_H\Gamma(3/2-H)}t^{2H-1} =: c_\psi t^{2H-1}, \quad \text{for } t \geq 0.$$

As shown in [35], The relative L^2 error between the fractional Brownian motion W^H and its approximation Z is less than 12% for $H \in (0.4, 1)$.

Conus and Wildman [28] applied this approximation process to options pricing theory. They proved that there exists a Brownian motion $(B_t)_{t \geq 0}$ adapted to the filtration $(\mathcal{F}_t^H)_{t \geq 0}$ such that the process Z satisfies

$$dZ_t = Ct^d dB_t + 2dc_\psi t^{2d-1} M_t dt,$$

where $d := H - 1/2 \in (-1/2, 1/2)$ and $C := c_\psi \sqrt{c_M(1-2d)}$. The authors then proposed a modified version of the Dobrić-Ojeda process, denoted now by $(Z_t^\varepsilon)_{t \in [0, T]}$:

$$dZ_t^\varepsilon = Ct^d dB_t + 2dc_\psi t^{2d-1} M_t \mathbf{1}_{[\varepsilon, \infty)}(t) dt, \quad (3.1)$$

for some fixed ε in $(0, T)$. * Following this, the stock price $(S_t)_{t \geq 0}$ in [28] is given by $dS_t = S_t(\mu dt + \sigma dZ_t^\varepsilon)$, $S_0 = s_0 > 0$, where μ and σ are drift and diffusion constants under the physical measure. Conus and Wildman proved the existence of a risk-neutral measure $\mathbb{Q}$, and derived a Black-Scholes type pricing formula for the European call option:

$$\mathbb{E}^\mathbb{Q} [S_t - S_0 e^k]^+ = S_0 \{ \mathcal{N}(d_1) - e^k \mathcal{N}(d_2) \},$$

where $d_1 := -\frac{k}{\sigma C} \sqrt{\frac{2d+1}{t^{2d+1}}} + \frac{\sigma C}{2} \sqrt{\frac{t^{2d+1}}{2d+1}}$ and $d_2 := d_1 - \sigma C \sqrt{\frac{t^{2d+1}}{2d+1}}$. We assume as usual that the risk free interest rate is zero.

While the stock price process now allows for past dependency, the assumption of a constant volatility $\sigma > 0$ suggests that the volatility smile of a

*The introduction of ε is a technical assumption needed to provide a risk-neutral pricing measure.

European option is still flat (see Figure 3.1), which is inconsistent with market observations. In this chapter we consider a stochastic volatility version of the model proposed by Conus and Wildman [28]. We prove the existence of a risk-neutral pricing measure, then we provide numerical evidence showing that such a stochastic volatility setting is capable of generating desirable volatility smiles.

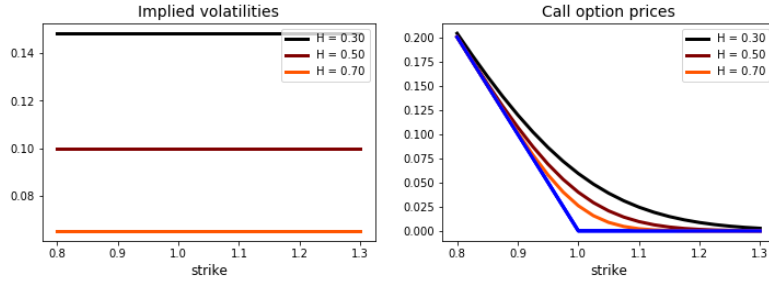


Figure 3.1: Flat implied volatilities in a standard Conus-Wildman setting with different Hurst parameters. Time to maturity is one year and $\sigma = 0.1$.

3.2 GENERAL STOCHASTIC VOLATILITY ENVIRONMENT

Assume that the process $(S_t, \sigma_t)_{t \in [0, T]}$ satisfies the following stochastic differential equation under the physical measure $\mathbb{P}$:

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dZ_t^\varepsilon, & S_0 &= 1, \\ d\sigma_t &= \alpha(t, \sigma_t) dt + \beta(t, \sigma_t) \left(\rho dB_t + \sqrt{1 - \rho^2} dW_t \right), & \sigma_0 &> 0, \end{aligned} \quad (3.2)$$

where the process W is a Brownian motion independent of B (the driving process of Z^ε as in (3.1)), $\rho \in [-1, 1]$, and functions α and β satisfy usual conditions such that the solution to (3.2) exists. Assume that the risk free interest rate is zero, and that the price of the volatility is constant, i.e. there exists $\eta > 0$ such that $\mu = \eta \sigma_t$, for any $t \geq 0$. We define measure $\mathbb{Q}$ by

$$\left. \frac{d\mathbb{Q}}{d\mathbb{P}} \right|_{\mathcal{F}_t} := \Xi_t = \exp \left\{ - \int_0^t \lambda_s dB_s - \frac{1}{2} \int_0^t \lambda_s^2 ds \right\},$$

where

$$\lambda_s := \frac{\eta}{Cs^d} + \frac{2c_\psi d}{C}s^{d-1}M_s 1_{[\varepsilon, \infty)}(s) =: \frac{A}{s^d} + Bs^{d-1}M_s 1_{[\varepsilon, \infty)}(s), \quad \text{for } s \in [0, T].$$

The representation of the process $(\lambda_t)_{t \geq 0}$ suggests that the dynamics of the volatility change dramatically from $\mathbb{P}$ to $\mathbb{Q}$ for non-zero ρ . Hence we focus on the uncorrelated case ($\rho = 0$).

Proposition 49. *For any $\varepsilon > \exp\{-T/(2B^2c_M)\}$, $\mathbb{E}^\mathbb{P}[\exp(\frac{1}{2} \int_0^T \lambda_s^2 ds)] < \infty$. As a result, the measure $\mathbb{Q}$ is an equivalent local martingale measure (ELMM), and the process $(S_t)_{t \in [0, T]}$ is a $\mathbb{Q}$ -supermartingale.*

Proof. Direct computations yield

$$\begin{aligned} \mathbb{E}^\mathbb{P} \left[e^{\frac{1}{2} \int_0^T \lambda_s^2 ds} \right] &= \mathbb{E}^\mathbb{P} \left[\exp \left\{ \frac{1}{2} \int_0^T \left(\frac{A}{s^d} + BM_s 1_{[\varepsilon, \infty)}(s) s^{d-1} \right)^2 ds \right\} \right] \\ &\leq \mathbb{E}^\mathbb{P} \left[\exp \left\{ A^2 \int_0^T \frac{ds}{s^{2d}} + B^2 \int_\varepsilon^T M_s^2 s^{2d-2} ds \right\} \right] \\ &= \exp \left\{ \frac{A^2 T^{1-2d}}{1-2d} \right\} \mathbb{E}^\mathbb{P} \left[\exp \left\{ B^2 \int_\varepsilon^T M_s^2 s^{2d-2} ds \right\} \right]. \end{aligned}$$

[28, Proposition 3.6] states that the second term above is finite for $\varepsilon > \exp\{-T/(2B^2c_M)\}$ from which we immediately have the result. The Girsanov theorem and the Novikov's condition imply that

$$dS_t = S_t \sigma_t C t^d (dB_t + \lambda_t dt) =: S_t \sigma_t C t^d d\hat{B}_t,$$

where $(\hat{B}_t)_{t \geq 0}$ is a Brownian motion under $\mathbb{Q}$. The process S is a positive local martingale, hence it is a supermartingale under $\mathbb{Q}$. $\square$

Remark 50. *The dynamics of the volatility determines whether the price process is a true martingale under $\mathbb{Q}$. For example, if $(\sigma_s)_{s \in [0, t]}$ is bounded almost surely, then $\mathbb{E}^\mathbb{Q} \left[\exp\{\frac{1}{2} C^2 \int_0^t \sigma_s^2 s^{2d} ds\} \right] < \infty$, so that the stock price is a true martingale under $\mathbb{Q}$.*

Remark 51. *Applying a similar procedure as in [95, Proposition 2], define $d\mathbb{Q}'/d\mathbb{P}|_{\mathcal{F}_t} := \Xi_t \exp\{W_t - t/2\}$. The independence of W and B guarantees that $\mathbb{Q}'$ is also an ELMM, hence the market is incomplete.*

Following Proposition 49, for the remainder of this chapter we move to the risk-neutral pricing measure $\mathbb{Q}$, then for brevity we omit the notation $\mathbb{Q}$ when taking expectations. The log-price $(X_t)_{t \geq 0}$ in the risk-neutral world is given by

$$\begin{aligned} dX_t &= -\frac{C^2 \sigma_t^2 t^{2d}}{2} dt + C \sigma_t t^d d\widehat{B}_t, \quad X_0 = 0, \\ d\sigma_t &= \alpha(t, \sigma_t) dt + \beta(t, \sigma_t) dW_t, \quad \sigma_0 > 0. \end{aligned} \quad (3.3)$$

Example 52 (OU process). *Assume that the volatility is an Ornstein-Uhlenbeck process:*

$$d\sigma_t = (a + b\sigma_t)dt + \xi dW_t, \quad \sigma_0 > 0.$$

Applying Monte-Carlo we generate a volatility surface with symmetric smiles.

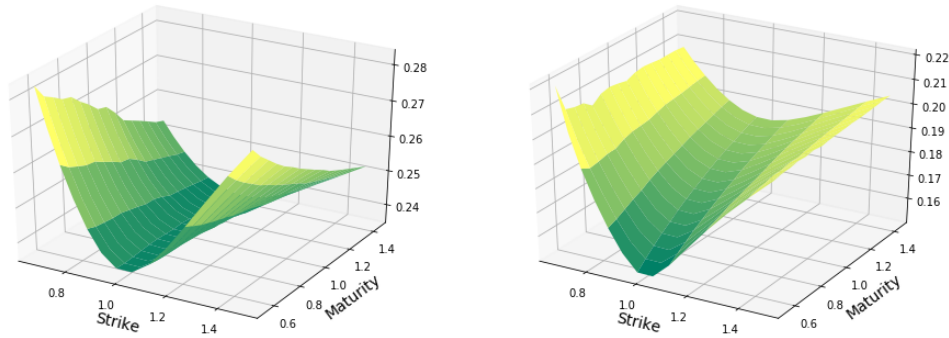


Figure 3.2: We generate a volatility surface using Monte-Carlo with $(a, b, \xi, \sigma_0) = (0.4, -2, 0.2, 0.1)$, $d = -0.2$ on the left and $d = 0$ on the right. The simulation is run for 5×10^5 times.

3.3 CONUS-WILDMAN WITH A CIR VARIANCE PROCESS

Assume that the process $(X_t, V_t)_{t \geq 0}$ satisfies the following SDE,

$$\begin{aligned} dX_t &= -\frac{C^2 t^{2d} V_t}{2} dt + C t^d \sqrt{V_t} d\widehat{B}_t, \quad X_0 = 0, \\ dV_t &= \kappa(\theta - V_t) dt + \xi \sqrt{V_t} dW_t, \quad V_0 = v_0 > 0, \end{aligned} \quad (3.4)$$

where the parameters satisfy the Feller condition $2\kappa\theta > \xi^2$. We term this setting as the ‘CW-CIR’ model. Direct computations yield $C^2 \mathbb{E}(\int_0^t s^{2d} V_s ds) = C^2 \int_0^t s^{2d} (\theta + e^{-\kappa s} (v_0 - \theta)) ds < \infty$ for $d \in (-1/2, 1/2)$, hence the existence and uniqueness of the solution to (3.4) are guaranteed. The following theorem, the proof of which is postponed to Section 3.5, provides the martingale property of the price process under the pricing measure.

Theorem 53. *For any $d \in (-1/2, 1/2)$ there exists $T^* > 0$ such that*

$$\mathbb{E} \left[e^{\frac{1}{2} C^2 \int_0^{T^*} V_s s^{2d} ds} \right] < \infty,$$

so that the price process $(S_t)_{t \in [0, T^]}$ is a true martingale under $\mathbb{Q}$.*

Proposition 54. *For any $(t, u) \in [0, T^*] \times \mathbb{R}$ such that $\mathbb{E}(e^{uX_t}) < \infty$, the moment generating function of X_t is given by*

$$M(t, u) = \mathbb{E}(e^{uX_t}) = \exp \{ C(t, u) + D(t, u) v_0 \},$$

where functions C and D satisfy the following PDE for any $0 \leq s < t$,

$$\begin{aligned} \partial_s C(s, u) - \kappa \theta D(s, u) &= 0, \\ \partial_s D(s, u) + \kappa D(s, u) - \frac{1}{2} \xi^2 D(s, u)^2 + \frac{u(1-u)C^2}{2} (t-s)^{2d} &= 0, \end{aligned} \quad (3.5)$$

with the initial condition $C(0, u) = D(0, u) = 0$.

Proof. Denote $(\mathcal{F}_t^V)_{t \geq 0}$ as the filtration that is generated by $(W_t)_{t \geq 0}$ only. Applying the tower property, and notice that the processes W and $\widehat{B}$ are

independent, the mgf has the form

$$\begin{aligned} M(t, u) &= \mathbb{E} \left[\mathbb{E} \left[\exp \left\{ -\frac{uC^2}{2} \int_0^t s^{2d} V_s ds + uC \int_0^t s^d \sqrt{V_s} d\widehat{B}_s \right\} \middle| \mathcal{F}_t^V \right] \right] \\ &= \mathbb{E} \left[e^{\frac{1}{2}u(u-1)C^2 \int_0^t V_s s^{2d} ds} \right]. \end{aligned}$$

For fixed u denote $\psi(s, v) := \mathbb{E}[\exp\{\frac{1}{2}C^2u(u-1) \int_s^t V_\tau \tau^{2d} d\tau\} | V_s = v]$. The Feynman-Kac formula [76, Theorem 2.5.1.1] implies that the function ψ satisfies

$$\partial_s \psi(s, v) + \kappa(\theta - v) \partial_v \psi(s, v) + \frac{\xi^2}{2} v \partial_{vv} \psi(s, v) + \frac{C^2 u(u-1)}{2} v s^{2d} \psi(s, v) = 0$$

with the initial condition $\psi(t, v) \equiv 1$. Define $\phi(s, v) := \psi(t-s, v)$, and plug in the ansatz $\phi(s, v) := \exp\{C(s, u) + D(s, u)v\}$ we immediately have the result. $\square$

The term $(t-s)^{2d}$ rules out an explicit solution to the Riccati equation (3.5). Nevertheless we can numerically solve the PDE and apply the Fourier transform method to compute options prices.

3.4 NUMERICS

This section gathers some numerical examples of a CW-CIR model. In Figure 3.3 we apply Proposition 54 and numerically compute the mgf of X_t . Figure 3.4 presents three sample paths for the price process with different Hurst parameters. Finally we generate volatility surfaces in a CW-CIR environment. Notice that in Equation (3.4) the volatility of X has a factor t^d , then intuitively as t tends to zero the implied volatility will blow up ($d < 0$) or tend to zero ($d > 0$). This is confirmed in Figure 3.5.

3.5 PROOF OF THEOREM 53

Denote $(\rho(t))_{t \geq 0}$ as a $BESQ^\delta$ process (squared δ -dimensional Bessel process) starting from $\rho(0) > 0$. The following result stated in [89, Chapter 11] is

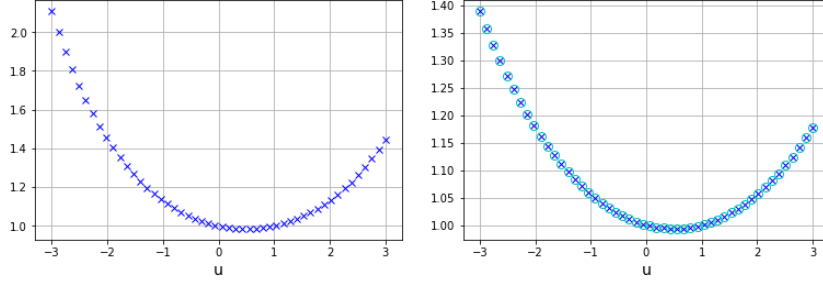


Figure 3.3: On the left ($d = -0.2$) we have the mgf of the log-price in the CW-CIR framework with $(\kappa, \theta, \xi, v_0) = (2.1, 0.05, 0.2, 0.06)$. On the right ($d = 0$) we plot the numerically solved CW-CIR mgf (crosses) and the explicit solution (standard Heston, circles) given in [52, Chapter 2]. Time to maturity is one year.

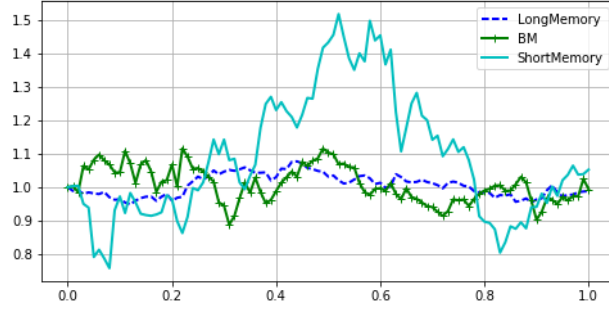


Figure 3.4: In the CW-CIR setting we simulate three sample paths for the price process with $d = -0.42$ (short memory, cyan), $d = 0$ (standard Brownian motion, green) and $d = 0.3$ (long memory, blue dashed). $(\kappa, \theta, \xi, v_0) = (2.1, 0.06, 0.4, 0.06)$.

essential to prove the desired martingale property.

Proposition 55. ([89, Chapter 11, Theorem 1.7]) Denote by μ a positive Radon measure on $\mathbb{R}_+$ satisfying $\int_0^\infty (1+y)\mu(dy) < \infty$, and denote ϕ as the solution to the Sturm-Liouville equation $\phi'' = \phi\mu$ with the initial condition $\phi(0) = 1$. Then

$$\mathbb{E} \left[\exp \left(-\frac{1}{2} \int_0^\infty \rho(t) \mu(dt) \right) \right] = \phi(\infty)^{\delta/2} \exp \left(\frac{\rho(0)}{2} \phi'(0) \right). \quad (3.6)$$

Remark 56. Revuz and Yor [89, Appendix 8] also provided the representa-

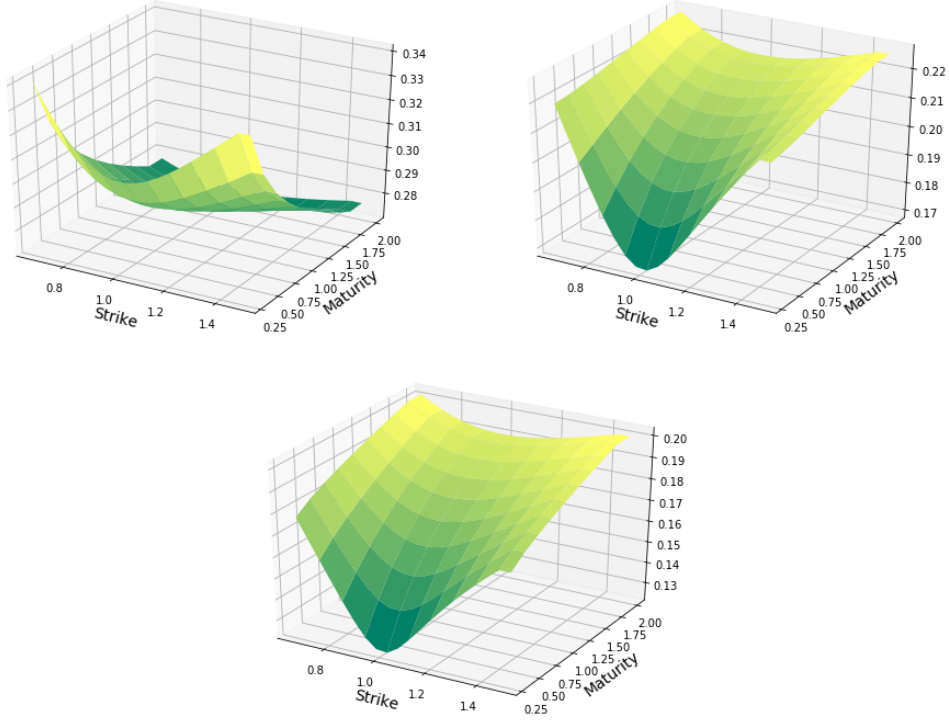


Figure 3.5: A CW-CIR volatility surface with $(\kappa, \theta, \xi, v_0) = (2.1, 0.06, 0.4, 0.02)$, $d = -0.2$ (top left), $d = 0$ (top right) and $d = 0.1$ at the bottom.

tion of ϕ : define function g as the solution to the Riccati equation

$$g(x) = 1 + \mu([0, x]) - \int_0^x g^2(y) dy, \quad g(0) = 1.$$

Further define $\psi(x) := \exp \left\{ \int_0^x g(y) dy \right\}$, then the map ϕ is given by $\phi(x) := \psi(x) \int_x^\infty \psi(y)^{-2} dy$.

We separate the proof into 3 different cases.

Case $d = 0$ This is the standard Heston environment and the moment explosion feature has been well analysed in the literature, see [5] for example.

Case $d \in (0, 1/2)$ In this case the following inequality is trivial,

$$\mathbb{E} \left[\exp \left\{ \frac{C^2}{2} \int_0^t V_s s^{2d} ds \right\} \right] < \mathbb{E} \left[\exp \left\{ \frac{C^2 t^{2d}}{2} \int_0^t V_s ds \right\} \right], \quad \text{for } d > 0.$$

The mgf of an integrated CIR process is provided in [36, Theorem 4.2], and the right-hand-side above is finite if

$$\frac{C^2 t^{2d}}{2} \leq \frac{\kappa^2}{2\xi^2} \implies t \leq \left(\frac{\kappa}{C\xi} \right)^{1/d} =: T^*.$$

Finally, the Novikov's condition implies that the process $(S_t)_{t \in [0, T^*]}$ is a $\mathbb{Q}$ -martingale.

Case $d \in (-1/2, 0)$ [76, Proposition 6.3.1.1] states that the CIR process $(V_t)_{t \geq 0}$ is a space-time changed $BESQ^\delta$ process described by $V_t = e^{-\kappa t} \rho(\xi^2(e^{\kappa t} - 1)/4\kappa)$, where $(\rho(t))_{t \geq 0}$ is a $BESQ^\delta$ process with dimension $\delta = 4\kappa\theta/\xi^2$ that starts from $\rho(0) = v_0 > 0$. For any $t > 0$,

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ \frac{C^2}{2} \int_0^t V_s s^{2d} ds \right\} \right] \\ &= \mathbb{E} \left[\exp \left\{ \frac{C^2}{2} \int_0^t e^{-\kappa s} \rho \left(\frac{\xi^2}{4\kappa} (e^{\kappa s} - 1) \right) s^{2d} ds \right\} \right] \\ &= \mathbb{E} \left[\exp \left\{ \frac{C^2}{2} \int_0^{y^*} \left(1 + \frac{4\kappa y}{\xi^2} \right)^{-1} \rho(y) \left(\log \left(1 + \frac{4\kappa y}{\xi^2} \right) \right)^{2d} \frac{4\kappa^{2d} dy}{\xi^2 + 4\kappa y} \right\} \right] \\ &= \mathbb{E} \left[\exp \left\{ 2C^2 \kappa^{2d} \xi^2 \int_0^{y^*} \left(\log \left(1 + \frac{4\kappa y}{\xi^2} \right) \right)^{2d} \frac{\rho(y) dy}{(\xi^2 + 4\kappa y)^2} \right\} \right], \end{aligned}$$

where $y^* := \xi^2(e^{\kappa t} - 1)/4\kappa$. Define now a measure μ on $\mathbb{R}_+$ by

$$\mu(dy) := \lambda \left(\log \left(1 + \frac{4\kappa y}{\xi^2} \right) \right)^{2d} (\xi^2 + 4\kappa y)^{-2} 1_{[0, y^*]}(y) dy, \quad \text{for any } \lambda > 0.$$

Fix constant $l > \max(\xi^{-2}, (4\kappa)^{-1})$ such that $l(\xi^2 + 4\kappa y) > 1 + y$ holds for

any $y \geq 0$. Then

$$\begin{aligned}
\int_0^\infty (1+y)\mu(dy) &= \lambda \int_0^{y^*} \frac{1+y}{(\xi^2 + 4\kappa y)^2} \left(\log \left(1 + \frac{4\kappa y}{\xi^2} \right) \right)^{2d} dy \\
&< \frac{\lambda l}{4\kappa} \int_0^{y^*} \left(\log \left(1 + \frac{4\kappa y}{\xi^2} \right) \right)^{2d} \frac{4\kappa dy}{\xi^2 + 4\kappa y} \\
&= \frac{\lambda l}{4\kappa} \int_0^{\kappa t} s^{2d} ds \\
&= \frac{\lambda l (\kappa t)^{2d+1}}{4\kappa(2d+1)} < \infty,
\end{aligned}$$

and Proposition 55 implies that the expectation $\mathbb{E}[\exp\{-\frac{1}{2} \int_0^{y^*} \rho(s)\mu(ds)\}]$ has a representation given by (3.6), for any $\lambda > 0$, $y^* > 0$. We prove that there exists (small) $y^* > 0$ such that the function ϕ in (3.6) is still well-defined for $\lambda^* := -4C^2\xi^2\kappa^{2|d|} < 0$. Following the representation of ϕ detailed in Remark 56 (see also [89, Appendix 8]), we first show that there exists $y^* > 0$ such that the Riccati equation

$$g'(y) = \frac{\lambda^*}{(\xi^2 + 4\kappa y)^2} \left(\log \left(1 + \frac{4\kappa y}{\xi^2} \right) \right)^{2d} - g^2(y), \quad g(0) = 1 \quad (3.7)$$

does not explode on the interval $[0, y^*]$. Fix $d^* \in (-1/2, d)$, and choose $y^+ > 0$ small enough such that the inequality $\log(1 + 4\kappa y/\xi^2) > y^{d^*/d}$ holds for any y in $[0, y^+]$. The existence of y^+ is guaranteed by the convexity of the power function $f(x) = x^\gamma$ for $\gamma > 1$. Then the following inequality holds:

$$\frac{\lambda^*}{\xi^4} y^{2d^*} - g^2(y) < g'(y) < -g^2(y), \quad \text{for any } y \in [0, y^+]. \quad (3.8)$$

As a result, the comparison principle implies that the explosion time of g as the solution to (3.7) on $[0, y^+]$ is no less than the explosion of the solution to the ode

$$f'(y) = -f^2(y) + \frac{\lambda^*}{\xi^4} y^{2d^*}, \quad f(0) = 1.$$

Since $d^* < 0$, the function f' is again bounded from below by $h(y, f) := \xi^{-4}\lambda^*y^{2d^*}(1 + f^2)$ for small $y > 0$. Brauer [20, Section 3] proved that the solution to the (seperable) equation $f'(y) = h(y, f(y))$ exists on a small

interval $[0, y^{++}]$ for some $y^{++} > 0$, hence there exists $y^* \geq y^+ \wedge y^{++} > 0$ such that the solution to (3.7) is well-defined on $[0, y^*]$. Moreover, the continuity of the solution indicates that we can choose $y^{**} \in (0, y^*]$ such that the solution to the following ode

$$g'(y) = \frac{\lambda^* \mathbf{1}_{[0, y^{**}]}(y)}{(\xi^2 + 4\kappa y)^2} \left(\log \left(1 + \frac{4\kappa y}{\xi^2} \right) \right)^{2d} - g^2(y), \quad g(0) = 1,$$

is strictly positive on the interval $[0, y^{**}]$. Since for $y > y^{**}$ the solution has the form $g(y) = (y - y^{**} + 1/g(y^{**}))^{-1}$, then function g is actually well-defined, decreasing, and strictly positive on $\mathbb{R}_+$. Consequently, functions ψ and ϕ in Remark 56 are still well-defined on $\mathbb{R}_+$ with our choice of μ and a negative λ^* , and straightforward computations yield that ϕ is concave and increasing on $[0, y^{**}]$, and is constant on $[y^{**}, \infty)$, hence $\phi(\infty)$ is finite and positive. The rest of the proof is identical to the proof of [89, Chapter 11, Theorem 1.7], and finally we have $\mathbb{E} \left[\exp \left\{ \frac{C^2}{2} \int_0^{T^{**}} V_s s^{2d} ds \right\} \right] < \infty$, where the small $T^{**} > 0$ is determined by inverting y^{**} through $y^{**} = \xi^2(e^{\kappa T^{**}} - 1)/4\kappa$. The Novikov's condition is then satisfied, hence the process $(S_t)_{t \in [0, T^{**}]}$ is a $\mathbb{Q}$ -martingale.

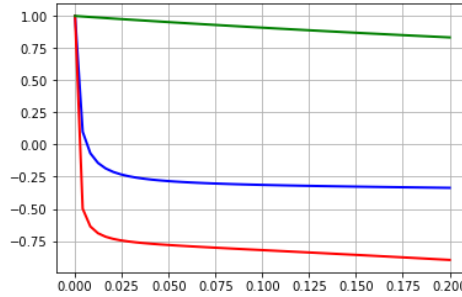


Figure 3.6: An example where we (numerically) solve Equation (3.7) (blue), and compare it with the boundaries given in (3.8).

3.6 CONCLUSION AND FUTURE WORK

We have shown that in the CW-CIR model the stock price $(S_t)_{t \in [0, T^*]}$ is a true martingale under the risk-neutral measure. The boundary T^* is not optimal, however, and in Theorem 53 it is not specified in the short memory

case ($d < 0$). We put this part of analysis, i.e. a more precise estimate of the upper bound T^* which could be dependent on the parameters of the model, into future work.

Considering the similarities between Theorem 45 and Proposition 54, it is reasonable to conjecture that the fractional Heston model and the CW-CIR model should have similar small-time asymptotic behaviours for the implied volatility. A rigorous asymptotic analysis will be carried out in the future.

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Reminder on large deviations and Regular variations

A.1 REGULAR VARIATIONS

We recall here some notions on regular variations, following the monograph [18].

Definition 57. Let $\mathfrak{a} > 0$. A function $f : (\mathfrak{a}, \infty) \rightarrow \mathbb{R}_+^*$ is said to be regularly varying with index $l \in \mathbb{R}$ (and we write $f \in \mathcal{R}_l$) if $\lim_{x \uparrow \infty} f(\lambda x)/f(x) = \lambda^l$, for any $\lambda > 0$. When $l = 0$, the function f is called slowly varying.

Lemma 58 (Bingham's Lemma, Theorem 4.12.10 in [18]). Let f be a regularly varying function with index $l > 0$; then, as x tends to infinity, the asymptotic equivalence $-\log \int_x^\infty e^{-f(y)} dy \sim f(x)$ holds.

Let Y be a random variable supported on $[0, \infty)$ with a smooth density f . The following lemma ensures that its moment generating function has unbounded support.

Lemma 59. If there exists $l > 1$ such that $|\log f| \in \mathcal{R}_l$, then $\sup\{u \in \mathbb{R} : \mathbb{E}(e^{uY}) < \infty\} = +\infty$.

Proof. Karamata's Characterisation Theorem [18, Theorem 1.4.1] implies that $|\log f(v)| = v^l g(v)$ for any $v > 0$, where the function g is slowly varying, and Karamata's Representation Theorem [18, Theorem 1.3.1] provides the following expression:

$$g(v) = c(v) \exp \left(\int_a^v \varepsilon(y) \frac{dy}{y} \right),$$

where the functions c and ε satisfy $\lim_{v \uparrow \infty} c(v) = c > 0$, $\lim_{v \uparrow \infty} \varepsilon(v) = 0$, and a is a fixed positive number. Then there exists $v_1 \geq a$ such that $c(v) > c/2$ for all $v \geq v_1$. Additionally, for any fixed ε_0 small enough satisfying that $l > 1 + \varepsilon_0$, there exists $v_2 \geq a$ such that $\int_{v_2}^v \varepsilon(y) dy/y > -\varepsilon_0 \log(v/v_2)$, for any $v \geq v_2$. Denote $d := \exp \left(\int_a^{v_2} \varepsilon(y) \frac{dy}{y} \right)$, then for any $v > \max(v_1, v_2)$, and any $u > 0$,

$$u - c(v) \exp \left(\int_a^v \varepsilon(y) \frac{dy}{y} \right) v^{l-1} < u - \frac{cd}{2} \exp \left(\int_{v_2}^v \varepsilon(y) \frac{dy}{y} \right) v^{l-1} < u - \frac{cd}{2} v_2^{\varepsilon_0} v^{l-1-\varepsilon_0}.$$

Thus there exists v_3 large enough so that $u - \frac{1}{2}cdv_2^{\varepsilon_0}v^{l-1-\varepsilon_0} < -1$, for $v \geq v_3$. With $v^* := \max(v_1, v_2, v_3)$,

$$\begin{aligned} \mathbb{E}(e^{uY}) &= \int_0^{v^*} e^{uv} f(v) dv + \int_{v^*}^{\infty} e^{v(u-v^{l-1}g(v))} dv \\ &< \int_0^{v^*} e^{uv} f(v) dv + \int_{v^*}^{\infty} e^{-v} dv < \infty. \end{aligned}$$

□

A.2 LARGE DEVIATIONS AND THE GÄRTNER-ELLIS THEOREM

In this section, we briefly recall the main definitions and results from large deviations theory, which we need in the thesis. For full details, the interested reader is advised to look at the excellent monograph by Dembo and Zeitouni [33]. Let $(Y_n)_{n \geq 0}$ denote a sequence of real-valued random variables. A map $I : \mathbb{R} \rightarrow \mathbb{R}_+$ is said to be a good rate function if it is lower semicontinuous and if the set $\{y : I(y) \leq \alpha\}$ is compact in $\mathbb{R}$ for each $\alpha \geq 0$.

Definition 60. Let $h : \mathbb{R} \rightarrow \mathbb{R}_+$ be a continuous function that tends to zero at infinity. The sequence $(Y_n)_{n \geq 0}$ satisfies a large deviations principle as n tends to infinity with speed $h(n)$ and rate function I (in our notations, $Y \sim \text{LDP}_\infty(h(n), I)$) if for each Borel measurable set $\mathcal{S} \subset \mathbb{R}$, the following inequalities hold:

$$-\inf_{y \in \mathcal{S}^\circ} I(y) \leq \liminf_{n \uparrow \infty} h(n) \log \mathbb{P}(Y_n \in \mathcal{S}) \leq \limsup_{n \uparrow \infty} h(n) \log \mathbb{P}(Y_n \in \mathcal{S}) \leq -\inf_{y \in \overline{\mathcal{S}}} I(y).$$

Let now Λ_h be the pointwise limit of the rescaled cgf of Y : $\Lambda_h(u) := \lim_{n \uparrow \infty} h(n) \log \mathbb{E}[\exp(uY_n/h(n))]$, whenever the limit exists, and denote by $\mathcal{D}_\Lambda := \{u \in \mathbb{R} : |\Lambda_h(u)| < \infty\}$ its effective domain. Then Λ_h is said to be essentially smooth if the interior $\mathcal{D}_\Lambda^\circ$ is non-empty, Λ_h is differentiable on $\mathcal{D}_\Lambda^\circ$, and $\lim_{u \rightarrow u_0} |\Lambda_h'(u)| = \infty$, for any $u_0 \in \partial \mathcal{D}_\Lambda$. Finally, for any $y \in \mathbb{R}$, define $\Lambda_h^*(y) := \sup_{u \in \mathcal{D}_\Lambda} \{uy - \Lambda_h(u)\}$, the convex conjugate of function Λ_h .

Theorem 61 (Gärtner-Ellis theorem, Theorem 2.3.6 in [33]). *If the function Λ_h is lower semicontinuous on $\mathcal{D}_\Lambda$ and essentially smooth, and $0 \in \mathcal{D}_\Lambda^\circ$, then $(Y_n)_{n \geq 0} \sim \text{LDP}_\infty(h(n), \Lambda_h^*)$.*

B

Notations and the moment generating function of the Heston model

B.1 NOTATIONS FROM THE HESTON MODEL

In this section we recall some notations and definitions of the Heston model. As proved in [1], the moment generating function (1.3) admits the closed-form representation $M(t, u) = \exp(C(t, u) + D(t, u)v_0)$, for any $u \in \mathcal{D}_M^t$, where

$$\left\{ \begin{array}{l} C(t, u) := \frac{\kappa\theta}{\xi^2} \left[(\kappa - \rho\xi u - d(u))t - 2 \log \left(\frac{1 - g(u)e^{-d(u)t}}{1 - g(u)} \right) \right], \\ D(t, u) := \frac{\kappa - \rho\xi u - d(u)}{\xi^2} \frac{1 - \exp(-d(u)t)}{1 - g(u) \exp(-d(u)t)}, \\ d(u) := ((\kappa - \rho\xi u)^2 + \xi^2 u(1 - u))^{1/2} \quad \text{and} \quad g(u) := \frac{\kappa - \rho\xi u - d(u)}{\kappa - \rho\xi u + d(u)}. \end{array} \right. \quad (\text{B.1})$$

In [44], the authors proved that the functions d and g have the following behaviour as t tends to zero:

$$d\left(\frac{u}{t}\right) = \mathbf{i} \frac{d_0 u}{t} + d_1 + \mathcal{O}(t) \quad \text{and} \quad g\left(\frac{u}{t}\right) = g_0 - \mathbf{i} \frac{g_1}{u} t + \mathcal{O}(t^2), \quad (\text{B.2})$$

with $d_0 := \xi \bar{\rho} \operatorname{sgn}(u)$, $d_1 := i \frac{2\kappa\rho - \xi}{2\bar{\rho}} \operatorname{sgn}(u)$, $g_0 := \frac{i\rho - \bar{\rho} \operatorname{sgn}(u)}{i\rho + \bar{\rho} \operatorname{sgn}(u)}$ and $g_1 := \frac{(2\kappa - \rho\xi) \operatorname{sgn}(u)}{\xi \bar{\rho} (\bar{\rho} + i\rho \operatorname{sgn}(u))^2}$. Define the two real numbers $u_- \leq 0$ and $u_+ \geq 1$ as

$$\begin{cases} u_- &:= \frac{2}{\xi \bar{\rho}} \arctan\left(\frac{\bar{\rho}}{\rho}\right) 1_{\{\rho < 0\}} - \frac{\pi}{\xi} 1_{\{\rho = 0\}} + \frac{2}{\xi \bar{\rho}} \left(\arctan\left(\frac{\bar{\rho}}{\rho}\right) - \pi \right) 1_{\{\rho > 0\}}, \\ u_+ &:= \frac{2}{\xi \bar{\rho}} \left(\arctan\left(\frac{\bar{\rho}}{\rho}\right) + \pi \right) 1_{\{\rho < 0\}} + \frac{\pi}{\xi} 1_{\{\rho = 0\}} + \frac{2}{\xi \bar{\rho}} \arctan\left(\frac{\bar{\rho}}{\rho}\right) 1_{\{\rho > 0\}}, \end{cases} \quad (\text{B.3})$$

as well as the function $\Lambda : (u_-, u_+) \rightarrow \mathbb{R}$ by

$$\Lambda(u) := \frac{u}{\xi(\bar{\rho} \cot(\xi \bar{\rho} u / 2) - \rho)}, \quad (\text{B.4})$$

which arises as the pointwise limit of the (rescaled) cumulant generating function of X_t :

$$\lim_{t \downarrow 0} t \log M\left(t, \frac{u}{t}\right) = \Lambda(u) v_0, \quad \text{for any } u \in (u_-, u_+).$$

From [44], the function Λ is well defined, smooth and strictly convex on (u_-, u_+) , and infinite elsewhere.

B.2 COMPONENTS OF THE MGF

We start by investigating the short-time behaviour of the function $D(t, u/h(t))$. For any $\beta \in \mathbb{R}$, define the function $D_0^\beta : (u_-, u_+) \rightarrow \mathbb{R}$ by

$$\begin{aligned} D_0^\beta(u) &:= \frac{1 - e^{-id_0 u}}{\xi^2(1 - g_0 e^{-id_0 u})} [(\rho\xi + id_0)\beta u + \kappa - d_1] + \frac{ig_1(\rho\xi + id_0)(1 - e^{-id_0 u})}{\xi^2(1 - g_0 e^{-id_0 u})^2 e^{id_0 u}} \\ &\quad - \frac{(\rho\xi + id_0)u}{\xi^2} \frac{d_1 - id_0 u \beta}{(1 - g_0 e^{-id_0 u})^2} (1 - g_0) e^{-id_0 u}, \end{aligned} \quad (\text{B.5})$$

where the functions d_0, d_1, g_0, g_1 are defined below (B.2) above.

Remark 62. *The function D_0^β is well defined: to see this, we only need to check that the β terms sum up to a real number, and the rest follows from [44,*

Remark 3.2]. The first term in (B.5) reads

$$\frac{1 - e^{-id_0 u}}{\xi^2(1 - g_0 e^{-id_0 u})}(\rho\xi + id_0)\beta u = -\beta\Lambda(u),$$

which is a real number, and the sum of the remaining terms with β reads (taking out the prefactor $id_0 u\beta$)

$$\begin{aligned} \frac{(\rho\xi + id_0)ue^{-id_0 u}(1 - g_0)}{\xi^2(1 - g_0 \exp(-id_0 u))^2} &= \frac{(g_0 - 1)e^{-id_0 u}\Lambda(u)}{(1 - g_0 e^{-id_0 u})(1 - e^{-id_0 u})} \\ &= \frac{i\bar{\rho} \operatorname{sgn}(u)\Lambda(u)}{\rho \cos(d_0 u) + \bar{\rho} \operatorname{sgn}(u) \sin(d_0 u) - \rho}, \end{aligned}$$

which is purely imaginary, so that the whole term is a real number.

The following lemma makes the effective domain of D_0^β precise, and shows that it arises as the second order of the short-time expansion of a rescaled version of the function D in (B.1).

Lemma 63. *Let $\beta \in \mathbb{R}$. As t tends to zero, the map $t \rightarrow D(t, u/h(t))$ behaves as*

$$D\left(t, \frac{u}{h(t)}\right) = \begin{cases} 0, & u = 0, & \text{for any } h, \\ \text{undefined}, & u \neq 0, & \text{if } h(t) = o(t), \\ t^{-1}\Lambda(u) + D_0^\beta(u) + o(1), & u \in (u_-, u_+), & \text{if } h(t) = t + \beta t^2 + o(t^2), \\ \frac{u^2 t}{2h^2(t)} \left(1 - \frac{h(t)}{u} + \frac{\rho\xi u t}{2h(t)}\right) & u \in \mathbb{R}, & \text{if } t = o(h(t)). \\ +\mathcal{O}\left(t + \frac{t^2}{h^2(t)} + \frac{t^3}{h^4(t)}\right). \end{cases}$$

Remark 64.

(i) *If $h(t) = t + o(t)$ without further information on higher-order terms (third case in the lemma), then only the leading order is available:*

$$D(t, u/h(t)) = t^{-1}\Lambda(u)(1 + o(1)).$$

(ii) *As in Remark 4(ii), one can consider $h(t) = ct + \beta t^2 + o(t^2)$, but by dilation, setting $c = 1$ is inconsequent.*

(iii) When $h(t) = t^{1/2}$, $D\left(t, \frac{u}{h(t)}\right) = \frac{1}{2}u^2 + \frac{1}{4}(\rho\xi u^2 - 2)ut^{1/2} + \mathcal{O}(t)$, which is consistent with [70].

The function $C_0 : (u_-, u_+) \rightarrow \mathbb{R}$ defined as

$$C_0(u) := -\frac{\kappa\theta}{\xi^2} \left[(\rho\xi + \mathbf{i}d_0)u + 2 \log \left(\frac{1 - g_0 \exp(-\mathbf{i}d_0 u)}{1 - g_0} \right) \right] \quad (\text{B.6})$$

is clearly real valued [44], and determines the asymptotic behaviour of the function C as follows.

Lemma 65. *The map $t \rightarrow C(t, u/h(t))$ has the following asymptotic behaviour as t tends to zero:*

$$C\left(t, \frac{u}{h(t)}\right) = \begin{cases} \text{undefined,} & u \neq 0, \quad h(t) = o(t), \\ C_0(u) + \mathcal{O}(t), & u \in (u_-, u_+), \quad h(t) = t + \mathcal{O}(t^2), \\ \frac{\kappa\theta u^2 t^2}{4h^2(t)} \left[1 + \mathcal{O}\left(h(t) + \frac{t}{h(t)}\right) \right] & \\ + \mathcal{O}(th(t) + h^3(t)), & u \in \mathbb{R}, \quad t = o(h(t)). \end{cases}$$

Proof of Lemma 63. Obviously $D(t, 0) \equiv 0$, so we assume from now on that $u \neq 0$. From (B.2), we have

$$d\left(\frac{u}{h(t)}\right) = \mathbf{i} \frac{d_0 u}{h(t)} + d_1 + \mathcal{O}(h(t)) \quad \text{and} \quad g\left(\frac{u}{h(t)}\right) = g_0 - \mathbf{i} \frac{g_1}{u} h(t) + \mathcal{O}(h^2(t)). \quad (\text{B.7})$$

Plugging these back into the expression of the function D in (B.1), we obtain

$$D\left(t, \frac{u}{h(t)}\right) = \frac{1}{\xi^2} \left[\frac{1 - \exp\left\{-\frac{\mathbf{i}ud_0 t}{h(t)} - d_1 t + \mathcal{O}(th(t))\right\}}{1 - \left[g_0 - \mathbf{i} \frac{g_1}{u} h(t) + \mathcal{O}(h^2(t))\right] e^{-\frac{\mathbf{i}ud_0 t}{h(t)} - d_1 t + \mathcal{O}(th(t))}} \right] \\ \left(\kappa - d_1 - \frac{\rho\xi u + \mathbf{i}d_0 u}{h(t)} + \mathcal{O}(h(t)) \right). \quad (\text{B.8})$$

If $h(t) = o(t)$, d_0 is a real number and d_1 is purely imaginary, then as $t/h(t)$ goes to infinity the term $\exp(-\mathbf{i}ud_0 t/h(t) - d_1 t)$ oscillates on the unit circle in the complex plane, thus no asymptotic can be derived.

Assume now that $h(t) = t + \beta t^2 + o(t^2)$. Then $t/h(t) = 1 - \beta t + o(t)$, and

Equation (B.8) yields

$$\begin{aligned}
& D\left(t, \frac{u}{h(t)}\right) \\
&= \frac{1}{\xi^2} \left(-\frac{(\rho\xi + \mathbf{i}d_0)u}{h(t)} + (\kappa - d_1) + \mathcal{O}(h(t)) \right) \\
&\quad \left[\frac{1 - \exp(-\mathbf{i}d_0ut/h(t) - d_1t + \mathcal{O}(th(t)))}{1 - (g_0 - \mathbf{i}tg_1/u + \mathcal{O}(t^2)) \exp(-\mathbf{i}d_0ut/h(t) - d_1t + \mathcal{O}(th(t)))} \right] \\
&= \frac{1}{\xi^2} \left(-\frac{(\rho\xi + \mathbf{i}d_0)u}{t} (1 - \beta t + o(t)) + (\kappa - d_1) + \mathcal{O}(t) \right) \\
&\quad (1 - e^{-\mathbf{i}d_0u} (1 - d_1t + \mathcal{O}(t^2)) (1 + \mathbf{i}\beta d_0ut + o(t))) (1 - g_0e^{-\mathbf{i}d_0u})^{-1} \\
&\quad \left(1 + \frac{(-\mathbf{i}g_1/u + g_0(\mathbf{i}d_0u\beta - d_1))e^{-\mathbf{i}d_0u}}{1 - g_0e^{-\mathbf{i}d_0u}} t + o(t) \right) \\
&= \frac{e^{-\mathbf{i}d_0u} - 1}{\xi^2 t} \frac{(\rho\xi + \mathbf{i}d_0)u}{1 - g_0e^{-\mathbf{i}d_0u}} + \frac{1 - e^{-\mathbf{i}d_0u}}{\xi^2 (1 - g_0e^{-\mathbf{i}d_0u})} ((\rho\xi + \mathbf{i}d_0)\beta u + \kappa - d_1) \\
&\quad + \frac{(\rho\xi + \mathbf{i}d_0)(\mathbf{i}g_1 - g_0u(\mathbf{i}\beta d_0u - d_1))(1 - e^{-\mathbf{i}d_0u})}{\xi^2 (1 - g_0e^{-\mathbf{i}d_0u})^2} e^{-\mathbf{i}d_0u} \\
&\quad - \frac{(\rho\xi + \mathbf{i}d_0)u}{\xi^2} \frac{(d_1 - \mathbf{i}\beta d_0u)e^{-\mathbf{i}d_0u}}{1 - g_0e^{-\mathbf{i}d_0u}} + o(1) \\
&= \frac{\Lambda(u)}{t} + D_0^\beta(u) + o(1).
\end{aligned}$$

The form of the effective domain is straightforward from these expressions.

If $h(t) = t + o(t)$ without further information on higher-order terms, then $t/h(t) = 1 + o(1)$. Following the same procedure as above then only the leading order can be derived, i.e. $D(t, u/h(t)) = t^{-1}\Lambda(u)[1 + o(1)]$.

Finally in the case $t = o(h(t))$,

$$\begin{aligned}
& \left[1 - \left(g_0 - \frac{\mathbf{i}g_1}{u} h(t) + \mathcal{O}(h^2(t)) \right) e^{-\frac{\mathbf{i}d_0ut}{h(t)} - d_1t + \mathcal{O}(th(t))} \right]^{-1} \\
&= \frac{1 - \frac{\mathbf{i}g_1h(t)}{u(1-g_0)} - \frac{\mathbf{i}d_0g_0ut}{(1-g_0)h(t)} + \mathcal{O}\left(t + h^2(t) + \frac{t^2}{h^2(t)}\right)}{1 - g_0}, \\
& 1 - \exp\left(\frac{-\mathbf{i}d_0ut}{h(t)} - d_1t + \mathcal{O}(th(t))\right) = \frac{\mathbf{i}d_0ut}{h(t)} + d_1t + \frac{d_0^2u^2t^2}{2h^2(t)} + \mathcal{O}\left(\frac{t^2}{h(t)} + th(t)\right).
\end{aligned}$$

Plugging these results into (B.8) yields

$$\begin{aligned}
& D\left(t, \frac{u}{h(t)}\right) \\
&= \frac{1}{\xi^2(1-g_0)} \left[1 - \frac{\mathbf{i}g_1h(t)}{u(1-g_0)} - \frac{\mathbf{i}d_0g_0ut}{(1-g_0)h(t)} + \mathcal{O}\left(t + h^2(t) + \frac{t^2}{h^2(t)}\right) \right] \\
&\quad \left[\frac{\mathbf{i}d_0ut}{h(t)} + d_1t + \frac{d_0^2u^2t^2}{2h^2(t)} + \mathcal{O}\left(\frac{t^2}{h(t)} + th(t)\right) \right] \left[-\frac{(\rho\xi + \mathbf{i}d_0)u}{h(t)} + (\kappa - d_1) + \mathcal{O}(h(t)) \right] \\
&= \frac{1}{\xi^2(1-g_0)} \left[1 - \frac{\mathbf{i}g_1h(t)}{u(1-g_0)} - \frac{\mathbf{i}d_0g_0ut}{(1-g_0)h(t)} + \mathcal{O}\left(t + h^2(t) + \frac{t^2}{h^2(t)}\right) \right] \\
&\quad \left[\frac{(d_0 - \mathbf{i}\rho\xi)d_0u^2t}{h^2(t)} + \frac{[\mathbf{i}d_0u(\kappa - d_1) - d_1u(\rho\xi + \mathbf{i}d_0)]t}{h(t)} - \frac{d_0^2u^3(\rho\xi + \mathbf{i}d_0)t^2}{2h^3(t)} \right] \\
&\quad + \mathcal{O}\left(t + \frac{t^2}{h^2(t)}\right) \\
&= \frac{u^2t}{2h^2(t)} \left[1 - \frac{h(t)}{u} + \frac{\rho\xi ut}{2h(t)} + \mathcal{O}\left(\frac{t^2}{h^2(t)} + h^2(t) + t\right) \right],
\end{aligned}$$

where we used the identity

$$\frac{(d_0 - \mathbf{i}\rho\xi)d_0}{(1-g_0)\xi^2} = \frac{(\xi\bar{\rho}\text{sgn}(u) - \mathbf{i}\rho\xi)\xi\bar{\rho}\text{sgn}(u)}{\xi^2 \left(1 - \frac{\mathbf{i}\rho - \bar{\rho}\text{sgn}(u)}{\mathbf{i}\rho + \bar{\rho}\text{sgn}(u)}\right)} = \frac{1}{2}.$$

□

Proof of Lemma 65. Assume that $u \neq 0$. Expand $d(u/h(t))$ and $g(u/h(t))$ to the third order,

$$\begin{aligned}
d\left(\frac{u}{h(t)}\right) &= \mathbf{i}\frac{d_0u}{h(t)} + d_1 - \mathbf{i}d_2h(t) + \mathcal{O}(h^2(t)), \\
g\left(\frac{u}{h(t)}\right) &= g_0 - \mathbf{i}\frac{g_1}{u}h(t) - \mathbf{i}\frac{g_2}{u^2}h^2(t) + \mathcal{O}(h^3(t)),
\end{aligned} \tag{B.9}$$

where $d_2 := (\kappa^2 - d_1^2)/(2d_0u)$ and $g_2 := [(\kappa^2 - d_1^2)\rho\xi/d_0 + (\kappa - d_1)(\rho\xi -$

$\mathbf{i}d_0)g_1](\rho\xi - \mathbf{i}d_0)^{-2}$. Combining these expansions with Equation (B.7) implies

$$\begin{aligned} \mathcal{C}\left(t, \frac{u}{h(t)}\right) &= \frac{\kappa\theta}{\xi^2} \left[(\kappa - d_1)t - \frac{(\rho\xi + \mathbf{i}d_0)ut}{h(t)} + \mathcal{O}(th(t)) \right] \\ &\quad - \frac{2\kappa\theta}{\xi^2} \log \left(\frac{1 - \left(g_0 - \frac{\mathbf{i}g_1h(t)}{u} - \frac{\mathbf{i}g_2h^2(t)}{u^2} + \mathcal{O}(h^3(t)) \right) e^{-\mathbf{i}d_0ut/h(t) - d_1t + \mathbf{i}d_2th(t) + \mathcal{O}(th^2(t))}}{1 - g_0 + \mathbf{i}g_1h(t)/u + \mathbf{i}g_2h^2(t)/u^2 + \mathcal{O}(h^3(t))} \right). \end{aligned} \quad (\text{B.10})$$

If $h(t) = o(t)$, no short-time asymptotics can be derived since $t/h(t)$ tends to infinity. For the proof of the case where $h(t) = t + \mathcal{O}(t^2)$ we refer to [44, Lemma 6.1]. Assume now that $t = o(h(t))$, then the following asymptotic expansions hold:

$$\begin{aligned} &\left(1 - g_0 + \frac{\mathbf{i}h(t)g_1}{u} + \frac{\mathbf{i}h^2(t)g_2}{u^2} + \mathcal{O}(h^3(t)) \right)^{-1} \\ &= \frac{1}{1 - g_0} \left(1 - \frac{\mathbf{i}g_1h(t)}{u(1 - g_0)} - \frac{g_3h^2(t)}{u^2(1 - g_0)^2} + \mathcal{O}(h^3(t)) \right), \\ &\exp \left(-\frac{\mathbf{i}d_0ut}{h(t)} - d_1t + \mathbf{i}d_2th(t) + \mathcal{O}(th^2(t)) \right) \\ &= 1 - \frac{\mathbf{i}d_0ut}{h(t)} - \frac{1}{2} \left(\frac{d_0ut}{h(t)} \right)^2 - d_1t + \mathbf{i}d_2th(t) + \mathcal{O} \left(th^2(t) + \frac{t^2}{h(t)} \right), \end{aligned}$$

where $g_3 := g_1^2 + ig_2(1 - g_0)$. Consequently,

$$\begin{aligned}
& \frac{1 - (g_0 - ig_1h(t)/u - ig_2h^2(t)/u^2 + \mathcal{O}(h^3(t))) e^{-id_0ut/h(t) - d_1t + \mathcal{O}(th(t))}}{1 - g_0 + ig_1h(t)/u + ig_2h^2(t)/u^2 + \mathcal{O}(h^3(t))} \\
&= \left[1 - \frac{id_0ut}{h(t)} - \frac{1}{2} \left(\frac{d_0ut}{h(t)} \right)^2 - d_1t + id_2th(t) + \mathcal{O} \left(th^2(t) + \frac{t^2}{h(t)} \right) \right] \\
& \left[1 - \left(g_0 - \frac{ig_1}{u}h(t) - \frac{ig_2}{u^2}h^2(t) + \mathcal{O}(h^3(t)) \right) \right] (1 - g_0)^{-1} \\
& \left(1 - \frac{ig_1h(t)}{u(1 - g_0)} - \frac{g_3h^2(t)}{u^2(1 - g_0)^2} + \mathcal{O}(h^3(t)) \right) \\
&= \left(1 + \frac{ig_0d_0ut}{(1 - g_0)h(t)} + \frac{ig_1h(t)}{u(1 - g_0)} + \frac{d_1g_0 + d_0g_1}{1 - g_0}t + \frac{ig_2h^2(t)}{u^2(1 - g_0)} + \frac{g_0d_0^2u^2t^2}{2(1 - g_0)h^2(t)} \right) \\
& \left(1 - \frac{ig_1h(t)}{u(1 - g_0)} - \frac{g_3h^2(t)}{u^2(1 - g_0)^2} + \mathcal{O}(h^3(t)) \right) + \mathcal{O} \left(th(t) + h^3(t) + \frac{t^2}{h(t)} \right) \\
&= 1 + \frac{ig_0d_0u}{1 - g_0} \frac{t}{h(t)} + \frac{u^2d_0^2g_0t^2}{2(1 - g_0)h^2(t)} + \left(\frac{d_1g_0 + d_0g_1}{1 - g_0} + \frac{d_0g_0g_1}{(1 - g_0)^2} \right) t \\
&+ \mathcal{O} \left(th(t) + h^3(t) + \frac{t^2}{h(t)} \right),
\end{aligned}$$

and therefore

$$\begin{aligned}
& \log \left(\frac{1 - [g_0 - ih(t)g_1/u - ih^2(t)g_2/u^2 + \mathcal{O}(h^3(t))] e^{-id_0ut/h(t) - d_1t + \mathcal{O}(th(t))}}{1 - g_0 + ih(t)g_1/u + ih^2(t)g_2/u^2 + \mathcal{O}(h^3(t))} \right) \\
&= \frac{ig_0d_0u}{1 - g_0} \frac{t}{h(t)} + \frac{u^2d_0^2g_0}{2(1 - g_0)^2} \frac{t^2}{h^2(t)} + \left(\frac{d_1g_0 + d_0g_1}{1 - g_0} + \frac{d_0g_0g_1}{(1 - g_0)^2} \right) t \\
&+ \mathcal{O} \left(th(t) + h^3(t) + \frac{t^2}{h(t)} + \frac{t^3}{h^3(t)} \right).
\end{aligned}$$

Plugging this into (B.10), the result follows by noticing that the coefficients of $\frac{t}{h(t)}$ and t are both zero. $\square$



Calibration of the randomised Heston model

We have shown that any explosion rate of t^γ with $\gamma \in [0, 1/2]$ for the implied variance can be captured by suitably choosing the distribution of the initial variance $\mathcal{V}$. It is also proved in Section 1.4.5 that the large-time (asymptotic) behaviours of the randomised and standard Heston models are identical. These results suggest two methods for calibration,

- Method I: calibrate the parameter set $(\kappa, \theta, \xi, v_0, \rho)$ in the standard Heston model, fix $(\kappa, \theta, \xi, \rho)$, then calibrate the parameters of $\mathcal{V}$ according to the choice of v_0 . This method is similar to the calibration procedure of the ‘stochastic volatility plus jump’ model described in [52, Chapter 5];
- Method II: preset the distribution and the initial parameters of $\mathcal{V}$, then calibrate the updated parameter set which consists of $(\kappa, \theta, \xi, \rho)$ and all parameters of $\mathcal{V}$. For instance, fix $\mathcal{V} = \Gamma(\alpha, \beta)$, then we calibrate $(\kappa, \theta, \xi, \rho, \alpha, \beta)$ at once.

In our test Method II outperforms Method I in terms of the calibration precision. Define the objective function by

$$G_{\mathcal{H}}(\Theta) := \sum_{j=1}^J \sum_{i=1}^{n(j)} \omega_{i,j} (\sigma(t_j, k_{i,j}) - \sigma_{\mathcal{H}}(t_j, k_{i,j}, \Theta))^2,$$

where $\omega_{i,j}$ are weights of importance and $n(j)$ are numbers of strikes available in the market for each maturity t_j . Then the calibration process is to minimise $G_{\mathcal{H}}$ over the parameter set Θ .

Similar to the standard Heston case, the output of Θ is very sensitive to the initial guess. In the thesis we apply the method described in [44, Theorem 5.1] to obtain an initial choice of $(\kappa, \theta, \xi, \rho, v_0)$. The essence is to calculate the model parameters so that the model-generated volatilities precisely match 4 small-time near the money market volatilities and the ATM implied volatility with the smallest time to maturity.