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DEPARTMENT OF ELECTRICAL AND ELECTRONIC ENGINEERING

A Mathematical Framework for Reconstructing Shapes from Tomographic Projections at Unknown Angles

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Statement of Originality

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To my parents...

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Abstract

Tomographic reconstruction is a critical tool in many scientific and medical applications, allowing the visualization of internal structures non-destructively using only external measurements. Standard reconstruction methods typically rely on the knowledge of projection angles. However, this information may not always be available in many practical scenarios. This problem frequently arises in fields such as biomedical imaging and material science where constraints on equipment, experimental setups, or the nature of the subject being imaged make angle measurements infeasible. For instance, in single particle cryogenic electron microscopy (cryo-EM), determining the 3D structure of molecules involves analyzing tomographic projections of particles that are randomly oriented within a thin layer of vitreous ice. Although this problem has been studied for practical implementations, several fundamental questions remain unresolved. Chief among these are issues related to the uniqueness of the reconstruction and the existence of algorithms capable of accurately estimating the underlying structure. Moreover, many existing approaches do not take into account the conversion from the continuous nature of the underlying physical process to its discrete measurements, which is fundamentally governed by a sampling procedure. Typically, existing methods rely on a large number of projections to achieve reliable reconstruction. However, there is no theoretical framework that establishes the minimum number of projections required to guarantee perfect reconstruction of the structure. This limitation is further compounded in many practical scenarios where obtaining a large number of projections is infeasible. Therefore, in this thesis: 1) we develop a novel sampling and reconstruction framework for the problem of unknown angle tomography, 2) we present both theoretical and algorithmic solutions to the reconstruction problem, and 3) we investigate advanced signal models that overcome the limitations imposed by a restricted number of projections, enabling more robust and accurate reconstructions in practical scenarios, and for different applications.

Keywords: Unknown angle tomography, sampling theory, finite rate of innovation (FRI), multi-dimensional signal, sampling at unknown location, sparse reconstruction, complex analysis, limited angle tomography, curve annihilation, image reconstruction, angle refinement

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Chapter 1

Introduction

路漫漫其修远兮，
吾将上下而求索。

——屈原·《离骚》

For on and on the road stretches far,
where I'll search high and low.

— Qu Yuan · *Li Sao*

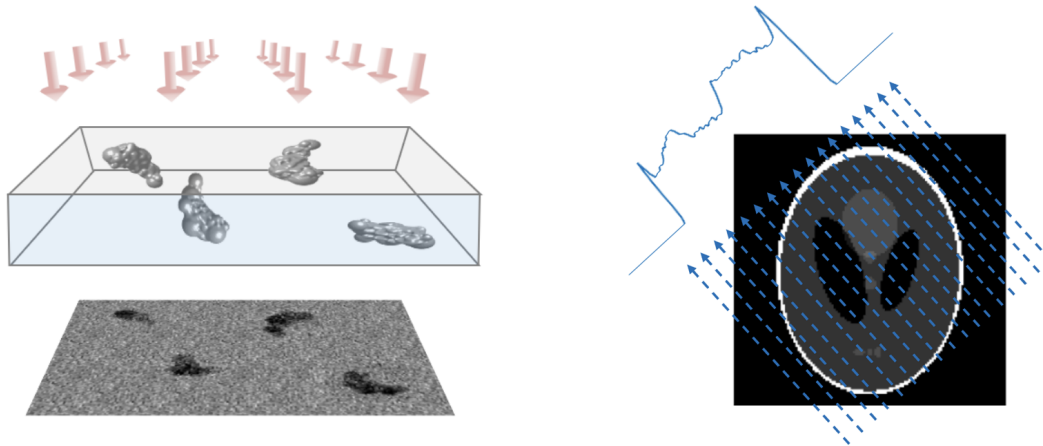


Figure 1.1: Left: Parallel-line tomographic projections of multiple identical copies of a specimen, as used in cryo-electron microscopy (cryo-EM). Right: One-dimensional parallel-line tomographic projections of a two-dimensional phantom brain image.

How does one see a shape without directly observing it? The idea may seem almost magical, reminiscent of detective works, like Sherlock Holmes or Hercule Poirot, characters who reconstruct complex scenarios from scattered fragments of information. In the real world, this is not fantasy but a challenge tackled by modern imaging techniques. Ask a scientist working in cryogenic electron microscopy (cryo-EM): they uncover the structure of molecules not through direct observation, but by assembling tomographic projections captured from arbitrary, unknown angles.

The trick lies in the interpretation of 2D projections of a 3D object. Conceptually, this can be likened to observing the shadows cast by a rotating object, such as a plastic toy on a wall. Each projection, or shadow, provides partial information about the object's geometry, yet no single view is sufficient to fully reconstruct its three-dimensional structure. This principle underpins the mathematical formulation of tomography, which originates from the seminal work of Radon in 1917, where he introduced what is now known as the Radon transform [79]. Despite its early theoretical foundations, it was not until the 1970s that computed tomography (CT) emerged as a widely adopted and powerful imaging modality, particularly in the field of medical diagnostics. CT scanners function by rotating an X-ray source around the subject, capturing multiple attenuation profiles from different angles, please see two examples of tomographic projections in Figure 3.2. These measurements are then computationally reconstructed to form cross-sectional images, commonly referred to as slices that reveal internal structures with high spatial resolution, distinguishing between tissues based on their differential X-ray absorption characteristics.

Now imagine you don't know the angle of rotation between each shadow you see. Can you

still reconstruct the object? This takes this concept to the next level, where the aim is to reconstruct an unknown structure from a collection of its lower dimensional tomographic projections taken at arbitrary, unknown angles. A tomographic projection differs from a shadow in that it is generated by capturing the cumulative density of an object along a specific angle using some penetrating waves. This task is central to applications like cryo-EM [96, 16], where it enables 3D reconstruction of biomolecule structures to help understanding diseases or developing drugs. Besides cryo-EM, this problem also arises in applications such as for example industrial radiography [12], where engineers try to inspect internal components of machinery without disassembling them, and biogeography, where bio-engineers analyze morphological variation of organisms across biogeographic zones using in-situ microscopic tomography imaging [58]. In the latter, high-throughput 2D imaging of live, freely moving specimens in their natural environment is leveraged to statistically reconstruct 3D morphological data, enabling structural and ecological insights without disturbing the organisms or their habitats.

The central question driving this thesis is: can one perfectly reconstruct a shape from its projections at unknown angles? The first keyword striking in the question is *perfect reconstruction*, a notion that stems from sampling theory, whose aim is to establish the conditions under which a continuous signal can be completely recovered from a set of its discrete samples. In the physical world, most phenomena and signals such as light, sound, natural images, electromagnetic waves etc. exist in the continuous domain. However, modern signal processing and computational tools rely on digital systems, which can only deal with discrete sequences. Therefore, it is crucial to obtain faithful discrete representations of the original continuous signal that carry all the essential information contained in the continuous domain. This is where sampling theory plays a central role. Sampling theory not only provides a comprehensive model of the data acquisition process, accounting for the conversion between continuous and discrete domains, but also establishes rigorous principles to ensure the fidelity of discrete samples.

The second keyword is *projections at unknown angles*. Geometrically, each tomographic projection represents the ‘footprint’ of a higher-dimensional structure onto a lower dimensional space. When the projection angles are unknown, the challenge lies in ‘piecing’ together these projections, much like solving a complex geometric puzzle. This requires understanding the intrinsic relationships between the unknown structure, its projections, and the transformations that connect them.

By considering the unknown angle tomographic problem from a sampling perspective, the goal of this thesis is to explore the interplay between geometry and tomography, exploring how structural features of the object can inform its reconstruction, and creating a natural bridge between theoretical frameworks and practical algorithms.

1.1 Background and motivation

In this thesis, we focus on the development of mathematical methods to address the problem of tomographic reconstruction with unknown projection angles. We first explore the fundamental sampling problem of when perfect reconstruction of an unknown 3D structure can be achieved from samples of its tomographic projections at unknown angles. Then, we address the problem of image reconstruction in the context of unknown angle tomography, with particular emphasis on scenarios involving a limited number of projections.

1.1.1 Sampling from tomographic projections at unknown angles

Since the advent of the Shannon sampling theorem, sampling has been crucial in modern signal processing as it underpins almost all modern digital technologies. Its importance stems from its fundamental role in enabling the conversion between continuous and discrete representations of signals. The core question in sampling is how can discrete samples be obtained in a way that guarantees faithful reconstruction of the original continuous signal? In other words, under what conditions can perfect reconstruction of a continuous signal be achieved from its discrete samples? For bandlimited signals, the answer is given by the famous Shannon sampling theorem [68, 84]. Later, the scope is extended to signals that belong to a fixed shift-invariant space [94, 14]. Finite rate of innovation [98, 15, 32] then extends to signals that are neither bandlimited nor reside in shift invariant subspaces.

The commonality of the above traditional sampling frameworks is that they assume fixed or predictable sampling patterns. While they provide conditions for perfect reconstruction when sampling locations are precisely known, they fall short in addressing the demands of many modern sensing and imaging techniques which require reconstruction from measurements taken at unknown or uncertain locations, posing challenges that go beyond the scope of classical sampling frameworks. An important example of such, which motivates this study is the unknown angle tomographic reconstruction problem, which arises in various fields ranging from bioimaging, industrial radiography, seismology to biogeography.

This data acquisition model poses various challenges to the traditional sampling framework. First of all, due to the unknown angle setup, the locations at which the measurements are taken are unknown. Secondly, the measurements are not directly sampled from the original signal or its filtered version; rather, they correspond to lower-dimensional tomographic projections, which inherently reduce the dimensionality and potentially distort the representation of the signal. Finally, the measurement modality introduces an uneven distribution of information across the frequency spectrum, as dictated by

the central slice theorem. These factors collectively demand innovative modifications to traditional sampling methods to ensure robust and faithful reconstruction under these constraints.

Within this study, the key problem that arises is to find methods to perfectly reconstruct the original continuous structure from samples of its lower dimensional tomographic projections at unknown angles. Specifically, we address the following questions: 1) Uniqueness of the solution: under what conditions can the original structure be perfectly reconstructed? 2) Existence of algorithms under uniqueness conditions: how to estimate the unknown structure and projection angles? 3) What is the minimum number of projections needed to achieve perfect reconstruction?

1.1.2 Image reconstruction in unknown angle tomography

In many tomography settings, the number of projections available is very limited. This may happen, for example, in pediatric scans, where minimizing radiation exposure to patients is critical. Techniques like low-dose computational tomography (CT) aim to reduce the number of projections while still providing diagnostically useful images. In radio astronomy, the positions of the telescopes limit the number of projections that can be obtained of a distant object. In archaeological imaging, delicate artifacts cannot withstand prolonged scanning, only a few projections may be feasible due to physical or temporal limitations.

The above applications pose an interesting question: can tomographic reconstruction be achieved using a few numbers of projections only? This makes us reflect on the intrinsic limitation that typically necessitates a large number of projections to achieve adequate reconstruction accuracy.

From the perspective of the central slice theorem (also known as the Fourier slice theorem), each projection of an image corresponds to a slice of its Fourier transform in the frequency domain. For an accurate tomographic reconstruction, these slices need to sufficiently cover the Fourier space of the object. With only a few projections, the Fourier space coverage becomes sparse, leaving significant gaps in the frequency domain. In particular, the uneven distribution of information within the Fourier domain adds further complexity. Low-frequency components are typically well represented even with a few projections due to their high concentration near the center of the Fourier space. In contrast, high-frequency components are more sparsely distributed and require denser projection measurements to be adequately captured. This uneven distribution of information makes it challenging to reconstruct high-resolution images with limited projections.

Therefore, within the study of image reconstruction in unknown angle tomography, there

are several key questions: 1) How can the unknown projection angles be accurately estimated from tomographic projections for arbitrary images? 2) In order to overcome the above stated limitation, what signal representation is most suitable for this problem? 3) What algorithms can feasibly achieve accurate reconstruction with a limited number of projections?

1.2 Outline of the thesis

The remainder of this thesis is organised as follows.

- **Chapter 2:** In this chapter, we provide background information on the main mathematical theory and sampling techniques used throughout this thesis.
- **Chapter 3:** We consider the problem of sampling from tomographic projections at unknown angles. We show that under certain settings, we can perfectly and simultaneously reconstruct the unknown 3D structure, as well as estimate the projection angles. Moreover, the minimum number of projections needed is independent of the complexity of the signal model. We also present alternative strategies to increase the robustness of the reconstruction algorithm to noise.
- **Chapter 4:** In this chapter, we first consider the general angle retrieval problem in 2D unknown angle tomography, where no model assumption is made to the 2D object being imaged. We then focus on the challenge posed by limited projection settings and propose to use images with a finite rate of innovation as the underlying signal model. This approach helps overcome the issue of uneven frequency distribution in the Fourier domain caused by a limited number of projections. Building on this, we introduce an image reconstruction algorithm that iteratively refines both the angle estimation and the reconstructed image.
- **Chapter 5:** We summarize the key contributions of this thesis and propose potential directions for future research.

1.3 List of publications

Peer Reviewd Journals

- **Renke Wang**, Francien Bossema, Thierry Blu, Pier Luigi Dragotti, “Reconstructing Classes of 3D FRI signals from Sampled Tomographic Projections at Unknown Angles”, IEEE Transactions on Image Processing, to be submitted.

Peer Reviewed Conferences

- **Renke Wang**, Roxana Alexandru and Pier Luigi Dragotti, “Perfect Reconstruction of Classes of Non-Bandlimited Signals from Projections with Unknown Angles”, International Conference on Acoustics, Speech, and Signal Processing (ICASSP), Singapore, 2022.
- **Renke Wang**, Jun-jie Huang and Pier Luigi Dragotti, “FRISPEE: FRI-Based Single Image Super-Resolution with Deep Recursive Residual Network”, European Signal Processing Conference (EUSIPCO), Belgrade, Serbia, 2022.
- **Renke Wang**, Thierry Blu and Pier Luigi Dragotti, “Reconstructing of Images with Finite Rate of Innovation from Noisy Tomographic Projections”, European Signal Processing Conference (EUSIPCO), Helsinki, Finland, 2023.
- **Renke Wang** and Pier Luigi Dragotti, “Perfect Reconstruction of Classes of 3D Non-Bandlimited Signals from Projections with Unknown Angles”, European Signal Processing Conference (EUSIPCO), Helsinki, Finland, 2023.
- Jiakang Chen, **Renke Wang**, Vincent Leung and Pier Luigi Dragotti, “Fast and Direct Angle Inference Using Masked Projection Modelling in 2D Tomography with Unknown Views”, European Signal Processing Conference (EUSIPCO), Lyon, France, 2024.
- Zhaoyang Wang, **Renke Wang** and Pier Luigi Dragotti, “Improved Identification of Natural Frequencies of Fault-Originated Travelling Waves”, IEEE PES Innovative Smart Grid Technologies Europe Conference (ISGT), Dubrovnik, Croatia, 2024.

Invited Talks

- **Renke Wang**, “From Shadows to Shapes: Sampling Scheme for Reconstructing 3D Shapes from Tomographic Projections at Unknown Angles”, winner of the Three Minute Thesis Competition, European Signal Processing Conference (EUSIPCO), Lyon, France, 2024.

1.4 Nomenclature

Acronyms

ACMP	Algebraically Coupled Matrix Pencil method
B-spline	Polynomial Spline

Cryo-EM	Cryogenic Electron Microscopy
CT	Computational Tomography
CRB	Cramér-Rao bound
E-spline	Exponential Spline
FISTA	Fast Iterative Shrinkage-Thresholding Algorithm
FRI	Finite Rate of Innovation
FBP	Filtered Backprojection
MSE	Mean Squared Error
PSNR	Peak Signal to Noise Ratio
RMSE	Root Mean Squared Error
SLAM	Simultaneous Localisation and Mapping
SfM	Structure from Motion
SNR	Signal to Noise Ratio
SSIM	Structural Similarity Index Measure
TV	Total Variation

Numbers

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integer numbers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers
$\mathbb{R}^d$	Vectors of dimension d with real entries
$\mathbb{C}^d$	Vectors of dimension d with complex entries

List of notations in Chapter 3

I	Convex hull of a polyhedron
K	Number of vertices
J	Number of projections
$\mathbf{v}_k$	Locations of vertices in $\mathbb{R}^3$
$g(\mathbf{r})$	Bilevel convex polyhedron in $\mathbb{R}^3$
$\mathbf{d}_j$	Projection direction vector in $\mathbb{R}^3$
$\mathbf{u}_{j,x}, \mathbf{u}_{j,y}$	Direction vectors of rectangular observation window in $\mathbb{R}^3$
$\mathbf{s}_j$	Unknown shifts in the observation window in $\mathbb{R}^2$
$\mathbf{O}_j$	Center of the j th observation window in $\mathbb{R}^3$
R_x, R_y	Sizes of the rectangular observation window
$\varphi(x, y)$	Sampling kernel
$\mathbf{p}_{k,j}$	Location of k th projected vertices on j th observation window in $\mathbb{R}^2$
$z_{k,j}$	Correspondence of $\mathbf{p}_{k,j}$ in the complex plane
$h(z)$	Analytic function

List of notations in Chapter 4

$I(x, y)$	2D image
θ	Angle in $[0, \pi)$ rad
$g_\theta(t)$	1D Radon transform at angle θ
J	Number of projections
$m_{k,j}$	Geometrical moment of order k of the j th projection
$d_{k,l}$	Weighted 2D complex moments of the image $I(x, y)$
$\tilde{\theta}_j$	Estimation of the j th projection angle
C	FRI curve
$\mu(x, y)$	Mask function in the trigonometric polynomial form

U_C	Piecewise constant region with boundary C
1_{U_C}	Indicator function
$\hat{I}$	Fourier transform of $I(x, y)$
$\tilde{g}_{\theta_j}(t)$	j th noisy projection measurement
$\bar{\mu}(x, y)$	Sum of squares mask function

Chapter 2

Preliminaries

Qualsiasi dato diventa importante quando si è connesso a un altro.

— Umberto Eco · *Il pendolo di Foucault*

Any fact becomes important when it's connected to another.

— Umberto Eco · *Foucault's Pendulum*

This chapter provides a comprehensive overview of the main mathematical theory and sampling techniques used throughout this thesis. We begin with an overview of traditional sampling, which typically ensures perfect reconstruction under conditions where sample locations are precisely known. In particular, we present the classical sampling results for bandlimited signals, and then discuss techniques for sampling signals with finite rate of innovation (FRI). The chapter then transitions to the problem of tomographic reconstruction under conditions of unknown projection angles, as introduced in Section 2.2. In Section 2.3, we demonstrate that the acquisition of tomographic projection data can be interpreted as a sampling process. In contrast to the classical sampling framework detailed in Section 2.1, this scenario aligns with the problem of sampling at unknown locations.

2.1 Classical sampling

The sampling process refers to the mechanism which converts continuous-time signals into discrete-time samples [94]. The core to this process is to preserve sufficient information to enable perfect reconstruction of the original continuous-time signal from its discrete samples.

In the conventional sampling framework, the continuous-time signal $x(t)$ is first filtered with a sampling kernel $\varphi\left(-\frac{t}{T}\right)$, and then sampled at uniformly-spaced time intervals $t = nT, n \in \mathbb{Z}$. Here, $\varphi(t)$ represents the time reversed and scaled version of the linear time-invariant impulse response $h(t)$ of the acquisition device, and T represents the uniform sampling period. Under the above settings, the uniform discrete samples are given by:

$$y(nT) := y_n = \left\langle x(t), \varphi\left(\frac{t}{T} - n\right) \right\rangle = \int_{-\infty}^{\infty} x(t) \varphi\left(\frac{t}{T} - n\right) dt, \quad (2.1)$$

where $\langle \cdot, \cdot \rangle$ denotes inner product.

The fundamental question in sampling is to determine whether the original continuous signal $x(t)$ can be perfectly reconstructed from the discrete measurements y_n , and to establish a framework that enables such reconstruction.

2.1.1 Bandlimited signals

A classical solution to the above question is given by the well-known Shannon's sampling theorem [84]¹, which states the following:

¹While Shannon is rightly credited with formalizing the sampling theorem and highlighting its significance for communication theory and signal processing, it is noteworthy that similar formulations

Theorem 2.1. (*Shannon, 1949 [84]*) *If a function $x(t)$ contains no frequencies higher than B cycles per second, then it can be completely determined from its ordinates at a sequence of points spaced less than $\frac{1}{2B}$ seconds apart.*

When the input signal $x(t)$ belongs to the subspace of bandlimited signals, Shannon sampling theorem gives the critical sampling rate $f_s = 2B$, which is also called the Nyquist rate, a term coined by Shannon in recognition of Nyquist's contribution in communication theory [68]. Under the conditions stated in Theorem 2.1, the original continuous signal $x(t)$ can be perfectly reconstructed from its discrete samples by employing the following equality:

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(nT) \operatorname{sinc}\left(\frac{t}{T} - n\right) = x(t), \quad (2.2)$$

where the kernel is the ideal low-pass filter: $\operatorname{sinc}(t) = \frac{\sin \pi t}{\pi t}$, and $\hat{x}(t)$ denotes the reconstructed signal. This linear reconstruction formulation indicates that if $x(t)$ belongs to the subspace of bandlimited functions, that is: $x(t) \in \operatorname{span} \left\{ \operatorname{sinc}\left(\frac{t}{T} - n\right) \right\}_{n \in \mathbb{Z}}$, then it can be perfectly reconstructed from discrete point samples $y_n = x(nT)$ using the sinc interpolation formula. At the same time, the sampling process can also be interpreted as projecting $x(t)$ onto the shift-invariant subspace of bandlimited functions. If the original signal happens to belong to the subspace of bandlimited signals, then the above interpolation formula can guarantee the perfect reconstruction of $x(t)$, namely $\hat{x}(t) = x(t)$. However, not all signals are bandlimited. If $x(t)$ is not bandlimited, the convolution with an ideal low pass filter $\operatorname{sinc}(t)$ allows an approximate reconstruction of $\hat{x}(t)$, which is the lowpass approximation of $x(t)$.

While Shannon's sampling theorem is elegant, it is rarely used in practice because the sinc function has infinite support and slow decay. Moreover, it falls short of achieving perfect reconstruction of many real-world signals, as those signals are never exactly bandlimited.

2.1.2 Sampling in shift invariant subspace

Interpreting the sampling process as a projection onto fixed subspaces enables an extension of Shannon's theorem beyond the confines of bandlimited signals. Research in [94, 14] have shown that even signals that are not bandlimited can still be perfectly reconstructed from their uniform samples, as long as they belong to a special class of spaces called shift-invariant subspaces. These spaces include signals that can be represented as a combination of shifted versions of a basic function, such as splines or wavelets. This means that perfect reconstruction is possible for a wider range of signals.

appeared independently in various scientific literatures: by Kotel'nikov (1933) in [53], E. T. Whittaker (1915) in [101, 18] and J. M. Whittaker (1929) in [102].

In this formulation, discrete samples are obtained in the same manner as described in Equation (2.1). The continuous signal $x(t)$ is then reconstructed using a synthesis function $\tilde{\varphi}(t)$, as given by:

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} y_n \tilde{\varphi}\left(\frac{t}{T} - n\right). \quad (2.3)$$

In order to achieve perfect reconstruction, the family of functions $\{\tilde{\varphi}(\frac{t}{T} - n)\}_{n \in \mathbb{Z}}$ needs to form a Riesz basis such that there exists a unique dual function $\varphi(t)$ that satisfies $\langle \varphi(\frac{t}{T} - n), \tilde{\varphi}(\frac{t}{T} - n) \rangle = \delta_{m-n}$, $m, n \in \mathbb{Z}$, where $\delta(\cdot)$ denotes the Dirac delta function.

This sampling process can be interpreted as a projection operator $\mathcal{P}_{\tilde{\varphi}} : L_2 \rightarrow V_{\tilde{\varphi}}$, which computes the minimum-error (in the least square sense) approximation of an input signal $x(t) \in L_2$ onto a shift-invariant subspace $V_{\tilde{\varphi}}$. This subspace $V_{\tilde{\varphi}}$ is generated by the synthesis function $\tilde{\varphi}(t)$ and its uniform shifts: $\text{span}\{\tilde{\varphi}(\frac{t}{T} - n)\}_{n \in \mathbb{Z}}$. Therefore perfect reconstruction of $x(t)$ is achieved when $x(t) \in V_{\tilde{\varphi}}$. It is worth noting that this extends the traditional sampling framework based on the bandlimited sinc space V_{sinc} , providing a more general perspective. The inclusion relationships between the relevant functional spaces can be expressed as: $V_{\text{sinc}} \subset V_{\tilde{\varphi}} \subset V_{L_2}$.

2.1.3 Sampling signals with finite rate of innovation

Finite rate of innovation (FRI) sampling framework [98, 15, 32] extends the scope to signals that are neither bandlimited nor belong to a fixed subspace. This broader framework allows for the recovery of a wider range of signals while maintaining the fundamental principles of sampling theory. In this section, we present an overview of recent advancements in the sampling and reconstruction framework developed for signals with a finite rate of innovation.

1D sampling framework

Continuous signals such as 1D streams of Diracs are obviously neither bandlimited nor belong to a fixed subspace, however, they can be fully characterized by the locations and amplitudes of their impulse components. This is a typical one-dimensional example of an FRI signal. Such signals are determined by a finite set of parameters per unit of time or space and are also known as signals with finite rate of innovation. Vetterli *et al.* [98] introduced a counting function $C_x(t_a, t_b)$ that counts the number of free parameters of $x(t)$ over a time interval $\tau = [t_a, t_b]$. Based on this concept, the rate of innovation of $x(t)$ is defined as:

$$\rho = \lim_{\tau \rightarrow \infty} \frac{1}{\tau} C_x(t_a, t_b). \quad (2.4)$$

1D signals with finite rate of innovation can usually be expressed as follows:

$$x(t) = \sum_{k \in \mathbb{Z}} \sum_{r=0}^R a_{r,k} g_r(t - t_k), \quad (2.5)$$

where the functions $\{g_r(t)\}_{r=0}^R$ are known. The free parameters in the signal $x(t)$ are therefore the time instants $\{t_k\}$ and the amplitudes $\{a_{r,k}\}$. Typical FRI signals include groups of Diracs, groups of pulses, piecewise polynomial signals and piecewise sinusoidal signals. It is noteworthy that bandlimited signals are encompassed by the definition of signals with finite rate of innovation as well. For a real bandlimited signal with a maximum non-zero frequency of B , it can be seen as having $\frac{1}{T} = 2B$ degrees of freedom per unit of time, which is the number of samples per unit of time that specify it. Therefore, its rate of innovation is calculated as: $\rho = \frac{1}{T} = f_s = 2B$.

In [98], the authors showed that it is possible to sample and reconstruct perfectly streams of Dirac impulses, nonuniform splines, and piecewise polynomials using infinite-support kernels such as sinc and Gaussian functions. Later on, authors of [32] identified limitations in the sampling schemes presented in [98], primarily due to the usage of infinite-support kernels, which introduces instability and complexity to the reconstruction algorithm. Specifically, the algorithm's complexity is affected by the global rate of innovation of the continuous signal. In contrast, the findings in [32] demonstrated that a variety of 1D FRI signals with a local finite rate of innovation can be also sampled and perfectly reconstructed using compactly supported sampling kernels, which offer a more robust and efficient solution. The local rate of innovation at time t is defined in [32] as follows:

$$\rho_\tau(t) = \frac{1}{\tau} C_x(t - t_a, t + t_b), \quad \tau = [t_a, t_b]. \quad (2.6)$$

The local rate of innovation plays a more important role when local reconstruction is desired, and is often more useful in practice. To this end, compactly supported kernels that satisfy Strang-Fix conditions [91] are introduced. A more detailed discussion on this topic will be provided later.

By definition, reconstructing the continuous signal $x(t)$ is effectively equivalent to estimating its free parameters. Consequently, the objective of the FRI sampling framework is to determine the free parameters of the continuous signal based on its discrete samples. For the sake of argument, assume that $x(t)$ is a stream of Diracs: $x(t) = \sum_{k=1}^K a_k \delta(t - t_k)$, instead of directly estimating $x(t)$, the FRI-based framework aim to reconstruct a smooth function $\mu(t)$ (typically a polynomial), which vanishes at the non-zero locations of the signal $x(t)$. We visualize this annihilation relation in Figure 2.1. From this perspective, $\mu(t)$ serves as a mask function that annihilates the FRI signal $x(t)$ in the continuous domain:

$$\mu(t)x(t) = 0. \quad (2.7)$$

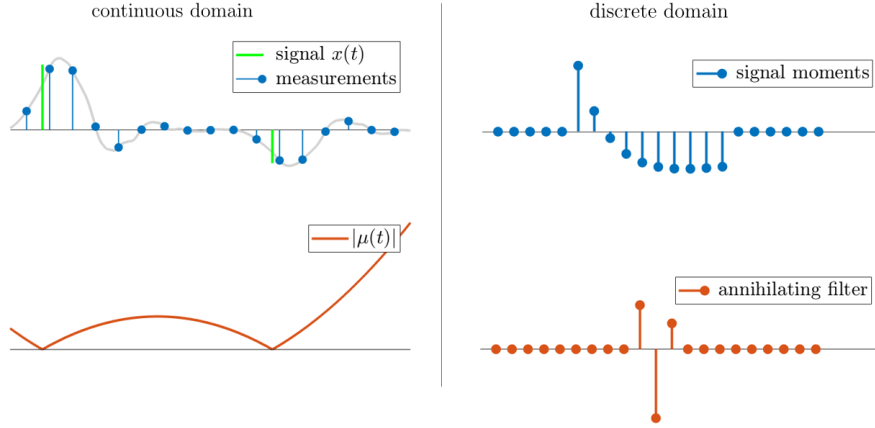


Figure 2.1: Annihilation relation in the continuous domain and discrete domain. In the continuous domain, the smooth function $\mu(t)$ serves as a mask function to the FRI signal $x(t)$: $\mu(t)x(t)=0$. In the discrete domain, the convolution between the coefficients of the smooth function and the signal moments is zero: $\mathbf{h} * \mathbf{s} = 0$.

In general, directly imposing the annihilation relation in Equation (2.7) in the continuous domain is a challenging task, because the continuous domain involves infinite dimensional signal spaces, and operations like convolution or derivative filtering are not easily defined in a finite, computationally tractable way without discretization. The annihilation relation, if applied directly, would require defining a filter that operates perfectly on all points of a continuum, which is not practical. Moreover, signals like Diracs or their derivatives are distributions, not classical functions. Operations like convolution with a filter or enforcing zero output need to be interpreted in the distributional sense, which complicates analysis and numerical implementation. The FRI framework instead demonstrates an equivalent discrete formulation. The core idea is to transform the discrete samples of the FRI signal in Equation (2.1) into a discrete power sum series whose ‘frequency’ components correspond to the free parameters $\{t_k\}$ in Equation (2.5). Such equivalence between the continuous and discrete domain is established by R. de Prony in [76]. More specifically, there exists a finite length filter $\mathbf{h}$ which corresponds to the coefficient of the mask function $\mu(t)$, such that its convolution with the discrete signal moments $\mathbf{s}$ is zero:

$$\mathbf{h} * \mathbf{s} = 0, \quad (2.8)$$

where the discrete filter $\mathbf{h}$ is also called the annihilating filter, and the signal moments $\mathbf{s} : s_m, m = 0, 1, \dots, M-1$ are obtained as the weighted sum of samples y_n , and can always be written in the form of a power sum series as follows:

$$s_m = \sum_{k=1}^K w_k (u_k)^m, \quad (2.9)$$

and $\{w_k, u_k\}$ are the K pairs of parameters related to the free parameters of the signal $x(t)$. These parameters are to be estimated from the signal moments s_m by using spectral estimation methods such as Prony's method or matrix pencil method. Details on the exact estimation of parameters of $\{w_k, u_k\}_{k=1}^K$ are presented in Appendix A.1.

Core to the FRI sampling scheme is the selection of an appropriate sampling kernel $\varphi(t)$ such that the samples y_n can be transformed into viable discrete signal moments s_m . The first kernel used for sampling FRI signals is the traditional ideal low-pass filter [98], i.e. the sinc function. Sampling a τ -periodic train of K Diracs with the sinc function produces samples y_n , and the discrete Fourier coefficients of y_n yield consecutive Fourier series coefficients that can serve as viable signal moments, and can be used to recover the free parameters using Prony's method. However, a fundamental limitation arises because the sampling kernel (the sinc function) is infinitely supported, making it physically non-realizable. Within the FRI framework, typical choices of compactly supported sampling kernels include the exponential or polynomial reproducing kernel, as introduced in [32]. In what follows, we analyze how these sampling kernels allow the transformation of samples y_n into signal moments s_m .

Polynomial reproducing kernel: A polynomial reproducing kernel of order M is a function that together with its shifted versions can reproduce polynomials of maximum degree $M - 1$. Namely, there exist coefficients c_n^m such that:

$$\sum_{n \in \mathbb{Z}} c_n^m \varphi\left(\frac{t}{T} - n\right) = t^m, \quad m = 0, 1, \dots, M - 1, \quad (2.10)$$

where the subscript n denotes the shift and the superscript m denotes the order of the polynomial. The coefficients are predetermined and given by:

$$c_n^m = \frac{1}{T} \int_{-\infty}^{\infty} t^m \tilde{\varphi}\left(\frac{t}{T} - n\right) dt, \quad (2.11)$$

where $\tilde{\varphi}(t)$ are chosen to form a quasi-biorthonormal set with $\varphi(t)$ [32]. This includes any functions satisfying the so-called Strang-Fix conditions [91], a typical example of such is the B-spline. A B-spline of order M can be obtained from the $(M + 1)$ -th order convolution with the zero-order B-spline $\beta_0(t)$:

$$\beta_M(t) = \underbrace{\beta_0(t) * \beta_0(t) \dots \beta_0(t)}_{M+1 \text{ times}}, \quad (2.12)$$

where the the zero-order B-spline $\beta_0(t)$ is defined as:

$$\beta_0(t) = \begin{cases} 1, & 0 \leq t < 1, \\ 0, & \text{otherwise} \end{cases} \quad \xrightarrow{\mathcal{F}} \quad \beta_0(\omega) = \frac{1 - e^{-i\omega}}{i\omega}, \quad (2.13)$$

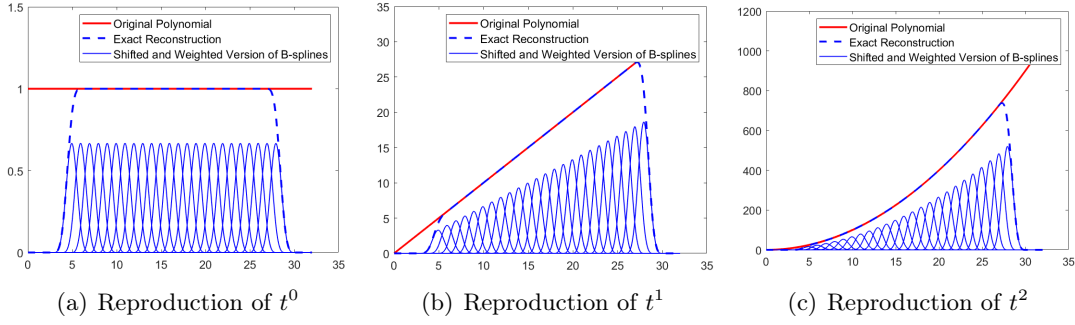


Figure 2.2: Reproduction of polynomials up to degree 2 using shifted and weighted versions of B-spline $\beta_3(t)$.

where i is the imaginary unit. A visual example of polynomial reproduction using B-splines is shown in Figure 2.2. More importantly, any kernel $\varphi(t)$ that can reproduce polynomials up to degree M can be decomposed into a B-spline of order M and a distribution $u(t)$: $\varphi(t) = \beta_M(t) * u(t)$. The Strang-Fix conditions also facilitate the use of scaling and wavelet functions within wavelet theory for the sampling and reconstruction of FRI signals [32]. For instance, as illustrated in Figure 2.1 (a), the sampling of the Dirac signal $x(t)$ is performed using the *daubechies-4* scaling function.

Take 1D streams of Diracs $x(t) = \sum_{k=1}^K a_k \delta(t - t_k)$ as an example, we show that by employing a polynomial reproducing kernel as the sampling kernel $\varphi(t)$, the samples y_n can be transformed into a power sum series (or signal moments):

$$\begin{aligned}
 s_m &= \sum_{n \in \mathbb{Z}} c_n^m y_n \\
 &\stackrel{(a)}{=} \left\langle x(t), \sum_{n \in \mathbb{Z}} c_n^m \varphi\left(\frac{t}{T} - n\right) \right\rangle \\
 &= \left\langle \sum_{k=1}^K a_k \delta(t - t_k), \sum_{n \in \mathbb{Z}} c_n^m \varphi\left(\frac{t}{T} - n\right) \right\rangle \\
 &\stackrel{(b)}{=} \int_{-\infty}^{\infty} \sum_{k=1}^K a_k \delta(t - t_k) t^m dt \\
 &= \sum_{k=1}^K a_k t_k^m, \quad m = 0, \dots, M-1,
 \end{aligned} \tag{2.14}$$

where (a) follows from the linearity of the inner product and (b) follows from the polynomial reproduction property stated in Equation (2.10). In Appendix A.1, we provide details showing that, for $M \geq 2K$, the pairs representing the rate of innovation $\{a_k, t_k\}$ can be exactly recovered from M such signal moments in Equation (2.14), by using Prony's method. When working with polynomial splines, other types of 1D FRI

signals, such as differentiated streams of Diracs and piecewise polynomials, can also be transformed into valid signal moments by leveraging finite differences of the samples [32].

Exponential reproducing kernel: An exponential reproducing kernel of order M is a function that together with its shifted versions can reproduce M different exponentials of the form $e^{\alpha_m t}$ with $\alpha_m = \alpha_0 + m\lambda$, where $\lambda \in \mathbb{C}$ and $m = 0, 1, \dots, M - 1$. Namely, there exist coefficients c_n^m such that:

$$\sum_{n \in \mathbb{Z}} c_n^m \varphi\left(\frac{t}{T} - n\right) = e^{\alpha_m t}, \quad m = 0, 1, \dots, M - 1. \quad (2.15)$$

Similar to the polynomial reproducing case, the predetermined coefficients can be computed as:

$$c_n^m = \frac{1}{T} \int_{-\infty}^{\infty} e^{-\alpha_m t} \tilde{\varphi}\left(\frac{t}{T} - n\right) dt. \quad (2.16)$$

The theory of exponential reproduction stems from the notion of exponential splines (E-splines), and an E-spline of order M can be obtained from the $(M + 1)$ -th order convolution with the zero-order E-spline:

$$\beta_{\vec{\alpha}}(t) = \beta_{\alpha_0}(t) * \beta_{\alpha_1}(t) * \dots * \beta_{\alpha_{M-1}}(t), \quad (2.17)$$

where $\vec{\alpha} = [\alpha_0, \alpha_1, \dots, \alpha_{M-1}]$, $\alpha_m = \alpha_0 + m\lambda$, and the zero-order E-spline is defined as:

$$\beta_{\alpha_m}(t) = \begin{cases} e^{\alpha_m t}, & 0 \leq t < 1, \\ 0, & \text{otherwise} \end{cases} \quad \xrightarrow{\mathcal{F}} \quad \hat{\beta}_{\alpha_m}(\omega) = \frac{1 - e^{\alpha_m - i\omega}}{i\omega - \alpha_m}. \quad (2.18)$$

We note that when $\alpha_m = 0$, the zero-order E-spline reduces to a classical zero-order B-spline. Moreover, since the exponential property is preserved through convolution, any composite function of the form is also able to reproduce exponentials. Therefore, kernels satisfying the exponential reproduction property in Equation (2.15) comprises all kernels that can be written as: $u(t) * \beta_{\vec{\alpha}}(t)$. An example of the reproduction of exponentials using E-splines is shown in Figure 2.3.

Following a similar derivation as Equation (2.14), we show that by using an exponential reproducing spline as the sampling kernel $\varphi(t)$, the samples y_n of a stream of Diracs can

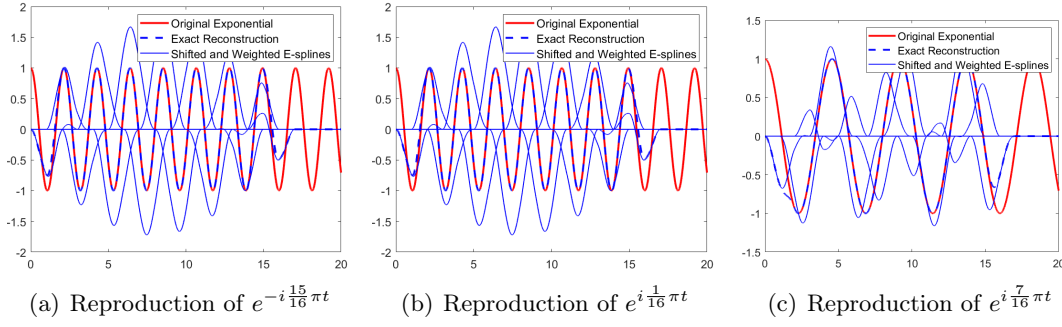


Figure 2.3: Reproduction of exponentials $\{e^{-i\frac{15}{16}\pi t}, e^{i\frac{1}{16}\pi t}, e^{i\frac{7}{16}\pi t}\}$ using shifted and weighted versions of a third order E-spline. Here only the real part of the signal is shown.

be transformed into signal moments in the form of a power sum series:

$$\begin{aligned}
 s_m &= \sum_{n \in \mathbb{Z}} c_n^m y_n \\
 &= \int_{-\infty}^{\infty} \sum_{k=1}^K a_k \delta(t - t_k) e^{\alpha_m t} dt \\
 &= \sum_k a_k e^{\alpha_m t_k}, \\
 &= \sum_{k=1}^K a_k \underbrace{e^{\alpha_0 t_k}}_{\hat{a}_k} \underbrace{e^{(\lambda t_k)m}}_{\hat{u}_k^m}, \quad m = 0, \dots, M-1.
 \end{aligned}$$

In Appendix A.1, we provide details showing that, for $M \geq 2K$, the pairs $\{\hat{a}_k, \hat{u}_k\}$ can be exactly recovered from M such signal moments using Prony's method.

Other sampling kernels: As mentioned in the previous section, a critical requirement in the FRI framework is the capability of the sampling kernel $\varphi(t)$ to reproduce exponential or polynomial functions. This property enables the formulation of the continuous signal reconstruction problem into a spectral estimation problem, which is well studied and can be solved by methods such as Prony's method.

Apart from the kernels that can reproduce exactly polynomials or exponentials, authors in [95] have considered the case where the sampling kernel does not reproduce exponentials or polynomials exactly. They proposed alternative strategies for achieving approximate reproduction and for finding the best coefficients c_n^m to retrieve FRI signals. These approximation strategies offer two key advantages: first, they provide greater adaptability to diverse acquisition devices where the sampling kernel cannot be controlled. Second, in scenarios with significant noise, the stability introduced by approximation enables the use

of a broader range of kernels, whose benefits may surpass the impact of approximation error.

2D sampling framework

For sampling a 2D signal $I(x, y)$, the procedure is similar to the one in Equation (2.1), where the samples $I_{m,n}$ are obtained as follows:

$$\begin{aligned} I(mT_x, nT_y) &:= I_{m,n} \\ &= \left\langle I(x, y), \varphi \left(\frac{x}{T_x} - m, \frac{y}{T_y} - n \right) \right\rangle \\ &= \iint_{-\infty}^{\infty} I(x, y) \varphi \left(\frac{x}{T_x} - m, \frac{y}{T_y} - n \right) dx dy, \quad m, n \in \mathbb{Z}, \end{aligned} \quad (2.19)$$

where $\varphi(x, y)$ is the 2D sampling kernel, and $T_x, T_y \in \mathbb{R}^+$ are the sampling intervals along x and y directions respectively. In the FRI sampling scheme, extending the 1D sampling kernel to a 2D sampling kernel is straightforward and can be achieved by taking the tensor product of the 1D kernels. For example, a 2D kernel $\varphi(x, y)$ that can reproduce polynomials up to degree $P - 1$ can be constructed by taking the tensor product of two 1D polynomial reproducing kernels that can reproduce polynomials up to order x^α and y^β respectively, where $\alpha, \beta = 0, 1, \dots, P - 1$. In this case, there exist coefficients $c_{m,n}^{\alpha,\beta}$ such that:

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} c_{m,n}^{\alpha,\beta} \varphi \left(\frac{x}{T_x} - m, \frac{y}{T_y} - n \right) = x^\alpha y^\beta, \quad \alpha, \beta = 0, \dots, P - 1. \quad (2.20)$$

Similarly, 2D exponential reproducing kernels of degree P can be constructed by taking the tensor product of two 1D exponential kernels of degree $P - 1$, and they satisfy:

$$\sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} c_{m,n}^{p,q} \varphi \left(\frac{x}{T_x} - m, \frac{y}{T_y} - n \right) = e^{\alpha_p x} e^{\beta_q y}, \quad p, q = 0, \dots, P - 1, \quad (2.21)$$

where $\alpha_p = \alpha_0 + \lambda_x p$ and $\beta_q = \beta_0 + \lambda_y q$.

A simple 2D FRI signal is the 2D Diracs, as shown in Figure 2.4 (a), represented as: $I(x, y) = \sum_{k=1}^K a_k \delta(x - x_k, y - y_k)$, and it is completely specified by the amplitudes $\{a_k\}_{k=1}^K$ and the locations of Diracs $\{x_k, y_k\}_{k=1}^K$. When sampled using a 2D exponential spline, its corresponding signal moments can be obtained as follows:

$$s_{p,q} = \sum_{k=1}^K a_k e^{\alpha_p x_k} e^{\beta_q y_k}, \quad p, q = 0, \dots, P - 1. \quad (2.22)$$

One might initially consider a straightforward extension of the 1D spectral estimation

method to this 2D setting. However, this extension fails because the 2D annihilating filter $\mathbf{h}$ which satisfies $\mathbf{h} * \mathbf{s} = 0$ is not unique. To address this issue, Vanpoucke *et al.* [97] introduced a matrix enhancement approach called algebraically coupled matrix pencils (ACMP) method, which retrieves the correctly paired coordinates by simultaneously solving two algebraically related generalized eigenvalue equations.

Later on, more complex 2D signal models have been explored [60, 86, 22, 4, 72]. Notably, Shukla *et al.* [86] introduced a global framework for the sampling and reconstruction of bilevel polygons using polynomial splines, where the free parameters of the signal correspond to the locations of its vertices, as shown in Figure 2.4 (b). Unlike Dirac signals, the geometrical signal moments $s_{\alpha,\beta}$ derived from the weighted samples do not conform to a power sum series form. However, the authors have shown that, complex geometrical moments synthesized from $s_{\alpha,\beta}$ adhere to the power sum series form, enabling the recovery of the signal's free parameters. Additional details on this method are provided in Appendix A.2.

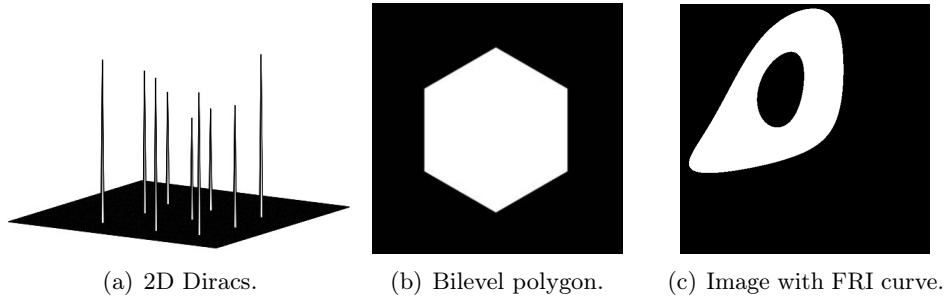


Figure 2.4: Some examples of 2D FRI signals.

A more intricate 2D FRI model, representing curves with finite rate of innovation, was later proposed by Pan *et al.* [72]. In this model, the curve is represented as the zero level-set of a 2D function $\mu(x, y)$, expressed as a weighted sum of a finite number of trigonometric functions:

$$\mu(x, y) = \sum_{k=-K}^K \sum_{l=-L}^L c_{k,l} e^{i \frac{2\pi k x}{\tau_x} + i \frac{2\pi l y}{\tau_y}}, \quad (2.23)$$

where the coefficients $c_{k,l}$ are the free parameters, i is the imaginary unit and τ_x, τ_y denotes the spatial periods. The function $\mu(x, y)$ serves as an annihilating function (or mask function) for a piecewise constant image $I(x, y)$ whose edgeset corresponds to the curve, as illustrated in Figure 2.4 (c). The authors demonstrated that the following continuous domain annihilation relation holds:

$$\mu(x, y) I'(x, y) = 0, \quad (2.24)$$

where $I'(x, y)$ denotes the derivative of $I(x, y)$. To align with the FRI framework, a

discrete annihilation relation is derived by sampling $I(x, y)$ using an ideal low pass filter. The 2D discrete Fourier coefficients $\hat{I}_{k,l}$ of the samples $I_{m,n}$ are then shown to satisfy the following discrete annihilation relation:

$$c_{k,l} * \hat{I}_{k,l} = 0. \quad (2.25)$$

Thus, the estimation of the free parameters lies primarily in solving the above annihilation relation. A more detailed description of this model will be provided later in Chapter 4.

2.1.4 Sampling and reconstruction of FRI signals in the presence of noise

In the previous section, we have reviewed the reconstruction of continuous FRI signals from their filtered and sampled discrete signals in ideal and noiseless setting. However, in practical scenarios, the acquired samples may be invariably subject to various sources of degradation, including measurement noise, quantization errors, and model mismatches arising from deviations between the assumed signal model and the true underlying process. These perturbations can significantly impact the stability and accuracy of the reconstruction process. Therefore, in this section, we review the development of robust algorithms that operates in the presence of noise.

In the presence of noise, the annihilation relation in Equation (2.7) and Equation (2.8) are no longer satisfied exactly. Nonetheless, it remains meaningful to estimate the annihilating filter $\mathbf{h}$ by seeking an approximate solution that minimizes the Euclidean norm of the linear system in Equation (A.3) associated with the discrete annihilation equation. It is known that this minimization can be solved using total-least square approach.

While the total least squares approach provides a viable estimate of the annihilating filter under moderate noise conditions, its performance tends to degrade significantly in low signal-to-noise ratio (SNR) regimes, where the reconstruction results may become unstable or unreliable. To address this limitation, Blu et al. [15] proposed an alternative denoising strategy that leverages the structural properties of the measurement data. Specifically, in the absence of noise, the Toeplitz matrix constructed from the signal moments s_m exhibits a rank equal to the rate of innovation K . This intrinsic low-rank structure can be exploited for denoising by enforcing both the Toeplitz structure and the rank constraint in an iterative fashion. To this end, the authors advocate the use of the Cadzow denoising algorithm, an iterative projection method that alternates between enforcing the low-rank condition and the Toeplitz structure, effectively filtering out noise while preserving the underlying signal structure.

Another effective approach to deal with noisy observations is represented by the subspace

estimator methods, e.g. matrix pencil [43, 61]. The core idea involves constructing two Hankel or Toeplitz matrices from a finite sequence of signal moments or samples, with a fixed shift or pencil. In the absence of noise, these matrices share a common column space, and the generalized eigenvalues of the matrix pencil formed by the pair encode the parameters of interest, such as the locations of Diracs. In the presence of noise this method remains effective due to its reliance on the signal subspace, which can be estimated via singular value decomposition (SVD). Its main advantage over Cadzow denoising method lies in its non-iterative nature which leads to fast computation. The matrix pencil approach is also extended to the 2D scenario, where algebraically coupled matrix pencil (ACMP) can be applied to reconstruct classes of 2D FRI signals (e.g., Diracs and exponentials) from multiple measurement samples that share common underlying parameters (e.g., locations of the 2D Diracs or exponential rates) [97]. ACMP jointly processes this information by forming a set of algebraically coupled matrix pencils by stacking structured matrices (e.g., Hankel or Toeplitz matrices) derived from signal moments into a larger composite pencil, and solving a generalized eigenvalue problem that yields the shared parameters.

More recently, the authors in [36] proposed a model-fitting approach that departs from traditional annihilation-based methods. Instead of explicitly enforcing the annihilation relation, their strategy involves directly fitting a parametric FRI model to the observed noisy measurements in order to estimate the underlying FRI samples. This approach has shown improved performance in the context of sampling and reconstruction of 1D Dirac streams and exponential signals, particularly in low signal-to-noise ratio scenarios, when compared to classical algebraic techniques. However, the method is inherently iterative and does not offer formal guarantees of convergence, which may limit its applicability in time-critical applications. An overview of this model-fitting approach and its implementation details is provided in Appendix B.5.

2.2 Unknown angle tomographic reconstruction

Tomography is a classical image reconstruction problem whose goal is to reconstruct an unknown structure from its lower dimensional tomographic projections at different angles. A tomographic projection is a shadow-like image created by exposing an object to penetrating radiation, capturing its internal density as viewed along a specific angle. This problem finds applications ranging from medical imaging, microscopy, seismology to biogeography [58].

In this thesis, we consider tomographic projections as parallel line projections of an object. Let $I(\mathbf{x})$ represent an object in two or three dimensional space, where $\mathbf{x} \in \mathbb{R}^k$, $k = 2, 3$. The parallel line projection of $I(\mathbf{x})$ along the projection direction $\mathbf{d} \in \mathbb{R}^k$ with $\|\mathbf{d}\| = 1$,

is defined as:

$$\mathcal{P}_d\{I\}(\mathbf{t}) = \int_{-\infty}^{\infty} I(\mathbf{x}) \prod_{n=1}^{k-1} \delta(\mathbf{x}^T \mathbf{u}_n - t_n) d^k \mathbf{x}, \quad (2.26)$$

where

- $\mathbf{t} = [t_1, t_2, \dots, t_{k-1}]^T$ represents the coordinate of the detection line or plane.
- $\mathbf{u}_1, \dots, \mathbf{u}_{k-1} \in \mathbb{R}^k$ are mutually orthogonal basis vectors defining the detection plane or line, satisfying $\mathbf{u}_n^T \mathbf{d} = 0$ for all $n = 1, \dots, k-1$, which means that these vectors are orthogonal to the projection direction $\mathbf{d}$.
- $\delta(\cdot)$ is the Dirac delta function.

When $k = 2$, the above expression is also known as the 2D Radon transform. When the projection is taken at angle $\theta \in [0, \pi)$ on the 2D plane, namely, when the projection direction is $\mathbf{d} = [\sin \theta, -\cos \theta]$, the direction vector of the detection line is given by $\mathbf{u}_1 = [\cos \theta, \sin \theta]^T$, as shown in Figure 2.5, then the above equation can be written as:

$$\begin{aligned} g_\theta(t) &:= \int_{-\infty}^{\infty} I(\mathbf{x}) \delta(\mathbf{x}^T \mathbf{u}_1 - t) d^2 \mathbf{x} \\ &= \mathcal{P}_d\{I\}(t) = \iint_{-\infty}^{\infty} I(x, y) \delta(x \cos \theta + y \sin \theta - t) dx dy. \end{aligned} \quad (2.27)$$

A visual illustration of the above equation can be seen in Figure 2.5. Given several

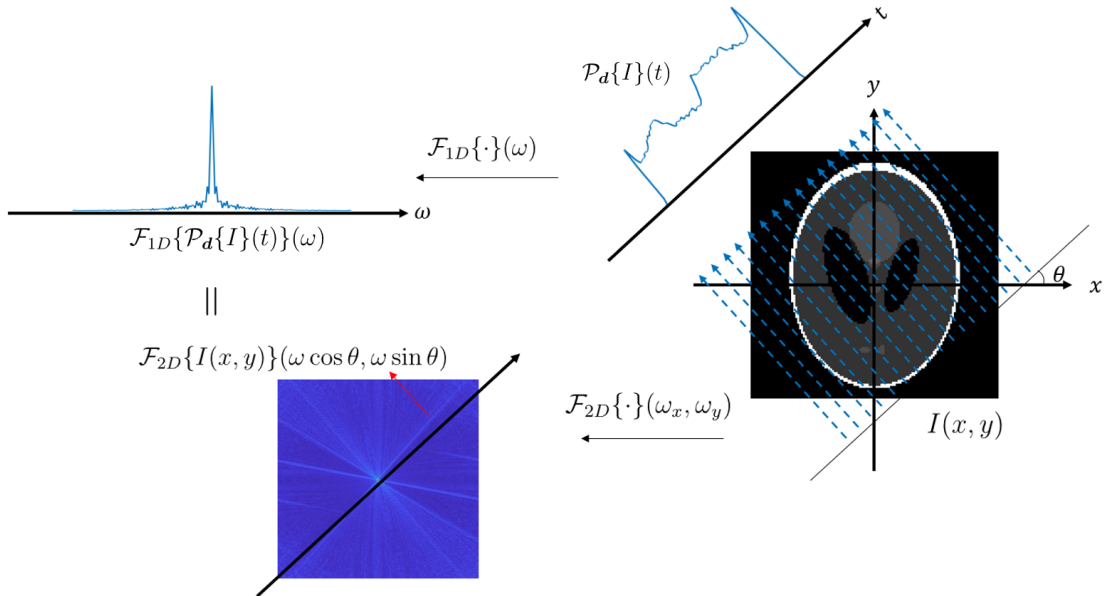


Figure 2.5: An illustration of the central slice theorem.

such projections at angles $\{\theta_j\}_{j=1}^J$, the goal is to reconstruct the 2D object $I(x, y)$. The classical approach is based on the famous central slice theorem (also known as the Fourier slice theorem). We first consider the Fourier transform of the image:

$$\mathcal{F}_{2D}\{I(x, y)\}(\omega_x, \omega_y) = \hat{I}(\omega_x, \omega_y) = \int_0^{\tau_x} \int_0^{\tau_y} I(x, y) e^{-i(\omega_x x + \omega_y y)} dx dy. \quad (2.28)$$

We then observe that, the 1D Fourier transform of the projection at projection angle θ can be written as:

$$\begin{aligned} \mathcal{F}_{1D}\{g_\theta(t)\}(\omega) &= \hat{g}_\theta(\omega) = \int_{-\infty}^{\infty} g_\theta(t) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} \int_0^{\tau_x} \int_0^{\tau_y} I(x, y) \delta(x \cos \theta + y \sin \theta - t) e^{-i\omega t} dx dy dt \\ &= \int_0^{\tau_x} \int_0^{\tau_y} I(x, y) e^{-i\omega(x \cos \theta + y \sin \theta)} dx dy \\ &= \int_0^{\tau_x} \int_0^{\tau_y} I(x, y) e^{-j\omega \cos \theta x - j\omega \sin \theta y} dx dy. \end{aligned} \quad (2.29)$$

The above equation is equivalent to evaluating the 2D Fourier transform of the image in Equation (2.28) on a slice with coordinates $(\omega_x, \omega_y) = (\omega \cos \theta, \omega \sin \theta)$. This leads to the central slice theorem which states that the Fourier transform of the 1D projection data is equal to the 2D Fourier transform of the object sliced through the origin in the parallel direction to the projection line, mathematically this can be expressed as:

$$\mathcal{F}_{1D}\{g_\theta(t)\}(\omega) = \hat{g}(\omega) = \mathcal{F}_{2D}\{I(x, y)\}(\omega \cos \theta, \omega \sin \theta) = \hat{I}(\omega \cos \theta, \omega \sin \theta). \quad (2.30)$$

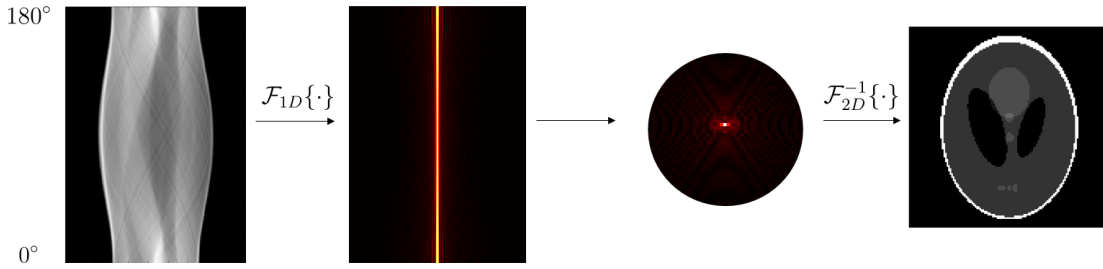


Figure 2.6: An illustration of the central slice theorem induced tomographic reconstruction method. The Fourier transform of the sinogram is first computed, then the rows of the Fourier transform are rearranged into diameters of the circle in the 2D plane. Finally, the reconstructed image is obtained by applying the inverse 2D Fourier transform of the rearranged Fourier plane.

The 2D data matrix which encapsulates all the projections (line integrals) at various angles is called a sinogram, effectively storing the Radon transform data needed to reconstruct the original image. The central slice theorem gives rise to a category of tomographic reconstruction methods, which essentially takes the Fourier transform of

the projection sinogram and rearrange them as diameters of a circle on the 2D Fourier plane. Finally, the image is reconstructed by taking the inverse 2D Fourier transform. This process is illustrated in Figure 2.6. While the central slice theorem is formulated here in 2D, its extension to higher dimensions, including 3D spaces, is straightforward. Methods that incorporate such ideas include direct Fourier methods [89] and filtered back projection (FBP) methods [47]. Image reconstruction error bounds have been derived for various types of FBP filters for piecewise smooth images within Sobolev or Besov spaces [11]. In general, within the classical tomographic reconstruction framework, where projection angles are known, the reconstruction error decreases as the number of projections increases.

More recently, the advent of advanced sensing and imaging techniques has introduced a novel class of tomographic reconstruction problems, namely, tomographic reconstruction from projections at unknown angles. This problem is primarily motivated by its occurrence in single particle cryogenic electron microscopy (also known as cryo-EM), where the goal is to reconstruct a high-resolution 3D structure of biomolecules from its 2D tomographic projections at unknown angles. During the data acquisition procedure, homogeneous biomolecules samples are rapidly frozen under cryogenic temperature in a thin layer of ice, an electron beam is then directed through the frozen layer, casting their tomographic projections on the detection plane. Because the orientations of the biomolecules are unknown upon freezing, the resulting tomographic projections are captured at various, unknown angles. The presence of unknown projection angles poses a significant challenge for the accurate reconstruction of 3D structures from projection measurements. The projection acquisition process for Cryo-EM is illustrated in Figure 2.7 (a). Beyond Cryo-EM, the problem of tomographic reconstruction at unknown angles has relevance in a range of other applications, including industrial radiography [12] and oceanography [58], among others.

Over the years, extensive research has been devoted to addressing the problem in both 2D and 3D settings. One class of methods focuses on first estimating the unknown projection angles, employing either common-line-based approaches [75] or moment-based techniques [10, 8, 9, 56]. Subsequently, the structure is reconstructed using either direct filtered back projection [78, 71] or regularized optimization-based methods [30, 65]. In particular, moment-based angle estimation approaches exploit data consistency constraint [42, 59], which establishes systems of equations linking the geometric moments of the structure to those of its projections. The constraint will be elaborated upon in Chapter 4.

Another category of methods iteratively alternate between refining the estimation of projection angles and the reconstruction of the unknown structure by either projection matching [74, 5, 6, 63] or expectation maximization (EM) [82]. The angular refinement in projection matching is achieved by comparing every projection image against the clean templates obtained from the current estimate of the structure and then assigns the angular parameters based on the closest match. These methods usually perform

angular assignment on a discretized space, where increased resolution incurs significant computational costs and may perform poorly in low signal-to-noise ratio (SNR) scenarios. To address the challenges posed by noise, EM-based methods adopt soft angular assignments through posterior distributions [81, 88, 77]. Then, the structure is computed as the maximum-a-posteriori (MAP) estimation using the EM algorithm. A key advantage of EM methods is that they do not limit the projections to a unique angular class. This leads to increased robustness in high-noise regimes compared to projection matching procedures. Other joint refinement methods have also been proposed, which often require *ab initio* initialization [6, 106].

A third category of methods completely bypass the estimation of projection angles by capturing rotational invariant features [44, 57, 104, 85, 48]. This is usually done by analyzing autocorrelations of projection images in spatial or frequency domain, leveraging first-, second-, and occasionally higher-order moments to infer the underlying structure.

Many of the aforementioned methods often adopt either a purely discrete or a purely continuous perspective when modeling the projection measurements, unknown angles, or underlying structure. While such approaches simplify the problem and facilitate practical implementation, they fail to fully capture the interplay between the continuous and discrete aspects of the underlying phenomena. This interplay, which can be more accurately represented through a sampling framework, is critical for theoretically understanding the data acquisition and reconstruction process. Interpreting this problem from a sampling perspective also facilitates the development of a rigorous theoretical framework to establish sufficient conditions for perfect reconstruction of both the unknown structure and projection angles. In the following section, we establish a sampling framework for the unknown angle tomographic reconstruction problem. Notably, this aligns with the broader category of problems of sampling at unknown locations, offering new insights into the problem’s theoretical and practical aspects.

2.3 Sampling at unknown locations

As we have seen in Section 2.1, from the foundational Shannon’s sampling theorem to the latest advancement in the finite rate of innovation, sampling theories have provided perfect reconstruction conditions when the precise locations of the samples are known.

However, in many applications, due to the limitations of the sampler or the nature of the problem, the sampling locations may not be available or only available partially. An example is the celebrated problem of simultaneous localization and mapping (SLAM), where a mobile sensor tries to infer the environment plan (mapping) while at the same time locating itself using the measurements from the environment. While taking samples from the environment, it is unaware of its own locations, and thus, the sampling locations

are unknown. Likewise in computer vision, the goal of a classic problem called structure from motion (SfM) is to reconstruct a 3D scene from a set of 2D images captured from different unknown view points. This process involves estimating the spatial and depth information of the scene based on the 2D measurements at unknown perspective. Another important application in the field of bioimaging is cryo-EM presented in the previous section. In many of the problems discussed, the measurement acquisition process can be effectively represented through an equivalent sampling framework. In particular, in the context of Cryo-EM, the point spread function of the microscopy lens can be modeled as a 2D sampling kernel $\varphi(x, y)$, while the sensor array on the detection plane can be represented as the sampling grid. This formulation leads to an equivalent sampling setup, as depicted in Figure 2.7 (b), which facilitates a unified analysis of the underlying continuous to discrete measurement process.

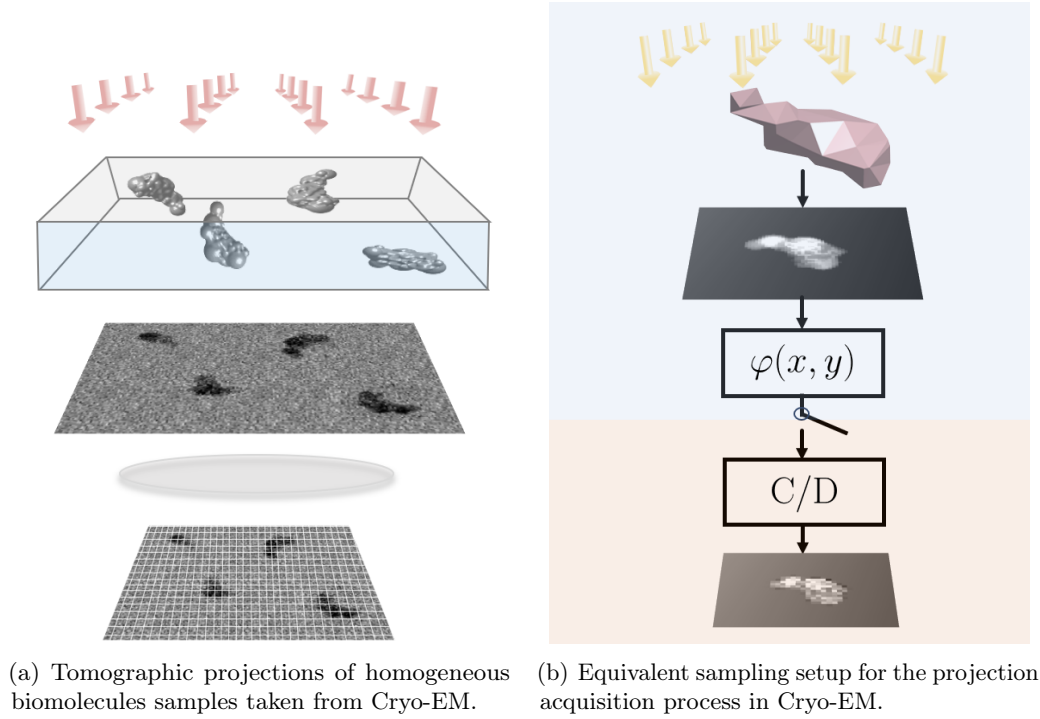


Figure 2.7: The process of acquiring tomographic projections in Cryo-EM and its equivalent sampling setup is illustrated. In subfigure (a): homogeneous biomolecules are frozen within a thin layer of ice, an electron beam is passed through the ice, and continuous projections of the biomolecules are formed. These projections are then filtered by the microscope lens’s point spread function, and discrete measurements of the filtered projections are recorded. Subfigure (b) presents the equivalent sampling setup for this acquisition process.

Looking at these problems from a sampling perspective, unlike the classical sampling scheme in Section 2.1 where the locations of the samples are predetermined and known, the sampling locations in the above problems are unknown. In other words, the quantity

mT_x and nT_y in Equation (2.19) cannot be obtained. In this case, it is unsurprising that we cannot solve these problems to get unique solutions for perfect reconstruction in general. Although the theoretical analysis of signal reconstruction from unknown sampling locations poses significant challenges, some works have explored this topic in the past decade [62, 55, 54, 17, 33], and they mainly focus on the 1D case. Apart from the work in [62], which addressed the problem in the discrete case, analyses in the continuous setting primarily focus on imposing constraints on the sampling locations. These constraints may require the sampling locations to exhibit statistical dependencies, as investigated in [55, 54], or to ensure that the trajectory of the sampling locations lies within a known space of functions [33]. In higher-dimensional settings, [2, 38] address the SLAM problem for a static field and a diffusion field, respectively. By imposing sparsity constraints on the field environment and the trajectory of the mobile sensor to be piecewise linear, the authors demonstrate that both the field and the trajectory of the mobile sensor can be reconstructed, up to an orthogonal transformation.

It is important to highlight a key distinction between the two most common high-dimensional sampling problems with unknown locations: SLAM and tomographic reconstruction. In SLAM, the sample values are directly obtained from the signal or its filtered version. In contrast, for tomographic reconstruction problems such as Cryo-EM, the sample values are derived from projections of the original signal rather than the signal itself. This is the core setting that will be studied in this thesis.

Chapter 3

Reconstructing classes of 3D FRI signals from sampled tomographic projections at unknown angles and its application

横看成岭侧成峰，远近高低各不同。
不识庐山真面目，只缘身在此山中。

——苏轼·《题西林壁》

Let eye go: near is high as far as near is low.
Sidewise you see a spine of ridges, outward peaks.
How can you hope to view Mt. Lu from every side
When your body is dwelling in her very eye?

— Su shi · *Inscribed on a wall of Xilin Temple*

Traditional sampling schemes often assume that the sampling locations are known. Motivated by the recent bioimaging technique known as cryogenic electron microscopy (cryo-EM), we consider a novel sampling setup where the aim is to reconstruct an unknown 3D structure from samples of its 2D tomographic projections at unknown angles. In this chapter, we focus on classes of 3D signals with finite rate of innovation, specifically, 3D convex bilevel polyhedra and 3D point sources, and show that the exact estimation of these 3D structures and of the projection angles can be achieved up to an orthogonal transformation. Moreover, we are able to show that the minimum number of projections needed to achieve perfect reconstruction is independent of the complexity of the signal model. By using the divergence theorem, we are able to retrieve the projected vertices of the polyhedron from the sampled tomographic projections, and then we show how to retrieve the 3D object and the projection angles from this information. The proof of our theorem is constructive and leads to a robust reconstruction algorithm, which we validate under various conditions. Finally, we apply aspects of the proposed framework to calibration of X-ray computed tomography (CT) data.

3.1 Introduction

Sampling plays an important role in signal processing. It refers to the mechanism which converts continuous signals into discrete sequences [94]. From the foundational Whittaker-Shannon theorem [84] to the latest advancements in compressive sensing [19, 31], finite rate of innovation [98, 15, 32], and super resolution [20, 39], sampling theories have provided perfect reconstruction conditions when the precise locations of the samples are known.

However, in many applications, the sampling locations may not be available or only available partially. This happens, for example, in a classical simultaneous localization and mapping (SLAM) problem in robotics where a location-unaware mobile sensor attempts to jointly determine its own locations and reconstruct a map using measurements from the environment [2, 38]. Likewise in computer vision, one tries to reconstruct a 3D scene from a series of images from different unknown viewpoints, as known as structure from motion (SfM) [41]. In structural biology, the latest technique to reconstruct a high resolution 3D structure of particles of interest is cryo-EM, where one has only access to parallel beam projections of the particle at unknown angles [13, 96, 9, 10]. In medical computed tomography (CT), projection angles are assumed to be known, however, they can be inaccurate due to patient motion or machine mis-calibration.

For the aforementioned reconstruction problems, it is not feasible to apply directly traditional sampling methods. Instead, we need to consider the problem of sampling at unknown locations. Compared to prior works, sampling at unknown locations has been so far less-explored. Marziliano *et al.* [62] considered the reconstruction of discrete-time

bandlimited signals with unknown sampling locations. Another discrete-time formulation is unlabeled sensing, where a linear sensing system contains unlabeled observation data [93]. As for the continuous-time counterpart, Browning [17] introduced an alternating least squares algorithm for bandlimited signals. Nordio *et al.* [67] also provided asymptotic performance analysis for the reconstruction of bandlimited signals using linear techniques. While Kumar [55, 54] took a statistical perspective and considered the unknown sampling locations to be statistically dependent. He also showed that the reconstruction error is asymptotically inversely proportional to the number of samples. Recently, Elhami *et al.* [33] showed that by constraining the sampling positions to belong to some known function spaces, it is possible to transform the location-unaware signal recovery problem to that of regular sampling of a composite of functions.

Despite the encouraging results, the theoretical works have so far only considered more traditional sampling setups where the samples are directly taken from the signal. Motivated by cryoEM, in this work, we consider a specific case for the location-unaware sampling, where the aim is to reconstruct an unknown 3D structure from samples of its 2D parallel beam tomographic projections at unknown projection angles.

Conventional methods directly retrieve the projection angles using common-line based algorithms [75] or moment based methods [9, 10, 8, 7], then the 3D structure is estimated through filtered back-projection [78, 71] or iterative regularized optimization [30]. Another category of methods alternate between the estimation of the projection angles and the refinement of the 3D structure. This can be achieved by projection matching [74, 5, 6, 63, 106] or formulating the 3D refinement step as a maximum marginalized *a posteriori* problem [27, 81, 82, 88]. The third category of methods completely bypass the process of estimating the projection angles by either distribution matching [40] or estimating rotational invariant features [57, 44, 104].

These methods often adopt a discrete perspective on the projection measurements or angles. While this simplification leads to practical implementation, it does not fully model the underlying continuous process. Moreover, there is still no theoretical analysis of sufficient conditions on e.g. the number of projections that ensures a perfect reconstruction.

In this work, we focus on specific classes of signals with finite rate of innovation (FRI), specifically convex bilevel polyhedra and 3D point sources, and address the fundamental sampling question of when perfect reconstruction of the 3D structure can be achieved, given only samples of a limited number of 2D projections at unknown angles. We present a method with a constructive proof that allows the simultaneous estimation of the 3D signal and projection angles. The estimation is exact up to an orthogonal transformation. Moreover, we show that the minimum number of projections needed is three and is irrespective of the complexity of the 3D signal.

In addition to its theoretical contribution, the proposed method can be applied to the geometrical calibration of real CT systems. During the calibration process, radiographs of reference objects (usually spherical markers) at equidistant angles are often used to determine the system geometry and to correct the geometrical offsets. Nevertheless, in certain instances, acquiring equidistant radiographs may prove unattainable due to misalignment or the absence of advanced equipment. We show that the proposed method can be applied to the calibration of projection angles of real CT data obtained at the FleX-ray laboratory at the Center for Mathematics & Computer Science, Amsterdam.

The rest of the chapter is organized as follows. In Section 3.2, we define the signal model and formulate the reconstruction problem from sampled projections at unknown angles. Then, in Section 3.3, we present an exact reconstruction method that achieves the simultaneous estimation of the 3D signal and the projection angles. In Section 3.4, we propose alternative strategies to robustify the method in case the samples are corrupted with noise. We then provide numerical results on synthetic data in Section 3.5, and show that the proposed framework can be applied for calibration in real computed tomography problems. Finally, we conclude in Section 3.6.

3.2 Image model and problem formulation

We consider a 3D convex, non-degenerate polyhedron with K vertices $\{\mathbf{v}_k\}_{k=1}^K$, where $K \in \mathbb{N}$ and $K \geq 4$, $\mathbf{v}_k \in \mathbb{R}^3$ denotes the coordinate of the k th vertex. The origin is assumed to be the geometric center of the vertices, in other words: $\sum_{k=1}^K \mathbf{v}_k = \mathbf{0}$, and all vertices lie within a sphere region of known radius $R \in \mathbb{R}^+$ centered at the origin, as shown in Figure 3.1.

The convex hull Γ of the vertices is then specified by :

$$\Gamma = \left\{ \sum_{k=1}^K \lambda_k \mathbf{v}_k : \forall \lambda_k \in \mathbb{R}, \lambda_k \geq 0 \text{ and } \sum_{k=1}^K \lambda_k = 1 \right\}. \quad (3.1)$$

We define its corresponding 3D bilevel, convex and simply connected polyhedron model $g(\mathbf{r})$ as follows:

$$g(\mathbf{r}) = \begin{cases} 1, & \text{for } \mathbf{r} \in \Gamma, \\ 0, & \text{otherwise,} \end{cases} \quad \forall \mathbf{r} \in \mathbb{R}^3. \quad (3.2)$$

Assume $J \geq 3$ parallel-beam projections of the 3D volume $g(\mathbf{r})$ are taken. The projection angles are assumed to be distinct, and are represented by unit direction vectors $\mathbf{d}_j \in$

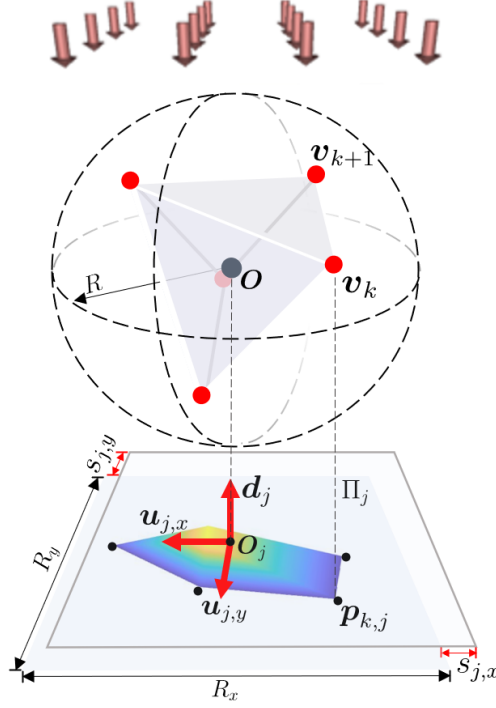


Figure 3.1: An example of a bilevel convex polyhedron model $g(\mathbf{r})$ with $K = 5$ vertices. The figure also depicts the parallel beam projection onto the observation window Π_j . Due to an unknown amount of shift, the center of the actual observation window (grey boundary) deviates from the projection of the origin $\mathbf{O}_j$ by $s_{j,x}$, $s_{j,y}$ along the two edges $\mathbf{u}_{j,x}$, $\mathbf{u}_{j,y}$ respectively.

$\mathbb{R}^3$, $j = 1, \dots, J$. On the j th projected 2D plane, we consider a rectangular observation window Π_j whose center $\mathbf{O}_j \in \mathbb{R}^3$ corresponds to the projection of the origin $\mathbf{O}$, as shown in Figure 3.1. We also assume that the observation window Π_j containing the 2D projection is of known dimension (R_x, R_y) such that $R_x, R_y > 2R$. The 2D tomographic projection $I_j(x, y)$ can then be expressed as the Radon transform of $g(\mathbf{r})$ onto the j th observation window Π_j as follows:

$$I_j(x, y) = \int_{\mathbb{R}^3} g(\mathbf{r}) \delta(x - \mathbf{r}^T \mathbf{u}_{j,x} - s_{j,x}) \delta(y - \mathbf{r}^T \mathbf{u}_{j,y} - s_{j,y}) d^3 \mathbf{r}, \quad j = 1, \dots, J \quad (3.3)$$

where $\mathbf{u}_{j,x}, \mathbf{u}_{j,y} \in \mathbb{R}^3$ denote the two orthogonal unit direction vectors representing the two edges of the rectangular window Π_j , and they satisfy:

$$\mathbf{u}_{j,x} \times \mathbf{u}_{j,y} = \mathbf{d}_j \quad (3.4)$$

$$\mathbf{u}_{j,x}^T \mathbf{u}_{j,y} = 0, \quad j = 1, \dots, J \quad (3.5)$$

where $\times$ denotes the vector product. Moreover, vectors in $\{\mathbf{u}_{j,x}\}_{j=1}^J$ are assumed to be pairwise distinct. The same assumption holds true for $\{\mathbf{u}_{j,y}\}_{j=1}^J$. Furthermore, $\mathbf{s}_j = [s_{j,x}, s_{j,y}]^T \in \mathbb{R}^2$ denotes the shifts of the observation window Π_j along $\mathbf{u}_{j,x}$, $\mathbf{u}_{j,y}$

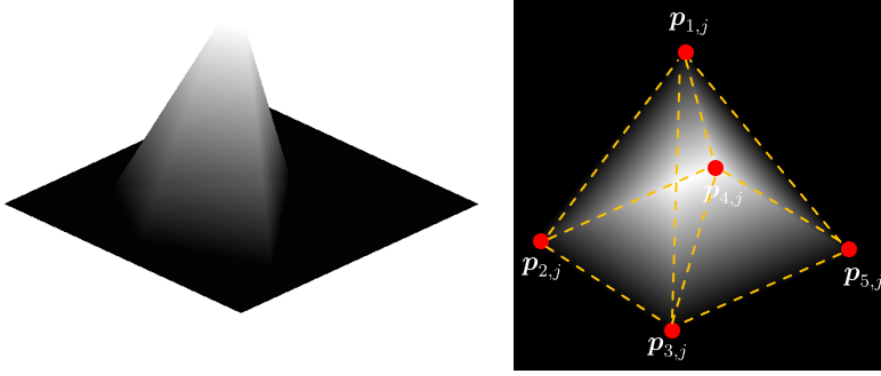


Figure 3.2: An example of a tomographic projection $I_j(x, y)$ of the polyhedron shown in Figure 3.1. The projection can be decomposed into 6 disjoint triangular regions with linear changes of intensity. The projected vertices $\mathbf{p}_{k,j}$ are denoted with red dots.

directions respectively. Thus, the center of the actual observation window is $\mathbf{O}'_j = \mathbf{O}_j + s_{j,x}\mathbf{u}_{j,x} + s_{j,y}\mathbf{u}_{j,y}$.

A visual example of the projection acquisition process is shown in Figure 3.1. It is of interest to note that due to the constant intensity assumption of the polyhedron model, the 2D projection $I_j(x, y)$ is actually piecewise linear and composed of triangular regions with linear changes of intensity. An example of a 2D projection is shown in Figure 3.2.

To avoid degeneracy, we assume $\forall k \neq k', \mathbf{v}_k - \mathbf{v}_{k'}$ is not colinear with $\mathbf{d}_j$ to avoid degenerate arrangements of the projected vertices with respect to the direction vector, where two projected vertices coincide with each other.

Moreover, to model the blurring effect of point spread functions and discrete sensors during the measurement stage, we consider the projection $I_j(x, y)$ to be filtered with a 2D sampling kernel $\varphi(x, y)$, and the measurements are taken on a uniform grid with step sizes T_x and T_y along the directions $\mathbf{u}_{j,x}$ and $\mathbf{u}_{j,y}$ respectively. Hence, we observe the following 2D samples:

$$\begin{aligned} I_j[m, n] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} I_j(x, y) \varphi\left(\frac{x}{T_x} - m, \frac{y}{T_y} - n\right) dx dy \\ &= \left\langle I_j(x, y), \varphi\left(\frac{x}{T_x} - m, \frac{y}{T_y} - n\right) \right\rangle, \end{aligned} \quad (3.6)$$

where $m, n \in \mathbb{N}$, and $T_x, T_y \in \mathbb{R}^+$ are the sampling intervals. Here $\langle \cdot, \cdot \rangle$ denotes inner product. For the rest of this thesis, we assume $T_x = T_y = T$ without loss of generality.

Given only 2D samples of the projections $I_j[m, n]$, $j = 1, \dots, J$, the problem we consider is the simultaneous estimation of the 3D polyhedron $g(\mathbf{r})$, the projection direction vectors $\{\mathbf{d}_j\}_{j=1}^J$ and the shifts $\{\mathbf{s}_j\}_{j=1}^J$.

In what follows, we show that the above problem admits a unique solution up to a 3D orthogonal transformation for polyhedra with $K \geq 4$ vertices, when $J \geq 3$ and the sampling kernel $\varphi(x, y)$ is a polynomial reproducing kernel up to order $2K - 4$ along both axes.

3.3 Exact estimation using samples on projections from unknown projection angles

There is a unique way in which the vertices $\mathbf{v}_k$ can be connected to form the convex polyhedron $g(\mathbf{r})$ [64]. Hence, estimating $g(\mathbf{r})$ is equivalent to estimating the locations of its vertices in $\mathbb{R}^3$ space. We denote the projected location of $\mathbf{v}_k$ on the plane Π_j as $\mathbf{p}_{k,j} = [p_{k,j}^x, p_{k,j}^y]^T \in \mathbb{R}^2$ (see Figure 3.3).

Moreover, from geometrical properties of the parallel beam projection, $p_{k,j}^x$ and $p_{k,j}^y$ can be expressed as the projection of the vertex $\mathbf{v}_k$ onto the direction vectors $\mathbf{u}_{j,x}$ and $\mathbf{u}_{j,y}$, then shifted by $s_{j,x}$ and $s_{j,y}$ along the two directions respectively:

$$\begin{aligned} \mathbf{p}_{k,j} &= [p_{k,j}^x, p_{k,j}^y]^T \\ &= [\mathbf{v}_k^T \mathbf{u}_{j,x} + s_{j,x}, \mathbf{v}_k^T \mathbf{u}_{j,y} + s_{j,y}]^T. \end{aligned} \quad (3.7)$$

The above geometric relation is illustrated in Figure 3.3. It links the desired 3D vectors $\mathbf{v}_k$, $\mathbf{u}_{j,x}$ and $\mathbf{u}_{j,y}$ with 2D parameters $\mathbf{p}_{k,j}$, and provides valuable insights on how we can unveil the 3D geometric information from lower dimensional features.

In what follows, we provide a constructive solution to the stated reconstruction problem. As a first step, we address the 2D problem of extracting the 2D parameters $\mathbf{p}_{k,j}$ from the samples $I_j[m, n]$. Then, we show that the desired 3D vectors can be estimated simultaneously by leveraging the geometric relations in Equation (3.7). An overview of the proposed method is illustrated in Figure 3.4.

3.3.1 Estimation of $\mathbf{p}_{k,j}$

We first establish a connection between the projection $I_j(x, y)$ and the projected locations $\mathbf{p}_{k,j}$ by introducing an auxiliary analytic function and state the following lemma:

Lemma 3.1. *There exist complex coefficients $\rho_{1,j}, \rho_{2,j}, \dots, \rho_{K,j} \in \mathbb{C}$, such that for any complex function $h(z)$, $z \in \mathbb{C}$ analytic within the support of $I_j(x, y)$ obtained from*

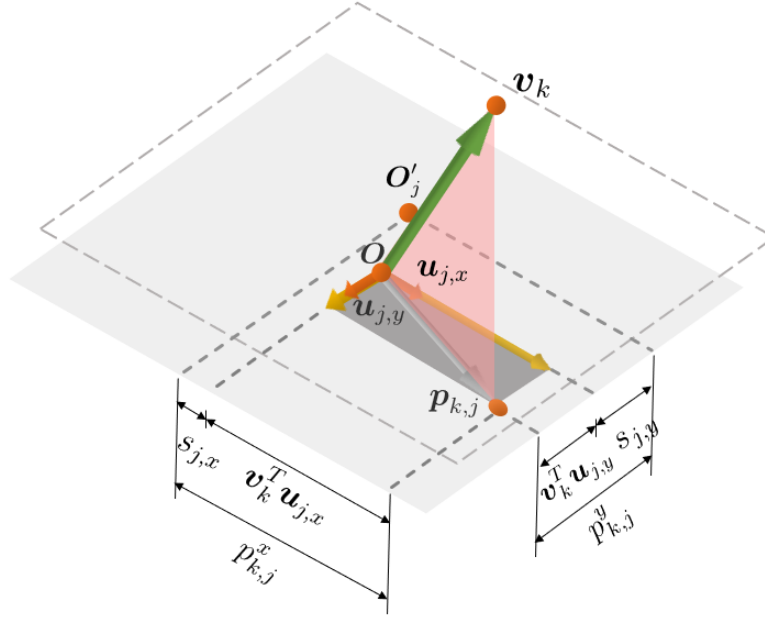


Figure 3.3: An illustration of the geometric relation. The grey plane represents a parallel plane to Π_j passing through the origin $\mathbf{O}$. The transparent plane with dashed line boundaries represents the shifted observation window whose center $\mathbf{O}'_j$ is shifted by $\mathbf{s}_j$. The inner product of $\mathbf{v}_k$ (green vector) with $\mathbf{u}_{j,x}$ (orange vector) gives $p_{k,j}^x$ up to the shift $s_{j,x}$. One can similarly obtain $p_{k,j}^y$.

Equation (3.3) with $g(\mathbf{r})$ given in Equation (3.2), the following equality holds:

$$\iint_{\Pi_j} I_j(x, y) h'''(z) dx dy = \sum_{k=1}^K \rho_{k,j} h(z_{k,j}), \quad (3.8)$$

where $z_{k,j} \in \mathbb{C}$ and $z_{k,j} = p_{k,j}^x + ip_{k,j}^y$ corresponds to the locations of $\mathbf{p}_{k,j}$ in the complex plane, and $h'''(\cdot)$ is the third order derivative of $h(\cdot)$.

Proof. See Appendix B.1. □

From Lemma 3.1, we understand the importance of finding a proper analytic function $h(z)$ that enables the estimation of $z_{k,j}$ from the *rhs* expression in Equation (3.8), as well as to approximate exactly the *lhs* integral of Equation (3.8) using the samples $I_j[m, n]$.

For this purpose, we choose $h(z) = z^r$. By doing so, the estimation task is converted to a spectral analysis problem. While for the second problem, we show that as long as the sampling kernel $\varphi(x, y)$ in Equation (3.6) can reproduce exactly 2D polynomials up to a certain order, the double integral on the *lhs* of Equation (3.8) can be obtained exactly from the 2D samples [98, 32]. An alternative choice of $h(z)$ is discussed in Appendix B.2.

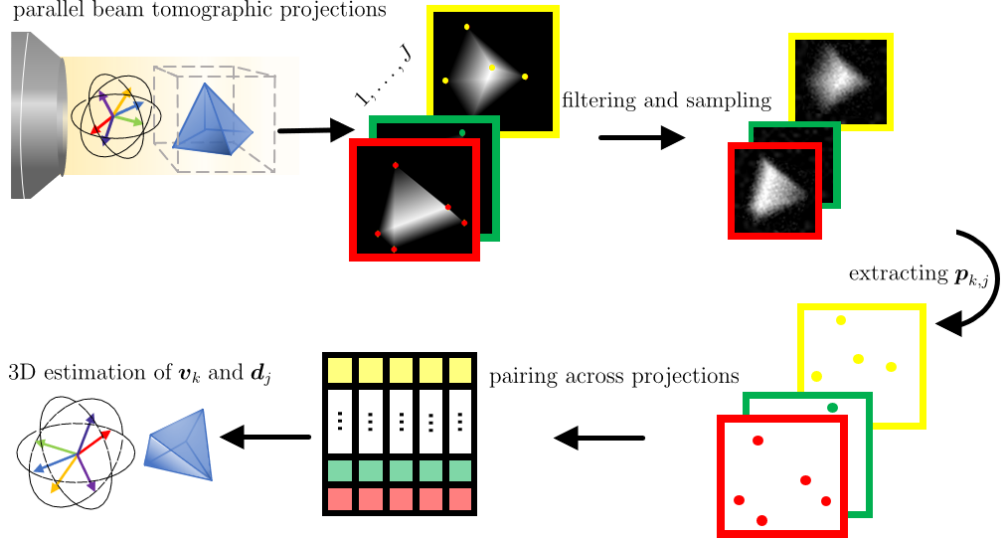


Figure 3.4: An illustration of the overall method. J projections of the convex polyhedron is taken at different projection angles (indicated with arrows of different colors). The projections are then filtered and sampled. The features $p_{k,j}$ are first extracted from the sampled projections (Sec 3.3.1). Then they are paired across different projections (Sec 3.3.2). Finally, the 3D polyhedron and projection directions are estimated from the ordered features using an algebraic approach described in Sec 3.3.3.

By choosing $h(z) = z^r, r \in \mathbb{N}$, Equation (3.8) becomes:

$$\begin{aligned}
 \iint_{\Pi_j} I_j(x, y) h'''(z) dx dy &= \iint_{\Pi_j} I_j(x, y) (z^r)''' dx dy \\
 &= r(r-1)(r-2) \iint_{\Pi_j} I_j(x, y) (x + iy)^{r-3} dx dy \\
 &= r(r-1)(r-2) \tau_{r-3,j} = \tilde{\tau}_{r,j} \\
 &= \sum_{k=1}^K \rho_{k,j} z_{k,j}^r,
 \end{aligned} \tag{3.9}$$

where

$$\tau_{r,j} = \iint_{\Pi_j} I_j(x, y) (x + iy)^r dx dy \tag{3.10}$$

is the complex moment of $I_j(x, y)$, while $\tilde{\tau}_{r,j}$ is the weighted version of $\tau_{r,j}$ with weight $r(r-1)(r-2)$.

Note that by construction $\tilde{\tau}_{0,j} = \tilde{\tau}_{1,j} = \tilde{\tau}_{2,j} = 0$. From Equation (3.9), it is clear that $\tilde{\tau}_{r,j}$ are the linear combinations of exponentials of $z_{k,j}$. Thus, we can apply Prony's method (also known as annihilating filter method) [98] to retrieve the K values of $z_{k,j}$ from $2K$ weighted complex moment $\tilde{\tau}_{r,j}$, $r = 0, 1, \dots, 2K - 1$, or equivalently from $(2K - 3)$

complex moments $\tau_{r,j}$, $r = 0, 1, \dots, 2K - 4$.

The remaining issue is to obtain the complex moments $\tau_{r,j}$ from the 2D samples $I_j[m, n]$. In order to achieve this, we require the sampling kernel $\varphi(x, y)$ in Equation (3.6) to be able to reproduce polynomials up to degree $x^{2K-4}y^{2K-4}$. In other words, there exist coefficients $c_{m,n}^{\alpha,\beta}$ such that for $0 \leq \alpha, \beta \leq 2K - 4$:

$$\sum_{m,n} c_{m,n}^{\alpha,\beta} \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) = x^\alpha y^\beta. \quad (3.11)$$

Kernels satisfying the above conditions include polynomial splines, and we refer to [32] for more details. Leveraging the polynomial reproducing property of the sampling kernel, we can retrieve the geometric moments $\mu_{\alpha,\beta,j}$ from the 2D samples $I_j[m, n]$ as follows [86, 100]:

$$\begin{aligned} & \sum_{m,n} c_{m,n}^{\alpha,\beta} I_j[m, n] \\ & \stackrel{(a)}{=} \sum_{m,n} c_{m,n}^{\alpha,\beta} \iint_{\Pi_j} I_j(x, y) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \\ & = \iint_{\Pi_j} I_j(x, y) \sum_{m,n} c_{m,n}^{\alpha,\beta} \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \\ & \stackrel{(b)}{=} \iint_{\Pi_j} I_j(x, y) x^\alpha y^\beta dx dy \\ & = \mu_{\alpha,\beta,j}, \end{aligned} \quad (3.12)$$

where (a) and (b) follows from Equation (3.6) and Equation (3.11) respectively. Finally, the geometric moments $\mu_{\alpha,\beta,j}$ relate to the complex moments $\tau_{r,j}$ through binomial expansion:

$$\tau_{r,j} = \sum_{\beta=0}^r \binom{r}{\beta} i^\beta \mu_{\alpha,\beta,j}, \quad \alpha + \beta = r. \quad (3.13)$$

Based on the above derivations, we now have an algorithm to estimate $\mathbf{p}_{k,j}$ from the sampled projection $I_j[m, n]$:

1. Compute the geometric moments $\mu_{\alpha,\beta,j}$ from the samples $I_j[m, n]$ using Equation (3.12).
2. Compute the complex moment $\tau_{r,j}$ using Equation (3.13).

3. Using the relationship in Equation (3.9):

$$r(r-1)(r-2)\tau_{r-3,j} = \tilde{\tau}_{r,j} = \sum_{k=1}^K \rho_{k,j} z_{k,j}^r,$$

apply Prony's method to $\tilde{\tau}_{r,j}$ to retrieve the locations $\mathbf{p}_{k,j}$.

Once the 2D parameters $\{\mathbf{p}_{k,j}\}_{k=1}^K$ are retrieved, we can estimate the unknown shifts $\mathbf{s}_j$ on each projected plane by summing the retrieved parameters for all vertices as follows:

$$\begin{aligned} \sum_{k=1}^K p_{k,j}^x &\stackrel{(c)}{=} \left(\sum_{k=1}^K \mathbf{v}_k^T \right) \mathbf{u}_{j,x} + K s_{j,x} \stackrel{(d)}{=} K s_{j,x} \\ \sum_{k=1}^K p_{k,j}^y &\stackrel{(c)}{=} \left(\sum_{k=1}^K \mathbf{v}_k^T \right) \mathbf{u}_{j,y} + K s_{j,y} \stackrel{(d)}{=} K s_{j,y}, \end{aligned} \quad (3.14)$$

where (c) follows from Equation (3.3) and (d) follows from the assumption that the geometric center of the vertices is at the origin. We denote the shift corrected parameters as $\tilde{\mathbf{p}}_{k,j}$ with $\tilde{\mathbf{p}}_{k,j} = \mathbf{p}_{k,j} - \mathbf{s}_j$.

3.3.2 Pairing $\tilde{\mathbf{p}}_{k,j}$ with $\mathbf{v}_k$

Given the values of $\tilde{\mathbf{p}}_{k,j}$, we need to identify whether two parameters $\tilde{\mathbf{p}}_{k,j}$ and $\tilde{\mathbf{p}}_{k',l}$ retrieved from two arbitrary projections $I_j[m,n]$ and $I_l[m,n]$ correspond to the same vertex $\mathbf{v}_k$.

We now show that this can be done in a pairwise manner. Given two sets of unpaired parameters $\{\tilde{\mathbf{p}}_{k,j}\}_{k=1}^K$ and $\{\tilde{\mathbf{p}}_{k',l}\}_{k'=1}^K$ related to arbitrary projections $I_j(x,y)$ and $I_l(x,y)$, we build the following matrix $\Upsilon \in \mathbb{R}^{K \times 4}$:

$$\Upsilon = \begin{bmatrix} \tilde{\mathbf{p}}_{1,j}^T & \tilde{\mathbf{p}}_{1,l}^T \\ \tilde{\mathbf{p}}_{2,j}^T & \tilde{\mathbf{p}}_{2,l}^T \\ \vdots & \vdots \\ \tilde{\mathbf{p}}_{K,j}^T & \tilde{\mathbf{p}}_{K,l}^T \end{bmatrix} \quad (3.15)$$

and we make the following observation:

Lemma 3.2. *Let $\Upsilon \in \mathbb{R}^{K \times 4}$ be a matrix constructed from permutations of estimated parameters as defined in Equation (3.15) with $K \geq 4$. Then for almost all such matrices Υ , the matrix is rank deficient only when the entries along the same row k correspond to the same vertex $\mathbf{v}_k$.*

Proof. When the correct correspondence is achieved, the matrix $\Upsilon \in \mathbb{R}^{K \times 4}$ admits the

factorization:

$$\mathcal{T} \stackrel{(e)}{=} \underbrace{\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_K^T \end{bmatrix}}_{K \times 3} \underbrace{\begin{bmatrix} \mathbf{u}_{j,x} & \mathbf{u}_{j,y} & \mathbf{u}_{l,x} & \mathbf{u}_{l,y} \end{bmatrix}}_{3 \times 4} = \mathbf{V}\mathbf{U},$$

where (e) follows from Equation (3.7) and $\mathbf{V} \in \mathbb{R}^{K \times 3}$ and $\mathbf{U} \in \mathbb{R}^{3 \times 4}$, both of rank 3, Hence, $\mathcal{T}$ has at most rank 3 in this case.

Now consider arbitrary permutations of the entries across each row of $\mathcal{T}$. For the matrix to remain rank-deficient (i.e., rank 3), the permuted columns must still satisfy a specific linear dependency. However, in the high-dimensional space $\mathbb{R}^{K \times 4}$, the set of matrices where such dependencies persist under arbitrary nontrivial permutations lies in a measure-zero subset. Thus, any random permutation of the columns almost surely results in a full rank matrix (i.e., rank 4). Therefore, $\mathcal{T}$ is rank deficient almost only when the each row corresponds to the same vertex. $\square$

Lemma 3.2 provides a way to pair the retrieved parameters $\{\tilde{\mathbf{p}}_{k,j}\}_{k=1}^K$ and $\{\tilde{\mathbf{p}}_{k',l}\}_{k'=1}^K$ from two projections. Specifically, when $K \geq 4$, we can perform the pairing between two projections by permuting the order of one set of parameters, e.g. permuting $\{\tilde{\mathbf{p}}_{k',l}\}_{k'=1}^K$. For each permutation, we can construct a matrix $\mathcal{T}$ using Equation (3.15). There are in total $K!$ possible permutations, and we choose the permutation that corresponds to the matrix $\mathcal{T}$ with a deficient rank.

3.3.3 Estimation of d_j and v_k

We are now in the position to state the sampling theorem for an exact estimation of the 3D polyhedron in Equation (3.2) from samples of its 2D tomographic projections at unknown angles in Equation (3.6):

Theorem 3.1. *Given uniform samples $I_j[m, n]$, $j = 1, \dots, J$, of $J \geq 3$ projections at unknown projection angles, the projection angles and any 3D bilevel convex polyhedron with $K \geq 4$ vertices can be exactly reconstructed up to a 3D orthogonal transformation when the sampling kernel $\varphi(x, y)$ is a polynomial reproducing kernel up to order $2K - 4$ along both axes.*

Proof. Given that the sampling kernel $\varphi(x, y)$ can reproduce 2D polynomials up to degree $2K - 4$ along both axes, by considering Lemma 3.1 with a specific choice of the analytic function $h(z) = z^r$, the 2D parameters $\tilde{\mathbf{p}}_{k,j}$ can be retrieved exactly for every

2D projection as in Section 3.3.1. Then, given $\tilde{\mathbf{p}}_{k,j}$, we perform the pairing by building the matrix $\Upsilon \in \mathbb{R}^{K \times 4}$ in Equation (3.15) and applying the rank criterion in Lemma 3.2.

After pairing the parameters $\tilde{\mathbf{p}}_{k,j}$, we now show that it is possible to estimate exactly the projection directions $\mathbf{d}_j$ and the locations of the vertices $\mathbf{v}_k$ up to an orthogonal transformation, using a factorization approach similar to the one proposed in [92]. In what follows, we provide a detailed explanation of the steps involved in the approach.

Using only the first element of the paired parameters $\{\tilde{p}_{k,j}^x\}_{k=1,j=1}^{K,J}$, we build the following matrix $\boldsymbol{\Omega}_x \in \mathbb{R}^{K \times J}$:

$$\begin{aligned} \boldsymbol{\Omega}_x &= \begin{bmatrix} \tilde{p}_{1,1}^x & \tilde{p}_{1,2}^x & \cdots & \tilde{p}_{1,J}^x \\ \tilde{p}_{2,1}^x & \tilde{p}_{2,2}^x & \cdots & \tilde{p}_{2,J}^x \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{p}_{K,1}^x & \tilde{p}_{K,2}^x & \cdots & \tilde{p}_{K,J}^x \end{bmatrix}_{K \times J} \\ &= \underbrace{\begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_K^T \end{bmatrix}}_{K \times 3} \underbrace{\begin{bmatrix} \mathbf{u}_{1,x} & \mathbf{u}_{2,x} & \cdots & \mathbf{u}_{J,x} \end{bmatrix}}_{3 \times J} = \mathbf{V} \mathbf{U}_x. \end{aligned} \quad (3.16)$$

By construction, the matrices $\mathbf{V}$ and $\mathbf{U}_x$ are of dimensions $K \times 3$ and $3 \times J$ respectively. Therefore, the matrix $\boldsymbol{\Omega}_x$ has rank ≤ 3 . Similarly, we build another rank deficient matrix $\boldsymbol{\Omega}_y$ using the second element in the paired parameters $\{\tilde{p}_{k,j}^y\}_{k=1,j=1}^{K,J}$:

$$\boldsymbol{\Omega}_y = \begin{bmatrix} \mathbf{v}_1^T \\ \mathbf{v}_2^T \\ \vdots \\ \mathbf{v}_K^T \end{bmatrix} \begin{bmatrix} \mathbf{u}_{1,y} & \mathbf{u}_{2,y} & \cdots & \mathbf{u}_{J,y} \end{bmatrix} = \mathbf{V} \mathbf{U}_y. \quad (3.17)$$

Similar to $\boldsymbol{\Omega}_x$, $\boldsymbol{\Omega}_y$ is also of rank ≤ 3 .

We first concatenate the two matrices: $\boldsymbol{\Omega} = [\boldsymbol{\Omega}_x \mid \boldsymbol{\Omega}_y]$, such that $\boldsymbol{\Omega} \in \mathbb{R}^{K \times 2J}$. It follows that

$$\boldsymbol{\Omega} = \mathbf{V} [\mathbf{U}_x \mid \mathbf{U}_y], \text{ and } \text{rank}(\boldsymbol{\Omega}) \leq 3, \quad (3.18)$$

where $[\mathbf{U}_x \mid \mathbf{U}_y] \in \mathbb{R}^{3 \times 2J}$. Then we perform singular value decomposition on the matrix: $\boldsymbol{\Omega} = \mathcal{U} \mathcal{S} \mathcal{V}^T$. The initial estimation of the direction vector matrices $\mathbf{U}_x, \mathbf{U}_y$ can be obtained as:

$$[\tilde{\mathbf{U}}_x \mid \tilde{\mathbf{U}}_y] = \mathcal{S} \mathcal{V}^T = [\tilde{\mathbf{u}}_{1,x} \cdots \tilde{\mathbf{u}}_{J,x} \mid \tilde{\mathbf{u}}_{1,y} \cdots \tilde{\mathbf{u}}_{J,y}].$$

Clearly, the initial estimation $\begin{bmatrix} \tilde{\mathbf{U}}_x & | & \tilde{\mathbf{U}}_y \end{bmatrix}$ relates to the true direction vectors $\begin{bmatrix} \mathbf{U}_x & | & \mathbf{U}_y \end{bmatrix}$ by an unknown linear transformation, which we denote with $\mathbf{Q} \in \mathbb{R}^{3 \times 3}$:

$$\mathbf{Q} \begin{bmatrix} \tilde{\mathbf{U}}_x & | & \tilde{\mathbf{U}}_y \end{bmatrix} = \begin{bmatrix} \mathbf{U}_x & | & \mathbf{U}_y \end{bmatrix}. \quad (3.19)$$

We identify the linear transformation $\mathbf{Q}$ by considering the fact that the matrix $\begin{bmatrix} \mathbf{U}_x & | & \mathbf{U}_y \end{bmatrix}$ has unit norm columns: $\mathbf{u}_{j,x}^T \mathbf{u}_{j,x} = 1$ and $\mathbf{u}_{j,y}^T \mathbf{u}_{j,y} = 1$. We replace $\mathbf{u}_{j,x}$, $\mathbf{u}_{j,y}$ with $\mathbf{Q}\tilde{\mathbf{u}}_{j,x}$ and $\mathbf{Q}\tilde{\mathbf{u}}_{j,y}$, and this yields:

$$\tilde{\mathbf{u}}_{j,x}^T \underbrace{\mathbf{Q}^T \mathbf{Q}}_{\mathbf{M}} \tilde{\mathbf{u}}_{j,x} = 1 \text{ and } \tilde{\mathbf{u}}_{j,y}^T \underbrace{\mathbf{Q}^T \mathbf{Q}}_{\mathbf{M}} \tilde{\mathbf{u}}_{j,y} = 1, \quad (3.20)$$

where $\mathbf{M} = \mathbf{Q}^T \mathbf{Q}$ is a symmetric matrix of dimension 3×3 . Hence, it satisfies the form

$$\mathbf{M} = \begin{bmatrix} a & b & c \\ b & e & d \\ c & d & f \end{bmatrix},$$

and it has six unknown values. Under the assumption that the vectors in $\{\mathbf{d}_j\}_{j=1}^J$, $\{\mathbf{u}_{j,x}\}_{j=1}^J$ and $\{\mathbf{u}_{j,y}\}_{j=1}^J$ are pairwise distinct, given $J \geq 3$ projections, we can solve uniquely for $\mathbf{M}$ using the system of $2J$ linear equations in Equation (3.20). The linear transformation $\mathbf{Q}$ is then given by $\mathbf{B}(\mathbf{M})^{\frac{1}{2}}$, where $\mathbf{B}$ is an arbitrary orthogonal matrix of dimension 3×3 , since $\mathbf{B}^T \mathbf{B} = \mathbf{I}$. Then, the final estimation of $\tilde{\mathbf{u}}_{j,x}$ and $\tilde{\mathbf{u}}_{j,y}$ can be obtained by applying the linear transformation $\mathbf{Q}$ using Equation (3.19).

Consequently, the vertices $\tilde{\mathbf{v}}_k$ can be retrieved by solving a linear system of equations using Equation (3.16) and (3.17). Finally, given $\tilde{\mathbf{u}}_{j,x}$ and $\tilde{\mathbf{u}}_{j,y}$, the direction vectors $\tilde{\mathbf{d}}_j$ can be retrieved using the relation in Equation (3.4).

Due to the existence of an arbitrary orthogonal matrix $\mathbf{B}$ in the final solution, the estimation of the locations of $\tilde{\mathbf{v}}_k$ and the projection directions $\tilde{\mathbf{d}}_j$ will be up to an orthogonal transformation from the true values. $\square$

It is worth noting that, in the above estimation algorithm, geometrical moments of the 2D projection images are retrieved as an intermediate step for the estimation of 2D feature points (2D projected angles). In fact, besides geometrical moments, exponential moments can also be used. Please refer to Appendix B.2 for more details. This is in contrast to moment-based angle estimation algorithms [10, 8, 7] which estimate unknown 3D angles by solving a polynomial system using the statistics of the 2D projections. Moreover, by leveraging the model assumptions, we significantly reduce the amount of projection data required for unique recovery, which represents a substantial improvement over the general analysis.

We summarize the complete method in Algorithm 3.1.

Algorithm 3.1 Algorithm for reconstructing the 3D polyhedron

Input: 2D samples $\{I_j[m, n]\}_{j=1}^J$, $J \geq 3$ and $K \geq 4$

Output: 3D vertices $\{\tilde{\mathbf{v}}_k\}_{k=1}^K$, projection directions $\{\tilde{\mathbf{d}}_j\}_{j=1}^J$ and 2D shifts $\tilde{\mathbf{s}}_j$

- 1: For every projection, compute geometric moments $\mu_{\alpha, \beta, j}$ using Equation (3.12).
 - 2: Compute weighted complex moment $\tilde{\tau}_{r, j}$ according to Equation (3.13) and Equation (3.9).
 - 3: Apply annihilating filter method to $\tilde{\tau}_{r, j}$ to retrieve 2D parameters $\mathbf{p}_{k, j}$.
 - 4: Estimate in-plane shifts $\mathbf{s}_j$ according to Equation (3.14).
 - 5: Pair corrected parameters $\tilde{\mathbf{p}}_{k, j}$ using the rank criterion in Lemma 3.2.
 - 6: Estimate the direction vectors $\{\tilde{\mathbf{u}}_{j, x}\}_{j=1}^J$ and $\{\tilde{\mathbf{u}}_{j, y}\}_{j=1}^J$ by performing SVD on $\mathbf{\Omega}$ in Equation (3.18).
 - 7: Estimate the linear transform $\mathbf{Q}$ by solving the linear system of equations in Equation (3.20).
 - 8: Apply the transform $\mathbf{Q}$ to the initial estimation of $\{\tilde{\mathbf{u}}_{j, x}\}_{j=1}^J$ and $\{\tilde{\mathbf{u}}_{j, y}\}_{j=1}^J$ to get the final estimation as in Equation (3.19).
 - 9: Estimate 3D vertices $\{\tilde{\mathbf{v}}_k\}_{k=1}^K$ by solving the linear system of equations in Equation (3.17) and Equation (3.16).
 - 10: Compute the projection direction $\{\tilde{\mathbf{d}}_j\}_{j=1}^J$ according to Equation (3.4).
-

Remark 1. The above reconstruction method can be applied to the estimation of 3D point sources from projections at unknown angles. The locations and amplitudes of the projected point sources can be retrieved from the 2D samples using techniques in the area of finite rate of innovation [86, 3, 73]. Given the assumptions as in Theorem 3.1, the projection directions and the locations of point sources can be estimated up to an orthogonal transformation using the proposed method.

Remark 2. The convexity constraint on the 3D polyhedral model is imposed to ensure the unique reconstruction of the polyhedron from the recovered vertices. However, this constraint does not influence the feasibility of the vertex recovery process itself.

Remark 3. The perfect reconstruction of 2D signals from their 1D tomographic projections can also be achieved. For 2D point sources and bilevel convex polygon models, given that there are $K \geq 3$ non-colinear vertices and the projections are taken at $J \geq 3$ different angles, the estimation can be achieved up to a 2D orthogonal transformation.

Remark 4. If three direction vectors in $\mathbf{U}_x$ or $\mathbf{U}_y$ are known, we can determine the orthogonal matrix $\mathbf{B}$ and estimate the locations of the vertices and the projection angles exactly.

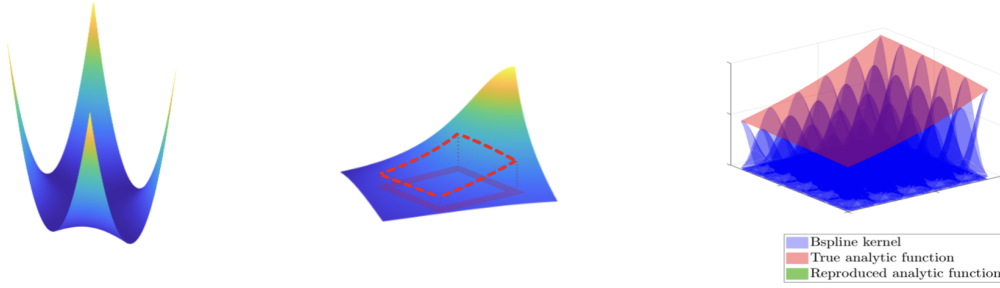


Figure 3.5: An illustration of the analytic functions $h(z) = z^r$ (left) and $h(z) = \frac{1}{(z-a_w)^4}$ (middle). In perfect reconstruction setting, we require that the sampling kernel to reproduce exactly the polynomial function $h(z) = z^r$. However, in noisy scenarios, its polynomial growth rate will cause instability in the estimation result, therefore, we propose to consider alternative analytic function $h(z) = \frac{1}{z-a_w}$, which requires the sampling kernel to approximately reproduce $\frac{1}{(z-a_w)^4}$. The rightmost panel illustrates the reproducing behavior of a 2D B-spline sampling kernel applied to the function $\frac{1}{(z-a_w)^4}$. The reproduced surface closely approximates the original function.

3.4 Noisy setting

We now assume the 2D samples $I_j[m, n]$ are corrupted by additive, white noise. Therefore, we measure $\tilde{I}_j[m, n] = I_j[m, n] + \epsilon_{m,n}$, where $\epsilon_{m,n}$ is the additive noise. In this section, we introduce alternative strategies to mitigate the effect of noise, including robust estimation of the 2D parameters $\mathbf{p}_{k,j}$ and the 3D vectors $\mathbf{v}_k$ and $\mathbf{d}_j$.

When the samples are contaminated by noise, this corruption propagates into the estimated signal moments. Furthermore, due to the polynomial or exponential growth of the coefficients $c_{m,n}^{\alpha,\beta}$, the noise is similarly amplified at a corresponding rate. In the context of the annihilating filter method used in the previous section, these noise-perturbed moments are structured into a Vandermonde matrix. However, Vandermonde matrices are notoriously ill-conditioned, especially as the number of vertices increases. This growth leads to a corresponding increase in the size of the Vandermonde matrix, causing its condition number to grow. Consequently, the associated linear systems become highly sensitive to noise, resulting in numerical instability and significant inaccuracies in the recovered parameters.

3.4.1 2D parameter estimation through analytic approximation

We developed Equation (3.8) in Section 3.3.1 with the analytic functions $h(z)$ to be complex polynomials.

To improve stability, we now relax the constraint of an exact reproduction of polynomials

and opt for a more localized family of analytic functions: $h_{a_w}(z) = \frac{1}{z-a_w}$, where $a_w \in \mathbb{C}$, and $a_w = a_0 e^{i2\pi \frac{w}{W}}$, $w = 0, 1, \dots, W-1$. A visual comparison between the analytic functions is shown in Figure 3.5. Furthermore, $a_0 \in \mathbb{R}$ is chosen to satisfy $|a_0| > R$ so that the function $h_{a_w}(z)$ is analytic in the projection area. By considering Lemma 3.1, we substitute $h_{a_w}(z)$ in Equation (3.8) and get the following expression:

$$\begin{aligned} & \iint_{\Pi_j} I_j(x, y) h_{a_w}'''(z) dx dy \\ &= -6 \iint_{\Pi_j} I_j(x, y) \frac{1}{(z - a_w)^4} dx dy \\ &= \sum_{k=1}^K \frac{\rho_{k,j}}{z_{k,j} - a_w}. \end{aligned} \quad (3.21)$$

In order to approximate the integral in the above equation, we find coefficients $c_{m,n}^w$ such that a linear combination of the sampling kernel $\varphi(x, y)$ and its uniform shifts gives the best approximation of the analytic function $\frac{1}{(z-a_w)^4}$:

$$\sum_{m,n} c_{m,n}^w \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) \cong \frac{1}{(z - a_w)^4}. \quad (3.22)$$

In order to find the optimal coefficients $c_{m,n}^w$, we apply the *least squares approximation method* introduced in [95]. Given the coefficients of $c_{m,n}^w$, we have the following:

$$\begin{aligned} & -6 \sum_{m,n} c_{m,n}^w I_j[m, n] \\ &= -6 \iint_{\Pi_j} I_j(x, y) \sum_{m,n} c_{m,n}^w \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \\ &\cong -6 \iint_{\Pi_j} I_j(x, y) \frac{1}{(z - a_w)^4} dx dy \\ &= \sum_{k=1}^K \frac{\rho_{k,j}}{z_{k,j} - a_w} = \eta_{w,j}. \end{aligned} \quad (3.23)$$

Given $\{\eta_{w,j}\}_{w=0}^{W-1}$, an annihilating filter based solution can be applied to retrieve the positions of the projected vertices $z_{k,j}$ when $W > 2K$ [49] (please refer to Appendix B.4 for more details). Compared to the choice of analytic functions in Section 3.3.1, the advantage of choosing $h_{a_w}(z) = \frac{1}{z-a_w}$ is that, when the number of the vertices K increases, we only need to place more poles on the circle instead of increasing the polynomial orders as in Equation (3.9). Moreover, the specific form of the chosen analytic function facilitates a robust estimation algorithm. Since $h_{a_w}(z)$ is a rational function,

$\eta_{\omega,j}$ can be expressed as the ratio of two polynomials:

$$\eta_{\omega,j} = \frac{\sum_{k=0}^{K-1} \lambda_{k,j} a_{\omega}^k}{\prod_{k=1}^K (a_{\omega} - z_{k,j})} = \frac{P_{K-1}(a_{\omega})}{Q_K(a_{\omega})}, \quad (3.24)$$

where $\lambda_{k,j} \in \mathbb{C}$. The denominator polynomial $Q_K(a_{\omega})$ is the annihilating filter whose roots are the positions of the projected vertices $z_{k,j}$. Exploiting the ratio structure, a model-fitting approach can be applied to obtain a more robust estimation of $z_{k,j}$:

$$\min_{Q_K, P_{K-1}} \sum_{\omega=0}^{W-1} \left| \eta_{\omega,j} - \frac{P_{K-1}(a_{\omega})}{Q_K(a_{\omega})} \right|^2. \quad (3.25)$$

We note that the above minimization problem is non-linear. However, if the poles are placed uniformly on the circle: $a_{\omega} = a_0 e^{i2\pi \frac{\omega}{W}}$, then we can apply the scheme proposed in [36]. Specifically, the minimization scheme estimates the solution of (3.25) in a linear manner by considering an iterative quadratic minimization as follows:

$$\min_{Q_K^{(n)}, P_{K-1}^{(n)}} \sum_{w=0}^{\omega_{\max}-1} \left| \frac{Q_K^{(n)}(a_{\omega}) \eta_{\omega,j} - P_{K-1}^{(n)}(a_{\omega})}{Q_K^{(n-1)}(a_{\omega})} \right|^2.$$

Each iteration yields a candidate for $Q_K^{(n)}$, $n = 1, \dots, \text{iter}_{\max}$, and the final solution of Q_K is chosen such that the mean square error (MSE) between the resynthesized values of $\eta_{\omega,j}$ and the measured values is the smallest. Finally, the projected vertices are retrieved as the roots of Q_K . Please refer to [36] for more details.

3.4.2 Robust estimation of 3D parameters

In Section 3.3.3, once the direction vectors $\tilde{\mathbf{U}}_x$ are retrieved, the estimation of the 3D vertices $\tilde{\mathbf{V}}$ and the other direction vectors $\tilde{\mathbf{U}}_y$ is based on solving a linear system of equations as in Equation (3.16) and (3.17). In noiseless cases, the solution is accurate. However, in the presence of noise, this may not be the case. Hence, once the initial estimation of $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{U}}_x$ is obtained using the method in Section 3.3.3, we further refine it as follows.

For every projection j , we first fix $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{U}}_x$ and formulate the following convex optimization problem to estimate $\tilde{\mathbf{u}}_{j,y}$:

$$\begin{aligned} & \min_{\tilde{\mathbf{u}}_{j,y}} \left\| \tilde{\mathbf{V}} \tilde{\mathbf{u}}_{j,y} - \tilde{\mathbf{p}}_j^y \right\|^2, \\ & \text{subject to } \tilde{\mathbf{u}}_{j,x}^T \tilde{\mathbf{u}}_{j,y} = 0, \\ & \quad \|\tilde{\mathbf{u}}_{j,y}\| \leq 1, \end{aligned} \quad (3.26)$$

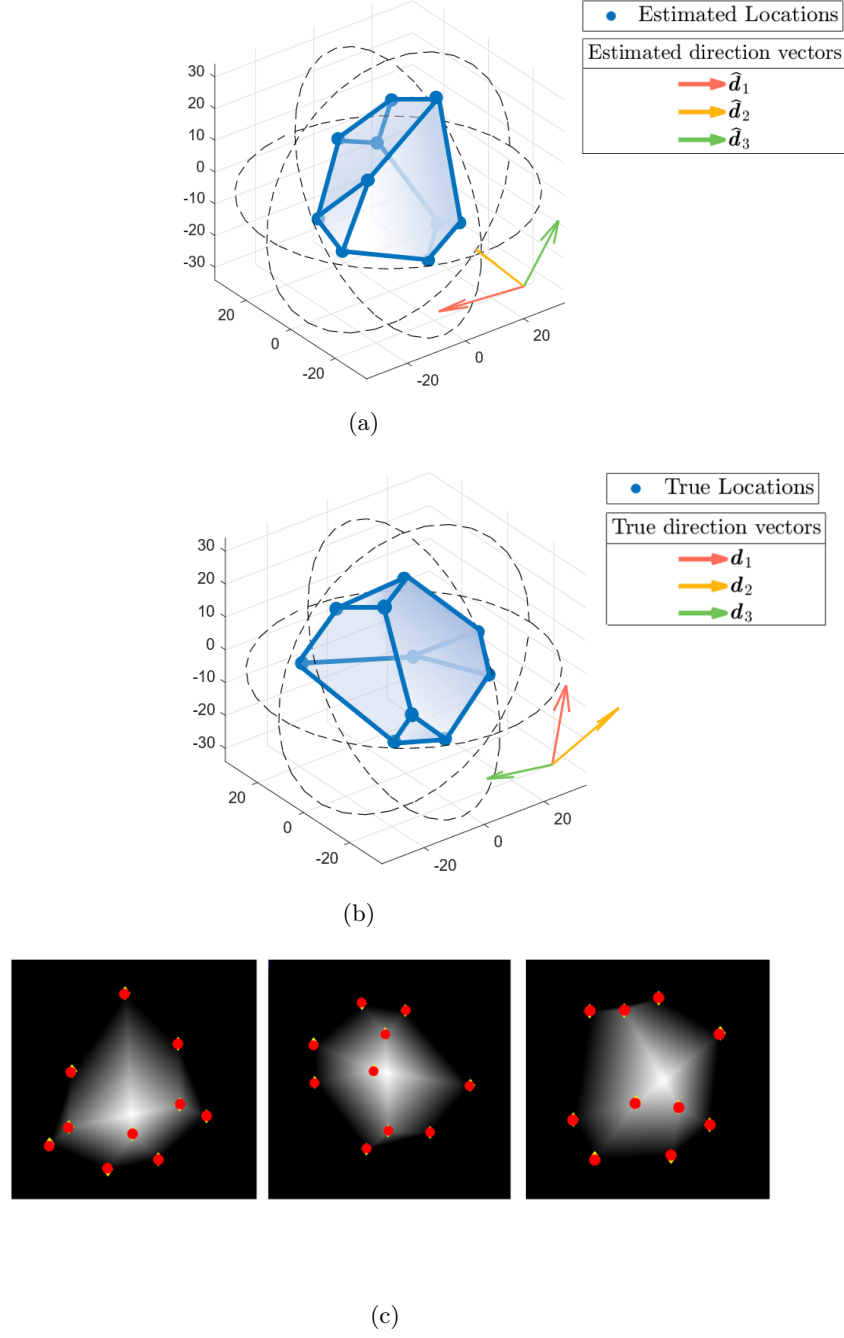


Figure 3.6: (a) The estimation result using our proposed method is up to an orthogonal transformation compared to the true values in (b). The 3 projections are shown in (c), while the estimation of the 2D locations of the projected vertices is shown with red dots which overlaps with the true vertices (yellow diamonds).

where $\tilde{\mathbf{p}}_j^y = [\tilde{p}_{1,j}^y, \tilde{p}_{2,j}^y, \dots, \tilde{p}_{K,j}^y]^T$. The equality constraint is used to enforce the geometric consistency of the problem and the inequality constraint is a convex relaxation of the

norm constraint. A feasible solution for the above minimization problem can be obtained using disciplined convex programming¹ [37]. Note that we solve the minimization in (3.26) for every projection j to estimate all the direction vectors in $\tilde{\mathbf{U}}_y$.

Next, given $\tilde{\mathbf{U}}_x$ and $\tilde{\mathbf{U}}_y$, we refine the estimation for $\tilde{\mathbf{V}}$ by solving the following minimization problem:

$$\min_{\tilde{\mathbf{V}}} \left\| \tilde{\mathbf{V}} \tilde{\mathbf{U}}_x - \tilde{\mathbf{p}}^x \right\|^2 + \left\| \tilde{\mathbf{V}} \tilde{\mathbf{U}}_y - \tilde{\mathbf{p}}^y \right\|^2, \quad (3.27)$$

where $\tilde{\mathbf{p}}^x = [\tilde{\mathbf{p}}_1^x, \tilde{\mathbf{p}}_2^x, \dots, \tilde{\mathbf{p}}_J^x]$ and $\tilde{\mathbf{p}}^y = [\tilde{\mathbf{p}}_1^y, \tilde{\mathbf{p}}_2^y, \dots, \tilde{\mathbf{p}}_J^y]$. The above minimization has a closed form solution. Finally, we update the other direction vectors $\tilde{\mathbf{U}}_x$ based on the refined $\tilde{\mathbf{V}}$ and $\tilde{\mathbf{U}}_y$ in a similar fashion as (3.26). The above updating process can be repeated to obtain a more precise reconstruction. Normally, a few such iterations lead to a reliable estimation.

3.4.3 Algorithmic complexity

In terms of computational cost, the model-fitting approach described in Section 3.4.1 requires the QR decomposition of matrix $\mathbf{Q}_{n-1}$ and $\mathbf{B}_{n-1}$ in Equation (B.43) as explained in Appendix B.5, which are of size $\mathbf{Q}_{n-1} \in \mathbb{C}^{WJ \times (K+1)}$ and $\mathbf{B}_{n-1} \in \mathbb{C}^{WJ \times KJ}$. While the matrix $\mathbf{B}_{n-1}$ may appear large, it exhibits structured sparsity and block-diagonal repetition, as detailed in Appendix B.5. This structural property significantly reduces the effective computational burden. As a result, the overall complexity of this stage scales primarily with the number of vertices K , while depending only weakly on the number of analytic functions and projections involved.

For vertex pairing across projections, a combinatorial pairwise matching strategy is employed, which in the worst case incurs a computational complexity of $O(K!)$. To address the sensitivity of this step to noise, we adopt a noise-tolerant matching strategy based on the rank criterion in Section 3.3.2. This criterion exploits the fact that increasing the number of projections improves the reliability of correct pairing by emphasizing rank deficiency in correctly paired matrices. To balance noise robustness and computational efficiency, we incorporate a RANSAC-inspired procedure [21]: we begin by evaluating the rank criterion over a small subset of projections (e.g., six), and incrementally add additional projections. This adaptive strategy mitigates computational overhead while improving pairing accuracy in noisy scenarios.

Finally, the robust 3D direction estimation is formulated as a constrained convex optimization problem solved independently for each projection. As described in Equation (3.26), the problem is small-scale and efficiently solvable using interior-point methods using CVX

¹We have used the MATLAB CVX solver (<http://cvxr.com/cvx/>) which implements the algorithm described in [37].

solver. Given that the optimization involves a vector of dimension K , the per-instance complexity is $O(K^3)$ per iteration. Since this problem is solved independently for each projection, the total computational cost scales linearly with the number of projections, and remains tractable for moderate values of K .

3.5 Numerical simulations and results

In this section, we validate the performance of the proposed algorithm, and show that it is able to effectively estimate the 3D polyhedron and projection angles from a small number of sampled and noisy projections. Moreover, we show that aspects of the proposed method can be applied to the geometrical calibration of X-ray scanning systems using real CT data.

3.5.1 Evaluation metrics

Since the solution is determined up to an orthogonal transformation, in all experiments, once we estimated the 3D vertices $\{\tilde{\mathbf{v}}_k\}_{k=1}^K$, we find the orthogonal transformation matrix $\mathbf{Q}$ between the estimated 3D locations and the true locations $\{\mathbf{v}_k\}_{k=1}^K$ by considering an orthogonal Procrustes problem [83]. Then, we quantify the error between the estimated locations and the true locations as:

$$E_{\text{vertex}} = \frac{1}{K} \sum_{k=1}^K \frac{\|\mathbf{Q}\tilde{\mathbf{v}}_k - \mathbf{v}_k\|_2}{\|\mathbf{v}_k\|_2}.$$

Using the same orthogonal matrix $\mathbf{Q}$, we define the estimation error for direction vectors as follows:

$$E_{\text{direction}} = \frac{1}{J} \sum_{j=1}^J \left(1 - \langle \mathbf{Q}\tilde{\mathbf{d}}_j, \mathbf{d}_j \rangle\right).$$

3.5.2 Noiseless setting

Figure 3.6 (a) shows the estimation result for a convex bilevel polyhedron with $K = 10$ vertices from 2D samples of $J = 3$ tomographic projections taken at unknown directions. The size of the rectangular observation window is assumed to be $(2.5R, 2.5R)$, $R = 32$, and the unknown shifts $\mathbf{s}_j$ are randomly drawn from the Uniform distribution $\mathcal{U}(-0.1R, 0.1R)$. The 2D projections are of size 1024×1024 pixels. Their corresponding 2D samples are generated from filtering the noise-free 2D tomographic projections in Figure 3.6 (c) with a 2D B-spline kernel of order 16. As expected, when compared to the true

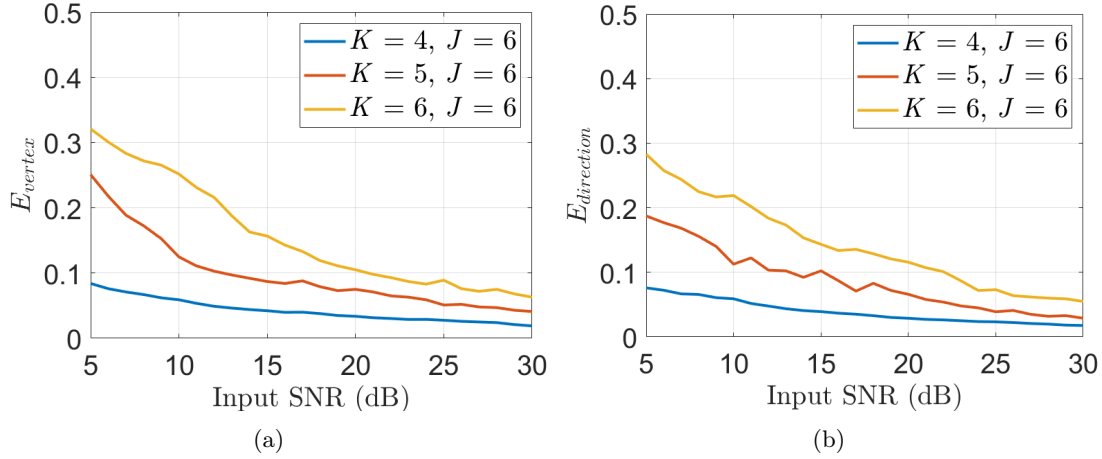


Figure 3.7: Average reconstruction error. In (a) we show the error of the estimated 3D vertices, and in (b) the error of the estimated direction vectors *wrt.* the SNR level of the 2D samples for $K = 4, 5$ and 6 , respectively. The number of projections is 6 .

polyhedron and direction vectors in Figure 3.6 (b), the reconstruction in Figure 3.6 (a) is up to an orthogonal transformation. After applying the orthogonal matrix $\mathbf{Q}$, the reconstruction errors in the locations and directions are $E_{\text{vertex}} = 8.10 \times 10^{-3}$ and $E_{\text{direction}} = 5.61 \times 10^{-3}$.

3.5.3 Noisy setting

In this section, we assess the performance of the proposed algorithm by analysing:

1. Dependence of the estimation error on the sample noise level.
2. Dependence of the estimation error on the number of vertices K .
3. Dependence of the estimation error on the number of projections J .

Effect of the noise level

We consider the case where the samples of the projections are corrupted by additive white Gaussian noise with zero mean and variance σ^2 , and the SNR level varies between 5 dB to 30 dB. We compare the estimation error for polyhedron models with $K = 4$ to $K = 6$ vertices using $J = 6$ noisy projections. The 2D projections are of size 1024×1024 pixels. Their corresponding blurred samples of size 256×256 pixels are obtained by first filtering with a 2D separable B-spline kernel of order 8 along each axis and downsampling. The coefficients $c_{m,n}^\omega$ in Equation (3.22) are generated with $a_\omega = 1.2R \times e^{i2\pi\frac{\omega}{50}}$, $\omega = 0, 1, \dots, 49$.

For every SNR level, the experiment is repeated 1000 times on randomly generated polyhedra and direction vectors.

To achieve robust reconstruction, we apply the algorithm proposed in Section 3.4.1 to estimate the 2D projected locations $\tilde{\mathbf{p}}_{k,j}$. Moreover, we employ a noise-tolerant pairing strategy based on the rank criterion introduced in Section 3.3.2 and the measurement consistency criterion. The latter criterion requires to reconstruct a test 3D polyhedron model based on the current pairing. It makes use of the fact that given correct pairing, the resynthesized parameters $\check{\mathbf{p}}_{k,j}$ from the test model should be consistent with the estimated parameters $\tilde{\mathbf{p}}_{k,j}$. Namely, the average difference between $\check{\mathbf{p}}_{k,j}$ and $\tilde{\mathbf{p}}_{k,j}$ should be within a small threshold. We operate by initiating the pairing using the rank criterion, and verify the paired result against the consistency criterion. For a final estimation result, we run the optimization scheme proposed in Section 3.4.2 using all paired 2D parameters.

Figure 3.7 illustrates the average reconstruction error E_{vertex} and $E_{\text{direction}}$ when the number of vertices $K = 4, 5, 6$ and the number of projections is $J = 6$. We notice that, although 2D samples are quite noisy (SNR = 5 dB), the proposed method can still retrieve accurately the polyhedron and direction vectors. Figure 3.8 depicts a visual example where the 2D samples are corrupted with noise at SNR level of 5 dB and 30 dB respectively and the number of vertices is $K = 4$.

In addition, we use Cramér-Rao bound (CRB) to assess the accuracy of the retrieval of the 2D locations of projected vertices $\mathbf{p}_{k,j}$ from the 2D samples. This bound enables us to assess the accuracy of the FRI estimation techniques proposed in Section 3.4.1. Please refer to Appendix B.3 for the derivation of the CRB bound. For the setting in Figure 3.9, we assume the $K = 4$ projected vertices are located at $\mathbf{p}_{1,j} = [-0.75R, 0]$, $\mathbf{p}_{2,j} = [0, 0.5R]$, $\mathbf{p}_{3,j} = [0.75R, 0]$ and $\mathbf{p}_{4,j} = [0, -0.5R]$.

Effect of the number of vertices K and number of projections J

The effect of the number K of vertices on the estimation accuracy of a polyhedron is shown in Figure 3.7 when we have only access to $J = 6$ projections. The figure shows that performance deteriorates with K especially at low SNRs. Theorem 3.1 indicates that the minimum number of projection needed for perfect reconstruction is $J = 3$ and is independent of K . In practice, one has often access to a number of projections much greater than $J = 3$. It is therefore natural to explore the use of this redundancy to improve performance.

First of all, more projections increase the parameter pairing accuracy since there are more parameters to be tested against. Moreover, the linear system in Equation (3.27) becomes more overdetermined, which yields better estimation results.

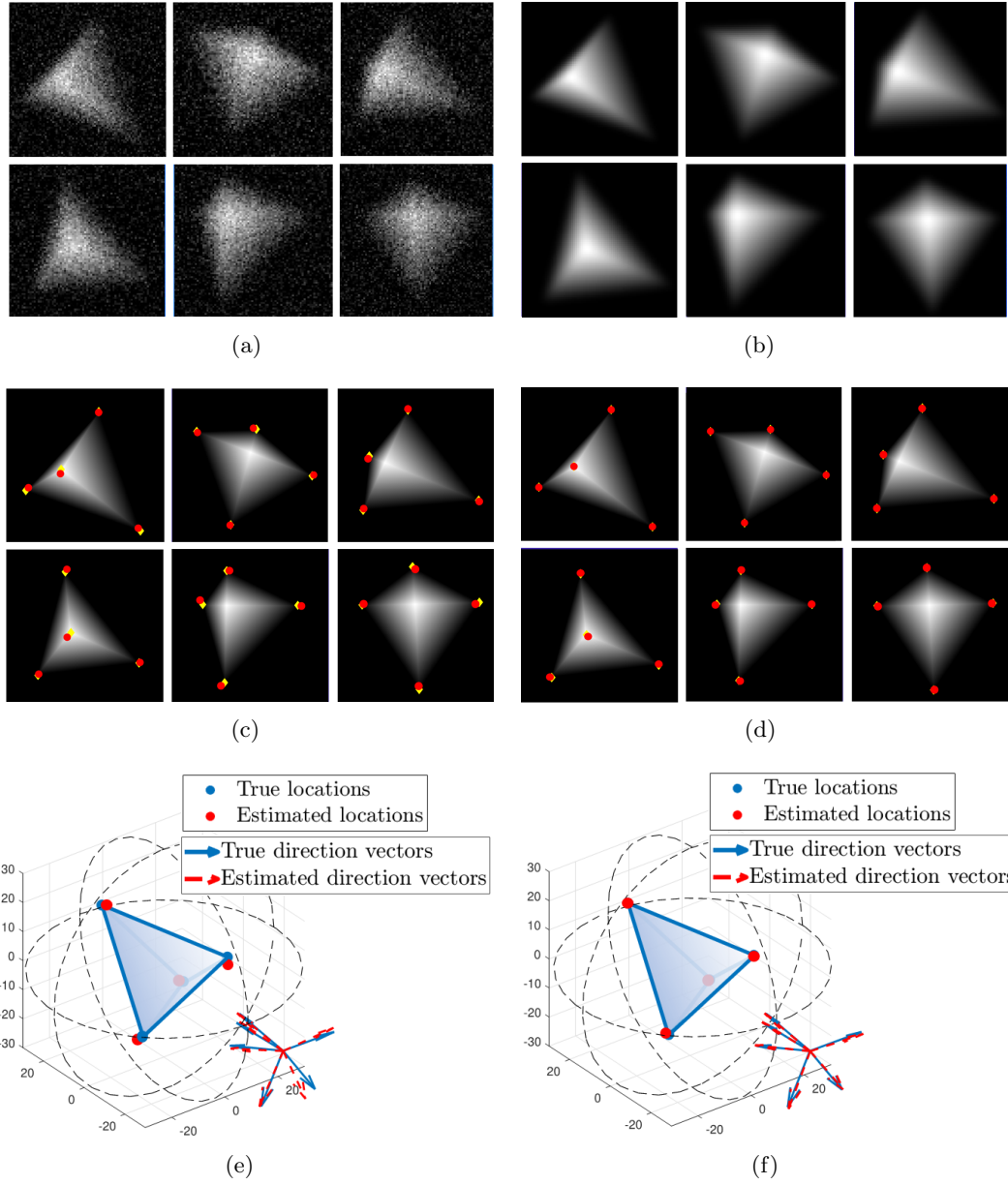


Figure 3.8: (a) and (b) depict the noisy sampled projections at SNR level of 5 and 30 dB respectively. The 2D locations of the projected vertices are estimated as shown in (c) and (d) respectively. The estimated locations are denoted with red dots while the true locations are denoted with yellow diamonds. (e) and (f) show the final estimation of the 3D polyhedron and direction vectors. The blue dots and arrows represent the true vertices and direction vectors, while the red ones represent the estimated values. For SNR = 5 dB, the reconstruction errors are $E_{\text{vertex}} = 0.076$ and $E_{\text{direction}} = 0.102$ respectively. When SNR = 30 dB, the reconstruction errors are $E_{\text{vertex}} = 0.018$ and $E_{\text{direction}} = 0.012$ respectively.

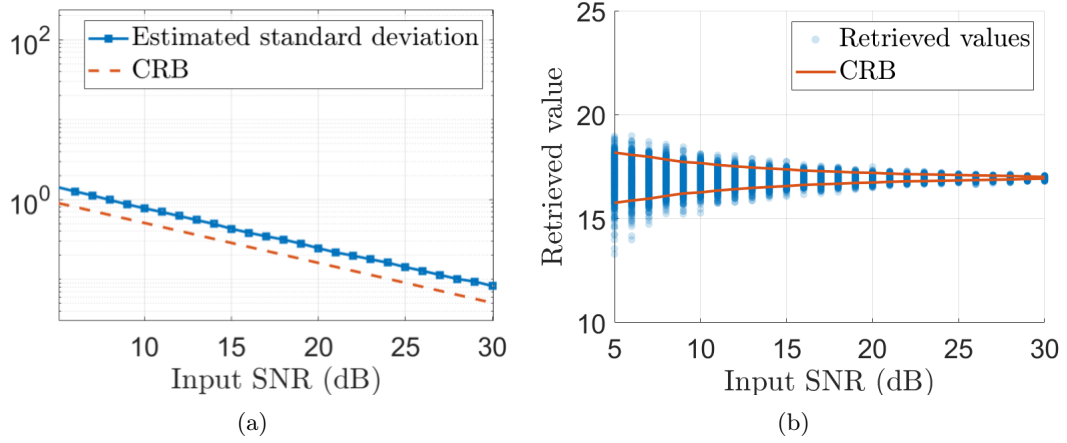


Figure 3.9: (a) The standard deviation in the estimation of $\mathbf{p}_{2,j}^y$ and the Cramér Rao bound for different levels of noise. (b) Scatter plot of retrieved values of $\mathbf{p}_{2,j}^y$.

Figure 3.10 compares the estimation error with respect to the number of projections J when $K = 5, 6$ and SNR equals to 5 dB, 10 dB and 15 dB respectively. For every J , the experiment is repeated 1000 times. It can be seen that increasing the number of projections yields more accurate estimation results especially when the SNR level is low. For example, it is of interest to note that the performance of our method for $K = 5$ when SNR = 5 dB and $J = 100$ is comparable to the case when $K = 4$ in Figure 3.7. This indicates that we can overcome the performance deterioration due to the complexity of the polyhedron by increasing J . A visual example is depicted in Figure 3.11 where $K = 5$ and the SNR level is 5 dB for 6 and 20 projections respectively. To further justify this claim, we show a visual experiment result in Figure 3.12 where the polyhedron has $K = 6$ vertices and its 2D projection samples are severely corrupted by noise (SNR = -2 dB). In order to obtain a faithful reconstruction, we increase the number of projections to $J = 300$. The reconstruction result is shown in Figure 3.12 (b). The estimation accuracy is $E_{\text{vertex}} = 0.196$ and $E_{\text{direction}} = 0.137$, which is comparable to Figure 3.11 (c), where $K = 5, J = 20$ and SNR = 5 dB.

3.5.4 Real computed tomography data

We apply our proposed method to the geometrical calibration of real CT scanning data, obtained at the FleX-ray laboratory at the Center for Mathematics & Computer Science, Amsterdam. Geometrical calibration is a critical step towards ensuring traceability in CT measurement. Often during the calibration process, a geometrically patterned reference object is imaged, then the system geometry, i.e. the projection angles, is deduced from the 2D projection measurement (radiographs). For this purpose, our proposed framework provides a theoretical guide and a practical tool to perform the calibration. Specifically,

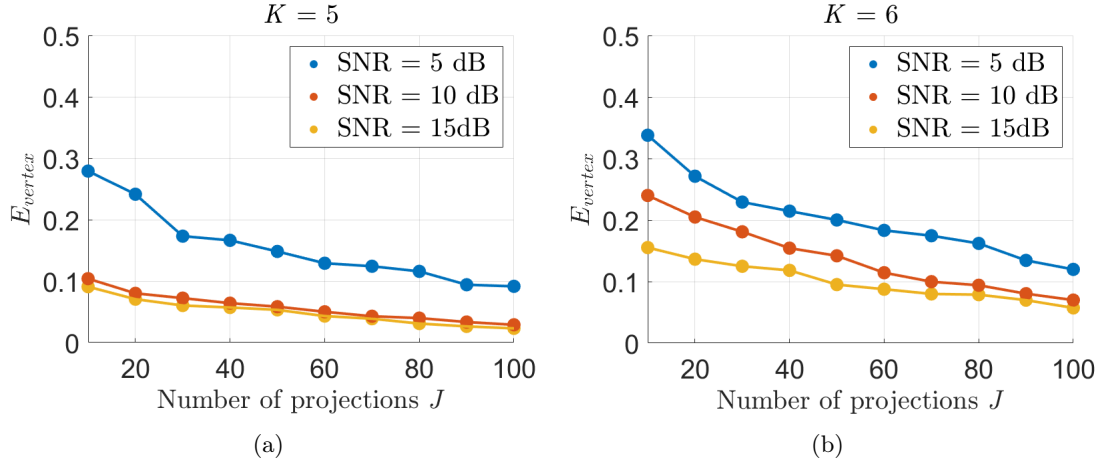


Figure 3.10: Average reconstruction error E_{vertex} wrt. the number of projections J ranging from 10 to 100 for polyhedra with $K = 5$ and 6 vertices respectively.

as a first step, one should extract paired projected locations of the reference objects. Then, the proposed robust algorithm in Section 3.4.2 can be applied for the estimation of the 3D system geometry. In what follows, we explain the experiment in details.

In the experiment, the projection measurements contain $J = 200$ 2D radiographs of a wooden block, surrounded by two pieces of foam in which 10 small metal balls (markers) are placed, as shown in Figure 3.13 (a). The metal balls serve as reference objects, facilitating the geometrical calibration of the scanning system, specifically, to estimate the projection angles. The radiographs are obtained using a micro-CT scanner. During the scanning process, the object is placed on a stable rotation stage, and subsequently rotates the object between a static source and a detector. In other words, the angular motion is only on a plane. The detector plane is placed vertically to the ground. Therefore, the projection directions $\{\mathbf{d}_j\}_{j=1}^J$ are co-planar on the horizontal plane. Moreover, the unit direction vectors $\{\mathbf{u}_{j,x}\}_{j=1}^J$ are the same across all projections, and they assume the value $[0, 0, 1]^T$. In Figure 3.13 (b), examples of the 2D radiographs at unknown projection angles are shown.

From the 2D radiographs taken at unknown angles, we aim to estimate the projection angles. We first find the projected marker locations $\mathbf{p}_{k,j}$ on each 2D radiograph, from which we extract the projection directions $\mathbf{d}_j$ using the method presented in Section 3.3.3. Since the direction vectors $\{\mathbf{u}_{j,x}\}_{j=1}^J$ are the same across all projections, given the 2D projected locations of the markers, the estimation of the projected directions is actually a degenerate case of our 3D estimation method, and we can directly apply a 2D method similar to the one that we proposed in [99] using $p_{k,j}^y$ only. We plot the estimated angles using our proposed method against the 2D angles given by the metadata of the scanner in Figure 3.13 (c). The average angle estimation error is 0.005 rad and the standard

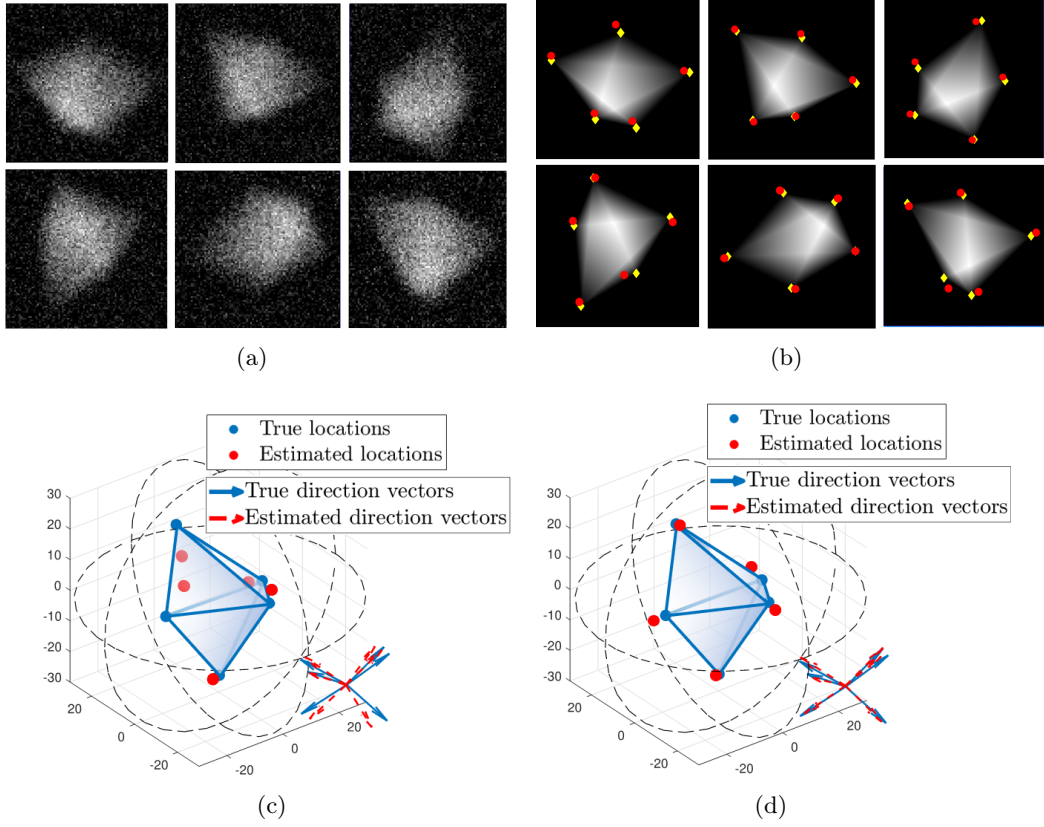


Figure 3.11: (a) Noisy 2D samples of projections at SNR level of 5dB. (b) Estimation of the 2D locations of the vertices (red dots) compared to the true locations (yellow diamonds). (c) Final estimation result of the polyhedron with $K = 5$ vertices from 6 projections only. $E_{\text{vertex}} = 0.392$ and $E_{\text{direction}} = 0.235$. (d) Final estimation result of the same polyhedron using 20 projections. $E_{\text{vertex}} = 0.187$ and $E_{\text{direction}} = 0.116$.

deviation for the estimation of angles is 0.013 rad.

For illustrative reasons, we also reconstruct a slice of the wooden block using the FISTA SWLS-TV algorithm provided by the ToMoBAR toolbox [50] with the estimated angles. The reconstruction result is shown in Figure 3.13 (d).

3.6 Conclusion

In this chapter, we have addressed the sampling problem of estimating an unknown 3D polyhedron from a small number of sampled 2D tomographic projections at unknown angles. The method is able to estimate exactly the projection directions and the 3D structure up to an orthogonal transformation. Interestingly, we are able to show that the minimum number of projections required is unrelated to the complexity of the 3D object.

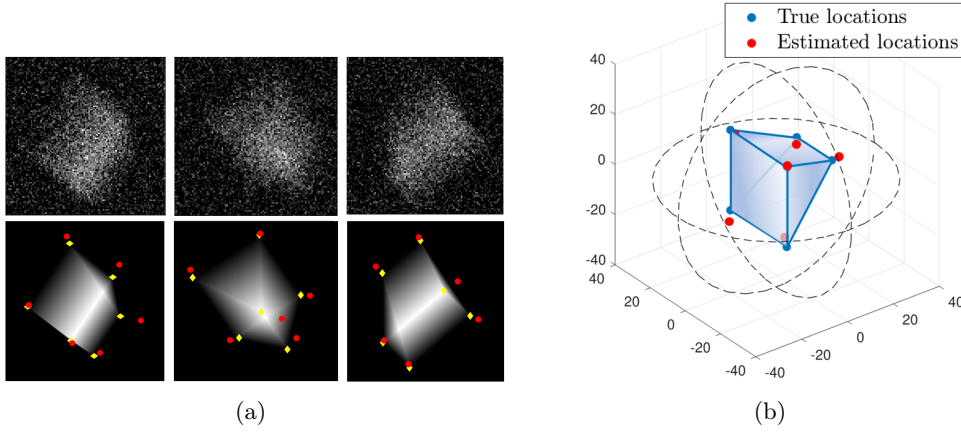


Figure 3.12: (a) Top row figures show examples of the noisy projection samples corrupted with additive Gaussian noise at $\text{SNR} = -2$ dB. While bottom row figures show the estimation of the 2D locations (red dots) of the vertices compared to the true vertices (yellow diamonds). (b) Estimation result of the polyhedron with $K = 6$ projections and $J = 300$ projections. $E_{\text{vertex}} = 0.196$ and $E_{\text{direction}} = 0.137$.

From various experiments, the algorithm is shown to achieve accurate estimation even when the samples are corrupted with noise. Moreover, experimental results on real data show the potential of the proposed approach.

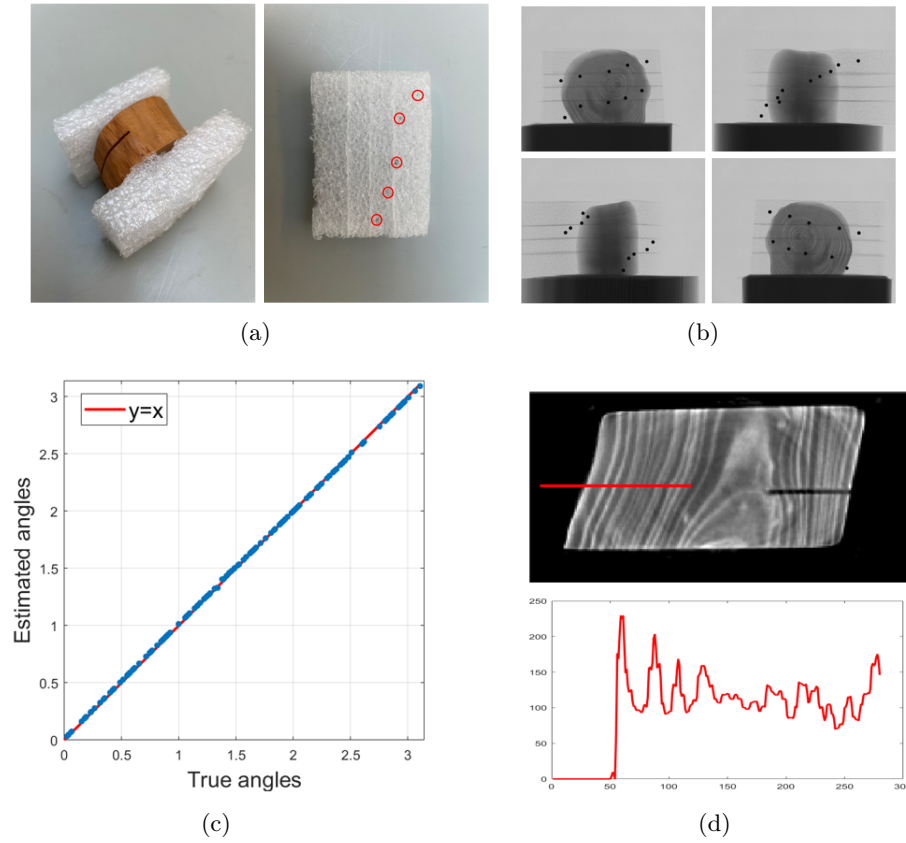


Figure 3.13: (a) The wooden block is surrounded by the foam, inside which the markers (red circle) are placed. (b) 2D radiographs of the wooden block with markers. (c) Estimation of the 2D projection angles vs. 2D angles given by the FleX-ray laboratory. (d) Reconstruction of a slice of the wooden block using the estimated angles.

Chapter 4

Moment-based angle retrieval and reconstruction of images with finite rate of innovation from noisy tomographic projections

天下难事，必作于易，
天下大事，必作于细。

——《老子·道德经》

All matters difficult start from the easy.
All matters great start from the minute.

— Laozi· *Tao Te Ching*

In the previous chapter, we explored the unknown angle tomographic reconstruction problem and introduced an algorithm for the simultaneous estimation of the unknown 2D/3D signal and the projection angles. The presented algorithm operates under a strong model assumption about the geometry of the unknown signal. While this approach leads to a solid sampling theorem and an efficient angle estimation algorithm based on explicit geometrical features, the restrictive assumptions may limit its applicability in more general scenarios. In this chapter, we relax these constraints and explore how angle estimation can be achieved under more general conditions, allowing for a broader range of signals. Moreover, we reflect on the inherent limitations present in the current tomographic reconstruction methods and propose the use of more appropriate signal models that better suits the specific imaging problem. More specifically, we consider the 2D unknown angle tomographic reconstruction problem and propose an angle estimation algorithm that requires only marginal constraints on the underlying 2D objects. Furthermore, when the projection angles are either accurately or inaccurately known, we claim that the primary limitation of the reconstruction of the signal lies in the uneven distribution of the Fourier information in the low and high frequency range, where the high frequency region remains much sparser than the low frequency region. Due to this intrinsic limitation, many of the current methods require a large number of projections to adequately capture the missing information. Based on this, we propose to use alternative signal models that allow estimation of high frequency content from their low frequency information.

4.1 Introduction

Much of the current research in tomography has focused on the problem of reconstructing an objects from a set of its line integration projections [29, 65].

In this chapter, we focus on the two-dimensional (2D) formulation of the problem, where the object being imaged is in 2D and its parallel beam projections are in 1D. As an initial step, we address the challenge of establishing the uniqueness of the angle retrieval problem in 2D unknown angle tomography. In this scenario, both the 2D object being imaged and the projection angles are unknown. Our aim is to investigate the uniqueness conditions under which the recovery of the projection angles can be achieved and present a practical algorithm to do so. We note that compared to the three dimensional case, in some ways the 2D unknown angle retrieval problem is simpler due to fewer degrees of freedom in the projection angles. However, in 3D scenarios, the central slice theorem (also known as Fourier slice theorem) implies that any two projections intersect along a common line, which can be leveraged for angle estimation. Unfortunately, in 2D case this cannot be applied since any 1D projections intersect at the origin, offering no useful information.

4.1.1 Overview of existing methods for 2D angle estimation

The 2D unknown angle tomographic problem has been discussed in [10, 8, 9, 80, 23, 34]. Among which, the uniqueness problem has been discussed in [10]. In this work, the connection between the object moments and the projection moment is leveraged and is followed by a numerical algorithm for the estimation problem in noisy scenarios assuming the unknown angles follow a uniform distribution in $[0, 2\pi)$, however, no theoretical analysis is provided [8]. The second category of methods focuses on establishing a circular ordering of projections, and assigns the estimated angles based on the uniform distribution assumption on the projection angles [23, 34]. In these works, the projections are first mapped to $\mathbb{R}^2$ space while preserving the neighborhood distance. The mapped vectors in 2D space are then transformed into a circular ordering based on their polar angles. Finally, the angle assignment is performed based on the uniform distribution assumption. Another category of methods proposes estimations based on the sparsity assumption on the underlying model, and they either bypass the estimation of angles by leveraging the rotational invariant features [105], or estimate the angles by leveraging the sparsity of the features [99].

In this work, we consider the scenario where no assumptions are made regarding the underlying model of the 2D object, and no prior knowledge of the distribution of projection angles is available. To this end, we provide a practical solution that is also validated through a series of experiments, demonstrating its effectiveness. Compared to the work in [10], our proposed algorithm reduces the minimum number of required projections by half. Notably, our method does not rely on any prior information about the distribution of the projection angles, making it highly adaptable and widely applicable to various settings where such information may not be readily available. This also leads to a reduction in computational and data requirements, particularly in situations where data acquisition is costly or limited.

4.1.2 Overview of existing methods in 2D tomographic reconstruction

The tomographic inversion problem with known angles is typically treated as a linear inverse problem. Conventional methods such as filtered back projection (FBP) and direct Fourier methods [90, 26] make use of central slice theorem. They work adequately when there is a substantial number of projections, provided these projections are relatively clean, and exact knowledge of the projection angles is available. Other methods formulate a regularized optimization problem to reconstruct the image by imposing prior knowledge of the image [87, 66, 28, 106], such as smoothness or sparsity. Yet the performance is limited by the number of available projections. Several angle refinement algorithms based on projection matching techniques have also been introduced to enhance the estimation accuracy [106, 35], they are either built based on the specific image model or expectation

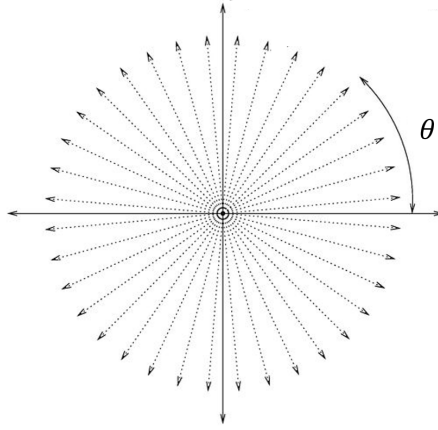


Figure 4.1: An illustration of the 2D frequency plane filling based on the central limit theorem.

maximization (EM) algorithm. More recently, deep learning methods have also been proposed which train on rich datasets. Some of these directly resynthesize 2D images from projections [1, 107, 46], while [103] presents a generative adversarial network (GAN) based approach to determine the unknown angle distribution and to reconstruct the 2D image. It is worth noting that the effectiveness of these methods depend largely on the availability of a large number of projections.

In situations where the number of projections is limited, or even the range of angles is limited, the reconstruction performance of these algorithms may deteriorate drastically. This may happen in applications where acquiring projections is time-consuming, or involves high radiation doses (such as X-ray tomography), the number of projections is intentionally limited to minimize exposure or reduce acquisition time.

We notice that in the tomographic reconstruction problem, the need of a large number of projections arises from one intrinsic limitation, which is implied from the central slice theorem. Since the 1D Fourier measurements fill the 2D plane radially, it results in high concentration of low frequency measurements and sparse high frequency measurements. A simple illustration is shown in Figure 4.1. This intrinsic limitation makes most reconstruction algorithms data demanding. To overcome this limitation, we propose to use a more suitable signal model that can be completely specified by its low frequency information.

4.1.3 Overview of our approach

In this chapter, we first consider the general uniqueness conditions for the angle estimation problem in 2D unknown tomography. By reviewing the data consistency conditions, we introduce a constructive and systematic way for estimating the unknown projection

angles by solving a system of trigonometric equations using the object moment and projection moments as parameters. This approach operates under very minor model constraints. Therefore, the method offers a practical solution to recover projection angles even in cases where prior information is scarce.

Then, we reflect on the choice of signal models that best suits this image acquisition model, where the aim is to reconstruct accurately the unknown 2D object from inaccurately estimated angles. Specifically, we consider piecewise constant images defined by curves with finite rate of innovation [72]. Given inaccurately estimated angles, instead of directly reconstructing the image from projections, we first retrieve the image curves from only the non-uniform low frequency information. The image is then recovered based on using the spatial domain annihilation relation for a regularized optimization problem. Finally, to enhance the performance, we introduce a method to refine the projection angles based on the previously reconstructed 2D image. Numerical results show that by using this image model, accurate reconstruction of images can be achieved even with very few projections, and under conditions when the projections are heavily corrupted by noise.

The rest of the chapter is organized as follows. We first introduce some preliminary concepts in 2D tomography, such as the data consistency conditions in Section 4.2. Building on these conditions, we present a general 2D angle estimation algorithm in Section 4.2.1. Section 4.3 then details the specific image model considered in this study, followed by an image reconstruction algorithm in Section 4.4. In Section 4.5, we provide a detailed numerical evaluation of the proposed methods, and we conclude with a summary of key findings in Section 4.6.

4.2 Parallel beam projection and data consistency condition

We begin by reviewing the concept of parallel line projection, or mathematically, the Radon transform and the data consistency condition associated with it. Following this, we introduce a moment-based data consistency condition that establishes the connection between the moments of the projections and that of the projected image.

We denote a 2D image with $I(x, y)$, $x \in [0, \tau_x)$, $y \in [0, \tau_y)$, $\tau_x, \tau_y \in \mathbb{R}^+$. Then, we consider the parallel line tomographic projections $g_\theta(t)$ of a 2D image $I(x, y)$ taken at angle θ , the expression is given by the Radon transform:

$$g_\theta(t) = \int_0^{\tau_y} \int_0^{\tau_x} I(x, y) \delta(x \cos \theta + y \sin \theta - t) dx dy, \quad x \in [0, \tau_x), \quad y \in [0, \tau_y), \quad (4.1)$$

where $\delta(\cdot)$ is the Dirac distribution.

A geometric moment is a type of mathematical measure used to describe various spatial properties of a signal. For example, the *zeroth* order moment of an image represents its total mass, and the ratio between the first and *zeroth* order moments can be used to determine its centroid.

From Equation (4.1), an immediate observation is that at any angle θ , the projection $g_\theta(t)$ has the same 0th order moment as the 2D image $I(x, y)$ since the quantity represents the total mass of the image. In fact, this is a straightforward example of a broader data consistency rule, which states the relationship between the moments of a 2D image and its tomographic projections. We now briefly review the data consistency condition. For every non-negative integer $k \in \mathbb{N}^+$, the k th order moment m_k of the projection $g_\theta(t)$ can be written as:

$$\begin{aligned}
 m_k &= \int_{-\infty}^{\infty} t^k g_\theta(t) dt \\
 &= \int_{-\infty}^{\infty} \int_0^{\tau_x} \int_0^{\tau_y} t^k I(x, y) \delta(x \cos \theta + y \sin \theta - t) dt dx dy \\
 &\stackrel{(a)}{=} \int_0^{\tau_x} \int_0^{\tau_y} (x \cos \theta + y \sin \theta)^k I(x, y) dx dy \\
 &= \int_0^{\tau_x} \int_0^{\tau_y} \left(\Re \left((x + iy) e^{-i\theta} \right) \right)^k I(x, y) dx dy \\
 &= \int_0^{\tau_x} \int_0^{\tau_y} \left(\frac{(x + iy) e^{-i\theta} + (x - iy) e^{i\theta}}{2} \right)^k I(x, y) dx dy \\
 &\stackrel{(b)}{=} \frac{1}{2^k} \sum_{l=0}^k e^{i(2k-l)\theta} \underbrace{\binom{k}{l} \int_0^{\tau_x} \int_0^{\tau_y} (x + iy)^l (x - iy)^{k-l} I(x, y) dx dy}_{d_{k,l}} \\
 &= \sum_{l=0}^k e^{i(2k-l)\theta} d_{k,l}, \tag{4.2}
 \end{aligned}$$

where i denotes the imaginary unit and

$$d_{k,l} = \frac{1}{2^k} \binom{k}{l} \int_0^{\tau_x} \int_0^{\tau_y} (x + iy)^l (x - iy)^{k-l} I(x, y) dx dy$$

denotes the weighted 2D complex moments of the image $I(x, y)$. The equality (a) follows from the property of the Dirac distribution and (b) follows from the binomial expansion. Equation (4.2) states that the k th order geometric moment of a projection $g_\theta(t)$ at angle θ of a given image $I(x, y)$ can be written as a trigonometric function of degree k of the projection angle θ , where the coefficients are the 2D weighted complex moments of the image. In the following section, we show how the above relation can lead to a step-by-step method to estimate unknown projection angles from 1D projections.

4.2.1 Moment-based angle retrieval

We suppose $J \geq 1$ tomographic projections $\{g_{\theta_j}(t)\}_{j=1}^J$ of the 2D image $I(x, y)$ are taken at unknown angles $\{\theta_j\}_{j=1}^J$, $\theta_j \sim U(0, \pi)$:

$$g_{\theta_j}(t) = \int_0^{\tau_y} \int_0^{\tau_x} I(x, y) \delta(x \cos \theta_j + y \sin \theta_j - t) dx dy. \quad (4.3)$$

In this section, we address the angle estimation problem where the aim is to retrieve exactly the unknown projection angles $\{\theta_j\}_{j=1}^J$, given only the projections $\{g_{\theta_j}\}_{j=1}^J$.

By leveraging the data consistency conditions, we propose a novel angle estimation algorithm based on solving a system of trigonometric equations, given the geometric moments of the imaged 2D object satisfy some mild conditions. Specifically, we consider the data consistency conditions with $k = 1, 2$, where we can rewrite Equation (4.2) as follows:

$$\begin{cases} m_{1,j} &= d_{1,0}e^{i\theta_j} + d_{1,1}e^{-i\theta_j} \\ m_{2,j} &= d_{2,0}e^{2i\theta_j} + d_{2,1} + d_{2,2}e^{-2i\theta_j} \end{cases}, \quad j = 1, \dots, J \quad (4.4)$$

where $m_{k,j}$ denotes the k th order moment of the projection at angle θ_j . Given J projections $g_{\theta_j}(t)$, $j = 1, \dots, J$, we have $2J$ such equations with $J + 5$ unknown variables $d_{1,0}$, $d_{1,1}$, $d_{2,0}$, $d_{2,1}$, $d_{2,2}$ and $\{\theta_j\}_{j=1}^J$. In what follows, we provide a constructive solution for the unknown system of equations when $J \geq 5$ under very mild constraints on the 2D image $I(x, y)$.

First of all, in order to construct $2J$ valid equations using the data consistency rule in Equation (4.2), we require that $\forall j$, $m_{1,j}^2$ is not linearly dependent on $m_{2,j}$, in other words, $\nexists \alpha, \beta \in \mathbb{R}$ such that $m_{2,j} = \alpha m_{1,j}^2 + \beta$, which is equivalent to saying:

$$\begin{vmatrix} d_{1,0}^2 & d_{2,0} & 0 \\ 2d_{1,0}d_{1,1} & d_{2,1} & 1 \\ d_{1,1}^2 & d_{2,2} & 0 \end{vmatrix} \neq 0. \quad (4.5)$$

This condition guarantees that the second order moment of the projections is not simply the square of the first order moment of the projections. It also implies that the first order moments are not identically zero and the second order moments vary as a function of different angle θ_j . This means that the 2D object is sufficiently asymmetric. For example, it has to be asymmetric with respect to any of its non-trivial orthogonal transformation. Simple cases that do not meet the criteria include regular polygons or circles. We also include one visual example of angle estimation on a fairly symmetric CT scan of brain, where the algorithm would fail in Appendix C.1. Under the above condition, we rewrite

Equation (4.2) as:

$$\begin{cases} m_{1,j}^2 &= d_{1,0}^2 e^{2i\theta_j} + 2d_{1,0}d_{1,1} + d_{1,1}^2 e^{-2i\theta_j} \\ m_{2,j} &= d_{2,0}e^{2i\theta_j} + d_{2,1} + d_{2,2}e^{-2i\theta_j} \end{cases}, \quad j = 1, \dots, J.$$

Hence, we have:

$$\begin{cases} d_{1,0}^2 e^{2i\theta_j} + d_{1,1}^2 e^{-2i\theta_j} &= m_{1,j}^2 - 2d_{1,0}d_{1,1} \\ d_{2,0}e^{2i\theta_j} + d_{2,2}e^{-2i\theta_j} &= m_{2,j} - d_{2,1} \end{cases}, \quad j = 1, \dots, J. \quad (4.6)$$

We can imply from the above equations that there exist complex-valued coefficients $a, b, c \in \mathbb{C}$ such that:

$$am_{1,j}^2 + bm_{2,j} + c = e^{2i\theta_j}, \quad j = 1, \dots, J. \quad (4.7)$$

Therefore, in order to estimate the projection angles $\{\theta_j\}_{j=1}^J$, it is important to determine the coefficients a, b and c . We use the fact that $|e^{2i\theta_j}| = 1$ and rewrite Equation (4.7) as follows:

$$|a|^2 m_{1,j}^4 + |b|^2 m_{2,j}^2 + 2\Re(ab^*)m_{1,j}^2 m_{2,j} + 2\Re(ac^*)m_{1,j}^2 + 2\Re(bc^*)m_{2,j} = 1 - |c|^2.$$

The above equation can be equivalently written in matrix as follows:

$$\underbrace{\begin{bmatrix} m_{1,1}^4 & m_{2,1}^2 & m_{1,1}^2 m_{2,1} & m_{1,1}^2 & m_{2,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{1,j}^4 & m_{2,j}^2 & m_{1,j}^2 m_{2,j} & m_{1,j}^2 & m_{2,j} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{1,J}^4 & m_{2,J}^2 & m_{1,J}^2 m_{2,J} & m_{1,J}^2 & m_{2,J} \end{bmatrix}}_{\mathbf{M} \in \mathbb{R}^{J \times 5}} \underbrace{\begin{bmatrix} \frac{|a|^2}{1-|c|^2} \\ \frac{|b|^2}{1-|c|^2} \\ \frac{2\Re(ab^*)}{1-|c|^2} \\ \frac{2\Re(ac^*)}{1-|c|^2} \\ \frac{2\Re(bc^*)}{1-|c|^2} \end{bmatrix}}_{\mathbf{x} \in \mathbb{R}^{5 \times 1}} = \underbrace{\begin{bmatrix} 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\mathbf{1}_J \in \mathbb{R}^{J \times 1}}.$$

Due to the linear independency assumption in Equation (4.5), the matrix $\mathbf{M}$ is full column rank. Therefore, from the above equation, the vector $\mathbf{x}$ can be calculated as:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \mathbf{M}^\dagger \mathbf{1}_J. \quad (4.8)$$

Given the entries of $\mathbf{x}$, we can construct the following matrix $\Gamma \in \mathbb{R}^{3 \times 3}$:

$$\Gamma = 2 \begin{bmatrix} |a|^2 & \Re(ab^*) & \Re(ac^*) \\ \Re(ab^*) & |b|^2 & \Re(bc^*) \\ \Re(ac^*) & \Re(bc^*) & |c|^2 \end{bmatrix} = (1 - |c|^2) \begin{bmatrix} 2x_1 & x_3 & x_4 \\ x_3 & 2x_2 & x_5 \\ x_4 & x_5 & \frac{2|c|^2}{1-|c|^2} \end{bmatrix}. \quad (4.9)$$

We can see that the above matrix Γ can be decomposed as follows:

$$\Gamma = 2 \begin{bmatrix} |a|^2 & \Re(ab^*) & \Re(ac^*) \\ \Re(ab^*) & |b|^2 & \Re(bc^*) \\ \Re(ac^*) & \Re(bc^*) & |c|^2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} a^* & b^* & c^* \end{bmatrix} + \begin{bmatrix} a^* \\ b^* \\ c^* \end{bmatrix} \begin{bmatrix} a & b & c \end{bmatrix},$$

hence, it is of rank 2. This implies:

$$\begin{vmatrix} 2x_1 & x_3 & x_4 \\ x_3 & 2x_2 & x_5 \\ x_4 & x_5 & \frac{2|c|^2}{1-|c|^2} \end{vmatrix} = 0,$$

thus,

$$\frac{2|c|^2}{1 - |c|^2} = - \frac{\begin{vmatrix} 2x_1 & x_3 & x_4 \\ x_3 & 2x_2 & x_5 \\ x_4 & x_5 & 0 \end{vmatrix}}{\begin{vmatrix} 2x_1 & x_3 & 0 \\ x_3 & 2x_2 & 0 \\ x_4 & x_5 & 1 \end{vmatrix}} = u \quad \Rightarrow \quad |c| = \sqrt{\frac{u}{2+u}}.$$

Since $c \in \mathbb{C}$, the coefficients are obtained with a phase ambiguity. Without loss of generality, we set the phase of c to zero, in other words, $\angle c = 0$ and $c = \sqrt{\frac{u}{2+u}}$. Consequently,

$$\begin{cases} |a| &= \sqrt{(1 - c^2) x_1} \\ \angle a &= \pm \arccos\left(\frac{x_4}{2c} \sqrt{\frac{1-c^2}{x_1}}\right) \end{cases} \text{ and } \begin{cases} |b| &= \sqrt{(1 - c^2) x_2} \\ \angle b &= \pm \arccos\left(\frac{x_5}{2c} \sqrt{\frac{1-c^2}{x_2}}\right) \end{cases},$$

where the signs of the phase are ruled by $2\Re(ab^*) = (1 - c^2)x_3$, as given in Equation (4.9).

Once the coefficients a, b and c have been obtained, we can retrieve the quantity $\{e^{2i\theta_j}\}_{j=1}^J$ by plugging back the values in Equation (4.7). We note that by setting $\angle c = 0$ in the solution, the estimation of $e^{2i\theta_j}$ is up to a phase and complex conjugate ambiguity. As a result, this introduces ambiguity in the estimation of the angles $\tilde{\theta}_j$ as well:

$$\tilde{\theta}_j = s\theta_j + l + k\pi, \quad (4.10)$$

where $s = \{1, -1\}$, $l \in [-\pi, \pi]$ and $k = \{0, 1\}$. Due to the same phase and complex conjugate ambiguity in the solution, the sign s and the constant shift l are the same for all retrieved angles, while k can differ from angle to angle. However, it can be determined by verifying the consistency of the linear systems based on the first order data consistency rule in Equation (4.2).

In summary, for an arbitrary 2D image $I(x, y)$ with geometrical moments that satisfy Equation (4.5), given $J \geq 5$ of its tomographic projections $\{g_{\theta_j}(t)\}_{j=1}^J$, we can estimate exactly the projection angles $\{\theta_j\}_{j=1}^J$ up to an unknown rotation and reflection. Specifically, the estimated angles satisfy $\hat{\theta}_j = s\theta_j + l$, where s and l represent unknown reflection and angle shift. In contrast to existing methods [10, 8], our approach not only establishes the uniqueness conditions for the 2D angle estimation problem but also provides a constructive solution for determining the projection angles. Additionally, our analysis demonstrates that the minimum number of required projection angles is reduced by half, significantly improving upon previous methods.

4.3 Image model and problem formulation

The angle estimation problem proposed in Section 4.2.1 is broadly applicable to a wide range of 2D images since its limitations are very mild, and does not rely on any restrictive signal models. However, recognizing the inherent challenges of 2D tomographic reconstruction, in this section, we consider a specific image model—those defined by curves with finite rate of innovation. Combined with the proposed angle estimation algorithm, we present a comprehensive pipeline for solving the 2D unknown angle tomographic reconstruction problem. More precisely, the whole proposed reconstruction method is a two-staged approach. In the first stage, we estimate the 2D projection angles using the algorithm introduced in Section 4.2.1, which is designed to handle a wide range of 2D images beyond the specific model considered here. Then, in the second stage, instead of directly reconstructing the unknown images, we propose an edge-based approach: we first recover the edge set of the image, followed by the full reconstruction of the image itself. This two-stage approach offers a robust solution to the 2D tomographic reconstruction problem when projection angles are unknown, which is demonstrated in later numerical evaluation sections.

4.3.1 Curve with finite rate of innovation and piecewise constant image model

We propose to use a piecewise constant image induced by the definition of curves with finite rate of innovation (FRI curve) [72], which offers two key advantages [73]:

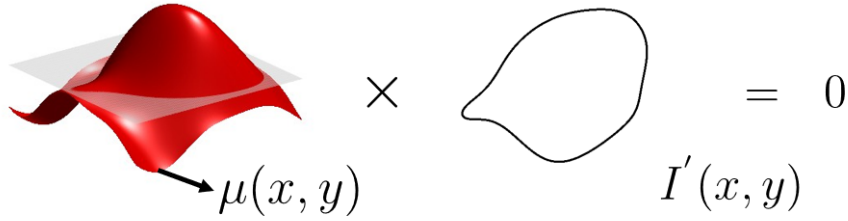
- it can effectively represent a wide range of image types using a relatively small number of parameters,
- it facilitates the recovery of high-frequency components by leveraging learned or prior structure, even when only low-frequency information is observed.

We first introduce the concept of FRI curves.

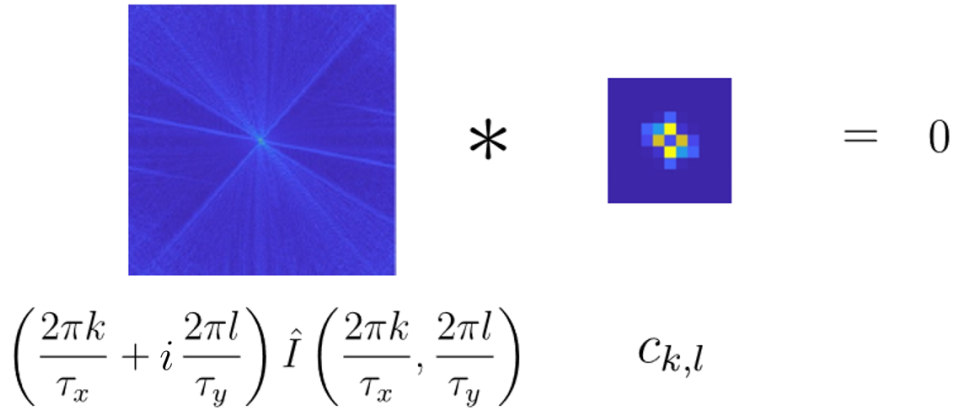


(a) The curve C is the zero level set of the trigonometric polynomial $\mu(x, y)$.

(b) Piecewise constant image $I(x, y)$ defined by curve C .



(c) Spatial domain interpretation of the annihilation relation.



(d) Frequency domain interpretation of the annihilation relation.

Figure 4.2: 2D image model induced by curves with finite rate of innovation.

This specific class of curves C are defined by the zero-level set of bandlimited periodic

trigonometric polynomials $\mu(x, y)$ [72]:

$$C : \underbrace{\sum_{k=-K_0}^{K_0} \sum_{l=-L_0}^{L_0} c_{k,l} e^{i \frac{2\pi k x}{\tau_x} + i \frac{2\pi l y}{\tau_y}}}_{\mu(x,y)} = 0, \text{ for } \begin{cases} x \in [0, \tau_x), \\ y \in [0, \tau_y), \end{cases} \quad (4.11)$$

where $K_0, L_0 \in \mathbb{N}^+$, and $c_{k,l} \in \mathbb{C}$ are coefficients of the complex exponentials. We define the degree of polynomial $\mu(x, y)$ to be the size of the smallest rectangle that contains the frequency support of μ , and it equals the rate of innovation of the curve C : $\deg(\mu) = \rho(C) = (2K_0 + 1, 2L_0 + 1)$. Moreover, we enforce the polynomial to be real-valued so that $\mu(x, y) = 0$ has always curves as solutions. This is equivalent to saying that the coefficients are Hermitian symmetric $c_{k,l} = c_{-k,-l}^*$ and an example is shown in Figure 4.2 (a).

It has been proven [72, 69] that for this class of curves, there exists a continuous domain piecewise constant image $I(x, y)$, such that the Wirtinger derivative of the image $\hat{I}'(\omega_x, \omega_y)$ is annihilated by convolving it with a filter made of the coefficients $c_{k,l}$, as shown in Figure 4.2 (d). Specifically, we define the image model $I(x, y)$ as follows:

$$I(x, y) = \sum_{n=1}^N a_n 1_{U_{C_n}}(x, y), \quad \forall (x, y) \in [0, \tau_x) \times [0, \tau_y), \quad (4.12)$$

where $U_{C_n} \subset [0, \tau_x) \times [0, \tau_y)$ is a region with boundaries defined by the curve C_n as shown in Figure 4.2 (b). Here $U_{C_n} = \{(x, y) : \mu_n(x, y) \geq 0\}$, and $1_{U_{C_n}}$ denotes the plane indicator function:

$$1_{U_{C_n}} = \begin{cases} 1, & \text{for } (x, y) \in U_{C_n}, \\ 0, & \text{otherwise.} \end{cases} \quad (4.13)$$

The Wirtinger derivative of the image $I'(x, y) = \partial_x I(x, y) + i \partial_y I(x, y)$ behaves like Dirac δ along the curve $C = \bigcup_{n=1}^N C_n$. Therefore, since $\mu(x, y) = \prod_{n=1}^N \mu_n(x, y)$ is zero on the curve C , we have that:

$$\mu(x, y) I'(x, y) = 0. \quad (4.14)$$

This is shown in Figure 4.2 (c). In frequency domain, the annihilation property is a convolution between the coefficients $c_{k,l}$ and the uniform discrete Fourier measurements of the image $\hat{I}'(\omega_x, \omega_y)$:

$$\sum_{k=-K_0}^{K_0} \sum_{l=-L_0}^{L_0} c_{k,l} \hat{I}'\left(\omega_x - \frac{2\pi k}{\tau_x}, \omega_y - \frac{2\pi l}{\tau_y}\right) = 0. \quad (4.15)$$

It is worth noting that such piecewise constant shape images actually play a significant role across a broad variety of tomographic applications. For instance, they are extensively utilized in non-destructive material analysis [45] to accurately identify material defects or structural characteristics from their CT scans without compromising the integrity of the material. In medical imaging, particularly in computed tomography, binary shape images are often used for the precise diagnosis of organ diseases; by converting conventional CT scans into binary format, they enhance the segmentation and differentiation of anatomical regions, allowing for accurate assessment and diagnosis [70].

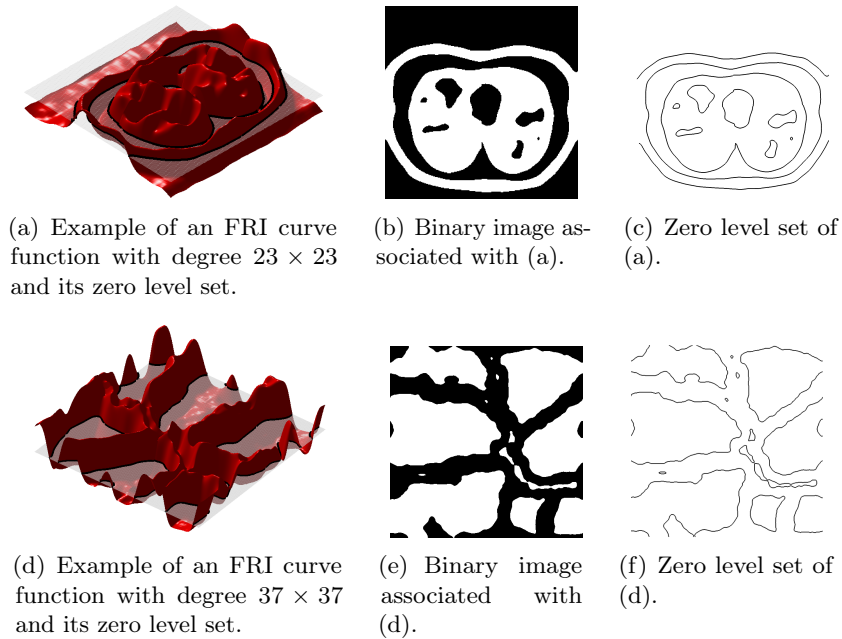


Figure 4.3: Examples of FRI curves. (a) shows a trigonometric function with coefficients of size 23×23 . Its associated binary image in (b) is an illustrative example of a binary CT image of a lung. Such binary images are often used to detect lung disease. The FRI curve, or the zero level set of the trigonometric function is shown in (c). (d) shows a trigonometric function with coefficients of size 37×37 . Its associated binary image in (e) is an illustrative example of a binary X-ray image of soil slice. These binary images are often used to distinguish between soil pores and solid materials. The FRI curve, or the zero level set of the trigonometric function is shown in (f).

Figure 4.3 exemplifies this by demonstrating how binary shape images encapsulate complex structural details, underscoring their adaptability and utility in various analytical and diagnostic contexts.

4.3.2 Projection acquisition

We consider the measurements to be a set of J tomographic projections $\{g_{\theta_j}(t)\}_{j=1}^J$ of the 2D image taken at unknown angles $\{\theta_j\}_{j=1}^J$, $\theta_j \sim U(0, \pi)$:

$$g_{\theta_j}(t) = \int_0^{\tau_y} \int_0^{\tau_x} I(x, y) \delta(x \cos \theta_j + y \sin \theta_j - t) dx dy, \quad (4.16)$$

Specifically, in this chapter, we focus on the challenging scenarios where the number of projections is substantially smaller than the image resolution-by at least an order of magnitude (e.g., $10\times$ fewer projections). Additionally, each projection measurement $\tilde{g}_{\theta_j}(t)$ is heavily corrupted by noise:

$$\tilde{g}_{\theta_j}(t) = g_{\theta_j}(t) + \epsilon_{\theta_j}(t), \quad (4.17)$$

where $\epsilon_j(t)$ is additive Gaussian white noise with zero mean and variance σ_ϵ^2 .

Given the few noise corrupted projections $\{\tilde{g}_{\theta_j}(t)\}_{j=1}^J$ and inaccurately estimated angles $\{\tilde{\theta}_j\}_{j=1}^J$, our aim is to reconstruct the image $I(x, y)$. In the following section, we introduce a two-stage reconstruction method.

4.4 Image reconstruction based on curve estimation

Given the estimated projection angles, instead of directly reconstructing the image from projections, we first retrieve the image curves using a robust FRI reconstruction algorithm [73]. The image is then recovered based on using the spatial domain annihilation relation for a regularized optimization problem. Finally, to enhance performance, we introduce a method to refine the projection angles based on the previously reconstructed 2D image. The overall method is depicted in Figure 4.4.

4.4.1 Curve recovery from inaccurate non-uniform Fourier measurements

Central slice theorem states that the Fourier transform of the 1D projection at angle θ_j is equal to the Fourier transform of its corresponding 2D image sliced through the origin in the direction parallel to the projection line. Therefore, given noisy projections and the inaccurately estimated projection angles $\{\tilde{\theta}_j\}_{j=1}^J$, we can obtain a set of Fourier measurements $\{\hat{I}(\omega \cos \tilde{\theta}_j, \omega \sin \tilde{\theta}_j)\}_{j=1}^J$. We stack all these measurements in a vector denoted with $\mathbf{a}$. Clearly, these measurements are non-uniformly taken in the frequency plane. The conventional reconstruction approach in the field of finite rate of innovation is to look for a discrete filter such that its convolution with the uniform Fourier samples

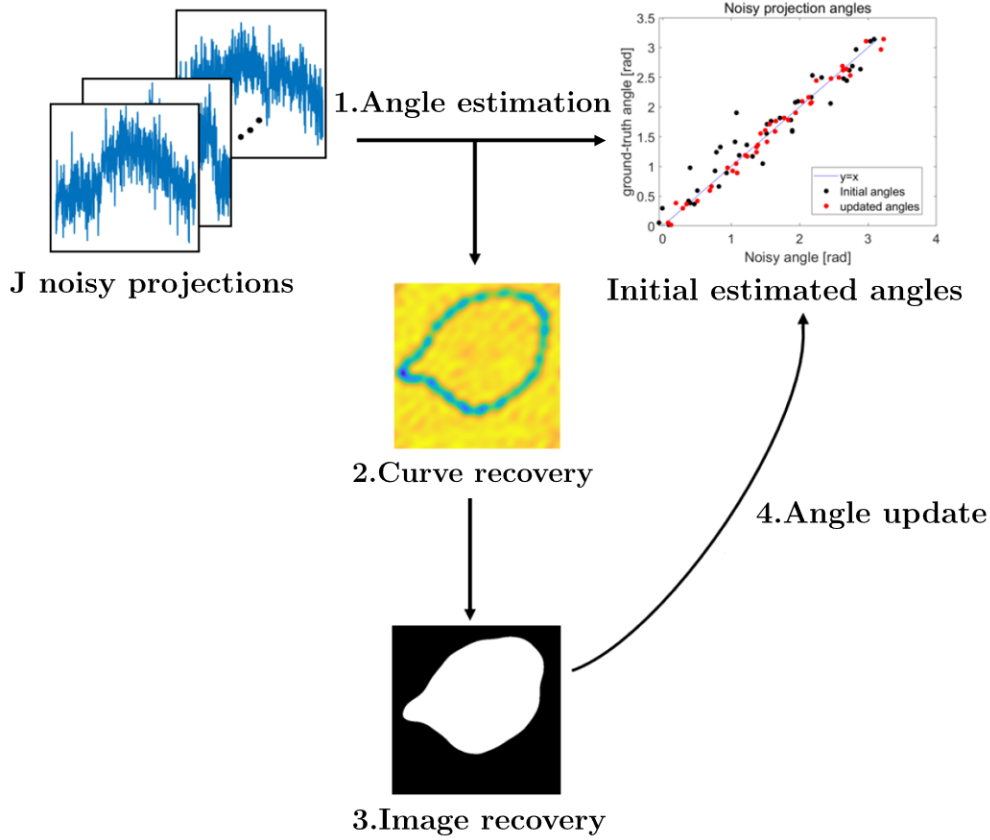


Figure 4.4: Overall scheme for the image reconstruction method. As a first step, projection angles are estimated from the noisy 1D projections using the proposed algorithm in Section 4.2.1. Then, the image associated FRI curve is recovered from the initial estimated angles and the projections by solving Equation (4.18). Based on the estimated curve, an FRI image is then retrieved. Lastly, an iterative angle update algorithm is applied to update both the image and angle estimation. The process is iterated a few times to obtain the final estimation result.

of the image is zero, as stated in Equation (4.15). Therefore, the question is whether we can recover curve coefficients $\mathbf{c} : \{c_{k,l}\}$ robustly from these non-uniform noisy Fourier measurements $\mathbf{a}$. We stack the curve coefficients in a vector $\mathbf{c}$.

The reconstruction problem can be reformulated as an approximation problem by fitting an FRI model that is consistent with the measurements. The FRI curve model requires the annihilation constraint to be satisfied. At the same time, after reconstructing the FRI signal, if we re-synthesize the measurements, they have to be consistent with the given measurements. Instead of looking for an optimal solution, we seek a valid solution that satisfies the above constraints. We adopt a constrained approximation formulation

to this problem as proposed in [73]:

$$\begin{aligned}
 & \text{find } \mathbf{b}, \mathbf{c} \in \mathcal{C} \\
 & \text{subject to } (\mathbf{R}(\mathbf{c}) \odot \mathbf{s}) \mathbf{b} = \mathbf{0} \\
 & \quad \|\mathbf{a} - \mathbf{G}\mathbf{b}\|_2^2 \leq \sigma_\epsilon^2,
 \end{aligned} \tag{4.18}$$

where

- $\mathbf{a}$ is the vector of non uniform Fourier measurements obtained from the projections;
- $\mathbf{b}$ is the vector of discrete uniform Fourier measurements that satisfies the annihilation constraint;
- $\mathbf{c}$ is the curve coefficients that annihilates $\mathbf{b}$, $\mathbf{R}(\cdot)$ is the Toeplitz structured convolution matrix;
- $\mathbf{s}$ is the frequency scaling factor $\left(\frac{2\pi k}{\tau_x} + j\frac{2\pi l}{\tau_y}\right)$ associated with the annihilation relation, and $\odot$ denotes the Hadamard product;
- $\mathbf{G}$ denotes the linear mapping between the measurements $\mathbf{a}$ and the uniform measurements $\mathbf{b}$ (uniform interpolation). In our case, it represents a 2D sinc kernel, because our image is support limited. The derivation for the interpolation formula is detailed in Appendix C.2.

The constrained approximation can be solved using an iterative scheme proposed in [73], which updates alternatively $\mathbf{c}$ and $\mathbf{b}$. More specifically, for a given $\mathbf{c}$, we can find the solution for $\mathbf{b}$ by considering a constrained minimization formulation of (4.18):

$$\begin{aligned}
 & \min_{\mathbf{b}} \|\mathbf{a} - \mathbf{G}\mathbf{b}\|_2^2 \\
 & \text{subject to } \tilde{\mathbf{R}}(\mathbf{c}) \mathbf{b} = \mathbf{0},
 \end{aligned} \tag{4.19}$$

where $\tilde{\mathbf{R}}(\mathbf{c}) = \mathbf{R}(\mathbf{c}) \odot \mathbf{s}$. On the other hand, by substituting the solution for $\mathbf{b}$ (in a function of $\mathbf{c}$), the optimization reduces to a problem involving only the annihilating filter coefficients $\mathbf{c}$. We denote the coefficients at the n th iteration $\mathbf{c}_n$ and $\mathbf{b}_n$. When fixing $\mathbf{c}_n$, the update for $\mathbf{b}$ can be solved by considering the constrained minimization in (4.19). By considering the associated Lagrangian, the solution can be obtained in the following matrix form (please see detailed derivation in Appendix C.3):

$$\begin{bmatrix} \mathbf{G}^H \mathbf{G} & \tilde{\mathbf{R}}^H(\mathbf{c}_n) \\ \tilde{\mathbf{R}}(\mathbf{c}_n) & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{b}_n \\ \mathbf{l} \end{bmatrix} = \begin{bmatrix} \mathbf{G}^H \mathbf{a} \\ \mathbf{0} \end{bmatrix}, \tag{4.20}$$

where $\mathbf{l}$ is an auxiliary variable. By substituting (4.20) into the objective function and

fixing $\mathbf{b}$, the updated $\mathbf{c}$ can be obtained by solving a quadratic minimization as follows:

$$\begin{aligned} & \min_{\mathbf{c}} \mathbf{c}^H \tilde{\mathbf{T}}^H(\beta) \left(\tilde{\mathbf{R}}(\mathbf{c}_{n-1}) (\mathbf{G}^H \mathbf{G})^{-1} \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1}) \right)^{-1} \tilde{\mathbf{T}}(\beta) \mathbf{c} \\ & \text{subject to } \mathbf{c}_0^H \mathbf{c} = 1, \end{aligned} \quad (4.21)$$

where $\beta = (\mathbf{G}^H \mathbf{G})^{-1} \mathbf{G}^H \mathbf{a}$ and $\mathbf{T}(\cdot)$ is the dual operator of $\mathbf{R}(\cdot)$ such that $\mathbf{T}(\mathbf{b})\mathbf{c} = \mathbf{T}(\mathbf{c})\mathbf{b}$. By construction, $\mathbf{T}(\mathbf{b})$ is also a Toeplitz-structured convolution matrix and $\tilde{\mathbf{T}}(\beta) = \mathbf{T}(\beta) \odot \mathbf{s}$. It can be shown that the solution of (4.21) is given by a solving a linear system as follows (please refer to Appendix C.3 for details):

$$\begin{bmatrix} \mathbf{0} & \tilde{\mathbf{T}}^H(\beta) & \mathbf{0} & \mathbf{c}_0 \\ \tilde{\mathbf{T}}(\beta) & \mathbf{0} & -\tilde{\mathbf{R}}(\mathbf{c}_{n-1}) & \mathbf{0} \\ \mathbf{0} & -\tilde{\mathbf{R}}^H(\mathbf{c}_{n-1}) & \mathbf{G}^H \mathbf{G} & \mathbf{0} \\ \mathbf{c}_0^H & \mathbf{0} & \mathbf{0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{c}_n \\ \mathbf{u} \\ \mathbf{v} \\ \lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ 1 \end{bmatrix}, \quad (4.22)$$

where $\mathbf{u}, \mathbf{v}, \lambda$ are introduced auxiliary variable. The FRI coefficient estimation algorithm is summarized in Algorithm 4.1 as follows:

Algorithm 4.1 Robust FRI coefficient estimation

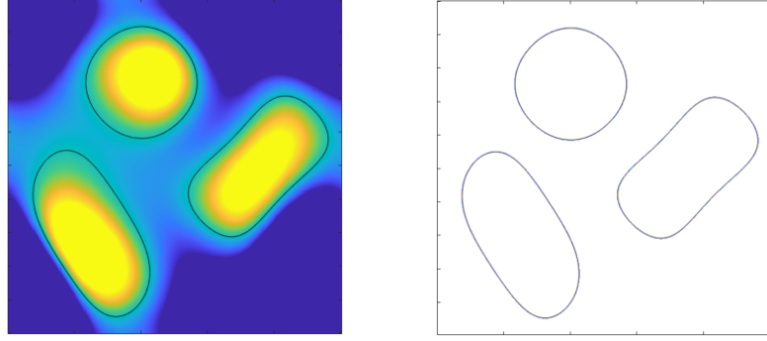
Input: non-uniform Fourier measurements $\mathbf{a}$, transformation matrix $\mathbf{G}$, noise level σ_ϵ^2

Output: Uniform Fourier measurements $\mathbf{b}$

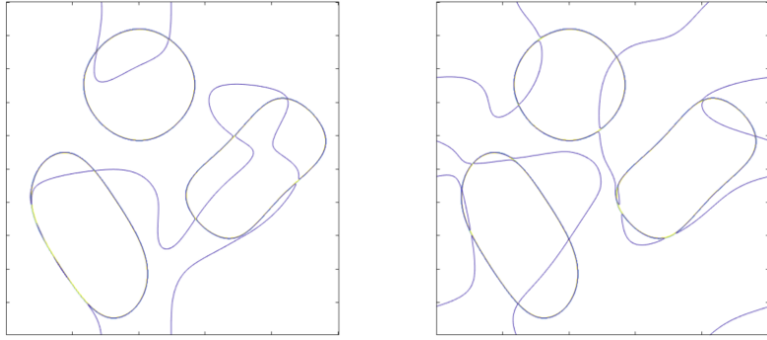
```

1: for loop  $\leftarrow 1$  to max. initialization do
2:   Randomly initialize  $\mathbf{c}_0$ 
3:   for  $n \leftarrow 1$  to max. iterations do
4:     Update  $\mathbf{c}_n$  by solving the linear system in (4.22)
5:     Update  $\mathbf{b}_n$  by solving the linear system in (4.20)
6:     if  $\|\mathbf{a} - \mathbf{G}\mathbf{b}_n\|^2 \leq \sigma_\epsilon^2$  then
7:       Terminate the loop
8:     end if
9:   end for
10: end for
    
```

Note that if the annihilation matrix $\mathbf{R}(\mathbf{c}) \odot \mathbf{s}$ is built with the correct filter support, i.e. $\deg(\mu) = \rho(C)$, the null space (a.k.a annihilating subspace) is one-dimensional, from which coefficients $\mathbf{c}$ can be retrieved using the annihilation relation. In cases when the correct filter support is unknown and the given measurements are very noisy, we can assume a filter support strictly larger than the correct one, i.e. $\deg(\mu) > \rho(C)$. In this case, the annihilating subspace will have dimension larger than 1. In this work, we leverage this redundancy to retrieve several polynomials $\mu_r(x, y)$. Each μ_r can be written in the form $\mu_r = \eta_r \mu$, where η_r is another trigonometric polynomial. Therefore, the desired curve is the common zeros of the set of polynomials μ_r and can be retrieved as



(a) Left: an FRI curve with coefficient support 7×7 . Right: recovery of the curve using 7×7 coefficients.



(b) Two recoveries of the curve using coefficient with support 9×9 .

Figure 4.5: A visual example of the recovery of an FRI curve using coefficients with a larger support than the critical value. (a) shows the original FRI curve of support 7×7 and the recovery curve using the critical size of the coefficient support. (b) shows two recoveries of the same curve using coefficient support larger than the critical one. It is not hard to observe that the common part of the two recovered curves in purple corresponds to the original curve. This is due to the existence of a common zero level set in their corresponding trigonometric polynomial.

the sum of squares (SoS):

$$\bar{\mu}(x, y) = \sqrt{\sum_{r=1}^R |\mu_r(x, y)|^2}. \quad (4.23)$$

A visual example is depicted in Figure 4.5.

4.4.2 Image recovery based on spatial formulation of the annihilation constraint

Once the polynomial $\bar{\mu}(x, y)$ is retrieved, we can estimate the image based on the spatial domain interpretation of the annihilation relation as stated in Equation (4.14). Therefore,

we propose to minimize objective function we want to minimize is:

$$\min_I \bar{\mu} \odot \|I'\|_{TV,2} + \lambda_1 \|\mathcal{A}I - \mathbf{b}\|_2^2 + \lambda_2 \|\Delta I\|_2^2, \quad (4.24)$$

where I is the reconstructed image, I' is approximated using finite differences and the first term imposes the annihilation constraint. $\mathcal{A}$ and Δ are the DFT and discrete Laplacian operator respectively, and they are used to impose data consistency and image smoothness. We propose to solve the above weighted total variation minimization problem by considering the following min-max formulation:

$$\min_I \max_{\mathbf{v} \in \mathbf{V}} \Phi(I, \mathbf{v}) := \mathbf{v}^T \mathcal{W}^T I + \lambda_1 \|\mathcal{A}I - \mathbf{b}\|_2^2 + \lambda_2 \|\Delta I\|_2^2, \quad (4.25)$$

where $\mathcal{W}$ is a linear operator that computes the weighted finite differences. The specific definition of $\mathbf{V}$ is given in Appendix C.4. We propose to use an iterative primal-dual approach to solve the above min-max problem [108]. In k th iteration, at the dual step we apply one step of gradient descent to update the auxiliary variable $\mathbf{v}^k$ with I^k fixed, while at the primal step, we update I^k with $\mathbf{v}^{k+1}$ fixed. The updating steps are summarized in Algorithm 4.2, and the detailed derivation is given in Appendix C.4.

Algorithm 4.2 Primal-dual algorithm

Input: step size $\tau_1, \tau_2 > 0$, λ_1, λ_2 , estimated uniform Fourier samples $\mathbf{b}$

Output: image I

- 1: **for** $k \leq K$ **do**
 - 2: $\mathbf{v}^{k+1} = \mathbf{P}_{\mathbf{v} \in \mathbf{V}}(\mathbf{v}^k + \tau_1 \mathcal{W}^T I^k)$
 - 3: $\mathbf{h} = I^k - \tau_2 \mathcal{W} \mathbf{v}^{k+1} + \tau_2 \lambda_1 \mathcal{A}^T \mathbf{b}$
 - 4: $I^{k+1} = \mathbf{h} / (1 + \tau_2 \lambda_1 \mathcal{A}^T \mathcal{A} + \tau_2 \lambda_2 \Delta^T \Delta)$
 - 5: **end for**
-

4.4.3 Angle refinement based on the reconstructed image

Given the first estimation of the image, we then refine the projection angles based on the previously reconstructed image. This can be done using approximate gradient descent method. Specifically, we set the measured projections $\tilde{\mathbf{g}} : \{\tilde{g}_{\theta_j}(t)\}_{j=1}^J$ as reference, and use $\mathbf{g}(\tilde{\boldsymbol{\theta}}) : \{g_{\tilde{\theta}_j}(t)\}_{j=1}^J$ to represent re-synthesized projections from the reconstructed image at estimated angles $\tilde{\boldsymbol{\theta}} : \{\tilde{\theta}_j\}_{j=1}^J$. We define the cost function at angles $\tilde{\theta}_j$ as $\mathcal{L}(\tilde{\theta}_j) = \|\tilde{g}_{\theta_j} - g_{\tilde{\theta}_j}\|_2^2$. The approximate gradient at angles $\tilde{\theta}_j$ is defined as

$$\nabla \mathcal{L}(\tilde{\theta}_j) = \frac{\mathcal{L}(\tilde{\theta}_j + \delta) - \mathcal{L}(\tilde{\theta}_j)}{\delta}, \quad (4.26)$$

where δ is a small enough angle. In each iteration, we operate by updating the angles in the direction of the approximate gradient that reduce the cost function, and the angle step is gradually reduced until all the angles get updated once. We summarize the

algorithm in Algorithm 4.3.

Algorithm 4.3 Angle updating algorithm

Input: step size $\tau^0 > 0$, $\gamma \in (0, 1)$, $\boldsymbol{\theta}^0 = \tilde{\boldsymbol{\theta}}, \tilde{\mathbf{g}}, k = 0, K$

Output: updated angles $\boldsymbol{\theta}$

```

1: for  $k \leq K$  do
2:    $\tau \leftarrow \tau^0$ ,  $\text{flag} \leftarrow 1$ ,  $\zeta \leftarrow \boldsymbol{\theta}^k$ 
3:   while  $\text{flag}$  do
4:      $\zeta_j^{tmp} = \zeta_j - \tau \nabla \mathcal{L}(\zeta_j)$ 
5:     update  $\theta_j^{k+1}$  if  $\mathcal{L}(\zeta_j^{tmp}) \leq \mathcal{L}(\zeta_j)$ 
6:     exclude updated angles from  $\zeta$ 
7:      $\text{flag} \leftarrow 0$  if  $\text{len}(\zeta) == 0$ 
8:      $\tau = \tau * \gamma$ 
9:   end while
10: end for

```

In summary, the overall scheme for image reconstruction is comprised of three stage: 1) Curve recovery given noisy projections and initial angle estimates, 2) curve aware image recovery and 3) angle refinement based on the estimated 2D image.

4.5 Numerical simulations and results

To evaluate the performance of our proposed method, we conduct experiments using a set of curves C with a fixed dimensional property $\rho(C) = 7 \times 7$. This degree ensures sufficient detail to accurately reflect the curve structure while balancing computational efficiency. The associated images for these curves are of dimension 501×501 .

In each experiment, we simulate the acquisition of projection data by generating line integrals of the image along $J = 40$ distinct angles $\{\theta_j\}_{j=1}^{40}$, and the angles are randomly drawn from a uniform distribution in $[0, \pi)$ rad. Moreover, to test the method's resilience under challenging conditions, we introduce controlled degradation by adding the Gaussian noise to each projection, such that each projection measurement achieves a signal-to-noise ratio (SNR) of 5 dB.

In what follows, we aim to assess the stability and accuracy of our method by examining the reconstructions under these controlled conditions, providing quantitative insights into its efficacy in recovering the original curve structures from a small number of noisy projections.

4.5.1 Performance evaluation of the angle estimation algorithm

As a first step in our reconstruction algorithm, we obtain an initial angle estimation from the projection measurements using the algorithm proposed in Section 4.2.1. Therefore, we first evaluate numerically the performance of the angle estimation algorithm under noisy conditions.

As discussed in Section 4.2.1, the accuracy of angle estimation in our framework is significantly influenced by the noise level present in the projection measurements. Consequently, the signal-to-noise ratio (SNR) of the projection data becomes a critical parameter in determining the precision of the estimated angles.

Beyond signal-to-noise ratio, angle estimation accuracy is also inherently affected by the structural properties of the projected image. Specifically, we require linear independence between the first and second order moments of the projections to achieve robust performance. This requirement reflects an intrinsic property of the image that is quantitatively captured by the condition number of the matrix formulated in Equation (4.5). Therefore, we use the condition number of the coefficient matrix to quantify the sensitivity of the solution to perturbations. For linearly independent equations, the condition number is usually lower, reflecting a more stable and accurate solution.

In our numerical evaluation, we analyze the root mean square error (RMSE) of the estimation error, defined as the difference between the estimated angles (corrected for the global unknown shifts) and the true angles:

$$\text{RMSE} = \sqrt{\frac{1}{J} \sum_{j=1}^J (\tilde{\theta}_j - \theta_j)^2} \quad (4.27)$$

The RMSE serves as a metric for quantifying the accuracy of angle estimation under varying SNR levels and condition numbers. We evaluate the performance with respect to the matrix condition number within the range of $[10, 1200]$. At each number, we generate 100 different images with $\rho(C) = 7 \times 7$.

In Figure 4.6, we present the relationship between the RMSE (in rad), the SNR of the projection data, and the condition number of the system coefficient matrix. The results demonstrate an expected trend: lower noise levels (i.e., higher SNR) and smaller condition numbers yield better accuracy of angle estimation.

This evaluation provides an initial quantitative assessment of the reconstruction performance, focusing primarily on the angle estimation accuracy of our methodology. While these results offer valuable insights into the effectiveness of our approach. It is important to emphasize that the angle estimation algorithm presented here represents only a foundational step in the overall reconstruction pipeline we propose. We have further

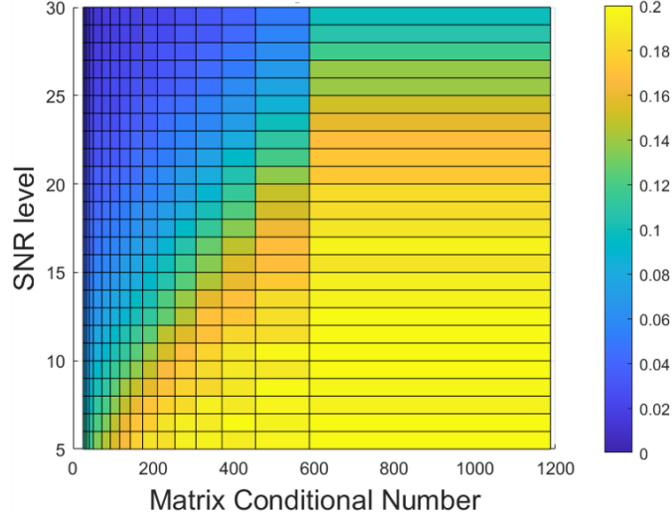


Figure 4.6: RMSE error in the angle estimation with respect to the SNR and matrix conditional number. The number of projections is 40.

integrated an additional angle refinement stage. This refinement process is designed to iteratively correct and improve the accuracy of the initial angle estimates, thereby enhancing the robustness of the final reconstruction. We will show in the next section that this approach leads to substantial improvements in estimation accuracy, even in cases where the initial estimates exhibit significant deviation from the ground truth.

4.5.2 Image reconstruction accuracy

As mentioned before, we consider curves with $\rho(C) = 7 \times 7$, and each of the 40 projections is corrupted with additive Gaussian noise such that the SNR in each measurement is 5 dB. To visualize the noisy projections, please refer to Figure 4.7. The projection angles are estimated based on the algorithm proposed in Section 4.2.1, and they deviate from the true angles in general. From the analysis in the previous section, we understand that such deviations also depend on the image structural property.

In implementing the robust FRI estimation methods detailed in Algorithm 4.1 for determining the coefficients of the curve, we select the 2D Sinc kernel as our interpolation kernel. Furthermore, to improve the robustness and resolution of the reconstructed curve, we employ a higher degree of curve support, specifically 9×9 . This extended support facilitates a richer null space within the annihilation matrix, enabling the application of the sum-of-squares strategy to enhance stability and to suppress the effect of noise.

For Algorithm 4.2, the choice of the regularization weights λ_1 and λ_2 balances the trade-off between encouraging piecewise smoothness of the image and preserving sharp edges.

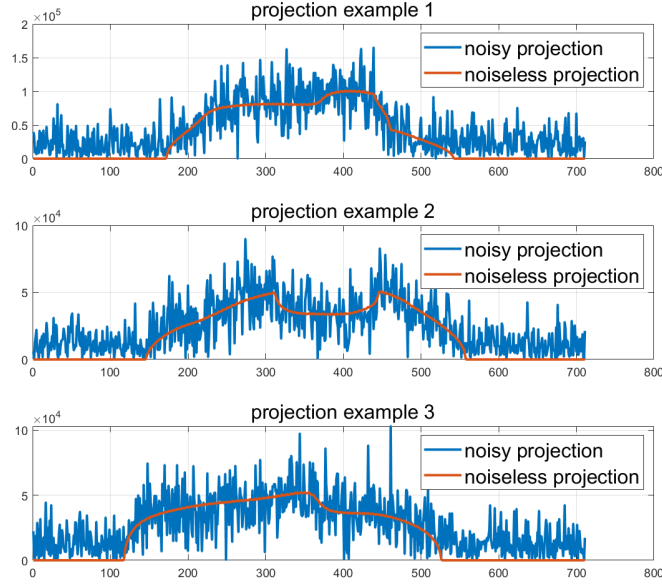


Figure 4.7: Examples of noisy projections of the synthetic data.

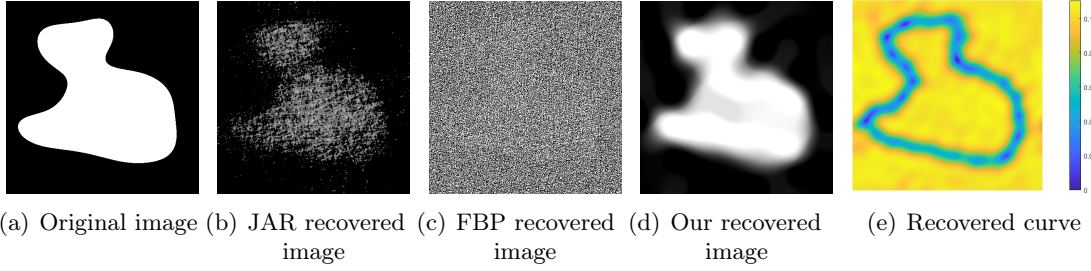


Figure 4.8: Visual comparison of reconstructed images by our proposed method, JAR method and FBP method. Compared to the original image, the SSIM and PSNR of our reconstructed image are 0.823 and 20.89 dB respectively (image size: 501×501 , number of projections: 40, PSNR on each projection: 5 dB, matrix conditional number 240).

We set $\lambda_1 = 5 \times 10^2$ and $\lambda_2 = 0.5$ for the following experiments¹. To quantitatively evaluate the reconstruction performance, we employ both Structural Similarity Index (SSIM) and Peak Signal-to-Noise Ratio (PSNR) metrics. The experiments are conducted over a set of 100 images, yielding an average PSNR of 21.72 dB and an average SSIM of 0.792. For comparison, we also show the reconstruction results for filtered backprojection (FBP) method and the joint angular refinement (JAR) method proposed by Zehni *et. al* [106] given that they know the correct projection angles, as shown from Figure 4.9 to Figure 4.11. It is clear that due to the correct choice of the image model, the estimated polynomial is able to successfully yield the correct curve even with limited number of noisy projections and inaccurate initial estimation of projection angles. Apart from the

¹From our experiments, the choice of the tunable weights is not critical. A faithful reconstruction can be achieved with $\lambda_1 \in [10^{-1}, 10^4]$ and $\lambda_2 \in [10^{-3}, 10^1]$ (if the image range is $[0,1]$).

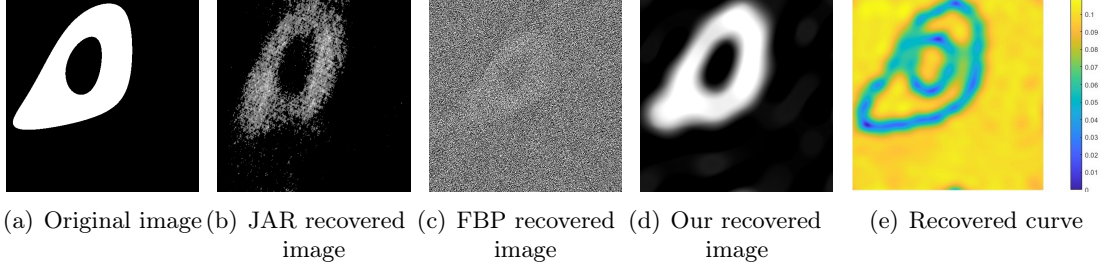


Figure 4.9: Visual comparison of reconstructed images by our proposed method, JAR method and FBP method. Compared to the original image, the SSIM and PSNR of our reconstructed image are 0.801 and 18.23 dB respectively (image size: 501×501 , number of projections: 40, PSNR on each projection: 5 dB, matrix conditional number 400).

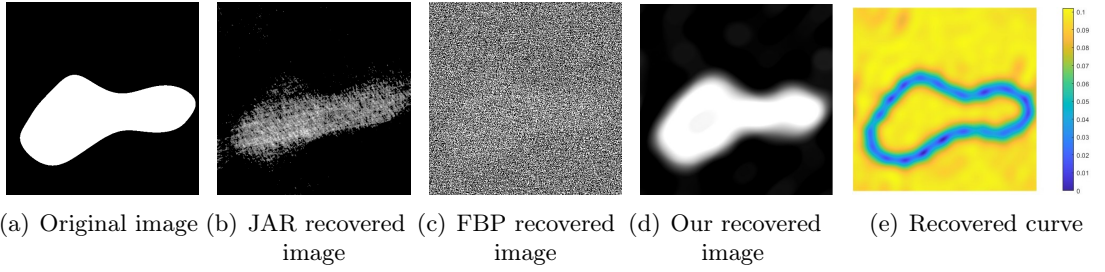


Figure 4.10: Visual comparison of reconstructed images by our proposed method, JAR method and FBP method. Compared to the original image, the SSIM and PSNR of our reconstructed image are 0.921 and 24.54 dB respectively (image size: 501×501 , number of projections: 40, PSNR on each projection: 5 dB, matrix conditional number 50).

robust FRI reconstruction algorithm, essential to the successful recovery of the curve is that we exploit the whole annihilating subspace, which makes the algorithm more robust under noisy scenarios. The FBP method fails due to heavy noise, while the JAR method can still recover the object despite blurring effect, due to model mismatch and lack of projections.

To highlight the impact of different matrix condition numbers on reconstruction accuracy, Table 4.1 presents the PSNR and SSIM values across different matrix condition numbers. Generally, higher reconstruction accuracy is achieved when the matrix is well-conditioned. As the matrix condition number increases moderately, PSNR and SSIM remain relatively stable due to the angle refinement algorithm, which adjusts the angle estimation. However, when the matrix becomes highly ill-conditioned, the accuracy of the reconstruction deteriorates significantly, attributed to the challenges in initial angle estimation under these conditions.

For the image presented in Figure 4.10, the associated matrix exhibits a condition number of approximately 400. This relatively high condition number indicates a negative impact

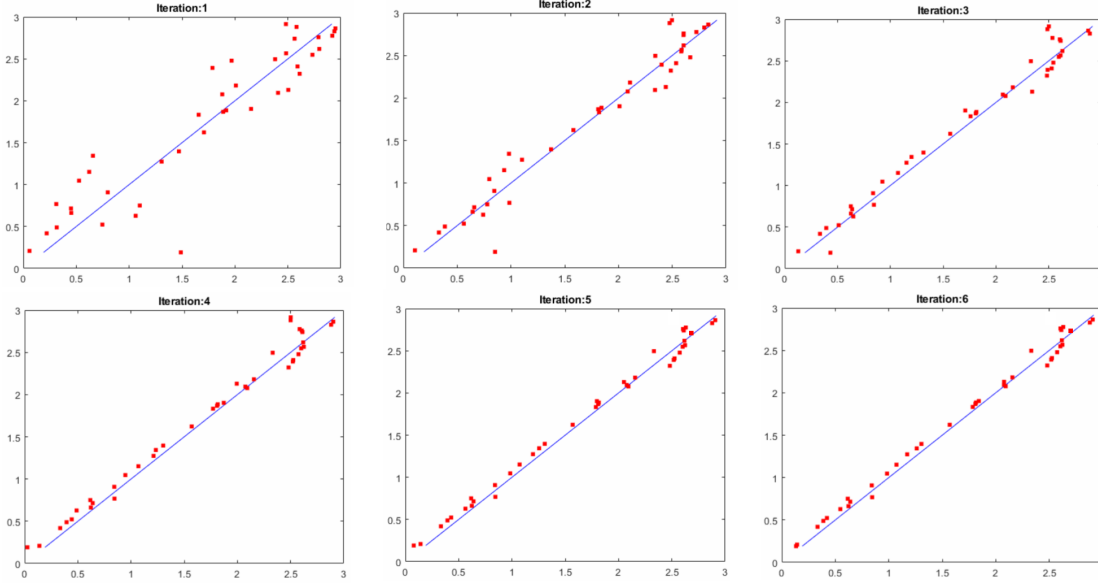


Figure 4.11: Angle update process during reconstruction based on Algorithm 4.3, corresponding to the image in Figure 4.9. The estimated angles (x-axis) progressively converge towards the true angles (y-axis) over 5 iterations, with the root mean square error decreasing significantly from 0.32 rad to 0.13 rad.

on the accuracy of initial angle estimation. To illustrate the improvement from initial to final angle estimation, we provide a detailed comparison in Figure 4.11, which captures the angle estimation trajectory across the iterative refinement process. The progression from initial estimation to final estimation reflects how iterative angle update algorithm facilitates the gradual convergence towards the true angle values. The large conditional number underscores the necessity of the iterative refinement stage in our algorithm, as the initial estimates alone are insufficiently reliable.

Metric	Matrix Condition Number					FBP	JAR [106]
	100	200	400	600	1200		
PSNR (dB)	23.26	22.64	22.17	16.27	15.76	3.451	9.012
SSIM	0.906	0.869	0.863	0.601	0.575	0.092	0.489

Table 4.1: PSNR and SSIM values for the reconstruction of a synthetic piecewise constant image with varying matrix condition numbers, with each projection’s SNR set to 5 dB. The last two columns report the reconstruction performance of FBP and JAR methods, which, giving the correct angle value, remain unaffected by the matrix condition number.

4.5.3 Test on real CT scan of lung

As mentioned previously, piecewise constant images, particularly in the context of medical imaging such as CT (computed tomography) scans, play a crucial role in the detection and diagnosis of organ diseases. Particularly in lung CT imaging, these often appear as the presence of abnormal regions. The ability to detect these boundaries precisely is critical, as nodules and lesions can be indicators of conditions such as lung cancer, tuberculosis, or other respiratory diseases. For this purpose, we apply our method on the real CT scan of a lung and show the reconstruction result².

The image considered in this analysis has size 1024×1024 . Consistent with previous sections, the number of projections remains significantly lower than the image resolution, with only 150 projections collected. These projections are subject to noise corruption, with an SNR of 5 dB. Examples of projections are shown in Figure 4.12. We then apply the proposed reconstruction method to estimate the finite rate of innovation (FRI) curve and reconstruct the lung image. Due to the complex nature of the lung boundary, which is not precisely captured by a standard FRI curve, we choose an abundant number of degree for the curve, $\rho(C) = 25 \times 25$. The reconstruction result is shown in Figure 4.13, where the reconstructed SSIM and PSNR are 0.721 and 15.13 dB respectively.

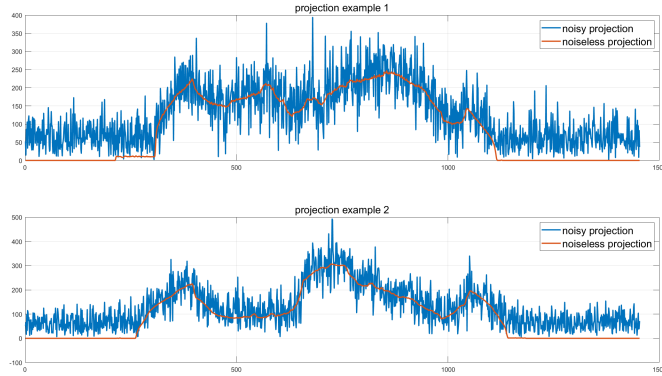


Figure 4.12: Examples of noisy projections.

4.6 Conclusions

In this work, we introduce a constructive and systematic solution to the problem of determining projection angles in 2D tomography with unknown angles. By leveraging data consistency conditions, we demonstrate that our proposed angle estimation algorithm remains effective with as few as 5 projections, even under broad and general conditions, highlighting the adaptability of the approach.

²image taken from <https://kaggle.com/competitions/data-science-bowl-2017>, Kaggle.

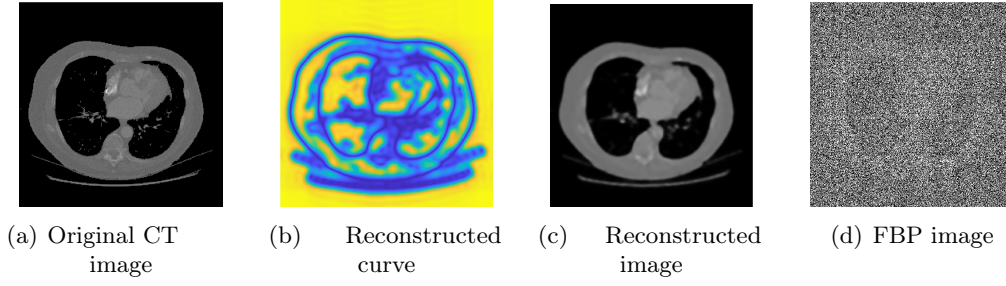


Figure 4.13: Reconstruction of a real CT scan image of lung. (a) The original CT scan of a lung. (b) The recovered curve with $\rho(C) = 25 \times 25$. (c) The recovered piecewise constant image. Compared to the original image in (a), the SSIM and PSNR are 0.721 and 15.13 dB respectively. (image size: 1024×1024 , number of projections: 150, PSNR on each projection: 5 dB) (d) Reconstructed image using FBP method.

In addition to this, we also carefully reflect on the choice of signal models that are appropriate for this specific type of image acquisition. Specifically, we focus on images defined by curves with a finite rate of innovation. Unlike conventional reconstruction methods that aim to directly estimate the 2D images from projection data, we employ a frequency-domain annihilation relationship that enables us to retrieve curve coefficients from projections, even in the presence of significant noise. This approach differs from traditional methods by capturing the curve coefficients in the frequency domain before transitioning back to the spatial domain, where we utilize a weighted total variation formulation of the spatial-domain annihilation relations to achieve a detailed and robust initial reconstruction.

The effectiveness of this approach is underscored by experimental results, which show that our method achieves satisfactory reconstruction quality, even when projection data are severely corrupted by noise and the initial projection angles deviate substantially from their true values. After only a few iterations of the refinement process, the algorithm successfully yields high-quality reconstructions that closely approximate the ground-truth. The iterative nature of our approach not only enhances robustness but also provides a framework adaptable to various levels of noise and angle estimation challenges, making it a promising method for accurate 2D tomographic reconstruction in complex and noisy environments.

Chapter 5

Conclusion

Nothing in life is to be feared,
it is only to be understood.
Now is the time to understand more,
so that we may fear less.

— Marie Curie

5.1 Main contributions

In the first part of this thesis, we introduced a novel sampling framework for reconstructing classes of 2D and 3D FRI signals, including 2D Diracs, bilevel polygons, 3D Diracs and bilevel polyhedra, from samples of tomographic projections at unknown angles. Our approach establishes uniqueness conditions for achieving perfect reconstruction. The sampling process models the conversion from continuous signals to discrete measurements inherent in projection acquisition. By leveraging the divergence theorem and exploring the polynomial and exponential reproduction property of the sampling kernel, we demonstrated that discrete 2D samples obtained at any projection angle retain critical geometrical features of the original signal. For any polyhedral shape, as few as 3 distinct projections suffice for exact reconstruction of the polyhedron and the unknown projection angles. This reconstruction is achieved via an algebraic approach applied to the extracted geometric features. We further proposed a robust algorithm based on analytic approximation which tackles severe measurement noise and allows arbitrary sampling kernels. Additionally, our methodology offers both theoretical insights and practical solutions for real-world applications such as the geometric calibration of X-ray imaging systems. Overall, this framework bridges the gap between theory and application in sampling and reconstructing geometrical structures, providing a powerful theoretical insight and tool for efficient and precise signal reconstruction in tomographic imaging contexts.

We later investigated the problem of general angle estimation in 2D unknown tomography and proposed a systematic and constructive algorithm for addressing it. By building on data consistency principles, we developed a practical method to estimate unknown projection angles through a system of trigonometric equations using image and projection moments. This approach operates under minimal model constraints, making it adaptable to a range of scenarios. Furthermore, recognizing the challenges posed by limited numbers of projections, we proposed a tailored image reconstruction algorithm based on piecewise constant image models defined by curves with finite rate of innovation. This method bypasses direct image reconstruction from noisy projections by retrieving image curves from non-uniform, inaccurate and low-frequency data and employing spatial domain annihilation for regularized optimization. To further improve the reconstruction accuracy, we introduced a refinement step for projection angles, informed by the initially reconstructed image. The numerical results also demonstrate the potential of combining innovative mathematical models with computational optimization to enhance image reconstruction quality in the field of unknown angle tomography with minimal prior information and noisy data environments.

5.2 Outlook and future work

We conclude this thesis with an overview of potential directions for future research.

5.2.1 Sampling and reconstruction of realistic molecular structure

Based on the sampling framework developed in this thesis, an intriguing direction for future research is the analysis and reconstruction of more complex and realistic structures. Realistic structures often exhibit smooth and non-uniform features that are challenging to model with idealized signal models such as polyhedra. However, they can be approximated as a sum of pulses where the kernel can be Gaussians or other types of isotropic functions. By approximating structures as a superposition of localized features, the locations and amplitudes of the Gaussians encapsulate the essential geometrical and structural information. The locations of the pulses can serve as key feature points, similar to the vertices in polyhedra, representing the structure’s geometry, while allowing for exact angle retrieval using an approach similar to our proposed sampling framework. Moreover, the Gaussian-based framework also provides a bridge to modern data-driven techniques [52], such as deep learning. Neural networks could help learning the mapping from projection data to Gaussian parameters to improve reconstruction in the presence of measurement noise.

5.2.2 3D surface recovery from tomographic projections at unknown angles with reference points

Building on our advancements in 2D unknown angle tomography, a natural progression is to extend these principles to the recovery of 3D surfaces from tomographic projections at unknown angles. 3D surface reconstruction enables the recovery of an object’s boundaries and shapes, which is often more important than the full volumetric data in applications such as structural analysis and object modeling. For instance, in industrial testing, 3D surface reconstruction can identify cracks and voids on material surfaces non-destructively. Another important application is 3D printing where surface recovery from tomography is a critical step in creating 3D printable models or reverse-engineering objects for replication. To achieve efficient and accurate reconstruction, 3D surfaces can be modeled as piecewise smooth manifolds with finite rate of innovation. This representation allows the use of annihilation-based methods to extract surface parameters directly from the projection data, bypassing heavy volumetric reconstructions and mitigating the effects of noise and undersampling. This represents a significant leap in complexity but holds immense potential for real-world applications.

Appendix A

Appendices for Chapter 2

A.1 Prony's method

Prony's method [76] (also known as annihilating filter method) was originally developed to retrieve the frequency, amplitude, phase and damping components of a signal from a series of damped sinusoids. When adapted to the context of finite rate of innovation (FRI), this method provides a precise framework for retrieving the parameters defining the parameters of the FRI signal. Specifically, it achieves this by leveraging the signal's moments, structured as a power sum series, to determine the desired parameters efficiently.

The name of this approach comes from the annihilation relations stated in Equation (2.7) and (2.8), where the power sum series $s_m = \sum_{k=1}^K w_k(u_k)^m$ built from samples of the FRI signal is annihilated by a filter $\mathbf{h}$ whose zeroes locate at u_k . Now we present the detailed method to retrieve the parameters $\{w_k, u_k\}_{k=1}^K$ from $\mathbf{s} : s_m, m = 0, \dots, M-1$.

The z -transform of the discrete annihilation filter $\mathbf{h}$ can be written as:

$$H(z) = \sum_{m=0}^K h_m z^{-m} \stackrel{(a)}{=} \prod_{k=1}^K (1 - u_k z^{-1}), \quad (\text{A.1})$$

where (a) comes from the definition of this filter. The discrete annihilation relation can be derived by:

$$\mathbf{h} * \mathbf{s} = \sum_{l=0}^K h_l s_{m-l} = \sum_{k=1}^K w_k(u_k)^m \underbrace{\sum_{l=0}^K h_l u_k^{-l}}_{=0} = 0. \quad (\text{A.2})$$

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When $M \geq 2K$, we can rewrite the above equation in a Yule-Walker system as follows:

$$\begin{bmatrix} s_K & s_{K-1} & \cdots & s_0 \\ s_{K+1} & s_K & \cdots & s_1 \\ \vdots & \vdots & \ddots & \vdots \\ s_{M-1} & s_{M-2} & \cdots & s_{K-1} \end{bmatrix} \begin{bmatrix} 1 \\ h_1 \\ \vdots \\ h_K \end{bmatrix} = S\mathbf{h} = \mathbf{0}, \quad (\text{A.3})$$

where h_0 is assumed to take the value 1. It can be shown that when $\{w_k\}_{k=1}^K$ are non zero and $\{u_k\}_{k=1}^K$ are distinct, the above matrix S is full row rank, therefore, the solution for the annihilating filter $\mathbf{h}$ is unique. Once the filter is retrieved, the ‘frequency’ component $\{u_k\}_{k=1}^K$ can be obtained from evaluating the roots of this filter. Subsequently, the amplitudes $\{w_k\}_{k=1}^K$ can be obtained by considering the linear system given by $s_m = \sum_{k=1}^K w_k (u_k)^m$.

A.2 Sampling and reconstruction of bilevel polygons using complex moment based method

Let $I(x, y)$ be a bilevel polygon with K vertices located at $\{x_k, y_k\}_{k=1}^K$. When its samples are obtained using a sampling kernel that can reproduce polynomials up to degree $P - 1$ on both x and y directions, the corresponding signal moments $s_{\alpha, \beta}$ can be represented as:

$$\begin{aligned} s_{\alpha, \beta} &= \sum_{m, n} c_{m, n}^{\alpha, \beta} I_{m, n} \\ &= \sum_m \sum_n c_{m, n}^{\alpha, \beta} \iint I(x, y) \varphi\left(\frac{x}{T_x} - m, \frac{y}{T_y} - n\right) dx dy \\ &= \iint I(x, y) \sum_m \sum_n c_{m, n}^{\alpha, \beta} \varphi\left(\frac{x}{T_x} - m, \frac{y}{T_y} - n\right) dx dy \\ &= \iint I(x, y) x^\alpha y^\beta dx dy, \quad \alpha, \beta = 0, \dots, P-1. \end{aligned} \quad (\text{A.4})$$

We notice that, the signal moments are not directly in the form of a power sum series. However, the authors [86] have shown that by leveraging the Davis’s theorem [25, 24], alternative complex signal moments synthesized from $s_{\alpha, \beta}$ can be written in the form of a power sum series. Specifically, the theorem states that, for any complex function $f(z)$, $z = x + iy \in \mathbb{C}$ analytic in the signal support, the following equality holds:

$$\iint I(x, y) f''(z) dx dy = \sum_{k=1}^K \rho_k f(z_k), \quad (\text{A.5})$$

where $\rho_k \in \mathbb{C}$ are some non-zero coefficients and $z_k = x_k + iy_k$.

In [86], the authors have shown that by choosing the analytic function $f(z) = z^r$, $r = 0, \dots, P - 1$, Equation (A.5) can be rewritten as follows:

$$r(r-1) \underbrace{\iint I(x, y)(x + iy)^{r-1} dx dy}_{\tau_{r-2}} = \sum_{k=1}^K \rho_k z_k^r, \quad (\text{A.6})$$

where τ_r denotes the complex moments of $I(x, y)$. It is not hard to note that the *rhs* of the above equation is in the form of a power sum series, with which we can apply spectra estimation method to estimate the free parameters z_k . On the *lhs*, the complex moments can be obtained from the geometrical moments $s_{\alpha, \beta}$ by simply applying binomial expansion. Hence, it follows that if the bilevel polygon $I(x, y)$ is convex, then it can be uniquely reconstructed from $2K$ weighted complex moments, or equivalently geometrical moments $s_{\alpha, \beta}$ up to order $2K - 3$, i.e. $P \geq 2K - 2$.

Appendix B

Appendices for chapter 3

B.1 Proof of Lemma 1 in Chapter 3

Proof. Let

$$L = \iint_{\Pi_j} I_j(x, y) h'''(z) \, dx dy.$$

By substituting $I_j(x, y)$ with the expression in Equation (3.3), L can be written as:

$$\begin{aligned} L &= \iint_{\Pi_j} \int_{\mathbb{R}^3} g(\mathbf{r}) \delta(x - \mathbf{u}_{j,x}^T \mathbf{r} - s_{j,x}) \delta(y - \mathbf{u}_{j,y}^T \mathbf{r} - s_{j,y}) \, d^3 \mathbf{r} h'''(z) \, dx dy \\ &\stackrel{(a)}{=} \int_{\mathbb{R}^3} g(\mathbf{r}) h'''(\mathbf{u}_{j,x}^T \mathbf{r} + s_{j,x} + i(\mathbf{u}_{j,y}^T \mathbf{r} + s_{j,y})) \, d^3 \mathbf{r} \\ &= \int_{\mathbb{R}^3} g(\mathbf{r}) h''' \left(\underbrace{(\mathbf{u}_{j,x} + i\mathbf{u}_{j,y})^T}_{\mathbf{u}_j} \mathbf{r} + \underbrace{(s_{j,x} + i s_{j,y})}_{s_j} \right) \, d^3 \mathbf{r} \\ &= \int_{\mathbb{R}^3} g(\mathbf{r}) h'''(\mathbf{u}_j^T \mathbf{r} + s_j) \, d^3 \mathbf{r}, \end{aligned} \tag{B.1}$$

where $\mathbf{u}_j \in \mathbb{C}^3$, $s_j \in \mathbb{C}$, and (a) follows from the property of the Dirac Delta function $\delta(\cdot)$.

For any vector $\mathbf{w} \in \mathbb{C}^3$, the following equality holds:

$$\text{div} \left(h''(\mathbf{u}_j^T \mathbf{r} + s_j) \mathbf{w} \right) = h'''(\mathbf{u}_j^T \mathbf{r} + s_j) \mathbf{u}_j^T \mathbf{w}, \tag{B.2}$$

where div denotes divergence.

Therefore, if we multiply both sides in Equation (B.1) with $\mathbf{u}_j^T \mathbf{w}$, it yields:

$$\begin{aligned}
 \mathbf{u}_j^T \mathbf{w} L &= \int_{\mathbb{R}^3} g(\mathbf{r}) h'''(\mathbf{u}_j^T \mathbf{r} + s_j) \mathbf{u}_j^T \mathbf{w} d^3 \mathbf{r} \\
 &\stackrel{(b)}{=} \int_{\mathbb{R}^3} g(\mathbf{r}) \operatorname{div} (h''(\mathbf{u}_j^T \mathbf{r} + s_j) \mathbf{w}) d^3 \mathbf{r} \\
 &\stackrel{(c)}{=} \int_{\Gamma} \operatorname{div} (h''(\mathbf{u}_j^T \mathbf{r} + s_j) \mathbf{w}) d^3 \mathbf{r} \\
 &\stackrel{(d)}{=} \oint_{\partial \Gamma} h''(\mathbf{u}_j^T \mathbf{r} + s_j) \mathbf{w}^T d\mathbf{S}.
 \end{aligned} \tag{B.3}$$

where (b) follows from Equation (B.2), (c) follows from Equation (3.2) and (d) follows from the divergence theorem.

We note that the surface of a polyhedron $\partial \Gamma$ is made of flat polygonal faces, and every such polygonal face can be decomposed into disjoint triangular areas whose vertices are the vertices of the polyhedron. Please see Figure B.1 for an example.

Let $k, l, m \in \mathcal{T}_\Gamma$ be distinct integers. We denote with $\triangle \mathbf{v}_k \mathbf{v}_l \mathbf{v}_m$ the triangular surface whose vertices are $\mathbf{v}_k$, $\mathbf{v}_l$ and $\mathbf{v}_m$, and $\mathcal{T}_\Gamma$ represents the set of triplets k, l, m such that all triangular faces forming $\partial \Gamma$ are included. The closed surface integral in Equation (B.3) can therefore be written as a sum of integral over triangular faces as follows:

$$\begin{aligned}
 \mathbf{u}_j^T \mathbf{w} L &= \oint_{\partial \Gamma} h''(\mathbf{u}_j^T \mathbf{r} + s_j) \mathbf{w}^T d\mathbf{S} \\
 &= \sum_{k, l, m \in \mathcal{T}_\Gamma} \iint_{\triangle \mathbf{v}_k \mathbf{v}_l \mathbf{v}_m} h''(\mathbf{u}_j^T \mathbf{r} + s_j) \mathbf{w}^T d\mathbf{S}_{k, l, m},
 \end{aligned} \tag{B.4}$$

where $d\mathbf{S}_{k, l, m} = \mathbf{n}_{k, l, m} dS$, and $\mathbf{n}_{k, l, m} \in \mathbb{R}^3$ is the outward pointing unit vector normal to the triangular face $\triangle \mathbf{v}_k \mathbf{v}_l \mathbf{v}_m$, as shown in Figure B.1.

We first consider the integral on a single triangular face $\triangle \mathbf{v}_k \mathbf{v}_l \mathbf{v}_m$. For $\forall \mathbf{r} \in \triangle \mathbf{v}_k \mathbf{v}_l \mathbf{v}_m$, $\mathbf{r}$ can be represented as:

$$\mathbf{r} = \mathbf{v}_k + \lambda(\mathbf{v}_l - \mathbf{v}_k) + \mu(\mathbf{v}_m - \mathbf{v}_k),$$

where $\lambda, \mu \in \mathbb{R}$ and $0 \leq \lambda, \mu, \lambda + \mu \leq 1$. Therefore,

$$\begin{aligned}
 d\mathbf{S}_{k, l, m} &= \frac{\partial \mathbf{r}}{\partial \lambda} \times \frac{\partial \mathbf{r}}{\partial \mu} d\lambda d\mu \\
 &= \underbrace{(\mathbf{v}_l - \mathbf{v}_k) \times (\mathbf{v}_m - \mathbf{v}_k)}_{\mathbf{v}_{k, l, m}} d\lambda d\mu.
 \end{aligned} \tag{B.5}$$

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The integral over a single triangular face can then be written as:

$$\begin{aligned}
& \iint_{\triangle \mathbf{v}_k \mathbf{v}_l \mathbf{v}_m} h''(\mathbf{u}_j^T \mathbf{r} + s_j) \mathbf{w}^T d\mathbf{S}_{k,l,m} \\
&= \iint_{\substack{0 \leq \lambda, \mu \leq 1 \\ \lambda + \mu \leq 1}} h''(\mathbf{u}_j^T (\mathbf{v}_k + \lambda(\mathbf{v}_l - \mathbf{v}_k) + \mu(\mathbf{v}_m - \mathbf{v}_k)) + s_j) \mathbf{w}^T \mathbf{v}_{k,l,m} d\lambda d\mu \\
&= \mathbf{w}^T \mathbf{v}_{k,l,m} \left(\frac{h(\mathbf{u}_j^T \mathbf{v}_k + s_j)}{\mathbf{u}_j^T (\mathbf{v}_k - \mathbf{v}_l) \mathbf{u}_j^T (\mathbf{v}_k - \mathbf{v}_m)} \right. \\
&\quad + \frac{h(\mathbf{u}_j^T \mathbf{v}_l + s_j)}{\mathbf{u}_j^T (\mathbf{v}_l - \mathbf{v}_k) \mathbf{u}_j^T (\mathbf{v}_l - \mathbf{v}_m)} \\
&\quad \left. + \frac{h(\mathbf{u}_j^T \mathbf{v}_m + s_j)}{\mathbf{u}_j^T (\mathbf{v}_m - \mathbf{v}_k) \mathbf{u}_j^T (\mathbf{v}_m - \mathbf{v}_l)} \right). \tag{B.6}
\end{aligned}$$

For the above equation to be meaningful, we require that $\forall k \neq l, \mathbf{u}_j^T (\mathbf{v}_k - \mathbf{v}_l) \neq 0$. This is coherent with our assumption that any two projected vertices should not coincide with each other. Then, by letting $\mathbf{w}$ be the conjugate of $\mathbf{u}_j : \mathbf{w} = \mathbf{u}_j^*$ and substituting Equation (B.6) into Equation (B.4), we obtain that:

$$\begin{aligned}
L &= \sum_{k,l,m \in \mathcal{T}_\Gamma} \frac{\mathbf{u}_j^\dagger \mathbf{v}_{k,l,m}}{2} \left(\frac{h(\mathbf{u}_j^T \mathbf{v}_k + s_j)}{\mathbf{u}_j^T (\mathbf{v}_k - \mathbf{v}_l) \mathbf{u}_j^T (\mathbf{v}_k - \mathbf{v}_m)} \right. \\
&\quad + \frac{h(\mathbf{u}_j^T \mathbf{v}_l + s_j)}{\mathbf{u}_j^T (\mathbf{v}_l - \mathbf{v}_k) \mathbf{u}_j^T (\mathbf{v}_l - \mathbf{v}_m)} \\
&\quad \left. + \frac{h(\mathbf{u}_j^T \mathbf{v}_m + s_j)}{\mathbf{u}_j^T (\mathbf{v}_m - \mathbf{v}_k) \mathbf{u}_j^T (\mathbf{v}_m - \mathbf{v}_l)} \right) \\
&\stackrel{(e)}{=} \sum_{k,l,m \in \mathcal{T}_\Gamma} \frac{\mathbf{u}_j^\dagger \mathbf{v}_{k,l,m}}{2} \left(\frac{h(z_{k,j})}{\mathbf{u}_j^T (\mathbf{v}_k - \mathbf{v}_l) \mathbf{u}_j^T (\mathbf{v}_k - \mathbf{v}_m)} \right. \\
&\quad + \frac{h(z_{l,j})}{\mathbf{u}_j^T (\mathbf{v}_l - \mathbf{v}_k) \mathbf{u}_j^T (\mathbf{v}_l - \mathbf{v}_m)} \\
&\quad \left. + \frac{h(z_{m,j})}{\mathbf{u}_j^T (\mathbf{v}_m - \mathbf{v}_k) \mathbf{u}_j^T (\mathbf{v}_m - \mathbf{v}_l)} \right), \tag{B.7}
\end{aligned}$$

where (e) follows from

$$\mathbf{u}_j^T \mathbf{v}_k + s_j = \mathbf{u}_{j,x}^T \mathbf{v}_k + s_{j,x} + i(\mathbf{u}_{j,y}^T \mathbf{v}_k + s_{j,y}) = z_{k,j}.$$

If we consider the contribution from each projected vertex $z_{k,j}$ in Equation (B.7), then L

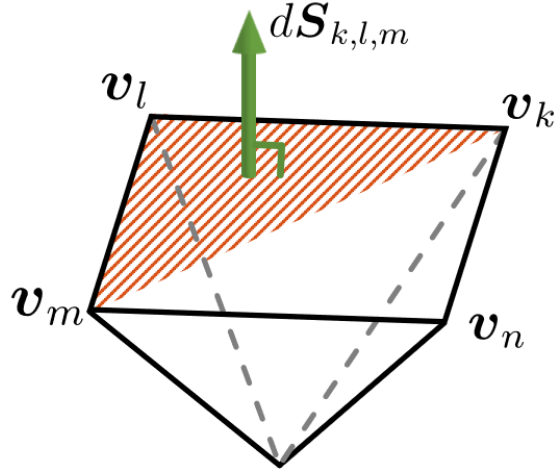


Figure B.1: The surface of a polyhedron is made of disjoint triangular faces.

can be expressed alternatively as:

$$L = \sum_{k,l,m \in \mathcal{T}_\Gamma} \frac{\mathbf{u}_j^\dagger \mathbf{v}_{k,l,m} h(z_{k,j})}{2\mathbf{u}_j^T (\mathbf{v}_k - \mathbf{v}_l) \mathbf{u}_j^T (\mathbf{v}_k - \mathbf{v}_m)} = \sum_{k=1}^K \rho_{k,j} h(z_{k,j}), \quad (\text{B.8})$$

which concludes the proof.

□

B.2 Estimation of $p_{k,j}$ using exponential reproducing kernels

By setting $h(z) = e^{\theta_r z}$, Equation (3.8) becomes:

$$\begin{aligned} \sum_{k=1}^K \rho_{k,j} e^{\theta_r z_{k,j}} &= \iint_{\Pi_j} I_j(x, y) h'''(z) dx dy \\ &= \iint_{\Pi_j} I_j(x, y) (e^{\theta_r z})''' dx dy \\ &= \theta_r^3 \iint_{\Pi_j} I_j(x, y) e^{\theta_r z} dx dy \\ &= \theta_r^3 \chi_{\theta_r} \\ &= \hat{\chi}_{\theta_r}, \end{aligned} \quad (\text{B.9})$$

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where $\theta_r = \theta_0 + \lambda r$, $\theta_0, \lambda \in \mathbb{C}$. Here χ_{θ_r} denotes the simple exponential moment of $I_j(x, y)$:

$$\chi_{\theta_r} = \iint_{\Pi_j} I_j(x, y) e^{\theta_r z} dx dy, \quad (\text{B.10})$$

and $\hat{\chi}_{\theta_r}$ is the weighted exponential moment $\hat{\chi}_{\theta_r} = \theta_r^3 \chi_{\theta_r}$. Following a similar observation as Equation (3.9), given $2K$ weighted exponential moments χ_{θ_r} , $r = 0, 1, \dots, 2K - 1$, we can retrieve the K projected vertices $z_{k,j}$ using Prony's method [32].

If $\varphi(x, y)$ can reproduce 2D exponentials $e^{\theta_r x} e^{i\theta_r y}$ with $r = 0, \dots, 2K - 1$, in other words, there exist coefficients $c_{m,n}^{r,r} \in \mathbb{C}$ such that:

$$\sum_{m,n} c_{m,n}^{r,r} \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) = e^{\theta_r(x+iy)} = e^{\theta_r z}, \quad (\text{B.11})$$

then from the samples $I_j[m, n]$, we can retrieve $2K$ simple exponential moments as follows:

$$\begin{aligned} & \sum_{m,n} c_{m,n}^{r,r} I_j[m, n] \\ &= \iint_{\Pi_j} I_j(x, y) \sum_{m,n} c_{m,n}^{r,r} \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \end{aligned} \quad (\text{B.12})$$

$$\begin{aligned} & \stackrel{(a)}{=} \iint_{\Pi_j} I_j(x, y) e^{\theta_r z} dx dy \\ &= \chi_{\theta_r}, \end{aligned} \quad (\text{B.13})$$

where (a) follows from Equation (B.11).

B.3 Derivation of the Cramér-Rao bounds

We derive the Cramér-Rao bound (CRB) associated to the estimation of the 2D parameters $\mathbf{p}_{k,j}$ for a given polyhedron with $K = 4$ vertices. Specifically, we consider the 4 vertices $\mathbf{v}_k$ of the polyhedron are located at $\mathbf{v}_1 = [-c, 0, 0]^T$, $\mathbf{v}_2 = [0, c, 0]^T$, $\mathbf{v}_3 = [c, 0, 0]^T$ and $\mathbf{v}_4 = [0, 0, c]^T$, where c is a known value: $c \in (0, R)$. For simplicity, we neglect the unknown shifts: $\mathbf{s}_j = [0, 0]^T$. We denote the unknown projection direction vector as $\mathbf{d}_j = [d_1, d_2, d_3]^T$. To ensure the 2D projection $I_j(x, y)$ is always differentiable with respect to d_2 and d_3 , we assume $d_2, d_3 \geq 0$. For simplicity, we fix one edge direction vector of the observation window $\mathbf{u}_{j_x} = [1, 0, 0]^T$. Then from Equation (3.4), the other edge direction vector can be expressed as $\mathbf{u}_{j_y} = [0, d_3, -d_2]^T$. Under this setting, the

desired 2D parameters can be expressed as:

$$\mathbf{p}_{1,j} = [-c, 0]^T \quad (\text{B.14})$$

$$\mathbf{p}_{2,j} = [0, cd_3]^T \quad (\text{B.15})$$

$$\mathbf{p}_{3,j} = [c, 0]^T \quad (\text{B.16})$$

$$\mathbf{p}_{4,j} = [0, -cd_2]^T \quad (\text{B.17})$$

Since c is fixed and known, estimating the 2D parameters is equivalent to estimating the unknown values d_2 and d_3 . Following Equation (3.7), we can form a vector of unknown parameters as follows:

$$\Theta = [\theta_1, \theta_2]^T = [d_2, d_3]^T. \quad (\text{B.18})$$

From the above assumptions, the 2D projection $I_j(x, y)$ can be expressed explicitly as:

$$\begin{aligned} I_j(x, y) &= \frac{1}{d_2 + d_3} \left(x + \frac{1}{d_2}y + c \right) u(-y)u(-x)u(y + cd_2 + d_2x) \\ &+ \frac{1}{d_2 + d_3} \left(-x + \frac{1}{d_2}y + c \right) u(-y)u(x)u(y + cd_2 - d_2x) \\ &+ \frac{1}{d_2 + d_3} \left(-x - \frac{1}{d_3}y + c \right) u(y)u(x)u(-y + cd_3 - d_3x) \\ &+ \frac{1}{d_2 + d_3} \left(x - \frac{1}{d_3}y + c \right) u(y)u(-x)u(-y + cd_3 + d_3x), \end{aligned} \quad (\text{B.19})$$

where $u(\cdot)$ denotes the Heaviside step function.

From Equation (3.6), we aim to retrieve Θ from the measured samples:

$$\begin{aligned} \tilde{I}_j[m, n] &= \left\langle I_j(x, y), \varphi \left(\frac{x}{T} - m, \frac{y}{T} - n \right) \right\rangle + \epsilon_{m,n} \\ &= f(\Theta, m, n) + \epsilon_{m,n}, \end{aligned} \quad (\text{B.20})$$

where $\epsilon_{m,n}$ is i.i.d additive Gaussian noise with zero mean and variance σ^2 , and its probability density function (pdf) is expressed as follows:

$$p_\epsilon(\epsilon_{m,n}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left(-\frac{\epsilon_{m,n}^2}{2\sigma^2} \right).$$

We assume the sampling kernel $\varphi(x, y)$ to be a 2D B-spline function of order 4.

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It is clear that the probability for $\tilde{I}_j[m, n]$ conditioned on Θ can be expressed as:

$$p\left(\tilde{I}_j[m, n]; \Theta\right) = p_\epsilon\left(\tilde{I}_j[m, n] - f(\Theta, m, n)\right) = p_\epsilon(\epsilon_{m,n}). \quad (\text{B.21})$$

For any unbiased estimator $\hat{\Theta}$, its performance is lower bounded by the Cramér-Rao bound: $\text{var}(\hat{\Theta}) \geq \mathcal{I}^{-1}(\Theta)$, where $\mathcal{I}(\Theta)$ is the Fisher Information Matrix (FIM), which is defined as follows:

$$\mathcal{I}(\Theta) = E\left(\nabla l(\Theta) \nabla l(\Theta)^T\right), \quad (\text{B.22})$$

where $l(\Theta)$ is the log-likelihood function, and can be calculated as follows:

$$\begin{aligned} l(\Theta) &= \ln p\left(\tilde{I}_j[0, 0], \tilde{I}_j[1, 0], \dots, \tilde{I}_j[M-1, M-1]; \Theta\right) \\ &\stackrel{(a)}{=} \sum_{m,n} \ln p_\epsilon\left(\tilde{\mathcal{P}}_j[m, n] - f(\Theta, m, n)\right), \end{aligned} \quad (\text{B.23})$$

where (a) follows from the independency of the noise samples. The partial derivative of the likelihood function can be written as:

$$\frac{\partial l(\Theta)}{\partial \theta_i} = \frac{1}{\sigma^2} \sum_{m,n} \epsilon_{m,n} \frac{\partial f(\Theta, m, n)}{\partial \theta_i}, \quad (\text{B.24})$$

equivalently,

$$\nabla l(\Theta) = \frac{1}{\sigma^2} \sum_{m,n} \epsilon_{m,n} \nabla f(\Theta, m, n). \quad (\text{B.25})$$

Therefore, the Fisher Information Matrix can be written as:

$$\begin{aligned} \mathcal{I}(\Theta) &= E\left(\nabla l(\Theta) \nabla l(\Theta)^T\right) \\ &= E\left(\frac{1}{\sigma^4} \sum_{m,n,p,q} \epsilon_{m,n} \epsilon_{p,q} \nabla f(\Theta, m, n) \nabla f(\Theta, p, q)^T\right) \\ &\stackrel{(b)}{=} \frac{1}{\sigma^2} \sum_{m,n} \nabla f(\Theta, m, n) \nabla f(\Theta, m, n)^T, \end{aligned} \quad (\text{B.26})$$

where (b) follows from the linearity of the expectation and the independency of the noise. Hence, the Cramér-Rao bound is given by the inverse of the Fisher Information Matrix:

$$\text{CRB}(\Theta) = \sigma^2 \left(\sum_{m=0}^{M-1} \sum_{n=0}^{M-1} \nabla f(\Theta, m, n) \nabla f(\Theta, m, n)^T \right)^{-1}. \quad (\text{B.27})$$

Using Equation (B.19), we obtain

$$\begin{aligned}
 & \frac{\partial f(\Theta, m, n)}{\partial d_2} \\
 &= \frac{1}{T^2} \frac{\partial}{\partial d_2} \iint I_j(x, y) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \\
 &= \kappa \left(- \int_{-c}^0 \int_{-d_2(x+c)}^0 \left(x + \frac{2d_2 + d_3}{d_2^2} y + c\right) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \right. \\
 &\quad + \int_0^c \int_{-d_2(c-x)}^0 \left(x - \frac{2d_2 + d_3}{d_2^2} y - c\right) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \\
 &\quad + \int_0^X \int_0^{d_3(c-x)} \left(x + \frac{1}{d_3} y - X\right) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \\
 &\quad \left. - \int_{-c}^0 \int_0^{d_3(c+x)} \left(x - \frac{1}{d_3} y + c\right) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \right), \tag{B.28}
 \end{aligned}$$

where $\kappa = \frac{1}{((d_2 + d_3)T)^2}$. In a similar fashion, we obtain

$$\begin{aligned}
 & \frac{\partial f(\Theta, m, n)}{\partial d_3} \\
 &= \frac{1}{T^2} \frac{\partial}{\partial d_3} \iint I_j(x, y) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \\
 &= \kappa \left(- \int_{-c}^0 \int_{-d_2(x+c)}^0 \left(x + \frac{1}{d_2} y + c\right) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \right. \\
 &\quad + \int_0^c \int_{-d_2(c-x)}^0 \left(x - \frac{1}{d_2} y - c\right) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \\
 &\quad + \int_0^c \int_0^{d_3(c-x)} \left(x + \frac{2d_3 + d_2}{d_3} y - c\right) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \\
 &\quad \left. - \int_{-c}^0 \int_0^{d_3(c+x)} \left(x - \frac{2d_3 + d_2}{d_3^2} y + c\right) \varphi\left(\frac{x}{T} - m, \frac{y}{T} - n\right) dx dy \right). \tag{B.29}
 \end{aligned}$$

Given the above two equations, we can evaluate (at least numerically) the Fisher Information Matrix for specific values of d_2 and d_3 :

$$I(\Theta)_{11} = \frac{1}{\sigma^2} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} \left(\frac{\partial f}{\partial d_2} \right)^2 \tag{B.30}$$

$$I(\Theta)_{22} = \frac{1}{\sigma^2} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} \left(\frac{\partial f}{\partial d_3} \right)^2 \tag{B.31}$$

$$I(\Theta)_{12} = I(\Theta)_{21} = \frac{1}{\sigma^2} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} \frac{\partial f}{\partial d_2} \frac{\partial f}{\partial d_3}. \tag{B.32}$$

Finally, the Cramér–Rao bounds can be computed by taking the inverse of the FIM.

B.4 Annihilating filter based method for retrieving the 2D parameters

From Equation (3.23) and Equation (3.24), the exact values of $\eta_{\omega,j}$ can be written as the ratio of two polynomials:

$$\eta_{\omega,j} = \sum_{k=1}^K \frac{\rho_{k,j}}{z_{k,j} - a_{\omega}} \stackrel{(a)}{=} \frac{\sum_k a_0 \lambda_{k,j} e^{i \frac{2\pi}{W} k \omega}}{Q_K(a_{\omega})} = \frac{\sum_k \tilde{\lambda}_{k,j} e^{i \alpha k \omega}}{Q_K(a_{\omega})} = \frac{r_{\omega,j}}{Q_K(a_{\omega})},$$

where (a) comes from $a_{\omega} = a_0 e^{i 2\pi \frac{\omega}{W}}$ and $\tilde{\lambda}_{k,j} = a_0 \lambda_{k,j}$. The goal is to retrieve $z_{k,j}$ from $\{\eta_{\omega,j}\}_{\omega=0}^W$. To achieve this, we consider a discrete filter $f = \{f_k\}_{k=0}^K$ whose zeros are located at $e^{ik\alpha}$ for $k = 0, \dots, K-1$, in other words:

$$F(z) = \sum_k f_k z^{-1} = \prod_k (1 - e^{ik\alpha} z^{-1}). \quad (\text{B.33})$$

Consequently, the discrete filter also annihilates the sequence $r_{\omega,j}$, namely $f * r = 0$, which can be proven as follows:

$$\begin{aligned} \{f * r\}_{\omega} &= \sum_l f_l r_{\omega-l,j} \\ &= \sum_l f_l \sum_m \tilde{\lambda}_{m,j} e^{i \alpha m (\omega-l)} \\ &= \sum_m \tilde{\lambda}_{m,j} \sum_l f_l e^{i \alpha m (\omega-l)} \\ &= \sum_m \tilde{\lambda}_{m,j} e^{i \alpha m \omega} \sum_l f_l e^{-i \alpha m l} \\ &= \sum_m \tilde{\lambda}_{m,j} e^{i \alpha m \omega} F(e^{i \alpha m}) \\ &= 0. \end{aligned} \quad (\text{B.34})$$

The above annihilation relation can be further written as:

$$\begin{aligned} \{f * r\}_{\omega} &= \sum_l f_{\omega-l} r_{l,j} = \sum_l f_{\omega-l} Q_K(a_l) \eta_{l,j} \\ &= \sum_l f_{\omega-l} \sum_k q_k a_l^k \eta_{l,j} \\ &= \sum_k q_k \underbrace{\sum_l f_{\omega-l} a_l^k}_{A_{\omega,k,j}} \eta_{l,j} \\ &= 0, \end{aligned} \quad (\text{B.35})$$

where q_k denotes the coefficients of the polynomial Q_K . Therefore, Equation (B.35) represents another annihilation relation:

$$\sum_k A_{\omega,k,j} q_k = 0, \quad (\text{B.36})$$

which can be written in the matrix form $\mathbf{A}\mathbf{q} = \mathbf{0}$, where $\mathbf{q} = [q_0, \dots, q_K]^T$ and $\mathbf{A} = \mathbf{F}\mathbf{\Lambda}\mathbf{a}$, with:

$$\mathbf{F} = \begin{bmatrix} f_K & \dots & f_0 & 0 & \dots & 0 \\ 0 & f_K & \dots & f_0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & f_K & \dots & f_0 \end{bmatrix}_{(W-K) \times W}, \quad (\text{B.37})$$

$$\mathbf{\Lambda} = \begin{bmatrix} \eta_{0,j} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \eta_{W-1,j} \end{bmatrix}_{W \times W}, \quad (\text{B.38})$$

and

$$\mathbf{a} = \begin{bmatrix} a_0^0 & \dots & a_0^K \\ \vdots & \ddots & \vdots \\ a_{W-1}^0 & \dots & a_{W-1}^K \end{bmatrix}_{W \times (K+1)}. \quad (\text{B.39})$$

The coefficients of the polynomial Q_K can be retrieved using the annihilation relation in Equation (B.36). Then by computing the polynomial, we are able to find the 2D locations of projected vertices $z_{k,j}$.

B.5 Model fitting approach

In terms of assessing the performance of an FRI recovery method, the accuracy of the estimated parameters is often used. In our case, this is equivalent as comparing the estimated parameters $z_{k,j}$ with the original values.

Authors in [36] have proposed an alternative criterion for the assessment, which focuses on the resynthesized FRI ‘moments’ $\eta_{\omega,j}$ in Equation (3.23). More precisely, the assessment of recovery is based on the $\text{MSE}_{\text{resyn}}$ between the resynthesized FRI moments $\eta_{\omega,j}^{\text{resyn}}$ and the measured moments $\eta_{\omega,j}$. Mathematically, the criterion can be written as:

$$\text{MSE}_{\text{resyn}} < \sigma_\eta^2, \quad (\text{B.40})$$

Appendix B

where σ_η^2 denotes the input MSE that represents the ‘noise’ level. Note that it is reasonable to assume prior knowledge about σ_η^2 from noise statistics.

The core idea of the model fitting approach is that, if the FRI ‘moments’ can be written as the ratio between two polynomials in a_ω , where $a_\omega = a_0 e^{i2\pi \frac{\omega}{W}}$, and W is the number of ‘moments’ and ω is the index. We have shown from Equation (3.24) that the nominator polynomial $P_{K-1}(a_\omega)$ is of order $K-1$, while the denominator polynomial Q_K is of order K and is in fact the annihilation filter whose zeros uniquely defines $z_{k,j}$. From Parseval’s identity, the ratio structure can be used as a model, and by fitting the model to the noisy ‘moments’, we obtain the best sparse representation of the ‘moments’ that fit the model. The fitting objective is as follows:

$$\min_{Q_K, P_{K-1}} \sum_{\omega=0}^{W-1} \left| \eta_{\omega,j} - \frac{P_{K-1}(a_\omega)}{Q_K(a_\omega)} \right|^2. \quad (\text{B.41})$$

The direct computation of the above minimization would require non linear based methods [51]. Instead, the model fitting approach in [36] estimates the solution of the above minimisation (B.41) in a linear manner by considering the following iterative minimisation:

$$\min_{Q_K^{(n)}, P_{K-1}} \sum_{\omega=0}^{W-1} \left| \frac{Q_{K-1}^{(n)}(a_\omega) \eta_{\omega,j} - P_{K-1}(a_\omega)}{Q_K^{(n)}(a_\omega)} \right|^2, \quad (\text{B.42})$$

where n denotes the iteration number. To solve for the iterative minimisation, we notice that the variable a_ω can be written as $a_\omega = a_0 e^{i2\pi \frac{\omega}{W}}$, $a_0 \in \mathbb{R}^+$, therefore, we can rewrite (B.42) in terms of Fourier matrices. Specifically, by defining $\mathbf{F}_{N \times M}$ as the $N \times M$ inverse DFT matrix, we can write the polynomials as follows:

$$\begin{aligned} \{P_{K-1}(a_\omega)\}_{\omega=0}^{W-1} &= \mathbf{F}_{W,K} \mathbf{p} \\ \{Q_K(a_\omega)\}_{\omega=0}^{W-1} &= \mathbf{F}_{W,K+1} \mathbf{q}, \end{aligned}$$

where $\mathbf{p}$ is a_0 times the K coefficients of P , similarly, $\mathbf{q}$ is a_0 times the $K+1$ coefficients of Q . Furthermore, we define the diagonal matrix $\mathbf{A}$, constructed using the vector $\boldsymbol{\eta} = [\eta_{0,j}, \eta_{1,j}, \dots, \eta_{W-1,j}]^T$, where the entries of $\boldsymbol{\eta}$ serve as the diagonal elements of $\mathbf{A}$. Using the above notations, (B.42) can be written as:

$$\min_{\mathbf{q}_n, \mathbf{p}} \|\mathbf{Q}_{n-1} \mathbf{q}_n - \mathbf{B}_{n-1} \mathbf{p}\|^2, \quad (\text{B.43})$$

where the matrices $\mathbf{Q}_{n-1} = \mathbf{A} \mathbf{R}_{n-1} \mathbf{F}_{W,K+1}$ and $\mathbf{B}_{n-1} = \mathbf{R}_{n-1} \mathbf{F}_{W,K}$, and $\mathbf{R}_{n-1} = [\text{diag}\{\mathbf{F}_{W,K+1} \mathbf{q}_{n-1}\}]^{-1}$. To solve the above problem, the trick is to decouple the minimisation with respect to $\mathbf{q}_n$ and $\mathbf{p}$ respectively.

Firstly, by constraining the Euclidean norm of $\|\mathbf{q}_n\|^2 = 1$, we can decouple (B.43) in just

$\mathbf{q}_n$ as follows:

$$\min_{\mathbf{q}_n} \left\| \left(\mathbf{I} - \mathbf{B}_{n-1} \mathbf{B}_{n-1}^\dagger \right) \mathbf{Q}_{n-1} \mathbf{q}_n \right\|^2, \text{ s.t. } \|\mathbf{q}_n\|^2 = 1. \quad (\text{B.44})$$

The above minimisation can be solved using total least square (TLS) methods, which involves performing QR decomposition on matrices $\mathbf{Q}_{n-1}$ and $\mathbf{B}_{n-1}$. The initialization of $\mathbf{q}_0$ can be obtained by performing TLS on the noisy Toeplitz ‘moments’ matrix as in classical FRI methods.

Once updated $\mathbf{q}_n$, $\mathbf{p}$ can then be determined by solving:

$$\min_{\mathbf{p}} \|\boldsymbol{\eta} - \mathbf{Q}_n \mathbf{F}_{W,K} \mathbf{p}\|^2. \quad (\text{B.45})$$

The iteration stops when the criterion in Equation (B.40) is reached.

Appendix C

Appendices for chapter 4

C.1 Visual example of a failure case of angle estimation algorithm proposed in Section 4.2.1

The efficacy of the angle estimation algorithm introduced in Section 4.2.1 relies on the assumption that the underlying two-dimensional image exhibits sufficient asymmetry. In this section, we present a visual example illustrating the performance of the angle estimation algorithm when applied to a relatively symmetric brain CT image. Figure C.1 shows the estimation results under both noiseless and noisy projection conditions. Due to the high degree of symmetry in the image, even a moderate level of noise (e.g., 30 dB signal-to-noise ratio) significantly impairs the accuracy of the angle recovery process. This example highlights a key limitation of the proposed algorithm when applied to highly symmetric structures.

C.2 Exact interpolation with Dirichlet kernel

We start with the time-limited signal $I(x, y)$ that has finite support between $[-\frac{\tau_x}{2}, \frac{\tau_x}{2}]$ and $[-\frac{\tau_y}{2}, \frac{\tau_y}{2}]$. This can be expressed using a rectangular window function and its periodic extension:

$$I(x, y) = \text{rect}\left(\frac{x}{\tau_x}, \frac{y}{\tau_y}\right) \sum_{m, n \in \mathbb{Z}} I(x - m\tau_x, y - n\tau_y) \quad (\text{C.1})$$

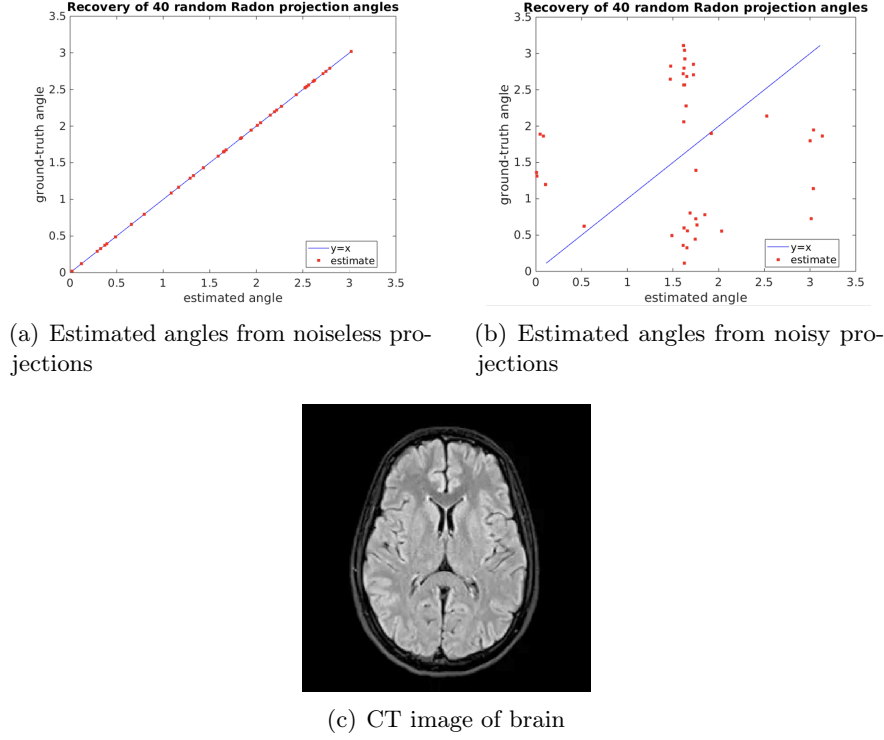


Figure C.1: An example of failure in angle estimation. The CT scan of the brain image is highly symmetric. As a result, when the projection data is slightly corrupted by noise (30 dB), the angle estimation becomes completely inaccurate.

The Fourier transform of $I(x, y)$ is:

$$\hat{I}(\omega_x, \omega_y) = \iint_{-\infty}^{\infty} \text{rect}\left(\frac{x}{\tau_x}, \frac{y}{\tau_y}\right) \sum_{m, n \in \mathbb{Z}} I(x - m\tau_x, y - n\tau_y) e^{-j\omega_x x - j\omega_y y} dx dy \quad (\text{C.2})$$

Applying the Poisson sum formula:

$$\begin{aligned} \hat{I}(\omega_x, \omega_y) &= \iint \text{rect}\left(\frac{x}{\tau_x}, \frac{y}{\tau_y}\right) e^{-j\omega_x x - j\omega_y y} \cdot \frac{1}{\tau_x \tau_y} \sum_{k, l \in \mathbb{Z}} I\left(\frac{2\pi k}{\tau_x}, \frac{2\pi l}{\tau_y}\right) e^{j\frac{2\pi k}{\tau_x} x + j\frac{2\pi l}{\tau_y} y} dx dy \\ &= \sum_{k, l \in \mathbb{Z}} I\left(\frac{2\pi k}{\tau_x}, \frac{2\pi l}{\tau_y}\right) \iint \frac{1}{\tau_x \tau_y} \text{rect}\left(\frac{x}{\tau_x}, \frac{y}{\tau_y}\right) e^{-j\left(\omega_x - \frac{2\pi k}{\tau_x}\right)x - j\left(\omega_y - \frac{2\pi l}{\tau_y}\right)y} dx dy \end{aligned} \quad (\text{C.3})$$

This simplifies to:

$$\hat{I}(\omega_x, \omega_y) = \sum_{k, l \in \mathbb{Z}} I\left(\frac{2\pi k}{\tau_x}, \frac{2\pi l}{\tau_y}\right) \text{sinc}\left(\frac{\tau_x}{2} \left(\omega_x - \frac{2\pi k}{\tau_x}\right), \frac{\tau_y}{2} \left(\omega_y - \frac{2\pi l}{\tau_y}\right)\right) \quad (\text{C.4})$$

Appendix C

Due to the periodicity of $\hat{I}(\omega_x, \omega_y)$, we can rewrite the infinite summation as:

$$\begin{aligned}
\hat{I}(\omega_x, \omega_y) &= \sum_{m, n \in \mathbb{Z}} \sum_{|k| \leq \frac{K\tau_x}{2}, |l| \leq \frac{L\tau_y}{2}} \hat{I}\left(\frac{2\pi k}{\tau_x}, \frac{2\pi l}{\tau_y}\right) \\
&\quad \text{sinc}\left(\frac{\tau_x}{2} \left(\omega_x - \frac{2\pi(k + mK\tau_x)}{\tau_x}\right), \frac{\tau_y}{2} \left(\omega_y - \frac{2\pi(l + nL\tau_y)}{\tau_y}\right)\right) \\
&= \sum_{|k| \leq \frac{K\tau_x}{2}, |l| \leq \frac{L\tau_y}{2}} I\left(\frac{2\pi k}{\tau_x}, \frac{2\pi l}{\tau_y}\right) \\
&\quad \sum_{m, n \in \mathbb{Z}} \text{sinc}\left(\frac{\tau_x}{2} \left(\omega_x - \frac{2\pi(k + mK\tau_x)}{\tau_x}\right), \frac{\tau_y}{2} \left(\omega_y - \frac{2\pi(l + nL\tau_y)}{\tau_y}\right)\right)
\end{aligned} \tag{C.5}$$

Finally, this results in:

$$\begin{aligned}
&\hat{I}(\omega_x, \omega_y) \\
&= \sum_{|k| \leq \frac{K\tau_x}{2}, |l| \leq \frac{L\tau_y}{2}} \hat{I}\left(\frac{2\pi k}{\tau_x}, \frac{2\pi l}{\tau_y}\right) \frac{\sin\left(\frac{\tau_x}{2} \left(\omega_x - \frac{2\pi k}{\tau_x}\right)\right)}{K\tau_x \sin\left(\frac{1}{2K} \left(\omega_x - \frac{2\pi k}{\tau_x}\right)\right)} \cdot \frac{\sin\left(\frac{\tau_y}{2} \left(\omega_y - \frac{2\pi l}{\tau_y}\right)\right)}{L\tau_y \sin\left(\frac{1}{2L} \left(\omega_y - \frac{2\pi l}{\tau_y}\right)\right)}.
\end{aligned} \tag{C.6}$$

The above exact interpolation formula applies where $\hat{I}(\omega_x, \omega_y)$ is periodic with period $2\pi K, 2\pi L$ for some K and L . In this case, the linear mapping $\mathbf{G}$ in Algorithm 4.1 can be established by evaluating Equation (C.6). More generally, other interpolation functions can be considered, for example in case the image is support limited, sinc functions or other spline functions can be used. In these situations, an actual mapping between the uniform and non-uniform Fourier samples is not reached but approximated. Consequently, we will have a model mismatch even in the noiseless cases. However, the linear mapping term in Algorithm 4.1 is only used to drive the data fidelity metric, instead of enforcing a hard constraint.

C.3 Derivation of the equivalent formulation

We consider the constrained minimization formulation of the FRI coefficients:

$$\begin{aligned}
&\min_{\mathbf{b}, \mathbf{c}} \|\mathbf{a} - \mathbf{G}\mathbf{b}\|_2^2 \\
&\text{subject to } \tilde{\mathbf{R}}(\mathbf{c}) \mathbf{b} = \mathbf{0}, \\
&\quad \mathbf{c}_0^H \mathbf{c} = 1.
\end{aligned} \tag{C.7}$$

For a fixed $\mathbf{c}_{n-1}$, the above minimization becomes a constrained quadratic minimization that can be solved using Lagrange multiplier:

$$\mathcal{L}(\mathbf{b}, \mathbf{l}) = \frac{1}{2} \|\mathbf{a} - \mathbf{G}\mathbf{b}\|_2^2 + \mathbf{l}^H \tilde{\mathbf{R}}(\mathbf{c}_{n-1})\mathbf{b} \quad (\text{C.8})$$

By considering the optimality conditions with respect to $\mathbf{b}$ and $\mathbf{l}$, we can derive (4.20):

$$\mathbf{b}_n = \boldsymbol{\beta} - (\mathbf{G}^H \mathbf{G})^{-1} \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1}) \left(\tilde{\mathbf{R}}(\mathbf{c}_{n-1}) (\mathbf{G}^H \mathbf{G})^{-1} \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1}) \right)^{-1} \tilde{\mathbf{R}}(\mathbf{c}_{n-1})\boldsymbol{\beta}, \quad (\text{C.9})$$

where $\boldsymbol{\beta} = (\mathbf{G}^H \mathbf{G})^{-1} \mathbf{G}^H \mathbf{a}$. When substituting the above solution for $\mathbf{b}_n$ into the objective function, we have that:

$$\begin{aligned} & \|\mathbf{a} - \mathbf{G}\mathbf{b}_n\|_2^2 \\ &= \boldsymbol{\beta}^H \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1}) \left(\tilde{\mathbf{R}}(\mathbf{c}_{n-1}) (\mathbf{G}^H \mathbf{G})^{-1} \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1}) \right)^{-1} \tilde{\mathbf{R}}(\mathbf{c}_{n-1})\boldsymbol{\beta} + \|\mathbf{a} - \mathbf{G}\boldsymbol{\beta}\|_2^2 \\ &= \mathbf{c}^H \tilde{\mathbf{T}}^H(\boldsymbol{\beta}) \left(\tilde{\mathbf{R}}(\mathbf{c}_{n-1}) (\mathbf{G}^H \mathbf{G})^{-1} \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1}) \right)^{-1} \tilde{\mathbf{T}}(\boldsymbol{\beta})\mathbf{c} \\ &+ \text{terms independent of } \mathbf{c} \end{aligned} \quad (\text{C.10})$$

To derive the equivalent formulation of (4.21), we consider its associated Lagrange:

$$\mathcal{L}(\mathbf{c}, \lambda) = \frac{1}{2} \mathbf{c}^H \tilde{\mathbf{T}}^H(\boldsymbol{\beta}) \left(\tilde{\mathbf{R}}(\mathbf{c}_{n-1}) (\mathbf{G}^H \mathbf{G})^{-1} \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1}) \right)^{-1} \tilde{\mathbf{T}}(\boldsymbol{\beta})\mathbf{c} + \lambda (\mathbf{c}_0^H \mathbf{c} - 1). \quad (\text{C.11})$$

The optimality conditions state that:

$$\begin{cases} \tilde{\mathbf{T}}^H(\boldsymbol{\beta}) \left(\tilde{\mathbf{R}}(\mathbf{c}_{n-1}) (\mathbf{G}^H \mathbf{G})^{-1} \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1}) \right)^{-1} \tilde{\mathbf{T}}(\boldsymbol{\beta})\mathbf{c} + \lambda \mathbf{c}_0 &= \mathbf{0}, \\ \mathbf{c}_0^H \mathbf{c} &= 1. \end{cases} \quad (\text{C.12})$$

By denoting $\mathbf{u} = \left(\tilde{\mathbf{R}}(\mathbf{c}_{n-1}) (\mathbf{G}^H \mathbf{G})^{-1} \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1}) \right)^{-1} \tilde{\mathbf{T}}(\boldsymbol{\beta})\mathbf{c}$, the above conditions can be equivalently written as:

$$\begin{cases} \tilde{\mathbf{T}}^H(\boldsymbol{\beta})\mathbf{u} + \lambda \mathbf{c}_0 &= \mathbf{0}, \\ \tilde{\mathbf{R}}(\mathbf{c}_{n-1}) (\mathbf{G}^H \mathbf{G})^{-1} \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1})\mathbf{u} &= \tilde{\mathbf{T}}(\boldsymbol{\beta})\mathbf{c}, \\ \mathbf{c}_0^H \mathbf{c} &= 1, \end{cases} \quad (\text{C.13})$$

then, by introducing another auxiliary variable $\mathbf{v} = (\mathbf{G}^H \mathbf{G})^{-1} \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1})\mathbf{u}$, the above

linear system becomes equivalent to:

$$\begin{cases} \tilde{\mathbf{T}}^H(\beta)\mathbf{u} + \lambda\mathbf{c}_0 &= \mathbf{c}_0, \\ \tilde{\mathbf{R}}(\mathbf{c}_{n-1})\mathbf{v} &= \tilde{\mathbf{T}}(\beta)\mathbf{c}, \\ \mathbf{G}^H\mathbf{G}\mathbf{v} &= \tilde{\mathbf{R}}^H(\mathbf{c}_{n-1})\mathbf{u}, \\ \mathbf{c}_0^H\mathbf{c} &= 1, \end{cases} \quad (\text{C.14})$$

which can be rearranged to obtain the matrix form in (4.22). In a similar fashion, once $\mathbf{c}_n$ is obtained, $\mathbf{b}_n$ can be updated by rewriting the optimality conditions associated with (C.8) as a linear system.

C.4 Primal-dual algorithm for weighted total variation

The weighted total variation objective function in (4.24) can be equivalently written as:

$$\min_I \bar{\mu} \odot \mathcal{TV}(I) + \frac{\lambda_1}{2} \|\mathcal{A}I - \mathbf{b}\|_2^2 + \frac{\lambda_2}{2} \|\Delta I\|_2^2, \quad (\text{C.15})$$

where $\mathcal{TV}$ is the total variation operator. We adopt a discrete definition of total variation. Suppose I is the aimed image of size $n \times n$ on a discrete rectangular grid. The discrete operator $\mathcal{D}$ takes the finite differences, for $1 \leq i, j \leq n$:

$$\mathcal{D}(I) = [\mathcal{D}_1(I), \mathcal{D}_2(I)] \quad (\text{C.16})$$

where

$$\begin{aligned} \mathcal{D}_1(I) &= I(i+1, j) - I(i, j), \\ \mathcal{D}_2(I) &= I(i, j+1) - I(i, j). \end{aligned}$$

Using the above notations, the discrete total variation operator can be written as:

$$\mathcal{TV}(I) = \sum_{i=1}^n \sum_{j=1}^n \sqrt{\mathcal{D}_1^2(I) + \mathcal{D}_2^2(I)} \quad (\text{C.17})$$

To better describe this problem in matrix form, we reorder the image I row-wisely into a vector $y \in \mathbb{R}^N$, $N = n^2$, such that:

$$I(i, j) = y_{(j-1)n+i}$$

Then the (i, j) th component of the gradient $\mathcal{D}(I)$ can thus be represented as a multipli-

cation of the vector y by a matrix $\mathcal{W}_l \in \mathbb{R}^{N \times 2}$, for $l = 1, 2, \dots, N$:

$$\mathcal{W}_l^T y = \begin{cases} (y_{l+1} - y_l, y_{l+n} - y_l)^T, & \text{if } l \bmod n \neq 0 \text{ and } l \leq N - n, \\ (0, y_{l+n} - y_l)^T, & \text{if } l \bmod n = 0 \text{ and } l \leq N - n \\ (y_{l+1} - y_l, 0)^T, & \text{if } l \bmod n \neq 0 \text{ and } l > N - n \\ (0, 0)^T, & \text{if } l \bmod n = 0 \text{ and } l > N - n \end{cases}$$

Therefore, we can express $\mathcal{TV}(I)$ as $\sum_{l=1}^N \|\mathcal{W}_l^T y\|$. For any vector b , as a direct consequence of Cauchy-Schwartz inequality $\|a\| = \max_{\|b\| \leq 1} a^T b$, as a result, we can rewrite $\mathcal{TV}(I)$ into:

$$\begin{aligned} \sum_{l=1}^N \|\mathcal{W}_l y\| &= \max_{\|v_l\| \leq 1} \sum_{l=1}^N (\mathcal{W}_l y)^T v_l \\ &= \max_{\|v_l\| \leq 1} y^T \sum_{l=1}^N \mathcal{W}_l v_l \\ &= \max_{\mathbf{v} \in \mathbf{V}} y^T \mathcal{W} \mathbf{v}, \end{aligned} \tag{C.18}$$

where $\mathcal{W} = [\mathcal{W}_1, \dots, \mathcal{W}_N] \in \mathbb{R}^{N \times 2N}$, $v_l = [v_l^1, v_l^2]^T \in \mathbb{R}^2$, $\mathbf{v} = [v_1, \dots, v_N]^T \in \mathbb{R}^{2N}$, and $\mathbf{V} = \{\mathbf{v} : \mathbf{v} \in \mathbb{R}^{2N}, \|v_l\| \leq 1 \text{ for } l = 1, 2, \dots, N\}$.

Using the derivation from above, we can rewrite our objective function as:

$$\min_I \max_{\mathbf{v} \in \mathbf{V}} \Phi(I, \mathbf{v}) := \mathbf{v}^T \mathcal{W}^T I + \frac{\lambda_1}{2} \|\mathcal{A}I - \mathbf{b}\|_2^2 + \frac{\lambda_2}{2} \|\Delta I\|_2^2, \tag{C.19}$$

this min-max or max-min problem can be solved through iterative primal-dual approach as follows [108]. At the k th iteration, suppose $I = I^k$, then

$$\nabla_{\mathbf{v}} \Phi(I^k, \mathbf{v}) = \mathcal{W}^T I^k.$$

Thus $\mathbf{v}^{k+1}$ is updated as $\mathbf{v}^{k+1} = \mathbf{v}^k + \tau_1 \mathcal{W}^T I^k$, where $\tau_1 > 0$ is the step size. To make sure $\mathbf{v}^{k+1} \in \mathbf{V}$, it is further projected: $\mathbf{v}^{k+1} = \mathcal{P}_{\mathbf{V}}(\mathbf{v}^k + \tau_1 \mathcal{W}^T I^k)$.

Then at the primal step, we consider for fixed $\mathbf{v} = \mathbf{v}^{k+1}$:

$$\nabla_I \Phi(I, \mathbf{v}^{k+1}) = \lambda_1 \mathcal{A}^T (\mathcal{A}I - \mathbf{b}) + \lambda_2 \Delta^T \Delta I + \mathcal{W} \mathbf{v}^{k+1}.$$

Thus, I^{k+1} is updated as

$$I^{k+1} = \frac{I^k - \tau_2 \mathcal{W} \mathbf{v}^{k+1} + \tau_2 \lambda_1 \mathcal{A}^T \mathbf{b}}{1 + \tau_1 \lambda_1 \mathcal{A}^T \mathcal{A} + \tau_2 \lambda_2 \Delta^T \Delta}.$$

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