

Impact of multiple curve dynamics in Credit Valuation Adjustments under collateralization*

Giacomo Bormetti[†] Damiano Brigo[‡] Marco Francischello[§] Andrea Pallavicini[¶]

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Abstract

We present a detailed analysis of interest rate derivatives valuation under credit risk and collateral modeling. We show how the credit and collateral extended valuation framework in Pallavicini et al. (2011), and the related collateralized valuation measure, can be helpful in defining the key market rates underlying the multiple interest rate curves that characterize current interest rate markets. A key point is that spot Libor rates are to be treated as market primitives rather than being defined by no-arbitrage relationships. We formulate a consistent realistic dynamics for the different rates emerging from our analysis and compare the resulting model performances to simpler models used in the industry. We include the often neglected margin period of risk, showing how this feature may increase the impact of different rates dynamics on valuation. We point out limitations of multiple curve models with deterministic basis considering valuation of particularly sensitive products such as basis swaps. We stress that a proper wrong way risk analysis for such products requires a model with a stochastic basis and we show numerical results confirming this fact.

1 Introduction

After the onset of the crisis in 2007, all market instruments are quoted by taking into account, more or less implicitly, credit and collateral related adjustments. As a consequence, when approaching modeling problems one has to carefully check standard theoretical assumptions which often ignore credit and liquidity issues. One has to go back to market processes and fundamental instruments by limiting oneself to value only derivative contracts that can be replicated by means of market instruments. Referring to market observables and processes is the only means we have to validate our theoretical assumptions, so as to drop them if in contrast with observations. This general recipe is what is guiding us in this paper, where we try to adapt interest rate models for valuation to the current landscape.

A detailed analysis of the updated valuation problem one faces when including credit risk and collateral modeling (and further funding costs) has been presented in [Brigo et al.(2014b)Brigo, Liu, Pallavicini

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[†]Scuola Normale Superiore Pisa and QUANTLab Pisa, giacomo.bormetti@sns.it

[‡]Imperial College London, damiano.brigo@imperial.ac.uk

[§]Imperial College London, m.francischello14@imperial.ac.uk

[¶]Imperial College London and Banca IMI Milan, a.pallavicini@imperial.ac.uk

Brigo et al.(2016)Brigo, Francischello, and Pallavicini]. We refer to those papers and references therein for a detailed discussion. Here we specialize our updated valuation framework to consider the following key points:

- (i) Focus on interest rate derivatives.
- (ii) Understand how the updated valuation framework involving collateralized valuation measures can be helpful in defining the key market rates; these are the rates underlying the multiple interest rate curves that characterize current interest rate markets; in this setting, spot Libor rates are taken as primitives and are not defined by no-arbitrage relationships, while both Libor and overnight indexed swaps (OIS) forward rates are obtained by zeroing specific contract values.
- (iii) Formulate a consistent realistic dynamics for the rates emerging from the above analysis and compare valuation based on these models to valuation based on more standard market models.
- (iv) Show how the framework can be applied to valuation of particularly sensitive products such as basis swaps under credit risk, collateral and margin period of risk. By margin period of risk we mean the risk that the position changes value dramatically during the time between the relevant counterparty default and the actual liquidation rather than to instantaneous contagion (for the latter please refer to [Brigo et al.(2014a)Brigo, Capponi, and Pallavicini]); while the latter is usually called gap risk, in this paper we will use “gap risk” in the sense of margin period of risk above.
- (v) Illustrate numerically how gap risk may help appreciate the impact of different interest-rate dynamics on valuation.
- (vi) Point out limitations in some current market practices such as explaining the multiple curves through deterministic fudge factors or shifts where the option embedded in the CVA calculation would be priced without any volatility.

Overall, this paper presents a rigorous attempt to connect the updated credit and collateral inclusive valuation paradigms to multiple-curves interest rate theory, thus encompassing two of the key post-2007 developments in derivatives markets. This paper is an extended and refined version of ideas originally appeared in [Pallavicini and Brigo(2013)].

2 Valuation Equation with Credit and Collateral

Classical interest-rate models were formulated to satisfy no-arbitrage relationships by construction, which allowed one to price and hedge forward-rate agreements in terms of risk-free zero-coupon bonds

2.1 Spot Libor Rates as Market Primitives

Starting from summer 2007, with the spreading of the credit crunch, market quotes of forward rates and zero-coupon bonds began to violate usual pre-crisis no-arbitrage relationships. The main driver of such behavior was the liquidity crisis reducing the credit lines along with the fear of an imminent systemic break-down. As a result the impact of counterparty risk on market prices could not be considered negligible any more.

This is the first of many examples of relationships that broke down with the crisis. Assumptions and approximations stemming from valuation theory should be replaced by strategies implemented with market instruments. For instance, inclusion of credit valuation adjustments (CVA) for interest-rate instruments, such as those analyzed in [Brigo and Pallavicini(2007)], breaks the relationship between risk-free zero coupon bonds and Libor forward rates. Also, funding in domestic currency on different time horizons must include counterparty risk adjustments and liquidity issues, see [Filipovic and Trolle(2012)], breaking again this relationship. We thus have, against the earlier standard theory,

$$L(T_0, T_1) \neq \frac{1}{T_1 - T_0} \left(\frac{1}{P_{T_0}(T_1)} - 1 \right), \quad F_t(T_0, T_1) \neq \frac{1}{T_1 - T_0} \left(\frac{P_t(T_0)}{P_t(T_1)} - 1 \right), \quad (1)$$

where $P_t(T)$ is a zero coupon bond price at time t for maturity T , L is the Libor rate and F is the related Libor forward rate. A direct consequence is the impossibility to describe all Libor rates in terms of a unique zero-coupon yield curve. Indeed, since 2009 and even earlier, we had evidence that the money market for the Euro area was moving to a multi-curve setting, as widely described in the literature. See, for instance, [Ametrano and Bianchetti(2009)], [Henrard(2007), Henrard(2010)], [Mercurio(2009)], [Pallavicini and Tarenghi(2010)], and [Fujii and Takahashi(2010)].

A further example of evolving assumptions is given by the use of collateralized contracts. The growing attention on counterparty credit risk is transforming Over The Counter (OTC) derivatives money markets. An increasing number of derivative contracts is cleared by CCPs, while most of the remaining contracts are traded under collateralization, regulated by Credit Support Annex (CSA). Both cleared and CSA deals require collateral posting, as default insurance, along with its remuneration. We cannot neglect such effects. We refer to the extensive work in [Brigo and Pallavicini(2014)] for an analysis of the CCP case.

2.2 The Valuation Framework

In order to value a financial product (for example a derivative contract), we follow the approach of [Pallavicini et al.(2011)Pallavicini, Perini, and Brigo], and we take the risk-neutral expectation of all the discounted cash flows occurring after the trading position is entered. In particular, we include in the cash flow computation all coupons, dividends and premiums listed in the contract and occurring upon default (close-out procedure), along with the cash flows required by the collateral margining procedure. Here, we do not analyze the impact of additional costs due to the funding or hedging procedure. The impact of these costs in pricing interest-rate derivatives is discussed in [Pallavicini and Brigo(2013)].

2.2.1 The Master Formula

We refer to the two names involved in the financial contract and subject to default risk as investor (also called name “I”, usually the bank) and counterparty (also called name “C”, for example a corporate client, but also another bank). In all the following we will take the perspective of the investor “I”. We denote by τ_I and τ_C respectively the default times of the investor and counterparty. We fix the portfolio time horizon $T > 0$, and fix the risk-neutral valuation model $(\Omega, \mathcal{G}, \mathbb{Q})$, with a filtration $(\mathcal{G}_t)_{t \in [0, T]}$ such that τ_C, τ_I are $(\mathcal{G}_t)_{t \in [0, T]}$ -stopping times. More precisely we define τ_I and τ_C as first jump time of Cox processes of intensities λ^I, λ^C as in the so called *canonical construction* of example 9.1.5 of [Bielecki and Rutkowski(2002)]. We suppose the measure $\mathbb{Q}$ to be a martingale measure with

respect to a risk-free rate process r_t . We denote by $\mathbb{E}_t[\cdot]$ the conditional expectation under $\mathbb{Q}$ given $\mathcal{G}_t$, and by $\mathbb{E}_{\tau_i}[\cdot]$ the conditional expectation under $\mathbb{Q}$ given the stopped filtration $\mathcal{G}_{\tau_i}$. We exclude the possibility of simultaneous defaults, and define the first default event between the two parties as the stopping time

$$\tau := \tau_C \wedge \tau_I.$$

We will also consider the market sub-filtration $(\mathcal{F}_t)_{t \geq 0}$ that one obtains implicitly by assuming a separable structure for the complete market filtration $(\mathcal{G}_t)_{t \geq 0}$. $\mathcal{G}_t$ is then generated by the pure default-free market filtration $\mathcal{F}_t$ and by the filtration generated by all the relevant default times monitored up to t (see [Bielecki and Rutkowski(2002)]).

Here, without an explicit derivation, we write the pricing formula for a derivative contract inclusive of collateralized credit and debit valuation adjustments (CVA/DVA) and margining costs. We address the readers to [Pallavicini et al.(2011)Pallavicini, Perini, and Brigo, Pallavicini et al.(2012)Pallavicini, Perini, and Brigo] for a complete discussion that includes funding costs, while for the cases examined in this paper it would suffice chapter 12 in [Brigo et al.(2013)Brigo, Morini, and Pallavicini] or chapter 3 of [Crépey et al.(2014)Crépey, Bielecki, and Brigo]. The pricing master formula is the following:

$$V_t := \mathbb{E}[\Pi(t, T \wedge \tau) + \gamma(t, T \wedge \tau) + 1_{\{t < \tau < T\}} D(t, \tau) \theta_\tau | \mathcal{G}_t], \quad (2)$$

where

- $\Pi(t, T)$ is the sum of all discounted payoff terms in the interval $(t, T]$, without credit or debit risk and without collateral cash flows. In other terms, these are the financial instrument cash flows without additional risks.
- $\gamma(t, T)$ are the collateral margining costs discounted cash flows within the interval $(t, T]$,
- θ_τ is the on-default cash flow. It is primarily this term that originates the CVA and DVA terms, that may also embed collateral and contagion risk, see for instance the case of credit derivatives described in [Brigo et al.(2014a)Brigo, Capponi, and Pallavicini].

When we say discounting above, we mean discounting at the risk-free rate r associated with the risk-neutral measure. We therefore need to define the related stochastic discount factor $D(t, \tau)$. Notation $D(t, u; r)$, or simply $D(t, u)$, in general will denote the risk-neutral discount factor, given by the ratio

$$D(t, u; r) := D(t, u) := B_t / B_u, \quad dB_t = r_t B_t dt,$$

where B is the bank-account numeraire, driven by the risk-free instantaneous interest rate r and associated to the risk-neutral measure $\mathbb{Q}$. This rate r is assumed to be $(\mathcal{F}_t)_{t \in [0, T]}$ adapted and it is the key variable in all pre-crisis term structure modeling, since all other interest rates and bonds could be defined as a function of r , see for example [Brigo and Mercurio(2006)].

Common market procedures, as we will see later on, may link the paths of the additional valuation terms γ, θ in Equation (2) to the future paths of the price V itself, specifying further Equation (2) as a recursive relationship. This feature is hidden in simplified approaches based on adding a spread to the discount curve to accommodate collateral and funding costs. A different approach is followed by [Crépey(2015)], who extends the usual risk-neutral evaluation framework to include many cash accounts accruing at different rates.

2.2.2 Cash-Flows in Continuous Time

The pricing master equation depends on the specific form of the contractual cash-flows and the margining procedure. In [Pallavicini et al.(2011)Pallavicini, Perini, and Brigo, Pallavicini et al.(2012)Pallavicini, these terms are defined as a sum of individual flows over a discrete time-grid. Since we are going to adapt the master equation to interest-rate derivatives, where collateralization usually happens on a daily basis, we prefer to model cash flows as happening in a continuous time-grid, since this simplifies notation and calculations. Furthermore, we assume that collateral re-hypothecation is allowed, as done in practice (see [Brigo et al.(2011)Brigo, Capponi, Pallavicini, and Papatheodorou for a discussion of re-hypothecation]). Thus, we define the unknown terms in Equation (2) as

$$\Pi(t, u) := \int_t^u D(t, v) d\pi_v, \quad \gamma(t, u) = \int_t^u (r_v - c_v) C_v D(t, v) dv, \quad (3)$$

where π_t is the contractual coupon process, and the collateral rate is defined as

$$c_t := c_t^+ 1_{\{C_t > 0\}} + c_t^- 1_{\{C_t < 0\}}$$

with $c^\pm$ defined in the CSA contract. In this paper, since we are not dealing with credit derivatives, we assume that the collateral account, the on-default processes and the collateral rate c_t are $\mathcal{F}$ -predictable and that the product cash-flows are $\mathcal{F}$ -adapted. Note that in particular this means that they cannot jump at τ (see for example chapter 13 of [Crépey et al.(2014)Crépey, Bielecki, and Brigo]).

We can plug the above definitions into Equation (2), and we are able to write the following continuous-time master equation for valuation.

Proposition 2.1. (Master equation for valuation under credit risk and collateralization). *Under the assumptions above, the valuation master equation in presence of collateral (3) for the payout Π in (3) is given by*

$$V_t = \mathbb{E} \left[\int_t^T D(t, u; r) (1_{\{u < \tau\}} d\pi_u + 1_{\{\tau \in du\}} \theta_u + 1_{\{u < \tau\}} (r_u - c_u) C_u du) \mid \mathcal{G}_t \right] \quad (4)$$

where the discount factors are defined as given by

$$D(t, T; x) := \exp \left\{ - \int_t^T x_u du \right\}.$$

Notice that the above Equation (4) is not suited for explicit numerical evaluations, since the right-hand side is still depending on the derivative price via the indicators within the collateral rates, and possibly via the on-default term, leading to recursive non-linear features. We could resort to numerical solutions, as in [Crépey et al.(2013)Crépey, Gerboud, Grbac, and Ngor], but, since our goal is valuing interest-rate derivatives, we prefer to further specialize the valuation equation for such deals.

In this first work we develop our analysis without considering a dependence between the default times if not through their spreads, or more precisely by assuming that the default times are $\mathcal{F}$ -conditionally independent. Thus, we can simplify the valuation equation given by (4) by switching to the default-free market filtration. By following the filtration switching

formula in [Bielecki et al.(2008)], we introduce for any $\mathcal{G}_t$ -adapted process X_t a unique $\mathcal{F}_t$ -adapted process $\tilde{X}_t$, defined such that

$$1_{\{\tau > t\}}X_t = 1_{\{\tau > t\}}\tilde{X}_t.$$

Hence, we can write the pre-default price process as given by $1_{\{\tau > t\}}\tilde{V}_t = V_t$ where the right hand side is given in Equation (4) and where $\tilde{V}_t$ is $\mathcal{F}_t$ adapted.

2.3 Close-Out and Margining Procedures

We continue this section by specifying the close-out procedure occurring at default time, so that we can define the on-default cash flow θ_u appearing in Equation (4). We consider two possibilities: (i) at default time the close-out procedure is completed without delay, or (ii) the procedure takes δ days to be completed. Moreover from now on we assume that the collateral rate is the risk-free rate, i.e. $c_t = r_t$. This can be a reasonable approximation since the collateral is exchanged daily and hence is subject to a low risk.

2.3.1 The Close-Out Procedure without Delay

The derivative pricing Equation (4) requires a specification of the close-out procedure occurring at default time. A first possibility is given by a close-out procedure that at default time is completed without delay. In the following section we relax this assumption.

The analysis of the ISDA documentation describing the close-out procedure in presence of collateralization is described in the approximation of no delay in [Brigo et al.(2011)Brigo, Capponi, Pallavicini, and Papatheodorou]. On default event τ the surviving party computes the close-out value ε_τ , which represents the value of the remaining cash flows. If the surviving party is a net debtor, then she must pay the whole close-out value ε_τ to the defaulting party. On the other hand, if the surviving party is a net creditor, then she is able to recover only a fraction of her credits. The collateral account is used to reduce the exposure, but if it is not enough an unsecured claim is needed to get back the remaining part. Here, we do not derive the analysis of [Brigo et al.(2011)Brigo, Capponi, Pallavicini, and Papatheodorou], but we simply quote their result to write the on-default term θ_τ as

$$\theta_\tau(C, \varepsilon) := \varepsilon_\tau - 1_{\{\tau_C < \tau_I\}}\text{LGD}_C(\varepsilon_\tau - C_\tau)^+ - 1_{\{\tau_I < \tau_C\}}\text{LGD}_I(\varepsilon_\tau - C_\tau)^- \quad (5)$$

where ε is the exposure computed by the surviving party (close-out amount), $\text{LGD} \in [0, 1]$ is the loss given default, and $(x)^+$ indicates the positive part of x while $(x)^- := -(-x)^+$. A generalization of the on-default term in the case of initial margins is given in [Brigo and Pallavicini(2014)].

The close-out amount is not tightly defined by ISDA documentation. Usually a risk-free or a pre-default specification is adopted. See [Brigo et al.(2013)Brigo, Morini, and Pallavicini] for a discussion. Here, we choose a risk-free version, and we define

$$\varepsilon_t := \mathbb{E} \left[\int_t^T D(t, u; r) d\pi_u \mid \mathcal{G}_t \right]. \quad (6)$$

We complete this list of assumptions with a description of the margining procedure. For ease of discussion we adopt the choice of [Pallavicini and Brigo(2013)], and we write

$$C_t := \alpha_t \varepsilon_t \quad (7)$$

where $\alpha_t \in [0, 1]$ is $\mathcal{F}_t$ adapted. Notice that interest-rate derivatives have $\mathcal{F}$ -adapted cash flows, so that in this paper we can also write

$$\varepsilon_t = \mathbb{E} \left[\int_t^T D(t, u; r) d\pi_u \mid \mathcal{F}_t \right] \quad (8)$$

2.3.2 Switching to Market Filtration

Then, if we can disregard contagion risk, as discussed in [Brigo et al.(2011)Brigo, Capponi, Pallavicini, and Papat], we have that the value of the contract is continuous at default time, namely

$$\tilde{V}_{\tau-} = \tilde{V}_{\tau}$$

and we have the possibility to simplify Equation (4) by switching from the filtration $\mathcal{G}$ to the market filtration $\mathcal{F}$.

In general for any $\mathcal{F}_u$ -predictable process we can write for any time t and u such that $t \leq u$

$$\mathbb{E} \left[1_{\{\tau \in du\}} 1_{\{\tau_C < \tau_I\}} \phi_u \mid \mathcal{G}_t \right] = 1_{\{\tau > t\}} du \mathbb{E} \left[\lambda_u^C D(t, u; \lambda) \phi_u \mid \mathcal{F}_t \right]$$

and

$$\mathbb{E} \left[1_{\{\tau \in du\}} 1_{\{\tau_I < \tau_C\}} \phi_u \mid \mathcal{G}_t \right] = 1_{\{\tau > t\}} du \mathbb{E} \left[\lambda_u^I D(t, u; \lambda) \phi_u \mid \mathcal{F}_t \right] .$$

Furthermore, we introduce the pre-default intensity λ_t^I of the investor and the pre-default intensity λ_t^C of the counterparty as

$$\begin{aligned} 1_{\{\tau_I > t\}} \lambda_t^I dt &:= \mathbb{Q} \{ \tau_I \in dt \mid \tau_I > t, \mathcal{F}_t \} \\ 1_{\{\tau_C > t\}} \lambda_t^C dt &:= \mathbb{Q} \{ \tau_C \in dt \mid \tau_C > t, \mathcal{F}_t \} \end{aligned} \quad (9)$$

along with their sum λ_t

$$1_{\{\tau > t\}} \lambda_t := 1_{\{\tau > t\}} \lambda_t^I + 1_{\{\tau > t\}} \lambda_t^C \quad (10)$$

and we omit the tilde over the intensity symbols to lighten the notation. The result above follows from an application of the so called *Key Lemma* (for further details please refer to Section 3.8 of [Bielecki et al.(2009)]). We can now switch to the default-free market filtration $\mathcal{F}$ by means of the proposition which follows. For the detailed derivation and further discussion see again [Bielecki et al.(2009)] or [Crépey(2015)] (see Lemma 3.10(i)), [Crépey et al.(2014) Crépey, Bielecki, and Br], (see Proposition 5.2.10), and [Brigo et al.(2016)Brigo, Francischello, and Pallavicini] for applications to funding costs.

Proposition 2.2. (Master equation without gap risk). *By changing filtration to $\mathcal{F}$, we can specialize the valuation Equation (4) when the close-out procedure given by Equation (6) is completed without delay at default time (no gap risk) obtaining:*

$$\begin{aligned} \tilde{V}_t &= \varepsilon_t \\ &- \mathbb{E} \left[\int_t^T D(t, u; r + \lambda) \lambda_u^C (1 - \alpha_u) L_{GDC}(\varepsilon_u)^+ du \mid \mathcal{F}_t \right] \\ &- \mathbb{E} \left[\int_t^T D(t, u; r + \lambda) \lambda_u^I (1 - \alpha_u) L_{GDI}(\varepsilon_u)^- du \mid \mathcal{F}_t \right] . \end{aligned} \quad (11)$$

Remark 2.3. (The perfect collateralization case) *We can further specialize the above equation to a relevant case: when the margining procedure is able to track the exposure in continuous time, namely when*

$$\alpha_t = 1.$$

This case is an approximation for interest-rate liquid instruments widely adopted in the literature. See, for instance, [Pallavicini and Brigo(2013)]. In this case from Equations (7), (8), and (11) we get

$$\tilde{V}_t = \varepsilon_t = C_t = \mathbb{E} \left[\int_t^T D(t, u; r) d\pi_u \mid \mathcal{F}_t \right]. \quad (12)$$

2.3.3 The Close-Out Procedure with Delay: Gap Risk

Although the margining procedure reduces the counterparty risk, there are still some risks that can arise on default due to some mismatch between the margin account and the contract valuation. This problem is discussed in detail in [Brigo and Pallavicini(2014)] for both bilateral and centrally cleared contracts. These risks contribute to what is called “gap risk”. We give examples of some of them:

- we can have a risk due to mismatch between the collateral account and the value of the contract just before the default, in the sense that the margining procedure is not perfect and do not match exactly the value of the contract;
- we can have a risk because the contract value jumps at default, so diminishing the impact of our collateral on default risk;
- we can have a risk due to delays in the default procedure. In fact usually there is a delay of δ days, called “cure period”, between the default event τ and the close-out cash-flows at time $\tau + \delta$. Thus, the change in value of the contract over this period is not accounted for in the collateral account.

In practice to mitigate such risks bilateral and centrally cleared contracts implements margining procedures which includes many collateral accounts with specific margining procedures. We refer to [Brigo and Pallavicini(2014)] for details.

A detailed description of the close-out procedure occurring in presence of a cure period, when the margining procedure includes of variation and initial margin accounts, is presented in [Brigo and Pallavicini(2014)]. Here, we write the final expression without an explicit derivation in the case where the initial margins are put to 0.

$$\begin{aligned} \theta_\tau(C, \varepsilon) &:= \mathbb{E}[\varepsilon_{\tau+\delta}(\tau, T) \mid \mathcal{G}_{\tau-}] \\ &- \mathbb{E}[1_{\{\tau_C < \tau_I + \delta\}} \text{LGD}_C(\varepsilon_{\tau+\delta}(\tau, T) - C_\tau)^+ \mid \mathcal{G}_{\tau-}] \\ &- \mathbb{E}[1_{\{\tau_I < \tau_C + \delta\}} \text{LGD}_I(\varepsilon_{\tau+\delta}(\tau, T) - C_\tau)^- \mid \mathcal{G}_{\tau-}] \end{aligned} \quad (13)$$

where we change the definition of close-out amount to deal with coupon paid during the cure period, namely

$$\varepsilon_s(t, T) = \mathbb{E} \left[\int_t^T D(t, u; r) d\pi_u \mid \mathcal{G}_s \right]. \quad (14)$$

Then, as in the previous section we make an assumption for the margining procedure. For ease of discussion we adopt the simple choice

$$C_t := \alpha_t \varepsilon_t \quad (15)$$

where $\alpha_t \in [0, 1]$ is $\mathcal{F}_t$ adapted. A concrete definition of the initial margin accounts can be found in [Brigo and Pallavicini(2014)]. Then, we can obtain a formula similar to Equation (11).

Proposition 2.4. (Master equation with gap risk) *We can specialize the valuation Equation (4) when the close-out procedure given by Equation (14) is completed with a delay of δ days at default time (gap risk) obtaining:*

$$\begin{aligned} V_0 &= \varepsilon_0(0, T) \\ &\quad - \mathbb{E} \left[1_{\{\tau \leq T\}} 1_{\{\tau_C < \tau_I + \delta\}} D(0, \tau; r) L_{GDC}(\varepsilon_{\tau+\delta}(\tau, T) - \alpha_t \varepsilon_\tau(\tau, T))^+ \right] \\ &\quad - \mathbb{E} \left[1_{\{\tau \leq T\}} 1_{\{\tau_I < \tau_C + \delta\}} D(0, \tau; r) L_{GDI}(\varepsilon_{\tau+\delta}(\tau, T) - \alpha_t \varepsilon_\tau(\tau, T))^- \right]. \end{aligned} \quad (16)$$

Proof. We start from Equation (4) with $c_u = r_u$.

$$V_0 = \mathbb{E} \left[\int_0^T D(0, u; r) (1_{\{u < \tau\}} d\pi_u + 1_{\{\tau \in du\}} \theta_u) \right].$$

If we plug in Equation (13) we obtain

$$\begin{aligned} V_0 &= \mathbb{E} \left[\int_0^T D(0, u; r) 1_{\{u < \tau\}} d\pi_u \right] \\ &\quad + \mathbb{E} \left[1_{\{\tau \leq T\}} D(0, \tau; r) \mathbb{E}[\varepsilon_{\tau+\delta}(\tau, T) | \mathcal{G}_{\tau-}] \right] \\ &\quad - \mathbb{E} \left[1_{\{\tau \leq T\}} D(0, \tau; r) \mathbb{E} \left[1_{\{\tau_C < \tau_I + \delta\}} L_{GDC}(\varepsilon_{\tau+\delta}(\tau, T) - \alpha_t \varepsilon_\tau(\tau, T))^+ | \mathcal{G}_{\tau-} \right] \right] \\ &\quad - \mathbb{E} \left[1_{\{\tau \leq T\}} D(0, \tau; r) \mathbb{E} \left[1_{\{\tau_I < \tau_C + \delta\}} L_{GDI}(\varepsilon_{\tau+\delta}(\tau, T) - \alpha_t \varepsilon_\tau(\tau, T))^- | \mathcal{G}_{\tau-} \right] \right]. \end{aligned}$$

Then, we use equation (14) and we have

$$\begin{aligned} V_0 &= \mathbb{E} \left[\int_0^T D(0, u; r) 1_{\{u < \tau\}} d\pi_u \right] \\ &\quad + \mathbb{E} \left[1_{\{\tau \leq T\}} \int_\tau^T D(0, u; r) d\pi_u \right] \\ &\quad - \mathbb{E} \left[1_{\{\tau \leq T\}} \mathbb{E} \left[1_{\{\tau_C < \tau_I + \delta\}} D(0, \tau; r) L_{GDC}(\varepsilon_{\tau+\delta}(\tau, T) - \alpha_t \varepsilon_\tau(\tau, T))^+ | \mathcal{G}_{\tau-} \right] \right] \\ &\quad - \mathbb{E} \left[1_{\{\tau \leq T\}} \mathbb{E} \left[1_{\{\tau_I < \tau_C + \delta\}} D(0, \tau; r) L_{GDI}(\varepsilon_{\tau+\delta}(\tau, T) - \alpha_t \varepsilon_\tau(\tau, T))^- | \mathcal{G}_{\tau-} \right] \right]. \end{aligned}$$

So, by recombining the first two lines and using again Equation (14) and the tower property for conditional expectations, we obtain

$$\begin{aligned} V_0 &= \varepsilon_0(0, T) \\ &\quad - \mathbb{E} \left[1_{\{\tau \leq T\}} 1_{\{\tau_C < \tau_I + \delta\}} D(0, \tau; r) L_{GDC}(\varepsilon_{\tau+\delta}(\tau, T) - \alpha_t \varepsilon_\tau(\tau, T))^+ \right] \\ &\quad - \mathbb{E} \left[1_{\{\tau \leq T\}} 1_{\{\tau_I < \tau_C + \delta\}} D(0, \tau; r) L_{GDI}(\varepsilon_{\tau+\delta}(\tau, T) - \alpha_t \varepsilon_\tau(\tau, T))^- \right]. \end{aligned}$$

□

Even though formula (16) does not make any reference to the market filtration $\mathcal{F}$, it is sufficient for the assessment of the impact of multiple curve dynamics on valuation in presence of gap risk. Considering funding costs and slightly different definitions of V [Crépey and Song(2016)] perform the switching to a reference filtration $\mathcal{F}$, and, similarly to what is done here, consider the problem of valuation under wrong way and gap risk.

3 Valuing Collateralized Interest-Rate Derivatives

As we mentioned in the introduction, we will base our analysis on real market processes. All liquid market quotes on the money market (MM) correspond to instruments with daily collateralization at overnight rate (e_t), both for the investor and the counterparty, hence from now on we will choose

$$r_t = c_t \doteq e_t.$$

As noted at the beginning of section 2.3, this makes sense because being e_t an overnight rate, it embeds a low counterparty risk and can be considered a good proxy for the risk-free rate r_t . In principle one could instead use the reference rate for secured transactions (e.g. Eurosto in the Euro area) but there are fewer liquid products with long maturity and hence it presents some difficulties in the calibration to market data. Here, we do not consider funding costs. We refer to [Pallavicini and Brigo(2013)] for details.

We will describe some of these MM instruments, such as overnight indexed swaps (OIS) and interest-rate swaps (IRS), along with their underlying market rates, in the following sections. At the moment of writing this paper an important part of the market is moving from OTC contracts regulated by a (standardized) bilateral CSA to a market cleared by CCPs. We refer to the extensive treatment in [Brigo and Pallavicini(2014)] for details. Here, we assume, as in most of the literature, that the collateralization procedure operates in continuous time for liquid MM instruments being able to remove all credit risk, and, moreover, the gap risk is not present, so that we can use the perfect collateralization approximation of Equation (12) to price these derivatives. See [Henrard(2014)] for details. See [Brigo et al.(2011)Brigo, Capponi, Pallavicini, and Papatheodorou] for a discussion on the impact of discrete-time collateralization on interest-rate derivatives.

3.1 Overnight Rates and OIS

The MM usually quotes the prices of collateralized instruments. In particular, a daily collateralization procedure is assumed, so that CSA contracts require that the collateral account is remunerated at the overnight rate (e_t). In particular, the MM usually quotes the price of OIS. Such contracts exchange a fix-payment leg with a floating leg paying the same overnight rate used for their collateralization, compounded daily. Since we are going to price OIS under the assumption of perfect collateralization, namely we are assuming that daily collateralization may be viewed as done on a continuous basis, we approximate also daily compounding in OIS floating leg with continuous compounding, which is reasonable when there is no gap risk. Hence the discounted payoff of an OIS with tenor x and maturity $T = nx$ is given by

$$\sum_{i=1}^n D(t, T - (n-i)x; e) \left(1 + xK - \exp \left\{ \int_{T-(n-i-1)x}^{T-(n-i)x} e_u du \right\} \right)$$

In particular in the one-period case we have:

$$D(t, T; e) \left(1 + xK - \exp \left\{ \int_{T-x}^T e_u du \right\} \right)$$

where K is the fixed rate paid by the OIS. Furthermore, we can introduce the (par) fix rates $K \doteq E_t(T, x; e)$ that make the one-period OIS contract fair, namely priced 0 at time t . They

are implicitly defined as

$$\tilde{V}_t^{\text{OIS}}(K) := \mathbb{E} \left[\left(1 + xK - \exp \left\{ \int_{T-x}^T e_u du \right\} \right) D(t, T; e) \mid \mathcal{F}_t \right]$$

with

$$\tilde{V}_t^{\text{OIS}}(E_t(T, x; e)) = 0$$

leading to

$$E_t(T, x; e) := \frac{1}{x} \left(\frac{P_t(T-x; e)}{P_t(T; e)} - 1 \right) \quad (17)$$

where we define collateralized zero-coupon bonds¹ as

$$P_t(T; e) := \mathbb{E}[D(t, T; e) \mid \mathcal{F}_t] . \quad (18)$$

One-period OIS rates $E_t(T, x; e)$, along with multi-period ones, are actively traded on the market. Notice that we can bootstrap collateralized zero-coupon bond prices from OIS quotes.

3.2 Libor Rates, IRS and Basis Swaps

Libor rates ($L_t(T)$) used to be linked to the term structure of default-free interbank interest rates in a fundamental way. In the classical term structure theory, Libor rates would satisfy fundamental no-arbitrage conditions with respect to zero-coupon bonds that we no longer consider to hold, as we pointed out earlier in Equation 1. We now deal with a new definition of forward Libor rates that may take into account collateralization. Libor rates are still the indices used as reference rate for many collateralized interest-rate derivatives (IRS, basis swaps, ...). IRS contracts swap a fix-payment leg with a floating leg paying simply compounded Libor rates. IRS contracts are collateralized at overnight rate e_t . Thus, an IRS discounted payoff with maturity $T = nx$ and tenor x is given by

$$\sum_{i=1}^n x D(t, T - (n-i)x; e) (K - L_{T-(n-i-1)x}(T - (n-i)x)) .$$

In particular in the one-period case we have:

$$D(t, T; e)x(K - L_{T-x}(T))$$

where K is the fix rate paid by the IRS. Furthermore, we can introduce the (par) fix rates $K = F_t(T, x; e)$ that make the one-period IRS contract fair, namely priced 0 at time t . They are implicitly defined via

$$\tilde{V}_t^{\text{IRS}}(K) := \mathbb{E}[(xK - xL_{T-x}(T)) D(t, T; e) \mid \mathcal{F}_t]$$

with

$$\tilde{V}_t^{\text{IRS}}(F_t(T, x; e)) = 0$$

¹Notice that we are only defining a price process for hypothetical collateralized zero-coupon bond. We are not assuming that collateralized bonds are assets traded on the market.

leading to the following definition of collateralized forward Libor rate

$$F_t(T, x; e) := \frac{\mathbb{E}[L_{T-x}(T)D(t, T; e) \mid \mathcal{F}_t]}{\mathbb{E}[D(t, T; e) \mid \mathcal{F}_t]}.$$

The above definition may be simplified by a suitable choice of the measure under which we take the expectation. In particular, we can consider the collateralized T -forward measure $\mathbb{Q}^{T;e}$ defined by means of the following Radon-Nikodym derivative

$$Z_t(T; e) := \frac{d\mathbb{Q}^{T;e}}{d\mathbb{Q}} \Big|_{\mathcal{F}_t} := \frac{\mathbb{E}[D(0, T; e) \mid \mathcal{F}_t]}{P_0(T; e)} = \frac{D(0, t; e)P_t(T; e)}{P_0(T; e)}$$

which is a positive $\mathbb{Q}$ -martingale, normalized so that $Z_0(T; e) = 1$.

Thus, for any payoff ϕ_T , perfectly collateralized at overnight rate e_t , we can express prices as expectations under the collateralized T -forward measure, and we get

$$\mathbb{E}[\phi_T D(t, T; e) \mid \mathcal{F}_t] = P_t(T; e) \mathbb{E}^{T;e}[\phi_T \mid \mathcal{F}_t].$$

In particular, we can write collateralized forward Libor rates as

$$F_t(T, x; e) = \mathbb{E}^{T;e}[L_{T-x}(T) \mid \mathcal{F}_t]. \quad (19)$$

One-period forward rates $F_t(T, x; e)$, along with multi-period ones (swap rates), are actively traded on the market. Once collateralized zero-coupon bonds are derived, we can bootstrap forward rate curves from such quotes. See, for instance, [Ametrano and Bianchetti(2009)] or [Pallavicini and Tarenghi(2010)] for a discussion on bootstrapping algorithms.

Basis swaps are an interesting product that became more popular after the marked switched to a multicurve structure. In fact in a basis swap there are two floating legs, one pays a Libor rate with a certain tenor and the other pays the Libor rate with a shorter tenor plus a spread that makes the contract fair at inception. More precisely the discounted payoff of a basis swap which legs pay respectively a Libor rate with tenors $x < y$ with maturity $T = nx = my$ is given by

$$\begin{aligned} & \sum_{i=1}^n D(t, T - (n-i)x; e) x (L_{T-(n-i-1)x}(T - (n-i)x) + K) \\ & - \sum_{j=1}^m D(t, T - (m-j)y; e) y L_{T-(m-j-1)y}(T - (m-j)y). \end{aligned}$$

It is clear that a part from being traded per se, this instrument is naturally present in the banks portfolios as result of the netting of opposite swap positions with different tenors.

3.3 Modeling Constraints

Our aim is to setup a multiple-curve dynamical model starting from collateralized zero-coupon bonds $P_t(T; e)$, and Libor forward rates $F_t(T, x; e)$. As we have seen we can bootstrap the initial curves for such quantities from directly observed quotes in the market. Now, we wish to propose a dynamics that preserves the martingale properties satisfied by such quantities.

Thus, without loss of generality, we can define collateralized zero-coupon bonds under the $\mathbb{Q}$ measure as

$$\frac{dP_t(T; e)}{P_t(T; e)} = e_t dt - \sigma_t^P(T; e)^* dW_t$$

and Libor forward rates under the $\mathbb{Q}^{T;e}$ measure as

$$dF_t(T, x; e) = \sigma_t^F(T, x; e)^* dZ_t^{T;e}$$

where W s and Z s are correlated standard (column) vector² Brownian motions with correlation matrix ρ , and the volatility vector processes σ^P and σ^F may depend on bonds and forward Libor rates themselves.

The following definition of $f_t(T; e)$ is not strictly necessary, and we could keep working with bonds $P_t(T; e)$, using their dynamics. However, as it is customary in interest rate theory to model rates rather than bonds, we may try to formulate quantities that are closer to the standard HJM framework. In this sense we can define instantaneous forward rates $f_t(T; e)$, by starting from (collateralized) zero-coupon bonds, as given by

$$f_t(T; e) := -\frac{\partial}{\partial T} \log P_t(T; e)$$

We can derive instantaneous forward-rate dynamics by Itô lemma, and we obtain the following dynamics under the $\mathbb{Q}^{T;e}$ measure

$$df_t(T; e) = \sigma_t(T; e) dW_t^{T;e}, \quad \sigma_t(T; e) := \frac{\partial}{\partial T} \sigma_t^P(T; e)$$

where partial differentiation is meant to be applied component-wise.

Hence, we can summarize our modeling assumptions in the following way.

1. Since linear products (OIS, IRS, basis swaps...) can be expressed in terms of simpler quantities, namely collateralized zero-coupon bonds $P_t(T; e)$ and Libor forward rates $F_t(T, x; e)$, we focus on their modeling.
2. Initial term structures for collateralized products may be bootstrapped from market data.
3. For volatility and dynamics, we can write rates dynamics by enforcing suitable no-arbitrage martingale properties, namely

$$df_t(T; e) = \sigma_t(T; e)^* dW_t^{T;e}, \quad dF_t(T, x; e) = \sigma_t^F(T, x; e)^* dZ_t^{T;e}. \quad (20)$$

As we explained in the introduction, this is where the multiple-curve picture finally shows up: we have a curve with Libor based forward rates $F_t(T, x; e)$, that are collateral adjusted expectation of Libor market rates $L_{T-x}(T)$ we take as primitive rates from the market, and we have instantaneous forward rates $f_t(T; e)$ that are OIS based rates. OIS rates $f_t(T; e)$ are driven by collateral fees, whereas Libor forward rates $F_t(T, x; e)$ are driven both by collateral rates and by the primitive Libor market rates.

Now, the framework for multiple curves is ready. In the following section we propose a specific dynamics within this framework.

²In the following we will consider N -dimensional vectors as $N \times 1$ matrices. Moreover given a matrix A , we will indicate A^* its transpose, and if B is another conformable matrix we indicate AB the usual matrix product.

4 Interest-Rate Modeling

We can now specialize our modeling assumptions to define a model for interest-rate derivatives which is on one hand flexible enough to calibrate the quotes of the MM, and on the other hand robust. Our aim is to use a HJM framework using a single family of Markov processes to describe all the term structures and interest rate curves we are interested in.

In the literature many authors proposed generalizations of interest-rate models to include multiple yield curves. In particular we cite the first tentatives of [Kijima et al.(2009)Kijima, Tanaka, and Wong] and [Fujii et al.(2010)Fujii, Shimada, and Takahashi], followed by the extensions of the Libor Market Models proposed by [Mercurio(2010), Mercurio and Xie(2012)], short-rate models based on multiplicative basis as in [Henrard(2013)], the extensions of the HJM framework proposed by [Moreni and Pallavicini(2010), Moreni and Pallavicini(2014)], [Crépey et al.(2012)Crépey, Grbac, and L. Crépey et al.(2015)Crépey, Grbac, Ngor, and Skovmand], and [Cuchiero et al.(2014)Cuchiero, Fontana, and Gnani]. A survey of the literature can be found in [Henrard(2014)].

In such works the problem is faced in a pragmatic way by considering each forward rate as a single asset without investigating the microscopical dynamics implied by liquidity and credit risks. However, the hypothesis of introducing different underlying assets may lead to over-parametrization issues that affect the calibration procedure. Indeed, the presence of swap and basis-swap quotes on many different yield curves is not sufficient, as the market quotes swaption premia only on few yield curves. For instance, even if the Euro market quotes one-, three-, six- and twelve-month swap contracts, liquidly traded swaptions are only those indexed to the three-month (maturity one-year) and the six-month (maturities from two to thirty years) Euribor rates. Swaptions referring to other Euribor tenors or to Eonia are not actively quoted. A similar line of reasoning holds also for caps/floors and other interest-rate options.

In order to solve such problem [Moreni and Pallavicini(2010)] introduces a parsimonious model to describe a multi-curve setting by starting from a limited number of (Markov) processes, so to extend the logic of the HJM framework to describe with a unique family of Markov processes all the curves we are interested in.

4.1 Multiple-Curve Collateralized HJM Framework

Let us summarize the basic requirements our model must fulfill:

- i) existence of OIS rates, which we can describe in terms of instantaneous forward rates $f_t(T; e)$;
- ii) existence of Libor rates assigned by the market, typical underlyings of traded derivatives, with associated forwards $F_t(T, x; e)$;
- iii) no arbitrage dynamics of the $f_t(T; e)$ and the $F_t(T, x; e)$ (both being (T, e) -forward measure martingales);
- iv) possibility of writing both $f_t(T; e)$ and $F_t(T, x; e)$ as functions of a common family of Markov processes, so that we are able to build parsimonious yet flexible models.

We stress that our approach models only quantities whose initial conditions can be bootstrapped in a model independent way from market quotes, and it includes implicitly the margining procedure within valuation equations.

We reformulate the results of [Moreni and Pallavicini(2010)] by taking into account that interest-rate products are collateralized. Hence, we choose under $\mathbb{Q}^{T;e}$ measure, the following dynamics.

$$\begin{aligned} df_t(T; e) &= \sigma_t(T)^* dW_t^{T;e} \\ \frac{dF_t(T, x; e)}{k(T, x) + F_t(T, x; e)} &= \Sigma_t(T, x)^* dW_t^{T;e} \end{aligned} \quad (21)$$

where we introduce the families of (stochastic N -dimensional) volatility processes $\sigma_t(T)$ and $\Sigma_t(T, x)$, the vector of N independent $\mathbb{Q}^{T;e}$ -Brownian motions $W_t^{T;e}$, and the set of deterministic shifts $k(T, x)$, such that

$$\lim_{x \downarrow 0} xk(T, x) = 1.$$

This limit condition ensures that the model approaches a standard default and liquidity free HJM model when the tenor goes to zero. We bootstrap $f_0(T; e)$ and $F_0(T, x; e)$ from market quotes.

In order to satisfy the fourth of above requirements, getting a model with a reduced number of common driving factors in the spirit of HJM approaches, it is sufficient to conveniently tie together the volatility processes $\sigma_t(T)$ and $\Sigma_t(T, x)$ through a third process $\sigma_t(u, T, x)$.

$$\sigma_t(T) := \sigma_t(T; T, 0), \quad \Sigma_t(T, x) := \int_{T-x}^T \sigma_t(u; T, x) du. \quad (22)$$

Under this parametrization the OIS curve dynamics is the very same as the risk-free curve in an ordinary HJM framework. Indeed, we have for linearly-compounding forward rates

$$\frac{dE_t(T, x; e)}{1/x + E_t(T, x; e)} = \int_{T-x}^T \sigma_t(u)^* du dW_t^{T;e}.$$

In the generalized version of the HJM framework proposed by [Moreni and Pallavicini(2010)] we have an explicit expression for both the collateralized zero-coupon bonds $P_t(T; e)$ and the Libor forward rates $F_t(T, x; e)$. The first result is a direct consequence of modeling the OIS curve as the risk-free curve in a standard HJM framework, while the second result can be achieved only if a particular form of the volatilities is selected. We obtain this if we generalize the approach of [Ritchken and Sankarasubramanian(1995)] by introducing the following separability constraint

$$\sigma_t(u, T, x) := h(t)q(u, T, x)g(t, u), \quad (23)$$

$$g(t, u) := \exp \left\{ - \int_t^u a(s) ds \right\}, \quad q(u; u, 0) := Id$$

where h_t is a $N \times N$ matrix process, $q(u, T, x)$ is a deterministic $N \times N$ diagonal matrix function, and $a(s)$ is a deterministic N dimensional vector function. The condition on $q(u; T, x)$ being the identity matrix, when $T = u$ ensures that a standard HJM framework holds for collateralized zero-coupon bonds.

We can work out an explicit expression for the Libor forward rates, by plugging the expression of the volatilities into equation (21). We obtain

$$\begin{aligned} \log \left(\frac{k(T, x) + F_t(T, x; e)}{k(T, x) + F_0(T, x; e)} \right) = \\ G(t, T - x, T; T, x)^* \left(X_t + Y_t \left(G_0(t, t, T) - \frac{1}{2} G(t, T - x, T; T, x) \right) \right) \end{aligned} \quad (24)$$

where the stochastic vector process X_t and the auxiliary matrix process Y_t are defined under the $\mathbb{Q}$ measure as in the ordinary HJM framework

$$X_t^i = \sum_{k=1}^N \int_0^t g_i(s, t) \left(h_{ik,s} dW_{k,s} + (h_s^* h_s)_{ik} \int_s^t g_k(s, y) dy ds \right), \quad i = 1 \dots N$$

$$Y_t^{ik} = \int_0^t g_i(s, t) (h_s^* h_s)_{ik} g_k(s, t) ds, \quad i, k = 1 \dots N$$

and

$$G_0(t, T_0, T_1) = \int_{T_0}^{T_1} g(t, s) ds, \quad G(t, T_0, T_1, T, x) = \int_{T_0}^{T_1} q(s, T, x) g(t, s) ds$$

Furthermore it can be shown that the processes X^i and Y^{ik} follow

$$dX^i = \left(\sum_{k=1}^N Y_t^{ik} - a^i(t) X_t^i \right) dt + h_t^* dW_t$$

$$dY_t^{ik} = \left((h_t^* h_t)_{ik} - (a^i(t) + a^k(t)) Y_t^{ik} \right) dt. \quad (25)$$

It is worth noting that the integral representation of forward Libor volatilities given by equation (22), together with the common separability constraint given in equation (23) are sufficient conditions to ensure the existence of a reconstruction formula for all OIS and Libor forward rates based on the very same family of Markov processes. Indeed, we can write the reconstruction formula for OIS rates as given by

$$\log \left(\frac{\frac{1}{x} + E_t(T, x; e)}{\frac{1}{x} + E_0(T, x; e)} \right) =$$

$$G_0(t, T - x, T)^* \left(X_t + Y_t \left(G_0(t, t, T) - \frac{1}{2} G_0(t, T - x, T) \right) \right). \quad (26)$$

Moreover we have the following formula for the OIS instantaneous forward rate

$$f_t(T) = f_0(T) + g(t, T)^* (X_t + Y_t G_0(t, t, T)). \quad (27)$$

In the next section we derive three different dynamics within the HJM collateralized framework, and we will use them in the numerical experiments of Section 4.3.

4.2 Model Specifications

We are interested in some specification of this model, in particular a variant of the Hull and White model, a variant of the Cheyette model and the Moreni and Pallavicini model.

The Hull and White model (hereafter HW) described in [Hull and White(1990)] is the simplest one, and is obtained by choosing

$$h(t) \doteq R, \quad q(u, T, x) \doteq Id, \quad a(s) \doteq a, \quad \kappa(T, x) \doteq \frac{1}{x} \quad (28)$$

where a is a constant vector, and R is the Cholesky decomposition of the correlation matrix that we want our X_t vector to have. In this case we obtain as $\sigma_t(u; T, x)$ process

$$\sigma_t(u; T, x) = R \cdot e^{-a(u-t)}$$

where the exponential is intended to be component-wise. Then we note that X_t is a mean reverting Gaussian process while the Y_t process is deterministic.

In order to model implied volatility smiles, we can add a stochastic volatility process to our model, as shown in [Moreni and Pallavicini(2013)]. In particular, we can obtain a variant of the Cheyette model (hereafter Ch) described in [Cheyette(1992)] by considering a common square-root process for all the entries of h , as in [Trolle and Schwartz(2009)]. More precisely we replace $h(t)$ in (28) with

$$h(t) \doteq \sqrt{v_t} R$$

where a and R are defined as before. The volatility v_t is a process with the following dynamics:

$$dv_t = \eta(1 - v_t) dt + \nu_0(1 + (\nu_1 - 1)e^{-\nu_2 t}) \sqrt{v_t} dZ_t, \quad v_0 = \bar{v} \quad (29)$$

where Z_t is a Brownian motion correlated to W_t , so that we obtain as $\sigma_t(u; T, x)$ process

$$\sigma_t(u; T, x) = \sqrt{v_t} R \cdot e^{-a(u-t)}.$$

As the last specification of the framework we consider the model [Moreni and Pallavicini(2013)] (hereafter MP) which adopts a different shift $k(T, x)$, and introduces a dependence on the tenor in the volatility process.

$$h(t) \doteq \sqrt{v_t} R, \quad q(u, T, x)^{i,i} \doteq e^{x\eta^i}, \quad a(s) \doteq a, \quad \kappa(T, x) \doteq \frac{e^{-\gamma x}}{x} \quad (30)$$

where a and R are defined as before. The volatility v_t is defined by (29). Here, we have for the $\sigma_t(u; T, x)$ process

$$\sigma_t(u; T, x) = \sqrt{v_t} R \cdot e^{\eta x - a(u-t)}$$

To better appreciate the difference between the Ch and MP models it is useful to compute the quantity $\frac{P_t(t+x; e)}{P_t(t+x; x)}$, where x is a tenor and

$$P_t(t+x; x) := \frac{1}{1 + xF_t(t+x, x; e)}.$$

By means of equations (24) and (26), we obtain that for the HW model and the Ch model the quantity $\frac{P_t(t+x; e)}{P_t(t+x; x)}$ is deterministic and equals to

$$\frac{P_t(t+x, e)}{P_t(t+x, x)} = \frac{1 + xF_0(t+x, x)}{1 + xE_0(t+x, x)}$$

while for the MP model we have that

$$\frac{P_t(t+x, e)}{P_t(t+x, x)}$$

is a stochastic process. Hence, if we consider the following quantity, which represents the time normalized difference between two forward rate with different tenors,

$$\beta_t(x_1, x_2; e) := \frac{1}{x_2} \log \left(\frac{\frac{1}{x_2} + E_t(t+x_2, x_2; e)}{\frac{1}{x_2} + F_t(t+x_2, x_2; e)} \right) - \frac{1}{x_1} \log \left(\frac{\frac{1}{x_1} + E_t(t+x_1, x_1; e)}{\frac{1}{x_1} + F_t(t+x_1, x_1; e)} \right) \quad (31)$$

we have that in the HW model and in the Ch model $\beta_t(x_1, x_2; e)$ is deterministic while in the MP model is a stochastic quantity. This suggests that the MP model should be able to better capture the dynamics of the basis between two rates with different tenors.

Model	Factors	Pars	Rate	Volatility	Basis
HullWhite	2	5	Stoch.	Constant	Time Dep.
Cheyette	3	12	Stoch.	Stoch.	Time Dep.
MoreniPallavicini	3	14	Stoch.	Stoch.	Stoch.

Table 1: Summary of the characteristics of the models used for the numerical experiments. In particular for each model is displayed the number of stochastic factors, the number of free parameters for the calibration, and the properties of rate, volatility and basis processes.

4.3 Numerical Results

We apply our framework to simple but relevant products: an IRS and a basis swap. We analyze the impact of the choice of an interest rate model on the portfolio valuation, in particular we measure:

1. the dependency of the price on the correlations between interest-rates and credit spreads, the so-called wrong-way risk;
2. the impact of the so called gap risk in an otherwise perfectly collateralized deal, due to the presence of a cure period of δ days.

We model the market risks by simulating the following processes in a multiple curve HJM model under the pricing measure $\mathbb{Q}$. The overnight rate e_t and the Libor forward rates $F_t(T; e)$ are simulated according to the dynamics given in Section 4.1. Maintaining the same notation of the aforementioned section, we choose $N = 2$, and for our numerical experiments we use a HW model, a Ch model and a MP model, all calibrated to swaption at-the-money volatilities listed on the Euro market. The swaption prices in the specified models are computed as shown in [Moreni and Pallavicini(2014)]. Table 1 summarizes the properties of the considered models. As we have already noted, the Ch model introduces a stochastic volatility and hence has an increased number of parameters with respect to the HW model. The MP model aims at better modeling the basis between rates with different tenors, while keeping the model parsimonious in terms of extra parameters with respect to the Ch model.

4.3.1 Calibration Results

The result of the calibration are shown in Figures 1 and 2. From these figures it is clear that the HW model is able to reproduce the at-the-money quotes but is not able to correctly reproduce the volatility smile. On the other hand the introduction of a stochastic volatility process helps recovering the market data smile and thus the Ch and the MP models have similar results in properly fitting the smile.

For what concerns the credit part, the default intensities of the investor and the counterparty are given by two CIR++ processes λ_t^i

$$\begin{aligned}\lambda_t^i &= y_t^i + \psi^i(t) \\ dy_t^i &= \zeta^i(\mu^i - y_t^i) dt + \nu^i \sqrt{y_t^i} dZ_t^{i,c}, \quad i \in \{I, C\}\end{aligned}$$

and they are calibrated to the market data shown in [Brigo et al.(2011)]Brigo, Capponi, Pallavicini, and Papatheo. In particular, two different market settings are used in the numerical examples: the medium

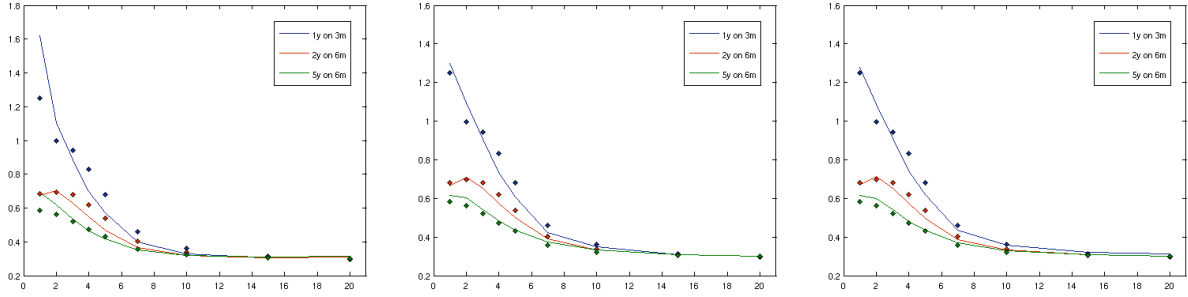


Figure 1: At-the-money swaption volatilities. On the horizontal axis swaption expiries, on the vertical axis swaption volatilities. The curves correspond to swaptions with tenors of 1y (on 3m Euribor rate), and 2y, 5y (on 6m Euribor rate). The dots are market quotes. Left panel: HW model. Central panel: Ch model. Right panel: MP model.

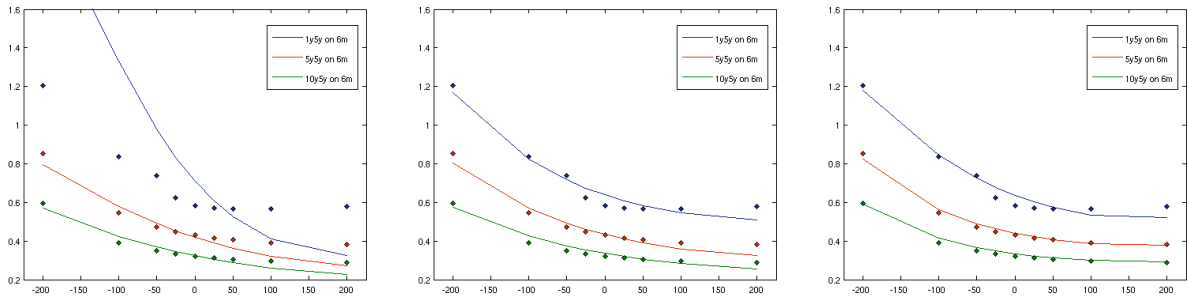


Figure 2: Swaption volatility smile. On the horizontal axis differences between swaption strike rate and underlying forward swap-rate, on the vertical axis swaption volatilities. The curves correspond to swaptions with tenor of 5y and expiries of 1y, 5y and 10y. The dots are market quotes. Left panel: HW model. Central panel: Ch model. Right panel: MP model.

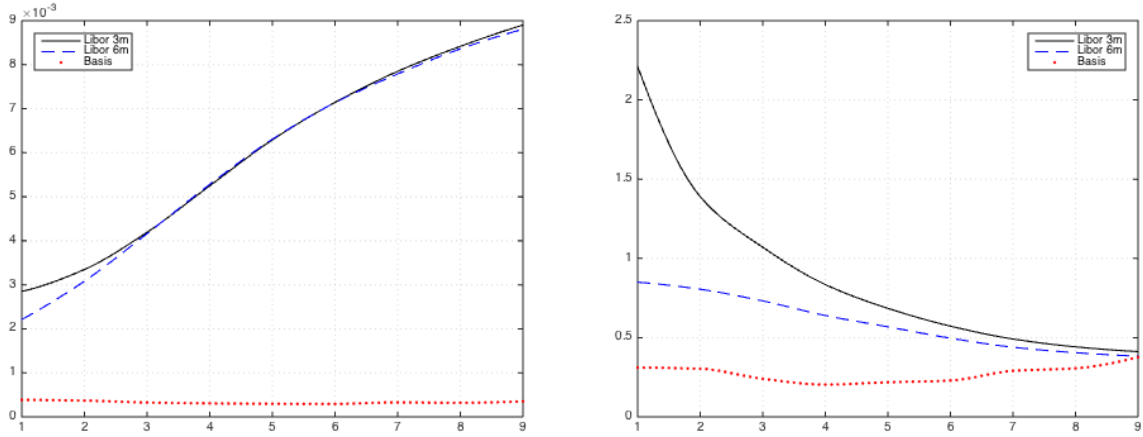


Figure 3: Volatility of 6m and 3m Libor rates and of $\beta_t(x_{3m}, x_{6m}; e)$ the proxy for the 6m/3m Libor basis. On the horizontal axis expiry t , on the vertical axis volatilities obtained via a Monte Carlo simulation. Left panel: normal volatilities. Right panel: log-normal volatilities.

risk and the high risk settings. The correlations among the risky factors are induced by correlating the Brownian motions as in [Brigo and Pallavicini(2007)].

4.3.2 Wrong-Way Risk

The wrong-way risk (WWR) represents how the derivative price changes by varying the dependency between market and credit risks. In our case we calculate WWR for IRS and basis swaps. In the first case the dependency between market and credit risks is expressed in term of the correlation between the default intensities and the overnight rate, while in the second case we use the correlation between β_t , as given by Equation (31), and the default intensities.

We start analyzing the impact of wrong-way risk on the bilateral adjustment, namely CVA plus DVA, of IRS and basis swaps when collateralization is switched off, namely we want to evaluate Equation (11) when

$$\alpha_t \doteq 0.$$

In Figure 3 we show the volatilities of both Libor rates relative to the studied products and the basis β_t , while in Figure 4 we show the wrong way risk (WWR) as variation of the bilateral adjustment with respect to market-credit correlation for a ten years IRS receiving a fix rate yearly and paying 6m Libor twice a year. We limit correlation to low values such as 0.3 also because it is unnatural to expect a higher correlation between a central market rate such as Libor or OIS and a specific counterparty credit spread. From our analysis, it is clear that for a product not subject to the basis dynamics we have that the big difference among the model is the presence of a stochastic volatility. In fact we can see that the Ch model and the MP model are almost indistinguishable while the results of the HW model are far from the stochastic volatility ones. Moreover we can observe that all the models have the same trend, i.e. the bilateral adjustment grows as correlation increase. In fact this can be explained by the fact that an higher correlation means that the deal will be more profitable

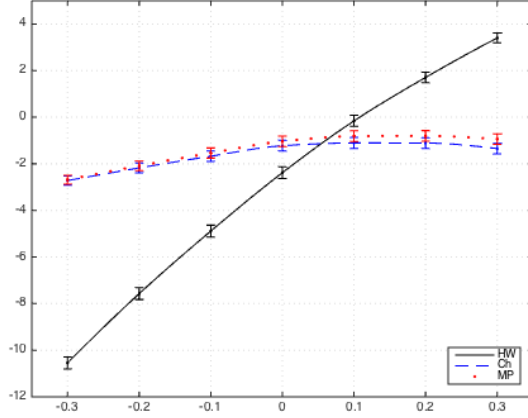


Figure 4: Wrong-way risk for different models. On the horizontal axis correlation among credit and market risks; on the vertical axis the bilateral adjustment, namely CVA + DVA for a 10y IRS receiving a fix rate and paying 6m Libor. Values are in basis points.

when it will be more risky (since we are receiving the fixed rate and paying the floating one), hence the bilateral adjustment will be bigger.

In Figure 5 we show instead the variation of the bilateral adjustment for a ten years basis swap receiving 3m Libor plus spread and paying 6m Libor. In this case we see that as said before the HW model and the Ch model don't have a basis dynamic and hence the curve represented is almost flat. On the other hand the MP model is able to capture the dynamics of the basis and hence we can see that the more the basis is correlated with the credit risk the smaller becomes the bilateral adjustment. In Figure 6 we focus on the MP model and we show how the wrong way risk depends on the correlation between the default intensities and $\beta_t(x_{3m}, x_{6m}; e)$.

4.3.3 Gap Risk

As a final analysis, we study the gap risk of IRS and basis swaps when collateralization is switched on, but initial margin is not exchanged. Namely, we use Equation 16 with various values for α_t . We generate the default times τ_I and τ_C and then we apply an American Monte Carlo at random times to compute the conditional expectations at τ and $\tau + \delta$. As cure period, we set $\delta = 10$ days.

First in Figures 7 and 8 we show how the collateralization procedure impacts the price of our products, i.e. how changes in α_t affect the CVA and DVA component of the price in presence of a delay. In particular, we see that as expected high levels of collateralization reduce default risk adjustments, but even with perfect collateralization we have a residual risk due to the cure period.

Then, we specialized our analysis to the case:

$$\alpha_t \doteq 1.$$

The results for IRS are reported in Figure 9. We see that prices computed with the HW model are flat with respect to the correlation. This happens because, when pricing with full

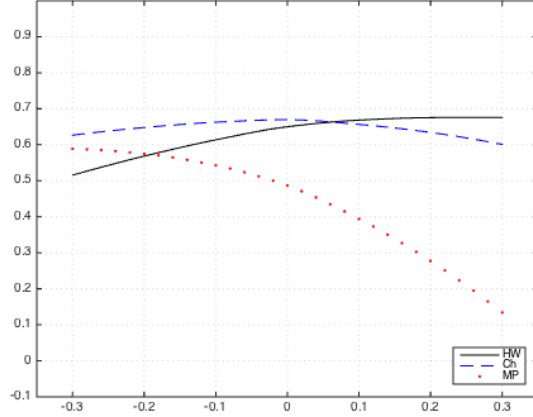


Figure 5: Wrong-way risk for different models. On the horizontal axis correlation among credit and market risks; on the vertical axis the bilateral adjustment, namely CVA + DVA for a 10y basis swap receiving 3m Libor plus spread and paying 6m Libor. Values are in basis points.

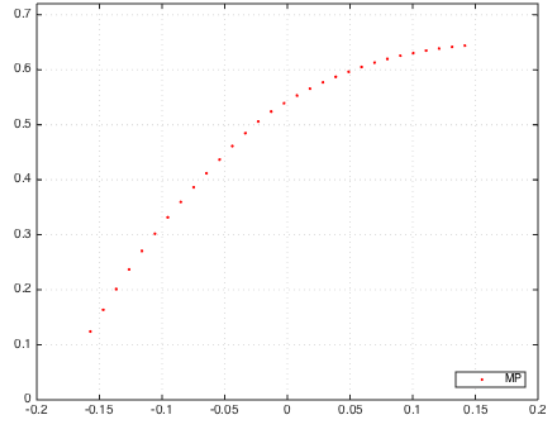


Figure 6: Wrong-way risk for MP model. On the horizontal axis correlation among default intensities and $\beta_t(x_{3m}, x_{6m}; e)$; on the vertical axis the bilateral adjustment, namely CVA + DVA for a 10y basis swap receiving 3m Libor plus spread and paying 6m Libor. Values are in basis points. Right

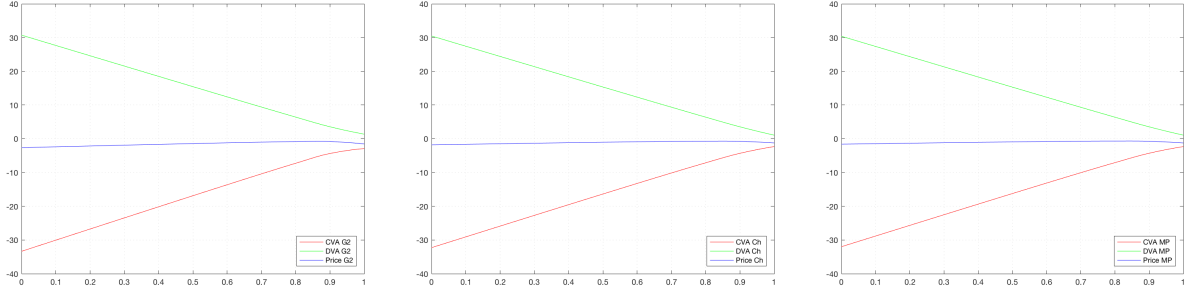


Figure 7: CVA and DVA for an IRS in presence of cure period at various levels of collateralization. On the horizontal α_t , i.e. the fraction of the close-out value covered by the collateral account, on the vertical axis the values of CVA, DVA and the price of the product all expressed in basis points. Left panel: HW model. Central panel: Ch model. Right panel: MP model.

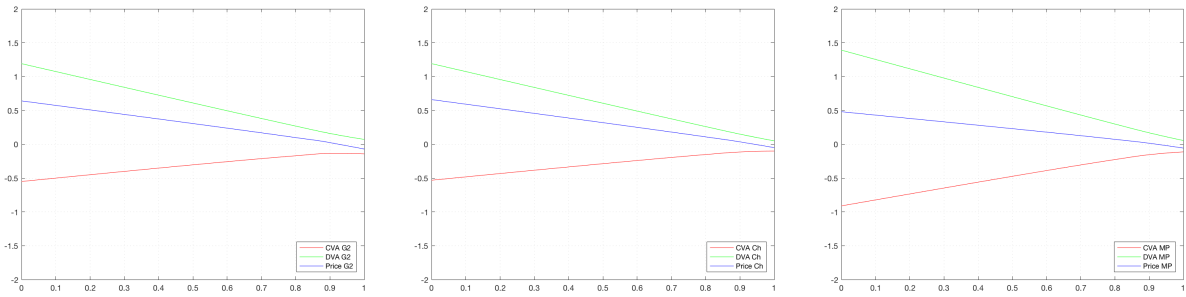


Figure 8: CVA and DVA for a basis swap in presence of cure period at various levels of collateralization. On the horizontal α_t , i.e. the fraction of the close-out value covered by the collateral account, on the vertical axis the values of CVA, DVA and the price of the product all expressed in basis points. Left panel: HW model. Central panel: Ch model. Right panel: MP model.

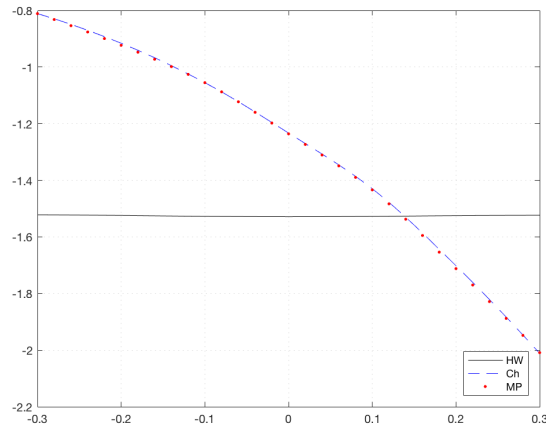


Figure 9: Residual bilateral adjustment due to gap risk. On the horizontal axis correlation among credit and market risks; on the vertical axis the bilateral adjustment, namely CVA + DVA for a 10y IRS receiving a fix rate and paying 6m Libor. Values are in basis points

collateralization in presence of a margin period of risk, the volatility of the product in the cure period is not hedged. Since Libor rates within the HW model have static volatilities, the volatility of the IRS payoff during the cure period is also static. Then, it shows little variation with respect to the changes in correlation among the market and credit risk factors. On the other hand models like Ch and MP, with a dynamic volatility structure, are susceptible to wrong-way risk and they show very similar results.

Lastly, we analyze the gap risk of a basis swap. In Figure 10 we report the result for the MP model. Since the volatility of the basis between the rates is low, the remaining risk is very small too, one order of magnitude smaller than the IRS. The gap risk is so small that one can neglect it and hence there is no need of implementing an initial margin procedure.

5 Conclusions and Perspectives

In this paper we present a detailed analysis of interest rate derivatives valuation under credit risk and collateral modeling. In particular, we follow the approach in [Pallavicini et al.(2011)]Pallavicini, Perini, and to define a pricing framework where we can specify different collateralization policies and we can include the possibility of a delay in the close-out procedure. Within this framework we reformulate the multiple-curve model of [Moreni and Pallavicini(2010)]. In the final numerical section we investigate the impact of a stochastic basis in pricing credit valuation adjustments for IRS and basis swaps.

This work is a starting point to understand which interest-rate products require a stochastic basis model. In particular, we could extend the analysis to compare different stochastic basis models.

In such a context funding costs enter the picture in a more comprehensive way. Some initial suggestions in this respect were given in [Pallavicini and Brigo(2013)] and [Brigo and Pallavicini(2014)].

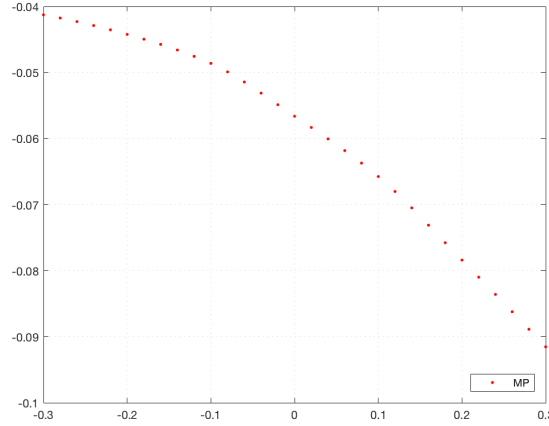


Figure 10: Residual bilateral adjustment due to gap risk. On the horizontal axis correlation among credit and market risks; on the vertical axis the bilateral adjustment, namely CVA + DVA for a 10y basis swap receiving 3m Libor plus spread and paying 6m Libor. Values are in basis points

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